Topology

Stefan Friedl

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[Ae2008] J. M. Aarts. *Plane and solid geometry*, Universitext, Springer Verlag (2008).

[AGK2018] A. Abrams, D. Gay and R. Kirby. *Group trisections and smooth 4-manifolds*, Geom. Topol. 22 (2018), no. 3, 1537–1545.

[Adac1993] M. Adachi. *Embeddings and immersions*, Translations of Mathematical Monographs 124. American Mathematical Society (1993).

[Adam1960] J. F. Adams. *On the non-existence of elements of Hopf invariant one*, Ann. of Math. 72 (1960), 20–104.

[Adam1962] J. F. Adams. *Vector fields on spheres*, Ann. of Math. (2) 75 (1962), 603–632.

[Adam1978] J. F. Adams. *Infinite Loop Spaces*, Ann. of Math. Studies 90, 1978.

[AdA1966] J. F. Adams and M. F. Atiyah. *Theory and the Hopf Invariant*, Quarterly Journal of Mathematics, 17 (1966), 31–38.

[Ade1952] J. Ádem. *The iteration of the Steenrod squares in algebraic topology*, Proceedings of the National Academy of Sciences of the United States of America, 38 (1952), 720–726.

[Ade1957] J. Ádem. *The relations on Steenrod powers of cohomology classes*, Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 191-238. Princeton University Press (1957).

[Ady1955] S. I. Adyan. *Algorithmic unsolvability of problems of recognition of certain properties of groups*, Dokl. Akad. Nauk SSSR (N.S.) 103 (1955), 533–535.
`https://arxiv.org/pdf/2208.08560.pdf`

[AGP2002] M. Aguilar, S. Gitler and C. Prieto. *Algebraic topology from a homotopical viewpoint*, Universitext, Springer Verlag (2002)

[APr1999] M. Aguilar and C. Prieto. *Quasifibrations and Bott periodicity. II.* Iberoamerican Conference on Topology and its Applications (Morelia, 1997). Topology Appl. 98 (1999), 3–17.

[AS1960] L. V. Ahlfors and L. Sario. *Riemann Surfaces*, Princeton Mathematical Series 26, Princeton University Press (1960).

[AZ2014] M. Aigner and G. Ziegler. *Proofs from The Book*, Fifth edition. Springer-Verlag, Berlin (2014).

[AiR2004] I. R. Aitchison and J. H. Rubinstein. *Localising Dehn's lemma and the loop theorem in 3-manifolds*, Math. Proc. Camb. Philos. Soc. 137 (2004), 281–292.

[AkM1990] S. Akbulut and J. D. McCarthy. *Casson's invariant for oriented homology 3-spheres*, volume 36 of Mathematical Notes, Princeton University Press, Princeton, NJ, 1990.

[AK1979] S. Akbulut and R. Kirby. *Mazur manifolds*, Mich. Math. J. 26 (1979), 259–284.

[AP2008] A. Akhmedov and D. Park. *Exotic smooth structures on small 4-manifolds*, Invent. Math. 173 (2008), 209–223.

[AP2010] A. Akhmedov and D. Park. *Exotic smooth structures on small 4-manifolds with odd signatures*, Invent. Math. 181 (2010), 577–603.

[Al1920] J. W. Alexander. *Note on Riemann spaces*, Bull. Amer. Math. Soc. 26, Number 8 (1920), 370–372.

[Al1922] J. W. Alexander. *A proof and extension of the Jordan-Brouwer Separation Theorem*, Trans. Amer. Math. Soc. 23 (1922), 333–349.

[Al1923] J. W. Alexander. *On the deformation of an n-cell*, Proc. Natl. Acad. Sci. USA 9 (1923), 406–407.

[Al1924a] J. W. Alexander. *An example of a simply connected surface bounding a region which is not simply connected*, Proc. Nat. Acad. Sci. USA 10 (1924), 8–10.

[Al1924b] J. W. Alexander. *On the subdivision of space by a polyhedron*, Proc. Nat. Acad. Sci. USA 10 (1924), 6–8.

[Al1930] J. W. Alexander. *The combinatorial theory of complexes*, Ann. of Math. 31 (1930) 292–320.

[AU1929] P. Alexandrov and P. Urysohn. *Mémoire sur les espaces topologiques compacts*, Koninklijke Nederlandse Akademie van Wetenschappen te Amsterdam, Proceedings of the section of mathematical sciences 14 (1929)

[AW1986] N. Alon and D. B. West. *The Borsuk-Ulam theorem and bisection of necklaces*, Proc. Amer. Math. Soc. 98 (1986), 623–628.

[And2005] J. Anderson. *Hyperbolic geometry*, Second edition. Springer Undergraduate Mathematics Series. Springer Verlag (2005)

[AC1965] J. Andrews and M. Curtis. *Free groups and handlebodies*, Proc. Amer. Math. Soc. 16 (1965), 192–195.

[AnR1965] J. Andrews and L. Rubin. *Some spaces whose product with E is E^4*, Bull. Amer. Math. Soc. 71 (1965), 675–677.

[ABK1972] P. Antonelli, D. Burghelea and P. Kahn. *The non-finite homotopy type of some diffeomorphism groups*, Topology 11 (1972), 1–49.

[Ap1987] F. Apéry. *Models of the real projective plane. Computer graphics of Steiner and Boy surfaces*, Computer Graphics and Mathematical Models. Vieweg Verlag (1987).

[Are1946] R. Arens. *Topologies for homeomorphism groups*, Amer. J. Math.68 (1946), 593–610.

[Arf1941] C. Arf. *Untersuchungen über quadratische Formen in Körpern der Charakteristik 2, I*, J. Reine Angew. Math. 183 (1941), 148–167.

[Arm1967] M. A. Armstrong. *Extending triangulations*, Proc. Amer. Math. Soc. 18 (1967), 701–704.

[Arm1970] M. A. Armstrong. *Collars and concordances of topological manifolds*, Comment. Math. Helv. 45 (1970), 119–128.

[Arm1983] M. A. Armstrong. *Basic topology*, Undergraduate Texts in Mathematics, Springer-Verlag (1983).

[Arm1988] M. A. Armstrong. *Groups and symmetry*, Undergraduate Texts in Mathematics, Springer Verlag (1988)

[ArM1991] P. Arnoux and A. Marin. *The Kühnel triangulation of the complex projective plane from the view point of complex crystallography. II*, Mem. Fac. Sci. Kyushu Univ. Ser. A 45 (1991), no. 2, 167–244.

[ARM2017] J. Arredondo and C. Ramírez-Maluendas. *On the Infinite Loch Ness monster*, Comment. Math. Univ. Carolin. 58 (2017), 465–479.

[Art1925] E. Artin. *Zur Isotopie zweidimensionaler Flächen im R_4*, Abh. Math. Seminar Univ. Hamburg 4 (1925), 174–177.

[AFW2015] M. Aschenbrenner, S. Friedl and H. Wilton. *Decision problems for 3-manifolds and their fundamental groups* Baykur, R. Inanc (ed.) et al., Interactions between low dimensional topology and mapping class groups. Geometry and Topology Monographs 19 (2015), 201–236.

[As1998] N. Askitas. *On pairs of locally flat 2-spheres in simply connected 4-manifolds*, Math. Z. 228 (1998), no. 1, 131–153.

[AMB2007] T. Asselmeyer-Maluga and C. H. Brans. *Exotic Smoothness in Physics*, World Scientific Singapore (2007).

[At1961] M. F. Atiyah. *Immersions and embeddings of manifolds*, Topology 1 (1961), 125–132.

[At1987] M. Atiyah. *On the work of Simon Donaldson*, Proc. Int. Congr. Math., Berkeley/Calif. 1986 (1987), 3–6.

[APS1975] M. Atiyah, V. Patodi, I. Singer. *Spectral asymmetry and Riemannian geometry II*, Math. Proc. Camb. Phil. Soc 78 (1975), 405–432.

[Aub2001] T. Aubin. *A course in differential geometry*, Graduate Studies in Mathematics 27. American Mathematical Society (2001).

[Aum1956] R. Aumann. *Asphericity of alternating knots*, Ann. of Math. (2) 64 (1956), 374–392.

[AKM2022] F. Ayaz, M. Kegel and K. Mohnke. *The classification of surfaces via normal curves*, preprint (2022).

[Bad2013] M. Bader. *Space-filling curves. An introduction with applications in scientific computing*, Texts in Computational Science and Engineering 9. Berlin: Springer (2013).

[Baer1928] R. Baer. *Isotopie von Kurven auf orientierbaren, geschlossenen Flächen und ihr Zusammenhang mit der topologischen Deformation der Flächen*, J. reine u. angew. Math. 159 (1928), 101-116.

[Baez2001] J. Baez. *The Octonions*, Bull. Amer. Math. Soc. 39 (2001), 145–205.

[Baez2018] J. Baez. *From the Icosahedron to E_8*, Lond. Math. Soc. Newsl. 476 (2018), 18–23.
[Bai1899] R. Baire. *Sur les fonctions de variables réelles*, Ann. Di Mat. 3 (1899), 1–123.
[Bak2002] A. Baker. *Matrix groups. An introduction to Lie group theory*, Springer Undergraduate Mathematics Series (2002).
[Bal2006] W. Ballmann. *Lectures on Kähler manifolds*, ESI Lectures in Mathematics and Physics. Zürich: European Mathematical Society Publishing House (2006).
[BK2019] R. Bamler and B. Kleiner. *Ricci flow and contractibility of spaces of metrics*, Preprint, arXiv:1909.08710 (2019).
[Ban2007] M. Banagl. *Topological invariants of stratified spaces*, Springer Monographs in Mathematics. Springer (2007).
[BaH2004] A. Banyaga and D. Hurtubise. *Lectures on Morse homology*, Kluwer Texts in the Mathematical Sciences, 29. Kluwer Academic Publishers Group (2004).
[Bard1965] D. Barden. *Simply Connected Five-Manifolds*, Ann. Math. 82 (1965), 365–385.
[BaT2003] D. Barden and C. Thomas. *An introduction to differential manifolds*, Imperial College Press (2003).
[Barm2011] J. Barmak. *Algebraic topology of finite topological spaces and applications*, Lecture Notes in Mathematics, 2032. Springer (2011).
[BaM1962] M. Barratt and J. Milnor. *An example of anomalous singular homology*, Proc. Amer. Math. Soc. 13 (1962), 293–297.
[BW1996] J. W. Barrett and B. W. Westbury. *Invariants of piecewise-linear 3-manifolds*, Trans. Amer. Math. Soc. 348 (1996) 3997–4022.
[Barr1981] J. Barrington. *15 new ways to catch a lion*, in "seven years of manifold 1968-1980", edited by Ian Stewart and John Jaworski, Shiva Publishing (1981).
[BLR2008] A. Bartels, W. Lück and H. Reich. *The K-theoretic Farrell-Jones conjecture for hyperbolic groups*, Invent. Math. 172, No. 1 (2008), 29–70.
[BHS1964] H. Bass, A. Heller and R. G. Swan. *The Whitehead group of a polynomial extension*, Publ. Math., Inst. Hautes Etud. Sci. 22 (1964), 61–80.
[Bats2014] J. Batson. *Nonorientable four-ball genus can be arbitrarily large*, Math. Res. Lett. 21, No. 3 (2014), 423–436.
[Batu1971] J. Batude. *Singularité générique des applications différentiables de la 2-sphère dans une 3-variété différentiable*, Ann. Inst. Fourier (Grenoble) 21 (1971), no. 3, 155–172.
[Bau1993] G. Baumslag. *Topics in combinatorial group theory*, Lectures in Mathematics, ETH Zürich. Basel: Birkhäuser Verlag. (1993).
[BaS1962] G. Baumslag and D. Solitar. *Some two-generator one-relator non-Hopfian groups*, Bull. Amer. Math. Soc. 68 (1962), 199–201.
[BHHM17] M. Behrens, M. Hill, M. Hopkins and M. Mahowald. *Detecting exotic spheres in low dimensions using coker J*, Preprint (2017)
[BeL18] R. Benedetti and P. Lisca. *Framing 3-manifolds with bare hands*, Enseign. Math. (2) Journal Profile 64 (2018), 395–413.
[BP1992] R. Benedetti and C. Petronio. *Lectures on hyperbolic geometry*, Universitext, Springer Verlag (1992)
[Berge1990] J. Berge. *Some knots with surgeries yielding lens spaces*, unpublished preprint (1990). `https://arxiv.org/abs/1802.09722`
[Berger2009] M. Berger. *Geometry. I.*, Universitext, Springer Verlag (2009).
[BG1988] M. Berger and B. Gostiaux. *Differential geometry: manifolds, curves, and surfaces*, Graduate Texts in Mathematics 115, Springer-Verlag (1988).
[Bers1968] I. Berstein. *On imbedding numbers of differentiable manifolds*, Topology 7 (1968), 95–109.
[BB2001] S. Bigelow and R. Budney. *The mapping class group of a genus two surface is linear*, Algebr. Geom. Topol. 1 (2001), 699–708.
[Bin1951] R. H. Bing. *Metrization of topological spaces*, Canadian Journal of Mathematics. 3 (1951), 175–186.

[Bin1952] R. H. Bing. *A homeomorphism between the 3-sphere and the sum of two solid horned spheres*, Ann. of Math. (2) 56 (1952), 354–362.

[Bin1957] R. H. Bing. *Approximating surfaces with polyhedral ones*, Ann. of Math. (2) 65 (1957), 465–483.

[Bin1959] R. H. Bing. *An alternative proof that 3-manifolds can be triangulated*, Ann. of Math. (2) 69 (1959), 37–65.

[Bin1964] R. H. Bing. *Some aspects of the topology of 3-manifolds related to the Poincaré conjecture*, Lectures on Modern Mathematics II, Wiley (1964).

[Bin1983] R. H. Bing. *The geometric topology of 3-manifolds*, American Mathematical Society Colloquium Publications 40, American Mathematical Society (1983).

[Bir1974] J. Birman. *Braids, links, and mapping class groups*, Annals of Mathematics Studies 82. Princeton University Press (1974).

[BGAM1976] J. Birman, F. Gonzalez-Acuna and J. M. Montesinos. *Heegard splittings of prime 3-manifolds are not unique*, Mich. Math. J. 23 (1976), 97-103.

[Bj1995] A. Björner. *Topological methods*, Handbook of combinatorics, Vol. 1, 2, 1819–1872, Elsevier Sci. B. V., Amsterdam, 1995.

[Bj2003003] A. Björner. *Nerves, fibers and homotopy groups*, J. Combin. Theory Ser. A 102 (2003), no. 1, 88–93.

[BlM1952] A. L. Blakers and W. S. Massey. *The homotopy groups of a triad II*, Ann. of Math. 55 (1952), 192–201.

[Blo1979] D. Bloom. *Linear algebra and geometry*, Cambridge University Press (1979).

[Boar2010] M. Boardman. *Some common* Tor *and* Ext *groups*, lecture notes (2010). `https://math.jhu.edu/~jmb/note/torext.pdf`

[Boas1995] R. Boas. *Lion hunting & other mathematical pursuits. A collection of mathematics, verse and stories*, the Dolciani Mathematical Expositions, 15. Mathematical Association of America (1995).

[Bog2008] O. Bogopolski. *Introduction to group theory*, EMS Textbooks in Mathematics (2008).

[BHK2002] C. Bohr, B. Hanke and D. Kotschick. *Cycles, submanifolds, and structures on normal bundles*, Manuscr. Math. 108 (2002), 483–494.

[BF2020] M. Boileau and S. Friedl. *The profinite completion of* 3*-manifold groups, fiberedness and the Thurston norm*, Thurston, Dylan P. (ed.), What's next? The mathematical legacy of William P. Thurston. Princeton University Press. Ann. Math. Stud. 205 (2020), 21-44.

[BCFHLN2019] I. Bokor, D. Crowley, S. Friedl, F. Hebestreit, M. Land and J. Nicholson. *Connected sum decompositions of high-dimensional manifolds*, Preprint (2019)

[Bon2009] F. Bonahon. *Low-dimensional geometry. From Euclidean surfaces to hyperbolic knots*, Student Mathematical Library, 49. IAS/Park City Mathematical Subseries. American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton (2009).

[BoM2008] J. Bondy and U. Murty. *Graph Theory*, Graduate Texts in Mathematics 244, Springer Verlag (2008)

[BD1973] F. F. Bonsall and J. Duncan. *Complete normed algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete 80, Springer Verlag (1973).

[Boo1958] W. Boone. *The word problem*, Proc. Nat. Ac. Sci. 44 (1958), 1061–1065.

[BHP1968] W. Boone, W. Haken and V. Poenaru. *On recursively unsolvable problems in topology and their classification*, 1968 Contributions to Math. Logic (1968), 37–74.

[BoS1953] A. Borel and J.-P. Serre. *Groupes de Lie et puissances réduites de Steenrod*, Amer. J. Math. 75 (1953), 409–448.

[BoS1973] A. Borel and J.-P. Serre. *Corners and arithmetic groups*, Comment. Math. Helv. 48 (1973), 436–491.

[BDR2017] M. Borodzik, S. Dangskul and A. Ranicki. *Solid angles and Seifert hypersurfaces*, Preprint (2017), arXiv:1706.06405.

[BSh1966] Z. I. Borevich and I. R. Shafarevich. *Number theory*, Academic Press (1966).

[Bor1932] K. Borsuk. *Über eine Klasse von lokal zusammenhängenden Räumen*, Fund. Math., 19 (1932), 220–240.

[Bor1933] K. Borsuk. *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fund. Math. 20 (1933), 177–190.

[Bor1948] K. Borsuk. *On the imbedding of systems of compacta in simplicial complexes*, Fund. Math. 35 (1948), 217–234.

[Bor1967] K. Borsuk. *Theory of retracts*, Monografie Matematyczne, Tom 44 Państwowe Wydawnictwo Naukowe (1967).

[Bot1957] R. Bott. *The stable homotopy of the classical groups*, Proc. Nat. Acad. Sci. USA 43 (1957), 933–935.

[Bot1959] R. Bott. *The stable homotopy of the classical groups*, Ann. of Math. 70 (1959), 313–337.

[Bot1970] R. Bott. *The periodicity theorem for the classical groups and some of its applications*, Adv. Math. 4 (1970), 353–411.

[BoT1982] R. Bott and L. Tu. *Differential forms in algebraic topology*, Graduate Texts in Mathematics 82, Sprinter Verlag (1982)

[Bou1954] N. Bourbaki. *Théorie des ensembles*, Éléments de mathématique Premiére Partie, Livre I, Chapitres I, II (1954).

[Bou1998] N. Bourbaki. *General topology. Chapters 1–4*, Elements of Mathematics, Springer Verlag (1998)

[Bou2007] N. Bourbaki. *Elements of mathematics. Algebra. Chapters 1–3*, Springer Verlag (2007).

[Bou2016] N. Bourbaki. *Éléments de mathématique. Topologie algébrique. Chapitres 1 à 4*, Springer Verlag (2016)

[Bou1981] N. Bourbaki. *Groupes et Algébres de Lie*, Chapitres 4, 5 et 6 (1981).

[Boy1903] W. Boy. *Über die Curvatura integra und die Topologie geschlossener Flächen*, Math. Ann. 57 (1903), 151–184.

[Boye2002] S. Boyer. *Dehn surgery on knots*, Handbook of geometric topology, 165–218, North-Holland (2002).

[BTT2020] T.-D. Bradley, B. Tyler and J. Terilla. *Topology. A categorical approach*, MIT Press (2020).

[Bra2007] J. P. Brasselet. *Poincaré-Hopf theorems on singular varieties*, Singularities in geometry and topology, World Sci. Publ. (2007), 57–80.

[Bre1993] G. Bredon. *Geometry and Topology*, Graduate Texts in Mathematics 139, Sprinter Verlag (1993)

[Bre1997] G. Bredon. *Sheaf theory*, second edition, Graduate Texts in Mathematics 170, Springer Verlag (1997).

[BrH2007] M. R. Bridson and J. Howie. *Normalizers in Limit Groups*, Math. Ann. 337 (2007), 385–394.

[BR2020] M. R. Bridson and A. W. Reid. *Profinite rigidity, fibering, and the figure-eight knot*, What's next? The mathematical legacy of William P. Thurston. Princeton University Press. Annals of Mathematics Studies 205 (2020), 45–94.

[BMc2002] G. Brinkmann and B.D. McKay. *Posets on up to 16 points*, Order 19 (2002), 147–179.

[BJ1982] T. Bröcker and K. Jänich. *Introduction to differential topology*, Cambridge University Press. VII (1982).

[BtD1995] T. Bröcker and T. tom Dieck. *Representations of Compact Lie Groups*, Graduate Texts in Mathematics 98, Springer Verlag (1995)

[Brooke2017] A. Brooke-Taylor. *Products of CW complexes*, Preprint (2017).

[Brooks1981] R. Brooks. *Some remarks on bounded cohomology*, In Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), 53–63, Princeton, N.J., 1981. Princeton Univ. Press.

[Brou1911] L. Brouwer. *Beweis der Invarianz der Dimensionenzahl*, Math. Ann. 70 (1911), 161–165.

[Brou1912a] L. Brouwer. *Über Abbildung von Mannigfaltigkeiten*, Math. Ann. 71 (1911), 97–115.

[Brou1912b] L. Brouwer. *Beweis des Jordanschen Satzes fü den n-dimensionalen Raum*, Math. Ann. 71 (1912), 314–319.

[Brou1921] L. Brouwer. *Aufzählung der Abbildungsklassen endlichfach zusammenhängender Flächen*, Math. Ann. 82 (1921), 280–286.

[Brou1976] L. Brouwer. *Collected works*, in: Freudenthal, H. (ed.) Geometry, Analysis, Topology and Mechanics 2, North-Holland Publishing Co (1976)

[Browd1960] W. Browder. *Homology and homotopy of H-spaces*, Proc. Nat. Acad. Sci. U.S.A. 46 (1960), 543–545.

[Browd1961] W. Browder. *Torsion in H-spaces*, Ann. of Math. (2) 74 (1961), 24–51.

[Browd1972] W. Browder. *Surgery on simply connected manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete 65, Springer (1972).

[BrownA1935] A. Brown. *Functional dependence*, Trans. Amer. Math. Soc. 38 (1935), 379–394.

[BrownE1962] E. H. Brown. *Cohomology theories*, Ann. Math. 75 (1962), 467-484.

[BrownM1960] M. Brown. *A proof of the generalized Schoenflies theorem*, Bull. Amer. Math. Soc. 66 (1960), 74–76.

[BrownM1962] M. Brown. *Locally flat embeddings of topological manifolds*, Topology of 3-Manifolds and Related Topics (Proc. Univ. of Georgia Inst., 1961), Prentice-Hall, Englewood Cliffs, N. J. (1962), 83-91.

[BrownR1963] R. F. Brown. *Ten topologies for $X \times Y$*, Quart. J.Math. (2) 14 (1963), 303–319.

[BrownR1968] R. F. Brown. *Elements of modern topology*, McGraw-Hill Book Co. (1968).

[BrownR1971] R. F. Brown. *The Lefschetz fixed point theorem*, Scott, Foresman and Company VI (1971).

[BrownR2006] R. F. Brown. *Topology and groupoids*, 3rd revised, updated and extended ed. Published by groupoids.org (2006).
`http://groupoids.org.uk/pdffiles/topgrpds-e.pdf`

[Bruc1944] R. H. Bruck. *Some results in the theory of linear nonassociative algebras*, Trans. Amer. Amer. Math. SOC. 56 (1944), 141–199.

[Brun2005] M. Brunnbauer. *Über eine Signaturinvariante für Mannigfaltigkeiten mit symmetrischem Rand*, Diploma thesis, University of Regensburg (2005).
`https://www.maths.ed.ac.uk/~v1ranick/papers/brunnbauer.pdf`

[Bry2002] J. L. Bryant. *Piecewise linear topology*, Daverman, R. J. (ed.) et al., Handbook of geometric topology. Amsterdam: Elsevier (2002), 219–260.

[BrM1999] J. L. Bryant and W. Mio. *Embeddings of homology manifolds in codimension ≥ 3*, Topology 38 (1999), 811–821.

[Bud2007] R. Budney. *Little cubes and long knots*, Topology 46 (2007), 1–27.

[BuM1982] S. R. Bullett and I. G. MacDonald. *On the Adem relations*, Topology 21 (1982), 329–332.

[BuS2010] U. Bunke and T. Schick. *Uniqueness of smooth extensions of generalized cohomology theories*, J. Topol. 3 (2010), 110–156.

[BZH2014] G. Burde, H. Zieschang and M. Heusener. *Knots*, 3rd fully revised and extended edition, De Gruyter Studies in Mathematics 5 (2014).

[BuL1974] D. Burghelea and R. Lashof. *The homotopy type of the space of diffeomorphisms. II*, Trans. Amer. Math. Soc. 196 (1974), 37–50. De Gruyter Studies in Mathematics 5 (2014).

[Cai1934] S. S. Cairns. *On the Triangulation of Regular Loci*, Ann. Math. 35, (1934), 579–587.

[Cai1961] S. S. Cairns. *A simple triangulation method for smooth manifolds*, Bull. Amer. Math. Soc. 67 (1961), 389–390.

[Cal2009] D. Calegari. *scl - stable commutator length*, Mathematical Society of Japan Monographs (2009)

[Cal2014] D. Calegari. *Notes on 3-manifolds*, lecture notes (2014)
`https://math.uchicago.edu/~dannyc/courses/3manifolds_2014/3_manifolds_notes.pdf`

[CR1980] C. M. Campbell and E. F. Robertson. *A Deficiency Zero Presentation for* $\mathrm{SL}(2,p)$, Bull. Lond. Math. Soc. 12 (1980), 17–20.

[Cam1990] R. Campbell. *Notes on the Baumslag-Solitar non-residually finite examples*, Proc. Amer. Math. Soc. 109 (1990), no. 1, 59–62.

[CgRR2008] T. Camps, V. große Rebel and G. Rosenberger. *Einführung in die kombinatorische und die geometrische Gruppentheorie*, Berliner Studienreihe zur Mathematik 19. Heldermann Verlag (2008).

[Cann1979] J. W. Cannon. *Shrinking cell-like decompositions of manifolds. Codimension three*, Annals of Mathematics 110 (1979), 83–112.

[Cann1994] J. W. Cannon. *Mathematical Parables*, BYU Studies Quarterly: Vol. 34 : Iss. 4 , Article 5.
`https://scholarsarchive.byu.edu/byusq/vol34/iss4/5`

[CFP1996] J. W. Cannon, W. J. Floyd and W. R. Parry. *Introductory notes on Richard Thompson's groups*, Enseign. Math. (2) 42 (1996), no. 3-4, 215–256.

[Cant1883] G. Cantor. *Über unendliche, lineare Punktmannigfaltigkeiten*, Math. Ann. 21 (1883), 545–586.

[Cant1892] G. Cantor. *Über eine elementare Frage der Mannigfaltigskeitslehre*, Deutsche Math. Ver. I (1892), 75–78.

[Cap1974] S. E. Cappell. *On connected sums of manifolds*, Topology 13 (1974), 395–400.

[Cap1976] S. E. Cappell. *A splitting theorem for manifolds*, Inventiones Math. 33 (1976), 69–170.

[Cara1913] C. Carathéodory. *Über die gegenseitige Beziehung der Ränder bei der konformen Abbildung des Inneren einer Jordanschen Kurve auf einen Kreis*, Math. Ann. 73 (1913), 305–320.

[Cara1954] C. Carathéodory. *Theory of functions of a complex variable. Vol. II*, Chelsea Publishing Company (1954).

[Carl2009] G. Carlsson. *Topology and data*, Bull. Amer. Math. Soc. 46 (2009), 255–308.

[Caro2000] N. L. Carothers. *Real analysis*, Cambridge University Press (2000).

[Carta1936] E. Cartan. *La topologie des groupes de Lie*, Actualités Scientifiques et Industrielles, no. 358, Paris, 1936, also L'Enseignement Mathématique vol. 35 (1936), 177–200.

[Carta1945] E. Cartan. *Les systèmes différentiels exterieurs et leurs applications geométriques*, Actualités Scientifiques et Industrielle 994. Hermann & Cie (1945).

[Carte1995] S. Carter. *How surfaces intersect in space*, An introduction to topology. Second edition. Series on Knots and Everything, 2. World Scientific Publishing Co. (1995)

[CaG1986] A. Casson and C. Gordon. *Cobordism of classical knots*, Progr. Math. 62, A la recherche de la topologie perdue, 181-199, Birkhäuser, Boston (1986).

[Cat2003] F. Catanese. *Fibred Kähler and quasi-projective groups*, Adv. Geom. 2003, Spec. Issue (2003), 13–27.

[Cau1992] R. Cauty. *Sur les ouverts des CW-complexes et les fibrés de Serre*, Colloq. Math. 63 (1992), no. 1, 1–7.

[CSC2010] T. Ceccherini–Silberstein and M. Coornaert. *Cellular Automata and Groups*, Springer Monographs in Mathematics, Springer Verlag (2010)

[Čec1932] E. Čech. *Höherdimensionale Homotopiegruppen*, Verhandlungen Kongreß Zürich 1932, Vol. 2 (1932), 203.

[Cer1961] J. Cerf. *Topologie de certains espaces de plongements*, Bulletin de la Soc. Math. France 89 (1961), 227–380.

[Cer1968] J. Cerf. *Sur les difféomorphismes de la sphère de dimension trois* ($\Gamma_4 = 0$), Lecture Notes in Mathematics 53, Springer Verlag (1968)

[Cer1970] J. Cerf. *La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie*, Publ. Math., Inst. Hautes Étud. Sci. 39 (1970), 5–173.

[Chab1950] C. Chabauty. *Limite d'ensembles et géométrie des nombres*, Bulletin de la Société Mathématique de France 78 (1950), 143–151.

[CM1982] B. Chandler and W. Magnus. *The history of combinatorial group theory*, A case study in the history of ideas. Studies in the History of Mathematics and Physical Sciences, 9. Springer-Verlag, (1982).

[Chan1993] K.-C. Chang. *Infinite-dimensional Morse theory and multiple solution problems*, Progress in Nonlinear Differential Equations and their Applications. 6. Birkhäuser (1993).

[CW2021] S. Chang and S. Weinberger. *A Course on Surgery Theory*, Annals of Mathematics Studies 211, Princeton University Press (2021).

[Chap1974] T. Chapman. *Topological invariance of Whitehead torsion*, Am. J. Math. 96 (1974), 488–497.

[Char2019] G. Charalambous. *Dimension theory. A selection of theorems and counterexamples*, Atlantis Studies in Mathematics 7. Springer/Atlantis Press (2019).

[CK1970] J. Cheeger and J. M. Kister. *Counting topological manifolds*, Topology 9 (1970), 149–151.

[Che1979] S. S. Chern. *Complex manifolds without potential theory*, (With an appendix on the geometry of characteristic classes), Second edition. Universitext. Springer-Verlag (1979).

[Cherna1969a] A. V. Chernavskii. *Local contractibility of the group of homeomorphisms of a manifold*, Math. USSR-Sb., 8 : 3 (1969), 287–333 Mat. Sb., 79 (121) : 3 (1969), 307–356.

[Cherna1969b] A. V. Chernavskii. *Topological embeddings of manifolds*, Dokl. Akad. Nauk SSSR, 187:6 (1969), 1247–1250. Sov. Math. 10, No. 4, (1969).

[Cherna2008] A. V. Chernavskii. *Local contractibility of the homeomorphism group of* $\mathbb{R}^n$, Proc. Steklov Inst. Math. 263 (2008), no. 1, 189–203. Universitext, Springer Verlag (1995).

[Chf1992] P. R. Chernoff. *Simple Proof of Tychonoff's Theorem Via Nets*, The American Mathematical Monthly 99 (1992), 932–934.

[Chic1980] G. Chichilnisky. *Social choice and the topology of spaces of preferences*, Adv. in Math. 37 (1980), no. 2, 165–176.

[Chig1996] A. Chigogidze. *Inverse spectra*, North-Holland Mathematical Library 53. North-Holland Publishing Co. (1996).

[Chis1999] I. M. Chiswell. *Length functions and* Λ*-trees*, Summer School in Group Theory in Banff, 1996, 11–44, CRM Proc. Lecture Notes, 17, Amer. Math. Soc., Providence, RI, 1999.

[Chr1950] C. M. Christensen. *A square inscribed in a convex figure*, Matematisk Tidsskrift B 1950 (1950), 22–26.

[Cib2016] D. Cibotaru. *A short proof of a theorem by Hopf*, Expo. Math. 34 (2016), 335–338.

[Cie1997] K. Ciesielski. *Set theory for the working mathematician*, London Mathematical Society Student Texts 39. Cambridge University Press (1997).

[Cîr2021] F. C. Cîrstea. *Proofs of Urysohn's Lemma and the Tietze Extension Theorem via the Cantor function*, Bull. Aust. Math. Soc 103 (2021), 326–332.

[CoheD1957] D. Cohen. *Products and carrier theory*, Proc. London Math. Soc. (3) 7 (1957), 219–248.

[CoheD1989] D. Cohen. *Combinatorial group theory: a topological approach*, London Mathematical Society Student Texts, 14. Cambridge University Press (1989).

[CoheM1973] M. Cohen. *A course in simple-homotopy theory*, Graduate Texts in Mathematics 10, Springer Verlag (1973).

[CELR1994] R. Cohen, P. Eades, T. Lin and F. Ruskey. *Three-dimensional graph drawing*, in Graph Drawing: DIMACS International Workshop, GD '94 Princeton, New Jersey, USA, October 10–12, 1994, Proceedings, Lecture Notes in Computer Science, 894, Springer (1994), 1–11.

[CKMRV2017] H. Cohn, A. Kumar, S. Miller, D. Radchenko and M. Viazovska. *The sphere packing problem in dimension 24*, Ann. of Math. (2) 185 (2017), 1017–1033.

[Cohn1956] P. M. Cohn. *The complement of a finitely generated direct summand of an abelian group*, Proc. Amer. Math. Soc. 7 (1956), 520–521.

[Cohn2003] P. M. Cohn. *Basic algebra. Groups, rings and fields*, Springer-Verlag (2003).

[CZ1993] D. J. Collins and H. Zieschang. *Combinatorial group theory and fundamental groups*, in: Algebra, VII, pp. 1–166, 233–240, Encyclopaedia of Mathematical Sciences, vol. 58, Springer, Berlin, 1993.

[Conn1971] R. Connelly. *A new proof of Brown's collaring theorem*, Proc. Am. Math. Soc. 27 (1971), 180–182.

[CFH2016] A. Conway, S. Friedl and G. Herrmann. *Linking forms revisited*, Pure and Applied Mathematics Quarterly 12 (2016), 493–515.

[Conw1970] J. H. Conway. *An enumeration of knots and links, and some of their algebraic properties*, Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967), Pergamon (1970), 329–358.

[Conw78] J. H. Conway. *Functions of One Complex Variable I*, Graduate Texts in Mathematics 159, Springer Verlag (1978).

[CCNPW1985] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson. *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups*, Oxford University Press (1985).

[CoG1983] J. H. Conway and C. McA. Gordon. *Knots and links in spatial graphs*, J. Graph Th. 7 (1983), 446–453.

[CSl1999] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, third edition. Fundamental Principles of Mathematical Sciences 290. Springer Verlag (1999)

[CSm2003] J. H. Conway and D. A. Smith. *On quaternions and octonions: their geometry, arithmetic, and symmetry*, Natick, MA: A K Peters (2003).

[CLOT2003] O. Cornea, G. Lupton, J. Oprea and D. Tanré. *Lusternik-Schnirelmann category* Mathematical Surveys and Monographs 103. American Mathematical Society (2003).

[CSB1954] E. Corominas and F. Sunyer Balaguer. *Conditions for an infinitely differentiable function to be a polynomial*, Rev. Mat. Hisp.-Amer. (4) 14 (1954), 26–43.

[Cox1948] H. S. M. Coxeter. *Regular polytopes*, Methuen & Co (1948).

[CF1977] R. Crowell and R. Fox. *Introduction to knot theory*, Graduate Texts in Mathematics, No. 57. Springer-Verlag (1977).

[CLM2021] D. Crowley, W. Lück and T. Macko. *Surgery Theory: Foundations*, preprint (2021). `https://www.him.uni-bonn.de/lueck/data/sb.pdf`

[CGLS1987] M. Culler, C. Gordon, J. Luecke, P. Shalen. *Dehn surgery on knots*, Ann. of Math. (2) 125 (1987), 237–300.

[DLP1980] Z. D. Dai, T. Y. Lam and C. K. Peng. *Levels in algebra and topology*, Bull. Amer. Math. Soc. 3 (1980), 845–848.

[Dav1986] R.J. Daverman. *Decompositions of Manifolds*, Academic Press, Orlando (1986).

[DV2009] R. Daverman and G. Venema. *Embeddings in manifolds*, Graduate Studies in Mathematics 106. American Mathematical Society (2009).

[DaK2001] J. Davis and P. Kirk. *Lecture notes in algebraic topology*, Graduate Studies in Mathematics. 35. Providence, RI: AMS, American Mathematical Society (2001).

[DaP2015] J. F. Davis and N. Petrosyan. *Manifolds and Poincaré complexes*, lecture notes (2015) `http://www.indiana.edu/~jfdavis/notes/dp.pdf`

[DGMM2019] J. De Loera, X, Goaoc, F. Meunier and N. Mustafa. *The discrete yet ubiquitous theorems of Carathéodory, Helly, Sperner, Tucker, and Tverberg*, Bull. Am. Math. Soc., New Ser. 56, No. 3 (2019), 415–511.

[Ded1962] P. Dedecker. *Variétés linéaires par morceaux et variétés combinatoires*, Ann. Mat. Pura Appl. (4) 60 (1962), 365–383.

[Deh1910] M. Dehn. *Über die Topologie des dreidimensionalen Raumes*, Math. Ann. 69 (1910), no. 1, 137-168.

[Deh1911] M. Dehn. *Über unendliche diskontinuierliche Gruppen*, Math. Ann. 71 (1911), 116–144.

[Deh1912] M. Dehn. *Transformation der Kurven auf zweiseitigen Flächen*, Math. Ann. 72 (1912), 413–421.

[Deh1938] M. Dehn. *Die Gruppe der Abbildungsklassen*, Acta Math. 69 (1938), 135–206.

[Deh1987] M. Dehn. *Papers on Group Theory and Topology*, Springer, New York (1987).

[Deo2018] S. Deo. *Algebraic Topology: A Primer*, Texts and Readings in Mathematics 27, Second Edition, Springer (2018).

[Der1966] J. Derwent. *Handle decompositions of manifolds*, J. Math. Mech. 15 (1966), 329–345.

[Dev1993] K. Devlin. *The joy of sets*, Fundamentals of contemporary set theory. Second edition. Undergraduate Texts in Mathematics. Springer-Verlag (1993).

[tD2000] T. tom Dieck. *Topologie*, 2., völlig neu bearb. und erw. Aufl. de Gruyter Lehrbuch. de Gruyter (2000).

[tD2008] T. tom Dieck. *Algebraic topology*, EMS Textbooks in Mathematics. (2008).

[tDKP1970] T. tom Dieck, K. H. Kamps and D. Puppe. *Homotopietheorie*, Lecture Notes in Mathematics 157. Springer Verlag (1970).

[Die1944] J. Dieudonné. *Une généralisation des espaces compacts*, Journal de Mathématiques Pures et Appliquées, Neuvième Série 23 (1944), 65–76.

[Dim2004] A. Dimca. *Sheaves in topology*, Universitext. Springer-Verlag (2004).

[Dij2005] J. Dijkstra. *On homeomorphism groups and the compact open topology*, Am. Math. Mon. 112 (2005), 910–912.

[doC2016] M. do Carmo. *Differential geometry of curves and surfaces*, Revised and updated 2nd edition of the 1976 edition published by Prentice-Hall. Dover Publications (2016).

[Dol1956] A. Dold. *Erzeugende der Thomschen Algebra $\mathcal{N}$*, Math. Z. 65 (1956), 25–35.

[Dol1980] A. Dold. *Lectures on algebraic topology*, Springer Verlag (1980)

[Don1983] S. K. Donaldson. *An application of gauge theory to four-dimensional topology*, J. Diff. Geom. 18 (1983), 279–315.

[Don1987a] S. Donaldson. *The orientation of Yang–Mills moduli spaces and 4-manifold topology*, J. Differential Geometry 26 (1987), 397–428.

[Don1987b] S. Donaldson. *Irrationality and the h-cobordism conjecture*, J. Differential Geom. 26 (1987), 141–168.

[DR2018] C. L. Douglas and D. J. Reutter. *Fusion 2-categories and a state-sum invariant for 4-manifolds*, Preprint, arXiv: 1812.11933 (2018)

[Dow1952] C. Dowker. *Topology of metric complexes*, Amer. J. Math. 74 (1952), 555–577.

[DoP2009] A. K. Doxiàdis and C. Papadimitriou. *Logicomix*, Bloomsbury Publishing (2009).

[DM1968] P. H. Doyle and D. A. Moran. *A short proof that compact 2-manifolds can be triangulated*, Invent. Math. 5 (1968), 160–162.

[DrK2018] C. Druţu and M. Kapovich. *Geometric group theory*, With an appendix by Bogdan Nica. American Mathematical Society Colloquium Publications, 63. American Mathematical Society (2018).

[DFN1985] B. A. Dubrovin, A. T. Fomenko and S. P. Novikov. *Modern Geometry - Methods and Applications: Part II: The Geometry and Topology of Manifolds*, Graduate Texts in Mathematics 104, Springer Verlag (1985)

[Dug1951] J. Dugundji. *An extension of Tietze's theorem*, Pac. J. Math 1 (1951), 353-367.

[Dug1966] J. Dugundji. *Topology*, Allyn & Bacon (1966).

[DF2004] D. Dummit and R. Foote. *Abstract algebra*, 3rd ed. Wiley International Edition (2004).

[Dun2018] B. I. Dundas. *A short course in differential topology*, Cambridge Mathematical Textbooks. Cambridge University Press (2018).

[Dup1978] J. L. Dupont. *Curvature and characteristic classes*, Lecture Notes in Mathematics 640. Springer Verlag, Berlin, 1978.

[DGGK2020] S. Durst, H. Geiges, J. Gonzalo Pérez and M. Kegel. *Parallelisability of 3-manifolds via surgery*, Expo. Math. 38 (2020), 131–137.

[DGK2019] S. Durst, H. Geiges and M. Kegel. *Handle homology of manifolds*, Topology Appl. 256 (2019), 113–127.

[Dye1978] R. H. Dye. *On the Arf invariant*, J. Algebra 53 (1978), no. 1, 36–39.

[DL61] E. Dyer and R. Lashof. *A topological proof of the Bott periodicity theorems*, Ann. Mat. Pura Appl. (4) 54 (1961), 231–254.

[EHHKMNPR1991] H.-D. Ebbinghaus, H. Hermes, F. Hirzebruch, M. Koecher, K. Mainzer, J. Neukirch, A. Prestel and R. Remmert. *Numbers*, Graduate Texts in Mathematics 123, Springer Verlag (1991).

[Ebe2007] W. Ebeling. *Functions of several complex variables and their singularities*, Graduate Studies in Mathematics 83. American Mathematical Society (2007).

[Eckh1993] J. Eckhoff. *Helly, Radon, and Carathéodory type theorems*, Handbook of convex geometry, Vol. A, B, 389–448, North-Holland, Amsterdam, 1993.

[Eckm1954] B. Eckmann. *Räume mit Mittelbildungen*, Comment. Math. Helv. 28 (1954), 329–340.

[Eckm2004] B. Eckmann. *Social Choice and Topology. A Case of Pure and Applied Mathematics*, Exp. Math. 22 (2004), 385–393.

[Eckm2006] B. Eckmann. *Is algebraic topology a respectable field?*, in Mathematical survey lectures 1943–2004, Springer (2006).

[EGH1962] B. Eckmann, T. Ganea and P. J. Hilton. *Generalized means*, Studies in mathematical analysis and related topics, Stanford Univ. Press (1962), 82-92.

[EcH1962] B. Eckmann and P. J. Hilton. *Group-like structures in general categories. I. Multiplications and comultiplications*, Math. Ann. 145 (1962), 227–255.

[EKa2000] K. Eda and K. Kawamura. *The singular homology of the Hawaiian earring*, J. London Math. Soc. 62 (2000), 305–310.

[EdH2010] H. Edelsbrunner and J. Harer. *Computational Topology: An Introduction*, Amer. Math. Soc. (2010)

[Edm2020] A. Edmonds. *Aspherical 4-manifolds of odd Euler characteristic*, Proc. Am. Math. Soc. 148, No. 1 (2020), 421–434.

[Edw1980] R. D. Edward. *The topology of manifolds and cell-like maps*, Proceedings of the International Congress of Mathematicians, Helsinki, 1978 ed. O. Lehto, Acad. Sci. Fenn (1980), 111–127.

[Edw1984] R. D. Edwards. *The solution of the 4-dimensional Annulus conjecture* (*after Frank Quinn*), in "Four-manifold Theory", Gordon and Kirby ed., Contemporary Math. 35 (1984), 211–264.

[Edw2006] R. D. Edwards. *Suspensions of homology spheres*, unpublished manuscripts from the 1970's, arXiv:math/0610573.

[Edw2009] R. D. Edwards. *Topological regular neighborhoods*, unpublished paper from the 1970s, arXiv:0904.4665.

[EKi1971] R. D. Edwards and R. C. Kirby. *Deformations of spaces of imbeddings*, Ann. of Math. (2) 93 (1971), 63–88.

[Eil1949] S. Eilenberg. *On the Problems of Topology*, Ann. Math. 50 (1949), 247–260.

[EMc1945] S. Eilenberg and S. Mac Lane. *General Theory of Natural Equivalences*, Trans. Amer. Math. Soc. 58 (1945), 231-294.

[EMc1953] S. Eilenberg and S. MacLane. *On the groups $H(\Pi, n)$. I*, Ann. of Math. (2) 58 (1953), 55–106.

[ES1952] S. Eilenberg and N. Steenrod. *Foundations of Algebraic Topology*, Princeton Univ. Press, Princeton, New Jersey (1952).

[EN1985] D. Eisenbud and W. Neumann. *Three-dimensional link theory and invariants of plane curve singularities*, Annals of Mathematics Studies, 110. Princeton University Press (1985).

[EG1979] M. Eisenberg and R. Guy. *A proof of the hairy ball theorem*, Am. Math. Mon. 86 (1979), 572–574.

[Eis2007] M. Eisermann. *Knot colouring polynomials*, Pac. J. Math. 231 (2007), 305–336.

[Eli1992] Y. Eliashberg. *Contact 3-manifolds twenty years since J. Martinet's work*, Ann. Inst. Fourier (Grenoble) 42 (1992), 165-192.

[Engm1970] R. Engmann. *Nicht-homöomorphe Heegaard-Zerlegungen vom Geschlecht* 2 *der zusammenhängenden Summe zweier Linsenräume*, Abh. Math. Sem. Univ. Hamburg 35 (1970), 33–38.

[Enge1978] R. Engelking. *Dimension theory*, North-Holland Mathematical Library 19. Polish Scientific Publishers (1978).

[Enge1989] R. Engelking. *General topology*, Rev. and compl. ed. Sigma Series in Pure Mathematics 6. Heldermann Verlag (1989).

[Enge1995] R. Engelking. *Theory of dimensions, finite and infinite*, Sigma Series in Pure Mathematics 10. Heldermann Verlag (1995).

[Epp1999] M. Epple. *Die Entstehung der Knotentheorie. Kontexte und Konstruktionen einer modernen mathematischen Theorie*, Vieweg Verlag (1999)

[Eps1961] D. B. A. Epstein. *Finite presentations of groups and 3-manifolds*, Quart. J. Math. Oxford Ser. (2) 12 (1961), 205–212.

[Eps1966] D. B. A. Epstein. *Curves on* 2*-manifolds and isotopies*, Acta Math. 115 (1966), 83–107.

[EF1997] D. B. A. Epstein and K. Fujiwara. *The second bounded cohomology of work-hyperbolic groups*, Topology 36 (1997), 1275–1289.

[Er1969] D. Erle. *Quadratische Formen als Invarianten von Einbettungen der Kodimension zwei*, Topology 8 (1969), 99–114.

[Eul1736] L. Euler. *Solutio problematis ad geometriam situs pertinentis*, Comment. Acad. Sci. U. Petrop 8 (1736), 128–40.
`http://eulerarchive.maa.org//docs/originals/E053.pdf`

[Eul1758] L. Euler. *Elementa doctrinae solidorum*, Novi Commentarii academiae scientiarum Petropolitanae (1758), 109–140.
`https://scholarlycommons.pacific.edu/euler-works/230/`

[Eve1963] L. Evens. *A generalization of the transfer map in the cohomology of groups*, Trans. Am. Math. Soc. 108 (1963), 54–65.

[Ewa1986] G. Ewald. *Über stellare Unterteilung von Simplizialkomplexen*, Arch. Math. 46 (1986), no. 2, 153–158.

[Fab2005] P. Fabel. *The topological Hawaiian earring group does not embed in the inverse limit of free groups*, Algebr. Geom. Topol. 5 (2005), 1585–1587.

[Fab2011] P. Fabel. *Multiplication is discontinuous in the Hawaiian earring group (with the quotient topology)*, Bull. Pol. Acad. Sci. Math. 59 (2011), no. 1, 77–83.

[FaM2011] B. Farb and D. Margalit. *A primer on mapping class groups*, Princeton Mathematical Series. Princeton, NJ: Princeton University Press (2011).

[Far2004] M. Farber. *Topology of closed one-forms*, Mathematical Surveys and Monographs 108. American Mathematical Society (2004)

[Far2008] M. Farber. *Invitation to topological robotics*, Zurich Lectures in Advanced Mathematics. Zürich: European Mathematical Society (EMS) (2008).

[FaH1981] F. Farrell and W.-C. Hsiang. *The Whitehead group of poly-(finite or cyclic) groups*, J. London Math. Soc. 24 (1981), 308–324.

[FJ1986] F. Farrell and L. Jones. *K-theory and dynamics*, I, Ann. of Math. (2) 124 (1986), 531–569.

[FT1943] L. Fejes Tóth. *Über die dichteste Kugellagerung*, Math. Z. 48 (1943), 676–684.

[FT1972] L. Fejes Tóth. *Lagerungen in der Ebene, auf der Kugel und im Raum*, 2. Auflage, Grundlehren der mathematischen Wissenschaften 65 (1972).

[Fen1970] R. Fenn. *The table theorem*, Bull. Lond. Math. Soc. 2 (1970), 73–76.

[Fen2001] R. Fenn. *Geometry*, Springer Undergraduate Mathematics Series. Springer Verlag (2001).

[Fera2004] A. Feragen. *A topological manifold is homotopy equivalent to some CW-complex*, University of Copenhagen (2004).
`http://image.diku.dk/aasa/oldpage/aasa.pdf`

[FePi2011] D. Ferrario and R. Piccinini. *Simplicial structures in topology* CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. (2011)

[FeR2001] S. Ferry and A. Ranicki. *A survey of Wall's finiteness obstruction*, Surveys on surgery theory, Vol. 2, 63–79, Ann. of Math. Stud., 149, Princeton Univ. Press (2001).

[Feru2007] D. Ferus. *Analysis III*, lecture notes at TU Berlin, winter semester 2007/8.
`http://page.math.tu-berlin.de/~ferus/ANA/Ana3.pdf`

[Fi1947] N. J. Fine. *Binomial Coefficients Modulo a Prime*, The American Mathematical Monthly 54 (1947), 589–592.

[FS1984] R. Fintushel and R. Stern. *A μ-invariant one homology 3-sphere that bounds an orientable rational ball*, Contemp. Math. 35 (1984), 265–268.

[FS1998] R. Fintushel and R. Stern. *Knots, links, and 4-manifolds*, Invent. Math. 134 (1998), 363–400.

[FiG1991] P. A. Firby and C. F. Gardiner. *Surface topology*, Second edition. Ellis Horwood Series: Mathematics and its Applications (1991).

[FZ2007] H. Fischer and A. Zastrow. *Generalized universal covering spaces and the shape group*, Fundam. Math. 197 (2007), 167–196.

[Flo2020] R. Floris. *Topological manifolds are Euclidean neighborhood retracts*, University of Bonn (2020).
`http://people.mpim-bonn.mpg.de/aruray/documents/top_manifolds.pdf`

[Fol1999] G. Folland. *Real analysis. Modern techniques and their applications*, 2nd ed. Pure and Applied Mathematics. A Wiley-Interscience Series of Texts, Monographs, and Tracts (1999).

[FF2016] A. Fomenko and D. Fuchs. *Homotopical topology*, Second edition. Graduate Texts in Mathematics 273, Springer Verlag (2016).

[FM1997] A. Fomenko and S. Matveev. *Algorithmic and computer methods for three-manifolds*, Mathematics and its Applications 425. Kluwer Academic Publishers (1997).

[Form1995] R. Forman. *A discrete Morse theory for cell complexes*, Yau, S.-T. (ed.), Geometry, topology and physics for Raoul Bott. Lectures of a conference in honor of Raoul Bott's 70th birthday, Harvard University, Cambridge, MA, USA 1993. Cambridge, MA: International Press. Conf. Proc. Lect. Notes Geom. Topol. 4 (1995), 112–125.

[Form1998] R. Forman. *Morse theory for cell complexes*, Adv. Math. 134, No. 1 (1998), 90–145.

[Form02] R. Forman. *A user's guide to discrete Morse theory*, Sémin. Lothar. Comb. 48, B48c (2002).

[Fors2008] O. Forster. *Analysis 2*, Vieweg und Teubner, 8. Auflage (2008)

[Fox1941] R. H. Fox. *On the Lusternik-Schnirelmann category*, Ann. of Math. (2) 42 (1941), 333–370.

[Fox1953] R. H. Fox. *Free differential calculus I, Derivation in the free group ring*, Ann. Math. 57 (1953), 547–560.

[Fox1957] R. H. Fox. *Covering spaces with singularities*, Algebraic Geometry and Topology: A Symposium in Honor of S. Lefschetz (1957), 243–257.

[FMi1966] R. H. Fox and J. W. Milnor, *Singularities of 2-spheres in 4-space and cobordism of knots*, Osaka J. Math 3 (1966), 257–267.
[FW1999] G. Francis and J. Weeks. *Conway's ZIP Proof*, Amer. Math. Monthly 106 (1999), 393–399.
[FPo1930] F. Frankl and L. Pontrjagin. *Ein Knotensatz mit Anwendung auf die Dimensionstheorie*, Math. Annalen. 102 (1930), 785–789.
[Fraz1935] W. Franz. *Über die Torsion einer Überdeckung*, J. Reine Angew. Math. 173 (1935), 245–254.
[Frai2012] G. Franzoni. *The Klein bottle: variations on a theme*, Notices Am. Math. Soc. 59 (2012), 1076–1082.
[Fre1982] M. Freedman. *The topology of four-dimensional manifolds*, J. Diff. Geom. 17 (1982), 357–453.
[Fre2021] M. Freedman. *PC4 at age 40*, preprint, arXiv:2101.10519.
[FK1978] M. Freedman and R. Kirby. *A geometric proof of Rochlin's theorem*, Algebraic and geometric topology, Part 2, pp. 85-97, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc, 1978.
[FQ1990] M. Freedman and F. Quinn. *Topology of 4-manifolds*, Princeton Mathematical Series, 39. Princeton University Press (1990).
[Freu1937] H. Freudenthal. *Über die Klassen der Sphärenabbildungen*, Comp. Math. 5 (1937), 299–314.
[Freu1939] H. Freudenthal. *Die Triangulation der differenzierbaren Mannigfaltigkeiten*, Nederl. Akad. Wetensch. 42 (1939), 880–901.
[FYHLMO1985] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett and A. Ocneanu. *A new polynomial invariant of knots and links*, Bull. Am. Math. Soc., New Ser. 12 (1985), 239–246.
[FJ2008] M. Fried and M. Jarden. *Field arithmetic*. Revised by Moshe Jarden. 3rd revised ed. Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer Verlag (2008).
[Frie2004] S. Friedl. *Eta invariants as sliceness obstructions and their relation to Casson-Gordon invariants*, Alg. Geom. Top. 4 (2004), 893–934.
[Frie2012] S. Friedl. *On the uniqueness of prime, Jacobi functors*, accepted for publication by the Journal of Algebra and Number Theory Academia (2012)
`https://www.uni-regensburg.de/Fakultaeten/nat_Fak_I/friedl/papers/4121106-AN23-S.Friedl.pdf`
[Frie2014] S. Friedl. *Twisted Reidemeister torsion, the Thurston norm and fibered manifolds*, Geom. Ded. 172, (2014), 135-145.
[Frie2016a] S. Friedl. *Analysis IV*, lecture notes, University of Regensburg (2016)
`https://www.uni-regensburg.de/Fakultaeten/nat_Fak_I/friedl/papers/analysis-IV-open-source.pdf`
[Frie2022] S. Friedl. *Analysis III*, lecture notes, University of Regensburg (2022)
`https://www.uni-regensburg.de/Fakultaeten/nat_Fak_I/friedl/papers/analysis-IV-open-source.pdf`
[FrH2018] S. Friedl and G. Herrmann. *Graphical neighborhoods of spatial graphs*, Preprint (2018)
[FrH2019] S. Friedl and G. Herrmann. *Spatial graphs*, to appear in the Encyclopedia of Knot Theory to be published by CRC Press (2019).
[FI2022] S. Friedl and K. Ince. *A note on the table theorem*, in preparation (2022).
[FNOP2019] S. Friedl, M. Nagel, P. Orson and M. Powell. *A survey of the foundations of* 4*-manifold theory*, Preprint (2019).
[FO2015] S. Friedl and P. Orson. *Twist spinning of knots and metabolizers of Blanchfield pairings*, Ann. Fac. Sci. Toulouse Math. (6) 24 (2015), no. 5, 1203–1218.
[FrG2005] E. Friedlander and D. Grayson. *Handbook of K-theory. Vol. 1 and 2*, Springer Verlag (2005).
[Friedm2018] M. Friedman. *A History of Folding in Mathematics: Mathematizing the Margins*, Birkhäuser 2018)
[FPr1985] U. Friedrichsdorf and A. Prestel. *Mengenlehre für den Mathematiker*, Vieweg Studium: Grundkurs Mathematik (1985).
[Frig2017] R. Frigerio. *Bounded Cohomology of Discrete Groups*, Mathematical Surveys and Monographs 227, Amer. Math. Soc. (2017)
[FrPi1990a] R. Fritsch and R. Piccinini. *Cellular structures in topology*, Cambridge Studies in Advanced Mathematics, 19. Cambridge University Press (1990)

[FrPi1990b] R. Fritsch and R. Piccinini. *CW-complexes and Euclidean spaces*, Rend. Circ. Mat. Palermo (2) Suppl. No. 24 (1990), 79–95.

[FuR1984] D. B. Fuks and V. A. Rokhlin. *Beginner's course in topology. Geometric chapters*, Springer Series in Soviet Mathematics. Universitext. Springer-Verlag (1984).

[Ful1995] W. Fulton. *Algebraic topology. A first course*, Graduate Texts in Mathematics 153, Springer-Verlag (1995)

[Furs1955] H. Furstenberg. *On the infinitude of primes*, Amer. Math. Monthly. 62 (1955), 353.

[Furu1990] M. Furuta. *Homology cobordism group of homology 3–spheres*, Invent. Math. 100 (1990), 339–355.

[Furu2001] M. Furuta. *Monopole Equation and the 11/8-Conjecture*, Math. Res. Letters 8 (2001), 279–291.

[Gab1983] D. Gabai. *Foliations and the topology of 3-manifolds*, J. Differential Geometry 18 (1983), no. 3, 445–503.

[Gal1979] D. Gale. *The Game of Hex and Brouwer Fixed-Point Theorem*, Amer. Monthly 86 (1979), 818–827.

[Gal1987] D. Gale. *The Classification of 1-manifolds: a take-home exam*, Amer. Math. Monthly 94 (1987), 170–175.

[GX2013] J. Gallier and D. Xu. *A guide to the classification theorem for compact surfaces*, Geometry and Computing 9. Berlin: Springer Verlag (2013).

[GNR2021] D. Galvin, W. Niu and B. Ruppik. *Explicit immersions of a punctured n-torus in $\mathbb{R}^n$ and Kirby's torus trick for surfaces and 3-manifolds*, University of Bonn (2021).
`https://maths.dur.ac.uk/users/mark.a.powell/low%20dim%20torus%20trick.pdf`

[Garda2021] G. Gardam. *A counterexample to the unit conjecture for group rings*, Preprint (2021), arXiv:2102.11818.

[Gardn1989] M. Gardner. *Mathematical carnival*, Second edition. Mathematical Association of America (1989)

[GM2005] J. Garnett and D. Marshall. *Harmonic measure*, New Mathematical Monographs 2. Cambridge: Cambridge University Press (2005).

[Gaul1982] D. Gauld. *Differential topology. An introduction*, Monographs and Textbooks in Pure and Applied Mathematics, 72. Marcel Dekker (1982).

[Gaul2014] D. Gauld. *Non-metrisable manifolds*, Springer (2014).

[Gaus1813] C. F. Gauss. *Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum methodo nova tractata*, Commentationes societatis regiae scientiarium Gottingensis recentiores, 2 (1813), 355–378.

[Gaus1877] C. F. Gauss. *Zur Mathematischen Theorie der Electrodynamischen Wirkungen*, Königliche Gesellschaft der Wissenschaften zu Göttingen (1877), 602-630.

[GK2016] D. Gay and R. Kirby. *Trisecting 4-manifolds*, Geom. Topol. 20, No. 6 (2016), 3097–3132.

[Gei2008] H. Geiges. *An Introduction to Contact Topology*, Cambridge University Press (2008).

[Gei2009] H. Geiges. *A contact geometric proof of the Whitney-Graustein theorem*, Enseign. Math. (2) 55 (2009), 93–102.

[Geo2008] R. Geoghegan. *Topological methods in group theory*, Graduate Texts in Mathematics 243, Springer Verlag (2008)

[Gh2014] R. Ghrist. *Elementary Applied Topology*, ed. 1.0, Createspace, 2014.

[Gib2010] P. Giblin. *Graphs, surfaces and homology.*, Third edition. Cambridge University Press 2010.

[Gif1966] C. Giffen. *The generalized Smith conjecture*, Amer. J. Math. 88 (1966), 187–198.

[Gil1981] P. Gilmer. *Configurations of surfaces in 4-manifolds*, Trans. Am. Math. Soc. 264 (1981), 353–380.

[GiS1965] S. Gitler and J. D. Stasheff. *The first exotic class of BF*, Topology 4 (1965), 257–266.

[Gla1970] L. Glaser. *Geometrical combinatorial topology*, Vol. I. Van Nostrand Reinhold Mathematics Studies, 27. Van Nostrand Reinhold Co. (1970).

[Glea1950] A. Gleason. *Spaces with a compact Lie group of transformations*, Proc. Amer. Math. Soc. 1 (1950), 35–43.

[Gle] D. Gleeson. *A Rigorous Treatment of Conway's ZIP Proof*, University College Cork project report
`https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.119.4749&rep=rep1&type=pdf`

[Glue1962] H. Gluck. *The embedding of two-spheres in the four-sphere*, Trans. Amer. Math. Soc. 104 (1962), 308–333.

[God1973] R. Godement. *Topologie algébrique et théorie des faisceaux*, Troisième édition revue et corrigée. Publications de l'Institut de Mathématique de l'Université de Strasbourg, XIII. Actualités Scientifiques et Industrielles 1252. Hermann, (1973).

[GW1985] C. H Goldberg and D. B. West. *Bisection of circle colorings*, SIAM J. Algebraic Discrete Methods 6 (1985), 93–106.

[GG1973] M. Golubitsky and V. Guillemin. *Stable mappings and their singularities*, Graduate Texts in Mathematics 14, Springer-Verlag (1973).

[Gom1984] R. Gompf. *Stable diffeomorphism of compact 4-manifolds*, Topology Appl. 18 (1984), 115–120.

[Gom1991] R. Gompf. *Killing the Akbulut-Kirby 4-sphere, with relevance to the Andrews-Curtis and Schoenflies problems*, Topology 30 (1991), no. 1, 97–115.

[GoS1999] R. Gompf and A. Stipsicz. *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics 20, AMS (1999).

[GT1998] S. Gonnord and N. Tosel. *Thèmes d'analyse pour l'Agrégation : Calcul différentiel1*, Ellipses (1998)

[Gor1972] C. McA. Gordon. *Knots whose branched cyclic coverings have periodic homology*, Trans. Amer. Math. Soc. 168 (1972), 357–370.

[Gor1991] C. McA. Gordon. *Dehn surgery on knots*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 631–642, Math. Soc. Japan (1991).

[GL1989] C. McA. Gordon and J. Luecke. *Knots are determined by their complements*, J. Amer. Math. Soc. 2 (1989), no. 2, 371–415.

[GD2003] A. Granas and J. Dugundji. *Fixed point theory*, Springer Monographs in Mathematics. Springer-Verlag (2003).

[Gra1975] B. Gray. *Homotopy Theory*, An Introduction to Algebraic Topology, Pure and Applied Mathematics, Academic Press (1975)

[Greenb1967] M. Greenberg. *Lectures on algebraic topology*, W. A. Benjamin, Reading Mass (1967).

[GrH1981] M. Greenberg and J. Harper. *Algebraic Topology: A First Course*, Benjamin/Cummings (1981).

[Green1828] G. Green. *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* (1828).

[Greene2002] J. Greene. *A new short proof of Kneser's conjecture*, Am. Math. Mon. 109 (2002), 918–920.

[Greene2013] J. Greene. *The lens space realization problem*, Ann. Math. 177 (2013), 449–511.

[GrL2020] J. Greene and A. Lobb. *The Rectangular Peg Problem*, preprint (2020), arXiv:2005.09193.

[Grom1983] M. Gromov. *Volume and bounded cohomology*, Inst. Hautes Etudes Sci. Publ. Math. 56 (1983), 5–99.

[Grom1987] M. Gromov. *Hyperbolic groups*, in Essays in group theory, volume 8 of Math. Sci. Res. Inst. Publ., 75–263, Springer, 1987.

[Grom1990] M. Gromov. *Convex sets and Kähler manifolds*, Advances in differential geometry and topology (1990), 1–38.

[Gron2010] D. Gronau. *Paul Urysohn in Batz sur Mer*, Mitt. Dtsch. Math.-Ver. 18 (2010), 236–239.

[Gros1964] O. Gross. *The rendezvous value of metric space*, 1964 Advances in game theory, Ann. Math. Stud. Journal Profile 52 (1964), 49–53.

[Grote1969] K. P. Grotemeyer. *Topologie*, Bibliographisches Institut Mannheim (1969).

[Groth1957] A. Grothendieck. *Sur quelques points d'algèbre homologique*, Tohoku Math. J. (2) 9 (1957), 119–221.

[Groth1971] A. Grothendieck. *Revêtements étales et groupe fondamental*, Séminaire de Géométrie Algébrique du Bois-Marie, Lecture Notes in Mathematics 224 (1971).

[Gru1940] I. A. Grushko. *On the bases of a free product of groups*, Matematicheskii Sbornik 8 (1940), 169–182.

[GuH2019] V. Guillemin and P. Haine. *Differential forms*, World Scientific (2019).

[GP1974] V. Guillemin and A. Pollack. *Differential topology*, Englewood Cliffs, N.J.: Prentice-Hall, Inc. XVI (1974).

[Hab1982] N. Habegger. *Une varieté de dimension 4 avec forme d'intersection paire et signature* -8, Comm. Math. Helv. 57 (1982), 22–24.

[Had1910] J. Hadamard. *Note sur quelques applications de l'indice de Kronecker*, Introduction á la théorie des fonctions d'une variable (Volume 2), 2nd edition, A. Hermann & Fils (1910), 437–477.

[Hae1961a] A. Haefliger. *Plongements différentiables de variétés dans variétés*, Comment. Math. Helv. 36 (1961), 47–82.

[Hae1961b] A. Haefliger. *Differentiable imbeddings*, Bull. Amer. Math. Soc., 67 (1961), 109-112.

[Hae1962a] A. Haefliger. *Plongements différentiables dans le domaine stable*, Comment. Math. Helv. 37 (1962), 155–176.

[Hae1962b] A. Haefliger. *Knotted* $(4k-1)$*-spheres in* $6k$*-space*, Ann. of Math., 75 (1962), 452–466.

[Hae1967] A. Haefliger. *Enlacements de sphères en codimension supérieure à 2*, Comment. Math. Helv. 41 (1966–67), 51–72.

[HiH1963] A. Haefliger and M. Hirsch. *On the existence and classification of differentiable embeddings*, Topology 2 (1963), 129–135.

[Haj2019] P. Hajlasz. *Linking topological spheres*, to appear in Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. (2019)

[Hak1973] W. Haken. *Connections between topological and group theoretical decision problems*, Word problems: decision problems and the Burnside problem in group theory, Studies in Logic and the Foundations of Math. 71 (1973), 427-441.

[Hale20000] T. Hales. *Cannonballs and honeycombs*, Notices Amer. Math. Soc. 47 (2000), 440–449.

[Hale2005] T. Hales. *A proof of the Kepler conjecture*, Annals of Mathematics 162 (2005), 1065–1185.

[Hale2017] T. Hales et al. *A Formal Proof of the Kepler Conjecture*, Forum Math. Pi 5 (2017), e2, 29 pp.

[HallB2015] B. Hall. *Lie groups, Lie algebras, and representations. An elementary introduction*, Second edition. Graduate Texts in Mathematics 222 (2015)

[HallM1976] M. Hall. *The theory of groups*, Chelsea Publishing Company New York, N.Y. (1976)

[Halm1960] P. Halmos. *Naive set theory*, The University Series in Undergraduate Mathematics, D. Van Nostrand Co. (1960).

[HalW2012] D. M. Halverson and D. G. Wright. *The Homogeneous Property of the Hilbert Cube*, unpublished note, arXiv:1211.1363 (2012).

[Ham1976] A. J. S. Hamilton. *The triangulation of* 3*-manifolds*, Quart. J. Math. Oxford Ser. (2) 27 (1976), 63–70.

[Han1951] O. Hanner. *Some theorems on absolute neighborhood retracts*, Arkiv Mat. 1 (1951), 389–408.

[Har1992] J. Harris. *Algebraic geometry. A first course*, Graduate Texts in Mathematics 133, Springer Verlag (1992).

[dlH2000] P. de la Harpe. *Topics in geometric group theory*, Chicago Lectures in Mathematics. Chicago: The University of Chicago Press (2000).

[HNV2004] K. Hart, J. Nagata and J. Vaughan. *Encyclopedia of general topology.* Elsevier Science Publishers (2004).

[Harz1978] E. Harzheim. *Einführung in die kombinatorische Topologie*, Die Mathematik. Wissenschaftliche Buchgesellschaft, Darmstadt (1978).

[Hat1978] A. Hatcher. *Concordance spaces, higher simple-homotopy theory, and applications*, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1, pp. 3–21, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc. (1978).

[Hat1983] A. Hatcher. *A proof of the Smale conjecture,* $\mathrm{Diff}(S^3) \cong O(4)$, Ann. of Math. (2) 117 (1983), 553–607.

[Hat2002] A. Hatcher. *Algebraic topology*, Cambridge University Press (2002)
`http://pi.math.cornell.edu/~hatcher/AT/AT.pdf`

[Hat] A. Hatcher. *Spectral Sequences*
`https://pi.math.cornell.edu/~hatcher/AT/ATch5.pdf`

[Hat2] A. Hatcher. *Vector bundles and* K*-theory*
`http://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf`

[Hat3] A. Hatcher. *Notes on basic 3-manifold topology*
http://pi.math.cornell.edu/~hatcher/3M/3Mfds.pdf
[Hat2012] A. Hatcher. *A 50-Year View of Diffeomorphism Groups*
http://pi.math.cornell.edu/~hatcher/Papers/Diff(M)2012.pdf
[Hat2013] A. Hatcher. *The Kirby Torus Trick for surfaces*, preprint (2013) https://arxiv.org/abs/1312.3518
[Hat2021] A. Hatcher. *The topology of numbers*, Preprint (2021).
http://pi.math.cornell.edu/~hatcher/TN/TNbook.pdf
[HT1980] A. Hatcher and W. Thurston. *A presentation for the mapping class group of a closed orientable surface*, Topology 19 (1980), 221–237.
[HatW1973] A. Hatcher and J. Wagoner. *Pseudo-isotopies of compact manifolds*, Astérisque, No. 6. Société Mathématique de France (1973).
[Hau1987] J.-C. Hausmann. *Geometric Hopfian and non-Hopfian situations*, Geometry and topology (Athens, Ga., 1985), 157–166, Lecture Notes in Pure and Appl. Math., 105, Dekker (1987).
[Hau2014] J.-C. Hausmann. *Mod two homology and cohomology*, Universitext. Springer Verlag (2014).
[HGK2010] M. Hazewinkel, N. Gubareni and V. V. Kirichenko. *Algebras, rings and modules. Lie algebras and Hopf algebras*, Mathematical Surveys and Monographs, 168. American Mathematical Society (2010).
[Hee1898] P. Heegaard. *Forstudier til en topologisk Teori for de algebraiske Fladers Sammenhang*, University of Copenhagen (1898)
[Hel1923] E. Helly. *Über Mengen konvexer Körper mit gemeinschaftlichen Punkten*, J. Deutsche Math.-Ver. 32 (1923), 175–176.
[Hem1972] J. Hempel. *Residual finiteness of surface groups*, Proc. Amer. Math. Soc. 32 (1972), 323.
[Hem1976] J. Hempel. *3-manifolds*, Annals of Mathematics Studies 86. Princeton, New Jersey: Princeton University Press and University of Tokyo Press. XII (1976).
[Hem87] J. Hempel. *Residual finiteness for 3-manifolds*, in: Combinatorial Group Theory and Topology, Annals of Mathematics Studies, vol. 111, Princeton University Press (1987), 379-396.
[Herl1996] H. Herrlich. *Compactness and the axiom of choice*, Categorical topology (L'Aquila, 1994). Appl. Categ. Structures 4 (1996), 1–14.
[Herr2019] G. Herrmann. *Sutured manifolds, L^2-Betti numbers and an upper bound on the leading coefficient*, Doctoral Thesis, University of Regensburg (2019).
[Hig1940] G. Higman. *The units of group-rings*, Proc. London math. Soc. (2) 46 (1940), 231–248.
[HNN1950] G. Higman, B. Neumann and H. Neumann. *Embedding theorems for groups*, J. Lond. Math. Soc. 24 (1950), 247–254.
[Hilb1891] D. Hilbert. *Über die stetige Abbildung einer Linie auf ein Flächenstück*, Math. Ann. 38 (1891), 459–460.
[Hild1976] H. Hilden. *Three-Fold branched coverings of S^3*, Amer. J. Math 98 (1976), 989–997.
[HHR2016] M. A. Hill, M. J. Hopkins, and D. C. Ravenel. *On the nonexistence of elements of Kervaire invariant one*, Ann. of Math. (2) 184 (2016), no. 1, 1–262.
[Hill2012] J. Hillman. *Algebraic invariants of links*, second edition, Series on Knots and Everything, World Scientific (2012).
[Hilt1953] P. J. Hilton. *An introduction to homotopy theory*, Cambridge Tracts in Mathematics and Mathematical Physics 43 (1953).
[Hilt1955] P. J. Hilton. *On the homotopy groups of the union of spheres*, J. London Math. Soc. 30 (1955), 154–172.
[HS1997] P. J. Hilton and U. Stammbach. *A course in homological algebra*, 2nd ed. Graduate Texts in Mathematics 4, Springer Verlag (1997).
[Hirs1961] M. Hirsch. *On imbedding differentiable manifolds in euclidean space*, Ann. of Math. (2), 73 (1961), 566–571.
[Hirs1976] M. Hirsch. *Differential topology*, Graduate Texts in Mathematics 33, Springer Verlag (1976).
[HiM1974] M. Hirsch and B. Mazur. *Smoothings of piecewise linear manifolds*, Annals of Mathematics Studies 80, Princeton University Press and University of Tokyo Press (1974)

[Hirz1953] F. Hirzebruch. *On Steenrod's reduced powers, the index of inertia and the Todd genus*, Proc. Nat. Acad. Sci. U.S.A. 39 (1953), 951–956.

[Hirz1969] F. Hirzebruch. *The signature of ramified coverings*, 1969 Global Analysis (Papers in Honor of K. Kodaira) Univ. Tokyo Press (1969), 253–265.

[HZ1974] F. Hirzebruch and D. Zagier. *The Atiyah-Singer Theorem and Elementary Number Theory*, Mathematical Lecture Series, No. 3, Publish or Perish, Inc. (1974).

[HY1988] J. Hocking and G. Young. *Topology*, Second edition. Dover Publications (1988)

[Hoe2017] R. Hoekzema. *Manifolds with odd Euler characteristic and higher orientability*, preprint (2017), arXiv:1704.06607.

[HoM2020] K. Hofmann and S. Morris. *The structure of compact groups. A primer for the student - a handbook for the expert*, 4th revised and expanded edition. De Gruyter Studies in Mathematics 25 (2020).

[Hopf1925] H. Hopf. *Über die Curvatura integra geschlossener Hyperflächen*, Math. Ann. 95, 340–367 (1925); Philos. Diss. Berlin (1925).

[Hopf1927a] H. Hopf. *Abbildungsklassen n-dimensionaler Mannigfaltigkeiten*, Math. Ann. 96 (1927), 209–224.

[Hopf1927b] H. Hopf. *Vektorfelder in n-dimensionalen Mannigfaltigkeiten*, Math. Ann. 96 (1927), 225–249.

[Hopf1929] H. Hopf. *Über die algebraische Anzahl von Fixpunkten*, Math. Z. 29 (1929), 493–524.

[Hopf1931] H. Hopf. *Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche*, Math. Ann. 104 (1931), 637–665.

[Hopf1940a] H. Hopf. *Ein topologischer Beitrag zur reellen Algebra*, Comm. Math. Helvetici 13 (1940/41), 219–239.

[Hopf1940b] H. Hopf. *Systeme symmetrischer Bilinearformen und euklidische Modelle der projektiven Räume*, Vjschr. naturf.Ges. Zürich 85, Beibl. Nr. 32, 165-176. (Rudolf Fueter Festschr.) (1940).

[Hopf1948] H. Hopf. *Zur Topologie der komplexen Mannigfaltigkeiten*, Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, Interscience Publishers (1948), 167–185.

[Hopf1964] H. Hopf. *Selecta Heinz Hopf*, Herausgegeben zu seinem 70. Geburtstag von der Eidgenössischen Technischen Hochschule Zürich. Springer-Verlag (1964).

[Hopk1989] M. Hopkins. *Minimal atlases of real projective spaces*, Algebraic topology, Proc. Int. Conf., Arcata/Calif. 1986, Lect. Notes Math. 1370 (1989), 171–192.

[Hopk2015] M. Hopkins. *Quadratic and bilinear forms*, Harvard University (2015) `http://math.harvard.edu/~mjh/northwestern.pdf`

[HLSX2018] M. Hopkins, J. Lin, X. Shi and Z. Xu. *Intersection Forms of Spin 4-Manifolds and the* Pin(2)*-Equivariant Mahowald Invariant*, Preprint (2018)

[Hör1990a] L. Hörmander. *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*, second edition. Springer Verlag (1990).

[Hör1990b] L. Hörmander. *An introduction to complex analysis in several variables*, Third edition. North-Holland Mathematical Library, 7. North-Holland Publishing Co. (1990).

[HJa2013] R. Horn and C. Johnson. *Matrix analysis*, Second edition. Cambridge University Press, Cambridge, 2013.

[How1982] J. Howie, *On locally indicable groups*, Math. Z. 180 (1982), no. 4, 445–461.

[HoS1983] J. Howie and H. R. Schneebeli, *Homological and topological properties of locally indicable groups*, Manuscripta Math. 44 (1983), no. 3, 71–93.

[HsH1967] W. C. Hsiang and W. Y. Hsiang. *Differentiable actions of compact connected classical groups I*, Am. J. Math. 89 (1967), 705–786.

[HsSh1976] W. C. Hsiang and R. W. Sharpe. *Parametrized surgery and isotopy*, Pacific J. Math. 67 (1976), no. 2, 401–459.

[HsSz1971] W. C. Hsiang and R. H. Szczarba. *On embedding surfaces in four-manifolds*, Algebraic Topology, Proc. Sympos. Pure Math. 22 (1971), 97–103.

[HSL1965] W. C. Hsiang, J. Levine and R. H. Szczarba. *On the normal bundle of a homotopy sphere embedded in Euclidean space*, Topology 3 (1965), 173–181.

[Hu1959] S.-T. Hu. *Homotopy theory*, Pure and Applied Mathematics 8. Academic Press (1959).

[Hu1965] S.-T. Hu. *Theory of retracts*, Wayne State University Press (1965).
[Hu1966] S.-T. Hu. *Homology theory: A first course in algebraic topology*, Holden-Day Series in Mathematics. Holden-Day, Inc. XII (1966).
[Hu1969] S.-T. Hu. *Differentiable manifolds*, Holt, Rinehart and Winston (1969).
[Hub1999] J. R. Hubbuck. *A Short History of H-spaces*, History of topology (1999), 747–755.
[Hud1969] J. F. P. Hudson. *Piecewise linear topology*, University of Chicago Lecture Notes prepared with the assistance of J. L. Shaneson and J. Lees W. A. Benjamin, Inc. (1969)
[Hud1972] J. F. P. Hudson. *Embeddings of bounded manifolds*, Proc. Camb. Philos. Soc. 72 (1972), 11–20.
[Hug2018] C. Hugelmeyer. *Every smooth Jordan curve has an inscribed rectangle with aspect ratio equal to* $\sqrt{3}$, Preprint, arXiv:1803.07417.
[HR1996] B. Hughes and A. Ranicki. *Ends of complexes*, Cambridge Tracts in Mathematics, 123. Cambridge University Press (1996).
[Hun1980] T. Hungerford. *Algebra*, Graduate Texts in Mathematics 73, Springer Verlag (1980).
[HuW1948] W. Hurewicz and H. Wallman. *Dimension theory*, Princeton University Press (1948).
[Hus1993] D. Husemoller. *Fibre bundles*, third edition, Graduate Texts in Mathematics 20, Springer Verlag (1993).
[Huy2005] D. Huybrechts. *Complex geometry, An introduction*, Universitext, Springer-Verlag (2005).
[Huy2016] D. Huybrechts. *Lectures on K3 surfaces*, Cambridge Studies in Advanced Mathematics 158 (2016).
[Ig1979] K. Igusa. *The* $\mathrm{Wh}(\pi, 3)$ *obstruction for pseudoisotopy*, Thesis, Princeton University. ProQuest LLC, Ann Arbor, MI (1979).
[Ig1984] K. Igusa. *What happens to Hatcher and Wagoner's formulas for* $\pi_0 C(M)$ *when the first Postnikov invariant of* M *is nontrivial?*, Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982), 104–172, Lecture Notes in Math., 1046, Springer (1984).
[IR1990] K. Ireland and M. Rosen. *A Classical Introduction to Modern Number Theory*, Second edition, Springer Verlag (1990)
[Isa1994] M. Isaac. *Algebra: a graduate course*, Graduate Studies in Mathematics 100, American Mathematical Society (1994).
[IWX2001] D. C. Isaksen, G. Wang and Z. Xu. *Stable homotopy groups of spheres*, Preprint (2020), arXiv:2001.04247.
[Iv1987] N. Ivanov. *Foundations of the theory of bounded cohomology.* J. Soviet Math., 37 (1987), 1090–1114.
[Iw1998] N. Iwase. *Ganea's conjecture on Lusternik-Schnirelmann category*, Bull. London Math. Soc. 30 (1998), 623–634.
[Iw2002] N. Iwase. A_∞*-method in Lusternik–Schnirelmann category*, Topology 41 (2002), 695–723.
[Jaco1969] W. Jaco. *Heegaard splittings and splitting homomorphisms*, Trans. Amer. Math. Soc. 144 (1969), 365–379.
[Jacb1953] N. Jacobson. *Lectures in abstract algebra. Vol. 2. Linear algebra*, D. Van Nostrand Co., Inc. XII (1953).
[Jam1976] I. M. James. *The topology of Stiefel manifolds*, London Mathematical Society Lecture Note Series, No. 24. Cambridge University Press (1976).
[Jä1968] K. Jänich. *Charakterisierung der Signatur von Mannigfaltigkeiten durch eine Additivitätseigenschaft*, Invent. Math. 6 (1968), 35–40.
[Jä1969] K. Jänich. *On invariants with the Novikov additive property*, Math. Ann. 184 (1969), 65–77.
[Jä93] K. Jänich. *Vektoranalysis*, 2. Auflage, Springer Lehrbuch (1993).
[Jä2005] K. Jänich. *Topology*, Springer Lehrbuch (2005).
[Jä2011] K. Jänich. *Funktionentheorie*, 6. Auflage, Springer Lehrbuch (2011).
[JP2015] M. Jarnicki and P. Pflug. *Continuous nowhere differentiable functions. The monsters of analysis*, Springer Monographs in Mathematics (2018).
[Je1973] T. Jech. *The axiom of choice*, Studies in Logic and the Foundations of Mathematics 75. North-Holland Publishing Co (1973).

[Je2003] T. Jech. *Set theory*, The third millennium edition, revised and expanded. Springer Monographs in Mathematics (2003).

[JM2006] J. Jezierski and W. Marzantowicz. *Homotopy methods in topological fixed and periodic points theory*, Topological Fixed Point Theory and Its Applications, 3. Springer (2006).

[JohD1980] D. Johnson. *Spin structures and quadratic forms on surfaces*, J. Lond. Math. Soc., II. Ser. 22 (1980), 365–373.

[JohF1972] F. E. A. Johnson. *Lefschetz duality and topological tubular neighborhoods*, Trans. Amer. Math. Soc. 172 (1972), 95–110.

[JohJ200707] J. Johnson. *Notes on Heegaard splittings*, lecture notes, Yale University (2007) `http://users.math.yale.edu/~jj327/notes.pdf`

[JonL1973] L. Jones. *Patch spaces: A geometric representation for Poincaré spaces*, Ann. Math. (2) 97 (1973), 306–343.

[JonV1985] V. Jones. *A polynomial invariant for knots via von Neumann algebra*, Bull. Amer. Math. Soc. 12 (1985), 103–111.

[Jor1887] C. Jordan. *Cours d'analyse de l'École Polytechnique*, Gauthier-Villars (1887), 587–594.

[Jos1983] K. D. Joshi. *Introduction to general topology*, Wiley Eastern Limited (1983).

[Joy2012] D. Joyce. *On manifolds with corners* Janeczko, Stanislaw (ed.) et al., Advances in geometric analysis. Collected papers of the workshop on geometry in honour of Shing-Tung Yau's 60th birthday, Warsaw, Poland, April 6-8, 2009. International Press and Higher Education Press, Advanced Lectures in Mathematics (ALM) 21 (2012), 225–258.

[JZ2018] A. Juhász and I. Zemke. *Stabilization distance bounds from link Floer homology*, Preprint (2018), arXiv:1810.09158.

[Kah1970] P. Kahn. *Codimension-one imbedded spheres*, Invent. Math. 10 (1970), 44–56.

[KaW1961] D. Kan and G. Whitehead. *On the realizability of singular cohomology groups*, Proc. Amer. Math. Soc. 12 (1961), 24-25.

[KaS1989] I. L. Kantor and A. S. Solodovnikov. *Hypercomplex numbers. An elementary introduction to algebras*, Springer-Verlag (1989).

[Kap1979] S. Kaplan. *Constructing of framed 4-manifolds with given almost framed boundaries*, Trans. Amer. Math. Soc. 254 (1979), 237–263.

[Kapy1957] I. Kaplansky. *Problems in the theory of rings*, Report of a conference on linear algebras, June, 1956, pp. 1-3. National Academy of Sciences-National Research Council, Washington, Publ. 502 (1957).

[Kapy1969] I. Kaplansky. *Linear algebra and geometry*, A second course. Allyn and Bacon (1969).

[Kapy1970] I. Kaplansky. *Problems in the theory of rings*, revisited. Amer. Math. Monthly, 77 (1970), 445–454.

[KaM2002] M. Kapovich and J. Millson. *Universality theorems for configuration spaces of planar linkages*, Topology 41 (2002), 1051–1107.

[Karo2008] M. Karoubi. *K-theory. An introduction*, Classics in Mathematics, Springer Verlag (2008).

[KaL1987] M. Karoubi and C. Leruste. *Algebraic topology via differential geometry*, London Mathematical Society Lecture Note Series, 99. Cambridge University Press, Cambridge, 1987.

[Karu1958] T. Karube. *On the local cross-sections in locally compact groups*, J. Math. Soc. Japan 10 (1958), 343–347.

[Kas2014] D. Kasprowski. *On the K-theory of groups with finite decomposition complexity*, PhD thesis (2014).
`http://www.math.uni-bonn.de/people/daniel/ThesisKasprowski.pdf`.

[Katz1979] V. Katz. *The history of Stokes' theorem*, Math. Mag. 52 (1979), 146–156.

[Kau1987] . L. H. Kauffman. *On knots*, Annals of Mathematics Studies 115. Princeton University Press (1987).

[Kaw1996] A. Kawauchi. *A survey of knot theory*, Birkhäuser Verlag (1996).

[KK1980] A. Kawauchi and S. Kojima. *Algebraic classification of linking pairings on 3-manifolds*, Math. Ann. 253 (1980), 29–42.

[Kel1975] J. Kelley. *General Topology*, Graduate Texts in Mathematics 27, Springer Verlag (1975)

[Kere1923] B. Kerékjártó. *Vorlesungen über Topologie. I*, Springer Verlag (1923).

[Kerv1958] M. Kervaire. *Non-parallelizability of the n-sphere for $n > 7$*, Proc. N.A.S. 44 (1958), 280–283.
[Kerv1959] M. Kervaire. *An interpretation of G. Whitehead's generalization of Hopf's invariant*, Ann. Math. 69 (1959), 345–365.
[Kerv1960] M. Kervaire. *A manifold which does not admit any differentiable structure*, Comment. Math. Helv. 34 (1960), 257–270.
[Kerv1969] M. Kervaire. *Smooth homology spheres and their fundamental groups*, Trans. Amer. Math. Soc. 144 (1969), 67–72.
[KeM1961] M. Kervaire and J. Milnor. *On 2-spheres in 4-manifolds*, Proc. Nat. Acad. Sci. U.S.A. 47 (1961), 1651–1657.
[KeM1963] M. Kervaire and J. Milnor. *Groups of homotopy spheres: I*, Ann. of Math. 77 (1963), 504–537.
[Kin2003] O. King. *A mass formula for unimodular lattices with no roots*, Math. Comp. 72 (2003), 839–863.
[Kir1969] R. C. Kirby. *Stable homeomorphisms and the annulus conjecture*, Annals of Math. 89(1969), 575–582.
[Kir1978] R. C. Kirby. *A calculus for framed links in S^3*, Inventiones Mathematicae. 45 (1978), 35–56.
[Kir1989] R. C. Kirby. *The topology of 4-manifolds*, Lecture Notes in Mathematics, 1374. Springer Verlag (1989).
[Kir1997] R. C. Kirby. *Problems in low-dimensional topology*, Kazez, William H. (ed.), Geometric topology. 1993 Georgia international topology conference, 1993, Athens, GA, USA. American Mathematical Society. AMS/IP Stud. Adv. Math. 2(pt.2), 35–473 (1997).
[Kir2007] R. C. Kirby. *Boy's surface*, Notices of the Amer. Math. Soc. 54 (2007), 1306–1307.
[Kir2013] R. C. Kirby. *The Whitney trick*, Celebration Mathematica (2013)
`http://celebratio.org/cmmedia/essaypdf/76_main.pdf`
[KSc1979] R. C. Kirby and M. Scharlemann. *Eight faces of the Poincaré homology 3-sphere*, Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977), pp. 113–146, Academic Press, New York-London, 1979.
[KSi1969] R. C. Kirby and L. C. Siebenmann. *On the triangulation of manifolds and the Hauptvermutung.* Bull. Amer. Math. Soc. 75 (1969), 742–749.
[KSi1977] R. C. Kirby and L. C. Siebenmann. *Foundational essays on topological manifolds, smoothings, and triangulations*, Princeton University Press (1977).
[Kiri2008] A. Kirillov. *Introduction to Lie groups and Lie algebras*, Cambridge Studies in Advanced Mathematics, 113. Cambridge University Press (2008).
[KiW2006] F. Kirwan, and J. Woolf. *An introduction to intersection homology theory*, Second edition. Chapman & Hall/CRC (2006).
[Kis1963] J. M. Kister. *Microbundles are fibre bundles*, Bull. Am. Math. Soc. 69 (1963), 854–857.
[Kis1964] J. M. Kister. *Microbundles are fibre bundles*, Ann. of Math. (2), 80 (1964), 190–199.
[Kle2018] P. Kleinschmidt. *Udo Pachner (1947-2002) - A "Hidden Champion" in Mathematics*, arXiv:1802.09357.
[KR1970] D. Kleitman and B. Rothschild. *The number of finite topologies*, Proc. Amer. Math. Soc. 25 (1970), 276–282.
[Kneb1977] M. Knebusch. *Some open problems*, Conference on Quadratic Forms-1976, Queen's Papers in Pure and Appl. Math. 46 (1977), 361–370.
[Kneb1982] M. Knebusch. *An algebraic proof of the Borsuk-Ulam theorem for polynomial mappings*, Proc. Am. Math. Soc. 84 (1982), 29–32.
[KnesH1929] H. Kneser. *Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten*, Jber. Deutsch. Math.-Verein. 38 (1929), 248–260.
[KnesH1958] H. Kneser. *Sur les variétés connexes de dimension* 1, Bull. Soc. Math. Belg. 10 (1958), 19–25.
[KnesM1955] M. Kneser. *Aufgabe 360*, Jahresbericht der Deutschen Mathematiker-Vereinigung 58 (1955) 27.
[Knu2015] K. Knudson. *Morse theory. Smooth and discrete*, World Scientific Publishing (2015)
[Koj1986] S. Kojima. *Determining knots by branched covers*, in: D. B. A. Epstein (ed.), Low-Dimensional Topology and Kleinian Groups, London Math. Soc. Lecture Note Series 112, Cambridge University Press (1986), 193–207.

[Kön2004] K. Königsberger. *Analysis 1*, 6. Auflage, Springer Verlag (2004).
[KT2006] A. Kono and D. Tamaki. *Generalized cohomology*, Translations of Mathematical Monographs, 230. Iwanami Series in Modern Mathematics. American Mathematical Society (2006).
[Kos1993] A. Kosinski. *Differentiable manifolds*, Pure and Applied Mathematics, 138. Academic Press. xvi (1993).
[Kot1989] D. Kotschick. *On manifolds homeomorphic to* $\mathbb{C}\mathrm{P}^2\#8\overline{\mathbb{C}\mathrm{P}}^2$, Invent. Math. 95 (1989), 591–600.
[Koz2008] D. Kozlov. *Combinatorial algebraic topology*, Algorithms and Computation in Mathematics 21, Springer Verlag (2008).
[Kra2009] S. Krantz. *A guide to topology*, The Dolciani Mathematical Expositions, 40. MAA Guides, 4. Mathematical Association of America (2009).
[KP2002] S. Krantz and H. Parks. *A primer of real analytic functions*, 2nd ed. Birkhäuser Advanced Texts. Basler Lehrbücher (2002).
[Kre1999] M. Kreck. *Surgery and duality*, Ann. of Math. 149 (1999), 707–754.
[KrL2005] M. Kreck and W. Lück. *The Novikov conjecture. Geometry and algebra*, Oberwolfach Seminars, 33. Birkhäuser Verlag (2005).
[KLT1995] M. Kreck, W. Lück and P. Teichner. *Counterexamples to the Kneser conjecture in dimension four*, Comment. Math. Helvetici 70 (1995), 423–433.
[KrS1984] M. Kreck and J. A. Schafer. *Classification and stable classification of manifolds: some examples*, Comment. Math. Helv. 59 (1984), no. 1, 12–38.
[KrM1994] P. Kronheimer and T. Mrowka. *The Genus of Embedded Surfaces in the Projective Plane*, Math. Res. Let. 1 (1994), 797–808.
[KrM2004] P. Kronheimer and T. Mrowka. *Witten's conjecture and Property P*, Geometry & Topology 8 (2004), 295–310.
[Kui1999] N. H. Kuiper. *A short history of triangulation and related matters*, in: History of Topology, pp. 491–502, North-Holland (1999).
[Kupe2017a] S. Kupers. *Zeeman's Conjecture*, lecture notes (2017)
`http://math.harvard.edu/~kupers/notes/zeemansconjecture.pdf`
[Kupe2017b] S. Kupers. *Three lectures on topological manifolds*, lecture notes (2017)
`http://people.math.harvard.edu/~kupers/notes/toplectures.pdf`
[Kupe2017c] S. Kupers. *Oriented bordism: calculation and application*, lecture notes (2017)
`http://people.math.harvard.edu/~kupers/notes/orientedbordism.pdf`
[Kupe2019] S. Kupers. *Lectures on diffeomorphism groups of manifolds*
`https://people.math.harvard.edu/~kupers/teaching/272x/book.pdf`
[Kupe2021] S. Kupers. *Diffeomorphisms of disks*, talk at the University of Pennsylvania (2021)
`https://www.utsc.utoronto.ca/people/kupers/wp-content/uploads/sites/50/2021/01/upenntalk.pdf`
[Kupk1963] I. Kupka. *Contribution à la théorie des champs génériques*, Contributions to Differential Equations 2 (1963), 457–484.
[Kura1922] C. Kuratowski. *Une méthode d'élimination des nombres transfinis des raisonnements mathématiques*, Fund. Math. 3 (1922), 76–108.
[Kuro1934] A. G. Kurosh. *Die Untergruppen der freien Produkte von beliebigen Gruppen*, Math. Annalen 109 (1934), 647–660.
[Kuro1960] A. G. Kurosh. *The theory of groups*, Volume 2, Chelsea Publishing Co. (1960).
[KuS2004] H. Kurzweil and B. Stellmacher. *The theory of finite groups. An introduction*, Universitext. Springer Verlag (2004).
[KwS2018] S. Kwasik and F. Sun. *Local coefficients revisited*, Enseign. Math. (2) 64 (2018), 283–303.
[Lacke1996] M. Lackenby. *The Whitney Trick*, Top. Appl. 71 (1996), 115–118.
[Lacke1999] M. Lackenby. *Three-dimensional manifolds*, lecture notes, University of Oxford (1999)
`http://people.maths.ox.ac.uk/lackenby/`
[Lackm2012] M. Lackmann. *Octonions and octonionic projective spaces*, University of Bonn (2012).
[Lak1976] I. Lakatos. *Proofs and refutations. The logic of mathematical discovery*, Edited by John Worrall and Elie Zahar. Cambridge University Press (1976).

[Lam1991] T. Y. Lam. *A first course in noncommutative rings*, Graduate Texts in Mathematics 131, Springer Verlag (1991).

[Lam1999] T. Y. Lam. *Lectures on modules and rings*, Graduate Texts in Mathematics, 189. Springer Verlag (1999).

[Lam2005] T. Y. Lam. *Introduction to Quadratic Forms over Fields*, Graduate Studies in Mathematics 67. American Mathematical Society (2005).

[Lanc2000] T. Lance. *Differentiable structures on manifolds*, Surveys on surgery theory, Vol. 1, 73–104, Ann. of Math. Stud., 145, Princeton Univ. Press (2000).

[Lang1965] S. Lang. *Algebra*, Addison-Wesley Publishing Company, Inc., XVIII (1965).

[Lang1984] S. Lang. *Algebra*, 2nd ed. The Advanced Book Program. Addison-Wesley Publishing Company, Inc. XV (1984).

[Lang1985] S. Lang. *Differential manifolds*, Springer (1985).

[Lang1993] S. Lang. *Algebra*, third edition, Addison Wesley (1993).

[Lang1999] S. Lang. *Complex analysis*, Graduate Texts in Mathematics 103, Springer Verlag (1999).

[Lang2001] S. Lang. *Fundamentals of differential geometry*, Graduate Texts in Mathematics 191, Springer Verlag (2001).

[Lang2002a] S. Lang. *Introduction to differentiable manifolds*, 2nd ed. Universitext. Springer Verlag (2002).

[Lang2002b] S. Lang. *Algebra*, 3rd revised ed. Springer (2002).

[Lat1979] F. Latour. *Double suspension d'une sphère d'homologie*, Seminaire Bourbaki, Vol. 1977/78, Expose No. 515, Lect. Notes. Math. 710 (1979), 169–186.

[Lau1992] F. Laudenbach. *On the Thom-Smale complex*, appendix to J.-M. Bismut and W. Zhang, An extension of a theorem by Cheeger and Miller, Astérisque 235 (1992).

[Lau2011] F. Laudenbach. *Transversalité, courants et théorie de Morse*, Éditions de l'école polytechnique (2011).

[Law1985] H. B. Lawson. *The theory of gauge fields in four dimensions*, Reg. Conf. Ser. Math. 58 (1985).

[LaM1989] H. B. Lawson and M.-L. Michelsohn. *Spin Geometry*, Princeton University Press (1989).

[Leb1903] H. Lebesgue. *Sur le problème des aires*, Bull. Soc. Math. France 31 (1903), 197–203.

[Lee1997] J. Lee. *Riemannian manifolds: an introduction to curvature*, Graduate Texts in Mathematics 176, Springer Verlag (1997).

[Lee2000] J. Lee. *Introduction to topological manifolds*, Graduate Texts in Mathematics 202, Springer Verlag (2000).

[Lee2002] J. Lee. *Introduction to smooth manifolds*, Graduate Texts in Mathematics 218, second edition, Springer Verlag (2002).

[Lee2002b] J. Lee. *Corrections to "Introduction to smooth manifolds"* `https://sites.math.washington.edu/~lee/Books/ISM/errata.pdf`

[Lee2009] J. Lee. *Manifolds and differential geometry*, Graduate Studies in Mathematics 107 (2009).

[Lef1926] S. Lefschetz. *Intersections and transformations of complexes and manifolds*, Trans. Amer. Math. Soc. 28 (1926), 1–49.

[Ler1950] J. Leray. *L'anneau spectral et l'anneau filtré d'homologie d'un espace localement compact et d'une application continue*, J. Math. Pures Appl. (9) 29 (1950), 1–139.

[Levin1965a] J. Levine. *Unknotting spheres in codimension two*, Topology 4 (1965), 9–16.

[Levin1965b] J. Levine. *A classification of differentiable knots*, Ann. Math. (2) 82 (1965), 15–50.

[Levin1970] J. Levine. *An algebraic classification of some knots of codimension two*, Comment. Math. Helv. 45 (1970), 185–198.

[Levit1972] N. Levitt. *Poincaré duality cobordism*, Ann. Math. (2) 96 (1972), 211–244.

[Lic1962] W. B. R. Lickorish. *A representation of orientable combinatorial 3-manifolds*, Ann. of Math. (2) 76 (1962), 531–540.

[Lic1963] W. B. R. Lickorish. *Homeomorphisms of non-orientable two-manifolds*, Proc. Cambridge Philos. Soc. 59 (1963), 307–317.

[Lic1964] W. B. R. Lickorish. *A finite set of generators for the homeotopy group of a 2-manifold*, Proc. Cambridge Philos. Soc. 60 (1964), 769–778. 1964.

[Lic1997a] W. B. R. Lickorish. *An introduction to knot theory*, Graduate Texts in Mathematics 175, Springer Verlag (1997).

[Lic1985] W. B. Lickorish. *The unknotting number of a classical knot*, Combinatorial methods in topology and algebraic geometry, Proc. Conf. Hon. A. M. Stone, Rochester/N.Y. 1982, Contemp. Math. 44, 117–121 (1985).

[Lic1997b] W. B. R. Lickorish. *Piecewise linear manifolds and Cerf theory*. Geometric topology (Athens, GA, 1993), 375–387, AMS/IP Stud. Adv. Math., 2.1, Amer. Math. Soc. (1997)

[Lic1999] W. B. R. Lickorish. *Simplicial moves on complexes and manifolds*, Proceedings of the Kirbyfest (Berkeley, CA, 1998), Geom. Topol. Monogr. 2, Geom. Topol. Publ. (1999), 299–320.

[Lin1970] C. E. Linderholm. *A Group Epimorphism is Surjective*, Amer. Math. Monthly 77 (1970), 176–177.

[Lip2014] S. Lipscomb. *Art Meets Mathematics in the Fourth Dimension*, second edition, Springer Verlag (2014)

[LSVJ2011] D. Lipsky, P. Skraba and M. Vejdemo-Johansson. *A spectral sequence for parallelized persistence*, Preprint, arXiv:1112.1245 (2011).

[LiM] C. Livingston and A. Miller. *KnotInfo: Table of Knot Invariants* `http://www.indiana.edu/~knotinfo`, August 26, 2020.

[Liv2021] C. Livingston. *Connected sums of codimension two locally flat submanifolds*, Preprint, arXiv:2104.03930 (2021).

[Loe2014] N. Loehr. *Advanced linear algebra*, Textbooks in Mathematics Series, CRC Press (2014).

[Löh2010] C. Löh. *Group Cohomology & Bounded Cohomology: An introduction for topologists*, lecture notes, University of Münster (2010) `http://www.mathematik.uni-regensburg.de/loeh/teaching/ggt_ws1415/lecture_notes.pdf`

[Löh2015] C. Löh. *Geometric group theory, an introduction*, lecture notes, University of Regensburg (2015).

[Löh2019] C. Löh. *Algebraic topology*, lecture notes, University of Regensburg (2019). `http://www.mathematik.uni-regensburg.de/loeh/teaching/topologie1_ws1819/lecture_notes.pdf`

[dLon2013] M. de Longueville. *A course in topological combinatorics*, Universitext, Springer (2013).

[Lov1978] L. Lovász. *Kneser's conjecture, chromatic number, and homotopy*, J. Comb. Theory, Ser. A 25 (1978), 319–324.

[Loy2017] P. Loya. *Amazing and aesthetic aspects of analysis*, Springer Verlag (2017).

[Luc1978] D. T. Luc. *One remark on the realizability of singular cohomology groups*, Math. Nachr. 82 (1978), 87–88.

[Lü1989] W. Lück. *Transformation groups and algebraic K-theory*, Lecture Notes in Mathematics 1408, Springer Verlag (1989).

[Lü1997] W. Lück. *Hilbert modules and modules over finite von Neumann algebras and applications to L^2-invariants*, Math. Ann. 309 (1997), 247–285.

[Lü1998] W. Lück. *Dimension theory of arbitrary modules over finite von Neumann algebras and L^2-Betti numbers. I: Foundations*, J. Reine Angew. Math. 495 (1998), 135–162.

[Lü2002] W. Lück. *L^2-invariants: theory and applications to geometry and K-theory*, Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics, 44. Springer Verlag, Berlin, 2002.

[Lü2005] W. Lück. *Algebraische Topologie*, Vieweg Verlag (2005)

[Lü2009] W. Lück. *L^2-invariants from the algebraic point of view*, 2009 Proceedings of the Symposium "Geometry and Cohomology in Group Theory", LMS Lecture Notes Series 358, Cambridge University Press (2009), 63–161.

[Lü2010] W. Lück. *Survey on aspherical manifolds*, Ran, André (ed.) et al., European congress of mathematics. Proceedings of the 5th ECM congress, Amsterdam, Netherlands, July 14-18, 2008. European Mathematical Society (2010).

[LW1969] A. Lundell and S. Weingram. *The topology of CW complexes*, The University Series in Higher Mathematics. Van Nostrand Reinhold Company. VIII (1969).

[Lur2009] J. Lurie. *Comparison of Smooth and PL Structures*, Harvard University lecture notes (2009) https://www.math.ias.edu/~lurie/937notes/937Lecture23.pdf

[Lur2014] J. Lurie. *The Wall Finiteness Obstruction*, Harvard University lecture notes (2014) http://www.math.harvard.edu/~lurie/281notes/Lecture2-Wall.pdf

[LuS1930] L. Lusternik and L. Schnirelmann. *Méthodes topologiques dans les problèmes variationnels*, Moscow: Gosudarstvennoe Izdat. (1930).

[LyS1977] R. Lyndon and P. Schupp. *Combinatorial group theory*, Springer Verlag (1977).

[Mac1975] S. Mac Lane. *Homology*, Reprint of the 3rd corr.rint. 1975. Classics in Mathematics. Springer Verlag (1995).

[MadM1979] I. Madsen and R. J. Milgram. *The classifying spaces for surgery and cobordism of manifolds*, Annals of Mathematics Studies, 92. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo (1979).

[MTo1997] I. Madsen and J. Tornehave. *From Calculus to Cohomology–De Rham Cohomology and Characteristic Classes*, Cambridge University Press (1997).

[Mag1974] W. Magnus. *Noneuclidean tesselations and their groups*, Pure and Applied Mathematics. 61. Academic Press (1974).

[MKS1976] W. Magnus, A. Karrass and D. Solitar. *Combinatorial group theory. Presentations of groups in terms of generators and relations*, Reprint of the 1976 second edition. Dover Publications (2004).

[Mah1962] M. Mahowald. *On the embeddability of the real projective spaces*, Proc. Am. Math. Soc. 13 (1962), 763–764.

[MahM1968] M. Mahowald and R. J. Milgram. *Embedding real projective spaces*, Ann. Math. (2) 87 (1968), 411–422.

[Mal1940] A. Malcev. *On faithful representations of infinite groups of matrices*, Mat. Sb. 8 (1940), 405–422.

[Mal1945] A. Malcev. *On the theory of the Lie groups in the large*, Rec. Math. [Mat. Sbornik] N.S. 16 (1945), 163–190.

[Mal1965] A. Malcev. *On faithful representations of infinite groups of matrices*, Amer. Math. Soc. Transl. 45 (1965), no. 2, 1–18.

[Man2016a] C. Manolescu. *Pin(2)-equivariant Seiberg-Witten Floer homology and the triangulation conjecture*, J. Amer. Math. Soc. 29 (2016), 147-176.

[Man2016b] C. Manolescu. *Lectures on the triangulation conjecture*, Proceedings of the Gökova Geometry / Topology Conference 2015, Gökova (2016), 1-38.

[MaS1982] S. Mardešić and J. Segal. *Shape theory. The inverse system approach*, North-Holland Mathematical Library, 26. North-Holland Publishing Co. (1982).

[Mark1958] A. A. Markov. *The insolubility of the problem of homeomorphy*, Dokl. Akad. Nauk SSSR 121 (1958), 218–220.

[dMar2002] R. P. C. de Marrais. *Flying Higher Than a Box-Kite: Kite-Chain Middens, Sand Mandalas, and Zero-Divisor Patterns in the 2n-ions Beyond the Sedenions* (2002) https://arxiv.org/abs/math/0207003

[Mart2016] B. Martelli. *An Introduction to Geometric Topology* (2016) https://arxiv.org/abs/1610.02592

[Mask1971] B. Maskit. *On Poincaré's theorem for fundamental polygons*, Adv. Math. 7 (1971), 219–230.

[Mass1978a] W. Massey. *Homology and cohomology theory. An approach based on Alexander-Spanier cochains*, Monographs and Textbooks in Pure and Applied Mathematics 46 (1978).

[Mass1978b] W. Massey. *How to give an exposition of the Cech-Alexander-Spanier type homology theory*, The American Mathematical Monthly 85 (1978), 75–83.

[Mass1981a] W. Massey. *Algebraic topology: An introduction*, Graduate Texts in Mathematics 56, Springer Verlag (1981).

[Mass1981b] W. Massey. *A generalization of the Alexander duality theorem*, Indiana Univ. Math. J. 30 (1981), 467–477.

[Mass1991] W. Massey. *A basic course in algebraic topology*, Graduate Texts in Mathematics 127, Springer Verlag (1991)

[Mato2002] J. Matoušek. *Lectures on discrete geometry. Graduate Texts in Mathematics, 212*, Springer-Verlag (2002).

[Mato2008] J. Matoušek. *Using the Borsuk-Ulam theorem. Lectures on topological methods in combinatorics and geometry*, Written in cooperation with Anders Björner and Günter M. Ziegler. 2nd corrected printing. Universitext. Berlin: Springer (2008).

[Matsc2014] B. Matschke. *A survey on the square peg problem*, Notices Am. Math. Soc. 61, No. 4 (2014), 346–352.

[Mata2002] Y. Matsumoto. *An introduction to Morse theory*, Translations of Mathematical Monographs. Iwanami Series in Modern Mathematics. 208. Providence, (2002).

[Mats1989] H. Matsumura. *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, 8 (1989).

[Matv1987] S. V. Matveev. *The Zeeman conjecture for nonthickenable special polyhedra is equivalent to the Andrews-Curtis conjecture*, Sibirsk. Mat. Zh. 28 (1987), 66–80, 218 and Sib. Math. J. 28 (1987), 917–928.

[Matv2006] S. V. Matveev. *Lectures on algebraic topology*, EMS Series of Lectures in Mathematics. European Mathematical Society (2006).

[Matv2007] S. V. Matveev. *Algorithmic topology and classification of 3-manifolds*, Algorithms and Computation in Mathematics 9, Springer (2007)

[MR1993] S. V. Matveev and D. Rolfsen. *Zeeman's collapsing conjecture*, Two-dimensional homotopy and combinatorial group theory, London Math. Soc. Lecture Note Ser. 197, Cambridge Univ. Press (1993), 335–364.

[Mau1980] C. R. F. Maunder. *Algebraic topology*, Cambridge University Press (1980).

[Max1870] J. C. Maxwell. *On hills and dales*, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 40:269 (1870), 421–427.

[May1999a] J. P. May. *A concise course in algebraic topology*, Chicago Lectures in Mathematics (1999).

[May1999b] J. P. May. *Stable algebraic topology, 1945-1966*, History of topology, 665–723, North-Holland, Amsterdam (1999).

[Maye1989] K. H. Mayer. *Algebraische Topologie*, Birkhäuser Verlag (1989).

[Maz1959] B. Mazur. *On embeddings of spheres*, Bull. Amer. Math. Soc. 65 (1959), 59–65.

[Maz1961] B. Mazur. *A note on some contractible 4-manifolds*, Ann. of Math. 73 (1961), 221–228.

[Maz1963] B. Mazur. *Differential topology from the point of view of simple homotopy theory*, Inst. Hautes Études Sci. Publ. Math. No. 15, (1963).

[McCoy2017] D. McCoy. *Alternating knots with unknotting number one*, Adv. Math. 305 (2017), 757–802.

[McCor1967] M. McCord. *Homotopy type comparison of a space with complexes associated with its open covers*, Proc. Amer. Math. Soc. 18 (1967), 705–708.

[McCor1969] M. McCord. *Classifying spaces and infinite symmetric products*, Trans. Am. Math. Soc. 146 (1969), 273–298.

[McS2016] D. McDuff and D. Salomon. *Introduction to symplectic topology*, 3rd edition, Oxford Graduate Texts in Mathematics 27 (2016).

[McG2016] P. McGrath. *An extremely short proof of the hairy ball theorem*, Am. Math. Mon. 123, No. 5 (2016), 502–503.

[Mc2002] C. McMullen. *The Alexander polynomial of a 3-manifold and the Thurston norm on cohomology*, Ann. Sci. Éc. Norm. Supér. (4) 35, No. 2 (2002), 153–171.

[Meie] L. Meier. *The classification of surfaces.*
`https://webspace.science.uu.nl/~meier007/ClassificationSurfaces.pdf`

[Mein2013] E. Meinrenken. *Clifford algebras and Lie theory*, Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics 58. Springer (2013).

[Mel2018] S. A. Melikhov. *Combinatorics of combinatorial topology*, Preprint (2018), arXiv: 1208.6309.

[MeS2006] S. A. Melikhov and E. V. Shchepin. *The telescope approach to embeddability of compacta*, 2006 arXiv: math.GT/0612085.

[Mes1972] S. Meskin. *Nonresidually finite one-relator groups*, Trans. Amer. Math. Soc. 164 (1972), 105–114.

[Met1967] W. Metzler. *Beispiele zu Unterteilungsfragen bei CW- und Simplizialkomplexen*, Arch. Math. (Basel) 18 (1967), 513–519.

[Meu2008] F. Meunier. *Discrete Splittings of the Necklace*, Mathematics of Operations Research 33 (2008), 678–688.

[Meyr2000] C. Meyer. *Matrix analysis and applied linear algebra*, Society for Industrial and Applied Mathematics (2000).

[Meye1981] M. Meyerson. *Convexity and the table theorem*, Pacific J. Math. 97 (1981), 167–169.

[Meye1982] M. Meyerson. *Balancing acts*, Topology, Proc. Conf., Vol. 6, No.1, Blacksburg/Va. 1981 (1982), 59–75.

[Mic1980] P. W. Michor. *Manifolds of differentiable mappings*, Shiva Mathematics Series 3, Shiva Publishing Limited. IV (1980).

[Mic2008] P. W. Michor. *Topics in differential geometry*, Graduate Studies in Mathematics 93. American Mathematical Society (2008).

[MW2017] F. Michel and C. Weber. *Higher-dimensional knots according to Michel Kervaire*, EMS Series of Lectures in Mathematics. Zürich: European Mathematical Society (2017).

[MillC1992] C. F. Miller III. *Decision problems for groups-survey and reflections*, Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989), 1–59, Math. Sci. Res. Inst. Publ., 23, Springer 1992.

[MillH1989] H. Miller. *Homotopy theory of the vector field problem*, Course notes from MIT course, 1989.
`http://www-math.mit.edu/~hrm/papers/vf1.pdf`
`http://www-math.mit.edu/~hrm/papers/vf2.pdf`

[MillH2021] H. Miller. *Lectures on Algebraic Topology*, Course notes from MIT cours, 2021.
`https://math.mit.edu/~hrm/papers/lectures-905-906.pdf`

[MillR1972] R. T. Miller. *Approximating codimension 3 embeddings*, Ann. Math. (2) 95 (1972), 406–416.

[Miln1956a] J. Milnor. *On manifolds homeomorphic to the 7-sphere*, Annals of Mathematics 64 (1956), 399–405.

[Miln1956b] J. Milnor. *Construction of universal bundles. I*, Ann. of Math. 63 (1956), 272–284.

[Miln1957a] J. W. Milnor. *Lectures on characteristic classes*, notes by J. Stasheff, Princeton University, 1957.

[Miln1957b] J. W. Milnor. *Groups which act on S^n without fixed points*, Amer. J. Math. 79 (1957), 623–630.

[Miln1958a] J. Milnor. *Some consequences of a theorem of Bott*, Ann. of Math. 68 (1958), 444–449.

[Miln1958b] J. Milnor. *On simply connected 4-manifolds*, Sympos. Int. Topologia Algebraica (1958), 122–128.

[Miln1959a] J. Milnor. *On spaces having the homotopy of a CW-complex*, Trans. Am. Math. Soc. 90 (1959), 272–280.

[Miln1959b] J. Milnor. *Differentiable manifolds which are homotopy spheres*, Princeton University (1959).

[Miln1961] J. Milnor. *Two complexes which are homeomorphic but combinatorially distinct*, Ann. Math. 74 (1961), 575–590.

[Miln1962a] J. Milnor. *A unique factorization theorem for 3-manifolds*, Amer. J. Math. 84 (1962), 1-7.

[Miln1962b] J. Milnor. *On axiomatic homology theory*, Pacific J. Math. 12 (1962), 337–341.

[Miln1963a] J. Milnor. *Morse Theory*, Annals of Mathematics Studies. No. 51. Princeton, N.J.: Princeton University Press (1963).

[Miln1963b] J. Milnor. *Spin structures on manifolds*, L'Enseignement Math. 9 (1963), 198–203. Reprinted in Collected papers of John Milnor. III. Differential topology. American Mathematical Society (2007), 293–298.

[Miln1964a] J. Milnor. *Microbundles*, Topology 3, Suppl. 1 (1964), 53–80.

[Miln1964b] J. Milnor. *Differential topology*, 1964 Lectures on Modern Mathematics, Vol. II (1964), 165–183.

[Miln1964c] J. Milnor. *Remarks Concerning Spin Manifolds*, Differential and combinatorial topology, a symposium in honor of Marston Morse, Princeton University Press (1965), 55–62.

[Miln1965a] J. Milnor. *Topology from the differentiable viewpoint*, based on notes by David W. Weaver. Revised 2nd ed. Princeton Landmarks in Mathematics (1997).

[Miln1965b] J. Milnor. *Lectures on the h-cobordism theorem*, Notes by L. Siebenmann and J. Sondow. Princeton Mathematical Notes. Princeton University Press (1965).

[Miln1965c] J. Milnor. *Remarks Concerning Spin Manifolds*, Differential and combinatorial topology, a symposium in honor of Marston Morse, Princeton University Press (1965), 55–62.

[Miln1966] J. Milnor. *Whitehead torsion*, Bull. Amer. Math. Soc. 72 (1966), 358–426.

[Miln1968] J. Milnor. *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, No. 61 Princeton University Press (1968).

[Miln1972] J. Milnor. *On the construction FK*, in F. Adams: Algebraic topology-a student's guide, Cambridge University Press (1972), 118–136.

[Miln1978] J. Milnor. *Analytic proofs of the "Hairy ball theorem" and the Brouwer fixed point theorem*, Am. Math. Mon. 85 (1978), 521–524.

[Miln2000] J. Milnor. *Classification of $(n-1)$-connected $2n$-dimensional manifolds and the discovery of exotic spheres*, Cappell, Sylvain (ed.) et al., Surveys on surgery theory. Vol. 1: Papers dedicated to C. T. C. Wall on the occasion to his 60th birthday. Princeton University Press. Ann. Math. Stud. 145 (2000), 25–30.

[Miln2009] Milnor, John. *Collected papers of John Milnor. IV. Homotopy, homology and manifolds*, Edited by John McCleary. American Mathematical Society (2009).

[MH1973] J. Milnor and D. Husemoller. *Symmetric bilinear forms*, Springer Verlag 1973.

[MiS1974] J. Milnor and J. Stasheff. *Characteristic classes*, Annals of Mathematics Studies. No. 76. Princeton, N.J.: Princeton University Press (1974).

[Miy1952] H. Miyazaki. *The paracompactness of CW-complexes*, Tohoku Math. J. (2) 4 (1952), 309–313.

[Möb1863] A. F. Möbius. *Theorie der elementaren Verwandtschaft*, Berichte über die Verhandlungen der königlich sächsischen Gesellschaft der Wissenschaften zu Leipzig, mathematisch-physische Classe 15 (1863), 18–57. A. F. Möbius, Gesammelte Werke t.2 (1886), p.433–471.

[MTh2001] B. Mohar and C. Thomassen. *Graphs on surfaces*, Johns Hopkins Studies in the Mathematical Sciences (2001).

[Moi1952] E. Moise. *Affine structures in 3-manifolds V. The triangulation theorem and Hauptvermutung*, Annals of Mathematics 56 (1952), 96–114.

[Moi1977] E. Moise. *Geometric Topology in Dimensions 2 and 3*, Graduate Texts in Mathematics 47, Springer Verlage (1977)

[Mon1976] J. Montesinos. *Three-manifolds as 3-fold branched coverings of S^3*, Quart. J. Math. Oxford Ser. (2) 27 (1976), 85–94.

[MoR2009] S. Montiel and A. Ros. *Curves and Surfaces*, Graduate Studies in Mathematics 69. American Mathematical Society (2009).

[Moo2008] G. Moore. *The emergence of open sets, closed sets, and limit pointsin analysis and topology*, Historia Mathematica 35 (2008), 220–241.

[More1997] R. Moreno. *The zero divisors of the Cayley-Dickson algebras over the real numbers*, unpublished paper (1997), arXiv:q-alg/971001.

[Morg1984] J. Morgan. *On Thurston's uniformization theorem for three-dimensional manifolds*, pp. 37-125 in The Smith conjecture (New York, 1979), edited by J. W. Morgan and B. Hyman, Pure Appl. Math. 112, Academic Press (1984).

[Morg1996] J. Morgan. *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*, Mathematical Notes 44. Princeton Univ. Press. (1996).

[MB1984] J. Morgan and H. Bass. *The Smith conjecture*, Pure and Applied Mathematics, 112, Boston, MA: Academic Press, (1984).

[MoS1988] J. Morgan and P. Shalen. *Degenerations of hyperbolic structures. II*, Measured laminations in 3-manifolds. Ann. of Math. (2) 127 (1988), no. 2, 403–456.

[MoM1986] J. Morgan and I. Morrison. *A van Kampen theorem for weak joins*, Proc. Lond. Math. Soc., III. Ser. 53 (1986), 562–576.

[MTi2007] J. Morgan and G. Tian. *Ricci flow and the Poincaré conjecture*, Clay Mathematics Monographs 3. Providence, RI: American Mathematical Society (2007).

[MorsA1939] A. Morse. *The behaviour of a function on its critical set*, Ann. Math 40 (1939), 62–70.

[MorsM1934] M. Morse. *The Calculus of Variations in the Large*, American Mathematical Society (1934).
[Mos1971] L. Moser. *Elementary surgery along a torus knot*, Pac. J. Math. 38 (1971), 737–745.
[MTa1968] R. E. Mosher and M. C. Tangora. *Cohomology operations and applications in homotopy theory.* Harper's Series in Modern Mathematics, Harper and Row (1968).
[Mos1949] G. D. Mostow. *A new proof of E. Cartan's theorem on the topology of semi-simple groups*, Bull. Am. Math. Soc. 55 (1949), 969–980.
[Mró1959] S. Mrówka. *Compactness and product spaces*, Colloq. Math. 7 (1959), 19–22.
[Muk2015] A. Mukherjee. *Differential topology*, 2nd ed. Hindustan Book Agency, Birkhäuser/Springer Verlag (2015).
[Mun1959a] J. Munkres. *Obstructions to the smoothing of piecewise-differentiable homeomorphisms*, Bull. Amer. Math. Soc. 65 (1959), 332–334.
[Mun1960a] J. Munkres. *Obstructions to the smoothing of piecewise-differentiable homeomorphisms*, Ann. of Math. (2) 72 (1960), 521–554.
[Mun1960b] J. Munkres. *Differentiable isotopies on the* 2*-sphere*, Michigan Math. J. 7 (1960), 193–197.
[Mun1966a] J. Munkres. *Elementary Differential Topology*, revised edition, Annals of Mathematics Studies 54, Princeton University Press, 1966.
[Mun1966b] J. Munkres. *Concordance is equivalent to smoothability*, Topology 5 (1966), 371–389.
[Mun1967] J. Munkres. *Concordance of differentiable structures-two approaches*, Mich. Math. J. 14 (1967), 183–191.
[Mun1975] J. Munkres. *Topology. A first course*, Englewood Cliffs, N.J.: Prentice-Hall, Inc. XVI (1975).
[Mun1984] J. Munkres. *Elements of algebraic topology*, Advanced Book Program, Addison-Wesley Publishing Company, Inc. (1984).
[Mur2021] A. G. Murray. *More Counterexamples to the Unit Conjecture for Group Rings*, preprint, arXiv:2106.02147 (2021).
[Naga1950] J. Nagata. *On a necessary and sufficient condition for metrizability*, J. Inst. Polytech. Osaka City Univ. Ser. A. 1 (1950), 93–100.
[Naga1983] J. Nagata. *Modern dimension theory*,. Rev. and ext. ed. Sigma Series in Pure Mathematics 2, Heldermann Verlag (1983).
[Nago2007] A. Nagórko. *Carrier and nerve theorems in the extension theory*, Proc. Am. Math. Soc. 135, No. 2 (2007), 551-558.
[Nak1981] Y. Nakanishi. *Primeness of links*, Math. Sem. Notes Kobe Univ. 9 (1981), 415–440.
[NR2004] T. Napier and M. Ramachandran. *Elementary construction of exhausting subsolutions of elliptic operators*, Enseign. Math., II. Sér. 50, No. 3-4 (2004), 367–390.
[Nel] P. Nelson. *Free product factorization*
`http://www.math.uchicago.edu/~may/VIGRE/VIGRE2008/REUPapers/Nelson.pdf`
[Neuk1999] J. Neukirch. *Algebraic number theory*, Springer Verlag (1999)
[Neum1943] B. H. Neumann. *On the number of generators of a free product*, J. London Math. Soc. 18, (1943), 12–20.
[Neum1953] B. H. Neumann. *On a problem of Hopf*, J. London Math. Soc. 28 (1953), 351–353.
[NN1957] A. Newlander and L. Nirenberg. *Complex analytic coordinates in almost complex manifolds*, Ann. Math. 65 (1957), 391–404.
[New1926] M. H. A. Newman. *On the foundations of combinatorial Analysis Situs*, Proc. Royal Acad. Amsterdam 29 (1926) 610–641.
[New66] M. H. A. Newman. *The engulfing theorem for topological manifolds*, Ann. Math. 84 (1966), 555–571.
[NewM1972] M. Newman. *Integral matrices*, Pure and Applied Mathematics, 45. Academic Press. XVII (1972).
[NewM1985] M. Newman. *A note on Fuchsian groups*, Illinois J. Math. 29 (1985), no. 4, 682–686.
[Ni2007] Y. Ni. *Knot Floer homology detects fibred knots*, Inv. Math. 170 (2007), 577–608.
[Nic2000] L. Nicolaescu. *Notes on Seiberg-Witten theory*, Graduate Studies in Mathematics 28. American Mathematical Society (2000).
[Nic2011] L. Nicolaescu. *An invitation to Morse theory*, Universitext, Springer Verlag (2011).

[Nic2020] L. Nicolaescu. *Lectures on the geometry of manifolds*, 3rd edition. World Scientific (2020).
[Nie1924] J. Nielsen. *Die Isomorphismengruppe der freien Gruppen*, Math. Ann. 91 (1924), 169–209.
[Nor2012] C. Norman. *Finitely generated Abelian groups and similarity of matrices over a field*, Springer Undergraduate Mathematics Series (2012).
[Nog1987] A. Noga. *Splitting Necklaces*, Adv. Math. 63 (1987), 247–253.
[Nov1955] S. Novikov. *On the algorithmic unsolvability of the word problem in group theory*, Trudy Mat. Inst. Steklov 44 (1955)
[Nov1965] S. Novikov. *Topological invariance of rational Pontryagin classes*, Dokl. Akad. Nauk SSSR 163 (1965), 298–300.
[Nov1970] S. Novikov. *Pontrjagin classes, the fundamental group and some problems of stable algebra*, 1970 Essays on Topology and Related Topics (MÃ©moires dÃ©diés Ã Georges de Rham) Springer (1970), 147–155.
[Nov1981] S. Novikov. *Multi-valued functions and junctionals. An analogue of Morse theory*, Soviet Math. Doklady, 24(1981), 222–226.
[Nov1982] S. Novikov. *The Hamiltonian formalism and a multi-valued analogue of Morse theory*, Russian Math. Surveys 37 (1982), 1–56.
[NRG2004] S. Novikov, V. Rokhlin and R. Gamkrelidze. *Topology II. Homotopy and homology. Classical manifolds*, Encyclopaedia of Mathematical Sciences 24. Springer-Verlag (2004).
[Nyi1992] P. Nyikos. *Various smoothings of the long line and their tangent bundles*, Adv. Math. 93 (1992), 129–213.
[Ol2005] Y. Ollivier. *A January 2005 invitation to random groups*, Ensaios Matemáticos 10. Rio de Janeiro: Sociedade Brasileira de Matemática (2005).
`http://www.yann-ollivier.org/rech/publs/randomgroups.pdf`
[OTL2016] K. Ono and S. Trebat-Leder. *The 1729 K3 surface*, Res. Number Theory 2, Paper No. 26 (2016)
[Osb1962] J. M. Osborn. *Quadratic division algebras*, Trans. Amer. Math. Soc. 105 (1962), 202–221.
[Osg1903] W. Osgood. *A Jordan Curve of Positive Area*, Trans. Amer. Math. Soc. 4 (1903), 107–112.
[Ost1826] M. Ostragradsky. *Démonstration d'un th'eorème du calcul intégral* (1826).
[OR2009] E. Outerelo and J. Ruiz. *Mapping degree theory*, Graduate Studies in Mathematics 108 (2009).
[Ox1971] J. C. Oxtoby. *Measure and Category*, Graduate Texts in Mathematics 2, Springer Verlag (1971).
[OS2003] P. Ozsváth and Z. Szabó. *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Advances in Mathematics 173 (2003), 179–261.
[OSS2015] P. Ozsváth, A. Stipsicz and Z. Szabó. *Grid homology for knots and links*, Mathematical Surveys and Monographs Series Profile 208. American Mathematical Society (2015).
[Pac1978] U. Pachner. *Bistellare Äquivalenz kombinatorischer Mannigfaltigkeiten*, Arch. Math. (Basel) 30 (1978), no. 1, 89–98.
[Pac1991] U. Pachner. *Homeomorphic manifolds are equivalent by elementary shellings*, European J. Combin. 12 (1991), 129–145.
[Paj2006] A. Pajitnov. *Circle-valued Morse theory*, De Gruyter Studies in Mathematics 32 (2006)
[Pal1960a] R. Palais. *Extending diffeomorphisms*, Proc. Am. Math. Soc. 11 (1960), 274–277.
[Pal1960b] R. Palais. *Local triviality of the restriction map for embeddings*, Comment. Math. Helv. 34 (1960), 305-312.
[Pal1961] R. Palais. *On the existence of slices for actions of non-compact Lie groups*, Ann. Math. (2) 73 (1961), 295–323.
[Pal1963] R. Palais. *Morse theory on Hilbert manifolds*, Topology 2 (1963), 299–340.
[Pal1968] R. Palais. *The classification of real division algebras*, Am. Math. Mon. 75 (1968), 366–368.
[Pap1946] C. Papakyriakopoulos. *A new proof for the invariance of the homology groups of a complex*, Bull. Soc. Math. Greece 2 (1946), 1–150.
[Pap1957a] C. Papakyriakopoulos. *On Dehn's lemma and the asphericity of knots*, Annals of Mathematics 66 (1957), 1–26.
[Pap1957b] C. Papakyriakopoulos. *On solid tori*, Proc. London Math. Soc. (3) 7 (1957), 281-299.
[Pare1970] B. Pareigis. *Categories and functors*, Pure and Applied Mathematics 39, Academic Press (1970).

[Park2005] J. Park. *Simply connected symplectic 4-manifolds with $b_2^+ = 1$ and $c_1^2 = 2$*, Invent. Math. 159 (2005), 657–667.

[Pat2009] C. W. Patty. *The Nagata-Smirnov Metrization Theorem*, Foundations of Topology (2nd ed.), Jones & Bartlett (2009), 257–262.

[Pav2020] P. Pavešić. *Universal covering spaces, a footnote*, Preprint, arXiv:2012.15264 (2020).

[Pean1890] G. Peano. *Sur une courbe, qui remplit toute une aire plane*, Math. Ann. 36 (1890), 157–160.

[Pear1975] A. R. Pears. *Dimension Theory of General Spaces,* Cambridge University Press (1975).

[PedE1977] E. K. Pedersen. *Regular neighborhoods in topological manifolds*, Mich. Math. J. 24 (1977), 177–183.

[PedS2015] S. Pedersen. *From Calculus to Analysis*, Springer International Publishing (2015).

[Per2002] G. Perelman. *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math.DG/0211159

[Per2003a] G. Perelman. *Ricci flow with surgery on three-manifolds*, arXiv:math.DG/0303109.

[Per2003b] G. Perelman. *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv:math.DG/0307245.

[PS2008] C. Peters and J. Steenbrink. *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge 52. Springer Verlag (2008).

[Pets1984] S. Peters. *Cheeger's finiteness theorem for diffeomorphism classes of Riemannian manifolds*, J. reine und angew. Math. 349 (1984), 77–82.

[Petn1979] M. Peterson. *Dante and the 3-sphere*, American Journal of Physics 47, Issue 12 (1979), 1031–1035.

[Pf1965] A. Pfister. *Zur Darstellung von -1 als Summe von Quadraten in einem Körper*, J. London Math. Soc. 40 (1965), 159–165.

[Pf1995] A. Pfister. *Quadratic forms with applications to algebraic geometry and topology*, London Mathematical Society Lecture Note Series. 217. Cambridge Univ. Press. (1995).

[Pic1992] R. Piccinini. *Lectures on homotopy theory*, North-Holland Mathematics Studies, 171. North-Holland Publishing Co. (1992)

[Pie1984] J.-P. Pier. *Amenable Locally Compact Groups*, Pure and Applied Mathematics, John Wiley & Sons, Inc., New York, 1984.

[Pl1980] S. Plotnick. *Vanishing of Whitehead groups for Seifert manifolds with infinite fundamental group*, Comment. Math. Helv. 55 (1980), no. 4, 654–667.

[Poi1885] H. Poincaré. *Mémoire sur les courbes définies par une équation différentielle*, Jour. Math. Pures et Appl. 4, (1885), 167-244.

[Poi1895] H. Poincaré. *Analysis situs*, Journal de l'École Polytechnique 1 (1895), 1–123.

[Poi1900] H. Poincaré. *Second complément l'Analysis Situs*, Proc. Lond. Math. Soc. 32 (1900), 277–308.

[Poi1904] H. Poincaré. *Cinquième complément à l'analysis situs*, Rendiconti del Circolo Matematico di Palermo. 18 (1904), 45–110.

[Poi2010] H. Poincaré. *Papers on Topology: Analysis Situs and Its Five Supplements*, translated by John Stillwell, Amer. Math. Soc. (2010)

[PGL1953] H. Poincaré, R. Garnier and J. Leray. *Oeuvres de Henri Poincaré. Tome VI*, Académie des sciences, Section de géométrie. Gauthier-Villars (1953).

[Pol2001] B. T. Polyak. *Convexity of nonlinear image of a small ball with applications to optimization*, Wellposedness in optimization and related topics (Gargnano, 1999). Set-Valued Anal. 9 (2001), no. 1-2, 159–168.

[Pon38] L. S. Pontryagin. *A classification of continuous transformations of a complex into a sphere I, II*, C. R. (Doklady) Acad. Sci. URSS (N.S.) 19 (1938), 147–149, 361–363.

[Pon1950a] L. S. Pontryagin. *Characteristic cycles on differentiable manifolds*, Mat. Sbornik N. S. 21(63) (1947), 233–284; A.M.S. Translation 32 (1950).

[Pon1950b] L. S. Pontryagin. *Homotopy classification of mappings of an $(n+2)$- dimensional sphere on an n-dimensional one*, Doklady Akad. Nauk SSSR (N. S.), 19 (1950), 957–959.

[Pon1952] L. S. Pontryagin. *Foundations of combinatorial topology*, Graylock Press (1952).

[Pon1959] L. S. Pontryagin. *Smooth manifolds and their applications in homotopy theory*, Am. Math. Soc., Transl., II. Ser. 11 (1959), 1–114.

[PoR2021] M. Powell, A. Ray et al. *Topological manifolds*, lecture notes (2021). `https://maths.dur.ac.uk/users/mark.a.powell/topological-manifolds-lecture-notes.pdf`

[Pra2006] V. V. Prasolov. *Elements of combinatorial and differential topology*, Graduate Studies in Mathematics, 74. American Mathematical Society (2006).

[Pra2007] V. V. Prasolov. *Elements of homology theory*, Graduate Studies in Mathematics 81. Providence, RI: American Mathematical Society (2007).

[PrS1997] V. V. Prasolov and A. B. Sossinsky. *Knots, links, braids and 3-manifolds. An introduction to the new invariants in low-dimensional topology*, Translations of Mathematical Monographs. 154. American Mathematical Society (1997)

[Pre2002] G. Preuß. *Foundations of Topology. An Approach to Convenient Topology*, Kluwer Academic Publishers (2002)

[PM2007] A. O. Prishlyak and K. I. Mischenko. *Classification of noncompact surfaces with boundary*, Methods Funct. Anal. Topol. 13 (2007), 62–66.

[Pro2007] C. Procesi. *Lie groups. An approach through invariants and representations*, Universitext, Springer Verlag (2007).

[PW2019] J. Prüss and M. Wilke. *Gewöhnliche Differentialgleichungen und dynamische Systeme*, 2nd, revised and extended edition, Grundstudium Mathematik, Birkhäuser Verlag (2019).

[PT1987] J. Przytycki and P. Traczyk. *Invariants of links of Conway type*, Kobe J. Math. 4 (I987), 115-139.

[Put] A. Putman. *Homotopy groups of spheres and low-dimensional topology* `https://www3.nd.edu/~andyp/notes/HomotopySpheresLowDimTop.pdf`

[Put2] A. Putman. *A quick proof of the classification of surfaces* `https://www3.nd.edu/~andyp/notes/ClassificationSurfaces.pdf`

[Qu1982] F. Quinn. *Ends of maps. III: Dimensions 4 and 5*, J. Differ. Geom. 17 (1982), 503–521.

[Qu1983] F. Quinn. *The stable topology of 4-manifolds*, Topology Appl. 15 (1983), 71–77.

[Qu2010] F. Quinn. *A controlled-topology proof of the product structure theorem*, Geom. Dedicata 148 (2010), 303–308.

[Rabi1958] M. Rabin. *Recursive unsolvability of group theoretic problems*, Ann. of Math. (2) 67 (1958), 172–194.

[Rabo2010] D. Rabouin. *What Descartes knew of mathematics in 1628*, Historia Mathematica 37 (2010), 428–459.

[Rad1926] T. Radó. *Über den Begriff der Riemannschen Fläche*, Acta Szeged 2 (1926), 101–121.

[Ram2005] S. Ramanan. *Global calculus*, Graduate Studies in Mathematics 65 (2005).

[Ran1996] A. Ranicki. *On the Hauptvermutung*, The Hauptvermutung book, 3–31, K-Monogr. Math., 1, Kluwer Acad. Publ. (1996)

[Ran2002] A. Ranicki. *Algebraic and geometric surgery*, Oxford University Press (2002).

[Rap2016] G. Raptis. *Homotopy invariance of pushouts along cofibrations*, Algebraic topology II, University of Regensburg (2016).

[Rat2019] J. G. Ratcliffe. *Foundations of hyperbolic manifolds*, 3rd expanded edition. Graduate Texts in Mathematics 149. Springer (2019).

[RW1980] D. C. Ravenel and W. S. Wilson. *The Morava K-theories of Eilenberg-Mac Lane spaces and the Conner-Floyd conjecture*, Amer. J. Math. 102 (1980), 691–748.

[Reeb1946] G. Reeb. *Sur les points singuliers d'une forme de Pfaff complètement intégrable ou d'une fonction numérique*, C. R. Acad. Sci. Paris 222 (1946), 847–849.

[Rees1983] E. Rees. *Notes on geometry*, Universitext, Springer Verlag (1983).

[Rei1932] K. Reidemeister. *Knotentheorie*, Ergebnisse der Mathematik (1932).

[Rei1935] K. Reidemeister, *Homotopieringe und Linsenräume*, Abh. Math. Sem. Univ. Hamburg 11 (1935), no. 1, 102–109.

[Rei1936] K. Reidemeister. *Kommutative Fundamentalgruppen*, Monatsh. f. Math. und Physik, 43 (1936), 20–28.

[RN2011] R. Ricca and B. Nipoti. *Derivation and interpretation of the Gauss linking number*, In Introductory Lectures on Knot Theory. Series on Knots and Everything 46. World Scientific (2011), 482-501.

[Ric1963] I. Richards. *On the classification of noncompact surfaces*, Trans. Amer. Math. Soc. 106 (1963), 259–269.

[Rie2016] E. Riehl. *Category theory in context*, Aurora Dover Modern Math Originals. Dover Publications (2016).

[Ris2006] V. Rispoli. *Il "valore rendez-vous" di uno spazio metrico*, Università di Pisa (2006) `http://people.dm.unipi.it/acquistp/rispoli.pdf`

[Rob1995] D. Robinson. *A course in the theory of groups*, 2nd ed, Graduate Texts in Mathematics 80, Springer Verlag (1995).

[Rok1951] V. A. Rohlin. *A three-dimensional manifold is the boundary of a four-dimensional one*, Doklady Akad. Nauk SSSR (N.S.) 81 (1951), 355–357.

[Rok1952] V. Rokhlin. *New results in the theory of four-dimensional manifolds*, Doklady Acad. Nauk. SSSR (N.S.) 84 (1952), 221–224.

[Rok1986a] V. Rokhlin. *Nouveaux resultats dans la théorie des varietés de dimension 4*, in L. Guillou, A. Marin: A la recherche de la topologie perdue. Progress in Mathematics 62. Birkhäuser (1986).

[Rok1986b] V. Rokhlin. *Classification des applications de la $(n+3)$-sphère dans la n-sphère*, in L. Guillou, A. Marin: A la recherche de la topologie perdue. Progress in Mathematics 62. Birkhäuser (1986).

[Rolf1990] D. Rolfsen. *Knots and links*, Mathematics Lecture Series. 7. Houston, TX: Publish or Perish. (1990).

[Roll2009] P. Rolli. *Quasi-morphisms on Free Groups*, preprint, arXiv: 0911.4234 (2009).

[Rosb1994] J. Rosenberg. *Algebraic K-theory and its applications*, Graduate Texts in Mathematics 147, Springer Verlag (1994).

[Rosm2000] D. Roseman. *Projections of codimension two embeddings*, Gordon, Cameron McA. (ed.) et al., Knots in Hellas'98. Proceedings of the international conference on knot theory and its ramifications, European Cultural Centre of Delphi, Greece, August 7–15, 1998. Singapore: World Scientific. Ser. Knots Everything 24 (2000), 380–410.

[Ross2013] K. A. Ross. *Elementary analysis. The theory of calculus*, in collaboration with Jorge M. López. 2nd revised and updated ed. Undergraduate Texts in Mathematics. Springer (2013).

[Rot1988] J. Rotman. *An introduction to algebraic topology*, Graduate Texts in Mathematics 119, Springer Verlag (1988).

[Rot2002] J. Rotman. *Advanced modern algebra*, Prentice Hall/Pearson Education. xv (2002).

[Rot2009] J. Rotman. *An introduction to homological algebra*, 2nd edition, Universitext, Springer Verlag (2009).

[Rou1985] C. P. Rourke. *A new proof that Ω_3 is zero*, J. London Math. Soc. 31 (1985), 373–376.

[RS1968] C. P. Rourke and B. J. Sanderson. *Block bundles. III. Homotopy theory*, Ann. of Math. (2) 87 (1968), 431-483.

[RS1972] C. P. Rourke and B. J. Sanderson. *Introduction to piecewise-linear topology*, Ergebnisse der Mathematik und ihrer Grenzgebiete 69 (1972).

[Rub1997] D. Ruberman. *Null-homotopic embedded spheres of codimension one*, Tight and taut submanifolds (Berkeley, CA, 1994), 229–232, Math. Sci. Res. Inst. Publ., 32, Cambridge Univ. Press (1997).

[RudiM1969] M. E. Rudin. *A new proof that metric spaces are paracompact*, Proc. Amer. Math. Soc. 20 (1969), 603.

[RudiW1976] W. Rudin. *Principles of mathematical analysis*, Third edition. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co. (1976)

[RudiW1987] W. Rudin. *Real and complex analysis*, Third edition. McGraw-Hill Book Co. (1987)

[Rudy1998] Y. Rudyak. *On Thom spectra, orientability, and cobordism*, Springer Monographs in Mathematic (1998).

[Rudy2016] Y. Rudyak. *Piecewise linear structures on topological manifolds*, World Scientific (2016).

[Rudy2017] Y. Rudyak. *Maps of degree 1 and Lusternik-Schnirelmann category*, Topology Appl. 221 (2017), 225–230.

[Rus1973] B. T. Rushing. *Topological embeddings*, Pure and Applied Mathematics 52. Academic Press (1973).

[Sag1994] H. Sagan. *Space-Filling Curves*, Universitext, Springer Verlag (1994).

[Saka2013] K. Sakai. *Geometric aspects of general topology*, Springer Monographs in Mathematics (2013).

[Sakk1990] T. Sakkalis. *On a theorem by Hopf*, Int. J. Math. Math. Sci. 13 (4) (1990), 813–816.

[Saku1981] M. Sakuma. *Periods of composite links*, Math. Sem. Notes Kobe Univ. 9 (1981), 445–452.

[Sam1941] H. Samelson. *Beiträge zur Topologie der Gruppenmannigfaltigkeiten*, Ann. of Math. 2, 42, (1941), 1091 - 1137.

[Sam1952] H. Samelson. *Topology of Lie groups*, Bull. Amer. Math. Soc. 58, (1952), 2–37.

[Sar1942] A. Sard. *The measure of the critical values of differentiable maps*, Bull. Amer. Math. Soc. 48 (1942), 883–890.

[Sav2002] N. Saveliev. *Invariants for homology 3-spheres*, Encyclopaedia of Mathematical Sciences 140. Low-Dimensional Topology 1. Springer Verlag (2002).

[Sav2012] N. Saveliev. *Lectures on the topology of 3-manifolds. An introduction to the Casson invariant*, Second revised edition. de Gruyter Textbook (2012)

[SchafJ1970] J. Schafer. *Topological Pontrjagin classes*, Comment. Math. Helv. 45 (1970), 315–332.

[SchafR1966] R. D. Schafer. *An introduction to nonassociative algebras*, Pure and Applied Mathematics, 22. Academic Press. (1966).

[Schar2002] M. Scharlemann. *Heegaard splittings of compact 3-manifolds*, Daverman, R. J. (ed.) et al., Handbook of geometric topology. Amsterdam: Elsevier (2002), 921–953.

[Schle2017] S. Schleimer. *Waldhausen's theorem*, Gordon, Cameron (ed.) et al., Proceedings of the Technion workshop on Heegaard splittings, Haifa, Israel, summer 2005. Geometry and Topology Monographs 12 (2017), 299–317.

[Schoe1938] I. J. Schoenberg. *On the Peano curve of Lebesgue*, Bull. Amer. Math. Soc. 44 (1938), 519.

[Schni1944] L. G. Schnirelman. *On some geometric properties of closed curves*, Usp. Mat. Nauk 10 (1944), 34–44.

[Schol2019] P. Scholze. *Lectures on Condensed Mathematics*, lecture notes, University of Bonn (2019). `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf`

[Schol2019b] P. Scholze. *Lectures on Analytic Geometry*, lecture notes, University of Bonn (2019). `https://www.math.uni-bonn.de/people/scholze/Analytic.pdf`

[Schre1927] O. Schreier. *Die Untergruppen der freien Gruppen*, Abhandlungen Hamburg 5 (1927), 161–183.

[SchubH1956] H. Schubert. *Knoten mit zwei Brücken*, Math. Zeitschr. 65 (1956), 133–170.

[SchubH1968] H. Schubert. *Topology*, Allyn and Bacon (1968).

[SchubJ2017] J. Schubert. *Klassifikation triangulierter kompakter Flächen*, Bachelor thesis, University of Regensburg (2017).

[Schul2014] J. Schultens. *Introduction to 3-manifolds*, Graduate Studies in Mathematics 151. American Mathematical Society (2014).

[Schwe1979] H. Schwerdtfeger. *Geometry of complex numbers. Circle geometry, Moebius transformation, non-Euclidean geometry*, A corrected reprinting of the 1962 edition. Dover Books on Advanced Mathematics. Dover Publications (1979).

[Scor2005] A. Scorpan. *The wild world of 4-manifolds*, Providence, RI: American Mathematical Society (2005).

[Scot1978] P. Scott. *Subgroups of surface groups are almost geometric*, J. London Math. Soc. 17 (1978), 555–565.

[ScW1979] P. Scott and C. T. C. Wall. *Topological methods in group theory*, Homological group theory, Proc. Symp., Durham 1977, Lond. Math. Soc. Lect. Note Ser. 36 (1979), 137–203.

[Scov2019] N. A. Scoville. *Discrete Morse theory*, Student Mathematical Library 90. American Mathematical Society (2019).

[Sei1933] H. Seifert. *Verschlingungsinvarianten*, Sitzungsber. Preuß. Akad. Wiss., Phys.-Math. Kl. 1933, No.26-29 (1933), 811–828.

[Sei1934] H. Seifert. *Über das Geschlecht von Knoten*, Math. Annalen 110 (1934), 571–592.

[ST1931] H. Seifert and W. Threlfall. *Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes*, Math. Ann. 104 (1931), no.1, 1–70.
[ST1934] H. Seifert and W. Threlfall. *Lehrbuch der Topologie*, Teubner Verlag (1934).
[ST1980] H. Seifert and W. Threlfall. *A textbook of topology*, Pure and Applied Mathematics, 89 (1980).
[SeW1933] C. Weber and H. Seifert. *Die beiden Dodekaederräume*, Math. Z. 37 (1933), no.1, 237–253.
[Sep2007] M. Sepanski. *Compact Lie Groups*, Graduate Texts in Mathematics 235, Springer Verlag (2007)
[Serd2010] D. Serre. *Matrices. Theory and Applications*, Second Edition, Graduate Texts in Mathematics 216 (2010).
[Ser1953] J.-P. Serre. *Groupes d'homotopie et classes de groupes abéliens*, Ann. of Math. (2) 58 (1953), 258–294.
[Ser1973] J.-P. Serre. *A course in arithmetic*, Graduate Texts in Mathematics 7. Springer Verlag (1973).
[Ser1980] J.-P. Serre. *Trees*, Springer Verlag, Berlin-New York, 1980.
[Shah2016] S. Shahshahani. *An introductory course on differentiable manifolds*, Aurora Dover Modern Math Originals. Dover Publications (2016).
[Shal1984] P. Shalen. *A "piecewise-linear" method for triangulating 3-manifolds*, Adv. in Math. 52 (1984), no. 1, 34–80.
[Shark1993] V. V. Sharko. *Functions on manifolds. Algebraic and topological aspects*, Translations of Mathematical Monographs 131. American Mathematical Society (1993).
[Sharm1985] B. L. Sharma. *Topologically invariant integral characteristic classes*, Topology Appl. 21 (1985), 135–146.
[Shas2011] A. Shastri. *Elements of differential topology*, CRC Press (2011).
[Shas2014] A. Shastri. *Basic algebraic topology*, CRC Press (2014)
[Sie1969] L. Siebenmann. *Are nontriangulable manifolds triangulable?*, 1970 Topology of Manifolds. Proc. Inst., Univ. of Georgia (1969), 77-84.
[Sko] A. Skopenkov. *Embeddings in Euclidean space: an introduction to their classification* `http://www.map.mpim-bonn.mpg.de/Embeddings_in_Euclidean_space:_an_introduction_to_their_classification`
[Sma1959] S. Smale. *Diffeomorphisms of the 2-sphere*, Proc. Am. Math. Soc. 10 (1959), 621-626.
[Sma1961a] S. Smale. *Generalized Poincaré's conjecture in dimensions greater than four*, Annals of Mathematics. 74 (1961), 391–406.
[Sma1961b] S. Smale. *On gradient dynamical systems*, Ann. of Math. (2) 74 (1961), 199–206.
[Sma1962] S. Smale. *On the structure of manifolds*, Amer. J. Math. 84 (1962), 387–399.
[Sma1963] S. Smale. *Stable manifolds for differential equations and diffeomorphisms*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) 17 (1963), 97–116.
[Smi1951] Y. Smirnov. *A necessary and sufficient condition for metrizability of a topological space*, Dokl. Akad. Nauk SSSR 77 (1951), 197–200.
[Smit1992] B. de Smit. *The fundamental group of the Hawaiian earring is not free*, Int. J. of Alg. and Comp. 2 (1992), 33–37.
[SKKT2000] K. Smith, L. Kahanpää, P. Kekäläinen and W. Traves. *An invitation to algebraic geometry*, Universitext, Springer Verlag (2000).
[So1997] T. Soma. *Bounded cohomology and topologically tame Kleinian groups*, Duke Math. J. 88 (1997), 357–370.
[Spa1963] E. Spanier. *Quasi-topologies*, Duke Math. J. 30 (1963), 1–14.
[Spa1993] E. Spanier. *Singular homology and cohomology with local coefficients and duality for manifolds*, Pacific J. Math. 160 (1993), no. 1, 165–200.
[Spa1995] E. Spanier. *Algebraic topology*, Springer Verlag (1995).
[Spe1928] E. Sperner. *Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes*, Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität 6 (1928), 265–272.
[Spr1957] G. Springer. *Introduction to Riemann surfaces*, Addison-Wesley Publishing Company (1957).
[Staf2015] M. Stafa. *The Mayer-Vietoris spectral squence*, unpublished note (2015) `http://www.math.tulane.edu/~mstafa/papers/mayer.vietoris.ss.pdf`
[Stal1960a] J. Stallings. *On the loop theorem*, Ann. of Math. (2) 72 (1960), 12–19.

[Stal1960b] J. Stallings. *Polyhedral homotopy spheres*, Bull. Amer. Math. Soc. 66 (1960), 485–488.
[Stal1961] J. Stallings. *The piecewise-linear structure of Euclidean space*, Proc. Cambridge Philos. Soc. 58 (1961), 481–488.
[Stal1963] J. Stallings. *On topologically unknotted spheres*, Ann. of Maths. 77 (1963), 490–503.
[Stal1965a] J. Stallings. *How not to prove the Poincaré conjecture*, in Topology seminar, Wisconsin, Ann. of Math. Studies 60, Princeton Univ. Press (1965)
[Stal1965b] J. Stallings. *On infinite processes leading to differentiability in the complement of a point*, 1965 Differential and Combinatorial Topology, Princeton Univ. Press (1965), 245-254.
[Stal1965c] J. Stallings. *Whitehead torsion of free products*, Ann. Math. (2) 82 (1965), 354–363.
[Stal1967] J. Stallings. *Lectures on polyhedral topology*, Notes by G. Ananda Swarup. Tata Institute of Fundamental Research Lectures on Mathematics, No. 43 Tata Institute of Fundamental Research (1967).
[Stal1968] J. Stallings. *On torsion free groups with infinitely many ends*, Ann. of Math. 88 (1968), 312–334.
[Stal1975] J. Stallings. *Coherence of 3-manifold fundamental groups*, Séminaire N. Bourbaki 481 (1975-1976), 167–173.
[Stan1973] M. A. Stan'ko. *Approximation of imbeddings of compacta in codimenskons greater than two*, Sov. Math., Dokl. 12 (1971), 906-909.
[SS1978] L. Steen and A. Seebach. *Counterexamples in topology*, Second ed. Springer Verlag (1978).
[Stee1947] N. E. Steenrod. *Products of cocycles and extensions of mappings*, Ann. Math. 48 (1947), 290–320.
[Stee1951] N. E. Steenrod. *The Topology of Fibre Bundles*, Princeton Mathematical Series 14. Princeton University Press (1951).
[Stee1967] N. E. Steenrod. *A convenient category of topological spaces*, Mich. Math. J. 14 (1967), 133–152.
[SE1962] N. E. Steenrod and D. B. A. Epstein. *Cohomology operations*, Annals of Mathematics Studies 50, Princeton University Press (1962).
[Stein1978] M. Stein. *Whitehead groups of finite groups*, Bull. Am. Math. Soc. 84 (1978), 201–212.
[StS2005] E. M. Stein and R. Shakarchi. *Real Analysis III: Measure Theory, Integration, and Hilbert Spaces*, Princeton University Press (2005).
[StW1984] M. Steinberger and J. West. *Covering homotopy properties of maps between C.W. complexes or ANRs*, Proc. Amer. Math. Soc. 92 (1984), no. 4, 573–577.
[Steini1908] E. Steinitz. *Beiträge zur Analysis situs*, Sitzungsber. Berl. Math. Ges. 7 (1908), 29–49.
[Steinl1985] H. Steinlein. *Borsuk's antipodal theorem and its generalizations and applications: a survey*, Topological methods in nonlinear analysis, 166–235, Sém. Math. Sup., 95, Presses Univ. Montréal, (1985)
[Steinl1993] H. Steinlein. *Spheres and symmetry: Borsuk's antipodal theorem*, Topol. Methods Nonlinear Anal. 1 (1993), no. 1, 15–33.
[Ster1975] R. J. Stern. *On topological and piecewise linear vector fields*, Topology 14 (1975), 257–269.
[Stie1935] E. Stiefel. *Richtungsfelder und Fernparallelismus in n-dimensionalen Mannigfaltigkeiten*, Comm. Math. Helv. 8 (1935), 305-353.
[Stil1982] J. Stillwell. *The word problem and the isomorphism problem for groups*, Bull. Amer. Math. Soc. (N.S.) 6 (1982), 33–56.
[Sti1993] J. Stillwell. *Classical topology and combinatorial group theory*, second edition, Graduate Texts in Mathematics 72, Springer Verlag (1993).
[Sti2013] J. Stillwell. *The real numbers. An introduction to set theory and analysis*, Undergraduate Texts in Mathematics. Springer (2013).
[SZ1994] R. Stöcker and H. Zieschang. *Algebraische Topologie: Eine Einführung*, B. G. Teubner. xi (1994).
[StT1942] A. Stone and J. Tukey. *Generalized "sandwich" theorems*, Duke Math. J. 9 (1942), 356–359.
[SHP2016] B. Stolz, H. A. Harrington and M. Porter. *The Topological "Shape" of Brexit*, Preprint (2016) `https://arxiv.org/abs/1610.00752`
[Stong1968] R. Stong. *Notes on cobordism theory*, Mathematical notes. Princeton Univ. Press (1968)
[Stone1948] A. H. Stone. *Paracompactness and product spaces*, Bull. Amer. Math. Soc. 54 (1948), 977–982.

[Stri] N. Strickland. *The category of* CGWH *spaces.* `https://neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf`

[Strl2004] S. Strle. *Bounds on genus and geometric intersections from cylindrical end moduli spaces*, J. Differ. Geom. 65 (2004), 469–511.

[Stro2011] J. Strom. *Modern classical homotopy theory*, Graduate Studies in Mathematics 127 (2011).

[Strø1966] A. Strøm. *Note on cofibrations*, Math. Scand. 19 (1966), ll–14.

[Strø1968] A. Strøm. *Note on cofibrations II*, Math. Scand. 22 (1968), 130–142.

[Strg1981] K. Stromberg. *An introduction to classical real analysis*, Wadsworth & Brooks/Cole Mathematics Series (1981).

[Su1999] F. E. Su. *Rental harmony: Sperner's lemma in fair division*, Am. Math. Mon. 106 (1999), 930–942.

[Sul1979] D. Sullivan. *Hyperbolic geometry and homeomorphisms*, Geometric topology, Proc. Conf., Athens/Ga. 1977 (1979), 543–555.

[Sul2004] D. Sullivan. *Ren'e Thom's work on geometric homology and bordism*, Bull. Am. Math. Soc., New Ser. 41 (2004), 341–350.

[Sun2017] F. Sun. *An elementary proof for Poincaré Duality with local coefficients*, preprint (2017), arXiv:1709.00569.

[Suz1955] M. Suzuki. em On finite groups with cyclic Sylow subgroups for all odd primes, Amer. J. Math. 77 (1955), 657–691.

[Swa1961] R. G. Swan. *The nontriviality of the restriction map in the cohomology of groups*, Proc. Am. Math. Soc. 11 (1961), 885–887.

[Swa1969] R. G. Swan. *Groups of cohomological dimension one*, J. of Algebra. 12 (1969), 585–610.

[Sw1975] R. Switzer. *Algebraic topology–homotopy and homology*, Die Grundlehren der mathematischen Wissenschaften 212. Springer Verlag (1975)

[Szy1997] K. Szymiczek. *Bilinear algebra. An introduction to the algebraic theory of quadratic forms*, Algebra, Logic and Applications, 7. Gordon and Breach Science Publishers (1997).

[Tao2017] T. Tao. *An integration approach to the Toeplitz square peg problem*, Forum Math. Sigma 5, Article ID e30, 63 p. (2017).

[Tau1987] C. Taubes. *Gauge theory on asymptotically periodic 4-manifolds*, J. Diff. Geom. 25 (1987), 363–430.

[TSH1998] N. Tatsuuma, H. Shimomura and T. Hirai. *On group topologies and unitary representations of inductive limits of topological groups and the case of the group of diffeomorphisms*, J. Math. Kyoto Univ. 38 (1998), no. 3, 551–578.

[Tay1984] L. Taylor. *Relative Rochlin invariants*, Topology Appl. 18 (1984), 259–280.

[Tei1992] P. Teichner. *Topological 4-manifolds with finite fundamental group*, PhD thesis, University of Mainz. Shaker Verlag (1992).

[Tho1952] R. Thom. *Espaces fibrés en sphères et carrés de Steenrod*, Ann. Sci. Éc. Norm. Supér., III. Sér. 69 (1952), 109–182.

[Tho1954a] R. Thom. *Sous-variétés et classes d'homologie des variétés différentiables*, Séminaire N. Bourbaki, 1951-1954, exp. no 78, 271–277.

[Tho1954b] R. Thom. *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. 28:1 (1954), 17–86.

[Tho1958] R. Thom. *Les classes caractéristiques de Pontrjagin des variétés triangulées*, Sympos. internac. Topología algebráica (1958), 54–67.

[Tho2007] R. Thom. *Some "global" properties of differentiable manifolds*, in Topological library. Part 1: Cobordisms and their applications. Series on Knots and Everything 39. World Scientific (2007).

[Tn1992] C. Thomassen. *The Jordan-Schönflies theorem and the classification of surfaces*, Am. Math. Mon. 99 (1992), 116–130.

[Thu1978] W. P. Thurston. *The Geometry and Topology of 3-Manifolds* Lecture notes, Princeton University, 1978.

[Thu1982] W. P. Thurston. *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc., New Ser. 6 (1982), 357–379.

[Thu1986] W. P. Thurston. *A norm for the homology of* 3*-manifolds*, Mem. Amer. Math. Soc. 59 (1986), no. 339, i–vi and 99–130.

[Thu1997] W. P. Thurston. *Three-dimensional geometry and topology. Vol. 1*, Edited by Silvio Levy. Princeton Mathematical Series, 35. Princeton University Press (1997).

[Tie1908] H. Tietze. *Über die topologische Invarianten mehrdimensionaler Mannigfaltigkeiten*, Monatshefte für Mathematik und Physik 19 (1908), 1–118.

[Tie1915] H. Tietze. *Über Funktionen, die auf einer abgeschlossenen Menge stetig sind*, J. für die reine und angewandte Mathematik 145 (1915), 9–14.

[Tie1923] H. Tietze. *Beiträge zur allgemeinen Topologie. I. Axiome für verschiedene Fassungen des Umgebungsbegriffs*, Math. Ann. 88 (1923), 290–311.

[Toen2017] F. Toenniessen. *Topologie. Ein Lesebuch von den elementaren Grundlagen bis zur Homologie und Kohomologie*, Springer Spektrum (2017).

[Toep1911] O. Toeplitz. *Über einige Aufgaben der Analysis situs*, Verhandlungen der Schweizerischen Naturforschenden Gesellschaft in Solothurn 4 (1911), 197.

[TC2019] Á. Torras Casas. *Distributing Persistent Homology via Spectral Sequences*, Preprint, arXiv: 1907.05228 (2019)

[Tod1962] H. Toda. *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies 49, Princeton University Press (1962).

[Tu2011] L. Tu. *An introduction to manifolds*, 2nd revised ed. (2011).

[Tu2017] L. Tu. *Differential geometry. Connections, curvature, and characteristic classes*, Graduate Texts in Mathematics 275 (2017). Universitext, Springer Verlag (2011)

[Tur1990] V. Turaev. *Euler structures, nonsingular vector fields, and Reidemeister-type torsions*, Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 3, 607–643, 672 Math. USSR-Izv. 34 (1990), no. 3, 627–662.

[Tur2001] V. Turaev. *Introduction to combinatorial torsions*, Notes taken by Felix Schlenk. Lectures in Mathematics (2001).

[Tur2002] V. Turaev. *A homological estimate for the Thurston norm*, unpublished note (2002), arXiv:math.GT/0207267.

[TV1992] V. G. Turaev and O. Y. Viro. *State sum invariants of 3–manifolds and quantum 6j-symbols*, Topology 31 (1992), 865–902.

[Tyc1935] A. Tychonoff. *Über einen Funktionenraum*, Math. Ann. 111 (1935), 762–766.

[Ury1925a] P. Urysohn. *Über die Mächtigkeit der zusammenhängenden Mengen*, Math. Ann. 94 (1925), 262–295.

[Ury1925b] P. Urysohn. *Zum Metrisationsproblem*, Math. Ann. 94 (1925), 309–315.

[Val2013] R. Vallin. *The elements of Cantor sets. With applications*, John Wiley & Sons (2013).

[vdW1948] B. L. van der Waerden. *Author Profile Free products of groups*, Am. J. Math. 70 (1948), 527–528.

[vGl1985] R. J. van Glabbeek. *Good coverings*, Master thesis, University of Leiden (1985). `http://kilby.stanford.edu/~rvg/pub/good.pdf`

[Var1989] K. Varadarajan. *The Finiteness Obstruction of C.T.C. Wall*, Wiley-lnterscience Publication (1989).

[Veb1905] O. Veblen. *Theory on Plane Curves in Non-Metrical Analysis Situs*, Trans. Amer. Math. Soc. 6 (1): 83–98.

[Verb1969] A. Verbeek. *Homogeneity of the Hilbert cube*, Notes taken by A. Verbeek from lectures by J. de Groot. Stichting Mathematisch Centrum. Zuivere Wiskunde. Stichting Mathematisch Centrum (1969).

[Vero1984] A. Verona. *Stratified mappings – structure and triangulability*, Lecture Notes in Mathematics. 1102. Subseries: Mathematisches Institut der Universität und Max-Planck-Institut für Mathematik, Bonn, Vol. 4. Springer-Verlag (1984).

[Via2017] M. Viazovska. *The sphere packing problem in dimension 8*, Ann. of Math. 185 (2017), 991–1015.

[Vic1994] J. Vick. *Homology theory. An introduction to algebraic topology*, second edition, Graduate Texts in Mathematics 145, Springer Verlag (1994).

[VZ2014] Ž. Virk and A. Zastrow. *The comparison of topologies related to various concepts of generalized covering spaces*, Topology Appl. 170 (2014), 52–62.

[VogI] I. Vogt. *Vector fields on spheres*, expository note
`http://web.stanford.edu/~vogti/VectorFieldsonSpheres.pdf`

[VogR1971] R. Vogt. *Convenient categories of topological spaces for homotopy theory*, Arch. Math. 22 (1971), 545–555.

[Vol2013] K. Volkert. *Poincaré's homology sphere*, Bull. Manifold Atlas 40 (2013)
`http://www.boma.mpim-bonn.mpg.de/data/40screen.pdf`

[vQu2001] B. von Querenburg. *Mengentheoretische Topologie* , third edition, Springer (2001).

[Waj1983] B. Wajnryb. *A simple presentation for the mapping class group of an orientable surface*, Israel J. Math. 45 (1983), 157–174.

[Wald1968a] F. Waldhausen. *Heegaard-Zerlegungen der 3-Sphäre*, Topology 7 (1968), 195–203.

[Wald1968b] F. Waldhausen. *On irreducible 3-manifolds which are sufficiently large*, Ann. Math. (2) 87 (1968), 56–88.

[Wald1978] F. Waldhausen. *Algebraic K-theory of generalized free products*, II, Ann. of Math. 108 (1978), 205–256.

[WalkE1956] E. A. Walker. *Cancellation in direct sums of groups*, Proc. Amer. Math. Soc. 7(1956), 898–902.

[WalkK1985] K. Walker. *Configuration Spaces of Linkages*, undergraduate thesis, Princeton University, 1985.
`http://canyon23.net/math/1985thesis.pdf`

[Wall1960] C. T. C. Wall. *Determination of the cobordism ring*, Ann. of Math. 7 (1960), 292–311.

[Wall1962] C. T. C. Wall. *Classification of* $(n-1)$*-connected* $2n$*-manifolds*, Ann. of Maths. 75 (1962), 163–189.

[Wall1964] C. T. C. Wall. *On simply-connected 4-manifolds*, J. London Math. Soc. 39 (1964), 141–149.

[Wall1965a] C. T. C. Wall. *Finiteness conditions for CW-complexes*, Ann. Math. 81 (1965), 56–69.

[Wall1965b] C. T. C. Wall. *All 3-manifolds imbed in 5-space*, Bull. Amer. Math. Soc., 71 (1965), 564–567.

[Wall1966a] C. T. C. Wall. *Finiteness conditions for CW-complexes II*, Proc. Roy. Soc. London, Ser. A 295 (1966), 129–139.

[Wall1966b] C. T. C. Wall. *Classification problems in Differential Topology - IV Thickenings*, Topology 5 (1966), 73–94.

[Wall1967] C. T. C. Wall. *Poincaré complexes I*, Ann. of Math. 86 (1967), 213–245.

[Wall1969] C. T. C. Wall. *Non-additivity of the signature*, Invent. Math. 7 (1969), 269–274.

[Wall1970] C. T. C. Wall. *Surgery on compact manifolds*, Academic Press (1970).

[Wall2016] C. T. C. Wall. *Differential topology*, Cambridge Studies in Advanced Mathematics 156. Cambridge University Press (2016).

[Walla1960] A. H. Wallace. *Modifications and cobounding manifolds*, Can. J. Math. 12 (1960), 503–528.

[Wals1973] J. L. Walsh. *History of the Riemann mapping theorem*, Amer. Math. Monthly 80 (1973), 270–276.

[WX2010] G. Wang and Z. Xu. *A Survey of Computations of Homotopy Groups of Spheres and Cobordisms*, survey paper (2010)
`http://math.mit.edu/~guozhen/homotopy%20groups.pdf`

[WX2017] G. Wang and Z. Xu. *The triviality of the 61-stem in the stable homotopy groups of spheres*, Ann. Math. (2) 186 (2017), 501–580.

[War1983] F. Warner. *Foundations of differentiable manifolds and Lie groups*, Graduate Text in Mathematics, Springer Verlag (1983).

[Wat2018] T. Watanabe. *Some exotic nontrivial elements of the rational homotopy groups of* $\mathrm{Diff}(S^4)$, Preprint, arXiv: 1812.02448 (2018).

[Weh1973] B. A. F. Wehrfritz. *Infinite Linear Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 76, Springer, New York-Heidelberg, 1973.

[Weib1994] C. Weibel. *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics 38, Cambridge University Press (1994).

[Weil1952] A. Weil. *Sur les théorèmes de de Rham*, Comment. Math. Helv. 26 (1952), 119–145.

[Weil1958] A. Weil. *Final report on contract AF 18(603)-57*, Scientific works. Collected papers, II, Berlin, New York: Springer Verlag, pp. 390–395, 545–547 (1958).

[Weinb2004] S. Weinberger. *On the topological social choice model*, J. Econom. Theory 115 (2004), no. 2, 377–384.

[Weint2014] S. Weintraub. *Differential forms. Theory and practice*, 2nd ed. Amsterdam: Elsevier/Academic Press (2014).

[Wel2008] R. O. Wells. *Differential analysis on complex manifolds*, Third edition. Graduate Texts in Mathematics 65 (2008).

[Wes1977] J. West. *Mapping Hilbert Cube Manifolds to ANR's: A Solution of a Conjecture of Borsuk*, Ann. of Math. 106 (1977), 1–18.

[Wey1913] H. Weyl. *Die Idee der Riemannschen Fläche*, B. G. Teubner, Math. Vorles. a. d. Univ. Göttingen Nr. V (1913).

[Wey1923] H. Weyl. *Análisis situs combinatorio*, Revista Matem. Hispano-Americana, 43 p. (1923). `https://www.maths.ed.ac.uk/~v1ranick/papers/weyl.pdf`

[Wey1964] H. Weyl. *The concept of a Riemann surface, from the German original*, International Series in Mathematics, Addison-Wesley (1964).

[WhdG1950] G. W. Whitehead. *The $(n+2)$-nd homotopy group of the n-sphere*, Ann. Math. 52 (1950), 245–247.

[WhdG1978] G. W. Whitehead. *Elements of homotopy theory*, Graduate Texts in Mathematics 61. Springer Verlag (1978).

[WhdJ1939] J. H. C. Whitehead. *Simplicial spaces, nuclei and m-groups*, Proc. London math. Soc. (2) 45 (1939), 243–327.

[WhdJ1940] J. H. C. Whitehead. *On C^1-complexes*, Ann. Math. (2) 41 (1940), 809–824.

[WhdJ1941a] J. H. C. Whitehead. *On adding relations to homotopy groups*, Ann. of Math. 42 (1941), 409–428.

[WhdJ1941b] J. H. C. Whitehead. *On incidence matrices, nuclei and homotopy types*, Ann. of Math. (2) 42 (1941), 1197–1239.

[WhdJ1944] J. H. C. Whitehead. *On the groups $\pi_r(V_{n,m})$ and sphere-bundles*, Proc. London Math. Soc. 48 (1944), 243–291.

[WhdJ1948] J. H. C. Whitehead. *Note on a theorem due to Borsuk*, Bull. Amer. Math. Soc. 54 (1948), 1125–1132.

[WhdJ1949a] J. H. C. Whitehead. *On simply connected 4-dimensional polyhedra*, Comment. Math. Helv. 22 (1949), 48–92.

[WhdJ1949b] J. H. C. Whitehead. *Combinatorial homotopy I*, Bull. Amer. Math. Soc. 55 (1949), 213–245.

[WhdJ1950] J. H. C. Whitehead. *A certain exact sequence*, Ann. of Math. (2) 52 (1950), 51–110.

[WhdJ1958a] J. H. C. Whitehead. *On 2-spheres in 3-manifolds*, Bull. Amer. Math. Soc. 64 (1958), 161–166.

[WhdJ1958b] J. H. C. Whitehead. *On finite cocycles and the sphere theorem*, Colloq. Math. 6 (1958), 271–281.

[WhdJ1961a] J. H. C. Whitehead. *Manifolds with transverse fields in euclidean space*, Ann. of Math. (2) 73 (1961), 154–212.

[WhdJ1961b] J. H. C. Whitehead. *The immersion of an open 3-manifold in Euclidean 3-space*, Proc. London Math. Soc. 11 (1961), 81–90.

[Why1934] H. Whitney. *Analytic extensions of differentiable functions defined on closed sets*, Trans. Amer. Math. Soc. 36 (1934), 63–89.

[Why1936] H. Whitney. *Differentiable manifolds*, Ann. of Math. 37 (1936), 645–680.

[Why1937] H. Whitney. *On regular closed curves in the plane*, Compositio Mathematica. 4 (1937), 276–284.

[Why1944a] H. Whitney. *The self-intersections of a smooth n-manifold in $2n$-space*, Ann. of Math. (2) 45 (1944), 220–246.

[Why1944b] H. Whitney. *The singularities of a smooth n-manifold in $(2n-1)$-space*, Ann. of Math. (2) 45 (1944), 247–293.

[Why1957] H. Whitney. *Geometric integration theory*, Princeton Mathematical Series (1957).

[Whn1987] W. Whitten. *Knot complements and groups*, Topology 26 (1987), no. 1, 41–44.

[Wie1969] J. Wiegold. Ext($\mathbb{Q}, \mathbb{Z}$) *is the additive group of real numbers*, Bull. Aust. Math. Soc. 1 (1969), 341–343.

[Wiki] Wikipedia contributors. *Homotopy groups of spheres*, Wikipedia, The Free Encyclopedia (2018), accessed 2-January-2019.
`https://en.wikipedia.org/w/index.php?title=Homotopy_groups_of_spheres&oldid=855933540`

[Wild2010] N. Wildberger. *An algebraic ZIP proof of the classification*, lecture at the University of New South Wales, Sydney (2010).
`https://www.youtube.com/watch?v=-GJs7_NdLm8&`

[Will1970] S. Willard. *General topology*, Addison-Wesley Series in Mathematics (1970).

[Wit1982] E. Witten. *Supersymmetry and Morse theory*, J. Diff. Geom. 17 (1982), 661–692.

[Wri1994] D. Wright. *Tychonoff's theorem*, Proc. Amer. Math. Soc. 120 (1994), 985–987.

[Yok1955] I. Yokota. *On the cell structures of* SU(n) *and* Sp(n), Proc. Japan Acad. 31 (1955), 673–677.

[Yos2013] K. Yoshida. *The minimal volume orientable hyperbolic 3-manifold with 4 cusps*, Pacific J. Math. 266 (2013), no. 2, 457–476.

[Zee1961] E. C. Zeeman. *The generalised Poincaré conjecture*, Bull. Amer. Math. Soc. 67 (1961), 270.

[Zee1962] E. C. Zeeman. *The Poincaré conjecture for* $n = 5$, Topology of 3-manifolds and related topics, M.K. Fort (1962), 198-203.

[Zee1963a] E. C. Zeeman. *On the dunce hat*, Topology 2 (1963), 341–358.

[Zee1963b] E. C. Zeeman. *Seminar on combinatorial topology*, Paris: Institut des Hautes Études Scientifiques 1963. Chap. 1-6; Chap. 7, (1965); Chap. 8 (1966).

[Zee1963c] E. C. Zeeman. *Unknotting combinatorial balls*, Ann. of Maths. 78 (1963), 501–526.

[Zee1965] E. C. Zeeman. *Twisting spun knots*, Trans. Amer. Math. Soc. 115 (1965), 471–495.

[Zee1966] E. C. Zeeman. *An introduction to topology: The classification theorem for surfaces.* University of Warwick (1966).
`https://www.maths.ed.ac.uk/~v1ranick/surgery/zeeman.pdf`

[Zer1904] E. Zermelo. *Beweis, daß jede Menge wohlgeordnet werden kann*, Math. Ann. 59 (1904), 514–516.

[Zi1921] K. Zindler. *Über konvexe Gebilde*, Monatshefte für Mathematik und Physik 31 (1921), 25–56.

[Zo1935] M. Zorn. *A remark on method in transfinite algebra*, Bull. Amer. Math. Soc. 41 (1935), 667–670.

Introduction

These are the lecture notes for a sequence of courses on algebraic and geometric topology at the University of Regensburg that started in the winter term of 2016/17 and ended in the summer term 2022. A new sequence is starting in the winter term of 2022/23 and the hope is to revise some of the lecture notes and to add some material (e.g. tubular neighborhoods, de Rham cohomology, bundles, characteristic classes, Reidemeister torsion).

Some references point to future material that is either not written yet or not presentable yet. If you have any comments, especially if you find mistakes or typos, please let me know.

sfriedl@gmail.com

Part I

General Topology

0. Preliminaries

In this preliminary chapter we introduce some concepts and results from set theory and linear algebra which we will use throughout these notes, but which otherwise have little to do with topology itself. The reader might want to skip this chapter and directly move to the next chapter.

0.1. **Basic set theory.** Throughout these notes we work with "naive set theory". Like many working mathematicians this author does not want to get drawn into a discussion about what a set really is. Furthermore, when we describe sets we sometimes use creative notation and language. The hope is that the language is chosen in such a way that any good-willed reader will understand what is meant.

In this short section we recall a few basic definitions and results from set theory which will frequently use throughout these notes. Usually we will not refer to the statements explicitly, unless it helps to clarify an argument.

First we introduce the following notation which should mostly be standard.

Notation.

(1) We write $\mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$ and $\mathbb{N} := \{1, 2, 3, \dots\}$.
(2) We denote by $\varnothing$ the empty set.
(3) Given a set X we write $A \subset X$ if A is a subset of X and we write $A \subsetneq X$ if A is a subset of X with $A \neq X$.
(4) Given a set X we denote by $\mathcal{P}(X)$ its power set, i.e. $\mathcal{P}(X)$ is the set of all subsets of X.
(5) Given a map $f\colon X \to Y$ between sets and given a subset $A \subset Y$ we write
$$f^{-1}(A) \;=\; \{x \in X \mid f(x) \in A\}.$$
Furthermore, given $y \in Y$ we write $f^{-1}(y) := f^{-1}(\{y\})$.[1]

Definition. Let X be a set. A family of subsets of X consists of a set I and a map $f\colon I \to \mathcal{P}(X)$.

Remark. When it comes to families of subsets we often use suggestive notation, e.g. we might write "let $\{U_i\}_{i\in I}$ be a family of subsets of X", which just means that I is a set and the map $I \to \mathcal{P}(X)$ is denoted by $i \mapsto U_i$.

Convention. Let X be a set. We will view any subset $\mathcal{U} \subset \mathcal{P}(X)$ as a family of subsets of X via the index set $I = \mathcal{U}$ and the map $I = \mathcal{U} \to \mathcal{P}(X)$ given by $U \mapsto U$. This convention might sound silly, but it will be useful on many occasions.

We continue with two of the most important definitions of set theory.

Definition. Let $\{U_i\}_{i\in I}$ be a family of subsets of a given set X. We write
$$\begin{aligned} \bigcup_{i\in I} U_i &= \{x \in X \mid \text{ there exists an } i \in I \text{ with } x \in X_i\} \\ \bigcap_{i\in I} U_i &= \{x \in X \mid \text{ for every } i \in I \text{ we have } x \in X_i\}. \end{aligned}$$

[1] Purists might decry this notation, but it does make notation much more readable.

In particular, if $I = \varnothing$, then $\bigcup_{i\in I} U_i = \varnothing$ and $\bigcap_{i\in I} U_i = X$.[2]

Notation. Let $\{U_i\}_{i\in I}$ be a family of subsets of a given set X. If for any $i \neq j \in I$ we have $U_i \cap U_j = \varnothing$, then at times we write

$$\bigsqcup_{i\in I} U_i \;:=\; \bigcup_{i\in I} U_i.$$

The very first statement of these notions are e Morgan's Laws from elementary set theory.

Lemma 0.1. (De Morgan's Laws) Let X be a set and let $\{A_i\}_{i\in I}$ be a family of subsets of X. If given a subset $B \subset X$ we denote its complement by $B^c := X \setminus B$, then

$$\Big(\bigcap_{i\in I} A_i\Big)^c \;=\; \bigcup_{i\in I} A_i^c \quad \text{and} \quad \Big(\bigcup_{i\in I} A_i\Big)^c \;=\; \bigcap_{i\in I} A_i^c.$$

Or equivalently, in a more cumbersome notation, we have

$$X \setminus \Big(\bigcap_{i\in I} A_i\Big) \;=\; \bigcup_{i\in I} (X \setminus A_i) \quad \text{and} \quad X \setminus \Big(\bigcup_{i\in I} A_i\Big) \;=\; \bigcap_{i\in I} (X \setminus A_i).$$

Proof. The lemma follows easily from the definitions and elementary logic. ■

The following lemma concerns the interaction between operations on sets and maps between sets.

Lemma 0.2. (Image-Preimage Lemma) Let $f\colon X \to Y$ be a map between two sets X and Y. Then

(1)	for every subset $U \subset X$ we have	U	$\subset$	$f^{-1}(f(U))$
	furthermore, if f is injective[3], then we have	U	$=$	$f^{-1}(f(U))$
(2)	for every subset $V \subset Y$ we have	V	$\supset$	$f(f^{-1}(V))$
	furthermore, if f is surjective, then we have	V	$=$	$f(f^{-1}(V))$
(3)	for every family $\{U_i\}_{i\in I}$ of subsets of X we have	$f\Big(\bigcup_{i\in I} U_i\Big)$	$=$	$\bigcup_{i\in I} f(U_i)$
(4)	for every family $\{U_i\}_{i\in I}$ of subsets of X we have	$f\Big(\bigcap_{i\in I} U_i\Big)$	$\subset$	$\bigcap_{i\in I} f(U_i)$
(5)	for every subset $U \subset X$ we have	$f(X \setminus U)$	$\supset$	$f(X) \setminus f(U)$
	furthermore, if $f^{-1}(f(U)) = U$, then we have	$f(X \setminus U)$	$=$	$f(X) \setminus f(U)$
(6)	for every family $\{V_i\}_{i\in I}$ of subsets of Y we have	$f^{-1}\Big(\bigcup_{i\in I} V_i\Big)$	$=$	$\bigcup_{i\in I} f^{-1}(V_i)$
(7)	for every family $\{V_i\}_{i\in I}$ of subsets of Y we have	$f^{-1}\Big(\bigcap_{i\in I} V_i\Big)$	$=$	$\bigcap_{i\in I} f^{-1}(V_i)$
(8)	for every subset $V \subset Y$ we have	$f^{-1}(Y \setminus V)$	$=$	$X \setminus f^{-1}(V)$.

Proof. The lemma follows again from the definitions and elementary logic. ■

Definition.

[2] This simple observation will be useful at times.

[3] As a totally irrelevant side remark we would like to point out that the fundamental terms "surjective", "injective" and "bijective" are actually fairly recent additions to the mathematical language. They were first introduced on page 80 of Nicolas Bourbaki's "Théorie des ensembles" first published in 1954, see [**Bou1954**]. The reader who has not read the biography of the great mathematician Nicolas Bourbaki is strongly encouraged to do so.

(1) Given finitely many sets $X_1, \ldots, X_m$ we define the product of $X_1, \ldots, X_n$ to be

$$X_1 \times \cdots \times X_n \ := \ \{(x_1, \ldots, x_n) \,|\, \text{ for } i = 1, \ldots, n \text{ we have } x_i \in X_i\}.$$

(2) Given a set X and given $n \in \mathbb{N}$ we write

$$X^n \ := \ \underbrace{X \times \cdots \times X}_{n \text{ times}}.$$

With this notation we can also formulate the following addendum to the above lemmas.

Lemma 0.3. Let X be a set and let $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$ be two families of subsets of X. Then

$$\left(\bigcup_{i \in I} A_i \right) \cap \left(\bigcup_{j \in J} B_j \right) \ = \ \bigcup_{(i,j) \in I \times J} (A_i \cap B_j).$$

Proof. The lemma follows - surprise! - easily from the definitions and elementary logic. ■

We turn to the product of not just finitely many sets but the product of a family of sets.

Definition. Let $\{X_i\}_{i \in I}$ be a family of sets. The product of these sets is defined as the set[4] [5]

$$\prod_{i \in I} X_i \ := \ \left\{ f \colon I \to \bigcup_{i \in I} X_i \,\middle|\, \text{ for every } i \in I \text{ we have } f(i) \in X_i \right\}.$$

Furthermore, given a set X and a set I we write

$$X^I \ := \ \prod_{i \in I} X_i.$$

In this lecture notes we will use the Axiom of Choice, even though for the most part it is rather irrelevant. We refer to [**Je1973**] for a detailed discussion of the Axiom of Choice.

Axiom of Choice 0.1. For every family of sets $\{X_i\}_{i \in I}$ the product $\prod_{i \in I} X_i$ is non-empty.

We conclude this section with one last piece of notation.

Notation.

(1) If $\{A_i\}_{i \in I}$ is a family of sets, then we define the disjoint union as the set

$$\bigsqcup_{i \in I} A_i \ := \ \bigcup_{i \in I} (A_i \times \{i\}).$$

(2) If $I = \{1, \ldots, k\}$, then we write $A_1 \sqcup \cdots \sqcup A_k := \bigsqcup_{i=1}^{k} A_i$.

(3) We use obvious variations on (1) and (2), for example given sets A and B we refer to

$$A \sqcup B \ := \ (A \times \{1\}) \cup (B \times \{2\})$$

as the disjoint union of A and B

Remark. Let $\{U_i\}_{i \in I}$ be a family of subsets of a given set X such that for any $i \neq j \in I$ we have $U_i \cap U_j = \varnothing$. The expression $\bigsqcup_{i \in I} U_i$ now describes two different sets, namely it describes the subset of X defined on page 87 and it describes the disjoint union as defined above. There is a natural bijection between these two sets and thus we will freely go back and forth between these notations. In practice should not pose any problems.

[4] This definition is now possibly beyond the realm of "naive set theory", since it is perhaps a little unclear what the union of arbitrary sets is supposed to mean.

[5] If $I = \{1, \ldots, n\}$ then we will not distinguish between $X_1 \times \cdots \times X_n$ and $\prod_{i \in \{1, \ldots, n\}} X_i$.

0.2. **Relations and equivalence relations on sets.** We start out with recalling the notion of a relation.

Definition. A relation on a set S is a subset of $S \times S$.

The definition of a relation in its full generality is pretty useless. It becomes more interesting once we introduce several adjectives.

Definition. Let S be a set and let $R \subset S \times S$ be a relation on S.

(1) The relation is called reflexive, if for every $s \in S$ we have $(s, s) \in R$.
(2) The relation is called transitive, if for every $(s, t), (t, u) \in R$ we also have $(s, u) \in R$.
(3) The relation is called symmetric, if for every (s, t) in R we also have $(t, s) \in R$.

We adopt the following notation.

Notation. Let S be a set and let $R \subset S \times S$ be a relation on S. Given $x, y \in S$ we write $x \, R \, y$ if and only if $(x, y) \in R$.[6]

Example. We consider the set $\mathbb{R}$ together with the relation

$$\leq \ := \ \{(x, y) \in \mathbb{R}^2 \,|\, y - x \text{ is not negative}\}.$$

One can easily verify that this relation $\leq$ is reflexive and transitive, but that it is not symmetric. Note that given $x, y \in \mathbb{R}$ the above expression "$x \leq y$" has the usual meaning that we are all used to.

Most of the relations that we will consider are going to be reflexive and transitive. There is a big difference though in the roles of symmetric and non-symmetric relations. Let us consider the symmetric ones first. We will turn to the (possibly) non-symmetric relations in the next section.

Definition. An equivalence relation on a set S is a relation that is reflexive, transitive and symmetric.

Convention. Equivalence relations on a set S are almost invariably denoted by "$\sim$" or some variation thereof.

To practice our conventions and notations, the three properties of an equivalence relation "$\sim$" on a set S are precisely the following three statements:

(1) for any $s \in S$ we have $s \sim s$ (reflexivity),
(2) for any $s, t, u \in S$ with $s \sim t$ and $t \sim u$ we have $s \sim u$ (transitivity),
(3) for any $s, t \in S$ with $s \sim t$ we also have $t \sim s$ (symmetry).

We continue with the following definition.

Definition. Let S be a set together with an equivalence relation "$\sim$". An equivalence class of $(S, \sim)$ is a subset C of S with the following three properties:

(1) the subset is non-empty,
(2) given any two $s, t \in C$ we have $s \sim t$, and
(3) given $s \in C$ and $t \in S$ with $s \sim t$ we have $t \in C$.

We refer to [**Halm1960**, Section 7] for a confirmation of the above definitions.

The following lemma summarizes the key properties of equivalence classes.

[6] Most relations are denoted by symbols like "$\sim$", "$\leq$", "$\cong$".

Lemma 0.4. (Equivalence Class Lemma) Let S be a set together with an equivalence relation "$\sim$".

(1) Every element of S is contained in a unique equivalence class, namely
$$[s] := \{t \in S \,|\, t \sim s\}.$$
(2) Two equivalence classes are either disjoint or the same.

Proof. Surely the reader will have seen the proof of the lemma before. Nonetheless we give the proof, in particular to highlight the fact that we do indeed need the symmetry condition.

(1) Let $s \in S$. Since $\sim$ is reflexive we see that $s \in [s]$. Thus it remains to show that $[s] := \{t \in S \,|\, t \sim s\}$ is indeed an equivalence class.
   (a) We already showed that $s \in [s]$, in particular we see that $[s]$ is non-empty.
   (b) Let $t, t' \in [s]$. By definition this means that $t \sim s$ and $t' \sim s$. By symmetry we $s \sim t'$. By transitivity we deduce from $t \sim s$ and $s \sim t'$ that $t \sim t'$.
   (c) Let $t \in [s]$ and suppose that $t \sim u$. Since by definition we have $s \sim t$ we obtain from transitivity that $s \sim u$, i.e. we see that $u \in [s]$.

(2) Let C and D be two equivalence classes. We suppose that $C \cap D \neq \varnothing$. Thus there exists an $x \in C \cap D$. We need to show that $C = D$. Thus let $d \in D$. By Condition (2) of an equivalence class applied to D we know that $x \sim d$. By condition (1) of an equivalence class, applied to C, we see that $d \in C$. Thus we have shown that $D \subset C$. Evidently the same argument shows that $C \subset D$. ∎

Throughout these notes we adopt the following standard notation.

Notation. Let S be a set together with an equivalence relation "$\sim$".

(1) We denote by $S/\!\sim$ the set of equivalence classes.
(2) Given $s \in S$ we denote by $[s]$ the unique equivalence class that contains $[s]$.
(3) We refer to the map
$$\begin{array}{rccc} p\colon X & \to & X/\!\sim \\ x & \mapsto & [x] \end{array}$$
as the natural projection.

Example. Let $n \in \mathbb{N}$. We consider the set $\mathbb{Z}$ together with the equivalence relation that is given by
$$x \sim y \quad :\Longleftrightarrow \quad (x-y) \text{ is divisible by } n.$$
There are exactly n equivalence classes, namely
$$\begin{array}{rcl} [0] & = & \{\ldots, -2n, -n, 0, n, 2n, 3n, \ldots\} \\ [1] & = & \{\ldots, 1-2n, 1-n, 1, 1+n, 1+2n, 1+3n, \ldots\} \\ & \vdots & \\ [n-1] & = & \{\ldots, -2n-1, -n-1, -1, n-1, 2n-1, 3n-1, \ldots\}. \end{array}$$
To the despair of many of my cultured friends we write $\mathbb{Z}_n := \mathbb{Z}/\!\sim$.

Lemma 0.5. (Quotient-Factorization Lemma) Let $\sim$ be an equivalence relation on a set X. We denote by $p\colon X \to X/\sim$ the natural projection. Let $f\colon X \to Y$ be a map to

some set Y with the property that $f(x) = f(y)$ whenever $x \sim y$. Then the map

$$\begin{array}{rccc} g\colon & X/\!\sim & \to & Y \\ & [x] & \mapsto & f(x) \end{array}$$

is well-defined and it is the unique map $g\colon X/\!\sim \to Y$, such that $f = g \circ p$, i.e. such that the following diagram of maps commutes:

$$\begin{array}{ccc} X & \xrightarrow{p} & X/\!\sim \\ & \searrow^{f} & \downarrow g \\ & & Y. \end{array}$$

Proof. Let $\sim$ be an equivalence relation on a set X and let $f\colon X \to Y$ be a map with the property that $f(x) = f(y)$ whenever $x \sim y$. Let a be an element of $X/\!\sim$. By definition there exists an $x \in X$ with $a = [x]$. We set

$$g(a) \;:=\; f(x).$$

Evidently this definition does not depend on the choice of x. The map g satisfies $f = g \circ p$. Since the projection $X \to X/\!\sim$ is surjective we see that g is uniquely determined by this property. ■

We will now see that every relation gives rise to an equivalence relation.

Definition. Let S be a set and let R be a relation on S. We say $x, y \in S$ are equivalent, if there exists a sequence $x_0, \ldots, x_k$ of elements in S such that $x = x_0$, $x_k = y$ and such that for each $i \in \{0, \ldots, k-1\}$ we have $x_i \, R \, x_{i+1}$ or $x_{i+1} \, R \, x_i$. It is straightforward to verify that "$\sim$" defines an equivalence relation on S. We refer to it as the equivalence relation generated by the relation R.

Example. We consider the set $S = \mathbb{R}$ and the relation that is defined by $x \, R \, y$ if and only $y = x + 1$. The equivalence relation generated by R is the equivalence relation that is given by $x \sim y$ if and only $x - y \in \mathbb{Z}$.

Convention. At times we are a little sloppy in our notation and we do not distinguish in our notation between a relation and the equivalence relation it generates.

0.3. **Orders on sets.** In this section we turn our gaze towards relations which are not required to be symmetric. We have the following harmless definition.

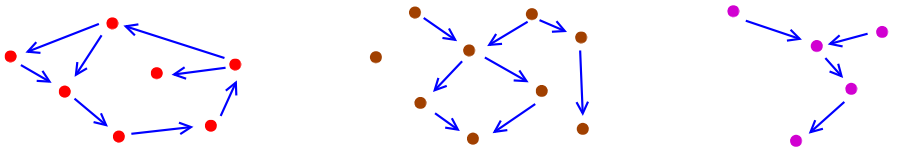
Definition.

(1) A relation "$\leq$" on a set S is called antisymmetric if $a \leq b$ and $b \leq a$ implies $a = b$.
(2) A relation that is reflexive, transitive and antisymmetric is called a partial order.
(3) A set together with a partial order is called a partially ordered set.
(4) Let "$\leq$" be a partial order on a set A. Given $a, b \in A$ we write $a < b$ if $a \leq b$ and if $a \neq b$.
(5) Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. A map $f\colon A \to B$ is called order-preserving, if for every two elements $a_1, a_2 \in A$ with $a_1 \leq a_2$ we also have $f(a_1) \leq f(a_2)$.[7]

(6) We say that two partially ordered sets $(A, \leq)$ and $(B, \leq)$ have the same order type if there exists a bijection $f\colon A \to B$ which is order-preserving and such that the inverse $f^{-1}\colon B \to A$ is also order-preserving.

Example.

(1) Let X be a subset of $\mathbb{R}$ and denote by $\leq$ the usual "less or equal" relation on the real numbers. Then $(X, \leq)$ is a partially ordered set.
(2) We equip $\mathbb{R} \cup \{\pm\infty\}$ with the relation $\leq$ that is given by the usual relation on $\mathbb{R}$ together with $-\infty \leq a \leq \infty$ for any $a \in \mathbb{R}$. This turns $\mathbb{R} \cup \{\pm\infty\}$ into a partially ordered set.
(3) Let M be a set. Recall that we denote by $\mathcal{P}(M)$ the power set of M, i.e. the set of all subsets of M. Then "being a subset", i.e. the relation given by "$\subset$", is a partial order on $\mathcal{P}(M)$.
(4) Let X be a set. The trivial partial order $\leq_{\text{tr}}$ is the partial order where $x \leq_{\text{tr}} y$ if and only if $x = y$.
(5) We consider the three reflexive transitive relations that are illustrated in the figure below. Here it is understood that $P \leq Q$ if there exists a "directed path" from Q to P.
   (a) The example on the left has a "loop". Given any two points P, Q on the loop we have $P \leq Q$ and $Q \leq P$, hence this example is not a partial order.
   (b) This example is a partial order.
   (c) This example is also a partial order.



Example (a) Example (b) Example (c)

The following construction lets us build new examples of partially ordered sets out of given ones.

Definition. Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. Given (a, b) and (a', b') in $A \times B$ we define[8]

$$(a, b) < (a', b') \quad :\Longleftrightarrow \quad \text{either } a < a' \text{ or } a = a' \text{ and } b < b'$$

and we define $(a, b) \leq (a', b')$ if either $(a, b) = (a', b')$ or $(a, b) < (a', b')$. One can easily verify that "$\leq$" is a partial order on $A \times B$. We refer to this partial order on $A \times B$ as the lexicographic order on $A \times B$.

We need a few more definitions before we can state the first non-trivial result.

Definition. Let $(P, \leq)$ be a partially ordered set.

(1) A chain in P is a subset Q such that for any two elements $q, q' \in Q$ we have $q \leq q'$ or $q' \leq q$.

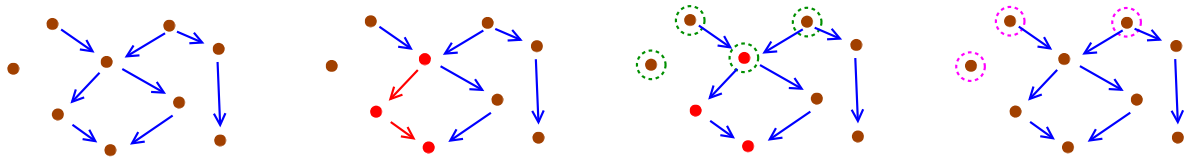
[7]Note that "order-preserving" does not mean that $a_1 < a_2$ implies $f(a_1) < f(a_2)$.
[8]Note that if $a < a'$, then we do not demand that $b \leq b'$.

(2) An upper bound of a subset Q is an element $p \in P$ such that $q \leq p$ for all $q \in Q$.
(3) A maximal element of a subset Q is an element $q \in Q$ such that there is no $r \in Q$ with $q < r$.

We define the notion of a lower bound and a minimal element in the obvious analogous way.

Example.

(1) In the figure below we once again consider the partially ordered set in the middle of the previous figure. We show a chain, we show upper bounds for the chain and we show maximal elements for P.



partially ordered set P | chain Q | upper bounds for the chain Q | maximal elements of the set P

(2) We consider the interval $P = [0, \infty)$ with the usual partial ordering. Every subset is a chain, but not every subset has an upper bound. For example P does not have an upper bound.
(3) We consider the interval $P = (-\infty, 0]$ with the usual partial ordering. Every subset is a chain and 0 is an upper bound for every subset furthermore it is a maximal element of P.

Next we formulate Zorn's Lemma. Its modest name conceals its true importance in mathematics.

Lemma 0.6. (Zorn's Lemma) Suppose a partially ordered set $(P, \leq)$ has the property that every chain has an upper bound in P. Then the set P contains at least one maximal element.

Remark.

(1) Zorn's Lemma even made it into "The Simpsons", see

`http://www.simonsingh.net/Simpsons_Mathematics/zorns-lemma/`

(2) The lemma is named after Max Zorn [**Zo1935**] who published a proof in 1935. In fact the same statement was already proved by Casimir Kuratowski [**Kura1922**] in 1922.

Proof. A nice discussion of the proof and its historical context is given in [**Sti2013**, Section 7.2]. Alternatively see also [**Cie1997**, Theorem 4.3.4] and [**Je1973**, p. 9] for a proof, assuming the Axiom of Choice 0.1. In fact, as is shown in [**Sti2013**, p. 154] or alternatively [**Je1973**, Theorem 2.1], Zorn's Lemma is equivalent to the Axiom of Choice 0.1. ■

We continue with the next adjective that we can associate to a relation.

Definition.

(1) A relation "$\leq$" on a set S is called connex if for every $a, b \in S$ we have $a \leq b$ or $b \leq a$.

(2) A partial order that is connex is called a total order.
(3) A set together with a total order is called a totally ordered set.

Examples.

(1) Evidently the usual "$\leq$"-relation on $\mathbb{R}$ is connex.
(2) None of the partial orders considered in the figure on page 92 is connex.
(3) Given two totally ordered sets $(A, \leq)$ and $(B, \leq)$ one can show easily that the lexicographic order on $A \times B$ is again a total order.

Finally we introduce one last type of partial orders.

Definition.

(1) A total order $\leq$ on a set S is called a well-order if every non-empty subset T has a minimal element.
(2) A set together with a well-order is called a well-ordered set.

Examples.

(1) The usual order on $\mathbb{Q}$ is a total order but it is not a well-order, for example the subset $T := \{x \in \mathbb{Q} \,|\, x > 2\}$ does not have a minimal element. But since $\mathbb{Q}$ is in bijection to $\mathbb{N}$ we see that $\mathbb{Q}$ admits a well-order.
(2) The usual order on $\mathbb{R}$ is a total order but it is not a well-order. It is a priori not so clear how one can define a well-order on $\mathbb{R}$.
(3) Given two well-ordered sets $(A, \leq)$ and $(B, \leq)$ one can again show easily that the lexicographic order on $A \times B$ is a well-order.

Perhaps somewhat shockingly the following theorem, due to Ernst Zermelo [**Zer1904**], says that every set admits a well-ordering.

Theorem 0.7. (Well-Ordering Theorem) Every set admits a well-order.

Proof. We refer to [**Cie1997**, Theorem 4.3.3] or alternatively to [**Je1973**, p. 9] for a proof assuming the Axiom of Choice 0.1.In fact, as for Zorn's Lemma it is shown in [**Je1973**, Theorem 2.1] that the Well-Ordering Theorem is equivalent to the Axiom of Choice 0.1. ■

0.4. **The cardinality of sets.** In this section we introduce the notion of the cardinality of a set.

Definition.

(1) We say that two given sets S and T are equinumerous if there exists a bijection $\varphi\colon S \to T$.[9]
(2) Given a set S we define the cardinality $\#S$ of S as the equinumerosity equivalence class of S in the class of all sets. Put differently we have

$$\#S \;:=\; \text{class of all sets } T \text{ that are equinumerous with } S.$$

(3) If S is a finite set, then we write $\#S = n$ for the unique $n \in \mathbb{N}_0$ which satisfies $\#S = \#\{1, \ldots, n\}$. If S infinite, then sometimes we write $\#S = \infty$[10]. At times we also write $|S| \in \mathbb{N}_0 \cup \{\infty\}$ instead of $\#S \in \mathbb{N}_0 \cup \{\infty\}$.

(4) Let S be a set. If $\#S = n$ for some $n \in \mathbb{N}_0$ or if $\#S = \#\mathbb{N}$, then we say that S is countable, otherwise we say that S is uncountable.

Example. We have $\#(\mathbb{R} \setminus \{0\}) = \#\mathbb{R}$. Indeed a bijection is given by

$$\begin{array}{rcl} \mathbb{R} \setminus \{0\} & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} x, & \text{if } x \in \mathbb{R} \setminus \mathbb{N}_0, \\ x - 1, & \text{if } x \in \mathbb{N}. \end{cases} \end{array}$$

Remark. Another way of putting the definition of cardinality is that given two sets S and T we have $\#S = \#T$ if and only if there exists a bijection from S to T.

The following lemma gets used on many occasions in these lecture notes.

Lemma 0.8. (Countability Lemma)

(1) Every subset of a countable set is again countable.
(2) Let S be a countable set and let $f\colon S \to T$ be a map. The image $f(S)$ is again countable.
(3) The union of countably many countable sets is again countable.
(4) Let T be a set and let S be a subset such that $T \setminus S$ is countable. Then S is countable if and only if T is countable.
(5) The product of finitely many countable sets is again countable.
(6) If S is a finite set and if T is a countable set, then the set of maps from S to T is also countable.

Example. We consider the map

$$\begin{array}{rcl} \mathbb{Z} \times \mathbb{N} & \to & \mathbb{Q} \\ (m, n) & \mapsto & \frac{m}{n}. \end{array}$$

Evidently this map is surjective. It follows from the Countability Lemma 0.8 (2) and (5) that $\mathbb{Q}$ is countable.

Sketch of proof.

(1) After excluding the trivial case that S is finite and the case that T is finite one sees that one has to prove the following: given any infinite subset T of $\mathbb{N}$ there exists a bijection $\varphi\colon \mathbb{N} \to T$. Such a bijection is given by defining $\varphi(i)$ as the i-th smallest element of T.
(2) Let S be a countable set and let $f\colon S \to T$ be a map. We only really need to consider the case that $S = \mathbb{N}$ and that $f(S)$ is infinite. In this case the bijection $\varphi\colon \mathbb{N} \to f(S)$ is given by defining $\varphi(1) = f(1)$ and by defining iteratively

$$\varphi(i+1) \ := \ \left\{ f(k) \,\middle|\, \begin{array}{c} k \in \mathbb{N} \text{ is the smallest number such} \\ \text{that } f(k) \notin \{\varphi(1), \ldots, \varphi(i)\} \end{array} \right\}.$$

(3) It is fairly straightforward, albeit notationally slightly messy to give a direct proof. For completeness we refer to [**Cie1997**, Corollary 5.2.6] for a proof.
(4) This statement follows immediately from (2) and (3).

[9] It follows immediately from the definitions that being equinumerous defines an equivalence relation on the class of all sets.

[10] This notation is a little risque. The sets $\mathbb{N}$ and $\mathbb{R}$ are both infinite, but in Corollary 0.10 we will see that $\mathbb{N}$ and $\mathbb{R}$ are not equinumerous. Thus we write $\#\mathbb{N} = \infty$ and $\#\mathbb{R} = \infty$, but $\#\mathbb{N} \neq \#\mathbb{R}$.

(5) By induction and by ignoring the rather dull case of finite sets it remains to show that $\mathbb{N}_0 \times \mathbb{N}_0$ is countable. This can be shown by the well-known argument sketched in the figure below.



(6) Let $S = \{v_1, \ldots, v_n\}$ be a finite set and let T be a countable set. We have an obvious bijection between T^n and the set of maps from S to T. Thus it follows from (5) that the set of maps from S to T is countable. ■

We move on to the next definition.

Definition. Let S and T be two sets.

(1) We write $\#S \leq \#T$ if there exists an injection $\varphi\colon S \to T$.
(2) We write $\#S < \#T$ if $\#S \leq \#T$ but $\#S \neq \#T$.

Example. It follows from the Countability Lemma 0.8 (1) that a set S is countable if and only if $\#S \leq \#\mathbb{N}$.

The following theorem was first proved by Georg Cantor [**Cant1892**]. The ration of significance of the statement to length of the proof seems to be unbeatable.

Theorem 0.9. (Cantor's Theorem) Recall that given a set X we denote by $\mathcal{P}(X)$ its power set, i.e. the set of all subsets of X. For every X we have $\#X < \#\mathcal{P}(X)$.

Proof. First note that the map

$$\begin{array}{rcl} X & \to & \mathcal{P}(X) \\ x & \mapsto & \{x\} \end{array}$$

is evidently injective, hence $\#X \leq \#\mathcal{P}(X)$. Now let $f\colon X \to \mathcal{P}(X)$ be a map. We have to show that it cannot be surjective. We consider the set

$$Y \;:=\; \{x \in X \,|\, x \notin f(x)\}.$$

We claim that Y is not in the range of f. Indeed, if there did exist a $z \in X$ with $f(z) = Y$, then we would have

$$z \in Y \quad \underset{\text{by definition of } Y}{\overset{\uparrow}{\iff}} \quad z \notin f(z) \quad \underset{\text{since } f(z) = Y}{\overset{\uparrow}{\iff}} \quad z \notin Y.$$

We have thus obtained a contradiction. ■

Corollary 0.10. The set $\mathbb{R}$ of real numbers is uncountable.

Proof. [11] We consider the maps

$$\begin{array}{rcl} f\colon \mathcal{P}(\mathbb{N}) & \to & \{\mathbb{Z}_2\}^{\mathbb{N}} \\ X & \mapsto & \left(\begin{array}{rcl} \mathbb{N} & \to & \mathbb{Z}_2 \\ k & \mapsto & \left\{ \begin{array}{ll} 1, & \text{if } k \in X \\ 0, & \text{otherwise} \end{array} \right. \end{array} \right) \end{array}$$

and

$$\begin{array}{rcl} g\colon \{\mathbb{Z}_2\}^{\mathbb{N}} & \to & \mathbb{R} \\ f & \mapsto & \sum_{k=1}^{\infty} \frac{f(k)}{3^k} \end{array}$$

[11] There are many proofs that the real numbers are uncountable. Perhaps the most beautiful and astounding proof that $\mathbb{Q}$ is countable is given via the amazing Farey sequence, see [**Hat2021**, Chapter 3].

One can easily verify that $f\colon \mathcal{P}(\mathbb{N}) \to \{\mathbb{Z}_2\}^{\mathbb{N}}$ is a bijection and that $g\colon \{\mathbb{Z}_2\}^{\mathbb{N}} \to \mathbb{R}$ is injective. Thus it follows from Cantor's Theorem 0.9 together with the Countability Lemma 0.8 (1) that $\mathbb{R}$ is uncountable. ■

Now we can formulate the final result of this chapter.

Theorem 0.11. (Bernstein-Schröder)[12] Let S and T be two sets. Then

$$\#S \leq \#T \quad \text{and} \quad \#T \leq \#S \quad \Longrightarrow \quad \#S = \#T.$$

Example. Let U be a non-empty subset of $\mathbb{R}$ which contains an open interval (a, b) with $a < b$. Then $\#U = \#\mathbb{R}$. Indeed, the inclusion $U \to \mathbb{R}$ is of course injective, thus we see that $\#U \leq \#\mathbb{R}$. Furthermore the map $\varphi\colon (a, b) \to \mathbb{R}$ given by $x \mapsto \frac{x}{(x-a)(x-b)}$ defines a bijection between (a, b) and $\mathbb{R}$. Thus $\mathbb{R} \xrightarrow{\varphi^{-1}} (a, b) \to U$ is also an injection. This means that $\#\mathbb{R} \leq \#U$. Therefore it follows from the Bernstein-Schröder-Theorem 0.11 that there exists a bijection from U to $\mathbb{R}$.

Proof. Let $f\colon A \to B$ and $g\colon B \to A$ be two injective maps between two sets A and B. Without loss of generality we can assume that A and B are disjoint.[13] We denote by $\sim$ the equivalence relation on $A \sqcup B$ that is generated by $a \sim f(a)$ for $a \in A$ and by $b \sim g(b)$ for $b \in B$. Note that if $a \in A$ lies in the image of g, then by the injectivity of g there exists a unique element $g^{-1}(a) \in B$ and we have $a \sim g^{-1}(a)$. The same remark works with the roles of f and g reversed.

Claim. For each equivalence class I there exists a bijection $A \cap I \to B \cap I$.

Proof. So let I be an equivalence class and let $a \in A \cap I$.[14] The subset $I \subset A \sqcup B$ consists of all elements equivalent to a. The equivalence class I is then of one of the following three types:[15]

			A	B	A	B	A	B		
(1)	a'	$\dots$	$f^{-1}(g^{-1}(a))$,	$g^{-1}(a)$,	a,	$f(a)$,	$g(f(a))$,	$f(g(f(a))$,	$\dots$	with $a' \in A \setminus g(B)$
(2)	b'	$\dots$	$f^{-1}(g^{-1}(a))$,	$g^{-1}(a)$,	a,	$f(a)$,	$g(f(a))$,	$f(g(f(a))$,	$\dots$	with $b' \in B \setminus f(A)$
(3)	$\dots$	$\dots$	$f^{-1}(g^{-1}(a))$,	$g^{-1}(a)$,	a,	$f(a)$,	$g(f(a))$,	$f(g(f(a))$,	$\dots$	

In case (1) the map f defines a bijection $A \cap I \to B \cap I$, in case (2) the map g^{-1} defines a bijection $A \cap I \to B \cap I$ and in case (3) both f and g^{-1} define a bijection $A \cap I \to B \cap I$.

Proof.

Recall that by the Equivalence Class Lemma 0.4 we know that equivalence classes always give a disjoint decomposition of a set. In our case, if I_j, $j \in J$ denote the set of

[12] The theorem was initially stated by Georg Cantor without a proof and it was first proved by Felix Bernstein (1878-1956) and Ernst Schröder (1841-1902) in 1896. Sometimes it is also called the Cantor-Bernstein-Schröder Theorem.

[13] If A and B are not disjoint, then we replace A by $A' = A \times \{0\}$ and B by $B' = B \times \{1\}$, then A' and B' are disjoint, since any two elements in A' and B' now have different second coordinate.

[14] There always exists an $a \in A \cap I$. Indeed, I is by definition non-empty. If it contains an element in A we are done. Otherwise it contains an element in $b \in B$, but then it also contains $g(b) \in A$, since $b \sim g(b)$.

[15] Note that we do not claim that the elements in the given sequences are distinct, for example $f\colon A \to B$ could be a bijection and $g = f^{-1}$ its inverse, then the equivalence class consists only of the two elements a and $g(a)$.

equivalence classes of $\sim$ on $A \sqcup B$, then we have

$$A \sqcup B = \bigsqcup_{j \in J} I_j \quad \text{and therefore also} \quad A = \bigsqcup_{j \in J} (A \cap I_j) \text{ and } B = \bigsqcup_{j \in J} (B \cap I_j).$$

Since for each $j \in J$ we have a bijection $A \cap I_j \to B \cap I_j$ we get a bijection $A \to B$. ■

Remark. The *continuum hypothesis*, first formulated by Georg Cantor in 1878, states that there exists no set S with $\#\mathbb{Q} < \#S < \#\mathbb{R}$. Any attempt at proving this statement relies on first of all making precise what "sets" are supposed to be or what axioms they are supposed to satisfy. As wikipedia writes "the answer to this problem is independent of ZFC set theory (that is, Zermelo-Fraenkel set theory with the Axiom of Choice 0.1 included), so that either the continuum hypothesis or its negation can be added as an axiom to ZFC set theory, with the resulting theory being consistent if and only if ZFC is consistent." This independence was proved in 1963 by Paul Cohen for which he obtained a Fields medal in 1966. More useful information can as always be found on wikipedia:

`https://en.wikipedia.org/wiki/Continuum_hypothesis`

0.5. **Conventions for (linear) algebra.** Next we want to introduce the notion of rings. Since the definition of a "ring" in the literature can differ considerably let us state precisely what we mean by a ring in these notes.

Definition.

(1) A ring is a triple $(R, +\colon R \times R \to R,\ \cdot\ \colon R \times R \to R)$ where $(R, +)$ is an abelian group and such that the following conditions are satisfied:
   (a) the multiplication "$\cdot$" is associative,
   (b) the multiplication "$\cdot$" is distributive,
   (c) there exists a multiplicatively neutral element $1 = 1_R$ with $1 \neq 0$.

(2) A ring R is called commutative, if given any $r, s \in R$ we have $r \cdot s = s \cdot r$.

(3) A ring homomorphism between two rings $(R, +, \cdot)$ and $(S, +, \cdot)$ is a map $R \to S$ that preserves the additive and multiplicative structures and that sends 1_R to 1_S.

(4) A field is a commutative ring such that every non-zero element has a multiplicative inverse.

Remark. Recall that "associative" means that for any $a, b, c \in R$ we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. It is well-known that this means that we can unambiguously define the product of finitely many elements using any choice of parentheses. Even though this statement is completely familiar, if one thinks about it, one realizes that this requires a proof. Such a proof is for example provided in [**DF2004**, p. 19].

We move on to the next definition.

Definition. Let R be a commutative ring.

(1) Let M be an R-module. A basis for M is a subset $\mathcal{B} \subset M$ such that every element in M can be written uniquely as a linear combination of elements in $\mathcal{B}$.

(2) An R-module is called free if it admits a basis.

(3) Let V and W be two free R-modules which are equipped with finite bases $\mathcal{B} = \{b_1, \ldots, b_m\}$ and $\mathcal{C} = \{c_1, \ldots, c_n\}$ and let $\varphi\colon V \to W$ be an R-homomorphism. We

refer to the unique $(n \times m)$-matrix $P = (p_{ij})$ with $\varphi(v_i) = \sum_{j=1}^{n} p_{ji} \cdot w_j$ as the matrix representing φ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$.

The previous definition is the standard definition well-known from earlier courses in linear algebra. Towards the end of the notes we will work with module homomorphisms over non-commutative rings. At that point we will see that at times it is convenient to switch to a different convention.

One of the problems haunting anybody trying to write coherent lecture notes are the "extreme cases" of the empty set and 0-dimensional vector spaces. In an attempt to avoid some issues from occurring we introduce the following convention.

Convention. The determinant of a 0×0-matrix over a ring R is $1_R \in R$.

We conclude this chapter with the following definition.

Definition. Let R be a commutative domain. Given an R-module M we define

$$\operatorname{rank}(M) \quad := \quad \dim_{K(R)}(M \otimes_R K(R))$$

($\otimes_R$: tensor product over R as defined in Section 165.7)

where $K(R)$ denotes the quotient field of R.

0.6. The Cauchy-Schwarz Inequality.

Definition. Let R be a commutative ring and let V and W be R-modules.

(1) A pairing on $V \times W$ is a map
$$\varphi\colon V \times W \ \to \ R$$
such that for all $v, v', w, w' \in W$ we have $\varphi(v + v', w) = \varphi(v, w) + \varphi(v', w)$ and $\varphi(v, w + w') = \varphi(v, w) + \varphi(v, w')$.

(2) A pairing is called R-bilinear if for all $\lambda \in R$ and all $v \in V$ and $w \in W$ we have $\varphi(\lambda \cdot v, w) = \lambda \cdot \varphi(v, w)$ and $\varphi(v, \lambda \cdot w) = \varphi(v, w) \cdot \lambda$.

(3) We say that an R-bilinear pairing $\varphi\colon V \times V \to S$ is symmetric if $\varphi(v, w) = \varphi(w, v)$ for all $v, w \in V$.

We now specialize to the case that the ring R are the real numbers.

Definition. Let V be a real vector space.

(1) We say a pairing φ is positive-definite if $\varphi(v, v) > 0$ for all $v \in V \setminus \{0\}$.

(2) We say a pairing φ is positive-semidefinite if $\varphi(v, v) \geq 0$ for all $v \in V \setminus \{0\}$.

Example. Let $n \in \mathbb{N}_0$. The standard pairing on $\mathbb{R}^n$ is given by

$$\begin{aligned} \langle -, - \rangle\colon \mathbb{R}^n \times \mathbb{R}^n \ &\to \ \mathbb{R} \\ (v, w) \ &\mapsto \ \sum_{i=1}^{n} v_i \cdot w_i. \end{aligned}$$

Here we use the standard convention that for $v \in \mathbb{R}^n$ we denote by $v_1, \ldots, v_n$ its coordinates. This standard pairing on $\mathbb{R}^n$ is evidently positive-definite.

Proposition 0.12. (Cauchy-Schwarz Inequality) Let V be a real vector space and let

$$\begin{aligned} \varphi\colon V \times V \ &\to \ \mathbb{R}_{\geq 0} \\ (v, w) \ &\mapsto \ \varphi(v, w) \end{aligned}$$

be an $\mathbb{R}$-bilinear form.

(1) If φ is positive semidefinite, then for any $v, w \in V$ we have
$$\varphi(v,w)^2 \ \leq \ \varphi(v,v) \cdot \varphi(w,w).$$
(2) Now suppose that φ is positive-definite. In this equality in (1) holds if and only if v and w are linearly dependent.

Proof. First we just assume that φ is positive semidefinite. Let $v, w \in V$. Given $t \in \mathbb{R}$ we consider
$$f(t) \ := \ \varphi(v + t \cdot w, v + t \cdot w) \underset{\substack{\uparrow \\ \text{since } \varphi \text{ is } \mathbb{R}\text{-bilinear and symmetric}}}{=} \varphi(v,v) + 2t \cdot \varphi(v,w) + t^2 \cdot \varphi(w,w).$$

Since φ is positive semidefinite we see that f assumes only non-negative values. This implies that f assumes a minimum, which in turn implies that there exists a $t_0 \in \mathbb{R}$ with $f'(t_0) = 0$. It follows from the second description of $f(t)$ that
$$f'(t) \ = \ 2 \cdot \varphi(v,w) + 2t \cdot \varphi(w,w).$$
This implies that $t_0 \cdot \varphi(w,w) = -\varphi(v,w)$. Now note that
$$\begin{aligned} \varphi(v + t_0 \cdot w, v + t_0 \cdot w) \cdot \varphi(w,w) \ &= \ \varphi(w,w) \cdot \varphi(v,v) + 2\overbrace{t_0 \cdot \varphi(w,w)}^{=-\varphi(v,w)} \cdot \varphi(v,w) + \overbrace{t_0^2 \cdot \varphi(w,w) \cdot \varphi(w,w)}^{=\varphi(v,w)^2} \\ &\underset{\substack{\uparrow \\ \text{follows from } t_0 \cdot \varphi(w,w) = -\varphi(v,w)}}{=} \varphi(w,w) \cdot \varphi(v,v) - \varphi(v,w) \cdot \varphi(v,w). \end{aligned}$$

Since φ is positive semidefinite we see that $\varphi(v + t_0 \cdot w, v + t_0 \cdot w) \cdot \varphi(w,w) \geq 0$. It follows from the little calculation above that $\varphi(v,w)^2 \leq \varphi(v,v) \cdot \varphi(w,w)$.

Now suppose that $\varphi(v,w)^2 = \varphi(v,v) \cdot \varphi(w,w)$. This implies, by the above calculation, that $\varphi(v + t_0 \cdot w, v + t_0 \cdot w) \cdot \varphi(w,w) = 0$. Since φ is positive-definite this means that $v + t_0 \cdot w = 0$ or $w = 0$. In either case, v, w are linearly dependent. ■

On one occasion we will need the following corollary to the Cauchy-Schwarz Inequality 0.12.

Corollary 0.13. (Cauchy-Schwarz Inequality) Given $v_1, \ldots, v_n \in \mathbb{R}^n$ and given $w_1, \ldots, w_n \in \mathbb{R}$ we have the following inequality:
$$(v_1 \cdot w_1 + \cdots + v_n \cdot w_n)^2 \ \leq \ (v_1^2 + \cdots + v_n^2) \cdot (w_1^2 + \cdots + w_n^2).$$

Proof. The statement follows immediately from the Cauchy-Schwarz Inequality 0.12 applied to the standard pairing on $\mathbb{R}^n$. ■

Finally on some occasions we will need a variation of the Cauchy-Schwarz Inequality 0.12 where we work with pairings over $\mathbb{C}$. This leads us to the following definitions.

Definition. Let V be a complex vector space.

(1) A pairing $\varphi\colon V \times V \to \mathbb{C}$ is called $\mathbb{C}$-sesquilinear if for all $\lambda \in \mathbb{C}$ and all $v, w \in V$ we have $\varphi(\lambda \cdot v, w) = \lambda \cdot \varphi(v,w)$ and $\varphi(v, \lambda \cdot w) = \varphi(v,w) \cdot \overline{\lambda}$.
(2) A $\mathbb{C}$-sesquilinear pairing $\varphi\colon V \times V \to \mathbb{C}$ is called hermitian if $\varphi(v,w) = \overline{\varphi(w,v)}$ for all $v, w \in V$.
(3) We say a pairing φ is positive-definite if $\varphi(v,v) \in \mathbb{R}_{>0}$ for all $v \in V \setminus \{0\}$.
(4) We say a pairing φ is positive-semidefinite if $\varphi(v,v) \in \mathbb{R}_{\geq 0}$ for all $v \in V \setminus \{0\}$.

Example. Let $n \in \mathbb{N}_0$. The standard hermitian pairing on $\mathbb{C}^n$ is given by

$$\begin{array}{rccl} \langle -,- \rangle \colon & \mathbb{C}^n \times \mathbb{C}^n & \to & \mathbb{R} \\ & (v,w) & \mapsto & \sum_{i=1}^{n} v_i \cdot \overline{w}_i. \end{array}$$

This standard pairing on $\mathbb{C}^n$ is evidently positive-definite.

Notation. Given $z = x + y\mathrm{i} \in \mathbb{C}$ with $x, y \in \mathbb{R}$ we write $|z| := \sqrt{x^2 + y^2} = \sqrt{z \cdot \overline{z}}$.

Proposition 0.14. (Cauchy-Schwarz Inequality) Let V be a complex vector space and let

$$\begin{array}{rccl} \varphi \colon & V \times V & \to & \mathbb{R}_{\geq 0} \\ & (v,w) & \mapsto & \varphi(v,w) \end{array}$$

be a hermitian form.

(1) If φ is positive semidefinite, then for any $v, w \in V$ we have

$$|\varphi(v,w)|^2 \leq \varphi(v,v) \cdot \varphi(w,w).$$

(2) Now suppose that φ is positive-definite. In this case equality in (1) holds if and only if v and w are linearly dependent.

Proof. The proof is very similar to the proof of the above Cauchy-Schwarz Inequality 0.14. We leave it to the reader to make the necessary minor modifications. ∎

0.7. **Normed vector spaces.** We continue with another definition from linear algebra.

Definition. Let V be a real vector space. A norm on V is a map $\| - \| \varphi \colon V \to \mathbb{R}_{\geq 0}$ that satisfies the following three properties:

(1) For every $v \in V$ we have $\|v\| = 0 \iff v = 0$.
(2) For every $v \in V$ and $\lambda \in \mathbb{R}$ we have $\|\lambda \cdot v\| = |\lambda| \cdot \|v\|$.
(3) For every $v, w \in V$ we have $\|v + w\| \leq \|v\| + \|w\|$.

Lemma 0.15. Let V be a vector space. If $\langle -,- \rangle \colon V \times V \to \mathbb{R}_{\geq 0}$ is a positive definite $\mathbb{R}$-linear pairing, then the map

$$\begin{array}{rccl} & V \times V & \to & \mathbb{R}_{\geq 0} \\ & v & \mapsto & \|v\| := \sqrt{\langle v, v \rangle} \end{array}$$

is a norm.

Proof. It is straightforward to verify that the first two properties of a norm are satisfied. It remains to prove the third property. Given $v, w \in \mathbb{R}^n$ we see that

$$\begin{array}{rcl} \|v+w\|^2 & = & \langle v+w, v+w \rangle \overset{\text{follows from the } \mathbb{R}\text{-bilinearity of } \langle -,- \rangle}{=\!=\!=} \langle v,v \rangle + \langle v,w \rangle + \langle w,v \rangle + \langle w,w \rangle \overset{\text{since } \langle w,v \rangle = \langle v,w \rangle}{=\!=\!=} \langle v,v \rangle + 2 \cdot \langle v,w \rangle + \langle w,w \rangle \\ & \underset{\text{by the Cauchy-Schwarz Inequality 0.13 we have } \langle v,w \rangle^2 \leq \langle v,v \rangle \cdot \langle w,w \rangle}{\leq} & \langle v,v \rangle + 2 \cdot \sqrt{\langle v,v \rangle \cdot \langle w,w \rangle} + \langle w,w \rangle = \left(\sqrt{\langle v,v \rangle} + \sqrt{\langle w,w \rangle} \right)^2 = (\|v\| + \|w\|)^2. \end{array}$$

By taking square roots on both sides we obtain that $\|v + w\| \leq \|v| + \|w\|$. ∎

Definition. The standard pairing on $\mathbb{R}^n$ gives rise to the Euclidean norm

$$\begin{array}{rcl} \|-\|_{\text{Eucl}}\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ v = (v_1, \ldots, v_n) & \mapsto & \sqrt{\langle v, v, \rangle} = \sqrt{v_1^2 + \cdots + v_n^2} \end{array}.$$

Usually, when there is no source of confusion, we just write $\|v\|$ instead of $\|v\|_{\text{Eucl}}$.
We now give more examples of norms $\mathbb{R}^n$.

Example. In Exercise 0.12 we will see that the following maps are norms and we will see that for $n \geq 2$ they do *not* arise from a positive definite $\mathbb{R}$-bilinear pairing:

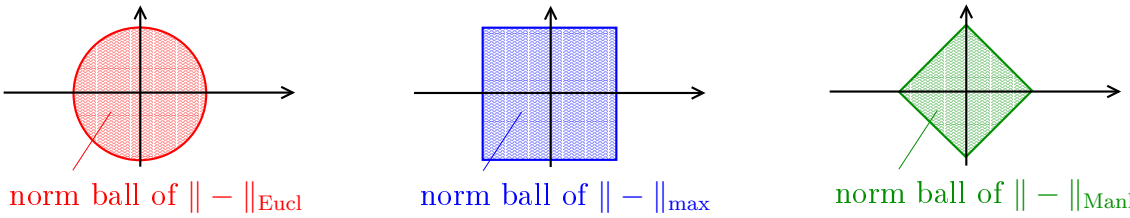
$$\underbrace{\begin{array}{rcl} \|-\|_{\max}\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x_1, \ldots, x_n) & \mapsto & \max\{|x_1|, \ldots, |x_n|\} \end{array}}_{\text{maximum norm}} \quad \text{and} \quad \underbrace{\begin{array}{rcl} \|-\|_{\text{Manh}}\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x_1, \ldots, x_n) & \mapsto & \sum_{i=1}^{n} |x_i|. \end{array}}_{\text{Manhattan norm}}$$

Since the human brain is equipped with a great image processor it is usually helpful to turn an abstract definition into an image. The key to doing so in the present context is given by the following definition.

Definition. Given a norm $\|-\|\colon V \to \mathbb{R}_{\geq 0}$ we define

$$\text{norm ball of } \|-\| \;:=\; \{v \in V \mid \|v\| \leq 1\}.$$

Example. In the figure below we show the norm balls of the Euclidean norm, the maximum norm and the Manhattan norm.

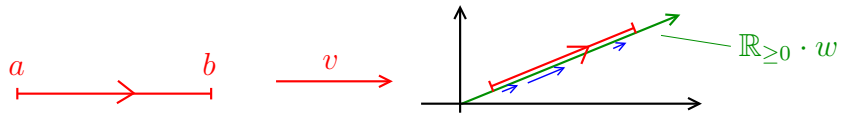


We conclude our sections on norms with the following lemma which we will use on several occasions.

Lemma 0.16. (Integral Inequality Lemma) For every continuous map $v\colon [a,b] \to \mathbb{R}^n$ the following inequality of real numbers holds:

$$\left\| \int_a^b v(t)\, dt \right\| \;\leq\; \int_a^b \|v(t)\|\, dt.$$

Furthermore equality holds if and only if u there exists a $w \in \mathbb{R}^n$ and a map $\lambda\colon [a,b] \to \mathbb{R}_{\geq 0}$ such that for every $t \in [a,b]$ we have $v(t) = \lambda(t) \cdot w$.



Proof. We first show that the inequality holds. We set

$$u \;:=\; \int_a^b v(t)\, dt \quad \in\; \mathbb{R}^n.$$

If $\|u\| = 0$, then there is nothing to show. Thus we might as well assume that $\|u\| > 0$. We obtain the following (in-) equalities:

$$\begin{aligned}
\|u\| \cdot \left\| \int_a^b v(t)\,dt \right\| \;=\; \|u\|^2 \;=\; \langle u, u\rangle \;&=\; \left\langle u, \int_a^b v(t)\,dt \right\rangle \\
&=\; \left\langle (u_1,\dots,u_n), \left(\int_a^b v_1(t)\,dt, \dots, \int_a^b v_n(t)\,dt \right) \right\rangle \\
&=\; \sum_{i=1}^n u_i \cdot \int_a^b v_i(t)\,dt \;=\; \int_a^b \sum_{i=1}^n u_i \cdot v_i(t)\,dt \\
&=\; \int_a^b \langle u, v(t)\rangle\,dt \;\underset{\uparrow}{\leq}\; \int_a^b \|u\| \cdot \|v(t)\|\,dt \;=\; \|u\| \cdot \int_a^b \|v(t)\|\,dt.
\end{aligned}$$

by the Cauchy-Schwarz Inequality 0.12 we know that
for any $x, y \in \mathbb{R}^n$ we have $\langle x, y\rangle \leq \|x\| \cdot \|y\|$

The desired inequality follows from canceling $\|u\|$.

Now we turn to the "furthermore statement". In fact this statement follows from the above considerations together with the "furthermore statement" of Proposition 0.12. We leave it to the reader to fill in the details. ■

0.8. **Metric spaces.** Finally we recall the notion of a metric space which is most likely familiar from an earlier course on real analysis.

Definition. A metric space is a pair (X, d) consisting of a set X and a metric d on X, i.e. a map

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} := \{x \in \mathbb{R} \mid x \geq 0\} \\ (x, y) & \mapsto & d(x, y) \end{array}$$

with the following properties:

(1)		$d(x,y)$	$=$	$0 \iff x = y$	
(2)	for all $x, y \in X$ we have	$d(x,y)$	$=$	$d(y,x)$	(symmetry)
(3)	for all $x, y, z \in X$ we have	$d(x,z)$	$\leq$	$d(x,y) + d(y,z)$	(triangle inequality).

Sometimes we refer to $d(x, y)$ as the distance between x and y.

The following lemma shows that any norm on a vector space V give rise to a metric on V.

Lemma 0.17. Let V be a real vector space and let $\| - \| : V \to \mathbb{R}_{\geq 0}$ be a norm. Then

$$\begin{array}{rcl} V \times V & \to & \mathbb{R}_{\geq 0} \\ (v, w) & \mapsto & d(v, w) := \|v - w\| \end{array}$$

is a metric on V.

Proof. The three properties of a metric follow easily from the three properties of a norm. For example, given $v, w \in V$ we see that

$$d(v,w) \;=\; \|v - w\| \;=\; \|(-1) \cdot (w - v)\| \;\underset{\uparrow}{=}\; |-1| \cdot \|w - v\| \;=\; \|w - v\| \;=\; d(w,v).$$

property (2) of a norm

The other two properties of a metric are verified in a similar fashion. ■

It follows from Lemma 0.17 that the Euclidean norm $\| - \|_{\text{Eucl}}$ gives rise to a metric, called the Euclidean metric. This is perhaps the most frequently used example of a metric.

Definition. Given $x, y \in \mathbb{R}^n$ we set

$$d(x,y) \;:=\; \|x-y\| \quad \text{where} \quad \|z\| \;=\; \|(z_1,\dots,z_n)\| \;:=\; \sqrt{z_1^2+\cdots+z_n^2}\,.$$

We refer to the map

$$\begin{array}{rcl} d\colon \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x,y) & \mapsto & d(x,y) \end{array}$$

as the Euclidean metric on $\mathbb{R}^n$.[16]. Usually we consider $\mathbb{R}^n$ as a metric space with respect to this Euclidean metric, unless we say something else.

Of course there are many other examples of metrics. In the following we will see two examples, the first one corresponds to the Manhattan norm and the other one is an example of a metric that does not come from a norm.

Example. Let $n \in \mathbb{R}^n$. We fix a point $P \in \mathbb{R}^n$. Besides the ever-popular Euclidean metric on $\mathbb{R}^n$ we can also consider the following two metrics:

$$\begin{array}{rcl} d_{\text{Manh}}\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ ((x_1,\dots,x_n),(y_1,\dots,y_n)) & \mapsto & \sum_{i=1}^{n} |x_i - y_i| \end{array} \qquad \text{and} \qquad \begin{array}{rcl} d_{\text{SNCF}}\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x,y) & \mapsto & \|x-P\| + \|y-P\|. \end{array}$$

The metric d_{Manh} is called the Manhattan metric since it measures the distance between two points if one can go only vertically (avenues) and horizontally (streets). The metric d_{SNCF} is called the SNCF metric since it models the distance in France where every train between two points necessarily has to go through Paris.

In Exercise 0.14 we will see that both maps do indeed form metrics on $\mathbb{R}^n$. Furthermore we will see in Exercise 0.14 that the SNCF-metric does not arise from a norm.

Manhattan metric



SNCF-metric



Of course there are many other examples of metric spaces. The last definition shows that we can define a metric on any set.

Definition. Let X be a set. The map

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} \\ (x,y) & \mapsto & \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y \end{cases} \end{array}$$

defines a metric on X called the trivial metric.

Exercises for Chapter 0.

[16] It follows from Lemma 0.15 together with Lemma 0.17 that this is indeed a metric on $\mathbb{R}^n$.

Exercise 0.1. Show that for statements (1), (2), (4) and (5) of the Image-Preimage Lemma 0.2 there are indeed examples such that the stated inclusion is indeed a strict inclusion. More precisely:

(a) Give an example of a map $f\colon X \to Y$ and a subset $U \subset X$ such that $U \subsetneq f^{-1}(f(U))$.
(b) Give an example of a map $f\colon X \to Y$ and a subset $V \subset Y$ such that $f(f^{-1}(f(V))) \subsetneq V$.
(c) Give an example of a map $f\colon X \to Y$ and a family $\{U_i\}_{i\in I}$ of subsets of X such that

$$f\Big(\bigcap_{i\in I} U_i\Big) \;\subsetneq\; \bigcap_{i\in I} f(U_i)$$

(d) Give an example of a map $f\colon X \to Y$ and a subset $U \subset X$ such that we have $f(X) \setminus f(U) \subsetneq f(X \setminus U)$.

Exercise 0.2. Let $f\colon X \to Y$ be a map between two sets. Let $A \subset X$ and let $B \subset Y$ be two subsets. Prove the following equivalence of statements:

$$f(A) \subset B \qquad \Longleftrightarrow \qquad A \subset f^{-1}(B).$$

Exercise 0.3. Let $f\colon X \to Y$ be a map between two sets. Let $A \subset X$ and let $B \subset Y$. Show that

$$f(A \cap f^{-1}(B)) \;=\; f(A) \cap B.$$

Exercise 0.4. Let $(V, \leq)$ be a totally ordered set of cardinality $n \in \mathbb{N}$. We equip the set $\{1, \ldots, n\}$ with the usual total order $\leq$. Show that there exists a *unique* order-preserving bijection $\{1, \ldots, n\} \to V$.

Exercise 0.5. Let $f\colon X \to Y$ be a map between two sets. Show that there exist total orders "$\leq_X$" on X and "$\leq_Y$" on Y such that the map f is order-preserving.

Exercise 0.6. Show that there exists an uncountable well-ordered set $(X, \leq)$ such that for every $y \in X$ the set $\{x \in X \,|\, x \leq y\}$ is countable.

Exercise 0.7. We consider $\mathbb{N}$ and $\mathbb{Z}$ with the usual partial order $\leq$.

(a) Does the lexicographically ordered set $\mathbb{N} \times \mathbb{N}$ have the same order type as $\mathbb{N}$?
(b) Does the lexicographically ordered set $\mathbb{Z} \times \mathbb{Z}$ have the same order type as $\mathbb{Z}$?

Exercise 0.8. We consider the half-open interval $[0, 1)$, the set $\mathbb{R}_{\geq 0}$ and the set $\mathbb{N}_0$, each equipped with the usual partial order $\leq$.

(a) Show that the lexicographically ordered set $[0, 1) \times \mathbb{N}_0$ has the same order type as $\mathbb{R}_{\geq 0}$.
(b) Show that the lexicographically ordered set $\mathbb{N}_0 \times [0, 1)$ does not have the same order type as $\mathbb{R}_{\geq 0}$.

Exercise 0.9.

(a) Let $n \in \mathbb{N}$. Show that $\mathbb{R}$ and $\mathbb{R}^n$ have the same cardinality.
(b) Show that $\mathbb{R}$ and $\mathbb{R}^{\mathbb{N}}$ have the same cardinality.

Exercise 0.10. Show that $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$ have the same cardinalitiy.
Hint. Use the Bernstein-Schröder Theorem 0.11.

Exercise 0.11. Let X be a set and let $\mathcal{P}(X)$ be the power set. Given $A, B \in \mathcal{P}(X)$ we define the symmetric difference

$$A \triangle B \;:=\; (A \setminus B) \cup (B \setminus A).$$

What axioms of a ring does $(\mathcal{P}(X), \triangle, \cap)$ satisfy?

Exercise 0.12. We consider the following two maps:

$$\begin{array}{rcl} \|-\|_{\max}\colon \mathbb{R}^k n \to & \mathbb{R}_{\geq 0} & \\ (x_1,\ldots,x_n) & \mapsto & \max\{|x_1|,\ldots,|x_n|\} \end{array} \quad \text{and} \quad \begin{array}{rcl} \|-\|_{\text{Manh}}\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x_1,\ldots,x_n) & \mapsto & \sum_{i=1}^{n} |\, x_i|. \end{array}$$

(a) Show that both maps define norms on $\mathbb{R}$.
(b) We now suppose that $n \geq 2$. Show that the norms do not arise from a positive definite $\mathbb{R}$-bilinear pairing, i.e. show that there is no positive-definite $\mathbb{R}$-bilinear pairing $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that for all $v \in \mathbb{R}^n$ we have $\|v\|_{\max} = \sqrt{\varphi(v,v)}$ and similarly for $\|-\|_{\text{Manh}}$.

Exercise 0.13. Let V be a real vector space Recall that given a norm $n|\colon V \to \mathbb{R}_{\geq 0}$ its norm ball is defined as $B(n) := \{v \in V \,|\, n(v) \leq 1\}$. Show that a norm is determined by its norm ball.

Exercise 0.14. Let $n \in \mathbb{N}$. We fix a point $P \in \mathbb{R}^n$ and we consider the maps

$$\begin{array}{rcl} d_M\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ ((x_1,\ldots,x_n),(y_1,\ldots,y_n)) & \mapsto & \sum_{i=1}^{n} |x_i - y_i| \end{array} \quad \text{and} \quad \begin{array}{rcl} d_S\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x,y) & \mapsto & \|x-P\| + \|y-P\| \end{array}$$

which on page 104 we called the Manhattan metric respectively the SNCF metric.

(a) Show that both maps define metrics on $\mathbb{R}^n$.
(b) Show that there is no norm $\|-\|$ on $\mathbb{R}^n$ such that $d_S(v,w) = \|v-w\|$ for all $v, w \in \mathbb{R}^n$.

1. Topological spaces: basic definitions and properties

In this chapter we give the definition of topological spaces, we provide many examples and we discuss a long list of basic properties. Many of these results might well be familiar to the reader. Nonetheless we attempt to give a fairly complete list of basic statements so that later on we have a firm common foundation to work with.

1.1. Topological spaces.

In this section we give the definition of a topological space and we discuss some very basic definitions and statements. Presumably most readers will have seen this material. Throughout these lecture notes, in the proofs, we strive for giving precise references to earlier results. The main exceptions are the lemmas from this Section 1.1. Usually we use them without referring to them explicitly.
The following definition will keep us occupied for the rest of these notes.

Definition.

(1) Let X be a set. A topology on X is a subset $\mathcal{T}$ of the power set $\mathcal{P}(X)$[17] with the following properties:
   (a) the empty set and the entire set X lie in $\mathcal{T}$,
   (b) the intersection of *finitely many* sets in $\mathcal{T}$ is again a set in $\mathcal{T}$,
   (c) the union of *arbitrarily many* sets in $\mathcal{T}$ is again a set in $\mathcal{T}$.
   The sets in $\mathcal{T}$ are called open with respect to $\mathcal{T}$. If $\mathcal{T}$ is understood from the context, then we just say open.
(2) A topological space is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a topology on X.

Remark.

(1) Usually we suppress the topology from the notation, e.g. usually we just write "let X be a topological space".
(2) If X is a topological space, then the properties of a topology imply immediately the following three statements:
   (a) the empty set and the entire set X are open,
   (b) the intersection of *finitely many* open sets is again open,
   (c) the union of *arbitrarily many* open sets is again open.
(3) It took mathematicians several decades to settle on the above definition, the long and tortuous history of the definition of a topological space is presented in [**Moo2008**].

We start out with some very basic examples of topological spaces.

Examples.

(1) Let X be a set, then $\mathcal{T} = \{\varnothing, X\}$ is a topology on X. This topology is sometimes called the trivial topology on X.
(2) Let X be a set. Then $\mathcal{T} = \mathcal{P}(X)$ is also a topology on X. In other words, $\mathcal{T}$ is the topology such that every subset of X is open. This topology is usually referred to as the discrete topology on X. Sometimes we say that a topological space is discrete, if the topology is indeed the discrete topology.

[17] In more humble words, $\mathcal{T}$ is a set of subsets of X.

(3) Let X be a set.
   (a) We define $\mathcal{T} \subset \mathcal{P}(X)$ as follows:

$$U \in \mathcal{T} \iff \text{either } U = \varnothing \text{ or } U \text{ is the complement of a finite subset of } X.$$

   It follows easily from de Morgan's Laws 0.1, that $\mathcal{T}$ is a topology on X. We refer to it as the cofinite topology. Sometimes this topology is also called the finite-complement topology.
   (b) Similarly we can define the cocountable topology $\mathcal{T} \subset \mathcal{P}(X)$ as follows:

$$U \in \mathcal{T} \iff \text{either } U = \varnothing \text{ or } U \text{ is the complement of a countable subset of } X.$$

   It follows from de Morgan's Laws together with the Countability Lemma 0.8 that this is a topology on X.
(4) Let X be the empty set. Together with $\mathcal{T} = \{\varnothing\}$ this is a topological space, called the empty topological space. It is a never ending source of counterexamples to carelessly formulated statements.
(5) (a) Let $X = \{*\}$ be a set with a single element $*$. It follows immediately from the definition of a topology there is a unique topology on $X = \{*\}$, namely the topology given by $\mathcal{T} = \{\varnothing, \{*\}\}$. Together with the empty set this is the only case where the topology is already determined by the set.
   (b) Given a set $X = \{a, b\}$ with two elements it is straightforward to see that there exist four distinct topologies on X. More precisely, there is the trivial and the discrete topology, and there are the two topologies where precisely one of the subsets $\{a\}$ and $\{b\}$ is open. The latter two topologies correspond to the Sierpiński space which we will discuss in greater detail in Exercise 2.22.
   Not surprisingly the number of topologies on a finite set grows rapidly with the cardinality of the set, we refer to **[KR1970, BMc2002]** for details.

Recall that on page 103 we introduced the notion of a metric space. We will now see that metric spaces give rise to topological spaces. To do so we will need the following notation.

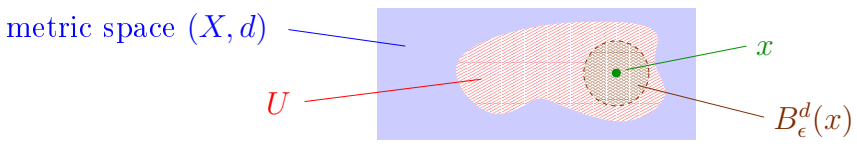
Notation. Let (X, d) be a metric space. Given $x \in X$ and $r \in \mathbb{R}$ we write

$$B_r^d(x) := \{y \in X \mid d(x, y) < r\}$$

If the metric d is understood from the context, then sometimes we drop it from the notation.

The following lemma shows that metric spaces give rise to topological spaces.

Lemma 1.1. (Metric-Topology Lemma) Let (X, d) be a metric space. A subset U of X is called open if for every $x \in U$ there exists an $\epsilon > 0$ such that $B_\epsilon^d(x)$ is contained in U. These open sets define a topology on X.



Proof.

(1) It is clear that the empty set and X itself are open.

(2) Suppose $U_1, \dots, U_k$ are open subsets of X. We need to show that the intersection $U_1 \cap U_2 \cap \dots \cap U_k$ is also open. Thus let $x \in U_1 \cap U_2 \cap \dots \cap U_k$. For each $i \in \{1, \dots, k\}$ there exists an $\epsilon_i > 0$ with $B_{\epsilon_i}(x) \subset U_i$. We set $\epsilon := \min\{\epsilon_1, \dots, \epsilon_k\}$. Then

$$\begin{array}{ccccc} B_\epsilon(x) & \subset & B_{\epsilon_1}(x) \cap \dots \cap B_{\epsilon_k}(x) & \subset & U_1 \cap U_2 \cap \dots \cap U_k. \\ & \uparrow & & \uparrow & \\ & \text{since } \epsilon \leq \epsilon_i & & \text{since } B_{\epsilon_i}(x) \subset U_i & \end{array}$$

(3) Let $\{U_i\}_{i \in I}$ be a family of open sets. We want to show that $\bigcup_{i \in I} U_i$ is also open. Thus let $x \in \bigcup_{i \in I} U_i$. This means that there exists an $i \in I$ with $x \in U_i$. Since U_i is open there exists an $\epsilon > 0$ with $B_\epsilon(x) \subset U_i$. Thus we have

$$B_\epsilon(x) \subset U_i \subset \bigcup_{i \in I} U_i.$$ ■

Definition. Let (X, d) be a metric space.

(1) We refer to the topology introduced in Metric-Topology Lemma 1.1 as the metric topology.
(2) Unless we say something else we always view (X, d) with the metric topology.

Example. Let X be a set and let $d\colon X \times X \to \mathbb{R}_{\geq 0}$ be the trivial metric that we introduced on page 104. It is straightforward to see that the corresponding metric topology is the discrete topology.

Convention. Let $n \in \mathbb{N}$.

(1) Unless we say something else we consider $\mathbb{R}^n$ as a metric space with the Euclidean metric given by

$$d(x, y) := \|x - y\| \quad \text{where} \quad \|z\| = \|(z_1, \dots, z_n)\| := \sqrt{z_1^2 + \dots + z_n^2}.$$

Furthermore, unless we say something else we equip $\mathbb{C}^n$ with the metric given by[18]

$$d(x, y) := \|x - y\| \quad \text{where} \quad \|z\| = \|(z_1, \dots, z_n)\| := \sqrt{z_1 \cdot \overline{z_1} + \dots + z_n \cdot \overline{z_n}}.$$

(2) In particular, unless we say explicitly otherwise, we equip $\mathbb{R}^n$ with the topology arising from the Euclidean metric and we $\mathbb{C}^n$ with the topology arising from the above metric. We call these the Euclidean topologies on $\mathbb{R}^n$ and $\mathbb{C}^n$.

We have now seen that metrics give rise to topologies. In general different metrics will give rise to different topologies, for example the Euclidean metric and the trivial metric on $\mathbb{R}^n$ give rise to wildly different topologies. But different metrics might also give rise to the same topology. This leads us to the following definition.

Definition. Let X be a set. We say that two metrics d and e on X are equivalent if they give rise to the same topology.

[18] It follows from the observation that $(x + \mathrm{i}y) \cdot \overline{x + \mathrm{i}y} = (x + \mathrm{i}y) \cdot (x - \mathrm{i}y) = x^2 + y^2$ that the metric on $\mathbb{C}^n$ is the Euclidean metric on $\mathbb{C}^n = \mathbb{R}^{2n}$ in slight disguise. In particular it is indeed a metric.

Lemma 1.2. (Equivalent Metric Lemma) Let X be a set and suppose that we have two metrics d and e on X such that the following two conditions are satisfied:

(1) For each $x \in X$ and each $\mu > 0$ there exists a $\nu > 0$ with $B^e_\nu(x) \subset B^d_\mu(x)$.

(2) For each $x \in X$ and each $\nu > 0$ there exists a $\mu > 0$ with $B^d_\mu(x) \subset B^e_\nu(x)$.

Then the two metrics are equivalent.

Example. Let $n \in \mathbb{R}^n$ and let $P \in \mathbb{R}^n$. On page 104 we introduced the following metrics:

$$\begin{array}{rcl} d_{\text{Manh}} \colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ ((x_1,\ldots,x_n),(y_1,\ldots,y_n)) & \mapsto & \sum_{i=1}^{n} |x_i - y_i| \end{array} \qquad \text{and} \qquad \begin{array}{rcl} d_{\text{SNCF}} \colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x,y) & \mapsto & \|x-P\| + \|y-P\|. \end{array}$$

In Exercise 1.7 we will see that the Manhattan metric is equivalent to the Euclidean metric and we will see that the SNCF-metric is not equivalent to the Euclidean metric.

Proof. We need to show that a set $U \subset X$ which is open with respect to d is open with respect to e and vice versa. So suppose that U is open with respect to d. We need to show that U is open with respect to e, i.e. we need to show that for each $x \in U$ there exists a $\nu > 0$ with $B^e_\nu(x) \subset U$. Thus let $x \in U$. Since U is open with respect to d there exists a $\mu > 0$ such that $B^d_\mu(x) \subset U$. By hypothesis there exists a $\nu > 0$ such that $B^e_\nu(x) \subset B^d_\mu(x)$. This ν has the desired property. The same argument evidently also works with the roles of d and e reversed. ■

Metric spaces are popular source of examples of topological spaces. This leads us to the following definition.

Definition. A topological space X is called metrizable if there exists a metric d on X such that the given topology agrees with the topology coming from d.

Even though many interesting topological spaces are indeed metrizable, we will see in Exercise 1.1 that a finite topological space is metrizable if and only if the topology is discrete.

1.2. Basic definitions.

We proceed with the next definition.

Definition. Let X be a topological space. We say a subset A of X is closed if the complement $X \setminus A$ is open.

Example. By an *interval* we mean any subset of $\mathbb{R}$ of the form

$$\begin{array}{rcll} (a,b) & := & \{x \in \mid a < x < b\} & \text{where } a,b \in \mathbb{R} \cup \{\pm\infty\} \\ (a,b] & := & \{x \in \mid a < x \leq b\} & \text{where } a \in \mathbb{R} \cup \{-\infty\} \text{ and } b \in \mathbb{R} \text{ with } a < b \\ [a,b) & := & \{x \in \mid a \leq x < b\} & \text{where } a \in \mathbb{R} \text{ and } b \in \mathbb{R} \cup \{\infty\} \text{ with } a < b \\ [a,b] & := & \{x \in \mid a \leq x \leq b\} & \text{where } a,b \in \mathbb{R} \text{ with } a \leq b. \end{array}$$

The intervals of the form (a,b), $(-\infty,b)$ and (a,∞) with $a,b \in \mathbb{R}$ are open subsets of $\mathbb{R}$ and the intervals of the form $[a,b]$, $(-\infty,b]$ and $[a,\infty)$ with $a,b \in \mathbb{R}$ are closed subsets of $\mathbb{R}$. Note that $\varnothing$ and $\mathbb{R}$ are also intervals, they are both open and closed in $\mathbb{R}$.

The following lemma mirrors the remark on page 107.

Lemma 1.3. Let X be a topological space.

(1) the empty set and the entire set X are closed,

(2) the intersection of *arbitrarily many* closed sets is again closed,
(3) the union of *finitely many* closed sets is again closed.

Proof. The lemma is an immediate consequence of the definition of a topology and de Morgan's Laws 0.1. ■

Remark. Let X be a set. If we specify what we mean by closed sets such that the three properties of Lemma 1.3 are satisfied, then it follows once again from de Morgan's Laws that the complements of the closed sets define a topology.

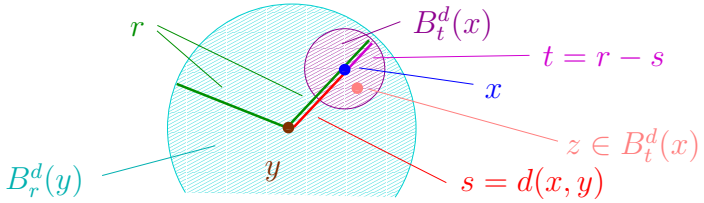
The following proposition gives us important examples of closed and open subsets.

Proposition 1.4. Let (X, d) be a metric space, let $y \in X$ and let $r \in \mathbb{R}$.
(1) the set $B_r^d(y) := \{x \in X \mid d(x, y) < r\}$ is open in X,
(2) the set $\overline{B}_r^d(y) := \{x \in X \mid d(x, y) \leq r\}$ is closed in X,
(3) the set $S_r^d(y) := \{x \in X \mid d(x, y) = r\}$ is closed in X.

Proof. Let (X, d) be a metric space, let $y \in X$ and let $r \in \mathbb{R}$.
(1) Let $x \in B_r^d(y)$. We set $s := d(x, y)$. By definition of $B_r^d(y)$ we have $s < r$. We set $t := r - s$. Since $t > 0$ it suffices to show that $B_t^d(x) \subset B_r^d(y)$. Thus let $z \in B_t^d(x)$. We see that

$$d(y, z) \underset{\substack{\uparrow \\ \text{triangle inequality}}}{\leq} d(x, y) + d(x, z) < s + t = r.$$

Thus we see that $z \in B_r^d(y)$, which is what we needed to show.



(2) Almost the same argument as in (1) shows that $X \setminus \overline{B}_r(y)$ is open, which implies that $\overline{B}_r(y)$ is closed.
(3) We have $X \setminus S_r^d(y) = B_r^d(y) \cup (X \setminus \overline{B}_r(y))$. By (1) and (2) we see that $X \setminus S_r^d(y)$ is the union of two open sets, thus by the third property of a topological space we see that $X \setminus S_r^d(y)$ is open, thus $S_r^d(y)$ is indeed closed. ■

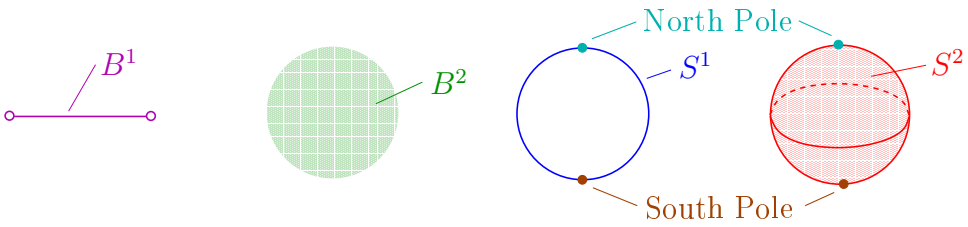
The most frequent application of Proposition 1.4 is to the metric space $\mathbb{R}^n$. Thus we introduce the following notation.

Notation. Let $n \in \mathbb{N}_0$. Given $y \in \mathbb{R}^n$ and $r \in \mathbb{R}$ we write[19] [20]

$$\begin{array}{rcll} B_r^n(y) & := & \{x \in \mathbb{R}^n \mid \|x - y\| < r\} & \text{called open } n\text{-ball} \\ \overline{B}_r^n(y) & := & \{x \in \mathbb{R}^n \mid \|x - y\| \leq r\} & \text{called closed } n\text{-ball} \\ S_r^{n-1}(y) & := & \{x \in \mathbb{R}^n \mid \|x - y\| = r\} & \text{called } (n-1)\text{-sphere.} \end{array}$$

We adopt the following conventions:
(1) A 2-ball is often called a disk and a 1-sphere is often referred to as a circle.
(2) If the parameter n is clear from the context we sometimes drop it from the notation.

(3) If $y = 0$, then we drop y from the notation and if $r = 1$, then we drop r from the notation.
(4) Sometimes we write rB^n instead of $B^n_r(0)$ and sometimes we write $r\overline{B}^n$ instead of $\overline{B}^n_r(0)$.
(5) We refer to the point $(0, \ldots, 0, 1) \in S^n = \{x \in \mathbb{R}^{n+1} \,|\, \|x\| = 1\}$ as the North Pole of S^n and we refer to the point $(0, \ldots, 0, -1) \in S^n = \{x \in \mathbb{R}^{n+1} \,|\, \|x\| = 1\}$ as the South Pole of S^n.

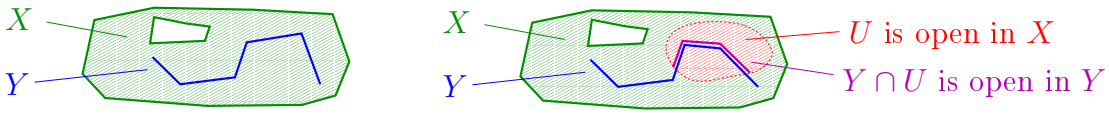


1.3. **The subspace topology.** The next definition gives us a rich source of new examples of topological spaces.

Definition. Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$ be a subset. It follows almost immediately from Lemma 0.3 that

$$\mathcal{S} := \{Y \cap U \,|\, U \in \mathcal{T}\}$$

is a topology on Y. We refer to $\mathcal{S}$ as the subspace topology on Y induced by Y.



Convention.
(1) Unless we say something else, we consider each subset Y of $\mathbb{R}^n$ always as a topological space with respect to the subspace topology.
(2) Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$ be a subset. If Y is indeed equipped with the subspace topology, then sometimes we refer to Y as a subspace of X.

We continue with the following useful lemma.

Lemma 1.5. (Neighborhood Openness Criterion) Let X be a topological space and let $Y \subset X$ be a subset. We equip Y with the subspace topology.

(1) $V \subset Y$ is open in $Y \iff$ there exists an open subset $U \subset X$ with $V = U \cap Y$
(2) $B \subset Y$ is closed in $Y \iff$ there exists a closed subset $A \subset X$ with $B = A \cap Y$.

Proof. The first statement is just a reformulation of the definition of the subspace topology. We turn to the proof of the second statement. Now let $B \subset Y$. Elementary set theory

[19] These definitions also make sense for $r < 0$, in that case all of the sets are the empty set. Also note that for $n = 0$ we obtain that $S^{n-1}_r(y) = S^{-1}_r(y) = \varnothing$.
[20] Of course one cannot easily sketch the 3-sphere S^3. But it is worth pointing out that in some form it has been around for a while. In [**Lip2014**, Chapter 2] and [**Petn1979**] the argument is made that the universe in Dante's Divina Commedia is precisely the 3-sphere.

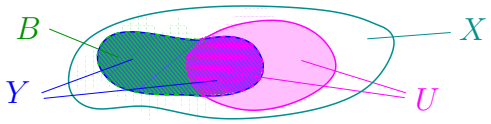
shows that for any subset U of X we have

$$(*) \qquad U \cap Y = Y \setminus B \iff Y \setminus (U \cap Y) = B \iff (X \setminus U) \cap Y = B.$$

Now the second statement is basically obvious. Indeed, we have

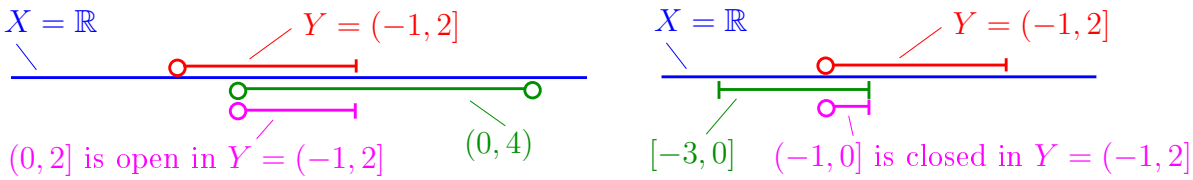
$$\begin{aligned} B \text{ is closed in } Y &\iff Y \setminus B \text{ is open in } Y \\ &\iff \text{there exists } U \subset Y \text{ open with } U \cap Y = Y \setminus B \\ &\underset{\text{by } (*)}{\iff} \text{there exists } U \subset Y \text{ open with } (X \setminus U) \cap Y = B \\ &\iff \text{there exists } A \subset Y \text{ closed with } A \cap Y = B. \end{aligned}$$

■



Example. We consider $Y = (-1, 2]$ equipped with the subspace topology coming from $\mathbb{R}$.

(1) The Neighborhood Openness Criterion 1.5 (1) implies that $V = (0, 2]$ is an open subset of Y, since we can write $V = (0, 4) \cap Y$ and $(0, 4)$ is an open subset of $\mathbb{R}$.
(2) The Neighborhood Openness Criterion 1.5 (2) implies that $B = (-1, 0]$ is a closed subset of Y, since we can write $B = [-3, 0] \cap Y$ and $[-3, 0]$ is a closed subset of $\mathbb{R}$.



We conclude this short discussion of the subspace topology with the following lemma which subconsciously gets used all the time.

Lemma 1.6. Let X be a topology space and let $Z \subset Y \subset X$ be two subsets. The following two topologies are the same:

(1) The subspace topology on Z induced by X.
(2) The subspace topology on Z induced by the subspace topology on Y, which is in turn induced by X.

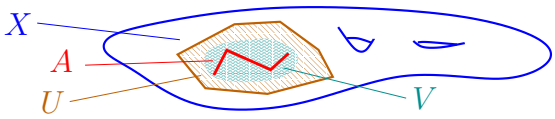
Proof. This lemma follows almost immediately from the definitions. ■

We continue with the next basic definition.

Definition. Let X be a topological space.

(1) Let $A \subset X$ be a subset. We say $U \subset X$ is a neighborhood of A if there exists an open set V such that $A \subset V \subset U$. We say U is an open neighborhood of A, if U is furthermore open. This definition is illustrated in the figure below.
(2) An (open) neighborhood of a point $x \in X$ is an (open) neighborhood of $\{x\}$.

Example. If $X = \mathbb{R}$ and $A = [0, 2)$, then $U = (-1, 3]$ and $V = (-2, \infty)$ are neighborhoods of A in X.



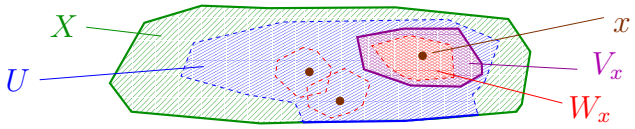
The following lemma gives a useful criterion for showing that a given subset is open.

Lemma 1.7. (JH-Lemma[21]) Let X be a topological space and let $U \subset X$ be a subset. If for every $x \in U$ there exists a neighborhood V of x that is contained in U, then U is open.

Proof. Let X be a topological space and let $U \subset X$ be a subset. Suppose that for every $x \in U$ there exists a neighborhood V_x of x that is contained in U. By definition of a neighborhood there exists for each $x \in U$ an open subset W_x with $x \in W_x \subset V_x$. We see that

$$U = \bigcup_{x \in U} \{x\} \subset \bigcup_{x \in U} W_x \underset{\text{since each } W_x \text{ is contained in } U}{\subset} U.$$

It follows that $U = \bigcup_{x \in U} W_x$, i.e. U is the union of open subsets of X, thus by the third property of a topology, U itself is open. ■



1.4. **Covers of topological spaces.** On many occasions it will be useful to "break" a topological space into smaller pieces. This leads us to the following definition.

Definition. Let X be a topological space.

(1) A cover[22] of a subset $A \subset X$ is a family[23] $\{U_i\}_{i \in I}$ of subsets of X with $A \subset \bigcup_{i \in I} U_i$.
(2) We say a cover $\{U_i\}_{i \in I}$ is open if each U_i is open. Similarly we define a closed cover.
(3) We say a cover $\{U_i\}_{i \in I}$ is finite if I is finite. Otherwise we say that the cover is infinite.
(4) We say a family $\{U_i\}_{i \in I}$ of subsets is locally finite if each $x \in X$ admits an open neighborhood W such that $\{i \in I \,|\, U_i \cap W \neq \varnothing\}$ is finite.

Example. In the figure below we sketch on the right a closed cover $\{U_i\}_{i \in I}$ of $X = \mathbb{R}^2$ which has the property that each $x \in X$ is contained in only finitely many U_i, but which nonetheless fails to be locally finite.

The following lemma is crucial for many arguments.

[21] A student, let's call him JH, pointed out to me in an earlier version of these notes, that I was implicitly frequently using the statement of this lemma.
[23] What we call a "cover" is often also called a "covering". We reserve the name "covering" for a completely different concept that we will introduce on page 1067.
[23] Recall that according to the definition on page 86 a family of subsets of a given set X consists of a set I and a map $f\colon I \to \mathcal{P}(X)$. We also recall the convention from page 86 which says that we will view any subset $\mathcal{U} \subset \mathcal{P}(X)$ as a family of subsets of X via the index set $I = \mathcal{U}$ and the map $I = \mathcal{U} \to \mathcal{P}(X)$ given by $U \mapsto U$.

locally finite closed cover of $X = \mathbb{R}^2$ not locally finite

Lemma 1.8. Let X be a topological space and let M be a subset of X.

(1) Let $\{U_i\}_{i\in I}$ be an open cover of X. The following two statements hold:
   (a) M is open if and only if each intersection $M \cap U_i$ is open in U_i,
   (b) M is closed if and only if each intersection $M \cap U_i$ is closed in U_i.

(2) Let $\{A_i\}_{i\in I}$ be a locally finite closed cover of X. (In most applications we will deal with a finite closed cover.) The following two statements hold:
   (a) M is open if and only if each intersection $M \cap A_i$ is open in A_i,
   (b) M is closed if and only if each intersection $M \cap A_i$ is closed in A_i.

Example. Let X be a topological space and let $\{A_i\}_{i\in I}$ be a family of closed subsets of X. If the family is locally finite, then it follows from Lemma 1.8 (2b) together with Lemma 1.3 (2) that $\bigcup\limits_{i\in I} A_i$ is also a closed subset of X.

Remark. Note that in Lemma 1.8 we allow an arbitrary family of open subsets but we only allow a finite family of closed subsets. One can easily see that this restriction is necessary. For example, consider $X = \mathbb{R}$ and the infinite family $\{A_i\}_{i\in I}$ of closed subsets of $X = \mathbb{R}$ given by all subsets consisting of a single point. Let $M \subset \mathbb{R}$ be a subset. For each $i \in I$ the intersection $M \cap A_i$ is either the empty set or all of A_i, in particular the intersection $M \cap A_i$ is closed in A_i. But of course there is no reason why M should be a closed subset of $\mathbb{R}$.

Proof. Let X be a topological space and let M be a subset of X.

(1) Let $\{U_i\}_{i\in I}$ be an open cover of X. The "only if"-directions of (a) and (b) are an immediate consequence of the Neighborhood Openness Criterion 1.5 (1). It remains to prove the "if"-directions.

   (a) Suppose that for every $i \in I$ the intersection $M \cap U_i$ is an open subset of U_i. By definition of the subspace topology this means that for each $i \in I$ there exists an open subset Z_i of X with $Z_i \cap U_i = M \cap U_i$. We have

$$M \underset{\substack{\uparrow \\ \text{since } \bigcup_{i\in I} U_i = X}}{=} \bigcup_{i\in I} (M \cap U_i) = \bigcup_{i\in I} \overbrace{(Z_i \cap U_i)}^{\text{intersection of open subsets of } X\text{, hence open in } X} \underset{\substack{\uparrow \\ \text{union of open subsets, hence open}}}{=} \text{open subset of } X.$$

   (b) Now suppose that each intersection $M \cap U_i$ is closed in U_i. In other words, each set $U_i \setminus (M \cap U_i)$ is open in U_i. Since $(X \setminus M) \cap U_i = U_i \setminus (M \cap U_i)$ we obtain from (a) that $X \setminus M$ is open in X. In other words, M is closed in X.

(2) First note that, as in (1) it remains to prove the "if"-directions.

(b) First we deal with the case that we are given a finite closed cover $\{A_i\}_{i=1,\dots,m}$ of X such that for each i the intersection $M \cap A_i$ is a closed subset of A_i. It follows from the Neighborhood Openness Criterion 1.5 (2) that for each $i \in \{1, \dots, m\}$ there exists a closed subset Z_i of X with $Z_i \cap A_i = M \cap A_i$. We have

$$M \underset{\substack{\uparrow \\ \text{since } A_1 \cup \dots \cup A_m = X}}{=} \bigcup_{i=1}^{m} (M \cap A_i) = \bigcup_{i=1}^{m} \overbrace{(Z_i \cap A_i)}^{\substack{\text{intersection of closed subsets of } X, \text{ hence closed in } X}} \underset{\substack{\uparrow \\ \text{union of finitely many closed subsets, hence closed}}}{=} \text{closed subset of } X$$

Now suppose that we are given a locally finite closed cover $\{B_i\}_{i \in I}$ of X such that for each $i \in I$ the intersection $M \cap B_i$ is a closed subset of B_i. Since the cover is locally finite there exists for each $x \in X$ an open neighborhood W_x such that $I_x := \{i \in I \,|\, W_x \cap B_i \neq \varnothing\}$ is finite. Since $\{W_x \cap B_i\}_{i \in I_x}$ is a finite closed cover of W_x we obtain from the above that $M \cap W_x$ is closed. But since $\{W_x\}_{x \in X}$ is an open cover of X we obtain from (1b) that M is a closed subset of X.

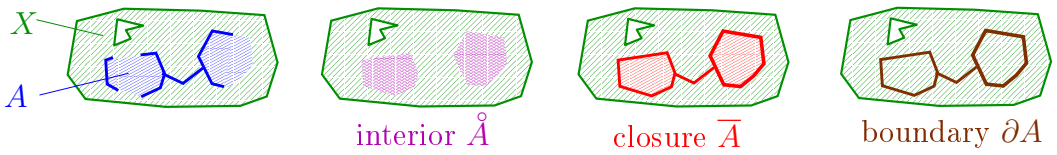
(a) As in (1) we can reduce the proof of (a) to the proof of (b). ■

1.5. The interior and the closure of a subset.

Definition. Let X be a topological space and let A be a subset of X. We make the following two definitions:[24]

$$\text{interior } \mathring{A} := \begin{array}{c}\text{union of all open sets of } X \\ \text{that are contained in } A\end{array} \qquad \text{closure } \overline{A} := \begin{array}{c}\text{intersection of all closed sets} \\ \text{of } X \text{ that contain } A\end{array}$$

Sometimes, to avoid confusion, we will also write $\operatorname{int}(A)$ for the interior of a subset A. Furthermore we define the boundary of A in X as $\partial A := \overline{A} \setminus \mathring{A}$.



In the following we will state general properties of the interior, the closure and the boundary of a subset. Evidently the point is not to memorize the following lemmas. Instead they are meant as a resource to turn to, when one is dealing with these objects. Therefore it is warmly recommended to move on to the next section and to return to the present section when the need arises.

Lemma 1.9. (Interior-Closure Lemma) Let X be a topological space and let A be a subset of X.

(1) The interior $\mathring{A}$ is an open subset of X.
(2) The closure $\overline{A}$ of A is a closed subset of X.
(3) We have $\mathring{A} \subset A \subset \overline{A}$.
(4) A is open $\iff \mathring{A} = A$.
(5) A is closed $\iff \overline{A} = A$.
(6) We have $\mathring{A} = A \setminus \partial A$ and $\overline{A} = A \cup \partial A$.

(7) If we set $B := X \setminus A$, then the following statements hold:

$$\begin{array}{lrcl} \text{(a)} & \overline{B} & = & X \setminus \mathring{A}, \\ \text{(b)} & \mathring{B} & = & X \setminus \overline{A}, \\ \text{(c)} & \partial B & = & \partial A. \end{array}$$

Proof.

(1) The union of arbitrarily many open sets is, by definition of a topology, again an open set. Thus the interior $\mathring{A}$ is an open subset of X.

(2) In Lemma 1.3 we saw that the intersection of arbitrarily many closed sets is again a closed set. It follows immediately that the closure $\overline{A}$ is a closed subset of X.

(3) This statement follows immediately from the definitions.

(4) The "$\Leftarrow$"-statement follows immediately from (1). We turn the proof of the "$\Rightarrow$"-statement. Given any subset $A \subset X$ we always have $\mathring{A} \subset A$. If A is open, then by definition we also have $A \subset \mathring{A}$. Thus we see that $\mathring{A} = A$.

(5) The "$\Leftarrow$"-statement follows immediately from (2). We turn the proof of the "$\Rightarrow$"-statement. Given any subset $A \subset X$ we always have $A \subset \overline{A}$. If A is closed, then by definition we also have $\overline{A} \subset A$. Thus we see that $\overline{A} = A$.

(6) We note that:

$$\begin{array}{rcccccl} A \setminus \partial A & = & A \setminus (\overline{A} \setminus \mathring{A}) & = & A \setminus (A \setminus \mathring{A}) & = & \mathring{A}. \\ & & \uparrow & & \uparrow & & \\ & & \text{definition of } \partial A & & \text{since } \mathring{A} \subset A \subset \overline{A} & & \\ & & \downarrow & & \downarrow & & \\ \text{and} \quad A \cup \partial A & = & A \cup (\overline{A} \setminus \mathring{A}) & = & \overline{A}. & & \end{array}$$

(7) We set $B := X \setminus A$. First we prove (a):

$$\overline{B} \overset{\text{definition of closure}}{\downarrow} = \bigcap_{\substack{Y \subset X \text{ closed} \\ \text{with } B \subset Y}} Y \overset{\text{substitution } U = X \setminus Y}{\downarrow} = \bigcap_{\substack{U \subset X \text{ open} \\ \text{with } U \subset A}} (X \setminus U) \overset{\text{de Morgan's Laws 0.1}}{\downarrow} = X \setminus \bigcup_{\substack{U \subset X \text{ open} \\ \text{with } U \subset A}} U \overset{\text{definition of interior}}{\downarrow} = X \setminus \text{interior of } A.$$

Next note that (b) is an immediate consequence of (a). Finally note that (c) is an immediate consequence of (a) and (b). ■

The above definition of the interior and the closure of a subset is succinct and precise. But just by using the definition it is perhaps difficult to find the interior and the closure of a given subset of a topological space. In many situations the following lemma takes care of that difficulty.

Lemma 1.10. Let X be a topological space and let A be a subset of X. We have the following three equalities:

$$\begin{array}{lrcl} (1) & \mathring{A} & = & \{x \in A \mid \text{there exists a neighborhood } U \text{ of } x \text{ that is contained in } A\} \\ (2) & \overline{A} & = & \{x \in X \mid \text{every neighborhood of } x \text{ contains at least one point in } A\} \\ (3) & \partial A & = & \{x \in X \mid \text{every neighborhood of } x \text{ contains at least one point in } A \\ & & & \qquad \text{and one point that does not lie in } A\} \end{array}$$

The same statements also hold with "neighborhood" replaced by "open neighborhood".

Proof. The proof of the lemma is Exercise 1.6. ■

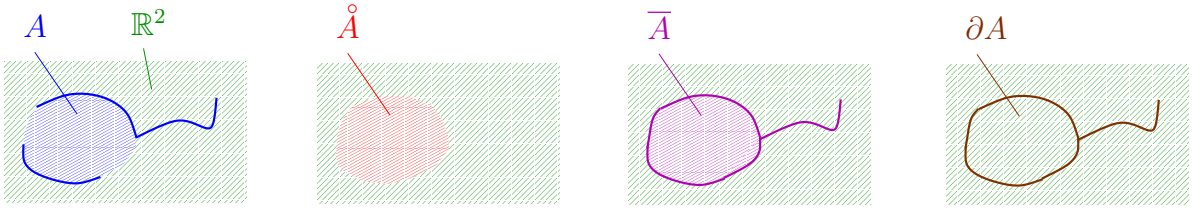
Examples.

(1) We consider the topological space $X = \mathbb{R}$ and the subset $A = [-1, 2)$. Using Lemma 1.10 one can easily see that the interior of A is the open interval $(-1, 2)$ and that the closure of A is the closed interval $[-1, 2]$. Furthermore $\partial A = \{-1, 2\}$.

(2) Let $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$. We view $B_r^n(x)$, $\overline{B}_r^n(x)$ and $S_r^{n-1}(x)$ as subsets of $\mathbb{R}^n$. Using Lemma 1.10 one can show fairly easily that

$$\overline{B_r^n(x)} = \overline{B}_r^n(x), \quad \operatorname{int}(\overline{B}_r^n(x)) = B_n^r(x), \quad \partial\overline{B}_r^n(x) = \partial B_r^n(x) = \partial S_r^{n-1}(x) = S_r^{n-1}(x).$$

(3) In the figure below we show a subset of $\mathbb{R}^2$, and we also show its interior $\mathring{A}$, its closure $\overline{A}$ and its boundary ∂A.



(4) It is rather unfortunate that the notations $\mathring{A}$, $\overline{A}$ and ∂A hide the role of X. For example let us consider $A = [0, 2]$. If we view it as a subset of $X = \mathbb{R}$, then $\partial A = \partial[0, 2] = \{0, 2\}$. On the other hand, if we view $A = [0, 2]$ as a subset of $X = \mathbb{R}_{\geq 0}$, then $\partial A = \partial[0, 2] = \{2\}$. This ambiguity in the notation can sometimes be a source of confusion. Thus it would be more appropriate to specify the role of X in the above notation. But mathematicians, as all humans, tend to be lazy and thus tend to simplify the notation.

(5) We consider $X = \mathbb{R}$ and the subset $A = \mathbb{Q}$. Using Lemma 1.10 one can easily show, using standard facts about real numbers, that $\mathring{\mathbb{Q}} = \varnothing$ and that $\overline{\mathbb{Q}} = \mathbb{R}$. The same conclusion also holds for $Q = \mathbb{R} \setminus \mathbb{Q}$.

Finally we consider the relationship between closures and interiors under taking unions and intersections. As the following examples show, this is a tricky topic.

Example. Throughout we work with the topological space $X = \mathbb{R}$.

(1) Let $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$. We see that

$$\begin{aligned} \text{interior of } \underbrace{\mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})}_{=\mathbb{R}} &= \mathbb{R} \neq \varnothing = \underbrace{\text{interior of } \mathbb{Q}}_{=\varnothing} \cup \underbrace{\text{interior of } \mathbb{R} \setminus \mathbb{Q}}_{=\varnothing} \\ \text{closure of } \underbrace{\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q})}_{=\mathbb{R}} &= \varnothing \neq \mathbb{R} = \underbrace{\text{closure of } \mathbb{Q}}_{=\mathbb{R}} \cap \underbrace{\text{closure of } \mathbb{R} \setminus \mathbb{Q}}_{=\mathbb{R}}. \end{aligned}$$

This discussion shows that taking interiors does not commute even with just taking *finite* unions and analogously that taking closures does not even commute with taking *finite* intersections.

(2) We consider the subsets $[0, \frac{1}{n})$ and $[0, 1 - \frac{1}{n})$ of $\mathbb{R}$. Now we see that

$$\text{interior of } \bigcap_{n\in\mathbb{N}} [0, \tfrac{1}{n}) = \text{interior of } \{0\} = \varnothing \subsetneq \{0\} = \bigcap_{n\in\mathbb{N}} \underbrace{\text{interior of } [0, \tfrac{1}{n})}_{=[0,\frac{1}{n})}$$

$$\text{closure of } \bigcup_{n\in\mathbb{N}} [0, 1-\tfrac{1}{n}) = \text{closure of } [0,1) = [0,1] \supsetneq [0,1) = \bigcup_{n\in\mathbb{N}} \underbrace{\text{closure of } [0, 1-\tfrac{1}{n})}_{=[0,1-\frac{1}{n})}$$

These examples show that taking interiors does not commute even with taking infinite intersections and analogously that taking closures does not commute with taking infinite unions.

After this exhaustive list of counter examples the reader will be relieved to see that one can in fact formulate a lemma regarding the interaction of interiors and closures with unions and intersections.

Lemma 1.11. Let X be a topological space and let $\{U_i\}_{i\in I}$ be a family of subsets. The following inclusions hold:

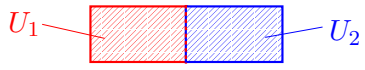
(1a) $\text{interior of } \bigcup_{i\in I} U_i \supset \bigcup_{i\in I} \mathring{U}_i$ and (1b) $\text{closure of } \bigcap_{i\in I} U_i \subset \bigcap_{i\in I} \overline{U_i}$

(2a) $\text{interior of } \bigcap_{i\in I} U_i \subset \bigcap_{i\in I} \mathring{U}_i$ and (2b) $\text{closure of } \bigcup_{i\in I} U_i \supset \bigcup_{i\in I} \overline{U_i}$.

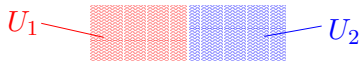
Furthermore, if the family $\{U_i\}_{i\in I}$ is locally finite (e.g. if it is finite), then the following equalities hold:

(3a) $\text{interior of } \bigcap_{i\in I} U_i = \bigcap_{i\in I} \mathring{U}_i$ and (3b) $\text{closure of } \bigcup_{i\in I} U_i = \bigcup_{i\in I} \overline{U_i}$.

Example. In the figure below we show two subsets U_1, U_2 of $\mathbb{R}^2$ such that the inclusions in Lemma 1.11 (1a) and (1b) are actually proper inclusions.



interior of $U_1 \cup U_2 \supsetneq \mathring{U}_1 \cup \mathring{U}_2$



closure of $U_1 \cap U_2 \subsetneq \overline{U}_1 \cup \overline{U}_2$

Proof.

(1a) By the Interior-Closure Lemma 1.9 we know that each $\mathring{U}_i$ is open. Therefore $\bigcup_{i\in I} \mathring{U}_i$ is open. Evidently this set is contained in $\bigcup_{i\in I} U_i$. By definition of the interior the set $\bigcap_{i\in I} \mathring{U}_i$ is now also contained in the interior of $\bigcup_{i\in I} U_i$.

(1b) It is straightforward to prove (1b) directly. In the following we show alternatively, that (1b) is in fact a formal consequence of (1a). Indeed, we have

$$\begin{array}{ccccc} \text{closure of } \bigcap_{i\in I} U_i & \overset{\text{Interior-Closure Lemma 1.9 (7)}}{\downarrow =} & X \setminus \text{interior of } \Big(X \setminus \bigcap_{i\in I} U_i\Big) & \overset{\text{de Morgan's Laws 0.1}}{\downarrow =} & X \setminus \text{interior of } \Big(\bigcup_{i\in I} (X \setminus U_i)\Big) \\ & \overset{\text{follows from (1a)}}{\uparrow \subset} & X \setminus \bigcup_{i\in I} \text{interior of } (X \setminus U_i) & \overset{\text{de Morgan's Laws 0.1}}{\uparrow =} & \bigcap_{i\in I} X \setminus \text{interior of } (X \setminus U_i) \overset{\text{Interior-Closure Lemma 1.9 (7)}}{\uparrow =} \bigcap_{i\in I} \overline{U}_i. \end{array}$$

(2a) The interior V of $\bigcap_{i\in I} U_i$ is an open subset which is contained in each U_i. By definition of the interior of U_i we see that V is contained in each $\mathring{U}_i$. The promised inclusion is an immediate consequence of this observation.

(2b) The closure A of $\bigcup_{i\in I} U_i$ is a closed set that contains each U_i. By definition of the closure of U_i we see that A contains each $\overline{U}_i$. The promised inclusion is an immediate consequence of this observation. observation.

We now assume that the family $\{U_i\}_{i\in I}$ is locally finite.

(3a) By (2a) We already know that the "$\subset$"-inclusion holds. Thus it suffices to prove the "$\supset$"-inclusion. By definition of the interior it suffices to show that $\bigcap_{i\in I} \mathring{U}_i$ is open. By Lemma 1.8 it remains to prove the following claim.

Claim. For each $x \in \bigcap_{i\in I} \mathring{U}_i$ there exists an open neighborhood W of $x \in X$ such that $W \subset \bigcap_{i\in I} \mathring{U}_i$.

Proof. Let $x \in \bigcap_{i\in I} \mathring{U}_i$. Since $\{U_i\}_{i\in I}$ is a locally finite family of open subsets there exists an open neighborhood V of x such that $J := \{i \in I \,|\, V \cap U_i = \varnothing\}$ is finite. We see that

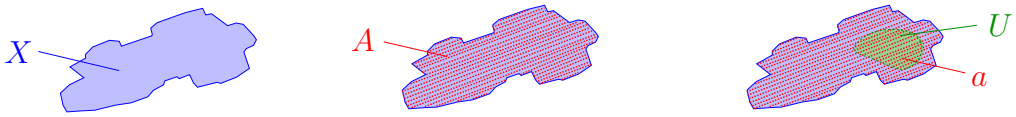
$$W \;:=\; \underbrace{V \cap \bigcap_{i\in J} \mathring{U}_i}_{\substack{\text{intersection of finitely}\\ \text{many open sets}}} \;=\; \bigcap_{i\in J} (V \cap \mathring{U}_i) \;\subset\; \underbrace{\bigcap_{i\in I} (V \cap \mathring{U}_i)}_{\uparrow \text{ by definition of } J} \;\subset\; \bigcap_{i\in I} \mathring{U}_i.$$

Since the intersection of finitely many open sets is open we see that W is an open neighborhood of $x \in X$. Thus we are done.

(3b) By the same argument as in the proof of (1b) we can use de Morgan's Laws 0.1 to reduce the proof of (3b) to the proof of (3a). ■

We conclude this section with the discussion of the following definition, which will show up every now and then.

Definition. Let X be a topological space. We say a subset A of X is dense in X if every open non-empty subset U of X contains a point $a \in A$.



Example.

(1) Let $n \in \mathbb{N}$. It follows from basic facts in real analysis that $A = \mathbb{Q}^n$ is a dense subset of $X = \mathbb{R}^n$.

(2) It is very tempting to think that dense *open* subsets of say $X = [0,1]$ must be "almost the whole set". But the following example might dent one's optimism. First we pick a bijection $f\colon \mathbb{N} \to \mathbb{Q} \cap [0,1]$. For $i \in \mathbb{N}$ we write $x_i := f(i)$. We consider the set

$$U \;:=\; \bigcup_{i\in\mathbb{N}} \underbrace{(x_i - \tfrac{1}{3^i}, x_i + \tfrac{1}{3^i}) \cap [0,1]}_{\text{open in } [0,1]} \;\subset\; [0,1].$$

The same argument as in (1) shows that U is a dense subset of $[0,1]$. Since U is the union of open subsets we see that U is also an open subset of $[0,1]$. On the other hand its Lebesgue measure satisfies

$$\mathrm{vol}(U) = \mathrm{vol}\left(\bigcup_{i\in\mathbb{N}} \left(x_i - \tfrac{1}{3^i}, x_i + \tfrac{1}{3^i}\right) \cap [0,1]\right) \leq \sum_{i\in\mathbb{N}} \mathrm{vol}\left((x_i - \tfrac{1}{3^i}, x_i + \tfrac{1}{3^i})\right) \leq \sum_{i\in\mathbb{N}} \frac{2}{3^i} = \frac{2}{3}.$$

Thus we have shown that the Lebesgue measure of U is less than the Lebesgue measure of $[0,1]$. So by no means can one reasonably claim that the dense open U subset is "almost all of $[0,1]$".

(3) Here is an immediate, perhaps not entirely serious application of topology to the eternal problem for how to capture a lion in the desert [**Barr1981**]: We give the desert the *leonine topology*, in which a subset is closed if it is the whole desert or if it contains no lions.[25] The set of lions is now a dense subset.[26] Now put an open cage in the desert. By density it contains a lion. Once we shut the cage we have caught the lion![27]

In the following we relate the notions of closure and interiors to some of notions that we had introduced earlier.

Lemma 1.12. Let X be a topological space and let $A \subset X$ be a subset. The following three statements are equivalent:

(1) A is dense in X.
(2) We have $\overline{A} = X$.
(3) The interior of $X \setminus A$ is empty.

Proof. Let $A \subset X$. The equivalence of (2) and (3) is a consequence of the Interior-Closure Lemma 1.9. Thus it remains to show that (1) and (3) are equivalent. In fact we have

$$\begin{aligned} A \text{ is dense} &\iff \text{for every open non-empty subset } U \text{ of } X \text{ we have } U \cap A \neq \varnothing \\ &\iff \text{for every open non-empty subset } U \text{ of } X \text{ we have } U \subsetneq X \setminus A \\ &\iff \text{there is no open non-empty subset } U \text{ of } X \text{ with } U \subset X \setminus A \\ &\iff \text{the interior of } X \setminus A \text{ is empty.} \end{aligned}$$

■

1.6. The Hausdorff property.

Definition.

(1) We say a topological space X is Hausdorff, if given any two distinct points $x \neq y$ there exist disjoint open neighborhoods U of x and V of y.
(2) A Hausdorff space is a topological space that is Hausdorff.[28]

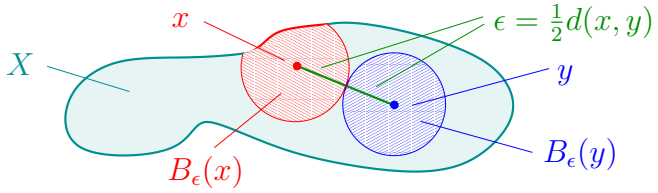


[25] Why is this a topology?
[26] Why is that?
[27] This and many other mathematical approaches to hunting lions can be found in [**Boas1995**].
[28] We use this language for all other properties of topological spaces, for example we will soon introduce the very important notion of a compact topological space. We will often write "let X be a compact space" instead of the more precise "let X be a compact topological space".

The following proposition, which very likely is well-known to most readers, says that topological spaces arising from metric spaces are Hausdorff.

Proposition 1.13. Let (X, d) be a metric space. The corresponding topological space is Hausdorff.

Proof. Let $x, y \in X$ be two different points. We set $\epsilon := \frac{1}{2}d(x, y)$. It follows from Proposition 1.4 that $U := B_\epsilon(x)$ and $V := B_\epsilon(y)$ are neighborhoods of x respectively y. An elementary argument using the triangle inequality shows that U and V are disjoint. ■



We now introduce two more examples of topological spaces. Both are of the type "$\mathbb{R}^n$ together with an extra point". These two topological spaces are often good candidates for (counter-) examples in topology.

Definition.

(1) Let $n \in \mathbb{N}$. We consider the set

$$X := \mathbb{R}^n \cup \{\infty\},$$

i.e. X consists of $\mathbb{R}^n$ and an extra point ∞. We say $U \subset X$ is open, if both of the following two conditions are satisfied:
(a) for each point $x \in U \cap \mathbb{R}^n$ there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subset U$,
(b) if $\infty \in U$, then there exists a $C > 0$ such that $\{x \in \mathbb{R}^n \mid \|x\| > C\} \subset U$.
It is straightforward to see that this defines indeed a topology on X. We refer to $\mathbb{R}^n \cup \{\infty\}$ as $\mathbb{R}^n$ with a point at infinity. For $n = 1$ we refer to $\mathbb{R} \cup \{\infty\}$ also as the line with a point at infinity.

(2) We consider the set

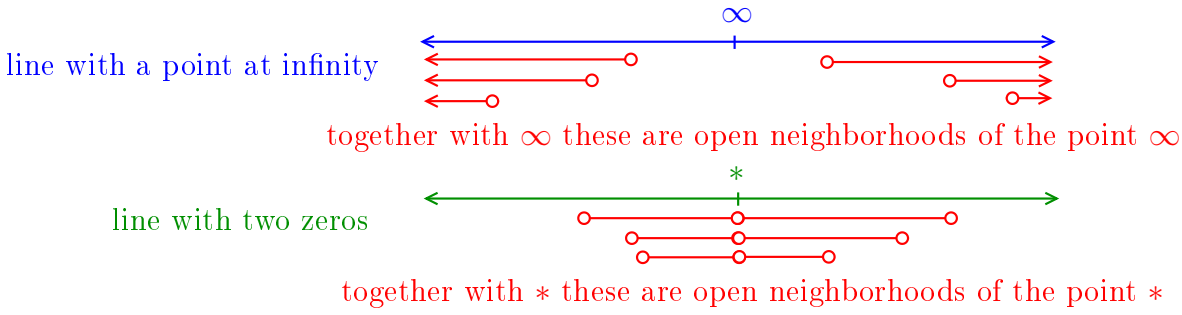
$$X := \mathbb{R} \cup \{*\},$$

i.e. X consists of $\mathbb{R}$ and an extra point $*$. We say $U \subset X$ is open, if the following two conditions are satisfied:
(a) for each point $x \in U \cap \mathbb{R}$ there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$,
(b) if $* \in U$, then there exists an $\epsilon > 0$ such that $(-\epsilon, 0) \cup (0, \epsilon) \subset U$.
One can easily verify that this is indeed a topology on X. We refer to this topological space as the line with two zeros.

Even though the "line with two zeros" looks at first glance similar to the "line with a point at infinity", the next example shows that they are strikingly different topological spaces.

Example. In Exercise 1.18 we will see that "$\mathbb{R}^n$ with a point at infinity" is Hausdorff and that on the other hand "the line with two zeros" is not Hausdorff.

The following lemma is basically trivial, but since it gets used on many occasions it is still worth spelling it out.



Lemma 1.14. Let X be a topological space and let A be a subset of X. If X is Hausdorff, then A is also Hausdorff.

Proof. The lemma follows almost immediately from the definitions. ■

The following gives us a modest but still quite useful consequence of being Hausdorff.

Lemma 1.15. (Points-in-Hausdorff Space Lemma) Let X be a topological space.

(1) If X is Hausdorff, then for every $x \in X$ the corresponding subset $\{x\}$ is closed.
(2) If X is Hausdorff, then for every $x \neq y \in X$ there exists an open neighborhood U of x such that $y \notin \overline{U}$.

Proof. Let X be a topological space which is Hausdorff.

(1) Let $x \in X$. Since X is Hausdorff we know that for every $y \neq x$ there exist open disjoint neighborhoods U_y of x and V_y of y. Note that

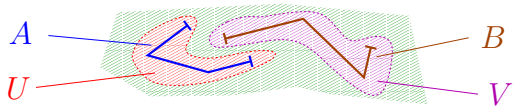
$$X \setminus \{x\} = \bigcup_{y \in X \setminus \{x\}} \{y\} \underset{\substack{\uparrow \\ \text{since } y \in V_y}}{\subset} \bigcup_{y \in X \setminus \{x\}} V_y \underset{\substack{\uparrow \\ \text{since } x \notin V_y}}{\subset} X \setminus \{x\}.$$

Thus we see that the inclusions are equalities. Since each V_y is open we see that $X \setminus \{x\}$ is the union of open sets, in particular $X \setminus \{x\}$ is open. In other words, $\{x\}$ is a closed subset of X.

(2) Let $x \neq y \in X$. Since X is Hausdorff there exist disjoint open neighborhoods U of x and V of y. Since $U \subset X \setminus V$ and since $X \setminus V$ is closed we obtain from the definition of the closure that $\overline{U} \subset X \setminus V$. In particular $y \notin \overline{U}$. ■

There are many variations on the definition of the Hausdorff property. We will discuss some of these in Chapters 8 and 9, and especially in Section 9.4. For the time being we just want to introduce one variation on being Hausdorff:

Definition. A topological space X is called normal if any two disjoint closed subsets A and B of X can be separated by open neighborhoods, i.e. if there exist disjoint open neighborhoods U of A and V of B.



We will discuss this notion of begin normal in greater detail in Section 8.1. For the time being we only want to prove the following lemma and a corollary, both of which can be quite useful.

Lemma 1.16. (Compact Hausdorff Spaces-Normal Lemma) Every compact Hausdorff space is normal.

Proof. Let X be a Hausdorff space. We start out with the following claim.

Claim. Let B be a compact subspace of X. Given any $a \in X \setminus B$ there exists an open neighborhood U_a of $a \in X$ and an open neighborhood V_a of B that are disjoint.



Proof. Let $a \in A$. Since X is Hausdorff we can find for any $b \in B$ disjoint open neighborhoods R_b and S_b of a and b. Furthermore, since B is compact there exist $b_1, \ldots, b_n \in B$ such that $B \subset S_{b_1} \cup \cdots \cup S_{b_n}$. The open sets

$$U_a := R_{b_1} \cap \cdots \cap R_{b_n} \quad \text{and} \quad V_a := S_{b_1} \cup \cdots \cup S_{b_n}$$

have the desired properties. ⊞

Now assume that X is also compact. We need to show that X is normal. Thus let A and B be two disjoint closed subsets of X. Note that the Compact-Closed Lemma 1.21 (1) implies that A and B are also compact.

Since B is compact we can apply the above claim to every $a \in A \subset X \setminus B$ and B. In the following we use the notation of the above claim. Since A is compact we can find $a_1, \ldots, a_m \in A$ such that $A \subset U_{a_1} \cup \cdots \cup U_{a_m}$. The open sets

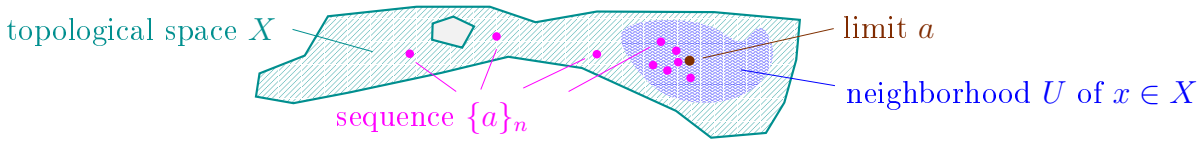
$$U := U_{a_1} \cup \cdots \cup U_{a_m} \quad \text{and} \quad V := V_{a_1} \cap \cdots \cap V_{a_m}$$

have the desired properties. ■

1.7. **Convergence of sequences.** In this short section we will introduce the notion of convergence of a sequence of points in a topological spaces and we will see how the Hausdorff property comes in handy. We will barely make use of the results of this section. For a reader who is used to analysis it is often quite natural to argue with limits. But when it comes to topological spaces it is in most cases more elegant to avoid arguing with limits.

After this preamble, let us finally introduce the notion of a convergent sequence in a topological space.

Definition. Let X be a topological space. We say that a sequence $\{a_n\}_{n\in\mathbb{N}}$ converges if there exists an $a \in X$ such that for every neighborhood U of a there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $a_n \in U$. We refer to a as a limit of the sequence.

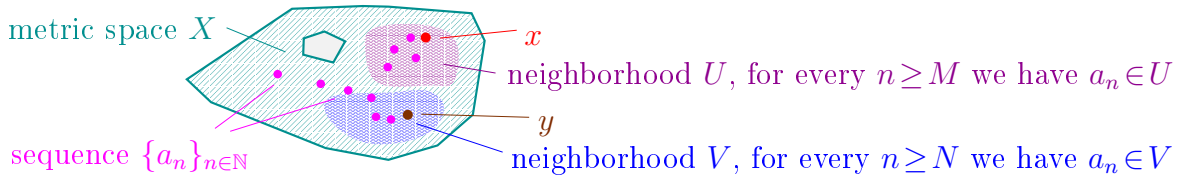


Example. We consider the line with two zeros, i.e. the topological space $X = \mathbb{R} \cup \{*\}$ defined above. The sequence $a_n = \frac{1}{n}$ converges, more or less evidently with limit 0. But strikingly enough, it follows easily from the definitions that the sequence also converges to $*$. Thus we see that the limit is not unique.

Lemma 1.17. (Unique Limit Lemma) Let X be a topological space. If X is Hausdorff, then every convergent sequence in X has a unique limit.

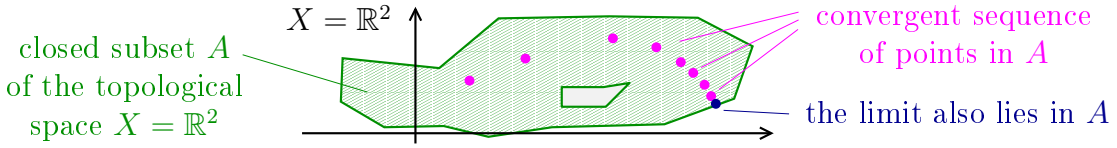
Proof. Assume there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ in X that converges to two different points x and y. Since X is Hausdorff there exist disjoint neighborhoods U and V of x and y. By our hypothesis we know the following:

- There exists an $M \in \mathbb{N}$ such for every $n \geq M$ we have $a_n \in U$.
- There exists an $N \in \mathbb{N}$ such for every $n \geq N$ we have $a_n \in V$.



But this means that for $n \geq \max\{M, N\}$ have to lie in U and V at the same time. But that is not possible and we have thus obtained a contradiction. ■

Lemma 1.18. (Closed Subset Limit Lemma) Let X be a topological space, let $A \subset X$ be a closed subset and let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of points in A. Every limit of the sequence lies in A.



Proof. We suppose that the sequence has a limit $x \in X \setminus A$. Since A is closed we know that $U := X \setminus A$ is open. But if we now apply the definition of a limit to this open neighborhood $U = X \setminus A$ of x we see that almost all of the a_n have to lie in $U = X \setminus A$. We have thus obtained a contradiction. ■

1.8. **Compact topological spaces.** In this section we introduce the notion of compactness. Usually it takes a while till one appreciates the brilliance of the definition.

Definition. We say a topological space X is compact if for each open cover $\{U_i\}_{i\in I}$ of X there exist finitely many indices $i_1, \ldots, i_k \in I$ such that

$$X = U_{i_1} \cup \cdots \cup U_{i_k}.$$

Remark. The introduction of the notion of compact space by Pavel Alexandrov and Pavel Urysohn [**AU1929**] in 1929 shows that often in mathematics the really clever thing is not necessarily the proofs, but coming up with just the right definition. It is difficult to exaggerate the role of compactness in topology.

Example.

(1) Every finite topological space (regardless of the topology) is compact, since there exist only finitely many distinct open subsets. This simple example can sometimes be quite useful.
(2) We consider the topological space $X = \mathbb{R}$. The sets $\{(-n, n)\}_{n\in\mathbb{N}}$ form an open cover of $X = \mathbb{R}$, but evidently we cannot cover $X = \mathbb{R}$ by finitely many of these sets. This shows that $X = \mathbb{R}$ is not compact.

In the following we will often use the following convention.

Convention. Let P be a property of a topological space (e.g. being discrete, Hausdorff, compact). We say that a subset Y of a topological space X has the property P if Y, equipped with the subspace topology, has the property P.

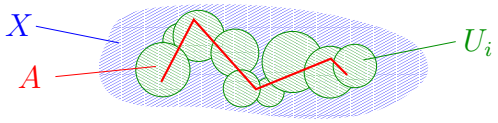
Examples.

(1) The set $\mathbb{Z} \subset \mathbb{R}$ is easily seen to be a discrete subset.
(2) Any finite set of a topological space is, by the above, a compact subset.

Frequently and often subconsciously the following elementary lemma is employed.

Lemma 1.19. (Compact Subspace Covering Lemma) Let X be a topological space, let $A \subset X$ be a subset and let $\{U_i\}_{i\in I}$ be an open cover of the subset $A \subset X$. If A is compact, then there exist finitely many indices $i_1, \ldots, i_k \in I$ such that

$$A \subset U_{i_1} \cup \cdots \cup U_{i_k}.$$



Proof. Given $i \in I$ we set $V_i := U_i \cap A$. By definition of the subspace topology each V_i is an open subset of A. Thus $\{V_i\}_{i\in I}$ is an open cover of A. Since A is compact there exist $i_1, \ldots, i_k \in I$ with $A = V_{i_1} \cup \cdots \cup V_{i_k}$. But this implies that $A \subset U_{i_1} \cup \cdots \cup U_{i_k}$. ■

We continue with the following frequently employed lemma.

Lemma 1.20. (Compact-Union-Intersection Lemma) Let X be a topological space.
(1) The union of finitely many compact subsets of X is again a compact subset.
(2) If X is Hausdorff, then the intersection of arbitrarily many compact subsets of X is again a compact subset.

Remark. Note that in the Compact-Union-Intersection Lemma 1.20 we cannot drop the hypothesis that X is Hausdorff. For example consider the set $X = \mathbb{N} \cup \{x_0, x_1\}$ where we equip $\mathbb{N}$ with the discrete topology and where the only other open sets are $\mathbb{N} \cup \{x_i\}$ and $\mathbb{N} \cup \{x_0, x_1\}$. The subsets $A_0 = \mathbb{N} \cup \{x_0\}$ and $A_1 = \mathbb{N} \cup \{x_1\}$ are easily seen to be compact, but the intersection $A_0 \cap A_1$ is certainly not compact.

Proof.

(1) This statement follows easily from the definitions, we leave it to the reader to fill in the details.

(2) Let $\{A_i\}_{i\in I}$ be a family of compact subsets of X. We pick $j \in I$. By the Compact-Closed Lemma 1.21 (2) know that each $A_i \cap A_j$ is a closed subset of A_i. Thus $\bigcap_{i\in I}(A_i \cap A_j)$ is a closed subset of the compact space A_j. Hence $\bigcap_{j\in I} A_i$ is compact by (1) Compact-Closed Lemma 1.21. ■

Before we give some more interesting examples of compact spaces we want to discuss a few properties of compact spaces and compact subsets. The following lemma is frequently used, often subconsciously, in topology.

Lemma 1.21. (Compact-Closed Lemma) Let X be a topological space.
(1) If X is compact, then any closed subset of X is also compact.
(2) Let $A \subset X$ be a compact subset. If X is Hausdorff, then A is a closed subset of X.

Example.

(1) Let X be a topological space. As we pointed out on page 126, any finite subset of X is compact. Thus if X is Hausdorff, then it follows from the Compact-Closed Lemma 1.21 (2) that any finite subset of X, in particular any point in X, is a closed subset of X. Thus we have found a new proof for the Points-in-Hausdorff Space Lemma 1.15 (1).
(2) Let X be a topological space, let A be a closed subset and let K be a compact subset of X. It follows from the Compact-Closed Lemma 1.21 (1) that $A \cap K$ is also compact. This simple observation will get used frequently.
(3) We consider the topological space $X = \{0, 1\}$ that is equipped with the trivial topology $\{\varnothing, X\}$. The subset $A = \{0\}$ is evidently compact, but it is not a closed subset of X. This shows that in the formulation of Compact-Closed Lemma 1.21 (2) we cannot drop the hypothesis that X is Hausdorff.

Proof.

(1) Let X be a compact space and let A be a closed subset. Let $\{U_i\}_{i\in I}$ be an open cover of A. By definition of the subset topology there exists for every $i \in I$ an open subset V_i of X such that $U_i = V_i \cap A$. Since A is closed the complement $X \setminus A$ is open. The open subsets $\{V_i\}_{i\in I}$ together with $X \setminus A$ are an open cover of X. Since X is compact we can cover X by finitely many of these open sets. But then we can also cover A by finitely many of the U_i.



(2) Let X be a Hausdorff space and let $A \subset X$ be a compact subset. We want to show that $X \setminus A$ is open. By the JH-Lemma 1.7 it suffices to prove the following claim.

Claim. Let $x \in X \setminus A$. Then there exists an open neighborhood V of x that is contained in $X \setminus A$.

Proof. We apply the Hausdorff property to x and every $y \in A$. For every $y \in A$ we obtain disjoint open neighborhoods U_y of y and V_y of x. Evidently we have

$$A = \bigcup_{y\in A} \{y\} \subset \bigcup_{y\in A} (U_y \cap A) \subset A.$$

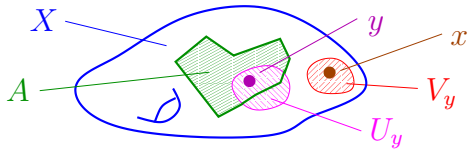
Thus we see that $\{U_y \cap A\}_{y\in A}$ is an open cover of A. Since A is compact there exist $y_1, \ldots, y_k \in A$ such that

$$A = \bigcup_{i=1}^{k} (U_{y_i} \cap A).$$

Now we consider

$$V := \bigcap_{i=1}^{k} V_{y_i}.$$

Since V is the intersection of finitely many open sets, it is open itself. Furthermore V does not intersect any of the U_{y_i}, $i = 1, \ldots, k$. This implies that V is disjoint from $A \subset U_{y_1} \cup \cdots \cup U_{y_k}$. ■



Before we move on to giving more interesting examples of compact spaces we state and prove the following elementary lemma.

Lemma 1.22. (Compact-Discrete Lemma)

(1) Every discrete space that is compact is finite.
(2) Let X be a compact space and let A be a closed subset. If A is discrete, then A is finite.

Remark. Note that the conclusion of the Compact-Discrete Lemma 1.22 (2) does not necessarily hold, if we drop the hypothesis that A is closed. For example, consider the topological space $X = [0, 1]$. The subset $A := \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$ is a discrete subset, but evidently it is not finite.

Proof.

(1) Let X be a discrete space. Each point thus defines an open set. Evidently X is covered by all the open sets given by the points in X. Since X is compact we can cover X by finitely many of those open sets. But this means that X contains only finitely many points.
(2) Let X be a compact space and let A be a closed subset. It follows from the Compact-Closed Lemma 1.21 (1) that A is also compact. So if A is discrete, then it follows from (1) that A is finite. ■

We continue with the following almost self-explanatory definition.

Definition. Let (X, d) be a metric space. A subset A of $\mathbb{R}^n$ is called bounded if there exists a $C \geq 0$ such that for all $x_0, x_1 \in A$ we have $d(x_0, x_1) \leq C$.

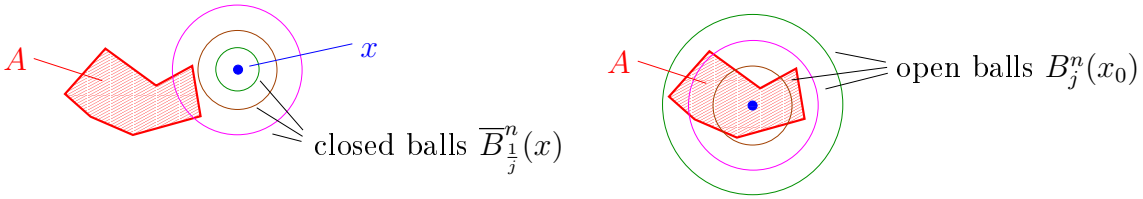
Example. Let X be a metric space, let $y \in X$ and let $r \geq 0$. It follows easily from the triangle inequality that the balls $B_r^d(y)$, $\overline{B}_r^d(y)$ and the sphere $S_r^d(y)$ are bounded.

Lemma 1.23. Let (X, d) be a metric space and let $A \subset X$ be a subset. If A is compact, then A is closed and it is bounded.

Proof. Let (X,d) be a metric space and let $A \subset X$ be a subset.

(1) We want to show that $X \setminus A$ is open. Let $x \in X \setminus A$. We need to show that there exists an $\epsilon > 0$ with $B^d_\epsilon(x) \subset X \setminus A$. Given $j \in \mathbb{N}$ we consider the subset $U_j = X \setminus \overline{B}^d_{\frac{1}{j}}(x)$. By Proposition 1.4 we know that the U_j are open. Evidently these open sets cover A. Since A is compact we obtain from the Compact Subspace Covering Lemma 1.19 that there exist $j_1, \dots, j_k \in \mathbb{N}$ with $A \subset U_{j_1} \cup \dots \cup U_{j_k}$. We set $\epsilon := \min\{\frac{1}{j_1}, \dots, \frac{1}{j_k}\}$. We obtain that $A \subset X \setminus \overline{B}^d_\epsilon(x)$. Thus we have found the desired ϵ.

(2) We need to show that A is bounded. If A is empty, then there is nothing to show. Now suppose that A is non-empty. We pick $x_0 \in A$. It follows from the triangle inequality that it suffices to show that there exists an $R \geq 0$ with $A \subset \overline{B}^d_R(x_0)$. We consider the open subsets $U_j = B^d_j(x_0)$ with $j \in \mathbb{N}$. Evidently these sets cover A. Since A is compact we obtain again from the Compact Subspace Covering Lemma 1.19 that there exist $j_1, \dots, j_k \in \mathbb{N}$ such that $A \subset U_{j_1} \cup \dots \cup U_{j_k}$. We set $R := \max\{j_1, \dots, j_k\}$. We obtain that $A \subset B^d_R(x_0) \subset \overline{B}^d_R(x_0)$. We have thus found the desired R. ∎



The ever-popular Heine-Borel Theorem says that for the metric space $X = \mathbb{R}^n$ mercifully the converse to Lemma 1.23 holds.

Theorem 1.24. (Heine-Borel) A subset of $\mathbb{R}^n$ is compact if and only if it is bounded and closed.

In the following we first provide several examples before we turn to the chore of proving the Heine-Borel Theorem.

Example.

(1) Let $y \in \mathbb{R}^n$ and $r \geq 0$. It follows from the Heine-Borel Theorem 1.24 and Proposition 1.4 that the closed ball $\overline{B}^n_r(y)$ and the sphere $S^{n-1}_r(y)$ are compact.

(2) It is an amusing exercise, see Exercise 1.18, to show, using the Heine-Borel Theorem 1.24, that "$\mathbb{R}^n$ with a point at infinity" is compact.

(3) It is an equally entertaining exercise, see Exercise 1.18, to show that the "line with two zeros" is not compact.

(4) Since the concepts of being bounded and closed are initially much more intuitive than being compact it is not uncommon that many, especially students, like to replace "compact" by "bounded and closed". But this equality only holds for subsets of $\mathbb{R}^n$, it does not hold for other metric spaces. For example, the set $[0,1] \cap \mathbb{Q}$ is a bounded and closed subset of the metric space $(\mathbb{Q}, d(x,y) = |x-y|)$, but as we will see in Exercise 1.26, it is not compact. Furthermore in Exercise 1.27 we will see that in general the conclusion of the Heine-Borel Theorem 1.24 does not hold for the infinite-dimensional real vector space

$$\mathbb{R}^\infty := \{(x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}} \,|\, \text{there exists an } i \in \mathbb{N} \text{ such that } x_j = 0 \text{ for } j \geq i\}$$

equipped with the obvious generalization of the Euclidean metric.

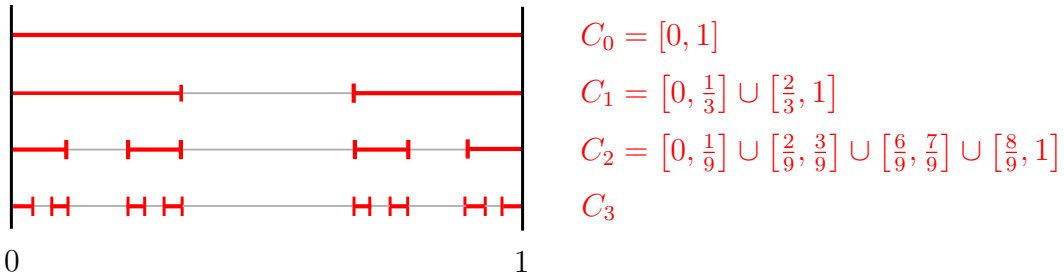
Our final example deserves to be introduced as a proper definition.

Definition. We set $C_0 := [0,1]$. Iteratively we now define sets $C_2, C_3, \ldots$ as follows:

$$C_i \;:=\; \underbrace{\{\tfrac{1}{3}\cdot x \,|\, x \in C_{i-1}\}}_{=\frac{1}{3}\cdot C_{i-1}} \;\cup\; \underbrace{\{\tfrac{2}{3}+\tfrac{1}{3}\cdot x \,|\, x \in C_{i-1}\}}_{=\frac{2}{3}+\frac{1}{3}\cdot C_{i-1}}.$$

Note that each C_i consists of 2^i closed intervals of length $\frac{1}{3^i}$ and that we obtain C_i from C_{i-1} by removing the "open middle third" of each interval. We define

$$\text{Cantor set } C \;:=\; \bigcap_{i\in\mathbb{N}_0} C_i.$$



Lemma 1.25. The Cantor set is compact.

Proof. Each C_i is the union of 2^i closed intervals, hence each C_i is closed. It follows from Lemma 1.3 that C itself is a closed subset of $\mathbb{R}$. It is evidently bounded, hence the Heine-Borel Theorem 1.24 implies that the Cantor set is compact. ■

The following proposition is the key ingredient in the proof of the Heine-Borel Theorem 1.24.

Proposition 1.26. (Hyperrectangle-Compact Proposition) Let $n \in \mathbb{N}$ and let $a_i < b_i$, $i = 1, \ldots, n$ be real numbers. The hyperrectangle

$$\underbrace{[a_1,b_1] \times \ldots \times [a_n,b_n]}_{:=\{(x_1,\ldots,x_n)\in\mathbb{R}^n \,|\, x_i\in[a_i,b_i] \text{ for } i=1,\ldots,n\}} \;\subset\; \mathbb{R}^n$$

is compact.

Proof. To simplify the notation we only treat the case $n = 1$. The general case is treated in a very similar fashion.[29] To simplify the notation further we assume that $a = 0$ and $b = 1$. Thus we will now show that $I := [0,1]$ is compact. In an attempt at a proof by contradiction we suppose that I is not compact. This means that there exists an open cover $\{U_j\}_{j\in J}$ such that I cannot be covered by finitely many of the U_j.

Claim 1. There exists a sequence of closed intervals

$$I \;=\; I_0 \;\supset\; I_1 \;\supset\; I_2 \;\supset\; \ldots$$

such that for any $m \in \mathbb{N}_0$ the following two conditions hold:

(1) I_m cannot be covered by finitely many of the U_j,
(2) the length of I_m equals 2^{-m}.

[29] The full details are for example provided in [**RudiW1976**, Theorem 2.4.1]. Alternatively one can also use the Product Topology Properties Proposition 5.6 (2).

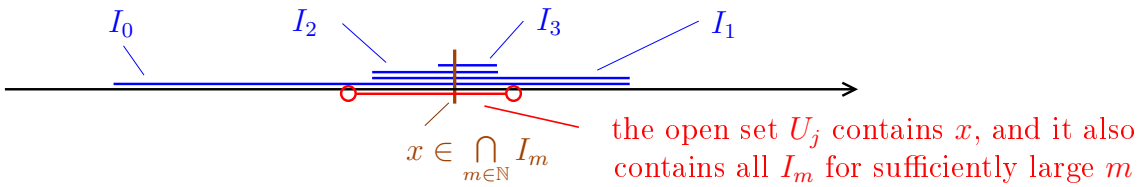
Proof. We prove the claim by induction. We set $I_0 := I$. Suppose we have already found $I_0, I_1, \ldots, I_m$. We divide the interval I_m into two closed intervals I' and I'' of half the length. Since I_m cannot be covered by finitely many U_j it follows that at least one of the intervals I' and I'' cannot be covered by finitely many U_j either. We declare such interval to be I_{m+1}. ⊞

Claim 2. There exists an $x \in I$ with $x \in \bigcap_{m\in\mathbb{N}} I_m$.

Proof. For every $m \in \mathbb{N}$ we pick an $a_m \in I_m$. For any $m \geq n$ we know that a_m and a_n both lie in I_n. In particular

$$|a_m - a_n| \;\leq\; \text{length of } I_n \;=\; \frac{1}{2^n}.$$

It follows that $(a_m)_{m\in\mathbb{N}}$ is a Cauchy sequence of real numbers. Since $\mathbb{R}$ is complete we know that the sequence converges. We denote by x the limit of this Cauchy sequence. We want to show that $x \in \bigcap_{m\in\mathbb{N}} I_m$. Put differently, we want to show that for every $m \in \mathbb{N}$ we have $x \in I_m$. Thus let $m \in \mathbb{N}$. For all $n \geq m$ we have $a_n \in I_n \subset I_m$. In particular, $(a_n)_{n\geq m}$ is a sequence that converges in I_m. Since I_m is closed we know by the Closed Subset Limit Lemma 1.18 that $\lim_{n\to\infty} a_n \in I_m$. ⊞



Since $\{U_j\}_{j\in J}$ is an open cover of I there exists a $j \in J$ with $x \in U_j$. Since U_j is open there exists an $\epsilon > 0$ with $(x-\epsilon, x+\epsilon) \cap I \subset U_j$. We pick $m \in \mathbb{N}$ with $2^{-m} < \epsilon$. Now note that

$$I_m \underset{\substack{\uparrow\\ \text{since } x\in I_m \text{ and since}\\ \text{length of } I_m \text{ equals } 2^{-m}}}{\subset} [x-2^{-m}, x+2^{-m}] \cap I \underset{\substack{\uparrow\\ \text{choice of } m}}{\subset} (x-\epsilon, x+\epsilon) \cap I \underset{\substack{\uparrow\\ \text{choice of } U_j}}{\subset} U_j$$

i.e. I_m is contained in a unique U_j. But this contradicts property (1) of the intervals $I_m, m \in \mathbb{N}$. ■

Now we can finally give the proof of the Heine-Borel Theorem 1.24.

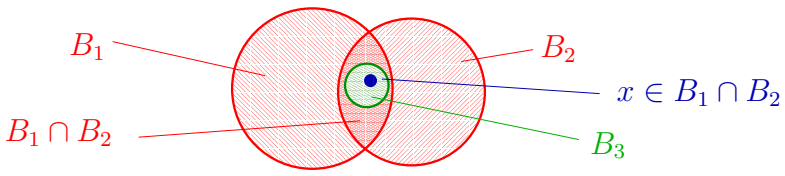
Proof of the Heine-Borel Theorem 1.24. In light of Lemma 1.23 we only need to show that if A is a bounded and closed subset of $\mathbb{R}^n$, then it is compact. Since A is bounded there exists a $C \geq 0$ such that $A \subset [-C, C]^n$. By the Hyperrectangle-Compact Proposition 1.26 we know that $A \subset [-C, C]^n$ is compact. Since A is closed in $\mathbb{R}^n$ it follows from the Neighborhood Openness Criterion 1.5 that is also a closed subset of $[-C, C]^n$. It follows from the Compact-Closed Lemma 1.21 that A itself is compact. ■

1.9. **Bases of a topology.** We start out with the following definition.

Definition. Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$ be a subset of the power set of X. We say $\mathcal{B}$ has the basis property if the following two conditions are satisfied:

(B1) For every $x \in X$ there exists a $B \in \mathcal{B}$ with $x \in B$.

(B2) Given any $B_1, B_2 \in \mathcal{B}$ and given any $x \in B_1 \cap B_2$ there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.



Example. Let (X, d) be a metric space. It follows from an elementary argument, see e.g. the proof of Proposition 1.4 (1), that the set of all open balls

$$\mathcal{B} \;=\; \{\text{all balls } B_r(x) \text{ with } r > 0 \text{ and } x \in X\} \;\subset\; \mathcal{P}(X)$$

satisfies (B1) and (B2).

The following lemma justifies our sudden interest in this new definition.

Lemma 1.27. (Topology-from-Basis Lemma) Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$. We define

$$\mathcal{T}(\mathcal{B}) \;:=\; \{V \subset X \,|\, \text{for every } x \in V \text{ there exists a } B \in \mathcal{B} \text{ with } x \in B \subset V\}.$$

If $\mathcal{B}$ has the basis property, then $\mathcal{T}(\mathcal{B})$ is a topology on X.

Definition. Let X be a set.

(1) If $\mathcal{B} \subset \mathcal{P}(X)$ has the basis property, then we refer to $\mathcal{T}(\mathcal{B})$ as the topology on X generated by $\mathcal{B}$.

(2) Conversely, if $\mathcal{T}$ is a given topology on X and if $\mathcal{B} \subset \mathcal{P}(X)$ has the basis property and if $\mathcal{T} = \mathcal{T}(\mathcal{B})$, then we say that $\mathcal{B}$ is a basis of the topology $\mathcal{T}$ of X.

Examples.

(1) If (X, d) is a metric space, then the topology generated by the set of all open balls, is by definition precisely the metric topology we already introduced on page 109.

(2) Let X be a set and let $\mathcal{T}$ be a topology on X. It follows easily from the definitions that $\mathcal{T}$ has the basis property. Furthermore it follows from the JH-Lemma 1.7 that the topology generated by $\mathcal{T}$ is in fact $\mathcal{T}$.

(3) Let $n \in \mathbb{N}$ and let $\mathbb{R}^n \cup \{\infty\}$ be the topological space defined on page 122. It follows immediately from the definition of the topology that a basis for the topological space $\mathbb{R}^n \cup \{\infty\}$ is given by

$$\mathcal{B} \;=\; \big\{B_r^n(x) \,\big|\, r > 0 \text{ and } x \in \mathbb{R}^n\big\} \;\cup\; \big\{\{\infty\} \cup (\mathbb{R}^n \setminus \overline{B}_C^n) \,\big|\, C \in \mathbb{R}\big\}.$$

For completeness we provide the elementary proof of the Topology-from-Basis Lemma 1.27, which turns out to be quite similar to the proof of the Metric-Topology Lemma 1.1.

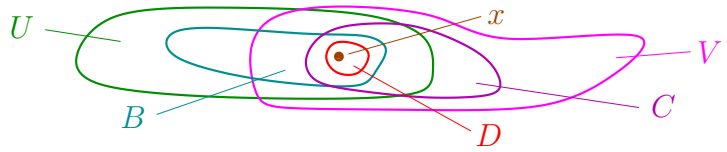
Proof. Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$. We write

$$\mathcal{T} \;:=\; \mathcal{T}(\mathcal{B}) \;:=\; \{V \subset X \,|\, \text{for every } x \in V \text{ there exists a } B \in \mathcal{B} \text{ with } x \in B \subset V\}.$$

We need to show that $\mathcal{T}$ satisfies the three properties (1), (2) and (3) in the definition of a topological space, see page 107.

(1) First we need to show that $\varnothing, X \in \mathcal{T}$. It is clear that the empty set $\varnothing$ lies in $\mathcal{T}$. Furthermore given any $x \in X$ there exists by (B1) a $B \in \mathcal{B}$ with $x \in B$. Thus it follows from the definitions that $X \in \mathcal{T}$.

(2) We need to show that the intersection of finitely many sets in $\mathcal{T}$ is again a set in $\mathcal{T}$.
First we consider the case of two sets in $\mathcal{T}$. Thus let $U, V \in \mathcal{T}$. We need to show that $U \cap V \in \mathcal{T}$. Let $x \in U \cap V$. Since U and V lie in $\mathcal{T}$ there exist $B, C \in \mathcal{B}$ with $x \in B \subset U$ and $x \in C \subset V$. By (B2) there exists a $D \in \mathcal{B}$ with $x \in D$ and $D \subset B \cap C$. In particular we have $x \in D \subset U \cap V$. This shows that $U \cap V \in \mathcal{T}$.



Now let $U_1, \ldots, U_k$ be sets in $\mathcal{T}$. Using the above statement and an elementary induction argument we see that $U_1 \cap \cdots \cap U_k$ lies in $\mathcal{T}$ as well.

(3) Let $U_j, j \in J$ be a family of sets in $\mathcal{T}$. We need to show that the union $\bigcup_{j \in J} U_j$ lies in $\mathcal{T}$. Thus let $x \in \bigcup_{j \in J} U_j$. This means that there exists a $k \in J$ with $x \in U_k$. Since $U_k \in \mathcal{T}$ there exists a $B_x \in \mathcal{B}$ with $x \in B_x \subset U_k$. But then we also have

$$x \in B_x \subset U_k \subset \bigcup_{j \in J} U_j.$$

■

The following lemma can be used to show that two topologies on a given set agree.

Lemma 1.28. (Subbasis Lemma) Let X be a set and let $\mathcal{S}$ and $\mathcal{T}$ be topologies for X. Suppose that $\mathcal{C}$ is a basis for $\mathcal{T}$. (As discussed above we could take $\mathcal{C} = \mathcal{T}$.) If for every $x \in X$ and every $V \in \mathcal{C}$ there exists a $U \in \mathcal{S}$ with $x \in U \subset V$, then $\mathcal{T} \subset \mathcal{S}$.

Example. Let $n \in \mathbb{N}$. We consider the two metrics

$$\begin{array}{rcl} \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (v, w) & \mapsto & \|v - w\| \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (v, w) & \mapsto & \max\{|v_i - w_i| \,|\, i \in \{1, \ldots, n\}\}. \end{array}$$

The former is the Euclidean metric that we introduced on page 104. We refer to the second metric as the *maximum's metric*. We leave it as a task to the dedicated reader to show, using the Subbasis Lemma 1.28, that the two topologies on $\mathbb{R}^n$ agree.

Proof. It follows immediately from the JH-Lemma 1.7 that any set in $\mathcal{C}$ is open in $\mathcal{S}$. But then, once again using the JH-Lemma 1.7 we see that any set in $\mathcal{T}$ is open in $\mathcal{S}$, i.e. $\mathcal{T} \subset \mathcal{S}$. ■

Sometimes the following two lemmas can be useful.

Lemma 1.29. Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$. If $\mathcal{B}$ has the basis property, then

$$V \subset X \text{ is open with respect to } \mathcal{T}(\mathcal{B}) \iff V \text{ is the union of sets in } \mathcal{B}.$$

Proof of Lemma 1.29. We first prove the "$\Leftarrow$"-direction. Thus let V be the union of sets in $\mathcal{B}$. Since sets in $\mathcal{B}$ are open with respect to $\mathcal{T}(\mathcal{B})$ and since $\mathcal{T}(\mathcal{B})$ is a topology we see that $V \in \mathcal{T}(\mathcal{B})$.

Now we prove the "$\Rightarrow$"-direction. Thus we suppose that $V \subset X$ is open with respect to $\mathcal{T}(\mathcal{B})$. For each $x \in V$ there exists, by definition of $\mathcal{T}(\mathcal{B})$, a set $B_x \in \mathcal{B}$ with $x \in B_x \subset V$. We see that

$$V = \bigcup_{x\in V}\{x\} \subset \bigcup_{x\in V} B_x \subset V, \quad \text{thus it follows that } V = \bigcup_{x\in V} B_x. \qquad \blacksquare$$

Lemma 1.30. Let $\mathcal{B}$ be a basis for the topology of a topological space X. Furthermore let $\{U_j\}_{j\in J}$ be an open cover of X, i.e. $\{U_j\}_{j\in J}$ is a family of open sets with $\bigcup_{j\in J} U_j = X$. Then

$$\{B \in \mathcal{B} \,|\, \text{ there exists some } j \in J \text{ with } B \subset U_j\}$$

is also a basis for the topology of X.

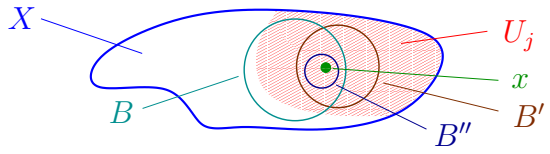
Proof. We set

$$\mathcal{C} := \{B \in \mathcal{B} \,|\, \text{ there exists some } j \in J \text{ with } B \subset U_j\}.$$

Since $\mathcal{C} \subset \mathcal{B}$ we have $\mathcal{T}(\mathcal{C}) \subset \mathcal{T}(\mathcal{B})$. It remains to show the reverse inclusion $\mathcal{T}(\mathcal{B}) \subset \mathcal{T}(\mathcal{C})$. It follows easily from the JH-Lemma 1.7 that it suffices to show that $\mathcal{B} \subset \mathcal{T}(\mathcal{C})$. In other words, it suffices to prove the following claim.

Claim. Given any $x \in X$ and any $B \in \mathcal{B}$ with $x \in B$ there exists a $C \in \mathcal{C}$ such that $x \in C \subset B$.

Proof. So let $x \in X$ and let $B \in \mathcal{B}$ with $x \in B$. By hypothesis there exists a j such that $x \in U_j$. Since U_j is open and since $\mathcal{B}$ is a basis for the topology of X there exists a $B' \in \mathcal{B}$ with $x \in B' \subset U_j$. By (B2) there exists a $B'' \subset B$ with $x \in B''$ and $B'' \subset B \cap B' \subset U_j$. So we are done since by definition we have $B'' \in \mathcal{C}$. $\blacksquare$



Example. We consider the topological space $\mathbb{R}^n$. As we pointed out on page 132, the set of open balls $\mathcal{B} = \{B_r^n(x) \,|\, r > 0 \text{ and } x \in \mathbb{R}^n\}$ is a basis for the topology of $\mathbb{R}^n$. We denote by $\mathcal{U}$ the family of all open sets that are contained in an open ball of radius 1. Then the open sets in $\mathcal{U}$ cover $\mathbb{R}^n$. It follows from Lemma 1.30 that

$$\{B_r^n(x) \,|\, r \in (0,1) \text{ and } x \in \mathbb{R}^n\}$$

is also a basis for $\mathbb{R}^n$.

The following lemma gives us in particular another source of topologies.

Lemma 1.31. Let X be a set. For any $\mathcal{C} \subset \mathcal{P}(X)$ the set

$$\mathcal{B}(\mathcal{C}) := \begin{array}{c}\text{all subsets of } X \text{ that can be written as} \\ \text{intersections of a finite family of sets in } \mathcal{C}\end{array}$$

has the basis property.

Proof.

(1) On page 87 we pointed out that if we take the empty family of subsets of X, then the intersection is all of X. In other words, we see that $X \in \mathcal{B}(\mathcal{C})$. It follows that $\mathcal{B}(\mathcal{C})$ satisfies (B1).

(2) It is easy to verify that $\mathcal{B}(\mathcal{C})$ satisfies (B2). $\blacksquare$

The previous lemma leads us to the following definition.

Definition. Let X be a set and let $\mathcal{C} \subset \mathcal{P}(X)$.

(1) We define $\mathcal{B} = \mathcal{B}(\mathcal{C})$ as in Lemma 1.31 and we refer to $\mathcal{T}(\mathcal{B})$ as the topology generated by $\mathcal{C}$.
(2) Conversely, if $\mathcal{T}$ is a topology on X and if the topology generated by $\mathcal{C}$ agrees with $\mathcal{T}$, then we say that $\mathcal{C}$ is a subbasis for $\mathcal{T}$.

Example.

(1) Let $\mathcal{C}$ be the set of all intervals in $\mathbb{R}$ of length one. One can easily verify that $\mathcal{C}$ generates the usual topology on $\mathbb{R}$.
(2) It follows easily from the definitions that a basis of a given topology is also a subbasis of the topology.

The following lemma gives us a convenient characterization of (sub-) bases for a given topology.

Lemma 1.32. (Basis Characterization Lemma) Let X be a topological space and let $\mathcal{C}$ be a family of subsets of X.

(1) The following two statements are equivalent:
   (a) Every $C \in \mathcal{C}$ is open and given any open set U and given any $x \in U$ there exists a $C \in \mathcal{C}$ with $x \in C \subset U$.
   (b) $\mathcal{C}$ is a basis for the topology of X.
(2) Suppose $\mathcal{B}$ is a basis for the topology of X. The following two statements are equivalent:
   (a) Every $C \in \mathcal{C}$ is open and given any $x \in X$ and given any $B \in \mathcal{B}$ there exist finitely many $C_1, \ldots, C_m \in \mathcal{C}$ with $x \in C_1 \cap \cdots \cap C_m \subset B$.
   (b) $\mathcal{C}$ is a subbasis for the topology of X.

Example. In Exercise 1.8 we will show, using the Basis Characterization Lemma 1.32, that the set of open *rational* intervals forms a basis for the usual topology on $\mathbb{R}$. In particular $\mathbb{R}$ has a countable basis for its topology.

Proof.

(1) We denote by $\mathcal{T}$ the topology of X. The implication (b) $\Rightarrow$ (a) follows immediately from the definitions. So let us turn to the proof of (a) $\Rightarrow$ (b). First we show that $\mathcal{C}$ has the basis property:
   (B1) If we apply the property of $\mathcal{C}$ to $U = X$ then we obtain immediately that given any $x \in X$ there exists a $C \in \mathcal{C}$ with $x \in C$.
   (B2) Let $x \in C_1 \cap C_2$ with $C_1, C_2 \in \mathcal{C}$. Since C_1 and C_2 are open the intersection $U = C_1 \cap C_2$ is also open. It follows from the hypothesis on $\mathcal{C}$, applied to U, that there exists a $C \in \mathcal{C}$ with $x \in C \subset C_1 \cap C_2$.

   Let $\mathcal{S}$ be the topology on X that is generated by $\mathcal{C}$. We need to show that $\mathcal{S} = \mathcal{T}$.

   First we show the inclusion $\mathcal{S} \subset \mathcal{T}$. Since all sets in $\mathcal{C}$ are open with respect to $\mathcal{T}$ it follows from Lemma 1.29 that $\mathcal{S} \subset \mathcal{T}$.

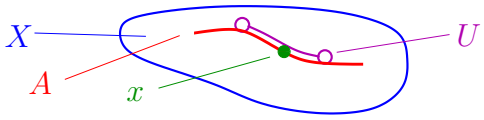
Finally we show the reverse inclusion $\mathcal{T} \subset \mathcal{S}$. Thus let $U \in \mathcal{T}$. We need to show that $U \in \mathcal{S}$. Thus let $x \in U$. By our hypothesis there exists a $C \in \mathcal{C}$ with $x \in C \subset U$. Therefore by definition we have $U \in \mathcal{S}$. We have thus shown that $\mathcal{T} \subset \mathcal{S}$.

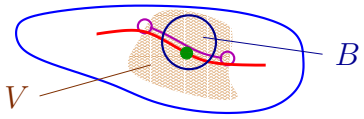
(2) We leave the verification of this statement to the reader. ■

We conclude this section with the following not terribly inspiring lemma.

Lemma 1.33. Let X be a topological space and let $A \subset X$ be a subset. If $\mathcal{B}$ is a basis for the topology of X, then $\{B \cap A \,|\, B \in \mathcal{B}\}$ is a basis for the topology of A, equipped with the subspace topology.

Proof. We prove the lemma using the Basis Characterization Lemma 1.32 (1). Thus let $U \subset A$ be an open subset in A and let $x \in U$. By definition of the subspace topology there exists an open subset $V \subset X$ of X with $U = V \cap A$. Since $\mathcal{B}$ is a basis for the topology of X there exists a $B \in \mathcal{B}$ with $x \in B \subset V$. This implies that $x \in B \cap A \subset V \cap A = U$. It follows from the Basis Characterization Lemma 1.32 (1) that $\{B \cap A \,|\, B \in \mathcal{B}\}$ is indeed a basis for the topology of A. ■



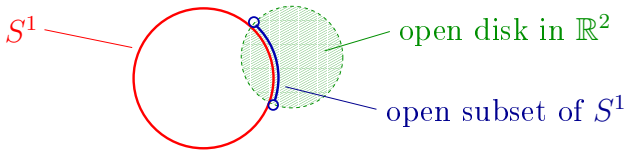


Example. We consider $$S^1 \;=\; \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\} \;\subset\; \mathbb{R}^2.$$

As we saw on page 132, a basis for the topology on $\mathbb{R}^2$ is given by

$$\mathcal{B} \;=\; \{B_r(x,y) \,|\, (x,y) \in \mathbb{R}^2 \text{ and } r > 0\}.$$

The intersection of such an open ball with S^1 is an "open interval on S^1", i.e. it is a subset of the form $\{\exp(\mathrm{i}\varphi) \,|\, \varphi \in (a,b)\}$. It follows from Lemma 1.33 that these "open intervals" form a basis for the topology on S^1.



Exercises for Chapter 1.

Exercise 1.1. Let X be a finite set. Show that a topology $\mathcal{T}$ on X is metrizable if and only if it is the discrete topology.

Exercise 1.2. Recall that a topological space X is called *metrizable* if there exists a metric d on X such that the given topology agrees with the topology coming from d. We say that a metric space (X,d) is *complete* if every Cauchy sequence in (X,d) converges. Furthermore we say that a topological space X is *completely metrizable* if there exists a complete metric d on X such that the given topology agrees with the topology coming from d.

(a) Is $(0,1)$ completely metrizable?
(b) Is $\mathbb{Q}$ completely metrizable?

Exercise 1.3. Let (X,d) be a metric space and let $A \subset X$ be a subset. The restriction of the metric $d\colon X \times X \to \mathbb{R}_{\geq 0}$ to the map $d\colon A \times X \to \mathbb{R}_{\geq 0}$ defines a metric on A. We equip the metric spaces (X,d) and (A,d) with the corresponding topologies $\mathcal{X}$ and $\mathcal{A}$ as defined in Metric-Topology Lemma 1.1. Show that the topology $\mathcal{A}$ on A is the subspace topology of $A \subset X$ as defined on page 112.

Exercise 1.4. Let X be a set and let $\mathcal{S}, \mathcal{T}$ be two topologies on X. We assume that the following statements hold:

(1) For each $U \in \mathcal{S}$ there exists a $V \in \mathcal{T}$ with $V \subset U$.
(2) For each $V \in \mathcal{T}$ there exists a $U \in \mathcal{S}$ with $U \subset V$.

Does it follow that $\mathcal{S} = \mathcal{T}$?

Exercise 1.5. Let $X \subset \mathbb{R}^n$.

(a) Show that if the subspace topology on X is the discrete topology, then X is countable.
(b) Show that the converse to (a) does not hold.

Exercise 1.6. Let X be a topological space and let A be a subset of X. Show that the following three equalities hold:

$$\begin{array}{lrcl}
(1) & \mathring{A} & = & \{x \in A \mid \text{there exists a neighborhood } U \text{ of } x \text{ that is contained in } A\} \\
(2) & \overline{A} & = & \{x \in X \mid \text{every neighborhood of } x \text{ contains at least one point in } A\} \\
(3) & \partial A & = & \{x \in X \mid \text{every neighborhood of } x \text{ contains at least one point in } A \\
 & & & \qquad \text{and one point that does not lie in } A\}
\end{array}$$

Exercise 1.7. Let $n \in \mathbb{N}$. We fix a point $P \in \mathbb{R}^n$ and we consider the maps

$$\begin{array}{rcl} d_M\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ ((x_1,\ldots,x_n),(y_1,\ldots,y_n)) & \mapsto & \sum_{i=1}^{n} |x_i - y_i| \end{array} \quad \text{and} \quad \begin{array}{rcl} d_S\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x,y) & \mapsto & \|x-P\| + \|y-P\| \end{array}$$

which on page 104 we called the Manhattan metric respectively the SNCF metric.

(a) Show that the Manhattan metric is equivalent to the Euclidean metric, i.e. show that they define the same topology on $\mathbb{R}^n$.
(b) Show that the SNCF metric is not equivalent to the Euclidean metric.

Exercise 1.8.

(a) Show that the open *rational* intervals, i.e. the open intervals (a,b) with $a,b \in \mathbb{Q}$, form a basis for the usual topology on $\mathbb{R}$.
(b) Let $n \in \mathbb{N}_0$. Show that the set of open balls $B_r(x)$ with $r \in \mathbb{Q}_{>0}$ and $x \in \mathbb{Q}^n$ form a basis for the usual topology on $\mathbb{R}^n$.

Exercise 1.9. Let U be an open subset of $\mathbb{R}^2$. Is it true that the interior of $\overline{U}$ equals U?

Exercise 1.10. Let $\{U_i\}_{i \in I}$ be a family of subsets. If I is finite, then we know by Lemma 1.11 that the following two equalities hold:

$$\text{(a)} \quad \text{interior of } \bigcap_{i \in I} U_i = \bigcap_{i \in I} \mathring{U}_i \qquad \text{and} \qquad \text{(b)} \quad \text{closure of } \bigcup_{i \in I} U_i = \bigcup_{i \in I} \overline{U_i}.$$

Do these equalities also hold if I is infinite?

Exercise 1.11. Are the topological spaces $\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$ (equipped with the subspace topology coming from $\mathbb{R}$) homeomorphic?

Exercise 1.12. Is every open subset of $\mathbb{R}$ the disjoint union of countably many open intervals?

Exercise 1.13. Let $\{U_s\}_{s \in S}$ be an open cover of the interval $[0,1]$. Show that there exist $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k$ such that each interval $[t_i, t_{i+1}]$ is contained in one of the U_s.
Remark. Note that we do not assume that the U_s are connected.

Exercise 1.14. Let (X,d) be a compact metric space. A rather surprising theorem, see **[Gros1964]** or alternatively **[Ris2006]**, says that there exists a unique $r \in \mathbb{R}_{\geq 0}$ such that for any finite sequence of points $x_1, \ldots, x_n \in X$ there exists an $x' \in X$ with $\frac{1}{n}\sum_{i=1}^{n} d(x_i, x') = r$. This number is called the *rendezvous value of* (X,d).

(a) Determine the rendezvous value of the interval $X = [0,1]$ equipped with the metric $d(x,y) = |x-y|$.
(b) Determine the rendezvous value of the circle $X = S^1$ equipped with the Euclidean metric of $\mathbb{R}^2$.

Exercise 1.15. Let X be a Hausdorff space, let $x_1, \ldots, x_k$ be distinct points in X and let $U_1, \ldots, U_k$ be neighborhoods of $x_1, \ldots, x_k$. Show that there exist disjoint open neighborhoods $V_1, \ldots, V_k$ of $x_1, \ldots, x_k$.

Exercise 1.16. Let X be a topological space that is Hausdorff.

(a) Let $K_1 \supset K_2 \supset K_3 \supset \ldots$ be a sequence of non-empty compact subsets. Show that $\bigcap_{n \in \mathbb{N}} K_n$ is also non-empty.
(b) Does the conclusion of (a) hold, if we only assume that the K_n are closed?

Exercise 1.17. Let X be a topological space that is Hausdorff and let $x, y \in X$ be two distinct points. Can you always find open neighborhoods U of x and V of y such that the closures are disjoint, i.e. such that $\overline{U} \cap \overline{V} = \varnothing$?

Exercise 1.18.

(a) Show that "$\mathbb{R}^n$ with a point at infinity" is Hausdorff and compact.
(b) Show that "the line with two zeros" is neither Hausdorff nor compact.

Exercise 1.19. Let X be a topological space.

(1) A point $x_0 \in X$ is called *isolated* if $\{x_0\}$ itself is an open subset.
(2) Let $U \subset X$ be a subset. We say $x \in X$ is an *accumulation point of* U, if every neighborhood of x contains a point in $U \setminus \{x\}$.

With these definitions we can formulate our exercise:

(a) Let $x \in U$. Show that x is an accumulation point of U if and only if x is not isolated in U, in the sense of the definition on page 235.
(b) We denote by $A(U)$ the accumulation points of U and we denote by $I(U)$ the isolated points of U. Show that the closure $\overline{U}$ equals $A(U) \cup I(U)$.
(c) Let $f, g \colon X \to \mathbb{R}$ be maps which agree on U. Show that for every accumulation point x of U we also have $f(x) = g(x)$.

Remark. In the literature accumulation points are sometimes also called limit points or cluster points.

Exercise 1.20. Given $a, b \in \mathbb{N}$ we define

$$U_a(b) \ := \ \{b + n \cdot a \in \mathbb{N} \,|\, n \in \mathbb{Z}\} \ \subset \ \mathbb{N}.$$

Furthermore we set

$$\mathcal{C} \ := \ \{U_p(b) \,|\, p, b \in \mathbb{N} \text{ and } p \text{ prime}\} \ \subset \ \mathcal{P}(\mathbb{N}).$$

The topology $\mathcal{T}$ on $\mathbb{N}$ generated by $\mathcal{C}$ is called the *prime integer topology*.

(a) Show that $(\mathbb{N}, \mathcal{T})$ is Hausdorff.

(b) A topological space X is called *regular* if given any $P \in X$ and given any closed subset $A \subset X$ with $x \notin A$ there exist disjoint open subsets U and V with $P \in U$ and $A \subset V$. Show that $(\mathbb{N}, \mathcal{T})$ is not regular.

Exercise 1.21. Given $a \in \mathbb{Z} \setminus \{0\}$ and $b \in \mathbb{Z}$ we define

$$S(a, b) \ := \ \{a \cdot n + b \,|\, n \in \mathbb{Z}\} \ = \ a \cdot \mathbb{Z} + b.$$

We say that $U \subset \mathbb{Z}$ is open if for every $x \in U$ there exists a subset $S(a, b)$ as above with $x \in S(a, b) \subset U$. We denote by $\mathcal{T}$ the set of all open subsets of $\mathbb{Z}$.

(a) Show that $\mathcal{T}$ is a topology on $\mathbb{Z}$. (This topology is called the *evenly spaced integer topology* on $\mathbb{Z}$).

(b) Show that each set of the form $S(a, b)$ is also closed.

(c) Let P be the set of prime numbers. Show that $\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \in P} S(p, 0)$.

(d) Show that there exist infinitely many primes.

Remark. This proof that there exist infinitely many primes was found by Harry Furstenberg [**Furs1955**] in 1955.

Exercise 1.22. Let X be a topological space, let $A \subset X$ be a closed subset and let $K \subset X$ be a compact subset. Show that $A \cap K$ is also compact.

Exercise 1.23. Let X be a topological space and let $A \subset X$ be a subset.

(a) We suppose that X is Hausdorff. Show that $\partial A = \partial \overline{A}$.

(b) Does the conclusion in (a) also hold if X is not Hausdorff?

Exercise 1.24. Let X be a topological space.

(a) Show that the intersection of finitely many open and dense subsets is again open and dense.

(b) Show that the intersection of countably many open and dense subsets is in general not dense.

Exercise 1.25. Let X be a topological space. We say a subset $U \subset X$ is *large* if U is open and dense.

(a) Show that the intersection of finitely many large subsets of X is again large.

(b) Is it true that the intersection of countably many large subsets of X is necessarily large?

Exercise 1.26. Show that the topological space $[0, 1] \cap \mathbb{Q}$, equipped with the subspace topology coming from $\mathbb{R}$, is not compact.

Exercise 1.27. We consider

$$\mathbb{R}^\infty := \{(x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}} \mid \text{there exists an } i \in \mathbb{N} \text{ such that } x_j = 0 \text{ for } j \geq i\}.$$

We equip $\mathbb{R}^\infty$ with the Euclidean metric

$$\begin{array}{rcl} d\colon \mathbb{R}^\infty \times \mathbb{R}^\infty & \to & \mathbb{R}^\infty \\ ((x_1, x_2, \dots), (y_1, y_2, \dots)) & \mapsto & \sqrt{\sum_{i \in \mathbb{N}} |x_i - y_i|^2}. \end{array}$$

Show that $(\mathbb{R}^\infty, d)$ is not compact.

Exercise 1.28. Let X be a topological space. Recall that a subset $A \subset X$ is called *precompact in* X if the closure $\overline{A}$ is compact. Let X be a subset of some topological space Y and let $A \subset X$ be a precompact subspace.

(a) Show that A is not necessarily precompact in Y.
(b) Show that if Y is Hausdorff, then A is in fact precompact in Y.

Remark. Dealing with closures can be quite subtle. For example, it seems to me that for (b) you want to use the Neighborhood Openness Criterion 1.5 (2) at some point.

Exercise 1.29. Let X be a topological space. We say that a family $\{A_i\}_{i \in I}$ of subsets has the *finite intersection property* if given any finite collection of indices $i_1, \dots, i_r \in I$ the corresponding intersection $A_{i_1} \cap \dots \cap A_{i_r}$ is non-empty.

(a) We assume that X is a compact space and that all the A_i are closed subsets. Show that $\bigcap_{i \in I} A_i \neq \varnothing$.
(b) Show that in (a) we cannot drop the hypothesis that X is compact.

Exercise 1.30. Show that every open subset of $\mathbb{R}^n$ is the union of countably many open balls.

Exercise 1.31. Let X be a Hausdorff space, let A be a subset of X and let B be a compact subset of X. Show that if $A \cap B = \varnothing$, then there exists an open subset U of X with $A \subset U$ and $U \cap B = \varnothing$.

Hint. Look at the (proof of) the Compact Hausdorff-Normal Lemma 1.16.



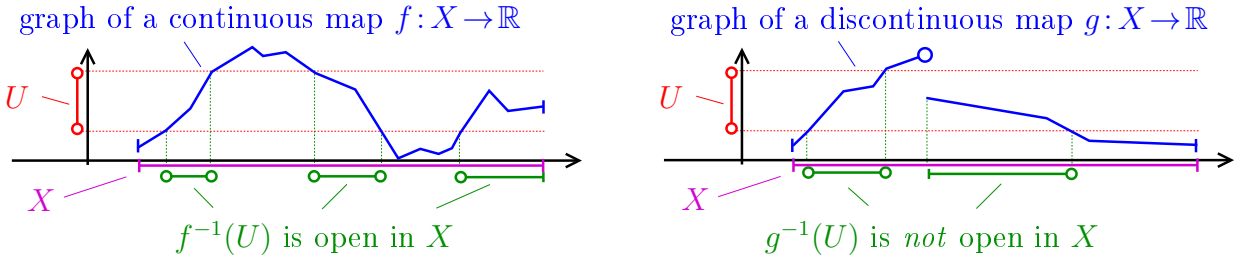
2. Continuity

2.1. **Continuous maps.** The following is the key definition in the context of topological spaces.

Definition. We say a map $f\colon X \to Y$ between two topological spaces X and Y is continuous, if for each open set U in Y the preimage $f^{-1}(U)$ is open in X. Otherwise the map is called discontinuous.

Examples.

(1) For subsets $X \subset \mathbb{R}$ and maps $\mathbb{R} \to \mathbb{R}$ we see in the figure below that the above definition of continuous matches our intuition.



(2) Let X and Y be two topological spaces. If Y is equipped with the trivial topology or if X is equipped with the discrete topology, then every map $f\colon X \to Y$ is continuous.

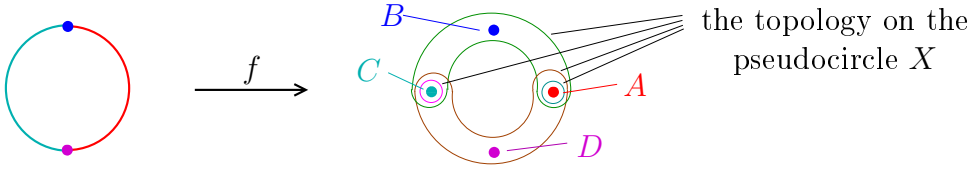
(3) We consider the set $X = \{A, B, C, D\}$ where the topology is given by the set

$$\mathcal{T} := \{\varnothing, \{A\}, \{C\}, \{A, C\}, \{A, B, C\}, \{A, C, D\}, X\}.$$

We refer to this to topological spaces as the *pseudocircle*. It follows easily from the definitions that the map

$$\begin{array}{rcl} f\colon S^1 & \to & X \\ \exp(it) & \mapsto & \begin{cases} A, & \text{if } t \in (-\frac{\pi}{4}, \frac{\pi}{4}), \\ B, & \text{if } t = \frac{\pi}{4}, \\ C & \text{if } t \in (\frac{\pi}{4}, \frac{3\pi}{4}), \\ D & \text{if } t = \frac{3\pi}{4}. \end{cases} \end{array}$$

is continuous.



(4) Let X and Y be topological spaces and let $y \in Y$. The constant map

$$\begin{array}{rcl} c_y\colon X & \to & Y \\ x & \mapsto & y \end{array}$$

is continuous since the preimage of any set in Y is either $\varnothing$ or all of X. This simple example will come in handy on many occasions.

Lemma 2.1.

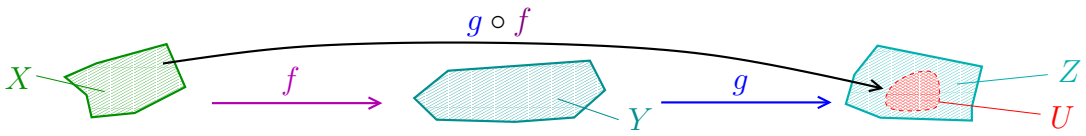
(1) If $f\colon X \to Y$ and $g\colon Y \to Z$ are two continuous maps between topological spaces, then the composition $g \circ f\colon X \to Z$ is also continuous.
(2) Let X be a topological space and let A be a subset. If we equip A with the subspace topology, then the inclusion map $i\colon A \to X$ is continuous.
(3) Let $f\colon X \to Y$ be a continuous map between topological spaces and let A be a subset. If we equip A with the subspace topology, then the restriction of f to the map $f|_A\colon A \to Y$ is also continuous.
(4) Let $f\colon X \to Y$ be a continuous map. If $Z \subset Y$ is a subset with $f(X) \subset Z$, then the map $f\colon X \to Z$ is also continuous.

Proof.

(1) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two continuous maps between topological spaces. We need to show that the composition $g \circ f\colon X \to Z$ is also continuous. Let $U \subset Z$ be an open subset. Then

$$(g \circ f)^{-1}(U) \;=\; f^{-1}(g^{-1}(U)) \;\underset{\uparrow}{=}\; f^{-1}(\text{the open set } g^{-1}(U)) \;\underset{\uparrow}{=}\; \text{open}.$$

since g is continuous and U is open; since f is continuous



(2) This statement follows immediately from the definitions.
(3) This statement is an immediate consequence of (1) and (2) since the restriction of f to A is precisely the same as $f \circ i$ where $i\colon A \to X$ denotes the inclusion.
(4) For clarity we denote by $g\colon X \to Z$ the map given by $x \mapsto f(x)$. Let $U \subset Z$ be an open subset. We need to show that $g^{-1}(U)$ is an open subset of X. By definition of the subspace topology there exists an open subset $V \subset X$ with $U = Z \cap V$. Thus we see that

$$g^{-1}(U) \;=\; g^{-1}(Z \cap V) \;\underset{\uparrow}{=}\; f^{-1}(Z \cap V) \;\underset{\uparrow}{=}\; f^{-1}(V) \;\underset{\uparrow}{=}\; \text{open subset of } X.$$

definition of g; since $f(X) \subset Z$; since f is continuous

■

We continue with several examples of continuous maps.

Lemma 2.2. Let $n \in \mathbb{N}$.

(1) For each $i \in \{1, \dots, n\}$ the projection

$$\begin{array}{rcl} \mathbb{R}^n & \to & \mathbb{R} \\ (x_1, \dots, x_n) & \mapsto & x_i \end{array}$$

is continuous.
(2) Let $f = (f_1, \dots, f_n)\colon X \to \mathbb{R}^n$ be a map from a topological space X to $\mathbb{R}^n$. Then f is continuous if and only if each $f_i\colon X \to \mathbb{R}$ is continuous.

Proof. We leave the slightly messy proof of the lemma to the reader. ■

Lemma 2.3. Let X be a topological space. If $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ are continuous maps, then the maps

$$\begin{array}{lll} X \to \mathbb{R} & X \to \mathbb{R} & X \to \mathbb{R} \\ x \mapsto f(x)+g(x) & x \mapsto f(x)\cdot g(x) & x \mapsto \max\{f(x),g(x)\} \end{array} \qquad \begin{array}{l} X \to \mathbb{R} \\ x \mapsto \min\{f(x),g(x)\} \end{array}$$

are also continuous.

Example. Let A be a real $(m\times n)$–matrix. It follows easily from Lemma 2.2 (1), Lemma 2.1 (1) and Lemma 2.3 that the following map is continuous:

$$\begin{array}{rcl} f\colon \mathbb{R}^n & \to & \mathbb{R}^m \\ v & \mapsto & A \cdot v. \end{array}$$

Proof. In Exercise 2.31 we will show that the maps

$$\begin{array}{rcl} \alpha\colon \mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & x+y \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta\colon \mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & x\cdot y \end{array}$$

are continuous. The map $x \mapsto f(x)+g(x)$ equals the map

$$X \xrightarrow{x\mapsto(f(x),g(x))} \mathbb{R}^2 \xrightarrow{\alpha} \mathbb{R}.$$

By Lemma 2.2 (2) the first map is continuous. By the above we know that α is continuous. From Lemma 2.1 (1) we deduce that the map $f+g$ is continuous. The same argument also takes care of the map $x \mapsto f(x)\cdot g(x)$. Finally note that for any $a, b \in \mathbb{R}$ we have

$$\max\{a,b\} \;=\; \tfrac{1}{2}(a+b+|a-b|) \quad \text{and} \quad \min\{a,b\} \;=\; \tfrac{1}{2}(a+b-|a-b|).$$

Using this observation one can now easily show, by a slight modification of the previous arguments, that the maps $x \mapsto \max\{f(x),g(x)\}$ and $x \mapsto \min\{f(x),g(x)\}$ are continuous. ■

At times it is also useful to have a characterization of continuous maps in terms of closed sets.

Lemma 2.4. (Continuity-via-Closed Subsets Lemma) A map $f\colon X \to Y$ between two topological spaces X and Y is continuous if and only if for each closed set A in Y the preimage $f^{-1}(A)$ is closed in X.

Proof. This lemma is a straightforward consequence of the Image-Preimage Lemma 0.2 (8) and the definitions. We leave it to the reader to fill in the details. ■

For completeness' sake we also state the following lemma.

Lemma 2.5. (Closure-Interior Mapping Lemma) Let $f\colon X \to Y$ be a map between topological spaces.

(1) The following statements are equivalent:
   (a) The map is continuous.
   (b) For every subset $B \subset Y$ we have $f^{-1}(\overline{B}) \subset \overline{f^{-1}(B)}$.
   (c) For every subset $B \subset Y$ we have $f^{-1}(\mathring{B}) \subset$ interior of $f^{-1}(B)$.
   The analogous statement hold with the closures replaced by the interiors.

(2) If f is continuous, then the following statements hold:
   (a) For every subset $A \subset X$ we have $\overline{f(\overline{A})} = \overline{f(A)}$.

(b) For every subset $A \subset X$ we have $\operatorname{int}(f(\operatorname{int} A)) = \operatorname{int}(f(A))$.

Proof.

(1) We prove (a) $\Rightarrow$ (b). Thus we assume that $f\colon X \to Y$ is continuous. Let $B \subset Y$. By the Continuity-via-Closed Subsets Lemma 2.4 we know that $f^{-1}(\overline{B})$ is a closed subset of X that contains $f^{-1}(B)$. By the definition of the closure this means that $f^{-1}(\overline{B}) \subset \overline{f^{-1}(B)}$. We will prove the remaining statements in Exercise 2.11.

(2) (a) Evidently we have $f(A) \subset f(\overline{A})$. Thus we have $\overline{f(A)} \subset \overline{f(\overline{A})}$. We now show the reverse inclusion $\overline{f(\overline{A})} \subset \overline{f(A)}$. Since $\overline{f(A)}$ is a closed subset it suffices to show that $f(\overline{A}) \subset \overline{f(A)}$. Let Z be a closed subset of Y that contains $f(A)$. We need to show that $f(\overline{A}) \subset Z$. First note that we have $f(A) \subset Z$. Thus we see that $A \subset f^{-1}(f(A)) \subset f^{-1}(Z)$. Since Z is closed and f is continuous we obtain from the Continuity-via-Closed Subsets Lemma 2.4 that $f^{-1}(Z)$ is closed. Thus we have $\overline{A} \subset f^{-1}(Z)$. This finally implies that $f(\overline{A}) \subset f(f^{-1}(Z)) \subset Z$.

(b) Evidently we have $f(\operatorname{int}(A)) \subset f(A)$. Thus we have $\operatorname{int}(f(\operatorname{int}(A))) \subset \operatorname{int}(f(A))$. We now show the reverse inclusion $\operatorname{int}(f(A)) \subset \operatorname{int}(f(\operatorname{int}(A)))$. Since $\operatorname{int}(f(\operatorname{int} A))$ is an open subset it suffices to show that $\operatorname{int}(f(A)) \subset f(\operatorname{int}(A))$. Let U be an open subset subset of Y that is contained in $f(A)$. We need to show that $U \subset f(\operatorname{int}(A))$. We have $f^{-1}(U) \subset f^{-1}(f(A)) \subset A$. Since f is continuous we know that $f^{-1}(U)$ is an open subset of X. Thus we see that $f^{-1}(U) \subset \operatorname{int}(A)$. Since U is contained in the image of X we see that $U = f(f^{-1}(U))$. In summary we see that $U = f(f^{-1}(U)) \subset f(\operatorname{int}(A))$. ■

By Lemma 2.1 (3) we know that the restriction of a continuous map to any subset is still continuous. The following lemma can be viewed as a converse: if "enough" restrictions of a given map are continuous, then the original map is also continuous.
Initially the main application of Lemma 2.7 is the following proposition which we will use abundantly.

Proposition 2.6. (Pasting Proposition) Let $f\colon X \to Y$ be a map between topological spaces.

(1) If there exists an open subsets $\{W_i\}_{i\in I}$ of X such that each restriction $f|_{W_i}$ is continuous, then f itself is continuous.

(2) If there exists a *locally finite* closed cover $\{A_i\}_{i\in I}$ of X such that each restriction $f|_{A_i}$ is continuous, then f itself is continuous.

In practice often the following variation on (2) gets used:

(2′) If X is Hausdorff and if there exists a locally finite cover $\{A_i\}_{i\in I}$ of X such that each A_i is compact and such that each restriction $f|_{A_i}$ is continuous, then f itself is continuous.

Remark.

(1) In practice the Pasting Proposition 2.6 (2) often gets used in the following way: Suppose we are given two topological spaces X and Y and closed subsets $A_1, \ldots, A_m$ of X such that $X = A_1 \cup \cdots \cup A_m$. Furthermore we are given continuous maps

$f_i\colon A_i \to Y$, $i = 1, \dots, m$. If the f_i agree on the overlaps, then the map

$$\begin{array}{ccl} X & \to & Y \\ P & \mapsto & f_i(P) \quad \text{if } P \in A_i \end{array}$$

is well-defined and it follows from the Pasting Proposition 2.6 (2) that it is continuous.

(2) Note that in Pasting Proposition 2.6 (1) we are allowed to deal with arbitrarily many open subsets whereas in Pasting Proposition 2.6 (2) we are restricted to finitely many closed subsets. It is clear that Pasting Proposition 2.6 (2) cannot be generalized to arbitrarily many closed subsets, after all, the restriction of a map $f\colon \mathbb{R} \to \mathbb{R}$ to any one-point subset $\{x\} \subset \mathbb{R}$ is continuous, but clearly not every map $f\colon \mathbb{R} \to \mathbb{R}$ is continuous.

The proof of the Pasting Proposition 2.6 rests on the shoulders of the following lemma, which will at times also be of independent interest.

Auxiliary Lemma 2.7. Let X be a topological space and let $\{X_i\}_{i\in I}$ be a family of subsets. The following two statements are equivalent:

(1) A subset $U \subset X$ is open if each $U \cap X_i$ is open in X_i.
(2) A subset $U \subset X$ is closed if each $U \cap X_i$ is closed in X_i.

Furthermore, if (1) or equivalently (2) holds, then also the following statement holds:

(3) A map $f\colon X \to Y$ to some topological space Y is continuous if each restriction $f|_{X_i}\colon X_i \to Y$ is continuous.

Proof of Lemma 2.7.

(1)$\Rightarrow$(2) We assume that (1) holds. Now let $U \subset X$ be a subset. We have

$$\begin{array}{l} \text{for every } i \in I \text{ the set} \\ X_i \cap U \text{ is closed in } X_i. \end{array} \quad \begin{array}{l} \overset{\text{since } (X\setminus U)\cap X_i = X_i \setminus (X_i \cap U)}{\downarrow} \\ \Longrightarrow \quad \text{for every } i \in I \text{ the set} \\ \quad\ (X\setminus U)\cap X_i \text{ is open in } X_i \\ \Longrightarrow \ X \setminus U \text{ is open in } X \implies U \text{ is closed in } X. \\ \overset{\text{since (1) holds}}{\uparrow} \end{array}$$

(2)$\Rightarrow$(1) This proof is basically the same as the above argument.

(1)$\Rightarrow$(3) Let $f\colon X \to Y$ be a map to some topological space Y. We suppose that each $f|_{X_i}$ is continuous. Let $U \subset Y$ be an open subset. We need to show that $f^{-1}(U)$ is an open subset of X. By (1) it suffices to show that each intersection $f^{-1}(U) \cap X_i$ is an open subset of X_i. But this follows from the observation that $f^{-1}(U) \cap X_i = (f|_{X_i})^{-1}(U)$ and the hypothesis that $f|_{X_i}$ is continuous. ■

Proof of the Pasting Proposition 2.6. Let $f\colon X \to Y$ be a map between topological spaces.

(1) This statement follows immediately from the combination of Lemma 1.8 (1a) and Lemma 2.7.
(2) Similar to (1) this statement follows immediately from the combination of Lemma 1.8 (2a) and Lemma 2.7.
(2′) This statement follows from (2) and the Compact-Closed Lemma 1.21 (2). ■

The notion of continuity is of course already familiar from earlier analysis courses. The following proposition shows that for maps between metric spaces the earlier "$\epsilon-\delta$"–definition of continuity agrees with the above definition of continuity.

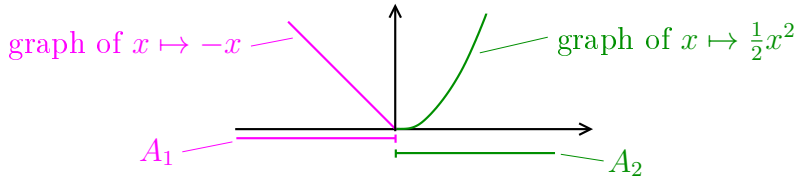
Proposition 2.8. (Metric Continuity Proposition) Let $f\colon X \to Y$ be a map between metric spaces. Then the following holds:

$$f \text{ is continuous} \quad \Longleftrightarrow \quad \forall_{x\in X}\ \forall_{\epsilon>0}\ \exists_{\delta>0}\ \forall_{x'\in B_\delta(x)}\ d(f(x),f(x')) < \epsilon.$$

Remark. We use the Metric Continuity Proposition 2.8 to import from real analysis the various continuous functions on $\mathbb{R}$ and $\mathbb{C}$, e.g. polynomials, trigonometric functions and the exponential function. The replacement of the classic "$\epsilon - \delta$"–definition of continuity by the definition in terms of open sets can even be helpful in the context of real analysis. For example, let us consider the function

$$\begin{array}{rcl} f\colon \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} \frac{1}{2}x^2, & \text{if } x \geq 0, \\ -x, & \text{if } x \leq 0. \end{cases} \end{array}$$

With the "$\epsilon - \delta$"–definition it is slightly annoying to prove continuity at 0. But applying Pasting Proposition 2.6 (2) to the closed subsets $A_1 = (-\infty, 0]$ and $A_2 = [0, \infty)$ it is totally evident that this function is continuous.



The previous remark leads us straight to the following convention.

Convention. The purist will correctly complain that the definition of the map in the previous remark is irregular, since the two cases treated in the definition of the map f are not disjoint. Nonetheless, on numerous occasions we will use such irregular definitions, usually to highlight the fact that we can use Pasting Proposition 2.6 to show that the map is continuous.

The proof of the Metric Continuity Proposition 2.8 requires some preparations. In particular it builds on the following proposition that frequently makes it much easier to show that a given map is continuous.

Proposition 2.9. (Subbasis-Continuity Proposition) Let $f\colon X \to Y$ be a map between topological spaces. Let $\mathcal{C}$ be a subbasis for the topology of Y (e.g. $\mathcal{C}$ could be a basis for the topology). Then the following holds

$$f \text{ is continuous} \quad \Longleftrightarrow \quad \text{for each } C \in \mathcal{C} \text{ the preimage } f^{-1}(C) \text{ is open in } X.$$

Proof of the Subbasis-Continuity Proposition 2.9. Let $f\colon X \to Y$ be a map between topological spaces and let $\mathcal{C}$ be a subbasis for the topology of Y. As in Lemma 1.31 we denote by $\mathcal{B} := \mathcal{B}(\mathcal{C})$ all subsets of Y that can be obtained by intersecting finitely many sets of $\mathcal{C}$.

The "⇒"-direction of the proposition follows from the observation that elements of a subbasis are in fact open. Thus let us turn to the "⇐"-direction. Now we assume that for each $C \in \mathcal{C}$ the preimage $f^{-1}(C)$ is open in X. We start out with a little observation. Suppose we are given $C_1, \ldots, C_k \in \mathcal{C}$, then

$$f^{-1}(C_1 \cap \cdots \cap C_k) \overset{\text{Image-Preimage Lemma 0.2}}{=} \underbrace{f^{-1}(C_1)}_{\text{open in } X} \cap \cdots \cap \underbrace{f^{-1}(C_k)}_{\text{open in } X}$$

is also open in X. This shows, by definition, that for each $B \in \mathcal{B} = \mathcal{B}(\mathcal{C})$ the preimage $f^{-1}(B)$ is open in X.

Now we turn to the actual proof that f is continuous. Thus let $U \subset Y$ be an open subset. By Lemma 1.29 there exists a family $\{B_j\}_{j \in J}$ of sets in $\mathcal{B}$ with

$$U = \bigcup_{j \in J} B_j.$$

Then we see that

$$f^{-1}(U) = f^{-1}\Big(\bigcup_{j \in J} B_j\Big) \overset{\text{Image-Preimage Lemma 0.2}}{=} \underbrace{\bigcup_{j \in J} \underbrace{f^{-1}(B_j)}_{\text{open in } X}}_{\substack{\text{union of open} \\ \text{sets, thus open in } X}}.$$

■

Now that we have the Subbasis-Continuity Proposition 2.9 under the belt we can easily provide a proof of the Metric Continuity Proposition 2.8.

Proof of the Metric Continuity Proposition 2.8. Let $f\colon X \to Y$ be a map between metric spaces. We need to show that

$$f \text{ is continuous} \iff \forall_{x \in X} \forall_{\epsilon > 0} \exists_{\delta > 0} \forall_{y \in B_\delta(x)} d(f(x), f(y)) < \epsilon.$$

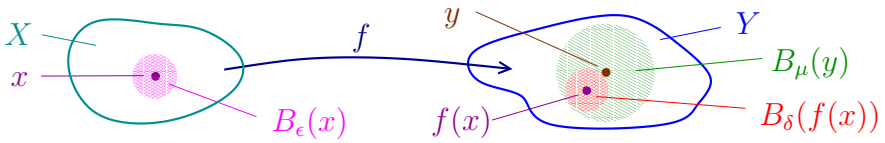
We prove the two directions separately:

"⇒" Let $x \in X$ and let $\epsilon > 0$. We consider $V := f^{-1}(B_\epsilon(f(x))) \subset X$. Since f is continuous we know that V is open. By the definition of the topology on a metric space there exists a $\delta > 0$ such that $B_\delta(x) \subset V$. This delta has the desired property.

"⇐" On page 132 we saw that the open balls form a basis for the topology of metric spaces. Thus by the Subbasis-Continuity Proposition 2.9 it suffices to show that if $y \in Y$ and $\mu > 0$, then $f^{-1}(B_\mu(y))$ is an open subset of X. By the definition of the topology on X it suffices to prove the following claim.

Claim. Given any $x \in f^{-1}(B_\mu(y))$ there exists an $\epsilon > 0$ with $B_\epsilon(x) \subset f^{-1}(B_\mu(y))$.

Proof. We set $\delta = \mu - d(f(x), y)$. Note that $\delta > 0$. By the triangle inequality we have $B_\delta(f(x)) \subset B_\mu(y)$. Furthermore by our hypothesis on f there exists an $\epsilon > 0$ such that $f(B_\epsilon(x)) \subset B_\delta(f(x))$. Thus we see that $B_\epsilon(x) \subset f^{-1}(B_\mu(y))$. ■



We continue with the following straightforward definition.

Definition. Let $f\colon X \to Y$ be a map between topological spaces.
(1) We say f is open if the image of every open subset of X is open in Y.
(2) We say f is closed if the image of every closed subset of X is closed in Y.

Examples.

(1) The projection $\mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto x$ is easily seen to be an open map, but it is not a closed map.[30]
(2) On the other hand the inclusion $i\colon \mathbb{R} \to \mathbb{R}^2$, $i(x) = (x, 0)$ is evidently not an open map, but it is a closed map.

Lemma 2.10. (Open-Closed Inclusion Lemma) Let X be a topological space.
(1) If $U \subset X$ is an open subset, then the inclusion map $i\colon U \to X$ is open.
(2) If $A \subset X$ is a closed subset, then the inclusion map $i\colon A \to X$ is closed.

Proof.

(1) Suppose that V is an open subset of U. By definition of the subspace topology there exists an open subset $W \subset X$ with $V = U \cap W$. But then $i(V) = V = U \cap W$ is the intersection of two open subsets of X, thus it is an open subset of X.
(2) The proof of this statement is analogous to the proof of Statement (1), we just not to appeal to the Neighborhood Openness Criterion 1.5 (2) to make the necessary adjustments. ■

We have the following analogue to the Subbasis-Continuity Proposition 2.9.

Lemma 2.11. Let $f\colon X \to Y$ be a map between topological spaces. Let $\mathcal{C}$ be a subbasis for the topology of X (e.g. $\mathcal{C}$ could be a basis for the topology). Then the following holds:

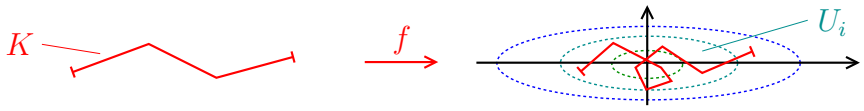
$$f \text{ is open} \quad \Longleftrightarrow \quad \text{for each } C \in \mathcal{C} \text{ the image } f(C) \text{ is open in } Y.$$

Proof. The logic of the proof is very similar to the proof of the Subbasis-Continuity Proposition 2.9. Eventually the proof boils down to Lemma 1.29 and the Image-Preimage Lemma 0.2. ■

On many occasions we will use the following elementary lemma.

Lemma 2.12. Let $f\colon K \to X$ be a continuous map from a compact space K to a topological space X.[31] Suppose there exists a sequence $U_1, U_2, \ldots$ of subsets of X with the following properties:
(1) each U_i is open,
(2) the sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $U_i \subset U_{i+1}$,
(3) we have $X = \bigcup_{i \in \mathbb{N}} U_i$.
Then there exists a $k \in \mathbb{N}$ with $f(K) \subset U_k$.

[30] Why is not a closed map?
[31] A common application is that K is a compact subset of X and f is the inclusion map.



Proof. For $i \in \mathbb{N}$ we write $V_i = f^{-1}(U_i)$. Since f is continuous it follows from (1) that each V_i is an open subset of K. By (3) the V_i cover all of K. Since K is compact there exist $i_1, \ldots, i_n \in \mathbb{N}$ such that $K = V_{i_1} \cup \cdots \cup V_{i_n}$. We set $k := \max\{i_1, \ldots, i_n\}$. Since the U_i are nested the V_i are also nested. It follows that K is already contained in V_k, put differently, $f(K)$ is contained in U_k. ■

In most instances we are dealing with continuous maps. Since constantly writing and reading "continuous maps" is tiresome for the author and the reader we adopt throughout the remainder of these lecture notes the following convention:

Convention. Henceforth any map between two topological spaces is assumed to be continuous, unless we say explicitly otherwise.

2.2. **The Compact-Image Lemma.** The first statement of the following lemma is perhaps the most important feature of compact spaces.

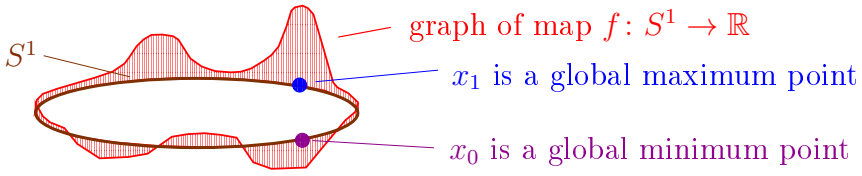
Lemma 2.13. (Compact Image Lemma)

(1) Let $f\colon X \to Y$ be a continuous map. If X is compact, then $f(X)$ is also compact.

(2) Let $f\colon X \to \mathbb{R}$ be a continuous map. If X is compact and non-empty, then f assumes its maximum and its minimum, i.e. there exist $x_0, x_1 \in X$ such that

$$f(x_0) \;\leq\; f(x) \;\leq\; f(x_1) \quad \text{for all } x \in X.$$

(3) Let $f\colon X \to Y$ be a continuous map. If X is compact and if Y is Hausdorff, then $f(X)$ is a closed subset of Y.



Example. Let X be a topological space, let Y be a subset and let $K \subset Y$ be another subset. If K is a compact subset of Y, then K is also a compact subset of X. This elementary observation follows immediately from the Compact Image Lemma 2.13 and the fact, observed above, that inclusion maps are continuous.

Remark. The statement of the Compact Image Lemma 2.13 (1) is really quite astounding. More precisely, we already saw that the image of an open set under a continuous map is not necessarily open. Similarly, the image of a closed set under a continuous map is not necessarily closed. For example consider the inclusion map $i\colon X = (0,1] \to Y = \mathbb{R}$. Evidently X is a closed subset of itself, but $i(X)$ is not a closed subset of $\mathbb{R}$. We will discuss such examples in more detail in Exercise 2.29.

Proof.

(1) Let $\{U_i\}_{i\in I}$ be an open cover of the topological space $f(X)$. We need to show that $f(X)$ is contained in the union of finitely many U_i's. We have

$$X \underset{\uparrow}{=} f^{-1}(f(X)) \subset f^{-1}\Big(\bigcup_{i\in I} U_i\Big) \underset{\uparrow}{=} \bigcup_{i\in I} f^{-1}(U_i).$$

Image-Preimage Lemma 0.2 (1) — Image-Preimage Lemma 0.2 (6)

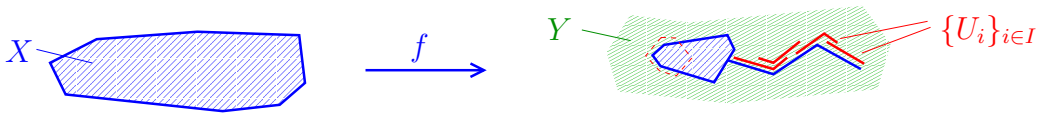
By hypothesis the map $f\colon X \to Y$ is continuous. It follows from the elementary Lemma 2.1 that the map $f\colon X \to f(X)$ is also continuous, in particular each $f^{-1}(U_i)$ is an open subset of X. This observation together with the above discussion shows that $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of X. Since X is compact there exist $i_1, \dots, i_k \in I$ such that

$$X = f^{-1}(U_{i_1}) \cup \dots \cup f^{-1}(U_{i_k}).$$

Therefore

$$\begin{aligned} f(X) &= f\left(f^{-1}(U_{i_1}) \cup \dots \cup f^{-1}(U_{i_k})\right) \\ &\underset{\uparrow}{=} f\left(f^{-1}(U_{i_1})\right) \cup \dots \cup f\left(f^{-1}(U_{i_k})\right) \underset{\uparrow}{\subset} U_{i_1} \cup \dots \cup U_{i_k}. \end{aligned}$$

Image-Preimage Lemma 0.2 (3) — Image-Preimage Lemma 0.2 (2)



(2) It follows from (1) and the Heine-Borel Theorem 1.24 that $f(X) \subset \mathbb{R}$ is bounded and closed. Since X is non-empty we thus see that $y_0 := \inf(f(X))$ and $y_1 := \sup(f(X))$ exist and they lie in $f(X)$. Any choice of points $x_0 \in f^{-1}(y_0)$ and $x_1 \in f^{-1}(y_1)$ have the desired properties.

(3) This statement is an immediate consequence of the Compact-Closed Lemma 1.21 (1) and the first statement. ■

The Compact Image Lemma 2.13 is one of the results that we will cite most often. In the remainder of this section we want to give a first application to the study of norms on finite-dimensional real vector spaces. This result is somewhat tangential to the main story of the lecture notes, but it falls into the category of "nice to know".

For the reader's convenience let us recall the following definition from page 101.

Definition. Let V be a real vector space. A norm on V is a map $n\colon V \to \mathbb{R}_{\geq 0}$ that satisfies the following three properties:

(1) For every $v \in V$ we have $n(v) = 0 \iff v = 0$.
(2) For every $v \in V$ and $\lambda \in \mathbb{R}$ we have $n(\lambda \cdot v) = |\lambda| \cdot n(v)$.
(3) For every $v, w \in V$ we have $n(v + w) \leq n(v) + n(w)$.

Recall that in Lemma 0.17 and Metric-Topology Lemma 1.1 we saw that a norm gives rise to a metric and hence to a topology.

Definition. Let V be a real vector space. We say that two norms n and $\widetilde{n}$ on V are equivalent if there exist $c, C \in \mathbb{R}_{>0}$ such that for all $v \in V$ we have

$$c \cdot n(v)| \leq \widetilde{n}(v) \leq C \cdot n(v).$$

Lemma 2.14. Let V be a real vector space.

(1) The notion of "equivalence" of norms on V is indeed an equivalence relation.
(2) Equivalent norms give rise to the same topology on V

Proof. The first statement is basically trivial. The second statement follows from the elementary the Equivalent Metric Lemma 1.2. ■

The following proposition is the promised result on norms on finite-dimensional real vector spaces.

Proposition 2.15. (Norm Equivalence Proposition) Any two norms on a finite-dimensional real vector space are equivalent.

Remark. In Exercise 2.5 we will see that the analogous statement for infinite-dimensional vector spaces does not necessarily hold.

Proof. Evidently it suffices to consider $\mathbb{R}^k$. Furthermore note that by Lemma 2.14 it suffices to show that every norm n on $\mathbb{R}^k$ is equivalent to the Euclidean norm $\|-\|$. As usual we denote by $e_1, \dots, e_k$ the standard basis of $\mathbb{R}^k$. We set

$$C \ := \ \sqrt{n(e_1)^2 + \cdots + n(e_k)^2}.$$

Note that $C > 0$.

Claim 1. For every $v \in \mathbb{R}^k$ we have $n(v) \leq C \cdot \|v\|$.

Proof. Let $v = (v_1, \dots, v_k) \in \mathbb{R}^k$. We see that

$$\begin{aligned} n(v) \ = \ n(v_1 \cdot e_1 + \cdots + v_k \cdot e_k) &\overset{\text{since } n \text{ is a norm}}{\leq} |v_1| \cdot n(e_1) + \cdots + |v_k| \cdot n(e_k) \\ &\underset{\text{Cauchy-Schwarz Inequality 0.13 applied to } |v_1|, \dots, |v_k| \text{ and } n(e_1), \dots, n(e_k)}{\leq} \sqrt{(v_1^2 + \cdots + v_k^2) \cdot (n(e_1)^2 + \cdots + n(e_k)^2)} \ = \ \|v\| \cdot C. \end{aligned}$$

⊞

Since n is a norm it follows from Claim 1 that $n \colon \mathbb{R}^k \to \mathbb{R}$ is continuous with respect to the standard topology on $\mathbb{R}^k$ given by the Euclidean norm. By the Heine-Borel Theorem 1.24 we know that $S^{n-1} = \{v \in \mathbb{R}^k \,|\, \|v\| = 1\}$ is compact. Therefore we know by the Compact Image Lemma 2.13 that there exists a $u \in S^{n-1}$ such that $n(u) \leq n(w)$ for all $w \in S^{n-1}$. Note that by Property (1) of the norm n we know that $n(u) > 0$. We set $c := \frac{1}{n(u)}$. Evidently $c > 0$. At this stage it remains to prove the following claim.

Claim 2. For every $v \in \mathbb{R}^k$ we have $\|v\| \leq c \cdot n(v)$.

Proof. Let $v \in \mathbb{R}^k$. If $v = 0$, then there is nothing to show. So suppose that $v \neq 0$. We see that

$$\|v\| \ \underset{\text{definition of } c}{=} \ \|v\| \cdot c \cdot n(u) \ \underset{\text{choice of } u}{\leq} \ \|v\| \cdot c \cdot n\Big(\frac{v}{\|v\|}\Big) \ \underset{\text{since } n \text{ is a norm}}{=} \ c \cdot n(v).$$

■

2.3. Homeomorphisms.

We introduce the main objects of interest of this section.

Definition.

(1) We say a map $f\colon X \to Y$ between two topological spaces X and Y is a homeomorphism if the following three properties are satisfied:
   (a) f is continuous,
   (b) f is bijective,
   (c) the inverse map $f^{-1}\colon Y \to X$ is also continuous.
   If there exists a homeomorphism between X and Y we say that X and Y are homeomorphic and sometimes we write $X \cong Y$.

(2) We say a map $f\colon X \to Y$ between two topological spaces X and Y is an embedding if $f\colon X \to f(X)$, where $f(X)$ is equipped with the subspace topology, is a homeomorphism. We say the embedding is closed if $f(X)$ is a closed subset of Y and we say the embedding is open if $f(X)$ is an open subset of Y.

Remark.

(1) If two topological spaces are homeomorphic, then they have the same topological properties, i.e. they share all properties that are defined purely in terms of the topology. For example, if X and Y are homeomorphic, then X is Hausdorff if and only if Y is Hausdorff, X is compact if and only if Y is compact and so on.

(2) It follows from the Open -Closed Inclusion Lemma 2.10 that an open (respectively closed) embedding is also an open (respectively closed) map.

Convention. At times we refer to a topological space that is homeomorphic to a ball or to a sphere also as a ball or a sphere.

Before we turn to our first examples we introduce some standard notation.

Notation. Given $n_0 \in \mathbb{N}$ we write

$$\begin{array}{lcll} S^n_{\geq 0} & := & \{(x_1,\dots,x_{n+1}) \in S^n \mid x_{n+1} \geq 0\} & \text{the upper hemisphere} \\ S^n_{\leq 0} & := & \{(x_1,\dots,x_{n+1}) \in S^n \mid x_{n+1} \leq 0\} & \text{the lower hemisphere} \\ S^n_{=0} & := & \{(x_1,\dots,x_{n+1}) \in S^n \mid x_{n+1} = 0\} & \text{the equator.} \end{array}$$

Similarly we define

$$\begin{array}{lcl} B^n_{\geq 0} & := & \{(x_1,\dots,x_n) \in B^n \mid x_n \geq 0\} \\ B^n_{\leq 0} & := & \{(x_1,\dots,x_n) \in B^n \mid x \leq 0\} \end{array}$$

Finally, the same way we also define $\overline{B}^n_{\geq 0}$ and $\overline{B}^n_{\leq 0}$.



Examples.

(1) The maps

$$\begin{array}{rcl} \varphi\colon \overline{B}^n & \to & S^n_{\geq 0} \\ x & \mapsto & \left(x, \sqrt{1-\|x\|^2}\right) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} p\colon S^n_{\geq 0} & \to & \overline{B}^n \\ (x_1,\dots,x_n,x_{n+1}) & \mapsto & (x_1,\dots,x_n) \end{array}$$

are easily seen to be continuous and inverses to one another. Thus both maps are homeomorphisms. The same way we see that the lower hemisphere is also homeomorphic to the closed ball $\overline{B}^n$.

(2) Let $A \in \mathrm{GL}(n,\mathbb{R})$ be an invertible real matrix. It follows from the discussion on page 143 that the map $\mathbb{R}^n \to \mathbb{R}^n$ given by multiplication by A is a homeomorphism. This simple observation has a useful consequence: Let V be an n-dimensional real vector space. We pick an isomorphism $\varphi\colon \mathbb{R}^n \to V$ and we equip V with the unique topology that turns φ into a homeomorphism. By the above discussion this definition of a topology does not depend on the choice of φ. Unless we say something else any finite-dimensional real vector space will be equipped with this topology. Note that with these topologies any isomorphism between real vector spaces of the same finite dimension is also a homeomorphism.

(3) Let $n \in \mathbb{N}$. It follows immediately from the definitions of the standard metrics on $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ that the map

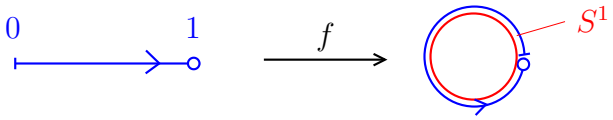
$$\begin{array}{rcl} \mathbb{R}^{2n} & \to & \mathbb{C}^n \\ (x_1, y_1, \ldots, x_n, y_n) & \mapsto & (x_1 + \mathrm{i}y_1, \ldots, x_n + \mathrm{i}y_n) \end{array}$$

is an isometry of metric metric spaces. Recall that on page 109 we defined the standard topology on $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ to be the metric topology corresponding to the standard metrics. Thus we see that the above map is a homeomorphism. We will use this homeomorphism to freely go back and forth between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$.

(4) We consider the map

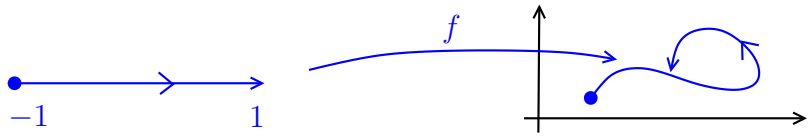
$$\begin{array}{rcl} f\colon [0,1) & \to & S^1 \\ t & \mapsto & \exp(2\pi \mathrm{i} t). \end{array}$$

The map f is a continuous bijection but it is not a homeomorphism. One can see this by either directly showing that f^{-1} is not continuous or by noting that $[0,1)$ is not compact whereas S^1 is compact. The importance of this example cannot be overestimated. It is an unfortunate fact of life that all too often one writes down a continuous map $f\colon X \to Y$ between topological spaces, checks that it is a bijection, and walks away under the warm, but mistaken, impression that one now has found a homeomorphism.



f is a continuous bijection but not a homeomorphism

(5) To drive home the point we made in (3) we show in the figure below an injective continuous map $f\colon [-1,1) \to \mathbb{R}^2$ which is *not* an embedding. Especially once we move to more abstract topological spaces it is easy to loose track of the fact that an injective homeomorphism is not necessarily an embedding.



In many situations one is given an explicit continuous bijection $f\colon X \to Y$ between two topological spaces and one would like to show that f is in fact a homeomorphism. Unfortunately usually it is very painful to write down the explicit inverse map $f^{-1}\colon Y \to X$, let alone to verify that it is continuous. Fortunately we have several results which often help us in circumventing this difficulty.

Lemma 2.16. (Open-Injective Map Lemma) Let $f\colon X \to Y$ be a map between topological spaces that is continuous. We assume that f is open or that f is closed.

(1) If f is an injection, then f is an open (respectively closed) embedding.
(2) If f is a bijection, then f is a homeomorphism.

Proof. Let $f\colon X \to Y$ be a map between topological spaces that is continuous

(2) We assume that f is a bijection. Furthermore we now assume that f is open. We want to show that f is a homeomorphism. It remains to show that $g := f^{-1}\colon Y \to X$ is continuous. Thus let $U \subset X$ be an open set. We need to show that $g^{-1}(U)$ is an open subset of Y. But this is obvious, since $g^{-1}(U) = f(U)$ is an open subset since we assume that f is an open map.

    Now suppose that the map f is closed. The proof is almost identical to the above argument, except that now we have to use the continuity criterion provided by the Continuity-via-Closed Subsets Lemma 2.4.

(1) We assume that f is an injection and we assume that f is open. We write $Z := f(X)$. Since $f\colon X \to Y$ is open it follows basically immediately from the definition of the subspace topology that $f\colon X \to Z$ is also open. Thus we obtain from (2) that f is an embedding. Furthermore, since f is open we see that f is an open embedding. The proof in the case that f is closed is basically identical. ■

Example. Let U be an open subset of $\mathbb{R}$ and let $f\colon U \to \mathbb{R}$ be a strictly monotonously increasing map, i.e. for any $x < y$ in U we have $f(x) < f(y)$. Note that open intervals form a basis for the topology of U. By the Intermediate Value Theorem the image of an open interval under f is again an open interval. Thus it follows from Lemma 2.11 that $f\colon U \to \mathbb{R}$ is an open map. The Open-Injective Map Lemma 2.16 implies that $f\colon U \to f(U)$ is a homeomorphism.

The next proposition is certainly one of the most frequently used results in topology.

Proposition 2.17. (Compact-Hausdorff Proposition) Let $f\colon X \to Y$ be a continuous map between topological spaces. If X is compact and if Y is Hausdorff, then the following statements hold:

(1) The map $f\colon X \to Y$ is closed.
(2) If f is an injection, then f is a closed embedding.
(3) If f is a bijection, then f is a homeomorphism.

Remark.

(1) On page 153 we gave an example of a continuous bijective map $f\colon [0,1) \to S^1$ that is not a homeomorphism,. This shows that we cannot drop the condition that X is compact from the proposition. But what about the "Hausdorff condition"? In Exercise 2.42 we will come up with an example of a bijective continuous map $f\colon X \to Y$ between two topological spaces where X is compact but such that f is nonetheless not a homeomorphism.
(2) If X is non-compact, then it is often a major nuisance to try to show that a given map $f\colon X \to Y$ is in fact a closed embedding. Later on, in the Closed Embedding-to-$\mathbb{R}^n$ Proposition 2.19 and the Proper-Closed Proposition 2.43 we give two criteria

which can often be used to show that an injective continuous map starting from a non-compact space to a Hausdorff space is a closed embedding after all.

Proof. Let $f\colon X \to Y$ be a continuous map between topological spaces. We assume that X is compact and that Y is Hausdorff.

(1) Let A be a closed subset of X. We need to show that $f(A)$ is a closed subset of Y. In fact we see that

$$A \subset X \text{ closed} \underset{\substack{\uparrow\\ \text{by the Compact-Closed Lemma 1.21 since } X \text{ is compact}}}{\Longrightarrow} A \text{ is compact} \underset{\substack{\uparrow\\ \text{by the Compact Image Lemma 2.13 (1)}}}{\Longrightarrow} f(A) \text{ is compact} \underset{\substack{\uparrow\\ \text{by the Compact-Closed Lemma 1.21 (2) since } Y \text{ is Hausdorff}}}{\Longrightarrow} f(A) \subset Y \text{ is closed.}$$



(2) If f is an injection, then it follows immediately from (1) together with the Open -Injective Map Lemma 2.16 (1) that f is a closed embedding.

(3) This follows immediately from (2). ∎

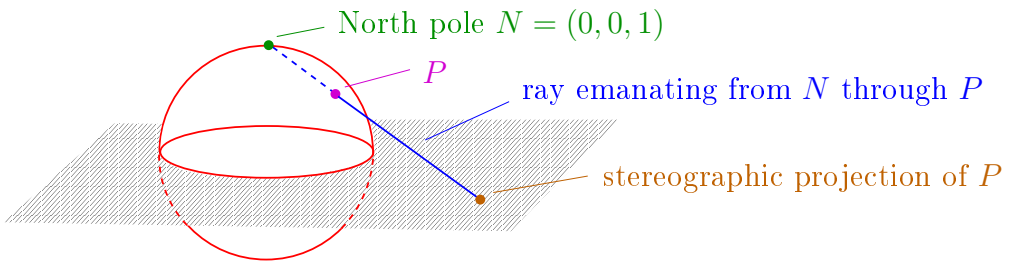
We formulate our first application of the Compact-Hausdorff Proposition 2.17 (3) as a lemma.

Lemma 2.18. Let $n \in \mathbb{N}$. We consider the map

$$\begin{array}{rcl} \Phi\colon S^n & \to & \mathbb{R}^n \cup \{\infty\} \\ (x_1, \ldots, x_{n+1}) & \mapsto & \begin{cases} \left(\frac{x_1}{1-x_{n+1}}, \ldots, \frac{x_n}{1-x_{n+1}}\right), & \text{if } x_{n+1} < 1, \\ \infty, & \text{if } x_{n+1} = 1. \end{cases} \end{array}$$

where we equip $\mathbb{R}^n \cup \{\infty\}$ with the topology that we introduced on page 122. We refer to Φ as the stereographic projection. This map has the following properties:

(1) The map Φ sends the North Pole $N = (0, \ldots, 0, 1) \in S^n$ to ∞.
(2) For any $P \in S^n$ that does not equal the North Pole N the point $\Phi(P) \in \mathbb{R}^n$ is the unique point such that the ray emanating from N and that goes through P intersects the plane $\mathbb{R}^n \times \{0\}$ in $(\Phi(P), 0)$.
(3) For any $(v_1, \ldots, v_n, 0) \in S^n \cap (\mathbb{R}^n \times \{0\})$ we have $\Phi(v_1, \ldots, v_n, 0) = (v_1, \ldots, v_n)$.
(4) The map Φ is continuous.
(5) The map Φ is a homeomorphism.
(6) The restriction of Φ to a map $S^n \setminus \{N\} \to \mathbb{R}^n$ is also a homeomorphism.



Convention. Let $n \in \mathbb{N}$. On many occasions we will use the homeomorphism from Lemma 2.18 to make the identification $S^n = \mathbb{R}^n \cup \{\infty\}$. In particular we have the identification $S^2 = \mathbb{C} \cup \{\infty\}$.

Proof.

(1) This statement follows immediately from the definition of Φ.
(2) This statement follows from an elementary calculation which we leave to the reader.
(3) This statement follows immediately from the definition of Φ.
(4) We will prove this statement in Exercise 2.47.
(5) We leave it to the reader to verify that Φ is a bijection. Note that S^n is compact by the Heine-Borel Theorem 1.24. Furthermore note that on page 122 we pointed out that $\mathbb{R}^n \cup \{\infty\}$ is Hausdorff. Hence it follows from the Compact-Hausdorff Proposition 2.17 (3) that Φ is a homeomorphism.
(6) This statement follows immediately from (5). ■

The existence of non-compact spaces is one of those unfortunate facts of life. As we saw, in general the conclusion of the Compact-Hausdorff Proposition 2.17 does not hold if the domain is non-compact. The following proposition gives us a criterion for an injective map to be a closed embedding which will deal with most cases that we will encounter.

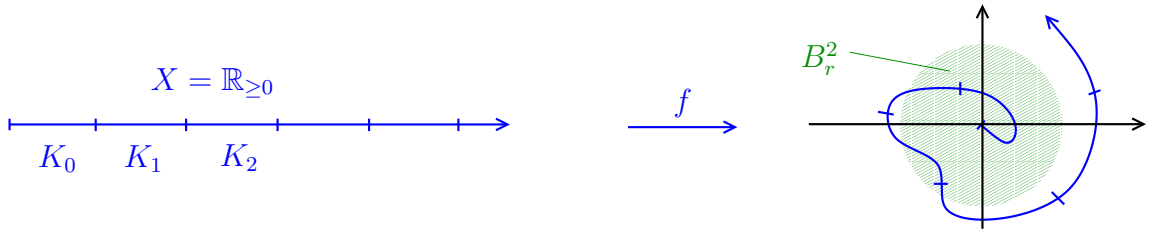
Proposition 2.19. (Closed Embedding-to-$\mathbb{R}^n$ Proposition) Let X be a topological space, let $n \in \mathbb{N}$ and let $f\colon X \to \mathbb{R}^n$ be an injective continuous map. Furthermore suppose that X is the union of compact subsets $\{K_i\}_{i\in I}$ such that the following condition is satisfied:

($*$) Given any $r > 0$ there exist only finitely many $i \in I$ with $f(K_i) \cap \overline{B}^n_r \neq \varnothing$.

Then f is a closed embedding.

Remark. Later we will prove the Proper-Closed Proposition 2.43 which gives a more general criterion for an injective map to be a closed embedding. On page 182 we will use it to give a different, arguably more conceptual proof of the Closed Embedding-to-$\mathbb{R}^n$ Proposition 2.19.

Example. Let $X = \mathbb{R}_{\geq 0}$. We consider the "outward spiraling" map $f\colon X \to \mathbb{R}^2$ that is sketched in the figure below. Given $i \in \mathbb{N}_0$ we set $K_i := [i, i+1]$. These compact subsets cover all of $X = \mathbb{R}_{\geq 0}$ and they satisfy the condition ($*$) of the proposition. Thus we obtain from the Closed Embedding-to-$\mathbb{R}^n$ Proposition 2.19 that the map f is a closed embedding.



Proof. By the Open -Injective Map Lemma 2.16 (1) it remains to show that the map $f\colon X \to \mathbb{R}^n$ is closed. Thus let A be a closed subset of X. We need to show that $f(A)$ is a closed subset of $\mathbb{R}^n$. Given $s \in \mathbb{N}_0$ we consider $C_s := \overline{B}^n_{s+1} \setminus B^n_s$. Clearly $\{C_s\}_{s\in\mathbb{N}_0}$ is a

locally finite closed cover of $\mathbb{R}^n$. By Lemma 1.8 (2b) it suffices to show that each $C_s \cap f(A)$ is closed in $\mathbb{R}^n$. Thus let $s \in \mathbb{N}_0$. By hypothesis we know that $J := \{i \in I \,|\, f(K_i) \cap \overline{B}_s^n \neq \varnothing\}$ is a finite set. Now we see that

$$\begin{array}{lcccl} C_s \cap f(A) & = & C_s \cap f\Big(\bigcup_{i\in I} K_i\Big) & \overset{\downarrow}{=} & C_s \cap f\Big(\bigcup_{j\in J} K_j\Big) \quad \overset{\downarrow}{=} \quad C_s \cap f(\text{compact subset}) \\ & = & C_s \cap \text{compact subset of } \mathbb{R}^n & = & C_s \cap \text{closed subset of } \mathbb{R}^n \quad = \quad \text{closed subset of } C_s. \\ & \uparrow & & \uparrow & \uparrow \end{array}$$

definition of J; by the Compact-Union-Intersection Lemma 1.20 (1), since each K_j is compact, and since J is finite

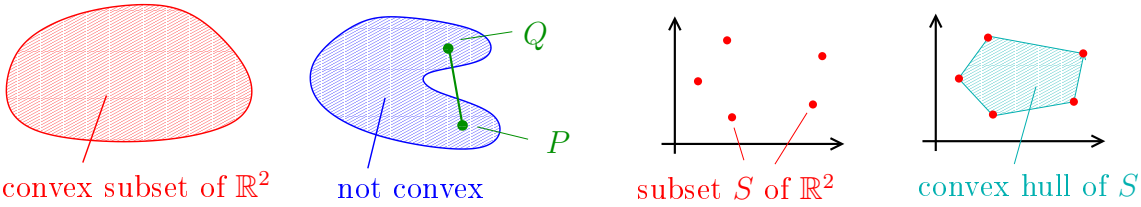
Compact Image Lemma 2.13; Compact-Closed Lemma 1.21; definition of subspace topology ■

2.4. **Convex subsets.** We start out with the definition of a convex subset.

Definition.

(1) We say that a subset $A \subset \mathbb{R}^n$ is convex, if for every two points P and Q in A the segment $\overline{PQ} := \{t \cdot P + (1-t) \cdot Q \,|\, t \in [0,1]\}$ also lies in A.

(2) Given a subset S of $\mathbb{R}^k$ the convex hull of S is defined as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S. Since the intersection of convex sets is again convex we see that the convex hull of S is a convex subset of $\mathbb{R}^n$.



convex subset of $\mathbb{R}^2$ | not convex | subset S of $\mathbb{R}^2$ | convex hull of S

Examples.

(1) For any point $x \in \mathbb{R}^n$ and any $\epsilon > 0$ the open ball $B_\epsilon^n(x)$ is convex. Indeed, let $P, Q \in B_\epsilon^n(x)$. Then for any $t \in [0,1]$ we have

$$\begin{array}{rcll} \|t\cdot P + (1-t)\cdot Q - x\| & = & \|t\cdot(P-x) + (1-t)\cdot(Q-x)\| & \\ & \underset{\uparrow}{\leq} & t\cdot\|P-x\| + (1-t)\cdot\|Q-x\| & \underset{\uparrow}{<} \; t\cdot\epsilon + (1-t)\cdot\epsilon = \epsilon. \end{array}$$

triangle inequality and homogeneity of $\|-\|$; since $P, Q \in B_\epsilon^n(x)$

We have thus shown that $t \cdot P + (1-t) \cdot Q \in B_\epsilon^n(x)$. Basically the same way one shows that the closed ball $\overline{B}_\epsilon^n(x)$ is also convex.

(2) It is straightforward to verify that any hyperrectangle $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is convex.

(3) In Exercise 2.38 we will show that the convex hull of points $P_1, \ldots, P_n \in \mathbb{R}^k$ is given by the set

$$\Big\{\sum_{i=1}^n t_i \cdot P_i \,\Big|\, t_1, \ldots, t_n \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^n t_i = 1\Big\}.$$

(4) One of the most useful properties of convex subsets is that the intersection of arbitrarily many convex subsets is, basically by definition, again convex.

The following proposition gives a useful criterion for showing that subsets of $\mathbb{R}^n$ are homeomorphic to an open or to a closed ball.

Proposition 2.20. (Convex-to-Ball Proposition)

(1) Let A be a bounded *open* convex subset of $\mathbb{R}^n$ and let $Q \in A$.[32] We consider the map

$$\begin{array}{rcl} \rho_A \colon A \setminus \{Q\} & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & \sup\left\{ \|r \cdot x\| \,\middle|\, r \in \mathbb{R}_{>0} \text{ and } r \cdot (x-Q) + Q \in A \right\}. \end{array}$$

This map is well-defined in the sense that the supremum exists for each $x \in A \setminus \{Q\}$. Furthermore the map ρ_A is continuous and the map

$$\begin{array}{rcl} f_Q \colon A & \to & B^n \\ x & \mapsto & \begin{cases} (x-Q) \cdot \frac{1}{\rho_A(x)}, & \text{if } x \neq Q, \\ 0, & \text{if } x = Q. \end{cases} \end{array}$$

is a homeomorphism with $f_Q(Q) = 0$.

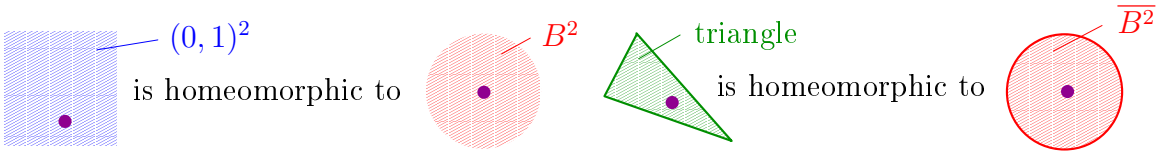
(2) Let A be a bounded *closed*[33] convex subset of $\mathbb{R}^n$. If Q is a point in the interior[34] of A, then the same expression as in (1) defines a homeomorphism[35]

$$f_Q \colon A \to \overline{B}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$$

with the following properties:

(a) the homeomorphism f_Q sends Q to the origin,

(b) the homeomorphism f_Q restricts to a homeomorphism $\partial A \to S^{n-1}$.



Example.

(1) For $i = 1, \ldots, n$ let $a_i < b_i$ be real numbers. We consider the "hyperrectangle" $[a_1, b_1] \times \cdots \times [a_n, b_n]$. It follows from the discussion on page 157 and from the Hyperrectangle-Compact Proposition 1.26 that the hyperrectangle is convex and compact. The Convex-to-Ball Proposition 2.20 (2) gives us an explicit homeomorphism

$$f_Q \colon [a_1, b_1] \times \cdots \times [a_n, b_n] \to \overline{B}^n$$

that sends the boundary of the hyperrectangle to the sphere S^{n-1} and that sends the "midpoint" $Q := (\frac{a_1+b_1}{2}, \ldots, \frac{a_n+b_n}{2})$ to the origin. Throughout this course we will use this homeomorphism to identify the left-hand side with the right-hand side. It follows easily from the explicit description of f_Q that the restriction of f_Q to each open cone on a side is smooth.[36]

---

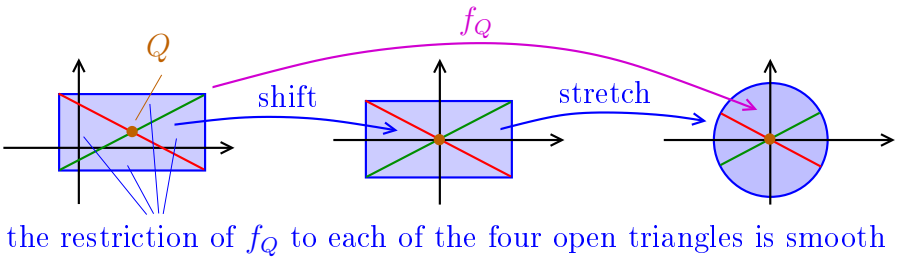
[32] The statement becomes slightly more readable if one sets $Q = 0$.

[33] Of course "bounded and closed" is in this instance, by the Heine-Borel Theorem 1.24 equivalent to "compact". In this instance we prefer to talk of "bounded closed" to stress the analogy to Statement (1) of the lemma.

[34] Note that the interior might well be empty, i.e. such a point Q might not exist.

[35] This says in particular that if A is a bounded closed convex subset of $\mathbb{R}^n$ with *non-empty interior*, then A is homeomorphic to $\overline{B}^n$.

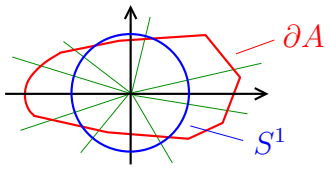
[36] We leave it to the reader to make it precise what "open cone on a side" means. We will make use of this observation only once, and then it should be clear what we mean.



(2) Let A be a bounded convex subset of $\mathbb{R}^n$ such that the origin 0 lies in the interior of A. The Convex-to-Ball Proposition 2.20, applied to $Q = 0$, implies in particular that the map

$$\begin{array}{rcl} f\colon \partial A & \to & S^{n-1} \\ x & \mapsto & \frac{x}{\|x\|} \end{array}$$

is a homeomorphism. This statement is illustrated in the figure below.



Remark. The reader might be familiar with the concept of a diffeomorphism. In the Star Shaped-Diffeomorphism Type Theorem **??** we will show that every open convex subset of $\mathbb{R}^n$ is in fact diffeomorphic to $\mathbb{R}^n$, and thus diffeomorphic to B^n. The diffeomorphism that we will provide in that later instance is, in contrast to the above proposition, not explicit.

The proof of the Convex-to-Ball Proposition 2.20 relies on the following lemma.

Lemma 2.21. Let A be a bounded closed convex subset of $\mathbb{R}^n$ such that the origin 0 lies in the interior of A.

(1) The map $\rho_A\colon A \setminus \{0\} \to \mathbb{R}_{\geq 0}$ defined in the Convex-to-Ball Proposition 2.20 is continuous.

(2) The map

$$\begin{array}{rcl} f_0\colon A & \to & \overline{B}^n \\ x & \mapsto & \begin{cases} x \cdot \frac{1}{\rho_A(x)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases} \end{array}$$

is continuous and it satisfies $f_0^{-1}(S^{n-1}) = \partial A$.

Proof of Lemma 2.21. We start out with the following claim.

Claim. The map $\rho_A\colon A \setminus \{0\} \to \mathbb{R}$ is defined, it is continuous, it is bounded from above and it is bounded from below by 1.

Proof. Since A is bounded it follows that ρ_A is defined and that it is bounded from above. It follows from the definition of ρ_A that ρ_A is bounded from below by 1. Therefore it remains to show that ρ_A is continuous. Now let $x \in A \setminus \{0\}$ and let $\epsilon > 0$. It suffices to show the following two statements:

(1) there exists a neighborhood U of $x \in A \setminus \{0\}$ with $\rho_A(y) > \rho_A(x) - \epsilon$ for all $y \in U$,

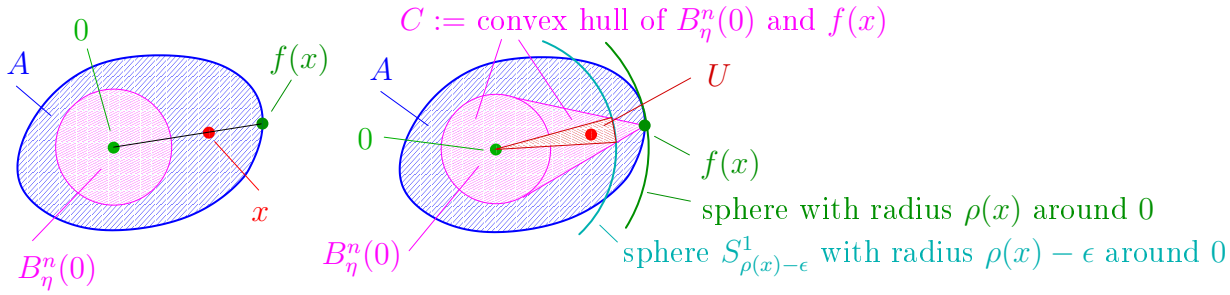
(2) there exists a neighborhood V of $x \in A \setminus \{0\}$ with $\rho_A(y) < \rho_A(x) + \epsilon$ for all $y \in V$.

We first show the existence of U. After possibly replacing ϵ by $\min\{\frac{1}{2}\rho_A(x), \epsilon\}$ we can suppose that $\epsilon \in (0, \rho_A(x))$.

Since A is closed we have $f_0(x) \in A$. Since 0 lies in the interior of A there exists an $\eta > 0$ such that $B^n_\eta(0) \subset A$. Since A is convex, the convex hull C of $B^n_\eta(0) \cup \{f_0(x)\}$ is still contained[37] in A. Furthermore, since $B^n_\eta(0)$ is open it is straightforward to see that $C' := C \setminus \{f_0(x)\}$ is an open subset of $\mathbb{R}^n$. We denote by $S^{n-1}_{\rho_A(x)-\epsilon}$ the sphere of radius $\rho_A(x) - \epsilon$ around 0. The point $x \cdot \frac{\rho_A(x)-\epsilon}{\|x\|}$ lies on $S^{n-1}_{\rho_A(x)-\epsilon}$ and it lies in C'. Since C' is open there exists an open neighborhood U' on $S^{n-1}_{\rho_A(x)-\epsilon}$ that is contained in C'. We set

$$U \;:=\; \{r \cdot z \,|\, z \in U' \text{ and } r \in (0,1)\}.$$

This is an open neighborhood of x and for any $y \in U$ we have $(\rho_A(x)-\epsilon)\cdot y \in A$, i.e. for any $y \in A$ we have $\rho_A(y) \geq \rho_A(x) - \epsilon$. Thus we have shown that the desired neighborhood U exists. The existence of V is proved in a very similar way. We refer to [**Berger2009**, Chapter 11.3] for full details. ⊞



From the continuity of ρ_A it follows that f is continuous on $A \setminus \{0\}$. From the fact that ρ_A is bounded it follows easily that f_Q is continuous in 0. Finally one can easily verify, using the convexity of A, that $f_0^{-1}(S^{n-1}) = \partial A$. We leave it to the reader to fill in the details. ■

Proof of the Convex-to-Ball Proposition 2.20.

(2) Let A be a bounded closed convex subset of $\mathbb{R}^n$ and let Q be a point in the interior of A. After a translation we can assume that $Q = 0$ is the origin, in particular we can assume that the origin 0 lies in the interior of A. By Lemma 2.21 we know that the map $\rho_A \colon A \setminus \{0\} \to \mathbb{R}_{\geq 0}$ is defined and continuous. Next we consider the map

$$\begin{array}{rcl} f_0 \colon A & \to & \overline{B}^n \\ x & \mapsto & \begin{cases} x \cdot \frac{1}{\rho_A(x)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \end{array}$$

It is straightforward to verify that this map is bijective[38] and by Lemma 2.21 we know that f_0 is continuous and that $f_0^{-1}(S^{n-1}) = \partial A$. Note that our hypothesis on A and

[37] Why is that the case?

[38] It follows easily from the convexity of A, the hypothesis that $\mathring{A} \neq \varnothing$ and the definition of f_0 that for each $v \in S^{n-1}$ the map f_0 restricts to a bijection on the "ray defined by v", i.e. it restricts to a bijection

$$A \cap \{r \cdot v \,|\, r \in \mathbb{R}_{>0}\} \;\to\; \overline{B}^n \cap \{r \cdot v \,|\, r \in \mathbb{R}_{>0}\}.$$

the Heine-Borel Theorem 1.24 imply that A is compact. Thus we obtain from the Compact-Hausdorff Proposition 2.17 (3) that f_0 is in fact a homeomorphism. Finally note that it follows from $f_0^{-1}(S^{n-1}) = \partial A$ that f_0 restricts to a homeomorphism $\partial A \to S^{n-1}$.

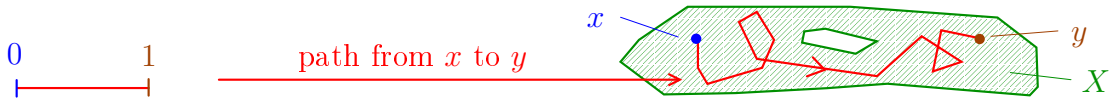
(1) We denote by $C = \overline{A}$ the closure of A. Since A is open and convex we know by Exercise 2.8 that $A = \mathring{C}$. One verifies fairly easily that C is convex. It is clear that C is bounded and it follows from the Interior-Closure Lemma 1.9 that C is closed. One verifies easily that the map f_0 defined for $C = \overline{A}$ restricts to the map f_0 defined for A. Thus it follows from (2) that the map $f_0 \colon C = \overline{A} \to \overline{B}^n$ is a homeomorphism and it restricts to a homeomorphism $f_0 \colon \mathring{C} \to B^n$. Since $\mathring{C} = A$ we get the desired homeomorphism. ■

2.5. **The two notions of connected spaces.** In this section we will see that there are two notions of connectedness of a topological space. As we will see, for most "reasonable" topological spaces the two notions mercifully coincide. We start out with the more obvious definition.

Definition.

(1) Let X be a topological space. A path is a map $p \colon [0,1] \to X$. Often we say that p is a path from $p(a)$ to $p(b)$.

(2) We say that a topological space X is path-connected if given any two points x and y there exists a path from x to y.



Before we discuss examples let us immediately provide the other, less obvious definition of connectedness.

Definition. We say that a topological space X is connected if the following holds[39]

$$\text{if } X = U \sqcup V \text{ is the disjoint union of open sets } U, V \quad \Longrightarrow \quad X = U \text{ or } X = V.$$

A topological space which is not connected is called disconnected.

Example.

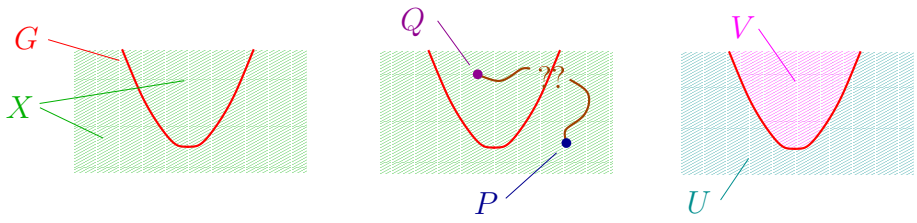
(1) It follows basically from the definitions that any convex subset of some $\mathbb{R}^n$ is path-connected. But it is much less clear whether say $\mathbb{R}^n$ is connected in the above sense.

(2) Next let $G \subset \mathbb{R}^2$ be the graph of the function $f \colon \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^2$. As in the figure below we consider the complement $X = \mathbb{R}^2 \setminus G$. It is pretty clear that X is not path-connected, surely there is no path from a point P below G to a point above G. But how can one show the non-existence of such a path?[40] But it is clear that X is not connected, since X is the disjoint union of the two open non-empty sets $U := \{(x,y) \in \mathbb{R}^2 \,|\, y < x^2\}$ and $V := \{(x,y) \in \mathbb{R}^2 \,|\, y > x^2\}$.

(3) It follows from the Intermediate Value Theorem that the rational numbers $\mathbb{Q}$ are not path-connected. But $\mathbb{Q}$ is also not connected, for example we can write the topological

[39] Recall that we write $X = U \sqcup V$ if X is the disjoint union of U and V, i.e. if $X = U \cup V$ and if $U \cap V = \varnothing$.

[40] The fact that such a path does not exist can be shown, with some dexterity, using real analysis. How?



space $\mathbb{Q}$ as the disjoint union of the two open non-empty subsets $\{x \in \mathbb{Q} \,|\, x^2 < 2\}$ and $\{x \in \mathbb{Q} \,|\, x^2 > 2\}$.

So we see that both notions of connectedness have their advantages and disadvantages. It is often easy to show that a given topological space is path-connected but it can be a pain to show that a given topological space is not path-connected. With connectedness it is precisely the other way round.

It is basically obvious that any interval is path-connected. But as we will see, it takes some effort to show that the interval $[0,1]$ is connected.

Proposition 2.22. (Interval-Connectedness Proposition) The interval $[0,1]$ is connected.

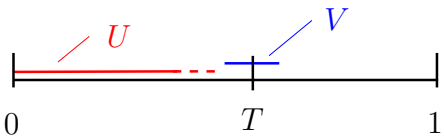
Proof. Let U and V be two disjoint open subsets of $[0,1]$ with $U \cup V = [0,1]$. Without loss of generality we can assume that $0 \in U$. We want to show that $U = [0,1]$. We consider

$$A \;:=\; \big\{t \in [0,1] \,\big|\, [0,t] \subset U\big\}.^{41}$$

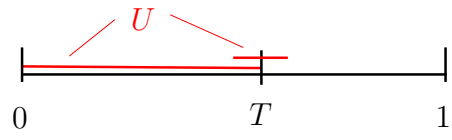
The set A contains 0, hence it is non-empty. Evidently the set is bounded thus we can consider $T := \sup A$. We need to show that $T = 1$ and that $T \in U$.

First let us suppose that $T \notin U$. This implies that $T \in V$. Since V is open it follows that there exists an $\epsilon > 0$ such that $(T-\epsilon, T+\epsilon) \cap [0,1]$ is also contained in V. In particular we have $(T-\epsilon, T+\epsilon) \cap A = \varnothing$ which implies that $\sup A \leq T - \epsilon$. Thus we have obtained a contradiction.

Now we know that $T \in U$. We still need to show that $T = 1$. Since U is open it follows that there exists an $\epsilon > 0$ such that $(T-\epsilon, T+\epsilon) \cap [0,1] \subset U$. Thus $[0, T+\epsilon) \cap [0,1] \subset U$. In light of the definition of T that is only possible if $T = 1$. ■



if $T \in V$ then $(T-\epsilon, T+\epsilon) \cap [0,1]$ is also contained in V



since $T \in U$ we see that $[0, T+\epsilon) \cap [0,1]$ is contained in U

Corollary 2.23. (Path-Connected Implies Connected Corollary) Every path-connected topological space is also connected.

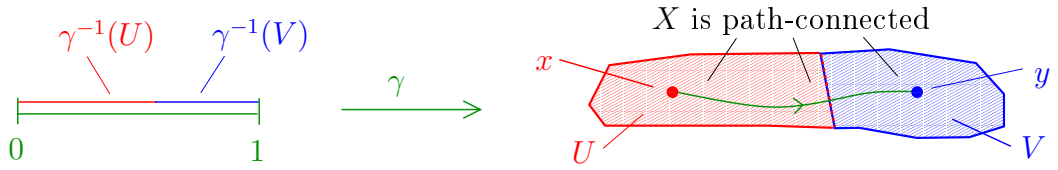
[41] Put differently,

$$A \;=\; \text{the set of all } t \in [0,1] \text{ such that the interval } [0,t] \text{ is contained in } U.$$

Remark.

(1) It follows from the Path-Connected Implies Connected Corollary 2.23 that in fact every interval is connected.
(2) In Exercise 2.18 we will prove the converse to (1), namely we will show that every connected subset of $\mathbb{R}$ is actually an interval.

Proof. Let X be a topological space that is path-connected. Suppose that X is not connected. This means that there exist disjoint open non-empty subsets U and V with $X = U \cup V$. Since U and V are non-empty we can find points $x \in U$ and $y \in V$. Since X is path-connected there exists a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Then $\gamma^{-1}(U) \cup \gamma^{-1}(V)$ is a decomposition of $[0,1]$ into two disjoint open non-empty subsets. But that is a contradiction to the Interval-Connectedness Proposition 2.22. ■

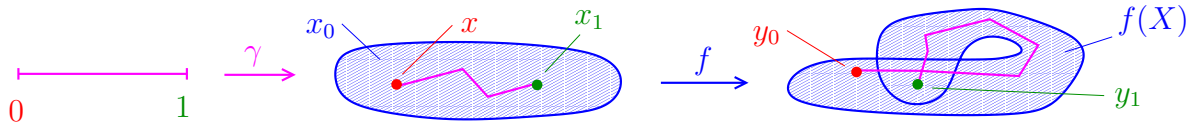


Lemma 2.24. (Image-Connected Lemma) Let $f\colon X \to Y$ be a map between topological spaces.

(1) If X is path-connected, then $f(X)$ is also path-connected.
(2) If X is connected, then $f(X)$ is also connected.

Proof.

(1) We suppose that X is path-connected. Let $y_0 = f(x_0)$ and $y_1 = f(x_1)$ be two points in $f(X)$. Since X is path-connected there exists a path $\gamma\colon [0,1] \to X$ from x_0 to x_1. But then $f \circ \gamma\colon [0,1] \to f(X)$ is a path from $f(x_0)$ to $f(x_1)$.



(2) We suppose that $f(Y)$ is disconnected. This means that there exist disjoint open non-empty subsets U and V of $f(Y)$ with $f(Y) = U \cup V$. By the Image-Preimage Lemma 0.2 we know that $f^{-1}(U)$ and $f^{-1}(V)$ are also disjoint and we know that $X = f^{-1}(U) \cup f^{-1}(V)$. By definition of a continuous map $f^{-1}(U)$ and $f^{-1}(V)$ are open. Finally, since $U, V \subset f(X)$ we see that $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty. Thus we have shown that X is also disconnected. ■

In the following lemma we show that most of the topological spaces that we had encountered so far are path-connected and connected. We will use this lemma on numerous occasions without explicitly referring to it.

Lemma 2.25. (Ball-Sphere-Connected Lemma)

(1) Let $n \geq 1$. Every open ball $B_r^n(y)$, every closed ball $\overline{B}_r^n(y)$ and $\mathbb{R}^n$ are path-connected and connected.

(2) Every sphere $S_r^n(y)$ with $n \geq 1$ is path-connected and connected.

Proof. By the Path-Connected Implies Connected Corollary 2.23 it suffices to show that all these topological spaces are path-connected.

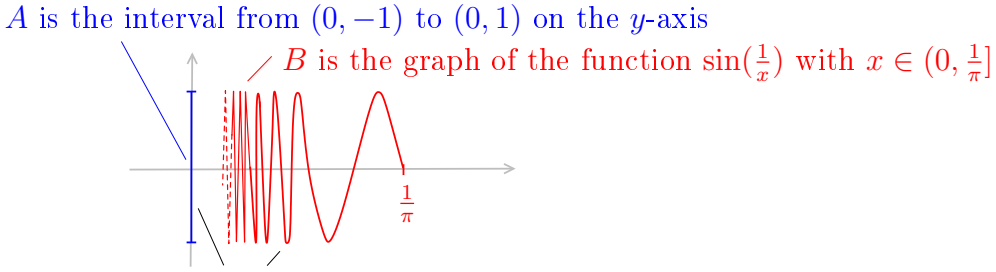
(1) It follows from the example on page 157 that all balls (open or closed) and $\mathbb{R}^n$ are convex. But as we already pointed out on page 161, convex subsets are basically by definition path-connected.
(2) We will show in Exercise 2.55 that spheres of dimension ≥ 1 are path-connected. ■

For better or worse, the converse to the Path-Connected Implies Connected Corollary 2.23 is not always correct.

Example. The *Topologist's Sine Curve* is defined as the following topological space

$$X \;:=\; \big\{(0,y)\,\big|\,y\in[-1,1]\big\} \cup \big\{(x,\sin(\tfrac{1}{x}))\,\big|\,x\in(0,\tfrac{1}{\pi}]\big\} \;\subset\; \mathbb{R}^2,$$

equipped with the subspace topology. In other words, the Topologist's Sine Curve is the union of the graph of $x \mapsto \sin(\frac{1}{x})$ for $x \in (0,\pi]$ and an interval on the y-axis. It is a very instructive exercise, see Exercise 2.48, to show that X is connected and it is an equally instructive exercise to show that X is *not* path-connected.



the Topologist's Sine Curve $X = A \cup B$ is connected but not path-connected

Remark. The fact that an example of a topological space that is connected but not path-connected is so "wild" indicates that perhaps for most "reasonable" spaces the two notions of connectedness and path-connectedness agree. In fact in basically all examples of topological spaces that we encounter throughout these notes the two notions of connectedness will agree.

By a clever use of the notion of connectedness one can sometimes prove statements which at first glance have nothing to do with connectedness. For example we can now prove the following lemma.

Lemma 2.26. Let $n \in \mathbb{N}_{\geq 2}$.

(1) The topological spaces $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic.
(2) The topological spaces $[0,1]$ and $[0,1]^n$ are not homeomorphic.

Remark.

(1) The statement of Lemma 2.26 is not overly surprising. But note that in Theorem 6.13 we will see that for any $n \in \mathbb{N}$ there exists a *surjective* map $[0,1] \to [0,1]^n$.
(2) Later on, in Proposition 74.6, we will study the question whether for $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ can be homeomorphic.

Proof.

(1) Let us suppose that there exists a homeomorphism $f\colon \mathbb{R} \to \mathbb{R}^n$. Let $P \in \mathbb{R}$ be a point. Then f restricts to a homeomorphism from $\mathbb{R} \setminus \{P\}$ to $\mathbb{R}^2 \setminus \{f(P)\}$. The topological space $\mathbb{R}\setminus\{P\} = (-\infty, P)\cup(P, \infty)$ is not path-connected[42], while one can easily verify that $\mathbb{R}^n \setminus \{f(P)\}$ is path-connected. But as we remarked on page 152, homeomorphic spaces have the same topological properties. In particular two homeomorphic spaces are either both path-connected or they are both not path-connected. Thus we have obtained a contradiction.

(2) The proof of this statement is fairly similar to the proof of (1). We leave it to the reader to fill in the details. ■

We continue with the following elementary but useful lemma.

Lemma 2.27. (Connected-to-Discrete Lemma) Every map $f\colon X \to Y$ from a connected space X to a discrete space is constant.

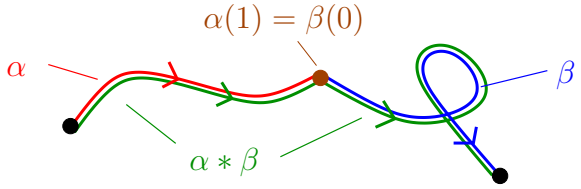
Remark. One could argue that the Connected-to-Discrete Lemma 2.27 is the lemma in topology that is most relevant to real life. More precisely, an immediate consequence of the Connected-to-Discrete Lemma 2.27 is that any *non-constant* map $(a, b) \to A$ from an interval to a finite set is not continuous. This is of huge significance since humanity likes to squeeze a real-parameter (e.g. time, intelligence, performance) into a finite set of values (e.g. historical stages, grades in school and exams, hired-or-fired). Any such function is non-continuous and hence very problematic.

Proof. We need to show that for any $x \in X$ we have $f(\{x\}) = f(X)$. Since Y is a discrete space we know that $\{f(x)\}$ and $Y \setminus \{f(x)\}$ are open subsets of Y. The preimages $U := f^{-1}(f(x))$ and $V := f^{-1}(Y \setminus \{f(x)\})$ are also open and they are evidently disjoint. Since U is non-empty and since X is connected we see that $V = \varnothing$. But this means that f is constant. ■

Before we continue we introduce the concatenation of two paths.

Definition. Let X be a topological space and let $\alpha\colon [0,1] \to X$ and $\beta\colon [0,1] \to X$ be two paths with $\alpha(1) = \beta(0)$. We define the concatenation of α and β as the path $\alpha * \beta$ which is given by[43]

$$\begin{array}{rcl} \alpha * \beta\colon [0,1] & \to & X \\ t & \mapsto & \begin{cases} \alpha(2t), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta(\frac{1}{2} + (2t-1)), & \text{if } t \in \left(\frac{1}{2}, 1\right]. \end{cases} \end{array}$$



The concatenation of paths will play a major role throughout these notes, especially when we introduce the fundamental group of a pointed topological space. In a more humble

[42] Why is this topological space not path-connected? How can one show this?

[43] It follows from the Pasting Proposition 2.6 (2) and our hypothesis $\alpha(1) = \beta(0)$ that $\alpha * \beta\colon [0,1] \to X$ is a continuous map.

fashion it also plays a minor role in the proof of the following lemma, which gives a criterion for the union of (path-) connected subsets to be (path-) connected.

Lemma 2.28. (Union Connected Lemma) Let X be a topological space and let $\{B_i\}_{i\in I}$ be a family of subsets such that the intersection $\bigcap_{i\in I} B_i$ is non-empty.

(1) If each B_i, $i \in I$ is path-connected, then $\bigcup_{i\in I} B_i$ is path-connected.

(2) If each B_i, $i \in I$ is connected, then $\bigcup_{i\in I} B_i$ is connected.

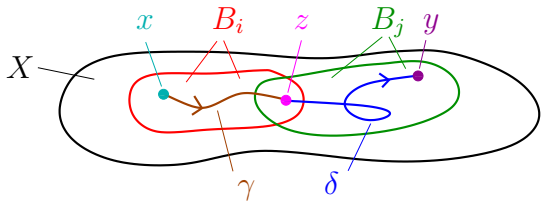
Remark. In Exercise 6.11 we will provide a partial converse to the Union Connected Lemma 2.28.

Proof. Let X be a topological space and let $\{B_i\}_{i\in I}$ be a family of subsets such that the intersection $\bigcap_{i\in I} B_i$ contains a point z. We write $Z := \bigcup_{i\in I} B_i$.

(1) We suppose that each B_i, $i \in I$ is path-connected. So let $x, y \in Z$. There exists an $i \in I$ with $x \in B_i$ and there exists a $j \in I$ with $y \in B_j$. Since B_i and B_j are path-connected we can pick a path $\gamma\colon [0,1] \to B_i$ from x to z and we can pick a path $\delta\colon [0,1] \to B_j$ from z to y. Note that the concatenation of the paths γ and δ is a path from x to y.

(2) We suppose that each B_i, $i \in I$ is connected. We write $Z = U \sqcup V$ where U and V are open subsets. Without loss of generality we can assume that $z \in U$. For each $i \in I$ we have a decomposition $B_i = (U \cap B_i) \sqcup (V \cap B_i)$ into disjoint open subsets (here we use the definition of the subspace topology). Since B_i is connected and since $U \cap B_i \neq \varnothing$ we see that $B_i = U \cap B_i$. Thus we have shown that

$$U = \bigcup_{i\in I}(B_i \cap U) = \bigcup_{i\in I} B_i = Z. \qquad \blacksquare$$



Sometimes it is also useful to have criteria which show that removing a subset does not change connectedness.

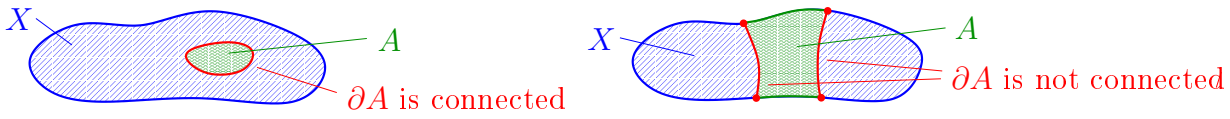
Lemma 2.29. (Complement Connected Lemma) Let X be a topological space and let $A \subset X$ be a subset.

(1) If X is path-connected and if ∂A is path-connected, then $X \setminus \mathring{A}$ is also path-connected.

(2) If X is connected and if ∂A is connected, then $X \setminus \mathring{A}$ is also connected.

Proof. We will prove the Complement Connected Lemma 2.29 in Exercise 2.50. $\blacksquare$

2.6. **The (path-) components of a topological space.** This section is a follow-up on the previous section. Let us start with the following simple-minded definition.



Definition. Let x and y be two points in a topological space X. We say that the points x and y are path-equivalent, if there exists a map $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Lemma 2.30. For every topological space X the notion of path-equivalence is indeed an equivalence relation.

Proof. All properties of an equivalence relation are trivial except possibly for transitivity. The fact that path-equivalence is transitive can be shown by concatenating paths. More precisely, if $\alpha\colon [0,1] \to X$ is a path from $P = \alpha(0)$ to $Q = \alpha(1)$ and if $\beta\colon [0,1] \to X$ is a path from $Q = \beta(0)$ to $R = \beta(1)$, then the concatenation $\alpha * \beta\colon [0,1] \to X$ is a path from P to R. ■

Definition. Let X be a topological space.

(1) We call the path-equivalence classes of points the path-components of X.
(2) We denote by $\pi_0(X)$ the set of path-components of X.[44] Furthermore we write $\pi_0(X) = 0$ if X is non-empty and if it consists of a single path-component.

Example. It is a straightforward consequence of the Intermediate Value Theorem that each point in $\mathbb{Q}$ is in fact already a path-component of $\mathbb{Q}$.

Lemma 2.31. (Path-Component Lemma) Let X be a topological space.

(1) As a set, X is the disjoint union of its path-components.
(2) Each path-component of X is path-connected.
(3) Each path-component C of X is a maximal path-connected subset, i.e. if $C \subsetneq D$, then D is not path-connected.
(4) Let A be a subset of X.
   (a) If no point in A is path-equivalent to a point in $X \setminus A$, then A is the union of path-components of X.
   (b) If A is path-connected, then A is contained in a path-component of X.
   (c) If A is path-connected and if no point in A is path-equivalent to a point in $X \setminus A$, then A is a path-component of X.

Proof. The lemma follows immediately from the definitions and basic properties of equivalence relations, see the Equivalence Class Lemma 0.4. We leave it to the reader to fill in the few remaining details. ■

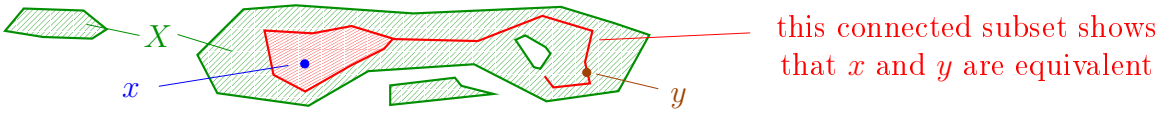
Example. We consider the topological space $\mathbb{R} \setminus \{0\}$. The subsets $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$ are evidently path-connected. It follows from the Intermediate Value Theorem that no point in $\mathbb{R}_{<0}$ is path-equivalent to a point in $\mathbb{R}_{>0}$. Thus it follows from the Path-Component Lemma 2.31 (4c) that $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$ are the path-components of $\mathbb{R} \setminus \{0\}$.

[44] The notation "$\pi_0(X)$" might look rather odd at the moment, but it will become clearer later on, when we introduce the fundamental groups and the higher homotopy groups, why this is a perfectly sensible notation. At that point the weird definition of "$\pi_0(X) = 0$" will start making more sense.

Our goal now is to define the "components" of a topological space that should play the same role as the path-components, i.e. we want to introduce the notion of "components" such that the statements of the Path-Component Lemma 2.31 hold, but with "path-connected" replaced by "connected" and "path-component" replaced by "component".

We start out with the following definition.

Definition. Let x and y be two points in a topological space X. We say that the points x and y are equivalent, if there exists a connected subset of X that contains both x and y.



Lemma 2.32. For every topological space X the above notion of equivalence is indeed an equivalence relation.

Proof. All properties of an equivalence relation are trivial except for transitivity. So suppose that x and y are equivalent and that y and z are equivalent. By definition there exists a connected subset A that contains x and y and there exists a connected subset B that contains y and z. By the Union Connected Lemma 2.28 (2) the union $A \cup B$ is connected and it thus follows that x and z are equivalent. ■

Definition. Let X be a topological space. We call the equivalence classes of points on X the components of X.

Examples.

(1) In Exercise 2.48 we will show that the Topologist's Sine Curve, that we illustrated in the figure on page 164, has two path-components but only a single component.
(2) We consider $X = \mathbb{Q}$ with the subspace topology coming from $\mathbb{R}$. In Exercise 2.23 we show that the only connected subsets of $\mathbb{Q}$ are the subsets consisting of at most a point. It follows that the components of $\mathbb{Q}$ are precisely the one-point subsets.

The following is in particular the promised analogue of the Path-Component Lemma 2.31.

Lemma 2.33. (Component Lemma) Let X be a topological space.

(1) As a set X is the disjoint union of its components.
(2) Each component of X is connected.
(3) Each component C of X is a maximal connected subset of X, i.e. if $C \subsetneq D$, then D is not connected.
(4) Let A be a subset of X.
    (a) If C is a component of X, then $A \cap C$ is a union of components of A.
    (b) If A is closed and open in X, then A is the union of components of X.
    (c) If A is closed and open in X and if A is connected, then A is a component of X.
(5) Each component of X is the union of path-components of X.

Example. We consider the topological space $\mathbb{R} \setminus \{0\}$. The subsets $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$ are evidently path-connected. By the Path-Connected Implies Connected Corollary 2.23 we know that $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$ are also connected. Since these two subsets are closed and open

in $\mathbb{R} \setminus \{0\}$ we deduce from the Component Lemma 2.33 (4c) that the two subsets are the components of $\mathbb{R} \setminus \{0\}$.

Proof. Let X be a topological space.

(1) This statement is clear since components are defined to be equivalence classes, see also the Equivalence Class Lemma 0.4.

(2) Now let C be a component of X. We pick $c \in C$. For each $d \in C$ there exists by definition a connected subset Y_d of X that contains c and d. We have

$$C = \bigcup_{c \in C} \{c\} \subset \bigcup_{c \in C} Y_c \underset{\uparrow}{\subset} C \quad \text{which implies that} \quad C = \bigcup_{c \in C} Y_c.$$

by definition all points in Y_d are equivalent to c, thus we have $Y_d \subset C$

It follows from the Union Connected Lemma 2.28 (2) that C is connected.

(3) This statement follows immediately from the definitions.

(4) Let A be a subset of X.
   (a) Let C be a component of X. Note that if two points of A are equivalent in A, then they are also equivalent in X. This implies that no point in $A \cap C$ is equivalent to a point of $A \setminus (A \cap C)$. The statement now follows from (1) applied to A.
   (b) Suppose that A is closed and open. Let C be a component of X. Note that $A \cap C$ is a subset of C that is closed and open. By (2) we know that C is connected, hence we see that either $A \cap C = \varnothing$ or $A \cap C = C$. This shows that A is precisely the union of all the components of X that A intersects.
   (c) This statement follows immediately from (b).

(5) Let C be a component of X and let P be a component of X with $P \cap C \neq \varnothing$. We need to show that $P \subset X$. We do so as follows:
   (a) Note that P is connected by the Path-Connected Implies Connected Corollary 2.23.
   (b) It follows from (a) and the Union Connected Lemma 2.28 that $P \cup C$ is connected.
   (c) It follows from (b) and (3) that $P \cup C \subset C$. This of course implies that $P \subset C$. ■

We continue our discussion of the components of a topological space with the following, possibly slightly counterintuitive, lemma.

Lemma 2.34. (Components-are-Closed Lemma) Let X be a topological space.

(1) Every component of X is a closed subset of X.

(2) If X has only finitely many components, then each component is also an open subset of X.[45]

Example. We consider $X = \mathbb{Q}$. As we mentioned on page 168, in this case every point is already a component. We see that every component is closed, but in this case no component is an open subset. Therefore we cannot drop the condition on the number of components in the second statement of the lemma.

[45] In the Locally Path-Connected Lemma 2.40 we give a criterion which often can be used to show that components are in fact open.

Proof. Let X be a topological space.

(1) Let C be a component of X. We need to show that $X \setminus C$ is open. By the JH-Lemma 1.7 it suffices to show that every $P \in X \setminus C$ admits a neighborhood that does not intersect C. Thus let $P \in X \setminus C$. By the Component Lemma 2.33 (3) we know that $C \cup \{P\}$ is not connected. Therefore there exists a decomposition $C \cup \{P\} = U \sqcup V$ with U and V open subsets such that both U and V are non-empty. Without loss of generality $C \cap U \neq \varnothing$. Since C is connected and since $C = (U \cap C) \sqcup (V \cap C)$ we see that $C = U \cap C$. Since V is also non-empty we see that $V = \{P\}$. By the definition of the subspace topology that means that there exists an open subset W of X such that $W \cap (C \cup \{P\}) = V = \{P\}$. But this shows that W is an open neighborhood of P that does not intersect C. We have thus shown that the complement of C is open.



(2) Now suppose that X has only finitely many components. Let C be a component of X. We already know from (1) that C is closed. By definition the complement of C is the union of all other components. Since there are only finitely many and since they are closed by (1) it follows that the complement of C is closed, i.e. C is open. ■

Remark. Let X be a topological space. We say that the points x and y are *quasi-equivalent*, if the following holds

$$\text{if } X = U \sqcup V \text{ is the disjoint union with } U, V \text{ open } \implies \text{ either } x, y \in U \text{ or } x, y \in V.$$

It is straightforward to see that this indeed an equivalence relation on X. The equivalence classes of this equivalence relation are called *quasi-components*. It follows immediately from the definitions that components are contained in quasi-components. But the notions are different, for example it is shown in [**SS1978**, p. 137] that there exist topological spaces such that quasi-components are not connected. We will not make use of the concept of quasi-equivalence and quasi-components.

We conclude this section with the following lemma that can be viewed as a generalization of the Image-Connected Lemma 2.24 and Lemmas 2.27.

Lemma 2.35. Let $f\colon X \to Y$ be a map between two topological spaces.

(1) Let C be a (path)-component C of Y. The preimage $f^{-1}(C)$ is the union of (path)-components of X.
(2) Let B be a (path)-component B of X. There exists a (path)-component C of Y with $f(B) \subset C$.

Proof.

(1) We only need to show that if a point $a \in f^{-1}(C)$ is (path-) equivalent to a point $b \in X$, then a and b are (path-) equivalent. If a and b are (path-) equivalent, then as in the Image-Connected Lemma 2.24 we see that $f(a)$ and $f(b)$ are (path-) equivalent, thus $f(a)$ and $f(b)$ both lie in C. Therefore a and b both lie in $f^{-1}(C)$.
(2) This statement is proved by the same elementary argument as in (1). ■

2.7. **The (path-) components of matrix groups.** In this section we will discuss the (path-) connectedness of certain matrix groups over $\mathbb{R}$ and over $\mathbb{C}$. First we introduce the main players.

Notation. Given a commutative ring R we write

$$\begin{aligned} \mathrm{GL}(n,R) &:= \{A \in \mathrm{M}(n\times n,R) \,|\, \det(A) \text{ a multiplicative unit in } R\} \\ \mathrm{SL}(n,R) &:= \{A \in \mathrm{M}(n\times n,R) \,|\, \det(A) = 1\}. \end{aligned}$$

For $\mathbb{K} = \mathbb{R}$ we have the following extra definitions:

$$\begin{aligned} \mathrm{GL}_+(n,\mathbb{R}) &:= \{A \in \mathrm{M}(n\times n,\mathbb{R}) \,|\, \det(A) > 0\} \\ \mathrm{GL}_-(n,\mathbb{R}) &:= \{A \in \mathrm{M}(n\times n,\mathbb{R}) \,|\, \det(A) < 0\} \\ \mathrm{O}(n) &:= \{A \in \mathrm{M}(n\times n,\mathbb{R}) \,|\, A^TA = \mathrm{id}\} \\ \mathrm{SO}(n) &:= \mathrm{O}(n) \cap \mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{M}(n\times n,\mathbb{R}) \,|\, A^TA = \mathrm{id} \text{ and } \det(A) = 1\}. \end{aligned}$$

Furthermore for $\mathbb{K} = \mathbb{C}$ we have the following extra definitions:

$$\begin{aligned} \mathrm{U}(n) &:= \{A \in \mathrm{M}(n\times n,\mathbb{C}) \,|\, \overline{A}^TA = \mathrm{id}\} \\ \mathrm{SU}(n) &:= \mathrm{U}(n) \cap \mathrm{SL}(n,\mathbb{C}) = \{A \in \mathrm{M}(n\times n,\mathbb{C}) \,|\, \overline{A}^TA = \mathrm{id} \text{ and } \det(A) = 1\}. \end{aligned}$$

The following convention explains in what sense these matrix groups are topological spaces.

Convention. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Given $m, n \in \mathbb{N}$ we make the identification

$$\mathrm{M}(m\times n,\mathbb{K}) = \mathbb{K}^{m\cdot n}$$

by "juxtaposing" the rows of the matrix. We equip $\mathbb{K}^{m\cdot n}$ and thus also $\mathrm{M}(m\times n,\mathbb{K})$ with the standard topology that we introduced on page 109. We equip each subset of $\mathrm{M}(m\times n,\mathbb{K})$ with the corresponding subspace topology that we introduced on page 112.

Lemma 2.36. (Matrix Group Lemma) Let $n \in \mathbb{N}$.

(1) $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}_+(n,\mathbb{R})$ and $\mathrm{GL}_-(n,\mathbb{R})$, are open subsets of $\mathrm{M}(n\times n,\mathbb{R})$.
(2) $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{O}(n)$ and $\mathrm{SO}(n)$, are closed subsets of $\mathrm{M}(n\times n,\mathbb{R})$.
(3) $\mathrm{O}(n)$ is compact.
(4) $\mathrm{GL}(n,\mathbb{C})$ is an open subset of $\mathrm{M}(n\times n,\mathbb{C})$.
(5) $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$, are closed subsets of $\mathrm{M}(n\times n,\mathbb{C})$.
(6) $\mathrm{U}(n)$ is compact.

Remark. We will generalize the Matrix Group Lemma 2.36 (1) in Exercise 2.41.

Proof. We consider the following two maps

$$\det\colon \begin{array}{ccc} \mathrm{M}(n\times n,\mathbb{R}) & \to & \mathbb{R} \\ A & \mapsto & \det(A) \end{array} \quad \text{and} \quad \varphi\colon \begin{array}{ccc} \mathrm{M}(n\times n,\mathbb{R}) & \to & \mathbb{R} \\ A & \mapsto & A^T\cdot A. \end{array}$$

It follows from the Leibniz formula and the rather elementary Lemmas 2.2 and 2.3 that both maps are continuous.

(1) It follows from the above that $\mathrm{GL}(n,\mathbb{R}) = \det^{-1}(\mathbb{R}\setminus\{0\})$, $\mathrm{GL}_+(n,\mathbb{R}) = \det^{-1}((0,\infty))$ and $\mathrm{GL}_-(n,\mathbb{R}) = \det^{-1}((-\infty,0))$ are open subset of $\mathrm{M}(n\times n,\mathbb{R})$.
(2) It follows from the above and the Continuity-via-Closed Subsets Lemma 2.4 that $\mathrm{SL}(n,\mathbb{R}) = \det^{-1}(\{1\})$, $\mathrm{O}(n) = \varphi^{-1}(\{\mathrm{id}\})$ and $\mathrm{SO}(n) = \mathrm{SL}(n,\mathbb{R}) \cap \mathrm{O}(n,\mathbb{R})$ are closed subsets of $\mathrm{M}(n\times n,\mathbb{R})$.

(3) Note that each column of an orthogonal matrix has norm one. This implies that $\mathrm{O}(n) \subset \mathbb{R}^{n^2}$ is a bounded subset of $\mathrm{M}(n \times n, \mathbb{R}) = \mathbb{R}^{n^2}$. Thus it follows from the Heine-Borel Theorem 1.24 that $\mathrm{O}(n)$ is compact.

The proofs of Statements (4), (5) and (6) are almost identical to the above proofs. ■

The following proposition will hopefully quench the reader's thirst for information on the (path-) connectedness of the matrix groups.

Proposition 2.37. (Matrix Group Path-Component Proposition) Let $n \in \mathbb{N}$.

	topological space	path-components
(1)	$\mathrm{GL}(n, \mathbb{R})$	path-components are $\mathrm{GL}_+(n, \mathbb{R})$ and $\mathrm{GL}_-(n, \mathbb{R})$
(2)	$\mathrm{SL}(n, \mathbb{R})$	path-connected
(3)	$\mathrm{O}(n)$	path-components are $\mathrm{SO}(n)$ and $\mathrm{O}(n) \cap \mathrm{GL}_-(n, \mathbb{R})$
(4)	$\mathrm{GL}(n, \mathbb{C})$	path-connected
(5)	$\mathrm{SL}(n, \mathbb{C})$	path-connected
(6)	$\mathrm{U}(n)$	path-connected
(7)	$\mathrm{SU}(n)$	path-connected.

Furthermore the path-components are also the components.

Remark.

(1) In Exercise 156.12 we will provide a somewhat different proof of some of the statements of the Matrix Group Path-Component Proposition 2.37. An alternative proof of the Matrix Group Path-Component Proposition 2.37 is also given in [**Bak2002**, Chapter 9.2].

(2) In the Matrix Groups (Smooth) Deformation Retract Propositions 15.7 and 33.10 we collect more information on the relationships between the topological spaces that are given by the various matrix groups.

In the proof of the Matrix Group Path-Component Proposition 2.37 we will make use of the Gram-Schmidt Orthogonalization Process which we now recall.

Lemma 2.38. (Gram-Schmidt Orthogonalization Process) Let V be a real vector space and let $\langle\, , \rangle \colon V \times V \to \mathbb{R}$ be a positive-definite symmetric $\mathbb{R}$-bilinear[46] form. Given $v \in V$ we write $\|v\| := \sqrt{\langle v, v \rangle}$. Let $v_1, \ldots, v_n$ be a basis of V. We consider the following sequence of n-tuples of vectors in V:[47]

$$
\begin{aligned}
&u_{0,1}, .., u_{0,n} := v_1, .., v_n \\
&u_{1,1}, .., u_{1,n} := \frac{v_1}{\|v_1\|}, \|v_1\| \cdot v_2, v_2, .., v_n \\
&\qquad\vdots \qquad\qquad\qquad \vdots \\
&u_{i,1}, .., u_{i,n} := u_{i-1,1}, .., u_{i-1,i-1}, \frac{z}{\|z\|}, \|z\| \cdot v_{i+1}, v_{i+2}, .., v_n \text{ with } z := v_i - \sum_{j=1}^{i-1} \langle v_i, u_{i-1,j} \rangle \cdot u_{i-1,j} \\
&\qquad\vdots \qquad\qquad\qquad \vdots \\
&u_{n,1}, .., u_{n,n} := u_{n-1,1}, .., u_{n-1,n-1}, \frac{z}{\|z\|} \qquad\qquad \text{with } z := v_n - \sum_{j=1}^{n-1} \langle v_n, u_{n-1,j} \rangle \cdot u_{n-1,j}.
\end{aligned}
$$

This sequence has the following properties:

(1) For each $i \in \{1, \ldots, n\}$ the vectors $\{u_{i,1}, \ldots, u_{i,n}\}$ form a basis of V.

(2) For each $i \in \{1, \ldots, n-1\}$ the base change matrix from the basis $v_1, \ldots, v_n$ to the basis $u_{i,1}, \ldots, u_{i,n}$ has determinant 1.
(3) The base change matrix from the basis $v_1, \ldots, v_n$ to the basis $u_{n,1}, \ldots, u_{n,n}$ has positive determinant.
(4) For each $i \in \{1, \ldots, n\}$ the vectors $u_{i,1}, \ldots, u_{i,i}$ are orthonormal.
(5) If $v_1, \ldots, v_n$ are orthonormal, then the sequence of n-tuples of vectors is constant, i.e. for any $i, j \in \{1, \ldots, n\}$ we have $u_{i,j} = v_j$.

The analogous statement also holds if we are dealing with a positive-definite hermitian $\mathbb{C}$-sesquilinear form over a complex vector space.

Proof. The lemma follows from elementary arguments which we leave to the readers who suffer from linear algebra withdrawal syndrome. ■

Now we turn to the actual proof of the Matrix Group Path-Component Proposition 2.37. In fact we break the proof of the Matrix Group Path-Component Proposition 2.37 into two parts. First we consider the matrix groups defined over $\mathbb{R}$, afterwards we consider the matrix groups defined over $\mathbb{C}$.

Proof for the Matrix Group Path-Component Proposition 2.37 (1), (2) and (3). The proof of the three statements rests mostly on the following three claims.

Claim 1. Given any matrix $A \in \mathrm{GL}_+(n, \mathbb{R})$ there exists a path $\alpha\colon [0,1] \to \mathrm{GL}_+(n, \mathbb{R})$ with $\alpha(0) = A$ and with $\alpha(1) \in \mathrm{SL}(n, \mathbb{R})$.

Proof. Let $A \in \mathrm{GL}_+(n, \mathbb{R})$. Since $\det(A) > 0$ we can consider the path that is given by $\alpha(t) = \det(A)^{-t/n} \cdot A$. Evidently $\alpha(0) = A$ and one can easily verify that $\det(\alpha(1)) = 1$ ⊞

Claim 2. Given any $A \in \mathrm{SL}(n, \mathbb{R})$ there exists a path $\beta\colon [0,1] \to \mathrm{SL}(n, \mathbb{R})$ with $\beta(0) = A$ and $\beta(1) \in \mathrm{SO}(n)$.

Proof. We apply the Gram-Schmidt Orthonormalization Process 2.38 to the column vectors $v_1, \ldots, v_n$ of the matrix $A \in \mathrm{SL}(n, \mathbb{R})$. For $i = 0, \ldots, n$ we obtain sets of vectors $u_{i,1}, \ldots, u_{i,n} \in \mathbb{R}^n$. For $i = 0, \ldots, n$ we write $A_i := (u_{i,1} \ \ldots \ u_{i,n}) \in \mathrm{M}(n \times n, \mathbb{R})$. By construction $A_0 = A$. Note that it follows from Properties (1), (2), (3) and (4) stated in the Gram-Schmidt Orthonormalization Process, together with the fact that the determinant of an orthogonal matrix is ± 1, that each matrix A_i lies in $\mathrm{SL}(n, \mathbb{R})$ and that $A_n \in \mathrm{SO}(n)$.

Let $i \in \{1, \ldots, n\}$. We will now show that there exists path in $\mathrm{SL}(n, \mathbb{R})$ from A_{i-1} to A_i. To simplify the notation we write $u_1, \ldots, u_n$ instead of $u_{i-1,1}, \ldots, u_{i-1,n}$. First we consider the path

$$\begin{aligned} \varphi\colon [0,1] &\to \mathrm{SL}(n, \mathbb{R}) \\ t &\mapsto \left(u_1 \ \ldots \ u_{i-1} \ \ \frac{u_i}{\lambda(t)} \ \ \lambda(t) \cdot u_{i+1} \ \ u_{i+2} \ \ldots \ u_n\right) \text{ where } \lambda(t) = (1-t) + t \cdot \|u_i\|. \end{aligned}$$

[46] Recall that according, to the definition that we gave on page 99, we say that a symmetric $\mathbb{R}$-bilinear form $\langle -, - \rangle\colon V \times V \to \mathbb{R}$ is positive-definite if for every $v \neq 0$ we have $\langle v, v \rangle > 0$.

[47] Our formulation is more complicated than the standard formulation of the Gram-Schmidt Orthogonalization Process. The reason is that we want to have control over the base change matrices. This is important when we show, in the proof of the Matrix Group Path-Component Proposition 2.37 (2), that $\mathrm{SL}(n, \mathbb{R})$ is path-connected.

We write $\widetilde{u}_i := \frac{u_i}{\|u_i\|}$ and $\widetilde{u}_{i+1} := \|u_i\| \cdot u_{i+1}$. Next we consider the path

$$\begin{array}{rcl} \psi\colon [0,1] & \to & \mathrm{SL}(n,\mathbb{R}) \\ t & \mapsto & \left(u_1 \ \dots \ u_{i-1} \ \ \frac{z(t)}{\|z(t)\|} \ \ \|z(t)\| \cdot \widetilde{u}_{i+1} \ \dots \ u_n\right) \text{ where } z(t) = \widetilde{u}_i - t \cdot \sum_{j=1}^{i-1} \langle \widetilde{u}_i, u_j \rangle \cdot u_j. \end{array}$$

Note that $\varphi(0) = A_{i-1}$ and note that $\varphi(1) = \psi(0)$. A short moment's thought shows that $\psi(1) = A_n$. Thus the concatenation of the paths φ and ψ is a path in $\mathrm{SL}(n,\mathbb{R})$ from A_{i-1} to A_i. The concatenation of these finitely many paths is the desired path $\beta\colon [0,1] \to \mathrm{SL}(n,\mathbb{R})$ from A to the orthogonal matrix $A_n \in \mathrm{SO}(n)$. ⊞

Claim 3. Given any $A \in \mathrm{SO}(n)$ there exists a path $\gamma\colon [0,1] \to \mathrm{SO}(n)$ with $\gamma(0) = A$ and $\gamma(1) = \mathrm{id}_n$.

Proof. We prove this claim by induction on n. The case $n = 1$ is clear since $\mathrm{SO}(1)$ consists of precisely the matrix $(1) = \mathrm{id}_1$. Now suppose that the statement holds for some $n \geq 1$. Let $A \in \mathrm{SO}(n+1)$. We denote by $u \in \mathbb{R}^{n+1}$ the first column of the matrix A. We write $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$. Since $n + 1 \geq 2$ there exists a vector $v \in \mathbb{R}^{n+1}$ with $\|v\| = 1$, such that $u \perp v$ and such that $e_1 \in \mathrm{span}\{u, v\}$. We set $U := \mathrm{span}\{u, v\}$. Since u and v are orthonormal there exists an $\alpha \in \mathbb{R}$ such that $e_1 = \cos(\alpha) \cdot u + \sin(\alpha) \cdot v$. We extend u, v to an orthonormal basis $u, v, w_1, \dots, w_{n-1}$. Given $t \in [0,1]$ we consider the matrix

$$P(t) \ := \ Q \cdot \underbrace{\begin{pmatrix} \cos(\alpha \cdot t) & -\sin(\alpha \cdot t) & 0 \\ \sin(\alpha \cdot t) & \cos(\alpha \cdot t) & 0 \\ 0 & 0 & \mathrm{id}_{n-1} \end{pmatrix}}_{\in \mathrm{O}(n+1)} \cdot Q^{-1} \qquad \text{where } Q = \underbrace{(u \ v \ w_1 \ \dots \ w_{n-1})}_{\in \mathrm{O}(n+1)}.$$

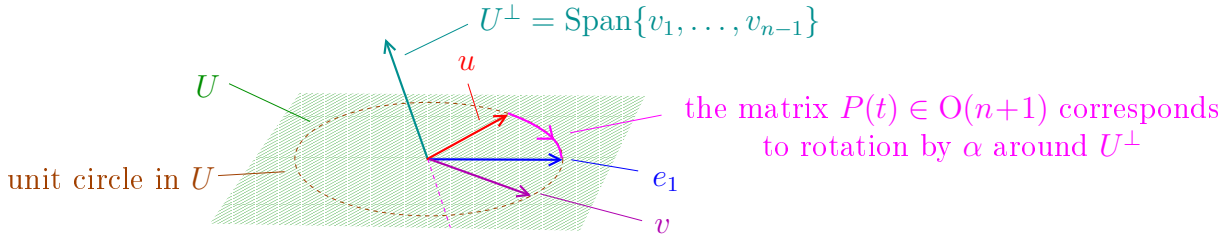
Next we consider the path

$$\begin{array}{rcl} \gamma\colon [0,1] & \to & \mathrm{SO}(n+1) \\ t & \mapsto & P(t) \cdot A. \end{array}$$

Evidently $\gamma(0) = A$. An elementary calculation shows that the first column of the matrix $\gamma(1) \in \mathrm{SO}(n+1)$ equals e_1. Since $\gamma(1)$ is orthogonal this implies that there exists an $\widetilde{A} \in \mathrm{SO}(n)$ with

$$\gamma(1) \ = \ \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{A} \end{pmatrix}.$$

By induction hypothesis there exists a path $\widetilde{\delta}\colon [0,1] \to \mathrm{SO}(n)$ from $\widetilde{A}$ to id_n. By "adding a 1 at the top left" this path $\widetilde{\delta}$ gives rise to a path δ in $\mathrm{SO}(n+)$ from $\gamma(1)$ to id_{n+1}. The concatenation of the two paths γ and δ is the promised path in $\mathrm{SO}(n+1)$ from A to id_{n+1}. ⊞



Now we turn to the actual proofs of the statements (1), (2) and (3) of the Matrix Group Path-Component Proposition 2.37.

(1) It follows from Claims 1, 2 and 3, together with the fact that path-equivalence is an equivalence relation, that $\mathrm{GL}_+(n,\mathbb{R})$ is path-connected. Next note that multiplication by a matrix in $\mathrm{GL}(n,\mathbb{R})$ with negative determinant gives rise to a homeomorphism $\mathrm{GL}_+(n,\mathbb{R}) \to \mathrm{GL}_-(n,\mathbb{R})$. Thus it follows that $\mathrm{GL}_-(n,\mathbb{R})$ is also path-connected.

Next we consider the map $f\colon \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}\setminus\{0\}$ that is given by the determinant. On page 167 we saw that $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$ are the path-components of $\mathbb{R}\setminus\{0\}$. It follows from the above together with Lemma 2.35 that $\mathrm{GL}_-(n,\mathbb{R}) = f^{-1}(\mathbb{R}_{<0})$ and $\mathrm{GL}_+(n,\mathbb{R}) = f^{-1}(\mathbb{R}_{>0})$ are the path-components of $\mathrm{GL}(n,\mathbb{R})$.

This discussion, together with the Path-Connected Implies Connected Corollary 2.23 and the Component Lemma 2.33 shows that $\mathrm{GL}_-(n,\mathbb{R})$ and $\mathrm{GL}_+(n,\mathbb{R})$ are also the components of $\mathrm{GL}(n,\mathbb{R})$.

(2) It follows from Claims 2 and 3 that $\mathrm{SL}(n,\mathbb{R})$ is path-connected. By the Path-Connected Implies Connected Corollary 2.23 this implies that $\mathrm{SL}(n,\mathbb{R})$ is connected.

(3) It follows from Claim 3 that $\mathrm{SO}(n,\mathbb{R})$ is path-connected. By basically the same argument as in (1) we see that $\mathrm{O}(n)$ has precisely two path-components, namely $\mathrm{SO}(n)$ and $\mathrm{O}(n)\cap\mathrm{GL}_-(n,\mathbb{R})$, and by the same argument as above one sees that these are also the components of $\mathrm{O}(n)$. ■

Now we turn to the proof of Proposition 2.37 for the matrix groups that are defined over $\mathbb{C}$.

Proof of Proposition 2.37 (4), (5), (6) and (7).

Claim 4.

(1) Given any $A \in \mathrm{GL}(n,\mathbb{C})$ there exists a path $\delta\colon [0,1] \to \mathrm{GL}(n,\mathbb{C})$ with $\gamma(0) = A$ and $\gamma(1) \in \mathrm{SL}(n,\mathbb{C})$.

(2) If $A \in \mathrm{U}(n)$, then we can find such a path that takes values in $\mathrm{U}(n)$.

Proof. Let $A \in \mathrm{GL}(n,\mathbb{C})$. We pick $r \in \mathbb{R}_{>0}$ and $\varphi \in \mathbb{R}$ with $\det(A) = r\cdot\exp(\mathrm{i}\varphi)$. The map

$$\begin{array}{rcl} \delta\colon [0,1] & \to & \mathrm{GL}(n,\mathbb{C}) \\ t & \mapsto & r^{-t/n}\cdot\exp(-\mathrm{i}\varphi\cdot t/n)\cdot A \end{array}$$

is a path from A to a matrix in $\mathrm{SL}(n,\mathbb{C})$. If $A \in \mathrm{U}(n)$, then $r = 1$ and we see that the path takes values in $\mathrm{U}(n)$. ⊞

Using Claim 4 one sees that it suffices to show that $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{SU}(n)$ are connected. This is achieved by making straightforward modifications to the Claims 2 and 3 that we considered in the proof of the real case. We leave it to the reader to fill in the details. ■

2.8. Local and regional properties.

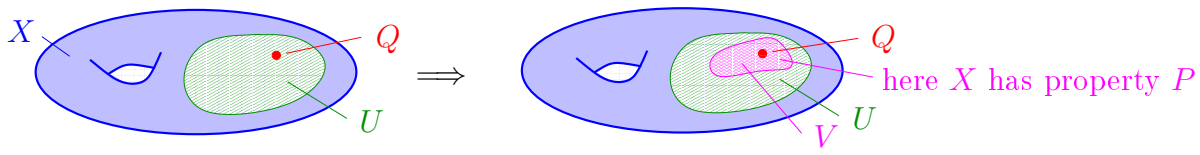
As so often, let us start out with a definition.

Definition. Let P be a property of topological spaces and let X be a topological space.

(1) We say X is regionally P if given any $Q \in X$ and any neighborhood U of Q there exists a neighborhood V of Q that is contained in U that has the property P.

(2) We say X is locally P if given any $Q \in X$ and any neighborhood U of Q there exists an *open* neighborhood V of Q that is contained in U that has the property P.

Note that by definition a topological space that is locally P is also regionally P.



Warning. Our usage of the expression "locally P" is non-standard. In fact most textbooks, see e.g. **[Hat2002, Bre1993, Rot1988]**, define "locally P" as what we call "regionally P". (Note though that, as is apparent from **[Mun1975**, p. 161], in **[Mun1975]** our notion of "locally P" is being used.) The property that we now call "locally P" does not seem to have an established name. For our purpose it is useful to give names to both concepts since both have their uses and justifications. Later on we will also introduce notion of a local property of a map, in this context most books implicitly or explicitly seem to work with open neighborhoods. Thus it seems reasonable to us to use the word "local" whenever we deal with open neighborhoods and to rename the concept when any neighborhood is allowed.

Examples.

(1) Every open subset of $\mathbb{R}^n$ is locally path-connected. Indeed, suppose A is an open subset of $\mathbb{R}^n$. Let $Q \in A$ and let U be a neighborhood of Q in A. Then there exists an $\epsilon > 0$ such that $B^n_\epsilon(x) \subset U$. The ϵ-ball $B^n_\epsilon(x)$ is of course path-connected.
(2) The topological space $X = (-2, 1) \cup [3, 6] \subset \mathbb{R}$ with the usual topology is not path-connected but it is locally path-connected.[48] On the other hand $X = \mathbb{Q}$ is not even locally path-connected.
(3) The line with two zeros, i.e. the topological space $\mathbb{R} \cup \{*\}$ with the topology from page 122, is not Hausdorff, but one can easily show that $\mathbb{R} \cup \{*\}$ is locally Hausdorff.

The following lemma can at times be useful.

Lemma 2.39. Let P be a property of topological spaces and let X be a topological space. If X is locally P, then the open subsets of X that have P form a basis for the topology of X.

Proof. This lemma follows immediately from the Basis Characterization Lemma 1.32 (1). ■

The following lemma is the key to showing that for "reasonable" topological spaces the notions of being connected and being path-connected are actually the same.

Lemma 2.40. (Locally Path-Connected Lemma) Let X be a topological space.

(1) If X is locally path-connected, then the following statements hold:
  (a) X is connected if and only if X is path-connected.
  (b) The components of X are precisely the path-components of X.
(2) If X is locally path-connected, then it is also locally connected.
(3) If X is locally connected, then every component of X is a closed and an open subset of X.[49]

[48] Why is that?

[49] Recall that on page 168 we saw that for the topological space $X = \mathbb{Q}$ every point defines a component. Thus in general components are, very annoyingly, not necessarily open.

Proof.

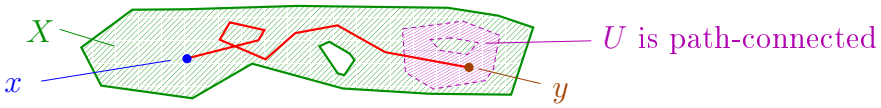
(1) First note that if X is path-connected, then we know by the Path-Connected Implies Connected Corollary 2.23 that X is connected. This gives us the cheap "if"-direction of (a).

We now assume that X is locally path-connected. Note that the "only if"-direction of (a) is a special case of (b). Thus it remains to prove (b). We start out with the following claim.

Claim 1. Each path-component is open.

Proof. Let P be a path-component. We pick $x \in X$. To show that P is open it suffices by the JH-Lemma 1.7 to show that each $y \in P$ admits a neighborhood that is contained in P.

Thus let $y \in P$. We pick a path $p\colon [0,1] \to X$ with $p(0) = x$ and $p(1) = y$. Since X is locally path-connected there exists an open neighborhood U of y that is path-connected. It follows from the Image-Connected Lemma 2.24 (1) together with the Union Connected Lemma 2.28 that $p([0,1]) \cup U$ is path-connected. It follows from the Path-Component Lemma 2.31 (4b) that $p([0,1]) \cup U \subset P$. Finally note that since U is open we know that $p([0,1]) \cup U$ is a neighborhood of x. ⊞



Claim 2. Each path-component is closed.

Proof. By Claim 1 we now know that each path-component is open. Since the complement of a path-component is the union of path-components and since the union of open sets is open we see that each path-component is also closed. ⊞

Now let C be a component of X. By the Component Lemma 2.33 (2) and (5) we know that C is connected and that it is the union of a family of path-component $\{P_i\}_{i\in I}$. By Claim 1 and Claim 2 we know that each P_i is open and connected. But since C is connected this implies that I consists of a single element, i.e. C is a path-component of X.

(2) This statement follows from the Path-Connected Implies Connected Corollary 2.23.

(3) Let C be a component of X. By the Components-are-Closed Lemma 2.34 we know that C is a closed subset of X. We now assume that X is locally connected. We want to show that C is an open subset of X. By the Neighborhood Openness Criterion 1.5 it suffices to show that each $x \in C$ admits an open neighborhood that is contained in C. Thus let $x \in C$. Since X is locally connected we see that there exists an open neighborhood U of $x \in X$ such that U is connected. It follows immediately from the definition of a component, see page 168, that every point in U is also contained in x. ■

In the following we discuss the notion of being regionally compact.

Examples.

(1) Every open subset of $\mathbb{R}^n$ is regionally compact. Indeed, suppose A is an open subset of $\mathbb{R}^n$. Let $Q \in A$ and let U be a neighborhood of Q in A. Then there exists an

$\epsilon > 0$ such that $B_\epsilon^n(x) \subset U$. The closed ϵ-ball $\overline{B}_{\frac{\epsilon}{2}}^n(x)$ is a neighborhood of x and it is compact by the Heine-Borel Theorem 1.24.

(2) Similar to (1) one can easily show that every interval is regionally compact. In fact later on we will mostly need the fact that the interval $[0, 1]$ is regionally compact.

In Exercise 2.58 we will see that there exists a topological space that is compact but not regionally compact. The following lemma shows that this slightly unintuitive phenomenon cannot occur if we restrict ourselves to topological spaces that are Hausdorff.

Lemma 2.41. Let X be a Hausdorff space.

(1) If X is compact, then X is also regionally compact.

(2) If given any point $P \in X$ there exists a compact neighborhood of P, then X is regionally compact.

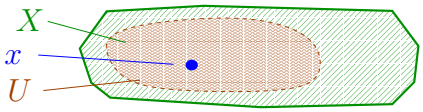
Proof.

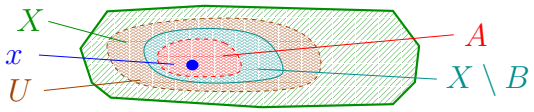
(1) Let X be a compact Hausdorff space. We need to show that X is regionally compact. Thus let $x \in X$ and let U be an neighborhood of x.

Claim. There exists an open neighborhood A of x with $\overline{A} \subset U$.

Proof. Note that it follows from the hypothesis that X is Hausdorff and the Points-in-Hausdorff Space Lemma 1.15 that $\{x\}$ is a closed subset of X. Furthermore note that by the Compact Hausdorff Spaces-Normal Lemma 1.16 we know that the compact Hausdorff X is normal. Thus we can separate the closed subsets $\{x\}$ and $X \setminus \mathring{U}$ by open neighborhoods A and B. More precisely, there exist disjoint open neighborhoods A of $\{x\}$ and B of $X \setminus \mathring{U}$.

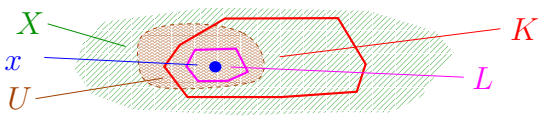
It remains to show that $\overline{A} \subset U$. Note that $A \subset X \setminus B \subset U$. Since B is open we know that $X \setminus B$ is closed. By definition of the closure $\overline{A}$ we have $\overline{A} \subset X \setminus B$. ⊞





Finally note, since $\overline{A}$ is a closed subset of the compact space X we know by the Compact-Closed Lemma 1.21 (1) that $\overline{A}$ is compact. We are done.

(2) Let X be a Hausdorff space such that every $P \in X$ admits a compact neighborhood. We need to show that X is regionally compact. Thus let $P \in X$ and let U be a neighborhood of P. By hypothesis we know that P admits a compact neighborhood K. By (1) we know that K is regionally compact, which implies that there exists a compact neighborhood L of $P \in K$ with $P \in L \subset U \cap K$. ■



Our next definition is a closely related concept.

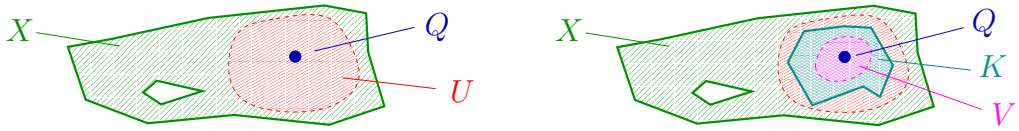
Definition. Let X be a topological space. A subset $A \subset X$ is called precompact if the closure $\overline{A}$ is compact.

Lemma 2.42. Let X be a Hausdorff space that is regionally compact.

(1) Given any $Q \in X$ and given any open neighborhood U of Q there exists an open neighborhood V of Q such that $\overline{V}$ is compact and such that $Q \in V \subset \overline{V} \subset U$.
(2) X is locally precompact.

Proof. Let X be a Hausdorff space that is regionally compact.

(1) Let $Q \in X$ and let U be a neighborhood of Q. Since X is regionally compact we know that there exists a compact neighborhood K of Q with $K \subset U$. Since K is a neighborhood of Q we know that there exists an open neighborhood V of Q with $Q \in V \subset K$. Since X is Hausdorff we know by the Compact-Closed Lemma 1.21 (2) that K is a closed subset of X. Hence $\overline{V} \subset K$. Since $\overline{V}$ is a closed subset of the compact space K we obtain from the Compact-Closed Lemma 1.21 (1) that $\overline{V}$ is compact.



(2) It follows immediately from (1) that X is locally precompact. ■

We conclude the discussion of regionally compact spaces with another health warning.

Warning. Recall that what we call "regionally P" is usually called "locally P". Even more confusing, in the literature frequently "usually compact" is used with different definitions. For example in [**HY1988**, p. 71] and [**Dug1966**, p. 237] a topological space is called "regionally compact" if every point admits an open neighborhood such that its closure is compact. Furthermore in [**SS1978**, p. 20], [**Bre1993**, p. 31] and [**Jos1983**, p. 266] a topological space is called "regionally compact" if every point is contained in a compact neighborhood. The latter two notions of "regionally compact" are in general different, see e.g. [**SS1978**, Example 52]. Furthermore note that for the above two concepts of "regionally compact" every compact space is already regionally compact. But as we mentioned above, in Exercise 2.58 we will see that there exists a compact space that is not regionally compact topological. Nonetheless, it is a "metatheorem" that all concepts of "locally compact" in the literature agree for Hausdorff spaces and they agree with our notion of "regionally compact".

In the following definition we turn from properties of topological spaces to properties of maps.

Definition. Let P be a property of maps between topological spaces and let $f\colon X \to Y$ be a map between topological spaces.

(1) We say that $f\colon X \to Y$ is locally P at $x \in X$ if there exists an open neighborhood U of x and an open neighborhood V of $f(x)$ such that $f\colon U \to V$ has the property P.
(2) We say that f is locally P if it is locally P at any $x \in X$.

Example.

(1) One easily verifies that the map $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is a local homeomorphism at all points $x \neq 0$.

(2) The map

$$\begin{array}{rccl} p\colon & \mathbb{R} & \to & S^1 \\ & t & \mapsto & \exp(2\pi\,\mathrm{i}t) \end{array}$$

is a local homeomorphism. Indeed, let $t \in \mathbb{R}$. We set $U = (t-\frac{1}{2}, t+\frac{1}{2})$ and $V = p(U)$. The restriction of p to $p\colon U \to V$ is continuous and a bijection. Furthermore it is not difficult to show that p is an open map, see also Exercise 2.36. It follows from the Open -Injective Map Lemma 2.16 (2) that $p\colon U \to V$ is a homeomorphism.

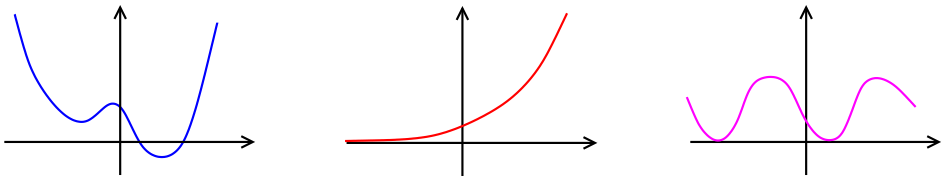
(3) The statement of the Pasting Proposition 2.6 can be reformulated as saying that a map between two topological spaces is continuous, if it is locally continuous.

2.9. **Proper maps and closed embeddings.** Perhaps the best loved result in topology is the Compact-Hausdorff Proposition 2.17 (3) which says that a bijective continuous map $f\colon X \to Y$ is actually a homeomorphism if X is compact and Y is Hausdorff. In this section we will prove a result which at times can deal with the case that X is non-compact. First we need to introduce the following definition which will appear time and again.

Definition. A map $f\colon X \to Y$ between topological spaces is called proper if the preimage of every compact subset of Y is a compact subset of X.

Example.

(1) In the figure below we show graphs of maps $\mathbb{R} \to \mathbb{R}$, some of the maps are proper and some are not proper.



map is proper map is not proper map is not proper

(2) A map from a non-compact space to a compact space *cannot* be proper. In particular inclusion maps are not necessarily proper, for example the inclusion $i\colon (0,1) \to [-2,2]$ is not proper.

(3) Evidently every homeomorphism is proper.

(4) It follows easily from the Compact-Closed Lemma 1.21 (1) and (2) that every map from a compact space to a Hausdorff space is proper.

Now we can formulate and prove the following variation on the Compact-Hausdorff Proposition 2.17.

Proposition 2.43. (Proper-Closed Proposition) Let $f\colon X \to Y$ be a map from a topological space X to a topological space Y. We assume that Y is Hausdorff and regionally compact.

(1) If f is proper, then $f\colon X \to Y$ is a closed map.

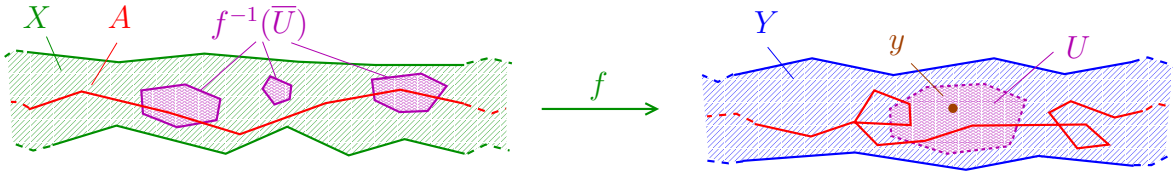
(2) The following two statements are equivalent:

  (a) The map $f\colon X \to Y$ is injective and proper.

  (b) The map $f\colon X \to Y$ is a closed embedding.

Proof.

(1) Let $A \subset X$ be a closed subset. We need to show that $f(A)$ is a closed subset of Y. By the JH-Lemma 1.7 it suffices to show that every $y \in Y \setminus f(A)$ admits an open neighborhood V with $V \subset Y \setminus f(A)$ or equivalently with $V \cap f(A) = \varnothing$. Thus let $y \in Y$. The proof proceeds in the following steps, which are mostly forced on us:
- Since Y is regionally compact and Hausdorff we know by Lemma 2.42 that y admits an open neighborhood U such that the closure $\overline{U}$ is compact.
- Since f is proper we know that $f^{-1}(\overline{U})$ is compact.
- Since A is a closed subset we obtain from the Compact-Closed Lemma 1.21 (1) that $A \cap f^{-1}(\overline{U})$ is also compact.
- By the Compact Image Lemma 2.13 we know that $f(A \cap f^{-1}(\overline{U}))$ is compact.
- By the elementary Exercise 0.3 we know that $f(A \cap f^{-1}(\overline{U})) = f(A) \cap \overline{U}$.
- Since Y is Hausdorff we obtain immediately from the Compact-Closed Lemma 1.21 (2) that $f(A \cap f^{-1}(\overline{U})) = f(A) \cap \overline{U}$ is a closed subset of Y.
- Therefore $V := U \setminus (f(A) \cap \overline{U}) = U \setminus f(A)$ is an open neighborhood of $y \in Y$ that does not intersect $f(A)$.



(2) First we prove the "(a)⇒(b)"-direction. Thus we assume that $f\colon X \to Y$ is injective and proper. Since Y is Hausdorff and regionally compact we know by Statement (1) that $f\colon X \to Y$ is a closed map. Since f is furthermore injective we obtain from the Compact-Hausdorff Proposition 2.17 that $f\colon X \to Y$ is indeed a closed embedding.

Now we prove the "(b)⇒(a)"-direction. We only need to show that $f\colon X \to Y$ is proper. Thus let $K \subset Y$ be a compact subset.
- Since $f\colon X \to Y$ is a closed map we know by the Compact-Closed Lemma 1.21 (1) that $K \cap f(X)$ is also compact.
- Since $f\colon X \to f(X)$ is a homeomorphism we know that $f^{-1}(K \cap f(X))$ is compact.
- Since $f^{-1}(K \cap f(X)) = f^{-1}(K)$ we have indeed shown that $f^{-1}(K)$ is compact. ■

We can use the Proper-Closed Proposition 2.43 to give a new proof of the following popular proposition.

Proposition 2.19. (Closed Embedding-to-$\mathbb{R}^n$ Proposition) Let X be a topological space, let $n \in \mathbb{N}$ and let $f\colon X \to \mathbb{R}^n$ be an injective continuous map. Furthermore suppose that X is the union of compact subsets $\{K_i\}_{i \in I}$ such that the following condition is satisfied:

(∗) Given any $r > 0$ there exist only finitely many $i \in I$ with $f(K_i) \cap \overline{B}_r^n \neq \varnothing$.

Then f is a closed embedding.

For the new proof of the Closed Embedding-to-$\mathbb{R}^n$ Proposition 2.19 we will need the following lemma, which often simplifies the task of showing that a certain map is proper.

Auxiliary Lemma 2.44. Let $f\colon X \to Y$ be a map between two topological spaces. We suppose that Y is Hausdorff.

(1) We suppose that we are given a family $\{L_j\}_{j\in J}$ of compact subsets of Y such that each compact subset of Y is contained in some L_j. If each $f^{-1}(L_j)$ is compact, then f is proper.

(2) Suppose that X is covered by compact subsets $\{K_i\}_{i\in I}$. Let $L \subset Y$ be compact. If there exist only finitely many $i \in I$ with $f(K_i) \subset L$, then $f^{-1}(L)$ is compact.

Proof of Lemma 2.44.

(1) Let L be a compact subset of Y. By hypothesis there exists a $j \in J$ with $L \subset L_j$. Note that

$$\begin{array}{rcl} f^{-1}(L) & \overset{\text{since } L\subset L_j}{\underset{\downarrow}{=}} & f^{-1}(L\cap L_j) \overset{\text{Image-Preimage Lemma 0.2 (7)}}{\underset{\downarrow}{=}} f^{-1}(L)\cap f^{-1}(L_j) \\ & & = \text{ closed subset } \cap \text{ compact subset } \underset{\uparrow}{=} \text{ compact.} \end{array}$$

since Y is Hausdorff we know by the Compact-Closed Lemma 1.21 that L is closed, therefore $f^{-1}(L)$ is closed, furthermore by hypothesis we know that $f^{-1}(L_j)$ is compact

by the Compact-Closed Lemma 1.21

(2) Let $L \subset Y$ be a compact subspace with the property that there exist only finitely many $i \in I$ with $f(K_i) \subset L$. Note that

$$\begin{array}{rcl} f^{-1}(L) & = & f^{-1}(L)\cap \bigcup_{i\in I \text{ with } f(K_i)\subset L} K_i \\ & \underset{\uparrow}{=} & \text{closed subset } \cap \text{ compact subset } \underset{\uparrow}{=} \text{ compact.} \end{array}$$

since Y is Hausdorff we know by the Compact-Closed Lemma 1.21 that L is closed, therefore $f^{-1}(L)$ is closed, furthermore by the Compact-Union-Intersection Lemma 1.20 we know that union of finitely many compact subspaces is again compact

by the Compact-Closed Lemma 1.21

■

Proof of Proposition 2.19. (Closed Embedding-to-$\mathbb{R}^n$ Proposition)] Since $\mathbb{R}^n$ is Hausdorff it follows from Lemma 2.44 (2) that for each $r > 0$ the preimage $f^{-1}(\overline{B}^n_r)$ is compact. It follows from Lemma 2.44 (1) that f is actually proper. On page 178 we pointed out that $\mathbb{R}^n$ is regionally compact. We now obtain from the Proper-Closed Proposition 2.43 that f is in fact a closed embedding. ■

Exercises for Chapter 2.

Exercise 2.1.

(a) Let X be a topological space that is locally connected. Show that the continuous maps $f\colon X \to \mathbb{R}$ that are locally constant are precisely the maps that are constant on all of the components of X.

(b) Show that the conclusion in (a) does not necessarily hold if we drop the hypothesis that X is locally connected.

Exercise 2.2. We consider

$$\mathcal{B} \;:=\; \{[a,b) \,|\, a,b \in \mathbb{R}\} \;\subset\; \mathcal{P}(\mathbb{R}).$$

It is clear that $\mathcal{B}$ has the basis property introduced on page 132. Let $\mathcal{T} := \mathcal{T}(\mathcal{B})$ be the topology generated by $\mathcal{B}$. This topology on $\mathbb{R}$ is called the *lower limit topology* and the topological space $(\mathbb{R}, \mathcal{T})$ is called the *Sorgenfrey line*.

(a) Let $\mathcal{S}$ be the standard topology on $\mathbb{R}$. Which, if any, of the two identity maps $(\mathbb{R}, \mathcal{S}) \to (\mathbb{R}, \mathcal{T})$ and $(\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{S})$ is continuous?
(b) Determine the closure of the intervals $(1, \sqrt{2})$ and $(\sqrt{2}, 3)$ in the Sorgenfrey line.
(c) Is the Sorgenfrey line path-connected?
(d) Is the Sorgenfrey line regionally compact?

Exercise 2.3. Let X be a set, let Y be a topological space and let $f\colon X \to Y$ be a map. Now suppose that we are given two topologies $\mathcal{T}$ and $\mathcal{T}'$ on X such that $f\colon (X, \mathcal{T}) \to Y$ and $f\colon (X, \mathcal{T}') \to Y$ are local homeomorphisms. Show that $\mathcal{T} = \mathcal{T}'$.
Hint. Use the Pasting Proposition 2.6.

Exercise 2.4.

(a) Show that every self-homeomorphism of B^1 extends to a self-homeomorphism of $\overline{B}^1$.
(b) Give an example self-homeomorphism of B^2 that does not extend to a self-homeomorphism of $\overline{B}^2$.

Exercise 2.5. Let $C([0,1])$ be the real vector space of continuous maps from the interval $[0,1]$ to $\mathbb{R}$. We consider the

$$\text{supremum norm } s(f) := \sup\{|f(x)| \,|\, x \in [0,1]\} \quad \text{and the integral norm} \quad i(f) := \int_0^1 |f(x)|\, dx.$$

We denote the topologies corresponding to these norms by $\mathcal{S}$ and $\mathcal{I}$. Show that one topology is contained in the other and show that the two topologies are not the same.
Remark. This shows in particular that the identity map $\mathrm{id}\colon (C([0,1], \mathcal{S}) \to (C([0,1], \mathcal{I})$ is not a homeomorphism.

Exercise 2.6. Let $f\colon [a,b] \to \mathbb{R}$ be a continuous map. Show that if f is injective, then $f([a,b])$ is an interval with endpoints $f(a)$ and $f(b)$ and show that the map $f\colon [a,b] \to f([a,b])$ is a homeomorphism.
Hint. Use the Intermediate Value Theorem.

Exercise 2.7. Let $C([0,1])$ be the real vector space of continuous maps from the interval $[0,1]$ to $\mathbb{R}$ and let $C^1([0,1])$ be the real vector space of continuously differentiable maps from the interval $[0,1]$ to $\mathbb{R}$. In this exercise we equip both vector spaces with the supremum norm $s(f) := \sup\{f(x) \,|\, x \in [0,1]\}$. We consider the two maps

$$\begin{array}{rcl} D\colon C^1([0,1]) & \to & C([0,1]) \\ f & \mapsto & f' \end{array} \qquad \text{and} \qquad \begin{array}{rcl} I\colon C^1([0,1]) & \to & C([0,1]) \\ f & \mapsto & \left(x \mapsto \int_0^x f(t)\, dt\right). \end{array}$$

Note that D and I are inverses of one another, in particular both are isomorphisms of real vector spaces.

(a) Show that I is continuous.
(b) Show that D is not continuous.

Remark. This exercise shows that integration is usually a well-behaved operation whereas differentiation can be problematic.
Remark. This exercise is a variation on Exercise 2.5.

Exercise 2.8. Let U be an open subset of $\mathbb{R}^n$. We denote by $C = \overline{U}$ the closure of U.

(a) Suppose that U is convex. Show that $U = \mathring{C}$.
(b) Given an example of an open subset U of $\mathbb{R}^n$ where $U \neq \mathring{C}$.

Exercise 2.9. Let V be a finite-dimensional real vector space. Recall that given a norm $n\colon V \to \mathbb{R}_{\geq 0}$ its norm ball is defined as the subset $B(n) := \{v \in V \mid n(v) \leq 1\}$.

(a) Show that for any norm $n\colon V \to \mathbb{R}_{\geq 0}$ the norm ball $B(n)$ is a convex subset with non-empty interior.
(b) Show that for any convex subset $C \subset \mathbb{R}^n$ with non-empty interior there exists a norm $n\colon V \to \mathbb{R}_{\geq 0}$ with $B(n) = C$.

Exercise 2.10. Let $n \in \mathbb{N}$. Show that there are precisely $n + 2$ homeomorphism types of compact convex subsets of $\mathbb{R}^n$.
Remark. Use the Convex-to-Ball Proposition 2.20 (2).

Exercise 2.11. Let $f\colon X \to Y$ be a map between topological spaces.

(a) Show that the following statements are equivalent:
  (i) The map is continuous.
  (ii) For every subset $B \subset Y$ we have $f^{-1}(\overline{B}) \subset \overline{f^{-1}(B)}$.
  (iii) For every subset $B \subset Y$ we have $f^{-1}(\mathring{B}) \subset$ interior of $f^{-1}(B)$.
  Remark. We showed (i)⇒(ii) in the Closure-Interior Mapping Lemma 2.5.
(b) Given an example of a continuous map $f\colon X \to Y$ that shows that in general we do not have an equality in (ii).
(c) Given an example of a continuous map $f\colon X \to Y$ that shows that in general we do not have an equality in (iii).

Exercise 2.12. Does there exist a continuous map $f\colon S^2 \to S^2$ with a single fixed point, i.e. with a unique point x such that $f(x) = x$?

Exercise 2.13. Let (X, d) and (Y, e) be two metric spaces and let $\varphi\colon X \to Y$ be a map.

(a) Suppose that there exists some $C > 0$ such that for every choice of $x, x' \in X$ we have $e(\varphi(x), \varphi(x')) < C \cdot d(x, x')$. Show that d is continuous.
(b) Suppose that there exists some $c > 0$ such that for every choice of $x, x' \in X$ we have $e(\varphi(x), \varphi(x')) > c \cdot d(x, x')$. Furthermore suppose that φ is surjective. Does it follow that d is open?

Exercise 2.14. Let (a, b) be an interval and let $f\colon (a, b) \to X$ be a map to a topological space. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences of real numbers in (a, b) such that $\lim\limits_{n\to\infty} a_n = a$ and such that $\lim\limits_{n\to\infty} b_n = b$. We suppose that the restriction of f to each $[a_n, b_n]$ is continuous. Show that f itself is continuous.
Remark. An analogous statement also holds if we deal with a half-open interval $[a, b)$.

Exercise 2.15. Let (X, d) be a metric space and let A be a subset of X. Show that the map

$$\begin{array}{rcl} A & \to & [0,1] \\ a & \mapsto & \min\{1, \sup\{r \in \mathbb{R}_{\geq 0} \,|\, B_r(a) \subset A\}\} \end{array}$$

is continuous.

Exercise 2.16. Let X be a set and let d and e be two metrics on X that are equivalent, i.e. that induce the same topology on X. We use this common topology to view X as a topological space.

(a) Suppose that X is compact. Show that given any $\mu > 0$ there exists a $\nu > 0$ such that the following conclusion holds:

$$d(x,y) < \mu \qquad \Longrightarrow \qquad e(x,y) < \nu.$$

(b) Let $\alpha\colon X \to \mathbb{R}_{>0}$ be a map. Show that there exists a map $\beta\colon X \to \mathbb{R}_{>0}$ such that the following conclusion holds:

$$d(x,y) < \alpha(x,y) \qquad \Longrightarrow \qquad e(x,y) < \beta(x,y).$$

Remark. As always all maps are understood to be continuous.

Hint. Use Exercise 2.15

Exercise 2.17. Let C be the Cantor set as defined on page 130.

(a) Show that for every $x \in C$ there exists a unique sequence $a_1, a_2, \ldots$ of elements of $\{0,1\}$ such that

$$x = \sum_{k=1}^{\infty} \frac{2a_k}{3^k}.$$

(b) We define the *Cantor function* as follows:

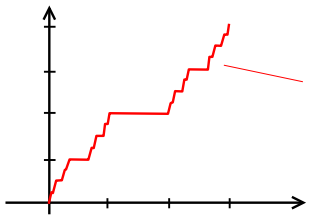
$$\begin{array}{rcl} c\colon [0,1] & \to & [0,1] \\ x & \mapsto & \begin{cases} \sum_{k=1}^{\infty} \frac{a_k}{2^k}, & \text{if } x = \sum_{k=1}^{\infty} \frac{2a_k}{3^k} \in C, \\ \sup\{c(y) \,|\, y \in C \text{ and } y \leq x\}, & \text{if } x \notin C. \end{cases} \end{array}$$

Show that the Cantor function is continuous.

(c) Show that the Cantor function is surjective.

(d) Show that the Cantor set is uncountable.

Remark. The graph of the Cantor function is sometimes called the *Devil's Staircase*.



Exercise 2.18. Show that every connected non-empty subset of $\mathbb{R}$ is an interval.

Exercise 2.19. Let X be a topological space and let $A \subset X$ be a subset.

(a) Show that if A is connected, then so is its closure $\overline{A}$.

(b) If A is path-connected, does it follow, that its closure $\overline{A}$ is also path-connected?

Exercise 2.20. Let X be a topological space and let $A \subset X$ be a subspace. We say that A *can be disconnected in* X if there exist disjoint open subsets U and V of X such that $U \cap A \neq \varnothing$, $V \cap A \neq \varnothing$ and $A \subset U \cup V$.

(a) Show that if X is a metric space, then any disconnected subspace A can be disconnected in X.
(b) Give an example of a topological space and a disconnected subspace A that cannot be disconnected in X.

Exercise 2.21.

(a) Show that every self-homeomorphism of $\mathbb{R}$ is monotonously increasing or monotonously decreasing.
(b) Is every self-homeomorphism of $\mathbb{Q}$ is monotonously increasing or monotonously decreasing?

Exercise 2.22. Let $X = \{0, 1\}$ be the set with two elements which we equip with the topology $\mathcal{T} = \{\varnothing, X, \{0\}\}$. The topological space $X = (X, \mathcal{T})$ is called the *Sierpiński space*.

(a) Is every continuous maps $X \to \mathbb{R}$ constant?
(b) Is every continuous map $\mathbb{R} \to X$ constant?
(c) Is X Hausdorff?
(d) Is X connected?
(e) Is X path-connected?
(f) Determine all self-homeomorphisms of X.

Exercise 2.23. We consider $X = \mathbb{Q}$ with the subspace topology coming from $\mathbb{R}$. Show that the only connected subsets of $\mathbb{Q}$ are the subsets consisting of at most a point.

Exercise 2.24. Let $f\colon X \to Y$ be a continuous map between topological spaces, let $U \subset X$ be a subset and let $A \subset Y$ be a subset.

(a) Suppose that A is closed and suppose that $f(U) \subset A$. Show that $f(\overline{U}) \subset A$.
(b) Suppose that A is open and that $A \subset f(U)$. Show that $A \subset f(\mathring{U})$.

Exercise 2.25. Let X be a topological space. We say a subset $A \subset X$ is a *retract of* X if there exists a *retraction* $r\colon X \to A$, i.e. a continuous map with $r(a) = a$ for all $a \in A$. (Evidently we demand that r is continuous).

(a) Suppose that X is Hausdorff. Show that A is a closed subset of X.
(b) Show that the conclusion of (a) does not necessarily hold if we do not assume that X is Hausdorff.

Exercise 2.26. Let $f\colon (X, \mathcal{S}) \to (Y, \mathcal{T})$ be a (possibly non-continuous) map between two topological spaces.

(a) Show that if f is continuous, then for every $(x_n)_{n\in\mathbb{N}}$ sequence in X we have
$$\lim_{n\to\infty} x_n = x \qquad \Longrightarrow \qquad \lim_{n\to\infty} f(x_n) = f(x).$$
(b) We suppose that for every sequence $(x_n)_{n\in\mathbb{N}}$ in X we have
$$\lim_{n\to\infty} x_n = x \qquad \Longrightarrow \qquad \lim_{n\to\infty} f(x_n) = f(x).$$

(i) Give an example where this condition is satisfied but f is not continuous.
(ii) A topological space is called *first-countable*, if for every $x \in X$ there exists a countable family of open neighborhoods $\{U_i\}_{i\in I}$ of x such that each open neighborhood of x contains at least one U_i. Show that if X is first-countable, then f is continuous.

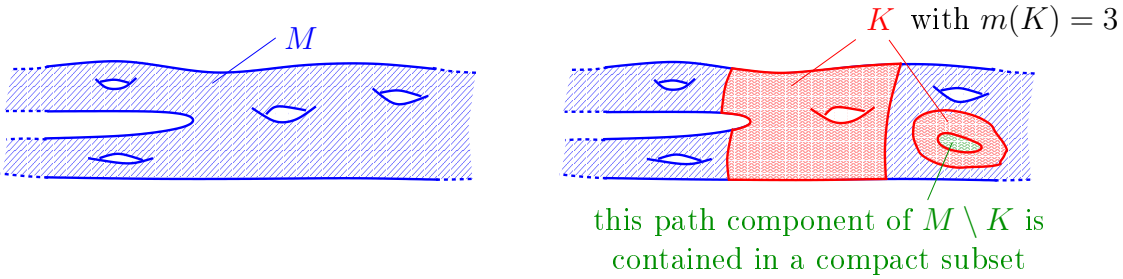
Exercise 2.27. Let X be a topological space. Given a compact subset $K \subset X$ we define $m(K)$ to be the number of non-compact path-components of $X \setminus K$ that are *not* contained in a compact subset of X. We refer to

$$\mathrm{End}(X) \;:=\; \max\{m(K) \,|\, K \text{ is a compact subset of } X\} \in \mathbb{N}_0 \cup \{\infty\}$$

as the *number of ends of* X.

(a) Given $n \in \mathbb{N}$ determine $m(\mathbb{R}^n)$.
Remark. This exercise shows that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.
(b) Can you find connected spaces with arbitrarily large number of ends?

Remark. We will introduce the "ends" of a topological space in Exercise 56.13. We will see that what we now call the "number of ends" equals the cardinality of the set of ends.



Exercise 2.28. Let $C \subset \mathbb{R}$ be the Cantor set. We equip C with the subspace topology. Determine the components of C.

Exercise 2.29. Let $f\colon X \to Y$ be a continuous map between topological spaces. By the Compact Image Lemma 2.13 we know that the image of a compact subset of X under f is a compact subset of Y.

(a) Give an example of a continuous map $f\colon \mathbb{R} \to \mathbb{R}$ and an open subset $U \subset \mathbb{R}$ such that $f(U)$ is not an open subset of $\mathbb{R}$.
(b) Give an example of a continuous map $f\colon \mathbb{R} \to \mathbb{R}$ and a closed subset $U \subset \mathbb{R}$ such that $f(U)$ is not a closed subset of $\mathbb{R}$.

Exercise 2.30. In this exercise we will see that for maps between subsets of $\mathbb{R}^2$ the notions of "open", "closed" and "continuous" are completely independent. More precisely for (a)-(f) provide an example of a map $\varphi\colon X \to Y$ between subsets of $\mathbb{R}^2$, equipped with the subspace topology coming from $\mathbb{R}^2$, that have the stated property.

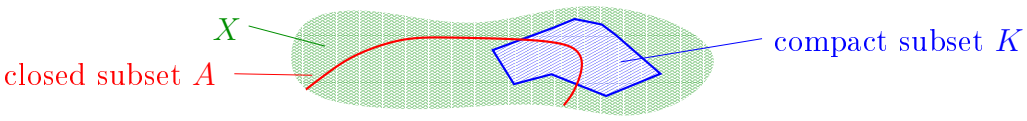
(a) The map φ is open but neither closed nor continuous.
(b) The map φ is closed but neither open nor continuous.
(c) The map φ is continuous but neither open nor closed.
(d) The map φ is continuous and open but not closed.
(e) The map φ is continuous and closed but not open.

(f) The map φ is open and closed but not continuous.

Exercise 2.31. Show that the maps

$$\begin{array}{rcl} \alpha\colon \mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & x+y \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \beta\colon \mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & x\cdot y \end{array}$$

are continuous.



Exercise 2.32. Let $f\colon X \to Y$ be a continuous map from a topological space X to some metric space Y, e.g. $Y = \mathbb{R}^n$. Let $P \in Y$ be a point that is not contained in $f(X)$. We suppose that X is compact. Show that there exists an $r > 0$ such that $B_r(P) \cap f(X) = \varnothing$.

Exercise 2.33. Let $f, g\colon X \to Y$ be two continuous maps between topological spaces.

(a) Suppose that Y is Hausdorff. Show that $\{x \in X \mid f(x) = g(x)\}$ is a closed subset of X.

(b) Show that the conclusion of (a) does not necessarily hold if we do not assume that Y is Hausdorff.

(c) Show that if f and g agree on a subset A and if Y is Hausdorff, then f and g also agree on the closure $\overline{A}$ of A.

Exercise 2.34. Let $\mathbb{R}^n \cup \{\infty\}$ be "$\mathbb{R}^n$ with a point at infinity". Let $v \in \mathbb{R}^n$. Show that the map

$$\begin{array}{rcl} \mathbb{R}^n \cup \{\infty\} & \to & \mathbb{R}^n \cup \{\infty\} \\ Q & \mapsto & \begin{cases} Q+v, & \text{if } Q \in \mathbb{R}^n, \\ \infty, & \text{if } Q = \infty \end{cases} \end{array}$$

is continuous.

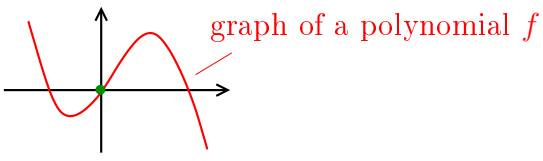
Exercise 2.35. Let f be a complex polynomial of degree ≥ 1. Show that the map

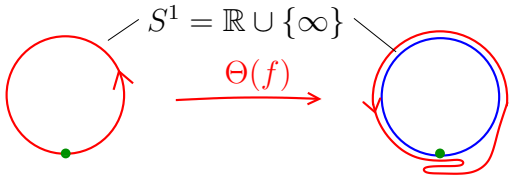
$$\begin{array}{rcl} \Theta(f)\colon \mathbb{C} \cup \{\infty\} & \to & \mathbb{C} \cup \{\infty\} \\ t & \mapsto & \begin{cases} f(t), & \text{if } t \in \mathbb{C}, \\ \infty, & \text{if } t = \infty \end{cases} \end{array}$$

is continuous.

Remark. By Lemma 2.18 we can make the identification $S^2 = \mathbb{C} \cup \{\infty\}$. Thus we see that complex polynomials of degree ≥ 1 give rise to interesting self-maps of S^2.

Remark. The analogous statement also holds for a polynomial f with real coefficients, it gives rise to a self-map $\Theta(f)$ of $\mathbb{R} \cup \{\infty\} = S^1$.





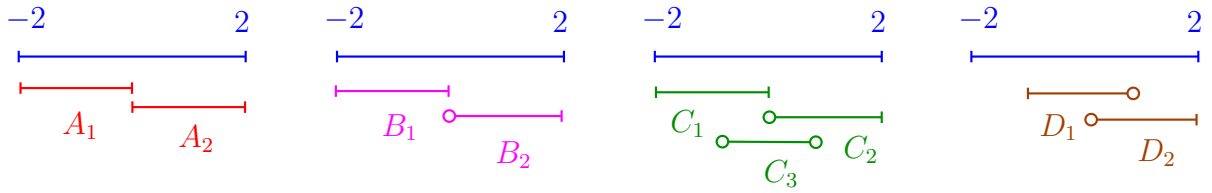
Exercise 2.36. We consider the map

$$\begin{array}{rcl} f\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & \exp(2\pi\,\mathrm{i}t) \end{array}$$

(a) Show that f is an open map.
(b) Is f also a closed map?

Exercise 2.37. Let X be a set and let $\{A_i\}_{i\in I}$ be a family of subsets. Suppose that each A_i is equipped with a topology.

(a) We say $U \subset X$ is open if for each $i \in I$ the intersection $U \cap A_i$ is an open subset of A_i. Show that this defines a topology on X.
(b) We consider the set $X = [-2, 2]$.
  (i) We consider the subsets $A_1 = [-2, 0]$ and $A_2 = [0, 2]$.
  (ii) We consider the subsets $B_1 = [-2, 0]$ and $B_2 = (0, 2]$.
  (iii) We consider the subsets $C_1 = [-2, 0]$, $C_2 = (0, 2]$ and $C_3 = (-1, 1)$.
  (iv) We consider the subsets $D_1 = [-1, 1)$ and $D_2 = (0, 2]$.
  For each of these four settings we equip the subsets with the obvious topology and we equip $X = [-2, 2]$ with the corresponding topology. Describe the resulting four topological spaces. Are these topological spaces that we are familiar with?



Exercise 2.38. Let $P_1, \ldots, P_n \in \mathbb{R}^k$.

(a) Show that the convex hull of the set $\{P_1, \ldots, P_n\}$ is given by the set

$$\left\{ \sum_{i=1}^{n} t_i \cdot P_i \,\middle|\, t_1, \ldots, t_n \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^{n} t_i = 1 \right\}.$$

Note: We make no hypothesis on the points $P_1, \ldots, P_n$, in particular some of them could be identical and we do not assume say that the vectors $P_2 - P_1, \ldots, P_n - 1$ are linearly independent.
(b) Show that the set in (a) is compact.
(c) Is the convex hull of a compact subset of $\mathbb{R}^k$ necessarily compact?
(d) Is the convex hull of an open subset of $\mathbb{R}^k$ necessarily an open subset?

Exercise 2.39. Let $f, g\colon X \to Y$ be two maps between two topological spaces. Let $A \subset X$ be a subset such that f and g agree on A. Show that if A is dense, if Y is Hausdorff and if f and g are continuous, then f and g agree on all of X.

Exercise 2.40.

(a) Show that the half-open interval $[0, 1)$ and the open interval $(0, 1)$ are not homeomorphic.
(b) Is the topological space $[0, 1) \cap \mathbb{Q}$ homeomorphic to the topological space $(0, 1) \cap \mathbb{Q}$?

Exercise 2.41. Let $m, n \in \mathbb{N}$.

(a) Let $A = (a_1 \ldots a_n) \in \mathrm{M}(m \times n, \mathbb{R}) = (\mathbb{R}^m)^n$. Show that A has rank m if and only there exist $k_1, \ldots, k_m \in \{1, \ldots, n\}$ with $\det(a_{k_1} \ldots a_{k_m}) \neq 0$.
(b) Show that the set of matrices of rank m is an open subset of $\mathrm{M}(m \times n, \mathbb{R})$.
Remark. In the Matrix Group Lemma 2.36 we dealt with the case $m = n$.

Exercise 2.42. Give an example of a bijective continuous map $f\colon X \to Y$ between two topological spaces where X is compact but such that f is nonetheless not a homeomorphism.

Exercise 2.43. Let X and Y be topological spaces and let $i\colon X \to Y$ be a map. We suppose that there exists a map $r\colon Y \to X$ with $r \circ i = \mathrm{id}_X$. Show that i is an embedding.

Exercise 2.44. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Let $a < b$ be two real numbers. Show that f can only go finitely many times from a to b. More precisely, show that it is not possible to find numbers $x_1 < y_1 < x_2 < y_2 < x_3 < y_3 < \ldots$ in $[0,1]$ such that for each $i \in \mathbb{N}$ we have $f(x_i) = a$ and we have $f(y_i) = b$.

Exercise 2.45. Let $A \in \mathrm{M}(m \times n, \mathbb{R})$. We define the *norm of* A as follows:

$$\|A\| \;:=\; \max\{\|Av\| \mid v \in S^{n-1}\} \;\in\; \mathbb{R}_{\geq 0}.$$

(Note that by the Compact Image Lemma 2.13 (2) it does indeed make sense to take the maximum.)

(a) Show that for all $v \in \mathbb{R}^n$ we have $\|Av\| \leq \|A\| \cdot \|v\|$.
(b) (i) Show that for all $A, B \in \mathrm{M}(m \times n, \mathbb{R})$ we have $\|A + B\| \leq \|A\| + \|B\|$.
(ii) Show that for any $A \in \mathrm{M}(m \times n, \mathbb{R})$ and $\lambda \in \mathbb{R}$ we have $\|\lambda \cdot A\| = |\lambda| \cdot \|A\|$.
Note that this shows that $\| - \|$ turns $\mathrm{M}(m \times n, \mathbb{R})$ into a normed vector space.
(c) Show that for $A \in \mathrm{O}(n)$ we have $\|A\| = 1$.
(d) Show that for $A \in \mathrm{M}(l \times m, \mathbb{R})$ and $B \in \mathrm{M}(m \times n, \mathbb{R})$ we have $\|A \cdot B\| \leq \|A\| \cdot \|B\|$.
(e) Give examples of matrices $A, B \in \mathrm{M}(m \times m, \mathbb{R})$ with $\|A \cdot B\| \neq \|A\| \cdot \|B\|$.
(f) Show that for $A \in \mathrm{GL}(n, \mathbb{R})$ we have $\|A^{-1}\| = \|A\|^{-1}$.
(g) Let $A \in \mathrm{GL}(n, \mathbb{R})$ be an invertible matrix. Show that there exists a $\lambda > 0$ such that $\|Av\| \geq \lambda \cdot \|v\|$ for all $v \in \mathbb{R}^n$.
(h) Show that the map

$$\begin{array}{rcl} \mathrm{M}(m \times n, \mathbb{R}) & \to & \mathbb{R}_{\geq 0} \\ A & \mapsto & \|A\| \end{array}$$

is continuous.
(i) Let $f\colon [a,b] \to \mathrm{M}(m \times n, \mathbb{R})$ be a continuous map. Show that

$$\Big\| \int_a^b A(t)\, dt \Big\| \;\leq\; \int_a^b \|A(t)\|\, dt.$$

Hint. Make use of the Integral Inequality Lemma 0.16.

Exercise 2.46. Let A be an invertible real $(m \times m)$–matrix. Show that the map

$$\begin{array}{rcl} \mathbb{R}^m \cup \{\infty\} & \to & \mathbb{R}^m \cup \{\infty\} \\ x & \mapsto & \begin{cases} A \cdot x, & \text{if } x \in \mathbb{R}^m, \\ \infty, & \text{if } x = \infty \end{cases} \end{array}$$

is continuous.
Hint. Use Exercise 2.45 (g).

Exercise 2.47. Let $n \in \mathbb{N}$. We consider the stereographic projection

$$\begin{array}{rcl} \Phi\colon S^n & \to & \mathbb{R}^n \cup \{\infty\} \\ (x_1, \ldots, x_{n+1}) & \mapsto & \begin{cases} \left(\frac{x_1}{1-x_{n+1}}, \ldots, \frac{x_n}{1-x_{n+1}}\right), & \text{if } x_{n+1} < 1, \\ \infty, & \text{if } x_{n+1} = 1 \end{cases} \end{array}$$

that we introduced in Lemma 2.18. Show that Φ is continuous.

Exercise 2.48. We consider the Topologist's Sine Curve

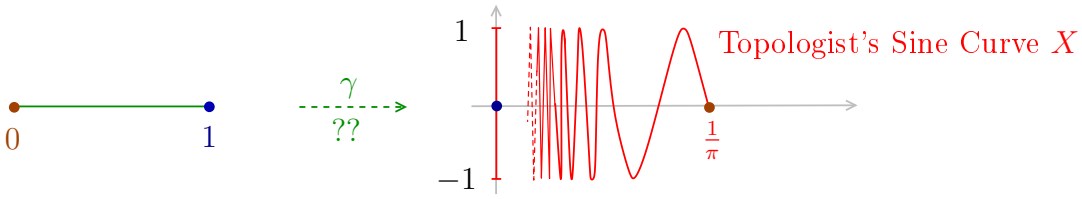
$$X \;:=\; \left\{(0,y)\,\middle|\, y \in [-1,1]\right\} \cup \left\{(x, \sin(\tfrac{1}{x}))\,\middle|\, x \in (0, \tfrac{1}{\pi}]\right\} \;\subset\; \mathbb{R}^2.$$

which we introduced on page 164.

(a) Show that X is connected.

(b) Show that there does not exist a continuous map $\gamma\colon [0,1] \to X$ with $\gamma(0) = (\frac{1}{\pi}, 0)$ and with $\gamma(1) = (0,0)$.
Hint. Assume such a map exists. We denote by $q\colon X \to \mathbb{R}$ the projection onto the second coordinate. Use the Intermediate Value Theorem to construct sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in $[0,1]$ which both converge to the same number but such that for each $n \in \mathbb{N}$ we have $q(\gamma(a_n)) = -1$ and $q(\gamma(b_n)) = +1$.
Remark. This exercise implies of course that X is *not* path-connected.

(c) Show that X has two path components.

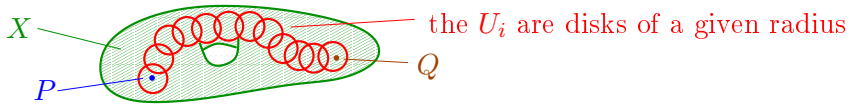


Exercise 2.49. Let X be a topological space and let $\mathcal{U} \subset \mathcal{P}(X)$ be an open cover of X. Show that the following two statements are equivalent:

(1) The topological space X is connected.

(2) Given any two points $P, Q \in X$ there exist $U_0, \ldots, U_k \in \mathcal{U}$ with $P \in U_0$, $Q \in U_k$, with $U_i \cap U_{i+1} \neq \varnothing$ for $i = 0, \ldots, k-1$ and such that $U_i \cap U_j = \varnothing$ whenever $|i-j| \geq 2$.

Remark. Some people refer to this statement as the *Sausage Lemma*.



Exercise 2.50. Let X be a topological space and let $A \subset X$ be a subset.

(a) Suppose that X is path-connected and that ∂A is path-connected. Show that $X \setminus \mathring{A}$ is also path-connected.

(b) Suppose that X is connected and that ∂A is connected. Show that $X \setminus \mathring{A}$ is also connected.

Remark. This exercise is the content of the Complement Connected Lemma 2.29.

Exercise 2.51. We set $K := \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$. The *K-topology* on $\mathbb{R}$ is defined as the topology $\mathcal{K}$ that is generated by the open intervals (a, b) and by all sets of the form $(a, b) \setminus K$ where $a, b \in \mathbb{R}$.

(a) Show that $(\mathbb{R}, \mathcal{K})$ is connected.
(b) Show that $(\mathbb{R}, \mathcal{K})$ is not path-connected, more precisely, show that it has precisely two path-components, namely $(-\infty, 0]$ and $(0, \infty)$.

Exercise 2.52. Let X be a topological space and let $A \subset X$ be a subset. Show that if A is connected, then its closure $\overline{A}$ is also connected.

Exercise 2.53. A topological space X is called *arc-connected* if given any two points $P, Q \in X$ there exists a map $f\colon [0, 1] \to X$ with $f(0) = P$, $f(1) = Q$ and which has the property that f is an embedding.[50]

(a) Show that the line with two zeros is not arc-connected.
(b) Let $n \geq 2$ and let X be the "$\mathbb{R}^n$ with two origins", which is defined in the hopefully obvious way. Is X arc-connected?

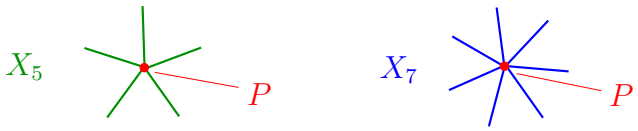
Remark. A rather amazing (non-trivial!) theorem says that any topological space that is path-connected *and which is Hausdorff* is in fact arc-connected. We refer to [**Will1970**, Corollary 31.6], [**Enge1989**, p. 381] and [**HY1988**, p. 118 and p. 129] for a proof of this statement.

Exercise 2.54.

(a) Let $n \in \mathbb{N}_{\geq 3}$. We consider

$$X_n \;:=\; \{r \cdot \exp(2\pi\,\mathrm{i}t \cdot \tfrac{k}{n}) \,|\, r \in [0, 1) \text{ and } k \in \{0, \ldots, n-1\}\}.$$

We write $P := 0$. Show that every self-homeomorphism $f\colon X_n \to X_n$ preserves P.
(b) Does the statement in (a) also hold for $n = 1$ and/or $n = 2$?



Exercise 2.55.

(a) Let $n \geq 1$. Show that the sphere S^n is path-connected.
Remark. This is the content of the Ball-Sphere-Connected Lemma 2.25 (2).
(b) Is the line with two zeros connected?

Exercise 2.56. A subset of a topological space is sometimes called *clopen* if it is open and closed. Let $n \in \mathbb{N}$. Determine all clopen subsets of $\mathbb{R}^n$.

Exercise 2.57. Let X be a path-connected subset of $\mathbb{R}^2$. Is it also necessarily regionally path-connected?

[50] Sometimes, see e.g. [**Will1970, Enge1989**], such topological spaces are also called *arcwise connected*. Rather confusingly some other books, e.g. [**Bre1993**], use the term *arcwise connected* for what we call path-connected. The lesson is, that in any given book one always should look up the definitions.

Exercise 2.58. We consider the set $X = (0, 1)$ with the topology where the open sets are precisely the sets $\varnothing, X$ and the intervals $(0, 1 - \frac{1}{n})$. Show that X is compact and show that X is not regionally compact.

Exercise 2.59. Let X be a topological space. We consider the set $X^* := X \cup \{\infty\}$. We say $U \subset X^*$ is open if one of the following conditions is satisfied:

(1) U is an open subset of X, or
(2) $\infty \in U$ and there exists a compact subset K of X such that $U = X^* \setminus K$.

The inclusion $c\colon X \to X^*$ is called the *Alexandroff extension*. Show that the following statements hold:

(a) The open sets do indeed define a topology on X.
(b) The map c is continuous and open.
(c) Show that the topological space X^* is compact.
(d) Show that if X is non-compact, then $c(X) \subset X^*$ is dense.
(e) Show that X^* is Hausdorff if and only if X is Hausdorff and regionally compact.

Remark. The topological space X^* is called the *one-point compactification of* X or *Alexandroff compactification of* X.[51]

Remark. One can show easily that the one-point compactification of $\mathbb{R}^n$ equals the "$\mathbb{R}^n$ with a point at infinity" that we introduced on page 122.

Exercise 2.60. Let $X = \mathbb{N}$ with the discrete topology. Show that the one-point compactification of X, as defined in Exercise 2.59, is homeomorphic to a subset of $\mathbb{R}$.

Exercise 2.61. Give an example of a subspace of $\mathbb{R}^2$ that is path-connected but that is not locally path-connected.

Exercise 2.62. Let X be a topological space.

(a) Let $f\colon X \to \mathbb{R}^n$ and $g\colon X \to \mathbb{R}^n$ be two maps. We assume that f is proper and we assume that there exists an $\epsilon \in \mathbb{R}$ such that for every $x \in X$ we have $\|f(x) - g(x)\| < \epsilon$. Show that g is also proper.
(b) Does the conclusion of (a) hold if we replace the metric space $(\mathbb{R}^n, d(x, y) := \|x - y\|)$ by any other metric space (Y, d)?

[51] Some authors use the expression "one-point compactification" only if X is non-compact, Hausdorff and regionally compact.

3. Categories, functors and natural transformations

In the last chapter we introduced the concept of topological spaces, we studied several natural examples and we discussed various basic concepts. We will now take a short break from discussing topological spaces.

In this chapter we introduce the notion of a category, of a functor and of a natural transformation that many readers will already have encountered in different courses. In the following chapter we will introduce the notions of (co-) products and (co-) limits in categories. These notions might initially appear to be rather abstract, but they play an essential role in topology and they are regularly used in all branches of pure mathematics. These concepts will also guide us for how to proceed with the discussion of topological spaces.

Nonetheless, readers who want to learn more about topological spaces as quickly as possible can also safely skip these two chapters. Having a background in categories is helpful for the later chapters on topological spaces, but not completely essential.

3.1. Definition and examples of categories.

Definition. A category $\mathcal{C}$ consists of the following data:

(1) A class[52] $\mathrm{Ob}(\mathcal{C})$, the elements of which are called the objects of the category.

(2) For each pair (X, Y) of objects there exists a set[53] [54] $\mathrm{Mor}_{\mathcal{C}}(X, Y)$, the elements of which are called the morphisms from X to Y.

(3) For any three objects X, Y and Z we are provided a map

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{C}}(X,Y) \times \mathrm{Mor}_{\mathcal{C}}(Y,Z) & \to & \mathrm{Mor}_{\mathcal{C}}(X,Z) \\ (f,g) & \mapsto & g \circ f, \end{array}$$

that satisfies the following axioms:

(A) (associativity): For every $f \in \mathrm{Mor}_{\mathcal{C}}(W,X), g \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ and $h \in \mathrm{Mor}_{\mathcal{C}}(Y,Z)$ we have

$$(h \circ g) \circ f \;=\; h \circ (g \circ f) \;\in\; \mathrm{Mor}_{\mathcal{C}}(W,Z).$$

(B) (identity): For every object X there exists a morphism $\mathrm{id}_X \in \mathrm{Mor}_{\mathcal{C}}(X,X)$ with the property that

$$\begin{array}{rcll} \mathrm{id}_X \circ f & = & f & \text{for all } W \in \mathrm{Ob}(\mathcal{C}) \text{ and all } f \in \mathrm{Mor}_{\mathcal{C}}(W,X), \quad \text{and} \\ f \circ \mathrm{id}_X & = & f & \text{for all } Y \in \mathrm{Ob}(\mathcal{C}) \text{ and all } f \in \mathrm{Mor}_{\mathcal{C}}(X,Y). \end{array}$$

Before we give a long list of examples let us first state a lemma which will look familiar from a basic algebra course.

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[52] As many "working mathematicians" the author has a slightly guilty conscience for not knowing what a "set" and what a "class" is. Therefore the reader is referred to **[Cie1997, FPr1985]** for details. Truth be told, most mathematicians learn from logicians at some point when to say "class" instead of "set", but many mathematicians do not know exactly what they are doing at that point.

[53] One could ask why $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ has to be a set, why not just a class? To the best of my knowledge the axioms on morphisms that we write down also make sense if $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ is only required to be a class. Here's a long discussion which gives a satisfactory answer to people in category theory:

`http://mathoverflow.net/questions/48810/why-need-the-morphisms-to-form-a-set`

I myself do not understand the answer.

[54] If the category is clear from the context, then we just write $\mathrm{Mor}(X,Y)$ instead of $\mathrm{Mor}_{\mathcal{C}}(X,Y)$.

Lemma 3.1. Let $\mathcal{C}$ be a category. For every object $X \in \mathrm{Ob}(\mathcal{C})$ there exists a *unique* morphism $\mathrm{id}_X \in \mathrm{Mor}_{\mathcal{C}}(X,X)$ with the property that

(i) $\mathrm{id}_X \circ f = f$ for all $W \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathrm{Mor}_{\mathcal{C}}(W,X)$,
(ii) $f \circ \mathrm{id}_X = f$ for all $X \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$.

Proof. The existence of id_X is given by property (3B) of a category. Now suppose that $I \in \mathrm{Mor}_{\mathcal{C}}(X,X)$ has the same property. Then we have

$$\mathrm{id}_X \underset{\substack{\uparrow \\ \text{property (i) of } I \text{ applied to } f=\mathrm{id}_X}}{=} I \circ \mathrm{id}_X \underset{\substack{\uparrow \\ \text{property (ii) of } \mathrm{id}_X \text{ applied to } f=I}}{=} I.$$

■

Lemma 3.1 allows us to introduce the following definition.

Definition. Let $\mathcal{C}$ be a category and let $X \in \mathrm{Ob}(\mathcal{C})$. We refer to the morphism that is uniquely defined in Lemma 3.1 as the identity morphism id_X.

In the following we introduce all kinds of categories. Many of those will be faithful companions throughout these notes.

Definition.

(a) We refer to the category $\mathcal{Set}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Set}) &:= \text{all sets}, \\ \mathrm{Mor}(X,Y) &:= \text{all maps from } X \text{ to } Y, \end{aligned}$$

together with the usual composition of maps as the category of sets.

(b) We refer to the category $\mathcal{Top}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Top}) &:= \text{all topological spaces}, \\ \mathrm{Mor}(X,Y) &:= C(X,Y) := \text{all continuous maps from } X \text{ to } Y, \end{aligned}$$

with the usual composition of maps[55] as the category of topological spaces.

(c) Let $\mathbb{K}$ be a field. We refer to the category $\mathcal{Vec}_{\mathbb{K}}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Vec}_{\mathbb{K}}) &:= \text{all } \mathbb{K}\text{-vector spaces}, \\ \mathrm{Mor}(X,Y) &:= \mathrm{Hom}_{\mathbb{K}}(X,Y) = \text{all } \mathbb{K}\text{-homomorphisms from } X \text{ to } Y, \end{aligned}$$

with the usual composition of homomorphisms as the category of $\mathbb{K}$-vector spaces.

(d) We refer to the category $\mathcal{Group}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Group}) &:= \text{all groups}, \\ \mathrm{Mor}(X,Y) &:= \mathrm{Hom}(X,Y) = \text{all group homomorphisms from } X \text{ to } Y, \end{aligned}$$

with the usual composition of group homomorphisms as the category of groups. Similarly we define the category $\mathcal{AbGroup}$ of abelian groups.

(e) We refer to the category $\mathcal{Ring}$ with[56]

$$\begin{aligned} \mathrm{Ob}(\mathcal{Ring}) &:= \text{all rings}, \\ \mathrm{Mor}(R,S) &:= \mathrm{Hom}(R,S) = \text{all ring homomorphisms from } R \text{ to } S, \end{aligned}$$

with the usual composition of ring homomorphisms as the category of rings. Similarly we define the category $\mathcal{CRing}$ of commutative rings.

(f) A pair of sets is a pair (X, A), where X is a set and A is a subset of X. We refer to the category $\mathcal{PairSet}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{PairSet}) &:= \text{all pairs of sets,} \\ \mathrm{Mor}((X,A),(Y,B)) &:= \text{all maps } f \text{ from } X \text{ to } Y \text{ with } f(A) \subset B \end{aligned}$$

with the usual composition of maps as the category of pairs of sets. With the same approach we could also introduce the category of pairs of groups and so on.

The name "morphisms" suggests that morphisms are necessarily maps. But as we will see in the following two examples, this is not necessarily always the case.

Example.

(g) We consider the category $\mathcal{C}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}) &:= \text{all sets,} \\ \mathrm{Mor}(X,Y) &:= \begin{cases} \{\mathrm{id}_X\}, & \text{if } X = Y \\ \varnothing, & \text{if } X \neq Y. \end{cases} \end{aligned}$$

The composition map only needs to be defined if $X = Y = Z$[57], and in this case we define $\mathrm{id}_X \circ \mathrm{id}_X := \mathrm{id}_X$. One can easily convince oneself that this defines a category.

(h) Let G be an arbitrary group. We consider the category $\mathcal{C}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}) &:= \text{the set with the unique element } *, \\ \mathrm{Mor}(*,*) &:= G. \end{aligned}$$

We define

$$\begin{aligned} \mathrm{Mor}(*,*) \times \mathrm{Mor}(*,*) &\to \mathrm{Mor}(*,*) \\ (f,g) &\mapsto g \circ f := \underbrace{g \cdot f}_{\text{multiplication in } G} \end{aligned}$$

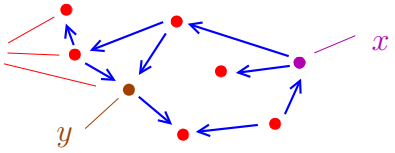
via the group structure on $\mathrm{Mor}(*,*) = G$. The axioms of a category can be easily deduced from the group axioms of G.

(i) Let $(P, \leq)$ be a partially ordered set, as defined on page 91. We fix a set $\{*\}$ consisting of a single element. We define the order category $\mathcal{Order}_{(P,\leq)}$ that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{Order}_{(P,\leq)}) &= P \\ \mathrm{Mor}(x,y) &= \begin{cases} \{*\}, & \text{if } x \leq y, \\ \varnothing, & \text{otherwise.} \end{cases} \end{aligned}$$

together with the only possible way to compose morphisms.

partially ordered set where $a \geq b$ if a is connected to b via a sequence of arrows



$\mathrm{Mor}(x,y) = \{*\}$
$\mathrm{Mor}(y,x) = \varnothing$
$\mathrm{Mor}(x,x) = \varnothing$
$\mathrm{Mor}(y,y) = \{*\}$

The following definition is almost self-explanatory.

[56] We refer to page 98 for our definition of a ring.

[57] If $X \neq Y$ then $\mathrm{Mor}(X,Y) = \varnothing$, but then $\mathrm{Mor}(X,Y) \times \mathrm{Mor}(Y,Z)$ is also the empty set for any Z.

Definition. Let $\mathcal{C}$ be a category. A subcategory is a category $\mathcal{D}$ such that $\mathrm{Ob}(\mathcal{D}) \subset \mathrm{Ob}(\mathcal{C})$ and such that for each $X, Y \in \mathrm{Ob}(\mathcal{D})$ we have $\mathrm{Mor}_{\mathcal{D}}(X,Y) \subset \mathrm{Mor}_{\mathcal{C}}(X,Y)$. We say $\mathcal{D}$ is a full subcategory if for each $X, Y \in \mathrm{Ob}(\mathcal{D})$ we have $\mathrm{Mor}_{\mathcal{D}}(X,Y) = \mathrm{Mor}_{\mathcal{C}}(X,Y)$.

Examples.

(1) The category *AbGroup* of abelian groups is a full subcategory of the category of groups *Group*.
(2) Let $\mathcal{C}$ be the category where the objects are groups but where the only morphisms are group isomorphisms. This is a subcategory of *Group*, but it is certainly not a full subcategory of *Group*.

Definition. Let $\mathcal{C}$ be a category.

(1) An object $X \in \mathrm{Ob}(\mathcal{C})$ is called initial if for each object $Y \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism from X to Y.
(2) An object $X \in \mathrm{Ob}(\mathcal{C})$ is called terminal if for each object $Y \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism from Y to X.

Example.

(1) In the category *Set* of sets an initial object is given by the empty set and a terminal object is given by any set consisting of a single element. Similarly in the category *Top* of topological spaces an initial object is given by the empty topology space and a terminal object is given by an topological space consisting of a single element.
(2) In the category *Group* of groups any group consisting of a single element is an initial and also a terminal object. The same applies to the category *AbGroup* of abelian groups and the analogous statement holds in the category $\mathcal{Vec}_{\mathbb{K}}$ of vector spaces over a field $\mathbb{K}$.

Lemma 3.2. Let $\mathcal{C}$ be a category.

(1) If an initial object exists, then it is unique up to isomorphism.
(2) If a terminal object exists, then it is unique up to isomorphism.

Proof. The statements follow easily from the definition. We will fill in the details in Exercise 3.4. ■

Next we introduce the dual of a category $\mathcal{C}$. The idea is very simple, we keep the objects, but we "invert the direction of the morphisms". This simple concept can be very convenient in future discussions.

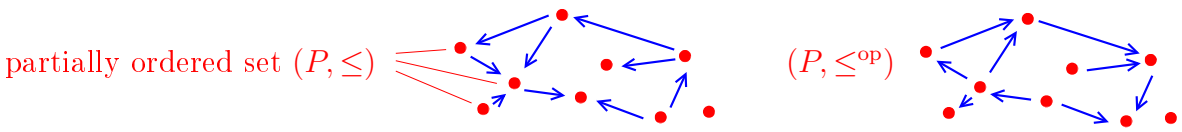
Definition. Let $\mathcal{C}$ be a category. The opposite category $\mathcal{C}^{\mathrm{op}}$ of $\mathcal{C}$ is defined as the category where

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) &:= \mathrm{Ob}(\mathcal{C}) \\ \mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(X,Y) &:= \mathrm{Mor}_{\mathcal{C}}(Y,X) \end{aligned}$$

with the composition given by $f \circ_{\mathcal{C}^{\mathrm{op}}} g = g \circ_{\mathcal{C}} f$.

Example.

(1) Let $(P, \leq)$ be a partially ordered set. We consider the corresponding order category $\mathcal{O}rder_{(P,\leq)}$. The dual of the order category $\mathcal{O}rder_{(P,\leq)}$ is the order category corresponding to the partially ordered set $(P, \leq^{\mathrm{op}})$ where $x \leq^{\mathrm{op}} y$ if and only if $y \leq x$.



(2) For any category $\mathcal{C}$ we evidently have $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$.
(3) Let X be the initial object of a category $\mathcal{C}$. It follows easily from the definitions that it is also the terminal object of the category $\mathcal{C}^{\mathrm{c}}$.

We conclude this introduction to categories and morphisms with the following definition and a short discussion thereof.

Definition. Let $\mathcal{C}$ be a category.
(1) In the following we call a morphism $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ invertible if there exists a morphism $g \in \mathrm{Mor}_{\mathcal{C}}(Y, X)$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. Sometimes we refer to invertible morphisms also as isomorphisms.
(2) We denote by $\mathrm{Mor}_{\mathcal{C}}(X, Y)_{\mathrm{inv}}$ the set of all invertible morphisms.

Examples.
(1) In the category of vector spaces the invertible morphisms are precisely the isomorphisms which are familiar from linear algebra.
(2) In the category considered in (g) every morphism is an isomorphism. In fact categories in which every morphism is an isomorphism are called groupoids.

Lemma 3.3. Let $\mathcal{C}$ be a category.
(1) Let $X, Y \in \mathrm{Ob}(\mathcal{C})$ and let $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$. If $g \in \mathrm{Mor}_{\mathcal{C}}(Y, X)$ is a morphism with $f \circ g = \mathrm{id}_Y$ and if $\widetilde{g} \in \mathrm{Mor}_{\mathcal{C}}(Y, X)$ is a morphism with $\widetilde{g} \circ f = \mathrm{id}_X$, then $g = \widetilde{g}$.
(2) Let $X, Y \in \mathrm{Ob}(\mathcal{C})$ and let $g, \widetilde{g} \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ with $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$ and with $f \circ \widetilde{g} = \mathrm{id}_Y$ and $\widetilde{g} \circ f = \mathrm{id}_X$. Then $g = \widetilde{g}$.
(3) Let $X \in \mathrm{Ob}(\mathcal{C})$. The set of invertible morphisms $\mathrm{Mor}_{\mathcal{C}}(X, X)_{\mathrm{inv}}$ forms a group under the composition of morphisms.

Proof.
(1) We have

$$g \underset{\substack{\uparrow \\ \text{defining property of } \mathrm{id}_X}}{=} \mathrm{id}_X \circ g \underset{\substack{\uparrow \\ \text{defining property of } g}}{=} (\widetilde{g} \circ f) \circ g \underset{\substack{\uparrow \\ \text{associativity of } \circ}}{=} \widetilde{g} \circ (f \circ g) \underset{\substack{\uparrow \\ \text{defining property of } \widetilde{g}}}{=} \widetilde{g} \circ \mathrm{id}_Y \underset{\substack{\uparrow \\ \text{defining property of } \mathrm{id}_Y}}{=} \widetilde{g}.$$

(2) This statement follows immediately from (1).
(3) Let $X \in \mathrm{Ob}(\mathcal{C})$. first note that by definition $\mathrm{Mor}_{\mathcal{C}}(X, X)$ is actually a set. Next note that by definition of a category we can compose any two morphisms that lie in $\mathrm{Mor}_{\mathcal{C}}(X, X)$. It follows easily from the associativity of "$\circ$" that the composition of two invertible morphisms is again invertible. It now follows from Properties (3A) and (3B) of a category that $\mathrm{Mor}_{\mathcal{C}}(X, X)_{\mathrm{inv}}$ forms a group. ∎

Definition. Let $\mathcal{C}$ be a category, let $X, Y \in \mathrm{Ob}(\mathcal{C})$ and let $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)_{\mathrm{inv}}$ be an invertible morphism. By Lemma 3.3 there exists a *unique* morphism $g \in \mathrm{Mor}_{\mathcal{C}}(Y, X)$ with

$f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. We refer to this morphism as the inverse of f and usually we denote it by f^{-1}.

Lemma 3.4. Let $\mathcal{C}$ be a category and let $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ be an invertible morphism. For every $Z \in \mathrm{Ob}(\mathcal{C})$ the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{C}}(Y, Z) & \to & \mathrm{Mor}_{\mathcal{C}}(X, Z) \\ g & \mapsto & g \circ f \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{Mor}_{\mathcal{C}}(Z, X) & \to & \mathrm{Mor}_{\mathcal{C}}(Z, Y) \\ g & \mapsto & f \circ g \end{array}$$

are bijections.

Proof. One can easily show that precomposition respectively postcomposition with the inverse morphism f^{-1} defines inverses to the given maps. ∎

3.2. **Functors.** In this section we introduce the all-important notions of covariant and contravariant functors.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A covariant functor $F \colon \mathcal{C} \to \mathcal{D}$ consists of a map

$$F \colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$$

and for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ a map

$$F \colon \mathrm{Mor}_{\mathcal{C}}(X, Y) \to \mathrm{Mor}_{\mathcal{D}}(F(X), F(Y)),$$

such that the following axioms are satisfied:

(F1) For all $X \in \mathrm{Ob}(\mathcal{C})$ we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

(F2) For all $\phi \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ and all $\psi \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)$ the following equality holds

$$F(\psi \circ \phi) = F(\psi) \circ F(\phi) \in \mathrm{Mor}_{\mathcal{D}}(F(X), F(Z)).$$

Convention. For a covariant functor F and $\phi \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ we often write ϕ_* instead of $F(\phi)$. The first axiom (F1) then becomes $(\mathrm{id}_X)_* = \mathrm{id}_{F(X)}$ and the second axiom (F2) becomes $(\psi \circ \phi)_* = \psi_* \circ \phi_*$.

Examples.

(1) The map

$$\begin{array}{rcl} F \colon \mathrm{Ob}(\mathcal{V}ec_{\mathbb{R}}) & \to & \mathrm{Ob}(\mathcal{S}et) \\ (V, +, \cdot) & \mapsto & V, \\ \mathrm{Mor}_{\mathcal{V}ec_{\mathbb{R}}}(V, W) & \to & \mathrm{Mor}_{\mathcal{S}et}(V, W) \\ (\varphi \colon V \to W) & \mapsto & (\varphi \colon V \to W) \end{array}$$

together with the maps

defines a covariant functor from the category $\mathcal{V}ec_{\mathbb{R}}$ of real vector spaces to the category $\mathcal{S}et$ of sets. This example can be generalized to many contexts. Functors of this type are colloquially called *forgetful functor*.

(2) Given any category there is the identity functor with the obvious definition.

(3) Let $\mathcal{Top}$ be the category of topological spaces and let $\mathcal{Set}$ be the category of sets. It follows easily from follows easily from Lemma 2.35 (2) that the map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{Top}) & \to & \mathrm{Ob}(\mathcal{Set}) \\ X & \mapsto & \pi_0(X) := \text{set of path-components of } X, \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Top}}(X,Y) & \to & \mathrm{Mor}_{\mathcal{Set}}(\pi_0(X),\pi_0(Y)) \\ f & \mapsto & \begin{pmatrix} \pi_0(X) & \to & \pi_0(Y) \\ C & \mapsto & f(C) \end{pmatrix} \end{array}$$

is a covariant functor from the category $\mathcal{Top}$ of topological spaces to the category $\mathcal{Set}$ of sets.

The next lemma says that covariant functors can be composed in an obvious way.

Lemma 3.5. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be three categories and let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be covariant functors. The map

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{C}) & \to & \mathrm{Ob}(\mathcal{E}) \\ X & \mapsto & G(F(X)) \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{C}}(X,Y) & \to & \mathrm{Mor}_{\mathcal{E}}(G(F(X)),G(F(Y))) \\ (\varphi\colon X \to Y) & \mapsto & G(F(\varphi))\colon G(F(X)) \to G(F(Y)) \end{array}$$

is a covariant functor from $\mathcal{C}$ to $\mathcal{E}$. We denote this functor by $G \circ F$.

Proof. The lemma follows easily from the definitions. ∎

Example. We consider the covariant functors $F\colon \mathcal{Set} \to \mathcal{PairSet}$ and $G\colon \mathcal{PairSet} \to \mathcal{Set}$ that are given by

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{Set}) & \to & \mathrm{Ob}(\mathcal{PairSet}) \\ X & \mapsto & (X,\varnothing) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G\colon \mathrm{Ob}(\mathcal{PairSet}) & \to & \mathrm{Ob}(\mathcal{Set}) \\ (X,A) & \mapsto & X \end{array}$$

together with the obvious maps on morphisms. The composition $G \circ F$ is just the identity functor on $\mathcal{Set}$ and the composition $F \circ G$ is the covariant functor that sends (X,A) to $(X,\varnothing)$.

We state our next example of a covariant functor as a lemma.

Lemma 3.6. Let $\mathcal{D}$ be a category and let $\mathcal{Set}$ be the category of sets. Furthermore let $W \in \mathrm{Ob}(\mathcal{D})$. The map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{D}) & \to & \mathrm{Ob}(\mathcal{Set}) \\ X & \mapsto & \mathrm{Mor}_{\mathcal{D}}(W,X) \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{D}}(Y,Z) & \to & \mathrm{Mor}_{\mathcal{Set}}(\mathrm{Mor}_{\mathcal{D}}(W,Y),\mathrm{Mor}_{\mathcal{D}}(W,Z)) \\ \phi & \mapsto & \begin{pmatrix} \mathrm{Mor}_{\mathcal{D}}(W,Y) & \to & \mathrm{Mor}_{\mathcal{D}}(W,Z) \\ g & \mapsto & \phi \circ g \end{pmatrix} \end{array}$$

is a covariant functor.

Proof. The statement follows easily from the axioms of a category. ∎

Example. The example of a covariant functor in Lemma 3.6 comes with many variations, for example if $\mathrm{Mor}_{\mathcal{D}}(W,X)$ has more structure than that of a set. For instance let W be a

real vector space. Then the above map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{V}ec_{\mathbb{R}}) & \to & \mathrm{Ob}(\mathcal{V}ec_{\mathbb{R}}) \\ V & \mapsto & \underbrace{\mathrm{Hom}_{\mathbb{R}}(W,V)}_{\text{real vector space}}, \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{V}ec_{\mathbb{R}}}(U,V) & \to & \mathrm{Mor}_{\mathcal{V}ec_{\mathbb{R}}}(F(U),F(V)) \\ (\phi\colon U \to V) & \mapsto & \begin{pmatrix} \mathrm{Hom}_{\mathbb{R}}(W,U) & \to & \mathrm{Hom}_{\mathbb{R}}(W,V) \\ g & \mapsto & \phi \circ g \end{pmatrix} \end{array}$$

defines a covariant functor from the category $\mathcal{V}ec_{\mathbb{R}}$ of real vector spaces to itself.

We also have the following definition which looks almost the same as the previous definition.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A contravariant functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of a map

$$F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$$

and for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ a map

$$\mathrm{Mor}_{\mathcal{C}}(X,Y) \to \mathrm{Mor}_{\mathcal{D}}(F(Y),F(X))$$

such that the following axioms are satisfied:

(F1) For all $X \in \mathrm{Ob}(\mathcal{C})$ we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.
(F2) For all $\phi \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ and all $\psi \in \mathrm{Mor}_{\mathcal{C}}(Y,Z)$ we have

$$F(\psi \circ \phi) = F(\phi) \circ F(\psi) \in \mathrm{Mor}_{\mathcal{D}}(F(Z),F(X)).$$

The definition of a contravariant functor looks very much the same as the definition of a covariant functor. The difference is that a contravariant functor "reverses the direction of morphisms" and the orders of $F(\psi)$ and $F(\phi)$ in the second axiom are swapped.

Convention. For a contravariant functor F and $\phi \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ we often write ϕ^* instead of $F(\phi)$. The first axiom then becomes $(\mathrm{id}_X)^* = \mathrm{id}_{F(X)}$ and the second axiom becomes $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

We state our first example as a lemma, namely we have the following analogue of Lemma 3.6.

Lemma 3.7. Let $\mathcal{D}$ be a category and let $\mathcal{S}et$ be the category of sets. Furthermore let $W \in \mathrm{Ob}(\mathcal{D})$. The map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{D}) & \to & \mathrm{Ob}(\mathcal{S}et) \\ X & \mapsto & \mathrm{Mor}_{\mathcal{D}}(X,W) \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{D}}(Y,Z) & \to & \mathrm{Mor}_{\mathcal{S}et}(\mathrm{Mor}_{\mathcal{D}}(Z,W),\mathrm{Mor}_{\mathcal{D}}(Y,W)) \\ f & \mapsto & \begin{pmatrix} \mathrm{Mor}_{\mathcal{D}}(Z,W) & \to & \mathrm{Mor}_{\mathcal{D}}(Y,W) \\ g & \mapsto & g \circ f \end{pmatrix} \end{array}$$

is a contravariant functor.

Proof. Again the statement follows easily from the axioms of a category. ∎

Examples. Lemma 3.7 can be viewed as the mother of all contravariant functors. As in the case of Lemma 3.6 it spawns many variations if $\mathrm{Mor}_{\mathcal{D}}(X,W)$ has more structure than just being a set.

(1) The map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{Set}) & \to & \mathrm{Ob}(\mathcal{Vec}_{\mathbb{R}}) \\ X & \mapsto & \mathrm{Map}(X,\mathbb{R}) := \underbrace{\text{set of maps } X \to \mathbb{R}}_{\text{vector space in the obvious way}}, \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Set}}(X,Y) & \to & \mathrm{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(\mathrm{Map}(Y,\mathbb{R}),\mathrm{Map}(X,\mathbb{R})) \\ (\phi\colon X \to Y) & \mapsto & \begin{pmatrix} \mathrm{Map}(Y,\mathbb{R}) & \to & \mathrm{Map}(X,\mathbb{R}) \\ f & \mapsto & f\circ\phi \end{pmatrix} \end{array}$$

is a contravariant functor from the category $\mathcal{Set}$ of sets to the $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces.

(2) The map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{Vec}_{\mathbb{R}}) & \to & \mathrm{Ob}(\mathcal{Vec}_{\mathbb{R}}) \\ V & \mapsto & V^* := \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}) =: \text{dual vector space of } V, \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(U,V) & \to & \mathrm{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(V^*,U^*) \\ (\phi\colon U \to V) & \mapsto & \begin{pmatrix} \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}) & \to & \mathrm{Hom}_{\mathbb{R}}(U,\mathbb{R}) \\ f & \mapsto & f\circ\phi \end{pmatrix} \end{array}$$

is a contravariant functor from the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces to itself.

We have the following analogue of Lemma 3.5, but note that it comes with a twist.

Lemma 3.8. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be three categories.

(1) Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be *two contravariant functors*. The map

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{C}) & \to & \mathrm{Ob}(\mathcal{E}) \\ X & \mapsto & G(F(X)) \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{C}}(X,Y) & \to & \mathrm{Mor}_{\mathcal{E}}(G(F(X)),G(F(Y))) \\ (\varphi\colon X \to Y) & \mapsto & F(G(\varphi))\colon G(F(X)) \to G(F(Y)) \end{array}$$

is a *covariant functor* from the category $\mathcal{C}$ to the category $\mathcal{E}$.

(2) Similarly we can define the composition of a covariant functor with a contravariant functor and we obtain a contravariant functor.

Proof. The lemma again follows easily from the definitions. Nonetheless it is worth going through the argument to make sue that all maps actually make sense and go into the desired direction. ■

The following lemma shows that contravariant functors are covariant functors in disguise.

Lemma 3.9. If

$$\begin{array}{c} F\colon \mathcal{C} \to \mathcal{D} \\ X \mapsto F(X) \\ (X \xrightarrow{\varphi} Y) \mapsto (F(Y) \xrightarrow{F(\varphi)} F(X)) \end{array}$$

is a contravariant functor between two categories, then

$$\begin{array}{c} \mathcal{C} \to \mathcal{D}^{\mathbf{c}} \\ X \mapsto F(X) \\ (X \xrightarrow{\varphi} Y) \mapsto (F(Y) \xrightarrow{F(\varphi)} F(X)) \end{array}$$

is a covariant functor. The same statement also holds with the roles of "covariant" and "contravariant" swapped.

Proof. As so often in category theory, the statement follows immediately from the definitions. ■

Lemma 3.9 can be useful for developing the general study of functors. When we work with "real life" categories, then personally we find that Lemma 3.9 is not all that useful and we prefer to work with both notions of functors.
We conclude this section with the following basically obvious lemma.

Lemma 3.10. Every functor (regardless of whether it is covariant or contravariant) sends invertible morphisms to invertible morphisms.

3.3. **Natural transformations.** In this section we introduce the notion of a natural transformation between two functors. This topic might eventually look even more esoteric, but once one keeps one's eyes open one realizes that the mathematical world is stuffed with natural transformations and natural isomorphisms. They just suffer from the sad fate that often they do not recognized as such.
After this long preamble, let us give the definition of natural transformations.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two covariant functors.

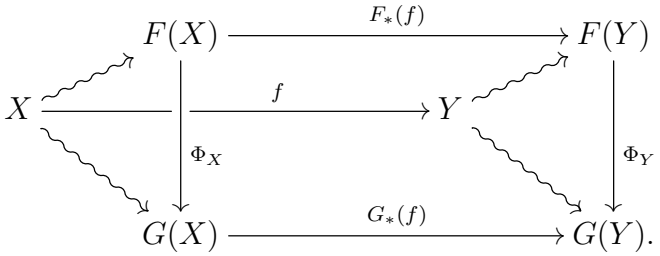
(1) A natural transformation between the functors F and G assigns to each $X \in \mathrm{Ob}(\mathcal{C})$ a morphism $\Phi_X\colon F(X) \to G(X)$ in $\mathcal{D}$ such that for each morphism $f\colon X \to Y$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F_*(f)} & F(Y) \\ {\scriptstyle \Phi_X}\downarrow & & \downarrow{\scriptstyle \Phi_Y} \\ G(X) & \xrightarrow{G_*(f)} & G(Y). \end{array}$$

(2) If for every $X \in \mathrm{Ob}(\mathcal{C})$ the morphism $\Phi_X\colon F(X) \to G(X)$ is an isomorphism, then we refer to the natural transformation as a natural isomorphism. In this case, given $X \in \mathrm{Ob}(\mathcal{C})$ we often say that $F(X)$ and $G(X)$ are naturally isomorphic.[58]

The same way we can of course also define the notion of a natural transformation and a natural isomorphism between two contravariant functors.

Remark. The notion of a natural transformation can also be summarized in the following diagram:



[58] In the literature the words "natural isomorphism" and "canonical isomorphism" are often used indiscriminately, but whereas the expression "natural isomorphism" has a precise meaning, the expression "canonical isomorphism" is not a well-defined mathematical concept. However, in the lecture notes we will on occasions write that "some object exists canonically". This means that in the proof we show the existence of the object by giving an explicit construction that does not rely on any choices.

Here the wiggly arrows are not morphisms but they are just meant to remind us which objects a functor associates to the objects X and Y.

Now we will see that we have encountered natural transformations and natural isomorphisms much more often than one might have initially thought.

Examples.

(1) Let *CRing* be the category of commutative rings and let *Group* be the category of groups. Let $n \in \mathbb{N}$. We consider the covariant functors

$$\begin{array}{rcl} F\colon \mathcal{CRing} & \to & \mathcal{Group} \\ R & \mapsto & \mathrm{GL}(n,R) \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon \mathcal{CRing} & \to & \mathcal{Group} \\ R & \mapsto & R^* := \text{units in } R. \end{array}$$

For each commutative ring R we consider the group homomorphism

$$\begin{array}{rcl} \Phi_R\colon \mathrm{GL}(n,R) & \to & R^* \\ A & \mapsto & \det(A). \end{array}$$

We claim that these group homomorphisms define a natural transformation from F to G. Thus let $f\colon R \to S$ be a homomorphism between commutative rings and let $A \in F(R)$, i.e. $A = (a_{ij})$ is a matrix in $\mathrm{GL}(n,R)$. We have

$$\begin{aligned} \Phi_S(F_*(f)(A)) &= \det((f(a_{ij}))_{i,j=1,\dots,n}) \overset{\text{Leibniz formula}}{\underset{\downarrow}{=}} \sum_{\sigma\in S_n} \operatorname{sign}(\sigma)\cdot\prod_{i=1}^{n} f(a_{i,\sigma(i)}) \\ &\underset{\substack{\uparrow \\ f \text{ is a ring homomorphism}}}{=} f\Big(\sum_{\sigma\in S_n} \operatorname{sign}(\sigma)\cdot\prod_{i=1}^{n} a_{i,\sigma(i)}\Big) \underset{\substack{\uparrow \\ \text{Leibniz formula}}}{=} f(\det(A)) = G_*(f)(\Phi_R(A)). \end{aligned}$$

But that means, by definition, that the Φ_R define a natural transformation from F to G.

(2) Let $\mathbb{K}$ be a field. We consider again the category $\mathcal{Vec}_{\mathbb{K}}$ of vector spaces over $\mathbb{K}$. By Lemma 3.8 we have a covariant functor $F\colon \mathcal{Vec}_{\mathbb{K}} \to \mathcal{Vec}_{\mathbb{K}}$ that is given by the maps

$$\begin{array}{rcl} V & \mapsto & (V^*)^* \\ (f\colon V \to W) & \mapsto & ((f^*)^*\colon (V^*)^* \to (W^*)^*). \end{array}$$

For each real vector space V we denote by Φ_V the homomorphism

$$\begin{array}{rcl} V & \to & (V^*)^* \\ v & \mapsto & \left(\begin{array}{rcl} V^* & \to & \mathbb{K} \\ (\varphi\colon V \to \mathbb{K}) & \mapsto & \varphi(v) \end{array}\right). \end{array}$$

It is straightforward to verify that these maps define a natural transformation from the identity functor $\mathrm{id}\colon \mathcal{Vec}_{\mathbb{K}} \to \mathcal{Vec}_{\mathbb{K}}$ to the functor $F\colon \mathcal{Vec}_{\mathbb{K}} \to \mathcal{Vec}_{\mathbb{K}}$. We will discuss this example in greater detail in Exercise 3.4.

(3) Let $\mathbb{K}$ be a field. We denote by *Epi* the following category:

- The objects are epimorphisms $\varphi\colon V \to W$ between vector spaces over $\mathbb{K}$.

- A morphism between objects $\varphi\colon V \to W$ and $\varphi'\colon V' \to W'$ consists of two $\mathbb{K}$-linear maps $\alpha\colon V \to V'$ and $\beta\colon W \to W'$ such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ V' & \xrightarrow{\varphi'} & W'. \end{array}$$

  The composition of two morphisms (α, β) and $(\widetilde{\alpha}, \widetilde{\beta})$ is defined as $(\widetilde{\alpha} \circ \alpha, \widetilde{\beta} \circ \beta)$, provided $\widetilde{\alpha} \circ \alpha$ and $\widetilde{\beta} \circ \beta$ are defined.

The following maps define covariant functors from the category $\mathcal{E}pi$ to the category $\mathcal{V}ec_{\mathbb{K}}$ of vector spaces over $\mathbb{K}$:

$$\begin{array}{rcl} F\colon \mathcal{E}pi & \to & \mathcal{V}ec_{\mathbb{K}} \\ (\varphi\colon V \to W) & \mapsto & V/\ker(\varphi) \\ (\alpha, \beta) & \mapsto & \begin{pmatrix} V/\ker(\varphi) & \to & V'/\ker(\varphi') \\ [v] & \mapsto & [\alpha(v)] \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon \mathcal{E}pi & \to & \mathcal{V}ec_{\mathbb{K}} \\ (\varphi\colon V \to W) & \mapsto & W \\ (\alpha, \beta) & \mapsto & \beta. \end{array}$$

The maps

$$\begin{array}{rcl} V/\ker(\varphi) & \to & W \\ [v] = v + \ker(\varphi) & \mapsto & \varphi(v) \end{array}$$

form a natural isomorphism from the functor F to the functor G.

(4) We consider the covariant functors

$$\begin{array}{rcl} F\colon \mathcal{R}ing & \to & \mathcal{R}ing \\ R & \mapsto & \mathbb{Z} \\ (f\colon R \to S) & \mapsto & (\mathrm{id}\colon \mathbb{Z} \to \mathbb{Z}) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{id}\colon \mathcal{R}ing & \to & \mathcal{R}ing \\ R & \mapsto & R \\ (f\colon R \to S) & \mapsto & (f\colon R \to S). \end{array}$$

For each commutative ring R we consider the ring homomorphism

$$\begin{array}{rcl} \Phi_R\colon \mathbb{Z} & \to & R \\ n & \mapsto & n \cdot 1_R. \end{array}$$

Note by our definition of a ring homomorphism on page 98 it follows that for each ring homomorphism $f\colon R \to S$ we have $f(1_R) = 1_S$. This implies that the following diagram commutes:

$$\begin{array}{ccc} F(R) = \mathbb{Z} & \xrightarrow{F_*(f)=\mathrm{id}} & \mathbb{Z} = F(S) \\ {\scriptstyle\Phi_R}\downarrow & & \downarrow{\scriptstyle\Phi_S} \\ \mathrm{id}(R) = R & \xrightarrow{\mathrm{id}_*(f)=f} & S = \mathrm{id}(S). \end{array}$$

But this shows that the ring homomorphisms Φ_R define a natural transformation from the covariant functor F to the identity functor id. Thus, given a commutative ring, we refer to this ring homomorphism $\mathbb{Z} \to R$ as the *natural ring homomorphism*.

Perhaps in this instance it is also important to give a "non-example".

Non-Example. Let $\mathcal{FGAbGroup}$ be the category of finitely generated abelian groups. Given an abelian group A we denote by $\mathrm{Tor}(A) := \{g \in A \mid \text{there exists an } n \in \mathbb{N} \text{ with } n \cdot g = 0\}$

its torsion subgroup and we denote by $\mathrm{F}A := A/\operatorname{Tor}(A)$ the so maximal torsion-free quotient. We consider the following two covariant functors that are given by[59]

$$\begin{array}{rcl} F\colon \mathcal{FGAbGroup} & \to & \mathcal{FGAbGroup} \\ A & \mapsto & A \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon \mathcal{FGAbGroup} & \to & \mathcal{FGAbGroup} \\ A & \mapsto & \operatorname{Tor}(A) \oplus \mathrm{F}A \end{array}$$

together with the obvious maps on morphisms. It follows from the Finitely Generated Abelian Group Classification Theorem 51.4 that for any finitely generated abelian group A the groups $F(A) = A$ and $G(A) = \operatorname{Tor}(A) \oplus \mathrm{F}A$ are isomorphic. Now we claim that nonetheless there is *no* natural isomorphism between F and G. Indeed, suppose there was such a natural isomorphism Φ. Let us consider the group $A = \mathbb{Z}_2 \oplus \mathbb{Z}$ and the homomorphism

$$\begin{array}{rcl} f\colon A = \mathbb{Z}_2 \oplus \mathbb{Z} & \to & \mathbb{Z}_2 \oplus \mathbb{Z} = A \\ ([a], b) & \mapsto & ([b], 0). \end{array}$$

Since we assume that Φ is a natural isomorphism we obtain the following commutative diagram

$$\begin{array}{ccc} F(A) & \xrightarrow[\cong]{\Phi_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(A) & \xrightarrow[\cong]{\Phi_A} & G(A) \end{array} \quad \text{which specializes in our setting to the following commutative diagram} \quad \begin{array}{ccc} \mathbb{Z}_2 \oplus \mathbb{Z} & \xrightarrow[\cong]{\Phi_A} & \mathbb{Z}_2 \oplus \mathbb{Z} \\ {\scriptstyle F(f)=f}\downarrow & & \downarrow{\scriptstyle G(f)=0} \\ \mathbb{Z}_2 \oplus \mathbb{Z} & \xrightarrow[\cong]{\Phi_A} & \mathbb{Z}_2 \oplus \mathbb{Z}. \end{array}$$

Note that the vertical map to the left is non-trivial whereas a close inspection shows that the vertical map to the right is trivial. But this collides with the statement that the horizontal maps are isomorphisms and that the diagram commutes.

On many occasions we, and many authors, say that a certain map is "natural". This means that the map is in fact a natural transformation between two suitable functors between two suitable categories. Sometimes it takes some thought to figure out what the right context is.

Example. Let G be a group and let A and B be normal subgroups of G with $A \subset B$. The Third Isomorphism Theorem in group theory, see [**Hun1980**, Corollary 5.10], says that the map

$$\begin{array}{rcl} \Phi_{(G,A,B)}\colon G/B & \to & (G/A)/(B/A) \\ g \cdot B & \mapsto & (g \cdot A) \cdot B/A \end{array}$$

is an isomorphism. This isomorphism is a natural isomorphism in the following sense. We consider the category $\mathcal{TripGr}$ of triples of groups which is defined as follows:

- The objects are the triples (G, A, B) where G is a group and A and B are normal subgroups of G with $A \subset B$.
- The morphisms from (G, A, B) to (G', A', B') are the homomorphisms $\varphi\colon G \to G'$ with $\varphi(A) \subset A'$ and $\varphi(B) \subset B'$.

[59] Let us explain why G is in fact a covariant functor. Let $f\colon A \to B$ be a homomorphism between two finitely generated abelian groups. Since a homomorphism sends torsion elements to torsion elements we see that f restricts to a homomorphism $\operatorname{Tor}(A) \to \operatorname{Tor}(B)$. This also implies that f descends to a homomorphism $\mathrm{F}\,A = A/\operatorname{Tor}(A) \to \mathrm{F}\,B = B/\operatorname{Tor}(B)$. Summarizing we see that $f\colon A \to B$ induces a homomorphism $G(f)\colon G(A) = \operatorname{Tor}(A) \oplus \mathrm{F}\,A \to G(B) = \operatorname{Tor}(B) \oplus \mathrm{F}\,B$.

We consider the two functors

$$\begin{array}{rcl} F\colon \mathcal{TripGr} & \to & \mathcal{Group} \\ (G,A,B) & \mapsto & G/B \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon \mathcal{TripGr} & \to & \mathcal{Group} \\ (G,A,B) & \mapsto & (G/A)/(B/A). \end{array}$$

One can easily verify that the above isomorphisms $\Phi_{(G,A,B)}$ from the third Isomorphism Theorem define a natural isomorphism from F to G.

All of the above examples of natural transformations are perhaps not particularly impressive. But as we will see, the concept of a natural transformation will eventually become one of the key notions in this course.

3.4. Equivalence of categories.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories.

(1) An equivalence of the categories $\mathcal{C}$ and $\mathcal{D}$ is a tuple (F, G, μ, ν) where $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ are two covariant functors and $\mu\colon G \circ F \to \mathrm{id}_{\mathcal{C}}$ and $\nu\colon F \circ G \to \mathrm{id}_{\mathcal{D}}$ are natural isomorphisms.
(2) A weak equivalence of the categories $\mathcal{C}$ and $\mathcal{D}$ is a covariant functor $F\colon \mathcal{C} \to \mathcal{D}$ that admits a covariant functor $G\colon \mathcal{D} \to \mathcal{C}$ and two natural isomorphisms $\mu\colon G \circ F \to \mathrm{id}_{\mathcal{C}}$ and $\nu\colon F \circ G \to \mathrm{id}_{\mathcal{D}}$ such that (F, G, μ, ν) is an equivalence of categories.[60]
(3) We say $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories, if there exists an equivalence of categories.

Remark. Given a weak equivalence F of categories $\mathcal{C}$ and $\mathcal{D}$ the corresponding G, μ, ν are not uniquely defined.

Proposition 3.11. (Weak Equivalence of Categories Proposition) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a covariant functor. The following two statements are equivalent:

(1) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a weak equivalence of categories.
(2) The following three statements hold:
   (a) The functor F is full, i.e. for any two objects C_1, C_2 of the category $\mathcal{C}$ the map $\mathrm{Mor}_{\mathcal{C}}(C_1, C_2) \to \mathrm{Mor}_{\mathcal{D}}(F(C_1), F(C_2))$ is surjective.
   (b) The functor F is faithful, i.e. for any two objects C_1, C_2 of the category $\mathcal{C}$ the map $\mathrm{Mor}_{\mathcal{C}}(C_1, C_2) \to \mathrm{Mor}_{\mathcal{D}}(F(C_1), F(C_2))$ is injective.
   (c) The functor F is essentially surjective (or *dense*), i.e. each object D in the category $\mathcal{D}$ is isomorphic to an object of the form $F(C)$ where C is an object of the category $\mathcal{C}$.

Proof. The "(1) $\Rightarrow$ (2)"-direction is elementary. The "(2) $\Rightarrow$ (1)" builds on the Axiom of Choice 0.1 and can be proved "by hand" with a little bit of an effort. The details are worked out in [**Rie2016**, Theorem 1.5.9]. ■

Example. Let $\mathcal{C}$ be the category given by the objects $\{\mathbb{R}^n \mid n \in \mathbb{N}_0\}$, where the morphisms $\mathrm{Mor}_{\mathcal{C}}(\mathbb{R}^m, \mathbb{R}^n)$ are all real $(n \times m)$-matrices and where composition of morphisms is given by multiplication of matrices. Furthermore let $\mathcal{D}$ be the category where the objects are all finite dimensional real vector spaces and where the morphisms are all homomorphisms. It

[60] In other words, a weak equivalence of categories is the same as an equivalence of categories, except that G, μ and ν are not part of the date. In particular an equivalence of categories exists if and only if a weak equivalence of categories exists.

follows easily from Proposition 3.11 and very elementary linear algebra that the obvious functor $\mathcal{C} \to \mathcal{D}$ is an equivalence of categories.

We conclude this section with the following lemma that will come in handy at times.

Lemma 3.12. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a weak equivalence of categories between two categories and let $V\colon \mathcal{C} \to \mathcal{V}$ be a covariant functor.

(1) There exists a covariant functor $W\colon \mathcal{D} \to \mathcal{V}$ such that $W \circ F$ is naturally isomorphic to V.
(2) If $W\colon \mathcal{D} \to \mathcal{V}$ and $\widetilde{W}\colon \mathcal{D} \to \mathcal{V}$ are two covariant functors such that $W \circ F$ and $\widetilde{W} \circ F$ are naturally isomorphic to V, then W and $\widetilde{W}$ are naturally isomorphic.

Proof. Proofs in basic category theory are almost invariably elementary and a little dull. Thus we outsource the proof to Exercise 3.8. ■

Exercises for Chapter 3.

Exercise 3.1. Let $\mathcal{C}$ be a category.

(1) A morphism $g \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ is called a *monomorphism* if for every two morphisms $f_1, f_2 \in \mathrm{Mor}_{\mathcal{C}}(W, X)$ with $g \circ f_1 = g \circ f_2$ we also have $f_1 = f_2$.
(2) A morphism $g \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ is called an *epimorphism* if for every two morphisms $h_1, h_2 \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)$ with $h_1 \circ g = h_2 \circ g$ we also have $h_1 = h_2$.

We consider these notions in two different categories.

(a) Show that in the category *AbGroup* of abelian groups the monomorphisms are precisely the injective homomorphisms and that the epimorphisms are precisely the surjective homomorphisms.
Remark. The same statements also hold in the category *Group* of all groups. The proof for monomorphisms is straightforward, but the proof for epimorphisms is non-trivial, we refer to [**Lin1970**] and [**Pare1970**, p. 16] for details.
(b) Show that in the category *Ring* of rings the inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism.
(c) An abelian group G is called *divisible* if for every $g \in G$ and $n \in \mathbb{N}$ there exists an $h \in G$ with $n \cdot h = g$. We define the category of divisible abelian groups in the obvious way. Show that in this category the obvious homomorphism $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a monomorphism.

Exercise 3.2. Let $\mathcal{C}$ be a category. We recall the following definition from Exercise 3.1:

(1) A morphism $g \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ is called a *monomorphism* if for every two morphisms $f_1, f_2 \in \mathrm{Mor}_{\mathcal{C}}(W, X)$ with $g \circ f_1 = g \circ f_2$ we also have $f_1 = f_2$.
(2) A morphism $g \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ is called an *epimorphism* if for every two morphisms $h_1, h_2 \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)$ with $h_1 \circ g = h_2 \circ g$ we also have $h_1 = h_2$.

Now solve the following two exercises:

(a) Show that in the category *Top* of topological spaces a monomorphism is not necessarily injective.
(b) Show that in the category *Top* of topological spaces an epimorphism is not necessarily surjective.

Exercise 3.3. Let $\mathbb{K}$ be a field. Given a $\mathbb{K}$-vector space V we write $V^* := \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$. Given a homomorphism $f\colon V \to W$ we denote by $f^*\colon W^* \to V^*$ the dual map. As on page 204 we consider the covariant functor $F\colon \mathcal{V}ec_{\mathbb{K}} \to \mathcal{V}ec_{\mathbb{K}}$ F that is given by the maps

$$\begin{array}{rcl} V & \mapsto & (V^*)^* \\ (f\colon V \to W) & \mapsto & ((f^*)^*\colon (V^*)^* \to (W^*)^*). \end{array}$$

Furthermore for each $\mathbb{K}$-vector space V we denote by Φ_V the homomorphism

$$\begin{array}{rcl} V & \to & (V^*)^* \\ v & \mapsto & \left(\begin{array}{rcl} V^* & \to & \mathbb{K} \\ (\varphi\colon V \to \mathbb{K}) & \mapsto & \varphi(v) \end{array} \right). \end{array}$$

On page 204 we saw that the maps Φ_V define a natural transformation from $F\colon \mathcal{V}ec_{\mathbb{K}} \to \mathcal{V}ec_{\mathbb{K}}$ to the identity morphism $\mathrm{id}\colon \mathcal{V}ec_{\mathbb{K}} \to \mathcal{V}ec_{\mathbb{K}}$.

(a) Show that the maps Φ_V are in general not isomorphisms.
(b) We denote by $\mathcal{V}ec_{\mathbb{K}}^{\mathrm{fin}}$ the full subcategory of $\mathcal{V}ec_{\mathbb{K}}$ that is given by the finite dimensional $\mathbb{K}$-vector spaces. Note that F restricts to a covariant functor $\mathcal{V}ec_{\mathbb{K}}^{\mathrm{fin}} \to \mathcal{V}ec_{\mathbb{K}}^{\mathrm{fin}}$. Show that this functor is naturally isomorphic to the identity functor.

Exercise 3.4.

(a) Let $\mathcal{C}$ be a category. Show that if an initial object exists, then it is unique up to isomorphism.
(b) Which of the following categories admits an initial object and which ones admit a terminal object?
  (i) The category *Ring* of rings.
  (ii) The full subcategory *Field* of the category *Ring* that consists only of fields.
  (iii) Let $(P, \leq)$ be a partially order set. We consider the corresponding order category $\mathcal{O}rder_{(P,\leq)}$.
(c) Give an example of a category that contains an initial object but not a terminal object.
(d) Give an example of a category that contains a terminal object but that does not contain an initial object.

Exercise 3.5. Let $\mathbb{K}$ be a field. We denote by $\mathcal{V}ec_{\mathbb{K}}^n$ the category where the objects are all vector spaces over $\mathbb{K}$ of dimension n and where the morphisms are isomorphisms of vector spaces. We consider the following two covariant functors:

$$\begin{array}{rcl} F\colon \mathcal{V}ec_{\mathbb{K}}^n & \to & \mathcal{V}ec_{\mathbb{K}}^n \\ V & \mapsto & \mathbb{K}^n \\ (\varphi\colon V \to W) & \mapsto & (\mathrm{id}\colon \mathbb{K}^n \to \mathbb{K}^n) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G\colon \mathcal{V}ec_{\mathbb{K}}^n & \to & \mathcal{V}ec_{\mathbb{K}}^n \\ V & \mapsto & V \\ (\varphi\colon V \to W) & \mapsto & (\varphi\colon V \to W). \end{array}$$

Are the two functors naturally isomorphic?

Exercise 3.6. Let $\mathbb{K}$ be a field, let W be a $\mathbb{K}$-vector space and let $U, V \subset W$ be vector subspaces. In what sense is the isomorphism

$$\begin{array}{rcl} V/(U \cap V) & \to & (U+V)/U \\ v + U \cap V & \mapsto & v + U \end{array}$$

a natural isomorphism?

Exercise 3.7. Let $G = (G, *)$ be a group. We define the *opposite group* $G^{\mathrm{op}} = (G, *^{\mathrm{op}})$ as the group

$$\begin{array}{rcl} G \times G & \to & G \\ (g, h) & \mapsto & g *^{\mathrm{op}} h := h * g. \end{array}$$

(a) Show that the maps $G \mapsto G^{\mathrm{op}}$ and $f \mapsto f$ define a covariant functor from the category *Group* of groups to itself.

(b) Show that the functor in (a) is naturally isomorphic to the identity functor on the category *Group* of groups.

Exercise 3.8. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an equivalence of categories between two categories and let $V\colon \mathcal{C} \to \mathcal{V}$ be a covariant functor.

(1) Show that there exists a covariant functor $W\colon \mathcal{D} \to \mathcal{V}$ such that $W \circ F$ is naturally isomorphic to V.

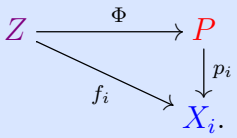
(2) Let $W\colon \mathcal{D} \to \mathcal{V}$ and $\widetilde{W}\colon \mathcal{D} \to \mathcal{V}$ be two covariant functors such that $W \circ F$ and $\widetilde{W} \circ F$ are naturally isomorphic to V. Show that W and $\widetilde{W}$ are naturally isomorphic.

4. Products and Coproducts – Limits and colimits

In this chapter we will continue with our general discussion of categories. First we will introduce the notion of products and coproducts. Afterwards we will discuss the even more general notions of limits and colimits. These notions will guide us in our exploration of topological spaces.

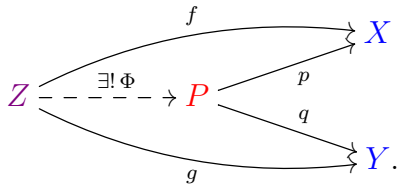
4.1. Products.

Definition. Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. A product of $\{X_i\}_{i\in I}$ in the category $\mathcal{C}$ is an object $P \in \mathrm{Ob}(\mathcal{C})$ together with a family of morphisms $\{p_i\colon P \to X_i\}_{i\in I}$ such that the following universal property is satisfied: Whenever we are given a $Z \in \mathrm{Ob}(\mathcal{C})$ and a family of morphisms $\{f_i\colon Z \to X_i\}_{i\in I}$ there exists a unique morphism $\Phi\colon Z \to P$ such that for each $i \in I$ we have $f_i = p_i \circ \Phi$, i.e. such that the following diagram commutes:[61]



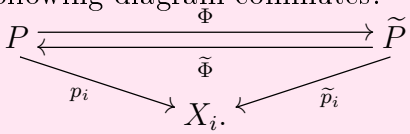
Example. Let $\mathcal{C}$ be a category.

(1) Let $X, Y \in \mathrm{Ob}(\mathcal{C})$ be two objects. A product of X and Y in the category $\mathcal{C}$ is some object $P \in \mathrm{Ob}(\mathcal{C})$ together with two morphisms $p\colon P \to X$ and $q\colon P \to Y$ such that the following universal property is satisfied: Whenever we are given an object $Z \in \mathrm{Ob}(\mathcal{C})$ and two morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ there exists a unique morphism $\Phi\colon Z \to P$ such that $f = p \circ \Phi$ and $g = q \circ \Phi$. The situation is summarized in the following diagram:



(2) If we take the empty family of objects in $\mathcal{C}$, then it follows easily from the definitions that the corresponding product is a terminal object in the category $\mathcal{C}$.

Lemma 4.1. (Product Uniqueness Lemma) Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. If we are given two products $(P, \{p_i\colon P \to X_i\}_{i\in I})$ and $(\widetilde{P}, \{\widetilde{p}_i\colon \widetilde{P} \to X_i\}_{i\in I})$ then the universal properties of both products define uniquely determined isomorphisms $\Phi\colon P \to \widetilde{P}$ and $\widetilde{\Phi}\colon \widetilde{P} \to P$ which are inverses of one another and such that for each $i \in I$ the following diagram commutes:



[61] Sometimes we refer to P also as the *categorical product* of $\{X_i\}_{i\in I}$ if we want to distinguish it from the usual set-theoretic product.

Proof. The lemma follows easily from the definitions. We will work out the details in Exercise 4.1. ■

Notation. Let $\mathcal{C}$ be a category. Let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$ such that a product exists.

(1) We usually denote the product by $\prod_{i\in I} X_i$ and we refer to the morphisms $\prod_{i\in I} X_i \to X_i$ as natural projections. Furthermore, given morphisms $f_i\colon Z \to X_i$ we denote the corresponding morphism $Z \to \prod_{i\in I} X_i$ by $\prod_{i\in I} f_i$.

(2) If $I = \{1, \dots, k\}$, then often we denote the product by $X_1 \times \cdots \times X_k$. Furthermore given morphisms $f_i\colon Z \to X_i$, $i = 1, \dots, k$ we denote the corresponding morphism $Z \to X_1 \times \cdots \times X_k$ by $(f_1, \dots, f_k)$.

Lemma 4.2. (Product-Sets-and-Groups Lemma)

(1) Let *Set* be the category of sets and let $\{X_i\}_{i\in I}$ be a family of sets. The product in *Set* is precisely the product

$$\prod_{i\in I} X_i \ := \ \left\{ f\colon I \to \bigcup_{i\in I} X_i \,\middle|\, \text{for every } i \in I \text{ we have } f(i) \in X_i \right\}$$

as defined on page 88, together with the obvious maps to X_i that is given by $f \mapsto f(i)$.

(2) Let *Group* be the category of groups and let $\{X_i\}_{i\in I}$ be a family of groups. The product in *Group* is the set

$$\prod_{i\in I} X_i \ = \ \left\{ f\colon I \to \bigcup_{i\in I} X_i \,\middle|\, \text{for every } i \in I \text{ we have } f(i) \in X_i \right\}$$

as in (1), which we now view as a group by the obvious multiplication, together with the obvious homomorphisms to X_i.

(3) The same approach as in (2) also applies to the category *AbGroup* of abelian groups and the category $\mathcal{V}ec_{\mathbb{K}}$ of vector spaces over a fixed field $\mathbb{K}$.

Proof.

(1) So let $Z \in \mathrm{Ob}(\mathcal{S}et)$ and let $\{f_i\colon Z \to X_i\}_{i\in I}$ be a family of maps between sets. We define

$$\begin{array}{rcl} \Phi\colon Z & \to & \prod_{i\in I} X_i \\ z & \mapsto & \begin{pmatrix} I & \to & \bigcup_{i\in I} X_i \\ i & \mapsto & f_i(z) \end{pmatrix}. \end{array}$$

It follows immediately from the definition that for each $i \in I$ we have $f_i = p_i \circ \Phi$. It remains to show that Φ is unique. So let $\Psi\colon Z \to \prod_{i\in I} X_i$ be a map such that for each $i \in I$ we have $f_i = p_i \circ \Psi$. We need to show that $\Phi = \Psi$, i.e. we need to show that for each $z \in Z$ and $i \in I$ we have $\Phi(z)(i) = \Psi(z)(i) \in X_i$. But this follows from the observation that $\Phi(z)(i) = p_i(\Phi(z)) = f_i(z) = p_i(\Psi(z)) = \Psi(z)(i) \in X_i$.

(2) The same argument as in (1) also works for *Group*.

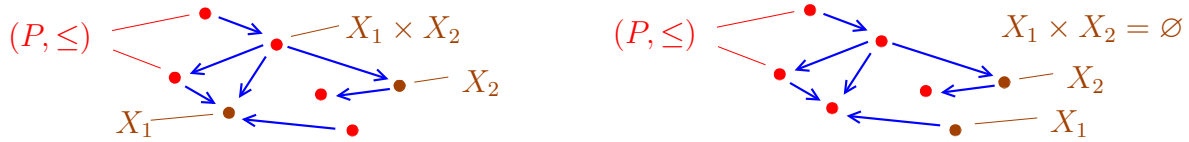
(3) The argument of (2) also works in (3). ■

After these examples one might assume that one gets the hang of the product in categories. The following examples might rob the reader of this illusion.

Example.

(1) Let $(P, \leq)$ be a partially ordered set. We consider the corresponding order category $\mathcal{Order}_{(P,\leq)}$ as defined on page 196. Given a family $\{X_i\}_{i\in I}$ of objects in $\mathcal{Order}_{(P,\leq)}$, i.e. given a family $\{X_i\}_{i\in I}$ of elements of P, one can easily show that

$$\prod_{i\in I} X_i \;:=\; \begin{cases} \text{the smallest } Y \in P \text{ such that } X_i \leq Y \text{ for all } i \in I, \text{ if such } Y \text{ exists} \\ \varnothing \qquad\quad \text{if no such } Y \text{ exists.} \end{cases}$$



(2) We introduce a new category. A *pointed set* is a pair (X, x_0), where X is a set and x_0 is a point in X. We refer to the category $\mathcal{PSet}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{PSet}) &:= \text{all pointed sets,} \\ \mathrm{Mor}((X,x_0),(Y,y_0)) &:= \text{all maps } f \text{ from } X \text{ to } Y \text{ with } f(x_0) = y_0 \end{aligned}$$

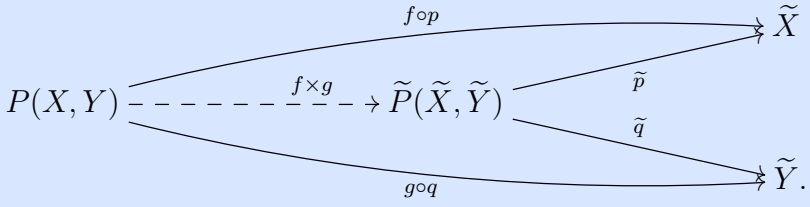
with the usual composition of maps as the *category of pointed sets*. Now let (X, x_0) and (Y, y_0) be two pointed sets. In Exercise 4.6 we will see that the product of (X, x_0) and (Y, y_0) in the category $\mathcal{PSet}$ is given by the set

$$(X \times Y)/\underbrace{((X \times \{y_0\}) \cup (\{x_0\} \times Y))}_{\substack{\text{the points in this subset get} \\ \text{identified to a single point } *}} \quad \text{together with the point} \quad *$$

and the obvious morphisms to (X, x_0) and (Y, y_0).

We conclude our discussion of products with the following notation.

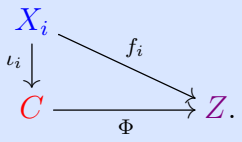
Notation. Let $\mathcal{C}$ be a category. Furthermore let $f\colon X \to \widetilde{X}$ and $g\colon Y \to \widetilde{Y}$ be two morphisms in $\mathcal{C}$. If $(P(X,Y), p\colon P \to X, q\colon P(P,X) \to Y)$ is a product of X and Y and if $(\widetilde{P}(\widetilde{X},\widetilde{Y}), \widetilde{p}\colon \widetilde{P}(\widetilde{X},\widetilde{Y}) \to \widetilde{X}, \widetilde{q}\colon \widetilde{P}(\widetilde{X},\widetilde{Y} \to \widetilde{Y})$ is a product of $\widetilde{X}$ and $\widetilde{Y}$, then there exists a unique morphism $f \times g$ that makes the following diagram commute



4.2. **Coproducts.** We now reverse the direction of the morphisms in the definition of a product and we obtain a new interesting definition.

Definition. Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. A coproduct of $\{X_i\}_{i\in I}$ in the category $\mathcal{C}$ is an object $C \in \mathrm{Ob}(\mathcal{C})$ together with a family of morphisms $\{\iota_i\colon X_i \to C\}_{i\in I}$ such that the following property is satisfied: Whenever we are given a $Z \in \mathrm{Ob}(\mathcal{C})$ and a family of morphisms $\{f_i\colon X_i \to Z\}_{i\in I}$ there exists a unique morphism

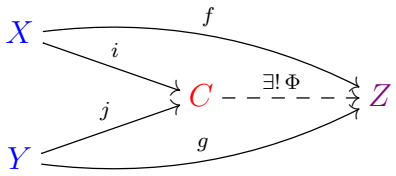
$\Phi\colon C \to Z$ such that for each $i \in I$ we have $f_i = \Phi \circ \iota_i$, i.e. such that the following diagram commutes:[62]



Remark. Note that by definition coproducts in $\mathcal{C}$ are products in the opposite category $\mathcal{C}^{\text{op}}$. So we did not introduce a new concept at all.

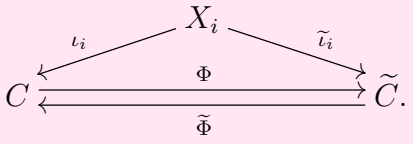
Example. Let $\mathcal{C}$ be a category.

(1) Let $X, Y \in \text{Ob}(\mathcal{C})$ be two objects. A coproduct of X and Y in the category $\mathcal{C}$ is some object $C \in \text{Ob}(\mathcal{C})$ together with two morphisms $i\colon X \to C$ and $j\colon Y \to C$ such that the following universal property is satisfied: Whenever we are given an object $Z \in \text{Ob}(\mathcal{C})$ and two morphisms $f\colon X \to Z$ and $g\colon Y \to Z$ there exists a unique morphism $\Phi\colon C \to Z$ such that $f = \Phi \circ i$ and $g = \Phi \circ j$. The situation is summarized in the following diagram:



(2) If we take the empty family of objects in $\mathcal{C}$, then it follows easily from the definitions that the corresponding coproduct is an initial object in the category $\mathcal{C}$.

Lemma 4.3. (Coproduct Uniqueness Lemma) Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. If we are given two coproducts $(C, \{\iota_i\colon X_i \to C\}_{i\in I})$ and $(\widetilde{C}, \{\widetilde{\iota}_i\colon X_i \to \widetilde{C}\}_{i\in I})$, then there exist unique isomorphisms $\Phi\colon C \to \widetilde{C}$ and $\widetilde{\Phi}\colon \widetilde{C} \to C$ which are inverses of one another and such that for each $i \in I$ the following diagram commutes:



Proof. The lemma is just a reformulation of the Product Uniqueness Lemma 4.1 since coproducts in $\mathcal{C}$ are products in the opposite category $\mathcal{C}^{\text{c}}$. Alternatively the statement also follows easily from the definitions. ∎

Notation. Let $\mathcal{C}$ be a category. Let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$ such that a coproduct exists. Often[63] we denote the coproduct by $\coprod_{i\in I} X_i$ and we refer to the morphisms $X_i \to \coprod_{i\in I} X_i$ as natural injections.[64]

[63] In fact, as we will see shortly, in many settings we will see that the coproduct already has an established different notation.

[63] The name *natural injection* is somewhat unfortunate since in general for a morphism it makes no sense to say that is injective. Even if there is a reasonable notion of injective, the natural injection does not

Lemma 4.4. (Coproduct-Sets-and-Abelian Groups Lemma)

(1) Let *Set* be the category of sets and let $\{X_i\}_{i\in I}$ be a family of sets. The coproduct in *Set* is the disjoint union

$$\bigsqcup_{i\in I} X_i \;:=\; \bigcup_{i\in I}(A_i \times \{i\}),$$

as defined on page 88, together with the natural injections of the X_i.

(2) Let *AbGroup* be the category of *abelian* groups and let $\{X_i\}_{i\in I}$ be a family of groups. The coproduct in *AbGroup* is given by

$$\bigoplus_{i\in I} X_i \;:=\; \left\{ (g_i)_{i\in I} \in \prod_{i\in I} X_i \,\middle|\, \text{all but finitely many } g_i \text{ are trivial} \right\}$$

together with the group homomorphisms

$$\begin{array}{rcl} X_i & \to & \bigoplus_{i\in I} X_i \\ x & \mapsto & \left(\begin{array}{lcl} I & \to & \bigcup_{i\in I} X_i \\ j & \mapsto & \begin{cases} x_i, & \text{if } i = j, \\ 0_{X_j}, & \text{if } i \neq j. \end{cases} \end{array} \right) \end{array}$$

(3) The same approach as in (2) also applies to the category $\mathcal{V}ec_{\mathbb{K}}$ of vector spaces over a fixed field $\mathbb{K}$.

Notation. Let $\{X_i\}_{i\in I}$ be objects in the category *AbGroup* of abelian groups. We write

$$\bigoplus_{i\in I} X_i \;:=\; \left\{ (g_i)_{i\in I} \in \prod_{i\in I} X_i \,\middle|\, \text{all but finitely many } g_i \text{ are trivial} \right\}$$

and we refer to it as the direct sum of the X_i. We proceed similarly with objects in the category $\mathcal{V}ec_{\mathbb{K}}$ of vector spaces over a fixed field $\mathbb{K}$.

Remark. In Exercise 4.7 we will see that the construction of the Coproduct-Sets-and-Abelian Groups Lemma 4.4 does *not* give the coproduct in the category *Group* of all groups. It turns out that for the construction of the coproduct in the category *Group* will require a more clever construction.

Proof.

(1) So let $Z \in \mathrm{Ob}(\mathcal{S}et)$ and let $\{f_i \colon X_i \to Z\}_{i\in I}$ be a family of maps between sets. We define

$$\begin{array}{rcl} \Phi \colon \bigsqcup_{i\in I} X_i & \to & Z \\ (z_i, i) & \mapsto & f_i(z_i). \end{array}$$

It follows immediately from the definition that for each $i \in I$ we have $f_i = \Phi \circ \iota_i$. It remains to show that Φ is unique. So let $\Psi \colon \bigsqcup_{i\in I} X_i \to Z$ be a map such that for each $i \in I$ we have $f_i = \Psi \circ \iota_i$. We need to show that $\Phi = \Psi$, i.e. we need to show that for each $i \in I$ and each $x_i \in X_i$ we have $\Phi((x_i, i)) = \Psi((x_i, i)) \in Z$. But this follows from the observation that $\Phi((x_i, i)) = \Phi(\iota_i(x_i)) = f_i(z) = \Psi(\iota_i(x_i)) = \Psi((x_i, i)) \in Z$.

need to be injection. On the other hand, in all examples we ever care about the natural injection will be an injective map.

(2) Let a $Z \in \mathrm{Ob}(\mathcal{AbGroup})$ and let$\{f_i \colon X_i \to Z\}_{i\in I}$ be a family of homomorphisms. We set

$$\begin{aligned} \Phi \colon \Big\{(g_i)_{i\in I} \in \prod_{i\in I} X_i \,\Big|\, \text{all but finitely many } g_i \text{ are trivial}\Big\} &\to Z \\ (g_i)_{i\in I} &\mapsto \prod_{\substack{i\in I \text{ with} \\ g_i \neq e}} f_i(g_i). \end{aligned}$$

Note that this map makes sense since we take the product only of finitely elements of Z. Furthermore note that this map is a homomorphism since we work with the category of abelian groups.

It follows immediately from the definition that for each $i \in I$ we have $f_i = \Phi \circ \iota_i$. It remains to show that Φ is unique. So let Ψ be another homomorphism such that for each $i \in I$ we have $f_i = \Psi \circ \iota_i$. We need to show that $\Phi = \Psi$. So suppose we are given $(g_i)_{i\in I} \in \prod_{i\in I} X_i$ such that there exist only finitely many $i_1, \dots, i_k \in I$ such that the corresponding group element is non-trivial. Then

$$\Phi((g_i)_{i\in I}) = \Phi\Big(\prod_{r=1}^{k} \iota_{i_r}(g_{i_r})\Big) = \prod_{r=1}^{k} \Phi(\iota_{i_r}(g_{i_r})) = \prod_{r=1}^{k} f_{i_r}(g_{i_r}) \underset{\text{same argument backwards}}{\uparrow} = \Psi((g_i)_{i\in I}).$$

(3) This statement is proved the same way as we proved (2). ■

4.3. Limits.

Definition. Let $(P, \leq)$ be a partially ordered set and let $\mathcal{C}$ be a category. A diagram in $\mathcal{C}$ over $(P, \leq)$ is a covariant functor from the order category $\mathcal{Order}_{(P,\leq)}$ to $\mathcal{C}$.

Remark. In plain English a diagram in a category $\mathcal{C}$ over a partially ordered set $(P, \leq)$ is the following data:

(1) for each $x \in P$ we are given any object $F(x) \in \mathcal{C}$,
(2) for each $x < y$ we are given a morphism $F(x) \to F(y)$,
(3) if $x < y < z$, then the following diagram needs to commute:

$$\begin{array}{ccc} F(x) & \longrightarrow & F(z). \\ & \searrow \quad \nearrow & \\ & F(y) & \end{array}$$

Example.

(1) We consider the partially ordered set $(P, \leq_{\mathrm{tr}})$ that is given by a set P with the trivial partial order $\leq_{\mathrm{tr}}$, i.e. $x \leq_{\mathrm{tr}} y$ if and only if $x = y$. In this setting a diagram over $(P, \leq_{\mathrm{tr}})$ is the same data as a family of objects $\{X_i\}_{i\in P}$.
(2) We consider the partially ordered set $(\mathbb{N}, \leq)$. In this setting a diagram over $(\mathbb{N}, \leq)$ is the same data as a sequence

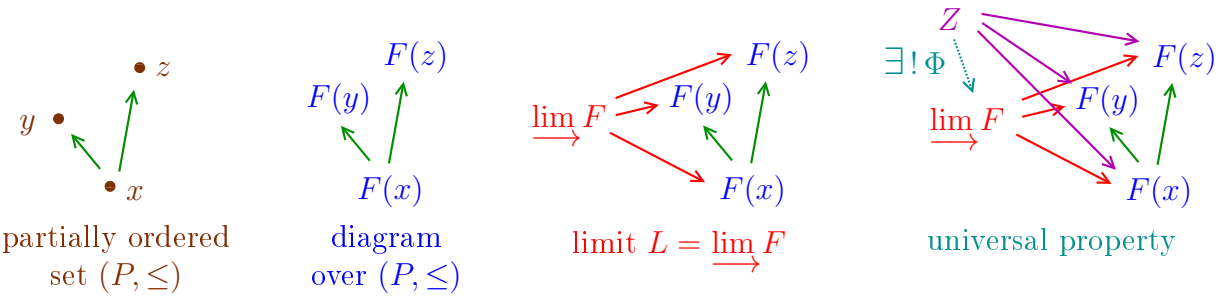
$$X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} X_3 \xrightarrow{\varphi_3} \dots$$

of morphisms $\varphi_i \colon X_i \to X_{i+1}$. For $i < j$ the morphisms $X_i \to X_j$ are understood to be given by $\varphi_{j-1} \circ \dots \circ \varphi_{i+1} \circ \varphi_i$.

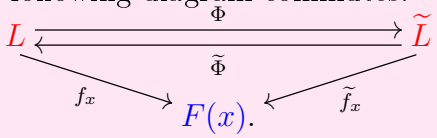
Definition. Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$. A limit of F consists of an object $L \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $f_x \colon L \to F(x)$ for each $x \in P$ such that the following two conditions are satisfied:

(1) For any $x \leq y$ we have $f_y = F(x \to y) \circ f_x \colon L \to F(x)$.
(2) The following universal property holds: If Z is an object in $\mathcal{C}$ and if we are given morphisms $g_x \colon Z \to F(x)$ for each $x \in P$ such that (1) is satisfied, then there exists a unique morphism $\Phi \colon Z \to L$ such that $g_x = f_x \circ \Phi$ for all $x \in P$.

If a limit exists, then often we denote it by $\varinjlim F$, or by sensible variations thereof.



Lemma 4.5. (Limit Uniqueness Lemma) Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$. Suppose that we are given two limits $(L, \{f_x \colon L \to F(x)\}_{x \in P})$ and $(\widetilde{L}, \{\widetilde{f}_x \colon \widetilde{L} \to F(x)\}_{x \in P})$. Then there exist uniquely determined isomorphisms $\Phi \colon L \to \widetilde{L}$ and $\widetilde{\Phi} \colon \widetilde{L} \to L$ which are inverses of one another and such that for each $x \in P$ the following diagram commutes:



Proof. As in the case of the Product Uniqueness Lemma 4.1 the proof follows easily from the definitions. ∎

Example.

(1) If we consider a trivial partially ordered set $(P, \leq_{\mathrm{tr}})$, then the limit is by definition precisely the product.
(2) We consider $(\mathbb{N}, \leq)$ and we consider a diagram in the category $\mathcal{S}et$ of sets where each $\varphi_i \colon X_i \to X_{i+1}$ is an inclusion of subsets of a given set Y. In Exercise 4.8 we will see that in this case the limit is the union of the X_i.

We conclude this discussion of limits with a particularly frequently used special case.

Definition.

(1) We consider the set $P = \{0, x, y\}$ with the partial order that is given by the trivial relations $0 \leq 0, x \leq x, y \leq y$ and the non-trivial relations $x \leq 0$ and $y \leq 0$:

the partially ordered set

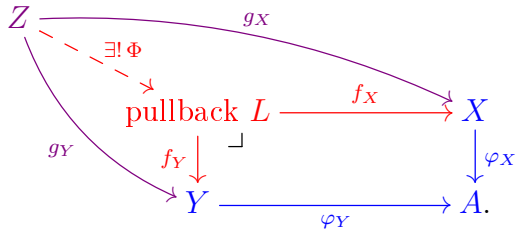
$$\begin{array}{ccc} & & x \\ & & \downarrow \\ y & \longrightarrow & 0 \end{array}$$

gives rise to the pullback diagram

$$\begin{array}{ccc} & & F(x) \\ & & \downarrow{\scriptstyle \varphi_x} \\ F(y) & \xrightarrow{\varphi_y} & F(0). \end{array}$$

(2) The limit of a pullback diagram is called a pullback.

Remark. The following diagram summarizes the definition of a pullback:



(1) We start out with a pullback diagram.
(2) The pullback L comes with morphisms $f_X \colon L \to X$ and $f_Y \colon L \to Y$ such that the rectangle commutes.
(3) If we are given an object Z of $\mathcal{C}$ with morphisms $g_X \colon Z \to X$ and $g_Y \colon Z \to Y$ such that the outer big quadrilateral commutes, then there exists a unique morphism $\Phi \colon Z \to L$ that makes the diagram commute.

Here the symbol "$\lrcorner$" indicates that L is the pullback of the diagram.

Example. We consider the following pullback diagram in the category of groups:

$$\begin{array}{ccc} & & G \\ & & \downarrow{\scriptstyle \varphi} \\ \{e\} & \longrightarrow & H \end{array}$$

one can easily show that the corresponding pullback is given by

$$\begin{array}{ccc} \ker(\varphi) & \longrightarrow & G \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \varphi} \\ \{e\} & \longrightarrow & H \end{array}$$

Lemma 4.6. (Pullback Lemma) Let $\mathcal{C}$ be a category.

(1) Let

$$\begin{array}{ccc} L & \xrightarrow{f_X} & X \\ {\scriptstyle f_Y}\downarrow & \lrcorner & \downarrow{\scriptstyle \varphi_X} \\ Y & \xrightarrow[\varphi_Y]{} & A. \end{array}$$

be a pullback.
(a) If φ_X is an isomorphism, then so is the "opposite" morphism f_Y.
(b) If φ_Y is an isomorphism, then so is the "opposite" morphism f_X.

(2) If we are given pullbacks

$$\begin{array}{ccc} L & \xrightarrow{f_X} & X \\ {\scriptstyle f_Y}\downarrow & \lrcorner & \downarrow{\scriptstyle \varphi_X} \\ Y & \xrightarrow{\varphi_Y} & A \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{\theta} & Z \\ {\scriptstyle \varphi_X}\downarrow & \lrcorner & \downarrow{\scriptstyle \psi_Z} \\ A & \xrightarrow{\psi_A} & B \end{array}$$

then the composition

$$\begin{array}{ccc} L & \xrightarrow{\theta \circ f_X} & Z \\ {\scriptstyle f_y}\downarrow & \lrcorner & \downarrow{\scriptstyle \psi_Z} \\ Y & \xrightarrow{\psi_A \circ \varphi_Y} & B \end{array}$$

is also a pullback.

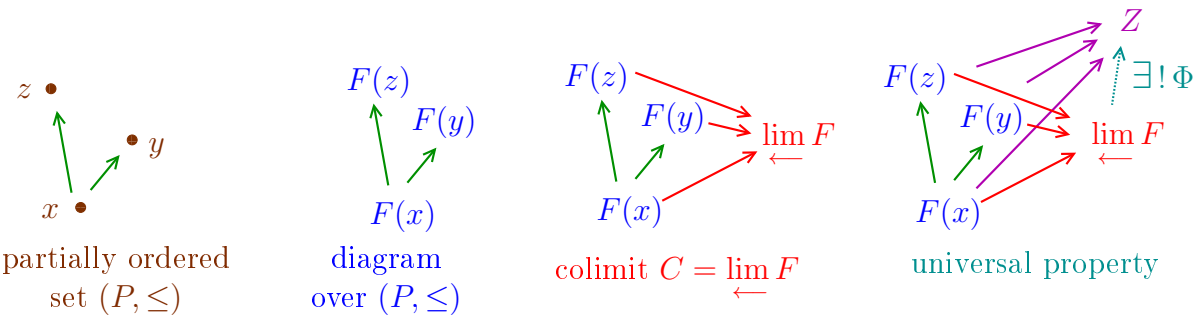
Proof. All statements follow from purely formal arguments which do not become clearer by writing them down. Thus we leave it to the reader to go through the argument. ■

4.4. **Colimits.** In this section we introduce the notion of a colimit, which is very similar to the notion of a limit, except that the directions of the morphisms are reversed.

Definition. Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$. A colimit of F consists of an object $C \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $f_x \colon F(x) \to C$ for each $x \in P$ such that the following two conditions are satisfied:

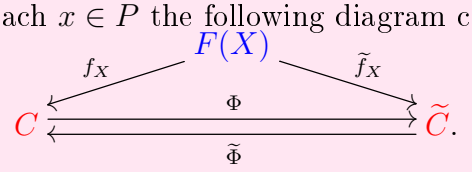
(1) For any $x \leq y$ we have $f_y = F(x \to y) \circ f_x \colon F(x) \to C$.
(2) The following universal property holds: If Z is an object in $\mathcal{C}$ and if we are given morphisms $g_x \colon F(x) \to Z$ for each $x \in P$ such that (1) is satisfied, then there exists a unique morphism $\Phi \colon C \to Z$ such that $g_x = f_x \circ \Phi$ for all $x \in P$.

If a colimit exists, then often we denote it by $\varinjlim F$, or by sensible variations thereof.



Not surprisingly we have the following analogue of the Limit Uniqueness Lemma 4.5.

Lemma 4.7. (Colimit Uniqueness Lemma) Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$. Suppose that we are given two colimits $(C, \{f_x \colon F(x) \to C\}_{x \in P})$ and $(\widetilde{C}, \{\widetilde{f}_x \colon F(X) \to \widetilde{C}\}_{x \in P})$. Then there exist uniquely determined isomorphisms $\Phi \colon C \to \widetilde{C}$ and $\widetilde{\Phi} \colon \widetilde{C} \to C$ which are inverses of one another and such that for each $x \in P$ the following diagram commutes:



Proof. As always the proof follows easily from the definitions. ■

Example.

(1) If we consider a trivial partially ordered set $(P, \leq_{\mathrm{tr}})$, then the colimit is by definition precisely the coproduct.
(2) We consider $(\mathbb{N}, \leq)$ and we consider a diagram in the category *Set* of sets where each $\varphi_i \colon X_i \to X_{i+1}$ is an inclusion of subsets of a given set Y. In Exercise 4.8 we will see that in this case the colimit is the intersection of the X_i.

Next we introduce the analogue of the pullback.

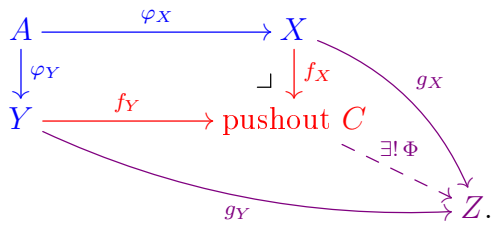
Definition.

(1) We consider the set $P = \{0, x, y\}$ with the partial order that is given by the trivial relations $0 \leq 0, x \leq x, y \leq y$ and the non-trivial relations $0 \leq x$ and $0 \leq y$:

the partially ordered set $\begin{array}{ccc} 0 & \longrightarrow & x \\ \downarrow & & \\ y & & \end{array}$ gives rise to the pushout diagram $\begin{array}{ccc} F(0) & \xrightarrow{\varphi_x} & F(x) \\ \downarrow{\scriptstyle \varphi_y} & & \\ F(y) & & \end{array}$

(2) The colimit of a pushout diagram is called a pushout.

Remark. The following diagram summarizes the definition of a pushout:



(1) We start out with a pushout diagram.
(2) The pushout C comes with morphisms $f_X \colon X \to C$ and $f_Y \colon Y \to C$ such that the rectangle commutes.
(3) If we are given Z with morphisms $g_X \colon X \to Z$ and $g_Y \colon Y \to Z$ such that the big quadrilateral commutes, then there exists a unique morphism $\Phi \colon C \to Z$ that makes the diagram commute.

Here the symbol "⌟" indicates that C is the pushout of the diagram.

Remark. Note that by definition coproducts in $\mathcal{C}$ are products in the opposite category $\mathcal{C}^{\mathrm{op}}$. So we did not introduce a new concept at all.

Example. Let G be a group and let A be a normal subgroup. We consider the following pushout diagram in the category of groups:

$\begin{array}{ccc} A & \longrightarrow & \{e\} \\ \downarrow & & \\ G & & \end{array}$ one can easily show that the corresponding pushout is given by $\begin{array}{ccc} A & \longrightarrow & \{e\} \\ \downarrow & & \downarrow \\ G & \longrightarrow & G/A. \end{array}$

In Exercise 4.9 we will consider a generalization of this situation.

We conclude this section with the following analogue of the Pullback Lemma 4.6.

Lemma 4.8. (Pushout Lemma) Let $\mathcal{C}$ be a category.

(1) Let

$$\begin{array}{ccc} A & \xrightarrow{\varphi_X} & X \\ \downarrow{\scriptstyle \varphi_Y} & & \downarrow{\scriptstyle f_X} \\ Y & \xrightarrow{f_Y} & C. \end{array}$$

be a pushout.

(a) If φ_X is an isomorphism, then so is the "opposite" morphism f_Y.

(b) If φ_Y is an isomorphism, then so is the "opposite" morphism f_X.

(2) If we are given pushouts

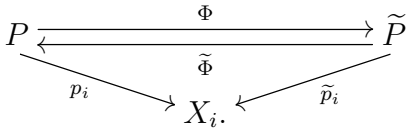
$$\begin{array}{ccc} A & \xrightarrow{\varphi_X} & X \\ {\scriptstyle \varphi_Y}\downarrow & \lrcorner & \downarrow{\scriptstyle f_X} \\ Y & \xrightarrow{f_Y} & L \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{\theta} & Z \\ {\scriptstyle f_X}\downarrow & \lrcorner & \downarrow{\scriptstyle g_Z} \\ L & \xrightarrow{g_L} & W \end{array} \quad \text{then the composition} \quad \begin{array}{ccc} A & \xrightarrow{\theta\circ\varphi_X} & Z \\ {\scriptstyle \varphi_Y}\downarrow & \lrcorner & \downarrow{\scriptstyle g_Z} \\ Y & \xrightarrow{g_L\circ f_Y} & W \end{array}$$

is also a pushout.

Proof. Again all statements follow from purely formal arguments which do not become clearer by writing them down. Thus we leave it to the reader to go through the argument. ■

Exercises for Chapter 4.

Exercise 4.1. Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. Suppose we are given two products $(P, \{p_i\colon P \to X_i\}_{i\in I})$ and $(\widetilde{P}, \{\widetilde{p}_i\colon \widetilde{P} \to X_i\}_{i\in I})$. Show that there exist uniquely determined isomorphisms $\Phi\colon P \to \widetilde{P}$ and $\widetilde{\Phi}\colon \widetilde{P} \to P$ which are inverses of one another and such that for each $i \in I$ the following diagram commutes:



Remark. This exercise is the content of the Product Uniqueness Lemma 4.1.

Exercise 4.2. Show that the category *Field* of fields does not admit products.

Exercise 4.3. Let *Metric* be the category of metric spaces where the objects are metric spaces and the morphisms are continuous maps.

(a) Do any two metric spaces X, Y admit a product in *Metric*?
(b) Do any two metric spaces X, Y admit a coproduct in *Metric*?

Exercise 4.4. Let $(P, \leq)$ be a partially ordered set. We consider the corresponding order category $\mathit{Order}_{(P,\leq)}$ as defined on page 196. What is the coproduct of a family $\{X_i\}_{i\in I}$ of objects in $\mathit{Order}_{(P,\leq)}$, if it exists.

Exercise 4.5. Does the coproduct of the rings $\mathbb{Z}_2$ and $\mathbb{Z}_3$ in the category *Ring* of rings exist, and if yes, what is it?

Exercise 4.6. A *pointed set* is a pair (X, x_0), where X is a set and x_0 is a point in X. We refer to the category *PSet* with

$$\begin{array}{rcl} \mathrm{Ob}(\mathit{PSet}) & := & \text{all pointed sets,} \\ \mathrm{Mor}((X,x_0),(Y,y_0)) & := & \text{all maps } f \text{ from } X \text{ to } Y \text{ with } f(x_0) = y_0 \end{array}$$

with the usual composition of maps as the *category of pointed sets*.

(a) Does the category of pointed sets admit an initial and a terminal object?

(b) Let (X, x_0) and (Y, y_0) be two pointed sets. Show that a product of (X, x_0) and (Y, y_0) in the category $\mathcal{PSet}$ is given by the set

$$(X \times Y)/\underbrace{((X \times \{y_0\}) \cup (\{x_0\} \times Y))}_{\substack{\text{the points in this subset get} \\ \text{identified to a single point } *}} \quad \text{together with the point} \quad *$$

and the obvious morphisms to (X, x_0) and (Y, y_0).

(c) Let (X, x_0) and (Y, y_0) be two pointed sets. What is the coproduct of (X, x_0) and (Y, y_0)?

Exercise 4.7. We consider the category $\mathcal{Group}$ of all groups. Show that in general for two groups A, B the product $A \times B$ is not the coproduct in the category $\mathcal{Group}$.
Remark. This exercise shows that the construction in the Coproduct-Sets-and-Abelian Groups Lemma 4.4 does not provide the coproduct in the category $\mathcal{Group}$.

Exercise 4.8. We consider the partially ordered set $(\mathbb{N}, \leq)$. Let Y be a set. Furthermore let $X_1 \subset X_2 \subset \ldots$ be a nested sequence of subsets of Y. For each $i \in \mathbb{N}$ we denote by $\varphi_i \colon X_i \to X_{i+1}$ the inclusion. The X_i and φ_i define a diagram over $(\mathbb{N}, \leq)$.

(a) Show that the limit of this diagram equals the union of the X_i.

(b) Show that the colimit of this diagram equals the intersection of the X_i.

Exercise 4.9. Let $\varphi \colon A \to G$ be a group homomorphism. What is the pushout of the following pushout diagram in the category of groups:

$$\begin{array}{ccc} A & \longrightarrow & \{e\} \\ {\scriptstyle\varphi}\downarrow & & \\ G & & \end{array}$$

Exercise 4.10. We consider the category $\mathcal{AbGroup}$ of abelian groups.

(a) Show that pullbacks always exist.

(b) Show that pushouts always exist.

Exercise 4.11. Let

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & \lrcorner & \downarrow{\scriptstyle r} \\ C & \xrightarrow{s} & X. \end{array}$$

be a pushout in the category $\mathcal{Set}$ of sets.

(a) We suppose that $f \colon A \to B$ is injective. Show that the opposite map $s \colon C \to X$ is also injective.

(b) If $f \colon A \to B$ is surjective, does it follow that the opposite map $s \colon C \to X$ is also surjective?

Exercise 4.12. Let $\mathcal{C}$ be some category. We consider the following commutative diagram of morphisms in $\mathcal{C}$:

$$\begin{array}{ccccc} A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ A' & \xrightarrow{\varphi'} & B' & \xrightarrow{\psi'} & C'. \end{array}$$

We suppose that the big rectangle forms a pushout square.

(a) We suppose that the square to the left is a pushout square. Show that the square to the right is also a pushout square.
(b) We suppose that the square to the right is a pushout square. Is the square to the left also a pushout square?

5. Constructions of topological spaces

In Chapter 4 we introduced the concepts of (co-) products and more generally (co-) limits in categories. In this chapter we will see how these concepts play out in the category of topological spaces.

5.1. The product of topological spaces.

In this section we will see that the category of topology spaces has products. It is convenient to recall the following definition from page 88.

Definition. Let $\{X_i\}_{i\in I}$ a family of sets. The product of these sets is defined as the set

$$\prod_{i\in I} X_i \;:=\; \left\{ f\colon I \to \bigcup_{i\in I} X_i \;\middle|\; \text{for every } i\in I \text{ we have } f(i)\in X_i \right\}.$$

Proposition 5.1. (Topological-Product Proposition) Let $\{X_i\}_{i\in I}$ be a family of topological spaces. We set

$$\mathcal{B} \;:=\; \left\{ \prod_{i\in I} U_i \;\subset\; \prod_{i\in I} X_i \;\middle|\; \begin{array}{ll} (1) & \text{each } U_i \text{ is open in } X_i \text{ and} \\ (2) & \text{there exist only finitely many } i\text{'s with } U_i \neq X_i \end{array} \right\}.$$

The following statements hold:

(0) The set $\mathcal{B}$ has the basis property that we introduced on page 132.

(1) (a) For any $i\in I$ the map

$$\begin{array}{rcl} p_i\colon \prod_{i\in I} X_i & \to & X_i \\ f & \mapsto & f(i) \end{array}$$

is continuous with respect to the topology from (0).

(b) Given a topological space Z and given a family $\{f_i\colon Z\to X_i\}_{i\in I}$ of continuous maps the map

$$\begin{array}{rcl} F\colon Z & \to & \prod_{i\in I} X_i \\ z & \mapsto & \begin{pmatrix} I & \to & \bigcup_{i\in I} X_i \\ i & \mapsto & f_i(z) \end{pmatrix}. \end{array}$$

is continuous with respect to the topology from (0).

(2) The topological space $\prod_{i\in I} X_i$ together with the natural projections p_i is a product (in the sense of the definition on page 211 of $\{X_i\}_{i\in I}$ in the category $\mathcal{T}op$ of topological spaces.

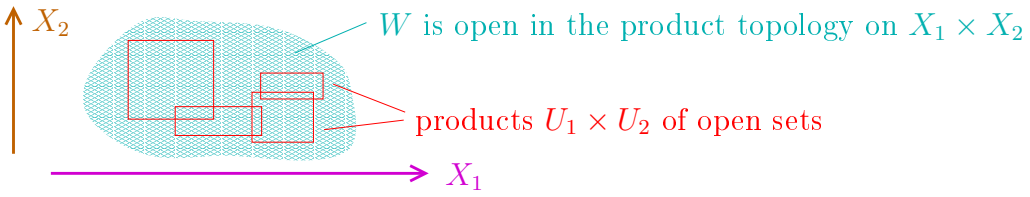
Example. It is worth considering the case we will use most frequently, namely the case that we are given finitely many topological spaces. Thus let $X_1,\dots,X_k$ be topological spaces. In this case the product topology on $X_1\times\cdots\times X_k$ is the topology on $X_1\times\cdots\times X_k$ that is generated by

$$\mathcal{B} \;=\; \{U_1\times\cdots\times U_k \,|\, \text{ each } U_i \text{ is open in } X_i\}.$$

Proof. Let $\{X_i\}_{i\in I}$ be a family of topological spaces.

(0) We need to verify that

$$\mathcal{B} \;:=\; \left\{ \prod_{i\in I} U_i \;\subset\; \prod_{i\in I} X_i \;\middle|\; \begin{array}{ll} (1) & \text{each } U_i \text{ is open in } X_i \text{ and} \\ (2) & \text{there exist only finitely many } i\text{'s with } U_i \neq X_i \end{array} \right\}$$



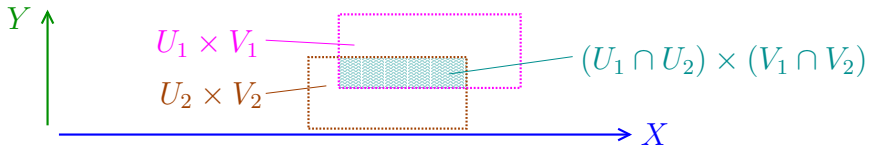
satisfies the conditions (B1) and (B2) formulated on page 132.

(B1) Evidently we have $\prod_{i\in I} X_i \in \mathcal{B}$. It follows immediately that $\mathcal{B}$ satisfies (B1).

(B2) Let $\prod_{i\in I} U_i$ and $\prod_{i\in I} V_i$ be two sets in $\mathcal{B}$. Then

$$\prod_{i\in I} U_i \;\cap\; \prod_{i\in I} V_i \;=\; \prod_{i\in I} \underbrace{(U_i \cap V_i)}_{\substack{\text{only finitely many} \\ \text{are} \neq X_i}}$$

is again a set in $\mathcal{B}$. Thus $\mathcal{B}$ also satisfies (B2).



(1) (a) It follows immediately from the definitions that the projection maps p_i are continuous with respect to the given topology.

(b) Suppose we are given a topological space Z and continuous maps $f_i\colon Z \to X_i$. We denote by $F\colon Z \to \prod_{i\in I} X_i$ the map that is given by $z \mapsto (i \mapsto f_i(z))$. We claim that F is continuous. By the Subbasis-Continuity Proposition 2.9 it suffices to verify that the preimage of any set in $\mathcal{B}$ is open. So let $\prod_{i\in I} U_i$ be a set in $\mathcal{B}$. By definition there exists a finite set $J \subset I$ such that $U_i = X_i$ for any $i \notin J$. Then

$$F^{-1}\Big(\prod_{i\in I} U_i\Big) \;=\; \bigcap_{i\in I} f_i^{-1}(U_i) \;=\; \bigcap_{j\in j} \underbrace{f_j^{-1}(U_j)}_{\text{open in } Z} \;=\; \begin{array}{c}\text{intersection of } \textit{finitely} \\ \text{many open subsets of } Z.\end{array}$$

This concludes the proof that F is continuous.

(2) Let Z be a topological space and let $\{f_i\colon Z \to X_i\}_{i\in I}$ be a family of continuous maps. It follows from the Product-Sets-and-Groups Lemma 4.2 that the map

$$\begin{array}{rcl} \Phi\colon Z & \to & \prod_{i\in I} X_i \\ z & \mapsto & (f_i(z))_{i\in I} \end{array}$$

is the unique map (of sets) $\Phi\colon Z \to \prod_{i\in I} X_i$ such that for each $i \in I$ we have $f_i = p_i \circ \Phi$. In (1b) we just showed that this map is fortunately continuous, so this shows that there exists a unique continuous map $\Phi\colon Z \to \prod_{i\in I} X_i$ such that for each $i \in I$ we have $f_i = p_i \circ \Phi$. This shows that the topological space $\prod_{i\in I} X_i$ together with the natural projections p_i is a product of $\{X_i\}_{i\in I}$ in the category $\mathcal{T}op$ of topological spaces. ■

Definition. Let $\{X_i\}_{i\in I}$ a family of topological spaces. We refer to the topology on the set $\prod_{i\in I} X_i$ that we defined in the Topological-Product Proposition 5.1 (0) as the product topology on $\prod_{i\in I} X_i$.

Example. It follows from the Topological-Product Proposition 5.1 (1a) and (1b) that given any topological space X the

diagonal map $\begin{array}{rcl} d\colon X & \to & X\times X \\ x & \mapsto & (x,x) \end{array}$ and the projection $\begin{array}{rcl} p\colon X\times X & \to & X \\ (x,y) & \mapsto & x \end{array}$

are continuous. Since $p\circ d = \mathrm{id}_X$ we see that $d\colon X\to X\times X$ is an embedding. This fact will come in handy on many occasions.

The following remark can be viewed as a sibling of the future Remark 5.17.

Remark 5.2. We consider maps in and out of a product space $X_1\times X_2$:

(1) The Topological-Product Proposition 5.1 gives us an easy to show that a given map $X\to X_1\times X_2$ to a product space is continuous. More precisely, by the Topological-Product Proposition 5.1 (1b) we "just" need to show that the two coordinate maps $X\to X_1$ and $X\to X_2$ are continuous.

(2) There is usually no easy way to show that a map $X_1\times X_2\to X$ out of a product space is continuous.

The following lemma gives us a reinterpretation of the product topology in some of the most frequently studied cases.

Lemma 5.3. (Product Metric Lemma)

(1) Let $(X_1,d_1),\dots,(X_m,d_m)$ be metric spaces. We define the product metric d on $X_1\times\cdots\times X_m$ as follows:

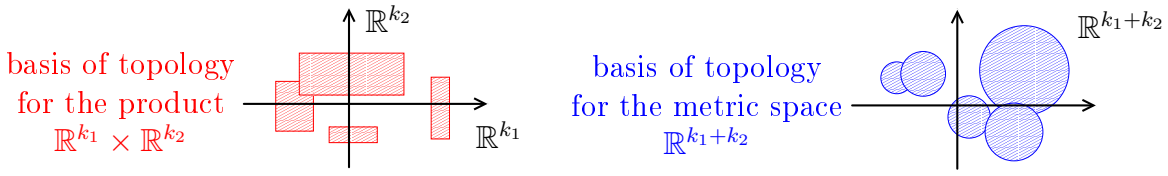
$$d((x_1,\dots,x_m),(y_1,\dots,y_m)) \;:=\; \sum_{i=1}^{m} d_i(x_i,y_i).$$

The product topology on $X_1\times\cdots\times X_m$ agrees with the metric topology coming from the product metric.

(2) Let $k_1,\dots,k_m\in\mathbb{N}_0$. The map

$$\begin{array}{ccc} \overbrace{\mathbb{R}^{k_1}\times\cdots\times\mathbb{R}^{k_m}}^{\text{product of Euclidean topologies}} & \to & \overbrace{\mathbb{R}^{k_1+\cdots+k_m}}^{\text{Euclidean topology}} \\ ((x_1,\dots,x_{k_1}),\dots,(x_1,\dots,x_{k_m})) & \mapsto & (x_1,\dots,x_{k_1},\dots,x_1,\dots,x_{k_m}) \end{array}$$

is a homeomorphism. The analogous statement also holds if we replace "$\mathbb{R}$" by "$\mathbb{C}$".



Remark. As we just pointed out, by design the product topology is set up in such a way that it is usually easy to show that a map into a product is continuous. On the other hand, except for the projections there is no obvious way to show that a map out of a product of

topological spaces is continuous. This is where the Product Metric Lemma 5.3 comes in. If we want to show that a map out of the product topological space $\mathbb{R}^k \times \mathbb{R}^l$ is continuous, it suffices to show that the map from $\mathbb{R}^{k+l}$ is continuous, which is often easier, since we can use the tools that we acquired in analysis.

Proof.

(1) This statement can be proved using the Subbasis Lemma 1.28. We will fill in the details in Exercise 5.1.

(2) Note that it follows from (1) that it suffices to show that the product metric on $\mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_m}$ and the Euclidean metric on $\mathbb{R}^{k_1+\cdots+k_m}$ define the same topology. This in turn follows easily from the Equivalent Metric Lemma 1.2. We leave it to the reader to fill in the details. ■

Convention. Henceforth, given $k_1, \ldots, k_m \in \mathbb{N}_0$ we will always identify the product topological space $\mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_m}$ with $\mathbb{R}^{k_1+\cdots+k_m}$. In particular, given $X_i \subset \mathbb{R}^{k_i}$, $i = 1, \ldots, m$, we view $X_1 \times \cdots \times X_m$ as a subspace of $\mathbb{R}^{k_1+\cdots+k_m}$.

Example. It follows almost immediately from the definition of the topology on the matrix $\mathrm{M}(r \times s, \mathbb{R})$ on page 171, the examples on page 141 and the Product Metric Lemma 5.3 (2) that for any $k, l, m \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{M}(k \times l, \mathbb{R}) \times \mathrm{M}(l \times m, \mathbb{R}) & \to & \mathrm{M}(k \times m, \mathbb{R}) \\ (A, B) & \mapsto & A \cdot B \end{array}$$

is continuous. The same also applies to matrices over $\mathbb{C}$.

After these examples let us turn to the study of the more formal properties of the product topology. On several occasions we will use, consciously or unconsciously, the following lemma.

Lemma 5.4. (Product Topology-Basics Lemma)

(1) Let $\{Y_i\}_{i \in I}$ be a family of topological spaces.

(a) Let $j \in I$ and for $k \neq j$ let $y_k \in Y_k$. The inclusion map

$$\begin{array}{rcl} Y_i & \to & \prod\limits_{i \in I} Y_i \\ y & \mapsto & \left(\begin{array}{rcl} I & \to & \bigcup\limits_{i \in I} X_i \\ i & \mapsto & \begin{cases} y, & \text{if } i = j, \\ y_j & \end{cases} \end{array} \right). \end{array}$$

is continuous and it is an embedding.[64]

(b) If $f_i \colon Y_i \to Z_i$, $i \in I$ are continuous maps between topological spaces, then the map

$$\begin{array}{rcl} \prod\limits_{i \in I} f_i \colon \prod\limits_{i \in I} Y_i & \to & \prod\limits_{i \in I} Z_i \\ (y_i)_{i \in I} & \mapsto & (f(y_i))_{i \in I} \end{array}$$

is continuous.

(2) If we are given families of topological spaces $\{\{X_i\}_{i \in I_j}\}_{j \in J}$, then the obvious map

$$\prod_{j \in J} \left(\prod_{i \in I_j} X_i \right) \to \prod_{i \in \bigsqcup_{j \in J} I_j} X_i$$

is a homeomorphism.

(3) If $\{X_i\}_{i\in I}$ is a family of topological spaces and if for each $i \in I$ we are given a basis $\mathcal{B}_i$ of the topological space X_i, then

$$\mathcal{C} \;=\; \left\{ \prod_{i\in I} U_i \;\subset\; \prod_{i\in I} X_i \;\middle|\; \begin{array}{ll} (1) & \text{each } U_i \neq X_i \text{ is contained in } \mathcal{B}_i \text{ and} \\ (2) & \text{there exist only finitely many } i\text{'s with } U_i \neq X_i \end{array} \right\}$$

is a basis for $\prod_{i\in I} X_i$. If I is finite, then we can restrict ourselves to the case that the U_i are always in $\mathcal{B}_i$.

(4) The product topology commutes with subspace topology. More precisely, let $\{X_i\}_{i\in I}$ be a family of topological spaces and for each $i \in I$ let A_i be a subset. Then the subspace topology of $\prod_{i\in I} A_i \subset \prod_{i\in I} X_i$ agrees with product topology on $\prod_{i\in I} A_i$

Example.

(1) On page 135 we saw that the *rational* intervals form a basis for the usual topology on $\mathbb{R}$. It follows from the Product Topology-Basics Lemma 5.4 (4) that the set $\mathcal{B}$ of rational hyperrectangles, i.e. that the set

$$\mathcal{B} \;=\; \{\text{hyperrectangle } [a_1, b_1] \times \cdots \times [a_k, b_k] \,|\, a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{Q}\} \;\subset\; \mathcal{P}(\mathbb{R}^k)$$

forms a basis for the standard topology on $\mathbb{R}^k$. It follows from this observation together with the Countability Lemma 0.8 (5) that $\mathbb{R}^k$ admits a basis for the topology that consists of only countably many sets.

(2) Let $s_1, \ldots, s_n \in \mathbb{N}_0$. We set $s = s_1 + \cdots + s_n$. It follows from the Product Topology-Basics Lemma 5.4 (4) and the Product Metric Lemma 5.3 (2) that we can view $B^{s_1} \times \cdots \times B^{s_n}$ as a subspace of $\mathbb{R}^s$. By definition this is an open subset of $\mathbb{R}^s$ and one verifies easily that it is convex and bounded. Thus it follows from the Convex-to-Ball Proposition 2.20 that there exists a canonical homeomorphism $B^{s_1} \times \cdots \times B^{s_n}$ to the ball B^s that sends $(0, \ldots, 0)$ to the origin. The same statement holds for the corresponding closed balls.

Proof.

(1) (a) This statement follows from the Topological-Product Proposition 5.1 (1b) and the observation that the identity map and the constant map are both continuous.

(b) It follows from Lemma 2.1 and the Topological-Product Proposition 5.1 (1a) that the maps $\prod_{i\in I} Y_i \to Z_j$ given by $(y_i)_{i\in I} \mapsto f_j(y_j)$ are continuous. Thus it follows from Topological-Product Proposition 5.1 (1b) that the given map is continuous.

(2) It now follows from the Topological-Product Proposition 5.1 (1b) that the given map $Y_1 \times \cdots \times Y_k \to Z_1 \times \cdots \times Z_k$ is continuous.

(3) The main idea is to prove the lemma using the Basis Characterization Lemma 1.32 (1). First note that by definition the maps in $\mathcal{C}$ are open. Next let $W \subset \prod_{i\in I} X_i$ be an open subset and let $(x_i)_{i\in I} \in W$. By definition of the product topology there exist open sets $V_i \subset X_i$, $i \in I$ such that $X_i = V_i$ for all but finitely many $i \in I$ and

[64] Recall that "embedding" means that the map is a homeomorphism onto its image, in our setting this means that the inclusion map defines a homeomorphism $Y_i \to \{y_1\} \times \cdots \times \{y_{i-1}\} \times Y_i \times \{y_{i+1}\} \times \cdots \times \{y_k\}$.

such that $(x_i)_{i\in I} \subset \prod_{i\in I} V_i \subset W$. Since $\mathcal{B}_i$ is a basis for X_i we see that for each i with $V_i \neq X_i$ there exists a $U_i \in \mathcal{B}_i$ with $x \in U_i \subset V_i$. For all other i we set $U_i = V_i = X_i$. It follows that

$$(x_i)_{i\in I} \in \prod_{i\in I} U_i, \subset \prod_{i\in I} V_i \subset W.$$

We deduce from the Basis Characterization Lemma 1.32 (1) that the given set $\mathcal{C}$ is indeed a basis for $\prod_{i\in I} X_i$. ■

Example. Let X be a topological space and let $\{P\}$ be the topological space consisting of a single point P (note that $\{P\}$ has a unique topological structure!). It follows from the Product Topology-Basics Lemma 5.4 (1a) that the natural bijection $X \to X \times \{P\}$ is a homeomorphism. On many occasions we will use this homeomorphism to make the identification $X = X \times \{P\}$.

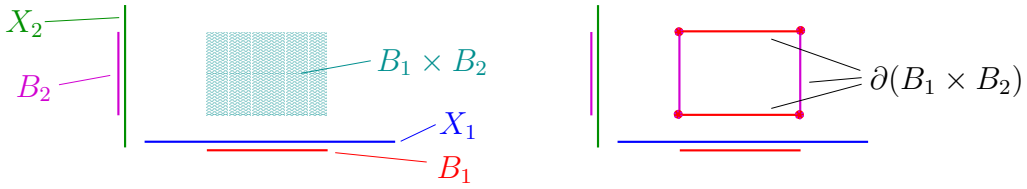
5.2. **Properties of products of topological spaces I.** Throughout these notes we will almost invariably only work with the product of finitely many topological spaces. In this section we will therefore discuss properties of the product of finitely many topological spaces. This simplifies some of the arguments and it makes the notation arguably more natural. In the following section we will, for completeness sake, also discuss some of the properties of products of arbitrarily many topological spaces.

Lemma 5.5. (Product-of-Subsets Lemma) Let $X_1, \ldots, X_k$ be topological spaces.

(1) If $A_i \subset X_i$, $i = 1, \ldots, k$ are closed subsets, then $A_1 \times \cdots \times A_k$ is a closed subset of $X_1 \times \cdots \times X_k$.

(2) Let $B_i \subset X_i$, $i = 1, \ldots, k$ be subsets. The following equalities hold:

$$\begin{aligned}
\text{closure of } B_1\times\ldots\times B_k \text{ in } X_1\times\ldots\times X_k &= \overline{B}_1\times\ldots\times\overline{B}_k\\
\text{interior of } B_1\times\ldots\times B_k \text{ in } X_1\times\ldots\times X_k &= \mathring{B}_1\times\ldots\times\mathring{B}_k\\
\text{boundary of } B_1\times\ldots\times B_k \text{ in } X_1\times\ldots\times X_k &= \bigcup_{i=1}^{k} (\overline{B}_1\times\ldots\times\partial B_i\times\ldots\times\overline{B}_k).
\end{aligned}$$



Proof.

(1) We have

$$(X_1 \times \cdots \times X_k) \setminus (A_1 \times \cdots \times A_k) \overset{\text{basic set theory}}{=} \bigcup_{i=1}^{k} \underbrace{X_1 \times \cdots \times (X_i \setminus A_i) \times \cdots \times X_k}_{\text{open in } X_1\times\cdots\times X_k \text{ since } A_i \text{ is closed}}.$$

Thus we see that $(X_1 \times \cdots \times X_k) \setminus (A_1 \times \cdots \times A_k)$ is the intersection of finitely many open sets, i.e. the set itself is open, which implies by definition that $A_1 \times \cdots \times A_k$ is closed.

(2) We will verify this statement in Exercise 5.8. ■

The next proposition deals with some of our favorite properties of topological spaces.

Proposition 5.6. (Product Topology Properties Proposition) Let $X_1, \ldots, X_k$ be non-empty topological spaces.

(1) The product $X_1 \times \cdots \times X_k$ is Hausdorff if and only if each X_i is Hausdorff.
(2) The product $X_1 \times \cdots \times X_k$ is compact if and only if each X_i is compact.
(3) The product $X_1 \times \cdots \times X_k$ is (path)-connected if and only if each X_i is (path)-connected.

Remark. The Product Topology Properties Proposition 5.6 (2) also shows how the case $n = 1$ in the Hyperrectangle-Compact Proposition 1.26 implies the general case of the Hyperrectangle-Compact Proposition 1.26.

Before we head to the proof of the Product Topology Properties Proposition 5.6 let us first discuss some examples in detail.

Definition.

(1) We refer to $(S^1)^n$ as the n-dimensional torus. For $n = 2$ we just say the torus.
(2) We refer to $S^1 \times \overline{B}^2$ as the solid torus and we refer to $S^1 \times B^2$ as the open solid torus.

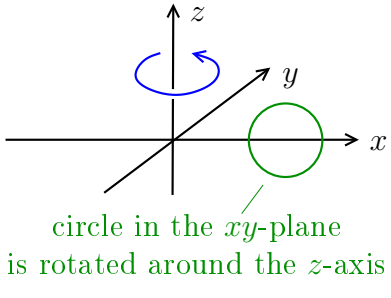
We deduce from the Product Topology Properties Proposition 5.6 that the n-dimensional torus $(S^1)^n$ and the solid torus $S^1 \times \overline{B}^2$ are compact.

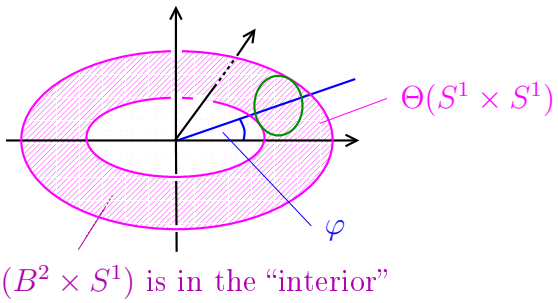
The following lemma allows us to view (solid) tori as subspaces of $\mathbb{R}^3$.

Lemma 5.7. We consider the map[65]

$$\begin{array}{rcl} \Theta\colon \mathbb{R}^2 \times S^1 & \to & \mathbb{R}^3 \\ ((x,y),\exp(\mathrm{i}\varphi)) & \mapsto & \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation around } z\text{-axis}} \cdot \underbrace{\begin{pmatrix} 1+\frac{1}{2}x \\ 0 \\ -\frac{1}{2}y \end{pmatrix}}_{\substack{\text{describes circle} \\ \text{in } xz\text{-plane}}} \end{array}$$

The restrictions of Θ to $S^1 \times S^1$, $\overline{B}^2 \times S^1$ and $B^2 \times S^1$ are embeddings.



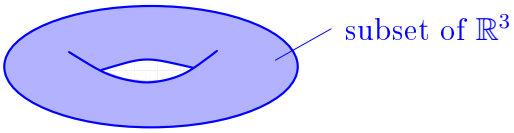


Proof. It follows easily from the example on page 227 that the map $\Theta\colon S^1 \times \mathbb{R}^2 \to \mathbb{R}^3$ is continuous. An elementary argument shows that the restriction of Θ to $\overline{B}^2 \times S^1$ is injective. Since the solid torus $\overline{B}^2 \times S^1$ is compact we obtain from the Compact-Hausdorff

[65] The map is chosen in such a way that it is orientation-preserving (we will introduce the notions of orientation and orientation-preserving later on page 682) and such that $\Theta(\{(0,0)\} \times S^1) = S^1 \times \{0\}$.

Proposition 2.17 (3) that the restriction of Θ to $\Theta\colon \overline{B}^2 \times S^1 \to \Theta(S^1 \times \overline{B}^2)$ is a homeomorphism. But then it follows also that the restrictions of Θ to $S^1 \times S^1$ and $B^2 \times S^1$ are embeddings. ■

Remark. Lemma 5.7 allows us to view (solid) tori as subspaces of $\mathbb{R}^3$. We refer to the images under these maps as the *standard* (*solid*) (*open*) *torus in* $\mathbb{R}^3$. Unfortunately, in a picture, it is hard to distinguish between tori, solid tori and open solid tori.

$S^1 \times S^1$ is homeomorphic to 

Now the time of procrastination is over, we need to provide the proof of the Product Topology Properties Proposition 5.6.

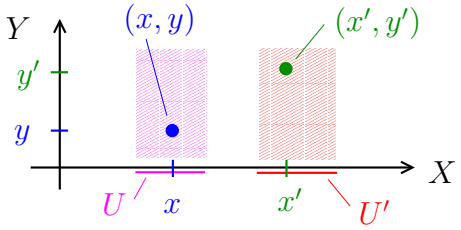
Proof of the Product Topology Properties Proposition 5.6 (1). By induction we only have to deal with the case $k = 2$. Thus let X and Y be two non-empty topological spaces.

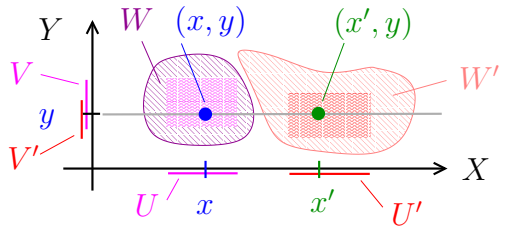
First we deal with the "$\Leftarrow$"-direction of Statement (1). Thus we assume that X and Y are Hausdorff. Let (x, y) and (x', y') be two distinct points in $X \times Y$. Without loss of generality we can assume that $x \neq x'$. By our hypothesis there exist open neighborhoods U of x and U' of x' with $U \cap U' = \varnothing$. Note that

$$(U \times Y) \cap (U' \times Y) \;=\; (U \cap U') \times Y \;=\; \varnothing \times Y \;=\; \varnothing.$$

By definition of the product topology the sets $U \times Y$ and $U' \times Y$ are open in $X \times Y$. We have thus found disjoint open neighborhoods of (x, y) and (x', y'). Therefore $X \times Y$ is Hausdorff. (This argument is sketched in the figure below to the left.)

Now we deal with the "$\Rightarrow$"-direction of Statement (1). Therefore we suppose that $X \times Y$ is Hausdorff. We want to show that X is Hausdorff. Thus let x and x' be two different points in X. Since Y is non-empty there exists a $y \in Y$. Furthermore, since $X \times Y$ is Hausdorff there exist disjoint open neighborhoods W and W' of (x, y) and (x', y). By definition of the product topology there exist open neighborhoods U of x and U' of x' and open neighborhoods V of y and V' of y such that $(U \times V) \cap (U' \times V') = \varnothing$. In particular we see that $(U \times \{y\}) \cap (U' \times \{y\}) = \varnothing$. But this implies that $U \cap U' = \varnothing$. This implies that U and U' are disjoint open neighborhoods of x and x'. Thus we have shown that X is Hausdorff. (This argument is sketched in the figure below to the right.) The proof that Y is also Hausdorff is evidently basically the same. ■





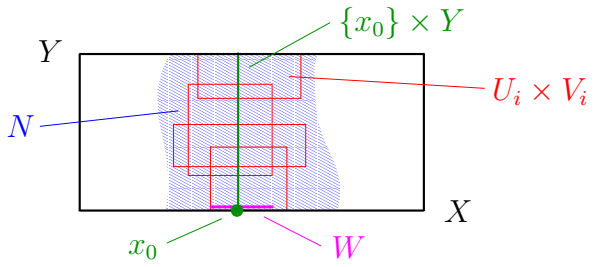
Proof of the Product Topology Properties Proposition 5.6 (2). As in the proof of Statement (1) we only need to deal with the case $k = 2$. Thus let X and Y be two non-empty topological spaces. The "$\Rightarrow$"-direction of Statement (2) is fortunately elementary: if $X \times Y$ is compact, then it follows immediately from the Topological-Product Proposition 5.1 (1a) and the Compact Image Lemma 2.13 (1).
Next we deal with "$\Leftarrow$"-direction of Statement (2). This is by far the hardest part of the proposition. So we now assume that X and Y are both compact. We need to show that $X \times Y$ is compact. We start out with the following claim.

Claim. Let $x_0 \in X$. If N is an open set of $X \times Y$ that contains $\{x_0\} \times Y$, then there exists a neighborhood W of $x_0 \in X$ such that $W \times Y \subset N$.

Proof. By definition of the product topology there exist sets of the form $U_i \times V_i$, $i \in I$ with the following properties:

(1) each U_i is open in X and each V_i is open in Y,
(2) each U_i contains x_0, i.e. for each i we have $\{x_0\} \times V_i \subset U_i \times V_i$,
(3) each $U_i \times V_i$ is contained in N,
(4) $\{x_0\} \times Y$ is contained in the union of the $U_i \times V_i$.

Since $\{x_0\} \times Y$ is compact we can arrange that I is a finite set, i.e. we can assume that $I = \{1, \ldots, n\}$. We set $W = U_1 \cap \cdots \cap U_n$. This is an open subset of X that contains x_0. It follows immediately from (3) and the definition of W that $W \times Y \subset N$. ⊞



Now we turn to the actual proof that $X \times Y$ is compact. Thus suppose we are given an open cover $\{A_i\}_{i \in I}$ of $X \times Y$. Let $x \in X$. Since Y is compact there exists a finite set $I(x) \subset I$ such that $\{x\} \times Y \subset \bigcup_{i \in I(x)} A_i$. By the above claim there exists an open neighborhood $W(x)$ of x such that $W(x) \times Y \subset \bigcup_{i \in I(x)} A_i$.

All these open sets $\{W(x)\}_{x \in X}$ are an open cover of X. Since X is compact there exist $x_1, \ldots, x_k \in X$ such that $X = W(x_1) \cup \cdots \cup W(x_k)$. But then we see that

$$X \times Y \;=\; (W(x_1) \cup \cdots \cup W(x_k)) \times Y \;=\; \bigcup_{j=1}^{k} W(x_j) \times Y \;\subset\; \bigcup_{j=1}^{k} \bigcup_{i \in I(x_j)} A_i \;\subset\; X \times Y.$$

Thus we have successfully covered $X \times Y$ by finitely many sets in the given open cover. ■

Proof of the Product Topology Properties Proposition 5.6 (3). As in the proof of Statement (1) we only need to deal with the case $k = 2$. Thus let X and Y be two non-empty topological spaces.

The "$\Rightarrow$"-direction of Statement (3) is once again basically elementary, Indeed, if $X \times Y$ is (path)-connected, then it follows immediately from the Image-Connected Lemma 2.24

applied to the projections $X \times Y \to X$ and $X \times Y \to Y$, that X and Y are (path)-connected.[66]

Next we deal with "$\Leftarrow$"-direction of Statement (3). First we suppose that X and Y are path-connected. We need to show that $X \times Y$ is also path-connected. Thus let (x, y) and (x', y') be two points in $X \times Y$. Since X is path-connected we can find a path $p\colon [0,1] \to X$ from x to x' and similarly we can find a path $q\colon [0,1] \to Y$ from y to y'. Then

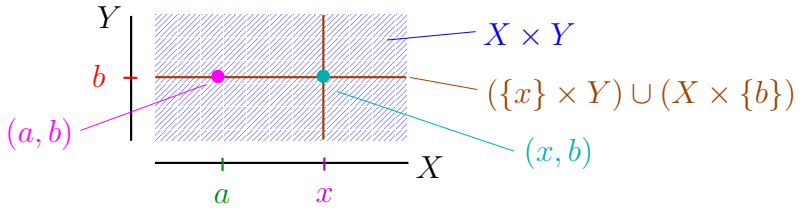
$$\begin{array}{rcl} [0,1] & \to & X \times Y \\ t & \mapsto & \begin{cases} (p(2t), y), & \text{if } t \in [0, \frac{1}{2}], \\ (x', q(2t-1)), & \text{if } t \in (\frac{1}{2}, 1] \end{cases} \end{array}$$

is a path from (x, y) to (x', y').[67]

Finally suppose that X and Y are connected. We want to show that $X \times Y$ is also connected. We pick $a \in X$ and $b \in Y$. Let $x \in X$. Note that the subsets $\{x\} \times Y$ and $X \times \{b\}$ are connected by our hypothesis. Since these have the point (x, b) in common we deduce from the Union Connected Lemma 2.28 (2) that $(\{x\} \times Y) \cup (X \times \{b\})$ is also connected. Note that

$$X \times Y = \bigcup_{x \in X} (\{x\} \times Y) \cup (X \times \{b\}).$$

All the sets on the right are connected and they have the point (a, b) in common. Thus it follows again from the Union Connected Lemma 2.28 (2) that the union on the right-hand side is connected. ■



5.3. **Properties of products of topological spaces II.** In this section we now return to the study of the product of arbitrarily many topological spaces. We will not bother with formulating the generalization of the Product-of-Subsets Lemma 5.5. Instead we jump straight to the following generalization of the Product Topology Properties Proposition 5.6.

Proposition 5.8. (Product Topology Properties Proposition) Let $\{X_i\}_{i \in I}$ be a family of non-empty topological spaces.

(1) The product $\prod_{i \in I} X_i$ is Hausdorff if and only if each X_i is Hausdorff.

(2) The product $\prod_{i \in I} X_i$ is compact if and only if each X_i is compact.

(3) The product $\prod_{i \in I} X_i$ is (path)-connected if and only if each X_i is (path)-connected.

[66] Note that the projections are surjective since we assume that X and Y are non-empty.

[67] It follows from the Topological-Product Proposition 5.1 (1b) and the Pasting Proposition 2.6 (2) that this map is indeed continuous.

Proof. The proof of all statements of the Product Topology Properties Proposition 5.8 is basically the same as the proof of the corresponding statements of the Product Topology Properties Proposition 5.6, *except*, for the "if"-direction of the Product Topology Properties Proposition 5.8 (2). This proof is now considerably harder and thus the proposition gets upgraded to a theorem, which we formulate below. ■

Theorem 5.9. (Tychonoff's Theorem[68]) The product of arbitrarily many compact spaces is again a compact space.

Remark. The theorem has its origin in the work of Andrey Tikhonov [**Tyc1935**]. The spelling for the theorem is due to the fact that Tikhonov initially published in German and his name got transcribed from the Cyrillic alphabet to the Latin alphabet as Tychonoff.

Sketch of proof. We will not really make use of Tychonoff's Theorem, therefore we limit ourselves to a proof for $I = \mathbb{N}$. Thus let $\{X_i\}_{i\in\mathbb{N}}$ be a family of compact spaces. Furthermore let $\{U_j\}_{j\in J}$ be a family of open sets of $\prod\limits_{i\in\mathbb{N}} X_i$ such that we cannot cover $\prod\limits_{i\in\mathbb{N}} X_i$ by finitely many sets in the given open cover $\{U_j\}_{j\in J}$. We need to show that $\{U_j\}_{j\in J}$ is not a cover of $\prod\limits_{i\in\mathbb{N}} X_i$. In other words, we need to show that $\bigcup\limits_{j\in J} U_j$ is a proper subset of $\prod\limits_{i\in\mathbb{N}} X_i$.

Claim 1. There exist $x_i \in X_i$, $i \in \mathbb{N}$ with the following property: Given any $k \in \mathbb{N}$ and given any open neighborhoods V_i of x_i, $i = 1, \dots, k$ the product

$$V_1 \times \cdots \times V_k \times X_{k+1} \times X_{k+2} \times \dots$$

cannot be covered by finitely many sets in the given open cover.

Proof. Suppose that for some $k \in \mathbb{N}_0$ we have already found $x_i \in X_i$, $i = 1, \dots, k$ with the desired properties. Suppose there is no such $x_{k+1} \in X_{k+1}$. This means that for every $y \in X_{k+1}$ there exist open neighborhoods $V_1^y, \dots, V_k^y, V_y^y$ of $x_1, \dots, x_k, y$ such that $\underbrace{V_1^y \times \cdots \times V_k^y \times V_y^y}_{=:W^y} \times X_{k+2} \times \dots$ can be covered by finitely many of the U_j, i.e. there exists a finite subset $J_y \subset J$ such that $W^y \times X_{k+2} \times \cdots \subset \bigcup\limits_{j\in J_y} U_j$.

Note that $\{W^y\}_{y\in V_{k+1}}$ is an open cover of $X_1 \times \cdots \times X_{k+1}$. By the original Product Topology Properties Proposition 5.6 (2) we know that $X_1 \times \cdots \times X_{k+1}$, as the product of *finitely many* compact topological spaces is compact. This implies that there exists a finite subset $Y \subset V_{k+1}$ such that $\bigcup\limits_{y\in Y} W_y = X_1 \times \cdots \times X_{k+1}$. But this means that

$$\begin{aligned}\prod_{i\in\mathbb{N}} X_i \;&=\; (X_1 \times \cdots \times X_{k+1}) \times \prod_{i=k+2}^{\infty} X_i \;=\; \Big(\bigcup_{y\in Y} W_y\Big) \times \prod_{i=k+2}^{\infty} X_i \\ &= \bigcup_{y\in Y} \Big(W_y \times \prod_{i=k+2}^{\infty} X_i\Big) \;\subset\; \bigcup_{y\in Y} \bigcup_{j\in J_y} U_j.\end{aligned}$$

Recall that Y is finite and that each J_y is finite. We have thus obtained contradiction to our hypothesis that $\prod\limits_{i\in\mathbb{N}} X_i$ cannot be covered by finitely many of the U_j. ⊞

The following claim concludes the proof of the theorem for $I = \mathbb{N}$.

Claim 2. The point $(x_1, x_2, \dots) \in \prod\limits_{i\in\mathbb{N}} X_i$ does not lie in any of the U_j.

Proof. Suppose there exists a $j \in J$ with $(x_1, x_2, \dots) \in U_j$. By definition of the topology on $\prod_{i\in\mathbb{N}} X_i$, (see the Topological-Product Proposition 5.1 (0) together with the Topology-from-Basis Lemma 1.27) there exists a $k \in \mathbb{N}$ and open neighborhoods V_i of x_i, $i = 1, \dots, k$ such that

$$V_1 \times \cdots \times V_k \times X_{k+1} \times X_{k+2} \times \dots \subset U_j.$$

But by construction of $(x_1, x_2, \dots)$ this is impossible. ⊞

The proof of the general case of the Tychonoff Theorem can be done in a very similar way by equipping the index set I with a total order, which we can do by the Well-Ordering Theorem 0.7. We refer to [**Wri1994**] for details. Alternatively proofs of Tychonoff's Theorem are provided in most textbooks on general topology, see e.g. [**Mun1975**, Chapter 5.1] or [**Jä2005**, Chapter X]. A particularly concise proof is also given in [**Chf1992**]. Note that all proofs of the Tychonoff Theorem rely on the Axiom of Choice 0.1, in fact, as is shown in [**Je1973**, p. 26] or [**BTT2020**, Theorem 3.14], the Tychonoff Theorem is equivalent to the Axiom of Choice 0.1. ■

We continue with a very instructive example.

Example. We equip $\{0, 2\}$ with the discrete topology. We consider the map

$$\begin{aligned} \varphi\colon \prod_{i\in\mathbb{N}} \{0,2\} &\to C = \text{Cantor set from page 130} \\ (f\colon \mathbb{N} \to \{0,2\}) &\mapsto \sum_{i=1}^{\infty} \frac{f(i)}{3^i}. \end{aligned}$$

It is elementary to see that φ takes values in C. It is a refreshing exercise, left to the enthusiastic reader, to show that φ is a bijection. The left-hand side is compact by Tychonoff's Theorem 5.9 and the right-hand is evidently Hausdorff. In Exercise 5.21 we verify that φ is continuous. It thus follows from the Compact-Hausdorff Proposition 2.17 (3) that φ is a homeomorphism.

Initially the definition of the Cantor set is nicely explicit, but the definition looks rather arbitrary. The previous example gives a much more natural definition of the Cantor set, viewed as a topological space. Amazingly there exists a characterization of the Cantor set, up to homeomorphism, in terms of basic topological properties. To state the proposition we need two easy definitions.

Definition. Let X be a topological space. A point $x_0 \in X$ is called isolated if $\{x_0\}$ itself is an open subset.

Examples.

(1) We consider $X = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$. The only point in X that is not isolated is 0.
(2) One can easily verify that no point in the Cantor set is isolated.
(3) It follows immediately from the definitions that no point in $\prod_{i\in\mathbb{N}} \{0, 2\}$ is isolated.
(4) The topological spaces $\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$ equipped with the subspace topology coming from $\mathbb{R}$ do not contain any isolated points.
(5) Let X be a topological space. If every point is isolated, then by definition X has the discrete topology. If X is furthermore finite, then it follows from the Compact-Discrete Lemma 1.22 (1) that X is finite.

Definition. A topological space X is called totally disconnected if the only connected subsets are the empty set and the subsets consisting of a single point.

Example. The following topological spaces are totally disconnected:

(1) Discrete spaces.
(2) The Cantor set.T
(3) the infinite product $\prod_{i\in\mathbb{N}}\{0,2\}$.
(4) The topological spaces $\mathbb{Q}$ and $\mathbb{Q}\setminus\{0\}$ equipped with the subspace topology coming from $\mathbb{R}$.

Here (1) is obvious and (4) was dealt with in Exercise 2.23. We will deal with the remaining two examples in Exercise 5.22.

We conclude this section with the following amusing proposition.

Proposition 5.10. Let X be a topological space that is compact and totally disconnected. If X contains no isolated points, then X is homeomorphic to the Cantor set.

Proof. We will not make use of the proposition, hence we will not prove it. A proof is for example given in [**Val2013**, Theorem 7.8] or [**Will1970**, Theorem 30.3]. ■

Example. By the above examples Proposition 5.10 implies not only that the Cantor set is homeomorphic to $\prod_{i\in\mathbb{N}}\{0,2\}$ but also to $\mathbb{Q}$ and $\mathbb{Q}\setminus\{0\}$. In particular we obtain the weird result that $\mathbb{Q}$ and $\mathbb{Q}\setminus\{0\}$ are homeomorphic.[69]

5.4. **The disjoint union of topological spaces.** In this section we will discuss the coproduct in the category of topological spaces. Before we get to topological spaces it is worth recalling the following definitions from page 88 and the Coproduct-Sets-and-Abelian Groups Lemma 4.4.

Definition. Let $\{A_i\}_{i\in I}$ be a family of sets.

(1) We define the disjoint union as the set

$$\bigsqcup_{i\in I} A_i := \bigcup_{i\in I}(A_i\times\{i\}).$$

As usual we take some liberties with the notation if we have finitely many sets.

(2) Given $i\in I$ we refer to the map $A_i\to\bigsqcup_{i\in I}A_i$ given by $a\mapsto(a,i)$ as the natural injection.

In the Coproduct-Sets-and-Abelian Groups Lemma 4.4 we saw that the disjoint union, together with the obvious natural injections, gives us the coproduct in the category of sets. In a second we will see that a very similar statement holds in the context of the category of topological spaces. But before we get to this point, let us introduce the following convention.

Convention. Let $\{A_i\}_{i\in I}$ be a family of sets. For any $i\in I$ we use the natural injection $A_i\to\bigsqcup_{i\in I}A_i$ to think of A_i as a subset of the disjoint union.

[69] In Exercise 1.11 you were challenged to prove directly that $\mathbb{Q}$ and $\mathbb{Q}\setminus\{0\}$ are homeomorphic?

Definition. Let $\{X_i\}_{i\in I}$ be a family of of topological spaces. The disjoint union topology on the disjoint union $\bigsqcup_{i\in I} X_i$ is defined by the condition that $U \subset \bigsqcup_{i\in I} X_i$ is open if and only if for each $i \in I$ the intersection $X_i \cap U$ is an open subset of X_i.[70]

The following proposition is the analogue of the Topological-Product Proposition 5.1.

Proposition 5.11. (Topological-Coproduct Proposition) Let $\{X_i\}_{i\in I}$ be a family of topological spaces. We equip the disjoint union $\bigsqcup_{i\in I} X_i$ with the disjoint union topology. For each $i \in I$ we denote by $\iota_i \colon X_i \to \bigsqcup_{i\in I} X_i$ the natural injection.

(1) (a) Each ι_i is continuous.
   (b) If Z is a topological space and if $\{f_i \colon X_i \to Z\}_{i\in I}$ is a family of continuous maps, then the map
$$\begin{array}{rcl} \bigsqcup_{i\in I} X_i & \to & Z \\ (x,i) & \mapsto & f_i(x) \end{array}$$
   is continuous.
(2) The topological space $\bigsqcup_{i\in I} X_i$ together with the natural injections ι_i is a coproduct of $\{X_i\}_{i\in I}$ in the category $\mathcal{T}op$ of topological spaces.

Proof.

(1) Both statements follow immediately from writing down the definitions.
(2) Let Z be a topological space and let $\{f_i \colon X_i \to Z\}_{i\in I}$ be a family of continuous maps. By the definition of the coproduct on page 214 we have to show that there exists a *unique* continuous map $\Phi \colon \bigsqcup_{i\in I} X_i \to Z$ such that for each $i \in I$ we have $f_i = \Phi \circ \iota_i$. In (1b) we defined such a map Φ and we saw that it is continuous. It remains to prove uniqueness. The proof of uniqueness is basically identical to the argument in the proof of the Coproduct-Sets-and-Abelian Groups Lemma 4.4 (1) where we dealt with the coproduct in the category of sets. ■

Convention. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. Unless we say something else we equip the disjoint union $\bigsqcup_{i\in I} X_i$ with the disjoint union topology. We refer to this topological space as the disjoint union of the X_i.[71]

Lemma 5.12. Let $\{X_i\}_{i\in I}$ be a family of topological spaces.

(1) A subset $A \subset \bigsqcup_{i\in I} X_i$ is closed if and only if for each $i \in I$ the intersection $A \cap X_i$ is a closed subset of X_i.
(2) For each $i \in I$ the natural injection $\iota_i \colon X_i \to \bigsqcup_{i\in I} X_i$ is an embedding which is open and closed.

Proof.

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[70] It follows very very easily from the definitions that this defines a topology on the set $\bigsqcup_{i\in I} X_i$.

[71] It would perhaps make more sense to refer to $\bigsqcup_{i\in I} X_i$ as the coproduct of the X_i. But we prefer to stick to the more classical name.

(1) The lemma follows immediately from the definition of the disjoint union topology together with Lemma 2.7.
(2) Let $i \in I$. It follows easily from the definition of the subspace topology that the natural injection $\iota_i \colon X_i \to \bigsqcup_{i\in I} X_i$ is an open map. Furthermore it follows from (1) that the natural injection is a closed map. It now follows from the Open -Injective Map Lemma 2.16 that the natural injection is an embedding which is open and closed. ■

The following lemma plays the role of the Product Topology Properties Proposition 5.6.

Lemma 5.13. (Disjoint Union Topology-Properties Lemma) Let $\{X_i\}_{i\in I}$ be a family of topological spaces.

(1) The disjoint union $\bigsqcup_{i\in I} X_i$ is Hausdorff if and only if each X_i is Hausdorff.
(2) The disjoint union $\bigsqcup_{i\in I} X_i$ is compact if and only if each X_i is compact and if there are only finitely many $i \in I$ such that $X_i \neq \varnothing$.
(3) The (path-) components of the disjoint union $\bigsqcup_{i\in I} X_i$ are precisely the (path-) components of the X_i.

Proof.

(1) This statement follows easily from the definitions
(2) We first prove the "if"-direction. Thus we assume that each X_i is compact and that there are only finitely many $i_1, \ldots, i_k \in I$ such that the corresponding topological space is non-empty. Let $\{U_j\}_{j\in J}$ be an open cover of $\bigsqcup_{i\in I} X_i$. Since each X_i is compact we can find for each $i \in I$ a finite subset $J_i \subset J$ such that $\bigsqcup_{j\in J_i} (U_j \cap X_i) = X_i$. It is now clear that $X = \bigcup_{r=1}^{k} \bigsqcup_{j\in J_{i_r}} U_j$.

We turn to the proof of the "only if"-direction. Thus we assume that $\bigsqcup_{i\in I} X_i$ is compact. It follows from the Disjoint Union Topology-Properties Lemma 5.13 (2) together with the Compact-Closed Lemma 1.21 (1) that each X_i is compact. Since the $\{X_i\}_{i\in I}$ form an open cover of the compact topological space $\bigsqcup_{i\in I} X_i$ we see that there exists a finite subset $J \subset I$ with $\bigsqcup_{j\in J} X_j = \bigsqcup_{i\in I} X_i$. In particular $X_i = \varnothing$ for $i \notin J$.
(3) This statement follows without too much effort from the Component Lemma 2.33. ■

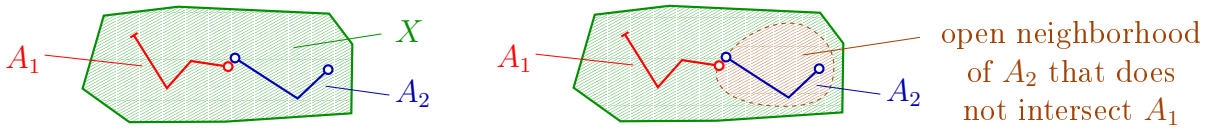
The following lemma gives a criterion for showing that a union of subspaces is homeomorphic to the disjoint union of the subspaces.

Lemma 5.14. (Disjoint Union Embedding Lemma) Let X be a topological space and let $\{A_i\}_{i\in I}$ be a family of disjoint subsets of X. The following two statements are equivalent:

(1) For each $i \in I$ there exists an open neighborhood of $A_i \subset X$ that does not intersect any of the A_j with $j \neq i$.[72]
(2) The map

$$\begin{array}{ccc} \overbrace{\bigsqcup_{i\in I} A_i}^{\text{disjoint union topology}} & \to & \overbrace{\bigcup_{i\in I} A_i}^{\text{subspace of } X} \\ (a,i) & \mapsto & a \end{array}$$

is a homeomorphism.



Proof. First we point out that the given map

$$\begin{array}{rccc} \Phi\colon & \bigsqcup_{i\in I} A_i & \to & \bigcup_{i\in I} A_i \\ & (a,i) & \mapsto & a \end{array}$$

is a bijection since the A_i are disjoint subsets of X. Since the inclusion maps $A_i \to \bigcup_{i\in I} A_i$ are continuous it follows from the Topological-Coproduct Proposition 5.11 that the above map Φ is continuous.

We first prove (1)⇒(2). Thus we assume that for each $i \in I$ there exists an open neighborhood V_i of $A_i \subset X$ that does not intersect any of the A_j with $j \neq i$. We need to show that Φ is a homeomorphism. By the above discussion and the Open-Injective Map Lemma 2.16 it remains to show that Φ is open. Thus let U be an open subset of $\bigsqcup_{i\in I} A_i$. By definition of the disjoint union topology each $A_i \cap U$ is an open subset of A_i. By the definition of the subspace topology on A_i this means that there exists an open subset W_i of X with $A_i \cap U = A_i \cap W_i$. We now see that

$$\begin{array}{rcccl} \Phi(U) & = & \Phi\Big(\bigsqcup_{i\in I}(A_i\cap U)\Big) & \overset{\text{definition of }\Phi}{=} & \bigsqcup_{i\in I}(A_i\cap U) \overset{\text{since } A_i\subset V_i \text{ and since } A_j\cap V_i=\varnothing \text{ for } i\neq j}{=} \bigsqcup_{i\in I}((A_i\cap U)\cap V_i) \\ & = & \bigsqcup_{i\in I}((A_i\cap(V_i\cap W_i)) & = & \Big(\bigcup_{i\in I} A_i\Big)\cap\Big(\bigcup_{j\in J}(V_j\cap W_j\Big). \\ & & \uparrow \text{choice of } W_i & & \uparrow \text{since } A_i\subset V_i\cap W_i \text{ and since } A_i\cap(V_j\cap W_j)=\varnothing \text{ for } i\neq j \end{array}$$

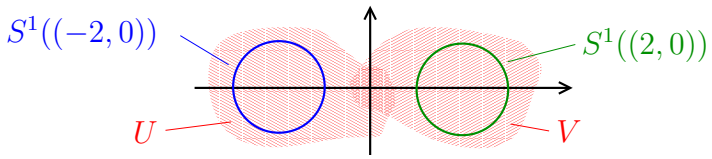
Since $\bigcup_{j\in J}(V_j\cap W_j)$ is an open subset of X we obtain, by definition of the subspace topology, that U is indeed an open subset of $\bigcup_{i\in I} A_i$.

Finally we prove (2)⇒(1). Thus we suppose that Φ is a homeomorphism. Let $i \in I$. Evidently A_i is an open subset of $\bigsqcup_{j\in I} A_j$. Thus $\Phi(A_i) = A_i$ is an open subset of $\bigcup_{j\in I} A_j$. By definition of the subspace topology this means that there exists an open subset V_i such that $V_i \cap \bigcup_{j\in J} A_j = A_i$. But this means that V_i is an open neighborhood V_i of $A_i \subset X$ that does not intersect any other A_j. ■

[72] Of course this hypothesis is satisfied if the A_i are disjoint open subsets of X.

Examples.

(1) Using Lemma 5.14 one can easily verify that the topological space $S^1 \sqcup S^1$ is homeomorphic to the topological space $S^1((-2,0)) \cup S^1((2,0)) \subset \mathbb{R}^2$.



(2) Using Lemma 5.14 one can also easily show that the topological space $[-1,0] \sqcup (0,1]$ is homeomorphic to $[-2,-1] \cup (0,1] \subset \mathbb{R}$. But note that the topological space $[-1,0] \sqcup (0,1]$ is not homeomorphic to $[-1,1]$ since the former is non-compact whereas the latter is compact.

(3) Let X be a topological space and let $\{X_i\}_{i \in I}$ be its set of components. Note that the Component Lemma 2.33 (1) says that as a set we have $X = \bigsqcup_{i \in I} X_i$, thus the natural map $\bigsqcup_{i \in I} X_i \to X$ is a bijection.

(a) If I is finite, then we know by the Components-are-Closed Lemma 2.34 that each X_i is closed and open, thus it follows from the Disjoint Union Embedding Lemma 5.14 that the natural map $\bigsqcup_{i \in I} X_i \to X$ is in fact a homeomorphism.

(b) If I is infinite, this logic does not necessarily apply. An easy counterexample is given by $X = \mathbb{Q}$, where, as saw on page 168, the components are precisely the one-point subsets.

5.5. **Quotients of topological spaces.** On page 89 we recalled the notion of an equivalence relation. We recall the following notation from page 90.

Notation. Let $\sim$ be an equivalence relation $\sim$ on a set X.

(1) Given $x \in X$ we denote by $[x]$ the unique equivalence class that contains x.

(2) We refer to the map

$$\begin{array}{rccc} p\colon & X & \to & X/\!\sim \\ & x & \mapsto & [x] \end{array}$$

as the natural projection.

(3) If $f\colon X \to Y$ is a map with the property that $f(x) = f(y)$ whenever $x \sim y$, then by the Quotient-Factorization Lemma 0.5 the map

$$\begin{array}{rccc} g\colon & X/\!\sim & \to & Y \\ & [x] & \mapsto & f(x) \end{array}$$

is well-defined and it is the unique map $g\colon X/\!\sim \to Y$, such that $f = g \circ p$, i.e. such that the following diagram of maps commutes:

$$\begin{array}{ccc} X & \xrightarrow{p} & X/\!\sim \\ & \searrow^{f} & \downarrow g \\ & & Y. \end{array}$$

We refer to this map g as the induced map.

The following proposition is the analogue of the Topological-Product Proposition 5.1 and of the Topological-Coproduct Proposition 5.11.

Proposition 5.15. (Topological-Quotient Proposition) Let $\sim$ be an equivalence relation on a topological space X. We denote by $p\colon X \to X/\sim$ the natural projection from X onto the set of equivalence classes $X/\sim$. The following statements hold:

(0) (a) The set
$$\mathcal{T} := \{U \subset X/\sim \mid p^{-1}(U) \text{ is open in } X\}$$
is a topology on $X/\sim$.[73]

(b) A subset $A \subset X/\sim$ is closed if and only if the preimage $p^{-1}(A)$ is closed in X.

(1) (a) The natural projection $p\colon X \to X/\sim$ is continuous.

(b) Let Y be a topological space and let $f\colon X \to Y$ be a continuous map with the property that $f(x) = f(y)$ whenever $x \sim y$. The induced map
$$\begin{array}{rccc} g\colon & X/\sim & \to & Y \\ & [x] & \mapsto & f(x) \end{array}$$
is also continuous.

(2) If Z is a topological space and if $q\colon X \to Z$ is a map such that the pair (Z, q) has the same properties as $(X/\sim, p)$ as formulated in (1a) and (1b), then there exists a unique homeomorphism $\Phi\colon X/\sim \to Z$ with $q = \Phi \circ p$.

Proof. Let $\sim$ be an equivalence relation on a topological space X.

(1) (a) This statement follows easily from the Image-Preimage Lemma 0.2 (6) and (7).

(b) This statement follows easily from the Image-Preimage Lemma 0.2 (8).

(2) (a) It follows immediately from the definition of the topology on $X/\sim$ that the natural projection $p\colon X \to X/\sim$ is continuous.

(b) Let $f\colon X \to Y$ be a continuous map such that $f(x) = f(y)$ whenever $x \sim y$. We need to show that the induced map $g\colon X/\sim \to Y$ is continuous as well. Let $U \subset Y$ be open. We need to show that $g^{-1}(U) \subset X/\sim$ is open. By definition of the quotient topology we need to show that $p^{-1}(g^{-1}(U))$ is open in X. We observe that
$$p^{-1}(g^{-1}(U)) \;=\; (g \circ p)^{-1}(U) \underset{\text{by the commutative diagram}}{\overset{\uparrow}{=}} f^{-1}(U) \underset{\text{since } f \text{ continuous}}{\overset{\uparrow}{=}} \text{open subset}\,.$$
We have thus shown that g is continuous.

(3) Let Z be a topological space and let $q\colon X \to Z$ be a map such that the following conditions are satisfied:

(a′) The map q is continuous.

(b′) Whenever we are given a topological space Y and a continuous map $f\colon X \to Y$ with the property that $f(x) = f(y)$ whenever $x \sim y$, then there exists a unique continuous map $h\colon Z \to Y$ with $h \circ q = f$.

We apply (b) and (a′) to obtain a continuous map $\Phi\colon X/\sim \to Z$ with $q = \Phi \circ p$. Furthermore we apply (a) and (b′) to obtain a continuous map $\Psi\colon Z \to X/\sim$ with $p = \Psi \circ p$. It follows from the uniqueness statement of the Quotient-Factorization Lemma 0.5 that $\Psi \circ \Phi = \mathrm{id}_{X/\sim}$ and it follows from the uniqueness statement of (b′) that $\Phi \circ \Psi = \mathrm{id}_Z$. ∎

[73] In other words, $U \subset X/\sim$ is open if and only if $p^{-1}(U) \subset X$ is open.

The following definition will not come as a surprise.

Definition. Let $\sim$ be an equivalence relation on a topological space X. We refer to the topology on $X/\sim$ from the Topological-Quotient Proposition 5.15 (0a) as the quotient topology and we refer to $X/\sim$ as a quotient space of X.

Definition. Let $f\colon X \to Y$ be a map as in the Topological-Quotient Proposition 5.15 (1b).

(1) We refer to the map $g\colon X/\sim \to Y$ as the induced map.
(2) At times we denote the induced map by $\overline{f}$. Sometimes, when there is no danger of confusion we denote the induced map just by f itself.
(3) Sometimes we also say that the map $f\colon X \to Y$ descends to a map $X/\sim \to Y$.

Lemma 5.16. (Quotient-Compact Connected Lemma) Let $\sim$ be an equivalence relation on a topological space X.

(1) If X is compact, then $X/\sim$ is also compact.
(2) If X is (path)-connected, then $X/\sim$ is also (path)-connected.

Proof.

(1) This statement follows immediately from (3) and from the Compact Image Lemma 2.13.
(2) This statement follows immediately from (3) together with the Image-Connected Lemma 2.24. ■

The following remark can be viewed as a sibling of Remark 5.2.

Remark 5.17. We consider maps in and out of a quotient space $X/\sim$:

(1) In practice it is usually easy to show that a given map $X/\sim \to Y$ out of the quotient space is continuous. More precisely, in practice it is often fairly clear that the map $X \to X/\sim \to Y$ is continuous, but then it follows from the Topological-Quotient Proposition 5.15 (1b) that the original map $X/\sim \to Y$ is continuous.
(2) There is usually no easy way to show that a map $Y \to X/\sim$ to a quotient space is continuous, unless the map factors through the projection $X \to X/\sim$.

We continue with a particularly important source of quotient spaces.

Definition. Let X be a topological space and let $A \subset X$ be a subset.

(1) (a) First we suppose that A is a non-empty subset. For $P, Q \in X$ we define

$$P \sim Q \quad :\Longleftrightarrow \quad P = Q \text{ or } P, Q \text{ both lie in } A.$$

This is easily seen to be an equivalence relation on X. We write $X/A := X/\sim$ and we always equip X/A with the quotient topology. We refer to X/A as the quotient of X by A. Note that $A \subset X$ is an equivalence class, thus it defines a point X/A. In an attempt to avoid confusion we write $[A]$ for the point in X/A that is given by the equivalence class A.

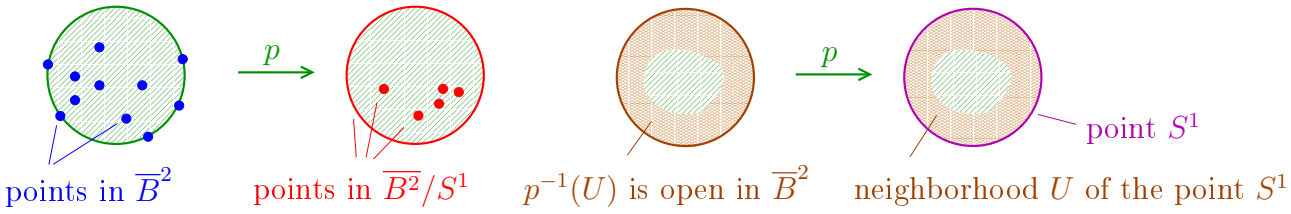
(b) If $A = \varnothing$ is the empty set, then we define $X/\varnothing := X \sqcup \{\varnothing\}$, i.e. $X/\varnothing$ equals the disjoint union of the topological space X with the topological space consisting of a single point $\varnothing$. For consistency we denote this point by $[A] = [\varnothing]$ as well. Note that $\varnothing/\varnothing$ is the topological space consisting of the single point $\{\varnothing\}$.[74]

(2) We refer to the obvious map $p\colon X \to X/A$ as the natural projection .

Remark. Let us attempt to digest the definition of X/A. As a set X/A corresponds precisely to the points in $X \setminus A$ together with the point $[A]$. Furthermore, by definition of the quotient topology a set $U \subset X/A$ is open if and only if $p^{-1}(U)$ is an open subset of X. Note that given any subset $Z \subset X/A = (X \setminus A) \cup \{[A]\}$ we have[75]

$$p^{-1}(Z) = \begin{cases} Z, & \text{if } [A] \notin Z, \\ Z \cup A, & \text{if } [A] \in Z. \end{cases}$$

This shows that the open sets in X/A correspond precisely to the open sets of X that lie in $X \setminus A$ and the open sets of X that contain A.



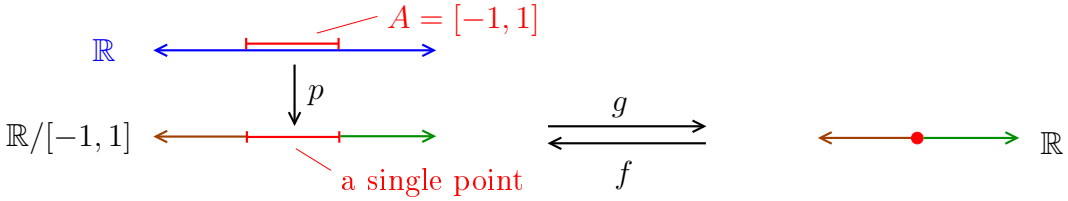
Examples.

(1) We consider the quotient of $X = \mathbb{R}$ by the compact interval $A = [-1, 1]$. This example has two interesting features:

- The quotient $X/A = \mathbb{R}/[-1, 1]$ is actually homeomorphic to an old favorite, namely one can easily verify that the maps

$$f : \mathbb{R} \to \mathbb{R}/[-1,1] \quad x \mapsto \begin{cases} [x+1], & \text{if } x \geq 0 \\ [x-1], & \text{if } x < 0 \end{cases} \quad \& \quad g : \mathbb{R}/[-1,1] \to \mathbb{R} \quad P \mapsto \begin{cases} x-1, & \text{if } P = [x] \text{ with } x \geq 1 \\ x+1, & \text{if } P = [x] \text{ with } x < -1 \end{cases}$$

  are continuous and inverses of one another.
- Above we saw that projections $X \to X/\sim$ are basically by definition, continuous. One might think that projections should also be open. But in this example the projection $p\colon X = \mathbb{R} \to X/A = \mathbb{R}/[-1, 1]$ is *not* an open map, for example it follows immediately from the previous remark that the image of the open set $(-1, 1) \subset \mathbb{R}$ is not an open subset of $\mathbb{R}/[-1, 1]$.

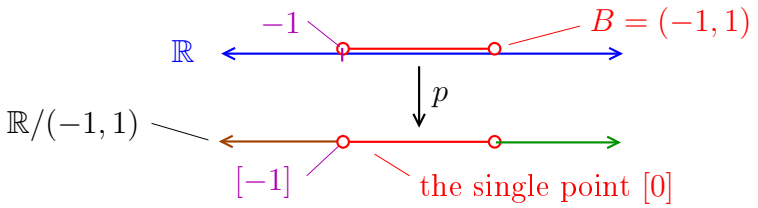


(2) We consider the quotient of $X = \mathbb{R}$ by the open interval $B = (-1, 1)$. We denote by $p\colon \mathbb{R} \to \mathbb{R}/(-1, 1)$ the projection . Even though this example looks very similar

[74]The reader will have spotted that the case $A = \varnothing$ is a major nuisance. It is conceivable that later on in proofs we will subconsciously suppress this special case. Whenever this occurs we leave it to the reader to deal with the case $A = \varnothing$ separately.

[75]In an attempt to lighten the notation, given $x \in X \setminus A$ we do not distinguish between the point $x \in X \setminus A$ and the equivalence class $\{x\} \subset X$.

to the previous one, it shows very different behavior. For example it turns out that the quotient $X/B = \mathbb{R}/(-1,1)$ is not Hausdorff. More precisely we consider the two points $[-1]$ and $[0]$ in $\mathbb{R}/(-1,1)$. The previous remark shows that any open neighborhood U of $[-1]$ in $\mathbb{R}/(-1,1)$ has the property that $p^{-1}(U)$ is an open subset of $\mathbb{R}$ that contains -1. In particular $p^{-1}(U)$ has non-empty intersection with B. But this implies that $U = p(p^{-1}(U))$ contains the point $[0]$. Thus $[-1]$ and $[0]$ cannot be separated by open neighborhoods.[76]



The following lemma is elementary but useful. On numerous occasions it will allow us to slightly shift our point of view.

Lemma 5.18. Let X be a topological space and let $A \subset X$ be a subset. Furthermore let Y be another topological space and let $y_0 \in Y$. The map

$$\Phi\colon \left\{\begin{array}{c}\text{set of continuous maps } f\colon X \to Y \\ \text{such that } f(A) = \{y_0\}\end{array}\right\} \to \left\{\begin{array}{c}\text{set of continuous maps } f\colon X/A \to Y \\ \text{such that } f([A]) = y_0\end{array}\right\}$$
$$(f\colon X \to Y) \mapsto (\overline{f}\colon X/A \to Y)$$

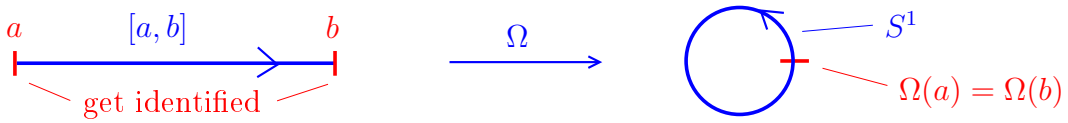
is well-defined and it is a bijection.

Proof. It follows from the Topological-Quotient Proposition 5.15 (1b) that given a continuous map $f\colon X \to Y$ with $f(A) = \{y_0\}$ the corresponding map $\overline{f}\colon X/A \to Y$ is continuous. This shows that the map Φ is well-defined. It is clear that Φ is injective. Finally suppose we are given a map $g\colon X/A \to Y$ with $g([A]) = y_0$. We denote by $f\colon X \to X/A \to Y$ the map that is given by precomposing g with the natural projection $X \to X/A$. It follows from the Topological-Quotient Proposition 5.15 (1a) that f is continuous. Clearly we have $\Phi(f) = g$. This shows that Φ is also a surjection. ■

Example. Let $a < b$ be two real numbers. It follows easily from the Quotient-Compact Connected Lemma 5.16 (1), the Topological-Quotient Proposition 5.15 (1b) and the Compact-Hausdorff Proposition 2.17 (3) that the map

$$\Omega\colon [a,b]/\{a,b\} \to S^1$$
$$[t] \mapsto \exp\left(i \cdot \tfrac{2\pi(t-a)}{b-a}\right)$$

is a homeomorphism.



We continue with the following innocuous lemma.

[76] Note that in the Quotient Hausdorff Lemma 5.21 we will give a criterion that ensures that the quotient of a topological space is Hausdorff.

Lemma 5.19.

(1) The map

$$\begin{array}{rcl} ([0,1]\times S^n)/(\{0\}\times S^n) & \to & \overline{B}^{n+1} \\ [(r,v)] & \mapsto & r\cdot v \end{array}$$

is a homeomorphism.

(2) Let X be a topological space and let $f\colon [0,1]\times S^n \to X$ be a continuous map. If $f(0,v)=f(0,w)$ for all $v,w\in S^n$, then the map

$$\begin{array}{rcl} \overline{B}^{n+1} & \to & X \\ r\cdot v & \mapsto & f(r,v) \text{ with } r\in[0,1] \text{ and } v\in S^n \end{array}$$

is well-defined and continuous.

Proof.

(1) It follows from the Topological-Quotient Proposition 5.15 (1b) that the given map is continuous. It is basically clear that the map is a bijection. The left-hand side is compact by the Topological-Quotient Proposition 5.15 (1b) (4). It follows from the Compact-Hausdorff Proposition 2.17 (3) that the map is a homeomorphism.

(2) It follows from the hypothesis that the map is well-defined. It follows from (1) and the Topological-Quotient Proposition 5.15 (1b) that the map is continuous. ■

We record our next example as a lemma.

Lemma 5.20. (Ball-Quotient Sphere Lemma) The map

$$\begin{array}{rcl} \Omega\colon \overline{B}^n/S^{n-1} & \to & S^n \\ [r\cdot v] & \mapsto & \underbrace{\begin{pmatrix} 0 & (-1)^{n-1} & \dots \\ 0 & 0 & \mathrm{id}_{n-1} \\ -1 & \dots & 0 \end{pmatrix}}_{\in \mathrm{SO}(n+1)} \cdot \underbrace{\begin{pmatrix} \cos(\pi r) \\ \sin(\pi r)\cdot v \end{pmatrix}}_{\in \mathbb{R}\times\mathbb{R}^n=\mathbb{R}^{n+1}} \text{ with } r\in[0,1] \text{ and } v\in S^{n-1} \end{array}$$

is a homeomorphism. (Note that for $n=1$ and $n=2$ this homeomorphism is illustrated in the figure below.) This homeomorphism has the following properties:

(1) The map Ω sends the point $[S^{n-1}]\in \overline{B}^n/S^{n-1}$ to the North Pole $(0,\dots,0,1)$.

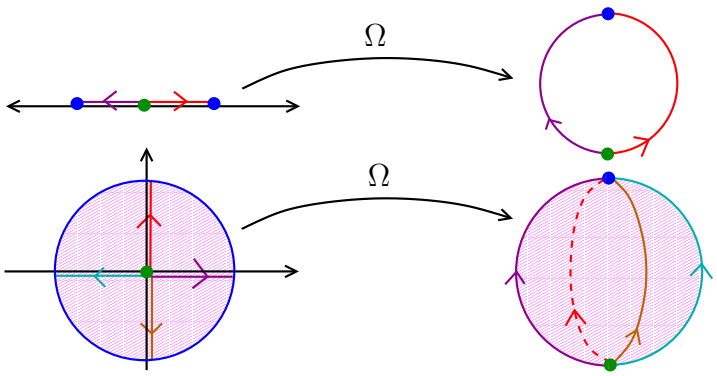
(2) The map Ω sends the origin to the South Pole $(0,\dots,0,-1)$.

(3) The map $B^n \to \overline{B}^n/S^{n-1} \xrightarrow{\Omega} S^n$ is a smooth embedding in the sense of the definition on page 652. Furthermore, if we equip B^n and S^n with the standard orientations from page 677 and page 681, then the smooth embedding is orientation-preserving in the sense of the definition on page 682.[77]

Proof. It follows from Lemma 5.19, the Topological-Quotient Proposition 5.15 (1b) and the Compact-Hausdorff Proposition 2.17 (3) that the map $\Omega\colon \overline{B}^n/S^{n-1}\to S^n$ is a homeomorphism. It is clear that $\Omega([S^{n-1}])=(0,\dots,0,1)$ and that $\Omega(0)=(0,\dots,0,-1)$.

Thus it remains to prove Statement (3). To show smoothness it is convenient to rewrite the definition of Ω. First note that by elementary real analysis we know that there exists a smooth map $f\colon \mathbb{R}\to\mathbb{R}$ such that for any $x\neq 0$ we have $f(x)=\frac{\sin(\pi\cdot x)}{x}$. One can now

[77] We hope that the reader is not too fazed by the fact that we use some expressions that we will introduce only later. But we think it is useful to state all properties of Ω now.



easily verify that the map $\Omega\colon \overline{B}^n/S^{n-1} \to S^n$ has the following, in some sense cleaner, description:

$$\begin{array}{rcl} \Omega\colon \overline{B}^n/S^{n-1} & \to & S^n \\ [w] & \mapsto & \underbrace{\begin{pmatrix} 0 & (-1)^{n-1} & \dots \\ 0 & 0 & \mathrm{id}_{n-1} \\ -1 & \dots & 0 \end{pmatrix}}_{\in \mathrm{SO}(n+1)} \cdot \underbrace{\begin{pmatrix} \cos(\pi \cdot \|w\|) \\ f(\|w\|) \cdot w \end{pmatrix}}_{\in \mathbb{R}\times\mathbb{R}^n = \mathbb{R}^{n+1}} \quad \text{with } w \in \overline{B}^n. \end{array}$$

It is now clear that the map $B^n \to \overline{B}^n/S^{n-1} \xrightarrow{\Omega} S^n$ is smooth. One can easily verify that the differential at each $w \in B^n$ is invertible, i.e. the map $B^n \to S^n$ is an immersion in the sense of the definition on page 651. Using the above one can also easily verify that $B^n \to S^n$ is an embedding. In summary, we have shown that the map $B^n \to S^n$ is a smooth embedding.

It remains to show that the smooth embedding $B^n \to S^n$ is orientation-preserving. By the Basics-of-Orientations Lemma 25.13 it suffices to show that the map is orientation-preserving at the origin. At the origin the standard basis $e_1, \dots, e_n$ of $\widetilde{\mathrm{T}}_0 B^n = \mathbb{R}^n$ is a positive basis. An elementary calculation shows that it gets sent to the basis $(-1)^{n-1} \cdot e_1, e_2, \dots, e_n$ of the tangent space of S^n at $\Phi(0) = -e_{n+1}$. Since $\det(-e_{n+1}\ (-1)^{n-1} \cdot e_1\ e_2 \dots e_n) > 0$ we see, by the definition on page 681 of the standard orientation on S^n, that $(-1)^{n-1} \cdot e_1, e_2, \dots, e_n$ is a positive basis of the tangent space of S^n at $\Phi(0) = -e_{n+1}$. We have thus shown that the smooth embedding $B^n \to S^n$ is orientation-preserving at the origin and thus everywhere. ■

We conclude this section with a rather technical discussion. Namely, on page 244 we saw that the quotient of a topological space that is Hausdorff is not necessarily again Hausdorff. It is pretty clear that if one puts enough conditions on the equivalence relation, then one can avoid such misdemeanor. This leads us to the following definition.

Definition. Let X be a topological space and let $\sim$ be an equivalence relation on X. Recall that an equivalence relation is, by the definition on page 89, a subset of $X \times X$ with suitable properties. We say that $\sim$ is closed if $\sim$ is a closed subset of $X \times X$.

Now we can formulate the next lemma.

Lemma 5.21. (Quotient Hausdorff Lemma) Let X be a topological space that is Hausdorff and compact. If $\sim$ is a closed relation, then $X/\sim$ is Hausdorff.

Proof. Let X be a topological space that is Hausdorff and compact and let $\sim$ be a closed relation. For psychological reasons we denote by $G \subset X \times X$ the subset given by the relation $\sim$. By hypothesis we know that G is a closed subset of $X \times X$. We denote by $p\colon X \to X/\sim$ the projection. We start out with the following claim.

Claim. Given any closed subset C of X the projection $p(C)$ is a closed subset of $X/\sim$.

Proof. Let C be a closed subset of X. We need to show that $p^{-1}(p(C))$ is closed in X. We denote by $p_1, p_2\colon X \times X \to X$ the projection onto the first respectively the second factor. Then one easily sees that

$$p^{-1}(p(C)) \;=\; \{y \in X \,|\, \text{there exists an } x \in C \text{ with } y \sim x\} \;=\; p_2(p_1^{-1}(C) \cap G).$$

Note that $p_1^{-1}(C) = C \times X$ is the product of two closed subsets of X, hence it follows from the Product-of-Subsets Lemma 5.5 that $p_1^{-1}(C) = C \times X$ is a closed subset of $X \times X$. Furthermore by our hypothesis G is a closed subset of $X \times X$. Thus we see that $p_1^{-1}(C) \cap G$ is closed. By the Product Topology Properties Proposition 5.6 (2) and our hypothesis on X we know that $X \times X$ is compact. It follows from the Compact-Closed Lemma 1.21 (1) that $p_1^{-1}(C) \cap G$ is a compact subset of $X \times X$. It is now a consequence of the Compact Image Lemma 2.13 (1) that $p_2(p_1^{-1}(C) \cap G)$ is a compact subset of X. But since X is Hausdorff we obtain from the Compact-Closed Lemma 1.21 (2) that $p_2(p_1^{-1}(C) \cap G)$ is also a closed subset of X. ⊞

Now let $[x], [y] \in X/\sim$ be two disjoint points in $X/\sim$. Since X is Hausdorff we know from the Compact-Closed Lemma 1.21 (2) that $\{x\}$ and $\{y\}$ are closed subsets of X. By the claim we see that $\{[x]\}$ and $\{[y]\}$ are closed subsets of $X/\sim$. Since the projection $p\colon X \to X/\sim$ is continuous we see that $p^{-1}([x])$ and $p^{-1}([y])$ are two disjoint closed subsets of X. Since X is compact and Hausdorff we know by the Compact Hausdorff Spaces-Normal Lemma 1.16 that X is normal, i.e. that there exist open disjoint neighborhoods U and V of $p^{-1}([x])$ and $p^{-1}([y])$. We write $U' = X \setminus U$ and $V' = X \setminus V$. These are now closed subsets of X. By the claim their images are closed subsets of $X/\sim$. Now we see that $(X/\sim) \setminus p(U')$ and $(X/\sim) \setminus p(V')$ are the desired open disjoint sets containing $[x]$ respectively $[y]$. ■

We conclude this section on quotient spaces with three technical lemmas.

Lemma 5.22. Let X be a topological space and let $\sim$ be an equivalence relation on X. Furthermore let $\mathcal{B}$ be a basis of the topology of X. If the projection $p\colon X \to X/\sim$ is an open map, then

$$p(\mathcal{B}) \;=\; \{p(B) \,|\, B \in \mathcal{B}\}$$

is a basis for the topology of $X/\sim$.

Proof. First note that the sets in $p(\mathcal{B})$ are indeed open subsets of $X/\sim$ since we assume that the projection is open. We use the criterion provided by the Basis Characterization Lemma 1.32 (1) to show that $p(\mathcal{B})$ is a basis for the topology of $X/\sim$. Thus let $U \subset X/\sim$ be an open set and let $y \in U$. We need to show that there exists a $B \in \mathcal{B}$ with $y \in p(B) \subset U$.

We choose an $x \in X$ with $p(x) = y$. Since $\mathcal{B}$ is a basis for the topology of X and since $p^{-1}(U)$ is open in X there exists by the Basis Characterization Lemma 1.32 (1) a $B \in \mathcal{B}$ with $x \in B \subset p^{-1}(U)$. But then we also have that $y = p(x) \in p(B) \subset p(p^{-1}(U)) = U$. ■

A careful reading of these notes shows that the following lemma gets used on many occasions without giving its due credit.

Lemma 5.23. (Twice Quotient Lemma)

(1) Let X be a topological space and let $\sim_1$ be an equivalence relation on X. Furthermore let $\sim_2$ be an equivalence relation on $X/\sim_1$. Given $x, y \in X$ we define

$$x \sim y \quad :\Longleftrightarrow \quad [x]_{\sim_1} \sim_2 [y]_{\sim_1}.$$

Then the following two statements hold:

(a) "$\sim$" is an equivalence relation.

(b) The map

$$\begin{array}{rcl} X/\sim & \to & (X/\sim_1)/\sim_2 \\ [x] & \mapsto & [[x]_{\sim_1}]_{\sim_2} \end{array}$$

is a homeomorphism.

(2) Let $f\colon X \to Y$ be a map between two topological spaces and let $\sim_X$ and $\sim_Y$ be equivalence relations on X respectively Y. If $x_1 \sim_x x_2$ implies that $f(x_1) \sim_y f(x_2)$, then the map

$$\begin{array}{rcl} X/\sim_X & \to & Y/\sim_Y \\ [x] & \mapsto & [f(x)] \end{array}$$

is well-defined and continuous.

Proof.

(1) We leave the proof of this lemma as a gift to our dear reader.

(2) It follows from the Topological-Quotient Proposition 5.15 (1a) that the map $X \xrightarrow{f} Y \xrightarrow{y \mapsto [y]} Y/\sim_Y$ is continuous. It follows from our hypothesis on f and the Topological-Quotient Proposition 5.15 (1b) that this map descends to a continuous map $X/\sim_X \to Y/\sim_Y$. ■

5.6. **Group actions on topological spaces.** We continue with the following definition that eventually will provide a rich source of examples of topological spaces.

Definition. Let X be a set and let G be a group with trivial element e.

(1) An action of G on X is a map

$$\begin{array}{rcl} G \times X & \to & X \\ (g, x) & \mapsto & g \cdot x \end{array}$$

with the following properties

$$\begin{array}{rcll} e \cdot x & = & x, & \text{for all } x \in X, \\ \underset{\substack{\uparrow \\ \text{group action}}}{g} \cdot (\underset{\uparrow}{h} \cdot x) & = & (\underset{\substack{\uparrow \\ \text{multiplication in the group } G}}{g \cdot_G h}) \cdot x, & \text{for all } x \in X \text{ and } g, h \in G. \end{array}$$

(2) The action is called free, if $g \cdot x = x$ for some $x \in X$ implies that $g = e$.

(3) We say G acts transitively, if for every x and y in X there exists a $g \in G$ with $g \cdot x = y$.

(4) If X is a topological space, then we say that the action is continuous, if for every $g \in G$ the map

$$\begin{array}{rcl} X & \to & X \\ x & \mapsto & g \cdot x \end{array}$$

is continuous.[78]

Remark. Sometimes what we call an "action" is referred to as a "left-action". A "right-action" is exactly the same as a left-action except that the second property gets replaced by $g \cdot (h \cdot x) = (h \cdot_G g) \cdot x$. Or, more intuitively, a right-action is a map $X \times G \to X$ such that we have $(x \cdot h) \cdot g = x \cdot (h \cdot_G g)$. Basically all the statements and definitions that we give for left-actions carry over, with minuscule modifications, to the setting of right-actions.

Examples.

(1) Given any group G the map

$$\begin{array}{rcl} G \times G & \to & G \\ (g, h) & \mapsto & ghg^{-1} \end{array}$$

defines an action of G on itself. Let G be a non-trivial group. Considering $h = e$ we see that this action is not free and it is also straightforward to see that the action is not transitive.

(2) Let $\mathbb{F}$ be a field and let V be a k-dimensional vector space over $\mathbb{F}$. We denote by B the set of all ordered bases of V. The map

$$\begin{array}{rcl} \mathrm{GL}(k, \mathbb{F}) \times B & \to & B \\ \big((a_{ij})_{i,j=1,\dots,k}, (v_1, \dots, v_k)\big) & \mapsto & \Big(\sum\limits_{i=1}^{k} a_{i1} v_i, \dots, \sum\limits_{i=1}^{k} a_{ik} v_i\Big) \end{array}$$

is a transitive action. It is an amusing linear algebra exercise to figure out under what circumstances this action is free.

(3) The group $G = \mathbb{Z}^n$ acts on $X = \mathbb{R}^n$ by addition. This action is evidently free and continuous but it is not transitive.

We continue with the following definition.

Definition. Let X be a set and let G be a group that acts on X. One can easily verify that

$$x \sim y \quad :\Longleftrightarrow \quad \text{there exists a } g \in G \text{ such that } g \cdot x = y$$

is an equivalence relation on X. We write[79]

$$X/G \;:=\; X/\!\sim .$$

If X is a topological space, then we view X/G as a topological space equipped with the quotient topology.

Lemma 5.24. (Group-Action Projection Lemma) Let X be a topological space and let G be a group that acts on X.

(1) The projection $p\colon X \to X/G$ is continuous.
(2) We suppose that G acts continuously on X.
   (a) The projection p is open.

[78] If the action is continuous, then the map $x \mapsto g \cdot x$ is in fact a homeomorphism with inverse map given by $x \mapsto g^{-1} \cdot x$.

[79] It is arguably more logical to write $G\backslash X$ since we are dealing with a left-action and to reserve the notation X/G for right-actions. We stick to X/G since it is closer to our language "X mod G" and since we will not deal with right-actions at all.

(b) If $A \subset X$ is a closed subset with $G \cdot A = A$, then $p(A)$ is a closed subset of X/G.

Remark. Note that even if G acts continuously on a topological space, it is not true in general that the projection $p\colon X \to X/G$ is closed. For example, let $X = \mathbb{R}$ and $G = \mathbb{Z}$ with the additive action on $\mathbb{R}$. The set $A := \{n + \frac{1}{n} \,|\, n \in \mathbb{Z}\}$ is a closed subset of $X = \mathbb{R}$, but using the Topological-Quotient Proposition 5.15 (0b) one can easily show that $p(A)$ is not a closed subset of $X/G = \mathbb{R}/\mathbb{Z}$.

Proof.

(1) The first statement follows immediately from the Topological-Quotient Proposition 5.15 (1a).

(2) Now we assume that G acts continuously on X.

(a) Let $U \subset X$ be an open subset. We need to show that $p(U)$ is open in X/G. Thus we need to show that $p^{-1}(p(U))$ is an open subset of X. It follows easily from the definitions that

$$p^{-1}(p(U)) \;=\; \bigcup_{g \in G} g \cdot U.$$

Since G acts continuously the map $x \mapsto g \cdot x$ is a homeomorphism. Thus we see that $p^{-1}(p(U))$ is the union of open sets, thus it is open itself.

(b) Let $A \subset X$ be a closed subset with $G \cdot A = A$. We see that

$$(X/G) \setminus p(A) \;\underset{\substack{\uparrow\\ \text{by the Image-Preimage Lemma 0.2 (5), since } G\cdot A = A \\ \text{implies that } p^{-1}(p(A)) = A}}{=}\; p(X \setminus A) \;\underset{\substack{\uparrow\\ \text{since } A \text{ closed}}}{=}\; p(\text{open set of } X) \;\underset{\substack{\uparrow\\ \text{by (2a)}}}{=}\; \text{open subset of } X/G.$$

But this means that $p(A)$ is indeed a closed subset of X/G. ∎

The following lemma discusses one example in greater detail.

Lemma 5.25. The map

$$\begin{array}{rcl} \mathbb{R}^n/\mathbb{Z}^n & \to & (S^1)^n \\ [(t_1, \ldots, t_n)] & \mapsto & \big(\exp(2\pi \mathrm{i} t_1), \ldots, \exp(2\pi \mathrm{i} t_n)\big) \end{array}$$

is a homeomorphism.

Convention. Often we use the homeomorphism of Lemma 5.25 to make the identification $\mathbb{R}^n/\mathbb{Z}^n = (S^1)^n$.

Proof. We start out with the following observations:

(1) The map is continuous by the Topological-Product Proposition 5.1 (1b) and. the Topological-Quotient Proposition 5.15 (1b). Furthermore it is straightforward to see that it is a bijection.

(2) The obvious map $[0,1]^n \to \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is continuous and it is easily seen to be surjective. By the Hyperrectangle-Compact Proposition 1.26 we know that $[0,1]^n$ is compact. It follows from the Compact Image Lemma 2.13 that $\mathbb{R}^n/\mathbb{Z}^n$ is compact.

(3) From the Product Topology Properties Proposition 5.6 we obtain that $(S^1)^n$ is Hausdorff.

Now it follows from the Compact-Hausdorff Proposition 2.17 (3) that the given map is indeed a homeomorphism. ∎

We continue with more examples of group actions and the resulting quotient topological spaces. The first example doubles as a definition

Definition. Let $X = S^n$ and $G = \{\pm 1\}$. The map

$$\begin{array}{rcl} \{\pm 1\} \times S^n & \to & S^n \\ (\epsilon, P) & \mapsto & \epsilon \cdot P \end{array}$$

defines an action that is free and continuous. We refer to the quotient space $S^n/\{\pm 1\}$ as the n-dimensional real projective space $\mathbb{R}\mathrm{P}^n$. [80][81]

We continue with one more example of a group action.

Example. Let $X = \mathbb{R}$ and $G = \{\pm 1\}$. The map

$$\begin{array}{rcl} \{\pm 1\} \times \mathbb{R} & \to & \mathbb{R} \\ (\epsilon, x) & \mapsto & \epsilon \cdot x \end{array}$$

defines an action that is continuous. But this action is not free, since $(-1) \cdot 0 = 0$, but -1 is not the trivial element in the group $G = \{\pm 1\}$. In Exercise 5.29 we will show that $\mathbb{R}/\{\pm 1\}$ is homeomorphic to the half-open interval $[0, \infty)$.

We formulate the next example as a lemma.

Lemma 5.26. (Orthogonal Action Lemma) Let $n \in \mathbb{N}_0$.

(1) The maps

$$\begin{array}{rcl} \mathrm{O}(n+1) \times S^n & \to & S^n \\ (A, v) & \mapsto & A \cdot v \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{O}(n+1) \times \overline{B}^{n+1} & \to & \overline{B}^{n+1} \\ (A, v) & \mapsto & A \cdot v \end{array}$$

define continuous actions of $\mathrm{O}(n+1)$ on S^n and on $\overline{B}^{n+1}$.

(2) The action of $\mathrm{SO}(n+1)$ on S^n is free if and only if $n = 1$.

(3) The action of $\mathrm{O}(n+1)$ on S^n is transitive.

(4) If $n \geq 1$, then the action of $\mathrm{SO}(n+1)$ on S^n is also transitive.

(5) Let $k \in \{0, \ldots, n\}$. The obvious action of $\mathrm{SO}(n)$ on the set of k-dimensional vector subspaces of $\mathbb{R}^n$ is transitive.

We can use the above to draw the following two conclusions:

(6) For every $P \in S^n$ the complement $S^n \setminus \{P\}$ is homeomorphic to $\mathbb{R}^n$.

(7) Given any $x, y \in S^n$ there exists a homeomorphism $\varphi \colon \overline{B}^{n+1} \to \overline{B}^{n+1}$, which restricts to a homeomorphism $S^n \to S^n$, with $\varphi(x) = y$

Proof. Statements (1) and (2) can be verified easily.

We turn to the proof of Statements (3) and (4). The statements fall in the realm of linear algebra, but for the reader's convenience we provide the argument. First we prove Statement (3). It suffices to show that given any $v \in S^n$ there exists an $A \in \mathrm{O}(n+1)$ with $A \cdot e_1 = v$. To produce such a matrix we find vectors $v_2, \ldots, v_{n+1}$ such that $v, v_2, \ldots, v_{n+1}$ form a basis for $\mathbb{R}^{n+1}$. We apply the Gram-Schmidt Orthonormalization Process 2.38 to the ordered basis $(v, v_2, \ldots, v_{n+1})$. Since $\|v\| = 1$ this process leaves the first vector invariant

[80] For $n = 2$ we sometimes refer to $\mathbb{R}\mathrm{P}^2$ as the real projective plane.

[81] In Exercise 5.25 we will show that $\mathbb{R}\mathrm{P}^1$ is homeomorphic to S^1.

and we obtain an orthonormal basis $(v, \widetilde{v}_2, \dots, \widetilde{v}_{n+1})$. We denote by A the orthogonal matrix with columns given by $v, \widetilde{v}_2, \dots, \widetilde{v}_{n+1}$. Clearly $A \cdot e_1 = v$.

We turn to the proof of Statement (4). We argue as in Statement (3). The only difference is, that if $\det(A) = -1$ and if $n \geq 1$, then we replace $\widetilde{v}_2$ by $-\widetilde{v}_2$, and now we are happy.

We move on to the proof of Statement (5). Now let $k \in \{0, \dots, n\}$ and let $V, W \subset \mathbb{R}^n$ be two k-dimensional vector subspaces of $\mathbb{R}^n$. The case $k = n$ is trivial. So we can assume that $k \in \{0, \dots, n-1\}$. By the Gram-Schmidt Orthogonalization Process 2.38 we can pick orthogonal bases $v_1, \dots, v_k$ and $w_1, \dots, w_k$. It suffices to show that for $i = 1, \dots, k$ there exists an $A_i \in \mathrm{SO}(n)$ such that $A \cdot v_j = \pm w_j$ for $j = 1, \dots, i$. These matrices $A_1, \dots, A_k$ can be found iteratively by applying (4) to the vectors v_j and w_j in the orthogonal complement of $v_1, \dots, v_{j-1}$, which has dimension $n - (j-1) \geq n - (k-1) \geq n - ((n-1)-1) \geq 2$.

Next note that Statement (6) follows from Statement (3) together with Lemma 2.18. Finally Statement (7) follows from Statements (1) and (3). ■

We consider our next example in greater detail.

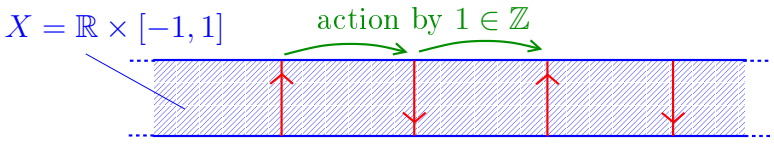
Definition.

(1) Let $X = \mathbb{R} \times [-1, 1]$ and $G = \mathbb{Z}$. The map

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n, (x, y)) & \mapsto & (x + n, (-1)^n \cdot y) \end{array}$$

defines an action that is free and continuous. We refer to the quotient $X/\mathbb{Z}$ as the Möbius band. The same argument as in the proof of Lemma 5.25 shows that the Möbius band is compact.

(2) On occasion we will also be interested in the open Möbius band which is basically defined as above, except that we define it as the quotient of $\mathbb{R} \times (-1, 1)$ instead of the quotient of $\mathbb{R} \times [-1, 1]$.



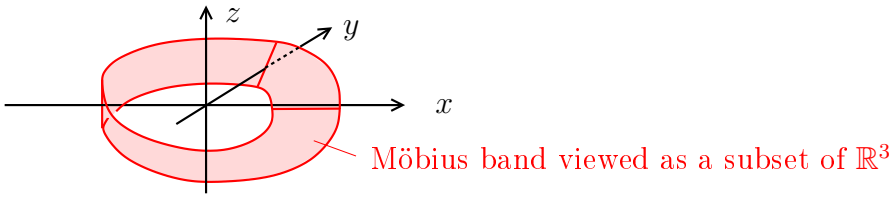
Remark. The name "Möbius band" conjures up a certain picture. We will now see that our name is justified. Given $\varphi, \psi \in \mathbb{R}$ and $r \in [-1, 1]$ we set

$$A(\varphi) := \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad v(r, \psi) := \begin{pmatrix} 2 + r \cdot \sin(\psi) \\ 0 \\ r \cdot \cos(\psi) \end{pmatrix}.$$

We consider the map

$$\begin{array}{rcl} \Psi \colon (\mathbb{R} \times [-1,1])/\mathbb{Z} & \to & \mathbb{R}^3 \\ [(\varphi, r)] & \mapsto & A(2\pi \cdot \varphi) \cdot v\big(r, \pi\varphi\big). \end{array}$$

The argument of Lemma 5.25 shows that Ψ is an embedding. Thus the Möbius band, defined abstractly as above, is indeed homeomorphic to a subset of $\mathbb{R}^3$ that looks like what we had in mind.



If X is a Hausdorff space and if G acts continuously on X, then it is unfortunately not necessarily true that X/G itself is Hausdorff. For example, consider the obvious action of $G = (\mathbb{Q}, +)$ on $X = \mathbb{R}$. It is a moderately insightful exercise, see Exercise 5.27, to verify that the quotient space $\mathbb{R}/\mathbb{Q}$ is not Hausdorff.
This example leads us to the following definition.

Definition. Let X be a topological space and let G be a group that acts on X. We say that G acts properly if for every two points x and y in X there exist open neighborhoods U of x and V of y such that the set $\{g \in G \,|\, gU \cap V \neq \varnothing\}$ is finite.[82]

Examples.

(1) The action of any finite group is proper by definition.
(2) One can show easily that the action of $\mathbb{Z}$ on $\mathbb{R} \times [-1, 1]$ that we had considered on page 252 to define the Möbius band, is proper.
(3) It is straightforward to show that the action of $\mathrm{O}(n+1)$ on S^n from the Orthogonal Action Lemma 5.26 is not proper.

Proposition 5.27. (Group Action-Hausdorff Quotient Proposition) Let G be a group that acts on a topological space X. If X is Hausdorff and if the action is proper and continuous, then the quotient space X/G is also Hausdorff.

Example. Note that it follows from the Group Action-Hausdorff Quotient Proposition 5.27 and the above examples that the real projective spaces $\mathbb{R}\mathrm{P}^n/\{\pm 1\}$ and the Möbius band $(\mathbb{R} \times [-1, 1])/\mathbb{Z}$ are Hausdorff.

Unless one has an insatiable appetite for technical proofs it is best to move on to the next section.

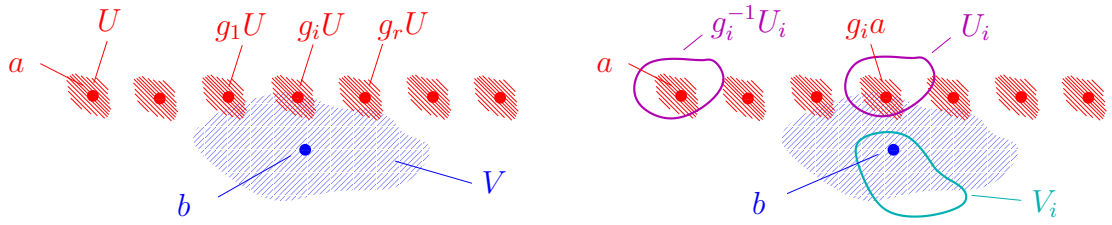
The proof of the Group Action-Hausdorff Quotient Proposition 5.27 relies on the following two lemmas.

Lemma 5.28. Let G be a group that acts continuously and properly on a topological space. If X is Hausdorff, then given any two points a and b in X there exist open neighborhoods A of a and B of b with the following property: for every $g \in G$ with $ga \neq b$ we also have $gA \cap B = \varnothing$.

Proof of Lemma 5.28. Let G be a group that acts continuously and properly on a topological space X that is Hausdorff. Let a and b be two points in X. Since G acts properly on X there exist open neighborhoods U of a and V of b such that $\{g \in G \,|\, gU \cap V \neq \varnothing\}$ is a finite set. We denote the elements of this finite set by $g_1, \ldots, g_r$. We set

$$I \;:=\; \{i = 1, \ldots, r \,|\, g_i a \neq b\}.$$

[82] The notion of a "proper action" is defined in different ways by different authors.



Since X is Hausdorff we can find for every $i \in I$ an open neighborhood U_i of $g_i a$ and an open neighborhood V_i of b such that $U_i \cap V_i = \varnothing$. We set

$$A \;:=\; U \cap \bigcap_{i\in I} g_i^{-1}U_i \quad \text{and} \quad B \;:=\; V \cap \bigcap_{i\in I} V_i.$$

The sets A and B are the intersection of finitely many open subsets,[83] thus they are open. In particular A is an open neighborhood of a and B is an open neighborhood of b.

Now one can easily verify that A and B have the desired properties. Indeed, let $g \in G$ with $ga \neq b$. If $g \notin \{g_1, \dots, g_r\}$, then $gU \cap V = \varnothing$, therefore $gA \cap B = \varnothing$. On the other hand, if $g \in \{g_1, \dots, g_r\}$, then there exists an $i \in I$ with $g = g_i$. It follows that $g_i(g_i^{-1}U_i) \cap V_i = U_i \cap V_i = \varnothing$, hence $g_i A \cap B = \varnothing$. ∎

Lemma 5.29. Let G be a group that acts on a set X. We denote by $p\colon X \to X/G$ the projection. Let A and B be subsets of X. Then

$$p(A) \cap p(B) \;=\; \varnothing \quad \Longleftrightarrow \quad \text{for every } g \in G \text{ we have } gA \cap B = \varnothing,$$

or equivalently,

$$p(A) \cap p(B) \;\neq\; \varnothing \quad \Longleftrightarrow \quad \text{there exists a } g \in G \text{ with } gA \cap B \neq \varnothing.$$

Proof of Lemma 5.29. We prove the second statement. The first statement is easily seen to be equivalent to the second statement. Note that

$$\begin{aligned} p(A) \cap p(B) \;\neq\; \varnothing \;&\Longleftrightarrow\; \text{there exist } a \in A \text{ and } b \in B \text{ with } p(a) = p(b) \in X/G \\ &\Longleftrightarrow\; \text{there exist } a \in A,\ b \in B \text{ and } g \in G \text{ with } ga = b \\ &\Longleftrightarrow\; \text{there exists a } g \in G \text{ with } gA \cap B \neq \varnothing. \end{aligned}$$

∎

Now we turn to the actual proof of the Group Action-Hausdorff Quotient Proposition 5.27.

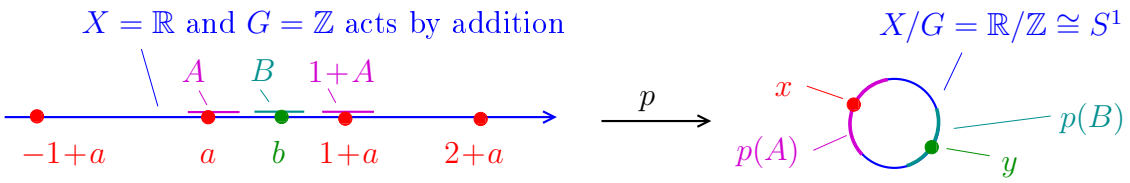
Proof of the Group Action-Hausdorff Quotient Proposition 5.27. Let G be a group that acts on a topological space X. We assume that X is Hausdorff and that the action is proper and continuous. We want to show that the quotient space X/G is also Hausdorff. We denote by $p\colon X \to X/G$ the projection .

Let x and y be two distinct points in X/G. We pick a and b in X with $p(a) = x$ and $p(b) = y$. Since $p(a) = x \neq y = p(b)$ we see that for every $g \in G$ we have $ga \neq b$. By Lemma 5.28 there exist open neighborhoods A of a and B of b such that $gA \cap B = \varnothing$ for every $g \in G$. It follows from Lemma 5.29 that $p(A)$ and $p(B)$ are disjoint.

Since A and B are open it follows from the Group-Action Projection Lemma 5.24 that $p(A)$ and $p(B)$ are open subsets of X/G, in particular they are open neighborhoods of

[83] Here we use that G acts continuously, since this implies that the sets $h^{-1} \cdot U_i$ are also open.

$x = p(a)$ and $y = p(b)$. We have thus found the desired disjoint open neighborhoods of x and y in X/G. ■



5.7. **Projective spaces.** Let $n \in \mathbb{N}$. On page 251 we introduced the real projective space $\mathbb{R}\mathrm{P}^n = S^n/\{\pm 1\}$. In this section we will give an alternative definition and we will introduce the complex projective spaces. Even though these projective spaces appear initially rather arbitrary they will on several occasions play an essential role in these notes.

Convention. Let $X = \mathbb{R}^{n+1} \setminus \{0\}$ and let $G = \mathbb{R} \setminus \{0\}$, where we view G as a group via multiplication. The map

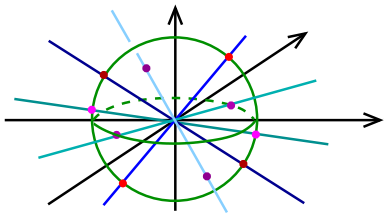
$$\begin{array}{rcl} (\mathbb{R} \setminus \{0\}) \times (\mathbb{R}^{n+1} \setminus \{0\}) & \to & \mathbb{R}^{n+1} \setminus \{0\} \\ (r, P) & \mapsto & r \cdot P \end{array}$$

defines an action that is free and continuous. It is straightforward to see that the map

$$\begin{array}{rcl} \mathbb{R}\mathrm{P}^n = S^n/\{\pm 1\} & \to & (\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\}) \\ [P] & \mapsto & [P] \end{array}$$

defines a homeomorphism[84]. We will use this homeomorphism to make the identification $\mathbb{R}\mathrm{P}^n = (\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\})$.

$(\mathbb{R}^3 \setminus \{0\})/(\mathbb{R} \setminus \{0\})$ is the "set of lines through 0"



$\mathbb{R}\mathrm{P}^2 = S^2/x \sim -x$ is the "set of pairs of antipodal points on S^2"

Our new point of view on real projective spaces has the advantage that we can create a different class of examples of topological spaces by replacing $\mathbb{R}$ by $\mathbb{C}$. More precisely, this discussion leads us to the following definition:

Definition. Let $X = \mathbb{C}^{n+1} \setminus \{0\}$ and let $G = \mathbb{C} \setminus \{0\}$. The map

$$\begin{array}{rcl} (\mathbb{C} \setminus \{0\}) \times (\mathbb{C}^{n+1} \setminus \{0\}) & \to & \mathbb{C}^{n+1} \setminus \{0\} \\ (z, P) & \mapsto & z \cdot P \end{array}$$

defines an action that is free and continuous. The quotient space $(\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$ is called the n-dimensional complex projective space $\mathbb{C}\mathrm{P}^n$.

In our treatments of real and complex projective spaces we will often use the following notation.

[84] For example an inverse is given by the map $(\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\}) \to S^n/\{\pm 1\}$ that is defined by $[x] \mapsto [\frac{x}{\|x\|}]$. More precisely, it follows from the Topological-Quotient Proposition 5.15 (1b) and the Topological-Quotient Proposition 5.15 (1a) that both maps are continuous. Clearly they are inverses of one another.

Notation. Let $n \in \mathbb{N}_0$. For $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$ we denote by $[x_0 : \ldots : x_n]$ the corresponding point in $\mathbb{R}\mathrm{P}^n$. We also use the same notation for points in complex projective spaces.

The following lemma shows that real and complex projective spaces have some of our favorite properties.

Proposition 5.30. (Projective Spaces-Compact+Hausdorff Proposition) The real projective and the complex projective spaces are compact and Hausdorff.

Example. Let $n \in \mathbb{N}_0$ and let $m \leq n$. We consider the maps

$$\begin{array}{rcl} i\colon \mathbb{R}\mathrm{P}^m & \to & \mathbb{R}\mathrm{P}^n \\ [x_0 \colon \ldots \colon x_m] & \mapsto & [x_0 \colon \ldots \colon x_m \colon 0 \colon \ldots \colon 0] \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon \mathbb{C}\mathrm{P}^m & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 \colon \ldots \colon z_m] & \mapsto & [z_0 \colon \ldots \colon z_m \colon 0 \colon \ldots \colon 0]. \end{array}$$

It follows from the Twice Quotient Lemma 5.23, the Projective Spaces-Compact+Hausdorff Proposition 5.30 and the Compact-Hausdorff Proposition 2.17 (2) that both maps are embeddings. Sometimes we will use these maps to view $\mathbb{R}\mathrm{P}^m$ as a subset of $\mathbb{R}\mathrm{P}^n$ and similarly we will view $\mathbb{C}\mathrm{P}^m$ as a subspace of $\mathbb{C}\mathrm{P}^n$.

In the proof of the Projective Spaces-Compact+Hausdorff Proposition 5.30 it is convenient to have the following description of complex projective spaces.

Auxiliary Lemma 5.31. Let $n \in \mathbb{N}_0$. We make the identification

$$S^{2n+1} \;=\; \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \,|\, |z_1|^2 + \cdots + |z_{n+1}|^2 = 1\}.$$

We consider the action

$$\begin{array}{rcl} S^1 \times S^{2n+1} & \to & S^{2n+1} \\ (w, (z_1, \ldots, z_{n+1})) & \mapsto & (w \cdot z_1, \ldots, w \cdot z_{n+1}). \end{array}$$

The maps

$$\begin{array}{l} f\colon S^{2n+1}/S^1 \to \mathbb{C}\mathrm{P}^n \\ [z_1, \ldots, z_{n+1}] \mapsto [z_1 \colon \ldots \colon z_{n+1}] \end{array} \quad \text{and} \quad \begin{array}{l} p\colon \mathbb{C}\mathrm{P}^n \to S^{2n+1}/S^1 \\ [z_1 \colon \ldots \colon z_{n+1}] \mapsto \left[\frac{1}{\sqrt{|z_1|^2 + \cdots + |z_{n+1}|^2}}(z_1, \ldots, z_{n+1})\right] \end{array}$$

are continuous and inverses of one another, in particular both maps are homeomorphisms. We will use these maps to make the identification $\mathbb{C}\mathrm{P}^n = S^{2n+1}/S^1$.

Proof. Lemma 5.31] It follows easily from the definition of the quotient topology on the complex projective space $\mathbb{C}\mathrm{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$ and from the Topological-Quotient Proposition 5.15 (1a) together with the Topological-Quotient Proposition 5.15 (1b) that both maps are continuous. It is easy to verify that the maps are inverses of one another. ■

Proof of the Projective Spaces-Compact+Hausdorff Proposition 5.30. Let $n \in \mathbb{N}$. It follows from the Compact Image Lemma 2.13 and the Group Action-Hausdorff Quotient Proposition 5.27 that the real projective space $\mathbb{R}\mathrm{P}^n = S^n/\{\pm 1\}$ is compact and Hausdorff.

For the complex projective space $\mathbb{C}\mathrm{P}^n$ we need to work just a little harder. Throughout the remainder of this proof we will use the identification $\mathbb{C}\mathrm{P}^n = S^{2n+1}/S^1$ from Lemma 5.31. Note that since S^{2n+1} is compact it follows again from the Compact Image Lemma 2.13 that $\mathbb{C}\mathrm{P}^n = S^{2n+1}/S^1$ is compact.

It remains to show that $\mathbb{C}\mathrm{P}^n = S^{2n+1}/S^1$ is Hausdorff. A direct argument is sketched in Exercise 5.44. In the following we will provide a less direct but more general argument.

Note that it follows from the Quotient Hausdorff Lemma 5.21 that S^{2n+1}/S^1 is Hausdorff once we have shown that the equivalence relation is closed. Thus we need to show that

$$G \;=\; \{(v_1, v_2) \in S^{2n+1} \times S^{2n+1} \,|\, \text{there exists } w \in S^1 \text{ with } w \cdot v_1 = v_2\},$$

is a closed subset of $S^{2n+1} \times S^{2n+1}$. Note that G is the image of the map

$$\begin{array}{rcl} \Phi\colon S^{2n+1} \times S^1 & \to & S^{2n+1} \times S^{2n+1} \\ (v, w) & \mapsto & (v, w \cdot v). \end{array}$$

This map is continuous and the left-hand side is compact by the Product Topology Properties Proposition 5.6 (2). Thus the image of Φ is compact by the Compact Image Lemma 2.13. By the Product Topology Properties Proposition 5.6 (1) we know that $S^{2n+1} \times S^{2n+1}$ is Hausdorff. Thus we obtain from the Compact-Closed Lemma 1.21 that the image of Φ is a closed subset of $S^{2n+1} \times S^{2n+1}$. ■

In the following lemma we give alternative descriptions of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^1$ that will be used on many occasions.

Lemma 5.32.

(1) Let $n \in \mathbb{N}$. If we denote by $\sim$ the equivalence relation on $\overline{B}^n$ that is generated by $P \sim -P$ for $P \in S^{n-1} = \partial\overline{B}^n$, then the map

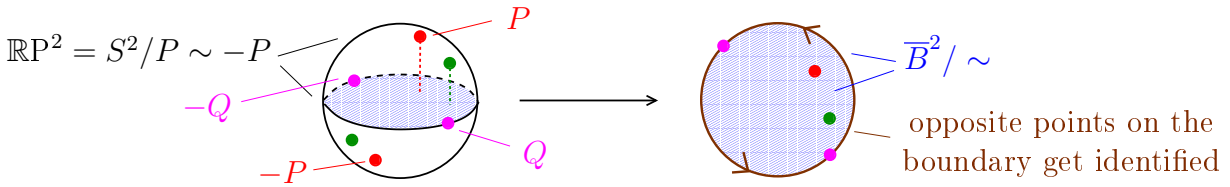
$$\begin{array}{rcl} \mathbb{R}\mathrm{P}^n = S^n/\{\pm 1\} & \xrightarrow{\cong} & \overline{B}^n/\!\sim \\ [(x_1, \ldots, x_n, x_{n+1})] & \mapsto & \begin{cases} [(x_1, \ldots, x_n)], & \text{if } x_{n+1} \geq 0, \\ [-(x_1, \ldots, x_n)], & \text{if } x_{n+1} < 0 \end{cases} \end{array}$$

is a homeomorphism.

(2) The map

$$\begin{array}{rcl} \mathbb{C}\mathrm{P}^1 & \to & \mathbb{C} \cup \{\infty\} = \mathbb{R}^2 \cup \{\infty\} \\ [z_0 : z_1] & \mapsto & \begin{cases} z_0 \cdot z_1^{-1}, & \text{if } z_1 \neq 0, \\ \infty, & \text{if } [z_0 : z_1] = [1 : 0] \end{cases} \end{array}$$

is a homeomorphism. Similarly we see that $\mathbb{R}\mathrm{P}^1$ is homeomorphic to $\mathbb{R} \cup \{\infty\}$.



Proof. First note that one can easily verify that both maps are actually well-defined and that they are bijections. Next we explain why the two maps are continuous:

(1) By the Topological-Quotient Proposition 5.15 (1b) it suffices to show that the corresponding map $S^n \to \overline{B}^n/\!\sim$ is continuous. One can easily verify that the restriction to the closed subsets $S^n_{\geq 0}$ and $S^n_{\leq 0}$ are continuous. Thus it follows from the Pasting Pasting Proposition 2.6 (2) that the map $S^n \to \overline{B}^n$ is continuous.

(2) It follows again from the Topological-Quotient Proposition 5.15 (1b) that it suffices to show that the corresponding map $\Theta\colon \mathbb{C}^2 \setminus \{(0,0\} \to \mathbb{C} \cup \{\infty\}$ is continuous. Note that for any $C \in \mathbb{R}$ we have

$$\Theta^{-1}(\text{complement of } \overline{B}^n_C) \;=\; \{(z_0, z_1) \in \mathbb{C}^2 \setminus \{(0,0)\} \,|\, |z_0| > |z_1|\}$$

which is an open subset of $\mathbb{C}^2 \setminus \{(0,0)\}$. It follows quite easily from this discussion together with the Subbasis-Continuity Proposition 2.9 and together with the discussion of the basis of $\mathbb{R}^2 \cup \{\infty\}$ on page 132 that the map Θ is continuous.

By the Projective Spaces-Compact+Hausdorff Proposition 5.30 we know that each $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ is compact. Using the Quotient Hausdorff Lemma 5.21 one can show that $\overline{B}^n/\sim$ is Hausdorff. Furthermore we know from Exercise 1.18 that $\mathbb{R}^2 \cup \{\infty\}$ is Hausdorff. Thus it follows from the Compact-Hausdorff Proposition 2.17 (3) that both maps are homeomorphisms. ■

Convention.

(1) On many occasions we will use the homeomorphism from Lemma 5.32 to make the identification $\mathbb{R}\mathrm{P}^n = \overline{B}^n/\sim$.

(2) In Lemma 2.18 we gave an explicit homeomorphism $S^2 \to \mathbb{R}^2 \cup \{\infty\}$. Combining this homeomorphism with the homeomorphism from Lemma 5.32 we obtain a homeomorphism $S^2 \to \mathbb{C}\mathrm{P}^1$ which we will often use to make the identification $S^2 = \mathbb{C}\mathrm{P}^1$. Note that under this identification the North Pole $(0,0,1) \in S^2$ corresponds to the point $[1:0] \in \mathbb{C}\mathrm{P}^1$.

We conclude this short section on projective spaces with two somewhat tangential remarks.

Remark.

(1) Here is a fun fact: if we consider $\mathbb{C}\mathrm{P}^2/\sim$, where $\sim$ denotes the equivalence relation defined by complex conjugation, i.e. where $[z:w] \sim [\overline{z}:\overline{w}]$, then the quotient space is homeomorphic (in fact diffeomorphic in a suitable sense) to S^4. This statement is known as the Arnold–Kuiper–Massey Theorem, see e.g. [**?, ?, ?**].

(2) The definition of projective spaces makes sense for any field, even for fields of non-zero characteristic. For example, if $\mathbb{F}$ is a field of characteristic p we can still define the set $\mathbb{F}\mathrm{P}^n = (\mathbb{F}^{n+1} \setminus \{0\})/(\mathbb{F} \setminus \{0\})$. A priori this is just a set, but together with the Zariski topology, see [**Har1992**, Lecture 2] or [**SKKT2000**, Chapter 1], it actually becomes an unusual, but interesting topological space.
For any field the map

$$\begin{array}{rcl} \mathbb{F}\mathrm{P}^n = (\mathbb{F}^{n+1} \setminus \{0\})/(\mathbb{F} \setminus \{0\}) & \to & \{\text{all one-dimensional subspaces of } \mathbb{F}^{n+1}\} \\ [v] & \mapsto & \mathbb{F} \cdot v \end{array}$$

is a bijection. Thus we can view $\mathbb{F}\mathrm{P}^n$ as the set of all lines through the origin.

5.8. **The pushout in the topological category.** In this section we consider a particularly important special case of a quotient topology which deserves its own section. Before we delve into topology we recall the following definition.

Definition. Suppose we are given a relation $\sim$ on a set X. We say $x, y \in X$ are equivalent if there exists a sequence $x = x_1, \ldots, x_k = y$ of elements in X such that for all $i = 1, \ldots, k-1$ the following holds: either $x_i \sim x_{i+1}$ or $x_{i+1} \sim x_i$. We write again $x \sim y$ if x and y are equivalent. It is straightforward to see that this is now indeed an equivalence relation. We say it is generated by the initial relation.

Example. The relations $x \sim (x+1)$ with $x \in \mathbb{R}$ on $\mathbb{R}$ generate the familiar equivalence relation $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$.

Definition. Let $f\colon X \to Y$ and $g\colon X \to Z$ be two maps between topological spaces. We define

$$Y \cup_X Z \;:=\; (Y \sqcup Z)/\!\sim \qquad \text{where } f(x) \sim g(x) \text{ for all } x \in X.$$

We refer to $Y \cup_X Z$ as the pushout of f and g. Furthermore we refer to the maps

$$\begin{array}{rcl} i\colon Y & \to & Y \cup_X Z \\ y & \mapsto & [y] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} j\colon Z & \to & Y \cup_X Z \\ z & \mapsto & [z] \end{array}$$

as the two natural maps.

The notion of a pushout will play an important role throughout the lecture notes. We first summarize some key properties of pushouts, afterwards we will discuss an important special case.

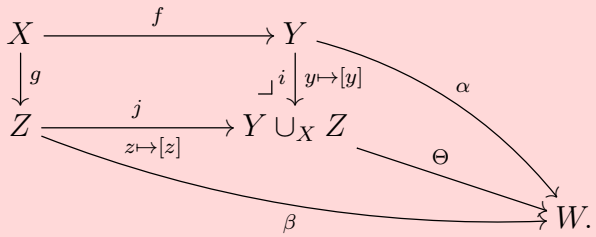
Proposition 5.33. (Topological Pushout Proposition) Let $f\colon X \to Y$ and $g\colon X \to Z$ be two maps between topological spaces. The following statements hold:

(1) (a) The natural maps $i\colon Y \to Y \cup_X Z$ and $j\colon Z \to Y \cup_X Z$ are continuous.

(b) Suppose we are given two maps $\alpha\colon Y \to W$ and $\beta\colon Z \to W$ to some topological space W such that $\alpha \circ f = \beta \circ g\colon X \to W$. The map

$$\begin{array}{rcl} \Theta\colon Y \cup_X Z & \to & W \\ [P] & \mapsto & \begin{cases} \alpha(x) & \text{if } P = [y] \text{ for some } y \in Y, \\ \beta(y) & \text{if } P = [z] \text{ for some } z \in Z \end{cases} \end{array}$$

is well-defined and it is the unique map $h\colon Y \cup_X Z \to W$ with $\Theta \circ i = \alpha$ and $\Theta \circ j = \beta$. Furthermore, if α and β are continuous, then Θ is also continuous. In particular we obtain the following commutative diagram:



(2) The topological space $Y \cup_X Z$ with the natural maps $Y \to Y \cup_X Z$ and $Z \to Y \cup_X Z$ is the pushout in the sense of the definition on page 220.

Proof. Throughout the proof we denote by $p\colon Y \sqcup Z \to Y \cup_X Z$ the natural projection.

(1) (a) It follows from the Topological-Coproduct Proposition 5.11 (1a) and the Topological-Quotient Proposition 5.15 (1a) that the natural maps $i\colon Y \to Y \cup_X Z$ and $j\colon Z \to Y \cup_X Z$ are continuous.

(b) This statement is an immediate consequence of the definitions and the Topological-Quotient Proposition 5.15 (1b).

(2) This statement follows easily from the definitions and Statement (1). We leave it to the reader to fill in the details. ∎

Lemma 5.34. (Topological Pushout-Maps Lemma) Let $f\colon X \to Y$ and $g\colon X \to Z$ be two maps between topological spaces. We obtain the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle g} & \lrcorner & \downarrow{\scriptstyle i}\ {\scriptstyle y\mapsto[y]} \\ Z & \xrightarrow[z\mapsto[z]]{j} & Y \cup_X Z. \end{array}$$

The following statements hold:

(1) If f is a homeomorphism, then the "opposite map" j is a homeomorphism.
(2) If f is injective, then the "opposite map" j is injective. The same statement holds for "surjective" and "bijective".[85]
(3) If f is an embedding, then the "opposite map" j is an embedding.
(4) Suppose that at least one of the two maps f or g is injective.
   (α) If the image of f is a closed subset of Y, then the image of the "opposite map" j is a closed subset of $Y \cup_X Z$.
   (β) If the image of g is a closed subset of Z, then the image of the "opposite map" i is a closed subset of $Y \cup_X Z$.
   Both statements also hold with "closed" replaced by "open".

By symmetry the statements in (1), (2) and (3) hold also for g and its "opposite map" i.

Proof. Throughout the proof we denote by $p\colon Y \sqcup Z \to Y \cup_X Z$ the natural projection.

(1) Suppose $f\colon X \to Y$ is a homeomorphism. It follows immediately from the Pushout Lemma 4.8 (1) that $j\colon Z \to Y \cup_X Z$ is a homeomorphism.
(2) Let $z \in Z$. It follows basically from the definitions that the set of elements in Z that are equivalent to z with respect to the equivalence relation "$\sim$" on $Y \sqcup Z$ equals precisely

$$\bigcup_{k\in\mathbb{N}_0} \underbrace{g(f^{-1}(f(g^{-1}(\dots g(f^{-1}(f(g^{-1}(}_{\text{applied } k \text{ times}}\{z\}))))\dots)))).$$

If f is injective, then it follows immediately from the Image-Preimage Lemma 0.2 (1) and (2) that this set consists precisely of the element z. In other words, the map j is injective. The proof for "surjective" is very similar.
(3) We suppose that $f\colon X \to Y$ is an embedding. By the Topological Pushout Proposition 5.33 we know that j is continuous and by (1) we know that $j\colon Z \to Y \cup_X Z$ is injective. By the Open -Injective Map Lemma 2.16 (2) and by definition of the subspace topology on $j(U)$ it remains to prove the following claim.

Claim. If U is an open subset of Z, then there exists an open subset W of $Y \cup_X Z$ such that $j(U) = W \cap j(Z)$.

Proof. Let U be an open subset of Z. Since g is continuous we know that $g^{-1}(U)$ is an open subset of X. Since f is an embedding we know that there exists an open subset $V \subset Y$ with $f(g^{-1}(U)) = f(X) \cap V$. We set $W := p(U \sqcup V)$. A short argument shows that $p^{-1}(W) = U \sqcup V$. Thus it follows from the definition of the quotient

[85] This statement is very close to the content of Exercise 4.11.

topology and the disjoint union topology that W is indeed an open subset of $Y \cup_X Z$. Basically by construction we have $j(U) = W \cap j(Z)$. ⊞

(4) Without loss of generality we can assume that f is injective. The statements for "closed" follow almost immediately from the following observations:
- By the Topological-Quotient Proposition 5.15 (0b) a subset W of $Y \cup_X Z$ is closed if and only if $p^{-1}(W)$ is closed.
- Since f is injective it follows from the argument in (2a) that
$$p^{-1}(i(Y)) = Y \sqcup g(X) \quad \text{and} \quad p^{-1}(j(Z)) = f(g^{-1}(Z)) \sqcup Z = f(X) \sqcup Z.$$
- By Lemma 5.12 a subset $U \sqcup V$ of $Y \sqcup Z$ is closed if and only if U and V are closed.

The proof for "open" is basically the same. ■

Proposition 5.35. (Pushout Properties Proposition) Let $f\colon X \to Y$ and $g\colon X \to Z$ be two maps between topological spaces.

(1) If Y and Z are compact, then so is the pushout $Y \cup_X Z$.
(2) If Y and Z are (path-) connected and if X is non-empty, then so is the pushout $Y \cup_X Z$.

Proof.

(1) If Y and Z are compact, then it follows from the Disjoint Union Topology-Properties Lemma 5.13 (2) and the Quotient-Compact Connected Lemma 5.16 (1) that $Y \cup_X Z$ is compact.
(2) If Y and Z are (path-) connected, then it follows easily from the Image-Connected Lemma 2.24 and the Union Connected Lemma 2.28 that $Y \cup_X Z$ is also (path-) connected. ■

Example. Let $n \in \mathbb{N}_0$. We consider the maps[86]
$$\begin{array}{rcl} g_\pm\colon \overline{B}^n = \overline{B}^n_\pm & \to & S^n \\ (x_1, \dots, x_n) & \mapsto & \big((-1)^n \cdot x_1, \dots, x_n, \pm\sqrt{1 - x_1^2 - \dots - x_n^2}\big). \end{array}$$
It follows from the Topological Pushout Proposition 5.33 (1b) that the map
$$\begin{array}{rcl} \overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_- & \to & S^n \\ [P] & \mapsto & \begin{cases} g_+(P), & \text{if } P \in \overline{B}^n_+, \\ g_-(P), & \text{if } P \in \overline{B}^n_- \end{cases} \end{array}$$
is continuous. One can easily verify that it is a bijection. It follows from the the Pushout Properties Proposition 5.35 (1) combined with the Compact-Hausdorff Proposition 2.17 (3) that the map is a homeomorphism. We use this homeomorphism to make the identification $\overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_- = S^n$.[87]

Now, as promised we will discuss an important class of pushouts, namely where the map $f\colon X \to Y$ is just the inclusion of a subset. This leads us to the following definition.

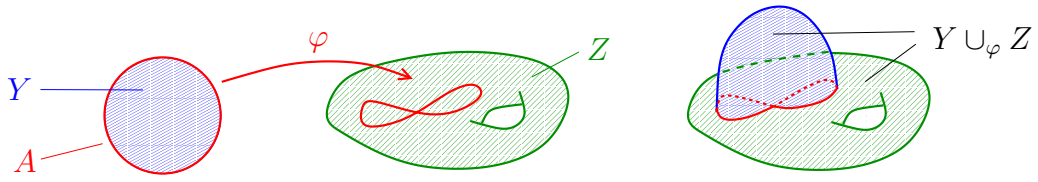
[86] We put in the superfluous looking sign $(-1)^n$ for the following reason: once we have introduced orientations of smooth manifolds and once we have introduced the standard orientations of $\overline{B}^n$ and S^n this sign assures us that the map $g_+\colon \overline{B}^n \to S^n$ is orientation-preserving and it means that $g_-\colon \overline{B}^n \to S^n$ is orientation-reversing.

[87] In Exercise **??** we will see that we can also identify $\overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_-$ and S^n as "smooth manifolds".

Definition. Let Y and Z be topological spaces, let $A \subset Y$ be a subset and let $\varphi\colon A \to Z$ be a continuous map. (Note that φ does not need to be injective.) We write

$$Y \cup_\varphi Z \;:=\; Y \cup_{\varphi\colon A \to Z} Z \;:=\; (Y \sqcup Z)/\!\sim$$

where $\sim$ is the equivalence relation generated by $a \sim \varphi(a)$ for $a \in A$. We call $Y \cup_\varphi Z$ the topological space obtained by gluing Y to Z via the map f.



Example. Let Y be a topological space and let A be a subset of Y. If Z is the topological space that consists of a single point, then there exists evidently precisely one map $\varphi\colon A \to Z$ and it follows basically immediately from the definitions that $Y \cup_\varphi Z$ is homeomorphic to Y/A.

The following lemma is quite subtle and gets used, often inadvertently, on many occasions.

Lemma 5.36. Let X be a topological space and let A and B be two subsets of X with $X = A \cup B$. We denote by $\varphi\colon A \cap B \to A \cap B$ the identity map. We suppose that one of the following two conditions hold:

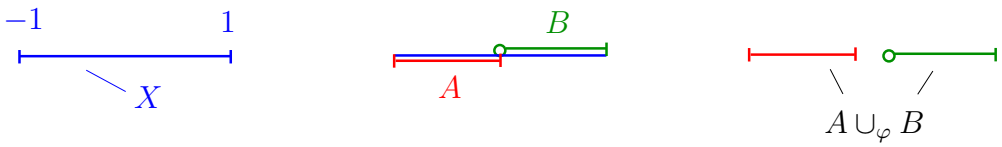
(1) Both subsets A and B are closed.
(2) Both subsets A and B are open.

Then the map

$$\begin{array}{rcl} A \cup_{\varphi\colon A\cap B \to A\cap B} B & \to & X \\ [x] & \mapsto & x \end{array}$$

is a homeomorphism.

Remark. In general the conclusion of the lemma does not hold without some assumptions on A and B. For example, consider $X = [-1, 1]$ together with the subsets $A = [-1, 0]$ and $B = (0, 1]$. In this case $A \cap B = \varnothing$, thus $A \cup_\varphi B$ is the disjoint union of A and B. It follows from the Disjoint Union Topology-Properties Lemma 5.13 (2) and the Heine-Borel 1.24 Theorem that $A \sqcup B$ is not compact. On the other hand $X = [-1, 1]$ is compact by the Heine-Borel 1.24. Thus the two topological spaces $A \cup_\varphi B$ and X are not homeomorphic.



Proof. Let X be a topological space and let A and B be two subsets of X with $X = A \cup B$. We denote by $\varphi\colon A \cap B \to A \cap B$ the identity map. In the following we consider the case that A and B are both closed subsets of X. The case that both A and B are open subsets is dealt with in almost entirely the same way.

It follows from the Topological Pushout Proposition 5.33 (1b) that the given map $h\colon A \cup_{\varphi\colon A\cap B \to A\cap B} B \to X$ is continuous. Furthermore, basically by definition the map h is

a bijection. By the Open -Injective Map Lemma 2.16 (2) it remains to show that the map h is closed. Thus let $W \subset A \cup_\varphi B$ be a closed subset. Note that $h(W) = (W \cap A) \cup (W \cap B)$. It follows from the Topological-Quotient Proposition 5.15 (0b), applied to the projection

$$A \sqcup B \ \to \ A \cup_\varphi B = (A \sqcup B)/\!\sim,$$

that $A \cap W$ is closed in A and that $B \cap W$ is closed in B. Since A and B are by hypothesis closed in X we obtain from the Open -Closed Inclusion Lemma 2.10 (2) that $A \cap W$ and $B \cap W$ are closed in X. Therefore $h(W) = (W \cap A) \cup (W \cap B)$ is a closed subset of X. ■

We conclude this section with an amusing description of spheres. We introduce the following notation:

(1) We write $I = [-1, 1]$.
(2) It follows from the Product-of-Subsets Lemma 5.5 (2) that given $k \in \mathbb{N}_0$ we have

$$\partial I^k \ = \ \{(x_1, \ldots, x_k) \in I^k \mid \text{ at least one } x_i \text{ lies in } \{\pm 1\}\}.$$

(3) We recall that given $k \in \mathbb{N}_0$ there exists by the Convex-to-Ball Proposition 2.20 a canonical homeomorphism

$$\varphi_k \colon (\overline{B}^k, S^{k-1}) \ \xrightarrow{\cong} \ (I^k, \partial I^k)$$

with $\varphi_k(0) = 0$.

Now let $m, n \in \mathbb{N}_0$. An elementary set-theoretic argument shows that

$$((\partial I^m) \times I^n) \ \cup \ (I^m \times \partial I^n) \ = \ \partial I^{m+n}$$

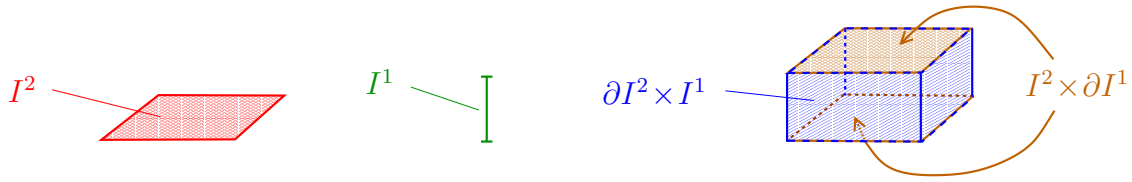
and the two subsets to the left coincide in $(\partial I^m) \times (\partial I^n)$. It follows from Lemma 5.36 that the inclusions induce a homeomorphism

$$((\partial I^m) \times I^n) \cup_{\mathrm{id}_{\partial I^m \times \partial I^n}} (I^m \times \partial I^n) \ \xrightarrow{\cong} \ \partial I^{m+n}.$$

Together with the above homeomorphisms we obtain a canonical homeomorphism

$$(S^{m-1} \times \overline{B}^n) \cup_{\mathrm{id}_{S^{m-1} \times S^{n-1}}} (\overline{B}^m \times S^{n-1}) \ \xrightarrow{\cong} \ S^{m+n-1}.$$

For example, if we take $m = n = 2$, then we see that we can write S^3 as the union of two solid tori. We will discuss this particular case in greater detail later in Section 59.1.



5.9. **Surfaces.** In this section we introduce some of my favorite topological spaces, namely surfaces.

(1) We consider $X = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ and the equivalence relation which is generated by

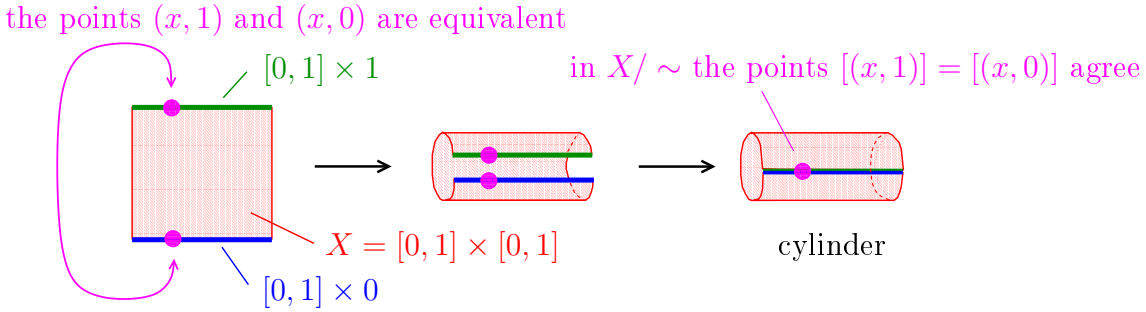
$$(x, 0) \ \sim \ (x, 1) \quad \text{for all } x \in [0, 1].$$

The quotient topological space $X/\!\sim$ is obtained from the square $X = [0, 1] \times [0, 1]$ by identifying each point on the upper edge with the corresponding point on the lower

edge. Put differently, "we glue the upper edge to the lower edge". Using the Compact-Hausdorff Proposition 2.17 (3) one can easily show that the map

$$\begin{array}{rcl} X/\sim & \to & S^1 \times [0,1] \\ [(s,t)] & \mapsto & \big(\exp(2\pi\,\mathrm{i}t), s\big) \end{array}$$

is a homeomorphism. We refer to $X/\sim$ and also to $S^1 \times [0,1]$ as the cylinder or the annulus. Sometimes we refer to $S^1 \times (0,1)$, which is homeomorphic to $(0,1) \times [0,1]/\sim$, as the open cylinder or open annulus.



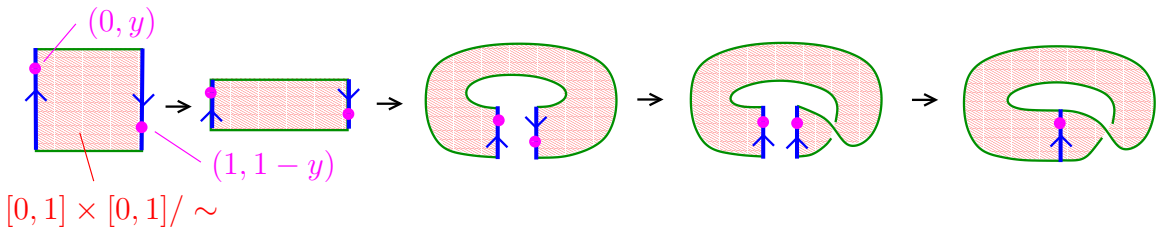
(2) Now we consider $X = [0,1] \times [0,1] \subset \mathbb{R}^2$, this time with the equivalence relation which is generated by

$$(0,y) \sim (1,1-y) \quad \text{for all } y \in [0,1].$$

The quotient space $X/\sim$ is obtained from the square X by gluing the edge on the left to the edge on the right, but this time "we glue the edges with a twist". We will now see that this topological space is homeomorphic to the Möbius band as defined on page 252. We consider the map

$$\begin{array}{rcl} ([0,1] \times [0,1])/\sim & \to & (\mathbb{R} \times [-1,1])/\mathbb{Z} \\ [(x,y)] & \mapsto & [(x, 2y-1)] \end{array}$$

The left-hand side is compact by the Quotient-Compact Connected Lemma 5.16 (1) and the right-hand side is Hausdorff by the Group Action-Hausdorff Quotient Proposition 5.27 and the discussion on page 253. The map is clearly continuous and a bijection. Thus it follows from the ever-popular the Compact-Hausdorff Proposition 2.17 (3) that the map is a homeomorphism. We will use this homeomorphism to identify $([0,1] \times [0,1])/\sim$ with the Möbius band. It is straightforward to see that this homeomorphism restricts to a homeomorphism from $([0,1] \times (0,1))/(0,y) \sim (1,1-y)$ to the open Möbius band $(\mathbb{R} \times (-1,1))/\sim$.



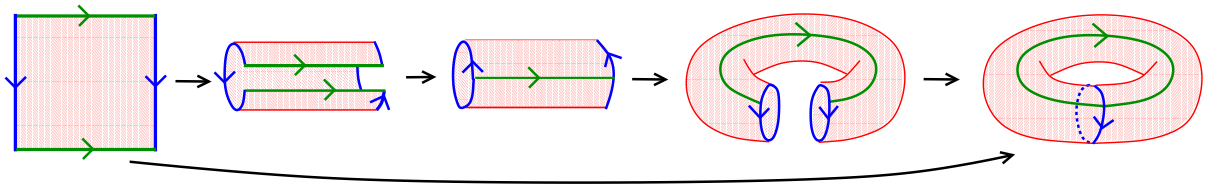
(3) Now we consider again $X = [0,1] \times [0,1] \subset \mathbb{R}^2$, but this time with the equivalence relation which is generated by

$$(x,0) \sim (x,1) \quad \text{for all } x \in [0,1]$$

and by

$$(0,y) \sim (1,y) \quad \text{for all } y \in [0,1].$$

Put differently, we obtain the quotient topological space $X/\sim$ by gluing the top edge to the bottom edge and the left-hand edge to the right-hand edge, each time without adding a twist. The figure below illustrates that the quotient space $X/\sim$ is indeed a torus. As in the previous example one can show that the inclusion $X \to \mathbb{R}^2$ descends



by gluing of the opposite edges of a square we obtain a torus

to a homeomorphism $X/\sim \to \mathbb{R}^2/\mathbb{Z}^2$. In other words, we can make the identification $\mathbb{R}^2/\mathbb{Z}^2 = ([0,1] \times [0,1])/\sim$

(4) Now we consider $X = [0,1] \times [0,1] \subset \mathbb{R}^2$, this time with the equivalence relation which is generated by

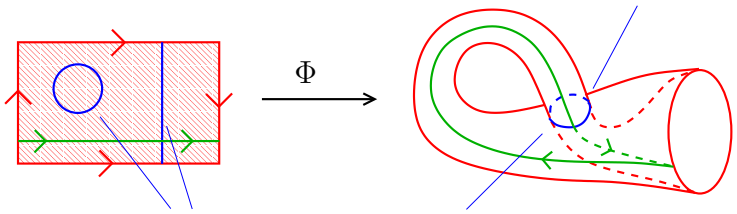
$$(x,0) \sim (x,1) \quad \text{for all } x \in [0,1]$$

and by

$$(0,y) \sim (1,1-y) \quad \text{for all } y \in [0,1].$$

Thus we obtain the quotient space $X/\sim$ by gluing the upper edge to the lower edge and by gluing the edge on the left to the edge on the right with a twist. The resulting topological space is called the Klein bottle. In the figure below we show the Klein bottle to the left. Furthermore we indicate a map from the Klein bottle to $\mathbb{R}^3$. This particular map is not injective.

the given map from the Klein bottle to $\mathbb{R}^3$ has a self-intersection

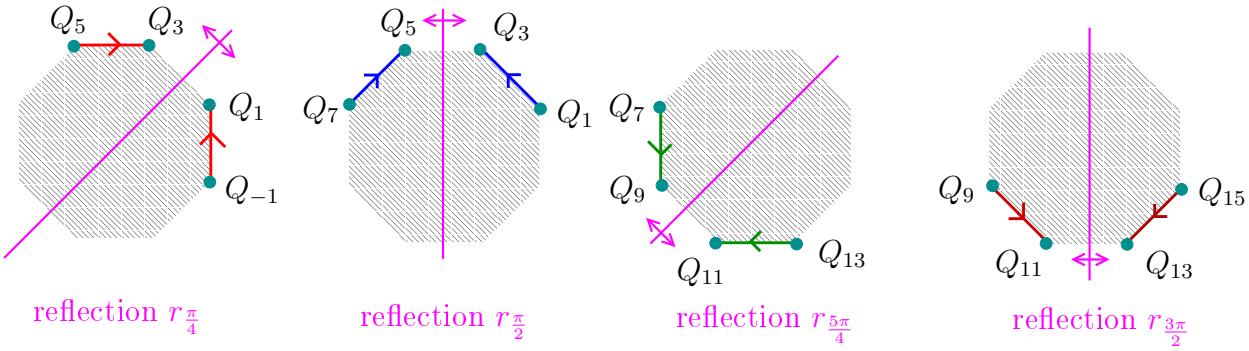


the images of these two circles are the same

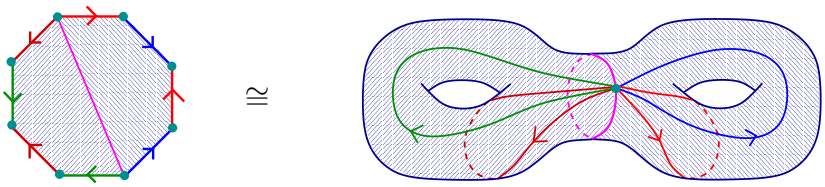
(5) We denote by E_8 the regular octagon in $\mathbb{C}$ with the vertices $Q_k = \exp(2\pi \mathrm{i} k/16)$ where $k = 1, 3, \ldots, 15$, i.e. E_8 is the convex hull of these eight points in $\mathbb{R}^2$. Given two points $A, B \in \mathbb{R}^2 = \mathbb{C}$ we denote by $\overline{AB}$ the Euclidean segment from A to B. Furthermore, given $\varphi \in \mathbb{R}$ we denote by $r_\varphi \colon \mathbb{C} \to \mathbb{C}$ the reflection in the Euclidean line $\{t \exp(\mathrm{i}\varphi) \mid t \in \mathbb{R}\}$. We denote by $\sim$ the equivalence relation on E_8 which is generated by

$$P \in \overline{Q_{2k-1}Q_{2k+1}} \;\sim\; r_{2\pi(2k+2)/16}(P) \;\in\; \overline{Q_{2k+3}Q_{2k+5}}$$

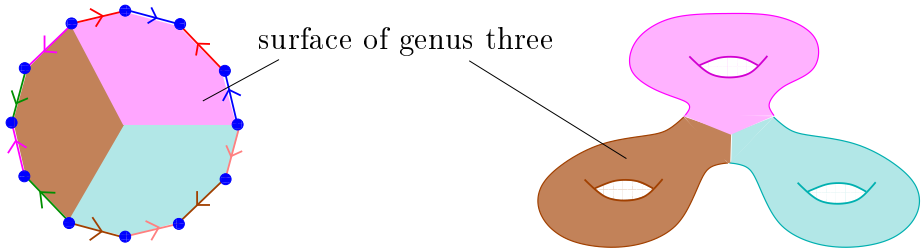
for $k = 0, 1, 4, 5$. We refer to the topological space $E_8/\sim$ as the surface of genus 2. Furthermore, in the figure below we indicate how $E_8/\sim$ is related to the actual



"physical" surface of genus 2 in $\mathbb{R}^3$. The same way we can define the surface Σ_g of

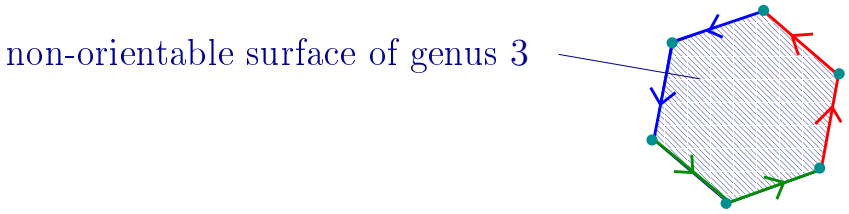


genus g for any $g \geq 3$. More precisely, we take a regular $4g$-gon in $\mathbb{C}$ and for $j = 1, \ldots, g$ we identify the $(4j+1)$-st edge with the $(4j+3)$-rd edge and the $(4j+2)$-nd edge with the $(4j+4)$-th edge. Here each identification is given by a reflection.



Finally we refer to the sphere S^2 as the surface of genus 0 and we refer to the torus $S^1 \times S^1$ as the surface of genus 1.

(6) Given $g \in \mathbb{N}_{\geq 2}$ we consider the regular $2g$-gon E_{2g} with the identification of the boundary segments shown in the figure below. We refer to this topological space as the non-orientable surface N_g of genus g. (Later on in the Surface Non-Orientability Lemma 25.20 we will see why N_g deserves its epithet.) Furthermore, we refer to the real projection place $\mathbb{R}\mathrm{P}^2$ as the non-orientable surface of genus 1. In Exercise 5.50 we will see that the Klein bottle is homeomorphic to the non-orientable surface of genus 2. Furthermore, as a sneak preview, let us point out that later in Lemma **??** we will see that N_g is diffeomorphic to the connected sum of g copies of $\mathbb{R}\mathrm{P}^2$.



We have now introduced many examples of topological spaces, some of them, for example the Möbius band, the torus and the surface of genus 2, can easily be pictured as subsets of "our universe", i.e. they can be viewed as subsets of $\mathbb{R}^3$. This raises the following question.

Question 5.37. Can the real projective space $\mathbb{R}\mathrm{P}^2$ and the Klein bottle, or more generally the non-orientable surfaces of some genus ≥ 3, be viewed[88] as subsets of $\mathbb{R}^3$?

5.10. **Topological groups.** Now that we notion of the product topology we can introduce the notion of a topological group. This notion does not play an essential role in the lecture notes, but it will pup up every now and then.

Definition. A topological group is a topological space X together with a group structure, such that the two maps[89]

$$\begin{array}{ccc} X\times X & \to & X \\ (x,y) & \mapsto & x\cdot y \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \to & X \\ x & \mapsto & x^{-1} \end{array}$$

are continuous.

The most basic example of a topological group is given by taking any group and equipping it with the discrete topology. As we will see in Lemma 5.39 (2), somewhat surprisingly even this banal example can turn out to be interesting.

The following lemma gives us much more interesting examples of topological groups.

Lemma 5.38. For any $n \in \mathbb{N}$ the matrix groups $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$ are topological groups.

Proof. As we pointed out on page 227, matrix multiplication is continuous. Furthermore, basic linear algebra, see e.g. [**HJa2013**, Chapter 0.8.2], says that taking the inverse is given by the map

$$\begin{array}{ccc} \mathrm{GL}(n,\mathbb{R}) & \to & \mathrm{GL}(n,\mathbb{R}) \\ A & \mapsto & \frac{1}{\det(A)} \cdot \underbrace{\begin{array}{c}\text{matrix whose } (i,j)\text{-entry is given by taking the determinant} \\ \text{of the matrix that is obtained from } A \text{ by removing} \\ \text{the } i\text{-th column and the } j\text{-th row of } A.\end{array}}_{=\text{called the adjugate of } A} \end{array}$$

But this map is clearly continuous. This shows that $\mathrm{GL}(n,\mathbb{R})$ is a topological group. Basically the same argument also shows that $\mathrm{GL}(n,\mathbb{C})$ is a topological group. ∎

After this first list of examples let us formalize the setting a little bit by introducing the category of topological groups.

Definition. We refer to the category $\mathcal{TopGr}$ with

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{TopGr}) & := & \text{all topological groups,} \\ \mathrm{Mor}(X,Y) & := & \text{all homomorphisms from } X \text{ to } Y \text{ that are continuous,} \end{array}$$

with the usual composition of maps as the category of topological groups.

We obtain more examples of topological groups by taking subgroups and products:

[88] The word "view" is intentionally vague.

[89] Here we evidently equip $X \times X$ with the product topology.

Lemma 5.39.

(1) Let X be a topological group and let Y be a subgroup. If we equip Y with the subspace topology, then Y is also a topological group.

(2) Let $\{X_i\}_{i\in I}$ be a family of topological groups. If we equip $\prod_{i\in I} X_i$ with the product topology and the obvious group structure, then it is again a topological group.

Proof.

(1) This statement follows immediately from Lemma 2.1 together with the Product Topology-Basics Lemma 5.4 (4).

(2) This statement follows easily from the Topological-Product Proposition 5.1 and the Product Topology-Basics Lemma 5.4 (2). ■

Example. It follows from Lemmas 5.38 and 5.39 (1) that $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are topological groups.

Lemma 5.40. Let G be a topological group.

(1) Given any $g \in G$ the two maps

$$\begin{array}{rcl} l_g\colon G & \to & G \\ h & \mapsto & g\cdot h \end{array} \quad \text{and} \quad \begin{array}{rcl} r_g\colon G & \to & G \\ h & \mapsto & h\cdot g \end{array}$$

are continuous.

(2) If G is a topological group, then any two path-components of G are homeomorphic.

Proof.

(1) The map l_g is the composition of the two maps

$$\begin{array}{rcl} G & \to & G\times G \\ h & \mapsto & (g,h) \end{array} \quad \text{and} \quad \begin{array}{rcl} G\times G & \to & G \\ (x,y) & \mapsto & x\cdot y. \end{array}$$

The first map is continuous by the Product Topology-Basics Lemma 5.4 (1) and the second map is continuous since G is a topological group. Thus l_g itself is continuous. The same argument shows that r_g is continuous.

(2) Let X, Y be two path-components of G. We pick $x \in X$ and $y \in Y$. We consider the map $\varphi\colon G \to G$ given by $g \mapsto yx^{-1}g$. By (1) we know that φ is continuous. Furthermore it follows from Lemma 2.35 that $\varphi(X) \subset Y$.

We also consider the map $\psi\colon G \to G$ given by $g \mapsto xy^{-1}g$. The same argument as above shows that ψ is continuous and that $\psi(Y) \subset X$. Clearly φ and ψ are inverses of one another. But this implies that φ and ψ are homeomorphisms between X and Y. ■

5.11. **Continuity of zeros.** We conclude this long chapter with a slightly peripheral discussion on the zeros of complex polynomials. This discussion shows in particular that group actions can appear in unexpected situations. We start out with the following slightly vague question.

Question 5.41. Let $n \in \mathbb{N}$. Do the zeros of a polynomial of degree n vary continuously with the coefficients of the polynomial?

The first question is, what does it mean that the "zeros of a polynomial" vary "continuously"? What are the topological spaces involved so that we can talk about continuity? The following proposition kills two birds at once: it makes the question precise and it gives a complete answer.

Proposition 5.42. (Zeros Continuity Proposition) Let $n \in \mathbb{N}$. We consider the action by the permutation group S_n on $\mathbb{C}^n$ that is given by permuting the coordinates. The map[90]

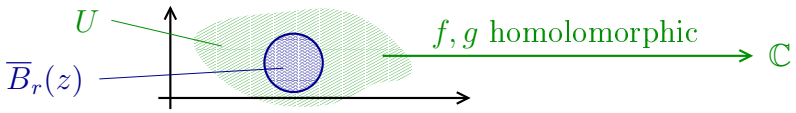
$$\begin{array}{rcl} f\colon \mathbb{C}^n & \to & \mathbb{C}^n/S_n \\ (v_0,\dots,v_{n-1}) & \mapsto & \left[\begin{array}{c} \text{set of zeros of the polynomial} \\ v_0 + v_1\cdot z + \dots + v_{n-1}\cdot z^{n-1} + z^n, \\ \text{counted with multiplicities} \end{array}\right] \end{array}$$

is continuous, in fact it is a homeomorphism.

Remark. For $n = 1, 2, 3, 4$ one can hope to prove the Zeros Continuity Proposition 5.42 by the formulas for determining zeros of polynomials of degree n. But since no such formula exists for general polynomials of higher degree this approach is doomed for $n \geq 5$.

The proof of the Zeros Continuity Proposition 5.42 makes use Rouché's Theorem from complex analysis.

Theorem 5.43. (Rouché's Theorem) Let $U \subset \mathbb{C}$ be an open set. Furthermore let $f, g\colon U \to \mathbb{C}$ be two holomorphic[91] functions and let $z \in U$ and $r \in \mathbb{R}_{>0}$ be such that $\overline{B}_r(z) = \{w \in \mathbb{C} \,|\, |w - z| \leq r\}$ is contained in U. If for all $w \in \partial\overline{B}_r(z)$ we have the inequality $|g(w)| < |f(w)|$, then f and $f + g$ have the same number of zeros (counted with multiplicities) within the open disk $B_r(z)$.



Proof. This theorem is proved in every self-respecting book on complex analysis, see e.g. [**Lang1999**, Theorem VI.1.6]. ■

Example. We consider $f(z) = z^5$ and the disk $\overline{B}_1(0)$. The polynomial $f(z)$ has one zero of multiplicity five in $\overline{B}_1(0)$. For any $a, b \in \mathbb{C}$ with $|a| + |b| < 1$ we have $|az + b| < |z^5|$ for all points $z \in \partial\overline{B}_1(0) = S^1$. Thus by Rouché's Theorem 5.43 we see that for such a, b the polynomial $z^5 + az + b$ also has five zeros (counted with multiplicities) in $\overline{B}_1(0)$.

We will also need the following definition:

Definition. Given $w = [(w_1,\dots,w_n)]$ and $z = [(z_1,\dots,z_n)]$ in $\mathbb{C}^n/S_n$ we define

$$d(w,z) \;:=\; \min_{\sigma\in S_n} \max_{1\leq j\leq n} \big|w_j - z_{\sigma(j)}\big|.$$

The following lemma summarizes two key statements regarding the above definition.

[90] Slightly more rigorously the map f is defined as follows: given $(v_0,\dots,v_{n-1}) \in \mathbb{C}^n$ we define $f(v_0,\dots,v_{n-1}) := [(\alpha_1,\dots,\alpha_n)]$ where $\alpha_1,\dots,\alpha_n \in \mathbb{C}$ are determined uniquely (up to reordering) by the condition that $v_0 + v_1 z + \dots + v_{n-1}z^{n-1} + z^n = (z-\alpha_1)\cdot\dots\cdot(z-\alpha_n) \in \mathbb{C}[z]$.

[91] We will introduce the notion of a holomorphic function on page 789. But at this point it suffices to know that polynomial functions are holomorphic.

Lemma 5.44.

(1) The map

$$\begin{array}{rcl} d\colon \mathbb{C}^n/S_n \times \mathbb{C}^n/S_n & \to & \mathbb{R}_{\geq 0} \\ (w,z) & \mapsto & d(w,z) \end{array}$$

is a metric on $\mathbb{C}^n/S_n$.

(2) The topology defined by the metric d on $\mathbb{C}^n/S_n$ agrees with the quotient topology on $\mathbb{C}^n/S_n$.

Proof. We hand over the task providing the proof of the lemma to the surely very meticulous reader. ■

Now we are in a position to prove the Zeros Continuity Proposition 5.42.

Proof of the Zeros Continuity Proposition 5.42. Recall that we need to show that the map

$$\begin{array}{rcl} f\colon \mathbb{C}^n & \to & \mathbb{C}^n/S_n \\ (v_0,\ldots,v_{n-1}) & \mapsto & \left[\begin{array}{c} \text{set of zeros of the polynomial} \\ v_0 + v_1\cdot z + \cdots + v_{n-1}\cdot z^{n-1} + z^n, \\ \text{counted with multiplicities} \end{array}\right] \end{array}$$

is a homeomorphism. To do so we consider the map[92]

$$\begin{array}{rcl} g\colon \mathbb{C}^n/S_n & \to & \mathbb{C}^n \\ [(\alpha_1,\ldots,\alpha_n)] & \mapsto & \text{coefficients of the polynomial } (z-\alpha_1)\cdot\cdots\cdot(z-\alpha_n). \end{array}$$

It is straightforward to check that g is continuous and that f and g are inverses of one another.

It remains to show that f is continuous. We introduce the following definitions: We denote by d the above metric on $\mathbb{C}^n/S_n$. It follows from Lemma 5.44 that we can view f as a map between the two metric spaces $(\mathbb{C}^n, \|-\|)$ and $(\mathbb{C}^n/S_n, d)$. Therefore it follows from the Metric Continuity Proposition 2.8 that it suffices to prove the following claim.

Claim.
$$\forall_{v\in\mathbb{C}^n}\ \forall_{\epsilon>0}\ \exists_{\delta>0}\ \forall_{\substack{w\in\mathbb{C}^n \text{ with} \\ \|w\|<\delta}}\quad d(f(v), f(v+w)) < \epsilon.$$

Proof. Before we start with the proof of the claim we introduce the following definition: Given $v = (v_0,\ldots,v_{n-1}) \in \mathbb{C}^n$ we denote by $p_v(z) := v_0 + v_1\cdot z + \cdots + v_{n-1}\cdot z^{n-1} + z^n$ the corresponding polynomial. Now we turn to the actual proof of the claim. Thus let $v = (v_0,\ldots,v_{n-1}) \in \mathbb{C}^n$ and let $\epsilon > 0$. We denote by $\xi_1,\ldots,\xi_k$ the *distinct* zeros of the corresponding polynomial $p_v(z)$. (Note that if $p_v(z)$ has zeros of higher multiplicity, then $k < n$.) After possibly replacing ϵ by a smaller non-negative number we can assume, without loss of generality, that $\epsilon < \frac{1}{2}\cdot|\xi_i - \xi_j|$ for all $i \neq j$.

We denote by Γ the union of the circles of the radius ϵ around $\xi_1,\ldots,\xi_k$. Furthermore we set

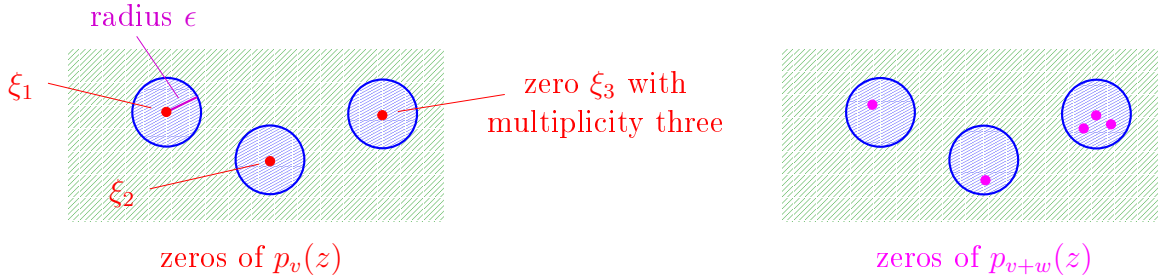
$$\eta \ := \ \inf_{z\in\Gamma} |p_v(z)| \ \in \ \mathbb{R}_{>0}.$$

[92] Slightly more precisely the map g is defined as follows: given $[(\alpha_1,\ldots,\alpha_n)] \in \mathbb{C}^n/S_n$ we have $f([(\alpha_1,\ldots,\alpha_n)]) = (v_0,\ldots,v_{n-1})$ where $v_0,\ldots,v_{n-1} \in \mathbb{C}$ are uniquely determined by the condition that $v_0 + v_1 z + \cdots + v_{n-1}z^{n-1} + z^n = (z-\alpha_1)\cdot\cdots\cdot(z-\alpha_n) \in \mathbb{C}[z]$.

It is elementary to show that there exists a $\delta > 0$ such that for all $w = (w_0, \ldots, w_{n-1}) \in \mathbb{C}^n$ such that $\|w\| < \delta$ we have

$$|w_0 + w_1 \cdot z + \cdots + w_{n-1} \cdot z^{n-1}| \ < \ \eta \quad \text{for all } z \in \Gamma.$$

We claim that this δ has the desired property. Thus let $w \in \mathbb{C}^n$ with $|w| < \delta$. It follows from the definition of δ and Rouché's Theorem 5.43 that the polynomials $p_v(z)$ and $p_{v+w}(z)$ have the same number of zeros (counted with multiplicities) in the disks $B_\epsilon(\xi_1), \ldots, B_\epsilon(\xi_k)$. This means that the zeros of $p_v(z)$ and can be matched up with the zeros of $p_{v+w}(z)$ such that each zero is less then ϵ removed from its match. This statement implies immediately that $d(f(v), f(v+w)) < \epsilon$. ■



Remark. With some effort one can also prove the Zeros Continuity Proposition 5.42 without using Rouché's Theorem 5.43. We outline the key steps, using the notation that we introduced in the above proof of Proposition 5.42.

(1) Note that by the Open -Injective Map Lemma 2.16 (2) it suffices to show that the map $g\colon \mathbb{C}^n/S_n \to \mathbb{C}^n$ is open.
(2) By Lemma 2.11 it suffices to show that for each $\epsilon > 0$ and each $z \in \mathbb{C}^n/S_n$ the image of the open ball $B_\epsilon(z) \subset (\mathbb{C}^n/S_n, d)$ under g is open.
(3) Thus let $\epsilon > 0$ and $z \in \mathbb{C}^n/S_n$.
(4) We pick $C \in \mathbb{R}_{\geq 0}$ such that $g(\overline{B}_\epsilon(z)) \subset B_C(0) \subset \mathbb{C}^n$.
(5) There exists a $D \in \mathbb{R}_{\geq 0}$ such that for any $v \in \mathbb{C}^n$ which satisfies $\|v\| \leq C$ we have $f(v) \in B_D(0) \subset \mathbb{C}^n/S_n$. (This statement requires some unraveling and a little bit of thought. It can be proved using Rouché's Theorem 5.43, but with a worse bound it can also be proved by hand.)
(6) Note that by the Compact-Hausdorff Proposition 2.17 (3) we know that g restricted to $\overline{B}_D(0) \to g(\overline{B}_D(0))$ is a homeomorphism. Thus $g(B_\epsilon(z))$ is an open subset of $g(\overline{B}_D(0))$.
(7) By (5) and (6), and since f and g are inverses of one another, we know that $g(B_\epsilon(z))$ is also open in $B_D(0)$. But $B_D(0)$ is an open subset of $\mathbb{C}^n$, hence $g(B_\epsilon(z))$ is also open in $\mathbb{C}^n$.

Exercises for Chapter 5.

Exercise 5.1. Let $(X_1, d_1), \ldots, (X_m, d_m)$ be metric spaces. We define the product metric d on $X_1 \times \cdots \times X_m$ as follows:

$$d((x_1, \ldots, x_m), (y_1, \ldots, y_m)) \ := \ \sum_{i=1}^{m} d_i(x_i, y_i).$$

Show that the product topology on $X_1 \times \cdots \times X_m$ agrees with the metric topology coming from the product metric.

Exercise 5.2. Let $C \subset \mathbb{R}$ be the Cantor set, as defined on page 130. Is the disjoint union $C \sqcup C$ homeomorphic to C itself?

Exercise 5.3.

(a) Let $\mathcal{C}$ be a category $\mathcal{C}$ that admits products and coproducts. Let X, Y, A be objects of $\mathcal{C}$. Show that there exists a natural map $(X \times A) \amalg (Y \times A) \to (X \amalg Y) \times A$.

(b) Show that the product topology is "distributive" with respect to the disjoint union topology. More precisely, let X and Y be two topological spaces and let A be another topological space. Show that the natural map $(X \times A) \sqcup (Y \times A) \to (X \sqcup Y) \times A$ from (a) is a homeomorphism.

(c) Show that the statement of (b) is not a consequence of the formal properties of products and coproducts. More precisely give an example of a category $\mathcal{C}$ that admits products and coproducts and objects X, Y, A of $\mathcal{C}$ such that the natural map $(X \times A) \amalg (Y \times A) \to (X \amalg Y) \times A$ is not an isomorphism.

Exercise 5.4. Let X be a topological space and let I be a set. We equip I with the discrete topology. Show that the natural map

$$\begin{array}{ccc} \overbrace{\bigsqcup_{i \in I} X}^{\text{disjoint union topology}} & \to & \overbrace{X \times I}^{\text{product topology}} \\ (x, i) & \mapsto & (x, i) \end{array}$$

is a homeomorphism.

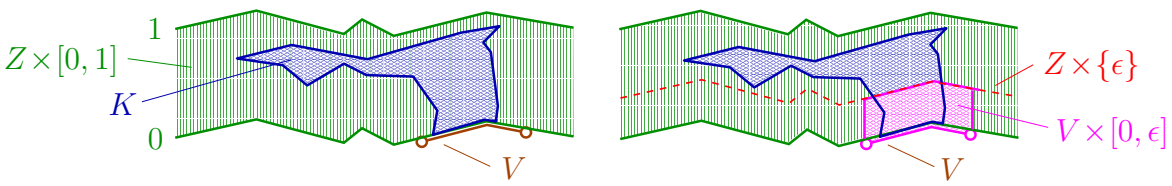
Exercise 5.5. Show that there are non-homeomorphic topological spaces A and B and a non-empty topological space X such that $A \times X$ is homeomorphic to $B \times X$.
Hint. you could take $A = \{0\}$ and $B = \{0, 1\}$.

Exercise 5.6. Let X be a topological space. We consider the "diagonal"

$$\Delta \ := \ \{(x, x) \in X \times X \,|\, x \in X\}.$$

We equip $X \times X$ with the product topology and we equip Δ with the subspace topology. Show that the map $X \to \Delta$ given by $x \mapsto (x, x)$ is a homeomorphism.

Exercise 5.7. Let Z be a topological space and let $K \subset Z \times [0, 1]$ be a compact subset. Furthermore let $V \subset Z$ be an open subset such that $(K \cap (Z \times \{0\}) \subset V \times \{0\}$. Show that there exists an $\epsilon > 0$ such that $K \cap (Z \times [0, \epsilon]) = K \cap (V \times [0, \epsilon])$.



Exercise 5.8. Let $X_1, \ldots, X_k$ be topological spaces and let $B_i \subset X_i$, $i = 1, \ldots, k$ be subsets. Show that the following equalities hold:

(a) closure of $B_1 \times \ldots \times B_k$ in $X_1 \times \ldots \times X_k$ $= \overline{B}_1 \times \ldots \times \overline{B}_k$

(b) interior of $B_1 \times \ldots \times B_k$ in $X_1 \times \ldots \times X_k$ $= \mathring{B}_1 \times \ldots \times \mathring{B}_k$

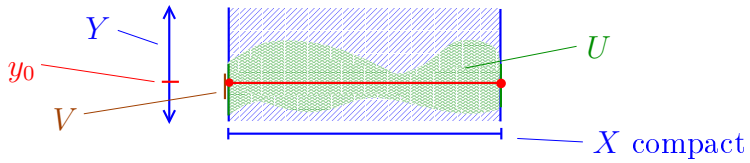
(c) boundary of $B_1 \times \ldots \times B_k$ in $X_1 \times \ldots \times X_k$ $= \bigcup_{i=1}^{k} \overline{B}_1 \times \ldots \times \partial B_i \times \ldots \times \overline{B}_k$.

Remark. This exercise provides the proof of the Product-of-Subsets Lemma 5.5.

Exercise 5.9. Let X and Y be topological spaces, let $y_0 \in Y$ and let U be an open subset of $X \times Y$ that contains $X \times \{y_0\}$.

(a) We assume that X is compact. Show that there exists an open neighborhood V of y_0 such that $X \times V \subset U$.
Remark. This statement is sometimes called the *Tube Lemma*.

(b) Show that the conclusion of (a) in general does not hold if we do not assume that X is compact.



Exercise 5.10. Let X and Y be topological spaces and let $W \subset X \times Y$ be an open subset.

(a) Let $y \in Y$. Show that $U_y := \{x \in X \,|\, (x, y) \in W\}$ is an open subset of X.

(b) Show that the natural projection $p\colon X \times Y \to X$ is an open map.

(c) Let K be a compact subset of Y. Show that $U := \{x \in X \,|\, \{x\} \times K \subset W\}$ is an open subset of X.

Exercise 5.11. Let X and Y be topological spaces and let $K \subset X \times Y$ be a compact subset. Let $y \in Y$. We set $K_y := \{x \in X \,|\, (x, y) \in K\}$.

(a) Show that if Y is Hausdorff, then K_y is a compact subspace of X.

(b) Show that in general it is not true that K_y is a compact subspace of X.

Exercise 5.12. Let X and Y be topological spaces.

(a) Show that if X is compact, then the natural projection $p\colon X \times Y \to Y$ is a closed map.
Remark. This exercise is related to Exercise 5.9.

(b) Give an example that shows that in general in (b) we cannot drop the hypothesis that X is compact.

Somewhat surprisingly, the converse to (a) holds. More precisely, if a topological space X has the property, that for any other topological space Y the projection $p\colon X \times Y \to Y$ is a closed map, then X is actually compact. This non-trivial statement is sometimes known as the *Kuratowski-Mrówka Theorem*, it was first proved by Stanislaw Mrówka [**Mró1959**, p. 21] in 1959. See also [**Herl1996**, p. 3] for a proof and see [**Enge1989**, Theorem 3.1.16] for a proof under the extra hypothesis that X is Hausdorff.

Exercise 5.13. Let $S := (\mathbb{R}, \mathcal{T})$ be the Sorgenfrey line that we introduced on page 182. We refer to the product $S \times S$ as the *Sorgenfrey plane*.

(a) Show that the diagonal $\Delta := \{(x, x) \,|\, x \in \mathbb{R}\}$ is a closed subset of the Sorgenfrey plane.

(b) Show that the subspace topology on the anti-diagonal $A := \{(x, -x) \,|\, x \in \mathbb{R}\}$ is the discrete topology.

Exercise 5.14. Let $\{X_i\}_{i \in I}$ be a family of topological spaces. The *box topology* on the set $\prod\limits_{i \in I} X_i$ is the topology that is generated by the set

$$\Big\{ \prod_{i \in I} U_i \,\Big|\, U_i \subset X_i \text{ open} \Big\}.$$

(a) Show that the product of compact topological spaces, equipped with the box topology, is not necessarily compact.

(b) Let X be a topological space. We equip $X^{\mathbb{N}} = \prod\limits_{i \in \mathbb{N}} X$ with the box topology. Is the diagonal map

$$\begin{array}{ccc} X & \to & X^{\mathbb{N}} \\ x & \mapsto & (x, x, \dots) \end{array}$$

continuous?

Exercise 5.15. Let $X \subset \mathbb{R}^n$ be a compact convex non-empty subset. Show that there exists a $k \in \mathbb{N}_0$ such that X is homeomorphic to $\overline{B}^k$.

Exercise 5.16. Let $\{X_i\}_{i \in I}$ be a family of topological spaces.

(a) Suppose that for each $i \in I$ we are given a closed subset $A_i \subset X_i$. If I is finite, then it follows from the Product-of-Subsets Lemma 5.5 that $\prod\limits_{i \in I} A_i$ is a closed subset of $\prod\limits_{i \in I} X_i$. Does this conclusion also hold if I is infinite?

(b) Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Suppose that for each $i \in I$ we are given a subset $A_i \subset X_i$. Show that

$$\prod_{i \in I} \overline{A_i} \;=\; \overline{\prod_{i \in I} A_i} \;\subset\; \prod_{i \in I} X_i.$$

Exercise 5.17. Let X and Y be topological spaces and let $M \subset X \times Y$ be a compact subset. Show that there exist compact subsets $K \subset X$ and $L \subset Y$ such that $M \subset K \times L$.

Exercise 5.18. Let X and Y be two topological spaces. Show that the product topology on $X \times Y$ is the smallest topology that has the property that the two natural projections $X \times Y \to X$ and $X \times Y \to Y$ are continuous.

Exercise 5.19. Let A be the annulus and let M be the Möbius band. Are the product topological spaces $A \times [0, 1]$ and $M \times [0, 1]$ homeomorphic?

Exercise 5.20. Let X be a topological space that is compact and Hausdorff. Furthermore let $f \colon X \to X$ be a map. We denote by $F := \{x \in X \,|\, f(x) = x\}$ the fixed point set of f. Show that F is a discrete subset of X if and only if F is finite.

Exercise 5.21. Show "by hand" that the map

$$\begin{aligned} \varphi\colon \prod_{i\in\mathbb{N}} \{0,2\} &\to C = \text{Cantor set from page 130} \\ (f\colon \mathbb{N}\to\{0,2\}) &\mapsto \sum_{i=1}^{\infty} \frac{f(i)}{3^i}, \end{aligned}$$

which we defined on page 235, is continuous.

Exercise 5.22.

(a) Show that the Cantor set is totally disconnected.

(b) Show that $\prod_{i\in\mathbb{N}} \{0,2\}$ is totally disconnected.

Exercise 5.23. Let X be a topological space.

(a) Show that the following two statements are equivalent:
   (i) X is Hausdorff,
   (ii) the diagonal $\Delta = \{(x,x) \mid x \in X\}$ is a closed subset of $X \times X$.

(b) Let $A \subset X$ be a subset that admits a retraction $r\colon X \to A$.
   (i) We denote by $f\colon X \to X \times X$ the map that is given by $x \mapsto (x, r(x))$. Show that f is continuous and that $f^{-1}(\Delta) = A$.
   (ii) Show that if X is Hausdorff, then A is closed.
   Remark. This gives a new solution of Exercise 2.25 (1).

Exercise 5.24. Let C be the Cantor set as defined on page 130. We recall two facts:

(1) On page 235 we showed that C is homeomorphic to the product $\{0,1\}^{\mathbb{N}}$.

(2) In Exercise 2.17 we showed that there exists a surjective map $\varphi\colon C \to [0,1]$.

Use the above to prove the following statements:

(a) The Cantor set C is homeomorphic to $C \times C$.

(b) Any continuous map $C \to \mathbb{R}$ extends to a continuous map $[0,1] \to \mathbb{R}$.
   Remark. This statement follows from the Tietze Extension Theorem 8.5. But it also allows for an elementary proof.

(c) Use the above statements to conclude that there exists a surjective continuous map $[0,1] \to [0,1] \times [0,1]$.
   Remark. Such a map is called a *space-filling curve*. We will provide alternative constructions of space-filling curves in Theorem 6.13 and Exercise 6.15.

Exercise 5.25.

(a) Show that the real projective space $\mathbb{R}\mathrm{P}^1$ is homeomorphic to S^1.

(b) Do you think that the real projective space $\mathbb{R}\mathrm{P}^2 = (\mathbb{R}^3 \setminus \{0\})/(\mathbb{R} \setminus \{0\})$ is homeomorphic to S^2? As of now we lack the tools to answer this question. Nonetheless, what do you think the answer will be?

Exercise 5.26. Let $\sim$ be an equivalence relation on a topological space X. Furthermore let $f\colon X \to Y$ be a map with the property that $f(x) = f(y)$ whenever $x \sim y$. Show that if $f\colon X \to Y$ is an open map, then the induced map $X/\!\sim \to Y$ is also an open map.

Exercise 5.27. Let the group $(\mathbb{Q}, +)$ act on the topological space $\mathbb{R}$ by addition. Show that the quotient topology on $\mathbb{R}/\mathbb{Q}$ is actually the trivial topology.
Remark. This of course shows in particular that $\mathbb{R}/\mathbb{Q}$ is not Hausdorff.

Exercise 5.28. We consider the map

$$\begin{array}{rcl} f\colon [0,1) & \to & \mathbb{R}/\mathbb{Z}. \\ x & \mapsto & [x]. \end{array}$$

Give an example of an subset of $[0,1)$ such that $f(U)$ not open in $\mathbb{R}/\mathbb{Z}$.

Exercise 5.29. Show that the topological space $\mathbb{R}/\{\pm 1\} = \mathbb{R}/x \sim -x$ is homeomorphic to the half-open interval $[0,\infty)$.

Exercise 5.30. Let

$$S \;:=\; \big\{(0,y)\,\big|\,y\in[-1,1]\big\} \cup \big\{(x,\sin(\tfrac{1}{x}))\,\big|\,x\in(0,\tfrac{1}{\pi}]\big\} \;\subset\; \mathbb{R}^2$$

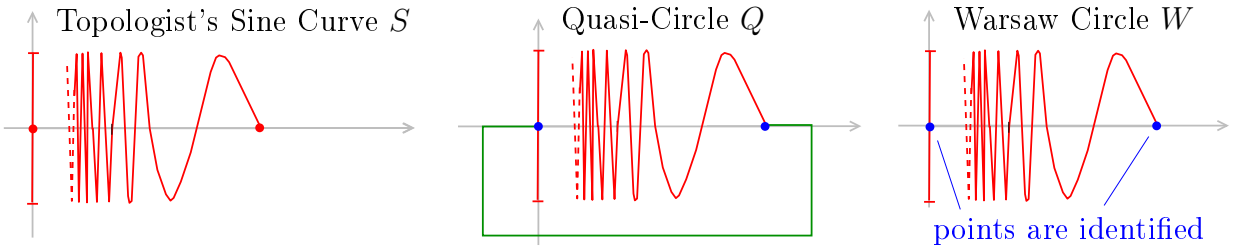
be the "Topologist's Sine Curve" from page 164.

- We turn the Topologist's Sine curve into the *Quasi-Circle* Q by connecting the point $(0,0)$ with $(\frac{1}{\pi},0)$ via the green curve shown in the figure below.
- We refer to quotient space $W := S/(0,0)\sim(\frac{1}{\pi},0)$ as the *Warsaw Circle*.

Now perform the following tasks:

(a) Show that the Topologist's Sine Curve, the Quasi-Circle and the Warsaw Circle are compact.
(b) Show that the Warsaw Circle is homeomorphic to the Quasi-Circle.
(c) Show that the Warsaw Circle and the Quasi-Circle are path-connected.

Remark. The Warsaw Circle is sometimes also called the *Polish Circle*.



Exercise 5.31. Let $n\in\mathbb{N}$.

(a) Let $p\colon \mathbb{R}^n\setminus\{0\}\to S^{n-1}$ be the map given by $x\mapsto \frac{x}{\|x\|}$. Show that the preimage of a dense subset of S^{n-1} is a dense subset of $\mathbb{R}^n\setminus\{0\}$.
(b) Let $q\colon S^{n-1}\to \mathbb{R}\mathrm{P}^{n-1}$ be the projection . Show that the preimage of a dense subset of $\mathbb{R}\mathrm{P}^{n-1}$ is a dense subset of S^{n-1}.

Exercise 5.32. Let $\sim$ be the equivalence relation on $\mathbb{R}_{\geq 0}\times\mathbb{R}$ that is generated by

(1) $(0,\varphi)\sim(0,\varphi')$ for all $\varphi,\varphi'\in\mathbb{R}$ and by
(2) $(r,\varphi)\sim(r,\varphi+k\cdot 2\pi)$ for all $r\in\mathbb{R}_{\geq 0}$, $\varphi\in\mathbb{R}$ and $k\in\mathbb{Z}$.

We equip $(\mathbb{R}_{\geq 0}\times\mathbb{R})/\!\sim$ with the quotient topology.

(a) Show that the map

$$\begin{array}{rcl} \Theta\colon (\mathbb{R}_{\geq 0}\times\mathbb{R})/\!\sim & \to & \mathbb{C} \\ [(r,\varphi)] & \mapsto & r\cdot\exp(\mathrm{i}\varphi) \end{array}$$

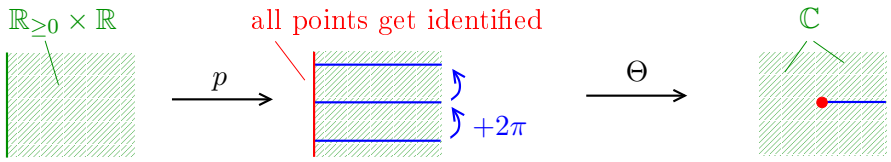
is a homeomorphism.
Hint. To show that the map is a homeomorphism you might want to use the Closed Embedding-to-$\mathbb{R}^n$ Proposition 2.19.

(b) Let $f\colon \mathbb{R}_{\geq 0} \times \mathbb{R} \to X$ be a continuous map to some topological space X such that $f(P) = f(Q)$ whenever $P \sim Q$. Show that the map

$$\begin{array}{rcl} \mathbb{C} & \to & X \\ r \cdot \exp(\mathrm{i}\varphi) & \mapsto & f(r, \varphi) \end{array}$$

is continuous.

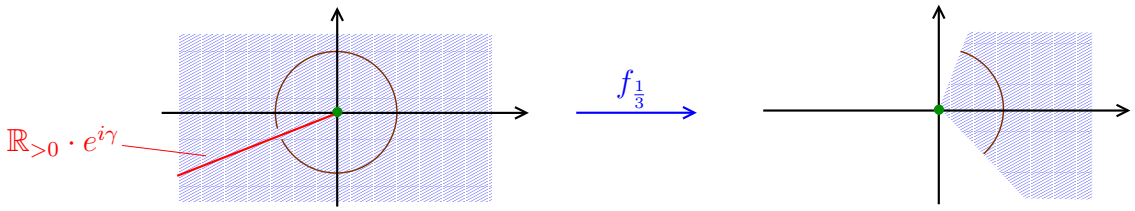
Remark. This exercise is a variation on Lemma 5.19.



Exercise 5.33. Let $\gamma \in \mathbb{R}$ and let $s \in \mathbb{R}$. We consider the map

$$\begin{array}{rcl} f_s\colon \mathbb{C} \setminus \{t \cdot \exp(\mathrm{i}\gamma) \,|\, t \in \mathbb{R}_{>0}\} & \to & \mathbb{C} \\ r \cdot \exp(\mathrm{i}\varphi) & \mapsto & r \cdot \exp(\mathrm{i}\varphi \cdot s). \end{array}$$

(a) Show that f_s is well-defined and continuous.
(b) Suppose that $s \neq 0$. Show that f_s is a local homeomorphism. (See the discussion on page 179 for the definition of a local homeomorphism.)



Exercise 5.34.

(a) Show that any matrix $A \in \mathrm{O}(2n)$ with $\det(A) = -1$ has 1 as an eigenvalue.
Hint. Show that if $\lambda \in \mathbb{C}$ is an eigenvalue of A, then so is $\overline{\lambda}$.
(b) Let $G \subset \mathrm{O}(2n)$ be a subgroup that acts freely on S^{2n-1}. Show that G is contained in $\mathrm{SO}(2n)$.
(c) Show that the analogue of (b) does not hold for subgroups of $G \subset \mathrm{O}(2n-1)$.

Exercise 5.35. Let $n \in \mathbb{N}$ and let $\varphi\colon S^n \to S^n$ be a homeomorphism. We consider the map

$$\begin{array}{rcl} \Phi\colon \overline{B}^{n+1} & \to & \overline{B}^{n+1} \\ x & \mapsto & \begin{cases} \|x\| \cdot \varphi(\frac{x}{\|x\|}), & \text{if } x \neq 0, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

(a) Show that Φ is continuous.
(b) Show that Φ is a homeomorphism.
(c) Give any example of a *diffeomorphism* $\varphi\colon S^n \to S^n$ such that the corresponding map $\Phi\colon \overline{B}^{n+1} \to \overline{B}^{n+1}$ is *not* a diffeomorphism.

Hint. You might want to use Lemma 5.19.

Remark 1. The fact that we can extend φ to a homeomorphism of $\overline{B}^{n+1}$ is sometimes

referred to as the Alexander trick. We will get to know a more fancy Alexander trick later on page 421.

Remark 2. We have thus shown that every *homeomorphism* of S^n extends to a *homeomorphism* of $\overline{B}^{n+1}$. In contrast, in Proposition **??** we will see that exists a *diffeomorphism* $\varphi\colon S^6 \to S^6$ that does not extend to a *diffeomorphism* of $\overline{B}^7$.

Exercise 5.36. Given $v, w \in \mathbb{R}^2 \setminus \{(0,0)\}$ we define

$$\sphericalangle(v,w) := \text{unique } \varphi \in (-\pi,\pi] \text{ such that } w = r \cdot \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \cdot v \text{ for some } r \in \mathbb{R}_{>0}.$$

(a) Show that the map

$$\begin{array}{rcl} \Phi\colon (\mathbb{R}^2 \setminus \{(0,0)\}) \times (\mathbb{R}^2 \setminus \{(0,0)\}) & \to & (-\pi,\pi] \\ (v,w) & \mapsto & \sphericalangle(v,w) \end{array}$$

is not continuous.

(b) Show that the map

$$\begin{array}{rcl} \Phi\colon (\mathbb{R}^2 \setminus \{(0,0)\}) \times (\mathbb{R}^2 \setminus \{(0,0)\}) & \to & \mathbb{R}/2\pi\mathbb{Z} \\ (v,w) & \mapsto & [\sphericalangle(v,w)] \end{array}$$

is continuous. (Here $\mathbb{R}/2\pi\mathbb{Z}$ denotes the quotient of the topological space $\mathbb{R}$ by the group $(2\pi\mathbb{Z},+)$.)

Exercise 5.37. We consider the following subsets of $[0,1]$:

$$A = \{0, \tfrac{1}{3}, \tfrac{2}{3}, 1\}, \quad B = \{\tfrac{1}{n} \mid n \in \mathbb{N}\} \quad \text{and} \quad C = \{0\} \cup B.$$

(a) Show that the quotient $[0,1]/A$ is homeomorphic to a subset of $\mathbb{R}^2$.
(b) Is the quotient $[0,1]/B$ homeomorphic to a subset of $\mathbb{R}^2$?
(c) Is the quotient $[0,1]/C$ homeomorphic to a subset of $\mathbb{R}^2$?

Exercise 5.38. Let X be topological space and let A and B be two subsets with $A \cup B = X$. We write $C := A \cap B$. We consider the disjoint union $A \sqcup B = (A \times \{1\}) \cup (B \times \{2\})$ and we consider $Y = (A \sqcup B)/\!\sim$ where for each $c \in C$ we identify the point $(c,1) \in A \times \{1\}$ with the corresponding point $(c,2) \in B \times \{2\}$. We consider the map

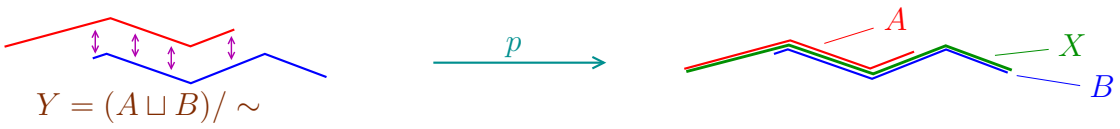
$$p\colon Y = (A \sqcup B)/\!\sim \ \to \ X$$

that is given by

$$\begin{array}{rcll} [(a,1)] & \mapsto & a & \text{for } a \in A, \\ [(b,2)] & \mapsto & b & \text{for } b \in B. \end{array}$$

(a) Is the map p necessarily continuous?
(b) Is the map p necessarily open?
(c) Suppose that A and B are open subsets of X. Show that p is a homeomorphism.

Remark. The fact that p is not necessarily a homeomorphism can lead to nasty surprises.



Exercise 5.39. Let Σ be the surface of genus 4.

(a) Show that Σ admits a free and continuous action by $\mathbb{Z}_2$.
(b) Show that Σ admits a free and continuous action by $\mathbb{Z}_3$.
(c) Show that Σ admits a free and continuous action by $\mathbb{Z}_6$.

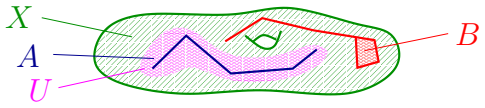
Remark. In Proposition 87.11 we will see that if a finite group G admits a free and continuous action on Σ, then the order of G divides 6.

Exercise 5.40. Give an example of maps $f\colon X \to Y$ and $X \to Z$ between infinite topological spaces such that the pushout $Y \cup_X Z$ consists of a single point.

Exercise 5.41. Let X be a topological space and let $A \subset X$ be a subset. We denote by $\{*\}$ the topological space with a single point. Let $i\colon A \to X$ be the inclusion map and let $p\colon A \to \{*\}$ be the unique map there is. Show that the corresponding pushout $X \cup_A \{*\}$ is homeomorphic to the quotient X/A.
Remark. Note that we also allow the case $A = \varnothing$.

Exercise 5.42. Let X be a topological space and let A and B be two disjoint subsets. We equip A with the subspace topology and we equip X/B with the quotient topology. We denote by $i\colon A \to X$ the inclusion map and we denote by $p\colon X \to X/B$ the projection.

(a) Show that the map $p \circ i\colon A \to X/B$ is continuous and injective.
(b) Show that the map $p \circ i\colon A \to X/B$ is not necessarily an embedding.
(c) Suppose that there exists an open subset $U \subset X$ with $A \subset U$ and $U \cap B = \varnothing$. Show that the map $p \circ i\colon A \to X/B$ is an embedding.



Exercise 5.43. Similar to the definition of the Möbius band on page 252 we consider the action of the group $G = \mathbb{Z}$ on $X = \mathbb{R} \times \{-1, 1\}$ that is given by

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times \{-1,1\}) & \to & \mathbb{R} \times \{-1,1\} \\ (n, (x,y)) & \mapsto & (x+n, (-1)^n \cdot y). \end{array}$$

Show that $X/G = X/\mathbb{Z}$ is homeomorphic to S^1.

Exercise 5.44. We consider the map

$$\begin{array}{rcl} (\mathbb{C} \setminus \{0\}) \times (\mathbb{C}^{n+1} \setminus \{0\}) & \to & \mathbb{C}^{n+1} \setminus \{0\} \\ (z, P) & \mapsto & z \cdot P. \end{array}$$

By definition we have $\mathbb{C}\mathrm{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$.

(1) Show that the obvious map $p\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}\mathrm{P}^n$ is open.
(2) In the Projective Spaces-Compact+Hausdorff Proposition 5.30 we showed that $\mathbb{C}\mathrm{P}^n$ is Hausdorff. In this exercise we want to give a more direct, down-to-earth proof that $\mathbb{C}\mathrm{P}^n$ is Hausdorff. Thus let $[x], [y] \in \mathbb{C}\mathrm{P}^n$ be two disjoint points. Let $\langle -, - \rangle$ be the standard hermitian inner product on $\mathbb{C}^{n+1}$ that we introduced on page 101. We consider

$$U_x = \big\{ z \in \mathbb{C}^{n+1} \setminus \{0\} \,\big|\, |\langle z, y\rangle| < |\langle z, x\rangle| \big\} \;\&\; U_y = \big\{ z \in \mathbb{C}^{n+1} \setminus \{0\} \,\big|\, |\langle z, x\rangle| < |\langle z, y\rangle| \big\}.$$

(a) Show that U_x and U_y are open subsets of $\mathbb{C}^{n+1} \setminus \{0\}$.

(b) Show that $p(U_x)$ and $p(U_y)$ are disjoint. (Note that they are open by (1) and (2a).)
(c) Show that $[x] \in p(U_x)$ and $[y] \in p(U_y)$.
Hint. Use the Cauchy-Schwarz Inequality 0.14.

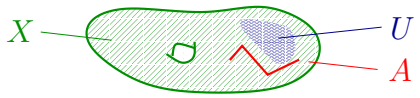
Exercise 5.45. We consider the map

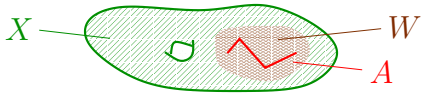
$$\begin{array}{rcl} f\colon \mathbb{R}^2 & \to & \mathbb{R}\mathrm{P}^2 \\ (x,y) & \mapsto & [x:y:1]. \end{array}$$

(a) Show that f is an embedding, i.e. show that f is a homeomorphic onto its image.
(b) What is the closure of $f(\mathbb{Z}^2) \subset \mathbb{R}\mathrm{P}^2$?

Exercise 5.46. Let X be a topological space and let A be a closed subset of X.

(a) Show that for every open subset $U \subset X \setminus A$ the obvious map $U \to X/A$ is an open embedding.
(b) Show that for every open neighborhood W of A the map $W \setminus A \to (W/A) \setminus (A/A)$ is a homeomorphism.
(c) Show that in (a) and (b) we cannot drop the hypothesis that A is closed.

(a) 

(b) 

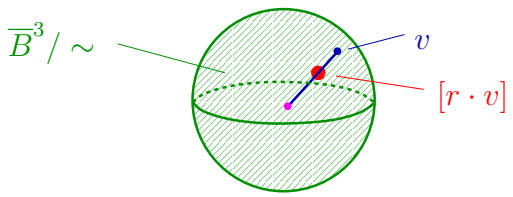
Exercise 5.47. Let $\sim$ be the equivalence relation on $\overline{B}^3$ that is generated by $x \sim -x$ for $x \in S^2$. We consider the map

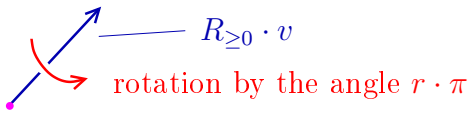
$$\begin{array}{rcl} F\colon \overline{B}^3/\!\sim & \to & \mathrm{SO}(3) \\ [\underbrace{r \cdot v}_{r \in [0,1] \text{ and } v \in S^2}] & \mapsto & \text{matrix in SO(3) which represents rotation by the angle } \pi \cdot r \\ & & \text{around the axis } \mathbb{R}_{\geq 0} \cdot v \text{ where the rotation} \\ & & \text{direction is given by the right-hand rule} \end{array}$$

We refer to the figure below for an illustration.

(a) Give a rigorous description of F and show that F is continuous.
(b) Use your linear algebra skills to show that F is a bijection.
(c) Show that F is a homeomorphism.

Remark. On page 258 we showed that $\overline{B}^3/\!\sim$ is homeomorphic to $\mathbb{R}\mathrm{P}^3$. Thus in combination we have just shown the amusing and surprising fact that $\mathbb{R}\mathrm{P}^3$ is homeomorphic to SO(3). We will give a different proof of this fact later on in the SU(2)-SO(3)-Covering Theorem 83.2.



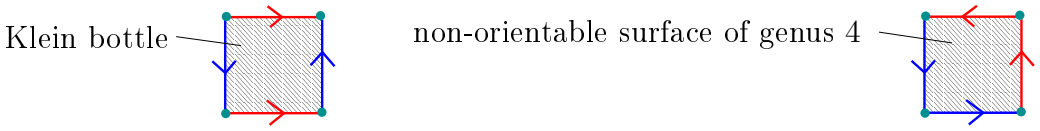


Exercise 5.48. We say a group G *acts discretely* on a topological space X, if it acts continuously and if for each $x \in X$ there exists an open neighborhood U such that we have $U \cap gU$ for all $g \neq e$. Can the group $\mathbb{Z}$ act discretely on S^2?

Exercise 5.49. Let X be a Hausdorff space and let G be a group that acts discretely on X. (See Exercise 5.48 for the notion of a discrete action.) Furthermore let K be a compact subset of X.

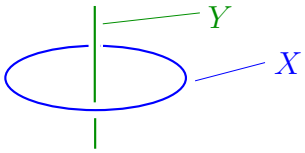
(a) We assume that X is regionally compact. Show that $\{g \in G \,|\, gK \cap K \neq \varnothing\}$ is finite.
(b) Does the conclusion of (a) also hold if we drop the hypothesis that X is regionally compact? This statement is [**Scot1978**, Lemma 1.2].

Exercise 5.50. Show that the Klein bottle, as defined on page 265, is homeomorphic to the non-orientable surface N_2 of genus 2, as defined on page 266.



Exercise 5.51. Show that for any choice of $m, n \in \mathbb{N}_0$ there exists a homeomorphism $\overline{B}^m * \overline{B}^n \xrightarrow{\cong} \overline{B}^{m+n+1}$.

Exercise 5.52. We let $X = S^1$ be the standard circle in the xy-plane of $\mathbb{R}^3$ and we let Y be the circle in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ that is given by the z-axis together with the point $\{\infty\}$. Show that we can "continuously" connect every point on X to every point Y such that the paths fill out S^3. In other words, try to visualize the fact that $S^1 * S^1 \cong X * Y$ is homeomorphic to S^3.

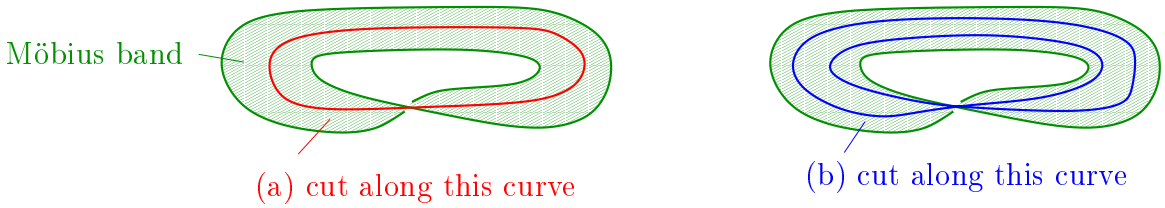


Exercise 5.53. Construct the Möbius band $M = ([0,1] \times [0,1])/(0,y) \sim (1, 1-y)$ by gluing two of the opposite sides of a strip of paper with a twist.

(a) Cut M along the central curve, i.e. start cutting at the point $[(0, \frac{1}{2})]$ and continue along the Möbius band till you reach the starting point. What topological space do you obtain that way?
(b) Start cutting at the point $[(0, \frac{1}{3})]$ and continue along the Möbius band till you reach the starting point. What topological space do you obtain that way?
(c) Let $n \in \mathbb{N}$. Start cutting a $[(0, \frac{1}{n})]$ and continue along the Möbius band till you reach the starting point. What topological space do you obtain that way?

Exercise 5.54. Let G be a topological group. We denote by X the set of subgroups of G that are closed subsets. We consider the following types of subsets of G:

$$\begin{aligned} \text{Miss} &:= \{H \in X \,|\, \text{there exists a compact subset } K \subset G \text{ with } H \cap K = \varnothing\}, \\ \text{Hit} &:= \{H \in X \,|\, \text{there exists an open subset } U \subset G \text{ with } H \cap U \neq \varnothing\}. \end{aligned}$$



The topology on X that is generated by Miss and Hit is called the *Chabauty topology on* X, see [**Chab1950**].

(a) Show that the Chabauty topology is Hausdorff.
(b) Let $G = \mathbb{R}$.
  (i) Determine the closed subgroups of $G = \mathbb{R}$.
  (ii) Show that in this case X, equipped with the Chabauty topology, is homeomorphic to $[0, 1]$.

6. Topology of metric spaces

In this chapter we will discuss several topics that are germane to metric spaces. For the reader's convenience we recall the definition of a metric space from page 103.

Definition. A metric space is a pair (X, d) consisting of a set X and a metric d on X, i.e. a map

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} := \{x \in \mathbb{R} \,|\, x \geq 0\} \\ (x, y) & \mapsto & d(x, y) \end{array}$$

with the following properties:

(1)		$d(x,y)$	$=$	0	$\iff$ $x = y$	
(2)	for all $x, y \in X$ we have	$d(x,y)$	$=$	$d(y,x)$		(symmetry)
(3)	for all $x, y, z \in X$ we have	$d(x,z)$	$\leq$	$d(x,y) + d(y,z)$		(triangle inequality).

Example. The most familiar example of a metric space is given by $X = \mathbb{R}^n$ equipped with the euclidean metric.

As always we equip a metric space (X, d) with the topology that we introduced in Lemma 1.1 and we equip $X \times X$ with the product topology that we introduced on page 224. With these conventions it now makes sense to formulate the following lemma.

Lemma 6.1. (Metric-is-Continuous Lemma) Let (X, d) be a metric space. The metric $d\colon X \times X \to \mathbb{R}_{\geq 0}$ is continuous.

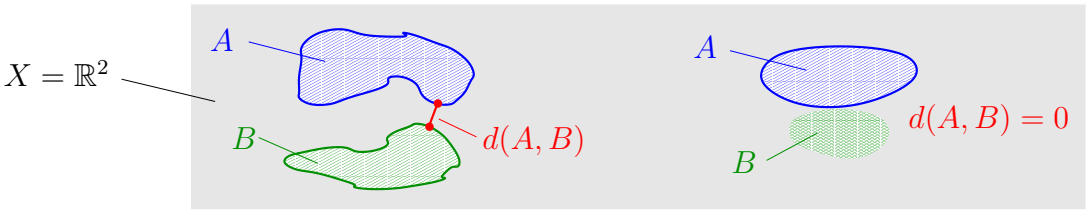
Proof. We prove the lemma in Exercise 6.2. ∎

6.1. Distance in metric spaces.

Definition. Let X be a metric space. Given two non-empty subsets A and B we refer to

$$d(A, B) \;:=\; \inf\{d(a, b) \,|\, a \in A \text{ and } b \in B\}$$

as the distance between A and B. If $A = \{a\}$ consists of a single point, then we write $d(a, B) := d(\{a\}, B)$.



Example. If two subsets A and B intersect, then we evidently have $d(A, B) = 0$. But in general the converse does not hold. For example if we consider $X = \mathbb{R}$, then

$$d\big(\underbrace{\text{open interval } (0,1)}_{=A}, \underbrace{\text{closed interval } [1,2]}_{=B}\big) \;=\; 0,$$

but evidently the intervals $A = (0, 1)$ and $B = [1, 2]$ do not intersect.

The reader might justly suspect that the phenomenon of the last example is due to the fact that one of the subspaces is not compact. In fact the next two lemmas, which will play a useful role time and again, confirms this suspicion.

Lemma 6.2. (Distance-to-Subspace Lemma) Let X be a metric space and let $B \subset X$ be a non-empty subset.

(1) The map
$$\begin{array}{rcl} X & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & d(x,B) := \inf\{d(x,b) \,|\, b \in B\} \end{array}$$
is continuous.

(2) If B is closed, then given $x \in X$ we have
$$d(x,B) = 0 \qquad \Longleftrightarrow \qquad x \in B.$$

Proof. We will prove Statement (1) in Exercise 6.5. Thus we can immediately proceed to the proof of Statement (2). Let B be a closed subset of X. If $x \in B$, then evidently $d(x,B) = 0$. Now suppose that $x \notin B$. Since B is closed we know that $X \setminus B$ is open. This implies that there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subset X \setminus B$. But this implies that $d(x,B) \geq \epsilon$. ■

Lemma 6.3. Let X be a metric space and let K and L be two non-empty disjoint subsets. If one of the two subsets is compact and if the other subset is a closed subset of X, then $d(K,L) > 0$.



Proof. We can assume that K is compact and that L is a closed subset of X. We consider the map
$$\begin{array}{rcl} K & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & d(x,L). \end{array}$$
By the Distance-to-Subspace Lemma 6.2 we know that the map is continuous. Since K is compact we know by the Compact Image Lemma 2.13 (1) that f assumes a minimum, i.e. there exists an $x \in K$ such that for all $y \in K$ we have $d(x,L) \leq d(y,L)$. This implies that

$$d(K,L) \underset{\text{by definition}}{=} \inf\{d(y,L) \,|\, y \in K\} \underset{\text{by choice of } x}{=} d(x,L) \overset{\text{by the Distance-to-Subspace Lemma 6.2 (2), since } L \text{ closed}}{>} 0.$$

■

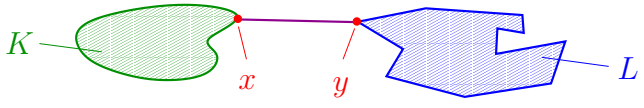
Sometimes it is nice to know that points of minimal distance exist. This leads us to the following lemma.

Lemma 6.4. (Distance-Minimizer Lemma) Let X be a metric space and let K and L be two non-empty subsets. We assume that one of the following holds:

(i) Both K and L are compact, or

(ii) $X = \mathbb{R}^n$, K is compact and L is a closed subset of $X = \mathbb{R}^n$.

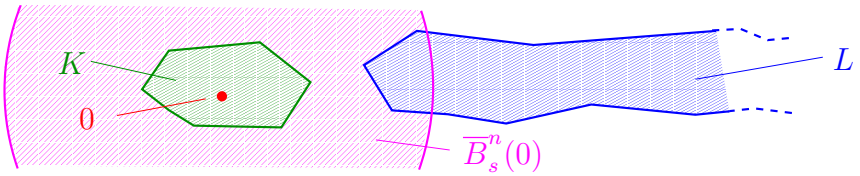
There exist $x \in K$ and $y \in L$ with $d(x,y) = d(K,L)$.



Proof. First we suppose that we are in the setting (i). Thus let K and L be two non-empty compact subsets of X. By the Product Topology Properties Proposition 5.6 we know

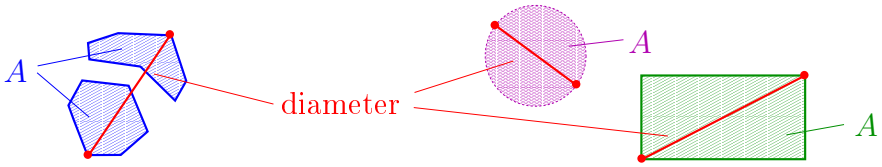
that $K \times L$ is compact. By the Metric-is-Continuous Lemma 6.1 we know that the metric $d\colon K \times L \to \mathbb{R}_{\geq 0}$ is continuous. It follows from the above, together with the fact that $K \times L$ is non-empty and the Compact Image Lemma 2.13 (2) that $d\colon K \times L \to \mathbb{R}_{\geq 0}$ assumes a minimum at some point $(x, y) \in K \times L$. But this means precisely that $d(x, y) = d(K, L)$.

Now suppose that we are in the setting (ii). We will reduce this case to the setting (i). To do so we set $s := \max\{\|x\| \,|\, x \in K\} + 1 + d(K, L)$. Next we set $L' := L \cap \overline{B}^n_s(0)$. Note that it follows from the hypothesis that L is closed and the Heine-Borel Theorem 1.24 together with the Compact-Closed Lemma 1.21 (2) that L' is compact. We leave it to the reader to verify that it follows easily from the choice of s and the triangle inequality that $L' \neq \varnothing$ and that $d(K, L') = d(K, L)$. We apply (i) to $X = \mathbb{R}^n$ together with the compact subsets K and L'. We obtain $x \in K$ and $y \in L' \subset L$ with $d(x, y) = d(K, L') = d(K, L)$. ■



6.2. **The Lebesgue Lemma.** In this section we intend to formulate and prove the Lebesgue Lemma. Despite its humble name it does actually play an important role in topology. The formulation of the Lebesgue Lemma requires the following definition.

Definition. Let X be a metric space. The diameter of a non-empty subset A of X is defined as $\operatorname{diam}(A) := \sup\{d(a, b) \,|\, a, b \in A\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}$.
To avoid any issues later on we also define $\operatorname{diam}(\varnothing) := 0$.



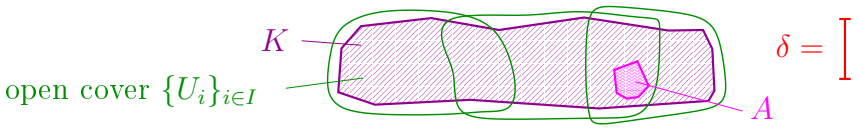
Example.

(1) Let (X, d) be a metric space, let $x \in X$ and $r \in \mathbb{R}_{\geq 0}$. Using the triangle inequality one can easily show that $\operatorname{diam}(\overline{B}^d_r(x)) = \operatorname{diam}(B^d_r(x)) \leq 2r$. Furthermore, if we consider balls in some $\mathbb{R}^n$, then it is straightforward to see that the above inequality is in fact an equality.

(2) We consider the metric space $(\mathbb{R}^n, d_{\text{Eucl}})$. We leave it to the reader to show that

$$\operatorname{diam}\big(\text{hyperrectangle } [a_1, b_1] \times \cdots \times [a_n, b_n]\big) = \sqrt{(b_1 - a_1)^2 + \cdots + (b_n - a_n)^2}.$$

Lemma 6.5. (Lebesgue Lemma) Let K be a compact metric space and let $\{U_i\}_{i \in I}$ be an open cover of K. There exists a $\delta > 0$ such that for every subset A of K with $\operatorname{diam}(A) < \delta$ there exists an $i \in I$ with $A \subset U_i$.



Remark. Any $\delta > 0$ that has the property stated in the Lebesgue Lemma is usually called a *Lebesgue number*.

Proof. Since K is compact we can cover K with finitely many of the U_i's. Put differently, without loss of generality we can assume that $I = \{1, \dots, n\}$ is a finite set.

If there exists an $i \in \{1, \dots, n\}$ with $K = U_i$, then any $\delta > 0$ has the desired property. Now suppose that this is not the case, i.e. we suppose that for every i we have $U_i \subsetneq K$.

Claim 1. The map

$$\begin{array}{rcl} f\colon K & \to & \mathbb{R} \\ x & \mapsto & f(x) := \frac{1}{n} \cdot \sum_{i=1}^{n} d(x, \underbrace{K \setminus U_i}_{\neq \varnothing}) \end{array}$$

is continuous and positive for every $x \in K$.

Proof. By the Distance-to-Subspace Lemma 6.2 (1) we know that f is continuous. Now let $x \in K$. We want to show that $f(x) > 0$. Since $K = U_1 \cup \dots \cup U_n$ there exists an i with $x \in U_i$. Since U_i is open and since $x \in U_i$ we obtain from the Distance-to-Subspace Lemma 6.2 (2) that $d(x, K \setminus U_i) > 0$. Evidently this implies that $f(x) > 0$. ⊞

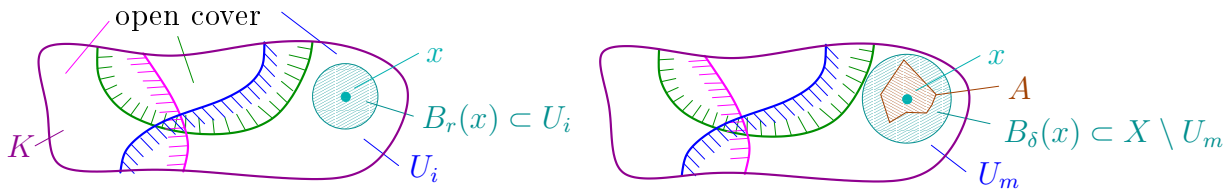
Since f is continuous and since K is compact it follows from the Compact Image Lemma 2.13 that the map f has a global minimum δ on K. It follows from Claim 1 that this minimum δ is greater than 0. Now we want to show that this δ has the desired property. More precisely, we want to prove the following claim.

Claim 2. Let A be a subset of K. If $\operatorname{diam}(A) < \delta$, then there exists an $i \in \{1, \dots, n\}$ with $A \subset U_i$.

Proof. If $A = \varnothing$, then there is nothing to show. If $A \neq \varnothing$, then we pick $x \in A$. We choose an $m \in \{1, \dots, n\}$ such that $d(x, K \setminus U_m)$ is maximal. We want to show that $A \subset U_m$. Since the diameter of A is less than δ we have $A \subset B_\delta(x)$. Thus it suffices to show that $B_\delta(x) \subset U_m$. Put differently, we need to show that $d(x, K \setminus U_m) \geq \delta$. Indeed we have

$$d(x, K \setminus U_m) \underset{\substack{\uparrow \\ \text{since } d(x,K\setminus U_m) \geq d(x,K\setminus U_i)}}{\geq} \frac{1}{n} \cdot \sum_{i=1}^{n} d(x, K \setminus U_i) = f(x) \underset{\substack{\uparrow \\ \text{by the choice of } \delta}}{\geq} \delta.$$

■

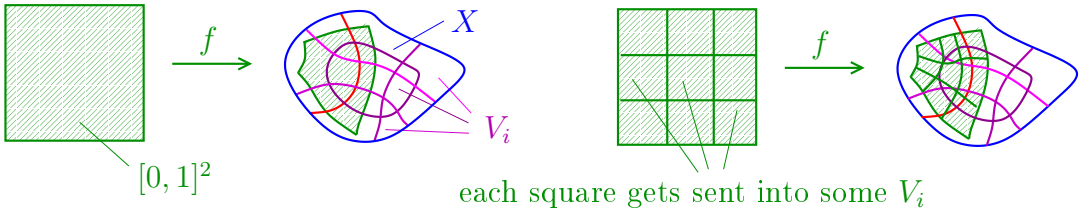


In many cases the following corollary is even more useful.

Corollary 6.6. (Lebesgue Corollary) Let $f\colon [0,1]^n \to X$ be a map from the cube $[0,1]^n$ to a topological space X and let $\{V_i\}_{i\in I}$ be an open cover of X. Then there exists an $N > 0$ such that for any $a_1, \ldots, a_n \in \{0, \ldots, N-1\}$ there exists an $i \in I$ such that

$$f\big(\big[\tfrac{a_1}{N}, \tfrac{a_1+1}{N}\big] \times \cdots \times \big[\tfrac{a_n}{N}, \tfrac{a_n+1}{N}\big]\big) \subset V_i.$$

More pictorially speaking the corollary says that if $f\colon [0,1]^n \to X$ is a map, and if we are given an open cover of X, then we can always find a small grid on the cube, such that each cube of the grid gets mapped into one of the open sets cover X.



Proof. Let $f\colon [0,1]^n \to X$ be a map from the cube $[0,1]^n$ to a topological space X and let $\{V_i\}_{i\in I}$ be an open cover of X. By the Hyperrectangle-Compact Proposition 1.26 we know that $[0,1]^n$ is compact. Thus we can apply Lemma 6.5 to the open cover $U_i := f^{-1}(V_i)$, $i \in I$ of $[0,1]^n$ and we obtain a $\delta > 0$ such that for each subset A of $[0,1]^n$ with $\operatorname{diam}(A) < \delta$ there exists an $i \in I$ with $A \subset U_i = f^{-1}(V_i)$, which means that $f(A) \subset V_i$. It follows from the example on page 285 that the diameter of the cube of side length $\frac{1}{m}$ equals $\frac{\sqrt{n}}{m}$. Thus we see that for any $N \in \mathbb{N}_0$ with $N \geq \frac{\sqrt{n}}{\delta}$ any cube of side length $\frac{1}{N}$ has diameter less than δ. ∎

6.3. **Uniformly continuous maps.** In this section we introduce the concept of uniformly continuous maps that the reader might be familiar with from a course on real analysis. We will show that a continuous map $X \to Y$ between metric spaces is actually uniformly continuous. This is a rather technical result which we will make use of only once, namely in Section 70.6 when we discuss "CW-complexes". Most readers will be best served by passing on to the next section.

Definition. Let $f\colon X \to Y$ be a map between two metric spaces X and Y. By the Metric Continuity Proposition 2.8 we know that

$$f \text{ is continuous} \iff \underset{x\in X}{\forall}\ \underset{\epsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{y\in B_\delta(x)}{\forall}\ d(f(x),f(y)) < \epsilon.$$

Now we define

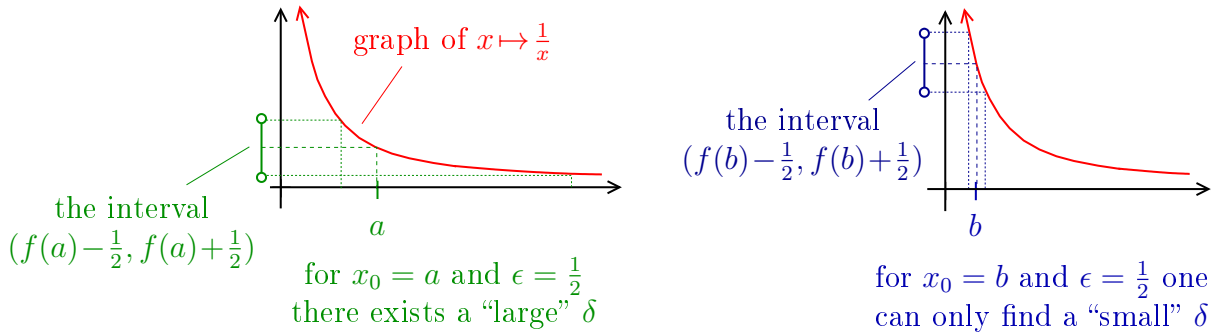
$$f \text{ is uniformly continuous} :\iff \underset{\epsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in X}{\forall}\ \underset{y\in B_\delta(x)}{\forall}\ d(f(x),f(y)) < \epsilon.$$

Example. The map

$$\begin{array}{rcl} f\colon (0,\infty) & \to & \mathbb{R} \\ x & \mapsto & \frac{1}{x} \end{array}$$

is not uniformly continuous. For example for $\epsilon = \frac{1}{2}$ there is no "universal" δ that works for all $x \in (0,\infty)$.

Proposition 6.7. Let $f\colon X \to Y$ be a continuous map between two metric spaces. If X is compact, then f is uniformly continuous.



Proof. Let $f\colon X \to Y$ be a continuous map between two metric spaces and let $\epsilon > 0$. The "$\Leftarrow$"-direction of the Metric Continuity Proposition 2.8 tells us in particular that given any $P \in X$ there exists a $\delta(P) > 0$ such that for all $y \in B_{\delta(P)}(P)$ we have $d(f(P), f(y)) < \frac{\epsilon}{2}$. Since the open balls $B_{\delta(P)/2}(x)$ form an open cover of X and since X is compact there exist $P_1, \ldots, P_k \in X$ such that $B_{\delta(P_1)/2}(P_1) \cup \cdots \cup B_{\delta(P_1)/2}(P_1) = X$. We claim that

$$\delta \;:=\; \min\left\{\frac{\delta(P_1)}{2}, \ldots, \frac{\delta(P_k)}{2}\right\}$$

has the desired properties. Thus let $x, y \in X$ with $d(x,y) < \delta$. There exists a $j \in \{1, \ldots, k\}$ with $x \in B_{\delta(P_j)/2}(P_j)$. From the triangle inequality and from $d(x,y) < \delta \le \frac{\delta(P_j)}{2}$ we obtain that $y \in B_{\delta(P_j)}(P_j)$. It follows that

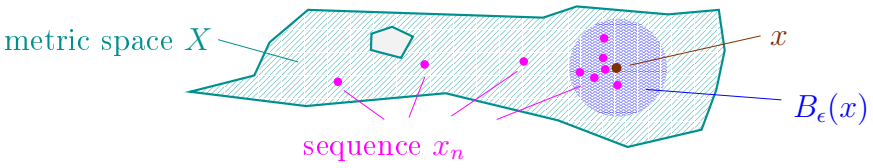
$$d(f(x), f(y)) \underset{\text{triangle inequality}}{\overset{\uparrow}{\le}} d(f(P_j), f(x)) + d(f(P_j), f(y)) \underset{\text{since } x \in B_{\delta(P_j)}(P_j) \text{ and } y \in B_{\delta(P_j)}(P_j)}{\overset{\uparrow}{<}} \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon.$$

We have thus shown that δ does indeed have the desired properties. ∎

6.4. **Complete metric spaces.** On page 124 we defined for any topological space X what it means for a sequence of points in X to converge to some other point. In metric spaces the following lemma gives an alternative characterization of convergent sequences.

Lemma 6.8. Let (X,d) be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. For each $x \in X$ we have

$$\begin{array}{c}x \text{ is a limit of the sequence } (x_n)_{n\in\mathbb{N}} \\ \text{according to the definition on page 124}\end{array} \quad\Longleftrightarrow\quad \underset{\epsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{n\ge N}{\forall}\ \underbrace{d(x_n, x) < \epsilon.}_{\text{i.e. } x_n \in B_\epsilon(x)}$$



Proof. We need to show

$$\underset{\text{neighborhoods } U \text{ of } x}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{n\ge N}{\forall}\ x_n \in U \quad\Longleftrightarrow\quad \underset{\epsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{n\ge N}{\forall}\ \underbrace{d(x_n, x) < \epsilon.}_{\text{i.e. } x_n \in B_\epsilon(x)}$$

Evidently that is just a dull routine exercise which is best left to the reader's office mate. ∎

Definition. By Proposition 1.13 we know that metric spaces are Hausdorff. It follows from the Unique Limit Lemma 1.17 that if a sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space X converges, then its limit is unique. Thus it makes sense to denote its limit by $\lim_{n\to\infty} x_n \in X$.

Definition. Let X be a metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is called Cauchy sequence if

$$\forall_{\epsilon>0}\ \exists_{N\in\mathbb{N}}\ \forall_{m,n\geq N}\ d(x_n,x_m)<\epsilon.$$

One can easily show that every convergent sequence is a Cauchy sequence. In general the converse does of course not hold though. For example, if we consider the metric space $(\mathbb{R}\setminus\{0\}, d(x,y)=|x-y|)$, then we see that the sequence $x_n=\frac{1}{n}$ is a Cauchy sequence, but it does not converge in the metric space $\mathbb{R}\setminus\{0\}$.

Definition. We say that a metric space (X,d) is complete if every Cauchy sequence in (X,d) converges. Otherwise we say that (X,d) is incomplete.

Examples.

(1) It is one of the defining properties of the real numbers that the metric space given by $\mathbb{R}$ and the usual metric $d(x,y)=|x-y|$ is complete.
(2) It follows without too much sweat from (1) that the metric space given by $\mathbb{R}^n$ and the Euclidean metric is complete.

Lemma 6.9. (Closed Subspace-Completeness Lemma) Let (X,d) be a complete metric space. If $A\subset X$ is a closed subset, then (A,d) is also complete.

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (A,d). Since (X,d) is complete we know that $(a_n)_{n\in\mathbb{N}}$ converges to some point $x\in X$. But since A is closed we obtain from the Closed Subset Limit Lemma 1.18 that $x\in A$. ■

Proposition 6.10. (Compact-Complete Proposition) Every compact metric space is complete.

Proof. Let (X,d) be a compact metric space and let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (X,d). We want to show that the sequence converges, i.e. we want to show that

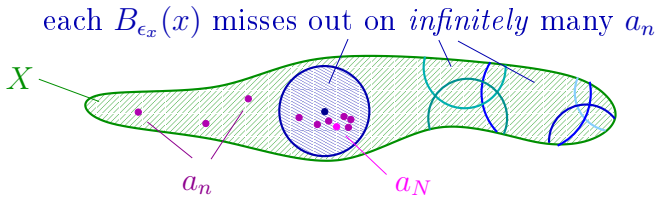
$$\exists_{x\in X}\ \forall_{\epsilon>0}\ \exists_{N\in\mathbb{N}}\ \forall_{n\geq N}\ a_n\in B_\epsilon(x).$$

Let us assume that no such x exists. Thus we negate the above statement and we obtain that

$$\forall_{x\in X}\ \exists_{\epsilon_x>0}\ \forall_{N\in\mathbb{N}}\ \exists_{n\geq N}\ a_n\notin B_{\epsilon_x}(x).$$

Evidently the family $\{B_{\epsilon_x}(x)\}_{x\in X}$ of subsets is an open cover of X. Since X is compact it follows from the Lebesgue Lemma 6.5 that there exists a $\mu>0$ such that any subset A of X of diameter less than μ is contained in some $B_{\epsilon_x}(x)$.

Since $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence there exists an $N\in\mathbb{N}$ such that for all $m,n\geq N$ we have $d(a_m,a_n)<\frac{\mu}{4}$. This means in particular that for all $m\geq N$ we have $d(a_N,a_m)<\frac{\mu}{4}$. It follows from the triangle inequality that the diameter of the set $A:=\{a_n\,|\,n\geq N\}$ is $\leq\frac{\mu}{2}<\mu$. By the above there exists an $x\in X$ such that $A\subset B_{\epsilon_x}(x)$. But this means that for all $n\geq N$ we have $a_n\in B_{\epsilon_x}(x)$. But this contradicts the defining property of ϵ_x. ■



The following lemma shows that one can replace a metric by a bounded metric without affecting the topology.

Lemma 6.11. Let (M,d) be a metric space.

(1) The map

$$\begin{array}{rcl} \widetilde{d}\colon M\times M & \to & \mathbb{R}_{\geq 0} \\ (x,y) & \mapsto & \widetilde{d}(x,y) := \min\{1, d(x,y)\} \end{array}$$

also defines a metric on M.

(2) This new metric $\widetilde{d}$ induces the same topology as d on M.

(3) (M,d) is complete if and only if $(M,\widetilde{d})$ is complete.

Proof. All three statements are elementary and they follow easily from the definitions. We refrain from boring the reader with details. ∎

Notation. Let (M,d) be a metric space and let X be a topological space. We denote by $C(X,M)$ the set of continuous maps from X to M.

Proposition 6.12. Let (M,d) be a metric space and let X be a non-empty topological space. The map

$$\begin{array}{rcl} \widetilde{d}\colon C(X,M)\times C(X,M) & \to & \mathbb{R}_{\geq 0} \\ (f,g) & \mapsto & \widetilde{d}(f,g) := \sup \underbrace{\left\{\widetilde{d}(f(x),g(x)) \,|\, x\in X\right\}}_{\text{non-empty and bounded}} \end{array}$$

has the following properties:

(1) The map $\widetilde{d}$ defines a metric on $C(X,M)$.

(2) If X is compact, then the map

$$\begin{array}{rcl} d\colon C(X,M)\times C(X,M) & \to & \mathbb{R}_{\geq 0} \\ (f,g) & \mapsto & d(f,g) := \sup \underbrace{\left\{d(f(x),g(x)) \,|\, x\in X\right\}}_{\substack{\text{non-empty and bounded} \\ \text{by the Compact Image Lemma 2.13}}} \end{array}$$

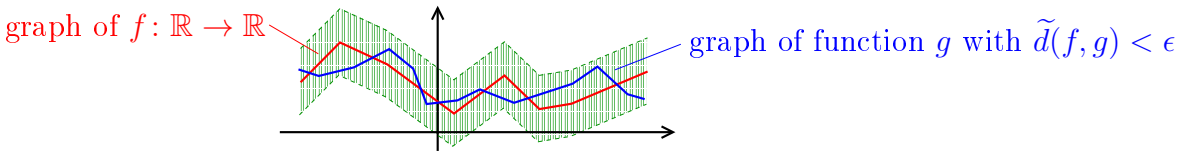
is also a metric on $C(X,M)$ which is equivalent, in the sense of the definition on page 109 to the above metric $\widetilde{d}$

(3) If (M,d) is complete, then given any Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $(C(X,M),\widetilde{d})$ the map

$$\begin{array}{rcl} f\colon X & \to & M \\ x & \mapsto & \lim\limits_{n\to\infty} f_n(x) \end{array}$$

exists (i.e. each limit exists), it is a continuous map (i.e. it defines an element of $C(X,M)$) and in the metric space $(C(X,M),\widetilde{d})$ it is the limit of the sequence $(f_n)_{n\in\mathbb{N}}$.

(4) If (M,d) is complete, then so is $(C(X,M),\widetilde{d})$.



Remark. The proposition might be familiar from a real analysis course, at least for the case that $X\subset\mathbb{R}^n$ and $M=\mathbb{R}$. The proof of the general case does not differ one iota from the special case.

Proof. The first statement follows easily from the definitions and the properties of the supremum. We leave it to the reader to fill in the details. The second statement follows easily from the definitions and the Equivalent Metric Lemma 1.2. Furthermore the last statement is contained in the third statement.

Thus it remains to prove the third statement of the proposition. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(C(X,M),\widetilde{d})$. The following three claims imply the third statement of the proposition.

Claim 1. For each $x\in X$ the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges.

Proof. Let $x\in X$. Note that we have $\widetilde{d}(f_n(x),f_m(x))\leq\widetilde{d}(f_n,f_m)$. Thus we see that $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in the metric space $(M,\widetilde{d})$. By our hypothesis (M,d) is complete. By Lemma 6.11 we know that $(M,\widetilde{d})$ is also complete, thus the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges in $(M,\widetilde{d})$ to a point which we call $f(x)$. ⊞

We can now consider the map

$$\begin{array}{rcl} f\colon X & \to & M \\ x & \mapsto & \lim\limits_{n\to\infty} f_n(x). \end{array}$$

Claim 2. $$\forall_{\epsilon>0}\ \exists_{N\in\mathbb{N}}\ \forall_{n\geq N}\ \forall_{x\in X}\ \widetilde{d}(f_n(x),f(x))<\epsilon.$$

Proof. Let $\epsilon>0$. Since $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(C(X,M),\widetilde{d})$ we know that there exists an $N\in\mathbb{N}$ such that for each $m,n\geq N$ we have $\widetilde{d}(f_n,f_m)<\frac{\epsilon}{2}$. Now let $n\geq N$ and let $x\in X$. We see that

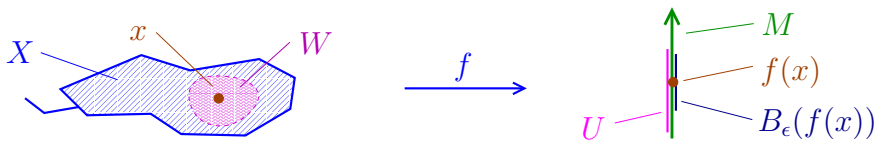
$$\widetilde{d}(f_n(x),f(x)) \;=\; \widetilde{d}\Big(f_n(x),\lim_{m\to\infty}f(x_m)\Big) \;\overset{\text{by continuity of metrics, see the Metric-is-Continuous Lemma 6.1, and Exercise 2.26}}{=}\; \lim_{m\to\infty}\underbrace{\widetilde{d}(f_n(x),f_m(x))}_{\leq\widetilde{d}(f_n,f_m)<\frac{\epsilon}{2}} \;\leq\; \frac{\epsilon}{2} \;<\; \epsilon.$$

⊞

Claim 3. The map $f\colon X\to M$ is continuous.

Proof. Let $U\subset M$ be an open subset. We need to show that $f^{-1}(U)$ is open. By the JH-Lemma 1.7 it suffices to show that for every $x\in f^{-1}(U)$ there exists an open neighborhood W of $x\in X$ with $f(W)\subset U$. Thus let $x\in f^{-1}(U)$. Since $U\subset M$ is open there exists an $\epsilon>0$ such that $B_\epsilon(f(x))\subset U$.

Next note that by Claim 2 and by definition of $\widetilde{d}$ there exists an $n\in\mathbb{N}$ such that for every $x\in X$ we have $d(f_n(x),f(x))<\frac{\epsilon}{3}$. Furthermore, by hypothesis $f_n\colon X\to M$ is continuous,



which implies that there exists an open neighborhood W of x with $f_n(W) \subset B_{\frac{\epsilon}{3}}(f_n(x))$. Now we see that for each $y \in W$ we have

$$d(f(x), f(y)) \leq \underbrace{d(f(x), f_n(x))}_{\leq \frac{\epsilon}{3} \text{ by choice of } n} + \underbrace{d(f_n(x), f_n(y))}_{\leq \frac{\epsilon}{3} \text{ by choice of } W} + \underbrace{d(f_n(y), f(y))}_{\leq \frac{\epsilon}{3} \text{ by choice of } n} < \epsilon.$$

But this means precisely that $f(W) \subset B_\epsilon(f(x)) \subset U$. ■

6.5. **Space-filling curves.** The goal of this section is to prove the following, possibly rather counterintuitive theorem.

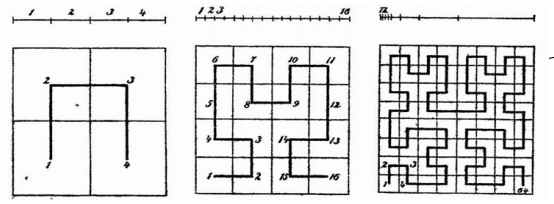
Theorem 6.13. (Space-Filling Curve Theorem) For every $m \in \mathbb{N}$ there exists a map[93] $f\colon [0,1] \to [0,1]^m$ that is surjective.

Definition. Let $m \geq 2$. A surjective map from $[0,1]$ to $[0,1]^m$ is called a space-filling curve.[94]

Remark.

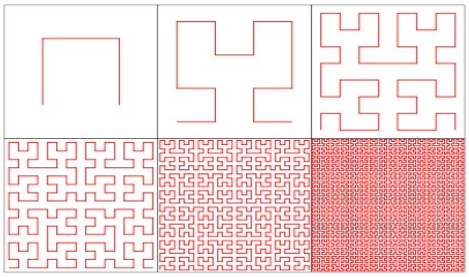
(1) The first example of a space-filling curve was provided by Giuseppe Peano [**Pean1890**] in 1890. Shortly afterwards David Hilbert [**Hilb1891**] gave the first "pictorial" description of a space-filling curve, see the figure below. Nowadays there are many many constructions of space-filling curves. In fact two such constructions are given in Exercises 5.24 and 5.24. Further constructions are considered in [**Sag1994**, Theorem 2.1] and [**Bad2013**].

(2) In Exercise 6.13 we will use the Space-Filling Curve Theorem 6.13 to show that for every $n \in \mathbb{N}$ there exists a surjective map $\mathbb{R} \to \mathbb{R}^n$.



first stages in the construction
of the Hilbert space-filling curve

– [**Hilb1891**]



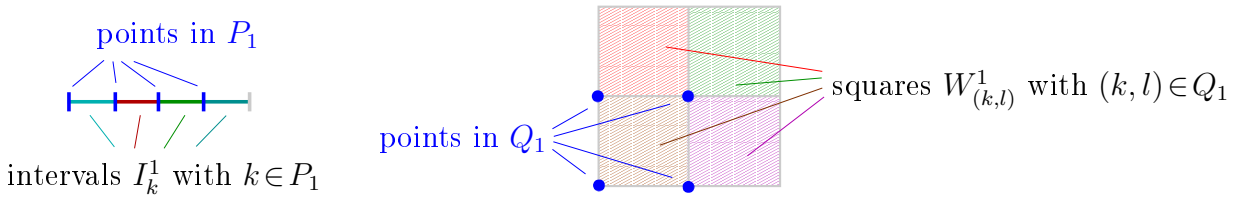
Proof of Theorem 6.13 for $m = 2$**.** For $n \in \mathbb{N}$ we set

$P_n := \{0, \frac{1}{4^n}, \ldots, \frac{4^n-1}{4^n}\}$ and for $k \in P_n$ we set $I_k^n := [k, k + \frac{1}{4^n}]$,

$Q_n := \{0, \frac{1}{2^n}, .., \frac{2^n-1}{2^n}\} \times \{0, \frac{1}{2^n}, .., \frac{2^n-1}{2^n}\}$ and for $(k,l) \in Q_n$ we set $W_{(k,l)}^n := [k, k+\frac{1}{2^n}] \times [l, l+\frac{1}{2^n}]$.

[93] Just to make things clear, here we demand of course as always that f is continuous.

[94] The same name is also used for obvious variations on this concept.

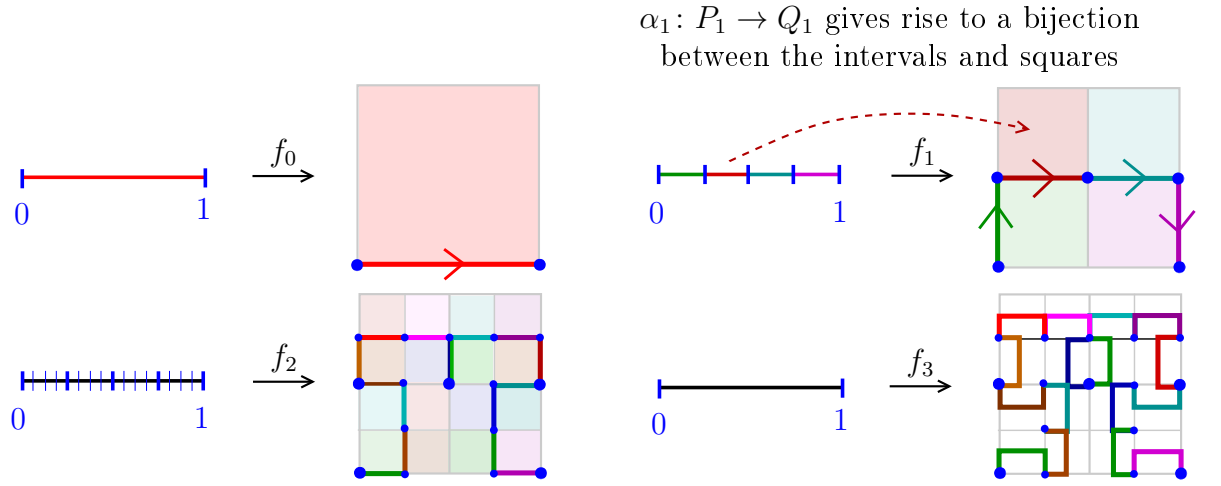


Claim 1. For $n = 0, 1, 2, \ldots$ we can find a bijection $\alpha_n \colon P_n \to Q_n$ and a (continuous) map $f_n \colon [0,1] \to [0,1] \times [0,1]$ such that the following conditions are satisfied:

(a) For each $n \in \mathbb{N}_0$ and each $k \in P_n$ we have $f_n(I_k^n) \subset W_{\alpha_n(k)}^n$.

(b) For each $n \in \mathbb{N}_0$ and each $k \in P_n$ the two endpoints of the interval I_k^n get sent to adjacent vertices of the square $W_{\alpha_n(k)}^n$.

(c) For each $n \geq m \in \mathbb{N}$ and each $k \in P_m$ and $l \in P_l$ with $I_k^n \subset I_l^m$ we have the inclusion $f_n(I_k^n) \subset W_{\alpha_m(l)}^m$.[95]



Proof. We define the maps iteratively.

- For $n = 0$ we have $P_1 = \{0\}$ and $Q_n = \{(0,0)\}$, thus we let $\alpha_0 \colon P_1 \to Q_0$ be the unique bijection. Furthermore we set

$$\begin{array}{rcl} f_1 \colon I_1^0 = [0,1] & \to & W_{(1,1)}^0 = [0,1] \times [0,1] \\ t & \mapsto & (t,0). \end{array}$$

  It is clear that (a) and (b) are satisfied.

- Now suppose that we have constructed bijections $\alpha_0, \ldots, \alpha_n$ and maps $f_0, \ldots, f_n$ with the desired properties. Next we need to construct the bijection $\alpha_{n+1} \colon P_{n+1} \to Q_{n+1}$ and the map $f_{n+1} \colon [0,1] \to [0,1] \times [0,1]$.

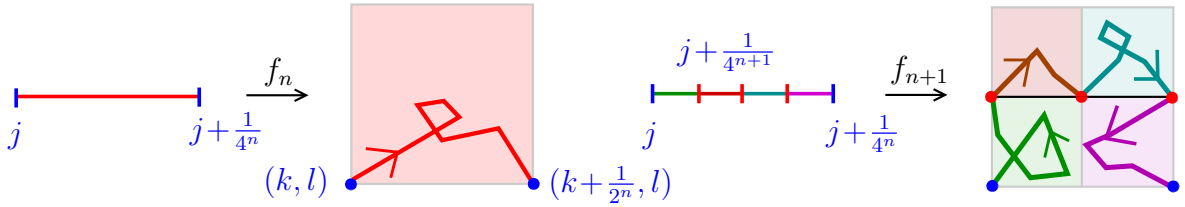
  Our first goal is, given $j \in P_n$, to define f_{n+1} on $I_j^n = [j, j + \frac{1}{4^n}]$ and to define α_{n+1} on $j, j + \frac{1}{4^{n+1}}, j + \frac{2}{4^{n+1}}, j + \frac{3}{4^{n+1}} \in P_{n+1}$. We write $\alpha_n(j) =: (k,l) \in Q_n$. It follows easily from (b) that we can find a bijection

$$\alpha_{n+1} \colon \{j, j+\tfrac{1}{4^{n+1}}, j+\tfrac{2}{4^{n+1}}, j+\tfrac{3}{4^{n+1}}\} \to \{(k,l), (k+\tfrac{1}{2^{n+1}}, l), (k, l+\tfrac{1}{2^{n+1}}), (k+\tfrac{1}{2^{n+1}}, l+\tfrac{1}{2^{n+1}})\}$$

[95] Note that Condition (a) is the special case of (c) with $n = m$ and $l = k$. But It seems to us that it is easier to first digest (a) and then to try to digest (c).

and a map $f_{n+1}\colon I_j^n \to W_k^n$ with the following three properties:
(1) The map $f_{n+1}\colon I_j^n \to W_k^n$ agrees with the map f_n on the endpoints.
(2) Each interval $I^{n+1}_{j+\frac{i}{4^{n+1}}}$ gets sent to the square $W^{n+1}_{\alpha_{n+1}(j+\frac{i}{4^{n+1}})}$.
(3) The endpoints of each interval $I^{n+1}_{j+\frac{i}{4^{n+1}}}$ get sent to adjacent vertices of the square $W^{n+1}_{\alpha_{n+1}(j+\frac{i}{4^{n+1}})}$.

This can be done in a systematic way[96] (as is shown in the figure above) but does not have to be so (as is shown in the figure below). We will not clutter the pages with writing down explicit maps. These maps $f_{n+1}\colon I_j^n \to [0,1]\times[0,1]$ agree on the endpoints.



f_n sends endpoints to adjacent vertices f_{n+1} sends endpoints to adjacent vertices

Therefore it follows from the Pasting Proposition 2.6 that they define a continuous map $f_{n+1}\colon [0,1] \to [0,1]\times[0,1]$. Furthermore the bijections α_{n+1} combined give a bijection $\alpha_{n+1}\colon P_{n+1} \to Q_{n+1}$. We leave it to the reader to verify that f_{n+1} and α_{n+1} have the desired properties. ⊞

As above we consider the set $C([0,1],[0,1]\times[0,1])$ of continuous maps from $[0,1]$ to $[0,1]\times[0,1]$. We view this as a metric space with respect to the metric

$$d(f,g) \;:=\; \sup\{\|f(t)-g(t)\|_{\text{Eucl}} \,|\, t\in[0,1]\}$$

that we introduced in Proposition 6.12 (2).

Claim 2. The sequence $(f_n)_{n\in\mathbb{N}_0}$ of maps $[0,1]\to[0,1]\times[0,1]$ is a Cauchy sequence in the metric space $C([0,1],[0,1]\times[0,1])$.

Proof. Let $\epsilon>0$. We pick $N\in\mathbb{N}_0$ with $\frac{1}{2^N}<\frac{\epsilon}{2}$. Let $m,n\geq N$. For any $t\in[0,1]$ we have

$$\|f_m(t)-f_n(t)\|_{\text{Eucl}} \;\underset{\uparrow}{\leq}\; \text{diameter of square of side length } \tfrac{1}{2^N} \;\underset{\uparrow}{=}\; \sqrt{2}\cdot\tfrac{1}{2^N} \;\underset{\uparrow}{<}\; \tfrac{\epsilon}{\sqrt{2}}.$$

it follows from Property (c) that for $m,n\geq N$ and $t\in[0,1]$ $f_m(t), f_n(t)$ are contained in a square of side length $\frac{1}{2^N}$ — see page 285 — choice of N

This implies that for $m,n\geq N$ we have $d(f_m,f_n)\leq\frac{\epsilon}{\sqrt{2}}<\epsilon$. ⊞
It follows from Claim 2 together with Proposition 6.12 that the resulting map

$$\begin{array}{rcl} f\colon [0,1] & \to & [0,1]\times[0,1] \\ t & \mapsto & f(t)=\lim\limits_{n\to\infty} f_n(t) \end{array}$$

is continuous.[97]

[96] In fact one can easily see that α_{n+1} is uniquely determined. A canonical way of defining f_{n+1} is now given by connecting the endpoints via a line segment.
[97] Note though that f is nowhere differentiable.

Claim 3. The image of the map $f\colon [0,1] \to [0,1] \times [0,1]$ is dense.

Proof. Let U be an open subset of $[0,1] \times [0,1]$. We need to show that $f([0,1]) \cap U \neq \varnothing$. It follows from an elementary argument, similar to the discussion on page 228, that there exists an $N \in \mathbb{N}_0$ and a $(k,l) \in Q_N$ such that $W^N_{(k,l)} \subset U$. By Property (a) there exists an $x \in [0,1]$ with $f_N(x) \in W^N_{(k,l)}$. By Property (c) we know that for each $n \geq N$ we have $f_n(x) \in W^N_{(k,l)}$. Since $W^N_{(k,l)}$ is closed we obtain from the Closed Subset Limit Lemma 1.18 that $f(x) = \lim\limits_{n\to\infty} f_n(x) \in W^N_{(k,l)} \subset U$. We have thus shown that $f([0,1]) \cap U \neq \varnothing$. ⊞

Claim 4. The map $f\colon [0,1] \to [0,1] \times [0,1]$ is surjective.

Proof. Since $[0,1]$ is compact and since $[0,1]\times[0,1]$ is Hausdorff we obtain from the Compact Image Lemma 2.13 together with the Compact-Closed Lemma 1.21 (2) that $f([0,1])$ is a closed subset of $[0,1] \times [0,1]$. By Claim 3 we know that $f([0,1])$ is dense in $[0,1] \times [0,1]$. It follows immediately from the definition of a dense subset that the only closed subset that is dense is the total space. Thus we have shown that $f([0,1]) = [0,1] \times [0,1]$. ■

Proof of Theorem 6.13 for $m \geq 2$. We need to show that for each $n \geq 2$ there exists a surjective map $[0,1] \to [0,1]^m$. We prove this statement by induction on m. By the above we know that there exists a surjective map $f\colon [0,1] \to [0,1]^2$. Now suppose we have found a surjective map $\varphi\colon [0,1] \to [0,1]^m$ for some $m \geq 2$. The map[98]

$$[0,1] \xrightarrow{f} [0,1] \times [0,1] \xrightarrow{\mathrm{id}\times\varphi} [0,1] \times [0,1]^m = [0,1]^{m+1}$$

is evidently surjective. ■

Let $m \geq 2$. Recall that in Lemma 2.26 we showed that $[0,1]$ and $[0,1]^m$ are not homeomorphic. By the Compact-Hausdorff Proposition 2.17 (3) this implies that there is no map $[0,1] \to [0,1]^m$ that is surjective and injective. Nonetheless, the following variation on the Space-Filling Curve Theorem 6.13 holds:

Proposition 6.14. There exists an *injective* map $f\colon [0,1] \to [0,1]^2$ such that $f([0,1])$ has non-zero 2-dimensional Lebesgue measure.

Remark. An injective map $f\colon S^1 \to [0,1]^2$ such that $f([0,1])$ has non-zero 2-dimensional Lebesgue measure is called an *Osgood curve*.

Proof. In 1903 William Osgood [**Osg1903**] and Henri Lebesgue [**Leb1903**] independently showed that there exist injective maps $f\colon [0,1] \to [0,1]^2$ such that $f([0,1])$ has non-zero 2-dimensional Lebesgue measure. Many more examples of such maps are given in [**Sag1994**, Chapter 8.2]. In Exercise 6.14 we will see how this fact can be used to find injective maps $g\colon S^1 \to [0,1]^2$ such that $g(S^1)$ has non-zero 2-dimensional Lebesgue measure. ■

6.6. **The Baire Category Theorem.** The formulation of the Baire Category Theorem requires the following basic definition.

[98] Note that it follows from THE Product Topology-Basics Lemma 5.4 (3) that the middle map is indeed continuous.

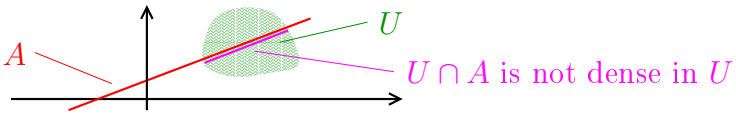
Definition. Let X be a topological space and let $A \subset X$.

(1) As on page 120 we say A is dense in X if every open non-empty subset of X has a non-empty intersection with A.

(2) We say A is nowhere dense in X if there is no open non-empty subset U of X such that $U \cap A$ is dense in U.

Examples.

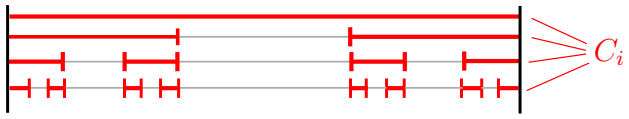
(1) Let $A \subset \mathbb{R}^n$ be an affine hyperplane of codimension ≥ 1. In Exercise 6.16 we will see that A is a nowhere dense subset of $\mathbb{R}^n$.



(2) In the following we recall the definition of the Cantor set from page 130. We set $C_0 := [0,1]$ and iteratively we define sets C_i, $i \in \mathbb{N}$ as follows:

$$C_i \;:=\; \tfrac{1}{3} \cdot C_{i-1} \cup \left(\tfrac{2}{3} + \tfrac{1}{3} \cdot C_{i-1}\right).$$

The Cantor set is defined as the intersection $C := \bigcap_{i \in \mathbb{N}_0} C_i$. Note that for $i \in \mathbb{N}_0$ the set C_i does not contain an interval of length $> \frac{1}{3^i}$. It follows that the Cantor set is nowhere dense. On the other hand we saw in Exercise 2.17 that the Cantor set is uncountable. This shows that $\mathbb{R}$ contains an uncountable nowhere dense subset.



We have the following analogue to Lemma 1.12.

Lemma 6.15. Let X be a topological space and let $A \subset X$ be a subset. The following two statements are equivalent:

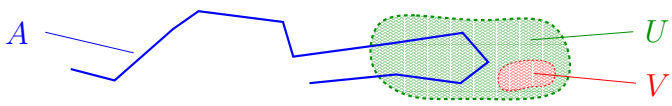
(1) A is a nowhere dense in X.

(2) The interior of the closure $\overline{A}$ is the empty set, or more succinctly, if $\operatorname{int}(\overline{A}) = \varnothing$.

Proof. We have the following equivalences of statements:

$$\begin{aligned}
& A \text{ is nowhere dense in } X \\
\iff & \forall_{\varnothing \neq U \subset X \text{ open}} \quad U \cap A \text{ is not dense in } U \\
\iff & \forall_{\varnothing \neq U \subset X \text{ open}} \; \exists_{\varnothing \neq V \subset U \text{ open}} \quad A \cap V = \varnothing \\
\iff & \forall_{\varnothing \neq U \subset X \text{ open}} \; \exists_{\varnothing \neq V \subset U \text{ open}} \quad A \subset (X \setminus V) \\
\iff & \forall_{\varnothing \neq U \subset X \text{ open}} \; \exists_{B \subsetneq X \text{ closed with } X \setminus U \subset B} \quad A \subset B \qquad \text{substitution } B = X \setminus V \text{ or } V = X \setminus B \\
\iff & \forall_{\varnothing \neq U \subset X \text{ open}} \; \exists_{C \subset X \text{ closed with } A \subset C} \quad U \not\subset C \qquad \text{substitution } C = (X \setminus U) \cup B \text{ or } B = C \\
\iff & \text{the interior of the closure } \overline{A} \text{ is the empty set.}
\end{aligned}$$

∎



Theorem 6.16. (Baire Category Theorem) Let X be one of the following:
(1) X is a complete metric space, or
(2) X is a regionally compact Hausdorff space.[99]
Then the union of countably many nowhere dense subsets of X has empty interior.

Remark. The Baire Category Theorem 6.16 has its origins in the work of René-Louis Baire [**Bai1899**]. The name of the theorem predates the advent of category theory. In particular the theorem has absolutely nothing to do with category theory.

Examples.

(1) It follows from the Baire Category Theorem 6.16 (1) and the example on page 296 that the union of countably many affine hyperplanes of $\mathbb{R}^n$ of codimension ≥ 1 has empty interior.
(2) The union of countable many nowhere dense sets might well be dense. For example take $X = \mathbb{R}$ and let $\varphi\colon \mathbb{N} \to \mathbb{Q}$ be a bijection. Then the sets $A_n := \{\varphi(n)\}$ are nowhere dense, but the union of the A_ns's is $\mathbb{Q}$, which is of course dense.

To clarify the exposition we prove the two cases of the Baire Category Theorem separately.

Proof of Theorem 6.16 for complete metric spaces. After possibly taking duplicates we only have to deal with the case that we have a countably *infinite* family $\{A_i\}_{i\in\mathbb{N}}$ of nowhere dense subsets of X. We need to show that no open non-empty subset U of X is contained in $\bigcup_{i\in\mathbb{N}} A_i$. Thus let U be an open non-empty subset of X. Our goal now is to find a point x in $U \setminus \bigcup_{i\in\mathbb{N}} A_i$.

Claim 1. There exists a sequence of open non-empty subsets $V_1, V_2, V_3, \ldots$ of X with the following three properties:
(0) $V_1 = U$,
(1) for each $i \in \mathbb{N}$ we have $\overline{V}_{i+1} \subset V_i \setminus \overline{A}_i$, and
(2) for each $i \in \mathbb{N}_{\geq 2}$ we have $\operatorname{diam}(V_i) \leq 2^{-i}$.

Proof. We construct the sequence iteratively. We set $U = V_1$ and we suppose that we have already constructed $V_1, \ldots, V_i$. Since A_i is nowhere dense and since V_i is non-empty and open we know by Lemma 6.15 that $V_i \setminus \overline{A}_i$ is non-empty. By the Interior-Closure Lemma 1.9 we know that $\overline{A}_i$ is a closed subset. Thus we see that $V_i \setminus \overline{A}_i$ is in fact an open non-empty subset of X. Therefore there exists an open ball $B_\epsilon(x) \subset V_i \setminus \overline{A}_i$. We can arrange that

[99] We will only make us of Statement (1). Nonetheless it is nice to have the second statement. It is a fun little exercise to show that the two hypothesis given in Statements (1) and (2) are independent of one another.

$\epsilon \leq \frac{1}{2^i}$. We set $V_{i+1} = B_{\frac{\epsilon}{2}}(x)$. One can now easily verify that $V_1, \ldots, V_{i+1}$ have the desired properties:

(1) Since $\overline{B}_{\frac{\epsilon}{2}}(x)$ is closed we see that $\overline{V}_{i+1} \subset \overline{B_{\frac{\epsilon}{2}}(x)} = \overline{B}_{\frac{\epsilon}{2}}(x) \subset B_\epsilon(x) \subset V_i \setminus \overline{A}_i$.

(2) By the discussion on page 285 we have $\operatorname{diam}(V_{i+1}) = \dim(B_{\frac{\epsilon}{2}}(x)) = 2 \cdot \frac{\epsilon}{2} \leq 2^{-i}$. ⊞

Let $i \in \mathbb{N}$. Since V_i is non-empty we can pick a point $x_i \in V_i$. Note that $\{x_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence since for $2 \leq i \leq j$ we have $d(x_i, x_j) \leq \operatorname{diam}(V_i) \leq 2^{-i}$. By hypothesis the metric space (X, d) is complete. This means that the Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ has a limit point $x \in X$.

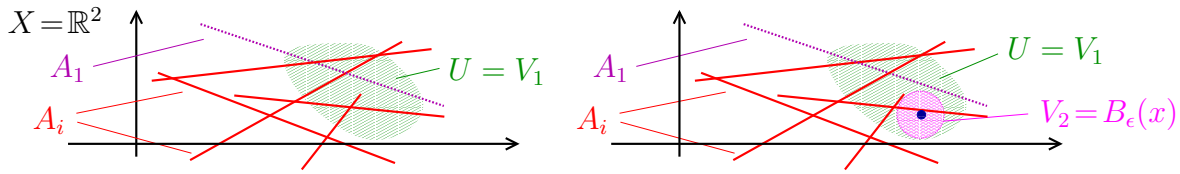
Claim 2. The point x lies in each $\overline{V}_i$.

Proof. Let $i \in \mathbb{N}$. By construction we know that for all $n \geq i$ we have $x_n \in \overline{V}_i$. Since $\overline{V}_i$ is closed we obtain from the Closed Subset Limit Lemma 1.18 that $x \in \overline{V}_i$. ⊞

Finally note that

$$\begin{array}{ccccccccc} x & \in & \bigcap_{i\in\mathbb{N}} \overline{V}_{i+1} & \subset & \bigcap_{i\in\mathbb{N}} (V_i \setminus \overline{A}_i) & \subset & \bigcap_{i\in\mathbb{N}} (U \setminus A_i) & \subset & U \setminus \bigcup_{i\in\mathbb{N}} A_i. \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ & \text{by the above} & & \text{by construction of } V_{i+1} & & \text{since } V_i \subset U & & \text{de Morgan's Laws 0.1} & \end{array}$$

This shows that $U \not\subset \bigcup_{i\in\mathbb{N}} A_i$. ■



Proof of Theorem 6.16 for regionally compact Hausdorff spaces. As in the previous setting we need to show that if $\{A_i\}_{\in\mathbb{N}}$ is a family of nowhere dense subsets and if U is an open non-empty subset U of X, then $U \cap \bigcup_{i\in\mathbb{N}} A_i \neq \varnothing$. Thus let U be an open non-empty subset of X.

Claim. There exists a sequence of open non-empty subsets $V_1, V_2, V_3, \ldots$ with the following three properties:

(0) $V_1 = U$,

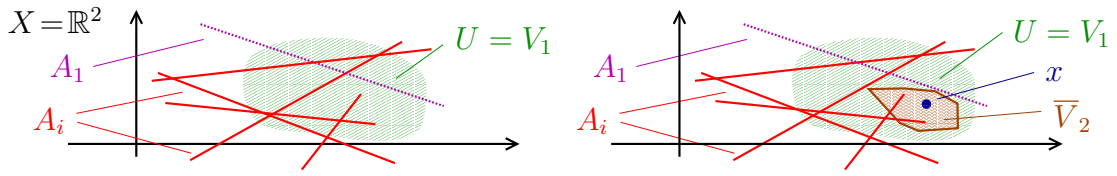
(1) for each $i \in \mathbb{N}$ we have $\overline{V}_{i+1} \subset V_i \setminus \overline{A}_i$, and

(2) for each $i \in \mathbb{N}_{\geq 2}$ the subspace $\overline{V}_i$ is compact.

Proof. We construct the sequence iteratively. We set $U = V_1$ and we suppose that we have already constructed $V_1, \ldots, V_i$. Since A_i is nowhere dense and since V_i is open we know by Lemma 6.15 that $V_i \setminus \overline{A}_i$ is non-empty. Since $\overline{A}_i$ is a closed subset we see that $V_i \setminus \overline{A}_i$ is in fact an open non-empty subset of X. We pick $x \in V_i \setminus \overline{A}_i$. Since X is regionally compact and Hausdorff we know by Lemma 2.42 (1) that there exists an open neighborhood V_{i+1} of x such that $\overline{V}_{i+1}$ is compact and such that $\overline{V}_{i+1}$ is contained in $V_i \setminus \overline{A}_i$. Evidently $V_1, \ldots, V_{i+1}$ have the desired properties. ⊞

We make the following observations:

(1) The intersection of finitely many $\overline{V}_i$ is non-empty by construction.



(2) Since each $\overline{V}_i$ is closed and since X is Hausdorff we obtain from the Compact-Closed Lemma 1.21 that each $\overline{V}_i$ is a closed subset of $\overline{V}_1$.

Since $\overline{V}_1$ is compact it follows from the above, together with Exercise 1.29[100] that $\bigcap_{i\in\mathbb{N}} \overline{V}_i$ is non-empty. As in the previous case we note that we have $\bigcap_{i\in\mathbb{N}} \overline{V}_i \subset U \setminus \bigcup_{i\in\mathbb{N}} \overline{A}_i$. This shows that $U \subsetneq \bigcup_{i\in\mathbb{N}} \overline{A}_i$. ■

6.7. **Baire spaces and completely metrizable spaces.** The Baire Category Theorem 6.16 motivates several definitions.

Definition. A topological space that has the property that the union of countably many nowhere dense subsets of X has empty interior is called a Baire space.

Lemma 6.17. (Baire Space Characterization Lemma) Let X be a topological space. The following two statements are equivalent:

(1) X is a Baire space.

(2) For each countable collection of open dense subsets $\{U_n\}_{n\in\mathbb{N}}$ the intersection $\bigcap_{n\in\mathbb{N}} U_n$ is dense.

Proof. We have the following equivalences of statements:

X is a Baire space

$\iff$ for every family $\{W_n\}_{n\in\mathbb{N}}$ of nowhere dense subsets the interior of $\bigcup_{n\in\mathbb{N}} W_n$ is $\varnothing$

$\overset{6.15}{\iff}$ for every family $\{W_n\}_{n\in\mathbb{N}}$ of subsets with $\operatorname{int}(\overline{W}_n) = \varnothing$ the interior of $\bigcup_{n\in\mathbb{N}} W_n$ is $\varnothing$

$\overset{1.9(2)}{\iff}$ for every family $\{A_n\}_{n\in\mathbb{N}}$ of closed subsets with $\operatorname{int}(A_n) = \varnothing$ the interior of $\bigcup_{n\in\mathbb{N}} A_n$ is $\varnothing$

$\underset{U_n=X\setminus A_n}{\overset{1.12}{\iff}}$ for every family $\{U_n\}_{n\in\mathbb{N}}$ of dense open subsets the interior of $\bigcup_{n\in\mathbb{N}} (X \setminus U_n)$ is $\varnothing$

$\overset{1.12}{\iff}$ for every family $\{U_n\}_{n\in\mathbb{N}}$ of dense open subsets the subset $X \setminus \bigcup_{n\in\mathbb{N}} (X \setminus U_n)$ is dense

$\overset{0.1}{\iff}$ for every family $\{U_n\}_{n\in\mathbb{N}}$ of dense open subsets the intersection $\bigcap_{n\in\mathbb{N}} U_n$ is dense. ■

Lemma 6.18. If X is a Baire space, then every open subset V is also a Baire space.

Proof. The statement follows easily from the Open -Closed Inclusion Lemma 2.10 and the definitions. We leave it to the reader to fill in the details. ■

Definition. A topological space X is called completely metrizable if there exists a *complete* metric d on X such that the given topology agrees with the topology coming from d.

[100] The content Exercise 1.29 is frankly just reformulation of the definition of a compact space using de Morgan's Laws 0.1.

Example.

(1) By the Compact-Complete Proposition 6.10 we know that every compact metrizable space is already completely metrizable.
(2) The metric space $(\mathbb{R}, d^{\text{eucl}})$ is of course complete, hence $\mathbb{R}$ is completely metrizable.
(3) If a topological space X is homeomorphic to a completely metrizable topological space, then X is evidently also completely metrizable.
(4) We consider $X = (-1, 1)$. If we equip $(-1, 1)$ with the usual Euclidean metric d^{eucl}, then $((-1, 1), d^{\text{eucl}})$ is of course not complete. But the map $(-1, 1) \to \mathbb{R}$ given by $t \mapsto \tan(\frac{\pi}{2}t)$ is a homeomorphism. Thus it follows from (2) and (3) that $(-1, 1)$ is completely metrizable.

With these definitions we can now reformulate the Baire Category Theorem 6.16 (1).

Theorem 6.16 (Baire Category Theorem)
(1) If X is a topological space that is completely metrizable, then X is a Baire space.

We conclude this section with the following amusing corollary. It is an amusing exercise to try to prove the corollary from scratch, i.e. without appealing to the Baire Category Theorem 6.16 (1).

Corollary 6.19. The topological space $\mathbb{Q}$ is metrizable but not completely metrizable.

Proof. Evidently $\mathbb{Q}$ is metrizable. But $\mathbb{Q}$ is not a Baire space. This can be seen as follows. Since $\mathbb{Q}$ is countable there exists a bijection $\varphi\colon \mathbb{N} \to \mathbb{Q}$. For each $n \in \mathbb{N}$ we set $U_n := \{\varphi(n)\}$. Each U_n is nowhere dense, but $\bigcup_{n\in\mathbb{N}} U_n = \mathbb{Q}$. So we have found a countably many nowhere dense subsets such that the interior of the union is non-empty. Since $\mathbb{Q}$ is not a Baire space it follows from the Baire Category Theorem 6.16 (1) that $\mathbb{Q}$ is not completely metrizable. ■

6.8. Application of the Baire Category Theorem 6.16 to sequences of maps.

We conclude this chapter with an application of the Baire Category Theorem 6.16 to analysis. We will not make use of the concepts and results of this section. But hopefully Proposition 6.20 gives the reader an idea of what the Baire Category Theorem 6.16 "is good for" in analysis.

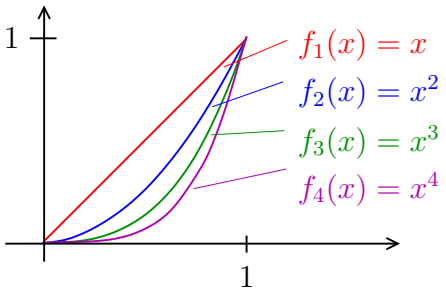
Definition. Let X be a topological space and let Y be a Hausdorff space. Furthermore let $\{f_n\colon X \to Y\}_{n\in\mathbb{N}}$ be a sequence of continuous maps. We say the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise, if for each $x \in X$ the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges in Y. We refer to the map[101]

$$\begin{array}{ccc} X & \to & Y \\ x & \mapsto & \lim\limits_{n\to\infty} f_n(x) \end{array}$$

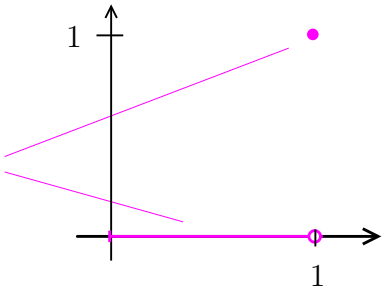
as the limit map.

Example. A popular way to shock students in a real analysis course is to show that the limit map of a pointwise convergent sequence of continuous maps is not necessarily continuous.

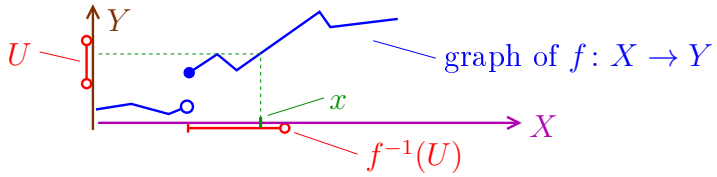
[101] Since Y is Hausdorff we know by the Unique Limit Lemma 1.17 that we can unambiguously talk about the limit of the sequence $(f_n(x))_{n\in\mathbb{N}}$.



the sequence of maps $f_n(x) = x^n$ converges pointwise, but the limit map is not continuous at $x=1$



Definition. Let $f\colon X \to Y$ be a map between two topological spaces X and Y and let $x \in X$. We say f is continuous at the point $x \in X$, if for each open neighborhood U of $f(x) \in Y$ the preimage $f^{-1}(U)$ is a neighborhood of $x \in X$.



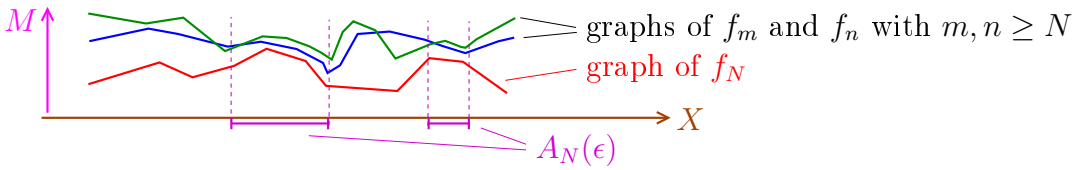
Remark. It follows easily from the JH-Lemma 1.7 that a map $f\colon X \to Y$ is continuous at every point $x \in X$ if and only if f is continuous in the sense of the definition on page 141.

The following proposition shows that the set of discontinuities of a limit map cannot get completely out of hand.

Proposition 6.20. Let X be a topological space, let (M, d) be a metric space and let $(f_n\colon X \to M)_{n\in\mathbb{N}}$ be a sequence of continuous maps which converges pointwise. If X is a Baire space, then the set of points at which the limit map $f\colon X \to M$ is continuous is dense in X.

Proof. Given $N \in \mathbb{N}$ and $\epsilon > 0$ we set

$$A_N(\epsilon) \;:=\; \{x \in X \,|\, d(f_m(x), f_n(x)) \le \epsilon \text{ for all } m, n \ge N\}$$



Claim 1.

(a) Each $A_N(\epsilon)$ is closed.

(b) $\bigcup_{N\in\mathbb{N}} A_N(\epsilon) = X$.

Proof.

(a) Note that $A_N(\epsilon) \;=\; \bigcap_{m,n\ge N} (d \circ ((f_m, f_n)\colon X \to M \times M)\colon X \to \mathbb{R})^{-1}([-\epsilon, \epsilon])$.

By the Metric-is-Continuous Lemma 6.1 we know that d is continuous. It follows from this description that $A_N(\epsilon)$ is closed.

(b) Let $x \in X$. By hypothesis $(f_n(x))_{n\in\mathbb{N}}$ is a convergent sequence. This implies that it is in particular a Cauchy sequence, which by definition implies that there exists an $N \in \mathbb{N}$ such that $x \in A_N(\epsilon)$. ⊞

Given $\epsilon > 0$ we set

$$U(\epsilon) \ := \ \bigcup_{N\in\mathbb{N}} \text{interior of } A_N(\epsilon).$$

By hypothesis X is a Baire space. Thus it follows from the Baire Space Characterization Lemma 6.17 that it suffices to prove the following claim.

Claim 2.

(a) For each $\epsilon > 0$ the set $U(\epsilon)$ is open and dense in X.

(b) The map f is continuous at every point in $\bigcap_{m\in\mathbb{N}} U(\frac{1}{m})$.

Proof.

(a) Since interiors of subsets are open we see that $U(\epsilon)$ is open. To show that $U(\epsilon)$ is dense it suffices, by definition of $U(\epsilon)$, to show that for each open subset $V \subset X$ there exists an $N \in \mathbb{N}$ such that $V \cap \text{int}(A_N(\epsilon)) \neq \varnothing$. Thus let $V \subset X$ be an open subset.

(i) It follows from Claim 1 (a) that each $V \cap A_N(\epsilon)$ is a closed subset of V.

(ii) Since X is a Baire space and since $V \subset X$ is open we know by Lemma 6.18 that V itself is a Baire space.

(iii) By Claim 1 (b) we know that the sets $V \cap A_N(\epsilon)$ form a cover of V. Since V is a Baire space we see that not all $V \cap A_N(\epsilon)$ can be nowhere dense in V. In other words there exists an $N \in \mathbb{N}$ such that $V \cap A_N(\epsilon)$ is not nowhere dense in V.

(iv) By Lemma 6.15 we know that the interior of the closure of $V \cap A_N(\epsilon)$ in V is non-empty.

(v) By (i) and (iv) we know that the interior of $V \cap A_N(\epsilon)$ is non-empty.

In summary, we have shown that there exists an $N \in \mathbb{N}$ such that $V \cap A_N(\epsilon)$ contains a subset $W \subset V$ that is open in V. Since V is open in X it follows from the elementary the Open -Closed Inclusion Lemma 2.10 that W is open in X. In particular we see that W is contained in the interior of $A_N(\epsilon)$.

(b) Let $x \in \bigcap_{m\in\mathbb{N}} U(\frac{1}{m})$. We need to show that f is continuous at the point x. It follows easily from the definition, see page 301, that it suffices to show that for every $\epsilon > 0$ there exists a neighborhood W of x such that for all $y \in W$ we have $d(f(x), f(y)) < \epsilon$. We make a few preparations:

(i) We pick $k \in \mathbb{N}$ with $\frac{1}{k} < \frac{\epsilon}{3}$.

(ii) Note that we know for free that $x \in U(\frac{1}{k})$. Therefore there exists an $n \in \mathbb{N}$ such that $x \in \text{int}(A_n(\frac{1}{k}))$.

(iii) Since f_n is continuous and since $\text{int}(A_n(\frac{1}{k}))$ there exists a neighborhood W of $x \in X$ with $W \subset A_n(\frac{1}{k})$ such that $d(f_n(x), f_n(y)) < \frac{\epsilon}{3}$ for all $y \in W$.

Now we see that for all $z \in W$ we have

$$d(f(z), f_n(z)) = \Big(\lim_{m\to\infty} f_m(z), f_n(z)\Big) \overset{\text{by continuity of metrics, see the Metric-is-Continuous Lemma 6.1, and Exercise 2.26}}{\overset{\downarrow}{=}} \lim_{m\to\infty} \underbrace{d(f_m(z), f_n(z))}_{\substack{< \frac{1}{k} \text{ for } m \geq n \text{ since} \\ z \in W \subset A_n(\frac{1}{k})}} \leq \frac{1}{k} < \frac{\epsilon}{3}.$$

Finally note that for all $y \in W$ we have the following (in-) equalities:

$$d(f(x), f(y)) \overset{\text{triangle inequality}}{\overset{\downarrow}{=}} \underbrace{d(f(x), f_n(x))}_{\substack{< \frac{\epsilon}{3} \text{ by the above,} \\ \text{since } x \in W}} + \underbrace{d(f_n(x), f_n(y))}_{\substack{< \frac{\epsilon}{3}, \text{ since } y \in W, \\ \text{see (iii)}}} + \underbrace{d(f_n(y), f(y))}_{\substack{< \frac{\epsilon}{3} \text{ by the above,} \\ \text{since } y \in W}} < \epsilon.$$

■

Remark. The Baire Category Theorem 6.16 has many other applications to analysis. For example one can use it to show that "most" continuous maps $\mathbb{R} \to \mathbb{R}$ are nowhere differentiable. We refer to [**JP2015**, Chapter 7] or alternatively to [**Ross2013**, Theorem 38.3] for details.

Exercises for Chapter 6.

Exercise 6.1. Let (X, d) be a metric space, let $K \subset X$ be a compact subspace and let U be an open neighborhood of $K \subset X$. Show that there exists an $r > 0$ such that for each $x \in K$ we have $B_r(x) \subset U$.

Exercise 6.2. Let (X, d) be a metric space. Show that the map

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} \\ (x, y) & \mapsto & d(x, y) \end{array}$$

is continuous.

Exercise 6.3. Let (X, d) be a metric space. We suppose that X is compact. Furthermore let $f\colon X \to X$ be a map with $f(x) \neq x$ for all $x \in X$. Show that there exists an $\epsilon > 0$ such that $d(f(x), x) \geq \epsilon$ for all $x \in X$.

Exercise 6.4. Let (X, d) be a metric space and let $f\colon X \to \mathbb{R}$ be a continuous map. We now define

$$\begin{array}{rcl} \widetilde{d}\colon X \times X & \to & \mathbb{R}_0 \\ (x, y) & \mapsto & d(x, y) + |f(x) - f(y)|. \end{array}$$

Show that $\widetilde{d}$ is a metric and show that $\widetilde{d}$ induces the same topology as d.

Exercise 6.5. Let (X, d) be a metric space and let $B \subset X$ be a non-empty subset. Show that the map

$$\begin{array}{rcl} X & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & d(x, B) := \inf\{d(x, b) \,|\, b \in B\} \end{array}$$

is continuous.



Exercise 6.6. Suppose that X is a metrizable topological space. Show that given any closed set $A \subset X$ there exists a map $f\colon X \to \mathbb{R}$ such that $f^{-1}(0) = A$.
Hint. Use the Distance-to-Subspace Lemma 6.2.

Exercise 6.7. Show that the statement of the Distance-Minimizer Lemma 6.4 does not generalize to general metric spaces. More precisely, give an example of a metric space X, a compact subset $K \subset X$ and a closed subset $L \subset X$ such that there do not exist $x \in K$ and $y \in L$ with $d(x, y) = d(K, L)$.

Exercise 6.8. Let (X, d) be a metric space. We denote by $\mathcal{P}(X)$ its power set. We consider the map

$$\begin{array}{rcl} d\colon \mathcal{P}(X) \setminus \{\varnothing\} \times \mathcal{P}(X) \setminus \{\varnothing\} & \to & \mathbb{R}_{\geq 0} \\ (A, B) & \mapsto & d(A, B). \end{array}$$

Which properties of a metric does this map satisfy?

Exercise 6.9. We consider the metric space $X = \mathbb{R}$. Let K and L be two closed non-empty disjoint subsets of $\mathbb{R}$. Is it necessarily true that $d(K, L) > 0$?

Exercise 6.10. Let X be a metric space and let A, B be two non-empty subsets of X. Suppose that $\operatorname{diam}(A) + \operatorname{diam}(B) < d(A, B)$. Show that $A \cap B = \varnothing$.

Exercise 6.11. Let X be a topological space and let $\{B_i\}_{i \in I}$ be a family of *open* subsets such that the intersection $Y := \bigcap_{i \in I} B_i$ is non-empty.

(a) We suppose that X and Y are path-connected. Show that each B_i is path-connected.
Hint. Use the Lebesgue Lemma 6.5.
(b) We suppose that X and Y are connected. Show that each B_i is connected.
(c) Show that statements (a) and (b) are wrong if we drop the hypothesis that the B_i are open.
(d) Do statements (a) and (b) hold if each B_i is a closed subset?

Remark. This statement can be viewed as a partial converse to the Union Connected Lemma 2.28.

Exercise 6.12. Let $A \subset \mathbb{R}^n$ be a subset. We equip A with the usual Euclidean metric. Show that A is complete if and only if A is a closed subset of $\mathbb{R}^n$.

Exercise 6.13. Show that for for every $n \in \mathbb{N}$ there exists a surjective map $\mathbb{R} \to \mathbb{R}^n$.

Exercise 6.14. Recall that by Proposition 6.14 we know that there exists an injective map $f\colon [0,1] \to [0,1]^2$ be an injective map such that $f([0,1])$ has non-zero 2-dimensional Lebesgue measure. Show that there exists an injective map $g\colon S^1 \to [0,1]^2$ such that $g(S^1)$ has non-zero 2-dimensional Lebesgue measure.

Exercise 6.15. Let $f\colon \mathbb{R} \to [0,1]$ be a continuous map that satisfies the following three conditions:

(1) for every $t \in \mathbb{R}$ we have $f(t+2) = f(t)$,

(2) for every $t \in [0, \frac{1}{3}]$ we have $f(t) = 0$,
(3) for every $t \in [\frac{2}{3}, 1]$ we have $f(t) = 1$.

Given $t \in \mathbb{R}$ we set

$$\varphi(t) := \sum_{n=1}^{\infty} 2^{-n} \cdot f(3^{2n-1} \cdot t), \qquad \psi(t) := \sum_{n=1}^{\infty} 2^{-n} \cdot f(3^{2n} \cdot t).$$

Prove the following two statements:

(a) The maps $\varphi, \psi \colon \mathbb{R} \to \mathbb{R}$ are continuous.
(b) The map $t \mapsto (\varphi(t), \psi(t))$ is a surjection $[0,1] \to [0,1] \times [0,1]$.
(c) Let $n \in \mathbb{N}$. How can you modify the above construction of maps to get a space-filling curve $[0,1] \to [0,1]^n$?

Remark. This explicit description of a space-filling curve was found by Isaac Schoenberg [**Schoe1938**]. It nicely complements the constructions given in Exercise 5.24 and the Space-Filling Curve Theorem 6.13.

Exercise 6.16. Show that every affine hyperplane in $\mathbb{R}^n$ of codimension ≥ 1 is nowhere dense.

Exercise 6.17. The Baire Category Theorem 6.16 (1), together with Lemma 6.15, implies that $\mathbb{R}$ cannot be written as a countable union of *closed* subsets having empty interiors. Show that this conclusion does not hold, if the subsets are not required to be closed.

Exercise 6.18. Let X be a topological space.

(a) Show that the union of finitely many nowhere dense subsets is again nowhere dense.
(b) Is the union of countably many nowhere dense necessarily nowhere dense?

Exercise 6.19. Show that the conclusion of the Baire Category Theorem 6.16 does not necessarily hold for incomplete metric spaces. More precisely, provide an example of a metric space X, a countable family $\{U_i\}_{i \in I}$ of nowhere dense subsets such that the union $\bigcup_{i \in I} U_i$ has non-empty interior.

Exercise 6.20. Let (M, d) be a complete metric space with no isolated points. Show that the set M is uncountable.
Remark. Recall that a point $x_0 \in X$ is called *isolated* if $\{x_0\}$ itself is an open subset.

Exercise 6.21. Let $f \colon (0,1) \to \mathbb{R}$ be an infinitely differentiable function such that for each $x \in (0,1)$ there exists an $n \in \mathbb{N}_0$ such that $f^{(n)}(x) = 0$. We want to show that f is a polynomial function. In the following we outline a proof by contradiction. Suppose that f is not a polynomial function.

(a) We consider the sets

$$\begin{aligned} S_n &:= \{x \in (0,1) \mid f^{(n)}(x) = 0\} \\ X &:= \left\{ x \in (0,1) \,\middle|\, \begin{array}{c} \text{there is no open neighborhood of } x \in (0,1) \text{ on} \\ \text{which } f \text{ is a polynomial function} \end{array} \right\} \end{aligned}$$

Show that S_n and X are closed and show that X has no isolated points.
(b) Apply the Baire Category Theorem 6.16 (1) to the covering $\{X \cap S_n\}_{n \in \mathbb{N}}$ of X to show that there exists an open interval (a, b) and an $n \in \mathbb{N}$ such that $(a, b) \cap X \neq \varnothing$ and such that $(a, b) \cap X \subset S_n$.

(c) Show that $(a,b) \cap X \subset S_m$ for all $m \geq n$.
Hint. Use that every point in X is an accumulation point and use the definition of derivatives.
(d) Let (s,t) be a maximal interval in $X \setminus (a,b)$. Note that f is described by a polynomial function of some degree d on (s,t). Use the fact that at least one of s,t lies in X to show that $d < n$.
Hint. Consider the d-th derivative.
(e) Use (d) to show that $f^{(n)} = 0$ on (a,b).
(f) Argue why (e) is a contradiction to $(a,b) \cap X$ being non-empty.

Remark. This statement was first proved in [**CSB1954**]. The details for this challenging exercise are worked out in [**Ross2013**, Theorem 38.5].

Exercise 6.22. We say that a map $f\colon \mathbb{R} \to \mathbb{R}$ is *nowhere monotonic* if there is no open non-empty subset $U \subset \mathbb{R}$ such that $f|_U\colon U \to \mathbb{R}$ is monotonic. Show that the set of nowhere monotonic maps is dense in the topological space $C(\mathbb{R},\mathbb{R})$.

Exercise 6.23. Give an example of a sequence $\{f_n\colon [0,1] \to \mathbb{R}\}_{n\in\mathbb{N}}$ of continuous maps that converges pointwise and such that the limit map $f\colon [0,1] \to \mathbb{R}$ is discontinuous at infinitely many points.

Exercise 6.24. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous map with the following property: for each $x \in \mathbb{R}_{>0}$ we have $\lim\limits_{n\to\infty} f(nx) = 0$. Show that $\lim\limits_{n\to\infty} f(x) = 0$.
Hint.
(a) Let $\epsilon > 0$. We set $D_n := \{x \in \mathbb{R}_{>0} \,|\, |f(mx)| \leq \epsilon \text{ for all } m \geq n\}$. Use the Baire Category Theorem 6.16 (1) to show that there exists an $n_0 \in \mathbb{N}$ such that D_{n_0} has non-empty interior.
(b) Use (a) to show that there exists an $r > 0$ such that $|f(x)| < \epsilon$ for all $x > r$.

Exercise 6.25. A topological space that has the property that every sequence admits a convergent subsequence is called *sequentially compact*.

(a) Show that every compact metric space is sequentially compact.
Remark. In fact a metric space is compact if and only if it is compact. The "only if"-direction is shown in [**Mun1975**, §28].
Remark. This exercise can be viewed as a generalization of the Bolzano-Weierstraß-theorem from real analysis.
(b) By Tychonoff's Theorem 5.9 we know that $[0,1]^{\mathbb{N}}$ is compact. Show that $[0,1]^{\mathbb{N}}$ is not sequentially compact.

Exercise 6.26. Let X be a metric space. We say that a map $f\colon X \to X$ is an *isometry* if for every $x,y \in X$ we have $d(f(x),f(y)) = d(x,y)$. Show that if $f\colon X \to X$ is an isometry and if X is compact, then f is a homeomorphism.
Hint. It is clear that f is injective. By the Compact-Hausdorff Proposition 2.17 (3) it suffices to show that f is a surjection. Suppose that there exists an $a \in X \setminus f(X)$. There exists an $\epsilon > 0$ with $B_\epsilon(a) \subset X \setminus f(X)$. We consider the sequence given by $x_0 := a$ and $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}_0$. Show that for every $m \neq n$ we have $d(x_m,x_n) > \epsilon$. Why is this not possible?

7. Graphs

In this chapter we will first introduce the purely combinatorial concept of an (undirected) abstract graph. Afterwards we introduce the corresponding topological realizations. These notions are of interest for various reasons. Abstract graphs are ubiquitous in mathematics and at times they will also play a useful role in our discussions. The corresponding topological realizations form fairly simple, but nonetheless interesting examples of topological spaces.

7.1. Abstract graphs and topological graphs.

We start out with the following rather abstract definition.

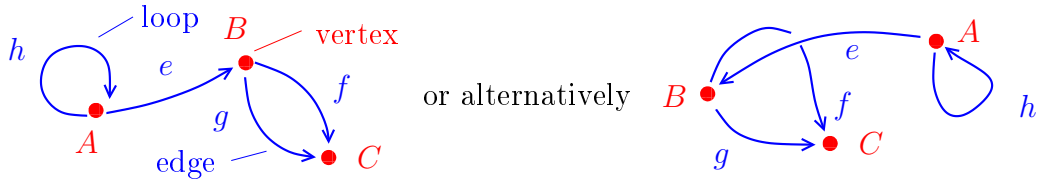
Definition. An abstract graph G is a quadruple (V, E, i, t) where V is a set, E is a set and $i, t\colon E \to V$ are maps from E to V.

(1) The elements of V are called the vertices of G and the elements of E are called the edges of G.
(2) Given $e \in E$ the point $i(e)$ is called the initial point of e and $t(e)$ is called the terminal point of e. We refer to $i(e)$ and $t(e)$ as the endpoints of e.
(3) We say an edge $e \in E$ is a loop if $i(e) = t(e)$. Otherwise we call e a non-loop.
(4) We say G is finite (countable) if both V and E are finite (countable).

Remark. The origins of graph theory go back to Leonhard Euler[**Eul1736**] and his work in 1736 on the problem on the seven bridges of Königsberg, see Exercise 7.2.

Examples.

(A) We take $V = \{A, B, C\}$ and $E = \{e, f, g, h\}$. Furthermore let $i\colon E \to V$ be the map that is defined by $i(e) = A$, $i(f) = B$, $i(g) = B$ and $i(h) = A$ and let $t\colon E \to V$ be the map that is defined by $t(e) = B$, $t(f) = C$, $t(g) = C$ and $t(h) = A$. The quadruple (V, E, i, t) is an abstract graph. We can try to visualize such an abstract graphs by drawing a point for each vertex and by drawing an arrow for each edge e that points from the initial point $i(e)$ to the endpoint $t(e)$.



(B) Let $a, b \in \mathbb{Z}$. We denote by $G_{a,b} = (V, E, i, t)$ the abstract graph with vertex set $V = \mathbb{Z}$, edge set $E = \mathbb{Z} \times \{0, 1\}$ and where $i(n, 0) = i(n, 1) = n$ and $t(n, 0) = n + a$ and $t(n, 1) = n + b$. We leave it to the reader to draw visualizations of $G_{a,b}$ for different values of a and b.

The above figure already indicates that abstract graphs give rise to topological space. The following definition is already somewhat topology-like.

Definition. We say that an abstract graph (V, E, i, t) is connected if for every two vertices $v \neq w \in V$ there exist vertices $v = v_0, v_1, \ldots, v_n = w$ and edges $e_0, \ldots, e_{n-1}$ such that for each $i \in \{0, \ldots, n-1\}$ the endpoints of e_i are v_i and v_{i+1}.

Examples. We want to determine whether or not the above examples (A) and (B) of abstract graphs are connected.

(A) This abstract graph is connected, for example if we take $v = A$ and $w = C$, then $v_0 = A$, $v_1 = B$ and $v_2 = C$ and $e_0 = e$ and $e_1 = f$ have the desired property.

(B) Let $a, b \in \mathbb{Z}$. Basically by definition the abstract graph $G_{a,b}$ is connected if and only if for every two integers $x, y \in \mathbb{Z}$ there exist $n, m \in \mathbb{Z}$ such that $x - y = ma + nb$. Elementary number theory says that this is the case if and only if a and b are coprime.

Definition.

(1) Let $G = (V, E, i\colon E \to V, t\colon E \to V)$ and $G' = (V', E', i'\colon E' \to V', t'\colon E' \to V')$ be two abstract graphs in the sense of the definition on page 307. We define a morphism $G \to G'$ of abstract graphs to be a pair $(\alpha\colon V \to V', \beta\colon E \to E')$ of maps such that for each $e \in E$ we have $i'(\beta(e)) = \alpha(i(e))$ and $t'(\beta(e)) = \alpha(t(e))$.

(2) We refer to the category $\mathcal{AbsGraph}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{AbsGraph}) &:= \text{all abstract graphs,} \\ \mathrm{Mor}(G, G') &:= \text{all maps from } G \text{ to } G' \text{ in the above sense} \end{aligned}$$

with the usual composition of maps as the category of abstract graphs.

Example. The maps

$$\begin{aligned} (V, E, i\colon E \to V, t\colon E \to V) &\mapsto (V, E, t\colon E \to V, i\colon E \to V) \\ (f\colon G \to G') &\mapsto (f\colon G \to G') \end{aligned}$$

defines a covariant functor from the category $\mathcal{AbsGraph}$ to itself. The effect on each abstract graph is that it swaps the roles of the initial points and the endpoints. Pictorially speaking the functor "inverts the orientations of the arrows".

Surely the reader is bothered by the absence of topology. We will change this sorry state of affairs by introducing the topological realization of a graph. The idea is, basically, that the topological realization of an abstract graph $G = (V, E, i, t)$ is given by one point for each vertex $v \in V$ and one interval for each edge $e \in E$ with endpoints given by $i(e)$ and $t(e)$. The definition below is the formalization of that idea.

Definition. Let $G = (V, E, i, t)$ be an abstract graph.

(1) We define the topological realization of G as the topological space

$$|G| := \big(V \sqcup (E \times [0,1])\big)/\!\sim \quad \text{where for any } e \in E \text{ we have } (e,0) \sim i(e) \text{ and } (e,1) \sim t(e).$$

$\uparrow$ here E is equipped with the *discrete* topology

Note that on many occasions we will use implicitly that by Exercise 5.4 we know that $[0,1] \times E = \coprod_{e \in E} [0,1]$.

(2) Given $e \in E$ we refer to the map $\Phi_e\colon [0,1] \xrightarrow{x \mapsto (e,x)} \{e\} \times [0,1] \to |G|$ as the characteristic map of the edge e.

(3) Let $e \in E$. By a slight abuse of language we refer to $|e| = \Phi_e([0,1])$ also as an edge of $|G|$. Furthermore we refer to $\langle e \rangle := \Phi_e((0,1))$ as an open edge of $|G|$.

Before we attempt to give a better intuition for the definition of the topological realization we need to get the following lemma out of the way.

Lemma 7.1. (Abstract Graph-Topological Realization Lemma) Let $G = (V, E, i, t)$ be an abstract graph.

(1) The obvious map $V \to |G|$ is an inclusion and the image is a discrete subset of $|G|$.
(2) $|G|$ is Hausdorff.
(3) Let $e \in E$.
   (a) The map $\Phi_e \colon [0,1] \to |G|$ is continuous and closed.
   (b) The map $\Phi_e \colon (0,1) \to |G|$ is an open embedding.
   (c) If $i(e) = t(e)$, then Φ_e induces a homeomorphism $[0,1]/0 \sim 1 \to |e|$.
   (d) If $i(e) \neq t(e)$, then Φ_e restricts to a homeomorphism $[0,1] \to |e|$.

   The same statements (a), (b), (c) and (d) also hold for the map $\widetilde{\Phi}_e \colon [0,1] \to |G|$ given by $t \mapsto \Phi_e(1-t)$. This map satisfies $\widetilde{\Phi}_e(0) = t(e)$ and $\widetilde{\Phi}_e(1) = i(e)$. For convenience we sometimes refer to $\widetilde{\Phi}_e$ also as a *characteristic map of the edge* e.
(4) G is finite if and only if $|G|$ is compact.
(5) The following statements are equivalent:
   (a) G is connected,
   (b) given any two distinct vertices v, v' there exists an embedding $\gamma \colon [0,1] \to |G|$ with $\gamma(0) = v$, $\gamma(1) = v'$ and such that $\gamma([0,1]) \subset |G|$ is the union of edges,
   (c) the topological realization $|G|$ is path-connected,
   (d) the topological realization $|G|$ is connected.

Proof.

(1) This statement follows fairly easily from the definitions of the quotient topology and the disjoint union topology. We leave it to the reader to fill in the details.
(2) Ditto.
(3) Let $e \in E$.
   (a) It follows immediately from Lemma 5.12 (2) and the Topological-Quotient Proposition 5.15 (1a) that Φ_e is continuous. Furthermore it follows easily from the definitions together with Lemma 5.12 and the Topological-Quotient Proposition 5.15 (1a) that Φ_e is closed.
   (b) One can easily verify that $\Phi_e \colon (0,1) \to |G|$ is injective and that it is an open map. Thus it follows from the Open -Injective Map Lemma 2.16 that it is an embedding.
   (c) We suppose that $i(e) = t(e)$. It follows easily from the Twice Quotient Lemma 5.23 that Φ_e induces a continuous map $[0,1]/0 \sim 1 \to |e|$ and that this map is a bijection. It follows from (2) and the Compact-Hausdorff Proposition 2.17 (3) that the map $\Phi_e \colon [0,1]/0 \sim 1 \to |e|$ is in fact a homeomorphism.
   (d) Now we suppose that $i(e) \neq t(e)$. In this case the map is clearly injective. Thus it follows from (2) together with the Compact-Hausdorff Proposition 2.17 (3) that $\Phi_e \colon [0,1] \to |e|$ is a homeomorphism.

   The statements regarding $\widetilde{\Phi}_e$ follow from the fact that the map $t \mapsto 1-t$ is a homeomorphism of $[0,1]$.

(4) First assume that G is finite. First note that it follows almost immediately from the Disjoint Union Topology-Properties Lemma 5.13 (2) together with the Compact Image Lemma 2.13 that $|G|$ is compact. Conversely, in Exercise 7.5 we will show that any compact subset of the topological realization of a graph is contained in the topological realization of a finite graph contained in G. This statement immediately implies that if $|G|$ is finite, then G is finite.

(5) We will prove this statement in Exercise 7.4. ■

The following lemma makes the unsurprising assertion that the above assignment of a topological space to an abstract simplicial complex is functorial.

Lemma 7.2. The following two statements hold:

(1) Let $G = (V, E, i, t)$ and $G' = (V', E', i', t')$ be two abstract graphs. Furthermore let $f := (\alpha\colon V \to V', \beta\colon E \to E')$ be a map between the two abstract graphs. Recall that, by the definition on page 308, this means that for each $e \in E$ we have the equalities $i'(\beta(e)) = \alpha(i(e))$ and $t'(\beta(e)) = \alpha(t(e))$. Now we consider the following map between the topological realizations of G and G':

$$\begin{array}{rcl} & \text{where } (e,0)\sim i(e) \text{ and } (e,1)\sim t(e) & \text{similarly } (e,0)\sim i'(e) \text{ and } (e,1)\sim t'(e) \\ |f|\colon |G| = \big(V \sqcup (E\times[0,1])\big)\big/\overset{\downarrow}{\sim} & \to & |G'| = \big(V' \sqcup (E'\times[0,1])\big)\big/\overset{\downarrow}{\sim} \\ [x] & \mapsto & \begin{cases} \alpha(x), & \text{if } x \in V, \\ (\beta(e), t), & \text{if } x = (e,t) \in E\times[0,1]. \end{cases} \end{array}$$

This map $|f|\colon |G| \to |G'|$ is well-defined and continuous.

(2) The map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{AbsGraph}) & \to & \mathrm{Ob}(\mathcal{Top}) \\ G & \mapsto & |G|, \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{AbsGraph}}(G, G') & \to & \mathrm{Mor}_{\mathcal{Top}}(|G|, |G'|) \\ f & \mapsto & |f| \end{array}$$

is a covariant functor from the category $\mathcal{AbsGraph}$ of abstract graphs to the category $\mathcal{Top}$ of topological spaces.

Proof.

(1) It follows almost immediately from the definitions that the map is well-defined. We leave it to the reader to verify, e.g. using the Twice Quotient Lemma 5.23, that the map is continuous.

(2) This statement follows easily from the definitions. ■

Even though the above definition of the topological realization of an abstract graph is a perfectly fine, rigorous and precise definition, it is perhaps not the most intuitive definition. It would be much more fun to work with topological spaces that we can "see". This leads us to the following definition.

Definition. A topological graph is a triple $(X, G = (V, E, i, t), \Theta\colon |G| \to X)$ where the following holds:

(1) X is a topological space,

(2) $G = (V, E, i, t)$ is an abstract graph,

(3) $\Theta\colon |G| \to X$ is a homeomorphism between the topological realization $|G|$ of G and the topological space X.

Given a topological graph $(X, G = (V, E, i, t), \Theta\colon |G| \to X)$ we introduce the following extra definitions:

(1) Given $v \in V$ we refer to $\Theta(v)$ as a vertex of the topological graph.
(2) Given $e \in E$ we refer to $\Theta(|e|)$ as an edge of the topological graph. Given a loop e we refer to $|e|$ as a loop of the topological graph.
(3) We refer to the abstract graph $G = (V, E, i, t)$ as the underlying abstract graph.
(4) Sometimes we refer, somewhat incorrectly, to X as a topological realization of G.

Sometimes we will just say that X is a topological graph. Furthermore on occasions we will suppress $G = (V, E, i, t)$ and $\Theta\colon |G| \to X$ from the notation.

The following lemma gives us a way to construct topological graphs with a given underlying abstract graph.

Lemma 7.3. Let $G = (V, E, i, t)$ be an abstract graph and let $n \in \mathbb{N}_0$. Suppose we are given the following data:

(1) An injective map $\alpha\colon V \to \mathbb{R}^n$.
(2) For each $e \in E$ we are given a map $\beta_e\colon [0,1] \to \mathbb{R}^n$ which is injective on $(0,1)$ and which satisfies $\beta_e(0) = \alpha(i(e))$ and $\beta_e(1) = \alpha(t(e))$.
(3) For $e \neq f \in E$ we have $\beta_e((0,1)) \cap \beta_f((0,1)) = \varnothing$.

If G is infinite, then we also demand that the following condition is satisfied:

(4) For every $r \in \mathbb{R}_{\geq 0}$ there exist only finitely many $v \in V$ with $\alpha(v) \cap B_r^n(0) \neq \varnothing$ and there exist only finitely many $e \in E$ with $\beta_e([0,1]) \cap B_r^n(0) \neq \varnothing$.

Then the following map is a closed embedding:

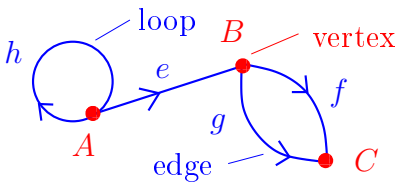
$$\begin{array}{rcl} |G| = \big(V \sqcup (E \times [0,1])\big)/\overset{\downarrow}{\sim} & \to & \mathbb{R}^n \\ & & \text{where } (e,0) \sim i(e) \text{ and } (e,1) \sim t(e) \\ [x] & \mapsto & \begin{cases} \alpha(x), & \text{if } x \in V, \\ \beta_e(t), & \text{if } x = (e,t) \in E \times [0,1]. \end{cases} \end{array}$$

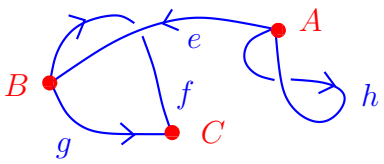
Proof. It follows easily from our hypotheses that the map is well-defined and that it is an injection. Furthermore we obtain from the Topological-Quotient Proposition 5.15 (1b) that the map is continuous. If G is finite, then it follows from the Abstract Graph-Topological Realization Lemma 7.1 together with the Compact-Hausdorff Proposition 2.17 (3) that the given map $|G| \to \mathbb{R}^n$ is an embedding. In the general case the same conclusion follows from the Closed Embedding-to-$\mathbb{R}^n$ Proposition 2.19. ■

Examples. We return to the two examples of abstract graphs that we gave on page 307.

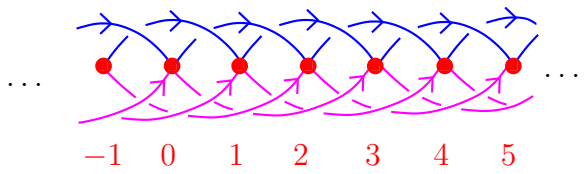
(A) As on page 307 we consider $V = \{A, B, C\}$ and $E = \{e, f, g, h\}$ together with the maps $i, t\colon E \to V$ given by $i(e) = A$, $i(f) = B$, $i(g) = B$ and $i(h) = A$ and given by $t(e) = B$, $t(f) = C$, $t(g) = C$ and $t(h) = A$. In the figure below we show two topological realizations[102] of this abstract graph.

[102] Here evidently we use the slightly sloppy language from above. More precisely, we do not show the actual topological realization $|G|$ but we show a topological graph X such that the underlying abstract graph equals G.

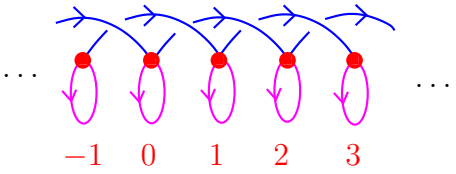




(B) Let $a, b \in \mathbb{Z}$. As on page 307 we consider the abstract graph with vertex set $V = \mathbb{Z}$, edge set $\mathbb{Z} \times \{0, 1\}$ and where $i(n, 0) = i(n, 1) = n$ and $t(n, 0) = n + a$ and $t(n, 1) = b$. In the figure below we show topological realizations of the graphs $G_{2,3}$ and $G_{0,2}$.



topological realization of $G_{2,3}$

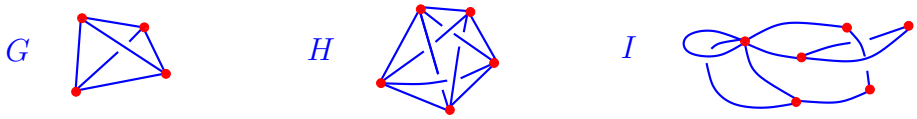


topological realization of $G_{0,2}$

In Exercise 7.9 we will see that given any finite abstract graph G there exists an embedding of $|G|$ into $\mathbb{R}^3$. (Alternatively see [**CELR1994**, p. 2] or [**Mun1975**, p. 309] for a proof of this fact). We conclude this section with the following question.

Question 7.4. Let G be a finite abstract graph. Does there necessarily exist an embedding of $|G|$ into $\mathbb{R}^2$?

Example. We consider the finite abstract graphs G, H and I that are (implicitly) illustrated in the figure below. Do their topological realizations admit embeddings into $\mathbb{R}^2$?



Surely the answer to Question 7.4 must be no, but it is not clear how one can prove this statement rigorously. We will return to this question in Section 88.3.

7.2. **Undirected abstract graphs and their topological realizations .** In this section we will introduce the notion of an undirected abstract graph and the corresponding topological realizations. In many settings undirected abstract graphs are more suitable than the abstract graphs from the previous section. But as we will see, the topological realization of an undirected abstract graph is more unwieldy. Thus in these notes we will mostly work with abstract graphs and we will leave it to the reader to generalize some of the later results to the setting of undirected abstract graphs.

Without further ado, here is the definition.

Definition. An undirected abstract graph G is a triple (V, E, φ) where V is a set, E is a set and φ is a map

$$\varphi\colon E \to \{\text{subsets of } V \text{ with one or two elements}\}.$$

Similar to the discussion on page 307 we introduce the following extra definitions:

(1) The elements of V are called the vertices of G and the elements of E are called the edges of G.
(2) We say an edge $e \in E$ is a loop if $\#\varphi(e) = 1$. Otherwise we call e a non-loop.
(3) Given $e \in E$ the points in $\varphi(e)$ are called the endpoints of e.
(4) We say G has multi-edges if there exist distinct edges e and e' with $\varphi(e) = \varphi(e')$.
(5) We say G is finite (countable) if both V and E are finite (countable).

Example. We take V to be the set of all subway stations in München and we take E to be the set of all segments of the subway system. The map φ then assigns to each $e \in E$ the two stations that are connected by the segment e. The triple (V, E, φ) is an undirected abstract graph.

The following definition is the analogue of the definition on page 308.

Definition.

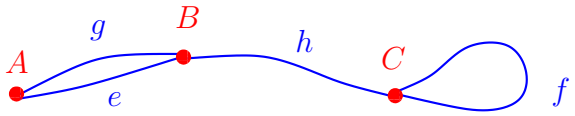
(1) Let $G = (V, E, \varphi\colon E \to \mathcal{P}(V))$ and $G' = (V', E', \varphi'\colon E' \to \mathcal{P}(V'))$ be two undirected abstract graphs. We define a morphism $G \to G'$ between the two undirected abstract graphs to be a pair $(\alpha\colon V \to V', \beta\colon E \to E')$ of maps such that for every $e \in E$ we have the equality $\varphi'(\beta(e)) = \alpha(\varphi(e))$.
(2) We refer to the category $\mathcal{UndirAbsGraph}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{UndirAbsGraph}) &:= \text{all undirected abstract graphs,} \\ \mathrm{Mor}(G, G') &:= \text{all maps from } G \text{ to } G' \text{ in the above sense} \end{aligned}$$

with the hopefully obvious composition of maps as the category of undirected abstract graphs.

Example.

(1) Let $V = \{A, B, C\}$, $E = \{e, f, g, h\}$ and $\varphi(e) = \{A, B\}$ and let $\varphi\colon E \to \mathcal{P}(V)$ be the map that is given by $\varphi(f) = \{C\}$, and $\varphi(g) = \{A, B\}$ and $\varphi(h) = \{B, C\}$. This defines an undirected abstract graph with loop f and multi-edges. As before we can visualize such an undirected abstract graph by drawing points for the vertices and by either drawing a segment that either connects the two endpoints of an edge or by drawing a loop if the edge has a single endpoint.



(2) Let $\mathcal{AbsGraph}$ be the category of abstract graphs and let $\mathcal{UndirAbsGraph}$ be the category of undirected abstract graphs. It follows easily from the definitions that the map

$$\begin{aligned} F\colon \mathrm{Ob}(\mathcal{AbsGraph}) &\to \mathrm{Ob}(\mathcal{UndirAbsGraph}) \\ (V, E, i\colon E \to V, t\colon E \to V) &\mapsto \begin{pmatrix} V, E, & E & \to & \mathcal{P}(V) \\ & e & \mapsto & \{i(e), t(e)\} \end{pmatrix} \end{aligned}$$

together with the hopefully obvious maps from the morphisms in *AbsGraph* to the morphisms in *UndirAbsGraph* is a covariant functor from the category *AbsGraph* of abstract graphs to the category *UndirAbsGraph* of undirected abstract graphs.[103]

Now comes the slightly intimidating definition of the topological realization of an undirected abstract graph.

Definition. Let $G = (V, E, \varphi)$ be an undirected abstract graph. We define the topological realization of G as the topological space[104]

$$|G| \;:=\; \Big(V \;\sqcup \bigsqcup_{\substack{e\,\in\,E\\ \#\varphi(e)\,=\,1}} [0,1] \;\sqcup \bigsqcup_{\substack{e\,\in\,E\\ \#\varphi(e)\,=\,2}} [0,1]\times \mathrm{Bij}(\{0,1\},\varphi(e))\Big)\Big/\!\sim$$

where $\sim$ is generated by the following two relations:

(1) For each $e \in E$ with $\#\varphi(e) = 1$ we identify 0 and 1 in the corresponding copy of $[0,1]$ with the unique point $\varphi(e) \in V$.

(2) For each $e \in E$ with $\#\varphi(e) = 2$ we make the following identifications:
   (a) For each $\gamma \in \mathrm{Bij}(\{0,1\}, \varphi(e))$ we set $(0,\gamma) \sim \gamma(0)$ and $(1,\gamma) \sim \gamma(1)$.
   (b) For $\alpha \neq \beta \in \mathrm{Bij}(\{0,1\}, \varphi(e))$ and each $t \in [0,1]$ we set $(t,\alpha) \sim (1-t,\beta)$.

Example. Using a variation on Lemma 7.3, or alternatively using Lemma 7.5 below one can see that the subset of $\mathbb{R}^2$ shown in the figure below is a topological realization of the subway system of München.



Lemma 7.5. Let $G = (V, E, i, t)$ be an abstract graph. As above we consider the undirected abstract graph $\widetilde{G} = (V, E, \varphi)$ where $\varphi\colon E \to \mathcal{P}(V)$ is the map that is given by

[103] It follows from the Axiom of Choice 0.1 that given any undirected abstract graph G there exists an abstract graph G' with $F(G') = G$.

[104] Let $e \in E$ be a vertex with $\#\varphi(e) = 2$. Note that in this case $\mathrm{Bij}(\{0,1\}, \varphi(e))$ has precisely two elements. The problem is that there is no "preferred" bijection, so we need to consider both in our definition.

$\varphi(e) = \{i(e), t(e)\}$. The map

$$\begin{aligned} \Theta_G\colon |G| &\to |\widetilde{G}| \\ [x] &\mapsto \begin{cases} [x], & \text{if } x \in V, \\ [t \in [0,1]_e], & \text{if } x = (e,t) \in E \times [0,1] \text{ with } i(e) = t(e), \\ [(t,\gamma)], & \text{if } x = (e,t) \in E \times [0,1] \text{ with } i(e) \neq t(e), \text{ where } \gamma \text{ is} \\ & \text{given by } \gamma(0) = i(e),\ \gamma(1) = t(e) \end{cases} \end{aligned}$$

is a homeomorphism.

Proof. It follows fairly easily from the definitions that the map is well-defined and a bijection. Furthermore one can deduce easily from the Twice Quotient Lemma 5.23 that the map is continuous. Working directly with the definitions one sees that the map is open. In summary it follows from the Open -Injective Map Lemma 2.16 that the map is indeed a homeomorphism. ■

Remark. Perhaps by now it has become clear why the definition of the topological realization of an *undirected* abstract graph is more painful than the definition of the topological realization of an abstract graph. In both cases we want to attach one interval for each edge. If we are given maps $i, t\colon E \to V$, then we have a preferred way of doing so. If we are only given a map $\varphi\colon E \to \mathcal{P}(V)$ none of the two ways is preferred, thus we need to attach an interval in both ways and then we need to "anti-identify" the two intervals.

The following definition is almost verbatim the same as the definition of a connected abstract graph that we provided on page 307.

Definition. We say that an undirected abstract graph (V, E, φ) is connected if for every two vertices $v \neq w \in V$ there exist vertices $v = v_0, v_1, \ldots, v_n = w$ and edges $e_0, \ldots, e_{n-1}$ such that for each $i \in \{0, \ldots, n-1\}$ the endpoints of e_i are v_i and v_{i+1}.

The following totally elementary lemma can be viewed as an analogue of the Connected-to-Discrete Lemma 2.27.

Lemma 7.6. Let $G = (V, E, i, t)$ be an abstract graph or let $G = (V, E, \varphi)$ be an undirected abstract graph. Let $f\colon V \to X$ be a map to some set X such that for any $e \in E$ the values of f on the endpoints of e agree. If G is connected, then f is constant.

Proof. This lemma follows almost immediately from the definitions. ■

7.3. **Trees.** In this last section we study a basic but surprisingly useful class of abstract graphs.

Definition. Let $G = (V, E, i, t)$ be an abstract graph or let (G, V, φ) be an undirected abstract graph.

(1) If G is finite, then we refer to

$$\chi(G) \;=\; \#V - \#E \;=\; \text{number of vertices} - \text{number of edges}$$

as the Euler characteristic of G.

(2) A tree is a finite connected abstract graph with Euler characteristic 1.

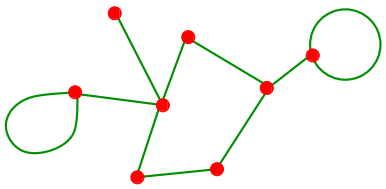
(3) (a) If $G = (V, E, i, t)$ is an abstract graph, then the valence of a vertex v is defined as

$$\text{valence}(v) \;:=\; \#\{e \in E \,|\, i(e) = v\} + \#\{e \in E \,|\, t(e) = v\}.$$

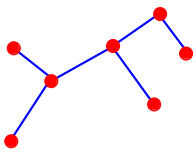
(b) If $G = (V, E, \varphi)$ is an undirected abstract graph, then the valence of a vertex v is defined as

$$\begin{aligned}\text{valence}(v) \quad := \quad & \#\{e \in E \,|\, v \in \varphi(e) \text{ and } \#\varphi(e) = 2\} \\ & +2 \cdot \#\{e \in E \,|\, v \in \varphi(e) \text{ and } \#\varphi(e) = 1\}.\end{aligned}$$

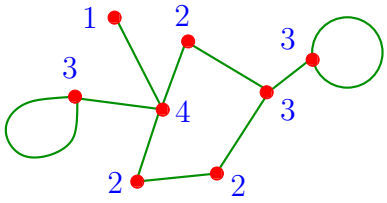
In a slightly less formal way, the valence of a vertex v is the number of edges at v, but hereby we have to count any loop at v twice, since it "goes twice into v".



Euler characteristic of G equals $\chi(G) = 8 - 10 = -2$



example of a tree



valences of the vertices

Lemma 7.7. Let $G = (V, E, i, t)$ be an abstract graph or let $G = (V, E, \varphi)$ be an undirected abstract graph. If G is a tree and if G has at least one edge, then G admits at least two vertices of valence one.

Proof. We prove Lemma 7.7 for abstract graphs, the proof for undirected abstract graphs is verbatim the same. Thus let $G = (V, E, i, t)$ be a tree that has at least one edge. Since G has at least one edge and since it is by definition connected, we see that the tree G does not have a vertex of valence 0. We calculate that

$$\begin{aligned}\sum_{\text{vertex } v} \underbrace{\text{valence}(v)}_{\geq 1} \quad &= \quad \sum_{\text{vertex } v} (\#\{e \in E \,|\, i(e) = v\} + \#\{e \in E \,|\, t(e) = v\}) \\ &= \quad 2 \cdot \#\text{ edges} \quad = \quad 2 \cdot \#\text{vertices} - 2.\end{aligned}$$

each edge contributes 2 to the sum — since $\chi(T) = 1$

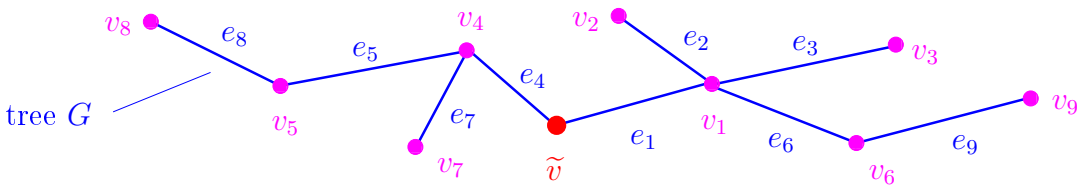
Recall that we just observed that the valence of each vertex is at least one. Thus the above equality can only hold if there are at least two vertices with valence one. ∎

On one occasion the following lemma will come in handy.

Lemma 7.8. Let $G = (V, E, i, t)$ be an abstract graph and let $\widetilde{v} \in V$ be a vertex. If G is a tree, then there exist distinct edges $e_1, \ldots, e_k$ and distinct vertices $v_1, \ldots, v_k$ with the following properties:

(1) We have $V = \{\widetilde{v}, v_1, \ldots, v_k\}$ and $E = \{e_1, \ldots, e_k\}$.
(2) For each $i \in \{1, \ldots, k\}$ the vertex v_i is an endpoint of the edge e_i.
(3) If some vertex v_i is an endpoint of some e_j, then $i \leq j$.

The same statement also holds for undirected abstract graphs.



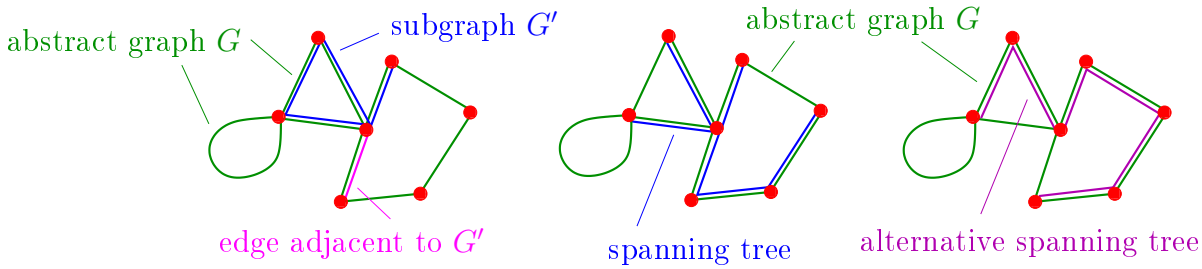
Proof. For abstract graphs we will provide the proof of the corollary in Exercise 7.7. For undirected abstract graphs the proof is verbatim the same. ■

Now we provide a few more definitions on (undirected) abstract graphs.

Definition. Let $G = (V, E, i, t)$ be an abstract graph.

(1) A subgraph of $G = (V, E, i, t)$ is an abstract graph $G' = (V', E', i', t')$ such that $V' \subset V$ and $E' \subset E$ are subsets such that $i' = i|_{E'}$ and $t' = t|_{E'}$. If G' is a tree, then we refer to G' as a subtree. Since i' and t' are determined by V' and E' we often suppress both of them from the notation.
(2) We say an edge $e \in E$ is adjacent to a subgraph G' if e has two endpoints where one of the endpoints lies on G' and the other does not.
(3) A spanning tree for G is a subgraph that is a tree and which does not admit an adjacent edge.

For undirected abstract graphs we introduce the obvious analogues of the above notions.



Remark. Let $G = (V, E, i, t)$ be an abstract graph.

(1) Every $v \in V$ gives rise to the rather trivial subgraph with a single vertex v and no edge.
(2) If an edge e of E is adjacent to some subgraph G', then it is not part of the subgraph. Indeed, by definition of a subgraph, for any edge that is part of the subgraph also its endpoints have to lie in the subgraph.
(3) The example of the figure above shows that an abstract graph usually has many different spanning trees.
(4) A spanning tree is often also called a *maximal tree*.

The following lemma summarizes a few basic properties of subgraphs.

Lemma 7.9. Let $G = (V, E, i, t)$ be an abstract graph and let $G' = (V', E', i', t')$ be a subgraph.

(1) The inclusions $V' \to V$ and $E' \to E$ define a map $\iota\colon G' \to G$ of abstract graphs in the sense of the definition on page 308.
(2) The induced map $|\iota|\colon |G'| \to |G|$ between the topological realization is a closed embedding.

In the following we will sometimes use (2) to view $|G'|$ as a closed subset of $|G|$.

Proof.

(1) This statement follows immediately from the definitions.
(2) By the discussion on page 310 we know that the map $|\iota|$ is continuous. By the Open -Injective Map Lemma 2.16 it remains to show that $|\iota|$ is a closed map. We leave it to

the reader to show, using Lemma 5.12 and the Topological-Quotient Proposition 5.15 (1a), that $|\iota|$ is closed. ■

We conclude this section with the following proposition.

Proposition 7.10. (Spanning Tree Existence Proposition) Let $G = (V, E, i, t)$ be a finite connected non-empty abstract graph.

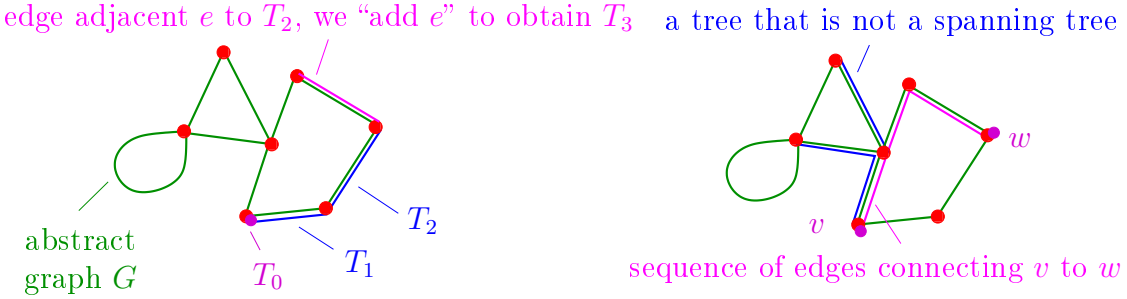
(1) The graph G admits a spanning tree.
(2) Every spanning tree of G contains all vertices of G.

The same statement also holds for undirected abstract graphs.

Remark. On page 1520 we will generalize the notion of a (spanning) tree to infinite abstract graphs and in the Spanning Tree Existence Proposition 69.7 we will prove a generalization of the Spanning Tree Existence Proposition 7.10 to infinite abstract graphs.

Proof. In the following we provide the proof for abstract graphs. The proof for undirected abstract graphs is verbatim the same. Thus let $G = (V, E, i, t)$ be a finite connected abstract graph.

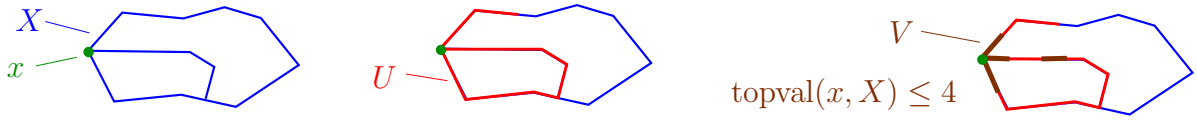
(1) We pick a vertex $v \in V$. We define $T_0 = \{v\}$ and view T_0 as a subtree of G. Now we construct iteratively a sequence of subtrees $T_0 \subset T_1 \subset \dots$ as follows: Suppose we have a subtree $T_i = (V_i, E_i)$. If there exists an edge $e \in E$ that is adjacent to T_i, then we "add e to T_i", more precisely, we define $T_{i+1} = (V_i \cup \varphi(e), E_i \cup e)$. Note that by definition of an adjacent edge we just added precisely one edge and precisely one vertex to T_i. Thus we see that $\chi(T_{i+1}) = \chi(T_i) = \dots = \chi(T_0) = 1$, i.e. T_{i+1} is indeed a subtree. Since G is finite this process stops after finitely many steps. Let T be the final subtree of our sequence. Note that by definition T is in fact a spanning tree. This argument is illustrated in the figure below on the left.
(2) Let T be a subgraph of G that is a tree. Suppose that there exists a vertex w that does not lie in T. We will show that T is not a spanning tree. Let v be a vertex in T. Since G is connected there exists a sequence of edges $e_0, \dots, e_n$ with the following properties:
   (a) v is a vertex of e_0,
   (b) for each $i \in \{0, \dots, n-1\}$ a vertex of e_i is also a vertex of e_{i+1}, and
   (c) w is a vertex of e_n.
   Then there exists an $i \in \{0, \dots, n-1\}$ such that a vertex of e_i lies in T and the other one does not. But then e_i is an edge adjacent to T. This shows that T was not a spanning tree. This argument is illustrated in the figure below on the right. ■



7.4. **Topological valence.** On page 315 we introduced the valence of a vertex in an (undirected) abstract graph. In this section we introduce an analogue for points of topological spaces.

Definition. Let X be a topological space and let $x \in X$. We define the topological valence of x as

$$\operatorname{topval}(x, X) := \min \left\{ n \in \mathbb{N} \cup \{\infty\} \;\middle|\; \begin{array}{c} \text{for every open neighborhood } U \text{ of } x \text{ there} \\ \text{exists an open neighborhood } V \text{ of } x \text{ with } V \subset U \\ \text{such that } V \setminus \{x\} \text{ has at least } n \text{ components} \end{array} \right\}.$$



The following lemma summarizes a few basic properties about the topological valence of points.

Lemma 7.11. Let X be a topological space and let $x \in X$.

(1) For every neighborhood W of x we have $\operatorname{topval}(x, W) = \operatorname{topval}(x, X)$.
(2) For every homeomorphism $f\colon X \to Y$ we have $\operatorname{topval}(x, X) = \operatorname{topval}(f(x), f(X))$.

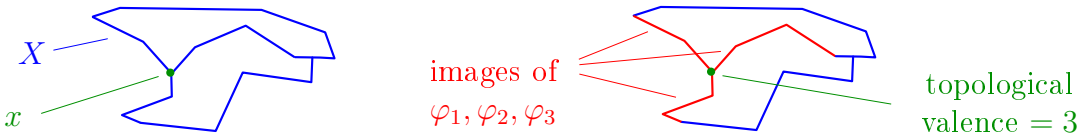
Proof. The lemma follows immediately from the definitions. ∎

The following proposition allows for the calculation of the valence in many interesting settings.

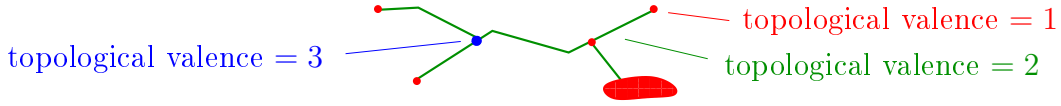
Proposition 7.12. (Topological Valence Determination Proposition) Let X be a Hausdorff space and let $x \in X$ be a point. We now suppose that there exist embeddings $\varphi_1, \ldots, \varphi_n\colon [0,1] \to X$ with the following properties:

(a) $\varphi_1(0) = \cdots = \varphi_n(0) = x$,
(b) the images $\varphi_i((0,1])$, $i = 1, \ldots, n$ are pairwise disjoint,
(c) $\varphi_1([0,1]) \cup \cdots \cup \varphi_n([0,1])$ is a neighborhood of x.

Then $\operatorname{topval}(X, x) = n$.



Example. Let I be an interval that consists of more than one point. It follows easily from the Topological Valence Determination Proposition 7.12 that the points in the interior of I have topological valence 2 and that the endpoints of I, if they are included, have topological valence 1. This observation implies for example that the half-open interval $[0,1)$ and the open interval $(0,1)$ are not homeomorphic. This gives in particular a solution to Exercise 2.40.



Proof. We start out with the following claim.

Claim. There exists a $\lambda \in (0,1]$ such that for every $\epsilon \in (0,\lambda]$ the following holds:
(1) For each $i \in \{1,\dots,n\}$ the set $\varphi_i((0,\epsilon))$ is open in X.
(2) For each $i \in \{1,\dots,n\}$ the set $\varphi_i((0,\epsilon))$ is closed in $\varphi_1((0,\epsilon)) \cup \dots \cup \varphi_n((0,\epsilon))$.
(3) $\varphi_1([0,\epsilon)) \cup \dots \cup \varphi_n([0,\epsilon))$ is an open neighborhood of x.

Proof. Note that by Hypothesis (c) we know that there exists an open neighborhood W of x that is contained $\varphi_1([0,1]) \cup \dots \cup \varphi_n([0,1])$. By looking at the preimages of W under the maps $\varphi_1,\dots,\varphi_n$ we see that there exists a $\lambda \in (0,1]$ such that $\varphi_1([0,\lambda]),\dots,\varphi_n([0,\lambda])$ are contained in W. Now let $\epsilon \in (0,\lambda]$.

(1) Note that

$$\varphi_i((0,\epsilon)) \;=\; W \cap \Big(X \setminus \Big(\underbrace{\varphi_i([\epsilon,1]) \cup \bigcup_{j\neq i} \varphi_j([0,1])}_{\text{compact by the Compact Image Lemma 2.13, since } X \text{ is Hausdorff we obtain from the Compact-Closed Lemma 1.21 (2), that the set is closed}} \Big)\Big).$$

choice of x and Hypothesis (b)

Thus we see that $\varphi_i((0,\epsilon))$ is an open subset of X.
(2) This statement follows immediately from (1).
(3) Almost the same argument as in (1) also implies (3). ⊞

We now turn to the actual proof that $\operatorname{topval}(x,x) = n$. First we want to show that $\operatorname{topval}(X,x) \leq n$. Thus let U be a neighborhood of x. There exists a $\mu \in (0,\epsilon)$ such that for each $i \in \{1,\dots,n\}$ we have $\varphi_i((0,\mu)) \subset U$. By the above claim we know that $V := \varphi_1([0,\mu)) \cup \dots \cup \varphi_1([0,\mu))$ is an open neighborhood of X. Next note that $V \setminus \{x\}$ consists of the n open and closed sets $\varphi_1((0,\mu)),\dots,\varphi_n((0,\mu))$. By the Path-Connected Implies Connected Corollary 2.23 we know that each of these sets is connected. Thus we obtain from the Component Lemma 2.33 (4c) that $V \setminus \{x\}$ has precisely n components.

Now let us show that $\operatorname{topval}(X,x) \geq n$. We set $U := \varphi_1((0,\epsilon)),\dots,\varphi_n((0,\epsilon))$. Let V be an open neighborhood of x that is contained in U. For each $i \in \{1,\dots,n\}$ we know that $\varphi_i^{-1}([0,\epsilon]) \cap V$ is open and that it contains 0. This implies that $\varphi_i^{-1}((0,\epsilon]) \cap (V \setminus \{x\})$ is also non-empty. By (2) this set is open and closed in $V \setminus \{x\}$. It follows from the Component Lemma 2.33 (4b) that $V \setminus \{x\}$ has at least n components. ■

The final proposition of this section shows that given an undirected abstract graph the valence of a vertex agrees with the topological valence of the corresponding point on the topological realization.

Proposition 7.13.
(1) Let $G = (V,E,\varphi)$ be an undirected abstract graph. For every vertex $v \in V$ we have

$$\operatorname{valence}(v) \;=\; \operatorname{topval}(v,|G|) \;\in\; \mathbb{N}_0 \cup \{\infty\}.$$

(2) If $f\colon X = |G| \to Y = |H|$ is a homeomorphism between the topological realizations of undirected abstract graphs G and H that sends vertices to vertices, then this homeomorphism preserves valences, viewed as numbers in $\mathbb{N}_0 \cup \{\infty\}$.

Remark. An alternative proof of Proposition 7.13 will be given in Exercises 75.2 and 75.9.

Proof.

(1) If the valence(v) is finite, then the statement follows easily from the Topological Valence Determination Proposition 7.12. We leave it to the reader to modify the argument if valence(v) is infinite.
(2) This statement follows immediately from (1) and Lemma 7.11 (2). ■

7.5. **Quotients of abstract graphs.** We conclude this chapter on graphs with the definition and discussion of quotients of abstract graphs and the corresponding topological realizations.

Definition. Let $G = (V, E, i, t)$ be an abstract graph and let $H = (W, F)$ be a non-empty subgraph.

(1) We define $\sim$ to be the equivalence relation generated by $v \sim v'$ if v and v' are both vertices in H. We denote by $*$ the equivalence class defined by the vertices in H.
(2) We denote by $\rho\colon V \to V/\!\sim$ the map given by $v \mapsto [v]$.
(3) Finally we define the abstract graph

$$G/H \;:=\; (V/\!\sim, E \setminus F, \rho \circ i\colon E \setminus F \to V/\!\sim, \rho \circ t\colon E \setminus F \to V/\!\sim).$$

We refer to G/H as the quotient of the abstract graph G by the abstract graph H.

The following lemma summarizes a few properties of the abstract graph G/H.

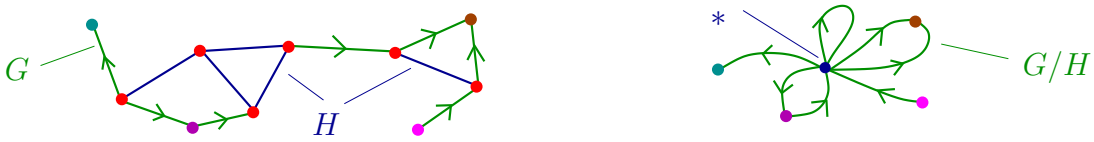
Lemma 7.14. (Quotient Abstract Graph Lemma) Let $G = (V, E, i, t)$ be an abstract graph and let $H = (W, F)$ be a non-empty subgraph. We continue with the above notation.

(1) We have $\chi(G/H) = \chi(G) - \chi(H) + 1$.
(2) We consider the map

$$p\colon |G| = (V \sqcup (E \times [0,1]))/\overset{\text{where } (e,0)\sim i(e) \text{ and } (e,1)\sim t(e)}{\sim} \to |G/H| = (V/\!\sim \sqcup ((E \setminus F) \times [0,1]))/\overset{\text{similarly } (e,0)\sim \rho(i(e)) \text{ and } (e,1)\sim \rho(t(e))}{\sim}$$

$$[x] \mapsto \begin{cases} [*], & \text{if } x \in |H|, \\ [\rho(x)], & \text{if } x \in V, \\ [(e,t)], & \text{if } x = (e,t) \text{ with } e \in E \setminus F \text{ and } t \in [0,1]. \end{cases}$$

This map has the following two properties:
(a) The map $p\colon |G| \to |G/H|$ is well-defined and continuous.
(b) If the abstract graph G is finite, then the map $p\colon |G| \to |G/H|$ descends to a homeomorphism[105] $|G|/|H| \to |G/H|$.



Proof.

(1) We calculate that

$$\chi(G/H) = \#(V/\!\sim) - \#(E \setminus F) = (\#V - \#W + 1) - (\#E - \#F) = \chi(G) - \chi(H) + 1.$$

[105]Note that it follows from Lemma 7.9 (2) that it is legitimate to view $|H|$ as a subset of $|G|$.

(2) (a) Once one has entangled all the definitions one sees that the map is indeed well-defined and one sees, using the Twice Quotient Lemma 5.23, that the map is indeed continuous.

(b) Since the abstract graph G is finite we know by the Abstract Graph-Topological Realization Lemma 7.1 (4) and the Compact Image Lemma 2.13 that $|G|/|H|$ is compact. Furthermore, by the Abstract Graph-Topological Realization Lemma 7.1 (2) we know that $|G/H|$ is Hausdorff. It is very elementary to show that the induced map $p\colon |G|/|H| \to |G/H|$ is a bijection. Thus we obtain from the Compact-Hausdorff Proposition 2.17 (3) that $p\colon |G|/|H| \to |G/H|$ is a homeomorphism. ■

Exercises for Chapter 7.

Exercise 7.1. We consider the category *AbsGraph* of abstract graphs. Let X, Y be abstract graphs.

(a) Does the product of X and Y exist in the category *AbsGraph*?
(b) Does the coproduct of X and Y exist in the category *AbsGraph*?

Exercise 7.2. In the below we see the seven bridges of Königsberg. Is it possible to cross all seven bridges in a walk without crossing a single bridge twice?

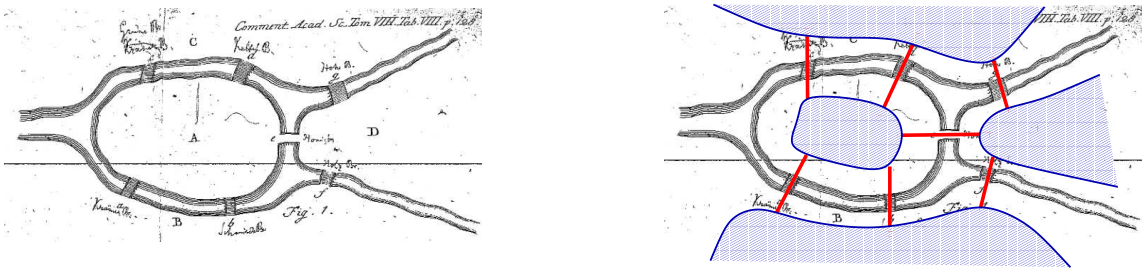


Illustration from Euler's paper [**Eul1736**]

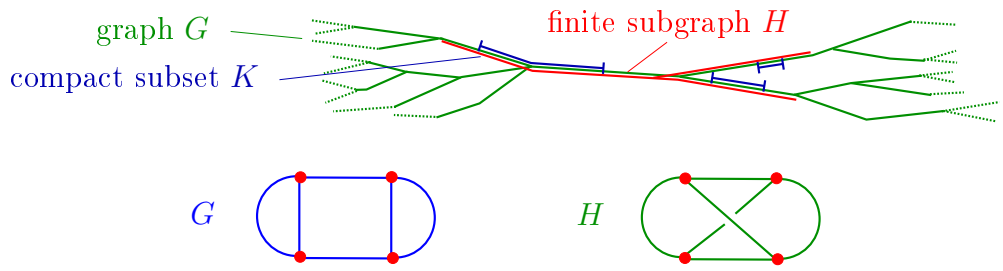
Exercise 7.3. Let $G = (V, E, \varphi)$ and $G' = (V', E', \varphi')$ be two undirected abstract graphs. If the topological realizations $|G|$ and $|G'|$ are homeomorphic, does it follow that the undirected abstract graphs G and G' are isomorphic?

Exercise 7.4. Let G be an abstract graph. Show that the following are equivalent:

(a) G is connected,
(b) given any two distinct vertices v, v' there exists an embedding $\gamma\colon [0,1] \to |G|$ with $\gamma(0) = v$, $\gamma(1) = v'$ and such that $\gamma([0,1]) \subset |G|$ is the union of edges,
(c) the topological realization $|G|$ is path-connected,
(d) the topological realization $|G|$ is connected.

Exercise 7.5. Let G be an abstract graph and let $K \subset |G|$ be compact subset. Show that there exists a finite subgraph H of G with $K \subset |H|$.

Exercise 7.6. Are the two topological graphs G and H shown in the figure below homeomorphic?



Exercise 7.7. Let $G = (V, E, i, t)$ be an abstract graph and let $\widetilde{v} \in V$ be a vertex. We suppose that G is a tree. Show that there exist distinct edges $e_1, \ldots, e_k$ and distinct vertices $v_1, \ldots, v_k$ with the following properties:

(1) We have $V = \{\widetilde{v}, v_1, \ldots, v_k\}$ and $E = \{e_1, \ldots, e_k\}$.
(2) For each $i \in \{1, \ldots, k\}$ the vertex v_i is an endpoint of the edge e_i.
(3) If some vertex v_i is an endpoint of some e_j, then $i \leq j$.

We refer to the figure on page 316 for an illustration.

Exercise 7.8. Let $G = (V, E, i, t)$ be a countable connected non-empty abstract graph. We pick a vertex $v \in V$. Show that there exists a sequence $\{G_j = (V_j, E_j, i, t)\}_{j \in \mathbb{N}}$ of subgraphs with the following properties:

(1) Each G_j is finite and connected.
(2) Each G_j contains v.
(3) We have $G = \bigcup\limits_{j \in \mathbb{N}} G_j$ in the sense that $V = \bigcup\limits_{j \in \mathbb{N}} V_j$ and $E = \bigcup\limits_{j \in \mathbb{N}} G_j$.

Exercise 7.9.

(a) Let $A \subset \mathbb{R}^3$ be a finite set. We say the points in A are in *general position* if no distinct three points in A are collinear and if no four points in A are coplanar. Show that if $t_1 < \cdots < t_k$ are real numbers, then the points

$$\{(t_i, t_i^2, t_i^3) \,|\, i = 1, \ldots, k\}$$

are in general position.

(b) Show that given any finite abstract graph G there exists an embedding of the topological realization $|G|$ into $\mathbb{R}^3$.

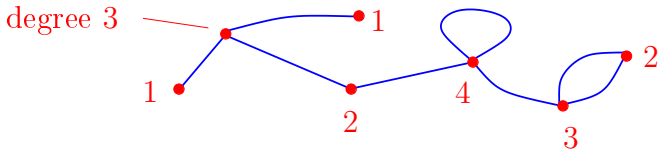
Exercise 7.10. As above let $\mathcal{Top}$ be the category of topological spaces, let $\mathcal{AbsGraph}$ be the category of abstract graphs and let $\mathcal{UndirAbsGraph}$ be the category of undirected abstract graphs. In Lemma 7.2 we defined a functor $|-|$ from $\mathcal{AbsGraph}$ to $\mathcal{Top}$. Furthermore on page 313 we defined a functor F from $\mathcal{AbsGraph}$ to $\mathcal{UndirAbsGraph}$ such that for any abstract graph the homeomorphism $\Theta_G \colon |G| \to |F(G)|$ between the topological realizations is a homeomorphism.

Is it possible to define a functor Ψ from $\mathcal{UndirAbsGraph}$ to $\mathcal{Top}$ such that the homeomorphisms Θ_G are natural?
Remark. Beware of loops.

Exercise 7.11. Let $G = (V, E, i, t)$ be a finite abstract graph. The degree of a vertex v is defined as

$$\deg(v) \;:=\; \#\{e \in E \,|\, i(e) = v\} + \#\{e \in E \,|\, t(e) = v\}.$$

Show that the number of vertices of odd degree is even.
Remark. This statement is known as the *Handshaking Lemma*, the name comes from the observation that the statement means that at an event where some people shake hands, an even number of people have shaken an odd number of other people's hands.



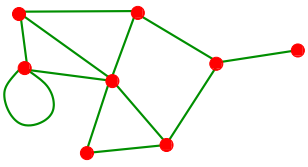
Exercise 7.12. We consider the topological space $X = \mathbb{Q}$. What is the topological valence $\operatorname{topval}(0, \mathbb{Q})$?

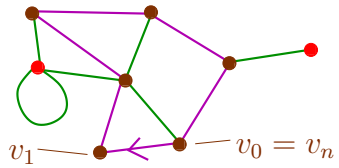
Exercise 7.13. Let $G = (V, E, \varphi)$ be an unoriented abstract graph. First we introduce some definitions.

(1) A *circuit of length* n in G is a sequence of vertices $v_0, v_1, v_2, \ldots, v_n = v_0$ such that the vertices $v_0, \ldots, v_{n-1}$ are distinct and such for each $i \in \{0, \ldots, n-1\}$ there exists an $e \in E$ with $\varphi(e) = \{v_i, v_{i+1}\}$.
(2) We say G is a *tree* if G is connected and if G contains no circuits.
(3) We say a subgraph $T \subset G$ is a *spanning tree* if T is a tree and if T contains all vertices of G.

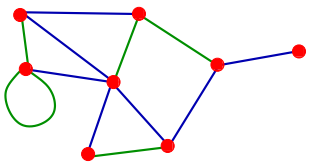
Now we can state the actual problems.

(a) Show that for a finite unoriented abstract graph the above definition of a tree is equivalent to the definition on page 315.
(b) Show that for a finite unoriented abstract graph the above definition of a spanning tree is equivalent to the definition on page 317.
(c) Show that every connected unoriented abstract graph G admits a spanning tree.
Hint. Apply Zorn's Lemma 0.6 to the set of trees in G with the partial order given by inclusion.


unoriented abstract graph G



circuit C


example of a spanning tree

Exercise 7.14. Let $G = (V, E, \varphi)$ be an unoriented abstract graph and let π be a group. A *π-action on G* is a homomorphism $\Phi\colon \pi \to \operatorname{Aut}(G)$. We say $v \in V$ is a *fixed vertex of the π-action* if for every $g \in \pi$ we have $\Phi(g)(v) = v$.

(a) Let T be an unoriented abstract graph that is a tree in the sense of the definition on page 324. Show that there exists a finite subtree $\widetilde{T}$ such that the π-action restricts to a π-action on $\widetilde{T}$.
(b) Let π be a finite group that acts on an undirected finite tree $\widetilde{T}$. Show that this action admits a fixed vertex.

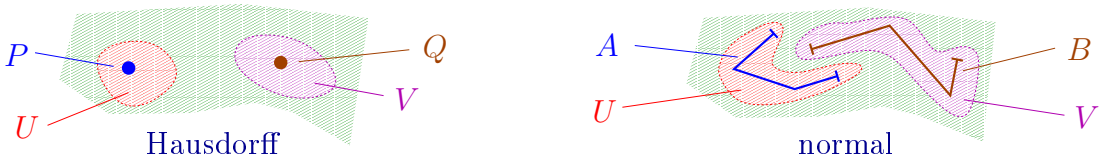
Remark. You can prove the statement by induction on $\#T$. You can perform the induction step by considering the vertices of valency ≥ 2.

8. Urysohn's Lemma and the Tietze Extension Theorem

In this chapter we state and prove Urysohn's Lemma and the Tietze Extension Theorem. For the most part the proofs of the results we stated so far are fairly straightforward and could be assigned as (at times slightly challenging) homework exercises. The results of this chapter are the first really non-trivial statements.

8.1. **Normal topological spaces.** Recall that according to the definition on page 121 a topological space X is called *Hausdorff* if any two distinct points $P \neq Q$ can be separated by disjoint open subsets, i.e. if there exist disjoint open neighborhoods U of P and V of Q. As we will see in the following two chapters, and especially in Section 9.4, there are many variations on this definition. In this chapter we will mostly care about the following concept, which we already introduced on page 123.

Definition. A topological space X is called normal if any two disjoint closed subsets A and B of X can be separated by open neighborhoods, i.e. if there exist disjoint open neighborhoods U of A and V of B.



Remark.

(1) Note that the definitions of a normal topological space can differ slightly in the literature. For example [**Mun1975**, p. 195] imposes the extra condition that one-point sets are closed.
(2) Let X be a normal topological space. If every one-point subset is closed, then X is also Hausdorff.[106] But, as we will see in Exercise 8.1, without this condition there do in fact exist normal spaces that are not Hausdorff.
(3) Let X be the line with two zeros that we introduced on page 122. One can easily verify that every one-point subset is closed. We saw that X is not Hausdorff. Thus it follows from (2) that X is not normal either.
(4) In Exercise 8.3 we will give an example of a topological space that is Hausdorff but which is not normal.
(5) It follows easily from the Open -Closed Inclusion Lemma 2.10 (2) that any closed subset of a normal space is again normal. A topological space that has the property that every subspace is normal is called *hereditarily normal*. As we will see in Proposition 9.12 together with Exercise 9.18, there exist normal spaces that are not hereditarily normal.

For the reader's convenience we recall the Compact Hausdorff Spaces-Normal Lemma 1.16. Together with the subsequent the Metric Spaces are Normal Proposition 8.1 it provides us with a long list of examples of normal spaces.

[106] On page 349 we define a T_0-*space* to be a topological space that has the property that each one-point subset is closed.

Lemma 1.16. (Compact Hausdorff-Normal Lemma) Every compact Hausdorff space is normal.



The following proposition gives us an often convenient approach to showing that a topological space is normal. It implies, among many other things, that every subspace of some $\mathbb{R}^n$ is normal.

Proposition 8.1. (Metric Spaces-Normal Proposition) Let (X, d) be a metric space. If we equip X with the corresponding topology, then X is normal.

Proof. Let (X, d) be a metric space. We recall the following notation from page 283: given $x \in X$ and given a non-empty subset C of X we write

$$d(x, C) \ := \ \inf\{d(x, c) \,|\, c \in C\}.$$

If C is closed, then we know by the Distance-to-Subspace Lemma 6.2 (2) that $d(x, C) = 0$ if and only if $x \in C$.

Now let A and B be two disjoint closed subsets of X. If one of A or B is empty, then the desired disjoint neighborhoods are given by the empty set and X. Thus it remains to consider the case that both A and B are non-empty.



In our context, since A and B are disjoint and closed, we conclude from the above that for every $x \in X$ we have $d(x, A) + d(x, B) > 0$. Thus it makes sense to consider the map

$$\begin{array}{rcl} X & \to & \mathbb{R} \\ x & \mapsto & \dfrac{d(x, A)}{d(x, A) + d(x, B)}. \end{array}$$

By the Distance-to-Subspace Lemma 6.2 (1) we know that f is continuous. Evidently $f|_A \equiv 0$ and $f|_B \equiv 1$. It follows that $U := f^{-1}((-\infty, \frac{1}{4}))$ and $V := f^{-1}((\frac{3}{4}, \infty))$ have the desired properties. ■

The following lemma, which is an analogue of the Points-in-Hausdorff Space Lemma 1.15, contains a convenient and frequently used property of normal spaces.

Lemma 8.2. (Normal-Open-Closed-Neighborhood Lemma) Let X be a topological space and let $A \subset X$ be a closed subset and $U \subset X$ be an open subset with $A \subset U$. If X is normal, then there exists an open subset $V \subset X$ with $A \subset V \subset \overline{V} \subset U$.



Proof. Let X be a topological space, let $A \subset X$ be a closed subset and let $U \subset X$ be an open subset with $A \subset U$. We set $B := X \setminus U$. This is a closed subset with $A \cap B = \varnothing$. Since X is normal there exist disjoint neighborhoods V of A and W of B. It remains to show that $\overline{V} \subset U$. In fact we have $V \subset X \setminus W \subset U$. But since W is open we know that $X \setminus W$ is closed. By definition of the closure $\overline{V}$ we have $\overline{V} \subset X \setminus W$. ■

Remark. The combination of the Paracompact-Criterion Lemma 10.2 and Dieudonné's Theorem 10.5 gives us many more examples of normal topological spaces.

We conclude this short introduction to normal topological spaces with the following lemma which often gets used subconsciously.

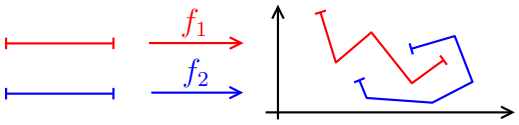
Lemma 8.3. Let Y be a normal space.

(1) Let $A_1, \dots, A_k \subset Y$ be disjoint subsets. If the A_i are all closed, then the natural map $A_1 \sqcup \dots \sqcup A_k \to A_1 \cup \dots \cup A_k$ is a homeomorphism.

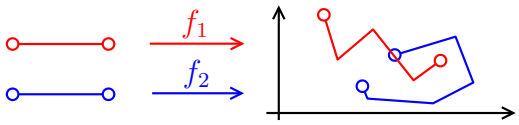
(2) Let $X_1, \dots, X_k$ be compact spaces and let $\varphi_i \colon X_i \to Y$ be embeddings with disjoint images. If Y is Hausdorff, then the map

$$\coprod_{i=1}^{k} f_i \colon \coprod_{i=1}^{k} X_i \;\to\; Y$$

is an embedding.



$f_1 \sqcup f_2 \colon X_1 \sqcup X_2 \to \mathbb{R}^2$ is an embedding



$f_1 \sqcup f_2 \colon X_1 \sqcup X_2 \to \mathbb{R}^2$ is *not* an embedding

Proof. Let Y be a normal space.

(1) Let $A_1, \dots, A_k \subset Y$ be closed disjoint subsets. Note that for any $i \in \{1, \dots, k\}$ the union of the A_j with $i \neq j$ is a finite union of closed subsets, thus it is a closed subset itself. Since X is normal we see that there exists an open neighborhood of A_i that does not intersect any of the other A_j. It follows from this observation and the Disjoint Union Embedding Lemma 5.14 that the natural map $A_1 \sqcup \dots \sqcup A_k \to A_1 \cup \dots \cup A_k$ is a homeomorphism.

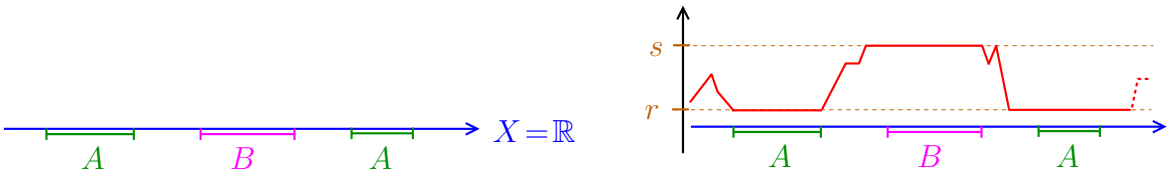
(2) Since $X_1, \dots, X_k$ are compact and since Y is in particular Hausdorff we know by the Compact Image Lemma 2.13 and the Compact-Closed Lemma 1.21 (2) that the images $\varphi_1(X_1), \dots, \varphi_k(X_k)$ are closed subsets of Y. It now follows from (1) that $\coprod_{i=1}^{k} f_i \colon \coprod_{i=1}^{k} X_i \to Y$ is indeed an embedding. ■

8.2. **Urysohn's Lemma.** In this section we will state and prove Urysohn's Lemma. Despite its modest name it is actually very much non-trivial.

Lemma 8.4. (Urysohn's Lemma) Let X be a topological space. The following two statements are equivalent:

(1) For any two disjoint closed subsets A and B and any $r < s \in \mathbb{R}$ there exists a continuous map $f \colon X \to [r, s]$ with $f|_A \equiv r$ and $f|_B \equiv s$.

(2) X is normal.



The statement of Urysohn's Lemma 8.4 is really quite amazing, just using the existence of suitable open subsets it builds up an interesting map.

Remark.

(1) Urysohn's Lemma 8.4 was first proved by Paul Urysohn [**Ury1925a**, p. 290] in 1924. The paper was finished three days before Paul Urysohn died in a swimming accident [**Gron2010**].
(2) A very nice motivation for the standard proof is given in [**Jä2005**, p. 109]. A somewhat different proof, using the Cantor function from page 185, is given in [**Cîr2021**].
(3) Note that Statement (1) in Urysohn's Lemma 8.4 does *not* say that there exists a continuous map $f\colon X \to [r,s]$ with $f^{-1}(r) = A$ and $f^{-1}(s) = B$. In fact, as we will see in Exercise 8.4, in general we cannot demand to find such a map on a normal space.
(4) The proof of the Metric Spaces-Normal Proposition 8.1 shows that the conclusion of Urysohn's Lemma 8.4 holds trivially for metric spaces.
(5) Urysohn's Lemma 8.4 motivates the interest in normal spaces. The combination of the Paracompact-Criterion Lemma 10.2 and the Dieudonné Theorem 10.5 shows that every topological space that is regionally compact, "second-countable" and Hausdorff is normal. In many settings this result will be strong enough to conclude that a given topological space is normal.
(6) In Lemma 21.7 we will prove a "smooth" version of Urysohn's Lemma 8.4.

Proof. The "(1)⇒(2)"-implication is straightforward. Indeed, suppose that A and B are two disjoint closed subsets of X. If (1) holds, then there exists a continuous function $f\colon X \to [0,1]$ with $f|_A \equiv 0$ and $f|_B \equiv 1$. Then $U := f^{-1}((-\infty, \frac{1}{2}))$ and $V := f^{-1}((\frac{1}{2}, \infty))$ have the desired properties.

The "(2)⇒(1)"-implication is the non-trivial direction of Urysohn's Lemma and its proof will occupy the remainder of the proof. Let A and B be two disjoint closed subsets. An elementary rescaling argument shows that it suffices to deal with the case $r = 0$ and $s = 1$, i.e. it suffices to show that there exists a continuous map $f\colon X \to [0,1]$ with $f|_A \equiv 0$ and $f|_B \equiv 1$.

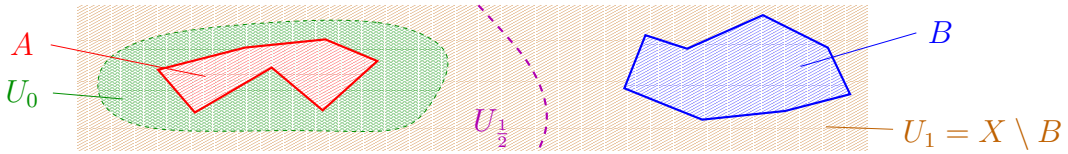
Claim 1. We set $P = [0,1] \cap \mathbb{Q}$. There exists a family $\{U_s\}_{s\in P}$ of open subsets of X with the following three properties:

(i) $A \subset U_0$,
(ii) for each $p \in P$ with $p < 1$ we have $U_p \subset X \setminus B$,
(iii) for each $s,t \in P$ with $s < t$ we have $\overline{U}_s \subset U_t$.

Proof. Since P is countable there exists a bijection $\varphi\colon \mathbb{N}_0 \to P$. We can arrange that $\varphi(0) = 0$ and $\varphi(1) = 1$. We set $U_1 := X \setminus B$. Since B is closed we know that U_1 is

open. Since X is normal and since A is closed we obtain from the Normal-Open-Closed-Neighborhood Lemma 8.2 that there exists an open neighborhood U_0 of A with $\overline{U}_0 \subset U_1$. Now suppose that for some $n \geq 1$ we have already defined $U_{\varphi(0)}, U_{\varphi(1)}, \ldots, U_{\varphi(n)}$ with the required properties. We set $P_n = \{\varphi(0), \ldots, \varphi(n)\}$ and we set $y := \varphi(n+1)$. Since $\varphi(0) = 0$ and $\varphi(1) = 1$ we see that $y \in (0,1)$. Since P_n is a finite subset of $[0,1]$ and since $\varphi(0) = 0$ and $\varphi(1) = 1$ we see that the set P_n contains a largest number x less than $y = \varphi(n+1)$ and it contains a smallest number z larger than $y = \varphi(n+1)$.

Note that our induction hypothesis implies that $\overline{U}_x \subset U_z$. Thus, since X is normal, we obtain from the Normal-Open-Closed-Neighborhood Lemma 8.2 that there exists an open subset U_y with $\overline{U}_x \subset U_y \subset \overline{U}_y \subset U_z$. One can now easily verify that $U_{\varphi(0)}, U_{\varphi(1)}, \ldots, U_{\varphi(n)}, U_{\varphi(n+1)}$ have the required properties. ⊞



For each $p \in (-\infty, 0) \cap \mathbb{Q}$ we set $U_p := \varnothing$ and for each $p \in (1, \infty) \cap \mathbb{Q}$ we set $U_p := X$. Next we consider

$$\begin{array}{rcl} f\colon X & \to & \mathbb{R} \\ x & \mapsto & \inf\{p \in \mathbb{Q} \,|\, x \in U_p\}. \end{array}$$

It follows easily from Properties (i) and (ii) that $f|_A \equiv 0$, that $f|_B \equiv 1$ and that f takes values in $[0,1]$. Thus it remains to prove that f is continuous. The proof of continuity requires some preparations.

Claim 2. Let $r \in \mathbb{Q}$.

$$\begin{array}{llll} \text{(a)} & x \in \overline{U}_r \implies f(x) \leq r & \text{(a')} & f(x) > r \;\Rightarrow\; x \notin \overline{U}_r. \\ \text{(b)} & x \notin U_r \implies f(x) \geq r & \text{(b')} & f(x) < r \;\Rightarrow\; x \in U_r. \end{array}$$

Proof. Statements (a') and (b') are evidently just reformulations of Statements (a) and (b). Statements (a) and (b) are trivial if $r < 0$ or $r > 1$. Now assume that $r \in P = [0,1] \cap \mathbb{Q}$. In this case the statements follow from the following straightforward arguments:

$$\begin{array}{lcl} & \overset{\text{by Property (iii)}}{\Longrightarrow} & \overset{\text{by definition of infimum and since } P \text{ is dense in } [0,1]}{\Longrightarrow} \\ x \in \overline{U}_r & \Longrightarrow & x \in U_s \text{ for every } s \in P \text{ with } s > r \;\Longrightarrow f(x) \leq r \\ x \notin U_r & \Longrightarrow & x \notin U_s \text{ for every } s \in P \text{ with } s < r \;\Longrightarrow f(x) \geq r. \end{array}$$ ⊞

Claim 3. Given any $r \in \mathbb{Q}$ we have

$$f^{-1}([0,r]) = \bigcap_{n\in\mathbb{N}} \overline{U}_{r+\frac{1}{n}} \qquad \text{and} \qquad f^{-1}([0,r)) = \bigcup_{n\in\mathbb{N}} U_{r-\frac{1}{n}}.$$

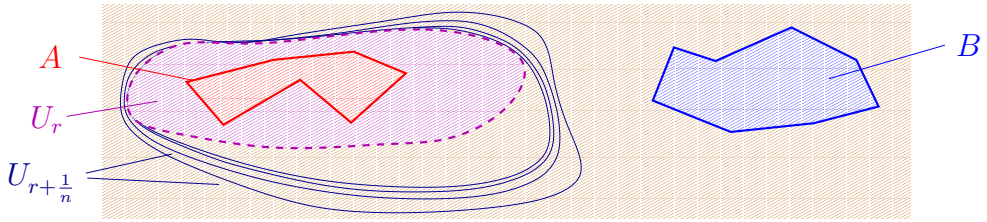
Proof. Let $r \in \mathbb{Q}$. We see that

$$\begin{array}{lcl} f^{-1}([0,r]) & = & f^{-1}\Big(\bigcap_{n\in\mathbb{N}} [0, r+\frac{1}{n})\Big) \overset{\text{Image-Preimage Lemma 0.2 (7)}}{=} \bigcap_{n\in\mathbb{N}} f^{-1}([0,r+\frac{1}{n})) \overset{\text{by Claim 2 (b')}}{\subset} \bigcap_{n\in\mathbb{N}} U_{r+\frac{1}{n}} \subset \bigcap_{n\in\mathbb{N}} \overline{U}_{r+\frac{1}{n}} \\ & \subset & \bigcap_{n\in\mathbb{N}} f^{-1}([0,r+\frac{1}{n}]) = f^{-1}\Big(\bigcap_{n\in\mathbb{N}} [0,r+\frac{1}{n}]\Big) = f^{-1}([0,r]). \\ & \uparrow \text{ by Claim 2 (a)} & \uparrow \text{ Image-Preimage Lemma 0.2 (7)} \end{array}$$

Note that at the beginning and at the end we have the same set. Thus all inclusions have to be equalities. Therefore we obtain the first equality of the claim. The proof of the second equality is very similar to the proof of the first equality. For completeness' sake we carry out the argument:

$$\begin{array}{ccccccc} f^{-1}([0,r)) & = & f^{-1}\Big(\bigcup_{n\in\mathbb{N}}[0,r-\frac{1}{n})\Big) & \overset{\text{Image-Preimage Lemma 0.2 (6)}}{\downarrow =} & \bigcup_{n\in\mathbb{N}} f^{-1}([0,r-\frac{1}{n})) & \overset{\text{by Claim 2 (b')}}{\downarrow \subset} & \bigcup_{n\in\mathbb{N}} U_{r-\frac{1}{n}} \subset \bigcup_{n\in\mathbb{N}} \overline{U}_{r-\frac{1}{n}} \\ & \subset & \bigcup_{n\in\mathbb{N}} f^{-1}([0,r-\frac{1}{n}]) & = & f^{-1}\Big(\bigcup_{n\in\mathbb{N}}[0,r-\frac{1}{n}]\Big) & = & f^{-1}([0,r)). \\ & \underset{\text{by Claim 2 (a)}}{\uparrow} & & \underset{\text{Image-Preimage Lemma 0.2 (6)}}{\uparrow} & & & \end{array}$$

As above we deduce the desired equality. ⊞



Claim 4. The map $f\colon X \to \mathbb{R}$ is continuous.

Proof. On page 135 we pointed out that the set of rational intervals forms a basis for the topology of $\mathbb{R}$. It follows from the Subbasis-Continuity Proposition 2.9 that it suffices to show that for every choice of $c, d \in \mathbb{Q}$ the preimage $f^{-1}((c,d))$ is open. Thus let $c, d \in \mathbb{Q}$. We see that

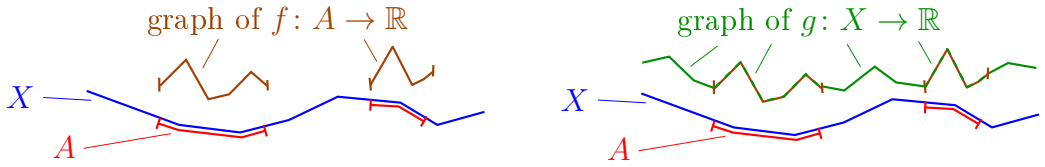
$$f^{-1}((c,d)) \overset{\text{by the Image-Preimage Lemma 0.2 (8) and since } f(X)\subset[0,1]}{\downarrow =} f^{-1}([0,d)) \setminus f^{-1}([0,c]) \overset{\text{by Claim 3}}{\downarrow =} \underbrace{\bigcup_{n\in\mathbb{N}} U_{d-\frac{1}{n}}}_{\substack{\text{open, since each} \\ U_p \text{ is open}}} \setminus \underbrace{\bigcap_{n\in\mathbb{N}} \overline{U}_{c+\frac{1}{n}}}_{\substack{\text{closed by} \\ \text{Lemmas 1.9 (2)} \\ \text{and 1.3 (2)}}}.$$

■

8.3. **The Tietze Extension Theorem.** Using Urysohn's Lemma 8.4 we can now prove the following major theorem.

Theorem 8.5. (Tietze Extension Theorem) Let X be a topological space, let $A \subset X$ be a closed subset and let $f\colon A \to \mathbb{R}$ be a continuous map. If X is normal, then there exists a continuous map $g\colon X \to \mathbb{R}$ with $g|_A = f$. If A is non-empty then one can find such g such that

$$\inf g(X) = \inf f(A) \in \mathbb{R}\cup\{-\infty\} \quad \text{and} \quad \sup g(X) = \sup f(A) \in \mathbb{R}\cup\{\infty\}.$$



Remark.

(1) The Tietze Extension Theorem was first proved for metric spaces by Heinrich Tietze [**Tie1915**] in 1915. The extension to normal topological spaces was proved by Paul Urysohn [**Ury1925a**, p. 293].
(2) Let X be a topological space and let A and B be two disjoint closed subsets. We consider the map on $A \sqcup B$ that is given by $f|_A \equiv 0$ and $f|_B \equiv 1$. By the Pasting Proposition 2.6 we know that f is continuous. If X is normal, then we can extend f by the Tietze Extension Theorem 8.5 to a continuous map on all of X. This shows that the Tietze Extension Theorem 8.5 is an extension of Urysohn's Lemma 8.4.
(3) In Exercise 8.10 we will see, borrowing some ideas from (2), that the Tietze Extension Theorem 8.5 is optimal in the sense, that if the conclusion holds for a topological space X, then X is normal.
(4) In Proposition 21.9 we will prove a "smooth" version of the Tietze Extension Theorem 8.5.

For many applications the following formulation of the Tietze Extension Theorem 8.5 is more convenient.

Theorem 8.6. (Tietze Extension Theorem) Let X be a topological space and let A be a closed subset of X. Suppose that for $i = 1, \ldots, n$ we are given a closed interval $I_i \subset \mathbb{R}$ and suppose that we are given a map $f\colon A \to I_1 \times \cdots \times I_n \subset \mathbb{R}^n$. If X is normal, then there exists a continuous map $g\colon X \to I_1 \times \cdots \times I_n$ with $g|_A = f$.

Proof. We apply the earlier Tietze Extension Theorem 8.5 to the various coordinate functions $f_k\colon A \to I_k$, $k = 1, \ldots, n$ and we find maps $g_1, \ldots, g_k\colon X \to \mathbb{R}$ with the same infima and suprema as $f_1, \ldots, f_k$. But since each I_k is closed we see that the infima and suprema lie in the corresponding I_k. This shows that each g_k takes again values in I_k. ■

The proof of the Tietze Extension Theorem 8.5 relies on the following proposition.

Proposition 8.7. Let X be a non-empty topological space. First we introduce some notation: given a bounded map[107] $g\colon X \to \mathbb{R}$ we set

$$\|g\| \;:=\; \sup\{|g(x)| \,|\, x \in X\} \;\in\; \mathbb{R}.$$

Let $(g_n\colon X \to \mathbb{R})_{n\in\mathbb{N}}$ be a sequence of bounded maps. If $\sum_{n=1}^{\infty} \|g_n\| < \infty$, then for each $x \in X$ the series $\sum_{n=1}^{\infty} g_n(x)$ converges in $\mathbb{R}$ and the map $x \mapsto \sum_{n=1}^{\infty} g_n(x)$ is continuous.

Proof of Proposition 8.7. Let X be a non-empty topological space. As in Proposition 6.12 we consider the set $C(X, \mathbb{R})$ of continuous maps $X \to \mathbb{R}$ and given $f, g \in C(X, \mathbb{R})$ we write again

$$\widetilde{d}(f, g) \;=\; \sup\big\{\, \min\{|f(x) - g(x)|, 1\} \,\big|\, x \in X\big\}.$$

[107] We say $g\colon X \to \mathbb{R}$ is bounded if there exists a $C \in \mathbb{R}$ such that $|g(x)| \leq C$ for all $x \in X$.

Now let $(g_n\colon X \to \mathbb{R})_{n\in\mathbb{N}}$ be a sequence of bounded maps such that $\sum_{n=1}^{\infty} \|g_n\| < \infty$. Note that for each $k \geq l \in \mathbb{N}$ we have

$$\widetilde{d}\Big(\sum_{n=1}^{k} g_n\,,\sum_{n=1}^{l} g_n\Big) \;\leq\; \Big\|\sum_{n=1}^{k} g_n \;-\; \sum_{n=1}^{l} g_n\Big\| \;=\; \Big\|\sum_{n=l+1}^{k} g_n\Big\| \;\underset{\substack{\uparrow\\ \text{triangle inequality}}}{\leq}\; \sum_{n=l+1}^{k} \|g_n\|.$$

It follows easily from this observation and from the hypothesis that the series $\sum_{n=1}^{\infty} \|g_n\|$ converges in $\mathbb{R}$ that the partial sums $\sum_{n=1}^{k} g_k$ define a Cauchy sequence in $(C(X,\mathbb{R}),\widetilde{d})$. The desired statements now follow from Proposition 6.12. ■

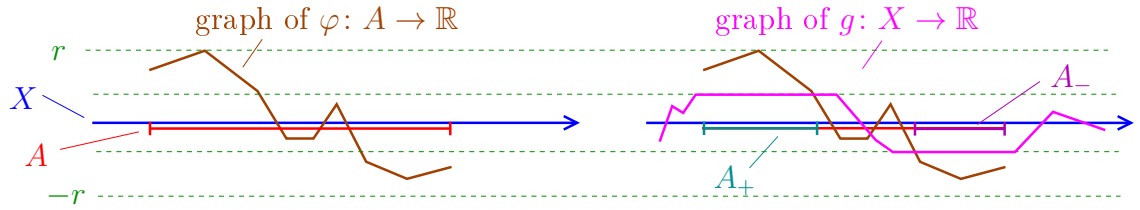
Proof of the Tietze Extension Theorem 8.5 if f is bounded. Let X be a topological space which is normal and let $A \subset X$ be a closed subset. If X is empty, then there is nothing to show. Thus we can assume that X is non-empty. Given a bounded map $\psi\colon X \to \mathbb{R}$ we write again

$$\|\psi\| \;:=\; \sup\{|\psi(x)|\,|\,x \in X\} \;\in\; \mathbb{R}.$$

Claim. Let $r \in \mathbb{R}_{\geq 0}$. Given any continuous map $\varphi\colon A \to [-r,r]$ there exists a continuous map $g\colon X \to \mathbb{R}$ such that

$$\|g\| \leq \tfrac{1}{3}\cdot r \qquad \text{and such that} \qquad |\varphi(a) - g(a)| \;\leq\; \tfrac{2}{3}\cdot r \qquad \text{for all } a \in A.$$

Proof. We set $A_- = \varphi^{-1}([-r,-\frac{1}{3}r])$, $A_0 = \varphi^{-1}([-\frac{1}{3}r,\frac{1}{3}r])$ and $A_+ = \varphi^{-1}([\frac{1}{3}r,r])$. Since A_- and A_+ are disjoint closed subsets of X it follows from Urysohn's Lemma 8.4 that there exists a continuous map $g\colon X \to [-\frac{1}{3}r,\frac{1}{3}r]$ such that $g|_{A_-} \equiv -\frac{1}{3}r$ and $g|_{A_+} \equiv \frac{1}{3}r$. In particular we have $\|g\| \leq \frac{1}{3}r$. Now let $a \in A$. In each of the three cases $a \in A_-$, $a \in A_0$ and $a \in A_+$ one can easily verify, using the triangle inequality, that $|\varphi(a)-g(a)| \leq \frac{2}{3}\cdot r$. ⊞



Now let $f\colon A \to \mathbb{R}$ be a bounded map. This means that there exists an $s \geq 0$ such that f takes values in $[-s,s]$. Thus we can apply the claim to $\varphi = f$ and $r = s$. We obtain a continuous map $g_1\colon X \to \mathbb{R}$ such that

$$\|g_1\| \;\leq\; \tfrac{1}{3}\cdot s \qquad \text{and} \qquad |f(a) - g_1(a)| \;\leq\; \tfrac{2}{3}\cdot s \qquad \text{for all } a \in A.$$

Next we apply the claim to the map $\varphi = f - g_1$ and $r = \frac{2}{3}\cdot s$. We obtain a continuous map $g_2\colon X \to \mathbb{R}$ such that

$$\|g_2\| \;\leq\; \tfrac{1}{3}\cdot\tfrac{2}{3}\cdot s \qquad \text{and} \qquad |\underbrace{(f(a) - g_1(a))}_{=\varphi(a)} - g_2(a)| \;\leq\; (\tfrac{2}{3})^2\cdot s \;\text{ for all } a \in A.$$

Iterating this procedure we obtain a sequence of continuous maps $g_1, g_2, \ldots : X \to \mathbb{R}$ such that for each $n \in \mathbb{N}$ we have

$$\|g_n\| \;\leq\; \tfrac{1}{3}\cdot(\tfrac{2}{3})^{n-1}\cdot s \qquad \text{and} \qquad |f(a) - g_1(a) - \cdots - g_n(a)| \;\leq\; (\tfrac{2}{3})^n\cdot s \;\text{ for all } a \in A.$$

Note that $\sum_{n=1}^{\infty} \|g_n\| \leq \sum_{n=1}^{\infty} \frac{1}{3} \cdot (\frac{2}{3})^{n-1} \cdot s = \frac{1}{3} \cdot s \cdot \sum_{n=0}^{\infty} (\frac{2}{3})^n = s < \infty$. Thus it follows from Proposition 8.7 that the map

$$\begin{array}{rcl} X & \to & \mathbb{R} \\ x & \mapsto & g(x) := \sum_{n=1}^{\infty} g_n(x) \end{array}$$

is defined and continuous. Next note that for any $a \in A$ we have

$$|f(a)-g(a)| = \left|f(a)-\sum_{n=1}^{\infty} g_n(a)\right| = \left|f(a)-\lim_{n\to\infty}\sum_{k=1}^{n} g_k(a)\right| = \lim_{n\to\infty} \underbrace{\left|f(a)-\sum_{k=1}^{n} g_k(a)\right|}_{\leq (\frac{2}{3})^n \cdot s \text{ by the second property}} = 0.$$

In other words, the maps g and f agree on A.

Finally assume that A is non-empty. Since $f\colon X \to \mathbb{R}$ is by hypothesis bounded it makes sense to set $a := \inf f(A) \in \mathbb{R}$ and to set $b := \sup f(A) \in \mathbb{R}$. By Lemma 2.3 we know that the map

$$x \mapsto \min(\max(g(x), a), b)$$

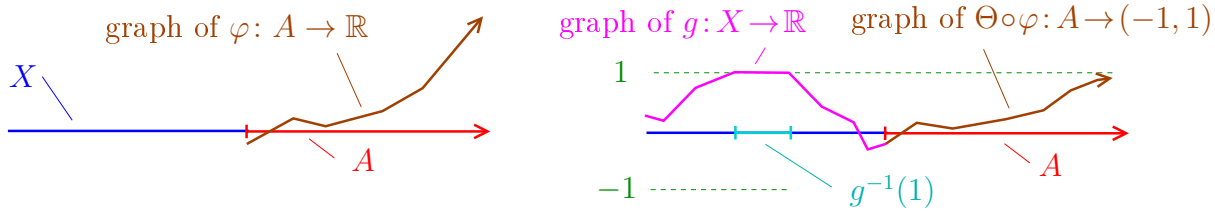
is continuous. Evidently this map has all the desired properties. ∎

Proof of the Tietze Extension Theorem 8.5 if f is unbounded. Let X be a topological space which is normal, let $A \subset X$ be a closed subset and let $f\colon A \to \mathbb{R}$ be a possibly unbounded continuous map.

We pick a homeomorphism $\Theta\colon \mathbb{R} \to (-1,1)$, e.g. we could pick $x \mapsto \frac{2}{\pi} \cdot \arctan(x)$. The map $\Theta \circ f\colon A \to \mathbb{R}$ is bounded and continuous. By the previous case of the Tietze Extension Theorem we know that there exists a continuous map $g\colon X \to [-1,1]$ which agrees with $\Theta \circ f$ on A. If $g(X) \subset (-1,1)$, then $\Theta^{-1} \circ g\colon X \to \mathbb{R}$ is the desired extension of f.

It remains to consider the case that $g^{-1}(-1) \cup g^{-1}(1)$ is non-empty. Note that A and $g^{-1}(-1) \cup g^{-1}(1)$ are two disjoint closed subsets of X. Therefore we can apply Urysohn's Lemma 8.4 and we obtain a continuous map $\gamma\colon X \to [0,1]$ with $\gamma|_A \equiv 1$ and which vanishes on $g^{-1}(-1) \cup g^{-1}(1)$. The map $x \mapsto \gamma(x) \cdot g(x)$ takes values in $(-1,1)$. The map $x \mapsto \Theta^{-1}(\gamma(x) \cdot g(x))$ is therefore well-defined and it is a continuous extension of $f\colon A \to \mathbb{R}$ to all of X.

By basically the same argument as in the bounded case we can arrange that f and g have the same infimum and the same supremum. ∎



Exercises for Chapter 8.

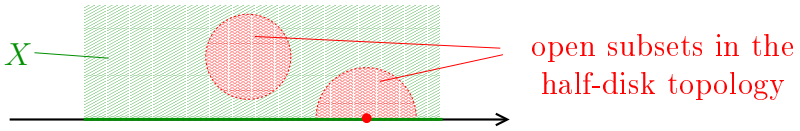
Exercise 8.1. Give an example of a topological space that is normal but which is not Hausdorff.

Exercise 8.2. Let Y be a topological space and let $X_1, \ldots, X_k$ be disjoint closed subsets of Y. Show that if Y is normal, then the X_i are contained in disjoint open subsets.

Exercise 8.3. We consider the upper half-plane $X := \{(x, y) \in \mathbb{R}^2 \,|\, y \geq 0\}$ together with the *half-disk topology* $\mathcal{H}$ which is generated by the usual open disks $B_r(x, y)$ with $r < y$ and the subsets of the form

$$\{(x, 0)\} \;\cup\; (B_r(x, 0) \cap \{(x, y) \in \mathbb{R}^2 \,|\, y > 0\}).$$

(a) Show that $(X, \mathcal{H})$ is Hausdorff.
(b) Show that $(X, \mathcal{H})$ is not normal.



Exercise 8.4. Give an example of a normal topological space X and two closed subsets A and B such that there exists no continuous map $f\colon X \to \mathbb{R}$ with $A = f^{-1}(0)$ and $B = f^{-1}(1)$.

Exercise 8.5. Show that every topological graph, as defined on page 310, is normal.

Exercise 8.6. Let $S := (\mathbb{R}, \mathcal{T})$ be the Sorgenfrey line that we introduced on page 182.

(a) Show that the Sorgenfrey line is normal.

As in Exercise 5.13 we refer to the product $S \times S$ as the *Sorgenfrey plane*.

(b) Show that $A := \{(x, -x) \,|\, x \in \mathbb{Q}\}$ and $B := \{(x, -x) \,|\, x \in \mathbb{R} \setminus \mathbb{Q}\}$ are closed subsets of the Sorgenfrey plane.
Remark. You could make use of Exercise 5.13.
(c) Show that the Sorgenfrey plane is not normal.

Exercise 8.7. Let $A \subset \mathbb{R}^n$ be a subset. Show that if A is not closed, then there exists a continuous map $f\colon A \to \mathbb{R}$ which cannot be extended to a continuous map on all of $\mathbb{R}^n$.

Exercise 8.8. Let X be a topological space that is normal. Let $U_1, \ldots, U_m$ be open subsets such that $X = \bigcup_{i=1}^{m} U_i$. Show that there exist open subsets $V_i \subset U_i$ such that $\overline{V}_i \subset U_i$ and such that $X = \bigcup_{i=1}^{m} V_i$.
Remark. A much stronger version of the statement of this exercise can be found in [**Dug1966**, Theorem VII.6.1] or alternatively in [**Grote1969**, Satz 43].

Exercise 8.9. Let X be a metric space.

(a) Let $A, B \subset X$ be two disjoint closed subsets of X. Does there exist a continuous map $f\colon X \to [0, 1]$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$?
(b) Let $A, B \subset X$ be two disjoint open subsets of X. Does there exist a continuous map $f\colon X \to \mathbb{R}$ and disjoint open intervals I and J such that $A = f^{-1}(I)$ and $B = f^{-1}(J)$?
(c) Let $A, B \subset X$ be two subsets of X such that $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$. Do there exist disjoint open neighborhoods U of A and V of B?

Exercise 8.10. Let X be a topological space that satisfies the conclusion of the Tietze Extension Theorem 8.5, i.e. we assume that given any closed subset A of X and given any

continuous map $f\colon A \to \mathbb{R}$ there exists a continuous map $g\colon X \to \mathbb{R}$ with $g|_A = f$. Show that X is normal.

9. Second-countable topological spaces and the Urysohn Metrization Theorem

In this chapter we will first introduce the notion of a second-countable topological space. This notion is used in the formulation of the Urysohn Metrization Theorem. Depending on the reader's taste, even more importantly, the notion of a second-countable topological space plays an essential role in the definition of topological manifolds on page 511.

9.1. Second-countable topological spaces.

Recall that on page 132 we introduced the notion of a basis of a topological space. With this notion we can define what it means for a topological space to be second-countable.

Definition. A topological space is called second-countable if there exists a countable basis for the topology.

Non-Example. Of course not every topological space is second-countable. For example consider the uncountable set $\mathbb{R}$, but for once equipped with the discrete topology $\mathcal{D}$. It follows easily from the definitions that $X = (\mathbb{R}, \mathcal{D})$ is *not* second-countable. There are also more subtle examples, for example we will see in Exercise 9.6 that there exist *compact* topological spaces that are not second-countable. Furthermore in Theorem 22.13 we will see that there exists a path-connected space that is not second-countable.

Using the next lemma we will see that many of the topological spaces that we are interested in are actually second-countable.

Lemma 9.1. (Second Countability Lemma)

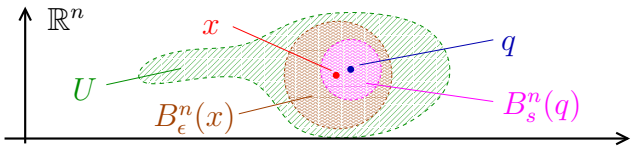
(1) For every $n \in \mathbb{N}_0$ the topological space $\mathbb{R}^n$ is second-countable.
(2) If X is a topological space that is second-countable, then every subspace is also second-countable.
(3) (a) If $X_1, \dots, X_k$ are topological spaces that are second-countable, then $X_1 \times \cdots \times X_k$ is second-countable.
    (b) Let $\{X_i\}_{i \in I}$ be a countable family of topological spaces. If each X_i is second-countable, then $\prod_{i \in I} X_i$ is also second-countable.
(4) Let X be a topological space. If there exists a countable family of open subsets $\{U_i\}_{i \in I}$ such that $\bigcup_{i \in I} U_i = X$ and such that each U_i is second-countable, then X itself is second-countable.
(5) Let X be a topological space and let $\sim$ be an equivalence relation on X. If X is second-countable and if the projection $X \to X/\!\sim$ is an open map, then $X/\!\sim$ is also second-countable.

Proof.

(1) This statement is basically the content of Exercise 1.8. For completeness' sake we provide the argument. Let $n \in \mathbb{N}_0$. We denote by $\mathcal{B}$ the set of all open balls around points in $\mathbb{Q}^n$ with rational radii. By the Countability Lemma 0.8 we know that $\mathcal{B}$ is countable. Thus it remains to show that $\mathcal{B}$ is a basis for the standard topology on $\mathbb{R}^n$. Note that each set in $\mathcal{B}$ is an open subset of $\mathbb{R}^n$. By the definition of the basis of a topology, see page 132, it now suffices to prove the following claim.

Claim. Given any open subset $U \subset \mathbb{R}^n$ and given any $x \in U$ there exists a $B \in \mathcal{B}$ with $x \in B \subset U$.

Proof. Let $U \subset \mathbb{R}^n$ be an open subset and let $x \in U$. Since U is open there exists an $\epsilon > 0$ such that $B^n_\epsilon(x) \subset U$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ there exists a $q \in \mathbb{Q}^n$ with $\|x - q\| < \frac{1}{3}\epsilon$. Next we pick an $s \in \mathbb{Q}$ with $s \in (\frac{1}{3}\epsilon, \frac{2}{3}\epsilon)$. A straightforward argument, using the triangle inequality, shows that $x \in B^n_s(q) \subset B^n_\epsilon(x) \subset U$. ⊞



(2) This statement follows from Lemma 1.33 and the Countability Lemma 0.8 (3).

(3) (a) This statement follows from the Product Topology-Basics Lemma 5.4 (3) and the Countability Lemma 0.8 (5).

(b) This statement follows from the Product Topology-Basics Lemma 5.4 (3) and the Countability Lemma 0.8 (3) and (5).

(4) For each U_i we pick a countable basis $\mathcal{B}_i$ of the topology of U_i. Using the Basis Characterization Lemma 1.32 it is straightforward to verify that $\bigcup_{i \in I} \mathcal{B}_i$ is a basis for the topology of X. By the Countability Lemma 0.8 (2) this is a countable set.

(5) This statement follows from Lemma 5.22 and the Countability Lemma 0.8 (3). ■

The following lemma gives us a convenient property of second-countable topological spaces.

Lemma 9.2. (Countable Covering Lemma) Let X be a topological space and let $\{U_i\}_{i \in I}$ be an open cover of X. If X is second-countable, then there exists a countable subset $J \subset I$ with $X = \bigcup_{j \in J} U_j$.

Proof. Let $\mathcal{B}$ be a countable basis for the topology of X. We write

$$\mathcal{B}' := \{V \in \mathcal{B} \,|\, V \subset U_i \text{ for some } i \in I\}.$$

Given $V \in \mathcal{B}'$ we pick $j \in I$ with $V \subset U_j$. It follows from the Countability Lemma 0.8 (1) that the set $J \subset I$ of all such j is countable. It remains to show that $X = \bigcup_{j \in J} U_j$.

Thus let $x \in X$. By hypothesis there exists an $i \in I$ with $x \in U_i$. Since $\mathcal{B}$ is a basis for the topology of X there exists a $V \in \mathcal{B}$ with $x \in V \subset U_i$. Note that $V \in \mathcal{B}'$. But this implies that $x \in U_j$ for some $j \in J$. ■

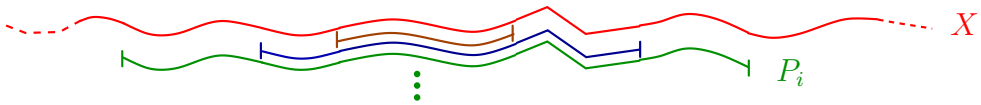
We conclude this introduction to second-countable topological spaces with the discussion of exhaustions of topological spaces, which play a role later on.

Definition. Let X be a topological space. We say that a sequence $\{P_i\}_{i \in \mathbb{N}}$ of subsets of X is an exhaustion if the following two conditions are satisfied:

(1) We have $X = \bigcup_{i \in \mathbb{N}} P_i$.

(2) For each $i \in \mathbb{N}$ we have $\overline{P}_i \subset \overset{\circ}{P}_{i+1}$.

We say that the exhaustion is compact, if each P_i is compact.



Example. A compact exhaustion of $\mathbb{R}^n$ is given by $P_i := \overline{B}_i^n(0)$ with $i \in \mathbb{N}$.

Lemma 9.3. (Compact Exhaustion Lemma) Let X be a topological space which is second countable, Hausdorff and regionally compact.

(1) There exists an exhaustion of X by compact subsets.
(2) Given any compact subset $K \subset X$ there exists an open neighborhood P of K such that the closure $\overline{P}$ is compact.

Remark. In the Exhaustion Function Lemma 10.7 we will prove a refinement of the Compact Exhaustion Lemma 9.3 (1).

Proof.

(1) First recall that on page 178 we said that a subset $A \subset X$ is *precompact in* X if the closure $\overline{A}$ of A in X is compact. By hypothesis we know that X is regionally compact and Hausdorff. It follows from Lemma 2.42 that X is locally precompact, which implies by Lemma 2.39 that the topology of X admits a basis of precompact open subsets. Since X is second-countable we know by the Countable Covering Lemma 9.2 that X is in fact covered by a countable family $\{U_i\}_{i\in\mathbb{N}}$ of precompact open subsets. It remains to prove the following claim.

Claim. There exists a sequence $\{K_i\}_{i\in\mathbb{N}}$ of compact subsets of X such that for each $i \in \mathbb{N}$ we have $U_i \subset K_i$ and such that for each $i \geq 2$ we have $K_{i-1} = \overline{K}_{i-1} \subset \mathring{K}_i$.

Proof. We set $K_1 := \overline{U}_1$. Suppose we have already constructed $K_1, \ldots, K_n$ with the above properties. Since K_n is compact there exists an $r \in \mathbb{N}$ with $K_n \subset U_1 \cup \cdots \cup U_r$. We set

$$K_{n+1} := \overline{U}_1 \cup \cdots \cup \overline{U}_r \cup \overline{U}_{n+1}.$$

Since the $U_1, \ldots, U_r, U_{n+1}$ are precompact, their closures are by definition compact. Thus it follows from the Compact-Union-Intersection Lemma 1.20 that K_{n+1} is compact. Evidently $U_{n+1} \subset K_{n+1}$. Finally observe that

$$\overline{K}_n \underset{\uparrow}{=} K_n \subset U_1 \cup \cdots \cup U_r \subset \mathring{K}_{n+1} \subset K_{n+1}.$$

since X is Hausdorff we know by the Compact-Closed Lemma 1.21 (2) that compact subsets are closed

By iterating this procedure we get the desired sequence.

(2) Let K be a compact subset of X. We continue with the notation from (1). The sets $\mathring{K}_n$ form an open cover of K. Since K is compact and since we have $\mathring{K}_n \subset \mathring{K}_{n+1}$ we see that there exists an $n \in \mathbb{N}$ with $K \subset \mathring{K}_n$. But K_n is compact, thus we are done. ∎

9.2. **The Urysohn Metrization Theorem.** In this section we will state and prove a major consequence of Urysohn's Lemma 8.4, namely the Urysohn Metrization Theorem. Before we can state the theorem, let us recall the following definition from page 110.

Definition. A topological space X is called metrizable if there exists a metric d on X such that the given topology agrees with the topology coming from d.

For second-countable topological spaces the following theorem gives a complete characterization of metrizability.

Theorem 9.4. (Urysohn Metrization Theorem) Let X be a topological space. If X is second-countable, then the following two statements are equivalent:

(1) X is normal and every one-point subset is closed[108].
(2) X is metrizable.

Remark.

(1) The Urysohn Metrization Theorem 9.4 was proved by Paul Urysohn in 1924. Urysohn's proof was written up posthumously by Paul Alexandroff [**Ury1925b**].
(2) The later Completely Metrizable Criterion Proposition 10.8 can often be used to show that a given topological space is in fact completely metrizable.
(3) In Exercise 9.21 we will show that a compact metrizable space is second-countable.

The proof of the Urysohn Metrization Theorem 9.4 relies on a proposition which is interesting in its own right. To formulate this proposition we need to introduce or recall the following notation.

Notation. We write

$$\mathbb{R}^{\mathbb{N}} := \{\text{all maps } \mathbb{N} \to \mathbb{R}\} = \{(x_1, x_2, \dots) \,|\, \text{all } x_i \in \mathbb{R}\}.$$

We equip $\mathbb{R}^{\mathbb{N}}$ with the product topology introduced on page 226, namely this is the topology which is defined by the condition that a basis for the topology is given by the sets

$$U_1 \times \cdots \times U_k \times \mathbb{R} \times \mathbb{R} \times \dots \quad \subset \mathbb{R}^{\mathbb{N}}$$

where $U_1, \dots, U_k$ are open sets in $\mathbb{R}$.

Proposition 9.5. The topological space $\mathbb{R}^{\mathbb{N}}$ is metrizable.

Proof. Throughout the proof we adopt the following notation:

- Given $a, b \in \mathbb{R}$ we write $d(a,b) := \min\{|b-a|, 1\}$.
- Given $x \in \mathbb{R}^{\mathbb{N}}$ we denote by $x_1, x_2, \dots$ the "coordinates" of x.
- Given $x, y \in \mathbb{R}^{\mathbb{N}}$ we set[109]

$$D(x,y) := \sup_{k\in\mathbb{N}}\{\tfrac{1}{k} \cdot d(x_k, y_k)\} \in \mathbb{R}.$$

Claim 1. The map D defines a metric on $\mathbb{R}^{\mathbb{N}}$.

Proof. It is clear that D is symmetric and that $D(x,y) = 0$ if and only if $x = y$. Thus it remains to prove the triangle inequality. Let $x, y, z \in \mathbb{R}^{\mathbb{N}}$. We have

$$D(x,z) = \sup_{k\in\mathbb{N}}\{\tfrac{1}{k}\cdot d(x_k,z_k)\} \underset{\uparrow}{\leq} \sup_{k\in\mathbb{N}}\{\tfrac{1}{k}\cdot d(x_k,y_k) + \tfrac{1}{k}\cdot d(y_k,z_k)\} \underset{\uparrow}{\leq} D(x,y) + D(y,z).$$

follows from the straightforward observation that d is a metric

by definition of the supremum

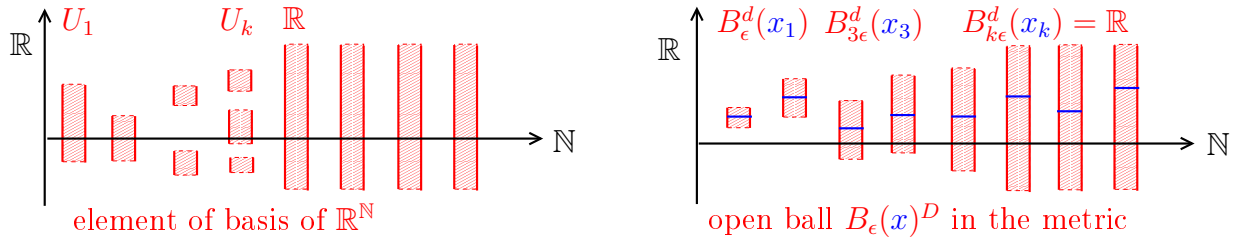
⊞

[108] By the Points-in-Hausdorff Space Lemma 1.15 (1) this condition is satisfied if X is Hausdorff.

[109] By definition of d we see that the set $\{\frac{1}{k}\cdot d(x_k, y_k) \,|\, k \in \mathbb{N}\}$ is bounded by 1. It follows that the supremum does in fact exist.

It remains to prove the following claim.

Claim 2. The topology defined by the metric D equals the product topology on $\mathbb{R}^{\mathbb{N}}$.



Proof. First we show that the metric topology is contained in the product topology. Thus let $U \subset \mathbb{R}^{\mathbb{N}}$ be open in the metric topology. We want to show that U is open in the product topology. By the JH-Lemma 1.7 it suffices to show that for each $x \in U$ there exists a set V, which is open in the product topology, with $x \in V \subset U$. Note that since U is open in the metric topology we know that there exists an $\epsilon > 0$ with $B^D_\epsilon(x) \subset U$. We pick $n \in \mathbb{N}$ with $\frac{1}{n} < \epsilon$ and we set

$$V \;=\; (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon) \times \mathbb{R} \times \mathbb{R} \times \ldots \qquad \subset\; \mathbb{R}^{\mathbb{N}}.$$

It follows easily from the definitions that $V \subset B^D_\epsilon(x)$. By the above discussion we know that V is open in the product topology. Thus we have verified that U is open in the product topology.

Now we need to show that the product topology is contained in the metric topology. By the Subbasis Lemma 1.28 it suffices to show that for each subset

$$U \;:=\; U_1 \times \cdots \times U_k \times \mathbb{R} \times \mathbb{R} \times \ldots \qquad (\text{where } U_1, \ldots, U_k \text{ are open sets in } \mathbb{R})$$

and each $x \in U$ there exists an open set V of the metric topology with $x \in V \subset U$. Note that since the U_i are open we know that for $i = 1, \ldots, k$ there exists a $\mu_i \in (0, 1)$ with $(x_i - \mu_i, x_i + \mu_i) \subset U_i$. We set

$$\epsilon \;:=\; \min\left\{\tfrac{\mu_1}{1}, \tfrac{\mu_2}{2}, \ldots, \tfrac{\mu_k}{k}\right\}.$$

It follows easily from the definitions that $V := B^D_\epsilon(x) \subset U$. ∎

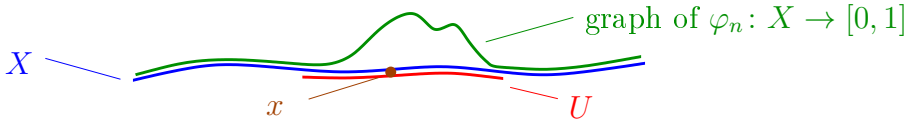
Now we can finally provide the proof of the Urysohn Metrization Theorem 9.4.

Proof of the Urysohn Metrization Theorem 9.4. Let X be a topological space which is second-countable. The "(2)⇒(1)"-direction is basically clear. Indeed, if X is metrizable, then it follows from Proposition 1.13 that X is Hausdorff which in turn implies by the Points-in-Hausdorff Space Lemma 1.15 (1) that every one-point subset is closed. Furthermore by the Metric Spaces-Normal Proposition 8.1 we know that X is normal.

We turn to the proof of the significantly more interesting "(1)⇒(2)"-direction. Thus we now assume that X is normal and that every one-point set is closed.

> The idea, at this stage not very surprising, is to attempt to find an embedding $\Phi\colon X \to \mathbb{R}^{\mathbb{N}}$ and to use the fact that by Proposition 9.5 we know that $\mathbb{R}^{\mathbb{N}}$ is metrizable. A map $\Phi\colon X \to \mathbb{R}^{\mathbb{N}}$ is basically the same as a sequence of maps $\{\varphi_n\colon X \to \mathbb{R}\}_{n\in\mathbb{N}}$.

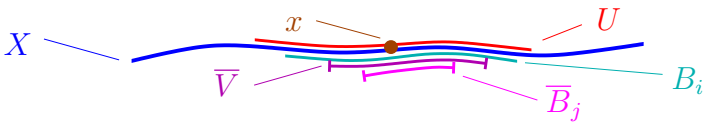
Claim 1. There exists a family $\{\varphi_n\colon X \to [0,1]\}_{n\in\mathbb{N}}$ of maps with the following property: for each $x \in X$ and each open neighborhood U of x there exists an $n \in \mathbb{N}$ such that $\varphi_n(x) = 1$ and such that $\varphi_n|_{X\setminus U} \equiv 0$.



Proof. Since X is by hypothesis second-countable there exists a countable basis $\{B_j\}_{j\in J}$ for the topology of X. For every choice of $i, j \in J$ with $\overline{B}_i \subset B_j$ we can and will apply Urysohn's Lemma 8.4[110] to the disjoint closed subsets $\overline{B}_i$ and $X \setminus B_j$ and we obtain a map $g_{i,j}\colon X \to [0,1]$ with $g_{i,j}|_{\overline{B}_i} \equiv 1$ and $g_{i,j}|_{X\setminus B_j} \equiv 0$. Since J is countable we know by the Countability Lemma 0.8 that the set

$$K \;:=\; \{(i,j) \in J \times J \,|\, \overline{B}_i \subset B_j\} \;\subset J \times J$$

is again countable, thus in bijection with a subset of $\mathbb{N}$. Therefore it remains to show that these maps $\{g_{i,j}\}_{(i,j)\in K}$ have the desired properties. Now let $x \in X$ and let U be a neighborhood of x. By definition of a basis of the topology there exists a $j \in J$ with $x \in B_j \subset U$. By hypothesis X is normal and every one-point subset is closed. Therefore we obtain from the Normal-Open-Closed-Neighborhood Lemma 8.2 an open subset V of X with $x \in V \subset \overline{V} \subset B_j$. By definition of a basis there exists an $i \in J$ with $x \in B_i \subset V$. But then we also have $\overline{B}_i \subset \overline{V} \subset B_j$. Note that the map $g_{i,j}$ satisfies $g_{i,j}(x) = 1$ and it vanishes on $X \setminus B_j$. From $B_j \subset U$ we obtain that $g_{i,j}$ vanishes in particular on $X \setminus U$. ⊞



We consider the product $\mathbb{R}^{\mathbb{N}} = \prod_{n\in\mathbb{N}} \mathbb{R}$, equipped with the product topology. By Proposition 9.5 we know that $\mathbb{R}^{\mathbb{N}}$ is metrizable. Thus it suffices to prove the following claim.

Claim 2. The map

$$\begin{array}{rccc} \Phi\colon & X & \to & \mathbb{R}^{\mathbb{N}} \\ & x & \mapsto & (\varphi_1(x), \varphi_2(x), \dots) \end{array}$$

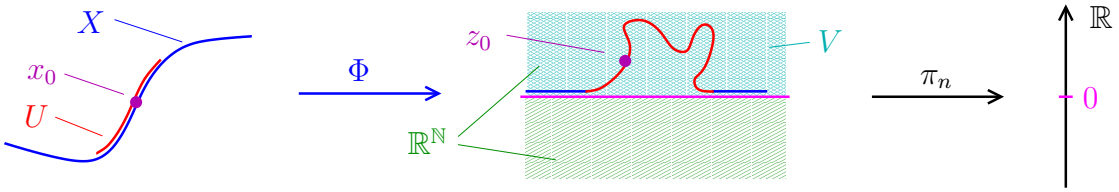
is an embedding, i.e. it is a homeomorphism onto its image.

Proof. It follows immediately from the continuity of the φ_n and the Topological-Product Proposition 5.1 that Φ is continuous. Next we show that Φ is injective. Thus let $x \neq y$ be two points in X. Since X is normal and since each one-point set is closed there exists an open neighborhood U of x that does not contain y. By construction there exists an $n \in \mathbb{N}$ with $\varphi_n(x) = 1$ and with $\varphi_n(y) = 0$. This shows that Φ is injective.

We have now shown that Φ is an injective continuous map. We want to show that Φ is actually a homeomorphism onto its image. By the Open -Injective Map Lemma 2.16 it now suffices to show that Φ is an open map.

[110] Here we use our hypothesis that X is normal

Thus let $U \subset X$ be an open subset. We need to show that $\Phi(U)$ is open in $\Phi(X)$. By the JH-Lemma 1.7 it remains to show that for every $z_0 \in \Phi(U)$ there exists an open set $W \subset \Phi(U)$ with $z_0 \in W \subset \Phi(U)$. We pick $x_0 \in U$ with $\Phi(x_0) = z_0$. By construction there exists an $n \in \mathbb{N}$ with $\varphi_n(x_0) = 1$ and with $\varphi_n|_{X \setminus U} \equiv 0$. We denote by $\pi_n \colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ the projection onto the n-th coordinate. By the Topological-Product Proposition 5.1 we know that π_n is continuous.



We set $V := \pi_n^{-1}((0, \infty))$ and we set $W := V \cap \Phi(X)$. By definition of the subspace topology we know that W is open in $\Phi(X)$. It remains to show that $z_0 \in W$ and that $W \subset \Phi(U)$.

- It follows from $\pi_n(z_0) = (\pi_n \circ \Phi)(x_0) = \varphi_n(x_0) = 1 > 0$ that $z_0 \in W$.
- Let $z \in W$. We denote by $x \in X$ the unique element with $\Phi(x) = z$. Note that $\pi_n(z) = (\pi_n \circ \Phi)(x) = \varphi_n(x) > 0$. Since $\varphi_n|_{X \setminus U} \equiv 0$ we see that $x \in U$. In particular $z = \Phi(x) \in \Phi(U)$. ■

9.3. **Regular topological spaces.** It turns out that one can formulate the Urysohn Metrization Theorem 9.4 with an a priori weaker hypothesis. This weaker hypothesis is used in most formulations of the Urysohn Metrization Theorem 9.4 in the literature. Therefore, for completeness' sake we will discuss this weaker hypothesis, even though we will not really make use of the results of this section.
We start out with the following notion.

Definition. A topological space X is called regular if given any $x \in X$ and given any closed subset $A \subset X$ there exist disjoint open subsets U and V with $x \in U$ and $A \subset V$.

Remark.

(1) Note that in the definition of a regular topological space we do not demand, in contrast say to [**Mun1975**, p. 195], that one-point sets are closed.
(2) In Section 9.4 we will discuss the relationship between the various "separation axioms", i.e. between the notions "Hausdorff", "regular" and "'normal".

We have the following analogue of Lemmas 1.15 and 8.2.

Lemma 9.6. (Regular-Open-Closed-Neighborhood Lemma) Let X be a topological space, let $x \in X$ and let $U \subset X$ be an open neighborhood of x. If X is regular, then there exists an open subset V with $x \in V \subset \overline{V} \subset U$.



Proof. The proof of the lemma is almost identical to the proof of the Normal-Open-Closed-Neighborhood Lemma 8.2. For peace of mind we nonetheless provide the argument. Let U be an open neighborhood of x. We set $B := X \setminus U$. This is a closed subset with $x \notin B$. Since X is regular there exist disjoint neighborhoods V of x and W of B. It remains to show that $\overline{V} \subset U$. In fact we have $V \subset X \setminus W \subset U$. But since W is open we know that $X \setminus W$ is closed. By definition of $\overline{V}$ we have $\overline{V} \subset X \setminus W$. ■

The following proposition, together with the previous remark shows that for a large class of topological spaces the notions of being regular and normal coincide.

Proposition 9.7. Let X be a topological space. If X is regular and if X is second-countable, then X is also normal.

Proof. Let X be a topological space that is regular and that is second-countable. The latter condition means that there exists a countable basis $\{B_j\}_{j\in J}$ for the topology of X. Now let C and D be two disjoint closed subsets of X.

Claim 1. There exist open subsets $\{U_m\}_{m\in\mathbb{N}}$ such that $C \subset \bigcup_{m\in\mathbb{N}} U_m \subset \bigcup_{m\in\mathbb{N}} \overline{U}_m \subset X \setminus D$.

Proof. Since J is countable we know that the set $\{j \in J \,|\, \overline{B}_j \subset X\setminus D\} \subset J$ is also countable. It remains to show that the union of the corresponding B_j covers C. Thus let $c \in C$. Since X is regular and since D is closed we know by the Regular-Open-Closed-Neighborhood Lemma 9.6 that there exists an open neighborhood U of c such that $\overline{U} \subset X \setminus D$. By definition of a basis of the topology there exists a $j \in J$ with $c \in B_j \subset U$. By construction $\overline{B}_j \subset \overline{U} \subset X \setminus D$, thus j lies in the given index set. ⊞

We swap the roles of C and D in the previous claim and we obtain open subsets $\{V_n\}_{n\in\mathbb{N}}$ such that $D \subset \bigcup_{n\in\mathbb{N}} V_n \subset \bigcup_{n\in\mathbb{N}} \overline{V}_n \subset X \setminus C$. Note that we do not know whether $\bigcup_{m\in\mathbb{N}} U_m$ and $\bigcup_{n\in\mathbb{N}} V_n$ are disjoint. Thus we have to go into extra time and do some more work. Given $m, n \in \mathbb{N}$ we now set

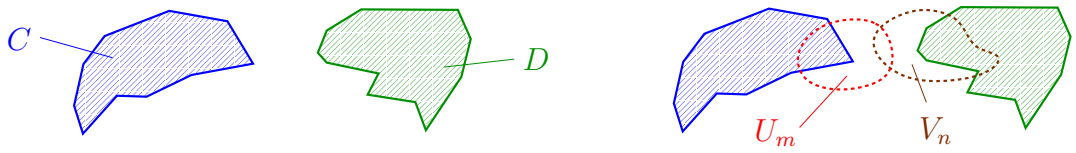
$$\widetilde{U}_m \;:=\; U_m \;\setminus\; \underbrace{\bigcup_{i=1}^{m} \overline{V}_i}_{\substack{\textit{finite}\text{ union of}\\ \text{closed subsets}}} \qquad \text{and} \qquad \widetilde{V}_n \;:=\; V_n \;\setminus\; \underbrace{\bigcup_{i=1}^{n} \overline{U}_i}_{\substack{\textit{finite}\text{ union of}\\ \text{closed subsets}}}.$$

Since the *finite* union of closed subsets is closed we see that the subsets $\widetilde{U}_m$ and $\widetilde{V}_n$ are open. Furthermore, since each $\overline{V}_i$ is disjoint from C we see that the $\widetilde{U}_m$ cover C. Similarly we see that the $\widetilde{V}_n$ cover D. It remains to prove the following claim.

Claim 2. The open sets $\bigcup_{m\in\mathbb{N}} \widetilde{U}_m$ and $\bigcup_{n\in\mathbb{N}} \widetilde{V}_n$ are disjoint.

Proof. It suffices to show that for each $m, n \in \mathbb{N}$ we have $\widetilde{U}_m \cap \widetilde{V}_n = \varnothing$. If $m \leq n$, then we have by construction that even $\widetilde{U}_m \cap \overline{V}_n = \varnothing$. Similarly, if $n \leq m$, we have by construction that even $\widetilde{V}_n \cap \overline{U}_m = \varnothing$. ■

Now we can give the standard formulation of the Urysohn Metrization Theorem.



Theorem 9.8. (Urysohn Metrization Theorem) Let X be a topological space. If X is second-countable, the following two statements are equivalent:

(1) X is regular and every one-point subset is closed.
(2) X is metrizable.

Proof. Let X be a topological space which is second-countable.

First we consider again the basically trivial "(2)⇒(1)"-direction. If X is metrizable, then it follows from Proposition 1.13 that X is Hausdorff which in turn implies by the Points-in-Hausdorff Space Lemma 1.15 (1) that every one-point subset is closed. Furthermore by the Metric Spaces-Normal Proposition 8.1 we know that X is normal. Since one-point subsets are closed this also implies that X is regular.

Now we consider the "(1)⇒(2)"-direction. Thus we assume that X is regular and that every one-point subset is closed. Since X is second-countable it follows from Proposition 9.7 that X is normal. Thus it follows from the "(1)⇒(2)"-direction of the original Urysohn Metrization Theorem 9.4 that X is metrizable. ■

The Urysohn Metrization Theorems 9.4 and 9.8 give a satisfactory characterization of metrizable topological spaces among all second-countable topological spaces.

For completeness' sake we state the following theorem which gives, rather astonishingly, a complete characterization of metrizable topological spaces among all topological spaces:

Theorem 9.9. (Nagata-Smirnov Metrization Theorem) Let X be a topological space. The following two statements are equivalent:

(1) X is metrizable.
(2) The following three conditions are satisfied:
  (a) X is regular.
  (b) X is Hausdorff.
  (c) The topology of X admits a basis that is the union of countably many locally finite families of subsets of X.In other words, there exists a countable set I and for each $i \in I$ there exists a locally finite family $\mathcal{B}_i$ of subsets of X such that $\bigcup_{i\in I}\mathcal{B}_i$ is a basis for the topology of X.

Remark. The theorem was proved independently by Junichi Nagata [**Naga1950**] and Yurii Smirnov [**Smi1951**]. A variation on the metrization theorem was proved by R. H. Bing [**Bin1951**].[111]

Example.

[111]The first and middle name of Bing are "R." and "H.".

(1) Let X be a topological space that is second-countable, i.e. it admits a countable basis $\{U_i\}_{i\in I}$ of the topology. We set $\mathcal{B}_i := \{U_i\} \subset \mathcal{P}(X)$ and we see that condition (c) is satisfied. This shows that the Nagata-Smirnov Metrization Theorem 9.9 implies the Urysohn Metrization Theorem 9.8.

(2) Let X be a set that is equipped with the discrete topology. The family of all one-point subsets is locally finite. Thus, if we set $I := \{*\}$ and $\mathcal{B}_* := \{\{x\}\}_{x\in X}\}$, then we see that X satisfies Properties (a), (b) and (c), thus X is metrizable. In fact in this case it is easy to provide an explicit example of a metric, namely we just need to consider the discrete metric that is given by

$$d(x,y) \;:=\; \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

Proof. Despite its beauty we will not make use of the Nagata-Smirnov Metrization Theorem. We use this as an excuse for not providing the proof. The theorem is of course proved in the original sources [**Naga1950**] and Yurii Smirnov [**Smi1951**]. A textbook proof is given in [**Mun1975**, Chapters 6.2 and 6.3], [**Enge1989**, Theorem 4.4.7] and [**Pat2009**, Chapter 7.3]. ■

9.4. **Separation Axioms.** We have now introduced the properties of being Hausdorff, normal and regular. Each such property states to what degree points and closed subsets can be separated by open subsets. The limitless imagination of mathematicians has produced many variations on these separation axioms. In this section we will state, for reference, several other time-honored separation axioms, even though we will not really make use of the axioms that we introduce in this section.

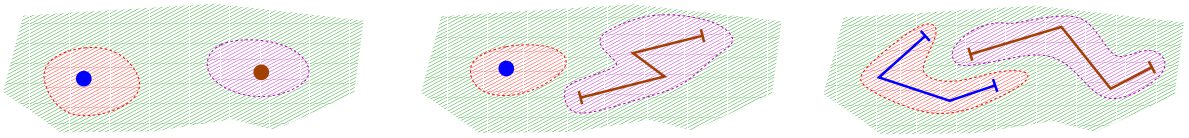
Definition. Let X be a topological space. Let A and B be two subsets of X. We say A and B are separated by subsets U and V if $A \subset U$ and $B \subset V$ and if $U \cap V = \varnothing$. We use the obvious extensions of these definitions if we are dealing with points instead of subsets.

With these definitions we can reformulate the definitions of being "Hausdorff", "regular" and "normal":

(1) A topological X is *Hausdorff* precisely if any two distinct points can be separated by open neighborhoods.

(2) A topological X is *regular* precisely if every point x can be separated from every closed subset A with $x \notin A$ by open neighborhoods.

(3) A topological X is *normal* if any two disjoint closed subsets of X can be separated by open neighborhoods.



Hausdorff regular normal

Definition. Let X be a topological space. Let A and B be two subsets of X.

(1) We say that A and B are separated by a map if there exists a continuous map $f\colon X \to \mathbb{R}$ such that $f|_A \equiv 0$ and $f|_B \equiv 1$.

(2) We say that A and B are precisely separated by a map if there exists a continuous map $f\colon X \to \mathbb{R}$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

We use the obvious extensions of these definitions if we are dealing with points instead of subsets.

By Urysohn's Lemma 8.4 we know that given a topological space X the following two statements are equivalent:

(1) Any two disjoint closed subsets can be separated by open neighborhoods.
(2) Any two disjoint closed subsets can be separated by maps.

It is less clear whether these statements are equivalent to the following statement:

(3) Any two disjoint closed subsets can be *precisely* separated by maps.

This leads us to the following refinement of the notion of being normal.

Definition. A topological space is called perfectly normal if any disjoint closed subsets can be precisely separated by a map.

Example. In the proof of the Metric Spaces-Normal Proposition 8.1 we implicitly showed that metric spaces are perfectly normal.

For some purposes it is convenient to work with a reformulation of being "perfectly normal and Hausdorff". To do so we will need the following definition.

Definition. Let X be a topological space. We say a subset is a G_δ-subset if it is the intersection of countably many open subsets.

Lemma 9.10. Let X be a Hausdorff space. The following three statements are equivalent:

(1) X is perfectly normal.
(2) For every closed subset $A \subset X$ there exists a map $f\colon X \to [0,1]$ with $A = f^{-1}(0)$.
(3) X is normal and every closed subset of X is a G_δ-subset.

Remark. In many, arguably most, textbooks, e.g. [**Kel1975**, p. 134], [**SS1978**, p. 16] and [**Jos1983**, p. 187], the characterization of Lemma 9.10 (3) is used for the definition of a perfectly normal topological space.

Proof. First we prove the "(1)⇒(2)"-direction. Let A be some closed subset of X. If $A = X$, then we can take f to be the constant map. Now suppose that $A \subsetneq X$. We pick $x \in X \setminus A$. Since X is Hausdorff we know by the Points-in-Hausdorff Space Lemma 1.15 that $\{x\}$ is a closed subset. Since X is perfectly normal there exists a map $f\colon X \to [0,1]$ with $A = f^{-1}(0)$ and with $\{x\} = f^{-1}(1)$.

We turn to the proof of the "(2)⇒(1)"-direction. Let A and B be two disjoint closed subsets. By hypothesis there exists a map $f\colon X \to [0,1]$ with $A = f^{-1}(0)$ and a map $g\colon X \to [0,1]$ with $B = g^{-1}(0)$. Since A and B are disjoint we see that for every $x \in X$ we have $f(x) + g(x) > 0$. As in the proof of the Metric Spaces-Normal Proposition 8.1 we can consider the map

$$\begin{array}{rccc} h\colon & X & \to & [0,1] \\ & x & \mapsto & \dfrac{f(x)}{f(x)+g(x)}. \end{array}$$

Evidently $A = h^{-1}(0)$ and $B = h^{-1}(0)$.

Next we prove the "(2)⇒(3)"-direction. It follows from the "(2)⇒(1)"-direction that X is normal. Thus it suffices to prove that every closed subset of X is a G_δ-subset. Now let A

be a closed subset of X. By hypothesis there exists a map $f\colon X \to [0,1]$ with $A = f^{-1}(0)$. We now see that

$$A \;=\; f^{-1}(0) \;=\; f^{-1}\Big(\bigcap_{n\in\mathbb{N}} [0,\tfrac{1}{n})\Big) \;=\; \bigcap_{n\in\mathbb{N}} \underbrace{f^{-1}([0,\tfrac{1}{n}))}_{\text{open subset of } X}.$$

Image-Preimage Lemma 0.2 (7)

Finally we provide the proof of the "(3)⇒(2)"-direction. Thus we assume that X is normal and we assume that every closed subset of X is a G_δ-subset. Let A be a closed subset of X. By hypothesis there exists a countable family $\{U_n\}_{n\in\mathbb{N}}$ of open subsets such that $A = \bigcap_{n\in\mathbb{N}} U_n$. Let $n \in \mathbb{N}$. Since X is normal and since A and $X \setminus U_n$ are disjoint we obtain from Urysohn's Lemma 8.4 a continuous map $f_n\colon X \to [0,1]$ with $f_n|_A \equiv 0$ and $f_n|_{X\setminus U_n} \equiv 1$. It follows from Proposition 6.12 that the map

$$\begin{aligned} f\colon X &\to [0,1] \\ x &\mapsto \sum_{n\in\mathbb{N}} \frac{1}{2^n}\cdot f_n(x) \end{aligned}$$

is continuous. Furthermore note that

$$X \setminus A \;=\; X \setminus \bigcap_{n\in\mathbb{N}} U_n \;=\; \bigcup_{n\in\mathbb{N}} \underbrace{(X\setminus U_n)}_{\text{here } f_n \equiv 1}.$$

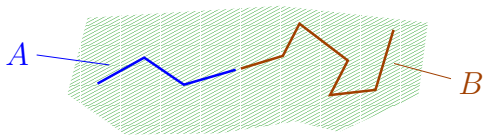
It follows almost immediately from this observation that $f^{-1}(0) = A$. ■

Since this was so much fun we want to introduce one more variation on being normal. For this we need the following definition.

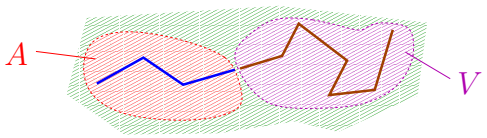
Definition. Let X be a topological space. We say two subsets A and B are separate if $A \cap \overline{B} = \varnothing$ and if $\overline{A} \cap B = \varnothing$.[112]

Example. Let $X = \mathbb{R}$. The subsets $A = (-1,0)$ and $B = (0,1)$ are separate.

Definition. A topological space is called completely normal if separate subsets are separated by open neighborhoods.



A and B are separate

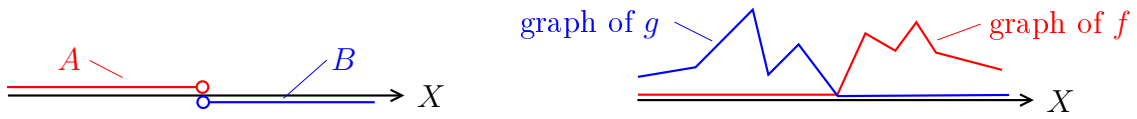


completely normal

Lemma 9.11. Let X be a topological space. If X is perfectly normal, then it is also completely normal.

Proof. Let X be a perfectly normal Hausdorff space and let A and B be two separate subsets of X. By Lemma 9.10 and the Interior-Closure Lemma 1.9 (2) there exist maps $f\colon X \to [0,1]$ and $g\colon X \to [0,1]$ with $f^{-1}(0) = \overline{A}$ and $g^{-1}(0) = \overline{B}$. We consider the map $h = f - g$. By construction $A \subset h^{-1}((-\infty,0))$ and $B \subset h^{-1}((0,\infty))$. ■

[112] Evidently disjoint closed subsets are separate.



Definition. Let X be a topological space. We say two points x and y are topologically indistinguishable if x and y have the same neighborhoods, i.e. if a subset $U \subset X$ is a neighborhood of x if and only if it is a neighborhood of y. Otherwise we say that the points are topologically distinguishable.

Example. If X is a set that is equipped with the trivial topology $\{\varnothing, X\}$, then any two points are topologically indistinguishable.

We can now formulate the following hierarchy of separations axioms, which has its origins in the work of Heinrich Tietze [**Tie1923**].[113]

Definition. Given a topological space X we introduce the following separability axioms:

T_0	Kolmogorov	distinct points are topologically distinguishable
T_1	Fréchet	every one-point subset is closed
T_2	Hausdorff	distinct points are separated by open neighborhoods
$T_{2\frac{1}{2}}$	Urysohn	distinct points are separated by closed neighborhoods
T_3	regular Hausdorff	X is Hausdorff & every point x is separated from any closed subset A with $x \notin A$ by open neighborhoods
$T_{3\frac{1}{2}}$	Tychonoff	X is Hausdorff & every point x is separated from every closed closed subset A with $x \notin A$ by a map
T_4	normal Hausdorff	X is Hausdorff & disjoint closed subsets are separated by open neighborhoods
T_5	completely normal Hausdorff	X is Hausdorff & separate subsets are separated by open neighborhoods
T_6	perfectly normal Hausdorff	X is Hausdorff & disjoint closed subsets are precisely separated by a map.

We say X is a T_i-space if the corresponding condition is satisfied.

Remark.

(1) Note that the definitions can differ slightly in the literature, for example in [**vQu2001**, Chapter 6] and in [**SS1978**, Section 2] the T_3 and the T_4-axiom is stated without the Hausdorff condition. The advantage of the above definition is that the separability axioms are increasingly stronger. More precisely, in Proposition 9.12 we will see that for T_i and T_j in the list with $i < j$ every T_j-space is also a T_i-space.

(2) In Exercise 9.18 we give an alternative characterization of complete normal Hausdorff spaces.

[113] The "T" in the names of the separation axioms goes back to the German word "Trennungsaxiom".

Proposition 9.12. For every two axioms T_i and T_j with $i < j$ in the above definition the following holds:

(1) If a topological space is a T_j-space, then it is also a T_i-space.
(2) There exists a topological space that is T_i but not T_j.

Remark. Proposition 9.12 says in particular that the obvious analogue of Urysohn's Lemma 8.4 for regular spaces does *not* hold. More precisely, Proposition 9.12 says in particular that there exists a topological space X which satisfies T_3 but does not satisfy $T_{3\frac{1}{2}}$. In plain English this means that the topological space X has the following two properties:

(1) X is regular space X and every one-point subset is closed.
(2) There exists a point $x \in X$ and a closed subset $A \subset X$ with $x \notin A$ such that there is no continuous map $f\colon X \to \mathbb{R}$ with $f(x) = 0$ and $f|_A \equiv 1$.

Proof.

(1) In the Points-in-Hausdorff Space Lemma 1.15 we showed that in a Hausdorff space every one-point subset is closed. In most cases this shows that for $i < j$ a T_j-space is also a T_i-space. The implications "$T_6 \Rightarrow T_5$" and "$T_{3\frac{1}{2}} \Rightarrow T_3$" rely on the simple observation, which we already used in the easy direction of Urysohn's Lemma 8.4, that if two subsets can be separated by a map, then they can also be separated by open subsets. Perhaps the only implication which is not entirely obvious is why a T_6-space is a T_5-space. Fortunately this implication follows immediately from Lemma 9.11.
(2) The following list of examples, exercises and references provides us with the promised topological spaces:

T_0	but not	T_1	Sierpiński space, see Exercise 2.22
T_1	but not	T_2	line with two zeros, see page 122
T_2	but not	$T_{2\frac{1}{2}}$	see Exercise 9.15
$T_{2\frac{1}{2}}$	but not	T_3	see Exercise 9.16 or alternatively Exercise 8.3
T_3	but not	$T_{3\frac{1}{2}}$	see [**SS1978**, Example 91]
$T_{3\frac{1}{2}}$	but not	T_4	see Exercise 9.17
T_4	but not	T_5	see [**SS1978**, Examples 87 and 105]
T_5	but not	T_6	see [**SS1978**, Example 32].

Many more examples regarding these and other separation axioms are discussed in [**SS1978**], which is a never-ending source of amusing topological spaces. ■

Exercises for Chapter 9.

Exercise 9.1. Let X be a topological space. Show that if X is second-countable, then X has only countably many components.

Exercise 9.2. Let X be a topological space. Show that if X is second-countable, then every locally finite open cover $\{U_i\}_{i\in I}$ of X is countable.

Exercise 9.3. Let $n \in \mathbb{N}$. We equip $\mathbb{R}^n$ with the standard topology $\mathcal{T}$.

(a) Show that the open balls with rational centers and rational radii, i.e. the set

$$\mathcal{C} \;:=\; \{B_r^n(q) \,|\, r \in \mathbb{Q}_{\geq 0} \text{ and } q \in \mathbb{Q}^n\},$$

is a basis for the topological space $\mathbb{R}^n$.

(b) Use (a) to show that for the cardinality of $\mathcal{T}$ we have $\#\mathcal{T} \leq \#\mathcal{P}(\mathbb{Q})$.

(c) Use the Bernstein-Schröder Theorem 0.11 to show that $\#\mathcal{T} = \#\mathcal{P}(\mathbb{Q})$.

Remark. By Exercise 0.10 we know that $\#\mathcal{P}(\mathbb{Q}) = \#\mathbb{R}$ and by Exercise 0.10 we know that $\#\mathbb{R} < \#\mathcal{P}(\mathbb{R})$. Evidently $\#\mathcal{P}(\mathbb{R}) \leq \#\mathcal{P}(\mathbb{R}^n)$. In summary we have thus shown that $\#\mathcal{T} < \#\mathcal{P}(\mathbb{R}^n)$. Informally speaking this means that open sets are "rare" among all subsets of $\mathbb{R}^n$.

Exercise 9.4. Given a non-empty subset $I \subset \mathbb{R}$ we denote by $C^b(I, \mathbb{R})$ the set of all bounded continuous maps $I \to \mathbb{R}$ and given $f \in C^b(I, \mathbb{R})$ we set

$$\|f\| \ := \ \sup\{|f(x)| \,|\, x \in I\}.$$

Note that $(C^b(I, \mathbb{R}), \|-\|)$ is a metric space. We will equip $C^b(I, \mathbb{R})$ with the corresponding topology.

(a) Show that if I is a compact interval, then $C^b(I, \mathbb{R})$ is second-countable.

(b) Show that if I is a non-compact interval, then $C^b(I, \mathbb{R})$ is not second-countable.

Exercise 9.5. Let X be a topological space that admits a compact exhaustion $\{P_i\}_{i\in\mathbb{N}}$ in the sense of the definition of page 338. Suppose that X is connected. Show that there exists also a compact exhaustion $\{Q_i\}_{i\in\mathbb{N}}$ such that each Q_i is connected.

Exercise 9.6. Let I be an uncountable set. Show that $[0, 1]^I$ is not second-countable.
Remark. By the Tychonoff Theorem 5.9 we know that $[0, 1]^I$ is compact. Thus the exercise shows that there exist compact spaces that are not second-countable.

Exercise 9.7. Let $S := (\mathbb{R}, \mathcal{T})$ be the Sorgenfrey line that we introduced on page 182. Show that the Sorgenfrey line is not second-countable.

Exercise 9.8. A topological space is called *separable* if it admits a subset that is countable and dense.

(a) Show that every second-countable topological space is separable.

(b) Let X be a set. We equip X with the cofinite topology that we defined on page 108.
(i) Show that X is separable.
(ii) Suppose that X is uncountable. Show that X is not second-countable.
Remark. This example is taken from [**SS1978**, Example 19].

(c) Let X be a topological space that is separable *and* Hausdorff. Show that X is second-countable.

Exercise 9.9. A topological space is called *separable* if it admits a subset that is countable and dense.

(a) Show that the Sorgenfrey line S, which we introduced on page 182, and the Sorgenfrey plane $S \times S$ are separable.

(b) Show that the diagonal $\Delta = \{(x, x) \in S \times S \,|\, x \in S\}$ is not separable.
Hint. Use Exercise 5.13.

Remark. This exercise shows that in general separability is not inherited by subsets.

Exercise 9.10. Let (X, d) be a metric space. Show that the corresponding topological space is second-countable if and only if X admits a countable dense subset.

Exercise 9.11. Show that every regionally compact Hausdorff space is regular.

Exercise 9.12. Let X be the *Sierpiński space* which we introduced on page 186, i.e. let $X = \{0,1\}$ be the set with two elements which we equip with the topology $\mathcal{T} = \{\varnothing, X, \{0\}\}$. Show that X is regular and show that X is not normal.

Exercise 9.13. Show that the product of two regular topological spaces is again regular.
Remark. Together with Exercise 8.6 we see that the Sorgenfrey plane is regular but not normal. One can easily show that the Sorgenfrey plane is Hausdorff. Thus we see that the Sorgenfrey plane is an example of a topological space that is T_3 but not T_4.

Exercise 9.14. Provide an example of a topological space that is not T_0.

Exercise 9.15. Let $D := B_1^2(0) \subset \mathbb{R}^2$ be the open disk of radius 1 centered on the origin and let

$$X \;:=\; D \cup \{(-1,0)\} \cup \{(0,1)\}.$$

We equip X with the topology that is generated by the following subsets:

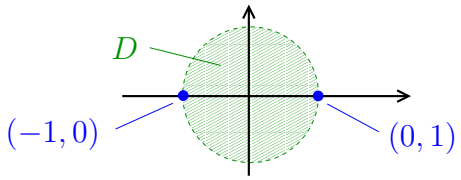
- For every $P \in D$ and every $r > 0$ we consider $B_r^2(P) \cap X$.
- For every $h \in (0,1)$ we consider the sets

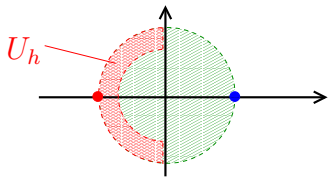
$$\begin{aligned} U_h &:= \{(x,y) \in D \,|\, x < 0 \text{ and } x^2 + y^2 > h\} \cup \{(0,-1)\} \\ V_h &:= \{(x,y) \in D \,|\, x > 0 \text{ and } x^2 + y^2 > h\} \cup \{(0,1)\}. \end{aligned}$$

The following two exercises show that $(X, \mathcal{T})$ is T_2 but not $T_{2\frac{1}{2}}$.

(a) Show that $(X, \mathcal{T})$ is Hausdorff.
(b) Show that the points $(0,-1)$ and $(0,1)$ cannot be separated by closed neighborhoods.

Remark. This exercise also solves Exercise 1.17.





Exercise 9.16. We set $K := \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$. As in Exercise 2.51 we define the *K-topology* on $\mathbb{R}$ as the topology $\mathcal{K}$ that is generated by the open intervals (a,b) and by all sets of the form $(a,b) \setminus K$ where $a, b \in \mathbb{R}$.

(a) Show that $(\mathbb{R}, \mathcal{K})$ is Hausdorff.
(b) Show that $(\mathbb{R}, \mathcal{K})$ is Urysohn.
(c) Show that $(\mathbb{R}, \mathcal{K})$ is not regular.
Hint. First show that K is closed in $(\mathbb{R}, \mathcal{K})$.

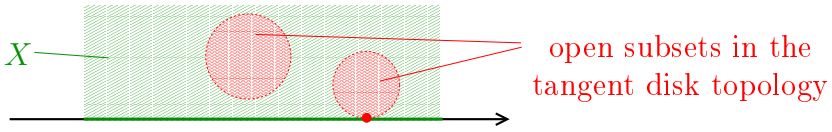
Remark. The exercises show that $(\mathbb{R}, \mathcal{K})$ is $T_{2\frac{1}{2}}$ but not T_3.

Exercise 9.17. We consider the upper half-plane $X := \{(x,y) \in \mathbb{R}^2 \,|\, y \geq 0\}$ together with the *tangent disk topology* $\mathcal{D}$, i.e. the topology which is generated by the usual open disks $B_r(x,y)$ with $r < y$ and the subsets $B_y(x,y) \cup \{(0,x)\}$.

(a) Show that $(X, \mathcal{D})$ is Hausdorff.
(b) Show that $(X, \mathcal{D})$ is Tychonoff.

(c) Show that $(X, \mathcal{D})$ is not normal.
Hint. Show that $\{(x,0) \mid x \in \mathbb{Q}\}$ and $\{(x,0) \mid x \notin \mathbb{Q}\}$ are closed subsets.

Remark. The exercises show that $(X, \mathcal{D})$ is $T_{3\frac{1}{2}}$ but not T_4.



Exercise 9.18. Let X be a Hausdorff space. Show that the following two statements are equivalent:

(1) X is T_5, i.e. X is completely normal and Hausdorff.
(2) X is *hereditarily normal*, i.e. if every subspace of X is normal.

Exercise 9.19. Let X be a perfectly normal topological space and let $A \subset X$ be a closed subset. By Lemma 9.10 we know that if X is Hausdorff, then there exists a map $f\colon X \to [0,1]$ with $A = f^{-1}(0)$. Do we need the hypothesis that X is Hausdorff?

Exercise 9.20. We consider the set

$$H \;:=\; \prod_{n\in\mathbb{N}} [0, \tfrac{1}{n}] \;=\; \big\{(x_1, x_2, \dots) \,\big|\, \text{for every } n \in \mathbb{N} \text{ we have } x_n \in [0, \tfrac{1}{n}]\big\}$$

with the metric

$$d(x,y) \;:=\; \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}.$$

This metric space, and the corresponding topological space, are called the *Hilbert Cube*.

(a) Show that d is defined, i.e. show that the series converges and show that d does indeed define a metric on H.
(b) Show that the Hilbert Cube, viewed as a topological space, is homeomorphic to the product space $[0,1]^{\mathbb{N}}$.
Remark. By Tychonoff's Theorem 5.9 this implies that the Hilbert Cube is compact.
(c) Let X be a metric space that is second-countable.[114] Show that there exists an embedding $f\colon X \to H$ of topological spaces.
Hint. Reread the proof of the Urysohn Metrization Theorem 9.4.
Remark. Statements (b) and (c) imply that any second-countable metric space admits an embedding, as a topological space, into a *compact* metrizable topological space.
(d) Show that every compact second-countable topological space is homeomorphic to a closed subset of the Hilbert Cube.

Exercise 9.21. Let X be a compact metrizable space.

(a) Show that X is second-countable.
Hint. Use the fact that given any $n \in \mathbb{N}$ there exists a finite subset $A_n \subset X$ with $\bigcup_{a\in A_n} B_{\frac{1}{n}}(a) = X$.

(b) Show that X is regionally compact.

[114] In Exercise 9.8 we showed that a metric space is second-countable if and only if it is separable. With this reformulation Statement (c) is commonly used in the literature.

10. Paracompact spaces and partitions of unity

10.1. **Paracompact spaces.** We recall the following definitions from page 114.

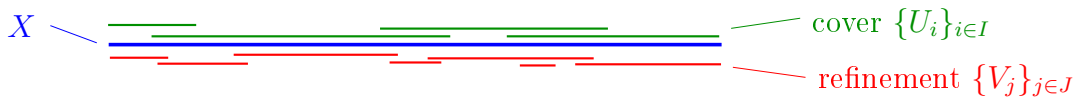
Definition. Let X be a topological space.

(1) A cover of a subset $A \subset X$ is a family $\{U_i\}_{i\in I}$ of subsets of X with $A \subset \bigcup_{i\in I} U_i$.
(2) We say a cover $\{U_i\}_{i\in I}$ is open if each U_i is open.
(3) We say a family $\{U_i\}_{i\in I}$ of subsets is locally finite if each $x \in X$ admits an open neighborhood W such that $\{i \in I \,|\, U_i \cap W \neq \varnothing\}$ is finite.

We add one and a half definitions in the context of covers.

Definition. Let X be a topological space and let $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ be two covers of X.

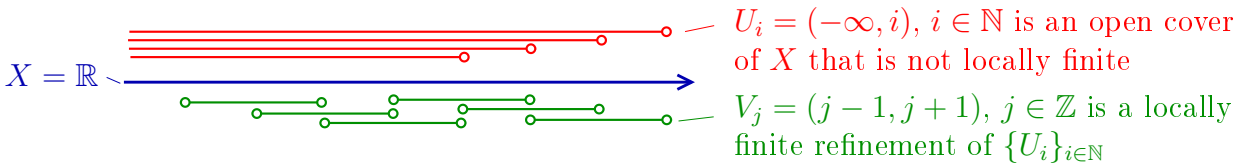
(1) We say that $\{V_j\}_{j\in J}$ is a refinement of $\{U_i\}_{i\in I}$ if each V_j is contained in some U_i.
(2) If furthermore each V_j is open, then we say that $\{V_j\}_{j\in J}$ is an open refinement of $\{U_i\}_{i\in I}$.



Now we can give the key definition of this chapter, which was introduced by Jean Dieudonné in [**Die1944**, p. 66].

Definition. A topological space is called paracompact if every open cover admits a locally finite open refinement.

Remark. Many books, e.g. [**Bre1993**, Definition I.12.3] and [**Kel1975**, p. 156] demand that a paracompact space is also Hausdorff.



As the name suggests, paracompact spaces are somewhat related to compact spaces. It is clear that a compact space is paracompact. The following lemma gives a useful example of a topological space that is not compact, but which is nonetheless paracompact.

Lemma 10.1. For every $n \in \mathbb{N}_0$ the topological space $\mathbb{R}^n$ is paracompact.

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of $\mathbb{R}^n$. Let $m \in \mathbb{N}$. We consider

$$\begin{aligned} \text{the closed annulus} \quad A_m &:= \{x \in \mathbb{R}^n \,|\, m \leq \|x\| \leq m+1\} \\ \text{the open annulus} \quad W_m &:= \{x \in \mathbb{R}^n \,|\, m-1 < \|x\| < m+2\}. \end{aligned}$$

By the Heine-Borel Theorem 1.24 we know that A_m is compact. Therefore there exists a finite subset $J_m \subset I$ such that the sets $\{U_j\}_{j\in J_m}$ form an open cover of A_m. One can now easily verify that

$$\{W_m \cap U_j\}_{m\in\mathbb{N}_0, j\in J_m}$$

is a locally finite open refinement of the open cover $\{U_i\}_{i\in I}$. ∎

In fact, according to the principle "what properties of $\mathbb{R}^n$ did we really use?" we can generalize the above lemma to the following statement.

Lemma 10.2. (Paracompact-Criterion Lemma) Let X be a topological space. If X is second countable, Hausdorff and regionally compact, then X is also paracompact.

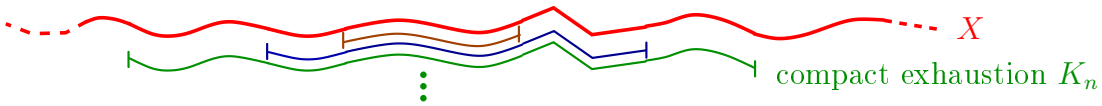
Remark. It follows from the discussion on pages 178 and 228 that Lemma 10.1 is actually a special case of the Paracompact-Criterion Lemma 10.2.

Proof. Let X be a topological space that is second countable, Hausdorff and regionally compact. We want to show that X is paracompact. Thus let $\{U_i\}_{i\in I}$ be an open cover of X.

By the Compact Exhaustion Lemma 9.3 (1) there exists a sequence of compact subsets $\{K_n\}_{n\in\mathbb{N}}$ of X such that the following two conditions are satisfied:

(1) for every $n \in \mathbb{N}$ we have $K_n \subset \overset{\circ}{K}_{n+1}$,
(2) we have $X = \bigcup_{n\in\mathbb{N}} K_n$.

We set $K_0 = \varnothing$. Let $n \in \mathbb{N}$. Since K_n is compact we obtain from the elementary the



Compact Subspace Covering Lemma 1.19 that there exists a finite subset $J_n \subset I$ such that the sets $\{U_j\}_{j\in J_n}$ form an open cover of K_n. One can now easily verify that[115]

$$\{(\overset{\circ}{K}_{n+1} \setminus K_{n-1}) \cap U_j\}_{n\in\mathbb{N}, j\in J_n}$$

is a locally finite open refinement of the open cover $\{U_i\}_{i\in I}$. ■

The following lemma can be viewed as an analogue of the Compact-Closed Lemma 1.21 (1). In combination with Lemmas 10.1 and 10.2 this new lemma gives us already a long list of paracompact spaces.

Lemma 10.3. (Closed Subset-Paracompact Lemma) Let X be a paracompact space. Every closed subset of X is also paracompact.

Remark. As is shown in [**Mun1975**, p. 254], in general a subspace of a paracompact space need not be paracompact.

Proof. The proof of the Closed Subset-Paracompact Lemma 10.3 is almost identical to the proof of the Compact-Closed Lemma 1.21 (1). Indeed, let X be a paracompact space and let $A \subset X$ be a closed subset. Let $\{U_i\}_{i\in I}$ be an open cover of A. By definition of the subspace topology there exists for each $i \in I$ an open subset V_i of X such that $U_i = V_i \cap A$. Since A is a closed subset of X we see that $\{V_i\}_{i\in I}$ together with $X \setminus A$ is an open cover of X. Since X is paracompact there exists a locally finite open refinement $\{W_j\}_{j\in J}$ of this cover of X. Evidently $\{W_j \cap A\}_{j\in J}$ is a locally finite open refinement of our original cover $\{U_i\}_{i\in I}$ of A. ■

[115] Since K_{n-1} is compact and since X is Hausdorff we obtain from the Compact-Closed Lemma 1.21 that K_{n-1} is a closed subset. Thus it follows that $\overset{\circ}{K}_{n+1} \setminus K_{n-1}$ is an open subset of X.

We conclude our initial list of examples of paracompact spaces with the following difficult theorem. We will not make use of it and thus we will not provide the proof.

Theorem 10.4. (Stone's Theorem) Every metric space is paracompact.

Proof. The theorem was first proved by Arthur Stone [**Stone1948**] in 1948. A much shorter proof can be found in [**RudiM1969**]. Alternatively a proof is given in [**Mun1975**, Theorem 41.4]. ■

Remark. As we will see, most topological spaces that are of interest to us, are in fact paracompact. For example we just saw that Stone's Theorem shows that metrizable topological spaces are paracompact. Furthermore in the Manifold Paracompact-Normal Proposition 18.9 (1) we will see, using the Paracompact-Criterion Lemma 10.2 that "topological manifolds" are paracompact. Furthermore in the CW-Complex Paracompactness Proposition 68.13 and in the Simplicial-Implies-CW Lemma 93.24 we will see that "CW-complexes" and "simplicial complexes" are paracompact. Nonetheless, elementary topological psychology tells us that surely not every topological space is paracompact. Indeed in Proposition 22.14 we will see that the "long line", that we will define on page 613, is not paracompact.

Next we recall the following definition from page 123.

Definition. A topological space X is called normal if any two disjoint closed subsets A and B of X are separated by open neighborhoods, i.e. if there exist disjoint open neighborhoods U of A and V of B.

In the Compact Hausdorff Spaces-Normal Lemma 1.16 (2) we showed that every compact Hausdorff space is normal. The following theorem, which is originally due to Jean Dieudonné [**Die1944**, Théorème 1], generalizes this result from compact to paracompact spaces.

Theorem 10.5. (Dieudonné's Theorem) Every paracompact Hausdorff space is normal.

The proof of Dieudonné's Theorem 10.5 rests on the following technical but useful lemma.

Lemma 10.6. (Closure-of-Locally Finite Cover Lemma) Let X be a topological space. If $\{U_i\}_{i\in I}$ is a locally finite cover of X, then $\{\overline{U}_i\}_{i\in I}$ is also a locally finite cover of X.

Proof of Lemma 10.6. Let X be a topological space and let $\{U_i\}_{i\in I}$ is a locally finite cover of X. The lemma follows immediately from the definitions and the following claim.

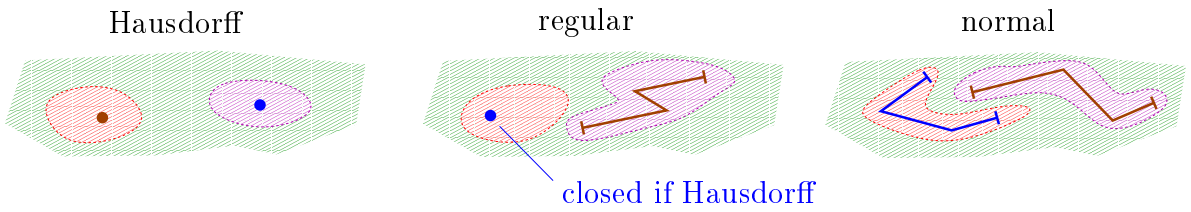
Claim. If W and U are two disjoint open subsets of X, then $\overline{U} \cap W = \varnothing$.

Proof. Let W and U be two disjoint open subsets of X. This means that U is contained in the closed subset $X \setminus W$. By definition of the closure this implies that $\overline{U} \subset X \setminus W$. In particular $\overline{U} \cap W = \varnothing$. ■

Proof of Dieudonné's Theorem 10.5. First recall that, according to the definition on page 343, a topological space X is called *regular* if given any $x \in X$ and given any closed

subset $B \subset X$ there exist disjoint open subsets U and V with $x \in U$ and $B \subset V$. Now let X be a paracompact Hausdorff space. In the following claim we first use paracompactness to upgrade from Hausdorff to regular.

Claim. The topological space X is regular.



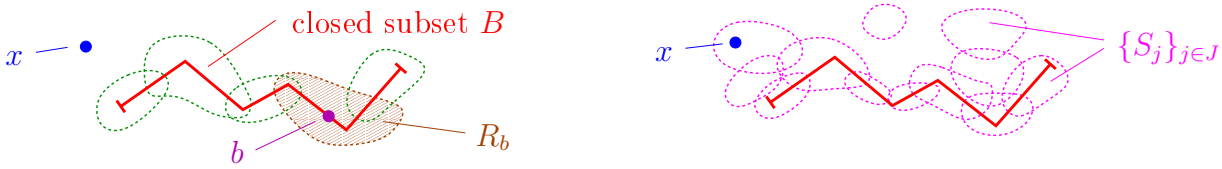
Proof. Let $x \in X$ and let $B \subset X$ be a closed subset. Since X is Hausdorff it follows from the Points-in-Hausdorff Space Lemma 1.15 (2) that we can find for each $b \in B$ an open neighborhood R_b of $b \in X$ such that $x \notin \overline{R}_b$. The open sets $\{R_b\}_{b\in B}$ together with the open set $X \setminus B$ form an open cover of X. Since X is paracompact there exists a locally finite open refinement $\{S_j\}_{j\in J}$ of this cover. We set

$$U := X \setminus \bigcup_{j\in J \text{ with } S_j\cap B\neq\varnothing} \overline{S}_j \qquad \text{and} \qquad V := \bigcup_{j\in J \text{ with } S_j\cap B\neq\varnothing} S_j.$$

We make the following observations:

- Evidently we have $U \cap V = \varnothing$.
- Since the S_j are an open cover of X we see that V is an open neighborhood of B.
- We claim that $x \in U$. If $j \in J$ with $S_j \cap B \neq \varnothing$, then we see that S_j is contained in one of the R_b's, since it cannot be contained in $X \setminus B$. Since $\overline{S}_j \subset \overline{R}_b$ we see that $x \notin \overline{S}_j$. This implies that $x \in U$.
- We need to show that U is open. Note that by construction the family $\{S_j\}_{j\in J}$ is locally finite. Next note that it follows from the Closure-of-Locally Finite Cover Lemma 10.6 that $\{\overline{S}_j\}_{j\in J}$ is also a locally finite family of subsets of X. By Lemma 1.3 (2b) we know that the union of a locally finite collection of closed subsets is again closed. This shows that U is open.

We have thus shown that U and V are disjoint open neighborhoods of x and B. ⊞



In the remainder we will use the argument of the claim to use paracompactness to upgrade from "regular" to "normal". Thus let A and B be two disjoint closed subsets of X. By the claim we know that X is regular. By the Regular-Open-Closed-Neighborhood Lemma 9.6 this implies that for each $a \in A$ there exists an open neighborhood R_a of a such that $\overline{R}_a \subset X \setminus B$.

The open sets $\{R_a\}_{a\in A}$ together with the open set $X \setminus A$ form an open cover of X. Since X is paracompact there exists a locally finite open refinement $\{S_j\}_{j\in J}$ of this cover.

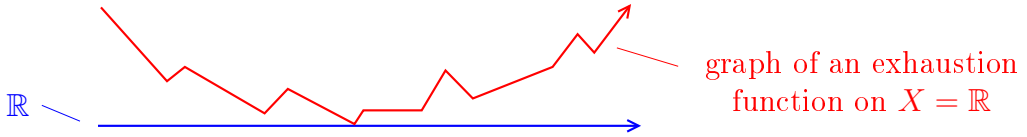
We set

$$U := \bigcup_{j\in J \text{ with } S_j\cap A\neq\varnothing} S_j \qquad \text{and} \qquad V := X \setminus \bigcup_{j\in J \text{ with } S_j\cap A\neq\varnothing} \overline{S}_j.$$

By basically the same argument as above we see that U is an open neighborhood of A, that V is an open neighborhood of B and that $U \cap V = \varnothing$. ■

We conclude this section with a short discussion of exhaustion functions.

Definition. Let X be a topological space. An exhaustion function is a map $f\colon X \to \mathbb{R}_{\geq 0}$ such that for every $c \in \mathbb{R}_{\geq 0}$ the preimage $f^{-1}([0,c])$ is a compact subset of X.[116]



Lemma 10.7. (Exhaustion Function Lemma) Let X be a topological space. If X is second countable, Hausdorff and regionally compact, then X admits an exhaustion function.

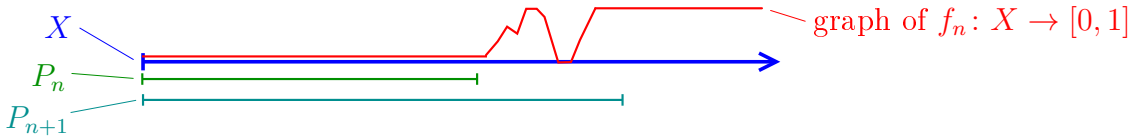
Remark. The Exhaustion Function Lemma 10.7 can be viewed as a refinement of the Compact Exhaustion Lemma 9.3 (1).

Proof. Let X be a topological space which is second countable, Hausdorff and regionally compact. By the Compact Exhaustion Lemma 9.3 (1) we know that X admits an exhaustion by compact subsets. In other words we know that there exists a sequence $\{P_n\}_{n\in\mathbb{N}}$ of compact subsets of X such that the following two conditions are satisfied:

(1) We have $X = \bigcup_{n\in\mathbb{N}} P_n$.

(2) For each $n \in \mathbb{N}$ we have[117] $P_n \subset \mathring{P}_{n+1}$.

Furthermore, the combination of the Paracompact-Criterion Lemma 10.2 and Dieudonné's Theorem 10.5 shows that X is normal. Since X is Hausdorff we know by the Compact-Closed Lemma 1.21 that each P_n is closed. Furthermore, by (2) we know that for each $n \in \mathbb{N}$ the closed sets P_n and $X \setminus \mathring{P}_{n+1}$ are disjoint. Since X is normal we obtain from Urysohn's Lemma 8.4 for each $n \in \mathbb{N}$ a map $f_n\colon X \to [0,1]$ such that $f_n|_{P_n} \equiv 0$ and $f_n|_{X\setminus\mathring{P}_{n+1}} \equiv 1$.



Claim 1. The sets $U_n = \{x \in X \mid f_n(x) \neq 0\}_{n\in\mathbb{N}}$ are a locally finite open cover of X.

Proof. Evidently each $U_n = f_n^{-1}((0,1])$ is open. Now let $x \in X$ be a point. According to the definition on page 114 we need to show that x admits an open neighborhood W such

[116] One can easily show that a map $X \to \mathbb{R}_{\geq 0}$ is an exhaustion function if and only if it is proper in the sense of the definition on page 180.

[117] Here we use the Compact-Closed Lemma 1.21 (2) which tells us that every compact subspace of our Hausdorff space X is a closed subset.

that W is contained in only finitely many U_n's. By (1) and (2) we know that x is contained in some open subset $\mathring{P}_k$. By (2) and by construction we know that $f_n(y) = 0$ for all $y \in P_k$ and all $n \geq k+1$. ⊞

By the claim it makes sense to consider the map

$$\begin{aligned} f\colon X &\to \mathbb{R}_{\geq 0} \\ x &\mapsto \sum_{k\in\mathbb{N}} f_n(x). \end{aligned}$$

Furthermore it follows from the Pasting Proposition 2.6 that f is continuous. It remains to prove the following claim.

Claim 2. For every $c \in \mathbb{R}$ the preimage $f^{-1}([0,c])$ is compact.

Proof. Let $c \in \mathbb{R}$. We pick $N \in \mathbb{N}$ with $N \geq c$. Then

$$\begin{aligned} f^{-1}([0,c]) &= f^{-1}([0,c]) \cap f^{-1}([0,N]) \\ &\subset f^{-1}([0,c]) \cap P_{N+1} \;=\; \text{closed subset} \cap \text{compact subset}. \end{aligned}$$

$\uparrow$ since $f(x) \geq N+1$ for all $x \in X \setminus \mathring{P}_{N+1}$

It follows from the Compact-Closed Lemma 1.21, see also the elementary Exercise 1.22, that $f^{-1}([0,c])$ is indeed compact. ■

Let X be a topological space that is second countable, Hausdorff and regionally compact. By the Paracompact-Criterion Lemma 10.2 we know that X is paracompact. It follows from Dieudonné's Theorem 10.5 that X is normal. Since X is second-countable, normal and Hausdorff it follows from the elementary the Points-in-Hausdorff Space Lemma 1.15 (1) and the much less elementary Urysohn Metrization Theorem 9.4 that X is metrizable. The concept of exhaustion functions allows us to prove the following proposition which upgrades the conclusion from "metrizable" to "completely metrizable".

Proposition 10.8. (Completely Metrizable Criterion Proposition) Let X be a topological space. If X is second countable, Hausdorff and regionally compact, then it is completely metrizable.

Proof. Let X be a topological space that is second countable, Hausdorff and regionally compact. As we just discussed, it follows from the Paracompact-Criterion Lemma 10.2, Dieudonné's Theorem 10.5, the Points-in-Hausdorff Space Lemma 1.15 (1) and the Urysohn Metrization Theorem 9.4 that X is metrizable.

Now let d be a metric on X which induces the given topology on X. Since X is second countable, Hausdorff and regionally compact we know by the Exhaustion Function Lemma 10.7 that X admits an exhaustion function, i.e there exists a map $f\colon X \to \mathbb{R}_{\geq 0}$ such that for every $c \in \mathbb{R}_{\geq 0}$ the preimage $f^{-1}([0,c])$ is a compact subset of X.

We now define

$$\begin{aligned} \widetilde{d}\colon X \times X &\to \mathbb{N}_0 \\ (x,y) &\mapsto d(x,y) + |f(x) - f(y)|. \end{aligned}$$

It follows from Exercise 6.4 that $\widetilde{d}$ is a metric and that $\widetilde{d}$ induces again the given topology.

Now let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(X, \widetilde{d})$. It follows immediately from the definition of $\widetilde{d}$ that there exists a $c \in \mathbb{R}_{\geq 0}$ such that $|f(x_n)| \leq c$ for all $n \in \mathbb{N}$. In other words, the sequence $(x_n)_{n\in\mathbb{N}}$ takes place in the compact subspace $f^{-1}([0,c])$. But by the

Compact-Complete Proposition 6.10 we know that any Cauchy sequence in a compact metric space converges. ■

10.2. **Partitions of unity.** In this section we introduce the notion of a "partition of unity" and we will see that for Hausdorff spaces the existence of suitable partitions of unity is equivalent to being paracompact. On numerous occasions we will see that "partitions of unity" are very useful for building interesting maps "out of" a given topological space.

Definition. Let X be a topological space.

(1) The support of a map $f\colon X \to \mathbb{R}$ is defined as

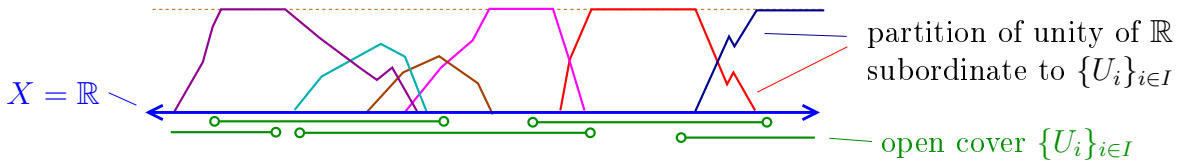
$$\operatorname{supp}(f) \;:=\; \overline{\{x \in X \mid f(x) \neq 0\}}.$$

(2) A partition of unity on X is a family of maps[118] $\{f_i\colon X \to [0,1]\}_{i\in I}$ with the following properties:
   (a) For each $x \in X$ there exists an open neighborhood U of x such that f_i vanishes on U for all but finitely many i.[119]
   (b) For every $x \in X$ we have[120]

$$\sum_{i\in I} f_i(x) \;=\; 1.$$

(3) Let $\{U_i\}_{i\in I}$ be an open cover. A partition of unity subordinate to the open cover $\{U_i\}_{i\in I}$ is a partition of unity $\{f_j\colon X \to [0,1]\}_{j\in J}$ such that for each $j \in J$ there exists an $i \in I$ with $\operatorname{supp}(f_j) \subset U_i$.

Example. In the figure below we consider the topological space $X = \mathbb{R}$ together with some open cover $\{U_i\}_{i\in I}$ and we illustrate a partition of unity subordinate to this open cover.



The following is the most interesting and useful theorem of this chapter.

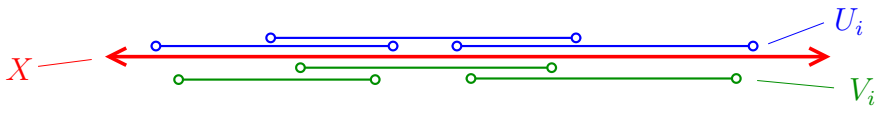
Theorem 10.9. (Paracompact-Partition of Unity Theorem) Let X be a topological space that is *Hausdorff*. The following three statements are equivalent:

(1) The topological space X is paracompact.
(2) For every open cover $\{U_i\}_{i\in I}$ of X we can find a partition of unity subordinate to the open cover $\{U_i\}_{i\in I}$.
(3) For every open cover $\{U_i\}_{i\in I}$ of X we can find a partition of unity $\{f_i\colon X \to [0,1]\}_{i\in I}$ such that for every $i \in I$ we have $\operatorname{supp}(f_i) \subset U_i$.

[118] As always, all maps are assumed to be continuous.
[119] If U is a subset on which a map $f\colon X \to \mathbb{R}$ vanishes, and if U is open, then it follows from the definition of the closure that $\operatorname{supp}(f) \cap U = \varnothing$. This observation shows that (a) implies that for each $x \in X$ there exist only finitely many $i \in I$ such that $x \in \operatorname{supp}(f_i)$.
[120] Note that by (a) there are only finitely many non-zero summands, hence the expression on the left-hand side makes sense even if I is infinite.

The proof of the "(1)⇒(3)"-direction of the Paracompact-Partition of Unity Theorem 10.9 rests on the shoulders of the following lemma which is interesting in its own right.

Lemma 10.10. (Shrinking Lemma) Let X be a paracompact Hausdorff space and let $\{U_i\}_{i\in I}$ be an open cover of X. There exists a locally finite open cover $\{V_i\}_{i\in I}$ of X such that for each $i \in I$ we have $\overline{V}_i \subset U_i$.



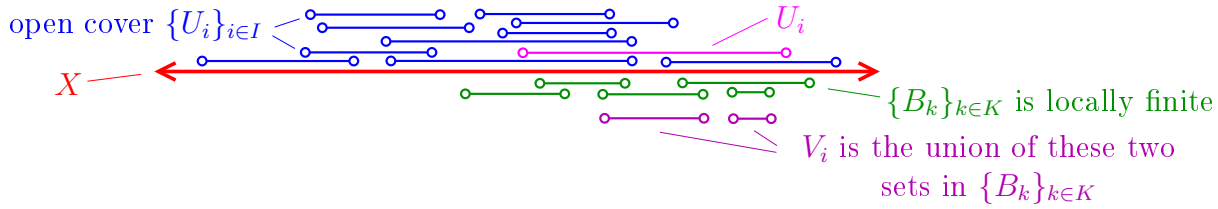
Proof of the Shrinking Lemma 10.10. Let X be a paracompact Hausdorff space and let $\{U_i\}_{i\in I}$ be an open cover of X. We define $\mathcal{A}$ to be the set of all open subsets A of X such that the closure $\overline{A}$ is contained in some subset U_i of the given open cover.

Claim 1. The sets in $\mathcal{A}$ form an open cover of X.

Proof. Since X is paracompact and Hausdorff we obtain from Dieudonné's Theorem 10.5 that X is normal. Furthermore since X is Hausdorff we know by the Points-in-Hausdorff Space Lemma 1.15 that every one-point subset of X is closed. It follows from these facts, together with the Normal-Open-Closed-Neighborhood Lemma 8.2, that given any U_i and given any $x \in U_i$ there exists an open subset A of X with $x \in A \subset \overline{A} \subset U_i$. Note that $A \in \mathcal{A}$. Since the U_i form an open cover of X we now see that the sets in $\mathcal{A}$ also form an open cover of X. ⊞

Since X is paracompact there exists a locally finite open cover $\{B_k\}_{k\in K}$ of X that is a refinement of $\mathcal{A}$. Note that every B_k is contained in some set in $\mathcal{A}$. Thus for every $k \in K$ we can and will pick an $f(k) \in I$ with $\overline{B}_k \subset U_{f(k)}$. Given $i \in I$ we set

$$V_i := \bigcup_{k\in K \text{ with } f(k)=i} B_k.$$



Claim 2. For each $i \in I$ we have $\overline{V}_i \subset U_i$.

Proof. Let $i \in I$. We see that

$$\overline{V}_i = \overline{\bigcup_{k\in K \text{ with } f(k)=i} B_k} = \bigcup_{k\in K \text{ with } f(k)=i} \overline{B}_k \subset U_i.$$

(The first $=$: follows from Lemma 1.11 (3b) since $\{B_k\}_{k\in K}$ is locally finite. The $\subset$: by definition of $f\colon K \to I$ we know that for each $k \in K$ with $f(k) = i$ we have $\overline{B}_k \subset U_i$.) ⊞

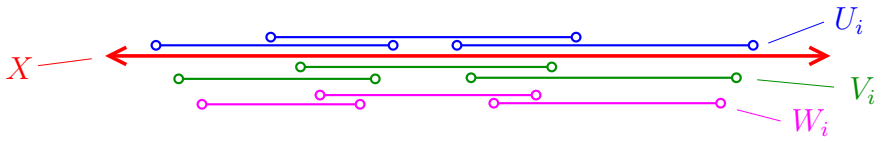
It remains to prove the following claim.

Claim 3. The family $\{V_i\}_{i\in I}$ is locally finite.

Proof. Let $x \in X$. Since $\{B_k\}_{k\in K}$ is locally finite there exists an open neighborhood W of $x \in X$ such that W has non-empty intersection with only finitely many B_k. Suppose that the corresponding indices are $k_1, \dots, k_m \in K$. If $W \cap V_i \neq \varnothing$, then by definition of V_i there exists a $k \in K$ with $f(k) = i$ and with $W \cap B_k \neq \varnothing$, but this implies that $i \in \{f(k_1), \dots, f(k_m)\}$. ■

Proof of the Paracompact-Partition of Unity Theorem 10.9. The "(2)⇒(1)"-direction is basically trivial. Indeed let $\{U_i\}_{i\in I}$ be an open cover of X. By hypothesis there exists a partition of unity $\{f_j \colon X \to [0,1]\}_{j\in J}$ subordinate to the open cover $\{U_i\}_{i\in I}$. It follows immediately from the definitions that $\{f_j^{-1}((0,\infty))\}_{j\in J}$ is a locally finite open cover that is a refinement of $\{U_i\}_{i\in I}$.

The "(3)⇒(2)"-direction is utterly trivial. Thus it remains to prove the "(1)⇒(3)"-direction. Let $\{U_i\}_{i\in I}$ be an open cover of X. By applying the Shrinking Lemma 10.10 twice we obtain locally finite open covers $\{V_i\}_{i\in I}$ and $\{W_i\}_{i\in I}$ such that for each $i \in I$ we have $\overline{W}_i \subset V_i \subset \overline{V}_i \subset U_i$.



Since X is paracompact and Hausdorff we know by Dieudonné's Theorem 10.5 that X is normal. Now let $i \in I$. Since X is normal we can apply Urysohn's Lemma 8.4 to the disjoint closed subsets $\overline{W}_i$ and $X \setminus V_i$ and we obtain a continuous map $g_i \colon X \to \mathbb{R}$ with $g|_{\overline{W}_i} \equiv 1$ and $g|_{X\setminus V_i} \equiv 0$.

Claim. For each $x \in X$ there exists an open neighborhood U of x such that g_i vanishes on U for all but finitely many i.

Proof. For every $i \in I$ we have $\{x \in X \mid g_i(x) \neq 0\} \subset V_i$. The claim now follows from the fact that the cover $\{V_i\}_{i\in I}$ is locally finite. ⊞

By the claim it makes sense to consider the map

$$\begin{array}{rcl} g \colon X & \to & \mathbb{R} \\ x & \mapsto & g(x) := \sum_{i\in I} g_i(x). \end{array}$$

It follows from the claim and the Pasting Proposition 2.6 that g is continuous. Furthermore note that it follows from the fact that the $\{W_i\}_{i\in I}$ form a cover of X and that $g_i|_{W_i} \equiv 1$ that for every $x \in X$ we have $g(x) > 0$. Therefore given $i \in I$ it makes sense to consider the map

$$\begin{array}{rcl} f_i \colon X & \to & \mathbb{R} \\ x & \mapsto & \frac{g_i(x)}{g(x)}. \end{array}$$

The maps f_i add up to the constant function 1. Together with the claim it follows that the maps $\{f_i\}_{i\in I}$ form a partition of unity.

Finally let $i \in I$. As we mentioned above, we have $\{x \in X \mid f_i(x) \neq 0\} \subset V_i$. It follows that $\operatorname{supp}(f_i) \subset \overline{V}_i \subset U_i$. ■

The following, slightly technical proposition, sometimes makes it possible to find a suitable *countable* open cover. We will only make use of it once, namely much later, when we study vector bundles. For the time being it is surely best to just ignore it.

Proposition 10.11. (Countable Cover Proposition) Let X be a topological space and let $\{U_i\}_{i\in I}$ be an open cover of X. If X is paracompact and Hausdorff, then there exists a *countable* locally finite open cover $\{V_j\}_{j\in J}$ such that each component of any V_j is contained in some U_i.

Proof. Since X is paracompact it follows from the Paracompact-Partition of Unity Theorem 10.9 that there exists a partition of unity $\{f_i\colon X \to [0,1]\}_{i\in I}$ such that for each $i \in I$ we have $\operatorname{supp}(f_i) \subset U_i$. Given a finite non-empty subset $S \subset I$ we define

$$V_S \;:=\; \{x \in X \,|\, f_s(x) > f_i(x) \text{ for all } s \in S \text{ and all } i \notin s\}.$$

Claim 1.

(1) Each V_S is open.
(2) Each $x \in X$ is contained in some V_S.
(3) Each V_S is contained in some U_i.
(4) For finite non-empty subsets S and T of I we have $V_S \subset V_T$ if and only if $S \subset T$.

Proof.

(1) Let $S \subset I$ be finite and non-empty. We need to show that V_S is open. By the JH-Lemma 1.7 it suffices to show that given any $x \in V_S$ there exists an open neighborhood U of x such that $V_S \cap U$ is open. Thus let $x \in V_S$. By definition of a partition of unity there exists a neighborhood U on which all but finitely many of the functions f_i vanish. We set $T := \{i \in I \setminus S \,|\, f_i(x) \neq 0\}$. We see that

$$V_S \cap U \;=\; \bigcap_{t\in T}\bigcap_{s\in S} \underbrace{\{x \in X \,|\, f_s(x) > f_t(x)\}}_{=(f_t-f_s)^{-1}((-\infty,0))}.$$

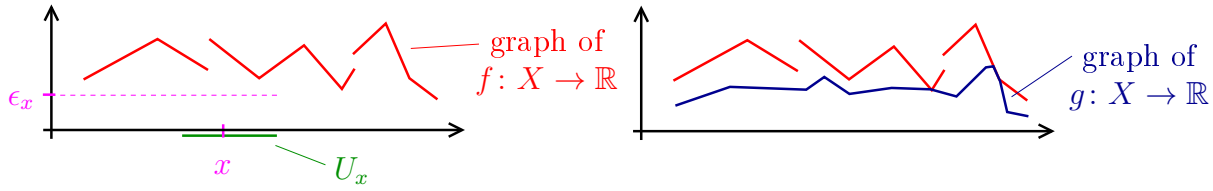
This is the intersection of finitely many open sets, thus open.
(2) Let $x \in X$. By definition of a partition of unity the set $S = \{i \in I \,|\, f_i(x) > 0\}$ is finite and non-empty. Evidently we have $x \in V_S$.
(3) Let $S \subset I$ be a finite non-empty subset. Given any $i \in S$ we have $V_S \subset \operatorname{supp}(f_i) \subset U_i$.
(4) This statement follows immediately from the definitions. ⊞

Now given $n \in \mathbb{N}_0$ we set $V_n := \bigcup_{\#S=n} V_S \subset X$. It follows from the above that the V_S with $\#S = n$ are disjoint and open. Therefore we deduce from the Component Lemma 2.33 (4b) that each component of V_n is contained in some V_S which in turn, as we saw above, is contained in some U_i. It is now clear that the family $\{V_n\}_{n\in\mathbb{N}_0}$ has all the desired properties. ■

10.3. **The Neighborhood Embedding Theorem.** We conclude this chapter with the Neighborhood Embedding Theorem which shows that under favorable circumstances one can extend an embedding from a closed subset to an open neighborhood.
Before we discuss the Neighborhood Embedding Theorem let us first state and prove a lemma, which is of independent interest, but which often gets used in conjunction with the Neighborhood Embedding Theorem.

Lemma 10.12. Let X be a topological space and let $f\colon X \to \mathbb{R}$ be a map (not necessarily continuous) with the property that for every $x \in X$ there exists an open neighborhood U_x

of x and an $\epsilon_x > 0$ such that for each $y \in U_x$ we have $f(y) \geq \epsilon_x$. If X is paracompact, then there exists a continuous map $g\colon X \to (0, \infty)$, such that for all $x \in X$ we have $g(x) \leq f(x)$.



Proof. Since X is paracompact we know by the Paracompact-Partition of Unity Theorem 10.9 that we can pick a partition of unity $\{\varphi_x\}_{x \in X}$ such that for each $x \in X$ we have $\operatorname{supp}(\varphi_x) \subset U_x$. We set

$$g := \sum_{x \in X} \varphi_x \cdot \epsilon_x.$$

As always it follows from the Pasting Proposition 2.6 that g is continuous. Since each $\epsilon_x > 0$ we see that $g(y) > 0$ for all $y \in X$. Finally, for every $y \in X$ we have

$$g(y) = \sum_{x \in X} \varphi_x(y) \cdot \epsilon_x \leq \sum_{x \in X \text{ with } y \in U_x} \varphi_x(y) \cdot \epsilon_x \leq \sum_{x \in X \text{ with } y \in U_x} \varphi_x(y) \cdot f(y) = f(y).$$

$\uparrow$ since $\operatorname{supp}(\varphi_x) \subset U_x$; $\uparrow$ property of ϵ_x; $\uparrow$ since $\{\varphi_x\}_{x \in X}$ is a partition of unity and since $\operatorname{supp}(\varphi_x) \subset U_x$

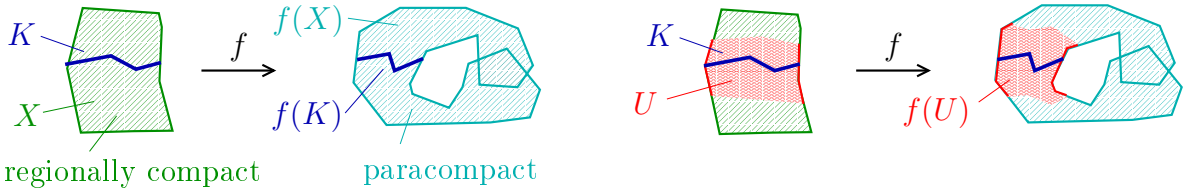
In summary, we have shown that $g(y) \leq f(y)$ for all $y \in X$. ■

Now we turn our attention to the actual hero of this section.

Theorem 10.13. (Neighborhood Embedding Theorem) Let $f\colon X \to Y$ be a map between two topological spaces and let $K \subset X$ be a subset. We assume that the following conditions are satisfied.

(1) The restriction of f to $f\colon K \to Y$ is an embedding.
(2) K is a closed subset of X and $f(K)$ is a closed subset of Y.
(3) The map $f\colon X \to Y$ is a local homeomorphism at each $P \in K$.
(4) The topological space X is regionally compact.
(5) The topological space Y is paracompact.

Then there exists an open neighborhood U of K such that the restriction of f to $f\colon U \to Y$ is an embedding and such that $f(U)$ is an open subset of Y.



Remark.

(1) This very useful theorem is based on [**Lang1985**, p. 97], [**Kos1993**, Lemma 9.2] and [**God1973**, p. 150]. Writing down the proof of the Neighborhood Embedding Theorem caused countless hours of pain for the author.
(2) The Neighborhood Embedding Theorem 21.13 gives the corresponding statement for smooth manifolds.

Proof. We start out with the first of several claims.

Claim 1. For each $x \in K$ there exists an open neighborhood A_x of $x \in X$ and an open neighborhood B_x of $f(x) \in Y$ such that $f|_{A_x} \colon A_x \to B_x$ is a homeomorphism and such that $f^{-1}(B_x \cap f(K)) = A_x \cap K$.

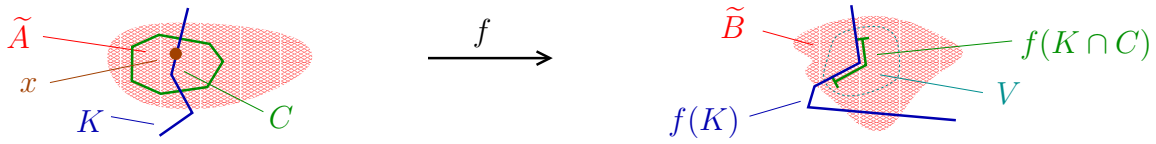
Proof. Let $x \in K$. Since f is a local homeomorphism at $x \in K$ we know that there exists an open neighborhood $\widetilde{A}$ of $x \in X$ and an open neighborhood $\widetilde{B}$ of $f(x) \in Y$ such that $f|_{\widetilde{A}} \colon \widetilde{A} \to \widetilde{B}$ is a homeomorphism.

Since X is regionally compact there exists a compact subset $C \subset X$ with $x \in C \subset \widetilde{A}$. Furthermore, since $K \subset X$ is closed we see that $K \cap C$ is a closed subset of C. It follows from the Compact-Closed Lemma 1.21 that $K \cap C$ is also compact. Next we consider $f(K \cap C)$. By the Compact Image Lemma 2.13 we know that $f(K \cap C)$ is compact. Since Y is paracompact it is in particular Hausdorff. Thus we obtain from the Compact-Closed Lemma 1.21 that $f(K \cap C)$ is a closed subset of Y.

Next note that $K \cap (X \setminus \widetilde{A})$ is a closed subset of K. Since f is an embedding we see that $f(K \cap (X \setminus \widetilde{A}))$ is a closed subset of $f(K)$. Since $f(K)$ is a closed subset of Y we obtain from the Open -Closed Inclusion Lemma 2.10 (2) that $f(K \cap (X \setminus \widetilde{A}))$ is a closed subset of Y.

Since f is in particular injective we see that the two closed subsets $f(K \cap C)$ and $f(K \cap (X \setminus \widetilde{A}))$ are actually closed disjoint. Since Y is paracompact we know by Dieudonné's Theorem 10.5 that Y is normal. Thus there exist disjoint open neighborhoods V of $f(K \cap C)$ and W of $f(K \cap (X \setminus \widetilde{A}))$.

We set $B_x := \widetilde{B} \cap V$ and we set $A_x := \widetilde{A} \cap f^{-1}(B_x)$. Basically by construction these have the desired properties. ⊞



Claim 2. There exists a family of open subsets $\{U_i\}_{i \in I}$ of X with the following properties:

(a) The maps $f|_{U_i} \colon U_i \to V_i = f(U_i)$ are homeomorphisms.
(b) For each $i \in I$ we have $f_i^{-1}(V_i \cap f(K)) = U_i \cap K$.
(c) The sets $\{V_i\}_{i \in I}$ form a locally finite open cover of $f(K)$ in the sense that given any $y \in f(K)$ there exists an open neighborhood $\widetilde{W}$ of $y \in Y$ such that $\widetilde{W}$ has non-empty intersection with only finitely many V_i.

Proof. For each $x \in K$ we pick A_x and B_x as in Claim 1. Note that the sets $\{B_x\}_{x \in K}$ form an open cover of $f(K)$. By hypothesis $f(K)$ is a closed subset of Y. Thus the sets $\{B_x\}_{x \in K}$ and $Y \setminus f(K)$ form an open cover of Y. Since Y is paracompact it follows from the Paracompact-Partition of Unity Theorem 10.9 that we can find a *locally finite* open cover $\{V_i\}_{i \in I'}$ that is a refinement of the given open cover of Y. In particular each V_i is contained in some B_x or it is contained in $Y \setminus f(K)$. We set $I := \{i \in I' \mid V_i \text{ is contained in some } B_x\}$. For each $i \in I$ we set $U_i = f^{-1}(V_i)$. These sets have, almost by definition, the desired properties. ⊞

For each $i \in I$ we write $f_i := f|_{U_i} \colon U_i \to V_i$ and we write $g_i := f_i^{-1} \colon V_i \to U_i$. It follows from the hypothesis that Y is paracompact and that $f(K)$ is a closed subset, by the argument of the Paracompact-Partition of Unity Theorem 10.9 that there exists an open cover $\{D_i\}_{i \in I}$ of $f(K) \subset Y$ such that for each $i \in I$ we have $\overline{D}_i \subset V_i$.

Claim 3. Let $y \in f(K)$. There exists a subset $W_y \subset Y$ with the following properties:

(a) $\{i \in I \,|\, \overline{D}_i \cap W_y \neq \varnothing\}$ is finite.
(b) W_y is an open neighborhood of $y \in Y$.
(c) If $\overline{D}_i \cap W_y \neq \varnothing$, then $W_y \subset V_i$.
(d) For each $i, j \in I$ with $\overline{D}_i \cap W_y \neq \varnothing$ and $\overline{D}_j \cap W_y \neq \varnothing$ we have[121] $g_i|_{W_y} = g_j|_{W_y}$.

Proof. Let $y \in f(K)$. Since the open cover $\{V_i\}_{i \in I}$ is locally finite and since for each $i \in I$ we have $\overline{D}_i \subset V_i$ we know that there exists an open neighborhood $\widetilde{W}$ of $y \in Y$ such that $J := \{i \in I \,|\, \overline{D}_i \cap \widetilde{W} \neq \varnothing\}$ is finite. We set $\widetilde{J} := \{j \in J \,|\, y \in \overline{D}_j\}$ and we set

$$W_y \;:=\; \Big(\widetilde{W} \setminus \bigcup_{j \,\in\, J \text{ with } y \,\notin\, \overline{D}_j} \overline{D}_j\Big) \cap f\Big(\bigcap_{j \in \widetilde{J}} U_j\Big).$$

We claim that W_y has the desired properties:

(a) By construction $\{i \in I \,|\, \overline{D}_i \cap W_y \neq \varnothing\}$ is finite.
(b) First we make the following observations:
- We pick some $i \in \widetilde{J}$. Note that $\bigcap_{j \in \widetilde{J}} U_j$ is the intersection of finitely many open subsets of Y. It follows that it is an open subset of U_i. Since $f \colon U_i \to V_i$ is a homeomorphism and since V_i is an open subset of Y we obtain, using the elementary the Open -Closed Inclusion Lemma 2.10, that $f\Big(\bigcap_{j \in \widetilde{J}} U_j\Big)$ is an open subset of Y.
- The set W_y is thus obtained by removing a finite number of closed sets from $\widetilde{W}$ and by taking afterwards the intersection with an open subset of Y.

This discussion shows that W_y is an open subset of Y.

It remains to show that $y \in W_y$. Since $f \colon K \to Y$ is an embedding there exists a unique $x \in K$ with $f(x) = y$. It remains to show that for each $j \in \widetilde{J}$ we have $x \in U_j$. Thus let $j \in \widetilde{J}$. By definition we have $y \in \overline{D}_j \subset V_j$. By Property (b) of Claim 2 we have $f_j^{-1}(V_j \cap f(K)) = U_j \cap K$. It follows that $f_j^{-1}(y) \in U_j \cap K$, which means that $f_j^{-1}(y) = x$. Thus we have shown that $x \in U_j$.

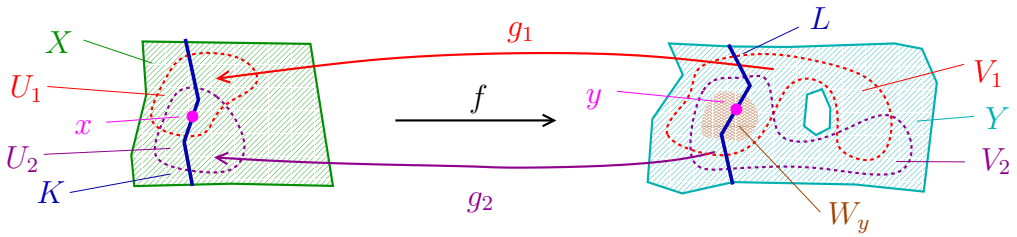
(c) Suppose that we have an $i \in I$ with $\overline{D}_i \cap W_y \neq \varnothing$. By construction of W_y we have $y \in \overline{D}_i$. This in turn implies that $i \in \widetilde{J}$ which then implies, by construction, that $W_y \subset f(U_i) = V_i$.

(d) Now let $Q \in W_y$. Let $i \in I$ with $\overline{D}_i \cap W_y \neq \varnothing$.
- By definition $g_i(Q)$ is the unique element $P \in U_i$ with $f(P) = Q$.
- In (c) we just saw that $i \in \widetilde{J}$.
- By construction of W_y we have $Q \in f\Big(\bigcap_{j \in \widetilde{J}} U_j\Big)$.

[121] Note that it follows from (c) that g_i and g_j are defined on W_y.

- Since $i \in \widetilde{J}$ and since $f\colon U_i \to Y$ is injective we obtain from the above that $P \in \bigcap_{j\in\widetilde{J}} U_j$.

In summary we have shown that $g_i(Q)$ is the unique $P \in \bigcap_{j\in\widetilde{J}} U_j$ with $f(P) = Q$. But this shows that $g_i(Q)$ is independent of the choice of $i \in J$. ⊞



Next we set

$$V := \bigcup_{y\in f(K)} W_y \qquad \text{and} \quad U := \bigcup_{y\in f(K)} \bigcup_{i\in I} g_i(W_y \cap D_i).$$

Since each W_y is open we see that V is an open neighborhood of $f(K)$. Furthermore, each $W_y \cap D_i$ is an open subset of V_i and thus each $g_i(W_y \cap D_i)$ is an open subset of U_i, in particular also of X. This shows that U is an open subset of X. Finally, since the D_i form an open cover of Y we see that U contains K. Thus we have shown that U is an open neighborhood of K.

The following claim now shows the desired statement, namely it shows that the restriction of f to $f\colon U \to Y$ is an embedding such that $f(U)$ is an open subset of Y.

Claim 4. The map

$$\begin{array}{rcl} g\colon V & \to & U \\ Q & \mapsto & g_i(Q) \text{ where } i \in I \text{ is chosen such that } Q \in D_i \end{array}$$

is well-defined, continuous and it satisfies $f \circ g = \mathrm{id}_V$ and $g \circ f = \mathrm{id}_U$.

Proof. It follows almost immediately from Claim 3 (d) that g is well-defined in the sense, that $g(Q)$ does not depend on the choice of $i \in I$. Furthermore, basically by definition of U we see that $g(V) \subset U$.

Next, note that on each subset $D_i \cap V$ the map g is given by the continuous map g_i. Since the $D_i \cap V$ are an open cover of V we obtain from the Pasting Proposition 2.6 that g is continuous.

Basically by definition we have $f \circ g = \mathrm{id}_V$. It remains to show that $g \circ f = \mathrm{id}_U$. Thus let $P \in U$. By definition P is of the form $g_i(Q) \subset U_i$ for some $Q \in W_y \cap D_i$. This implies that $f(P) = f(g_i(Q)) = Q \in W_y \cap D_i$. By definition we have $g(f(P)) = g_i(f(P))$. But $g_i(f(P))$ is the unique element of U_i which gets mapped to $f(P)$ under f. This element is P. We have thus shown that $g(f(P)) = P$. ■

Exercises for Chapter 10.

Exercise 10.1. Let X be a topological space and let $\{U_i\}_{i\in I}$ be an open cover of X. Show that if there exists a partition of unity subordinate to the open cover $\{U_i\}_{i\in I}$, then there exists a partition of unity $\{f_i\colon X \to [0,1]\}_{i\in I}$ such that for every $i \in I$ we have $\mathrm{supp}(f_i) \subset U_i$.

Remark. For Hausdorff spaces this is precisely the (2)⇒(3)-direction of the Paracompact-Partition of Unity Theorem 10.9. But even for Hausdorff spaces it is worth providing a direct proof.

Exercise 10.2. Let X be a paracompact space and let Y be a compact space. Show that the product $X \times Y$ is also paracompact.

Exercise 10.3. Let $S := (\mathbb{R}, \mathcal{T})$ be the Sorgenfrey line that we introduced on page 182. Show that the Sorgenfrey line is Hausdorff and paracompact.
Remark. We consider the product $S \times S$, which is often called the *Sorgenfrey plane*. Since S is Hausdorff we know by the Product Topology Properties Proposition 5.6 that the Sorgenfrey plane $S \times S$ is also Hausdorff. In Exercise 8.6 we saw that the Sorgenfrey plane $S \times S$ is not normal. It follows from these two observations, together with Dieudonné's Theorem 10.5, that the Sorgenfrey plane $S \times S$ is not paracompact. This shows that the property of being paracompact is not preserved under taking products.

Exercise 10.4. Give an example of a paracompact space X together with an open cover $\{U_i\}_{i \in I}$ such that there is no subset $J \subset I$ such that $\{U_i\}_{i \in J}$ is a locally finite open cover of X.

Exercise 10.5.

(a) Show that every discrete space is paracompact.
(b) Provide an example of a map $f\colon X \to Y$ such that X is paracompact, but such that the image $f(X)$ is not paracompact.

Remark. This exercise shows that the Compact Image Lemma 2.13 cannot be generalized from compact spaces to paracompact spaces.

11. The covering dimension of topological spaces

In this chapter we introduce the covering dimension of a topological space. This topic is not essential for the main flow of the lecture notes, but it is interesting and amusing in its own right. The most difficult result of this chapter is the Euclidean Embedding Theorem 11.13 for Topological Spaces which shows that, under reasonably mild extra hypotheses, a topological space of covering dimension m admits a closed embedding into $\mathbb{R}^{2m+1}$. Later in Chapter 27 we will use some of the results of this chapter to show that every (possibly non-compact) "topological manifold" admits a closed embedding into some $\mathbb{R}^n$ and every "smooth manifold" admits a "smooth embedding" into some $\mathbb{R}^n$. The exposition of this chapter is based on [**Mun1975**, Chapter 50] and [**Flo2020**].

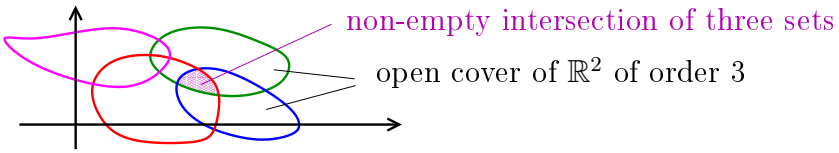
11.1. The definition of the covering dimension of a topological space.

Definition. Let X be a set and let $m \in \mathbb{N}_0$. A family $\{U_i\}_{i\in I}$ of subsets of X has order m if the following two statements hold:

(1) There exists $J \subset I$ with $\#J = m$ such that $\bigcap_{j\in J} U_j \neq \varnothing$.

(2) For every $J \subset I$ with $\#J > m$ we have $\bigcap_{j\in J} U_j = \varnothing$.

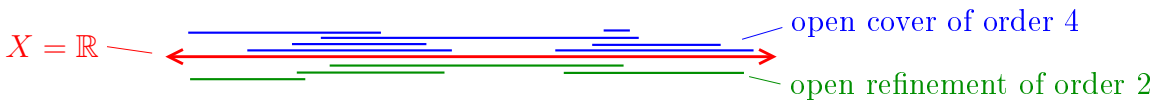
If no such m exists, then we say that the family has infinite order.



Definition. Let X be a topological space. If X is non-empty, then we refer to

$$\operatorname{cdim}(X) \;:=\; \min\left\{ m \in \mathbb{N}_0 \cup \{\infty\} \;\middle|\; \begin{array}{l}\text{every open cover of } X \text{ admits an}\\ \text{open refinement of order} \leq m+1\end{array} \right\}$$

as the[122] covering dimension of X. If X is empty, then we set $\operatorname{cdim}(X) := -1$.



Remark. The above definition of the covering dimension of a topological space is equivalent to the definition given in [**Mun1975**, Chapter 50].[123] In many other books, e.g.

[122] In the mathematical literature the words "cover" and "covering" often get used indiscriminately. In these notes we usually reserve the word "cover" for the concept introduced on page 114 and we reserve the word "covering" for the concept that we will introduce on page 1067. Thus it would make more sense to call the present concept "cover dimension". But since this term is not used at all in the literature we stick to the slightly illogical name "covering dimension".

[123] The definition is not entirely the same as the definition in [**Mun1975**, Chapter 50] since we consider families of subsets (where we allow repetitions), whereas [**Mun1975**, Chapter 50] works with subsets of the power set of X, in other words, families of subsets without repetition. Nonetheless, elementary logic shows that the definition of covering dimension that we give results in the same covering dimension as in [**Mun1975**, Chapter 50].

[**Enge1995**, p. 42], [**Pear1975**, p. 111] and [**Char2019**, p. 10], the covering dimension is defined by considering finite covers only. For many topological spaces these two notions agree, for example in [**Enge1995**, Proposition 3.2.2] it is shown that the two notions agree for paracompact spaces.

The following trivial lemma, which is an immediate consequence of the definitions, says that the covering dimension is a homeomorphism invariant.

Lemma 11.1. Let X and Y be two topological spaces. If X and Y are homeomorphic, then $\operatorname{cdim}(X) = \operatorname{cdim}(Y)$.

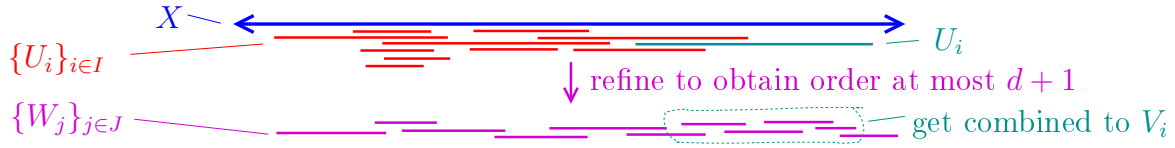
The next lemma simplifies life and the notation in applications significantly.

Lemma 11.2. Let X be a topological space with covering dimension $d := \operatorname{cdim}(X) \in \mathbb{N}_0$. Given any open cover $\{U_i\}_{i\in I}$ there exists an open cover $\{V_i\}_{i\in I}$ of order $\leq d+1$ such that for each $i \in I$ we have $V_i \subset U_i$.

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of X. Since $d := \operatorname{cdim}(X) \in \mathbb{N}_0$ we know that there exists an open refinement $\{W_j\}_{j\in J}$ of order $\leq d+1$. Since $\{W_j\}_{j\in J}$ is a refinement for the original cover we see that for each $j \in J$ we can pick an $i(j) \in I$ such that $W_j \subset U_{i(j)}$. Given $i \in I$ we set

$$V_i \;=\; \bigcup_{j\,\in\, J \text{ with } i(j)\,=\,i} W_j \quad\subset\; U_i.$$

Evidently the family $\{V_i\}_{i\in I}$ is an open cover of X that is a refinement of the original cover $\{U_i\}_{i\in I}$. It remains to show that the cover $\{V_i\}_{i\in I}$ has order $\leq d+1$. So let $x \in X$. Since $\{W_j\}_{j\in J}$ has order $\leq d+1$ we see that $\#\{j \in J \,|\, x \in W_j\} \leq d+1$. But then it follows follows that $\#\{i \in I \,|\, x \in V_i\} \leq d+1$. ∎



The following theorem hopefully justifies somewhat the term covering *dimension*.

Theorem 11.3. Let $n \in \mathbb{N}_0$. For every compact subspace $X \subset \mathbb{R}^n$ we have $\operatorname{cdim}(X) \leq n$.

Auxiliary Lemma 11.4. Let $n \in \mathbb{N}_0$. There exists an open cover $\{V_j\}_{j\in J}$ of $\mathbb{R}^n$ of order $n+1$ such that for each $j \in J$ we have $\operatorname{diam}(V_j) \leq \sqrt{n}$.

Remark. The proof we give of Lemma 11.4 is rather ad hoc. We will give an alternative proof using simplicial complexes in Exercise 94.8.

Proof of Lemma 11.4. Let $x = (x_1, \ldots, x_n) \in (\frac{1}{2}\cdot\mathbb{Z})^n$.

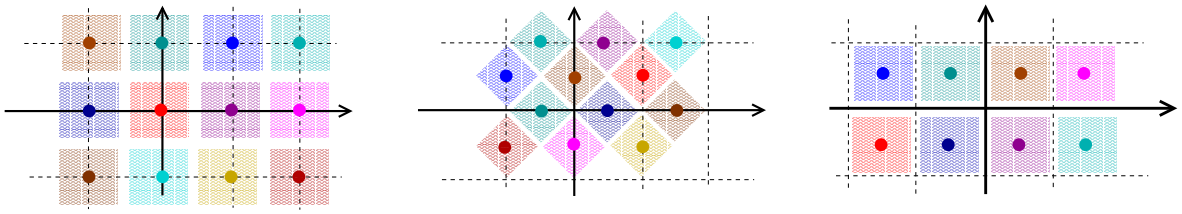
- We set $R_x := \{i \in \{1,\ldots,n\} \,|\, x_i \in \mathbb{Z} + \frac{1}{2}\}$ (the "rational coordinates" of x)
- We set

$$V(x) \;:=\; x + \left\{ \sum_{i=1}^{n} e_i \cdot \lambda_i \;\middle|\; \begin{array}{l} \text{for } i \in R_x \text{ we have } \lambda_i \in (-\frac{1}{2}, \frac{1}{2}), \\ \text{for } i \notin R_x \text{ we have } |\lambda_i| < \max\{\frac{1}{2} - |\lambda_j| \,|\, j \in R_x\} \end{array} \right\}$$

Given $d \in \{0,\ldots,n\}$ we consider the following families of open subsets of $\mathbb{R}^n$:

$$\mathcal{V}_d \;:=\; \{V(x)\}_{x\in X_d} \qquad \text{where} \qquad X_d \;:=\; \{x \in (\tfrac{1}{2}\cdot\mathbb{Z})^n \,|\, \#R_x = d\}.$$

We claim that $\mathcal{V}_0 \cup \cdots \cup \mathcal{V}_d$ has the desired properties.



open sets in $\mathcal{V}_0$ open sets in $\mathcal{V}_1$ open sets in $\mathcal{V}_2$

It remains to prove the following claim.

Claim.

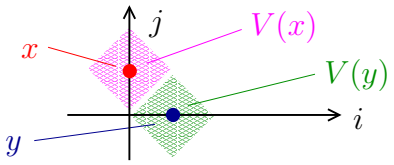
(1) Each $V(x)$ is an open subset of $\mathbb{R}^n$ with $\operatorname{diam}(V(x)) \leq \sqrt{n}$.
(2) $\mathcal{V}$ is a cover of $\mathbb{R}^n$.
(3) For every $d \in \{0, \ldots, n\}$ the sets in $\mathcal{V}_d$ are disjoint.

Proof.

(1) Let $x = (x_1, \ldots, x_n) \in (\frac{1}{2} \cdot \mathbb{Z})^n$. It is rather elementary to verify that $V(x)$ is open. We leave it to the reader to fill in the details. Furthermore note that for each $y = (y_1, \ldots, y_n) \in V(x)$ and each $i \in \{1, \ldots, n\}$ we have $y_i \in (x_i - \frac{1}{2}, x_i + \frac{1}{2})$. This implies that $\operatorname{diam}(V(x)) \leq \sqrt{n}$.
(2) Let $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. For $i = 1, \ldots, n$ we set

$$x_i := \begin{cases} y_i, & \text{if } y_i \in \mathbb{Z}, \\ \lfloor y_i \rfloor + \frac{1}{2}, & \text{if } y_i \notin \mathbb{Z} \end{cases}$$

and we set $x = (x_1, \ldots, x_n)$ Note that for $i \in \{1, \ldots, n\}$ with $y_i \notin \mathbb{Z}$ we have $i \in R_x$ and $y_i \in (x_i - \frac{1}{2}, x_i + \frac{1}{2})$. It is now clear that $y \in V(x)$.
(3) Let $d \in \{0, \ldots, n\}$ and let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in X_d$ with $x \neq y$.
- If there exists an $i \in \{1, \ldots, n\}$ with $|x_i - y_i| \geq 1$, then it follows from considering the i-th coordinate that $V(x) \cap V(y) = \varnothing$.
- Suppose there exists no $i \in \{1, \ldots, n\}$ with $|x_i - y_i| \geq 1$. Since $x \neq y$ we see that $R_x \neq R_y$. Since $\#R_x = \#R_y$ we see that there exists an $i \in \{1, \ldots, n\}$ with $i \in R_x$ and $i \notin R_y$ and there exists a $j \in \{1, \ldots, n\}$ with $j \notin R_x$ and $j \in R_y$. It is now elementary to see that the points in $V(x)$ and $V(y)$ differ in the i-th or the j-th coordinate. ■



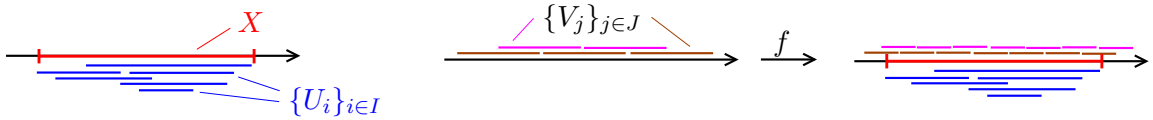
Proof of Theorem 11.3. Let $X \subset \mathbb{R}^n$ be a compact subspace and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. We equip X with the usual Euclidean metric coming from $\mathbb{R}^n$.[124] Since X is compact it follows from the Lebesgue Lemma 6.5 that there exists a $\delta > 0$ such that for every subset A of X with $\operatorname{diam}(A) < \delta$ there exists an $i \in I$ with $A \subset U_i$.

[124] By Exercise 1.3 we know that this metric gives the topology of X.

By Lemma 11.4 there exists an open cover $\mathcal{V} = \{V_j\}_{j\in J}$ of order $\leq n+1$ such that for each $j \in J$ we have $\operatorname{diam}(V_j) \leq \sqrt{n}$. We consider the homeomorphism

$$\begin{array}{rccc} f\colon \mathbb{R}^n & \to & \mathbb{R}^n \\ x & \mapsto & \frac{\delta}{2\sqrt{n}} \cdot x. \end{array}$$

It is clear that $\{f(V_j) \cap X\}_{j\in J}$ is an open cover of X of order $\leq n+1$ such that for each $j \in J$ we have $\operatorname{diam}(f(V_j) \cap X) \leq \delta$. But then it follows from the choice of δ that this is an open cover subordinate to the original open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X. ■



11.2. **Basic properties of the covering dimension.** The main goal of this section is to prove the Exhaustion-Covering Dimension Proposition 11.8 which allows us, in many cases, to obtain an upper bound on the covering dimension of a topological space in terms of covering dimensions of suitable subspaces. We start out with the following fairly straightforward lemma.

Lemma 11.5. (Closed Subspace Covering Dimension Lemma) Let X be a topological space and let $A \subset X$ be a subspace. If A is a closed subset, then $\operatorname{cdim}(A) \leq \operatorname{cdim}(X)$.

Proof. As we will see, the proof of the lemma is very similar to the proof of the Compact-Closed Lemma 1.21 (1) and of the Closed Subset-Paracompact Lemma 10.3. Indeed, let X be a topological space of covering dimension $d = \operatorname{cdim}(X)$ and let $A \subset X$ be a closed subset. If $d = \operatorname{cdim}(X) = \infty$, then the statement of the lemma is trivial. Thus we assume that $d = \operatorname{cdim}(X) < \infty$. We need to show that $\operatorname{cdim}(A) \leq d$. Let $\{U_i\}_{i\in I}$ be an open cover of A. By definition of the subspace topology there exists for each $i \in I$ an open subset V_i of X such that $U_i = V_i \cap A$. Since A is a closed subset of X see that $\{V_i\}_{i\in I}$ together with $X \setminus A$ is an open cover of X. Since $\operatorname{cdim}(X) = d$ there exists an open refinement $\{W_j\}_{j\in J}$ of this cover of X of order $\leq d+1$. Evidently $\{W_j \cap A\}_{j\in J}$ is an open refinement of our original cover of A of order $\leq d+1$. ■



Proposition 11.6. (Finite Union-Covering Dimension Proposition) Let X be a topological space and let $A_1, A_2, \ldots, A_k$ be finitely many closed subsets of X. Then

$$\operatorname{cdim}(A_1 \cup \cdots \cup A_k) \;=\; \max\{\operatorname{cdim}(A_1), \ldots, \operatorname{cdim}(A_k)\} \;\in\; \mathbb{N}_0 \cup \{\infty\}.$$

Example. Let $n \in \mathbb{N}_0$. We can write S^n as the union of the upper and lower hemispheres $S^n_{\geq 0}$ and $S^n_{\leq 0}$. We obtain that

$$\underset{\text{by Proposition 11.6 since } S^n_{\geq 0} \text{ and } S^n_{\leq 0} \text{ are closed subsets of } S^n}{\operatorname{cdim}(S^n) \;=\; \max\{\operatorname{cdim}(S^n_{\geq 0}), \operatorname{cdim}(S^n_{\leq 0})\}} \;\underset{\text{by Lemma 11.1 since on page 152 we saw that both hemispheres are homeomorphic to } \overline{B}^n}{=}\; \operatorname{cdim}(\overline{B}^n) \;\underset{\text{Theorem 11.3}}{\leq}\; n.$$

The proof of the Finite Union-Covering Dimension Proposition 11.6 requires some preparations.

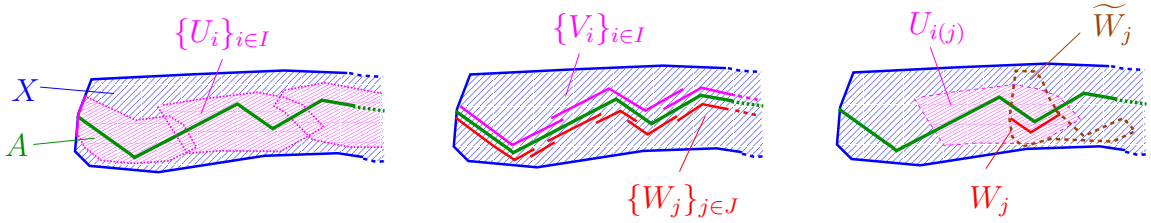
Definition. Let X be a topological space and let $A \subset X$ be a subset. We say that a family $\{U_i\}_{i\in I}$ of subsets of X has order $m \in \mathbb{N}_0 \cup \{\infty\}$ in A if the following two statements hold:[125]

(1) There exists a $J \subset I$ with $\#J = m$ such that $A \cap \bigcap_{j\in J} U_j \neq \varnothing$.

(2) For each $J \subset I$ with $\#J > m$ we have $A \cap \bigcap_{j\in J} U_j = \varnothing$.

Auxiliary Lemma 11.7. Let X be a topological space, let $d \in \mathbb{N}_0$ and let $A \subset X$ be a closed subset with $\operatorname{cdim}(A) \leq d$. If $\{U_i\}_{i\in I}$ is an open cover of X, then there exists an open refinement of $\{U_i\}_{i\in I}$ that has order at most $d+1$ in A.

Proof of Lemma 11.7. For each $i \in I$ we set $V_i = U_i \cap A$. Since $\{V_i\}_{i\in I}$ is an open cover of A and since $\operatorname{cdim}(A) \leq d$ there exists an open refinement $\{W_j\}_{j\in J}$ of $\{V_i\}_{i\in I}$ of order $d+1$. For each $j \in J$ we can and will pick an $i(j) \in I$ with $W_j \subset V_{i(j)}$.



By definition of the subspace topology we can pick for each $j \in J$ an open subset $\widetilde{W}_j \subset X$ with $W_j = \widetilde{W}_j \cap A$. Since A is a closed subset we see that $\{\widetilde{W}_j \cap U_{i(j)}\}_{j\in J}$ together with the sets $\{U_i \setminus A\}_{i\in I}$ is an open cover of X that is a refinement of the original cover $\{U_i\}_{i\in I}$ and that has order at most $d+1$ in A. ∎

Proof of the Finite Union-Covering Dimension Proposition 11.6. An elementary induction argument shows that it suffices to prove the proposition for $k = 2$. Furthermore we can assume that the two subspaces cover the total topological space. In other words, we can assume that we are in the following setting. We are given a topological space X and two closed subsets $A, B \subset X$ such that $X = A \cup B$. We set

$$d \;:=\; \max\{\operatorname{cdim}(A), \operatorname{cdim}(B)\} \quad \in\; \mathbb{N}_0 \cup \{\infty\}.$$

We need to show that $\operatorname{cdim}(X) = d$. Since A and B are closed subsets we obtain from the Closed Subspace Covering Dimension Lemma 11.5 that $\operatorname{cdim}(X) \geq d$. Thus it remains to show that $\operatorname{cdim}(X) \leq d$.

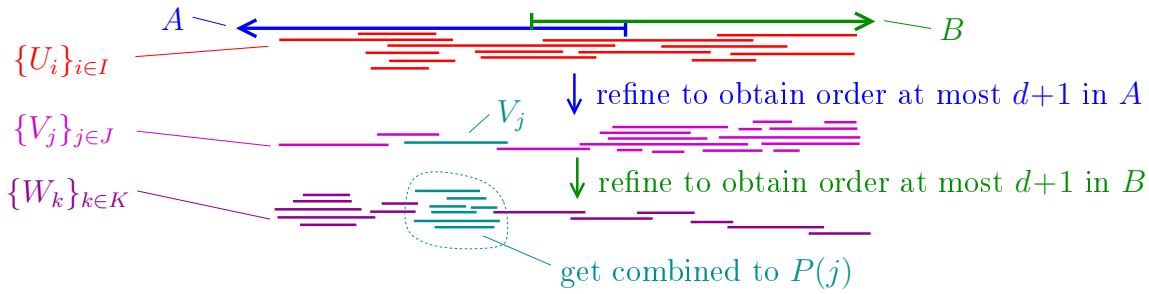
[125] In other words, the family $\{U_i\}_{i\in I}$ of subsets of X has order m in A if and only if the family $\{U_i \cap A\}_{i\in I}$ of subsets of A has order m.

If $d = \infty$, then the statement is trivial. Thus we assume that $d \in \mathbb{N}_0$. Let $\{U_i\}_{i\in I}$ be an open cover of X. By Lemma 11.7 there exists an open refinement $\{V_j\}_{j\in J}$ of $\{U_i\}_{i\in I}$ that has order at most $d+1$ in A and there exists an open refinement $\{W_k\}_{k\in K}$ of $\{V_j\}_{j\in J}$ that has order at most $d+1$ in B.

For each $k \in K$ we pick $j(k) \in J$ with $W_k \subset V_{j(k)}$. Given $j \in J$ we set[126]

$$P(j) \ := \bigcup_{k \in K \text{ with } j(k) = j} W_k \qquad \subset \ V_j.$$

The following claim says that $\{P(j)\}_{j\in J}$ has all the desired properties.



Claim.

(1) $\{P(j)\}_{j\in J}$ is an open cover of X.
(2) $\{P(j)\}_{j\in J}$ is a refinement of the original cover $\{U_i\}_{i\in I}$ of X.
(3) The order of the open cover $\{P(j)\}_{j\in J}$ is at most $d+1$.

Proof.

(1) It is clear that each $P(j)$ is open. Furthermore, given $x \in X$ there exists a $k \in K$ with $x \in W_k$ but this implies that $x \in P(j(k))$.
(2) For every $j \in J$ we have $P(j) \subset V_j$. Since $\{V_j\}_{j\in J}$ was a refinement of $\{U_i\}_{i\in I}$ we see that $\{P(j)\}_{j\in J}$ is a refinement of $\{U_i\}_{i\in I}$.
(3) Suppose we have distinct $j_1, \ldots, j_e \in J$ such that $P(j_1) \cap \cdots \cap P(j_e)$ contains a point x. We need to show that $e \leq d+1$. For $l = 1, \ldots, e$ we can and will pick a $k_l \in K$ with $j(k_l) = j_l$ and with $x \in W_{k_l}$. Note that $k_1, \ldots, k_l$ are also distinct. Now we see that
$$x \ \in \ W_{k_1} \cap \cdots \cap W_{k_e} \ \subset \ V_{j_1} \cap \cdots \cap V_{j_e}.$$
Since $X = A \cup B$ we know that $x \in A$ or $x \in B$.
- If $x \in A$, then $e \leq d+1$ since $\{V_j\}_{j\in J}$ has order at most $d+1$ in A.
- If $x \in B$, then $e \leq d+1$ since $\{W_k\}_{k\in K}$ has order at most $d+1$ in B. ■

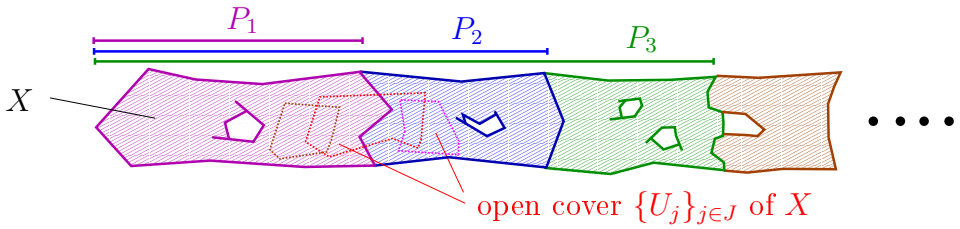
The following proposition can be viewed as a generalization of the Finite Union-Covering Dimension Proposition 11.6.

Proposition 11.8. (Exhaustion-Covering Dimension Proposition) Let X be a topological space, let $d \in \mathbb{N}_0$ and let $\{P_i\}_{i\in\mathbb{N}}$ be a sequence of closed subsets of X such that the following three conditions are satisfied:[127]

(1) We have $X = \bigcup_{i\in\mathbb{N}} P_i$.

[126] Here we use the same approach as in the proof of the Shrinking Lemma 10.10.

(2) For each $i \in \mathbb{N}$ we have $P_i \subset \mathring{P}_{i+1}$.
(3) For each $i \in \mathbb{N}$ we have $\operatorname{cdim}(P_{i+1} \setminus \mathring{P}_i) \leq d$.
Then $\operatorname{cdim}(X) \leq d$.



Remark. In many settings the combination of the Exhaustion-Covering Dimension Proposition 11.8, the Closed Subspace Covering Dimension Lemma 11.5 and the Finite Union-Covering Dimension Proposition 11.6 can be used to give an upper bound on the covering dimension of a non-compact space. For example in Proposition 11.11 we will use this trinity to show that the covering dimension of any subspace of $\mathbb{R}^n$ is $\leq n$. Furthermore in the Covering Dimension-of-Manifolds Proposition 27.7 we will use these three results to show that the covering dimension of any n-dimensional "topological manifold" is $\leq n$.

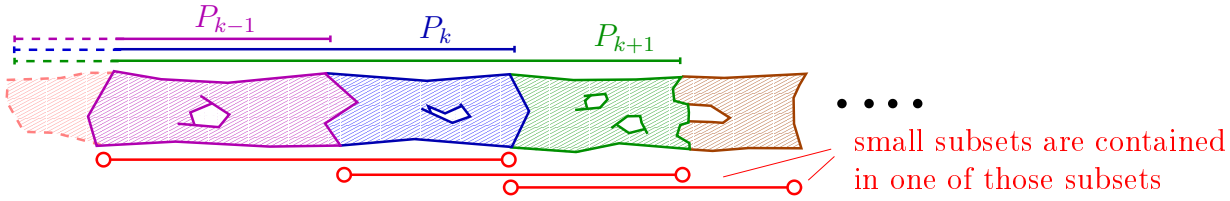
Proof. [128] Let X be a topological space, let $d \in \mathbb{N}_0$ and let $\{P_i\}_{i\in\mathbb{N}}$ be a sequence of closed subsets of X such that the following two conditions are satisfied:

(1) We have $X = \bigcup_{i\in\mathbb{N}} P_i$.
(2) For each $i \in \mathbb{N}$ we have $P_i \subset \mathring{P}_{i+1}$.

To be on the safe side we set $P_0 := \varnothing$. In the following we say that a subset $W \subset X$ is *small* if there exists a $j \in \mathbb{N}$ with $W \subset \mathring{P}_{j+1} \setminus P_{j-1}$.

Claim 1. Let $W \subset X$ be a small subset and let $k \in \mathbb{N}_0$.

(1) $\quad W \cap P_k \neq \varnothing \implies W \subset \mathring{P}_{k+1}$
(2) $\quad W \cap (X \setminus \mathring{P}_k) \neq \varnothing \implies W \cap \mathring{P}_{k-1} = \varnothing$.



[127] The first two conditions just say that $\{P_i\}_{i\in\mathbb{N}}$ is an exhaustion of X in the sense of the definition on page 338.
[128] As we mentioned above, the Exhaustion-Covering Dimension Proposition 11.8 can be viewed as a generalization of the Finite Union-Covering Dimension Proposition 11.6. In fact the proof of the Exhaustion-Covering Dimension Proposition 11.8 can be viewed as a refinement of the ideas of the proof of the Finite Union-Covering Dimension Proposition 11.6.

Proof. Note that Statement (2) is just the contrapositive of Statement (1). Thus it suffices to prove Statement (1). We assume that $W \cap P_k \neq \varnothing$. Since W is small there exists a $j \in \mathbb{N}$ with $W \subset \mathring{P}_{j+1} \setminus P_{j-1}$. Note that $W \cap P_{j-1} = \varnothing$. Since $W \cap P_k \neq \varnothing$ we see that $k \geq j$. But this implies that $W \subset \mathring{P}_{j+1} \subset \mathring{P}_{k+1}$. ⊞

Now we turn to the actual proof of the Exhaustion-Covering Dimension Proposition 11.8. Thus we make the following extra assumption:

(3) For each $i \in \mathbb{N}$ we have $\operatorname{cdim}(P_{i+1} \setminus \mathring{P}_i) \leq d$.

Recall that we need to show that $\operatorname{cdim}(X) \leq d$. Thus let $\mathcal{U} = \{U_j\}_{j \in J}$ be an open cover of X. It follows easily from (1) and (2) that the sets $\{\mathring{P}_{i+1} \setminus P_{i-1}\}_{i \in \mathbb{N}}$ form an open cover of X. Thus, after possibly replacing $\mathcal{U}$ by the open cover $\{U_j \cap (\mathring{P}_{i+1} \setminus P_{i-1})\}_{j \in J, i \in \mathbb{N}}$ we can assume that each set in the given open cover $\mathcal{U}$ is small in the above sense.

Claim 2. There exists a sequence $\mathcal{V}_i$, $i \in \mathbb{N}_0$ of families of subsets of X with the following properties:

(0) $\mathcal{V}_0 = \mathcal{U}$.
(1) Each $\mathcal{V}_i$ is an open cover of X.
(2) Each $\mathcal{V}_i$ with $i \geq 1$ is a refinement of $\mathcal{V}_{i-1}$.
(3) If $i \geq 1$ and $A \in \mathcal{V}_{i-1}$ is a set that is contained in $\mathring{P}_{i-2}$, then $A \in \mathcal{V}_i$.
(4) Each set in $\mathcal{V}_i$ is small in the above sense.
(5) Each $\mathcal{V}_i$ has order at most $d+1$ in P_i.

Proof. We set $\mathcal{V}_0 = \mathcal{U}$. By the discussion preceding the claim we know that (4) is satisfied. Otherwise there is nothing to show.

Now we suppose that we have already constructed $\mathcal{V}_0, \ldots, \mathcal{V}_i$ with the desired properties. By hypothesis P_{i+1} is a closed subset of X. It follows that $P_{i+1} \setminus \mathring{P}_i$ is also a closed subset of X. By hypothesis we know that $\operatorname{cdim}(P_{i+1} \setminus \mathring{P}_i) \leq d$. Thus it follows from Lemma 11.7 that there exists an open refinement $\mathcal{W}$ of $\mathcal{V}_i$ that has order at most $d+1$ in $P_{i+1} \setminus \mathring{P}_i$. Since $\mathcal{W}$ is a refinement of $\mathcal{V}_i$ we can and will pick for each $W \in \mathcal{W}$ a set $\Psi(W) \in \mathcal{V}_i$ with $W \subset \Psi(W)$. For $A \in \mathcal{V}_i$ we set

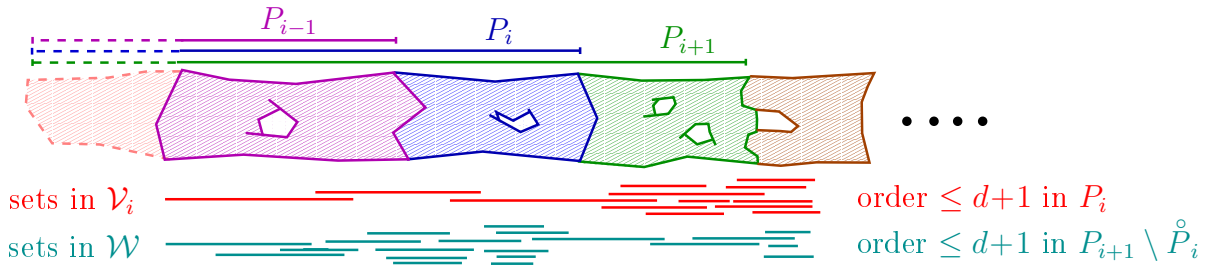
$$\Phi(A) \;:=\; (A \cap \mathring{P}_i) \cup \bigcup_{\substack{W \in \mathcal{W} \text{ with } \Psi(W) = A \\ \text{such that } W \cap P_i \neq \varnothing \\ \text{and such that } W \cap \mathring{P}_{i-1} = \varnothing}} W \;\subset A.$$

Note that for any $W \in \mathcal{W}$ with $W \cap P_i \neq \varnothing$ and with $W \cap \mathring{P}_{i-1} = \varnothing$ we have by definition $W \subset \Phi(\Psi(W))$. We define $\mathcal{V}_{i+1} \subset \mathcal{P}(X)$ to be the following two types of subsets of X:

(α) all $\Phi(A)$ for $A \in \mathcal{V}_i$,
(β) all $W \in \mathcal{W}$ such that $W \cap P_i = \varnothing$.

We need to verify that $\mathcal{V}_{i+1}$ has the desired properties:

(1) We want to show that $\mathcal{V}_{i+1}$ is an open cover of X. Evidently each set in $\mathcal{V}_{i+1}$ is open. Now let $x \in X$. We need to show that there exists a $U \in \mathcal{V}_{i+1}$ with $x \in U$.
   - First we suppose that $x \in \mathring{P}_i$. Since $\mathcal{V}_i$ is an open cover of X there exists an $A \in \mathcal{V}_i$ with $x \in A$. Since $x \in \mathring{P}_i$ we have $x \in A \cap \mathring{P}_i \subset \Phi(A)$. Thus we have shown that x is contained in a set of type (α).



- Now we suppose that $x \notin \mathring{P}_i$. Since $\mathcal{W}$ is an open cover of X we know that there exists a $W \in \mathcal{W}$ with $x \in W$. If $W \cap P_i = \varnothing$, then W is a set of type (β). Thus it remains to consider the case that $W \cap P_i \neq \varnothing$. Since $x \in W$ and $x \notin \mathring{P}_i$ we have $W \cap (X \setminus \mathring{P}_i) \neq \varnothing$, thus we obtain from Claim 1 (2), that $W \cap \mathring{P}_{i-1} = \varnothing$. This implies that $x \in W \subset \Phi(\Psi(W))$. We have thus shown that x is contained in a set of type (α).

(2) It is clear from the construction that $\mathcal{V}_{i+1}$ is a refinement of $\mathcal{V}_i$.

(3) Let $A \in \mathcal{V}_i$ be a set that is contained in $\mathring{P}_{i-1}$. By definition $\Phi(A) = A$. Thus we see that $A = \Phi(A) \in \mathcal{V}_i$.

(4) Since each set $\mathcal{V}_i$ is small and since $\mathcal{V}_{i+1}$ is a refinement of $\mathcal{V}_i$ we see that each set in $\mathcal{V}_{i+1}$ is also small.

(5) We need to show that $\mathcal{V}_{i+1}$ has order at most $d+1$ in P_{i+1}. Let $U_1, \ldots, U_k$ be distinct sets in $\mathcal{V}_{i+1}$ such that there exists an $x \in P_{i+1}$ with $x \in U_1 \cap \cdots \cap U_k$. We need to show that $k \leq d+1$.

- First we suppose that $x \in P_i$. In this case $U_1, \ldots, U_k$ are of type (α), i.e. there exist distinct $A_1, \ldots, A_k \in \mathcal{V}_i$ such that each U_j is given by $U_j = \Phi(A_j)$. By construction we have $\Phi(A_j) \subset A_j$. Thus we see that x is contained in each A_j. Since $\mathcal{V}_i$ has order at most $d+1$ in P_i we see that $k \leq d+1$.
- We suppose that $x \in P_{i+1} \setminus P_i$. After possibly reordering we can assume that $U_1, \ldots, U_l$ are of type (α) and $U_{l+1}, \ldots, U_k$ are of type (β). By definition of the two types of sets this implies the following:
  (a) There exist distinct $A_1, \ldots, A_l \in \mathcal{V}_i$ with $U_j = \Phi(A_j)$ for $j = 1, \ldots, l$. Note that by definition of the sets of type (α) and the fact that $x \notin \mathring{P}_i$ there exist $W_1, \ldots, W_l \in \mathcal{W}$ with $x \in W_j$ for $j = 1, \ldots, l$ and with $W_j \cap P_i \neq \varnothing$ for $j = 1, \ldots, l$. Since $\Psi(W_j) = A_j$ for $j = 1, \ldots, l$ are pairwise distinct we see that $W_1, \ldots, W_l$ are pairwise distinct.
  (b) We have $U_{l+1}, \ldots, U_k \in \mathcal{W}$ with $U_j \cap P_i = \varnothing$ for $j = l+1, \ldots, k$.
  It follows that $W_1, \ldots, W_l, U_{l+1}, \ldots, U_k$ are pairwise distinct sets in $\mathcal{W}$ and each contains x. Since $\mathcal{W}$ has order at most $d+1$ in $P_{i+1} \setminus \mathring{P}_i$ we see that $k \leq d+1$. ⊞

The following claim completes the proof of the proposition.

Claim 3. We consider

$$\mathcal{V} := \bigcup_{i \in \mathbb{N}} \{V \in \mathcal{V}_i \,|\, V \subset \mathring{P}_{i-1}\} \quad \subset \mathcal{P}(X).$$

We view $\mathcal{V}$ as a family of subsets. This family of subsets has the following properties:

(1) It is an open cover of X.
(2) It a refinement of $\mathcal{U}$.
(3) It has order at most $d+1$.

Proof.

(1) Let $x \in X$. Since $X = \bigcup_{i\in\mathbb{N}} P_i$ there exists an $i \in \mathbb{N}_{\geq 2}$ such that $x \in P_{i-2}$. Since $\mathcal{V}_i$ is an open cover of X there exists a $V \in \mathcal{V}_{i+2}$ with $x \in V$. By Claim 2 (4) we know that V is small. Therefore it follows from the fact that $V \cap P_{i-2} \neq \varnothing$ together with Claim 1 (1), that $V \subset \mathring{P}_{i-1}$, i.e. $V \in \mathcal{V}$. This shows that the sets in $\mathcal{V}$ form an open cover of X.
(2) By Claim 2 (2) each $\mathcal{V}_i$ is a refinement of $\mathcal{U}$, it follows immediately that $\mathcal{V}$ is a refinement of $\mathcal{U}$.
(3) Let $U_1, \ldots, U_k \in \mathcal{V}$ be distinct sets in $\mathcal{V}$ with $U_1 \cap \cdots \cap U_k \neq \varnothing$. We need to show that $k \leq d+1$.

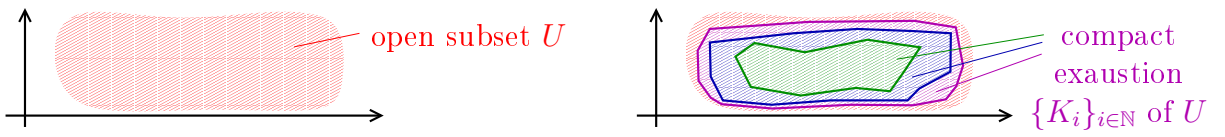
We pick $x \in U_1 \cap \cdots \cap U_k$ and we pick an $i_0 \in \mathbb{N}$ with $x \in P_{i_0}$. Next note that by definition of $\mathcal{V}$ there exists for each $j \in \{1, \ldots, k\}$ an $i_j \in \mathbb{N}$ with $U_j \in \mathcal{V}_{i_j}$ and with $U_j \subset \mathring{P}_{i_j-1}$. We set $i := \max\{i_0, i_1, \ldots, i_k\}$. It follows from Claim 2 (3), applied iteratively, that for each j we have $U_j \in \mathcal{V}_i$. By Claim 2 (5) we know that $\mathcal{V}_i$ has order at most $d+1$ in P_i. Since $x \in P_{i_0} \subset P_i$ we see that indeed $k \leq d+1$. ■

Corollary 11.9. For every open subset $U \subset \mathbb{R}^n$ we have $\operatorname{cdim}(U) \leq n$.

Proof. Let $U \subset \mathbb{R}^n$ be an open subset. By the Second Countability Lemma 9.1 we know that U is second-countable and by the discussion on page 178 we know that U is regionally compact. Evidently U is Hausdorff. It follows from the Compact Exhaustion Lemma 9.3 (1) that there exists a sequence of compact subsets $\{K_i\}_{i\in\mathbb{N}}$ of U such that the following two conditions are satisfied:

(1) for every $i \in \mathbb{N}$ we have $K_i \subset \mathring{K}_{i+1}$,
(2) we have $U = \bigcup_{i\in\mathbb{N}} K_i$.

Since $\mathbb{R}^n$ is Hausdorff we know by the Compact-Closed Lemma 1.21 that K_i is closed. Also note that each $K_{i+1} \setminus \mathring{K}_i$ is a closed subset of the compact subset K_{i+1}. In particular $K_{i+1} \setminus \mathring{K}_i$ is compact by the Compact-Closed Lemma 1.21. By Theorem 11.3 we know that for each $i \in \mathbb{N}$ we have $\operatorname{cdim}(K_{i+1} \setminus \mathring{K}_i) \leq n$. It follows from the Exhaustion-Covering Dimension Proposition 11.8 that $\operatorname{cdim}(U) \leq n$. ■



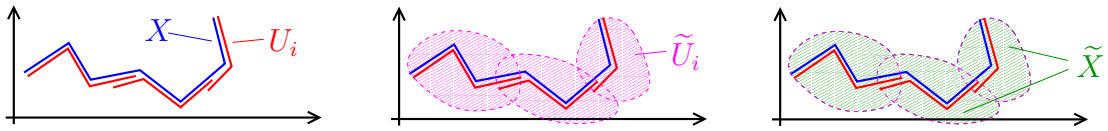
Question 11.10. Is the inequality from Corollary 11.9 always an equality? In other words, is it true that for every open subset $U \subset \mathbb{R}^n$ we have $\operatorname{cdim}(U) = n$?

We will answer the above question in Section 27.3. We conclude this discussion of the Finite Union-Covering Dimension Proposition 11.6 and its consequences with the following proposition. The statement of the proposition is perhaps not a surprise, but the proof is exceedingly neat.

Proposition 11.11. For every subset $X \subset \mathbb{R}^n$ we have $\mathrm{cdim}(X) \leq n$.

Remark. Later, in Theorem 77.6 we will state a characterization of subsets X of $\mathbb{R}^n$ with $\mathrm{cdim}(X) = n$.

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of X. By definition of the subspace topology we know that for each $i \in I$ there exists an open subset $\widetilde{U}_i$ of $\mathbb{R}^n$ with $U_i = X \cap \widetilde{U}_i$. We set $\widetilde{X} := \bigcup_{i\in I} \widetilde{U}_i$. Evidently $\widetilde{X}$ is an open subset of $\mathbb{R}^n$ and $\{\widetilde{U}_i\}_{i\in I}$ is an open cover of $\widetilde{X}$. By Corollary 11.9 we know that $\mathrm{cdim}(\widetilde{X}) \leq n$. This implies that there exists an open refinement $\{\widetilde{V}_j\}_{j\in J}$ of $\{\widetilde{U}_i\}_{i\in I}$ of order $\leq n+1$. But then $\{X \cap \widetilde{V}_j\}_{j\in J}$ is evidently an open refinement of $\{U_i\}_{i\in I}$ of order $\leq n+1$. ■

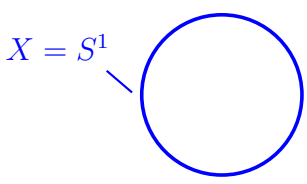


11.3. **The Covering Dimension Refinement Theorem.** The following theorem will play a crucial role in the proof of the Small Atlas Proposition 27.8 which says that every "smooth manifold" admits a "finite atlas". This result in turn is an essential ingredient in the proof of the Euclidean Embedding Theorem 27.1 that says that any "smooth manifold" admits a "smooth embedding" into some $\mathbb{R}^n$.

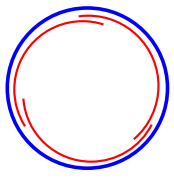
Theorem 11.12. (Covering Dimension Refinement Theorem) Let X be a Hausdorff space of covering dimension m and let $\{U_i\}_{i\in I}$ be an open cover of X.

(1) If X is paracompact, then there exists a locally finite refinement $\{V_j\}_{j\in J}$ of $\{U_i\}_{i\in I}$ and subsets $J_1, \ldots, J_{m+1}$ of J such that $J = J_1 \sqcup \cdots \sqcup J_{m+1}$ and such that for any $n \in \{1, \ldots, m+1\}$ and any $r, s \in J_n$ we have $V_r \cap V_s = \varnothing$.

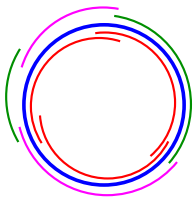
(2) If X is second-countable, then we can arrange that J is countable.



covering dimension = 1



open cover $\{U_i\}_{i\in I}$ of order 1

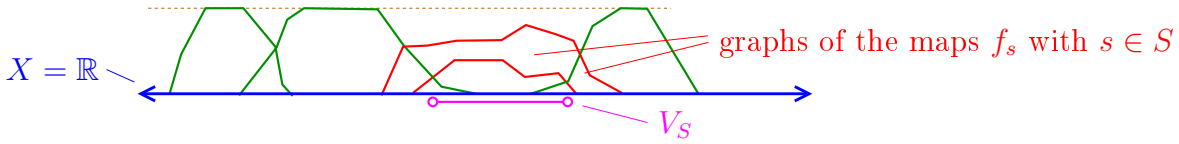


open cover $\{V_j\}_{j\in J}$ that splits into two families of disjoint sets

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of X. Since X has covering dimension m we might as well assume that the open cover $\{U_i\}_{i\in I}$ has order $m+1$. If X is second-countable we can assume, by the Countable Covering Lemma 9.2, that I is countable. Since X is paracompact

and Hausdorff it follows from the Paracompact-Partition of Unity Theorem 10.9 that there exists a partition of unity $\{f_j\colon X \to [0,1]\}_{i\in I}$ such that for each $i \in I$ we have $\operatorname{supp}(f_i) \subset U_i$. Given a finite non-empty subset $S \subset I$ we set

$$V_S \;:=\; \{x \in X \,|\, f_s(x) > f_i(x) \text{ for all } s \in S \text{ and all } i \notin S\}.$$



Claim 1.

(a) Each V_S is open.
(b) Each $x \in X$ is contained in some V_S.
(c) If $i \in S$, then $V_S \subset U_i$. In particular if S is non-empty and finite, then V_S is contained in some U_i.
(d) Let S and T of I be two finite non-empty subsets of J. If $\#S = \#T$ and $S \neq T$, then $V_S \cap V_T = \varnothing$.

Proof. We basically proved this claim already in the proof of the Countable Cover Proposition 10.11. For the reader's convenience we recall the argument.

(a) Let $S \subset I$ be non-empty and finite. We need to show that V_S is open. By the JH-Lemma 1.7 it suffices to show that given any $x \in V_S$ there exists an open neighborhood U of $x \in X$ such that $V_S \cap U$ is open. Thus let $x \in V_S$. By definition of a partition of unity there exists a neighborhood U on which all but finitely many of the functions f_i vanish. We set $T := \{i \in I \setminus S \,|\, f_i(x) \neq 0\}$. We see that

$$V_S \cap U \;=\; \bigcap_{t\in T}\bigcap_{s\in S} \underbrace{\{x \in X \,|\, f_s(x) > f_t(x)\}}_{=(f_s-f_t)^{-1}((0,\infty))}.$$

This is the intersection of finitely many open sets, thus open.
(b) Let $x \in X$. By definition of a partition of unity the set $S = \{i \in I \,|\, f_i(x) > 0\}$ is finite and non-empty. Evidently we have $x \in V_S$.
(c) Let $S \subset I$ be a finite non-empty subset. Given any $i \in S$ we have $V_S \subset \operatorname{supp}(f_i) \subset U_i$.
(d) Let S and T of I be two different finite non-empty subsets of J with $\#S = \#T$. We can pick $s \in S \setminus T$ and we can pick $t \in T \setminus S$. If $x \in V_S$, then $f_s(x) > f_t(x)$ and if $x \in V_T$, then $f_t(x) > f_s(x)$. This implies that $V_S \cap V_T = \varnothing$. ⊞

Beyond this claim which is already familiar we also have the following new claim.

Claim 2. If $S \subset I$ is a finite subset with $\#S > m+1$, then $V_S = \varnothing$.

Proof. If $S \subset I$ is a finite subset with $\#S > m+1$, then

$$\underset{\text{by Claim (1c)}}{V_S \;\underset{\uparrow}{\subset}\; \bigcap_{j\in S} U_j \;\underset{\uparrow}{=}\; \varnothing.}$$

since $\#S > m+1$ and $\{U_i\}_{i\in I}$ has order $m+1$ ⊞

Given $n \in \mathbb{N}$ we set

$$J_n \;:=\; \{\text{all subsets of } I \text{ with precisely } n \text{ elements}\}$$

and we set $J := \bigsqcup_{n\in\mathbb{N}} J_n$. Now we can conclude the argument:

(1) It follows from Claim 1 (a)-(c) that $\{V_j\}_{j\in J}$ is an open refinement of the original cover $\{U_i\}_{i\in I}$. Furthermore it follows from Claim 2 that $J = J_1 \sqcup \cdots \sqcup J_{m+1}$. Finally it follows from Claim 1 (d) that for any $n \in \{1,\dots,m+1\}$ and any $S,T \in J_n$ with $S \neq T$ the sets V_S and V_T are disjoint.

(2) Finally, if X is second-countable then we already arranged that I is countable. It follows from the Countability Lemma 0.8 (3) and (6) that J is also countable. ■

11.4. **The Euclidean Embedding Theorem for Topological Spaces.** We conclude this discussion of covering dimensions with the following Embedding Theorem.

Theorem 11.13. (Euclidean Embedding Theorem for Topological Spaces) Let X be a topological space such that the covering dimension $d := \operatorname{cdim}(X)$ is finite. If X is second countable, Hausdorff and regionally compact, then there exists a closed embedding $X \to \mathbb{R}^{2d+1}$, in other words, X is homeomorphic to a closed subset of $\mathbb{R}^{2d+1}$.

Before we turn to the proof of the Euclidean Embedding Theorem 11.13 for Topological Spaces it is worth pointing out a little corollary.

Corollary 11.14. Let X be a topological space with $\operatorname{cdim}(X) < \infty$. If X is second countable, Hausdorff and regionally compact, then X is completely metrizable.[129][130]

Proof. Let X be a topological space with $\operatorname{cdim}(X) < \infty$ that is second countable, Hausdorff and regionally compact. By the Euclidean Embedding Theorem 11.13 for Topological Spaces there exists an embedding $f\colon X \to \mathbb{R}^{2d+1}$ such that $f(X)$ is a closed subset. We equip $\mathbb{R}^{2d+1}$ with the usual Euclidean metric d. Since $f(X)$ is a closed subset we know by the Closed Subspace-Completeness Lemma 6.9 that $(f(X), d)$ is also complete and by Exercise 1.3 we know that the metric d on $f(X)$ gives the subspace topology.

Since $f\colon X \to f(X)$ is a homeomorphism we see that X admits a complete metric such that the metric topology agrees with the given topology. ■

The proof of the Euclidean Embedding Theorem 11.13 will require the remainder of this section. We start out with formulating a criterion for a map $f\colon X \to \mathbb{R}^N$ to be a closed embedding.

Definition. Let X be a topological space and let $f \in C(X,\mathbb{R}^N)$. We write

$$f(x) \xrightarrow{x\to\infty} \infty \quad :\Longleftrightarrow \quad \begin{array}{c}\text{for every } R>0 \text{ there exists a compact subset } K\subset X\\ \text{such that } \|f(x)\|_{\max} > R \text{ for all } x \in X\setminus K\end{array}$$

Lemma 11.15. Let X be a topological space and let $f\colon X \to \mathbb{R}^N$ be an injective map. If $f(x)\xrightarrow{x\to\infty}\infty$, then $f\colon X \to \mathbb{R}^N$ is a closed embedding.

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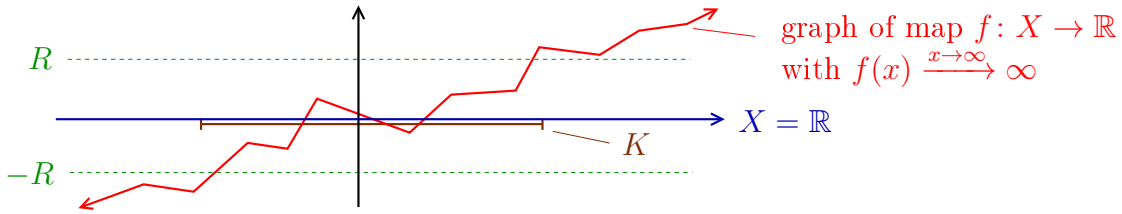
[128] It is perhaps worth recalling that the Urysohn Metrization Theorem 9.4 implies that a second-countable Hausdorff space is metrizable. This shows that (1) and (2) have a significant overlap. Nonetheless the two statements are independent.

[129] Recall that on page 299 we said that a topological space X is *completely metrizable* if there exists a complete metric d on X such that the given topology agrees with the topology coming from d.

[130] Recall that by the Compact-Complete Proposition 6.10 we know that every *compact* metrizable space is already completely metrizable.



Proof. Let $f\colon X \to \mathbb{R}^N$ be a map which satisfies $f(x) \xrightarrow{x\to\infty} \infty$.

Claim. The map $f\colon X \to \mathbb{R}^N$ is proper.[131][132]

Proof. Let $L \subset \mathbb{R}^N$ be a compact subset. By Lemma 1.23 we know that L is closed and we know that it is bounded, i.e. there exists an $R \geq 0$, such that $L \subset \overline{B}^N_R(0)$. Since $f(x) \xrightarrow{x\to\infty} \infty$ there exists a compact subset $K \subset X$ with $f^{-1}(\overline{B}^N_R(0)) \subset K$. Now we see that

$$\begin{aligned} f^{-1}(L) &= f^{-1}\big(\overline{B}^N_R(0) \cap L\big) = f^{-1}\big(\overline{B}^N_R(0)\big) \cap f^{-1}(L) \\ &\subset K \cap f^{-1}(L) \underset{\substack{\uparrow \\ \text{since } L \text{ is closed}}}{=} \text{compact subset} \cap \text{closed subset of } X \underset{\substack{\uparrow \\ \text{by the Compact-Closed Lemma 1.21 (1)}}}{=} \text{compact.} \end{aligned}$$

We have thus shown that $f\colon X \to \mathbb{R}^N$ is proper. ⊞

Now we assume that $f\colon X \to \mathbb{R}^N$ is furthermore injective. By the discussion on page 178 we know that $\mathbb{R}^N$ is regionally compact. Evidently $\mathbb{R}^N$ is also Hausdorff. Thus it follows from the Proper-Closed Proposition 2.43[133] that $f\colon X \to \mathbb{R}^N$ is in fact a closed embedding. ■

We proceed with introducing the following notation which we will use throughout the remainder of this chapter.

Notation. Let $N \in \mathbb{N}_0$.

(1) Given $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ we consider the maximum norm from page 102:

$$\|x\|_{\max} := \max\{|x_1|, \ldots, |x_N|\}.$$

(2) As in Lemma 6.11 we equip $\mathbb{R}^N$ with the metric that is given by

$$\rho(x, y) := \min\{1, \|x - y\|_{\max}\}.$$

It follows from the Equivalent Metric Lemma 1.2 and the Norm Equivalence Proposition 2.15 together with Lemma 6.11 that ρ induces the usual topology on $\mathbb{R}^N$. In particular $(\mathbb{R}^N, \rho)$ is also complete.

Let X be a topological space.

(3) We denote as usual by $C(X, \mathbb{R}^N)$ the set of continuous maps from X to $\mathbb{R}^N$.

---

[131] Recall that on page 180 we called a map $f\colon X \to Y$ between two topological spaces *proper* if the preimage of every compact subset of Y is a compact subset of X.

[132] Let $f\colon X \to \mathbb{R}^N$ be a map. The claim says that if $f(x) \xrightarrow{x\to\infty} \infty$, then f is proper. It is straightforward to show that the converse also holds. For our purpose though it is better work with the notion "$f(x) \xrightarrow{x\to\infty} \infty$" instead of the notion "proper".

[133] One can also deduce the statement from the Closed Embedding-to-$\mathbb{R}^n$ Proposition 2.19.

(4) If X is non-empty, then we equip $C(X,\mathbb{R}^N)$ with the metric

$$\rho(f,g) \;:=\; \sup\{\rho(f(x),g(x)) \,|\, x\in X\}.$$

Since $(\mathbb{R}^N,\rho)$ is complete we know by Proposition 6.12 that $(C(X,\mathbb{R}^N),\rho)$ is also complete.

Lemma 11.16. Let X be a topological space. Furthermore let $f,g\in C(X,\mathbb{R}^N)$ such that $\rho(f,g)<1$. If $f(x)\xrightarrow{x\to\infty}\infty$, then also $g(x)\xrightarrow{x\to\infty}\infty$.

Proof. Let $R>0$. Since $f(x)\xrightarrow{x\to\infty}\infty$ there exists a compact subset $K\subset X$ such that $\|f(x)\|_{\max}>R+1$ for all $x\in X\setminus K$. For $x\in X\setminus K$ we now obtain that

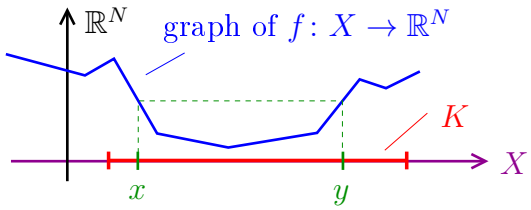
$$\|g(x)\|_{\max} \underset{\text{triangle inequality}}{\geq} \|f(x)\|_{\max}-\|f(x)-g(x)\|_{\max} \underset{\substack{\text{since } \rho(f,g)<1 \text{ implies that } \rho(f(x),g(x))<1\\ \text{which implies that } \|f(x)-g(x)\|_{\max}<1}}{>} \|f(x)\|_{\max}-1 \underset{\text{choice of } K}{>} R+1-1 \;=\; R.$$

■

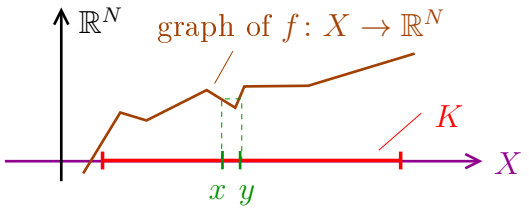
Definition. Let (X,d) be a metric space. Given $f\in C(X,\mathbb{R}^N)$ and given a non-empty compact subset $K\subset X$ we set

$$\Delta(f,K) \;:=\; \sup\{d(x,y)\,|\,x,y\in K \text{ with } f(x)=f(y)\}.$$

For safety, we extend this definition to $\Delta(f,\varnothing):=0$.



"large" $\Delta(f,K)$ means that $f\colon K\to\mathbb{R}^N$ is "far" from an injective map



"small" $\Delta(f,K)$ means that $f\colon K\to\mathbb{R}^N$ is "close" to an injective map

Lemma 11.17. Let (X,d) be a metric space. We equip $C(X,\mathbb{R}^N)$ with the metric ρ from above. Let $\epsilon>0$ and let $K\subset X$ be compact. Then

$$U_\epsilon(K) \;:=\; \{f\in C(X,\mathbb{R}^N)\,|\,\Delta(f,K)<\epsilon\}$$

is an open subset of $C(X,\mathbb{R}^N)$.

Proof. By definition of the topology in a metric space it suffices to show that for every $f\in U_\epsilon(K)$ there exists an $r>0$ with $B_r^\rho(f)\subset U_\epsilon(K)$. Thus let $f\in U_\epsilon(K)$. By definition of $U_\epsilon(K)$ we can pick a $b\in\mathbb{R}$ with $\Delta(f,K)<b<\epsilon$. We set

$$A \;:=\; \{(x,y)\in K\times K\,|\,d(x,y)\geq b\}.$$

In the Metric-is-Continuous Lemma 6.1 we showed that $d\colon K\times K\to\mathbb{R}$ is continuous. Thus it follows that A is a closed subset of the compact space $K\times K$. It follows from the Compact-Closed Lemma 1.21 that A itself is compact.

Claim 1. The map

$$\begin{array}{rcl} X\times X & \to & \mathbb{R} \\ (x,y) & \mapsto & \rho(f(x),f(y)) \end{array}$$

is strictly positive and continuous on A.

Proof. It follows from the Metric-is-Continuous Lemmas 6.1, Lemma 6.11 and the Product Topology-Basics Lemma 5.4 (3) that the map is continuous. Now let $(x, y) \in A \subset K \times K$. By definition of A we have $d(x, y) \geq b > \Delta(f, K)$. By definition of $\Delta(f, K)$ this implies that $f(x) \neq f(y)$. But this in turn implies that $d(f(x), f(y)) > 0$. ⊞

Since A is compact it follows from the claim and the Compact Image Lemma 2.13 that

$$r \;:=\; \tfrac{1}{2} \cdot \min\{\rho(f(x), f(y)) \,|\, (x, y) \in A\}$$

exists and that $r > 0$. It remains to prove the following claim.

Claim 2. We have $B_r^\rho(f) \subset U_\epsilon(K)$.

Proof. Let $g \in B_r^\rho(f)$, in other words, $\rho(f, g) < r$. We need to show that $g \in U_\epsilon(K)$. Since $b < \epsilon$ it suffices to show that for any $x, y \in K$ we have either $g(x) \neq g(y)$ or we have $d(x, y) \leq b$. Thus let $x, y \in K$. We distinguish two cases:

- Suppose that $(x, y) \in A$. In this case $\rho(f(x), f(y)) \geq 2r$. Note that we have the inequalities $\rho(f(x), g(x)) \leq \rho(f, g) < r$ and $\rho(f(y), g(y)) \leq \rho(f, g) < r$. In combination these show that $\rho(g(x), g(y)) > 0$, i.e. $g(x) \neq g(y)$.
- Suppose that $(x, y) \notin A$. In this case $d(x, y) < b$ by definition of A. ■

Definition.

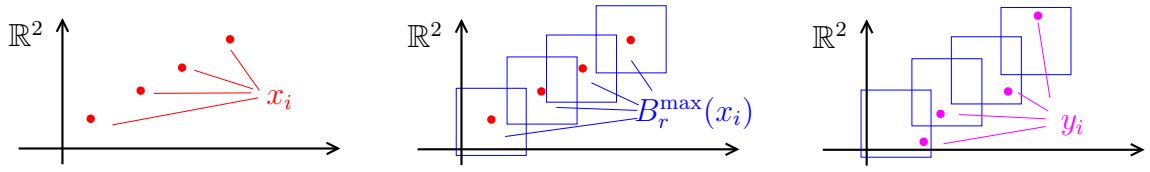
(1) We say that points $P_0, \ldots, P_k \in \mathbb{R}^N$ are affinely independent if for every choice of $\lambda_0, \ldots, \lambda_k \in \mathbb{R}$ we have[134]

$$\sum_{i=0}^{k} \lambda_i \cdot P_i \;=\; 0 \quad \text{and} \quad \sum_{i=0}^{k} \lambda_i \;=\; 0 \qquad \Longrightarrow \qquad \lambda_0 = \cdots = \lambda_k = 0.$$

(2) We say that a subset $S \subset \mathbb{R}^N$ is in general position if every collection of $\leq N + 1$ distinct points in S is affinely independent.[135]

Lemma 11.18. Let $x_1, \ldots, x_n \in \mathbb{R}^N$ be distinct points and let $r > 0$. There exist distinct points $y_1, \ldots, y_n \in \mathbb{R}^N$ with the following properties:

(1) For all $i = 1, \ldots, n$ we have $\|x_i - y_i\|_{\max} < r$.
(2) The set $\{y_1, \ldots, y_n\}$ is in general position.



Proof. Given $x \in \mathbb{R}^N$ and $r \geq 0$ we denote by $B_r^{\max}(x) = \{y \in \mathbb{R}^N \,|\, \|x - y\|_{\max} < r\}$ the r-ball with respect to the maximum norm. With this notation we can formulate the following claim.

[134] In other words, $P_0, \ldots, P_k$ are affinely independent if the affine vector subspace of $\mathbb{R}^N$ spanned by these points is k-dimensional. Or put differently, $P_0, \ldots, P_k$ are affinely independent if $P_1 - P_0, \ldots, P_k - P_0$ are linearly independent.

[135] We leave it to the reader to verify that for $N = 3$ this is precisely the notion of being in general position that we introduced on page 323.

Claim. Let $V_1, \dots, V_k \subset \mathbb{R}^N$ be affine subspaces each of which has codimension ≥ 1. For each $x \in \mathbb{R}^N$ and $r > 0$ the set $B_r^{\max}(x) \setminus (V_1 \cup \dots \cup V_k)$ is non-empty.

Proof. Recall that by the Norm Equivalence Proposition 2.15 the maximum norm and the usual Euclidean norm define equivalent metrics on $\mathbb{R}^N$. Thus the desired statement follows immediately from the Baire Category Theorem 6.16 (1), the fact that the metric space $\mathbb{R}^N$ is complete and the fact that hyperplanes of codimension ≥ 1 are nowhere dense. ⊞

Let $x_1, \dots, x_n \in \mathbb{R}^N$ be distinct points and let $r > 0$. We construct the points $y_1, \dots, y_n$ iteratively. We set $y_1 := x_1$. Now suppose that points $y_1, \dots, y_k$ have already been found with the desired properties.

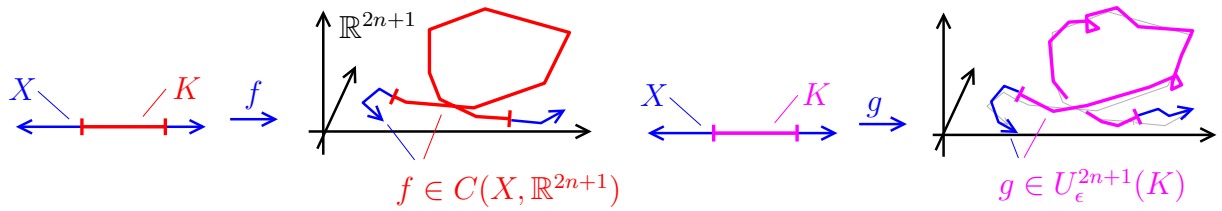
We denote by P the union of all affine subspaces of $\mathbb{R}^N$ that are generated by all subsets $A \subset \{y_1, \dots, y_k\}$ with $\#A \leq N$. Note that each such subspace in P has codimension $\geq N - \#A - 1 \geq 1$. By the above claim there exists a $y_{k+1} \in B_r^{\max}(x) \setminus P$. This point has the desired properties. ■

The following lemma is the key technical ingredient in the proof of the Euclidean Embedding Theorem 11.13 for Topological Spaces.

Lemma 11.19. Let (X, d) be a metric space that is second-countable and regionally compact such that $\operatorname{cdim}(X) \leq n$. If $K \subset X$ is a compact subspace, then for every $\epsilon > 0$ the set

$$U_\epsilon^{2n+1}(K) \;:=\; \{g \in C(X, \mathbb{R}^{2n+1}) \,|\, \Delta(g, K) < \epsilon\}$$

is dense in $C(X, \mathbb{R}^{2n+1})$ with respect to the metric ρ that we defined on page 383.



Proof. Let $f \in C(X, \mathbb{R}^{2n+1})$ and let $r \in (0, 1)$. We need to show that there exists a $g \in U_\epsilon^{2n+1}(K)$ such that $\rho(f, g) \leq r$.

Claim 1. There exist finitely many open non-empty subsets $U_1, \dots, U_m$ of K with the following properties:

(1) the sets $U_1, \dots, U_m$ are an open cover of K,
(2) for all $i = 1, \dots, m$ we have $\operatorname{diam}(U_i) < \frac{\epsilon}{2}$,
(3) for all $i = 1, \dots, m$ we have $\operatorname{diam}(f(U_i)) < \frac{r}{2}$,
(4) the cover $\{U_1, \dots, U_m\}$ has order at most $n + 1$.

Proof. For each $x \in K$ we consider

$$V_x \;:=\; \underbrace{B_{\frac{\epsilon}{2}}^X(x)}_{\substack{\text{open ball of radius } \frac{\epsilon}{2} \\ \text{around } x \in X}} \;\cap\; f^{-1}\Big(\underbrace{B_{\frac{r}{2}}^{\mathbb{R}^{2n+1}}(f(x))}_{\substack{\text{open ball of radius } \frac{r}{2} \\ \text{around } f(x) \in \mathbb{R}^{2n+1}}} \Big) \cap K.$$

These sets form an open cover of K. By hypothesis we have $\operatorname{cdim}(X) \leq n$. Since $K \subset X$ is compact and since X is Hausdorff we know by the Compact-Closed Lemma 1.21 (2) that

$K \subset X$ is a closed subset. Thus it follows from the Closed Subspace Covering Dimension Lemma 11.5 that we also have $\operatorname{cdim}(K) \leq n$. Therefore there exists an open refinement $\{U_i\}_{i \in I}$ of the open cover $\{V_x\}_{x \in K}$ that has order at most $n+1$. Since K is compact we know that we can cover K by finitely many of the U_i. Finally, since each U_i is contained in some V_x we see that each of non-empty U_i has the desired properties (2) and (3). ⊞

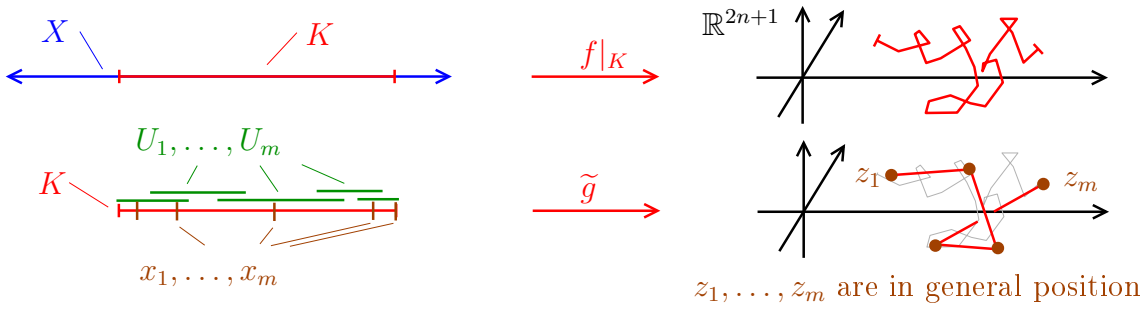
Since X is regionally compact and second-countable and since metric spaces are Hausdorff we know by the Paracompact-Criterion Lemma 10.2 that X is paracompact. As we just mentioned, $K \subset X$ is a closed subset. Thus we obtain from the Closed Subset-Paracompact Lemma 10.3 that K itself is also paracompact. Thus it follows from the Paracompact-Partition of Unity Theorem 10.9 that there exists a partition of unity $\Phi_1, \ldots, \Phi_m$ on K such that for each $i \in \{1, \ldots, m\}$ we have $\operatorname{supp}(\Phi_i) \subset U_i$.

For each $i \in \{1, \ldots, m\}$ we pick a point $x_i \in U_i$. By Lemma 11.18 there exist distinct points $z_1, \ldots, z_m \in \mathbb{R}^{2n+1}$ with the following two properties:

(1) For each $i \in \{1, \ldots, m\}$ we have $\|z_i - f(x_i)\|_{\max} < \frac{r}{2}$,
(2) The set $\{z_1, \ldots, z_m\}$ is in general position.

Next we consider the map

$$\begin{array}{rccc} \widetilde{g}\colon & K & \to & \mathbb{R}^{2n+1} \\ & x & \mapsto & \sum_{i=1}^{m} \Phi_i(x) \cdot z_i. \end{array}$$



Claim 2. For all $x \in K$ we have $\|\widetilde{g}(x) - f(x)\|_{\max} < r$.

Proof. Given $x \in K$ we see that

$$\begin{aligned} \|\widetilde{g}(x) - f(x)\|_{\max} &\overset{\text{since } \Phi_1(x) + \cdots + \Phi_m(x) = 1}{=} \left\| \sum_{i=1}^{m} \Phi_i(x) \cdot (z_i - f(x_i)) \; + \; \sum_{i=1}^{m} \Phi_i(x) \cdot (f(x_i) - f(x)) \right\|_{\max} \\ &\leq \sum_{i=1}^{m} \underbrace{\Phi_i(x) \cdot \|z_i - f(x_i)\|_{\max}}_{< \frac{r}{2} \text{ by choice of } z_i} + \sum_{i=1}^{m} \Phi_i(x) \cdot \underbrace{\|f(x_i) - f(x)\|_{\max}}_{\substack{\text{if } \Phi_i(x) \neq 0, \text{ then } x \in U_i, \\ \text{since } \operatorname{diam}(f(U_i)) < \frac{r}{2} \text{ this} \\ \text{implies that this term } < \frac{r}{2}}} \\ &< \sum_{i=1}^{m} \Phi_i(x) \cdot \tfrac{r}{2} + \sum_{i=1}^{m} \Phi_i(x) \cdot \tfrac{r}{2} \;=\; r \cdot \underbrace{\sum_{i=1}^{m} \Phi_i(x)}_{=1} \;=\; r. \end{aligned}$$

⊞

Claim 3. If $x, y \in K$ satisfy $\widetilde{g}(x) = \widetilde{g}(y)$, then $d(x, y) < \frac{\epsilon}{2}$.

Proof. Let $x, y \in K$ with $\widetilde{g}(x) = \widetilde{g}(y)$. Since each $\operatorname{diam}(U_i) < \frac{\epsilon}{2}$ it suffices to show that there exists an $i \in \{1, \dots, m\}$ with $x, y \in U_i$. We set

$$A := \{i \in \{1, \dots, m\} \mid \Phi_i(x) \neq 0\} \qquad \text{and} \qquad B := \{i \in \{1, \dots, m\} \mid \Phi_i(y) \neq 0\}.$$

Recall that the cover $\{U_1, \dots, U_m\}$ has order $n+1$. Furthermore recall that by choice we know that each $\operatorname{supp}(\Phi_i)$ is contained in the corresponding U_i. Thus we see that $\#A \leq n+1$ and we see that $\#B \leq n+1$. Next note that

$$\sum_{i \in A \cup B} (\Phi_i(x) - \Phi_i(y)) \cdot z_i \underset{\substack{\uparrow \\ \text{since the } i \notin A \cup B \\ \text{do not contribute}}}{=} \widetilde{g}(x) - \widetilde{g}(y) \underset{\substack{\uparrow \\ \text{since } \widetilde{g}(x) = \widetilde{g}(y)}}{=} 0. \qquad \text{and} \qquad \sum_{i \in A \cup B} (\Phi_i(x) - \Phi_i(y)) \underset{\substack{\uparrow \\ \text{since the } \Phi_i \text{ are a partition of unity and} \\ \text{since the } i \notin A \cup B \text{ do not contribute}}}{=} 0$$

Since the points $z_1, \dots, z_m$ are distinct, since the set $\{z_1, \dots, z_m\} \subset \mathbb{R}^{2n+1}$ is in general position and since

$$\#\{z_i \mid i \in A \cup B\} = \#(A \cup B) \leq \#A + \#B = 2n+2$$

we see that $\Phi_i(x) - \Phi_i(y) = 0$ for each $i \in \{1, \dots, m\}$. Since $\sum_{i=1}^{m} \Phi_i(x) = 1$ we see that there exists an $i \in \{1, \dots, m\}$ with $\Phi_i(x) > 0$. By the above we have $\Phi_i(x) = \Phi_i(y) > 0$. This implies that $x, y \in \operatorname{supp}(\Phi_i) \subset U_i$. ⊞

In Claim 1 and 2 we constructed an interesting map $\widetilde{g} \colon K \to \mathbb{R}^{2n+1}$. It remains to extend this map in a satisfactory way to a map $g \colon X \to \mathbb{R}^{2n+1}$. To do so we consider the map

$$\begin{aligned} \Theta \colon K &\to \mathbb{R}^{2n+1} \\ x &\mapsto f(x) - \widetilde{g}(x). \end{aligned}$$

It follows from Claim 2 that this map takes values in $[-r, r]^{2n+1}$.

Since X is metric we know by the Metric Spaces-Normal Proposition 8.1 that X is normal. Furthermore, since K is compact and since X is Hausdorff we know by the Compact-Closed Lemma 1.21 that $K \subset X$ is a closed subset. This implies that we can apply the Tietze Extension Theorem 8.6 to obtain an extension of the above map $\Theta \colon K \to [-r, r]^{2n+1}$ to a continuous map $\widetilde{\Theta} \colon X \to [-r, r]^{2n+1}$. Finally we consider the map

$$\begin{aligned} g \colon X &\to \mathbb{R}^{2n+1} \\ x &\mapsto f(x) - \widetilde{\Theta}(x). \end{aligned}$$

By construction we have $g|_K = \widetilde{g}$. By Claim 3 we have $\Delta(g, K) \leq \frac{\epsilon}{2}$, which implies that $g \in U_\epsilon^{2n+1}(K)$. Furthermore, since Θ takes values in $[-r, r]^{2n+1}$ we see that $\rho(f, g) \leq r$. ■

Now we turn to the actual proof of the Euclidean Embedding Theorem 11.13 for Topological Spaces.

Proof of the Euclidean Embedding Theorem 11.13. Let $n \in \mathbb{N}_0$ and let X be a topological space with $\operatorname{cdim}(X) \leq n$ that is second countable, Hausdorff and regionally compact. We need to show that X is homeomorphic to a closed subset of $\mathbb{R}^{2n+1}$. We make the following two preparations:

(1) Since X is second countable, Hausdorff and regionally compact we know by the Compact Exhaustion Lemma 9.3 (1) that there exists an exhaustion of X by compact subspaces. By the definition on page 338 this means that there exists a sequence

$\{K_i\}_{i\in\mathbb{N}}$ of compact subspaces of X such that the following two conditions are satisfied:

(a) We have $X = \bigcup\limits_{i\in\mathbb{N}} K_i$.

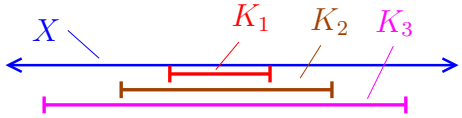
(b) For each $i \in \mathbb{N}$ we have $K_i \subset \mathring{K}_{i+1}$.

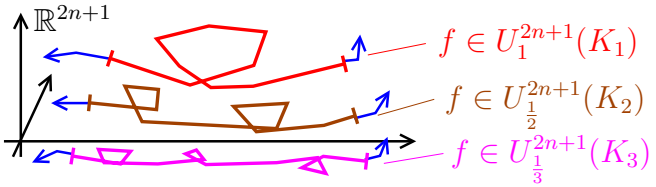
(2) Since X is second countable, Hausdorff and regionally compact we know by the Paracompact-Criterion Lemma 10.2 that X is paracompact, together with the Dieudonné Theorem 10.5 this implies that X is normal. It follows from the Urysohn Metrization Theorem 9.4 that X is metrizable. We pick a metric d on X such that the corresponding metric topology equals the given topology.

Claim 1. The set[136] $\bigcap\limits_{i=1}^{\infty} U^{2n+1}_{\frac{1}{i}}(K_i)$ is dense in $C(X,\mathbb{R}^{2n+1})$ with respect to the metric ρ that we defined on page 383.

Proof. It follows from our hypothesis that $\operatorname{cdim}(X) \leq n$, together with Lemmas 11.17 and 11.19, that each $U^{2n+1}_{\frac{1}{i}}(K_i)$ is open and dense in $C(X,\mathbb{R}^{2n+1})$. By Proposition 6.12 we know that the metric space $C(X,\mathbb{R}^{2n+1})$ is complete. The claim now follows from the Baire Category Theorem 6.16 (1) together with the Baire Space Characterization Lemma 6.17. ⊞

Claim 2. Every $f \in \bigcap\limits_{i=1}^{\infty} U^{2n+1}_{\frac{1}{i}}(K_i)$ is injective.





Proof. Let $x, y \in X$ such that $f(x) = f(y)$. There exists an $i \in \mathbb{N}$ such that for all $j \geq i$ we have $x, y \in K_j$. Now let $j \geq i$. By hypothesis $f \in U^{2n+1}_{\frac{1}{j}}(K_j)$. Thus it follows from $f(x) = f(y)$ that $d(x,y) \leq \frac{1}{j}$. Since this holds for all $j \geq i$ we see that $d(x,y) = 0$. ⊞

Since X is second countable, Hausdorff and regionally compact we know by the Exhaustion Function Lemma 10.7 that X admits an exhaustion function $\widetilde{f}\colon X \to \mathbb{R}_{\geq 0}$. Recall that according to the definition on page 358 this is a map $\widetilde{f}\colon X \to \mathbb{R}_{\geq 0}$ such that for every $c \in \mathbb{R}_{\geq 0}$ the preimage $f^{-1}([0,c])$ is a compact subset of X. We now consider the map

$$\begin{array}{rcl} f\colon X & \to & \mathbb{R}^{2n+1} \\ x & \mapsto & (\widetilde{f}(x), 0, \ldots, 0). \end{array}$$

By comparing definitions one sees that $f(x) \xrightarrow{x \to \infty} \infty$.

[136] Recall that in Lemma 11.17 we defined $U^{2n+1}_{\epsilon}(K) = \{f \in C(X,\mathbb{R}^N) \mid \Delta(f,K) < \epsilon\}$.

By Claim 1 the set $\bigcap_{i=1}^{\infty} U_{\frac{1}{i}}^{2n+1}(K_i)$ is dense in $(C(X,\mathbb{R}^{2n+1}),\rho)$. In particular we see that there exists a $\iota \in \bigcap_{i=1}^{\infty} U_{\frac{1}{i}}^{2n+1}(K_i)$ with $\rho(f,\iota) < 1$. The theorem now follows from the following observations:

(1) By Claim 2 we know that $\iota\colon X \to \mathbb{R}^N$ is injective.
(2) It follows from $\rho(f,\iota) < 1$ and Lemma 11.16 that $\iota(x) \xrightarrow{x \to \infty} \infty$.
(3) It follows from (1) and (2), together with Lemma 11.15, that $\iota\colon X \to \mathbb{R}^{2n+1}$ is a closed embedding. ■

Remark. There are many books that are completely dedicated to dimension theory, see e.g. [**HuW1948, Enge1978, Enge1995, Naga1983, Pear1975, Char2019**]. We refer the reader, whose appetite got whetted, to the above books for more results.

Exercises for Chapter 11.

Exercise 11.1. Give an example of a compact space X of covering dimension $\leq d$ and an open cover $\{U_i\}_{i\in I}$ such that there is no subset $J \subset I$ such that $\{U_j\}_{j\in J}$ is a cover of X of order $\leq d+1$.

Exercise 11.2. We consider the Topologist's Sine Curve

$$S \;:=\; \big\{(0,y)\,\big|\,y\in[-1,1]\big\} \cup \big\{(x,\sin(\tfrac{1}{x}))\,\big|\,x\in(0,\tfrac{1}{\pi}]\big\} \;\subset\; \mathbb{R}^2.$$

Show that $\operatorname{cdim}(S) = 1$.

Exercise 11.3. Let X be a connected space with at least two points. Show that if X is T_1, in the sense of the definition on page 349, then $\operatorname{cdim}(X) \geq 1$.

Exercise 11.4. Let X be a topological space. Show that if $\operatorname{cdim}(X) = 0$, then X is normal.

Exercise 11.5. What is the covering dimension of $\mathbb{Q}$? Here we equip $\mathbb{Q} \subset \mathbb{R}$ as always with the subspace topology.

Exercise 11.6. Let C be the Cantor set as defined on page 130. What is the covering dimension of the topological space $X := (C\times[0,1]) \cup ([0,1]\times\{0\}) \subset \mathbb{R}^2$?

Exercise 11.7. Let $X = \{a,b,c,d\}$. We equip X with the topology that is generated by $\{a\},\{d\},\{b,d\}$ and $\{c,d\}$. What is $\operatorname{cdim}(X)$?

Exercise 11.8. Let X be a topological space and let $\{U_i\}_{i\in I}$ be an open cover of finite order. Is the open cover necessarily locally finite?

Exercise 11.9. Show that there exists a surjective map $f\colon X \to Y$ between topological spaces such that $\operatorname{cdim}(X) < \operatorname{cdim}(Y)$.
Remark. You could use the Cantor function from page 185.

Exercise 11.10. Let X be a topological space and let A and B be disjoint subsets of X. We say a subset $P \subset X$ is a *partition between A and B*, if there exist disjoint open subsets U and V of X with $A \subset U$, $B \subset V$ and $X \setminus P = U \cup V$.

Next we iteratively define classes $\mathcal{C}_{-1},\mathcal{C}_0,\mathcal{C}_1,\dots$ of topological spaces:

(1) We define $\mathcal{C}_{-1}$ to be the class consisting only of the empty topological space.

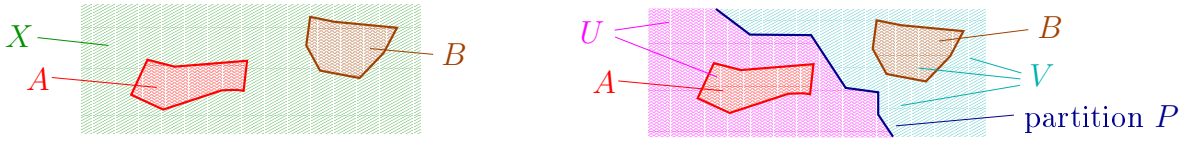
(2) Suppose that for some $n \in \mathbb{N}_0$ we have defined $\mathcal{C}_{-1}, \mathcal{C}_0, \ldots, \mathcal{C}_{n-1}$. Given a topological space X we say $X \in \mathcal{C}_n$ if for every $x \in X$ and every closed subset $B \subset X$ with $x \notin B$ there exists a partition P between $\{x\}$ and B with $P \in \mathcal{C}_{n-1}$.

We define $\mathcal{C}_\infty$ to be the class of all topological spaces. Note that $\mathcal{C}_{-1} \subset \mathcal{C}_0 \subset \mathcal{C}_1 \subset \ldots$. Given a topological space X we define the *small inductive dimension* $\operatorname{ind}(X)$ as follows:

$$\operatorname{ind}(X) \;:=\; \min\{n \in \mathbb{Z}_{\geq -1} \cup \{\infty\} \mid X \in \mathcal{C}_n\}.$$

Prove the following statements:

(a) If X is a non-empty discrete space, then $\operatorname{ind}(X) = 0$.
(b) The small inductive dimension of $\mathbb{R}$ equals 1.
(c) For every subspace A of a topological space X we have $\operatorname{ind}(A) \leq \operatorname{ind}(X)$.



Exercise 11.11. We iteratively define classes $\mathcal{D}_{-1}, \mathcal{D}_0, \mathcal{D}_1, \ldots$ of topological spaces:

(1) We define $\mathcal{D}_{-1}$ to be the class consisting only of the empty topological space.
(2) Suppose that for some $n \in \mathbb{N}_0$ we have defined $\mathcal{D}_{-1}, \mathcal{D}_0, \ldots, \mathcal{D}_{n-1}$. Given a topological space X we say $X \in \mathcal{D}_n$ if for any two closed disjoint subsets A and B of X there exists a partition P between A and B with $P \in \mathcal{D}_{n-1}$. (We refer to Exercise 11.10 for the definition of a partition.)

We define $\mathcal{D}_\infty$ to be the class of all topological spaces. Note that $\mathcal{D}_{-1} \subset \mathcal{D}_0 \subset \mathcal{D}_1 \subset \ldots$. Given a topological space X we define the *large inductive dimension* $\operatorname{ind}(X)$ as follows:

$$\operatorname{Ind}(X) \;:=\; \min\{n \in \mathbb{Z}_{\geq -1} \cup \{\infty\} \mid X \in \mathcal{D}_n\}.$$

Prove the following statements:

(a) The large inductive dimension of $[0,1]$ equals 1.
(b) For every closed subspace A of a topological space X we have $\operatorname{Ind}(A) \leq \operatorname{Ind}(X)$.

Exercise 11.12. Given a topological space X we denote by $\operatorname{ind}(X)$ the small inductive dimension that we defined in Exercise 11.10 and we denote by $\operatorname{Ind}(X)$ the large inductive dimension that we defined in Exercise 11.11.

(a) Let X be the Sierpiński space that we introduced on page 186, i.e. let $X = \{0,1\}$ with the topology $\mathcal{T} = \{\varnothing, X, \{0\}\}$. Show that $\operatorname{cdim}(X) = \operatorname{Ind}(X) = 0$ and show that $\operatorname{ind}(X) = \infty$.
(b) Let $X = \{1,2,3\}$ with the topology that is generated by the subsets $\{1,2\}$ and $\{2,3\}$. Show that $\operatorname{cdim}(X) = 1$ and show that $\operatorname{ind}(X) = \operatorname{Ind}(X) = \infty$.

Remark. This exercise shows that the covering dimension, the small inductive and the large inductive dimension are different "dimension functions".

Exercise 11.13. Let X be the topological space that is obtained from the open disk B^2 by removing three points. By Corollary 11.9 we know that $\operatorname{cdim}(X) \leq 2$. It follows from

the Euclidean Embedding Theorem 11.13 that there exists an embedding $f\colon X \to \mathbb{R}^5$ such that $f(X)$ is a closed subset. Determine

$$\min\{n \in \mathbb{N}_0 \,|\, \text{there exists an embedding } f\colon X \to \mathbb{R}^n \text{ such that } f(X) \text{ is a closed subset}\}.$$

Exercise 11.14. Let $P_1, \ldots, P_5$ be five points in $\mathbb{R}^2$ that are in general position. Show that there exist four points among $P_1, \ldots, P_5$ that span a convex quadrilateral.
Remark. This result was proved by George Szekeres and Esther Klein in 1935. Paul Erdös called this statement the Happy Ending Theorem since it lead to the marriage of George Szekeres and Esther Klein who were married for 68 years.

12. The compact-open topology

Given two topological spaces X and Y we will introduce the compact-open topology on the set of all continuous maps $X \to Y$.
First, recall that we saw on page 124 that a topology on a given set X makes it possible to talk of the convergence of a sequence in X. From real analysis we know that it is not only interesting to think about convergence of sequences of points but also to talk of convergence of functions. In fact, in real analysis one often encounters two different types of convergence that we now recall.

Definition. Let $U \subset \mathbb{R}^n$ be a subset, let $\{f_n \colon U \to \mathbb{R}\}_{n\in\mathbb{N}}$ be a sequence of functions and let $f \colon U \to \mathbb{R}$ be a function. We define

$$(f_n)_{n\in\mathbb{N}} \text{ converges pointwise to } f \quad :\Longleftrightarrow \quad \underset{x\in U}{\forall} \; \underset{\epsilon>0}{\forall} \; \underset{N\in\mathbb{N}}{\exists} \; \underset{n\geq N}{\forall} \; |f_n(x) - f(x)| < \epsilon$$

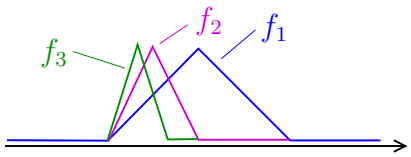
and

$$(f_n)_{n\in\mathbb{N}} \text{ converges uniformly to } f \quad :\Longleftrightarrow \quad \underset{\epsilon>0}{\forall} \; \underset{N\in\mathbb{N}}{\exists} \; \underset{x\in U}{\forall} \; \underset{n\geq N}{\forall} \; |f_n(x) - f(x)| < \epsilon.$$

Example. We consider the sequence of functions

$$\begin{array}{rccl} f_n \colon \mathbb{R} & \to & \mathbb{R} & \\ & & \begin{cases} 0, & \text{if } t < 0 \text{ or } t > \frac{2}{n}, \\ n \cdot t, & \text{if } t \in [0, \frac{1}{n}], \\ 2 - n \cdot t, & \text{if } t \in (\frac{1}{n}, \frac{2}{n}]. \end{cases} \\ t & \mapsto & & \end{array}$$

One easily verifies that this sequence converges pointwise to the zero function but it does not converge uniformly to the zero function.



At times one also wants to study convergence of sequences of continuous[137] maps between general topological spaces. This leads us to the following definition.

Definition. Let X and Y be topological spaces.

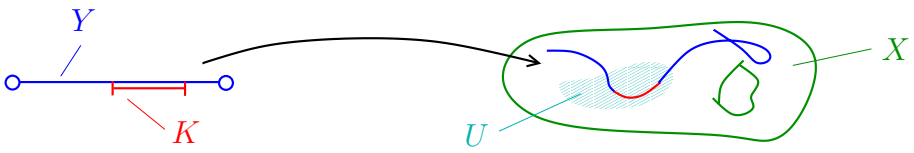
(1) We denote by $X^Y = C(Y, X)$ the set of all continuous maps from Y to X.
(2) The compact-open topology on X^Y is the topology generated, in the sense of the definition on page 135, by all the sets of the form

$$M(K, U) \;:=\; \{f \in X^Y \mid f(K) \subset U\}$$

where $K \subset Y$ is compact and where $U \subset X$ is open. This definition is illustrated in the figure below.

Examples.

[137] Here and in the following we sometimes talk explicitly about "continuous" maps, even though it is usually understood in these notes that all maps are continuous.



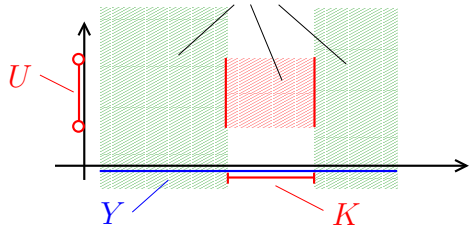
(1) Let $Y \subset \mathbb{R}$ be an interval. We want to consider the compact-open topology on $C(Y,\mathbb{R}) = \mathbb{R}^Y$. As usual we picture elements of $C(Y,\mathbb{R})$ by their graphs. In the figure below we show on the left an open set $M(K,U)$ as above and on the right we show the graph of some function $f \in C(Y,\mathbb{R})$ and an open neighborhood thereof.

Now let $f_n \in C(Y,\mathbb{R})$, $n \in \mathbb{N}$ be a sequence of functions and let $f \in C(Y,\mathbb{R})$. We leave it as an amusing, albeit somewhat lengthy, exercise to show that the following holds:

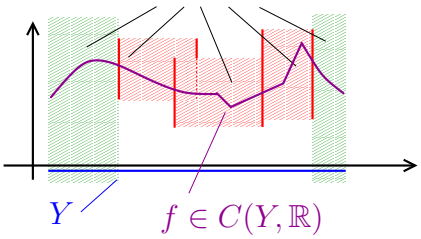
$$(f_n)_{n\in\mathbb{N}} \text{ converges}^{138} \text{ to } f \text{ with respect to the compact-open topology} \iff \text{for each compact subset } K \subset Y \text{ the sequence } (f_n|_K)_{n\in\mathbb{N}} \text{ converges uniformly to } f|_K$$

Alternatively a proof[139] for this statement is given in [**Will1970**, Theorem 43.7] or in [**Jos1983**, Theorem 15.1.2].

$M(K,U)$ are the continuous functions whose graph is contained in this set



open neighborhood of $f \in C(Y,\mathbb{R})$ with respect to the compact-open topology



(2) We consider $Y = \{1,\dots,n\}$ which we equip with the discrete topology. In Exercise 12.7 we will see that for any topological space X we have a homeomorphism $X^Y \cong X^n$.

The final example deserves its own lemma.

Lemma 12.1. Let $m,n \in \mathbb{N}$. We have the obvious inclusion

$$\begin{array}{ccc} \mathrm{M}(m\times n,\mathbb{R}) & \to & C(\mathbb{R}^n,\mathbb{R}^m) = (\mathbb{R}^m)^{\mathbb{R}^n} \\ A & \mapsto & \begin{pmatrix} \mathbb{R}^n & \to & \mathbb{R}^m \\ v & \mapsto & Av \end{pmatrix}. \end{array}$$

The compact-open topology on $C(\mathbb{R}^n,\mathbb{R}^m)$ induces the usual topology on $\mathrm{M}(m\times n,\mathbb{R})$ that we introduced on page 171.

Sketch of proof. This statement can be proved, with some effort, using the Subbasis Lemma 1.28. Alternatively the statement can also be deduced from [**Hat2002**, Proposition A.13] or [**Mun1975**, Theorem 5.1]. ■

[139] We refer to page 124 for the definition of convergence of a sequence in a topological space.

To get some practice with the compact-open topology we will prove the following two lemmas in Exercises 12.1 and 12.2.

Lemma 12.2. Let X and Y be topological spaces. If Y is Hausdorff, then X^Y is also Hausdorff.

Lemma 12.3. (Constant Map Lemma) If X and Y are non-empty topological spaces, then the map

$$\begin{array}{rcl} X & \to & X^Y \\ x & \mapsto & \text{constant map } y \mapsto x \end{array}$$

is continuous and it is an embedding.

We continue with a few technical statements on the compact-open topology. In most applications later on we will consider the case that $Y = [0,1]$. In particular the extra hypothesis in the subsequent propositions on the topological space Y will not be an issue in our future applications.

Proposition 12.4. (Evaluation Map Proposition) Let X and Y be topological spaces. If Y is regionally compact, then the following two statements hold:

(1) The evaluation map[140]

$$\begin{array}{rcl} e\colon X^Y \times Y & \to & X \\ ((f\colon Y \to X), y) & \mapsto & f(y) \end{array}$$

is continuous.

(2) For any $y \in Y$ the evaluation map

$$\begin{array}{rcl} e_y\colon X^Y & \to & X \\ (f\colon Y \to X) & \mapsto & f(y) \end{array}$$

is continuous.

Proof.

(1) Let U be an open subset of X. We need to show that $e^{-1}(U)$ is open. By the JH-Lemma 1.7 it suffices to show that given $(f,y) \in e^{-1}(U)$ there exists a neighborhood W of (f,y) with $W \subset e^{-1}(U)$. Thus let $f \in X^Y$ and $y \in Y$ with $e(f,y) = f(y) \in U$. Since f is continuous and since Y is regionally compact there exists a compact neighborhood K of y which is contained in the open neighborhood $f^{-1}(U)$ of y. In other words, we have found a compact neighborhood K of y with $f(K) \subset U$. Therefore we have $f \in M(K,U)$ and the map e sends $W := M(K,U) \times K$ into U. By the definition of the product topology the set $W = M(K,U) \times K$ is in fact a neighborhood of (f,y) in $X^Y \times Y$.

(2) This statement follows from (1) and the Product Topology-Basics Lemma 5.4 (2a). ∎

We continue with the following elementary but essential lemma.

[140] Here we equip $X^Y \times Y$ with the product topology from page 224.

Lemma 12.5. (Compact-Open Functoriality Lemma) Let X and Y be topological spaces.

(1) If $\varphi\colon X \to \widetilde{X}$ is a continuous map, then the induced map
$$\begin{array}{rcl} \varphi^*\colon X^Y & \to & \widetilde{X}^Y \\ (f\colon Y \to X) & \mapsto & (\varphi \circ f\colon Y \to \widetilde{X}) \end{array}$$
is continuous.

(2) If $\varphi\colon X \to \widetilde{X}$ is an embedding, then so is $\varphi^*\colon X^Y \to \widetilde{X}^Y$.

(3) Let $\varphi, \psi\colon X \to \widetilde{X}$ be continuous maps. If φ and ψ are homotopic[141], then so are $\varphi^*, \psi^*\colon X^Y \to \widetilde{X}^Y$.

(4) If $\widetilde{X}$ is a topological space that is homotopy equivalent to X, then $\widetilde{X}^Y$ is also homotopy equivalent to X^Y.

(5) If $\psi\colon Y \to \widetilde{Y}$ is a continuous map, then the induced map
$$\begin{array}{rcl} \psi_*\colon X^{\widetilde{Y}} & \to & X^Y \\ (f\colon \widetilde{Y} \to X) & \mapsto & (f \circ \psi\colon Y \to X) \end{array}$$
is continuous.

Remark. The Compact-Open Functoriality Lemma 12.5 (2) says in particular, that given topological spaces X and Y and a subset A of X, we can view A^Y as a subset of X^Y.

Proof. To clarify the statements we replace in this proof the notation $M(K,U)$ by the notation $M_{X^Y}(K,U)$, with the obvious interpretation thereof.

(1) We check continuity using the continuity criterion of the Subbasis-Continuity Proposition 2.9. Thus let $K \subset Y$ be a compact subset and let $U \subset \widetilde{X}$ be an open subset. We have
$$\begin{aligned} (\varphi^*)^{-1}(M_{\widetilde{X}^Y}(K,U)) &= \{f \in X^Y \mid \varphi^*(f) \in M_{\widetilde{X}^Y}(K,U)\} = \{f\colon Y \to X \mid \overbrace{(\varphi \circ f)(K)}^{=\varphi(f(K))} \subset U\} \\ &= \underset{\text{by Exercise 0.2}}{\uparrow} \{f\colon Y \to X \mid f(K) \subset \varphi^{-1}(U)\} \;=\; M_{X^Y}(K, \varphi^{-1}(U)). \end{aligned}$$
Since φ is continuous we see that $\varphi^{-1}(U)$ is an open subset of X. Since the sets $M(K,U)$ form by definition a basis for the topology of $\widetilde{X}^Y$ we deduce from the Subbasis-Continuity Proposition 2.9 that φ^* is indeed continuous.

(2) Since φ is an embedding we can identify X with its image in $\widetilde{X}$. Put differently, it suffices to consider the case of an inclusion $i\colon X \to \widetilde{X}$. By the Open -Injective Map Lemma 2.16 (1) together with Lemma 2.11 it suffices to show that for any compact subset $K \subset Y$ and any open subset $U \subset X$ the image $i_*(M_{X^Y}(K,U))$ is an open subset of $i_*(X^Y)$. Since $U \subset X$ is open there exists, by definition of the subspace

[141] We apologize to the reader for using the notion of being homotopic, even though it is only introduced on page 418. The reader who made it that far into this chapter will surely not be frightened by this inconvenience.

topology, an open subset $V \subset \widetilde{X}$ such that $U = V \cap X$. We have

$$\begin{array}{rcl} i_*(M_{X^Y}(K,U)) & = & \{i \circ f \colon Y \to \widetilde{X} \mid f \colon Y \to X \text{ and } f(K) \subset U\} \\ & = & \{f \colon Y \to X \mid f(K) \subset V\} \;=\; i_*(X^Y) \cap M_{\widetilde{X}^Y}(K,V). \end{array}$$

since $f(Y) \subset X$ implies for any $x \in X$ that $f(x) \in U$ if and only if $f(x) \in V$

By the definition of the subspace topology this is an open subset of $i_*(X^Y) \subset \widetilde{X}^Y$.

(3) Let $F \colon X \times [0,1] \to Y$ be a homotopy between $F_0 = \varphi$ and $F_1 = \psi$. Using the Subbasis-Continuity Proposition 2.9 and the JH-Lemma 1.7 it is not particularly challenging to show that the map

$$\begin{array}{rcl} X^Y \times [0,1] & \to & \widetilde{X}^Y \\ (g \colon Y \to X, t) & \mapsto & \begin{pmatrix} Y & \to & \widetilde{X} \\ y & \mapsto & F(g(x),t) \end{pmatrix} \end{array}$$

is continuous. This is the desired homotopy between φ^* and ψ^*.

(4) This statement follows almost immediately from (3).

(5) Let $K \subset Y$ be a compact subset and let $U \subset X$ be an open subset. We have

$$\begin{array}{rcl} (\psi_*)^{-1}(M_{X^Y}(K,U)) & = & \{f \in X^{\widetilde{Y}} \mid \psi_*(f) \in M_{X^Y}(K,U)\} \\ & = & \{f \colon \widetilde{Y} \to X \mid \underbrace{(f \circ \psi)(K)}_{=f(\psi(K))} \subset U\} \;=\; M_{X^{\widetilde{Y}}}(\psi(K),U). \end{array}$$

Since ψ is continuous we obtain from the Compact Image Lemma 2.13 that $\psi(K)$ is a compact subset of $\widetilde{Y}$. Thus we obtain again from the Subbasis-Continuity Proposition 2.9 that ψ_* is indeed continuous. ■

On many occasions we will also need the following rather technical proposition.

Proposition 12.6. (Compact-Open Exponential Proposition) Let Y be a topological space that is regionally compact. Furthermore let X and T be two topological spaces. Let $H \colon Y \times T \to X$ be a map. Given $t \in T$ we denote by $H_t \colon Y \to X$ the map defined by $H_t(y) := H(y,t)$. Then

$$H \colon Y \times T \to X \text{ is continuous} \iff \begin{array}{l} (1) \text{ each } H_t \colon Y \to X \text{ is continuous, and} \\ (2) \text{ the map}^{142} \; T \to X^Y \text{ given by } t \mapsto H_t \text{ is continuous.} \end{array}$$

In other words, the map

$$\begin{array}{rcl} X^{Y \times T} & \to & (X^Y)^T \\ (H \colon Y \times T \to X) & \mapsto & \begin{pmatrix} T & \to & X^Y \\ t & \mapsto & (y \mapsto H(y,t)) \end{pmatrix} \end{array}$$

is well-defined[143] and it is a bijection.

Proof. [144] Note that the "in other words" part of the proposition is indeed just a reformulation of the first part of the proposition. Thus it remains to prove the equivalence of the two statements in the beginning.

[142] Note that by (1) we know that $H_t \in X^Y$.

[143] Here "well-defined" means that the map $T \to X^Y$ actually defines an element in $(X^Y)^T$. More precisely, the map $T \to X^Y$ given by $z \mapsto (y \mapsto H(y,z))$ actually lies in $(X^Y)^T$.

[144] I strongly advise against reading the proof.

First we prove the "$\Leftarrow$"-direction. We point out that $H\colon Y\times T\to X$ is the composition of the two maps

$$\begin{array}{rcl} Y\times T & \to & X^Y\times Y \\ (y,t) & \mapsto & (H_t,y) \end{array} \qquad \text{and the evaluation map} \qquad \begin{array}{rcl} X^Y\times Y & \to & X \\ (f,y) & \mapsto & f(y). \end{array}$$

The first map is continuous by (2) and the Product Topology-Basics Lemma 5.4 (3). Furthermore, since Y is regionally compact we obtain from the Evaluation Map Proposition 12.4 that the second map is continuous. Thus H itself is continuous.

Now we turn to the "$\Rightarrow$"-direction. Thus we suppose that $H\colon Y\times T\to X$ is continuous.

(1) Let $t\in T$. The map H_t is the composition of the inclusion $Y\to Y\times T$, $y\mapsto(y,t)$ with the map H. The first map is continuous by the Product Topology-Basics Lemma 5.4 (2a) and the second map is continuous by our hypothesis. Hence H_t is also continuous. In particular we now know that H_t defines an element in X^Y.

(2) We need to show that the map $T\to X^Y$ given by $t\mapsto H_t$ is continuous. By the Subbasis-Continuity Proposition 2.9 and the JH-Lemma 1.7 it suffices to show that for any set of the form $M(K,U)\subset X^Y$ and any $t\in T$ with $H_t\in M(K,U)$ there exists a neighborhood W of $t\in T$ such that for all $t'\in W$ we have $H_{t'}\in M(K,U)$.

Thus let K be a compact subset of Y, let U be an open subset of X and let $t\in T$ with $H_t\in M(K,U)$. We start out with the following claim.

Claim. Given any $y\in K$ there exist open neighborhoods $V_y\subset Y$ of y and $W_y\subset T$ of t such that $H(V_y\times W_y)\subset U$.

Proof. By the continuity of $H\colon Y\times T\to X$ we know that $H^{-1}(U)$ is an open subset of $Y\times K$. Note that $H(y,t)=H_t(y)\in U$ since $H_t\in M(K,U)$ and $y\in K$. Thus $H^{-1}(U)$ is an open neighborhood of $(y,t)\in Y\times K$. The existence of V_y and W_y is now a consequence of the definition of the product topology of $Y\times T$. ⊞

It follows from the compactness of K and the Compact Subspace Covering Lemma 1.19 that there exist points $y_1,\dots,y_n\in K$ with $K\subset V_{y_1}\cup\dots\cup V_{y_n}$. Next we set $W:=W_{y_1}\cap\dots\cap W_{y_n}\subset T$. Note that this is an open neighborhood of t. Then

$$H(K\times W) \underset{\substack{\uparrow\\ \text{since } K\subset \bigcup_{i=1}^n V_{y_i}}}{\subset} H\Big(\bigcup_{i=1}^n V_{y_i}\times W\Big) \underset{\substack{\uparrow\\ \text{since } W\subset W_{y_i}}}{\subset} H\Big(\bigcup_{i=1}^n V_{y_i}\times W_{y_i}\Big) = \bigcup_{i=1}^n \underbrace{H(V_{y_i}\times W_{y_i})}_{\subset U} \subset U.$$

It follows from this observation and the definition of $H_{t'}$ and $M(K,U)$ that for any $t'\in W$ we have $H_{t'}\in M(K,U)$. ■

On one later occasion we will also need the following equally technical proposition.

Proposition 12.7. (Exponential Proposition) Let X,Y and Z be topological spaces. If Y and Z are both regionally compact and Hausdorff, then the map[145]

$$\begin{array}{rcl} X^{Y\times Z} & \to & (X^Y)^Z \\ (f\colon Y\times Z\to X) & \mapsto & \left(\begin{array}{rcl} Z & \to & X^Y \\ z & \mapsto & (y\mapsto f(y,z)) \end{array}\right) \end{array}$$

is a homeomorphism.

Remark. In fact a slightly stronger version of the Exponential Proposition 12.7 holds. By [**Dug1966**, Theorem XII.5.3] one only needs to assume that Y is regionally compact, with no further restrictions on X, Y or Z. We will not need this stronger statement.

Once again no sane person will ever voluntarily read the proof. For those readers who are still around we first provide the proof of the following lemma.

Lemma 12.8. Let X and Y be topological spaces. Furthermore let $\mathcal{V}$ be a subbasis for X. We consider

$$\mathcal{C} \;=\; \big\{M(K,V) \in \mathcal{P}\big(X^Y\big) \;\big|\; K \text{ a compact subset of } Y \text{ and } V \in \mathcal{V}\big\}.$$

If Y is Hausdorff, then $\mathcal{C}$ is subbasis[146] for the topology of X^Y.

Proof. Let K be a compact subset of Y, let U be an open subset of X and let f be a point in $M(K,U)$, i.e. $f\colon Y \to X$ is a map with $K \subset f^{-1}(U)$. By the Basis Characterization Lemma 1.32 (2) we need to show that there exist finitely many $C_1, \ldots, C_m \in \mathcal{C}$ with $f \in \bigcap_{i=1}^{m} C_i \subset M(K,U)$.

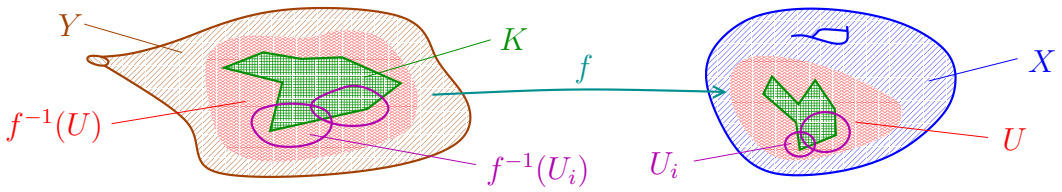
Since $\mathcal{V}$ is a subbasis for the topology of X we can write $U = \bigcup_{i\in I} U_i$ where each U_i is of the form $U_i = \bigcap_{V\in P_i} V$ where P_i is a finite subset of $\mathcal{V}$. Since K is compact and since f is continuous it follows from the Compact Subspace Covering Lemma 1.19 that there exists a finite subset $J \subset I$ such that $K \subset \bigcup_{j\in J} f^{-1}(U_j)$.

Since K is compact and Hausdorff we obtain from the Shrinking Lemma 10.10, together with the Compact-Closed Lemma 1.21 (1) and the hypothesis that Y is Hausdorff, that there exist compact subsets $K_j \subset f^{-1}(U_j)$, $j \in J$, with $K \subset \bigcup_{j\in J} K_j$.

Thus we see that

$$\begin{array}{ccccccc} f & \in & \bigcap\limits_{j\in J} M(K_j,U_j) & = & \bigcap\limits_{j\in J} M\Big(K_j, \bigcap\limits_{V\in P_j} V\Big) & \overset{\text{since } M(K,A\cap B) = M(K,A)\cap M(K,B)}{=} & \bigcap\limits_{j\in J}\bigcap\limits_{V\in P_j} \underbrace{M(K_j,V)}_{\in\mathcal{C}} \;\subset\; M(K,U). \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \text{since } K_j \subset f^{-1}(U_j) & & \text{definition of } P_j & & \text{since } K \subset \bigcup_{j\in J} K_j \text{ and since each } V \text{ lies in } U \end{array}$$

The finitely many sets $\{M(K_j,V)\}_{j\in J, V\in P_j}$ in $\mathcal{C}$ thus do the trick. ∎



Now we are finally ready to provide the proof of the Exponential Proposition 12.7.

[145] By the Compact-Open Exponential Proposition 12.6 we know that the map is well-defined and that it is a bijection.

[146] We refer to page 135 for the definition of a subbasis of a topological space.

Proof of Proposition 12.7. Let X, Y and Z be topological spaces. We assume that Y and Z are both regionally compact and Hausdorff. We need to show that the map

$$\begin{array}{rcl} \Phi\colon X^{Y\times Z} & \to & (X^Y)^Z \\ (f\colon Y\times Z\to X) & \mapsto & \begin{pmatrix} Z & \to & X^Y \\ z & \mapsto & (y\mapsto f(y,z)) \end{pmatrix} \end{array}$$

is a homeomorphism.

Claim. The set

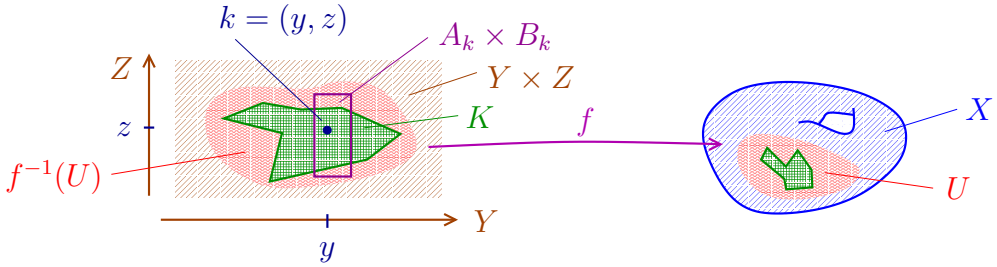
$$\mathcal{C} = \left\{ M(A\times B,U) \in \mathcal{P}\big(X^{Y\times Z}\big) \;\middle|\; \begin{array}{l} A \text{ a compact subset of } Y,\ B \text{ a compact subset of } Z \\ \text{and } U \text{ an open subset of } X \end{array} \right\}$$

is a subbasis for the topological space $X^{Y\times Z}$.

Proof. Let $K \subset Y \times Z$ be a compact subset, let U be an open subset of X and let f be a point in $M(K,U)$, i.e. $f\colon Y\times Z \to X$ is a map with $K \subset f^{-1}(U)$. By the Basis Characterization Lemma 1.32 (2) we need to show that there exist finitely many $C_1,\dots,C_m \in \mathcal{C}$ with $f \in \bigcap_{i=1}^m C_i \subset M(K,U)$.

Let $k \in K \subset Y\times Z$. We write $k=(y,z)$. Since $f^{-1}(U)$ is an open subset of $Y\times Z$ it follows from the definition of the product topology on $Y\times Z$ that there exist open neighborhoods $V_k \subset Y$ of y and $W_k \subset Z$ of z such that $k=(y,z) \in V_k \times W_k \subset f^{-1}(U)$. Since Y and Z are regionally compact we can find a compact neighborhood $A_k \subset Y$ of y and a compact neighborhood $B_k \subset Z$ of z such that $k \in A_k \times B_k \subset V_k \times W_k \subset f^{-1}(U)$. Since K is compact it follows from the Compact Subspace Covering Lemma 1.19 that there exist finitely many points $k_1,\dots,k_m \in K$ such that $K \subset \bigcup_{i=1}^m A_{k_i}\times B_{k_i}$. Now we see that

$$f \;\in\; \underset{\text{since } f(A_{k_i}\times B_{k_i})\subset U}{\uparrow} \bigcap_{i=1}^m \overbrace{M(A_{k_i}\times B_{k_I},U)}^{\in\mathcal{C}} \;\overset{\text{since } M(V\cup W,U)=M(V,U)\cap M(W,U)}{\underset{\downarrow}{=}}\; M\Big(\bigcup_{i=1}^m A_{k_i}\times B_{k_i},U\Big) \;\underset{\text{since } K\subset \bigcup_{i=1}^m A_{k_i}\times B_{k_i}}{\underset{\uparrow}{\subset}}\; M(K,U).$$



As we already remarked in the Compact-Open Exponential Proposition 12.6, the given map $\Phi\colon X^{Y\times Z} \to (X^Y)^Z$ is clearly a bijection. Furthermore under this bijection the sets $M(A\times B,U)$ are precisely the sets $M(B,M(A,U))$. By the claim the former sets form a subbasis for $X^{Y\times Z}$. Thus it suffices to show that the latter sets form a subbasis for $(X^Y)^Z$. But that is an immediate consequence of Lemma 12.8, applied twice, and our hypothesis that Y and Z are Hausdorff. ∎

The following proposition is also somewhat technical.

Proposition 12.9. (Compact-Open Composition Proposition) Let X, Y and Z be topological spaces. If Y is regionally compact, then the map

$$\begin{array}{rcl} \Phi\colon X^Y \times Y^Z & \to & X^Z \\ (f,g) & \mapsto & f \circ g \end{array}$$

is continuous.

The proof of the Compact-Open Composition Proposition 12.9 rests on the following little lemma.

Lemma 12.10. Let Y be a topological space that is regionally compact. If K is a compact subset that is contained in an open subset U of Y, then there exists a compact subset L with $K \subset \mathring{L} \subset L \subset U$.

Proof. Let $x \in K$. Since Y is regionally compact there exists a compact neighborhood $L_x \subset U$ of x. Since K is compact there exist $x_1, \ldots, x_m \in K$ with $K \subset \mathring{L}_{x_1} \cup \cdots \cup \mathring{L}_{x_m}$. We set $L := L_{x_1} \cup \cdots \cup L_{x_m}$. By the Compact-Union-Intersection Lemma 1.20 (1) we know that L is compact. We have

$$K \subset \bigcup_{i=1}^m \mathring{L}_{x_i} \subset \underset{\text{Lemma 1.11 (1a)}}{\text{interior of}} \bigcup_{i=1}^m L_x = \mathring{L}_{x_i} \subset L \subset U.$$

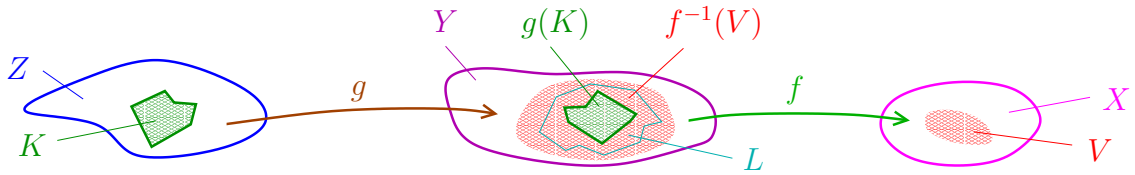
■

Proof of the Compact-Open Composition Proposition 12.9. By the Subbasis-Continuity Proposition 2.9 and the JH-Lemma 1.7 and the definition of the product topology on $X^Y \times Y^Z$ it suffices to prove the following claim.

Claim. Let K be a compact subset of Z and let V be an open subset of X. If $f \in X^Y$ and $g \in Y^Z$ satisfy $f \circ g \in M(K,V)$, then there exists an open subset $U \subset Y$ and a compact subset $L \subset Y$ with

$$\Phi\big(\underbrace{M(L,V)}_{\subset X^Y} \times \underbrace{M(K,U)}_{\subset Y^Z}\big) \subset M(K,V).$$

Proof. First note that $g(K) \subset Y$ is compact by the Compact Image Lemma 2.13 and that $f^{-1}(V) \subset Y$ is open since f is evidently continuous. Since Y is regionally compact we obtain from Lemma 12.10 that there exists a compact subset L with $g(K) \subset \mathring{L} \subset L \subset f^{-1}(U)$. It is clear that $U = \mathring{L}$ and L have the desired properties. ■



The following corollary gives us a very long list of new examples of topological groups.

Corollary 12.11. Let X be a topological space. If X is compact and Hausdorff, then the group

$$\text{Homeo}(X) := \text{the set of all self-homeomorphisms of } X,$$

equipped with the subspace topology coming from X^X, is a topological group.

Remark. Let X be a topological space that is regionally compact and Hausdorff. In light of the Compact-Open Composition Proposition 12.9 one might expect that in this case

Homeo(X) forms a group. In [**Dij2005**] it is shown that this is in general *not* the case. On the other hand it is shown in [**Are1946**, Theorem 4] that if X is furthermore regionally connected, then Homeo(X) is in fact a topological group.

Proof. Let X be a topological space that is compact and Hausdorff. First note that by Lemma 2.41 we know that X is regionally compact. Thus it follows from the Compact-Open Composition Proposition 12.9 that the multiplication map $\mathrm{Homeo}(X)\times\mathrm{Homeo}(X) \to \mathrm{Homeo}(X)$ is continuous. Therefore it suffices to show that the map

$$\begin{array}{rccc} \Phi\colon & \mathrm{Homeo}(X) & \to & \mathrm{Homeo}(X) \\ & f & \mapsto & f^{-1} \end{array}$$

is continuous. The proof of continuity rests on the following simple claim.

Claim 1. If $K \subset X$ and $U \subset X$ are subsets, then $\Phi^{-1}(M(K,U)) = M(X \setminus U, X \setminus K)$.

Proof. Let $h \in \mathrm{Homeo}(X)$. We have

$$\begin{array}{lll} & \overset{\text{since } \Phi \text{ is a bijection and } \Phi(h) = h^{-1}}{\overset{\downarrow}{\iff}} & \overset{\text{since } h \text{ is a bijection}}{\overset{\downarrow}{\iff}} \\ h \in \Phi^{-1}(M(K,U)) & \iff h^{-1} \in M(K,U) & \iff h^{-1}(K) \subset U \iff K \subset h(U) \\ & \iff \underbrace{X \setminus h(U)}_{=h(X\setminus U)} \subset X \setminus K & \iff h \in M(X \setminus U, X \setminus K). \end{array}$$

⊞

By the Subbasis-Continuity Proposition 2.9 it now suffices to prove the following claim.

Claim 2. If $K \subset X$ is a compact subset and if $U \subset X$ is an open subset, then $\Phi^{-1}(M(K,U))$ is an open subset of X.

Proof. By Claim 1 we know that $\Phi^{-1}(M(K,U)) = M(X \setminus U, X \setminus K)$. Since U is open we know that $X \setminus U$ is closed. Since X is compact we obtain from the Compact-Closed Lemma 1.21 (1) that $X \setminus U$ is compact. Furthermore, since K is compact and since X is Hausdorff we obtain from the Compact-Closed Lemma 1.21 (2) that $X \setminus K$ is open. Thus we see that $M(X \setminus U, X \setminus K)$ is an open subset of X^X. ■

We conclude this section with a different source of topological groups.

Lemma 12.12. Let X be a topological space.

(1) Let $\widetilde{X}$ and Y be topological spaces. If Y is regionally compact, then the map

$$\begin{array}{rccc} \Psi\colon & X^Y \times \widetilde{X}^Y & \to & (X \times \widetilde{X})^Y \\ & (f, \widetilde{f}) & \mapsto & \left(\begin{array}{rcl} f \times \widetilde{f}\colon Y & \to & X \times \widetilde{X} \\ y & \mapsto & (f(y), \widetilde{f}(y)) \end{array} \right) \end{array}$$

is continuous.

(2) Let G be a topological group. If G is regionally compact, then the topological space G^X together with the multiplication

$$\begin{array}{rccc} & G^X \times G^X & \to & G^X \\ & (f,g) & \mapsto & \left(\begin{array}{rcl} X & \to & G \\ x & \mapsto & f(x) \cdot g(x) \end{array} \right) \end{array}$$

is a topological group.

Proof.

(1) By the Subbasis-Continuity Proposition 2.9 it suffices to show that for any compact subset $K \subset Y$ and any open subset $W \subset X \times \widetilde{X}$ the preimage $\Psi^{-1}\big(M_{(X\times\widetilde{X})^Y}(K,W)\big)$ is an open subset of $X^Y \times \widetilde{X}^Y$. We will do so by appealing to the JH-Lemma 1.7.

Thus let $(f,\widetilde{f}) \in \Psi^{-1}(M_{(X\times\widetilde{X})^Y}(K,W))$. Note that this means that we are given $f\colon Y \to X$ and $\widetilde{f}\colon Y \to \widetilde{X}$ with $(f\times\widetilde{f})(K) \subset W$. For each $y \in K$ we can and will pick open subsets U_y of X and $\widetilde{U}_y$ of $\widetilde{X}$ such that $(f\times\widetilde{f})(y) \in U_y \times \widetilde{U}_y \subset W$. By the Product Topology-Basics Lemma 5.4 (1b) we know that $(f\times\widetilde{f})^{-1}(U_y\times\widetilde{U}_y)$ is an open subset of Y. Since Y is regionally compact there exists a compact neighborhood K_y of y such that $y \in K_y \subset (f\times\widetilde{f})^{-1}(U_y\times\widetilde{U}_y)$.

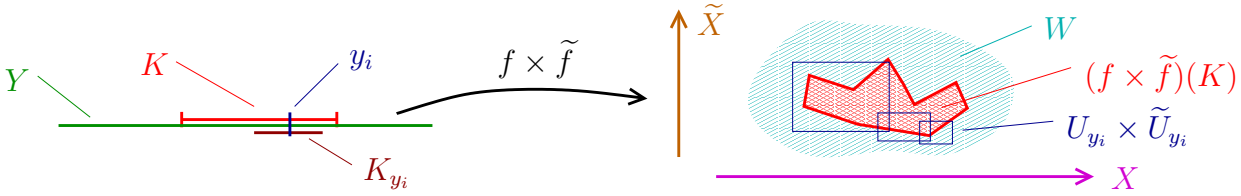
Since K is compact we obtain from the Compact Subspace Covering Lemma 1.19 that there exist $y_1,\dots,y_m \in Y$ such that $K \subset K_{y_1}\cup\cdots\cup K_{y_m}$. Now we see that

$$(f,\widetilde{f}) \in \underset{\uparrow}{\bigcap_{i=1}^{m}} M_{X^Y}(K_{y_i},U_{y_i})\times M_{\widetilde{X}^Y}(K_{y_i},\widetilde{U}_{y_i}) \subset \underset{\uparrow}{\Psi^{-1}\big(M_{(X\times\widetilde{X})^Y}(K,W)\big)} \subset X^Y\times\widetilde{X}^Y.$$

since $(f\times\widetilde{f})(K_{y_i}) \subset U_{y_i}\times\widetilde{U}_{y_i}$ — since $U_{y_i}\times\widetilde{U}_{y_i} \subset W$ and since $K \subset \bigcup K_{y_i}$

It follows from the JH-Lemma 1.7 that $\Psi^{-1}(M_{(X\times\widetilde{X})^Y}(K,U))$ is indeed an open subset of $X^Y \times \widetilde{X}^Y$.

(2) At this stage there is not much left that needs to be done, we outsource the argument to Exercise 12.4. ■



Exercises for Chapter 12.

Exercise 12.1. Let X and Y be topological spaces. Suppose that Y is Hausdorff. Show that X^Y is also Hausdorff.

Exercise 12.2. Let X and Y be non-empty topological spaces. We consider the following map:

$$\begin{array}{rcl} \varphi\colon X & \to & X^Y \\ x & \mapsto & \text{constant map } y \mapsto x \end{array}$$

(a) Show that φ is continuous.
(b) Show that φ is an embedding.

Exercise 12.3. Let X be a topological space that is compact and Hausdorff. Show that the group $\mathrm{Homeo}(X)$ of self-homeomorphisms of X, equipped with the compact-open topology, is a topological group.

Exercise 12.4. Let $X, \widetilde{X}$ and Y be topological spaces. We suppose that Y is regionally compact and Hausdorff. Show that the map

$$\begin{array}{rcl} X^Y \times \widetilde{X}^Y & \to & (X \times \widetilde{X})^Y \\ (f,g) & \mapsto & \left(\begin{array}{ccc} Y & \to & X \times \widetilde{X} \\ y & \mapsto & (f(y), \widetilde{f}(y)) \end{array} \right) \end{array}$$

is a homeomorphism.

Remark. We refer to [**Enge1989**, Exercise 3.4B, p. 165] for a similar, and for the most part, much more general statement.

Exercise 12.5. Let X be a topological space and let G be a topological group. We suppose that G is regionally compact. Show that G^X with the multiplication

$$\begin{array}{rcl} G^X \times G^X & \to & G^X \\ (f,g) & \mapsto & \left(\begin{array}{ccc} X & \to & G \\ x & \mapsto & f(x) \cdot g(x) \end{array} \right) \end{array}$$

is a topological group.

Exercise 12.6. Let $n \in \mathbb{N}$. We consider the following map

$$\begin{array}{rcl} \Theta\colon \mathrm{SO}(n) & \to & \mathrm{Homeo}(S^{n-1}) := \text{self-homeomorphisms of } S^{n-1} \\ A & \mapsto & (P \mapsto A \cdot P). \end{array}$$

(a) Show that the map Θ is continuous.
(b) Show that the map Θ is an embedding.

Hint. Use Lemma 2.1 (4) and Lemma 12.1.

Exercise 12.7. Let $Y = \{1, \ldots, n\}$ be a finite discrete space with n elements and let X be a topological space. Show there exists a homeomorphism $X^Y \cong X^n$.

13. Mixing product and quotient topologies

In this, rather technical chapter, we will see that mixing product and quotient topologies can be quite delicate. In particular we will give rather disturbing examples of natural continuous bijections that are not homeomorphisms. Fortunately we will also give some criteria which let us circumvent these issues.

13.1. **Mixing product and quotient topologies: a disturbing example.** We start out with the following elementary lemma.

Lemma 13.1. Let X and Y be two topological spaces. Suppose we are given an equivalence relation $\sim$ on X. We denote by $\sim_\times$ the equivalence relation on $X \times Y$ that is generated by $(x, y) \sim_\times (x', y)$ whenever $x \sim x'$. The following map is a bijection and it is continuous:

$$\begin{array}{rcl} \varphi\colon (X \times Y)/\!\sim_\times & \to & (X/\!\sim) \times Y \\ [(x, y)] & \mapsto & ([x], y). \end{array}$$

Proof. It follows basically immediately from the definition of $\sim_\times$ that the map φ is a bijection. We will walk cautiously through the continuity argument.

We denote by $p\colon X \times Y \to (X \times Y)/\!\sim_\times$ and $q\colon X \to X/\!\sim$ the two natural projections. We have the following commutative diagram

$$\begin{array}{ccc} & X \times Y & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q\times \mathrm{id}_Y} \\ (X \times Y)/\!\sim_\times & \xrightarrow{\quad\varphi\quad} & (X/\!\sim) \times Y. \end{array}$$

Let $W \subset (X/\!\sim) \times Y$ be an open subset. We need to show that $\varphi^{-1}(W)$ is an open subset of $(X \times Y)/\!\sim_\times$. By definition of the quotient topology we need to show that $p^{-1}(\varphi^{-1}(W))$ is an open subset of $X \times Y$.

By the JH-Lemma 1.7 and the definition of the product topology it suffices to prove the following claim.

Claim. For each $(x, y) \in p^{-1}(\varphi^{-1}(W)) \subset X \times Y$ there exists an open neighborhood U_x of $x \in X$ and an open neighborhood V_y of $y \in Y$ such that $U_x \times V_y \subset p^{-1}(\varphi^{-1}(W))$.

Proof. Since $([x], y) \in W$ and since W is an open subset of $(X/\!\sim) \times Y$ we obtain, by definition of the product topology, an open neighborhood $\widetilde{U}_{[x]}$ of $[x] \in X/\!\sim$ and an open neighborhood V_y of $y \in Y$ with $\widetilde{U}_{[x]} \times V_y \subset W$. We set $U_x := q^{-1}(\widetilde{U}_{[x]})$. By definition of the quotient topology this is an open subset of X. Therefore it remains to show that $U_x \times V_y \subset p^{-1}(\varphi^{-1}(W))$. In other words, we need to show that $\varphi(p(U_x \times V_y)) \subset W$. This inclusion follows from the following short argument:

$$\varphi(p(U_x \times V_y)) = (\varphi \circ p)(U_x \times V_y) \underset{\substack{\uparrow \\ \text{by the above commutative triangle}}}{=} (q \times \mathrm{id}_Y)(U_x \times V_y) \underset{\substack{\uparrow \\ \text{since } U_x = q^{-1}(\widetilde{U}_{[x]})}}{=} q(q^{-1}(\widetilde{U}_{[x]})) \times V_y = \widetilde{U}_{[x]} \times V_y \subset W.$$

This concludes the proof of the claim and thus also of the lemma. ∎

One would expect that the map from Lemma 13.1 surely must be a homeomorphism. But perhaps rather shockingly, the next lemma shows that in general this is not the case:

Lemma 13.2. There exist topological spaces X and Y and an equivalence relation $\sim$ on X with the following property: If we denote by $\sim_\times$ the equivalence relation on $X \times Y$ generated by $(x, y) \sim_\times (x', y)$ whenever $x \sim x'$, then the map

$$\begin{array}{rcl} \varphi\colon (X \times Y)/\!\sim_\times & \to & (X/\!\sim) \times Y \\ [(x, y)] & \mapsto & ([x], y) \end{array}$$

is a bijection and it is continuous, but it is *not* a homeomorphism.

Remark. We refer to [**Fab2005**, Theorem 1], [**BrownR2006**, p. 111] and [**Bou1998**, p. 128] for other examples which also show that mixing product and quotient topologies is in general a very dangerous idea.

Proof. We consider $X = \mathbb{R} \times \mathbb{N} = \{(x, n) \,|\, x \in \mathbb{R}, n \in \mathbb{N}\}$, equipped with the product topology. Let $\sim$ be the equivalence relation on X that is generated by $(0, m) \sim (0, n)$ for all $m, n \in \mathbb{N}$. We set $Y := \prod_{i \in \mathbb{N}} \mathbb{R}$, equipped with the product topology introduced on page 226. By Lemma 13.1 we know that the map

$$\begin{array}{rcl} \varphi\colon (X \times Y)/\!\sim_\times & \to & (X/\!\sim) \times Y \\ [(x, y)] & \mapsto & ([x], y) \end{array}$$

is a bijection and that it is continuous. It remains to show that φ is not a homeomorphism. We will do so by exhibiting an open subset V of $(X \times Y)/\!\sim_\times$ such that the image $\varphi(V)$ is not an open subset of $(X/\!\sim) \times Y$.

Before we can proceed we need to introduce the following notation:

(1) We denote by $q\colon X \to X/\!\sim$ the natural projection.
(2) We write $f = q \times \mathrm{id}_Y\colon X \times Y \to (X/\!\sim) \times Y$.
(3) Given $n \in \mathbb{N}$ we set

$$U_n := \{((t, n), (y_1, y_2, y_3, \dots)) \in X \times Y \,|\, |t \cdot y_n| < 1\}.$$

and we set $U := \bigcup_{n \in \mathbb{N}} U_n$.

The key to proving the lemma is the following claim.

Claim 1. The set U has the following properties:
(1) U is an open subset of $X \times Y$,
(2) $f^{-1}(f(U)) = U$,
(3) $f(U)$ is not open in $(X/\!\sim) \times Y$.

Proof.

(1) We want to show that U is an open subset of $X \times Y$. It suffices to show that each U_n is open. First note that the map

$$\begin{array}{rcl} \gamma\colon X \times Y = \mathbb{R} \times \mathbb{N} \times \prod_{i \in \mathbb{N}} \mathbb{R} & \to & \mathbb{R} \\ (t, n, y_1, y_2, \dots) & \mapsto & t \cdot y_n \end{array}$$

is continuous by the Topological-Product Proposition 5.1. Furthermore note that $\{n\}$ is an open subset of $\mathbb{N}$. Now we see that

$$U_n = \gamma^{-1}((-1, 1)) \cap \underbrace{\mathbb{R} \times \{n\} \times \prod_{i \in \mathbb{N}} \mathbb{R}}_{\text{open by definition of the product topology}}$$

is the intersection of two open sets, thus it is open itself.

(2) We need to show $f^{-1}(f(U)) = U$. By the Image-Preimage Lemma 0.2 (1) it remains to show that $f^{-1}(f(U)) \subset U$. In other words, we need to show that if v is an element of $X \times Y$ with $f(v) \in f(U)$, then $v \in U$. But this statement follows from the following claim.

Subclaim. Let $v = ((s,m),(y_1,y_2,\dots))$ and $w = ((t,n),(z_1,z_2,\dots))$ be two elements in $X \times Y$. If $f(v) = f(w)$ and if $w \in U$, then we also have $v \in U$.

Proof. If $v = w$, then there is nothing to show. On the other hand, if $v \neq w$, then it follows from the definition of the map f and the equivalence relation $\sim$ that $s = t = 0$. But this implies that v and w lie in every U_n, in particular they both lie in U. ⊞

(3) We prove this statement by contradiction. Thus suppose that $f(U)$ is, contrary to our hopes, an open subset of $(X/\sim) \times Y$. Note that $((0,0),(0,0,\dots)) \in U$. This implies that $f((0,0),(0,0,\dots)) = ([(0,0)],(0,0,\dots))$ is contained in $f(U)$. By definition of the product topology of $(X/\sim) \times Y$ and of the product topology of $Y = \prod_{i\in\mathbb{N}} \mathbb{R}$ there exists an open subset $V \subset X/\sim$ and open subsets $W_i \subset \mathbb{R}$ with $([(0,0)],(0,0,\dots)) \in V \times \prod_{i\in\mathbb{N}} W_i \subset f(U)$ and such that all but finitely W_i are equal to $\mathbb{R}$. We pick an $N \in \mathbb{N}$ with $W_N = \mathbb{R}$. Since $[(0,0)] \in V$ we see that $q^{-1}(V)$ is an open subset of $X = \mathbb{R} \times \mathbb{N}$ that contains the subset $q^{-1}([(0,0)]) = \{0\} \times \mathbb{N}$. In particular $q^{-1}(V)$ contains some point (s,N) with $s \neq 0$. We pick a real number $y_N > \frac{1}{|s|}$ and we consider the point

$$P \;:=\; ((s,N),(0,\dots,0,\underset{\substack{\uparrow\\ N\text{-th entry}}}{y_N},0,\dots)) \;\in X \times Y.$$

Note that

$$P \underset{\substack{\uparrow\\ \text{since } (s,N)\in q^{-1}(V)\\ \text{and since } W_N=\mathbb{R}}}{\in} q^{-1}(V) \times \prod_{i\in\mathbb{N}} W_i \underset{\substack{\uparrow\\ \text{since } f = q\times \mathrm{id}_Y}}{=} f^{-1}\Big(V \times \prod_{i\in\mathbb{N}} W_i\Big) \underset{\substack{\uparrow\\ \text{see above}}}{\subset} f^{-1}(f(U)) \underset{\substack{\uparrow\\ \text{by (2)}}}{=} U = \bigcup_{n\in\mathbb{N}} U_n.$$

By definition of the U_n's we see that the only possibility is that $P \in U_N$. But we have $|s \cdot y_N| > 1$, i.e. P cannot possibly lie in U_N. We have obtained a contradiction. ⊞

It remains to prove the following claim.

Claim 2. We set $V := p(U)$.

(1) V is an open subset of $(X \times Y)/\sim_\times$,

(2) $\varphi(V)$ is not an open subset of $(X/\sim) \times Y$.

Proof. We denote by $p\colon X \times Y \to (X \times Y)/\sim_\times$ the natural projection. We have the following commutative diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{\quad f = q\times \mathrm{id}_Y \quad} & (X/\sim) \times Y \\ & {\scriptstyle p}\searrow \quad \nearrow{\scriptstyle \varphi} & \\ & (X \times Y)/\sim_\times . & \end{array}$$

Note that

$$p^{-1}(V) \;=\; p^{-1}(p(U)) \;\underset{\uparrow}{=}\; f^{-1}(f(U)) \;\underset{\uparrow}{=}\; U \;\underset{\uparrow}{=}\; \text{an open subset of } X \times Y.$$

since $p = \varphi^{-1} \circ f$ and since φ is a bijection — by the previous claim

By the definition of the quotient topology on $(X \times Y)/\!\sim_\times$ this means that V is indeed an open subset of $(X \times Y)/\!\sim_\times$. Furthermore we have $\varphi(V) = \varphi(p(U)) = f(U)$. But by the previous claim we know that $f(U)$ is not an open subset of $(X/\!\sim) \times Y$. This concludes the proof of the claim and thus of the lemma. ■

The previous lemma leads us to the following cautionary remark:

Remark.

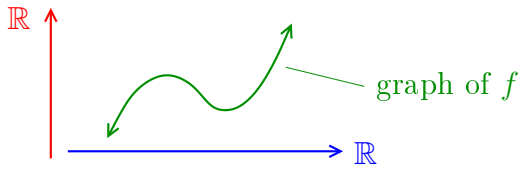
In general products and quotients of topological spaces do not mix well.

13.2. **Whitehead's Quotient Theorem.** In this section we will state Whitehead's Quotient Theorem. It allows us to prove the Product-Quotient Mixing Lemma 13.6 which shows that the disturbing situation of Lemma 13.1 cannot arise if Y is regionally compact. In many applications later on we will consider the case $Y = [0, 1]$ and the Product-Quotient Mixing Lemma 13.6 will ride to our rescue.

Definition. We say that a map $f\colon X \to Y$ between topological spaces is a quotient map if f is surjective and if a subset $U \subset Y$ is open in Y if and only if the preimage $f^{-1}(U)$ is open in X.

Examples.

(1) We consider the map $f\colon \mathbb{R} \to \mathbb{R}$ whose graph is shown in the figure below. This map is continuous, it is *not* an open map, but the reader is invited to verify that the map is in fact a quotient map.



(2) Let $X = \{0, 1\}$ be a set of two elements $0, 1$. Let $\mathcal{S} = \{\varnothing, X\}$ be the trivial topology and let $\mathcal{T} = \{\varnothing, X, \{0\}, \{1\}\}$ be the discrete topology. Evidently the identity map $(X, \mathcal{T}) \to (X, \mathcal{S})$ is a continuous bijection. But clearly it is not a quotient map.

The following lemma summarizes a few elementary properties of quotient maps.

Lemma 13.3. (Quotient Map Lemma)

(1) Let X be a topological space and let $\sim$ be an equivalence relation on X. The projection $X \to X/\!\sim$ is a quotient map.
(2) If $f\colon X \to Y$ is a quotient map, then it is also continuous.
(3) If $f\colon X \to Y$ is a bijection between topological spaces and if it is a quotient map, then f is a homeomorphism.

Proof.

(1) This statement follows immediately from the definition of the quotient topology on the quotient $X/\!\sim$.

(2) Let $f\colon X \to Y$ be a quotient map. It follows immediately from the "only if" bit of the definition of a quotient map that f is continuous.
(3) This statement follows immediately from (2) and the Open -Injective Map Lemma 2.16 (2). ■

The following is the key technical theorem which in most reasonable situations will help us to navigate the choppy waters involving product and quotient maps.

Theorem 13.4. (Whitehead Quotient Theorem) Let $f\colon X \to Z$ be a map between topological spaces that is a quotient map. Furthermore let Y be a topological space. If Y is regionally compact (e.g. if $Y = [0,1]$), then the map

$$f \times \mathrm{id}_Y\colon X \times Y \ \to \ Z \times Y$$

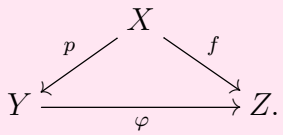
is also a quotient map.

Remark. This theorem is sometimes just called the "Whitehead Theorem" since it was first proved by John Whitehead [**WhdJ1948**, Lemma 4] in 1948. For disambiguation purposes we refer to it as the Whitehead Quotient Theorem and we reserve the name "Whitehead Theorem" for Theorem 162.1.

We will provide the proof of the Whitehead Quotient Theorem 13.4 in the next section. In the following we first state and prove some useful consequences of the Whitehead Quotient Theorem 13.4. In fairness to the reader one should perhaps say that it is arguably best to move on to the next chapter. It is only advisable to read the remainder of this section once one really faces acute technical problems involving the mix of product and quotient maps.

The following little lemma is the key to many applications of the Whitehead Quotient Theorem 13.4.

Lemma 13.5. Suppose we are given a commutative diagram of maps between topological spaces:



The following two statements hold:
(1) If p and f are quotient maps, then φ is also a quotient map.
(2) If p is a quotient map and if f is continuous, then φ is continuous.

Proof.

(1) Suppose that p and f are quotient maps. Let $V \subset Z$ be an open subset. We have the following implications:

$$\begin{array}{ccccc} V \text{ is open} & \Leftrightarrow & f^{-1}(V) \text{ is open} & \Leftrightarrow & p^{-1}(\varphi^{-1}(V)) \text{ is open} \\ & \uparrow & & \uparrow & \\ & \text{since } f \text{ is a quotient map} & & \text{since } f = \varphi \circ p \text{ we have } f^{-1}(V) = p^{-1}(\varphi^{-1}(V)) & \end{array} \quad \begin{array}{ccc} \Leftrightarrow & \varphi^{-1}(V) \text{ is open.} \\ \uparrow & \\ \text{since } p \text{ is a quotient map and since } p(p^{-1}(\varphi^{-1}(V))) = \varphi^{-1}(V) & \end{array}$$

Furthermore, it follows from the fact that the diagram commutes and the fact that f is surjective, that φ is surjective. We have thus verified that φ is a quotient map.
(2) Now we suppose that p is a quotient map and that f is continuous. We need to show that given an open subset $V \subset Z$ the preimage $\varphi^{-1}(V)$ is open. The argument is

basically exactly the same as in (1), except that the first arrow is only a right arrow, since we only assume that f is continuous. ■

We have the following addendum to Lemma 13.1.

Lemma 13.6. (Product-Quotient Mixing Lemma) Let X and Y be two topological spaces. Suppose we are given an equivalence relation $\sim$ on X. We denote by $\sim_\times$ the equivalence relation on $X \times Y$ that is generated by $(x, y) \sim_\times (x', y)$ whenever $x \sim x'$. If Y is regionally compact, then the following map is a homeomorphism:

$$\begin{array}{rcl} \varphi\colon (X \times Y)/\!\sim_\times & \to & (X/\!\sim) \times Y \\ [(x, y)] & \mapsto & ([x], y). \end{array}$$

Proof. As we pointed out in the proof of Lemma 13.1, it is basically clear that φ is a bijection. By the Quotient Map Lemma 13.3 it remains to show that φ is a quotient map. We denote by $p\colon X \times Y \to (X \times Y)/\!\sim_\times$ and $q\colon X \to X/\!\sim$ the two natural projections. We have the following commutative diagram

$$\begin{array}{ccc} & X \times Y & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle f=q\times \mathrm{id}_Y} \\ (X \times Y)/\!\sim_\times & \xrightarrow{\quad\varphi\quad} & (X/\!\sim) \times Y. \end{array}$$

By the Quotient Map Lemma 13.3 (1) we know that p and q are quotient maps. Since Y is regionally compact we obtain from the Whitehead Quotient Theorem 13.4 that $q \times \mathrm{id}_Y$ is a quotient map. It follows from Lemma 13.5 (1) that φ is also a quotient map. ■

Later on we will use the following special case of the Product-Quotient Mixing Lemma 13.6.

Lemma 13.7. Let X and Y be two topological spaces, let $A \subset X$ be a subset and let $f\colon A \to Y$ be a map. Using the notation from page 262, we see that the map

$$\begin{array}{rcl} (X \times [0,1]) \cup_{f\times \mathrm{id}_{[0,1]}} (Y \times [0,1]) & \to & (X \cup_f Y) \times [0,1] \\ [(P, t)] & \mapsto & ([P], t) \end{array}$$

is a homeomorphism.

Proof. Once one unravels the definitions it is clear that this statement follows immediately from the Product-Quotient Mixing Lemma 13.6 together with the observation that $[0, 1]$ is regionally compact. ■

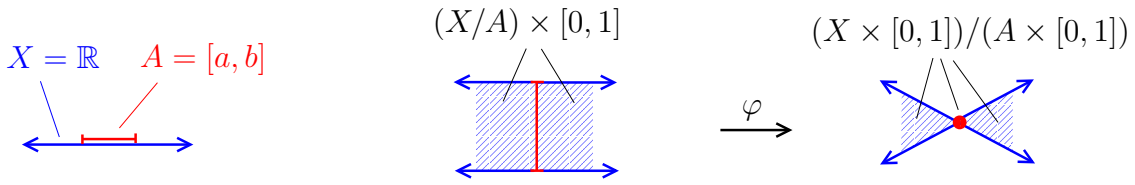
We conclude this list of applications of the Whitehead Quotient Theorem 13.4 with the following proposition that gives a fairly explicit illustration of how the Whitehead Quotient Theorem 13.4 can be used.

Proposition 13.8. Let X be a topological space, let A be a subset of X and let Y be a topological space that is regionally compact. Then the map

$$\begin{array}{rcl} \varphi\colon (X/A) \times Y & \to & (X \times Y)/(A \times Y) \\ ([x], y) & \mapsto & [(x, y)] \end{array}$$

is a quotient map, in particular it is continuous.

Example. In the figure below we illustrate the map studied in Proposition 13.8 in the case that $X = \mathbb{R}$, $A = [a, b]$ and $Y = [0, 1]$.



Proof. We denote by $p\colon X \times Y \to (X \times Y)/(A \times Y)$ and $q\colon X \to X/A$ the two natural projections. We have the following commutative diagram

$$\begin{array}{ccc} & X \times Y & \\ {\scriptstyle q\times \mathrm{id}_Y}\swarrow & & \searrow{\scriptstyle p} \\ (X/A) \times Y & \xrightarrow{\quad\varphi\quad} & (X \times Y)/(A \times Y). \end{array}$$

By the Quotient Map Lemma 13.3 the maps p and q are quotient maps. Since Y is regionally compact we obtain from the Whitehead Quotient Theorem 13.4 that $q \times \mathrm{id}_Y$ is a quotient map. It follows from Lemma 13.5 that φ is also a quotient map. By the Quotient Map Lemma 13.3 (2) this implies that φ is indeed continuous. ■

13.3. **Proof of the Whitehead Quotient Theorem 13.4.** Now that we procrastinated for as long as possible, we will finally run out of reasons not to turn to the proof of the Whitehead Quotient Theorem 13.4. In fact we provide two proofs.

The first proof of the Whitehead Quotient Theorem 13.4 cleverly reduces the proof to the Compact-Open Exponential Proposition 12.6. This approach pushes all the set-theoretic nastiness into the proof of the Compact-Open Exponential Proposition 12.6. The second proof of the Whitehead Quotient Theorem 13.4 is a bare-hands argument which can be read independently of the previous section on the compact-open topology.

Proof. First proof of Theorem 13.4, using the Compact-Open Exponential Proposition 12.6] We write $W = Z \times Y$ and we equip W with the topology that turns

$$\begin{array}{rcl} g\colon X \times Y & \to & W \\ (x, y) & \mapsto & (f(x), y) \end{array}$$

into a quotient map. In other words, as a set we have $W = Z \times Y$, but this time equipped with the topology that is defined by the condition that $U \subset W$ is open if and only if $g^{-1}(U)$ is open. Note that with this topology on W the map g is in particular continuous. Next we consider the following diagram

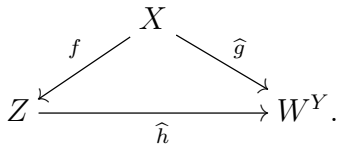
$$\begin{array}{ccc} X \times Y & \xrightarrow{\quad f\times \mathrm{id}_Y\quad} & Z \times Y \\ & {\scriptstyle g(x,y)=(f(x),y)}\searrow \quad \swarrow{\scriptstyle h(z,y)=(z,y)} & \\ & W & \end{array}$$

where $h\colon Z \times Y \to W$ is the identity. Since g is by definition a quotient map it remains to show that h is a homeomorphism. Evidently h is a bijection. Furthermore note that every open set in $Z \times Y$ is open in W since $f \times \mathrm{id}_Y$ is continuous. Thus it remains to show that h is continuous.

We consider the maps

$$\begin{array}{rcl} \widehat{g}\colon X & \to & W^Y \\ x & \mapsto & \begin{pmatrix} Y & \to & W \\ y & \mapsto & g(x,y) \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{rcl} \widehat{h}\colon Z & \to & W^Y \\ z & \mapsto & \begin{pmatrix} Y & \to & W \\ y & \mapsto & h(z,y) \end{pmatrix} \end{array}$$

These maps fit into the following commutative diagram:

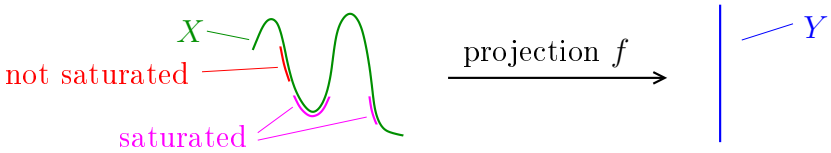


We make the following observations:

(1) Since $g\colon X \times Y \to W$ is continuous and since Y is by hypothesis regionally compact we obtain from the Compact-Open Exponential Proposition 12.6 that $\widehat{g}\colon X \to W^Y$ is continuous.
(2) By our hypothesis the map f is a quotient map.
(3) It follows from (1) and (2), together with Lemma 13.5 (2), that the map $\widehat{h}\colon Z \to W^Y$ is continuous.
(4) It follows from the Compact-Open Exponential Proposition 12.6 and (3) that h is indeed continuous. ■

Our second proof of the Whitehead Quotient Theorem 13.4 is independent of earlier results. But it does mean that we will have to get our hands somewhat dirty.

Definition. Let $f\colon X \to Y$ be a map between two sets. We say $U \subset X$ is saturated with respect to f if $U = f^{-1}(f(U))$. If f is understood, then we just say U is saturated.



Remark. Let $f\colon X \to Y$ be a map between two sets.

(1) Let $U \subset Y$ be a subset. By the Image-Preimage Lemma 0.2 (1) we know that $U \subset f^{-1}(f(U))$. Thus U is saturated if and only if $f^{-1}(f(U)) \subset U$.
(2) If f is an injection, then it follows from the Image-Preimage Lemma 0.2 (1) that every subset of X is saturated.

The following is, besides the Image-Preimage Lemma 0.2, the only purely set-theoretic lemma of these notes.

Lemma 13.9. Let $f\colon X \to Z$ be a map between two sets and let Y be another set. Let $\widetilde{W}$ be a subset of $X \times Y$ that is saturated with respect to the map $f \times \mathrm{id}_Y\colon X \times Y \to Z \times Y$. Let $K \subset Y$. We consider the set

$$U \;:=\; \{x' \in X \,|\, \{x'\} \times K \subset \widetilde{W}\}.$$

The set U is saturated with respect to f.

Proof. We write $h = f \times \mathrm{id}_Y$. By the above we need to show that $f^{-1}(f(U)) \subset U$. Thus let $x \in f^{-1}(f(U)) \subset X$. We have

$$h(\{x\}\times K) \underset{\substack{\uparrow\\ \text{definition of } h}}{=} (f\times \mathrm{id}_Y)(\{x\}\times K) = \{f(x)\}\times K \underset{\substack{\uparrow\\ \text{since } f(x)\in f(U) \text{ we know that}\\ f(x)=f(x') \text{ for some } x'\in U}}{=} \{f(x')\}\times K \underset{\substack{\uparrow\\ \text{definition of}\\ h = f\times \mathrm{id}_Y}}{=} h(\{x'\}\times K) \underset{\substack{\uparrow\\ \text{since } x'\in U}}{\subset} h(\widetilde{W}).$$

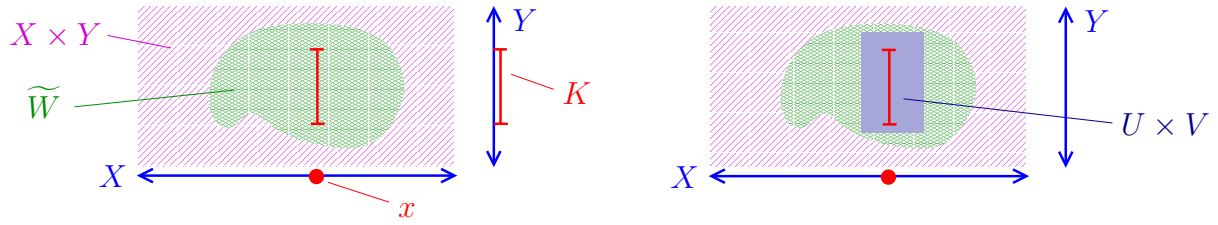
It follows that

$$\{x\}\times K \underset{\substack{\uparrow\\ \text{Image-Preimage Lemma 0.2 (1)}}}{\subset} h^{-1}(h(\{x\}\times K) \underset{\substack{\uparrow\\ \text{by the above argument}}}{\subset} h^{-1}(h(\widetilde{W})) \underset{\substack{\uparrow\\ \text{since } \widetilde{W} \text{ is saturated}}}{=} \widetilde{W}.$$

We have thus shown that $x \in U$. ∎

We will also need the following lemma which is again in the realm of topological spaces. It can be viewed as a minuscule generalization of the Tube Lemma from Exercise 5.9.

Lemma 13.10. Let X and Y be topological spaces, let $x \in X$ and let $K \subset Y$ be a compact subset. Let $\widetilde{W} \subset X \times Y$ be an open subset that contains $\{x\} \times K$. There exist open neighborhoods $U \subset X$ of x and $V \subset Y$ of K such that $\{x\} \times K \subset U \times V \subset \widetilde{W}$.



Proof of Lemma 13.10. Since $\widetilde{W} \subset X \times Y$ is open and since $\{x\} \times K \subset \widetilde{W}$ we can find for any $k \in K$ open neighborhoods A_k of $x \in X$ and B_k of $k \in Y$ such that $A_k \times B_k \subset \widetilde{W}$. Since K is compact there exist $k_1, \ldots, k_m \in K$ such that $K \subset V := B_{k_1} \cup \cdots \cup B_{k_m}$. We define $U := U_{k_1} \cap \cdots \cap U_{k_m}$. Since U is the intersection of finitely many open sets each of which contain x we see that U is an open neighborhood of x. Clearly we have the inclusions $\{x\} \times K \subset U \times V \subset \widetilde{W}$. ∎

For convenience we remind the reader of the following lemma.

Lemma 1.7. Let X be a topological space and let $U \subset X$ be a subset. If given any $x \in U$ there exists a neighborhood V of x that is contained in U, then U is open.

Now we can provide our alternative proof of the Whitehead Quotient Theorem 13.4.

Second proof of Theorem 13.4. Let $f\colon X \to Z$ be a map between topological spaces that is a quotient map. Furthermore let Y be a topological space that is regionally compact. We need to show that the map

$$f \times \mathrm{id}_Y\colon X \times Y \;\to\; Z \times Y$$

is also a quotient map.

Since f is a quotient map we know by the Quotient Map Lemma 13.3 (2) that f is continuous. Therefore it follows from the Product Topology-Basics Lemma 5.4 (3) that the map $f \times \mathrm{id}_Y$ is continuous. Therefore it now remains to show that if $W \subset Z \times Y$ is a subset

such that the preimage $\widetilde{W} := (f \times \mathrm{id}_Y)^{-1}(W)$ is an open subset of $X \times Y$, then W itself is open.

We intend to use the JH-Lemma 1.7 to show that W is open. Thus let $(z, y) \in W$. We pick $x_0 \in f^{-1}(z)$. We equip $\{x_0\} \times Y \subset X \times Y$ with the subspace topology. By the Product Topology-Basics Lemma 5.4 (2a) the map

$$\begin{array}{rccc} i\colon & Y & \to & \{x_0\} \times Y \\ & y & \mapsto & (x_0, y) \end{array}$$

is an embedding. In particular $Y_{x_0} := i^{-1}(\widetilde{W} \cap (\{x_0\} \times Y))$ is an open subset of Y. Note that $\{x_0\} \times Y_{x_0} \subset \widetilde{W}$. Since Y is regionally compact we can find a compact neighborhood K_{x_0} of x_0 that is contained in Y_{x_0}. By definition of a neighborhood there exists an open subset Y'_{x_0} of Y with $x_0 \in Y'_{x_0} \subset K_{x_0}$. We define

$$U_{x_0} \ := \ \{x \in X \,|\, \{x\} \times K_{x_0} \subset \widetilde{W}\}.$$

Claim 1. The set $f(U_{x_0})$ is an open subset of Z.

Proof. First we intend to use the JH-Lemma 1.7 to show that U_{x_0} is indeed an open subset of Y_{x_0}. Thus let $x \in U_{x_0}$. Since $\widetilde{W}$ is an open subset of $X \times Y$ and since K_{x_0} is compact it follows from Lemma 13.10 that there exist open subsets $A \subset X$ and $B \subset Y$ with $x \in A$ and $K_{x_0} \subset B$ such that $A \times B \subset \widetilde{W}$. Evidently this implies that $A \subset U_{x_0}$. Since $x \in A$ we have verified that the openness criterion of the JH-Lemma 1.7 is satisfied. We have thus shown that U_{x_0} is an open subset of Y_{x_0}.

Now we want to show that $V := f(U_{x_0})$ is an open subset of Z. By Lemma 13.9 we know that U_{x_0} is saturated with respect to f. In other words, we know that $U_{x_0} = f^{-1}(V)$. Now we use that by hypothesis $f\colon X \to Z$ is a quotient map. Since $f^{-1}(V) = U_{x_0}$ is open by (1) we obtain that V itself is open. But that is what we had wanted to show. ⊞

By definition of the product topology we obtain that $f(U_{x_0}) \times Y'_{x_0}$ is an open subset of $X \times Y$. Evidently we have $(x, y) \in f(U_{x_0}) \times Y'_{x_0}$. It remains to prove the following claim.

Claim 2. We have $f(U_{x_0}) \times Y'_{x_0} \subset W$.

Proof. The claim follows from the following equalities and inclusions:

$$f(U_{x_0}) \times Y'_{x_0} \ = \ (f \times \mathrm{id}_Y)(U_{x_0} \times Y'_{x_0}) \ \underset{\substack{\uparrow \\ \text{since } Y'_{x_0} \subset K_{x_0}}}{\subset} \ (f \times \mathrm{id}_Y)(U_{x_0} \times K_{x_0}) \ \overset{\substack{\text{definition of } U_{x_0} \\ \downarrow}}{\subset} \ (f \times \mathrm{id}_Y)(\widetilde{W}) \ \underset{\substack{\uparrow \\ \text{by the Image-Preimage Lemma 0.2 (2)} \\ \text{since by definition } \widetilde{W} = (f \times \mathrm{id}_Y)^{-1}(W)}}{=} \ W.$$

■

Remark. A different proof of the Whitehead Quotient Theorem 13.4, based on Propositions 12.4 and 12.6, is given in [**Hat2002**, Lemma A.17].

13.4. **Mixing product and quotient topologies II (∗).** In this section we prove a second statement regarding the interplay between product and quotient topologies. More precisely, we will prove the following proposition.

Proposition 13.11. Let X and Y be topological spaces. Let B be a subset of Y. We denote by $q\colon Y \to Y/B$ the projection. If B is a compact subset of Y and if Y is Hausdorff, then the map

$$\begin{array}{rccc} \mathrm{id}_X \times q\colon & X \times Y & \to & X \times (Y/B) \\ & (x,y) & \mapsto & (x,[y]) \end{array}$$

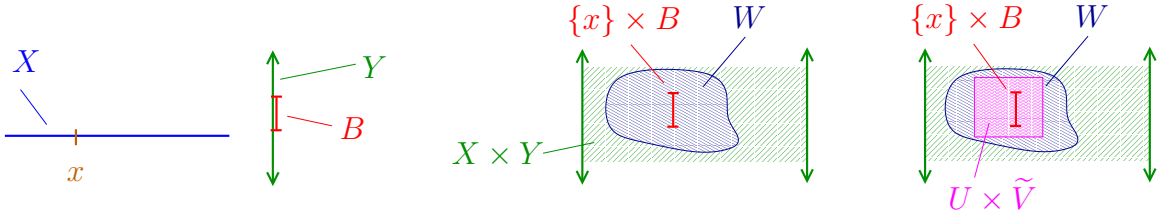
is a quotient map.

Remark.

(1) At first glance the statement of Proposition 13.11 might sound quite similar to the statement of the Whitehead Quotient Theorem 13.4. But in a way it is quite different, this time there are no conditions at all on the "unaffected" factor X, but we do put a condition on the projection $Y \to Y/B$.
(2) According to [**tD2008**, Chapter 2.2, Exercise 16] one does not need to assume that Y is Hausdorff.
(3) Finally we refer to [**BrownR2006**, Chapter 4.3] for more information on the interplay between product and quotient topologies.

In the proof of Proposition 13.11 we need the following fairly elementary lemma.

Auxiliary Lemma 13.12. Let X and Y be topological spaces and let B be a compact subset of Y. Let $x \in X$. If W is an open subset of $X \times Y$ that contains $\{x\} \times B$, then there exists an open subset U of X and an open subset $\widetilde{V} \subset Y$ with $\{x\} \times B \subset U \times \widetilde{V} \subset W$.



Proof. Let W be an open subset of $X \times Y$ that contains $\{x\} \times B$. By definition of the product topology we can find for each $b \in B$ an open neighborhood U_b of x in X and an open neighborhood V_b of b in Y such that $U_b \times V_b \subset W$. Since B is compact there exist $b_1, \ldots, b_m$ such that $B \subset V_{b_1} \cup \cdots \cup V_{b_m}$. One can easily verify that $U := U_{b_1} \cap \cdots \cap U_{b_m}$ and $\widetilde{V} := V_{b_1} \cup \cdots \cup V_{b_m}$ have the desired properties. ■

Proof of Proposition 13.11. Let X and Y be topological spaces and let B be a compact subset of Y. We assume that Y is Hausdorff. We denote by $q\colon Y \to Y/B$ the natural projection. Note that $p = \mathrm{id}_X \times q$. First note that it follows from the Topological-Quotient Proposition 5.15 (1b) and the Product Topology-Basics Lemma 5.4 (3) that p is continuous. This implies the "only if"-direction. It remains to show that "if"-direction. Thus let $W \subset X \times (Y/B)$ be a subset that has the property that $p^{-1}(W)$ is an open subset of $X \times Y$. We need to show that W is an open subset of $X \times (Y/B)$. By the JH-Lemma 1.7 and the definition of the product topology it suffices to prove the following claim.

Claim. For each $(x,[y]) \in W \subset X \times (Y/B)$ there exists an open neighborhood U_x of $x \in X$ and an open neighborhood $V_{[y]}$ of $[y] \in Y/B$ such that $U_x \times V_{[y]} \subset W$.

Proof. As in the proof of Proposition 13.8 we consider two cases separately. First we consider the case that $[y] = [B] \in Y/B$. It follows that $p^{-1}((x,[y])) = \{x\} \times B$. By hypothesis $p^{-1}(W)$ is an open subset of $X \times Y$ that contains $p^{-1}((x,[y])) = \{x\} \times B$. Therefore it follows from Lemma 13.12 there exists an open subset U_x of X and an open

subset $\widetilde{V}_y \subset Y$ with $\{x\} \times B \subset U_x \times \widetilde{V}_y \subset p^{-1}(W)$. We denote by $V_{[y]}$ the image of $\widetilde{V}_y$ under the projection $q\colon Y \to Y/B$. Note that $V_{[y]}$ is an open neighborhood of $[y] \in Y/B$.[147] Finally note that

$$U_x \times V_{[y]} \;=\; (\mathrm{id}_X \times q)(U_x \times \widetilde{V}_y) \;=\; p(U_x \times \widetilde{V}_y) \subset p(p^{-1}(W)) \;=\; W.$$

It remains to consider the case that $[y] \neq [B] \in Y/B$. We see that $(x, y) \in W$. Since B is compact and since Y is Hausdorff we know by the Compact-Closed Lemma 1.21 that B is a closed subset of Y, i.e. $Y \setminus B$ is open. It follows immediately from the definition of the product topology on $X \times B$, the fact that $Y \setminus B$ is open, the fact that $p^{-1}(W)$ is an open neighborhood of (x, y) and the fact that $y \notin B$ that there exists an open neighborhood U_x of x and an open neighborhood $\widetilde{V}_y$ of y with $\widetilde{V}_y \subset X \setminus B$ and with $U_x \times \widetilde{V}_y \subset p^{-1}(W)$. One easily verifies that U_x and $V_y = q(\widetilde{V}_y)$ have the desired properties. ■

[147] Indeed, since $B \subset \widetilde{V}_y$ we see that $q^{-1}(V_{[y]}) = q^{-1}(q(\widetilde{V}_y)) = \widetilde{V}_y \cup B = \widetilde{V}_y$, which is an open set. By the definition of the quotient topology on Y/B we see that $q^{-1}(V_{[y]})$ is indeed open.

Part II

Basics of homotopy theory

14. Homotopies

Loosely speaking two maps $f, g\colon X \to Y$ are called homotopic if one can be deformed into the other. In the subsequent chapter we will use to this notion of homotopic maps to explain what it means for two topological spaces to be "homotopy equivalent".

14.1. **Homotopic maps.** In its most basic form a homotopy is a continuous deformation of maps between two topological spaces. To make this notion precise it is convenient to introduce the following notation.

Notation. Let X and Y be two topological spaces and let $I \subset \mathbb{R}$ be a subset. Given a map $F\colon X \times I \to Y$[148] and given $t \in I$ we often denote by F_t the map[149]

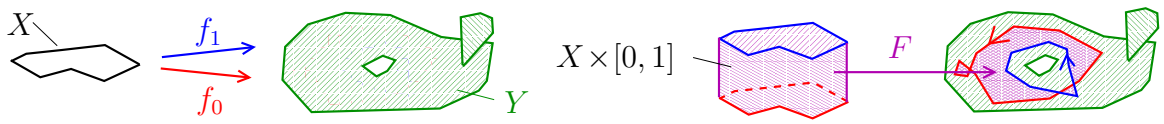
$$\begin{array}{rccc} F_t\colon & X & \to & Y \\ & x & \mapsto & F(x,t). \end{array}$$

Now we move on to the actual definition we are interested in.

Definition. Let X and Y be two topological spaces.

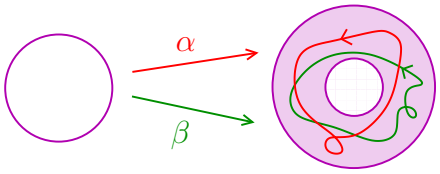
(1) Let $f_0, f_1\colon X \to Y$ be two maps. A homotopy between the maps f_0 and f_1 is a map $F\colon X \times [0,1] \to Y$ with $F_0 = f_0$ and $F_1 = f_1$. If there exists a homotopy between the maps f_0 and f_1, then we say that f_0 and f_1 are homotopic and sometimes we write $f_0 \simeq f_1$.

(2) A map $f\colon X \to Y$ is called null-homotopic if it is homotopic to a constant map.

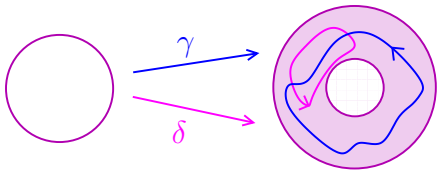


Examples.

(1) In the figure below we show four maps α, β, γ and δ from the circle S^1 to the annulus $Y = B_2(0) \setminus B_1(0)$. The maps α and β are homotopic whereas the maps γ and δ look as if they are not homotopic. But only much later, namely in Chapter 48, will we develop methods for showing that these maps are indeed not homotopic.



the maps α and β are homotopic



the maps γ and δ appear not to be homotopic

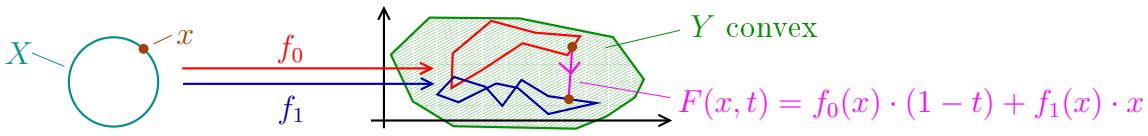
[148] Recall that all maps between topological spaces are understood to be continuous.

[149] Note that it follows from the Product Topology-Basics Lemma 5.4 (1a) together with Lemma 2.1 (1) that the map F_t is indeed continuous.

(2) Given an orthogonal matrix $A \in \mathrm{O}(n)$ we denote by $\rho(A)$ the map $f\colon S^{n-1} \to S^{n-1}$ that is given $x \mapsto A \cdot x$.
   (a) Let $\gamma\colon [0,1] \to \mathrm{SO}(n)$ be a path from some matrix A to some matrix B. The map[150]
   $$\begin{array}{rcl} S^{n-1} \times [0,1] & \to & S^{n-1} \\ (x,t) & \mapsto & \gamma(t) \cdot x \end{array}$$
   is a homotopy from $\rho(A)$ to $\rho(B)$.
   (b) Let $A \in \mathrm{O}(n)$. If $\det(A) = 1$, i.e. if $A \in \mathrm{SO}(n)$, then it follows from the Matrix Group Path-Component Proposition 2.37 (3) that there exists a path $\gamma\colon [0,1] \to \mathrm{SO}(n)$ with $\gamma(0) = A$ and $\gamma(1) = \mathrm{id}$. It follows from (a) that $\rho(A)$ is homotopic to the identity. Much later, on page 1686, we will see that if $\det(A) = -1$, then f is actually not homotopic to the identity.
   (c) It follows immediately from the definitions that a topological space X is path-connected if any two maps $\{*\} \to X$ are homotopic.

The next example of a homotopy is written down as a lemma.

Lemma 14.1. (Convex Homotopy Lemma) Let X be a topological space and let $Y \subset \mathbb{R}^n$ be a subset. Furthermore let $f_0, f_1\colon X \to Y$ be two maps. If Y is convex, then
$$\begin{array}{rcl} F\colon X \times [0,1] & \to & Y \\ (x,t) & \mapsto & \underbrace{f_0(x) \cdot (1-t) + f_1(x) \cdot t}_{\in Y \text{ since } Y \text{ is convex}} \end{array}$$
is a homotopy between f_0 and f_1.



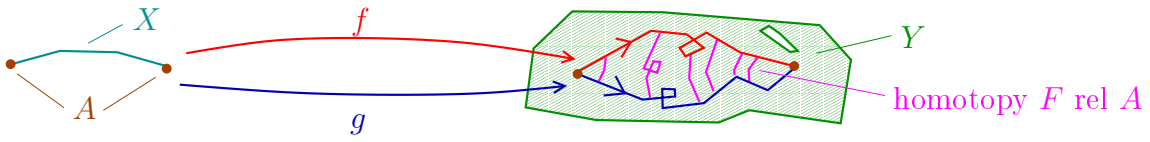
Proof. Basically by definition we have $F_0 = f_0$ and $F_1 = f_1$. But we are not done yet, the conscientious reader will not have failed to notice that we need to show that the map F is actually continuous. The map F can be written as the composition of the following two maps:
$$X \times [0,1] \xrightarrow{(x,t) \mapsto (f_0(x), f_1(x), t)} \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \xrightarrow{(x,y,t) \mapsto (x \cdot (1-t) + y \cdot t)} \mathbb{R}^n.$$
The first map is continuous by the Product Topology-Basics Lemma 5.4 (2) and the Product Topology-Basics Lemma 5.4 (1) and the second map is continuous by Lemmas 2.3 and the Product Topology-Basics Lemma 5.4 (2a). ■

Often it is more interesting to consider homotopies that fix a given subset. This leads us to the following definition.

Definition. Let X and Y be topological spaces and let $A \subset X$ be a subset.
   (1) We say a map $F\colon X \times [0,1] \to Y$ is a homotopy rel A if for all $x \in A$ and $s, t \in [0,1]$ we have $F(x,s) = F(x,t)$.

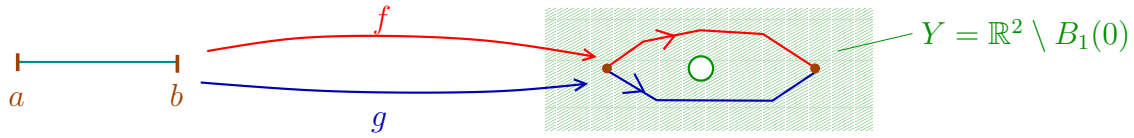
[150] It follows from various standard results, see e.g. Lemma 2.3, the Product Metric Lemma 5.3 and the Product Topology-Basics Lemma 5.4 (4) that the map is continuous.

(2) We say that two maps $f, g\colon X \to Y$ with $f|_A = g|_A$ are homotopic rel A if there exists a homotopy F rel A with $F_0 = f$ and $F_1 = g$.

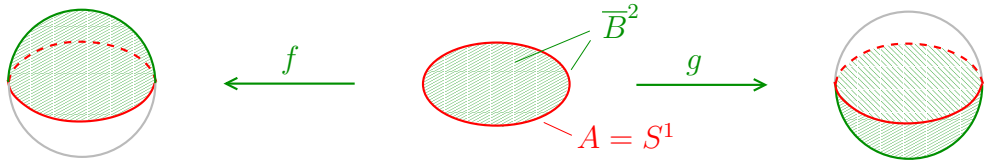


Examples.

(1) Let $f, g\colon [a,b] \to Y = \mathbb{R}^2 \setminus B_1(0)$ be the two maps from an interval to the topological space Y that are shown in the figure below. Note that $f(a) = g(a)$ and $f(b) = g(b)$. As we will see in Exercise 14.2, any two maps from an interval into a path-connected space are homotopic. On other hand it looks rather unlikely that the two maps are homotopic rel $\{a,b\}$, i.e. it does not look like we can deform f into g while keeping the endpoints of the interval fixed. Unfortunately we will have to wait till Chapter 48 to develop tools for showing that these two maps f and g are indeed not homotopic rel $\{a,b\}$.



(2) We consider the two maps $f, g\colon \overline{B}^2 \to S^2$ that are shown in the figure below. These maps agree on S^1. It is fairly easy to see that the two maps are homotopic but it seems rather unlikely that they are homotopic rel S^1. Unfortunately, as in (1), right now we lack the tools for showing the latter statement. Only much later, namely in Exercise 74.24, will we see that the two maps f and g are not homotopic rel S^1.



We continue with the next variation on the notion of homotopy.

Definition. Let X and Y be two topological spaces.

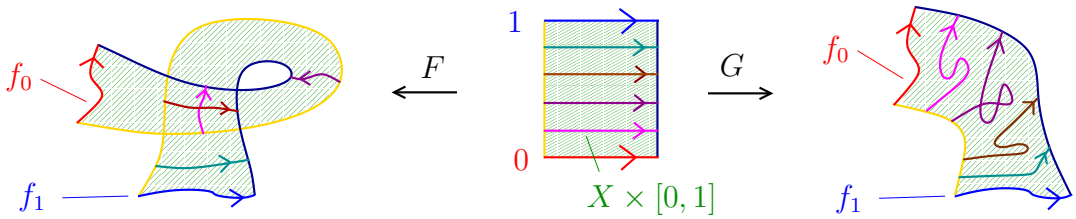
(1) (a) An isotopy is a homotopy $F\colon X \times [0,1] \to Y$ such that for each $t \in [0,1]$ the map $F_t\colon X \to Y$ is an embedding.

(b) A homeotopy is a homotopy $F\colon X \times [0,1] \to Y$ such that for each $t \in [0,1]$ the map $F_t\colon X \to Y$ is a homeomorphism.

(2) Let $f_0, f_1\colon X \to Y$ be two maps that are embeddings. We say f_0 and f_1 are isotopic if there exists an isotopy $F\colon X \times [0,1] \to Y$ with $F_0 = f_0$ and $F_1 = f_1$. Similarly we define homeotopic.

Examples.

(1) In the figure below we show two maps $f_0, f_1\colon [0,1] \to \mathbb{R}^2$ that are embeddings. We also indicate two homotopies F and G between f_0 and f_1. The map F is an isotopy, the map G is not.

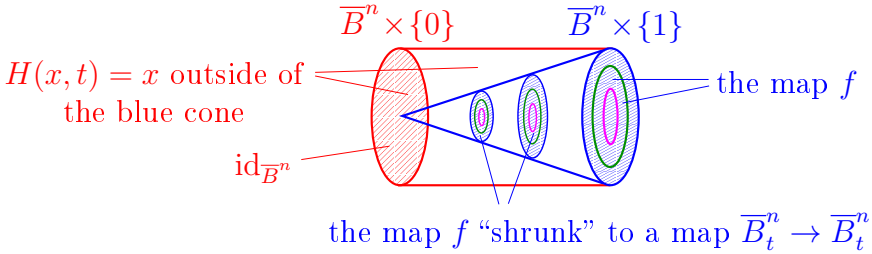


F is an isotopy $\qquad$ G is not an isotopy

(2) Let $n \in \mathbb{N}$. Let $f\colon \overline{B}^n \to \overline{B}^n$ be a homeomorphism. If $f|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$, then f is isotopic to $\mathrm{id}_{\overline{B}^n}$. Indeed, in Exercise 14.5 we will see that the map

$$\begin{array}{rcl} H\colon \overline{B}^n \times [0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \begin{cases} t \cdot f(\frac{x}{t}), & \text{if } 0 \le \|x\| < t, \\ x, & \text{otherwise} \end{cases} \end{array}$$

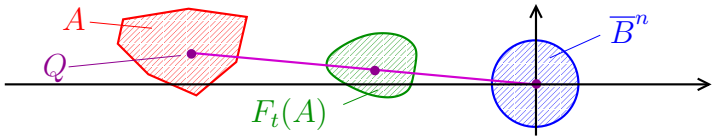
is continuous and that each H_t is a homeomorphism. It is clear that $H_0 = \mathrm{id}$ and $H_1 = f$. Thus we see that H is the desired isotopy. The fact that f is isotopic to $\mathrm{id}_{\overline{B}^n}$ is often referred to as the *Alexander trick*.[151]



The next lemma is a refined version of the Convex-to-Ball Proposition 2.20 (2).

Lemma 14.2. Let A be a bounded closed convex subset of $\mathbb{R}^n$ with non-empty interior. Let Q be a point in the interior of A. There exists a canonical isotopy $F\colon A \times [0,1] \to \mathbb{R}^n$ with the following three properties:

(1) we have $F_0 = \mathrm{id}_A$,
(2) the map F_1 is a homeomorphism from A to $\overline{B}^n$ and this map restricts to a homeomorphism $\partial A \to S^{n-1}$,
(3) for any $t \in [0,1]$ we have $F(Q,t) = (1-t) \cdot Q$.[152]



Remark. If A is "diffeomorphic" to $\overline{B}^n$, then the Diffeotopies-of-$\mathbb{R}^n$-Classification Proposition 34.3 gives a significant refinement of the conclusion of Lemma 14.2.

[151] The existence of such a homotopy was first observed by James Alexander in 1923, see [**Al1924a**].
[152] In particular, if Q is already the origin, then F is a homotopy rel the origin.

Proof. First consider the case that Q is the origin. As in the statement of the Convex-to-Ball Proposition 2.20 (2) we consider the map.

$$\begin{array}{rcl} \rho\colon A \setminus \{Q\} & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & \sup\big\{\|rx\| \,\big|\, r \in \mathbb{R}_{>0} \text{ and } r(x-Q)+Q \in A\big\}. \end{array}$$

In Lemma 2.21 (1) we showed that ρ is continuous. Similar to the discussion on page 160 we consider the map

$$\begin{array}{rcl} F\colon A \times [0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \begin{cases} x \cdot \frac{1}{(1-t)+t\cdot\rho(x)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \end{array}$$

As in Lemma 2.21 (2) one can show that F is continuous. It is straightforward to verify that this map has all the desired properties.

We leave it to the reader to modify the above proof to deal with the case that Q is not the origin. ■

After these examples we return to developing the general theory of homotopies. On many occasions it will be useful to combine two homotopies or isotopies. This leads us to the following lemma.

Lemma 14.3. (Homotopy Stacking Lemma) Let X and Z be topological spaces.

(a) Let Y be a topological space and let $f\colon X \times Y \to Z$ be a map. Suppose we are given finitely many closed subsets $A_1, \ldots, A_m$ of Y with $\bigcup_{i=1}^{m} A_i = Y$. If each map $X \times A_i \to X \times Y \to Z$ is continuous, then f itself is continuous.

(b) Let $a < b < c$ be real numbers and let $f\colon X \times [a,c] \to Z$ be a map. If the restrictions of f to $X \times [a,b]$ and to $X \times [b,c]$ are continuous, then f itself is continuous.

Proof.

(1) By the Product Topology-Basics Lemma 5.4 (4) the identity

$$X \times \begin{array}{c} A_i \text{ equipped with the} \\ \text{subspace topology from } Y \end{array} \longrightarrow \begin{array}{c} X \times A_i \text{ equipped with the subspace} \\ \text{topology from } X \times Y \end{array}$$

is actually a homeomorphism. Together with our hypothesis this shows that for each $i \in \{1, \ldots, m\}$ the map

$$f|_{X\times A_i}\colon \underbrace{X \times A_i}_{\substack{\uparrow \\ \text{equipped with subspace topology from } X \times Y}} \longrightarrow Z$$

is continuous. By the Product-of-Subsets Lemma 5.5 (1) we know that $A_i \times Y$ is a closed subset of $X \times Y$. Now it follows from the Pasting Proposition 2.6 (2) that f is indeed continuous.

(2) This statement is just a frequently used special case of (1). ■

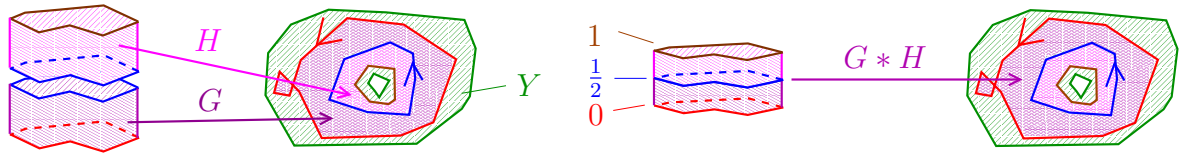
Now we can introduce the promised lemma that allows us to combine homotopies.

Lemma 14.4. (Homotopy Combination Lemma) Let X and Y be topological spaces and let $A \subset X$ be a (possibly empty) subset of X. Let $G, H\colon X \times [0,1] \to Y$ be two

homotopies rel A. If $G_1 = H_0$, then the map

$$\begin{array}{rcl} G * H \colon X \times [0,1] & \to & Y \\ (P,t) & \mapsto & \begin{cases} G(P,2t), & \text{if } t \in [0,\frac{1}{2}], \\ H(P,2t-1), & \text{if } t \in [\frac{1}{2},1]. \end{cases} \end{array}$$

is a homotopy rel A from G_0 to H_1. Furthermore, if G and H are isotopies (homotopies), then $G * H$ is also an isotopy (homeotopy). We refer to $G * H$ as the combination of G and H.



Proof. We only need to verify that $G * H$ is indeed continuous, but that is a consequence of the Homotopy Stacking Lemma 14.3 together with our hypothesis that $G_1 = H_0$. ■

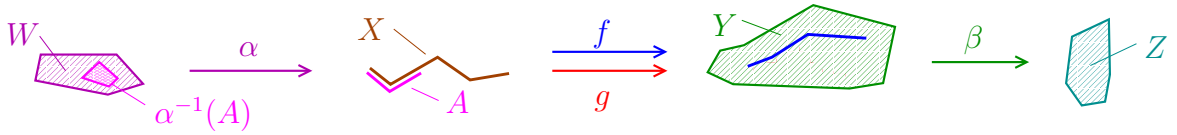
The following lemma says in particular that being homotopic is an equivalence relation on the set of maps between two topological spaces.

Lemma 14.5.

(1) Let X be and Y topological spaces and let A be a (possibly empty) subset of X. Then "homotopy rel A of maps" forms an equivalence relation on the set of maps from X to Y. The same applies for "isotopy" and "homeotopy".

(2) Let $f, g \colon X \to Y$, let $\alpha \colon W \to X$ and $\beta \colon Y \to Z$ be maps between topological spaces. Furthermore let $A \subset X$ be a (possibly empty) subset. The following conclusion holds:

$$\begin{array}{rcl} f \text{ is homotopic to } g \text{ rel } A & \implies & f \circ \alpha \text{ is homotopic to } g \circ \alpha \text{ rel } \alpha^{-1}(A) \\ f \text{ is homotopic to } g \text{ rel } A & \implies & \beta \circ f \text{ is homotopic to } \beta \circ g \text{ rel } A \end{array}$$



Proof.

(1) We have to show that "homotopy rel A" satisfies the three properties of an equivalence relation. To simplify the notation we only deal with the case $A = \varnothing$. In the following let $f, g, h \colon X \to Y$ be three maps.

- The "constant homotopy" given by $F(x,t) := f(x)$ defines a homotopy between f and f.
- Let F be a homotopy from f to g. Then it is clear that the map F' that is defined by $F'(x,t) := F(x, 1-t)$ is a homotopy from g to f.
- Let F be a homotopy from f to g and let G be a homotopy from g to h. We consider the map

$$\begin{array}{rcl} X \times [0,1] & \to & Y \\ (x,t) & \mapsto & \begin{cases} F(x,2t), & \text{if } t \in \left[0,\frac{1}{2}\right], \\ G(x,2t-1), & \text{if } t \in \left(\frac{1}{2},1\right] \end{cases} \end{array}$$

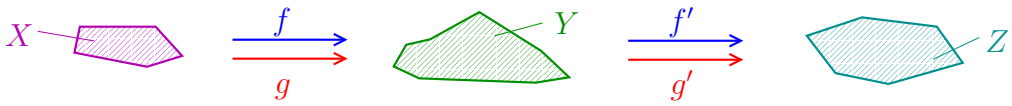
It follows from the Homotopy Stacking Lemma 14.3 that this map is continuous. It is now clear that this map is a homotopy from f to h.

(2) This statement is basically obvious, we just need to precompose and postcompose a homotopy $H\colon X \times [0,1] \to Y$ between f and g by $\alpha \times \mathrm{id}_{[0,1]}\colon W \times [0,1] \to X \times [0,1]$ respectively by $\beta\colon Y \to Z$. ■

Lemma 14.6. Let X, Y and Z be topological spaces. Let $f, f'\colon X \to Y$ and $g, g'\colon Y \to Z$ be maps.

(1) If f and f' are homotopic and if g and g' are homotopic, then the map $g \circ f$ is homotopic to $g' \circ f'$.

(2) If there exists a subset A of X such that f and f' are homotopic rel A and if there exists a subset B of Y such that $f(A) = f'(A) \subset B$ and such that g and g' are homotopic rel B, then $g \circ f$ is homotopic to $g' \circ f'$ rel A.



Proof.

(1) Let $F\colon X \times [0,1] \to Y$ be a homotopy between f and f' and let $G\colon Y \times [0,1] \to Z$ be a homotopy between g and g'. We denote by $\pi\colon X \times [0,1] \to [0,1]$ the projection onto the second factor. We now consider the map

$$\begin{array}{rccc} H\colon X \times [0,1] & \xrightarrow{(F,\pi)} & Y \times [0,1] & \xrightarrow{G} Z \\ (x,t) & \mapsto & (F(x,t),t) & \\ & & (y,t) & \mapsto G(y,t). \end{array}$$

It follows from Lemma 2.1 (1) and the Product Topology-Basics Lemma 5.4 (1b) that the map $H(x,t) = G(F(x,t),t)$ is actually continuous. By plugging in $t = 0$ and $t = 1$ we see that $H\colon X \times [0,1] \to Z$ is a homotopy between $g \circ f$ and $g' \circ f'$.

(2) One can easily verify that the homotopy in (1) also does the trick for (2). ■

14.2. **Homotopies on quotients and pushouts.** In this section we collect several subtle lemmas which show that, under the right circumstances, homotopies descend to quotient spaces and that given a "gluing $X \cup_f Y$" we can also glue homotopies. This section is much less pleasant to read than the previous section and it is arguably best if the reader only takes a peek once the results are actually used.

Before we state these two technical statements, we point out that the proof of these two statements eventually boils down to the rather subtle and delicate Whitehead Quotient Theorem 13.4.

After this preamble, let us move to the key result of this section which shows that homotopies descend to quotients.

Theorem 14.7. (Homotopy Quotient Theorem) Let X and Y be topological spaces. Furthermore let $\sim_X$ be an equivalence relation on X and let $\sim_Y$ be an equivalence relation on Y. If $F\colon X \times [0,1] \to Y$ is a homotopy such that $F(x,t) \sim_Y F(x',t)$ whenever $x \sim_X x'$, then the map[153]

$$\begin{array}{rcl} \varphi\colon (X/\!\sim_X) \times [0,1] & \to & Y/\!\sim_Y \\ ([x],t) & \mapsto & [F(x,t)] \end{array}$$

is continuous.

Remark. We will deal with the important special case where the equivalence relation on X is given by a group action in Exercise 48.23.

Proof. We denote by $p\colon X \to X/\!\sim_X$ and $q\colon Y \to Y/\!\sim_Y$ the natural projections. We have the following commutative diagram

$$\begin{array}{ccc} & X \times [0,1] & \\ {\scriptstyle p\times \mathrm{id}_{[0,1]}}\swarrow & & \searrow{\scriptstyle q\circ F} \\ (X/\!\sim_X) \times [0,1] & \xrightarrow[([x],t)\mapsto [F(x,t)]]{\varphi} & Y/\!\sim . \end{array}$$

By the Quotient Map Lemma 13.3 (1) the map p is a quotient map. Since $[0,1]$ is regionally compact we obtain from the Whitehead Quotient Theorem 13.4 that $p \times \mathrm{id}_{[0,1]}$ is also a quotient map. Note that by the Topological-Quotient Proposition 5.15 (1a) and by hypothesis the map $q \circ F$ is continuous. Thus we obtain from Lemma 13.5 (1) that φ is continuous. ∎

For the reader who is wary of the Whitehead Quotient Theorem 13.4 we provide a second argument that works in many, but certainly not all cases.

Proof of Theorem 14.7 if X is compact and if $X/\!\sim_X$ is Hausdorff. On $X \times [0,1]$ we consider the equivalence relation $\sim$ given by $(x,t) \sim (x',t') \iff x \sim_X x'$. We denote by $h\colon X \times [0,1] \to (X \times [0,1])/\!\sim$ the natural projection. We consider the maps

$$\begin{array}{rcl} f\colon X \times [0,1] & \to & X/\!\sim_X \times [0,1] \\ (x,t) & \mapsto & ([x],t) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \pi\colon (X \times [0,1])/\!\sim & \to & X/\!\sim_X \times [0,1] \\ [(x,t)] & \mapsto & ([x],t). \end{array}$$

By the Topological-Quotient Proposition 5.15 (1a) the projection $p\colon X \to X/\!\sim_X$ is continuous. This shows that the map to the left is continuous. By the Topological-Quotient Proposition 5.15 (1b) this implies that the induced map π to the right is also continuous. One can easily verify that π is a bijection. By hypothesis X is compact. By the Product Topology Properties Proposition 5.6 (2) this implies that $X \times [0,1]$ is compact. It follows from Topological-Quotient Proposition 5.15 (1a) and the Compact Image Lemma 2.13 that $(X \times [0,1])/\!\sim$ is compact. Next recall that by hypothesis $X/\!\sim_X \times$ is Hausdorff. By the Product Topology Properties Proposition 5.6 (1) this implies that $X/\!\sim_X \times [0,1]$ is Hausdorff. In summary it follows from the Compact-Hausdorff Proposition 2.17 (3) that π is a homeomorphism.

Now we consider the following diagram:

$$\begin{array}{ccccc} X \times [0,1] & & \xrightarrow{\quad F \quad} & & Y \\ \downarrow{\scriptstyle f} & \searrow{\scriptstyle h} & & & \downarrow{\scriptstyle q} \\ & & (X \times [0,1])/\!\sim & & \\ & \swarrow{\scriptstyle \pi}_{\cong} & & \searrow{\scriptstyle g} & \\ (X/\!\sim_X) \times [0,1] & & \xrightarrow{\quad \varphi \quad} & & Y/\!\sim_Y . \end{array}$$

[153] It follows from our hypothesis on F that the map φ is well-defined.

We make the following clarifications and observations:

(1) Note that the map $q \circ F\colon X \times [0,1] \to Y$ factors by our hypothesis through the map $h\colon X \times [0,1] \to (X \times [0,1])/\sim$. It follows again from the Topological-Quotient Proposition 5.15 (1b) that the induced map $g\colon (X \times [0,1]) \sim \to Y/\sim_Y$ is continuous.
(2) We denote by $q\colon Y \to Y/\sim_Y$ the natural projection.
(3) The map $\varphi\colon (X/\sim_X) \times [0,1] \to Y/\sim_Y$ equals the map $q \circ \overline{F} \circ \pi^{-1}$. We just showed that each of these three maps is continuous. Thus φ is continuous. ■

Example. Let $n \in \mathbb{N}$ and let $\varphi\colon [0,1] \to \mathrm{GL}(n,\mathbb{R})$ be a map. It follows from the Homotopy Quotient Theorem 14.7 that the map

$$\begin{array}{rcl} \mathbb{R}\mathrm{P}^{n-1} \times [0,1] & \to & \mathbb{R}\mathrm{P}^{n-1} \\ ([P],t) & \mapsto & [\varphi(t) \cdot P] \end{array}$$

is continuous. The same conclusion also holds if we replace "$\mathbb{R}$" by "$\mathbb{C}$".

The following lemma is a hands-on special case of the Homotopy Quotient Theorem 14.7.

Lemma 14.8. Let X be a topological space. Furthermore let $A \subset B$ be two subsets of X. If $F\colon B \times [0,1] \to X$ is a homotopy with $F(A \times [0,1]) \subset A$, then the induced map[154]

$$\begin{array}{rcl} (X/A) \times [0,1] & \to & Y/B \\ ([x],t) & \mapsto & [F(x,t)] \end{array}$$

is continuous.

Example. Let $\gamma\colon [0,1] \to \mathrm{O}(n)$ be a continuous map. We consider the maps

$$\begin{array}{rcl} F\colon \overline{B}^n \times [0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \gamma(t) \cdot x \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \overline{F}\colon (\overline{B}^n/S^{n-1}) \times [0,1] & \to & \overline{B}^n/S^{n-1} \\ ([x],t) & \mapsto & [\gamma(t) \cdot x]. \end{array}$$

It follows from Lemma 2.3, the Product Metric Lemma 5.3 and the Product Topology-Basics Lemma 5.4 (4) that the map F to the left is continuous. We now obtain from Lemma 14.8 that the map $\overline{F}$ to the right is also continuous.

Proof. This statement follows immediately from the Homotopy Quotient Theorem 14.7 and the fact that quotients of topological spaces are defined via equivalence relations. ■

We conclude this section with the following lemma which shows how to define homotopies on pushouts.

Lemma 14.9. (Homotopy Pushout Lemma) Let $f\colon X \to Y$ and $g\colon X \to Z$ be maps between topological space. As on page 259 we define the pushout

$$Y \cup_X Z \;:=\; (Y \sqcup Z)/\sim \qquad \text{where } f(x) \sim g(x) \text{ for all } x \in X.$$

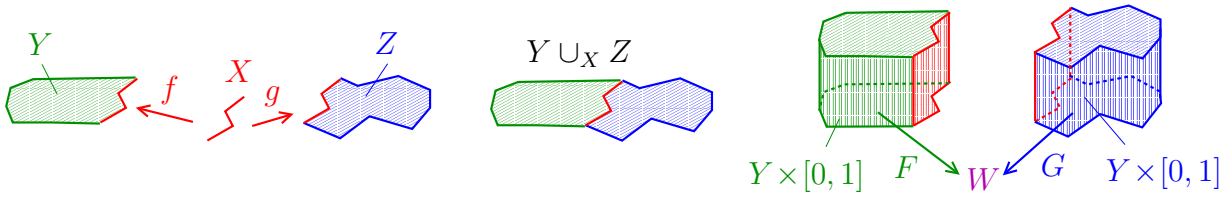
Furthermore let W be another topological space. Suppose that we are given two homotopies $G\colon Y \times [0,1] \to W$ and $H\colon Z \times [0,1] \to W$ such that for every $x \in X$ and every

[154] Since F is a homotopy rel A we see that the map is well-defined.

$t \in [0,1]$ we have $G(f(x),t) = H(g(x),t)$. Then the map

$$\begin{array}{rcl} F\colon (Y \cup_X Z) \times [0,1] & \to & W \\ ([P],t) & \mapsto & \begin{cases} G(P,t), & \text{if } P \in Y, \\ H(P,t), & \text{if } P \in Z \end{cases} \end{array}$$

is well-defined and it is continuous.



Proof. It follows from Exercise 5.3 that the map

$$\begin{array}{rcl} (Y \sqcup Z) \times [0,1] & \to & W \\ (P,t) & \mapsto & \begin{cases} G(P,t), & \text{if } P \in Y, \\ H(P,t), & \text{if } P \in Z \end{cases} \end{array}$$

is continuous. The statement now follows from the Homotopy Quotient Theorem 14.7 and the definition of the pushout as a quotient of $Y \sqcup Z$. ■

Exercises for Chapter 14.

Exercise 14.1. Let X be a topological space and let $f, g\colon X \to S^n$ be two maps such that for any $x \in X$ we have $\|f(x) - g(x)\| < 2$. Show that f and g are homotopic.

Exercise 14.2. Let Y be a path-connected space.

(a) Show that any two maps from an interval into Y are homotopic.

(b) Let $k \in \mathbb{N}_0$. Show that any two maps $\overline{B}^k \to Y$ are homotopic.

Exercise 14.3. Let X be a topological space and let $f, g\colon X \to S^1$ be maps. We assume that for any $x \in X$ we have $f(x) \neq -g(x) \in S^1$. Show that f and g are homotopic.
Hint. Consider the map $S^1 \to S^1$ that is given by $z \mapsto f(z) \cdot g(z)^{-1}$.

Exercise 14.4. Let X be a topological space and let $f\colon X \to \mathbb{R}^n \setminus \{0\}$ be a map. Show that f is homotopic to the map $X \to \mathbb{R}^n \setminus \{0\}$ given by $x \mapsto \frac{f(x)}{\|f(x)\|}$.

Exercise 14.5. Let $n \in \mathbb{N}$ and let $f\colon \overline{B}^n \to \overline{B}^n$ be a homeomorphism with $f|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$. We consider the map

$$\begin{array}{rcl} H\colon \overline{B}^n \times [0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \begin{cases} t \cdot f(\frac{x}{t}), & \text{if } 0 \leq \|x\| < t, \\ x, & \text{otherwise.} \end{cases} \end{array}$$

(a) Show that H is continuous.

(b) Show that for each $t \in [0,1]$ the map $H_t\colon \overline{B}^n \to \overline{B}^n$ is a homeomorphism.

Remark. The fact that f is isotopic to $\mathrm{id}_{\overline{B}^n}$ is often referred to as the *Alexander trick*.

Exercise 14.6. Let X and Y be topological spaces and let $F\colon X \times [0,1] \to Y$ be a homotopy. Show that there exists a homotopy $G\colon X \times [0,1] \to Y$ with the following properties:

(1) $G_0 = F_0$ and $G_1 = F_1$,
(2) there exists an $\epsilon \in (0,1)$ such that for each $t \in [0,\epsilon]$ we have $G_t = G_0$ and such that for each $t \in [1-\epsilon, 1]$ we have $G_t = G_1$
(3) For each $s \in [0,1]$ there exists a $t \in [0,1]$ with $G_s = F_t$.

Exercise 14.7.

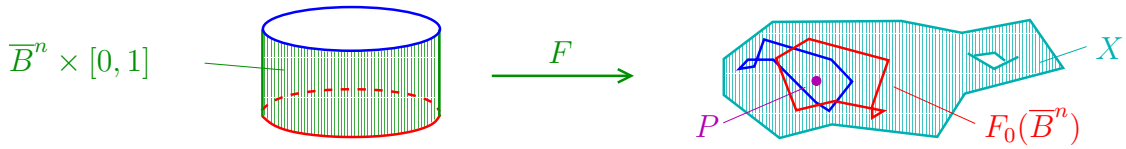
(a) Let $f\colon [0,1] \to [0,1]$ be a self-homeomorphism.
  (i) Show that f is monotonously increasing or monotonously decreasing.
  (ii) Suppose f is monotonously increasing. Show that f is isotopic to the map that is given by $g(x) = x$.
  (iii) Suppose f is monotonously decreasing. Show that f is isotopic to the map that is given by $g(x) = -x$.

(b) Show that any self-homeomorphism of S^1 is isotopic to the map $f(z) = z$ or to the map $g(z) = \overline{z}$.
Hint. Let $f\colon S^1 \to S^1$ be a self-homeomorphism. First consider the case that $f(1) = 1$.

Remark. We will provide a different, and arguably more satisfying, proof of Statement (a) in Exercise 48.24.

Exercise 14.8. Let X be a Hausdorff space, let $P \in X$ and let $F\colon \overline{B}^n \times [0,1] \to X$ be a map such that for every $t \in [0,1]$ the image $F_t(\overline{B}^n)$ is a neighborhood of P.

(a) Show that there exists a connected open neighborhood U of P such that for every $t \in [0,1]$ we have $F_t(S^{n-1}) \cap U = \varnothing$.
(b) Let U be as in (a). Show that for each $t \in [0,1]$ we have $U \subset F_t(B^n)$.



Exercise 14.9. Let h be a complex polynomial of degree ≥ 1. In Exercise 2.35 we showed that the map

$$\begin{array}{rcl} \Theta(h)\colon \mathbb{C} \cup \{\infty\} & \to & \mathbb{C} \cup \{\infty\} \\ z & \mapsto & \begin{cases} h(z), & \text{if } z \in \mathbb{C}, \\ \infty, & \text{if } z = \infty \end{cases} \end{array}$$

is continuous. Now let f and g be two polynomials with complex coefficients of degree ≥ 1. Suppose that $\deg(f) = \deg(g)$. Show that the maps $\Theta(f)$ and $\Theta(g)$ are homotopic.
Remark. It is quite easy to write down a candidate for a homotopy. But why is the map you wrote down actually continuous?

Exercise 14.10. Using Lemma 2.18 we can make the identification $S^1 = \mathbb{R} \cup \{\infty\}$. Now let f be a polynomial with real coefficients of degree $n \geq 1$. In Exercise 2.35 we showed that the map

$$\begin{array}{rcl} F\colon S^1 = \mathbb{R}\cup\{\infty\} & \to & \mathbb{R}\cup\{\infty\} = S^1 \\ t & \mapsto & \begin{cases} f(t), & \text{if } t\in\mathbb{R}, \\ \infty, & \text{if } t=\infty \end{cases} \end{array}$$

is continuous. (See the figure on page 188 for an illustration.) Show that F is homotopic to one of the following three types of self-maps of S^1:

$$(1)\qquad a(z) = z \qquad\qquad (2)\qquad b(z) = \overline{z} \qquad\qquad (3)\qquad c(z) = 1.$$

Exercise 14.11. Let X be a topological space and let Y be a topological space that is regionally compact. Let $f_0, f_1\colon Y \to X$ be two maps. Show that the following holds:

$$f_0 \text{ and } f_1 \text{ are homotopic} \iff f_0 \text{ and } f_1 \text{ lie in the same path component of } Y^X.$$

Remark. Use the Compact-Open Exponential Proposition 12.6.

Exercise 14.12. We consider the two maps $f, g\colon [0,1] \to \mathbb{R}^3$ shown in the figure below. Note that $f(0) = g(0)$ and $f(1) = g(1)$.

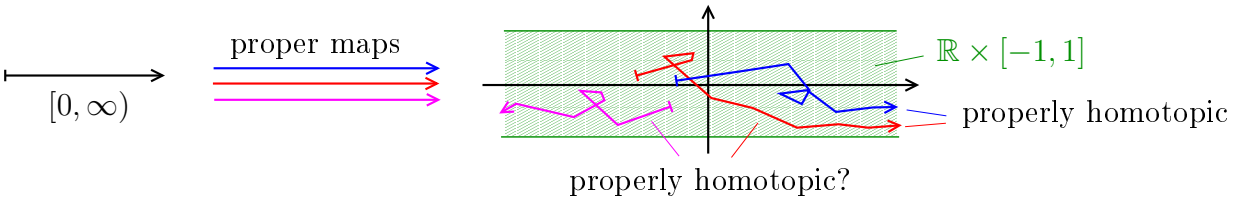
(a) Show that the maps f and g are homotopic rel $\{0\}$.
(b) Are the two maps f and g homotopic rel $\{0,1\}$?



Exercise 14.13. Recall that according to the definition on page 180 a map $f\colon A \to B$ between topological spaces is called *proper*, if the preimage of each compact subset of B is a compact subset of A. Let X and Y be topological spaces. Provide an example of a homotopy $F\colon X\times[0,1] \to Y$ such that each $F_t\colon X \to Y$ is proper but such that the map $F\colon X\times[0,1] \to Y$ is not proper itself.
Hint. You can find examples with $X = Y = \mathbb{R}$ and $F_0 = F_1 = \mathrm{id}$.

Exercise 14.14. Recall that according to the definition on page 180 a map $f\colon A \to B$ between topological spaces is called *proper*, if the preimage of each compact subset of B is a compact subset of A. Let X and Y be topological spaces. We say that two maps $f, g\colon X \to Y$ are *properly homotopic* if there exists a *proper homotopy* $F\colon X\times[0,1] \to Y$ between f and g, i.e. F is a homotopy between f and g and the map $F\colon X\times[0,1] \to Y$ is also proper.

(a) Show that any two proper maps $[0,\infty) \to \mathbb{R}^2$ are properly homotopic.
(b) Show that there exist proper maps $f, g\colon [0,\infty) \to \mathbb{R}\times[-1,1]$ that are not properly homotopic.



15. Homotopy equivalences and homotopy equivalent topological spaces

This chapter is a direct continuation of the previous chapter. In the last chapter we defined what it means for *maps* between topological spaces to be *homotopic*. Now in this chapter we will use this notion to define what it means for *topological spaces* to be *homotopy equivalent*. This notion is rather subtle and - at least for this author - not entirely intuitive.

15.1. Homotopy equivalent topological spaces.

For the reader's convenience we recall the following definition from page 418.

Definition. Let X and Y be two topological spaces. We say that two maps $f_0, f_1 \colon X \to Y$ are homotopic, written $f_0 \simeq f_1$, if there exists map $F \colon X \times [0,1] \to Y$ with $F_0 = f_0$ and with $F_1 = f_1$.

The following definition is one of the key definitions in topology. Unfortunately it usually takes some time to fully digest it.

Definition. Let X be and Y topological spaces.

(1) A map $f \colon X \to Y$ is a homotopy equivalence between X and Y, if f admits a homotopy inverse, i.e. a map $g \colon Y \to X$ with $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.
(2) If there exists a homotopy equivalence between X and Y, then we say that X and Y are homotopy equivalent.
(3) Given a topological space X we refer to the class of all topological spaces that are homotopy equivalent to X as the homotopy type of X.

Remark.

(1) Basically by definition a homeomorphism is a homotopy equivalence.
(2) In this chapter we study basic properties and basic examples of homotopy equivalences. Much much later we will state and prove the Whitehead Theorem 162.1 which in many cases gives a fairly practical criterion for showing that a given map is a homotopy equivalence.

The name "homotopy equivalence" already suggests that this is an equivalence relation on the class of topological spaces. This is indeed the case:

Lemma 15.1. (Homotopy-Equivalence Basics Lemma)

(1) Let $f \colon X \to Y$ and $g \colon Y \to Z$ be maps between topological spaces. If two out of the three maps f, g and $g \circ f$ are homotopy equivalences, then so is the third.
(2) The relation "homotopy equivalence" is an equivalence relation on the class of all topological spaces.[155]
(3) Let $f, g \colon X \to Y$ be two maps between topological spaces. If f is a homotopy equivalence and if g is homotopic to f, then g is also a homotopy equivalence.
(4) Let X, Y and Z be topological spaces.
   (a) Let $f \colon X \to Y$ and let $g_0, g_1 \colon Y \to Z$ be maps such that $g_0 \circ f \simeq g_1 \circ f$. If f is a homotopy equivalence, then $g_0 \simeq g_1$.
   (b) Let $f_0, f_1 \colon X \to Y$ and let $g \colon Y \to Z$ be maps such that $g \circ f_0 \simeq g \circ f_1$. If g is a homotopy equivalence, then $f_0 \simeq f_1$.

Proof.

(1) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be maps between topological spaces. There are three slightly different statements that need to be proved.

(a) First suppose that the maps $f\colon X \to Y$ and $g\colon Y \to Z$ admit homotopy inverses $f'\colon Y \to X$ and $g'\colon Z \to Y$. Then we see that

$$(f' \circ g') \circ (g \circ f) \;=\; f' \circ (g' \circ g) \circ f \;\underset{\uparrow}{\simeq}\; f' \circ \mathrm{id}_Y \circ f \;=\; f' \circ f \;\simeq\; \mathrm{id}_X\,.$$

by Lemma 14.5 (2) since $g' \circ g \simeq \mathrm{id}_Y$

Similarly one shows that $(g \circ f) \circ (f' \circ g') \simeq \mathrm{id}_Z$. Thus we have shown that $g \circ f$ is a homotopy equivalence.

(b) Next suppose that $f\colon X \to Y$ and $g \circ f\colon X \to Z$ admit homotopy inverses $f'\colon Y \to X$ and $h'\colon Z \to X$. Then we see that

$$(f \circ h') \circ g \;\underset{\uparrow}{\simeq}\; f \circ h' \circ (g \circ f) \circ f' \;\underset{\uparrow}{\simeq}\; f \circ \mathrm{id}_X \circ f' \;\simeq\; \mathrm{id}_Y\,.$$

by Lemma 14.5 (2) since $f \circ f' \simeq \mathrm{id}_Y$ — since h' is a homotopy inverse of $g \circ f$

Similarly one shows that $g \circ (f \circ h') \simeq \mathrm{id}_Z$. Thus we have shown that g is a homotopy equivalence.

(c) A slight variation on the argument of (1b) shows that if g and $g \circ f$ are homotopy equivalence, then so is f.

(2) It is clear that the only property of an equivalence relation that requires any thought is transitivity. But this follows from (1).

(3) We leave the elementary verification of this statement to the reader.

(4) The proof is very similar to the proof of (1). Again we leave the details to the reader. ■

Before we give examples of homotopy equivalent topological spaces we introduce the following definition.

Definition. Let X be a topological space. We say X is contractible, if X is homotopy equivalent to a topological space which consists of precisely one point.

Remark. Note that the empty topological space is *not* contractible.

The following elementary lemma gives a handy criterion for showing that a topological space is contractible.

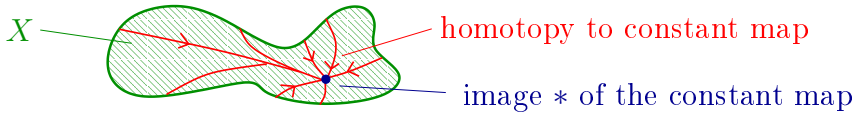
[155]The perspicacious reader might notice that here things get logically a little dicey. More precisely, on page 89 we had recalled the notion of an equivalence relation. But we had only defined the notion of an "equivalence relation" on a set and "all topological spaces" do not form a set they "only" form a class. One can also define the notion of an "equivalence relation" on a class, see e.g. [**FPr1985**, Kapitel 2]. But we do not want to get into the discussion "what is a class?". As a cheap way out we interpret the second statement of the lemma as saying that reflexivity, symmetry and transitivity hold for "homotopy equivalence" with the obvious interpretation.

Lemma 15.2. (Contractibility Criterion Lemma) Let X be a topological space.

(1) X is contractible $\iff$ (2) there exists an $x_0 \in X$ such that the inclusion $\{x_0\} \to X$ is a homotopy equivalence

$\iff$ (3) there exists an $x_0 \in X$ such that the identity map id_X is homotopic to the constant map c_{x_0}.

Furthermore, if statement (2) or (3) of the Contractibility Criterion Lemma 15.2 hold for a single point, then the statements also hold for all points in X.



Proof.

(1)⇒(2) First suppose that X is contractible. Let $Y = \{*\}$ be the topological space consisting of a single element. Since X is contractible we know that there exists a map $g\colon \{*\} \to X$ that is a homotopy equivalence. We set $x_0 := g(*)$. Basically by definition the inclusion map $\{x_0\} \to X$ is also a homotopy equivalence.

(2)⇒(3) Let $x_0 \in X$ be a point such that the inclusion map $f\colon \{x_0\} \to X$ is a homotopy equivalence. Let $g\colon X \to \{x_0\}$ be the unique map. Since f is a homotopy equivalence we know that $f \circ g$ is homotopic to id_X. But $f \circ g = c_{x_0}$. Thus we have shown that id_X is homotopic to c_{x_0}.

(3)⇒(1) Finally we suppose that the identity map of X is homotopic to a constant map $f\colon X \to X$. Let $*$ be the unique point in the image of $*$. We set $Y := \{*\}$. Let $g\colon \{*\} \to X$ be the inclusion map. By hypothesis we have $g \circ f \simeq \mathrm{id}_X$. Furthermore we evidently have $f \circ g = \mathrm{id}_Y$. This shows that X is homotopic equivalent to $Y = \{*\}$. ■

Finally the "furthermore statement" is proved in Exercise 15.10.

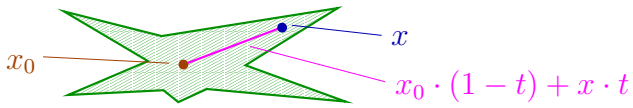
We give two basic examples of homotopy equivalent topological spaces.

Examples.

(1) We say a subset X of $\mathbb{R}^n$ is star-shaped if there exists a point $x_0 \in X$ such that for any $x \in X$ the segment from x_0 to x lies in X. Basically by the same argument as in the Convex Homotopy Lemma 14.1 we see that the map

$$\begin{array}{rcl} f\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & x_0 \cdot (1-t) + x \cdot t \end{array}$$

is a homotopy from id_X to a constant map. By the Contractibility Criterion Lemma 15.2 this shows that X is contractible.



(2) It follows from the previous example together with the fact that being homotopy equivalence is transitive, see the Homotopy-Equivalence Basics Lemma 15.1 (2), that for any $n \in \mathbb{N}$ and any $k \in \mathbb{N}$ the topological spaces $\mathbb{R}^n$ and $\mathbb{R}^k$ are homotopy equivalent.

We recall the following definition from page 167.

Definition. Let X be a topological space. We denote by $\pi_0(X)$ the set of path-components of X.

Lemma 15.3. (π_0-Homotopy Equivalence Lemma)

(1) Let $f\colon X \to Y$ be a map between topological spaces. If f is a homotopy equivalence, then f induces a bijection[156] $f_*\colon \pi_0(X) \to \pi_0(Y)$.
(2) If X and Y are homotopy equivalent topological spaces, then X is path-connected if and only if Y is path-connected.
(3) If X is a contractible topological space, then it is path-connected.

Proof. We leave the proof of the following lemma as an edifying exercise to the reader, see Exercise 15.12. ■

15.2. **Deformation retractions.** Before we give the definition of a deformation retraction let us give the definition of a retraction.

Definition. Let X be a topological space. We say a subset $A \subset X$ is a retract of X if there exists a retraction $r\colon X \to A$, i.e. a map with $r(a) = a$ for all $a \in A$.



retraction r from X to A

We immediately move on to the related notion of a deformation retraction.

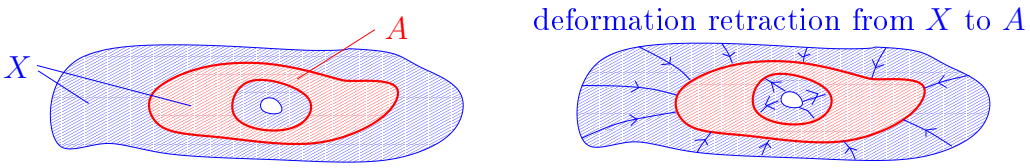
Definition. Let X be a topological space and let A be a subset.

(1) A homotopy $F\colon X \times [0,1] \to X$ is called a homotopy rel A, if $F(a,t) = a$ for all $a \in A$ and all $t \in [0,1]$.
(2) A deformation retraction from X to A is a homotopy $F\colon X \times [0,1] \to X$ which has the following three properties:[157]
   (a) $F_0 = \mathrm{id}_X$,
   (b) F is a homotopy rel A,
   (c) $F_1(X) \subset A$.
(3) We say A is a deformation retract of X if there exists a deformation retraction from X to A.[158]
(4) We say a retraction $r\colon X \to A$ is realized by a deformation retraction if there exists a deformation retraction F from X to A with $F_1 = r$.

We say a point $a \in X$ is a deformation retract of X if the subset $\{a\}$ if a deformation retract of X.

Examples.

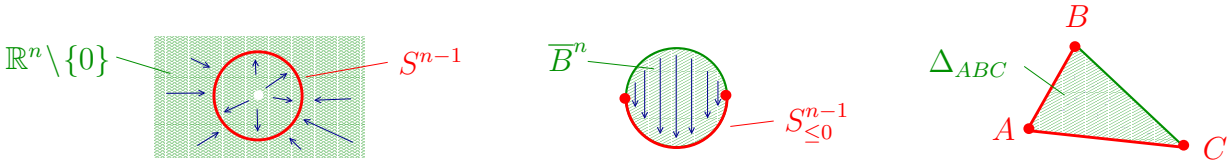
[156] On page 200 we saw that $f\colon X \to Y$ induces a map $f_*\colon \pi_0(X) \to \pi_0(Y)$.
[157] Note that $F_1\colon X \to A$ is retraction from X to A.
[158] Unfortunately in the literature the term "deformation retract" is used differently by different authors. Our notion of a "deformation retract" agrees with the definition in [**Hat2002**, p. 2] and [**Mun1984**, p. 208]. What we call a "deformation retract" is called a "strong deformation retract" in [**Bre1993**, p. 45]



(1) For any topological space X, any interval $I \subset \mathbb{R}$ and any $a \in I$ the subspace $X \times \{a\}$ is a deformation retract of $X \times I$. In fact the deformation retraction is given[159] by

$$\begin{array}{rcl} (X \times I) \times [0,1] & \to & X \times I \\ ((x,s),t) & \mapsto & (x, s \cdot (1-t) + a \cdot s). \end{array}$$

(2) Let $n \in \mathbb{N}$. In Exercise 15.13 we will prove the following two statements:
   (a) The sphere S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$.
   (b) The hemispheres $S^{n-1}_{\leq 0}$ is a deformation retract of $\overline{B}^n$.
   (c) Given any three non-collinear points $A, B, C \in \mathbb{R}^n$ the triangle Δ_{ABC} admits a deformation retraction to the union of any two sides of the triangle.



Lemma 15.4. (Iterated Deformation Retracts Lemma) Let X be a topological space and let $A \subset B \subset X$ be subspaces. If A is a deformation retract of B and if B is a deformation retract of X, then A is a deformation retract of X.

Proof. Let X be a topological space and let $A \subset B \subset X$ be subspaces. Furthermore let $F\colon B \times [0,1] \to B$ be a deformation retraction from B to A and let $G\colon X \times [0,1] \to X$ be a deformation retraction from X to B. We consider the map

$$\begin{array}{rcl} X \times [0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} G(x,2t), & \text{if } t \in [0,\frac{1}{2}], \\ F(G_1(x), 2t-1), & \text{if } t \in [\frac{1}{2},t]. \end{cases} \end{array}$$

It follows from the Homotopy Stacking Lemma 14.3 that this map is continuous. Once continuity is verified it is clear that this map is indeed a deformation retraction from X to A. ■

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and [**Rot1988**, p. 209]. In these books this notion is used to distinguish it from the notation of a "weak deformation retract", which means that there exists a map $F\colon X \times [0,1] \to X$ such that

(1) for every $x \in X$ we have $F(x,0) = x$ and $F(x,1) \in A$,
(2) for every $a \in A$ we have $F(a,1) = a$.

It turns out that the two concepts are indeed different. For example, let Z be the infinite zigzag comb which is shown below the Point Deformation Retract Lemma 15.6. the Point Deformation Retract Lemma 15.6 (2) says that every one-point subset of Z is a weak deformation retract of Z but the Point Deformation Retract Lemma 15.6 (3) says no one-point subset of Z is a strong deformation retract of X.

[159] It follows from the discussion on page 142 and the Product Topology-Basics Lemma 5.4 (3) that this map is actually continuous.

Lemma 15.5. (Deformation Retract-Homotopy Equivalence Lemma) Let A be a subset of a topological space X which admits a deformation retraction F from X to A. Then the inclusion map

$$\begin{array}{rcl} i\colon A & \to & X \\ a & \mapsto & a \end{array} \qquad \text{and the map} \qquad \begin{array}{rcl} r\colon X & \to & A \\ x & \mapsto & F(x,1) \end{array}$$

are homotopy inverses of one another. In particular both maps are homotopy equivalences and in particular the deformation retract A is homotopy equivalent to the total space X.

Proof. By definition of a deformation retraction the map $r \circ i = F_1 \circ i\colon A \to A$ is the identity. Thus it remains to show that $i \circ r\colon X \to X$ is homotopic to the identity. A homotopy between id and $i \circ r$ is given by F. ■

Example. It follows from the Deformation Retract-Homotopy Equivalence Lemma 15.5 and the previous example the sphere S^{n-1} and $\mathbb{R}^n \setminus \{0\}$ are homotopy equivalent.

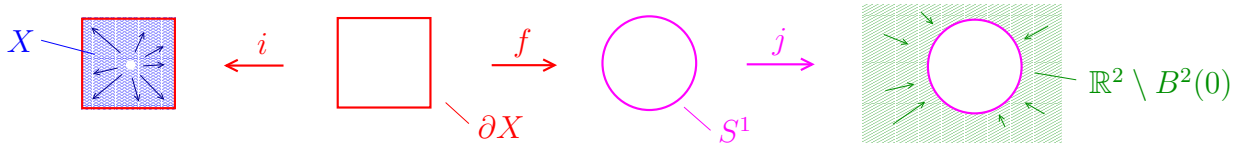
Remark. We now have three methods for showing that topological spaces are homotopy equivalent:

(1) Homeomorphisms are almost by definition homotopy equivalences.
(2) the Homotopy-Equivalence Basics Lemma 15.1 (2) says that being homotopy equivalent is an equivalence relation.
(3) the Deformation Retract-Homotopy Equivalence Lemma 15.5 says that if A is a deformation retract of some topological space X, then the inclusion $i\colon A \to X$ is a homotopy equivalence.

The combination of these three methods can be very powerful. For example let us consider $X = ([-1,1] \times [-1,1]) \setminus \{(0,0)\}$ and $Y = \mathbb{R} \setminus B^2(0)$. We consider the maps

$$X \xhookleftarrow{i} \partial X \xrightarrow[f]{x \mapsto \frac{x}{\|x\|}} S^1 \xhookrightarrow{j} \mathbb{R}^2 \setminus B^2(0).$$

As above one can easily show that ∂X is a deformation retract of X and that S^1 is a deformation retract of $\mathbb{R}^2 \setminus B^2(0)$. Furthermore f is a homeomorphism. It follows from the combination of (1), (2) and (3) that X is homotopy equivalent to $\mathbb{R}^2 \setminus B^2(0)$. In the Point



Deformation Retract Lemma 15.6 and much later in Exercise **??** we will see that it is in general not true that homotopy equivalent topological spaces admit deformation retracts that are homeomorphic. Nonetheless in Corollary 17.17 we will show that (1), (2) and (3) are in fact enough to characterize homotopy equivalence. More precisely we will show that given two topological X and Y the following two statements are equivalent:

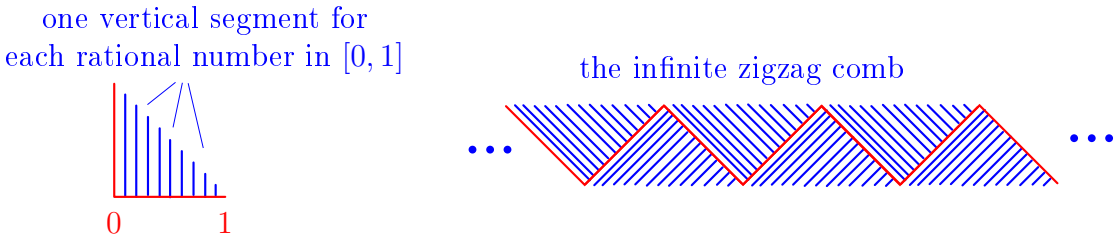
- X and Y are homotopy equivalent.

- There exists a topological space W and closed embeddings $i\colon X \to W$ and $j\colon Y \to W$ such that $i(X)$ is a deformation retract of W and $j(Y)$ is a deformation retract of W.

The notions of a topological space being contractible and admitting a deformation retraction to a point sound quite similar. The following lemma gives the precise relationship between these two notions.

Lemma 15.6. (Point Deformation Retract Lemma)

(1) If a topological space X admits a deformation retraction to a point $x_0 \in X$, then X is homotopy equivalent to $\{x_0\}$, in particular X is contractible.
(2) If a topological space X is contractible, then given any point $x_0 \in X$ there exists a homotopy $H\colon X \times [0,1] \to X$ such that $H_0 = \mathrm{id}_X$ and $H_1(x) = x_0$ for all $x_0 \in X$.
(3) The "infinite zigzag comb" shown in the figure below is contractible, but it does not admit a deformation retraction to a point.



Proof.

(1) This statement follows immediately from the Deformation Retract-Homotopy Equivalence Lemma 15.5.
(2) Suppose X is a topological space that is contractible. By the Contractibility Criterion Lemma 15.2 we know that there exists a homotopy $G\colon X \times [0,1] \to X$ from id_X to a constant map. Let x_1 be the image of this constant map. Recall that we know from the π_0-Homotopy Equivalence Lemma 15.3 that X is path-connected. Thus there exists a path $\gamma\colon [0,1] \to X$ from x_1 to x_0. The desired homotopy is now given by the map[160]
$$\begin{array}{rcl} H\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} G(x, 2\cdot t), & \text{if } t \in [0, \frac{1}{2}], \\ \gamma(2\cdot t - 1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$
(3) This statement will be proved in Exercise 15.2. ■

We conclude this discussion of deformation retractions with an extremely useful example that gets formulated as a proposition.

Proposition 15.7. (Matrix Groups Deformation Retract Proposition) Let $n \in \mathbb{N}_0$ and let $k \in \{1, \ldots, n\}$. We write
$$\begin{array}{rcl} \mathrm{GL}(n,k) & = & \text{set of } k\text{-tuples of linearly independent vectors of } \mathbb{R}^n \\ \mathrm{O}(n,k) & = & \text{set of } k\text{-tuples of orthonormal vectors of } \mathbb{R}^n. \end{array}$$

[160] It follows from the Homotopy Stacking Lemma 14.3 that this map is continuous.

(Note that for $k = n$ we have $\mathrm{GL}(n,k) = \mathrm{GL}(n)$ and $\mathrm{O}(n,k) = \mathrm{O}(n)$.) We denote by $r\colon \mathrm{GL}(n,k) \to \mathrm{O}(n,k)$ the map given by the Gram-Schmidt Orthonormalization Process 2.38. The following statements hold:

(1) The above retraction $r\colon \mathrm{GL}(n,k) \to \mathrm{O}(n,k)$ is realized by a deformation retraction $R\colon \mathrm{GL}(n,k) \times [0,1] \to \mathrm{O}(n,k)$ which has the following extra property:
($*$) For every k-dimensional vector subspace V of $\mathbb{R}^n$ and every $t \in [0,1]$ we have

$$R_t(\text{basis of } V) \;=\; \text{basis of } V.$$

(2) The inclusion $\mathrm{O}(n,k) \to \mathrm{GL}(n,k)$ is a homotopy equivalence.

The analogous statements hold if we replace the pair $\mathrm{O}(n,k)$ and $\mathrm{GL}(n,k)$ by one of the following four pairs:

(a)	$\mathrm{GL}_+(n,\mathbb{R})$	and	$\mathrm{SO}(n)$
(b)	$\mathrm{SL}(n,\mathbb{R})$	and	$\mathrm{SO}(n)$
(c)	$\mathrm{GL}(n,\mathbb{C})$	and	$\mathrm{U}(n)$
(d)	$\mathrm{SL}(n,\mathbb{C})$	and	$\mathrm{SO}(n)$.

Remark. In Proposition 33.10 we will see that the deformation retraction can be chosen to be "smooth".

Proof.

(1) On page 173, in the proof of the Matrix Group Path-Component Proposition 2.37, we implicitly already showed that each of the $2k$ steps of the Gram-Schmidt Orthogonalization Process is realized by a deformation retraction that satisfies ($*$). It follows from the Iterated Deformation Retracts Lemma 15.4 that we can combine these $2k$ deformation retractions to obtain the desired deformation retraction

$$R\colon \mathrm{GL}(n,k) \times [0,1] \;\to\; \mathrm{GL}(n,k)$$

from $\mathrm{GL}(n,k)$ to $\mathrm{O}(n,k)$ such that R that satisfies ($*$). If $n = k$, then it follows from the Gram-Schmidt Orthonormalization Process 2.38 (2) and (3) that this deformation retraction R restricts to a deformation retraction from $\mathrm{GL}_+(n,\mathbb{R})$ to $\mathrm{SO}(n)$ and from $\mathrm{SL}(n,\mathbb{R})$ to $\mathrm{SO}(n)$.

Finally note that the complex cases are treated in a very similar fashion.

(2) This statement follows immediately from (1) and the Deformation Retract-Homotopy Equivalence Lemma 15.5. ■

In the following we will see that deformation retractions descend to quotients under suitable hypotheses.

Theorem 15.8. (Deformation Retraction Quotient Theorem) Let X be a topological space and let $A \subset X$ be a subset. Furthermore let $F\colon X \times [0,1] \to X$ be a deformation retraction from X to A. Suppose that "$\sim$" is an equivalence relation on X which is compatible with the deformation retraction F in the sense that $F(x,t) \sim F(x',t)$ whenever $x \sim x'$. Then the map

$$\begin{array}{rcl} \overline{F}\colon (X/\sim) \times [0,1] & \to & X/\sim \\ ([x],t) & \mapsto & [F(x,t)] \end{array}$$

is a deformation retraction from $X/\sim$ to $A/\sim$.

Proof. It follows from the Homotopy Quotient Theorem 14.7 that $\overline{F}$ is continuous. It is now clear that $\overline{F}$ is a deformation retraction from $X/\sim$ to $A/\sim$. ■

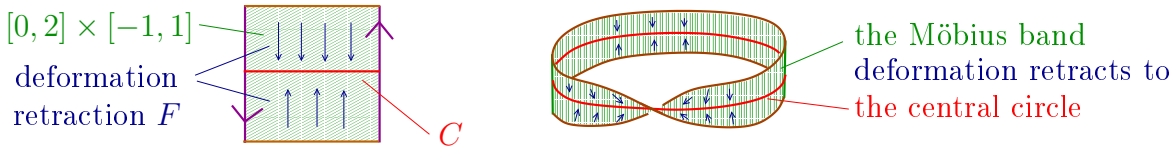
Example. We consider $X = [0,2] \times [-1,1]$ together with the equivalence relation "$\sim$" that is generated by

$$(0,y) \sim (1,-y) \quad \text{for all } y \in [-1,1].$$

Basically the same argument as on page 264 shows that we can view $X/\sim$ as the Möbius band. We consider the maps

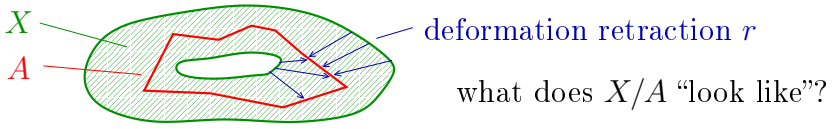
$$\begin{array}{rcl} F\colon X \times [0,1] & \to & X \\ ((x,y),t) & \mapsto & (x, y \cdot (1-t)) \end{array} \quad \text{and} \quad \begin{array}{rcl} \overline{F}\colon (X/\sim) \times [0,1] & \to & X/\sim \\ ([(x,y)],t) & \mapsto & [(x, y \cdot (1-t))]. \end{array}$$

Note that F is a deformation retraction from X to $C := [0,2] \times \{0\}$. One can easily verify that F is compatible with the equivalence relation "$\sim$". Thus we obtain from the Deformation Retraction Quotient Theorem 15.8 that $\overline{H}$ is a deformation retraction from the Möbius band to $C/\sim$. Note that $C/\sim$ is homeomorphic to $[0,2]/0 \sim 2$ which is in turn homeomorphic to S^1. It follows from this discussion together with the Deformation Retract-Homotopy Equivalence Lemma 15.5 that the Möbius band is homotopy equivalent to S^1.



We conclude our discussion of deformation retracts with the following proposition which might sound innocuous, but whose proof relies eventually on the tricky Whitehead Quotient Theorem 13.4.

Proposition 15.9. (Quotient Deformation Retract Proposition) Let X be a topological space and let A be a subset of X. If A is a deformation retract of X, then the point A/A is a deformation retract of X/A. In particular X/A is contractible.



Proof. Let $r\colon X \times [0,1] \to X$ be a deformation retraction from X to A. We consider the equivalence relation "$\sim$" on X that is generated by the condition that all points in A are equivalent. Note that by definition of a deformation retraction the homotopy r is compatible with the equivalence relation "$\sim$". Therefore it follows from the Deformation Retraction Quotient Theorem 15.8 that the map

$$\begin{array}{rcl} (X/\sim) \times [0,1] & \to & X/\sim \\ ([x],t) & \mapsto & [r(x,t)]. \end{array}$$

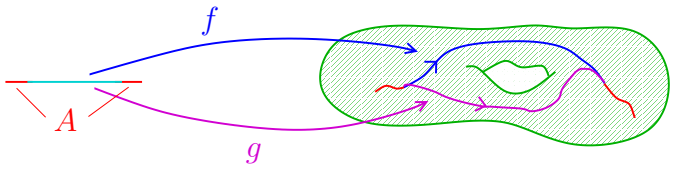
is a deformation retraction from $X/\sim = X/A$ to $A/\sim = A/A$. ■

15.3. **Homotopies and homotopy equivalences for pairs of topological spaces.** On page 418 we defined what it means for two maps $f, g\colon X \to Y$ between topological spaces to be homotopic and on page 430 we defined what it means for two topological spaces to be homotopy equivalent.

On many occasions one would like to restrict the freedom of a homotopy by fixing some subset pointwise or setwise. This idea leads us to two relative versions of "homotopy". The first version we already saw on page 419. For the reader's convenience we recall the definition.

Definition. Let $f, g\colon X \to Y$ be two maps between topological spaces and let $A \subset X$ be a subset such that $f|_A = g|_A$.

(1) A homotopy between f and g rel A is a homotopy $H\colon X \times [0,1] \to Y$ between f and g such that $H(a,t) = f(a) = g(a)$ for all $t \in [0,1]$ and all $a \in A$.
(2) We say f and g are homotopic rel A, if such a homotopy rel A exists.



Before we continue it is convenient to introduce the following harmless definition which is an analogue of the category of pairs of sets that we had introduced on page 196.

Definition.

(1) A pair of topological spaces is a pair (X, A), where X is a topological space and A is a subset of X.
(2) A map $f\colon (X, A) \to (Y, B)$ between pairs of topological spaces is a map $f\colon X \to Y$ of topological spaces with $f(A) \subset B$.
(3) We refer to the category $\mathcal{PairTop}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{PairTop}) &:= \text{all pairs of topological spaces,} \\ \mathrm{Mor}((X,A),(Y,B)) &:= \text{all continuous maps } f \text{ from } X \text{ to } Y \text{ with } f(A) \subset B \end{aligned}$$

with the usual composition of maps as the category of pairs of topological spaces.

We continue with another notion of a "relative homotopy".

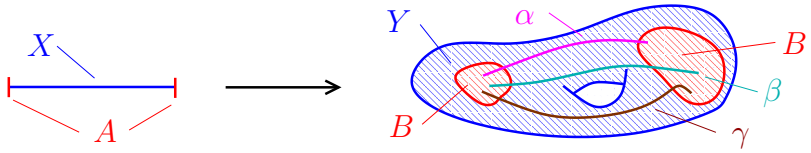
Definition. Let X and Y be topological spaces and let $A \subset X$ and $B \subset Y$ be subsets. Let $f, g\colon (X, A) \to (Y, B)$ be two maps of pairs of topological spaces.

(1) A homotopy between f and g is a map[161]

$$H\colon X \times [0,1] \to Y,$$

such that the following two conditions are satisfied:
(a) $H_0 = f$ and $H_1 = g$, and
(b) $H(a,t) \in B$ for all $a \in A$ and all $t \in [0,1]$.
(2) If there exists a homotopy between f and g, then we say that f and g are homotopic and we write $f \simeq g$.

[161] For $A = B = \varnothing$ we get the usual notion of a homotopy between maps $f, g\colon X \to Y$.



$\alpha, \beta\colon (X, A) \to (Y, B)$ are homotopic but $\alpha, \gamma\colon (X, A) \to (Y, B)$ are not homotopic

For completeness we also mention the following definition which is the obvious relative analogue of the concepts that we introduced on page 430.

Definition. Let (X, A) and (Y, B) be two pairs of topological spaces.

(1) A map $f\colon (X, A) \to (Y, B)$ is a homotopy equivalence between (X, A) and (Y, B), if there exists a map $g\colon (Y, B) \to (X, A)$ such that the following hold:
   (a) $g \circ f$ and id_X are homotopic as maps of (X, A) to itself,
   (b) $f \circ g$ and id_Y are homotopic as maps of (Y, B) to itself.

(2) If there exists a homotopy equivalence between (X, A) and (Y, B), then we say that (X, A) and (Y, B) are homotopy equivalent.

Examples.

(1) The two pairs of topological spaces (X, A) and (Y, B) shown in the figure below are homotopy equivalent.

(2) Let X be a topological space and let $A \subset B$ be subsets of X. If A is a deformation retract of B, then the inclusion map $(X, A) \to (X, B)$ is easily seen to be a homotopy equivalence of pairs of topological spaces.

(3) Let $A \subset Y$ be subsets of X. If Y is a deformation retract of X, then the inclusion map $(Y, A) \to (X, A)$ is again easily seen to be a homotopy equivalence of pairs of topological spaces.



The above notions can be generalized even further. For example, one could define triples of topological spaces and there is an obvious definition of what it means for maps $f, g\colon (X, A, B) \to (Y, C, D)$ between triples of topological spaces to be homotopic. Similarly, given pairs of topological spaces (X, A) and (Y, B) and a subset Z of A there is an obvious definition for what it means for two maps $f, g\colon (X, A) \to (Y, B)$ to be homotopic rel Z. We will not spell out all the definitions. From the context it should always be clear what we mean.

15.4. **The set of homotopy classes of maps between topological spaces.** In this short section we study the set of homotopy classes of maps between topological spaces. This set will play a crucial much much later in Chapter 164. But at first reading one can still safely ignore this section.

Definition. Given two topological spaces X and Y we define

$$[X,Y] := \{\text{maps } X \to Y\}/\sim \quad \text{where } f \sim g \text{ if there exists a homotopy between } f \text{ and } g.$$

Sometimes, given a map $f\colon X \to Y$ we denote by $[f]$ the equivalence class it represents in $[X,Y]$.

Example. We write $I = [0,1]$. In Exercise 15.1 we will see that for any path-connected topological space X the set $[I,X]$ consists of a single element, i.e. all maps $I = [0,1] \to X$ are homotopic.

The following lemma is an immediate consequence of Lemma 14.6.

Lemma 15.10. Let X, Y and Z be topological spaces. The map

$$\begin{array}{rcl} [X,Y] \times [Y,Z] & \to & [X,Z] \\ ([f],[g]) & \mapsto & [g \circ f] \end{array}$$

is well-defined.

Lemma 15.10 leads us to the following definition.

Definition. We define the homotopy category $\mathcal{HomTop}$ of topological spaces by[162]

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{HomTop}) & := & \text{all topological spaces,} \\ \mathrm{Mor}(X,Y) & := & [X,Y] \end{array}$$

with the composition

$$\begin{array}{rcl} \mathrm{Mor}(X,Y) \times \mathrm{Mor}(Y,Z) & \to & \mathrm{Mor}(X,Z) \\ ([f],[g]) & \mapsto & [g \circ f]. \end{array}$$

We continue with the following harmless definition.

Definition. Let $f\colon X \to Y$ be a map between topological spaces. A map $g\colon Y \to X$ is called a homotopy left inverse of f if $g \circ f$ is homotopic to id_X. Furthermore, a map $h\colon Y \to X$ is called a homotopy right inverse of f if $f \circ h$ is homotopic to id_Y.

Taking the categorical view point now allows us to prove the following lemma by a completely approach.

Lemma 15.11. Let $f\colon X \to Y$ be a map between topological spaces. If f admits a homotopy left inverse g and a homotopy right inverse h, then f is in fact a homotopy equivalence and both g and h are homotopy inverses.

Proof. This lemma follows immediately from applying Lemma 3.3 to the homotopy category $\mathcal{HomTop}$. ∎

We conclude this section with the following elementary but useful lemma.

Lemma 15.12.

(1) Let $f\colon X \to Y$ be a map between topological spaces. The element $[f] \in \mathrm{Mor}(X,Y)$ is an isomorphism[163] in the category $\mathcal{HomTop}$ if and only if f is a homotopy equivalence.

[162] Note that in this category the morphisms are not maps but they are equivalence classes of maps.

(2) Let $f\colon X_1 \to X_2$ be a map between topological spaces and let A be a topological space. If f is a homotopy equivalence, then the induced maps

$$\begin{array}{rcl} [X_2, A] & \to & [X_1, A] \\ [g] & \mapsto & [g \circ f] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} [A, X_1] & \to & [A, X_2] \\ [g] & \mapsto & [f \circ g] \end{array}$$

are bijections.

Proof.

(1) This statement is a tautology, it follows immediately from the definitions.
(2) This statement follows immediately from (1) together with Lemma 3.4. ■

Exercises for Chapter 15.

Exercise 15.1. Let X be a topological space and let Y be a contractible topological space.

(a) Show that all maps $X \to Y$ are homotopic.
(b) We suppose that X is path-connected. Show that all maps $Y \to X$ are homotopic.

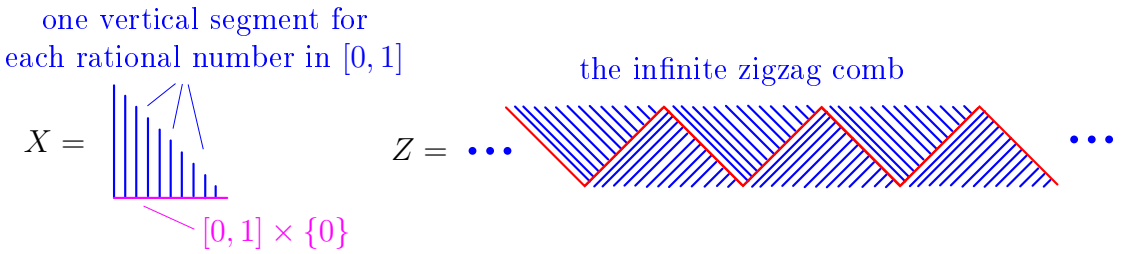
Exercise 15.2.

(a) We consider the topological space

$$X \;=\; \{(x,y) \,|\, x \in [0,1] \cap \mathbb{Q}, y \in [0, 1-x]\} \cup ([0,1] \times \{0\})$$

that is shown in the figure below to the left.
  (i) Show that given any point P in $[0,1] \times \{0\}$ there exists a deformation retraction from X to P.
  (ii) Show that for any point Q in X that does not lie on $[0,1] \times \{0\}$ there is no deformation retraction from X to Q.

(b) We consider the infinite zigzag comb Z that is shown in the figure below to the right.
  (i) Show that Z is contractible.
  (ii) Show that Z does not admit a deformation retraction to any point in Z.



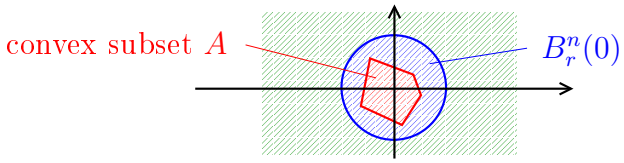
Exercise 15.3. Let $X = \{0,1\}$ be the Sierpiński space, i.e. let X be the topological space with two elements 0 and 1 and where the open sets are precisely $\varnothing, \{0\}, X$.

(a) Is $\{0\}$ a deformation retract of X?
(b) Is $\{1\}$ a deformation retract of X?

Exercise 15.4. Let $0 < r < s$ be two real numbers. Show that the obvious inclusion map $B_r^n(0) \setminus \{0\} \to B_s^n(0) \setminus \{0\}$ is a homotopy equivalence.

[163] We refer to page 198 for the definition of an isomorphism in a category.

Exercise 15.5. Let A be a convex subset of $\mathbb{R}^n$ which contains the origin 0. Let $r > 0$ such that $A \subset B_r^n(0)$. Show that $\mathbb{R}^n \setminus B_r^n(0)$ is a deformation retract of $\mathbb{R}^n \setminus A$.
Remark. This exercise is a generalization of Exercise 15.4.



Exercise 15.6. Let $T = (V, E, i, t)$ be a tree in the sense of the definition on page 315. As on page 308 we consider the corresponding topological realization $|T|$. Show that the topological realization $|T|$ is contractible.
Remark. You could make use of Lemma 7.7. You should also explain carefully why the homotopies you write down are actually continuous.

Exercise 15.7. Let X be a topological space and let $H\colon X \times [0,1] \to X$ be a deformation retraction to a subset A. We consider

$$\begin{array}{rcl} \phi\colon X & \to & [0,1] \\ x & \mapsto & \inf\{t \in [0,1] \,|\, H(x,t) \in A\}. \end{array}$$

Is the map ϕ necessarily continuous?

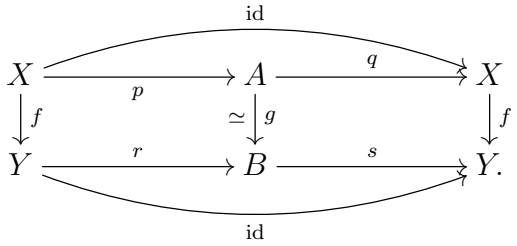
Exercise 15.8. Show that the topological spaces

$$X = \{\tfrac{1}{n} \,|\, n \in \mathbb{N}\} \cup \{0\} \quad \text{and} \quad Y = \{\tfrac{1}{n} \,|\, n \in \mathbb{N}\} \cup \{-1\}$$

are not homotopy equivalent.



Exercise 15.9. Let $f\colon X \to Y$ and $g\colon A \to B$ be maps between topological spaces. We say *f is a retract of g* if there exists a commutative diagram of maps



such that $q \circ p = \mathrm{id}_X$ and $s \circ r = \mathrm{id}_Y$. Show that if g is a homotopy equivalence, then f is also a homotopy equivalence.
Hint. Use Lemma 15.11.

Exercise 15.10. Let X be a topological space.

(a) Suppose that there exists an $x_0 \in X$ such that the inclusion $\{x_0\} \to X$ is a homotopy equivalence. Show that for every $x_1 \in X$ the inclusion $\{x_1\} \to X$ is a homotopy equivalence.
(b) Suppose that there exists an $x_0 \in X$ such that the identity map id_X is homotopic to the constant map c_{x_0}. Show that for every $x_1 \in X$ the identity map id_X is homotopic to the constant map c_{x_1}.

Exercise 15.11. Does there exist a topological space with three elements that is homotopy equivalent to the topology space $X = \{*\}$ given by a single point?

Exercise 15.12. Let $f\colon X \to Y$ be a map between topological spaces. Show that if f is a homotopy equivalence, then f induces a bijection $\pi_0(X) \to \pi_0(Y)$.[164]

Exercise 15.13. Let $n \in \mathbb{N}$. Prove the following three statements:

(a) The sphere S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$.
(b) The hemispheres $S^{n-1}_{\leq 0}$ is a deformation retract of $\overline{B}^n$.
(c) Given any three non-collinear points $A, B, C \in \mathbb{R}^n$ the triangle Δ_{ABC} admits a deformation retraction to the union of any two sides of the triangle.
Hint. You could first consider the case that $A = (0,0)$, $B = (0,1)$ and $C = (1,0)$.

These three examples are illustrated in the figure on page 434.

Exercise 15.14. Let

$$S \;:=\; \big\{(0,y)\,\big|\,y \in [-1,1]\big\} \cup \big\{(x,\sin(\tfrac{1}{x}))\,\big|\,x \in (0,\tfrac{1}{\pi}]\big\} \;\subset\; \mathbb{R}^2.$$

be the "Topologist's Sine Curve" from page 164. As on page 276 we refer to the quotient space $W := S/(0,0) \sim (\frac{1}{\pi},0)$ as the *Warsaw Circle*. Our goal is to show that the circle is not contractible. To do so we introduce the following notation:

- We denote by $p\colon S \to W$ the projection .
- We set $Q = p(0,0)$ and given $n \in \mathbb{N}$ we set $Q_n = p((\frac{1}{\pi n},0))$.
- We set $L := \{p(0,y)\,|\,y \in [-1,1]\} \subset W$ and $U := \big\{p(x,\sin(\frac{1}{x}))\,\big|\,x \in (0,\frac{1}{2\pi}]\big\} \subset W$.

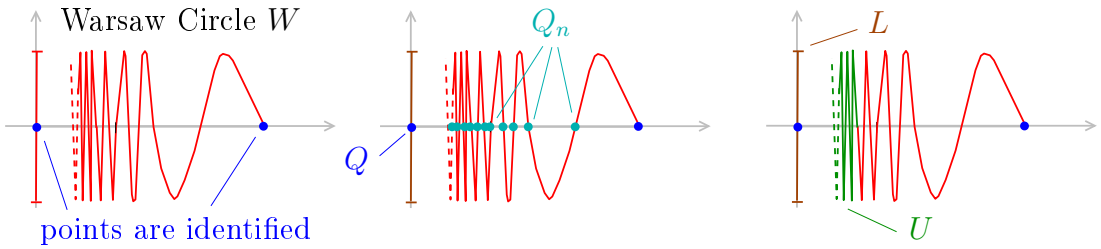
Now go through the following steps:

(a) Let $g\colon W \to W$ be a map. We suppose that there exists an $n \in \mathbb{N}$ such that for all $m \geq n$ we have $g(Q_n) \in U$. Show that $g(Q) \in L \cup U$.
(b) Let $F\colon W \times [0,1] \to W$ be a homotopy. Show that

$$T \;:=\; \left\{ t \in [0,1] \,\middle|\, \begin{array}{c} F_t(Q) \in L \text{ and there exists an } n \in \mathbb{N} \text{ such} \\ \text{that for all } m \geq n \text{ we have } F_t(Q_m) \in U \end{array} \right\} \;\subset\; [0,1]$$

is open and closed.
(c) Show that W is not contractible.
Hint. Suppose that W is contractible. By the Contractibility Criterion Lemma 15.2 this means that there exists a homotopy $F\colon W \times [0,1] \to W$ with $F_0 = \mathrm{id}$ and such that $F_1(w) = Q$ for all $w \in W$. Use (b) to show that such F cannot exist.

[164] Recall that given a topological space Z we denote by $\pi_0(Z)$ the set of path-components of Z.



Exercise 15.15. We consider $X = [-1,1]/-1 \sim 1$ and we denote by $p\colon [-1,1] \to X$ the projection . We consider the two subspaces $A = p((-1,1)) \subset X$ and $B = p(\{0\}) \subset X$. Are the two pairs (X,A) and (X,B) of topological spaces homotopy equivalent?

Exercise 15.16. Let $U \subset \mathbb{R}^n$ be a neighborhood of 0 and let $f\colon U \to \mathbb{R}^n$ be a map with $f^{-1}(0) = \{0\}$. Furthermore let $g\colon U \to \mathbb{R}^n$ be a map with $f(x) - g(x) = O(\|x\|^2)$, i.e. we assume that there exists a $C \geq 0$ and a $\delta > 0$ such that $\|f(x) - g(x)\| < C \cdot \|x\|^2$ for all $x \in \overline{B}^n_\delta$. Show that there exists an $s > 0$ such that the maps f and g are homotopic as maps of pairs $(\overline{B}^n_s, \overline{B}^n_s \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$.

16. Joins, mapping cones, cylinders and tori

In this section we first introduce two systematic ways to construct new topological spaces out of a given topological space and we present three systematic ways to construct new topological spaces out of a map $f\colon X \to Y$ between given topological spaces. Initially it might not be entirely apparent why we should be interested, but all five examples will play an important role at various points.

For example we will associate to a map $f\colon X \to Y$ between topological spaces a topological space $\mathrm{Cyl}(f\colon X \to Y)$, called the *mapping cylinder of* $f\colon X \to Y$. The mapping cylinder translates questions about the map $f\colon X \to Y$ to questions about the topological space $\mathrm{Cyl}(f\colon X \to Y)$.

16.1. **The join of two topological spaces.** In this section we introduce the join of two topological spaces. This construction will be useful on a few occasions, but it is difficult to argue that at this stage this is an essential topic in these notes.

Loosely speaking the join $X * Y$ is obtained by connecting each point in X to each point in Y via an interval. The following definition makes this idea precise.

Definition. Let X and Y be two non-empty topological spaces. We define the join $X * Y$ to be the topological space obtained from $X \times [0,1] \times Y$, equipped with the product topology, by performing the following two types of identifications:

(1) for every $x \in X$ we identify all points in $\{x\} \times \{0\} \times Y$ to a single point and
(2) for every $y \in Y$ we identify all points in $X \times \{1\} \times \{y\}$ to a single point.

Furthermore, if one of X or Y is the empty topological space, then we define the join $X * Y$ to be the other topological space.

For completeness we state the following lemma.

Lemma 16.1. Let X_1, Y_1, X_2, Y_2 be four topological spaces and let $f\colon X_1 \to X_2$ and $g\colon Y_1 \to Y_2$ be maps. The corresponding map

$$\begin{array}{rcl} f * g\colon X_1 * Y_1 & \to & X_2 * Y_2 \\ [(x,t,y)] & \mapsto & [(f(x),t,g(y))] \end{array}$$

is well-defined and continuous.

Proof. It is straightforward to verify that the map $f * g$ is well-defined. It follows easily from the Topological-Quotient Proposition 5.15 (1b), the Topological-Quotient Proposition 5.15 (1a) and the Product Topology-Basics Lemma 5.4 (3) that the map $f * g$ is continuous. ∎

Before we deal with a few examples we give a criterion for showing that a map of the form $f\colon X * Y \to Z$ is a homeomorphism.

Lemma 16.2. If X and Y are compact spaces, then the following two statements hold:

(1) The join $X * Y$ is compact.
(2) Let $f\colon X \times [0,1] \times Y \to Z$ be a map to a topological space Z such that for any $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$ we have the equalities $f(x,0,y_1) = f(x,0,y_2)$ and $f(x_1,1,y) = f(x_2,1,y)$. If Z is Hausdorff and if the induced map[165]

$$\begin{array}{rcl} \Psi\colon X * Y & \to & Z \\ [(x,t,y)] & \mapsto & f(x,y,t) \end{array}$$

is a bijection, then the map Ψ is a homeomorphism.

Proof.

(1) It follows from our hypothesis and the Product Topology Properties Proposition 5.6 together with the Quotient-Compact Connected Lemma 5.16 (1) that the join $X*Y = (X \times [0,1] \times Y)/\sim$ is compact.
(2) The map $\Psi\colon X*Y \to Z$ is continuous by the Topological-Quotient Proposition 5.15 (1b). Thus we obtain from our hypotheses on Ψ and Z together with the Compact-Hausdorff Proposition 2.17 (3) that Ψ is indeed a homeomorphism. ■

The following, otherwise broadly irrelevant lemma, justifies somewhat the idea that the join $X*Y$ is obtained by connecting each point in X to each point in Y via an interval.

Lemma 16.3. Given $P, Q \in \mathbb{R}^n$ we denote by $\overline{PQ} = \{x\cdot(1-t)+y\cdot t \,|\, t\in[0,1]\}$ the Euclidean segment determined by P and Q. Let X and Y be two compact subsets of $\mathbb{R}^n$ which have the property that for any x, x' and y, y' the segments $\overline{xy}$ and $\overline{x'y'}$ intersect at most in the endpoints. Then the map

$$\begin{array}{rcl} \Theta\colon X*Y=(X\times[0,1]\times Y)/\sim & \to & \bigcup\limits_{x\in X, y\in Y}\overline{xy} \\ [(x,t,y)] & \mapsto & x\cdot(1-t)+y\cdot t \end{array}$$

is a homeomorphism.

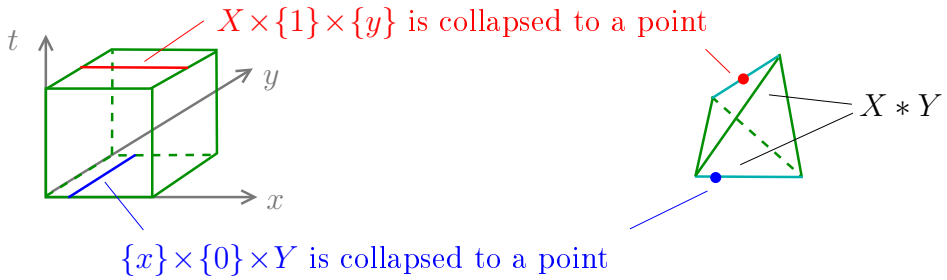
Proof. It follows almost immediately from Lemma 16.2 and our hypothesis that the map Θ is a homeomorphism. ■

We continue with another example of joins.

Example. We consider the join of the two intervals $X=[0,1]$ and $Y=[0,1]$. The figure below suggests that the join $X*Y=[0,1]*[0,1]$ should be homeomorphic to a 3-dimensional pyramid. In fact let us consider the map

$$\begin{array}{rcl} \overbrace{([0,1]\times[0,1]\times[0,1])/\sim}^{=[0,1]*[0,1]} & \to & \Delta=\{(t_1,t_2,t_3)\in[0,1]^3 \,|\, t_1+t_2+t_3\le 1\} \\ [(x,t,y)] & \mapsto & \frac{1}{2}\cdot\big(x\cdot t, y\cdot(1-t), t\big). \end{array}$$

It follows easily from Lemma 16.2 (2) that this map is a homeomorphism.



We record the next example of joins as a lemma.

[165] It follows from our hypothesis on f that the map f descends to a map on $X*Y=(X\times[0,1]\times Y)/\sim$.

Lemma 16.4. (Join of Spheres Lemma) Given any $m, n \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} S^m * S^n & \to & S^{m+n+1} \\ [(x,t,y)] & \mapsto & \underbrace{\left(x \cdot \cos\left(\frac{\pi t}{2}\right), y \cdot \sin\left(\frac{\pi t}{2}\right)\right)}_{\in \mathbb{R}^{m+1} \times \mathbb{R}^{n+1} = \mathbb{R}^{m+n+2}} \end{array}$$

is a homeomorphism.

Proof. Using Lemma 16.2 (2) one can fairly easily verify that the given map is a homeomorphism. ■

Convention. Let $m, n \in \mathbb{N}_0$. We use the homeomorphism from the Join of Spheres Lemma 16.4 to add the identification $S^m * S^n = S^{m+n+1}$ to our long list of avatars of the spheres.

For completeness we state the following fairly straightforward lemma.

Lemma 16.5. Let X and Y be topological spaces. If X and Y are both Hausdorff, then the join $X * Y$ is also Hausdorff.

Proof. The proof will be given in Exercise 16.2. ■

Finally we consider the delicate question whether the join is associative.

Proposition 16.6. (Join Associativity Proposition) Let X, Y and Z be non-empty topological spaces that are compact and Hausdorff. Then there exists a homeomorphism

$$(X * Y) * Z \xrightarrow{\cong} X * (Y * Z).$$

Remark.

(1) The join $X * Y$, viewed as a set, can be equipped with another reasonable topology, namely the "initial topology", see [**BrownR2006**, Chapter 5.7]. By [**BrownR2006**, Corollary 5.7.2] these two topologies agree if X and Y are compact and Hausdorff. When dealing with joins it seems that it is safest to stick to the case that X and Y are compact and Hausdorff. For example the Join Associativity Proposition 16.6 says that in this context the join is associative. But this statement is not true for all topological spaces. We refer to [**CoheD1957**], [**BrownR2006**, Chapter 5.7] and [**FF2016**, Chapter 2.4] for details.[166][167]

(2) Let $X_1, \ldots, X_k$ be finitely many topological spaces that are non-empty compact and Hausdorff. It follows from Join Associativity Proposition 16.6 and the discussion on page 98 that we can unambiguously define the join $X_1 * \cdots * X_k$.

Proof. We define the join $X * Y * Z$ as follows: we consider

$$\{(r,x,s,y,t,z) \in [0,1] \times X \times [0,1] \times Y \times [0,1] \times Z \,|\, r+s+t = 1\}/\sim$$

where "$\sim$" is generated by the following relations:

- $(0,x,s,y,t,z) \sim (0,x',s,y,t,z)$,
- $(r,x,0,y,t,z) \sim (r,x,0,y',t,z)$,
- $(r,x,s,y,0,z) \sim (r,x,s,y,0,z')$.

[166] As we will see in Lemma 13.2, products and quotients of topological spaces do not mix well. It is easy to write down "continuous looking" maps that are not continuous.

[167] In light of Lemma 13.2 it is perhaps not too shocking that the join is not associative.

In the following we will use "$\sim$" with several means, we leave it to the reader to figure out any given moment what is meant by this symbol.

Claim. The map

$$\begin{array}{rccc} \Theta\colon & \overbrace{(((X\times[0,1]\times Y)/\!\sim)\times[0,1]\times Z)/\!\sim}^{=(X*Y)*Z} & \to & \overbrace{(X\times[0,1]\times Y*[0,1]\times Z)/\!\sim}^{=X*Y*Z} \\ & [([(x,s,y)],t,z)] & \mapsto & [((1-t)\cdot(1-s),x,(1-t)s,y,t,z)] \end{array}$$

is a homeomorphism.

Proof. We consider the following diagram

$$\begin{array}{ccc} (X\times[0,1]\times Y)\times([0,1]\times Z) & \xrightarrow{\text{id}} & (X\times[0,1]\times Y)\times([0,1]\times Z) \\ \downarrow & & \downarrow \\ ((X\times[0,1]\times Y)\times([0,1]\times Z)/\!\sim & & \\ {\scriptstyle ([u],v)\mapsto[(u,v)]}\uparrow\varphi & \overset{\Phi}{\searrow} & \\ ((X\times[0,1]\times Y)/\!\sim\times([0,1]\times Z) & \xrightarrow{\Psi} & X*Y*Z. \\ \downarrow & \overset{\Theta}{\nearrow} & \\ (((X\times[0,1]\times Y)/\!\sim)\times[0,1]\times Z)/\!\sim & & \end{array}$$

We make the following clarifications and observations:

(1) All the maps are the obvious (projection) maps, except for φ which is the map given by Lemma 13.2. The diagram commutes basically by definition.
(2) It follows from the Twice Quotient Lemma 5.23 that Φ is continuous.
(3) Our hypothesis that Z is Hausdorff and compact implies by the Product Topology Properties Proposition 5.6 together with Lemma 2.41 that $[0,1]\times Z$ is regionally compact. Thus we obtain from the Product-Quotient Mixing Lemma 13.6 that φ is actually a homeomorphism.
(4) It follows from (2) and (3) that Ψ is continuous.
(5) It follows from (4) together with the Topological-Quotient Proposition 5.15 (1b) that Θ is continuous.
(6) It follows easily from the definitions that Θ is a bijection.
(7) By our hypotheses on X, Y and Z we obtain from the Product Topology Properties Proposition 5.6 and the Compact Image Lemma 2.13 that the domain $(X*Y)*Z$ of the map Θ is compact. Furthermore , by an argument as in Exercise 16.2 one sees that the target $X*Y*Z$ of the map Θ is Hausdorff.
(8) It follows from (5), (6) and (7) together with the Compact-Hausdorff Proposition 2.17 (3) that Θ is a homeomorphism. ⊞

Basically the same argument as in the above claim also shows that there exists a homeomorphism $X*(Y*Z)\to X*Y*Z$. In summary we see that there exists a homeomorphism $X*(Y*Z)\to(X*Y)*Z$. ■

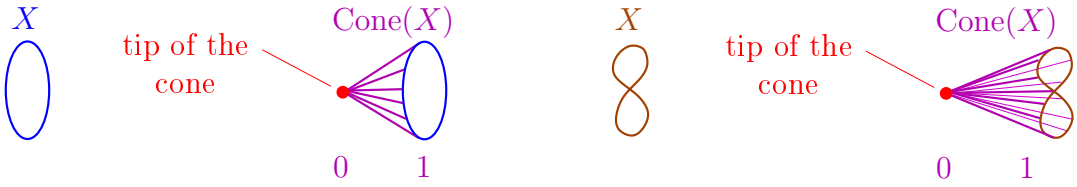
16.2. **The cone and suspension of a topological space.** In this section we define the cone and suspension of a topological space. The cone of a topological space is a particularly easy construction.

Definition. Given a non-empty topological space X the cone on X is defined to be the topological space

$$\mathrm{Cone}(X) := (X \times [0,1])/(X \times \{0\}) = (X \times [0,1])/\sim \quad \text{where } (x,0) \sim (x',0) \text{ for all } x, x' \in X.$$

We refer to the point $X \times \{0\}$ as the tip of the cone $\mathrm{Cone}(X)$ or alternatively as the cone point. For completeness we define the cone of the empty topological space to be the one point topological space $\{*\}$. In this setting we refer to $*$ as the tip of the cone.

Remark. As a set $\mathrm{Cone}(X)$ consists of X, the tip of the cone, and for each $x \in x$ a segment from the tip of the cone to x.



Examples.

(1) Let $n \in \mathbb{N}_0$. It follows from the Topological-Quotient Proposition 5.15 (1b) that the map

$$\begin{array}{rcl} \mathrm{Cone}(S^n) & \to & \overline{B}^{n+1} \\ [(v,t)] & \mapsto & v \cdot t \end{array}$$

is continuous. One can easily verify that the map is a bijection. It follows immediately from the Quotient-Compact Connected Lemma 5.16 (1) and the Product Topology Properties Proposition 5.6 that $\mathrm{Cone}(S^n)$ is compact. Since $\overline{B}^{n+1}$ is Hausdorff we obtain from the Compact-Hausdorff Proposition 2.17 (3) that the given map is a homeomorphism. On a few occasions we will use this specific homeomorphism to make the identification $\mathrm{Cone}(S^n) = \overline{B}^{n+1}$.

(2) Let X be a subset of $\mathbb{R}^n$. We define its *physical cone* as

$$\widetilde{\mathrm{Cone}}(X) \;:=\; \text{union of all line segments from } \underbrace{(x,0)}_{\in \mathbb{R}^{n+1}} \text{ to } \underbrace{(0,1)}_{\in \mathbb{R}^{n+1}} \;\subset\; \mathbb{R}^{n+1}.$$

(a) We consider the map

$$\begin{array}{rcl} f_X\colon \mathrm{Cone}(X) & \to & \widetilde{\mathrm{Cone}}(X) \\ [(x,t)] & \mapsto & t \cdot (x,0) + (1-t) \cdot (0,1). \end{array}$$

In Exercise 16.3 we will see that this map is continuous. If X is a *compact* subset of $\mathbb{R}^n$, then it follows from the Compact-Hausdorff Proposition 2.17 that the map is a homeomorphism.

(b) After looking at the figure above and considering Examples (1) and (2a) one might walk away with the feeling that one has a pretty good understanding of the cone construction. But this optimism is potentially ill-founded. For example in Exercise 16.3 we see will see that for $X = (0,1) \subset \mathbb{R}$ the above map $f_X\colon \mathrm{Cone}(X) \to \widetilde{\mathrm{Cone}}(X)$ from (2a) is *not* a homeomorphism.

The definition of a cone does not seem to be particularly interesting, but we have one lemma whose proof is perhaps slightly more subtle than one might initially think.

Lemma 16.7. (Cone Contractibility Lemma) Let X be a topological space. The tip of the cone $\mathrm{Cone}(X)$ is a deformation retract of $\mathrm{Cone}(X)$.

Proof. The lemma follows easily from the Quotient Deformation Retract Proposition 15.9. For the reader's convenience we recall its statement: Let W be a topological space and let A be a subset of W. If A is a deformation retract of W, then the point A/A is a deformation retract of X/A. We return to our present setting. As we discussed on page 434, the map

$$\begin{array}{rcl} F\colon (X\times[0,1])\times[0,1] & \to & X\times[0,1] \\ ((x,s),t) & \mapsto & (x,s\cdot(1-t)) \end{array}$$

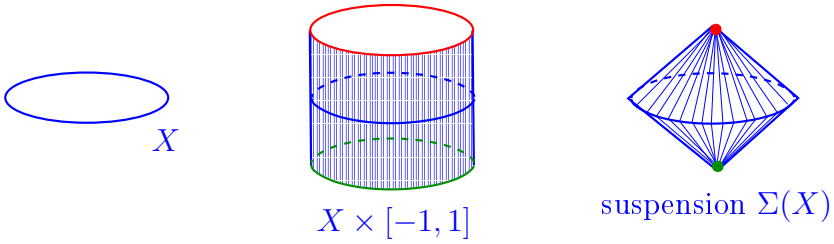
is a deformation retraction from $X\times[0,1]$ to $X\times\{0\}$. It follows from the Quotient Deformation Retract Proposition 15.9 that the tip of the cone, namely $(X\times\{0\})/(X\times\{0\})$ is a deformation retract of $\mathrm{Cone}(X)=(X\times[0,1])/(X\times\{0\})$. ■

We will state a few basic properties of the cone in the next lemma. But before we do so it is convenient to first introduce a close relative of the cone.

Definition. Let X be a non-empty topological space. We define the suspension of X to be the topological space

$$\Sigma(X) \;:=\; (X\times[-1,1])/\sim$$

where we identify all points in $X\times\{-1\}$ to a single point and we identify all points in $X\times\{1\}$ to a single point. Furthermore, we define the suspension of the empty topological space to be the topological space $S^0=\{\pm 1\}$, equipped with the discrete topology.



Before we give an example we state a basic lemma regarding cones and suspensions.

Lemma 16.8. (Cone-Suspension Basics Lemma) Let X be a topological space.

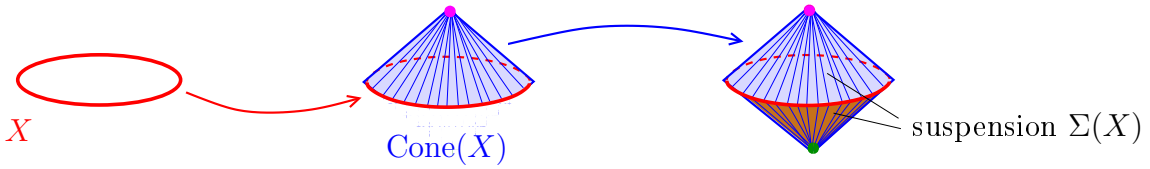
(0) Let X be a topological space.
  (a) The cone $\mathrm{Cone}(X)$ is naturally homeomorphic to the join $X*\{*\}$.
  (b) The suspension $\Sigma(X)$ is naturally homeomorphic to the join $X*S^0$.
(1) If X is compact, then $\mathrm{Cone}(X)$ and the suspension $\Sigma(X)$ are also compact.
(2) If X is Hausdorff, then $\mathrm{Cone}(X)$ and the suspension $\Sigma(X)$ are also Hausdorff.
(3) The maps

$$\begin{array}{rcl} X & \to & \mathrm{Cone}(X) \\ x & \mapsto & [(x,1)] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathrm{Cone}(X) & \to & \Sigma(X) \\ [(x,t)] & \mapsto & [(x,1-t)] \end{array}$$

are closed embeddings.

Sometimes we will use the embeddings in (3) to identify the domains with their images. For example we will use these maps to view X as a closed subset of $\mathrm{Cone}(X)$ and to view

$\mathrm{Cone}(X)$ as a closed subset of $\Sigma(X)$. In particular we can and will at times view X as a closed subset of $\Sigma(X)$.



Proof.

(0) We will prove this statement in Exercise 16.5.
(1) This statement follows from the Quotient-Compact Connected Lemma 5.16 (1) and the Product Topology Properties Proposition 5.6.
(2) In Exercise 16.7 we will show that if X is Hausdorff, then $\Sigma(X)$ is Hausdorff as well. The proof for $\mathrm{Cone}(X)$ is basically identical.
(3) We will prove this statement in Exercise 16.8. ■

Example. It is pretty clear from the figure on page 451 that the suspension of S^n is homeomorphic to S^{n+1}. In fact throughout the course we will use the following homeomorphism[168] to identify $\Sigma(S^n)$ with S^{n+1}:

$$\begin{array}{rcl} \Sigma(S^n) & \overset{\cong}{\to} & S^{n+1} \subset \mathbb{R}^{n+1} \times \mathbb{R} \\ [(P,t)] & \mapsto & \big(\underbrace{P \cdot \cos\left(\frac{\pi}{2}t\right)}_{\in \mathbb{R}^{n+1}}, \underbrace{\sin\left(\frac{\pi}{2}t\right)}_{\in \mathbb{R}}\big). \end{array}$$

Note that this homeomorphism sends the "North Pole" of $\Sigma(S^n)$ to the "North Pole" of S^{n+1}.

We conclude this discussion of suspensions with the fairly obvious notion of the suspension of a map.

Definition. Given a map $f\colon X \to Y$ between topological spaces we denote by

$$\begin{array}{rcl} \Sigma(f)\colon \Sigma(X) & \to & \Sigma(Y) \\ [(x,t)] & \mapsto & [(f(x),t)] \end{array}$$

the suspension of f.

Most of the statements of the following lemma will not come as a surprise.

Lemma 16.9. (Suspension Functor Lemma)

(1) If $f\colon X \to Y$ is a map between topological spaces, then $\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)$ is also continuous.
(2) If $f\colon X \to Y$ and $g\colon Y \to Z$ are maps between topological spaces, then the maps $\Sigma(g \circ f)$ and $\Sigma(g) \circ \Sigma(f)$ from $\Sigma(X)$ to $\Sigma(Z)$ agree.
(3) The maps

$$\begin{array}{rcl} X & \mapsto & \Sigma(X) \\ (f\colon X \to Y) & \mapsto & (\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)) \end{array}$$

[168] Using the Compact-Hausdorff Proposition 2.17 (3) and the Cone-Suspension Basics Lemma 16.8 (1) one easily verifies that this map is indeed a homeomorphism.

define a covariant functor from the category of topological spaces to the category of topological spaces. We refer to this functor as the suspension functor.

(4) Let $f, g\colon X \to Y$ be two maps between topological spaces. If f and g are homotopic rel a (possibly empty) subset A, then $\Sigma(f)$ and $\Sigma(g)$ are homotopic rel $\Sigma(A)$.

(5) If $f\colon X \to Y$ is a homotopy equivalence of topological spaces, then the suspension $\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)$ is also a homotopy equivalence.

Remark. We consider the topological space $X = [-1, 1]$, the subset $A = (-1, 1)$ and the inclusion map $i\colon A \to X$. In Exercise 16.14 we will prove the slightly scary statement that the induced map $\Sigma(i)\colon \Sigma(A) \to \Sigma(X)$ is *not* an embedding.

Proof.

(1) We consider the following commutative diagram

$$\begin{array}{ccc} X \times [-1,1] & \xrightarrow{f \times \mathrm{id}_{[-1,1]}} & Y \times [-1,1] \\ \downarrow & \nearrow & \downarrow \\ \Sigma(X) & \xrightarrow[\Sigma(f)]{} & \Sigma(Y). \end{array}$$

By the Product Topology-Basics Lemma 5.4 (3) we know that the top horizontal map is continuous. By the Topological-Quotient Proposition 5.15 (1a) the left and right vertical maps are continuous. It follows that the diagonal map is continuous. Finally we obtain from the Topological-Quotient Proposition 5.15 (1b) that the bottom horizontal map is continuous.

(2) This statement follows immediately from the definitions.

(3) This statement follows immediately from (1) and (2) and the definitions.

(4) Clearly it suffices to prove the following claim.

Claim. If $F\colon X \times [0, 1] \to Y$ is a map, then the map

$$\begin{array}{rccc} G\colon & (\overbrace{(X \times [-1,1])/\!\sim}^{=\Sigma(X)}) \times [0,1] & \to & \overbrace{(Y \times [-1,1])/\!\sim}^{=\Sigma(Y)} \\ & ([(x,t)], s) & \mapsto & [(F(x,s),t)] \end{array}$$

is continuous.

Proof. Fortunately the claim follows immediately from the Homotopy Quotient Theorem 14.7. ⊞

(5) It suffices to show that if $f\colon X \to Y$ and $g\colon Y \to X$ are two maps such that $g \circ f \simeq \mathrm{id}_X$, then $\Sigma(g) \circ \Sigma(f) \simeq \mathrm{id}_{\Sigma(X)}$. Then we see that

$$\begin{array}{ccccccc} \Sigma(g) \circ \Sigma(f) & \underset{\text{by (3)}}{=} & \Sigma(g \circ f) & \underset{\text{by (4)}}{\simeq} & \Sigma(\mathrm{id}_X) & \underset{\text{by (3)}}{=} & \mathrm{id}_{\Sigma(X)}\,. \end{array}$$

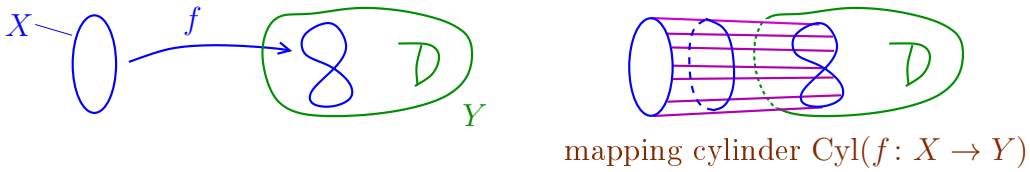
■

16.3. **Mapping cylinder and mapping cone.** In this section we associate to a map $f\colon X \to Y$ between two topological spaces two different topological spaces. The main interest in these topological spaces is that these topological spaces encode some of the information contained in the actual map. On numerous occasions this will allow us to translate questions about maps between topological spaces into questions about topological spaces.

We start out with the definition of a mapping cylinder.

Definition. Let $f\colon X \to Y$ be a map between topological spaces. We define the corresponding mapping cylinder to be the topological space

$$\mathrm{Cyl}(f) := \mathrm{Cyl}(f\colon X \to Y) := ((X \times [0,1]) \sqcup Y)/\sim \text{ where } (x,1) \sim f(x) \text{ for all } x \in X.$$



mapping cylinder $\mathrm{Cyl}(f\colon X \to Y)$

Remark. In many, arguably most, textbooks on algebraic topology the mapping cylinder is defined by identifying $(x,0)$ with $f(x)$ instead of identifying $(x,1)$ with $f(x)$. The definitions are evidently equivalent, but as the author knows from experience, the usage of two different definitions can cause some confusion. The totally non-mathematical advantage of our definition is that it leads to "horizontal" pictures which take up less space.

Example. If X is a topological space and if $f\colon X \to \{y\}$ is the constant map to the topological space consisting of a single element y, then the mapping cylinder $\mathrm{Cyl}(f)$ is homeomorphic to $\mathrm{Cone}(X)$. This statement is almost, but not completely[169] evident. We invite the general-topology-aficionado to fill in the details.

Definition. Let $f\colon X \to Y$ be a map between two topological spaces. We refer to the maps

$$\begin{array}{ccc} X & \to & \mathrm{Cyl}(f\colon X \to Y) \\ x & \mapsto & [(x,0)] \end{array} \quad \text{and} \quad \begin{array}{ccc} Y & \to & \mathrm{Cyl}(f\colon X \to Y) \\ y & \mapsto & [y] \end{array}$$

as the natural inclusions. Sometimes we use these inclusions to identify X and Y with the corresponding subsets of the mapping cylinder.

The following lemma says that the natural inclusions are closed embeddings. This means that we can legitimately identify X and Y with their images in the mapping cylinder $\mathrm{Cyl}(f\colon X \to Y)$. On many occasions we will implicitly use the lemma without actually referring to it.

Lemma 16.10. (Mapping Cylinder Basics Lemma) Let $f\colon X \to Y$ be a map between topological spaces.

(1) If X and Y are compact, then $\mathrm{Cyl}(f)$ is also compact.
(2) If X and Y are Hausdorff, then $\mathrm{Cyl}(f)$ is also Hausdorff.
(3) The natural inclusions $X \to \mathrm{Cyl}(f)$ and $Y \to \mathrm{Cyl}(f)$ are embeddings, in particular they are continuous.
(4) The images of the natural inclusions $X \to \mathrm{Cyl}(f)$ and $Y \to \mathrm{Cyl}(f)$ are closed subsets.

Proof.

(1) We suppose that X and Y are compact. It follows from the Product Topology Properties Proposition 5.6 (2) and the Pushout Properties Proposition 5.35 (1) that $\mathrm{Cyl}(f)$ is also compact.

[169] Note that the statement is *not* true for the empty topological space. This observation shows that the statement requires at least a little bit of thought.

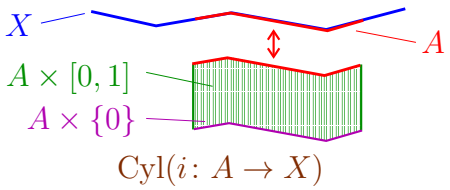
(2) We suppose that X and Y are Hausdorff. We will prove Exercise 16.9 that in this case the mapping cylinder is Hausdorff.
(3) We will provide the proof of the third statement in Exercise 16.10.
(4) The preimage of $i(X)$ under the projection $(X \times [0,1]) \sqcup Y \to \mathrm{Cyl}(f)$ is $X \times \{0\}$. It follows from Lemma 5.12 and the Product-of-Subsets Lemma 5.5 that this is a closed subset. Therefore it follows from the Topological-Quotient Proposition 5.15 (0b) that $i(X)$ is indeed a closed subset of $\mathrm{Cyl}(f)$. The proof that the image of the natural inclusion $Y \to \mathrm{Cyl}(f)$ is closed is almost the same. ■

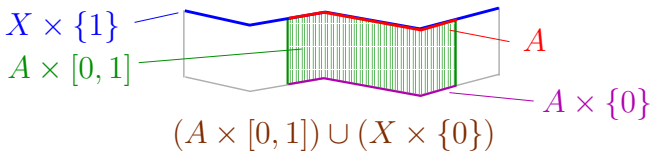
We continue with a (non-) example. We will not really make use of the following lemma, but it should warn the reader that the topology on a mapping cylinder is not necessarily what one might think it is.

Lemma 16.11. Let X be a topological space and let $A \subset X$ be a subset. We denote by $i\colon A \to X$ the inclusion map and we consider the map

$$\begin{array}{rcl} \Theta\colon \mathrm{Cyl}(i) = ((A \times [0,1]) \sqcup X)/(a,1) \sim i(a) & \to & \overbrace{(A \times [0,1]) \cup (X \times \{1\})}^{\text{equipped with the subspace topology coming from the product } X \times [0,1]} \\ [P] & \mapsto & \begin{cases} P, & \text{if } P \in A \times [0,1], \\ (P,1), & \text{if } P \in X. \end{cases} \end{array}$$

The following three statements hold:
(1) The map Θ is continuous and it is a bijection.
(2) If A is a closed subset of X, then Θ is a homeomorphism.
(3) There exist situations, e.g. $X = [0,\infty)$ and $A = [0,1)$, in which Θ is *not* a homeomorphism.





Proof.

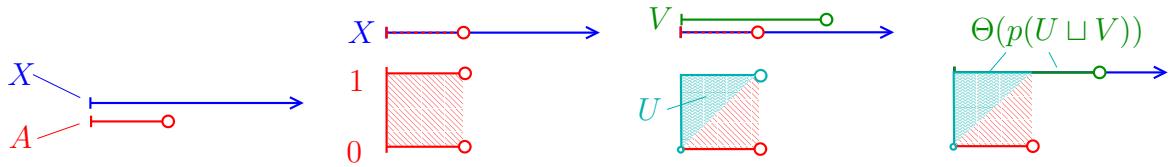
(1) It follows easily from the Topological-Quotient Proposition 5.15 (1b) that the given map Θ is continuous. Furthermore it is basically clear that Θ is a bijection.
(2) Suppose that A is a closed subset of X. Note that $A \times \{1\}$ is a closed subset of $A \times [0,1]$. It follows basically from Lemma 5.36 that Θ is a homeomorphism.
(3) We consider the case $X = [0,\infty)$ and $A = [0,1)$. We denote by

$$p\colon (A \times [0,1)) \sqcup X \ \to \ \mathrm{Cyl}(i) = ((A \times [0,1)) \sqcup X)/\sim$$

the projection . We consider the open set $U := \{(x,t) \in [0,1) \times [0,1] \,|\, x > t\}$ of $A \times [0,1)$. Furthermore we consider the open subset $V := [0,2)$ of $X = [0,\infty)$. Note that $p^{-1}(p(U \sqcup V)) = U \sqcup V$. Thus it follows immediately from the definition of the quotient topology on $\mathrm{Cyl}(i) = ((A \times [0,1)) \sqcup X)/\sim$ that $p(U \sqcup V)$ is an open subset of $\mathrm{Cyl}(i)$. On the other hand we have

$$\Theta(p(U \sqcup V)) \ = \ U \cup \{(x,1) \,|\, x \in V\},$$

but considering the point $(1,1)$ one can easily verify that this is *not* an open subset of $(A \times [0,1]) \cup (X \times \{1\})$. ■



The following lemma summarizes some basic properties of the mapping cylinder.

Lemma 16.12. (Mapping Cylinder Basics Lemma II) Let $f\colon X \to Y$ be a map between topological spaces.

(1) The subspace Y is a deformation retract of $\mathrm{Cyl}(f\colon X \to Y)$, in fact an explicit deformation retraction is given by the following map

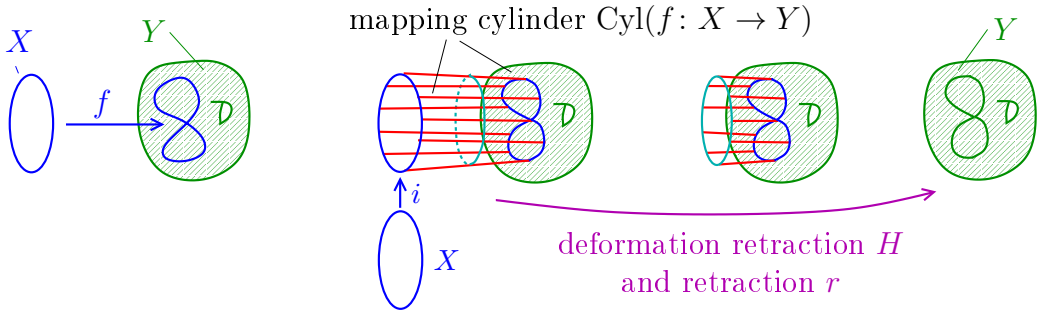
$$H\colon \mathrm{Cyl}(f\colon X \to Y) \times [0,1] \;\to\; \mathrm{Cyl}(f\colon X \to Y)$$
$$(P,t) \;\mapsto\; \begin{cases} P, & \text{if } P \in Y, \\ [(Q, s\cdot(1-t)+t)], & \text{if } P = [(Q,s)], \text{ where } Q \in X \text{ and } s \in [0,1]. \end{cases}$$

(2) We denote by $r = H_1$ the retraction $\mathrm{Cyl}(f\colon X \to Y) \to Y$ given by (1). If we denote by $i\colon X \to \mathrm{Cyl}(f)$ and $j\colon Y \to \mathrm{Cyl}(f)$ the natural inclusions, then the following statements hold:
   (a) The map $j\colon Y \to \mathrm{Cyl}(f\colon X \to Y)$ and $r\colon \mathrm{Cyl}(f\colon X \to Y) \to Y$ are homotopy equivalences.
   (b) The map $i\colon X \to \mathrm{Cyl}(f\colon X \to Y)$ is a homotopy equivalence if and only if the map $f\colon X \to Y$ is a homotopy equivalence.

Remark.

(1) On page 454 we saw that the mapping cylinder of a constant map $f\colon X \to \{y\}$ is homeomorphic to the cone on X. Thus we see that the Mapping Cylinder Basics Lemma 16.12 (1) can be viewed as a generalization of the Cone Contractibility Lemma 16.7.

(2) In Corollary 17.16 we will strengthen the statement of the Mapping Cylinder Basics Lemma 16.12 (2b), namely we will show that if $f\colon X \to Y$ is a homotopy equivalence, then $i(X)$ is in fact a deformation retract of $\mathrm{Cyl}(f\colon X \to Y)$.



Proof.

(1) We need to show that the map H is continuous. First note that the two maps

$$\begin{array}{rcl}(X\times[0,1])\times[0,1] & \to & \mathrm{Cyl}(f)\\ (Q,t) & \mapsto & [(Q,s\cdot(1-t)+t)]\end{array} \quad \text{and} \quad \begin{array}{rcl}Y\times[0,1] & \to & \mathrm{Cyl}(f)\\ (P,t) & \mapsto & P\end{array}$$

are continuous[170]. But this implies, by the somewhat delicate Homotopy Pushout Lemma 14.9, that

$$H\colon ((X\times[0,1])\cup_{(x,1)\mapsto f(x)} Y)\times[0,1] \to \mathrm{Cyl}(f)$$

itself is continuous. Once continuity is out of the way it is basically clear that H is a deformation retraction. Therefore it follows from the Deformation Retract-Homotopy Equivalence Lemma 15.5 that the inclusion $Y \to \mathrm{Cyl}(f\colon X\to Y)$ is a homotopy equivalence.

(2) (a) It follows from the Deformation Retract-Homotopy Equivalence Lemma 15.5 that j and r are homotopy equivalences.

(b) We consider the map

$$\begin{array}{rcl}G\colon X\times[0,1] & \to & \mathrm{Cyl}(f\colon X\to Y)\\ (x,t) & \mapsto & [(x,t)].\end{array}$$

This map is continuous by the Topological-Quotient Proposition 5.15 (1b). We make the following observations:

$$\begin{array}{ccccc}i=G_0\colon X\to\mathrm{Cyl}(f) \text{ is a homotopy equivalence} & \Leftrightarrow & i=G_1\colon X\to\mathrm{Cyl}(f) \text{ is a homotopy equivalence} & \Leftrightarrow & \overbrace{r\circ G_1}^{=f}\colon X\to Y \text{ is a homotopy equivalence}\\ \uparrow & & \uparrow & & \uparrow\\ \text{by the Homotopy-Equivalence Basics Lemma 15.1 (3), since } G_0 \text{ and } G_1 \text{ are homotopic} & & & & \text{by the Homotopy-Equivalence Basics Lemma 15.1 (1), since by (2a) we know that } r \text{ is a homotopy equivalence}\end{array}$$

We have thus shown that i is a homotopy equivalence if and only if $f = r\circ G_1$ is a homotopy equivalence. ∎

In the above discussion we associated to a map $f\colon X\to Y$ between topological spaces a new topological space, namely the mapping cylinder $\mathrm{Cyl}(f\colon X\to Y)$. But we see in the Mapping Cylinder Basics Lemma 16.12 (1) that the mapping cylinder, as a topological space, is in many ways the same as Y. Therefore we move on to the definition of the mapping cone which is a much more interesting construction.

Definition. Given a map $f\colon A\to X$ between topological spaces the corresponding mapping cone is defined as

$$\mathrm{Cone}(f\colon A\to X) \;:=\; (\mathrm{Cone}(A)\sqcup X)/\!\sim \text{ where } [(a,1)]\sim f(a) \text{ for all } a\in A.$$

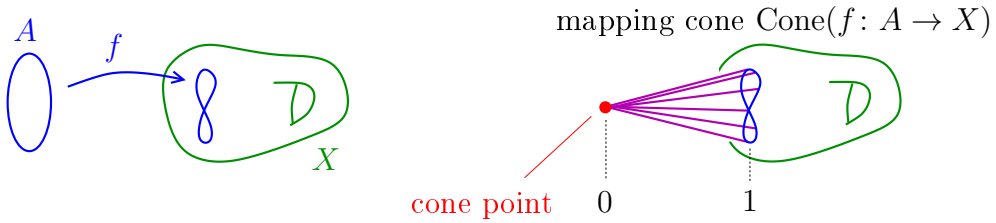
We refer to the point $[A\times\{0\}]$ as the cone point of the mapping cone.

Remark. Let $f\colon A\to X$ be a map between topological spaces. If A is non-empty, then it follows almost immediately from the definitions that

$$\mathrm{Cone}(f\colon A\to X) \;=\; \mathrm{Cyl}(f\colon A\to X)/(A\times\{0\}).$$

The following lemma summarizes some properties of mapping cones.

[170] For once this is basically obvious.



Lemma 16.13. (Mapping Cone Basics Lemma) Let $f\colon A \to X$ be a map between topological spaces.

(1) We denote by $i\colon A \to \mathrm{Cone}(A)$ the map that is given by $a \mapsto [(a,1)]$. The mapping cone $\mathrm{Cone}(f\colon A \to X)$ is the same[171] as the pushout of the maps $i\colon A \to \mathrm{Cone}(A)$ and $f\colon A \to X$, i.e. we have the pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad i \quad} & \mathrm{Cone}(A) \\ \downarrow{\scriptstyle f} & & \lrcorner \downarrow \\ X & \longrightarrow & \mathrm{Cone}(f\colon A \to X). \end{array}$$

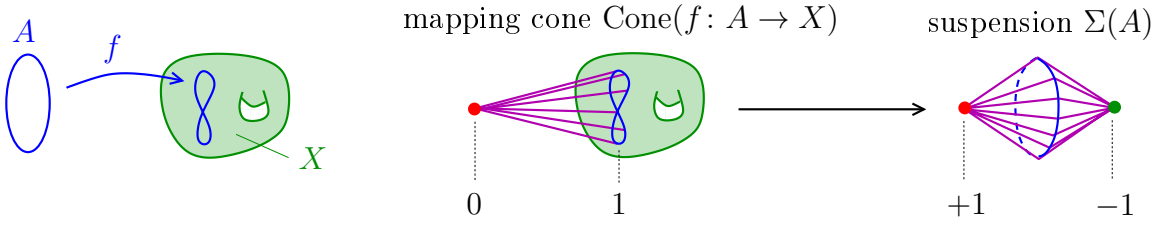
(2) The natural inclusion
$$\begin{array}{ccc} X & \to & \mathrm{Cone}(f\colon A \to X) \\ x & \mapsto & [x] \end{array}$$
is a closed embedding. Often we will use these natural inclusions to identify X with its image in $\mathrm{Cone}(f\colon A \to X)$.

(3) If the map $f\colon A \to X$ is a (closed or open) embedding, then the natural map $\mathrm{Cone}(A) \to \mathrm{Cone}(f\colon A \to X)$ is a (closed or open) embedding.

(4) If A is non-empty, then the map
$$\begin{array}{ccc} \mathrm{Cone}(f\colon A \to X)/X & \to & \Sigma(A) \\ [(a,t)] & \mapsto & [(a,1-2t)] \end{array}$$
is a natural homeomorphism.

(5) The cone point $*$ defines a closed subset of $\mathrm{Cone}(f\colon A \to X)$.

(6) There exists a natural deformation retraction from $\mathrm{Cone}(f\colon A \to X) \setminus \{*\}$ to X. In particular the subset X is a deformation retract of the complement of the cone point.



Sketch of proof.

(1) This statement follows basically immediately from the definitions.

[171] At this point one can now argue whether they are the same on the nose or whether the obvious map is just a natural homeomorphism.

(2) This statement follows immediately from (1) together with the Topological Pushout-Maps Lemma 5.34 (3) and (4).
(3) This statement also follows immediately from (1) together with the Topological Pushout-Maps Lemma 5.34 (3) and (4).
(4) It is clear that the given map is a bijection. One can easily verify, without any extra wizardry, that the given map is indeed a homeomorphism.
(5) This statement follows quite easily from the definition of the topologies involved.
(6) The proof of this statement is basically identical to the proof of the Mapping Cylinder Basics Lemma 16.12 (1). ■

So what have we done so far? We have associated to a map $f\colon X \to Y$ between topological spaces the topological space $\mathrm{Cone}(f\colon X \to Y)$. This construction will allow us to turn questions about maps between topological spaces into questions about topological spaces. As always in algebraic topology, not only do we want a map

$$\text{set of maps between topological spaces} \;\to\; \text{set of topological spaces},$$

but we really want a functor. It is clear what the category should be on the right. It is a little less clear what the category should be on the left. Here it is.

Definition. We refer to the category *MapTop* with

$$\begin{aligned} \mathrm{Ob}(\mathit{MapTop}) &:= \text{all maps between topological spaces,} \\ \mathrm{Mor}(f\colon X \to Y, \widetilde{f}\colon \widetilde{X} \to \widetilde{Y}) &:= \begin{array}{c}\text{all pairs of maps} \\ (\varphi\colon X \to \widetilde{X}, \psi\colon Y \to \widetilde{Y}) \\ \text{such that the diagram}\end{array} \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \end{array} \quad \text{commutes} \end{aligned}$$

with the (hopefully) obvious composition of morphisms as the category of maps between topological spaces.

The following lemma says that we can view "taking the mapping cone" and "taking the mapping cylinder" as covariant functors.

Lemma 16.14. (Mapping Cone Functor Lemma)

(1) The maps

$$(f\colon X \to Y) \mapsto \mathrm{Cyl}(f\colon X \to Y)$$

and

$$(\varphi\colon X \to \widetilde{X}, \psi\colon Y \to \widetilde{Y}) \mapsto \left(\begin{array}{l} \mathrm{Cyl}(f\colon X \to Y) \to \mathrm{Cyl}(\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}) \\ \qquad [P] \mapsto \begin{cases} [(\varphi(x), t)], & \text{if } P = (x,t) \text{ with,} \\ & \quad x \in X,\, t \in [0,1] \\ [\psi(y)], & \text{if } P = y \text{ with } y \in Y \end{cases} \end{array} \right).$$

define a covariant functor from the category *MapTop* of maps between topological spaces to the category *Top* of topological spaces.
(2) If, in the setting of (1), the maps φ and ψ are embeddings, then the corresponding map $\mathrm{Cyl}(f\colon X \to Y) \to \mathrm{Cyl}(\widetilde{f}\colon \widetilde{X} \to \widetilde{Y})$ is also an embedding.
(3) The analogues of (1) and (2), with obvious modifications, hold if we replace mapping cylinders by mapping cones.

(4) Given a morphism[172]

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \end{array}$$

the map from (3) for mapping cones fits into the following commutative diagram:

$$\begin{array}{ccccc} Y & \hookrightarrow & \mathrm{Cone}(f\colon X \to Y) & \longrightarrow & \Sigma(X) \\ {\scriptstyle\psi}\downarrow & & \downarrow & & \downarrow{\scriptstyle\Sigma(\varphi)} \\ \widetilde{Y} & \hookrightarrow & \mathrm{Cone}(\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}) & \longrightarrow & \Sigma(\widetilde{X}) \end{array}$$

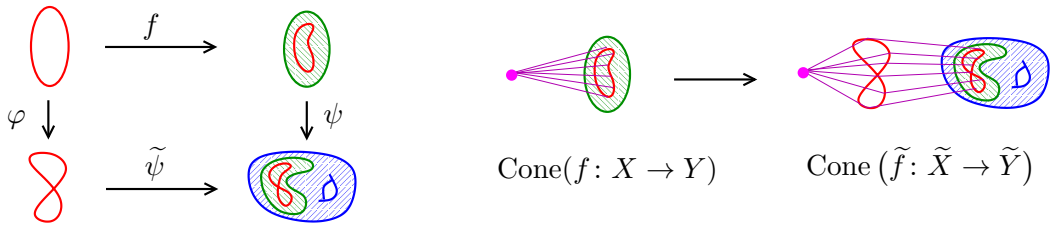
where the horizontal maps to the left are the inclusion maps that we introduced in the Mapping Cone Basics Lemma 16.13 (2) and where the horizontal maps to the right are given by the projection and the homeomorphism from the Mapping Cone Basics Lemma 16.13 (4).

Proof. One needs to convince oneself that the map between the mapping cylinders and the mapping cones is in fact continuous. For the mapping cylinders that is an immediate consequence of the Topological Pushout Proposition 5.33 (1b). For the mapping cones first note that the same argument as in the proof of the Suspension Functor Lemma 16.9 (1) shows that the map

$$\begin{array}{rcl} \mathrm{Cone}(X) & \to & \mathrm{Cone}(\widetilde{X}) \\ [(x,t)] & \mapsto & [(\varphi(x),t)] \end{array}$$

is continuous. The continuity of the map between the mapping cones now follows again from the Topological Pushout Proposition 5.33 (1b).

The second statement will be proved in Exercise 16.13 (3). Finally Statement (4) of the lemma follows basically from the definitions. ■

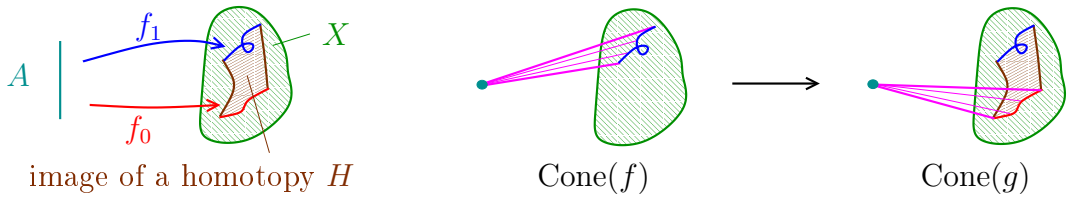


The following lemma shows in particular that homotopic maps give rise to homotopy equivalent mapping cones. In particular at times this lemma allows us to translate the question whether two maps are homotopic to the question whether two topological spaces are homotopy equivalent.

Proposition 16.15. (Mapping Cone Homotopy Equivalent Proposition) Let $f_0, f_1\colon A \to X$ be maps between topological spaces. If f_0, f_1 are homotopic, then there exists a homotopy equivalence $\mathrm{Cone}(f_0) \to \mathrm{Cone}(f_1)$ rel the union of X and the cone point.

It is not that difficult to write down explicit maps between the cones and to write down explicit homotopies. But we prefer to go take a more conceptual approach. Namely, instead

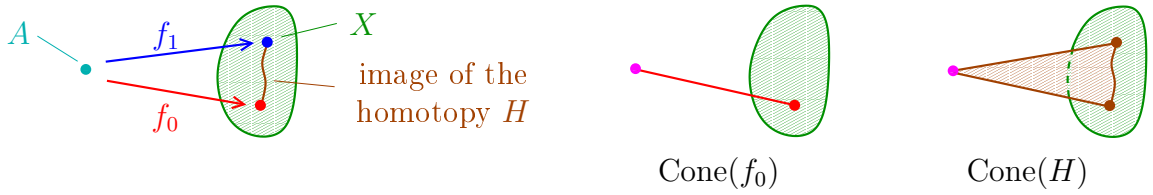
[172] In fancy speak this commutative diagram says that the inclusion maps into the mapping cone and the projection from the mapping cone to the suspension are natural transformations between suitable functors.



of proving the Mapping Cone Homotopy Equivalent Proposition 16.15 directly we will prove the following more technical lemma.

Lemma 16.16. (Mapping Cone Inclusion Lemma) Let $f_0, f_1 \colon A \to X$ be maps between topological spaces. Furthermore let $H \colon A \times [0,1] \to X$ be a homotopy between f_0 and f_1. Let $j \in \{0,1\}$. We denote by $i_j \colon A \to A \times \{j\}$ the obvious inclusion map.

(1) The inclusion map i_j induces by the Mapping Cone Functor Lemma 16.14 (3) a map $\mathrm{Cone}(f_j) \to \mathrm{Cone}(H)$. This map is an embedding.
(2) If we use the map from (1) to view $\mathrm{Cone}(f_j)$ as a subset of $\mathrm{Cone}(H)$, then it is a deformation retract of $\mathrm{Cone}(H)$.



Proof of Proposition 16.15 assuming Lemma 16.16. Let H be a homotopy between f_0 and f_1. By the Mapping Cone Inclusion Lemma 16.16 we can view $\mathrm{Cone}(f_0)$ and $\mathrm{Cone}(f_1)$ as deformation retracts of $\mathrm{Cone}(H)$. It follows from the Deformation Retract-Homotopy Equivalence Lemma 15.5 and the Homotopy-Equivalence Basics Lemma 15.1 that $\mathrm{Cone}(f_0)$ and $\mathrm{Cone}(f_1)$ are homotopy equivalent. A quick inspection of the maps involved shows that the two cones are homotopy equivalent rel the union of X and the cone point. ■

Proof of the Mapping Cone Inclusion Lemma 16.16. Let $f_0, f_1 \colon A \to X$ be maps between topological spaces. Furthermore let $H \colon A \times [0,1] \to X$ be a homotopy between f_0 and f_1. For notational convenience we only deal with the case $j = 0$. We denote by $i \colon A \to A \times \{0\}$ the obvious inclusion map. We get a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f_0} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle =} \\ A \times [0,1] & \xrightarrow{H} & X. \end{array}$$

Thus we get from the Mapping Cone Functor Lemma 16.14 (3) a map $\mathrm{Cone}(f_0) \to \mathrm{Cone}(H)$. It follows almost immediately from the Mapping Cone Functor Lemma 16.14 (3) that this map is an embedding.

Since this proof suffers from an abundance of intervals playing different roles we rescale two of the intervals:

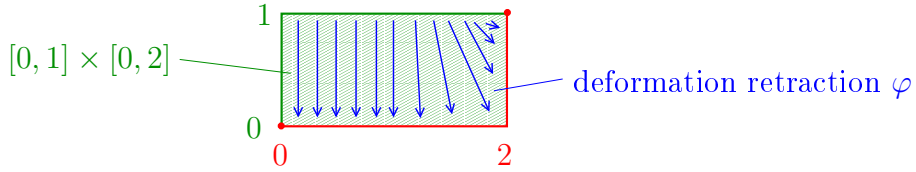
(1) We use the interval $[0,1]$ for the homotopy H.

(2) We use the interval $[0,2]$ for the definition of the mapping cone.
(3) We use the interval $[0,3]$ for the definition of the desired deformation retraction.

With this convention out of the way we pick a deformation retraction

$$\varphi\colon ([0,1]\times[0,2])\times[0,3] \;\to\; [0,1]\times[0,2]$$

rel $([0,2]\times\{0\})\cup(\{2\}\times[0,1])$ such that $\varphi(r,s,t) = (r, \frac{1}{3}st)$ whenever $r \in [0,\frac{1}{2}]$. It is straightforward to see that such a deformation retraction exists. We refer to the figure below for an illustration.



Now we consider the map

$$G\colon \overbrace{(((A\times[0,1]\times[0,2])\sqcup X)/\!\sim)}^{=\mathrm{Cone}(H\colon A\times[0,1]\to X)}\times[0,3] \;\to\; \mathrm{Cone}(H)$$
$$([P],z) \;\mapsto\; \begin{cases} [(a,\varphi(x,y,z))], & \text{if } P=(a,x,y),\\ P, & \text{if } P\in X.\end{cases}$$

Using the Topological-Quotient Proposition 5.15 (1a) and the Homotopy Pushout Lemma 14.9 one can easily show that this map G is continuous. It is straightforward to verify that this map has the desired properties. ■

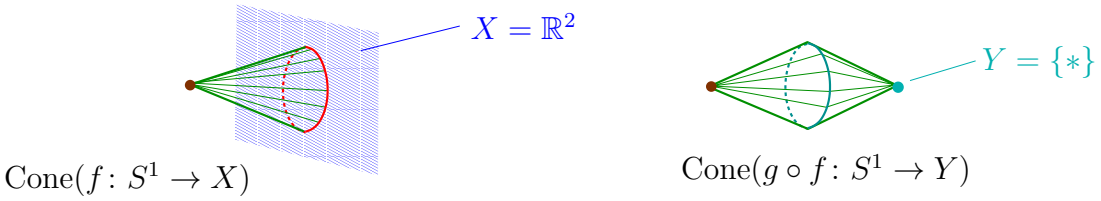
For completeness' sake we also state the following corollary.

Corollary 16.17. Let $f\colon A\to X$ be a map between topological spaces. Furthermore let $g\colon X\to Y$ be a homotopy equivalence. The map

$$\mathrm{Cyl}(f\colon A\to X) \;\to\; \mathrm{Cyl}(g\circ f\colon A\to Y)$$
$$[P] \;\mapsto\; \begin{cases} [(a,t)], & \text{if } P=(a,t)\in A\times[0,1],\\ [g(P)], & \text{if } P\in X\end{cases}$$

is a homotopy equivalence rel $A\times\{0\}$. The same way one sees that the mapping cones of f and $g\circ f$ are homotopy equivalent.

Example. Let $f\colon S^1\to\mathbb{R}^2$ be the inclusion and let $g\colon\mathbb{R}^2\to\{*\}$ be the retraction to the origin. In the figure below we show the corresponding mapping cones, which we now know are homotopy equivalent by Corollary 16.17.

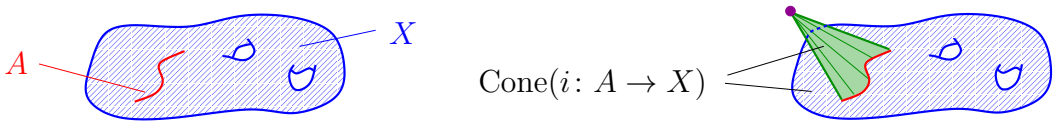


Proof. The corollary is a straightforward consequence of the Mapping Cone Homotopy Equivalent Proposition 16.15. We leave it to the reader to fill in the details. ■

We conclude our discussion of mapping cones with the following proposition.

Proposition 16.18. Let X be a topological space and let A be a subset of X. We denote by $i\colon A \to X$ the inclusion map. We use the Mapping Cone Basics Lemma 16.13 (2) to view X as a subset of $\mathrm{Cone}(i\colon A \to X)$.

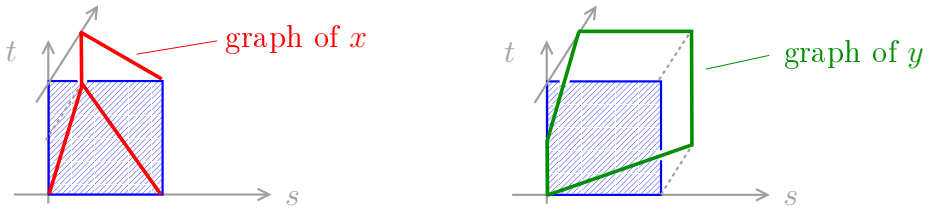
(1) If A is contractible, then given any $a_0 \in A$ there exists a deformation retraction G from $\mathrm{Cone}(i\colon A \to X)$ to X such that $G_1(\text{cone point } [A \times \{0\}]) = a_0$.
(2) If $a_0 \in A$ is a deformation retract of A, then there exists a deformation retraction G from $\mathrm{Cone}(i\colon A \to X)$ to X such that for any $t \in [0,1]$ we have $G_1([(a_0,t)]) = a_0$.



Proof. We first prove Statement (1). Thus let $a_0 \in A$. Since A is contractible it follows from the Point Deformation Retract Lemma 15.6 (2) that there exists a map $r\colon A \times [0,1] \to A$ such that $r_0 = \mathrm{id}$ and such that $r_1(a) = a_0$ for all $a \in A$. We pick continuous functions $x, y\colon [0,1] \times [0,1] \to [0,1]$ with the following properties:

(a) For any $t \in [0,1]$ we have $x(0,t) = 1$ or $y(0,t) = 0$.
(b) We have $x(s,0) = 0$ and $y(s,0) = s$ for all $s \in [0,1]$.
(c) We have $y(s,1) = 1$ for all $s \in [0,1]$.
(d) We have $x(1,t) = 0$ and $y(1,t) = 1$ for all $t \in [0,1]$.

We sketch graphs of such functions x and y in the figure below. Now we consider the map



$$F\colon \overbrace{((A \times [0,1])/(A \times \{0\}))}^{=\mathrm{Cone}(A)} \times [0,1] \to \overbrace{(A \times [0,1])/(A \times \{0\})}^{=\mathrm{Cone}(A)}$$
$$([(a,s)],t) \mapsto [(r(a,x(s,t)),y(s,t))].$$

We make the following observations:

(0) Since x and y are continuous and since $[0,1]$ is regionally compact we obtain from the Topological-Quotient Proposition 5.15 (1b) and the Product-Quotient Mixing Lemma 13.6 that F is continuous.
(1) It follows from (a) that F is actually well-defined.
(2) It follows from (b) that $F_0 = \mathrm{id}$.
(3) It follows from (c) that F_1 takes values in $A \times \{1\}$.
(4) It follows from (d) that F is a homotopy rel $A \times \{1\}$.

In summary this shows that F is a deformation retraction from $\mathrm{Cone}(A)$ to $A \times \{1\}$. Now we consider the map

$$\begin{array}{rcl} G\colon \overbrace{(\mathrm{Cone}(A) \cup_{(a,1)=a} X)}^{=\mathrm{Cone}(i\colon A \to X)} \times [0,1] & \to & \mathrm{Cone}(i\colon A \to X) \\ ([P],t) & \mapsto & \begin{cases} [F(P,t)] & \text{if } P \in \mathrm{Cone}(A), \\ [P], & \text{if } P \in X. \end{cases} \end{array}$$

It follows from the Homotopy Pushout Lemma 14.9 that this map is continuous. Clearly G is a deformation retract from $\mathrm{Cone}(i\colon A \to X)$ to X. By construction it sends the cone point to a_0.

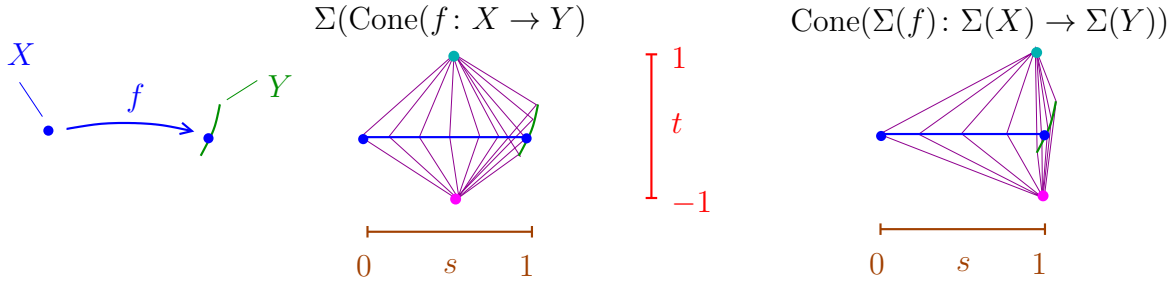
It remains to prove Statement (2). If we start out with a deformation retraction r from A to a_0, then the above homotopy F has the desired property. ■

16.4. **The mapping cone of a suspension.** In this section we show that under favorable circumstances the operation of taking a mapping cone (or mapping cylinder) commutes with the operation of taking the suspension of a map. This technical proposition will play a useful role later on. The proposition is also a good way to test whether we really understand the various topologies involved in the definition of mapping cones, mapping cylinders and suspensions.

Proposition 16.19. (Mapping Cone-Suspension Proposition) If $f\colon X \to Y$ is a map between topological spaces, then the map[173]

$$\begin{array}{rcl} \Phi\colon \Sigma(\mathrm{Cone}(f\colon X \to Y)) & \to & \mathrm{Cone}(\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)) \\ P & \mapsto & \begin{cases} [([[(x,t)],s)], & \text{if } P = [([[(x,s)],t)] \text{ with } x \in X,\ t \in [-1,1] \\ & \quad \text{and with } s \in [0,1], \\ [([y],t)], & \text{if } P = [([y],t)] \text{ with } y \in Y \text{ and } t \in [-1,1]. \end{cases} \end{array}$$

is a homeomorphism.



In the proof of the Mapping Cone-Suspension Proposition 16.19 we will need the following lemma which is quite similar to Proposition 13.8.

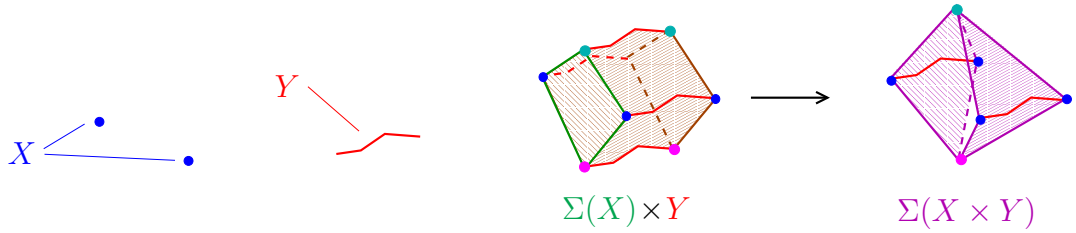
[173] It takes a major effort to disentangle the notation. We write $[-]$ to indicate the equivalence class corresponding to a suspension and we write $[-]$ to indicate the equivalence class corresponding to the equivalence relation used to define mapping cones.

Lemma 16.20. (Suspension-Product Lemma) Let X be a topological space. If Y is a regionally compact space (e.g. if $Y = [0,1]$), then the map

$$\begin{array}{rccc} \varphi\colon & \Sigma(X) \times Y & \to & \Sigma(X \times Y) \\ & ([(x,t)], y) & \mapsto & [((x,y),t)] \end{array}$$

is continuous.

In the figure below we illustrate the Suspension-Product Lemma 16.20. This cute picture is not meant to hide the grim reality that for arbitrary topological spaces X it actually requires a non-trivial input to prove the Suspension-Product Lemma 16.20.



Proof. We denote by $p\colon (X \times Y) \times [-1,1] \to \Sigma(X \times Y)$ and $q\colon X \times [-1,1] \to \Sigma(X)$ the two natural projections. We have the following commutative diagram

$$\begin{array}{ccc} (X \times [-1,1]) \times Y & \xrightarrow[\cong]{(x,t,y)\mapsto(x,y,t)} & (X \times Y) \times [-1,1] \\ \downarrow{\scriptstyle q \times \mathrm{id}_Y} & & \downarrow{\scriptstyle p} \\ \Sigma(X) \times Y & \xrightarrow{\qquad\varphi\qquad} & \Sigma(X \times Y). \end{array}$$

By the Product Topology-Basics Lemma 5.4 (2) the top horizontal map is a homeomorphism. By the Quotient Map Lemma 13.3 the maps p and q are quotient maps. Since Y is regionally compact we obtain from the Whitehead Quotient Theorem 13.4 that $q \times \mathrm{id}_Y$ is a quotient map. It follows from Lemma 13.5 that φ is also a quotient map. By the Quotient Map Lemma 13.3 (2) this implies that φ is indeed continuous. ■

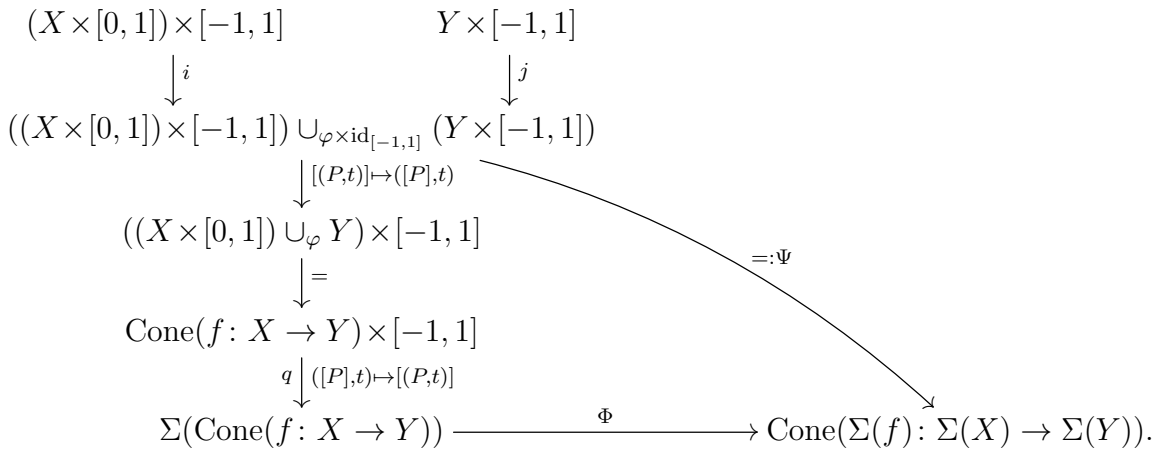
Proof of the Mapping Cone-Suspension Proposition 16.19. Let $f\colon X \to Y$ be a map between topological spaces. One can easily verify that the given map Φ is a bijection. It remains to show that Φ and Φ^{-1} are continuous. We will do so in the following two claims. The domain and the target of Φ are both a scary mix of product and quotient topologies. Thus we need to tread super carefully.

Claim 1. The map Φ is continuous.

Proof. To prove the claim we introduce the following notation:

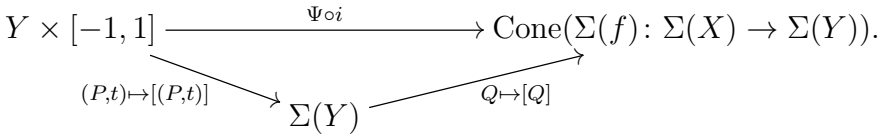
(a) We consider the map $\varphi\colon X \times \{1\} \to Y$ that is given by $\varphi(x,1) = f(x)$.
(b) We denote by $p\colon \mathrm{Cone}(f\colon X \to Y) \times [-1,1] \to \Sigma(\mathrm{Cone}(f\colon X \to Y))$ the projection
.

We consider the following diagram:



We make the following observations:

(1) By Lemma 13.7 we know that the vertical map between the second and the third row is a homeomorphism.
(2) It follows from (1) together with the Topological-Quotient Proposition 5.15 (1b), applied to the projection q, that Φ is continuous if Ψ is continuous.
(3) It follows from the Topological Pushout Proposition 5.33 (1b) that Ψ is continuous if $\Psi\circ i$ and $\Psi\circ j$ are continuous.
(4) We have the following commutative diagram:



The diagonal maps are continuous by the Topological-Quotient Proposition 5.15 (1a) and the Topological Pushout Proposition 5.33 (1b). Thus we see that $\Psi\circ i$ is continuous. The same argument shows that $\Psi\circ j$ is continuous.

The combination of (2), (3) and (4) shows that Φ is indeed continuous. ⊞

Claim 2. The map

$$\overbrace{\mathrm{Cone}(\Sigma(f)\colon\Sigma(X)\to\Sigma(Y))}^{=((\Sigma(X)\times[0,1])\sqcup\Sigma(Y))/\sim}\xrightarrow{\Phi^{-1}}\Sigma(\mathrm{Cone}(f\colon X\to Y))$$

$$P\mapsto\begin{cases}[([(x,s)],t)], & \text{if } P=[([(x,t)],s)] \text{ with } s\in[0,1] \text{ and}\\ & \text{with } [(x,t)]\in\Sigma(X), \text{ i.e. } x\in X,\ t\in[-1,1],\\ [([y],t)], & \text{if } P=[([y],t)] \text{ with } t\in Y,\ t\in[-1,1].\end{cases}$$

is continuous.

Proof. By the Topological Pushout Proposition 5.33 (1b) it suffices to show that the maps

$$\begin{array}{ccc}\Sigma(X)\times[0,1] & \to & \Sigma(\mathrm{Cone}(f\colon X\to Y))\\ ([(x,t)],s) & \mapsto & [([(x,s)],t)]\end{array}\quad\text{and}\quad\begin{array}{ccc}\Sigma(Y) & \to & \Sigma(\mathrm{Cone}(f\colon X\to Y))\\ [(y,t)] & \mapsto & [([y],t)]\end{array}$$

are continuous. The second map is continuous by the Suspension Functor Lemma 16.9 (1). Furthermore it follows from the combination of the Suspension-Product Lemma 16.20 with the Suspension Functor Lemma 16.9 (1) that the first map is continuous. ■

16.5. **The mapping torus.** In this last section we discuss the mapping torus. This construction gives rise to many interesting topological spaces. We will also see that determining the fundamental group of a mapping torus is somewhat challenging, but not impossible.

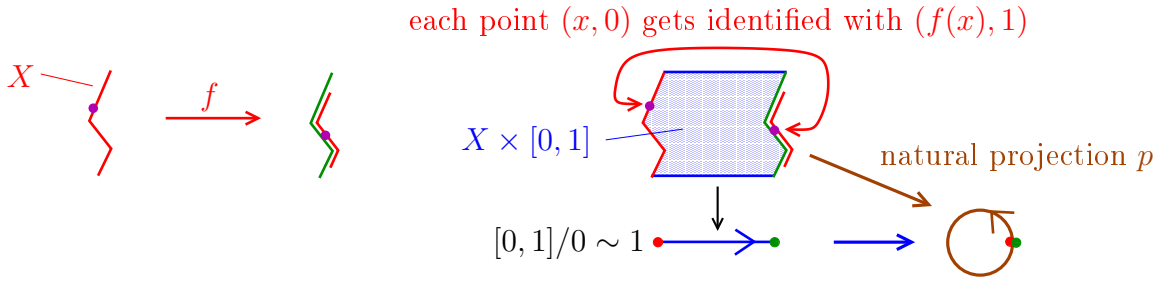
Definition. Let X be a topological space and let $f\colon X \to X$ be a map. We refer to

$$\mathrm{Tor}(X,f) \;:=\; (X\times[0,1])/\!\sim \qquad \text{where } (x,0)\sim(f(x),1) \text{ for all } x\in X,$$

as the mapping torus of (X,f). We refer to the map

$$\begin{array}{rcl}\mathrm{Tor}(X,f)=(X\times[0,1])/(x,0)\sim(f(x),1) & \to & S^1\\ [(x,t)] & \mapsto & \exp(2\pi\mathrm{i}t)\end{array}$$

as the natural projection onto S^1.

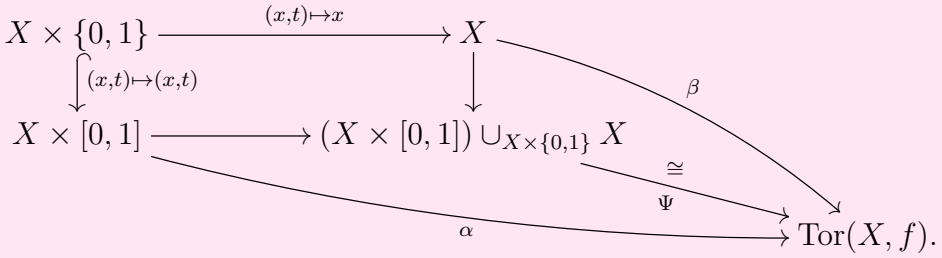


The following lemma shows that the mapping torus is a pushout. This is very useful since it allows us to use all the results from the Topological Pushout-Maps Lemma 5.34.

Lemma 16.21. Let X be a topological space and let $f\colon X\to X$ be a map. The natural maps

$$\begin{array}{rcl}\alpha\colon X\times[0,1] & \to & \mathrm{Tor}(X,f)\\ (x,t) & \mapsto & [(x,t)]\end{array} \qquad \text{and} \qquad \begin{array}{rcl}\beta\colon X & \to & \mathrm{Tor}(X,f)\\ x & \mapsto & [(x,0)]\end{array}$$

induce the following natural homeomorphism:



Proof. First note that it follows from the the Topological Pushout Proposition 5.33 (1b) that Ψ is continuous. Next we consider the natural map

$$\begin{array}{rcl}\Phi\colon (X\times[0,1])/\!\sim & \to & (X\times[0,1])\cup_{X\times\{0,1\}}X\\ [(x,t)] & \mapsto & [(x,t)].\end{array}$$

[173] It follows from the Topological-Quotient Proposition 5.15 (1b) that the natural projection is indeed continuous.

It follows from the the Topological Pushout Proposition 5.33 (1a) and the Topological-Quotient Proposition 5.15 (1b) that Φ is continuous. One can easily verify that Φ and Ψ are inverses of one another. Thus we have shown that both maps are homeomorphisms. ■

For completeness' sake we state the following almost self-evident lemma which gets used, often subconsciously, on numerous occasions.

Lemma 16.22. Let X be a topological space and let $f\colon X \to X$ be a map.

(1) The map

$$\begin{array}{rcl} \Theta\colon (X\times\mathbb{R})/(x,r) \overset{\text{with } x\in X \text{ and } r\in\mathbb{R}}{\sim} (f(x),r+1) & \to & \mathrm{Tor}(X,f) = (X\times[0,1])/(x,0)\sim(f(x),1) \\ [(x,t)] & \mapsto & [(x,t)] \end{array}$$

is a homeomorphism.

(2) For every interval $I \subset \mathbb{R}$ of length less than one, the map

$$\begin{array}{rcl} \varphi\colon X\times I & \to & (X\times\mathbb{R})/\sim \ \xrightarrow{\Theta}\ \mathrm{Tor}(X,f) \\ (x,t) & \mapsto & [(x,t)] \end{array}$$

is an embedding. Furthermore, if I is closed respectively open, then φ is a closed respectively open embedding.

Proof. First note that it is straightforward to write down an inverse to Θ. The second statement follows from elementary arguments. We leave it to the reader to fill in the details. ■

The following lemma takes care of our two favorite properties of topological spaces, namely compactness and being Hausdorff.

Lemma 16.23. (Mapping Torus-Compact-Hausdorff Lemma) Let X be a topological space and let $f\colon X \to X$ be a homeomorphism.

(1) If X is Hausdorff, then the mapping torus $\mathrm{Tor}(X,f)$ is Hausdorff.
(2) If X is compact, then the mapping torus $\mathrm{Tor}(X,f)$ is compact.

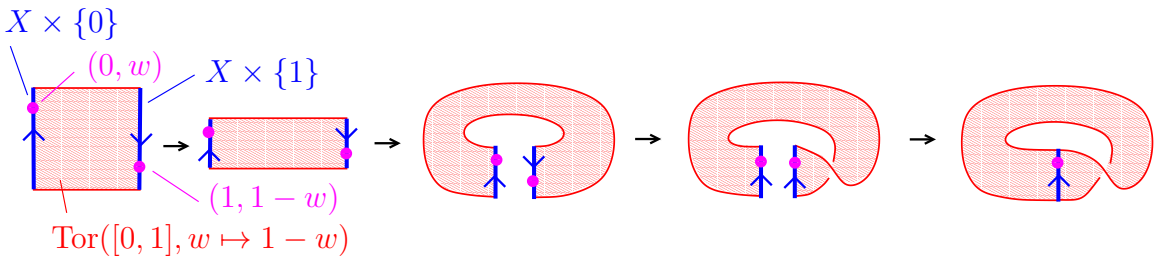
Proof.

(1) This statement will be proved in Exercise 16.16.
(2) This statement follows immediately from the Product Topology Properties Proposition 5.6 (2) and the Quotient-Compact Connected Lemma 5.16 (1). ■

Next we will see that several well-established topological spaces are actually mapping tori.

Examples.

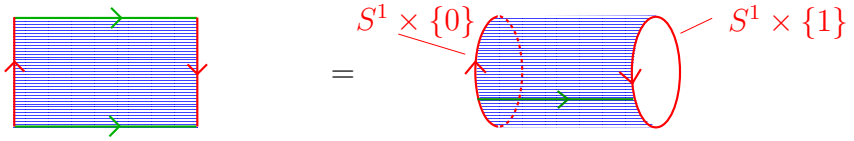
(1) We consider the topological space $[0,1]$ and let $f\colon [0,1] \to [0,1]$ be the self-homeomorphism of $[0,1]$ that is given by $f(w) = 1 - w$. It follows immediately from the definitions that the corresponding mapping torus $\mathrm{Tor}([0,1],f)$ is precisely the Möbius band as defined on page 264.
(2) Let $X = S^1$ and let $f\colon X \to X$ be the homeomorphism that is given by $f(z) = \overline{z}$. It follows easily from the Compact-Hausdorff Proposition 2.17 (3) and the Mapping



Torus-Compact-Hausdorff Lemma 16.23 (1) that the map

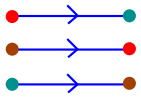
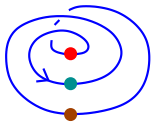
$$\begin{array}{rcl} \overbrace{([0,1]\times[0,1])\,/\,(x,0)\sim(x,1),(0,y)\sim(1,1-y)}^{\text{Klein bottle}} & \to & \mathrm{Tor}(S^1,f) \\ [(x,y)] & \mapsto & [\exp(2\pi\,\mathrm{i}\,y),x)] \end{array}$$

is a homeomorphism from the Klein bottle to the mapping torus $\mathrm{Tor}(S^1, f)$.[174]



Klein bottle $[0,1]\times[0,1]/\sim$ mapping torus $\mathrm{Tor}(S^1, z \mapsto \overline{z})$

(3) As a final example we consider $X = \mathbb{Z}_3 = \{0,1,2\}$, equipped with the discrete topology. Let $f\colon X \to X$ be the homeomorphism that is given by $f(x) = x+1 \bmod 3$. As we illustrate in the figure below, the mapping torus $\mathrm{Tor}(X, f)$ is homeomorphic to S^1.

the mapping torus $\mathrm{Tor}(\mathbb{Z}_3, f(x) = x+1)$  is homeomorphic to 

We slow down a little to consider the case $\varphi = \mathrm{id}$ in more detail. On page 264 we already saw that $\mathrm{Tor}(S^1, \mathrm{id})$ is homeomorphic to the torus $S^1 \times S^1$. Thus the following lemma does not come as a surprise.

Lemma 16.24. (Mapping Torus-of-id Lemma) Given any topological space X the map

$$\begin{array}{rcl} \Phi\colon \mathrm{Tor}(X,\mathrm{id}) & \to & X \times S^1 \\ [(x,t)] & \mapsto & (x, \exp(2\pi\,\mathrm{i}\,t)) \end{array}$$

is a homeomorphism.

Convention. Let X be a topological space. We use the homeomorphism from the Mapping Torus-of-id Lemma 16.24 to make the identification $\mathrm{Tor}(X, \mathrm{id}) = X \times S^1$.

Proof. It is easy to verify that Φ is a bijection. Using Remark 5.17 (1) one can easily show that Φ is continuous. It still remains to show that Φ is actually a homeomorphism.

First we provide an argument based on Proposition 13.11. First note that it follows from the Compact-Hausdorff Proposition 2.17 (3) that the map $[0,1]/\{0,1\} \to S^1$ induced

[174] What does the mapping torus $\mathrm{Tor}(S^1, z \mapsto -z)$ look like?

by $[t] \mapsto \exp(2\pi \mathrm{i} t)$ is a homeomorphism. We use this homeomorphism to replace S^1 by $[0,1]/\{0,1\}$. Next we consider the following diagram:

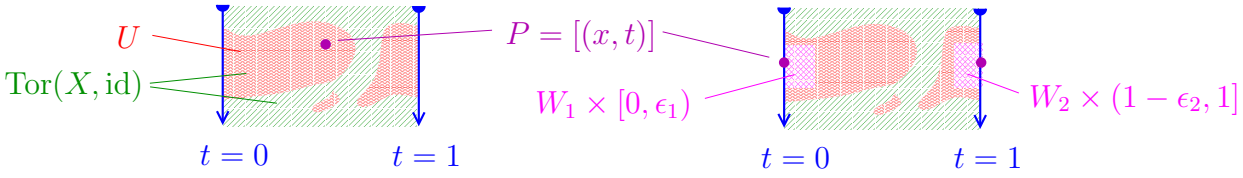
$$\begin{array}{ccc} & X \times [0,1] & \\ {\scriptstyle (x,t)\mapsto(x,[t])} \swarrow^{p} & & \searrow^{q} {\scriptstyle (x,t)\mapsto[(x,t)]} \\ X \times ([0,1]/\{0,1\}) & \xrightarrow[(x,[t])\mapsto[(x,t)]]{\Phi^{-1}} & (X \times [0,1])/(x,0) \sim (x,1). \end{array}$$

By Proposition 13.11, applied to $Y = [0,1]$ and the compact subset $B = \{0,1\}$ we know that p is a quotient map. By the Topological-Quotient Proposition 5.15 (1a) we know that q is continuous. Since the diagram commutes we obtain from Lemma 13.5 that Φ^{-1} is continuous. We have thus shown that Φ is a homeomorphism.

In the rather unlikely event that the reader skipped Proposition 13.11 we also give a more direct argument. By the Open -Injective Map Lemma 2.16 (2) it remains to show that Φ is an open map. So let U be an open subset of $\mathrm{Tor}(X, \mathrm{id}) = (X \times [0,1])/\!\sim$. Let $P := [(x,t)] \in U$ be a point. We can assume that $t \in [0,1)$. We need to show that $Q := \Phi([(x,t)])$ admits an open neighborhood Y that is contained in $V := \Phi(U)$. In the following we denote by $p\colon X \times [0,1] \to \mathrm{Tor}(X, \mathrm{id})$ the projection .

If $t \in (0,1)$, then by the definition of the topology on $\mathrm{Tor}(X, \mathrm{id})$ there exists an open subset $W \subset X$ and an open interval $(a,b) \subset (0,1)$ with $P \in W \times (a,b) \subset U$. We set $Y := \Phi(W \times (a,b)) = W \times \{\exp(2\pi \mathrm{i} t) \,|\, t \in (a,b)\}$. Note that $p^{-1}(Y) = Y$ is an open subset of $X \times [0,1]$. Thus Y has the desired property

Now suppose that $t = 0$. In $X \times [0,1]$ we pick an open neighborhood $W_1 \times [0, \epsilon_1)$ of the point $(x,0)$ that is contained in $p^{-1}(U)$. Similarly we pick an open neighborhood $W_2 \times (1-\epsilon_2, 1]$ of the point $(x,1)$ that is also contained in $p^{-1}(U)$. It is now straightforward to see that $Y := (W_1 \cap W_2) \times \{\exp(2\pi \mathrm{i} s) \,|\, s \in (-\epsilon_2, \epsilon_1)\}$ has the desired property. ■



Now we continue the discussion of mapping tori with a few more examples.

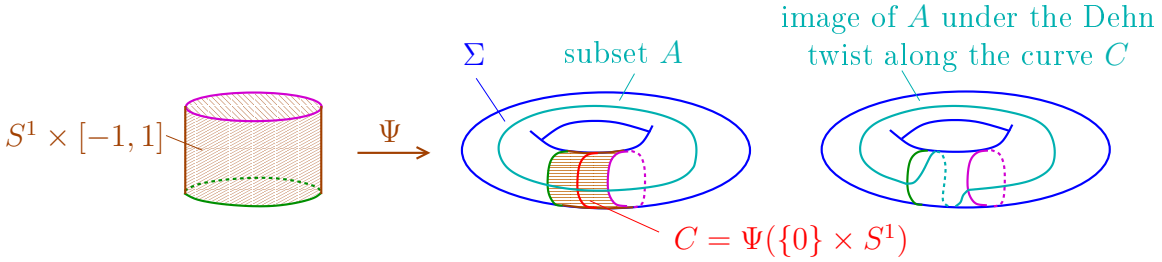
Examples.

(1) In [**Rolf1990**, Chapter 10.I.10] and [**BZH2014**, Section 5.C] it is shown that the complement of the trefoil and the complement of the figure-8 knot are homeomorphic to a mapping torus, where the fiber is given by removing a point from the torus.

(2) If one wants to construct new interesting topological spaces out of a given topological space we need to find "non-trivial" self-homeomorphisms. For example let Σ be a surface and let $\Psi\colon [-1,1] \times S^1 \to \Sigma$ be an embedding The map

$$\begin{array}{rcl} \Sigma & \to & \Sigma \\ P & \mapsto & \begin{cases} P, & \text{if } P \notin \Psi([-1,1] \times S^1), \\ \Psi(t, z \cdot \exp(\mathrm{i} \cdot ((t+1) \cdot \pi))) & \text{if } P = \Psi(t,z) \text{ for some } t \in [-1,1] \text{ and } z \in S^1, \end{cases} \end{array}$$

is called a *Dehn twist*[175] *twist along the curve* $C := \Psi(\{0\} \times S^1)$. We will discuss Dehn twists in greater detail on page 3126.



For completeness we also state the following lemma.

Lemma 16.25. (Manifold-Mapping Torus Lemma) Let M be an n-dimensional smooth manifold and let $f\colon M \to M$ be a diffeomorphism. The following statements hold:

(1) The mapping torus $\mathrm{Tor}(M,f)$ is naturally an $(n+1)$-dimensional smooth manifold such that the obvious inclusion map $M \times (0,1) \to \mathrm{Tor}(M,f)$ is a smooth map.
(2) The boundary $\partial\,\mathrm{Tor}(M,f)$ is given by the mapping torus $\mathrm{Tor}(\partial M, f|_{\partial M})$, in particular, if M is closed, then $\mathrm{Tor}(M,f)$ is also closed.
(3) If M is orientable and if f is orientation-preserving, then $\mathrm{Tor}(M,f)$ is also an orientable smooth manifold.

Proof. First consider the case that M has no boundary. By the Smooth Manifold Product Proposition 19.31 (4) we know that $M \times [0,1]$ is an $(n+1)$-dimensional smooth manifold with boundary given by $M \times \{0,1\}$. Note that we obtain the mapping torus $\mathrm{Tor}(M,f)$ from the smooth manifold $M \times [0,1]$ by identifying the boundary component $M \times \{0\}$ with $M \times \{1\}$. Thus the statements are an immediate consequence of the Gluing-Smooth Manifolds-Proposition **??**.

We will not make use of the Manifold-Mapping Torus Lemma 16.25 when M has non-empty boundary. Thus we leave it to the curious reader to modify the above argument to the general case. ■

The lemma says in particular that self-diffeomorphisms of surfaces give rise to 3-dimensional smooth manifolds. This construction is an important source of examples of 3-dimensional smooth manifolds.

Definition. Let X and Y be topological spaces. Two homeomorphisms $f,g\colon X \to Y$ are called isotopic, if there exists an isotopy from f to g, i.e. a map

$$\begin{array}{rcl} H\colon X \times [0,1] & \to & Y \\ (x,t) & \mapsto & H(x,t) \end{array}$$

such that the following hold:

(1) We have $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$,
(2) for each $t \in [0,1]$ the map $X \to Y$ given by $x \mapsto H(x,t)$ is a homeomorphism.

Examples.

[175] Max Dehn (1878-1952) was a German–Jewish mathematician. In 1939 he fled to Norway. After the German occupation of Norway he fled to America via Siberia and Japan. He died in 1952 in Black Mountain, North Carolina.

(1) We consider $X = [0,2]$ and the two self-homeomorphisms given by $f(x) = x$ and by $g(x) = \frac{1}{2} \cdot x^2$. Then[176]

$$\begin{array}{rcl} H\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & f(x) \cdot (1-t) + g(x) \cdot t \end{array}$$

is an isotopy from f to g.

(2) Lickorish [**Lic1962**] showed that any self-homeomorphism of a compact orientable 2-dimensional smooth manifold is isotopic to the concatenation of finitely many Dehn twists.

Lemma 16.26. Let X be a topological space. If $f, g\colon X \to X$ are two homeomorphisms that are isotopic, then the mapping tori $\mathrm{Tor}(X,f)$ and $\mathrm{Tor}(X,g)$ are homeomorphic.

Proof. The lemma will be proved in Exercise 16.17. ■

Example. We consider $X = S^1$ and the self-homeomorphism that is given by $g(z) = -z$. Note that the map $H(z,t) = \exp(\pi \mathrm{i} t) \cdot z$ defines an isotopy from $f = \mathrm{id}$ to g. It follows from Lemma 16.26 that the mapping torus $\mathrm{Tor}(S^1, g(z) = -z)$ is homeomorphic to the torus $\mathrm{Tor}(S^1, \mathrm{id}) = S^1 \times S^1$.

Now we can determine which surfaces are mapping tori.

Proposition 16.27. The annulus, the Möbius band, the torus and the Klein bottle are, up to homeomorphism, the only connected, compact 2-dimensional topological manifolds that can be written as a mapping torus $\mathrm{Tor}(X,f)$ of a connected 1-dimensional topological manifold X.

Proof. By the examples (1), (2) on page 468 and by the Mapping Torus-of-id Lemma 16.24 we know that there exist the following types of homeomorphisms:

$$\begin{array}{rcl} \text{annulus } S^1 \times [0,1] & \cong & \mathrm{Tor}([0,1], \mathrm{id}) \\ \text{Möbius band} & \cong & \mathrm{Tor}([0,1], x \mapsto 1-x) \end{array} \quad \text{and} \quad \begin{array}{rcl} \text{torus } S^1 \times S^1 & \cong & \mathrm{Tor}(S^1, \mathrm{id}) \\ \text{Klein bottle} & \cong & \mathrm{Tor}(S^1, z \mapsto \overline{z}). \end{array}$$

In particular all these four smooth manifolds are homeomorphic to mapping tori. It remains to show that these are the only connected, compact 2-dimensional topological manifolds that can be written as a mapping torus.

It follows from the Topological 1-Dimensional Manifold Classification Theorem 22.1 that any compact connected 1-dimensional topological manifold is homeomorphic to $[0,1]$ or to S^1. We know from Lemma 50.1 that any self-homeomorphism is isotopic to one of the self-homeomorphisms that we had written down above. But then it follows from Lemma 16.26 that any mapping torus is homeomorphic to one from the above list. ■

For proving statements about mapping tori the following lemma gives a useful alternative point of view.

Lemma 16.28. (Mapping Torus-of-Homeomorphism Lemma) Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. Then the following hold:

(1) The map[177]

$$\begin{array}{rcl} \mathbb{Z} \times (X \times \mathbb{R}) & \to & X \times \mathbb{R} \\ (n, (x,t)) & \mapsto & (f^n(x), t+n) \end{array}$$

defines a continuous, proper and discrete action[178] of the group $\mathbb{Z}$ on $X \times \mathbb{R}$.

[176] Let $t \in [0,1]$. Why is the map $[0,2] \to [0,2]$ given by $x \mapsto H(x,t)$ a homeomorphism?

(2) The map

$$\begin{array}{rcl} \mathrm{Tor}(X,f) = (X\times[0,1])/\!\sim & \to & (X\times\mathbb{R})/\mathbb{Z} \\ [(x,t)] & \mapsto & [(x,t)] \end{array}$$

is a homeomorphism.

Proof. We leave the task of providing the elementary proof to the reader. ■

In the following, given a topological space X and a homeomorphism $f\colon X\to X$ we denote the corresponding quotient space by $(X\times\mathbb{R})/\mathbb{Z}_f$ and we will identify the mapping torus $\mathrm{Tor}(X,f)$ with $(X\times\mathbb{R})/\mathbb{Z}_f$ using the homeomorphism of the Mapping Torus-of-Homeomorphism Lemma 16.28.

Remark. It follows from the Mapping Torus-of-Homeomorphism Lemma 16.28 together with the Manifold Quotient-by-Group Action Proposition 19.32 that if X is an n-dimensional topological (respectively) smooth manifold and f is a homeomorphism, then the corresponding mapping torus $\mathrm{Tor}(X,f) = (X\times\mathbb{R})/\mathbb{Z}_f$ is a $(n+1)$-dimensional topological (respectively smooth) manifold. Furthermore, if X is in fact an n-dimensional smooth manifold and $f\colon X\to X$ is a diffeomorphism, then almost the same argument shows that the mapping torus $\mathrm{Tor}(X,f) = (X\times\mathbb{R})/\mathbb{Z}_f$ is an $(n+1)$-dimensional smooth manifold.

Lemma 16.29. (Mapping Torus Covering Lemma) Let X be a topological space and let $f\colon X\to X$ be a homeomorphism. The map

$$\begin{array}{rcl} X\times\mathbb{R} & \to & (X\times\mathbb{R})/\mathbb{Z}_f = \mathrm{Tor}(X,f) \\ (x,t) & \mapsto & [(x,t)] \end{array}$$

is a covering of infinite degree. Furthermore for $k\in\mathbb{N}$ the map

$$\begin{array}{rcl} \mathrm{Tor}(X,f^k) = (X\times\mathbb{R})/\mathbb{Z}_{f^k} & \to & (X\times\mathbb{R})/\mathbb{Z}_f = \mathrm{Tor}(X,f) \\ [(x,t)] & \mapsto & [(x,k\cdot t)] \end{array}$$

is a k-fold covering.

Proof. Let X be a topological space and let $f\colon X\to X$ be a homeomorphism. The first statement is an immediate consequence of the Mapping Torus-of-Homeomorphism Lemma 16.28 and Proposition 48.10.

Now let $k\in\mathbb{N}$. We consider the following commutative diagram of maps:

$$\begin{array}{ccc} (X\times\mathbb{R})/\mathbb{Z}_{f^k} & \xrightarrow{[(x,t)]\mapsto[(x,k\cdot t)]} & (X\times\mathbb{R})/\mathbb{Z}_f. \\ & {\scriptstyle [(x,t)]\mapsto[(x,k\cdot t)]}\searrow \quad \nearrow{\scriptstyle [(x,t)]\mapsto[(x,t)]} & \\ & (X\times\mathbb{R})/(k\cdot\mathbb{Z})_f & \end{array}$$

The left diagonal map is easily seen to be well-defined and to be a homeomorphism. By Proposition 48.10 the right diagonal map is a k-fold covering. It follows that the top horizontal map is also a k-fold covering. ■

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[177] Here for $n\geq 1$ we write $f^n(x) = (\overbrace{f\circ\cdots\circ f}^{n\text{-times}})(x)$, for $n=0$ we write $f^0(x)=x$ and for $n\leq -1$ we write $f^n(x) = (\underbrace{f^{-1}\circ\cdots\circ f^{-1}}_{|n|\text{-times}})(x)$.

[178] Why is it an action?

Examples.

(1) We apply Mapping Torus Covering Lemma 16.29 to $X = [0,1]$ and the homeomorphism that is given by $f(x) := 1 - x$. Note that for all x we have

$$f^2(x) \;=\; f(f(x)) \;=\; f(1-x) \;=\; 1-(1-x) \;=\; x.$$

This shows that $f^2 = \mathrm{id}$. Thus we recover the statement from page 1070 that the annulus $\mathrm{Tor}([0,1], \mathrm{id})$ is a 2-fold covering of the Möbius band $\mathrm{Tor}([0,1], f(x) = 1-x)$.

(2) We apply Mapping Torus Covering Lemma 16.29 to $X = S^1$ and the self-homeomorphism of $X = S^1$ that is given by $f(z) := \overline{z}$. As in the previous example we see that $f^2 = \mathrm{id}$. Thus we get, after the discussion on page 1078, a new proof that the torus $\mathrm{Tor}(S^1, \mathrm{id})$ is a 2-fold covering of the Klein bottle $\mathrm{Tor}(S^1, f(z) = \overline{z})$.

Exercises for Chapter 16.

Exercise 16.1. Let X and Y be topological spaces. Show that the join $X * Y$ is homeomorphic to the pushout of the maps

$$\begin{array}{ccc} X \times \{0,1\} \times Y & \longrightarrow & X \times [0,1] \times Y \\ \downarrow & & \\ X \sqcup Y & & \end{array}$$

where the horizontal map is the inclusion and the vertical map is defined by the two projections $X \times \{0\} \times Y \to X$ and $X \times \{1\} \times Y \to Y$.
Remark. This exercise explains in particular the initially slightly odd convention in the definition of the join for the special case that at least one of X or Y is the empty topological space.

Exercise 16.2. Let X and Y be topological spaces. Show that if X and Y are Hausdorff, then the join $X * Y$ is also Hausdorff.

Exercise 16.3. Given a subset X of $\mathbb{R}^n$ we define its *physical cone* as

$$\widetilde{\mathrm{Cone}}(X) \;:=\; \text{union of all line segments from } \underbrace{(x,1)}_{\in \mathbb{R}^{n+1}} \text{ to } \underbrace{(0,0)}_{\in \mathbb{R}^{n+1}} \;\subset\; \mathbb{R}^{n+1}$$

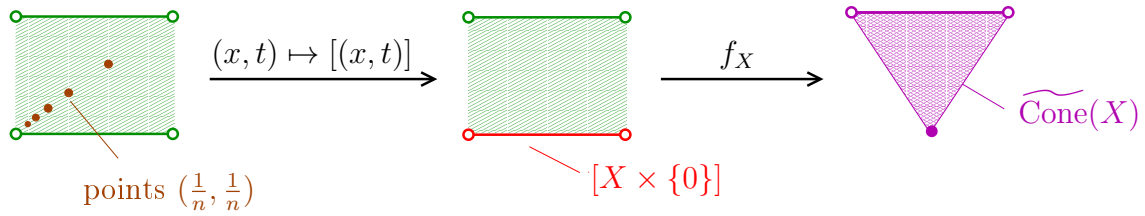
and we consider the map

$$\begin{array}{rccl} f_X \colon & \mathrm{Cone}(X) & \to & \widetilde{\mathrm{Cone}}(X) \\ & [(x,t)] & \mapsto & t \cdot (x,1) + (1-t) \cdot (0,1). \end{array}$$

(a) Let X be a compact subset of $\mathbb{R}^n$. Show that the map f_X is a homeomorphism.

(b) Let $X = (0,1) \subset \mathbb{R}$. Show that the map f_X is not a homeomorphism.
Remark. It is actually true that f_X is continuous and a bijection. To show that f_X is not a homeomorphism we need to show that there exists an open subset U of $\mathrm{Cone}(X)$ such that the image under f_X is not open. You could try your hand at

$$U \;:=\; \mathrm{Cone}(X) \setminus \{[(\tfrac{1}{n}, \tfrac{1}{n})] \,|\, n \in \mathbb{N}_{\geq 2}\}.$$



Exercise 16.4. Let X be a topological space. Show that $\mathrm{Cone}(X)$ respectively the suspension $\Sigma(X)$ are homeomorphic to the pushout of the diagram

$$\begin{array}{ccc} X & \xrightarrow{x\mapsto(x,1)} & X \times [0,1] \\ \downarrow & & \\ \{*\} & & \end{array} \qquad \text{respectively} \qquad \begin{array}{ccc} X & \xrightarrow{x\mapsto[(x,0)]} & \mathrm{Cone}(X) \\ {\scriptstyle x\mapsto[(x,0)]}\downarrow & & \\ \mathrm{Cone}(X) & & \end{array}$$

Remark. These descriptions of the cone and the suspension explain our definitions for the case that X is the empty topological space.

Exercise 16.5. Let X be a topological space.

(a) Show that the cone $\mathrm{Cone}(X)$ is naturally homeomorphic to the join $X * \{*\}$.
(b) Show that the suspension $\Sigma(X)$ is naturally homeomorphic to the join $X * S^0$.

Remark. We refer to page 446 for the definition of the join of two topological spaces.

Exercise 16.6. Let X be a topological space. Show that the suspension $\Sigma(X)$ is path-connected.

Exercise 16.7. Let X be a topological space. Show that if X is Hausdorff, then the suspension $\Sigma(X)$ is also Hausdorff.

Exercise 16.8. Let X be a topological space. Show that the injections

$$\begin{array}{ccc} X & \to & \mathrm{Cone}(X) \\ x & \mapsto & [(x,1)] \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Cone}(X) & \to & \Sigma(X) \\ [(x,t)] & \mapsto & [(x,1-t)] \end{array}$$

are embeddings.

Exercise 16.9. Let $f\colon X \to Y$ be a map between topological spaces. Show that if X and Y are Hausdorff, then the mapping cylinder $\mathrm{Cyl}(f)$ is also Hausdorff.
Remark. This exercise is the content of the Mapping Cylinder Basics Lemma 16.10 (2).

Exercise 16.10. Let $f\colon X \to Y$ be a map between topological spaces. Show that the natural inclusions

$$\begin{array}{ccc} X & \to & \mathrm{Cyl}(f\colon X \to Y) \\ x & \mapsto & (x,0) \end{array} \quad \text{and} \quad \begin{array}{ccc} Y & \to & \mathrm{Cyl}(f\colon X \to Y) \\ y & \mapsto & [y] \end{array}$$

are embeddings.
Remark. This exercise is the content of the Mapping Cylinder Basics Lemma 16.10 (3).

Exercise 16.11. Let $f\colon X \to Y$ be a map between topological spaces and let Z be a topological space.

(a) Show that the map

$$\Theta\colon \mathrm{Cyl}(f\times \mathrm{id}\colon X\times Z\to Y\times Z) \to \mathrm{Cyl}(f\colon X\to Y)\times Z$$
$$P \mapsto \begin{cases} ([(Q,t)],s), & \text{if } P=[(Q,t,s)] \text{ with } Q\in X, t,s\in[0,1],\\ ([Q],s), & \text{if } P=[(Q,s)] \text{ with } Q\in Y, s\in[0,1]. \end{cases}$$

is continuous.

(b) Suppose that Z is regionally compact. Show that the map Θ from (a) is a homeomorphism.
Hint. Use the Product-Quotient Mixing Lemma 13.6.

Exercise 16.12. Let X be a topological space and let A be a subset of X.

(a) We denote by $i\colon A\to X$ the inclusion. Show that the map

$$\begin{array}{rcl} A\times[0,1] & \to & \mathrm{Cyl}(i) = ((A\times[0,1])\sqcup X)/\!\sim \\ (a,t) & \mapsto & [(a,t)] \end{array}$$

is an embedding.

(b) Let $f\colon A\to X$ be an embedding. Is the map $A\times[0,1]\to \mathrm{Cyl}(f)$ necessarily an embedding?

Exercise 16.13. Give a proof of the Mapping Cone Functor Lemma 16.14 (3).

Exercise 16.14. We consider the topological space $X=[-1,1]$, the subset $A=(-1,1)$ and the inclusion map $i\colon A\to X$. Show that the induced map $\Sigma(i)\colon \Sigma(A)\to\Sigma(X)$ is *not* an embedding.
Remark. It might help to have a look at the equally scary Exercise 16.3.

Exercise 16.15. Let X and Y be topological spaces. We consider the corresponding join $X*Y=(X\times[0,1]\times Y)/\!\sim$ as defined on page 446. Show that the map

$$\begin{array}{rcl} X*Y=(X\times[0,1]\times Y)/\!\sim & \cong & (\mathrm{Cone}(X)\times Y)\cup_{(X\times\{1\})\times Y = X\times(Y\times\{1\})} (X\times \mathrm{Cone}(Y)) \\ [(x,t,y)] & \mapsto & \begin{cases} (x,[(y,2t)]), & \text{if } t\in[0,\frac{1}{2}],\\ ([(x,2(t-1))],y), & \text{if } t\in[\frac{1}{2},1]. \end{cases} \end{array}$$

is a homeomorphism.
Remark. Evidently you need to verify that all the maps are actually continuous.

Exercise 16.16. Let X be a topological space and let $f\colon X\to X$ be a homeomorphism. We assume that X is Hausdorff. Show that the mapping torus $\mathrm{Tor}(X,f)$ is also Hausdorff.

Exercise 16.17. Let X be a topological space. Show that if $f,g\colon X\to X$ are two homeomorphisms that are isotopic, then the mapping tori $\mathrm{Tor}(X,f)$ and $\mathrm{Tor}(X,g)$ are homeomorphic.

Exercise 16.18. Let $\varphi\colon \overline{B}^2\to\overline{B}^2$ be the map that is given by $(x,y)\mapsto(x,-y)$. Is the corresponding mapping torus $\mathrm{Tor}(\overline{B}^2,\varphi)$ homeomorphic to the solid torus?

Exercise 16.19. Let $f\colon X\to Y$ be a map between two topological spaces. Show that if X and Y are both locally contractible, then so is the mapping cylinder $\mathrm{Cyl}(f\colon X\to Y)$.

17. Cofibrations

Loosely speaking a map $i\colon A \to X$ between topological spaces is called a cofibration if it has the homotopy extension property, i.e. if a homotopy on A extends to a homotopy on X. As we will see many "reasonable" inclusion maps, e.g. the inclusion of a "submanifold" into a "smooth manifold", or of a "subcomplex" of a "CW-complex" into the CW-complex are cofibrations. The main technical achievement of this chapter is the daunting proof of the Homotopy Pushout Theorem 17.19 which will come in handy on several occasions throughout these notes.

Nonetheless, for the most part the results in this chapter are not essential for the understanding of the subsequent chapters, thus not much harm is done by initially skipping this chapter. It is best to return to it, when some of the results are actually needed.
Throughout this chapter we will on numerous occasions make use of the notion of a mapping cylinder and of a mapping cone. Thus we start out by recalling the definition of a mapping cylinder and a mapping cone that we had initially given on pages 454 and page 457.

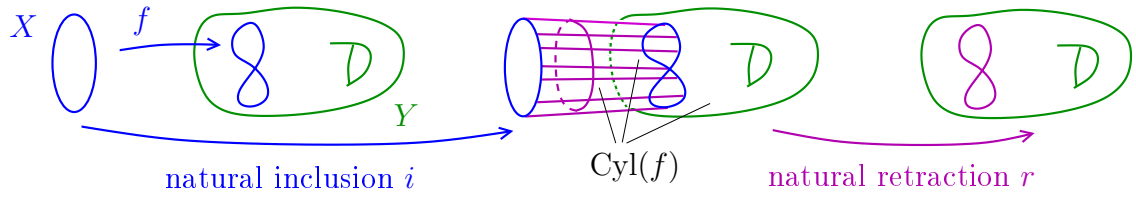
Definition. Let $f\colon X \to Y$ be a map between topological spaces. We define the corresponding mapping cylinder to be the topological space

$$\mathrm{Cyl}(f\colon X \to Y) \;:=\; ((X \times [0,1]) \sqcup Y)/\!\sim \quad \text{where } (x,1) \sim f(x) \text{ for all } x \in X.$$

We refer to the maps

$$\begin{array}{l} X \xrightarrow{i} \mathrm{Cyl}(f) \\ x \mapsto (x,0) \end{array} \quad \text{respectively} \quad \begin{array}{l} \mathrm{Cyl}(f) \xrightarrow{r} Y \\ P \mapsto \begin{cases} P, & \text{if } P \in Y, \\ f(Q), & \text{if } P = [(Q,t)] \text{ with } Q \in X, t \in [0,1]. \end{cases} \end{array}$$

as the natural inclusion respectively the natural retraction.[179]



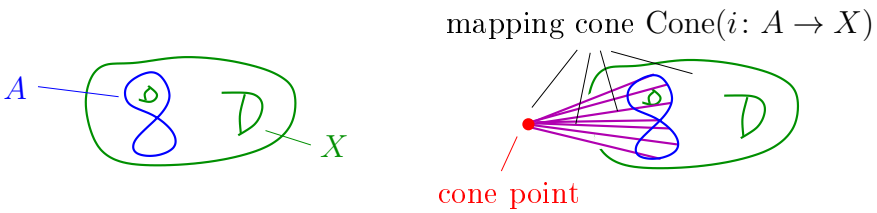
Definition. Given a map $f\colon A \to X$ between topological spaces the corresponding mapping cone is defined as

$$\mathrm{Cone}(f\colon A \to X) \;:=\; (A \times [0,1]) \sqcup X/\!\sim \qquad \text{where} \quad \begin{array}{l} (a,0) \sim (b,0) \text{ for all } a,b \in A \text{ and} \\ (a,1) \sim f(a) \text{ for all } a \in A. \end{array}$$

We refer to the point $[A \times \{0\}] \in \mathrm{Cone}(f)$ as the cone point. We illustrate the definition for an inclusion map $i\colon A \to X$ in the figure below.

17.1. **Cofibrations: definition and basic examples.** We immediately introduce the main player of this section.

[179] Note that by the Mapping Cylinder Basics Lemma 16.12 we know that there exists in fact a natural deformation retraction $R\colon \mathrm{Cyl}(f) \times [0,1] \to \mathrm{Cyl}(f)$ from $\mathrm{Cyl}(f)$ to Y, viewed as a subspace of $\mathrm{Cyl}(f)$, such that $R_1 = r$.



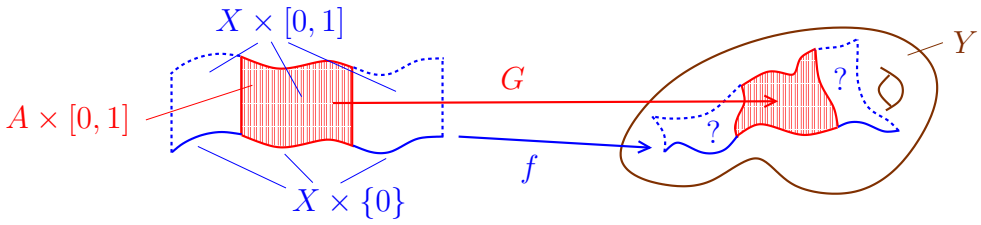
Definition.

(1) A map $i\colon A \to X$ between topological spaces is called a cofibration if it has the homotopy extension property, i.e. if given any topological space Y and any two maps $f\colon X \times \{0\} \to Y$ and $G\colon A \times [0,1] \to Y$ with $f(i(a),0) = G(a,0)$ for all $a \in A$ there exists a map $F\colon X \times [0,1] \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} A \times \{0\} & \hookrightarrow & A \times [0,1] \\ \downarrow{\scriptstyle i} & \overset{f}{\nearrow} & \downarrow{\scriptstyle i\times \mathrm{id}} \\ X \times \{0\} & \hookrightarrow & X \times [0,1]. \end{array} \qquad Y,\ G,\ F$$

We illustrate the definition in the figure below.

(2) If $i\colon A \to X$ is a cofibration such that $i(A)$ is a closed subset of X, then we refer to i as a closed cofibration.



Remark.

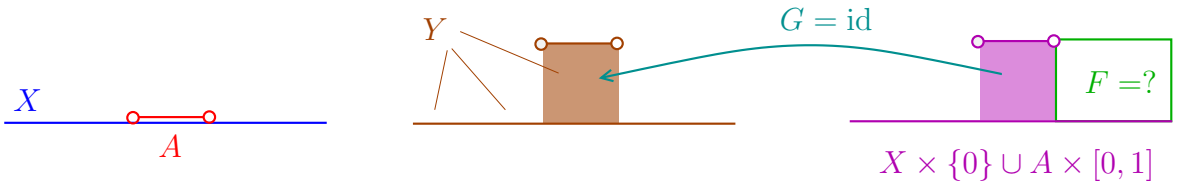
(1) As we will see in the Cofibration-Closed Injective Lemma 17.6, in most cases of interest the map $i\colon A \to X$ is for all intents and purposes an inclusion. For such situations the definition of the homotopy extension property is illustrated in the figure above.

(2) In some books what we call a cofibration is called a "Hurewicz cofibration". Also, in some books, see e.g. [**Pic1992**], a cofibration is defined as what we call a closed cofibration.

Non-Example. Let us start out with a non-example. Namely let us consider $X = \mathbb{R}$ and the subspace $A = (0,1)$. We write $Y = \mathbb{R} \times \{0\} \cup (0,1) \times [0,1]$ and let $f\colon X \to Y$ be the obvious inclusion map and let $G\colon (0,1) \times [0,1] \to Y$ be the obvious inclusion map. (See the figure below for an illustration.) It is an amusing exercise to show that G cannot be extended to a homotopy on all of $\mathbb{R}$.[180] It is an equally amusing exercise to show that the

[180] Here is a hint: if there was an extension of G to a homotopy $F\colon X \times [0,1] \to Y$, what would/should $F(1,1)$ be?

homotopy can be extended if throughout we replace the open interval $A = (0,1)$ by the closed interval $A = [0,1]$.



We continue with the following fairly elementary lemma.

Lemma 17.1. (Cofibration Composition Lemma) If $f\colon X \to Y$ and $g\colon Y \to Z$ are two (closed) cofibrations, then the composition $g \circ f\colon X \to Z$ is also a (closed) cofibration.

Proof. The lemma will be proved in Exercise 17.2. ■

Before we give examples of cofibrations it is convenient to add a characterization of cofibrations to our repertoire.

Lemma 17.2. (Cofibration Characterization Lemma) If $i\colon A \to X$ is a map between two topological spaces, then the following two statements are equivalent:

(1) The map $i\colon A \to X$ is a cofibration.

(2) The map[181]

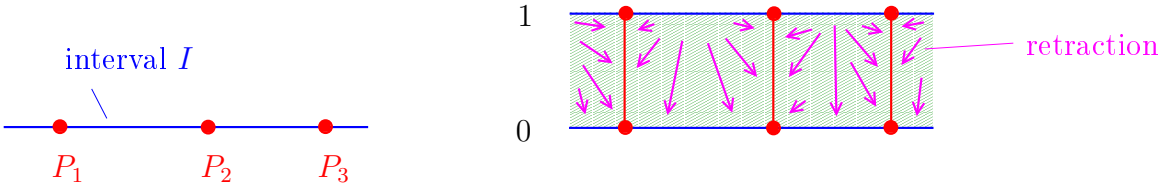
$$\begin{array}{rccl} s\colon & \mathrm{Cyl}(i) & \to & X \times [0,1] \\ & P & \mapsto & \begin{cases} (i(a), 1-t), & \text{if } P = [(a,t)] \text{ with } a \in A \text{ and } t \in [0,1], \\ (P, 0), & \text{if } P \in X \end{cases} \end{array}$$

admits a left-inverse, in other words, there exists a map $r\colon X \times [0,1] \to \mathrm{Cyl}(i)$ such that $r \circ s = \mathrm{id}_{\mathrm{Cyl}(i)}$.

If A is a subset of X and if $i\colon A \to X$ is the inclusion map, then (1) and (2) are equivalent to

(3) The set $(X \times \{0\}) \cup (A \times [0,1])$ is a retract of $X \times [0,1]$.

Example. Let I be an interval in $\mathbb{R}$ and let $P_1, \ldots, P_k$ be finitely many distinct points in I. In the figure below we show a retraction from $I \times [0,1]$ to $(I \times \{0\}) \cup (\{P_1, \ldots, P_k\} \times [0,1])$. Together with the Cofibration Characterization Lemma 17.2 this shows that the inclusion $\{P_1, \ldots, P_k\} \to I$ is a cofibration.



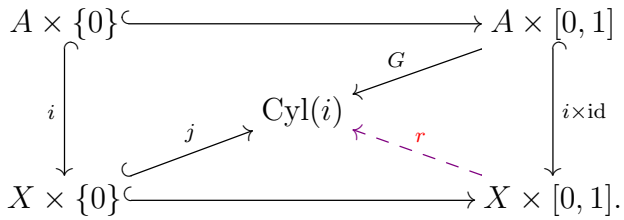
Proof. First we prove the "(1) ⇒ (2)"-direction. Thus we now suppose that the map $i\colon A \to X$ is a cofibration. We consider the inclusion map $j\colon X \times \{0\} \to \mathrm{Cyl}(i)$ which is

[181] It follows from the Topological Pushout Proposition 5.33 (1b) that this map is continuous.

given by $j(x,0) = x$ and we consider the map

$$\begin{array}{rcl} G\colon A \times [0,1] & \to & \mathrm{Cyl}(i) \\ (a,s) & \mapsto & [(a, 1-s)]. \end{array}$$

We obtain the following diagram:

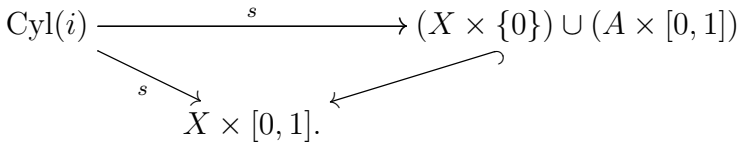


One easily verifies that the black diagram commutes. Since $i\colon A \to X$ is a cofibration we obtain a map $r\colon X \times [0,1] \to \mathrm{Cyl}(i)$ which makes the diagram commute. It is straightforward to verify that $r \circ s = \mathrm{id}_{\mathrm{Cyl}(i)}$.

The proof of the converse, i.e. the proof of the "(2) $\Rightarrow$ (1)"-direction is also quite straightforward. We will provide the proof in Exercise 17.3.

Now we assume that A is a subset of X and that $i\colon A \to X$ is the inclusion map. We need to provide the proof for the equivalence of (2) and (3). This turns out to be quite subtle.

(a) First we assume that A is a closed subset of X.[182] Note that the image of $\mathrm{Cyl}(i)$ in $X \times [0,1]$ under the map s is precisely $(X \times \{0\}) \cup (A \times [0,1])$. Thus we have the following commutative diagram



To show that (2) and (3) are equivalent it suffices to show that the top horizontal map is a homeomorphism. It is basically clear that the top horizontal map is a bijection. Furthermore it follows easily from the Topological Pushout Proposition 5.33 (1b) that the map is continuous. Finally it follows from our hypothesis that A is a *closed* subset, together with Lemma 16.11 (2), that the top horizontal map is indeed a homeomorphism.

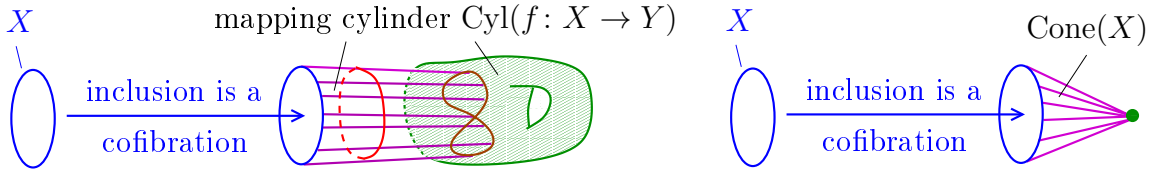
(b) The logic of (a) breaks down if A is not a closed subset. More precisely, in Lemma 16.11 (3) we saw that in general the top horizontal map is *not* a homeomorphism. Rather surprisingly it is nonetheless true that (2) and (3) are equivalent. This delicate argument is given in [**Strø1968**, Lemma 4] or alternatively in [**Hat2002**, Proposition A.18]. ■

The next proposition gives another rich source of examples of cofibrations.

Proposition 17.3. (Cone-Mapping Cylinder Inclusion Cofibration Proposition)

[182] This is also the only case we will ever need in practice.

(1) If $f\colon X \to Y$ is a map between two topological spaces, then the natural inclusion $i\colon X \to \mathrm{Cyl}(f)$ is a closed cofibration.
(2) Let X be a non-empty topological space. The natural inclusion $X \to \mathrm{Cone}(X)$ is a closed cofibration.



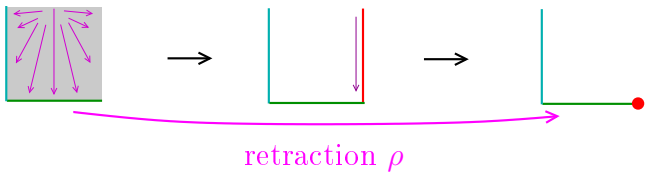
The proof of the Cone-Mapping Cylinder Inclusion Cofibration Proposition 17.3 relies on the following elementary lemma.

Lemma 17.4. There exists a retraction

$$\rho\colon [0,1]\times[0,1] \;\to\; (\{0\}\times[0,1]) \cup ([0,1]\times\{0\})$$

that has the property that $\rho(\{1\}\times[0,1]) = \{(1,0)\}$.

Proof. In the figure below we show how such a retraction ρ can be constructed, namely we first retract to the $\sqcup$-shaped set $(\{0,1\}\times[0,1]) \cup ([0,1]\times\{0\})$ and then we retract to $(\{0\}\times[0,1]) \cup ([0,1]\times\{0\})$ by collapsing $\{1\}\times[0,1]$ to the point $\{0,1\}$. ■



Proof of the Cone-Mapping Cylinder Inclusion Cofibration Proposition 17.3. First note that if X is a non-empty topological space and if $f\colon X \to \{*\}$ is the constant map, then, as we pointed out on page 454, the mapping cylinder $\mathrm{Cyl}(f)$ is basically the cone of X. Thus we see that Statement (2) is a special case of Statement (1).

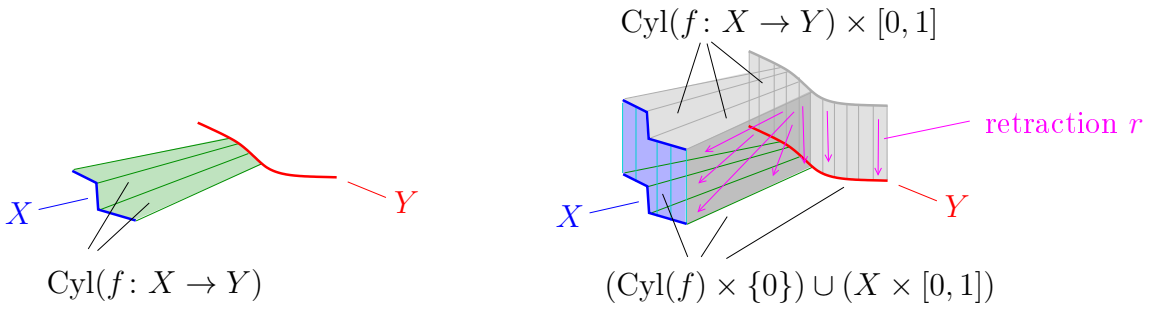
It remains to prove Statement (1). Let $f\colon X \to Y$ be a map between topological spaces. By the Mapping Cylinder Basics Lemma 16.10 (4) we know that the image of X in $\mathrm{Cyl}(f)$ is a closed subset. Thus it remains to show that $i\colon X \to \mathrm{Cyl}(f)$ is a cofibration. By the Mapping Cylinder Basics Lemma 16.10 (3) the map i is an embedding, thus we can identify X with its image in $\mathrm{Cyl}(f)$.

By the Cofibration Characterization Lemma 17.2 (3)⇒(1) it suffices to prove the following claim:

Claim. There is a retraction r from $\mathrm{Cyl}(f)\times[0,1]$ to its subset $(\mathrm{Cyl}(f)\times\{0\})\cup(X\times[0,1])$.

Proof. To construct the desired retraction r we recall that in Lemma 17.4 we saw that there exists a retraction

$$s\colon [0,1]\times[0,1] \;\to\; (\{0\}\times[0,1]) \cup ([0,1]\times\{0\})$$

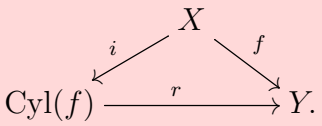


that has the property that $s(\{1\} \times [0,1]) = \{(1,0)\}$. Now the desired retraction is given by the map[183]

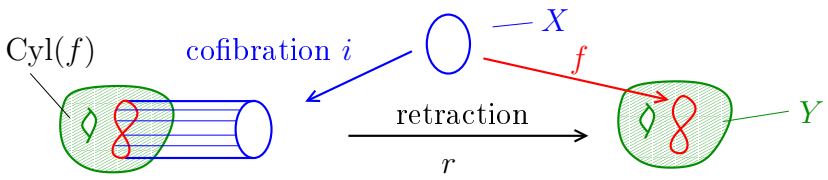
$$r\colon \operatorname{Cyl}(f) \times [0,1] \to (\operatorname{Cyl}(f) \times \{0\}) \cup (X \times [0,1])$$
$$P \mapsto \begin{cases} [(x, s(t,u))], & \text{if } P = [(x,t,u)] \text{ with } x \in X \text{ and } t,u \in [0,1], \\ [(y,0)], & \text{if } P = [(y,u)] \text{ with } y \in Y \text{ and } u \in [0,1]. \end{cases}$$ ■

We conclude this section with a proposition which says that any map $f\colon X \to Y$ between topological spaces can be "replaced" by a cofibration $X \to \operatorname{Cyl}(f)$.

Proposition 17.5. (Replace-by-Cofibration Proposition) Let $f\colon X \to Y$ be a map between topological spaces. We denote by $i\colon X \to \operatorname{Cyl}(f)$ the natural inclusion. The natural retraction $r\colon \operatorname{Cyl}(f) \to Y$ is a homotopy equivalence and the following diagram commutes:



By the Cone-Mapping Cylinder Inclusion Cofibration Proposition 17.3 (1) we know that the left diagonal map $i\colon X \to \operatorname{Cyl}(f)$ is a cofibration.



Proof. We showed in the Mapping Cylinder Basics Lemma 16.12 (2a) that the natural retraction $r\colon \operatorname{Cyl}(f) \to Y$ is a homotopy equivalence. It is clear that the given diagram commutes. Thus we are done by the Cone-Mapping Cylinder Inclusion Cofibration Proposition 17.3 (1). That was quick. ■

17.2. **Properties of cofibrations.** The following lemma summarizes a few key properties of cofibrations.

[183] This map is continuous by Pasting Proposition 2.6 (2).

Lemma 17.6. (Cofibration-Closed Injective Lemma) If $i\colon A \to X$ is a cofibration, then the following three statements hold:

(1) the map $i\colon A \to X$ is injective,
(2) the map $i\colon A \to X$ is an embedding,
(3) if X is Hausdorff, then $i(A)$ is a closed subset of X, i.e. i is a closed cofibration.

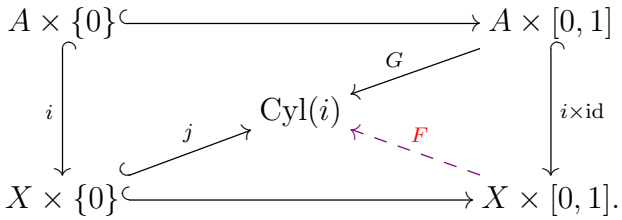
Remark. In Exercise 17.6 we will see that in the Cofibration-Closed Injective Lemma 17.6 (3) we cannot drop the hypothesis that X is Hausdorff.

Proof. Let $i\colon A \to X$ be a cofibration.

(1) Similar to the proof of the Cofibration Characterization Lemma 17.2 we consider the inclusion map $j\colon X \times \{0\} \to \mathrm{Cyl}(i)$ which is given by $j(x,0) = x$ and we consider the map

$$\begin{array}{rccl} G\colon & A \times [0,1] & \to & \mathrm{Cyl}(i) \\ & (a,s) & \mapsto & [(a,1-s)]. \end{array}$$

We obtain the following diagram:



One easily verifies that the black diagram commutes. Since i is a cofibration we obtain a map $F\colon X \times [0,1] \to \mathrm{Cyl}(i)$ which makes the diagram commute. Note that by design for any $t \in (0,1]$ the map $G\colon A \times \{t\} \to \mathrm{Cyl}(i)$ is injective. Since the diagram commutes this implies that $i \times \mathrm{id}_{\{t\}}\colon A \times \{t\} \to X \times \{t\}$ is also injective, i.e. $i\colon A \to X$ is injective.

(2) We continue with the argument and the notation in (1). Let $t \in (0,1]$. The map $G|_{A\times\{t\}}\colon A \times \{t\} \to A \times \{1-t\}$ is evidently a homeomorphism. Thus we see that

$$(G|_{A\times\{t\}})^{-1} \circ F|_{X\times\{t\}}\colon i(A) \times \{t\} \ \to \ A \times \{t\}$$

is an inverse to the map $i \times \mathrm{id}_{\{t\}}\colon A \times \{t\} \to X \times \{t\}$. This implies that the map $i \times \mathrm{id}_{\{t\}}\colon A \times \{t\} \to X \times \{t\}$ is an embedding, but this also implies that the map $i\colon A \to X$ is an embedding.

(3) Note that

$$\begin{array}{ccl} & & \text{for } x \in A \text{ we have } F(x,0) = G(x,0) = [(x,1)] = [x] \text{ by construction of } F, \text{ furthermore} \\ & & \text{for } x \notin A \text{ the point } [x] \text{ does not lie in } \mathrm{Cyl}(i), \text{ thus it does not lie in the image of } F \\ A & \overset{\downarrow}{=} & \{x \in X \mid F(x,0) = [x]\} \\ & = & \left\{ \begin{array}{l} \text{the points in } X \text{ on} \\ \text{which the maps} \end{array} \begin{array}{ccc} X & \to & \mathrm{Cyl}(i) \\ x & \mapsto & F(x,0) \end{array} \text{ and } \begin{array}{ccc} X & \to & \mathrm{Cyl}(i) \\ x & \mapsto & [x] \end{array} \text{ agree} \right\}. \end{array}$$

It follows from this description of A, our hypothesis that X is Hausdorff (which implies by (2) and the Mapping Cylinder Basics Lemma 16.10 (2) that $\mathrm{Cyl}(i)$ is also Hausdorff) and Exercise 2.33 that A is indeed a closed subset of X. ∎

The following lemma shows that inclusion maps that are cofibrations are particularly nicely behaved.

Lemma 17.7. (Cofibration-Quotient Lemma) Let X be a topological space and let $A \subset X$ be a non-empty subset. We denote by $i\colon A \to X$ the inclusion map. If i is a closed cofibration, then the following three statements hold:

(1) The map[184]

$$\begin{array}{rcl} \varphi\colon \mathrm{Cone}(i\colon A \to X) & \to & X/A \\ P & \mapsto & \begin{cases} [P], & \text{if } P \in X, \\ [A], & \text{if } P = [(a,t)] \text{ with } a \in A \text{ and } t \in [0,1] \end{cases} \end{array}$$

is a homotopy equivalence.

(2) If A is contractible, then the projection $X \to X/A$ is a homotopy equivalence.

(3) If $a_0 \in A$ is a point such that $\{a_0\}$ is a deformation retract of A, then the projection $X \to X/A$ is a homotopy equivalence between the pointed topological spaces (X, a_0) and $(X/A, [A])$.

Remark. The statement of the Cofibration-Quotient Lemma 17.7 (2) is related in spirit to the statement of Proposition 70.5.

Proof. Let X be a topological space and let $A \subset X$ be a closed subset such that the inclusion $i\colon A \to X$ is a cofibration.

(1) Similar to the proof of the Cofibration-Closed Injective Lemma 17.6 (1) we consider the following two maps:

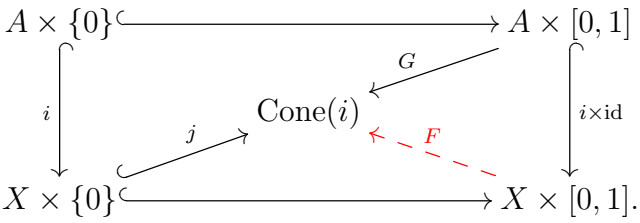
(a) First of all we consider the obvious inclusion map $j\colon X \times \{0\} \to \mathrm{Cone}(i)$ which is given by $j(x,0) = x$.

(b) We consider the map

$$\begin{array}{rcl} G\colon A \times [0,1] & \to & \mathrm{Cone}(i) \\ (a,s) & \mapsto & [(a, 1-s)]. \end{array}$$

Note that G_1 sends all points in A to the cone point.

We obtain the following diagram:



One easily verifies that the diagram commutes. Since i is a cofibration we obtain a map $F\colon X \times [0,1] \to \mathrm{Cone}(i)$ which makes the diagram commute. Note that the map $F_1\colon X \to \mathrm{Cone}(i)$ sends all points in A to the cone point. Thus F_1 descends to a map $\psi\colon X/A \to \mathrm{Cone}(i)$. It remains to prove the following claim.

Claim. The maps $\varphi\colon \mathrm{Cone}(i) \to X/A$ and $\psi\colon X/A \to \mathrm{Cone}(i)$ are homotopy inverses of one another.

[184]It follows from the ever popular Pasting Proposition 2.6 (2) and the fact that A is a closed subset of X that the map φ is continuous.

Proof. Note that $F(A \times [0,1]) = G(A \times [0,1]) \subset A$. Therefore it follows from Lemma 14.8 that the map

$$H\colon (X/A) \times [0,1] \xrightarrow{([x],t)\mapsto[(x,t)]} (X \times [0,1])/(A \times [0,1]) \xrightarrow{[(x,t)]\mapsto[F(x,t)]} X/A$$

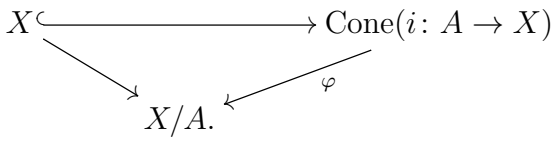
is continuous. It follows immediately from the definitions that H is a homotopy between $\mathrm{id}_{X/A}$ and $\varphi \circ \psi$.

Conversely we consider the map

$$\begin{array}{rcl} H\colon \mathrm{Cone}(i) \times [0,1] & \to & \mathrm{Cone}(i) \\ (P,s) & \mapsto & \begin{cases} F(P,s), & \text{if } P \in X, \\ [a,(1-s)\cdot t], & \text{if } P = [(a,t)] \text{ with } a \in A \text{ and } t \in [0,1]. \end{cases} \end{array}$$

One easily verifies that this is a homotopy from $\mathrm{id}_{\mathrm{Cone}(i)}$ to $\psi \circ \varphi$. ⊞

(2) We consider the following diagram



It follows immediately from the definition of φ that this diagram commutes. Now suppose that A is contractible. By Proposition 16.18 (1) there exists a deformation retraction G from $\mathrm{Cone}(i\colon A \to X)$ to X. By Statement (1) we know that φ is a homotopy equivalence. Thus it follows from the Homotopy-Equivalence Basics Lemma 15.1 (1) that the map $X \to X/A$ is a homotopy equivalence.

(3) Now assume that $a_0 \in A$ is a point which has the property that $\{a_0\}$ is a deformation retract of A. By Proposition 16.18 (2) there exists a deformation retraction G from $\mathrm{Cone}(i\colon A \to X)$ to X such that for any $t \in [0,1]$ we have $G_1([(a_0,t)]) = a_0$. Next we point out that the homotopy equivalence H from (1) has the property that $H(a_0,t) = [(a_0,1-t)]$. If one combines these observations with the argument in (2) it is not difficult to see that the projection is in fact a homotopy equivalence of the pointed topological spaces (X,a_0) and $(X/A,[a_0])$. ■

17.3. **"Real-life" Examples of cofibration.** In future chapters we will discuss many interesting types of topological spaces, e.g. smooth and topological manifolds, CW-complexes and simplicial complexes. To make the search for results less confusing we now list several settings that give rise to cofibrations, even though we have not introduced the concepts yet.

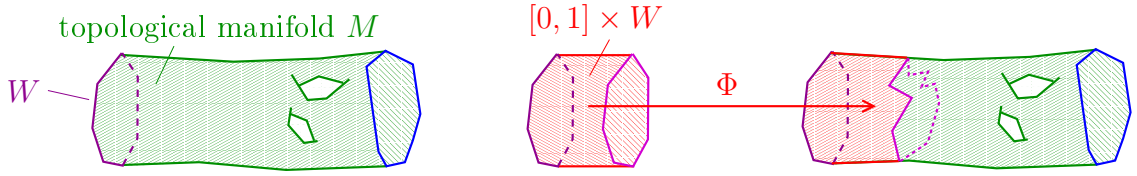
On page 511 we will introduce the notion of a topological manifold M and its boundary ∂M and on page 543 we will introduce the notion of a smooth manifold M and its boundary ∂M. At this state we will not give the definitions, but in the figure below we show a cartoon version of a 2-dimensional topological manifold and of a 2-dimensional smooth manifold which hopefully gives the reader an idea of the concept.

Proposition 17.8. (Manifold Boundary-Cofibration Proposition) Let M be a topological manifold. For any union W of components of ∂M the inclusion $i\colon W \to M$ is a closed cofibration.



Proof. Let M be a topological manifold and let W be a union of components of ∂M. It follows from the Topological Collar Neighborhood Theorem 75.5[185] that there exists a map $\Phi\colon [0,1]\times W \to M$ with the following four properties:

(1) The map Φ is an embedding.
(2) The image of $[0,1)\times W$ is an open subset of M.
(3) For every $P\in W$ we have $\Phi(0,P)=P$.
(4) The image of $[0,1]\times W$ is a closed subset of M.



Recall that in Lemma 17.4 we saw that there exists a retraction

$$\varphi=(\varphi_1,\varphi_2)\colon [0,1]\times[0,1] \;\to\; (\{0\}\times[0,1])\cup([0,1]\times\{0\})$$

that has the property that $\varphi(\{1\}\times[0,1])=\{(1,0)\}$. We consider the map[186]

$$r\colon M\times[0,1] \;\to\; \mathrm{Cyl}(i)=((W\times[0,1])\sqcup M)/\!\sim \quad \text{where } (x,1)\sim x \text{ for } x\in W$$

$$(P,t) \;\mapsto\; \begin{cases} (\Phi(\underbrace{z\cdot\varphi_1(s,t),Q}_{\in\Phi([0,1]\times W)}),\varphi_2(s,t)), & \text{if } P=\Phi(s,Q) \text{ with } s\in[0,1] \text{ and } Q\in W,\\ P, & \text{if } P\in M\setminus\Phi([0,1)\times W). \end{cases}$$

(We refer to the figure below for an illustration of the map r.) One can easily verify that r is a left-inverse to the map $s\colon \mathrm{Cyl}(i)\to M\times[0,1]$ that we had considered in the Cofibration Characterization Lemma 17.2 (2). Thus it follows from the Cofibration Characterization Lemma 17.2 that $i\colon W\to M$ is a cofibration. Finally it follows from the Topological Manifold Boundary Proposition 75.2 (4a) that the union W of boundary components of the topological manifold M is a closed subset.[187] It follows that $i\colon W\to M$ is in fact a closed cofibration. ∎

Next we turn to "proper submanifolds" of smooth manifolds. We will introduce this concept on page 550. In the figure below we again provide a cartoon to give the reader an idea what the concept of a "proper submanifold" is about.

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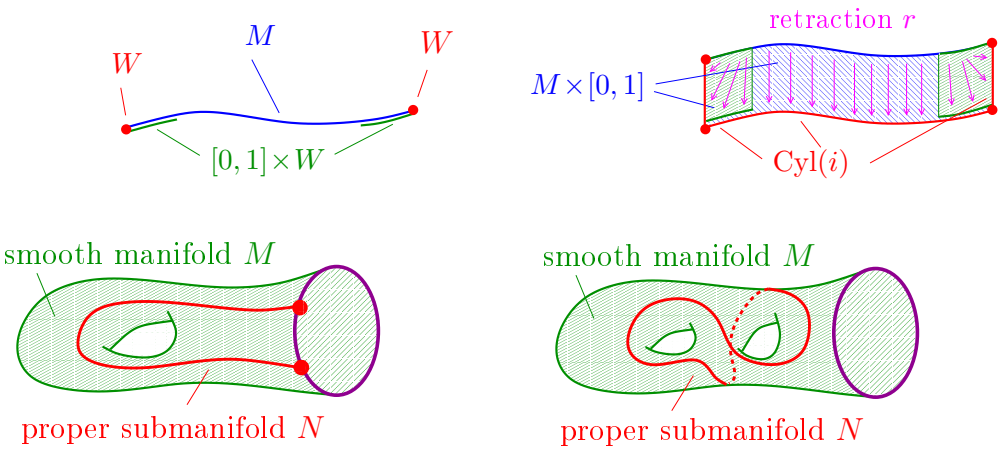
[185] For smooth manifolds we could also use the Smooth Collar Neighborhood Theorem 28.3.

[186] Recall that by (2) and (4) the sets $\Phi([0,1]\times W)$ and $M\setminus\Phi([0,1)\times W)$ are actually closed subsets of M. It now follows from this observation together with the Basics-of-Collars Lemma 28.4 and Pasting Proposition 2.6 (2) that the map r is indeed continuous.

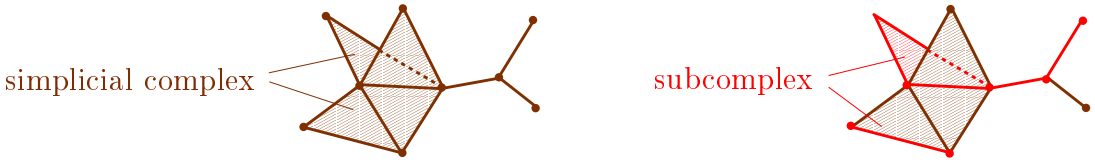
[187] See also the Smooth Manifold Boundary Proposition 19.26 (3a) for the corresponding statement for smooth manifolds which is much easier to prove.



Proposition ??. (Submanifold-Cofibration) Let M be a smooth manifold. For any compact proper submanifold N of M the inclusion $i\colon N \to M$ is a closed cofibration.

Proof. The proof of the Submanifold-Cofibration Proposition **??** relies on the General Tubular Neighborhood Theorem **??** which can be viewed as a submanifold analogue of the Topological Collar Neighborhood Theorem 75.5 and the Smooth Smooth Collar Neighborhood Theorem 28.3. Since the General Tubular Neighborhood Theorem **??** is more delicate to formulate we refrain from doing so right now and we refer to the impatient reader to the proof of the Submanifold-Cofibration Proposition **??** on page **??**. ■

Finally we turn to CW-complexes and simplicial complexes. We will define the notion of a CW-complex on page 1478 and we will define the notion of a simplicial complex on page 1994. As we will see on pages 1484 and 1995 there are also natural notions of subcomplexes. We will not give any of the definitions. But as an illustration we show in the figure below a simplicial complex together with a subcomplex.



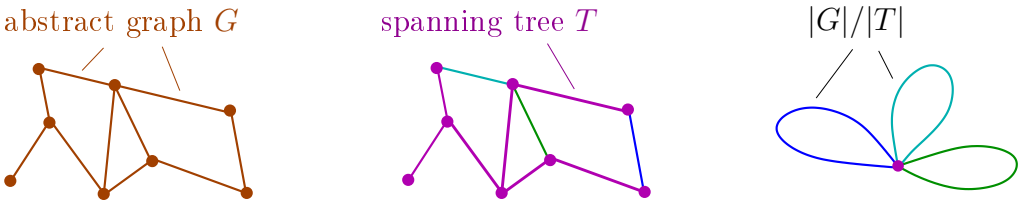
We also point out that a topological graph, as defined on page 310, is essentially the same as a 1-dimensional CW-complex and that by the Simplicial-Implies-CW Lemma 93.24 every simplicial (sub-) complex is a CW- (sub-) complex.
The following theorem gives us one of the most important examples of a cofibration.

Theorem 17.9. (Subcomplex Cofibration Theorem) Let X be a CW-complex or a simplicial complex or a topological graph. Given any subcomplex A (or subgraph A) the corresponding inclusion map $i\colon A \to X$ is a closed cofibration.

Proof. By the Simplicial-Implies-CW Lemma 93.24 we only need to consider the case of a CW-complex. In this setting the desired statement is precisely the statement of the Homotopy Extension Theorem 70.1. The special case that X is the topological realization of a graph will also be dealt with in Exercise 17.4. ■

Proposition 17.10. (Graph-mod-Tree Proposition) Let $G = (V, E, i, t)$ be a finite connected non-empty abstract graph in the sense of the definition on page 307.

(1) There exists spanning tree T for G.
(2) The quotient $|G|/|T|$ is naturally homeomorphic to the topological realization of a finite abstract graph that has a single vertex.
(3) The projection $|G| \to |G|/|T|$ is a homotopy equivalence.



Remark. In Proposition 50.15 we will give a hands-on proof of the Graph-mod-Tree Proposition 17.10 that, for better or worse, does not use the language of cofibrations. In Proposition 70.4 we will generalize the Graph-mod-Tree Proposition 17.10 in a suitable sense to infinite graphs.

Proof. Let G be a finite connected non-empty abstract graph.

(1) By the Spanning Tree Existence Proposition 7.10 we know that G admits a spanning tree T and that T contains all vertices of G.
(2) As on page 321 we consider the corresponding quotient abstract graph G/T. Since T contains all vertices of G we see that G/T is an abstract graph with a single vertex. By the Quotient Abstract Graph Lemma 7.14 we know that $|G/T|$ is homeomorphic to $|G|/|T|$.
(3) In Exercise 15.6 we showed, using Lemma 7.7, that topological realizations of trees are contractible. Furthermore by the Subcomplex Cofibration Theorem 17.9, or alternatively by Exercise 17.4, we know that the inclusion map $i\colon |T| \to |G|$ is a closed cofibration. Thus it follows from the above together with the Cofibration-Quotient Lemma 17.7 that the projection $|G| \to |G|/|T|$ is a homotopy equivalence. ■

17.4. **Products and cofibrations.** In this and the following two sections we collect several technical useful results on cofibrations. As stand-alone sections these are arguably not much fun to read. Thus it is better to move on and to come back to these three sections when the results are needed.

Proposition 17.11. (Product-Cofibration Proposition)

(1) If $f\colon A \to X$ and $g\colon B \to Y$ are two (closed) cofibrations, then the product map $f \times g\colon A \times B \to X \times Y$ is also a (closed) cofibration.
(2) If $f\colon A \to X$ is a (closed) cofibration and if Z is a topological space, then the resulting map $f \times \mathrm{id}_Z\colon A \times Z \to X \times Z$ is also a (closed) cofibration.

Example. Let I be an interval and let $P \in I$. On page 479 we saw that the inclusion $\{P\} \to I$ is a cofibration. It follows from the Product-Cofibration Proposition 17.11 that given any topological space Z the inclusion $\{P\} \times Z \to I \times Z$ is a cofibration.

Proof. It is clear that Statement (2) is an immediate consequence of Statement (1). Thus it remains to prove Statement (1). Now let $f\colon A \to X$ and $g\colon B \to Y$ be two cofibrations,

By the Cofibration-Closed Injective Lemma 17.6 we can assume that $f\colon A \to X$ and $g\colon B \to Y$ are both inclusions. We point out that by the Cofibration Characterization Lemma 17.2 (1) $\Rightarrow$ (3) there exist retractions $r\colon X \times [0,1] \to (X \times \{0\}) \cup (A \times [0,1])$ and $s\colon Y \times [0,1] \to (Y \times \{0\}) \cup (B \times [0,1])$. We denote by $r =: (r_X, r_{[0,1]})$ the coordinate functions of r viewed as a map with values in $X \times [0,1]$. Similarly we define $s =: (s_Y, s_{[0,1]})$. Next we consider the map

$$\begin{array}{rcl} (X \times Y) \times [0,1] & \to & ((X \times Y) \times \{0\}) \cup ((A \times B) \times [0,1]) \\ ((x,y),t) & \mapsto & \big(\underbrace{(r_X(x,t), s_Y(y,t))}_{\in X \times Y}, \underbrace{\tfrac{1}{2}(r_{[0,1]}(x,t) + s_{[0,1]}(y,t))}_{\in [0,1]}\big). \end{array}$$

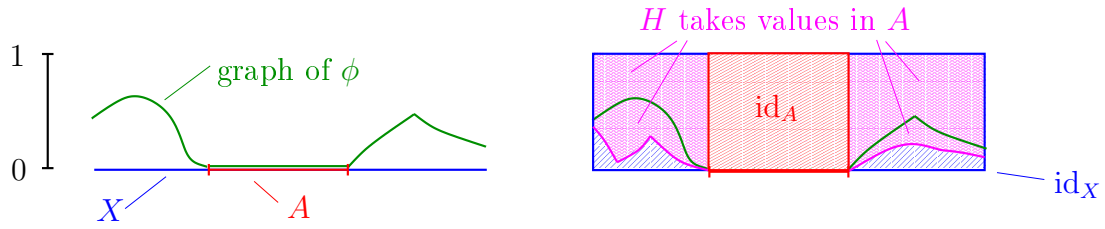
One can easily verify that this map is a retraction. It follows from the Cofibration Characterization Lemma 17.2 (3) $\Rightarrow$ (1) that $f \times g\colon A \times B \to X \times Y$ is indeed a cofibration.

If f and g are in fact closed cofibrations, then it follows from the Product-of-Subsets Lemma 5.5 (1) and the above that $f \times g$ is also a closed cofibration. ∎

We continue with the following characterization of closed cofibrations that we initially proved in [**Strø1966, Strø1968**].

Theorem 17.12. (Strom's Cofibration Characterization Theorem) Let X be a topological space and let A be a closed subset. The following two statements are equivalent:

(1) The inclusion $i\colon A \to X$ is a cofibration.
(2) There exists a map $\phi\colon X \to [0,1]$ such that $A = \phi^{-1}(0)$ and there exists a homotopy $H\colon X \times [0,1] \to X$ rel A such that $H_0 = \mathrm{id}_X$ and such that $H(x,t) \in A$ whenever $t > \phi(x)$.



Remark.

(1) We continue with the notation of the above Strom's Cofibration Characterization Theorem 17.12. We consider the open subset $U := \phi^{-1}([0,1))$. The homotopy H restricts to a homotopy $K\colon U \times [0,1] \to X$ rel A with $K_0 = \mathrm{id}$ and with $K_1(U) \subset A$. This almost sounds like A is a deformation retract of U, but the given situation is somewhat looser, since K can take values outside of U.
(2) Sometimes a topological space X together with a closed subset A that admit H and ϕ as in Strom's Cofibration Characterization Theorem 17.12 (2) is called a *neighborhood deformation retract pair*, or short, *NDR-pair*.
(3) Let X be a topological space and let $x_0 \in X$. The pointed space (X, x_0) is called *well-pointed* if the inclusion $\{x_0\} \to X$ is a cofibration. If (X, x_0) is well-pointed, then in most practical situations the point $x_0 \in X$ is good in the sense of the definition

on page 1171. Nonetheless, as discussed in

https://mathoverflow.net/questions/87387/

it is not necessarily the case that x_0 is good.

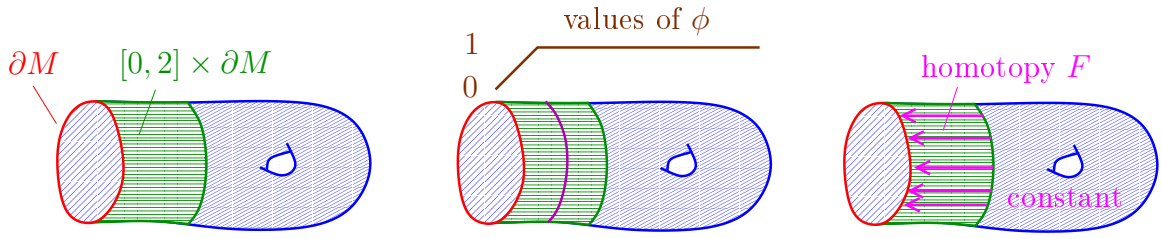
Example. Let M be a topological manifold. As in the proof of the Manifold Boundary-Cofibration Proposition 17.8 we appeal to the Topological Collar Neighborhood Theorem 75.5 to obtain a collar neighborhood $[0,2] \times \partial M$ of ∂M. Next we consider the maps

$$\begin{array}{rcl} \phi\colon M & \to & [0,1] \\ P & \mapsto & \begin{cases} s, & \text{if } P=(Q,s) \text{ with } Q \in \partial M \text{ and } t \in [0,1], \\ 1, & \text{otherwise} \end{cases} \end{array}$$

and

$$\begin{array}{rcl} F\colon M \times [0,1] & \to & M \\ (P,t) & \mapsto & \begin{cases} (Q, \frac{2(s-t)}{2-t}), & \text{if } P=(Q,s) \text{ with } Q \in \partial M \text{ and } s \in [t,2], \\ (Q,0), & \text{if } P=(Q,s) \text{ with } Q \in \partial M \text{ and } s \in [0,t], \\ P, & \text{otherwise.} \end{cases} \end{array}$$

One can easily verify that the conditions of Strom's Cofibration Characterization Theorem 17.12 are satisfied. This gives basically a new proof of the Manifold Boundary-Cofibration Proposition 17.8.



Proof. First we deal with the "(1)⇒(2)"-direction. Thus we suppose that $i\colon A \to X$ is a cofibration. Note that by the Cofibration Characterization Lemma 17.2 (1)⇒(3) we know that there exists a retraction $r\colon X \times [0,1] \to (X \times \{0\}) \cup (A \times [0,1])$. We denote the two coordinate functions of r, viewed as a map with values in $X \times [0,1]$, by $r_X\colon X \times [0,1] \to X$ and $r_{[0,1]}\colon X \times [0,1] \to [0,1]$. We consider the following two maps:

$$\begin{array}{rcl} \phi\colon X & \to & [0,1] \\ x & \mapsto & \max\limits_{t \in [0,1]} |t - r_{[0,1]}(x,t)| \end{array} \quad \text{and} \quad \begin{array}{rcl} H\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & r_X(x,t). \end{array}$$

Note that by definition of a retraction we have $H_0 = \mathrm{id}_X$.

Claim 1. The map ϕ is well-defined and continuous.

Proof. We consider the map $\theta\colon X \times [0,1] \to [0,1]$ that is given by $\theta(x,t) := |t - r_{[0,1]}(x,t)|$. Since $[0,1]$ is compact and since θ is continuous we obtain from the Compact Image Lemma 2.13 (2) that θ assumes a maximum, hence ϕ is well-defined.

It remains to show that $\phi\colon X \to [0,1]$ is continuous. Let V be an open subset of $[0,1]$. By the JH-Lemma 1.7 it suffices to show that given any $x_0 \in \phi^{-1}(V)$ there exists an open neighborhood U of x_0 with $U \subset \phi^{-1}(V)$. First we consider the case that $\phi(x_0) \in (0,1)$. This implies that we can pick $\epsilon > 0$ such that $(\phi(x_0) - \epsilon, \phi(x_0) + \epsilon) \subset V$. Note that by

definition of ϕ and θ we have $\theta(\{x_0\} \times [0,1]) \subset [0, \phi(x_0)]$. Since $[0,1]$ is compact and since $[0, \phi(x_0) + \epsilon)$ is an open subset of $[0,1]$ we obtain from Exercise 5.9 that there exists an open neighborhood U_1 of $x_0 \in X$ such that

$$\{x_0\} \times [0,1] \ \subset \ U_1 \times [0,1] \ \subset \ \theta^{-1}([0, \phi(x_0) + \epsilon)).$$

Hence, for every $x \in U_1$ we have $\phi(x) \leq \phi(x_0) + \epsilon$. Next suppose we are given $t_0 \in [0,1]$ such that $\theta(x_0, t_0) \in (\phi(x_0) - \epsilon, \phi(x_0) + \epsilon)$. By definition of the product topology there exists an open subset $U_2 \subset X$ such that

$$(x_0, t_0) \ \in \ U_2 \times \{t_0\} \ \subset \ \theta^{-1}((\phi(x_0) - \epsilon, \phi(x_0) + \epsilon)).$$

Note that this implies that for every $x \in U_2$ we have $\phi(x) \geq \phi(x_0) - \epsilon$. In particular we see that $U_1 \cap U_2 \subset \phi^{-1}((\phi(x_0) - \epsilon, \phi(x_0) + \epsilon)) \subset V$. Basically the same argument also works if $\phi(x_0) = 0$ or $\phi(x_0) = 1$. ⊞

Claim 2. We have $A = \phi^{-1}(0)$.

Proof. First let $x \in A$. Since r is a retraction we know that $r(x,t) = (x,t)$ for all $t \in [0,1]$. This means that $r_{[0,1]}(x,t) = t$ for all $t \in [0,1]$. Thus we see that $\phi(x) = 0$. Conversely, suppose that we have $x \in X$ with $\phi(x) = 0$. One easily verifies that this implies that for every $t > 0$ we have $r(x,t) \in A \times [0,1]$, in other words, we have $\{x\} \times (0,1] \subset r^{-1}(A \times [0,1])$. Since $A \times [0,1]$ is a closed subset of $X \times [0,1]$ we see that the closure of $\{x\} \times (0,1]$ is contained in $r^{-1}(A \times [0,1])$. But the closure is precisely $\{x\} \times [0,1]$. Thus we have shown that $\{x\} \times [0,1]$ also belongs to $r^{-1}(A \times [0,1])$. But this implies that $x \in A$. ⊞

It remains to prove the following claim.

Claim 3. Given $(x,t) \in X \times [0,1]$ with $t > \phi(x)$ we have $H(x,t) \in A$.

Proof. So suppose we are given $(x,t) \in X \times [0,1]$ with $t > \phi(x)$. Note that this implies that $t > \phi(x) \geq |t - r_{[0,1]}(x,t)|$. This is only possible if $r_{[0,1]}(x,t) > 0$. Thus we see that $r(x,t) \in A \times [0,1]$. This implies that $H(x,t) \in A$. This concludes the proof of the claim and thus completes the proof of the "(2)⇒(1)"-direction.

Finally we provide the proof for the "(1)⇒(2)"-direction. Given $\phi\colon X \to [0,1]$ and the homotopy $H\colon X \times [0,1] \to X$ we consider the map

$$\begin{array}{rcl} r\colon X \times [0,1] & \to & (X \times \{0\}) \cup (A \times [0,1]) \\ (x,t) & \mapsto & (H(x,t), \max\{t - \phi(x), 0\}). \end{array}$$

One can easily verify that this map is indeed a retraction. Thus it follows from the Cofibration Characterization Lemma 17.2 (3)⇒(1) that the inclusion $i\colon A \to X$ is a cofibration. ■

Proposition 17.13. Let $A \subset X$ and $B \subset Y$ be subsets of topological spaces. If the inclusion maps $i\colon A \to X$ and $j\colon B \to Y$ are closed cofibrations, then the inclusion map

$$(A \times Y) \cup (X \times B) \ \to \ X \times Y$$

is also a closed cofibration.

Remark. In Exercise 17.7 we will see that the conclusion of Proposition 17.13 does not hold, if neither A nor B are closed subsets.

Proof. Since i and j are closed cofibrations we can pick maps $\phi\colon X \to [0,1]$, $\psi\colon Y \to [0,1]$ and homotopies $G\colon X \times [0,1] \to X$ and $H\colon Y \times [0,1] \to Y$ as in Strom's Cofibration Characterization Theorem 17.12. We consider the maps

$$\begin{array}{rcl} \gamma\colon X \times Y & \to & [0,1] \\ (x,y) & \mapsto & \min\{\phi(x), \psi(y)\} \end{array}$$

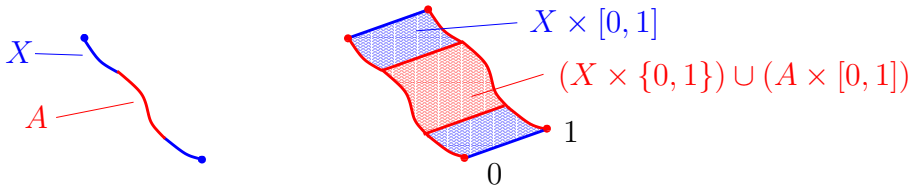
and

$$\begin{array}{rcl} K\colon (X \times Y) \times [0,1] & \to & X \times Y \\ ((x,y),t) & \mapsto & (G(x, \min\{t, \psi(y)\}), H(y, \min\{t, \phi(x)\})). \end{array}$$

It is easy to see that $\gamma^{-1}(0) = (X \times B) \cup (A \times Y)$. Next note that it is basically clear that $K_0 = \mathrm{id}$. Finally note that by construction we have $K((x,y),t) \in (A \times Y) \cup (X \times B)$, whenever $t > \gamma(x,y)$. Therefore it follows from Strom's Cofibration Characterization Theorem 17.12 that the inclusion map $(A \times X) \cup (X \times B) \to X \times Y$ is a closed cofibration. ■

Corollary 17.14. Let X be a topological space and let $A \subset X$ be a subset. If the inclusion $i\colon A \to X$ is a closed cofibration, then the inclusion $(X \times \{0,1\}) \cup (A \times [0,1]) \to X \times [0,1]$ is also a closed cofibration.

Proof. On page 479 we saw that the inclusion map $\{0,1\} \to [0,1]$ is a cofibration. It follows from this observation together with Proposition 17.13 that the given inclusion map is a closed cofibration. ■

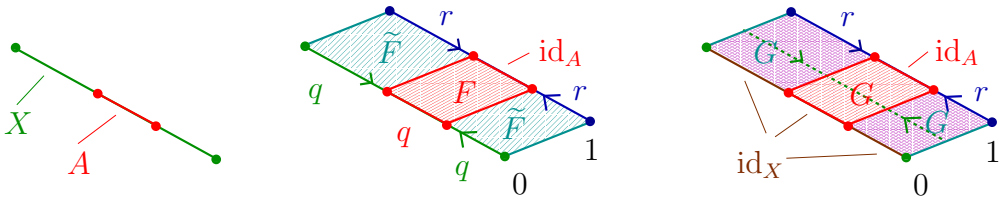


Proposition 17.15. (Homotopy Equivalence-Cofibration Proposition) Let $i\colon A \to X$ be a closed cofibration. If i is a homotopy equivalence, then $i(A)$ is a deformation retract of X.

Proof. Let $i\colon A \to X$ be a cofibration. By the Cofibration-Closed Injective Lemma 17.6 we can assume that A is a closed subset of X and that $i\colon A \to X$ is the inclusion. Let $q\colon X \to A$ be a homotopy inverse of i. Recall that this means that $q \circ i = q\colon A \to A$ is homotopic to id_A and $i \circ q$ is homotopic to id_X.

We pick a homotopy $F\colon A \times [0,1] \to A$ from $q \circ i = q\colon A \to A$ to id_A. Since $i\colon A \to X$ is a cofibration we can extend the map $q\colon X \to A$ on X and the homotopy $F\colon A \times [0,1] \to A$ to a homotopy $\widetilde{F}\colon X \times [0,1] \to A$. Note that $\widetilde{F}_1|_A = F_1|_A = \mathrm{id}_A$, in other words, $\widetilde{F}_1\colon X \to A$ is a retraction. We write $r := \widetilde{F}_1\colon X \to A \subset X$.

Note that $r\colon X \to X$ is homotopic to $q = i \circ q\colon X \to X$ which in turn is homotopic to id_X. Thus it follows from Lemma 14.5 (1) that there exists a homotopy G from id_X to r.

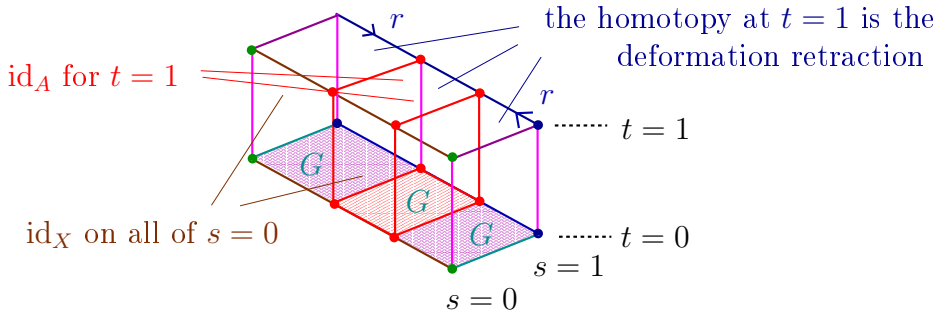


Now we consider the following map

$$\begin{array}{rcl} K\colon ((X\times\{0,1\})\cup(A\times[0,1]))\times[0,1] & \to & X \\ (x,s,t) & \mapsto & \begin{cases} x, & \text{if } s=0, \\ G(r(x),1-t), & \text{if } s=1, \\ G(x,(1-t)s), & \text{if } x\in A, \\ G(x,s), & \text{if } t=0. \end{cases} \end{array}$$

It follows easily from the hypothesis that A is a closed subset together with Pasting Proposition 2.6 (2) that the map is continuous.

By Corollary 17.14 we know that $(X\times\{0,1\})\cup(A\times[0,1])\to X\times[0,1]$ is a cofibration. Thus we can extend the homotopy K to a homotopy $\widetilde{K}\colon (X\times[0,1])\times[0,1]\to X$. The map $\widetilde{K}_1\colon X\times[0,1]\to X$ is now a homotopy rel A from id_X to the retraction r. We have thus shown that A is a deformation retract of X. ∎



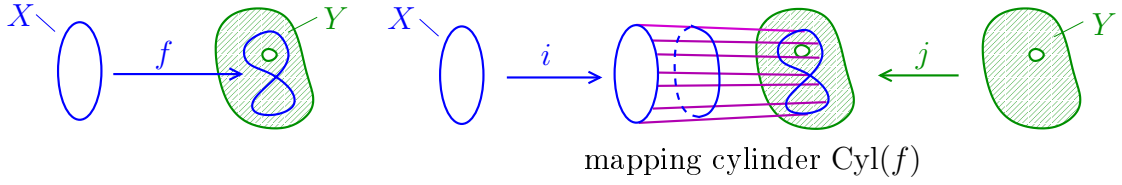
The Homotopy Equivalence-Cofibration Proposition 17.15 admits two interesting corollaries. Since both involve mapping cylinders, let us recall the definition. Given a map $f\colon X\to Y$ between topological spaces we set

$$\mathrm{Cyl}(f) := ((X\times[0,1])\sqcup Y)/\!\sim \text{ where } (x,1)\sim f(x) \text{ for all } x\in X.$$

Furthermore we consider the natural inclusions

$$\begin{array}{rcl} i\colon X & \to & \mathrm{Cyl}(f) \\ x & \mapsto & [(x,0)] \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon Y & \to & \mathrm{Cyl}(f) \\ y & \mapsto & [y] \end{array}$$

The first corollary to the Homotopy Equivalence-Cofibration Proposition 17.15 is a strengthening of the Mapping Cylinder Basics Lemma 16.12 (2b).



Corollary 17.16. Let $f\colon X\to Y$ be a map between two topological spaces. The following two statements are equivalent:

(1) f is a homotopy equivalence,
(2) $i(X)$ is a deformation retract of $\mathrm{Cyl}(f)$.

Proof. By the Mapping Cylinder Basics Lemma 16.12 (2b) we know that the following two statements are equivalent:

(1) f is a homotopy equivalence,
(2′) $i\colon X \to \mathrm{Cyl}(f)$ is a homotopy equivalence.

Thus it remains to show that (2) and (2′) are equivalent. By the Deformation Retract-Homotopy Equivalence Lemma 15.5 we know that the "(2)⇒(2′)"-implication holds. It remains to prove the "(2′)⇒(2)"-direction. Thus we assume that $i\colon X \to \mathrm{Cyl}(f)$ is a homotopy equivalence. By the Cone-Mapping Cylinder Inclusion Cofibration Proposition 17.3 we know that the natural inclusion $i\colon X \to \mathrm{Cyl}(f)$ is a closed cofibration. But now it follows from the Homotopy Equivalence-Cofibration Proposition 17.15 that $i(X)$ is indeed a deformation retract of X. ■

Corollary 17.17. Let X and Y be two topological spaces. The following two statements are equivalent:

(1) X and Y are homotopy equivalent.
(2) There exists a topological space W and two closed embeddings $i\colon X \to W$ and $j\colon Y \to W$ such that $i(X)$ is a deformation retract of W and $j(Y)$ is a deformation retract of W.

Proof. The "(2)⇒(1)"-direction is just a consequence of the Deformation Retract-Homotopy Equivalence Lemma 15.5 and the fact that being homotopy equivalent is an equivalence relation, see the Homotopy-Equivalence Basics Lemma 15.1 (2).

We turn to the proof of the "(1)⇒(2)"-direction. Thus we now assume that X and Y are homotopy equivalent. Let $f\colon X \to Y$ be a homotopy equivalence. We consider the corresponding mapping cylinder $\mathrm{Cyl}(f)$. By the Mapping Cylinder Basics Lemma 16.10 we know that the natural inclusions $i\colon X \to \mathrm{Cyl}(f)$ and $j\colon Y \to \mathrm{Cyl}(f)$ are closed embeddings. In the Mapping Cylinder Basics Lemma 16.12 we showed that $j(Y)$ is always, regardless of the choice of f, a deformation retract of $\mathrm{Cyl}(f)$. Finally, since $f\colon X \to Y$ is by hypothesis a homotopy equivalence we now know by Corollary 17.16 that $i(X)$ is a deformation retract of $\mathrm{Cyl}(f)$. ■

17.5. **Push-outs and cofibrations.** In this section we will consider the relationship between cofibrations and pushouts. We recall the following notation from page 1260.

Notation. Let $f\colon A \to X$ and $g\colon A \to Y$ be two maps between topological spaces. We write

$$X \cup_A Y \;:=\; (X \sqcup Y)/\!\sim \qquad \text{where } f(a) \sim g(a) \text{ for all } a \in A.$$

We refer to $X \cup_A Y$ as the pushout of f and g.

The next proposition continues a theme that we already encountered in the Pushout Lemma 4.8 and the Topological Pushout-Maps Lemma 5.34.

Proposition 17.18. (Opposite Map-Cofibration Proposition) Let $f\colon A \to X$ and $g\colon A \to Y$ be two maps between topological spaces. We consider the corresponding

pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \lrcorner\downarrow{\scriptstyle i} \\ Y & \xrightarrow{j} & X \cup_A Y. \end{array}$$

The following two statements hold:

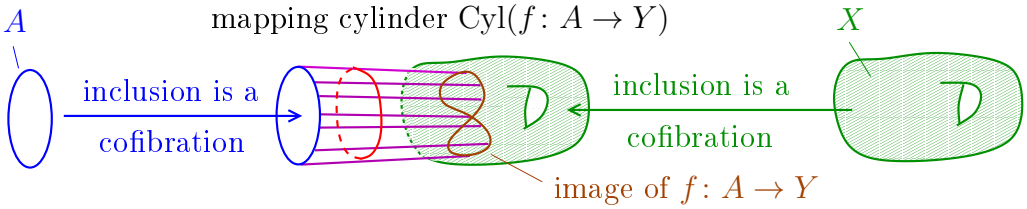
(1) If f is a (closed) cofibration, then the "opposite map" $j\colon Y \to X \cup_A Y$ is also a (closed) cofibration.
(2) If f is a closed cofibration and if f is a homotopy equivalence, then the "opposite map" $j\colon Y \to X \cup_A Y$ is also a homotopy equivalence.

The same statements hold, by symmetry, for g and the "opposite map" i.

Example. Let $f\colon A \to Y$ be a map between topological spaces. We consider the pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{a\mapsto(a,0)} & A \times [0,1] \\ {\scriptstyle f}\downarrow & & \lrcorner\downarrow{\scriptstyle i} \\ Y & \xrightarrow{j} & \underbrace{(A \times [0,1]) \cup_A Y}_{=\mathrm{Cyl}(f)}. \end{array}$$

By the discussion on page 488 we know that the top horizontal map is a cofibration. Thus it follows from the Opposite Map-Cofibration Proposition 17.18 that the natural inclusion $Y \to \mathrm{Cyl}(f\colon A \to Y)$ is a cofibration. Thus, together with the Cone-Mapping Cylinder Inclusion Cofibration Proposition 17.3 (1) we see that both the natural inclusions $A \to \mathrm{Cyl}(f\colon A \to Y)$ and $Y \to \mathrm{Cyl}(f\colon A \to Y)$ are cofibrations.



Proof. Throughout the proof we assume that $f\colon A \to X$ is a cofibration. By the Cofibration-Closed Injective Lemma 17.6 we can assume that A is a subspace of X and that $f\colon A \to X$ is the inclusion.

First we want to show that $j\colon Y \to X \cup_A Y$ is also a cofibration. Since f is a cofibration we know by the Cofibration Characterization Lemma 17.2 (1)⇒(3) that there exists a retraction

$$r\colon X \times [0,1] \;\to\; (X \times \{0\}) \cup (A \times [0,1]).$$

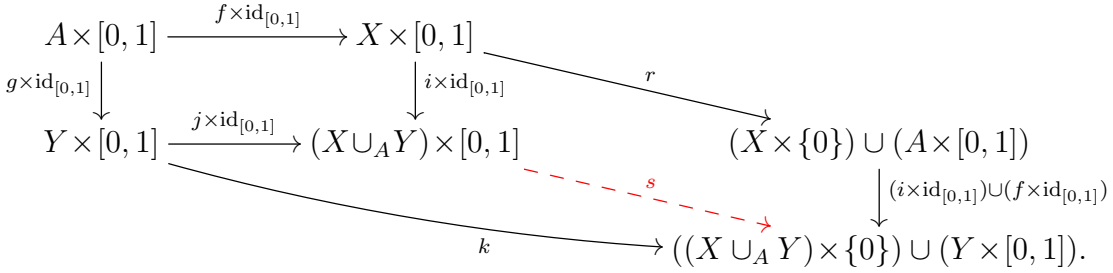
We introduce the following extra maps:

(1) We denote by $i\colon X \to X \cup_A Y$ and $j\colon Y \to X \cup_A Y$ the two natural maps.

(2) We consider the pushout diagram

$$\begin{array}{ccc} Y \times \{0\} & \longrightarrow & Y \times [0,1] \\ {\scriptstyle j\times \mathrm{id}}\downarrow & & \lrcorner \downarrow {\scriptstyle k} \\ ((X \cup_A Y) \times \{0\}) & \longrightarrow & ((X \cup_A Y) \times \{0\}) \cup (Y \times [0,1]). \end{array}$$

We denote by $k\colon Y \times [0,1] \to ((X \cup_A Y) \times \{0\}) \cup (Y \times [0,1])$ the natural map. Now we consider the following diagram:



We make the following observations:

(1) It follows immediately from the Topological Pushout-Maps Lemma 5.34 (3) that we can view $Y \times [0,1]$ as a subspace of $(X \cup_A Y) \times [0,1]$.

(2) The diagonal dashed map s comes from the fact, discussed on page 1264, that the pushout commutes with taking products with the regionally compact space $[0,1]$.

(3) It follows from (1) and (2) that we can view the map s as a retraction. Thus it follows from the Cofibration Characterization Lemma 17.2 that the map $Y \to X \cup_A Y$ is indeed a cofibration.

Finally note that if f is a closed cofibration, then it follows from the above together with the Topological Pushout-Maps Lemma 5.34 (4) that the map $Y \to X \cup_A Y$ is also a closed cofibration.

Now we assume that f is a closed cofibration and that it is a homotopy equivalence. We need to show that $j\colon Y \to X \cup_A Y$ is a homotopy equivalence. By the Topological Pushout-Maps Lemma 5.34 (3) we know that j is an embedding. By the Deformation Retract-Homotopy Equivalence Lemma 15.5 it now suffices to show that $j(Y)$ is a deformation retract of $X \cup_A Y$. By the Homotopy Equivalence-Cofibration Proposition 17.15 we know that there exists a deformation retraction $G\colon X \times [0,1] \to X$ from X to A. We consider the map

$$\begin{array}{rcl} H\colon (X \cup_A Y) \times [0,1] & \to & X \cup_A Y \\ ([P],t) & \mapsto & \begin{cases} i(G(Q),t), & \text{if } P = i(Q) \text{ with } Q \in X, \\ j(Q), & \text{if } P = j(Q) \text{ with } Q \in Y. \end{cases} \end{array}$$

It follows from the Homotopy Pushout Lemma 14.9 that this map is actually continuous. Now that we put our worries regarding continuity to rest it is clear that this map is a deformation retraction from $X \cup_A Y$ to $j(Y)$. ■

The following theorem is the main result of this chapter.

Theorem 17.19. (Homotopy Pushout Theorem) Suppose we are given the following commutative diagram of maps between topological spaces:

$$\begin{array}{ccccc} X & \xleftarrow{\ i\ } & A & \xrightarrow{\ f\ } & Y \\ \downarrow{\scriptstyle \varphi_X} & & \downarrow{\scriptstyle \varphi_A} & & \downarrow{\scriptstyle \varphi_Y} \\ X' & \xleftarrow{\ i'\ } & A' & \xrightarrow{\ f'\ } & Y'. \end{array}$$

If the vertical maps are homotopy equivalences and if the maps i and i' to the left are closed cofibrations, then the induced map

$$\begin{array}{rcl} X \cup_A Y & \to & X' \cup_{A'} Y' \\ [P] & \mapsto & \begin{cases} \varphi_X(P), & \text{if } P \in X, \\ \varphi_Y(P), & \text{if } P \in Y \end{cases} \end{array}$$

between the pushouts is a homotopy equivalence.

Remark. We outsource the proof, which follows quite closely the argument in [**Rap2016**], to the next section. Alternative proofs are given in [**BrownR1968**, p. 249], [**BrownR2006**, Proposition 7.5.7] and [**tD2008**, Proposition 5.3.4].

Proposition 17.20. Let $\varphi\colon Y \to Y'$ be a homotopy equivalence between two topological spaces. Furthermore let A be a topological space and let $f\colon A \to Y$ and $f'\colon A \to Y'$ be two maps such that $\varphi \circ f$ is homotopic to f'. Then there is a homotopy equivalence

$$\mathrm{Cone}(f\colon A \to Y) \ \to\ \mathrm{Cone}(f'\colon A \to Y')$$

Example. Let $f_0, f_1\colon A \to Y$ be two maps between topological spaces. If f_0 and f_1 are homotopic, then it follows immediately from Proposition 17.20, applied to the map $\varphi = \mathrm{id}\colon Y \to Y' = Y$, that the mapping cones $\mathrm{Cone}(f\colon A \to Y)$ and $\mathrm{Cone}(f'\colon A \to Y)$ are homotopy equivalent. Note that this recovers for the most part the statement of the Mapping Cone Homotopy Equivalent Proposition 16.15.

Proof. Let $F\colon A \times [0,1] \to Y'$ be a homotopy between $\varphi \circ f$ and f'. We consider the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Cone}(A) & \xleftarrow{\quad} & A & \xrightarrow{\ f\ } & Y \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varphi} \\ \mathrm{Cone}(A) & \xleftarrow{\quad} & A & \xrightarrow{\ \varphi\circ f\ } & Y' \\ \downarrow{\scriptstyle x\mapsto(x,0)} & & \downarrow{\scriptstyle a\mapsto(a,0)} & & \downarrow{\scriptstyle y\mapsto(y,0)} \\ \mathrm{Cone}(A)\times[0,1] & \xleftarrow{\quad} & A\times[0,1] & \xrightarrow{\ (a,t)\mapsto(F(a,t),t)\ } & Y'\times[0,1] \\ \uparrow{\scriptstyle x\mapsto(x,1)} & & \uparrow{\scriptstyle a\mapsto(a,1)} & & \uparrow{\scriptstyle y\mapsto(y,1)} \\ \mathrm{Cone}(A) & \xleftarrow{\quad} & A & \xrightarrow{\ f'\ } & Y'. \end{array}$$

The vertical maps are evidently homotopy equivalences. Furthermore we know by the Cone-Mapping Cylinder Inclusion Cofibration Proposition 17.3 (2) and the Product-Cofibration Proposition 17.11 that the left horizontal maps are closed cofibrations. It follows from the Homotopy Pushout Theorem 17.19 together with the Homotopy-Equivalence Basics Lemma 15.1 (1) that the pushout on top is homotopy equivalence to the pushout at the

bottom. But the pushout on top is precisely $\mathrm{Cone}(f\colon A \to Y)$ and the pushout at the bottom is precisely $\mathrm{Cone}(f'\colon A \to Y')$. ■

Remark. Another typical application of the Homotopy Pushout Theorem 17.19 is given in the proof of the Handles-Cells Proposition 136.18 where we show that "handle attachments" and "cell attachments" lead to homotopy equivalent topological spaces.

17.6. **Proof of Theorem 17.19.** In this section we provide a proof of Theorem 17.19, following the argument of [**Rap2016**]. The least that can be said is that the proof is not for the faint of heart.

The following lemma is a key step towards the proof of Theorem 17.19.

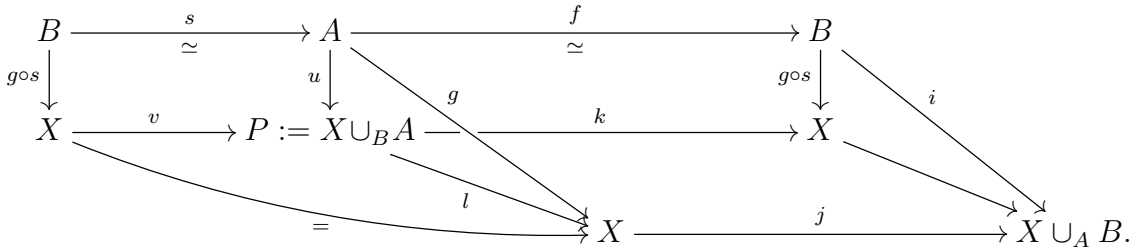
Lemma 17.21. Let $f\colon A \to X$ and $g\colon A \to Y$ be two maps between topological spaces. We consider the corresponding pushout diagram

$$\begin{array}{ccc} A & \underset{f}{\overset{s}{\rightleftarrows}} & B \\ {\scriptstyle g}\downarrow & & \lrcorner\downarrow{\scriptstyle i} \\ X & \xrightarrow{\ j\ } & X \cup_A B. \end{array}$$

Let f be a homotopy equivalence. If g is a closed cofibration, and if there exists a map $s\colon B \to A$ with $f \circ s = \mathrm{id}_B$ and which is a closed cofibration, then the map "opposite to f", i.e. the map j is a homotopy equivalence.

Remark. Sometimes it takes a while to spot the differences to earlier results. In the similar looking the Opposite Map-Cofibration Proposition 17.18 we had assumed that f is not only a homotopy equivalence but also a closed cofibration. Now we assume g is a closed cofibration and that we have the extra map s to play with.

Proof of Lemma 17.21. First note that it follows from the hypotheses that $f \circ s = \mathrm{id}_B$ and that f is a homotopy equivalence, together with the Homotopy-Equivalence Basics Lemma 15.1 (1), that s is also a homotopy equivalence. We consider the following diagram:



We make the following clarifications and observations:

(1) To the top left we have a pushout square. In particular the maps $u\colon A \to X \cup_B A$ and $v\colon X \to X \cup_B A$ are the corresponding natural maps.
(2) The map $k\colon X \cup_B A \to X$ is defined via the identity on X and the map $g \circ s \circ f$ on A.

(3) It follows easily from the Pushout Lemma 4.8 (1) and (2)[188] that the square to the top right is a pushout diagram.
(4) The map $l\colon X \cup_B A \to X$ is defined via the identity on X and the map g on A.
(5) It follows easily from the definitions that the diagram commutes.
(6) Since s is a closed cofibration and a homotopy equivalence it follows from the Opposite Map-Cofibration Proposition 17.18 (2) that the map $v\colon X \to X \cup_B A$ is also a closed cofibration and a homotopy equivalence.
(7) Note that $k \circ v = \mathrm{id}_X$ and $l \circ v = \mathrm{id}_X$. Since v and id_X are homotopy equivalences we obtain from the Homotopy-Equivalence Basics Lemma 15.1 (1) that k and l are also homotopy equivalences.
(8) Since $g \circ s$ is a closed cofibration we obtain from the Opposite Map-Cofibration Proposition 17.18 (1) that the map u is also a closed cofibration.

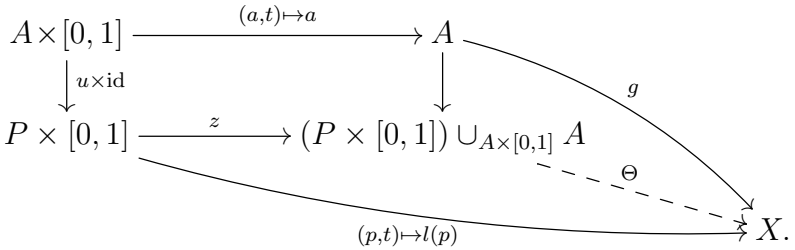
Next we consider the following two pushout diagrams:

$$\begin{array}{ccc} P & \xrightarrow{l} & X \\ \downarrow{\scriptstyle p\mapsto(p,1)} & \simeq & \downarrow{\scriptstyle t} \\ (P\times[0,1]) \cup_{A\times[0,1]} A & \to & \underbrace{((P\times[0,1])\cup_{A\times[0,1]} A)\cup_P X}_{=:\mathrm{Cyl}(l)_A} \end{array} \quad \text{and} \quad \begin{array}{ccc} P = X\cup_B A & \xrightarrow{k} & X \\ \downarrow{\scriptstyle \varphi(p)=(p,1)} & \simeq & \downarrow{\scriptstyle m} \\ \mathrm{Cyl}(l)_A & \xrightarrow{w} & \mathrm{Cyl}(l)_A \cup_P X. \end{array}$$

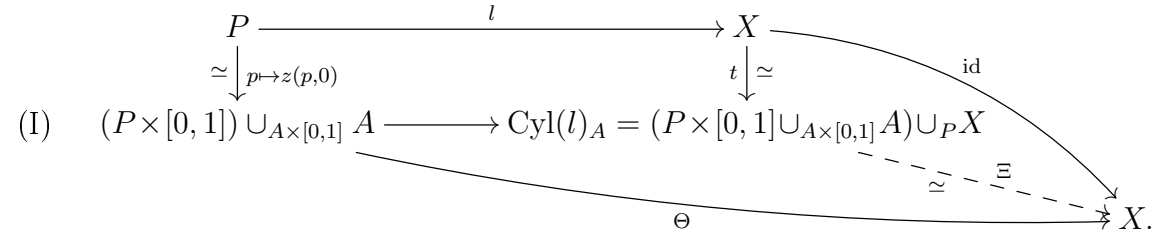
Providing the proof of the following claim is the main difficulty in the proof of the lemma.

Claim. The map $w\colon \mathrm{Cyl}(l)_A \to \mathrm{Cyl}(l)_A \cup_P X$ is a homotopy equivalence.

Proof. To prove the claim we first consider the following pushout diagram:



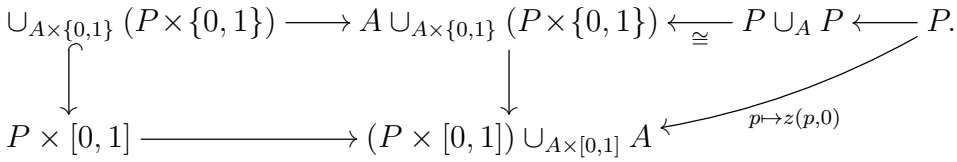
We take the resulting map Θ and use it to produce the following even more confusing pushout diagram



[188]The Pushout Lemma 4.8 (2) implies that the natural map $X \cup_B B \to (X \cup_B \cup A) \cup_A \cup B$ is a homeomorphism and the Pushout Lemma 4.8 (1) implies that the natural map $X \to X \cup_B B$ is a homeomorphism. We will apply this type of argument on several occasions.

We continue with the following comments:

(9) We claim that the left-hand vertical map is a closed cofibration. To verify this statement we consider the following commutative diagram



We make a few comments:

(a) The left top horizontal map is induced by the projection $A \times [0,1] \to A$ and the identity on $P \times \{0,1\}$.

(b) The left vertical map is the "obvious" map. It is a closed cofibration by the combination of Corollary 17.14 with Lemma 5.36.

(c) The middle vertical map and the bottom horizontal map are the obvious maps.

(d) We leave it to the reader to verify that the square is in fact a pushout diagram.

(e) It follows from (b) and (d) together with the Opposite Map-Cofibration Proposition 17.18 that the middle vertical map is a closed cofibration.

(f) The map $P \cup_A P \to A \cup_{A\times\{0,1\}} (P \times \{0,1\})$ is given by sending the "left copy" of P to $P \times \{0\}$ and the "right copy" of P to $P \times \{1\}$. There is an obvious inverse which shows that this map is actually a homeomorphism.

(g) The map $P \to P \cup_A P$ is given by sending P to the "left copy" of P. Since $A \to P$ is a closed cofibration we obtain again from the Opposite Map-Cofibration Proposition 17.18 that the map $P \to P \cup_A P$ is a closed cofibration.

(h) It follows immediately from the definitions that the triangle to the right commutes. Thus we obtain from (e), (f) and (g) together with the Cofibration Composition Lemma 17.1 that the diagonal map is a closed cofibration. But that is precisely what we wanted to show.

(10) We claim that the left-hand vertical map $P \to (P \times [0,1]) \cup_{A\times[0,1]} A$ of (I) is a homotopy equivalence. This can be seen as follows. By (9) and the Cofibration-Closed Injective Lemma 17.6 we can identify P with its image under the inclusion. It follows from the Homotopy Pushout Lemma 14.9 that the obvious deformation retraction on $P\times[0,1]$ and the constant map on A give rise to a deformation retraction from $(P \times [0,1]) \cup_{A\times[0,1]} A$ to the image of P. Thus it follows from the elementary the Deformation Retract-Homotopy Equivalence Lemma 15.5 that the inclusion is indeed a homotopy equivalence.

(11) We consider the pushout square in (I). It follows from (9), (10) and the Opposite Map-Cofibration Proposition 17.18 (2) that the right-hand vertical map t in (I) is also closed cofibration and a homotopy equivalence.

(12) We deduce from (11) and the Homotopy-Equivalence Basics Lemma 15.1 (1) that Ξ is a homotopy equivalence.

Next we consider the following diagram:

(II)
$$\begin{array}{ccc} P = X \cup_B A & \xrightarrow[\simeq]{k} & X \\ {\scriptstyle\simeq}\downarrow{\scriptstyle \varphi(p)=(p,0)} & & \downarrow{\scriptstyle m} \\ \mathrm{Cyl}(l)_A & \xrightarrow{w} & \mathrm{Cyl}(l)_A \cup_P X \\ {\scriptstyle\simeq}\downarrow{\scriptstyle \Xi} & & \\ X. & & \end{array} \qquad (l : \mathrm{Cyl}(l)_A \to X)$$

We make the following clarifications and observations:

(13) $\mathrm{Cyl}(l)_A \cup_P X$ is the pushout of the pushout diagram to the to top.

(14) The composition $P \xrightarrow{\varphi} \mathrm{Cyl}(l)_A \xrightarrow{\Xi} X$ equals the map l.

(15) Since l is a homotopy equivalence and since we just saw that $\Xi\colon \mathrm{Cyl}(l)_A \to X$ is a homotopy equivalence we obtain again from the Homotopy-Equivalence Basics Lemma 15.1 (1) that $\varphi\colon P \to \mathrm{Cyl}(l)_A$ is a homotopy equivalence.

(16) We will now argue that the map $\varphi\colon P \to \mathrm{Cyl}(l)_A$ is a closed cofibration. To do so we consider the following diagram:

$$\begin{array}{ccc} & & P \\ & & \downarrow \\ (A \times [0,1]) \cup_{A\times\{0,1\}} (P \times \{0,1\}) & \longrightarrow & X \cup_A P \\ \downarrow & & \downarrow \\ P \times [0,1] & \longrightarrow & \mathrm{Cyl}(l)_A. \end{array} \qquad \varphi$$

We make the following observations:

(a) The top vertical map is a closed cofibration by the Opposite Map-Cofibration Proposition 17.18 and our hypothesis that $A \to X$ is a closed cofibration.

(b) The map $X \cup_A P \to \mathrm{Cyl}(l)_A$ is defined by t on X and by φ on P.

(c) The vertical map to the left is induced by the inclusions.

(d) The horizontal map on the top is induced by the projection $A \times [0,1] \to A$, the obvious identification $P \times \{0\} \to P$ and the map l on $P = P \times \{1\} \to X$.

(e) We consider the maps

$$(P \times [0,1]) \cup_{(A\times[0,1])\cup_{A\times\{0,1\}}(P\times\{0,1\})} (X \cup_A P) \;\leftrightarrow\; \underbrace{((P \times [0,1]) \cup_{A\times[0,1]} A) \cup_P X}_{=\mathrm{Cyl}(l)_A}$$

which are the obvious maps on $P \times [0,1]$, the obvious map on X and which going left-to-right is given by the obvious map $P \to P \times \{0\}$ and which right-to-left is given by the natural map $A \to X \cup_B A = P$. These maps are well-defined and inverses of one another, hence they are isomorphisms. These maps show that the square is a pushout diagram.

(f) It follows from Corollary 17.14 together with Lemma 5.36 that the left vertical map in the square is a closed cofibration.

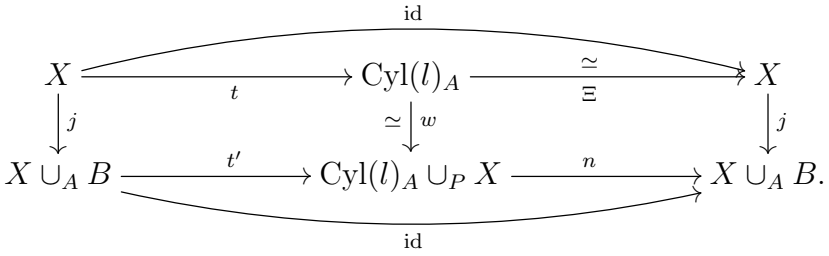
(g) It follows (f) together with the Opposite Map-Cofibration Proposition 17.18 (1) that the right vertical map in the square is also a closed cofibration.

(h) By (a) and (g) the map of our desire is a composition of two closed cofibrations, hence it is itself a closed cofibration by the Cofibration Composition Lemma 17.1.

(17) It follows from (15) and (16) together with the Opposite Map-Cofibration Proposition 17.18 (2), applied the diagram (II), that the map $m\colon X \to \mathrm{Cyl}(l)_A \cup_P X$ is a homotopy equivalence.

(18) By (7), (15) and (17) we know that m, k, φ are homotopy equivalences. By the diagram (II) we have $w \circ \varphi = m \circ k$. Thus we obtain from the Homotopy-Equivalence Basics Lemma 15.1 (1) that w is also a homotopy equivalence. ⊞

Finally we consider the following diagram:



Here are the customary clarifications and observations:

(19) Recall that the map $t\colon X \to \mathrm{Cyl}(l)_A$ is the natural inclusion.

(20) By definition of t and Ξ we have $\Xi \circ t = \mathrm{id}_X$.

(21) The map $t'\colon X \cup_A B \to \mathrm{Cyl}(l)_A \cup_P X$ is defined by $w \circ t\colon X \to \mathrm{Cyl}(l)_A$ and it is defined by $m \circ g \circ s\colon B \to X$. Furthermore the map $n\colon \mathrm{Cyl}(l)_A \cup_P X \to X \cup_A B$ is defined by Ξ on $\mathrm{Cyl}(l)_A$ and by the natural map $j\colon X \to X \cup_A B$ on X.

(22) One can easily verify that $n \circ t' = \mathrm{id}_{X \cup_A B}$.

(23) It follows immediately from the definitions that the diagram commutes.

It follows from (20), (22) and (23) and the above commutative diagram that j is a retract of the map w, in the sense of the definition in Exercise 15.9. Since w is a homotopy equivalence we obtain from Exercise 15.9 that j is also a homotopy equivalence. We are done! ■

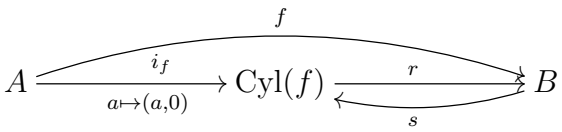
The next lemma is the same statement as Lemma 17.21 except that we can drop the hypothesis on the existence of s.

Lemma 17.22. (Opposite Map-Homotopy Equivalence Lemma) Let $f\colon A \to X$ and $g\colon A \to Y$ be two maps between topological spaces. We consider the corresponding pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{j} & X \cup_A B. \end{array}$$

Let f be a homotopy equivalence. If g is a closed cofibration, then the map "opposite to f", i.e. the map j is a homotopy equivalence.

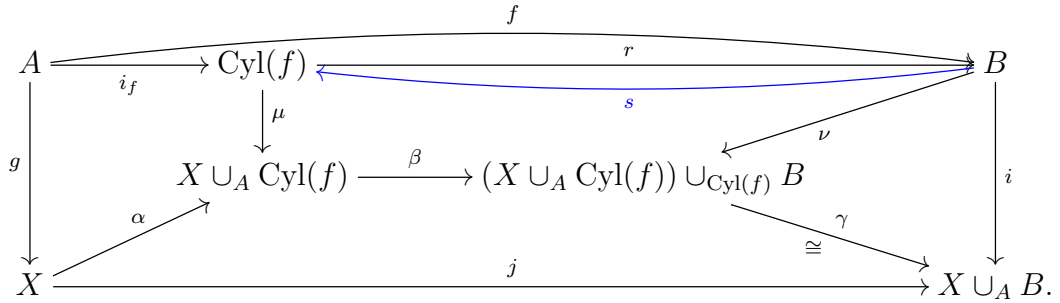
Proof of the Opposite Map-Homotopy Equivalence Lemma 17.22. We consider the maps



where $i_f\colon A \to \mathrm{Cyl}(f)$ and $s\colon B \to \mathrm{Cyl}(f)$ are the natural inclusions and $r\colon \mathrm{Cyl}(f) \to B$ is the natural retraction. We make the following observations:

(1) It follows from the Cone-Mapping Cylinder Inclusion Cofibration Proposition 17.3 and the discussion on page 495 that i_f and s are closed cofibrations.
(2) It follows immediately from the definitions that $f = r \circ i_f$ and $r \circ s = \mathrm{id}_B$.
(3) By the Mapping Cylinder Basics Lemma 16.10 (3) and the Mapping Cylinder Basics Lemma 16.12 we know that r is a homotopy equivalence.
(4) It follows from the hypothesis that f is a homotopy equivalence, together with (2) and (3) and together with the Homotopy-Equivalence Basics Lemma 15.1 (1), that i_f is a homotopy equivalence.

Next we consider the following diagram



We continue with the following clarifications and observations:

(5) The quadrilateral to the left is a pushout diagram. In particular α and μ are the natural maps of a pushout.
(6) The quadrilateral in the top center is also a pushout diagram. In particular β and ν are the natural maps of a pushout.
(7) The map γ is the natural homeomorphism given by the Pushout Lemma 4.8 (2).
(8) It follows immediately from the definitions that the diagram commutes.
(9) We consider the quadrilateral to the left.
   (a) By (1) and (4) we know that i_f is a homotopy equivalence and that it is a closed cofibration. Thus we obtain from (5) together with the Opposite Map-Cofibration Proposition 17.18 (2) that the opposite map $\alpha\colon X \to X \cup_A \mathrm{Cyl}(f)$ is a homotopy equivalence.
   (b) By hypothesis g is a closed cofibration, thus we obtain from (5) together with the Opposite Map-Cofibration Proposition 17.18 (1) that μ is also a closed cofibration.
(10) Next we obtain the quadrilateral in the top center. By (1), (3), (6) and (9b) we can apply Lemma 17.21 and we obtain that the map β is a homotopy equivalence.
(11) By (7), (9a) and (10) we know that α, β and γ are homotopy equivalences. Since the diagram commutes we see that the bottom map $j\colon X \to X \cup_A B$ is indeed a homotopy equivalence. ■

Proof of Theorem 17.19. We are given the following commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{\ i\ } & A & \xrightarrow{\ f\ } & Y \\ \downarrow{\scriptstyle \varphi_X} & & \downarrow{\scriptstyle \varphi_A} & & \downarrow{\scriptstyle \varphi_Y} \\ X' & \xleftarrow{\ i'\ } & A' & \xrightarrow{\ f'\ } & Y' \end{array}$$

where all of the vertical maps are homotopy equivalences and where the maps i and i' to the left are closed cofibrations. Recall that we need to show that the induced map

$$\Phi\colon X \cup_A Y \ \to\ X' \cup_{A'} Y'$$

is a homotopy equivalence.

Case 1. First we assume that f and f' are homotopy equivalences. We consider the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ u\ } & X \cup_A Y \\ \simeq\downarrow{\scriptstyle \varphi_X} & & \downarrow{\scriptstyle \Phi} \\ X' & \xrightarrow{\ u'\ } & X' \cup_{A'} Y' \end{array} .$$

We make the following comments:

(1) The horizontal maps u and u' are the obvious natural maps coming from the following pushout diagrams

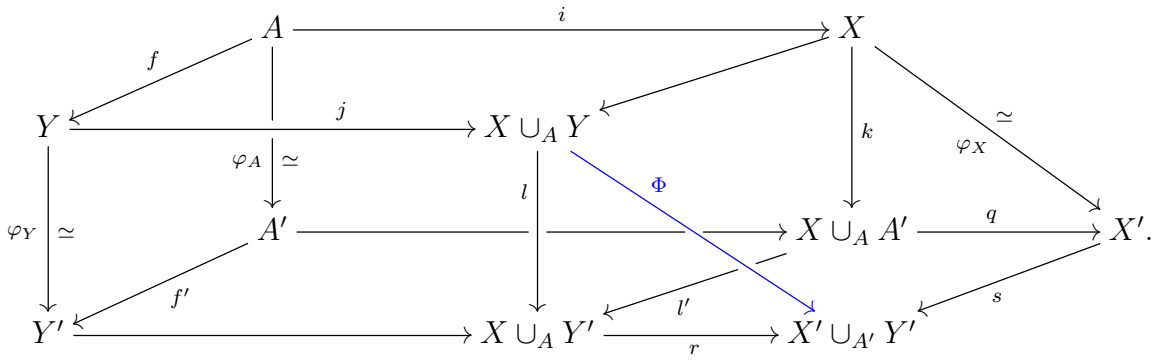
$$\begin{array}{ccc} A & \xrightarrow{\ f\ }_{\simeq} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{\ u\ } & X \cup_A Y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A' & \xrightarrow{\ f'\ }_{\simeq} & Y' \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle v'} \\ X' & \xrightarrow{\ u'\ } & X' \cup_{A'} Y'. \end{array}$$

Since f and f' are homotopy equivalences and since i and i' are closed cofibrations we obtain from the Opposite Map-Homotopy Equivalence Lemma 17.22, applied to these two pushout diagrams, that the horizontal maps u and u' are homotopy equivalences.

(2) The left vertical map of the diagram of the claim is a homotopy equivalence by hypothesis.

(3) It follows from (1) and (2), together with the Homotopy-Equivalence Basics Lemma 15.1 (1), that Φ is a homotopy equivalence.

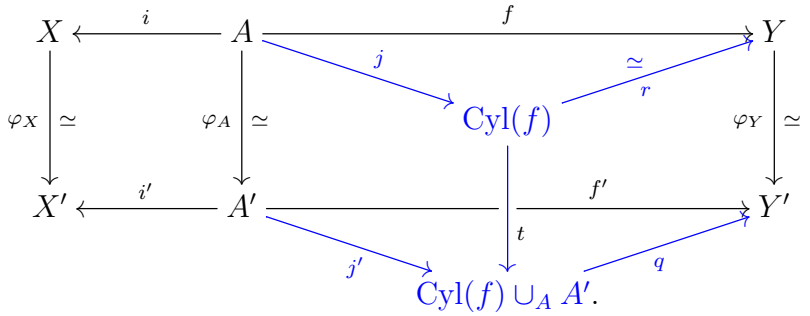
Case 2. Next we assume that f and f' are cofibrations. We consider the following diagram:



We make the following observations and clarifications:

(1) The top square and bottom left square are the obvious pushout diagrams.
(2) The vertical map $l\colon X \cup_A Y \to X \cup_A Y'$ is the map induced by φ_Y. The square in the front is a pushout diagram by the Pushout Lemma 4.8 (2).
(3) The square in the back is defined to be a pushout diagram.
(4) Since i is a closed cofibration and φ_A is a homotopy equivalence it follows from the Opposite Map-Homotopy Equivalence Lemma 17.22, applied to the pushout diagram in the back, that k is a homotopy equivalence.
(5) Since $i\colon A \to X$ is a closed cofibration we obtain, by applying the Opposite Map-Cofibration Proposition 17.18 to the pushout diagram on the top, that the "opposite map" $j\colon Y \to X \cup_A Y$ is a closed cofibration.
(6) Since j is a cofibration and since φ_Y is homotopy equivalence we obtain again from the Opposite Map-Homotopy Equivalence Lemma 17.22, this time applied to the pushout diagram in front, that l is a homotopy equivalence.
(7) Since f' is a closed cofibration we obtain again from the Opposite Map-Cofibration Proposition 17.18 that the "opposite map" l' is a closed cofibration.
(8) We turn to the right part of the diagram. The map $q\colon X \cup_A A' \to X'$ is defined by φ_X on X and i' on A. Furthermore the map $r\colon X \cup_A Y' \to X' \cup_{A'} Y'$ is defined by φ_X on X' and id on Y'. Finally $s\colon X' \to X' \cup_{A'} Y'$ is the natural map. One can easily verify that the right hand part of the diagram commutes.
(9) Since φ_X and k are homotopy equivalences and since $q \circ k = \varphi_X$ we obtain from the Homotopy-Equivalence Basics Lemma 15.1 (1) that q is a homotopy equivalence.
(10) As we mentioned in (2), the bottom left square is a pushout square. Furthermore it follows immediately from the definitions that the big square formed by the two squares at the bottom is also a pushout square. It follows from a fairly elementary argument, see e.g. Exercise 4.12, that the bottom right square is also a pushout square.
(11) It follows from (7), (9) and (10) together with the hard-fought the Opposite Map-Homotopy Equivalence Lemma 17.22 that r is a homotopy equivalence.
(12) Since $\Phi = r \circ l$ we obtain from (6) and (11) that Φ is indeed a homotopy equivalence.

Case 3. Finally we deal with the case that f and f' are arbitrary maps. We consider the following diagram



We make the following clarifications and observations:

(1) The black part of the diagram is precisely the commutative diagram of our setup.
(2) The top triangle is given by the usual factorization of the map $f\colon A \to Y$ into the natural inclusion $j\colon X \to \mathrm{Cyl}(f)$ and the natural retraction $r\colon \mathrm{Cyl}(f) \to Y$. By the Cone-Mapping Cylinder Inclusion Cofibration Proposition 17.3 we know that j is a

closed cofibration. Furthermore, by the Mapping Cylinder Basics Lemma 16.12 (2a) we know that r is a homotopy equivalence.

(3) The left parallelogram is defined to be the pushout of the maps $j\colon A \to \mathrm{Cyl}(f)$ and $\varphi_A\colon A \to A'$.

(4) Since j is a closed cofibration we obtain from (3) and the Opposite Map-Cofibration Proposition 17.18 that the map $j'\colon A' \to \mathrm{Cyl}(f) \cup_A A'$ is also a closed cofibration.

(5) Since φ_A is a homotopy equivalence and since j is a closed cofibration we obtain from the Opposite Map-Homotopy Equivalence Lemma 17.22 that t is a homotopy equivalence.

(6) The map $q\colon \mathrm{Cyl}(f) \cup_A A' \to Y'$ is defined via t on $\mathrm{Cyl}(f)$ and f' on A'.

(7) It follows easily from the definitions that the right parallelogram commutes, i.e. we have $q \circ t = \varphi_Y \circ r$. Since r, φ_Y and t are homotopy equivalences and since $q \circ t = \varphi_Y \circ r$ we obtain from the Homotopy-Equivalence Basics Lemma 15.1 (1) that q is a homotopy equivalence.

Next we consider the commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{\ i\ } & A & \xrightarrow{\ j\ } & \mathrm{Cyl}(f) \\ \downarrow{\scriptstyle \varphi_X} & & \downarrow{\scriptstyle \varphi_A} & & \downarrow{\scriptstyle t} \\ X' & \xleftarrow{\ i'\ } & A' & \xrightarrow{\ j'\ } & \mathrm{Cyl}(f) \cup_A A' \end{array}$$

Here we just make a single observations:

(8) By (2) and (4) and by hypothesis we know that all horizontal maps are closed cofibrations. Furthermore by hypothesis and (5) we know that all vertical maps are homotopy equivalences. Thus we obtain from Case 2 that the induced map

$$\Xi\colon X \cup_A \mathrm{Cyl}(f) \ \to\ X' \cup_{A'} (\mathrm{Cyl}(f) \cup_A A')$$

is a homotopy equivalence.

We move on to consider the following diagram:

$$(*) \qquad \begin{array}{ccccc} X \cup_A \mathrm{Cyl}(f) & & \xleftarrow{\quad\alpha\quad} & & \mathrm{Cyl}(f) \xrightarrow[\simeq]{r} Y \\ \simeq\downarrow{\scriptstyle \nu} & \overset{\simeq}{\searrow}{\scriptstyle \Xi} & & & \simeq\downarrow{\scriptstyle t} \qquad \simeq\downarrow{\scriptstyle \varphi_Y} \\ X' \cup_A \mathrm{Cyl}(f) & \underset{\cong}{\swarrow}{\scriptstyle \mu} & X' \cup_{A'} (\mathrm{Cyl}(f) \cup_A A') & \overset{\beta}{\nwarrow} & \\ & & \xleftarrow[\gamma]{\quad} \mathrm{Cyl}(f) \cup_A A' & \xrightarrow[q]{\simeq} & Y'. \end{array}$$

We make the following clarifications and observations:

(9) The lower left diagonal map μ is the natural homeomorphism that is provided by the Pushout Lemma 4.8 (2).

(10) The horizontal map α at the top left and the diagonal map β are the natural maps coming from pushouts. The vertical map ν to the left is induced by φ_X and the horizontal map γ at the bottom left map is induced by i'. One can easily verify that the diagram commutes.

(11) Recall that by hypothesis $i\colon A \to X$ and $i'\colon A' \to X'$ are closed cofibrations. Thus we obtain from the Opposite Map-Cofibration Proposition 17.18, applied to the pushouts

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & X \\ \downarrow & & \downarrow \\ \mathrm{Cyl}(f) & \xrightarrow{\ \alpha\ } & X \cup_A \mathrm{Cyl}(f) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A' & \xrightarrow{\ i'\ } & X' \\ \downarrow & & \downarrow \\ \mathrm{Cyl}(f)\cup_A A' & \xrightarrow{\ \beta\ } & X'\cup_{A'}(\mathrm{Cyl}(f)\cup_A A') \end{array}$$

that the maps α and β are closed cofibrations. Since $\gamma = \mu \circ \beta$ and since μ is a homeomorphism we see that γ is also a closed cofibrations.

Finally we consider the following diagram:

$$\begin{array}{ccc} (X \cup_A \mathrm{Cyl}(f)) \cup_{\mathrm{Cyl}(f)} Y & \longrightarrow & (X' \cup_A \mathrm{Cyl}(f)) \cup_{\mathrm{Cyl}(f)\cup_A A'} Y' \\ \downarrow\cong & & \downarrow\cong \\ X \cup_A Y & \xrightarrow{\ \Phi\ } & X' \cup_{A'} Y'. \end{array}$$

As usual we cannot be refrained from making comments.

(12) The top horizontal map is induced by ν and φ_Y.

(13) By (5), (8), (9) and our hypothesis we know that all the vertical maps in the diagram $(*)$ are homotopy equivalences. Furthermore by (2) and (7) we know that the right horizontal maps of $(*)$ are homotopy equivalences. Thus we obtain from Case 1 that the top horizontal map in our present diagram is a homotopy equivalence.

(14) The vertical maps are the natural homeomorphism provided by the Pushout Lemma 4.8 (2).

(15) One can easily verify that the diagram commutes. Since the top horizontal map is a homotopy equivalence and since the vertical maps are homeomorphisms we see that the bottom horizontal map Φ is also a homotopy equivalence. And that had been our goal since page 497. ■

This concludes our discussion of closed cofibrations. More information on closed cofibrations can be found in [**tD2008**, Chapter 5], [**BrownR2006**, Chapter 7], [**May1999a**, Chapter 6] and [**Pic1992**, Chapter 2.3].

Exercises for Chapter 17.

Exercise 17.1. Solve the following problems "by hand", i.e. without making use of any of above results. For example you should not use the Cofibration-Closed Injective Lemma 17.6.

(a) Let $A = [a, b]$ be a compact interval and let $X = \mathbb{R}$. Show that the inclusion $i\colon A \to X$ is a cofibration.

(b) Let $A = \mathbb{R} \setminus \{0\}$ and $X = \mathbb{R}$. Show that the inclusion $i\colon A \to X$ is not a cofibration.

(c) Let $A = (-1, 1)$ and $X = \mathbb{R}$. Show that the inclusion $i\colon A \to X$ is not a cofibration.

Exercise 17.2. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two (closed) cofibrations. Show that $g \circ f\colon X \to Z$ is a (closed) cofibration.

Exercise 17.3. Prove the "(2) $\Rightarrow$ (1)"-direction of the Cofibration Characterization Lemma 17.2.

Exercise 17.4. Let $X = (V, E, i, t)$ be an abstract graph as defined on page 307 and let A be a subgraph as defined on page 317. We denote by $|X|$ and $|A|$ the topological realizations of X and A as defined on page 308. Show that the corresponding inclusion map $i\colon |A| \to |X|$ is a closed cofibration.
Remark. This is a special case of the Subcomplex Cofibration Theorem 17.9.

Exercise 17.5. Given an example of a topological space X and a non-empty subset A such that the map $\mathrm{Cone}(i\colon A \to X) \to X/A$ given in the Cofibration-Quotient Lemma 17.7 (1) is not a homotopy equivalence.

Exercise 17.6. Let $X = \{x, y\}$ be a set with two elements and the trivial topology furthermore let $A = \{x\}$. Show that the inclusion $A \to X$ is a cofibration.
Remark. This is an example of a cofibration that is not closed.

Exercise 17.7. Let $X = \{a, b\}$ be the topological space with two elements a and b and where the open sets are precisely $\varnothing, \{a\}, X$. We set $A := \{a\}$.

(a) Show that the inclusion $A \to X$ is a cofibration.
(b) Show that the inclusion $(X \times A) \cup (A \times X) \to X \times X$ is not a cofibration.

Remark. This exercise shows that the conclusion of Proposition 17.13 does not hold if neither A nor B are closed subsets.

Exercise 17.8. Let $i\colon A \to X$ be a closed cofibration. Show that the induced map $\mathrm{Cone}(A) \to \mathrm{Cone}(X)$ is also a cofibration.
Hint. Make sense of the following diagram and use it is as a guideline:

$$\begin{array}{ccccc}
A \times \{0\} & \longrightarrow & X \times \{0\} & \longrightarrow & \{*\} \\
\downarrow & & \downarrow & & \downarrow \\
A \times [0,1] & \longrightarrow & (X \times \{0\}) \cup_{A \times \{0\}} (A \times [0,1]) & \longrightarrow & \mathrm{cone}(A) = * \cup_{A \times \{0\}} (A \times [0,1]) \\
 & & \downarrow & & \downarrow \\
 & & X \times [0,1] & \longrightarrow & \mathrm{Cone}(X).
\end{array}$$

Part III

Topological and smooth manifolds

In this part of the lecture notes we introduce the concept of topological and smooth manifolds. We develop all the basic notions, like tangent spaces and orientations, and we will prove some of the foundational results like the Inverse Mapping Theorem 24.1, the Smooth Embedding Theorem 24.10, the Regular Value Theorem 26.4, the Euclidean Embedding Theorem 27.1 and Sard's Theorem 31.1. Throughout we will allow manifolds to have a boundary, which makes it technically more demanding to write down the correct statements and precise proofs.

18. Topological manifolds

In this chapter we introduce topological manifolds and study some basic properties. Afterwards, in the next chapter, we will introduce smooth manifolds, which for most applications are even more interesting.

18.1. **Topological manifolds: definition and examples.** Before we can give the definition of topological manifolds we need to introduce the following notation.

Notation. Given $n \in \mathbb{N}_0$ we write

$$\begin{aligned} \mathbb{H}^n &= \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_n \geq 0\} \\ \partial\,\mathbb{H}^n &= \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_n = 0\}. \end{aligned}$$

We refer to $\mathbb{H}^n$ as the upper half-space in $\mathbb{R}^n$. We equip both sets with the subspace topology coming from $\mathbb{R}^n$. One can easily verify that $\partial\,\mathbb{H}^n$ is indeed the boundary of $\mathbb{H}^n$ viewed as a subset of $\mathbb{R}^n$. Furthermore it is straightforward to show that the map $\mathbb{R}^{n-1} \to \partial\,\mathbb{H}^n$ given by $x \mapsto (x, 0)$ is a homeomorphism.

Now we can move on to the definition of a topological manifold.

Definition. Let X be a topological space.

(1) An n-dimensional chart for X at a point $x \in X$ is a homeomorphism $\Phi\colon U \to V$ where U is an open neighborhood of x and
   (i) V is an open subset of $\mathbb{R}^n$ or
   (ii) V is an open subset of the upper half-space $\mathbb{H}^n$ and $\Phi(x)$ lies in $\partial\,\mathbb{H}^n$.
   In the former case we say that Φ is a chart of type (i), in the latter case we say that Φ is a chart of type (ii).
(2) We say X is an n-dimensional topological manifold, if X is second-countable and Hausdorff, and if for every $x \in X$ there exists an n-dimensional chart $\Phi\colon U \to V$ at x.[189]
(3) An n-dimensional atlas for X is a collection of n-dimensional charts such that the domains of the charts cover all of X.

For topological manifolds we can furthermore define the notion of the boundary (points):

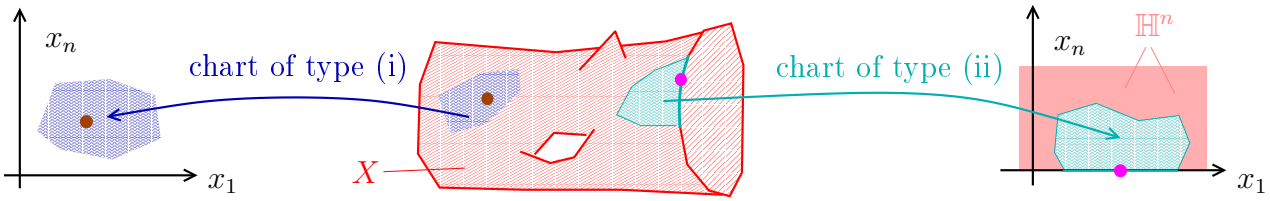
(4) We say that a point x on a topological manifold X is a boundary point if x does not admit a chart of type (i).[190] We denote by ∂X the set of all boundary points of X. We refer to ∂X as the boundary of X.
(5) We say a topological manifold X is closed if X is compact with $\partial X = \varnothing$.[191]

Remark.

[188] The definition of a (topological) manifold goes back to the work of Hermann Weyl [**Wey1913, Wey1964**, Chapter I.4].

[189] It might sound more natural to define a boundary point to be a point that admits a chart of type (ii). In fact we will see in the Topological Manifold Boundary Proposition 75.2 that a point on a topological manifold admits *either* a chart of type (i) or a chart of type (ii). But the proof of this statement is not trivial and as of now we cannot prove this statement. As we will see in the subsequent examples our definition works fairly well, even without making use of the Topological Manifold Boundary Proposition 75.2.

[190] The latter condition just means that all points in X admit a chart of type (i).



(1) In some references, see e.g. **[Hu1969, Aub2001]** and **[PoR2021]**, a topological manifold is defined as above, except that the condition that X is second-countable is replaced by the condition that X is paracompact. We will see in the Manifold-Partitions of Unity Existence Theorem 21.1 and Proposition 21.5 that these two definitions of topological manifolds are equivalent. Furthermore in **[Wall2016]** the condition of being second-countable is replaced by the condition that X is the union of countably many compact subsets. We will see in Lemmas 18.12 and 21.6 that these two definitions of topological manifolds are also equivalent.

(2) Note that only in the Topological Manifold-Dimension Proposition 75.4 will we show that the dimension of a non-empty topological manifold is uniquely determined.

In the following we will give several examples of topological manifolds. When it comes to verifying that a topological space is a topological manifold the main focus is usually on establishing that any point admits a chart. It is easy to forget that one also needs to verify that the topological space is second-countable and Hausdorff. In many cases this task is taken care of by the following lemma.

Lemma 18.1. Let $n \in \mathbb{N}$. Every subset of $\mathbb{R}^n$ is second-countable and Hausdorff.

Proof. Let A be a subset of $\mathbb{R}^n$. By the Second Countability Lemma 9.1 we know that A is second-countable. Furthermore, it follows from Proposition 1.13 and Lemma 1.14 that A is also Hausdorff. ■

For the record we also state the following elementary lemma.

Lemma 18.2. Let X be a topological space.

(1) If X admits a countable atlas, then X is second-countable.
(2) If X is compact and if it admits an atlas, then X is second-countable.

Proof.

(1) Suppose X admits a countable atlas $\{\Phi_i \colon U_i \to V_i\}_{i \in I}$. By Lemma 18.1 we know that each V_i, and thus each U_i, is second-countable. It follows from the Second Countability Lemma 9.1 (4) that X itself is second-countable.

(2) The second statement in turn is an immediate consequence of the first statement since every compact space that admits an atlas also admits a finite atlas. ■

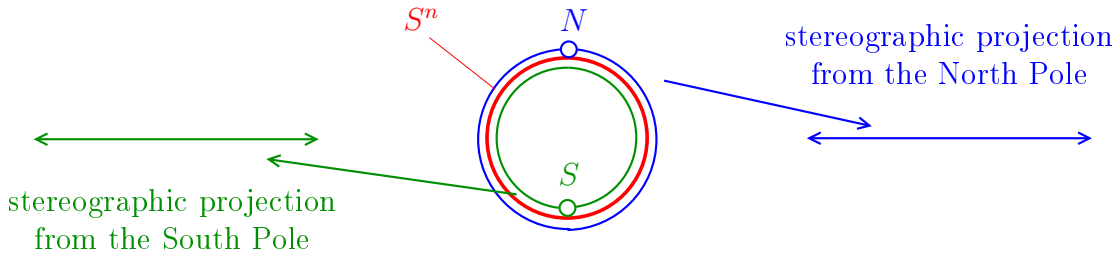
Now we turn to giving examples of topological manifolds.

Examples. Let $n \in \mathbb{N}$.

(1) We consider the sphere S^n. As usual we refer to $N := (0, \dots, 0, 1) \in S^n$ as the *North Pole* and similarly we refer to $S := (0, \dots, 0, -1) \in S^n$ as the *South Pole*. It follows quite easily from slight modifications of the arguments in the proof of Lemma 2.18 that the stereographic projection $S^n \setminus \{N\} \to \mathbb{R}^n$ from the North Pole together

with the stereographic projection $S^n \setminus \{S\} \to \mathbb{R}^n$ from the South Pole, as defined in Lemma 2.18, are two n-dimensional charts of type (i) that cover all of S^n. By Lemma 18.1 this shows that $S^n \subset \mathbb{R}^{n+1}$ is an n-dimensional topological manifold.

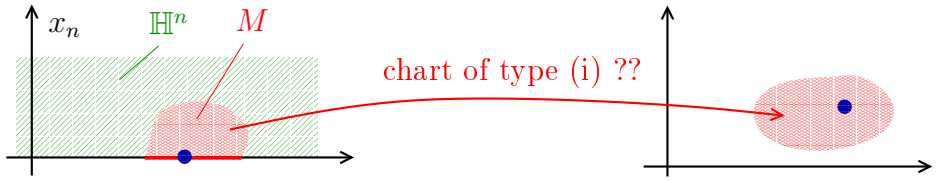
Note that both charts are of type (i), thus we see that the boundary of the topological manifold S^n is the empty set.[191] Finally note that S^n is compact, thus S^n is a closed n-dimensional topological manifold.



(2) Every open subset U of $\mathbb{R}^n$ is an n-dimensional topological manifold. This follows again from Lemma 18.1 and the observation that the identity map is a chart of type (i) for all points. In particular we see that the boundary of U is empty.

(3) Let M be an open subset of the upper half-space $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_n \geq 0\}$. Again, using Lemma 18.1, it is straightforward to see that M is an n-dimensional topological manifold:
   (a) Given any point $X = (x_1, \dots, x_n) \in M$ with $x_n > 0$ the identity map on the open subset $U := \{(y_1, \dots, y_n) \in M \,|\, y_n > 0\}$ is a chart of type (i) around x.
   (b) Given any point $(x_1, \dots, x_n) \in M$ with $x_n = 0$ the identity map $M \to M$ is a chart of type (ii) around x.

   In this example we run into a major, and rather painful, subtlety. Namely one would think that the boundary of M as a topological manifold should be precisely the points in M with $x_n = 0$. But at the moment we cannot draw this conclusion. After all, in principle, such points also might admit charts of type (i). Only after having introduced and studied homology groups we will see in the Topological Manifold Boundary Proposition 75.2 that no such charts of type (i) exist.



(4) Next we want to show that the closed n-ball $\overline{B}^n$ is an n-dimensional topological manifold. We provide the following charts:
   (a) We consider the chart of type (i) that is given by the identity on the open subset B^n of $\overline{B}^n$.

[191] The vigilant reader will have noticed that we have committed a major sin: we introduced an object called "boundary" twice, with quite different meanings:
   (a) On page 116 we introduced the boundary ∂A of a subset A of a topological space X.
   (b) On page 511 we introduced the boundary ∂M of a topological manifold M.

Usually it is clear from the context what we mean when we write ∂. This sorry state of affairs explain why some authors, e.g. [**Kel1975**], refer to the boundary of a subset of a topological space as the *frontier*.

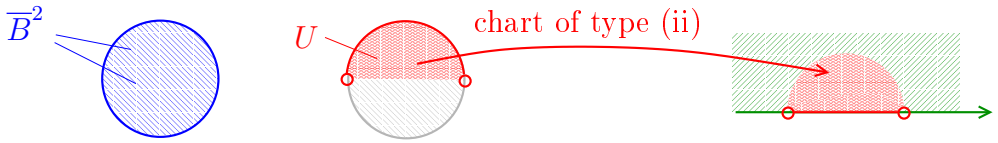
(b) We set $U := \{(x_1, \ldots, x_n) \in \overline{B}^n \,|\, x_n > 0\}$ and we consider the map

$$\begin{array}{rcl} U & \to & \mathbb{H}^n \\ (x_1, \ldots, x_n) & \mapsto & (x_1, \ldots, x_{n-1}, \sqrt{1 - x_1^2 - \cdots - x_{n-1}^2} - x_n). \end{array}$$

This is a chart of type (ii) for all points in $\{(x_1, \ldots, x_n) \in S^{n-1} \,|\, x_n > 0\}$. We refer to the figure below for an illustration of this chart.

(c) We consider the obvious variations of the previous chart for $x_n < 0$ and for other coordinates.

These charts form an atlas for $\overline{B}^n$, together with Lemma 18.1 this shows that $\overline{B}^n$ is an n-dimensional topological manifold. We have found a chart of type (i) for each point in B^n and for the points on S^{n-1} we have found charts of type (ii). This shows that $\partial\overline{B}^n \subset S^{n-1}$. Furthermore we have the strong suspicion that $\partial\overline{B}^n = S^{n-1}$ but as in (3), as of right now we cannot confirm this. We will only be able to confirm our hunch by the time we have proved the Topological Manifold Boundary Proposition 75.2.



(5) Given $a < b$ in $\mathbb{R}$ the interval $[a, b]$ is easily seen to be a 1-dimensional topological manifold. Every point in (a, b) admits a chart of type (i). It follows from Lemma 7.11 together with the Topological Valence Determination Proposition 7.12 that the two points $a, b \in [a, b]$ do *not* admit charts of type (i).[192] Thus the boundary of $[a, b]$ consists, as expected, of the two points a and b. Similarly the boundary of the half-open interval $[a, b)$ equals $\{a\}$.

(6) In Exercise 18.1 we will see that a topological space X is a 0-dimensional topological manifold if and only if X is countable and X is equipped with the discrete topology. It follows from the Compact-Discrete Lemma 1.22 that a compact 0-dimensional topological manifold is precisely a finite set equipped with the discrete topology.

(7) We conclude this list of examples with the simplest of all examples: the empty set is a topological manifold of any dimension.

It is also worth looking at a few non-examples.

Non-Examples.

(1) In Exercise 18.2 we will verify that the line with two zeros admits a chart of type (i) for every point. It is also easy to see that the line with two zeros is second-countable. But since the line with two zeros is not Hausdorff we see that it is not a topological manifold.

(2) We consider the topological space X that is given by the set $X = \mathbb{R}$ but for once we equip $X = \mathbb{R}$ with the discrete topology. Every point of X admits a 0-dimensional chart and X is obviously Hausdorff. But X is not second-countable, thus X is not a topological manifold.

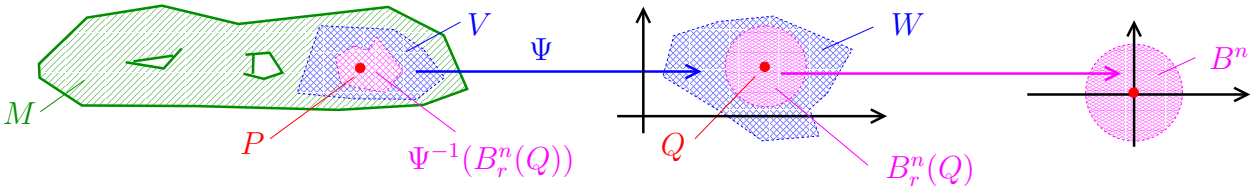
[192] One could also prove this fact directly by hand, see Exercise 2.40.

(3) In Section 22.4 we will see that there exists a *connected* Hausdorff space which admits a 1-dimensional atlas, but which is not second-countable. In particular this topological space is not a topological manifold.

The following lemma will get used, at least implicitly, on numerous occasions.

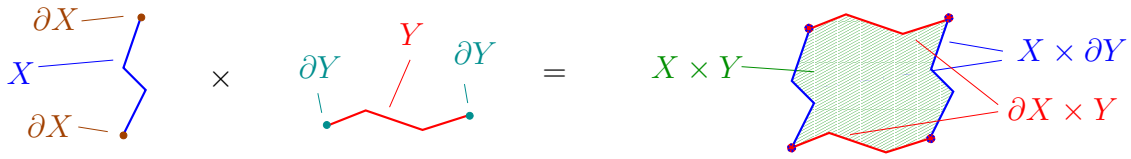
Lemma 18.3. Let X be an n-dimensional topological manifold and let $P \in X$.
(1) If $P \in X \setminus \partial X$, then there exists a chart $\Phi\colon U \to B^n$ with $\Phi(P) = 0$.
(2) If $P \in \partial X$, then there exists a chart $\Phi\colon U \to B^n_{\geq 0}$ with $\Phi(P) = 0$.[193]

Proof.
(1) If $P \in X \setminus \partial X$, then by definition of ∂X there exists a chart $\Psi\colon V \to W$ for the point P of type (i). We write $Q = \Psi(P)$. Note that W is an open subset of $\mathbb{R}^n$. This implies that there exists an $r > 0$ with $B^n_r(Q) \subset W$. The desired chart is given by composing $\Psi\colon \Psi^{-1}(B^n_r(Q)) \to B^n_r(Q)$ with the obvious homeomorphism $B^n_r(Q) \to B^n$.
(2) The proof of this statement is basically identical to the proof of (1). ∎



The following proposition gives a useful method for constructing more topological manifolds.

Proposition 18.4. (Topological Manifold Product Proposition) Let X be an m-dimensional topological manifold and let Y be an n-dimensional topological manifold. Then the following hold:
(1) The product $X \times Y$ is an $(m+n)$-dimensional topological manifold.
(2) (a) Each point in $(X \times Y) \setminus ((\partial X \times Y) \cup (X \times \partial Y))$ admits a chart of type (i).
(b) Each point in $(\partial X \times Y) \cup (X \times \partial Y)$ admits a chart of type (ii).
(c) We have[194] $\partial(X \times Y) \subset (\partial X \times Y) \cup (X \times \partial Y)$.
(3) If X and Y are closed topological manifolds, then so is $X \times Y$.



Examples.
(1) It follows from the Topological Manifold Product Proposition 18.4 and the above discussion of S^1 that for any $m \in \mathbb{N}$ the m-dimensional torus $(S^1)^m$ is a closed m-dimensional topological manifold.

[193] Here we use the notation introduced on page 152, namely $B^n_{\geq 0} = \{(x_1, \ldots, x_{n+1}) \in B^n \mid x_{n+1} \geq 0\}$.
[194] Note that right now we can only prove the inclusion "$\subset$". Later on in the Topological Manifold Boundary Proposition 75.2 we will see that in fact we have an equality.

(2) It follows from the Topological Manifold Product Proposition 18.4 and the above examples that the annulus $S^1 \times [0,1]$ is a 2-dimensional topological manifold with $\partial(S^1 \times [0,1]) \subset S^1 \times \{0,1\}$.

(3) Let $X = \overline{B}^2$ and $Y = [0,1]$, then $X \times Y = \overline{B}^2 \times [0,1]$ is the solid cylinder which is a 3-dimensional topological manifold and the boundary satisfies

$$\partial\big(\overline{B}^2 \times [0,1]\big) \underset{\uparrow}{\subset} (\partial\overline{B}^2 \times [0,1]) \cup (\overline{B}^2 \times \partial([0,1])) \underset{\uparrow}{\subset} (S^1 \times [0,1]) \cup (\overline{B}^2 \times \{0,1\}).$$

in the Topological Manifold Boundary Proposition 75.2 we will see that equality holds

since $\partial\overline{B}^2 \subset S^1$

Proof.

(1) Let X be an m-dimensional topological manifold and let Y be an n-dimensional topological manifold. We consider the topological space $X \times Y$. In the Product Topology Properties Proposition 5.6 we remarked that the product of two Hausdorff spaces is again Hausdorff. Thus we see that $X \times Y$ is Hausdorff. Furthermore note that it follows from the Second Countability Lemma 9.1 (3) that $X \times Y$ is second-countable.

It remains to find charts for $X \times Y$. So let $(x,y) \in X \times Y$. We distinguish three cases:

(a) If $x \in X \setminus \partial X$ and $y \in Y \setminus \partial Y$ then by taking "product charts" it is straightforward to find a chart of type (i) for $(x,y) \in X \times Y$.

(b) If $x \in X \setminus \partial X$ and $y \in \partial Y$ or if $x \in \partial X$ and $y \in Y \setminus \partial Y$ then it is equally straightforward to find a product chart of type (ii) for $(x,y) \in X \times Y$.

(c) Finally suppose that $x \in \partial X$ and $y \in \partial Y$. Our goal is to find a chart of type (ii) for $(x,y) \in X \times Y$. First note that using a product chart it is pretty straightforward to see that there exists an open neighborhood U of $(x,y) \in X \times Y$ and a homeomorphism $\Phi\colon U \to \mathbb{R}^{m+n-2} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. Next we consider the map

$$\begin{array}{rcl} \Psi\colon \mathbb{R}^{m+n-2} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} & \to & \mathbb{R}^{m+n-2} \times \mathbb{R} \times \mathbb{R}_{\geq 0} \\ (x_1, \dots, x_{m+n-2}, \underbrace{r\cos(\varphi), r\sin(\varphi)}_{=:z \in \mathbb{C}}) & \mapsto & (x_1, \dots, x_{m+n-2}, \underbrace{r^2 \cdot \cos(2\varphi), r^2 \cdot \sin(2\varphi)}_{=z^2}). \end{array}$$

Note that the map $\mathbb{C} \to \mathbb{C}$ that is given by $z = x + \mathrm{i}y \mapsto z^2 = (x^2 - y^2) + 2xy\mathrm{i}$ is continuous. It follows that Ψ is continuous. It is straightforward to show, say using the Closed Embedding-to-$\mathbb{R}^n$ Proposition 2.19, that Ψ is indeed a homeomorphism.

One can now easily verify that $\Psi \circ \Phi$ is a chart of type (ii) around (x,y).

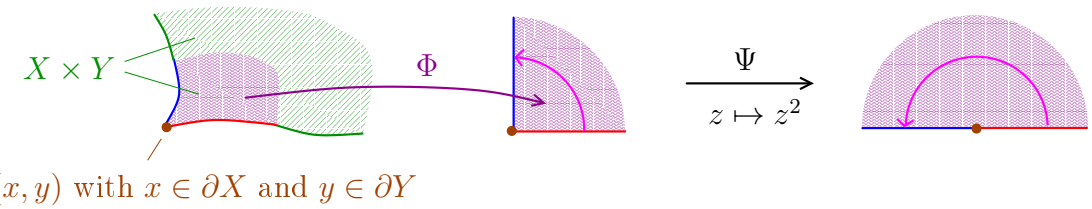
(2) We proved Statements (a) and (b) in (1). Furthermore Statement (c) is an immediate consequence of (b) and the definition of the boundary of a topological manifold.

(3) Now suppose that X and Y are closed. By (2) we know that $\partial(X \times Y) = \varnothing$ and from the Product Topology Properties Proposition 5.6 (2) we obtain that $X \times Y$ is compact. Thus we have shown that $X \times Y$ is a closed topological manifold. ■

We conclude this section with two technical lemmas.

Lemma 18.5. Let X and Y be two topological manifolds.

(1) If $f\colon X \to Y$ is a local homeomorphism, then $f(\partial X) = (\partial Y) \cap f(X)$.



(2) If $f\colon X \to Y$ is in fact a homeomorphism, then f restricts to a homeomorphism $\partial X \to \partial Y$.

Proof. The lemma follows from the elementary observation that the restriction of a chart for a point on a topological manifold has the same type as the original chart. We leave it to the reader to fill in the minuscule details. ■

The following lemma is slightly more interesting than the previous lemma.

Lemma 18.6. (Manifold Boundary-Closed Subset Lemma) Let X be a topological manifold.

(1) Any union of components of ∂X is a closed subset of X. In particular ∂X itself is a closed subset of X.
(2) If X is compact, then ∂X is also compact.
(3) $X \setminus \partial X$ is an open subset of X and if $\partial X \neq \varnothing$, then $X \setminus \partial X$ is non-compact.

Proof. Let X be a topological manifold.

(1) First note that, basically by definition, the set of points in X admitting a chart of type (i) is an open subset of X. It follows from the definition of ∂X that $X \setminus \partial X$ is an open subset. Thus the boundary ∂X is a closed subset of X. We denote by $\{Y_i\}_{i \in I}$ the components of ∂X. By the Components-are-Closed Lemma 2.34 each Y_i is an open subset of ∂X. Thus for any $i \in I$ there exists, by definition of the subspace topology, an open set $Z_i \subset X$ with $Z_i \cap \partial X = Y_i$. For any $J \subset I$ we have

$$X \setminus \Big(\bigcup_{j \in J} Y_j \Big) = \underbrace{(X \setminus \partial X)}_{\text{open in } X} \cup \underbrace{\bigcup_{j \notin J} Z_j}_{\text{open in } X}.$$

We have thus shown that the union of components of ∂X corresponding to J is indeed a closed subset of X.

(2) Now suppose that X is compact. Thus ∂X is a closed subset of the compact space X. We obtain from the Compact-Closed Lemma 1.21 (1) that ∂X itself is compact.

(3) It follows from (1) that $X \setminus \partial X$ is open. It remains to show that if $X \setminus \partial X$ is compact, $\partial X = \varnothing$. It follows from the elementary the Disjoint Union Topology-Properties Lemma 5.13 (2) that we only need to consider the case that X is connected. Thus assume that $Y := X \setminus \partial X$ is compact. Since X is Hausdorff we obtain from the Compact-Closed Lemma 1.21 that Y is closed. By (1) we know that ∂X is also closed. Since X is connected, since $X = Y \sqcup \partial X$ and since $Y \neq \varnothing$ we see that $\partial X = \varnothing$. ■

18.2. **Surfaces are topological manifolds.** On pages 265 and 266 we introduced the orientable surface of genus $g \in \mathbb{N}_0$ and the non-orientable surface of genus $g \in \mathbb{N}$. The

names already suggests that these should be 2-dimensional topological manifolds, and the following proposition confirms our hunch.

Proposition 18.7. (Surface-is-Topological Manifold Proposition)

(1) For every $g \in \mathbb{N}_0$ the surface Σ_g of genus g is a closed 2-dimensional topological manifold.

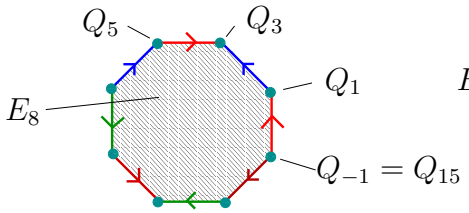
(2) For every $g \in \mathbb{N}$ the non-orientable surface N_g of genus g is a closed 2-dimensional topological manifold.



Proof. To simplify the discussion we will only prove Statement (1) of the proposition. We leave it to the reader to modify the argument to deal with the non-orientable case.

We start out this proof with the discussion of the two cases $g = 0$ and $g = 1$ that we had basically dealt with already. First recall that on page 513 we saw that any S^n is a closed n-dimensional topological manifold. The surface of genus 0 is by definition the sphere S^2, thus we are done by the above. Furthermore, the surface of genus 1 is by definition the torus $S^1 \times S^1$. The desired statement thus follows from the above and the Topological Manifold Product Proposition 18.4.

It remains to deal with the case $g \geq 2$. To simplify the notation we only deal with the case $g = 2$. The proof of the general case is totally analogous. Recall that the surface of genus 2 is defined as $E_8/\sim$ where E_8 is the octagon with the vertices $Q_k = \exp(2\pi \mathrm{i} k/16)$, $k = 1, 3, \ldots, 15$ and where "$\sim$" is the equivalence relation that we introduced on page 265.



$E_8/\sim$ is the surface of genus 2

Note that E_8 is compact by the Heine-Borel Theorem 1.24. Thus we obtain from the Quotient-Compact Connected Lemma 5.16 (1) that $E_8/\sim$ is also compact. In the following we will see that $E_8/\sim$ admits a finite atlas, by Lemma 18.2 this implies in particular that $E_8/\sim$ is second-countable.[195] Note that E_8 is compact by the Heine-Borel Theorem 1.24. Thus we obtain from the Compact Image Lemma 2.13 that $E_8/\sim$ is also compact. We leave it to the reader to verify that $E_8/\sim$ is Hausdorff.[196]

Thus, to show that $E_8/\sim$ is a closed 2-dimensional topological manifold it remains to give a finite atlas for $E_8/\sim$ that consists only of charts of type (i).

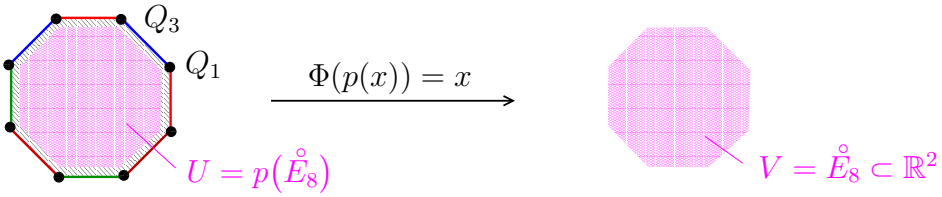
[195] It would also not be difficult to prove second-countability "by hand".

[196] This can either be done "by hand", or more elegantly but possibly with more work, by using the Quotient Hausdorff Lemma 5.21.

First we consider the map

$$\begin{array}{rcl} \Phi\colon U := p\big(\mathring{E}_8\big) & \to & V = \mathring{E}_8 \subset \mathbb{R}^2 \\ p(x) & \mapsto & x. \end{array}$$

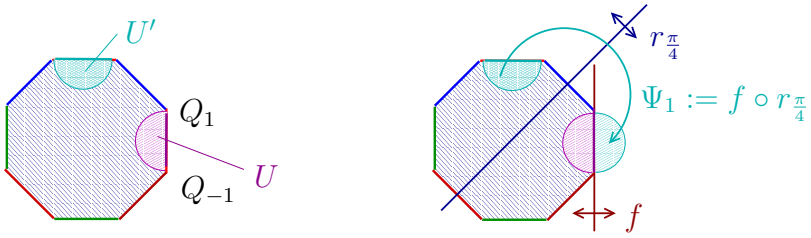
One can easily verify that this is a chart of type (i) for each point in the interior of E_8.



Next we introduce charts that cover the interiors of the edges. First we consider the edge given by Q_{-1} and Q_1. We denote by U and U' the half-disks that are illustrated in the figure below. (Note that it is understood that the line segment is included but that the circle is not included in U and U'.) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection in the line that goes through Q_{-1} and Q_1. It is straightforward to verify that the map

$$\begin{array}{rcl} \Psi_1\colon p(U \cup U') & \to & \text{the open disk } U \cup f(r_{\frac{\pi}{4}}(U)) \subset \mathbb{R}^2 \\ p(x) & \mapsto & \begin{cases} x, & \text{if } x \in U \\ f(r_{\frac{\pi}{4}}(x)), & \text{if } x \in U' \end{cases} \end{array}$$

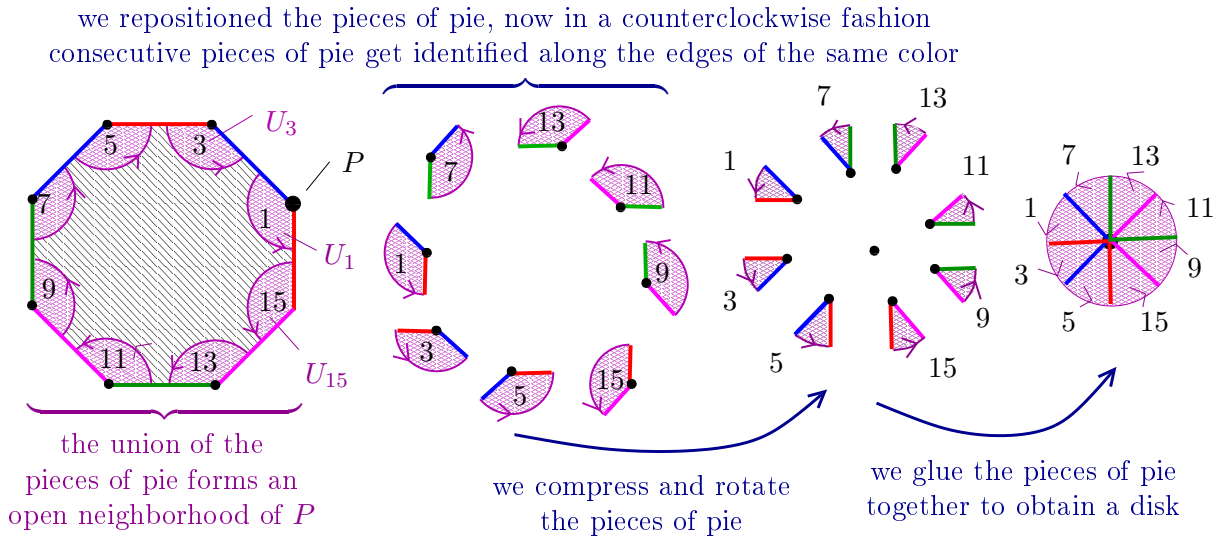
is well-defined. Furthermore, using Pasting Proposition 2.6 (2) one can show that Ψ_1 and its inverse are both continuous. Finally, it follows easily from the definitions that $p(U \cup U')$ is an open subset of $E_8/\!\sim$. Thus Ψ_1 is a chart of type (i) for all points in the interior of the edge from Q_{-1} to Q_1. Similarly we define charts Ψ_2, Ψ_3, Ψ_4 that take care of the other pairs of edges.



It remains to find a chart of type (i) that takes care of the point P in $E_8/\!\sim$ that is represented by the vertices. (It is a fun exercise to verify that all vertices get identified.) We pick an $\eta > 0$ such that the η-disks centered at $Q_1, \ldots, Q_{15}$ are disjoint. For $k = 1, 3, \ldots, 15$ we set $U_k := B_\eta(Q_k) \cap E_8$. The sets $U_1, U_3, \ldots, U_{15}$ are disjoint and one can easily verify that $p(U_1) \cup p(U_3) \cup \cdots \cup p(U_{15})$ is an open neighborhood of $p(Q_1) = p(Q_3) = \cdots = p(Q_{15})$.

Before we write down the chart rigorously it might help to visualize the situation. To the left of the figure below we see the sets $U_1, \ldots, U_{15}$. Next we arrange these "pieces of pie" as in the second second part of the above. Thus we see the "pieces of pie" with an angle of $\frac{3\pi}{4}$ where we show the various identifications of the edges. The map from the second to the third part is given by compressing the angle of the "pieces of pie" to a third and by rotating

the "pieces of pie". These eight "pieces of pie", each with an angle of $\frac{\pi}{4}$, can be combined to give a disk in $\mathbb{R}^2$.



Now we want to turn the above idea into formal mathematics. We consider the map

$$\Omega\colon p(U_1)\cup\cdots\cup p(U_{15}) \;\to\; B^2_\eta(0)$$

that is defined on each $p(U_k)$, $k\in\{1,3,\ldots,15\}$ to be the map

$$p\big(\underbrace{Q_k + r\cdot\exp(\mathrm{i}\alpha)}\big) \;\mapsto\; r\cdot\exp(\mathrm{i}(\tfrac{1}{3}(\alpha-\beta(k))+\gamma(k))).$$

where $r\in[0,\eta)$ and $\alpha\in[\beta(k),\beta(k)+\frac{3}{4}\pi]$

Here $\beta(k)$ and $\gamma(k)$ are given by the following table:

k	1	3	5	7	9	11	13	15
$\beta(k)$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	0	$\frac{1}{4}\pi$	$\frac{1}{2}\pi$
$\gamma(k)$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{1}{2}\pi$	$\frac{7}{4}\pi$	0	$\frac{1}{4}\pi$	$\frac{3}{2}\pi$

It is straightforward, albeit slightly cumbersome, to verify that Ω is a bijection. Furthermore, using Exercise 5.33 together with Pasting Proposition 2.6 (2) one can show fairly easily that Ω is continuous and that its inverse is also continuous. In other words, the map Ω is a homeomorphism. As pointed out above, it is straightforward to verify that $p(U_1)\cup\cdots\cup p(U_{15})$ is an open subset of $E_8/\!\sim$. In summary, we have shown that Ω is a chart of type (i) for P.

The eight charts $\Phi, \Psi_1, \Psi_2, \Psi_3, \Psi_4$ and Ω are now charts of type (i) that cover all of $E_8/\!\sim$. We have thus found the desired atlas. ■

18.3. **Properties of topological manifolds.** Topological manifolds are particularly important and also particularly nice topological spaces. For example the next lemma shows in particular that when it comes to topological manifolds, we do not have to worry about the two different definitions of connectedness.

Lemma 18.8. (Topological Manifolds-Local Properties Lemma)

(1) Every n-dimensional topological manifold is locally homeomorphic to an open ball in $\mathbb{R}^n$ or an open half-ball in $\mathbb{H}^n$, in particular it is locally homeomorphic to a convex subset of $\mathbb{R}^n$ or $\mathbb{H}^n$.
(2) Every topological manifold is locally contractibleand in particular locally path-connected.
(3) (a) The components of a topological manifold M are precisely the path-components of M. In particular M is connected if and only if M is path-connected.
(b) The components of a topological manifold are open and closed subsets of M.
(4) Every topological manifold is regionally compact.
(5) Every topological manifold is locally precompact.

Remark. Throughout the lecture notes we will on many occasions make use of the Topological Manifolds-Local Properties Lemma 18.8 (3) without referring to it explicitly.

Proof. Let X be an n-dimensional topological manifold.

(1) Let X be an n-dimensional topological manifold and let $P \in X$. First assume that $P \notin \partial X$. This means that there exists a chart $\Phi\colon U \to V$ of type (i) from some open neighborhood U of x to some open subset V of $\mathbb{R}^n$. We set $Q = \Phi(P)$. Since V is open there exists an $r > 0$ such that $B_r^n(Q) \subset V$. Evidently $B_r^n(Q)$ is convex and we see that $\Phi^{-1}(B_r^n(Q))$ is an open neighborhood of P that is homeomorphic to a convex subset of $\mathbb{R}^n$. Basically the same argument also works if $P \in \partial X$.
(2) On page 432 we saw that convex sets are contractible. Thus it follows from (1) that topological manifolds are locally contractible. In particular we see that topological manifolds are locally path-connected.
(3) This statement is an immediate consequence of (2) and of the Locally Path-Connectedness Lemma 2.40.
(4) On page 178 we saw that open subsets of $\mathbb{R}^n$ are regionally compact. The same argument shows that open subsets of $\mathbb{H}^n$ are regionally compact. Using the Compact Image Lemma 2.13 it is now easy to see that topological manifolds are regionally compact.
(5) Since topological manifolds are by definition Hausdorff this statement follows immediately from (4) together with Lemma 2.42. ■

The humble Topological Manifolds-Local Properties Lemma 18.8, together with previous work, begets several interesting consequences.

Proposition 18.9. (Manifold Paracompact-Normal Proposition)
(1) Every topological manifold is paracompact.
(2) Every topological manifold X is normal, i.e. given any two disjoint closed subsets A and B of X there exist disjoint open neighborhoods U of A and V of B.

Proposition 18.10. (Manifold Metrization Proposition) Every topological manifold X is completely metrizable, i.e. there exists a complete metric d on X such that the given topology agrees with the topology coming from d.

Remark.

(1) In the Atlas Improvement Proposition 21.3 we will give an alternative proof that topological manifolds are paracompact.
(2) In the Manifold Metrization Proposition 27.4 we will give a different proof of the Manifold Metrization Proposition 18.10 which does not require the Urysohn Metrization Theorem 9.4.

Proof of the Manifold Paracompact-Normal Proposition 18.9. Let X be a topological manifold.

(1) By the Topological Manifolds-Local Properties Lemma 18.8 we know that X is regionally compact. By definition X is Hausdorff and second-countable. Thus it follows from the Paracompact-Criterion Lemma 10.2 that X is paracompact.
(2) By definition topological manifolds are Hausdorff. We just saw in (1) that topological manifolds are paracompact. Therefore it follows from Dieudonné's Theorem 10.5 that topological manifolds are also normal. ■

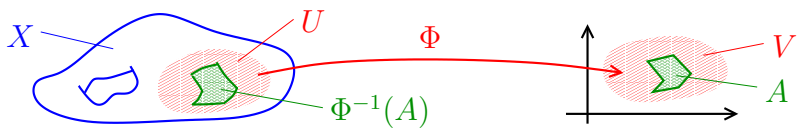
Proof of the Manifold Metrization Proposition 18.10. Let X be a topological manifold. By the Manifold Paracompact-Normal Proposition 18.9 (2) we know that X is normal. By definition X is second-countable and Hausdorff. The latter condition implies by the Points-in-Hausdorff Space Lemma 1.15 (1) that every one-point subset of X is closed. This shows that all the hypotheses of the Urysohn Metrization Theorem 9.4 are satisfied and we conclude that X is metrizable.

Furthermore, in the Topological Manifolds-Local Properties Lemma 18.8 we just saw that X is also regionally compact. Thus it follows from the Completely Metrizable Criterion Proposition 10.8 that we can upgrade "metrizable" to "completely metrizable". ■

On several occasions we will need the following, slightly technical lemma.

Lemma 18.11. Let X be a Hausdorff space (e.g. X could be a topological manifold), let U be an open subset of X and let $\Phi\colon U \to V$ be a homeomorphism (e.g. Φ could be a chart of type (i) or (ii).)

(1) If $A \subset V$ is a subset that is compact, then the corresponding subset $\Phi^{-1}(A)$ is a compact and in particular a closed subset of X.[197]
(2) If $B \subset V$ is a subset such that the closure of B is compact, then the closure of $\Phi^{-1}(B)$ is also compact.



Proof. Let X be a Hausdorff space, let U be an open subset of X and let $\Phi\colon U \to V$ be a homeomorphism. We denote by $\Psi = \Phi^{-1}\colon V \to U$ the inverse map.

(1) Let $A \subset V$ be a subset that is compact. It follows from the Compact Image Lemma 2.13 that $\Psi(A)$ is a compact subset of X. But by hypothesis X is Hausdorff.

[197] It does not suffice that A is a closed subset of the topological space V. Indeed, $A = V$ is a closed subset of V, but $U = \Phi^{-1}(V)$ is in general not a closed subset of X.

Hence it follows from the Compact-Closed Lemma 1.21 (2) that $\Psi(A) = \Phi^{-1}(A)$ is closed in X.

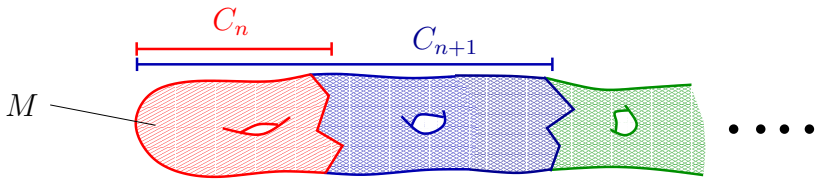
(2) Let $B \subset V$ be a subset such that the closure $A := \overline{B}$ is compact. By (1) we know that $\Phi^{-1}(A) = \Psi(A)$ is compact and closed. Since $\Phi^{-1}(B) \subset \Phi^{-1}(A)$ and the latter set is closed we see that the closure of $\Phi^{-1}(B)$ is contained in the compact set $\Phi^{-1}(A)$. But then it follows from the Compact-Closed Lemma 1.21 that the closure of $\Phi^{-1}(B)$ is also compact. ■

We continue with another technical lemma.

Lemma 18.12. (Topological Manifold Exhaustion Lemma) Let M be a topological manifold. There exists a compact exhaustion of M in the sense of the definition on page 338. More precisely, there exists a family of compact subsets $\{C_n\}_{n\in\mathbb{N}}$ of M such that the following two conditions are satisfied:

(1) we have $M = \bigcup_{n\in\mathbb{N}} C_n$,

(2) for every $n \in \mathbb{N}$ we have $\overset{\circ}{C}_n \subset C_{n+1}$.



Proof. The lemma follows immediately from the Compact Exhaustion Lemma 9.3 (1) together with the Topological Manifolds-Local Properties Lemma 18.8 (4). ■

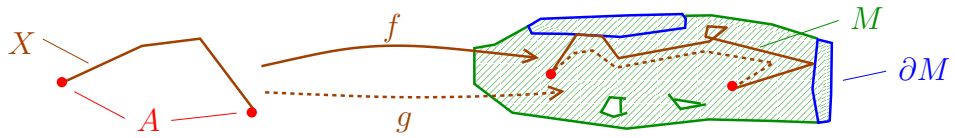
Remark. We will prove a refinement of the Topological Manifold Exhaustion Lemma 18.12 in the Manifold Exhaustion Lemma 21.2. For smooth manifolds we will prove a further refinement in the Compact Submanifold Exhaustion Proposition 31.2.

We conclude this section with the following technical lemma.

Lemma 18.13. (Push Away From Boundary Lemma) Let M be a topological manifold.

(1) Let X be a compact space, let $A \subset X$ be closed and let $f\colon X \to M$ be a map. If $f(A) \subset M \setminus \partial M$, then f is homotopic rel A to a map $g\colon X \to M \setminus \partial M$.

(2) If M is path-connected, then $M \setminus \partial M$ is also path-connected.



Remark. the Push Away From Boundary Lemma 18.13 foreshadows the Smooth Collar Neighborhood Theorems 28.3 and 75.5 that we will prove later. In Exercise 28.2 we will prove a "smooth" version of the Push Away From Boundary Lemma 18.13.

Proof.

(2) Let us first show how to deduce (2) from (1). This also serves as a good example for (1). So let us assume that M is path-connected. Let $P, Q \in M \setminus \partial M$. Since M is path-connected there exists a map $f\colon [0,1] \to M$ with $f(0) = P$ and $f(1) = Q$. We apply (1) to f and $A := \{0,1\} \subset X := [0,1]$ and we obtain a map $g\colon [0,1] \to M \setminus \partial M$ with $g(0) = f(0) = P$ and $g(1) = f(1) = Q$.

(1) Let X be a compact space, let $A \subset X$ be closed and let $f\colon X \to M$ be a map with $f(A) \subset M \setminus \partial M$. Since X is compact we know by the Compact Image Lemma 2.13 that $f(X)$ is compact. By the Manifold Boundary-Closed Subset Lemma 18.6 we know that ∂M is compact. It follows that $f(X) \cap \partial M$ is compact. As we pointed out on page 127, this implies that $f(X) \cap \partial M$ is also compact. It follows from the Compact Subspace Covering Lemma 1.19 and a variation on Lemma 18.3 that there exist charts $\Phi_i\colon U_i \to B_2^{n-1} \times [0,2)$, $i = 1, \dots, k$ of type (ii) such that

$$f(X) \cap \partial M \subset \bigcup_{i=1}^{k} \Phi_i^{-1}(B^{n-1} \times \{0\}) \quad \text{and such that} \quad A \subset M \setminus \bigcup_{i=1}^{k} \Phi_i^{-1}(B_2^{n-1} \times [0,2)).$$

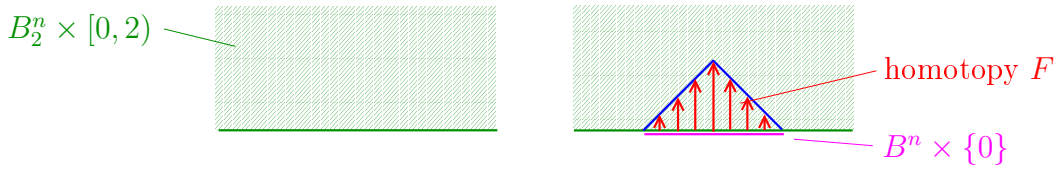
Next we consider the homotopy

$$\begin{aligned} F\colon B^{n-1} \times [0,2) \times [0,1] &\to B^{n-1} \times [0,2) \\ ((x,s),t) &\mapsto \begin{cases} (x,s), & \text{if } \|x\| > 1 \text{ or } s \geq 1 - \|x\|, \\ (x, s\cdot(1-t) + (1-\|x\|)\cdot t), & \text{if } \|x\| \leq 1 \text{ and } s \leq 1 - \|x\| \end{cases} \end{aligned}$$

which is illustrated in the figure below. Using the charts $\Phi_1, \dots, \Phi_k$ we can "transplant" the homotopy F and we obtain homotopies $F_1, \dots, F_k\colon M \times [0,1] \to M$. Now we apply these homotopies one after another to obtain a homotopy $M \times [0,1] \to M$. More precisely, we consider the homotopy that is given by

$$\begin{aligned} M \times [0,1] &\to M \\ (x,t) &\mapsto F_1(\dots F_2(F_1(x,t),t),\dots,t). \end{aligned}$$

One can easily verify that F has all the desired properties. ∎



18.4. **Submanifolds.** In this section we want to specify what we mean by a submanifold of a topological manifold.

Definition. Let X be an n-dimensional topological manifold.

(1) We say a subset $N \subset X$ is a k-dimensional submanifold if given any $P \in N$ one of the following holds:

(α) there exists a chart $\Phi\colon U \to V$ of type (i) for X and P such that[198]

$$\Phi(U \cap N) \;=\; V \cap \{(0,\dots,0,x_1,\dots,x_k) \,|\, x_i \in \mathbb{R}\},$$

(β) or there exists a chart $\Phi\colon U \to V$ of type (ii) for X and P such that $\Phi(P)$ lies in $\partial\,\mathbb{H}^n$ and[199]

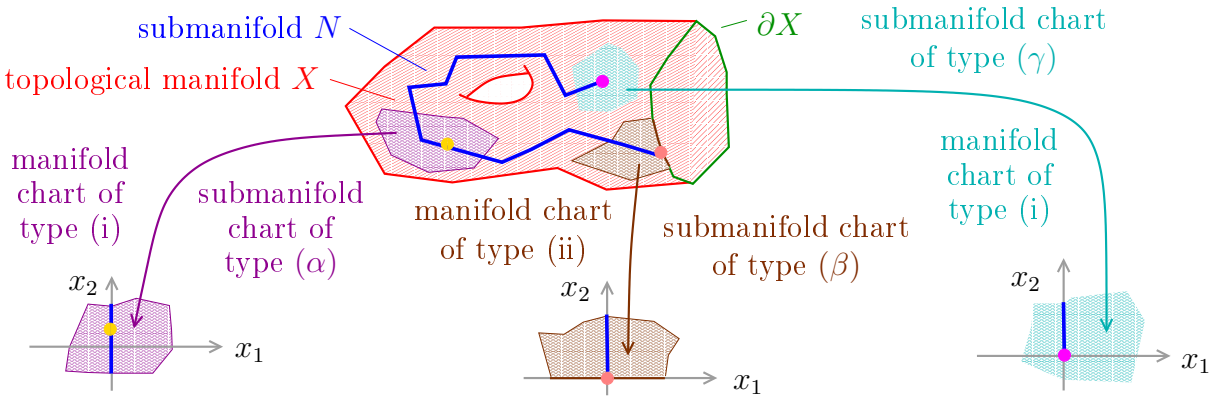
$$\Phi(U \cap N) \;=\; V \cap \{(0,\ldots,0,x_1,\ldots,x_k) \in \mathbb{R}^n \,|\, x_k \geq 0\},$$

(γ) or there exists a chart $\Phi\colon U \to V$ of type (i) for X and P such that $\Phi(P)$ lies in $\partial\,\mathbb{H}^n$ and[200]

$$\Phi(U \cap N) \;=\; V \cap \{(0,\ldots,0,x_1,\ldots,x_k) \in \mathbb{R}^n \,|\, x_k \geq 0\}.$$

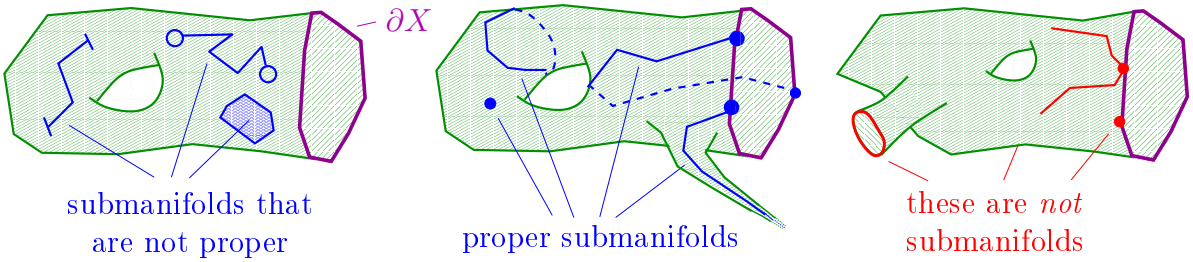
We refer to such charts as submanifold charts of N.

(2) We say a submanifold N is proper if N is a closed subset of X and if for each $P \in N$ we can find submanifold charts of type (α) and (β).



Examples.

(1) In the figure below we give typical examples of submanifolds, some of which are proper, some are not. We also show some subsets that are not submanifolds. In particular it is worth pointing out that a non-empty submanifold of X is never a subset of ∂X.[201]



(2) Let us show that S^1 is a proper 1-dimensional submanifold of $\mathbb{R}^2$. First we consider the case of a point $P = (x_0, y_0) \in S^1$ with $x_0 \in (-1, 1)$ and with $y_0 > 0$. We consider

[198] Why are the zeros to the left and not to the right?

[199] For $k = 0$ we interpret $\{(0,\ldots,0,x_1,\ldots,x_k) \in \mathbb{R}^n \,|\, x_k \geq 0\}$ as the empty set.

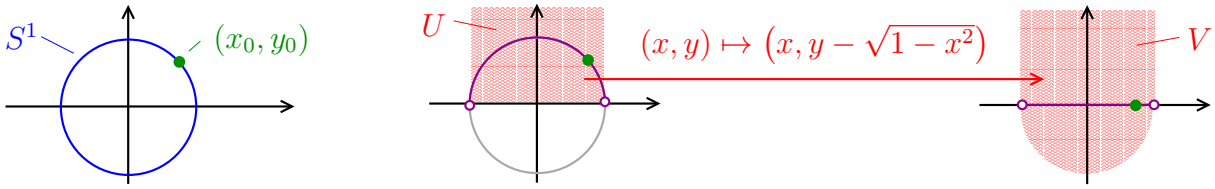
[200] See the previous footnote.

[201] Indeed, suppose that N is a non-empty submanifold of a topological manifold X. It follows immediately from the definitions of a submanifold that N contains a point P that is contained in a chart of type (i) for X. But by the definition of ∂X on page 511 this means that $P \notin \partial X$.

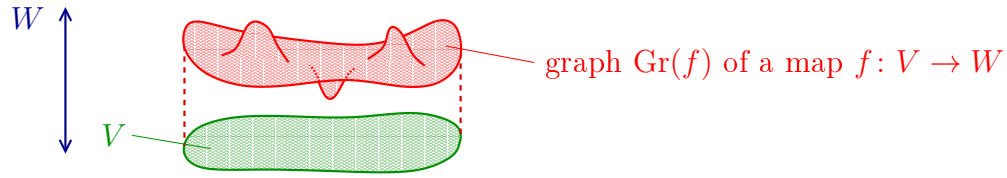
the map

$$\Phi\colon \overbrace{\{(x,y)\,|\,x\in(-1,1) \text{ and } y>0\}}^{=:U} \to \overbrace{\{(x,y)\,|\,x\in(-1,1) \text{ and } y>-\sqrt{1-x^2}\}}^{=:V}$$
$$(x,y) \mapsto \big(x, y-\sqrt{1-x^2}\big).$$

One can easily verify that this is a chart of type (α) for $P=(x_0,y_0)$. By precomposing Φ by a rotation one obtains a chart of type (α) for any other point on S^1. Since S^1 is a closed subset of $\mathbb{R}^2$ we see that S^1 is a 1-dimensional proper submanifold of $\mathbb{R}^2$. A very mild variation on this argument shows that S^n is a proper n-dimensional proper submanifold of $\mathbb{R}^{n+1}$.



(3) Let $f\colon V\to W$ be a map between two topological manifolds with $f(V)\subset W\setminus\partial W$. In Exercise 18.8 we will see that the graph $\mathrm{Gr}(f):=\{(x,f(x))\,|\,x\in V\}$ is a proper submanifold of the topological manifold $V\times W$. An especially intersection example



is given by the Cantor function $f\colon[0,1]\to\mathbb{R}$ that we introduced on page 185. The above discussion implies that the graph of the Cantor function is a proper 1-dimensional submanifold of $[0,1]\times\mathbb{R}$.

(4) Let X be a topological manifold. It follows easily from the definitions that a subset $N\subset X$ is a 0-dimensional submanifold of X if and only if N is a discrete subset of $X\setminus\partial X$.

(5) Let X be a topological manifold and let $U\subset X$ be an open subset. It follows almost immediately from the definitions that U is a codimension-zero submanifold of X. Furthermore, if we view U as a topological manifold in its own right, then it follows again basically immediately from the definitions, see also Lemma 18.5, that $\partial U = U\cap\partial X$.

Remark. The notion of a "submanifold" is used differently by different authors. Therefore, reading the literature it is a good idea to check what definition of a submanifold the authors use. In particular our definition of a "proper submanifold" is often used as the definition of a submanifold.

The following lemma shows that submanifolds are topological manifolds in their own right.

Lemma 18.14. (Submanifold-is-Manifold Lemma) Let X be a topological manifold. If $N\subset X$ is a k-dimensional submanifold, then N is a k-dimensional topological manifold.

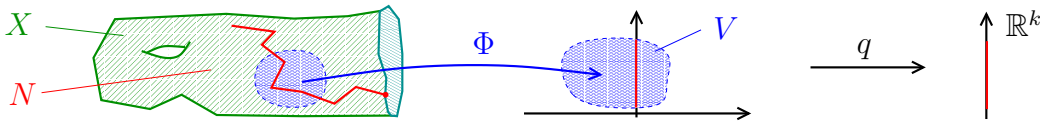
Remark. Let X be a topological manifold and let $N \subset X$ be a k-dimensional submanifold. In the Topological Manifold Boundary Proposition 75.2 we will see that N is a proper submanifold if and only if the following two conditions are satisfied:

(1) N is a closed subset of X.
(2) We have the equality $\partial N = (\partial X) \cap N$.[202] [203]

Proof. Let X be a topological manifold and let $N \subset X$ be a k-dimensional submanifold. It follows from Lemmas 1.14 and 9.1 that N is Hausdorff and second-countable. We denote by $q\colon \mathbb{R}^n \to \mathbb{R}^k$ the projection onto the last k coordinates. Let $P \in N$. We make the following slightly excessive observations:

(a) If P admits a submanifold chart $\Phi\colon U \to V$ of type (α), then this submanifold chart is a chart of type (i) for $P \in X$ and the map $q \circ \Phi|_{U \cap N}$ is a k-dimensional chart of type (i) for $P \in N$.
(b) If P admits a submanifold chart $\Phi\colon U \to V$ of type (β), then this submanifold chart is a chart of type (ii) for $P \in X$ and the map $q \circ \Phi|_{U \cap N}$ is a k-dimensional chart of type (ii) for $P \in N$.
(c) If P admits a submanifold chart $\Phi\colon U \to V$ of type (γ), then this submanifold chart is a chart of type (i) for $P \in X$ and the map $q \circ \Phi|_{U \cap N}$ is a k-dimensional chart of type (ii) for $P \in N$.

It follows from these observations that N is indeed a k-dimensional topological manifold. ■

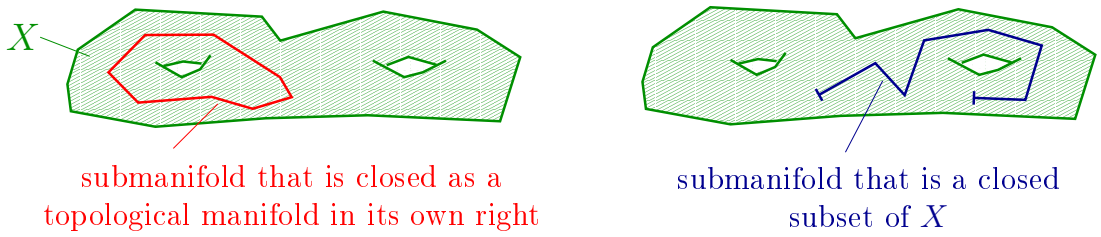


Remark. Unfortunately, when it comes to submanifolds the double usage of the word "closed" can lead to confusion. More precisely, let X be a topological manifold and let N be a submanifold.

(1) If N, viewed as a topological manifold in its own right, is closed, then it is in particular compact, and it follows from the Compact-Closed Lemma 1.21 (2) that N is actually a closed subset of X.
(2) If N is a closed subset of X, then it does not follow that N is a closed topological manifold. For example take $X = \mathbb{R}^2$ and $N = \{0\} \times \mathbb{R}$. In this case N is a closed subset and it is actually even a proper submanifold. But N is evidently not compact, hence it is not a closed topological manifold.

---

[202] Here by ∂N we mean the boundary of N, viewed as a topological manifold in its own right.
[203] Note that if N is a proper submanifold, then as of right now we lack the means to show that $\partial N = (\partial X) \cap N$, in fact we even lack the means to show either of the "$\subset$" and "$\supset$" inclusions.



Exercises for Chapter 18.

Exercise 18.1. Let X be a topological space. Show that the following two statements are equivalent:

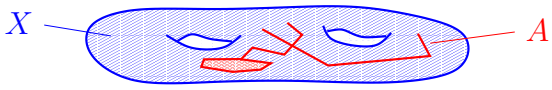
(1) X is a 0-dimensional topological manifold.
(2) The set X is countable and X is equipped with the discrete topology.

Exercise 18.2. Show that the line with two zeros admits a chart of type (i) for every point.

Exercise 18.3.

(a) Let $X = \mathbb{R} \cup \{*\}$ be the line with two zeros. Does there exist a map $f\colon X \to \mathbb{R}$ with $f(0) = 0$ and $f(*) = 1$.
(b) Let X be a topological manifold and let $P, Q \in X$ be two disjoint points. Show that there exists a map $f\colon X \to \mathbb{R}$ with $f(P) = 0$ and $f(Q) = 1$.

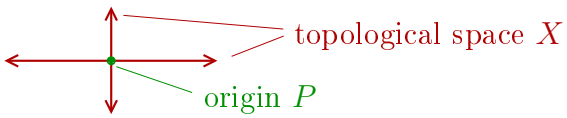
Exercise 18.4. Let X be a connected topological manifold and let $A \subset X$ be a non-empty closed subset with $A \neq X$. Show that $X \setminus A$ is not compact.



Exercise 18.5. Does there exist a compact space that satisfies all axioms of a closed 1-dimensional topological manifold except for the Hausdorff axiom?

Exercise 18.6. Let $f\colon X \to Y$ be a map between two topological manifolds. Show that given any $x \in X$ there exist neighborhoods U of x and V of $f(x)$ with $f(U) \subset V$ such that both U and V are homeomorphic to closed balls.

Exercise 18.7. We consider the topological space $X = \{(x, y) \in \mathbb{R}^2 \,|\, x \cdot y = 0\}$ shown in the figure below. Show that X is not a topological manifold.
Remark. It is clear that things go awry at the origin P. But it is somewhat tricky to nail down why there is no chart around that point.



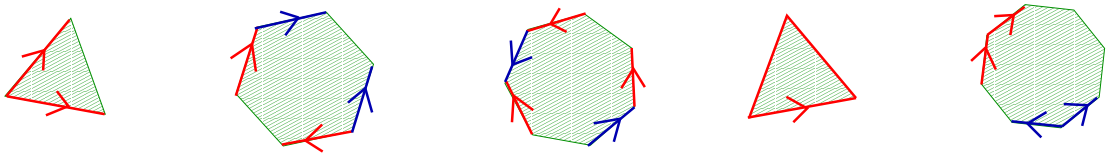
Exercise 18.8. Let $f\colon V \to W$ be a map between two topological manifolds such that $f(V) \subset W \setminus \partial W$. Show that the corresponding graph $\mathrm{Gr}(f) := \{(x, f(x)) \,|\, x \in V\}$ is a proper submanifold of the topological manifold $V \times W$.

Exercise 18.9. Can you find an atlas for S^n that consists of a single chart?

Exercise 18.10. Let W be an n-gon with sides $F_1, \dots, F_n$. Suppose we identify some of the sides in an orientation-preserving or orientation-reversing way.

(a) Under what conditions on the identification scheme do we get a topological manifold?
(b) If we get a topological manifold, what is its boundary?
(c) If you have a sense of what an orientable topological manifold should be, under what conditions do we get an orientable topological manifold?

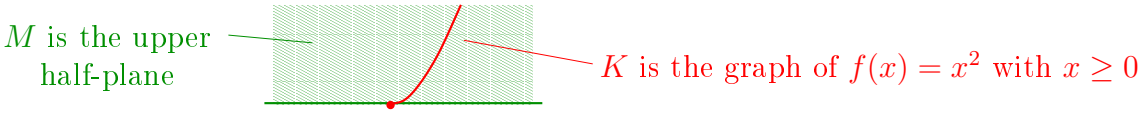
Remark. Strictly speaking we do not have the tools right now to rigorously prove the statements. So the exercise is to make an educated guess.



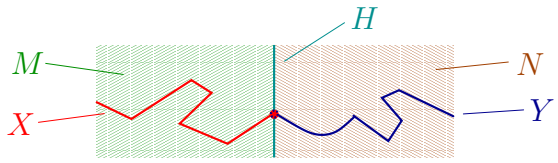
Exercise 18.11. In the figure below we show the topological manifold

$$X \;=\; \{(x,y) \in \mathbb{R}^2 \,|\, y \geq 0\}$$

together with the subset $K := \{(x, x^2) \,|\, x \geq 0\}$. Is K a submanifold of X?



Exercise 18.12. Let $M := \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_1 \leq 0\}$, $N := \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_1 \geq 0\}$ and $H := \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_1 = 0\}$. Let $X \subset M$ be a k-dimensional topological submanifold with $\partial X = X \cap H$ and let $Y \subset N$ be a k-dimensional topological submanifold with $\partial Y = Y \cap H$. We suppose that $X \cap H = Y \cap H$. Show that $X \cup Y$ is a k-dimensional topological submanifold of $\mathbb{R}^n$.



Exercise 18.13. Let X be a topological manifold, let $P \in X$ and let $f\colon K \to X$ be a map from a compact space K to X. Show that if $P \notin f(K)$, then there exists an open neighborhood U of P with $f(K) \cap U = \varnothing$.

Exercise 18.14. Let X be a topological manifold, let K be a compact subset and let U be an open neighborhood of K. Show that there exists an open subset V of X with $K \subset V \subset U$ such that $\overline{V} \subset U$ and such that $\overline{V}$ is compact.

Exercise 18.15. Let $n \in \mathbb{N}$. Let $\Phi_+\colon S^n \to \mathbb{R}^n \cup \{\infty\}$ be the stereographic projection from the North Pole $(0, \dots, 0, 1)$ and let $\Phi_-\colon S^n \to \mathbb{R}^n \cup \{\infty\}$ be the stereographic projection from the South Pole $(0, \dots, 0, -1)$. Show that the map $\Phi_- \circ \Phi_+^{-1}\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is given by $x \mapsto \frac{1}{\|x\|^2} \cdot x$.

Exercise 18.16. Let X be a connected topological manifold of dimension ≥ 2 and let $P_1, \ldots, P_k \in X$ be finitely many points. Show that $X \setminus \{P_1, \ldots, P_k\}$ is connected.
Remark. You could make use of the Union Connected Lemma 2.28.

Exercise 18.17. Let $g \in \mathbb{N}_0$.

(a) Show that the surface of genus g admits an atlas with two charts.
(b) Show that the surface of genus g admits an atlas consisting of three charts whose domains are diffeomorphic to 2-balls.

Exercise 18.18. Let X be a connected topological manifold.

(a) Suppose that X is compact. Show that there exists a surjective map $[0,1] \to X$.
(b) Suppose that X is non-compact. Show that there exists a surjective map $\mathbb{R} \to X$.

Hint. You might want to use the Space-Filling Curve Theorem 6.13.
Remark. Any surjective map from a connected one-dimensional topological manifold to a non-empty topological manifold of dimension ≥ 2 is called a *space-filling curve*.

Exercise 18.19. Let X be a topological manifold. Show that if X is connected, then $X \setminus \partial X$ is also connected.
Remark. Suppose that U and V are open disjoint subsets of $X \setminus \partial X$ with $X \setminus \partial X = U \cup V$. Show that there exists a decomposition $\partial M = A \sqcup B$ into open subsets such that $U \cup A$ and $V \cup B$ are open subsets of X.

Exercise 18.20. Let X be a topological manifold and let $A \subset X$ be a submanifold of codimension ≥ 2. Show that if X is path-connected, then so is $X \setminus A$.
Remark. You could try to modify the argument of the Push Away From Boundary Lemma 18.13.

Exercise 18.21. Given a subset $I \subset \mathbb{R}$ we define

$$H(I) \;:=\; \{(x,y) \in \mathbb{R}^2 \,|\, y > 0 \text{ or } x \in I \text{ and } y = 0\}.$$

In other words, $H(I)$ is the open upper half-plane of $\mathbb{R}^2$ together with the points on the x-axis corresponding to I.

(a) Let $I \subset \mathbb{R}$ be a subset. Show that $H(I)$ is a submanifold of $\mathbb{R}^2$ if and only if I is an open subset of $\mathbb{R}$.
(b) We consider

$$\begin{aligned} X &:= H((-\infty, 0) \cup (0, \infty)) \\ Y &:= H((-\infty, -1) \cup (1, \infty)). \end{aligned}$$

Are the two topological manifolds X and Y homeomorphic?



19. Smooth manifolds

In this chapter we introduce smooth structures on topological manifolds. A topological manifold equipped with a smooth structure will just be called a smooth manifold. In the literature the term "smooth manifold" can be used with quite different conventions and meanings. So we try to state clearly the various definitions used throughout these notes.

19.1. **Smooth functions on subsets of $\mathbb{R}^n$.** In the next section we want to define the notion of a "smooth structure" on a topological manifold. In this section we want to recall some basic definitions and facts regarding "classical" smooth maps.
Before we discuss properties of smooth maps we have to agree on what we mean by smooth maps on a subset of $\mathbb{R}^n$. The first definition is surely uncontroversial.

Definition. Let U be an open subset of $\mathbb{R}^n$.

(1) Let $f\colon U \to \mathbb{R}$ be a map, let $i \in \{1,\dots,n\}$ and let $P \in U$. We denote by e_i as usual the i-th vector of the standard basis of $\mathbb{R}^n$. We define the *i-th partial derivative of f at P* as follows:
$$\frac{\partial}{\partial x_i} f\big|_P := \lim_{h\to 0} \frac{f(P+h\cdot e_i)-f(P)}{h},$$
if the limit exists.[204]

(2) We say a map $f = (f_1,\dots,f_m)\colon U \to \mathbb{R}^m$ is *smooth*, if for every coordinate function f_i all iterated partial derivatives $\frac{\partial}{\partial x_{k_1}} \dots \frac{\partial}{\partial x_{k_d}} f_i$, of arbitrarily large degree, exist.

Example. One can easily verify that given any $n \in \mathbb{N}_0$ the following maps are smooth:
$$\begin{array}{ccc} \mathbb{R}\times\mathbb{R} & \to & \mathbb{R} \\ (x,y) & \mapsto & x\cdot y \end{array} \qquad \begin{array}{ccc} \mathbb{R}\setminus\{0\} & \to & \mathbb{R} \\ x & \mapsto & \frac{1}{x} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{R}^n\times\mathbb{R}^n & \to & \mathbb{R} \\ (x,y) & \mapsto & x+y. \end{array}$$

Remark. Since partial derivatives are just classical derivatives in a slightly different context we see that all the classical rules for derivatives hold. For example on several occasions we will use the Product Rule for partial derivatives which says that for an open subset $U \subset \mathbb{R}^n$ and two smooth functions $f, g\colon U \to \mathbb{R}$ and $P \in U$ and $i \in \{1,\dots,n\}$ we have
$$\frac{\partial(f\cdot g)}{\partial x_i}(P) = \frac{\partial f}{\partial x_i}(P)\cdot g(P) + f(P)\cdot \frac{\partial g}{\partial x_i}(P).$$

Theorem 19.1. (Schwarz Theorem) Let U be an open subset of $\mathbb{R}^n$ and let $f\colon U \to \mathbb{R}$ be a map. Furthermore let $i, j \in \{1,\dots,n\}$. For any $P \in U$ we have
$$\frac{\partial^2}{\partial x_i\,\partial x_j}(P) = \frac{\partial^2}{\partial x_i\,\partial x_j}(P).$$

Proof. This fundamental theorem is for example proved in [**RudiW1976**, Theorem 9.41]. ∎

On many occasions it will also be interesting to consider maps which are not defined on open subsets of $\mathbb{R}^n$. This leads us to the following more risqué definition.

[204] As usual the notation for partial derivatives can change slightly, depending on the circumstances, for example for $n = 1$ we might write $\frac{\partial}{\partial x}$ instead of $\frac{\partial}{\partial x_i}$.

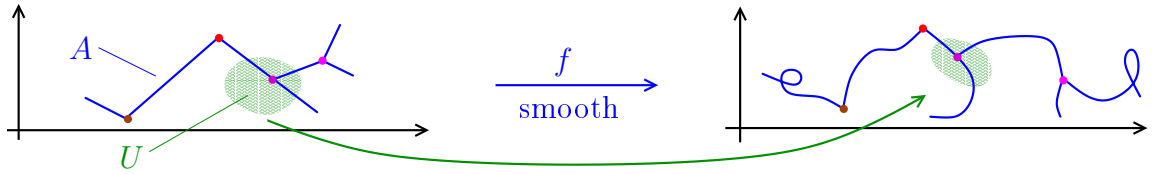
Definition. Let A be any subset of $\mathbb{R}^n$. We say a map $f\colon A \to \mathbb{R}^m$ is smooth, if for every $a \in A$ there exists an open neighborhood U (in $\mathbb{R}^n$) of a, viewed as a point in $\mathbb{R}^n$, and a smooth map $\widetilde{f}\colon U \to \mathbb{R}^m$ (now "smooth" in the sense above) which coincides with f on $A \cap U$.[205]

Examples.

(1) Let $A = \{(x,y) \in \mathbb{R}^2 \,|\, x \cdot y = 0\} \subset \mathbb{R}$. The map $f\colon A \to \mathbb{R}$ given by $f(x,y) = x^2 - y^2$ is smooth, since it can be extended in an obvious way to a smooth map on all of $\mathbb{R}^2$.

(2) The map $f\colon [0,\infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is not smooth, since it cannot be extended to a smooth function on any open interval (a,b) that contains 0.

(3) In the figure below we show a subset A of $\mathbb{R}^2$ and a smooth map $f\colon A \to \mathbb{R}^2$ which is more exotic than the lame example given in (1).



smooth map $\widetilde{f}$ on the open subset U that agrees with f on $A \cap U$

Definition. A subset of a topological space is called regular closed if $A = \overline{\mathring{A}}$, i.e. if it equals the closure of its interior.[206]

Example.

(1) By the discussion on page 118 every closed ball $\overline{B}_r(x)$ of radius > 0 is regular closed.

(2) One can easily show that every closed interval with non-empty interior is regular closed.

Lemma 19.2. Let $A \subset \mathbb{R}^n$ be a subset that is regular closed and let $f\colon A \to \mathbb{R}$ be a smooth map. Given any $i \in \{1,\dots,n\}$ there exists a unique continuous map $A \to \mathbb{R}$ that agrees with the partial derivatives $\frac{\partial f}{\partial x_i}$ on the open subset $\mathring{A}$.

Convention. Let $A \subset \mathbb{R}^n$ be a subset that is regular closed and let $f\colon A \to \mathbb{R}$ be a smooth map. Given $i \in \{1,\dots,n\}$ we refer to the map $A \to \mathbb{R}$ from Lemma 19.2 also as the partial derivative $\frac{\partial f}{\partial x_i}$.

Proof. Let $A \subset \mathbb{R}^n$ be a subset that is regular closed, let $f\colon A \to \mathbb{R}$ be a smooth map and let $i \in \{1,\dots,n\}$. By the above definition of a smooth map on A these partial derivatives extend locally to continuous maps on A. Since $\mathbb{R}$ is Hausdorff and since $\overline{\mathring{A}}$ we obtain from Exercise 2.33 that these extensions are unique. ■

The following proposition can be viewed as the smooth analogue of the Pasting Proposition 2.6 for continuous maps. It gives a convenient criterion for showing that a given map

[205] Note that if A is an open subset of $\mathbb{R}^n$, then this definition of a smooth map $A \to \mathbb{R}^n$ actually agrees with the previous definition.

[206] The name "regular closed" is for example used in [**SS1978**, p. 6].

is smooth. This proposition will be used throughout the lecture notes, mostly without referring to it explicitly.

Proposition 19.3. (Smooth Pasting Proposition I) Let A be a subset of $\mathbb{R}^n$, let $f\colon A \to \mathbb{R}^m$ be a map and let $\{V_i\}_{i\in I}$ be an open cover of A. Then the following holds:

$$f\colon A \to \mathbb{R}^m \text{ is smooth} \iff \text{every restriction } f|_{V_i}\colon V_i \to \mathbb{R}^m \text{ is smooth.}$$

Proof. The lemma follows immediately from the definitions. ∎

Notation. Let $I \subset \mathbb{R}$ be an interval. Given a smooth map $f = (f_1, \dots, f_n)\colon I \to \mathbb{R}^n$ and given $x_0 \in \mathring{I}$ we set

$$f'(x_0) := \left(\frac{\partial}{\partial x}\Big|_{x_0} f_1, \dots, \frac{\partial}{\partial x}\Big|_{x_0} f_n \right) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}.$$

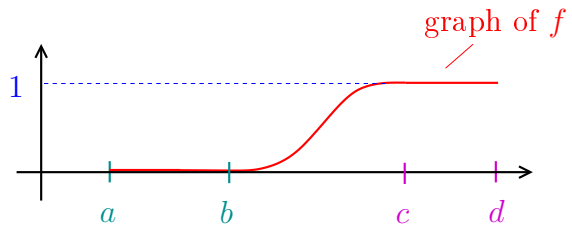
If I is closed and if I has non-empty interior, then we see that I is regular closed. Thus, by the above discussion, we can unambiguously extend $f'\colon \mathring{I} \to \mathbb{R}^n$ to a continuous map $f'\colon I \to \mathbb{R}^n$.

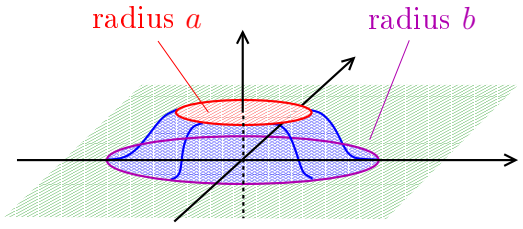
The following lemma is one of the key tools for constructing interesting smooth functions.

Lemma 19.4. (Smooth Transition Function Lemma)

(1) Suppose that we are given real numbers $a \le b < c \le d$ and $\epsilon > 0$. There exists a smooth function $f\colon [a,d] \to [0,1]$ with the following properties:
   (a) $f|_{[a,b]} \equiv 0$ and $f|_{[c,d]} \equiv 1$.
   (b) f is strictly monotonously increasing on $[b,c]$.
   (c) Given any $t \in (a,d)$ we have $f'(t) < (1+\epsilon)\cdot \frac{1}{c-b}$.

(2) Let $a < b$ be two positive real numbers and let $n \in \mathbb{N}$. There exists a smooth function $f\colon \mathbb{R}^n \to [0,1]$ such that $f(x) = 1$ whenever $\|x\| \le a$ and such that $f(x) = 0$ whenever $\|x\| > b$.





Sketch of proof.

(1) We consider the map

$$\begin{aligned} \varphi\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \begin{cases} 0, & \text{if } x \le 0, \\ \exp(-\frac{1}{x^2}), & \text{if } x > 0. \end{cases} \end{aligned}$$

An induction argument, using l'Hopital's rule, shows that φ is in fact smooth, see [**Kön2004**, Chapter 9] for details. Next we consider the map

$$\begin{aligned} \Phi\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \int_{t=0}^{t=x} \varphi(t)\cdot\varphi(1-t)\,dt. \end{aligned}$$

Using a little bit of massaging this map Φ can be turned into the desired map f. We leave it to the reader to fill in the remaining details.

(2) By (1) there exists a smooth function $\varphi\colon [0,\infty) \to [0,1]$ such that $\varphi(x) = 0$ for $x \in [0,a]$ and such that $\varphi(x) = 1$ for $x \in [b,\infty)$. Now consider the map

$$\begin{array}{rccc} f\colon \mathbb{R}^n & \to & [0,1] \\ x & \mapsto & 1-\varphi(\|x\|). \end{array}$$

Since $x \mapsto \|x\|$ is smooth on $\mathbb{R}^n\backslash\{0\}$ we see that f is smooth on $\mathbb{R}^n\backslash\{0\}$. Furthermore, since f is constant on a neighborhood of the origin it follows from the Smooth Pasting Proposition 19.3 that f is in fact smooth on all of $\mathbb{R}^n$. ■

In the following we give characterizations of smooth maps on intervals and certain subsets of $\mathbb{R}^n$. To the best of our knowledge we do not make use of the characterizations, neither explicitly nor implicitly. Nonetheless, it is good to know that they exist.

Proposition 19.5. (Smooth Functions on Compact Interval Proposition) Let $a < b$ be two real numbers and let $f\colon [a,b] \to \mathbb{R}$ be a function. The following two statements are equivalent:

(1) The function $f\colon [a,b] \to \mathbb{R}$ is smooth in the above sense.
(2) All derivatives of $f\colon (a,b) \to \mathbb{R}$ are defined on the open interval (a,b) and they extend to continuous maps on the closed interval $[a,b]$.

Proof. The "(1)⇒(2)"-implication is basically clear. The "(2)⇒(1)"-implication is a consequence of Borel's Theorem, see [**Strg1981**, Theorem 4.32]. ■

In fact an even more general statement holds:

Theorem 19.6. (Whitney Extension Theorem) Let $a < b$ be two real numbers, let U be an open subset of $\mathbb{R}^{n-1} \times [a,b]$ and let $f\colon U \to \mathbb{R}^m$ be a function. The following statements are equivalent:

(1) The map $f\colon U \to \mathbb{R}^m$ is smooth in the sense of the definition on page 531.
(2) The map f is smooth on the open subset $U \cap (\mathbb{R}^{n-1} \times (a,b))$ and all its partial derivatives extend to continuous maps on $U \cap (\mathbb{R}^{n-1} \times \{a,b\})$.

If $U = \mathbb{R}^{n-1} \times [a,b]$, the two statements are also equivalent to:

(3) The map f can be extended to a smooth map on all of $\mathbb{R}^n$.

The above statements also hold for half-spaces $\mathbb{R}^{n-1} \times (-\infty,a]$ and $\mathbb{R}^{n-1} \times [a,\infty)$.

Remark. The theorem was first proved by Hassler Whitney [**Why1934**] in 1934.

Proof. The "(1)⇒(2)"-implication and the "(3)⇒(1)"-implication are basically clear. The "(2)⇒(3)"-implication is a consequence of a generalization of Borel's Theorem, see [**Hör1990a**, Theorem 2.3.6]. An outline of the argument is also given in [**KP2002**, Theorem 2.3.2]. Finally note that the "(1)⇒(3)"-implication also follows from the Tietze Extension Theorem 8.6 combined with the Smooth Approximation Proposition 21.10. ■

19.2. **Taylor's Theorem.** In this short section we formulate Taylor's Theorem, so that later on we can refer to it. Since the reader is most likely aware of its existence it is arguably best to move on to the next section.

Definition. Let $n \in \mathbb{N}_0$.

(1) Let $m \in \mathbb{N}_0$ and let $I = (i_1, \ldots, i_m)$ be a sequence of length m with values in the set $\{1, \ldots, n\}$.
   (a) We write $|I| = m$.
   (b) We write $\partial_I = \frac{\partial^m}{\partial x_{i_1} \cdots \partial x_{i_m}}$.
   (c) Given $x, a \in \mathbb{R}^n$ we write $(x-a)^I = (x_{i_1} - a_{i_1}) \cdot \cdots \cdot (x_{i_m} - a_{i_m})$ and similarly we define $|x-a|^I$.

(2) Let $U \subset \mathbb{R}^n$ be an open subset. Given a smooth map $f\colon U \to \mathbb{R}$, given $a \in U$ and given $k \in \mathbb{N}_0$ we define the k-th Taylor polynomial of f at a as follows:

$$p_k \;:=\; p(f,a)_k \;:=\; \sum_{m=0}^{k} \frac{1}{m!} \sum_{|I|=m} \partial_I f|_a \cdot (x-a)^I.$$

Furthermore, given any $x \in \mathbb{R}^m$ such that the segment $\{(1-t)\cdot a + t \cdot x \,|\, t \in [0,1]\}$ lies in A we define the k-th remainder term of f at a as follows:

$$r_k \;:=\; r(f,a)_k \;:=\; \frac{1}{k!} \sum_{|I|=k+1} (x-a)^I \cdot \int_{t=0}^{t=1} (1-t)^k \cdot \partial_I f|_{(1-t)\cdot a + t\cdot x}\, dt.$$

Theorem 19.7. (Taylor's Theorem) Let $U \subset \mathbb{R}^n$ be an open subset, let $f\colon U \to \mathbb{R}$ be a smooth map, let $a \in U$ and let $k \in \mathbb{N}_0$. If W is a convex neighborhood of a in U, then for every $x \in W$ we have

$$f(x) \;=\; p(f,a)_k(x) + r(f,a)_k(x).$$

Proof. Taylor's Theorem gets proved in basically every single book on multivariable analysis. We refer to [**Lee2002**, p. 648] for a short proof. ∎

Corollary 19.8. Let $U \subset \mathbb{R}^n$ be an open subset, let $f\colon U \to \mathbb{R}$ be a smooth map, let $a \in U$, let $k \in \mathbb{N}_0$ and let W be a convex neighborhood of a in U. Finally suppose that we are given $M \in \mathbb{R}$ such that

$$M \;\geq\; \text{absolute value of all } (k+1)\text{-st partial derivatives of } f \text{ on } W.$$

Then for every $x \in W$ we have

$$\big|f(x) - p(f,a)_k(x)\big| \;\leq\; \frac{n^{k+1}\cdot M}{(k+1)!} \cdot \|x-a\|^{k+1}.$$

Proof. The corollary follows easily from Taylor's Theorem 19.7, the definitions, the observation that for any I we have $|(x-a)^I| \leq \|x-a\|^{|I|}$ and some elementary estimates. ∎

19.3. **The differential of a smooth map.** We start out with the following definition that should also be familiar from a course in real analysis.

Definition. Let $U \subset \mathbb{R}^n$ be an open subset and let $f\colon U \to \mathbb{R}^m$ be a map. We say that f is differentiable in P if there exists a matrix $\mathrm{D}f_P$ such that

$$\lim_{v \to 0} \frac{\|f(P+v) - f(P) - \mathrm{D}f_P \cdot v\|}{\|v\|} \;=\; 0$$

We refer to the $(m \times n)$-matrix $\mathrm{D}f_P$ (which is easily shown to be unique) as the differential of f at P.

Intuitively speaking a map $f\colon U \to \mathbb{R}^m$ is differentiable in P if in a neighborhood of P the map f is "close" to the affine linear map $w \mapsto \mathrm{D}f_P(P - w) + f(P)$.

Example. If $A \in \mathrm{M}(m \times n, \mathbb{R})$ and $v \in \mathbb{R}^m$, then it follows almost immediately from the definitions that for every $P \in \mathbb{R}^n$ the affine map $x \mapsto A \cdot x + v$ is differentiable with differential $\mathrm{D}f_P = A$.

The following proposition is essential for showing differentiability and computing differentials. We will use it all the time without acknowledging it.

Proposition 19.9. (Differential-from-Partial Derivatives Proposition) Let $U \subset \mathbb{R}^n$ be an open subset and let $f = (f_1, \ldots, f_m)\colon U \to \mathbb{R}^m$ be a map. If f is smooth, then the map f is differentiable in every $P \in U$ and for every $P \in U$ we have

$$\mathrm{D}f_P \;:=\; \left(\frac{\partial f_i}{\partial x_j}\Big|_P \right)_{i=1,\ldots,m, j=1,\ldots,n}$$

Proof. This statement is contained in every book on multivariable real analysis, see e.g. [**Lee2002**, Proposition C.8] or [**Fors2008**, p. 70]. ∎

Example. Let $I \subset \mathbb{R}$ be an open subset, let $f = (f_1, \ldots, f_m)\colon I \to \mathbb{R}^m$ be a smooth map and let $P \in I$. It follows immediately from the definitions that

$$f'(P) \;=\; \mathrm{D}\, f_P \cdot \underbrace{1}_{\in \mathbb{R}} \;=\; \text{the unique column vector of } \mathrm{D}f_P \;\in\; \mathbb{R}^m.$$

Put differently, the vector $f'(P) \in \mathbb{R}^m$, viewed as an $(m \times 1)$-matrix in the obvious way, equals $\mathrm{D}f_P$.

We also recall the chain rule for differentials.

Proposition 19.10. (Chain Rule) Let $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^l$ be open subsets and let $f\colon U \to \mathbb{R}^l$ and $g\colon V \to \mathbb{R}^m$ be two smooth maps such that $f(U) \subset V$. The composition $g \circ f\colon U \to \mathbb{R}^m$ is smooth and given any $P \in U$ we have

$$\underbrace{\mathrm{D}(g \circ f)_P}_{m \times k\text{-matrix}} \;=\; \underbrace{\mathrm{D}g_{f(P)}}_{m \times l\text{-matrix}} \cdot \underbrace{\mathrm{D}f_P}_{l \times k\text{-matrix}} \qquad \text{"Chain Rule".}$$

Proof. This proposition is proved in every book on multivariable real analysis, see e.g. [**Lee2002**, Proposition C.3]). ∎

Example. Let $U \subset \mathbb{R}^k$ be an open subset and let $f, g\colon U \to \mathbb{R}$ be smooth maps. It follows from Chain Rule 19.10 and the discussion on page 531 that the following maps are also smooth:[207]

$$\begin{array}{ccc} U & \to & \mathbb{R} \\ x & \mapsto & f(x) \cdot g(x) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} U & \to & \mathbb{R} \\ x & \mapsto & f(x) + g(x). \end{array}$$

We move on to the following definition.

[207] This example is the smooth analogue of Lemma 2.3.

Definition. Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$.[208] We say that a map $f\colon U \to V$ is a diffeomorphism if f is smooth, if f is a bijection and if $f^{-1}\colon V \to U$ is also smooth. If such a diffeomorphism exists, then we say that U and V are diffeomorphic.

Examples.

(1) Let $n \in \mathbb{N}$ and $r, s > 0$ and $x, y \in \mathbb{R}^n$. It is straightforward to show that the open balls $B_r^n(x)$ and $B_s^n(y)$ are diffeomorphic.

(2) We consider the maps

$$\begin{array}{rcl} B^n & \to & \mathbb{R}^n \\ x & \mapsto & \dfrac{x}{\sqrt{1-\|x\|^2}} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathbb{R}^n & \to & B^n \\ x & \mapsto & \dfrac{x}{\sqrt{1+\|x\|^2}}. \end{array}$$

One can easily verify that they are smooth and that they are inverses of one another. This shows that both maps are diffeomorphisms. In particular we have shown that B^n is diffeomorphic to $\mathbb{R}^n$.[209]

Next we recall a proposition from the analysis courses. Since this proposition is one of the reasons why later on dealing with smooth manifolds is easier than dealing with topological manifolds we also provide the proof.

Proposition 19.11. (Smooth Invariance of Domain Proposition) Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets.

(1) If $f\colon U \to V$ is a diffeomorphism with inverse $f^{-1}\colon V \to U$, then for any $P \in U$ we have

$$\mathrm{D}(f^{-1})_{f(P)} \cdot \mathrm{D}f_P = \mathrm{id}_m \qquad \text{and} \qquad \mathrm{D}f_P \cdot \mathrm{D}(f^{-1})_{f(P)} = \mathrm{id}_n\,.$$

In particular the differential $\mathrm{D}f_P$ is an invertible matrix.

(2) If U and V are diffeomorphic and non-empty, then $m = n$.

Example. Suppose U is an open subset, let's say connected, of $\mathbb{R}^n$ and that $f\colon U \to \mathbb{R}^n$ is a smooth map such that $\mathrm{D}f_P$ is invertible for every $P \in U$. One might be tempted to think that $f\colon U \to f(U)$ is already a diffeomorphism itself. But this is not necessarily the case, a counterexample is for example given by

$$\begin{array}{rcl} f\colon \mathbb{C}\setminus\{0\} & \to & \mathbb{C}\setminus\{0\} \\ z & \mapsto & z^2. \end{array}$$

One can easily verify that $\mathrm{D}f_z$ is invertible for every $z \in \mathbb{C}\setminus\{0\}$. But f is evidently not injective.

Proof.

(1) Let $f\colon U \to V$ be a diffeomorphism between open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$. We denote by $f^{-1}\colon V \to U$ the inverse of f. Let $P \in U$. We see that

$$\mathrm{D}(f^{-1})_{f(P)} \cdot \mathrm{D}f_P \underset{\uparrow}{=} \mathrm{D}(f^{-1}\circ f)_P = \mathrm{D}(\mathrm{id}_U)_P = \mathrm{id}_m \in \mathrm{M}(m\times m, \mathbb{R}).$$

Chain Rule 19.10

[208] Note that we do *not* demand that U and V are open subsets.

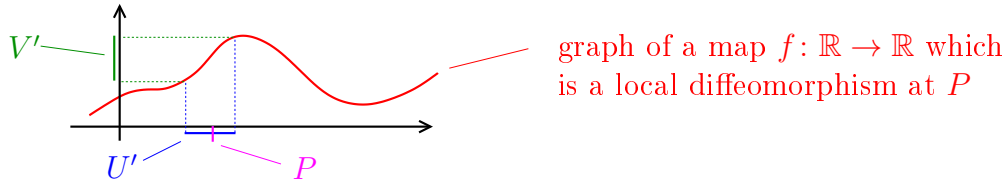
[209] In the Star Shaped-Diffeomorphism Type Theorem **??** we will prove a much more general statement, namely we will see that every open star-shaped subset of $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$.

The same way one shows that $\mathrm{D}f_P \cdot \mathrm{D}(f^{-1})_{f(P)} = \mathrm{id}_n \in \mathrm{M}(n \times n, \mathbb{R})$. In other words, the matrices $\mathrm{D}f_P$ and $\mathrm{D}(f^{-1})_{f(P)}$ are inverses of one another. This implies that $\mathrm{D}f_P$ is invertible.

(2) Since U is non-empty we can find a $P \in U$. By (1) the $(m \times n)$–matrix $\mathrm{D}f_P$ is invertible, this means that the matrix is a square matrix, i.e. $m = n$. ■

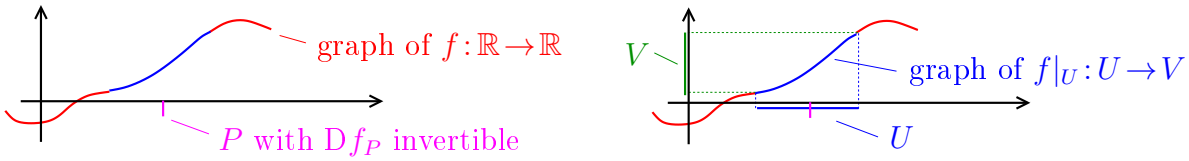
Before we state the next theorem let us introduce the following harmless definition, which is in line with our definition of a local property of a map that we gave on page 179.

Definition. Let $f\colon U \to V$ be a smooth map between two open subsets of $\mathbb{R}^n$ and let $P \in U$. We say f is a local diffeomorphism at P if there exist open neighborhoods U' of P in U and V' of $f(P)$ in V such that $f|_{U'}\colon U' \to V'$ is a diffeomorphism.



Now we can formulate the following theorem which can be viewed as a partial inverse to the Smooth Invariance of Domain Proposition 19.11 (1).

Theorem 19.12. (Inverse Function Theorem) Let $U \subset \mathbb{R}^n$ be an open subset and let $f\colon U \to \mathbb{R}^n$ be a smooth map. Furthermore let $P \in U$ be a point. If $\mathrm{D}f_P$ is invertible, then f is a local diffeomorphism at P.



Proof. This theorem is proved in most courses on multivariable real analysis. A proof is for example given in [**Lee2002**, Corollary C.34]. ■

The Inverse Function Theorem 19.12 is one of the main results in real analysis. For example the proof of the following proposition rests heavily on the Inverse Function Theorem 19.12.

Theorem 19.13. (Smooth Invariance of Domain Theorem) Let $U \subset \mathbb{R}^n$ be an open set and let $f\colon U \to \mathbb{R}^n$ be a smooth map such that the differential $\mathrm{D}f_P$ is invertible at every point P in the open set U. Then the following two statements hold:

(1) The map f is open, i.e. given any open subset $V \subset U$ the image $f(V)$ is an open subset of $\mathbb{R}^n$.

(2) If f is injective, then $f\colon U \to f(U)$ is a diffeomorphism.

Remark. In the Topological Invariance of Domain Theorem 82.7 we will see that the statement of the Smooth Invariance of Domain Theorem 19.13 (1) also holds if we only assume that f is continuous and injective.

Proof. Let $U \subset \mathbb{R}^n$ be an open set and let $f\colon U \to \mathbb{R}^n$ be a smooth map such that the differential $\mathrm{D}f_P$ is invertible at every point P in the open set U.

(1) Let $V \subset U$ be an open subset. We need to show that $f(V)$ is an open subset of $\mathbb{R}^n$. Let $Q \in f(V)$. We pick $P \in V$ with $f(P) = Q$. Since $\mathrm{D}f_P$ is invertible it follows from the Inverse Function Theorem 19.12, applied to the restriction $f\colon V \to \mathbb{R}^n$, that there exists an open neighborhood V' of P and an open neighborhood W' of $Q = f(P)$ such that $f\colon V' \to W'$ is a diffeomorphism. But this implies that $W' \subset f(V)$. It follows from this discussion, together with the elementary the JH-Lemma 1.7, that $f(V)$ is indeed an open subset of $\mathbb{R}^n$.

(2) We now suppose that $f\colon U \to \mathbb{R}^n$ is injective. By (1) we know that $f(U)$ is an open subset of $\mathbb{R}^n$. We consider the inverse $f^{-1}\colon f(U) \to U$. We need to show that $f^{-1}\colon f(U) \to U$ is smooth. Since f is a local diffeomorphism we see that f^{-1} is also a local diffeomorphism. This implies that f^{-1} is locally smooth which implies by the Smooth Pasting Proposition 19.3 that f^{-1} itself is smooth. ■

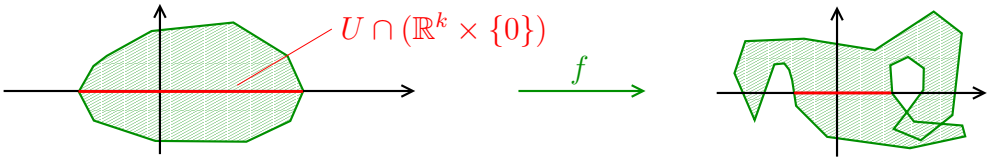
19.4. **Special topics in the theory of smooth functions.** In this short section we collect a ragtag of results on smooth functions, each of which will have its moment of glory later on. Nonetheless, in a first reading of this chapter, this section can be skipped.

The first proposition says that locally, up to translation, one can describe any smooth map as matrix multiplication, where the coefficients of the matrix are allowed to vary.

Proposition 19.14. (Map-as-Matrix Multiplication Proposition) Let $U \subset \mathbb{R}^n$ be some convex subset that contains the origin. Furthermore let $f\colon U \to \mathbb{R}^m$ be a smooth map with $f(0) = 0$.

(1) There exists a smooth map $A\colon U \to \mathrm{M}(m \times n, \mathbb{R})$ such that $A(0) = \mathrm{D}f_0$ and such that
$$f(x) \;=\; A(x) \cdot x \qquad \text{for all } x \in U.$$

(2) If there exist $k \in \{0, \ldots, n\}$ and $l \in \{0, \ldots, m\}$ with $f((\mathbb{R}^k \times \{0\}) \cap U) \subset \mathbb{R}^l \times \{0\}$, then we can arrange that for each $x \in (\mathbb{R}^k \times \{0\}) \cap U$ the corresponding matrix $A(x)$ is of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, where the top left corner is an $l \times k$-matrix.



Proof. First we take care of Statement (1). By considering the coordinate functions of a given map $f = (f_1, \ldots, f_m)\colon U \to \mathbb{R}^m$ separately we see that it suffices to deal with the case $m = 1$. Thus let $f\colon U \to \mathbb{R}$ be a smooth map with $f(0) = 0$. Note that in this case we need to show that there exist smooth maps $a_1, \ldots, a_n\colon U \to \mathbb{R}$ such that $a_i(0) = \frac{\partial f}{\partial x_i}(0)$ and with

$$f(x) \;=\; \sum_{i=1}^{n} a_i(x_i) \cdot x_i \qquad \text{for all } x = (x_1, \ldots, x_n) \in U.$$

We observe that given any $x = (x_1, \dots, x_n) \in U$ we have

$$f(x) = f(x) - \underbrace{f(0)}_{=0} = \int_{t=0}^{t=1} \frac{d}{dt} f(\underbrace{t \cdot x}_{\in U})\, dt = \int_{t=0}^{t=1} \sum_{i=1}^{n} \frac{\partial f(t\cdot x)}{\partial x_i} \cdot x_i \, dt = \sum_{i=1}^{n} \underbrace{\int_{t=0}^{t=1} \frac{\partial f(tx)}{\partial x_i}\, dt}_{=:a_i(x)} \cdot x_i.$$

Fundamental Theorem of Calculus — Chain Rule 19.10

If U is open, then it follows from standard methods in analysis, e.g. the Leibniz Integral Rule [**PedS2015**, Theorem 9.2.11], that each a_i is smooth. If U is not open, then this fact can be shown with more effort by hand. Alternatively one can use the Smooth Extension Proposition 21.9 to reduce the problem to the open case.

Finally note that it follows from the above equality and the Product Rule that we have the equality $a_i(0) = \frac{\partial f}{\partial x_i}(0)$. This shows that we have indeed $A(0) = \mathrm{D}f_0$. This concludes the proof of Statement (1).

Now we turn to the proof of Statement (2). In fact it is straightforward to see that the map $A\colon U \to \mathrm{M}(m \times n, \mathbb{R})$ we constructed above has the desired property. Indeed, as above we only have to deal with the case $m = 1$ and $l = 1$. But if $f((\mathbb{R}^k \times \{0\}) \cap U) = \{0\}$, then it follows immediately from the construction of the matrix that for each $x \in (\mathbb{R}^k \times \{0\}) \cap U$ we obtain the zero matrix. ■

On very few occasions we will need the following proposition.

Proposition 19.15. Let $U \subset \mathbb{R}^n$ be an open subset, let $f\colon U \to \mathbb{R}^n$ be a smooth map and let $P \in U$. If $\mathrm{D}f_P$ is invertible, then there exists an $\epsilon > 0$ such that $f(B^n_\epsilon(P))$ is convex.

Proof. The proposition is proved in [**Pol2001**, Theorem 2.1]. ■

We conclude this section with a harmless lemma which is elementary, but which time and again puzzles the author.

Lemma 19.16. Let $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^l$ be two open non-empty subsets and suppose we are given a map of the form

$$\begin{aligned} \Theta\colon U \times V &\to \mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^{k+l} \\ (x, y) &\mapsto (f(x, y), g(y)). \end{aligned}$$

that is a diffeomorphism onto its image. Then $g\colon V \to \mathbb{R}^l$ is also a diffeomorphism onto its image and the inverse of Θ is of the form

$$\begin{aligned} \Theta^{-1}\colon \Theta(U \times V) &\to U \times V \subset \mathbb{R}^k \times \mathbb{R}^l \\ (x, y) &\mapsto (\varphi(x, y), g^{-1}(y)) \end{aligned}$$

for some smooth map $\varphi\colon \Theta(U \times V) \to \mathbb{R}^k$.

Proof. We denote by $p\colon \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k$ and $q\colon \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^l$ the obvious projections. We set $\varphi = p \circ \Theta^{-1}$ and $\psi = q \circ \Theta^{-1}$. Note that φ and ψ are compositions of smooth maps, hence smooth. Furthermore we have $\Theta^{-1} = (\varphi, \psi)$. Now let $x \in U$. We denote by $j_x\colon V \to U \times V$ the inclusion map that is given by $y \mapsto (x, y)$. Given any $y \in V$ we perform the following calculation:

$$\psi(x, g(y)) = (q \circ \Theta^{-1} \circ j_x \circ g)(y) = (q \circ \Theta^{-1} \circ j_x \circ q \circ \Theta \circ j_x)(y) = (q \circ \Theta^{-1} \circ \Theta \circ j_x)(y) = y.$$

by definition of ψ — definition of g — since $q \circ j_x = \mathrm{id}_{\mathbb{R}^l}$

This shows that $y \mapsto \psi(x,y)$ is an inverse to g. In particular g is a diffeomorphism and $\psi(x,y) = g^{-1}(y)$ for all $y \in V$. ■

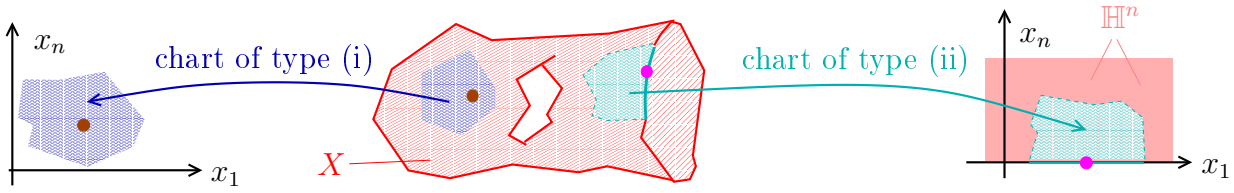
19.5. **Smooth manifolds.** In this section we will finally introduce one of the most important notions of these lecture notes, namely the concept of a "smooth manifold".
For the reader's convenience we first recall the notion of a topological manifold that we introduced on page 511.

Definition. Let X be a topological space.

(1) An *n-dimensional chart for X at a point $x \in X$* is a homeomorphism $\Phi\colon U \to V$ where U is an open neighborhood of x and
 (i) V is an open subset of $\mathbb{R}^n$ or
 (ii) V is an open subset of the upper half-space $\mathbb{H}^n$ and $\Phi(x)$ lies in $\partial\mathbb{H}^n$.
 In the former case we say that Φ is a chart of type (i), in the latter case we say that Φ is a chart of type (ii). We refer to the figure below for an illustration.

(2) We say X is an *n-dimensional topological manifold*, if X is second-countable and Hausdorff, and if for every $x \in X$ there exists an n-dimensional chart $\Phi\colon U \to V$ at x.

(3) An *n-dimensional atlas* for X is a collection of n-dimensional charts such that the domains of the charts cover all of X.



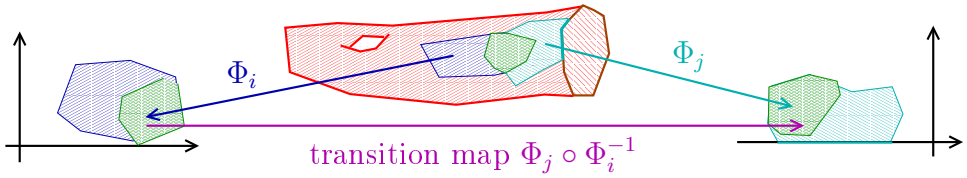
Definition. Let X be a topological manifold.

(1) We say that two n-dimensional charts $\Phi_1\colon U_1 \to V_1$ and $\Phi_2\colon U_2 \to V_2$ are *smoothly compatible* if the *transition map*[210]

$$\Phi_2 \circ \Phi_1^{-1}\colon \underbrace{\Phi_1(U_1 \cap U_2)}_{\text{open set in } \mathbb{R}^n \text{ or } \mathbb{H}^n} \to \underbrace{\Phi_2(U_1 \cap U_2)}_{\subset \mathbb{R}^n}$$

is a diffeomorphism.[211]

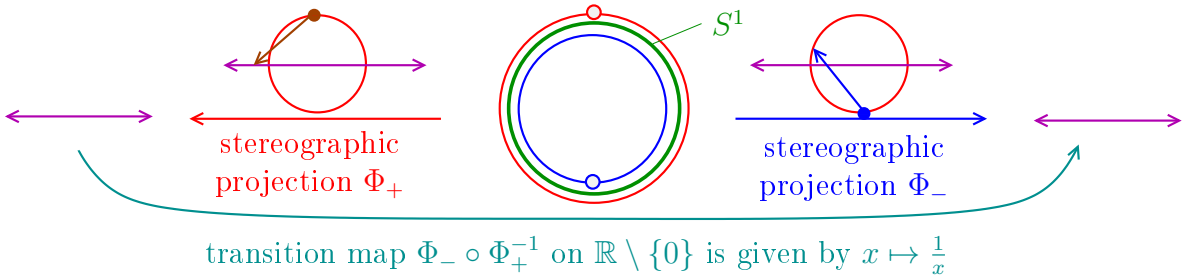
(2) An n-dimensional atlas of an topological manifold for X is called *smooth* if all charts in the atlas are smoothly compatible to one another.



[210] Once again we refer to page 532 for the definition of a smooth map $U \to \mathbb{R}^n$ where U is an open subset of $\mathbb{H}^n$. We also refer to the Whitney Extension Theorem 19.6 for an alternative characterization of smooth maps on $\mathbb{H}^n$.

[211] Note that this condition is indeed symmetric in i and j.

Example. Let $n \in \mathbb{N}$. We denote by $N := (0, \dots, 0, 1)$ the North Pole of S^n and we denote by $S := (0, \dots, 0, -1)$ the South Pole of S^n. Let $\Phi_+ \colon S^n \setminus \{N\} \to \mathbb{R}^n$ be the stereographic projection from the North Pole N and let $\Phi_- \colon S^n \setminus \{S\} \to \mathbb{R}^n$ be the stereographic projection from the South Pole S, as defined in Lemma 2.18. In Exercise 18.15 we saw that the maps $\Phi_- \circ \Phi_+^{-1} \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is given by the diffeomorphism $x \mapsto \frac{1}{\|x\|^2} \cdot x$. This shows that Φ_+ and Φ_- are smoothly compatible. Thus the atlas for S^n given by Φ_+ and Φ_- is a smooth atlas.



Definition. Let M be a topological manifold and let $\mathcal{A}$ be a smooth atlas for M. We say $\mathcal{A}$ is maximal if every chart Ψ that is smoothly compatible with all the charts in the smooth atlas $\mathcal{A}$ is already contained in $\mathcal{A}$.

Proposition 19.17. (Maximal Atlas Existence Proposition) Let M be a topological manifold and let $\mathcal{A}$ be a smooth atlas for M.

(1) The smooth atlas is contained in a unique maximal smooth atlas for M.
(2) The unique maximal smooth atlas from (1) is given by the set of all charts of type (i) and (ii) from page 511 that are smoothly compatible with all charts in the given smooth atlas $\mathcal{A}$.

Proof. Let $\mathcal{A}$ be a smooth atlas for M. We denote by $\mathcal{B}$ the set of all charts of M that are smoothly compatible with respect to all charts in $\mathcal{A}$.

Claim 1. The charts in $\mathcal{B}$ form a smooth atlas.

Proof. Let $\Psi_1 \colon U_1 \to V_1$ and $\Psi_2 \colon U_2 \to V_2$ be two charts in $\mathcal{B}$. We need to show that Ψ_1 and Ψ_2 are smoothly compatible, i.e. we need to show that the transition map

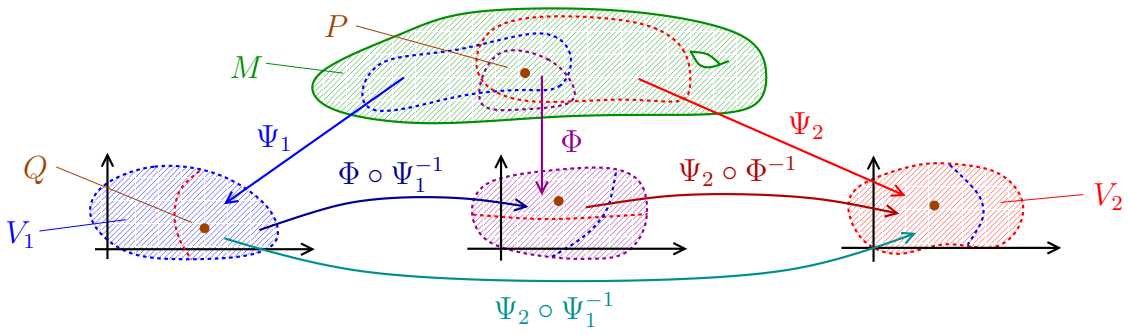
$$\Psi_2 \circ \Psi_1^{-1} \colon \Psi_1(U_1 \cap U_2) \;\to\; \Psi_2(U_1 \cap U_2)$$

is smooth. By the Smooth Pasting Proposition 19.3 it suffices to show that the map is locally smooth. Thus let $Q \in \Psi_1(U_1 \cap U_j)$. We write $P := \Psi_1^{-1}(Q) \in M$. We pick a chart $\Phi \colon A \to B$ from our smooth atlas $\mathcal{A}$ with $P \in A$. Since Ψ_1 and Ψ_2 are smoothly compatible with all charts in $\mathcal{A}$ we know that the maps $\Phi \circ \Psi_1^{-1} \colon \Psi_1(U_1 \cap A) \to \Phi(U_1 \cap A)$ and $\Psi_2 \circ \Phi^{-1} \colon \Phi(U_2 \cap A) \to \Psi_2(U_2 \cap A)$ are smooth. Thus we see that the restriction of $\Psi_2 \circ \Psi_1^{-1}$ to the open neighborhood $\Psi_1(U_1 \cap U_2 \cap A)$ of P is smooth. ⊞

It remains to prove the following claim.

Claim 2. The smooth atlas $\mathcal{B}$ is maximal and it is the unique maximal smooth atlas that contains $\mathcal{A}$.

Proof. If Ψ is a chart that is smoothly compatible with all charts in $\mathcal{B}$, then it is also smoothly compatible with all charts in $\mathcal{A}$, thus Ψ is already contained in $\mathcal{B}$. This shows that $\mathcal{B}$ is maximal.



Now let $\mathcal{C}$ be a maximal smooth atlas that contains $\mathcal{A}$. We need to show that $\mathcal{B} = \mathcal{C}$. First let $\Phi \in \mathcal{C}$. By definition Φ is smoothly compatible with all charts $\mathcal{A}$, thus $\Phi \in \mathcal{B}$. Now let $\Phi \in \mathcal{B}$. The proof of Claim 1 shows that Φ is also smoothly compatible with any chart that is smoothly compatible with all charts in $\mathcal{A}$. Thus we see that $\Phi \in \mathcal{C}$. ■

Definition. Let X be a topological manifold.

(1) An n-dimensional smooth structure on X is a choice of a maximal n-dimensional smooth atlas for X.
(2) Let $\mathcal{A}$ be an n-dimensional smooth atlas for X. By the Maximal Atlas Existence Proposition 19.17 the smooth atlas $\mathcal{A}$ is contained in a unique maximal n-dimensional smooth atlas $\mathcal{M}$. We refer to $\mathcal{M}$ as the smooth structure defined by $\mathcal{A}$.[212]
(3) An n-dimensional smooth manifold is a pair $(X, \mathcal{M})$ consisting of a topological manifold X together with an n-dimensional smooth structure $\mathcal{M}$, i.e. together with a maximal smooth atlas $\mathcal{M}$ for X. (As with many other definitions we usually suppress the choice of a smooth atlas from the notation.) We refer to the topological space X as the underlying topological manifold of the smooth manifold $(X, \mathcal{A})$.

Remark.

(1) In the literature the term "manifold" gets used with wildly different meanings, in particular often it gets to mean "topological manifold" or "smooth manifold". To make matters worse, there are also other perfectly reasonable notions of a "manifold", namely on page 2093 we will introduce "PL-manifolds" and on page 2104 we will get to know "combinatorial manifolds". To prevent any misunderstanding we only rarely write "manifold" without any decorations. We only do so in informal discussions where the precise meaning is irrelevant.
(2) Since the underlying topological space of a smooth manifold is a topological manifold, many results on topological manifolds hold by fiat also for smooth manifolds. For example by the Topological Manifolds-Local Properties Lemma 18.8 we know that every topological manifold, and thus also every smooth manifold, is locally contractible.

[212] In practice we might be a little sloppy every now and then and we might not distinguish in our notation between a smooth atlas and the corresponding maximal smooth atlas, i.e. the corresponding smooth structure.

(3) The question which topological manifolds admit smooth atlases is subtle. Perhaps rather shockingly in 1960 Michel Kervaire [**Kerv1960**] showed that there exists a 10-dimensional topological manifold that does not admit a smooth atlas. We will discuss this question in greater detail in Section **??**.
(4) A map $f\colon U \to \mathbb{R}^m$ from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^m$ is called a *C^k-map*, if all iterated k-fold partial derivatives exist and if they are continuous. Throughout the above discussion one could replace "smooth" by "C^k". This leads to the notion of a C^k-atlas, a C^k-structure and a C^k-manifold. In [**Hirs1976**, Theorem 2.9] it is shown that every C^1-structure gives rise to an essentially unique smooth structure.

The following proposition gives us interesting examples of smooth manifolds.

Proposition 19.18. The following topological manifolds admit a smooth atlas:
(1) open subsets of $\mathbb{R}^n$ and open subsets of $\mathbb{H}^n$,
(2) all closed balls $\overline{B}^n$,
(3) all spheres $S^n \subset \mathbb{R}^{n+1}$,
(4) the n-dimensional torus $(S^1)^n$,
(5) the surfaces of genus $g \in \mathbb{N}_0$, as defined on page 265,
(6) the non-orientable surfaces of genus $g \in \mathbb{N}$, as defined on page 266.

More precisely, the atlases that we provided in Section 18.1 and in the proof of the Surface-is-Topological Manifold Proposition 18.7 are smooth.

Proof. It is straightforward to verify that the atlases for examples (1) to (4) that we provided in Section 18.1 are smooth. For example on page 542, see also Exercise 18.15, we saw that the transition map for the atlas for S^n given by the stereographic projections from the Poles is smooth, which shows that the given atlas for S^n is smooth.

It remains to consider the (non-orientable) surfaces of genus $g \geq 2$. The atlas that we provided in the proof of the Surface-is-Topological Manifold Proposition 18.7 might initially look dubious. In fact all transition maps are compositions of (some of) the following types of maps:

(1) reflections in lines,
(2) translations,
(3) and for some $\eta > 0$ the map

$$\begin{array}{rcl} \{r \cdot \exp(\mathrm{i}\alpha) \,|\, r \in (0,\eta), \alpha \in (0, \frac{\pi}{4})\} & \to & \{r \cdot \exp(\mathrm{i}\alpha) \,|\, r \in (0,\eta), \alpha \in (0, \frac{3\pi}{4})\} \\ z = r \cdot \exp(\mathrm{i}\varphi) & \mapsto & r \cdot \exp(3\mathrm{i}\varphi) = \frac{1}{|z|^2} \cdot z^3, \end{array}$$

(4) the inverse of (3).

It is clear that the first three types of maps are smooth. It follows from the Smooth Invariance of Domain Theorem 19.13 (2) that the fourth map is also smooth. Thus the transition maps in the atlas are smooth. ■

Convention. As we discussed in Proposition 19.18, the atlases that we provided in Section 18.1 and in the proof of the Surface-is-Topological Manifold Proposition 18.7, give smooth atlases for the following topological manifolds:
(1) open subsets of $\mathbb{R}^n$ and open subsets of $\mathbb{H}^n$,
(2) all closed balls $\overline{B}^n$,

(3) all spheres $S^n \subset \mathbb{R}^{n+1}$,
(4) the n-dimensional torus $(S^1)^n$,
(5) the surfaces of genus $g \in \mathbb{N}_0$, as defined on page 265,
(6) the non-orientable surfaces of genus $g \in \mathbb{N}$, as defined on page 266.

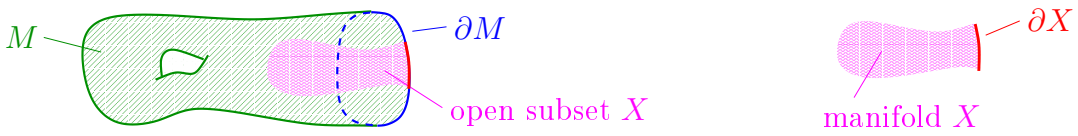
We will invariably work with the corresponding smooth structures, unless we say something different. We refer to these as the standard smooth structures on these topological manifolds.

For completeness we state the following lemma which is closely related to the discussion on page 526.

Lemma 19.19. (Open Subset-Submanifold Lemma) Let M be a smooth manifold and let X be an open subset of M. The restrictions of the charts of the smooth atlas of M to X define a smooth atlas for X.

Remark. In the Open Subset-Submanifold Lemma 19.19 we will prove an important addendum to Lemma 19.20.

Proof. The lemma follows easily from the definitions. ■



Convention. We will constantly use the Open Subset-Submanifold Lemma 19.19 to view open subsets of a smooth manifold as smooth manifolds in their own right.

On page 511 we defined the boundary of a topological manifold. We can now similarly define the notion of the boundary of a smooth manifold.

Definition. Let M be a smooth manifold.

(1) We say that a point $x \in M$ is a boundary point if x does not admit a chart of type (i) in the smooth structure. We denote by ∂M the set of all boundary points of M. We refer to ∂M as the boundary of M.
(2) We say a smooth manifold M is closed if M is compact with $\partial M = \varnothing$.

Remark. As we just mentioned, on page 511 we already defined the boundary of a topological manifolds. In the Topological Manifold Boundary Proposition 75.2 (2) we will see that the boundary of a smooth manifold in the above sense agrees with the boundary of a topological manifold.

The following lemma can be viewed as an addendum to the Open Subset-Submanifold Lemma 19.19.

Lemma 19.20. Let M be a smooth manifold and let X be an open subset of M. We use the Open Subset-Submanifold Lemma 19.19 to view X as a smooth manifold in its own right. The boundary ∂X of X viewed as a smooth manifold is given by $\partial X = X \cap \partial M$.

Proof. The lemma follows easily from the definitions. ■

19.6. **Smooth maps between smooth manifolds.** We give a few more standard definitions regarding smooth manifolds.

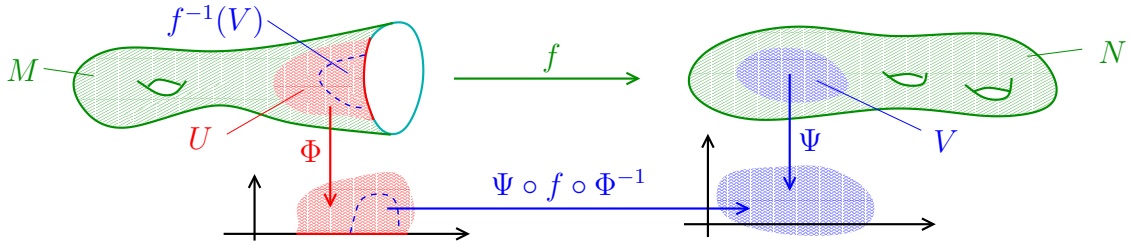
Definition. Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold.

(1) A map $f\colon M \to N$ is called smooth if for every chart $\Phi\colon U \to \widetilde{U}$ of M and every chart $\Psi\colon V \to \widetilde{V}$ of N, in the respective smooth structures, the concatenation

$$\Psi \circ f \circ \Phi^{-1}\colon \underbrace{\Phi(U \cap f^{-1}(V))}_{\text{open subset of } \mathbb{R}^m \text{ or } \mathbb{H}^m} \to \widetilde{V} \subset \mathbb{R}^n$$

is smooth, in the sense of our definition on page 532.

(2) We denote by $C^\infty(M,N)$ the set of all smooth maps from M to N.



Checking the condition in the definition of a smooth manifold for all charts in the smooth structure is evidently very very painful. Fortunately the following lemma simplifies the life of a working mathematician greatly.

Lemma 19.21. (Smooth-via-Atlas Lemma) Let $f\colon M \to N$ be some map between an m-dimensional smooth manifold M and an n-dimensional smooth manifold N. Furthermore let $\{\Phi_i\colon U_i \to \widetilde{U}_i\}_{i\in I}$ be charts in the smooth structure with $M = \bigcup_{i\in I} U_i$ and let $\{\Psi_j\colon V_j \to \widetilde{V}_j\}_{J\in J}$ be charts in the smooth structure $N = \bigcup_{j\in J} V_j$. If for all $i \in I$ and $j \in J$ the map

$$\Psi_j \circ f \circ \Phi_i^{-1}\colon \Phi(U_i \cap f^{-1}(V_j)) \to \widetilde{V}_j \subset \mathbb{R}^n$$

is smooth, then f is smooth.

Proof. The proof of the lemma is somewhat similar to the proof of Claim 1 in the proof of the Maximal Atlas Existence Proposition 19.17. Namely, the lemma follows easily from the Smooth Pasting Proposition 19.3 together with the fact, see Proposition 19.10, that the composition of smooth maps between open subsets of some $\mathbb{R}^n$'s is again smooth. We leave it to the reader to fill in the details. ■

Example.

(1) We consider maps $f\colon S^n \to \mathbb{R}^m$. As on page 542 we equip S^n with the smooth atlas that is given by the stereographic projections and we equip $\mathbb{R}^m$ with the smooth atlas given by the identity chart. An entertaining calculation shows that the inverses of

the stereographic projections $\Phi_\pm$ are given by

$$\begin{array}{rcl} \Phi_\pm^{-1}\colon \mathbb{R}^n & \to & \mathbb{R}^n \setminus \{(0,\dots,0,1)\} \\ (x_1,\dots,x_n) & \mapsto & \left(\frac{2x_1}{x_1^2+\cdots+x_n^2+1}, \dots, \frac{2x_1}{x_1^2+\cdots+x_n^2+1}, \pm 1 \mp \frac{2}{x_1^2+\cdots+x_n^2+1} \right) \end{array}$$

the Smooth-via-Atlas Lemma 19.21 implies that $f\colon S^n \to \mathbb{R}^m$ is smooth precisely if $f \circ \Phi_\pm{}^{-1}\colon \mathbb{R}^n \to \mathbb{R}^m$ are smooth. For example it follows immediately that the inclusion map $S^n \to \mathbb{R}^{n+1}$ is smooth.

(2) Let U be an open subset of $\mathbb{R}^n$ and let $f\colon U \to \mathbb{R}^m$ be a map. It follows easily from the definitions that the map f is smooth as a map between smooth manifolds if and only if it is smooth in the "classical" sense of page 531.

(3) Let $n \in \mathbb{Z}$. In Exercise 19.5 we will see that the following map is smooth:

$$\begin{array}{rcl} S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}$$

We continue with a very straightforward lemma which gets used implicitly all the time.

Lemma 19.22.

(1) Every smooth map between two smooth manifolds is continuous.

(2) The composition of two smooth maps between smooth manifolds is again smooth.

Proof.

(1) Let $f\colon M \to N$ be a smooth map between two smooth manifolds M and N. Let $\{\Phi_i\colon U_i \to \widetilde{U}_i\}_{i\in I}$ be the charts in the smooth structure of M. Furthermore let $\{\Psi_j\colon V_j \to \widetilde{V}_j\}_{J\in J}$ be the charts in the smooth atlas of N. By hypothesis all the maps

$$\Psi_j \circ f \circ \Phi_i^{-1}\colon \Phi(U_i \cap f^{-1}(V_j)) \to \widetilde{V}_j \subset \mathbb{R}^n$$

are smooth. In particular they are continuous. Since Φ_i and Ψ_j are by definition homeomorphisms we see that the restriction of f to each $U_i \cap f^{-1}(V_j)$ is continuous. Since these form an open cover of M we obtain from the Pasting Proposition 2.6 that f is indeed continuous.

(2) By the Chain Rule 19.10 this statement holds for smooth maps between open subsets of some $\mathbb{R}^n$'s. The statement for smooth maps between smooth manifolds follows fairly easily from this observation, together with the Smooth-via-Atlas Lemma 19.21 and the Smooth Pasting Proposition 19.3. We leave it to the reader to fill in the details. ∎

Definition.

(1) We say a map $f\colon M \to N$ between smooth manifolds is a diffeomorphism if f is smooth, if f is a bijection and if $f^{-1}\colon N \to M$ is also smooth. If such a diffeomorphism exists, then we say that M and N are diffeomorphic.

(2) Let $f\colon M \to N$ be a map between two smooth manifolds and let $P \in M$. We say f is a local diffeomorphism at P if there exist open neighborhoods U of P in M and V of $f(P)$ in N such that[213] $f|_U\colon U \to V$ is a diffeomorphism.

[213] By the Open Subset-Submanifold Lemma 19.19 we can and will view U and V as smooth manifolds in their own right.

Example.

(1) Let $A \in \mathrm{O}(n)$. It is true, but slightly nasty to verify directly from the definitions, that the map

$$\begin{array}{rcl} S^{n-1} & \to & S^{n-1} \\ v & \mapsto & A \cdot v \end{array}$$

is a diffeomorphism. We will give a satisfactory argument on page 554.

(2) Let $n \in \mathbb{N}$. In Exercise 19.6 we will see that the map

$$\begin{array}{rcl} \mathbb{R}^n & \to & (S^1)^n = S^1 \times \cdots \times S^1 \\ (t_1, \ldots, t_n) & \mapsto & (\exp(2\pi \mathrm{i} t_1), \ldots, \exp(2\pi \mathrm{i} t_n)) \end{array}$$

is a local diffeomorphism.

Remark. The Inverse Function Theorem 19.12 gives a tremendously convenient way for showing that a smooth map between open subsets of $\mathbb{R}^n$ is a local diffeomorphism. Later on, in the Inverse Mapping Theorem 24.1 we will state and prove a generalization of the Inverse Function Theorem 19.12 to the more general context of smooth maps between smooth manifolds.

The following harmless proposition contains two basic statements about smooth maps between smooth manifolds. The first statement can be viewed as a smooth analogue of the Pasting Proposition 2.6

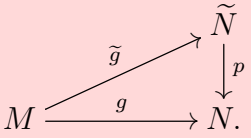
Proposition 19.23. (Smooth Pasting Proposition II) Let $f\colon M \to N$ be a map between two smooth manifolds.

(1) (a) If there exists an open cover $\{U_i\}_{i \in I}$ of M such that for every $i \in I$ the restriction $f|_{U_i}\colon U_i \to N$ is smooth, then f itself is smooth.

(b) If there exists a smooth atlas $\{\Phi_i\colon U_i \to \widetilde{U}_i\}_{i \in I}$ for M and a smooth atlas $\{\Psi_i\colon V_j \to \widetilde{V}_j\}_{j \in J}$ for N such that $\bigcup_{i,j} (U_i \cap f^{-1}(V_j)) = M$ and such that each $\Psi_j \circ f \circ \Phi_i^{-1}\colon \Phi_i(U_i \cap f^{-1}(V_j)) \to V_j$ is smooth, then f is smooth.

(2) If $f\colon M \to N$ is a bijection (e.g. if it is a homeomorphism) and if it is a local diffeomorphism, then it is in fact a diffeomorphism.

(3) Let $\widetilde{N}$ be a smooth manifold and let $p\colon \widetilde{N} \to N$ be a local diffeomorphism. Furthermore let $\widetilde{f}\colon M \to \widetilde{N}$ be a map such that the following diagram commutes:



Then $f\colon M \to N$ is smooth if and only if $\widetilde{f}\colon M \to \widetilde{N}$ is smooth.

Remark.

(1) In contrast to the Pasting Proposition 2.6 it does not suffice to check smoothness on finitely many *closed* subsets. For example, consider the map $\mathbb{R} \to \mathbb{R}$ that is given by $x \to |x|$. The restrictions to the closed subsets $(-\infty, 0]$ and $[0, \infty)$ are smooth, but the map is evidently not smooth on all of $\mathbb{R}$.

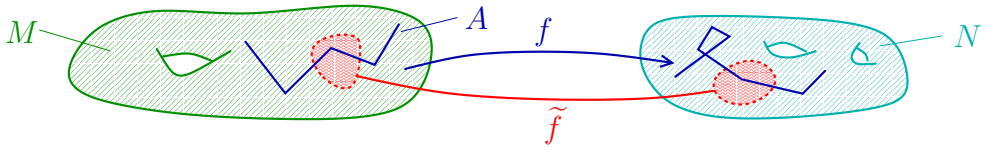
(2) In practice one can often control a map $f\colon M \to N$ on compact codimension-zero submanifolds instead of open subsets. As we just pointed out, one cannot conclude that f is smooth. But in Proposition **??** we will see that one can modify f slightly to turn f into a smooth map.

Proof.

(1) Both statements follow immediately from the Smooth Pasting Proposition 19.3.
(2) Let $f\colon M \to N$ be a local diffeomorphism that is a bijection. We need to show that $f^{-1}\colon N \to M$ is also smooth. But that follows immediately from (1) and our hypothesis that f is a local diffeomorphism.
(3) This statement follows immediately from (1a). ■

For completeness' sake we provide the following definition which is a just a generalization of the definition on page 532.

Definition. Let M and N be smooth manifolds. Furthermore let $A \subset M$ be some subset and let $f\colon A \to N$ be a map. We say that f is smooth if given any $a \in A$ there exists an open neighborhood U in M of a and a smooth map $\widetilde{f}\colon U \to N$[214] and which coincides with f on A..



Definition. Let M be a topological manifold. We say that two smooth structures $\mathcal{A}$ and $\mathcal{B}$ are equivalent if there exists a diffeomorphism between $(M, \mathcal{A})$ and $(M, \mathcal{B})$.

Remark.

(1) There exist topological manifolds that admits inequivalent smooth structures. For example, as we will see in Corollary **??**, John Milnor [**Miln1956a**] showed in 1956 that S^7 admits at least two inequivalent smooth structures.
(2) In Theorem **??** we will see that in dimensions $n \neq 4$ every closed topological manifold admits at most a finite number of inequivalent smooth structures. Amazingly, this statement is wrong in dimension 4. More precisely there is a plethora of closed 4-dimensional smooth manifolds that admit an infinite number of pairwise inequivalent smooth structures, see e.g. Theorem 135.13 for details.

19.7. **Submanifolds of smooth manifolds.** In Section 18.4 we introduced the notion of a submanifold of a topological manifold. The definition of a submanifold of a smooth manifold is basically the same, the only difference is that now we demand that the submanifold charts are smooth. For the reader's convenience we write down the detailed definition, even though there is a serious overlap with our previous definition.

[214] Let $U \subset M$ be an open subset and let $\widetilde{f}\colon U \to N$ be a map. We use the convention on page 545 to view U as a smooth manifold in its own right, and thus we have defined what it means for $\widetilde{f}$ to be smooth.

Definition. Let M be an n-dimensional smooth manifold.

(1) We say a subset $N \subset M$ is a k-dimensional submanifold if given any $P \in N$ one of the following holds:

(α) there exists a chart $\Phi\colon U \to V$ of type (i) in the smooth structure for M and P such that

$$\Phi(U \cap N) \;=\; V \cap \{(0,\dots,0,x_1,\dots,x_k) \,|\, x_i \in \mathbb{R}\},$$

(β) or there exists a chart $\Phi\colon U \to V$ of type (ii) in the smooth structure for M and P such that $\Phi(P)$ lies in $\partial\mathbb{H}^n$ and[215]

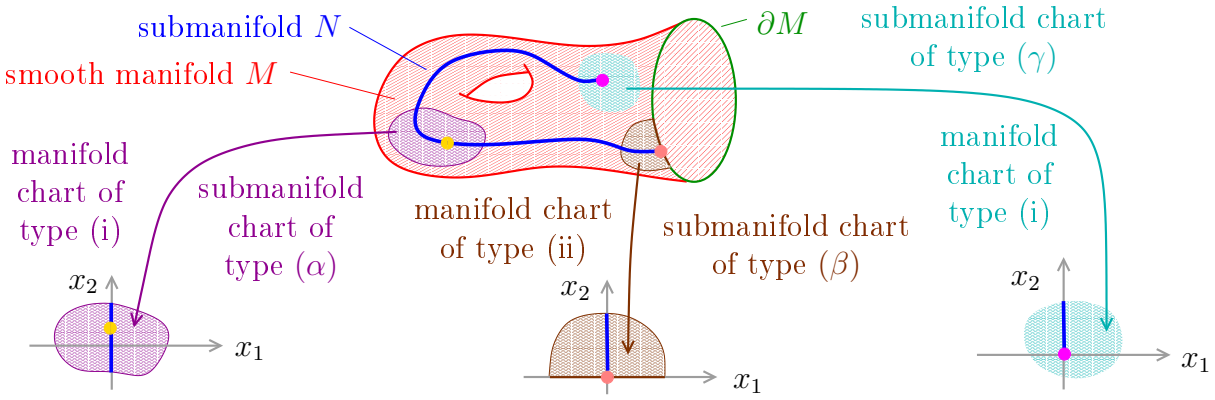
$$\Phi(U \cap N) \;=\; V \cap \{(0,\dots,0,x_1,\dots,x_k) \in \mathbb{R}^n \,|\, x_k \geq 0\},$$

(γ) or there exists a chart $\Phi\colon U \to V$ of type (i) in the smooth structure for M and P such that $\Phi(P)$ lies in $\partial\mathbb{H}^n$ and[216]

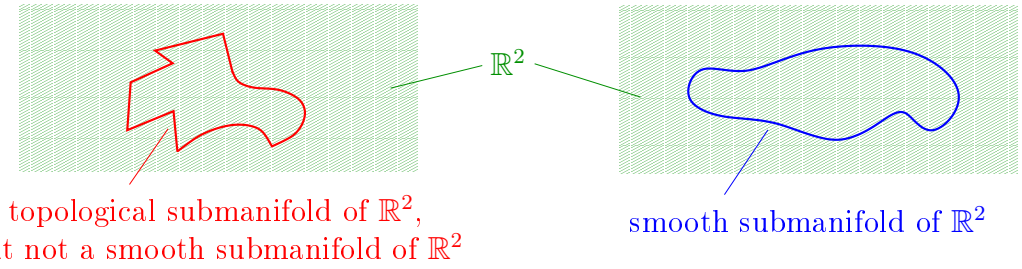
$$\Phi(U \cap N) \;=\; V \cap \{(0,\dots,0,x_1,\dots,x_k) \in \mathbb{R}^n \,|\, x_k \geq 0\}.$$

We refer to such charts as submanifold charts of N. Any collection of submanifold charts of N that covers N is called a submanifold atlas.

(2) We say a submanifold N is proper if N is a closed subset of M and if for each $P \in N$ we can find submanifold charts of type (α) and (β).



Remark. Sometimes the above language can be ambiguous. For example, if we consider $M = \mathbb{R}^2$ it is not clear whether we view M as a topological manifold or as a smooth manifold. If from the context it is not clear what we mean, then we distinguish in our language between a *topological submanifold* and a *smooth submanifold*. This distinction is illustrated in the figure below.

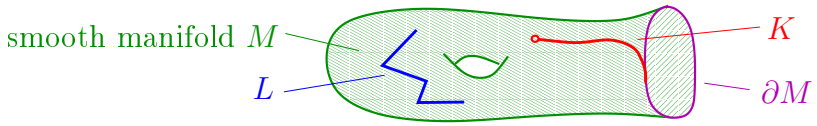


[216] For $k = 0$ we interpret $\{(0,\dots,0,x_1,\dots,x_k) \in \mathbb{R}^n \,|\, x_k \geq 0\}$ as the empty set.

Examples.

(1) We consider the topological manifold $M = \mathbb{R}^2$ with the smooth atlas $\mathcal{A}$ given by the identity $\Phi = \mathrm{id}_M$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function. We consider the subset $N = \{(x, f(x) \,|\, x \in \mathbb{R}\} \subset \mathbb{R}^2$. We cannot use the unique chart Φ in our atlas $\mathcal{A} = \{\Phi\}$ to show that N is indeed a submanifold of the smooth manifold $(\mathbb{R}^2, \mathcal{A})$. But the chart $\Psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $\Psi(x, y) = (x, y - f(x))$ lies in the smooth structure, i.e. in the maximal smooth atlas, defined by $(\mathbb{R}^2, \mathcal{A})$ and we can use it to verify that N is indeed a submanifold of the smooth manifold $(\mathbb{R}^2, \mathcal{A})$.

(2) In the figure below we show a smooth manifold M with boundary together with two subsets K and L. In Exercise 19.22 we will make the setting more precise and we will see that both K and L are submanifolds of the topological manifold M but that they are not submanifolds if we view M as a smooth manifold.



Remark. As we remarked on page 526, there are various subtly different definitions of a "submanifold" out in the literature. This is even more pronounced when it comes to submanifolds of smooth manifolds, especially if M and N are allowed to have boundary.

(1) Surely there is no textbook that would consider L in the above figure as a submanifold of the smooth manifold M. As we remarked above, if we follow our convention or the convention of [**Wall2016**, p. 31], then K is not a submanifold of M. On the other hand the definition used in [**Lee2002**, p. 98] or [**Hirs1976**, p. 30] views K as a smooth submanifold of the smooth manifold M.

(2) Also note that what we call a proper submanifold is called a neat submanifold in [**Hirs1976**, p. 30].

The following lemma is an analogue of the Submanifold-is-Manifold Lemma 18.14 .

Lemma 19.24. (Submanifold Smooth Structure Lemma) Let M be a smooth manifold and let N be a smooth submanifold of M. The collection of all submanifold charts of M gives naturally rise to a smooth atlas for N. The corresponding smooth structure has the following properties:

(1) The inclusion map $N \to M$ is a smooth map.
(2) If we start out with some submanifold atlas for M, then this gives rise to the same smooth structure on N.
(3) Let $f\colon W \to M$ be a smooth map. If $f(W) \subset N$, then $f\colon W \to N$ is also smooth.
(4) We assume that N is a submanifold of codimension-zero. Let $f\colon N \to W$ be a map to a smooth manifold. The following two statements are equivalent:
   (a) The map $f\colon N \to W$ is smooth as a map between smooth manifolds.
   (b) The map $f\colon N \to W$ is smooth in the sense of the definition on page 549, where we think of N as a subset of the smooth manifold M.

Convention. In the setting of the Submanifold Smooth Structure Lemma 19.24 we will refer to the smooth structure on the submanifold N, which we construct in the proof of the Submanifold Smooth Structure Lemma 19.24, as the standard smooth structure on N.[217]

Proof. We start out with the following claim.

Claim. Let U and V be open subsets of $\mathbb{R}^n$ and let $f\colon U \to V$ be a smooth map.

(a) If for every $y \in \mathbb{R}^k$ we have $f(0,y) \in \{0\} \times \mathbb{R}^k$, then the map

$$\underbrace{\{y \in \mathbb{R}^k \,|\, (0,y) \in U\}}_{\text{open subset of } \mathbb{R}^k} \xrightarrow{f} V \cap (\{0\} \times \mathbb{R}^k) \xrightarrow{(0,y)\mapsto y} \underbrace{\{y \in \mathbb{R}^k \,|\, (0,y) \in V\}}_{\text{open subset of } \mathbb{R}^k}$$

is smooth.

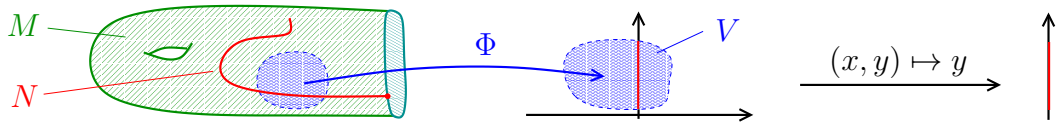
(b) The analogue of (a) holds if we replace the $\mathbb{R}^k$ to the left by $\mathbb{H}^k$.

Proof. The claim follows from the observation that the map to the right is the restriction of the projection $\mathbb{R}^{n-k} \times \mathbb{R}^k \to \mathbb{R}^k$, which evidently is smooth, to a suitable subset. ⊞

Let M be an n-dimensional smooth manifold and let $N \subset M$ be a k-dimensional submanifold. It follows from Lemma 1.14 and the Second Countability Lemma 9.1 that N is Hausdorff and second-countable. Now let $\Phi\colon U \to V$ be a submanifold chart of type (α). As we pointed out in the proof of the Submanifold-is-Manifold Lemma 18.14 it is pretty straightforward to verify that the map

$$U \cap N \xrightarrow{\Phi} V \cap \{(0,x) \in \mathbb{R}^{n-k} \times \mathbb{R}^k\} \xrightarrow{(0,y)\mapsto y} \{x \in \mathbb{R}^k \,|\, (0,x) \in V\}$$

is a chart of type (i) for N. In the same fashion submanifold charts of type (β) and (γ) give rise to charts of type (ii) for N.



(1) It follows easily from the claim that the above charts define a smooth atlas on N.

(2) Again using the claim one can easily show that any submanifold atlas for N gives rise to the same smooth structure on N.

(3) Let $f\colon W \to M$ be a smooth map between smooth manifolds with $f(W) \subset N$. It follows easily from the fact that the projection $\mathbb{R}^n \xrightarrow{(x,y)\,\mapsto\, x} \mathbb{R}^k$ is smooth that the map $f\colon W \to N$ is also smooth.

(4) This statement follows easily, but not immediately, from the definitions. We leave it to the reader to fill in the details. ■

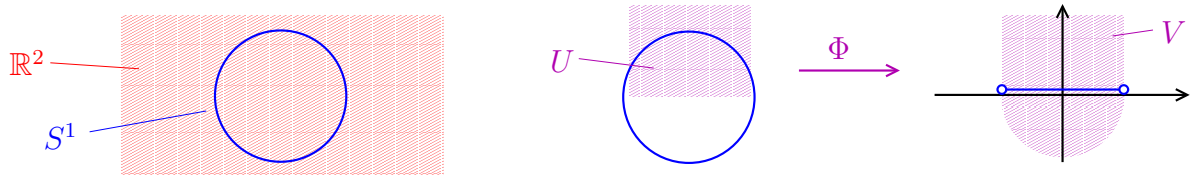
Examples.

[217] The reader might be worried that we equip certain topological manifolds with different "standard smooth structures". But, as we will see on page 553, for S^n this new standard smooth structure equals the standard smooth structure introduced earlier on.

(1) Let $n \in \mathbb{N}$. We consider the n-dimensional sphere $S^n \subset \mathbb{R}^{n+1}$. Similar to the discussion on page 513 we consider the map

$$\begin{array}{rcl} \Phi\colon U := B^n \times (0,\infty) & \to & V = \{(x,y) \in B^n \times \mathbb{R} \,|\, y > -\|x\|^2\} \\ (x,y) & \mapsto & (x, y - \|x\|^2). \end{array}$$

One can easily verify that this is a submanifold chart of type (α) from the smooth structure of $\mathbb{R}^{n+1}$. By changing the roles of the coordinates and changing signs we see that we can cover S^n by altogether $2(n+1)$ such submanifold charts which shows that S^n is an n-dimensional submanifold of the smooth manifold $\mathbb{R}^{n+1}$. In particular we obtain from the Submanifold Smooth Structure Lemma 19.24 a smooth atlas for S^n. We leave it to the reader to verify, using the Submanifold Smooth Structure Lemma 19.24 (2), that this smooth atlas is smoothly compatible to the smooth atlas that we gave on page 513. In particular both define the same smooth structure.



(2) Let M be a smooth manifold and let $U \subset M$ be an open subset. As on page 526 one sees easily that U is a codimension-zero submanifold of M. Furthermore, if we use the Submanifold Smooth Structure Lemma 19.24 to view U as a smooth manifold in its own right, then we obtain, basically by definition, the same smooth structure as we obtained in the Open Subset-Submanifold Lemma 19.19. Finally note that it follows again basically immediately from the definitions that $\partial U = U \cap \partial M$.

Remark. If one is mathematically very honest, then one realizes that it is actually a pain to work directly with the definition of the standard smooth structure on a submanifold. For example, just using the definition, it is not that easy to show that a given map $f\colon N \to W$ is smooth. Life will improve once we have proved the Smooth Embedding Theorem 24.10.

It is worth stressing the following special case of the Submanifold Smooth Structure Lemma 19.24.

Lemma 19.25. (Restrict-to-Submanifold-or-Boundary Lemma)

(1) Let M be a smooth submanifold of some $\mathbb{R}^n$ and let $f\colon M \to N$ be a map to some smooth manifold N. If there exists an open neighborhood U of $M \subset \mathbb{R}^n$ and a map $g\colon U \to N$ that is smooth in the sense of page 531 and which satisfies $g|_M = f$, then f itself is smooth.

(2) Let M be a smooth manifold and let N be a smooth submanifold of some $\mathbb{R}^n$. Let $f\colon M \to N$ be a map such that the composition $f\colon M \to N \to \mathbb{R}^n$ is smooth. Then f itself is smooth.

Proof.

(1) We denote by $i\colon M \to U$ the inclusion map. It follows from the Submanifold Smooth Structure Lemma 19.24 (1) and (2) that i is smooth. By the discussion on page 547 we know that $f\colon U \to N$ is smooth as a map between smooth manifolds. Finally it follows from Lemma 19.22 that $f \circ i\colon M \to N$ is also smooth.

(2) This statement is a special case of the Submanifold Smooth Structure Lemma 19.24 (3). ■

Example. Let $A \in \mathrm{O}(n)$. We consider again the map

$$\begin{array}{rcl} f\colon S^{n-1} & \to & S^{n-1} \\ v & \mapsto & A \cdot v. \end{array}$$

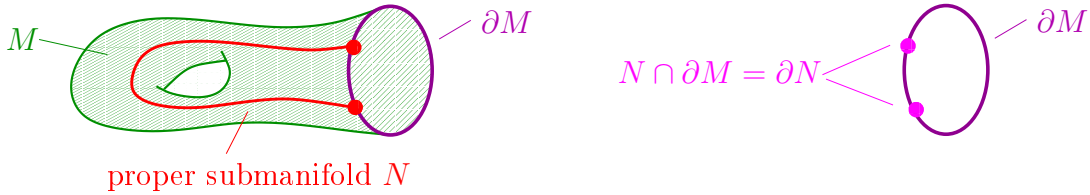
We denote by $g\colon \mathbb{R}^n \to \mathbb{R}^n$ the map that is also given by multiplication by A. By the discussion on page 536 we know that this defines a smooth map on $\mathbb{R}^n$. It follows from the Restrict-to-Submanifold-or-Boundary Lemma 19.25 (1) that the restriction $S^{n-1} \to \mathbb{R}^n$ is smooth. But this map takes values in S^{n-1}, so it follows from the Restrict-to-Submanifold-or-Boundary Lemma 19.25 (2) that the original map $f\colon S^{n-1} \to S^{n-1}$ is actually smooth. The same argument shows of course that f^{-1}, which is given by multiplication with A^{-1} is smooth. Thus we now have a convincing argument that for any $A \in \mathrm{O}(n)$ the map f is a diffeomorphism, solving the slight conundrum from page 548.

19.8. **Topological properties of smooth manifolds.** In this section we will employ several results from Section 19.1 to draw conclusions on smooth manifolds. The fact that real analysis provides us with such powerful tools, like the Inverse Function Theorem 19.12 and the resulting Smooth Invariance of Domain Theorem 19.13, is one of the reasons why dealing with smooth manifolds is much easier than dealing with topological manifolds.
We start out with the following, slightly subtle proposition.

Proposition 19.26. (Smooth Manifold Boundary Proposition) Let M be an n-dimensional smooth manifold. The following statements hold:

(1) Every point on M admits either a chart of type (i) in the smooth structure or it admits a chart of type (ii) in the smooth structure.
(2) The restrictions of all of the charts of the smooth atlas to the boundary ∂M give rise to a natural smooth atlas on ∂M and turn ∂M into a smooth $(n-1)$-dimensional smooth manifold with $\partial(\partial M) = \varnothing$. The corresponding natural smooth structure has the following properties:[218]
   (a) The inclusion map $\partial M \to M$ is a smooth map.
   (b) Restrictions of any set of charts of M that cover ∂M define the same smooth structure on ∂M.
   (c) Let $f\colon W \to M$ be a smooth map between smooth manifolds. If $f(W) \subset \partial M$, then $f\colon W \to \partial M$ is also smooth.
(3) (a) Any union of components of ∂M is a closed subset of M. In particular ∂M itself is a closed subset of M.
   (b) If M is compact, then ∂M is also compact.
(4) $M \setminus \partial M$ is an open subset of M, furthermore, if ∂M is non-empty, then $M \setminus \partial M$ is non-compact.
(5) Let N be a submanifold of M. The following two statements are equivalent:
   (a) N is a proper submanifold.
   (b) The following two conditions are satisfied:
      (i) N is a closed subset of M.
      (ii) We have the equality $\partial N = N \cap \partial M$.

(6) If N is a proper submanifold of M, then[219] $\partial N = N \cap \partial M$ is a proper submanifold of ∂M.



Convention. Let M be an n-dimensional smooth manifold. We refer to the smooth structure on ∂M that we obtain in the proof of the Smooth Manifold Boundary Proposition 19.26, by restricting charts from M, as the standard smooth structure on ∂M.

Remark.

(1) Let M be a smooth manifold. By the Smooth Manifold Boundary Proposition 19.26 (1) a point $x \in M$ is a boundary point if and only if x admits a chart of type (ii).
(2) Here again we would expect that a similar statement as in the Smooth Manifold Boundary Proposition 19.26 should hold for topological manifolds. But since for topological manifolds we have fewer tools, e.g. right now we are missing an analogue of the Smooth Invariance of Domain Theorem 19.13, we have to postpone a proof of the analogue of the Smooth Manifold Boundary Proposition 19.26 (1) for topological manifolds to the Topological Manifold Boundary Proposition 75.2.
(3) In practice the description of the smooth structure on ∂M can be a nuisance to work with. An alternative point of view is given in Exercise 24.12.

Example. Let $n \in \mathbb{N}$. On page 513 we constructed an explicit smooth atlas for $\overline{B}^n$ which gives charts of type (ii) precisely for the points in S^{n-1}. Thus it follows from the Smooth Manifold Boundary Proposition 19.26 that the boundary of the smooth manifold $\overline{B}^n$ is indeed given by S^{n-1}. We leave it to the reader to verify, the Smooth Manifold Boundary Proposition 19.26 (2b), that the smooth structure on $\partial\overline{B}^n = S^{n-1}$ agrees with the standard smooth structure on S^{n-1} that we introduced on page 544.

Proof. Let M be an n-dimensional topological manifold that is equipped with a smooth atlas $\{\Phi_i \colon U_i \to V_i\}_{i \in I}$.

(1) Suppose there exists a point P that admits a chart Φ of type (i) and a chart Ψ of type (ii). By intersecting the two domains, which are both open neighborhoods of P, we can assume that the charts have a common domain U. We set $x_0 = \Phi(P)$ and $y_0 = \Psi(P)$.

We set $\tau := \Psi \circ \Phi^{-1}$. Since Φ and Ψ belong to a smooth atlas we see that τ and τ^{-1} are smooth. Note that by definition on page 532 of a smooth map on a (not necessarily open subset) of $\mathbb{R}^n$ there exists an open neighborhood Y of $y_0 \in \mathbb{R}^n$ and a smooth map $\eta \colon Y \to \mathbb{R}^n$ such that η agrees with τ^{-1} on $Y \cap \Psi(U)$.

[218] This statement is similar in nature to the statement of the Submanifold Smooth Structure Lemma 19.24.
[219] Here ∂N denotes the boundary of N, where we use the Submanifold Smooth Structure Lemma 19.24 to view N as a smooth manifold in its own right.

Claim. There exists an open subset $X \subset \mathbb{R}^n$ with $x_0 \in X \subset \tau^{-1}(Y) \cap \Phi(U)$.

Proof. Note that $\tau\colon \Phi(U) \to \Psi(U)$ is in particular continuous and that, by definition of the subspace topology, $Y \cap \Psi(U)$ is an open neighborhood of $y_0 \in \Psi(U)$. Thus we see that $\tau^{-1}(Y)$ is an open neighborhood of x_0. Thus $X := \tau^{-1}(Y) \cap \Phi(U)$ has the desired property. ⊞

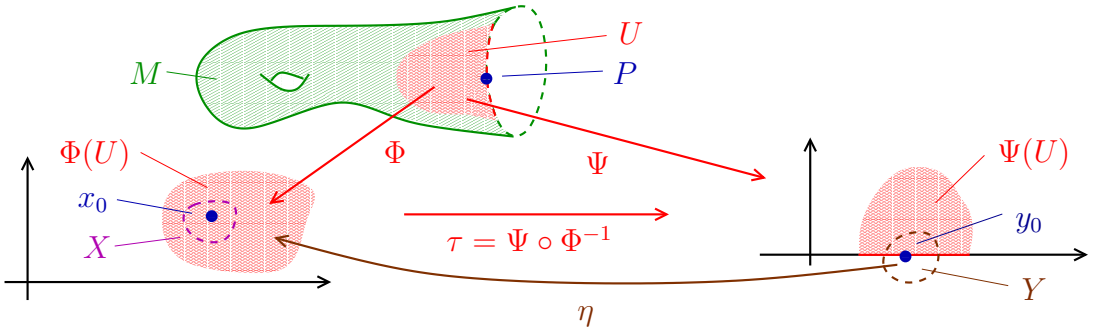
We make the following observations:

(a) Let $x \in X$. Note that $\eta \circ \tau = \mathrm{id}_X$. Since X is an open subset of $\mathbb{R}^n$ we can take differentials at x of both sides. Since X is open and since Y is open we can apply the Chain Rule 19.10 to the left-hand side and we see that $\mathrm{D}\tau_x$ is invertible.

(b) It follows from (a) and the Smooth Invariance of Domain Theorem 19.13 that $\tau(X)$ is an open subset of $\mathbb{R}^n$.

(c) Since $X \subset \Phi(U)$ we have $\tau(X) \subset \tau(\Phi(U)) = \Psi(U) \subset \mathbb{H}^n$. Furthermore we know that $y_0 \in \Phi(U)$ and that $y_0 \in \partial\,\mathbb{H}^n$. In summary, we have that $\tau(X) \subset \mathbb{H}^n$ and that $\tau(X) \cap \partial\,\mathbb{H}^n \neq \varnothing$.

(d) Finally note that (b) and (c) are contradictory.



The remainder of the proof consists mostly of cleverly applying (1). In the unlikely case that the reader does not feel like reading more technical proofs it is a good moment to move on.

(2) First note that it follows from Lemma 1.14 and from the Second Countability Lemma 9.1 (2) that ∂M is Hausdorff and second-countable.

Next we want to show that the smooth atlas on M gives rise to a natural smooth atlas on ∂M. We start out with the following claim.

Claim. Let U and V be open subsets of $\mathbb{H}^n$ and let $f\colon U \to V$ be a smooth map. If for every $y \in \mathbb{R}^{n-1}$ we have $f(y,0) \in \mathbb{R}^{n-1} \times \{0\}$, then the map

$$\underbrace{\{x \in \mathbb{R}^k \,|\, (x,0) \in U\}}_{\text{open subset of } \mathbb{R}^{n-1}} \xrightarrow{f} V \cap (\mathbb{R}^{n-1} \times \{0\}) \xrightarrow{(x,0)\mapsto x} \underbrace{\{x \in \mathbb{R}^{n-1} \,|\, (x,0) \in V\}}_{\text{open subset of } \mathbb{R}^{n-1}}$$

is smooth.

Proof. This statement follows easily from the definition of a smooth map on open subsets of $\mathbb{H}^n$ and the fact that the composition of smooth maps is again smooth. ⊞

We continue with the following observation. Let $\Phi\colon U \to V$ be a chart of type (ii) for M. Note that it is a chart of type (i) precisely for the points in $\Phi^{-1}(\partial\,\mathbb{H}^n)$. It follows from (1) and the definition of ∂M that $\Phi^{-1}(\partial\,\mathbb{H}^n) = \partial M \cap U$.

Given $i \in I$ we now set $\widetilde{V}_i = V_i \cap \partial\,\mathbb{H}^n$, we set $\widetilde{U}_i := \Phi_i^{-1}(\widetilde{V}_i)$ and we set $\widetilde{\Phi}_i := \Phi_i|_{\widetilde{U}_i}$. Furthermore we denote by $p\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ the map given by the projection onto the first $n-1$ coordinates. It follows fairly easily from the above discussion and the claim that $\{p \circ \widetilde{\Phi}_i\colon \widetilde{U}_i \to p(\widetilde{V}_i)\}_{i\in I}$ is a smooth atlas for ∂M and that with respect to this atlas every point in ∂M admits a chart of type (i). In other words, $\partial(\partial M) = \varnothing$.

(a) It follows easily from the fact that the inclusion $\mathbb{R}^{n-1} \xrightarrow{x \mapsto (x,0)} \mathbb{R}^n$ is smooth that the inclusion $\partial M \to M$ is a smooth map.

(b) It follows easily from the claim that if we start out with some set of charts covering ∂M, that these define the same smooth structure on ∂M.

(c) Let $f\colon W \to M$ be a smooth map between smooth manifolds with $f(W) \subset \partial M$. It follows easily from the fact that the projection $\mathbb{H}^n \xrightarrow{(x,y) \mapsto x} \mathbb{R}^{n-1}$ is smooth that the map $f\colon W \to \partial M$ is also smooth.

(3) The proof of the third statement is basically identical to the proof of the Manifold Boundary-Closed Subset Lemma 18.6.[220]

(4) The proof of this statement is basically identical to the proof of the corresponding statement for topological manifolds, see the Manifold Boundary-Closed Subset Lemma 18.6 (3).

(5) Let $N \subset M$ be a k-dimensional submanifold. Recall that N is called *proper* if N is a closed subset of M and if every point in N admits a submanifold chart of type (α) or (β), as defined on page 524. As in the proof of the Submanifold-is-Manifold Lemma 18.14 we denote by $q\colon \mathbb{R}^n \to \mathbb{R}^k$ the projection onto the last k coordinates and we make the following observations:

- If P admits a submanifold chart of type (α), then this submanifold chart is a chart of type (i) for $P \in M$ and the map $q \circ \Phi|_{U\cap N}$ is a k-dimensional chart of type (i) for $P \in N$.
- If P admits a submanifold chart of type (β), then this submanifold chart is a chart of type (ii) for $P \in M$ and the map $q \circ \Phi|_{U\cap N}$ is a k-dimensional chart of type (ii) for $P \in N$.
- If P admits a submanifold chart of type (γ), then this submanifold chart is a chart of type (i) for $P \in M$ and the map $q \circ \Phi|_{U\cap N}$ is a k-dimensional chart of type (ii) for $P \in N$.

The equivalence of (a) and (b) now follows from these three observations together with (1).

(6) Let N be a proper submanifold of M. We want to show that ∂N is a proper submanifold of ∂M.

- As we saw in (5), a point $P \in \partial N$ admits a submanifold chart of type (β). But such a submanifold chart restricts to a submanifold chart of type (α) for ∂N viewed as a subset of ∂M.
- In (5) we just saw that $\partial N = M \cap \partial M$. Since N is a closed subset of M we obtain from the definition of the subspace topology that ∂N is a closed subset of ∂M.

[220] Note that this statement does not follow directly from the Manifold Boundary-Closed Subset Lemma 18.6 since our definition of the boundary of a smooth manifold, given on page 545, is a priori different from the definition of the boundary of a topological manifold. Only later in the Topological Manifold Boundary Proposition 75.2 (2) will we see that there is no difference.

These two observations put together show that ∂N is indeed a proper submanifold of ∂M. ■

The following lemma is the obvious analogue of Lemma 18.5.

Lemma 19.27. Let M and N be two smooth manifolds.

(1) If $f\colon M \to N$ is a local diffeomorphism, then $f(\partial M) = (\partial N) \cap f(M)$.
(2) If $f\colon M \to N$ is in fact a diffeomorphism, then f restricts to a diffeomorphism $\partial M \to \partial N$, where we equip ∂M and ∂N with the above standard smooth structure.

Proof. As in the case of the sister Lemma 18.5, the statement follows easily from the definitions and is left to the reader. ■

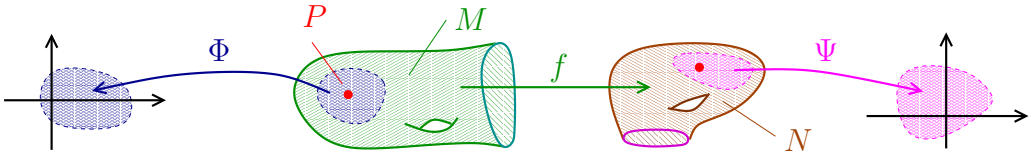
The following proposition says that the dimension of a non-empty smooth manifold is well-defined. Eventually, using homology groups, we will show in the Topological Manifold-Dimension Proposition 75.4 that the analogous statements also holds for topological manifolds.

Proposition 19.28. (Smooth Manifold-Dimension Proposition) Let M be a non-empty m-dimensional smooth manifold and let N be an n-dimensional smooth manifold. If M and N are diffeomorphic, then $m = n$.

Proof. Let M be a non-empty m-dimensional smooth manifold, let N be an n-dimensional smooth manifold and let $f\colon M \to N$ be a diffeomorphism.

It follows from the hypothesis that M is non-empty, together with the Smooth Manifold Boundary Proposition 19.26 (4), that $M \setminus \partial M$ is non-empty. This implies that the smooth manifold M admits a chart $\Phi\colon U \to \widetilde{U}$ of type (i), where $\widetilde{U}$ is an open subset of $\mathbb{R}^m$.

Since f is a diffeomorphism we see that $\Phi \circ f^{-1}\colon f(U) \to V$ is a chart of type (i) for N. In particular we see that $f(P) \in N$ admits a chart of type (i). Now let $\Psi\colon V \to \widetilde{V}$ be a chart of type (i) from the given smooth atlas of N for $f(P)$. Note that $\widetilde{V}$ is an open subset of $\mathbb{R}^n$.



By the Smooth Manifold Boundary Proposition 19.26 we know that Ψ also has to be a chart of type (i). Furthermore note that $U \cap f^{-1}(V)$ contains P, thus it is non-empty. In summary we see that

$$\Psi \circ f \circ \Phi^{-1}\colon \Phi(U \cap f^{-1}(V)) \;\to\; \Psi(f(U) \cap V)$$

is a diffeomorphism between an open non-empty subset of $\mathbb{R}^m$ and an open non-empty subset of $\mathbb{R}^n$. It follows from the Smooth Invariance of Domain Proposition 19.11 that $m = n$. ■

19.9. **The Smooth Path-Connectivity Proposition.** We continue with the following slightly proposition, which is somewhat technical, but which will have surprisingly many uses.

Proposition 19.29. (Smooth Path-Connectivity Proposition) Let M be a smooth manifold.

(1) Every map $\gamma\colon [a,b] \to M$ is homotopic rel the endpoints $\{a,b\}$ to a smooth map.
(2) If M is path-connected, then given any $P, Q \in M$ there exists a smooth path from P to Q.

The proof of the Smooth Path-Connectivity Proposition 19.29 requires the remainder of this section. The reader might be well advised to move on to the next section. For the reader who sticks around, we first formulate and prove the following useful lemma, which will be a key ingredient in the proof of the Smooth Path-Connectivity Proposition 19.29.

Lemma 19.30. Let M be a smooth manifold. Let $\alpha\colon [a,b] \to M$ and $\beta\colon [b,c] \to M$ be two smooth paths with $\alpha(b) = \beta(b)$. There exists a smooth map $\gamma\colon [a,c] \to M$ which is homotopic to the map $[a,c] \to M$ that is defined by α and β.

Proof of Lemma 19.30. To simplify the notation we assume that $a = 0$, $b = 1$ and $c = 2$. By the Smooth Transition Function Lemma 19.4 there exists a smooth map $\tau\colon [0,1] \to [0,1]$ such that $\tau(t) = 0$ for $t \in [0, \frac{1}{3}]$ and such that $\tau(t) = 1$ for $t \in [\frac{2}{3}, 1]$. We consider the map

$$\begin{array}{rcl} \gamma\colon [0,2] & \to & M \\ t & \mapsto & \begin{cases} \alpha(\tau(t)), & \text{if } t \in [0,1], \\ \beta(\tau(t-1)), & \text{if } t \in [1,2]. \end{cases} \end{array}$$

Note that γ is constant on the interval $[\frac{2}{3}, \frac{4}{3}]$. Thus it follows easily from the Smooth Pasting Proposition 19.23 that γ is smooth. Finally the desired homotopy is given by the map

$$\begin{array}{rcl} [0,2] \times [0,1] & \to & M \\ (t,s) & \mapsto & \begin{cases} \alpha((1-s)\cdot t + s \cdot \tau(t)), & \text{if } t \in [0,1], \\ \beta((1-s)\cdot (t-1) + s \cdot \tau(t-1)), & \text{if } t \in [1,2]. \end{cases} \end{array}$$

■

Proof of the Smooth Path-Connectivity Proposition 19.29. Note that Statement (2) is an immediate consequence of Statement (1). Thus it suffices to prove Statement (1).

Let M be an n-dimensional smooth manifold. Let $\gamma\colon [a,b] \to M$ be a map. We need to show that γ is homotopic rel $\{a,b\}$ to a smooth map $\widetilde{\gamma}\colon [a,b] \to M$.

Since $\gamma([a,b])$ is compact it follows easily from Lemmas 1.19 and 18.3 that we can find points $a = t_0 < t_1 < \cdots < t_k = b$ and charts $\{\Phi_i\colon U_i \to V_i\}_{0=1,\ldots,k-1}$ with the following properties:

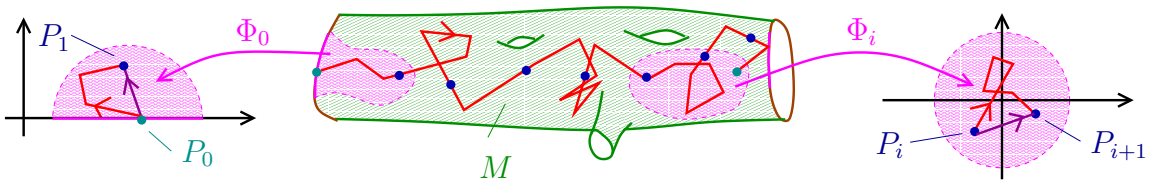
(a) For each $i \in \{0,\ldots,k-1\}$ we have $\gamma([t_i, t_{i+1}]) \subset U_i$.
(b) Each V_i is a convex open subset either of $\mathbb{R}^n$ or of $\mathbb{H}^n$.

For $i = 0,\ldots,k-1$ we set $P_i = \Phi_i(\gamma(t_i))$ and for $i = 0,\ldots,k-1$ we consider the homotopy

$$\begin{array}{rcl} F_i\colon [t_i, t_{i+1}] \times [0,1] & \to & \mathbb{R}^n \\ (\lambda, s) & \mapsto & (1-s)\cdot \Phi_i(\gamma(\lambda)) + s \cdot \underbrace{\left(\frac{t_{i+1}-\lambda}{t_{i+1}-t_i} \cdot P_i + \frac{\lambda - t_i}{t_{i+1}-t_i} \cdot P_{i+1} \right)}_{\text{“straight path” from } P_i \text{ to } P_{i+1}}. \end{array}$$

Note that F_i is a homotopy rel $\{t_i, t_{i+1}\}$ from $\Phi_i \circ \gamma|_{[t_i,t_{i+1}]}$ to the “straight path” from P_i to P_{i+1}. Since V_i is a convex subset we see that this homotopy takes values in V_i. We now see that $\Phi_i^{-1} \circ F_i\colon [t_i, t_{i+1}] \times [0,1] \to M$ is a homotopy from $\gamma|_{[t_i,t_{i+1}]}$ rel $\{t_i, t_{i+1}\}$ to a smooth path $[t_i, t_{i+1}] \to M$.

Finally it follows from these results, together with Lemma 19.30, that there exists in fact a homotopy rel $\{a, b\}$ from $\gamma\colon [a, b] \to M$ to a smooth map $[a, b] \to M$. ■



19.10. **Products of smooth manifolds I.** In the Topological Manifold Product Proposition 18.4 we saw that the product of any two topological manifolds is again a topological manifold. The proof of the Topological Manifold Product Proposition 18.4 was mostly straightforward, except that if both topological manifolds have non-empty boundary, then we had to use a little ruse to find a chart. In the following proposition, where we will deal with smooth manifolds, we will sidestep this tricky business.

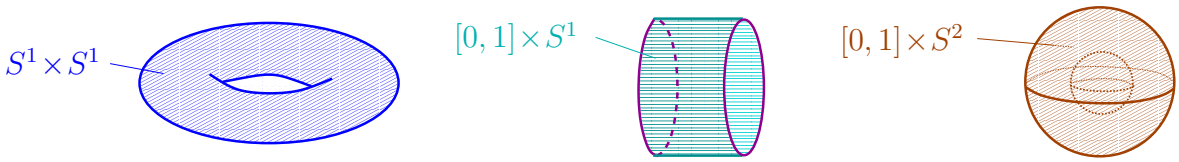
Proposition 19.31. (Smooth Manifold Product Proposition) Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold. We assume that $\partial M = \varnothing$ or $\partial N = \varnothing$. We denote by

$$\begin{array}{rcl} p\colon M \times N & \to & M \\ (x, y) & \mapsto & x \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon M \times N & \to & N \\ (x, y) & \mapsto & y \end{array}$$

the two projections. The following statements hold:

(1) The product $M \times N$ is an $(m + n)$-dimensional topological manifold that admits a *unique* smooth manifold structure that satisfies the following two conditions:
   (a) The projections $p\colon M \times N \to M$ and $q\colon M \times N \to N$ are smooth.
   (b) For any smooth manifold W and any map $f\colon W \to M \times N$ the following holds: If the two maps $p \circ f\colon W \to M$ and $q \circ f\colon W \to N$ are smooth, then f itself is smooth.

(2) (a) If $\partial M = \varnothing$, then $\partial(M \times N) = M \times (\partial N)$.
   (b) If $\partial N = \varnothing$, then $\partial(M \times N) = (\partial M) \times N$.[221]



Remark. In Propositions 20.6 and **??** we will deal with the product of two smooth manifolds where both smooth manifolds are allowed to have non-empty boundary.

Proof. We first consider the case that $\partial M = \varnothing$. In the Topological Manifold Product Proposition 18.4 we already saw that $M \times N$ is an $(m+n)$-dimensional topological manifold.

Now we turn to the construction of a smooth atlas. Let $\{\Phi_i\colon U_i \to \widetilde{U}_i\}_{i \in I}$ be the smooth atlas of M and let $\{\Psi_j\colon V_j \to \widetilde{V}_j\}_{j \in J}$ be the smooth atlas of N. One can easily verify that

[221] Note that here, when we are dealing with smooth manifolds, we can determine the boundary. This is in contrast to the slightly unfortunate situation in the Topological Manifold Product Proposition 18.4.

the maps

$$\begin{array}{rcl} \Phi_i \times \Psi_j \colon U_i \times V_j & \to & \widetilde{U}_i \times \widetilde{V}_j \\ (x, y) & \mapsto & (\Phi_i(x), \Psi_j(y)) \end{array}$$

form a *smooth* atlas for $M \times N$. We equip $M \times N$ with this smooth atlas and with the corresponding smooth structure. We make the following observations:

- It follows easily from the definitions that this smooth atlas has the properties (1a) and (1b).
- Since $\partial M = \varnothing$ it is also clear that $\Phi_i \times \Psi_j$ is of type (ii) if and only if Ψ_j is of type (ii). It follows from this observation, together with the Smooth Manifold Boundary Proposition 19.26 (1), that $\partial(M \times N) = M \times (\partial N)$.

The fact that the smooth structure is uniquely determined by (1a) and (1b) follows from a very general, formal argument. Suppose that $M \times N$ is equipped with two smooth atlases $\mathcal{A}$ and $\mathcal{B}$ that satisfy (1a) and (1b). We need to show that the identity map $\mathrm{id} \colon (M \times N, \mathcal{A}) \to (M \times N, \mathcal{B})$ is smooth. Since $\mathcal{A}$ satisfies (1a) we know that $p \circ \mathrm{id}$ and $q \circ \mathrm{id}$ are smooth. But since $\mathcal{B}$ satisfies (1b) this implies that id is smooth.

Finally we consider the case that $\partial M \neq \varnothing$ and $\partial N = \varnothing$. Recall that a chart of type (ii) takes values in the upper half-space which is defined by restricting the *last* coordinate to ≥ 0. Thus we proceed as above, except that we consider the product charts $\Phi_i \times \Psi_j$ to make sure that the product of a chart of type (i) and a chart of type (ii) is in fact a chart of type (ii). ■

Convention. Let M and N be two smooth manifolds, where at most one has non-empty boundary. Unless we say explicitly something else we will always equip $M \times N$ with the smooth structure introduced in the Smooth Manifold Product Proposition 19.31. We will refer to this as the standard smooth structure on $M \times N$.

Example.

(1) We equip each $\mathbb{R}^k$ with the standard smooth structure that we introduced on page 544. In Exercise 19.27 we will see that the map

$$\begin{array}{rcl} \mathbb{R}^{m+n} & \to & \mathbb{R}^m \times \mathbb{R}^n \\ (x_1, \ldots, x_m, y_1, \ldots, y_n) & \mapsto & ((x_1, \ldots, x_m), (y_1, \ldots, y_n)) \end{array}$$

is a diffeomorphism. This observation can be viewed as a variation on the Product Metric Lemma 5.3.

(2) We now see that products of spheres are naturally smooth manifolds.

(3) Let M and N be two smooth manifolds, where at most one has non-empty boundary. Let $Q \in N$ and $P \in M$. It follows from the Smooth Manifold Product Proposition 19.31 (1b) that the two inclusion maps

$$\begin{array}{rcl} i_Q \colon M & \to & M \times N \\ x & \mapsto & (x, Q) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} j_P \colon N & \to & M \times N \\ y & \mapsto & (P, y) \end{array}$$

are smooth.

(4) Let $f\colon V \to W$ be a smooth map between two smooth manifolds with $\partial V = \varnothing$ and with $\partial W = \varnothing$. As on page 526 one sees that the graph $\mathrm{Gr}(f) := \{(x, f(x)) \,|\, x \in V\}$ is a proper submanifold of the smooth manifold $V \times W$.[222]

19.11. **Manifolds and group actions.** In this section we will see that group actions can be used to construct interesting new examples of smooth manifolds.

For the reader's convenience we recall some definitions from pages 248 and 253.

Definition. Let X be a set and let G be a group with trivial element e that acts on X.

(1) An action is called free, if $g \cdot x = x$ for some $x \in X$ implies that $g = e$.
(2) Suppose that X is a topological space.
   (a) We say that the action is continuous, if for every $g \in G$ the map $X \to X$ given by $x \mapsto g \cdot x$ is continuous.
   (b) We say the action is proper if for every two points x and y in X there exist open neighborhoods U of x and V of y such that the set $\{g \in G \,|\, gU \cap V \neq \varnothing\}$ is finite.

We add the following almost self-explanatory definition.

Definition. Let M be a smooth manifold which is equipped with an action by a group G. We say that the action is smooth if for every $g \in G$ the map $M \to M$ given by $x \mapsto g \cdot x$ is smooth.

The following proposition is the main result of this section.

Proposition 19.32. (Manifold Quotient-by-Group Action Proposition) If a group G acts freely, properly and continuously on an n-dimensional topological manifold M, then the following hold:

(1) The quotient space M/G is also an n-dimensional topological manifold and the projection $p\colon M \to M/G$ is a local homeomorphism.
(2) If M is a smooth manifold and if G acts smoothly, then there exists a unique smooth structure on M/G such that the projection $p\colon M \to M/G$ is a local diffeomorphism.

Furthermore, in both cases the following four statements hold:

(a) The G-action restricts to an action on ∂M and we have $\partial(M/G) = (\partial M)/G$.
(b) If M is compact, then M/G is compact.
(c) If M is closed, then M/G is closed.
(d) If $N \subset M$ is a (proper) submanifold such that the G-action restricts to an action on N, then N/G is a (proper) submanifold of M/G.

Convention. In the setting of the Manifold Quotient-by-Group Action Proposition 19.32 (2) we will refer to the smooth structure on M/G as the standard smooth structure on M/G.

Before we provide the proof of the Manifold Quotient-by-Group Action Proposition 19.32 let us first discuss some consequences and examples.

In the Topological-Quotient Proposition 5.15 (1b) we saw that the quotient topology has a universal property. The following proposition shows that the standard smooth structure on M/G also has a universal property:

[222] In the Local Graph Lemma 24.4 we will prove a partial converse to this statement.

Corollary 19.33. (Smooth Map-Descends-to-Quotient Corollary) Let G is a group that acts freely, properly and smoothly on an n-dimensional smooth manifold M. Furthermore let $f\colon M \to N$ be a smooth map such that for every $x \in X$ and every $g \in G$ we have $f(g \cdot x) = x$. The induced map

$$\begin{array}{rcl} \overline{f}\colon M/G & \to & N \\ [x] & \mapsto & f(x) \end{array}$$

is also smooth. If f is in fact a local diffeomorphism, then $\overline{f}$ is also a local diffeomorphism.

Proof. Let $[x] \in M/G$. Since the projection $p\colon M \to M/G$ is a local diffeomorphism there exists an open neighborhood U of x and an open neighborhood V of $[x]$ such that $p\colon U \to V$ is a diffeomorphism. Note that $\overline{f}|_V = f \circ p^{-1}\colon V \to N$ is smooth. It follows from the Smooth Pasting Proposition 19.23 that $\overline{f}$ is smooth. If f is a local diffeomorphism then a very mild modification of the above argument shows that $\overline{f}$ is also a local diffeomorphism. ■

We state our first example as a lemma.

Lemma 19.34. (Torus-as-Quotient Lemma)

(1) The action

$$\begin{array}{rcl} \mathbb{Z}^n \times \mathbb{R}^n & \to & \mathbb{R}^n \\ (z, v) & \mapsto & z + v \end{array}$$

from page 249 is free, proper and smooth.

(2) It follows from (1) and the Manifold Quotient-by-Group Action Proposition 19.32 that $\mathbb{R}^n/\mathbb{Z}^n$ has a standard smooth structure as an n-dimensional smooth manifold. We equip $(S^1)^n = S^1 \times \ldots S^1$ with the standard smooth structure coming from Proposition 19.18 and the Smooth Manifold Product Proposition 19.31. The map

$$\begin{array}{rcl} \mathbb{R}^n/\mathbb{Z}^n & \to & (S^1)^n = S^1 \times \cdots \times S^1 \\ [(t_1, \ldots, t_n)] & \mapsto & (\exp(2\pi \mathrm{i} t_1), \ldots, \exp(2\pi \mathrm{i} t_n)). \end{array}$$

is a diffeomorphism.

Proof. The proof of Statement (1) is elementary. Thus let us turn to the proof of Statement (2). As we mentioned on page 548, the map

$$\begin{array}{rcl} \mathbb{R}^n & \to & (S^1)^n = S^1 \times \cdots \times S^1 \\ (t_1, \ldots, t_n) & \mapsto & (\exp(2\pi \mathrm{i} t_1), \ldots \exp(2\pi \mathrm{i} t_n)) \end{array}$$

is a local diffeomorphism. By Lemma 5.25 the map

$$\begin{array}{rcl} f\colon \mathbb{R}^n/\mathbb{Z}^n & \to & (S^1)^n = S^1 \times \cdots \times S^1 \\ [(t_1, \ldots, t_n)] & \mapsto & (\exp(2\pi \mathrm{i} t_1), \ldots, \exp(2\pi \mathrm{i} t_n)) \end{array}$$

is a homeomorphism. By the Smooth Map-Descends-to-Quotient Corollary 19.33 (2) the map f is also a local diffeomorphism. Thus it follows from the Smooth Pasting Proposition 19.23 that the map f is in fact a diffeomorphism. ■

We continue with two more examples.

Examples.

(1) Using the action from page 251 of $G = \{\pm 1\}$ on $M = S^n$ we obtain that the real projective space $\mathbb{R}\mathrm{P}^n = S^n/\{\pm 1\}$ admits a canonical structure of a closed n-dimensional

smooth manifold.[223] Now let $m \leq n$. We have the embedding

$$\begin{array}{rcl} \iota\colon \mathbb{R}\mathrm{P}^m & \to & \mathbb{R}\mathrm{P}^n \\ [x] & \mapsto & [(x,0)]. \end{array}$$

It follows from the Manifold Quotient-by-Group Action Proposition 19.32 (2d) that $\iota(\mathbb{R}\mathrm{P}^m)$ is in fact a submanifold of the smooth manifold $\mathbb{R}\mathrm{P}^n$. As on page 256 we will at times drop ι from the notation and we will view $\mathbb{R}\mathrm{P}^m$ as a subspace and submanifold of $\mathbb{R}\mathrm{P}^n$.

(2) We consider $M = \mathbb{R} \times [-1,1]$ which we endow with the obvious smooth structure. Using the action of $G = \mathbb{Z}$ on $M = \mathbb{R} \times [-1,1]$ from page 252 we obtain that the Möbius band $(\mathbb{R} \times [-1,1])/\mathbb{Z}$ has a canonical structure of a 2-dimensional smooth manifold. Furthermore, we have

$$\partial(\text{Möbius band}) \underset{\substack{\uparrow \\ \text{by the Manifold Quotient-by-Group} \\ \text{Action Proposition 19.32 (a)}}}{=} (\mathbb{R} \times \{-1,1\})/\mathbb{Z} \underset{\substack{\uparrow \\ \text{see Exercise 5.43}}}{\cong} S^1.$$

Convention. In line with the above convention we refer to the above smooth structures on $\mathbb{R}^m/\mathbb{Z}^m$, on $\mathbb{R}\mathrm{P}^n$ and on the Möbius band as the standard smooth structures on $\mathbb{R}^m/\mathbb{Z}^m$, on $\mathbb{R}\mathrm{P}^n$ and on the Möbius band.

Remark. On page 255 we made the identification

$$\begin{array}{rcl} (\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\}) & \to & \mathbb{R}\mathrm{P}^n := S^n/\{\pm 1\} \\ [x] & \mapsto & \left[\frac{x}{\|x\|}\right]. \end{array}$$

Under this identification we obtain a standard smooth structure on $(\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\})$. Note that it follows from the Submanifold Smooth Structure Lemma 19.24 (3) that the map $\mathbb{R}^{n+1} \setminus \{0\} \to S^n$ given by $x \mapsto \frac{x}{\|x\|}$ is smooth. Thus it follows from the Manifold Quotient-by-Group Action Proposition 19.32 that the map $\mathbb{R}^{n+1} \setminus \{0\} \to (\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\})$ is also smooth.

The proof of the Manifold Quotient-by-Group Action Proposition 19.32 relies on the following two somewhat technical lemmas.

Lemma 19.35. Let X be a topological space and let G be a group that acts freely, properly and continuously on X. If X is Hausdorff, then for every $x \in X$ there exists an open neighborhood U such that $gU \cap U = \varnothing$ for all $g \neq e$.

Proof of Lemma 19.35. Let X be a topological space that is Hausdorff. Furthermore let G be a group that acts freely, properly and continuously on X. Finally let $x \in X$. Since G acts freely we know that for every non-trivial $g \in G$ we have $gx \neq x$.

Since the action is proper and continuous we can apply Lemma 5.28 to $a = b = x$ and we obtain open neighborhoods A and B of x such that $gA \cap B = \varnothing$ for every non-trivial $g \in G$. Thus $U := A \cap B$ has the desired property. ∎

[223] We will give an explicit smooth atlas for $\mathbb{R}\mathrm{P}^n$ in Exercise 19.16.

Lemma 19.36. Let X be a topological space that is Hausdorff. Furthermore let G be a group that acts continuously on X. We denote by $p\colon X \to X/G$ the projection. If $U \subset X$ is an open subset such that the map $p\colon U \to X/G$ is injective, then the map $p\colon U \to p(U)$ is a homeomorphism.

Proof of Lemma 19.36. By the Group-Action Projection Lemma 5.24 the projection $p\colon X \to X/G$ is continuous. Thus the restriction of p to U is also continuous. By our hypothesis the map $p\colon U \to p(U)$ is injective and evidently it is surjective. Thus the map is a bijection. It follows from Lemma 2.10 and the fact that U is open that $U \to X$ is an open map. Furthermore by the Group-Action Projection Lemma 5.24 (2a) the map $p\colon X \to X/G$ is also open. In summary we see that $p\colon U \to p(U)$ is open. It follows from the Open -Injective Map Lemma 2.16 (2) that the map $p\colon U \to p(U)$ is in fact a homeomorphism. ■

Now we are in a position to prove the Manifold Quotient-by-Group Action Proposition 19.32. For clarity we deal with topological manifolds and smooth manifolds separately.

Proof of Proposition 19.32 for topological manifolds. Let M be an n-dimensional topological manifold and let G be a group that acts on M. We suppose that the action is continuous, free and proper. We want to show that M/G is an n-dimensional topological manifold. Thus we need to prove the following claim.

Claim.
(1) M/G is second-countable,
(2) M/G is Hausdorff, and
(3) every point $y \in M/G$ admits an n-dimensional chart.

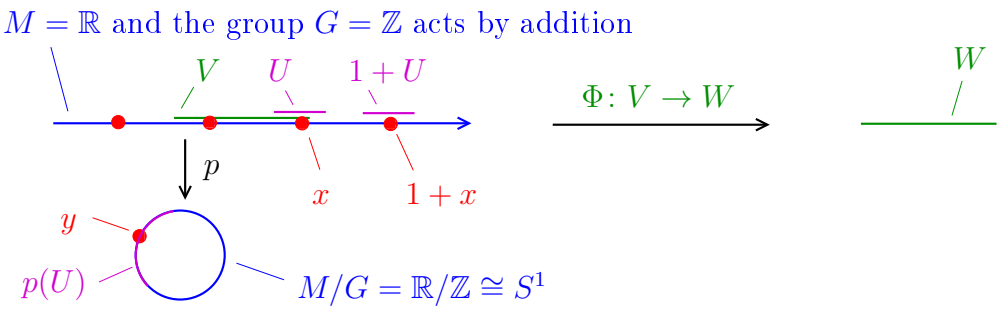
Proof. The first statement of the claim follows immediately from the Group-Action Projection Lemma 5.24 and 9.1 (5). The second statement follows from the Group Action-Hausdorff Quotient Proposition 5.27. Thus it remains to deal with the third statement. As usual we denote by $p\colon M \to M/G$ the projection. Let $y \in M/G$. We pick an $x \in M$ with $p(x) = y$. We make the following two observations:

- Since M is an n-dimensional topological manifold there exists an n-dimensional chart $\Phi\colon V \to W$ with $x \in V$.
- By Lemma 19.35 there exists an open neighborhood U of x such that $gU \cap U \neq \varnothing$ for every $g \neq e$. Note that this implies that the restriction of $p\colon M \to M/G$ to U is injective.

After possibly replacing U and V by $U \cap V$ we can assume that $U = V$. Note that by the Group-Action Projection Lemma 5.24 (2a) the projection $p(U)$ is an open neighborhood of x. Finally we consider the maps

$$p(U) \xrightarrow{p^{-1}} U \xrightarrow{\Phi} W.$$

The map Φ is by definition a homeomorphism. Furthermore it follows from Lemma 19.36 that the map $p\colon U \to p(U) \subset M/G$ is a homeomorphism as well. Thus the above map $\Phi \circ p^{-1}\colon p(U) \to W$ is a homeomorphism, in particular it is an n-dimensional chart for M/G that contains y. ⊞



We have now shown that M/G is an n-dimensional topological manifold. It remains to prove the following claim.

Claim.

($*$) The projection $p\colon M \to M/G$ is a local homeomorphism.
(a) The G-action on M restricts to an action on ∂M and we have $\partial(M/G) = (\partial M)/G$.
(b) If M is compact, then M/G is compact.
(c) If M is closed, then M/G is closed.
(d) If $N \subset M$ is a (proper) submanifold such that the G-action restricts to an action on N, then N/G is a (proper) submanifold of M/G.

Proof.

($*$) Implicitly we had already proved the first statement, indeed, given $x \in M$ we run through the above argument and we see, with the same notation, that $p(U)$ is an open subset of M/G and that $p\colon U \to p(U)$ is a homeomorphism.
(a) It follows from Lemma 18.5 (1) that the G-action on M restricts to an action on ∂M. Furthermore it follows easily from Lemma 18.5 (2) and the above statement ($*$) that $\partial(M/G) = (\partial M)/G$. We leave it to the reader to fill in the details.
(b) The third statement is of course just an immediate consequence of the Compact Image Lemma 2.13.
(c) This statement is an immediate consequence of statements (a) and (b).
(d) We now construct submanifold charts for $N/G \subset M/G$ from submanifold charts for $N \subset M$ the same way that we constructed manifold charts for M/G from manifold charts of M. Note that these submanifold charts for N/G have the same type as the submanifold charts for N.

Finally suppose that N is a proper submanifold. Recall that this means that N is a closed subset of M and that we only require submanifold charts of type (α) and (β). As we just mentioned, the type of the submanifold charts involved does not change. Finally it follows from the hypothesis that the G-action on M restricts to a G-action on N together withe Group-Action Projection Lemma 5.24 (2b) that N/G is a closed subset of M/G. ■

Proof of Proposition 19.32 for smooth manifolds. Let $(M, \mathcal{A})$ be an n-dimensional smooth manifold. Let G be a group that acts freely, properly and smoothly on M. We

need to show that M/G admits a smooth atlas such that the projection $p\colon M \to M/G$ is a local diffeomorphism.

(1) We run through the proof of the Manifold Quotient-by-Group Action Proposition 19.32 (1), but this time only using charts for M that lie in $\mathcal{A}$, and we obtain an atlas $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ for M/G. It is straightforward to see that any transition map $\Phi_j \circ \Phi_i^{-1}\colon \Phi_i(U_i \cap U_j) \to \Phi_j(U_i \cap U_j)$ can be written as the composition of (some) the following maps:
   - charts[224] and their inverses,
   - restrictions to open subsets,
   - the restriction of the action of $g \in G$ on M to some open subset.

   All these maps are smooth maps between smooth manifolds.[225] Thus the transition maps are smooth.

(2) Note that in the above proof of the Manifold Quotient-by-Group Action Proposition 19.32 (1) we already saw that the projection $p\colon M \to M/G$ is a local homeomorphism. It is straightforward to verify that, with our choice of a smooth atlas for M/G, the projection is in fact a local diffeomorphism.

The proof of the remaining statements (a), (b), (c) and (d) is basically identical to the above proof for topological manifolds.

Finally, the statement that the smooth structure on M/G is uniquely determined by the smooth structure on M and the property that the projection $M \to M/G$ is a local diffeomorphism is a consequence of the Smooth Pasting Proposition 19.23 (1) and the fact that the action is proper. We leave it to the reader to fill in the details. ■

We conclude this section with the following amusing lemma. It gives an example of a diffeomorphism that will pop up on several occasions.

Lemma 19.37. (Maps-on-Torus Lemma) Let $n \in \mathbb{N}$. If $A \in \mathrm{GL}(n,\mathbb{Z})$ is a matrix, then the map

$$\begin{array}{rcl} f(A)\colon \mathbb{R}^n/\mathbb{Z}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ [v] & \mapsto & [A\cdot v] \end{array}$$

is a diffeomorphism. (In Exercise 25.8 we will see that this diffeomorphism is "orientation-preserving" if and only if $A \in \mathrm{SL}(n,\mathbb{Z})$.)

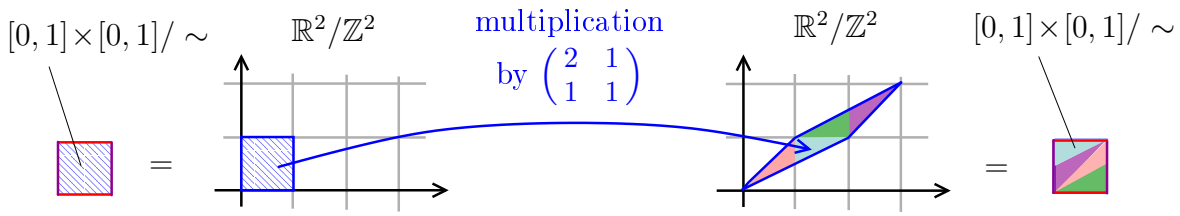
Proof. We will provide the proof in Exercise 19.14. ■

Example. We consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$. The corresponding self-diffeomorphism $f(A)$ of $\mathbb{R}^2/\mathbb{Z}^2 = ([0,1]\times[0,1])/\!\sim$ is illustrated in the figure below.

19.12. **Codimension-Zero submanifolds and decompositions of smooth manifolds.** We start out by introducing the following notation.

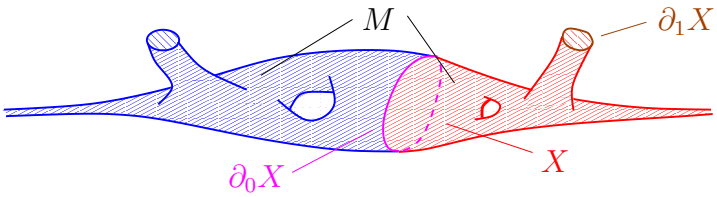
[224] Recall that we view open subsets of $\mathbb{R}^n$ as smooth manifolds. With this convention charts are in fact diffeomorphisms between smooth manifolds.

[225] Here the smooth manifolds are open subsets of $\mathbb{R}^n$ or of M.



Notation. Let M be an n-dimensional smooth manifold and let X be a codimension-zero submanifold. We write[226]

$$\partial_0 X \;:=\; \partial X \setminus \partial M \quad \text{and} \quad \partial_1 X \;:=\; \partial X \cap \partial M.$$



The following proposition is used frequently, often without noticing it.

Proposition 19.38. (Codimension-Zero Submanifold Proposition) Let M be an n-dimensional smooth manifold and let X be a codimension-zero submanifold. The following statement holds:

(1) $\partial_0 X$ and $\partial_1 X$ are unions of components of ∂X.

In the remainder we assume that X is a closed subset of M (e.g. X could be a compact subset). Then the following statements hold:

(2) The boundary of X as a subset of M agrees with $\partial_0 X$.

(3) The interior $\mathring{X}$ of X as a subset of M agrees with $X \setminus \partial_0 X$.

We can furthermore make some statements about the complement $Y := M \setminus \mathring{X}$ of the interior:

(4) $Y = M \setminus \mathring{X}$ is a codimension-zero submanifold with $\partial_0 Y = \partial_0 X$ and $\partial_1 Y = \partial M \setminus \partial_1 X$.

(5) Y is a closed subset of M and if M is compact, then $Y = M \setminus \mathring{X}$ is also compact.

(6) If M is connected and if $\partial_0 X$ is connected, then $Y = M \setminus \mathring{X}$ is also connected.

Remark. In Proposition 75.3 we will prove an analogue of the Codimension-Zero Submanifold Proposition 19.38 for topological manifolds.

Proof. Let X be a codimension-zero submanifold of M and let $P \in X$. By the Submanifold Smooth Structure Lemma 19.24 we can and will view the submanifold X as a smooth manifold in its own right. We start out with the following observation:

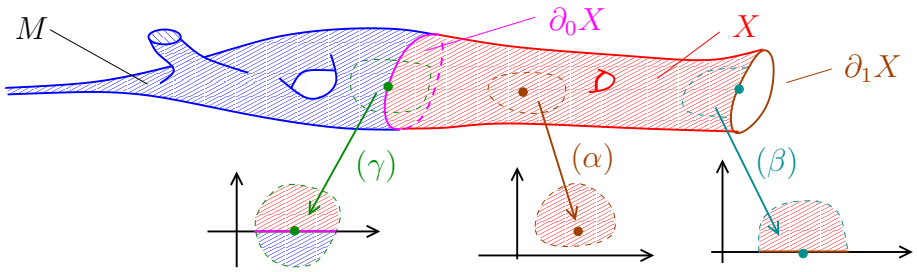
(α) A submanifold chart of type (α) for P is a manifold chart of type (i) for $P \in M$ and a manifold chart of type (i) for $P \in X$.

(β) A submanifold chart of type (β) for P is a manifold chart of type (ii) for $P \in M$ and a manifold of type (ii) for $P \in X$.

[226] Recall that by the Submanifold Smooth Structure Lemma 19.24 we can view the submanifold X as a smooth manifold in its own right. Here ∂X denotes the corresponding boundary of X viewed as a smooth manifold.

(γ) A submanifold chart of type (γ) for P is a manifold chart of type (i) for $P \in M$ and a manifold chart of type (ii) for $P \in X$.

Note that by the Smooth Manifold Boundary Proposition 19.26 (1) the chart type for a point is uniquely determined.



(1) From the above observations we see that the following hold:

$$\begin{aligned} \partial_0 X &= \text{the set of all } P \in X \text{ that admit a submanifold chart of type } (\gamma) \\ \partial_1 X &= \text{the set of all } P \in X \text{ that admit a submanifold chart of type } (\beta) \end{aligned}$$

It follows immediately that $\partial_0 X$ and $\partial_1 X$ are both open subsets of ∂X. Thus it follows from the Component Lemma 2.33 (4) that they are unions of components of ∂X.

Now we assume that X is a closed subset of M.

(2) Let $P \in M$. The desired statement follows from the following observations:

$$\begin{array}{l} \text{this follows from the hypothesis that } X \text{ is a closed subset and Lemma 1.10} \\ \downarrow \\ P \text{ lies in the boundary of } X \text{ as a subset of } M \iff P \text{ lies on } X \text{ and every open neighborhood of } P \text{ contains a point in } M \setminus X \\ \iff P \text{ admits a submanifold chart of type } (\gamma) \iff P \in \partial_0 X. \\ \uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\ \text{since } X \text{ is a submanifold of codimension zero} \qquad\qquad\qquad\qquad \text{see above} \end{array}$$

(3) This statement follows immediately from (2) together with the Interior-Closure Lemma 1.9 (6).

Now we turn to the proofs of the statements about $Y = M \setminus \mathring{X}$.

(4) By the Interior-Closure Lemma 1.9 (7) together with the above Statements (2) and (3) we know that $\mathring{Y} = Y \setminus \partial_0 X$. Now let $P \in Y$.
   (a) If $P \in \mathring{Y}$, then the charts for M, intersected with $\mathring{Y}$, give rise to codimension-zero submanifold charts of type (α) respectively (β) for $P \in Y$.
   (b) If $P \in \partial_0 X$, then by the above we know that there exists a submanifold chart of type (γ) for $P \in X$. But flipping the last coordinate we see that this gives rise to a submanifold chart of type (γ) for $P \in Y = M \setminus \mathring{X}$.

   It follows from this discussion that Y is indeed a codimension-zero submanifold of M. Furthermore, using the discussion in the beginning of the proof we see easily that $\partial_0 Y = \partial_0 X$ and $\partial_1 Y = \partial M \setminus \partial_1 X$.

(5) By the Interior-Closure Lemma 1.9 (1) we know that $\mathring{X}$ is an open subset of M. Thus we see that $Y = M \setminus \mathring{X}$ is a closed subset of M. If M is compact, then we obtain from the Compact-Closed Lemma 1.21 that $Y = M \setminus \mathring{Y}$ is also compact.

(6) This statement follows immediately from Statement (2) together with the Complement Connected Lemma 2.29. ■

the Codimension-Zero Submanifold Proposition 19.38 partially motivates the following definition.

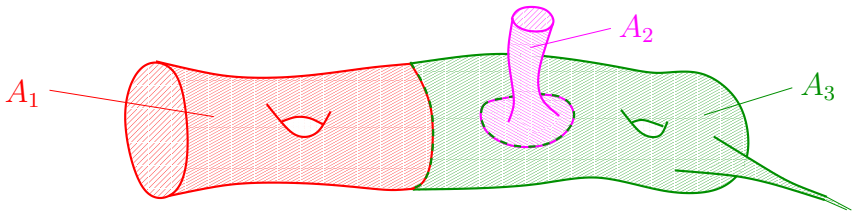
Definition. Let M be an n-dimensional smooth manifold. We say M is decomposed into submanifolds $\{A_i\}_{i \in I}$ if the following conditions hold:

(1) We have $M = \bigcup_{i \in I} A_i$.

(2) Each A_i is a codimension-zero submanifold of M.

(3) Each A_i is a closed subset of M.

(4) For any $i, j \in I$ with $i \neq j$ the intersection $A_i \cap A_j$ is a union of boundary components of A_i and it is a union of boundary components of A_j.



On page 564 we equipped the real projective plane with a standard smooth structure. In Exercise 19.20 we will see that the Klein bottle $([0,1] \times [0,1])/\sim$ comes with a canonical smooth structure. The following cute lemma says that the Klein bottle and the real projective plane admit some interesting decompositions into submanifolds.

Lemma 19.39.

(1) We can decompose the Klein bottle into two submanifolds M and N where M and N are both diffeomorphic to the Möbius band.

(2) We can decompose the real projective plane into two submanifolds D and M where D is diffeomorphic to the disk $\overline{B}^2$ and M is diffeomorphic to the Möbius band.

Proof.

(1) The proof of the first statement is sketched in Figure A. In Figure B we attempt to sketch the equivalent statement that the Klein bottle is obtained from gluing two copies of the Möbius band along the common boundary.

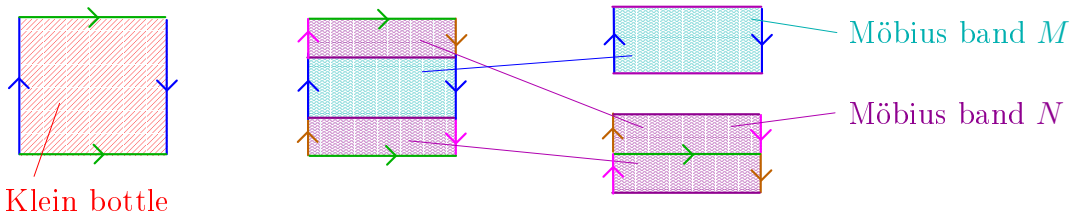


Figure A: Illustration of the proof of Lemma 19.39 (1).

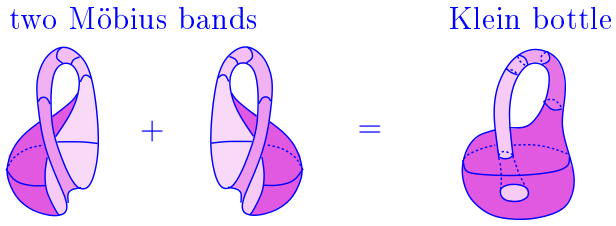


Figure B: Illustration of the proof of Lemma 19.39 (1).

(2) Let $p\colon S^2 \to \mathbb{R}\mathrm{P}^2 = S^2/x \setminus -x$ be the projection. To simplify the notation we think in the following of $\overline{B}^2 \subset \mathbb{C}$ and $S^2 \subset \mathbb{C} \times \mathbb{R}$. We consider the following two subsets of S^2 which are illustrated in the figure below:

$$\widetilde{D} := \{(z,t) \in S^2 \,|\, t \geq \tfrac{1}{2}\} \qquad \text{and} \qquad \widetilde{M} := \{(z,t) \in S^2 \,|\, t \in [0, \tfrac{1}{2}]\}.$$

Furthermore we set $D := p(\widetilde{D})$ and $M := p(\widetilde{M})$. We consider the Möbius band

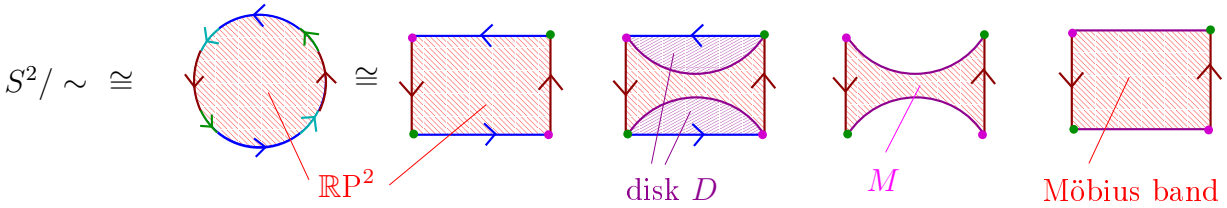


$\mathbb{R} \times [-1,1]/\!\sim$ as defined on page 252. We leave it to the reader to verify that the two maps

$$\begin{aligned} \overline{B}^2 &\to D \\ z &\mapsto p\Big(\tfrac{\sqrt{3}}{2}z, \sqrt{1 - \tfrac{3}{4}\cdot|z|^2}\Big) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathbb{R} \times [-1,1]/\!\sim &\mapsto M \\ [(x,t)] &\mapsto p\Big(\sqrt{1 - \tfrac{1}{4}t^2}\cdot\exp(\pi\,\mathrm{i}x), \tfrac{1}{2}t\Big) \end{aligned}$$

are diffeomorphisms.[227] ■

Remark. If one is not bothered too much about smooth structures, then another decomposition of the real projective plane into a Möbius band and a real disk is sketched in the figure below. Here, for $\mathbb{R}\mathrm{P}^2$ we use the homeomorphism $\mathbb{R}\mathrm{P}^2 \cong \overline{B}^2/\!\sim$ that we gave in Lemma 5.32.

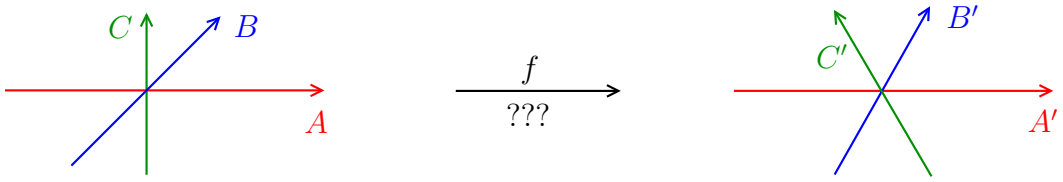


[227] This can clearly be done, but it not particularly pleasant. The work becomes easier if one uses the Smooth Embedding Theorem 24.10.

Exercises for Chapter 19.

Exercise 19.1. Show that there exists a diffeomorphism $[0,1) \to [0,\infty)$ such that $f(x) = x$ for all $x \in [0, \frac{1}{2})$.
Hint. Use the Smooth Transition Function Lemma 19.4 and some creativity.

Exercise 19.2. Let $M := A \sqcup B \sqcup C \subset \mathbb{R}^2$ be the subset shown in the figure below. Does there exist a smooth map $f\colon M \to \mathbb{R}^2$, as illustrated in the figure below? which restricts to diffeomorphisms $A \to A'$, $B \to B'$ and $C \to C'$?



Exercise 19.3. We consider the topological space $X = \mathbb{R}$ and the following three charts:

$$\begin{array}{rcl} \Phi_1\colon \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x \end{array} \qquad \begin{array}{rcl} \Phi_2\colon \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3 \end{array} \qquad \begin{array}{rcl} \Phi_1\colon \mathbb{R} & \to & (-\frac{\pi}{2}, \frac{\pi}{2}) \\ x & \mapsto & \arctan(x). \end{array}$$

(a) Is the atlas given by $\{\Phi_1\}$ a smooth atlas for $X = \mathbb{R}$?
(b) Is the atlas given by $\{\Phi_2\}$ a smooth atlas for $X = \mathbb{R}$?
(c) Is the atlas given by $\{\Phi_1, \Phi_2\}$ a smooth atlas for $X = \mathbb{R}$?
(d) Is the atlas given by $\{\Phi_1, \Phi_3\}$ a smooth atlas for $X = \mathbb{R}$?

Exercise 19.4. Let $x, y \in B^n$. Show that there exists a diffeomorphism $f\colon B^n \to B^n$ with $f(x) = y$.

Exercise 19.5. Let $n \in \mathbb{Z}$. We consider the following map:

$$\begin{array}{rcl} S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}$$

(a) Show, just using the definitions and the Smooth-via-Atlas Lemma 19.21, that f is smooth.
(b) Show that if $n \neq 0$, then f is a local diffeomorphism.

Exercise 19.6.

(a) Let $r \in \mathbb{R}$. Show that the map

$$\begin{array}{rcl} (r - \frac{1}{2}, r + \frac{1}{2}) & \to & S^1 \setminus \{-\exp(2\pi i \cdot r)\} \\ t & \mapsto & \exp(2\pi \mathrm{i} \cdot t) \end{array}$$

is a diffeomorphism.
(b) Let $n \in \mathbb{N}$. Show that the map

$$\begin{array}{rcl} \mathbb{R}^n & \to & (S^1)^n = S^1 \times \cdots \times S^1 \\ (t_1, \ldots, t_n) & \mapsto & (\exp(2\pi \mathrm{i} \cdot t_1) \ldots, \exp(2\pi \mathrm{i} \cdot t_n)) \end{array}$$

is a local diffeomorphism.

Remark. The point of this exercise is that one really needs to look up what are the standard smooth structures on the various smooth manifolds involved.

Exercise 19.7.

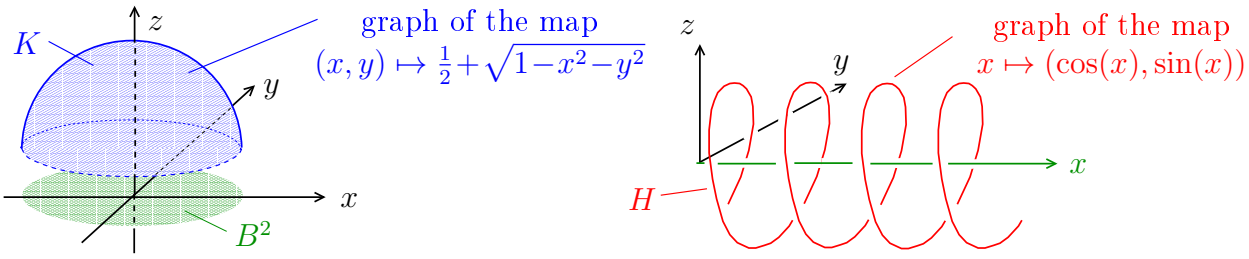
(1) Show that the topological space $X = \{(x, y) \in \mathbb{R}^2 \,|\, x \in [0, 1] \text{ and } y \in [0, x]\}$ is a topological manifold.
(2) Does the topological manifold X admit a smooth atlas?

Exercise 19.8. Give an example of two smooth atlases $\{\Phi_i\}_{i\in I}$ and $\{\Psi_j\}_{j\in J}$ for $X = \mathbb{R}$ such that for all $i \in I$ and $j \in J$ the map $\Psi_j \circ \Phi_i^{-1}$ is smooth, but such that the two atlases are not equivalent.

Exercise 19.9. Show that the following two statements are equivalent:

(1) S^n admits two inequivalent smooth structures.
(2) There exists a smooth manifold that is homeomorphic to S^n but not diffeomorphic to S^n, where we equip S^n with the standard smooth structure.

Exercise 19.10. Let V be some smooth manifold and let W be some smooth manifold with $\partial W = \varnothing$. Furthermore let $f\colon V \to W$ be a smooth map. Show that the corresponding graph $\mathrm{Gr}(f) := \{(x, f(x)) \,|\, x \in V\}$ is a proper submanifold of the smooth manifold $V \times W$.
Remark. This is the smooth analogue of Exercise 18.8.

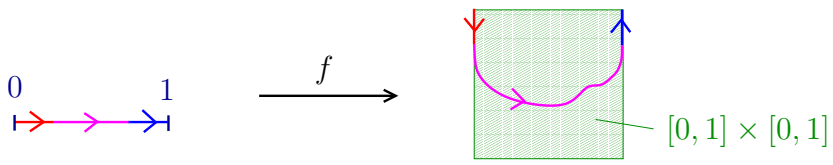


Exercise 19.11. Let M be a smooth manifold and let W be a submanifold of M. We denote by ∂W the boundary of W viewed as a smooth manifold in its own right. Show that for every component C of ∂W we have either $C \subset \partial M$ or $C \subset M \setminus \partial M$.

Exercise 19.12. Show that there exists a smooth map $f = (f_1, f_2)\colon [0, 1] \to [0, 1] \times [0, 1]$ with the following properties:

(a) For $t \in [0, \frac{1}{4}]$ we have $f(t) = (0, 1 - t)$.
(b) For $t \in [\frac{3}{4}, 1]$ we have $f(t) = (1, t)$.
(c) The map $f_1\colon [0, 1] \to [0, 1]$ is monotonously increasing.
(d) For each $t \in [0, 1]$ we have $f'(t) \neq 0 \in \mathbb{R}^2$.
(e) For $t \neq \{0, 1\}$ we have $f_2(t) \in (0, 1)$.

Hint. Use the Smooth Transition Function Lemma 19.4.



Exercise 19.13. Let U be an open convex neighborhood of $0 \in \mathbb{R}^n$ and let $f\colon U \to \mathbb{R}$ be a smooth map with $f(0) = 0$. We assume that 0 is a critical point, i.e. we assume that the

differential $\mathrm{D}f_0$ is zero. Show that there exist smooth functions $h_{ij}\colon U \to \mathbb{R}$, $i, j = 1, \ldots, n$ such that $h_{ij} = h_{ji}$ and such that

$$f(x) \;=\; \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij}(x) \cdot x_i \cdot x_j \qquad \text{for all } x = (x_1, \ldots, x_n) \in U.$$

Hint. Make use of the Map-as-Matrix Multiplication Proposition 19.14.

Exercise 19.14. Let $n \in \mathbb{N}$ and let $A \in \mathrm{GL}(n, \mathbb{Z})$ be a matrix. Show that the map

$$\begin{array}{rcl} f(A)\colon \mathbb{R}^n/\mathbb{Z}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ [v] & \mapsto & [A \cdot v] \end{array}$$

is a diffeomorphism.

Exercise 19.15. Let M be a smooth manifold and let $N \subset M$ be a smooth submanifold with $\dim(N) < \dim(M)$.

(a) Show that the interior of N in M is the empty set.
(b) Show that the closure of $M \setminus N$ in M equals M.

Exercise 19.16. Let $n \in \mathbb{N}_0$. As on page 255 we have the identification

$$\begin{array}{rcl} \Phi\colon \mathbb{R}\mathrm{P}^n = S^n/\{\pm 1\} & \to & \mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\}) \\ [x] & \mapsto & [x]. \end{array}$$

We use the description on the right to define a smooth structure on $\mathbb{R}\mathrm{P}^n$.

(a) Let $i \in \{0, \ldots, n\}$. Show that $V_i := \{[x_0 : \cdots : x_n] \in \mathbb{R}\mathrm{P}^n \,|\, x_i \neq 0\}$ is an open subset of $\mathbb{R}\mathrm{P}^n$.
(b) Show that the maps

$$\begin{array}{rcl} \Phi_i\colon \{[x_0 : \cdots : x_n] \in \mathbb{R}\mathrm{P}^n \,|\, x_i \neq 0\} & \to & \mathbb{R}^n \\ [x_0 : \cdots : x_n] & \mapsto & \big(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\big). \end{array}$$

form a smooth atlas for $\mathbb{R}\mathrm{P}^n$.
(c) Show that the smooth structure defined in (b) is equivalent to the standard smooth structure on $\mathbb{R}\mathrm{P}^n$ that we introduced on page 564.

Exercise 19.17. We consider the stereographic projection

$$\begin{array}{rcl} \Phi\colon S^2 & \to & \mathbb{R}^2 \cup \{\infty\} = \mathbb{C} \cup \{\infty\} \\ (x_1, x_2, x_3) & \mapsto & \begin{cases} \Big(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\Big), & \text{if } x_3 < 1, \\ \infty, & \text{if } x_3 = 1. \end{cases} \end{array}$$

Let $q(z)$ be a polynomial with complex coefficients. We denote by $N := (0, 0, 1)$ the North Pole. Show that the map

$$\begin{array}{rcl} f\colon S^2 & \to & S^2 \\ P & \mapsto & \begin{cases} \Phi^{-1}(q(\Phi(P))), & \text{if } P \neq N, \\ N, & \text{if } P = N \end{cases} \end{array}$$

is smooth.
Hint. Use Exercise 18.15.

Exercise 19.18. Let M be a smooth manifold and let $U \subset M$ be an open subset. Furthermore let $f\colon U \to \mathbb{R}^k$ be a smooth map and let $P \in U$. Show that there exists an open neighborhood V of P and a smooth map $g\colon M \to \mathbb{R}^k$ such that f and g agree on V.
Hint. Use the Smooth Transition Function Lemma 19.4.

Exercise 19.19. Let M be a compact smooth manifold and let N be a proper codimension-zero submanifold of M. Show that if M is connected and if $N \neq \varnothing$, then $M = N$.

Exercise 19.20. We consider the square $X = [0,1] \times [0,1]$ with the equivalence relation that is generated by $(0,y) \sim (1,1-y)$ and $(x,0) \sim (x,1)$. On page 265 we defined the Klein bottle to be the quotient $X/\!\sim$.

(a) Show that X admits a smooth structure such that the projection $X \to X/\!\sim$ is smooth.
(b) Show that X is diffeomorphic to the non-orientable surface of genus one.

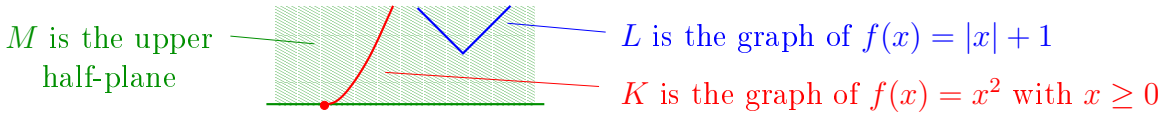
Remark. Note that part (b) is a refinement of Exercise 5.50. Also note that in Exercise 25.11 we will give an alternative, but equivalent, description of the smooth structure on the Klein bottle.

Exercise 19.21. Let $n \in \mathbb{N}_0$ and let $P, Q \in B^n = \overline{B}^n \setminus S^{n-1}$.

(a) Show that there exists a self-diffeomorphism φ of $\overline{B}^n$ with $\varphi(P) = Q$.
(b) Show that there exists a self-diffeomorphism φ of $\overline{B}^n$ with $\varphi(P) = Q$ and which is the identity in some neighborhood of S^{n-1}.

Exercise 19.22. In the figure below we show $M = \{(x,y) \in \mathbb{R}^2 \,|\, y \geq 0\}$ together with two subsets $K := \{(x,x^2) \,|\, x \geq 0\}$ and $L := \{(x,|x|+1) \,|\, x \in \mathbb{R}\}$.

(a) Show that both K and L are submanifolds of the topological manifold M.
Remark. You might have already shown in Exercise 18.11 that K is a submanifold.
(b) Show that K and L are not submanifolds if we view M as a smooth manifold.



Exercise 19.23. Let M be a smooth manifold and let $f\colon M \to N$ be a map to some smooth manifold N. Furthermore let W be a smooth manifold and let $\varphi\colon W \to M$ be a smooth injective map. Show that if $f \circ \varphi\colon W \to N$ is smooth, then it is nonetheless in general not true that $f\colon M \to N$ is smooth[228] on the subset $\varphi(W) \subset M$.
Remark. In Exercise 19.24 we will see that in many important settings we can in fact deduce that f is smooth on the subset $\varphi(W) \subset M$.

Exercise 19.24. Let M be a smooth manifold and let $f\colon M \to N$ be a map to some smooth manifold N. Recall that given a subset $A \subset M$ we defined on page 549 what it means for f to be smooth on A.

(a) We use the Smooth Manifold Boundary Proposition 19.26 to view the boundary ∂M as a smooth manifold in its own right. We denote by $\iota\colon \partial M \to M$ the inclusion map. Show that $f \circ \iota\colon \partial M \to N$ is smooth if and only if f is smooth on ∂M.

[228] Recall that given a subset $A \subset M$ we defined on page 549 what it means for f to be smooth on A.

(b) Let W be a submanifold of M. We use the Submanifold Smooth Structure Lemma 19.24 to view W as a smooth manifold in its own right. We denote by $\iota\colon W \to M$ the inclusion map. Show that $f \circ \iota\colon W \to N$ is smooth if and only if f is smooth on W.

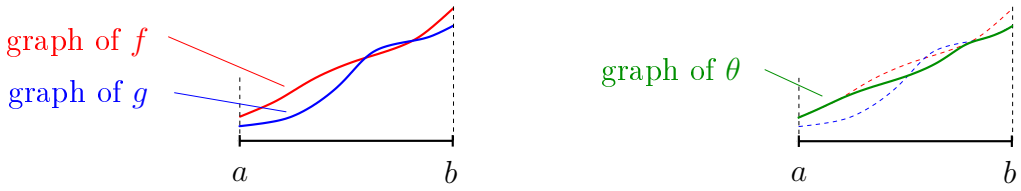
Remark. Exercise 19.23 shows that the results of this exercise cannot be generalized ad infinitum.

Exercise 19.25. Show that the Klein bottle can be decomposed into two submanifolds that are diffeomorphic to the annulus $[0,1] \times S^1$.

Exercise 19.26. Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be smooth maps such that $f'(x) > 0$ and $g'(x) > 0$ for all $x \in [a,b]$. We assume that $g(b) > f(a)$. Show that there exists an $\eta > 0$ and a smooth map $\theta\colon [a,b] \to \mathbb{R}$ with the following properties:

(1) $\theta'(x) > 0$ for all $x \in [a,b]$,
(2) θ agrees with f on $[a, a+\eta]$,
(3) θ agrees with g on $[b-\eta, b]$.

Hint. You could use the Smooth Transition Function Lemma 19.4 and consider the derivatives of f and g.



Exercise 19.27. We equip each $\mathbb{R}^k$ with the standard smooth structure that we introduced on page 544 and we equip $\mathbb{R}^m \times \mathbb{R}^n$ with the product smooth structure introduced in the Smooth Manifold Product Proposition 19.31. Show that the map

$$\begin{array}{rcl} \mathbb{R}^{m+n} & \to & \mathbb{R}^m \times \mathbb{R}^n \\ (x_1, \dots, x_m, y_1, \dots, y_n) & \mapsto & ((x_1, \dots, x_m), (y_1, \dots, y_n)) \end{array}$$

is a diffeomorphism.

Exercise 19.28. Show that $N := \{(x,y) \in \mathbb{R}^2 \,|\, y \geq x^2\}$ is not a smooth submanifold of the smooth manifold $M = \{(x,y) \in \mathbb{R}^2 \,|\, y \geq 0\}$.

Exercise 19.29. Let M be an m-dimensional submanifold of $\mathbb{R}^n$. Show that

$$T \;:=\; \{(P,w) \,|\, P \in M, w \in \mathrm{T}_P M\} \;\subset\; \mathbb{R}^n \times \mathbb{R}^n$$

is an $2m$-dimensional smooth submanifold of $\mathbb{R}^n \times \mathbb{R}^n$.

Exercise 19.30. Let $A \subset \mathbb{R}^k$ and $B \subset \mathbb{R}^l$ be smooth submanifolds such that at most one has non-empty boundary. Show that $A \times B \subset \mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^{k+l}$ is a smooth submanifold of dimension $\dim(A) + \dim(B)$.

Exercise 19.31. Let M be a smooth manifold and let W be a (proper) submanifold.

(a) Show that $W \cap \partial M$ is a (proper) submanifold of ∂M.
(b) Let X be a (proper) submanifold of W. Show that X is also a (proper) submanifold of M.

Exercise 19.32. Let M be an n-dimensional smooth manifold with empty boundary. We consider the product $M \times M$.

(a) Show that the diagonal $\Delta := \{(x, x) \,|\, x \in M\}$ is an n-dimensional submanifold of $M \times M$.

(b) Show that the diagonal Δ is *diffeomorphic* to M.
Remark. There is an obvious bijection, but why is this bijection a diffeomorphism?

Exercise 19.33. Let X be a topological space and let M be a smooth manifold. Furthermore let $f\colon X \to M$ be a map. Now suppose that we are given two smooth structures $\mathcal{A}$ and $\mathcal{A}'$ on X such that $f\colon (X, \mathcal{A}) \to M$ and $f\colon (X, \mathcal{A}') \to M$ are local diffeomorphisms. Show that the two smooth structures on X are the same.
Hint. Use the Smooth Pasting Proposition 19.23.
Remark. This exercise can be viewed as the smooth manifold version of Exercise 2.3.

Exercise 19.34. Let $n \in \mathbb{N}$. A *lattice in* $\mathbb{R}^n$ is a subgroup of $(\mathbb{R}^n, +)$ that is a discrete subset of $\mathbb{R}^n$.

(a) Let $v_1, \ldots, v_k \in \mathbb{R}^n$ be linearly independent vectors. Show that $\Lambda = \mathbb{Z}\cdot v_1 + \cdots + \mathbb{Z}\cdot v_k$ is a lattice in $\mathbb{R}^n$. We refer to k as the rank of the lattice.

(b) Show that any lattice in $\mathbb{R}^n$ is of the form described in (a).

(c) Show that the obvious action of a lattice $\Lambda \subset \mathbb{R}^n$ on $\mathbb{R}^n$, i.e. the action that is given by addition, is proper.

If Λ is a lattice in $\mathbb{R}^n$, then it follows from (c) and the Manifold Quotient-by-Group Action Proposition 19.32 that $\mathbb{R}^n/\Lambda$ is naturally an n-dimensional smooth manifold.

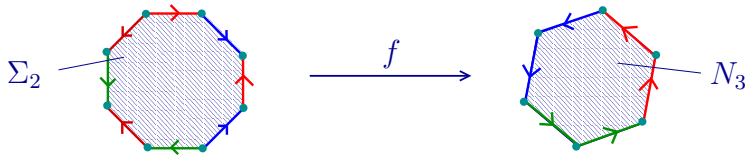
(d) Show that if Λ and $\widetilde{\Lambda}$ are two lattices in $\mathbb{R}^n$ of the same rank, then $\mathbb{R}^n/\Lambda$ and $\mathbb{R}^n/\widetilde{\Lambda}$ are diffeomorphic.

Exercise 19.35. Let $g \in \mathbb{N}$. Let M be the smooth manifold that we obtain from removing a single point from the surface of genus g. Does there exist a smooth map $M \to S^2$ that is a local diffeomorphism?

Exercise 19.36. In the figure below we show to the left the usual way to depict the Möbius band M as submanifold in $\mathbb{R}^3$. Does there exist an embedding of the Möbius band in $\mathbb{R}^3$ such that the boundary is the curve K shown to the right?



Exercise 19.37. Given $g \in \mathbb{N}_0$ we denote by Σ_g the surface g as defined on page 266 and given $k \in \mathbb{N}$ we denote by N_k the non-orientable surface of genus k as defined on page 266. Show that given any $k \in \mathbb{N}$ there exists a smooth map $f\colon \Sigma_{k-1} \to N_k$ that is a local diffeomorphism and that has the property that for each $x \in N_k$ we have $\#f^{-1}(x) = 2$.


Σ_2
f
N_3

20. Smooth manifolds with corner

In this chapter we introduce smooth manifolds with corner. This might be viewed as a rather obscure topic, but it naturally arises in various situations, for example when one considers smooth homotopies of smooth manifolds with non-empty boundary.

20.1. **Definition of smooth manifolds with corner.** In the Topological Manifold Product Proposition 18.4 we saw that the product of two topological manifolds X and Y is again a topological manifold. If both X and Y have non-empty boundary we had to make use of a delicate chart (see also the Corner Smooth Lemma 20.1 below.) It turns out that for smooth atlases this approach does not necessarily lead to a smooth atlas on $X \times Y$. This debacle requires us to introduce smooth manifolds with corner.

Notation. Given $n \in \mathbb{N}_{\geq 2}$ we refer to

$$\mathbb{R}^n_{++} := \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_{n-1} \geq 0 \text{ and } x_n \geq 0\}$$

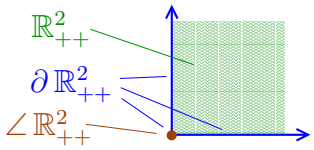
as the upper right quadrant of $\mathbb{R}^n$. We refer to

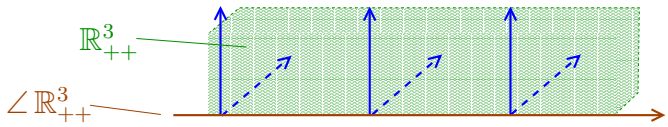
$$\partial\mathbb{R}^n_{++} := \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_{n-1} = 0 \text{ or } x_n = 0\}$$

as the boundary of $\mathbb{R}^n_{++}$ and we refer to

$$\angle\mathbb{R}^n_{++} := \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_{n-1} = 0 \text{ and } x_n = 0\}$$

as the corner of $\mathbb{R}^n_{++}$.





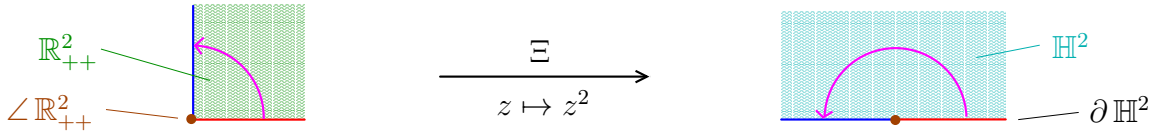
Lemma 20.1. (Corner Smooth Lemma) Let $n \in \mathbb{N}_{\geq 2}$.

(1) The map[229]

$$\begin{array}{rcl} \Xi\colon \mathbb{R}^n_{++} & \to & \mathbb{H}^n \\ (x_1, \dots, x_{n-2}, \underbrace{r \cdot \cos(\varphi), r \cdot \sin(\varphi)}_{=:z\in\mathbb{C}}) & \mapsto & (x_1, \dots, x_{n-2}, \underbrace{r^2 \cdot \cos(2\varphi), r^2 \cdot \sin(2\varphi)}_{=z^2}). \end{array}$$

is a homeomorphism which restricts to a homeomorphism $\partial\mathbb{R}^n_{++} \to \partial\mathbb{H}^n$.

(2) Let U be an open subset of $\mathbb{R}^n_{++}$ that contains a point of the corner $\angle\mathbb{R}^n_{++}$. There is no diffeomorphism $f\colon U \to V$ from U to some open subset $V \subset \mathbb{H}^n$ such that $f(\partial\mathbb{R}^n_{++} \cap U) \subset \partial\mathbb{H}^n$.



Proof.

[228] We first encountered this nice map on page 516.

[229] Note that the definition of a diffeomorphism on page 537 also makes sense for subsets of $\mathbb{R}^n$ that are not open.

(1) The map $\mathbb{C} \to \mathbb{C}$ that is given by $z = x + \mathrm{i}y \mapsto z^2 = (x^2 - y^2) + 2xy\mathrm{i}$ is continuous. It follows that Ξ is continuous. It is straightforward to show, say using the Closed Embedding-to-$\mathbb{R}^n$ Proposition 2.19, that Ξ is indeed a homeomorphism.

(2) The proof of this statement is similar to the proof of the Smooth Manifold Boundary Proposition 19.26 (1). We outsource the proof to Exercise 20.1. ■

In the following we recall the definition of charts of type (i) and (ii) from page 511 and we introduce the notion of a corner chart.

Definition. Let X be a topological space and let $x \in X$. An n-dimensional chart at the point $x \in X$ is a homeomorphism $\Phi\colon U \to V$ where U is an open neighborhood of x and

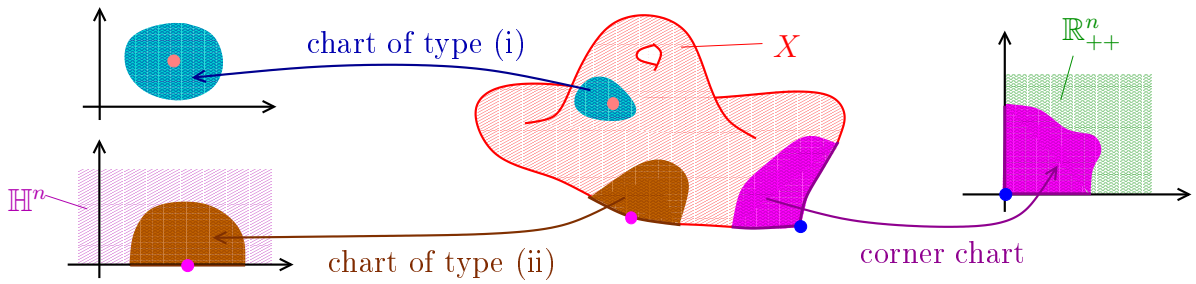
(i) V is an open subset of $\mathbb{R}^n$ or

(ii) V is an open subset of the upper half-space $\mathbb{H}^n$ and $\Phi(x)$ lies in $\partial\,\mathbb{H}^n$.

An n-dimensional corner chart at the point $x \in X$ is a homeomorphism $\Phi\colon U \to V$ where U is an open neighborhood of x and

(c) V is an open subset of $\mathbb{R}^n_{++}$ and $\Phi(x)$ lies in $\angle\,\mathbb{R}^n_{++}$.

To simplify the discussion we refer to these as charts of type (i), (ii) and (c). Sometimes we call a chart of type (c) also a corner chart.



It follows from the Corner Smooth Lemma 20.1 (1) that if a point $x \in X$ admits a corner chart, then it also admits a chart of type (ii). Thus if we are only interested in topological manifolds such charts do not add anything new and they are basically irrelevant. As the Corner Smooth Lemma 20.1 (2) indicates, smoothly the situation is different. This leads us to the following definition.

Definition. Let X be a topological space.

(1) An n-dimensional corner atlas for the topological space X is a collection of n-dimensional charts of type (i), (ii) and (c) such that the domains cover all of X.

(2) We say that two n-dimensional charts $\Phi_1\colon U_1 \to V_1$ and $\Phi_2\colon U_2 \to V_2$ for X are smoothly compatible if the transition map

$$\Phi_2 \circ \Phi_1^{-1}\colon \underbrace{\Phi_1(U_1 \cap U_2)}_{\substack{\text{open set in} \\ \mathbb{R}^n,\ \mathbb{H}^n \text{ or } \mathbb{R}^n_{++}}} \to \underbrace{\Phi_2(U_1 \cap U_2)}_{\subset \mathbb{R}^n}$$

is smooth, in the sense of our definition on page 532.

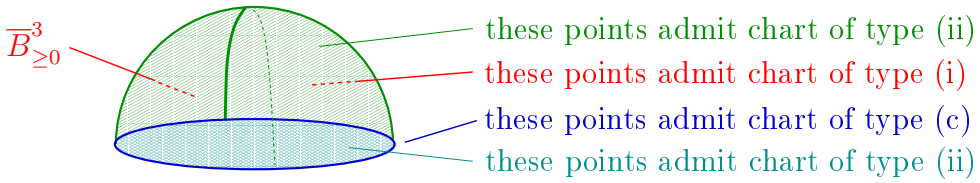
(3) A corner atlas $\mathcal{A}$ is called smooth if all charts in the corner atlas are smoothly compatible to one another.

(4) A smooth corner $\mathcal{A}$ for M is called maximal if every chart Ψ that is smoothly compatible with all the charts in the smooth atlas $\mathcal{A}$ is already contained in $\mathcal{A}$. The proof of the Maximal Atlas Existence Proposition 19.17 shows that every smooth corner atlas is contained in a unique maximal smooth corner atlas.
(5) An n-dimensional smooth manifold with corner is a pair $(X, \mathcal{A})$ consisting of a second-countable Hausdorff space together with a maximal n-dimensional smooth corner atlas $\mathcal{A}$ for X.

Example. Let $n \geq 2$. We consider the upper half-ball

$$\overline{B}^n_{\geq 0} \;=\; \{(x_1, \ldots, x_n) \in \overline{B}^n \,|\, x_n \geq 0\}.$$

In Exercise 20.2 we will see that $\overline{B}^n_{\geq 0}$ is a smooth manifold with corner, with charts as shown in the figure below.



The following lemma says, as we already hinted at, that as topological manifolds we did not introduce anything new.

Lemma 20.2. If $(X, \mathcal{A})$ is an n-dimensional smooth manifold with corner, then the underlying topological space X is an n-dimensional topological manifold.

Proof. This lemma follows, as will be clear to the reader, from the observation that the composition of a corner chart with the homeomorphism given in the Corner Smooth Lemma 20.1 is a chart of type (ii). ■

Definition. Many of the definitions for smooth manifolds extend in an obvious way to smooth manifolds with corner:

(1) As in Lemmas 19.19 and 19.20, and as in the convention on page 545, we can view an open subset of a smooth manifold with corner as a smooth manifold with corner.
(2) A smooth map between smooth manifolds with corner is defined as on page 546.

For completeness' sake we state the following generalization of Lemma 19.22.

Lemma 20.3.

(1) Every smooth map between two smooth manifolds with corner is continuous.
(2) The composition of two smooth maps between smooth manifolds with corner is again smooth.

Proof. The proof is basically identical to the rather trivial proof of Lemma 19.22. ■

The following proposition is a generalization of the Smooth Manifold Boundary Proposition 19.26 (1).

Proposition 20.4. (Unique Type of Chart Proposition) Let M be a smooth manifold with corner and let $x \in M$. Precisely one of the following three statements holds:

(1) The point x admits a chart of type (i) from the smooth structure.

(2) The point x admits a chart of type (ii) from the smooth structure.
(3) The point x admits a chart of type (c) from the smooth structure.

Proof. The fact that a point that admits charts of type (i) in the smooth structure cannot also admit a chart of type (ii) in the smooth structure is the content of the Smooth Manifold Boundary Proposition 19.26 (1). The same argument also shows that such a point cannot admit a chart of type (c) in the smooth structure.

The fact that charts of type (ii) and (c) in the smooth structure are mutually exclusive follows easily from the Corner Smooth Lemma 20.1 (2). We outsource the proof to Exercise 20.3. Alternatively see also [**Lee2002**, Proposition 16.20] for details. ■

The Unique Type of Chart Proposition 20.4 leads us to the following analogue and generalization of the definitions on pages 511 and 545.

Definition. Let M be a smooth manifold with corner.

(1) We say that a point $x \in M$ is a boundary point if x admits a chart of type (ii) or (c) in the smooth structure. We denote by ∂M the set of all boundary points of M. We refer to ∂M as the boundary of M.
(2) We say that a point $x \in M$ is a corner point if x admits a chart of type (c) in the smooth structure. We denote by $\angle M$ the set of all corner points of M. We refer to $\angle M$ as the corner of M.



We conclude this section with the following proposition which is an analogue of the Smooth Manifold Boundary Proposition 19.26 (2) and (3).

Proposition 20.5. Let M be an n-dimensional smooth manifold with corner. The following statements hold:

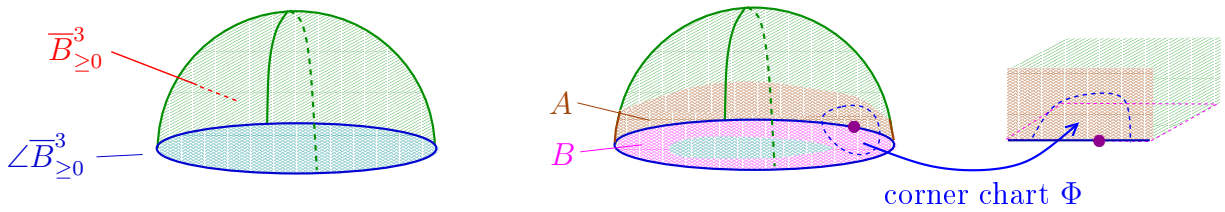
(1) The corner $\angle M$ has a natural structure of an $(n-2)$-dimensional smooth manifold with $\partial(\angle M) = \varnothing$. Furthermore the inclusion map $\angle M \to M$ is a smooth map.
(2) Any union of components of ∂M is a closed subset of M.
(3) Any union of components of $\angle M$ is a closed subset of M.

Proof. The proof of these statements is very similar to the proof of the Smooth Manifold Boundary Proposition 19.26 (2) and (3). All we really have to do is to replace the Smooth Manifold Boundary Proposition 19.26 (1) by the Unique Type of Chart Proposition 20.4. We leave it to the reader to fill in the details. ■

Of course one could now develop the theory of smooth manifolds with corner in great detail. For example one could now introduce the notion of a submanifold of a smooth manifold with corner. We will spare the author and the reader to go through this discussion which would mostly be a case of "l'art pour l'art".
Instead, to pique the reader's interest we give a sneak preview of a later result. To formulate the result we will need the following definition.

Definition. Let M be an n-dimensional smooth manifold with corner. We say that the corner $\angle M$ is two-sided if there exist two disjoint open subsets A and B of ∂M such that the following conditions are satisfied:

(1) $A \cup \angle M \cup B$ is an open subset of ∂M,
(2) For every chart $\Phi\colon U \to V \subset \mathbb{R}^n_{++}$ of type (c) one of the following holds:
- $\Phi(A \cap U) \subset \{(x_1, \dots, x_n) \,|\, x_{n-1} = 0\}$ and $\Phi(B \cap U) \subset \{(x_1, \dots, x_n) \,|\, x_n = 0\}$, or
- $\Phi(B \cap U) \subset \{(x_1, \dots, x_n) \,|\, x_{n-1} = 0\}$ and $\Phi(A \cap U) \subset \{(x_1, \dots, x_n) \,|\, x_n = 0\}$.



The following proposition now says that smooth manifolds with corner can be turned into smooth manifolds without corner.

Proposition ?? (Simplified). Let M be a smooth manifold with corner $\angle M$. If $\angle M$ is two-sided, then any choice of a "bicollar" gives rise to a smooth atlas on M which consists only of charts of type (i) and (ii).

20.2. **Products of smooth manifolds II.** In the Topological Manifold Product Proposition 18.4 we saw that the product of two topological manifolds is again a topological manifold. The case that both topological manifolds have non-empty boundary was somewhat tricky.

Thus, when in Section 19.10 we started to look at the product of two smooth manifolds we sidestepped the issue by only considering the product of two smooth manifolds where at least one has empty boundary. We now turn to the product of two smooth manifolds where both are allowed to have non-empty boundary.

Proposition 20.6. (Smooth Manifold Product Proposition) Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold. We denote by

$$\begin{array}{rcl} p\colon M \times N & \to & M \\ (x,y) & \mapsto & x \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon M \times N & \to & N \\ (x,y) & \mapsto & y \end{array}$$

the two projections. The following statements hold:

(1) The product $M \times N$ is an $(m+n)$-dimensional topological manifold that admits a *unique* structure as a smooth manifold with corner that satisfies the following two conditions:
   (a) The projections $p\colon M \times N \to M$ and $q\colon M \times N \to N$ are smooth.
   (b) For every smooth manifold with corner W and every map $f\colon W \to M \times N$ the following holds: If the two maps $p \circ f\colon W \to M$ and $q \circ f\colon W \to N$ are smooth, then f itself is smooth.
(2) $\partial(M \times N) = (M \times \partial N) \cup (N \times \partial M)$.
(3) $\angle(M \times N) = \partial M \times \partial N$.

Proof. Similar to the proof of the Smooth Manifold Product Proposition 19.31 we obtain a corner atlas for $M \times N$ by taking products of charts of M and N. The proof of the

proposition is thus just a straightforward modification of the already very straightforward proof of the Smooth Manifold Product Proposition 19.31. The only real difference is that for the proofs of Statement (2) and (3) we need to replace the Smooth Manifold Boundary Proposition 19.26 (1) by the Unique Type of Chart Proposition 20.4. ■

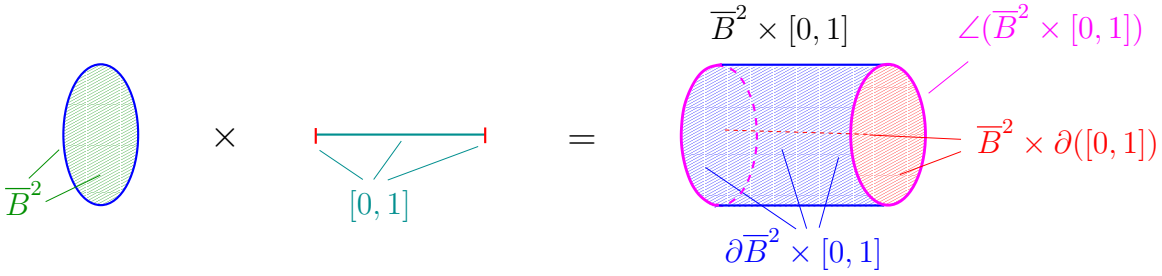
Example.

(1) If $\partial M = \varnothing$ or $\partial N = \varnothing$, then the Smooth Manifold Product Proposition 20.6 is of course just a special case of the Smooth Manifold Product Proposition 19.31.

(2) Let $M = \overline{B}^2$ and $N = [0,1]$, then $M \times N = \overline{B}^2 \times [0,1]$ is the solid cylinder which is a 3-dimensional smooth manifold with corner such that

$$\begin{aligned} \text{boundary } \partial\big(\overline{B}^2 \times [0,1]\big) &= (S^1 \times [0,1]) \cup (\overline{B}^2 \times \{0,1\}) \\ \text{corner } \angle\big(\overline{B}^2 \times [0,1]\big) &= S^1 \times \{0,1\}. \end{aligned}$$

(3) Later, on page 811, given two smooth manifolds M and N and given smooth maps $f, g\colon M \to N$ we will introduce the notation of a smooth homotopy between f and g. This is just a smooth map $F\colon M \times [0,1] \to N$ with $F_0 = f$ and $F_1 = g$. At this point we will be very interested in products of the form $M \times [0,1]$.



We continue with the following lemma.

Lemma 20.7. Let $f\colon M \to \widetilde{M}$ and $g\colon N \to \widetilde{N}$ be smooth maps between smooth manifolds. The map

$$\begin{aligned} f \times g\colon M \times N &\to \widetilde{M} \times \widetilde{N} \\ (x,y) &\mapsto (f(x), g(y)) \end{aligned}$$

is a smooth map between smooth manifolds with corner.

Proof. The proof is formally similar to the proof of the Product Topology-Basics Lemma 5.4 (3). We encourage the reader to fill in the details. ■

By Proposition **??** we know that, upon the choice of a bicollar, we can usually dispose of corners. For products of smooth manifolds this leads to the following proposition.

Proposition ?? (Simplified). Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold. The corner $\angle(M \times N)$ is two-sided and any choice of a "bicollar" gives rise to a smooth atlas on $M \times N$ which consists only of charts of type (i) and (ii).

If one keeps one's eyes open, smooth manifolds with corner occur naturally in many settings. Nonetheless, the theory of smooth manifolds with corner is an unloved (and some would say unlovable) topic which did not receive much treatment in the literature. We refer to

[**Lee2002**, Chapter 16] and [**Joy2012**] for more information on smooth manifolds with corner and the obvious generalization where one allows charts that take values in open subsets of $\mathbb{R}^k \times (\mathbb{R}_{\geq 0})^{n-k}$.

Exercises for Chapter 20.

Exercise 20.1. Let $n \in \mathbb{N}_{\geq 2}$. Let U be an open subset of $\mathbb{R}^n_{++}$ that contains a point of the corner $\angle \mathbb{R}^n_{++}$. Show that there is no diffeomorphism $f\colon U \to V$ from U to some open subset $V \subset \mathbb{H}^n$ such that $f(\partial \mathbb{R}^n_{++} \cap U) \subset \partial \mathbb{H}^n$.

Exercise 20.2. Let $n \geq 2$. Show that the upper half-ball

$$\overline{B}^n_{\geq 0} \;=\; \{(x_1, \ldots, x_n) \in \overline{B}^n \,|\, x_n \geq 0\}$$

admits the structure of a smooth manifold with corner such that

$$\angle \overline{B}^n_{\geq 0} \;=\; \{(x_1, \ldots, x_{n-1}, 0) \in \overline{B}^n \,|\, (x_1, \ldots, x_{n-1}) \in S^{n-2}\} \;=\; S^{n-2} \times \{0\}.$$

Exercise 20.3. Let M be an n-dimensional smooth manifold with corner and let $x \in M$. Show that x cannot admit both a chart of type (ii) and a chart of type (c).

Exercise 20.4. Give an example of a smooth manifold M with corner such that the corner $\angle M$ is not two-sided.

21. Partitions of unity on manifolds

In this short chapter we show in particular that (smooth) manifolds admit (smooth) partitions of unity and that they admit (smooth) exhaustion functions. For topological manifolds these results also follow from earlier results, like the Paracompact-Partition of Unity Theorem 10.9 which builds on Urysohn's Lemma 8.4. But for smooth manifolds the results of this chapter are new and for topological manifolds the arguments from this chapter are more elementary than the earlier ones.

The existence of (smooth) partitions of unity will be a key technical tool for constructing interesting (smooth) functions on (smooth) manifolds. In particular they will allow us to prove the Smooth Extension Proposition 21.9 and the Smooth Approximation Proposition 21.10.

21.1. Existence on partitions of unity on manifolds.

First we recall the following definitions from page 360.

Definition. Let X be a topological space.

(1) The support of a function $f\colon X \to \mathbb{R}$ is defined as

$$\operatorname{supp}(f) \;:=\; \overline{\{x \in X \mid f(x) \neq 0\}}.$$

(2) A partition of unity on X is a family of maps $\{f_i\colon X \to [0,1]\}_{i\in I}$ with the following properties:

(a) For each $x \in X$ there exists an open neighborhood U of x such that f_i vanishes on U for all but finitely many i.[230]

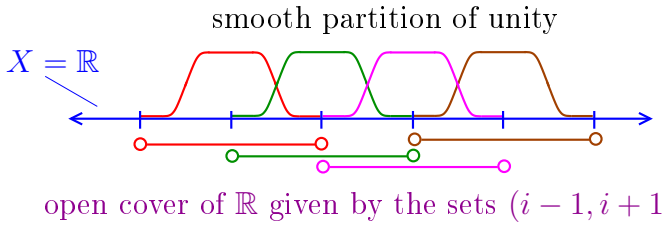
(b) For every $x \in X$ we have $\sum_{i\in I} f_i(x) = 1$.

The following theorem guarantees the existence of many interesting (smooth) partitions of unity.

Theorem 21.1. (Manifold-Partitions of Unity Existence Theorem) Let X be a topological manifold.

(1) Given any open cover $\{U_i\}_{i\in I}$ there exists a partition of unity $\{f_i\}_{i\in I}$ such that for each $i \in I$ we have $\operatorname{supp}(f_i) \subset U_i$.

(2) If X is in fact a smooth manifold, then we can arrange that the maps f_i are smooth.



Remark.

[230] It follows from the elementary Closure-of-Locally Finite Cover Lemma 10.6 that Condition (1) is satisfied if and only if the supports $\{\operatorname{supp}(f_i)\}_{i\in I}$ form a locally finite family of subsets of X. In other words, Condition (1) is satisfied if and only if given any $x \in X$ there exists an open neighborhood U of x such that $U \cap \operatorname{supp}(f_i) = \varnothing$ for all but finitely many i.

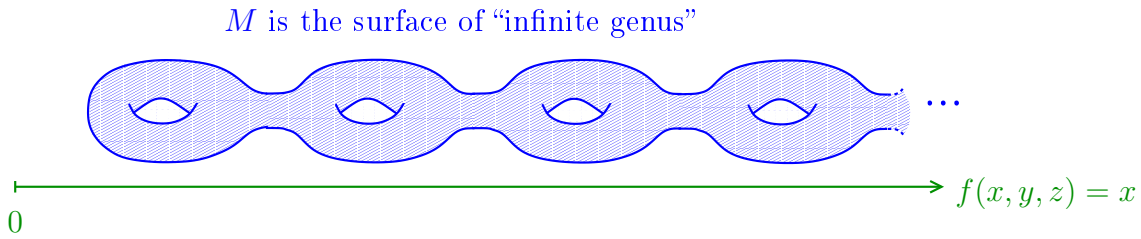
(1) the Manifold-Partitions of Unity Existence Theorem 21.1 is a great tool for constructing (smooth) maps *out* of a topological or smooth manifold. For constructing maps *into* a smooth manifold a particularly useful tool is given by the Neighborhood Projection Theorem 28.1.
(2) In the Topological Manifolds-Local Properties Lemma 18.8 we saw that topological manifolds are regionally compact. Since topological manifolds are by definition Hausdorff and second-countable we deduce from the Paracompact-Criterion Lemma 10.2 that topological manifolds are paracompact. In the Paracompact-Partition of Unity Theorem 10.9 we used Urysohn's Lemma 8.4 to show that any paracompact space satisfies the conclusion of the Manifold-Partitions of Unity Existence Theorem 21.1 (1). Or, to cut a long story short, the above earlier results already provide a proof of the Manifold-Partitions of Unity Existence Theorem 21.1. In this chapter we give a different proof, that does not make use of Urysohn's Lemma 8.4, and that has the distinct advantage that it also provides a proof of the Manifold-Partitions of Unity Existence Theorem 21.1 (2).
(3) The combination of the Manifold-Partitions of Unity Existence Theorem 21.1 and the Paracompact-Partition of Unity Theorem 10.9 shows that topological manifolds are paracompact. This provides a new proof of the Manifold Paracompact-Normal Proposition 18.9 (1).

We postpone the proof of the Manifold-Partitions of Unity Existence Theorem 21.1 to the next section. In the following we provide a typical application of the existence of partitions of unity. To do so we recall the following definition from page 358.

Definition. Let X be a topological space. An exhaustion function is a map $f\colon X \to \mathbb{R}_{\geq 0}$ such that for every $c \in \mathbb{R}_{\geq 0}$ the preimage $f^{-1}([0,c])$ is a compact subset of X.[231]

Examples.

(1) Let $M = [0,1)$ be the half-open interval, viewed as a 1-dimensional smooth manifold. In this case the function defined by $f(x) = \frac{1}{1-x}$ is a smooth exhaustion function.
(2) We consider the surface of infinite genus illustrated in the figure below. In this case the function $f(x,y,z) = x$ is a smooth exhaustion function.



Lemma 21.2. (Manifold Exhaustion Lemma)
(1) Every topological manifold admits an exhaustion function.

[231] One can easily show that a map $X \to \mathbb{R}_{\geq 0}$ is an exhaustion function if and only if it is proper in the sense of the definition on page 180.

(2) Every smooth manifold admits a smooth exhaustion function.

Remark.

(1) In the Exhaustion Function Lemma 10.7 we used Urysohn's Lemma 8.4 to show that every topological space that is regionally compact, second-countable and Hausdorff admits an exhaustion function. Thus, as in the previous remark, we see that we already gave a proof of the Manifold Exhaustion Lemma 21.2 (1). The interest of the present proof is that it does not make use of Urysohn's Lemma 8.4 and that it gives a stronger conclusion for smooth manifolds.

(2) The Manifold Exhaustion Lemma 21.2 can be viewed as a refinement of the Topological Manifold Exhaustion Lemma 18.12.

Proof. In the following we provide the proof for smooth manifolds. The proof for topological manifolds is almost verbatim the same.[232] Thus let M be a smooth manifold. By the Topological Manifolds-Local Properties Lemma 18.8 (5) we know that M is locally precompact, i.e. given any $x \in M$ there exists a neighborhood U of x such that the closure of U in M is compact. It follows from the observation that M is by definition second-countable, together with the Countable Covering Lemma 9.2, that there exists a countable cover $\{U_k\}_{k\in\mathbb{N}}$ of M by precompact sets. Furthermore by the Manifold-Partitions of Unity Existence Theorem 21.1 we now know that there exists a smooth partition of unity $\{\psi_k\}_{k\in\mathbb{N}}$ such that for each $k \in \mathbb{N}$ we have $\operatorname{supp}(\psi_k) \subset U_k$. We define

$$\begin{aligned} f\colon M &\to \mathbb{R}_{\geq 0} \\ x &\mapsto \sum_{k\in\mathbb{N}} k \cdot \psi_k(x). \end{aligned}$$

By definition of a partition of unity we know that locally this is a finite sum of smooth functions. It follows from the Smooth Pasting Proposition 19.23 that f is smooth. It remains to prove the following claim.

Claim. For every $c \in \mathbb{R}$ the preimage $f^{-1}([0,c])$ is compact.

Proof. Let $c \in \mathbb{R}$. We pick $N \in \mathbb{N}$ with $N \geq c$. We start out with the following observation: Given any point $x \notin \overline{U}_1 \cup \cdots \cup \overline{U}_N$ we have

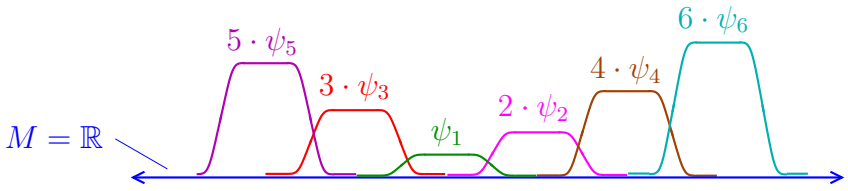
$$f(x) = \sum_{k=1}^{\infty} k \cdot \psi_k(x) \underset{\substack{\uparrow \\ \text{since } x \notin \overline{U}_1 \cup \cdots \cup \overline{U}_N \text{ and since } \operatorname{supp}\psi_k \subset U_k \subset \overline{U}_k}}{=} \sum_{k=N+1}^{\infty} k \cdot \psi_k(x) \geq \sum_{k=N+1}^{\infty} N \cdot \psi_k(x) \underset{\uparrow}{=} N \cdot \underbrace{\sum_{k=1}^{\infty} \psi_k(x)}_{=1} = N > c.$$

It follows from this observation that

$$f^{-1}([0,c]) = \underbrace{f^{-1}([0,c])}_{\text{closed}} \cap \big(\underbrace{\overline{U}_1 \cup \cdots \cup \overline{U}_N}_{\text{compact}}\big).$$

It follows from the Compact-Closed Lemma 1.21, see also Exercise 1.22, that $f^{-1}([0,c])$ is indeed compact. ∎

[232] Also, as we just mentioned in the remark above, the statement for topological manifolds is a special case of the Exhaustion Function Lemma 10.7. In fact our proof for smooth manifolds is almost identical to the proof of the Exhaustion Function Lemma 10.7, the biggest difference is that we need to replace the Paracompact-Criterion Lemma 10.2 and the Paracompact-Partition of Unity Theorem 10.9 by the Manifold-Partitions of Unity Existence Theorem 21.1.



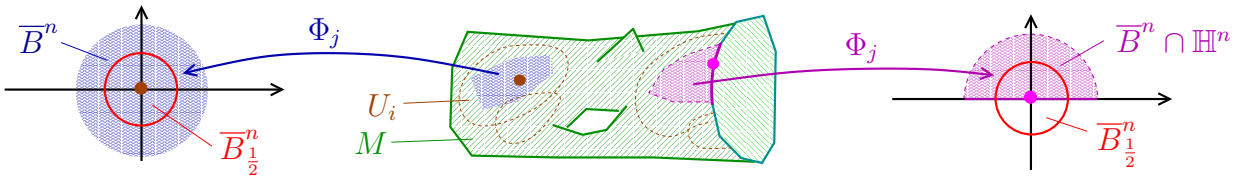
21.2. **Proof of the Manifold-Partitions of Unity Existence Theorem 21.1.** In this section we now provide the long overdue proof of the Manifold-Partitions of Unity Existence Theorem 21.1. The key ingredient is the following proposition which sounds technical, but which is interesting in its own right.

Proposition 21.3. (Atlas Improvement Proposition) Given an n-dimensional topological manifold M and given an open cover $\{U_i\}_{i\in I}$ there exists an atlas $\{\Phi_j \colon V_j \to W_j\}_{j\in J}$ with the following properties:

(1) The index set J is countable. If M is compact, then J can be chosen to be finite.
(2) The open cover $\{V_j\}_{j\in J}$ is a refinement of the open cover $\{U_i\}_{i\in I}$. In other words, given any $j \in J$ there exists an $i \in I$ with $V_j \subset U_i$.
(3) Each W_j is either the open n-ball B^n or the open half-ball $B^n \cap \mathbb{H}^n$.
(4) The open cover $\{V_j\}_{j\in J}$ is locally finite, i.e. given any point $x \in M$ there exists an open neighborhood X of x which has the property that $X \cap V_j = \varnothing$ for all but finitely many $j \in J$.
(5) We have $M = \bigcup\limits_{j\in J} \Phi_j^{-1}(W_j \cap \overline{B}^n_{\frac{1}{2}})$.

If M is in fact a smooth manifold, then we can also arrange that the following extra condition is satisfied:

(6) The charts are smooth, i.e. they lie in the maximal atlas of M.



Remark.

(1) the Atlas Improvement Proposition 21.3 implies in particular that every topological manifold is paracompact. This provides a new proof of Manifold Paracompact-Normal Proposition 18.9 (1).
(2) the Atlas Improvement Proposition 21.3 implies in particular that every topological manifold admits a locally finite cover by compact subsets.
(3) In the Atlas Improvement Proposition 21.3 we showed in particular that an n-dimensional topological manifold M admits a locally finite atlas. In the Covering Dimension-of-Manifolds Proposition 27.7 we will see that the covering dimension of M is $\leq n$. In the Small Atlas Proposition 27.8 we will use this fact to refine some of the conclusions of the Atlas Improvement Proposition 21.3.

Now it is time to provide the proof of the Atlas Improvement Proposition 21.3.

Proof. To simplify the discussion we only treat the case that M is a topological manifold with empty boundary. The proof of all other cases can is basically the same and is left to the reader.

Thus let M be an n-dimensional topological manifold with empty boundary and let $\{U_i\}_{i\in I}$ be an open cover of M. By the Topological Manifold Exhaustion Lemma 18.12 there exists a family of compact subsets $\{C_k\}_{k\in\mathbb{N}}$ of M such that the following two conditions are satisfied:

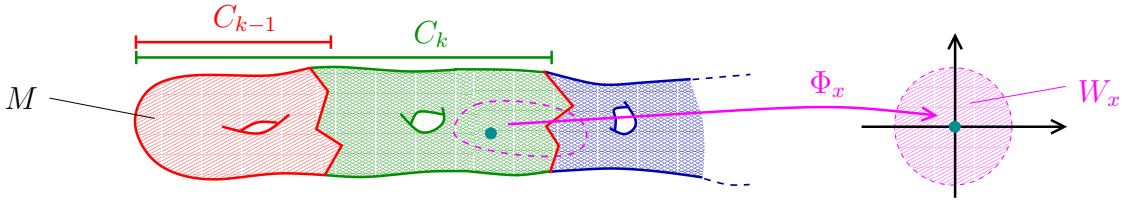
(1) we have $M = \bigcup_{k\in\mathbb{N}} C_k$,

(2) for every $k \in \mathbb{N}$ we have $C_k \subset \mathring{C}_{k+1}$.

Note that it follows from the Compact-Closed Lemma 1.21 (2) that each $\mathring{C}_{k+1} \setminus C_{k-1}$ is an open subset of M. Furthermore, by design these open subsets cover all of M. Using this observation it is elementary to see that given any $x \in M$ there exists a chart $\Phi_x \colon V_x \to W_x$ with the following properties:

- V_x is an open neighborhood of x which is contained in some U_i,
- there exists a $k \in \mathbb{N}$ with $V_x \subset \mathring{C}_{k+1} \setminus C_{k-1}$,
- W_x is the open n-ball B^n of radius 1.

For each $x \in M$ we set $\widetilde{U}_x := \Phi_x^{-1}(B^n_{\frac{1}{2}})$. Evidently the $\widetilde{U}_x$ are open sets which cover all of M. It follows from the Compact Subspace Covering Lemma 1.19 that for each $k \in \mathbb{N}$ there exists a finite set $J_k \subset M$ such that $C_k \subset \bigcup_{x\in J_k} \widetilde{U}_x$. We set $J := \bigcup_{k\in\mathbb{N}} J_k$. We now claim that the atlas $\{\Phi_j \colon V_j \to W_j\}_{j\in J}$ has all the desired properties. In fact all statements basically hold by construction. Perhaps Property (4) is the least obvious one, so we provide the argument. Thus let $x \in M$. There exists a $k \in \mathbb{N}$ such that $x \in \mathring{C}_{k+1} \setminus C_{k-1}$. We set $X := \mathring{C}_{k+1} \setminus C_{k-1}$. By construction we have $X \cap V_j = \varnothing$ unless $j \in J_k \cup J_{k-1}$. ■



At this point it is convenient to recall the following definition from page 360.

Definition. Let $\{U_i\}_{i\in I}$ be an open cover of a topological space X. A partition of unity subordinate to the open cover is a partition of unity $\{f_j \colon X \to [0,1]\}_{j\in J}$ such that for each $j \in J$ there exists an $i \in I$ with $\operatorname{supp}(f_j) \subset U_i$.

With this definition we can now formulate the following lemma.

Lemma 21.4. Let X be a topological space (smooth manifold) and let $\{U_i\}_{i\in I}$ be an open cover of X. Suppose that there exists a (smooth) partition of unity $\{\varphi_j \colon X \to [0,1]\}_{j\in J}$ subordinate to the given open cover, then there exists in fact a (smooth) partition of unity $\{\psi_i \colon X \to [0,1]\}_{i\in I}$ such that for every $i \in I$ we have $\operatorname{supp}(\psi_i) \subset U_i$.

Remark. For topological spaces the statement of Lemma 21.4 is precisely the (2)⇒(3)-direction of the Paracompact-Partition of Unity Theorem 10.9. But even for topological spaces our proof of Lemma 21.4 is of interest, since it is a direct proof, whereas in the Paracompact-Partition of Unity Theorem 10.9 we took an indirect approach via Urysohn's Lemma 8.4.

Proof. Let $\{\varphi_j\colon X \to [0,1]\}_{j\in J}$ be a (smooth) partition of unity subordinate to the given open cover $\{U_i\}_{i\in I}$ of the topological space X (smooth manifold X). For each $j \in J$ there exists therefore an $i(j) \in I$ with $\operatorname{supp}(\varphi_j) \subset U_{i(j)}$. Given $i \in I$ we set

$$\begin{array}{rcl} \psi_i\colon X & \to & [0,1] \\ x & \mapsto & \displaystyle\sum_{j\in J \text{ with } i(j)=i} \varphi_j(x). \end{array}$$

Note that ψ_i is locally a finite sum of (smooth) maps, hence the map ψ_i is locally continuous (locally smooth), which implies by the Pasting Proposition 2.6 (respectively the Smooth Pasting Proposition 19.23) that ψ_i is continuous (respectively smooth). Evidently the maps ψ_i add up to 1. Thus it remains to prove the following claim.

Claim. For each $i \in I$ we have $\operatorname{supp}(\psi_i) \subset U_i$.

Proof. Let $i \in I$. The claim follows from the following slightly delicate steps:

(1) By the definition of a partition of unity, see page 360, we know that the family $\{\varphi_j^{-1}((0,1])\}_{j\in J}$ is locally finite.
(2) It follows easily from (1) and the Closure-of-Locally Finite Cover Lemma 10.6, that the family $\{\operatorname{supp}(\varphi_j)\}_{j\in J}$ is also locally finite.
(3) It follows from (2) and Lemma 1.11 (3b) that $\displaystyle\bigcup_{j\in J \text{ with } i(j)=i} \operatorname{supp}(\varphi_j)$ is a closed subset of X.
(4) By definition of ψ_i we know that $\psi_i^{-1}((0,1])$ is contained in $\displaystyle\bigcup_{j\in J \text{ with } i(j)=i} \operatorname{supp}(\varphi_j)$.

Now we obtain the following inclusions:

$$\operatorname{supp}(\psi_i) \;=\; \overline{\psi_i^{-1}((0,1])} \;\subset\; \bigcup_{j\in J \text{ with } i(j)=i} \underbrace{\operatorname{supp}(\varphi_j)}_{\subset U_{i(j)}=U_i} \;\subset\; U_i.$$

$\uparrow$ follows from (3) and (4)

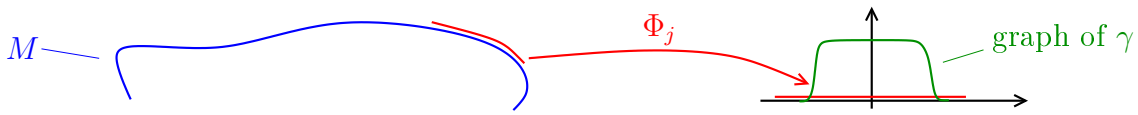
■

Now we can finally provide the proof of the Manifold-Partitions of Unity Existence Theorem 21.1.

Proof of the Manifold-Partitions of Unity Existence Theorem 21.1. Let M be a topological or a smooth manifold with a given open cover $\{U_i\}_{i\in I}$ of M. We apply the Atlas Improvement Proposition 21.3 and we obtain a (smooth) atlas $\{\Phi_j\colon V_j \to W_j\}_{j\in J}$ with Properties (1)–(6). Note that according to the Smooth Transition Function Lemma 19.4 there exists a smooth function $\gamma\colon B^n \to [0,1]$ such that $\gamma(x) = 1$ whenever $\|x\| \leq \frac{1}{2}$ and such that $\gamma(x) = 0$ whenever $\|x\| > \frac{3}{4}$. For each $j \in J$ we consider the function

$$\begin{array}{rcl} g_j\colon M & \to & [0,1] \\ x & \mapsto & \begin{cases} \gamma(\Phi_j(x)), & \text{if } x \in V_j, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

We make the following observations:



(a) Let $j \in J$. Note that the restriction of g_j to the open sets V_j and $M \setminus \Phi_j^{-1}(\overline{B}^n_{\frac{3}{4}})$ is continuous, hence g_j is continuous by the Pasting Proposition 2.6 (2). Using Property (6) of the atlas and using the Smooth Pasting Proposition 19.23 (1) we also see that the g_j are smooth, if we start out with a smooth manifold.
(b) It follows from Property (4) of the atlas that given any $x \in M$ there exists an open neighborhood W of x such that $g_j(x) = 0$ for all but finitely many $j \in J$.
(c) It follows from Property (2) of the atlas that for each $j \in J$ there exists an $i \in I$ with $\operatorname{supp}(g_j) \subset U_i$.
(d) It follows from Property (5) of the atlas that for each $x \in M$ there exists a $j \in J$ with $g_j(x) \neq 0$.

It follows from (b) that it makes sense to consider the function $G\colon M \to \mathbb{R}_{\geq 0}$ that is given by $G(x) := \sum_{j \in J} g_j(x)$. By (d) we know that G is non-zero everywhere. Furthermore, in the setting of a smooth manifold this function is smooth. It is now clear that the functions $x \mapsto \varphi_j(x) := \frac{g_j(x)}{G(x)}$ are a (smooth) partition of unity subordinate to the given open cover.

It follows from this discussion together with Lemma 21.4 that there exists a (smooth) partition of unity $\{\psi_i\colon X \to [0,1]\}_{i \in I}$ such that for every $i \in I$ we have $\operatorname{supp}(\psi_i) \subset U_i$. ■

21.3. **Paracompactness and second-countability.** On page 511 we defined an n-dimensional topological manifold as a topological space that satisfies the following three conditions:

(a) X is Hausdorff,
(b) X is locally homeomorphic to an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$,
(c) X is second-countable.

We now consider also the following two properties of topological spaces:

(d) X is paracompact,
(e) X is the union of countably many compact subsets.

In some references, see e.g. [**Hu1969, Aub2001**] and [**PoR2021**], a topological manifold is defined as a topological space that satisfies (a), (b) and (d). Furthermore in [**Wall2016**] a topological manifold is defined as a topological space that satisfies (a), (b) and (e). Our goal in this section is to show that if a topological space already satisfies (a) and (b), and if X is connected, then (c) is equivalent to (d) and also to (e).

In fact in the Manifold Paracompact-Normal Proposition 18.9 (1) we just saw that every topological manifold is paracompact. Furthermore in in the Topological Manifold Exhaustion Lemma 18.12 we showed in particular that every topological manifold satisfies (e). The following two results show that a connected space that satisfies (a), (b) and (d) respectively (e), also satisfies (a), (b) and (c).

We will not really make use of these results, except that it allows us to quote results from other resources without inhibition and with complete easy of mind. First we have

the following proposition which can be viewed as a converse to the Manifold-Partitions of Unity Existence Theorem 21.1.

Proposition 21.5. Let $n \in \mathbb{N}$ and let X be a topological space that is Hausdorff, connected and that is locally homeomorphic to an open subset of $\mathbb{R}^n$ or to an open subset of $\mathbb{H}^n$. If X is paracompact, then it is also second-countable.

The proof of Proposition 21.5 rests on the following lemma, which can be viewed as the promised partial converse to the Topological Manifold Exhaustion Lemma 18.12.

Lemma 21.6. Let $n \in \mathbb{N}$ and let X be a topological space that is locally homeomorphic to an open subset of $\mathbb{R}^n$ or to an open subset of $\mathbb{H}^n$. If X can be written as a union of countably many compact subsets, then X is second-countable.

Proof of Lemma 21.6. Let $\{K_i\}_{i\in I}$ be a countable family of compact subsets of X that cover X.

Claim. Each K_i is contained in an open subset V_i that is second-countable.

Proof. Let $i \in I$. Since K_i is compact and since X is locally homeomorphic to an open subset of $\mathbb{R}^n$ or to an open subset of $\mathbb{H}^n$ we see that K_i can be written as the union of finitely many sets $A_1, \ldots, A_n$ each of which is contained in an open subset U_i that is homeomorphic to a subset of $\mathbb{R}^n$ or $\mathbb{H}^n$. By the Second Countability Lemma 9.1 (1) and (2) we know that each U_i is second-countable and by the Second Countability Lemma 9.1 (4) we know that $V_i := U_1 \cup \cdots \cup U_n$ is second-countable. ⊞

It follows from the claim that X is the union of countably many second-countable open subsets. Thus we obtain from the Second Countability Lemma 9.1 (4) that X itself is second-countable. ■

Proof of Proposition 21.5. Let X be a topological space that is Hausdorff and connected and that is furthermore paracompact. To simplify the notation we only deal with the case that X is locally homeomorphic to an open subset of $\mathbb{R}^n$. The proof of the general case is almost identical. The goal is to show, using Lemma 21.6, that X is second-countable.

Claim 1. There exists a locally finite open cover $\mathcal{W}$ of X by subsets such that the closure of each $W \in \mathcal{W}$ is compact.

Proof. Note that by hypothesis for each $a \in X$ there exists an open neighborhood U_a and a homeomorphism $\Phi_a \colon U_a \to V_a$ where V_a is an open subset of $\mathbb{R}^n$. We pick an $r > 0$ such that $\overline{B}_r^n(\Phi_a(a)) \subset V_a$ and we set $W_a := \Phi_a^{-1}(B_r^n(a))$. These sets form an open cover of X. Note that it follows from our hypothesis that X is Hausdorff, from our choice of r, and from Lemma 18.11 that the closure of W_a is compact. Since X is paracompact there exists a locally finite refinement $\mathcal{W}$ of the open cover $\{W_a\}_{a\in X}$. This refinement has all the desired properties. ⊞

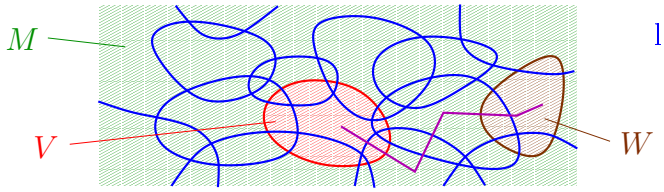
We pick a non-empty $V \in \mathcal{W}$.

Claim 2. Given any $W \in \mathcal{W}$ there exists a $k \in \mathbb{N}$ and sets $V = W_1, W_2, \ldots, W_k = W$ in $\mathcal{W}$ such that for each $i \in \{1, \ldots, k-1\}$ we have $W_i \cap W_{i+1} \neq \varnothing$.

Proof. We consider

$$U := \left\{ P \in M \,\middle|\, \begin{array}{c} \text{there exist sets } V = W_1, W_2, \ldots, W_k = W \text{ in } \mathcal{W} \text{ such} \\ \text{that each } W_i \cap W_{i+1} \neq \varnothing \text{ and such that } P \in W_k \end{array} \right\}.$$

Evidently $V \subset U$, thus U is non-empty. Since M is connected it remains to show that U is open and closed. First let $P \in U$. We pick a sequence $V = W_1, W_2, \ldots, W_k = W$ in $\mathcal{W}$ as in the definition of U. Evidently $P \in W_k \subset U$. This shows that U is open. Conversely, let $P \notin U$. We pick $W \in \mathcal{W}$ with $P \in W$. If $U \cap W$ was non-empty, it would follow basically immediately from the definition of U that we have $W \subset U$ and thus also $P \in U$, which is impossible. This shows that U is closed. ⊞



Given $W \in \mathcal{W}$ we define

$$\varphi(W) := \text{minimal } k \in \mathbb{N} \text{ such that there exists a sequence as in Claim 2}$$

and given $k \in \mathbb{N}$ we set

$$Z_k := \overline{\bigcup_{W \in \mathcal{W} \text{ with } \varphi(W) \leq k} W}.$$

Claim 3.
(1) For each $k \in \mathbb{N}$ there exists only finitely many $W \in \mathcal{W}$ with $\varphi(W) \leq k$.
(2) For each $k \in \mathbb{N}$ the set Z_k is compact.

Proof. First we make the following observation. If for some $k \in \mathbb{N}$ there exist only finitely many $W \in \mathcal{W}$ with $\varphi(W) \leq k$, then we obtain from Lemmas 1.20, 1.11 and 1.21 that Z_k is compact.

Thus it suffices to show Statement (1) of the claim. We prove this statement by induction. First let us deal with the case $k = 1$. Note that $\varphi(W) = 1$ if and only if $W = V$. Thus we see that $\varphi(1) = 1$. Now assume that the statement holds for some $k \in \mathbb{N}$.

Since $\mathcal{W}$ is locally finite there exists for each $z \in Z_k$ an open neighborhood U_z which has non-empty intersection with at most finitely many sets in $\mathcal{W}$. By our induction hypothesis and the above observation we know that Z_k is compact. Hence we can cover Z_k by finitely many of such U_z. But this implies that Z_k has non-empty intersection only with finitely many $W \in \mathcal{W}$. By definition we know that each $W \in \mathcal{W}$ with $\varphi(W) = k + 1$ satisfies $W \cap Z_k \neq \varnothing$. Thus we have shown that there are only finitely many $W \in \mathcal{W}$ with $\varphi(W) \leq k + 1$. ⊞

It follows from Claim 2 and Claim 3 that X is the union of countably many compact subsets. It follows from Lemma 21.6 that X is indeed second-countable. ■

21.4. **Smooth extensions and approximations.** In this section we will see that partitions of unity can be very helpful for constructing maps on manifolds. We start out with the following variation on Urysohn's Lemma 8.4 for topological and smooth manifolds.

Lemma 21.7. (Urysohn's Lemma for Manifolds) Let M be a smooth manifold and let A and B be two disjoint closed subsets of M.

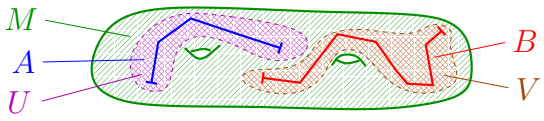
(1) There exists a map $f\colon M \to [0,1]$ with $f|_A \equiv 0$ and $f|_B \equiv 1$.
(2) If M is in fact a smooth manifold, then we can arrange in (1) that f is smooth.

Proof. Let M be a topological manifold. By the Manifold Paracompact-Normal Proposition 18.9 (2) we know that M is normal. Thus Statement (1) is an immediate consequence of Urysohn's Lemma 8.4.

Note that this approach to proving Statement (1) does not generalize to Statement (2). In the following we will therefore provide a different proof of Statement (1), which does not appeal to Urysohn's Lemma 8.4, and which generalizes to smooth manifolds.

Thus let M be a topological manifold and let A and B be two disjoint closed subsets of M. As we mentioned above, we know by the Manifold Paracompact-Normal Proposition 18.9 (2) that M is normal. By definition this means that there exist open neighborhoods U of A and V of B such that $U \cap V = \varnothing$.

Note that $M \backslash B, V$ is an open cover of M. By the Manifold-Partitions of Unity Existence Theorem 21.1 (1) there exists a corresponding partition of unity f, g with $\operatorname{supp}(f) \subset V$, $\operatorname{supp}(g) \subset M \setminus B$. It follows almost immediately from the various definitions that f has the desired properties.



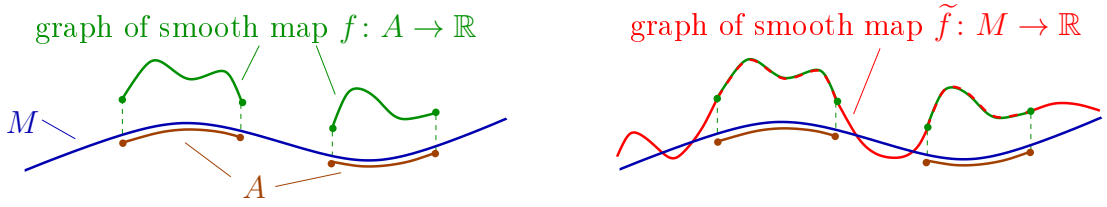
Now assume that M is in fact a smooth manifold. We use the same argument as above. The only difference is that this time we use the Manifold-Partitions of Unity Existence Theorem 21.1 (2). ■

Proposition 21.8. (Continuous Extension Proposition) Let M be a topological manifold, let $A \subset M$ be a closed subset and let $f\colon A \to \mathbb{R}^n$ be a map. Given any open neighborhood U of A there exists a smooth map $\widetilde{f}\colon M \to \mathbb{R}^n$ such that $\widetilde{f}|_A = f$ and such that $\operatorname{supp}(\widetilde{f}) \subset U$.

Proof. The Continuous Extension Proposition 21.8 is an immediate consequence of the Manifold Paracompact-Normal Proposition 18.9 (2), Lemma 21.7 and the Tietze Extension Theorem 8.6. ■

In the remainder of this section we will use partitions of unity to construct useful smooth functions on smooth manifolds. First we have the smooth analogue of the Continuous Extension Proposition 21.8.

Proposition 21.9. (Smooth Extension Proposition) Let M be a smooth manifold, let $A \subset M$ be a closed subset and let $f\colon A \to \mathbb{R}^n$ be a smooth map. Given any open neighborhood U of A there exists a smooth map $\widetilde{f}\colon M \to \mathbb{R}^n$ such that $\widetilde{f}|_A = f$ and such that $\operatorname{supp}(\widetilde{f}) \subset U$.



Proof. It follows almost immediately from the definition of a smooth map $f\colon A \to \mathbb{R}^n$ such that for each $P \in A$ there exists an open neighborhood $W_P \subset U$ and a smooth map $\widetilde{f}_P\colon W_P \to \mathbb{R}^n$ that agrees with f on $W_P \cap A$.

Since A is closed we see that the sets $\{W_P\}_{P\in A}$ and $M \setminus A$ form an *open* cover of M. By the Manifold-Partitions of Unity Existence Theorem 21.1 there exists a smooth partition of unity $\{\varphi_P\}_{P\in M}$ and ψ such that for each $P \in M$ we have $\operatorname{supp}(\varphi_P) \subset W_P$ and such that $\operatorname{supp}(\psi) \subset M \setminus A$. In particular we have $\psi|_A \equiv 0$.



Given $P \in M$ we set

$$\begin{array}{rcl} \Theta_P\colon M & \to & \mathbb{R}^n \\ x & \mapsto & \begin{cases} \varphi_P(x)\cdot \widetilde{f}(P), & \text{if } x \in W_P, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

It follows almost immediately from the Smooth Pasting Proposition 19.23 that Θ_P is smooth. We consider the map

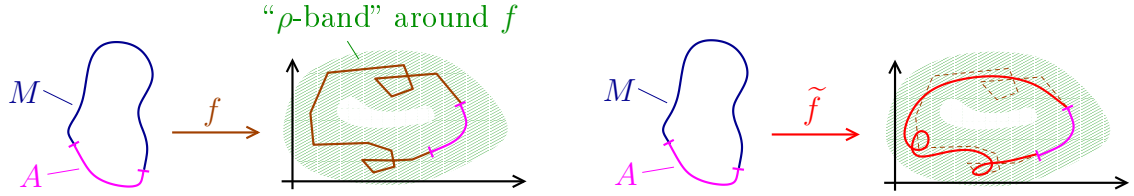
$$\begin{array}{rcl} \widetilde{f}\colon M & \to & \mathbb{R}^n \\ x & \mapsto & \displaystyle\sum_{P\in A} \Theta_P(x). \end{array}$$

It follows easily from $\psi|_A \equiv 0$ and the fact that we are dealing with a partition of unity that $\widetilde{f}$ agrees with f on A. Furthermore it follows easily from Lemma 1.11 that $\operatorname{supp}(\widetilde{f}) \subset U$. ■

Proposition 21.10. (Smooth Approximation Proposition) Let M be a smooth manifold and let $f\colon M \to \mathbb{R}^n$ be a map that is smooth on a (possibly empty) closed subset A. Given any map[233] $\rho\colon M \to \mathbb{R}_{\geq 0}$ with $\rho(x) > 0$ for all $x \notin A$ there exists a *smooth* map $\widetilde{f}\colon M \to \mathbb{R}^n$ with the following two properties:

(1) The maps f and $\widetilde{f}$ agree on A.

(2) For every $x \in M$ we have $\|f(x) - \widetilde{f}(x)\| \leq \rho(x)$.



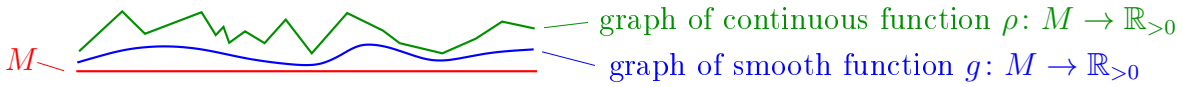
[233] Evidently we demand, as always, that ρ is continuous.

Remark. In the Whitney Approximation Theorem 29.1 we will generalize the above proposition from the target $\mathbb{R}^n$ to any smooth manifold N.

Before we turn to the proof of the Smooth Approximation Proposition 21.10 let us formulate two useful corollary.

Corollary 21.11. (Replace-by-Smooth Non-Negative Function Corollary) Let M be a smooth manifold.

(1) For every map $\rho\colon M \to \mathbb{R}_{>0}$ there exists a smooth map $g\colon M \to \mathbb{R}_{>0}$ such that $g(x) \in (0, \rho(x)]$ for all $x \in M$.
(2) Let $f\colon M \to \mathbb{R}$ be a map (not necessarily continuous) with the property that for every $x \in M$ there exists an open neighborhood U_x of x and an $\epsilon_x > 0$ such that for each $y \in U_x$ we have $f(y) \geq \epsilon_x$. Then there exists a smooth map $g\colon M \to (0, \infty)$, such that for all $x \in M$ we have $g(x) \leq f(x)$.



Proof of the Replace-by-Smooth Non-Negative Function Corollary 21.11.

(1) This statement follows immediately from Smooth Approximation Proposition 21.10 applied to the zero function $f \equiv 0$, to $A = \varnothing$ and to the given map $\rho\colon M \to \mathbb{R}_{>0}$.
(2) In the Manifold Paracompact-Normal Proposition 18.9 (1) we saw that topological manifolds are paracompact. Thus Statement (2) follows immediately from Lemma 10.12 together with Statement (1). ■

Proof of the Smooth Approximation Proposition 21.10. By Proposition 21.9 there exists a smooth map $g\colon M \to \mathbb{R}^n$ that agrees with f on A. We consider the open[234] subset

$$V \;:=\; \{y \in M \,|\, |g(y) - f(y)| < \rho(y)\}.$$

Claim. There exists a countable set $I \subset M \setminus A$ together with an open neighborhood $U_P \subset M \setminus A$ for each $P \in I$ such that $\bigcup_{P \in I} U_P = M \setminus A$ and such that for every $y \in U_P$ we have

$$|f(y) - f(P)| \;<\; \rho(y).$$

Proof. Let $x \in M \setminus A$. Since $\rho(x) > 0$ we obtain from the continuity of ρ and f that there exists an open neighborhood U_x of x in the open set $M \setminus A$ such that for every $y \in U_x$ we have

$$|\rho(y) - \rho(x)| \;\leq\; \tfrac{1}{2} \cdot \rho(x) \qquad \text{and} \qquad |f(y) - f(x)| \;<\; \tfrac{1}{2} \cdot \rho(x).$$

It follows from the triangle inequality that given any $y \in U_x$ we have $|f(y) - f(x)| < \rho(y)$. It follows from the Second Countability Lemma 9.1 (2) and the Countable Covering Lemma 9.2 that there exists a countable $I \subset M \setminus A$ such that $M \setminus A$ is covered by the U_P. ⊞

[234]Note that $|g(y) - f(y)| < \rho(y)$ if and only if $|g(y) - f(y)| - \rho(y) \in (-\infty, 0)$. Using this observation it is indeed clear that $V \subset M$ is open.

By the Manifold-Partitions of Unity Existence Theorem 21.1 there exists a smooth partition of unity $\{\varphi_P\}_{P\in I}, \psi$ such that for each $P \in I$ we have $\operatorname{supp}(\varphi_P) \subset U_P$ and such that $\operatorname{supp}(\psi) \subset V$. We consider the map

$$\begin{aligned} \widetilde{f}\colon M &\to \mathbb{R}^n \\ x &\mapsto \psi(x)\cdot g(x) + \sum_{P\in I} \varphi_P(x)\cdot f(P). \end{aligned}$$

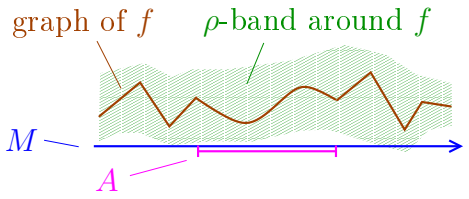
Note that $\widetilde{f}$ is smooth and that $\widetilde{f}$ agrees with f on A. It remains to prove the following claim.

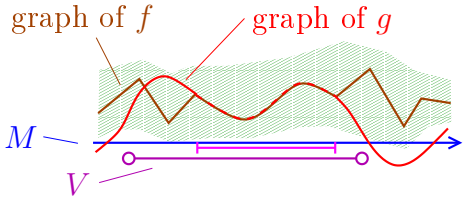
Claim. For every $x \in M$ we have $\|f(x) - \widetilde{f}(x)\| \le \rho(x)$.

Proof. Let $x \in M$. We perform the following calculation:

$$\begin{aligned} \|f(x) - \widetilde{f}(x)\| &\overset{\text{since } \psi+\sum\varphi_P\equiv 1}{=} \left\| \left(\psi(x) + \sum_{P\in I}\varphi_P(x)\right)\cdot f(x) - \left(\psi(x)\cdot g(x) + \sum_{P\in I}\varphi_P(x)\cdot f(P)\right)\right\| \\ &\le \psi(x)\cdot \underbrace{|f(x)-g(x)|}_{\le \rho(x) \text{ if } \psi(x)\neq 0} + \sum_{P\in I}\varphi_P(x)\cdot \underbrace{|f(x)-f(P)|}_{\le \rho(x) \text{ if } \varphi_P(x)\neq 0} \\ &\le \psi(x)\cdot\rho(x) + \sum_{P\in I}\varphi_P(x)\cdot\rho(x) \overset{\text{since } \psi+\sum\varphi_P\equiv 1}{=} \rho(x). \end{aligned}$$

■





Let M be a smooth manifold. For any function $f\colon M \to \mathbb{R}_{\ge 0}$ the preimage $f^{-1}(0)$ is of course a closed set. Rather surprisingly the following proposition shows that every closed subset of M can be realized as the zero set of a smooth function.

Proposition 21.12. (Zero Set-Realization Proposition) Let M be a smooth manifold. Given any closed subset $C \subset M$ there exists a smooth function $f\colon M \to \mathbb{R}_{\ge 0}$ such that $C = f^{-1}(0)$.

Proof. Let M be a smooth manifold and let $C \subset M$ be a closed subset. Recall that by the Manifold Metrization Proposition 18.10 we know that M is metrizable, i.e. there exists a metric d such that the metric topology agrees with the given topology.

If C is the empty set, then we take f to be the constant function $= 1$. Now assume that C is non-empty. This allows us to consider the map

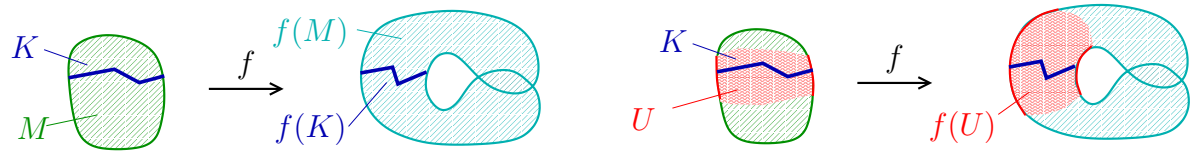
$$\begin{aligned} g\colon M &\to \mathbb{R}_{\ge 0} \\ x &\mapsto d(x,C) := \inf\{d(x,c) \mid b \in C\}. \end{aligned}$$

By the Distance-to-Subspace Lemma 6.2 we know that the map is continuous and, using that C is a closed subset, we know that $g(x) = 0$ if and only if $x \in C$.

Note that g is constant $= 0$ on C. In particular it is a smooth function on C. By the Smooth Approximation Proposition 21.10 we know that there exists a smooth map $f\colon M \to \mathbb{R}_{\geq 0}$ such that for any $x \in M$ we have $|f(x) - g(x)| \leq \frac{g(x)}{2}$. Clearly this map has the desired properties. ■

We conclude this chapter with the following useful theorem.

Theorem 21.13. (Neighborhood Smooth Embedding Theorem) Let $f\colon M \to N$ be a map between smooth manifolds. Furthermore let $K \subset M$ be a closed subset such that the map $f\colon K \to f(K)$ is a homeomorphism and such that $f(K)$ is a closed subset of N. If f is a local diffeomorphism at each $P \in K$, then there exists an open neighborhood U of K such that the restriction of f to U is a smooth embedding and such that $f(U)$ is an open subset of N.



Proof. By the Topological Manifolds-Local Properties Lemma 18.8 we know that M is regionally compact. Furthermore, by the Manifold Paracompact-Normal Proposition 18.9 (1) we know that N is paracompact. The proof of the Neighborhood Smooth Embedding Theorem 21.13 is now almost identical to the proof of the Neighborhood Embedding Theorem 10.13. All we have to do is to replace the Pasting Proposition 2.6 by the Smooth Pasting Proposition 19.23. ■

Exercises for Chapter 21.

Exercise 21.1. Provide an example of a connected space X that does not admit an exhaustion function $f\colon X \to \mathbb{R}_{\geq 0}$.

Exercise 21.2. Give an example of two topological manifolds M and N, a closed subset $A \subset M$ and a map $f\colon A \to N$ such that f cannot be extended to a map $\widetilde{f}\colon M \to N$.
Remark. This exercise shows that the Continuous Extension Proposition 21.8 cannot be extended from $N = \mathbb{R}^n$ to the completely general setting.

Exercise 21.3. Let $A \subset \mathbb{R}$ be a subset which is *not* closed.

(a) Show that there exists a smooth map $f\colon A \to \mathbb{R}$ which cannot be extended to a smooth map on all of $\mathbb{R}$.

(b) Show that there exists a continuous map $f\colon \mathbb{R} \to \mathbb{R}$ which is smooth on A but for which there does not exist a smooth map $\widetilde{f}\colon \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ we have $|f(x) - \widetilde{f}(x)| \leq 1$.

Remark. This exercise shows that in the Smooth Extension Proposition 21.9 and the Smooth Approximation Proposition 21.10 the hypothesis that A is closed is absolutely essential.

22. 1-dimensional manifolds and the long line

This chapter can be seen as a short break from developing the general theory of smooth manifolds. In this chapter we will only consider "1-dimensional objects". More precisely, we will do the following:

(1) In the Classification Theorems 22.1 and 22.7 we will give the classification of connected 1-dimensional topological respectively smooth manifolds.
(2) We will construct the *long line*. As we will see in Theorem 22.13, this is a path-connected topological space that satisfies all properties of a 1-dimensional topological manifold, except that it is not second-countable.

We will at times use these classification Theorems 22.1 and 22.7. But the proofs are somewhat technical and do not make for particularly leisurely reading. Thus the reader might be best served by having a quick glance at the, not very surprising, classification theorems and by moving on to the next chapter.

22.1. The classification of 1-dimensional topological manifolds.

One of the goals of topology is to completely classify suitable classes of topological spaces up to homeomorphism. For example one could try to classify all connected topological and smooth manifolds of a given dimension. In this section we give a complete classification in the 1-dimensional setting.

In the following we first deal with 1-dimensional *topological* manifolds.

Example. Some of the main examples of 1-dimensional topological manifolds are S^1, $\mathbb{R}$, the closed interval $[0,1]$ and the half-open interval $[0,1)$.

(1) Note that for S^1 and $\mathbb{R}$ we found atlases consisting only of charts of type (i), i.e. the boundary of S^1 and of $\mathbb{R}$ is empty. On the other hand, as we pointed out on page 514, it follows easily from Lemma 7.11 together with the Topological Valence Determination Proposition 7.12 that the boundary of $[0,1]$ equals $\{0,1\}$ and that the boundary of $[0,1)$ equals $\{0\}$.
(2) By the Heine-Borel Theorem 1.24 we know that S^1 and $[0,1]$ are compact, whereas $\mathbb{R}$ and $[0,1)$ are non-compact.

This discussion shows in particular that S^1, $\mathbb{R}$, $[0,1]$ and $[0,1)$ are pairwise non-homeomorphic.

The following theorem shows that the above list of 1-dimensional topological manifolds is in a suitable sense complete.

Theorem 22.1. (Topological 1-Dimensional Manifold Classification Theorem) Let M be a connected non-empty 1-dimensional topological manifold. Then the homeomorphism type of M is given by the following table

	without boundary	with non-empty boundary
compact	the circle S^1	the closed interval $[0,1]$
non-compact	the real line $\mathbb{R}$	the half-open interval $[0,1)$.

Remark. Note that the adjectives "Hausdorff" and "second-countable" in the definition of a topological manifold are both necessary:

(1) The line with two zeros, which we introduced on page 122, has all the properties of a 1-dimensional topological manifold except for the Hausdorff axiom.
(2) If one drops the condition that a topological manifold has to be second-countable, then there are two more examples of connected 1-dimensional topological manifolds, namely the "open long ray $\mathbb{L}^+$ and the "long line $\mathbb{L}$" which we will introduce on page 613.

Remark.

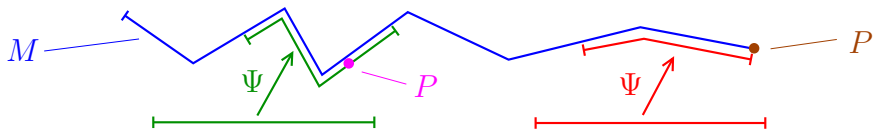
(1) Many proofs of the Topological 1-Dimensional Manifold Classification Theorem 22.1 exist in the literature, see e.g. [**Gal1987**], [**Lee2000**, Theorem 6.1] and [**FuR1984**, Sections 3.1.1.16-19].
(2) We will classify 2-dimensional topological manifolds in Theorem 55.4. On the other hand we will see in Theorem 60.4 that closed smooth manifolds of dimension ≥ 4 cannot be classified.

The proof of the Topological 1-Dimensional Manifold Classification Theorem 22.1 requires several preparations. We start out with the following definition.

Definition. Let M be a topological space.

(1) A c-parametrization for M is a homeomorphism $\Psi\colon [r,s] \to C$ from a compact interval $[r,s]$ with $r < s$ to a closed subset C of M such that $\Psi((r,s))$ is an open subset of M.
(2) A c-atlas is a family of c-parametrizations $\{\Psi_i\colon [r_i,s_i] \to C_i\}_{i\in I}$ of M such that the union of the C_i's equals M.

Lemma 22.2. Let M be a 1-dimensional topological manifold. Given any $P \in M$ and given any neighborhood W of P there exists a c-parametrization $\Psi\colon [r,s] \to C$ such that C is a neighborhood of P and such that $C \subset W$.

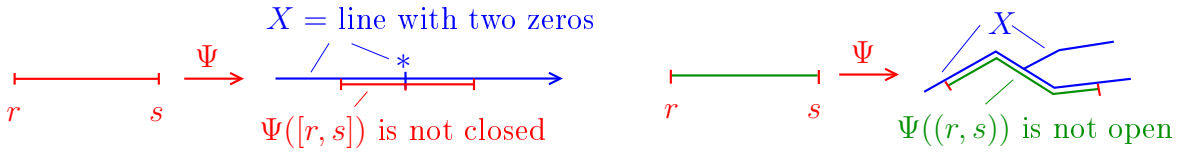


Proof. We distinguish the following cases.

- If $P \in M \setminus \partial M$, then we pick a chart $\Phi\colon U \to (-2,2)$ of type (i) with $\Phi(P) = 0$ and with $U \subset W$. We write $V = \Phi^{-1}((-1,1))$, $I = [-1,1]$ and $C = \Phi^{-1}([-1,1])$.
- If $P \in \partial M$, then we pick a chart $\Phi\colon U \to [0,2)$ of type (ii) with $\Phi(P) = 0$ and with $U \subset W$. We write $V = \Phi^{-1}([0,1))$, $I = [0,1]$ and $C = \Phi^{-1}([0,1])$.

It follows from Lemma 18.11 that C is closed in M. (Note that here we use that topological manifolds are Hausdorff.) Also note that it follows from the Open -Closed Inclusion Lemma 2.10 that V is open in M and that V is a neighborhood of P. Thus the map $\Phi^{-1}\colon I \to C$ has all the desired properties. ∎

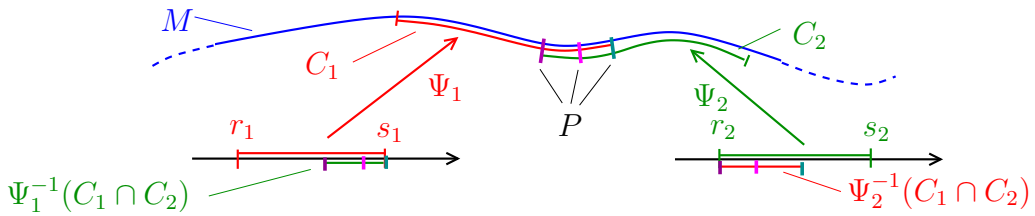
Example. In the figure below we also show two "non-examples" of c-parametrizations for wannabe 1-dimensional topological manifolds. In the situation to the left the image of the map $\Psi\colon [r,s] \to X$ is not closed whereas in the situation to the right $\Psi((r,s))$ is not an open subset of X.



In the following lemmas we study the "interaction" between two c-parametrizations whose images intersect.

Lemma 22.3. Let M be a topological space and let $\Psi_1\colon [r_1,s_1] \to C_1$ and $\Psi_2\colon [r_2,s_2] \to C_2$ be two c-parametrizations. Let $P \in C_1 \cap C_2$ and let $i \in \{1,2\}$. If $\Psi_i^{-1}(P)$ is not an endpoint of $[r_i,s_i]$, then one of the following happens:

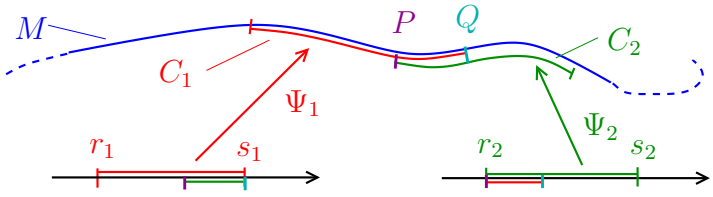
(1) $P \in \Psi_i((r_i,s_i)) \cap \Psi_{3-i}((r_{3-i},s_{3-i}))$, or
(2) $\Psi_{3-i}^{-1}(P)$ is an endpoint of $[r_{3-i},s_{3-i}]$.



Proof. Let $P \in C_i \cap C_{3-i}$ with $\Psi_i^{-1}(P) \in (r_i,s_i)$. If $\Psi_{3-i}^{-1}(P)$ is an endpoint of $[r_{3-i},s_{3-i}]$, then we are done. Now suppose that $\Psi_{3-i}^{-1}(P)$ is not an endpoint of $[r_{3-i},s_{3-i}]$, i.e. we suppose that $\Psi_{3-i}^{-1}(P) \in (r_{3-i},s_{3-i})$. But this means that $P \in \Psi_i((r_i,s_i)) \cap \Psi_{3-i}((r_{3-i},s_{3-i}))$. ■

Lemma 22.4. Let M be a topological space and let $\Psi_1\colon [r_1,s_1] \to C_1$ and $\Psi_2\colon [r_2,s_2] \to C_2$ be two c-parametrizations.

(1) For every component X of $C_1 \cap C_2$ there exist points $P,Q \in X$ (where we allow $P=Q$) such that for $i=1,2$ the following two statements hold:
  (a) The preimage $\Psi_i^{-1}(X)$ is the closed interval with endpoints $\Psi_i^{-1}(P)$ and $\Psi_i^{-1}(Q)$.
  (b) If $\Psi_i^{-1}(P)$ is not an endpoint of $[r_i,s_i]$, then $\Psi_{3-i}^{-1}(P)$ is an endpoint of $[r_{3-i},s_{3-i}]$. The same statement also holds for Q.
(2) If X is a component of $C_1 \cap C_2$ such that $\Psi_i^{-1}(X)$ is contained in (r_i,s_i), then C_{3-i} is contained in C_i.
(3) $C_1 \cap C_2$ has at most two components.



Proof. Let X be a component of $C_1 \cap C_2$. By the Component Lemma 2.33 we know that X is connected. Furthermore, since C_1 and C_2 are closed in M we know that $C_1 \cap C_2$ is closed in M and by the Components-are-Closed Lemma 2.34 this implies that the component X is also a closed subset of M.

We consider $W_1 = \Psi_1^{-1}(X)$. By the above discussion we know that W_1 is a closed connected subset of the interval $[r_1, s_1]$. Since W_1 is a connected subset of the interval $[r_1, s_1]$ we obtain from Exercise 2.18 that W_1 is closed interval contained in $[r_1, s_1]$. In other words $W_1 = [a_1, b_1]$ for some $r_1 \leq a_1 \leq b_1 \leq s_1$.

Claim. The points $P := \Psi_1(a_1)$ and $Q := \Psi_1(b_1)$ have the desired properties stated in (1).

Proof. We write $a_2 = \Psi_2^{-1}(P)$ and $b_2 = \Psi_2^{-1}(Q)$ and we set $\Theta := \Psi_2^{-1} \circ \Psi_1$. Note that $\Theta(a_1) = a_2$ and $\Theta(b_1) = b_2$. It follows from an elementary argument, see Exercise 2.6, that Θ sends the interval $[a_1, b_1]$ homeomorphically to the interval with endpoints a_2 and b_2.

(1a) It follows from the above that (1a) is satisfied for $i = 1, 2$.

(1b) Suppose that $a_i = \Psi_i^{-1}(P)$ is not an endpoint of $[r_i, s_i]$. Since $a_i \notin (r_i, s_i)$ we see that $P = \Psi_i(a_i) \notin \Psi_i((r_i, s_i)) \cap \Psi_{3-i}((r_{3-i}, s_{3-i}))$. It follows from Lemma 22.3 that $a_{3-i} = \Psi_{3-i}^{-1}(P)$ is an endpoint of $[r_{3-i}, s_{3-i}]$. ⊞

It remains to prove the remaining two statements of the lemma.

(2) We continue with the above notation. If neither a_i nor b_i is an endpoint of $[r_i, s_i]$, then it follows from the above that a_{3-i} and b_{3-i} are both endpoints of $[r_{3-i}, s_{3-i}]$. But this means that the interval given by a_{3-i} and b_{3-i} equals $[r_{3-i}, s_{3-i}]$, which implies that $C_{3-i} \subset C_i$.

(3) By (1) we know that each component of $C_1 \cap C_2$ contains at least two endpoints of the intervals $[r_1, s_1]$ and $[r_2, s_2]$. Since there are only four endpoints we see that there is only enough room for at most two components. ■

Lemma 22.5. Let M be a 1-dimensional topological manifold.[235] Let $\Psi_1 \colon [r_1, s_1] \to C_1$ and $\Psi_2 \colon [r_2, s_2] \to C_2$ be two c-parametrizations. If $C_1 \cap C_2$ has one component, then there exist $\widetilde{r} \leq r_1 < s_1 \leq \widetilde{s}$ and a c-parametrization

$$\widetilde{\Psi} \colon [\widetilde{r}, \widetilde{s}] \ \to \ C_1 \cup C_2$$

such that $\widetilde{\Psi}|_{[r_1, s_1]} = \Psi_1$.

Proof of Lemma 22.5 if C_1 is not contained in C_2. First we deal with the case that C_1 is not contained in C_2. Since $C_1 \cap C_2$ consists of one component and since C_1 is not contained in C_2 we obtain from Lemma 22.4 that $\Psi_1^{-1}(C_1 \cap C_2)$ contains precisely one endpoint of $[r_1, s_1]$. We will now deal with the case that the endpoint is s_1. The other case is treated in essentially the same way.

Next note that Lemma 22.4 now says that there exists a $b \in [r_1, s_1]$ with the following properties:

(1) $\Psi_1^{-1}(C_1 \cap C_2) = [b, s_1]$,
(2) $\Psi_2^{-1}(\Psi_1(b))$ is an endpoint of $[r_2, s_2]$.

[235] Note that in contrast to the previous Lemma 22.4 we now actually assume that we are dealing with a 1-dimensional topological manifold.

Note that after possibly replacing $t \mapsto \Psi_2(t)$ by $t \mapsto \Psi_2(r_2 + s_2 - t)$ we can assume that $\Psi_2^{-1}(\Psi_1(b)) = r_2$. (Note that now we are conveniently enough in a setting which corresponds to the situation shown in the figure below.) Next note that Lemma 22.4 now also says that $\Psi_2^{-1}(\Psi_1([b, s_1])) = [r_2, c]$ for some $c \in [r_2, s_2]$.

We consider the map

$$\begin{array}{rcl} \widetilde{\Psi}\colon [r_1, s_1 + (s_2 - c)] & \to & C_1 \cup C_2 \\ x & \mapsto & \begin{cases} \Psi_1(x), & \text{if } x \in [r_1, s_1], \\ \Psi_2(x - s_1 + c), & \text{if } x \in [s_1, s_1 + (s_2 - c)]. \end{cases} \end{array}$$

By construction we have $\widetilde{\Psi}|_{[r_1,s_1]} = \Psi_1$. Thus it suffices to prove the following claim.

Claim. The map $\widetilde{\Psi}$ is a c-parametrization.

Proof. We set $\widetilde{r} = r_1$ and $\widetilde{s} = s_1 + (s_2 - c)$. It follows from the Pasting Proposition 2.6 that $\widetilde{\Psi}\colon [\widetilde{r}, \widetilde{s}] \to C_1 \cup C_2$ is continuous and one can easily verify that $\widetilde{\Psi}\colon [\widetilde{r}, \widetilde{s}] \to C_1 \cup C_2$ is a bijection. Since M is Hausdorff we obtain from the Compact-Hausdorff Proposition 2.17 (3) that $\widetilde{\Psi}\colon [\widetilde{r}, \widetilde{s}] \to C_1 \cup C_2$ is in fact a homeomorphism and that $C_1 \cup C_2$ is a closed subset of M.

It remains to prove that $\widetilde{\Psi}((\widetilde{r}, \widetilde{s}))$ is an open subset of M. We set $P := \Psi_1(s_1)$. Note that

$$\widetilde{\Psi}((\widetilde{r}, \widetilde{s})) \;=\; \underbrace{\Psi_1((r_1, s_1))}_{\substack{\text{open since } \Psi_1 \\ c\text{-parametrization}}} \;\cup\; \underbrace{\Psi_2((r_2, s_2))}_{\substack{\text{open since } \Psi_2 \\ c\text{-parametrization}}} \;\cup\; \{P\} \;\subset\; M.$$

If $P \in \Psi_2((r_2, s_2))$, then we are done. Therefore it remains to consider the slightly delicate case that $P \notin \Psi_2((r_2, s_2))$. Note that this means that $P = \Psi_2(r_2)$. In this case we still need to show that

$$\widetilde{\Psi}((\widetilde{r}, \widetilde{s})) \;=\; \Psi_1((r_1, s_1]) \cup \Psi_2([r_2, s_2))$$

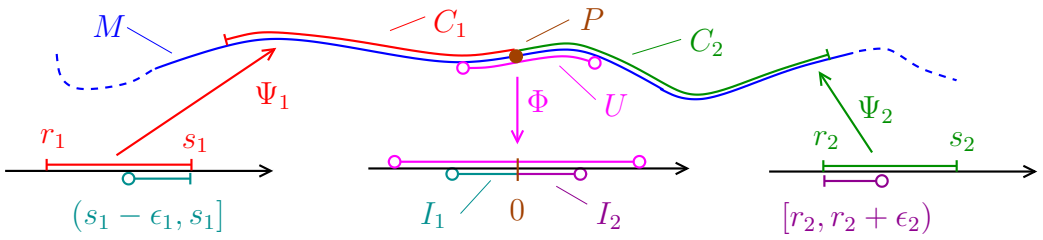
is a neighborhood of P.

(i) First we assume that P admits a chart of type (i). This means that we can pick a chart $\Phi\colon U \to (-1, 1)$ of type (i) around P with $\Phi(P) = 0$.
- Since $\Psi_1^{-1}(U)$ is an open subset of $[r_1, s_1]$ that contains s_1 we see that there exists an $\epsilon_1 > 0$ such that $\Psi_1((s_1 - \epsilon_1, s_1]) \subset U$.
- Similarly we see that there exists an $\epsilon_2 > 0$ such that $\Psi_2([r_2, r_2 + \epsilon_2)) \subset U$.

Note that $I_1 = \Phi(\Psi_1((s_1 - \epsilon_1, s_1]))$ is a half-open interval in $(-1, 1)$ that contains 0 and that $I_2 = \Phi(\Psi_2([r_2, r_2 + \epsilon_2)))$ is a half-open interval in $(-1, 1)$ that contains 0. Furthermore note that $I_1 \cap I_2 = \{0\}$. It follows that $I_1 \cup I_2$ is an open neighborhood of 0. But that implies that $\Psi_1((s_1 - \epsilon_1, s_1]) \cup \Psi_2([r_2, r_2 + \epsilon_2))$ is a neighborhood of P. Note that this implies that $\widetilde{\Psi}((\widetilde{r}, \widetilde{s})) = \Psi_1((r_1, s_1]) \cup \Psi_2([r_2, s_2))$ is a neighborhood of P.

(ii) A slight modification of the above argument also shows that P actually cannot admit a chart of type (ii). ∎

Proof of Lemma 22.5. We now deal with the case that C_1 is contained in C_2. We split C_2 into two c-parametrizations C_2' and C_2'' such that C_1 is contained in neither C_2' nor C_2'' and apply the previous case twice. ∎



Lemma 22.6. Let M be a 1-dimensional topological manifold. Let $\Psi_1\colon [r_1, s_1] \to C_1$ and $\Psi_2\colon [r_2, s_2] \to C_2$ be two c-parametrizations. If $C_1 \cap C_2$ has two components, then there exists a homeomorphism

$$\Theta\colon S^1 \to C_1 \cup C_2$$

such that $\Theta(S^1) = C_1 \cup C_2$ is an open subset of M.

Proof. The proof is a variation on the proof of Lemma 22.5. Since $C_1 \cap C_2$ consists of two components we obtain from Lemma 22.4 that there exist $a, b \in \mathbb{R}$ with $r_1 \le a < b \le s_1$ such that the following hold:

(1) $\Psi_1^{-1}(C_1 \cap C_2) = [r_1, a] \cup [b, s_1]$,
(2) $\Psi_2^{-1}(\Psi_1(a))$ and $\Psi_2^{-1}(\Psi_1(b))$ are endpoints of $[r_2, s_2]$.

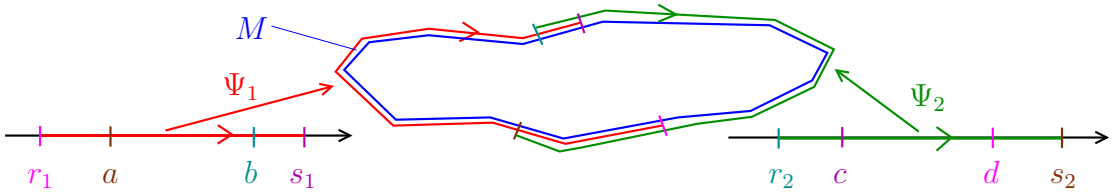
Note that after possibly replacing $t \mapsto \Psi_2(t)$ by $t \mapsto \Psi_2(r_2 + s_2 - t)$ we can assume that $\Psi_2(s_2) = \Psi_1(a)$ and $\Psi_2(r_2) = \Psi_1(b)$. Let $c \in [r_2, s_2]$ with $\Psi_2(c) = \Psi_1(s_1)$ and let $d \in [r_2, s_2]$ with $\Psi_2(d) = \Psi_1(r_1)$. Since C_2 is not contained in C_1 we see that $c < d$.

We consider the map

$$\begin{array}{rcl} \widetilde{\Psi}\colon [r_1, s_1 + (d-c)]/{\sim} & \to & C_1 \cup C_2 \\ & & \text{(we identify the two endpoints of the interval)} \\ [x] & \mapsto & \begin{cases} \Psi_1(x), & \text{if } x \in [r_1, s_1], \\ \Psi_2(x - s_1 + c), & \text{if } x \in [s_1, s_1 + (d-c)]. \end{cases} \end{array}$$

Note that $\widetilde{\Psi}$ is indeed well-defined.

As in the proof of Lemma 22.5 one can show fairly easily that $\widetilde{\Psi}$ is a homeomorphism, that the image is closed and that $C_1 \cup C_2$ is an open subset of M. Finally the desired homeomorphism $\Theta\colon S^1 \to C_1 \cup C_2$ is given by the observation, which we proved on page 244, that $[r_1, s_1 + (d-c)]/{\sim}$ is homeomorphic to S^1. ■



Now we can provide the proof of the Topological 1-Dimensional Manifold Classification Theorem 22.1. We break the proof into two parts. First we deal with the compact case, afterwards we will deal with the non-compact case.

Proof of Theorem 22.1 in the compact case. Let M be a compact connected 1-dimensional topological manifold. We start out with the following claim.

Claim. There exists a finite c-atlas for M.

Proof. Recall that by Lemma 22.2 we know that for each $P \in M$ there exists a c-parametrization $\Psi_P \colon I_P \to C_P$ such that C_P is a neighborhood of P, i.e. such that there exists an open subset V_P of M with $P \in V_P \subset C_P$. In particular $\{V_P\}_{P \in M}$ is an open cover of M. Since M is compact we see that there exist $P_1, \ldots, P_k \in M$ such that the corresponding open subsets cover M. The corresponding c-parametrizations form a finite c-atlas for M. ⊞

Claim. Either M is homeomorphic to S^1 or M is homeomorphic to $[0,1]$.

Proof. We prove the claim by induction on the number of elements of a finite c-atlas. If we have a c-atlas consisting of a single c-parametrization, then evidently M is homeomorphic to $[0,1]$. Now assume that we have $k \geq 2$ and that the statement holds whenever we have a c-atlas with $< k$ c-parametrizations. Let $\{\Psi_i \colon I_i \to C_i\}_{i=1,\ldots,k}$ be a c-atlas for M. Note that C_1 and $C_2 \cup \cdots \cup C_k$ are both non-empty closed subsets of M. Since M is connected we see that there exists a $j \in \{2, \ldots, k\}$ with $C_2 \cap C_j \neq \varnothing$. If $C_2 \subset C_j$ or $C_j \subset C_2$, then we can throw out of one of the c-parametrizations and we are done by induction.

Thus suppose that this is not the case. By Lemma 22.4 (3) we know that $C_2 \cap C_j$ consists of either one component or of two components. We deal with these two cases separately:

(1) If $C_2 \cap C_j$ consists of one component, then we know by Lemma 22.5 that there exists a c-parametrization $\widetilde{\Psi}$ with image $C_2 \cup C_j$. Thus $\{\Psi_i\}_{i \neq 2,j}$ together with $\widetilde{\Psi}$ is a c-parametrization of M with $k-1$ charts. Thus we are done by induction.

(2) Finally suppose that $C_2 \cap C_j$ consists of two components. By Lemma 22.6 we know that there exists a homeomorphism $\Theta \colon S^1 \to C_2 \cup C_j$ such that the image is a subset of M that is closed and open. Since M is connected we see that $C_2 \cup C_j = M$. ■

Now we turn to the classification of non-compact connected 1-dimensional topological manifolds.

Proof of Theorem 22.1 in the non-compact case. Let M be a non-compact connected 1-dimensional topological manifold. We start out with the following claim.

Claim. There exists a countable c-atlas $\{\Psi_i \colon [r_i, s_i] \to C_i\}_{i \in \mathbb{N}}$ and a family $\{V_i\}_{i \in \mathbb{N}}$ of open subsets of M with the following properties:

(1) The sets $\{C_i\}_{i \in \mathbb{N}}$ are a locally finite cover of M.[236]

(2) For each $i \in \mathbb{N}$ we have $V_i \subset C_i$ and the sets $\{V_i\}_{i \in \mathbb{N}}$ form an open cover of M.

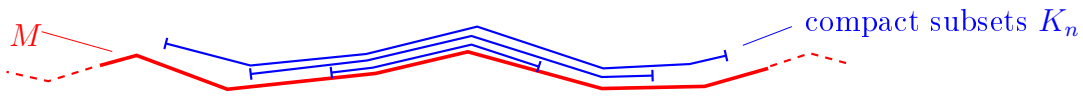
Proof. By the Topological Manifold Exhaustion Lemma 18.12 there exists a family of compact subsets $\{K_n\}_{n \in \mathbb{N}}$ of M and a family of open subsets $\{W_{n+\frac{1}{2}}\}_{n \in \mathbb{N}}$ of M such that the following two conditions are satisfied:

(a) for every $n \in \mathbb{N}$ we have $K_n \subset \overset{\circ}{K}_{n+1}$,

(b) we have $M = \bigcup_{n \in \mathbb{N}} K_n$.

Note that it follows from the Compact-Closed Lemma 1.21 (2) that each $\overset{\circ}{K}_{n+1} \setminus K_{n-1}$ is an open subset of M. Furthermore, by design these open subsets cover all of M.

[236] Recall, that according to the definition on page 114 this means that each $x \in M$ admits an open neighborhood W such that $\{i \in \mathbb{N} \,|\, C_i \cap W \neq \varnothing\}$ is finite.



It follows from the previous observation together with Lemma 22.2 that for each $P \in M$ there exists a c-parametrization $\Psi_P \colon I_P \to C_P$ with the following properties:

(i) C_P is a neighborhood of P, i.e. this means that there exists an open subset V_P of M with $P \in V_P \subset C_P$,

(ii) there exists an $n \in \mathbb{N}$ with $V_P \subset \mathring{K}_{n+1} \setminus K_{n-1}$.

Let $n \in \mathbb{N}$. It follows from the compactness of K_n, the fact that the V_P are an open cover of M and the Compact Subspace Covering Lemma 1.19 that there exists a finite set $X_n \subset M$ such that $K_n \subset \bigcup_{P \in X_n} V_P$. We set $X := \bigcup_{n \in \mathbb{N}} X_n$. One can now easily verify that the c-atlas $\{\Phi_P^{-1} \colon I_P \to C_P\}_{P \in X}$ together with the sets $\{V_P\}_{P \in X}$ has all the desired properties. ⊞

We move on to the next claim.

Claim. Given any $k, l \in \mathbb{N}$ there exist $r_0, \dots, r_m \in \mathbb{N}$ with $k = r_0$ and $l = r_m$ such that for every $i \in \{0, \dots, m-1\}$ we have $C_{r_i} \cap C_{r_{i+1}} \neq \varnothing$.

Proof. Let $k \in \mathbb{N}$. Recall that by construction the V_i introduced in the first claim form an open cover of M. We set

$$L := \{l \in \mathbb{N} \,|\, \text{there exist } k = r_0, \dots, r_m = l \text{ with } V_{r_i} \cap V_{r_{i+1}} \neq \varnothing \text{ for } i = 0, \dots, m-1\}.$$

Note that $A := \bigcup_{l \in L} V_l$ and $B := \bigcup_{l \notin L} V_l$ are both open subsets of M with $A \cup B = M$. It follows easily from the definitions that $A \cap B = \varnothing$. Since $k \in L$ we know that A is non-empty. Since M is connected we see that $A = M$. It follows almost immediately from this observation that $L = \mathbb{N}$. ⊞

We define a map $\varphi \colon \mathbb{N} \to \mathbb{N}$ as follows. We set $\varphi(1) := 1$. By the above claim it makes sense to iteratively set

$$\varphi(n) := \min\big\{k \in \mathbb{N} \,\big|\, k \notin \{\varphi(1), \dots, \varphi(n-1)\} \text{ and } C_k \cap (C_{\varphi(1)} \cup \dots \cup C_{\varphi(n-1)}) \neq \varnothing\big\}.$$

Given $n \in \mathbb{N}$ we also set $D_n := C_{\varphi(1)} \cup \dots \cup C_{\varphi(n)}$. Note that it follows immediately from the claim that $\bigcup_{n \in \mathbb{N}} D_n = \bigcup_{n \in \mathbb{N}} C_{\varphi(n)} = M$.

Claim. There exists a monotonously decreasing sequence $\{a_n\}_{n \in \mathbb{N}}$ of real numbers, a monotonously increasing sequence $\{b_n\}_{n \in \mathbb{N}}$ of real numbers and a sequence of homeomorphisms $\Phi_n \colon D_n \to [a_n, b_n]$ such that for each $n \in \mathbb{N}$ we have $\Phi_n|_{D_{n-1}} = \Phi_{n-1}$.

Proof. We set $\Phi_1 = \Psi_1 \colon [r_1, s_1] \to C_1 = D_1$. We construct the sequences of real numbers and of homeomorphisms iteratively by applying Lemma 22.5. ⊞

Note that $I := \bigcup_{n \in \mathbb{N}} [a_n, b_n]$ is an interval in $\mathbb{R}$ and that the maps Φ_n define a bijection

$$\begin{aligned} \Phi \colon M &\to I \\ x &\mapsto \Phi_n(x) \quad \text{if } x \in D_n. \end{aligned}$$

By construction the restriction of Φ_n to each D_n is continuous. This implies also that the restriction of Φ_n to each C_n is continuous. Since the $\{C_n\}_{n \in \mathbb{N}}$ form a locally finite

cover of M of closed subsets we obtain from the Pasting Proposition 2.6 that Φ is actually continuous.

It remains to show that the inverse $\Phi^{-1}\colon I \to M$ is continuous. Since each D_n is compact we obtain from the Compact-Hausdorff Proposition 2.17 (3) that the restriction of Φ^{-1} to each $[a_n, b_n]$ is continuous. It follows from Exercise 2.14 that $\Phi^{-1}\colon I \to M$ is continuous.

Since every non-compact interval is homeomorphic to either $\mathbb{R}$ or $[0,1)$ we are finally done. ■

22.2. **The classification of 1-dimensional smooth manifolds.** If we consider smooth manifolds instead of topological manifolds and diffeomorphisms instead of homeomorphisms we get the same answer. More precisely the following proposition holds.

Theorem 22.7. (Smooth 1-Dimensional Manifold Classification Theorem) Let M be a connected non-empty 1-dimensional smooth manifold. Then the diffeomorphism type of M is given by the following table

	without boundary	with non-empty boundary
compact	the circle S^1	the closed interval $[0,1]$
non-compact	the real line $\mathbb{R}$	the half-open interval $[0,1)$.

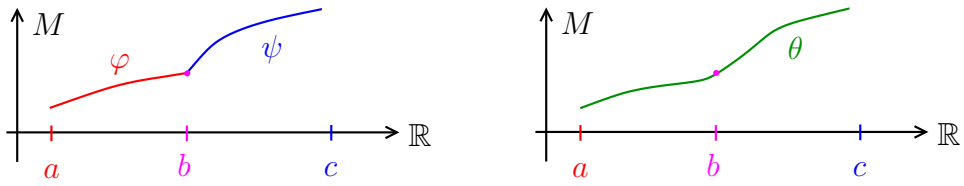
Remark.

(1) Other references for this theorem are [**GP1974**, p. 212], [**Shas2011**, Theorem 5.4.1] and [**Miln1965a**, appendix].
(2) We will provide a different proof of the Smooth 1-Dimensional Manifold Classification Theorem 22.7 in Proposition 142.1 and Exercise 142.2.

The proof of the Smooth 1-Dimensional Manifold Classification Theorem 22.7 relies on the following lemma.

Lemma 22.8. Let M be a 1-dimensional smooth manifold and suppose we have two smooth maps $\varphi\colon [a,b] \to M$ and $\psi\colon [b,c] \to M$ which are diffeomorphisms onto their images and which satisfy $\varphi([a,b]) \cap \psi([b,c]) = \{\varphi(b)\} = \{\psi(b)\}$. Then there exists a smooth map $\theta\colon [a,c] \to M$ which is a diffeomorphism onto its image with the following properties:

(1) $\theta([a,b]) = \varphi([a,b])$, with $\theta([b,c]) = \psi([b,c])$, and
(2) there exists an $\epsilon > 0$ such that $\theta = \varphi$ on $[a, a+\epsilon]$ and $\theta = \psi$ on $[c-\epsilon, c]$.



Proof. If $a = b$ then we can take $\theta = \psi$ and if $b = c$, then we can take $\theta = \varphi$. So we only need to consider the case that $a < b < c$. By picking a chart from the smooth atlas around $\varphi(b) = \psi(b)$ we might as well assume that $M = \mathbb{R}$. After possibly applying the reflection $t \mapsto -t$ to $M = \mathbb{R}$ we can furthermore assume that φ and ψ are strictly monotonously increasing.

We consider the map
$$\begin{array}{rccl} \chi\colon [a,c] & \to & \mathbb{R} \\ x & \mapsto & x - b + \varphi(b). \end{array}$$
It remains to show that the following hold:

(1) There exists an $\eta_a > 0$ and a smooth map $\theta_a\colon [a,b] \to \mathbb{R}$ which agrees with φ on $[a, a+\eta_a]$ and which agrees with χ on $[b-\eta_a, b]$.
(2) There exists an $\eta_c > 0$ and a smooth map $\theta_c\colon [b,c] \to \mathbb{R}$ which agrees with χ on $[b, b+\eta_c]$ and which agrees with ψ on $[c-\eta_c, c]$.

The existence of both was proved in Exercise 19.26, which in turn relies on the Smooth Transition Function Lemma 19.4. ■

Proof of the Smooth 1-Dimensional Manifold Classification Theorem 22.7. Let M be a connected non-empty 1-dimensional smooth manifold. Evidently the idea is modify the proof of the Topological 1-Dimensional Manifold Classification Theorem 22.1. In the following we discuss the case that M is non-compact with empty boundary. All other cases are treated in a similar fashion.

In the proof of Lemma 22.2 we now use smooth charts to obtain smooth c-parametrizations for M. If we now go through the proof of the Topological 1-Dimensional Manifold Classification Theorem 22.1 we see that there exists a homeomorphism $f\colon \mathbb{R} \to M$ which is smooth outside of a (possibly infinite) discrete subset $D \subset \mathbb{R}$. By applying Lemma 22.8 to each $x \in D$ we can turn f into a diffeomorphism. ■

22.3. **Interlude: The order topology.** In the next section we will introduce the "long line", which is a topological space, that has all the properties of a connected 1-dimensional topological manifold, except that it is not second-countable. The construction of the "long line" makes use of the order topology which we will introduce in this section.

First we recall some definitions. At least this particular author invariably forgets the precise meanings of the different type of orders that exist.

Definition. A partial order on a set X is a relation $\leq$ that has the following properties:

(1) For every $x \in X$ we have $x \leq x$.
(2) Whenever $x \leq y$ and $y \leq x$, then we have $x = y$.
(3) Whenever $x \leq y$ and $y \leq z$, then we also have $x \leq z$.

A total order is a partial order that also satisfies the following condition:

(4) For every $x, y \in X$ we have $x \leq y$ or $y \leq x$.

Finally a well-order is a total order that satisfies the following condition:

(5) Every non-empty subset has a minimal element.

Such a pair $(X, \leq)$ is called a partially ordered set, totally ordered set respectively well-ordered set.

Next we introduce some self-explanatory notation.

Notation. Let $(X, \leq)$ be a totally ordered set. As usual, given $x, y \in X$ we write $x < y$ if $x \leq y$ and $x \neq y$. Given $x, y \in X$ we write
$$\begin{array}{rclcrcl} (x,y) & := & \{a \in X \,|\, x < a < y\}, & & (x,y] & := & \{a \in X \,|\, x < a \leq y\} \\ [x,y) & := & \{a \in X \,|\, x \leq a < y\} & \text{and} & [x,y] & := & \{a \in X \,|\, x \leq a \leq y\}. \end{array}$$

We refer to these subsets in the obvious way as open intervals, half-open intervals and closed intervals.

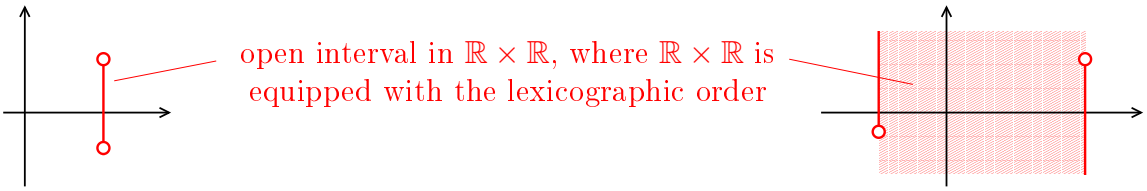
Definition. Let $(X, \leq)$ be a totally ordered set on X. Let $\mathcal{B}$ be the subset of the power set $\mathcal{P}(X)$ given by the subsets of X of the following four types:

(1) All intervals (a, b) in X.
(2) If x_0 is a smallest element of $(X, \leq)$, then we also consider the intervals $[x_0, b)$.
(3) If x_1 is a largest element of $(X, \leq)$, then we also consider the intervals $(a, x_1]$.
(4) The total set X.

We leave it to the reader to verify that $\mathcal{B}$ has the basis property, as defined on page 132.[237] By the Topology-from-Basis Lemma 1.27 the basis $\mathcal{B}$ defines a topology on X. We refer to this topology as the order topology on X.

Examples.

(1) If we equip $X = \mathbb{R}$ with the usual total order $\leq$, then the resulting order topology equals, basically by definition, the usual topology on X.
(2) We equip $X = \mathbb{R} \times \mathbb{R}$ with the lexicographic order induced from the usual well-order on the two $\mathbb{R}$-factors. Now we equip X with the corresponding order topology. In the figure below we show some open intervals in $\mathbb{R} \times \mathbb{R}$. We will discuss this fun example in greater detail in Exercise 22.4.



(3) We equip $\mathbb{N}_0$ and the half-open interval $[0, 1)$ with the usual total orders and we equip $\mathbb{N}_0 \times [0, 1)$ with the lexicographic order. In Exercise 22.3 we will show that if we equip $\mathbb{N}_0 \times [0, 1)$ with the corresponding order topology, then $\mathbb{N}_0 \times [0, 1)$ is homeomorphic to $[0, \infty)$.

Convention. In this chapter, given a totally ordered set $(X, \leq)$ we equip X with the corresponding order topology. On numerous occasions we will use the obvious statement that if two totally ordered sets $(X_1, \leq_1)$ and $(X_2, \leq_2)$ are of the same order type, then the corresponding topological spaces are homeomorphic.

On several occasions we will use the following lemma, without explicitly referring to it.

Lemma 22.9. Let $(X, \leq)$ be a totally ordered set. Given any interval $I = (a, b)$, $[a, b)$, $(a, b]$ and $[a, b]$ in X the order topology on $(I, \leq)$ agrees with the subspace topology coming from X.

Proof. We will prove the lemma in Exercise 22.2. ∎

[237] At first glance it might look unnecessary to explicitly throw in X into the set $\mathcal{B}$. But if X should happen to consist of a single element, then the sets of type (1), (2) and (3) actually do not satisfy property B1.

Example. Later in this section the half-open interval $[0,1) \subset \mathbb{R}$, equipped with the usual total order $\leq$, will play an important role. It follows from Lemma 22.9 that the resulting order topology equals the usual topology on $X \subset \mathbb{R}$.

We conclude this section with the following fairly straightforward lemma.

Lemma 22.10. Let $(X, \leq)$ be a totally ordered set. The topological space X that is given by the order topology is Hausdorff.

Proof. Let $(X, \leq)$ be a totally ordered set. We start out with the following claim.

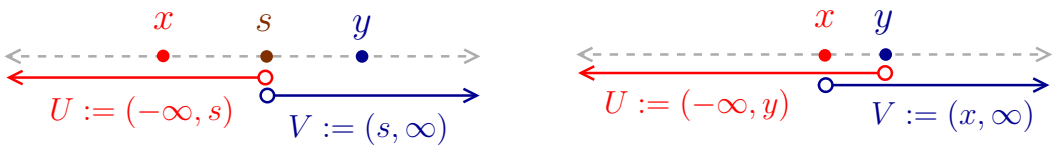
Claim. Given any $x \in X$ the sets

$$(-\infty, x) := \{a \in X \,|\, a < x\} \qquad \text{and} \qquad (x, \infty) := \{b \in X \,|\, x < b\}$$

are open subsets in the order topology.

Proof. If X has a minimal element x_0, then evidently we have $(-\infty, x) = [x_0, x)$. By definition this set is open. On the other hand, if X does not have a minimal element, then we have $(-\infty, x) = \bigcup_{a<x} (a, x)$ which is of course an open subset of X. Basically the same argument also shows that (x, ∞) is open. ⊞

Now let $x \neq y$ be two distinct points in X. Since $(X, \leq)$ is totally ordered we know that $x < y$ or $y < x$. Without loss of generality we can assume that $x < y$. If there exists an $s \in (x, y)$, then we set $U := (-\infty, s)$ and $V := (s, \infty)$. Otherwise we set $U := (-\infty, y)$ and $V := (x, \infty)$. By the claim U and V are open. It is now clear that these are disjoint open neighborhoods of x and y. ■



22.4. **The long line.** Let us start with the following dull definition.

Definition. Let $(X, \leq)$ be a partially ordered set. Given $x \in X$ we refer to

$$S_x := \{y \in X \,|\, y < x\}$$

as the x-section of $(X, \leq)$.

We consider the well-ordered set $(\mathbb{N}, \leq)$. Evidently the set is *countable*, but every section is actually *finite*. The following, quite remarkably, theorem says that the same type of mismatch between the cardinalities of all sections and the cardinality of the actual well-ordered set can also happen for other cardinalities.

Theorem 22.11.

(1) There exists an uncountable well-ordered set such that every section is countable.
(2) If X and Y are two well-ordered sets as in (1), then X and Y have the same order type.

We denote by ω_1 any well-ordered set as in (1).

Remark.

(1) The alert reader will have noticed that Statement (1) of Theorem 22.11 is precisely the content of Exercise 0.6.
(2) The order type of such a well-ordered set as in Theorem 22.11 is often called the *first uncountable ordinal*. We refer to [**Je2003**] and [**Dev1993**, Chapter 3] for an introduction into the concept of "ordinals".
(3) The construction of the well-ordered set ω_1 is a product of the ever-fertile mind of our friends the set-theorists. To the best of my knowledge there is no "down to earth" construction of ω_1.
(4) The set well-ordered set ω_1 is studied, in "everyday language", in [**Mun1975**, p. 67] and [**Gaul2014**, Chapter B.2].

Proof.

(1) By the Well-Ordering Theorem 0.7 there exists a well-order on any set. In particular there exists a well-order $\leq$ on the uncountable set $X = \mathbb{R}$. If every section of $(X, \leq)$ is countable, then we are done since $\mathbb{R}$ is uncountable. If not, then we consider the following non-empty subset

$$S := \{x \in X \mid \text{ the } x\text{-section is uncountable}\}.$$

Recall that by definition of a well-order we know that the non-empty subset S of X contains a minimal element z. Finally we consider the corresponding z-section $Y := S_z := \{x \in X \mid x < z\}$. Note that it follows from the construction of z that $Y = S_z$ is uncountable. Furthermore note that it follows immediately from the definition of z that every section of $Y = S_z$ is actually countable.
(2) We will not make use of this statement, thus we will not provide a proof. Instead we refer to [**Mun1975**, p. 73] or alternatively to [**Je2003**, Theorem I.2.12] for a proof. ■

Later on we will need the following simple lemma.

Lemma 22.12. The well-ordered set ω_1 does not have a maximal element.

Proof. Let $x \in \omega_1$. We need to show that x is not a maximal element. By one of the two defining properties of ω_1 we know that the section $S_x = \{y \in X \mid y < x\}$ is countable. It follows from the Countability Lemma 0.8 (4) that $\{y \in X \mid y \leq x\} = S_x \cup \{x\}$ is also countable. Since ω_1 is uncountable we see that x cannot be a maximal element. ■

Now we are finally ready to give the definition of the legendary long line.

Definition.

(1) We equip $[0, 1)$ with the usual well-order. Next we equip $\mathbb{L}_{\geq 0} := \omega_1 \times [0, 1)$ with the lexicographic order that we introduced on page 92 and we equip $\mathbb{L}_{\geq 0}$ with the corresponding order topology. We refer to this topological space $\mathbb{L}_{\geq 0}$ as the closed long ray.
(2) We denote by $* \in \omega_1$ the minimal element. We refer to $\mathbb{L}^+ := \mathbb{L}_{\geq 0} \backslash \{(*, 0)\}$, equipped with the subspace topology coming from $\mathbb{L}_{\geq 0}$, as the open long ray.

(3) We write $\mathbb{L}_{\leq 0} := \mathbb{L}_{\geq 0}$. We refer to the topological space $\mathbb{L} := \mathbb{L}_{\leq 0} \sqcup_{(*,0)=(*,0)} \mathbb{L}_{\geq 0}$ as the long line. (In the literature the long line is sometimes also called the Alexandroff line.)

Remark. In many references, see e.g. [**Mun1966a**, p. 158], [**SS1978**, p. 71] and [**Kra2009**, p. 66], the open long ray is called the long line.

The following proposition shows in particular that the open long ray $\mathbb{L}^+$ and the long line $\mathbb{L}$ both satisfy all properties of a 1-dimensional topological manifold, except that neither is second-countable.

Theorem 22.13. The open long ray $\mathbb{L}^+$ and the long line $\mathbb{L}$ both have the following properties:

(1) they are Hausdorff,
(2) they are locally homeomorphic to $\mathbb{R}$,
(3) they are path-connected and
(4) they are not second-countable.

Remark.

(1) The proof we give below is based on the sketch given in [**Mun1966a**, p. 158]. An alternative account of the proof is given in [**Gaul2014**, Chapter 1.2].
(2) The discovery of the long line $\mathbb{L}$ goes back to work of Georg Cantor [**Cant1883**] from 1883.
(3) A slightly less otherworldly example of a topological space that satisfies all properties of a path-connected topological manifold, except that it is not second-countable, is given in [**Bre1993**, Corollary 17.5] and [**Spr1957**, p. 56].

We will prove Theorem 22.13 in the next section. In the remainder of this section we will discuss the open long ray and long line in greater detail. First of all the following proposition lists several weird and surprising properties of the open long ray and the long line.

Proposition 22.14.

(1) Every map $f\colon \mathbb{L}^+ \to \mathbb{R}$ is eventually constant, i.e. there exists an $(x,s) \in \mathbb{L}^+$ such that $f((y,t)) = f(x,s)$ for every $(y,t) \geq (x,s)$.
(2) The topological space $\mathbb{L}^+$ is not paracompact.
(3) Every bounded map $f\colon \mathbb{L}^+ \to \mathbb{L}^+$ is homotopic to a constant map.
(4) Every unbounded map $f\colon \mathbb{L}^+ \to \mathbb{L}^+$ is homotopic to the identity map.
(5) An unbounded map $f\colon \mathbb{L}^+ \to \mathbb{L}^+$ is never homotopic to bounded map.
(6) The open long line $\mathbb{L}^+$ is not contractible.

Proof.

(1) It is a great exercise to try to prove this statement. If, inexplicably, the reader is not inclined to provide the arguments, then we refer to [**Gaul2014**, Proposition 1.17] for a proof.
(2) We need to show that $\mathbb{L}^+$ admits an open cover that does not admit a locally finite open refinement. In fact one can show that the open subsets $U_{(x,s)} := ((*,0),(x,s))$ of $\mathbb{L}^+$ is such an open cover. Since we will not make use of this statement we will

not show that this cover does not admit a locally finite open refinement. The statement can be proved using a slight variation on the argument given on [**Jos1983**, Theorem 15.3.2(vi)].

(3)-(5) These statements are proved in [**Gaul2014**, Proposition 1.12].

(6) This statement follows immediately from (5) and the Contractibility Criterion Lemma 15.2. ■

At this stage one might fear that now, when we no longer restrict ourselves to second-countable topological spaces, the genie is out of the bottle, and that thus any type of classification result is completely out of reach. Rather stunningly this is not the case.

The following theorem can now be viewed as a generalization of the Topological 1-Dimensional Manifold Classification Theorem 22.1.

Theorem 22.15. Let X be a topological space that is Hausdorff and path-connected.

(1) If X is locally homeomorphic to an open subset of $\mathbb{R}$, then X is homeomorphic to one of the following: S^1, $\mathbb{R}$, $\mathbb{L}^+$ and $\mathbb{L}$.

(2) If X is locally homeomorphic to an open subset of $\mathbb{R}_{\geq 0}$, then either X is homeomorphic to one of the above, or one of the following: $[0,1)$, $[0,1]$ and $\mathbb{L}_{\geq 0}$.

Proof.

(1) This statement is proved in [**KnesH1958**].

(2) This statement can be deduced easily from (1) by first removing the "boundary points". We leave it to the reader to provide the details. ■

By Theorem 22.13 we now know in particular that the long line $\mathbb{L}$ admits an atlas. In fact, as is shown in [**Gaul2014**, Proposition 7.9], there exists an atlas on $\mathbb{L}$ that is smooth, i.e. such that all transition maps are smooth. Much more impressively, it turns out that there are uncountable many mutually non-diffeomorphic smooth structures on $\mathbb{L}$. We refer to [**Gaul2014**, Theorem 7.14] and [**Nyi1992**] for a precise formulation and a proof of this assertion.

22.5. **Proof of Theorem 22.13.** In this section we finally provide the proof of Theorem 22.13. We start out with some general definitions and statements regarding well-ordered sets.

Definition. Let $(X, \leq)$ be a well-ordered set. A subset A of X is called inductive if for every $a \in X$ with $S_a \subset A$ we also have $a \in A$.

Examples.

(1) Let $X = \mathbb{R}_{\geq 0}$. The interval $A = [0,2)$ is not inductive whereas $A = [0,2]$ is inductive.

(2) One can easily show that the only inductive subset of $\mathbb{N}_0$ is $\mathbb{N}_0$.

The grandiose name of the following theorem is designed to impress friends and family.

Theorem 22.16. (Principle of Transfinite Induction) Let $(X, \leq)$ be a well-ordered set. If $A \subset X$ is an inductive subset of X, then $A = X$.

Proof. Let $A \subset X$ be an inductive subset. We suppose that $A \neq X$. Since X is well-ordered the set $X \setminus A$ has a minimal element y. Since y is the minimal element of $X \setminus A$

we see that for every $x < y$ we have $x \in A$. But this means that the y-section S_y is contained in A. But since A is inductive this implies that $y \in A$. We have thus obtained a contradiction. ■

In the proof of Theorem 22.13 we will need the following fairly straightforward lemma.

Definition. Let $(X, \leq)$ be a partially ordered set.

(1) Let $x \in X$. We say that $a \in X$ is an immediate predecessor of x if $a < x$ and if there is no $b \in X$ with $a < b < x$.
(2) Let $x \in X$. We say that $a \in X$ is an immediate successor of x if $x < a$ and if there is no $b \in X$ with $x < b < a$.
(3) Let $A \subset X$ be a subset. We say that $x \in X$ is an upper bound of A if for every $a \in A$ we have $a \leq x$.
(4) Let $A \subset X$ be a subset. We say that $x \in X$ is a supremum of A if x is an upper bound of A and if there is no upper bound of A that is less than x.
(5) Let $x_1 < x_2 < \dots$ be an increasing sequence of elements in X. If $b = \sup\{x_1, x_2, \dots\}$ then we say that b is a limit of the sequence.

Examples.

(1) Every element in $\mathbb{Z}$ has an immediate predecessor and an immediate successor.
(2) No element in $\mathbb{R}$ has an immediate predecessor and no element in $\mathbb{R}$ has an immediate successor.
(3) If $(X, \leq)$ is a well-ordered set, then it follows easily from the definitions that every proper subset $A \subsetneq X$ admits a unique supremum.

Lemma 22.17. Let $(X, \leq)$ be a well-ordered set and let $x \in X$ such that the corresponding section S_x is countable. Precisely one of the following three statements holds:

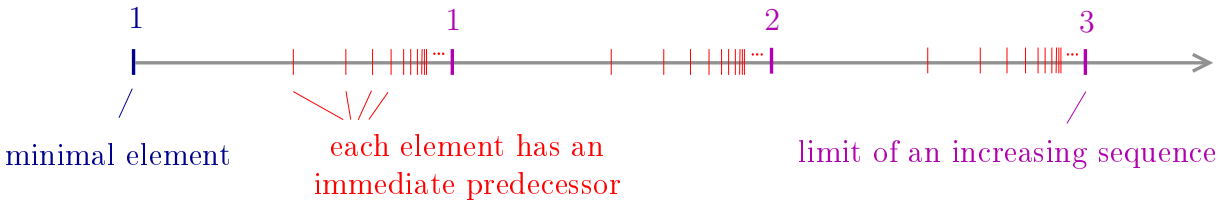
(1) x is the minimal element of X.
(2) There exists an immediate predecessor of x.
(3) x is the limit of an increasing sequence $x_0 < x_1 < x_2 <$ of elements of X.

Proof. Let $(X, \leq)$ be a well-ordered set and let $x \in X$ such that the corresponding section $S_x = \{w \in X \,|\, w < x\}$ is countable. We only need to consider the case that $x \in X$ is not the minimal element. This means that the section S_x is non-empty. If the section S_x is finite, then it admits a maximal element, which is clearly an immediate predecessor of x. Next we suppose that the section S_x is infinite. By hypothesis on x we know that S_x is also countable. Thus there exists a bijection $\varphi\colon \mathbb{N}_0 \to S_x$. Now given any $n \in \mathbb{N}_0$ we set $y_n := \max\{\varphi(0), \dots, \varphi(n)\}$. By construction we have $y_1 \leq y_2 \leq y_3 \leq \dots$. If the sequence $y_1, y_2, \dots$ is eventually constant, i.e. if there exists an N such that $y_n = Y_N$ for every $n \geq N$, then Y_N is easily seen to be an immediate predecessor of x. Otherwise there exists a subsequence $y_{s_1} < y_{s_2} < y_{s_3} < \dots$ which is strictly increasing. By construction x is the limit of this sequence.

Finally we leave it to the reader to verify that three possibilities are mutually exclusive. ■

Example. We consider the subset $X := \mathbb{N}_0 \cup \{n - \frac{1}{k} \,|\, n \in \mathbb{N}_0 \text{ and } k \in \mathbb{N}_{\geq 2}\}$ of $\mathbb{R}$, equipped with the usual partial order. One can easily verify that $(X, \leq)$ is actually well-ordered.

The number 0 is evidently the minimal element. Every x of the form $n-\frac{1}{k}$ with $n \in \mathbb{N}$ and $k \in \mathbb{N}_{\geq 2}$ has an immediate successor, namely $n-\frac{1}{k-1}$. Finally every $n \in \mathbb{N}$ is the limit of the increasing sequence $x_i = n - \frac{1}{1+i}$.



Lemma 22.18. Let $(X, \leq)$ be a totally ordered set.

(1) Let $a < b < c$ be three elements of X. The interval $\overbrace{[a,c)}^{\subset X}$ has the order type of $\overbrace{[0,1)}^{\subset \mathbb{R}}$ if and only if each of the two intervals $[a,b)$ and $[b,c)$ has the order type of $[0,1)$.

(2) Let $b \in X$ be the limit of an increasing sequence $x_0 < x_1 < \ldots$ of elements of X. If each $[x_n, x_{n+1})$ has the order type of $[0,1)$, then so does $[x_0, b)$.

Proof. Throughout the proof we will make use of the following simple observation: if $(A, \leq)$ and $(B, \leq)$ are two totally ordered sets, then they have the same order type if and only if there exists an order-preserving bijection $f\colon A \to B$. In other words, the fact that A and B are totally ordered implies that the inverse $f^{-1}\colon B \to A$ is automatically order-preserving. Now we turn to the proofs of the two statements.

(1) Let $a < b < c$ be three elements of X. First suppose that there exist order-preserving bijections $f\colon [a,b) \to [0,1)$ and $g\colon [b,c) \to [0,1)$. It is clear that the map

$$\begin{array}{rcl} [a,c) & \to & [0,1) \\ t & \mapsto & \begin{cases} \frac{1}{2} \cdot f(t), & \text{if } t \in [a,b), \\ \frac{1}{2} + \frac{1}{2} \cdot g(t), & \text{if } t \in [b,c) \end{cases} \end{array}$$

is an order-preserving bijection. Conversely suppose that there exists an order-preserving bijection $h\colon [a,c) \to [0,1)$. We point out that $h([a,b)) = [0, h(b))$ and $h([b,c)) = [h(b), 1)$. Evidently the intervals $[0,h(b))$ and $[h(b),1)$ have the order type of $[0,1)$. Thus we are done.

(2) Now let $b \in X$ be the limit of an increasing sequence $x_0 < x_1 < \ldots$ of elements of X. We suppose that for each $n \in \mathbb{N}_0$ there exists an order-preserving bijection $\varphi_n\colon [x_n, x_{n+1}) \to [0,1)$. One can now easily verify that the map

$$\begin{array}{ccccc} [x_0, b) & = & \bigsqcup_{n \in \mathbb{N}_0} [x_n, x_{n+1}) & \to & [0,1) \\ \uparrow & & & & \\ \text{since } b \text{ is the limit of the} & & t & \mapsto & \frac{1}{2^{n+1}} \cdot \varphi_n(t) + \sum_{i=0}^{n-1} \frac{1}{2^{i+1}} \quad \text{for } t \in [x_n, x_{n+1}) \\ \text{increasing sequence } x_0 < x_1 < \ldots & & & & \end{array}$$

is an order-preserving bijection. ∎

The next lemma contains the heart of the proof of Theorem 22.13.

Lemma 22.19. As above we denote by $*$ the minimal element of ω_1 and let $x \in \omega_1 \setminus \{*\}$.

(1) The half-open interval $[(*,0),(x,0)) \subset \mathbb{L}_{\geq 0}$ has the order type of $[0,1)$.

(2) The open interval $((*,0),(x,0)) \subset \mathbb{L}_{\geq 0}$ has the order type of $(0,1)$.

Proof. We consider the set

$$B := \big\{ b \in \omega_1 \setminus \{*\} \,\big|\, \text{the set } [(*,0),(b,0)) \text{ has order type of } [0,1) \big\}.$$

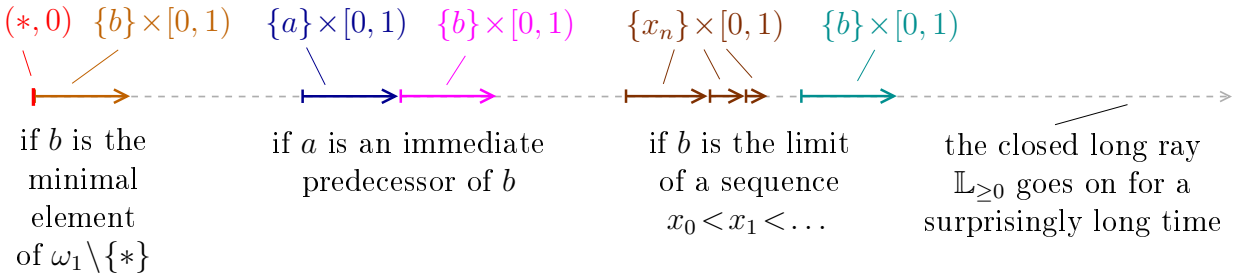
We want to show that $B = \omega_1 \setminus \{*\}$. Note that it follows from the fact that ω_1 is well-ordered that the complement of the minimal element $*$, i.e. the set $\omega_1 \setminus \{*\}$, is again well-ordered. Thus it follows from Theorem 22.16, i.e. it follows from the Principle of Transfinite Induction, that it suffices to show that B is inductive.

Now let $b \in B$ such that $S_b \subset B$. By Lemma 22.17 we know that one of the following holds:

(1) b is the minimal element of B.
(2) b has an immediate predecessor a.
(3) b is the limit of an increasing sequence $x_0 < x_1 < \dots$ of elements of X.

We will now treat these three cases separately.

(1) Suppose that b is the minimal element of $\omega_1 \setminus \{*\}$. In this case we see easily that $[(*,0),(b,0)) = \{(*,t) \,|\, t \in [0,1)\}$, which evidently has the order type of $[0,1)$.
(2) Suppose that b has an immediate predecessor a. Since $S_b \subset B$ and $a < b$ we see that $a \in B$, i.e. we see that $[(*,0),(a,0))$ has order type $[0,1)$. Next note that the fact that a is an immediate predecessor of b implies that $[(a,0),(b,0)) = \{(a,x) \,|\, x \in [0,1)\}$. Thus we see that this interval also has order type $[0,1)$. It follows from Lemma 22.18 (1) that $[(*,0),(b,0))$ has order type $[0,1)$.
(3) Now suppose that b is the limit of an increasing sequence $x_0 < x_1 < \dots$ of elements of X. Since $S_b \subset B$ and since for each x_n we have $x_n < b$ we see as in (2) that each $[(*,0),(x_n,0))$ has order type $[0,1)$. It follows from Lemma 22.18 (1) that each interval $[(x_n,0),(x_{n+1},0))$ has order type $[0,1)$. Finally it follows from Lemma 22.18 (2) that the interval $[(*,0),(b,0))$ has order type $[0,1)$. ∎



Now we can finally provide the proof of Theorem 22.13.

Proof of Theorem 22.13. We will prove the desired statements only for the open long ray $\mathbb{L}^+$. We leave it to the reader to make the minor modifications need to show that the analogous statements also hold for the long line $\mathbb{L}$. Now recall that we need to show the following four statements regarding the open long ray $\mathbb{L}^+$.

(1) $\mathbb{L}^+$ is Hausdorff,
(2) $\mathbb{L}^+$ is locally homeomorphic to an open subset of $\mathbb{R}$,
(3) $\mathbb{L}^+$ is path-connected,
(4) $\mathbb{L}^+$ is not second-countable.

We turn to the proofs of these four statements.

(1) This statement conveniently enough follows immediately from Lemma 22.10.

(2) Let $(x,t) \in \mathbb{L}^+$. By Lemma 22.12 we can pick some $y > x$. Since $y > x$ we see that $U := ((*,0),(y,0))$ is an open neighborhood of (x,t). Note that by Lemma 22.19 we know that the interval $[(*,0),(y,0))$ has the order type of $[0,1)$. It follows that the open interval $U := ((*,0),(y,0))$ is homeomorphic to $(0,1)$.

(3) Let (x,s) and (y,t) be two points in $\mathbb{L}^+$. By Lemma 22.12 we can pick $z \in X$ with $z > x$ and $z > y$. As in (2) we obtain from Lemma 22.19 that the open interval $U := ((*,0),(z,0))$ is homeomorphic to $(0,1)$. Since the open interval contains the points (x,s) and (y,t) and since $(0,1)$ is path-connected we see that there exists a path $\gamma\colon [0,1] \to \mathbb{L}^+$ that connects (x,s) and (y,t).

(4) Let $\mathcal{B}$ be a basis for the topological space $\mathbb{L}^+$. We need to show that $\mathcal{B}$ is uncountable. For each $x \in \omega_1$ we consider the open interval $U_x := ((x,\frac{1}{4}),(x,\frac{3}{4}))$. It follows from the definition of a basis that given any $x \in X$ there exists a set $B_x \in \mathcal{B}$ with $(x,\frac{1}{2}) \in B_x \subset U_x$. Since the U_x are disjoint we see that the B_x are distinct. Since X is uncountable we obtain from the Countability Lemma 0.8 (1) that $\mathcal{B}$ is uncountable. ■

Exercises for Chapter 22.

Exercise 22.1. Let X be a 1-dimensional topological manifold. Show that X is homeomorphic to a proper submanifold of $\mathbb{H}_2$.
Remark. Since X is second-countable we know by Exercise 9.1 that X has only countably many components.

Exercise 22.2.

(a) Let $(X,\leq)$ be a totally ordered set. Show that given any interval $I = (a,b)$, $[a,b)$, $(a,b]$ and $[a,b]$ in X the order topology on $(I,\leq)$ agrees with the subspace topology coming from X.

(b) Let $(X,\leq)$ be a totally ordered set and let $A \subset X$ be a subset. Show that the order topology on $(A,\leq)$ does not necessarily agree with the subspace topology coming from X.

Exercise 22.3. We equip $\mathbb{N}_0$ and the half-open interval $[0,1)$ with the usual total orders and we equip $\mathbb{N}_0 \times [0,1)$ with the lexicographic order. Finally we equip $\mathbb{N}_0 \times [0,1)$ with the corresponding order topology. Show that $\mathbb{N}_0 \times [0,1)$ is homeomorphic to $[0,\infty)$.

Exercise 22.4. We consider $X := \mathbb{R} \times \mathbb{R}$, for once equipped with the order topology $\mathcal{O}$.

(a) Let $\mathcal{T}$ be the usual topology on $X = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. Do we have an inclusion $\mathcal{O} \subset \mathcal{T}$? Do we have an inclusion $\mathcal{T} \subset \mathcal{O}$?

(b) Show that the topological space $(X,\mathcal{O})$ is Hausdorff.

(c) Is the topological space $(X,\mathcal{O})$ second-countable?

(d) Is $(X,\mathcal{O})$ locally homeomorphic to open subsets of $\mathbb{R}$?

(e) Is $(X,\mathcal{O})$ homeomorphic to the long line?

Exercise 22.5. Let ω_1 be the well-ordered set introduced in Theorem 22.11.

(a) Show that for every $x \in \omega_1$ the subset $\{y \in \omega_1 \,|\, x < y\}$ is uncountable.

(b) Let $X = \{x \in \omega_1 \,|\, x \text{ has no immediate successor}\}$. Show that X is uncountable.

Exercise 22.6.

(a) We consider the well-ordered set $\mathbb{L}^+ := \omega_1 \times [0,1)$. Show that every increasing sequence in $\mathbb{L}_{\geq 0}$ converges.

(b) A topological space that has the property that every sequence admits a convergent subsequence is called sequentially compact. Show that the closed long ray $\mathbb{L}^+$ is sequentially compact.

Exercise 22.7. Show that the open long ray $\mathbb{L}^+$ and the long line $\mathbb{L}$ are not homeomorphic. *Hint.* What happens if you remove a point?

Exercise 22.8.

(a) Let $(X, \leq)$ be a well-ordered set. Let $x \in X$. Show that either x is the largest element of X or that x has an immediate successor.

(b) Give an example of a partially ordered set $(X, \leq)$ that is not well-ordered but that nonetheless has the property that every $x \in X$ has an immediate successor.

Exercise 22.9. Let $(X, \leq)$ be a partially ordered set and let $x \in X$.

(1) We say x has *depth* -1 if x is a minimal element of X.

(2) We say x has *depth* 0 if x has an immediate predecessor.

Now we iteratively define what it means for $x \in X$ to have depth $k \in \mathbb{N}_{\geq 2}$.

(3) We say x has *depth* k if x is not of depth $k-1$ but if there exists an increasing sequence $x_1 < x_2 < \dots$ in X, such that each x_i has depth $< k$, and such that x is a limit of this sequence.

Show that there exists a countable well-ordered subset of $\mathbb{R}$ that admits elements of depth two.

Exercise 22.10. Does the long line $\mathbb{L}$ admit an embedding into $\mathbb{R}^2$?

Exercise 22.11. A topological space X is called homogeneous if given any $x, y \in X$ there exists a homeomorphism $f\colon X \to X$ with $f(x) = y$. Show that the long ling $\mathbb{L}$ is homogeneous.

23. Tangent spaces of smooth manifolds

In this chapter we introduce the notion of tangent spaces of smooth manifolds. As a warm-up we first consider tangent spaces of smooth submanifolds of some $\mathbb{R}^k$. In this setting the notion of a tangent space can be motivated easily. The notion of a tangent space for a general "abstract" manifold takes more time to get used to and to appreciate.

23.1. **Tangent spaces of smooth submanifolds of $\mathbb{R}^k$.** In this section we study smooth submanifolds of some $\mathbb{R}^k$, where, as always, we equip $\mathbb{R}^k$ with the standard smooth structure. Note that in the Submanifold Smooth Structure Lemma 19.24 we saw that every smooth submanifold of $\mathbb{R}^k$ inherits a standard smooth structure.

Definition. Let M be an n-dimensional smooth submanifold of $\mathbb{R}^k$ and let $P \in M$.

(1) A curve through P is a map $\gamma\colon I \to M$ where $I \subset \mathbb{R}$ is an open or half-open interval with $0 \in I$ and with $\gamma(0) = P$.

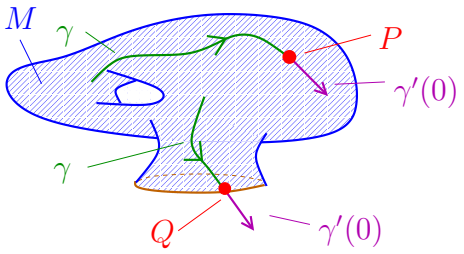
(2) Given a smooth curve $\gamma\colon I \to M$ through P we denote by $\gamma'(0) \in \mathbb{R}^k$ the derivative of γ viewed as a smooth map[238] $\gamma\colon I \to M \to \mathbb{R}^k$.

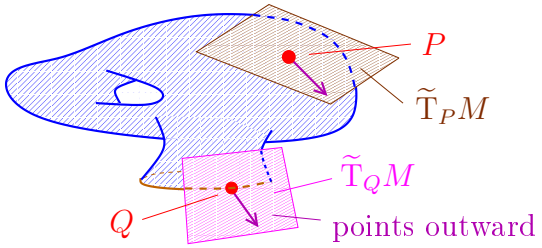
(3) We define the physical tangent space as the set of all derivatives of smooth curves on M through P, i.e. we define[239]

$$\widetilde{\mathrm{T}}_P M \;:=\; \{\gamma'(0)\,|\,\gamma\colon I \to M \text{ a smooth curve through } P\} \;\subset\; \mathbb{R}^k.$$

We refer to any vector in $\widetilde{\mathrm{T}}_P M$ as a tangent vector to M at P.

(4) Let $P \in \partial M$. We say a tangent vector $v \in \widetilde{\mathrm{T}}_P M$ points outward if $v = \gamma'(0)$ for some smooth map $\gamma\colon (-\epsilon, 0] \to M$ and if $v \notin \widetilde{\mathrm{T}}_P \partial M$. Similarly we also define the notion that v points inward.





Remark.

(1) Let M be an n-dimensional smooth submanifold of $\mathbb{R}^k$ and let $P \in M$. If $P \notin \partial M$, then using charts and using the definition of a smooth map on a half-open interval, see page 549, one can easily show that

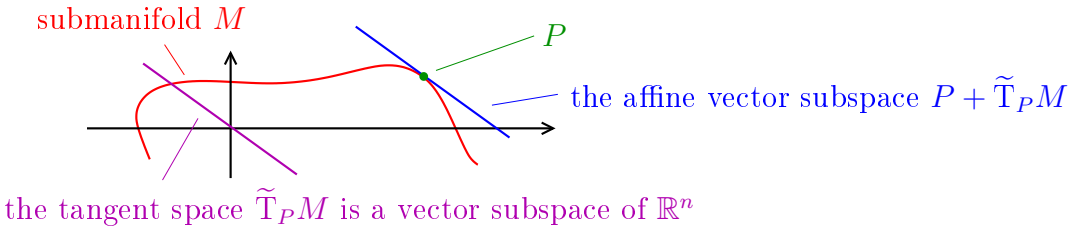
$$\widetilde{\mathrm{T}}_P M \;:=\; \text{all derivatives } \gamma'(0) \text{ of smooth maps } \gamma\colon (-\epsilon, \epsilon) \to M \text{ with } \gamma(0) = P.$$

(2) Let M be an n-dimensional smooth submanifold of $\mathbb{R}^k$ and let $P \in M$. In the Physical Tangent Space-Functor Proposition 23.2 (0) we will see that $\widetilde{\mathrm{T}}_P M$ is an n-dimensional vector subspace of $\mathbb{R}^k$. In pictures we usually show, for clarity, the

[238] Note that according to the Submanifold Smooth Structure Lemma 19.24 the inclusion $M \to \mathbb{R}^k$ is smooth, so the composition of the smooth map $\gamma\colon I \to M$ with the inclusion $M \to \mathbb{R}^k$ is also smooth.

[239] Note that if 0 is an endpoint of the interval I, then we use the convention from page 533 to define the derivative of γ at 0.

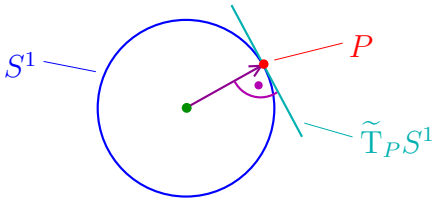
affine vector subspace $P + \widetilde{T}_P M$. This affine vector space is the "best possible" approximation of M by an affine vector subspace.

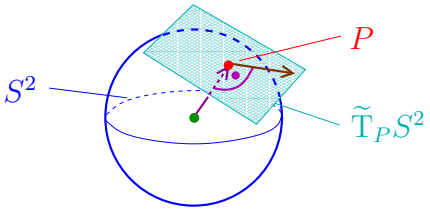


We formulate our first examples as a lemma.

Lemma 23.1. Let $n \in \mathbb{N}_0$.

(1) Let M be an open subset of $\mathbb{R}^n$ or of the half-plane $\mathbb{H}^n$. For every $P \in M$ we have the equality $\widetilde{\mathrm{T}}_P M = \mathbb{R}^n$.
(2) If $W \subset \mathbb{R}^n$ is a k-dimensional vector space and $Q \in \mathbb{R}^n$, then the affine space $Q + W = \{Q + w \,|\, w \in W\}$ is a k-dimensional smooth submanifold of $\mathbb{R}^n$ and for any $P \in W$ we have $\widetilde{\mathrm{T}}_P W = W$.
(3) Given any $P \in S^n \subset \mathbb{R}^{n+1}$ we have the equality $\widetilde{\mathrm{T}}_P S^n = \{v \in \mathbb{R}^{n+1} \,|\, \langle v, P\rangle = 0\}$.





Proof.

(1) First let M be an open subset of $\mathbb{R}^n$ and let $P \in M$. Note that by definition we have $\widetilde{\mathrm{T}}_P M \subset \mathbb{R}^n$. Conversely let $v \in \mathbb{R}^n$. Since M is open there exists an $\epsilon > 0$ such that $\{P + t \cdot v \,|\, t \in [0, \epsilon]\} \subset U$. Thus we can consider the smooth map $\gamma\colon [0, \epsilon) \to M$ given by $\gamma(t) = P + t \cdot v$. Evidently $\gamma'(0) = v$, which shows that $v \in \widetilde{\mathrm{T}}_P M$. The proof of the case that M is an open subset of $\mathbb{H}^n$ is very similar, we just have to notice that for any $v \in \mathbb{R}^n$ we have $v \in \mathbb{H}^n$ or $-v \in \mathbb{H}^n$. We leave it to the reader to fill in the few remaining details.
(2) It follows from very elementary linear algebra that there exists an invertible matrix $A \in \mathrm{GL}(n, \mathbb{R})$ such that $A \cdot W = \{0\} \times \mathbb{R}^k$. The map

$$\begin{array}{rcl} Q + W & \to & \mathbb{R}^n \\ x & \mapsto & A \cdot (x - Q) \end{array}$$

is now a submanifold chart. Thus $Q + W$ is a k-dimensional smooth submanifold of $\mathbb{R}^n$. Using this chart one can also easily verify that for any $P \in W$ we indeed have $\widetilde{\mathrm{T}}_P W = W$. We leave it to the reader to fill in the details.

(3) Let $P \in S^n$. We need to show that $\widetilde{\mathrm{T}}_P M = \{v \in \mathbb{R}^{n+1} \mid \langle v, P\rangle = 0\}$. We prove the two inclusions separately.

($\subset$) Let I be a (half-) open interval with $0 \in I$ and let $\gamma\colon I \to S^n$ be a smooth map with $\gamma(0) = P$. For any $t \in I$ we have

$$0 \underset{\uparrow}{=} \frac{d}{dt}\|\gamma(t)\|^2 = \frac{d}{dt}\langle \gamma(t), \gamma(t)\rangle \underset{\uparrow}{=} 2 \cdot \langle \gamma'(t), \gamma(t)\rangle.$$

since $\|\gamma(t)\| \equiv 1$ follows from the Product Rule

In particular for $t = 0$ we see that $0 = \langle \gamma'(0), \gamma(0)\rangle = \langle \gamma'(0), P\rangle$.

($\supset$) Suppose we are given $v \in \mathbb{R}^{n+1}$ with $\langle v, P\rangle = 0$. If $v = 0$, then v is the derivative of the constant curve through P. Now assume that $v \neq 0$. We consider the curve that is defined by $\gamma(t) := P \cdot \cos(t \cdot \|v\|) + \frac{v}{\|v\|} \cdot \sin(t \cdot \|v\|)$. Evidently we have $\gamma(0) = P$ and $\gamma'(0) = v$. Finally note that it follows from $\langle v, P\rangle = 0$ that γ defines a curve on S^n. Thus we have shown that $v \in \widetilde{\mathrm{T}}_P S^n$. ■

Proposition 23.2. (Physical Tangent Space-Functor Proposition)

(0) If M is a d-dimensional submanifold of $\mathbb{R}^n$, then for each $P \in M$ the physical tangent space $\widetilde{\mathrm{T}}_P M$ is a d-dimensional vector subspace of $\mathbb{R}^n$.

(1) Let $f\colon M \to N$ be a smooth map[240] between smooth submanifolds of $\mathbb{R}^m$ respectively $\mathbb{R}^n$. Given any $P \in M$ the map[241]

$$\begin{array}{rcl} \mathrm{D}f_P\colon \widetilde{\mathrm{T}}_P M & \to & \widetilde{\mathrm{T}}_{f(P)} N \\ \gamma'(0) & \to & (f \circ \gamma)'(0) \end{array}$$

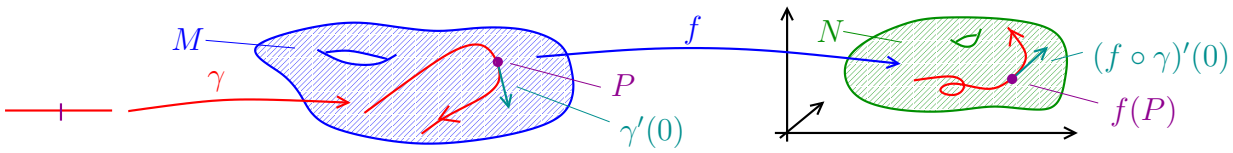
is well-defined and linear.

(2) The maps from (1) have the following two properties:[242]

(a) If $\mathrm{id}\colon M \to M$ is the identity map, then for any $P \in M$ the corresponding map $\mathrm{D}\,\mathrm{id}_P\colon \widetilde{\mathrm{T}}_P M \to \widetilde{\mathrm{T}}_P M$ is also the identity.

(b) For any two smooth maps $f\colon L \to M$ and $g\colon M \to N$ between smooth submanifolds and any $P \in L$ we have

$$\mathrm{D}(g \circ f)_P = \mathrm{D}g_{f(P)} \circ \mathrm{D}f_P\colon \widetilde{\mathrm{T}}_P L \to \widetilde{\mathrm{T}}_{g(f(P))} N.$$



[241] At the danger of repeating ourselves, we use the Submanifold Smooth Structure Lemma 19.24 to view M and N as smooth manifolds in their own right. Thus by a smooth map we mean a smooth map between these two smooth manifolds.

[242] By the Submanifold Smooth Structure Lemma 19.24 the inclusion map $N \to \mathbb{R}^n$ is smooth, thus $f \circ \gamma$ is indeed a smooth curve that takes values in $\mathbb{R}^n$, thus it makes sense to take its derivative.

Definition. Let $f\colon M \to N$ be a smooth map between smooth submanifolds of $\mathbb{R}^m$ respectively $\mathbb{R}^n$. Given any $P \in M$ we refer to the map

$$\begin{array}{rccc} \mathrm{D}f_P\colon & \widetilde{\mathrm{T}}_P M & \to & \widetilde{\mathrm{T}}_{f(P)} N \\ & \gamma'(0) & \to & (f\circ\gamma)'(0), \end{array}$$

that we introduced in the Physical Tangent Space-Functor Proposition 23.2 (2), as the differential of f at P.

Proof. We start with a preamble. In the proofs of (0) and (1) we will consider smooth maps $h\colon W \to \mathbb{R}^t$ where W is an open subset of some $\mathbb{R}^s$. For each $P \in W$ we denote by $\mathrm{D}h_P \in \mathrm{M}(t\times s, \mathbb{R})$ the differential as defined on page 536. Throughout the proofs of (0) and (1) we use this notation exclusively for such maps.

(0) Let M be a d-dimensional submanifold of $\mathbb{R}^m$ and let $P \in M$. First we consider the case that $P \in M \setminus \partial M$. Since M is a smooth submanifold of $\mathbb{R}^m$ we know that there exists a submanifold chart $\Phi\colon U \to V$ for $P \in M$. We write $Q := \Phi(P)$ and $\Psi := \Phi^{-1}\colon V \to U$. We write

$$\widetilde{V} \;:=\; \{x \in \mathbb{R}^d \,|\, (x,0) \in V\}.$$

This is an open subset of $\mathbb{R}^d$. We denote by $\iota\colon \widetilde{V} \to V$ the injective map given by $x \mapsto (x,0)$. Note that $Q = \iota(\widetilde{Q})$ for some $\widetilde{Q} \in \widetilde{V}$.

Claim.
$$\widetilde{\mathrm{T}}_P M \;=\; \underbrace{\mathrm{D}(\Psi\circ\iota)_{\widetilde{Q}}\cdot\mathbb{R}^d}_{\uparrow} \;\subset\; \mathbb{R}^m.$$
note that $\Psi\circ\iota\colon \widetilde{V} \to \mathbb{R}^m$ is indeed defined on an open subset of $\mathbb{R}^d$, thus it makes sense to consider the "classical" differential

Proof. We show the two inclusions "$\supset$" and "$\subset$" separately.

($\subset$) Let $\gamma\colon (-\epsilon,\epsilon) \to M$ be a smooth map with $\gamma(0) = P$. We need to show that $\gamma'(0) \in \mathrm{D}(\Psi\circ\iota)_{\widetilde{Q}}(\mathbb{R}^d)$. Note that after possibly reducing ϵ we can assume that $\gamma((-\epsilon,\epsilon)) \subset U$. We denote by $\pi\colon \mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d} \to \mathbb{R}^d$ the projection that is given by $(x,y) \mapsto x$. We see that

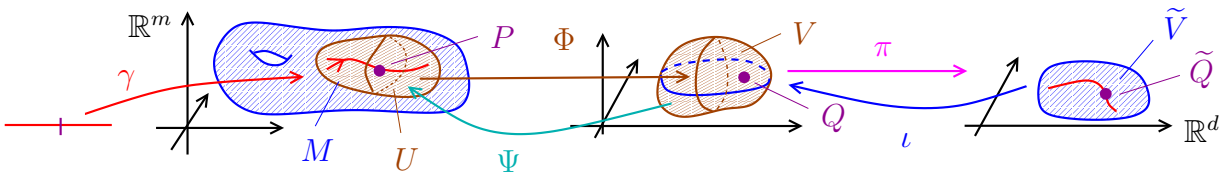
$$\gamma'(0) \;=\; (\Psi\circ\iota\circ\pi\circ\Phi\circ\gamma)'(0) \underset{\uparrow}{=} \mathrm{D}(\Psi\circ\iota)_{\widetilde{Q}}\cdot\underbrace{(\pi\circ\Phi\circ\gamma)'(0)}_{\text{vector in } \mathbb{R}^d}.$$
Chain Rule 19.10 together with the remark on page 536

($\supset$) Let $v \in \mathbb{R}^d$. Since $\widetilde{V}$ is open we there exists an $\epsilon > 0$ such that the corresponding curve $\gamma\colon (-\epsilon,\epsilon) \to \mathbb{R}^d$ given by $t \mapsto \widetilde{Q} + t\cdot v$ takes values in $\widetilde{V}$. Then

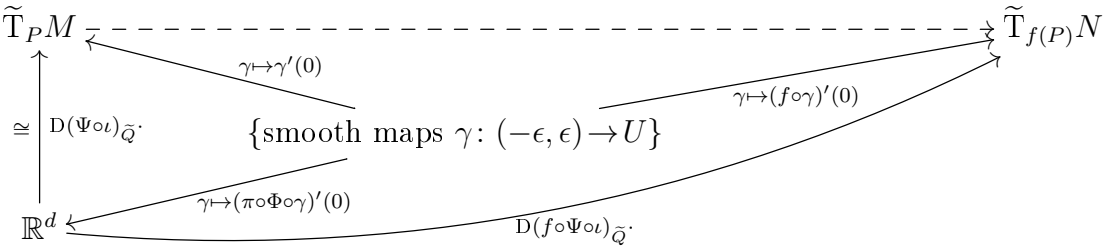
$$\mathrm{D}(\Psi\circ\iota)_{\widetilde{Q}}\cdot v \;=\; \mathrm{D}(\Psi\circ\iota)_{\widetilde{Q}}\cdot\gamma'(0) \underset{\uparrow}{=} (\Psi\circ\iota\circ\gamma)'(0).$$
Chain Rule 19.10

Since $\Psi\circ\iota\circ\gamma\colon (-\epsilon,\epsilon) \to M$ is a smooth map which sends 0 to P we have shown the "$\supset$"-inclusion. ⊞

This concludes the proof of Statement (0) if $P \in M \setminus \partial M$. The case that $P \in \partial M$ is treated in a very similar fashion. We leave it to the reader to fill in the details.



(1) To simplify the notation a little bit we consider the case that $P \notin \partial M$. The case that $P \in \partial M$ is dealt with in essentially the same way. We continue with the notation and setup that we introduced in (0) for $M \subset \mathbb{R}^m$. We consider the following diagram



We make the following observations:

(a) In the proof of the "$\subset$"-statement in (0) we already showed that the triangle to the left commutes.

(b) In (0) we showed that the left vertical map is an epimorphism.

(c) By the Chain Rule 19.10 we have $\mathrm{D}(\Psi \circ \iota)_{\widetilde{Q}} = \mathrm{D}\Psi_Q \cdot \mathrm{D}\iota_{\widetilde{Q}}$.

(d) Recall that Ψ is a diffeomorphism. It follows from this fact together with the Smooth Invariance of Domain Proposition 19.11 that $\mathrm{D}\Psi_Q$ is an invertible matrix.

(e) Since the inclusion ι is a linear map we obtain from the discussion on page 536 that multiplication by the matrix $\mathrm{D}\iota_{\widetilde{Q}}$ equals the inclusion $\mathbb{R}^d \to \mathbb{R}^m$.

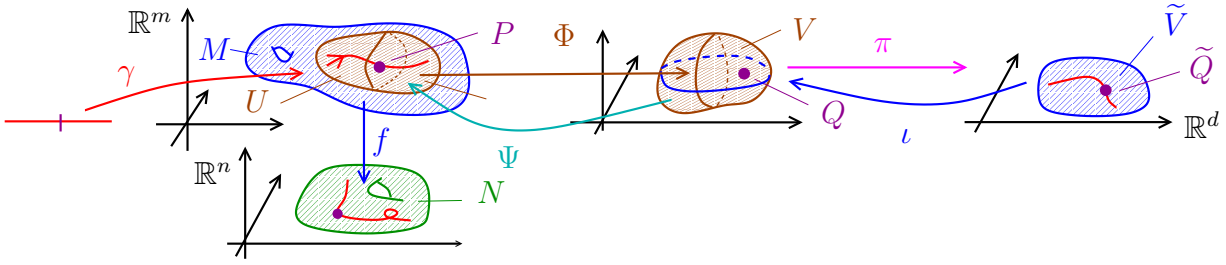
(f) It follows from (d) and (e) that the left vertical map is a monomorphism.

(g) The combination of (b) and (f) shows that the left vertical map is an isomorphism.

(h) It follows again from the Chain Rule 19.10 that the triangle to the bottom right commutes.

(i) Basically by definition of $\widetilde{\mathrm{T}}_P M$ the upper left diagonal map is a surjection.

The above observations imply that there is a unique horizontal map which makes the upper triangle commute. In other words, the map $\mathrm{D}f_P \colon \widetilde{\mathrm{T}}_P M \to \widetilde{\mathrm{T}}_{f(P)} N$ given by $\gamma'(0) \to (f \circ \gamma)'(0)$ is well-defined. Furthermore, since the above diagram commutes and since the left vertical map is an isomorphism and since the right diagonal map is a homomorphism we see that the top horizontal map is indeed a homomorphism.



In the following proof of (2) we denote by D again the differential as defined on page 623.

(2) (a) This statement is obvious.
(b) Let $f\colon L \to M$ and $g\colon M \to N$ be smooth maps between smooth submanifolds, let $P \in L$ and let $v \in \widetilde{\mathrm{T}}_P L$. Let $\gamma\colon I \to L$ be a smooth curve through P with $\gamma'(0) = v$. We see that

$$\mathrm{D}(g \circ f)_P(v) \;=\; ((g \circ f) \circ \gamma)'(0) \;=\; (g \circ (f \circ \gamma))'(0) \underset{\substack{\uparrow \\ \text{since } (f\circ\gamma)'(0) = \mathrm{D}f_P(v)}}{=} \mathrm{D}g_{f(P)}(\mathrm{D}f_P(v)).$$

■

Proposition 23.3. (Physical Tangent Space-Inclusion Proposition) Let M be a smooth submanifold of $\mathbb{R}^m$.

(1) Let W be a submanifold of M (e.g. W could be an open subset of M) and let $P \in W$. We denote by $i\colon W \to M$ the inclusion.
   (a) We have $\widetilde{\mathrm{T}}_P W \subset \widetilde{\mathrm{T}}_P M$ and the differential $\mathrm{D}i_P\colon \widetilde{\mathrm{T}}_P W \to \widetilde{\mathrm{T}}_P M$ is the inclusion.
   (b) If W is of codimension-zero, then we have $\widetilde{\mathrm{T}}_P W = \widetilde{\mathrm{T}}_P M$.
(2) We denote by $i\colon \partial M \to M$ the inclusion. We have $\widetilde{\mathrm{T}}_P \partial M \subset \widetilde{\mathrm{T}}_P M$ and the differential $\mathrm{D}i_P\colon \widetilde{\mathrm{T}}_P \partial M \to \widetilde{\mathrm{T}}_P M$ is the inclusion.
(3) Let N be a smooth submanifold of some $\mathbb{R}^n$. If $f\colon M \to N$ is a local diffeomorphism, then for every $P \in M$ the differential $\mathrm{D}f_P\colon \widetilde{\mathrm{T}}_P M \to \widetilde{\mathrm{T}}_{f(P)} N$ is an isomorphism.

Proof.

(1) (a) By the Submanifold Smooth Structure Lemma 19.24 we know that the inclusion $i\colon W \to M$ is smooth. Any smooth curve in W is thus a smooth curve in M. With this observation Statement (1a) is now obvious.
   (b) If W is of codimension-zero, then we know by the Physical Tangent Space-Functor Proposition 23.2 (0) that the vector spaces $\widetilde{\mathrm{T}}_P W$ and $\widetilde{\mathrm{T}}_P M$ have the same dimension. Thus it follows from (1a) that the vector spaces are actually the same.
(2) As discussed in the Smooth Manifold Boundary Proposition 19.26, the inclusion map $\partial M \to M$ is smooth. Thus the statement is obvious.
(3) Let $f\colon M \to N$ be a local diffeomorphism and let $P \in M$. Since f is a local diffeomorphism there exists an open neighborhood U of P and an open neighborhood V of $Q = f(P)$ such that the restriction $\varphi = f|_U\colon U \to V$ is a diffeomorphism. We write $\psi = \varphi^{-1}\colon V \to U$. We denote by $i\colon U \to M$ and $j\colon V \to N$ the inclusions. We obtain the following diagram:

$$\begin{array}{ccc}
\widetilde{\mathrm{T}}_P M & \xrightarrow{\quad \mathrm{D}f_P \quad} & \widetilde{\mathrm{T}}_{f(P)} N \\
{\scriptstyle \mathrm{D}i_P}\uparrow = & & = \uparrow{\scriptstyle \mathrm{D}j_{f(P)}} \\
\widetilde{\mathrm{T}}_P U & \underset{\mathrm{D}\psi_{f(P)}}{\overset{\mathrm{D}\varphi_P}{\rightleftarrows}} & \widetilde{\mathrm{T}}_{f(P)} V.
\end{array}$$

We make the following three observations:
   (a) We have $f \circ i = j \circ \varphi$. Thus it follows from the Physical Tangent Space-Functor Proposition 23.2 (2b) that the diagram commutes.

(b) It follows from $\psi \circ \varphi = \mathrm{id}_U$ and $\varphi \circ \psi = \mathrm{id}_V$, together with the Physical Tangent Space-Functor Proposition 23.2 (2) that $\mathrm{D}\varphi_P = \mathrm{D}\psi^{-1}_{f(P)}$. In particular we see that the bottom horizontal map is an isomorphism.

(c) By (1) we know that the vertical maps are the identity.

It follows from the above three observations that the top horizontal map is also an isomorphism. ■

The last lemma of this section relates the two notions of differentials that we have introduced so far.

Lemma 23.4. (Differential-equals-Differential Lemma) Let $f\colon M \to N$ be a smooth map between an *open subset of* $\mathbb{R}^m$ and a *smooth submanifold of* $\mathbb{R}^n$. For every $P \in M$ the following statement holds:

$\underbrace{\mathrm{D}f_P\colon \widetilde{\mathrm{T}}_P M = \mathbb{R}^m \to \widetilde{\mathrm{T}}_P N \subset \mathbb{R}^n}_{\text{differential as defined on page 623}}$ is given by multiplication by $\underbrace{\mathrm{D}f_P \in \mathrm{M}(n \times m, \mathbb{R}).}_{\text{the differential as defined on page 536}}$

Proof. Since $\mathrm{D}f_P\colon \widetilde{\mathrm{T}}_P M = \mathbb{R}^m \to \widetilde{\mathrm{T}}_P N = \mathbb{R}^n$ is linear we only have to check this statement for the standard basis $e_1, \ldots, e_m$ of $\widetilde{\mathrm{T}}_P M = \mathbb{R}^m$. Let $i \in \{1, \ldots, m\}$. Since M is an open subset of $\mathbb{R}^m$ there exists an $\epsilon > 0$ such that the map $\gamma\colon (-\epsilon, \epsilon) \to \mathbb{R}^m$ given by $t \mapsto P + t \cdot e_i$ takes values in M. Now we see:

$$(\mathrm{D}f_P)(e_i) \;=\; (\mathrm{D}f_P)(\gamma'(0)) \;\underset{\text{see page 623}}{\overset{\uparrow}{=}}\; (f \circ \gamma)'(0) \;\underset{\text{Chain Rule 19.10}}{\overset{\uparrow}{=}}\; \mathrm{D}f_P \cdot (\gamma'(0)) \;=\; \mathrm{D}f_P \cdot e_i$$

■

23.2. **Smooth manifolds and functors.** Before we move on to tangent spaces of general smooth manifolds it is convenient to formulate some of the above results in the language of category theory.

Definition.

(1) We refer to the category $\mathcal{SmMfd}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{SmMfd}) &:= \text{all smooth manifolds,} \\ \mathrm{Mor}(X, Y) &:= \text{all smooth maps from } X \text{ to } Y, \end{aligned}$$

with the usual composition of maps as the category of smooth manifolds.

(2) We refer to the category $\mathcal{SmMfd}_{\mathbb{R}^n}$ with[244]

$$\begin{aligned} \mathrm{Ob}(\mathcal{SmMfd}_{\mathbb{R}^n}) &:= \text{all smooth submanifolds of } \mathbb{R}^n \text{ for an arbitrary } n \in \mathbb{N}_0, \\ \mathrm{Mor}(X, Y) &:= \text{all smooth maps from } X \text{ to } Y, \end{aligned}$$

with the usual composition of maps as the category of smooth submanifolds of some $\mathbb{R}^n$.

In our setting we need a slight variation on the above categories.

Definition.

[244] Here the dimension n is variable, thus the category contains as objects that are submanifolds of $\mathbb{R}^0$, $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, and so on.

(1) A pointed smooth manifold is a pair (M, x_0) where M is a smooth manifold and x_0 is a point in M.
(2) We refer to the category $\mathcal{PSmMfd}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{PSmMfd}) &:= \text{all pointed smooth manifolds,} \\ \mathrm{Mor}((M,x_0),(N,y_0)) &:= \text{all smooth maps } f \text{ from } M \text{ to } N \text{ with } f(x_0) = y_0 \end{aligned}$$

with the usual composition of maps as the category of pointed smooth manifolds.
(3) We define the category $\mathcal{PSmMfd}_{\mathbb{R}^n}$ of pointed smooth submanifolds of some $\mathbb{R}^n$ exactly the same way.

With these definitions we can rephrase (most of) the Physical Tangent Space-Functor Proposition 23.2 as follows.

Proposition 23.5. The maps

$$\begin{aligned} (M,P) &\mapsto \widetilde{\mathrm{T}}_P M \\ (f\colon (M,P) \to (N,Q)) &\mapsto (\mathrm{D}f_P\colon \widetilde{\mathrm{T}}_P M \to \widetilde{\mathrm{T}}_Q N) \end{aligned}$$

define a covariant functor from the category $\mathcal{PSmMfd}_{\mathbb{R}^n}$ of pointed smooth submanifolds of some $\mathbb{R}^n$ to the category $\mathcal{Vec}_{\mathbb{R}}$ of $\mathbb{R}$-vector spaces.

Note that $\mathcal{SmMfd}_{\mathbb{R}^n}$ is a full subcategory of $\mathcal{SmMfd}$ in the sense of the definition of a full subcategory on page 197. Similar $\mathcal{PSmMfd}_{\mathbb{R}^n}$ is a full subcategory of $\mathcal{PSmMfd}$. Thus the following question arises:

Question 23.6.

(1) Can we "extend" the covariant functor

$$\begin{aligned} \mathcal{PSmMfd}_{\mathbb{R}^n} &\to \mathcal{Vec}_{\mathbb{R}} \\ (M,P) &\mapsto \widetilde{\mathrm{T}}_P M \\ (f\colon (M,Q) \to (N,Q)) &\mapsto (\mathrm{D}f_P\colon \widetilde{\mathrm{T}}_P M \to \widetilde{\mathrm{T}}_Q N) \end{aligned}$$

to a covariant functor $\mathcal{PSmMfd} \to \mathcal{Vec}_{\mathbb{R}}$?

(2) If we can extend the covariant functor, is the extension "essentially unique"?

We will make Question 23.6 (1) more precise in the following section. There are many reasons why one should hope for a positive answer to Question 23.6 (1). For example, even if we only care about smooth submanifolds of some $\mathbb{R}^n$, very often it is convenient to work with "abstract" descriptions. For example in Torus-as-Quotient Lemma 19.34 we showed that the torus $S^1 \times S^1$ is diffeomorphic to $\mathbb{R}^2/\mathbb{Z}^2$. In many situations it is much more pleasant to work with the smooth manifold $\mathbb{R}^2/\mathbb{Z}^2$ instead of working directly with the torus $S^1 \times S^1$. The reader will thus be pleased to hear that we will give a positive answer in the Tangent Space-Isomorphism Proposition 23.16 (4).

Finally note that once we have proved the Euclidean Embedding Theorem 27.1 we will be able to provide an answer to Question 23.6 (2) in Exercise 27.1.

23.3. Tangent spaces of smooth manifolds.

In this section we introduce the notion of tangent spaces to points on general smooth manifolds. We jump right into the fray by giving the definition. We will then spend some time trying to explain why this, initially invariably confusing, definition is actually meaningful.

Definition. Let M be a smooth manifold and let $P \in M$.[245]

(1) A derivation at P is a linear map[246] $w\colon C^\infty(M,\mathbb{R}) \to \mathbb{R}$ such that for every choice of $f, g \in C^\infty(M,\mathbb{R})$ we have

$$w(f \cdot g) \;=\; f(P) \cdot w(g) + w(f) \cdot g(P) \qquad \text{“Leibniz rule”}.$$

(2) The set of all derivations at P is called the tangent space $\mathrm{T}_P M$ at P. We refer to elements of $\mathrm{T}_P M$, i.e. we refer to derivations at P, also as tangent vectors. Sometimes, especially when we also talk about the physical tangent space $\widetilde{\mathrm{T}}_P M$, we refer to $\mathrm{T}_P M$ also the abstract tangent space at P.

Lemma 23.7. Let M be a smooth manifold. For every $P \in M$ the tangent space $\mathrm{T}_P M$ is a vector subspace of the vector space of linear maps $C^\infty(M,\mathbb{R}) \to \mathbb{R}$.

Proof. Let v, w be two derivations at P. We need to show that $v + w$ is also a derivation at P. Thus let $f, g \in C^\infty(M,\mathbb{R})$. We perform the following calculation:

$$\begin{aligned}(v+w)(f\cdot g) \;=\; v(f\cdot g) + w(f\cdot g) \;&=\; f(P)\cdot v(g) + v(f)\cdot g(P) + f(P)\cdot w(g) + w(f)\cdot g(P)\\ &=\; f(P)\cdot(v(g)+w(g)) + (v(f)+w(f))\cdot g(P)\\ &=\; f(P)\cdot(v+w)(g) + (v+w)(f)\cdot g(P).\end{aligned}$$

An even easier argument shows that if $\lambda \in \mathbb{R}$, then $\lambda \cdot v$ is also a derivation at P. Finally note that evidently the zero homomorphism is a derivation. ∎

Lemma 23.8. Let M be a smooth manifold, let $P \in M$ and let $w \in \mathrm{T}_P M$.

(1) If $f\colon M \to \mathbb{R}$ is a constant map, then $w(f) = 0$.
(2) Let $f, g \in C^\infty(M,\mathbb{R})$. If $f(P) = g(P) = 0$, then $w(f \cdot g) = 0$.
(3) Let $f, g \in C^\infty(M,\mathbb{R})$. If there exists some open neighborhood U of P such that $f|_U = g|_U$, then $w(f) = w(g)$.

Proof.

(1) Let $f\colon M \to \mathbb{R}$ be a constant map. We denote by c the map that is $= 1$ everywhere. We perform the following calculation:

$$w(f) \underset{\substack{\uparrow\\ \text{follows from } f = f(P)\cdot c,\\ \text{since } w \text{ linear}}}{=} f(P)\cdot w(c) \underset{\substack{\uparrow\\ \text{since } c = c\cdot c}}{=} f(P)\cdot w(c\cdot c) \underset{\substack{\uparrow\\ \text{Leibniz rule}}}{=} f(P)\cdot\big(\underbrace{c(P)}_{=1}\cdot w(c) + w(c)\cdot\underbrace{c(P)}_{=1}\big) = f(P)\cdot 2\cdot w(c).$$

Comparing the second and the last term we see that those terms, and hence all terms, are zero.

(2) Let $f, g \in C^\infty(M,\mathbb{R})$. If $f(P) = g(P) = 0$, then

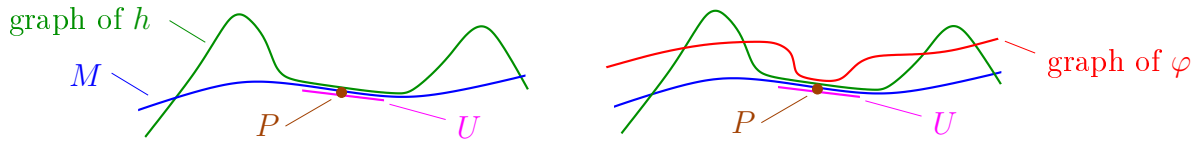
$$w(f\cdot g) \underset{\substack{\uparrow\\ \text{by definition of a derivation}}}{=} f(P)\cdot w(g) + w(f)\cdot g(P) \;=\; 0\cdot w(g) + w(f)\cdot 0 \;=\; 0.$$

[245] We should stress that we allow P to be in the boundary of M.
[246] Recall that according to the definition on page 546 we denote by $C^\infty(M,\mathbb{R})$ the vector space of all smooth maps $M \to \mathbb{R}$.

(3) Let $f, g \in C^\infty(M, \mathbb{R})$. We suppose that there exists some open neighborhood U of P such that $f|_U = g|_U$. We set $h := f - g$. Note that h vanishes on U. Using the Smooth Transition Function Lemma 19.4 one can easily show that there exists a smooth map $\varphi\colon M \to \mathbb{R}$ which is $\equiv 1$ on $M \setminus U$ and which satisfies $\varphi(P) = 0$. We see that

$$w(f) - w(g) \underset{\substack{\uparrow \\ \text{since } w \text{ linear}}}{=} w(h) \underset{\substack{\uparrow \\ \text{since } h = h\cdot\varphi}}{=} w(h \cdot \varphi) \underset{\substack{\uparrow \\ \text{by (2) since } h(P)=\varphi(P)=0}}{=} 0.$$

■



This definition of a tangent space might sound rather odd, but the following proposition should help in getting some intuition for the definition.

Proposition 23.9. Let M be an open subset of $\mathbb{R}^n$. For any $P \in M$ the map

$$\begin{aligned} \Xi_P\colon \mathbb{R}^n &\to \mathrm{T}_P M \\ v &\mapsto \Big(\begin{aligned} C^\infty(M,\mathbb{R}) &\to \mathbb{R} \\ f &\mapsto \underbrace{\tfrac{d}{dt}\big|_{t=0} f(P + t\cdot v))}_{= \mathrm{D}f_P \cdot v \text{ by the Chain Rule 19.10}} \end{aligned} \Big) \end{aligned}$$

is an isomorphism of vector spaces.

Proof. Let M be an open subset of $\mathbb{R}^n$ and let $P = (P_1, \ldots, P_n) \in M$. Our first claim implies that Ξ_P actually takes values in $\mathrm{T}_P M$.

Claim. For every $v \in \mathbb{R}^n$ the map

$$\begin{aligned} C^\infty(M,\mathbb{R}) &\to \mathbb{R} \\ f &\mapsto \mathrm{D}f_P \cdot v \end{aligned}$$

is a derivation.

Proof. It is clear that the map is linear in f. So it remains to show that it satisfies the Leibniz rule. Let $f, g \in C^\infty(M, \mathbb{R})$. Note that the map $f \cdot g$ equals the following composition of maps:

$$M \xrightarrow[=:d]{x \mapsto (x,y)} M \times M \xrightarrow{f \times g} \mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \xrightarrow[=:m]{(x,y)\mapsto x\cdot y} \mathbb{R}.$$

We see that

$$\begin{aligned} \mathrm{D}(f\cdot g)_P \cdot v &= \mathrm{D}(m \circ (f \times g) \circ d)_P \cdot v \overset{\text{Chain Rule 19.10}}{=} \mathrm{D}m_{(f(P),g(P))} \cdot (\mathrm{D}(f \times g)_{P\times P} \cdot \mathrm{D}d_P) \cdot v \\ &= (g(P)\ f(P)) \begin{pmatrix} \mathrm{D}f_P & 0 \\ 0 & \mathrm{D}g_P \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot v = g(P)\cdot \mathrm{D}f_P \cdot v + f(P) \cdot \mathrm{D}g_P \cdot v. \end{aligned}$$

We have thus successfully verified that the Leibniz rule holds. ⊞

It is clear that the map $\Xi_P\colon \mathbb{R}^n \to \mathrm{T}_P M$ given by $v \mapsto \mathrm{D}f_P \cdot v$ is linear. Thus it remains to show that Ξ_P is injective and surjective. Before we continue we introduce one piece of notation, namely given $i \in \{1, \ldots, n\}$ we denote by $c_i\colon M \to \mathbb{R}$ the map given by $(x_1, \ldots, x_n) \mapsto x_i$.

Claim. The map $\Xi_P\colon \mathbb{R}^n \to \mathrm{T}_P M$ is injective.

Proof. Let $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ with $\Xi_P(v) = 0$. Given any $i \in \{1, \dots, n\}$ we see that

$$0 \;=\; \Xi_P(v)(c_i) \;=\; \tfrac{d}{d}\big|_{t=0}\, c_i(P + t\cdot v) \;\underset{\text{definition of } c_i}{\overset{\uparrow}{=}}\; \tfrac{d}{dt}\big|_{t=0}\,(P_i + t\cdot v_i) \;=\; v_i.$$

We have now shown that $v = (0, \dots, 0) = 0 \in \mathbb{R}^n$. ⊞

Claim. The map $\Xi_P\colon \mathbb{R}^n \to \mathrm{T}_P M$ is surjective.

Proof. Let $w \in \mathrm{T}_P M$. We set $v_i := w(c_i) \in \mathbb{R}$ and we set $v := (v_1, \dots, v_n) \in \mathbb{R}^n$. We claim that $\Xi_P(v) = w$. Let $f \in C^\infty(M, \mathbb{R})$. By Taylor's Theorem 19.7 we have

$$f(x) \;=\; f(P) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\big|_P \cdot (c_i - P_i) + \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n} (c_i - P_i)(c_j - P_j)\cdot \int_0^1 (1-t)\frac{\partial^2}{\partial x_i \partial x_j}\big|_{P+t(x-P)} dt}_{\text{it follows from Lemma 23.8 (2) that } w(\dots) = 0}.$$

Finally we see that we have the following equalities:

$$w(f) \;\underset{\text{above observation}}{\overset{\uparrow}{=}}\; \overbrace{w(f(P))}^{=0 \text{ by Lemma 23.8}} + w\Big(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\big|_P \cdot (c_i - P_i)\Big) \;\underset{\text{linearity of } w}{\overset{\uparrow}{=}}\; \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\big|_P \cdot (\underbrace{w(c_i)}_{=v_i} - \underbrace{w(P_i)}_{=0}) \;\underset{\text{Chain Rule 19.10}}{\overset{\uparrow}{=}}\; \Xi_P(v).$$

■

The following proposition summarizes some basic properties of the tangent space. It can be viewed as an analogue of the Physical Tangent Space-Functor Proposition 23.2 (1) and (2) and of Proposition 23.5.

Proposition 23.10. (Tangent Space-Functor Proposition) Let M and N be two smooth manifolds.

(1) If $f\colon M \to N$ is a smooth map, then for every $P \in M$ the map

$$\begin{array}{rcl} \mathrm{D}f_P\colon \mathrm{T}_P M & \to & \mathrm{T}_{f(P)} N \\ v & \mapsto & \left(\begin{array}{rcl} C^\infty(N, \mathbb{R}) & \to & \mathbb{R} \\ g & \mapsto & v(g \circ f) \end{array}\right) \end{array}$$

is well-defined and linear.

(2) The maps from (1) have the following two properties:

(a) If $\mathrm{id}\colon M \to M$ is the identity map, then for any $P \in M$ the differential $\mathrm{D}\,\mathrm{id}_P\colon \mathrm{T}_P M \to \mathrm{T}_P M$ is also the identity.

(b) For any two smooth maps $f\colon L \to M$ and $g\colon M \to N$ between smooth manifolds and any $P \in L$ we have

$$\mathrm{D}(g \circ f)_P \;=\; \mathrm{D}g_{f(P))} \circ \mathrm{D}f_P\colon \mathrm{T}_P L \;\to\; \mathrm{T}_{g(f(P)} N \qquad \text{“Chain Rule”}.$$

(3) The maps

$$\begin{array}{rcl} (M, P) & \mapsto & \mathrm{T}_P M \\ (f\colon (M, Q) \to (N, Q)) & \mapsto & (\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_Q N) \end{array}$$

define a covariant functor from the category $\mathcal{PSmMfd}$ of pointed smooth manifolds to the category $\mathcal{Vec}_{\mathbb{R}}$ of $\mathbb{R}$-vector spaces.

Definition. Given a smooth map $f\colon M \to N$ between two smooth manifolds and given $P \in M$ we refer to the map

$$\begin{array}{rccl} \mathrm{D}f_P\colon & \mathrm{T}_P M & \to & \mathrm{T}_{f(P)}N \\ & v & \mapsto & (g \mapsto v(g \circ f)) \end{array}$$

from the Tangent Space-Functor Proposition 23.10 (1) as the differential of f at P.

Proof.

(1) The only statement one needs to think about for a second is, why does $\mathrm{D}f_P$ take values in $\mathrm{T}_{f(P)}N$? Thus let $v \in \mathrm{T}_P M$. We need to show that $g \mapsto v(g \circ f)$ is indeed a derivation. Now let $g, h \in C^\infty(N, \mathbb{R})$. We perform the following little calculation:

$$v((g \cdot h) \circ f) \;=\; v((g \circ f) \cdot (h \circ f)) \;=\; (g \circ f)(P) \cdot v(h \circ f) + v(g \circ f) \cdot (h \circ f)(P).$$

We have thus verified that the Leibniz rule is satisfied.

(2) This statement follows follow almost immediately from the definitions.

(3) This statement just summarizes the contents of (1) and (2) and of Lemma 23.7. ■

Example. Let M and N be two smooth manifolds. If $f\colon M \to N$ is a constant map, then it follows from Lemma 23.8 that for each $P \in M$ the differential $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}N$ is actually the zero homomorphism.

The following proposition is the analogue of the Physical Tangent Space-Functor Proposition 23.2 (0).

Proposition 23.11. (Tangent Space-Dimension Proposition) If M is an n-dimensional smooth manifold, then for every $P \in M$ the corresponding tangent space $\mathrm{T}_P M$ is an n-dimensional vector subspace of the vector space $\mathrm{Hom}(C^\infty(M, \mathbb{R}), \mathbb{R})$.

The proof of the Tangent Space-Dimension Proposition 23.11 relies on the following two lemmas.

Lemma 23.12. Let M be a smooth manifold and let $U \subset M$ be an open subset U. We denote by $\iota\colon U \to M$ the inclusion. For every $P \in U$ the map $\mathrm{D}\iota_P\colon \mathrm{T}_P U \to \mathrm{T}_P M$ is an isomorphism.

Proof. The lemma follows, with a little bit of thought, from Lemma 23.8 (3). We will fill in the details in Exercise 23.5. ■

Lemma 23.13. Let $n \in \mathbb{N}$ and let $V \subset \mathbb{H}^n$ be an open subset. We denote by $\iota\colon V \to \mathbb{R}^n$ the inclusion. For every $P \in \partial V$ the map $\mathrm{D}\iota_P\colon \mathrm{T}_P V \to \mathrm{T}_P \mathbb{R}^n$ is an isomorphism.

Proof. Let $P \in \partial V$. First we show that the map $\mathrm{D}\iota_P\colon \mathrm{T}_P V \to \mathrm{T}_P \mathbb{R}^n$ is injective. Thus let $w \in \mathrm{T}_P V$ with $\mathrm{D}\iota_P(w) = 0$. We need to show that $w(f) = 0$ for every smooth map $f\colon V \to \mathbb{R}$. Thus let $f\colon V \to \mathbb{R}$ be a smooth map. By definition of a smooth map on $\mathbb{H}^n$, seepage 532, there exists an open neighborhood $W \subset \mathbb{R}^n$ of P and a smooth map $g\colon W \to \mathbb{R}$ with $f|_{V \cap W} = g|_{V \cap W}$. By Exercise 19.18, applied to g, there exists furthermore an open neighborhood X of P and a smooth map $h\colon \mathbb{R}^n \to \mathbb{R}$ such that $h|_{W \cap X} = g|_{W \cap X}$.

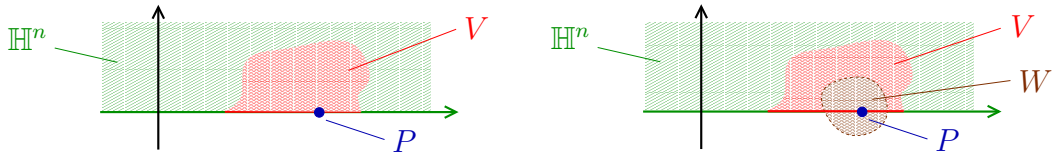
We calculate

$$w(f) \underset{\substack{\uparrow\\ \text{by Lemma 23.8 (3), since}\\ f \text{ and } h|_W \text{ agree on } W\cap X}}{=} w(h|_W) = w(h\circ\iota) \underset{\substack{\uparrow\\ \text{by definition of } \mathrm{D}\iota_P(w)}}{=} (\mathrm{D}\iota_P(w))(h) = 0.$$

It remains to show that $\mathrm{D}\iota_P\colon \mathrm{T}_P W \to \mathrm{T}_P\mathbb{R}^n$ is surjective. For $i=1,\dots,n$ we write

$$\begin{aligned} C^\infty(W,\mathbb{R}) &\xrightarrow{v_i} \mathbb{R} \\ f &\mapsto \lim_{h\to 0^+} \tfrac{1}{h}\cdot(f(\underbrace{P+h\cdot e_i}_{\in W \text{ for small } h>0}) - f(P)) \end{aligned} \quad \& \quad \begin{aligned} C^\infty(\mathbb{R}^n,\mathbb{R}) &\xrightarrow{w_i} \mathbb{R} \\ f &\mapsto \underbrace{\lim_{h\to 0} \tfrac{1}{h}\cdot(f(P+h\cdot e_i) - f(P))}_{=\frac{\partial}{\partial x_i}|_P}. \end{aligned}$$

One can easily verify that $v_1,\dots,v_n$ are derivations. Evidently we have $(\mathrm{D}\iota_P)(v_i) = w_i$. By Proposition 23.9 we know that $w_1,\dots,w_n$ form a basis for $\mathrm{T}_P\mathbb{R}^n$. We have thus shown that $\mathrm{D}\iota_P$ is a surjection. ∎



Proof of the Tangent Space-Dimension Proposition 23.11. Let M be an n-dimensional smooth manifold. Furthermore let $P\in M$. First we consider the case that $P\notin \partial M$. Thus we can pick a chart $\Phi\colon U\to V$ of type (i) where V is an open subset of $\mathbb{R}^n$. We denote by $i\colon U\to M$ the inclusion map. We have the following isomorphisms:

$$\mathrm{T}_P M \xleftarrow[\substack{\uparrow\\ \text{isomorphism by}\\ \text{Lemma 23.12}}]{\mathrm{D}i_P} \mathrm{T}_P U \xrightarrow[\substack{\uparrow\\ \text{by the Tangent Space-Functor}\\ \text{Proposition 23.10 this is an isomorphism}\\ \text{with inverse given by } \mathrm{D}(\Phi^{-1})_{\Phi(P)}}]{\mathrm{D}\Phi_P} \mathrm{T}_{\Phi(P)}V \xrightarrow[\substack{\uparrow\\ \text{isomorphism given by}\\ \text{Proposition 23.9}}]{\cong} \mathbb{R}^n.$$

Now suppose that $P\in\partial M$. This means that there exists a chart $\Phi\colon U\to V$ of type (ii) where V is an open subset of $\mathbb{H}^n$ and $\Phi(P)\in\partial\mathbb{H}^n$. We denote by $j\colon V\to\mathbb{R}^n$ the inclusion map. In this setting we have the following isomorphisms:

$$\mathrm{T}_P M \xleftarrow[\substack{\uparrow\\ \text{isomorphism by}\\ \text{Lemma 23.12}}]{\mathrm{D}i_P} \mathrm{T}_P U \xrightarrow[\substack{\uparrow\\ \text{by the Tangent Space-Functor}\\ \text{Proposition 23.10 this is an}\\ \text{isomorphism with inverse given}\\ \text{by } \mathrm{D}(\Phi^{-1})_{\Phi(P)}}]{\mathrm{D}\Phi_P} \mathrm{T}_{\Phi(P)}V \xrightarrow[\substack{\uparrow\\ \text{isomorphism}\\ \text{given by}\\ \text{Lemma 23.13}}]{\mathrm{D}j_P} \mathrm{T}_{\Phi(P)}\mathbb{R}^n \xrightarrow[\substack{\uparrow\\ \text{isomorphism given by}\\ \text{Proposition 23.9}}]{\cong} \mathbb{R}^n.$$

∎

The following proposition is the analogue of the Physical Tangent Space-Inclusion Proposition 23.3.

Proposition 23.14. Let M be a smooth manifold.

(1) Let W be a submanifold of M (e.g. W could be an open subset of M) and let $P\in W$. We denote by $i\colon W\to N$ the inclusion.
   (a) The differential $\mathrm{D}i_P\colon \mathrm{T}_P W\to \mathrm{T}_P M$ is a monomorphism.

(b) If W is of codimension-zero, then the differential $\mathrm{D}i_P\colon \mathrm{T}_P W \to \mathrm{T}_P M$ is an isomorphism.

(2) We denote by $i\colon \partial M \to M$ the inclusion. The differential $\mathrm{D}i_P\colon \mathrm{T}_P \partial M \to \mathrm{T}_P M$ is a monomorphism.

(3) Let N be a smooth manifold. If $f\colon M \to N$ is a local diffeomorphism, then for every $P \in M$ the differential $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ is an isomorphism.

Convention. Let M be a smooth manifold. Since it is notationally dreadful to carry around inessential information we will usually use the following two conventions.

(1) Let W be a submanifold of M. We denote by $i\colon W \to M$ the inclusion map. By the Physical Tangent Space-Inclusion Proposition 23.14 we know that the map $\mathrm{D}i_P\colon \mathrm{T}_P W \to \mathrm{T}_P M$ is a monomorphism. On many occasions we will just identify $\mathrm{T}_P W$ with its image under $\mathrm{D}i_P$ in $\mathrm{T}_P M$. In particular we will view $\mathrm{T}_P W$ as a vector subspace of $\mathrm{T}_P M$.

(2) Let $P \in \partial M$. By the same logic as in (1) we will view $\mathrm{T}_P \partial M$ as a vector subspace of $\mathrm{T}_P M$.

This convention brings us in line with corresponding situation for physical tangent spaces as described in the Physical Tangent Space-Inclusion Proposition 23.3 (1) and (2).

Proof. Let M be an n-dimensional smooth manifold.

(1) (a) Let W be a k-dimensional submanifold of M and let $P \in W$. In the following we deal with the case that P admits a submanifold chart of type (α) in the sense of the definition on page 550. Using Lemma 23.13 the other cases are treated in a similar fashion. Thus let $\Phi\colon U \to V$ be a submanifold chart of type (α). We write $H = \{0\} \times \mathbb{R}^k \subset \mathbb{R}^n$ and $Q = \Phi(P) \in H$. We denote by $\iota\colon H \to \mathbb{R}^n$ the inclusion and we denote by $\pi\colon \mathbb{R}^n \to H$ the projection that is given by $(x, y) \mapsto (0, y)$. We consider the following diagram:

$$\begin{array}{ccccccc}
\mathrm{T}_P W & \longleftarrow & \mathrm{T}_P(W \cap U) & \xrightarrow{\mathrm{D}(\Phi|_{W\cap U})_P} & \mathrm{T}_Q(V \cap H) & \longrightarrow & \mathrm{T}_Q H \\
\downarrow{\scriptstyle \mathrm{D}i_P} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{D}\iota_Q} \quad \circlearrowleft\ {\scriptstyle \mathrm{D}\pi_Q} \\
\mathrm{T}_P M & \longleftarrow & \mathrm{T}_P U & \xrightarrow{\mathrm{D}\Phi_P} & \mathrm{T}_Q V & \longrightarrow & \mathrm{T}_Q \mathbb{R}^n.
\end{array}$$

We make the following clarifications and observations:

- All maps that are not given explicitly are the differentials of the obvious inclusion maps.
- It follows from the Tangent Space-Functor Proposition 23.10 (2) that the diagram commutes.
- It follows from Lemma 23.12 that the horizontal maps to the left and to the right are isomorphisms.
- Since Φ and also $\Phi|_{W\cap U}$ are diffeomorphisms we obtain from from the Tangent Space-Functor Proposition 23.10 that the middle horizontal maps are isomorphisms.
- Note that $\pi \circ \iota\colon H \to H$ is the identity. It follows from the Tangent Space-Functor Proposition 23.10 (2) that $\mathrm{D}\pi_Q \circ \mathrm{D}\iota_Q = \mathrm{D}(\pi \circ \iota)_Q = \mathrm{D}\,\mathrm{id}_Q = \mathrm{id}$. Thus we see that $\mathrm{D}\iota_Q$ is a monomorphism.

The combination of the above shows that in fact all vertical maps are monomorphisms.

(b) This statement follows immediately from (a) and the fact, which we just proved in the Tangent Space-Dimension Proposition 23.11, that $\mathrm{T}_P W$ and $\mathrm{T}_P M$ have the same dimension.

(2) The proof of this statement is very similar to the proof of Statement (1). We leave it to the reader to fill in the details.

(3) The proof of this statement is basically identical to the proof of the analogous statement in the Physical Tangent Space-Inclusion Proposition 23.3, all we have to is to replace the Physical Tangent Space-Functor Proposition 23.2 (2) by the Tangent Space-Functor Proposition 23.10 (2). Again we leave it to the reader to fill in the details. ■

We conclude this section with a short discussion of outward and inward pointing tangent vectors. These notions are the analogues of the notions we introduced on page 620.

Definition. Let M be a smooth manifold and let $P \in \partial M$. We say a tangent vector $w \in \mathrm{T}_P M$ points outward, if the following two conditions are satisfied:

(1) For every smooth map $f\colon M \to \mathbb{R}_{\geq 0}$ with $f|_{\partial M} \equiv 0$ we have $w(f) \geq 0$.

(2) There exists a smooth map $f\colon M \to \mathbb{R}_{\geq 0}$ with $f|_{\partial M} \equiv 0$ such that $w(f) > 0$.

By replacing "$\geq$" and "$>$" by "$\leq$" and "$<$" we obtain the notion of an inward pointing tangent vector.

The following lemma says that for a point $P \in \partial M$ there is a trichotomy for the tangent vectors in $\mathrm{T}_P M$.

Lemma 23.15. Let M be a smooth manifold, let $P \in \partial M$ and let $w \in \mathrm{T}_P M$. Precisely one of the following three statements holds:

(1) w points outward,

(2) w points inward,

(3) $w \in \mathrm{D}\iota_P(\mathrm{T}_P\, \partial M \to \mathrm{T}_P M)$, where $\iota\colon \partial M \to M$ is the inclusion map.

Proof. We will prove the lemma in Exercise 23.7. ■

23.4. **The relationship between the definitions of tangent spaces.** Given a k-dimensional smooth submanifold M of some $\mathbb{R}^m$ and given $P \in M$ we have now introduced two notions of tangent spaces. On page 620 we introduced the physical tangent space $\widetilde{\mathrm{T}}_P M$, which is a k-dimensional vector subspace of $\mathbb{R}^m$. Later, on page 628 we introduced the tangent space $\mathrm{T}_P M$ in a significantly more abstract way.

The next proposition elucidates the relationship between the physical tangent space $\widetilde{\mathrm{T}}_P M$ and the tangent space $\mathrm{T}_P M$.

Proposition 23.16. (Tangent Space-Isomorphism Proposition)

(1) Let M be a k-dimensional submanifold of $\mathbb{R}^m$ and let $P \in M$. The map

$$\begin{array}{rcl} \Theta_M \colon \widetilde{\mathrm{T}}_P M & \to & \mathrm{T}_P M \\ v & \mapsto & \left(\begin{array}{rcl} C^\infty(M,\mathbb{R}) & \to & \mathbb{R} \\ g & \mapsto & \underset{\uparrow}{\mathrm{D}g_P(v)} \end{array} \right) \end{array}$$

where $\mathrm{D}g_P \colon \widetilde{\mathrm{T}}_P M \to \widetilde{\mathrm{T}}_{g(P)}\mathbb{R} = \mathbb{R}$

is well-defined and it is an isomorphism of vector spaces.

(2) Let M be a k-dimensional submanifold of $\mathbb{R}^m$. For $P \in \partial M$ the isomorphism Θ_M restricts to a bijection between outward tangent vectors in $\widetilde{\mathrm{T}}_P M$ (as defined on page 620) and outward tangent vectors in $\mathrm{T}_P M$ (as defined on page 634). The same statement also holds with "outward" replaced by "inward".

(3) If $f \colon M \to N$ is a smooth map between submanifolds of $\mathbb{R}^m$ respectively $\mathbb{R}^n$, then the following diagram commutes

$$\begin{array}{ccc} \widetilde{\mathrm{T}}_P M & \xrightarrow{\Theta_M} & \mathrm{T}_P M \\ {\scriptstyle \mathrm{D}f_P}\downarrow & & \downarrow{\scriptstyle \mathrm{D}f_P} \\ \widetilde{\mathrm{T}}_{f(P)} N & \xrightarrow{\Theta_N} & \mathrm{T}_{f(P)} N. \end{array}$$

Here to the left we consider the differential as defined on page page 623 and to the right we consider the differential as defined on page 631.

(4) The maps Θ_M defined in (1) give a natural isomorphism, in the sense of the definition on page 203, between the covariant functors

$$\begin{array}{rcl} \mathcal{PSmMfd}_{\mathbb{R}^n} & \to & \mathcal{Vec}_{\mathbb{R}} \\ (M,P) & \mapsto & \widetilde{\mathrm{T}}_P M \\ f & \mapsto & \mathrm{D}f_P \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathcal{PSmMfd}_{\mathbb{R}^n} & \to & \mathcal{Vec}_{\mathbb{R}} \\ (M,P) & \mapsto & \mathrm{T}_P M \\ f & \mapsto & \mathrm{D}f_P \end{array}$$

which we introduced in Propositions 23.5 and 23.10.

Remark.

(1) The Tangent Space-Isomorphism Proposition 23.16 (4) gives an affirmative answer to Question 23.6 (1).

(2) What matters in the end is that we managed to define a functor $(M,P) \to \mathrm{T}_P M$ that satisfies the Tangent Space-Isomorphism Proposition 23.16. The actual definition of $\mathrm{T}_P M$ is in the following utterly irrelevant. In fact in Section 23.5 we will sketch a different construction of such a functor, but at the end the actual definition of $\mathrm{T}_P M$ never gets used.

Proof. We prove the three statements in a different order than formulated.

(3) Let $f \colon M \to N$ be a smooth map between submanifolds of $\mathbb{R}^m$ respectively $\mathbb{R}^n$. Furthermore let $v \in \widetilde{\mathrm{T}}_P M$ and let $g \in C^\infty(N,\mathbb{R})$. We see that

$$(\mathrm{D}f_P \circ \Theta_M)(v)(g) \;=\; \mathrm{D}(g \circ f)_P(v) \;\underset{\uparrow}{=}\; (\mathrm{D}g_{f(P)} \circ \mathrm{D}f_P)(v) \;=\; (\Theta_N \circ \mathrm{D}f_P)(v)(g).$$

chain rule of the Physical Tangent Space-Functor Proposition 23.2 (2)

(1) As in the proof of Proposition 23.9 we see that Θ_M is well-defined in the sense that for any $v \in \widetilde{\mathrm{T}}_P M$ the map $C^\infty(M,\mathbb{R}) \to \mathbb{R}$ given by $g \mapsto \mathrm{D}g_P(v)$ is indeed a derivation.

It remains to show that the maps Θ_M are isomorphisms. Let M be a k-dimensional smooth submanifold of $\mathbb{R}^m$ and let $P \in M$. As always we use the Submanifold Smooth Structure Lemma 19.24 to view M as a smooth manifold in its own right. We pick a corresponding chart $\Phi\colon U \to V$ around P. We write $Q = \Phi(P)$. We denote by $i\colon U \to M$ and $j\colon V \to \mathbb{R}^k$ the inclusion maps. We consider the following diagram

$$\begin{array}{ccccccc}
\widetilde{\mathrm{T}}_P M & \xleftarrow{\mathrm{D}i_P} & \widetilde{\mathrm{T}}_P U & \xrightarrow{\mathrm{D}\Phi_P} & \widetilde{\mathrm{T}}_Q V & \xrightarrow{\mathrm{D}j_Q} & \widetilde{\mathrm{T}}_Q \mathbb{R}^k \\
\downarrow{\scriptstyle \Theta_M} & & \downarrow{\scriptstyle \Theta_U} & & \downarrow{\scriptstyle \Theta_V} & & \downarrow{\scriptstyle \Theta_{\mathbb{R}^k}} \\
\mathrm{T}_P M & \xleftarrow{\mathrm{D}i_P} & \mathrm{T}_P U & \xrightarrow{\mathrm{D}\Phi_P} & \mathrm{T}_Q V & \xrightarrow{\mathrm{D}j_Q} & \mathrm{T}_Q \mathbb{R}^k.
\end{array}$$

We make the following observations:

- It follows from (3) that the diagram commutes.
- It follows from Propositions 23.3 and 23.14 that the left and right horizontal maps are isomorphisms.
- Since Φ is a diffeomorphism we obtain from Propositions 23.2 and 23.10 that the middle horizontal maps are isomorphisms.
- By Proposition 23.9 we know that the right vertical map is an isomorphism.

The combination of the above shows that the left vertical map is indeed an isomorphism.

(2) Let M be a k-dimensional submanifold of $\mathbb{R}^m$ and let $P \in \partial M$. Let $v \in \mathrm{T}_P M$ be a vector that points outward. Recall that this means that $v = \gamma'(0)$ where $\gamma\colon (-\epsilon, 0] \to M$ is a smooth map with $\gamma(0) = P$ but with $\gamma'(0) \notin \widetilde{\mathrm{T}}_P(\partial M)$. Let $g\colon M \to \mathbb{R}_{\geq 0}$ be a smooth map with $g|_{\partial M} \equiv 0$. We obtain that

$$(\Theta_M)(v)(g) \;=\; \mathrm{D}g_P(v) \;=\; \mathrm{D}g_P(\gamma'(0)) \underset{\substack{\uparrow \\ \text{Chain Rule 23.2 (2)}}}{=} (g \circ \gamma)'(0) \underset{\substack{\uparrow \\ \text{since } (g\circ\gamma)(t)\geq 0 \text{ for } t\leq 0 \\ \text{and } (g\circ\gamma)(0) = 0}}{\leq} 0.$$

Furthermore, note that using charts it is not difficult to construct a smooth map $g\colon M \to \mathbb{R}_{\geq 0}$ with $g|_{\partial M} \equiv 0$ and with $(\Theta_M)(v)(g) > 0$. This shows that $\Theta_M(v)$ does indeed point outward. The converse can also be proved by considering charts. We leave it to the reader to fill in the details.

(4) This statement follows immediately from (1) and (3), and the definition of a natural isomorphism, see page 203. ■

the Tangent Space-Isomorphism Proposition 23.16 allows us to go back and forth between the two flavors of tangent spaces for a smooth submanifold of some $\mathbb{R}^k$. This leads us to the following convention.

Convention. Given a submanifold M of some $\mathbb{R}^k$ and $P \in M$ we will often, especially in the later chapters, use the natural isomorphism from the Tangent Space-Isomorphism Proposition 23.16 to make the identification $\widetilde{\mathrm{T}}_P M = \mathrm{T}_P M$.

As a special case we have the following convention.

Example. Let M be an open subset M of $\mathbb{R}^n$ or of $\mathbb{H}^n$ and let $P \in M$. The above convention says that at times we make the identification

$$\begin{array}{ccccc} \mathbb{R}^n & = & \widetilde{\mathrm{T}}_P M & = & \mathrm{T}_P M. \\ & \uparrow & & \uparrow & \\ & \text{Lemma 23.1} & & \text{identified via the natural isomorphism} & \\ & & & \text{provided by the Tangent Space-Isomorphism Proposition 23.16} & \end{array}$$

Going through the definitions one sees easily that this isomorphism $\mathbb{R}^n \to \mathrm{T}_P M$ is precisely the isomorphism provided by Proposition 23.9.

Definition. Let $I \subset \mathbb{R}$ be an open subset, let M be a smooth manifold and let $f\colon I \to M$ be a smooth map. Given any $t \in I$ we write

$$f'(t) \;:=\; \mathrm{D}f_t(\text{tangent vector } 1 \in \mathrm{T}_t\mathbb{R} = \widetilde{\mathrm{T}}_t\mathbb{R} = \mathbb{R}) \;\in\; \mathrm{T}_{f(t)}M.$$

Remark. If M is a smooth submanifold of some $\mathbb{R}^n$, then under above the identification $\widetilde{\mathrm{T}}_P M = \mathrm{T}_P M$ one sees that the above definition of $f'(t) \in \mathrm{T}_{f(t)}M = \widetilde{\mathrm{T}}_{f(t)}M \subset \mathbb{R}^n$ agrees with the definition on page 533.

23.5. **Alternative construction of tangent spaces.** In this short section we present an alternative approach to answering Question 23.6 (1). To simplify the discussion we will only work with smooth manifolds with empty boundary. Therefore we introduce the following notation.

Notation. We write:

$$\begin{array}{rcl} \mathcal{SmMfd}_{\partial=\varnothing} & := & \text{category of smooth manifolds without boundary,} \\ \mathcal{PSmMfd}_{\partial=\varnothing} & := & \text{category of pointed smooth manifolds without boundary.} \end{array}$$

Similarly we define the analogues where we only consider smooth submanifolds without boundary of some $\mathbb{R}^n$.

Definition.

(1) Let M be a smooth manifold and let $P \in M$. Let us consider

$$\widehat{\mathrm{T}}_P M \;:=\; \{\text{all smooth maps } \gamma\colon (-\epsilon,\epsilon) \to M \text{ with } \gamma(0) = P\}/\!\sim$$

where we define

$$\alpha \sim \beta \quad\Longleftrightarrow\quad \begin{array}{c} \text{there exists a chart } \Phi\colon U \to V \text{ and an } \eta > 0 \text{ with} \\ (\Phi \circ \alpha|_{(-\eta,\eta)})'(0) \;=\; (\Phi \circ \beta|_{(-\eta,\eta)})'(0) \;\in\; \mathbb{R}^n. \end{array}$$

(2) If $f\colon M \to N$ is a smooth map between two smooth manifolds without boundary, then for any $P \in M$ we define

$$\begin{array}{rcl} \mathrm{D}f_P\colon \widehat{\mathrm{T}}_P M & \to & \widehat{\mathrm{T}}_{f(P)} N \\ [\alpha] & \mapsto & [f \circ \alpha]. \end{array}$$

It is not that difficult to show that $\widehat{\mathrm{T}}_P M$ and $\mathrm{D}f_P$ have the same formal properties as the tangent spaces $\mathrm{T}_P M$ and the differentials $\mathrm{D}f_P$, in the sense that the fairly obvious analogues of Propositions 23.10, 23.11 and 23.14 hold. In particular we see that the maps

$$\begin{array}{rcl} (M, P) & \mapsto & \mathrm{T}_P M \\ (f\colon (M, Q) \to (N, Q)) & \mapsto & (\mathrm{D}f_P\colon \widehat{\mathrm{T}}_P M \to \widehat{\mathrm{T}}_Q N) \end{array}$$

define a covariant functor from the category $\mathcal{PSmMfd}_{\partial=\varnothing}$ of pointed smooth manifolds with empty boundary to the category $\mathcal{Vec}_{\mathbb{R}}$ of $\mathbb{R}$-vector spaces.

Furthermore, similar to the Tangent Space-Isomorphism Proposition 23.16 one can show that the maps

$$\begin{array}{rcl} \Theta_M \colon \widehat{\mathrm{T}}_P M & \to & \mathrm{T}_P M \\ [\gamma] & \mapsto & \left(\begin{array}{rcl} C^\infty(M,\mathbb{R}) & \to & \mathbb{R} \\ g & \mapsto & (g\circ\gamma)'(0) \end{array} \right) \end{array}$$

define a natural isomorphism.

Thus, at least in the absence of boundary, $\widehat{\mathrm{T}}_P M$ can be used as a replacement of $\mathrm{T}_P M$. One can also develop this theory for smooth manifolds with boundary, but as in the setting of the physical tangent space, this becomes rather messy. We refer to [**Jä93**, Chapter 3.3] and [**Nic2020**, Chapter 2] for details on $\widehat{\mathrm{T}}_P M$. We will not make use of this alternative point of view on tangent spaces of smooth manifolds. Another approach to defining the tangent space is given in [**Shah2016**, Chapter III.4.D].

23.6. **Vector fields.** In this section we will study vector fields on smooth manifolds and we will see that often they give rise to self-maps of smooth manifolds. These notions will play a role on many occasions throughout these notes.

Definition. Let $M \subset \mathbb{R}^n$ be a submanifold.

(1) A vector field on M is a continuous map $v\colon M \to \mathbb{R}^n$ such that for every $P \in M$ we have $v(P) \in \widetilde{\mathrm{T}}_P M$.
(2) A vector field on M is called smooth, if the map $v\colon M \to \mathbb{R}^n$ is smooth.[247]
(3) Given a vector field v on M we say that $x \in M$ is a zero of v if $v(x) = 0$.

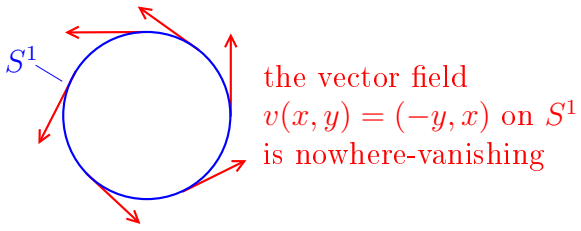
Example. Let $n \in \mathbb{N}$ and let P be a point on S^n. By Lemma 23.1 we have

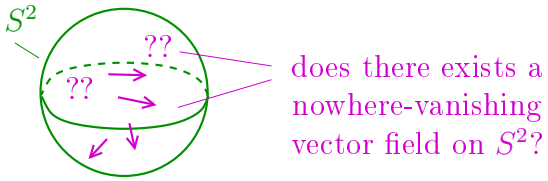
$$\widetilde{\mathrm{T}}_P S^n \;=\; \{v \in \mathbb{R}^{n+1} \mid \langle P, v\rangle = 0\}.$$

For an odd-dimensional sphere S^{2k-1} a smooth vector field is given by[248]

$$\begin{array}{rcl} v\colon S^{2k-1} & \to & \mathbb{R}^{2k} \\ (x_1, x_2, \ldots, x_{2k-1}, x_{2k}) & \mapsto & (-x_2, x_1, \ldots, -x_{2k}, x_{2k-1}). \end{array}$$

This vector field has the property that it has no zeros. In the case of the 1-sphere S^1 this is just the vector field of constant length 1 that points in the "counterclockwise direction".





The following question arises:

[247] As always we view M as a smooth manifold in its own right and we use the definition of smoothness from page 546.

[248] Indeed, for any $P \in S^{2k-1}$ we have $\langle P, v(P)\rangle = 0$, i.e. $v(P)$ does indeed lie in the tangent space $\mathrm{T}_P S^{2k-1}$. It follows easily from the Restrict-to-Submanifold-or-Boundary Lemma 19.25 (1) that v is indeed smooth.

Question 23.17. Does there exist a nowhere-vanishing vector field on even-dimensional spheres?

We will answer this question with the Hairy Ball Theorem 38.12[249] or alternatively in the Euler Characteristic-Vector Field Theorem 97.15.

Now we define the notion of a vector field on any smooth manifold.

Definition. Let M be a k-dimensional smooth manifold.

(1) A vector field on M is a map

$$v\colon M \to \bigsqcup_{P\in M} \mathrm{T}_P M$$

such that for each $P \in M$ we have $v(P) \in \mathrm{T}_P M$ and such that the following continuity condition is satisfied: If $\Phi\colon U \to V$ is a chart[250] from an open set U of M to an open set V of $\mathbb{R}^k$ or of $\mathbb{H}^k$, then the following map is continuous:

$$\begin{aligned} V &\to \mathbb{R}^k \\ Q &\mapsto \big(\Theta_V^{-1} \circ \mathrm{D}\Phi_{\Psi(Q)} \circ \big(\underbrace{\mathrm{D}i_{\Psi(Q)}}_{\substack{\text{isomorphism}\\ \text{by Proposition 23.14}}} \big)^{-1} \big(\underbrace{v(\Psi(Q))}_{\in \mathrm{T}_{\Phi^{-1}(Q)}M} \big)\big) \in \widetilde{\mathrm{T}}_Q V = \mathbb{R}^k. \end{aligned}$$

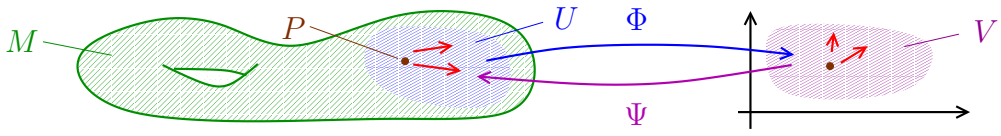
Here we use the following notation:

- We denote by $\Psi := \Phi^{-1}\colon V \to U$ the inverse of Φ.
- We denote by $i\colon U \to M$ the inclusion map.
- We denote by $\Theta_V\colon \widetilde{\mathrm{T}}_P V \to \mathrm{T}_P M$ the natural isomorphism from the Tangent Space-Isomorphism Proposition 23.16.

(2) We say that the vector field v is smooth if in the definition of (1) the resulting map $V \to \mathbb{R}^k$ is always smooth.

(3) Given a vector field v on M we say that $x \in M$ is a zero of v if $v(x) = 0$.

Remark. As in the Smooth-via-Atlas Lemma 19.21 one can show fairly easily that it suffices to check the condition in part (1) of the definition only for a smooth atlas and not for all charts.



On smooth submanifolds of $\mathbb{R}^n$ we now have a priori two different notions of (smooth) vector fields. The following lemma shows that these two notions are essentially the same.

Lemma 23.18. Let $M \subset \mathbb{R}^n$ be a smooth submanifold. Given $P \in M$ we denote by $\Theta_P\colon \widetilde{\mathrm{T}}_P M \to \mathrm{T}_P M$ the natural isomorphism from the Tangent Space-Isomorphism

[249] We give alternative proofs of the Hairy Ball Theorem on page 1689 and on page 2124.

[250] Recall that for a smooth manifold we mean by a "chart" a chart from the given smooth structure.

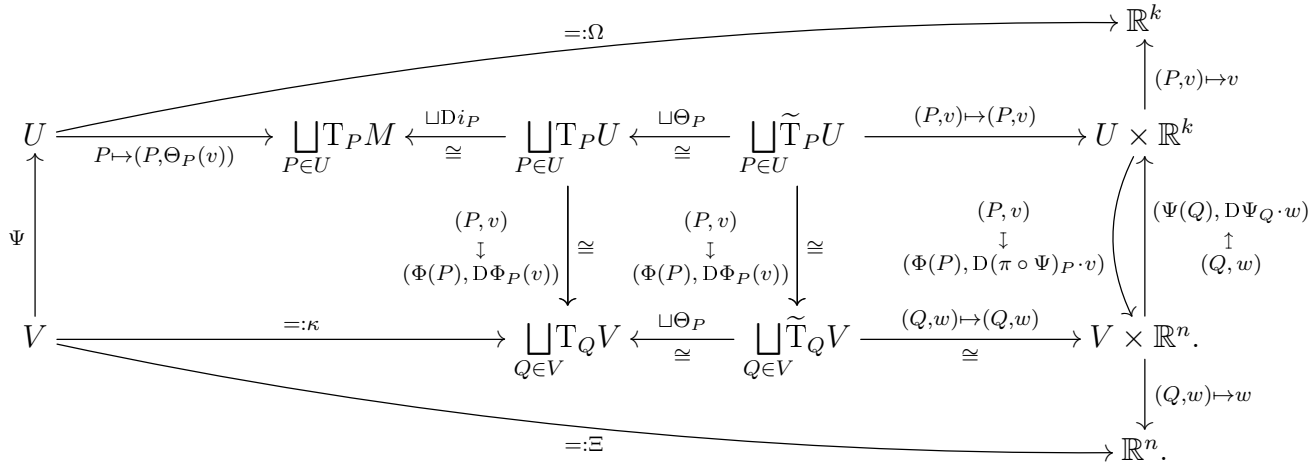
Proposition 23.16. Let $v\colon M \to \mathbb{R}^n$ be a map such that for each $P \in M$ we have $v(P) \in \widetilde{\mathrm{T}}_P M \subset \mathbb{R}^n$. The following statement holds:

(1) The map v is a vector field on M if and only if $P \mapsto \Theta_P(v)$ is a vector field on M.
(2) The map v is a smooth vector field on M if and only if $P \mapsto \Theta_P(v)$ is a smooth vector field on M.

Proof. Let $M \subset \mathbb{R}^n$ be a k-dimensional smooth submanifold. In the following we deal with (2), the proof for (1) follows exactly the same logic.

Let $\widetilde{\Phi}\colon \widetilde{U} \to \widetilde{V}$ be a submanifold chart for M. To simplify the notation we consider the case that Φ is a submanifold chart of type (α) in the sense of the definition on page 550. The other cases are treated the same way. We write $U := \widetilde{U} \cap M$ and $V := \{y \in \mathbb{R}^k \mid (0, y) \in \widetilde{V}\}$. We denote by $\iota\colon V \to \widetilde{V}$ the inclusion and we denote by $\pi\colon \mathbb{R}^k = \mathbb{R}^{n-k} \times \mathbb{R}^k \to \mathbb{R}^n$ the projection given by $(x, y) \mapsto y$. We denote by $\Phi\colon U \to V$ the diffeomorphism that is determined by the condition that $\iota \circ \Phi = \widetilde{\Phi}|_{\widetilde{U}}$.

We denote by $\Psi\colon V \to U$ the inverse of Φ and we denote by $i\colon U \to M$ the inclusion. We consider the following diagram:



We make the following clarifications and observations:

- The maps κ, Ω and Ξ are defined in such a way that the left square commutes and the top and bottom triangle commute.
- The middle square commutes by the naturality of the maps Θ.
- The right hand square commutes (in both possibilities) since $\Psi = \Phi^{-1}$ and by the chain rule.
- The two right vertical maps connecting the second and the third row are smooth since the maps $P \mapsto \mathrm{D}\Phi_P$, $P \mapsto \mathrm{D}(\pi \circ \Psi)_P$ and $Q \mapsto \mathrm{D}\Psi_Q$ are smooth.
- The question whether the vector field $P \mapsto \Theta_P(v)$ is smooth in the sense of the definition on page 639 boils down to the question whether $\Xi\colon V \to \mathbb{R}^n$ is smooth.
- Note that the naturality of the Θ maps implies that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathrm{T}}_PU & \xrightarrow{\Theta} & \mathrm{T}_PU \\ {\scriptstyle \mathrm{D}i_P}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \mathrm{D}i_P} \\ \widetilde{\mathrm{T}}_PM & \xrightarrow{\Theta} & \mathrm{T}_PM. \end{array}$$

If one goes through the definitions, and one uses the above observation, then one sees that the map Ω is precisely the map $P \mapsto v(P)$. Thus the question whether v is smooth vector field on M in the sense of the definition on page 638 boils down to the question whether Θ is smooth.

- It follows easily from the above that Ξ is smooth if and only if Ω is smooth. ■

As a teaser we state the following question.

Question 23.19. Which closed connected 2-dimensional smooth manifolds admit a nowhere-vanishing vector field?

We will address the question in Exercise 23.11. For orientable 2-dimensional smooth manifolds we will prove an answer in Corollary 45.23 (2). Furthermore we will give a complete answer on page 2125. While we are throwing out questions, let us also pose the following more general, but rather vague question.

Question 23.20. How can we tell whether a closed smooth manifold admits a nowhere-vanishing vector field?

Rather surprisingly we will be able to give a complete and satisfactory answer to this question in the Euler Characteristic-Vector Field Theorem 97.15 and the Nowhere Vanishing Vector Field Existence Theorem 129.6.

For completeness' sake we state the following basic lemma, which will get used on several occasions.

Lemma 23.21. (Vector Field-Pullback Lemma) Let $f\colon M \to N$ be a map between two smooth manifolds and let v be a (smooth) vector field on N. If f is a local diffeomorphism, then for each $x \in M$ we can define $(f^*v)(x) = (\mathrm{D}f_x)^{-1}(v(f(x)) \in \mathrm{T}_xM$. These vectors define a (smooth) vector field f^*v on M. We refer to f^*v as the pullback of v under f.

Example. Let $p\colon S^n \to \mathbb{R}\mathrm{P}^n = S^n/\{\pm 1\}$ be the usual projection. Given any vector field v on $\mathbb{R}\mathrm{P}^n$ the pullback f^*v is a vector field on S^n. The number of zeros of f^*v equals twice the number of zeros of v.

Proof. The lemma follows easily from the definitions. We leave it to the reader to fill in the details. ■

We will return to the study of vector fields on numerous occasions. They are obviously interesting in their own right. Furthermore, as we will see in the Flow Theorem 35.1, they can also be used to construct interesting self-maps of smooth manifolds.

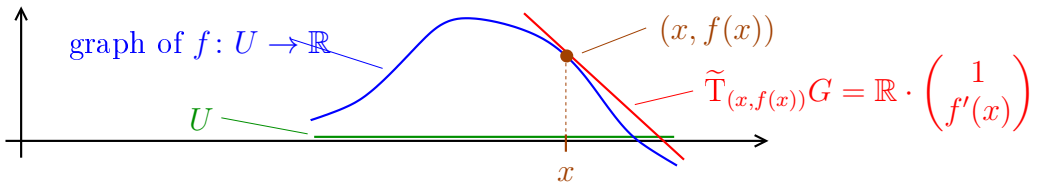
Exercises for Chapter 23.

Exercise 23.1. Let $U \subset \mathbb{R}^n$ be an open subset and let $f\colon U \to \mathbb{R}^k$ be a smooth map. On page 562 we discussed that the graph

$$G \;:=\; \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k \,|\, x \in U\}$$

is an n-dimensional smooth submanifold of $\mathbb{R}^{n+k}$. Show that for every $x \in U$ we have

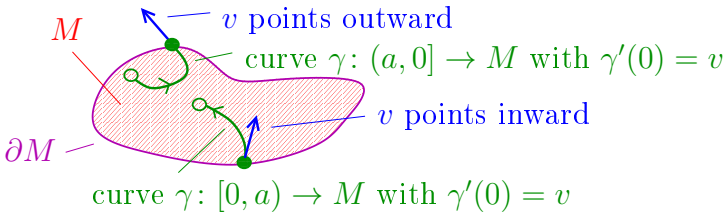
$$\widetilde{\mathrm{T}}_{(x,f(x))}G \;=\; \text{span of the columns of the } (n+k)\times n\text{-matrix } \begin{pmatrix} \mathrm{id}_n \\ \mathrm{D}f_x \end{pmatrix}.$$

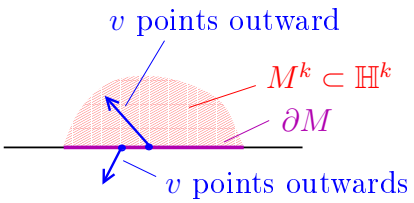


Exercise 23.2. Let M be a k-dimensional smooth submanifold of $\mathbb{R}^n$, let $P \in \partial M$ and let $v \in \widetilde{\mathrm{T}}_P M$. Show that precisely one of the following statements holds:

(1) $v \in \widetilde{\mathrm{T}}_P \partial M$,
(2) v points inward,
(3) v points outward.

Hint. First deal with the case that M is an open subset of the upper half-space $\mathbb{H}^k$.





Exercise 23.3. Let M be a smooth manifold, let $P \in \partial M$ and let $v \in \mathrm{T}_P M$. We denote by $i\colon \partial M \to M$ be the inclusion map. Show that precisely one of the following statements holds:

(1) v lies in the image of $\mathrm{D}i_P\colon \mathrm{T}_P\partial M \to \mathrm{T}_P M$,
(2) v points inward,
(3) v points outward.

Exercise 23.4. Let $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ be two closed smooth submanifolds. Show that $M \times N \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ is a smooth submanifold and show that for any $x \in M$ and $y \in N$ we have

$$\widetilde{\mathrm{T}}_{(x,y)}(M \times N) \;=\; \widetilde{\mathrm{T}}_x M \;\times\; \widetilde{\mathrm{T}}_y N \;\subset\; \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}.$$

Exercise 23.5. Let M be a smooth manifold and let $U \subset M$ be an open subset. We denote by $\iota\colon U \to M$ the inclusion. Show that for every $P \in U$ the map $\mathrm{D}\iota_P\colon \mathrm{T}_P U \to \mathrm{T}_P M$ is an isomorphism.
Hint. Use Lemma 23.8 (3).

Exercise 23.6. Let $M \subset \mathbb{R}^n$ be a k-dimensional smooth submanifold and let $P \in M \setminus \partial M$.

(a) Show that there exists an $A \in \mathrm{O}(n)$ with $A \cdot \widetilde{\mathrm{T}}_P M = \{(0,y) \in \mathbb{R}^n \,|\, y \in \mathbb{R}^k\}$.

(b) Show that there exists a neighborhood U of $P \subset M$ such that the map

$$\begin{array}{ccccccc} U & \to & \widetilde{\mathrm{T}}_P M & \xrightarrow{x \mapsto A \cdot x} & \{(x,0) \,|\, x \in \mathbb{R}^k\} & \xrightarrow{(x,y) \mapsto x} & \mathbb{R}^k \\ x & \mapsto & \text{orthogonal projection} & & & & \\ & & \text{of } x - P \text{ onto } \widetilde{\mathrm{T}}_P M & & & & \end{array}$$

is a chart for M.

Exercise 23.7. Let M be a smooth manifold, let $P \in \partial M$ and let $w \in \mathrm{T}_P M$.

(a) Let N be a smooth manifold and let $f\colon M \to N$ be a local diffeomorphism with $f(P) \in \partial N$. Show that $w \in \mathrm{T}_P M$ points outward if and only if $\mathrm{D}f_P(w) \in \mathrm{T}_P N$ points outward.

(b) Show that precisely one of the following three statements holds:
(i) w points outward,
(ii) w points inward,
(iii) $w \in \mathrm{D}\iota_P(\mathrm{T}_P\, \partial M \to \mathrm{T}_P M)$, where $\iota\colon \partial M \to M$ is the inclusion map.
Hint. First show that it suffices to consider the case that M is an open subset of some $\mathbb{H}^n$.

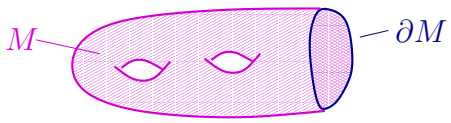
Exercise 23.8. Let M be a smooth manifold, let $X \subset M$ be a proper submanifold, let $P \in \partial X$ and let $w \in \mathrm{T}_P X$. We denote by $\iota\colon X \to M$ the inclusion map. Show that $w \in \mathrm{T}_P X$ points outward if and only if $\mathrm{D}\iota_P(w)$ points outward.

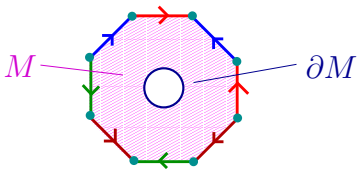
Exercise 23.9. We consider the following submanifold of $\mathbb{R}^3$:

$$M \;:=\; \{(x, y, x^2 + y^2) \,|\, x, y \in \mathbb{R}\}.$$

Determine $\widetilde{\mathrm{T}}_{(1,2,5)} M \subset \mathbb{R}^3$.

Exercise 23.10. We consider the smooth manifold M shown in the figure below. Show that it admits a nowhere vanishing smooth vector field.





Exercise 23.11.

(a) For which closed connected 2-dimensional smooth manifolds can you find a nowhere-vanishing vector field?

(b) For which compact connected 2-dimensional smooth manifolds M can you find a nowhere-vanishing vector field v with the additional property that for each $P \in \partial M$ we have $v(P) \in \mathrm{T}_P(\partial M)$?

Remark. At this stage you cannot give a complete answer to these questions. The goal is to find the largest collection of examples where such vector fields exist.

Exercise 23.12.

(a) Can you find a vector field on S^2 with two zeros?

(b) Can you find a vector field on S^2 with one zero?

Exercise 23.13. Let $n \in \mathbb{N}$. We consider $M = \mathbb{R}^n \cup \{\infty\}$. We denote by 0 the origin of $\mathbb{R}^n$.

(a) We consider the maps $\Phi = \mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^n$ and

$$\begin{array}{rcl} \Psi\colon M \setminus \{0\} & \to & \mathbb{R}^n \\ x & \mapsto & \begin{cases} \frac{1}{\|x\|^2} \cdot x, & \text{if } x \neq \infty, \\ 0, & \text{if } x = \infty. \end{cases} \end{array}$$

Show that Φ and Ψ form a smooth atlas for M.

(b) We consider the map

$$\begin{array}{rcl} f\colon M & \to & M \\ x & \mapsto & \begin{cases} 3x, & \text{if } x \neq \infty, \\ \infty, & \text{if } x = \infty \end{cases} \end{array}$$

Note that $f(0) = 0$ and $f(\infty) = \infty$.

(i) Show that f is smooth.

(ii) Show that $\mathrm{D}f_0 = 3 \cdot \mathrm{id}_{\mathrm{T}_0 M}$ and show that $\mathrm{D}f_\infty = \frac{1}{3} \cdot \mathrm{id}_{\mathrm{T}_\infty M}$.

24. The Inverse Mapping Theorem and the Smooth Embedding Theorem

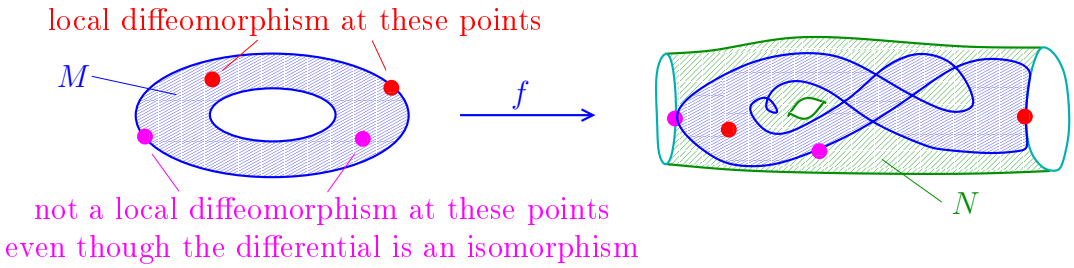
24.1. **The Inverse Mapping Theorem.** By the Physical Tangent Space-Inclusion Proposition 23.14 we know that for a local diffeomorphism the differential at any point is an isomorphism. The following theorem gives the much more interesting converse. The theorem gives in particular a convenient criterion for showing that a map between smooth manifolds is a (local) diffeomorphism.

Theorem 24.1. (Inverse Mapping Theorem) Let $f\colon M \to N$ be a smooth map between two smooth manifolds.

(1) Let $P \in M$ such that $\mathrm{D}f_P$ is invertible.
   (a) If $P \notin \partial M$, then f is a local diffeomorphism at P.
   (b) If $P \in \partial M$ and if furthermore there exists an open neighborhood U of P such that $f(U \cap \partial M) \subset f(U) \cap \partial N$, then f is a local diffeomorphism at P.

(2) If $\mathrm{D}f_P$ is invertible for every $P \in M$, if f is a bijection and if $f(\partial M) = \partial N$, then f is a diffeomorphism.

Remark. In Exercise 24.1 we will consider the case $P \in \partial M$ and $f(P) \notin \partial N$.



Examples.

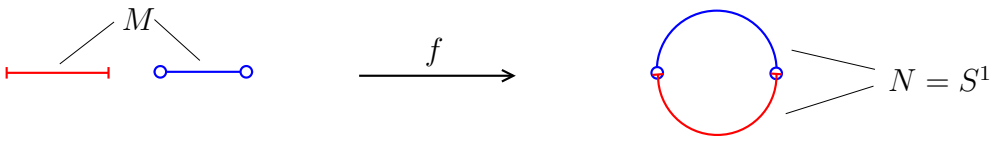
(1) The above figure indicates in particular a map $f\colon M \to N$ and two points $P \in \partial M$, such that $\mathrm{D}f_P$ is an isomorphism, but such f is not a local diffeomorphism at P. This shows that in general in the formulation of the Inverse Mapping Theorem 24.1 we do indeed have to put an extra condition on f if $P \in \partial M$.

(2) We consider the map

$$\begin{array}{rcl} f\colon \mathbb{R}^n & \to & B^n \\ (x_1, \dots, x_n) & \mapsto & \frac{x_1^2+\dots+x_n^2}{1+x_1^2+\dots+x_n^2} \cdot (x_1, \dots, x_n). \end{array}$$

A slightly heroic calculation shows that f satisfies the conditions of the Inverse Mapping Theorem 24.1 (2), thus we see that f is a diffeomorphism. This is actually quite interesting since at first glance it is not totally apparent why $f^{-1}\colon B^n \to \mathbb{R}^n$ is smooth at the origin.

(3) We consider the smooth map $f\colon M \to N$ that is illustrated in the figure below. It is a bijection and for every $x \in M$ the differential $\mathrm{D}f_x\colon \mathrm{T}_xM \to \mathrm{T}_{f(x)}N$ is an isomorphism. But evidently f is not a diffeomorphism. This shows that we cannot drop the condition $f(\partial M) = \partial N$ in the statement of the Inverse Mapping Theorem 24.1 (2).



Evidently the proof of the Inverse Mapping Theorem 24.1 rests on the shoulders of the Inverse Function Theorem 19.12, which we now recall for the reader's convenience.

Theorem 19.12. (Inverse Function Theorem) Let $U \subset \mathbb{R}^n$ be an open subset, let $f\colon U \to \mathbb{R}^n$ be a smooth map and let $P \in U$ be a point. If $\mathrm{D}f_P \in \mathrm{M}(n\times n, \mathbb{R})$ is invertible, then f is a local diffeomorphism at P.

For the purpose of this section it is convenient to introduce the following notation that is modelled on the notation introduced on page 152.

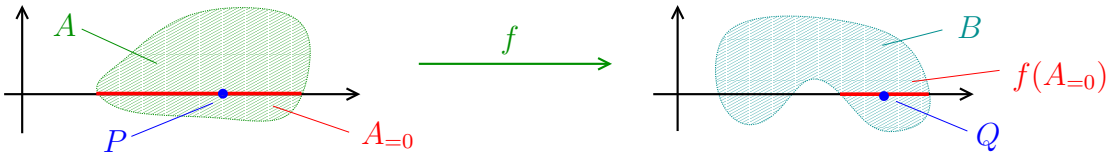
Notation. Given a subset A of $\mathbb{R}^n$ we write

$$A_{>0} \;=\; A \cap \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_n > 0\}.$$

We define $A_{<0}$, $A_{=0}$ and $A_{\neq 0}$ in the analogous way.

The proof of the Inverse Mapping Theorem 24.1 for the case $P \in \partial M$ rests on the following lemma, which might actually be interesting in its own right.

Lemma 24.2. Let A and B be open subsets of $\mathbb{R}^n$ and let $f\colon A \to B$ be a diffeomorphism. If $f(A_{=0}) \subset B_{=0}$, then $f(A_{=0})$ is an open subset of $B_{=0}$



Proof. Let $Q \in f(A_{=0})$. We pick $P \in A_{=0}$ with $Q = f(P)$. We denote by $i\colon A_{=0} \to A$ the inclusion map and we denote by $j\colon B_{=0} \to B$ the inclusion map. Finally we denote by $f_{=0}\colon A_{=0} \to B_{=0}$ the restriction of f to $A_{=0}$. Note that $f \circ i = j \circ f_{=0}\colon A_{=0} \to B$. It follows from the chain rule that

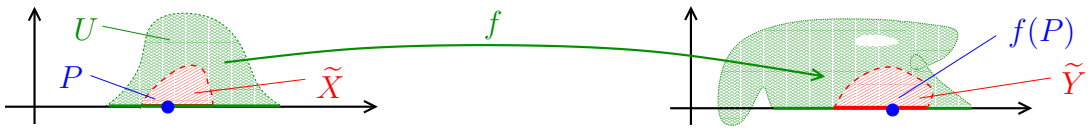
$$\mathrm{D}f_P \circ \mathrm{D}i_P \;=\; \mathrm{D}j_Q \circ \mathrm{D}(f_{=0})_P\colon \widetilde{\mathrm{T}}_P A_{=0} \to \widetilde{\mathrm{T}}_Q B = \mathbb{R}^n.$$

Since f is a diffeomorphism we know that $\mathrm{D}f_P$ is an isomorphism. Clearly $\mathrm{D}i_P$ has rank $n-1$. It follows from the above that the map to the right also has rank $n-1$. But this is only possible if $\mathrm{D}(f_{=0})_P$ has rank $n-1$. This shows that $\mathrm{D}(f_{=0})_P\colon \widetilde{\mathrm{T}}_P A_{=0} \to \widetilde{\mathrm{T}}_Q B_{=0}$ is an isomorphism. It follows from the Inverse Function Theorem 19.12 that the image of $A_{=0}$ contains an open neighborhood of $Q \in B_{=0}$. It follows from the JH-Lemma 1.7 that $f(A_{=0})$ is an open subset of $B_{n=0}$. ■

Using Lemma 24.2 we can now prove the following variation on the Inverse Function Theorem 19.12.

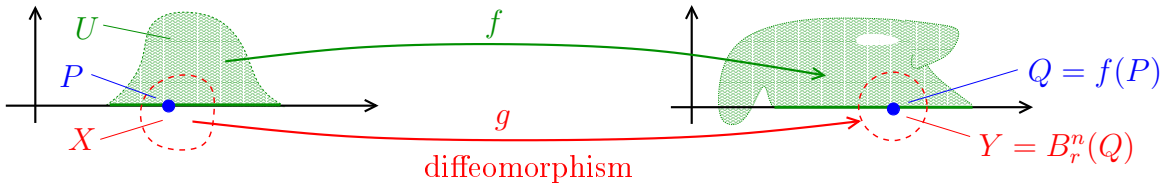
Lemma 24.3. Let $U \subset \mathbb{H}^n$ be an open subset and let $f\colon U \to \mathbb{R}^n$ be a smooth map with $f(U \cap \partial\,\mathbb{H}^n) \subset \partial\,\mathbb{H}^n$. Furthermore let $P \in U \cap \partial\,\mathbb{H}^n$ be a point. If $\mathrm{D}f_P$ is an isomorphism and if $\mathrm{D}f_P(\mathbb{H}^n) \subset \mathbb{H}^n$, then f is a local diffeomorphism at P in the sense that there exists

an open neighborhood $\widetilde{X}$ of $P \in U$ and an open neighborhood $\widetilde{Y}$ of $f(P) \in \mathbb{H}^n$ such that the map $f\colon \widetilde{X} \to \widetilde{Y}$ is a diffeomorphism between the two smooth manifolds $\widetilde{X}$ and $\widetilde{Y}$.



Proof. Note that by definition of a smooth map on an open subset of $\mathbb{H}^n$, see page 532, there exists an open neighborhood X of $P \in \mathbb{R}^n$ and a smooth map $g\colon X \to \mathbb{R}^n$ which agrees with f on $U \cap X$.

We set $Q := f(P)$. It follows easily from the Inverse Function Theorem 19.12 that, after possibly replacing X by a smaller open neighborhood of P, we can assume that there exists an open ball $Y := B^n_r(Q) \subset \mathbb{R}^n$ such that $g\colon X \to Y = B^n_r(Q)$ is a diffeomorphism.



Next recall that given a subset A of $\mathbb{R}^n$ we write

$$A_{>0} \;=\; A \cap \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_n > 0\}.$$

and we define $A_{<0}$, $A_{=0}$ and $A_{\neq 0}$ in the analogous way. With this notation it remains to prove the following claim.

Claim. We have $f(X_{\geq 0}) = Y_{\geq 0}$ and the map $f = g\colon X_{\geq 0} \to Y_{\geq 0}$ is a diffeomorphism.

Proof. We already know that $g\colon X \to Y$ is a diffeomorphism. Thus it remains to prove that $g(X_{\geq 0}) = Y_{\geq 0}$.

First note that by hypothesis we have $g(X_{=0}) \subset Y_{=0}$. It follows from Lemma 24.2 that $g(X_{=0})$ is an open subset of $Y_{=0}$. Furthermore note $X_{=0}$ is a closed subset of X, thus $g(X_{=0})$ is a closed subset of Y. But since $g(X_{=0})$ is contained in $Y_{=0}$ it follows from the elementary the Neighborhood Openness Criterion 1.5 that $g(X_{=0})$ is a closed subset of $Y_{=0}$. In summary we see that $g(X_{=0})$ is a subset of $Y_{=0}$ that is open and closed. Since $Y_{=0} \cong B^{n-1}_r(Q)$ is connected and since $P \in X_{=0}$ we see that $g(X_{=0}) = Y_{=0}$.

Since $g\colon X \to Y$ is a homeomorphism and since $g(X_{=0}) = Y_{=0}$ we see that $g\colon X_{\neq 0} \to Y_{\neq 0}$ is also a homeomorphism. Note that $Y_{\neq 0}$ has precisely two components namely $Y_{>0}$ and $Y_{<0}$. Thus $X_{\neq 0}$ also has precisely two components. Since $X_{>0}$ and $X_{<0}$ are open, closed and non-empty it follows from the elementary the Component Lemma 2.33 that the two components of $X_{\neq 0}$ are precisely $X_{>0}$ and $X_{<0}$. It follows from the fact that g gives a bijection of the components and from our hypothesis $\mathrm{D}f_P(\mathbb{H}^n) \subset \mathbb{H}^n$ that $g^{-1}(Y_{<0}) \subset X_{<0}$ and $g^{-1}(Y_{>0}) \subset X_{>0}$. ∎

Proof of the Inverse Mapping Theorem 24.1.

(1) Let $f\colon M \to N$ be a smooth map between smooth manifolds. We set $m := \dim(M)$ and $n := \dim(N)$.

(a) Let $P \in M \setminus \partial M$ be a point. We write $Q := f(P)$. In the following we assume that $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_Q N$ is invertible. We pick a chart $\Phi\colon U \to \widetilde{U}$ of type (i) around P and we pick a chart $\Psi\colon V \to \widetilde{V}$ around $f(P)$.[251] We write $\widetilde{P} := \Phi(P)$ and $\widetilde{Q} := \Psi(Q)$.

After possibly replacing U by $U \cap f^{-1}(V)$ we can assume that $f(U) \subset V$. We consider the map $g := \Psi \circ f \circ \Phi^{-1}\colon \widetilde{U} \to \widetilde{V} \subset \mathbb{R}^n$.

Claim. The differential $\mathrm{D}g_{\widetilde{P}} \in \mathrm{M}(m \times n, \mathbb{R})$ is an invertible matrix.

Proof. Let $i\colon U \to M$ and $j\colon V \to N$ be the inclusion maps. We consider the following diagram:

$$\begin{array}{ccccccccc}
\mathrm{T}_P M & \xleftarrow{\mathrm{D}i_P} & \mathrm{T}_P U & \xrightarrow{\mathrm{D}\Phi_P} & \mathrm{T}_{\widetilde{P}}\widetilde{U} & \xrightarrow{\Theta_{\widetilde{U}}} & \widetilde{\mathrm{T}}_{\widetilde{P}}\widetilde{U} & \xrightarrow{\mathrm{id}} & \mathbb{R}^m \\
\downarrow{\scriptstyle \mathrm{D}f_P} & & \downarrow{\scriptstyle \mathrm{D}f_P} & & \downarrow{\scriptstyle \mathrm{D}g_{\widetilde{P}}} & & \downarrow{\scriptstyle \mathrm{D}g_{\widetilde{P}}} & & \downarrow{\scriptstyle \mathrm{D}g_{\widetilde{P}}} \\
\mathrm{T}_Q N & \xleftarrow{\mathrm{D}j_P} & \mathrm{T}_Q V & \xrightarrow{\mathrm{D}\Psi_Q} & \mathrm{T}_{\widetilde{Q}}\widetilde{V} & \xrightarrow{\Theta_{\widetilde{V}}} & \widetilde{\mathrm{T}}_{\widetilde{Q}}\widetilde{V} & \xrightarrow{\mathrm{id}} & \mathbb{R}^n.
\end{array}$$

We make the following observations:

- It follows from Lemma 23.12 that the left horizontal maps are isomorphisms.
- Since Φ and Ψ are diffeomorphisms we obtain from the Tangent Space-Functor Proposition 23.10 that the center-left horizontal maps are invertible.
- It follows from the Tangent Space-Functor Proposition 23.10 that the two rectangles to the left commute.
- The horizontal maps $\Theta_{\widetilde{U}}$ and $\Theta_{\widetilde{V}}$ are the natural isomorphisms from the Tangent Space-Isomorphism Proposition 23.16. In particular the third rectangle commutes.
- By the Differential-equals-Differential Lemma 23.4 we know that the rectangle to the right commutes.

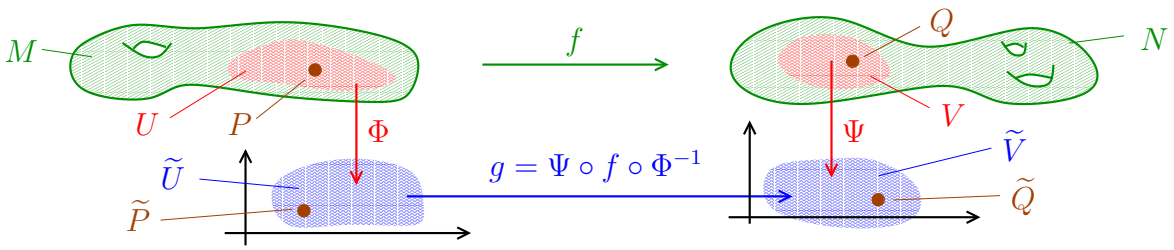
The combination of all of the above, together with our hypothesis that $\mathrm{D}f_P$ is invertible implies that $\mathrm{D}g_{\widetilde{P}}$ is indeed an invertible matrix. ⊞

It follows from the claim and the Inverse Function Theorem 19.12 that there exists an open neighborhood Y of $\widetilde{P} \in \widetilde{U}$ and an open neighborhood Z of $\widetilde{Q} \subset \widetilde{V}$ such that g restricts to a diffeomorphism $Y \to Z$. It follows that $f\colon \Phi^{-1}(Y) \to \Psi^{-1}(Z)$ is a diffeomorphism. We have thus shown that f is a local diffeomorphism at P.

(b) Now we assume that $P \in \partial M$ and we assume that there exists an open neighborhood U of P such that $f(U \cap \partial M) \subset f(U) \cap \partial N$, then f is a local diffeomorphism at P. The proof of this statement is almost identical to the proof of (a). We just need to replace the Inverse Function Theorem 19.12 by Lemma 24.3. (The condition that $\mathrm{D}g_P(\mathbb{H}^n) \subset \mathbb{H}^n$ is automatically satisfied since $g(P) \in \partial\mathbb{H}^n$ and $g(\widetilde{U}) \subset \mathbb{H}^n$.) We leave it to the reader to fill in the details.

(2) The second statement follows immediately from (1) and the Smooth Pasting Proposition 19.23 (2). ■

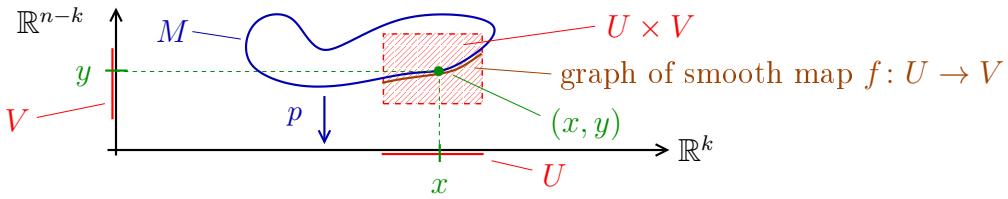
[251] Note that a priori we do not know yet whether $Q = f(P) \in \partial N$, so Ψ could be a chart of type (i) or of type (ii).



24.2. **The Local Graph Lemma.** On page 562 we saw that graphs of smooth maps between smooth manifolds give rise to submanifolds. Using the Inverse Mapping Theorem 24.1 we can now show that locally, after possibly a rotation, any submanifold of $\mathbb{R}^n$ without boundary is a graph.

Lemma 24.4. (Local Graph Lemma) Let $M \subset \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ be a k-dimensional smooth submanifold. We denote by $p\colon \mathbb{R}^n \to \mathbb{R}^k$ the projection onto the first k coordinates. Let $(x, y) \in M \setminus \partial M$ such that $\mathrm{D}p_{(x,y)}\colon \widetilde{\mathrm{T}}_{(x,y)}M \to \mathbb{R}^k$ is an isomorphism. There exists an open neighborhood U of $x \in \mathbb{R}^k$, an open neighborhood V of $y \in \mathbb{R}^{n-k}$ and a smooth map $f\colon U \to V$ such that

$$M \cap (U \times V) = \text{ Graph of } f\colon U \to V \ = \ \{(z, f(z)) \,|\, z \in U\}.$$



Remark.

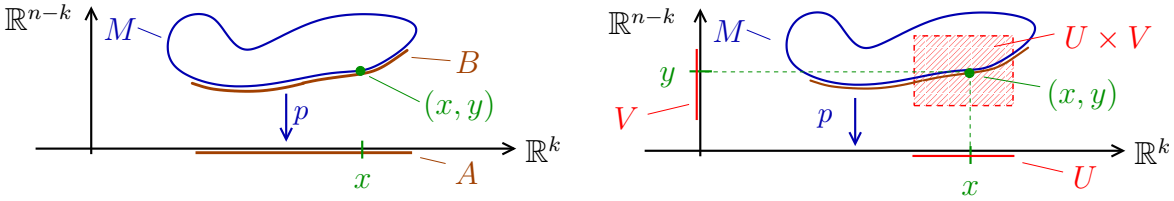
(1) Let $M \subset \mathbb{R}^n$ be a k-dimensional smooth submanifold and let $z \in M \setminus \partial M$. By the Orthogonal Action Lemma 5.26 (5) there exists an orthogonal matrix $A \in \mathrm{SO}(n)$ with $A \cdot \widetilde{\mathrm{T}}_z M = \mathbb{R}^k \times \{0\}$. It follows that, after a rotation, one can arrange that the hypotheses of the Local Graph Lemma 24.4 holds. For readers who come back to this discussion from a future chapter we pointed out that in the Diffeotopies-via-Matrices Lemma 33.8 we will see that a rotation can be realized by a diffeotopy of $\mathbb{R}^n$.

(2) In the Diffeomorphic-to-Euclidean Submanifold Corollary 27.2 we will see that every smooth manifold is diffeomorphic to a smooth submanifold of some $\mathbb{R}^d$. Thus we see that locally every smooth manifold without boundary is diffeomorphic to a graph.

Proof. We view M as a smooth manifold in its own right. By hypothesis the map $\mathrm{D}p_{(x,y)}\colon \widetilde{\mathrm{T}}_{(x,y)}M \to \widetilde{\mathrm{T}}_x\mathbb{R}^k = \mathbb{R}^k$ is an isomorphism. It follows from the Inverse Mapping Theorem 24.1 that there exists an open neighborhood A of $(x, y) \in M$ and an open neighborhood B of $x \in \mathbb{R}^k$ such that $p\colon A \to B$ is a diffeomorphism. Note that the inverse of $p\colon A \to B$ is necessarily of the form

$$\begin{array}{ccc} B & \to & A \\ z & \mapsto & (z, f(z)) \end{array}$$

where $f\colon B \to \mathbb{R}^{n-k}$ is a smooth map.



By the definition of the subspace topology on M and by the definition of the topology on $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ there exists an open neighborhood U of x and an open neighborhood V of y such that $(U \times V) \cap M \subset A$. It is now clear that U, V and f have the desired properties. ■

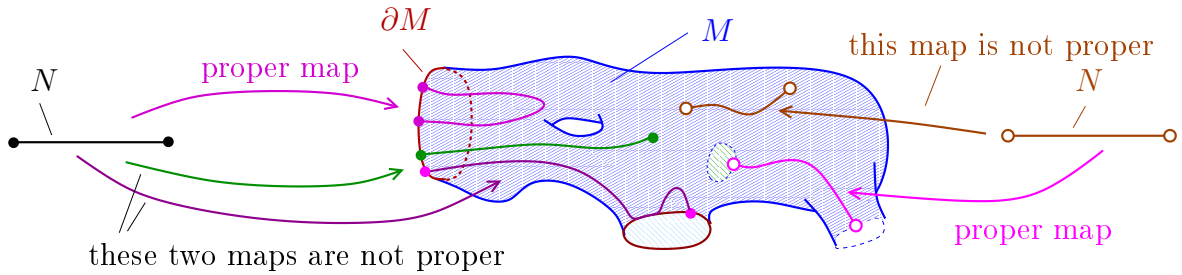
24.3. **Immersions and smooth embeddings.** For the reader's convenience we recall the following definition from page 152.

Definition. We say a map $f\colon X \to Y$ between two topological spaces X and Y is an embedding if $f\colon X \to f(X)$ is a homeomorphism, where $f(X)$ is equipped with the subspace topology.

Next we state a new definition.

Definition. Let N and M be topological manifolds. A map $\varphi\colon N \to M$ is called proper if the following two conditions are satisfied:

(1) $\varphi^{-1}(\partial M) = \partial N$,
(2) $\varphi(N)$ is a closed subset of M.



For an *embedding* of topological manifolds the following, rather technical, lemma gives an alternative characterization of the second condition of a proper map between topological manifolds.

Lemma 24.5. Let M and N be topological manifolds and let $\varphi\colon N \to M$ be a map. If φ is in fact an embedding, then the following two conditions are equivalent:[252]

(1) $\varphi(N)$ is a closed subset of M.
(2) For every compact subset K of N the preimage $\varphi^{-1}(K)$ is compact.

[252]By the Topological Manifolds-Local Properties Lemma 18.8 we know that topological manifolds are regionally compact. Thus the content of Lemma 24.5 is closely related to the Proper-Closed Proposition 2.43.

Remark. On page 180 we introduced the definition that a map $f\colon X \to Y$ between topological spaces is called *proper* if the preimage of every compact subset of Y is a compact subset of X. Unfortunately we now have two notions of the adjective "proper".

Let us start with the good news. Let M and N be two topological manifolds and let $\varphi\colon N \to M$ be a map. If φ is an embedding, then Lemma 24.5 says that the following two statements are equivalent:

(1) φ satisfies the second condition of a proper map between topological manifolds.
(2) $\varphi\colon N \to M$ is a proper map of topological spaces.

We continue with the bad news. In general the two notions of being "proper" diverge:

(1) The constant map $\mathbb{R} \to \{*\}$ is a proper map of topological manifolds but it is not a proper map of topological spaces.
(2) The inclusion map $f\colon N = [-1,1] \to \mathbb{R} = M$ is a proper map of topological spaces but it is not a proper map of topological manifolds.

Since both definitions of "proper" are commonly used in the literature we will have to live with this very unfortunate ambiguity. We will strive to make sure that, given a map between two topological manifolds, it is clear which of the two definitions of "proper" we are working with.

Proof. First we prove the "(1)⇒(2)"-implication. Thus we assume that $\varphi(M)$ is a closed subset of N. Let K be a compact subset of N. Since $\varphi(M)$ is a closed subset of N we know by the Neighborhood Openness Criterion 1.5 that $\varphi(M) \cap K$ is a closed subset of K. Thus we obtain from the Compact-Closed Lemma 1.21 that $\varphi(M) \cap K$ is compact. But $\varphi\colon M \to N$ is an embedding, i.e. $\varphi\colon M \to \varphi(M)$ is a homeomorphism. We denote by $\psi\colon \varphi(M) \to M$ the inverse map, which is a homeomorphism. We now see that $\varphi^{-1}(K) = \varphi^{-1}(\varphi(M) \cap K) = \psi(K)$ is compact.

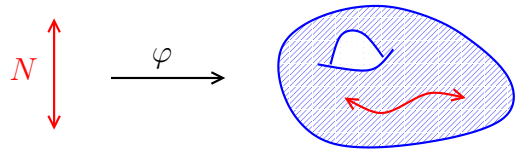
Now we prove the "(2)⇒(1)"-implication. We need to show that $\varphi(M)$ is a closed subset of N. By the Atlas Improvement Proposition 21.3 (which, even though it appears later is completely independent of the present discussion) there exists a locally finite cover $\{K_j\}_{j\in J}$ of N by compact subsets. Since N is Hausdorff we know that each K_j is also a closed subset of N. Thus by Lemma 1.8 it suffices to show that each $\varphi(M) \cap K_j$ is a closed subset of N. Now let $j \in J$. By hypothesis we know that $\varphi^{-1}(K_j)$ is compact. Note that $\varphi(M) \cap K_j = \varphi(\varphi^{-1}(K_j))$. It follows from this observation, together with the Compact Image Lemma 2.13 (1) and the Compact-Closed Lemma 1.21 (2) that $\varphi(M) \cap K_j$ is indeed closed. ■

We continue with the following definition that is specific to smooth manifolds.

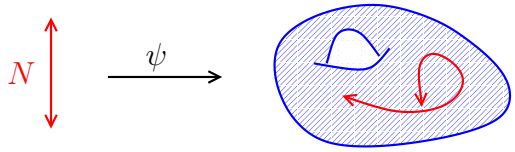
Definition. A map $\varphi\colon N \to M$ between two smooth manifolds is called an immersion if the following conditions are satisfied:

(1) The map φ is smooth.
(2) For each $P \in N$ the map $\mathrm{D}\varphi_P\colon \mathrm{T}_P N \to \mathrm{T}_{\varphi(P)} M$ is a monomorphism.
(3) If $\partial M \neq \varnothing$, then we require that also the following conditions are satisfied:
    (a) For each $P \in N \setminus \partial N$ we have $\varphi(P) \in M \setminus \partial M$.[253]
    (b) For each $P \in \partial N$ with $\varphi(P) \in \partial M$ the image $\mathrm{D}\varphi_P(\mathrm{T}_P N)$ is *not* contained in $\mathrm{T}_{\varphi(P)}(\partial M)$.[254][255]

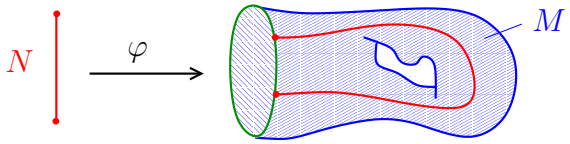
(c) For every component C of ∂N we have either $\varphi(C) \subset \partial M$ or $\varphi(C) \subset M \setminus \partial M$.[256]
An immersion that also satisfies the following extra condition
(5) $\varphi\colon N \to M$ is an embedding,[257]
is called a smooth embedding.



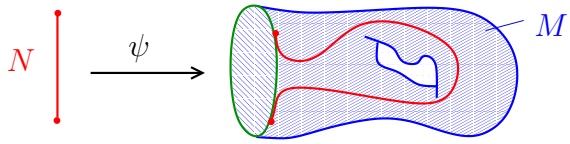
φ is a smooth embedding



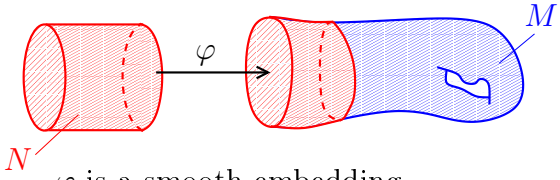
ψ is not a smooth embedding since
$\varphi\colon N \to \varphi(N)$ is not a homeomorphism



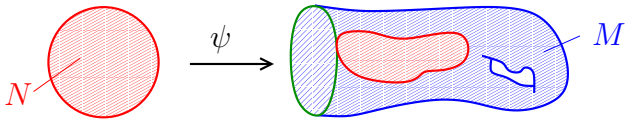
φ is a smooth embedding



ψ is not a smooth embedding since
condition (3b) is not satisfied



φ is a smooth embedding



ψ is not a smooth embedding since
condition (3c) is not satisfied

Remark. Let M be a smooth manifold and let $\varphi\colon M \to \mathbb{H}^n$ be a smooth map. We denote by $p\colon \mathbb{H}^n \to [0,\infty)$ the projection onto the last coordinate. One can show easily that Condition (b) in the definition of an immersion is satisfied if and only if for each $P \in \partial N$ with $\varphi(P) \in \partial M$ the map $\mathrm{D}(q \circ \varphi)(P)\colon \mathrm{T}_P M \to \mathbb{R}$ is an epimorphism.

The following lemma gives an instructive example of immersions and smooth embeddings.

Lemma 24.6. Given $\lambda \in \mathbb{R}$ we consider the following map

$$\begin{array}{rcl} \varphi_\lambda\colon \mathbb{R} & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ t & \mapsto & [(t, \lambda \cdot t)]. \end{array}$$

---

[253] Note that this condition is implied if $\varphi^{-1}(\partial M) = \partial N$, but the condition (3) is weaker than the condition that $\varphi^{-1}(\partial M) = \partial N$.

[254] Here we use the sanity preserving convention from page 633 which allows us to view $\mathrm{T}_{\varphi(P)}\partial M$ as a vector subspace of $\mathrm{T}_{\varphi(P)}M$.

[255] Another way of expressing this condition is that the map $\mathrm{T}_P N \to \mathrm{T}_{\varphi(P)}M/\mathrm{T}_{\varphi(P)}\partial M$ is an epimorphism.

[255] One can also state this condition as a local condition, namely using the fact that boundaries are closed subsets – see the Smooth Manifold Boundary Proposition 19.26 – one can easily show that this condition is equivalent to

(c') For every $P \in \partial N$ with $\varphi(P) \in \partial M$, there exists a neighborhood V of P such that $\varphi(P) \subset \partial M$.

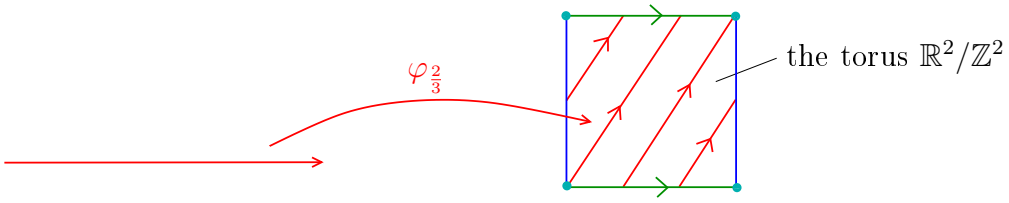
[256] If N is compact, then it follows from the Compact-Hausdorff Proposition 2.17 (2) that it suffices to verify that φ is injective and continuous.

The following statements hold:

(1) The map φ_λ is an immersion.
(2) The map φ_λ is injective if and only if λ is irrational.
(3) If λ is irrational, then $\varphi_\lambda \colon \mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$ is *not* an embedding, in particular φ_λ is not a smooth embedding.
(4) Suppose λ is rational. We write $\lambda = \frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that p and q are coprime. The map

$$\begin{array}{rcl} f_\lambda \colon S^1 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ \exp(2\pi \mathrm{i} t) & \mapsto & [(qt, \lambda \cdot qt)] \end{array}$$

is well-defined and it is a smooth embedding.



Proof. We write $\varphi = \varphi_\lambda$.

(1) We denote by $p \colon \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ the projection and we denote by $\Phi \colon \mathbb{R} \to \mathbb{R}^2$ the map that is given by $t \mapsto (t, \lambda \cdot t)$. Evidently we have $\varphi = p \circ \Phi$. Let $t \in \mathbb{R}$. We consider the following diagram:

$$\begin{array}{ccccc} \mathrm{T}_t\mathbb{R} & & \xrightarrow{\mathrm{D}\Phi_t} & & \mathrm{T}_{\Phi(t)}\mathbb{R}^2 \xrightarrow{\mathrm{D}p_{\Phi(t)}} \mathrm{T}_{\varphi(t)}(\mathbb{R}^2/\mathbb{Z}^2) \\ \Theta \uparrow \cong & & & & \cong \uparrow \Theta \\ \mathbb{R} = \widetilde{\mathrm{T}}_{\varphi(t)}\mathbb{R} & & \xrightarrow{\mathrm{D}\Phi_t} & & \widetilde{\mathrm{T}}_{\Phi(t)}\mathbb{R}^2 = \mathbb{R}^2 \end{array}$$

We make the following observations:
(a) It follows from $\varphi = p \circ \Phi$ and the functoriality of differentials that the top of the diagram commutes.
(b) By the Manifold Quotient-by-Group Action Proposition 19.32 we know that the map $p \colon \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is a local diffeomorphism, hence the differential of p at any point is an isomorphism.
(c) The vertical maps are the natural isomorphisms provided by the Tangent Space-Isomorphism Proposition 23.16. In particular the square commutes.
(d) By the Differential-equals-Differential Lemma 23.4 the bottom horizontal map is given by multiplication by the matrix $\mathrm{D}\Phi_t = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$. In particular the bottom map is a monomorphism.

It follows from all of the above the $\mathrm{D}\varphi_t \colon \mathrm{T}_t\mathbb{R} \to \mathrm{T}_{\varphi(t)}(\mathbb{R}^2/\mathbb{Z}^2)$ is a monomorphism.

(2) This statement is an amusing exercise that is left to the reader.

(3) The proof of this statement requires the simple number theoretic fact that given any $\alpha \in \mathbb{R}$ and any $N \in \mathbb{N}$ there exist $n, m \in \mathbb{Z}$ with $1 \leq n \leq N$ and with $|n\alpha - m| < \frac{1}{N}$.[257] We leave it to the reader to fill in the remaining details.

(4) Let λ be rational. We write $\lambda = \frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that p and q are coprime. We consider the following diagram

$$\begin{array}{ccccc} S^1 & \xrightarrow{\quad f_\lambda \quad} & \mathbb{R}^2/\mathbb{Z}^2 & & \\ \uparrow{\scriptstyle g([t])=\exp(2\pi i t)} & & \uparrow{\scriptstyle \varphi_\lambda} & & \\ \mathbb{R}/\mathbb{Z} & \xleftarrow[[t]\hookleftarrow t]{=:p} & \mathbb{R} & \xrightarrow[t\mapsto qt]{=:h} & \mathbb{R} \end{array} \quad \text{which gives rise to} \quad \begin{array}{ccccc} \mathrm{T}_{\exp(2\pi i t)}S^1 & \xrightarrow{\quad f_\lambda \quad} & \mathrm{T}_{\varphi(t)}\mathbb{R}^2/\mathbb{Z}^2 & & \\ \cong\uparrow{\scriptstyle g} & & \uparrow{\scriptstyle \varphi_\lambda} & & \\ \mathrm{T}_{[t]}\mathbb{R}/\mathbb{Z} & \xleftarrow[\cong]{\mathrm{D}p_t} & \mathrm{T}_t\mathbb{R} & \xrightarrow[\cong]{\mathrm{D}h_t} & \mathrm{T}_t\mathbb{R}. \end{array}$$

Considering the commutative diagram on the right we see that f_λ is indeed an immersion. Furthermore it is straightforward to see that $f_\lambda \colon S^1 \to \mathbb{R}^2/\mathbb{Z}^2$ is injective. Thus it follows from the Compact-Hausdorff Proposition 2.17 (3) that f_λ is a smooth embedding. ■

We continue our discussion of smooth embeddings with the following modest lemma.

Lemma 24.7. Let $\varphi \colon N \to M$ be a map between smooth manifolds.

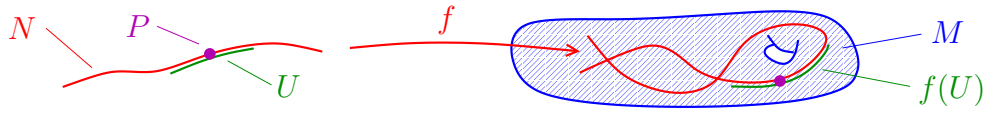
(1) If φ is a smooth embedding, then $\varphi|_{\partial N} \colon \partial N \to M$ is also a smooth embedding.

(2) If φ is a proper smooth embedding, then $\varphi|_{\partial N} \colon \partial N \to \partial M$ is also a proper smooth embedding.

Proof. The statement follows almost immediately from the Physical Tangent Space-Inclusion Proposition 23.14 (2) and the Smooth Manifold Boundary Proposition 19.26 (3a). ■

We conclude this section with the Local Smooth Embedding Theorem which can be viewed as a generalization of the Inverse Mapping Theorem 24.1.

Theorem 24.8. (Local Smooth Embedding Theorem) Let $f \colon N \to M$ be a smooth map between two smooth manifolds. Let $P \in N$ be a point with $f(P) \notin \partial M$. We suppose that $\mathrm{D}f_P \colon \mathrm{T}_P N \to \mathrm{T}_{f(P)} M$ is a monomorphism. Then there exists an open neighborhood U of P such that the restriction of f to a map $U \to M$ is a smooth embedding.



Proof. Since we were very meticulous in the proof of the Inverse Mapping Theorem 24.1, we feel that it is now OK to provide fewer details in the proof. Thus let $P \in N$ be a point such that $f(P) \notin \partial M$ and such that $\mathrm{D}f_P \colon \mathrm{T}_P N \to \mathrm{T}_{f(P)} M$ is a monomorphism.

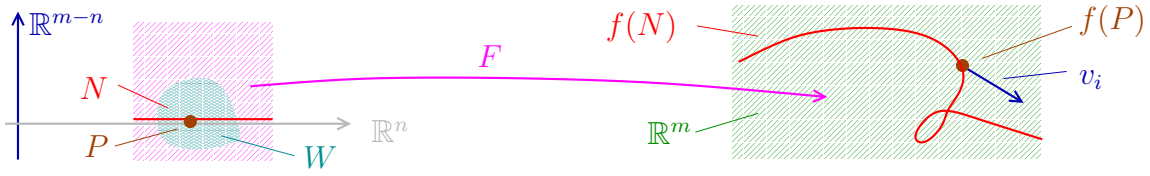
First we consider the case $P \in N \setminus \partial N$. Since $P \notin \partial N$ and $f(P) \notin \partial M$ we see that, after applying charts, we might as well assume that N is an open subset of $\mathbb{R}^n$ and that M is an open subset of $\mathbb{R}^m$. Since $\mathrm{D}f_P \colon \mathrm{T}_P N \to \mathrm{T}_{f(P)} M$ is a monomorphism we can pick $v_1, \ldots, v_{m-n} \in \mathbb{R}^m$ such that the image of $\mathrm{D}f_P$ together with the vectors $v_1, \ldots, v_{m-n}$

[257] This statement is known as Dirichlet's Approximation Theorem, but it is actually elementary to prove, see e.g. [**Lee2002**, Lemma 4.21].

span $\mathbb{R}^m$. Now we consider the map

$$\begin{aligned} F\colon M \times \mathbb{R}^{m-n} &\to \mathbb{R}^m \\ (x, (\lambda_1, \dots, \lambda_{m-n})) &\mapsto f(x) + \sum_{i=1}^{m-n} \lambda_i \cdot v_i. \end{aligned}$$

Basically by design the differential $\mathrm{D}F_{(P,0)}$ is an epimorphism, and thus an isomorphism. Therefore it follows from the Inverse Function Theorem 19.12 that there exists an open neighborhood U of $(P,0)$ such that $F|_W\colon W \to \Phi(W)$ is a diffeomorphism. It follows almost immediately that the restriction of f to $\{u \in N \,|\, (u,0) \in W\}$ has the desired property.



Finally we consider the case that $P \in \partial N$. After picking a chart we might as well assume that N is an open subset of $\mathbb{H}^n$. By definition of a smooth map on an open subset of $\mathbb{H}^n$, see pages 532 and 549, there exists an $r > 0$ such that $B^n_r(P) \cap \mathbb{H}^n \subset N$ and a smooth map $\widetilde{f}\colon B^n_r(P) \to M$ that agrees with f on $B^n_r(P) \cap \mathbb{H}^n$. Now that P lies in the interior of $B^n_r(P)$ it follows from the previous case that there exists an open neighborhood $\widetilde{U}$ of $P \in B^n_r(P)$ such that the restriction of $\widetilde{f}$ to a map $\widetilde{U} \to M$ is a smooth embedding. We now set $U := \widetilde{U} \cap \mathbb{H}^n$. Then the restriction of f to U is a smooth embedding. ■

24.4. **The Smooth Embedding Theorem.** We start out with a little lemma.

Lemma 24.9. Let M be a smooth manifold and let W be a (proper) submanifold. The inclusion $\iota\colon W \to M$ is a (proper) smooth embedding.

Proof.

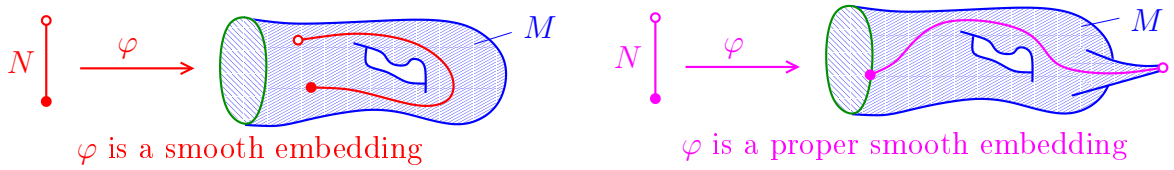
(1) By the Submanifold Smooth Structure Lemma 19.24 the inclusion ι is smooth.
(2) It follows from the Physical Tangent Space-Inclusion Proposition 23.14 that each $\mathrm{D}\iota_P$ is a monomorphism.
(3) (a) Basically by definition we have $W \setminus \partial W \subset M \setminus \partial M$.
    (b) For $P \in \partial N$ with ∂M we have a submanifold chart of type (β) which implies that $\mathrm{T}_P N \not\subset \mathrm{T}_P \partial M$.
    (a) In Exercise 19.11 we showed that for every component C of ∂W we have either $C \subset \partial M$ or $C \subset M \setminus \partial M$.
(4) By definition of the subspace topology ι is an embedding.

If W is a proper submanifold, then it follows from an easy comparison of the definitions that ι is indeed a proper embedding. ■

The following theorem can be viewed as a converse to the simple-minded Lemma 24.9.

Theorem 24.10. (Smooth Embedding Theorem) Let N be an n-dimensional smooth manifold and let M be a smooth manifold.

(1) If $\varphi\colon N \to M$ is a smooth embedding, then $\varphi(N)$ is an n-dimensional submanifold of M and the map $\varphi\colon N \to \varphi(N)$ is a diffeomorphism.[258]
(2) If $\varphi\colon N \to M$ is a proper smooth embedding, then $\varphi(N)$ is a proper n-dimensional submanifold of M and the map $\varphi\colon N \to \varphi(N)$ is a diffeomorphism.



Remark. Let M be a smooth manifold and let N be a smooth submanifold in the sense of the definition on page 550. The Smooth Embedding Theorem 24.10 implies that the standard smooth structure on N, which we introduced on page 552, is characterized by the property that it is the unique smooth structure that turns the inclusion $i\colon N \to M$ into a smooth embedding.

Proof. Let N be an n-dimensional smooth manifold and let M be an m-dimensional smooth manifold. Let $\varphi\colon N \to M$ be a smooth embedding. By definition this means that φ satisfies the following:

(1) The map φ is smooth.
(2) For each $P \in N$ the induced map $\mathrm{D}\varphi_P\colon \mathrm{T}_P N \to \mathrm{T}_{\varphi(P)} M$ is a monomorphism.
(3) If $\partial M \neq \varnothing$, then we require that also the following conditions are satisfied:
  (a) For each $P \in N \setminus \partial N$ we have $\varphi(P) \in M \setminus \partial M$.
  (b) For each $P \in \partial N$ with $\varphi(P) \in \partial M$ the image $\mathrm{D}\varphi_P(\mathrm{T}_P N)$ is *not* contained in $\mathrm{T}_{\varphi(P)}(\partial M)$.
  (c) For every component C of ∂N we have either $\varphi(C) \subset \partial M$ or $\varphi(C) \subset M \setminus \partial M$.
(4) $\varphi\colon N \to M$ is an embedding.

We need to show that $\varphi(N)$ is an n-dimensional submanifold of M and that the map $\varphi\colon N \to \varphi(N)$ is a diffeomorphism.

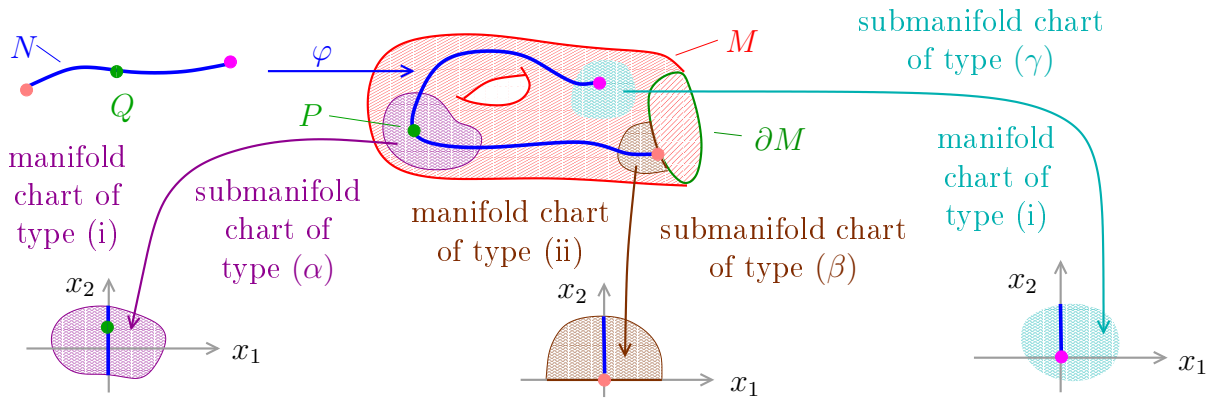
Claim. Let $Q \in N$. The following statements hold:

(α) $Q \notin \partial N$ and $\varphi(Q) \notin \partial M$ $\Rightarrow$ there exists a submanifold chart of type (α) for $\varphi(Q)$
(β) $Q \in \partial N$ and $\varphi(Q) \in \partial M$ $\Rightarrow$ there exists a submanifold chart of type (β) for $\varphi(Q)$
(γ) $Q \in \partial N$ and $\varphi(Q) \notin \partial M$ $\Rightarrow$ there exists a submanifold chart of type (γ) for $\varphi(Q)$.

We refer to page 550 for the definitions of the various submanifold charts. A cartoon version of the definitions is also given in the figure below.

Proof. Note that by hypothesis (3a) the above three cases are the only cases that can occur. Thus we see that the claim implies that $\varphi(N)$ is indeed a submanifold of M. Furthermore it follows from the Submanifold Smooth Structure Lemma 19.24 and the Smooth Pasting Proposition 19.23 that $N \to \varphi(N)$ is a diffeomorphism. Furthermore, if φ is proper in the sense of the definition on page 650, then $\varphi(N)$ is a closed subset of M and $\varphi^{-1}(\partial M) = \partial N$.

[258] Recall that a smooth submanifolds inherits by the Submanifold Smooth Structure Lemma 19.24 the structure of a smooth manifold. Thus it makes sense to say that $\varphi\colon N \to \varphi(N)$ is a diffeomorphism.



The latter fact implies, assuming the claim, that only submanifold charts of type (α) and (β) occur. Together with the fact that $\varphi(N)$ is closed we now know, by the definition on page 550, that $\varphi(N)$ is indeed proper submanifold. In other words, all that is left to do is to prove the claim.

So let us provide the proof of the claim. Let $Q \in N$. We treat the three cases separately.

(α) We assume that $Q \notin \partial M$ and we assume that $P := \varphi(Q) \notin \partial M$.

- Since $Q \notin \partial M$ we can pick a chart $\Theta\colon V \to \widetilde{V}$ of type (i) around Q.
- By hypothesis $\varphi\colon N \to M$ is an embedding which means that $\varphi\colon N \to \varphi(N)$ is a homeomorphism. This implies that $\varphi(V)$ is an open subset of $\varphi(N)$. By definition of the subspace topology this means that there exists an open subset $Z \subset M$ with $\varphi(V) = Z \cap \varphi(N)$.
- Since $P \notin \partial M$ we can pick a chart $\Xi\colon U \to \widetilde{U}$ of type (i) around P with $U \subset Z$.
- After possibly replacing V by $V \cap \varphi^{-1}(U)$ we can and will assume that $\varphi(U) \subset V$.

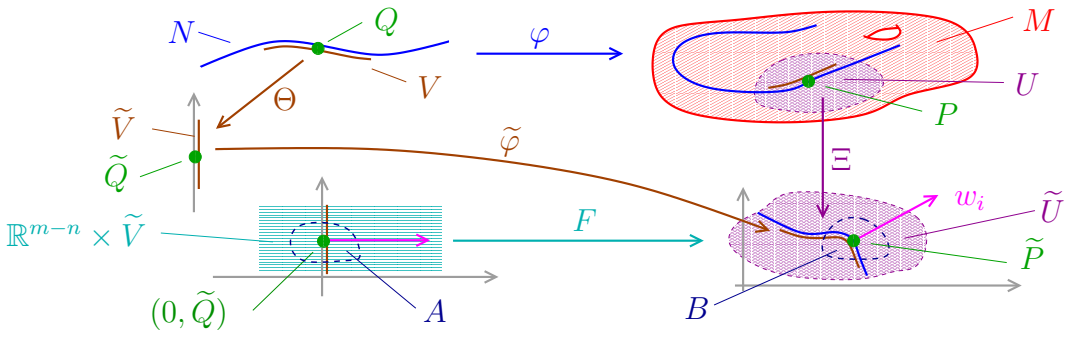
Since $\varphi(U) \subset V$ we can consider $\widetilde{\varphi} := \Xi \circ \varphi \circ \Theta^{-1}\colon \widetilde{V} \to \widetilde{U}$. Since φ is an immersion we know that $\mathrm{D}\varphi_Q\colon \mathrm{T}_Q N \to \mathrm{T}_P M$ is a monomorphism. It follows from the Tangent Space-Isomorphism Proposition 23.16 that the corresponding map $\mathrm{D}\widetilde{\varphi}_{\widetilde{Q}}\colon \mathbb{R}^n \to \mathbb{R}^m$ is a monomorphism. Thus there exist vectors $w_1, \ldots, w_{m-n} \in \mathbb{R}^m$ such that the image of $\mathrm{D}\varphi_Q$ together with these vectors span $\mathbb{R}^m$. We consider the map

$$\begin{array}{rcl} F\colon \mathbb{R}^{m-n} \times \widetilde{V} & \to & \mathbb{R}^m \\ ((t_1, \ldots, t_{m-n}), x) & \mapsto & \widetilde{\varphi}(x) + \sum_{i=1}^{m-n} t_i \cdot w_i. \end{array}$$

Note that

$$\mathrm{D}F_{(0,\widetilde{Q})} \;=\; \begin{pmatrix} w_1 & \ldots & w_{m-n} & \mathrm{D}\widetilde{\varphi}_{\widetilde{Q}} \end{pmatrix} \;\in\; \mathrm{M}(m \times m, \mathbb{R}).$$

Thus we see that $\mathrm{D}F_{(0,\widetilde{Q})}$ is invertible. It follows from the Inverse Function Theorem 19.12 that there exists an open neighborhood A of $(0, \widetilde{Q})$ and an open neighborhood B of $\widetilde{P} \in \widetilde{U}$ such that $F\colon A \to B$ is a diffeomorphism. It follows almost immediately from the construction that $(F|_B)^{-1} \circ \Xi\colon \Xi^{-1}(B) \to A$ is a submanifold chart of type (α) for P.



(β) The proof of this case is very similar to the proof of (α). We need to make only a few minor adjustments:

- This time we use charts of type (ii) for Q and for P. Our hypothesis (3c) implies that there exists an open neighborhood V of Q such that $\varphi(V \cap \partial N) \subset \partial M$.
- Our hypothesis (3b) implies that we can find $w_1, \ldots, w_{m-n} \in \mathbb{R}^{m-1} \times \{0\}$ such that the image of $\mathrm{D}\varphi_Q$ together with these vectors span $\mathbb{R}^m$.

Using these two observations one can now show easily that the approach taken in (α) leads to a submanifold chart of type (β). The only difference is that we need to use Lemma 24.3 instead of the Inverse Function Theorem 19.12.

(γ) The proof of this statement is again very similar to the proof of (α). We start out with a chart of type (ii) for Q and a chart of type (i) for P. We obtain a smooth map $\widetilde{\varphi}\colon \widetilde{V} \to V$ where $\widetilde{V}$ is an open subset of $\mathbb{H}^n$. By the definition of a smooth map on $\mathbb{H}^n$, see page 532, we can (after possibly first shrinking $\widetilde{V}$) extend this map to an open subset of $\mathbb{R}^n$. The rest of the argument is now basically identical to the proof of (α). ■

Examples.

(1) We consider the map

$$\begin{aligned} \Psi\colon \mathbb{R}^2 \times S^1 &\to \mathbb{R}^3 \\ ((x,y), \exp(\mathrm{i}\varphi)) &\mapsto \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation around } z\text{-axis}} \cdot \underbrace{\begin{pmatrix} 1+\frac{1}{2}x \\ 0 \\ -\frac{1}{2}y \end{pmatrix}}_{\substack{\text{describes circle} \\ \text{in } xz\text{-plane}}} \end{aligned}$$

which we introduced in Lemma 5.7.

(a) Using the natural isomorphism from the Tangent Space-Isomorphism Proposition 23.16 one can show fairly easily that Θ restricts to a smooth embedding $\Psi\colon \overline{B}^2 \times S^1 \to \mathbb{R}^3$.

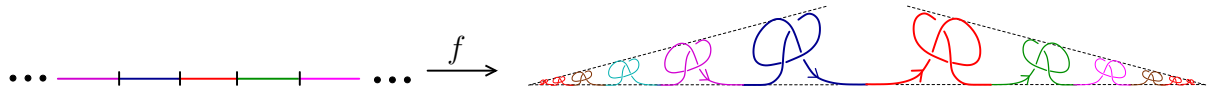
(b) It follows from (a) together with Lemma 24.7 that Ψ also restricts to a smooth embedding $\Psi\colon S^1 \times S^1 \to \mathbb{R}^3$.

By Theorem 24.10 this allows us to view $\overline{B}^2 \times S^1$ and $S^1 \times S^1$ as submanifolds of $\mathbb{R}^3$.

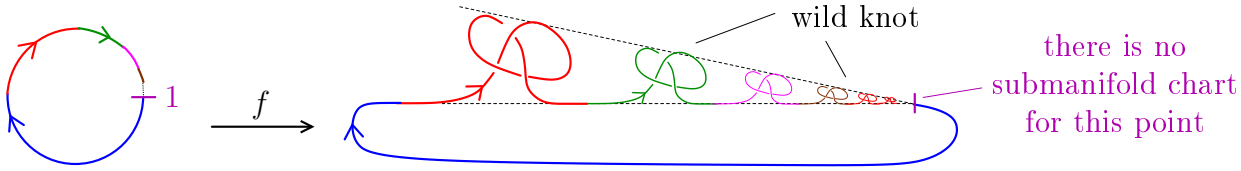
(2) Let $n \in \mathbb{N}_0$. In Lemma 2.18 we used the stereographic projection to give an explicit homeomorphism $\Phi\colon S^n \to \mathbb{R}^n \cup \{\infty\}$. We leave it to the reader to verify that the

map $\mathbb{R}^n \to \mathbb{R}^n \cup \{\infty\} \xrightarrow{\Phi^{-1}} S^n$ is a smooth embedding. We will use this embedding to view $\mathbb{R}^n$ as a submanifold of the smooth manifold S^n.

(3) We consider the map $f\colon \mathbb{R} \to \mathbb{R}^3$ that is shown in the figure below. The map f is a smooth embedding, hence its image is a smooth submanifold of $\mathbb{R}^3$.



(4) Note that in Theorem 24.10 it is essential that we are dealing with an immersion. For example consider the map $f\colon S^1 \to \mathbb{R}^3$ that is illustrated in the figure below. With a little bit of good will and effort one can arrange that f is smooth and that $\mathrm{D}f_P$ is a monomorphism at all points except for $P = 1$. Using the Compact-Hausdorff Proposition 2.17 (3) it is straightforward to show that this map is a homeomorphism onto its image, i.e. it is an embedding. But in Theorem 124.13 we will see that $f(S^1)$ is not even a topological submanifold of $\mathbb{R}^3$, more precisely we will see, that are no submanifold charts for the point $f(1) \in f(S^1)$.[259]

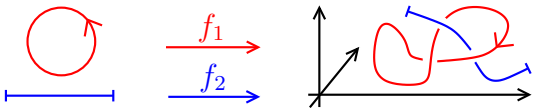


The following lemma is often used implicitly when dealing with smooth embeddings of disconnected manifolds.

Lemma 24.11. (Disjoint Union-Smooth Embedding Lemma) Let $M_1, \dots, M_k$ be compact smooth manifolds, let N be a smooth manifold and let $f_i\colon M_i \to N$ be smooth embeddings. If the images are disjoint, then the map

$$\bigsqcup_{i=1}^{k} f_i\colon \bigsqcup_{i=1}^{k} M_i \;\to\; N$$

is also a smooth embedding.



$f_1 \sqcup f_2\colon M_1 \sqcup M_2 \to \mathbb{R}^2$ is a smooth embedding

Proof. The only statement one really needs to verify is that the map is an embedding. This follows from Lemma 8.3 which we can apply by our hypothesis that $M_1, \dots, M_k$ are compact, the fact that N is by definition Hausdorff and the observation that by the Manifold Paracompact-Normal Proposition 18.9 we know that N is normal. ∎

We state our next example as a lemma.

[259] The image of an embedding $S^1 \to \mathbb{R}^3$ such that the image is not a topological submanifold is sometimes called a *wild knot*.

Lemma 24.12. (Möbius Smooth Embedding Lemma) As we discussed on pages 252 and 564 the map

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n,(x,y)) & \mapsto & (x+n, (-1)^n \cdot y) \end{array}$$

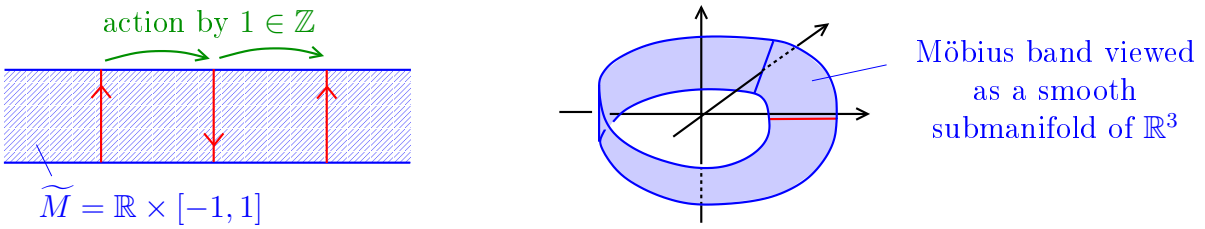
defines an action of $\mathbb{Z}$ on $\widetilde{M} := \mathbb{R} \times [-1,1]$ that is free and smooth. The quotient space $M := (\mathbb{R} \times [-1,1])/\mathbb{Z}$ is called the Möbius band. Given $\varphi, \psi \in \mathbb{R}$ and $r \in [-1,1]$ we set

$$A(\varphi) := \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad v(r,\psi) := \begin{pmatrix} 2 + r\sin\psi \\ 0 \\ r\cos\psi \end{pmatrix}.$$

The map

$$\begin{array}{rcl} \Psi\colon (\mathbb{R} \times [-1,1])/\mathbb{Z} & \to & \mathbb{R}^3 \\ [(\varphi, r)] & \mapsto & A(2\pi \cdot \varphi) \cdot v\big(r, \pi\varphi\big) \end{array}$$

is a smooth embedding. Thus the Smooth Embedding Theorem 24.10 allows us to view the Möbius band as a submanifold of $\mathbb{R}^3$.



Proof. On page 252 we already mentioned that Ψ is an embedding. It remains to show that it is an immersion. We denote by $\Phi\colon \widetilde{M} = \mathbb{R} \times [-1,1] \to \mathbb{R}^3$ the map that is given by $\Phi(\varphi, r) = A(2\pi \cdot \varphi) \cdot v(r, \pi\varphi)$ and we denote by $p\colon \widetilde{M} = \mathbb{R} \times [-1,1] \to (\mathbb{R} \times [-1,1])/\mathbb{Z}$ the projection. Given any $(\varphi, r) \in \widetilde{M} = \mathbb{R} \times [-1,1]$ we consider the following diagram:

$$\begin{array}{ccccc} \widetilde{\mathrm{T}}_{(\varphi,r)}\widetilde{M} & & \xrightarrow{\Theta} & & \mathrm{T}_{(\varphi,r)}\widetilde{M} \xrightarrow{\mathrm{D}p_{(\varphi,r)}} \mathrm{T}_{(\varphi,r)}M \\ \big\downarrow{\scriptstyle \mathrm{D}\Phi_{(\varphi,r)}} & & & & \big\downarrow{\scriptstyle \mathrm{D}\Phi_{(\varphi,r)}} \quad \swarrow{\scriptstyle \mathrm{D}\Psi_{[\varphi,r]}} \\ \widetilde{\mathrm{T}}_{\Phi(\varphi,r)}\mathbb{R}^3 & & \xrightarrow{\Theta} & & \mathrm{T}_{\Phi(\varphi,r)}\mathbb{R}^3 \end{array}$$

We make the following observations and clarifications:

(1) The horizontal maps to the left are the natural isomorphisms provided by the Tangent Space-Isomorphism Proposition 23.16.
(2) A slightly heroic, but ultimately elementary calculation using the Differential-equals-Differential Lemma 23.4 shows that the vertical map to the left is a monomorphism.
(3) It follows from the Manifold Quotient-by-Group Action Proposition 19.32 that the horizontal to the top right is an isomorphism.
(4) The triangle commutes by the functoriality of differentials.

The combination of the above shows that the diagonal map is a monomorphism. ■

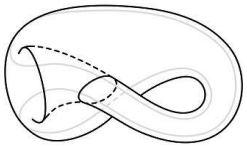
24.5. **Immersions and smooth embeddings of smooth manifolds into $\mathbb{R}^n$ I.** In Question 5.37 we had asked whether the projective space $\mathbb{R}\mathrm{P}^2$ or the Klein bottle can be viewed as subsets of $\mathbb{R}^3$. Now that we have more mathematical language at our disposal

we can turn this vague question into a precise question. In fact we can reinterpret this question in three different ways:

Question 24.13.
(1) Does $\mathbb{R}P^2$ (respectively the Klein bottle) admit an immersion into $\mathbb{R}^3$?
(2) Does $\mathbb{R}P^2$ (respectively the Klein bottle) admit a smooth embedding into $\mathbb{R}^3$?
(3) Does $\mathbb{R}P^2$ (respectively the Klein bottle) admit an embedding into $\mathbb{R}^3$?

We start out with Question 24.13 (1).

(a) We saw in the figure on page 265 that there exists an immersion of the Klein bottle K into $\mathbb{R}^3$. In the figure below we show the images of two other immersions of the Klein bottle into $\mathbb{R}^3$. This gives a non-rigorous positive answer to Question 24.13 (1) for the Klein bottle. In Exercise 24.17 we will give a rigorous positive answer to Question 24.13 (1) for the Klein bottle. Finally we point out that explicit immersions of the Klein bottle into $\mathbb{R}^3$ are discussed in detail in [**Frai2012**].

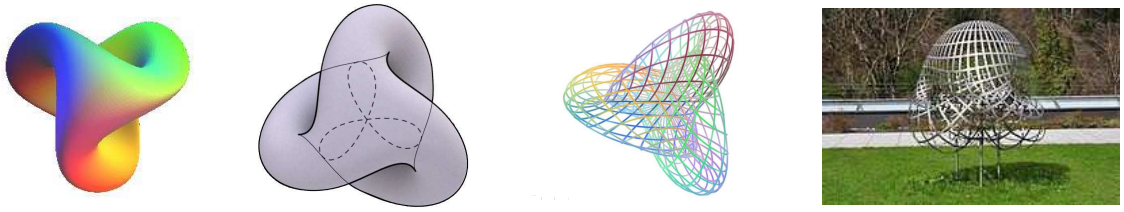



(b) The case of the real projective plane is much more interesting. In 1901 David Hilbert, arguably the greatest German mathematician of his generation, gave his PhD student Werner Boy [**Boy1903**][260] the task to prove that there is no immersion of $\mathbb{R}P^2$ into $\mathbb{R}^3$. To everyone's surprise Boy did the opposite, he showed that it was possible to immerse $\mathbb{R}P^2$ into $\mathbb{R}^3$. In the figure below we give several illustrations of the image of the immersion. A detailed description of the immersion is given in [**Kir2007, Ap1987**]. But perhaps it is more helpful to just watch the following video:

`https://www.youtube.com/watch?v=uiq-EcQz_uU`

A fun description of Boy's surface, and of many other concepts from topology, can also be found here:

`http://www.savoir-sans-frontieres.com/JPP/telechargeables/English/Topo_the_world_eng.pdf`



Boy's surface at Oberwolfach

As of right now we do not have the tools to answer Questions 24.13 (2) and (3). In Corollary 43.13 we will give an answer to Question 24.13 (2). Note that on pages 2705

[260] Werner Boy (1879-1914) was a German mathematician who died in the first weeks of World War I.

and 2705 we provide two different proofs of Corollary 43.13. Furthermore we will give an answer to Question 24.13 (3) in Corollary 124.6.
Since we cannot Question 24.13 (2) and (3), let us stick to statements we can actually prove. For example the next proposition says that any $\mathbb{R}\mathrm{P}^n$ can be smoothly embedded into some $\mathbb{R}^N$.

Proposition 24.14. (Real Projective Space-Smooth Embedding Proposition)
(1) The map

$$\begin{array}{rcl} \mathbb{R}\mathrm{P}^2 = S^2/\pm 1 & \to & \mathbb{R}^4 \\ [(x,y,z)] & \mapsto & (yz, xz, xy, x^2+2y^2+3z^2) \end{array}$$

is a smooth embedding.
(2) Let $n \in \mathbb{N}$. The map

$$\begin{array}{rcl} \mathbb{R}\mathrm{P}^n = S^n/\pm 1 & \to & \mathbb{R}^{\frac{n\cdot(n+1)}{2}+1} \\ [(x_1,\ldots,x_{n+1})] & \mapsto & \left((x_i\cdot x_j)_{i=1,\ldots,n+1, j=1,\ldots,i-1}, \sum_{k=1}^{n+1} k\cdot x_k^2\right) \end{array}$$

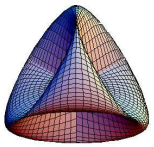
is a smooth embedding.

Proof. We will prove Statement (1) in Exercise 24.9. We leave the proof of Statement (2) to the intellectually intrepid reader. ■

Remark. We consider the map

$$\begin{array}{rcl} f\colon \mathbb{R}\mathrm{P}^2 & \to & \mathbb{R}^3 \\ [(x,y,z)] & \mapsto & (yz, xz, xy). \end{array}$$

In other words, we consider the composition of the smooth embedding $\mathbb{R}\mathrm{P}^2 \to \mathbb{R}^3$ from the Real Projective Space-Smooth Embedding Proposition 24.14 with the projection $\mathbb{R}^4 \to \mathbb{R}^3$ onto the first three coordinates. In Exercise 24.10 we will see that there are precisely 6 points on $\mathbb{R}\mathrm{P}^2$ such that $\mathrm{D}f_P$ is not a monomorphism. By [**Ap1987**, p. 40] the image $f(\mathbb{R}\mathrm{P}^2) \subset \mathbb{R}^3$ is the so-called *Steiner surface* or *Roman surface*, which was first discovered by the Swiss mathematician Jacob Steiner in Rome in 1844.



Steiner surface in $\mathbb{R}^3$

As always in mathematics, a result generates new questions. For example, now that by the Real Projective Space-Smooth Embedding Proposition 24.14 we know that we can smoothly embed each $\mathbb{R}\mathrm{P}^n$ into $\mathbb{R}^{\frac{n\cdot(n+1)}{2}+1}$, the following question naturally arises:

Question 24.15. Given $n \in \mathbb{N}$, what is the minimal dimension $d(n)$ such that $\mathbb{R}\mathrm{P}^n$ admits a smooth embedding into $\mathbb{R}^{d(n)}$?

By the Real Projective Space-Smooth Embedding Proposition 24.14 we know that $d(n) \leq \frac{n\cdot(n+1)}{2} + 1$. In the Weak Whitney Embedding Theorem 30.7 we will show that $d(n) \leq 2n+1$ and in Corollary 43.13 we will see that $d(2) \geq 4$. Furthermore it follows from Proposition 123.11 that $d(n) \geq n+2$. Over the years there have been lots of partial

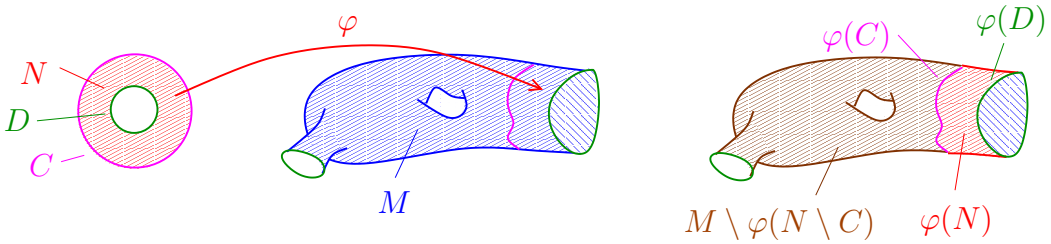
answers to Question 24.15, see e.g. [**Mah1962, MahM1968**]. Nonetheless it seems like Question 24.15 has not been fully answered yet. Some of the results are summarized in

`https://www.lehigh.edu/~dmd1/immtable`

24.6. **Immersions among smooth manifolds of the same dimension.** In the following proposition we consider immersions and smooth embeddings between smooth manifolds of the same dimension.

Proposition 24.16. (Codimension-Zero Smooth Embedding Proposition) Let M and N be smooth manifolds of the same dimension and let $\varphi\colon N \to M$ be an immersion. Recall that by the definition of an immersion on page 651 we know that for every component C of ∂N we have either $\varphi(C) \subset \partial M$ or $\varphi(C) \subset M \setminus \partial M$. We denote by C the union of all components of ∂N with $\varphi(C) \not\subset \partial M$ and we write $D := \partial N \setminus C$.

(1) The image $\varphi(N \setminus C)$ is an open subset of M.
(2) If $\varphi\colon N \to M$ is a smooth embedding and if $\varphi(N)$ is a closed subset of M, then the following five statements hold:
   (a) $\varphi(N)$ is a submanifold of M with[261] $\partial_0(\varphi(N)) = \varphi(C)$ and $\partial_1(\varphi(N)) = \varphi(D)$.
   (b) M is decomposed, in the sense of the definition on page 570, into the submanifolds $\varphi(N)$ and $X := M \setminus \varphi(N \setminus C)$.
   (c) The boundary of $M \setminus \varphi(N \setminus C)$ as a smooth manifold in its own right is given by $(\partial M \setminus \varphi(D)) \cup \varphi(C)$.
   (d) If ∂N is connected and if M is connected, then X is also connected.
   (e) If M is compact, then $M \setminus \varphi(N \setminus \partial N)$ is also compact.
   (We refer to the below for an illustration.)
(3) If $\varphi\colon N \to M$ is a smooth embedding, if N is closed and if M is connected, then φ is a diffeomorphism.



Example. It follows from the Codimension-Zero Smooth Embedding Proposition 24.16 (3) that it is impossible to smoothly embed a closed n-dimensional smooth manifold into $\mathbb{R}^n$.

Sketch of proof.

(1) This statement follows almost immediately from the Inverse Mapping Theorem 24.1.
(2) By the Smooth Embedding Theorem 24.10 we know that $\varphi(N)$ is a codimension-zero submanifold of N. The statement about $\partial_0(\varphi(N))$ and $\partial_1(\varphi(N))$ follows immediately from the definition on page 568.

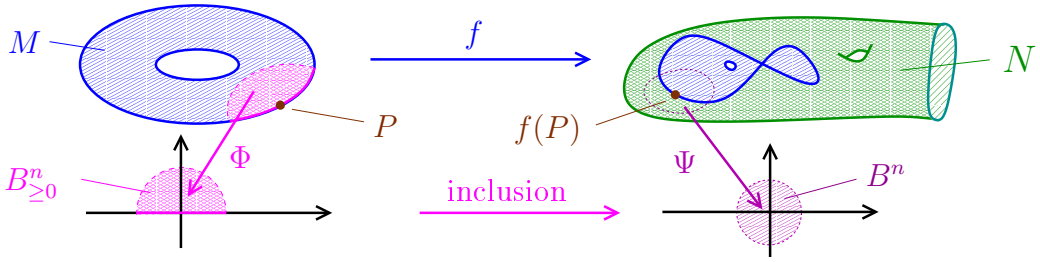
   All the remaining statements follow from our hypothesis that $\varphi(N)$ is a closed subset of M together with the Codimension-Zero Submanifold Proposition 19.38.

[261] We refer to page 568 for the definition of ∂_0 and ∂_1 of a codimension-zero submanifold.

(3) We already know by Theorem 24.10 that $\varphi\colon N \to \varphi(N)$ is a diffeomorphism. Thus it suffices to show that $\varphi(N) = M$. First note that it follows from the Compact Image Lemma 2.13 and the Compact-Closed Lemma 1.21 (2) that $\varphi(N)$ is a closed subset of M. Furthermore, it follows easily from the Smooth Invariance of Domain Theorem 19.13 that $\varphi(N)$ is an open subset of M. Since M is connected we see that $M = \varphi(N)$. ■

Exercises for Chapter 24.

Exercise 24.1. Let $f\colon M \to N$ be a smooth map between two n-dimensional smooth manifolds such that $f(\partial M) \subset N \setminus \partial N$. Let $P \in \partial M$ be a point such that $\mathrm{D}f_P$ is invertible. Show that there exists a chart $\Phi\colon U \to B^n_{\geq 0}$ for M around P and a chart $\Psi\colon V \to B^n$ around $f(P)$ such that the map $\Psi \circ f \circ \Phi^{-1}\colon B^n_{\geq 0} \to B^n$ is the inclusion.



Exercise 24.2. At which points is the map

$$\begin{array}{rcl} f\colon S^2 & \to & S^2 \\ (x,y,z) & \mapsto & \frac{1}{x^4+y^4+z^4}(x^2,y^2,z^2) \end{array}$$

a local diffeomorphism?

Exercise 24.3. Let X and Y be two topological spaces and let $\varphi\colon X \to Y$ be an embedding. We consider the following two conditions on φ.

(I) $\varphi(X)$ is a closed subset of Y.
(II) For every compact subset K of Y the preimage $\varphi^{-1}(K)$ is compact.

In Lemma 24.5 we saw that the two conditions (I) and (II) are equivalent if we are dealing with a map between topological manifolds.

(a) Given an example of a map $\varphi\colon X \to Y$ between two topological spaces such that (I) is satisfied but (II) is not satisfied.
(b) Given an example of a map $\varphi\colon X \to Y$ between two topological spaces such that (II) is satisfied but (I) is not satisfied.

Exercise 24.4. Let M and N be smooth manifolds and let $f\colon M \to N$ be an immersion. We denote by $\Delta = \{(x,x) \in M \times M \,|\, x \in M\}$ the "diagonal" of $M \times M$.

(a) Show that there exists an open neighborhood U of the diagonal Δ such that for every $(x,y) \in U \setminus \Delta$ we have $f(x) \neq f(y)$.
(b) We suppose that M is compact. Show that f admits only finitely many double points, i.e. that there exist only finitely many distinct pairs $(x_1,y_1),\dots,(x_n,y_n)$ in $(M \times M) \setminus \Delta$ such that $f(x_i) = f(y_i)$ for $i = 1,\dots,n$.

Exercise 24.5.

(1) Show that a closed n-dimensional smooth manifold cannot admit an immersion into $\mathbb{R}^n$.
(2) Let $\varphi\colon \overline{B}^2 \to T := S^1 \times S^1$ be a smooth embedding.
   (a) Does there exist an immersion of $M := T \setminus \varphi(B^2)$ to $\mathbb{R}^2$?
   (b) Does there exist a smooth immersion of $M := T \setminus \varphi(B^2)$ to $\mathbb{R}^2$?

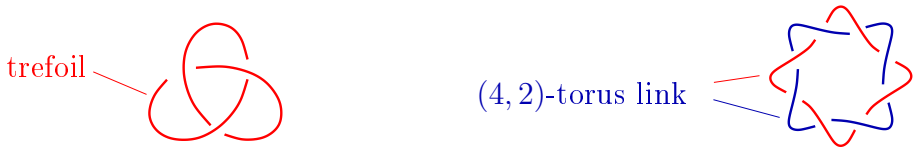
Exercise 24.6. Let $m, n \in \mathbb{N}_0$. Show that the smooth submanifold $S^m \times S^n$ is diffeomorphic to a smooth submanifold of $\mathbb{R}^{m+n+1}$.

Exercise 24.7. Let $A \in \mathrm{M}(m \times n, \mathbb{R})$ with rows $r_1, \dots, r_m$. Show that the map

$$\begin{array}{rcl} \mathbb{R}^n & \to & \mathbb{R}^m \\ v & \mapsto & A \cdot v \end{array}$$

is a monomorphism if and only if there exist $i_1, \dots, i_n \in \{1, \dots, m\}$ such that the determinant of the $n \times n$-matrix with rows $r_{i_1}, \dots, r_{i_n}$ has non-zero determinant.

Exercise 24.8. Consider the subsets of $\mathbb{R}^3$ that are shown in the figure below. Show that they are smooth submanifolds of $\mathbb{R}^3$ by giving explicit smooth embeddings of S^1 and $S^1 \sqcup S^1$ into $\mathbb{R}^3$ such that the images equals the shown subsets.



Exercise 24.9. Show that the map

$$\begin{array}{rcl} \mathbb{R}\mathrm{P}^2 = S^2/\pm 1 & \to & \mathbb{R}^4 \\ [(x,y,z)] & \mapsto & (yz, xz, xy, x^2 + 2y^2 + 3z^2) \end{array}$$

is a smooth embedding.

Exercise 24.10. We consider the map

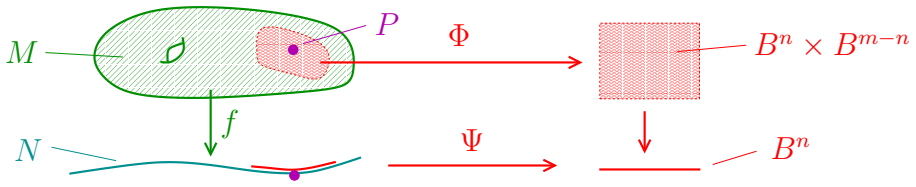
$$\begin{array}{rcl} f\colon \mathbb{R}\mathrm{P}^2 & \to & \mathbb{R}^3 \\ [(x,y,z)] & \mapsto & (yz, xz, xy). \end{array}$$

Show that there are precisely 6 points on $\mathbb{R}\mathrm{P}^2$ such that $\mathrm{D}f_P$ is not a monomorphism.

Exercise 24.11. Let $f\colon M \to N$ be a smooth map between two smooth manifolds M and N of dimensions m respectively n. Let $P \in M \setminus \partial M$ such that $f(P) \in N \setminus \partial N$. We suppose that the differential $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ is an epimorphism.[262] Show that there exists a chart $\Phi\colon U \to B^n \times B^{m-n}$ around P and a chart $\Psi\colon V \to B^n$ around $f(P)$ with $f(U) \subset V$ such that the map $\Psi \circ f \circ \Phi^{-1}\colon B^n \times B^{m-n} \to B^n$ is just the projection onto the first factor.
Remark. This statement can be viewed as a weak version of the Submersion Theorem **??**.

Exercise 24.12. Let M be a smooth manifold. We equip ∂M with the smooth structure that we introduced on page 555. Let W be a smooth manifold and let $f\colon W \to M$ be a smooth map with $f(W) \subset \partial M$.

[262] Note that this hypothesis implies that $m \geq n$.



(a) Show that if $\mathrm{D}f_P\colon \mathrm{T}_P W \to \mathrm{T}_P M$ is a monomorphism for each P, then $f\colon W \to \partial M$ is an immersion.
(b) Show that if f is a bijection and if $\mathrm{D}f_P\colon \mathrm{T}_P W \to \mathrm{T}_P \partial M \subset \mathrm{T}_P M$ is an isomorphism for each P, then $f\colon W \to \partial M$ is a diffeomorphism.

Remark. This exercise can be viewed as a variation on the Smooth Embedding Theorem 24.10.

Exercise 24.13. Let M be an m-dimensional smooth manifold, let N be an n-dimensional smooth manifold and let $f\colon M \to N$ to be a smooth map.

(a) Let $P \in M$ such that $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ is a monomorphism. Show that there exists a chart $\Phi\colon U \to \widetilde{U}$ around P and a chart $\Psi\colon V \to \widetilde{V}$ around $f(P)$ such that the map
$$\Psi \circ f \circ \Phi^{-1}\colon \Phi(f^{-1}(V) \cap U) \;\to\; \mathbb{R}^n$$
is the map
$$(x_1, \ldots, x_m) \;\mapsto\; (x_1, \ldots, x_m, 0, \ldots, 0).$$

(b) We now assume that there exists a $k \in \mathbb{N}$ such that for any $P \in M$ the differential $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ has rank k. Show that for each $P \in M$ there exists a chart $\Phi\colon U \to \widetilde{U}$ around P and a chart $\Psi\colon V \to \widetilde{V}$ around $f(P)$ such that the map
$$\Psi \circ f \circ \Phi^{-1}\colon \Phi(f^{-1}(V) \cap U) \;\to\; \mathbb{R}^n$$
is the map
$$(x_1, \ldots, x_m) \;\mapsto\; (x_1, \ldots, x_k, 0, \ldots, 0).$$

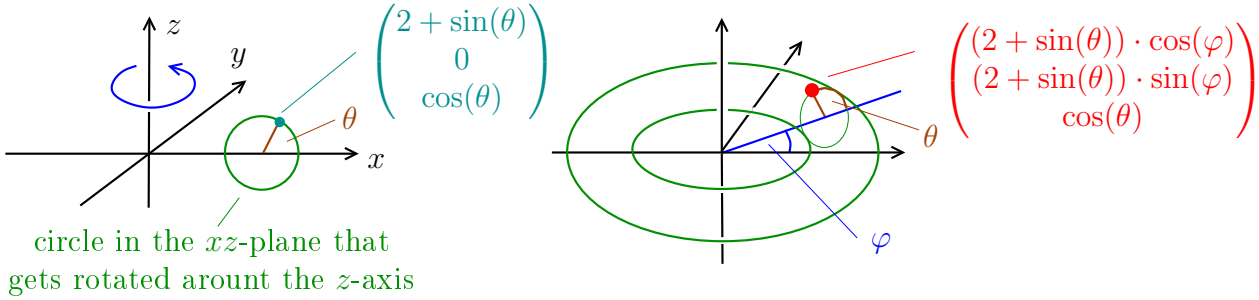
Remark. This statement is known as the *Constant Rank Theorem*.

Exercise 24.14. Let M be a connected m-dimensional smooth manifold and let $N \subsetneq M$ be a submanifold. Show that if there exists a smooth retraction $f\colon M \to N$, then $\dim(N) < \dim(M)$.
Hint. Show that if N is a smooth retract of M with $\dim(N) = \dim(M)$, then N is closed and open in M.

Exercise 24.15. Show that the map
$$\begin{array}{rcl} \mathbb{R}^2/(2\pi\mathbb{Z})^2 & \to & \mathbb{R}^3 \\ (\varphi, \theta) & \mapsto & \underbrace{\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation around the } z\text{-axis}} \cdot \underbrace{\begin{pmatrix} 2+\sin(\theta) \\ 0 \\ \cos(\theta) \end{pmatrix}}_{\substack{\text{describes a circle} \\ \text{in the } xz\text{-plane}}} \end{array}$$

is a smooth embedding.

Remark. By Torus-as-Quotient Lemma 19.34 we know that the torus $S^1 \times S^1$ is diffeomorphic to $\mathbb{R}^2/(2\pi\mathbb{Z})^2$. This fact, together with the fact that f is a smooth embedding and the Smooth Embedding Theorem 24.10 proves, the hardly surprising fact, that the torus is diffeomorphic to a submanifold of $\mathbb{R}^3$.



Exercise 24.16. We consider the map

$$\begin{array}{rcl} f\colon \mathbb{R}\times\mathbb{R}_{>0} & \to & \mathbb{R}^3 \\ (\varphi, r) & \mapsto & \begin{pmatrix} r\cdot\cos(\varphi) \\ r\cdot\sin(\varphi) \\ \varphi \end{pmatrix} \end{array}$$

(a) Sketch the image of f.
(b) Show that f is a smooth embedding.

Exercise 24.17. On page 265 we considered the Klein bottle

$$K \;:=\; ([0,1]\times[0,1])/\sim \qquad \text{where } (x,0)\sim(x,1) \text{ and } (y,0)\sim(1,1-y).$$

We now consider

$$\widetilde{K} \;:=\; \mathbb{R}^2/\sim \qquad \text{where } (u,v)\sim(u+2k\cdot\pi, (-1)^k\cdot v + 2l\pi) \text{ for all } k,l\in\mathbb{Z}.$$

(a) In the description of $\widetilde{K}$ we implicitly gave an action of $\mathbb{Z}^2$ on $\mathbb{R}^2$. This action is smooth. Show that this action is not free. Nonetheless show that $\widetilde{K}$ has a canonical smooth structure coming from $\mathbb{R}^2$.
(b) Show that K is diffeomorphic to $\widetilde{K}$.
(c) Show that the map

$$\begin{array}{rcl} \widetilde{K} & \to & \mathbb{R}^3 \\ [(u,v)] & \mapsto & \begin{pmatrix} \frac{\cos(2u)\cdot\sin(v)}{1-\frac{1}{\sqrt{2}}(\sin(u)\cdot\cos(u)+\sin(2u)\cdot\sin(v))} \\ \frac{\sin(2u)\cdot\sin(v)-\sin(u)\cdot\cos(v)}{\sqrt{2}(1-\frac{1}{\sqrt{2}}\sin(u)\cdot\cos(u)+\sin(2u)\cdot\sin(v))} \\ \frac{\cos(u)\cdot\cos(v)}{1-\frac{1}{2}(\sin(u)\cdot\cos(u)+\sin(2u)\cdot\sin(v))} \end{pmatrix} \end{array}$$

is an immersion.

25. Orientations of smooth manifolds

In this chapter we will introduce the notion of an orientation of a smooth manifold. This notion captures something very intuitive, even though the technical definition takes a little while to get used to.

25.1. **Orientations of real vector spaces I.** Before we can introduce and study orientations of smooth manifolds we need to introduce the notion of an orientation of a finite-dimensional real vector space. This section is pure linear algebra.

Definition. Let V be some finite-dimensional real vector space. If $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_n)$ are bases[263] for V, then there exists a unique matrix $P_{X,Y} = (p_{ij})$ such that given any $i \in \{1, \dots, n\}$ we have $x_i = \sum_{j=1}^{n} p_{ji} \cdot y_j$. We refer to $P_{X,Y}$ as the transition matrix from X to Y.

The following lemma is taught in every proper linear algebra course.

Lemma 25.1. (Transition Matrix Lemma) Let V be a finite-dimensional real vector space.

(1) If X and Y are two bases of V, then $P_{Y,X} = P_{X,Y}^{-1}$.[264]
(2) If X, Y and Z are bases of V, then $P_{X,Z} = P_{Y,Z} \cdot P_{X,Y}$.

Definition. Let V be a finite-dimensional real vector space of dimension > 0.

(1) We say that two bases of V are equivalent[265] if the determinant of the corresponding transition matrix is positive.[266]
(2) An equivalence class of bases is called an orientation for V. An oriented vector space is a vector space together with an orientation.
(3) Let $\mathcal{O}$ be an orientation for V. We say a basis $(v_1, \dots, v_k)$ of V is a positive basis, if $(v_1, \dots, v_k)$ lies in $\mathcal{O}$, otherwise we say that it is a negative basis.

Examples. Let $n \in \mathbb{N}$.

(1) We refer to the orientation on the real vector space $\mathbb{R}^n$ that is given by the equivalence class of the canonical basis $(e_1, \dots, e_n)$ as the standard orientation of $\mathbb{R}^n$. It follows immediately from the definition that

$$(v_1, \dots, v_n) \text{ is a positive basis for } \mathbb{R}^n \iff \det \underbrace{(v_1 \ \dots \ v_n)}_{n \times n\text{-Matrix}} > 0.$$

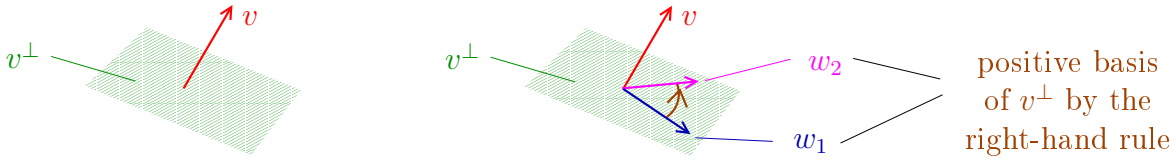
Unless we say something else we always view the real vector space $\mathbb{R}^n$ as equipped with the standard orientation.

[263] Here and throughout the notes a basis of a finite-dimensional vector space is always understood to be an ordered basis.
[264] To avoid discussions we define the inverse of the empty matrix to be the empty matrix.
[265] It follows easily from the Transition Matrix Lemma 25.1 and the multiplicativity of the determinant that this notion of "equivalent" does indeed define an equivalence relation.
[266] If V is 0-dimensional, then it has only the empty basis, we use the convention that the determinant of the empty matrix is $+1$.

(2) Let $v \in \mathbb{R}^n$ be a non-zero vector. We consider the orthogonal complement of v, i.e. we consider the vector space $v^\perp = \{w \in \mathbb{R}^n \mid \langle v, w\rangle = 0\}$. In Exercise 25.1 we will show, using elementary linear algebra, that

$$(w_1, \dots, w_{n-1}) \text{ is a positive basis of } v^\perp \quad :\Longleftrightarrow \quad \det(v\ w_1\ \dots\ w_{n-1}) > 0$$

does indeed define an orientation on $v^\perp$. We say that this orientation is obtained by the *right-hand rule*.



(3) Let V be a real vector space of dimension ≥ 1. If $(v_1, v_2, \dots, v_n)$ is a basis of V, then the basis $(-v_1, v_2, \dots, v_n)$ is inequivalent to the original basis.

(4) In the Complex Vector Space-Orientation Lemma 32.22 we will see that every *complex* vector space of dimension $n \in \mathbb{N}$, viewed as a $2n$-dimensional real vector space, comes with a natural orientation. The algebraically inclined reader might want to figure out what this means and how this works.

It follows easily from the previous example that any finite-dimensional real vector space of dimension ≥ 1 admits precisely two orientations. For a surprisingly large number of situations it is actually very convenient to extend the notion of an orientation to 0-dimensional real vector spaces in such a way that there are also precisely two orientations. This leads us to the following definition.

Definition. An orientation of a 0-dimensional real vector space is defined as a choice of a number in $\{-1, +1\}$.

Definition. Let V be a finite-dimensional real vector space. Given an orientation $\mathcal{O}$ for such a vector space V we denote by $-\mathcal{O}$ the other orientation. Sometimes we refer to it as the opposite orientation.

Definition. Let V and W be finite-dimensional real vector spaces.

(1) Let $f\colon V \to W$ be an isomorphism. If $\dim(V) = \dim(W) > 0$, then basic linear algebra shows that f induces a bijection

$$\begin{array}{rcl} f\colon \text{orientations of } V & \to & \text{orientations of } W \\ [(v_1, \dots, v_n)] & \mapsto & [(f(v_1), \dots, f(v_n))]. \end{array}$$

If $\dim(V) = \dim(W) = 0$, then given an orientation $\epsilon \in \{-1, 1\}$ for V we refer to ϵ also as the image of orientation under the isomorphism $f\colon V \to W$.

(2) Now assume that V and W are oriented. We say an isomorphism $\Phi\colon V \to W$ is orientation-preserving if the image of the orientation of V under the map f described in (1) equals the orientation of W.[267] Otherwise we say that Φ is orientation-reversing.

For reference we state the following straightforward fact as a lemma.

[267] In other words, the map is orientation-preserving if the image of a positive basis of V is a positive basis of W.

Lemma 25.2. Let $\Phi\colon V \to W$ be an isomorphism between two oriented finite-dimensional real vector spaces with $\dim(V) = \dim(W) > 0$. Furthermore let X be a basis for V and let Y be a basis for W. We make either following hypothesis:

(1) either both bases are positive or
(2) both bases are negative,

then

$$\Phi \text{ is orientation-preserving} \quad \Longleftrightarrow \quad \det(P_{\Phi(X),Y}) > 0.$$

The following lemma often gets used subconsciously:

Lemma 25.3. (Orientation-Composition Lemma) Let $V_0, \dots, V_k$ be oriented vector spaces. Furthermore let $f_i\colon V_i \to V_{i+1}$, $i = 0, \dots, k-1$ be isomorphisms. Then

$$\begin{array}{c}\text{the map } f_{k-1} \circ \cdots \circ f_0\colon V_0 \to V_k \\ \text{is orientation-preserving}\end{array} \quad \Longleftrightarrow \quad \begin{array}{c}\text{the number of } f_i \text{ which are} \\ \text{orientation-reversing is even.}\end{array}$$

In particular the composition of orientation-preserving isomorphisms is again orientation-preserving.

Proof. If all vector spaces are 0-dimensional, then the statement is completely elementary. Otherwise the lemma follows easily from the Transition Matrix Lemma 25.1 and Lemma 25.2 together with the multiplicativity of the determinant. ■

Definition. A short exact sequence of real vector spaces is a sequence of homomorphisms

$$U \xrightarrow{\varphi} V \xrightarrow{\psi} W$$

between real vector spaces U, V and W such that the following conditions are satisfied:

(1) $\varphi\colon U \to V$ is a monomorphism,
(2) $\psi\colon V \to W$ is an epimorphism,
(3) $\operatorname{im}(\varphi) = \ker(\psi)$.

Example. Let V be a real vector space and let $U \subset V$ be a vector subspace. Then the maps $U \to V \to V/U$, given by the inclusion and the natural projection, form a short exact sequence of real vector spaces. In fact every short exact sequence is essentially of this form. More precisely, if $U \xrightarrow{\varphi} V \xrightarrow{\psi} W$ is a short exact sequence, then we obtain the following commutative diagram

$$\begin{array}{ccccc} U & \xhookrightarrow{\quad\varphi\quad} & V & \xrightarrow{\quad\psi\quad} & W \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle w \mapsto [v] \text{ where } \psi(v) = w} \\ \varphi(U) & \xhookrightarrow{\quad\quad} & V & \xrightarrow{\quad\quad} & V/\varphi(U). \end{array}$$

where all vertical maps are isomorphisms.

Lemma 25.4. Let

$$U \xhookrightarrow{\quad\varphi\quad} V \xrightarrow{\quad\psi\quad} W$$

be a short exact sequence of finite-dimensional real vector spaces. We suppose that U and W are oriented and that $\dim(U) > 0$ and $\dim(W) > 0$.

(1) We pick a positive basis $(u_1, \dots, u_k)$ for U and a positive basis $(w_1, \dots, w_l)$ for W.

(2) We pick lifts $\widetilde{w}_1, \dots, \widetilde{w}_l \in V$ of $w_1, \dots, w_l \in W$.

Then $(u_1, \dots, u_k, \widetilde{w}_1, \dots, \widetilde{w}_l)$ is an ordered basis for V and the orientation it defines only depends on the orientations of U and W.

Sketch of proof. It is straightforward to verify that the given vectors define a basis for V. Furthermore note that any other choice of a positive basis for U, choice of positive basis for W and choice of lifts of $w_1, \dots, w_l \in W$ leads to a transition matrix of the form

$$\begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$$

with $\det(A) > 0$ and $\det(B) > 0$. Thus the two sets of choices lead to equivalent bases for V. ■

Definition. Let

$$U \xrightarrow{\varphi} V \xrightarrow{\psi} W$$

be a short exact sequence of finite-dimensional real vector spaces. We suppose that U, V and W are oriented.

(1) If $\dim(U) > 0$ and $\dim(W) > 0$, then we say that the orientations are compatible if the orientations of U and W define, in the sense of Lemma 25.4, the orientation of V.
(2) Suppose that $\dim(U) = 0$.
   (a) If the orientation of U is $+1$, then we say that the orientations are compatible if the isomorphism $\psi\colon V \to W$ is orientation-preserving.
   (b) If the orientation of U is -1, then we say that the orientations are compatible if the isomorphism $\psi\colon V \to W$ is orientation-reversing.
(3) The analogue of (2) holds is $\dim(W) = 0$.

If the orientations are not compatible, then we say that they are incompatible.

Lemma 25.5. (Compatible Orientations Lemma) Let

$$U \xrightarrow{\varphi} V \xrightarrow{\psi} W$$

be a short exact sequence of real vector spaces.

(1) If two out of U, V and W are oriented, then there exists a unique orientation on the third vector space such that the three orientations are compatible.
(2) Suppose U, V and W are oriented in a compatible way.
   (a) If we reverse an *even* number of the given three orientations, then the orientations are again *compatible*.
   (b) If we reverse an *odd* number of the given three orientations, then the orientations are *incompatible*.

Proof. The proof is straightforward and basically trivial. We will not bore the reader with details. ■

The Compatible Orientations Lemma 25.5 can be used to define various induced orientations.

Definition. Let U and V be oriented finite-dimensional real vector spaces. We define the direct sum orientation on $U \oplus V$ to be the unique orientation[268] that makes the orientations

in the following short exact sequence of real vector spaces compatible:

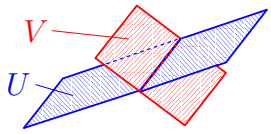
$$U \xrightarrow{u\mapsto(u,0)} U\oplus V \xrightarrow{u\oplus v\mapsto v} V.$$

Remark. If U and V are 0-dimensional real vector spaces with orientations $\mu\in\{-1,1\}$ and $\nu\in\{-1,1\}$, then our conventions imply that $U\oplus V$ has the orientation $\mu\cdot\nu\in\{-1,1\}$.

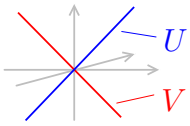
Definition. Let W be a real vector space and let U be a vector subspace of W.

(1) Let V be another vector subspace of W. We say U and V intersect transversally if $W=U+V$.

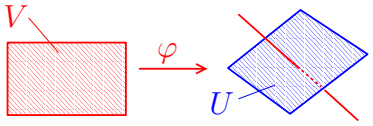
(2) Let V be a real vector space and let $\varphi\colon V\to W$ be a homomorphism. We say U and φ intersect transversally if $W=U+\varphi(V)$.



U and V intersect transversally in $W=\mathbb{R}^3$



U and V do not intersect transversally in $W=\mathbb{R}^3$



U and φ intersect transversally in $W=\mathbb{R}^3$

Remark. Let W be a real vector space and let U,V be oriented vector subspaces. We denote by $\iota\colon V\to W$ the inclusion map. It follows immediately from the definitions that U and V are transverse if and only if U and ι are transverse. On many occasions this little observation allows us to treat (1) as a special case of (2).

The following lemma is utterly elementary, but we use it so often that it is convenient to state it explicitly.

Lemma 25.6. Let W be a real vector space and let U be a vector subspace of W.

(1) Let V be another vector subspace of W. If U and V intersect transversally, then the map

$$\begin{array}{rcl} V/U\cap V & \to & W/U \\ [v]=v+U\cap V & \mapsto & [v]=v+U \end{array}$$

is an isomorphism.

(2) Let V be a real vector space and let $\varphi\colon V\to W$ be a homomorphism. If U and φ intersect transversally, then the map

$$\begin{array}{rcl} V/\varphi^{-1}(U) & \to & W/U \\ [v]=v+\varphi^{-1}(U) & \mapsto & [\varphi(v)]=\varphi(v)+U \end{array}$$

is an isomorphism.

Proof. As pointed out above, (1) is just a reader friendly special case of (2). Thus it remains to prove (2). It is clear that the map is a monomorphism. It follows almost immediately from the hypothesis that U and φ intersect transversally that the map is also an epimorphism. ■

[268] Here of course we use the Compatible Orientations Lemma 25.5.

Definition. Let W be an oriented real vector space and let U be an oriented vector subspace.

(1) Let V be another oriented vector subspace of W. We assume that U intersects V transversally. We consider the following diagram:

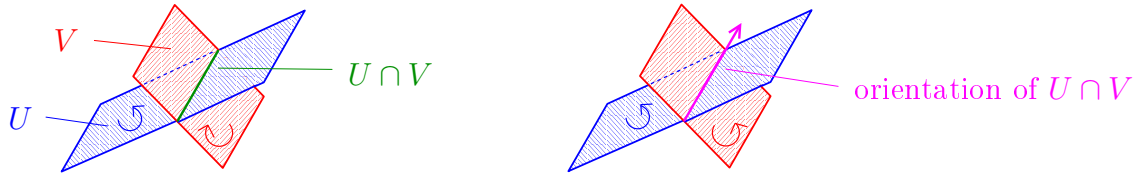
$$\begin{array}{ccccc} U & \hookrightarrow & W & \twoheadrightarrow & W/U \\ & & & & \cong\uparrow{\scriptstyle [v]\mapsto[v]} \\ U\cap V & \hookrightarrow & V & \twoheadrightarrow & V/(U\cap V). \end{array}$$

It follows from Lemma 25.6 that the right-hand vertical map is an isomorphism. Note that the horizontal sequences are short exact sequences. By the Compatible Orientations Lemma 25.5 the orientations on U and W induce an orientation on W/U and the isomorphism induces an orientation on $V/(U\cap V)$. Finally, again using the Compatible Orientations Lemma 25.5 we see that this orientation on $V/(U\cap V)$, together with the given orientation on V, induces an orientation on $U\cap V$. We refer to this as the intersection orientation on $U\cap V$.

(2) Let V be another real vector space and let $\varphi\colon V\to W$ be a homomorphism. We assume that φ intersects U transversally. We consider the following diagram:

$$\begin{array}{ccccc} U & \hookrightarrow & W & \twoheadrightarrow & W/U \\ & & & & \cong\uparrow{\scriptstyle [v]\mapsto[\varphi(v)]} \\ \varphi^{-1}(U) & \hookrightarrow & V & \twoheadrightarrow & V/\varphi^{-1}(U). \end{array}$$

By the same logic as in (1) we see that we obtain an orientation on $\varphi^{-1}(U)$. We refer to this as the preimage orientation on $\varphi^{-1}(U)$.



Remark.

(1) A priori there are many reasonable ways to define the intersection orientation on $U\cap V$. Our convention agrees with the convention chosen in [**GP1974**, p. 100].

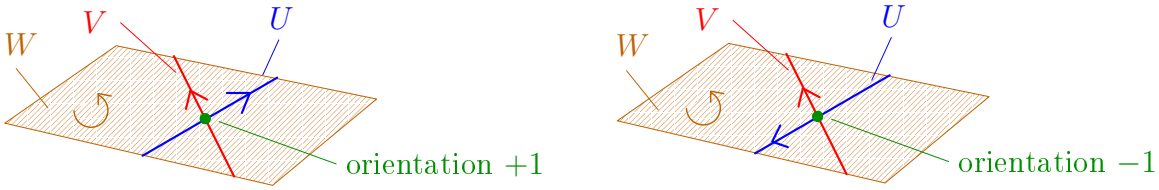
(2) Our convention of the intersection orientation is designed as follows: Let A, B and C be oriented real vector spaces. We set $W := A\oplus B\oplus C$ and we consider the vector subspaces $U = A\oplus B\oplus\{0\}$ and $V = \{0\}\oplus B\oplus C$. If we equip U, V, W with the obvious orientations, then the intersection orientation on $U\cap V = \{0\}\oplus B\oplus\{0\}$ is the obvious orientation.

(3) Let W be an oriented real vector space and let U, V be oriented vector subspaces. We denote by $\iota\colon V\to W$ the inclusion map. As we remarked above, it follows immediately from the definitions that U and V are transverse if and only if U and ι are transverse. Furthermore, if this is the case, then it follows again immediately from

the definitions that the intersection orientation on $U \cap V$ agrees with the preimage orientation on $\iota^{-1}(U) = U \cap V$.

Example. Let W be an oriented real vector space and let U and V be oriented vector subspaces with $W = U + V$ and with $U \cap V = 0$. We equip $U \oplus V$ with the direct sum orientation that we defined on page 672. It follows immediately from the definitions that the intersection-orientation of $U \cap V$ is given as follows:

$$\text{orientation of } U \cap V = \begin{cases} +1, & \text{if direct sum orientation of } U \oplus V \text{ equals orientation of } W, \\ -1, & \text{otherwise.} \end{cases}$$



We conclude this discussion of orientations on real vector spaces and of orientation-preserving and orientation-reversing isomorphisms with a short discussion of *automorphisms* of finite dimensional real vector spaces.

Definition. Let V be a finite dimensional real vector space and let $f\colon V \to V$ be an automorphism.

(1) If $\dim(V) > 0$, then we say that f is orientation-preserving if for one choice of a basis X for V we have $\det(P_{f(X),X}) > 0$. Otherwise we say that f is orientation-reversing.
(2) If $\dim(V) = 0$, then we always say that f is orientation-preserving.

We will use this notion at several occasions. At this point there is not much to say except that we have the following very elementary lemma which we will use many times, often without noticing.

Lemma 25.7. Let V be a finite dimensional real vector space.

(1) The composition of orientation-preserving automorphisms is again orientation-preserving.
(2) Let $\varphi\colon V \to W$ be an isomorphism and let $f\colon V \to V$ be an automorphism. Then the following two statements are equivalent:
   - $f\colon V \to V$ is orientation-preserving,
   - $\varphi \circ f \circ \varphi^{-1}\colon W \to W$ is orientation-preserving.

Proof. Both statements follow immediately from the Orientation-Composition Lemma 25.3. ■

25.2. **Orientations of smooth manifolds.** After the preparations from the last section we can now introduce the notion of an orientation on a smooth manifold.

[268] It follows immediately from the Transition Matrix Lemma 25.1 and the multiplicativity of the determinant that the sign of $\det(P_{f(X),X})$ does not depend on the choice of the basis X.

An orientation of a smooth manifold M is supposed to be an orientation for each tangent space $\mathrm{T}_P M$ which is "continuous". Turning the idea of "continuity" into a precise notion requires a little bit of thought.

Definition. Let M be a smooth manifold.

(1) An orientation for M is a map which assigns to every tangent space $\mathrm{T}_P M$, $P \in M$, an orientation such that the following continuity condition is satisfied:
   (∗) For every chart $\Phi\colon U \to V$ of the smooth structure of M and every component U' of U the map

$$\begin{array}{ccccccc} \mathrm{T}_P M & \xleftarrow[\uparrow]{\cong} & \mathrm{T}_P U & \xrightarrow[\cong]{\mathrm{D}\Phi_P} & \mathrm{T}_{\Phi(P)} V & \xrightarrow[\uparrow]{\Theta_V} & \widetilde{\mathrm{T}}_{\Phi(P)} V \quad = \quad \mathbb{R}^k \\ \text{isomorphism by} & & \text{by Proposition 23.14 (2)} & & \text{natural isomorphism} & & \text{Lemma 23.1} \\ & & & & \text{from Proposition 23.16} & & \end{array}$$

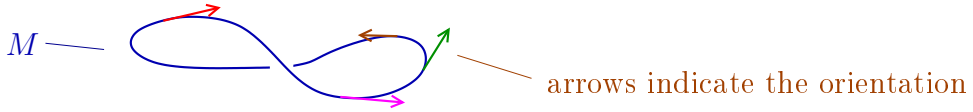
   is either orientation-preserving for all $P \in U'$ or it is orientation-reversing for all $P \in U'$.

(2) A smooth manifold together with an orientation is called an oriented smooth manifold.

(3) A smooth manifold that admits an orientation is called orientable, otherwise we say that the smooth manifold is non-orientable.

Remark. Let M be a 0-dimensional smooth manifold. For each $P \in M$ the tangent space $\mathrm{T}_P M$ is a 0-dimensional real vector space. By the definition on page 669 an orientation of $\mathrm{T}_P M$ is a choice of an element of $\{-1, 1\}$. By the discussion on page 514 we know that the 0-dimensional smooth manifold has the discrete topology. Thus we see that an orientation of 0-dimensional smooth manifold is the same as a map $M \to \{-1, 1\}$. Given a 0-dimensional smooth manifold M we equip it, unless we say something else, with the orientation given by assigning $+1$ to each point in M.

Convention. Given a picture of a 1-dimensional submanifold M of $\mathbb{R}^2$ or $\mathbb{R}^3$ we indicate an orientation by drawing a positive vector in $\mathrm{T}_P M = \widetilde{\mathrm{T}}_P M \subset \mathbb{R}^3$. (Note that in this case a non-zero vector is already a basis for the tangent space). On page 680 we will introduce two conventions for indicating orientations of 2-dimensional submanifolds of $\mathbb{R}^3$.



In practice the following lemma can be quite useful for verifying that orientations of tangent spaces actually define an orientation of a smooth manifold. In particular it says that instead of having to check the continuity condition for all charts, it suffices to consider charts that cover all of M.

Lemma 25.8. (Orientations-via-Atlas Lemma) Let M be a smooth manifold. Suppose that for each $P \in M$ we are given an orientation of $\mathrm{T}_P M$. The following statements are equivalent:

(1) The orientations of the tangent spaces define an orientation for M.

(2) Given any $x \in M$ there exists a chart $\Phi\colon U \to V$ in the given smooth structure of M which satisfies the above continuity condition, i.e. which has the property that for every component U' of U the map

$$\begin{array}{ccccccccc} \mathrm{T}_P M & \xleftarrow{\cong} & \mathrm{T}_P U & \xrightarrow[\cong]{\mathrm{D}\Phi_P} & \mathrm{T}_{\Phi(P)} V & \xrightarrow[\uparrow]{\Theta_V} & \widetilde{\mathrm{T}}_{\Phi(P)} V & = & \mathbb{R}^k \\ & & \uparrow & & & & \uparrow & & \uparrow \\ & & \text{isomorphism by} & & & & \text{natural isomorphism} & & \text{Lemma 23.1} \\ & & \text{by Proposition 23.14 (2)} & & & & \text{from Proposition 23.16} & & \end{array}$$

is either orientation-preserving for all $P \in U'$ or it is orientation-reversing for all $P \in U'$.

Proof. The implication (1) $\Rightarrow$ (2) is a tautology. Now let us turn to the proof of the reverse implication (2) $\Rightarrow$ (1).

Let M be a smooth manifold. The case $\dim(M) = 0$ is trivial. Therefore we now assume that $\dim(M) > 0$. Suppose that for each $P \in M$ we are given an orientation ω_P of $\mathrm{T}_P M$. We start out with introducing some notation.

Given a chart $\Phi\colon U \to V$ from the smooth structure we consider the map

$$\begin{array}{rcl} \chi_\Phi\colon U & \to & \{\pm 1\} \\ P & \mapsto & \begin{cases} +1, & \text{if } (\Theta_V \circ \mathrm{D}\Phi_P)(\omega_P) \text{ equals the standard orientation of } \mathbb{R}^k, \\ -1, & \text{otherwise.} \end{cases} \end{array}$$

We equip $\{\pm 1\}$ with the discrete topology. We make the following observations:

(a) The continuity condition for Φ is by definition satisfied if and only if χ_Φ is constant on each component of U.

(b) By the Connected-to-Discrete Lemma 2.27 a map $U \to \{\pm 1\}$ is continuous if and only if it is constant on each component of U.

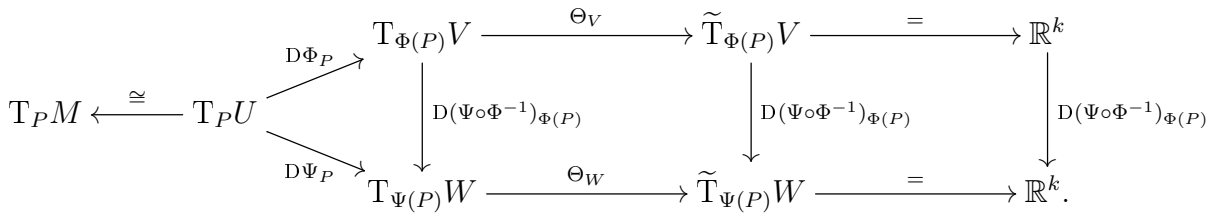
(c) By the Pasting Proposition 2.6 a map is continuous if and only if it is locally continuous.

(d) Let $\widetilde{U} \subset U$ be an open subset. We write $\widetilde{\Phi} := \Phi|_{\widetilde{U}}\colon \widetilde{U} \to \Phi(\widetilde{U})$. This is again a chart and one can easily verify that we have $\chi_\Phi|_{\widetilde{U}} = \chi_{\widetilde{\Phi}}$.

It is now straightforward to assemble a proof of the lemma from the above observations and the following claim.

Claim. Let $U \subset M$ be a connected subset and let $\Phi\colon U \to V$ and $\Psi\colon U \to W$ be two charts from the smooth atlas. Then $\chi_\Phi\colon U \to \{\pm 1\}$ is constant if and only if $\chi_\Psi\colon U \to \{\pm 1\}$ is constant.

Proof. Given $P \in U$ we consider the following diagram:



Here we also make a few observations:

(a) The diagram commutes by the functoriality of differentials and the naturality of the Θ maps.

(b) By the Differential-equals-Differential Lemma 23.4 the vertical map to the right is given by multiplication by the $(k \times k)$-matrix $\mathrm{D}(\Psi \circ \Phi^{-1})_{\Phi(P)}$.

(c) The map
$$\begin{array}{rcl} U & \to & \{\pm 1\} \\ P & \mapsto & \text{sign of } \det(\mathrm{D}(\Psi \circ \Phi^{-1})_{\Phi(P)}) \end{array}$$
is continuous. It follows from the connectedness of U and the Connected-to-Discrete Lemma 2.27 that this map is in fact constant.

The claim follows easily from combining all of the above. ■

As of right now there are not that many examples of oriented smooth manifolds that we can provide.

Example.

(1) Let $n \in \mathbb{N}$ and let U be an open subset of $\mathbb{R}^n$ or of the half-space $\mathbb{H}^n$. As always we equip U with the smooth structure given by the identity chart. We now use the Orientations-via-Atlas Lemma 25.8 in reverse to endow U with an orientation. More precisely, given any $P \in U$ we have the isomorphism
$$\begin{array}{ccccc} \mathbb{R}^n & = & \widetilde{\mathrm{T}}_P U & \xleftarrow{\Theta_U} & \mathrm{T}_P U. \\ & & \uparrow & & \uparrow \\ & & \text{Lemma 23.1} & & \text{natural isomorphism from Proposition 23.16} \end{array}$$
Now we use the standard orientation of the vector space $\mathbb{R}^n$ and the above isomorphism to equip $\mathrm{T}_P U$ with an orientation. It follows almost immediately from the Orientations-via-Atlas Lemma 25.8 that this defines an orientation on the smooth manifold U. We refer to this orientation on U as the standard orientation on U.

(2) We define the standard orientation on $\mathbb{R}^0 = \{0\}$ to be $+1$.

We will construct more examples of oriented smooth manifolds in the subsequent sections. But first let us conclude this section with the following definition.

Definition. Let M be an oriented smooth manifold. If for each tangent space we pick the opposite orientation then evidently we obtain again an orientation, which we refer to as the opposite orientation. We denote the resulting oriented smooth manifold by $-M$.

25.3. **Orientations of submanifolds of $\mathbb{R}^n$ and unit normal fields.** In this section we will consider "real-life" examples of smooth manifolds, namely we will consider submanifolds M of some $\mathbb{R}^n$. Throughout this section we will work with the physical tangent spaces $\widetilde{\mathrm{T}}_P M \subset \mathbb{R}^n$.

Observation 25.9. Let $M \subset \mathbb{R}^n$ be a smooth submanifold. Under the natural isomorphism $\widetilde{\mathrm{T}}_P M \to \mathrm{T}_P M$ from the Tangent Space-Isomorphism Proposition 23.16 we see that an orientation of M is the same as a map which assigns to every physical tangent space $\widetilde{\mathrm{T}}_P M$, $P \in M$, an orientation such that the following continuity condition is satisfied:

(∗) For every chart $\Phi\colon U \to V$ of the smooth structure of the smooth manifold M (with the smooth structure defined in the Submanifold Smooth Structure Lemma 19.24)

and every component U' of U the map

$$\widetilde{\mathrm{T}}_P M \xleftarrow[\uparrow]{=} \widetilde{\mathrm{T}}_P U \xrightarrow[\cong]{\mathrm{D}\Phi_P} \widetilde{\mathrm{T}}_{\Phi(P)} V \underset{\uparrow}{=} \mathbb{R}^k$$

Proposition 23.3 — Lemma 23.1

is either orientation-preserving for all $P \in U'$ or it is orientation-reversing for all $P \in U'$.

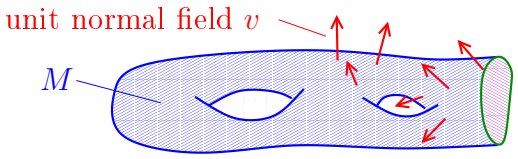
Furthermore, as in the Orientations-via-Atlas Lemma 25.8 the continuity condition only needs to be verified for charts in some smooth atlas of M.

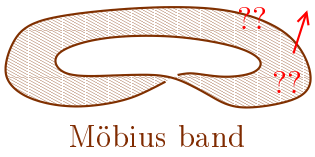
Definition. Let M be a codimension-one smooth submanifold of $\mathbb{R}^{n+1}$. A unit normal field is a map[269]

$$v\colon M \to \mathbb{R}^{n+1}$$

such that the following conditions are satisfied:

(1) For each $P \in M$ we have $\|v(P)\| = 1$,
(2) for each $P \in M$ we have $v(P) \perp \widetilde{\mathrm{T}}_P M$.





Examples.

(1) Let $n \in \mathbb{N}$. It follows from Lemma 23.1 that the map

$$\begin{aligned} v\colon S^n &\to \mathbb{R}^{n+1} \\ P &\mapsto P \end{aligned}$$

is a unit normal field.

(2) In the Möbius Smooth Embedding Lemma 24.12 we saw that we can view the Möbius band as a smooth submanifold M of $\mathbb{R}^3$. Does M admit a unit normal field? This seems unlikely, but how do you show the non-existence of a unit normal field?

Our next goal is to show that for a codimension-one smooth submanifold of $\mathbb{R}^{n+1}$ there is a natural correspondence between orientations and unit normal fields. The key tool for doing so is the following definition from linear algebra.

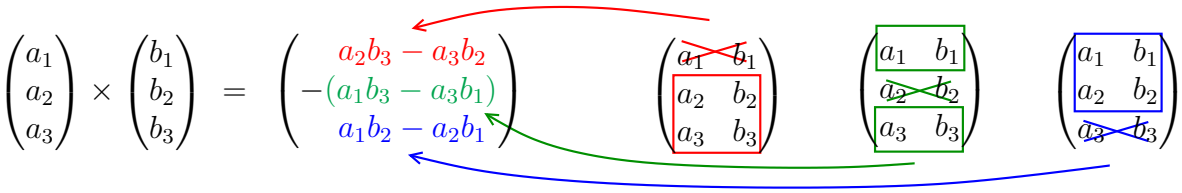
Definition. Let $w_1, \dots, w_n \in \mathbb{R}^{n+1}$. As in [**Blo1979**, Chapter 5.14] we define the cross product[270]

$$w_1 \times \cdots \times w_n := \sum_{k=1}^{n+1} (-1)^{k+1} \cdot e_k \cdot \det \begin{pmatrix} n \times n\text{-matrix obtained from} \\ \text{the } (n+1) \times n\text{-matrix } (w_1 \ \dots \ w_n) \\ \text{by crossing out the } k\text{-th row} \end{pmatrix}.$$

In the following lemma we summarize a few properties of the cross product.

[269] As always we expect maps to be continuous.

[270] By the convention on page 99 the determinant of a 0×0-matrix is $+1$. It follows that for $n = 1$ the cross product is always the vector $e_1 = 1 \in \mathbb{R}$.



Lemma 25.10. (Cross Product Lemma)

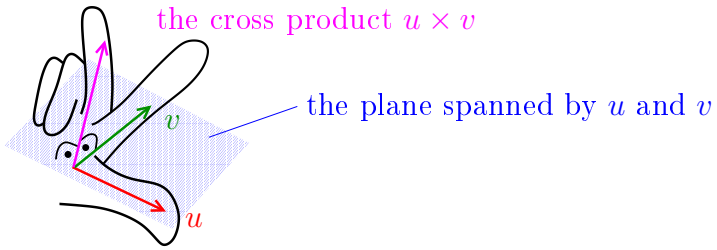
(1) The map

$$\begin{array}{rcl} \mathbb{R}^{n+1} \times \cdots \times \mathbb{R}^{n+1} & \to & \mathbb{R}^{n+1} \\ (v_1, \ldots, v_n) & \mapsto & \underbrace{v_1 \times \cdots \times v_n}_{\text{cross product}} \end{array}$$

is smooth.

(2) Let $v_1, \ldots, v_n \in \mathbb{R}^{n+1}$. The following statements hold:
   (a) $\det(v_1 \times \cdots \times v_n \; v_1 \; \ldots \; v_n) \geq 0$.
   (b) $v_1 \times \cdots \times v_n$ is orthogonal to each $v_1, \ldots, v_n$.
   (c) The cross product $v_1 \times \cdots \times v_n$ is non-zero if and only if the vectors $v_1, \ldots, v_n$ are linearly independent.[271]

(3) Suppose that $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are both collections of linearly independent vectors in $\mathbb{R}^{n+1}$. The following statements hold:
   (a) $v_1 \times \cdots \times v_n$ and $w_1 \times \cdots \times w_n$ are linearly dependent if and only if we have $\text{Span}(v_1, \ldots, v_n) = \text{Span}(w_1, \ldots, w_n)$.
   (b) There exists a $\lambda > 0$ with $v_1 \times \cdots \times v_n = \lambda \cdot (w_1 \times \cdots \times w_n)$ if and only if the vectors span the same vector subspace of $\mathbb{R}^{n+1}$ and if $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are equivalent bases.



Sketch of proof.

(1) This statement follows from the Leibniz formula for determinants.

(2) Let $v_1, \ldots, v_n \in \mathbb{R}^n$. For any $u \in \mathbb{R}^{n+1}$ it follows from the Leibniz expansion by the first row that

$$\det(u \; v_1 \; \ldots \; v_n) \; = \; \langle u, v_1 \times \cdots \times v_n \rangle.$$

   (a) Using the above equality we see that

$$\det(v_1 \times \cdots \times v_n \; v_1 \; \ldots \; v_n) \; = \; \langle v_1 \times \cdots \times v_n, v_1 \times \cdots \times v_n \rangle \; \geq \; 0.$$

[271] In the Cross Product Characterization Proposition **??** we will see that the length of the cross product $v_1 \times \cdots \times v_n$ equals the "n-dimensional volume" of the parallelotope spanned by $v_1, \ldots, v_n$.

(b) Let $i \in \{1, \dots, n\}$. We see that

$$\langle v_i, v_1 \times \cdots \times v_n \rangle \underset{\text{by the above equality}}{=} \det(v_i \ v_1 \ \dots \ v_n) \underset{\text{since two columns are the same}}{=} 0.$$

(c) This statement follows from the fact, proved in Exercise 24.7, that an $(n+1) \times n$-matrix has rank n if and only if there exist n rows that are linearly independent.

(3) (a) Statement (3a) follows from Statement (2b).

(b) It follows from (3a) that we only have to consider the case that $v_1, \dots, v_n$ and $w_1, \dots, w_n$ span the same vector space. Let A be the transition matrix. It follows easily from the definitions that $v_1 \times \cdots \times v_n = \det(A) \cdot (w_1 \times \cdots \times w_n)$ which implies the statement. ■

In the following proposition we will see that for a codimension-one smooth submanifold of $\mathbb{R}^{n+1}$ there is a natural bijection between orientations and unit normal fields.

Proposition 25.11. (Orientations-via-Normal Vector Fields Proposition) Let $n \in \mathbb{N}$ and let M be a codimension-one smooth submanifold of $\mathbb{R}^{n+1}$. The maps

$$\begin{array}{rcl} \{\text{unit normal fields on } M\} & \leftrightarrow & \{\text{orientations on } M\} \\ & & \text{given a point } P \in M \text{ we say that a} \\ v & \mapsto & \text{basis } w_1, \dots, w_n \text{ of } \widetilde{\mathrm{T}}_P M \text{ is positive} \\ & & \text{if } \det\left(v(P) \ w_1 \ \dots \ w_n\right) > 0 \\ \text{given a point } P \in M \text{ we pick a positive} & & \\ \text{basis } w_1, \dots, w_n \text{ for } \widetilde{\mathrm{T}}_P M \text{ and normalize} & \leftarrow\!\shortmid & \text{orientation on } M \\ \text{the vector } w_1 \times \cdots \times w_n \text{ to length one} & & \end{array}$$

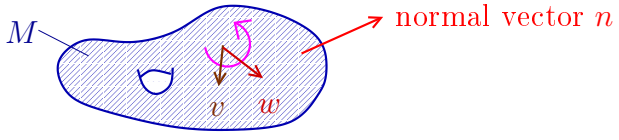
are well-defined and inverses of one-another.

Convention. Given a picture of a 2-dimensional submanifold M of $\mathbb{R}^3$ we have two conventions for indicating an orientation:

(1) We draw a part of a little circle together with a directional arrow. This indicates that a positive basis is given by (v, w) where w is obtained by rotating v into the indicated direction by an angle in $(0, \pi)$.

(2) Alternatively we draw a normal vector n, in this case a basis (v, w) of the tangent space is positive if (n, v, w) is a positive basis for $\mathbb{R}^3$.

both conventions indicate the same orientation of M



Proof. We start out with the following claim.

Claim. For every unit normal field v the construction defines an orientation on M.

Proof. It follows from the discussion on page 669 that for each $P \in M$ the construction does indeed define an orientation on $\widetilde{\mathrm{T}}_P M$. It remains to show that the continuity condition holds.

Let $\Phi\colon U \to V \subset \mathbb{R}^{n+1}$ be a submanifold chart with U connected. Recall that Φ gives rise to a chart $\Psi\colon U \cap M \to \widetilde{V} \subset \mathbb{R}^n$ of M viewed as an n-dimensional smooth manifold. For $Q \in U$ we now denote by $f(Q)$ the $(n+1)$-st coordinate of $\mathrm{D}\Phi_Q(v(Q)) \in \mathbb{R}^{n+1}$. Note that $f\colon U \to \mathbb{R}$ is continuous and note that f is non-zero everywhere. Now let again $Q \in U$. For $i = 1, \dots, n$ we set $w_i := \mathrm{D}\Psi^{-1}_{\Psi(Q)}(e_i)$. We have

$$\underbrace{\det(v(Q)\, w_1 \,\dots\, w_n)}_{\substack{\text{by definition} > 0 \\ \text{if } w_1,\dots,w_n \text{ is a} \\ \text{positive basis}}} \;=\; \det(\mathrm{D}\Phi_Q)^{-1} \cdot \begin{pmatrix} * & e_1 & \dots & e_n \\ f(Q) & 0 & \dots & 0 \end{pmatrix} \;=\; \det(\mathrm{D}\Phi_Q)^{-1} \cdot f(Q) \cdot (-1)^{n+1}.$$

$\uparrow$ follows from the Chain Rule 19.10

This shows that the isomorphism $\mathrm{D}\Psi^{-1}_{\Psi(Q)}\colon \mathbb{R}^n \to \widetilde{\mathrm{T}}_Q M$ is orientation-preserving if and only if $\det(\mathrm{D}\Phi_Q)^{-1} \cdot f(Q) \cdot (-1)^{n+1} > 0$. But this function in Q is non-zero everywhere. Hence, since U is connected, we see that the sign of this expression is independent of $Q \in U$. ⊞

Claim. For every orientation on M the construction defines a unit normal field on M.

Proof. It follows from the Cross Product Lemma 25.10 that for each $P \in M$ the vector $w_1 \times \dots \times w_n$ is indeed normal to $\widetilde{\mathrm{T}}_P M$ and that it is independent of the choice of the positive basis $w_1, \dots, w_n$ of $\widetilde{\mathrm{T}}_P M$. It only remains to show that the resulting map $v\colon M \to \mathbb{R}^{n+1}$ is continuous. Thus let $P \in M$. We pick a chart $\Phi\colon U \to V$ around P such that U is connected. After possibly composing Φ with a reflection we can assume that for each $Q \in V$ the map

$$\mathrm{D}\Phi^{-1}_Q\colon \mathbb{R}^n \;\to\; \widetilde{\mathrm{T}}_{\Phi^{-1}(Q)} M$$

is orientation-preserving. It now follows that on U the map v is given by

$$\begin{array}{rcl} U & \to & \mathbb{R}^{n+1} \\ x & \mapsto & \mathrm{D}\Phi^{-1}_{\Phi(x)}(e_1) \times \dots \times \mathrm{D}\Phi^{-1}_{\Phi(x)}(e_1) \text{ normalized to length } = 1. \end{array}$$

By the Cross Product Lemma 25.10 (1) we know that this map is continuous. This shows that v is locally continuous, hence it is continuous. ⊞

We have now shown that the maps given in the statement of current proposition are well-defined. It follows easily from the Cross Product Lemma 25.10 (2) that the two maps are also inverses of one another. ■

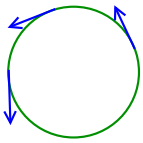
Example.

(1) Let $n \in \mathbb{N}$. As mentioned above it follows from Lemma 23.1 that the map

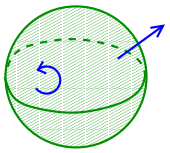
$$\begin{array}{rcl} v\colon S^n & \to & \mathbb{R}^{n+1} \\ P & \mapsto & P \end{array}$$

is a unit normal field. Given $P \in S^n$ we say that a basis $(v_1, \dots, v_n)$ of the tangent space $\widetilde{\mathrm{T}}_P S^n \subset \mathbb{R}^{n+1}$ is positive if $\det(P\, v_1 \,\dots\, v_n) > 0$. It follows from the Orientations-via-Normal Vector Fields Proposition 25.11 and Observation 25.9 that these orientations define an orientation of the smooth manifold S^n. We refer to this as the standard orientation of S^n. Unless we explicitly say something else we will always view S^n as an oriented smooth manifold with the above orientation.

(2) We equip the 0-dimensional sphere $S^0 = \{\pm 1\}$ with the obvious orientation given by $\pm 1 \mapsto \pm 1$.



orientation of S^1



orientation of S^2

25.4. **Orientation-preserving and reversing maps.** The next definition is basically self-explanatory.

Definition. Let $f\colon M \to N$ be a map smooth between two oriented smooth manifolds such that for every $P \in M$ the differential $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}N$ is an isomorphism.[272]

(1) We say the map f is orientation-preserving if for every $P \in M$ the induced isomorphism $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}N$ is orientation-preserving.
(2) We say the map f is orientation-reversing if for every $P \in M$ the induced isomorphism $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}N$ is orientation-reversing.

Examples.

(1) The map

$$\begin{array}{rccl} \Xi\colon \mathbb{R} & \to & S^1 \subset \mathbb{C} = \mathbb{R}^2 \\ t & \mapsto & \exp(\mathrm{i}t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \end{array}$$

is a local diffeomorphism. If we equip $\mathbb{R}$ and S^1 with the standard orientation, then Ξ is orientation-preserving. Indeed, for any $t \in \mathbb{R}$ we have

$$\mathrm{D}(\Xi_t)(1) = \underset{\substack{\uparrow \\ \text{positive basis of } \widetilde{\mathrm{T}}_t\mathbb{R}}}{\begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}} \in \widetilde{\mathrm{T}}_{\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}} S^1 \text{ which is positive since } \det \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} > 0.$$

(2) Let $A \in \mathrm{O}(n)$. In Exercise 25.5 we will see that the map $S^n \to S^n$ given by $x \mapsto A \cdot x$ is orientation-preserving if and only if $\det(A) = 1$.
(3) Let $f\colon M \to N$ be a local diffeomorphism between two oriented smooth manifolds. If M is non-empty and *connected*, then it follows from the Basics-of-Orientations Lemma 25.13 below that f is either orientation-preserving or orientation-reversing. On the other hand for disconnected M one can easily come up with maps that are neither orientation-preserving nor orientation-reversing.

The following lemma gives an easy criterion for showing that a diffeomorphism between two open subsets of $\mathbb{R}^n$ or of $\mathbb{H}^n$ is orientation-preserving.

Lemma 25.12. (Orientation Checking-via-Differentials Lemma) Let $n \in \mathbb{N}$ and let $f\colon X \to Y$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$ or $\mathbb{H}^n$. Recall that by the discussion on page 677 we equip X and Y with the standard orientations. Given $P \in X$

[272] In most applications f is a local diffeomorphism, but by the Physical Tangent Space-Inclusion Proposition 23.14 the inclusion of a codimension-zero submanifold into its "surrounding" smooth manifold also has this property.

we have[273]

$$\underset{\substack{\uparrow\\ \text{differential as defined on page 631}}}{\mathrm{D}f_P\colon \mathrm{T}_PX \to \mathrm{T}_{f(P)}Y} \text{ is orientation-preserving} \iff \det\Big(\underbrace{\text{differential } \mathrm{D}f_P}_{n\times n\text{-matrix}}\Big) > 0.$$

In particular we see

$$f\colon X \to Y \text{ is orientation-preserving} \iff \det(\mathrm{D}f_P) > 0 \text{ for all } P \in X.$$

Example. Let $U \subset \mathbb{R}^n$ be an open subset. Let $A \in \mathrm{GL}(n,\mathbb{R})$ with $\det(A) > 0$ and let $v \in \mathbb{R}^n$. It follows easily from the discussion on page 536 and the Orientation Checking-via-Differentials Lemma 25.12 that the map f given by $f(x) = x \mapsto A \cdot x + v$ defines an orientation-preserving diffeomorphism $U \to f(U)$.

Proof. By the Tangent Space-Isomorphism Proposition 23.16 we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{R}^n = \widetilde{\mathrm{T}}_PX & \xrightarrow[\cong]{\Theta_X} & \mathrm{T}_PX \\ \downarrow{\scriptstyle \mathrm{D}f_P} & & \downarrow{\scriptstyle \mathrm{D}f_P} \\ \mathbb{R}^n = \widetilde{\mathrm{T}}_{f(P)}Y & \xrightarrow[\cong]{\Theta_Y} & \mathrm{T}_{f(P)}Y. \end{array}$$

The lemma follows from combining the following observations:

(1) As on page 668 we equip $\mathbb{R}^n$ with the standard orientation that is defined by the standard basis $(e_1,\dots,e_n)$ of $\mathbb{R}^n$.
(2) The horizontal maps are orientation-preserving by definition of the orientations of the oriented smooth manifolds X and Y.
(3) By the Differential-equals-Differential Lemma 23.4 the vertical map to the left is given by multiplication by the differential $\mathrm{D}f_P \in \mathrm{M}(n\times n,\mathbb{R})$. It follows from Lemma 25.2 that the vertical map to the left is orientation-preserving if and only if $\det(\mathrm{D}f_P) > 0$.
(4) It follows from the Orientation-Composition Lemma 25.3 that the left-hand vertical map is orientation-preserving if and only if the right-hand vertical map is orientation-preserving. ∎

The following lemma summarizes a few elementary properties of orientations and of orientation-preserving maps.

Lemma 25.13. (Basics-of-Orientations Lemma) Let $n \in \mathbb{N}_0$ and let M be an n-dimensional smooth manifold.

(1) If two orientations on M agree at a point and if M is path-connected, then the two orientations agree everywhere.
(2) If M is path-connected and non-empty, then M is either non-orientable or it admits precisely two orientations, which are opposites of one another.
(3) Let $f\colon M \to N$ be a diffeomorphism. If M is path-connected and if there exists a $P \in M$, such that $\mathrm{D}f_P\colon \mathrm{T}_PM \to \mathrm{T}_{f(P)}N$ is orientation-preserving, then f is orientation-preserving.[274]

[273] Unfortunately the symbol $\mathrm{D}f_P$ has several, subtly different meanings. To the left it stands for the induced map on tangent spaces. To the right it stands for the $n\times n$–matrix given by the partial derivatives of f.

(4) Suppose M is equipped with an orientation $\mathcal{O}$. Let $f\colon M \to M$ be a self-diffeomorphism. Then

$$\begin{array}{c} f \text{ is orientation-preserving} \\ \text{with respect to } \mathcal{O} \end{array} \iff \begin{array}{c} f \text{ is orientation-preserving} \\ \text{with respect to } -\mathcal{O}. \end{array}$$

(5) Let $M_0, \dots, M_k$ be oriented smooth manifolds. Suppose that for $i = 0, \dots, k-1$ we are given a smooth map $f_i\colon M_i \to M_{i+1}$ such that for every $P \in M_i$ the differential $\mathrm{D}(f_i)_P\colon \mathrm{T}_P M_i \to \mathrm{T}_{f_i(P)} M_{i+1}$ is an isomorphism. Then

$$\begin{array}{c} \text{the map } f_{k-1} \circ \cdots \circ f_0\colon M_0 \to M_k \\ \text{is orientation-preserving} \end{array} \iff \begin{array}{c} \text{the number of } f_i \text{ which are} \\ \text{orientation-reversing is even.} \end{array}$$

In particular the composition of orientation-preserving diffeomorphisms is again orientation-preserving.

(6) (a) Let N be a smooth manifold and let $f\colon N \to M$ be a smooth map that is a local diffeomorphism. If M is oriented, then there exists a unique orientation on N such that f is an orientation-preserving map.

(b) Let N be a submanifold of codimension zero. (In many applications N is just an open subset of M.) If M is oriented, then N admits a unique orientation such that the inclusion map $N \to M$ is orientation-preserving.

(7) If $M \setminus \partial M$ admits an orientation, then M admits a unique orientation such that the inclusion $M \setminus \partial M \to M$ is orientation-preserving.

Convention. Let M be an oriented smooth manifold and let N be a submanifold of codimension zero. We refer to the orientation on N that we introduced in the Basics-of-Orientations Lemma 25.13 (6b), i.e. the unique orientation on N such that the inclusion map $N \to M$ is orientation-preserving, as the standard orientation of a codimension-zero submanifold.

Example. It follows from the Basics-of-Orientations Lemma 25.13 that for each $n \in \mathbb{N}$ the closed ball $\overline{B}^n$ inherits an orientation from $\mathbb{R}^n$. We refer to this as the standard orientation on $\overline{B}^n$. Given $P \in \overline{B}^n$ it follows immediately from the definitions that the image of the standard basis $e_1, \dots, e_n$ of $\mathbb{R}^n$ under the natural isomorphism $\Theta\colon \widetilde{\mathrm{T}}_P \overline{B}^n \to \mathrm{T}_P \overline{B}^n$ defines a positive basis for $\mathrm{T}_P \overline{B}^n$.

Proof. The case $n = 0$ follows from trivial considerations. Thus we now assume that $n \geq 1$.

(1) Suppose we are given two orientations on M. It follows immediately from the definitions that for any chart $\Phi\colon U \to V$ the two orientations either agree on all of U or they disagree on all of U.

But this implies that the set of points A where the orientations agree is open and that the set of points B where the orientations disagree is also open. If the orientations agree at a point we see that $A \neq \varnothing$. It follows from the Topological Manifolds-Local Properties Lemma 18.8 (3) that $A = M$.

[274] By definition $f\colon M \to N$ is orientation-preserving if $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ is orientation-preserving for every $P \in M$. If M is path-connected, then by Statement (3) says that it suffices to verify this condition for a single point.

(2) This statement follows easily from (1).
(3) The proof of this statement is very similar to the proof of the first statement. We leave it to the reader to fill in the details.
(4) This statement follows immediately from the definitions and the Orientation-Composition Lemma 25.3.
(5) Note only does the lemma smell like the Orientation-Composition Lemma 25.3, it is also an immediate consequence thereof.
(6) (a) Given $Q \in N$ we equip the vector space $\mathrm{T}_Q N$ with the orientation that turns $\mathrm{D}f_Q \colon \mathrm{T}_Q N \to \mathrm{T}_{f(Q)} M$ into an orientation-preserving isomorphism. Since f is a local diffeomorphism we see that charts of M gives rise to charts of N. It now follows easily from the Orientations-via-Atlas Lemma 25.8 that these orientations of the tangent spaces of N do indeed define an orientation of N.
    (b) Let N be a codimension-zero submanifold. We denote by $\iota \colon N \to M$ the inclusion. Let $Q \in N$. By the Physical Tangent Space-Inclusion Proposition 23.14 we know that $\mathrm{D}\iota_Q \colon \mathrm{T}_Q N \to \mathrm{T}_Q M$ is an isomorphism. As in (a) we equip the vector space $\mathrm{T}_Q N$ with the orientation that turns $\mathrm{D}f_Q \colon \mathrm{T}_Q N \to \mathrm{T}_{f(Q)} M$ into an orientation-preserving isomorphism. The charts of N, viewed as a smooth manifold in its own right, are given by restricting charts of M. Therefore it follows again from the Orientations-via-Atlas Lemma 25.8 that these orientations of the tangent spaces of N do indeed define an orientation of N.
(7) Using charts it suffices to prove this statement for smooth manifolds that are open subsets of $\mathbb{H}^n$, but in this case the statement is clear. ■

Definition. Let M be an oriented smooth manifold. We say an atlas $\{\Phi_i \colon U_i \to \widetilde{U}_i\}_{i \in I}$ is orientation-preserving if each Φ_i is orientation-preserving. Hereby we work with the following orientations:

(1) Each open subset U_i is a codimension-zero submanifold of M, thus we can equip it with the above standard orientation.
(2) Each $\widetilde{U}_i$ is an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$. Therefore it also comes with a standard orientation.

Remark. Note that if $\dim(M) > 0$, then any chart $\Phi \colon U \to V$ with V can be turned into an orientation-preserving respectively orientation-reversing chart by possibly composing Φ, on components of V, with the reflection in a hyperplane. In particular we see that any oriented smooth manifold of dimension ≥ 1 admits an orientation-preserving atlas.

The following lemma gives a convenient criterion for showing that a given map is orientation-preserving.

Lemma 25.14. Let $f \colon M \to N$ be a smooth map between two oriented n-dimensional smooth manifolds such that for each $P \in M$ the differential $\mathrm{D}f_P \colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ is an isomorphism. Let $\{\Phi_i \colon U_i \to \widetilde{U}_i\}_{i \in I}$ be an orientation-preserving atlas for M and let $\{\Psi_j \colon V_j \to \widetilde{V}_j\}_{j \in J}$ be an orientation-preserving atlas for N. The following two statements are equivalent:

(1) f is orientation-preserving.

(2) For each $i \in I$ and $j \in J$ and each $P \in U_i \cap f^{-1}(V_j)$ we have

$$\det \Big(\underbrace{\mathrm{D}(\Psi_j \circ f \circ \Phi_i^{-1})_{\Phi_i(P)}}_{\text{invertible } n \times n\text{-matrix}} \Big) \; > \; 0.$$

The analogous statement also holds for "orientation-reversing" and "< 0".

Proof. This lemma can be deduced easily from the Orientation Checking-via-Differentials Lemma 25.12 and the Basics-of-Orientations Lemma 25.13 (5) and the fact, shown in the Differential-equals-Differential Lemma 23.4 and the Tangent Space-Isomorphism Proposition 23.16 that the various definitions of differential are compatible. We leave it to the reader to fill in the few remaining details. ■

We continue with a lemma which gives another approach to defining an orientation on a given smooth manifold.

Lemma 25.15. (Define Orientations-via Transition Maps Lemma) Let $(M, \mathcal{A})$ be a smooth manifold of dimension ≥ 1. Suppose that all transition maps between charts in $\mathcal{A}$ are orientation-preserving. Given $P \in M$ we define

$$\text{orientation of } \mathrm{T}_P M \; := \; (\mathrm{D}\Phi_P)^{-1} \Big(\underbrace{\text{standard orientation of } \mathrm{T}_{\Phi(P)} V}_{\text{as defined on page 677}} \Big)$$

where $\Phi \colon U \to V$ is a chart around P from the given atlas. The following statements hold:

(1) The definition of the orientation of $\mathrm{T}_P M$ does not depend on the choice of the chart.
(2) The orientations of the tangent space $\mathrm{T}_P M$ define an orientation for M.
(3) With respect to this orientation the given atlas is orientation-preserving.

Sketch of proof. The first statement follows easily from the hypothesis that the transition maps are orientation-preserving together with the Orientation-Composition Lemma 25.3. The second statement is a consequence of the Orientations-via-Atlas Lemma 25.8. The third statement holds basically by definition. ■

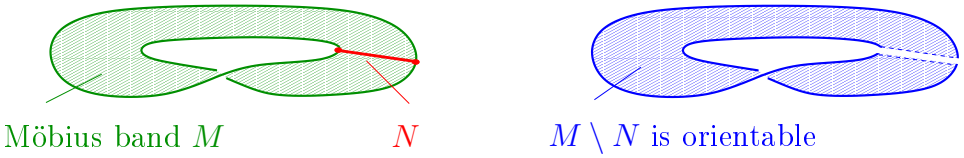
Example. We consider the surface Σ of genus 2. In the Surface-is-Topological Manifold Proposition 18.7 we gave an explicit atlas for Σ. In Proposition 19.18 we saw that the transition maps are smooth. In Exercise 25.12 we will verify that all the transition maps are in fact orientation-preserving. Thus it follows from the Define Orientations-via Transition Maps Lemma 25.15 that Σ is orientable. Basically the same argument shows that all surfaces of genus $g \geq 3$ are also orientable. Note that, as the name suggests, the same approach does not work for the non-orientable surfaces of some genus $g \in \mathbb{N}$.

Lemma 25.16. (Complement-of-Boundary Orientation Lemma) Let M be a smooth manifold.

(1) Given any orientation on $M \setminus \partial M$ there exists a unique orientation on M such that the inclusion $M \setminus \partial M \to M$ is orientation-preserving.
(2) Let N be a submanifold of M of codimension ≥ 2. Given any orientation on $M \setminus N$ there exists a unique orientation on M such that the inclusion $M \setminus N \to M$ is orientation-preserving.

Non-Example. Let M be the Möbius band and let N be the proper 1-dimensional submanifold that is shown in the figure below. Then $M \setminus N$ is evidently orientable, whereas M is

not orientable. Thus we see that in the Complement-of-Boundary Orientation Lemma 25.16 the condition on the codimension is necessary.



Proof. The proof can be found in the long version of the lecture notes. ■

Proof. Let M be an n-dimensional smooth manifold. In the following we assume that $n \geq 2$. The case $n = 1$ can be dealt with by a similar argument or one can directly appeal to the Smooth 1-Dimensional Manifold Classification Theorem 22.7.

(1) We start out with a general claim.

Claim 1. Let $U \subset M$ be an open subset. We set $\widetilde{U} := U \cap (M \setminus \partial M)$.
(a) If U is connected, then so is $\widetilde{U}$.
(b) The closure of $\widetilde{U}$ in M contains U.

Proof.
(a) The first statement follows from the Topological Manifolds-Local Properties Lemma 18.8 and the Push Away From Boundary Lemma 18.13.
(b) This statement follows easily from Lemma 1.10. ⊞

Now let $\{\Phi_i \colon U_i \to V_i\}_{i \in I}$ be an atlas for M such that each U_i is connected. We equip the V_i, which are open subsets of $\mathbb{R}^n$ or $\mathbb{H}_n$, with the standard orientation. Given $i \in I$ we write $\widetilde{U}_i := U_i \cap (M \setminus \partial M)$, $\widetilde{V}_i := \Phi(\widetilde{U}_i)$ and we denote by $\widetilde{\Phi}_i$ the restriction of Φ_i to the diffeomorphism $\widetilde{\Phi}_i \colon \widetilde{U}_i \to \widetilde{V}_i$.

Now suppose that we are given an orientation for $M \setminus \partial M$. By Claim 1 (a) we know that each $\widetilde{U}_i$ is connected. Thus we know, say by the Basics-of-Orientations Lemma 25.13 (3), that $\widetilde{\Phi}_i \colon \widetilde{U}_i \to \widetilde{V}_i$ is orientation-preserving or orientation-preserving. In the first case we set $\rho_i := \mathrm{id}$ and in the second case let ρ_i be the reflection in the $(x_1 = 0)$-hyperplane.[275] Note that in either case the map $\rho_i \circ \widetilde{\Phi}_i \colon \widetilde{U}_i \to \rho_i(\widetilde{V}_i)$ is orientation-preserving.

Claim 2. All transition maps of the atlas $\{\rho_i \circ \Phi_i \colon U_i \to \rho_i(V_i)\}_{i \in I}$ for M are orientation-preserving.

Proof. Let $i, j \in I$. By the Orientation Checking-via-Differentials Lemma 25.12 it suffices to show that the image of

$$\begin{aligned} f \colon U_i \cap U_j &\to \mathbb{R} \\ x &\mapsto \det(\underbrace{\mathrm{D}((\rho_i \circ \Phi_i) \circ (\rho_j \circ \Phi_j)^{-1})_{\rho_j(\Phi_j(x))}}_{n \times n\text{-matrix}}) \end{aligned}$$

consists only of positive real numbers. Note that this map f is smooth and non-zero everywhere. Furthermore, since $\rho_i \circ \widetilde{\Phi}_i$ and $\rho_j \circ \widetilde{\Phi}_j$ are orientation-preserving we see,

[275] Here we use that $n \geq 2$, since for $n \geq 2$ the reflection in the $(x_1 = 0)$-hyperplane is an orientation-reversing diffeomorphism of $\mathbb{H}_n$.

say using the Orientation Checking-via-Differentials Lemma 25.12, that f is positive on $\widetilde{U}_i \cap \widetilde{U}_j$. Since $U_i \cap U_j$ is an open neighborhood of x we obtain from Claim 1 that x is contained in the closure of $\widetilde{U}_i \cap \widetilde{U}_j = (U_1 \cap U_2) \cap (M \setminus \partial M)$ in M. Thus it follows from the Closure-Interior Mapping Lemma 2.5 that $f(x) > 0$. ⊞

It follows from the claim together with the Define Orientations-via Transition Maps Lemma 25.15 that this atlas defines an orientation on M. It is clear that this orientation restricts to the given orientation on $M \setminus \partial M$.

(2) The argument is basically the same as in (1). In the proof of Claim 1 (a) we just need to replace the Push Away From Boundary Lemma 18.13 by Exercise 18.20. The remainder of the argument is identical. ■

25.5. **Boundary orientation.** In this section we will see that the boundary of an oriented smooth manifold inherits an orientation. Before we can formulate the lemma let us recall the following definition from page 634.

Definition. Let M be a smooth manifold, let $P \in \partial M$ and let $w \in \mathrm{T}_P M$. We say w points outward, if the following two conditions are satisfied:

(1) For every smooth map $f\colon M \to \mathbb{R}_{\geq 0}$ with $f|_{\partial M} \equiv 0$ we have $w(f) \geq 0$.
(2) There exists a smooth map $f\colon M \to \mathbb{R}_{\geq 0}$ with $f|_{\partial M} \equiv 0$ such that $w(f) > 0$.

Lemma 25.17. (Boundary Orientation Lemma) Let M be an oriented n-dimensional smooth manifold and let $P \in \partial M$.

(a) If $n = 1$, then we define the orientation of P as follows:

$$\text{orientation of } \mathrm{T}_P \partial M := \begin{cases} -1, & \text{if a positive tangent vector of } \mathrm{T}_P M \text{ points inward} \\ +1, & \text{if a positive tangent vector of } \mathrm{T}_P M \text{ points outward.} \end{cases}$$

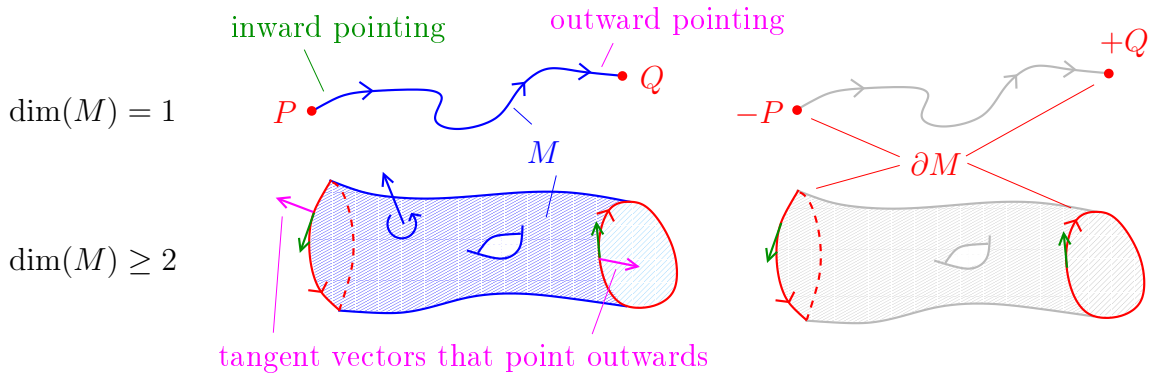
(b) Now we assume that $n \geq 2$. We pick a tangent vector $w \in \mathrm{T}_P M$ that points outward in the sense of the definition on page 634. We say that a basis $(v_1, \ldots, v_{n-1})$ for $\mathrm{T}_P(\partial M)$ is a positive basis of $\mathrm{T}_P(\partial M)$ if $(w, v_1, \ldots, v_{n-1})$ is a positive basis of $\mathrm{T}_P M$.

The following five statements hold:

(0) If M is a smooth submanifold of some $\mathbb{R}^k$, then the orientation convention in (a) and (b) also makes sense for the physical tangent spaces and it is compatible with the Tangent Space-Isomorphism Proposition 23.16.
(1) For $n \geq 2$ the definition of a positive basis for $\mathrm{T}_P M$ does not depend on the choice of w.
(2) These orientations of the tangent spaces of ∂M define an orientation on ∂M.
(3) We have $\partial(-M) = -\partial M$.
(4) Let $f\colon M \to N$ be an orientation-preserving (orientation-reversing) local diffeomorphism between oriented n-dimensional smooth manifolds. We equip ∂M and ∂N with the orientations defined in (a) and (b). The restriction of f to a diffeomorphism $\partial M \to \partial N$ is also orientation-preserving (orientation-reversing).

Example. The boundary of the 1-dimensional smooth manifold $[a, b]$, which we equip with the standard orientation coming from $\mathbb{R}$, is easily seen to be given by $(-\{a\}) \cup \{b\}$.

Proof. The proof in the case that $\dim(M) = 1$ is basically trivial. Thus in the remainder of the proof we only consider the case that $n = \dim(M) \geq 2$.



(0) This statement follows easily from the definitions together with Tangent Space-Isomorphism Proposition 23.16 (2).

(1) Let $P \in \partial M$ and let $(v_1, \dots, v_{n-1})$ for $\mathrm{T}_P(\partial M)$ be a basis. We pick an orientation-preserving chart $\Phi\colon U \to V$ of type (ii) around P. We denote by $i\colon U \to M$ the inclusion map. We consider the isomorphisms

$$\mathrm{T}_P M \xleftarrow[\uparrow]{\mathrm{D}i_P} \mathrm{T}_P U \xrightarrow{\mathrm{D}\Phi_P} \mathrm{T}_P V \xrightarrow[\uparrow]{\Theta} \widetilde{\mathrm{T}}_P V \underset{\uparrow}{=} \mathbb{R}^n.$$

isomorphism by Proposition 23.14 — Proposition 23.16 — Lemma 23.1

We denote by $\Omega\colon \mathrm{T}_P M \to \mathbb{R}^n$ the resulting isomorphism. Furthermore we denote by $p\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ the projection onto the first $(n-1)$ coordinates and we denote by $q\colon \mathbb{R}^n \to \mathbb{R}$ the projection onto the last coordinate. We make the following two observations:

(a) It follows fairly easily from the Tangent Space-Isomorphism Proposition 23.16 (2) that a vector $w \in \mathrm{T}_P M$ points outward if and only if $q(\Omega(w))$ is negative.

(b) Given a vector $w \in \mathrm{T}_P M$ the following hold:

$$\begin{aligned} &(w, v_1, \dots, v_{n_1}) \text{ is a positive basis for } \mathrm{T}_P M \\ &\iff (\Omega(w), \Omega(v_1), \dots, \Omega(v_{n-1})) \text{ is a positive basis for } \mathbb{R}^n \\ &\iff \det(\Omega(w)\,\Omega(v_1)\,\dots\,\Omega(v_{n-1})) > 0 \\ &\iff (-1)^n \cdot q(\Omega(w)) \cdot \det(p(\Omega(v_1))\,\dots\,p(\Omega(v_{n-1}))) > 0. \end{aligned}$$

It follows that the definition of a positive basis is independent of the choice of the outward pointing vector w.

(2) Using the approach of (1) it is not difficult to verify this statement.

(3) This statement is basically obvious.

(4) This statement follows immediately from the observation that a local diffeomorphism sends outward pointing vectors to outward pointing vectors. ■

Convention. Let M be an oriented n-dimensional smooth manifold. Unless we say something else we always equip ∂M with the orientation given by the Boundary Orientation Lemma 25.17. We refer to this orientation of ∂M as the boundary orientation.

Example. Let $n \in \mathbb{N}$. As on page 684 we equip the smooth manifold $\overline{B}^{n+1}$ with the standard orientation coming from $\mathbb{R}^{n+1}$. It is not hard to see that the boundary orientation of $\partial\overline{B}^{n+1} = S^n$ agrees with the standard orientation of S^n that we introduced on page 681.

At this point it makes sense to introduce the following definition.

Definition. Given $n \in \mathbb{N}_0$ we refer to the category $\mathcal{OrSmMfd}_n$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{OrSmMfd}_n) &:= \text{all oriented } n\text{-dimensional smooth manifolds,} \\ \mathrm{Mor}(X,Y) &:= \text{all orientation-preserving diffeomorphisms from } X \text{ to } Y, \end{aligned}$$

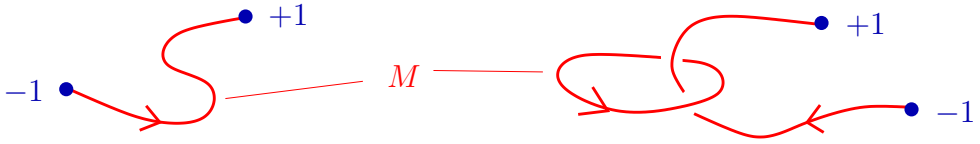
with the usual composition of maps as the category of oriented n-dimensional smooth manifolds.

Remark. Note that the Boundary Orientation Lemma 25.17 says in particular that the map $M \mapsto \partial M$ defines a functor from the category $\mathcal{OrSmMfd}_n$ of oriented n-dimensional smooth manifolds to the category $\mathcal{OrSmMfd}_{n-1}$ of oriented n-dimensional smooth manifolds.

We conclude this discussion of the orientations of the boundary of 1-dimensional smooth manifolds with the following subtle consequence of the Smooth 1-Dimensional Manifold Classification Theorem 22.7.

Proposition 25.18. (Boundary-of-Compact Smooth 1-Manifold Proposition) Let M be a compact oriented 1-dimensional smooth manifold. We equip ∂M with the orientation defined on page 688. Then

$$\sum_{P \in \partial M} \underbrace{\text{orientation of } P}_{\in\{-1,1\}} \;=\; 0.$$



Proof. It suffices to prove the statement for a compact oriented connected 1-dimensional smooth manifold with non-empty boundary. By the Smooth 1-Dimensional Manifold Classification Theorem 22.7 we know that there exists a diffeomorphism $f\colon M \to [0,1]$. After possibly composing this diffeomorphism with the orientation-reversing map $[0,1] \to [0,1]$ given by $t \mapsto 1-t$ we can assume that f is in fact orientation-preserving. Now we see that

$$\sum_{P\in\partial M} \epsilon(P) \;\underset{\uparrow}{=}\; \sum_{P\in\partial([0,1])} \epsilon(P) \;\underset{\uparrow}{=}\; -1+1 \;=\; 0.$$

by the Boundary Orientation Lemma 25.17 see the example on page 688

■

25.6. **Orientations and group actions.** We move on to considering the interplay between orientations and group actions.

Definition. Let M be an oriented smooth manifold that is equipped with a smooth action by a group G. We say that the action is orientation-preserving if for every $g \in G$ the diffeomorphism $M \to M$ given by $x \mapsto g \cdot x$ is orientation-preserving in the sense of the definition on page 682.

The first part of the following proposition can be viewed as a refinement of the Manifold Quotient-by-Group Action Proposition 19.32.

Proposition 25.19. (Orientation-Action Proposition) Let M be a smooth manifold. Furthermore let G be a group that acts freely properly and smoothly on M. By the Manifold Quotient-by-Group Action Proposition 19.32 the quotient M/G admits a standard smooth structure such that the quotient map $p\colon M \to M/G$ is a local diffeomorphism.

(1) If M is oriented and if the G-action is orientation-preserving, then the smooth manifold M/G admits an orientation such that the projection $p\colon M \to M/G$ is orientation-preserving.
(2) Conversely, if the smooth manifold M/G admits an orientation, then there exists a unique orientation on M such that the projection $p\colon M \to M/G$ is orientation-preserving. Furthermore this orientation has the property that the G-action on M is orientation-preserving.

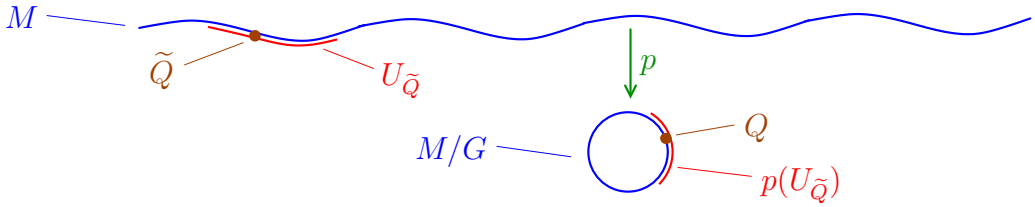
Remark. In Exercise 25.10 we will prove a partial converse to the Orientation-Action Proposition 25.19 (2).

Proof. Both statements follow fairly easily from the definitions. But for completeness' sake let us sketch the argument.

(1) We assume that M is oriented and that for each $g \in G$ the map $M \to M$ given by $Q \mapsto g \cdot Q$ is orientation-preserving.

For each $Q \in M/G$ we pick some $\widetilde{Q} \in M$ with $p(\widetilde{Q}) = Q$ and we equip $\mathrm{T}_Q(M/G)$ with the orientation given by applying the isomorphism $\mathrm{D}p_{\widetilde{Q}}\colon \mathrm{T}_{\widetilde{Q}}M \to \mathrm{T}_Q(M/G)$ to the orientation of $\mathrm{T}_{\widetilde{Q}}M$. Since the G-action is orientation-preserving we see that this definition of the orientation is independent of the choice of $\widetilde{Q}$. We claim that these orientations of the tangent spaces of M/G form an orientation of the smooth manifold M/G.

For each $Q \in M/G$ we pick a chart $\Phi_{\widetilde{Q}}\colon U_{\widetilde{Q}} \to V_{\widetilde{Q}}$ around $\widetilde{Q}$ which has the property that the restriction of p to $U_{\widetilde{Q}}$ is a diffeomorphism onto its image. The maps $\Phi_{\widetilde{Q}} \circ (p|_{U_{\widetilde{Q}}})^{-1}\colon p(U_{\widetilde{Q}}) \to V_{\widetilde{Q}}$ form a smooth atlas for M/G. It follows almost immediately from the definitions that the orientations we just defined are continuous with respect to these charts. Since these charts form a smooth atlas for M/G we obtain from the Orientations-via-Atlas Lemma 25.8 that we have defined an orientation on M/G. By construction the map $p\colon M \to M/G$ is orientation-preserving.



(2) Now we assume that M/G is equipped with an orientation. For each $\widetilde{Q} \in M$ there exists an open neighborhood $U_{\widetilde{Q}}$ of $\widetilde{Q}$ such that the restriction of p to $p\colon U_{\widetilde{Q}} \to p(U_{\widetilde{Q}})$ is a diffeomorphism and such that there exists a smooth chart $\Phi_{\widetilde{Q}}\colon p(U_{\widetilde{Q}}) \to V_{\widetilde{Q}}$ for $p(\widetilde{Q}) \in M/G$.

We equip $\mathrm{T}_{\widetilde{Q}}M$ with the orientation that is obtained by applying the isomorphism $\mathrm{D}(p|_{U_{\widetilde{Q}}}^{-1})_{\widetilde{Q}}\colon \mathrm{T}_{p(\widetilde{Q})}(M/G) \to \mathrm{T}_{\widetilde{Q}}M$ to the given orientation of $\mathrm{T}_{p(\widetilde{Q})}(M/G)$.

Since the charts $\Phi_{\widetilde{Q}} \circ p\colon U_{\widetilde{Q}} \to V_{\widetilde{Q}}$ define a smooth atlas for M we see that these orientations do indeed define an orientation on M. Basically by construction these orientations are invariant under the G-action. ■

Example. Let $v \in \mathbb{R}^n$. We denote by $t\colon \mathbb{R}^n \to \mathbb{R}^n$ the translation by v. Note that given any $x \in \mathbb{R}^n$ the differential $\mathrm{D}t_x \in \mathrm{M}(n \times n, \mathbb{R})$, in the sense of the definition on page 536, is the identity matrix. Thus it follows from the Orientation Checking-via-Differentials Lemma 25.12 that the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ that is given by addition, is orientation-preserving. It follows from the Orientation-Action Proposition 25.19 that $\mathbb{R}^n/\mathbb{Z}^n$ is orientable.

We formulate the next examples as a lemma.

Lemma 25.20. (Surface Non-Orientability Lemma) The following smooth manifolds are non-orientable:

(1) the Möbius band,
(2) the projective plane $\mathbb{R}\mathrm{P}^2$,
(3) the Klein bottle, and
(4) any non-orientable surface of genus ≥ 1.

Remark. In Exercises 25.6 and 25.11 we will give different, arguably more satisfactory proofs that the projective plane $\mathbb{R}\mathrm{P}^2$ and the Klein bottle are non-orientable.

Proof. Let $X = \mathbb{R} \times [-1, 1]$ and $G = \mathbb{Z}$. As on page 252 we consider the action

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n, (x, y)) & \mapsto & (x + n, (-1)^n \cdot y). \end{array}$$

As we remarked on pages 252 and 564, this action is free, proper and smooth. By definition the quotient $(\mathbb{R} \times [-1, 1])/\mathbb{Z}$ is the Möbius band.

Claim. The Möbius band is non-orientable.

Proof. Suppose that the Möbius band $X/\mathbb{Z}$ admits an orientation. It follows from the Orientation-Action Proposition 25.19 that $X = \mathbb{R} \times [-1, 1]$ admits an orientation such that the $\mathbb{Z}$-action is orientation-preserving. It follows from the Basics-of-Orientations Lemma 25.13 (3) and (4) that this $\mathbb{Z}$-action preserves the standard orientation on $X = \mathbb{R} \times [-1, 1]$. Given $n \in \mathbb{Z}$ we consider the diffeomorphism of $X = \mathbb{R} \times [-1, 1]$ that is given by the action by n, i.e. we consider the diffeomorphism

$$\begin{array}{rcl} \Phi_n\colon \mathbb{R} \times [-1,1] & \to & \mathbb{R} \times [-1,1] \\ (x, y) & \mapsto & (x + n, (-1)^n \cdot y). \end{array}$$

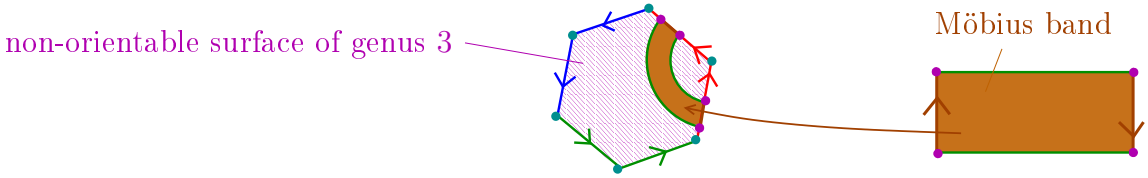
Given any $(x, y) \in X$ the differential of Φ_n is given by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & (-1)^n \end{pmatrix}$. It follows from the Orientation Checking-via-Differentials Lemma 25.12 that for n odd the diffeomorphism Φ_n is not orientation-preserving. We have thus obtained a contradiction. ⊞

We turn to the proof that the three given closed smooth manifolds are non-orientable. By the Basics-of-Orientations Lemma 25.13 (6) it suffices to prove the following claim.

Claim. The open Möbius band is diffeomorphic to open subsets of the Möbius band, of the projective plane $\mathbb{R}\mathrm{P}^2$, of the Klein bottle and for any non-orientable surface of genus $g \geq 1$.

Proof. It is clear that the open Möbius band is an open subset of the Möbius band. The statements regarding the projective plane $\mathbb{R}\mathrm{P}^2$ and the Klein bottle follow immediately from Lemma 19.39.

Recall that by definition on page 266 the real projective plane is precisely the non-orientable surface of genus ≥ 1 and that by Exercise 5.50 the Klein bottle is diffeomorphic to the non-orientable surface of genus two. Thus it remains to deal with non-orientable surfaces of genus $g \geq 3$. In the figure below we indicate that for any $g \geq 3$ the non-orientable surface of genus g admits an open subset that is diffeomorphic to the open Möbius band. We leave it to the reader to upgrade this proof to the desired level of rigor. ■



Remark. For most people starting out with mathematics the definition of a submanifold M of some $\mathbb{R}^n$ and the definition of a tangent space to M as a vector subspace of $\mathbb{R}^n$ are quite natural and reasonable. But initially it is perhaps not clear why one should introduce "abstract" smooth manifolds. Furthermore the definition on page 628 of the tangent space $\mathrm{T}_P M$ to such a smooth manifold traditionally leads to a lot of head scratching.

The above proof of the Surface Non-Orientability Lemma 25.20 is a good example why it is useful to introduce these abstract notions. In principle it is possible to show that the Möbius band, viewed as a submanifold of $\mathbb{R}^3$, is non-orientable purely by working in the explicit 3-dimensional setting, see e.g. [**Frie2016a**, Lemma 9.16]. But this proof is rather messy and not very illuminating. In contrast, the fact that we can work with abstract smooth manifolds, in this case with $(\mathbb{R} \times [-1,1])/\mathbb{Z}$, and their tangent spaces liberates us from the constraints of the 3-dimensional setting and using Lemmas 25.12 and 25.13 we obtain a much clearer and much more elegant proof that the Möbius band is non-orientable.

We conclude this section with the following lemma which generalizes the Surface Non-Orientability Lemma 25.20 (3).

Lemma 25.21. (Real Projective Space Orientability Lemma) Let $n \in \mathbb{N}$. The real projective space $\mathbb{R}\mathrm{P}^n$ is orientable if and only if n is odd.

Proof. We outsource the proof to Exercise 25.6. ■

25.7. **Products of smooth manifolds and orientations.** In this section we will study products of oriented smooth manifolds. As discussed in Section 19.10 and Chapter 20, at this point it is natural to also study smooth manifolds with corner.
The following definition is copy-pasted from page 628.

Definition. Let M be a smooth manifold with *corner*. Given $P \in M$ we define the tangent space $\mathrm{T}_P M$ at P as the set of all derivations at P, i.e. $\mathrm{T}_P M$ is the vector space of all

linear maps $w\colon C^\infty(M,\mathbb{R}) \to \mathbb{R}$ such that for every choice of $f, g \in C^\infty(M,\mathbb{R})$ we have

$$w(f \cdot g) \;=\; f(P) \cdot w(g) + w(f) \cdot g(P) \qquad \text{“Leibniz rule”.}$$

Remark. The obvious generalizations of the results obtained in Section 23.3 hold with minor changes to the proofs. Furthermore one can also define, with almost the same definition as on page 675, the notion of an orientation on a smooth manifold with corner. Since every author's duty is to strive for brevity we will not go into detail.

For the reader's convenience we recall the Smooth Manifold Product Proposition 20.6.

Proposition 20.6. (Smooth Manifold Product Proposition) Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold. We denote by

$$\begin{array}{rcl} p\colon M \times N & \to & M \\ (x,y) & \mapsto & x \end{array} \qquad \text{and} \qquad \begin{array}{rcl} q\colon M \times N & \to & N \\ (x,y) & \mapsto & y \end{array}$$

the two projections.

(1) The product $M \times N$ is an $(m+n)$-dimensional topological manifold that admits a *unique* structure as a smooth manifold with corner which that the following two conditions:
   (a) The projections $p\colon M \times N \to M$ and $q\colon M \times N \to N$ are smooth.
   (b) For any smooth manifold with corner W and any map $f\colon W \to M \times N$ the following holds: If the two maps $p \circ f\colon W \to M$ and $q \circ f\colon W \to N$ are smooth, then f itself is smooth.

(2) $\angle(M \times N) = \partial M \times \partial N$,[276]

(3) $\partial(M \times N) = (M \times \partial N) \cup (N \times \partial M)$.

We now have the following addendum to the Smooth Manifold Product Proposition 20.6.

Proposition 25.22. (Product Orientation Proposition) We continue with the notation of the Smooth Manifold Product Proposition 20.6. For the new statements we introduce the notation that given points $P \in M$ and $Q \in N$ we consider the inclusion maps

$$\begin{array}{rcl} i_Q\colon M & \to & M \times N \\ x & \mapsto & (x,Q) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} j_P\colon N & \to & M \times N \\ y & \mapsto & (P,y). \end{array}$$

The following statements hold:

(4) (a) All the inclusion maps i_P and j_Q are smooth.
   (b) Let $P \in M$ and $Q \in N$. The map

$$(\mathrm{D}i_Q)_P \oplus (\mathrm{D}i_P)_Q\colon \mathrm{T}_P M \oplus \mathrm{T}_Q N \;\to\; \mathrm{T}_{(P,Q)}(M \times N)$$

   is an isomorphism, the inverse is given by $\mathrm{D}p_{(P,Q)} \oplus \mathrm{D}q_{(P,Q)}$. (Sometimes we use this isomorphism to identify the vector spaces to the left and right.)
   (c) If M and N are oriented, then we equip the direct sum $\mathrm{T}_P M \oplus \mathrm{T}_Q N$ with the direct sum orientation that we defined on page 672. We use the isomorphism from (4b) to equip $\mathrm{T}_{(P,Q)}(M \times N)$ with the corresponding orientation. These orientations of tangent spaces define an orientation of $M \times N$.

[276] Recall that on page 582 we introduced the corner $\angle W$ of a smooth manifold with corner.

(d) If M and N are oriented, then

$$(-M) \times N \;=\; M \times (-N) \;=\; -(M \times N).$$

(5) We suppose that M and N are both oriented. If $\partial N = \varnothing$, then the above equality $\partial(M \times N) = (\partial M) \times N$ from (3) is in fact an equality of oriented smooth manifolds.[277]

Convention. Let M and N be two oriented smooth manifold. Unless we say something else we always equip the product $M \times N$ with the orientation given by the Product Orientation Proposition 25.22 (4). We refer to this orientation of $M \times N$ as the product orientation on $M \times N$.

Proof.

(4) (a) This statement follows from the Smooth Manifold Product Proposition 20.6 (1).
(b) This statement follows easily from the functoriality of the differentials.
(c) As so often, when it comes to verifying that orientations of tangent spaces define an orientation of the smooth manifold, we leave it to the reader to fill in the details.
(d) This statement follows from very elementary linear algebra.

(5) Let $P \in \partial M$ and $Q \in N$. Using Exercise 23.3 one can show that for an outward pointing vector $w \in \mathrm{T}_P M$ the image $\mathrm{D}(i_Q)_P(w) \in \mathrm{T}_{(P,Q)}(M \times N)$ is also outward pointing. We leave it to the reader to fill in the remaining details. ■

Remark. The definition of the product orientation is very slick, but it is perhaps worth spelling this out in more detail.

(1) If M and N are both of dimension ≥ 1, then it follows easily from the definitions that the orientation on $\mathrm{T}_{(P,Q)}(M \times N)$ is given by

$$\text{positive basis of } \mathrm{T}_{(P,Q)}(M \times N) \;:=\; \begin{matrix} (\mathrm{D}i_Q)_P(\text{positive basis of } \mathrm{T}_P M), \text{ followed by} \\ (\mathrm{D}j_P)_Q(\text{positive basis of } \mathrm{T}_Q N). \end{matrix}$$

(2) Let N be a smooth manifold and let $\{P\}$ be the 0-dimensional smooth manifold consisting of a single point P that is equipped with an orientation $\epsilon \in \{-1, 1\}$.
(a) If $\epsilon = 1$, then the product orientation on $\{P\} \times N$ is the orientation that turns the obvious map $N \to \{P\} \times N$ into an orientation-preserving diffeomorphism.
(b) If $\epsilon = -1$, then the production orientation $\{P\} \times N$ is the orientation that turns the obvious map $N \to \{P\} \times N$ into an orientation-reversing diffeomorphism.

(3) If $M = \{(P, \mu)\}$ and $N = \{(N, \nu)\}$ are two oriented 0-dimensional smooth manifolds, each consisting of a single point, then it follows from the above definition and the remark on page 672 that $M \times N = \{(P, Q)\}$ is equipped with the orientation $\mu \times \nu$.

Example.

(1) Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open subsets. We equip both with the standard orientation that we defined on page 677. It follows fairly easily from the definitions

[277] In Exercise 25.14 we will determine the relationship between the orientations of $\partial(M \times N)$ and $M \times \partial N$ if N is allowed to have non-empty boundary.

that the product orientation on $U \times V \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ equals the standard orientation of $U \times V$ viewed as an open subset of $\mathbb{R}^{m+n}$.

(2) If $N = \{P\}$ is the 0-dimensional smooth manifold consisting of a single point P, and if the orientation of that point is given by $+1$, then the product orientations on $N \times \{P\}$ and $\{P\} \times N$ are precisely the orientations that turn the maps

$$\begin{array}{ccc} M & \to & M \times \{P\} \\ x & \mapsto & (x, P) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} M & \to & \{P\} \times M \\ x & \mapsto & (P, x) \end{array}$$

into orientation-preserving diffeomorphisms.

(3) Recall that on page 681 we equipped the 0-dimensional sphere $S^0 = \{\pm 1\}$ with the orientation given by $\pm 1 \mapsto \pm 1$. It follows immediately from the definitions that for any oriented smooth manifold M we have a natural orientation-preserving diffeomorphism $M \times S^0 \xrightarrow{\cong} (-M) \sqcup M$.

(4) Given any oriented smooth manifold N without boundary and given $a < b \in \mathbb{R}$ we have the equality

$$\partial([a,b] \times N) \underset{\substack{\uparrow \\ \text{Product Orientation} \\ \text{Proposition 25.22 (5)}}}{=} \partial([a,b]) \times N \underset{\substack{\uparrow \\ \text{see page 688}}}{=} (-\{a\} \cup \{b\}) \times N = (-\{a\} \times N) \cup (\{b\} \times N).$$

We have one last, rather unsurprising lemma.

Lemma 25.23. (Product Orientation Preserving Lemma) Let $f\colon M \to \widetilde{M}$ and $g\colon N \to \widetilde{N}$ be maps between oriented smooth manifolds. We consider the map

$$f \times g\colon M \times N \to \widetilde{M} \times \widetilde{N}.$$

Let $x \in M$ and $y \in N$ be points such that $\mathrm{D}f_x\colon \mathrm{T}_x M \to \mathrm{T}_{f(x)}\widetilde{M}$ is an isomorphism and such that $\mathrm{D}g_y\colon \mathrm{T}_y N \to \mathrm{T}_{g(y)}\widetilde{N}$ is an isomorphism.

(1) The map $\mathrm{D}(f \times g)_{(x,y)}\colon \mathrm{T}_{(x,y)}(M \times N) \to \widetilde{\mathrm{T}}_{(\widetilde{x},\widetilde{y})}(\widetilde{M} \times \widetilde{N})$ is an isomorphism.

(2) The map $f \times g$ is orientation-preserving at (x, y) if and only if both or none of f and g are orientation-preserving at x respectively y.

Proof. The lemma can be proved easily using product charts. We leave it to the reader to fill in the details. ■

This concludes our introduction to the notion of an orientation of smooth manifolds. Perhaps somewhat unintuitively there does also exist a perfectly fine notion of the orientation of a topological manifold. We will introduce this notion in Chapter 105. In the Smooth-Topological Orientation Proposition 105.11 we will see that this generalizes the present notion of an orientation. Furthermore in the Topological Manifold-Boundary Orientation Proposition 105.19 we will generalize the Boundary Orientation Lemma 25.17 to topological manifolds. Finally in Propositions 113.10 and 113.11 (see also page 2502) we will generalize the Product Orientation Proposition 25.22 to topological manifolds.

Exercises for Chapter 25.

Exercise 25.1. Let $v \in \mathbb{R}^n$ be a non-zero vector. We consider the orthogonal complement of v, i.e. we consider the vector space $v^\perp = \{w \in \mathbb{R}^n \mid \langle v, w\rangle = 0\}$. Show that

$$(w_1, \dots, w_{n-1}) \text{ is a positive basis of } v^\perp \quad :\Longleftrightarrow \quad \det(v\ w_1\ \dots\ w_{n-1}) > 0$$

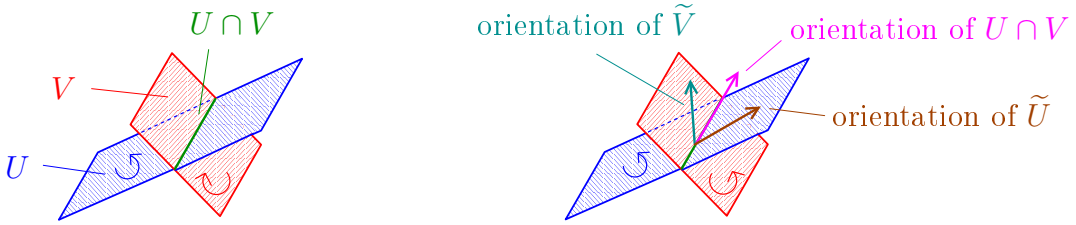
defines an orientation on $v^\perp$.

Exercise 25.2. Let W be an oriented finite dimensional vector space and let U, V be oriented vector subspaces of W with $W = U + V$. What is the relationship between the intersection orientation of $U \cap V$ and the intersection orientation of $V \cap U$?

Exercise 25.3. Let W be an oriented real vector space and let U and V be oriented vector subspaces with $W = U + V$. We pick a vector subspace $\widetilde{U} \subset U$ such that $U = \widetilde{U} \oplus (U \cap V)$ and we pick a vector subspace $\widetilde{V} \subset V$ such that $V = (U \cap V) \oplus \widetilde{V}$.

(a) Show that there exist unique orientations on $U \cap V$, $\widetilde{U}$ and $\widetilde{V}$ such that the direct sum orientations of $U = \widetilde{U} \oplus (U \cap V)$, $V = (U \cap V) \oplus \widetilde{V}$ and $W = \widetilde{U} \oplus (U \cap V) \oplus \widetilde{V}$ are the given orientation.

(b) Show that the orientation of $U \cap V$ is precisely the intersection orientation.



Exercise 25.4. Let $\{v_1, v_2\}$ be a basis for $\mathbb{R}^2$ and let φ be the unique real number in $(-\pi, \pi)$ such that there exists a $\lambda > 0$ with

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} v_1 \ = \ \lambda \cdot v_2.$$

Show that the basis (v_1, v_2) is a positive basis of $\mathbb{R}^2$ if and only if $\varphi \in (0, \pi)$.

Exercise 25.5. Let $A \in \mathrm{O}(n)$. Show that the map $S^n \to S^n$ given by $x \mapsto A \cdot x$ is orientation-preserving if and only if $\det(A) = 1$.

Exercise 25.6. Let $n \in \mathbb{N}$.

(a) Show that the action of $G = \{\pm 1\}$ on S^n is orientation-preserving if and only if n is odd.
Hint. Use the Tangent Space-Isomorphism Proposition 23.16 (2) and the Basics-of-Orientations Lemma 25.13 (3).

(b) Show that the real projective space $\mathbb{R}\mathrm{P}^n$ is orientable if and only if n is odd.
Hint. Use (a) together with the Basics-of-Orientations Lemma 25.13 and the Orientation-Action Proposition 25.19.

Exercise 25.7. Let M and N be two smooth manifolds of dimension ≥ 1. Show that $M \times N$ is orientable if and only if M and N are both orientable.

Exercise 25.8. Let $n \in \mathbb{N}$ and let $A \in \mathrm{GL}(n, \mathbb{Z})$ be a matrix. By the Maps-on-Torus Lemma 19.37 we know that the map

$$\begin{array}{rccc} f(A)\colon & \mathbb{R}^n/\mathbb{Z}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ & [v] & \mapsto & [A \cdot v] \end{array}$$

is a diffeomorphism. Show that this diffeomorphism is orientation-preserving if and only if $A \in \mathrm{SL}(n, \mathbb{Z})$, i.e. if and only if $\det(A) = 1$.

Exercise 25.9.

(a) Let $p\colon S^n \to \mathbb{R}^n \cup \{\infty\}$ be the stereographic projection from Lemma 2.18. It restricts to a diffeomorphism $\varphi\colon S^n \setminus \{(0, \dots, 0, 1)\} \to \mathbb{R}^n$. We equip both sides with the standard orientation. Show that φ is orientation-preserving if and only if n is odd.

(b) We consider the projection

$$\begin{array}{rccc} \psi\colon & S^n_{>0} = \{(x_1, \dots, x_{n+1}) \in S^n \mid x_{n+1} > 0\} & \mapsto & B^n \\ & (x_1, \dots, x_{n+1}) & \to & (x_1, \dots, x_n). \end{array}$$

We equip both sides with the standard orientation. Show that ψ is orientation-preserving if and only if n is even.

Exercise 25.10. Let G be a group that acts freely, properly and smoothly on an n-dimensional oriented smooth manifold M. Suppose that M/G is orientable.

(a) Suppose that M is path-connected. Show that every $g \in G$ acts on M in an orientation-preserving way.
Hint. Use the Orientation-Action Proposition 25.19 (2).

(b) Show that the conclusion of (a) does not hold if M is not path-connected.

Remark. This exercise can be viewed as a converse to the Orientation-Action Proposition 25.19 (1).

Exercise 25.11. We consider the following two self-diffeomorphisms of $\mathbb{R}^2$:

$$\begin{array}{rccc} A\colon & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & (x, y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rccc} B\colon & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & (x, y) & \mapsto & (x, y+1). \end{array}$$

We denote by G the subgroup of all diffeomorphisms of $\mathbb{R}^2$ that is generated by A and B. This means

$$G = \begin{array}{c} \text{all self-diffeomorphisms of } \mathbb{R}^2 \text{ that can be written} \\ \text{as a finite concatenation of the maps } A, B, A^{-1} \text{ and } B^{-1}. \end{array}$$

(a) For $n \in \mathbb{Z}$, what is $A^{-n}BA^n$?

(b) We write $C = A^2$. We denote by H the subgroup of all diffeomorphisms of $\mathbb{R}^2$ that is generated by C and B. Does A lie in H?
Hint. Consider which self-diffeomorphisms of $\mathbb{R}^2$ are orientation-preserving.

(c) Show that any element X of G lies in H or it is of the form $G = AY$ with $Y \in H$.

(d) Show that the group G acts freely and properly on $\mathbb{R}^2$.

(e) We consider the square $X = [0, 1] \times [0, 1]$ with the equivalence relation that is generated by $(0, y) \sim (1, 1-y)$ and $(x, 0) \sim (x, 1)$. Show that the map

$$\begin{array}{rccc} & X/\!\sim & \to & \mathbb{R}^2/G \\ & [(x, y)] & \mapsto & [(x, y)] \end{array}$$

is a diffeomorphism. (Recall that in Exercise 19.20 we equipped $X/\sim$ with a smooth structure.) This implies that $\mathbb{R}^2/G$ is diffeomorphic to the Klein bottle that we defined on page 265.

Remark. This discussion, together with the Orientation-Action Proposition 25.19, gives a new proof of the fact, already shown in the Surface Non-Orientability Lemma 25.20, that the Klein bottle is non-orientable.

Exercise 25.12. We consider the surface Σ of genus 2 and we consider the explicit atlas that we constructed in the proof of the Surface-is-Topological Manifold Proposition 18.7. Show that the transition maps are all orientation-preserving.

Exercise 25.13. Can a closed non-orientable connected smooth manifold be the boundary of a compact smooth manifold?

Exercise 25.14. Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold. We equip the product smooth manifolds with corner $M \times N$ and $N \times M$ with the product orientations as defined in the Product Orientation Proposition 25.22.

(a) Show that the diffeomorphism

$$\begin{array}{rcl} M \times N & \to & N \times M \\ (x, y) & \mapsto & (y, x) \end{array}$$

is orientation-preserving if and only if $m \cdot n$ is even.

(b) We suppose that $\partial N = \varnothing$. We equip ∂M and $N \times M$ with the boundary orientation and we equip $N \times \partial M$ with the product orientation. Show that we have the following equality of *oriented* smooth manifolds:

$$\partial(N \times M) \;=\; \begin{cases} N \times \partial M, & \text{if } n \text{ is even,} \\ -(N \times \partial M), & \text{if } n \text{ is odd.} \end{cases}$$

Exercise 25.15. Let M be an oriented smooth manifold without boundary. We denote by $-M$ the same smooth manifold with the opposite orientation. Show that the product orientations given by $M \times M$ and $(-M) \times (-M)$ are actually the same.

Exercise 25.16. Let M be a smooth codimension-one submanifold of $\mathbb{R}^n$, let $\gamma\colon [0,1] \to M$ be a curve and let $v \in (\widetilde{\mathrm{T}}_{\gamma(0)}M)^\perp$. Show that there exists a unique map $\widetilde{\gamma}\colon [0,1] \to \mathbb{R}^n$ such that $\widetilde{\gamma}(0) = \gamma(0) + v$ and such that for each $t \in [0,1]$ we have $\widetilde{\gamma}(t) - \gamma(t) \in (\widetilde{\mathrm{T}}_{\gamma(t)}M)^\perp$ with $\|\widetilde{\gamma}(t) - \gamma(t)\| = \|v\|$.

Remark. Evidently we want $\widetilde{\gamma}$ to be continuous. Note that we do not assume that M is orientable.



Exercise 25.17. We consider $\widetilde{M} := \mathbb{R} \times [-1,1]$ and the self-diffeomorphism

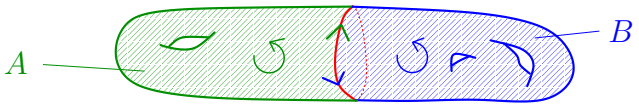
$$\begin{array}{rcl} g\colon \widetilde{M} = \mathbb{R} \times [-1,1] & \to & \widetilde{M} = \mathbb{R} \times [-1,1] \\ (r, s) & \mapsto & (r+1, -s). \end{array}$$

Recall that the Möbius band is defined as $M = \widetilde{M}/\sim$ where $P \sim g(P)$. We denote by $p\colon \mathbb{R}\times[-1,1] \to M$ the projection. We set $x := p(0,0)$. Show that there exists a diffeotopy $F\colon M \times [0,1] \to M$ with the following properties:

(1) $F_0 = \mathrm{id}$.
(2) $F_1(x) = x$.
(3) $\mathrm{D}(F_1)_x\colon \mathrm{T}_x M \to \mathrm{T}_x M$ is orientation-reversing.
(4) Bonus exercise: arrange that F is a diffeotopy rel ∂M.

Exercise 25.18. Let $f\colon \mathbb{R} \to \mathbb{R}$ be an orientation-preserving local diffeomorphism. Show that f is strictly monotonously increasing, i.e. show that for any real numbers $s > t$ we have $f(s) > f(t)$.

Exercise 25.19. Let M be an oriented smooth manifold and let A and B be codimension-zero submanifolds such that $A \cap B = \partial A = \partial B$. We equip the codimension-zero submanifolds A and B with the standard orientations defined on page 684 and we equip ∂A and ∂B with the orientations defined on page 689. Show that as oriented smooth manifolds we have $\partial A = -\partial B$.



Exercise 25.20.

(a) Let M be a smooth manifold and let $k \in \mathbb{N}_0$. Show that if $M \times B^k$ is orientable, then so is M.
(b) Let M and N be non-empty smooth manifolds. Show that if $M \times N$ is orientable, then so are M and N.

Remark. This exercise can be viewed as a converse to the Product Orientation Proposition 25.22 (4c).

Exercise 25.21. Does there exist a connected oriented 1-dimensional smooth manifold that does not admit an orientation-reversing diffeomorphism?
Remark. In Exercise 43.16 we will study the 2-dimensional case.

26. The Regular Value Theorem

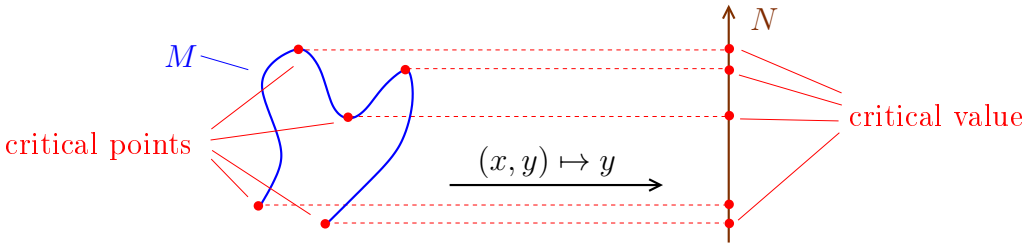
Let M be a smooth manifold, e.g. $M = \mathbb{R}^n$ or $M = \mathrm{M}(n \times n, \mathbb{R})$. In general it can be quite painful to show directly that a given subset, e.g. $\mathrm{O}(n) \subset \mathrm{M}(n \times n, \mathbb{R})$ is a smooth submanifold. Only the most enterprising reader will feel like writing down explicit submanifold charts for $\mathrm{O}(n) \subset \mathrm{M}(n \times n, \mathbb{R})$. In this chapter we will state and prove the Regular Value Theorem which can be used, among many other applications, to give an elegant proof that $\mathrm{O}(n)$ is a submanifold of $\mathrm{M}(n \times n, \mathbb{R})$.

26.1. **Regular values.** Not surprisingly the formulation of the Regular Value Theorem is preceded by the definition of a regular value. This notion might be familiar to the reader from an earlier course on real analysis.

Definition. Let M and N be smooth manifolds (possibly with boundary). Let $f\colon M \to N$ be a smooth map.

(1) Let $P \in M$.
   (a) If $P \in M \setminus \partial M$, then we say P is a regular point if $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}N$ is an epimorphism.
   (b) If $P \in \partial M$, then we say P is a regular point if $\mathrm{D}f_P\colon \mathrm{T}_P(\partial M) \to \mathrm{T}_{f(P)}N$ is an epimorphism.[278]

   Otherwise we say that P is a critical point.
(2) We say $Q \in N$ is a regular value if all points in the preimage $f^{-1}(Q)$ are regular. Otherwise we call Q a critical value. In other words, the critical values are precisely the images of the critical points.



The following elementary observation will be used several times without explicitly mentioning it.

Observation. Let $f\colon M \to N$ be a smooth map between smooth manifolds.

(1) If $Q \in N$ is a point with $f^{-1}(Q) = \varnothing$, then Q is a regular value.
(2) If $\dim(M) < \dim(N)$, then the converse to (1) holds: it follows from the Tangent Space-Dimension Proposition 23.11 and elementary linear algebra that a point $Q \in N$ is a regular value if and only if we have $f^{-1}(Q) = \varnothing$.
(3) If $\varphi\colon X \to M$ is a local diffeomorphism, then it follows easily from the Chain Rule 19.10 that

$$\text{set of critical points of } f \circ \varphi\colon X \to N \;=\; \varphi^{-1}(\text{set of critical points of } f\colon M \to N).$$

[278] Note that it follows from the Smooth Manifold Boundary Proposition 19.26 (2) that the restriction of a map on a smooth manifold to its boundary is also a smooth map.

From this observation together with the Image-Preimage Lemma 0.2 (2) it follows that

$$\text{set of critical values of } f\circ\varphi\colon X\to N \quad \underset{\substack{\uparrow\\ \text{equality holds if } \varphi \text{ is surjective}}}{\subset} \quad \text{set of critical values of } f\colon M\to N.$$

The following lemma is slightly less trivial than the above observation.

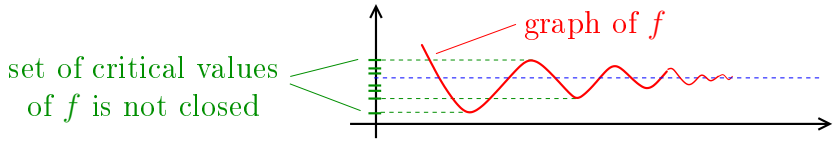
Lemma 26.1. (Regular Value-Open Subset Lemma) Let $f\colon M\to N$ be a smooth map between smooth manifolds.

(1) The set of regular points of f is an open subset of M.
(2) The set of critical points of f is a closed subset of M.

Example. We consider the smooth functions

$$\begin{array}{rcl} f\colon [1,\infty) & \to & \mathbb{R} \\ x & \mapsto & \frac{1}{x}\cdot\sin(x) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} f\colon \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} e^{-\frac{1}{x^2}}\cdot\sin(\frac{1}{x}), & \text{if } x\neq 0, \\ 0, & \text{if } x=0. \end{cases} \end{array}$$

The first example shows that the set of critical values is not necessarily closed and the second example shows that the set of critical points is not necessarily discrete. For purely artistic reasons we show in the figure below the graph of some other smooth function $f\colon (a,b)\to\mathbb{R}$ for which the set of critical values is not closed.



Proof. The second statement is of course just a reader-friendly reformulation of the first statement. We write $m := \dim(M)$ and $n := \dim(N)$. Let P be a regular point of f. We pick a chart $\Phi\colon U\to V$ around P and we pick a chart $\Psi\colon W\to X$ around $f(P)$. We consider the map

$$\begin{array}{rcl} \Xi\colon \Phi(U\cap f^{-1}(W)) & \to & \mathrm{M}(n\times m,\mathbb{R}) \\ Q & \mapsto & \mathrm{D}(\Psi\circ f\circ\Phi^{-1})_Q. \end{array}$$

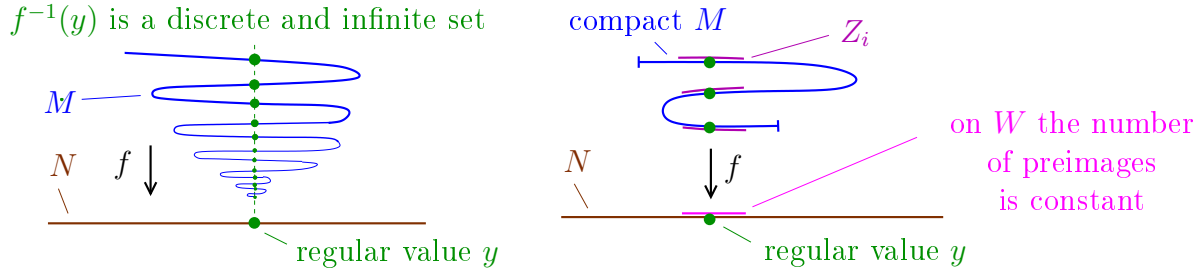
The lemma follows easily from the hypothesis that P is regular point and the following four observations:

(a) Since f is smooth we see that Ξ is continuous.
(b) A point $Q\in U\cap f^{-1}(W)$ is a regular point of f if and only if $\Xi(\Phi(Q))$ is a matrix of rank m.
(c) $U\cap f^{-1}(W)$ is an open neighborhood of P.
(d) By Lemma 42.8, see also Exercise 2.41, we know that the set of matrices of rank m is an open subset of $\mathrm{M}(n\times m,\mathbb{R})$. ■

In the following lemma we summarize a few basic facts about regular points and values of smooth maps between smooth manifolds *of the same dimension*.

Lemma 26.2. (Stack of Records Lemma)[279] Let $f\colon M \to N$ be a smooth map between two smooth manifolds of the same dimension.

(1) For every regular point $x \in M \setminus \partial M$ the map f is a local diffeomorphism around x.
(2) Let $y \in N$ be a regular value.
   (a) The preimage $f^{-1}(y)$ is a discrete subset of M
   (b) If M is compact, then $f^{-1}(y)$ is finite subset of M.
   (c) If M is compact, then there exists an open neighborhood W of $y \in N$ with the following properties:
      (i) $f^{-1}(W)$ consists of disjoint open sets $Z_1, \dots, Z_k$ and the restriction of f to each Z_i defines a diffeomorphism $Z_i \to W$.
      (ii) Each $y' \in W$ is a regular value.
      (iii) For each $y' \in W$ we have the equality $\#f^{-1}(y) = \#f^{-1}(y')$.



Proof.

(1) Let $x \in M \setminus \partial M$ be a regular point of f. This means that $\mathrm{D}f_x\colon \mathrm{T}_{f(x)}M \to \mathrm{T}_{f(x)}N$ is an epimorphism. Since M and N have the same dimension this implies that $\mathrm{D}f_x$ is an isomorphism. It follows from the hypothesis that $x \in M \setminus \partial M$ together with the Inverse Mapping Theorem 24.1 that f is a local diffeomorphism around x.
(2) Let y be a regular value of f.
   (a) It follows immediately from (1) that for each $x \in f^{-1}(y)$ there exists an open neighborhood U of x such that $f\colon U \to f(U)$ is a diffeomorphism. This implies that $U \cap f^{-1}(y) = \{x\}$. We have thus shown that $f^{-1}(y)$ is a discrete subset of M.
   (b) By (a) we know that $f^{-1}(y)$ is a discrete subset of M. If M is compact, then it follows from the Compact-Discrete Lemma 1.22 that $f^{-1}(y)$ is actually a finite set.
   (c) We denote the points in $f^{-1}(y)$ by $x_1, \dots, x_k \in M$. Since $x_1, \dots, x_k$ are regular points we obtain from (a) open neighborhoods $U_1, \dots, U_k$ of $x_1, \dots, x_k$ and open neighborhoods $V_1, \dots, V_k$ of $f(x_1), \dots, f(x_k)$ such that for each i the map $f_i\colon U_i \to V_i := f(U_i)$ is a diffeomorphism. Since M is Hausdorff we can and will

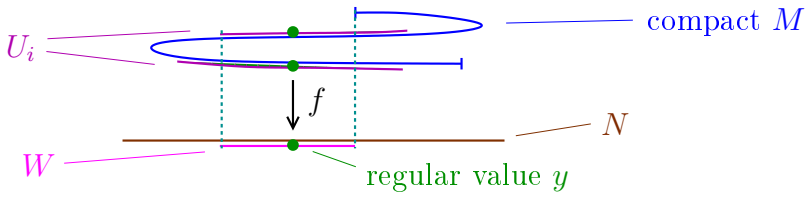
[279] The idea behind the name is that W is a disk ("record") and "above it" lie k disks ("records"). The name might make more sense for the readers over 50.

arrange that the U_i are pairwise disjoint. We set

$$W := \underbrace{\Big(\bigcap_{i=1}^{k} V_i\Big)}_{\text{open subset}} \quad \setminus \quad \underbrace{f\Big(M \setminus \bigcup_{i=1}^{k} U_i\Big)}_{\substack{\text{since } M \text{ is compact we}\\ \text{obtain from Lemma 2.13 that}\\ \text{this is a closed subset of } N}}.$$

Note that W is open and that it contains y. Furthermore note that by construction we have $f^{-1}(W) \subset \bigcup_{i=1}^{k} U_i$. For $i = 1, \dots, k$ we set $Z_i := f^{-1}(W) \cap U_i$. Evidently these sets are still disjoint.

- For each $i \in \{1, \dots, k\}$ the map $f\colon Z_i \to W$ is a bijection and it is given by the restriction of a diffeomorphism to an open subset. Hence the map $f\colon Z_i \to W$ is actually a diffeomorphism.
- It follows from (i) that each point in $f^{-1}(W)$ is a regular point, thus every point in W is a regular value.
- By (i) the preimage of each $y' \in V$ consists of precisely k points, namely one x'_i in each U_i. In other words, we have shown that $\#f^{-1}(y) = k$ for all $y \in W$. ∎



26.2. **The Fundamental Theorem of Algebra ❀.** Perhaps rather surprisingly the straightforward Lemma 26.2 allows us to prove the Fundamental Theorem of Algebra.

Theorem 26.3. (Fundamental Theorem of Algebra) Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a zero in $\mathbb{C}$.

Remark. We will provide two other proofs of the Fundamental Theorem of Algebra 26.3 on page 1094 and on page 1700.

Proof. Let $q(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a complex polynomial of degree $n \geq 1$ with complex coefficients. We will perform a proof by contradiction, so we now assume that $q(z)$ has no zeros. In our argument below we will use the following two inputs which we will not prove.

(i) Every complex polynomial of degree $d \geq 0$ has at most d zeros.
(ii) If p is a complex polynomial of degree ≥ 1, then the map $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto p(z)$ takes on infinitely many values.

We make a few preparations for the proof:

(1) We write $P_+ = (0,0,+1)$. As in Lemma 2.18 we consider the stereographic projection from P_+ which is given as follows:

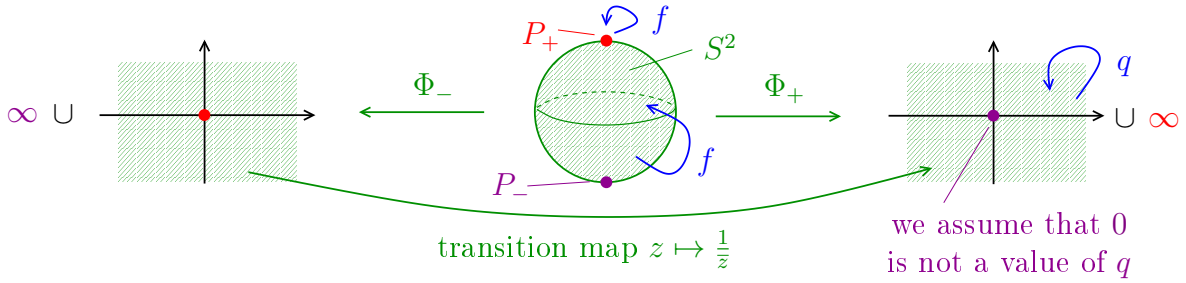
$$\begin{array}{rcl} \Phi_+ \colon S^2 & \to & \mathbb{R}^2 \cup \{\infty\} = \mathbb{C} \cup \{\infty\} \\ (x_1,x_2,x_3) & \mapsto & \begin{cases} \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\right), & \text{if } x_3 \neq +1, \\ \infty, & \text{if } x_3 = +1. \end{cases} \end{array}$$

Similarly we write $P_- = (0,0,-1)$ and we denote by $\Phi_- \colon S^2 \to \mathbb{C} \cup \{\infty\}$ the stereographic projection from P_-. Note that $\Phi_\pm(P_\pm) = \infty$ and $\Phi_\pm(P_\mp) = 0$.

(2) By Exercise 18.15 we know that the transition map between the two charts is given by

$$\begin{array}{rcl} \Phi_+ \circ \Phi_-^{-1} \colon \mathbb{C} \setminus \{0\} & \to & \mathbb{C} \setminus \{0\} \\ z & \mapsto & \frac{1}{|z|^2} \cdot z = \frac{1}{\bar{z}}. \end{array}$$

(3) As on page 513 and in Proposition 19.18 we equip S^2 with the usual smooth atlas given by Φ_+ and Φ_-.



Claim. The map

$$\begin{array}{rcl} f \colon S^2 & \to & S^2 \\ P & \mapsto & \begin{cases} \Phi_+^{-1}(q(\Phi_+(P))), & \text{if } P \neq P_+, \\ P_+, & \text{if } P = P_+ \end{cases} \end{array}$$

has the following two properties:

(1) the map f is smooth,
(2) the map f has at most $n+1$ critical points.

Proof.

(1)
- The map $\Phi_+ \circ f \circ \Phi_+^{-1} \colon \mathbb{C} \to \mathbb{C}$ is given by $z \mapsto q(z)$, hence it is smooth.
- Note that our assumption that q has no zeros implies that P_- does not lie in the image of f. This in turn implies that ∞ does not lie in the image of $\Phi_- \circ f$. Thus we can consider the map

$$\Phi_- \circ f \circ \Phi_-^{-1} \colon \mathbb{C} \to \mathbb{C}.$$

It follows from the discussion of the transition map and the fact that $f(P_+) = P_+$ that this map is given by

$$z \mapsto \begin{cases} 0, & \text{if } z = 0, \\ \dfrac{1}{p(\frac{1}{\bar{z}})} = \dfrac{z^n}{\bar{a}_0 z^n + \bar{a}_1 z^{n-1} + \cdots + \bar{a}_n}, & \text{if } z \neq 0. \end{cases}$$

But since $a_n \neq 0$ we see that this map is smooth, even at zero.

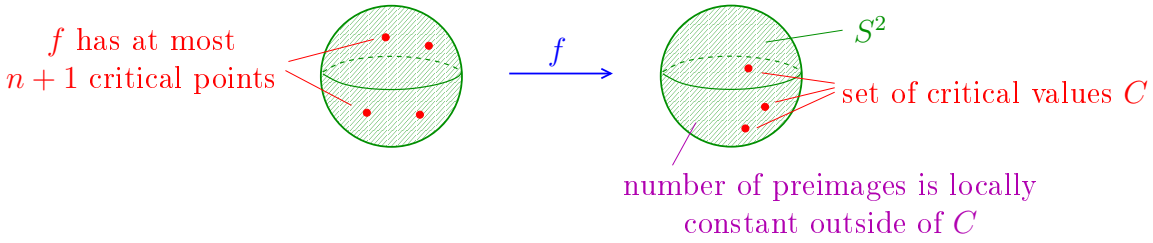
It follows from the above together, with the Smooth Pasting Proposition 19.23, that f is smooth.

(2) Note that by standard results in complex analysis, see e.g. the Smooth-Complex Differentiability Proposition 32.3, we know that for a holomorphic function the derivative corresponds to the differential. It follows almost immediately from this observation together with input (i) that the map $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto q(z)$ has at most n critical points. This implies that f has at most n critical points on $\Phi_+(\mathbb{C}) = S^2 \setminus \{P_+\}$. This implies that f has at most $n+1$ critical points. ⊞

Now we return to the main argument. First note that if $P_- = \Phi_+^{-1}(0)$ is a critical value of f, then this implies by the discussion on page 701 that 0 is a value of q, i.e. q has a zero. Now assume that $P_+ = \Phi_+^{-1}(0)$ is not a critical value of f. We denote by C the set of critical values of f. It follows from the claim that C is finite. Next we consider the map

$$\begin{array}{rcl} d\colon S^2 \setminus C & \to & \mathbb{N}_0 \\ P & \mapsto & \#f^{-1}(P). \end{array}$$

It follows from Lemma 26.2 that this map $d\colon S^2 \setminus C \to \mathbb{N}_0$ is locally constant, in particular it is continuous. Since C is finite we obtain from an elementary argument, see e.g. Exercise 18.16, that the left-hand side is connected. Since $\mathbb{N}_0$ is evidently discrete we obtain from the Connected-to-Discrete Lemma 2.27 that $d\colon S^2 \setminus C \to \mathbb{N}_0$ is actually a constant map.



Since q is not a constant polynomial it follows from input (ii) that the map $z \mapsto q(z)$, and thus also the map $f\colon S^2 \to S^2$, takes on infinitely many values. This implies that $d\colon S^2 \setminus C \to \mathbb{N}_0$ cannot be zero everywhere. But since this function is constant and since we assume that P_+ is a regular value we see that $d(P_+) = \#f^{-1}(P_+) = \#q^{-1}(0)$ is non-zero. But this implies that q has at least one zero. ■

26.3. **The Regular Value Theorem.** The following theorem is often a convenient way for showing that a given subset of a smooth manifold, e.g. a given subset of some $\mathbb{R}^n$ or some $\mathrm{M}(m \times n, \mathbb{R})$ is in fact a smooth submanifold.

Theorem 26.4. (Regular Value Theorem) Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold, let $f\colon M \to N$ be a smooth map and let $s \in N \setminus \partial N$ be a regular value of f. We denote by $f_\partial\colon \partial M \to N$ the restriction of f to ∂M.

(1) The preimage $X := f^{-1}(s)$ is a proper $(m-n)$-dimensional submanifold of M with $\partial f^{-1}(s) = (f_\partial)^{-1}(s)$.

(2) We denote by $i\colon X \to M$ the inclusion. For every $P \in X$ we have

$$\operatorname{im}(\mathrm{D}i_P\colon \mathrm{T}_P X \to \mathrm{T}_P M) \;=\; \ker(\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_s N).$$

In particular, if M is a submanifold of some $\mathbb{R}^k$ and N is a submanifold of some $\mathbb{R}^l$, then

$$\widetilde{\mathrm{T}}_P X \;=\; \ker(\mathrm{D}f_P\colon \widetilde{\mathrm{T}}_P M \to \widetilde{\mathrm{T}}_s N).$$

(3) Suppose we have an orientation on M and an orientation on $\mathrm{T}_s N$.
(a) Let $P \in X$. By (2) we have a short exact sequence[280]

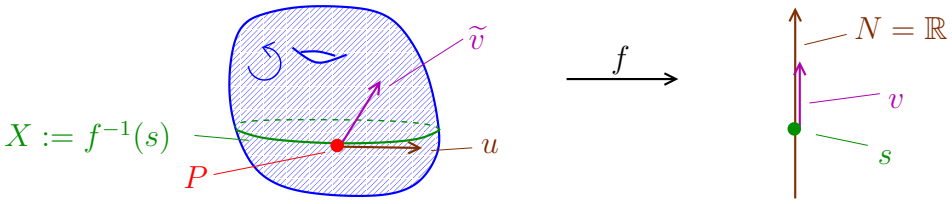
$$\mathrm{T}_P X \xrightarrow{\mathrm{D}i_P} \mathrm{T}_P M \xrightarrow{\mathrm{D}f_P} \mathrm{T}_s N.$$

By our hypothesis the vector spaces in the middle and to the right are oriented. It follows from the Compatible Orientations Lemma 25.5 that there exists a unique orientation on $\mathrm{T}_P X$ such that the orientations are compatiblein the sense of the definition on page 671. These orientations on the tangent spaces $\mathrm{T}_P X$ define an orientation on the smooth manifold X.

(b) If we equip $\partial f^{-1}(s)$ with the boundary orientation, then the orientation agrees with the orientation on $(f_\partial)^{-1}(s)$ defined by the definition of (3a) applied to $f_\partial\colon \partial M \to N$.

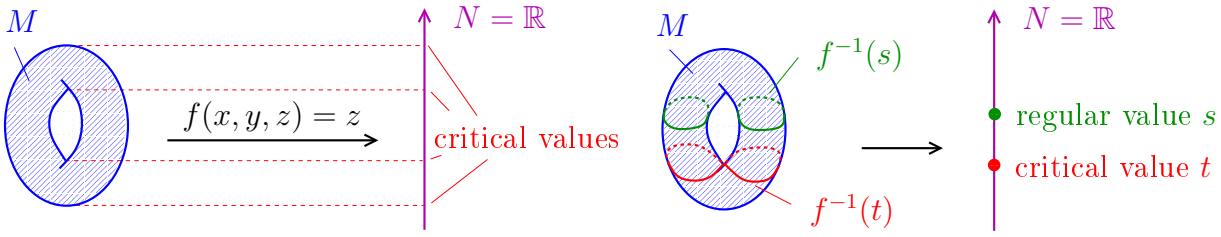
(4) The intersection of X with any compact set of M is compact. In particular, if M is compact, then X is also compact.

Convention. Recall that the Submanifold Smooth Structure Lemma 19.24 says that a submanifold of a smooth manifold has a natural smooth structure. We will use this structure to view preimages of regular values as smooth manifolds in their own right.



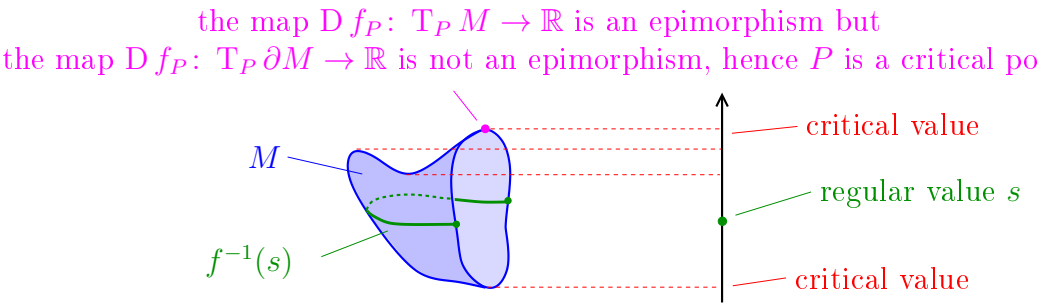
Example.

(1) In the figure below we show the torus M as a submanifold of $\mathbb{R}^3$ and we consider the "height function" $f\colon M \to \mathbb{R}$ given by $f(x, y, z) = z$. This function has precisely four critical points which are indicated in the figure on the left. On the right we show a regular value s and its preimage $f^{-1}(s)$ and we show a critical value t and its preimage $f^{-1}(t)$ which in this case is *not* a submanifold of M.



[280] We refer to page 670 for the definition of a short exact sequence.

(2) In the figure below we consider again a smooth manifold $M \subset \mathbb{R}^3$ together with the "height function". But this time M has boundary and $f^{-1}(s)$ is a proper submanifold of M with non-empty boundary.



(3) We consider the map

$$\begin{array}{rcl} f\colon \mathbb{R}^{n+1} & \to & \mathbb{R} \\ (x_1, \ldots, x_{n+1}) & \mapsto & x_1^2 + \cdots + x_{n+1}^2. \end{array}$$

Since $\mathrm{D}f_{(x_1,\ldots,x_{n+1})} = (2x_1 \ \ldots \ 2x_{n+1})$ we see that each point in $\mathbb{R}^{n+1} \setminus \{0\}$ is a regular point and each non-zero $r \in \mathbb{R}$ is a regular value of f. In particular it follows from the Regular Value Theorem 26.4 that $S^n = f^{-1}(1)$ is an n-dimensional submanifold of $\mathbb{R}^{n+1}$. This proof is significantly more pleasant than the original proof on page 553. Furthermore we see that for each $P \in S^n$ we have

$$\widetilde{\mathrm{T}}_P S^n \ = \ \ker(\mathrm{D}f_P\colon \mathbb{R}^{n+1} \to \mathbb{R}) \underset{\uparrow}{=} \{v \in \mathbb{R}^{n+1} \mid \langle v, P\rangle = 0\}.$$

by the above calculation of $\mathrm{D}f_P$

We have thus found a new proof of Lemma 23.1 (3). In Exercise 26.13 we determine for which $\in \mathbb{N}$ the orientation defined by the Regular Value Theorem 26.4 (3b) on S^n agrees with the standard orientation that we introduced on page 681.

(4) Let $g \in \mathbb{N}_0$. By [**Hirs1976**, p. 28] there exists a polynomial function $f\colon \mathbb{R}^3 \to \mathbb{R}$ such that 0 is a regular value and such that $f^{-1}(0)$ is diffeomorphic to the surface of genus g. For example, if we take

$$f(x,y,z) \ = \ (4x^2(1-x^2) - y^2)^2 + z^2 - \tfrac{1}{4},$$

then [**Hirs1976**, p. 28] states that $f^{-1}(0)$ is diffeomorphic to the surface of genus 2. This shows in particular that the surfaces of genus $g \in \mathbb{N}_0$ are diffeomorphic to submanifolds of $\mathbb{R}^3$. This was always "clear", but it is somewhat challenging to prove rigorously.

Proof of the Regular Value Theorem 26.4. Let M be an m-dimensional smooth manifold, let N be an n-dimensional smooth manifold, let $f\colon M \to N$ be a smooth map and let $s \in N \setminus \partial N$ be a regular value of f. We write $X = f^{-1}(s)$.

(1) Let $P \in X$. Since s is a regular value we know that P is a regular point.

Claim.

(α) If $P \in M \setminus \partial M$, then there exists an $(m-n)$-dimensional submanifold chart of type (α) around P.

(β) If $P \in \partial M$, then there exists an $(m-n)$-dimensional submanifold chart of type (β) around P.

Proof. First we consider the case that $P \in M \setminus \partial M$. This means that we can pick a chart of type (i) for $P \in M$. Since by hypothesis $s \in N \setminus \partial N$ we can also pick a chart of type (i) for $s \in N$. In fact this shows that we might as well assume that M is an open subset of $\mathbb{R}^m$ and that N is an open subset of $\mathbb{R}^n$. Furthermore after a translation we can assume that $s = 0$.

It follows from the fact that P is a regular point in $M \setminus \partial M$, together with the Tangent Space-Isomorphism Proposition 23.16, that the map $\mathrm{D}f_P \colon \mathbb{R}^m = \widetilde{\mathrm{T}}_P M \to \widetilde{\mathrm{T}}_s N = \mathbb{R}^n$ is an epimorphism. This implies that we can pick a linear map $\Theta \colon \mathbb{R}^m = \widetilde{\mathrm{T}}_P M \to \mathbb{R}^{m-n}$ such that the map

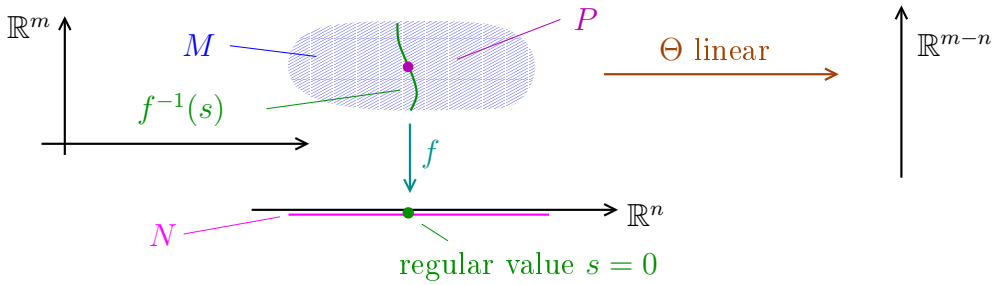
$$\mathrm{D}f_P \times \Theta \colon \underbrace{\widetilde{\mathrm{T}}_P M}_{=\mathbb{R}^m} \;\to\; \underbrace{\widetilde{\mathrm{T}}_s N}_{=\mathbb{R}^n} \times \mathbb{R}^{m-n} = \mathbb{R}^m$$

is an isomorphism. Next we consider the map

$$\Phi := f \times \Theta \colon M \;\to\; N \times \mathbb{R}^{m-n} \subset \mathbb{R}^m.$$

Note that $\mathrm{D}\Phi_P = \mathrm{D}f_P \times \Theta$ is an isomorphism. Thus it follows from the Inverse Mapping Theorem 24.1 that there exists an open neighborhood U of $P \in M$ and an open neighborhood V of $(0, \Theta(P)) \in N \times \mathbb{R}^{m-n}$ such that Φ restricts to a diffeomorphism $\Phi \colon U \to V$. We claim that Φ is a submanifold chart of type (α) for $P \in X = f^{-1}(0)$. In fact this follows immediately from the observation that given $Q \in U$ we have:

$$\Phi(Q) \in \{0\} \times \mathbb{R}^{m-n} \iff f(Q) = 0 \iff Q \in X = f^{-1}(0).$$



Now we consider the case that $P \in \partial M$. In this case we might as well assume that M is an open subset of $\mathbb{H}^m$ with $\partial M = M \cap \partial\, \mathbb{H}^m$. As above we can assume that N is an open subset of $\mathbb{R}^n$ and that $s = 0$. We denote by $\chi \colon \mathbb{R}^m \to \mathbb{R}$ the projection onto the last coordinate. Recall that it follows from the definition of a regular point on page 701 on the boundary, together with the Tangent Space-Isomorphism Proposition 23.16, that the map $\mathrm{D}f_P \colon \widetilde{\mathrm{T}}_P \partial M \to \widetilde{\mathrm{T}}_s N = \mathbb{R}^n$ is an epimorphism. Note that this implies that the map $\mathrm{D}f_P \times \chi \colon \mathbb{R}^m = \widetilde{\mathrm{T}}_P M \to \widetilde{\mathrm{T}}_s N \times \mathbb{R}$ is also an epimorphism. It follows that we can pick a linear map $\Theta \colon \mathbb{R}^m = \widetilde{\mathrm{T}}_P M \to \mathbb{R}^{m-n-1}$ such that the map

$$\mathrm{D}f_P \times \Theta \times \chi \colon \underbrace{\widetilde{\mathrm{T}}_P M}_{=\mathbb{R}^m} \;\to\; \underbrace{\widetilde{\mathrm{T}}_s N}_{=\mathbb{R}^n} \times \mathbb{R}^{m-n-1} \times \mathbb{R}$$

is an isomorphism. Next we consider the map

$$\Phi := f \times \Theta \times \chi \colon M \;\to\; N \times \mathbb{R}^{m-n-1} \times \mathbb{R}_{\geq 0} \;\subset\; \mathbb{H}^m.$$

The remainder of the argument is basically the same as above, except that this time we end up with a submanifold chart of type (β) for P. ⊞

We have thus found for each point in X a submanifold chart of type (α) or (β). This shows that X is an $(m-n)$-dimensional submanifold of M. Furthermore, note that $X = f^{-1}(s)$ is the preimage of a point, hence it is a closed subset of M. It follows from this observation and the fact that we only need submanifold charts of type (α) and (β) that X is in fact a proper submanifold in the sense of the definition on page 525.

(2) We denote by $i\colon X \to M$ the inclusion. Since $f \circ i\colon X \to N$ is a constant map it follows from the discussion on page 631 that for every $P \in X = f^{-1}(s)$ we have the inclusion

$$\operatorname{im}(\mathrm{D}i_P\colon \mathrm{T}_P X \to \mathrm{T}_P M) \;\subset\; \ker(\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_s N).$$

Furthermore we know the following:

- by the Physical Tangent Space-Inclusion Proposition 23.14 we know that $\mathrm{D}i_P$ is a monomorphism,
- by hypothesis we know that $\dim(\mathrm{T}_P M) = m$ and $\dim(\mathrm{T}_s N) = n$, furthermore by (1) we know that $\dim(\mathrm{T}_P X) = m - n$,
- by hypothesis $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_s N$ is an epimorphism.

It follows from the above observations that the above inclusion is in fact an equality.

Finally consider the case that M is a submanifold of some $\mathbb{R}^k$ and that N is a submanifold of some $\mathbb{R}^l$. By the Tangent Space-Isomorphism Proposition 23.16 we have the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{T}_P X & \xrightarrow{\mathrm{D}i_P} & \mathrm{T}_P M & \xrightarrow{\mathrm{D}f_P} & \mathrm{T}_s N \\
\cong\downarrow\Theta & & \cong\downarrow\Theta & & \cong\downarrow\Theta \\
\widetilde{\mathrm{T}}_P X & \xrightarrow{\mathrm{D}i_P} & \widetilde{\mathrm{T}}_P M & \xrightarrow{\mathrm{D}f_P} & \widetilde{\mathrm{T}}_s N.
\end{array}$$

The statement now follows from the above discussion and the fact, shown in the Physical Tangent Space-Inclusion Proposition 23.3, that the map $\mathrm{D}i_P\colon \widetilde{\mathrm{T}}_P X \to \widetilde{\mathrm{T}}_P M$ is an inclusion.

(3)

(a) We leave it to the reader to verify, using the chart from (1), that the given orientations on the tangent spaces $\mathrm{T}_P X$ do indeed define an orientation on the smooth manifold X.

(b) We need to prove that the two orientations on $\partial f^{-1}(s) = (f_\partial)^{-1}(s)$ agree. Thus let $P \in \partial f^{-1}(s)$. We work with the following orientations:

$\mathrm{T}_P(f^{-1}(s))$	with the orientation from (3a) applied to $f\colon M \to N$
$\mathrm{T}_P\,\partial(f^{-1}(s))$	with the boundary orientation from page 689
$\mathrm{T}_P(f_\partial)^{-1}(s))$	with the orientation from (3a) applied to $f_\partial\colon \partial M \to N$
$\mathrm{T}_P M$	with the given orientation
$\mathrm{T}_P\,\partial M$	with the boundary orientation from page 689.

To simplify the notation we view $\mathrm{T}_P f^{-1}(s)$ as a vector subspace of $\mathrm{T}_P M$. We pick an outward pointing vector $o \in \mathrm{T}_P f^{-1}(s)$. It follows from Exercise 23.8 that

o, viewed as a vector in $\mathrm{T}_P M$, is also outward pointing. Now let $u_1, \ldots, u_{m-n-1}$ be a basis of $\mathrm{T}_P(\partial f^{-1}(s)) = \mathrm{T}_P(f_\partial)^{-1}(s)$. We see:

$$\begin{array}{rll} & u_1, \ldots, u_{m-n-1} & \text{is a positive basis of } \mathrm{T}_P\, \partial(f^{-1}(s)) \\ \iff & o, u_1, \ldots, u_{m-n-1} & \text{is a positive basis of } \mathrm{T}_P f^{-1}(s) \\ \iff & o, u_1, \ldots, u_{m-n-1}, \widetilde{v}_1, \ldots, \widetilde{v}_n & \text{is a positive basis of } \mathrm{T}_P M \\ \iff & u_1, \ldots, u_{m-n-1}, \widetilde{v}_1, \ldots, \widetilde{v}_n & \text{is a positive basis of } \mathrm{T}_P\, \partial M \\ \iff & u_1, \ldots, u_{m-n-1} & \text{is a positive basis of } \mathrm{T}_P(f_\partial)^{-1}(s) \end{array}$$

(4) Let K be a compact subset of M. Since $X = f^{-1}(s)$ is a closed subset of M we know $X \cap K$ is a closed subset of K, hence $X \cap K$ is compact by the Compact-Closed Lemma 1.21. ■

26.4. **Matrix groups as smooth manifolds.** In this section we will see that several popular matrix groups are in fact smooth manifolds.

Convention.

(1) Given $m, n \in \mathbb{N}$ we will make the identification

$$\begin{array}{rcl} \mathrm{M}(m \times n, \mathbb{R}) & \to & \mathbb{R}^{m \cdot n} \\ A = (a_{ij}) & \mapsto & (a_{11}, \ldots, a_{1n}, \ldots, a_{m1}, \ldots, a_{mn}). \end{array}$$

In particular, given a smooth submanifold M of $\mathrm{M}(m \times n, \mathbb{R})$ we can consider the physical tangent space

$$\widetilde{\mathrm{T}}_P M \ \subset\ \mathrm{M}(m \times n, \mathbb{R}).$$

(2) We make the usual identification $\mathbb{C} = \mathbb{R}^2$ and combining these identifications we will identify $\mathrm{M}(m \times n, \mathbb{C})$ with $= \mathbb{C}^{m \cdot n} = \mathbb{R}^{2m \cdot n}$.

(3) Throughout this section we use the Tangent Space-Isomorphism Proposition 23.16, when appropriate, to identify the abstract tangent spaces $\mathrm{T}_P M$ with the physical tangent spaces $\widetilde{\mathrm{T}}_P M$.

We just saw made the identification $\mathrm{M}(m \times n, \mathbb{R}) = \mathbb{R}^{m \cdot n}$ which allows us to view $\mathrm{M}(m \times n, \mathbb{R})$ as an $m \cdot n$-dimensional smooth manifold. We can thus view the open subset $\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathrm{M}(n \times n, R) \,|\, \det(A) \neq 0\}$ also as an n^2-dimensional smooth manifold. The next results show that other matrix groups are also smooth manifolds, albeit in a less trivial way.

Proposition 26.5. (Matrix Group-Smooth Manifold Proposition) Let $n \in \mathbb{N}$.

(1) The special linear group

$$\mathrm{SL}(n, \mathbb{R}) \ = \ \{A \in \mathrm{M}(n \times n, \mathbb{R}) \,|\, \det(A) = 1\}$$

is an orientable proper (n^2-1)-dimensional submanifold of $\mathrm{M}(n \times n, \mathbb{R})$. Furthermore for every $A \in \mathrm{SL}(n, \mathbb{R})$ we have

$$\widetilde{\mathrm{T}}_A\, \mathrm{SL}(n, \mathbb{R}) \ = \ A \cdot \{B \in \mathrm{M}(n \times n, \mathbb{R}) \,|\, \mathrm{tr}(B) = 0\}.$$

(2) The special linear group

$$\mathrm{SL}(n, \mathbb{C}) \ = \ \{A \in \mathrm{M}(n \times n, \mathbb{C}) \,|\, \det(A) = 1\}$$

is an orientable proper $2(n^2-1)$-dimensional submanifold of $\mathrm{M}(n\times n,\mathbb{C})$. Furthermore for every $A\in \mathrm{SL}(n,\mathbb{C})$ we have

$$\widetilde{\mathrm{T}}_A\,\mathrm{SL}(n,\mathbb{C}) \;=\; A\cdot\{B\in \mathrm{M}(n\times n,\mathbb{C})\mid \mathrm{tr}(B)=0\}.$$

In the proof of the Matrix Group-Smooth Manifold Proposition 26.5 we will make use of the following lemma.

Lemma 26.6. Given any smooth map $\gamma\colon(-\epsilon,\epsilon)\to \mathrm{M}(n\times n,\mathbb{R})$ with $\gamma(0)\in \mathrm{GL}(n,\mathbb{R})$ we have
$$\tfrac{d}{dt}\det(\gamma(t))\big|_{t=0} \;=\; \det(\gamma(0))\cdot \mathrm{tr}\left(\gamma(0)^{-1}\cdot\gamma'(0)\right).$$

Proof of Lemma 26.6. We start out with the following claim.

Claim. The map $\det\colon \mathrm{M}(n\times n,\mathbb{R})\to\mathbb{R}$ is smooth and the differential at id is given by the map
$$\begin{array}{rcl}\mathrm{M}(n\times n,\mathbb{R}) & \to & \mathbb{R}\\ B & \mapsto & \mathrm{tr}(B).\end{array}$$

Proof. By the Leibniz formula the determinant of a matrix is a polynomial in the n^2 entries. It follows that the map $\det\colon \mathrm{M}(n\times n,\mathbb{R})\to\mathbb{R}$ is smooth. Next let $B=(b_{ij})\in \mathrm{M}(n\times n,\mathbb{R})$. Note that
$$\mathrm{D}\det_{\mathrm{id}}(B) \;=\; \tfrac{d}{dt}\det(\mathrm{id}+t\cdot B)\big|_{t=0}.$$
We now perform the following calculation:

$$\begin{aligned}\det(\mathrm{id}+t\cdot B) &= \sum_{\sigma\in S_n}(-1)^{\mathrm{sign}(\sigma)}\cdot\prod_{i=1}^{n}(\underbrace{\delta_{i\sigma(i)}}_{\substack{\text{Kronecker}\\ \text{delta}}}+t\cdot b_{i\sigma(i)})\\ &\quad\uparrow \text{ Laplace expansion}\\ &= \underbrace{\prod_{i=1}^{n}(1+t\cdot b_{ii})}_{\text{corresponds to }\sigma=\mathrm{id}} + \sum_{\mathrm{id}\neq\sigma\in S_n}(-1)^{\mathrm{sign}(\sigma)}\cdot\underbrace{\prod_{i=1}^{n}(\delta_{i\sigma(i)}+t\cdot b_{i\sigma(i)})}_{\substack{\text{since id}\neq\sigma\text{ we have }i\neq\sigma(i)\\ \text{for at least two } i\in\{1,\dots,n\}\\ \text{which implies that we can factor out } t^2}}\\ &= \prod_{i=1}^{n}(1+t\cdot b_{ii})+\text{terms of order at least } t^2\end{aligned}$$

It is now clear that if we take the derivative at $t=0$, then we obtain the trace of B. ⊞

Now let $\gamma\colon(-\epsilon,\epsilon)\to\mathrm{M}(n\times n,\mathbb{R})$ be a smooth map with $A:=\gamma(0)\in\mathrm{GL}(n,\mathbb{R})$. We see that
$$\tfrac{d}{dt}\det(\gamma(t))\big|_{t=0} \;=\; \det(A)\cdot\tfrac{d}{dt}\det(A^{-1}\cdot\gamma(t))\big|_{t=0} \;\underset{\substack{\uparrow\\ \text{by the Chain Rule 19.10 and the above claim}}}{=}\; \det(A)\cdot\mathrm{tr}(A^{-1}\cdot\gamma'(0)).$$
■

Proof of the Matrix Group-Smooth Manifold Proposition 26.5.

(1) First we show that $\mathrm{SL}(n,\mathbb{R})$ is an orientable proper (n^2-1)-dimensional submanifold of $\mathrm{M}(n\times n,\mathbb{R})$. By the Regular Value Theorem 26.4 it suffices to show that 1 is a regular value of the smooth map $\det\colon \mathrm{M}(n\times n,\mathbb{R})\to\mathbb{R}$. In fact we will show that any matrix $A\in\mathrm{M}(n\times n,\mathbb{R})$ with $\det(A)\neq 0$ is a regular point. This in turn follows from the observation that
$$\tfrac{d}{dt}\det((1+t)\cdot A)\big|_{t=0} \;=\; \tfrac{d}{dt}(1+t)^n\cdot\det(A)\big|_{t=0} \;=\; (n\cdot(1+t)^{n-1}\cdot\det(A))\big|_{t=0} = \underbrace{n\cdot\det(A)}_{\neq 0}.$$

Next let $A \in \mathrm{SL}(n,\mathbb{R})$. If $\gamma\colon (-\epsilon,\epsilon) \to \mathrm{SL}(n,\mathbb{R})$ is a smooth path with $\gamma(0) = A$, then it follows immediately from Lemma 26.6 that $\mathrm{tr}(A^{-1}\cdot\gamma'(0)) = 0$. In other words, we see that

$$\widetilde{\mathrm{T}}_A \,\mathrm{SL}(n,\mathbb{R}) \subset \{B \in \mathrm{M}(n\times n,\mathbb{R}) \,|\, \mathrm{tr}(A^{-1}\cdot B) = 0\} = A\cdot\{B \in \mathrm{M}(n\times n,\mathbb{R}) \,|\, \mathrm{tr}(B) = 0\}.$$

But both are vector spaces of dimension $n^2 - 1$, so we see that equality holds.

(2) First we show that $\mathrm{SL}(n,\mathbb{C}) = \{A \in \mathrm{M}(n\times n,\mathbb{C}) \,|\, \det(A) = 1\}$ is an orientable proper $2(n^2-1)$-dimensional submanifold of $\mathrm{M}(n\times n,\mathbb{C})$. This time we need to show that every matrix $A \in \mathrm{M}(n\times n,\mathbb{C})$ with $\det(A) \neq 0$ is in fact a regular point of $\det\colon \mathrm{M}(n\times n,\mathbb{C}) \to \mathbb{C} = \mathbb{R}^2$. Similarly to (1) we calculate that

$$\tfrac{d}{dt}\det((1+\mathrm{i}t)\cdot A)\big|_{t=0} = \tfrac{d}{dt}(1+\mathrm{i}t)^n\cdot\det(A)\big|_{t=0} = \mathrm{i}\cdot n\cdot\det(A).$$

Together with the calculation in (1) this shows that $\mathrm{D}\det_A\colon \mathrm{M}(n\times n,\mathbb{C}) \to \mathbb{C}$ is indeed an epimorphism. The remainder of the argument is similarly an adaptation of the argument provided in (1). We leave it to the reader to fill in the details. ■

The following proposition deals with much more interesting matrix groups.

Proposition 26.7. (Matrix Group-Smooth Manifold Proposition II) Let $n \in \mathbb{N}$.

(1) The set
$$\mathrm{O}(n) = \{A \in \mathrm{M}(n\times n,\mathbb{R}) \,|\, A^TA = \mathrm{id}\}$$
of orthogonal matrices is an orientable proper $\frac{1}{2}n(n-1)$-dimensional submanifold of $\mathrm{M}(n\times n,\mathbb{R}) = \mathbb{R}^{n^2}$. Furthermore for every $A \in \mathrm{O}(n)$ we have
$$\widetilde{\mathrm{T}}_A\,\mathrm{O}(n) = A\cdot\{B \in \mathrm{M}(n\times n,\mathbb{R}) \,|\, B = -B^T\}.$$
The analogous statements also hold for the component $\mathrm{SO}(n)$ of $\mathrm{O}(n)$.

(2) The set
$$\mathrm{U}(n) = \{A \in \mathrm{M}(n\times n,\mathbb{C}) \,|\, \overline{A}^TA = \mathrm{id}\}$$
of unitary matrices is similarly an orientable proper n^2-dimensional submanifold of $\mathrm{M}(n\times n,\mathbb{C}) = \mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$. Furthermore for every $A \in \mathrm{U}(n)$ we have
$$\widetilde{\mathrm{T}}_A\,\mathrm{U}(n) = A\cdot\{B \in \mathrm{M}(n\times n,\mathbb{C}) \,|\, B = -\overline{B}^T\}.$$

(3) The set
$$\mathrm{SU}(n) = \{A \in \mathrm{M}(n\times n,\mathbb{C}) \,|\, \overline{A}^TA = \mathrm{id} \text{ and } \det(A) = 1\}$$
of special unitary matrices is an orientable proper (n^2-1)-dimensional submanifold of $\mathrm{M}(n\times n,\mathbb{C}) = \mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$. Furthermore for every $A \in \mathrm{SU}(n)$ we have
$$\widetilde{\mathrm{T}}_A\,\mathrm{SU}(n) = A\cdot\{B \in \mathrm{M}(n\times n,\mathbb{C}) \,|\, B = -\overline{B}^T \text{ and } \mathrm{tr}(B) = 0\}.$$

All of the above examples, viewed as smooth manifolds in their own right, are closed.

Proof of the Matrix Group-Smooth Manifold Proposition 26.7 for $\mathrm{O}(n)$**.** We start out our proof of the Matrix Group-Smooth Manifold Proposition 26.7 (1) with the following claim.

Claim.

(i) The set
$$\mathrm{Sym}(n,\mathbb{R}) = \{B \in \mathrm{M}(n\times n,\mathbb{R}) \,|\, B^T = B\} = \{(b_{ij}) \in \mathrm{M}(n\times n,\mathbb{R}) \,|\, b_{ij} = b_{ji}\}$$

is a submanifold of $\mathrm{M}(n\times n,\mathbb{R}) = \mathbb{R}^{n^2}$ of dimension $\frac{1}{2}n(n+1)$.

(ii) The map
$$\begin{array}{rccc} f\colon & \mathrm{M}(n\times n,\mathbb{R}) & \to & \mathrm{Sym}(n,\mathbb{R}) \\ & A & \mapsto & A^TA \end{array}$$
is smooth and id is a regular value.

Proof. Statement (i) follows immediately from Lemma 23.1 (2). We turn to the proof of Statement (ii). It should be clear that f is smooth and that f does indeed take values in $\mathrm{Sym}(n,\mathbb{R})$. Furthermore by definition we have $\mathrm{O}(n) = f^{-1}(\mathrm{id})$. So it remains to show that id is a regular value.

In the following we again use Lemma 23.1 (2) which implies that the physical tangent spaces of the submanifolds $\mathrm{Sym}(n,\mathbb{R})$ and $\mathrm{M}(n\times n,\mathbb{R})$ are just given by these vector subspaces. So let $A\in\mathrm{M}(n\times n,\mathbb{R})$ with $A^TA=\mathrm{id}$ and let $C\in \mathrm{T}_{\mathrm{id}}\,\mathrm{Sym}(n,\mathbb{R}) = \mathrm{Sym}(n,\mathbb{R})$. We need to show that there exists a B in $\mathrm{T}_A\,\mathrm{M}(n\times n,\mathbb{R}) = \mathrm{M}(n\times n,\mathbb{R})$ with $\mathrm{D}f_A(B) = C$.

We start out with the following little calculation: Given $B\in\mathrm{M}(n,\times n,\mathbb{R})$ we have

$$\begin{aligned} \tfrac{d}{dt}f(A+tB)\big|_{t=0} &= \tfrac{d}{dt}(A^T+tB^T)\cdot(A+tB)\big|_{t=0} \\ &= \tfrac{d}{dt}\big(A^TA+tA^TB+tB^TA+t^2B^TB\big)\big|_{t=0} \;=\; A^TB+B^TA. \end{aligned}$$

Using that $A^T\cdot A = \mathrm{id}$ and $C = C^T$ it is now easy to see that $B := \frac{1}{2}A\cdot C$ satisfies $\mathrm{D}f_A(B) = C$. ⊞

It follows from the claim, together with the Regular Value Theorem 26.4, that $\mathrm{O}(n)$ is an orientable proper submanifold of $\mathrm{M}(n\times n,\mathbb{R}) = \mathbb{R}^{n^2}$ of dimension $n^2-\frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$. Since $\mathbb{R}^{n^2}$ has no boundary and since $\mathrm{O}(n)$ is a proper submanifold we see that the boundary of $\mathrm{O}(n)$ is also empty. Finally note that in the Matrix Group Lemma 2.36 (3) we showed that $\mathrm{O}(n)$ is compact. These last two observations imply that $\mathrm{O}(n)$ is in fact a closed smooth manifold.

It remains to prove the following claim.

Claim. For every $A\in\mathrm{O}(n)$ we have the following equality:
$$\widetilde{\mathrm{T}}_A\,\mathrm{O}(n) \;=\; A\cdot\{B\in\mathrm{M}(n\times n,\mathbb{R})\,|\,B=-B^T\}.$$

Proof. Note that both sides are vector subspaces of $\mathrm{M}(n\times n,\mathbb{R})$ of dimension $\frac{1}{2}n(n-1)$. Thus it suffices to prove the "$\subset$"-inclusion. Let $X\in\widetilde{\mathrm{T}}_A\,\mathrm{O}(n)$. We pick a smooth map $\gamma\colon(-\epsilon,\epsilon)\to\mathrm{O}(n)$ with $\gamma(0)=A$ and $\gamma'(0)=X$. We make the following observations:

$$\begin{aligned} \gamma(t)\in\mathrm{O}(n)\text{ for all }t &\implies \gamma(t)^T\cdot\gamma(t) = \mathrm{id}\text{ for all }t \\ &\implies (\gamma(t)^T)'\cdot\gamma(t)+\gamma(t)^T\cdot\gamma(t)' = 0\text{ for all }t \\ &\implies X^T\cdot A + A^T\cdot X = 0 \\ &\implies (A^T\cdot X)^T + A^T\cdot X = 0 \\ &\implies X = \underbrace{A\cdot A^T}_{=\mathrm{id},\text{ since }A\in\mathrm{O}(n)}\cdot X \text{ where } B := A^T\cdot X \text{ satisfies } B=-B^T. \end{aligned}$$

■

Proof of the Matrix Group-Smooth Manifold Proposition 26.7 for $\mathrm{U}(n)$ and $\mathrm{SU}(n)$. The proof for $\mathrm{U}(n)$ is almost verbatim the same as the proof for $\mathrm{O}(n)$. The proof for $\mathrm{SU}(n)$ follows from combining the ideas for the proof of $\mathrm{U}(n)$ with the ideas for the proof of the Matrix Group-Smooth Manifold Proposition 26.5 (2) where we showed that $\mathrm{SL}(n,\mathbb{C})$

is an (n^2-1)-dimensional submanifold of $\mathrm{M}(n\times n,\mathbb{R})$. We will deal with $\mathrm{SL}(n,\mathbb{C})$ in Exercise 26.12. We leave the case $\mathrm{SL}(n,\mathbb{C})$ to the reader. ■

Exercises for Chapter 26.

Exercise 26.1. Let M be a closed smooth manifold and let N be a connected smooth manifold. Let $f\colon M\to N$ be a map. Show that if f has no critical values, then f is surjective.
Hint. First consider the case that $\dim(M)=\dim(N)$.

Exercise 26.2. Let $p(z)=a_0+a_1z+\cdots+a_nz^n$ be a complex polynomial of degree n. We consider the map

$$\begin{array}{rcl} p\colon \mathbb{C} & \to & \mathbb{C} \\ w & \mapsto & p(w). \end{array}$$

Show that there exists a finite subset $Y\subset\mathbb{C}$ such that for every $z\notin Y$ we have the equality $\#p^{-1}(w)=n$.

Exercise 26.3. We consider

$$M := \{(x,y,z)\in\mathbb{R}^3 \mid x^2-y^2-z^2=1\} \qquad \text{and} \qquad N := \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2-z^2=1\}.$$

(a) Show that M and N are 2-dimensional smooth submanifolds of $\mathbb{R}^3$.
(b) Sketch M and N.

Exercise 26.4. Let A be a real symmetric $n\times$n-matrix.

(a) Show that $\{v\in\mathbb{R}^n \mid \langle v, A\cdot v\rangle = 1\}$ is an $(n-1)$-dimensional smooth submanifold of $\mathbb{R}^n$.
(b) Under what condition on the eigenvalues of A is the submanifold in (a) compact?

Exercise 26.5. We consider

$$T := \{(x,y,z)\in\mathbb{R}^3 \mid (\sqrt{x^2+y^2}-2)^2+z^2=1\}.$$

(a) Show that T is rotationally symmetric around the z-axis.
(b) Provide a sketch of T.
(c) Show that T is a submanifold of $\mathbb{R}^3$.
(d) In Exercise 24.15 we showed that the map

$$\begin{array}{rcl} f\colon \mathbb{R}^2/(2\pi\mathbb{Z})^2 & \to & \mathbb{R}^3 \\ (\varphi,\theta) & \mapsto & \underbrace{\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation around the } z\text{-axis}} \cdot \underbrace{\begin{pmatrix} 2+\sin(\theta) \\ 0 \\ \cos(\theta) \end{pmatrix}}_{\substack{\text{describes a circle} \\ \text{in the } xz\text{-plane}}} \end{array}$$

is a smooth embedding. Show that T is diffeomorphic to $f(\mathbb{R}^2/(2\pi\mathbb{Z})^2)$.

Exercise 26.6. Let M be an m-dimensional smooth manifold and let $f\colon M\to\mathbb{R}$ be a smooth function.

(a) Let $s \in \mathbb{R}$ be a regular value of f such that for each component C of ∂M we have either $f^{-1}(s) \cap C = C$ or $f^{-1}(s) \cap C = \varnothing$. Show that $X := f^{-1}((-\infty, s])$ is a smooth codimension-zero submanifold of M with $\partial_0 X = f^{-1}(s) \setminus \partial M$ and $\partial_1 X = X \cap \partial M$.

(b) Let $s < t \in \mathbb{R}$ be regular values such that for each component C of ∂M we have either $f^{-1}(s) \cap C = C$ or $f^{-1}(s) \cap C = \varnothing$ and such that for each component C of ∂M we have either $f^{-1}(t) \cap C = C$ or $f^{-1}(t) \cap C = \varnothing$. Show that $X := f^{-1}([s,t])$ is a smooth codimension-zero submanifold of M with $\partial_0 X = (f^{-1}(s) \cup f^{-1}(t)) \setminus \partial M$ and $\partial_1 X = X \cap \partial M$.

Exercise 26.7. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth map and let $z \in \mathbb{R}$ be a regular value. We set $M := f^{-1}(z)$. Given $P \in \mathbb{R}^n$ we denote by

$$\operatorname{grad} f_P \;:=\; (\mathrm{D}f_P)^T \;=\; \big(\tfrac{\partial f}{\partial x_1}\big|_P, \ldots, \tfrac{\partial f}{\partial x_n}\big|_P\big) \;\in\; \mathbb{R}^n$$

the gradient. Show that for every $P \in M$ we have

$$\widetilde{\mathrm{T}}_P M \;=\; (\operatorname{grad} f_P)^\perp \;=\; \{v \in \mathbb{R}^n \,|\, \langle v, \operatorname{grad} f_P\rangle = 0\}.$$

Exercise 26.8. Let $m, n \in \mathbb{N}$.

(a) Show that

$$\mathrm{O}(m,n) \;:=\; \text{set of } n\text{-tuples of orthonormal vectors of } \mathbb{R}^m$$

is a submanifold of $\mathrm{M}(m \times n, \mathbb{R})$.

(b) Given $A \in \mathrm{O}(m,n)$ determine $\widetilde{\mathrm{T}}_A \mathrm{O}(m,n) \subset \mathrm{M}(m \times n, \mathbb{R})$.

Exercise 26.9. Let $A \in \mathrm{M}(n \times n, \mathbb{R})$ be a symmetric matrix and let $v \in \mathbb{R} \setminus \{0\}$. Show that the quadric

$$M \;:=\; \{x \in \mathbb{R}^n \,|\, \langle x, A \cdot x\rangle = v\}$$

is an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$.

Exercise 26.10. Let $d \in \mathbb{N}_0$. Show that

$$M_d \;:=\; \Big\{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \,\Big|\, z_0^d + \cdots + z_n^d = 0 \text{ and } \sum_{k=0}^{n} |z_i|^2 = 2\Big\}$$

is a $(2n-1)$-dimensional submanifold of $\mathbb{C}^{2n+1}$.
Remark. These smooth manifolds are sometimes called *Brieskorn manifolds*.

Exercise 26.11. Any matrix $A \in \mathrm{M}(n \times n, \mathbb{C})$ defines an endomorphism of $\mathbb{C}^n = \mathbb{R}^{2n}$. Thus we can view A also as a matrix in $\mathrm{M}(2n \times 2n, \mathbb{R})$.

(a) Show that with the above convention $\mathrm{U}(n)$ is a subgroup of $\mathrm{O}(2n)$.

(b) Show that $\mathrm{U}(n)$ is a submanifold of $\mathrm{O}(2n)$.

Exercise 26.12.

(a) Show that

$$\mathrm{SU}(n) \;=\; \big\{A \in \mathrm{M}(n \times n, \mathbb{C}) \,\big|\, \overline{A}^T A = \mathrm{id} \text{ and } \det(A) = 1\big\}$$

is a smooth manifold of dimension $n^2 - 1$.

(b) Show that for every $A \in \mathrm{SU}(n)$ we have

$$\widetilde{\mathrm{T}}_A \mathrm{SU}(n) \;=\; A \cdot \{B \in \mathrm{M}(n \times n, \mathbb{C}) \,|\, B = -\overline{B}^T \text{ and } \operatorname{tr}(B) = 0\}.$$

Exercise 26.13. We consider the map

$$\begin{array}{rcl} f\colon \mathbb{R}^{n+1} & \to & \mathbb{R} \\ (x_1, \dots, x_{n+1}) & \mapsto & x_1^2 + \dots + x_{n+1}^2. \end{array}$$

We equip $\mathbb{R}^{n+1}$ and $\mathbb{R}$ with the standard orientations. For which $n \in \mathbb{N}$ does the orientation on the sphere $S^n = f^{-1}(1)$ that is defined by the Regular Value Theorem 26.4 (3b) agree with the standard orientation on S^n that we introduced on page 681.

Exercise 26.14.

(a) Let $A \subset \mathrm{M}(m \times n, \mathbb{R})$. Show that A has rank k if and only if there exist k columns and k rows such that the $k \times k$-matrix that is given by entries corresponding to these rows and columns has non-zero determinant.

(b) Show that $X := \{A \in \mathrm{M}(m \times n, \mathbb{R}) \mid A \text{ has rank } k\}$ is a submanifold of $\mathrm{M}(m \times n, \mathbb{R})$ of dimension $(m-k) \cdot (n-k)$.
Hint. Construct explicit charts around each $A \in X$. To preserve sanity it is helpful to start with $A \in X$ which has the property that the top-left $k \times k$-submatrix has non-zero determinant.

Exercise 26.15. Let $k \in \mathbb{N}$ and let M be a smooth manifold with empty boundary. We refer to

$$\mathrm{Conf}_k(M) \;:=\; \{(x_1, \dots, x_k) \in M^k \mid \text{ for all } i \neq j \text{ we have } x_i \neq x_j\}$$

as the *configuration space of k points in M*.

(a) Show that $\mathrm{Conf}_k(M)$ is an open subset of M. This shows that $\mathrm{Conf}_k(M)$ is a smooth manifold of dimension $\dim(M) \cdot k$.

(b) We now restrict ourselves to $k = 3$ and $M = \mathbb{R}^2$.

(i) We consider the map

$$\begin{array}{rcl} f\colon \mathrm{Conf}_3(\mathbb{R}^2) & \to & \mathbb{R} \\ (x_1, x_2, x_3) & \mapsto & \det \underbrace{(x_2 - x_1 \;\; x_3 - x_1)}_{2 \times 2\text{-matrix}}. \end{array}$$

Show that 0 is a regular value of f.

(ii) We say that $(x_1, x_2, x_3) \in \mathrm{Conf}_3(\mathbb{R}^2)$ are collinear if there exists a line that contains all three points. Show that the set of collinear (x_1, x_2, x_3) is a submanifold of $\mathrm{Conf}_3(\mathbb{R}^2)$ of dimension 5.

27. The Euclidean Embedding Theorem

27.1. **The statement of the Euclidean Embedding Theorem.** For the reader's convenience we recall the following definitions from pages 152, 651 and 650.

Definition.

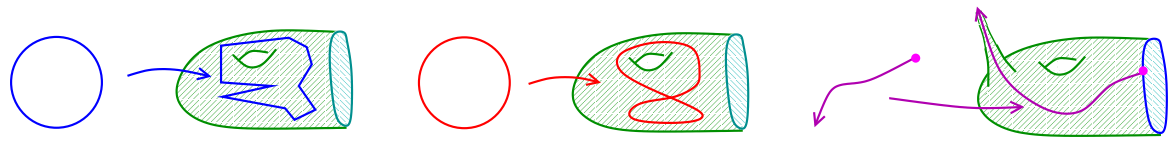
(1) A map $\varphi\colon Y \to X$ between topological spaces is called an embedding if $\varphi\colon Y \to \varphi(Y)$ is a homeomorphism. (Here we view $\varphi(Y)$ as a topological space equipped with the subspace topology coming from X.)

(2) A map $\varphi\colon N \to M$ between two smooth manifolds is called an immersion if the following conditions are satisfied:
   (a) φ is smooth,
   (b) for each $P \in N$ the differential $\mathrm{D}\varphi_P\colon \mathrm{T}_P N \to \mathrm{T}_{\varphi(P)} M$ is a monomorphism,
   (c) If $\partial M \neq \varnothing$, then we require that also the following conditions are satisfied:
      (i) For each $P \in N \setminus \partial N$ we have $\varphi(P) \in M \setminus \partial M$.
      (ii) For each $P \in \partial N$ with $\varphi(P) \in \partial M$ the image $\mathrm{D}\varphi_P(\mathrm{T}_P N)$ is *not* contained in $\mathrm{T}_{\varphi(P)}(\partial M)$.
      (iii) For every component C of ∂N we have $\varphi(C) \subset \partial M$ or $\varphi(C) \subset M \setminus \partial M$.

   An immersion that also satisfies the following extra condition
   (d) $\varphi\colon N \to M$ is an embedding,
   is called a smooth embedding.

(3) Let M and N be two topological manifolds. A map $\varphi\colon N \to M$ is called proper if the following two conditions are satisfied:[281]
   (a) $\varphi^{-1}(\partial M) = \partial N$,
   (b) $\varphi(N)$ is a closed subset of M.

(4) Given $n \in \mathbb{N}$ we write $\mathbb{H}^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_n \geq 0\}$.



embedding immersion proper smooth embedding

We can now formulate the main theorem of this chapter.

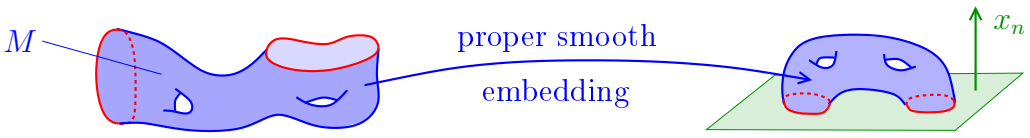
Theorem 27.1. (Euclidean Embedding Theorem)

(1) Given any topological manifold M there exists an $n \in \mathbb{N}$ and a proper embedding $M \to \mathbb{H}^n$.

(2) Given any smooth manifold M there exists an $n \in \mathbb{N}$ and a proper smooth embedding $M \to \mathbb{H}^n$.

We will provide the proof of the Euclidean Embedding Theorem 27.1 in the following sections.

[281] On page 651 we gave a short discussion of the unfortunate fact that the adjective "proper" is oversubscribed.



It is already worth pointing out that the proof of the Euclidean Embedding Theorem 27.1 breaks into two cases:

(1) In the compact case the proof of the Euclidean Embedding Theorem 27.1 is fairly straightforward: using a partition of unity subordinate to a finite atlas it is fairly straightforward to cobble together an argument. We refer to Section 27.2 for full details.

(2) The non-compact case is significantly harder. Our approach relies on the notion of the covering dimension that we had developed in Chapter 11. In Section 27.3 we will consider the covering dimension of topological manifolds. In particular in the Covering Dimension-of-Manifolds Proposition 27.7 we will show that the covering dimension of an n-dimensional topological manifold is $\leq n$. We will use this fact, together with the Covering Dimension Refinement Theorem 11.12, to show in the Small Atlas Proposition 27.8 that every n-dimensional topological manifold admits an atlas consisting of $\leq n+1$ charts. Using this result, together with the Covering Dimension Refinement Theorem 11.12, it is then straightforward to show that every non-compact topological manifolds admits a proper embedding into some $\mathbb{H}^n$. Furthermore the last two statements also work for smooth manifolds and provide us with a smooth atlas consisting of $\leq n+1$ charts and we obtain a proper smooth embedding $M \to \mathbb{H}^n$.

Remark. The fact that every non-compact smooth manifold (at least with empty boundary) admits a proper smooth embedding into some $\mathbb{H}^n$ is also proved, using a different approach, in [**Wall2016**, Corollary 4.7.8] and [**Lee2002**, Theorem 6.15]. We will sketch the argument provided in [**Lee2002**, Theorem 6.15] later on in Exercise **??**.

In the remainder of this section we take the Euclidean Embedding Theorem 27.1 for granted and we discuss a few consequences.

Corollary 27.2. (Diffeomorphic-to-Euclidean Submanifold Corollary) Given any smooth manifold M there exists an $n \in \mathbb{N}$ such that M is diffeomorphic to a proper smooth submanifold of $\mathbb{H}^n$.

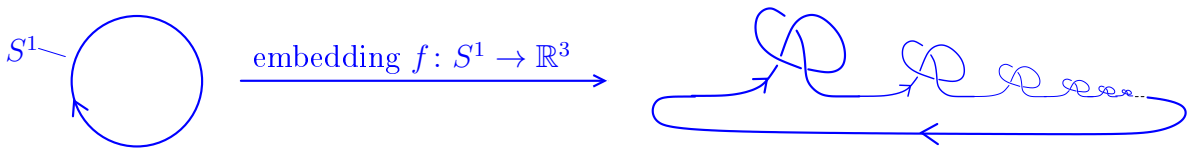
Proof. Let M be a smooth manifold. By the Euclidean Embedding Theorem 27.1 there exists an $n \in \mathbb{N}$ and a proper smooth embedding $M \to \mathbb{H}^n$. Furthermore by the Smooth Embedding Theorem 24.10 we know that $\varphi(N)$ is a proper n-dimensional smooth submanifold of $\mathbb{H}^n$ and that the map $\varphi\colon N \to \varphi(N)$ is a diffeomorphism. ■

Remark. Note that the logic of the Diffeomorphic-to-Euclidean Submanifold Corollary 27.2 does *not* apply to topological manifolds. For example let us consider the embedding $f\colon S^1 \to \mathbb{R}^3$ that is shown in the figure below and that we encountered already on page 659. The map $f\colon S^1 \to \mathbb{R}^3$ is an embedding but, as we will see in Proposition 124.15, the image

$f(S^1)$ is not a topological submanifold of $\mathbb{R}^3$. But it turns out that the situation can mostly be fixed:

(1) Let X and Y be topological manifolds with $\dim(Y) \geq \dim(X) + 3$. It was shown by Richard Miller [**MillR1972**, Theorem 5] (see also the work of Aleksei Chernavskii [**Cherna1969b**, Theorem 1] and M. A. Stan'ko [**Stan1973**, Theorem 1]) that given any embedding $f\colon X \to Y$, given any metric d on Y and given any $\epsilon > 0$ there exists an embedding $g\colon X \to Y$ such that $g(X)$ is a submanifold of Y and such that for all $x \in X$ we have $d(f(x), g(x)) < \epsilon$. It follows from this fact, together with the Euclidean Embedding Theorem 27.1, Exercise 2.62 and the Proper-Closed Proposition 2.43 that every topological manifold of dimension ≥ 2 is homeomorphic to a topological submanifold of $\mathbb{R}^{2n+1}$ that is a closed subset of $\mathbb{R}^{2n+1}$.[282]

(2) If X is a 1-dimensional topological manifold, then we know by the Topological 1-Dimensional Manifold Classification Theorem 22.1 that each component of X is homeomorphic to one of S^1, $[0,1]$, $[0,1)$ or $(0,1)$. Since topological manifolds are second-countable they have, see Exercise 9.1, only countably many components. So it is straightforward to show that X is homeomorphic to a proper submanifold of $\mathbb{H}^2$.



Corollary 27.3. As usual we denote by $\mathcal{SmMfd}$ the category of smooth manifolds and we denote by $\mathcal{SmMfd}_{\mathbb{R}^n}$ the category of smooth submanifolds of some $\mathbb{R}^n$. The functor $\mathcal{SmMfd}_{\mathbb{R}^n} \to \mathcal{SmMfd}$ that is given by

$$\begin{array}{rcl} M & \mapsto & (M, \text{standard smooth structure as defined on page 552}) \\ (f\colon M \to N) & \mapsto & (f\colon M \to N) \end{array}$$

is a weak equivalence of categories in the sense of the definition on page 207.

Proof. The Diffeomorphic-to-Euclidean Submanifold Corollary 27.2 says that the given functor is essentially surjective in the sense of the definition on page 207. The remaining conditions of the Weak Equivalence of Categories Proposition 3.11 are satisfied basically by definition. Thus we obtain from said proposition that the functor is a weak equivalence of categories. ■

In the Manifold Metrization Proposition 18.10 we showed, using the mighty Urysohn Metrization Theorem 9.4 and the less mighty the Completely Metrizable Criterion Proposition 10.8 that every topological manifold is completely metrizable. In the following proposition we will use the Euclidean Embedding Theorem 27.1 to provide a new proof of this statement, that does not rely on the Urysohn Metrization Theorem 9.4. In fact for a smooth manifold M we can even strengthen the earlier result, namely we can now find

[282] The literature does not give control over the behavior of the boundary so we cannot arrange that we get a proper embedding of $\mathbb{R}^{2n+1}$. This burning issue could surely be resolved if somebody went back to the original literature and upgraded the results.

a metric $d\colon M \times M \to \mathbb{R}_{\geq 0}$ that is smooth away from the "diagonal", i.e. away from $\{(x,x) \in M \times M \,|\, x \in M\}$.

Proposition 27.4. (Manifold Metrization Proposition)

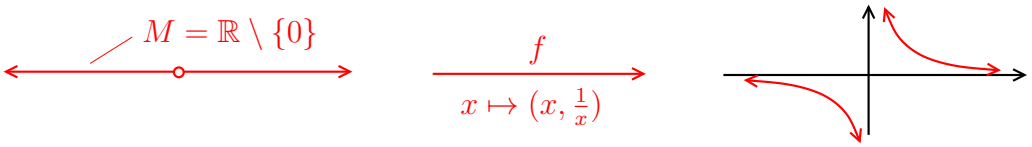
(1) Every topological manifold M is *completely* metrizable, i.e. there exists a *complete* metric d on M such that the given topology agrees with the topology coming from d.
(2) Every smooth manifold M admits a complete metric $d\colon M \times M \to \mathbb{R}_{\geq 0}$ such that the restriction of d to the open subset $\{(x,y) \in M \times M \,|\, x \neq y\}$ is smooth.

Remark. As we will see shortly, the Manifold Metrization Proposition 27.4 (1) is a straightforward consequence of the Euclidean Embedding Theorem 27.1 (1). As we mentioned above, for compact topological manifolds we provide in Section 27.2 an elementary proof of the Euclidean Embedding Theorem 27.1. On the other hand, for non-compact topological manifolds the proof of the Euclidean Embedding Theorem 27.1 relies on our detailed discussion of the covering dimension of topological manifolds in Chapter 11. For readers who are not familiar with the theory of covering dimensions we outline an alternative proof of the Manifold Metrization Proposition 27.4 (1) in Exercise 27.2.

Example. Let $M = \mathbb{R} \setminus \{0\}$. The usual Euclidean metric on $\mathbb{R}$ shows that M is metrizable, but this metric is of course not complete. The map

$$\begin{array}{rcl} f\colon \mathbb{R} \setminus \{0\} & \to & \mathbb{R}^2 \\ x & \mapsto & (x, \frac{1}{x}) \end{array}$$

is easily seen to be a proper smooth embedding. Thus $\mathbb{R} \setminus \{0\}$ is homeomorphic to a closed subset of $\mathbb{R}^2$. Now the Euclidean metric on $\mathbb{R}^2$ restricts to a *complete* metric on $f(\mathbb{R} \setminus \{0\})$ and thus it gives rise to a complete metric on $\mathbb{R} \setminus \{0\}$.



Proof. For each $n \in \mathbb{N}_0$ we consider the usual Euclidean metric

$$\begin{array}{rcl} e\colon \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ ((x_1,\dots,x_n),(y_1,\dots,y_n)) & \mapsto & \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}. \end{array}$$

(1) Let M be a topological manifold. By the Euclidean Embedding Theorem 27.1 (1) there exists an $n \in \mathbb{N}$ and a closed embedding $\varphi\colon M \to \mathbb{R}^n$. In other words, M is homeomorphic to the closed subset $\varphi(M) \subset \mathbb{R}^n$. It follows from the elementary Exercise 1.3 that the Euclidean metric e on $\mathbb{R}^n$ restricts to a metric e on $\varphi(M)$ such that the topology coming from e agrees with the topology given by $\varphi(M)$. Finally note that the fact that $\varphi(M)$ is closed in $\mathbb{R}^n$ implies, by the Closed Subspace-Completeness Lemma 6.9, that the metric space $\varphi(M)$ itself is complete.
(2) Let M be a smooth manifold. By the Euclidean Embedding Theorem 27.1 (2) there exists an $n \in \mathbb{N}$ and a closed smooth embedding $\varphi\colon M \to \mathbb{R}^n$. A very elementary argument shows that $e\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a smooth map on the open subset set $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, x \neq y\}$. In (1) we showed that the map $e \circ (\varphi \times \varphi)\colon M \times M \to \mathbb{R}_{\geq 0}$

is a complete metric. Since φ is smooth it follows from the above discussion together with Propositions 19.31 and 20.6 that $e \circ (\varphi \times \varphi)$ is smooth on the open subset $\{(x, y) \in M \times M \,|\, x \neq y\}$. ■

Remark.

(1) In the following chapters we will study several refinements of the Euclidean Embedding Theorem 27.1 for compact smooth manifolds:
   (a) In the Weak Whitney Embedding Theorem 30.7 we will show that any n-dimensional smooth manifold admits a proper smooth embedding into $\mathbb{H}^{2n+1}$.
   (b) In Proposition **??** we will show that any two proper smooth embeddings $M \to \mathbb{H}^n$ are "properly smoothly isotopic" as long as $n > 2\dim(M) + 1$.
   (c) In Proposition **??** we will consider a refinement of Theorem 27.1 for smooth manifolds M that are equipped with a proper submanifold W.

(2) The Euclidean Embedding Theorem 27.1 is not only interesting in its own right, for us the fact that every smooth manifold can be viewed as a proper submanifold of $\mathbb{H}^n$ will come in handy on many occasions. One of the nice consequences of considering submanifolds of $\mathbb{H}^n$ is that it makes it relatively easy to construct many interesting and useful maps. In particular we will use the Euclidean Embedding Theorem 27.1, together with the Neighborhood Projection Theorem 28.1, to prove the following results:
   (a) the Smooth Collar Neighborhood Theorem 28.3,
   (b) the Whitney Approximation Theorem 29.1 and
   (c) the Smooth Manifold-Simplicial Structure Theorem 96.2.

As happens so often in mathematics, the proof of a result immediately invites new questions. For example, in light of the Euclidean Embedding Theorem 27.1 and the Weak Whitney Embedding Theorem 30.7 the following question arises.

Question 27.5. For a given (topological) smooth manifold M, e.g. $M = \mathbb{R}\mathrm{P}^n$, what is the minimal $k \in \mathbb{N}$ such that M admits a (topological) smooth embedding into $\mathbb{R}^k$?

We will give some partial answers for general smooth manifolds in the Weak Whitney Embedding Theorem 30.7 and the Strong Whitney Embedding Theorem **??**. Furthermore we will consider some specific cases of topological and smooth manifolds in Propositions 43.12, 123.11 and the Topological Codimension-One Orientability Proposition 124.5.

27.2. **Proof of the Euclidean Embedding Theorem 27.1 in the compact case.** We start out with the following elementary lemma.

Lemma 27.6. Let M be a compact m-dimensional topological manifold. There exists a finite atlas $\{\Phi_i \colon U_i \to V_i\}_{i=1,\dots,s}$ such that each V_i is an open subset of $\mathbb{H}^m$. If M is a smooth manifold, then we can choose the atlas to be smooth.

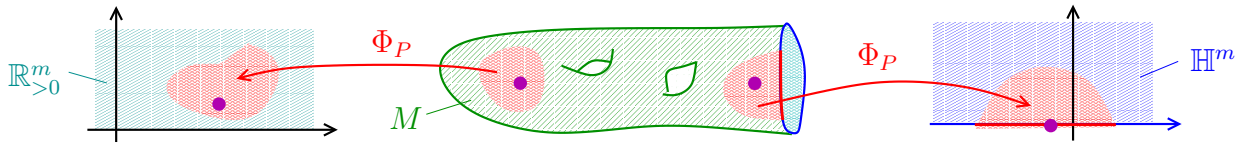
Remark. In the next section we will generalize Lemma 27.6 to the non-compact case.

Proof. Let M be a compact m-dimensional topological manifold.

(1) For each $P \in M \setminus \partial M$ we can and will pick a chart $\Phi_P \colon U_P \to V_P$ of type (i) around P with $V_P \subset \mathbb{R}^m_{>0} = \{(x_1, \dots, x_m) \in \mathbb{R}^m \,|\, x_m > 0\} \subset \mathbb{H}^m$.

(2) For each $P \in \partial M$ we pick a chart $\Phi_P \colon U_P \to V_P$ of type (ii) around P, where V_P is an open subset of $\mathbb{H}^m$.

Since the sets $\{U_P\}_{P\in M}$ form an open cover, the desired statement follows immediately from the hypothesis that M is compact. If M is in fact smooth, then we pick charts from the smooth atlas and we are again done. ■



Proof of the Euclidean Embedding Theorem 27.1 in the compact case. In the following we go out of order, we first deal with the slightly more difficult smooth case and then we argue that the topological case follows by essentially the same argument as the smooth case - except for a technical nuisance.

(2) Let M be a compact m-dimensional smooth manifold. By Lemma 27.6 there exists a finite smooth atlas $\{\Phi_i\colon U_i \to V_i\}_{i=1,\dots,s}$ such that each V_i is an open subset of $\mathbb{H}^m$. For $i = 1,\dots,s$ we denote by $\zeta_i\colon U_i \to \mathbb{R}$ the composition of Φ_i with the projection $\mathbb{H}^m \to \mathbb{R}_{\geq 0}$ onto the last coordinate.

By the Manifold-Partitions of Unity Existence Theorem 21.1 we can find a smooth partition of unity $\{\nu_i\colon M \to [0,1]\}_{i=1,\dots,s}$ such that for every $i \in \{1,\dots,s\}$ we have $\operatorname{supp}(\nu_i) \subset U_i$. For $i = 1,\dots,s$ we define

$$\widetilde{\Phi}_i\colon M \to \mathbb{R}^m,\quad x \mapsto \begin{cases} \Phi_i(x)\cdot\nu_i(x), & \text{if } x\in U_i,\\ 0, & \text{else}\end{cases} \qquad \text{and} \qquad \widetilde{\zeta}_i\colon M \to \mathbb{R},\quad x \mapsto \begin{cases} \zeta_i(x)\cdot\nu_i(x), & \text{if } x\in U_i,\\ 0, & \text{else.}\end{cases}$$

It follows easily from the Smooth Pasting Proposition 19.23 (1) that each of these maps $\widetilde{\Phi}_i$ and $\widetilde{\zeta}_i$ is smooth. It remains to prove the following claim.

Claim. The map

$$\begin{aligned}\varphi\colon M &\to \mathbb{H}^{s\cdot m+s+1}\\ x &\mapsto \Big(\widetilde{\Phi}_1(x),\dots,\widetilde{\Phi}_s(x),\nu_1(x),\dots,\nu_s(x),\sum_{i=1}^{s}\widetilde{\zeta}_i(x)\Big)\end{aligned}$$

is a proper smooth embedding.

Proof. For completeness' sake we provide the proof of the claim, even though the interest in the full details might be quite muted.

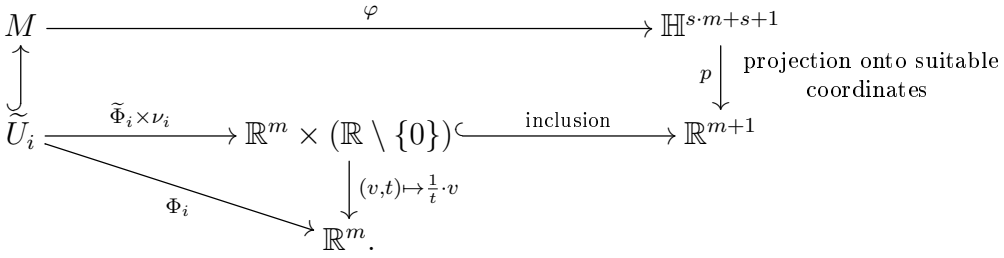
(i) First we show that φ is injective. Thus let $x, x' \in M$ be two distinct points. We pick i with $\nu_i(x) \neq 0$. If $\nu_i(x') \neq \nu_i(x)$, then we are done. Now suppose that $\nu_i(x') = \nu_i(x)$. In particular both are non-zero, thus $x, x' \in U_i$. Since $\Phi_i\colon U_i \to \mathbb{R}^m$ is injective we see that $\Phi_i(x) \neq \Phi_i(x')$. Since $\nu_i(x') = \nu_i(x) \neq 0$ we see that $\widetilde{\Phi}_i(x) \neq \widetilde{\Phi}_i(x')$. This implies that $\varphi(x) \neq \varphi(x')$.

(ii) It follows from (i), the hypothesis that M is compact and the Compact-Hausdorff Proposition 2.17 (2) that the map φ is an embedding.

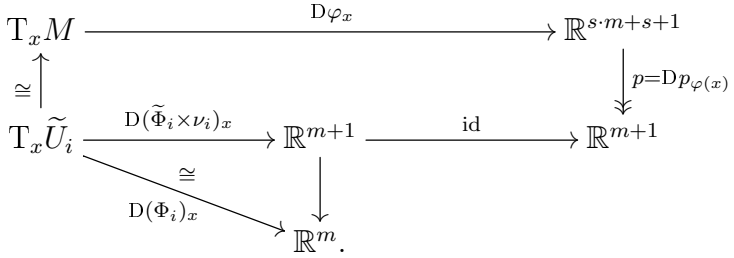
(iii) We need to show that for each $x \in M$ the differential

$$\mathrm{D}\varphi_x\colon \mathrm{T}_xM \to \mathrm{T}_{\varphi(x)}\mathbb{H}^{s\cdot m+s+1} = \mathbb{R}^{s\cdot m+s+1}$$

is a monomorphism. We pick i with $\nu_i(x) \neq 0$. We consider the open subset $\widetilde{U}_i := \{y \in U_i \,|\, \nu_i(y) \neq 0\}$. Next we consider the following diagram



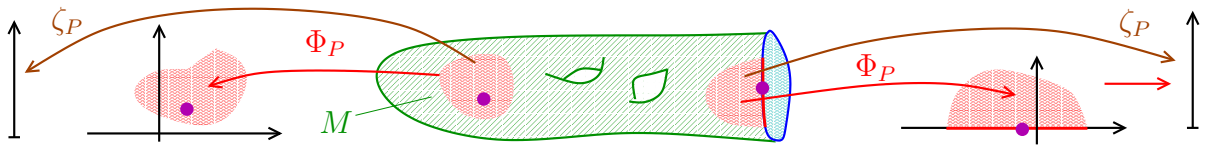
Here the right vertical map $p\colon \mathbb{H}^{s\cdot m+s+1} \to \mathbb{R}^{m+1}$ is the projection onto the coordinates that correspond to $\widetilde{\Phi}_i$ and ν_i. It is straightforward to see that the diagram commutes. We now now consider the corresponding tangent spaces and differentials:[283]



It follows from Lemma 23.12 that the left vertical map, the diagonal map and the right horizontal map are isomorphisms. The combination of these observations now shows that the top horizontal map is a monomorphism.

(iv) Recall that by the Smooth Manifold Boundary Proposition 19.26 (1) a point on the smooth manifold M admits either a smooth chart of type (i) or of type (ii). Furthermore note that it follows easily from our various definitions that the last coordinate of $\varphi(P)$ is zero if and only if $P \in \partial M$. In other words, we have $\varphi^{-1}(\partial\, \mathbb{H}^{s\cdot m+s+1}) = \partial M$.

(v) Using our careful definition of the maps $\widetilde{\zeta}_i$ and using the discussion on page 652 one can verify easily that for every $P \in \partial M$ the image $\mathrm{D}\varphi_P(\mathrm{T}_P M)$ is not contained in $\mathrm{T}_{\varphi(P)}(\partial\, \mathbb{H}^{s\cdot m+s+1})$. ⊞



[283] To preserve sanity we use the convention from page 636, namely given a submanifold Y of some $\mathbb{R}^N$ we do not distinguish between the tangent spaces $\mathrm{T}_y Y$ and $\widetilde{\mathrm{T}}_y Y$.

(1) The proof for compact topological manifolds is basically the same as in (2). There are only two differences:
   (a) In this setting we obtain from the Manifold-Partitions of Unity Existence Theorem 21.1 only a *continuous* partition of unity $\{\nu_i \colon M \to [0,1]\}_{i=1,\dots,s}$. In particular now the map $\varphi \colon M \to \mathbb{H}^{s\cdot m+s+1}$ that we constructed in (a) is an embedding, but evidently we cannot hope for any smoothness properties.
   (b) In the proof of (iv) we used that by the Smooth Manifold Boundary Proposition 19.26 (1) we know that a point on a smooth manifold admits either a smooth chart of type (i) or of type (ii). For the proof that $\varphi^{-1}(\partial\mathbb{H}^{s\cdot m+s+1}) = \partial M$ we need the corresponding statement for topological manifolds which, somewhat awkwardly, we only prove in the Topological Manifold Boundary Proposition 75.2.[284] ■

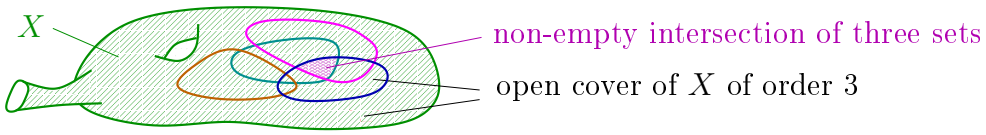
27.3. **The covering dimension of topological manifolds.** In this section we provide the proof of the Euclidean Embedding Theorem 27.1 in the non-compact case. This case is much more intricate, and thus just so much more fun, than the compact case. The key trick for dealing with the non-compact case is to make use of the concept of the covering dimension of a topological space. For the reader's convenience we recall the definition from page 369.

Definition. Let X be a topological space.
(1) A family $\{U_i\}_{i\in I}$ of subsets of X has order m if the following two statements hold:
   (a) There exists $J \subset I$ with $\#J = m$ such that $\bigcap_{j\in J} U_j \neq \varnothing$.
   (b) For every $J \subset I$ with $\#J > m$ we have $\bigcap_{j\in J} U_j = \varnothing$.

   If no such m exists, then we say that the family has infinite order.
(2) If X is non-empty, then we refer to

$$\operatorname{cdim}(X) \;:=\; \min\left\{ m \in \mathbb{N}_0 \cup \{\infty\} \;\middle|\; \begin{array}{c}\text{every open cover of } X \text{ admits an}\\ \text{open refinement of order} \le m+1\end{array} \right\}$$

as the covering dimension of X. If X is empty, then we set $\operatorname{cdim}(X) := -1$.



Proposition 27.7. (Covering Dimension-of-Manifolds Proposition) For any n-dimensional topological manifold M we have $\operatorname{cdim}(M) \le n$.

Remark.

[284] Note that we also use this statement to conclude that the boundary of the smooth manifold $\mathbb{H}^n$ is indeed $\mathbb{R}^{n-1}\times\{0\}$. If we view $\mathbb{H}^n$ as a topological manifold, then we also need the Topological Manifold Boundary Proposition 75.2 to obtain this result.

(1) Note that as of right now we do not know yet whether a non-empty topological manifold has a uniquely defined dimension. In principle a non-empty topological manifold might have several dimensions. We will verify the uniqueness of dimensions only later in Theorem 77.4 once we have introduced homology groups. But fortunately this issue does not have any impact on the Covering Dimension-of-Manifolds Proposition 27.7.

(2) Once we have introduced homology groups we can also prove Theorem 77.4 which says that the dimension and the covering dimension of a non-empty topological manifold agree.

Proof. Let M be an n-dimensional topological manifold. We start out with the following claim.

Claim. For every compact subspace $K \subset M$ we have $\operatorname{cdim}(K) \leq n$.

Proof. Since K is compact it follows from a straightforward argument that there exist finitely many charts $\Phi_i \colon U_i \to V_i$, $i = 1, \ldots, k$ and compact subsets $C_i \subset V_i$ such that $K \subset \Phi_1^{-1}(C_1) \cup \cdots \cup \Phi_k^{-1}(C_k)$.

(a) By Theorem 11.3 we know that for each i we have $\operatorname{cdim}(C_i) \leq n$.

(b) By (a) and Lemma 11.1 we know that for each i we have $\operatorname{cdim}(\Phi_i^{-1}(C_i)) \leq n$.

(c) From (b) and the Finite Union-Covering Dimension Proposition 11.6 we obtain that $\operatorname{cdim}(\Phi_1^{-1}(C_1) \cup \cdots \cup \Phi_k^{-1}(C_k)) \leq n$.

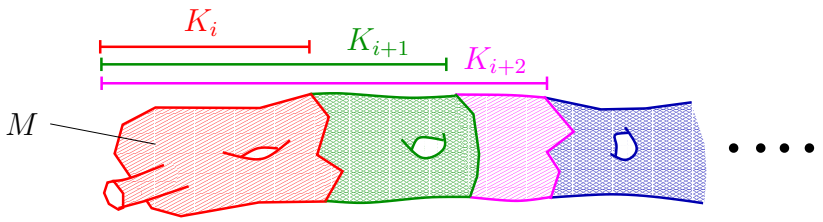
(d) Since K is compact and since M is by definition Hausdorff we know by the Compact-Closed Lemma 1.21 that K is a closed subset of $\Phi_1^{-1}(C_1) \cup \cdots \cup \Phi_k^{-1}(C_k)$. Therefore it follows from (c) together with the Closed Subspace Covering Dimension Lemma 11.5 that $\operatorname{cdim}(K) \leq n$. ⊞

We now proceed as in the proof of Corollary 11.9. By the Topological Manifold Exhaustion Lemma 18.12, respectively the Compact Exhaustion Lemma 9.3 (1) we know that there exists a sequence of compact subsets $\{K_i\}_{i \in \mathbb{N}}$ of M such that the following two conditions are satisfied:

(1) for every $i \in \mathbb{N}$ we have $K_i \subset \overset{\circ}{K}_{i+1}$,

(2) we have $M = \bigcup_{i \in \mathbb{N}} K_i$.

Let $i \in \mathbb{N}$. Since $K_{i+1} \setminus \overset{\circ}{K}_i$ is a closed subset of the compact space K_{i+1} we know by the Compact-Closed Lemma 1.21 that $K_{i+1} \setminus \overset{\circ}{K}_i$ is also compact. Therefore, by the above claim, we know that for each i we have $\operatorname{cdim}(K_{i+1} \setminus \overset{\circ}{K}_i) \leq n$. It follows from the Exhaustion-Covering Dimension Proposition 11.8 that $\operatorname{cdim}(M) \leq n$. ■



Remark. For topological manifolds without boundary we now have an off-the-shelves proof of the Euclidean Embedding Theorem 27.1. Indeed, let X be an n-dimensional topological manifold. In the Covering Dimension-of-Manifolds Proposition 27.7 we just showed that $\operatorname{cdim}(X) \leq n$. By definition X is Hausdorff and second-countable. Furthermore, by the Topological Manifolds-Local Properties Lemma 18.8 we know that X is regionally compact. Thus it follows from the Euclidean Embedding Theorem 11.13 for Topological Spaces that there exists a closed embedding $X \to \mathbb{R}^{2n+1}$. Note though that this embedding does not give us any control over ∂X. Also this approach does not give us anything meaningful for smooth manifolds.

27.4. **Proof of the Euclidean Embedding Theorem 27.1 in the non-compact case.** We can now generalize Lemma 27.6 to the non-compact case. Since the proof is no longer trivial we upgrade the statement from a lemma to a proposition. This proposition might well be interesting in its own right.

Proposition 27.8. (Small Atlas Proposition) Let $m \geq 2$ and let M be an m-dimensional topological manifold.[285]

(1) There exists an atlas $\{\Phi_i \colon U_i \to V_i\}_{i \in I}$ consisting of at most $m+1$ charts, such that each V_i is an open subset of $\mathbb{H}^m$.
(2) If M is a smooth manifold, then we can choose the atlas to be smooth.

Remark. Given an n-dimensional smooth manifold M we define, following [**Bers1968, Hopk1989**] the

$$\text{embedding covering number } N(M) \ := \ \begin{array}{c}\text{minimal cardinality of an open cover } \{U_i\}_{i \in I} \\ \text{of } M \text{ such that each } U_i \text{ admits a} \\ \text{smooth embedding into } \mathbb{R}^n.\end{array}$$

Note that the embedding covering number is precisely the minimal cardinality of a smooth atlas of M. By the Small Atlas Proposition 27.8 we know that $N(M) \leq \dim(M) + 1$. Evidently $N(M) \leq 1$ if and only if M is diffeomorphic to an open subset of some $\mathbb{R}^n$. It follows from this observation that for a non-empty closed smooth manifold we have $N(M) \geq 2$. We deduce from Proposition 19.18 that $N(S^n) = 2$. In general it is very difficult to determine $N(M)$ for a given smooth manifold M. For example let us consider the real projective space $\mathbb{R}\mathrm{P}^n$. We write $n+1 = 2^k \cdot m$ with m odd. If $n \neq 31$ and $n \neq 47$, then it is shown in [**Hopk1989**, Theorem 1] (see also [**Bers1968**, Theorem 1.5]) that

$$N(\mathbb{R}\mathrm{P}^n) \ = \ \begin{cases} \max\{2, m\}, & \text{if } k \leq 3, \\ \lceil \frac{n+1}{2(k+1)} \rceil, & \text{if } k \geq 4. \end{cases}$$

This result is surely not what was on everybody's lips. Later on we will prove a related result, namely we will show in Proposition 121.18 that any open cover of $\mathbb{R}\mathrm{P}^n$ by contractible open subsets (e.g. by open subsets that are diffeomorphic to $\mathbb{R}^n$) has cardinality at least $n+1$.

[285] Note that the condition $m \geq 2$ is indeed necessary, namely the conclusion does *not* hold for the 1-dimensional smooth manifold $M = [0,1] \sqcup [0,1]$.

Proof. In the following we only deal with the case of a smooth manifold. It is straightforward to modify the argument to deal with topological manifolds.

Let M be an m-dimensional smooth manifold. Similar to the compact case we make the following preparations:

(a) For each $P \in M \setminus \partial M$ we pick a smooth chart $\Phi_P \colon U_P \to B^m((0,\dots,0,1))$ of type (i) around P.

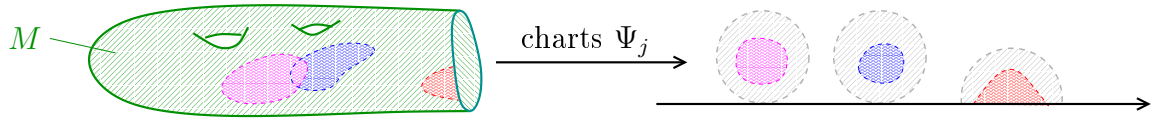
(b) For each $P \in \partial M$ we pick a smooth chart $\Phi_P \colon U_P \to B^m(0) \cap \mathbb{H}^m$ of type (ii) around P.

Note that $\{U_P\}_{P\in M}$ is an open cover of M. In the Covering Dimension-of-Manifolds Proposition 27.7 we just showed that $\operatorname{cdim}(M) \leq m$. By the Manifold Paracompact-Normal Proposition 18.9 (1) we know that M is paracompact and by definition we know that M is second-countable. Thus it follows from the Covering Dimension Refinement Theorem 11.12 that there exists a locally finite open refinement $\{V_j\}_{j\in J}$ of the open cover $\{U_P\}_{P\in M}$ and countable subsets $J_0,\dots,J_m$ of J with $J = J_0 \sqcup \dots \sqcup J_m$ such that for any $n \in \{0,\dots,m\}$ and any $r,s \in J_n$ we have $V_r \cap V_s = \varnothing$.

Since $\{V_j\}_{j\in J}$ is a refinement of $\{U_P\}_{P\in M}$ we can pick for each $j \in J$ a $P_j \in M$ with $V_j \subset U_{P_j}$. Note that the images of the charts Φ_{P_j} have diameter 2. Thus, since J is countable and since $m \geq 2$ we can find a family of points $\{Q_j\}_{j\in J}$ in $\mathbb{R}^{m-1}$ such the images of the maps

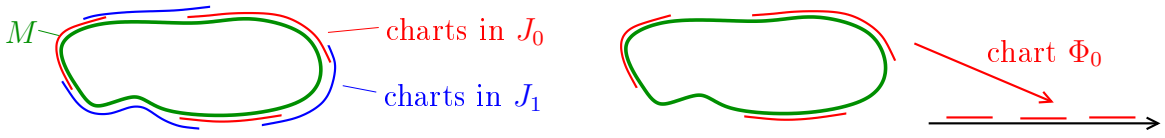
$$\begin{array}{rcl} \Psi_j \colon V_j & \to & \mathbb{R}^m \\ x & \mapsto & \Phi_{P_j}(x) + (Q_j, 0) \end{array}$$

are disjoint. Finally for $i \in \{1,\dots,m+1\}$ we now consider the map



$$\Phi_i := \bigsqcup_{j\in J_i} \Psi_j \colon V_j \to \mathbb{R}^m.$$

Note that, basically by construction, Φ_i is a chart and these charts $\Phi_1,\dots,\Phi_{m+1}$ form a smooth atlas for M. ∎

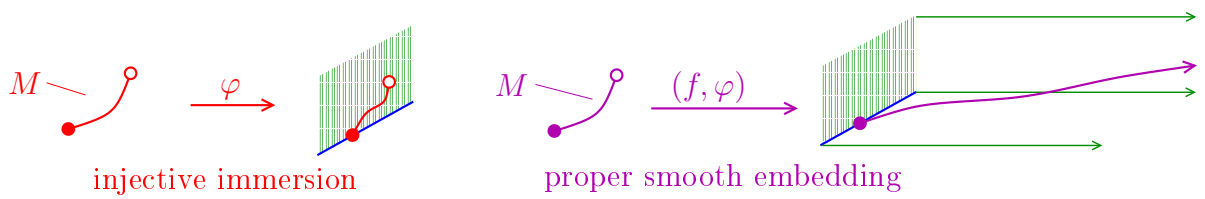


Our proof of the Euclidean Embedding Theorem 27.1 in the non-compact case also rests on the following lemma.

Lemma 27.9. Let M be a smooth manifold, let $\varphi \colon M \to \mathbb{H}^d$ be an injective immersion and let $f \colon M \to \mathbb{R}_{\geq 0}$ be a smooth exhaustion function, i.e. $f \colon M \to [0,\infty)$ is a smooth map such that the preimage of every compact interval is compact.[286] Then the map

$$\begin{array}{rcl} (f,\varphi) \colon M & \to & \mathbb{R} \times \mathbb{H}^d = \mathbb{H}^{d+1} \\ x & \mapsto & (f(x), \varphi(x)) \end{array}$$

is a proper smooth embedding. The same statement holds if we replace $\mathbb{H}^d$ by any other smooth manifold.



Proof of Lemma 27.9. It is clear that the map $(f,\varphi)\colon M \to \mathbb{R}\times\mathbb{H}^d$ is still an injective immersion with $\partial M = (f,\varphi)^{-1}(\partial\mathbb{H}^{d+1})$. Furthermore it follows easily from the Closed Embedding-to-$\mathbb{R}^n$ Proposition 2.19, or alternatively from the Proper-Closed Proposition 2.43, that the map $(f,\varphi)\colon M \to \mathbb{R}\times\mathbb{H}^d$ is a closed embedding. These two observations show that the map $(f,\varphi)\colon M \to \mathbb{R}\times\mathbb{H}^d$ is a proper smooth embedding. ■

Proof of the Euclidean Embedding Theorem 27.1 in the non-compact case. In the following we deal with the smooth case. The case of topological manifolds is dealt with in a very similar fashion.

Let M be a (possibly non-compact) m-dimensional smooth manifold. If $m = 0$, then it follows easily from the fact that M is second-countable that we can find a smooth embedding $M \to \mathbb{R}$. If $m = 1$, then the statement can be proved easily "by hand" using the Topological 1-Dimensional Manifold Classification Theorem 22.1 and using the fact that a smooth manifold, by virtue of being second-countable, has countably many components.

So now assume that $m \geq 2$. In the proof of the Euclidean Embedding Theorem 27.1 for compact manifolds we used the compactness so that we could appeal to Lemma 27.6. Now, that M is possibly non-compact, we appeal to the Small Atlas Proposition 27.8, and we obtain a smooth atlas $\{\Phi_i\colon U_i \to V_i\}_{i=1,\dots,m+1}$ such that each V_i is an open subset of $\mathbb{H}^m$. We apply the argument in the proof of the Euclidean Embedding Theorem 27.1 on page 723 and we see that there exists a $d \in \mathbb{N}$ and an injective immersion

$$\varphi\colon M \to \mathbb{H}^d$$

with $\partial M = \varphi^{-1}(\partial\mathbb{H}^d)$. If M is compact, then it follows from the Compact-Hausdorff Proposition 2.17 (2) that the map φ is an embedding.

If M is non-compact we have to put in some extra effort. First note that by the Manifold Exhaustion Lemma 21.2 we know that M admits a smooth exhaustion function, i.e. there exists a smooth map $f\colon M \to [0,\infty)$ such that the preimage of every compact interval is compact. By Lemma 27.9 we know that the map

$$\begin{array}{rcl}(f,\varphi)\colon M & \to & \mathbb{R}\times\mathbb{H}^d = \mathbb{H}^{d+1}\\ x & \mapsto & (f(x),\varphi(x))\end{array}$$

is a proper smooth embedding. ■

Exercises for Chapter 27.

[286]Note that by the Manifold Exhaustion Lemma 21.2 we know that every smooth manifold M admits a smooth exhaustion function.

Exercise 27.1. Recall that on page 627 we introduced the category $\mathcal{PSmMfd}$ of pointed smooth manifolds and we introduced the category $\mathcal{PSmMfd}_{\mathbb{R}^n}$ of pointed smooth submanifolds of some $\mathbb{R}^n$. In Question 23.6 (2) we asked whether the covariant functor

$$\begin{array}{rcl} \mathcal{PSmMfd}_{\mathbb{R}^n} & \to & \mathcal{Vec}_{\mathbb{R}} \\ (M,P) & \mapsto & \widetilde{\mathrm{T}}_P M \end{array}$$

can be extended in an "essentially unique" way to the category $\mathcal{PSmMfd}$. In the Tangent Space-Isomorphism Proposition 23.16 (4) we showed that such an extension exists. Turn the "essentially unique" into a precise statement and answer the question in the affirmative. *Hint.* Use the Euclidean Embedding Theorem 27.1 and reread Section 3.4.

Exercise 27.2. We consider

$$\ell^2(\mathbb{N}) \;:=\; \Big\{(x_1,x_2,\dots)\in\mathbb{R}^{\mathbb{N}} \;\Big|\; \sum_{i=1}^{\infty} x_i^2 < \infty\Big\}.$$

(a) We consider the map

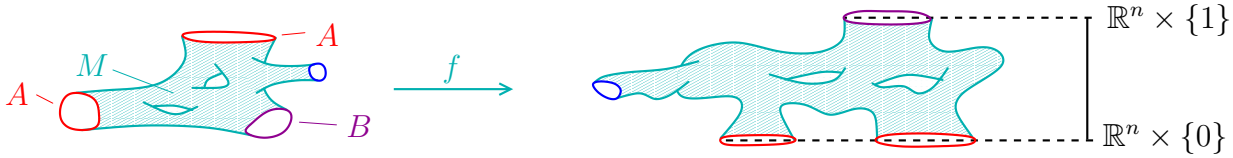
$$\begin{array}{rcl} d\colon \ell^2(\mathbb{N})\times\ell^2(\mathbb{N}) & \to & \mathbb{R}_{\geq 0} \\ ((x_1,x_2,\dots),(y_1,y_2,\dots)) & \mapsto & \sqrt{\sum_{i=1}^{\infty}(x_i-y_i)^2}. \end{array}$$

Show that this map does indeed take finite values and show that it defines a complete metric on $\ell^2(\mathbb{N})$.

(b) Let M be a topological manifold. Modify the argument of Section 27.2 to show that there exists a closed embedding $\varphi\colon M\to\ell^2(\mathbb{N})$.

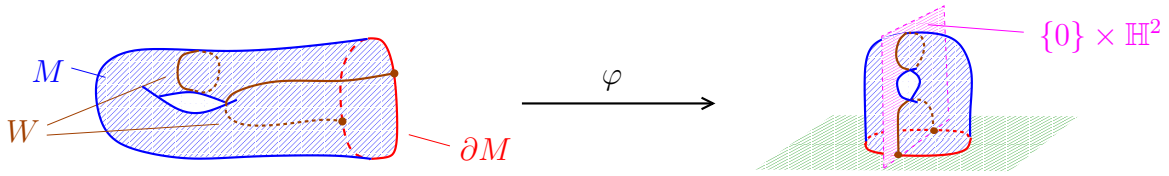
Remark. This exercise, together with the Closed Subspace-Completeness Lemma 6.9, provides a new proof that (possibly non-compact) topological manifolds are completely metrizable. In other words, it provides a proof of the Manifold Metrization Proposition 27.4 (1) that does not make use of the concept of covering dimension that we had developed in Chapter 11.

Exercise 27.3. Let M be a compact smooth manifold and let A and B be disjoint unions of components of ∂M such that $\partial M = A\sqcup B$. Show that there exists a proper smooth embedding $f\colon M\to\mathbb{R}^n\times[0,1]$ such that $A = f^{-1}(\mathbb{R}^n\times\{0\})$ and $B = f^{-1}(\mathbb{R}^n\times\{1\})$.

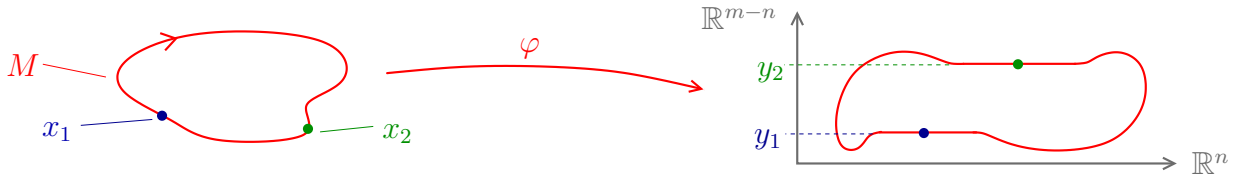


Exercise 27.4. Let M be a smooth manifold without boundary. Show that there is no metric $d\colon M\times M\to\mathbb{R}_{\geq 0}$ which is a smooth map from the smooth manifold $M\times M$ to the smooth manifold $\mathbb{R}_{\geq 0}$.

Exercise 27.5. Let M be a compact m-dimensional smooth manifold and let W be a (possibly empty) proper submanifold of M. Show that there exists a proper smooth embedding $\varphi\colon M\to\mathbb{H}^n$ and a $k\in\{0,\dots,n-1\}$ such that $\varphi(W) = \varphi(M)\cap(\{0\}\times\mathbb{H}^k)$.



Exercise 27.6. Let M be a closed n-dimensional smooth manifold and let $x_1, \dots, x_k \in M$ be disjoint points. Show that there exists a smooth embedding $\varphi\colon M \to \mathbb{R}^m$ such that $\varphi(M)$ is "flat" near $\varphi(x_1), \dots, \varphi(x_k)$, in the sense that there exist disjoint neighborhoods $U_1, \dots, U_k$ of $x_1, \dots, x_k$ and points $y_1, \dots, y_k \in \mathbb{R}^{m-n}$ such that for each $i \in \{1, \dots, k\}$ we have $\varphi(U_i) \subset \mathbb{R}^n \times \{y_i\}$.



Exercise 27.7. Let M be the smooth manifold that is obtained from the open disk B^2 by removing the three points $(0, -\frac{1}{2}), (0, 0)$ and $(0, \frac{1}{2})$. Give an explicit proper smooth embedding $f\colon M \to \mathbb{R}^3$.

Remark. This exercise is the "smooth version" of Exercise 11.13. In Exercise 43.12 we will see that there is no proper smooth embedding $f\colon M \to \mathbb{R}^2$.

Exercise 27.8. Given an n-dimensional smooth manifold M we define, following [**Bers1968, Hopk1989**] the

$$\text{embedding covering number } N(M) := \begin{array}{c}\text{minimal cardinality of an open cover } \{U_i\}_{i\in I} \\ \text{of } M \text{ such that each } U_i \text{ admits a} \\ \text{smooth embedding into } \mathbb{R}^n.\end{array}$$

In other words, the embedding covering number is precisely the minimal cardinality of a smooth atlas of M.

(a) Given $g \in \mathbb{N}$, what is the embedding covering number of the surface of genus g?
(b) Show that $N(\mathbb{R}\mathrm{P}^2) = 3$.
(c) What is the embedding covering number of the Klein bottle?

Remark. With the present knowledge one can give upper bounds to $N(M)$ but it is difficult to give very meaningful lower bounds on $N(M)$. So the task is to give the best possible upper bounds.

28. The Neighborhood Projection Theorem and Smooth Collar Neighborhoods

As the title of the chapter strongly suggests, in this chapter we will state, prove and discuss the Neighborhood Projection Theorem 28.1 and the Smooth Collar Neighborhood Theorem 28.3. Both are essential tools in the study of smooth manifolds and will be used frequently.

28.1. The Neighborhood Projection Theorem.

In this section we state the Neighborhood Projection Theorem, which is a key tool in many of our future constructions. To state the theorem it is necessary to introduce the following definition.

Definition. A map $f\colon M \to N$ between smooth manifolds is called a submersion if f is smooth and if given any $P \in M$ the differential $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}N$ is an epimorphism.

With this definition in our bag we can state the Neighborhood Projection Theorem.

Theorem 28.1. (Neighborhood Projection Theorem) Let M be a proper k-dimensional smooth submanifold of $\mathbb{R}^n$ with $\partial M = \varnothing$.[287] Given $P \in M$ we write

$$\big(\widetilde{\mathrm{T}}_P M\big)^{\perp} \;=\; \{w \in \mathbb{R}^n \,|\, \langle v, w\rangle = 0 \text{ for all } v \in \widetilde{\mathrm{T}}_P M\}.$$

There exists a smooth map $\epsilon\colon M \to (0,\infty)$ such that the following conditions are satisfied:

(1) The subset

$$Z(M,\epsilon) \;:=\; \big\{P + w \,|\, P \in M \text{ and } w \in \big(\widetilde{\mathrm{T}}_P M\big)^{\perp} \text{ with } \|w\| < \epsilon(P)\big\} \;\subset\; \mathbb{R}^n$$

is an open neighborhood of $M \subset \mathbb{R}^n$.

(2) The map

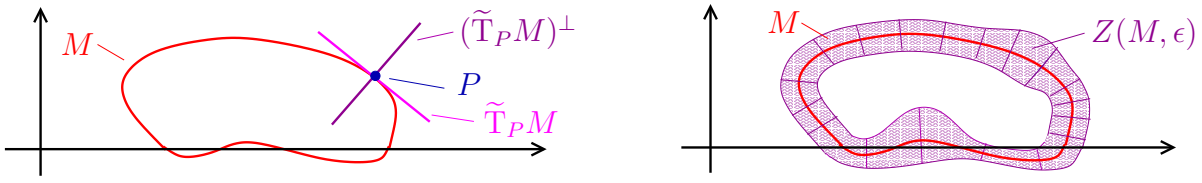
$$\begin{array}{rcll} \pi\colon Z(M,\epsilon) & \to & M & \\ P + w & \mapsto & P & \text{where } P \in M \text{ and } w \in \big(\widetilde{\mathrm{T}}_P M\big)^{\perp} \text{ with } \|w\| < \epsilon(P) \end{array}$$

is well-defined, it is smooth, it is a submersion, and it is a retraction $Z(M,\epsilon) \to M$. Furthermore there exists a smooth[288] deformation retraction

$$R\colon Z(M,\epsilon) \times [0,1] \;\to\; Z(M,\epsilon)$$

with $R_0 = \mathrm{id}$ and with $R_1 = \pi\colon Z(M,\epsilon) \to M$.

If M is compact, then we can choose ϵ to be a constant map.



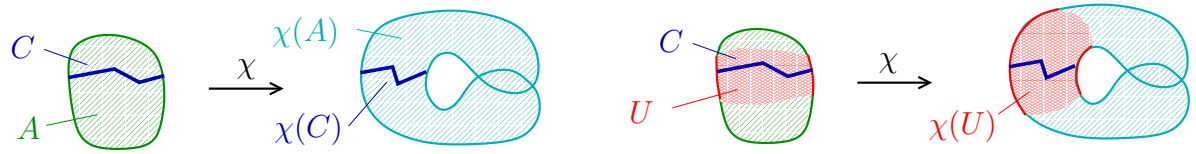
[287] Since $\partial M = \varnothing$ we see, by the definition on page 550, that in our context "proper" just means that M is a closed subset of $\mathbb{R}^n$.

[288] Note that $Z(M,\epsilon)$ is a smooth manifold with empty boundary. Thus by the Smooth Manifold Product Proposition 19.31 we know that $Z(M,\epsilon) \times [0,1]$ is a smooth manifold. So it makes sense for a map $Z(M,\epsilon) \times [0,1] \to Z(M,\epsilon)$ to be smooth.

Remark. In Borsuk's Theorem 104.3 we will prove a variation of the Neighborhood Projection Theorem 28.1 that also holds for closed subspaces of $\mathbb{R}^n$ that are topological manifolds.

Before we can provide the proof of the Neighborhood Projection Theorem 28.1 we need to make a few preparations. First, for the reader's convenience we recall the following theorem which will play a role on three occasions in this chapter.

Theorem 21.13. (Neighborhood Smooth Embedding Theorem) Let $\chi\colon A \to B$ be a map between smooth manifolds. Furthermore let $C \subset A$ be a closed subset such that the map $\chi\colon C \to \chi(C)$ is a homeomorphism and such that $\chi(C)$ is a closed subset of B. If χ is a local diffeomorphism at each $P \in C$, then there exists an open neighborhood U of C such that the restriction of χ to U is a smooth embedding and such that $\chi(U)$ is an open subset of B.



Our proof of the Neighborhood Projection Theorem 28.1 relies on a lemma that will also come in handy when we prove the Smooth Collar Neighborhood Theorem 28.3. Therefore we formulate the lemma in a more general fashion than is strictly needed for just the proof of the Neighborhood Projection Theorem 28.1.

Lemma 28.2. (Normal Vectors Lemma) Let M be a proper k-dimensional smooth submanifold of $\mathbb{H}^n$. We set

$$WM := \big\{(P,w) \in \mathbb{H}^n \times \mathbb{R}^n \,|\, P \in M \text{ and } w \in \big(\widetilde{\mathrm{T}}_P M\big)^\perp\big\}.$$

The following three statements hold:

(1) The subset WM is an n-dimensional submanifold of $\mathbb{H}^n \times \mathbb{R}^n$ where the boundary is given by $\partial(WM) = WM \cap (\partial\mathbb{H}^n \times \mathbb{R}^n)$.

(2) Given any point $P \in M$ and given any open neighborhood U of $P \in WM$ there exists an open neighborhood U_P of $P \in M$ and an $\epsilon_P > 0$ such that the subset $\big\{(x,w) \in \mathbb{H}^n \times \mathbb{R}^n \,|\, x \in U_P \text{ and } w \in \big(\widetilde{\mathrm{T}}_P M\big)^\perp \text{ with } \|w\| < \epsilon_P\big\}$ is contained in U.

(3) The map

$$\begin{aligned} \Pi\colon WM &\to M \\ (P,w) &\mapsto P \end{aligned}$$

is a submersion.

An analogous statement also holds if we replace $\mathbb{H}^n$ by $\mathbb{R}^n$.

Proof of the Normal Vectors Lemma 28.2. In the following we deal with the case that M is a proper k-dimensional smooth submanifold of $\mathbb{H}^n$. The case $\mathbb{R}^n$ is dealt with in almost the same way. Let $\Phi\colon U \to V$ be a submanifold chart of $M \subset \mathbb{H}^n$. We write $\Psi := \Phi^{-1}\colon V \to U$.

Note that $W(V \cap \Phi(M))$ is given by setting some coordinates of $V \times \mathbb{R}^n \subset \mathbb{H}^n \times \mathbb{R}^n$ to zero. Unfortunately the map $\Psi = \Phi^{-1}\colon V \to U$ does not quite induce a map $W(V \cap \Phi(M)) \to W(U \cap M) \subset WM$ since it does not preserve orthogonality. But this problem can be fixed by introducing a map that "makes vectors orthogonal".

We introduce the following map:[289]

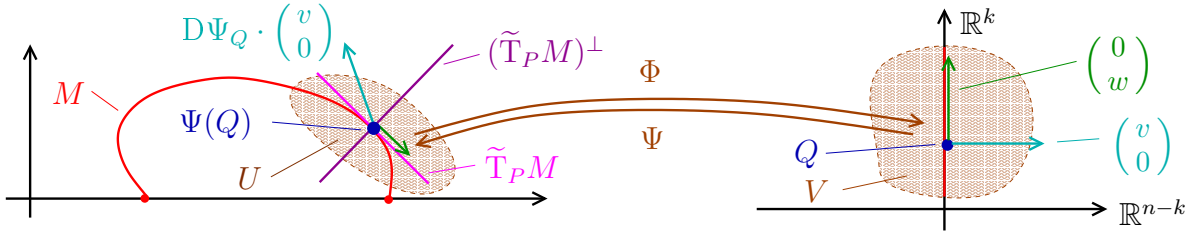
$$\begin{aligned}\Theta\colon (\mathbb{R}^n \setminus \{0\})^k \times \mathbb{R}^n &\to \mathbb{R}^n \\ (v_1, \dots, v_k, w) &\mapsto \underbrace{w - \sum_{i=1}^{k} \langle w, v_i\rangle \cdot \frac{v_i}{\|v_i\|^2}}_{\text{orthogonal to } v_1,\dots,v_k}\end{aligned}$$

Next we consider the map

$$\begin{aligned}G_\Psi\colon V \times \mathbb{R}^{n-k} \times \mathbb{R}^k &\to U \times \mathbb{R}^n \\ (Q, v, w) &\mapsto \Big(\Psi(Q), \underbrace{\Theta\big(\underbrace{\mathrm{D}\Psi_Q(e_{n-k+1}), \dots, \mathrm{D}\Psi_Q(e_n)}_{\text{basis of } \widetilde{\mathrm{T}}_{\psi(Q)}M}, \mathrm{D}\Psi_Q \cdot \begin{pmatrix} v \\ 0 \end{pmatrix}, \mathrm{D}\Psi_Q \cdot \begin{pmatrix} 0 \\ w \end{pmatrix}\big)}_{\in (\widetilde{\mathrm{T}}_{\psi(Q)}M)^\perp}\Big).\end{aligned}$$

One can easily show that the map G_Ψ is a bijection and smooth. Furthermore using the Smooth Pasting Proposition 19.23 (2) one can show fairly easily that G_Ψ is in fact a diffeomorphism. Furthermore, once one has digested the definitions it is also quite clear that the $G_\Psi^{-1}\colon U \times \mathbb{R}^n \to V \times \mathbb{R}^n$ defines (after a suitable permutation of the coordinates) a submanifold chart for WM. All the charts G_Ψ^{-1} show that WM is an n-dimensional submanifold of $\mathbb{H}^n \times \mathbb{R}^n$.

Furthermore note that it follows from the fact that M is a proper submanifold of $\mathbb{H}^n$ and the fact that the submanifold charts Ψ and G_Ψ have the same type, in the sense of the definition on page 550, that $\partial(WM) = WM \cap (\partial\mathbb{H}^n \times \mathbb{R}^n)$. In other words, we have completed the proof of Statement (1).



Next note that Statement (2) is a straightforward consequence of the definition of the product topology on $\mathbb{H}^n \times \mathbb{R}^n$ and the definition of the subspace topology on $WM \subset \mathbb{H}^n \times \mathbb{R}^n$.

Finally we turn to the proof of Statement (3). We need to show that $\Pi\colon WM \to M$ is a submersion. Given $(P, v) \in WM$ we pick a submanifold chart $\Phi\colon U \to V$ for $P \in M$ as above. We make use of the above notation and we obtain the following diagram

$$\begin{array}{ccc} (V \cap (\{0\} \times \mathbb{R}^k)) \times \mathbb{R}^{n-k} \times \{0\} & \xrightarrow[\cong]{G_\Psi} & W(U \cap M) \\ \downarrow{\scriptstyle (x,v)\mapsto x} & & \downarrow{\scriptstyle \Pi} \\ V \cap (\{0\} \times \mathbb{R}^k) & \xrightarrow[\cong]{\Psi} & U \cap M. \end{array}$$

It follows easily from the earlier discussion that the horizontal maps are diffeomorphisms and that the diagram commutes. Evidently the map to the left is a submersion, thus the

[289] Here $\langle -, - \rangle\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the usual Euclidean bilinear symmetric form on $\mathbb{R}^n$.

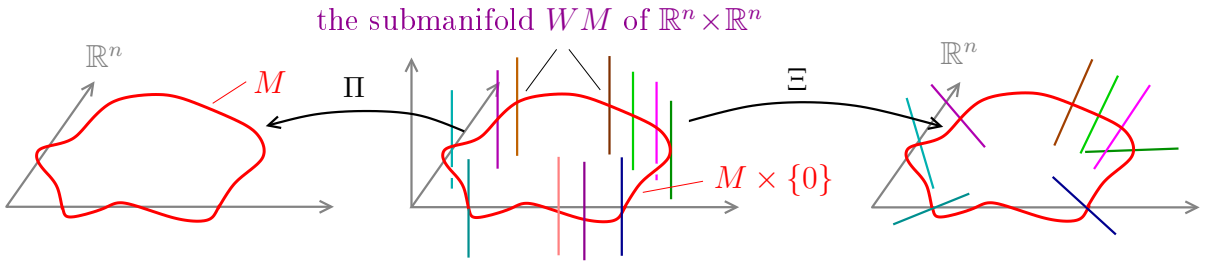
map to the right is also a submersion. This shows that Π is locally a submersion, but this implies evidently that Π is also a submersion as a whole. ■

Proof of the Neighborhood Projection Theorem 28.1. Let M be a proper k-dimensional smooth submanifold of $\mathbb{R}^n$. As in the Normal Vectors Lemma 28.2 we consider again

$$WM := \big\{(P,w) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, P \in M \text{ and } w \in \big(\widetilde{\mathrm{T}}_P M\big)^\perp\big\}.$$

By the Normal Vectors Lemma 28.2 we know that WM is an n-dimensional submanifold of $\mathbb{R}^n \times \mathbb{R}^n$. We consider the maps

$$\begin{array}{rcl} \Pi\colon WM & \to & M \\ (P,w) & \mapsto & P \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \Xi\colon WM & \to & \mathbb{R}^n \\ (P,w) & \mapsto & P+w. \end{array}$$



We make the following observations:

(a) The map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ that is given by $(P,w) \to P+w$ is evidently smooth. Since WM is a smooth submanifold we obtain from Lemmas 19.22 and 19.24 that the map $\Xi\colon WM \to \mathbb{R}^n$ is smooth.

(b) Let $P \in M$. It follows easily from $\widetilde{\mathrm{T}}_P M \oplus (\widetilde{\mathrm{T}}_P M)^\perp = \mathbb{R}^n$, together with the discussion of the differential of a linear map on page 536 and the fact that WM is an n-dimensional smooth manifold that the differential of $\Xi\colon WM \to \mathbb{R}^n$ at $(P,0) \in WM$ is an isomorphism.

(c) It follows almost immediately from (b), our hypothesis that $\partial M = \varnothing$, the Local Smooth Embedding Theorem 24.8 and the Normal Vectors Lemma 28.2 (2) that for each point $P \in M$ there exists an open neighborhood W_P of $(P,0) \in WM$ such that the restriction of $\Xi\colon WM \to \mathbb{R}^n$ to W_P is a smooth embedding.

(d) Note that the restriction of $\Xi\colon WM \to M$ to $M \times \{0\}$ is given by $(P,0) \mapsto P$. This is evidently an embedding. By hypothesis M is a proper smooth submanifold of $\mathbb{R}^n$ which means in particular, by definition of "proper", that $M = \Xi(M \times \{0\})$ is a closed subset of $\mathbb{R}^n$.

(e) It follows from (b), (c) and (d), together with the Neighborhood Smooth Embedding Theorem 21.13, that there exists an open neighborhood U of $M \times \{0\} \subset WM$ such that the restriction of Ξ to $\Xi|_U\colon U \to \mathbb{R}^n$ is a smooth embedding and such that $\Xi(U)$ is an open subset of $\mathbb{R}^n$.

(f) It follows from the Replace-by-Smooth Non-Negative Function Corollary 21.11 (2) together with the Normal Vectors Lemma 28.2 (2) that there exists a smooth map $\epsilon\colon M \to (0,\infty)$ such that for each $x \in M$ and each $w \in \big(\widetilde{\mathrm{T}}_x M\big)^\perp$ with $\|w\| < \epsilon(x)$ we have $(x,w) \in U$.

We set

$$W(\epsilon) \;:=\; \big\{(P,w) \in WM \,|\, P \in M \text{ and } w \in \big(\widetilde{\mathrm{T}}_P M\big)^\perp \text{ with } \|w\| < \epsilon(P)\big\} \;\subset\; WM.$$

The theorem is now a consequence of the following observations:

(i) It follows easily from the definition of the subspace topology that $W(\epsilon)$ is an open subset of WM.

(ii) It follows from (i) together with (e) and (f) that the restriction of $\Xi\colon WM \to \mathbb{R}^n$ to $\Xi\colon W(\epsilon) \to \mathbb{R}^n$ is a smooth embedding such that the image is open.

(iii) The image of $\Xi\colon W(\epsilon) \to \mathbb{R}^n$ is precisely $Z(M,\epsilon)$.

(iv) The map

$$\begin{array}{rcll} \pi\colon Z(M,\epsilon) & \to & M & \\ P+w & \mapsto & P & \text{where } P \in M \text{ and } w \in \big(\widetilde{\mathrm{T}}_P M\big)^\perp \text{ with } \|w\| < \epsilon(P) \end{array}$$

is precisely the map $\Pi \circ \Xi^{-1}\colon Z(M,\epsilon) \to M$ and thus well-defined.

(v) It follows from the Smooth Embedding Theorem 24.10 that $\Xi\colon U \to \Xi(U)$ is a diffeomorphism. Furthermore we know by the Normal Vectors Lemma 28.2 that $\Pi\colon WM \to M$ is a submersion. It follows that $\pi = \Pi \circ \Xi^{-1}\colon \Xi(U) \to M$ is a submersion.

(vi) There exists an obvious smooth deformation retraction from $W(\epsilon)$ to $M \times \{0\}$. Under the diffeomorphism Ξ is gives rise to the desired smooth deformation retraction $R\colon Z(M,\epsilon) \times [0,1] \to Z(M,\epsilon)$ with $R_0 = \mathrm{id}$ and $R_1 = \pi\colon Z(M,\epsilon) \to M$.

For the reader's and the author's convenience we summarize the various maps in the following commutative diagram:

$$\begin{array}{ccccc} WM & \hookleftarrow & U & \hookleftarrow & W(\epsilon) \\ \downarrow \Xi & & \cong \downarrow \Xi & & \cong \downarrow \Xi \\ \mathbb{R}^n & \hookleftarrow & \Xi(U) & \hookleftarrow & Z(M,\epsilon) \end{array} \qquad \begin{array}{c} \overset{\Pi}{\searrow} \\ \underset{\pi}{\rightleftarrows} \end{array} \; M.$$

Finally if M is compact, then it follows from the Compact Image Lemma 2.13 that the map $\epsilon\colon M \to \mathbb{R}_{>0}$ assumes a minimal value > 0. Thus we can replace ϵ throughout by this minimal value > 0. ∎

28.2. **Statement of the Smooth Collar Neighborhood Theorem.** In this section we will state the Smooth Collar Neighborhood Theorem which is another important tool in differential topology.

Definition. Let M be a smooth manifold. A collar of ∂M is defined as a smooth embedding $\chi\colon [0,1] \times \partial M \to M$ that has the following two properties:

(a) the restriction of χ to $\{0\} \times \partial M$ is the obvious map $(0,P) \mapsto P$,

(b) the image $\chi([0,1] \times \partial M)$ is a closed subset of M.

We refer to the image of χ as a collar neighborhood.

Example. Let $m \in \mathbb{N}$ and let W be a smooth manifold with $\partial W = \varnothing$. The map

$$\begin{array}{rcl} \chi\colon [0,1] \times (S^{m-1} \times W) & \to & \overline{B}^m \times W \\ (t,P,w) & \mapsto & ((1-\frac{1}{2}t)\cdot P, w) \end{array}$$

is evidently a collar for the smooth manifold $\overline{B}^m \times W$. We refer to this map as the standard collar of $\overline{B}^m \times W$ and we refer to the image of f as the standard collar neighborhood of $\overline{B}^m \times W$.



standard collar neighborhood for $\partial M = S^1$ collar neighborhood for ∂M

Convention.

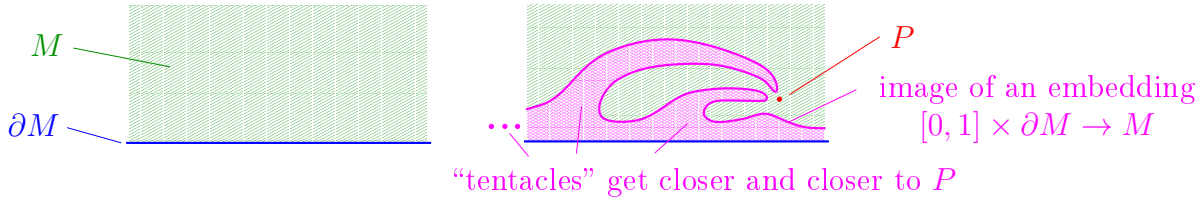
(1) We continue with the notation from the definition. By abuse of notation we sometimes denote the image of a collar $\chi\colon [0,1] \times \partial M \to M$ by $[0,1] \times \partial M$ as well. In particular sometimes we make the identification $\{0\} \times \partial M = \partial M$.
(2) Let $m \in \mathbb{N}$ and let W be a smooth manifold with $\partial W = \varnothing$. Unless we say something else we will equip $\overline{B}^m \times W$ with the above standard collar.

Warning. Suppose that the smooth manifold M is oriented. On page 689 we equipped ∂M with a natural orientation. As usual the smooth manifold $[0,1]$ comes with its standard orientation. Let us now equip $[0,1] \times \partial M$ with the product orientation that we introduced on page 695. It follows from the Product Orientation Proposition 25.22 (4d) that, as an oriented smooth manifold, $\partial([0,1] \times \partial M) = \partial([0,1]) \times M = -(\{0\} \times \partial M) \cup (\{1\} \times \partial M)$. Thus we see that every collar $[0,1] \times \partial M \to M$ is *orientation-reversing*. In principle it would thus be better to define a collar as a smooth embedding $[-1,0] \times \partial M \to M$ which is the "identity" on $\{0\} \times M$. But such a choice would be notationally awkward at times. So we stick with our somewhat suboptimal convention.

Remark. In the literature one can find several slightly different definitions of a collar and of a collar neighborhood.

(1) In [**Wall2016**, p. 32] the author works with $\partial N \times [0,1]$ instead of $[0,1] \times \partial M$. As we saw in the Product Orientation Proposition 25.22 (5), for products of two smooth manifolds it is usually preferable to have the smooth manifold with boundary as the first factor. As we just discussed, our orientation convention for collar neighborhoods is suboptimal, but working with $\partial N \times [0,1]$ makes the situation even worse.
(2) In some books, e.g. [**Kos1993**, p. 21] and [**BJ1982**, Definition 13.5], a collar is defined as a smooth embedding $\chi\colon [0,1) \times \partial M \to M$ such that $\chi(0,P) = P$ for all $P \in \partial M$. If we restrict a collar $\chi\colon [0,1] \times \partial M \to M$ to $[0,1) \times \partial M$ then this is evidently a smooth embedding. But as we will see in Exercise 28.3, not every smooth embedding $f\colon [0,1) \times \partial M \to M$ can be extended to a smooth embedding $[0,1] \times \partial M \to M$. We prefer to work with our definition since it makes it easier to formulate in what sense collars are unique, see the Uniqueness of Collar-Proposition **??**.
(3) Our condition (b), namely that a collar neighborhood needs to be closed, is non-standard. If ∂M is compact, which is anyway the case in most situations, the image of any smooth embedding $[0,1] \times \partial M \to M$ is a closed subset by the Product Topology

Properties Proposition 5.6, the Smooth Manifold Boundary Proposition 19.26 (3b), the Compact Image Lemma 2.13 and the Compact-Closed Lemma 1.21. Thus the condition only has content if ∂M is non-compact. It seems to the author that in the case of non-compact boundary it is often implicitly assumed that any subset that satisfies (a) is actually closed. But in general this is not the case. For example, in the figure below we consider $M = \{(x, y) \in \mathbb{R}^2 \,|\, y \geq 0\}$ and the image of a smooth embedding $[0, 1] \times \partial M$ that does not contain a certain point P, but such that P lies in the closure of the smooth embedding. In other words, the image of $[0, 1] \times \partial M$ is not a closed subset of M. In Lemma 28.7 we will see that this issue can be resolved in a suitable sense.



Now, after this lengthy discussion of the notion of a collar, we finally get a chance to state the Smooth Collar Neighborhood Theorem.

Theorem 28.3. (Smooth Collar Neighborhood Theorem) Let M be a smooth manifold. For any neighborhood V of the boundary ∂M there exists a collar $\chi\colon [0, 1]\times\partial M \to M$ with $\chi([0, 1] \times \partial M) \subset V$.

We will provide a proof of the Smooth Collar Neighborhood Theorem 28.3 in the next section. In the remainder of this section we will discuss the Smooth Collar Neighborhood Theorem 28.3 and some of its consequences.

Remark.

(1) The Smooth Collar Neighborhood Theorem 28.3, sometimes in slightly different flavors, is proved in almost every book dealing with the topology of smooth manifolds. In particular proofs are provided in [**Wall2016**, Theorem 1.5.5], [**Lee2002**, Theorem 9.25], [**BJ1982**, Theorem 13.6], [**Muk2015**, Theorem 7.2.13], [**Miln1965b**, Corollary 3.5] and [**Kos1993**, Theorem I.7.3]. As we mentioned above, for non-compact smooth manifolds our definition of a collar is stricter than in the literature. In Lemma 28.7 we will see that this is not really an issue.

(2) In Project 35.1 we will give an alternative proof of the Smooth Collar Neighborhood Theorem 28.3.

(3) In Theorem **??** we will state a refinement of the Smooth Collar Neighborhood Theorem 28.3 which can also take into account submanifolds. Furthermore in the Uniqueness of Collar-Proposition **??** we will see that collars of compact smooth manifolds are unique in a suitable sense.

(4) The proof of the Smooth Collar Neighborhood Theorem 28.3 that we will provide does not generalize to the setting of topological manifolds. Nonetheless, in the Topological Collar Neighborhood Theorem 75.5 we will generalize the above theorem to the setting of topological manifolds, using a different argument. The approach taken in the proof for topological manifolds can, after some finetuning, also be used to

prove the above Smooth Collar Neighborhood Theorem 28.3. On the other hand, the argument which we use in this chapter for smooth manifolds has the advantage that it can also be used to prove several refinements, e.g. Theorem **??** and the Uniqueness of Collar-Proposition **??**.

(5) At least for the case that ∂M is compact one can also prove the Smooth Collar Neighborhood Theorem 28.3 using the Smooth Manifold Product Theorem 138.2.

The following lemma summarizes a few more or less technical properties of collar neighborhood. The lemma in particular justifies the name "collar neighborhood".

Lemma 28.4. (Basics-of-Collars Lemma) Let M be an n-dimensional smooth manifold. Suppose we are given a collar $\chi\colon [0,1]\times\partial M \to M$. As above we identify $[0,1]\times\partial M$ with its image in M.

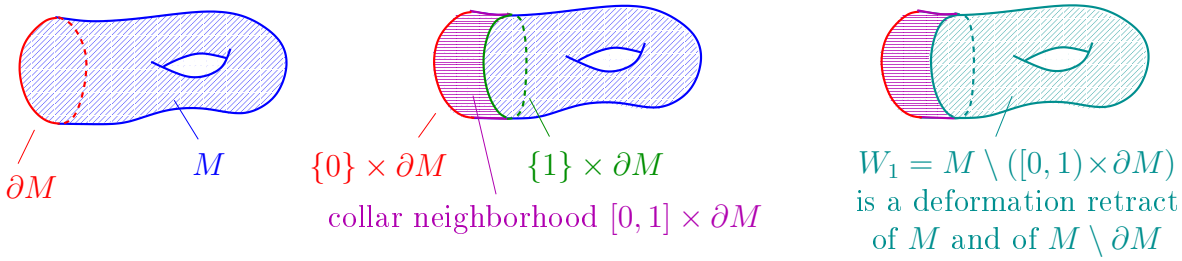
(1) Let $t \in (0,1]$ and let A be a union of components of ∂M. The subset $[0,t)\times A$ is an open subset of M. In particular $[0,t)\times A$ and $[0,t]\times A$ are neighborhoods of A in the sense of the definition on page 113.

(2) Given any $t \in (0,1]$ the subset $W_t := M \setminus ([0,t)\times\partial M)$ is a codimension-zero submanifold of M with boundary given by $\{t\}\times\partial M$.

(3) (a) For every $t \in (0,1]$ the submanifold $W_t = M \setminus ([0,t)\times\partial M)$ is a deformation retract of M and it is a deformation retract of $M \setminus \partial M$.

(b) For every $t \in [0,1)$ there exists a *smooth* homotopy[290] $F\colon M\times[0,1]\to M$ with the following properties:

- F is a homotopy rel $W_1 = M \setminus ([0,1)\times\partial M)$,
- $F_0 = \mathrm{id}$,
- for each $P \in \partial M$ and $s \in [0,1]$ we have $F_s(P) \subset [0,1]\times\{P\}$,
- $F_1(M) \subset W_t$.



Proof.

(1) This statement is an immediate consequence of the Codimension-Zero Smooth Embedding Proposition 24.16 (1).

(2) Let $t \in [0,1]$. By hypothesis $\chi([0,1]\times\partial M)$ is a closed subset of M. It follows from the Open -Closed Inclusion Lemma 2.10 that $\chi([0,t]\times\partial M)$ is also a closed subset of M. The desired statement follows from this observation together with the Codimension-Zero Smooth Embedding Proposition 24.16 (2).

[290] By the Smooth Manifold Product Proposition 20.6 we can view $M\times[0,1]$ as a smooth manifold (possibly with corner). Thus it makes sense to say that the map $F\colon M\times[0,1]\to M$ is smooth.

(3) (a) Let $t \in (0,1]$. We consider the map

$$\begin{array}{rcl} F\colon M \times [0,1] & \to & M \\ (Q,s) & \mapsto & \begin{cases} ((1-s)\cdot u + s\cdot t, R), & \text{if } Q = (u,R) \in [0,t] \times \partial M, \\ Q, & \text{if } Q \in W_t. \end{cases} \end{array}$$

Note that by definition the collar neighborhood $[0,1]\times\partial M$ is a closed subset of M and that we know by (1) that $M\setminus([0,1)\times\partial M)$ is also a closed subset of M. Since the restriction of F to the products of these closed subsets with $[0,1]$ is continuous we obtain from the Pasting Proposition 2.6 (2) that the map $F\colon M\times[0,1]\to M$ is indeed continuous. This map defines a deformation retraction from M to W_t and also from $M\setminus\partial M$ to W_t.

(b) Let $t \in [0,1)$. Using the Smooth Transition Function Lemma 19.4 one can show easily that there exists a smooth map $\eta\colon [0,1]\to[t,1]$ for which there exists an $\epsilon > 0$ such that $\eta(s) = s$ for all $s \in [1-\epsilon, 1]$. We consider the homotopy

$$\begin{array}{rcl} F\colon M \times [0,1] & \to & M \\ (Q,s) & \mapsto & \begin{cases} ((1-s)\cdot u + s\cdot \eta(u), R), & \text{if } Q = (u,R) \in [0,1] \times \partial M, \\ Q, & \text{if } Q \in W_1. \end{cases} \end{array}$$

It follows from the Smooth Pasting Proposition 19.23 that this map is smooth.[291] It is now clear that this map has all the desired properties. ■

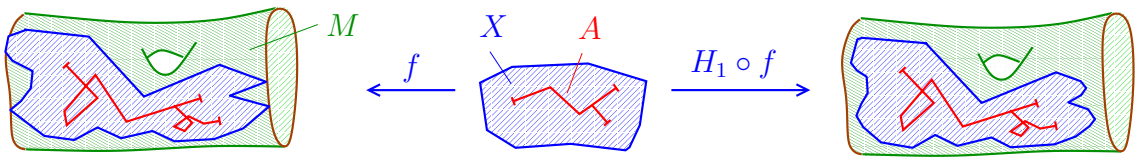
For technical purposes it is at times useful to arrange that maps take values "away from the boundary". In many cases this can be achieved by the following corollary.

Corollary 28.5. (Push-Away-from-Boundary Corollary) Let M be a smooth manifold.

(1) The smooth manifold M is homotopy equivalent to $M\setminus\partial M$. In particular, if ∂M is non-empty, then M is homotopy equivalent to a non-compact n-dimensional smooth manifold without boundary.

(2) (a) Let X be a topological space, let $f\colon X\to M$ be a map and let $A\subset X$ be a (possibly empty) compact subset with $f(A)\subset M\setminus\partial M$. There exists a smooth homotopy $H\colon M\times[0,1]\to M$ rel $f(A)$ with $H_0 = \mathrm{id}$ and with $(H_1\circ f)(X)\subset M\setminus\partial M$.

(b) If X is a smooth manifold and $f\colon X\to M$ is a smooth map, then f is smoothly homotopic to a map $f\colon X\to M\setminus\partial M$.



Proof. We set $M' := M\setminus\partial M$. By the Smooth Collar Neighborhood Theorem 28.3 there exists a collar $\chi\colon[0,1]\times\partial M\to M$.

(1) It follows from the Basics-of-Collars Lemma 28.4 (2a) and Lemmas 15.5 and 15.1 that M and M' are homotopy equivalent. Furthermore note that it follows from the

[291] The super vigilant reader will notice that we are really using the obvious extension of the Smooth Pasting Proposition 19.23 to smooth manifolds with corner.

Smooth Manifold Boundary Proposition 19.26 that M' is non-compact, that it is an open subset of M (in particular it is an n-dimensional smooth manifold) and that the smooth manifold M' does not have any boundary.

(2) (a) Since A is compact we obtain from the Compact Image Lemma 2.13 (1) that $f(A)$ is compact. Since $\chi([0,1] \times \partial M)$ is a closed subset we obtain from the elementary Lemma 1.21 that $f(A) \cap \chi([0,1] \times \partial M)$ is also compact. Finally note that it follows from the above discussion, the Compact Image Lemma 2.13 (2) and the hypothesis that $f(A) \subset M \setminus \partial M$ that there exists an $\epsilon \in (0,1]$ such that $\Phi([0,\epsilon] \times \partial M) \cap f(A) = \varnothing$.
By the Basics-of-Collars Lemma 28.4 (3b), applied to the collar $(t,x) \mapsto \chi(t \cdot \epsilon, x)$, there exists a smooth homotopy $H\colon M \times [0,1] \to M$ rel $M \setminus \chi([0,\epsilon] \times \partial M)$ such that $H_0 = \mathrm{id}$ and such that $H_1(M) \subset M \setminus \partial M$.

(b) We pick a smooth homotopy $H\colon M \times [0,1] \to M$ as in (a). The desired smooth homotopy is now given given the map[292]

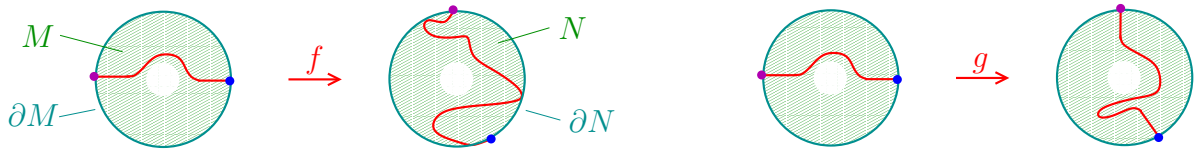
$$\begin{array}{rcl} X \times [0,1] & \to & M \\ (x,t) & \mapsto & H(f(x),t). \end{array}$$

■

We conclude this exhausting discussion of collar neighborhoods with the following technically quite useful lemma.

Lemma 28.6. (Homotopic-to-Product-Near-Boundary Lemma) Let M and N be smooth manifolds with collars $\mu\colon [0,1] \times \partial M \to M$ and $\nu\colon [0,1] \times \partial N \to N$. To simplify the notation we identify $[0,1] \times \partial M$ and $[0,1] \times \partial N$ with their images under μ and ν. Furthermore let $f\colon M \to N$ be a map with $f(\partial M) \subset \partial N$. There exists a homotopy $F\colon M \times [0,1] \to N$ from f to a map g such that the following holds:

(1) For each $t \in [0,1]$ we have $F_t(\partial M) \subset \partial N$.
(2) The map $g\colon M \to N$ has the following two properties:
  (a) For every $(t,P) \in [-\frac{1}{2},0] \times \partial M$ we have $g(t,P) = (t, f(P)) \in [-\frac{1}{2},0] \times \partial N$.
  (b) We have $g^{-1}([-\frac{1}{2},0] \times \partial N) = [-\frac{1}{2},0] \times \partial M$.



Proof. We consider the homotopy

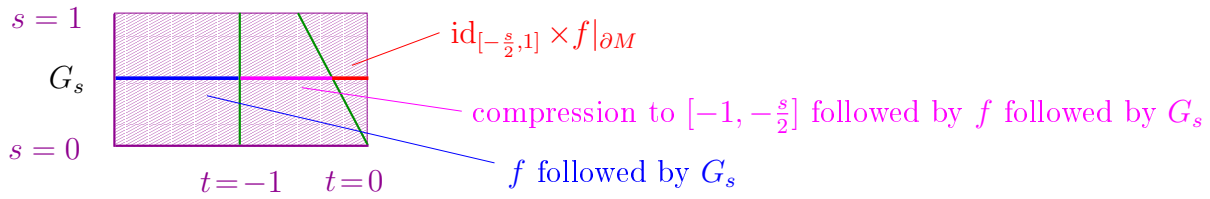
$$\begin{array}{rcl} G\colon N \times [0,1] & \to & N \\ (Q,s) & \mapsto & \begin{cases} (-\frac{s}{2}, \widetilde{Q}), & \text{if } Q = (t,\widetilde{Q}) \in [-\frac{s}{2},0] \times \partial N, \\ Q, & \text{if } Q \in N \setminus ((-\frac{s}{2},0] \times \partial N) \end{cases} \end{array}$$

Note that each G_s "pushes everything in $[-\frac{s}{2},0] \times \partial N$ to $\{-\frac{s}{2}\} \times \partial N$". Now we consider the map

[292] It follows from the Smooth Manifold Product Proposition 20.6 that this map is indeed smooth.

$$\begin{array}{rcl} F\colon M\times[0,1] & \to & N \\ (P,s) & \mapsto & \left(\begin{array}{ll} (t, f(0,\widetilde{P})), & \text{if } P=(t,\widetilde{P}) \text{ with } t\in[-\frac{1}{2}s,0], \\ G_s\big(f\big(\frac{t+\frac{1}{2}s}{1-\frac{1}{2}s},\widetilde{P}\big)\big), & \text{if } P=(t,\widetilde{P})\in[-1,-\frac{1}{2}s], \\ G_s(f(P)), & \text{if } P\notin(-1,0]\times\partial M. \end{array}\right) \end{array}$$

Note that $F_0 = f$. One can now easily verify that this homotopy F has the desired properties. ■



28.3. **Proof of the Smooth Collar Neighborhood Theorem 28.3.** This section is completely dedicated to the proof of the Smooth Collar Neighborhood Theorem 28.3. For non-compact smooth manifolds we will need the following technical lemma which is of interest only for diehard fans of noncompact smooth manifolds and their collars.

Lemma 28.7. Let M be a smooth manifold and let $f\colon [0,1]\times\partial M\to M$ be a smooth embedding such that for every $P\in\partial M$ we have $f(0,P)=P$. There exists a smooth map $\psi\colon \partial M\to(0,1]$ such that the map

$$\begin{array}{rcl} g\colon [0,1]\times\partial M & \to & M \\ (t,x) & \mapsto & f(\psi(x)\cdot t, x) \end{array}$$

is a collar, i.e. it is a smooth embedding such that $g(0,P)=P$ for all $P\in\partial M$ and such that $g([0,1]\times\partial M)$ is a closed subset of M.



Remark.

(1) We had alluded to Lemma 28.7 earlier on page 738 as it allows us navigate between the different notions of a collar for a non-compact smooth manifolds.
(2) Our proof of Lemma 28.7 is based on the proof provided in [**Lee2002b**].

Proof. By the Manifold Exhaustion Lemma 21.2 there exists a smooth map $\mu\colon M\to[0,\infty)$ with the property that given any $C\in\mathbb{R}$ the preimage $f^{-1}([0,C])$ is a compact subset of M. We consider the map

$$\begin{array}{rcl} \varphi\colon \partial M & \to & [0,1] \\ x & \mapsto & \sup\big\{t\in[0,1]\,\big|\,\mu(f(x,s))>\mu(\underbrace{f(x,0)}_{=x})-1 \text{ for all } s\in[0,t]\big\}. \end{array}$$

Using charts one can show fairly easily that the map φ is continuous and that $\varphi(x)>0$ for every $x\in\partial M$. By the Replace-by-Smooth Non-Negative Function Corollary 21.11 there

exists a smooth map $\psi\colon \partial M \to (0,1]$ such that $\psi(x) \in (0, \varphi(x)]$ for all $x \in \partial M$. It suffices to prove the following claim:

Claim. The map
$$\begin{array}{rcl} g\colon [0,1]\times\partial M & \to & M \\ (t,x) & \mapsto & f(\psi(x)\cdot t, x) \end{array}$$
is a smooth embedding such that $g(0,P) = P$ for all $P \in \partial M$ and such that $g([0,1]\times\partial M)$ is a closed subset of M.

Proof. The only statement which requires some thought is to show that $g([0,1]\times\partial M)$ is a closed subset of M. We have

$$M \setminus g([0,1]\times\partial M) = \underbrace{\Big(\bigcup_{C\in\mathbb{R}} \mu^{-1}([0,C))\Big)}_{=M} \setminus g([0,1]\times\partial M) = \bigcup_{C\in\mathbb{R}} \mu^{-1}([0,C)) \setminus g([0,1]\times\partial M)$$

$$= \bigcup_{C\in\mathbb{R}} \underbrace{\mu^{-1}([0,C))}_{\text{open}} \setminus \underbrace{g([0,1]\times(\,\partial M \cap \overbrace{\mu^{-1}([0,C+1])}^{\text{compact}}\,))}_{\substack{\text{compact by Proposition 19.26 (3a)}\\ \text{and Exercise 1.22}}}.$$

$\uparrow$ since $\mu(g(t,x)) > \mu(x,0) - 1$ for all $(t,x) \in [0,1]\times\partial M$

closed by the Compact Image Lemma 2.13 and the Compact-Closed Lemma 1.21

Thus we have shown that $M \setminus g([0,1]\times\partial M)$ is the union of open subsets of M, thus $M \setminus g([0,1]\times\partial M)$ itself is open, i.e. $g([0,1]\times\partial M)$ is closed. ∎

Now we can finally turn to the actual proof of the Smooth Collar Neighborhood Theorem 28.3.

Proof of the Smooth Collar Neighborhood Theorem 28.3. Let M be a smooth manifold and let V be an open neighborhood of ∂M. By the Euclidean Embedding Theorem 27.1 there exists an $n \in \mathbb{N}$ and a proper smooth embedding $\varphi\colon M \to \mathbb{H}^n$. Thus, by the Smooth Embedding Theorem 24.10, we might as well assume that M is a proper smooth submanifold of $\mathbb{H}^n$. By definition of "proper" this means in particular that the inclusion $M \to \mathbb{H}^n$ is a closed embedding. Note that it follows from the Smooth Manifold Boundary Proposition 19.26 that the inclusion $\partial M \to \mathbb{H}^n$ is also a closed embedding.
Our goal is to construct a collar $\chi\colon [0,1]\times\partial M \to M$. In the following three claims we first introduce three auxiliary maps. Afterwards we provide the actual construction of a collar.

Claim 1. The map
$$\begin{array}{rcl} \Omega\colon [0,\infty)\times\partial M & \to & \mathbb{H}^n \\ (t,P) & \mapsto & P + t\cdot e_n \end{array}$$
is smooth and it satisfies $\Omega^{-1}(\partial\mathbb{H}^n) = \{0\}\times\partial M$.

Proof. The two statements are basically obvious. ⊞

The idea now is to compose the map $\Omega\colon [0,\infty)\times\partial M \to \mathbb{H}^n$ with a projection onto M. This sounds like a case for the Neighborhood Projection Theorem 28.1. But we had formulated - for good reasons - the Neighborhood Projection Theorem 28.1 only for smooth manifolds with *empty boundary*. Therefore we will have to work quite a bit to make the original idea work.

Recall that by the Normal Vectors Lemma 28.2 we know that

$$WM \;:=\; \big\{(P,w)\in\mathbb{H}^n\times\mathbb{R}^n \,\big|\, P\in M \text{ and } w\in\big(\widetilde{\mathrm{T}}_P M\big)^{\perp}\big\}$$

is an n-dimensional submanifold of $\mathbb{H}^n\times\mathbb{R}^n$ with $\partial(WM) = WM\cap(\partial\mathbb{H}^n\times\mathbb{R}^n)$.

Claim 2. We consider the projections

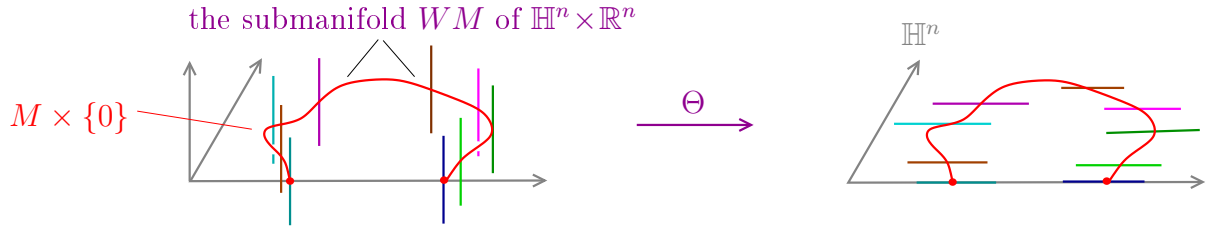
$$\begin{array}{rcl} p\colon \mathbb{H}^n & \to & \mathbb{R}^{n-1} \\ (x_1,\dots,x_n) & \mapsto & (x_1,\dots,x_{n-1}) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} q\colon \mathbb{H}^n & \to & \mathbb{R}_{\geq 0} \\ (x_1,\dots,x_n) & \mapsto & x_n. \end{array}$$

The map[293]

$$\begin{array}{rcl} \Theta\colon WM & \to & \mathbb{R}^{n-1}\times\mathbb{R}_{\geq 0} = \mathbb{H}^n \\ (P,w) & \mapsto & (p(P+w), q(P)) \end{array}$$

has the following properties:

(1) For every $P\in\partial M$ the differential $\mathrm{D}\Theta_{(P,0)}\colon \widetilde{\mathrm{T}}_{(P,0)}WM \to \mathbb{R}^n$ is an isomorphism.
(2) $\Theta^{-1}(\partial\mathbb{H}^n) = \partial(WM) = WM\cap(\partial\mathbb{H}^n\times\mathbb{R}^n)$.



Proof. Let $P\in\partial M$. Note that[294]

$$\widetilde{\mathrm{T}}_{(P,0)}WM \;=\; \{(v,w)\,|\,v\in\widetilde{\mathrm{T}}_P M \text{ and } w\in\big(\widetilde{\mathrm{T}}_P M\big)^{\perp}\}.$$

Recall that by definition of a proper smooth submanifold we know that $\widetilde{\mathrm{T}}_P M\subsetneq\mathbb{R}^{n-1}\times\{0\}$. Using these two observations one can easily prove (1). The second statements follow easily from the definitions. ⊞

Claim 3. The map

$$\begin{array}{rcl} \Pi\colon WM & \to & M \\ (P,w) & \mapsto & P \end{array}$$

is smooth and it satisfies $\Pi^{-1}(\partial M) = \partial(WM) = WM\cap(\partial\mathbb{H}^n\times\mathbb{R}^n)$.

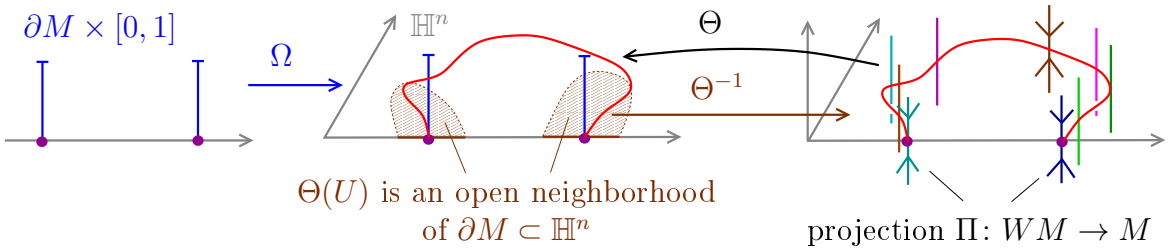
Proof. It follows from the Smooth Manifold Product Proposition 19.31 together with the Submanifold Smooth Structure Lemma 19.24 that $\Pi\colon WM\to M$ is smooth. The statement regarding $\Pi^{-1}(\partial\mathbb{H}^n)$ follows easily from the definitions. ⊞

As we mentioned above, the inclusion $\partial M\to\mathbb{H}^n$ is a closed embedding. Evidently for each $P\in\partial M$ we have $\Theta(P,0) = P$. It follows that the restriction of Θ to $\partial M\times\{0\}$ is a closed embedding. Furthermore note that it follows from Claim 2 (1) and (2), together with the Inverse Mapping Theorem 24.1, that $\Theta\colon WM\to\mathbb{H}^n$ is a local diffeomorphism at each $(P,0)$ with $P\in\partial M$. These arguments show that we can apply the Neighborhood Smooth Embedding Theorem 21.13 to $\Theta\colon WM\to\mathbb{H}^n$ and the closed subset $\partial M\times\{0\}$. We obtain

[293] The map $\Theta\colon WM\to\mathbb{H}^n$ is related to the map $\Xi\colon WM\to\mathbb{R}^n$, $(P,w)\mapsto P+w$ that we introduced in the Normal Vectors Lemma 28.2. The advantage of Θ is that it satisfies $\Theta^{-1}(\partial\mathbb{H}^n) = \partial(WM) = WM\cap(\partial\mathbb{H}^n\times\mathbb{R}^n)$ which is not true for Ξ.

[294] This equality can be seen as follows. One can easily verify that the right hand side is contained in the left hand side. Both are n-dimensional vector subspaces, thus they have to be the same.

an open neighborhood U of $\partial M \times \{0\} \subset WM$ such that $\Theta(U)$ is an open neighborhood of $\Theta(\partial M \times \{0\}) = \partial M \subset \mathbb{H}^n$ and such that $\Theta\colon U \to \Theta(U)$ is a diffeomorphism.



Claim 4. There exists a smooth map $\mu\colon \partial M \to (0,\infty)$ such that for each $P \in \partial M$ we have $\Omega([0,\mu(P)] \times \{P\}) \subset \Theta(U)$.

Proof. First note that it follows from the fact that $\Theta(U)$ is an open subset of $\mathbb{H}^n$ and the definition of the product topology on $[0,1] \times \partial M$ that for every $P \in \partial M$ there exists a $\mu > 0$ and an open neighborhood V of $P \in \partial M$ such that $\Omega([0,\mu] \times V) \subset \Theta(U)$. The claim follows from this observation together with the Replace-by-Smooth Non-Negative Function Corollary 21.11 (2). ⊞

For the reader's convenience we summarize all the maps involved so far:

$$\begin{array}{ccccccc} \{(t,P)\,|\,t \in [0,\mu(P)]\} & \xrightarrow{\Omega} & \Theta(U) & \underset{\cong}{\overset{\Theta}{\leftarrow}} & U & & \\ \downarrow & & \downarrow & \overset{\Theta^{-1}}{\curvearrowright} & \downarrow & & \\ [0,\infty) \times \partial M & \xrightarrow{\Omega} & \mathbb{H}^n & \xleftarrow{\Theta} & WM & \xrightarrow{\Pi} & M. \end{array}$$

The following claim now finally proclaims that we can find a collar.

Claim 5. There exists a smooth map $\nu\colon \partial M \to (0,\infty)$ such that for each $P \in \partial M$ we have $\nu(P) \le \mu(P)$ and such that the map[295]

$$\begin{array}{rcl} [0,1] \times \partial M & \to & M \\ (t,P) & \mapsto & (\Pi \circ \Theta^{-1} \circ \Omega)((\nu(P) \cdot t, P)) \end{array}$$

is a collar.

Proof. First we consider the map

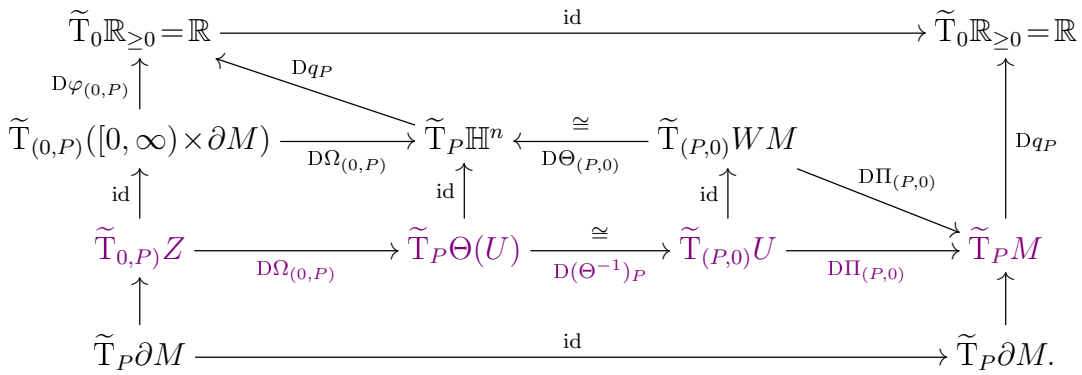
$$\begin{array}{rcl} g\colon \overbrace{\{(t,P) \in [0,\infty) \times \partial M\,|\,t \in [0,\mu(P)]\}}^{=:Z} & \to & M \\ (t,P) & \mapsto & (\Pi \circ \Theta^{-1} \circ \Omega)(t,P). \end{array}$$

We make the following observations:

(1) It follows immediately from the definitions of the various maps that for every $P \in \partial M$ we have $g(0,P) = P$. Since ∂M is a closed subset of $\mathbb{H}^n$ we see that the restriction of g to $\partial M \times \{0\}$ is a closed embedding.

[295] Note that it follows from $\nu(P) \le \mu(P)$ that $\Omega(\nu(P) \cdot t, P) \in \Theta(U)$. Thus $(\Pi \circ \Theta^{-1} \circ \Omega)((\nu(P) \cdot t, P))$ is indeed defined.

(2) We now want to show that for each $P \in \partial M$ the differential $\mathrm{D}g_{(0,P)} \colon \widetilde{\mathrm{T}}_{(0,P)}Z \to \widetilde{\mathrm{T}}_P M$ is an isomorphism. To do so we consider the following diagram:



We make the following clarifications and observations:

- As before we denote by $q \colon \mathbb{H}^n \to \mathbb{R}_{\geq 0}$ the projection onto the last coordinate.
- We denote by $\varphi \colon [0,\infty) \times \partial M \to \mathbb{R}_{\geq 0}$ the map that is given by the projection onto the first factor.
- The maps that have no names are the differentials of the obvious inclusions.
- It follows immediately from the definitions that $q \circ \Theta = q \circ \Pi \colon WM \to \mathbb{R}_{\geq 0}$ and that $q \circ \Omega = \varphi \colon [0,\infty) \times \partial M \to \mathbb{R}_{\geq 0}$. Using these two observations and using the chain rule one can easily verify that the diagram commutes.
- The composition of the purple horizontal maps is precisely the map $\mathrm{D}g_{(0,P)}$.
- It is straightforward to verify that the vertical sequence to the left is exact.[296]
- It follows easily from the definition of a proper submanifold that the vertical sequence to the right is exact.

A gentle stroll through the diagram now implies that $\mathrm{D}g_{(0,P)} \colon \widetilde{\mathrm{T}}_{(P,0)}Z \to \widetilde{\mathrm{T}}_P M$ is indeed an isomorphism.

(3) It follows immediately from the corresponding statements in Claim 1, 2 and 3 that $g^{-1}(\partial M) = \partial M \times \{0\}$.

It follows from (2) and (3), together with the Inverse Mapping Theorem 24.1, that for each $P \in \partial M$ the map g is a local diffeomorphism at $(0,P)$. Furthermore, using (1), we obtain from the Neighborhood Smooth Embedding Theorem 21.13 that there exists an open neighborhood Y of $\{0\} \times \partial M \subset Z = \{(t,P) \in [0,\infty) \times \partial M \,|\, t \in [0,\mu(P)]\}$ such that $g \colon Y \to M$ is a smooth embedding. After possibly replacing $Y \cap V$ where V is the given neighborhood of ∂M - the reader surely remembers that we are given such a neighborhood - we can assume that $Y \subset V$.

It follows again from Lemma 28.7 together with the Replace-by-Smooth Non-Negative Function Corollary 21.11 (2) that there exists a smooth map $\nu \colon \partial M \to (0,\infty)$ such that the following three conditions are satisfied:

(1) For each $P \in \partial M$ we have $\nu(P) \leq \mu(P)$.
(2) The image of $\{(t,P) \in [0,\infty) \times \partial M \,|\, t \in [0,\nu(P)]\}$ under g is a closed subset of M.

[296] Recall that according to the definition on page 670 a sequence of homomorphisms $U \xrightarrow{\varphi} V \xrightarrow{\psi} W$ between real vector spaces is exact if $\varphi \colon U \to V$ is a monomorphism, if $\psi \colon V \to W$ is an epimorphism and if $\mathrm{im}(\varphi) = \ker(\psi)$.

(3) For each $P \in \partial M$ we have $\{(t, P) \in [0, \infty) \times \partial M \mid t \in [0, \nu(P)]\} \subset Y$.

One can now easily verify that the map of the claim is indeed a collar. ■

Exercises for Chapter 28.

Exercise 28.1. Let X and Y be two topological spaces and suppose that Y is equipped with a metric d. Let $\epsilon > 0$. We say that two maps $f, g\colon X \to Y$ are ϵ-*close* if for all $x \in X$ we have $d(f(x), g(x)) < \epsilon$.

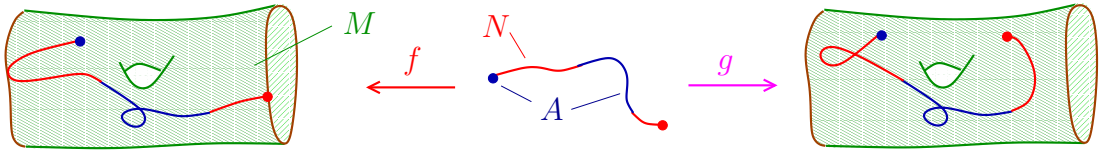
(a) Let M be a smooth submanifold of some $\mathbb{R}^n$ which is closed if viewed as a smooth manifold in its own right. Show that there exists an $\epsilon > 0$ with the following property: If two maps $f, g\colon X \to M$ are ϵ-close, then they are actually homotopic.

(b) Extend the result in (a) to the case that M is a compact smooth submanifold of $\mathbb{R}^n$

(c) Show that the conclusion of (a) and (b) does not hold if we do not assume that M is compact.

(d) Let M be a closed smooth manifold and let d be a metric on M that induces the topology on M. Show that there exists an $\epsilon > 0$ with the following property: If two maps $f, g\colon X \to M$ are ϵ-close, then they are actually homotopic.
Remark. Use Exercise 2.16.

Exercise 28.2. Let M be a smooth manifold, let N be a compact smooth manifold, let $A \subset N$ be closed and let $f\colon N \to M$ be a (smooth) map with $f(A) \subset M \setminus \partial M$. Show that f is (smoothly) homotopic rel A to a map $g\colon N \to M \setminus \partial M$.
Remark. This exercise can be viewed as a smooth analogue of the Push Away From Boundary Lemma 18.13. In particular one can use it to show again that if M is path-connected, then $M \setminus \partial M$ is also path-connected.

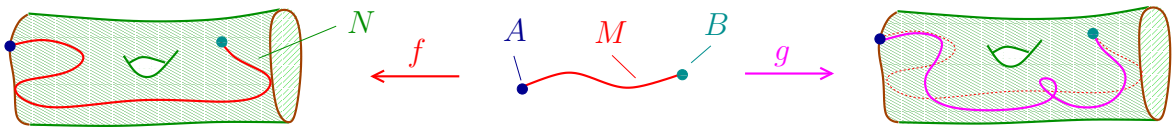


Exercise 28.3. Give an example of a smooth manifold M together with a smooth embedding $\chi\colon [0, 1) \times \partial M \to M$ such that $\chi(0, P) = P$ for all $P \in \partial M$, which cannot be extended to a smooth embedding $f\colon [0, 1] \times \partial M \to M$.

Exercise 28.4. Show that there is no smooth retraction from $[0, 2]$ to $[0, 1]$. More precisely, show that there is no smooth map $f\colon [0, 2] \to [0, 1]$ which satisfies $f(x) = x$ for all $x \in [0, 1]$.

Exercise 28.5. Let M be a compact smooth manifold and let N be any smooth manifold. Furthermore let A be a union of components of ∂M. We set $B := \partial M \setminus A$. Show that for every (smooth) map $f\colon M \to N$ with $f(A) \subset \partial N$ and with $f(B) \subset N \setminus \partial N$ there exists a (smooth) homotopy $F\colon M \times [0, 1] \to N$ rel ∂M from $f\colon M \to N$ to a map $g\colon M \to N$ such that $g^{-1}(\partial N) = A$.

Exercise 28.6. Let M be a smooth manifold and let A and B be two distinct boundary components. Show that there exists a map $f\colon M \to [0, 1]$ such that $f|_A \equiv 0$ and such that



$f|_B \equiv 1$.
Remark. One can prove this statement easily using Urysohn's Lemma 21.7 for Manifolds. But it is more fun to prove it using the Smooth Collar Neighborhood Theorem 28.3.

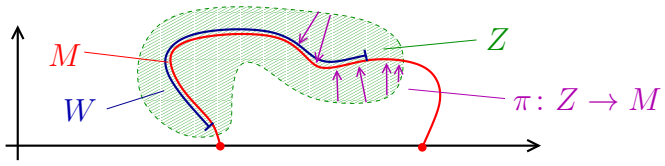
Exercise 28.7. Let M be a smooth manifold and let $f\colon \partial M \to S^n$ be a map. Show that there exists a smooth map $g\colon M \to \overline{B}^{n+1}$ with $g|_{\partial M} = f$.

Exercise 28.8. Show that given any n-dimensional smooth manifold M there exists an n-dimensional smooth manifold N with $\partial N = \varnothing$ and a smooth embedding $\varphi\colon M \to N$ such that $\varphi(M)$ is a closed subset of N.
Hint. You could take $N = M \setminus \partial M$.

Exercise 28.9. Let M be a proper smooth manifold of some $\mathbb{H}^n$ and let $W \subset M$ be a closed subset with $W \cap \partial M = \varnothing$. Show that there exists a smooth neighborhood Z of $W \subset \mathbb{R}^n$ with $Z \cap \partial M = \varnothing$ and a smooth map $\pi\colon Z \to M$ which has the following two properties:

(1) for each $P \in W$ we have $\pi(P) = P$,
(2) π is a submersion, i.e. for each $Q \in Z$ the differential $\mathrm{D}\pi_Q$ is an epimorphism.

Hint. You could try to modify the proof of the Neighborhood Projection Theorem 28.1.



29. The Whitney Approximation Theorem

29.1. **The statement of the Whitney Approximation Theorem.** As we will see later on, there are many ways to construct interesting continuous maps between two smooth manifolds. The following theorem often allows us to "replace" such a continuous map by a smooth map.

Theorem 29.1. (Whitney Approximation Theorem) Let M and N be smooth manifolds and let $f\colon M \to N$ be a map.[297] There exists a homotopy $F\colon M \times [0,1] \to N$ with the following properties:

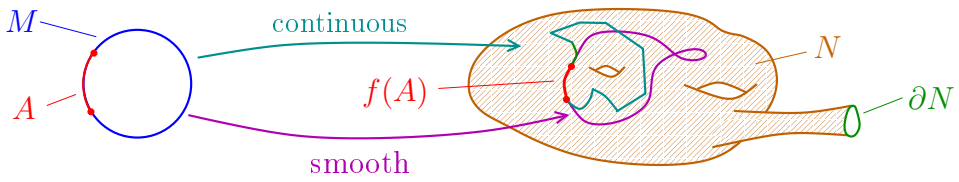
(1) We have $F_0 = f$.
(2) The map $F_1\colon M \to N$ is smooth.
(3) If there exists a union of components X of ∂M with $f(X) \subset \partial N$, then we can arrange that $F_t(X) \subset \partial N$ for all $t \in [0,1]$.

Suppose that f is smooth[298] on a (possibly empty) closed subset A and suppose that there exists an open neighborhood[299] of ∂N that does not intersect $f(A)$. Then we can find a homotopy as above which also has the following property:

(4) The homotopy is a homotopy rel A.

Finally suppose that we are given a map[300] $\rho\colon M \to \mathbb{R}_{\geq 0}$ such that $\rho(x) > 0$ for every $x \notin A$ and suppose that we are given a metric d on N which induces the given topology on N.[301] Then we can find a homotopy F as above that also satisfies the following extra condition:

(5) For every $(x,t) \in M \times [0,1]$ we have $d(F(x,t), f(x)) \leq \rho(x)$.



Remark. Suppose we are in the setting of the Whitney Approximation Theorem 29.1. Let $f\colon M \to N$ be a map that is injective (respectively surjective, bijective). Now we know that we can find a homotopy from f to some smooth map $F_1 = g\colon M \to N$, but the theorem does not give us any control over whether the smooth map $F_1 = g$ is also injective (respectively surjective, bijective). For example, by the Space-Filling Curve Theorem 6.13 we know that there exists a surjective map $[0,1] \to [0,1]^2$. It follows easily, see also Exercise 6.13, that

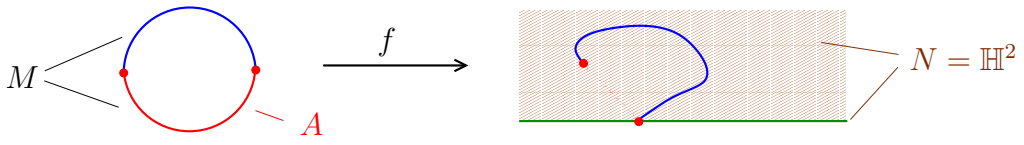
[297] Note that M and N are both allowed to have non-empty boundary.
[298] Let M and N be smooth manifolds and let $A \subset M$ be an arbitrary subset. On page 549 we defined what it means for a map $f\colon A \to N$ to be smooth.
[299] This condition is satisfied if A is compact and $f(A) \cap \partial N = \varnothing$. Indeed, by the Compact Image Lemma 2.13 and the Compact-Closed Lemma 1.21 we know that $f(A)$ is closed. Furthermore by the Smooth Manifold Boundary Proposition 19.26 we know that ∂N is closed. Finally note that by the Manifold Paracompact-Normal Proposition 18.9 (2) we know that N is normal. But this implies that there exists an open neighborhood of ∂N that does not intersect $f(A)$.
[300] Evidently we demand, as always, that ρ is continuous.
[301] Note that by the Manifold Metrization Proposition 18.10 we know that such a metric always exists.

there exists also a surjective map $\mathbb{R} \to \mathbb{R}^2$. But, as we will see in Propositions 30.2 and 30.3 there exists no surjective smooth map $\mathbb{R} \to \mathbb{R}^2$.

Example.

(1) Let M be a smooth manifold and let $f\colon [0,1] \to M$ be a path such that the endpoints $f(0)$ and $f(1)$ lie in $M \setminus \partial M$. By the Whitney Approximation Theorem 29.1, applied to f and the closed subset $\{0,1\} \subset [0,1]$, the map f is homotopic rel $\{0,1\}$ to a smooth path from $f(0)$ to $f(1)$. This gives a new proof of the Smooth Path-Connectivity Proposition 19.29 for smooth manifolds with empty boundary.

(2) In the setting of Whitney Approximation Theorem 29.1 (4) we demand that there exists an open neighborhood of ∂N that does not intersect $f(A)$. In the figure below we show a map $f\colon M = S^1 \to \mathbb{H}^2 = N$ which is smooth on $A = S^1_{\leq 0}$ but which cannot be turned into a smooth map without touching the values on A.



Remark. Proofs of various variations of the Whitney Approximation Theorem 29.1 are also provided in [**Lee2002**, Theorem 6.26], [**Lee2002**, Theorem 9.27], [**MTo1997**, Lemma A.9], [**Hirs1976**, Lemma 5.1.5] and [**DFN1985**, Theorem 10.1.2].

Before we turn to the proof of the Whitney Approximation Theorem 29.1 it is worth recalling the following proposition which basically takes care of the case $N = \mathbb{R}^n$.

Proposition 21.10. (Smooth Approximation Proposition) Let M be a smooth manifold and let $f\colon M \to \mathbb{R}^n$ be a map that is smooth on a (possibly empty) closed subset A. Given any map $\eta\colon M \to \mathbb{R}_{\geq 0}$ such that $\eta(x) > 0$ for every $x \notin A$ there exists a smooth map $\widetilde{f}\colon M \to \mathbb{R}^n$ with the following two properties:

(1) The maps f and $\widetilde{f}$ agree on A.

(2) For every $x \in M$ we have $\|f(x) - \widetilde{f}(x)\| \leq \eta(x)$.

We break the proof of the Whitney Approximation Theorem 29.1 into two parts. First we consider the case that $\partial N = \varnothing$. Afterwards we deal with the case that $\partial N \neq \varnothing$.

Proof of Theorem 29.1 if $\partial N = \varnothing$. Let M be a smooth manifold and let N be a smooth manifold with $\partial N = \varnothing$. Let $f\colon M \to N$ be a map that is smooth on a (possibly empty) closed subset A.

By the Euclidean Embedding Theorem 27.1, together with the Smooth Embedding Theorem 24.10, we can view N as a smooth submanifold of some $\mathbb{R}^n$. Since $\partial N = \varnothing$ we know by the Neighborhood Projection Theorem 28.1 there exists an open neighborhood U of $N \subset \mathbb{R}^n$ together with a smooth retraction $\pi\colon U \to N$.

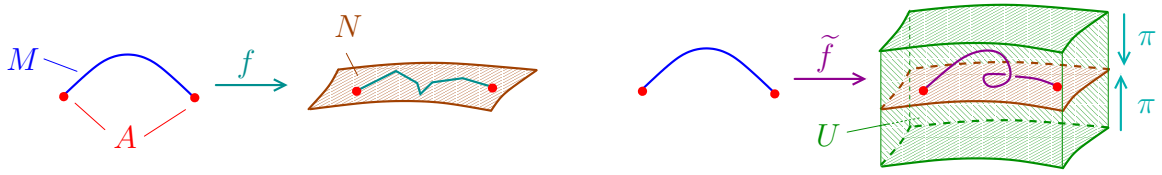
Since U is open it follows easily from Lemma 10.12 (see also Exercise 2.16) that there exists a continuous map $\mu\colon N \to \mathbb{R}_{>0}$ such that for each $y \in N$ we have $\overline{B}^n_{\mu(y)}(y) \subset U$. Next note that by the Smooth Approximation Proposition 21.10 we know that there exists a map $\widetilde{f}\colon M \to \mathbb{R}^n$ with the following two properties:

(a) The maps f and $\widetilde{f}$ agree on A.
(b) For every $x \in M$ we have $\|f(x) - \widetilde{f}(x)\| \leq \mu(f(x))$.

It is straightforward to verify that the map

$$\begin{array}{rcl} F\colon M \times [0,1] & \to & N \\ (x,t) & \mapsto & \pi\big(\underbrace{f(x)\cdot(1-t) + \widetilde{f}(x)\cdot t}_{\in U}\big) \end{array}$$

has all the desired properties (1), (2) and (4). Since $\partial N = \varnothing$ we see that in our current setting Statement (3) is vacuous.



Finally we turn to the proof of (5). Thus suppose that we are given a map $\rho\colon M \to \mathbb{R}_{\geq 0}$ such that $\rho(x) > 0$ for every $x \notin A$. Furthermore suppose that we are given a metric d on N that induces the given topology on N. We sketch how one needs to modify the above argument to obtain that Condition (5) is satisfied, i.e. we show that can arrange that for every $(x,t) \in M \times [0,1]$ we have $d(F(x,t), f(x)) \leq \rho(x)$. First we make the following two observations:

(a) Since $\|-\|$ and d give rise to the same topology we obtain from Lemmas 1.2 and 10.12 (see also Exercise 2.16) that there exists a continuous map $\nu\colon M \to \mathbb{R}_{\geq 0}$ such that for all $x \in M$ and all $y \in N$ with $\|y - f(x)\| \leq \nu(x)$ we have $d(y, f(x)) \leq \rho(x)$.
(b) Since π is smooth one can show without too much effort that there exists a continuous function $\chi\colon N \to \mathbb{R}_{>0}$ such that for every $y \in N$ and every $z \in \overline{B}_{\mu(y)}(y)$ we have $\|\pi(z) - y\| < \chi(y) \cdot \|z - y\|$.

Now we apply the Smooth Approximation Proposition 21.10 to the map

$$\eta(x) \;:=\; \max\Big\{\mu(f(x)), \tfrac{1}{\chi(f(x))} \cdot \nu(f(x))\Big\}$$

and we obtain a suitable smooth map $\widetilde{f}\colon M \to \mathbb{R}^n$. As above one can now easily verify that the map

$$\begin{array}{rcl} F\colon M \times [0,1] & \to & N \\ (x,t) & \mapsto & \pi\big(\underbrace{f(x)\cdot(1-t) + \widetilde{f}(x)\cdot t}_{\in U}\big) \end{array}$$

has all the desired properties. ■

Proof of Theorem 29.1 if $\partial N \neq \varnothing$ and $X = \varnothing$. Let M be a smooth manifold and let N be a smooth manifold with $\partial N \neq \varnothing$. Let A be a (possibly empty) closed subset such that there exists an open neighborhood V of ∂N which satisfies $f(A) \cap V = \varnothing$. Finally let $f\colon M \to N$ be a map that is smooth on A.

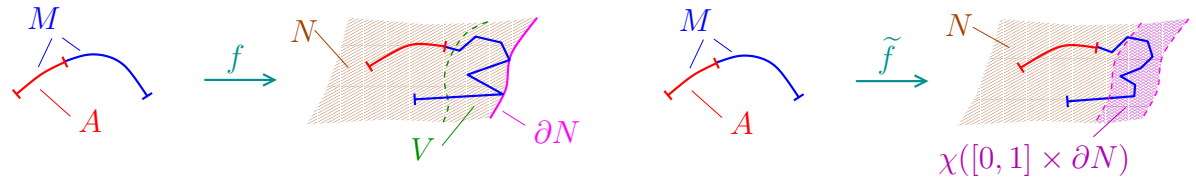
Note that by the Smooth Collar Neighborhood Theorem 28.3 there exists a collar $\chi\colon [0,1] \times \partial N \to N$ with $\chi([0,1] \times \partial N) \subset V$. As in the proof of the Basics-of-Collars

Lemma 28.4 (3a) we consider the homotopy

$$\begin{array}{rcl} G\colon N\times[0,1] & \to & N \\ (Q,s) & \mapsto & \begin{cases} \chi((1-s)\cdot u+s,R), & \text{if } Q=\chi(u,R)\in[0,1]\times\partial N, \\ Q, & \text{if } Q\in N\setminus\chi([0,1]\times\partial N) \end{cases} \end{array}$$

which describes the "obvious" deformation retraction from N to $N\setminus\chi([0,1)\times\partial N)$. Since $\chi([0,1]\times\partial N)\subset V$ and since $f(A)\cap V=\varnothing$ we see that G gives rise to a homotopy rel A from $f=G_0\circ f$ to $\widetilde{f}:=G_1\circ f$. Thus we have shown that f is homotopic rel A to a map $\widetilde{f}$ that takes values in $\widetilde{N}:=N\setminus\partial N$. By the Smooth Manifold Boundary Proposition 19.26 we know that $\widetilde{N}$ is a smooth manifold without boundary. Thus – by the above discussion of maps to smooth manifolds without boundary – we see that the map $\widetilde{f}\colon G_1\circ f\colon M\to\widetilde{N}$ is homotopic rel A to a smooth map $M\to\widetilde{N}$.

Finally suppose that we are given a map $\rho\colon M\to\mathbb{R}_{>0}$ and suppose that we are given a metric d on N that induces the given topology on N. Dealing with these extra constraints requires a more careful choice of γ. We leave it to the reader to fiddle around with our argument to produce a homotopy that also satisfies Condition (5). ■



Proof of Theorem 29.1 if $\partial N\neq\varnothing$ and $X\neq\varnothing$. We now suppose that there exists a union of components X of ∂M with $f(X)\subset\partial N$. We need to show that we can arrange that $F_t(X)\subset\partial N$ for all $t\in[0,1]$.

By the Smooth Collar Neighborhood Theorem 28.3 there exist collars $\mu\colon[0,1]\times X\to M$ and $\nu\colon[0,1]\times\partial N\to N$. As above we consider the homotopy

$$\begin{array}{rcl} G\colon N\times[0,1] & \to & N \\ (Q,s) & \mapsto & \begin{cases} \nu((1-s)\cdot u+s,R), & \text{if } Q=\nu(u,R)\in[0,1]\times\partial N, \\ Q, & \text{if } Q\in N\setminus\nu([0,1]\times\partial N). \end{cases} \end{array}$$

We write $\widetilde{f}:=f|_X$. Since $\widetilde{f}\colon X\to\partial M$ is a map between smooth manifolds without boundary we know by the above that there exists a homotopy $\widetilde{F}\colon X\times[0,1]\to\partial M$ with $\widetilde{F}_0=\widetilde{f}$ such that $\widetilde{F}_1\colon X\to\partial M$ is smooth.

It follows from an elementary argument, see Exercise 14.6 or the Trivial-Near-The-Ends-Lemma 33.4, that we can modify the homotopy $\widetilde{F}$ such that there exists an $\epsilon>0$ such that $\widetilde{F}_t=\widetilde{F}_1$ for all $t\in[1-\epsilon,1]$. Next we consider the homotopy[302]

$$\begin{array}{rcl} H\colon M\times[0,1] & \to & N \\ (x,t) & \mapsto & \begin{cases} G(x,t), & \text{if } x\notin\mu(X\times[0,1]), \\ G\big(f\big(\mu\big(y,\frac{s-\frac{t}{2}}{1-\frac{t}{2}}\big)\big),t\big) & \text{if } x=\mu(y,s) \text{ with } (y,s)\in X\times[\frac{t}{2},1], \\ \nu(\widetilde{F}(y,t-2s),2s) & \text{if } x=\mu(y,s) \text{ with } (y,s)\in X\times[0,\frac{t}{2}]. \end{cases} \end{array}$$

[302]This formula for H is clearly unreadable. Nonetheless, one can easily plug in the neuralgic values and see, using the Pasting Proposition 2.6, that this map is indeed continuous.

We set $h := H_1 \colon M \to N$. Note that h is smooth on the closed subset $\mu([0, \frac{\epsilon}{2}] \times \partial M)$. We set $\widetilde{M} := M \setminus \mu([0, \frac{\epsilon}{4}) \times \partial M)$ and consider the closed subset $\widetilde{A} := \mu([\frac{\epsilon}{4}, \frac{\epsilon}{2}] \times \partial M) \subset \widetilde{M}$.

Note that $h(\widetilde{M}) \subset N \setminus \partial N$ and note that h is smooth on $\widetilde{A}$. Thus we can apply the first case to $h|_{\widetilde{M}} \colon \widetilde{M} \to N$ and we see that $h|_{\widetilde{M}}$ is homotopic rel $\widetilde{A}$ to a smooth map. We extend this smooth map via h to $\partial M \times \mu([0, \frac{\epsilon}{4}])$ to obtain a map that is defined on all of M.

Note that the resulting map is smooth on the open subsets $M \setminus \mu([\frac{\epsilon}{4}, \frac{\epsilon}{2}) \times \partial M)$ and $\mu([0, \frac{\epsilon}{2}) \times \partial M)$. Thus it follows from the Smooth Pasting Proposition 19.23 that this map is smooth.

Some adjustments of the above argument can also take into account the presence of a closed subset A and a map $\rho \colon M \to \mathbb{R}_{>0}$ and a metric d on N. We leave this fun task to the reader who is surely tired of reading other people's proofs. ■

29.2. **Consequences of the Whitney Approximation Theorem 29.1.** In this section we formulate and prove several straightforward but useful consequences of the Whitney Approximation Theorem 29.1.

Proposition 29.2. (Replace-by-Smooth-Fixed-Point-Free Map Proposition) Let M be a smooth manifold. Every map $f \colon M \to M$ without fixed points is homotopic to a *smooth* map $g \colon M \to M$ without fixed points.

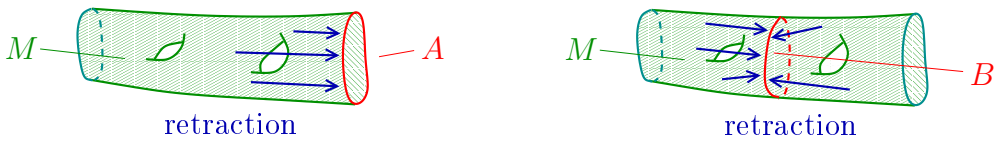
Proof. Let M be a smooth manifold and let $f \colon M \to M$ be a map without fixed points. By the Manifold Metrization Proposition 18.10 there exists a metric d on M that induces the given topology on M. Given $x \in M$ we set $\rho(x) := d(x, f(x))$. Note that it follows from the Metric-is-Continuous Lemma 6.1 that ρ is continuous. By the Whitney Approximation Theorem 29.1 there exists a homotopy $F \colon M \times [0,1] \to M$ with $f = F_0$ such that $g := F_1$ is a smooth map $g \colon M \to M$ that satisfies the inequality $d(f(x), g(x)) < \rho(x)$ for every $x \in M$. From the triangle inequality we obtain that $d(x, g(x)) > 0$ for every $x \in M$. In other words, g has no fixed points. ■

For the reader's convenience we recall the following definition from page 433.

Definition. Let X be a topological space. We say a subset $A \subset X$ is a retract of X if there exists a retraction $r \colon X \to A$, i.e. a map with $r(a) = a$ for all $a \in A$.

Proposition 29.3. (Replace-by-Smooth Retraction Proposition) Let M be a smooth manifold.

(1) Let A be a union of boundary components. If there exists a retraction $f \colon M \to A$, then there exists also a smooth retraction $g \colon M \to A$.
(2) Let B be a proper submanifold of M. If there exists a retraction $f \colon M \to B$, then there exists also a smooth retraction $g \colon M \to B$.



Example. Note that in the Replace-by-Smooth Retraction Proposition 29.3 (2) it is essential that we work with *proper* submanifolds. For example in Exercise 29.5 we will see that

for the smooth manifold $M = \mathbb{R}$ and the smooth submanifold $A = (-\infty, 0]$ there exists a retraction from M to A but there is no smooth retraction.

The proof of the Replace-by-Smooth Retraction 29.3 makes use of the following lemma, which is interesting in its own right.

Lemma 29.4. (Smooth-on-Boundary-Submanifold Lemma) Let $f\colon M \to N$ be a map between two smooth manifolds. We suppose that we are in one of the following two settings.

(1) We are given a union A of boundary components. We equip A with the standard smooth structure that we introduced on page 555
(2) We are given a submanifold A of M. We equip A with the standard smooth structure that we introduced on page 552.

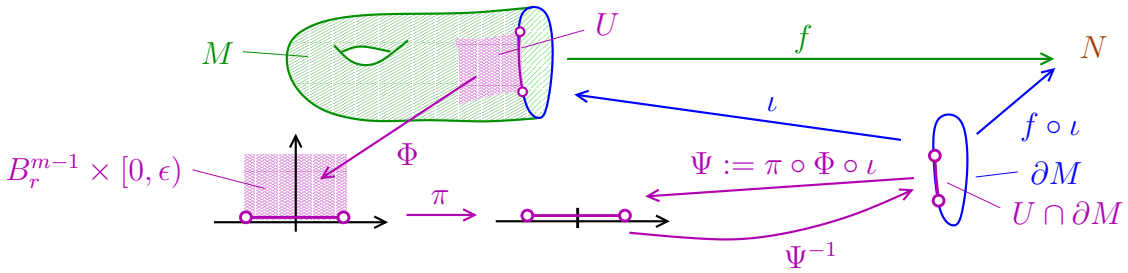
We denote by $\iota\colon A \to M$ the inclusion map. If $f \circ \iota\colon A \to N$ is a smooth map between smooth manifolds, then $f\colon M \to N$ is smooth on the subset $A \subset M$ in the sense of the definition on page 549.

Remark. the Smooth-on-Boundary-Submanifold Lemma 29.4 is basically the content of Exercise 19.24.

Proof of the Smooth-on-Boundary-Submanifold Lemma 29.4. We write $m = \dim(M)$. We provide the proof of Statement (1). It should be clear how to modify the proof to obtain a proof of Statement (2). Now let A be a union of boundary components and let $P \in A$. Note that basically by definition of a boundary point there exists a chart $\Phi\colon U \to B_r^{m-1} \times [0, \epsilon)$ of type (ii) for $P \in M$. We denote by $\pi\colon B_r^{m-1} \times [0, \epsilon) \to B_r^{m-1}$ the natural projection.

The map $\Psi := \pi \circ \Phi \circ \iota\colon U \cap \partial M \to B_r^{m-1}$ is, by definition of the smooth structure on ∂M, a chart for ∂M. We denote by $\Psi^{-1}\colon B_r^{m-1} \to U \cap \partial M$ the inverse.

Now we set $\widetilde{f} := (f \circ \iota) \circ \Psi^{-1} \circ \pi \circ \Phi$. This is a composition of smooth maps between smooth manifolds, thus it is smooth. Furthermore, by definition of Ψ we have $\widetilde{f}|_{U \cap \partial M} = f$. We have thus shown that the restriction of $f\colon M \to A$ to A is smooth in the sense of the definition on page 549. ■

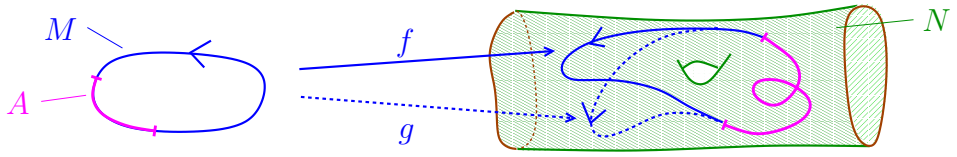


Proof of the Replace-by-Smooth Retraction 29.3. We provide the proof of Statement (1). Again it should be clear how to modify the proof to obtain a proof of Statement (2). Let M be an n-dimensional smooth manifold and let A be a union of boundary components. Furthermore let $f\colon M \to A$ be a retraction.

Note that $f|A = \mathrm{id}_A\colon A \to A$ is evidently smooth as a map from the smooth manifold A to itself. It follows from the Smooth-on-Boundary-Submanifold Lemma 29.4 that the

map $f\colon M \to A$ is smooth on A in the sense of the definition on page 549. It follows that we can apply the Whitney Approximation Theorem 29.1 and we obtain a homotopy rel A from f to a smooth map $g\colon M \to A$. Since g is homotopic rel A to the retraction f we see that g is also a retraction. Thus g is the desired smooth retraction from M to A. ■

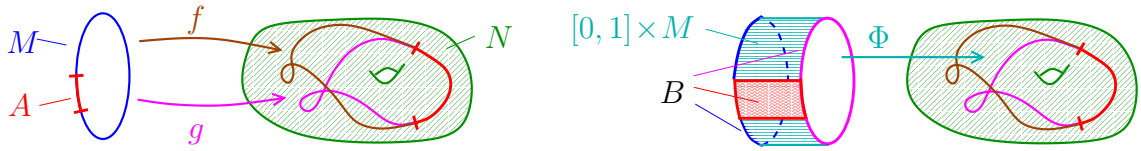
Proposition 29.5. (Replace-by-Smooth Homotopy Proposition) Let M be a smooth manifold without boundary and let N be a smooth manifold (with or without boundary). Let $f, g\colon M \to N$ be two smooth maps that agree on a (possibly empty) closed subset A. We suppose that there exists an open neighborhood V of ∂N such that $f(A) = g(A)$ is contained in $N \setminus V$.[303] If f and g are homotopic rel A, then there exists also a smooth homotopy rel A between f and g.



Proof. Let $f, g\colon M \to N \setminus \partial N$ be two smooth maps that agree on a closed subset A and let $\Phi\colon M \times [0,1] \to N$ be a homotopy rel A.

We first consider a special case, namely we suppose that there exists an open neighborhood V of ∂N such that actually $f(M)$ and $g(M)$ are contained in $N \setminus V$.

Note that $B := (M \times \{0\}) \cup (A \times [0,1]) \cup (M \times \{1\})$ is a closed subset of $M \times [0,1]$. Also note that $\Phi(B) \subset N \setminus V$. Since M has no boundary we see that $M \times [0,1]$ is a smooth manifold. Finally note that in Exercise 29.2, which is a slight generalization of the Smooth-on-Boundary-Submanifold Lemma 29.4 (2), we will see that Φ is smooth on B in the sense of the definition on page 549.



It follows from the above that we can apply the Whitney Approximation Theorem 29.1 to the map $\Phi\colon M \times [0,1] \to N$ and the closed subset B and we see that there exists a homotopy rel B from Φ to a smooth map $\Psi\colon M \times [0,1] \to N$. Since it is a homotopy rel B we see that $\Psi|_B = \Phi|_B$. But this implies that Ψ is now a smooth homotopy rel A from $\Psi_0 = \Phi_0 = f$ to $\Psi_1 = \Phi_1 = g$.

Now we consider the general case. Thus now we only suppose that there exists an open neighborhood V of ∂N such that $f(A) = g(A)$ is contained in $N \setminus V$. By a rather modest generalization of the Push-Away-from-Boundary Corollary 28.5 one sees that f and g are smoothly homotopic rel A to maps f' and g' such that there exists an open neighborhood W of ∂N such that $f'(M)$ and $g'(M)$ are contained in $N \setminus W$. By the above special case

[303] This condition is satisfied if M is compact and if f and g take values in $N \setminus \partial N$. Indeed, in this case we know by the Compact Image Lemma 2.13 and the Compact-Closed Lemma 1.21 that $f(M) \cup g(M)$ is a closed subset of N that is disjoint from the closed subset ∂N. Since topological manifolds are normal, see Manifold Paracompact-Normal Proposition 18.9 (2), there exists an open neighborhood V of ∂N that is disjoint from $f(M) \cup g(M)$.

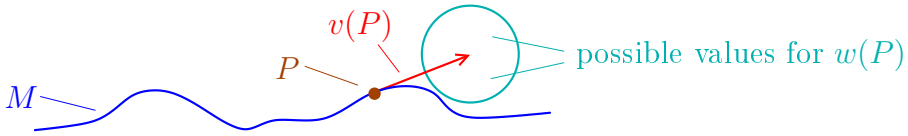
we know that f' and g' are smoothly homotopic rel A. Thus in summary we see that f and f', f' and g', g' and g are smoothly homotopic rel A. It is not hard to show that "smoothly homotopic rel A" is an equivalence relation[304], thus we see that f and g are indeed smoothly homotopic rel A. ■

Proposition 29.6. (Replace-by-Smooth Vector Field Proposition) Let M be a smooth manifold. If M admits a nowhere-vanishing vector field, then it also admits a smooth nowhere-vanishing vector field.

Proof. Let M be a smooth manifold and let v be a nowhere vanishing vector field. By the Euclidean Embedding Theorem 27.1 together with the Smooth Embedding Theorem 24.10 we can view M as a smooth submanifold of some $\mathbb{R}^n$. Furthermore by the Tangent Space-Isomorphism Proposition 23.16 and Lemma 23.18 we can take the view of (smooth) vector fields that take values in the physical tangent spaces $\widetilde{\mathrm{T}}_P M$. Note that each $\widetilde{\mathrm{T}}_P M$ is a subspace of $\mathbb{R}^n$. Thus the vector field v is in particular a map $M \to \mathbb{R}^n$.

We apply the Whitney Approximation Theorem 29.1 to the map $v\colon M \to \mathbb{R}^n$ and the map $\rho\colon M \to \mathbb{R}_{>0}$ that is given by $\rho(x) := \frac{1}{2}\cdot\|v(x)\|$. We obtain a smooth map $w\colon M \to \mathbb{R}^n$ with the following property:

(∗) For every $x \in M$ we have $\|v(x) - w(x)\| \leq \frac{1}{2} \cdot \|v(x)\|$.



Finally we consider the map

$$\begin{array}{rcl} \widetilde{w}\colon M & \to & \mathbb{R}^n \\ P & \mapsto & \text{orthogonal projection of } w(P) \text{ onto } \widetilde{\mathrm{T}}_P M. \end{array}$$

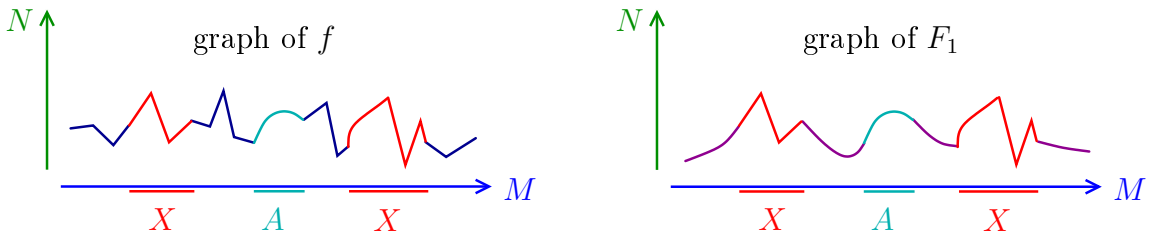
Using an argument as in the proof of the Normal Vectors Lemma 28.2 one can show that this map is in fact smooth. It is elementary to see, using (∗), that for every $P \in M$ we have $\widetilde{w}(P) \neq 0$. Thus we have found a nowhere-vanishing smooth vector field on M. ■

29.3. **More consequences of the Whitney Approximation Theorem 29.1.** We conclude this section with two and a half more technical refinements and consequences of the Whitney Approximation Theorem 29.1. These results are interesting in their own right, but the reader might want to skip them initially.

Proposition 29.7. Let M be a smooth manifold (with or without boundary) and let N be a smooth manifold without boundary. Furthermore let $f\colon M \to N$ be a map. Finally let A and X be disjoint closed subsets of M such that the restriction of f to A is smooth and let $U \subset N$ be an open subset that contains $f(M \setminus X)$. There exists a homotopy $F\colon M \times [0,1] \to N$ with the following properties:

(1) The homotopy is a homotopy rel $X \cup A$.
(2) We have $F_0 = f$.
(3) The restriction of F_1 to the open subset $M \setminus X$ is smooth.[305]
(4) For each $t \in [0,1]$ we have $F_t(M \setminus X) \subset U$.

[304] We will provide the details in the Smooth Isotopy Transitivity Proposition 33.3



Remark. If $X = \varnothing$ and $U = N$, then the statement of Proposition 29.7 is precisely the statement of the Whitney Approximation Theorem 29.1 (1)-(4).

Proof. First we consider the case that $U = N$. By the Manifold Metrization Proposition 18.10 we can pick a metric c on M such that the topology from the metric c agrees with the given topology on M. The same way we can pick a metric d on N. We consider the open subset $W := M \setminus X$ and we view it as a smooth manifold. Since $X \neq \varnothing$ we can consider the function

$$\begin{array}{rcl} \rho\colon W & \to & \mathbb{R}_{\geq 0} \\ w & \mapsto & \rho(w) := c(w, X) = \inf\{c(w, x) \,|\, x \in X\}. \end{array}$$

Since X is closed and since the topology defined by c agrees with the topology on M we see that $\rho(w) > 0$ for every $w \in W$. By the Distance-to-Subspace Lemma 6.2 we know that $\rho\colon M \to \mathbb{R}_{\geq 0}$ is continuous. Since N is a smooth manifold without boundary we can apply the Whitney Approximation Theorem 29.1 to the map $f\colon W \to N$ and we obtain that there exists a homotopy $G\colon W \times [0,1] \to N$ with the following properties:

(a) The homotopy is a homotopy rel A.
(b) We have $G_0 = f|_W$.
(c) The map $G_1\colon W \to N$ is smooth.
(d) For every $(w,t) \in W \times [0,1]$ we have $d(G(w,t), f(w)) \leq \rho(w)$.

It follows easily from (d) and the definition of ρ that the map

$$\begin{array}{rcl} F\colon M \times [0,1] & \to & N \\ (x,t) & \mapsto & \begin{cases} G(x,t), & \text{if } x \in W, \\ f(x), & \text{if } x \in X \end{cases} \end{array}$$

is continuous. Now it is straightforward to verify that this map has all the desired properties.

Finally we consider the case that $U \neq N$. In this case we consider the map

$$\begin{array}{rcl} \rho\colon W & \to & \mathbb{R}_{\geq 0} \\ w & \mapsto & \rho(w) := \min\{c(w, X), d(f(w), N \setminus U)\}. \end{array}$$

Since U is open and since $f(W) \subset U$ we see that ρ is again non-zero everywhere. The rest of the argument is basically the same and it follows from (d) that for each $t \in [0,1]$ we have $F_t(M \setminus X) \subset U$. ∎

[305]Since X is closed we see that $M \setminus X$ is an open subset of M, thus we can view $M \setminus X$ as a smooth manifold in its own right.

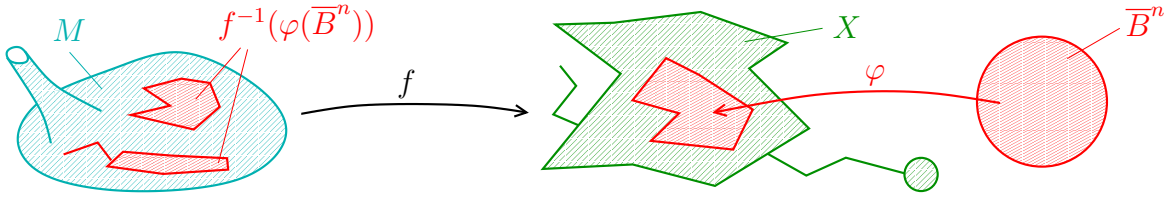
Loosely speaking the following proposition says that a map $M \to X$ from a smooth manifold to a topological space X can be made smooth with respect to open balls B^n contained in X.

Proposition 29.8. Let M be a smooth manifold and let $f\colon M \to X$ be a map to a Hausdorff space X. Let $\varphi\colon \overline{B}^n \to X$ be an injective map such that $\varphi(B^n)$ is an open subset of X. Given any $r \in (0,1)$ there exists a map $g\colon M \to X$ with the following properties:

(1) g is homotopic to f rel the complement of $f^{-1}(\varphi(B^n_r))$ in M,
(2) the map

$$\begin{array}{rcl} f^{-1}(\varphi(B^n_r)) & \to & B^n_r \\ x & \mapsto & \varphi^{-1}(g(x)) \end{array}$$

is smooth.



Proof. Since $\overline{B}^n$ is compact, since X is Hausdorff and since $\varphi\colon \overline{B}^n \to X$ is injective we know by the Compact-Hausdorff Proposition 2.17 (2) that φ is a closed embedding. Thus we might as well identify $\overline{B}^n$ with its image under the map $\varphi\colon \overline{B}^n \to X$. Now let $r \in (0,1)$. By the Open -Closed Inclusion Lemma 2.10 we know that B^n_r is also an open subset of X. Furthermore, note that it follows from the fact that X is Hausdorff, together with the Compact-Closed Lemma 1.21 (1), that $\overline{B}^n_r$ and $\overline{B}^n$ are a closed subsets of X. We consider the map

$$\begin{array}{rcl} \alpha\colon M & \to & \overline{B}^n/S^{n-1} \\ x & \mapsto & \begin{cases} f(x), & \text{if } x \in f^{-1}(B^n), \\ [S^{n-1}], & \text{if } x \in M \setminus f^{-1}(B^n). \end{cases} \end{array}$$

Note that α is continuous.[306] Next note that by the Ball-Quotient Sphere Lemma 5.20 we can view $\overline{B}^n/S^{n-1}$ as a smooth manifold such that B^n is a submanifold. Therefore we can apply Proposition 29.7 to the map $\alpha\colon M \to \overline{B}^n/S^{n-1}$ with $A = \varnothing$, with the closed subset $X = M \setminus f^{-1}(B^n_r)$[307] and with the open subset $U = B^n_r \subset N$. We obtain a homotopy $F\colon M \times [0,1] \to \overline{B}^n/S^{n-1}$ with the following properties:

(1) The homotopy is a homotopy rel $X = M \setminus f^{-1}(B^n_r)$.
(2) We have $F_0 = \alpha$.
(3) The restriction of F_1 to $M \setminus X = f^{-1}(B^n_r)$ is smooth.
(4) For each $t \in [0,1]$ we have $F_t(f^{-1}(B^n_r)) \subset B^n_r$.

[306] Indeed, the restrictions of α to the closed subset $M \setminus f^{-1}(B^n)$ is a constant map and the restriction of α to the closed subset $f^{-1}(\overline{B}^n)$ is given by the composition of the map f with the projection $\overline{B}^n \to \overline{B}^n/S^{n-1}$. Therefore we obtain from the Pasting Proposition 2.6 (2) that α is indeed continuous.
[307] Note the discrete subscript "r".

Finally we consider the map

$$\begin{array}{rcl} G\colon M\times[0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} F(x,t), & \text{if } x\in f^{-1}(B^n_r) \\ f(x), & \text{if } x\in M\setminus f^{-1}(B^n_r). \end{cases} \end{array}$$

Note that the restrictions of the map G to the two closed subsets $(M\setminus f^{-1}(B^n_r))\times[0,1]$ and $f^{-1}(\overline{B}^n_r)\times[0,1]$ are continuous.[308] Thus we obtain from the Homotopy Stacking Lemma 14.3 that G is continuous. It is now straightforward to verify that $g := G_1$ has all the desired properties. ■

The following proposition is a slight variation on Proposition 29.8.

Proposition 29.9. Let M be a smooth manifold and let $f\colon M\to X$ be a map to a Hausdorff space X. Let $n\in\mathbb{N}$ and let $*\in S^n$ be a point. Finally let $\varphi\colon S^n\to X$ be an injective map such that $\varphi(S^n\setminus\{*\})$ is an open subset of X. There exists a map $g\colon M\to X$ with the following properties:

(1) g is homotopic to f rel the complement of $f^{-1}(\varphi(S^n\setminus\{*\}))$ in M,

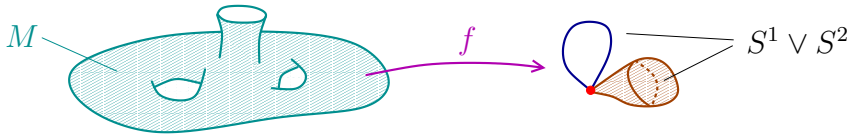
(2) the map

$$\begin{array}{rcl} f^{-1}(\varphi(S^n\setminus\{*\})) & \to & S^n\setminus\{*\} \\ x & \mapsto & \varphi^{-1}(g(x)) \end{array}$$

is smooth.

Proof. We leave it to the reader to modify the proof of Proposition 29.8 to obtain the desired result, see Exercise 29.1. As the reader will notice, in this case there is no need to work with some $r\in(0,1)$. In particular the conclusion we get is slightly more satisfactory. ■

Example. Let $\bigvee_{i=1}^{k} S^{n_i}$ be a wedge of spheres. We denote by $*$ the wedge point. Let M be a smooth manifold and let $f\colon M\to\bigvee_{i=1}^{k} S^{n_i}$ be a map. By iteratively applying Proposition 29.9 altogether k times we see that f is homotopic rel $f^{-1}(*)$ to a map that is smooth on the open sets $f^{-1}(S^{n_i}\setminus\{*\})$, $i=1,\dots,k$.



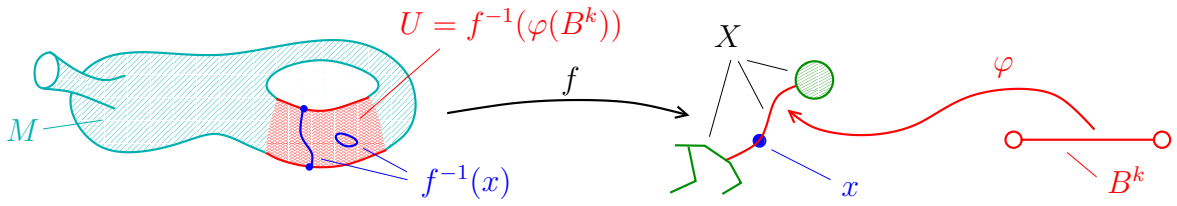
Lemma 29.10. (Preimage-of-Point-in-Ball Lemma) Let M be an orientable n-dimensional smooth manifold, let $f\colon M\to X$ be a map to some Hausdorff space X, let

[308] For nervous people, like the author, let us verify in slightly greater detail that the restriction of G to $f^{-1}(\overline{B}^n_r)\times[0,1]$ is indeed continuous. First note that by (4) the restriction of the map F to $f^{-1}(B^n_r)\times[0,1]$ takes values in B^n_r. It follows from the Closure-Interior Mapping Lemma 2.5 and Exercise 5.16 that the restriction of the map F to $f^{-1}(\overline{B}^n_r)\times[0,1]$ takes values in the closure of B^n_r in $\overline{B}^n/S^{n-1}$. Since $r\in(0,1)$ we see that the closure equals $\overline{B}^n_r\subset B^n$. Now we see that the restriction of the map G to $f^{-1}(\overline{B}^n_r)\times[0,1]$ equals the composition of the map $F\colon f^{-1}(\overline{B}^n_r)\times[0,1]\to\overline{B}^n_r$ with the inclusion $\overline{B}^n_r\to X$.

$\varphi\colon B^k \to X$ be a map and let $x \in \varphi(B^k)$ be a point such that the following conditions are satisfied:

(1) the image $\varphi(B^k)$ is an open subset of X,
(2) the restriction of f to $f^{-1}(\varphi(B^k))$ is a smooth map.

Then $f^{-1}(x)$ is a proper $(n-k)$-dimensional submanifold of M.



Proof. Note that $U := f^{-1}(\varphi(B^k))$ is an open subset of M, in particular, as discussed in the Open Subset-Submanifold Lemma 19.19, it is a smooth manifold in its own right. It follows from this observation and (3) that we can apply the Regular Value Theorem 26.4. We obtain that $f^{-1}(x)$ is a proper orientable $(n-k)$-dimensional submanifold of U. In particular $f^{-1}(x)$ admits submanifold charts of type (α) and (β), as defined on page 524, with respect to the smooth manifold U. Since U is an open subset of M we see that these submanifold charts are also submanifold charts of the same type for the set $f^{-1}(x)$, viewed as a subset of M.

Since X is by hypothesis Hausdorff we know, by the Points-in-Hausdorff Space Lemma 1.15 (1) that $\{x\}$ is a closed subset of X. Finally, since f is of course understood to be continuous on all of M we obtain that $f^{-1}(x)$ is a closed subset of M. In other words, we have shown that $f^{-1}(x)$ satisfies all the conditions, as set out on page 525, of being a proper submanifold of M. ■

Exercises for Chapter 29.

Exercise 29.1. Provide a proof for Proposition 29.9.

Exercise 29.2. Let M and N be two smooth manifolds. We assume that $\partial M = \varnothing$. Furthermore let $f, g\colon M \to N$ be two smooth maps that agree on a subset A and let $\Phi\colon M \times [0,1] \to N$ be a homotopy rel A. Show that Φ is smooth, in the sense of the definition on page 549, on the subset $(M \times \{0\}) \cup (A \times [0,1]) \cup (M \times \{1\})$ of $M \times [0,1]$.

Exercise 29.3. In the figure below we show an explicit smooth manifold M together with a boundary component A and a submanifold B.

(a) Is A a retract of M?
(b) Is B a retract of M?



Exercise 29.4. Let M be a smooth manifold. The Smooth Path-Connectivity Proposition 19.29 says that every map $\gamma\colon [a,b] \to M$ is homotopic rel the endpoints $\{a,b\}$ to a

smooth map. Give a new proof using the Whitney Approximation Theorem 29.1 and the Smooth Collar Neighborhood Theorem 28.3.

Exercise 29.5. We consider the smooth manifold $\mathbb{R}$ and the smooth submanifold $(-\infty, 0]$.

(a) Show that there exists a retraction from $\mathbb{R}$ to $(-\infty, 0]$.

(b) Show that there is no smooth retraction from $\mathbb{R}$ to $(-\infty, 0]$.

30. Subsets of measure zero

Evidently a smooth manifold does not come with a natural notion of a "measure" or "volume". But perhaps somewhat surprisingly we will see that it makes perfect sense to talk of subsets of "measure zero" and of "full measure" of a smooth manifold. We will see that the image of a smooth map $f\colon M \to N$ with $\dim(M) < \dim(N)$ has "measure zero", and that this implies, that $N \setminus f(M)$ is dense. This fact is surprisingly useful. In particular we will use it to show in the Weak Whitney Embedding Theorem 30.7 that every m-dimensional smooth manifold admits a smooth embedding $M \to \mathbb{R}^{2m+1}$.

The notion of subsets of measure zero also plays an essential role in the formulation of Sard's Theorem 31.1 that we will state and prove in the next chapter.

30.1. Sets of measure zero and the Weak Whitney Embedding Theorem.

Definition.

(1) Let $r \in \mathbb{R}_{\geq 0}$. An open cube of side length r is any subset of $\mathbb{R}^m$ of the form $(c_1, c_1 + r) \times \cdots \times (c_m, c_m + r)$. We define the volume of such an open cube as

$$\mathrm{vol}((c_1, c_1 + r) \times \cdots \times (c_m, c_m + r)) \;:=\; r^m.$$

(2) Given a countable family $\{r_i\}_{i \in I}$ of non-negative real numbers we set

$$\sum_{i \in I} r_i \;:=\; \sup\left\{ \sum_{j \in J} r_j \;\middle|\; J \subset I \text{ is a finite subset} \right\} \;\in\; \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

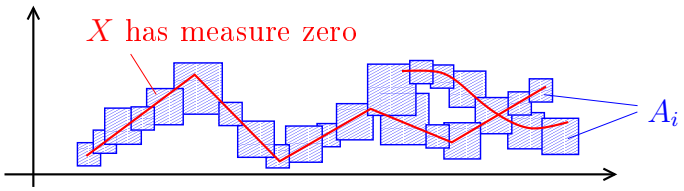
(3) Let $X \subset \mathbb{R}^m$ be a subset. We say that X has measure zero if given any $\epsilon > 0$ there exists a countable family $\{A_i\}_{i \in I}$ of open cubes such that $X \subset \bigcup\limits_{i \in I} A_i$ and such that $\sum\limits_{i \in I} \mathrm{vol}(A_i) < \epsilon$. This definition is illustrated in the figure below.

(4) Let M be an m-dimensional smooth manifold.

(a) We say that a subset $X \subset M$ is a subset of measure zero if for every chart $\Phi\colon U \to V$ from the smooth atlas the set $\Phi(X \cap U)$ is a subset of measure zero of $\mathbb{R}^m$.

(b) We say that a subset $X \subset M$ is a subset of full measure if $M \setminus X$ is a subset of measure zero.

(5) Let P be a property of points in M. We say that almost every point in M has property P if the set of points which satisfy P has full measure.



Example.

(1) Let $n \geq 1$ and let $\{P = (p_1, \ldots, p_n)\}$ be a one-point subset of $\mathbb{R}^n$. Since $\{P\}$ is contained in any open cube $(p_1 - \epsilon, p_1 + \epsilon) \times (p_n - \epsilon, p_n + \epsilon)$ of volume $(2\epsilon)^n$ we see that $\{P\}$ is a subset of measure zero.

(2) Let M be a smooth manifold of dimension ≥ 1. Using (a) it is straightforward to show that any finite subset of M has measure zero.

In the following we formulate three propositions which contain almost everything anybody ever wanted to know about subsets of measure zero. We will provide the proofs in the next section.

The definition of "measure zero" is not very practical since one needs to verify a certain condition for every chart. Our first proposition greatly simplifies life, namely it says, that it suffices to consider enough charts to cover all of M.

Proposition 30.1. (Measure Zero-via-Atlas Proposition) Let M be an m-dimensional smooth manifold and let $X \subset M$ be a subset. If there exists a family of smooth charts $\{\Phi_i \colon U_i \to V_i\}_{i\in I}$ with the following two properties:

(1) for each $i \in I$ the set $\Phi_i(X \cap U_i)$ is a subset of measure zero of $\mathbb{R}^m$,
(2) we have $X \subset \bigcup_{i\in I} U_i$,

then X has measure zero.

The second proposition summarizes a few basic properties of subsets of measure zero.

Proposition 30.2. (Measure-Zero Properties Proposition) Let M be an m-dimensional smooth manifold.

(1) If X is a subset of measure zero of M, then so is every subset of X.
(2) (a) The union of countably many subsets of measure zero is a subset of measure zero.
    (b) The intersection of countably many subsets of full measure is a subset of full measure.
(3) (a) Let $f \colon M \to N$ be a smooth map between smooth manifolds of the same dimension. If X is a subset of measure zero in M, then $f(X)$ has measure zero in N.
    (b) If $f \colon M \to N$ is a diffeomorphism, then $X \subset M$ has measure zero (respectively full measure) if and only if $f(X)$ does.[309]
(4) If $f \colon M \to N$ is a smooth map between smooth manifolds and if $\dim(M) < \dim(N)$, then $f(M)$ is a subset of measure zero in N. In other words, $N \setminus f(M)$ is a subset of full measure.

Example. Let N be a smooth manifold and let M be a submanifold of N of codimension at least one. By the Submanifold Smooth Structure Lemma 19.24 the inclusion map $\iota \colon M \to N$ is smooth. Therefore it follows from the Measure-Zero Properties Proposition 30.2 (4) that $M \subset N$ has measure zero.

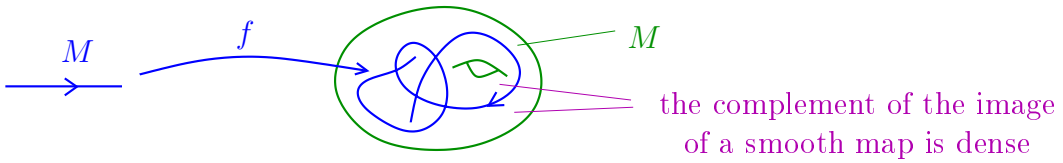
We conclude this section with one last proposition on subsets of measure zero.

Proposition 30.3. (Full Measure-Density Proposition) Let M be an m-dimensional smooth manifold.

(1) Given any subset X of measure zero of M the complement $M \setminus X$ is dense in M.[310]
(2) Every subset of M of full measure is dense in M.

[309] In Exercise 30.3 we will see that the conclusion does not hold if we replace "diffeomorphism" by "homeomorphism".

[310] By Lemma 1.12 this is equivalent to saying that the interior of X is the empty set.



Remark. The above propositions show that on smooth manifolds there is a sensible notion of subsets of measure zero. Denis Sullivan [**Sul1979**, p. 549] showed that for dimensions $n \neq 4$ any n-dimensional topological manifold has an essentially unique "Lipshitz structure", which makes it possible to again introduce a sensible notion of subsets of measure zero. We will not make use of this fact.

30.2. **Proof of Propositions 30.1, 30.2 and 30.3.** We start out with the following self-explanatory definition.

Definition. Let $r \in \mathbb{R}_{\geq 0}$. A closed cube of side length r is any subset of $\mathbb{R}^m$ of the form $[c_1, c_1 + r] \times \cdots \times [c_m, c_m + r]$. We define the volume of such a closed cube as

$$\operatorname{vol}([c_1, c_1 + r] \times \cdots \times [c_m, c_m + r]) \;:=\; r^m.$$

The following lemma contains some very basic statements about subsets of measure zero.

Lemma 30.4. (Measure Zero-Basics Lemma)

(1) If C is a subset of measure zero of $\mathbb{R}^m$, then so is every subset of C.
(2) If $\{C_i\}_{i \in I}$ is countable family of subsets of $\mathbb{R}^m$ of measure zero, then $\bigcup_{i \in I} C_i$ also has measure zero.
(3) Let C be a subset of measure zero of $\mathbb{R}^m$ and let $\sigma \in S_m$ be a permutation. Then
$$\{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid (x_{\sigma(1)}, \ldots, x_{\sigma(m)}) \in C\}$$
also has measure zero.
(4) Let C be a subset of $\mathbb{R}^m$. If given any $\epsilon > 0$ there exists a countable family $\{A_i\}_{i \in I}$ of closed cubes such that $X \subset \bigcup_{i \in I} A_i$ and such that $\sum_{i \in I} \operatorname{vol}(A_i) < \epsilon$, then C has measure zero.

Proof.

(1) This statement is trivial.
(2) Let $\{X_i\}_{i \in I}$ be a countable family of subsets of measure zero of $\mathbb{R}^m$. Using the fact that $\sum_{k \in \mathbb{N}} \frac{1}{2^k} = 1$ it is straightforward to show that $\bigcup_{i \in I} X_i$ is also a subset of measure zero.
(3) This statement is clear since permuting coordinates turns cubes into cubes of the same volume.
(4) This statement follows from the straightforward observation that any closed cube is contained in an open cube of at most twice the volume. ■

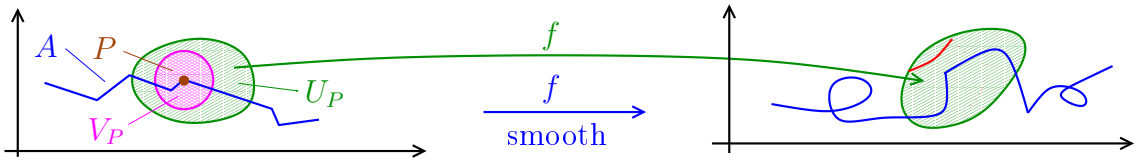
Lemma 30.5. Let $A \subset \mathbb{R}^n$ be a measure zero subset and let $f \colon A \to \mathbb{R}^n$ be a smooth map. Then $f(A)$ also has measure zero.

Remark. The conclusion of Lemma 30.5 does not hold if we replace "smooth" by "continuous":

(1) It is clear that $I = \{(x,0) \,|\, x \in [-1,1]\} \subset \mathbb{R}^2$ has measure zero, but in Theorem 6.13 we saw that there exists a space-filling curve, i.e. a map $I \to [0,1]^2$ that is surjective. But, as we will see in the Cubes-are-Not-Measure Zero Proposition 30.6, $[0,1]^2 \subset \mathbb{R}^2$ does not have measure zero.

(2) In Exercise 30.3 we will see that there exists a homeomorphism $[0,1] \to [0,2]$ and a measure zero subset $C \subset [0,1]$ such that $f(C)$ does not have measure zero.

Proof. Let $A \subset \mathbb{R}^n$ be a measure zero subset and let $f\colon A \to \mathbb{R}^n$ be a smooth map. We need to show that $f(A)$ also has measure zero. First let $P \in A$. By definition of a smooth map there exists an open neighborhood U_p and a smooth function $f\colon U_P \to \mathbb{R}^n$ which agrees with the original f on $A \cap U_P$. Next we pick $r > 0$ such that $V_P := \overline{B}^n_r(P) \subset U_P$. By the Second Countability Lemma 9.1 we know that A is second-countable. Thus it follows from the Countable Covering Lemma 9.2 that there exist countably many $P \in A$ such that A is covered by the V_P. Therefore, by the Measure Zero-Basics Lemma 30.4 it suffices to show that each $f(V_P \cap A)$ has measure zero.



So let $P \in A$, let $V_P = \overline{B}^n_r(P)$ and let $f\colon U_P \to \mathbb{R}^n$ be as above. Given a matrix $A \in \mathrm{M}(n \times n, \mathbb{R})$ we define its norm as

$$\|A\| \;:=\; \max\{\|Av\| \,|\, v \in S^{n-1}\} \;\in\; \mathbb{R}_{\geq 0}.$$

In Exercise 2.45 (h) we saw that the norm is a continuous map on $\mathrm{M}(n \times n, \mathbb{R})$. Thus, since V_P is compact, it follows that by the Compact Image Lemma 2.13 we can set

$$M \;:=\; \max\{\|\mathrm{D}f_P\| \,|\, P \in V_P\}.$$

Claim. For every $x, x' \in \overline{B}^n_r(P)$ we have

$$\|f(x) - f(x')| \;\leq\; M \cdot \|x - x'\|.$$

Proof. Let $x, x' \in \overline{B}^n_r(P)$. We set $v := x' - x$. We perform the following calculation:

$$\begin{array}{ccccccc}
 & \text{Fundamental Theorem of Calculus} & & \text{Chain Rule 19.10} & & \text{Integral Inequality Lemma 0.16} & \\
\|f(x) - f(x')\| & \overset{\downarrow}{=} & \left\| \int_{t=0}^{t=1} \frac{d}{dt} f(x+tv)\, dt \right\| & \overset{\downarrow}{=} & \left\| \int_{t=0}^{t=1} \mathrm{D}f_{x+tv} \cdot v\, dt \right\| & \overset{\downarrow}{\leq} & \int_{t=0}^{t=1} \|\mathrm{D}f_{x+tv} \cdot v\|\, dt \\
 & \leq & \int_{t=0}^{t=1} \|\mathrm{D}f_{x+tv}\| \cdot \|v\|\, dt & \leq & \int_{t=0}^{t=1} M \cdot \|v\|\, dt \;=\; M \cdot \|v\| \;=\; M \cdot \|x - x'\|. & & \\
 & \text{Exercise 2.45 (a)} & & \text{definition of } M & & & \boxplus
\end{array}$$

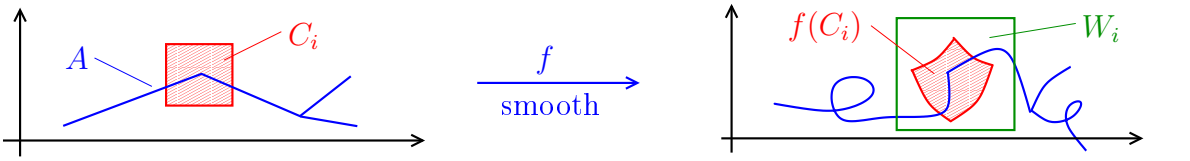
Let $\epsilon > 0$. We pick $\delta > 0$ with $\delta \cdot (2M)^n \leq \epsilon$. By the Measure Zero-Basics Lemma 30.4 (1) we know that $A \cap V_P$ is a subset of measure zero. Therefore there exists a countable set of closed cubes $\{C_i\}_{i \in \mathbb{N}}$ with $A \cap V_P \subset \bigcup_{i \in \mathbb{N}} C_i$ and with $\sum_{i \in \mathbb{N}} \mathrm{vol}(C_i) < \delta$. After possibly splitting each closed cube C_i into smaller closed cubes we can assume that each C_i is contained in

U_P.[311] It follows from the claim, together with an elementary geometric argument regarding diameters and cubes, that the image $f(C_i)$ of the cube C_i is contained in a cube W_i whose side length is $2M$-times larger than the side length of C_i. In other words, $f(C_i)$ is contained in a closed cube with $\mathrm{vol}(W_i) = (2M)^n \cdot \mathrm{vol}(C_i)$. We now see that

$$f(A \cap V_P) \subset f\Big(\bigcup_{i\in\mathbb{N}} C_i\Big) = \bigcup_{i\in\mathbb{N}} f(C_i) \subset \bigcup_{i\in\mathbb{N}} W_i.$$

The following calculation, together with the Measure Zero-Basics Lemma 30.4 (4), completes the proof:

$$\sum_{i=1}^{\infty} \mathrm{vol}(W_i) = \sum_{i=1}^{\infty} (2M)^n \cdot \mathrm{vol}(C_i) = (2M)^n \cdot \sum_{i=1}^{\infty} \mathrm{vol}(C_i) < (2M)^n \cdot \delta \leq \epsilon. \qquad \blacksquare$$



Now we can provide the proofs of Propositions 30.1 and 30.2.

Proof of the Measure Zero-via-Atlas Proposition 30.1. Let M be an m-dimensional smooth manifold ad let $X \subset M$ be a subset. Suppose that there exists a family of smooth charts $\{\Phi_i \colon U_i \to V_i\}_{i\in I}$ such that the following two conditions hold:

(1) for each $i \in I$ the set $\Phi_i(X \cap U_i)$ is a subset of measure zero of $\mathbb{R}^m$,

(2) we have $X \subset \bigcup_{i\in I} U_i$.

Note that it follows from the Second Countability Lemma 9.1 together with the Countable Covering Lemma 9.2 that X is already covered by countably many of the U_i. Thus we might as well assume that I is countable.

We need to show that X has measure zero. Thus let $\Psi \colon A \to B$ be a smooth chart. We need to show that $\Psi(A \cap X)$ has measure zero in $\mathbb{R}^m$. Note that

$$\Psi(A \cap X) = \Psi\Big(\bigcup_{i\in I}(U_i \cap X)\Big) = \Psi\Big(\bigcup_{i\in I}\Phi_i^{-1}(\Phi_i(U_i \cap X))\Big) = \bigcup_{i\in I} \underbrace{(\underbrace{\Psi \circ \Phi_i^{-1}}_{\text{smooth}})(\underbrace{\Phi_i(U_i \cap X)}_{\text{measure zero by (1)}})}_{\text{measure zero by Lemma 30.5}}$$

Since I is countable we now obtain from the Measure Zero-Basics Lemma 30.4 that $\Psi(A \cap X)$ does indeed have measure zero. $\blacksquare$

Proof of the Measure-Zero Properties Proposition 30.2. Let M be an m-dimensional smooth manifold.

(1) Let X be a subset of measure zero of M. It follows almost immediately from the Measure Zero-Basics Lemma 30.4 (1) that any subset of X also has measure zero.

[311] This can be seen as follows. Since U_P is open and since $V_P \subset U_P$ is compact we see that there exists an $\epsilon > 0$ such that every closed cube that intersects V_P and which has side length $< \epsilon$ is actually contained in U_P. Now let C be a cube of side length d. We pick $m \in \mathbb{N}$ such that $\frac{d}{m} < \epsilon$ and we break C into m^n closed cubes of side length $\frac{d}{m}$. Now we only take those cubes which have non-empty intersection with V_P.

(2) (a) It follows easily from the Measure Zero-Basics Lemma 30.4 (2) and the definitions that the union of countably many subsets of measure zero is again a subset of measure zero.
(b) It follows from (a) and de Morgan's Laws 0.1 that the intersection of countably many subsets of full measure is again a subset of full measure.

(3) (a) Let $f\colon M \to N$ be a smooth map between smooth manifolds of the same dimension and let X be a subset of measure zero in M. We need to show that $f(X)$ has measure zero in N. Let $\Psi\colon A \to B$ be a smooth chart for N. We need to show that $\Psi(A \cap f(X))$ has measure zero in $\mathbb{R}^n$. By the Countable Covering Lemma 9.2 we know that M admits a countable smooth atlas $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$. Now we see that

$$\begin{aligned}\Psi(A\cap f(X)) &= \Psi\Big(\bigcup_{i\in I} f\big(f^{-1}(A)\cap X\cap U_i\big)\Big) = \Psi\Big(\bigcup_{i\in I} f\big(\Phi_i^{-1}(\Phi_i(f^{-1}(A)\cap X\cap U_i))\big)\Big)\\ &= \underbrace{\bigcup_{i\in I} \underbrace{(\Psi\circ f\circ \Phi_i^{-1})}_{\text{smooth}}\underbrace{(\Phi_i(f^{-1}(A)\cap X\cap U_i))}_{\text{measure zero by (1)}}}_{\text{measure zero by Lemma 30.5}}\end{aligned}$$

It follows from the Measure Zero-Basics Lemma 30.4 (2) that $\Psi(A \cap f(X))$ does indeed have measure zero.
(b) This statement is an immediate consequence of (a).

(4) Let M and N be two smooth manifolds with $m := \dim(M) < \dim(N) =: n$. Furthermore let $f\colon M \to N$ be a smooth map. We need to show that $f(M)$ is a subset of measure zero in N. We consider the n-dimensional smooth manifold $W := M\times\mathbb{R}^{n-m}$.
It follows from $n - m > 0$ and the discussion on page 762 that $\{0\} \subset \mathbb{R}^{n-m}$ is a subset of measure zero. Using product charts and using the Measure Zero-via-Atlas Proposition 30.1 it follows easily that $M \times \{0\}$ is a subset of measure zero of $W = M \times \mathbb{R}^{n-m}$. Now consider the smooth map $g\colon W = M \times \mathbb{R}^{n-m} \to N$ that is given by $g(x, v) = f(x)$. Evidently $g(M \times \{0\}) = f(M)$. It follows from the above and (3a) that $g(M \times \{0\})$ has measure zero in N. ■

For the proof of the Full Measure-Density Proposition 30.3 we need the following subtle proposition.

Proposition 30.6. (Cubes-are-Not-Measure Zero Proposition) If Q is a closed cube with side length > 0, then Q does not have measure zero.

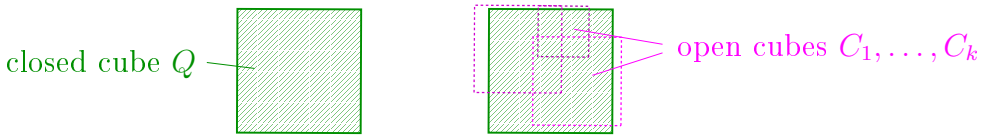
Proof. The statement of the proposition follows of course from standard theory of Lebesgue measures. In the following we give a self-contained proof which in [**GP1974**, p. 203] gets attributed to John von Neumann. Let Q be a closed cube with side length $q > 0$ in $\mathbb{R}^m$. Suppose that we have a countable covering $Q \subset \bigcup_{i\in I} C_i$ by open cubes. Given $i \in I$ we denote by c_i the side length of C_i. It suffices to prove the following claim.

Claim. We have the inequality $\sum_{i\in I} c_i^m \geq q^m$.

Proof. Since Q is compact we see that Q is covered by finitely many C_i. Thus we can assume that $I = \{1, \ldots, k\}$ and that $Q \subset C_1 \cup \cdots \cup C_k$. Given $n \in \mathbb{N}$ we set

$$X_n := (\tfrac{1}{n}\cdot\mathbb{Z})^m = \{(r_1, \ldots, r_m) \,|\, r_1, \ldots, r_m \in \tfrac{1}{n}\mathbb{Z}\}.$$

We make the following crucial observation: Given any cube $D \subset \mathbb{R}^m$, closed or open, of



side length d we have[312]

$$(*) \qquad \lceil d \cdot n \rceil^m \;\geq\; \#(D \cap X_n) \;\geq\; \lfloor d \cdot n \rfloor^m.$$

Now we see that

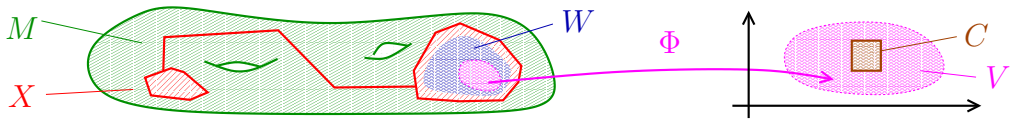
$$\sum_{i=1}^{k} \lceil c_i \cdot n \rceil^m \;\underset{\text{by } (*)}{\geq}\; \sum_{i=1}^{k} \#(X_n \cap C_i) \;\geq\; \#\Big(X_n \cap \bigcup_{i=1}^{k} C_i\Big) \;\underset{\text{since } Q \subset C_1 \cup \cdots \cup C_k}{\geq}\; \#(X_n \cap Q) \;\underset{\text{by } (*)}{\geq}\; \lfloor r \cdot n \rfloor^m.$$

Using the inequalities $x + 1 \geq \lceil x \rceil$ and $\lfloor x \rfloor \geq x - 1$ we see that

$$\sum_{i=1}^{k} (c_i \cdot n + 1)^m \;\geq\; (r \cdot n - 1)^m.$$

If we now divide both sides by n^m and if we take the limit $n \to \infty$, then we obtain the promised inequality $\sum_{i=1}^{k} c_i^m \geq q^m$. ■

Proof of the Full Measure-Density Proposition 30.3. Let M be an m-dimensional smooth manifold and let X be a subset. We will show that if $M \setminus X$ is not dense, then X cannot have measure zero. If $M \setminus X$ is not dense, then there exists an open non-empty subset W of M which has empty intersection with $M \setminus X$. In other words, $W \subset X$. Now let $\Phi\colon U \to V$ be a smooth chart with $U \subset W$. Since $V = \Phi(U)$ is an open non-empty subset of $\mathbb{R}^m$ it contains a closed cube C with side length > 0. It follows from the Cubes-are-Not-Measure Zero Proposition 30.6 that $V = \Phi(U)$ is not a subset of measure zero. Since $V = \Phi(U) \subset W \subset X$ we obtain from the Measure Zero-Basics Lemma 30.4 that X is not a subset of measure zero. ■



30.3. **The Weak Whitney Embedding Theorem.** In the Euclidean Embedding Theorem 27.1 we saw that every smooth manifold M can be properly smoothly embedded into some $\mathbb{R}^n$. For compact manifolds the following proposition nails down n in terms of the dimension of M.

Theorem 30.7. (Weak Whitney Embedding Theorem) Every m-dimensional smooth manifold M admits a smooth embedding $f\colon M \to \mathbb{R}^{2m+1}$ such that $f(M)$ is a closed subset of $\mathbb{R}^{2m+1}$.

[312] This observation follows from the fact that given any interval $I \subset \mathbb{R}$ of length d we have

$$\lceil d \cdot n \rceil \;\geq\; \#\big(I \cap \{x \in \tfrac{1}{n} \cdot \mathbb{Z}\}\big) \;\geq\; \lfloor d \cdot n \rfloor.$$

Remark.

(1) In the Covering Dimension-of-Manifolds Proposition 27.7 we showed that the covering dimension of an m-dimensional topological manifold M is $\leq m$. As we already mentioned on page 727 it follows almost immediately from the Euclidean Embedding Theorem 11.13 for Topological Spaces that there exists a closed embedding $M \to \mathbb{R}^{2m+1}$. The Weak Whitney Embedding Theorem 30.7 betters the result in the sense that for smooth manifolds it provides a smooth embedding into $\mathbb{R}^{2m+1}$.

(2) The fact that the dimension $2m+1$ appears in the Euclidean Embedding Theorem 11.13 for Topological Spaces and the Weak Whitney Embedding Theorem 30.7 suggests that it is a dimension barrier that is hard to break. In fact the Strong Whitney Embedding Theorem **??**, which is significantly harder to prove, shows that for smooth manifolds we can improve the dimension by 1, namely it says that any m-dimensional smooth manifold admits a smooth embedding into $\mathbb{R}^{2m}$.

(3) Later on we will prove some negative results, for example in Propositions 123.11 and the Topological Codimension-One Orientability Proposition 124.5 we will see that the real projective space $\mathbb{R}\mathrm{P}^n$ does not admit a smooth embedding into $\mathbb{R}^{n+1}$.

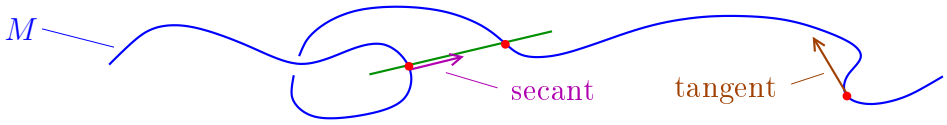
The proof of the Weak Whitney Embedding Theorem 30.7 requires the following lemma, which is perhaps of interest in its own right.

Lemma 30.8. Let M be an m-dimensional smooth submanifold of $\mathbb{R}^n$. We consider the vectors $v \in S^{n-1}$ with the following two properties:

(1) v does *not* lie in any tangent space $\widetilde{\mathrm{T}}_P M$.

(2) v is *not* secant to M, in other words, for any two distinct points $x, y \in M$ the difference $x - y$ is not parallel to v.

If $n > 2m+1$ the set of such vectors is dense in S^{n-1}.



Proof of Lemma 30.8. Let M be an m-dimensional submanifold of $\mathbb{R}^n$. We start out with the following claim.

Claim. The subsets[313]

$$\begin{array}{lrcll} (1) & X & := & \{(P,w) \,|\, P \in M, w \in \widetilde{\mathrm{T}}_P M \setminus \{0\}\} & \subset \mathbb{R}^n \times \mathbb{R}^n \\ (2a) & Y_a & := & \{(x,y) \in M \times (M \setminus \partial M) \,|\, x \neq y\} & \subset \mathbb{R}^n \times \mathbb{R}^n \\ (2b) & Y_b & := & \{(x,y) \in M \times \partial M \,|\, x \neq y\} & \subset \mathbb{R}^n \times \mathbb{R}^n \end{array}$$

are smooth submanifolds of $\mathbb{R}^n \times \mathbb{R}^n$ of dimension $\leq 2m$.

Proof. First we consider X. Thus let $(P,w) \in X$. We pick a submanifold chart $\Phi\colon U \to V$ around $P \in M$. The map

$$\begin{array}{rcl} \Psi\colon U \times (\mathbb{R}^n \setminus \{0\}) & \to & V \times (\mathbb{R}^n \setminus \{0\}) \\ (x,v) & \mapsto & (\Phi(x), \mathrm{D}\Phi_x(v)) \end{array}$$

[313] This claim is closely related to Exercises 19.29 and 19.30.

is easily seen to be smooth and, using Φ^{-1}, one can easily write down a smooth inverse. Thus we see that Ψ is a diffeomorphism. Up to a permutation of coordinates the map Ψ is a $2m$-dimensional smooth submanifold chart for $(P, w) \in X$.

Now we turn to Y_a and Y_b. Using products of submanifold charts one can easily show that the product of two smooth submanifolds of $\mathbb{R}^n$, where at most one has non-empty boundary, is a smooth submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ of the obvious dimension. The subsets Y_a and Y_b are open subsets of such products, thus they are smooth submanifolds with $\dim(Y_a) = 2m$ and $\dim(Y_b) = 2m - 1$. ⊞

In the following we suppose that $n > 2m + 1$. We consider the maps[314]

$$\begin{array}{rrcl} f\colon X = \{(P,w) \,|\, P \in M, w \in \widetilde{\mathrm{T}}_P M \setminus \{0\}\} & \to & \mathbb{R}\mathrm{P}^{n-1} = (\mathbb{R}^n \setminus \{0\})/(\mathbb{R} \setminus \{0\}) \\ (P,w) & \mapsto & [w] \\ \text{and} \quad g_a\colon Y_a = \{(x,y) \in M \times (M \setminus \partial M) \,|\, x \neq y\} & \to & \mathbb{R}\mathrm{P}^{n-1} \\ (x,y) & \mapsto & [x-y] \\ g_b\colon Y_b = \{(x,y) \in M \times \partial M \,|\, x \neq y\} & \to & \mathbb{R}\mathrm{P}^{n-1} \\ (x,y) & \mapsto & [x-y]. \end{array}$$

On page 564 we equipped $\mathbb{R}\mathrm{P}^{n-1}$ with a standard smooth structure and thus we can view $\mathbb{R}\mathrm{P}^{n-1}$ as an $(n-1)$-dimensional smooth manifold. As we just showed in the claim, the domains X, Y_a and Y_b of the above maps are smooth submanifolds of $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ of dimension $\leq 2m$. It follows from the discussion on page 564 that the maps f, g_a and g_b are smooth. Since $2m < n-1$ we obtain from Proposition 30.3 together with the Measure-Zero Properties Proposition 30.2 (2) that the complement of the union of the images of f, g_a and g_b is dense.

Let $p\colon S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}$ be the projection . One can easily show, see Exercise 5.31, that the preimage of a dense subset of $\mathbb{R}\mathrm{P}^{n-1}$ is a dense subset of S^{n-1}. The lemma now follows from the above discussion and the observation that we have the following equality:

$$\begin{array}{c}\text{set of vectors in } \mathbb{R}^n \setminus \{0\} \\ \text{that satisfy (1) and (2)}\end{array} = p^{-1}(\text{complement of the images of } f, g_a \text{ and } g_b).$$ ■

Now we turn to the proof of the Weak Whitney Embedding Theorem 30.7. As always, the case that M is compact is easier, so let us treat it first.

Proof of the Weak Whitney Embedding Theorem 30.7 if M is compact. Let M be an m-dimensional smooth manifold. We claim that we can smoothly embed M into $\mathbb{R}^{2m+1}$. By the Euclidean Embedding Theorem 27.1 together with the Smooth Embedding Theorem 24.10 we can assume that M is already a submanifold of some $\mathbb{R}^n$. If $n \leq 2m+1$ there is nothing to prove, so we may assume that $n > 2m+1$. By Lemma 30.8 there exists a non-zero vector $v \in \mathbb{R}^n$ which does not lie in any tangent space $\widetilde{\mathrm{T}}_P M$ and that is not secant to M, in other words, for any two distinct points $x, y \in M$ the difference $x - y$ is not parallel to v.

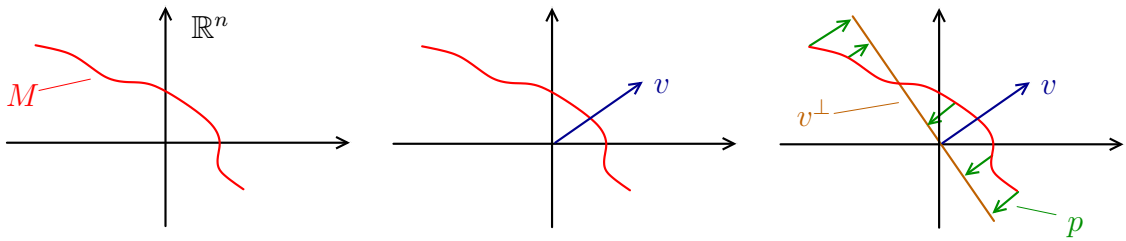
We consider the projection of $\mathbb{R}^n$ onto the orthogonal complement of v, i.e. we consider the projection

$$\begin{array}{rcl} p\colon \mathbb{R}^n & \to & v^\perp = \{w \in \mathbb{R}^n \,|\, \langle w, v\rangle = 0\} \\ u & \mapsto & u - \langle u, v\rangle \cdot \frac{v}{\|v\|^2}. \end{array}$$

[314] Here we use the identification $\mathbb{R}^n \setminus \{0\})/(\mathbb{R} \setminus \{0\}) = \mathbb{R}\mathrm{P}^{n-1} := S^{n-1}/\{\pm 1\}$ from page 255.

Since v does not lie in any tangent space $\widetilde{\mathrm{T}}_P M$ we see that $p\colon M \to v^\perp$ is an immersion. Furthermore, since v is not secant to M we see that $p\colon M \to v^\perp$ is injective. Since we assume that M is compact we obtain from the Compact-Hausdorff Proposition 2.17 (3) that $p\colon M \to v^\perp$ is actually an embedding.

In summary we have shown that the restriction of the projection p to M is a smooth embedding. Thus we have embedded M into a real vector space of dimension $n-1$. Iterating this process we get a smooth embedding of M into $\mathbb{R}^{2m+1}$. ■



Evidently we also want to prove the Weak Whitney Embedding Theorem 30.7 for possibly non-compact smooth manifolds. As an example, let us consider $\mathbb{Z}^2 \subset \mathbb{R}^2$. This is a non-compact 0-dimensional proper smooth submanifold of $\mathbb{R}^2$. Lemma 30.8 shows that there exist (many) vectors $v \in S^1$ such that the projection of $\mathbb{Z}^2$ onto $v^\perp$ is injective. But, as we will see in Exercise 30.6, none of these projections will be an embedding, i.e. a homeomorphism onto its image. Thus, to deal with non-compact smooth manifolds we will have to come up with a new idea.

We will need the following elementary lemma, which is the content of Exercise 28.8.

Lemma 30.9. Given any m-dimensional smooth manifold M there exists an m-dimensional smooth manifold N with $\partial N = \varnothing$ and a smooth embedding $\varphi\colon M \to N$ such that $\varphi(M)$ is a closed subset of N.

Proof. Let M be an m-dimensional smooth manifold. By the Smooth Collar Neighborhood Theorem 28.3 we can pick a collar neighborhood $[0,1] \times \partial M$. Using the Smooth Transition Function Lemma 19.4 one can show easily that there exists a diffeomorphism $f\colon [0,1] \to [\frac{1}{4}, 1]$ such that $\varphi(t) = t$ for $t \in [\frac{1}{2}, 1]$. It is now straightforward to verify that the map

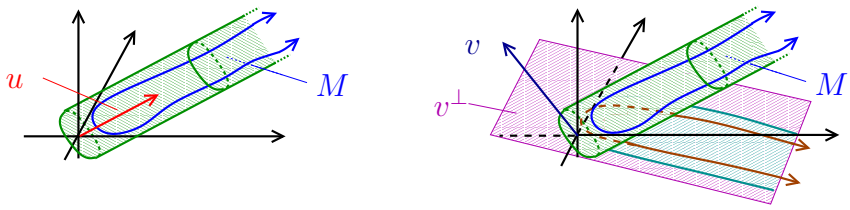
$$\begin{array}{rcl} \varphi\colon M & \to & N := M \setminus \partial M \\ x & \mapsto & \begin{cases} x, & \text{if } x \notin [0,1] \times \partial M, \\ (\varphi(t), y), & \text{if } x = (t,y) \in [0,1] \times \partial M \end{cases} \end{array}$$

is a smooth embedding such that the image is a closed subset of N. ■

Proof of the Weak Whitney Embedding Theorem 30.7 if M is non-compact. Let M be an m-dimensional smooth manifold. We need to show that there exists a smooth embedding $f\colon M \to \mathbb{R}^{2m+1}$ such that $f(M)$ is a closed subset of $\mathbb{R}^{2m+1}$. It follows from Lemma 30.9 that we might as well assume that $\partial M = \varnothing$.

Given $N \in \mathbb{N}$ we say that M *is N-tubular* if there exists a $u \in \mathbb{R}^N$, an $r > 0$ and a proper smooth embedding of M into

$$T(u,r) \;:=\; \{\mu \cdot u + w \,|\, \mu \in \mathbb{R}_{\geq 0}, \text{ and } w \in u^\perp \text{ with } \|w\| \leq r\}.$$



Note that if M is N-tubular, then it follows from the Euclidean Embedding Theorem 27.1 that M is indeed diffeomorphic to a proper submanifold of $\mathbb{R}^N$. Thus it remains to show that M is $(2m+1)$-tubular. We start out with the following claim.

Claim 1. The smooth manifold M is tubular for some $N \in \mathbb{N}$.

Proof. By the Euclidean Embedding Theorem 27.1 there exists a smooth embedding $f\colon M \to \mathbb{R}^k$ for some $k \in \mathbb{N}$. On page 537 we gave an explicit diffeomorphism $\alpha\colon \mathbb{R}^k \to B^k$. By the Manifold Exhaustion Lemma 21.2 we know that there exists a smooth exhaustion function $\varphi\colon M \to \mathbb{R}_{\geq 0}$, in other words, there exists a smooth map $\varphi\colon M \to \mathbb{R}_{\geq 0}$ such that the preimage of each compact interval $[0,c]$ is compact. Next we consider the map

$$\begin{array}{rcl} \Theta\colon M & \to & \mathbb{R} \times \mathbb{R}^k = \mathbb{R}^{k+1} \\ x & \mapsto & (\varphi(x), (\alpha \circ f)(x)). \end{array}$$

Note that $\Theta(M) \subset T(e_1, 1)$. Finally note that it follows from Lemma 27.9 that Θ is a proper smooth embedding. ⊞

We see that it remains to prove the following claim.

Claim 2. If M is N-tubular for some $N > 2m+1$, then M is also $(N-1)$-tubular.

Proof. We can assume that there exists a $u \in \mathbb{R}^N$ and an $r > 0$ such that M is a proper submanifold of $T(u,r) \subset \mathbb{R}^N$. By Lemma 30.8 there exists a $v \in \mathbb{R}^N$ with $v \notin \mathbb{R} \cdot u$ with the following two properties:

(1) v does *not* lie in any tangent space $\widetilde{\mathrm{T}}_P M$.
(2) v is *not* secant to M.

It follows from elementary linear algebra that there exists an isometry $\Theta\colon v^\perp \to \mathbb{R}^{N-1}$ of Euclidean real vector spaces. As on page 770 we consider the map

$$\begin{array}{rcl} p\colon \mathbb{R}^N & \to & v^\perp \\ x & \mapsto & x - \langle x, v\rangle \cdot \frac{v}{\|v\|^2} \end{array}$$

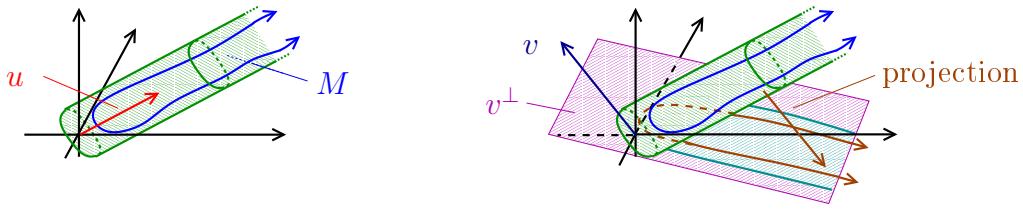
and now we also consider

$$\begin{array}{rcl} \Phi\colon \mathbb{R}^N & \to & \mathbb{R}^{N-1} \\ x & \mapsto & \Theta(p(v)). \end{array}$$

It follows easily from the fact that Θ is an isometry that $\Phi(T(v,r)) \subset T(\Phi(v), r)$. Thus it remains to show that $\Phi\colon M \to \mathbb{R}^{N-1}$ is a proper smooth embedding. Since v does not lie in any tangent space $\widetilde{\mathrm{T}}_P M$ we see, similar to the discussion on page 770, that $\Phi\colon M \to v^\perp$ is an immersion and we see, using that v is not secant to M, that $\Phi\colon M \to v^\perp$ is injective. By the Proper-Closed Proposition 2.43 and Lemma 2.44 (1) it remains to show that the preimage of each compact ball $\overline{B}_s^{N-1}(0)$ under the map $\Phi\colon M \to \mathbb{R}^{N-1}$ is compact. Thus let $s \geq 0$. Note that for each $x = \mu \cdot u + w \in T(u,r)$ with $w \in u^\perp$ and $\|w\| \leq r$ we have

$$\|\Phi(x)\| \underset{\substack{\uparrow \\ \text{since } \Phi \text{ is} \\ \text{an isometry}}}{=} \|p(x)\| \underset{\substack{\uparrow \\ \text{since } p \text{ is linear and} \\ \text{by the triangle inequality}}}{\geq} \|p(\mu \cdot u)\| - \|p(w)\| \underset{\substack{\uparrow \\ \text{since } p \text{ is linear and since } \|w\| \leq r}}{\geq} |\mu| \cdot \|p(u)\| - \max\big\{\|p(w)\| \,\big|\, w \in \overline{B}_r^N\big\}.$$

Recall that we cleverly picked $v \notin \mathbb{R} \cdot u$. Thus we see that $\|p(u)\| > 0$. It follows easily from this discussion that $\Phi^{-1}(\overline{B}_s^{N-1}(0)) \cap T(u,r)$ is bounded. Since $M \subset T(u,r)$ is closed we see that $\Phi^{-1}(\overline{B}_r^{N-1}(0))$ is a compact subset of M. ■

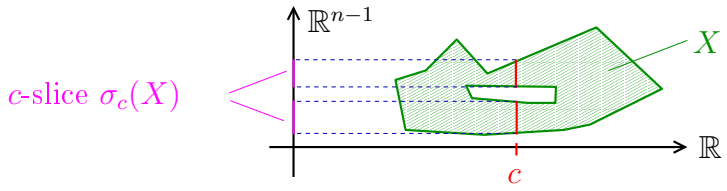


30.4. **The Mini-Fubini Theorem.** We conclude this chapter on subsets of measure zero with the "Mini-Fubini Theorem" which will play an important role in the proof of Sard's Theorem 31.1 in the next chapter. Our Mini-Fubini Theorem is a basic version of the Fubini Theorem [**StS2005**, Chapter 2.3.1] from measure and integration theory. To formulate the result we need to introduce the following harmless definition.

Notation. Let $X \subset \mathbb{R}^n$ be a subset. Given $c \in \mathbb{R}$ we refer to

$$\sigma_c(X) \;:=\; \{y \in \mathbb{R}^{n-1} \,|\, (c,y) \in X\}.$$

as the c-slice of X.



Now we can formulate the lemma which we will use in the proof of Sard's Theorem 31.1.

Lemma 30.10. (Mini-Fubini) Let $X \subset \mathbb{R}^n$ be a compact subset such that each slice $\sigma_c(X)$ is of measure zero in $\mathbb{R}^{n-1}$. Then X has measure zero in $\mathbb{R}^n$.

Remark. Lemma 30.10 follows from the Fubini Theorem in measure theory. But as we will see, one can also give a completely elementary proof, thus there is no need to cite high-powered results.

Proof. Given $U \subset \mathbb{R}^m$ we set

$$\mathrm{vol}_m(U) \;:=\; \inf\left\{ \sum_{i\in I} \mathrm{vol}(C_i) \;\middle|\; \begin{array}{c} \{C_i\}_{i\in I} \text{ is a countable} \\ \text{family of open cubes with } U \subset \bigcup_{i\in I} C_i \end{array} \right\} \;\in\; \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Claim 1. The following statements hold:

(i) A subset $U \subset \mathbb{R}^m$ has measure zero if and only if for every $\epsilon > 0$ we have $\mathrm{vol}_m(U) \leq \epsilon$.
(ii) Given $U \subset V \subset \mathbb{R}^m$ we have $\mathrm{vol}_m(U) \leq \mathrm{vol}_m(V)$.
(iii) For every interval (c,d) and every subset U of $\mathbb{R}^{m-1}$ we have

$$\mathrm{vol}_m((c,d) \times U) \;\leq\; (d-c) \cdot \mathrm{vol}_{m-1}(U).$$

(iv) For $U_1, \ldots, U_l \subset \mathbb{R}^m$ we have $\mathrm{vol}_m(U_1 \cup \cdots \cup U_l) \leq \mathrm{vol}_m(U_1) + \cdots + \mathrm{vol}_m(U_l)$.

Proof. Statements (i), (ii) and (iv) are very elementary. The only proof which is not a one-liner is the proof of (iii). We leave it to the reader to provide the proofs. ⊞

Now we start with the actual proof of the lemma. Thus let $X \subset \mathbb{R}^n$ be a compact subset such that each slice $\sigma_c(X)$ is of measure zero in $\mathbb{R}^{n-1}$. We need to show that X has measure zero in $\mathbb{R}^n$. Since X is compact there exist $a, b \in \mathbb{R}$ with $X \subset [a,b] \times \mathbb{R}^{n-1}$. We fix some $\delta > 0$. Next let $c \in [a,b]$. By hypothesis we know that $\sigma_c(X) \subset \mathbb{R}^{n-1}$ has measure zero. By (i) there exists an open set $U_c \subset \mathbb{R}^{n-1}$, given as the countable union of open cubes, with $\sigma_c(X) \subset U_c$ and $\mathrm{vol}_{n-1}(U_c) \leq \delta$.

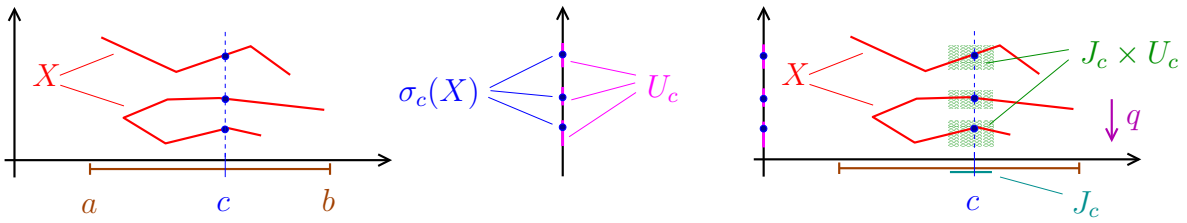
Claim 2. There exists an open interval J_c with $c \in J_c$ and with $X \cap (J_c \times \mathbb{R}^{n-1}) \subset J_c \times U_c$.

Proof. We consider $K := X \setminus (\mathbb{R} \times U_c)$. Since U_c is open we see that K is a closed subset of X, hence it follows from the Compact-Closed Lemma 1.21 that K is also compact. Next consider the map $q\colon K \to \mathbb{R}$ that is given by projection onto the first coordinate. By construction $c \notin q(K)$. Since K is compact we know by the Compact Image Lemma 2.13 and the Compact-Closed Lemma 1.21 (2) that $\mathbb{R} \setminus q(K)$ is open. This implies that there exists an open interval J_c that contains c and with $J_c \subset \mathbb{R} \setminus q(K)$. This open interval has the desired properties. ⊞

We have thus shown that for each $c \in [a,b]$ there exists an open subset $U_c \subset \mathbb{R}^{n-1}$ with $\mathrm{vol}_{n-1}(U_c) < \delta$ and an open interval J_c with $X \cap (J_c \times \mathbb{R}^{n-1}) \subset J_c \times U_c$. Since $[a,b]$ is compact we see that there exist $c_1, \ldots, c_l \in [a,b]$ such that $[a,b] \subset J_{c_1} \cup \cdots \cup J_{c_l}$. We write the intervals $J_{c_i} = (\alpha_i, \beta_i)$. An elementary argument[315] shows that after possibly shrinking the intervals we can arrange that $\sum_{i=1}^{l} (\beta_i - \alpha_i) \leq 2 \cdot (b-a)$. Altogether we see that

$$\mathrm{vol}_n(X) \underset{\text{by Claim 1 (ii)}}{\leq} \mathrm{vol}_n\Big(\bigcup_{i=1}^{l} J_{c_i} \times U_{c_i}\Big) \underset{\text{by Claim 1 (iv)}}{\leq} \sum_{i=1}^{l} \mathrm{vol}_n(J_{c_i} \times U_{c_i}) \underset{\text{by Claim 1 (iii)}}{\leq} \sum_{i=1}^{l} (\beta_i - \alpha_i) \cdot \underbrace{\mathrm{vol}_{n-1}(U_{c_i})}_{\leq \delta} \leq \delta \cdot 2 \cdot (b-a).$$

Since this inequality holds for every $\delta > 0$ we are done by Claim 1 (i). ■



On a later occasion we will also make use of the following converse to the Mini-Fubini Lemma 30.10.

Lemma 30.11. (Mini-Fubini II) Let $X \subset \mathbb{R}^n$ be a compact subset. If X has measure zero, then the set

$$\{c \in \mathbb{R}^{n-1} \mid \text{the } c\text{-slice } \sigma_c(X) \text{ has measure zero}\}$$

has full measure.

[315] For example one can arrange that for $i \in \{1, \ldots, l-1\}$ we have $\beta_i > \alpha_{i+1}$. Now one can easily modify the intervals by going "from left to right".

Proof. We will not provide a proof, we just point out that it follows from the usual Fubini Theorem in measure theory, see e.g. [**StS2005**, Chapter 2.3.1]. At least for $n = 2$ an elementary proof is given in [**Ox1971**, p. 51].[316] ■

Exercises for Chapter 30.

Exercise 30.1. Let $C \subset \mathbb{R}$ be the Cantor set as defined on page 130. Show that the Cantor set has measure zero.
Remark In Exercise 2.17 we saw that the Cantor set is uncountable. Thus we have shown that $\mathbb{R}$ contains uncountable subsets of measure zero.

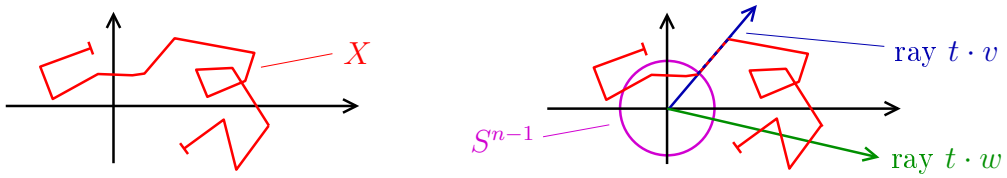
Exercise 30.2. Let $X \subset \mathbb{R}^n$ be a subset. Given $v \in S^{n-1}$ we write

$$\mu_v(X) \;:=\; \{t \in \mathbb{R}_{\geq 0} \,|\, t \cdot v \in X\}$$

Show that if X is compact with measure zero, then

$$\{v \in S^{n-1} \,|\, \mu_v(X) \text{ has measure zero}\} \;\subset\; S^{n-1}$$

has full measure in S^{n-1}.
Hint. Use the Mini-Fubini Lemma 30.11 and the Measure-Zero Properties Proposition 30.2 (3b).



Exercise 30.3. Let $c\colon [0,1] \to [0,1]$ be the Cantor function that we introduced on page 185. We consider the map

$$\begin{array}{rccc} f\colon & [0,1] & \to & [0,2] \\ & x & \mapsto & x + c(x). \end{array}$$

(a) Show that f is a homeomorphism.
(b) Let $C \subset [0,1]$ be the Cantor set. Show that $f(C)$ has Lebesgue measure one, in particular it is not a subset of measure zero.
Hint. Determine the Lebesgue measure of $f([0,1] \setminus C)$.

Remark. In principle this exercise requires familiarity with the Lebesgue measure. But one can also get joy out of the exercise by using a "naive" approach to length.

Exercise 30.4. Let $A \subset \mathbb{R}^n$ be a subset of diameter d.

(a) Does there exist a closed ball of radius $\frac{d}{2}$ that contains A?
(b) Does there exist a closed cube of side length d that contains A?

Exercise 30.5. Let M be an m-dimensional smooth manifold. Show that there exists an immersion $M \to \mathbb{R}^{2m}$.
Hint. Modify the proof of the Weak Whitney Embedding Theorem 30.7.

[316] The reader who has a self-contained proof of this lemma should communicate this proof to the author.

Exercise 30.6. We consider the non-compact 0-dimensional proper smooth submanifold $\mathbb{Z}^2 \subset \mathbb{R}^2$. Show that there is no $v \in S^1$ such that the projection of $\mathbb{Z}^2$ onto $v^\perp$ is a homeomorphism onto its image.

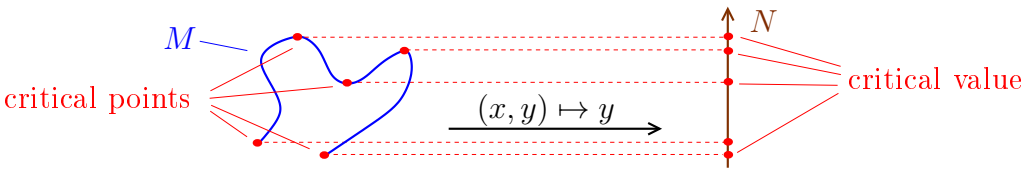
31. Sard's Theorem

We recall the following definition from page 701.

Definition. Let M and N be smooth manifolds and let $f\colon M \to N$ be a smooth map.

(1) Let $P \in M$.
   (a) If $P \in M \setminus \partial M$, then we say P is a regular point if $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}N$ is an epimorphism.
   (b) If $P \in \partial M$, then we say P is a regular point if $\mathrm{D}f_P\colon \mathrm{T}_P(\partial M) \to \mathrm{T}_{f(P)}N$ is an epimorphism.
   Otherwise we say that P is a critical point.

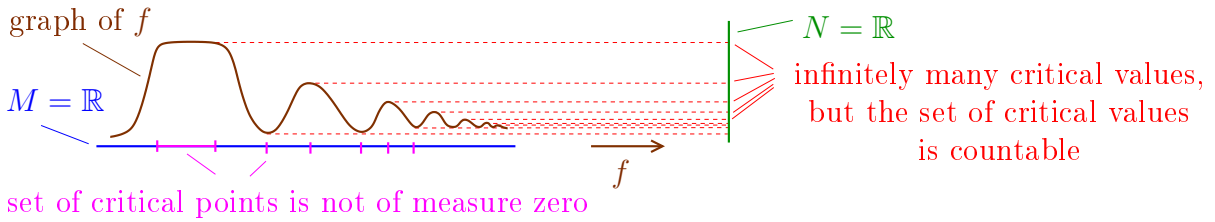
(2) We say $Q \in N$ is a regular value if all points in the preimage $f^{-1}(Q)$ are regular. Otherwise we call Q a critical value.



In this chapter we will prove Sard's Theorem 31.1 which says that for a smooth map $f\colon M \to N$ between smooth manifolds "almost every" point in N is a regular value. Sard's Theorem 31.1 is an absolutely essential tool in differential topology. As an amuse-bouche we will use it in this chapter to provide a proof of the Brouwer Fixed Point Theorem 31.4.

31.1. Sard's Theorem.

Theorem 31.1. (Sard's Theorem) If $f\colon M \to N$ is a smooth map between smooth manifolds (possibly with boundary)[317], then the set of critical values of f is of measure zero in N.



Remark.

(1) The theorem is named after Arthur Sard [**Sar1942**] who first proved a similar statement in 1942.

(2) As we pointed out on page 701, if $f\colon M \to N$ is a smooth map between smooth manifolds with $\dim(M) < \dim(N)$, then a point $Q \in N$ is a regular value if and only if $f^{-1}(Q) = \varnothing$. It follows from this observation that Sard's Theorem 31.1, together with the Measure-Zero Properties Proposition 30.2 (2), implies the Measure-Zero Properties Proposition 30.2 (4).

[317] We stress that M and N are allowed to have boundary, since (a) we need these cases, and (b) since many proofs in the literature assume that M and N do not have boundary.

(3) Sometimes Sard's Theorem 31.1 is also called *Morse-Sard Theorem*, since the case that $N = \mathbb{R}$ was first proved by Anthony Morse [**MorsA1939**] in 1939. Furthermore the fact that the set of regular values is dense was first proved by Arthur Brown [**BrownA1935**] in 1935.

The proof of Sard's Theorem 31.1 will be the content of the remainder of this section. Like many other textbooks we follow the account given in [**Miln1965a**, Chapter 3].

Proof. Let $f\colon M \to N$ be a smooth map between smooth manifolds. We need to show that the set of critical values of f is of measure zero in N. We set $m := \dim(M)$ and $n := \dim(N)$. We prove the theorem by induction on $m = \dim(M)$.

First let us consider the case $m = 0$. Note that this means that M is equipped with the discrete topology. Since smooth manifolds are in particular second-countable we see that M is a countable set. Thus the statement follows from the Measure-Zero Properties Proposition 30.2 (2).

Now assume that the conclusion of Sard's Theorem holds for smooth maps $M \to N$ with $\dim(M) < m$. Our goal is to prove Sard's Theorem for m-dimensional smooth manifolds M. From the observation that any smooth manifold is covered by a countable smooth atlas and from the definition of a smooth map on an open subset of $\mathbb{H}^m$ it follows easily, using the Measure-Zero Properties Proposition 30.2, that it suffices to deal with the case that M is an open subset of $\mathbb{R}^m$ and that $N = \mathbb{R}^n$. We denote by $f_1, \dots, f_n$ the coordinate functions of $f\colon M \to \mathbb{R}^n$.

We denote by $C \subset M \subset \mathbb{R}^m$ the set of critical points of $f\colon M \to N$. Given $k \in \mathbb{N}$ we set

$$C_k := \left\{ x \in C \;\middle|\; \begin{array}{c} \text{for all } j = 1, \dots, n \text{ and all } i = 1, \dots, k \text{ every} \\ \text{partial derivative of } f_j \text{ of order } i \text{ at } x \text{ vanishes} \end{array} \right\}.$$

Since f is smooth we know that all partial derivatives are continuous. Thus we see that C and all C_k's are closed in M. The proof of Sard's Theorem now proceeds in four steps.

Step 1. *The set $f(C \setminus C_1) \subset \mathbb{R}^n$ has measure zero.*
Let $a \in C \setminus C_1$. By definition of C_1 we know that there exists a $j \in \{1, \dots, n\}$ an $i \in \{1, \dots, k\}$ such that the partial derivative $\frac{\partial f_j}{\partial x_i}$ is non-zero at a. To simplify the notation we assume that $i = j = 1$. We write $\mathbb{R}^m = \mathbb{R} \times \mathbb{R}^{m-1}$ and we consider the map

$$\begin{array}{rcl} \Theta\colon M & \to & \mathbb{R}^m = \mathbb{R} \times \mathbb{R}^{m-1} \\ (\underbrace{x}_{\in \mathbb{R}}, \underbrace{y}_{\in \mathbb{R}^{m-1}}) & \mapsto & (f_1(x,y), y). \end{array}$$

Note that $\det(\mathrm{D}\Phi_a) = \frac{\partial f_1}{\partial x_1}(a) \neq 0$. Thus it follows from the Inverse Function Theorem 19.12 that Θ is a local diffeomorphism at a. This means that there exists an open neighborhood V_a of a in the open subset $M \setminus C_1$ such that $\Theta(V_a)$ is an open subset of $\mathbb{R}^m$ and such that $\Theta\colon V_a \to \Theta(V_a)$ is a diffeomorphism. By an elementary argument, similar to Lemma 19.16, we see that $f \circ \Theta^{-1}$ is of the form

$$(*) \qquad (f \circ \Theta^{-1})(x,y) = (x, \underbrace{g(x,y)}_{\in \mathbb{R}^{n-1}}) \qquad \text{for every } (x,y) \in \Theta(V_a)$$

where $g\colon \Theta(V_a) \to \mathbb{R}^{n-1}$ is a smooth map. Next we pick a closed interval $I \subset \mathbb{R}$ and a closed ball $J \subset \mathbb{R}^{m-1}$, each with non-empty interior, such that $\Theta(a) \in I \times J \subset \Theta(V_a)$.

Claim. For each $r \in I$ we have

$$\begin{array}{l} r\text{-slice of the set of critical values} \\ \text{of the map } f \circ \Theta^{-1} \colon I \times J \to \mathbb{R} \times \mathbb{R}^{n-1} \end{array} \quad = \quad \text{set of critical values of } \begin{array}{rcl} \varphi_r \colon J & \to & \mathbb{R}^{n-1} \\ y & \mapsto & g(r,y). \end{array}$$

Proof. Let $r \in I$. Given $y \in J$ it follows from $(*)$ that we have the following equality:

$$\mathrm{D}(f \circ \Theta^{-1})_{(r,y)} \;=\; \begin{pmatrix} 1 & 0 \\ * & \mathrm{D}(y \mapsto g(r,y))_y \end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 \\ * & \mathrm{D}(\varphi_r)_y \end{pmatrix}.$$

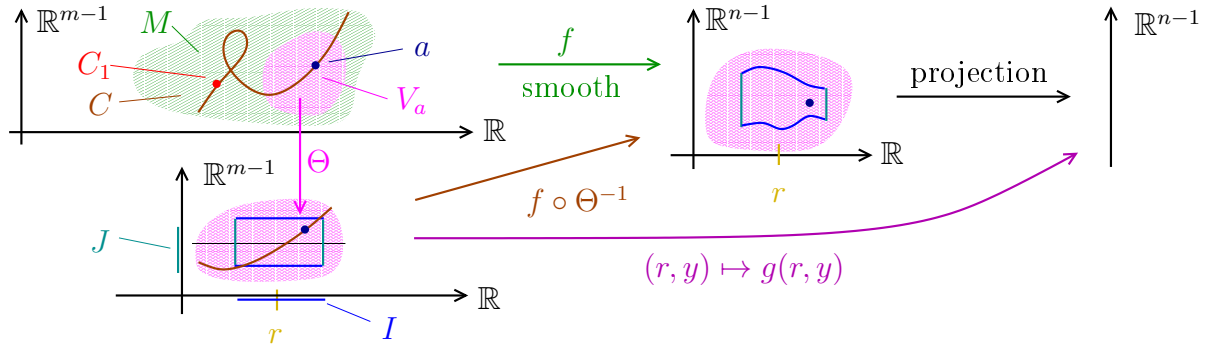
This shows that (r,y) is a critical point of $f \circ \Theta^{-1}$ if and only if y is a critical point of the map $\varphi_r \colon J \to \mathbb{R}^{n-1}$ given by $y \mapsto g(r,y)$. The claim now follows from the above and the following two observations:

(a) For any s, z we have $(f \circ \Theta^{-1})(s,z) = (s, g(s,z)) = (s, \varphi_s(z))$.
(b) It follows from (a) that for $w \in \mathbb{R}^{n-1}$ we have

$$w \in r\text{-slice of } f(I \times J) \quad \Longleftrightarrow \quad w \in \varphi_r(J). \qquad \boxplus$$

Now we can conclude the proof of Step 1:

(a) Since $\dim(J) = m-1$ we see that for each $r \in I$ the right hand side of the claim has measure zero by our induction hypothesis.
(b) As we mentioned above, C is closed. This implies that $C \cap (I \times J)$ is compact. Therefore it follows from (a) together with Lemma 30.10 that the set of critical values of $f \circ \Theta^{-1}$ in $I \times J$ has measure zero.
(c) Since Θ is a diffeomorphism it follows from (b) and the trivial observation on page 701 that the set of critical values of f in $W_a := \Theta^{-1}(I \times J)$ has measure zero.
(d) We have now shown that each $a \in C \setminus C_1$ admits a neighborhood $W_a \subset C \setminus C_1$ such that $f(W_a \cap (C \setminus C_1))$ has measure zero.
(e) By the Countable Covering Lemma 9.2 we can cover $C \setminus C_1$ by countably many of the W_a. Therefore we can cover $f(C \setminus C_1)$ by countably many of the $f(W_a)$.
(f) It follows from (d), (e) and the Measure Zero-Basics Lemma 30.4 that $f(C \setminus C_1)$ has indeed measure zero.



Step 2. *For each $k \in \mathbb{N}$ the set $f(C_k \setminus C_{k+1}) \subset \mathbb{R}^n$ has measure zero.*
Let $a \in C_k \setminus C_{k+1}$. Since $a \notin C_{k+1}$ there exists a k-th partial derivative $\varphi \colon M \to \mathbb{R}$ of some f_j such that some partial derivative of φ does not vanish at a. Note that this means that a is a regular point of $\varphi \colon M \to \mathbb{R}$. By the Regular Value-Open Subset Lemma 26.1 we know that the set of regular *points* of φ is an open subset of M, thus there exists an open neighborhood $V_a \subset M \setminus C_{k+1}$ of a that consists only of regular points of $\varphi \colon M \to \mathbb{R}$.

We set $Y := \varphi^{-1}(0)$. Note that it follows from the Regular Value Theorem 26.4 that Y is a codimension-one submanifold of V_a. By definition of C_k all k-th partial derivatives of f vanish on C_k, in particular φ vanishes on C_k, which implies that $C_k \cap V_a \subset Y = \varphi^{-1}(0)$.

Claim. $f(C_k \cap V_a) \subset$ critical values of $f|_Y \colon Y \to \mathbb{R}^n$.

Proof. Let $P \in C_k \cap V_a$. Since $P \in C$ we see that $\mathrm{D}f_P \colon \mathrm{T}_P M \to \mathbb{R}^n$ is not an epimorphism. But this implies that $\mathrm{D}(f|_Y)_P = (\mathrm{D}f_P)|_{\mathrm{T}_P Y} \colon \mathrm{T}_P Y \to \mathbb{R}^n$ is also not an epimorphism. Thus we see that P is a critical point of $f|_Y \colon Y \to \mathbb{R}^n$ which implies that $f(P)$ is a critical value of $f|_Y \colon Y \to \mathbb{R}^n$. ⊞

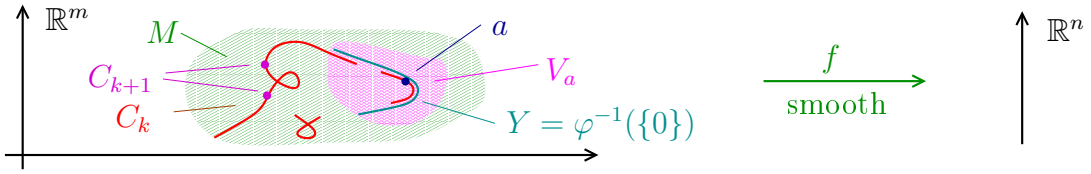
Now we can conclude the proof of Step 2:

(a) Since Y is a smooth manifold of dimension $m-1$ we obtain from our induction hypothesis that the right-hand side of the claim is a set of measure zero.

(b) It follows from (a) together with the Measure-Zero Properties Proposition 30.2 (1) that each $a \in C_k \setminus C_{k+1}$ admits an open neighborhood $V_a \subset C_k \setminus C_{k+1}$ such that $f(V_a \cap (C_k \setminus C_{k+1}))$ has measure zero.

(c) By the Countable Covering Lemma 9.2 we can cover $C_k \setminus C_{k+1}$ by countably many of the V_a. Therefore we can cover $f(C_k \setminus C_{k+1})$ by countably many of the $f(V_a)$.

(d) It follows from (b), (c) and the Measure Zero-Basics Lemma 30.4 that $f(C_k \setminus C_{k+1})$ has indeed measure zero.



Step 3. *If $k > \frac{m}{n} - 1$, then the set $f(C_k) \subset \mathbb{R}^n$ has measure zero.*

Let $a \in C_k$ and let V_a be an open cube that is a neighborhood of a in M and which satisfies $\overline{V}_a \subset M$.

Claim. There exists some $D \in \mathbb{R}$ such that for each closed cube $W \subset \overline{V}_a$ of side length s and which contains a point $b \in C_k$ we have

$$\|f(x) - f(b)\| \leq D \cdot s^{k+1} \qquad \text{for each } x \in W.$$

Proof. Since f is smooth and since $\overline{V}_a$ is compact we can set

$$M := \text{maximum of the absolute value of all } (k+1)\text{-st partial derivatives of } f \text{ on } \overline{V}_a.$$

We denote by $f_1, \ldots, f_n$ the coordinate functions of f.

Next let $W \subset \overline{V}_a$ be a closed cube of side length s which contains a point $b \in C_k$. Let $i \in \{1, \ldots, n\}$. Note that for $b \in C_k$ the k-th Taylor polynomial $p(f_i, b)_k$ of f_i at the point b, which we introduced on page 535, is actually the constant polynomial $f_i(b)$. Thus we see that for each $x \in W$ we have

$$|f_i(x) - f_i(b)| = |f_i(x) \underbrace{- p(f,b)_k(x)}_{= f_i(b)}| \overset{\text{Corollary 19.8}}{\leq} \frac{m^{k+1} \cdot M}{(k+1)!} \cdot \|x - b\|^{k+1} \overset{\substack{\text{since } x, b \text{ are contained in an} \\ m\text{-dimensional cube of side length } s}}{\leq} \frac{m^{k+1} \cdot M}{(k+1)!} \cdot (\sqrt{m} \cdot s)^{k+1}.$$

Next recall that given any $(y_1, \ldots, y_n) \in \mathbb{R}^n$ we have the elementary inequality

$$\|(y_1, \ldots, y_n)\| \;=\; \sqrt{y_1^2 + \cdots + y_n^2} \;\leq\; n \cdot \max\{|y_1|, \ldots, |y_n|\}.$$

Combining these two inequalities we see that

$$\|f(x) - f(b)\| \;\leq\; \underbrace{n \cdot \tfrac{m^{k+1} \cdot M}{(k+1)!} \cdot (\sqrt{m})^{k+1}}_{=:D} \cdot s^{k+1}$$

⊞

Note that it follows from the claim for each cube $W \subset \overline{V}_a$ of side length s and which contains a point in C_k we have

$$f(\underbrace{W}_{\text{vol}=s^m}) \;\subset\; \begin{array}{c}\text{closed ball in } \mathbb{R}^n \text{ of} \\ \text{radius } D \cdot s^{k+1}\end{array} \;\subset\; \underbrace{\text{closed cube in } \mathbb{R}^n \text{ of side length } D \cdot s^{k+1}}_{\text{vol}=D^n \cdot s^{(k+1)n}}.$$

Here comes the key observations:

> If $k > \frac{m}{n} - 1$, then it follows that $(k+1)n > m$, which implies that the smaller s, the more the volume shrinks by applying f.

We now have to turn this observation into a proper argument.

Claim. The set $f(C_k \cap \overline{V}_a)$ has measure zero.

Proof. Let $\epsilon > 0$. We denote by r the side length of the cube $\overline{V}_a$. Since $m - (k+1) \cdot n < 0$ there exists an $L \in \mathbb{N}$ such that

$$D^n \cdot r^{(k+1) \cdot n} \cdot L^{m-(k+1) \cdot n} \;<\; \epsilon.$$

We choose L as above and we break $\overline{V}_a$ into L^m closed cubes $W_1, \ldots, W_{L^m}$ of side length $\frac{r}{L}$. We set

$$I \;:=\; \{i \in \{1, \ldots, L^m\} \,|\, C_k \cap W_i \neq \varnothing\}.$$

We see that

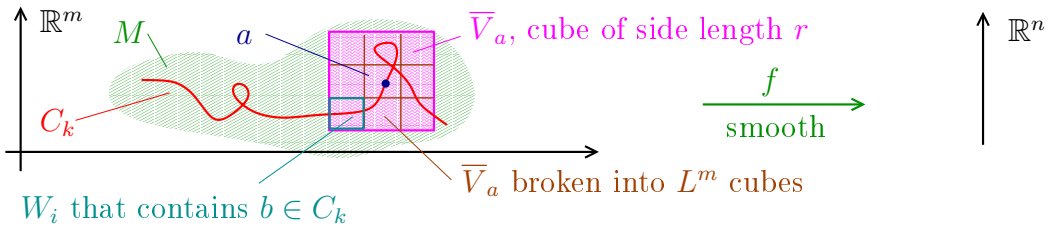
$$f(C_k \cap \overline{V}_a) \;\subset\; f\Big(\bigcup_{i \in I} W_i\Big) \;=\; \bigcup_{i \in I} f(W_i) \;\underset{\uparrow}{\subset}\; \bigcup_{i \in I} \text{closed cube of volume } D^n \cdot (\tfrac{r}{L})^{(k+1)n}.$$

by the above since each such W_i contains a point in C_k

It follows from the choice of L and the fact that $|I| \leq L^m$ that the sum of the volumes of the cubes to the right is at most $L^m \cdot D^n \cdot (\frac{r}{L})^{(k+1)n}$, which by choice of L is $< \epsilon$. We have now shown that we can cover $f(C_k \cap \overline{V}_a)$ by closed cubes whose volumes add up to less than ϵ. It follows from the Measure Zero-Basics Lemma 30.4 (4) that $f(C_k \cap \overline{V}_a)$ has indeed measure zero. ⊞

Now we can conclude the proof of Step 3:

(a) We have just shown that each $a \in C_k$ admits a neighborhood $\overline{V}_a \subset M$ such that $f(C_k \cap \overline{V}_a)$ has measure zero.
(b) By the Countable Covering Lemma 9.2 we can cover C_k by countably many of the $\overline{V}_a$.
(c) It follows from (a) and (b), together with the Measure Zero-Basics Lemma 30.4 (2), that $f(C_k)$ has indeed measure zero.



Step 4. *Conclusion of the argument.*
We pick some $k \in \mathbb{N}$ with $k > \frac{m}{n} - 1$. Then we have

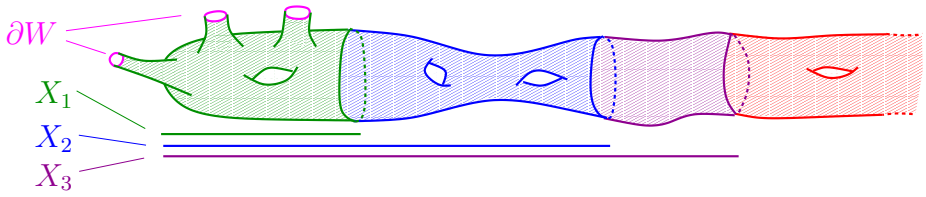
$$f(C) = f\Big((C \setminus C_1) \cup \bigcup_{i=1}^{k-1} (C_i \setminus C_{i+1}) \cup C_k\Big) = \underbrace{f(C \setminus C_1)}_{\substack{\text{measure zero} \\ \text{by Step 1}}} \cup \bigcup_{i=1}^{k-1} \underbrace{f(C_i \setminus C_{i+1})}_{\substack{\text{measure zero} \\ \text{by Step 2}}} \cup \underbrace{f(C_k)}_{\substack{\text{measure zero} \\ \text{by Step 3}}}.$$

It follows from the Measure Zero-Basics Lemma 30.4 (2) that $f(C)$, i.e. the set of critical values of f, has measure zero. ■

31.2. **Applications of Sard's Theorem.** In this section we give several applications of Sard's Theorem. Our first proposition of this section often allows us to reduce a statement about non-compact smooth manifolds to a statement about compact smooth manifolds.

Proposition 31.2. (Compact Submanifold Exhaustion Proposition) Let W be a connected n-dimensional smooth manifold with compact boundary. There exists a sequence $X_1, X_2, \ldots$ of n-dimensional smooth submanifolds of W with the following four properties:[318]

(0) We have $\partial W \subset X_1$.
(1) For each $i \in \mathbb{N}$ we have $X_i \subset \overset{\circ}{X}_{i+1}$.
(2) Each X_i is compact and connected.
(3) We have $\bigcup_{i \in \mathbb{N}} \overset{\circ}{X}_i = W$.



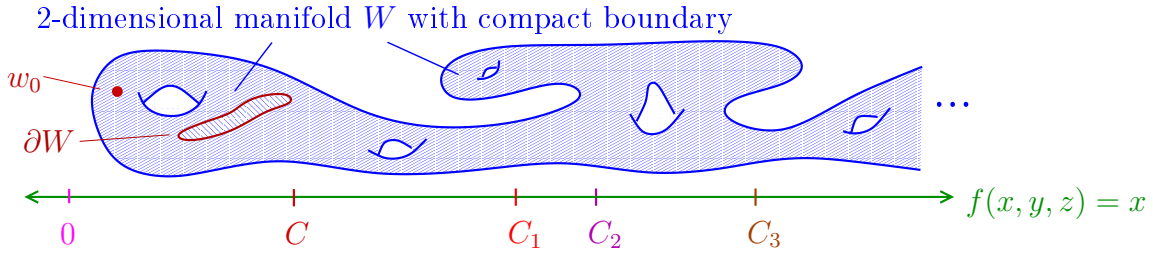
Remark. There is an analogue of the Compact Submanifold Exhaustion Proposition 31.2 if ∂W is non-compact. Since that statement is more technical to formulate we postpone it to Proposition **??**.

Proof. Let W be a connected n-dimensional smooth manifold. If W is empty, then there is nothing to show, so we might as well assume that W is non-empty. We pick a point $w_0 \in W$. By the Manifold Exhaustion Lemma 21.2 there exists a smooth exhaustion function on W,

[318] In particular we see that the X_i form a compact exhaustion of W in the sense of the definition on page 338.

i.e. there exists a smooth function $f\colon W \to \mathbb{R}_{\geq 0}$ with the property that for every $c \in \mathbb{R}_{\geq 0}$ the preimage $f^{-1}([0,c])$ is a compact subset of W. Since ∂W is compact it follows from the Compact Image Lemma 2.13 that there exists a $C \in \mathbb{R}_{\geq 0}$ such that $f(\partial W) \subset (-\infty, C]$ and such that $C \geq f(w_0)$.

It follows from Sard's Theorem 31.1 and the Full Measure-Density Proposition 30.3 (2) that there exists a strictly increasing sequence $C < C_1 < C_2 < C_3 < \dots$ of regular values of f such that $\lim_{i\to\infty} C_i = \infty$. For each $i \in \mathbb{N}$ we denote by X_i the path-component of $f^{-1}((-\infty, C_i])$ that contains w_0. By Exercise 26.6, which is a reasonably straightforward consequence of the Regular Value Theorem 26.4, and by our choice of f we know that each X_i is a compact n-dimensional submanifold with $\partial_0 X_i = X_i \cap f^{-1}(C_i)$. It follows from the Codimension-Zero Submanifold Proposition 19.38 that $\overset{\circ}{X}_i = X_i \cap f^{-1}((-\infty, C_i))$. From this observation, and from the fact that the C_i are strictly increasing we obtain that for each $i \in \mathbb{N}$ we have $X_i \subset \overset{\circ}{X}_{i+1}$. We have now shown that the X_i satisfy Properties (1) and (2).



Now let $P \in W$. Since W is path-connected there exists a path $\gamma\colon [0,1] \to W$ from w_0 to P. Since $[0,1]$ is compact it follows again from the Compact Image Lemma 2.13 and from $\lim_{i\to\infty} C_i = \infty$ that there exists an i with $\gamma([0,1]) \subset f^{-1}((-\infty, C_i))$. By the above this implies that $P \in \overset{\circ}{X}_i$. We have thus shown that (3) is also satisfied. ■

To start our next application we need to recall the following definition from page 433.

Definition. Let X be a topological space. We say a subset $A \subset X$ is a retract of X if there exists a retraction $r\colon X \to A$, i.e. a map with $r(a) = a$ for all $a \in A$.

Proposition 31.3. (Boundary Non-Retraction Proposition) Let M be a *compact* non-empty smooth manifold. There is no retraction from M to ∂M.[319]

Remark. In the Fundamental Class-of-Boundary Corollary 100.11 we will give an alternative proof of the Boundary Non-Retraction Proposition 31.3 and in Corollary 106.28 we will extend the conclusion of the Boundary Non-Retraction Proposition 31.3 to topological manifolds.

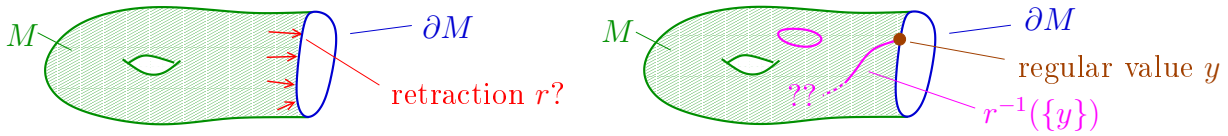
Proof. Let M be a compact non-empty smooth manifold. We suppose that there exists a retraction $M \to \partial M$. It follows from the Replace-by-Smooth Retraction 29.3 that there exists in fact a smooth retraction $r\colon M \to \partial M$. Since M is non-empty we obtain from Sard's Theorem 31.1 that the map $r\colon M \to \partial M$ admits a regular value y. Since M is

[319]The adjective "compact" is of course necessary, for example the smooth manifold $M = [0,\infty)$ admits a retraction onto its boundary $\partial M = \{0\}$.

compact we obtain from the Regular Value Theorem 26.4 that $W := r^{-1}(y)$ is a proper compact 1-dimensional submanifold of M. Note that

$$\partial W \underset{\text{since } W \text{ is a proper submanifold}}{=} W \cap \partial M = \{x \in \partial M \,|\, r(x) = y\} \underset{\text{since } r = \mathrm{id} \text{ on } \partial M}{=} \{y\}.$$

Thus we see that ∂W consists of a single point. But it follows from the Smooth 1-Dimensional Manifold Classification Theorem 22.7, that the boundary of a compact 1-dimensional smooth manifold consists of an *even* number of points. ■



31.3. **The Brouwer Fixed Point Theorem I.** The Boundary Non-Retraction Proposition 31.3 from the last section allows to prove the famous Brouwer Fixed Point Theorem which was first proved by Jacques Hadamard [**Had1910**] and Luitzen Brouwer [**Brou1912a**].

Theorem 31.4. (Brouwer Fixed Point Theorem) Every map $f\colon \overline{B}^n \to \overline{B}^n$ admits a fixed point, i.e. for every map $f\colon \overline{B}^n \to \overline{B}^n$ there exists a point $x \in \overline{B}^n$ such that $f(x) = x$.[320]

Remark. The Brouwer Fixed Point Theorem 31.4 admits many different proofs. We provide two more proofs on page 1618 and on page 2118. A fairly elementary proof is alternatively given in [**Miln1978**].

Proof. First let $f\colon \overline{B}^n \to \overline{B}^n$ be an arbitrary continuous map.

Claim 1. Given $x \in \overline{B}^n$ with $x \neq f(x)$ the open ray $r_x = \{f(x) + t \cdot (x - f(x)) \,|\, t > 0\}$ contains a unique point on S^{n-1}.

Proof. We consider the map

$$\begin{array}{rcl} \rho_x\colon \mathbb{R} & \to & \mathbb{R}_{\geq 0} \\ t & \mapsto & \|f(x) + t \cdot (x - f(x))\|^2 \underset{\text{follows from } \|v+w\|^2 = \langle v+w, v+w\rangle}{=} \|f(x)\|^2 + 2t \cdot \langle f(x), x - f(x)\rangle + t^2 \cdot \|x - f(x)\|^2. \end{array}$$

Since $x \neq f(x)$ we see that ρ_x is a quadratic function with $\lim_{t \to \pm\infty} \rho_x(t) = \infty$. Therefore we obtain from $\rho_x(0) = \|f(x)\| \leq 1$ together with an elementary calculus argument that $\rho_x^{-1}([0,1])$ is a compact interval.

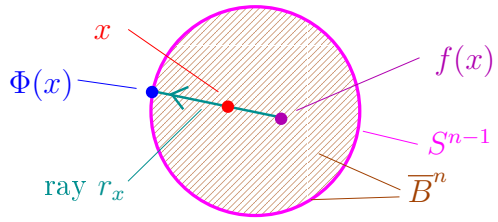
It follows from this observation together with $\rho_x(0) = \|f(x)\| \leq 1$ and $\rho_x(1) = \|x\| \leq 1$ that there exists a unique $t \in (0, \infty)$ with $\rho_x(t) = 1$. This shows that the open ray r_x contains a unique point on S^{n-1}. ⊞

Now we suppose that $f\colon \overline{B}^n \to \overline{B}^n$ has no fixed points. It follows from Claim 1 that for each $x \in \overline{B}^n$ the open ray $r_x = \{f(x) + t \cdot (x - f(x)) \,|\, t > 0\}$ contains a unique point

[320] How can one prove the statement for $n = 1$ using elementary methods from real analysis?

on S^{n-1}. Thus we can consider the following map:

$$\begin{array}{rcl} \Phi\colon \overline{B}^n & \to & \overline{B}^n \\ x & \mapsto & \text{the unique intersection point of } S^{n-1} \text{ with the} \\ & & \text{uniquely determined open ray from } f(x) \text{ to } x. \end{array}$$



Claim 2. The map Φ is continuous.

Proof. As we discussed in Claim 1, for every $x \in \overline{B}^n$ there exists a unique $t_x > 0$ with $\rho_x(t_x) = \|f(x)+t_x\cdot(x-f(x))\|^2 = 1$, and by definition we have $\Phi(x) = f(x)+t_x\cdot(x-f(x))$. Thus it suffices to show that the map $x \mapsto t_x$ is continuous. One can determine t_x through the quadratic equation

$$1 \;=\; \|f(x) + t_x \cdot (x - f(x))\|^2 \;=\; \|f(x)\|^2 + t_x \cdot 2\langle f(x, x - f(x))\rangle + t_x^2 \cdot \|x - f(x)\|^2.$$

The quadratic formula for polynomials of degree two is continuous in the coefficients, thus we see that the map $x \mapsto t_x \in \mathbb{R}_{>0}$ is indeed continuous.[321] ⊞

It is evident that $\Phi(x) = x$ for all $x \in S^{n-1}$ and that $\Phi(x) \in S^{n-1}$ for all $x \in S^{n-1}$. The map Φ is thus a retraction from $\overline{B}^n$ to S^{n-1}. But according to the Boundary Non-Retraction Proposition 31.3 this is not possible. Thus we have obtained a contradiction. ■

The Brouwer Fixed Point Theorem pops up many times in mathematics. For example we can use it to prove the following theorem in linear algebra.

Theorem 31.5. (Perron-Frobenius Theorem) Let $A = (a_{ij})$ be an $n \times n$-real matrix. If all entries are positive, then A has a positive eigenvalue λ with an eigenvector v such that all entries of v are non-negative.

Remark. There are more precise statements one can prove about matrices with positive entries we refer to [**Meyr2000**, Section 8.2] and also to

`https://en.wikipedia.org/wiki/Perron-Frobenius_theorem`

for details. Supposedly the Perron-Frobenius Theorem and some of the more refined results mentioned in [**Meyr2000**, Section 8.2] play a major role in Google's algorithm, see e.g.

`http://www.math.pku.edu.cn/teachers/yaoy/Fall2011/lecture07.pdf`

for more information.

Proof. We introduce two players:

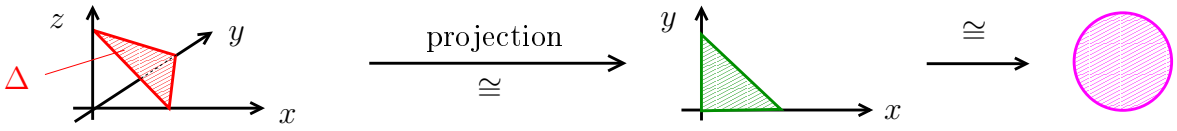
(1) Given $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ we write $|v| := \sum_{i=1}^{n} |v_i|$.

[321] One could also use the home grown Proposition 5.42, if one prefers more fancy arguments.

(2) We consider

$$\Delta := \{(t_1, \ldots, t_n) \in [0,1]^n \,\big|\, |v| = 1\}.$$

By projecting Δ onto $\mathbb{R}^{n-1}$ and using the Convex-to-Ball Proposition 2.20 one sees that Δ is homeomorphic to $\overline{B}^{n-1}$. It follows immediately from this observation and from the Brouwer Fixed Point Theorem 31.4 that any self-map $f\colon \Delta \to \Delta$ has a fixed point.



Now let $A = (a_{ij})$ be an $n \times n$-real matrix such that all entries are positive. We consider the map

$$\begin{array}{rcl} f\colon \Delta & \to & \Delta \\ v = (v_1, \ldots, v_n) & \mapsto & \frac{A \cdot v}{|A \cdot v|}. \end{array}$$

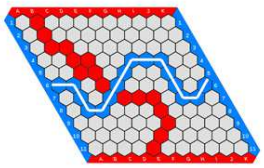
Note that this map is defined since our hypothesis that all a_{ij} are positive and the fact that all entries of v are non-negative and that at least one entry of v is positive imply that $A \cdot v \neq 0$. As we had just discussed above, the map $f\colon \Delta \to \Delta$ has a fixed point $v \in \Delta$. But this means that $A \cdot v = |A \cdot v| \cdot v$. Put differently, we have just shown that v is an eigenvector to the positive eigenvalue $|A \cdot v|$. ■

We conclude this discussion of the Brouwer Fixed Point Theorem 31.4 with two applications outside of mathematics:

(1) The board game "Hex" was invented in 1942 by Piet Hein[322] and it was re-invented in 1948 by John Nash.[323] The game is illustrated in the figure below and an online version can be found at

`http://www.lutanho.net/play/hex.html`

David Gale [**Gal1979**] showed in 1979 that the statement that the board game "Hex" cannot end in a draw, is equivalent to the Brouwer Fixed Point Theorem 31.4. We also refer to [**Gh2014**, p. 102] for a discussion of Hex and the Brouwer Fixed Point Theorem 31.4.



11×11 Hex gameboard showing a winning configuration for Blue

(2) The Brouwer Fixed Point Theorem 31.4 can also be used to show the existence of a "Nash equilibrium" in game theory. We refer to the following website for more information:

`https://en.wikipedia.org/wiki/Nash_equilibrium`.

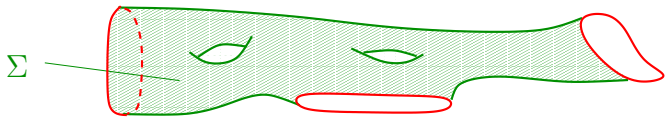
[322] Piet Hein (1905-1996) was a Danish mathematician.

[323] John Nash (1928-2015) was an American mathematician who was awarded the Nobel prize in economics in 1994. The movie "a beautiful mind" was based on his struggles with schizophrenia.

Exercises for Chapter 31.

Exercise 31.1. In the figure below we show Σ, the surface of genus two minus three open disks. By the Boundary Non-Retraction Proposition 31.3 we know that there is no retraction from Σ to $\partial\Sigma$.

(a) Does there exist a retraction to the union of two boundary components?
(b) Does there exist a retraction to a single boundary component?



Exercise 31.2. We say that a map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ has *bounded displacement* if there exists a $C \in \mathbb{R}$ such that for every $x \in \mathbb{R}^n$ we have $\|x - f(x)\| \leq C$. Show that every map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ of bounded displacement is onto.
Hint. Use the Brouwer Fixed Point Theorem 31.4.

Exercise 31.3. Let M be a smooth manifold and let $K, L \subset M$ be disjoint closed subsets of M. Show that there exist codimension-zero submanifolds A and B of M, which are closed subsets of M, such that $K \subset A$, $L \subset B$, $A \cup B = M$ and such that $A \cap B = \partial_0 A = \partial_0 B$.
Hint. Use Exercise 26.6 and use Urysohn's Lemma 21.7 for Manifolds.



Exercise 31.4. Let M be a smooth manifold, let U be an open subset of M and let K be a *compact* subset of M that is contained in U. Show that there exists a *compact* codimension-zero submanifold N of M with $K \subset N \subset U$.
Remark. The exercise is related to Exercise 31.3, but the compactness condition and requirement adds a welcome extra layer of difficulty.



Exercise 31.5. Let $n \in \mathbb{N}$. Show that the subset

$$\{A \in \mathrm{M}(n \times n, \mathbb{R}) \mid \det(A) = 0\} \subset \mathrm{M}(n \times n, \mathbb{R}) = \mathbb{R}^{n^2}$$

has measure zero.
Hint. There are many approaches to proving this statement. You could for example play with the map

$$\begin{aligned} \Phi\colon \overbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}^{n\text{-1 copies}} \times \mathbb{R}^{n-1} &\mapsto \mathrm{M}(n \times n, \mathbb{R}) \\ (v_1, \ldots, v_{n-1}, (\lambda_1, \ldots, \lambda_{n-1}) &\mapsto \Big(v_1 \ \ldots \ v_{n-1} \ \sum_{i=1}^{n-1} \lambda_i \cdot v_i\Big). \end{aligned}$$

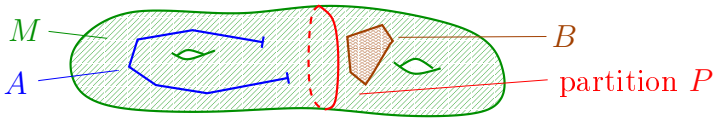
You could also make use of the fact that a matrix has determinant zero if and only if the column vectors are linearly dependent.

Exercise 31.6.

(a) Let X be a topological space and let A and B be disjoint subsets of X. As on page 389 we say a subset $P \subset X$ is a *partition between A and B*, if there exist disjoint open subsets U and V of X with $A \subset U$, $B \subset V$ and $X \setminus P = U \cup V$.

Show that if M is an n-dimensional smooth manifold, then given any two disjoint closed subsets A and B there exists a partition P between A and B that is an $(n-1)$-dimensional smooth manifold of M.

Remark. This part of the exercise is quite similar to Exercise 31.3.



(b) We iteratively define classes $\mathcal{D}_{-1}, \mathcal{D}_0, \mathcal{D}_1, \ldots$ of topological spaces:

- We define $\mathcal{D}_{-1}$ to be the class consisting only of the empty topological space.
- Suppose that for some $n \in \mathbb{N}_0$ we have defined $\mathcal{D}_{-1}, \mathcal{D}_0, \ldots, \mathcal{D}_{n-1}$. Given a topological space X we say $X \in \mathcal{D}_n$ if for any two closed disjoint subsets A and B of X there exists a partition P between A and B with $P \in \mathcal{D}_{n-1}$. (We refer to Exercise 11.10 for the definition of a partition.)

We define $\mathcal{D}_\infty$ to be the class of all topological spaces. Given a topological space X we define, as in Exercise 31.6, the *large inductive dimension* $\operatorname{ind}(X)$ as follows:

$$\operatorname{Ind}(X) \;:=\; \min\{n \in \mathbb{Z}_{\geq -1} \cup \{\infty\} \,|\, X \in \mathcal{D}_n\}.$$

Show that for every n-dimensional smooth manifold M we have $\operatorname{ind}(M) \leq n$.

32. Complex manifolds

As a slight detour from the main story, namely the development of the theory of smooth manifolds, let us now give a very short introduction to complex manifolds. Complex manifolds are topological manifolds that are equipped with a holomorphic atlas. As we will see, complex manifolds are smooth manifolds with particularly nice properties. Throughout these notes they will play a role every now and then.

Convention. Let $n \in \mathbb{N}_0$. Throughout these notes we will use the isomorphism

$$\begin{array}{rcl} \mathbb{R}^{2n} = (\mathbb{R}^2)^n & \to & \mathbb{C}^n \\ (x_1, y_1, \ldots, x_n, y_n) & \mapsto & (x_1 + y_1\,\mathrm{i}, \ldots, x_n + y_n\,\mathrm{i}) \end{array}$$

of real vector spaces to identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$.

The following elementary lemma is the essential tool for describing complex linear maps by real matrices.

Lemma 32.1. Given any $a, b \in \mathbb{R}$ the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{v \mapsto \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right) \cdot v} & \mathbb{R}^2 \\ {\scriptstyle (x,y)\mapsto x+y\mathrm{i}}\big\downarrow \cong & & \cong \big\downarrow {\scriptstyle (x,y)\mapsto x+y\mathrm{i}} \\ \mathbb{C} & \xrightarrow{z \mapsto (a+b\mathrm{i})\cdot z} & \mathbb{C}. \end{array}$$

Proof. The lemma follows immediately from an elementary calculation. ∎

32.1. Holomorphic functions.

First we recall the definition of a holomorphic function.

Definition. Let $U \subset \mathbb{C}$ be an open subset and let $f\colon U \to \mathbb{C}$ be a map.

(1) We say f is holomorphic in $z_0 \in U$, if the limit

$$\frac{d}{dz} f(z_0) \;:=\; f'(z_0) \;:=\; \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \;\in \mathbb{C}$$

exists. We refer to $\frac{d}{dz} f(z_0) = f'(z_0)$ as the derivative of f at z.

(2) We say f is holomorphic if f is holomorphic in each $z_0 \in U$.

Example. It follows immediately from the definition that the function $f(z) = z$ is holomorphic. Furthermore it is elementary to show products, fractions, sums and compositions of holomorphic functions are again holomorphic. This shows in particular that maps described by polynomials are holomorphic.

Proposition 32.2. (Complex Power Series Convergence Proposition) Let $(c_n)_{n \in \mathbb{N}_0}$ be a sequence of complex numbers and let $z_0 \in \mathbb{C}$. We consider the radius of convergence

$$R \;:=\; \sup\left\{ |z - z_0| \;\middle|\; \sum_{n=0}^{\infty} c_n \cdot (z - z_0)^n \text{ converges} \right\}.$$

The following two statements hold:

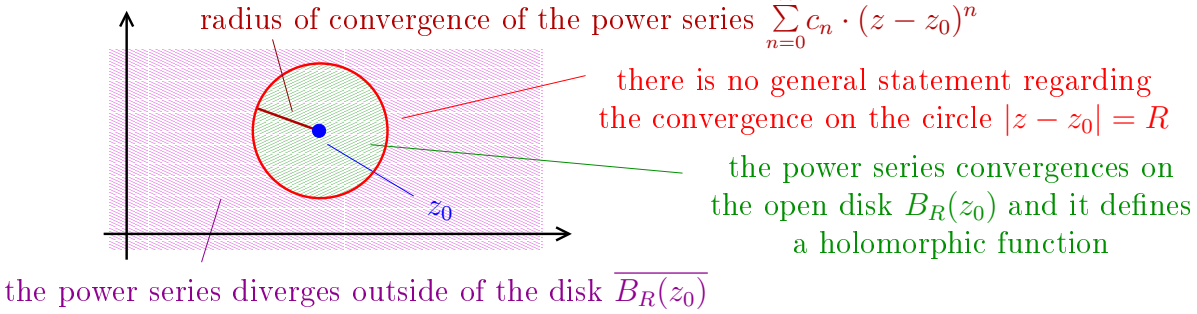
(1) Let $z \in \mathbb{C}$. The series $\sum_{n=0}^{\infty} c_n \cdot (z - z_0)^n$ converges if $|z - z_0| < R$ and it diverges if $|z - z_0| > R$.

(2) The map

$$\begin{aligned} B_R(z_0) &\to \mathbb{C} \\ z &\mapsto \sum_{n=0}^{\infty} c_n \cdot (z - z_0)^n \end{aligned}$$

is holomorphic with derivative

$$\begin{aligned} B_R(z_0) &\to \mathbb{C} \\ z &\mapsto \sum_{n=0}^{\infty} c_n \cdot n \cdot (z - z_0)^{n-1}. \end{aligned}$$



Proof. This proposition is proved in [**Lang1999**, Theorem II.5.1] or alternatively [**Jä2011**, Chapter 1.2]. ■

Via the above identification $\mathbb{C} = \mathbb{R}^2$ we can view a map $f\colon U \to \mathbb{C}$ as above also as a map from an open subset of $\mathbb{R}^2$ to $\mathbb{R}^2$. The following proposition clarifies the relationship between holomorphicity and smoothness in the sense of the definition on page 532.

Proposition 32.3. (Smooth-Complex Differentiability Proposition) Let $U \subset \mathbb{C}$ be an open subset and let $f\colon U \to \mathbb{C}$ be a map. Given $z \in U$ we write $u(z) := \mathrm{Re}(f(z))$ and $v(z) = \mathrm{Im}(f(z))$.

(1) For $z_0 \in U$ and $a, b \in \mathbb{R}$ the following two statements are equivalent:
   (a) The map $f\colon U \to \mathbb{C}$ is holomorphic in z_0 with derivative $f'(z_0) = a + b\mathrm{i}$.
   (b) The map
   $$\begin{aligned} U &\to \mathbb{C} = \mathbb{R}^2 \\ z &\mapsto (u(z), v(z)) \end{aligned}$$
   is differentiable in z_0, in the sense of the definition on page 535, with differential $\mathrm{D}f_{z_0} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.[324]

(2) The following two statements are equivalent:
   (a) The map $f\colon U \to \mathbb{C}$ is holomorphic.
   (b) The map $f\colon U \to \mathbb{C} = \mathbb{R}^2$ is smooth in the sense of the definition on page 532 and each differential is of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

[324] By the Differential-from-Partial Derivatives Proposition 19.9 the condition that $\mathrm{D}f_{z_0}$ is of this form is equivalent to the statement that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. These two equations are sometimes called the Cauchy-Riemann equations.

Sketch of proof. Let $U \subset \mathbb{C}$ be an open subset and let $f\colon U \to \mathbb{C}$ be a map.

(1) Given any $z_0 \in U$ and given any $a, b \in \mathbb{R}$ we see that:

$$\begin{aligned}
&f \text{ is holomorphic in } z_0 \text{ with derivative } a+b\mathrm{i} \\
&\iff \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = a+b\mathrm{i} \\
&\iff \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)-(a+b\mathrm{i})\cdot h}{h} = 0 \\
&\iff \underbrace{\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)-(a+b\mathrm{i})\cdot h}{|h|}}_{\text{complex limit in } \mathbb{C}} = 0 \\
&\iff \underbrace{\lim_{w\to 0} \frac{f(z_0+w)-f(z_0)-\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\cdot w}{\|w\|}}_{\text{real limit in } \mathbb{R}^2} = 0 \\
&\qquad \uparrow \text{ Lemma 32.1} \\
&\iff f \text{ is differentiable in } z_0 \text{ with } \mathrm{D}f_{z_0} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \\
&\qquad \uparrow \text{ see the definition on page 535}
\end{aligned}$$

(2) This statement follows from (1) and the fact, see the Power Series Expansion Theorem 32.5 below, that for every holomorphic map its derivative is again holomorphic. In other words, all higher derivatives exist. This corresponds to our convention, that by "smooth" we mean that all higher derivatives exist. ∎

The definition of a complex function being holomorphic is almost verbatim the same as the definition of a real function being differentiable, we just replace $\mathbb{R}$ by $\mathbb{C}$. But rather surprisingly being holomorphic has many consequences which one is not used to from the world of real valued differentiable functions. In the following we recall some of the main results from complex analysis. But before we can state the first theorem we need to introduce a few definitions.

Definition.

(1) Given a subset U of $\mathbb{C}$, we define a path in U to be a map $\gamma\colon [a,b] \to U$.

(2) Let $U \subset \mathbb{C}$ be an open subset, let $f\colon U \to \mathbb{C}$ be a map and let $\gamma\colon [a,b] \to U$ be a smooth path. We define the path integral of f along γ as follows:

$$\int_\gamma f(z)\,dz \;:=\; \int_{t=a}^{t=b} \underbrace{f(\gamma(t))}_{\in\mathbb{C}} \cdot \underbrace{\gamma'(t)}_{\in\mathbb{C}}\,dt \;\in \mathbb{C}.$$

$\uparrow$ product of complex numbers

Examples.

(1) Let $U = \mathbb{C}\setminus\{0\}$, let $f(z) = \frac{1}{z}$ and let $\gamma\colon [0,1] \to \mathbb{C}\setminus\{0\}$ be the path given by $\gamma(t) = \exp(2\pi\mathrm{i}t)$, i.e. γ is a path that "goes counterclockwise once around the origin". Then

$$\int_\gamma f(z)\,dz \;=\; \int_{t=0}^{t=1} f(\gamma(t))\cdot\gamma'(t)\,dt \;=\; \int_{t=0}^{t=1} \frac{1}{\exp(2\pi\mathrm{i}t)}\cdot\exp(2\pi\mathrm{i}t)\cdot 2\pi\mathrm{i}\,dt \;=\; 2\pi\mathrm{i}.$$

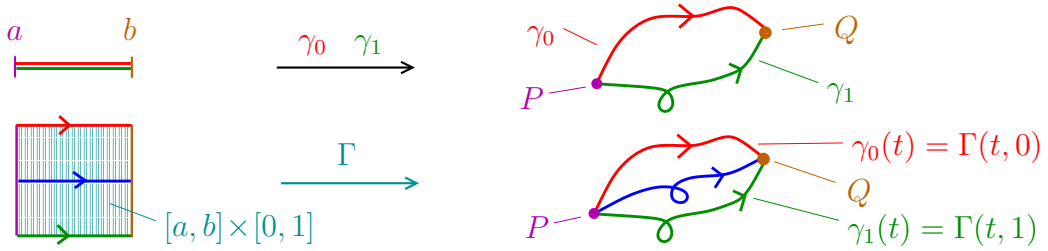
(2) If γ is a constant path, then $\gamma'(t) \equiv 0$. This shows that the path-integral along a constant path is zero.

Definition. Let $U \subset \mathbb{C}$ be a subset. Two paths $\gamma_0, \gamma_1 \colon [a,b] \to U$ with the same starting point $P := \gamma_0(a) = \gamma_1(a)$ and the same endpoint $Q := \gamma_0(b) = \gamma_1(b)$ are called path-homotopic in U, if there exists a homotopy rel $\{a,b\}$ from γ_0 to γ_1. In other words, if there exists a map

$$\begin{array}{rcl} \Gamma \colon [a,b] \times [0,1] & \to & U \\ (s,t) & \mapsto & \Gamma(s,t), \end{array}$$

with the following properties

(a) for every $s \in [a,b]$ we have $\Gamma(s,0) = \gamma_0(s)$ and $\Gamma(s,1) = \gamma_1(s)$,
(b) for every $t \in [0,1]$ we have $\Gamma(a,t) = P$ and $\Gamma(b,t) = Q$.



The following theorem is the foundational result in complex analysis.

Theorem 32.4. (Cauchy Theorem) Let $U \subset \mathbb{C}$ be an open subset, let $f \colon U \to \mathbb{C}$ be a holomorphic function and let $\gamma, \delta \colon [a,b] \to U$ be two smooth paths. If γ and δ are path-homotopic, then

$$\int_\gamma f(z)\,dz \;=\; \int_\delta f(z)\,dz.$$

Proof. A proof of Cauchy's Theorem is given in every book on complex analysis, see e.g. [**Lang1999**, Theorem III.5.1]. ∎

The Cauchy Theorem **??** is the mother of all results in complex analysis. It is at the heart of many other results. We now give a hotchpotch list of theorems, which are more or less immediate consequences of the Cauchy Theorem 32.4, and which we will make use of later on.

Theorem 32.5. (Power Series Expansion Theorem) Let $U \subset \mathbb{C}$ be an open subset and let $f \colon U \to \mathbb{C}$ be holomorphic.

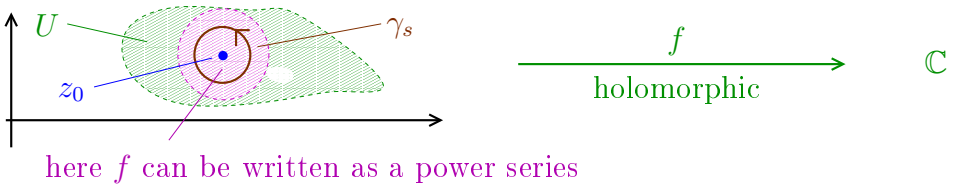
(1) If we are given a disk $B_r(z_0) \subset U$, then there exists a sequence $(c_n)_{n \in \mathbb{N}_0}$ of complex numbers such that

$$f(z) \;=\; \sum_{n=0}^{\infty} c_n \cdot (z - z_0)^n \qquad \text{for all } z \in B_r(z_0).$$

Furthermore for each $s \in (0,r)$ we have

$$c_n \;=\; \int_{\gamma_s} \frac{f(z)}{(z-z_0)^{n+1}}\,dz,$$

where $\gamma_s\colon [0,1] \to U$ is the path given by $\gamma_s(t) = z_0 + s \cdot \exp(2\pi i t)$.
(2) All derivatives of f are holomorphic.



Proof.

(1) This theorem is [**Lang1999**, Theorem III.7.2] or alternatively see [**Jä2011**, Chapter 3.2 Satz 3].
(2) This statement follows from Statement (1) together with the Complex Power Series Convergence Proposition 32.2. ■

Theorem 32.6. (Liouville's Theorem) Every bounded holomorphic map $f\colon \mathbb{C} \to \mathbb{C}$ is constant.

Proof. As we will see in Exercise 32.1, one can deduce Liouville's Theorem 32.6 from the Power Series Expansion Theorem 32.5 (1). Alternatively we refer to [**Lang1999**, Theorem III.7.5] or [**Jä2011**, Chapter 3.2] for a proof. ■

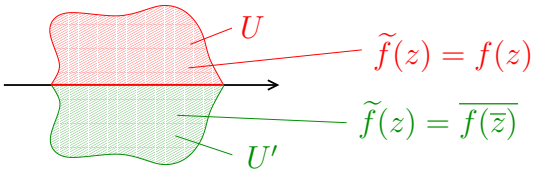
Proposition 32.7. (Schwarz Reflection Principle) Let U be an open subset of the upper half-plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0\}$ and let $f\colon U \to \mathbb{C}$ be a continuous function with the following properties:

(1) f is holomorphic on $\mathring{U} = \{z \in U \mid \mathrm{Im}(z) > 0\}$,
(2) f only assumes real values on $U \cap \mathbb{R}$.

We set $U' := \{\overline{z} \mid z \in U\}$, i.e. U' is the reflection of U in the real axis. Then the function[325]

$$\begin{array}{rcl} \widetilde{f}\colon U \cup U' & \to & \mathbb{C} \\ z & \mapsto & \begin{cases} f(z), & \text{if } z \in U, \\ \overline{f(\overline{z})}, & \text{if } z \in U', \end{cases} \end{array}$$

is holomorphic.



Proof. This proposition is proved in [**Lang1999**, Chapter IX.1] or alternatively in [**Jä2011**, Chapter 3.3]. ■

Theorem 32.8. (Locally-a-Power Theorem) Let U be an open subset of $\mathbb{C}$. Furthermore let $f\colon U \to \mathbb{C}$ be a holomorphic function and let $z_0 \in U$ be a zero of f.

(1) If all derivatives of f at z_0 vanish, then f is zero in an open neighborhood of z_0.

[325] It follows from Lemma 1.8 (2) that $U \cup U'$ is an open subset of $\mathbb{C}$.

(2) If there exists a $k \in \mathbb{N}$ such that $f(z_0) = \cdots = f^{(k-1)}(z_0) = 0$ but $f^{(k)}(z_0) \neq 0$, then there exists an open neighborhood $V \subset U$ of z_0 and a holomorphic function $g\colon V \to \mathbb{C}$ with $g'(z_0) \neq 0$ such that $f(z) = (g(z))^k$ for all $z \in V$.

Proof. The first statement is an immediate consequence of the Power Series Expansion Theorem 32.5 together with the Complex Power Series Convergence Proposition 32.2 (2). The second statement follows from [**Jä2011**, Kapitel 3 Satz 10 and 12]. ■

Proposition 32.9. Let U be an open subset of $\mathbb{C}$ and let $f\colon U \to \mathbb{C}$ be a holomorphic function. If $V \subset U$ is a connected open subset such that $f|_V$ is not constant, then $f(V)$ is an open subset of $\mathbb{C}$.

Proof. We will prove the proposition in Exercise 32.6 using the Locally-a-Power Theorem 32.8 and using the Inverse Function Theorem 19.12. Alternatively see [**Jä2011**, Kapitel 3 Satz 13]. ■

Theorem 32.10. (Maximum Principle) Let U be an open connected subset of $\mathbb{C}$ and let $f\colon U \to \mathbb{C}$ be a holomorphic function. If the function $|f|\colon U \to \mathbb{R}_{\geq 0}$ has a local maximum, then f is constant.

Proof. We will prove the Maximum Principle in Exercise 32.7 using Proposition 32.9. Alternatively see [**Lang1999**, p. 271] or [**Jä2011**, Kapitel 3 Satz 14]. ■

32.2. Biholomorphisms in the one-dimensional setting.

Definition. Let $f\colon U \to V$ be a map between two open subsets of $\mathbb{C}$. We say f is a biholomorphism if f is holomorphic, f is a bijection and $f^{-1}\colon V \to U$ is also holomorphic.

The following proposition shows that the only biholomorphisms of $\mathbb{C}$ are the obvious ones.

Theorem 32.11. Every biholomorphism $f\colon \mathbb{C} \to \mathbb{C}$ is an affine linear map, i.e. there exist $a, b \in \mathbb{C}$ such that $f(z) = a \cdot z + b$ for all $z \in \mathbb{C}$.

Proof. The proposition is proved in [**Lang1999**, Theorem V.5.3]. ■

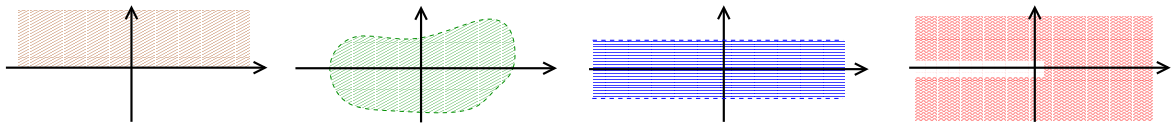
Example. In some cases one can explicitly write down a biholomorphism. For example, for the upper half-plane $\mathbb{H} = \{x + \mathrm{i}y \mid y > 0\}$ and the open disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ it is straightforward to verify that the maps

$$\begin{array}{rcl} \Phi\colon \mathbb{H} & \to & \mathbb{D} \\ z & \mapsto & \frac{z-\mathrm{i}}{z+\mathrm{i}} \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi\colon \mathbb{D} & \to & \mathbb{H} \\ w & \mapsto & \frac{\mathrm{i}+\mathrm{i}w}{1-w} \end{array}$$

are biholomorphisms that are inverse to one another.

In general it is hard to write down an explicit biholomorphism between two open subsets of $\mathbb{C}$. One of the main sources of biholomorphisms is the Riemann Mapping Theorem. To formulate this fantastic theorem we need to introduce the following definition.

Definition. Let $U \subset \mathbb{C}$ be a subset. We say U is simply connected if U is non-empty, if U is path-connected and if each path $\gamma\colon [0,1] \to U$ with $\gamma(0) = \gamma(1)$ is path-homotopic to a constant map.



these open subsets are biholomorphic to $\mathbb{D}$ by the Riemann Mapping Theorem 32.12

Theorem 32.12. (Riemann Mapping Theorem) Let $U \subset \mathbb{C}$ be a simply connected open subset U of $\mathbb{C}$. If $U \neq \mathbb{C}$, then U is biholomorphic to $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

Remark. The Riemann Mapping Theorem was first stated by Bernhard Riemann in 1851 in his doctoral thesis. The first rigorous proof was given by William Osgood in 1900, see [**Wals1973**].

Proof. The Riemann mapping theorem is proved in most textbooks on complex analysis, see e.g. [**Jä2011**, Chapter 10.1] or alternatively [**Lang1999**, Chapter X]. ■

In many applications it is also useful to know that the biholomorphism from the Riemann Mapping Theorem extends to the boundary of U and $\mathbb{D}$. To state the corresponding theorem we need one more definition.

Definition.

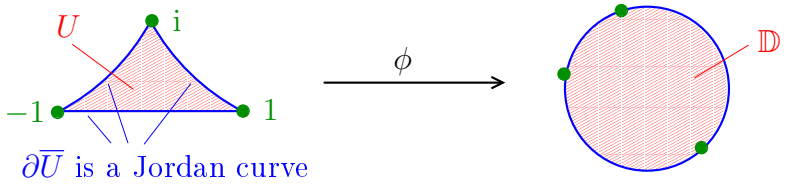
(1) We write $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, called "$\mathbb{C}$ with a point at infinity", where we equip the set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with the topology that we defined on page 122.

(2) Given $U \subset \overline{\mathbb{C}}$ we denote by $\overline{U}$ the closure of U in $\overline{\mathbb{C}}$.[326]

(3) A Jordan curve is the image of an embedding $S^1 \to \overline{\mathbb{C}}$.[327]

The following theorem was proved in 1913 by Constantin Carathéodory [**Cara1913**].[328]

Theorem 32.13. (Carathéodory Theorem) Let $U \subsetneq \mathbb{C}$ be a simply connected open subset U of $\mathbb{C}$ and let $\phi\colon U \to \mathbb{D}$ be a biholomorphism. If $\partial\overline{U}$ is a Jordan curve, then ϕ extends to a homeomorphism $\overline{U} \to \overline{\mathbb{D}}$.



Proof. A textbook proof of the theorem is provided in [**GM2005**, Chapter I.3]. ■

Example. We consider the open subset U of $\mathbb{C}$ that is shown in the figure above. It is a simply connected open proper subset of $\mathbb{C}$ and the boundary is clearly a Jordan curve. Thus it follows from the Riemann Mapping Theorem 32.12 and from Carathéodory's Theorem 32.13 that there exists a homeomorphism $\phi\colon \overline{U} \to \overline{\mathbb{D}}$ which restricts to a biholomorphism $\phi\colon U \to \mathbb{D}$.

[326] Let $U \subset \mathbb{C}$. It follows easily from the definition of $\overline{\mathbb{C}}$ that $\infty \in \overline{U}$ if and only if U is unbounded.

[327] On page 122 we mentioned that $\overline{\mathbb{C}}$ is Hausdorff. Therefore it follows from the Compact-Hausdorff Proposition 2.17 that a Jordan curve is the same as an injective map $S^1 \to \overline{\mathbb{C}}$.

[328] Constantin Carathéodory (1873-1950) was a Greek mathematician who was in particular a professor at the LMU in Munich.

32.3. **Holomorphic maps in the higher-dimensional setting.** We now move from the one-dimensional setting to the higher-dimensional case.

Definition. Let $U \subset \mathbb{C}^m$ be an open subset and let $f = (f_1, \dots, f_n) \colon U \to \mathbb{C}^n$ be a map.

(1) We say f is holomorphic if for every $i \in \{1, \dots, m\}$, every $j \in \{1, \dots, n\}$ and every point $(w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_k) \in \mathbb{C}^{m-1}$ the map

$$\begin{array}{rcl} \overbrace{\{z \in \mathbb{C} \mid (w_1, \dots, w_{i-1}, z, w_{i+1}, \dots, w_k) \in U\}}^{\text{open subset of } \mathbb{C} \text{ by Exercise 5.10 (a)}} & \to & \mathbb{C} \\ z & \mapsto & f_j(w_1, \dots, w_{i-1}, z, w_{i+1}, \dots, w_k) \end{array}$$

is holomorphic.

(2) We say $f \colon U \to f(U)$ is a biholomorphism if f is holomorphic, if f is a bijection, if $f(U)$ is an open subset of $\mathbb{C}^n$ and if $f^{-1} \colon f(U) \to U$ is also holomorphic.

(3) Let $P \in U$. We say $f \colon U \to \mathbb{C}^m$ is a local biholomorphism at P if there exists an open neighborhood V of P and an open neighborhood W of $f(P)$ such that $f \colon V \to W$ is a biholomorphism.

Remark. It follows from the Smooth-Complex Differentiability Proposition 32.3 that every holomorphic map from an open subset $U \subset \mathbb{C}^m = \mathbb{R}^{2m}$ to $\mathbb{C}^n = \mathbb{R}^{2n}$ is in particular smooth.

Proposition 32.14. (Smooth-Complex Differentiability Proposition II) In the following let $U \subset \mathbb{C}^m$ be an open subset and let $f = U \to \mathbb{C}^n$ be a map. The following two statements are equivalent:

(1) The map $f \colon U \to \mathbb{C}^n$ is holomorphic.

(2) The map $f \colon U \to \mathbb{C}^n$, viewed as a map from an open subset of $\mathbb{C}^m = \mathbb{R}^{2n}$ to $\mathbb{C}^n = \mathbb{R}^{2n}$ is smooth and for every $P \in U$ the differential $\mathrm{D}f_P \colon \mathbb{C}^m = \mathbb{R}^{2n} \to \mathbb{C}^n = \mathbb{R}^{2n}$ is a complex linear map.

Proof. The proposition can be deduced quite easily from the Smooth-Complex Differentiability Proposition 32.3. We leave it to the reader to fill in the details. ∎

The following theorem is the holomorphic analogue of the Inverse Function Theorem 19.12.

Theorem 32.15. (Holomorphic Inverse Function Theorem) Let $U \subset \mathbb{C}^n$ be an open subset and let $f \colon U \to \mathbb{C}^n$ be a holomorphic map. Furthermore let $P \in U$ be a point. If $\mathrm{D}f_P$ is invertible, then f is a local biholomorphism at P.

Example. The Holomorphic Inverse Function Theorem 32.15 is already interesting and non-trivial if $n = 1$. For example, let $k \in \mathbb{N}$. We consider the map

$$\begin{array}{rcl} f \colon \mathbb{C} \setminus \{0\} & \to & \mathbb{C} \setminus \{0\} \\ z & \mapsto & z^k. \end{array}$$

Note that for every $z \in \mathbb{C} \setminus \{0\}$ we have $f'(z) = k \cdot z^{k-1} \neq 0$. Therefore it follows from the Smooth-Complex Differentiability Proposition 32.3 and the Holomorphic Inverse Function Theorem 32.15 that the above map $f \colon \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is a local biholomorphism at every point in $\mathbb{C} \setminus \{0\}$.

Sketch of proof. First note that by the Inverse Function Theorem 19.12 we know that there exists an open neighborhood V of $P \in U \subset \mathbb{C}^n = \mathbb{R}^{2n}$ and an open neighborhood

W of $f(P) \in \mathbb{C}^n = \mathbb{R}^{2n}$ such that $f\colon V \to W$ is a bijection and such that $f^{-1}\colon W \to V$ is smooth. Next note that it follows from the Chain Rule 19.10 that for each $Q \in W$ we have $\mathrm{D}(f^{-1})_P = (\mathrm{D}f_{f^{-1}(Q)})^{-1}$. Since f is holomorphic we know by the Smooth-Complex Differentiability Proposition 32.14 that $\mathrm{D}f_{f^{-1}(Q)}\colon \mathbb{C}^n = \mathbb{R}^{2n} \to \mathbb{C}^n = \mathbb{R}^{2n}$ is a complex linear map. Thus $\mathrm{D}(f^{-1})_P$ is also a complex linear map. It follows from the Smooth-Complex Differentiability Proposition 32.14 that $f^{-1}\colon W \to V$ is holomorphic. ■

32.4. **Complex manifolds.** Given a topological manifold we now introduce the notion of a holomorphic atlas and of a complex manifold. The definition is almost the same as for a smooth atlas and a smooth manifold that we gave on page 541, except that we have to replace "smooth" by "holomorphic".

Definition.

(1) Let M be a $2n$-dimensional topological manifold M. An atlas $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ of charts of type (i), i.e. of charts where each V_i is an open subset of $\mathbb{R}^{2n}$, for M is called holomorphic if for all $i, j \in I$ the transition map

$$\Phi_j \circ \Phi_i^{-1}\colon \underbrace{\Phi_i(U_i \cap U_j)}_{\text{open in } \mathbb{R}^{2n}=\mathbb{C}^n} \to \underbrace{\Phi_j(U_i \cap U_j)}_{\subset \mathbb{R}^{2n}=\mathbb{C}^n}$$

is holomorphic.

(2) We define the maximal holomorphic atlas of a complex manifold the same way as we defined the maximal smooth atlas of a smooth manifold, see page 542. Note that the argument of the Maximal Atlas Existence Proposition 19.17 shows that every holomorphic atlas is contained in a unique maximal holomorphic atlas.

(3) An n-dimensional complex manifold is a pair $(M, \mathcal{A})$ consisting of a $2n$-dimensional topological manifold M, together with a maximal holomorphic atlas for M.

Remark.

(1) As we mentioned above, by the Smooth-Complex Differentiability Proposition 32.3 we know that every holomorphic map from an open subset $U \subset \mathbb{C}^m$ to $\mathbb{C}^n$ is in particular smooth. It follows now easily that every holomorphic atlas is also a smooth atlas. In particular we see that an n-dimensional complex manifold is in particular a $2n$-dimensional smooth manifold.

(2) If M is a complex manifold, then every point admits a chart of type (i) from a smooth atlas, which according to the definitions on page 511 and 545 means, that the boundary ∂M is empty. In particular a compact complex manifold is also closed in the sense of the definitions on page 511 and 545.

Example. Let U be an open subset of $\mathbb{C}^n$. The identity map is an atlas for U and it is evidently a holomorphic atlas. Thus we can view U as an n-dimensional complex manifold.

We move on straight to another set of closely related definitions, which are the holomorphic analogues of the smooth definitions on pages 546 and 543.

Definition.

(1) We say that a map $f\colon M \to N$ between two complex manifolds is holomorphic if f is continuous and if for every chart $\Phi\colon U \to V$ in the holomorphic atlas of M and every chart $\Psi\colon W \to Y$ in the holomorphic atlas of N the map
$$\Phi(f^{-1}(W) \cap U) \xrightarrow{\Psi \circ f \circ \Phi^{-1}} Y$$
is holomorphic.
(2) We say that a map $f\colon M \to N$ between two complex manifolds is a biholomorphism if f is a bijection and if both f and f^{-1} are holomorphic.
(3) We say that a map $f\colon M \to N$ between two complex manifolds is a local biholomorphism if given any $P \in M$ there exists an open neighborhood U of P and an open neighborhood V of $f(P)$ such that $f\colon U \to V$ is a biholomorphism.
(4) We say that two complex atlases for a given topological manifold are equivalent, if the identity map is a biholomorphism between the two complex manifolds corresponding to the complex atlases.
(5) A complex structure on a topological manifold is a choice of an equivalence class of holomorphic atlases.

The next lemma gives us an example of a complex manifold that will play a major role later on.

Lemma 32.16. (Complex Projective Space-is-Complex Manifold Lemma) Let $n \in \mathbb{N}$. On page 255 we introduced the complex projective space
$$\mathbb{C}\mathrm{P}^n \;=\; (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\}).$$
For $i = 0, \dots, n$ we consider the map[329]
$$\begin{aligned} \Phi_i\colon \{[z_0 : \cdots : z_n] \in \mathbb{C}\mathrm{P}^n \mid z_i \neq 0\} &\to \mathbb{C}^n \\ [z_0 : \cdots : z_n] &\mapsto \big(\tfrac{z_0}{z_i}, \dots, \tfrac{z_{i-1}}{z_i}, \tfrac{z_{i+1}}{z_i}, \dots, \tfrac{z_n}{z_i}\big). \end{aligned}$$
These maps form a holomorphic atlas for $\mathbb{C}\mathrm{P}^n$ that turn the complex projective space $\mathbb{C}\mathrm{P}^n$ into a compact n-dimensional complex manifold, in particular, as we discussed on page 797, we can view $\mathbb{C}\mathrm{P}^n$ as a closed $2n$-dimensional real smooth manifold.

Convention. We refer to the complex and smooth structure on $\mathbb{C}\mathrm{P}^n$ that we just introduced as the standard complex structure on $\mathbb{C}\mathrm{P}^n$ and as the standard smooth structure on $\mathbb{C}\mathrm{P}^n$.

Proof. In the Projective Spaces-Compact+Hausdorff Proposition 5.30 (see also Exercise 5.44) we saw that each $\mathbb{C}\mathrm{P}^n$ is compact and Hausdorff. As we will show below, $\mathbb{C}\mathrm{P}^n$ admits a finite atlas, thus it follows from Lemma 18.2 that $\mathbb{C}\mathrm{P}^n$ is second-countable. For the remaining statements it is easy to overlook that first one needs to prove the following claim.

Claim 1. If $i \in \{0, \dots, n\}$, then
$$V_i \;:=\; \{[z_0 : \cdots : z_n] \in \mathbb{C}\mathrm{P}^n \mid z_i \neq 0\}$$
is an open subset of $\mathbb{C}\mathrm{P}^n$.

[329] We use the notation from page 256, i.e. given a point $(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ we denote by $[z_0 : \cdots : z_n]$ the corresponding point in $\mathbb{C}\mathrm{P}^n$.

Proof. The claim is equivalent to showing that the complement $\mathbb{C}\mathrm{P}^n \setminus V_i$ is a closed subset of $\mathbb{C}\mathrm{P}^n$. We note that

$$\mathbb{C}\mathrm{P}^n \setminus V_i \;=\; \text{the image of the map } \begin{array}{rcl} f\colon \mathbb{C}\mathrm{P}^{n-1} & \to & \mathbb{C}\mathrm{P}^n \\ [w_0 : \ldots : w_{n-1}] & \mapsto & [w_0 : \ldots : w_{i-1} : 0 : w_i : \ldots : w_{n-1}]. \end{array}$$

It follows easily from the elementary the Twice Quotient Lemma 5.23 that the map f is continuous. Since $\mathbb{C}\mathrm{P}^{n-1}$ is compact we obtain from the Compact Image Lemma 2.13 that $f(\mathbb{C}\mathrm{P}^{n-1})$ is compact. But since $\mathbb{C}\mathrm{P}^n$ is Hausdorff we obtain from the Compact-Closed Lemma 1.21 (2) that $f(\mathbb{C}\mathrm{P}^{n-1}) = \mathbb{C}\mathrm{P}^n \setminus V_i$ is in fact a closed subset of $\mathbb{C}\mathrm{P}^n$. ⊞

Using this claim it is now straightforward to verify that the maps $\Phi_0, \ldots, \Phi_n$ form an atlas for $\mathbb{C}\mathrm{P}^n$. It remains to prove the following claim.[330]

Claim 2. The atlas is holomorphic.

Proof. First note that for $i \in \{0, \ldots, n\}$ the inverse of the chart Φ_i is given by the map

$$\begin{array}{rcl} \Phi_i^{-1}\colon \mathbb{C}^n & \to & \{[w_0 : \cdots : w_n] \in \mathbb{C}\mathrm{P}^n \,|\, w_i \neq 0\} \\ (w_1, \ldots, w_n) & \mapsto & [w_1 : \cdots : w_i : 1 : w_{i+1} : \cdots : w_n]. \end{array}$$

Now let $i, j \in \{0, \ldots, n\}$. By symmetry it suffices to consider the case $i < j$. Using the above calculation of Φ_i^{-1} one can easily show that the corresponding transition map is given by

$$\begin{array}{rcl} \Phi_j \circ \Phi_i^{-1}\colon \{(w_1, \ldots, w_n) \in \mathbb{C}^n \,|\, w_j \neq 0\} & \to & \{(w_1, \ldots, w_n) \,|\, w_{i+1} \neq 0\} \\ (w_1, \ldots, w_n) & \mapsto & \left(\frac{w_1}{w_j}, \ldots, \frac{w_i}{w_j}, \frac{1}{w_j}, \frac{w_{i+1}}{w_j}, \ldots, \frac{w_{j-1}}{w_j}, \frac{w_{j+1}}{w_j}, \ldots, \frac{w_n}{w_j}\right). \end{array}$$

This map is evidently a biholomorphism. We have thus shown that the given maps $\Phi_0, \ldots, \Phi_n$ form a holomorphic atlas for $\mathbb{C}\mathrm{P}^n$. ■

On page 562 we introduced the notion of a smooth action on a smooth manifold. Not surprisingly we have the following analogue in our present context.

Definition. Let M be a complex manifold which is equipped with an action by a group G. We say that the action is holomorphic if for every $g \in G$ the map $M \to M$ given by $x \mapsto g{\cdot}x$ is holomorphic.

The following proposition is the complex analogue of the Manifold Quotient-by-Group Action Proposition 19.32.

Proposition 32.17. (Complex Manifold Quotient-by-Group Action Proposition) If a group G acts freely, properly[331] and holomorphically on an n-dimensional complex manifold M, then M/G is an $2n$-dimensional topological manifold and there exists a unique complex structure on M/G such that the projection $p\colon M \to M/G$ is a local biholomorphism.

Proof. The proof of the proposition is verbatim the same as the proof of the corresponding statements in the Manifold Quotient-by-Group Action Proposition 19.32 for smooth manifolds. We leave it to the reader to fill in the details. ■

[330] The following claim is very similar to the content of Exercise 19.16.

[331] We refer to page 562 for the definition of a free action and a proper action.

Convention. In the setting of the Complex Manifold Quotient-by-Group Action Proposition 32.17 we will refer to the complex structure on M/G as the standard complex structure on M/G.

Example. We consider the action

$$\begin{array}{rcl} \mathbb{Z} \times \mathbb{C}^n \setminus \{0\} & \to & \mathbb{C}^n \setminus \{0\} \\ (k, (z_1, \ldots, z_n)) & \mapsto & (e^k \cdot z_1, \ldots, e^k \cdot z_n). \end{array}$$

One can easily show that this action is free, proper and holomorphic. Thus we see that the quotient $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$ has a standard complex structure. Furthermore it is fairly elementary to verify that the map

$$\begin{array}{rcl} (\mathbb{C}^n \setminus \{0\})/\mathbb{Z} & \to & S^{n-1} \times S^1 \\ z & \mapsto & \left(\frac{z}{\|z\|}, \exp(2\pi \mathrm{i} \cdot \ln(\|z\|))\right) \end{array}$$

is a diffeomorphism. Thus in summary we have shown that $S^{n-1} \times S^1$ admits a complex structure. This complex manifold is called the *Hopf manifold* [**Hopf1948**].

We formulate our next example as a lemma.

Lemma 32.18. Let $v, w \in \mathbb{C}$ be two complex numbers that are linearly independent over $\mathbb{R}$.

(1) The action

$$\begin{array}{rcl} (\mathbb{Z} \cdot v + \mathbb{Z} \cdot w) \times \mathbb{C} & \to & \mathbb{C} \\ ((m \cdot v + n \cdot w), z) & \mapsto & z + m \cdot v + n \cdot w \end{array}$$

is free, proper and holomorphic.

(2) The quotient $T(v, w) := \mathbb{C}/(\mathbb{Z} \cdot v + \mathbb{Z} \cdot w)$ is a 2-dimensional topological manifold and it admits a unique complex structure which has the property that the natural projection $\mathbb{C} \to T(v, w) = \mathbb{C}/(\mathbb{Z} \cdot v + \mathbb{Z} \cdot w)$ is a local biholomorphism.

(3) The complex manifold $T(v, w) = \mathbb{C}/(\mathbb{Z} \cdot v + \mathbb{Z} \cdot w)$ is compact and, viewed as a smooth manifold, it is diffeomorphic to $S^1 \times S^1$.

(4) Let $\widetilde{v}, \widetilde{w}$ be two complex numbers that are linearly independent over $\mathbb{R}$. If there exists an $r \in \mathbb{C} \setminus \{0\}$ such that $r \cdot (\mathbb{Z} \cdot v + \mathbb{Z} \cdot w) = \mathbb{Z} \cdot \widetilde{v} + \mathbb{Z} \cdot \widetilde{w}$, then the complex manifolds $T(v, w)$ and $T(\widetilde{v}, \widetilde{w})$ are biholomorphic.

Remark. In Proposition 63.1 we will see that the converse to Lemma 32.18 (4) holds. More precisely, we will see that if $T(v, w)$ and $T(\widetilde{v}, \widetilde{w})$ are biholomorphic, then there exists an $r \in \mathbb{C} \setminus \{0\}$ such that $r \cdot (\mathbb{Z} \cdot v + \mathbb{Z} \cdot w) = \mathbb{Z} \cdot \widetilde{v} + \mathbb{Z} \cdot \widetilde{w}$.

Proof.

(1) In Exercise 19.34 we showed that the action is proper. It is clear that the action is free and holomorphic.

(2) This statement follows from (1) together with the Complex Manifold Quotient-by-Group Action Proposition 32.17.

(3) We consider the map

$$\begin{array}{rcl} \mathbb{C}/(\mathbb{Z} \cdot v + \mathbb{Z} \cdot w) & \to & S^1 \times S^1 \\ [r \cdot v + s \cdot w] & \to & (\exp(2\pi \mathrm{i} s), \exp(2\pi \mathrm{i} t)). \end{array}$$

It follows from the Smooth Map-Descends-to-Quotient Corollary 19.33 that the map is a local diffeomorphism. Since v and w are linearly independent one sees that the map is a bijection. It follows from the Smooth Pasting Proposition 19.23 (2) that the map is in fact a diffeomorphism. Since $S^1 \times S^1$ is compact we see that $T(v, w)$ is also compact.

(4) Let $\widetilde{v}, \widetilde{w}$ be two complex numbers that are linearly independent over $\mathbb{R}$. We assume that there exists an $r \in \mathbb{C} \setminus \{0\}$ such that $r \cdot (\mathbb{Z} \cdot v + \mathbb{Z} \cdot w) = \mathbb{Z} \cdot \widetilde{v} + \mathbb{Z} \cdot \widetilde{w}$. In this case we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{z \mapsto r \cdot z} & \mathbb{C} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \widetilde{p}} \\ T(v, w) & \xrightarrow{[z] \mapsto [r \cdot z]} & T(\widetilde{v}, \widetilde{w}). \end{array}$$

Note that our hypothesis shows that bottom horizontal map is defined and a bijection. The top horizontal map is of course a biholomorphism. Since the vertical maps are local biholomorphisms we see that the bottom horizontal map is a bijection and a local biholomorphism. In other words, it is a biholomorphism. ■

We continue our discussion of complex manifolds with a pleasing proposition.

Proposition 32.19. Let M be a compact 1-dimensional complex manifold. Then any holomorphic function $f \colon M \to \mathbb{C}$ is constant.

Proof. We provide the proof of the proposition in Exercise 32.2. ■

Remark. If $M = T(v, w) = \mathbb{C}/(\mathbb{Z} \cdot v \oplus \mathbb{Z} \cdot w)$, then we can also give another, arguably easier proof of the conclusion of Proposition 32.19. Indeed, let $f \colon \mathbb{C}/(\mathbb{Z} \cdot v \oplus \mathbb{Z} \cdot w) \to \mathbb{C}$ be a holomorphic function. Since $\mathbb{C}/(\mathbb{Z} \cdot v \oplus \mathbb{Z} \cdot w) \cong S^1 \times S^1$ is compact we obtain from the Compact Image Lemma 2.13 that the function f is bounded. As usual we denote by $p \colon \mathbb{C} \to \mathbb{C}/(\mathbb{Z} \cdot v \oplus \mathbb{Z} \cdot w)$ the projection . Recall that p is holomorphic. Thus the composition $f \circ p \colon \mathbb{C} \to \mathbb{C}$ is a bounded holomorphic function. We deduce from the Liouville Theorem 32.6 that $f \circ p$ is constant. Since p is surjective it follows that f is also constant.

Next we introduce the notion of a submanifold of a complex manifold. The definition is almost verbatim the same as the definition of a submanifold of a smooth manifold on page 550. The only difference is that we do not need to worry about boundary points.

Definition. Let M be an n-dimensional complex manifold.

(1) We say a subset $N \subset M$ is a k-dimensional complex submanifold if given any $P \in N$ there exists a chart $\Phi \colon U \to V$ in the maximal complex atlas for M and P such that

$$\Phi(U \cap N) \;=\; V \cap \{(0, \ldots, 0, z_1, \ldots, z_k) \,|\, z_i \in \mathbb{C}\},$$

Any collection of charts covering N is called a holomorphic submanifold atlas.

(2) We say a submanifold N is proper if N is a closed subset of M.

Example. Let $n \in \mathbb{N}$ and let $k \in \{0, \dots, n\}$. Using a mild modification of the charts from the Complex Projective Space-is-Complex Manifold Lemma 32.16 one can easily show that

$$\{[z_0 : \cdots : z_k : 0 : \cdots : 0] \in \mathbb{C}\mathrm{P}^n \mid (z_0, \dots, z_k) \in \mathbb{C}^{k+1} \setminus \{0\}\}$$

is a submanifold of the complex manifold $\mathbb{C}\mathrm{P}^n$.

The following lemma is an analogue of the Submanifold Smooth Structure Lemma 19.24.

Lemma 32.20. (Submanifold Complex Structure Lemma) Let M be a complex manifold and let N be a complex submanifold of M. The collection of all submanifold charts of M gives naturally rise to a holomorphic atlas for N which we refer to as the standard complex structure on the submanifold. The standard complex structure has the following properties:

(1) The inclusion map $N \to M$ is a holomorphic map.
(2) If we start out with some submanifold atlas for M, then this gives rise to the same complex structure on N.
(3) Let $f\colon W \to M$ be a holomorphic map. If $f(W) \subset N$, then $f\colon W \to N$ is also holomorphic.

Proof. The proof is basically identical to the proof of the Submanifold Smooth Structure Lemma 19.24. We leave it to the reader to make the few necessary modifications. ■

The following theorem is the complex analogue of the Regular Value Theorem 26.4. As in the real case, it is a machine for providing examples of complex (sub-) manifolds.

Theorem 32.21. (Regular Value Theorem for Complex Manifolds) Let M be an m-dimensional complex manifold and let N be an n-dimensional complex manifold. Furthermore let $f\colon M \to N$ be a holomorphic map and let $s \in N$ be a regular value of f.

(1) The preimage $X := f^{-1}(s)$ is an $(m-n)$-dimensional complex submanifold of M.
(2) The intersection of X with any compact set of M is compact. In particular, if M is compact, then X is also compact.

Proof. The proof is almost identical to the proof of the "real" Regular Value Theorem 26.4. The main difference is that we need to replace the Inverse Function Theorem 19.12 by the Holomorphic Inverse Function Theorem 32.15. We refer to [**Huy2005**, p. 11 and p. 58] for some more details. ■

Examples.

(1) Let $n \in \mathbb{N}$. By basically the same argument as in the proof of the Matrix Group-Smooth Manifold Proposition 26.5 one can show that

$$\begin{array}{rcl} \mathrm{M}(n \times n, \mathbb{C}) & \to & \mathbb{C} \\ A & \mapsto & \det(A) \end{array}$$

is holomorphic and that $1 \in \mathbb{C}$ is a regular value. It follows from the Regular Value Theorem 32.21 for Complex Manifolds that $\mathrm{SL}(n, \mathbb{C})$ is an (n^2-1)-dimensional complex submanifold of $\mathrm{M}(n \times n, \mathbb{C}) = \mathbb{C}^{n^2}$. In particular by the Submanifold Complex Structure Lemma 32.20 we can view $\mathrm{SL}(n, \mathbb{C})$ as an (n^2-1)-dimensional complex submanifold.

(2) Let $n \in \mathbb{Z}$. In Exercise 32.12 we will use the Regular Value Theorem 32.21 for Complex Manifolds to show that $\{[x:y:z] \in \mathbb{C}\mathrm{P}^2 \,|\, x^n + y^n + z^n = 0\}$ is a compact 1-dimensional complex manifold.

32.5. The natural orientation of a complex manifold.

Lemma 32.22. (Complex Vector Space-Orientation Lemma)

(1) Let V be an n-dimensional complex vector space.
   (a) If $(v_1, \dots, v_n)$ is a basis of V viewed as a complex vector space, then the set $(v_1, \mathrm{i}v_1, \dots, v_n, \mathrm{i}v_n)$ is a basis of V viewed as a real vector space.
   (b) If $(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$ are two bases of V as a complex vector space, then the bases $(v_1, \mathrm{i}v_1, \dots, v_n, \mathrm{i}v_n)$ and $(w_1, \mathrm{i}w_1, \dots, w_n, \mathrm{i}w_n)$ are equivalent in the sense of page 668.

(2) Let $f\colon V \to W$ be an isomorphism of n-dimensional complex vector spaces. If we equip V and W, viewed as real vector spaces, with the orientation from (1), then $f\colon V \to W$ is an orientation-preserving isomorphism.

Definition. Given a finite-dimensional complex vector space we refer to the orientation introduced in the Complex Vector Space-Orientation Lemma 32.22 (1) as the natural orientation.

Example. Let $n \in \mathbb{N}_0$. If we make the usual identification $\mathbb{C}^n = (\mathbb{R}^2)^n = \mathbb{R}^{2n}$, then the corresponding natural orientation on $\mathbb{C}^n = \mathbb{R}^{2n}$ agrees with the standard orientation.

Proof.

(1) Let $(v_1, \dots, v_n)$ be a basis of V as a complex vector space. One can easily verify that $(v_1, \mathrm{i}v_1, \dots, v_n, \mathrm{i}v_n)$ is a generating set over $\mathbb{R}$ and that it is linearly independent over $\mathbb{R}$. This concludes the proof of (a).

We turn to the proof of (b). Let $(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$ be two bases of V as a complex vector space. We denote by $A \in \mathrm{GL}(n \times n, \mathbb{C})$ the transition matrix as defined on page 668. We write $A = B + \mathrm{i}C$, where $B, C \in \mathrm{M}(n \times n, \mathbb{R})$. We consider the bases

$$\begin{aligned} X &:= (v_1, \mathrm{i}v_1, \dots, v_n, \mathrm{i}v_n) \quad \text{and} \quad Y := (w_1, \mathrm{i}w_1, \dots, w_n, \mathrm{i}w_n) \\ \widetilde{X} &:= (v_1, \dots, v_n, \mathrm{i}v_1, \dots, \mathrm{i}v_n) \quad \text{and} \quad \widetilde{Y} := (w_1, \dots, w_n, \mathrm{i}w_1, \dots, \mathrm{i}w_n). \end{aligned}$$

We now calculate the determinant of the transition matrix $P_{X,Y}$ from X to Y:

$$\begin{aligned} \det(P_{X,Y}) &\overset{\text{see the Transition Matrix Lemma 25.1}}{=} \det(P_{\widetilde{X},X}) \cdot \det(P_{\widetilde{X},\widetilde{Y}}) \cdot \det(P_{\widetilde{Y},Y}) \overset{\text{since } P_{\widetilde{X},X} = P_{\widetilde{Y},Y}}{=} \underbrace{\det(P_{\widetilde{X},X})^2}_{=\pm 1} \cdot \det(P_{\widetilde{X},\widetilde{Y}}) \overset{\text{follows from Lemma 32.1}}{=} \det(P_{\widetilde{X},\widetilde{Y}}) \\ &= \det \begin{pmatrix} B & -C \\ C & B \end{pmatrix} \underset{\text{add } i \text{ times bottom to the top}}{=} \det \begin{pmatrix} B+\mathrm{i}C & -C+\mathrm{i}B \\ C & B \end{pmatrix} \underset{\text{subtract } i \text{ times left from the right}}{=} \det \begin{pmatrix} B+\mathrm{i}C & 0 \\ C & B-\mathrm{i}C \end{pmatrix} \\ &= \det(B+\mathrm{i}C) \cdot \det(B-\mathrm{i}C) = \det(B+\mathrm{i}C) \cdot \overline{\det(B+\mathrm{i}C)} \\ &= |\det(B+\mathrm{i}C)|^2 > 0. \end{aligned}$$

(2) This statement follows immediately from (1). ∎

The following proposition says that complex manifolds admit a natural orientation, in particular they are orientable.

Proposition 32.23. (Complex Manifold-Orientation Proposition)

(1) Let M be an n-dimensional complex manifold.
   (a) Given any $P \in M$ we can use charts from the holomorphic atlas to equip $\mathrm{T}_P M$ naturally with the structure of a complex n-dimensional vector space.
   (b) If we equip each complex vector space $\mathrm{T}_P M$ with the corresponding natural orientation, then these orientations define an orientation on M, viewed as a $2n$-dimensional real smooth manifold. We refer to this orientation as the natural orientation of the complex manifold M.

(2) If $f\colon M \to N$ is a holomorphic map between two complex manifolds, then for each $P \in M$ the differential $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ is a complex linear map.

(3) If $f\colon M \to N$ is a local biholomorphism between two complex manifolds, then f is an orientation-preserving diffeomorphism between M and N, viewed as oriented smooth manifolds.

Sketch of a proof.

(1) (a) Let M be an n-dimensional complex manifold and let $P \in M$. We pick a chart $\Phi\colon U \to V$ from the holomorphic atlas around P. We get an induced isomorphism $\mathrm{D}\Phi_P\colon \mathrm{T}_P M \to \mathrm{T}_P V = \mathbb{C}^n$. We use this isomorphism to equip $\mathrm{T}_P M$ with the structure of a complex vector space. Since Φ comes from a holomorphic atlas one obtains easily from the Smooth-Complex Differentiability Proposition 32.14 , see Exercise 32.8, that this complex vector space structure does not depend on the choice of the chart Φ.
   (b) This statement follows easily from the definitions. We leave it to the reader to fill in the details.

(2) This statement follows fairly easily from the definitions and the Smooth-Complex Differentiability Proposition 32.14. We outsource the details to Exercise 32.11.

(3) The proof is identical to the proof of (1b). ■

Definition. Let M be an n-dimensional complex manifold. We refer to the orientation on the $2n$-dimensional smooth manifold M that is provided by the Complex Manifold-Orientation Proposition 32.23 as the standard orientation.

Example. In the Complex Projective Space-is-Complex Manifold Lemma 32.16 we equipped $\mathbb{C}\mathrm{P}^n$ with a complex structure. We refer to the corresponding orientation on the $2n$-dimensional smooth manifold $\mathbb{C}\mathrm{P}^n$ as the standard orientation on $\mathbb{C}\mathrm{P}^n$.

We continue with the following corollary to the Complex Manifold-Orientation Proposition 32.23.

Corollary 32.24. The Klein bottle and the real projective plane $\mathbb{R}\mathrm{P}^2$ do not admit the structure of a complex manifold.

Proof. In the Surface Non-Orientability Lemma 25.20 we showed that neither the Klein bottle nor the real projective plane $\mathbb{R}\mathrm{P}^2$ are orientable. This is in contrast to the fact, shown in the Complex Manifold-Orientation Proposition 32.23, that complex manifolds are orientable. ■

As we had already pointed out on page 258, the one-dimensional projective space $\mathbb{C}\mathrm{P}^1$ is homeomorphic to S^2. This shows that S^2 can be viewed as a 1-dimensional complex manifold. We had just seen that the 2-sphere and the torus are complex manifolds and we saw that non-orientable smooth manifolds cannot be complex manifolds. The following question now arises:

Question 32.25. Let $g \geq 2$. Does the surface of genus g have the structure of a complex manifold?

We will discuss this question in Proposition 66.2.

Remark. It is often very hard to determine whether or not a given orientable even-dimensional smooth manifold admits that a complex structure. For example, Armand Borel and Jean-Pierre Serre [**BoS1953**][332] showed in 1953 that for $n \neq 2, 6$ the n-dimensional sphere S^n cannot be a complex manifold. On the other hand it is still an open question whether S^6 admits a complex structure.

Exercises for Chapter 32.

Exercise 32.1. Use the Power Series Expansion Theorem 32.5 (1) to show that every bounded holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ is constant.

Exercise 32.2. Let M be a compact connected 1-dimensional complex manifold. Show that any holomorphic function $f\colon M \to \mathbb{C}^n$ is constant.
Hint. Use the Maximum Principle, i.e. use Theorem 32.10.

Exercise 32.3. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic non-constant function. Use Liouville's Theorem 32.6 to show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Exercise 32.4. Use Liouville's Theorem 32.6 to show that every complex polynomial of degree ≥ 1 has a zero.

Exercise 32.5. Let $U \subset \mathbb{C}$ be an open connected subset and let $f, g\colon U \to \mathbb{C}$ be two holomorphic functions. Suppose that there exists a $z_0 \in U$ such that $f^{(k)}(z_0) = g^{(k)}(z_0)$ for all $k \in \mathbb{N}_0$. Use the Locally-a-Power Theorem 32.8 to show that $f = g$ on all of U.

Exercise 32.6. Let U be an open subset of $\mathbb{C}$, let $f\colon U \to \mathbb{C}$ be a holomorphic function and let $V \subset U$ is a connected open subset such that $f|_V$ is not constant. Show that $f(V)$ is an open subset of $\mathbb{C}$.
Hint. Use the Locally-a-Power Theorem 32.8 and use the Inverse Function Theorem 19.12.

Exercise 32.7. Let U be an open connected subset of $\mathbb{C}$ and let $f\colon U \to \mathbb{C}$ be a holomorphic function. Show that if the function $|f|\colon U \to \mathbb{R}_{\geq 0}$ has a local maximum, then f is constant.
Hint. Use Proposition 32.9.

Exercise 32.8. Let $U \subset \mathbb{C}^n$ be an open set and let $f\colon U \to \mathbb{C}^m$ be a holomorphic map. Show that given any $P \in U$ the differential $\mathrm{D}f_P$ lies in $\mathrm{M}(m \times n, \mathbb{C}) \subset \mathrm{M}(2m \times 2n, \mathbb{R})$.

[332] Armand Borel (1923–2003) was a Swiss mathematician, Jean-Pierre Serre (*1926) is a French mathematician who was awarded a Fields medal in 1954.

Exercise 32.9.

(a) We view S^2 as a subspace of $\mathbb{C}\times\mathbb{R}=\mathbb{R}^3$. We consider the maps

$$\begin{array}{rcl}\Phi_+\colon S^2\setminus\{(0+0\mathrm{i},1)\} & \to & \mathbb{C}\\ (z,t) & \mapsto & \frac{z}{1-t}\end{array}\qquad\text{and}\qquad\begin{array}{rcl}\Phi_-\colon S^2\setminus\{(0+0\mathrm{i},-1)\} & \to & \mathbb{C}\\ (z,t) & \mapsto & \frac{\overline{z}}{1+t}.\end{array}$$

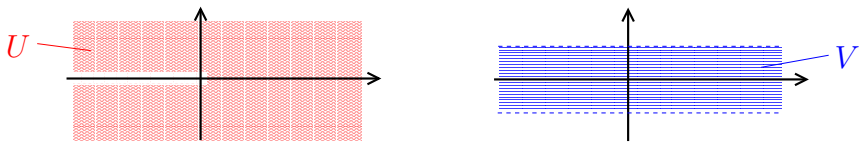
Note that Φ_+ is the stereographic projection from the North Pole $(0+0\mathrm{i},1)$ and that Φ_- is the stereographic projection the South Pole $(0+0\mathrm{i},-1)$ composed with complex conjugation. In Lemma 2.18 we showed that these two maps are homeomorphisms. Show that these two charts define a holomorphic atlas for S^2. We refer to this complex structure on S^2 as the *standard complex structure on* S^2.

(b) In Lemmas 2.18 and 5.32 we gave an explicit homeomorphism $S^2\to\mathbb{C}\mathrm{P}^1$. Show that this map is a biholomorphism between the complex manifolds S^2 and $\mathbb{C}\mathrm{P}^1$.

Exercise 32.10. Show, without using the Riemann Mapping Theorem 32.12, that the complex manifolds

$$\begin{array}{rcl}U & := & \{z\in\mathbb{C}\,|\,\mathrm{Re}(z)>0 \text{ or } \mathrm{Im}(z)\neq 0\}\\ V & := & \{z\in\mathbb{C}\,|\,\mathrm{Im}(z)\in(-\pi,\pi)\}\end{array}$$

are biholomorphic.



Exercise 32.11. Let $f\colon M\to N$ is a holomorphic map between two complex manifolds. Show that for each $P\in M$ the differential $\mathrm{D}f_P\colon \mathrm{T}_PM\to\mathrm{T}_{f(P)}N$ is a complex linear map.

Exercise 32.12. Let $n\in\mathbb{Z}$. Show that $\{[x:y:z]\in\mathbb{C}\mathrm{P}^2\,|\,x^n+y^n+z^n=0\}$ is a compact 1-dimensional complex submanifold of $\mathbb{C}\mathrm{P}^2$.

Exercise 32.13. Let $n\in\mathbb{N}$. Let $\mathbb{C}\mathrm{P}^n=(\mathbb{C}^{n+1}\setminus\{0\})/(\mathbb{C}\setminus\{0\})$ be the complex projective space.

(a) Show that

$$M \;:=\; \{([z],v)\in\mathbb{C}\mathrm{P}^n\times\mathbb{C}^{n+1}\,|\,v\in\mathbb{C}\cdot z\}$$

is a holomorphic submanifold of $\mathbb{C}\mathrm{P}^n\times\mathbb{C}^{n+1}$.

(b) Show that the map

$$\begin{array}{rcl}\pi\colon M & \to & \mathbb{C}^{n+1}\\ ([z],v) & \mapsto & v\end{array}$$

is holomorphic and show that $\pi^{-1}(\mathbb{C}^{n+1}\setminus\{0\})\to\mathbb{C}^{n+1}\setminus\{0\}$ is a biholomorphism.

Remark. The complex manifold M is called the *blow up of* $\mathbb{C}^{n+1}$ *at the origin.*

Exercise 32.14. Let $q(z)=$ be a non-constant complex polynomial. Show that the map

$$\begin{array}{rcl}\mathbb{C}\mathrm{P}^1 & \to & \mathbb{C}\mathrm{P}^1\\ [z:w] & \mapsto & \begin{cases}[q(a):1], & \text{if } [z:w]=[a:1] \text{ for some } a\in\mathbb{C},\\ [1:0], & \text{if } [z:w]=[1:0]\end{cases}\end{array}$$

is holomorphic.

Remark. On page 705 we showed that this map is smooth.

Exercise 32.15. We refer to the category $\mathcal{CplxMfd}$ with

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{CplxMfd}) & := & \text{all complex manifolds,} \\ \mathrm{Mor}(X,Y) & := & \text{all holomorphic maps from } X \text{ to } Y, \end{array}$$

with the usual composition of maps as the *category of complex manifolds*. Furthermore we define the category $\mathcal{PCplxMfd}$ of pointed complex manifolds in the obvious way. Show that the maps

$$\begin{array}{rcl} (M,P) & \mapsto & \mathrm{T}_P M \\ (f\colon (M,Q) \to (N,Q)) & \mapsto & (\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_Q N) \end{array}$$

define a covariant functor from the category $\mathcal{PCplxMfd}$ of pointed complex manifolds to the category $\mathcal{Vec}_{\mathbb{C}}$ of $\mathbb{C}$-vector spaces.

Remark. This exercise should be viewed as the complex analogue of the Tangent Space-Functor Proposition 23.10.

Part IV

Differential topology

In the last part of the lecture notes we introduced topological and smooth manifolds and we introduced many of the essential notions and we formulated and proved many of the basic theorems. In this part of the lecture notes we intensify our study of smooth manifolds. As the name "differential topology" suggests, we are interested in the topological properties of smooth manifolds. We will start out with a long discussion of smooth homotopies on smooth manifolds. Afterwards we will introduce the degree of a smooth map between smooth manifolds. This notion has many applications, for example it allows us to prove the Hairy Ball Theorem 38.12 and the Borsuk-Ulam 39.8.

33. Smooth homotopies

In this chapter we will introduce the notion of smooth homotopies, smooth isotopies and diffeotopies. Furthermore, in this and the following chapter we will discuss some basic properties and many examples.

33.1. **Smooth homotopies.** Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold. Suppose we are given two smooth maps $f_0, f_1 \colon M \to N$. On page 420 we defined a homotopy between f_0 and f_1 to be a map $F \colon M \times [0,1] \to N$ with $F_0 = f_0$ and $F_1 = f_1$. On many occasions one feels much happier if one actually has a "smooth homotopy". To make sense of the notion of a smooth homotopy we need to recall some concepts we introduced earlier.

(1) If M is a smooth manifold with empty boundary and if $a < b$ are two real numbers, then by the Smooth Manifold Product Proposition 19.31 (1) we can equip $M \times [a,b]$ with a natural structure of an $(m+1)$-dimensional smooth manifold.
(2) More generally, if M is any smooth manifold, possibly with non-empty boundary, then by the Smooth Manifold Product Proposition 20.6 we can view $M \times [a,b]$ as an $(m+1)$-dimensional smooth manifold with corner $\angle(M \times [a,b]) = \partial M \times \{a,b\}$.

Either way, by the definitions on page 546 and on page 581 we now have a notion of a map $F \colon M \times [a,b] \to N$ to be smooth.

For people who happen to be allergic to smooth manifolds with corner, the following lemma often makes it possible to avoid dealing with the notion of smooth maps defined on smooth manifolds with corner.

Lemma 33.1. Let M and N be smooth manifolds and let $F \colon M \times [a,b] \to N$ be a map.

(1) Let I be an interval with $[a,b] \subset I$. (We allow $I = [a,b]$).
(2) Let $\widetilde{M}$ be a smooth manifold that contains M as a submanifold. (We allow $\widetilde{M} = M$).

If there exists a smooth map[333] $G \colon \widetilde{M} \times I \to N$ such that the restriction of G to the subset $M \times [a,b]$ equals F, then $F \colon M \times [a,b] \to N$ is also smooth.

Proof. This lemma follows immediately from Lemmas 20.3 and 20.7. ∎

After this preamble we now introduce the main protagonist of this chapter.

Definition. Let M and N be smooth manifolds.

(1) A smooth homotopy is a smooth map $F \colon M \times [0,1] \to N$.
(2) We say that two smooth maps $f_0, f_1 \colon M \to N$ are smoothly homotopic if there exists a smooth homotopy $F \colon M \times [0,1] \to N$ with $F_0 = f_0$ and $F_1 = f_1$.[334]

There are various special types of smooth homotopies which we will now introduce.

[333] If either I is an open interval, or if $\partial \widetilde{M} = \varnothing$, then it follows from the Smooth Manifold Product Proposition 19.31 (1) that we can treat $\widetilde{M} \times I$ as a smooth manifold without corner.

[334] Recall that given a homotopy $F \colon X \times [a,b] \to Y$ and $t \in [a,b]$ we denote by $F_t \colon X \to Y$ the map that is given by $x \mapsto F(x,t)$.

Definition. Let M and N be smooth manifolds.

(1) A smooth isotopy of M in N is a smooth map

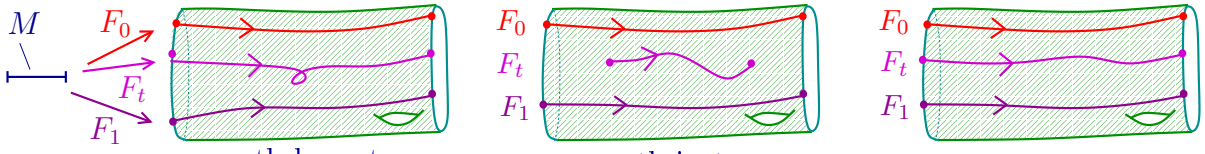
$$F\colon M \times [0,1] \;\to\; N$$

such that for each $t \in [0,1]$ the map $F_t\colon M \to N$ is a smooth embedding.[335]

(2) We say a smooth isotopy $F\colon M \times [0,1] \to N$ is proper if each F_t is a proper smooth embedding[336] *and* if the map $F\colon M \times [0,1] \to N$ is proper in the sense of the definition on page 180, i.e. the preimage of every compact subset of N is a compact subset of $M \times [0,1]$.

(3) We say that two (proper) smooth embeddings $f_0, f_1\colon M \to N$ are (properly) smoothly isotopic if there exists a (proper) smooth isotopy $F\colon M \times [0,1] \to N$ with $F_0 = f_0$ and $F_1 = f_1$.

If each F_t is a diffeomorphism, then we refer to a smooth isotopy as a diffeotopy and we say "diffeotopic" instead of "smoothly isotopic".



smooth homotopy but not a smooth isotopy

smooth isotopy but not a proper smooth isotopy

proper smooth isotopy

Example. We will soon give many examples of smooth isotopies. At this point it is perhaps better to give a non-example. More precisely, we consider the map $F\colon S^1 \times [0,1] \to \mathbb{R}^3$ that is sketched in the figure below. This map has the property that for each $t \in [0,1]$ the map $F_t\colon S^1 \to \mathbb{R}^3$ is a smooth embedding. But the map $F\colon S^1 \times [0,1] \to \mathbb{R}^3$ is not smooth. Thus this map is not a smooth isotopy.

Remark. The term "diffeotopic" is (alas!) used differently by different authors. Our definition agrees with the definition given in most textbooks, see e.g. [**Wall2016**, p. 49] and [**BJ1982**, Definition 9.3]. But it differs from the definition provided in [**Hirs1976**, Chapter 8], there the map on $M \times [0,1]$ only needs to be continuous, but each F_t needs to be a diffeomorphism. As we just saw in the previous example, this is a weaker condition.

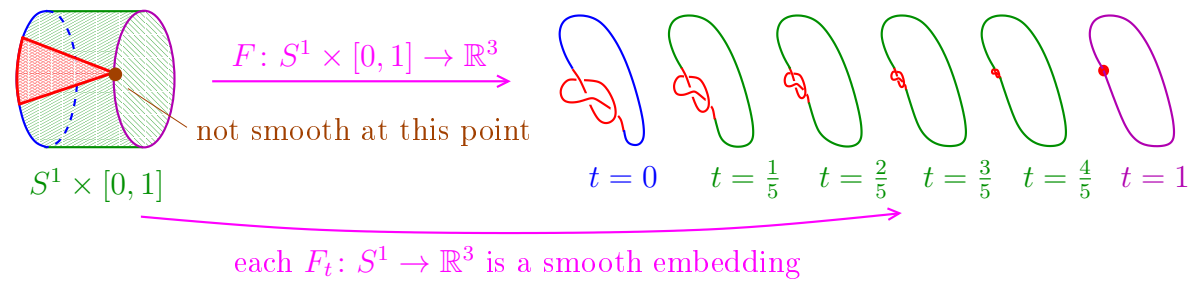
---

[335] Recall that by the definition on page 652 a map $f\colon M \to N$ between smooth manifolds is a *smooth embedding* if it is a homeomorphism onto its image, if it is smooth and if the following statements hold:

(1) For each $P \in M$ the map $\mathrm{D}\varphi_P\colon \mathrm{T}_P M \to \mathrm{T}_{\varphi(P)} N$ is a monomorphism.
(2) For each $P \in M \setminus \partial M$ we have $\varphi(P) \in N \setminus \partial N$.
(3) For each $P \in \partial M$ with $\varphi(P) \in \partial N$ the image $\mathrm{D}\varphi_P(\mathrm{T}_P M)$ is not contained in $\mathrm{T}_{\varphi(P)}(\partial N)$.
(4) For every component C of ∂M we have either $\varphi(C) \subset \partial N$ or $\varphi(C) \subset N \setminus \partial N$.

[336] By the definition on page 650 a smooth embedding is called *proper* if furthermore the following hold:

(5) $f^{-1}(\partial N) = \partial M$,
(6) $f(M)$ is a closed subset of N.

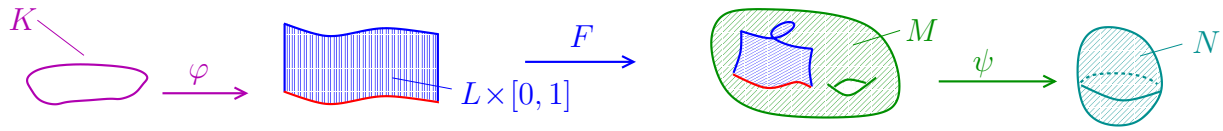


The following elementary lemma says in particular that one can precompose and postcompose smooth isotopies by smooth maps. It can be viewed as the analogue of Lemmas 14.5 (2).

Lemma 33.2. Let K, L, M and N be smooth manifolds. If $F\colon L \times [0,1] \to M$ is a smooth homotopy and if $\varphi\colon K \to L$ and $\psi\colon M \to N$ are smooth maps, then the maps

$$\begin{array}{rcl} K \times [0,1] & \to & M \\ (x,t) & \mapsto & F(\varphi(x),t) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} L \times [0,1] & \to & N \\ (x,t) & \mapsto & \psi(F(x,t)) \end{array}$$

are also smooth homotopies. If φ and ψ are diffeomorphisms, then a (proper) smooth isotopy (diffeotopy) $F\colon L \times [0,1] \to M$ becomes again a (proper) smooth isotopy (diffeotopy).



Proof. All of the statements are basically trivial, except that one needs to argue why the two maps $L \times [0,1] \to M$ and $K \times [0,1] \to N$ are smooth. Note that the map $L \times [0,1] \to M$ is the composition of the maps

$$K \times [0,1] \xrightarrow{\varphi \times \mathrm{id}_{[0,1]}} L \times [0,1] \xrightarrow{F} M.$$

It follows from the Smooth Manifold Product Proposition 20.6 that the map to the left is smooth. Thus the given map is the composition of two smooth maps between smooth manifolds with corner. Therefore it follows from Lemma 20.3 that it is smooth. Finally note that the map $K \times [0,1] \to N$ is precisely the map $\psi \circ F$, thus it is smooth by Lemma 20.3. ∎

In Lemma 14.5 (1) we saw that being homotopic is an equivalence relation. The analogous statement also holds in the smooth context.

Proposition 33.3. (Smooth Isotopy Transitivity Proposition) Let M and N be smooth manifolds.

(1) For maps $M \to N$ being "smoothly homotopic", being "smoothly (properly) isotopic" and being "diffeotopic" is an equivalence relation.
(2) If $A \subset M$ is a subset, then the statements in (1) also hold "rel A".[337]

[337] Recall that according to the definition on page 419 we say that a map $F\colon X \times [0,1] \to Y$ that is a *homotopy rel* A if for all $x \in A$ and $s, t \in [0,1]$ we have $F(x,s) = F(x,t)$.

At first glance the Smooth Isotopy Transitivity Proposition 33.3 might appear trivial, but the proof of the transitivity statement is technically slightly involved. The reader who is happy to swallow the Smooth Isotopy Transitivity Proposition 33.3 as it is, can also press the fast-forward button and move on to the next section.
Before we provide the proof of the Smooth Isotopy Transitivity Proposition 33.3 we need to make a few preparations. We start out with the following definition.

Definition. Let X and Y be topological spaces. We say a homotopy $F\colon X \times [0,1] \to Y$ is trivial near the ends if there exists an $\epsilon > 0$ such that for all $P \in X$ and all $t \in [0,\epsilon]$ we have $F(P,t) = F(P,0)$ and $F(P,1-t) = F(P,1)$.

The proof of the Smooth Isotopy Transitivity Proposition 33.3 rests on the following two lemmas.

Lemma 33.4. (Trivial-Near-The-Ends-Lemma) [338] Let M and N be smooth manifolds, let $F\colon M \times [0,1] \to N$ be a smooth homotopy and let $\epsilon \in (0, \frac{1}{2})$. There exists a smooth homotopy $G\colon M \times [0,1] \to N$ with the following properties:

(1) We have $G_0 = F_0$ and $G_1 = F_1$.
(2) For every $t \in [0,\epsilon)$ we have $G_t = F_0$ and for every $t \in [1-\epsilon, 1]$ we have $G_t = F_1$. In particular the smooth homotopy G is trivial near the ends.
(3) For each $s \in [0,1]$ there exists a $t \in [0,1]$ with $G_s = F_t$.

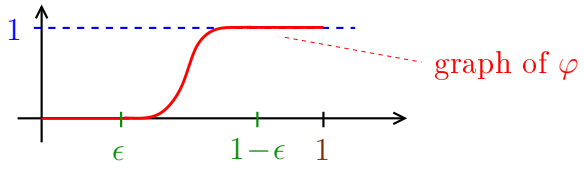
Furthermore, if F is a proper smooth homotopy or a diffeotopy, then so is G.

Proof of the Trivial-Near-The-Ends-Lemma 33.4. Let M and N be smooth manifolds and let $F\colon M \times [0,1] \to N$ be a smooth homotopy. Using the Smooth Transition Function Lemma 19.4 one can easily show that there exists a smooth map $\varphi\colon [0,1] \to [0,1]$ such that $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi(t) = 0$ for all $t \in [0,\epsilon]$ and $\varphi(t) = 1$ for all $t \in [1-\epsilon, 1]$. It follows from Lemmas 20.3 and 20.7 that the map

$$\begin{array}{rcl} F \circ (\mathrm{id}_M \times \varphi)\colon M \times [0,1] & \to & N \\ (x,y) & \mapsto & (x,t) \mapsto F(x,\varphi(t)) \end{array}$$

is smooth. It is clear that this map has the desired Properties (1), (2) and (3).

Finally note that if F is a proper smooth homotopy (diffeotopy), then it follows from Property (3) that G is also a proper smooth homotopy (diffeotopy). ■



We also need the following variation on the Homotopy Stacking Lemma 14.3.

[338] This lemma is the smooth analogue of Exercise 14.6.

Lemma 33.5. (Smooth Homotopy Stacking Lemma) Let M be a smooth manifold and let $a < b < c$ be real numbers. Let $f\colon M \times [a,c] \to N$ be a map to some smooth manifold N. We suppose that the following statements hold:

(1) The restrictions of f to $M \times [a,b]$ and to $M \times [b,c]$ (both viewed as smooth manifolds with corner) are smooth.
(2) There exists an $\epsilon > 0$ such that for all $t \in [a,b]$ with $t \in (b-\epsilon, b+\epsilon)$ and all $x \in M$ we have $f(x,t) = f(x,b)$.

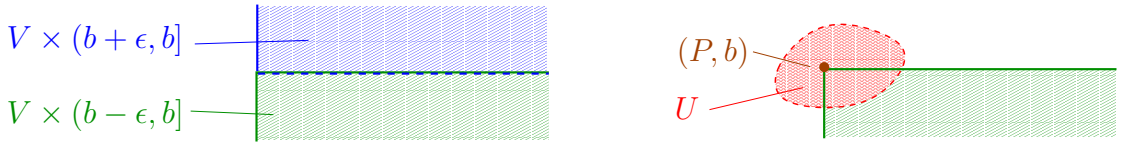
Then f itself is smooth.

Proof of the Smooth Homotopy Stacking Lemma 33.5. If $\partial M = \varnothing$, then the statement follows immediately from the Smooth Pasting Proposition 19.23. But as the meticulous reader will already have figured out, if $\partial M \neq \varnothing$, then there is one step which requires at least a little bit of thought. Namely, after taking charts for $P \in \partial M$ one sees that one still needs to prove the following claim.

Claim. Let V be an open subset of $\mathbb{H}^m$ and let $\varphi\colon V \times (b-\epsilon, b+\epsilon) \to \mathbb{R}^n$ be a map with the following properties:

(1) The restrictions of φ to $V \times (b-\epsilon, b] \to \mathbb{R}^n$ and $V \times [b, b+\epsilon) \to \mathbb{R}^n$ are smooth in the sense of the definition on page 532.
(2) For every $P \in V$ and $t \in [a,b]$ with $t \in (b-\epsilon, b+\epsilon)$ we have $\varphi(P,t) = \varphi(P,b)$.

Then φ is smooth in the sense of the definition on page 532.



Proof. As the reader can verify, this claim does *not* follow immediately from the definition. So let $P \in V$. By hypothesis there exists an open neighborhood U of $(P,b) \in \mathbb{R}^{m+1}$ and a smooth map $\widetilde{\varphi}\colon U \to \mathbb{R}^n$ with $\widetilde{\varphi} = \varphi$ on $U \cap (V \times (b-\epsilon, b])$. Now we consider the map

$$\begin{array}{rcl} \widetilde{\psi}\colon U & \to & \mathbb{R}^n \\ (P,t) & \mapsto & \widetilde{\varphi}(P,b). \end{array}$$

It follows easily from our hypothesis (2) that this map is smooth and that it agrees with φ on $U \cap (V \times (b-\epsilon, b+\epsilon))$. ∎

Proof of the Smooth Isotopy Transitivity Proposition 33.3.

(1) The only non-trivial aspect is to show transitivity. We show how to prove this statement for "properly smoothly isotopic". All other cases are dealt with in the same way. Thus let $f, g, h\colon M \to N$ be proper smooth embeddings. We assume that f is properly smoothly isotopic to g and that g is properly smoothly isotopic to h. We need to show that f is properly smoothly isotopic to h.

Let $\Phi\colon M \times [0,1] \to N$ be a proper smooth isotopy from f to g. Furthermore let $\Psi\colon M \times [0,1] \to N$ be a proper smooth isotopy from g to h. After applying the Trivial-Near-The-Ends-Lemma 33.4 to Φ and Ψ we can arrange that there exists an $\epsilon \in (0,1)$ such that for each $P \in M$ and each $t \in [0,\epsilon]$ we have $\Phi(P, t-\epsilon) = g(P) = \Psi(P,\epsilon)$. Note that by the "furthermore" part of Trivial-Near-The-Ends-Lemma 33.4 we can

guarantee that along the way we did not lose the property of Φ and Ψ being proper smooth isotopies. Now we consider the map

$$\begin{array}{rcl} M \times [0,1] & \to & N \\ (P,t) & \mapsto & \begin{cases} \Phi(P, 2\cdot t), & \text{if } t \in [0, \frac{1}{2}], \\ \Psi(P, 2\cdot t - 1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

By the Homotopy Stacking Lemma 33.5 we know that this map is smooth. It is now clear that this map is a proper smooth isotopy between f and h.

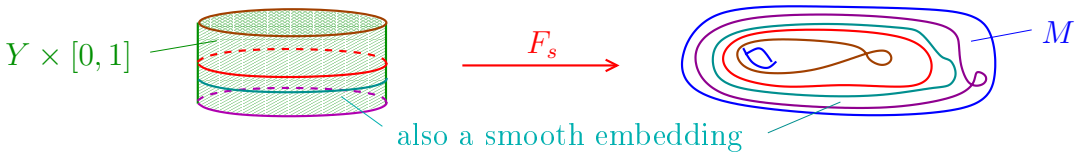
(2) If we are given a subset $A \subset M$ and if Φ and Ψ are smooth isotopies rel A, then it follows from the construction and Part (3) of the Trivial-Near-The-Ends-Lemma 33.4 that the above smooth isotopy is again rel A. ■

We conclude this section with a proposition that says, loosely speaking, that a smooth homotopy starting at a smooth embedding is for "small parameters" actually a smooth isotopy.

Proposition 33.6. (Deform Smooth Embedding Proposition) Let Y and M be smooth manifolds, let S be a smooth manifold without boundary and let $F\colon Y \times S \to M$ be a smooth map. We assume that Y is compact.

(1) Let $s \in S$ be a point such that the map $F_s\colon Y \to Y$ is a smooth embedding. There exists an open neighborhood U of s such that for every $t \in U$ the map $F_t\colon Y \to M$ is a smooth embedding.

(2) If in the setting of (1) the map F_s is actually a diffeomorphism and if Y and M are closed connected smooth manifolds, then there exists an open neighborhood U of s such that for every $t \in U$ the map $F_t\colon Y \to M$ is a diffeomorphism.



Proof.

(1) We consider the map

$$\begin{array}{rcl} \Phi\colon Y \times S & \to & M \times S \\ (y,t) & \mapsto & (F(y,t),t). \end{array}$$

Let $s \in S$ be a point such that the map $F_s\colon Y \to M$ is a smooth embedding. We observe that such $s \in S$ has the following two properties:

(a) The map $\Phi|_{Y\times\{s\}}\colon Y \times \{s\} \to M \times \{s\}$ is a smooth embedding.

(b) For each $y \in Y$ the differential $\mathrm{D}\Phi_{(y,s)}$ is a monomorphism. It follows from the Local Smooth Embedding Theorem 24.8 we obtain that for each $y \in Y$ the map Φ is locally a smooth embedding at (y,s).

We deduce from (a) and (b) together with the Neighborhood Smooth Embedding Theorem 10.13 that there exists an open neighborhood W of $Y \times \{s\} \subset Y \times S$ such that the restriction of Φ to $W \to M \times S$ is a smooth embedding. Since Y is compact we obtain from the elementary Exercise 5.9 that there exists an open neighborhood

U of $s \in S$ such that $Y \times U \subset W$. It follows that for each $t \in U$ the map

$$\begin{array}{rcl} \Phi_t \colon Y & \to & M \times \{t\} \subset M \times S \\ y & \mapsto & (F_t(y), y) \end{array}$$

is a smooth embedding. Finally note that this in turn implies immediately that for each $t \in U$ the map $F_t \colon Y \to M$ is also a smooth embedding.

(2) Now we assume that $F_s \colon Y \to M$ is a diffeomorphism and that Y and M are closed connected smooth manifolds. In particular we have $\dim(Y) = \dim(M)$. By (1) we know there exists an open neighborhood U of s such that for every $t \in U$ the map $F_t \colon Y \to M$ is a smooth embedding. Since Y is closed, since M is connected and since $\dim(Y) = \dim(M)$ we obtain from the Codimension-Zero Smooth Embedding Proposition 24.16 (3) that each F_t is actually a diffeomorphism. ■

33.2. **Smooth isotopies of (sub-) manifolds.** In this very short we introduce a few basic definitions which are slight variations on some of the definitions from the last section. The need for these variations will arise every now and then.

Definition. Let M_0, M_1 be two (proper) submanifolds of a smooth manifold N.

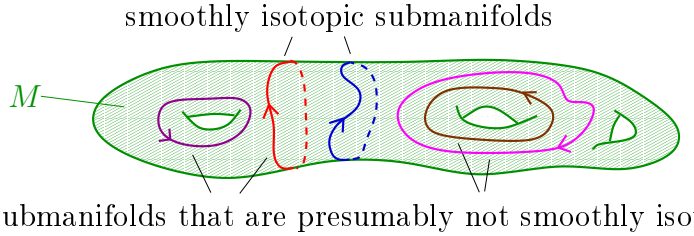
(1) A (proper) smooth isotopy between M_0 and M_1 is defined as a (proper) smooth isotopy $F \colon M_0 \times [0,1] \to N$ with $F_0 = \mathrm{id}_{M_0}$ and with $F_1(M_0) = M_1$.[339]

(2) We say that the two submanifolds M_0 and M_1 are (properly) smoothly isotopic if there exists a (proper) smooth isotopy between M_0 and M_1.

If M_0 and M_1 are oriented smooth manifolds, then we demand that the map $F_1 \colon M_0 \to M_1$ is orientation-preserving.

Remark. Let $F \colon M \times [0,1] \to N$ be a (proper) smooth isotopy. Recall that it follows from the Smooth Embedding Theorem 24.10 (1) that each $F_t(M)$ is a (proper) smooth submanifold of N. Thus, loosely speaking, two (proper) submanifolds are (properly) smoothly isotopic precisely if one submanifold can be smoothly deformed into the other through (proper) submanifolds.

Example. In the figure below we consider several oriented 1-dimensional submanifolds of a surface M. In some cases it is clear that submanifolds are smoothly isotopic, in other cases this looks dubious. In Chapter 43 we will get to know techniques which allow us to show that the dubious pairs of oriented 1-dimensional submanifolds are indeed not smoothly isotopic. In particular in Exercise 43.14 we will return to the given submanifolds.



For completeness' sake we state the following lemma which we will use implicitly on several occasions.

[339] Note that it follows from the Smooth Embedding Theorem 24.10 that $F_1 \colon M_0 \to M_1$ is a diffeomorphism.

Lemma 33.7. Let N be a smooth manifold. The property of being (properly) smoothly isotopic is an equivalence relation on the set of (proper) submanifolds of (oriented) submanifolds of M.

Proof. It follows almost immediately from the Smooth Isotopy Transitivity Proposition 33.3 that being (properly) smoothly isotopic is an equivalence relation. ■

33.3. **Examples of smooth homotopies.** In this section we will give several examples of smooth homotopies which will come in handy on numerous several occasions. We start out with an example of a diffeotopy.

Lemma 33.8. (Diffeotopies-via-Matrices Lemma) Let $n \in \mathbb{N}$.

(1) (a) If $\gamma\colon [0,1] \to \mathrm{GL}(n,\mathbb{R})$ is a smooth path[340], then the map

$$\begin{array}{rcl} F\colon \mathbb{R}^n \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & \gamma(t)\cdot x \end{array}$$

is a diffeotopy rel the origin.

(b) Given any $A \in \mathrm{GL}_+(n,\mathbb{R})$ there exists a diffeotopy of $\mathbb{R}^n$ from $\mathrm{id}_{\mathbb{R}^n}$ to the diffeomorphism of $\mathbb{R}^n$ that is given by multiplication by A.

(2) (a) If $\gamma\colon [0,1] \to \mathrm{O}(n)$ is a smooth path, then the map

$$\begin{array}{rcl} F\colon \overline{B}^n \times [0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \gamma(t)\cdot x \end{array}$$

is a diffeotopy rel the origin.

(b) Given any $A \in \mathrm{SO}(n)$ there exists a diffeotopy of $\overline{B}^n$ from $\mathrm{id}_{\overline{B}^n}$ to the diffeomorphism of $\overline{B}^n$ that is given by multiplication by A.

The same statements hold with $\overline{B}^n$ replaced by B^n and S^{n-1}.

Proof. We first prove Statement (2) which is slightly more subtle.

(2) (a) By the Matrix Group-Smooth Manifold Proposition 26.7 we know that $\mathrm{O}(n)$ is a smooth submanifold of $\mathrm{M}(n\times n,\mathbb{R})$. By the Submanifold Smooth Structure Lemma 19.24 (1) the map γ, viewed as a map $\gamma\colon [0,1] \to \mathrm{M}(n\times n,\mathbb{R})$ is also smooth. It follows from this observation that the obvious extension of F to $\mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ is smooth. We deduce from Lemma 33.1, that the given map $F\colon \overline{B}^n \times [0,1] \to \overline{B}^n$ is also smooth. It is now clear that it is in fact a diffeotopy.

(b) Let $A \in \mathrm{SO}(n)$. By the Matrix Group Path-Component Proposition 2.37 we know that $\mathrm{SO}(n)$ is path-connected. It follows from this fact, together with the Smooth Path-Connectivity Proposition 19.29, that there exists a smooth map $\gamma\colon [0,1] \to \mathrm{SO}(n)$ with $\gamma(0) = \mathrm{id}$ and $\gamma(1) = A$. It follows from (2a) that there exists a diffeotopy of $\overline{B}^n$ from $\mathrm{id}_{\overline{B}^n}$ to the diffeomorphism of $\overline{B}^n$ that is given by multiplication by A.

[340] Let $G = \mathrm{GL}(n,\mathbb{R})$ or $G = \mathrm{O}(n)$. By the discussion on page 711 and by the Matrix Group-Smooth Manifold Proposition 26.7 we know that G is a smooth submanifold of $\mathrm{M}(n\times n,\mathbb{R})$. Thus by the Submanifold Smooth Structure Lemma 19.24 we can and will view G as a smooth manifold in its own right. In this sense we demand that the map $\gamma\colon [0,1] \to G$ is smooth. In practice though we use the Submanifold Smooth Structure Lemma 19.24 which says that if $\gamma\colon [0,1] \to \mathrm{M}(n\times n,\mathbb{R})$ is smooth in the ordinary sense, and if γ takes values in G, then $\gamma\colon [0,1] \to G$ is smooth.

By the same argument the analogous statements also hold for B^n and S^{n-1}.

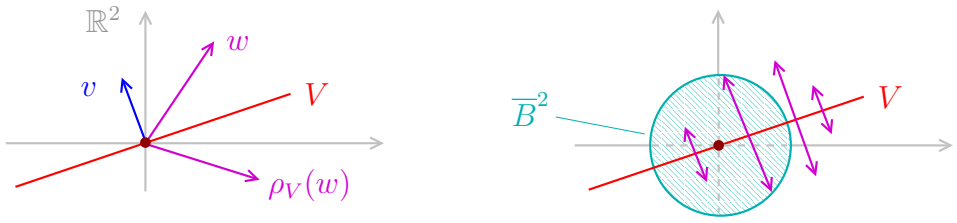
(1) The proof of Statement (1) is almost verbatim the same as the proof of (2). Note that by the Matrix Group Path-Component Proposition 2.37 we know that $\mathrm{GL}_+(n,\mathbb{R})$ is path-connected. ■

To formulate our next example we recall the following definition which should be familiar from a course in linear algebra.

Definition. Let $n \in \mathbb{N}$ and let $V \subset \mathbb{R}^n$ be a hyperplane, i.e. V is a vector subspace of $\mathbb{R}^n$ codimension one. We pick a vector $v \in \mathbb{R}^n$ with $\|v\| = 1$ that is orthogonal to V. We refer to the map[341][342]

$$\begin{aligned} \rho_V \colon \mathbb{R}^n &\to \mathbb{R}^n \\ w &\mapsto w - 2 \cdot \langle v, w \rangle \cdot v \end{aligned}$$

as the reflection in the hyperplane V.



Lemma 33.9. Let $n \in \mathbb{N}$.

(1) Any reflection in a hyperplane of $\mathbb{R}^n$ has the following properties:
   (a) it is an orientation-reversing diffeomorphism,
   (b) it is an isometry (i.e. it preserves the Euclidean distance) and preserves the origin,
   (c) it restricts to orientation-reversing diffeomorphisms of $\overline{B}^n$ and S^{n-1}.

(2) Any two reflections in hyperplanes of $\mathbb{R}^n$ are in fact related by a diffeotopy $F \colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ such that each F_t is also a reflection. In particular the diffeotopies are diffeotopies rel the origin. The diffeotopies restrict to diffeotopies of $\overline{B}^n$ and S^{n-1}.

Proof. Let $n \in \mathbb{N}$.

(1) Let ρ be a reflection in a hyperplane of $\mathbb{R}^n$. Clearly ρ is smooth and it satisfies $\rho \circ \rho = \mathrm{id}_{\mathbb{R}^n}$. This implies immediately that ρ is a diffeomorphism. Next we want to show that $\rho \colon \mathbb{R}^n \to \mathbb{R}^n$ is orientation-reversing. By the Tangent Space-Isomorphism Proposition 23.16 and by the discussion on page 536 we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{T}_0\mathbb{R}^n & \xrightarrow[\cong]{\Theta_{\mathbb{R}^n}} & \widetilde{\mathrm{T}}_0\mathbb{R}^n = \mathbb{R}^n \\ {\scriptstyle \mathrm{D}\rho_0}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \rho} \\ \mathrm{T}_0\mathbb{R}^n & \xrightarrow[\cong]{\Theta_{\mathbb{R}^n}} & \widetilde{\mathrm{T}}_0\mathbb{R}^n = \mathbb{R}^n. \end{array}$$

Elementary linear algebra shows that the right vertical map has negative determinant, which means that it is orientation-reversing. By Lemma 25.7 (2) this implies that the vertical map $\mathrm{D}\rho_0 \colon \mathrm{T}_0\mathbb{R}^n \to \mathrm{T}_0\mathbb{R}^n$ to the left is also orientation-reversing. It

[341] Note that v is well-defined up to sign, it follows easily that ρ_V does not depend on the choice of v.
[342] Here $\langle\ ,\ \rangle$ denotes the standard inner product on $\mathbb{R}^n$.

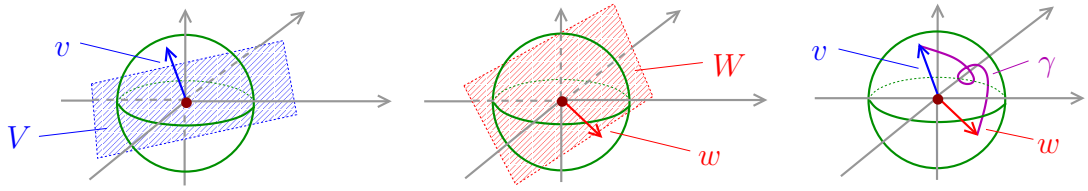
follows from the Basics-of-Orientations Lemma 25.13 that $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ is orientation-reversing.

A short amusing calculation shows that ρ preserves the scalar product, hence length. Since $\rho(0) = 0$ and since ρ preserves the Euclidean metric we see that ρ restricts to self-maps of $\overline{B}^n$ and S^{n-1}. It follows from the Submanifold Smooth Structure Lemma 19.24 that these self-maps are in fact diffeomorphisms of $\overline{B}^n$ and S^{n-1}. The same argument as above shows that the self-diffeomorphism of $\overline{B}^n$ is orientation-reversing. Finally it follows from the Boundary Orientation Lemma 25.17 (4) that the self-diffeomorphism of $S^{n-1} = \partial\overline{B}^n$ is also orientation-reversing

(2) Suppose we are given two hyperplanes V and W of $\mathbb{R}^n$. We pick vectors $v, w \in \mathbb{R}^n$ with $\|v\| = \|w\| = 1$ and with $v \perp V$ and $w \perp W$. By the Ball-Sphere-Connected Lemma 2.25 we know that S^{n-1} is path-connected. It follows from the Smooth Path-Connectivity Proposition 19.29 that there exists in fact a smooth path $\gamma\colon [0,1] \to S^{n-1}$ with $\gamma(0) = v$ and $\gamma(1) = w$. One can easily verify that the map

$$\begin{array}{rcl} F\colon \mathbb{R}^n \times [0,1] & \to & \mathbb{R}^n \\ (P,t) & \mapsto & P - 2\cdot\langle\gamma(t), P\rangle\cdot\gamma(t) \end{array}$$

is smooth. By construction $F_0 = \rho_V$ and $F_1 = \rho_W$. Thus F is the desired diffeotopy of $\mathbb{R}^n$. As in (1) we see that this diffeotopy of $\mathbb{R}^n$ restricts to a diffeotopy of $\overline{B}^n$ and to a diffeotopy of S^{n-1}. ■



Given a topological space X and a subset A we defined on page 433 the notion of a deformation retraction $R\colon X \times [0,1] \to X$ from X to A. If X is a smooth manifold, then it makes sense to talk of a smooth deformation retraction. The following proposition, which is the smooth version of the Matrix Groups Deformation Retract Proposition 15.7, gives interesting examples of smooth deformation retractions.

Proposition 33.10. (Matrix Groups Smooth Deformation Retract Proposition) Let $n \in \mathbb{N}_0$ and let $k \in \{1, \ldots, n\}$. We write

$$\begin{array}{rcl} \mathrm{GL}(n,k) & = & \text{set of } k\text{-tuples of linearly independent vectors of } \mathbb{R}^n \\ \mathrm{O}(n,k) & = & \text{set of } k\text{-tuples of orthonormal vectors of } \mathbb{R}^n. \end{array}$$

(Note that for $k = n$ we have $\mathrm{GL}(n,k) = \mathrm{GL}(n)$ and $\mathrm{O}(n,k) = \mathrm{O}(n)$.) We denote by $r\colon \mathrm{GL}(n,k) \to \mathrm{O}(n,k)$ the map given by the Gram-Schmidt Orthonormalization Process 2.38. The above retraction $r\colon \mathrm{GL}(n,k) \to \mathrm{O}(n,k)$ is realized[343] by a smooth deformation retraction[344] $R\colon \mathrm{GL}(n,k)\times[0,1] \to \mathrm{O}(n,k)$ which has the following extra property:

($*$) For every k-dimensional vector subspace V of $\mathbb{R}^n$ and every $t \in [0,1]$ we have

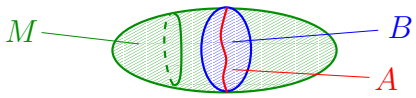
$$R_t(\text{basis of } V) \;=\; \text{basis of } V.$$

The analogous statements hold if we replace the pair $\mathrm{O}(n,k)$ and $\mathrm{GL}(n,k)$ by one of the following four pairs:

(a)	$\mathrm{GL}_+(n,\mathbb{R})$	and	$\mathrm{SO}(n)$
(b)	$\mathrm{SL}(n,\mathbb{R})$	and	$\mathrm{SO}(n)$
(c)	$\mathrm{GL}(n,\mathbb{C})$	and	$\mathrm{U}(n)$
(d)	$\mathrm{SL}(n,\mathbb{C})$	and	$\mathrm{SO}(n)$.

Our proof of Proposition 33.10 rests on the following elementary lemma.

Auxiliary Lemma 33.11. Let M be a smooth manifold and let $A \subset B \subset X$ be a submanifolds. If A is a smooth deformation retract of B and if B is a smooth deformation retract of M, then A is a smooth deformation retract of X.



Proof of Lemma 33.11. The proof of the lemma follows from combining the ideas of the proof of the Iterated Deformation Retracts Lemma 15.4 and the proof of the Smooth Isotopy Transitivity Proposition 33.3. We leave it to the reader to fill in the details. ■

Sketch of a proof of Proposition 33.10. In the Matrix Groups Deformation Retract Proposition 15.7 we gave the same statement as above, except that we did not worry about smoothness. We will now see that demanding that the deformation retractions are smooth is not a big deal.

On page 173, in the proof of the Matrix Group Path-Component Proposition 2.37, we implicitly already showed that each of the $2k$ steps of the Gram-Schmidt Orthogonalization Process is realized by a smooth deformation retraction that satisfies $(*)$. It follows from Lemma 33.11 that these smooth deformation retractions give rise to the desired smooth deformation retraction

$$R\colon\ \mathrm{GL}(n,k)\times[0,1]\ \to\ \mathrm{GL}(n,k)$$

from $\mathrm{GL}(n,k)$ to $\mathrm{O}(n,k)$ such that R that satisfies $(*)$. If $n=k$, then it follows from the Gram-Schmidt Orthonormalization Process 2.38 (2) and (3) that this deformation retraction R restricts to a deformation retraction from $\mathrm{GL}_+(n,\mathbb{R})$ to $\mathrm{SO}(n)$ and from $\mathrm{SL}(n,\mathbb{R})$ to $\mathrm{SO}(n)$.

Finally note that the cases of the complex matrix groups are treated in a very similar fashion. ■

33.4. **Homogeneous topological spaces.** We start out with the following general definition.

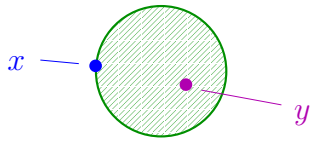
Definition. Let X be a topological space.

(1) We say X is homogeneous if given any two points $x,y\in X$ there exists a homeomorphism $f\colon X\to X$ with $\varphi(x)=y$.

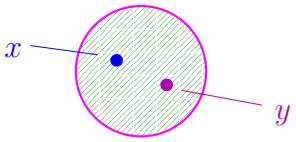
[343] Recall that by the definition on page 433 we say that a retraction $f\colon X\to A$ is realized by a deformation retraction $R\colon X\times[0,1]\to X$ if $R_1=f$.

[344] In Exercise 33.4 we will see that $\mathrm{GL}(n,k)$ is an open subset of $\mathbb{R}^{nk}$. Therefore the word "smooth" makes sense in the most naive sense.

(2) Let $A \subset X$ be a subset. We say X is homogeneous outside of A if given any two points $x, y \in X \setminus A$ there exists a homeomorphism $f\colon X \to X$ with $\varphi(x) = y$.



$\overline{B}^2$ is *not* homogenous



$\overline{B}^2$ is homogenous outside of S^1

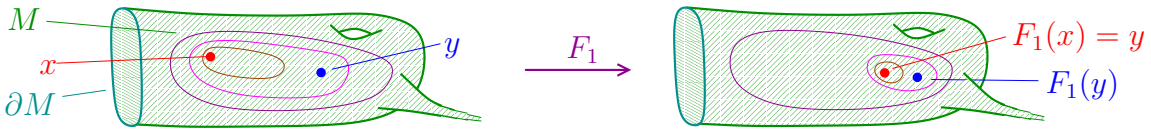
Examples.

(1) It is clear that the topological space $\mathbb{R}^n$ is homogeneous. Furthermore, an easy linear algebra argument shows that the sphere S^n is homogeneous.
(2) Every set that is equipped with the trivial topology or the discrete topology is homogeneous.
(3) In Exercise 22.11 we showed that the long ling $\mathbb{L}$ is homogeneous.
(4) We consider the infinite product $X = \prod_{k\in\mathbb{N}} \{0,1\} = \{0,1\}^{\mathbb{N}}$ of the discrete space $\{0,1\}$. In Exercise 33.7 we will see that X is homogeneous. Note that on page 235 we saw that the Cantor set is homeomorphic to $\prod_{k\in\mathbb{N}} \{0,1\} = \{0,1\}^{\mathbb{N}}$. This gives us the not entirely intuitive conclusion, that the Cantor set is homogeneous.
(5) It follows basically immediately from the definitions that a topological manifold with non-empty boundary and non-empty interior is not homogeneous. In particular $[0,1]^m$ is not homogeneous. It is thus quite stunning that the Hilbert Cube[345], i.e. the topological space $\prod_{k\in\mathbb{N}} [0,1] = [0,1]^{\mathbb{N}}$, is actually homogeneous. A proof of this fact is given in [**Verb1969, HalW2012**].

For topological and smooth manifolds we are interested in finer notions of homogeneity, which leads us to the following definition.

Definition. Let M be a topological manifold and let $x, y \in M \setminus \partial M$. We say x can be moved to y if there exists a homeotopy $F\colon M \times [0,1] \to M$ with the following properties:
(1) $F_0 = \mathrm{id}$,
(2) $F_1(x) = y$,
(3) F is a homotopy rel some neighborhood of ∂M.
(4) F is a homotopy with compact support, i.e. there exists a compact subset $K \subset X$ such that $F(x,t) = x$ for all $x \notin K$ and all $t \in [0,1]$.

If M is a smooth manifold, then we demand that F is actually a diffeotopy.



The following proposition shows in particular that each compact interval $[a,b]$ is homogeneous outside of the set of endpoints $\{a,b\}$, that every sphere is homogeneous and that the closed ball $\overline{B}^n$ is homogeneous outside of the boundary sphere $\partial\overline{B}^n = S^{n-1}$.

[345] We encountered the Hilbert Cube in a slightly different avatar in Exercise 9.20.

Proposition 33.12.
(1) Let $a < b \in \mathbb{R}$. Given any $x, y \in (a, b)$ we can move x to y.
(2) Let $n \in \mathbb{N}$. Given any $x, y \in S^n$ we can move x to y.
(3) Let $n \in \mathbb{N}$. Given any $x, y \in \overline{B}^n \setminus \partial\overline{B}^n = B^n$ we can move x to y.

Remark.
(1) Note that Proposition 33.12 (3) gives in particular a solution to the rather challenging Exercise 19.21.
(2) In Proposition 35.14 we will use the Flow Theorem 35.1 to provide an alternative proof of Proposition 33.12 (3).

The proof of Proposition 33.12 rests on the following little lemma.

Auxiliary Lemma 33.13. Let M be a smooth manifold and let $x, y, z \in M \setminus \partial M$. If we can move x to y and if we can move y to z, then we can move x to z.

Proof of Lemma 33.13. Let $F\colon M \times [0,1] \to M$ and $G\colon M \times [0,1] \to M$ be diffeotopies as in the above definition such that $F_1(x) = y$ and such that $G_1(y) = z$. Using the Trivial-Near-The-Ends-Lemma 33.4 we can arrange that F and G are trivial near the ends. One can now verify easily that the map

$$\begin{array}{rcl} H\colon \times [0,1] & \to & M \\ (P,t) & \mapsto & \begin{cases} F(P, 2t), & \text{if } t \in [0, \frac{1}{2}], \\ G(F_1(P), 2t-1), & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \end{array}$$

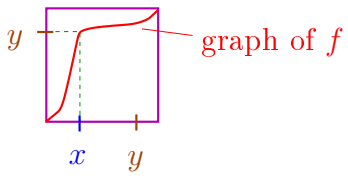
has the desired properties. ∎

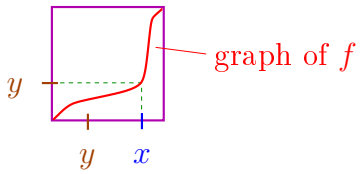
Proof of Proposition 33.12.

(1) Let $x, y \in (a, b)$. We leave it to the reader to show, say using the Smooth Transition Function Lemma 19.4, that there exists a diffeomorphism $f\colon [a,b] \to [a,b]$ with $f(x) = y$ and which is identity in a neighborhood of the two endpoints a and b. We consider the smooth homotopy

$$\begin{array}{rcl} F\colon [a,b] \times [0,1] & \to & [a,b] \\ (s,t) & \mapsto & s \cdot (1-t) + f(s) \cdot t \end{array}$$

which is evidently a homotopy rel an open neighborhood of $\{a, b\}$ with $F_0 = \mathrm{id}$ and $F_1 = f$. Since the derivative of f is strictly positive one sees easily, say using the Inverse Mapping Theorem 24.1 (2), that each F_t is a diffeomorphism of $[a, b]$, i.e. F is the promised diffeotopy.





(2) Let $n \in \mathbb{N}$ and let $x, y \in S^n$. By the Orthogonal Action Lemma 5.26 (4) we know that there exists an $A \in \mathrm{SO}(n+1)$ with $A \cdot x = y$. The desired orientation-preserving diffeomorphism is thus given by multiplication by A. By the Diffeotopies-via-Matrices Lemma 33.8 we know that this map is diffeotopic to the identity.

(3) The case $n = 1$ is a special case of (1). Thus let us now suppose that $n \geq 2$. Let $P, Q \in \overline{B}^n \setminus \partial\overline{B}^n = B^n$. First we deal with the case that $P, Q \neq 0$. We start out with the following claim.

Claim. There exists a diffeotopy $G\colon \overline{B}^n \times [0,1] \to \overline{B}^n$ with $G_0 = \mathrm{id}$, $G_1(\|P\|) = \|Q\|$ and such that G is a homotopy rel a neighborhood of the boundary.

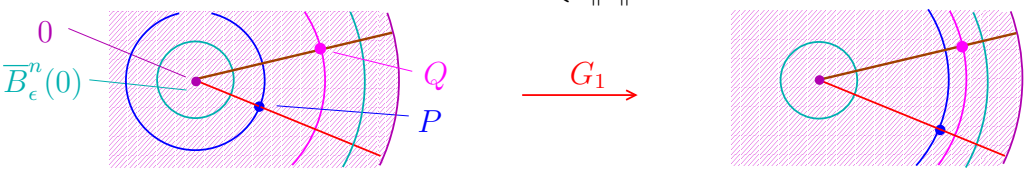
Proof. Since $P, Q \neq 0$ and since $P, Q \in B^n$ we see that $\|P\|, \|Q\| \in (0,1)$. Therefore it follows from (1) that there exists a diffeotopy $F\colon [0,1] \times [0,1] \to [0,1]$ with the following properties:

(a) We have $F_0 = \mathrm{id}$.
(b) The map $F_1\colon [0,1] \to [0,1]$ is a diffeomorphism with $F_1(\|P\|) = \|Q\|$.
(c) There exists an $\epsilon > 0$, such that $F(s,t) = s$ for all $s \in [0,\epsilon] \cup [1-\epsilon, 1]$ and all $t \in [0,1]$.

Using the Smooth Pasting Proposition 19.3 one can easily verify that the desired diffeotopy is given by

$$\begin{array}{rcl} G\colon \overline{B}^n \times [0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \begin{cases} x, & \text{if } \|x\| < \epsilon \\ \frac{x}{\|x\|} \cdot F(\|x\|, t), & \text{if } \|x\| > 0. \end{cases} \end{array}$$

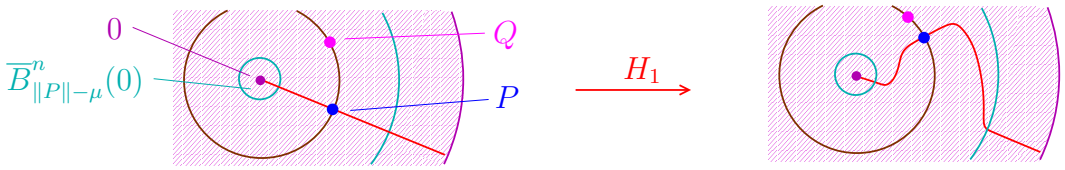
⊞



It follows from the claim and Lemma 33.13 that it remains to deal with the case that $\|P\| = \|Q\|$. We pick $\mu > 0$ with $0 < \|P\| - \mu < \|P\| + \mu < 1$. We write $v := \frac{P}{\|P\|}$ and $w := \frac{Q}{\|Q\|}$. Since $n \geq 2$ it follows from (2) and the Trivial-Near-The-Ends-Lemma 33.4 that there exists a diffeotopy $F\colon S^{n-1} \times [0,\mu] \to S^{n-1}$ with $F_0 = \mathrm{id}$, with $F_\mu(v) = w$ and which is trivial near the ends. Again one can easily verify that the desired diffeotopy is given by the map

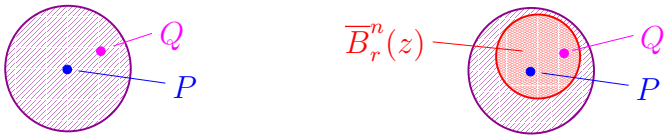
$$\begin{array}{rcl} H\colon \overline{B}^n \times [0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \begin{cases} x, & \text{if } \|x\| \in [0, \|P\| - \mu], \\ \|x\| \cdot F(\frac{x}{\|x\|}, t \cdot (\|x\| - \|P\| + \mu)), & \text{if } \|x\| \in [\|P\| - \mu, \|P\|], \\ \|x\| \cdot F(\frac{x}{\|x\|}, t \cdot (\|P\| + \mu - \|x\|)), & \text{if } \|x\| \in [\|P\|, \|P\| + \mu], \\ x, & \text{if } \|P\| \in [\|x\| + \mu, 1]. \end{cases} \end{array}$$

We have now settled the case that $P, Q \neq 0$.



Finally we need to deal with the case that one of two points $P, Q \in \overline{B}^n \setminus \partial\overline{B}^n = B^n$ is the origin. In this case we pick a closed ball $\overline{B}^n_r(z) \subset \overline{B}^n$ with $P, Q \in B^n_r(P)$ but with $z \neq P, Q$. We apply the above scheme to find a diffeotopy of $\overline{B}^n_r(z)$ that sends P to Q. Since the diffeotopy is a homotopy rel a neighborhood of the boundary

of $\overline{B}^n_r(z)$ we see that we can extend this diffeotopy via the constant homotopy to a diffeotopy of all of $\overline{B}^n$. This diffeotopy clearly has all the desired properties. ■



The following proposition implies in particular that connected topological manifolds without boundary are homogeneous.

Proposition 33.14. (Homogeneous-via-Diffeotopies Proposition)

(1) Let M be a connected topological manifold. Given any $x, y \in M \setminus \partial M$ we can move x to y via a homeotopy.
(2) Let M be a connected smooth manifold. Given any $x, y \in M \setminus \partial M$ we can move x to y via a diffeotopy.

Remark. For smooth manifolds we will prove in Proposition 36.4 a refinement of the Homogeneous-via-Diffeotopies Proposition 33.14.

Proof. We prove the Homogeneous-via-Diffeotopies Proposition 33.14 for smooth manifolds. The case of topological manifolds follows by almost the same argument. Thus let M be a connected smooth manifold and let $x \in M \setminus \partial M$.

We say that $y \in M \setminus \partial M$ is *connected to x via balls* if there exist smooth embeddings $\Phi_i \colon \overline{B}^n \to M \setminus \partial M$, $i = 1, \ldots, k$ such that $x \in \Phi_1(B^n)$, $y \in \Phi_k(B^n)$ and such that for each $i = 1, \ldots, k-1$ we have $\Phi_i(B^n) \cap \Phi_{i+1}(B^n) \neq \varnothing$.

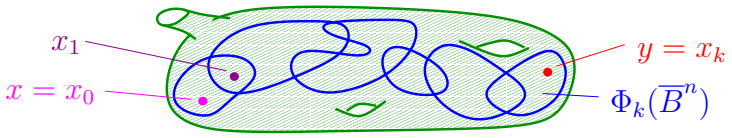
Claim. Every point in $M \setminus \partial M$ is connected to x via balls.

Proof. Recall that we assume that M is connected. It follows from the Topological Manifolds-Local Properties Lemma 18.8 and the Push Away From Boundary Lemma 18.13, see also Exercise 18.19, that $M \setminus \partial M$ is also connected. Now let U be the set of points in $M \setminus \partial M$ that are connected to x via balls.

- Since $x \in M \setminus \partial M$ we see that there exists a smooth embedding $\Phi \colon \overline{B}^n \to M \setminus \partial M$ with $x \in \Phi(B^n)$, thus $x \in U$, in particular U is non-empty.
- It is clear that U is open.
- Finally let $y \in \overline{U}$, where we take the closure of U in $M \setminus \partial M$. Again we see that there exists a smooth embedding $\Phi \colon \overline{B}^n \to M \setminus \partial M$ with $y \in \Phi(B^n)$. By the Codimension-Zero Smooth Embedding Proposition 24.16 we know that $\Phi(B^n)$ is an open neighborhood of y in $M \setminus \partial M$. Since $y \in \overline{U}$ we obtain from Lemma 1.10 (2) that there exists a point $\widetilde{y} \in \Phi(B^n) \cap U$. We can connect $\widetilde{y}$ to x via balls, but then we can also connect y to x via one extra ball. Thus we have shown that $y \in U$, i.e. we have shown that $U = \overline{U}$. By the Interior-Closure Lemma 1.9 (5) this implies that U is closed.

It follows from the above and the fact that $M \setminus \partial M$ is connected that $U = M \setminus \partial M$. ⊞

Now let $y \in M \setminus \partial M$. By the claim we know that y is connected to x via balls, i.e. there exist smooth embeddings $\Phi_i \colon \overline{B}^n \to M \setminus \partial M$, $i = 1, \ldots, k$ such that $x \in \Phi_1(B^n)$, $y \in \Phi_k(B^n)$ and such that for each $i = 1, \ldots, k-1$ there exists a point $x_i \in \Phi_i(B^n) \cap \Phi_{i+1}(B^n)$. We set $x_0 := x$ and $x_k := y$.



By Lemma 33.13 it remains to prove the following claim.

Claim 2. For each $i = 0, \dots, k-1$ we can move x_i to x_{i+1}.

Proof. Let $i \in \{0, \dots, k-1\}$. Note that by Proposition 33.12 (3) there exists an $r \in (0,1)$ and a diffeotopy $\Theta\colon \overline{B}^n \times [0,1] \to M$ rel $\{x \in \overline{B}^n \,|\, \|x\| \geq r\}$ with $\Theta_0 = \mathrm{id}$ and with $\Theta_1(\Phi_i^{-1}(x_i)) = \Phi_i^{-1}(x_{i+1})$. We set

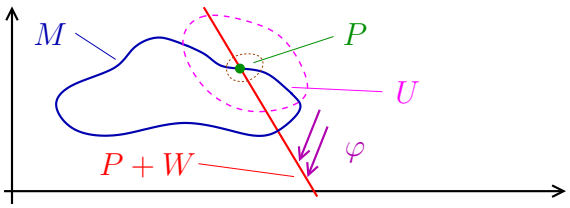
$$\begin{array}{rcl} F_i\colon M \times [0,1] & \to & M \\ (x,t) & \mapsto & \begin{cases} \Phi_i(\Theta(\Phi_i^{-1}(x),t)), & \text{if } x \in \Phi_i(B^n), \\ x, & \text{otherwise} \end{cases} \end{array}$$

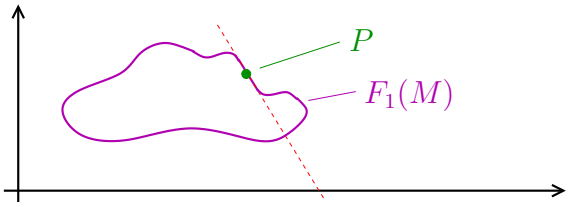
Note that F_i is well-defined and that it is smooth on the open subsets[346] $\Phi_i(B^n) \times [0,1]$ and $(M \setminus \Phi_i(\overline{B}^n_r)) \times [0,1]$ of $M \times [0,1]$. Thus F_i is smooth by the Smooth Pasting Proposition 19.23. One can now easily verify that F_i has the desired properties. ∎

33.5. **The Flattening Lemma.** We conclude this chapter with the following lemma, which is a little technical, but convenient at times.

Lemma 33.15. (Flattening Lemma) Let $M \subset \mathbb{R}^n$ be a smooth submanifold. Furthermore let $P \in M \setminus \partial M$, let $W \subset \mathbb{R}^n$ be a vector subspace and let $\varphi\colon \mathbb{R}^n \to W$ be a homomorphism such that $\varphi\colon \widetilde{\mathrm{T}}_P M \to W$ is a monomorphism. Given any open neighborhood U of $P \in M$ there exists a smooth isotopy $F\colon M \times [0,1] \to \mathbb{R}^n$ rel $(M \setminus U) \cup \{P\}$ and an open neighborhood $\widetilde{U}$ of $P \in M$ with the following properties:

(1) $F_0 = \mathrm{id}$,
(2) $\widetilde{\mathrm{T}}_P(F_1(M)) = \varphi(\widetilde{\mathrm{T}}_P M)$,
(3) $F_1(\widetilde{U}) \subset P + W$,
(4) for each $x \in M$ and each $t \in [0,1]$ the point $F(x,t)$ lies on the segment between x and $P + \varphi(x)$.





Proof. Let $M \subset \mathbb{R}^n$ be a k-dimensional smooth submanifold, let $P \in M \setminus \partial M$, let $W \subset \mathbb{R}^n$ be a vector subspace and let $\varphi\colon \mathbb{R}^n \to W$ be a homomorphism such that $\varphi\colon \widetilde{\mathrm{T}}_P M \to W$ is a monomorphism and let U be an open neighborhood of $P \in M$. We denote by $p\colon \mathbb{R}^n \to \mathbb{R}^k$

[346] It follows from the Codimension-Zero Smooth Embedding Proposition 24.16 that $\Phi_i(B^n) \times [0,1]$ is an open subset of $M \times [0,1]$. Furthermore it follows from the Compact Image Lemma 2.13, the Compact-Closed Lemma 1.21 and the fact that smooth manifolds are Hausdorff that $(M \setminus \Phi_i(\overline{B}^n_r)) \times [0,1]$ is also an open subset of $M \times [0,1]$.

the projection onto the first k coordinates. It follows from elementary linear algebra, see Exercise 34.5, that there exists a linear isomorphism $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ and a monomorphism $\chi\colon \mathbb{R}^k \to W$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathrm{T}}_P M & \xrightarrow[\cong]{\Phi} & \Phi(\widetilde{\mathrm{T}}_P M) \\ \downarrow{\scriptstyle \varphi} & & {\scriptstyle p}\downarrow{\scriptstyle (x,y)\mapsto x} \\ W & \xleftarrow{\quad\chi\quad} & \mathbb{R}^k. \end{array}$$

Since $\mathrm{D}\Phi_P = \Phi$ we can assume, after replacing M by $\Phi(M)$, that $\varphi = \chi \circ p$.

We write $P = (x_0, y_0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$. It follows almost immediately from the Local Graph Lemma 24.4 that there exists an $r > 0$ and an open neighborhood T of $y_0 \in \mathbb{R}^{n-k}$ and a smooth map $f\colon B^k_r(x_0) \to T$ such that $B^k_r(x_0) \times T \subset U$ and such that

$$M \cap (B^k_r(x_0) \times T) = \text{ Graph of } f\colon B^k_r(x_0) \to T \;=\; \{(x, f(x)) \,|\, x \in B^k_r(x_0)\}.$$

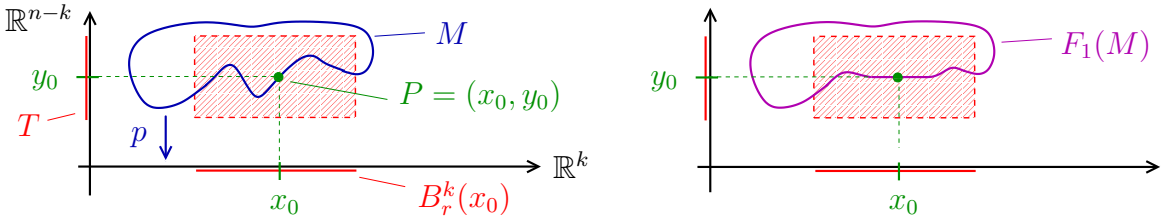
Next note that by the Smooth Transition Function Lemma 19.4 there exists a smooth map $\gamma\colon B^k_r(x_0) \to [0,1]$ such that

(a) if $\|x - x_0\| \geq \frac{r}{2}$, then $\gamma(x) = 0$,
(b) if $\|x - x_0\| \leq \frac{r}{4}$, then $\gamma(x) = 1$.

One can now easily verify that the map

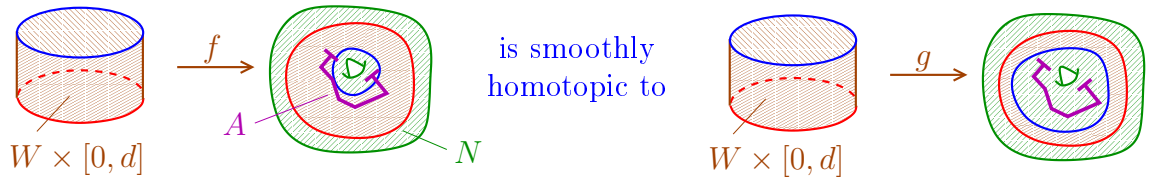
$$\begin{array}{rcl} F\colon M \times [0,1] & \to & M \\ (Q,t) & \mapsto & \begin{cases} Q, & \text{if } Q \notin (B^k_r(x_0) \times T \\ (x, f(x) + (y_0 - f(x)) \cdot \gamma(x) \cdot t), & \text{if } Q = (x, f(x)) \text{ with } x \in B^k_r(x_0) \end{cases} \end{array}$$

has the desired properties. ∎

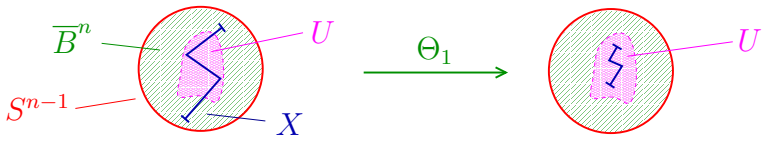


Exercises for Chapter 33.

Exercise 33.1. Let W be a compact smooth manifold without boundary and let N be a smooth manifold, possibly with boundary. Let $d > 0$ and let $f\colon W \times [0,d] \to N$ be a smooth embedding. Let A be a closed subset of M with $f(W \times \{0\}) \cap A = \varnothing$. Then the map f is smoothly isotopic rel $W \times \{0\}$ to a map $g\colon W \times [0,d] \to N$ with $g(W \times [0,d]) \cap A = \varnothing$.



Exercise 33.2. Let $n \in \mathbb{N}$, let $X \subset B^n$ be compact and let U be an open neighborhood of the origin $0 \in \overline{B}^n$. Show that there exists a diffeotopy $\Theta\colon \overline{B}^n \times [0,1] \to \overline{B}^n$ rel S^{n-1} such that $\Theta_0 = \mathrm{id}$ and such that $\Theta_1(X) \subset U$.



Exercise 33.3. Let M and N be two n-dimensional smooth manifolds. Furthermore let $F\colon M \times [0,1] \to N$ be a smooth isotopy.

(a) We suppose that there exists an open subset $V \subset N$ such that for each $t \in [0,1]$ we have $V \subset F_t(M \setminus \partial M)$. Show that the map

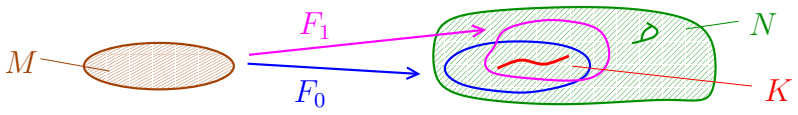
$$\begin{array}{rcl} \varphi\colon V \times [0,1] & \to & M \\ (P,t) & \mapsto & F_t^{-1}(P) \end{array}$$

is a smooth isotopy.
Hint. To show smoothness of φ you could use the Inverse Function Theorem 24.1 to show that the map $V \times [0,1] \to M \times [0,1]$ given by $(P,t) \mapsto (\varphi(P,t),t)$ is a local diffeomorphism.

(b) Let $K \subset N$ be a submanifold such that for each $t \in [0,1]$ we have $K \subset F_t(M \setminus \partial M)$. Show that the map

$$\begin{array}{rcl} K \times [0,1] & \to & M \\ (P,t) & \mapsto & F_t^{-1}(P) \end{array}$$

is a smooth isotopy.



Exercise 33.4. Let $n \in \mathbb{N}_0$ and let $k \in \{1,\dots,n\}$. We write

$$\mathrm{GL}(n,k) \;=\; \underbrace{\text{set of } k\text{-tuples of linearly independent vectors of } \mathbb{R}^n}_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n = \mathbb{R}^{nk}}.$$

Show that $\mathrm{GL}(n,k)$ is an open subset of $\mathbb{R}^{nk}$.

Exercise 33.5. Let M be a connected smooth manifold.

(a) Let P, Q be points in $M \setminus \partial M$.
  (i) Show that if $\dim(M) \geq 2$, then there exists a diffeomorphism $f\colon M \to M$ that swaps P and Q and that is diffeotopic to the identity.
  (ii) For which 1-dimensional smooth manifolds does the conclusion of (i) hold?

(b) Let C be a component of ∂M and let P, Q be two points in C. Show that there exists a diffeomorphism $f\colon M \to M$ that swaps P and Q.
Hint. Use the Smooth Collar Neighborhood Theorem 28.3.

Exercise 33.6. Show that $\mathbb{R}/\mathbb{Q}$ is compact, that it is homogeneous and that it is not a topological manifold.

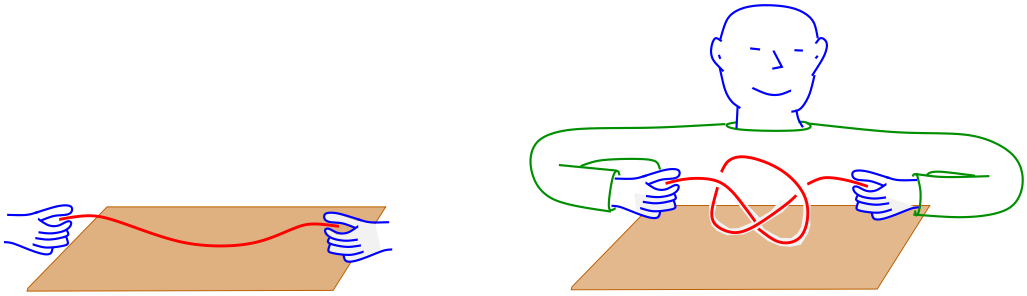
Exercise 33.7. Show that the topological space $\prod_{k\in\mathbb{N}} \{0,1\} = \{0,1\}^{\mathbb{N}}$ is homogeneous and show that it is not a topological manifold.
Remark. Note that by Tychonoff's Theorem 5.9 and the Product Topology Properties Proposition 5.8 we know that $\prod_{k\in\mathbb{N}} \{0,1\} = \{0,1\}^{\mathbb{N}}$ is compact and Hausdorff. Thus we see that there are compact Hausdorff spaces that are homogeneous but that are not topological manifolds.

Exercise 33.8.

(a) Let $n \geq 2$ and let M be a connected n-dimensional smooth manifold. Suppose we are given distinct points $P_1, \ldots, P_m$ and distinct points $Q_1, \ldots, Q_m$ in $M \setminus \partial M$. Show that there exists a diffeotopy with compact support, rel a neighborhood of ∂M, from the identity to a diffeomorphism $\Phi\colon M \to M$ with $\Phi(P_i) = Q_i$ for $i = 1, \ldots, m$.
Hint. Apply the Homogeneous-via-Diffeotopies Proposition 33.14 iteratively.
Remark. The same argument also works for topological manifolds where the conclusion is that we get an isotopy with compact support.

(b) Show that the conclusion of (a) is wrong for 1-dimensional smooth manifolds.

Exercise 33.9. Can you pick up a loose unknotted string with both hands and turn it into a knotted string as shown in the figure below, while holding on to the ends of the string? Either way, which smooth isotopy types of knotted strings can you produce this way?



Exercise 33.10. Let M and N be smooth manifolds and let $F\colon M \times \mathbb{R} \to N$ be a map such that each $F_t\colon M \to N$ is a (proper) smooth embedding. Show that

$$\begin{array}{rcl} M \times \mathbb{R} & \to & N \times \mathbb{R} \\ (P,t) & \mapsto & (F(P,t),t) \end{array}$$

is a (proper) smooth embedding.

34. Classification of certain diffeomorphisms up to diffeotopy

Given a smooth manifold one would like, ideally, to classify all diffeomorphisms up to diffeotopy and given two smooth manifolds one would like, ideally, to classify all smooth embeddings from one into the other up to smooth isotopy. Before we consider explicit examples let us introduce a few relevant definitions.

Definition. Let M and N be two smooth manifolds of the same dimension.

(1) We denote by $\mathrm{Diff}(M,N)$ the set of diffeomorphisms from M to N.
(2) By the Smooth Isotopy Transitivity Proposition 33.3 we know that "diffeotopy" defines an equivalence relation on $\mathrm{Diff}(M,N)$. We write

$$\widetilde{\mathrm{Diff}}(M,N) := \mathrm{Diff}(M,N)/\sim \qquad \text{where } f \sim g \text{ if } f \text{ and } g \text{ are diffeotopic.}$$

(3) If M and N are oriented, then it makes sense to restrict one selves to orientation-preserving diffeomorphisms and we obtain $\mathrm{Diff}^+(M,N)$ and $\widetilde{\mathrm{Diff}}^+(M,N)$.

Furthermore, given a single smooth manifold M we write $\mathrm{Diff}(M) := \mathrm{Diff}(M,N)$ and accordingly for all the related notions.

Remark. If M is a smooth manifold, then $\mathrm{Diff}(M)$ is evidently a group. Furthermore it is not difficult to see that this group structure descends to a group structure on $\widetilde{\mathrm{Diff}}(M)$. This group is also called the mapping class group $\mathrm{MCG}(M)$. We will study it in greater detail in Section **??**.

Definition. Let M and N be two smooth manifolds.

(1) We denote by $\mathrm{Emb}(M,N)$ the set of smooth embeddings $M \to N$.
(2) By the Smooth Isotopy Transitivity Proposition 33.3 we know that "smooth isotopy" defines an equivalence relation on $\mathrm{Emb}(M,N)$. We write

$$\widetilde{\mathrm{Emb}}(M,N) := \mathrm{Emb}(M,N)/\sim \qquad \text{where } f \sim g \text{ if } f \text{ and } g \text{ are smoothly isotopic.}$$

(3) If M and N are oriented and of the same dimension, then it makes sense to restrict one selves to orientation-preserving smooth embeddings and we obtain $\mathrm{Emb}^+(M,N)$ and $\widetilde{\mathrm{Emb}}^+(M,N)$.

For completeness' sake we state the following rather obvious lemma.

Lemma 34.1. Let M and N be two smooth manifolds and let $\varphi\colon A \to M$ and $\psi\colon N \to B$ be diffeomorphisms between smooth manifolds.

(1) The maps

$$\begin{array}{rcl} \mathrm{Diff}(M,N) & \to & \mathrm{Diff}(A,B) \\ f & \mapsto & \psi \circ f \circ \varphi \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \widetilde{\mathrm{Diff}}(M,N) & \to & \widetilde{\mathrm{Diff}}(A,B) \\ [f] & \mapsto & [\psi \circ f \circ \varphi] \end{array}$$

are bijections. If all smooth manifolds are oriented and if φ and ψ are orientation-preserving, then the obvious analogues for Diff^+ and $\widetilde{\mathrm{Diff}}^+$ also hold.
(2) The analogues for Emb instead of Diff also hold.

Proof. The lemma follows immediately from Lemma 33.2. ∎

Given two smooth manifolds M and N it is in general surely too difficult to describe $\mathrm{Diff}(M,N)$ and $\mathrm{Emb}(M,N)$ in a meaningful way. It is more realistic to try to get a handle on $\widetilde{\mathrm{Diff}}(M,N)$ and $\widetilde{\mathrm{Emb}}(M,N)$. But even this problem becomes very hard very quickly.

Example. In the figure below we see two smooth embeddings $f,g\colon S^1\to\mathbb{R}^3$. It is perhaps intuitively clear that the two smooth embeddings are not smoothly isotopic, i.e. that they represent different points in $\widetilde{\mathrm{Emb}}(S^1,\mathbb{R}^3)$. But a proper proof will have to wait till Proposition 59.6.

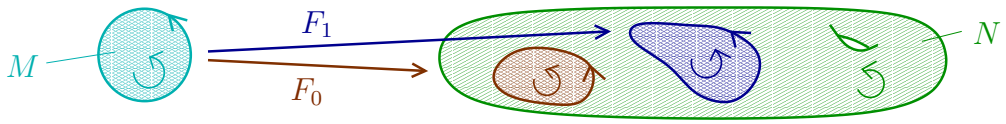


The example shows that if at this stage we want to give a classification we will need to restrict ourselves to fairly basic situations. In fact in this chapter we will consider the following:

(1) diffeomorphisms of $\mathbb{R}^n$,
(2) smooth embeddings $\overline{B}^n\to\mathbb{R}^n$,
(3) smooth embedding $\overline{B}^k\to\mathbb{R}^n$ for $k<n$,
(4) diffeomorphisms of S^1.

34.1. **Diffeotopies and orientations.** We start out with the following proposition, which might be intuitively obvious, but which is somewhat subtle to prove.

Proposition 34.2. (Smooth Isotopy-Orientation Proposition) Let M and N be oriented connected smooth manifolds of the same dimension and let $F\colon M\times[0,1]\to N$ be a smooth isotopy.[347] Then either all smooth embeddings $F_t\colon M\to N$ are orientation-preserving or all F_t are orientation-reversing.



Proof. Let M and N be oriented connected n-dimensional smooth manifolds and let $F\colon M\times[0,1]\to N$ be a smooth isotopy. We consider the sets

$$\begin{aligned} A &:= \{(x,t)\in M\times[0,1]\,|\,\mathrm{D}(F_t)_x\colon \mathrm{T}_xM\to\mathrm{T}_{F_t(x)}N \text{ is orientation-preserving}\}\\ B &:= \{(x,t)\in M\times[0,1]\,|\,\mathrm{D}(F_t)_x\colon \mathrm{T}_xM\to\mathrm{T}_{F_t(x)}N \text{ is orientation-reversing}\}. \end{aligned}$$

We need to show that $A=M\times[0,1]$ or $B=M\times[0,1]$. Since M is connected we know, by the Product Topology Properties Proposition 5.6 (3), that $M\times[0,1]$ is also connected. Thus it suffices to prove the following claim.

Claim. The subsets A and B are both open subsets of $M\times[0,1]$.

Proof. We will show that A is open. The proof that B is open is basically the same. Now let $(x,t)\in A$. We pick an orientation-preserving chart[348] $\Psi\colon V\to\widetilde{V}$ for $F_t(x)\in N$.

[347] The case of a diffeotopy is of course an important special case.

[348] Here we equip the open subset $\widetilde{V}$ of $\mathbb{R}^n$ with the standard orientation.

Since $F\colon M \times [0,1] \to N$ is in particular continuous we know that $F^{-1}(V)$ is open. Since $(x,t) \in F^{-1}(V)$ there exists an open subset of the form $U \times I$ of $M \times [0,1]$ such that $(x,t) \in U \times I \subset F^{-1}(V)$. After possibly making U even smaller we can assume that there exists an orientation-preserving chart $\Phi\colon U \to \widetilde{U}$ for M.

Claim. The map[349]

$$\begin{array}{rcl} \theta\colon U \times I & \to & \mathbb{R} \setminus \{0\} \\ (y,s) & \mapsto & \det \big(\underbrace{\mathrm{D}(\Psi \circ F_s \circ \Phi^{-1})_{\Phi(y)}}_{n \times n\text{-matrix}} \big) \end{array}$$

has the following two properties:

(1) For $(y,s) \in U \times I$ we have $(y,s) \in A$ if and only if $\theta(y,s) > 0$.
(2) The map θ is continuous.

Proof.

(1) Let $(y,s) \in U \times I$. By the Tangent Space-Functor Proposition 23.10 we have

$$\mathrm{D}(\Psi \circ F_s \circ \Phi^{-1})_{\Phi(y)} = \mathrm{D}\Psi_{F_s(y)} \circ \mathrm{D}(F_s)_y \circ \mathrm{D}(\Phi^{-1})_{\Phi(y)}.$$

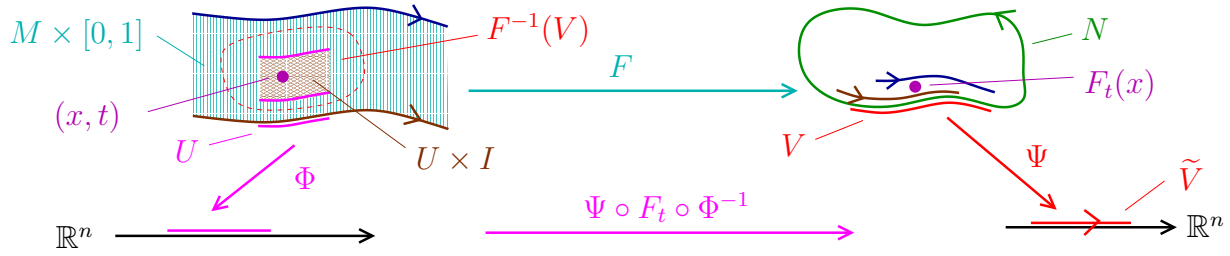
Since the charts Ψ and Φ are by our choice orientation-preserving we know that $\mathrm{D}\Psi_{F_s(y)}\colon \mathrm{T}_{F_s(y)}M \to \mathrm{T}_{\Psi(F_s(y))}\widetilde{V}$ and $\mathrm{D}(\Phi^{-1})_{\Phi(y)}\colon \mathrm{T}_{\Phi(y)}\widetilde{U} \to \mathrm{T}_y V$ are orientation-preserving. It follows from the elementary the Orientation-Composition Lemma 25.3 that the composition is orientation-preserving if and only if $\mathrm{D}(F_s)_y$ is orientation-preserving. But that is what we needed to show.

(2) Note that for each $\widetilde{y} \in \widetilde{U}$ and each $s \in I$ we have

$$(\Psi \circ F_s \circ \Phi^{-1})(\widetilde{y}) \;=\; (\Psi \circ F \circ \underbrace{(\Phi^{-1} \times \mathrm{id}_I)}_{\text{map } \widetilde{U} \times I \to U \times I})(\widetilde{y}, s).$$

Since $\Psi \circ F \circ (\Phi^{-1} \times \mathrm{id}_{[0,1]})$ is a smooth map from $\widetilde{U} \times I \subset \mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R}^n$ we see that the determinant of the differential is a continuous map on $\widetilde{U} \times I$. But this implies that θ is indeed continuous. ⊞

Since $(x,t) \in A$ we obtain from the claim that $\theta(x,t) > 0$. By the claim we also know that θ is continuous. Since $U \times I$ is connected we see that θ is positive on all of $U \times I$. But again by the claim this means that $U \times I \subset A$. It follows from the JH-Lemma 1.7 that A is indeed open. ■



[349] As usual we use the Differential-equals-Differential Lemma 23.4 and the Tangent Space-Isomorphism Proposition 23.16 to freely go back and forth between the different notions of differentials.

34.2. Diffeotopies of $\mathbb{R}^n$ and smooth embeddings of balls into $\mathbb{R}^n$.

Proposition 34.3. (Diffeotopies-of-$\mathbb{R}^n$-Classification Proposition) Let $n \in \mathbb{N}_0$.

(1) Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism.
   (a) If f is orientation-preserving, then f is diffeotopic to the identity map.
   (b) If f is orientation-reversing, then f is diffeotopic to the reflection $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ that is given by $(x_1, x_2, \dots, x_n) \mapsto (-x_1, x_2, \dots, x_n)$.

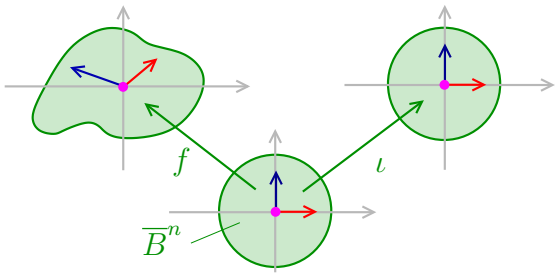
(2) Let $f\colon \overline{B}^n \to \mathbb{R}^n$ be a smooth embedding.
   (a) If f is orientation-preserving, then f is smoothly isotopic to the inclusion ι.
   (b) If f is orientation-reversing, then f is smoothly isotopic to the reflection $\rho|_{\overline{B}^n}$.
   The same conclusion also holds for smooth embeddings $f\colon \overline{B}^n \to B^n_r(0)$ into a smooth ball of radius $r > 1$.

(3) Let $k < n$. Every smooth embedding $f\colon \overline{B}^k \to \mathbb{R}^n$ is smoothly isotopic to the "standard" smooth embedding given by

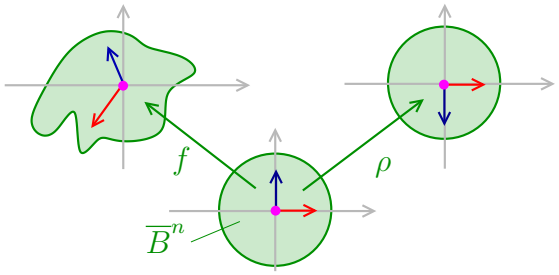
$$\begin{array}{rcl} \iota\colon \overline{B}^k & \to & \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k} \\ x & \mapsto & (x, 0). \end{array}$$

   The same conclusion also holds for smooth embeddings $f\colon \overline{B}^k \to B^n_r(0)$ into a smooth ball of radius $r > 1$.

In any of the above cases we have the following refinement: if $f(0) = 0$, then we can find a diffeotopy rel $\{0\}$ respectively a smooth isotopy rel $\{0\}$.



f orientation-preserving



f orientation-reversing

Remark. In the Smooth Ball Embedding Theorem 37.8 and in the Lower-Dimensional Smooth Ball Embedding Theorem 37.10 we will classify smooth embeddings of closed balls into smooth manifolds. This result can be viewed as a generalization of the Diffeotopies-of-$\mathbb{R}^n$-Classification Proposition 34.3 (2).

Proof. We start out with a few preparations:

- In the following we use on several occasions the fact, proved in the Smooth Isotopy Transitivity Proposition 33.3, that being diffeotopic and being smoothly isotopic is an equivalence relation.
- Given a matrix $A \in \mathrm{M}(m \times n, \mathbb{R})$ we denote by $\mu(A)\colon \mathbb{R}^n \to \mathbb{R}^m$ the map that is given by multiplication by A.
- We use the Differential-equals-Differential Lemma 23.4 as an excuse not to distinguish in our notation between the differential as a matrix and as a map.

Now we turn to the actual proof.

(1) First let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be an orientation-preserving diffeomorphism. Evidently there is a diffeotopy to a map that sends the origin to the origin. So by the Smooth Isotopy Transitivity Proposition 33.3 we might as well assume that $f(0) = 0$. We write $A := \mathrm{D}f_0$.

Claim. The map f is diffeotopic to the map $\mu(A)$.

Proof. Since $\mathbb{R}^n$ is a convex subset of $\mathbb{R}^n$ that contains the origin and since $f(0) = 0$ we know by the Map-as-Matrix Multiplication Proposition 19.14 that there exists a smooth map $\Theta\colon \mathbb{R}^n \to \mathrm{M}(n \times n, \mathbb{R})$ such that $\Theta(0) = A = \mathrm{D}f_0$ and such that for all $x \in \mathbb{R}^n$ we have $f(x) = \Theta(x) \cdot x$. We consider the map

$$\begin{array}{rcl} F\colon \mathbb{R}^n \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & \underbrace{\Theta(t \cdot x)}_{\in \mathrm{M}(n\times n,\mathbb{R})} \cdot x. \end{array}$$

This map is evidently smooth. Furthermore note that we have $F_1 = f$ and note that $F_0 = \mu(A) = \mu(\mathrm{D}f_0)$. It remains to show that each F_t is a diffeomorphism. This is clear for $t = 0$. For $t \neq 0$ we see that

$$F_t(x) \;=\; \Theta(t \cdot x) \cdot x \;=\; \tfrac{1}{t} \cdot \Theta(t \cdot x) \cdot (t \cdot x) \;=\; \tfrac{1}{t} \cdot f(t \cdot x).$$

Thus we see that F_t is the composition of the three diffeomorphisms $x \mapsto t \cdot x$, $y \mapsto f(y)$ and $z \mapsto \frac{1}{t} z$. Hence F_t is a diffeomorphism itself. ⊞

Next note that our hypothesis that the diffeomorphism f is orientation-preserving implies by the Basics-of-Orientations Lemma 25.13 that $\det(A) > 0$, in other words, $A \in \mathrm{GL}_+(n, \mathbb{R})$. By the Diffeotopies-via-Matrices Lemma 33.8 we know that multiplication by A is diffeotopic to the identity. Thus we are done by the Smooth Isotopy Transitivity Proposition 33.3.

Now we consider the case that f is orientation-reversing. There are two approaches to dealing with this situation. We can run exactly the same argument as above, except that we now use that $\det(\mathrm{D}f_0) < 0$ and that therefore $\mathrm{D}f_0$ lies in the same path-component of $\mathrm{GL}(n, \mathbb{R})$ as the orthogonal matrix representing the reflection ρ.

Alternatively we can note that $\rho \circ f$ is orientation-preserving. We can apply the above result to $\rho \circ f$ and we obtain a diffeotopy $F\colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ with $F_0 = \rho \circ f$ and $F_1 = \mathrm{id}$. Since $\rho \circ \rho = \mathrm{id}$ we see that $\rho \circ F$ is now the desired diffeotopy from f to ρ.

(2) Let $f\colon \overline{B}^n \to \mathbb{R}^n$ be a smooth embedding. It follows by basically exactly the same argument as in (1) that the smooth embedding is either smoothly isotopic to the standard inclusion or to the reflection. We leave the few remaining details to the reader.

Now assume that we are given an $r > 1$ and an orientation-preserving smooth embedding $f\colon \overline{B}^n \to B^n_r(0)$. As we saw in Exercise 19.1, using the Smooth Transition Function Lemma 19.4 one can easily find a diffeomorphism $\varphi\colon [0, r) \to [0, \infty)$ which is the identity on $[0, 1]$. This diffeomorphism φ gives rise to the orientation-preserving

diffeomorphism

$$\begin{array}{rcl}\Phi\colon B_r^n(0) & \to & \mathbb{R}^n \\ v & \mapsto & \varphi(\|v\|)\cdot v\end{array}$$

which is the identity on $\overline{B}^n$. By the above discussion there exists a smooth isotopy $H\colon \overline{B}^n \times [0,1] \to \mathbb{R}^n$ with $H_0 = \Phi \circ f$ and such that H_1 is the inclusion. The desired smooth isotopy is now given by $(x,t) \mapsto \Phi^{-1}(H(x,t))$. The case that f is orientation-reversing is handled in the same way.

(3) Let $k < n$ and let $f\colon \overline{B}^k \to \mathbb{R}^n$ be a smooth embedding. As in (1) we might as well assume that $f(0) = 0$. Throughout the following discussion we make use of the following two projections:

$$\begin{array}{rcl}p\colon \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k} & \to & \mathbb{R}^k \\ (x,y) & \mapsto & x\end{array} \quad \text{and} \quad \begin{array}{rcl}q\colon \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k} & \to & \mathbb{R}^{n-k} \\ (x,y) & \mapsto & y.\end{array}$$

Claim 1. The smooth embedding $f\colon \overline{B}^k \to \mathbb{R}^n$ is smoothly isotopic to a smooth embedding $g\colon \overline{B}^k \to \mathbb{R}^n$ such that $\mathrm{D}(p\circ g)_0\colon \mathrm{T}_0\,\overline{B}^k \to \mathbb{R}^k$ is an isomorphism.

Proof. Since f is a smooth embedding we know that the matrix $\mathrm{D}f_0 \in \mathrm{M}(n\times k)$ has rank k. It follows from $k \leq n$ and some elementary linear algebra that there exists a matrix $A \in \mathrm{GL}_+(n,\mathbb{R})$ such that the first k rows of $A \cdot \mathrm{D}f_P$ have rank k. This is equivalent to the statement that

$$p \circ \underbrace{\mathrm{D}(\mu(A)\circ f)_0}_{\text{represented by } A\cdot \mathrm{D}f_0} = \mathrm{D}(p\circ\mu(A)\circ f)_0\colon\ \mathrm{T}_0\,\overline{B}^k \ \to\ \mathbb{R}^k$$

is an isomorphism. By the Matrix Group Path-Component Proposition 2.37 we know that $\mathrm{GL}_+(n)$ is path-connected. It follows from this fact, together with the Smooth Path-Connectivity Proposition 19.29, that there exists a smooth map $\gamma\colon [0,1] \to \mathrm{GL}_+(n)$ with $\gamma(0) = \mathrm{id}$ and $\gamma(1) = A$. Similar to the Diffeotopies-via-Matrices Lemma 33.8 (1a) we know consider the smooth isotopy

$$\begin{array}{rcl}F\colon \overline{B}^k \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & \gamma(t)\cdot f(x).\end{array}$$

⊞

Claim 2. The smooth embedding $g\colon \overline{B}^k \to \mathbb{R}^n$ is smoothly isotopic to a smooth embedding $h\colon \overline{B}^k \to \mathbb{R}^n$ such that $p\circ h\colon \overline{B}^k \to \mathbb{R}^k$ is a smooth embedding.

Proof. Recall that the map $\mathrm{D}(p\circ g)_0\colon \mathrm{T}_0\,\overline{B}^k \to \mathbb{R}^k$ is an isomorphism. It follows almost immediately from the Inverse Mapping Theorem 24.1 that there exists an $r > 0$ such that $p\circ g\colon \overline{B}^k_r(0) \to \mathbb{R}^k$ is a smooth embedding. The desired smooth isotopy is now given by

$$\begin{array}{rcl}G\colon \overline{B}^k \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & g(x\cdot(1-t+t\cdot r)).\end{array}$$

⊞

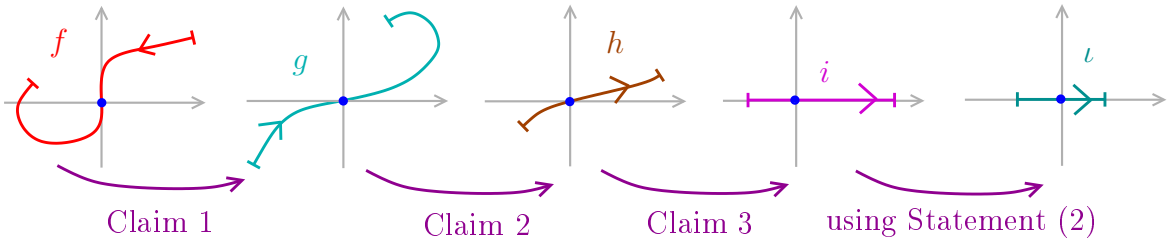
Claim 3. The smooth embedding $h\colon \overline{B}^k \to \mathbb{R}^n$ is smoothly isotopic to a smooth embedding $i\colon \overline{B}^k \to \mathbb{R}^n$ that takes values in $\mathbb{R}^k \times \{0\}$.

Proof. We consider the map

$$\begin{array}{rcl} H\colon \overline{B}^k \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & ((p\circ h)(x), (1-t)\cdot(q\circ h)(x)). \end{array}$$

Since $p\circ h\colon \overline{B}^k \to \mathbb{R}^k$ is a smooth embedding we see that each $H_t\colon \overline{B}^k \to \mathbb{R}^n$ is a smooth embedding. In particular H is a smooth isotopy. Note that $H_0 = h$. Finally note that H_1 takes values in $\mathbb{R}^k \times \{0\}$. ⊞

Note that it follows from (2) that there exists a smooth isotopy $I\colon \overline{B}^k \times [0,1] \to \mathbb{R}^k$ from $p\circ i$ to $\mathrm{id}_{\overline{B}^k}$. Finally note that this smooth isotopy I evidently gives rise to a smooth isotopy $\overline{B}^k \times [0,1] \to \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ from i to the standard smooth embedding $\iota\colon \overline{B}^k \to \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ that is given by $x \mapsto (x,0)$.



Now assume that we are given an $r > 1$ and an orientation-preserving smooth embedding $f\colon \overline{B}^k \to B^n_r(0)$. As in (2) we can reduce this case to the case of a smooth embedding $\overline{B}^k \to \mathbb{R}^n$, so we are done by the above discussion. ■

The following is our first classification result of this chapter.

Corollary 34.4. Given $n \in \mathbb{N}$ and given $k \in \{1,\dots,n\}$ we consider the maps

$$\begin{array}{rcl} \rho\colon \mathbb{R}^n & \to & \mathbb{R}^n \\ (x_1,\dots,x_n) & \mapsto & (-x_1,x_2,\dots,x_n) \end{array} \quad \text{and} \quad \begin{array}{rcl} \iota_k\colon \overline{B}^k & \to & \mathbb{R}^n \\ x & \mapsto & (x,0) \end{array}$$

With this notation the following statements hold:

(1) For $n \in \mathbb{N}$ we have $\widetilde{\mathrm{Diff}}^+(\mathbb{R}^n) = \{[\mathrm{id}]\}$ and $\widetilde{\mathrm{Diff}}(\mathbb{R}^n) = \{[\mathrm{id}],[\rho]\}$.

(2) For $n \in \mathbb{N}$ we have $\widetilde{\mathrm{Emb}}^+(\overline{B}^n,\mathbb{R}^n) = \{[\iota_n]\}$ and $\widetilde{\mathrm{Diff}}(\overline{B}^n,\mathbb{R}^n) = \{[\iota_n],[\rho|_{\overline{B}^n}]\}$.

(3) For $k < n$ we have $\widetilde{\mathrm{Emb}}(\overline{B}^k,\mathbb{R}^n) = \{[\iota_k]\}$.

Proof. The three statements follow immediately from the combination of Propositions 34.2 and 34.3. ■

34.3. **Smooth homotopies and diffeotopies of S^1.** In the Diffeotopies-of-$\mathbb{R}^n$-Classification Proposition 34.3 we obtained a very good understanding of diffeotopies of $\mathbb{R}^n$. In this section we will use the statement for $n = 1$ to classify the diffeomorphisms of S^1 up to diffeotopy. More precisely we will prove the following proposition.

Proposition 34.5. (Diffeotopies-of-S^1-Classification Proposition) Let $f\colon S^1 \to S^1$ be a diffeomorphism.

(1) If f is orientation-preserving, then f is diffeotopic to the identity.

(2) If f is orientation-reversing, then f is diffeotopic to the reflection $\rho\colon S^1 \to S^1$ that is given by $(x,y) \mapsto (x,-y)$.

Remark.

(1) In Lemma 50.1 we proved an analogous statement for self-homeomorphisms of S^1.
(2) In Exercise 35.5 we will use the Flow Theorem 35.13 to give an alternative proof of the Diffeotopies-of-S^1-Classification Proposition 34.5.
(3) The Smale Theorem 146.7 and the Cerf Theorem 58.5 say that the obvious analogues of the Diffeotopies-of-S^1-Classification Proposition 34.5 hold for S^2 and S^3. But in Proposition **??** and Theorem **??** we will see that in general the naive analogues in higher dimensions do *not* hold. More precisely, there exist infinitely many $n \in \mathbb{N}_{\geq 6}$ such that the sphere S^n admits an orientation-preserving self-diffeomorphism that is not diffeotopic to the identity.

Before we turn to the proof of the Diffeotopies-of-S^1-Classification Proposition 34.5 let us first state and prove the following almost immediate useful corollary.

Corollary 34.6. (Diffeomorphisms-of-S^1-Extension Corollary) Every diffeomorphism $f\colon S^1 \to S^1$ extends to a diffeomorphism of $\overline{B}^2$.

Remark.

(1) In Exercise 5.35 we encountered the Alexander trick, which shows that every homeomorphism of S^n extends to a homeomorphism of $\overline{B}^{n+1}$.
(2) In contrast to the setting of homeomorphisms the analogue of the Diffeomorphisms-of-S^1-Extension Corollary 34.6 does *not* hold in all dimensions. More precisely, in Proposition **??** we will see that there exists a diffeomorphism $f\colon S^6 \to S^6$ that does not extend to a diffeomorphism of $\overline{B}^7$.

Proof of Corollary 34.6 assuming Proposition 34.5. Let $f\colon S^1 \to S^1$ be a diffeomorphism. By (1) we know that there exists a diffeotopy $F\colon S^1 \times [0,1] \to S^1$ from $F_0 = f$ to the identity $F_1 = \mathrm{id}$ or to the reflection $F_1 = \rho$. By the Trivial-Near-The-Ends-Lemma 33.4 we can arrange that F is trivial near the ends. In particular we can assume that there exists an $\epsilon > 0$ such that $F_t = F_1$ for all $t \in [1-\epsilon, 1]$. We consider the following map:

$$\begin{array}{rcl} \Phi\colon \overline{B}^2 & \to & \overline{B}^2 \\ P = (x,y) & \mapsto & \begin{cases} P = (x,y), & \text{if } \|P\| \leq \frac{1}{2} \text{ and if } f \text{ is orientation-preserving,} \\ (x,-y), & \text{if } \|P\| \leq \frac{1}{2} \text{ and if } f \text{ is orientation-reversing,} \\ \|P\| \cdot F\big(\frac{P}{\|P\|}, 2 - 2\cdot\|P\|\big), & \text{if } \|P\| \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

It follows easily from the Smooth Pasting Proposition 19.23 that the above map $\Phi\colon \overline{B}^2 \to \overline{B}^2$ is indeed smooth. Furthermore in Exercise 34.6 we will write down an explicit inverse to Φ and we will show that the inverse is smooth. This implies that Φ is indeed a diffeomorphism. ■

The proof of the Diffeotopies-of-S^1-Classification Proposition 34.5 requires some preparations, which are IMHO actually interesting in their own right. First we introduce the following notation:

Notation. Throughout this section we work with the map

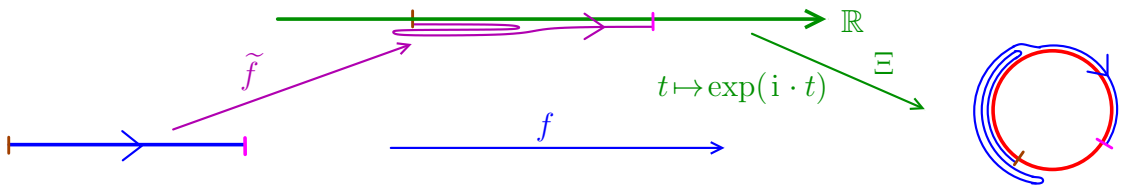
$$\begin{array}{rcl} \Xi\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & \exp(\mathrm{i}\cdot t). \end{array}$$

Lemma 34.7. Let $t \in \mathbb{R}$. The restriction of Ξ to $(t-\pi, t+\pi) \to S^1 \setminus \{-\Xi(t)\}$ is a diffeomorphism.

Proof. The lemma can be proved easily using the Smooth Embedding Theorem 24.10. Alternatively, as we saw in Exercise 19.6, it is not difficult to prove the lemma with one's bare hands. ■

Definition. Let $I \subset \mathbb{R}$ be an interval and let $f\colon I \to S^1$ be a map. A lift of f is a map[350] $\widetilde{f}\colon I \to \mathbb{R}$ such that $\Xi \circ \widetilde{f} = f$, in other words, such for all $t \in I$ we have

$$f(t) \;=\; \exp\big(\mathrm{i}\cdot \widetilde{f}(t)\big).$$



Remark. On pages 1078 and 1080 we will greatly generalize the notion of a lift.

Lemma 34.8. Let $I \subset \mathbb{R}$ be an interval and let $f\colon I \to S^1$ be a map. If f is smooth, then every lift is also smooth.

Proof. This statement follows from the Smooth Pasting Proposition 19.23 (3) together with the fact, implicitly shown in Lemma 34.7, that $\Xi\colon \mathbb{R} \to S^1$ is a local diffeomorphism. ■

In the proof of the Diffeotopies-of-S^1-Classification Proposition 34.5 we will also make use of the following very interesting lemma.

Lemma 34.9. (Lifting from S^1-to-$\mathbb{R}$ Lemma) Let $I \subset \mathbb{R}$ be an interval and let $x_0 \in I$. Given any map $f\colon I \to S^1$ and given any $\widetilde{y}_0 \in \mathbb{R}$ with $f(x_0) = \exp(\mathrm{i}\cdot \widetilde{y}_0)$ there exists a *unique* lift $\widetilde{f}\colon I \to \mathbb{R}$ of f with $\widetilde{f}(x_0) = \widetilde{y}_0$.

Remark. The Lifting from S^1-to-$\mathbb{R}$ Lemma 34.9 is an immediate consequence of the general theory of coverings that we will study in great detail in Chapter 48. More precisely, it follows from the discussion on page 1074 and the Actions-Covering Proposition 48.9 that the map $\Xi\colon \mathbb{R} \to S^1$ is a covering. The statement of the Lifting from S^1-to-$\mathbb{R}$ Lemma 34.9 follows now from a slight variation on Proposition 48.11. Since at this stage we do not assume that the reader is familiar with covering theory we will now give a self-contained proof of the Lifting from S^1-to-$\mathbb{R}$ Lemma 34.9.

Proof. Our first claim already deals with the uniqueness statement of the proposition.

[350] Evidently this map is required to be continuous.

Claim 1. Let J be an interval, let $f\colon J \to S^1$ be a map and let $\widetilde{\varphi}, \widetilde{\psi}\colon J \to \mathbb{R}$ be two lifts of f. If $\widetilde{\varphi}$ and $\widetilde{\psi}$ agree at some point $t_0 \in J$, then they agree everywhere.

Proof. We consider the map

$$\begin{array}{rccc} \theta\colon & J & \to & \mathbb{R} \\ & s & \mapsto & \widetilde{\varphi}(s) - \widetilde{\psi}(s). \end{array}$$

By hypothesis we have $\theta(t_0) = 0$. Furthermore note that for each $t \in J$ we have

$$\underbrace{\exp\big(\mathrm{i}\cdot(\widetilde{\varphi}(s) - \widetilde{\psi}(s))\big)}_{=\exp(\mathrm{i}\cdot\theta(t))} = \exp\big(\mathrm{i}\cdot\widetilde{\varphi}(s)\big)\cdot\exp\big(\mathrm{i}\cdot\widetilde{\psi}(s)\big)^{-1} \underset{\substack{\uparrow \\ \text{since } \widetilde{\varphi},\widetilde{\psi}\colon J\to\mathbb{R} \text{ are lifts of } f}}{=} f(s)\cdot f(s)^{-1} = 1.$$

It follows that for every $s \in J$ we have $\exp(2\pi\,\mathrm{i}\cdot(\widetilde{\varphi}(s) - \widetilde{\psi}(s))) = 1$. This shows that the map $s \mapsto \mu(s) := \widetilde{\varphi}(s) - \widetilde{\psi}(s)$ takes values in $\mathbb{Z}$. Since J is connected it follows from the Connected-to-Discrete Lemma 2.27 that the map μ is constant. Since the map μ vanishes at t_0 we see that it is zero everywhere. ⊞

Claim 2. Let $b \in \mathbb{R}$ and let $s \in \mathbb{R} \cup \{\infty\}$ with $s > b$. If for every $c \in [b, s)$ there exists a lift of $f|_{[b,c]}$, then there exists also a lift of $f_{[b,s)}$.

Proof. It follows from the previous claim that the lifts on the various intervals agree on the overlaps. Thus they define a lift on all of $[b, s)$. ⊞

To simplify the notation we now only deal with the case that the interval is given by $I = [x_0, \infty)$. The proof of the Lifting from S^1-to-$\mathbb{R}$ Lemma 34.9 for all other types of intervals is done in a very similar fashion. We set

$$S := \{s \in [x_0, \infty) \,|\, \text{there exists a lift of } \varphi|_{[x_0,s]}\}.$$

By Claim 2 it suffices to prove the following claim.

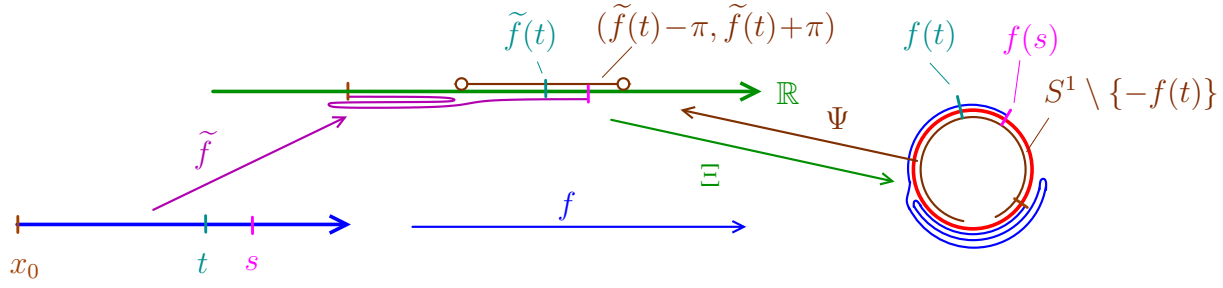
Claim 3. $S = [x_0, \infty)$.

Proof. Note that $x_0 \in S$. Since $[x_0, \infty)$ is connected if suffices to show that S is open and closed in $[x_0, \infty)$. It follows from elementary arguments, see the Interior-Closure Lemma 1.9 (5), that it suffices to show that for every $s \in [x_0, \infty)$ with $[x_0, s) \subset S$ there exists an $\epsilon > 0$, such that $[x_0, s + \epsilon) \subset S$.

Thus suppose we are given an $s \in [x_0, \infty)$ with $[x_0, s) \subset S$. By Claim 2 there exists a lift $\widetilde{f}\colon [x_0, s) \to \mathbb{R}$. We pick $t \in [x_0, s)$ such that $f([t, s]) \subset S^1 \setminus \{-f(s)\}$. Recall that by Lemma 34.7 we know that $\Xi\colon (\widetilde{f}(t) - \pi, \widetilde{f}(t) + \pi) \to S^1 \setminus \{-f(t)\}$ is a diffeomorphism. We denote its inverse by Ψ. Since f is continuous and since $f(s) \in S^1 \setminus \{-f(t)\}$ there exists an $\epsilon > 0$ such that $f([s, s + \epsilon)) \subset S^1 \setminus \{-f(t)\}$. Finally we consider the map

$$\begin{array}{rccc} & [x_0, s+\epsilon) & \to & \mathbb{R} \\ & x & \mapsto & \begin{cases} \widetilde{f}(x), & \text{if } x \le s, \\ \Psi(f(x)), & \text{if } x \in [s, s+\epsilon). \end{cases} \end{array}$$

Note that $\widetilde{f}$ and $\Psi \circ f$ are lifts of f on $[s, t]$ and note that they agree on t. Thus it follows from Claim 1 that they also agree on s. This shows that this map is in fact well-defined at s. It now follows from the Pasting Proposition 2.6 that this map is continuous. It is clear that it is a lift of $\varphi|_{[x_0,s+\epsilon)}$. Thus we see that $[x_0, s + \epsilon) \subset S$. ■



Now we can finally provide the long awaited proof of the Diffeotopies-of-S^1-Classification Proposition 34.5.

Proof of the Diffeotopies-of-S^1-Classification Proposition 34.5. Let $f\colon S^1 \to S^1$ be a diffeomorphism. First we consider the case that f is orientation-preserving. Note that by the Lifting from S^1-to-$\mathbb{R}$ Lemma 34.9, applied to the map $f \circ \Xi\colon \mathbb{R} \to S^1$, there exists a map $\widetilde{f}\colon \mathbb{R} \to \mathbb{R}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\widetilde{f}} & \mathbb{R} \\ {\scriptstyle t\mapsto \exp(it)}\big\downarrow{\scriptstyle \Xi} & & {\scriptstyle \Xi}\big\downarrow{\scriptstyle t\mapsto \exp(it)} \\ S^1 & \xrightarrow{f} & S^1. \end{array}$$

We make the following observations regarding $\widetilde{f}$:

(a) As we observed in Lemma 34.8, it follows from the fact that Ξ is a local diffeomorphism, together with the hypothesis that f is a smooth map, that $\widetilde{f}$ is also smooth.

(b) Recall that on page 682 we showed that Ξ is an orientation-preserving local diffeomorphism. Since f is orientation-preserving we see that $\widetilde{f}$ is also orientation-preserving. As we saw in the elementary Exercise 25.18, the above implies that $\widetilde{f}$ is a strictly monotonously increasing function.

(c) Since f is a diffeomorphism and since the above diagram commutes we see that for any $x \in \mathbb{R}$ the restriction of $\widetilde{f}$ to the interval $[x, x+2\pi)$ defines a bijection with $[f(x), f(x)+2\pi)$.

(d) Given $x \in \mathbb{R}$ it follows immediately from the commutativity of the above diagram, that the equality $\widetilde{f}(x+2\pi) - \widetilde{f}(x) \in \mathbb{Z} \cdot 2\pi$ holds.

(e) It follows from (b), (c) and (d) that for any $x \in \mathbb{R}$ we have $\widetilde{f}(x+2\pi) = \widetilde{f}(x) + 2\pi$.

Now we consider the map

$$\begin{array}{rcl} \widetilde{F}\colon \mathbb{R} \times [0,1] & \to & \mathbb{R} \\ (x,t) & \mapsto & \widetilde{f}(x)\cdot(1-t) + x \cdot t. \end{array}$$

This map is evidently smooth. Since the map $x \mapsto x$ has the above Properties (a) to (e) one can see easily that for any $t \in [0,1]$ the map $\widetilde{F}_t\colon \mathbb{R} \to \mathbb{R}$ has the above Properties (a) to (e). This implies in particular that the map $\widetilde{F}$ descends to a map

$$\begin{array}{rcl} F\colon S^1 \times [0,1] & \to & S^1 \\ (\Xi(x),t) & \mapsto & \Xi(\widetilde{f}(x)\cdot(1-t) + x \cdot t) \end{array}$$

where each $F_t\colon S^1 \to S^1$ is a homeomorphism. Since each $\widetilde{F}_t$ is a local diffeomorphism and since Ξ is a local diffeomorphism we obtain from the Smooth Pasting Proposition 19.23 (2) that each F_t is in fact a diffeomorphism. We have thus shown that F is indeed a diffeotopy from f to the identity.

Finally we consider the case that f is orientation-reversing. In this case $\rho \circ f$ is orientation-preserving. We apply the above result to $\rho \circ f$ and we obtain a diffeotopy $F\colon S^1 \times [0,1] \to S^1$ with $F_0 = \rho \circ f$ and $F_1 = \mathrm{id}$. Since $\rho \circ \rho = \mathrm{id}$ we see that $\rho \circ F$ is now the desired diffeotopy from f to ρ. ■

The following is our second classification result of this chapter.

Corollary 34.10. Let $\rho\colon S^1 \to S^1$ be the reflection that is given by $(x, y) \mapsto (x, -y)$. With this notation the following statements hold:

$$\widetilde{\mathrm{Diff}}^+(S^1) = \{[\mathrm{id}]\} \qquad \text{and} \qquad \widetilde{\mathrm{Diff}}(S^1) = \{[\mathrm{id}], [\rho]\}.$$

Proof. The statements follow immediately from the combination of Propositions 34.2 and 34.5. ■

Remark. In Exercise 34.1 we will pick up the last piece of low hanging fruit, namely we will determine $\widetilde{\mathrm{Diff}}^+(\overline{B}^1)$ and $\widetilde{\mathrm{Diff}}(\overline{B}^1)$.

Exercises for Chapter 34.

Exercise 34.1. Show that every diffeomorphism of $[-1, 1] = \overline{B}^1$ that preserves the boundary is diffeotopic to the identity.

Exercise 34.2. Let $I \subset \mathbb{R}$ be an interval and let $x_0 \in I$. Let $f, g\colon I \to S^1$ be two maps with $f(x_0) = g(x_0)$ and let $\widetilde{f}, \widetilde{g}\colon I \to \mathbb{R}$ be two lifts with $\widetilde{f}(x_0) = \widetilde{g}(x_0)$. Show that if $f, g\colon I \to S^1$ are homotopic rel $\{x_0\}$, then $\widetilde{f}$ and $\widetilde{g}$ are homotopic rel $\{x_0\}$.

Exercise 34.3. We consider the map

$$\begin{array}{rcl} \exp\colon \mathbb{C} & \to & \mathbb{C} \setminus \{0\} \\ z & \mapsto & \exp(z). \end{array}$$

Show that given any map $f\colon [a, b] \to \mathbb{C} \setminus \{0\}$ and given any $w \in \mathbb{C}$ with $\exp(z) = w$ there exists a unique map $\widetilde{f}\colon [a, b] \to \mathbb{C} \setminus \{0\}$ with $\widetilde{f}(a) = w$ and such that $\exp \circ \widetilde{f} = f$, i.e. such that for all $t \in [a, b]$ we have $\exp(\widetilde{f}(t)) = f(t)$.

Exercise 34.4.

(a) Show that every diffeomorphism of the half-open interval $[0, 1)$ is every orientation-preserving.

(b) Show that every diffeomorphism of the half-open interval $[0, 1)$ is diffeotopic to the identity.

Exercise 34.5. Let $T \subset \mathbb{R}^n$ be a k-dimensional vector subspace, let $W \subset \mathbb{R}^n$ be a vector space and let $\varphi\colon \mathbb{R}^n \to W$ be a homomorphism such that $\varphi\colon T \to W$ is a monomorphism. We denote by $p\colon \mathbb{R}^n \to \mathbb{R}^k$ the projection onto the first k coordinates. Show that there

exists a linear isomorphism $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ and a monomorphism $\chi\colon \mathbb{R}^k \to W$ such that $\varphi = \chi \circ p \circ \Phi$, i.e. such that the following diagram commutes:

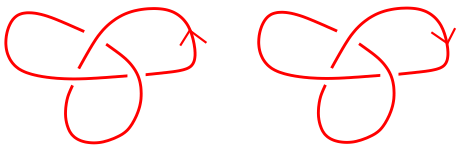
$$\begin{array}{ccc} T & \xrightarrow[\cong]{\Phi} & \Phi(T) \\ \downarrow{\scriptstyle \varphi} & & {\scriptstyle p}\downarrow{\scriptstyle (x,y)\mapsto x} \\ W & \xleftarrow{\chi} & \mathbb{R}^k. \end{array}$$

Exercise 34.6. Let $f\colon S^1 \to S^1$ be a diffeomorphism and let $F\colon S^1 \times [0,1] \to S^1$ be a diffeotopy with $F_0 = f$ and $F_1 = \mathrm{id}$. Furthermore let $\varphi\colon [\frac{1}{2}, 1] \to [0,1]$ be a smooth map with $\varphi(t) = 0$ for $t \in [\frac{1}{2}, \frac{5}{8}]$ and with $\varphi(t) = 1$ for $t \in [\frac{7}{8}, 1]$. We consider the following map:

$$\begin{array}{rcl} \Phi\colon \overline{B}^2 & \to & \overline{B}^2 \\ P = (x,y) & \mapsto & \begin{cases} P = (x,y), & \text{if } \|P\| \le \frac{1}{2}, \\ \|P\| \cdot F\big(\frac{P}{\|P\|}, f(2 - 2\cdot\|P\|)\big), & \text{if } \|P\| \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

Write down an explicit inverse of Φ and show that the inverse is smooth.

Exercise 34.7. For the purpose of this exercise an *oriented knot* is an oriented submanifold of $\mathbb{R}^3$ that is diffeomorphic to S^1. An oriented knot K is called *reversible* if there exists a smooth isotopy $F\colon K \times [0,1] \to \mathbb{R}^3$ such that $F_0\colon K \to \mathbb{R}^3$ is the inclusion and such that $F_1\colon K \to \mathbb{R}^3$ restricts to an orientation-reversing diffeomorphism $F_1\colon K \to K$. Show that the trefoil and the figure-8 knot are reversible. In other words, show that the two oriented knots shown in the figure to the left, respectively to the right, are smoothly isotopic.



trefoil with the two orientations

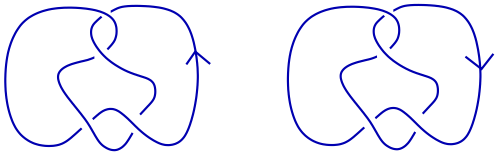
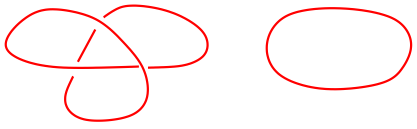


figure-8 knot with the two orientations

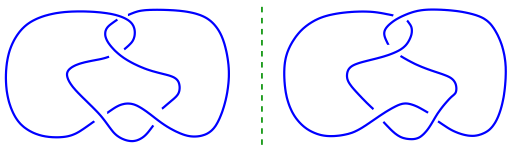
Exercise 34.8. For the purpose of this exercise a *knot* is a submanifold of $\mathbb{R}^3$ that is diffeomorphic to S^1. We say that two knots K and J are *smoothly isotopic* if there exists a smooth isotopy $F\colon S^1 \times [0,1] \to \mathbb{R}^3$ such that $F_0(S^1) = K$ and $F_1(S^1) = J$. We define the *mirror image* K^{mir} of a knot K to be the reflection of K in a hyperplane of $\mathbb{R}^3$.

(a) Show that any two mirror images of a given knot are smoothly isotopic.
(b) We say a knot is *amphichiral* if it is smoothly isotopic to its mirror image. Show that the figure-8 knot is amphichiral.

Remark. In Proposition 131.5 and also in Proposition 149.10 we will see that the trefoil is not amphichiral.



smoothly isotopic knots



the figure-8 knot K and its mirror K^{mir}

Exercise 34.9. What is the cardinality of the set $\mathrm{Emb}(\{1,\dots,n\}, S^1)$?

35. The Flow Theorem

On page 639 we introduced the notion of a smooth vector field on a smooth manifold. In this section we state the Flow Theorem that shows in particular that smooth vector fields give rise to diffeotopies. The Flow Theorem cuts both ways. It will help us to construct interesting diffeotopies using smooth vector fields, this occurs for example in the proof of the Isotopy Extension Theorem 36.1. But later on, using information on diffeotopies, it will also help us to prove results on what type of smooth vector fields can occur on a smooth manifold. One such instance is given in the proof of the Smooth Poincaré-Hopf Theorem 45.17 and much later in the proof of the Euler Characteristic-Vector Field Theorem 97.15.

35.1. Statement of the Flow Theorem.

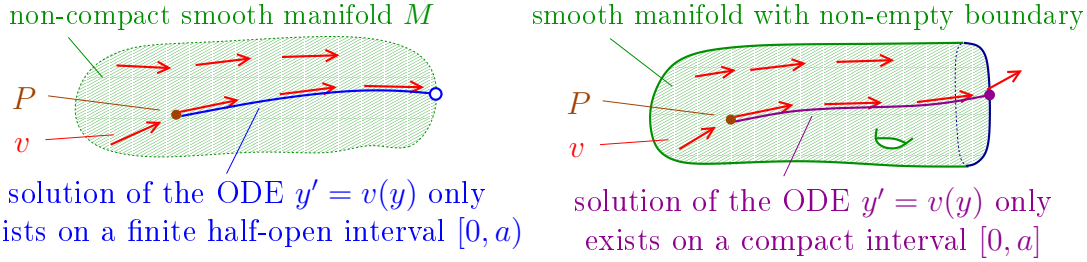
Definition. Let M be a smooth manifold and let v be a smooth vector field on M, as defined on page 638.

(1) Let I be an interval with non-empty interior. We say a smooth map $\varphi\colon I \to M$ is a[351] solution of the ODE $y' = v(y)$ if for every $t \in I$ we have the following equality:[352]

$$\varphi'(t) \;=\; v(\varphi(t)) \;\in\; \mathrm{T}_{\varphi(t)}M.$$

(2) A complete v-flow is a smooth map $F\colon M \times \mathbb{R} \to M$ that satisfies the following properties:
   (a) For each $P \in M$ we have $F(P, 0) = P$.
   (b) For each $P \in M$ the map $\mathbb{R} \to M$ given by $t \mapsto F(P, t)$ is a solution of the ODE $y' = v(y)$.

The reader might remember from an earlier course the Fundamental Theorem 35.7 for Autonomous ODEs which says, under mild hypotheses, that "short term" solutions of ODEs exist and that they are unique in a suitable sense. But note, as we can see in the figure below, these solutions might not be defined on all of $\mathbb{R}$. In particular there are hazards if the smooth manifold is non-compact and also if the smooth manifold has non-empty boundary.

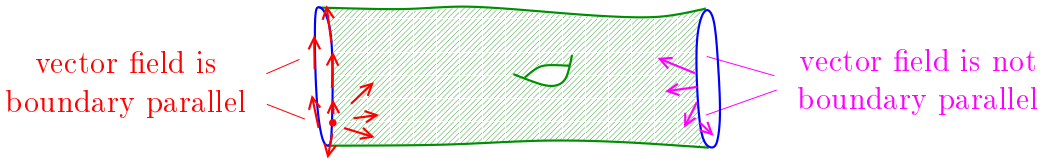


We will get around these issues by considering only compact smooth manifolds and by restricting ourselves to the following type of vector fields.

[351] Here ODE is short for "ordinary differential equation". Note that we do not define the notion of an ODE, we just define the expression "solution of an ODE $y' = v(y)$".

[352] We refer to page 637 for the definition of $\varphi'(t)$. As discussed on page 533, the fact that the interior of the interval is non-empty implies that we can talk sensibly of φ' at the endpoints of the interval.

Definition. Let M be a smooth manifold and let v be a vector field on M. We say v is boundary parallel if for each $P \in \partial M$ we have[353] $v(P) \in \mathrm{T}_P \partial M$.



Now we are in a position to state the main theorem of this chapter.

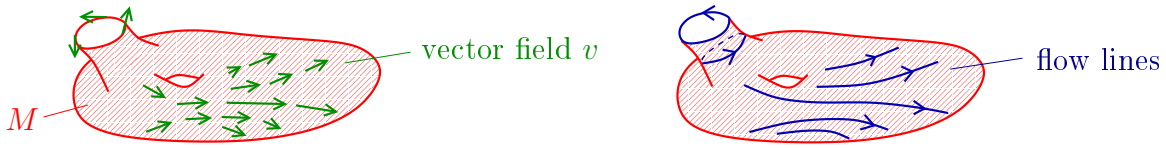
Theorem 35.1. (Flow Theorem) Let M be a compact smooth manifold and let v be a smooth vector field on M that is boundary parallel. There exists a unique complete v-flow $\Phi^v \colon M \times \mathbb{R} \to M$.

Remark. Statements that are very similar to our Flow Theorem 35.1 can also be found in [**Lee2002**, Theorem 9.12], [**Wall2016**, Theorem 1.4.2] and [**BJ1982**, Theorem 8.10].

The proof of the Flow Theorem 35.1 will consist of massaging the Fundamental Theorem for Autonomous ODEs 35.7, which, to most readers who made it that far, will be familiar from an earlier real analysis class. The proof of the Flow Theorem 35.1 is somewhat lengthy, thus we postpone the proof to the next section. In the following we prefer to discuss the Flow Theorem 35.1 and to consider an interesting addendum.

Definition. Let M be a compact smooth manifold and let v be a smooth vector field on M that is boundary parallel.

(1) For each $P \in M$ we refer to the subset $\Phi^v(\{P\} \times \mathbb{R}) \subset M$ as a flow line.[354]
(2) For $t \in \mathbb{R}$ the map $\Phi^v_t \colon M \to M$ that is given by $P \mapsto \Phi^v(P, t)$ is called the time-t map of the flow.



Example. We consider S^2 together with the smooth vector field that is given by

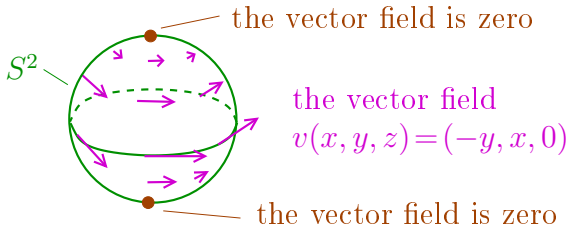
$$\begin{array}{rcl} v \colon S^2 & \to & \mathbb{R}^3 \\ (x, y, z) & \mapsto & (-y, x, 0). \end{array}$$

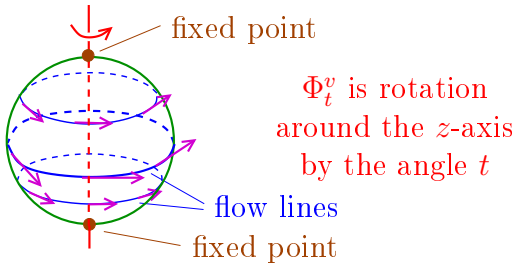
This vector field vanishes at the "North Pole" $(0, 0, 1)$ and at the "South Pole" $(0, 0, -1)$. By the Flow Theorem 35.1 this vector field gives rise to a complex v-flow $\Phi^v \colon S^2 \times \mathbb{R} \to S^2$. It is pretty straightforward to see that for any $t \in \mathbb{R}$ the corresponding time-t map $\Phi^v_t \colon S^2 \to S^2$ is just given by rotation around the z-axis by the angle t. In particular we see that the flow lines are the circles of latitude.

---

[353] Here and throughout this chapter we will use the convention from page 633 which says that given a smooth manifold M we can view the tangent spaces of the boundary ∂M and tangent spaces of submanifolds of M as vector subspaces of the tangent spaces $\mathrm{T}_P M$.

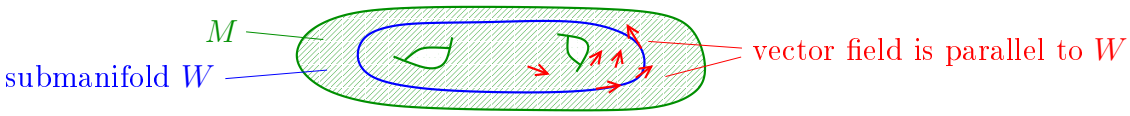
[354] We will not really make use of the concept of a flow line. But the concept will be useful in illustrations to draw pictures of flows.





Our next definition is a self-explanatory variation on the notion of a boundary parallel vector field.

Definition. Let M be a smooth manifold and let v be a vector field on M. Let W be a submanifold of M. We say v is parallel to the submanifold W if for each $P \in W$ we have $v(P) \in \mathrm{T}_P W$.



The following proposition summarizes a few properties of complete flows.

Proposition 35.2. (Flow-Diffeotopy Proposition) Let M be a compact smooth manifold and let v be a smooth vector field on M that is boundary parallel. We denote by $\Phi^v \colon M \times \mathbb{R} \to M$ the corresponding unique v-flow.

(1) For each $t \in \mathbb{R}$ the time-t map

$$\begin{array}{rccc} \Phi_t^v \colon & M & \to & M \\ & P & \mapsto & \Phi^v(P,t) \end{array}$$

is a diffeomorphism.

(2) Suppose that $W \subset M$ is a proper submanifold. If v is parallel to W, then the map Φ^v restricts to a map $W \times \mathbb{R} \to W$ and this map is the unique complete flow of the vector field $v|_W$ on the smooth manifold W.

Proof. We postpone the proof of the Flow-Diffeotopy Proposition 35.2 to The next section, more precisely, to page 853. ■

We continue with the following definition that will pop up time and again throughout these notes.

Definition. Let S be some set and let $f \colon S \to S$ be a map. We say $P \in S$ is a fixed point of f if $f(P) = P$.

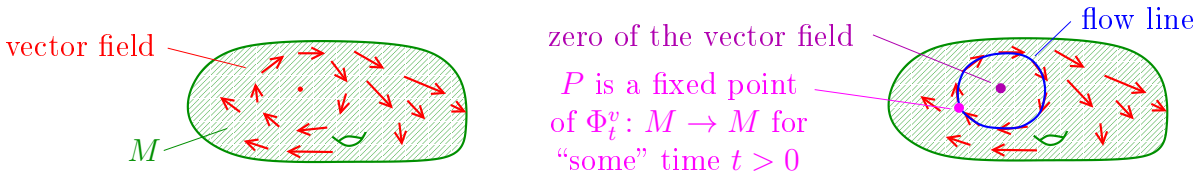
The following lemma says that zeros of vector fields give rise to fixed points of the time-t maps of the complete v-flows.

Lemma 35.3. Let M be a compact smooth manifold and let v be a smooth vector field on M that is boundary parallel. Let $\Phi^v \colon M \times \mathbb{R} \to M$ be the corresponding unique complete v-flow from the Flow Theorem 35.1. If P is a zero of the vector field, i.e. if $v(P) = 0$, then

for every $t \in \mathbb{R}$ we have $\Phi^v(P,t) = P$. Put differently, for every $t \in \mathbb{R}$ the point P is a fixed point of $\Phi^v_t \colon M \to M$.

Proof. This statement follows immediately from the uniqueness statement in the Fundamental Theorem 35.7 for Autonomous ODEs, applied to the map $t \mapsto \Phi^v(P,t)$ and the constant map $t \mapsto P$. ■

Now let M be a compact smooth manifold and let v be a smooth vector field on M that is boundary parallel. Let $\Phi^v \colon M \times \mathbb{R} \to M$ be the corresponding unique complete v-flow from the Flow Theorem 35.1. If $P \in M$ is a point with $v(P) \neq 0$ it can of course still happen that there exists a $t \in \mathbb{R}$ such that $\Phi^v_t(P) = P$. For example let us consider the vector field v on S^2 that we studied on page 845. In this case we see that $\Phi^v_{2\pi} = \mathrm{id}_{S^2}$.



The following theorem says that if a vector field has no zeros, then at least for a short time period each time-t map is fixed-point free.

Theorem 35.4. (Flow Fixed Point Theorem) Let M be a compact smooth manifold and let v be a smooth vector field on M that is boundary parallel. Let $\Phi^v \colon M \times \mathbb{R} \to M$ be the corresponding unique complete v-flow from the Flow Theorem 35.1. If v has no zeros, then there exists an $\epsilon > 0$ such that for each $t \in (0, \epsilon]$ the map $\Phi^v_t \colon M \to M$ is fixed-point free. In other words, given any $t \in (0, \epsilon]$ and given any $P \in M$ we have $\Phi^v_t(P) \neq P$.

Proof. To keep the discussion of the main results as clean as possible we postpone the proof of the Flow Fixed Point Theorem 35.4 to page 854. ■

Remark.

(1) In the Replace-by-Smooth Vector Field Proposition 29.6 we showed that if a smooth manifold admits a nowhere-vanishing vector field[355], then it also admits a smooth nowhere-vanishing vector field.
(2) Later, in the Euler Characteristic-Vector Field Theorem 97.15, we will use the combination of the Flow Fixed Point Theorem 35.4 and the Flow-Diffeotopy Proposition 35.2 to show that every smooth vector field on a closed smooth manifolds with non-zero "Euler characteristic" has at least one zero.

Most of the remainder of this chapter is dedicated to the proofs of Flow Theorem 35.1 and the Flow Fixed Point Theorem 35.4. Only in the very last section we give one more application of the Flow Theorem 35.1. Readers who happen to be allergic to technical proofs are advised to move on either to Section 35.6 or straight to the next chapter.

[355] Recall that according to our definition on page 639 a vector field is by definition continuous, not more and not less.

35.2. **Proof of the Flow Theorem 35.1.** In this section we will provide the proof of Flow Theorem 35.1. We will use one major ingredient from real analysis, namely the Fundamental Theorem 35.7 for Autonomous ODEs. Besides this input we will provide all details.

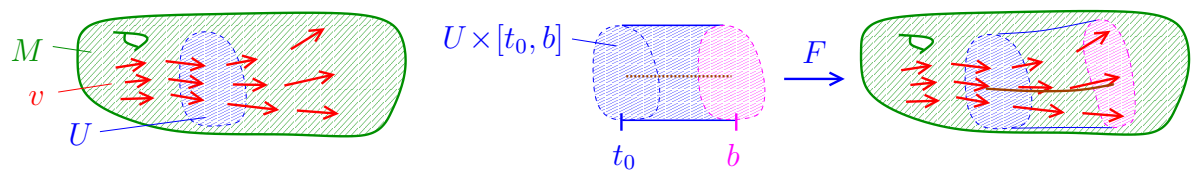
We start our discussion of the proof of the Flow Theorem 35.1 with the following sanity-preserving convention.

Convention. Given a smooth submanifold $U \subset \mathbb{R}^n$ we will use the Differential-equals-Differential Lemma 23.4 and the Tangent Space-Isomorphism Proposition 23.16 as an excuse for not distinguishing between the various notions of tangent spaces and differentials.

Throughout this section it is convenient to work with the following generalization of the notion of a complete flow.

Definition. Let M be a smooth manifold and let $t_0 \in \mathbb{R}$. A *v-flow based at t_0* consists of an open subset $U \subset M$ and an interval neighborhood I of t_0 together with a smooth map $F\colon U \times I \to M$ that satisfies the following properties:

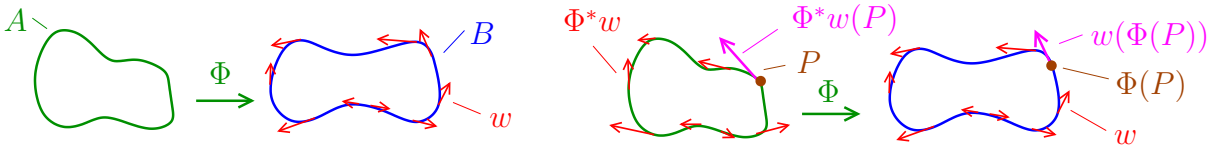
(1) For each $P \in U$ we have $F(P, t_0) = P$.
(2) For each $P \in U$ the map $I \to M$ that is given by $t \mapsto F(P,t)$ is a solution of the ODE $y' = v(y)$.



Notation. Let $\Phi\colon A \to B$ be a local diffeomorphism between two smooth manifolds and let w be a vector field on B. We denote by Φ^*w the vector field on A which assigns to each $P \in A$ the tangent vector

$$\Phi^*w(P) \;:=\; (\mathrm{D}\Phi_P)^{-1}(\underbrace{w(\Phi(P))}_{\in \mathrm{T}_{\Phi(P)}B}) \;\in\; \mathrm{T}_P A.$$

It follows easily from the definitions, and an argument say as in the proof of the Smooth-via-Atlas Lemma 19.21, that if w is a smooth vector field on B, then Φ^*w is a smooth vector field on A.



Lemma 35.5. Let $\Phi\colon A \to B$ and $\Psi\colon B \to C$ be local diffeomorphisms between smooth manifolds and let w be a vector field on C. Then $\Phi^*(\Psi^*w) = (\Psi \circ \Phi)^*w$.

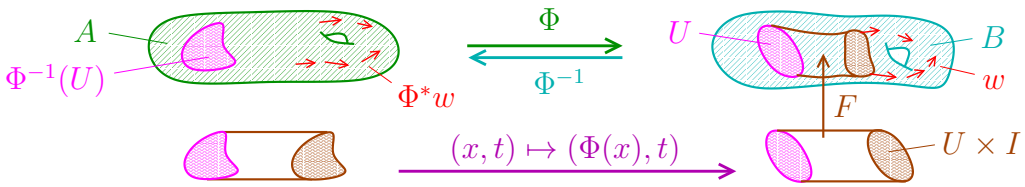
Proof. This statement follows immediately from the chain rule of differentials, see the Tangent Space-Functor Proposition 23.10 (2). ∎

The following lemma allows us to reduce questions on flows via diffeomorphisms to settings which are hopefully easier.

Lemma 35.6. (Flow-Diffeomorphism Lemma) Let $\Phi\colon A \to B$ be a diffeomorphism between two smooth manifolds and let w be a vector field on B. Furthermore let $U \subset B$ be an open subset and let $t_0 \in \mathbb{R}$. If $F\colon U \times I \to B$ is a w-flow on B based at t_0, then the map

$$\begin{array}{rcl} \Phi^{-1}(U) \times I & \to & A \\ (x,t) & \mapsto & \Phi^{-1}(F(\Phi(x),t)) \end{array}$$

is a $\Phi^* w$-flow based at t_0.



Proof. Let $x \in \Phi^{-1}(U)$. First we calculate

$$\frac{d}{dt}\Phi^{-1}(F(\Phi(x),t)) \overset{\text{by the chain rule, see the Tangent Space-Functor Proposition 23.10}}{=} \left(\mathrm{D}\Phi_{\Phi^{-1}(F(\Phi(x),t))}\right)^{-1} \cdot \frac{d}{dt}F(\Phi(x),t)$$
$$\overset{\text{since } F \text{ is a } w\text{-flow}}{=} \left(\mathrm{D}\Phi_{\Phi^{-1}(F(\Phi(x),t))}\right)^{-1} \cdot w(F(\Phi(x),t)) \overset{\text{by definition of } \Phi^* w \text{ at the point } \Phi^{-1}(F(\Phi(x),t))}{=} \Phi^*(w)(\Phi^{-1}(F(\Phi(x),t))).$$

Finally note that for each $x \in \Phi^{-1}(U)$ we have

$$\Phi^{-1}(F(\Phi(x),t_0)) \overset{\text{since the } w\text{-flow } F \text{ is a based at } t_0}{=} \Phi^{-1}(\Phi(x)) = x.$$

We have thus verified that the given map is indeed a $\Phi^* w$-flow based at t_0. ∎

The Flow-Diffeomorphism Lemma 35.6 gives us the tool to reduce many of our problems to the setting of open subsets of $\mathbb{R}^n$. This is the right moment for rolling out the following major theorem from real analysis.

Theorem 35.7. (Fundamental Theorem for Autonomous ODEs) Let $W \subset \mathbb{R}^n$ be an open subset and let $v\colon W \to \mathbb{R}^n$ be a smooth map.

(1) Given any $P \in W$ and any $t_0 \in \mathbb{R}$ there exists an open neighborhood U of P in W and an open neighborhood I of t_0 such that for every $x \in U$ there exists a solution $\varphi\colon I \to W$ of the ODE $y' = v(y)$ with $\varphi(t_0) = x$.

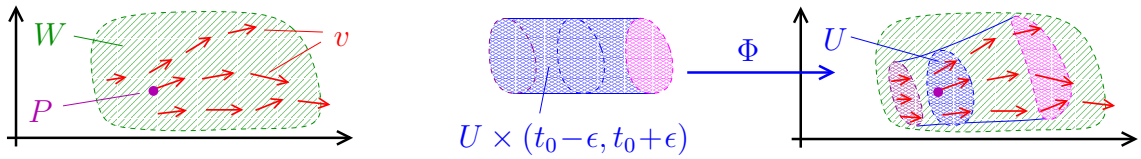
(2) Let $I \subset \mathbb{R}$ be an open interval. Furthermore let $f\colon I \to W$ and $g\colon I \to W$ be two solutions of the ODE $y' = v(y)$. If there exists a $t_0 \in I$ with $f(t_0) = g(t_0)$, then $f(t) = g(t)$ for all $t \in I$.

(3) Let $U \subset W$ be an open subset, let $t_0 \in \mathbb{R}$ and let $I \subset \mathbb{R}$ be an open neighborhood of t_0. Suppose that for each $x \in U$ we are given a map $\varphi_x\colon I \to W$ that is a solution of the ODE $y' = v(y)$ and such that $\varphi_x(t_0) = x$. Then the map

$$\begin{array}{rcl} \Phi\colon U \times I & \to & W \\ (x,t) & \mapsto & \varphi_x(t) \end{array}$$

is smooth. In other words, Φ is a v-flow based at t_0.

(4) If $F\colon U \times I \to W$ and $G\colon V \times J \to W$ are two v-flows based at t_0, then F and G agree on the intersection of their domains.



Proof. The first three statements of the theorem get proved in every respectable course on ordinary differential equations. For example a proof is provided in [**Lee2002**, Theorem D.1]. The last statement is an immediate consequence of (2) and the convenient fact that the intersection of two intervals is again an interval. ■

Lemma 35.8. Let M be a smooth manifold and let v be a smooth vector field on M. Let $F\colon M \times (a,b) \to M$ be a v-flow based at 0.

(1) For every μ, ν with $\mu \in (a,b)$ and $\mu + \nu \in (a,b)$ we have

$$F(F(P,\mu),\nu) \;=\; F(P,\mu+\nu).$$

(2) For every $\mu \in \mathbb{R}$ with $-\mu \in I$ the maps

$$\begin{array}{rcl} M & \to & M \\ P & \mapsto & F(P,-\mu) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} M & \to & M \\ P & \mapsto & F(P,\mu) \end{array}$$

are diffeomorphisms and they are inverses of one another.

Proof.

(1) Let $P \in M$ and let $\mu \in (a,b)$. We consider the map

$$\begin{array}{rcl} \psi\colon (a-\mu, b-\mu) & \to & M \\ t & \mapsto & F(P,\mu+t). \end{array}$$

Note that for each $t \in (a-\mu, b-\mu)$ we have

$$\psi'(t) \underset{\substack{\uparrow \\ \text{chain rule}}}{=} F'(P,\mu+t)\cdot 1 \;=\; v(F(P,\mu+t)) \;=\; v(\psi(t)).$$

This shows that $t \mapsto \psi(t) = F(P,\mu+t)$ is a solution of the ODE $y' = v(y)$. Note that $\psi(0) = F(P,\mu)$. Another solution of the ODE $y' = v(y)$ with the same initial condition is given by $t \mapsto \varphi(t) := F(F(P,\mu),t)$. It follows from the Fundamental Theorem 35.7 for Autonomous ODEs (3) that $F(P,t+\mu) = \psi(t) = \varphi(t) = F(F(P,\mu),t)$ for all $t \in (a-\mu, b-\mu)$. But that is what we needed to show.

(2) Let $\mu \in \mathbb{R}$ with $-\mu \in I$. We consider the maps

$$\begin{array}{rcl} \alpha\colon M & \to & M \\ P & \mapsto & F(P,-\mu) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \beta\colon M & \to & M \\ P & \mapsto & F(P,\mu) \end{array}$$

The map $F\colon M \times (a,b) \to M$ is a flow, in particular F is smooth. It follows easily from the Smooth Manifold Product Proposition 19.31 and Lemma 19.22 (2) that the two maps α and β are smooth. Thus it suffices to show that the maps α and β are inverses of one another. But this is an immediate consequence of (1). ■

Now we can finally provide the proof of the Flow Theorem 35.1. To improve readability we first deal with the case of smooth manifolds without boundary. Afterwards we will explain how one needs to adapt the argument to deal with smooth manifolds with non-empty boundary.

Proof of the Flow Theorem 35.1 if $\partial M = \varnothing$. Let M be a closed n-dimensional smooth manifold and let v be a smooth vector field on M. We need to show that there exists a unique complete v-flow on M. In fact the uniqueness follows almost immediately from the uniqueness statement of the Fundamental Theorem 35.7 for Autonomous ODEs together with the Flow-Diffeomorphism Lemma 35.6. Thus it remains to show the existence of a complete v-flow $\Phi\colon M \times \mathbb{R} \to M$.

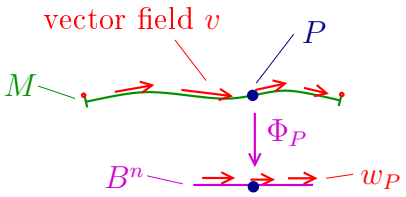
Claim 1. Given any $P \in M$ there exists an open neighborhood A_P of $P \in M$, an $\epsilon_P > 0$ and a v-flow $F_P\colon A_P \times (-\epsilon_P, \epsilon_P) \to M$ based at 0.

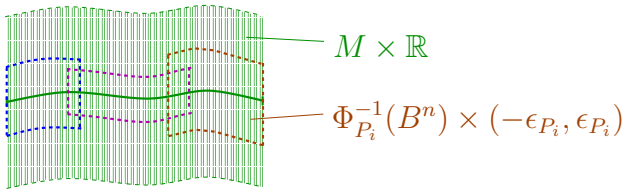
Proof. Let $P \in M$. Since we assume that $\partial M = \varnothing$ we know that there exists a chart $\Phi_P\colon U_P \to B^n$ with $\Phi_P(P) = 0$. We consider the smooth vector field $w := (\Phi_P^{-1})^*(v)$ on B^n. Recall that we assume that v is a smooth vector field on M. By the definition of a smooth vector field on a smooth manifold, see page 639, this implies that w, viewed as a map $w\colon B^n \to \mathbb{R}^n$ is actually a smooth map. Therefore it follows from Statements (1) and (3) of the Fundamental Theorem 35.7 for Autonomous ODEs there exists an open neighborhood N_P of $0 \in B^n$, an $\epsilon_P > 0$ and a w-flow $G\colon N_P \times (-\epsilon_P, \epsilon_P) \times B^n$ based at 0. We set $A_P := \Phi^{-1}(N_P)$. The claim now follows from the Flow-Diffeomorphism Lemma 35.6 together with Lemma 35.5. ⊞

Claim 2. There exists an $\epsilon > 0$ and a v-flow $F\colon M \times (-\epsilon, \epsilon) \to M$ based at 0.

Proof. By hypothesis the smooth manifold M is closed, in particular M is compact. It follows that there exist finitely many points $P_1, \dots, P_k \in M$ such that $M = A_{P_1} \cup \dots \cup A_{P_k}$. We set $\epsilon := \min\{\epsilon_{P_1}, \dots, \epsilon_{P_k}\}$.

It follows easily from Statement (4) of the Fundamental Theorem 35.7 for Autonomous ODEs, together with the Flow-Diffeomorphism Lemma 35.6, that the flows $F_{P_1}, \dots, F_{P_k}$ agree on the overlaps of their domains. In particular these v-flows combine to give a v-flow $F\colon M \times (-\epsilon, \epsilon) \to M$ based at 0. ⊞





By Lemma 35.8 we know that $\Theta := F_{\frac{\epsilon}{2}}\colon M \to M$ is a diffeomorphism. For each $k \in \mathbb{Z}$ we consider the map

$$\begin{array}{rcl} \widetilde{F}_k\colon M \times (\frac{k\cdot\epsilon}{2} - \epsilon, \frac{k\cdot\epsilon}{2} + \epsilon) & \to & M \\ (P, t) & \mapsto & \Theta^k(F(\Theta^{-k}(P), t - \frac{k\cdot\epsilon}{2})). \end{array}$$

It follows again from the Fundamental Theorem 35.7 for Autonomous ODEs (4) together with the Flow-Diffeomorphism Lemma 35.6 that these v-flows agree on the overlap of their

domains. Thus these maps can be combined to define a v-flow $F\colon M \times \mathbb{R} \to \mathbb{R}$ that is based at 0. ■

The proof of the Flow Theorem 35.1 if $\partial M \neq \varnothing$ will build on the following lemma.

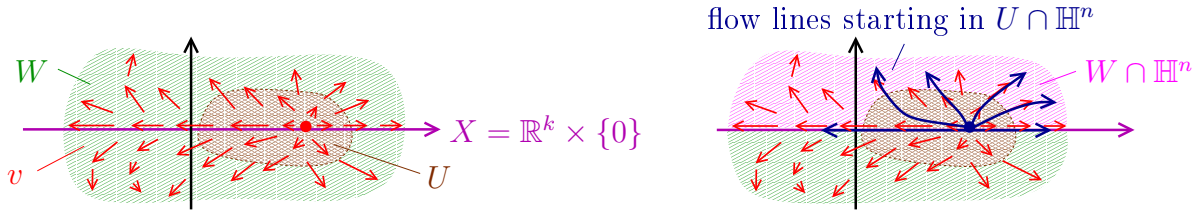
Lemma 35.9. Let $n \in \mathbb{N}$. We set $X := \mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$. Let W be an open subset of $\mathbb{R}^n$ and let $v\colon W \to \mathbb{R}^n$ be a smooth vector field on W that is parallel to $W \cap X$, i.e. we assume that for every $P \in X$ we have $v(P) \in \mathrm{T}_P X = \mathbb{R}^k \times \{0\}$. Furthermore let $U \subset W$ be an open subset and let $F\colon U \times I \to W$ be a v-flow on U based at some t_0. Let $k \in \{0, \dots, n\}$.

(1) The map $F\colon W \times I \to U$ restricts to a map

$$F\colon (U \cap X) \times I \ \to \ W \cap X.$$

(2) If $k = n - 1$, then the map $F\colon W \times I \to U$ restricts to a map

$$F\colon (U \cap \mathbb{H}^n) \times I \ \to \ W \cap \mathbb{H}^n.$$



Proof. The first statement follows immediately from the Mean Value Theorem. The second statement follows from the first statement and the Intermediate Value Theorem. We leave it to the reader to fill in the details. ■

Proof of the Flow Theorem 35.1 if $\partial M \neq \varnothing$. Let M be a compact n-dimensional smooth manifold with (possibly) non-empty boundary and let v be a smooth vector field on M that is boundary parallel. As in the proof for closed smooth manifolds it suffices to prove the following claim.

Claim 2. There exists an $\epsilon > 0$ and a v-flow $F\colon M \times (-\epsilon, \epsilon) \to M$ based at 0.

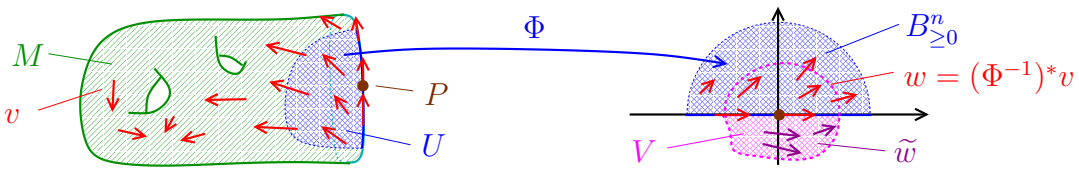
Furthermore, as in the proof for closed smooth manifolds, this Claim 2 follows easily from the following Claim 1.

Claim 1. Given any $P \in M$ there exists an open neighborhood A of $P \in M$, an $\epsilon > 0$ and a v-flow $F\colon A \times (-\epsilon, \epsilon) \to M$ based at 0.

Proof. Let $P \in M$. If $P \in M \setminus \partial M$, then we apply the same argument as on page 851. Now let $P \in \partial M$. This implies that there exists a chart $\Phi\colon U \to B^n_{\geq 0}$ with $\Phi(P) = 0$. We consider the vector field $w := (\Phi^{-1})^*(v)$ on $B^n_{\geq 0}$.

Since v is a smooth vector field on M we see, by the definition of a smooth vector field on a smooth manifold on page 639, that $(\Phi^{-1})^* v\colon B^n_{\geq 0} \to \mathbb{R}^n$ is a smooth map. By definition of a smooth map on $B^n_{\geq 0}$, see page 532, there exists an open neighborhood V of $0 \in \mathbb{R}^n$ and a smooth vector map $\widetilde{w}\colon V \to \mathbb{R}^n$ such that $\widetilde{w}$ and $(\Phi^{-1})^* v$ agree on the common subset $V \cap B^n_{\geq 0}$.

It follows again from Statements (1) and (3) of the Fundamental Theorem 35.7 for Autonomous ODEs that there exists an open neighborhood N of $0 \in V$, an $\epsilon > 0$ and a $\widetilde{w}$-flow $G\colon N \times (-\epsilon, \epsilon) \to V$ based at 0. Next note that it follows from our hypothesis that v is boundary parallel, together with Lemma 35.9, that the map G restricts to a map



$G\colon (N \cap B^n_{\geq 0}) \times (-\epsilon, \epsilon) \to V \cap B^n_{\geq 0}$. We set $A := \Phi^{-1}(N \cap B^n_{\geq 0})$. The desired statement now follows from the Flow-Diffeomorphism Lemma 35.6 together with Lemma 35.5. ■

Finally, with all the tools that we have acquired by now the proof of the Flow-Diffeotopy Proposition 35.2 is only a formality.

Proof of the Flow-Diffeotopy Proposition 35.2. Let M be a compact smooth manifold and let v be a smooth vector field on M that is boundary parallel. Let $\Phi^v\colon M \times \mathbb{R} \to M$ be the corresponding complete v-flow.

(1) Let $t \in \mathbb{R}$. We need to show that the map $\Phi^v_t\colon M \to M$ is a diffeomorphism that is diffeotopic to the identity. To do so we consider the map

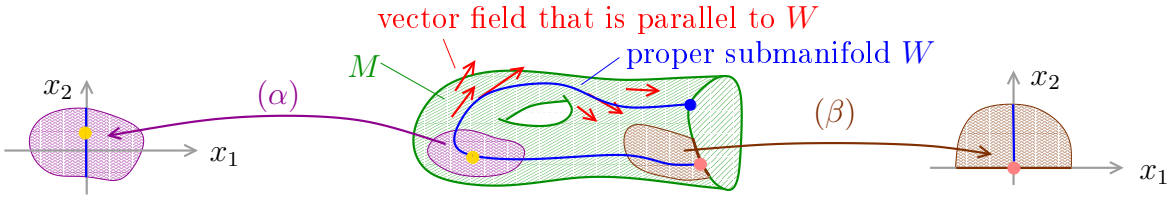
$$\begin{array}{rcl} H\colon M \times [0,1] & \to & M \\ (P, s) & \mapsto & \Phi^v(P, t \cdot s). \end{array}$$

Since $\Phi^v\colon M \times \mathbb{R} \to M$ is a v-flow we know that $\Phi^v\colon M \times \mathbb{R} \to M$ is in particular smooth. It follows that $H\colon M \times [0,1] \to M$ is smooth. Furthermore by Lemma 35.8 we know that each $H_s\colon M \to M$ is a diffeomorphism. This shows that H is a diffeotopy.

(2) Suppose that $W \subset M$ is a proper submanifold and suppose that v is parallel to W. We need to show that the map $\Phi^v\colon M \times \mathbb{R} \to M$ restricts to a map $W \times \mathbb{R} \to W$ and that this map is the unique complete flow of the vector field $v|_W$ on W. Recall that by the definition of a proper submanifold, see page 525, W is a closed subset of M and for each $P \in M$ we can find submanifold charts of type (α) or (β) as defined on page 525 and as illustrated in the figure below. Given $P \in W$ we consider the map

$$\begin{array}{rcl} \varphi\colon \mathbb{R} & \to & M \\ t & \mapsto & \Phi^v(P, t). \end{array}$$

It follows from Lemma 35.9 together with the Flow-Diffeomorphism Lemma 35.6 that $\varphi^{-1}(W)$ is an open subset of $\mathbb{R}$. Since W is a closed subset of M we know that $\varphi^{-1}(W)$ is a closed subset of $\mathbb{R}$. Since $\varphi(0) = P$ we see that $\varphi(W) \neq \varnothing$. By the Path-Connected Implies Connected Corollary 2.23 we know that $\mathbb{R}$ is connected. Thus we see that $\varphi^{-1}(W) = \mathbb{R}$. But this implies that $\Phi^v(W \times \mathbb{R}) \subset W$. Finally it follows from the uniqueness statement of the Flow Theorem 35.1 that $\Phi^v|_{W \times \mathbb{R}}$ is indeed the complete $v|_W$-flow. ■



35.3. **Proof of the Flow Fixed Point Theorem 35.4.** For the reader's convenience we recall the Flow Fixed Point Theorem 35.4.

Theorem 35.4. (Flow Fixed Point Theorem) Let M be a compact smooth manifold and let v be a smooth vector field on M that is boundary parallel. Let $\Phi^v\colon M\times\mathbb{R}\to M$ be the unique complete v-flow $\Phi^v\colon M\times\mathbb{R}\to M$ from the Flow Theorem 35.1. If v has no zeros, then there exists an $\epsilon>0$ such that for each $t\in(0,\epsilon]$ the time t-map $\Phi^v_t\colon M\to M$ is fixed-point free. In other words, given any $t\in(0,\epsilon]$ and given any $P\in M$ we have $\Phi^v_t(P)\neq P$.

Remark. We will give a bare hands proof of Flow Fixed Point Theorem 35.4. A more elegant proof can also be given using Proposition 35.11.

Our proof rests on the following auxiliary lemma.

Auxiliary Lemma 35.10. Let I be an interval and let $\varphi\colon I\to U$ be a smooth map. If there exists a linear map $\pi\colon\mathbb{R}^n\to\mathbb{R}$ such that for every $t\in I$ we have $\pi(\varphi'(t))>0$, then φ is injective.

Proof of Lemma 35.10. We consider the map $\pi\circ\varphi\colon I\to\mathbb{R}$. Note that for every $t\in I$ we have

$$\begin{array}{ccccc}(\pi\circ\varphi)'(t) & = & \mathrm{D}\pi_{\varphi(t)}(\varphi'(t)) & = & \pi(\varphi'(t)) \\ & \uparrow & & \uparrow & \uparrow \\ & \text{by the Chain Rule 19.10} & & \text{see page 536} & \text{by hypothesis}\end{array} \quad > \quad 0.$$

It follows from elementary real analysis that $\pi\circ\varphi$ is injective. But this implies of course that $\varphi\colon I\to U$ is injective. ■

Proof of the Flow Fixed Point Theorem 35.4. Let M be a compact smooth manifold and let v be a smooth vector field on M that is boundary parallel and which has no zeros. We denote by $\Phi^v\colon M\times\mathbb{R}\to M$ the corresponding unique complete v-flow.

Claim. For each $P\in M$ there exists an open neighborhood U_P of P and an $\epsilon_P>0$ such that for every $x\in U_P$ the map $(-\epsilon_P,\epsilon_P)\to M$ given by $t\mapsto\Phi^v(x,t)$ is injective.

Proof. Let $P\in M$. In the following we deal with the case that $P\in M\setminus\partial M$. The case $P\in\partial M$ is deal with in a very similar fashion.

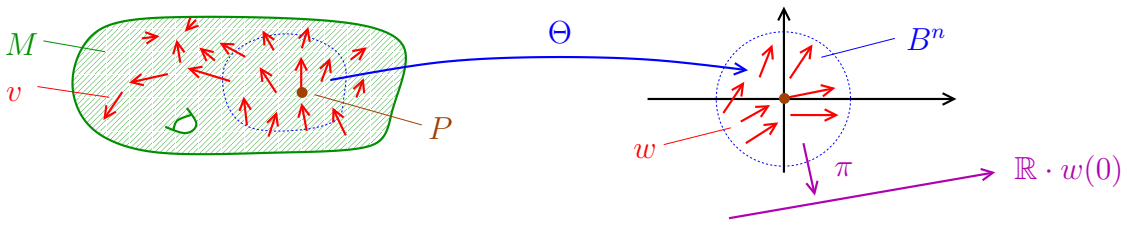
It follows easily from the hypothesis that $v(P)\neq 0$ that there exists a chart $\Theta\colon U\to B^n$ around P such that $\Phi(P)=0$ and such that $w:=(\Theta^{-1})^*v$ has the property that for each $x\in B^n$ we have $\|w(x)-w(0)\|<\frac{1}{2}\cdot\|w(0)\|$.

Since U is an open subset of M we obtain from the continuity of Φ^v that $\Phi^v(U)\subset M\times\mathbb{R}$ is an open neighborhood of $(P,0)$. Thus there exists an open neighborhood U_P of P and an $\epsilon_P>0$ such that $\Phi^v(U_P\times(-\epsilon_P,\epsilon_P))\subset U$.

Recall that $w(0)\neq 0$. Let $\pi\colon\mathbb{R}^n\to\mathbb{R}$ be the linear map that corresponds to the orthogonal projection onto $\mathbb{R}\cdot w(0)$. Since $\|w(x)-w(0)\|<\frac{1}{2}\cdot\|w(0)\|$ for all $x\in B^n$ we obtain from an elementary calculation that $\pi(w(x))>0$ for all $x\in B^n$. It follows from Lemma 35.10 that for each $x\in\Theta(U_P)$ the map

$$\begin{array}{ccc}(-\epsilon_P,\epsilon_P) & \to & B^n \\ x & \mapsto & \Theta(\Phi^v(\Theta^{-1}(x),t))\end{array}$$

is injective. This shows that ϵ_P together with the open neighborhood U_P of P has the desired properties. ⊞



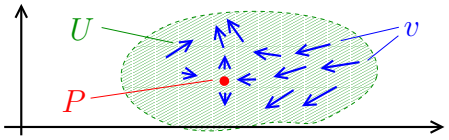
Since M is compact there exist $P_1, \dots, P_k \in M$ with $U_{P_1} \cup \dots \cup U_{P_k} = M$. It is clear that $\epsilon := \min\{\epsilon_{P_1}, \dots, \epsilon_{P_k}\}$ has the desired properties. ■

35.4. **Appendix I: Zeros of vector fields.** In this first appendix to this chapter we state a proposition which gives us information about the flow of a vector field near a zero of the vector field. We will not make use of it. But perhaps readers, who want to extend some of the earlier results, will find it useful.

Proposition 35.11. Let $U \subset \mathbb{R}^n$ be an open subset, let $v\colon U \to \mathbb{R}^n$ be a smooth vector field on U and let P be a zero of v. Furthermore let W be an open neighborhood of P and let $F\colon W \times (-\epsilon, \epsilon) \to U$ be a v-flow based at 0. Then for every $t \in (-\epsilon, \epsilon)$ we have[356]

$$\mathrm{D}(F_t)_P \;=\; \exp(t \cdot \mathrm{D}v_P).$$

Here, for a matrix $A \in \mathrm{M}(n \times n, \mathbb{R})$ we write $\exp(A) := \sum_{k=0}^{\infty} \frac{A^k}{k}$.



$$\mathrm{D}v_P = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Proof. The fact that P is a zero of v implies by Lemma 35.3 that $F(P,t) = P$ for all $t \in (-\epsilon, \epsilon)$. Therefore it follows from [**PW2019**, Satz 4.3.1], after some slight unraveling of the definitions, that the map

$$\begin{array}{rcl} (-\epsilon, \epsilon) & \to & \mathrm{M}(n \times n, \mathbb{R}) \\ t & \mapsto & \mathrm{D}(F_t)_P \end{array}$$

satisfies the ODE

$$y' \;=\; \mathrm{D}v_P \cdot y.$$

But by [**PW2019**, p. 58] this means that $\mathrm{D}(F_t)_P = \exp(t \cdot \mathrm{D}v_P)$ for every $t \in (-\epsilon, \epsilon)$. ■

35.5. **Appendix II: Dealing with non-compact smooth manifolds.** Let M be a smooth manifold and let v be a smooth vector field on M that is boundary parallel. As we saw on page 844, if M is non-compact, then there does not necessarily exist a complete v-flow $\Phi^v\colon M \times \mathbb{R} \to M$. If one wants to show that such a complete v-flow exists after all one needs some restrictions on the vector fields under consideration. In this appendix we will sketch one approach. We will not make use of the results of this appendix.

The proof of the Flow Theorem 35.1 was based on Statement (1) of the Fundamental Theorem 35.7 for Autonomous ODEs which says that if $W \subset \mathbb{R}^n$ be an open subset and if $v\colon W \to \mathbb{R}^n$ is a smooth map, then given any $P \in W$ and any $t_0 \in \mathbb{R}$ there exists an open

[356] Since $v(P) = 0$ we see in Lemma 35.3 that P is a fixed point of every F_t. Hence $\mathrm{D}(F_t)_P$ is an automorphism of $\mathrm{T}_P U = \widetilde{\mathrm{T}}_P U = \mathbb{R}^n$. In other words, it is given by a matrix in $\mathrm{M}(n \times n, \mathbb{R})$.

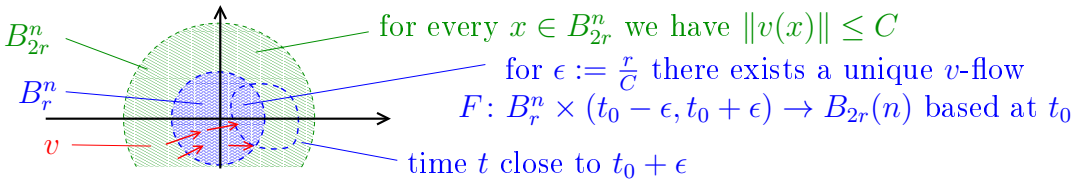
neighborhood U of P in W and an open neighborhood I of t_0 such that for every $x \in U$ there exists a solution $\varphi\colon I \to W$ of the ODE $y' = v(y)$ with $\varphi(t_0) = x$.

To deal with non-compact smooth manifolds we will need the following quantitative version of Statement (1) of the Fundamental Theorem 35.7 for Autonomous ODEs.

Proposition 35.12. Let $r > 0$ and let $v\colon B^n_{2r} \to \mathbb{R}^n$ be a smooth vector field on the open n-ball $B^n_{2r} \subset \mathbb{R}^n$ of radius $2r$. Suppose that we have a $C > 0$ such that for every $x \in B^n_{2r}$ we have $\|v(x)\| \leq C$. We set $\epsilon := \frac{r}{C}$.

(1) For every $x_0 \in B^n_r$ and every $t_0 \in \mathbb{R}$ there exists a solution $\varphi\colon (t_0 - \epsilon, t_0 + \epsilon) \to B^n_{2r}$ of the ODE $y' = v(y)$ with $\varphi(t_0) = x_0$.

(2) For every $t_0 \in \mathbb{R}$ there exists a unique v-flow $F\colon B^n_r \times (t_0 - \epsilon, t_0 + \epsilon) \to B^n_{2r}$ based at t_0.



Proof. Note that Statement (2) is an immediate consequence of Statement (1) together with Statements (3) and (4) of the Fundamental Theorem 35.7 for Autonomous ODEs. Thus it suffices to prove Statement (1).

We set $\epsilon := \frac{r}{C}$. Let $x_0 \in B^n_r$ and $t_0 \in \mathbb{R}$. In the following we will show that there exists a solution to the ODE $y' = v(y)$ with initial condition $\varphi(t_0) = y_0$ that is defined on $[t_0, t_0 + \epsilon)$. Basically the same argument can then be used to get the extension "to the left", i.e. to the interval $(t_0 - \epsilon, t_0]$. We consider

$$T := \sup\left\{ t \in (t_0, \infty) \;\middle|\; \begin{array}{c} \text{there exists a solution } \varphi\colon [t_0, t) \to B^n_{2r} \text{ of} \\ \text{the ODE } y' = v(y) \text{ with } \varphi(t_0) = x_0 \end{array} \right\}.$$

If $T > t_0 + \epsilon$ then we are evidently done. If $T = t_0 + \epsilon$, then for every $a \in (t_0, T)$ there exists a solution φ_a. But by Statement (2) of the Fundamental Theorem 35.7 for Autonomous ODEs we know that for any $b > a$ we have $\varphi_b|_{[t_0,a)} = \varphi_a$. This shows that these maps φ_a combine to define a solution φ on all of $[t_0, T) = [t_0, t_0 + \epsilon)$.

Finally suppose that $T < t_0 + \epsilon$. As in the previous paragraph we see that there exists a solution $\varphi\colon [t_0, T) \to B^n_{2r}$. Note that by Statement (2) of the Fundamental Theorem 35.7 for Autonomous ODEs we know at least that $T > t_0$.

Claim. The limit $P := \lim_{t \to T} \varphi(t)$ exists and it lies in B^n_{2r}.

Proof. We start with a little inequality, namely given $x, y \in [t_0, T)$ we have

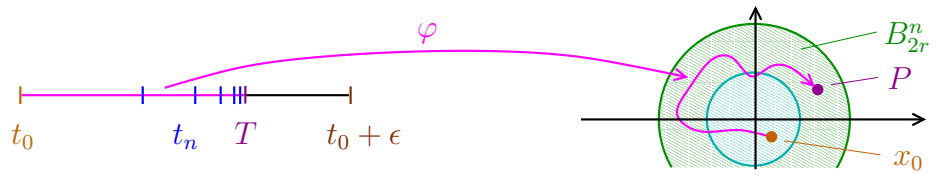
$$\|\varphi(y) - \varphi(x)\| \underset{\text{Fundamental Theorem of Calculus}}{=} \left\| \int_x^y \varphi'(s)\, ds \right\| \overset{\text{Integral Inequality Lemma 0.16}}{\leq} \int_x^y \|\varphi'(s)\|\, ds \underset{\text{since } \varphi \text{ is a solution of the ODE } y' = v(y)}{=} \int_x^y \|v(\varphi(s))\|\, ds \underset{\text{since } \varphi \text{ takes values in } B^n_{2r} \text{ we have } \|v(\varphi(s))\| \leq C}{\leq} |y - x| \cdot C.$$

We pick a monotonously increasing sequence $(t_n)_{n\in\mathbb{N}}$ in $[t_0, T)$ with $\lim_{n\to\infty} t_n = T$. It follows from the above inequality that $(\varphi(t_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in B^n_{2r}. We denote by $P \in \mathbb{R}^n$ its limit. It follows easily from the above inequality that in fact $\lim_{t\to T} \varphi(t) = P$. It remains to show that P actually lies in B^n_{2r}. This follows from the following (in-) equalities:

$$\|P\| \leq \|x_0\| + \|P - x_0\| = r + \left\| \lim_{t\to T} (\varphi(t) - \varphi(t_0)) \right\| \underset{\uparrow}{\leq} r + (T - t_0)\cdot C \underset{\uparrow}{<} r + \epsilon \cdot \frac{r}{\epsilon} = 2r.$$

by the above inequality we have $\|\varphi(t) - \varphi(t_0)\| \leq (t - t_0)\cdot C$

since $T < t_0 + \epsilon$ and since $\epsilon = \frac{r}{C}$ ⊞



Since P is contained in the open set B^n_{2r} it follows from Statement (1) of the Fundamental Theorem 35.7 for Autonomous ODEs that there exists an open neighborhood A of $P \in B^n_{2r}$, an open interval $(-\mu, \mu)$ and a v-flow

$$F\colon A \times (-\mu, \mu) \;\to\; B^n_{2r}$$

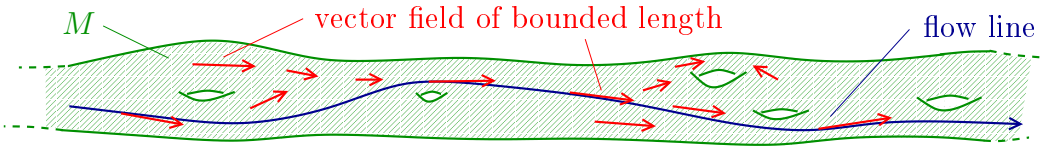
that is based at 0. Since $P = \lim_{t\to T} \varphi(t)$ and since A is an open neighborhood of U there exists a $t' \in (t_0, T)$ with $t' > T - \frac{\mu}{2}$ and with $\varphi(t') \in A$. We consider the map

$$\begin{array}{rcl} \psi\colon [t_0, t' + \mu) & \to & B^n_{2r} \\ s & \mapsto & \begin{cases} \varphi(s), & \text{if } s \in (t_0, T) \\ F(\varphi(t'), s - t'), & \text{if } s \in (t' - \mu, t' + \mu) \end{cases} \end{array}$$

Note that $s \mapsto \varphi(s)$ and $s \mapsto F(\varphi(t'), s - t')$ are both solutions to the ODE $y' = v(y)$ with $t' \mapsto \varphi(t')$. Thus it follows from Statement (2) of the Fundamental Theorem 35.7 for Autonomous ODEs that these two maps agree on the common domain, in other words, the above map ψ is in fact unambiguously defined. By construction it is a solution to the ODE $y' = v(y)$ with $\psi(t_0) = x_0$. Since $t + \mu > T - \frac{\mu}{2} + \mu > T$ we have obtained a contradiction to the assumption that $T < t_0 + \epsilon$. ■

Using Proposition 35.12 one can prove the following variation on the Flow Theorem 35.1.

Theorem 35.13. (Flow Theorem for Non-Compact Smooth Manifolds) Let M be a proper n-dimensional smooth submanifold of $\mathbb{R}^m$ and let v be a smooth vector field on M. If there exists a $C \in \mathbb{R}$, such that for each $x \in M$ we have $\|v(x)\| \leq C$, then there exists a unique complete v-flow $\Phi^v\colon M \times \mathbb{R} \to M$.



Proof. We leave the proof of the theorem as Project 35.2 to the reader. ■

35.6. **Moving points on smooth manifolds.** As a treat to the reader who made it that far we conclude this chapter with a neat application of the Flow Theorem 35.1, namely we can now give a new proof of Proposition 33.12 (2).

Proposition 35.14. Given any two points $P, Q \in \overline{B}^n \setminus \partial\overline{B}^n$ we can move P to Q in the sense of the definition on page 822.

Proof. Let $P, Q \in \overline{B}^n \setminus \partial\overline{B}^n$.

Claim. There exists a smooth vector field v on $\overline{B}^n$ which has the following two properties:
(1) The vector field vanishes on an open neighborhood of $\partial\overline{B}^n$.
(2) For every $t \in [0,1]$ we have $v(P \cdot (1-t) + t \cdot Q) = Q - P$.

Proof. We use Condition (2) as a definition of the vector field v on the line segment $\{P \cdot (1-t) + t \cdot Q \,|\, t \in [0,1]\}$. Since $P, Q \in \overline{B}^n \setminus \partial\overline{B}^n$ there exists an $\epsilon > 0$ such that this line segment is contained in $\overline{B}^n_{1-\epsilon}$. On $\overline{B}^n \setminus B^n_{1-\frac{\epsilon}{2}}$ we define v to be the zero vector field. We have thus defined v on a closed subset of $\overline{B}^n$. It is straightforward to show that v is smooth in the sense of the definition on page 532. Thus it follows from the Smooth Extension Proposition 21.9 that we can extend v to a smooth map $\overline{B}^n \to \mathbb{R}^n$. ⊞



Note that the vector field is in particular parallel to the boundary. Thus it follows from the Flow Theorem 35.1 that there exists a complete v-flow $\Phi^v \colon \overline{B}^n \times \mathbb{R} \to \overline{B}^n$. It follows easily from the uniqueness statement of the Fundamental Theorem 35.7 for Autonomous ODEs that $\Phi_1(P) = Q$ and that restriction of each Φ^v_t to $\overline{B}^n \setminus B^n_{1-\frac{\epsilon}{2}}$ is the identity.

As in the proof of the Flow-Diffeotopy Proposition 35.2 we see that it follows from Lemma 35.8 that the map

$$\begin{array}{rcl} H \colon \overline{B}^n \times [0,1] & \to & \overline{B}^n \\ (x,s) & \mapsto & \Phi^v(x,s) \end{array}$$

is a diffeotopy. From the above discussion it is clear that it has all the desired properties. ■

Exercises for Chapter 35.

Exercise 35.1. Let M be a smooth submanifold of $\mathbb{R}^n$ and let v be a "time dependent smooth vector field", i.e. let $v \colon M \times \mathbb{R} \to \mathbb{R}^n$ be a smooth map such that each $v_t \colon M \to \mathbb{R}^n$ is a smooth vector field on M.

- Let I be an interval with non-empty interior. We say a smooth map $\varphi \colon I \to M$ is a *solution of the ODE* $y' = v(y,t)$ if for every $t \in I$ we have the following equality:
$$\varphi'(t) \;=\; v(\varphi(t), t) \;\in\; \mathrm{T}_{\varphi(t)}M.$$
- A *complete v-flow* is a smooth map $F \colon M \times \mathbb{R} \to M$ that satisfies the following properties:
(a) For each $P \in M$ we have $F(P,0) = P$.

(b) For each $P \in M$ the map $\mathbb{R} \to M$ given by $t \mapsto F(P,t)$ is a solution of the ODE $y' = v(y,t)$.

Now suppose that M, viewed as a smooth manifold, is closed. Show that there exists a complete v-flow?

Exercise 35.2. Give an example of a vector field v on a closed smooth manifold that has only finitely man zeros and such there exists no $t > 0$ such that the map

$$\begin{array}{ccc} M & \to & M \\ P & \mapsto & \Phi^v(P,t) \end{array}$$

has infinitely many fixed-points.

Exercise 35.3. Let M be a compact smooth manifold.

(a) Let v be a smooth vector field on M that is boundary parallel. As usual we denote by $\Phi^v \colon M \times \mathbb{R} \to M$ the corresponding complete v-flow that exists by the Flow Theorem 35.1. Show that for every $s,t \in \mathbb{R}$ we have

$$\Phi^v_s \circ \Phi^v_t \;=\; \Phi^v_{s+t} \colon M \;\to\; M.$$

(b) We suppose that M is non-empty and that $\dim(M) > 0$. As on page 830 we denote by $\mathrm{Diff}(M)$ the group of self-diffeomorphism of M. Show that there exists a monomorphism $\mathbb{R} \to \mathrm{Diff}(M)$.

Exercise 35.4. We consider the smooth manifold $M = \mathbb{R}$. Does every smooth vector field v on M admit a complete v-flow?

Exercise 35.5. Use the Flow Theorem 35.13 to give a new proof of the Diffeotopies-of-S^1-Classification Proposition 34.5. More precisely, let $f \colon S^1 \to S^1$ be a diffeomorphism, show that the following two statements hold:

(1) If f is orientation-preserving, then f is diffeotopic to the identity.
(2) If f is orientation-reversing, then f is diffeotopic to the reflection $\rho \colon S^1 \to S^1$ that is given by $(x,y) \mapsto (x,-y)$.

Project 35.1. Let M be a compact smooth manifold.

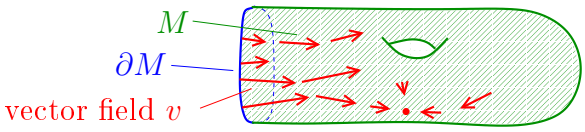
(a) Show that there exists a smooth vector field v on M such that for every $x \in \partial M$ the vector $v(x) \in \mathrm{T}_x M$ points inward in the sense of the definition on page 620.
Hint. Use partitions of unity.

(b) Show that there exists an $\epsilon > 0$ and a smooth map $F \colon \partial M \times [0,1] \to M$ such that for every $x \in M$ the following two statements hold:
(a) $F(x,0) = x$,
(b) for every $t_0 \in [0,1]$ we have $\frac{d}{dt}(t \mapsto F(x,t))|_{t=t_0} = v(F(x,t_0))$.

(c) Show that there exists a $\mu \in (0,\epsilon)$ such that $F \colon \partial M \times [0,\mu] \to M$ is a smooth embedding.
Hint. Use the Neighborhood Smooth Embedding Theorem 21.13.
Remark. This gives a new proof of the Smooth Collar Neighborhood Theorem 28.3 for smooth manifolds with compact boundary.

For readers with an insatiable appetite we have an extra exercise:

(d) Reprove the Smooth Collar Neighborhood Theorem 28.3 also for smooth manifolds with non-compact boundary.



Project 35.2. Prove the Flow Theorem 35.13 for Non-Compact Smooth Manifolds.

36. The Isotopy Extension Theorem

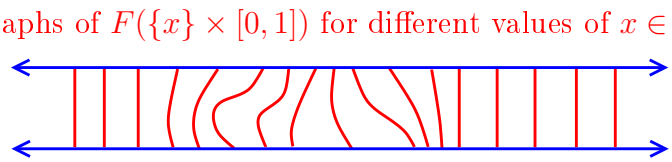
36.1. **The statement of the Isotopy Extension Theorem.** First let us recall the following definition from pages 419 and 822.

Definition. Let X and Y be topological spaces and let $F\colon X \times [0,1] \to Y$ be a homotopy.

(1) Let $A \subset X$ be a subset. We say F is a homotopy rel A if for all $x \in A$ and $s, t \in [0,1]$ we have $F(x,s) = F(x,t)$.
(2) If $X = Y$, then we say F has compact support if there exists a compact subset $K \subset X$ such that $F(x,t) = x$ for all $x \notin K$ and all $t \in [0,1]$.

Examples.

(1) Evidently every diffeotopy of a compact smooth manifold has compact support.
(2) The diffeotopy $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ given by $F(x,t) = x + t$ does *not* have compact support.
(3) In the figure below we sketch a non-trivial diffeotopy of $\mathbb{R}$ with compact support.



The following theorem is one of the key technical tools for dealing with smooth manifolds and their submanifolds.

Theorem 36.1. (Isotopy Extension Theorem) Let M be a compact smooth manifold and let N be a smooth manifold. Let A be a union of components of ∂M and let B be a union of components of ∂N. Let $F\colon M \times [0,1] \to N$ be a smooth isotopy that is trivial near the ends[357] and with the following property:

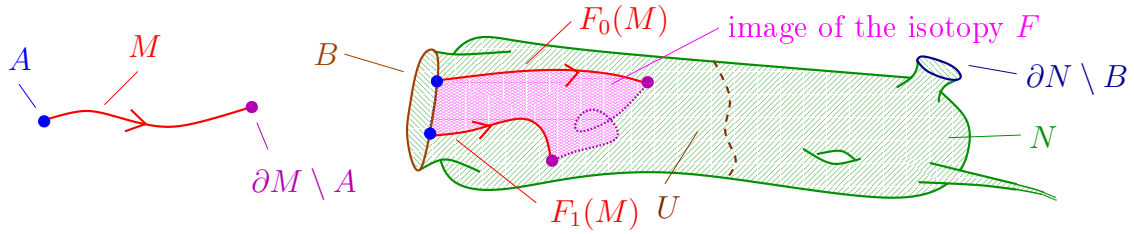
($*$) For each $t \in [0,1]$ we have $F_t(A) \subset B$ and we have $F_t(\partial M \setminus A) \subset N \setminus \partial N$.

Furthermore suppose that we are given an open subset U of N with $F(M \times [0,1]) \subset U$. Then there exists a diffeotopy $G\colon N \times [0,1] \to N$ with the following properties:

(1) $G_0 = \mathrm{id}$.
(2) We have $G \circ (F_0 \times \mathrm{id}) = F\colon M \times [0,1] \to N$, in other words, we have
$$G(F_0(x), t) \;=\; F(x,t) \quad \text{for all } x \in M \text{ and } t \in [0,1].$$
(3) (a) G is a diffeotopy rel $N \setminus U$.
(b) G is a diffeotopy rel some neighborhood of $\partial N \setminus B$.
(c) G has compact support.
(d) G is trivial near the ends.

Proof. We procrastinate a bit and provide the proof in later sections. More precisely, in Sections 36.3 and 36.4 we will prove the Isotopy Extension Theorem 36.1 in the special case that M and N are closed smooth manifolds. The general case is dealt with in Section 36.5. ∎

[357] By the Trivial-Near-The-Ends-Lemma 33.4 we know that any smooth isotopy $F\colon M \times [0,1] \to N$ can be modified easily to obtain a smooth isotopy $\widetilde{F}\colon M \times [0,1] \to N$ that is trivial near the ends.



Remark.

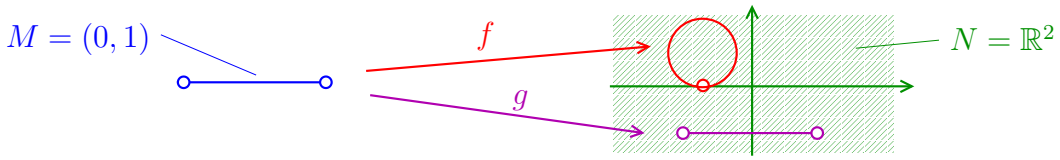
(1) Our proof of the Isotopy Extension Theorem 36.1 is modelled on the proofs of Theorem 2.4.2 and Theorem 2.4.6 of [**Wall2016**]. Our statement is somewhat more general though than the corresponding statements in [**Wall2016**] since we allow maps that take values in ∂N. We also refer to [**Kos1993**, Theorem II.5.2], [**Hirs1961**, Theorem 8.1.3] and [**Pal1960b**, Chapter 6] for related results.

(2) Note that there is a version of the Isotopy Extension Theorem 36.1 for topological manifolds and for "PL manifolds". We refer to [**EKi1971**, Corollary 1.4] and [**FNOP2019**, Theorem 2.10] respectively to [**RS1972**, Theorem 4.24] for details. In both cases one should carefully read the statements before applying them. We have no intention of making use of these results.

Example.

(1) Let N be a smooth manifold. The Isotopy Extension Theorem 36.1 often gets applied as follows: M is a (proper) submanifold in N and we start out with the inclusion map $i\colon M \to N$. Thus in this setting loosely speaking the Isotopy Extension Theorem 36.1 says, that if we deform M in N via (proper) submanifolds, then we can do so by "dragging N" along.

(2) In the statement of the Isotopy Extension Theorem 36.1 it is essential that M is compact. For example, let us consider the two smooth embeddings $f, g\colon (0,1) \to \mathbb{R}^2$ that are shown in the figure below. In Exercise 36.5 we will see that f and g are smoothly isotopic but we will also see that there is no diffeotopy $G\colon \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ with $G_0 = \mathrm{id}$ and $G_1 \circ f = g$. Another, even more subtle example, will be studied in Exercise 36.9.



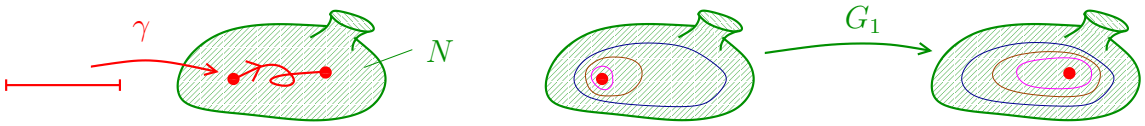
To formulate our last example of this section we need to introduce the following definition.

Definition. Let N be a smooth manifold and let $\gamma\colon [0,1] \to N$ be a smooth path. We say γ is trivial near the ends if there exists an $\epsilon > 0$ such that for each $t \in [0, \epsilon]$ we have $\gamma(t) = \gamma(0)$ and if for each $t \in [1-\epsilon, t]$ we have $\gamma(t) = \gamma(1)$.[358]

[358] As so often it follows from the Trivial-Near-The-Ends-Lemma 33.4 that we can turn any smooth path into a smooth path that is trivial near the ends.

Lemma 36.2. Let N be a smooth manifold and let $\gamma\colon [0,1] \to N \setminus \partial N$ be a smooth path that is trivial near the ends. There exists a diffeotopy $G\colon N \times [0,1] \to N$ with compact support, rel a neighborhood of ∂N, with $G_0 = \mathrm{id}$ and with $G(P,t) = \gamma(t)$ for all $t \in [0,1]$.

Proof. Let $\gamma\colon [0,1] \to N \setminus \partial N$ be a smooth path that is trivial near the ends. Note that the map $\{0\} \times [0,1] \to N \setminus \partial N$ that is given by $(0,t) \mapsto \gamma(t)$ is a smooth isotopy that is trivial near the ends. It follows from the Isotopy Extension Theorem 36.1 that there exists a diffeotopy $G\colon N \times [0,1] \to N$ with $G_0 = \mathrm{id}$ such that for each $t \in [0,1]$ we have $G(\gamma(0),t) = \gamma(t)$. ■



36.2. **Consequences of the Isotopy Extension Theorem 36.1.** The following corollary to the Isotopy Extension Theorem 36.1 is quite useful.

Proposition 36.3. (Smooth Isotopies-Complements Proposition) Let M be a compact smooth manifold and let N be a smooth manifold. Let $f\colon M \to N$ and $g\colon M \to N$ be two proper smooth embeddings. We suppose that there exists a proper smooth isotopy $F\colon M \times [0,1] \to N$ from f to g. Then the following two statements hold:

(1) There exists a diffeomorphism $\Phi\colon N \to N$ with $\Phi(f(M)) = g(M)$.

(2) There exists a diffeomorphism[359] from $N \setminus f(M)$ to $N \setminus g(M)$.

If N is oriented, then we can arrange the diffeomorphisms in (1) and (2) to be orientation-preserving.

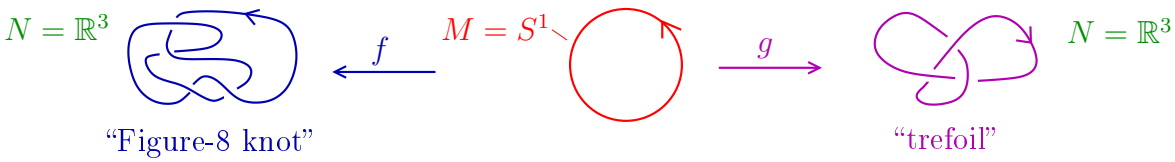
Proof. Let M be a compact smooth manifold and let N be a smooth manifold. Let $f\colon M \to N$ and $g\colon M \to N$ be two proper smooth embeddings. Note that here proper means that $f^{-1}(\partial N) = \partial M$ and $g^{-1}(\partial N) = \partial M$. We suppose that there exists a proper smooth isotopy $F\colon M \times [0,1] \to N$ from f to g. It follows from the Trivial-Near-The-Ends-Lemma 33.4 that we can assume that F is in fact trivial near the ends.

By the Isotopy Extension Theorem 36.1, applied with $A = \partial M$ and $B = \partial N$, we know that there exists a diffeotopy $G\colon N \times [0,1] \to N$ with $G_0 = \mathrm{id}$ such that $G_1(f(M)) = g(M)$. But this implies that $G_1\colon N \to N$ is a diffeomorphism with $G_1(f(M)) = g(M)$. Evidently G_1 restricts to a diffeomorphism $N \setminus f(M) \to N \setminus g(M)$. Finally note that if N is oriented, then it follows from $G_0 = \mathrm{id}$ and the Smooth Isotopy-Orientation Proposition 34.2 that G_1 is also orientation-preserving. ■

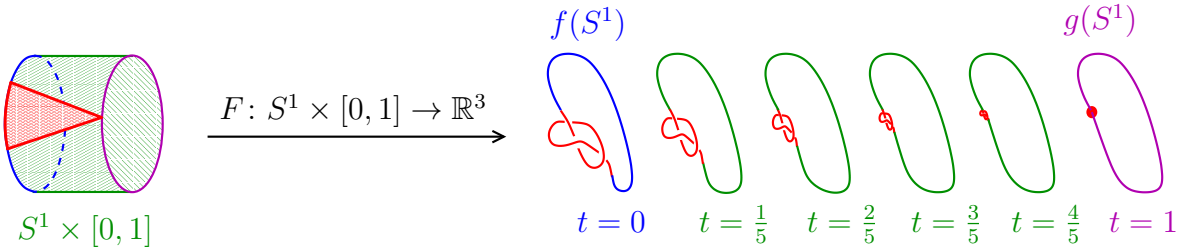
Example. A typical application of the Smooth Isotopies-Complements Proposition 36.3 is to consider smooth embeddings of S^1 into $\mathbb{R}^3$. In the figure below we see two such smooth embeddings $f, g\colon S^1 \to \mathbb{R}^3$. The question arises, whether they are smoothly isotopic or not. In Chapter 47.1 we will introduce the "fundamental group $\pi_1(X)$ of a path-connected topological X" whose isomorphism type is a homeomorphism invariant of X. In Chapter 59 we will see that the fundamental groups of the complements of $f(S^1)$ and $g(S^1)$ are not

[359] Since M is compact we know by the Compact Image Lemma 2.13 and the Compact-Closed Lemma 1.21 that $N \setminus f(M)$ and $N \setminus g(M)$ are open subsets of N, in particular they are smooth manifolds in their own right.

isomorphic. By the Smooth Isotopies-Complements Proposition 36.3 this implies that f and g are not smoothly isotopic.



Example. The statement of the Smooth Isotopies-Complements Proposition 36.3 and the statement of the Isotopy Extension Theorem 36.1 are quite subtle. In the figure below, which is basically identical to the figure on page 813, we illustrate two smooth embeddings $f, g\colon S^1 \to \mathbb{R}^3$ and we illustrate a map $F\colon S^1 \times [0,1] \to \mathbb{R}^3$ with $F_0 = f$, $F_1 = g$ and with the property that given any $t \in [0,1]$ the corresponding map $F_t\colon S^1 \to \mathbb{R}^3$ is a smooth embedding. But similar to the previous example, we will see in Section 59.2 that the complements $\mathbb{R}^3 \setminus f(S^1)$ and $\mathbb{R}^3 \setminus g(S^1)$ are *not* diffeomorphic. This shows that in the formulations of the Smooth Isotopies-Complements Proposition 36.3 and the Isotopy Extension Theorem 36.1 it is essential that the map F is smooth on all of $M \times [0,1]$.



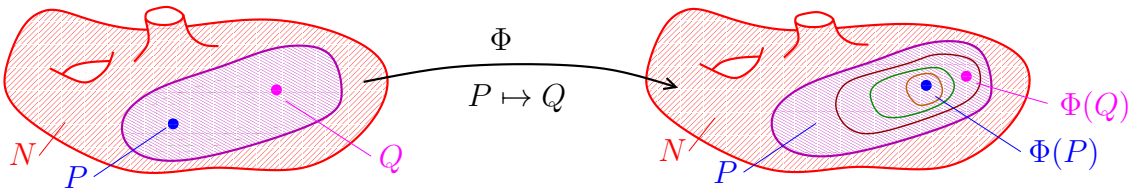
We conclude this section with the following refinement of the Homogeneous-via-Diffeotopies Proposition 33.14 for smooth manifolds.

Proposition 36.4. (Homogeneous-via-Diffeotopies Proposition) Let N be a connected n-dimensional smooth manifold.

(1) Given any two points P and Q in $N \setminus \partial N$ there exists a diffeotopy rel a neighborhood of ∂N from the identity to a diffeomorphism $\Phi\colon N \to N$ with $\Phi(P) = Q$.

(2) If $n \geq 2$, then given distinct points $P_1, \dots, P_m$ and distinct points $Q_1, \dots, Q_m$ in $N \setminus \partial N$ there exists a diffeotopy rel a neighborhood of ∂N from the identity to a diffeomorphism $\Phi\colon N \to N$ with $\Phi(P_i) = Q_i$ for $i = 1, \dots, m$.

Furthermore, in both cases we can furthermore arrange that the diffeotopy has compact support.[360]



[360]Note that if N is oriented, then it follows from the Smooth Isotopy-Orientation Proposition 34.2 that in both cases Φ is in fact orientation-preserving.

Remark.

(1) In Exercise 33.8 we gave a down-to-earth proof of Proposition 36.4. In fact the down-to-earth proof also works for topological manifolds.
(2) For *non-orientable* smooth manifolds we will prove in the Orientation-Flipping Proposition 37.4 a useful addendum to Proposition 36.4.

Proof. Let N be a connected n-dimensional smooth manifold.

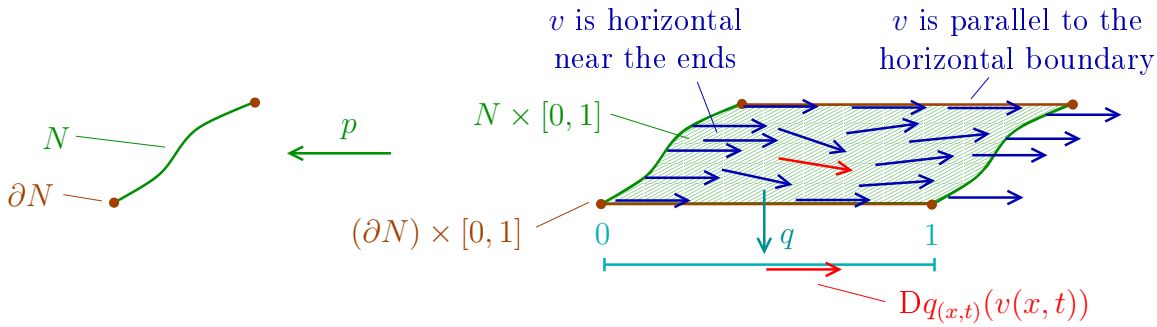
(1) Let P, Q be two points in $N \setminus \partial N$. By the Push Away From Boundary Lemma 18.13 we know that $N \setminus \partial N$ is also path-connected. It follows from the Smooth Path-Connectivity Proposition 19.29 that there exists a smooth path $F\colon [0,1] \to N \setminus \partial N$ from P to Q. As we saw in the proof of Lemma 36.2, a smooth path is basically the same as a smooth isotopy of a 0-dimensional manifold. By the Trivial-Near-The-Ends-Lemma 33.4 we can arrange that this smooth isotopy is trivial near the ends. The desired statement now follows from the Isotopy Extension Theorem 36.1.
(2) We prove this statement by induction on m. We dealt with the case $m = 1$ in (1). Now suppose that we know the statement for some $m \in \mathbb{N}_{\geq 2}$. Thus let $P_1, \dots, P_{m+1}$ and $Q_1, \dots, Q_{m+1}$ be two sets of distinct points in $N \setminus \partial N$. By induction we can without loss of generality already assume that $P_i = Q_i$ for $i = 1, \dots, m$.[361] Now pick disjoint closed n-balls around $P_1, \dots, P_k$ that are disjoint from $P := P_{m+1}$ and $Q := Q_{m+1}$. We denote by W the result of removing the corresponding open balls from M. By the Codimension-Zero Submanifold Proposition 19.38 we know, using the fact that $n \geq 2$, that W is a submanifold which is still connected. Thus we can apply (1) to P and Q in W to obtain a diffeotopy of W with all the desired properties. Since the diffeotopy is the identity near the boundary of W we can extend it via the identity to a diffeotopy of all of N. ∎

36.3. **Proof of the Isotopy Extension Theorem 36.1 I: Preparations.** In the following three subsection we will provide a proof of the Isotopy Extension Theorem 36.1. First we introduce some notation.

Definition. Let N be a smooth manifold. We consider the smooth manifold $N \times [0,1]$ with corner. We denote by $p\colon N \times [0,1] \to N$ and $q\colon N \times [0,1] \to [0,1]$ the two obvious projections. Let v be a vector field on[362] $N \times [0,1]$.

(1) We say v is horizontal near the ends if there exists an $\epsilon > 0$ such that for all $x \in N$ and all $t \in [0, \epsilon] \cup [1-\epsilon, 1]$ we have $v(x,t) \in \ker(\mathrm{D}p_{(x,t)}\colon \mathrm{T}_{(x,t)}(\partial N \times [0,1]) \to \mathrm{T}_x N)$.
(2) We say v is horizontally normal if for each $(x,t) \in N \times [0,1]$ we have
$$\mathrm{D}q_{(x,t)}(v(x,t)) \;=\; 1 \;\in\; \mathrm{T}_t\mathbb{R} = \widetilde{\mathrm{T}}_t\mathbb{R} = \mathbb{R}.$$
(3) We say v is parallel to the horizontal boundary if for each $(x,t) \in \partial N \times [0,1]$ we have $v(x,t) \in \mathrm{T}_{(x,t)}(\partial N \times [0,1])$.
(4) (a) Given $(x,t) \in N \times [0,1]$ we denote by $\overline{h}$ the unique vector in $\mathrm{T}_{(x,t)}(N \times [0,1])$ with $\mathrm{D}q_{(x,t)}(h) = 1$ and $\mathrm{D}p_{(x,t)}(h) = 0$.
(b) We say that v is $\overline{h}$-trivial on a set $U \subset N \times [0,1]$ if $v(x,t) = \overline{h}$ for all $(x,t) \in U$.

[361] The super vigilant reader will have noticed that implicitly we are using the Smooth Isotopy Transitivity Proposition 33.3.



The key to proving the Isotopy Extension Theorem 36.1 is the following consequence of the Flow Theorem 35.1.

Theorem 36.5. (Vector Fields-Diffeotopies Theorem) Let N be a smooth manifold. We denote by $p\colon N \times [0,1] \to N$ the projection. Suppose we are given a smooth vector field v on $N \times [0,1]$ with the following properties:

(1) The vector field is horizontal near the ends.
(2) The vector field is horizontally normal.
(3) The vector field is parallel to the horizontal boundary.
(4) The vector field is $\overline{h}$-trivial outside of a subset of the form $K \times [0,1]$ where K is a compact subset of N.

Then there exists a[363] diffeotopy $G\colon N \times [0,1] \to N$ of N with $G_0 = \mathrm{id}_N$ and with the property, that for each $(Q, t_0) \in N \times [0,1]$ we have

$$\frac{d}{dt}(t \mapsto G(Q,t))|_{t=t_0} = \quad \mathrm{D}p_{(G(Q,t_0),t_0)}(v(\underbrace{G(Q,t_0),t_0}_{\in N\times[0,1]})) \quad \in \ \mathrm{T}_{G(Q,t_0)}N.$$

Furthermore this diffeotopy is trivial near the ends and it is a diffeotopy rel $N \setminus K$.

Example. Suppose we are already given a diffeotopy $H\colon W \times [0,1] \to W$ on some compact smooth manifold W that has the properties (1)–(4) stated in the Vector Fields-Diffeotopies Theorem 36.5. We consider the vector field w on $W \times [0,1]$ that is given by

$$w(P,t_0) \ = \ \frac{d}{dt}(t \mapsto H(P,t))|_{t=t_0} \ \in \ \mathrm{T}_{(P,t_0)}(W \times [0,1]).$$

One can verify easily that this vector field satisfies the hypotheses of the Vector Fields-Diffeotopies Theorem 36.5 and that the original diffeotopy H has the properties that are promised in the Vector Fields-Diffeotopies Theorem 36.5.

The remainder of this section is dedicated to the proof of the Vector Fields-Diffeotopies Theorem 36.5. The proof consists mostly of massaging the setting till eventually one can apply the Flow Theorem 35.1. The proof does not contain any particularly interesting idea.

[362] Strictly speaking we never introduced the tangent space of a smooth manifold with corner. But literally the same definition as on page 628 also works for smooth manifolds with corner and all the obvious statements hold again. As always we strive for brevity and we will thus not develop the concept of tangent spaces of smooth manifolds with corner in detail.

[363] One can easily show, using the uniqueness statement of the Fundamental Theorem 35.7 for Autonomous ODEs, that the diffeotopy is uniquely determined by the stated properties. But we will not make use of the uniqueness statement and thus we do not claim it.

Proof. Let N be a smooth manifold. As before we denote by $p\colon N \times [0,1] \to N$ and $q\colon N \times [0,1] \to [0,1]$ the two projection s. Suppose we are given a smooth vector field v on $N \times [0,1]$ with the following properties:

(1) The vector field is horizontal near the ends.
(2) The vector field is horizontally normal.
(3) The vector field is parallel to the horizontal boundary.
(4) The vector field is $\overline{h}$-trivial outside of a subset of the form $K \times [0,1]$ where K is a compact subset of N.

We can view v as a "time dependent" vector field on N. The Flow Theorem 35.1 does not apply to time dependent vector fields. We will get around this issue by applying the Flow Theorem 35.1 instead to the compact smooth manifold $N \times (\mathbb{R}/\mathbb{Z})$ for an appropriate vector field $\widetilde{v}$. We get a diffeotopy $\widetilde{G}\colon N \times (\mathbb{R}/\mathbb{Z}) \times [0,1] \to N \times (\mathbb{R}/\mathbb{Z})$ on $N \times (\mathbb{R}/\mathbb{Z})$ such that each corresponding time-t map is just given by addition of t in the second factor. The desired diffeotopy is then given by the map

$$G\colon N \times [0,1] \xrightarrow{(Q,s)\mapsto \widetilde{G}(Q,[0],s)} N \times (\mathbb{R}/\mathbb{Z}) \xrightarrow{\text{projection}} N.$$

We consider the compact smooth manifold $N \times (\mathbb{R}/\mathbb{Z})$ and we consider the following two smooth maps

$$\begin{array}{rcl} \Psi\colon [0,1] & \to & \mathbb{R}/\mathbb{Z} \\ t & \mapsto & [t] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathrm{id}_N \times \Psi\colon N \times [0,1] & \to & N \times (\mathbb{R}/\mathbb{Z}) \\ (x,t) & \mapsto & (x,[t]). \end{array}$$

Claim 1.

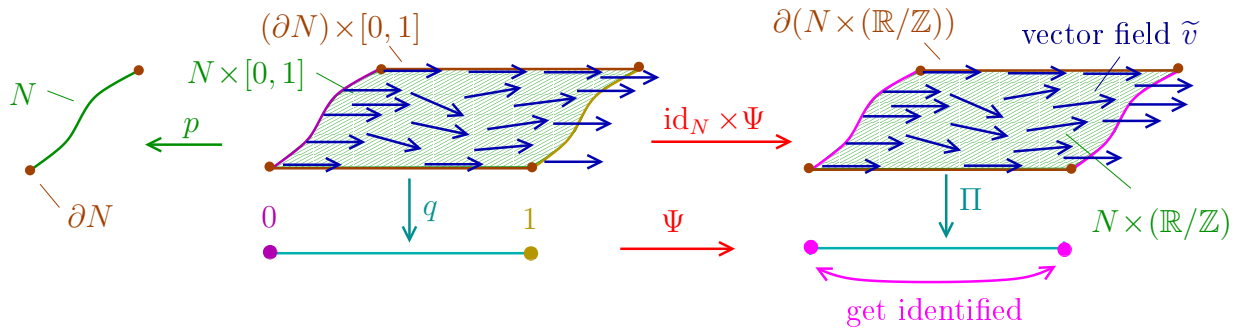
(1) There exists a unique smooth vector field $\widetilde{v}$ on $N \times (\mathbb{R}/\mathbb{Z})$ such that
$$\mathrm{D}(\mathrm{id}_N \times \Psi)_{(x,t)}(v(x,t)) = \widetilde{v}(x,[t]) \qquad \text{for each } (x,t) \in N \times [0,1].$$
(2) The vector field $\widetilde{v}$ on $N \times (\mathbb{R}/\mathbb{Z})$ is parallel to the boundary.

Proof.

(1) Note that it follows from our hypothesis that v is horizontal near the ends and that v is horizontally normal that there exists an $\epsilon > 0$ such that for all $x \in N$ and $t \in [0,\epsilon] \cup [1-\epsilon, 1]$ we have $v(x,t) = \overline{h}$. The promised statement follows easily from this observation. We leave it to the reader to fill in the few remaining details.
(2) It follows easily from the hypothesis that the vector field v on $N \times [0,1]$ is parallel to the horizontal boundary that the vector field $\widetilde{v}$ on $N \times (\mathbb{R}/\mathbb{Z})$ is parallel to the boundary.[364] ⊞

Claim 2. There exists a unique complete $\widetilde{v}$-flow
$$\widetilde{G}\colon N \times (\mathbb{R}/\mathbb{Z}) \times \mathbb{R} \to N \times (\mathbb{R}/\mathbb{Z}).$$

[364]Initially it might appear strange to work with $N \times (\mathbb{R}/\mathbb{Z})$. This smooth manifold has several nice properties though. It has "less boundary" than $N \times [0,1]$, so the vector field $\widetilde{v}$ is now horizontal to the boundary. We could have achieved the same result by working with $N \times \mathbb{R}$. But in our setting we will make use of the fact that $K \times (\mathbb{R}/\mathbb{Z})$ is compact, whereas $K \times \mathbb{R}$ is not compact.



Proof. As we saw in Claim 1, the vector field $\widetilde{v}$ on $N \times (\mathbb{R}/\mathbb{Z})$ is parallel to the boundary. Thus, if N is in fact compact, then the claim follows immediately from the above observation and the Flow Theorem 35.1 applied to the vector field $\widetilde{v}$ on the *compact* smooth manifold $N \times (\mathbb{R}/\mathbb{Z})$.

On the other hand, if N is non-compact, then unfortunately we cannot apply the Flow Theorem 35.1 as it is formulated, so we have to get our hands dirty a little bit. Recall that we assume that the vector field v is $\overline{h}$-trivial outside of a subset of the form $K \times [0,1]$ where K is a compact subset of N. This means that $v(x,t) = \overline{h}$ for all $(x,t) \in (N \times [0,1]) \setminus K$.

It follows from the Product Topology Properties Proposition 5.6 that $K \times (\mathbb{R}/\mathbb{Z})$ is also compact. One can easily verify that the following map defines a $\widetilde{v}$-flow:

$$\begin{array}{rcl} \widetilde{G}\colon (N \setminus K) \times (\mathbb{R}/\mathbb{Z}) \times \mathbb{R} & \to & N \times (\mathbb{R}/\mathbb{Z}) \\ ((x,[t]),s) & \mapsto & (x,[t+s]). \end{array}$$

Furthermore note that we can cover the compact set $K \times (\mathbb{R}/\mathbb{Z})$ by *finitely* many charts. It is now fairly straightforward to adapt the proof of the Flow Theorem 35.1 to give us the desired result.[365] We leave it to the reader, who is eager to show their technical prowess, to fill in the details. ⊞

Now we know that there exists a unique complete $\widetilde{v}$-flow $\widetilde{G}$. By definition of a complete $\widetilde{v}$-flow this means in particular that the following two statements hold:

(a) For each $(Q,[s]) \in N \times (\mathbb{R}/\mathbb{Z})$ we have $\widetilde{G}(Q,[s],0) = (Q,[s])$.

(b) For each $(Q,[s]) \in N \times (\mathbb{R}/\mathbb{Z})$ the map $\mathbb{R} \to N \times (\mathbb{R}/\mathbb{Z})$ given by $t \mapsto \widetilde{G}(Q,[s],t)$ is a solution of the ODE $y' = \widetilde{v}(y)$, in other words, for every $t_0 \in \mathbb{R}$ we have $\frac{d}{dt}(t \mapsto \widetilde{G}(Q,[s],t))|_{t=t_0} = \widetilde{v}(\widetilde{G}(Q,[s],t_0))$.

In the following we denote by $\Pi\colon N \times (\mathbb{R}/\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ the obvious projection.

Claim 3. For every $(Q,[s],t) \in N \times (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ we have $\Pi(\widetilde{G}(Q,[s],t)) = [s+t]$.

Proof. First note that we have the following commutative diagram

$$\begin{array}{ccc} N \times [0,1] & \xrightarrow{\mathrm{id}_N \times \Psi} & N \times (\mathbb{R}/\mathbb{Z}) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \Pi} \\ [0,1] & \xrightarrow{\Psi} & \mathbb{R}/\mathbb{Z}. \end{array}$$

[365] If $\partial N = \varnothing$, then we could apply the Compact Submanifold Exhaustion Proposition 31.2 to arrange that K is actually a compact smooth manifold.

Now let $(Q,[s],t_0) \in N \times (\mathbb{R}/\mathbb{Z}) \in \mathbb{R}$. We pick $(x,\tau) \in N \times [0,1]$ with $(x,[\tau]) = \widetilde{G}(Q,[s],t_0)$. We calculate

$$\begin{array}{lll}
\frac{d}{dt}\big(t \mapsto \Pi(\widetilde{G}(Q,[s],t))\big)\big|_{t=t_0} & \overset{\text{Chain Rule 23.10}}{=} & \mathrm{D}\Pi_{\widetilde{G}(Q,[s],t)}\big(\frac{d}{dt}(t \mapsto \widetilde{G}(Q,[s],t))\big|_{t=t_0}\big) \\
& = & (\mathrm{D}\Pi_{(x,[\tau])} \circ \mathrm{D}(\mathrm{id}_N \times \Psi)_{(x,\tau)})(v(x,\tau)) \\
& \overset{\text{by the definition of } \widetilde{v} \text{ and the choice of } (x,\tau)}{=} & \text{the image of } 1 \in \mathrm{T}_\tau\mathbb{R} \text{ in } \mathrm{T}_{[\tau]}(\mathbb{R}/\mathbb{Z}). \\
& \overset{\text{since } v \text{ is horizontally normal}}{=} &
\end{array}$$

$$\begin{array}{lll}
\overset{\text{by the above Statement (b) and the definition of } \tau}{=} & \mathrm{D}\Pi_{(x,[\tau])}(\widetilde{v}(\widetilde{G}(Q,[s],t_0))) \\
= & (\mathrm{D}\Psi_\tau \circ \mathrm{D}q_{(x,\tau)})(v(x,\tau)) \\
\overset{\text{by the above commutative diagram together with the Chain Rule 23.10}}{=} &
\end{array}$$

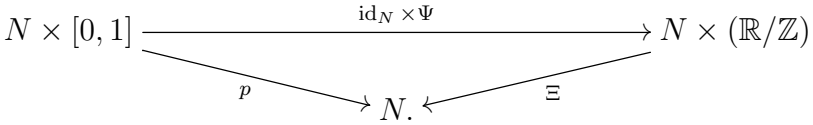
The claim follows almost immediately from this calculation and some elementary arguments from real analysis. ⊞

Finally we denote by $\Xi\colon N \times (\mathbb{R}/\mathbb{Z}) \to N$ the natural projection and we consider the map

$$G\colon N \times [0,1] \xrightarrow{(Q,s)\mapsto \widetilde{G}(Q,[0],s)} N \times (\mathbb{R}/\mathbb{Z}) \xrightarrow{\Xi} N.$$

Claim 4. The map $G\colon N \times [0,1] \to N$ has the desired properties.

Proof. First note that it follows from (a) that $G_0 = \mathrm{id}_N$. Next note that it follows easily from Claim 3 together with the Submanifold Smooth Structure Lemma 19.24 (3) that G is a diffeotopy. Next we want to show that the derivatives of $t \mapsto G(Q,t)$ are of the promised form. Note that we have the following simple minded commutative diagram:



Now let $(Q,t_0) \in N \times [0,1]$. By Claim 3 we know that $\widetilde{G}(Q,[0],t_0)$ is of the form $\widetilde{G}(Q,[0],t_0) = (x,[t_0])$ for some $x \in N$. We calculate

$$\begin{array}{lll}
\frac{d}{dt}(t \mapsto G(Q,t))|_{t=t_0} & \overset{\text{definition of } G}{=} & \frac{d}{dt}(t \mapsto \Xi(\widetilde{G}(Q,[0],t)))|_{t=t_0} \\
& = & \mathrm{D}\Xi_{(x,[t_0])}(\widetilde{v}(x,[t_0])) \\
& \overset{\text{by (b)}}{=} & \mathrm{D}p_{(x,t_0)}(v(x,t_0)) \\
& \overset{\text{by the above commutative diagram}}{=} &
\end{array}$$

$$\begin{array}{lll}
\overset{\text{Chain Rule 23.10}}{=} & \mathrm{D}\Xi_{(x,[t_0])}\big(\frac{d}{dt}(t \mapsto \widetilde{G}(Q,[0],t))\big|_{t=t_0}\big) \\
= & (\mathrm{D}\Xi_{(x,[t_0])} \circ \mathrm{D}(\mathrm{id}_N \times \Psi))(v(x,t_0)) \\
\overset{\text{by definition of } \widetilde{v}}{=} & \mathrm{D}p_{(G(x,t_0),t_0)}(v(G(x,t_0),t_0)). \\
\overset{\text{by the definition of } G \text{ and } x}{=} &
\end{array}$$

We conclude the proof of the claim with two observations:

(1) It follows from this calculation, the uniqueness statement of the Fundamental Theorem 35.7 for Autonomous ODEs and the fact that the vector field v is horizontal near the ends, that the diffeotopy $G\colon N \times [0,1] \to N$ is trivial near the ends.
(2) Recall that we assume that there exists a compact subset $K \subset N$ such that the vector field v is $\overline{h}$-trivial outside of $K \times [0,1]$. It follows from the above calculation and the uniqueness statement of the Fundamental Theorem 35.7 for Autonomous ODEs that the diffeotopy $G\colon N \times [0,1] \to N$ is a diffeotopy rel $N \setminus K$. ■

36.4. Proof of the Isotopy Extension Theorem 36.1 II: Closed smooth manifolds.

We now turn to the actual proof of the Isotopy Extension Theorem 36.1. In this section we will prove the theorem if M and N are closed. This setting makes the discussion much easier and the key idea of the proof becomes clearer. The general case will be proved in the subsequent section.

Evidently, in the setting of the Isotopy Extension Theorem 36.1, we want to use the Vector Fields-Diffeotopies Theorem 36.5 to construct the desired diffeotopy. Thus we need to construct suitable vector fields. We will do so locally and patch them together. This leads us to the following variation on the above definitions from page 865.

Definition. Let N be a smooth manifold. Let $U \subset N \times [0,1]$ be a subset and let v be a vector field on U. We denote by $p\colon N \times [0,1] \to N$ and $q\colon N \times [0,1] \to [0,1]$ the two obvious projections.

(1) We say v is horizontal near the ends if there exists an $\epsilon > 0$ such that for all $(x,t) \in U$ with $t \in [0,\epsilon] \cup [1-\epsilon, 1]$ we have $v(x,t) \in \ker(\mathrm{D}p_{(x,t)}\colon \mathrm{T}_{(x,t)}(\partial N \times [0,1]) \to \mathrm{T}_x N)$.

(2) We say v is horizontally normal if for each $(x,t) \in U$ we have

$$\mathrm{D}q_{(x,t)}(v(x,t)) \;=\; 1 \;\in\; \mathrm{T}_t\mathbb{R} \;=\; \widetilde{\mathrm{T}}_t\mathbb{R} \;=\; \mathbb{R}.$$

The following lemma is the key to patching together promising vector fields.

Lemma 36.6. Let N be a smooth manifold, let $U \subset N \times [0,1]$ be an open subset, let v_1, v_2 be two vector fields on U and let $f_1, f_2\colon U \to \mathbb{R}$ be two smooth maps.

(1) If v_1 and v_2 are horizontal near the ends, then so is $f_1 \cdot v_2 + f_2 \cdot v_2$.

(2) If $f_1 + f_2 \equiv 1$ and if v_1 and v_2 are horizontally normal, then so is $f_1 \cdot v_2 + f_2 \cdot v_2$.

Proof. We start out with the proof of Statement (1). We assume that the vector fields v_1 and v_2 are horizontal near the ends. With the hopefully obvious notation we pick suitable $\epsilon_1 > 0$ and $\epsilon_2 > 0$ and we set $\epsilon := \min\{\epsilon_1, \epsilon_2\}$. Now let $(x,t) \in U$ with $t \in [0,\epsilon] \cup [1-\epsilon, 1]$. We see that

$$\mathrm{D}p_{(x,t)}(f_1(x,t)\cdot v_2(x,t) + f_2(x,t)\cdot v_2(x,t)) \overset{\text{since differentials are linear}}{\downarrow} = f_1(x,t)\cdot\underbrace{\mathrm{D}p_{(x,t)}(v_1(x,t))}_{=0} + f_2(x,t)\cdot\underbrace{\mathrm{D}p_{(x,t)}(v_2(x,t))}_{=0}.$$

Next we turn to the proof of Statement (2). Now we assume that $f_1 + f_2 \equiv 1$. We see that

$$\begin{aligned}\mathrm{D}q_{(x,t)}(f_1(x,t)\cdot v_2(x,t) + f_2(x,t)\cdot v_2(x,t)) &= f_1(x,t)\cdot\underbrace{\mathrm{D}q_{(x,t)}(v_1(x,t))}_{=1} + f_2(x,t)\cdot\underbrace{\mathrm{D}q_{(x,t)}(v_2(x,t))}_{=1} \\ &= f_1(x,t) + f_2(x,t) \;=\; 1.\end{aligned}$$

■

Now we turn to the proof of the Isotopy Extension Theorem 36.1 in the special case that M and N are both closed smooth manifolds.

Proof of the Isotopy Extension Theorem 36.1 if M and N are closed smooth manifolds. Let M be a closed m-dimensional smooth manifold and let N be a closed n-dimensional smooth manifold. Let $F\colon M \times [0,1] \to N$ be a smooth isotopy which is trivial near the ends. We need to show that there exists a diffeotopy $G\colon N \times [0,1] \to N$ with the following properties:

(1) $G_0 = \mathrm{id}$.

(2) We have $G \circ (F_0 \times \mathrm{id}) = F\colon M \times [0,1] \to N$, in other words, we have

$$G(F_0(x),t) \;=\; F(x,t) \qquad \text{for all } x \in M \text{ and } t \in [0,1].$$

(3) G is trivial near the ends.

The idea is to construct the diffeotopy using the Vector Fields-Diffeotopies Theorem 36.5. We introduce the following notation:

(1) Since $F\colon M \times [0,1] \to N$ is a smooth isotopy it follows as in the fairly straightforward Exercise 33.10 that

$$\begin{array}{rcl} \widetilde{F}\colon M \times [0,1] & \to & N \times [0,1] \\ (P,t) & \mapsto & (F(P,t),t) \end{array}$$

is a proper smooth embedding. In particular we obtain from the Smooth Embedding Theorem 24.10 that $W := \widetilde{F}(M \times [0,1])$ is a proper submanifold of $N \times [0,1]$ such that for any $(P,t) \in M \times (0,1)$ the point $Q = \widetilde{F}(P,t)$ admits a submanifold chart of type (α).

(2) We denote by $p\colon N \times [0,1] \to N$ and $q\colon N \times [0,1] \to [0,1]$ the two projection s.

(3) Basically as on page 866 we consider the vector field w on W that for $(F(P,t),t) \in W \subset N \times [0,1]$ is uniquely determined by

$$\begin{array}{rcl} \mathrm{D}p_{(F(P,t),t)}(w(F(P,t),t)) & = & \frac{d}{ds}\big(s \mapsto F(P,s)\big)|_{s=t} \;\in\; \mathrm{T}_{F(P,t)}N \\ \mathrm{D}q_{(F(P,t),t)}(w(F(P,t),t)) & = & 1 \;\in\; \mathrm{T}_t\mathbb{R} = \mathbb{R}. \end{array}$$

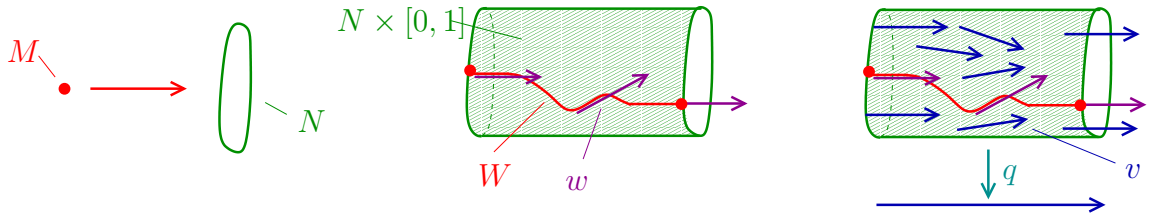
One can easily verify that w is horizontal near the ends and horizontally normal.

Claim 1. There exists a smooth vector field v on $N \times [0,1]$ with the following properties:

(1) The vector field v is horizontal near the ends.

(2) The vector field v is horizontally normal.

(3) On $W \cap (N \times [0,1])$ the vector field v agrees with w.



Proof.

The idea is to construct such a vector field v by constructing suitable vector fields on open subsets of $N \times [0,1]$ which form an open cover of $N \times [0,1]$. We paste these vector fields together using a partition of unity and using Lemma 36.6.

Since the smooth isotopy $F\colon M \times [0,1] \to N$ is by hypothesis trivial near the ends there exists an $\epsilon > 0$ such that $F(P,t) = F(P,0)$ for all $t \in [0,\epsilon]$ all $P \in N$ and such that $F(P,t) = F(P,1)$ for all $t \in [1-\epsilon,1]$ and all $P \in N$. On the open subset $U_0 := N \times [0,1] \setminus \widetilde{F}(M \times [\epsilon,1-\epsilon])$ we define the vector field v_0 to be given by the vector $\overline{h}$ at each point.

Now let $R = \widetilde{F}(P,t)$ with $P \in M$ and $t \in (\epsilon,1-\epsilon)$. As we mentioned above, the point $R = \widetilde{F}(P,t) \in W \subset N \times [0,1]$ admits a submanifold chart of type (α). In fact, similar to the Smooth Embedding Theorem 24.10 one can show that there exists even an

open neighborhood I of $t \in (0,1)$, an open neighborhood U_R of $R \in N \times [0,1]$ and a chart $\Phi_R \colon U_R \to V_R = B^{n-m} \times B^m \times I$ with the following properties:

- $\Phi_R(U_R \cap W) = \{0\} \times B^m \times I$,
- the map $U_R \xrightarrow{\Phi_R} B^{n-m} \times B^m \times I \xrightarrow{(x,y,t) \mapsto t} I$ equals q.

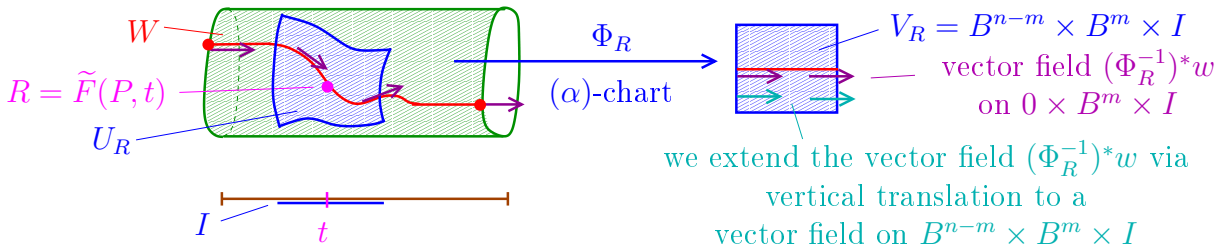
We now use Φ_R to extend w on $U_R \cap W$ to a vector field v_R on U_R as follows:

(x) We consider the vector field $(\Phi_R^{-1})^* w$ on $\Phi_R(U_R \cap W) = \{0\} \times B^m \times I$.
(y) We extend the vector field from (x) to a vector field on $B^{n-m} \times B^m \times I$ by "vertical translation".
(z) We use Φ_R to transport the vector field from (y) back to U_R.

One can can easily verify that the vector field u_0 and that the vector fields u_R are horizontal near the ends, that they are horizontally normal and that they agree with w on W.

Next note that by the Manifold-Partitions of Unity Existence Theorem 21.1 we can find a smooth partition of unity $f_0 \colon N \times [0,1] \to \mathbb{R}$ and $\{f_R \colon N \times [0,1] \to \mathbb{R}\}_{R \in \widetilde{F}(M \times (\epsilon, 1-\epsilon))}$ such that $\operatorname{supp}(f_0) \subset U_0$ and such that we always have $\operatorname{supp}(f_R) \subset U_R$. We denote by $\widetilde{f_0 \cdot v_0}$ the extension of the vector field $f_0 \cdot v_0$ on U_0 to all of $N \times [0,1]$ by setting it zero outside of U_0. The same way we define $\widetilde{f_R \cdot v_R}$. The vector field v is now given by the sum of $\widetilde{f_0 \cdot v_0}$ and of all the vector fields $\widetilde{f_R \cdot v_R}$.

It now follows from a very mild generalization of Lemma 36.6 that v does indeed have all the desired properties. ⊞



We now take the smooth vector field v on $N \times [0,1]$ from Claim 1 and we plug it into the Vector Fields-Diffeotopies Theorem 36.5. We obtain a diffeotopy $G \colon N \times [0,1] \to N$ with $G_0 = \mathrm{id}$ and such that for each $(Q,t) \in N \times [0,1]$ we have

$$\frac{d}{ds}(s \mapsto G(P,s))|_{s=t} \;=\; \mathrm{D}p_{(G(P,t),t)}(v(\underbrace{G(P,t),t}_{\in N \times [0,1]})) \;\in\; \mathrm{T}_{G(Q,t)}N.$$

Furthermore we saw in the Vector Fields-Diffeotopies Theorem 36.5 that G is trivial near the ends. Thus it remains to prove the following claim.

Claim 2. For each $x \in M$ and $t \in [0,1]$ we have $G(F_0(x),t) = F(x,t)$.

Proof. Let $x \in M$. We have $G(F_0(x),0) = F_0(x) = F(x,0)$. Furthermore, as we had just seen, for each $t \in [0,1]$ we have

$$\frac{d}{ds}\big(s \mapsto G(F_0(x),s)\big)\big|_{s=t} \;\underset{\text{see above}}{=}\; \mathrm{D}p_{(G(F_0(x),s),s)}(v(G(F_0(x),t),t).$$

Next note that we have

$$\frac{d}{ds}(s \mapsto F(x,s))|_{s=t} \underset{\substack{\uparrow \\ \text{definition of } w, \text{ see page 871}}}{=} \mathrm{D}p_{(F(x,t),t)}(w(F(x,t),t)) \underset{\substack{\uparrow \\ \text{since } w = v \text{ on } W}}{=} \mathrm{D}p_{(F(x,t),t)}(v(F(x,t),t)).$$

It follows from these observations that the maps

$$\begin{array}{rcl} [0,1] & \to & M \times [0,1] \\ t & \mapsto & (G(F_0(x),t),t) \end{array} \quad \text{and} \quad \begin{array}{rcl} [0,1] & \to & M \times [0,1] \\ t & \mapsto & (F(x,t),t) \end{array}$$

are both solutions of the ODE $(y,t)' = (\mathrm{D}p_{(y,t)}(v(y,t)),1)$ on $N \times [0,1]$ and both satisfy the same starting condition. Using charts one can apply the uniqueness statement of the Fundamental Theorem 35.7 for Autonomous ODEs and we see that for all $t \in [0,1]$ we have $G(F_0(x),t) = F(x,t)$. ■

36.5. **Proof of the Isotopy Extension Theorem 36.1 III: The general case.** In this section we finally provide the proof of the Isotopy Extension Theorem 36.1 in its fullest generality. The basic idea is identical to the idea of the proof in the special case considered in the previous section. But the presence of boundary and the fact that we allow N to be non-compact makes everything technically much more demanding and, let's face it, much less readable.

Definition. Let N be a smooth manifold. Let $U \subset N \times [0,1]$ be a subset and let v be a vector field on U.

(3) We say v is parallel to the horizontal boundary if for each $(x,t) \in (\partial N \times [0,1]) \cap U$ we have $v(x,t) \in \mathrm{T}_{(x,t)}(\partial N \times [0,1])$.

We will need the following addendum to Lemma 36.6.

Lemma 36.6. Let N be a smooth manifold, let $U \subset N \times [0,1]$ be an open subset, let v_1, v_2 be two vector fields on U and let $f_1, f_2 \colon U \to \mathbb{R}$ be two smooth maps.

(3) If v_1 and v_2 are parallel to the horizontal boundary, then so is $f_1 \cdot v_2 + f_2 \cdot v_2$.

Proof. If v_1 and v_2 are two vector fields that are parallel to the horizontal boundary, then for any $(x,t) \in (\partial N \times [0,1]) \cap U$ we have

$$\mathrm{D}p_{(x,t)}(f_1(x,t)\cdot v_2(x,t) + f_2(x,t)\cdot v_2(x,t)) = f_1(x,t)\cdot \underbrace{\mathrm{D}p_{(x,t)}(v_1(x,t))}_{\in \mathrm{T}_{(x,t)}(\partial N \times [0,1])} + f_2(x,t)\cdot \underbrace{\mathrm{D}p_{(x,t)}(v_2(x,t))}_{\in \mathrm{T}_{(x,t)}(\partial N \times [0,1])}.$$

Now we are done, since $\mathrm{T}_{(x,t)}(\partial N \times [0,1])$ is a vector subspace. ■

Proof of the Isotopy Extension Theorem 36.1. Let M be a compact n-dimensional smooth manifold and let N be an m-dimensional smooth manifold. Furthermore let A be a union of components of ∂M and let B be a union of components of ∂N. Finally let $F \colon M \times [0,1] \to N$ be a smooth isotopy which is trivial near the ends and which has the following property:

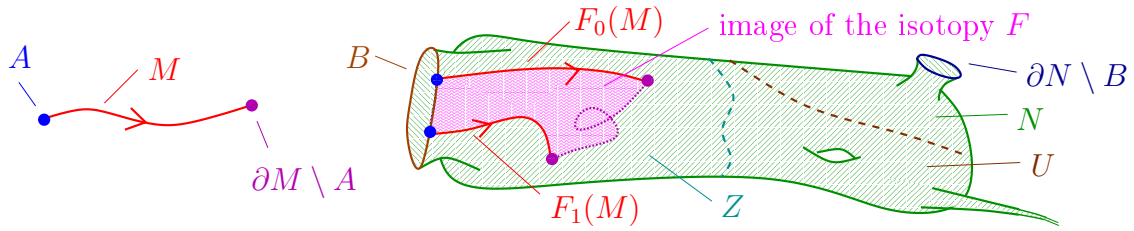
$(*)$ For each $t \in [0,1]$ we have $F_t(A) \subset B$ and we have $F_t(\partial M \setminus A) \subset N \setminus \partial N$.

Finally suppose that we are given an open subset U of N with $F(M \times [0,1]) \subset U$.

Claim 0. There exists an open subset Z of N with the following three properties:

(1) $F(M \times [0,1]) \subset Z \subset U$,

(2) $N \setminus Z$ is a neighborhood of $\partial N \setminus B$,
(3) the closure $\overline{Z}$ is compact.



Proof. By the Smooth Manifold Boundary Proposition 19.26 we know that $\partial N \setminus B$ is a closed subset of N. Furthermore since M is compact we obtain from the Compact-Closed Lemma 1.21, the Product Topology Properties Proposition 5.6 and the Compact Image Lemma 2.13 that $F(M \times [0,1])$ is a closed subset of N. Note that by hypothesis $F(M \times [0,1])$ and $\partial N \setminus B$ are disjoint. By the Manifold Paracompact-Normal Proposition 18.9 (2) we know that N is normal in the sense of the definition on page 123. Thus there exist open disjoint subsets S and T of N with $F(M \times [0,1]) \subset S$ and $\partial N \setminus B \subset T$.

The subset $U \cap S$ satisfies (1) and (2), but not necessarily (3). Thus we apply the Compact Exhaustion Lemma 9.3 (1) (2) to the topological space $U \cap S$, which is second-countable, Hausdorff and regionally compact, and the compact subspace $F(M \times [0,1])$, and we obtain a subset $Z \subset U \cap S$ with all the desired properties. ⊞

It remains to show that there exists a diffeotopy $G\colon N \times [0,1] \to N$ with the following properties:

(1) $G_0 = \mathrm{id}$.
(2) We have $G \circ (F_0 \times \mathrm{id}) = F\colon M \times [0,1] \to N$, in other words, we have

$$G(F_0(x), t) \;=\; F(x,t) \quad \text{for all } x \in M \text{ and } t \in [0,1].$$

(3) G is trivial near the ends.
(4) G is a diffeotopy rel $N \setminus Z$.[366]

Again the idea is to construct the diffeotopy using the Vector Fields-Diffeotopies Theorem 36.5. We introduce the following notation:[367]

(1) We extend the smooth isotopy $F\colon M \times [0,1] \to N \times [0,1]$ to a map $F\colon M \times \mathbb{R} \to N \times \mathbb{R}$ by setting $F(P,t) = F(P,0)$ for $t \le 0$ and $F(P,t) = F(P,1)$ for $t \ge 1$. Since F is a smooth isotopy that is trivial near the ends we obtain from the Smooth Pasting Proposition 19.23 that F is again smooth.
(2) Since F is a smooth isotopy it follows, as in the fairly elementary Exercise 33.10, that

$$\begin{array}{rcl} \widetilde{F}\colon M \times \mathbb{R} & \to & N \times \mathbb{R} \\ (P,t) & \mapsto & (F(P,t),t) \end{array}$$

[366] Since $G_0 = \mathrm{id}$, since G is a diffeotopy rel $N \setminus Z$ and since $\overline{Z}$ is compact we see that G is a diffeotopy with compact support.
[367] In the following we partially work with $N \times \mathbb{R}$ which has the big advantage that this is a smooth manifold with boundary $\partial N \times \mathbb{R}$, whereas $N \times [0,1]$ is a smooth manifold with corner $\partial N \times \{0,1\}$, for which we have fewer ready-made results. For example we never developed the concept of a submanifold of a smooth manifold with corner.

is a smooth embedding such that $\widetilde{F}(M\times\mathbb{R})$ is a closed subset of $N\times\mathbb{R}$. In particular we obtain from the Smooth Embedding Theorem 24.10 and the proof thereof that $W := \widetilde{F}(M\times\mathbb{R})$ is a proper submanifold of $N\times\mathbb{R}$ such that for any $(P,t)\in M\times\mathbb{R}$ the following three statements hold:
(a) If $P\in M\setminus\partial M$, then $Q=\widetilde{F}(P,t)$ admits a submanifold chart of type (α).
(b) If $P\in A$, then $Q=\widetilde{F}(P,t)$ admits a submanifold chart of type (β).
(c) If $P\in\partial M\setminus A$, then $Q=\widetilde{F}(P,t)$ admits a submanifold chart of type (γ).

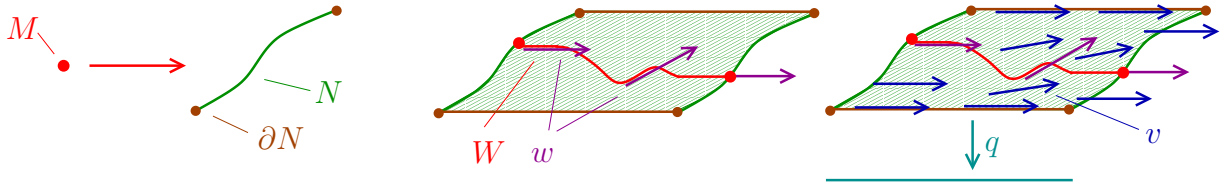
(3) We denote by $p\colon N\times[0,1]\to N$ and $q\colon N\times[0,1]\to[0,1]$ the two projection s.

(4) Basically as on page 866 we consider the vector field w on $W\cap(N\times[0,1])$ that for $(F(P,t),t)\in W\subset N\times[0,1]$ is uniquely determined by

$$\begin{aligned}\mathrm{D}p_{(F(P,t),t)}(w(F(P,t),t)) &= \tfrac{d}{ds}\big(s\mapsto F(P,s)\big)|_{s=t} \in \mathrm{T}_{F(P,t)}N\\ \mathrm{D}q_{(F(P,t),t)}(w(F(P,t),t)) &= 1 \in \mathrm{T}_t\mathbb{R}=\mathbb{R}.\end{aligned}$$

One can easily verify that w is horizontal near the ends, parallel to the horizontal boundary and horizontally normal.

Claim 1. There exists a smooth vector field v on $N\times[0,1]$ with the following properties:
(1) The vector field v is horizontal near the ends.
(2) The vector field v is horizontally normal.
(3) The vector field v is parallel to the horizontal boundary.
(4) The vector field v is $\overline{h}$-trivial outside of the compact set $\overline{Z}$.
(5) On $W\cap(N\times[0,1])$ the vector field v agrees with w.



Proof.

The idea is again to construct such a vector field v by constructing suitable vector fields around each point in $N\times[0,1]$ and by pasting them together using a partition of unity and using Lemma 36.6.

Thus let $R\in N\times[0,1]$.

(α) Suppose that $R=\widetilde{F}(P,t)$ with $P\in M\setminus\partial M$ and $t\in[0,1]$. As we mentioned above, the point $R=\widetilde{F}(P,t)\subset W\subset N\times\mathbb{R}$ admits a submanifold chart of type (α). In fact, similar to the Smooth Embedding Theorem 24.10 one can show that there exists even an open neighborhood I of $t\in\mathbb{R}$, an open neighborhood U_R of $R\in N\times[0,1]$, with $U_R\subset(N\setminus\partial N)\times[0,1]$ and with $U_R\subset Z\times[0,1]$, and a chart $\Phi_R\colon U_R\to V_R=B^{n-m}\times B^m\times I$ with the following properties:
- $\Phi_R(U_R\cap W)=\{0\}\times B^m\times I$,
- the map $U_R\xrightarrow{\Phi_R}B^{n-m}\times B^m\times I\xrightarrow{(x,y,t)\mapsto t}I$ equals q,
- $U_R\cap U=\varnothing$.

We now use Φ_R to extend w on $U_R \cap W$ to a vector field v_R on U_R as follows:
(x) The combination of Φ_R and w defines a vector field on $\Phi_R(U_R \cap W) = \{0\} \times B^m \times I$.
(y) We extend the vector field from (x) to a vector field on $B^{n-m} \times B^m \times I$ by "vertical translation".
(z) We use Φ_R to transport the vector field from (y) back to U_R.

(β) Suppose that $R = \widetilde{F}(P,t)$ with $P \in A$ and $t \in [0,1]$. We pick a submanifold chart $\Phi_R \colon U_R \to V_R = B^{n-m} \times B^m \times I$ of type (β) around R with $U_R \subset Z \times [0,1]$ and we define a vector field v_R on U_R in a similar fashion to (α).

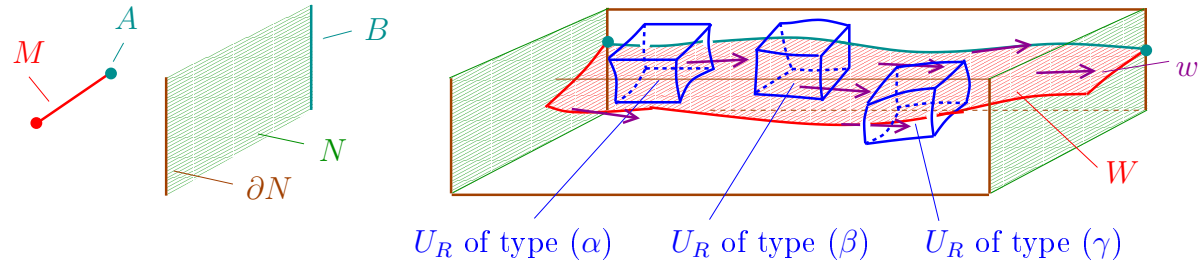
(γ) Suppose that $R = \widetilde{F}(P,t)$ with $P \in \partial M \setminus A$ and $t \in [0,1]$. We pick a submanifold chart $\Phi_R \colon U_R \to V_R = B^{n-m} \times B^m \times I$ of type (γ) around R with $U_R \subset Z \times [0,1]$. We need to modify the battle plan from (α).
(x) Since in our current setting we have a submanifold chart of type (γ) we have $\Phi_R(U_R \cap W) = \{0\} \times B^m_{\geq 0} \times I$. Therefore the combination of Φ_R and w now defines a smooth vector field on $\Phi_R(U_R \cap W) = \{0\} \times B^m_{\geq 0} \times I$.
(x̂) Using the Whitney Extension Theorem 19.6 we can extend the smooth vector field on $\{0\} \times B^m_{\geq 0}$ to a smooth vector field on $\{0\} \times B^m \times I$.
Now we perform (y) and (z) as above.

(δ) If $R \notin W = \widetilde{F}(M \times [0,1])$, then we set $U_R = (N \times [0,1]) \setminus W$ and we denote by v_R the unique $\overline{h}$-trivial vector field on U_R.

Note that one can easily verify that each of these four types of vector fields is horizontal near the ends, parallel to the horizontal boundary and horizontally normal on its domain.

By the Manifold-Partitions of Unity Existence Theorem 21.1 we can find a partition of unity $\{f_R \colon N \times [0,1] \to \mathbb{R}\}_{R \in N \times [0,1]}$ such that for each $R \in N \times [0,1]$ we have $\operatorname{supp}(f_R) \subset U_R$. For each $R \in N \times [0,1]$ we denote by $\widetilde{f_R \cdot v_R}$ the extension of the vector field $f_R \cdot v_R$ on $U_R \cap (N \times [0,1])$ to all of $N \times [0,1]$ by setting it zero outside of U_R. The vector field v is now given by the sum over all the vector fields $\widetilde{f_R \cdot v_R}$.

It follows from a very mild generalization of Lemma 36.6 that v does indeed have all the desired properties. ⊞



We now take the smooth vector field v on $N \times [0,1]$ from Claim 1 and we plug it into the Vector Fields-Diffeotopies Theorem 36.5. We obtain a diffeotopy $G \colon N \times [0,1] \to N$ with $G_0 = \mathrm{id}$ and such that for each $(Q,t) \in N \times [0,1]$ we have

$$\frac{d}{ds}(s \mapsto G(P,s))|_{s=t} \;=\; \mathrm{D}p_{(G(P,t),t)}(v(\underbrace{G(P,t),t}_{\in N \times [0,1]})) \;\in\; \mathrm{T}_{G(Q,t)}N.$$

Furthermore we saw in the Vector Fields-Diffeotopies Theorem 36.5 that G is trivial near the ends and that it is a diffeotopy rel $N \setminus Z$. In particular it is a diffeotopy rel $N \setminus U$, rel some neighborhood of $\partial N \setminus B$ and it has compact support.

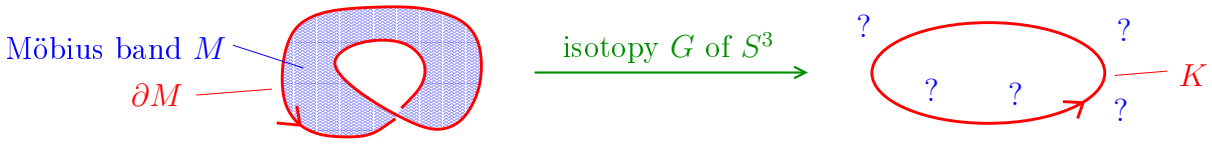
It remains to prove the following claim.

Claim 2. For each $x \in M$ and $t \in [0,1]$ we have $G(F_0(x), t) = F(x,t)$.

Proof. Fortunately we had proved this claim on page 872. ∎

Exercises for Chapter 36.

Exercise 36.1. We consider the Möbius band M in $\mathbb{R}^3$ as shown in the figure below. It should be pretty clear that there exists an isotopy F which turns ∂M into the unknot K. By the Isotopy Extension Theorem 36.1 this isotopy F can be extended to a diffeotopy G of $\mathbb{R}^3$. In particular we obtain a map $G_1 \colon \mathbb{R}^3 \to \mathbb{R}^3$ with $G_1(\partial M) = K$. What does $G_1(M)$ look like? Put differently, we just saw that there exists an embedded Möbius band in $\mathbb{R}^3$ with boundary given by K. What does it look like?



Exercise 36.2.

(a) Give an example of a connected smooth manifold M with empty boundary such that for every $P \in M$ the two smooth manifolds $M \setminus \{P\}$ and M are diffeomorphic.
(b) Give an example of a connected smooth manifold M with non-empty boundary such that for every $P \in M$ the two smooth manifolds $M \setminus \{P\}$ and M are diffeomorphic.

Exercise 36.3. Let $n \in \mathbb{N}_{\geq 2}$.

(a) Let $P \subset \mathbb{R}^n$ be a finite set. Show that there exists a smooth embedding $f \colon S^{n-1} \to \mathbb{R}^n$ with $P \subset f(S^{n-1})$.
(b) Show that the conclusion of (a) in general does not hold if P is a countable bounded subset of $\mathbb{R}^n$.

Exercise 36.4. Let M be a connected 2-dimensional smooth manifold without boundary. Furthermore let $P_1, \dots, P_k \in M$ be disjoint points. We denote by $\pi \colon M \times [0,1] \to [0,1]$ the natural projection. A *braid in* M is a smooth embedding $\varphi \colon \bigsqcup_{i=1}^{k} \{i\} \times [0,1] \to M \times [0,1]$ such that the following four conditions are satisfied:

(1) We have $\varphi(\{1,\dots,k\} \times \{0\}) = \{P_1, \dots, P_k\}$ and $\varphi(\{1,\dots,k\} \times \{1\}) = \{P_1, \dots, P_k\}$.
(2) There exists an $\epsilon > 0$ such that $\varphi(i,t) = \varphi(i,0)$ for $i = 1, \dots, k$ and every $t \in [0, \epsilon]$.
(3) There exists an $\epsilon > 0$ such that $\varphi(i,t) = \varphi(i,1)$ for $i = 1, \dots, k$ and every $t \in [1-\epsilon, 1]$.
(4) For each $i \in \{1, \dots, k\}$ and $t \in [0,1]$ we have $\pi(\varphi(i,t)) = t$.

We say that two braids φ_0, φ_1 as above are *equivalent* if there exists a smooth map $\Phi\colon \bigsqcup_{i=1}^{k} \{i\} \times [0,1] \times [0,1] \to M \times [0,1]$ such that $\Phi_0 = \varphi_0$, $\Phi_1 = \varphi_1$ and such that each Φ_t is a braid. We denote by $B_k(M) := B_k^{P_1,\dots,P_k}(M)$ the set of equivalence classes of braids.

(1) Show that the map

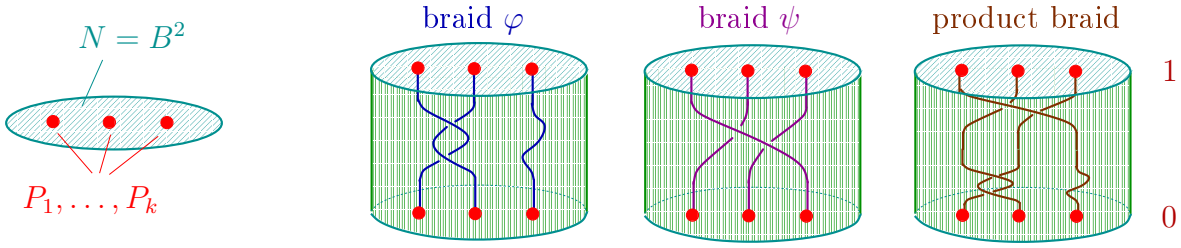
$$\begin{array}{rcl} B_k(M) \times B_k(M) & \to & B_k(M) \\ ([\varphi],[\psi]) & \mapsto & \left[\begin{array}{rcl} \{1,\dots,k\} \times [0,1] & \to & M \\ (i,t) & \mapsto & \begin{cases} \varphi(i,2t), & \text{if } t \in [0,\frac{1}{2}], \\ \psi(i,2t-1), & \text{if } t \in [\frac{1}{2},1] \end{cases} \end{array} \right] \end{array}$$

defines a group structure on $B_k(M)$.

(2) Show that the isomorphism type of $B_k(M)$ does not depend on the choice of the distinct points $P_1, \dots, P_k$.

(3) Why do we only consider 2-dimensional smooth manifolds?

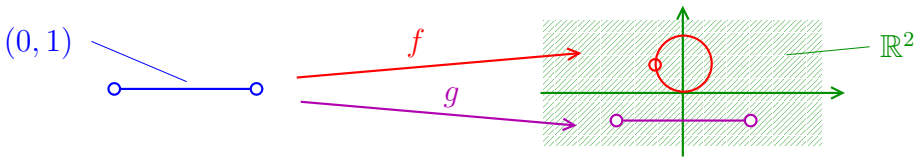
Remark. The group $B_k(M)$ is called the *k-braid group of M*. The group $B_k := B_k(B^2)$ is commonly called *the k-braid group*.



Exercise 36.5. We consider the two smooth embeddings

$$f\colon \begin{array}{rcl} (-1,1) & \to & \mathbb{R}^2 = \mathbb{C} \\ t & \mapsto & \exp(\pi i t) + \mathrm{i} \end{array} \qquad \text{and} \qquad g\colon \begin{array}{rcl} (-1,1) & \to & \mathbb{R}^2 \\ t & \mapsto & (2t,-1) \end{array}$$

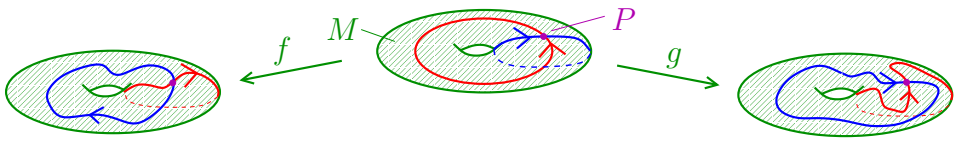
that are shown in the figure below. Show that f and g are smoothly isotopic and show that there is no diffeotopy $G\colon \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ with $G_0 = \mathrm{id}$ and $G_1 \circ f = g$.



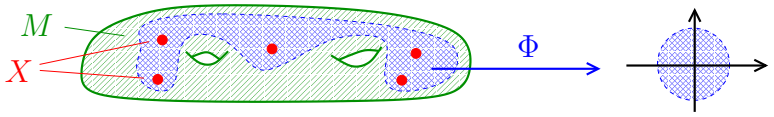
Exercise 36.6. Let $n \in \mathbb{N}$ and let $\varphi, \psi\colon \overline{B}^n \to S^n$ be smooth embeddings with disjoint images. Show that $S^n \setminus (\varphi(B^n) \cup \psi(B^n))$ is diffeomorphic to $S^{n-1} \times [0,1]$.

Exercise 36.7. Let M be a connected orientable smooth manifold and let $f\colon M \to M$ be an orientation-preserving diffeomorphism that fixes a point $P \in M \setminus \partial M$. Show that there exists a diffeotopy rel $\{P\}$ from f to a diffeomorphism $g\colon M \to M$ that is the identity on an open neighborhood of P.

Exercise 36.8. Let M be a connected smooth manifold.



(a) Let $X \subset M \setminus \partial M$ be a finite set. Show that there exists a chart $\Phi\colon U \to B^n$ of type (i) such that $X \subset U$.
(b) We now suppose that ∂M is non-empty and connected. Let $X \subset M \setminus \partial M$ be a finite set. Show that there exists a chart $\Phi\colon U \to B^n_{\geq 0}$ of type (ii) such that $X \subset U$.



Exercise 36.9. We consider the proper smooth embedding $f\colon \mathbb{R} \to \mathbb{R}^3$ that is shown in the figure below and we consider the standard smooth embedding $g\colon \mathbb{R} \to \mathbb{R}^3$ that is given by $x \mapsto (x, 0, 0)$. Show that there exists a *proper* smooth isotopy $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}^3$ with $F_0 = f$ and $F_1 = g$.
Remark. Recall that according to the definition on page 812 a map $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}^3$ is a proper smooth isotopy if each F_t is a proper smooth embedding and if $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}^3$ is proper, i.e. if the preimage of each compact subspace of $\mathbb{R}^3$ is a compact subspace of $\mathbb{R} \times [0,1]$. *Remark.* In Chapter 59 we will study the "fundamental groups" of the complements of $\mathbb{R}^3 \setminus f(\mathbb{R})$ and $\mathbb{R}^3 \setminus g(\mathbb{R})$ and we will see that they are not isomorphic. As in the Smooth Isotopies-Complements Proposition 36.3 this shows that there is no homeomorphism $\Phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ with $\Phi \circ f = g$. In particular there is no diffeotopy $G\colon \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ with $G(f(x), t) = F(x, t)$ for all $(x, t) \in \mathbb{R} \times [0,1]$. This shows again that in the formulation of the Isotopy Extension Theorem 36.1 it is difficult to replace the hypothesis that M is compact by a weaker condition.



37. Orientation-preserving and orientation-reversing loops

In this chapter we will apply the Isotopy Extension Theorem 36.1 to introduce and study the concept of orientation-preserving and orientation-reversing loops. We will use this concept and various results from earlier chapters to classify smooth embeddings of balls into smooth manifolds.

37.1. Orientation-preserving reversing and orientation-reversing loops.

Definition. Let X be a topological space and let $\gamma\colon [0,1] \to X$ be a path.

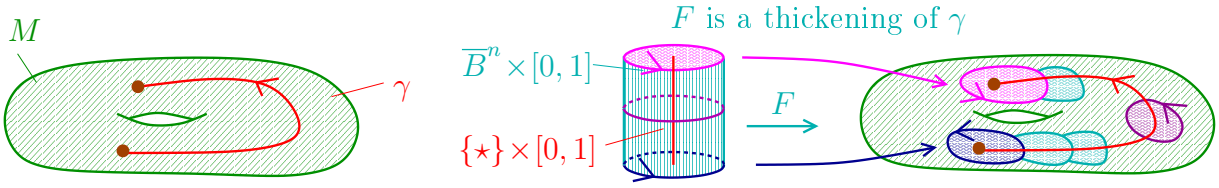
(1) We say γ is a loop if $\gamma(0) = \gamma(1)$.

(2) We say γ is trivial near the ends if there exists an $\epsilon > 0$ such that for each $t \in [0, \epsilon]$ we have $\gamma(t) = \gamma(0)$ and if for each $t \in [1-\epsilon, 1]$ we have $\gamma(t) = \gamma(1)$.[368]

Let M be an n-dimensional smooth manifold and let $\gamma\colon [0,1] \to M \setminus \partial M$ be a smooth path.

(3) Throughout this section we denote the origin of $\overline{B}^n$ by $\star$.

(4) A thickening of the smooth path γ is a smooth isotopy $F\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ such that for each $t \in [0,1]$ we have $F(\star, t) = \gamma(t)$.



Proposition 37.1. (Path-Thickening Proposition) Let M be a smooth manifold and let $\gamma\colon [0,1] \to M \setminus \partial M$ be a smooth path that is trivial near the ends.

(1) Given any smooth embedding $\varphi\colon \overline{B}^n \to M \setminus \partial M$ with $\varphi(\star) = \gamma(0)$ there exists a thickening $F\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ that is trivial near the ends with $F_0 = \varphi$.

(2) Given any smooth embedding $\psi\colon \overline{B}^n \to M \setminus \partial M$ with $\psi(\star) = \gamma(1)$ there exists a thickening $F\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ that is trivial near the ends with $F_1 = \psi$.

Proof. Let M be an n-dimensional smooth manifold and let $\gamma\colon [0,1] \to M \setminus \partial M$ be a smooth path that is trivial near the ends. We provide the proof of Statement (1). The proof of Statement (2) is very similar.

As in the proof of Lemma 36.2 we point out that the map $\{\star\} \times [0,1] \to M \setminus \partial M$ that is given by $(\star, t) \mapsto \gamma(t)$ is a smooth isotopy that is trivial near the ends. It follows from the Isotopy Extension Theorem 36.1 that there exists a diffeotopy $F\colon M \times [0,1] \to M$ that is trivial near the ends with $F_0 = \mathrm{id}$ and such that for each $t \in [0,1]$ we have $F(\star, t) = \gamma(t)$. Let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ be a smooth embedding with $\varphi(\star) = \gamma(0)$. The map

$$\begin{array}{rcl} F \circ (\varphi \times \mathrm{id}_{[0,1]})\colon \overline{B}^n & \to & M \\ (x,t) & \mapsto & F(\varphi(x), t) \end{array}$$

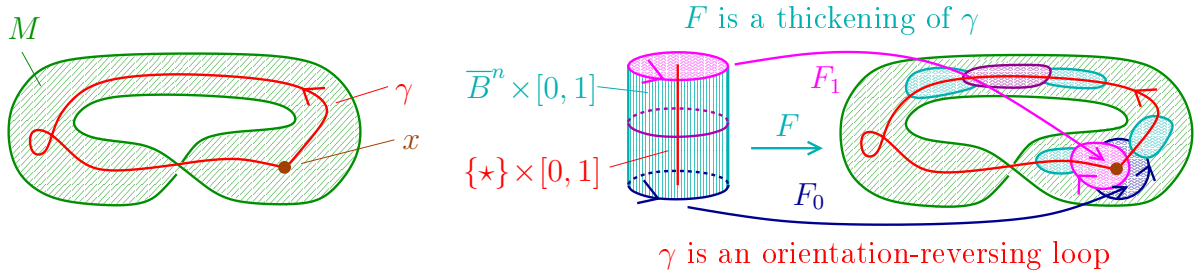
is a thickening of γ that is trivial near the ends with $F_0 = \varphi$. ∎

[368] As so often it follows from the Trivial-Near-The-Ends-Lemma 33.4 that we can turn any smooth path into a smooth path that is trivial near the ends.

Definition. Let M be an n-dimensional smooth manifold and let $\gamma\colon [0,1] \to M \setminus \partial M$ be a loop that is trivial near the ends. By the Path-Thickening Proposition 37.1 we can pick a thickening $F\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ of γ. We say γ is orientation-preserving if the automorphism

$$\mathrm{T}_\star \overline{B}^n \xrightarrow[\cong]{\mathrm{D}(F_0)_\star} \mathrm{T}_{\gamma(0)} M \xleftarrow[\cong]{\mathrm{D}(F_1)_\star} \mathrm{T}_\star \overline{B}^n$$

of the vector space $\mathrm{T}_\star \overline{B}^n$ is orientation-preserving. Otherwise we say that γ is orientation-reversing.



Remark. On page 1114 we will give an alternative and slightly more general definition of orientation-preserving and orientation-reversing loops. In Proposition 49.10 we will see that the two definitions are equivalent for smooth paths that take values in $M \setminus \partial M$.

Proposition 37.2. Let M be a smooth manifold. The above definition of a loop being orientation-preserving does not depend on the choice of the thickening.

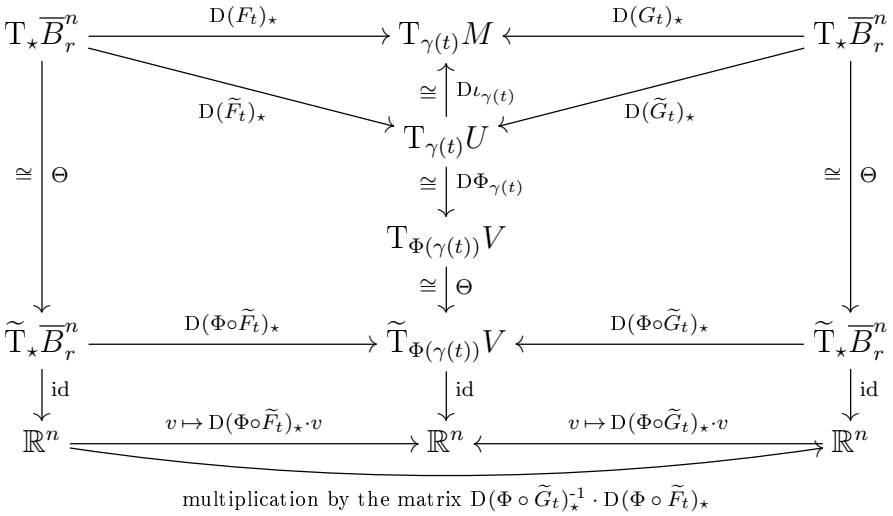
Proof. Let M be an n-dimensional smooth manifold and let $\gamma\colon [0,1] \to M \setminus \partial M$ be a loop that is trivial near the ends. Let $F\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ and $G\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ be two thickenings of γ.

Claim. The map
$$\begin{array}{rcl} [0,1] & \to & \mathbb{R} \setminus \{0\} \\ t & \mapsto & \det\left((\mathrm{D}(G_t)_\star)^{-1} \circ \mathrm{D}(F_t)_\star \colon \mathrm{T}_\star \overline{B}^n \to \mathrm{T}_{\gamma(t)} M \to \mathrm{T}_\star \overline{B}^n \right) \end{array}$$
is continuous. In particular, either all values are positive or all values are negative.

Proof. We need to show that the given map is continuous. By the Pasting Proposition 2.6 it suffices to show that the given map is locally continuous.

Let $t \in [0,1]$. We pick a chart $\Phi\colon U \to V$ around $\gamma(t)$. We denote by $\iota\colon U \to M$ the inclusion map. Since $F, G\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ are in particular continuous and since U is open there exists an open neighborhood I of $t \in [0,1]$ and an $r > 0$ such that $F(\overline{B}^n_r \times I) \subset U$ and such that $G(\overline{B}^n_r \times I) \subset U$. We denote by $\widetilde{F}_t\colon \overline{B}^n \to U$ and $\widetilde{G}_t\colon \overline{B}^n \to U$

the unique maps with $F_t = \iota \circ \widetilde{F}_t$ and with $G_t = \iota \circ \widetilde{G}_t$. We consider the following diagram



We make the following clarifications and observations:

(1) The maps Θ are the natural isomorphisms provided by the Tangent Space-Isomorphism Proposition 23.16.
(2) The pentagons commutes since the maps Θ are natural.
(3) The triangles commute by the functoriality of differentials.
(4) By the Differential-equals-Differential Lemma 23.4 the differential of a smooth map between the physical tangent spaces of open subsets of $\mathbb{R}^n$ is just given by multiplication by the $(n \times n)$-matrix that is given by the various partial derivatives. In other words, the two bottom squares commute.

It follows from the above discussion and the fact that F and G are "smooth in t", that the map of the claim is continuous on I. We have thus shown that the given map is locally continuous which, as we mentioned above, implies by the Pasting Proposition 2.6, that the given map is continuous. ⊞

Finally we consider the following little commutative diagram:

$$\begin{array}{ccc} \mathrm{T}_\star \overline{B}^n & \xrightarrow[\cong]{(\mathrm{D}(F_1)_\star)^{-1} \circ \mathrm{D}(F_0)_\star} & \mathrm{T}_\star \overline{B}^n \\ {\scriptstyle (\mathrm{D}(G_0)_\star)^{-1} \circ \mathrm{D}(F_0)_\star} \downarrow \cong & & \cong \downarrow {\scriptstyle (\mathrm{D}(G_1)_\star)^{-1} \circ \mathrm{D}(F_1)_\star} \\ \mathrm{T}_\star \overline{B}^n & \xrightarrow[\cong]{(\mathrm{D}(G_1)_\star)^{-1} \circ \mathrm{D}(G_0)_\star} & \mathrm{T}_\star \overline{B}^n \end{array}$$

It follows from the claim that the vertical isomorphism to the left is orientation-preserving if and only if the vertical isomorphic to the right is orientation-preserving. It follows from this observation and the Orientation-Composition Lemma 25.3 that the top horizontal map is orientation-preserving if and only if the bottom horizontal map is orientation-preserving. We have thus shown that it does not matter whether we work with the thickening F or the thickening G. ■

Proposition 37.3. (Orientability-via-Loops Proposition) Let M be a connected smooth manifold and let $P \in M \setminus \partial M$. The following four statements are equivalent:

(1) M is orientable,
(2) $M \setminus \partial M$ is orientable,
(3) every loop $\gamma\colon [0,1] \to M \setminus \partial M$ that is trivial near the ends is orientation-preserving,
(4) every loop $\gamma\colon [0,1] \to M \setminus \partial M$ with $\gamma(0) = \gamma(1) = P$ that is trivial near the ends is orientation-preserving.

Proof. Let M be a connected n-dimensional smooth manifold and let $P \in M \setminus \partial M$. The implication (1) $\Rightarrow$ (2) is basically obvious. The implication (2) $\Rightarrow$ (1) is the content of the Complement-of-Boundary Orientation Lemma 25.16.

We turn to the proof of the implication (2) $\Rightarrow$ (3). Thus we now assume that we can pick an orientation for $M \setminus \partial M$. Let $\gamma\colon [0,1] \to M \setminus \partial M$ be a loop that is trivial near the ends. We need to show that γ is orientation-preserving. We pick a thickening $F\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ of γ. By the Smooth Isotopy-Orientation Proposition 34.2 we know that each $\mathrm{D}F_t\colon \mathrm{T}_\star \overline{B}^n \to \mathrm{T}_{\gamma(t)} M$ is orientation-preserving or each such map is orientation-reversing. In particular the map is orientation-preserving for $t = 0$ if and only if it is for $t = 1$. It follows immediately that the loop γ is orientation-preserving.

The implication (3) $\Rightarrow$ (4) is a tautology. Therefore at this stage it remains to prove the implication (4) $\Rightarrow$ (2). This proof will require the remainder of this proof.

Thus, in the following we assume that every smooth smooth path $\gamma\colon [0,1] \to M \setminus \partial M$ with $\gamma(0) = \gamma(1) = P$ that is trivial near the ends is orientation-preserving. We need to show that $M \setminus \partial M$ is orientable.

We pick an orientation $\mathcal{O}_P$ of $\mathrm{T}_P M$. Let $Q \in M \setminus \partial M$. By the Push Away From Boundary Lemma 18.13 we know that $M \setminus \partial M$ is also path-connected. It follows from the Smooth Path-Connectivity Proposition 19.29 together with the Trivial-Near-The-Ends-Lemma 33.4 that there exists a smooth path $\gamma\colon [0,1] \to M \setminus \partial M$ from P to Q that is trivial near the ends. By the Path-Thickening Proposition 37.1 there exists a thickening $F\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ of γ that is trivial near the ends. We equip $\mathrm{T}_Q M$ with the orientation $\mathcal{O}_Q$ that is given by the image of the orientation $\mathcal{O}_P$ of $\mathrm{T}_P M$ under the isomorphism

$$\mathrm{T}_P M \xrightarrow{(\mathrm{D}(F_0)_\star)^{-1}} \mathrm{T}_\star \overline{B}^n \xrightarrow{\mathrm{D}(F_1)_\star} \mathrm{T}_Q M.$$

Claim 1. This orientation of $\mathrm{T}_Q M$ does not depend on the choice of the smooth path γ.

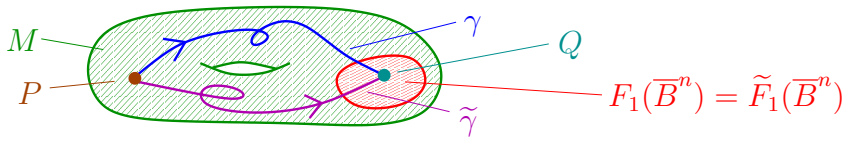
Proof. Let $\widetilde{\gamma}\colon [0,1] \to M \setminus \partial M$ be another smooth path from P to Q that is trivial near the ends. Recall that by the Path-Thickening Proposition 37.1 (2) there exists a thickening $\widetilde{F}\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ of $\widetilde{\gamma}$ that is trivial near the ends *and* which has the property that $\widetilde{F}_1\colon \overline{B}^n \to M \setminus \partial M$ equals $F_1\colon \overline{B}^n \to M \setminus \partial M$. We consider the concatenations

$$\delta\colon [0,1] \to M \setminus \partial M, \quad t \mapsto \begin{cases} \gamma(2t), & \text{if } t \in [0, \frac{1}{2}], \\ \widetilde{\gamma}(2-2t), & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \quad \text{and}$$

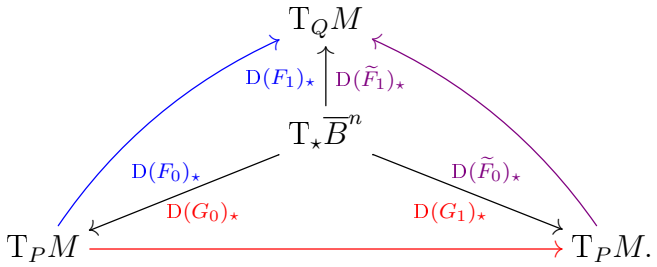
$$G\colon \overline{B}^n \times [0,1] \to M \setminus \partial M, \quad (x,t) \mapsto \begin{cases} F(x, 2t), & \text{if } t \in [0, \frac{1}{2}], \\ \widetilde{F}(x, 2-2t), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

It follows from the fact that $F_1 = \widetilde{F}_1$ that the maps are well-defined and it follows from the fact that γ, $\widetilde{\gamma}$, F and $\widetilde{F}$ are trivial near the ends, together with the Smooth Pasting

Proposition 19.23, that δ and G are actually smooth. Evidently δ is a loop and G is a thickening of δ.



Finally we consider the following diagram:



We make the following clarifications and observations:

(1) By construction we have $F_1 = \widetilde{F}_1$ and we have $G_0 = F_0$, $G_1 = \widetilde{F}_0$. Thus the two maps on the arrows are always the same map.
(2) The undecorated colored maps are defined to be the composition of the (inverse) of the two maps of the same color. In other words, the diagram commutes by definition.
(3) By hypothesis δ is orientation-preserving. This means that the horizontal red map at the bottom is orientation-preserving.
(4) It follows from the above that the blue map induces the same orientation on $\mathrm{T}_Q M$ as the violet map. ⊞

At this state it suffices to prove the following claim.

Claim 2. The orientations of the tangent spaces $\mathrm{T}_Q M$ define an orientation on $M \setminus \partial M$.

Proof. We will use the criterion from the Orientations-via-Atlas Lemma 25.8 to verify that these orientations define an orientation on $M \setminus \partial M$. Thus let $Q \in M \setminus \partial M$.

Before we do anything else we introduce the following little notation: for each $x \in B_2^n$ we denote by $\mathcal{S}_x \in \mathrm{T}_x \overline{B}^n$ the standard orientation.

Next we pick a chart $\Phi\colon U \to B_2^n$ around Q with $\Phi(Q) = \star$ and which has the property that the map $\mathrm{D}\Phi_Q\colon \mathrm{T}_Q U \to \mathrm{T}_\star \overline{B}_2^n$ sends the orientation $\mathcal{O}_Q$ to the standard orientation $\mathcal{S}_\star$ of $\mathrm{T}_\star \overline{B}_2^n$. As usual we denote the inverse of the chart Φ by $\Psi\colon B_2^n \to U$. By the Orientations-via-Atlas Lemma 25.8 it now suffices to show that Φ is orientation-preserving for every $R \in U$. Put differently, we need to show that for each $R \in U$ we have $\mathrm{D}\Phi_R(\mathcal{O}_R) = \mathcal{S}_{\Phi(R)}$.

We pick a smooth path $\gamma\colon [0,1] \to M \setminus \partial M$ from P to Q that is trivial near the ends. It follows from the Path-Thickening Proposition 37.1 (2) that we can pick a thickening $F\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ of γ that is trivial near the ends and which has the property that $F_1\colon \overline{B}^n \to M$ equals $\Psi\colon \overline{B}^n \to U \subset M$.

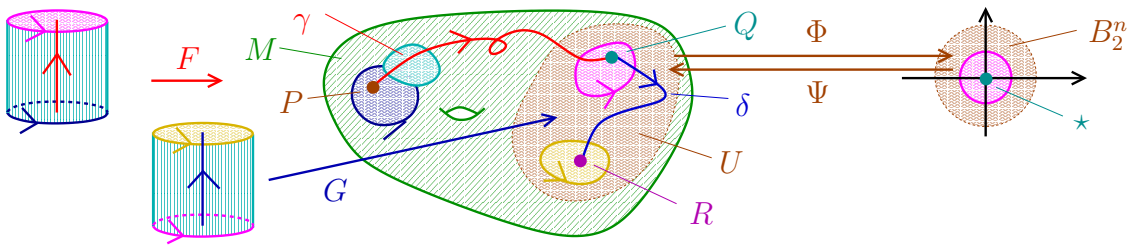
Given $R \in U$ we pick a path $\delta\colon [0,1] \to U$ from Q to R that is trivial near the ends. We use the Path-Thickening Proposition 37.1 (2) to find a thickening $G\colon \overline{B}^n \times [0,1] \to U$

of δ with $G_0 = F_1$. Note that

we use the path γ and the thickening F to define the orientation $\mathcal{O}_Q$, and
we use the path and the thickening that are given by concatenating γ and δ and F and G
to define the orientation $\mathcal{O}_R$,
equality then follows easily from the chain rule

$$\begin{array}{ccccc} \mathrm{D}\Phi_R(\mathcal{O}_R) & \overset{\downarrow}{=} & \mathrm{D}\Phi_R((\mathrm{D}(G_1)_\star \circ \mathrm{D}(G_0)_\star^{-1})(\mathcal{O}_Q)) & = & \mathrm{D}\Phi_R((\mathrm{D}(G_1)_\star \circ \mathrm{D}(G_0)_\star^{-1})((\mathrm{D}\Phi_\star^{-1})(\mathcal{S}_\star))) \\ & = & (\mathrm{D}(\Phi \circ G_1)_\star \circ (\mathrm{D}(\Phi \circ G_0)_\star)^{-1})(\mathcal{S}_\star) & = & \mathcal{S}_{\Phi(R)}. \\ & \overset{\uparrow}{} & & \overset{\uparrow}{} & \end{array}$$

chain rule — since $G_0 \circ \Phi$ and $G_1 \circ \Phi$ are smoothly isotopic maps from $\overline{B}^n \to B_2^n$ we obtain the equality from the Smooth Isotopy-Orientation Proposition 34.2 ■



Remark. In Section 49.4 we will give an alternative and slightly more general definition of orientation-preserving and orientation-reversing loops. In Proposition 49.10 we will see that the two definitions are equivalent for smooth paths that take values in $M \setminus \partial M$. In fact in Section 49.4 we will study orientation-preserving and reversing loops in greater detail. For example we will see that if two loops are homotopic, then one is orientation-preserving if and only if the other is. In principle one could try to prove this statement with the techniques of the current section, but that promises to be a technical mess.

37.2. Orientation flipping diffeotopies I.

Proposition 37.4. (Orientation-Flipping Proposition) Let M be a *non-orientable* connected smooth manifold and let $X, Y \subset M \setminus \partial M$ be two disjoint finite subsets. There exists a diffeotopy $F\colon M \times [0,1] \to M$ with the following properties:

(1) $F_0 = \mathrm{id}$.
(2) F has compact support.
(3) F is a diffeotopy rel $\partial M \cup Y$.
(4) $\Phi := F_1$ is the identity on $X \cup Y$.
(5) The map $\Phi := F_1$ has the following properties:
  (a) For each $x \in X$ the map $\mathrm{D}\Phi_x\colon \mathrm{T}_x M \to \mathrm{T}_x M$ is orientation-reversing.
  (b) For each $y \in Y$ the map $\mathrm{D}\Phi_y\colon \mathrm{T}_y M \to \mathrm{T}_y M$ is orientation-preserving.

Remark.

(1) It follows from the Smooth Isotopy-Orientation Proposition 34.2 that the conclusion of this proposition does *not* hold for *orientable* smooth manifolds.
(2) In Section 49.5 we will give a somewhat different proof of the Orientation-Flipping Proposition 37.4.

Example. We consider $\widetilde{M} := \mathbb{R} \times [-1,1]$ and the orientation-reversing self-diffeomorphism

$$\begin{array}{rccc} g\colon \widetilde{M} = \mathbb{R} \times [-1,1] & \to & \widetilde{M} = \mathbb{R} \times [-1,1] \\ (r,s) & \mapsto & (r+1,-s). \end{array}$$

Recall that the Möbius band is defined as $M = \widetilde{M}/\sim$ where $P \sim g(P)$. We denote by $p\colon \mathbb{R} \times [-1,1] \to M$ the projection. Let $x = p(0,0)$. We want to find a diffeotopy $F\colon M \times [0,1] \to M$ as in the Orientation-Flipping Proposition 37.4, with $X = \{x\}$ and $Y = \varnothing$. Using the Smooth Transition Function Lemma 19.4 we can find a smooth map $\tau\colon [-1,1] \to [0,1]$ such that $\tau(0) = 1$, such that $\tau(t) = 0$ for $s \geq \frac{1}{2}$ and such that $\tau(-s) = \tau(s)$ for all $s \in [-1,1]$. We consider the map

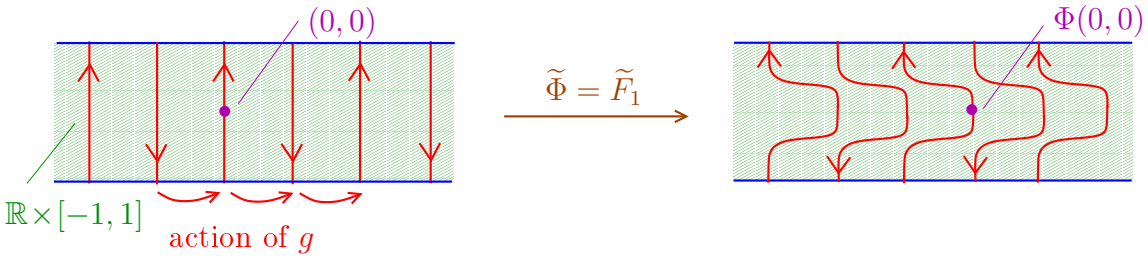
$$\begin{array}{rcl} \widetilde{F}\colon \widetilde{M} \times [0,1] & \to & \widetilde{M} \\ ((r,s),t) & \mapsto & ((r + t \cdot \tau(s), s) \end{array}$$

that is sketched in the figure below. Note that $\widetilde{F}$ descends to a map $F\colon M \times [0,1] \to M$ which is a diffeotopy rel ∂M with $F_0 = \mathrm{id}$ and with $F_1(x) = x$. We set $\widetilde{\Phi} := \widetilde{F}_1$ and $\Phi := F_1$. We claim that $\mathrm{D}\Phi_x\colon \mathrm{T}_x M \to \mathrm{T}_x M$ is orientation-reversing. To show this statement let us consider the following diagram

$$\begin{array}{ccccc} \mathrm{T}_{(0,0)}(\mathbb{R}\times[-1,1]) & \xrightarrow{\mathrm{D}\widetilde{\Phi}_{(0,0)}} & \mathrm{T}_{(1,0)}(\mathbb{R}\times[-1,1]) & \xrightarrow{\mathrm{D}(g^{-1})_{(1,0)}} & \mathrm{T}_{(0,0)}(\mathbb{R}\times[-1,1]) \\ \downarrow{\scriptstyle \mathrm{D}p_{(0,0)}} & & & \searrow^{\mathrm{D}p_{(1,0)}} & \downarrow{\scriptstyle \mathrm{D}p_{(0,0)}} \\ \mathrm{T}_{p(0,0)}M & & \xrightarrow{\quad\mathrm{D}\Phi_{p(0,0)}\quad} & & \mathrm{T}_{p(0,0)}M. \end{array}$$

Note that $p \circ \widetilde{\Phi} = \Phi \circ p$ and $p \circ g = p$. Thus the diagram commutes by the functoriality of the differential.

Since $F_0 = \mathrm{id}$ we know by the Smooth Isotopy-Orientation Proposition 34.2 that $F_1 = \Phi$ is also orientation-preserving. In particular $\mathrm{D}\widetilde{\Phi}_{(0,0)}$ is orientation-preserving. On the other hand $\mathrm{D}(g^{-1})_{(1,0)} = (\mathrm{D}g_{(0,0)})^{-1}$ is orientation-reversing. In summary we see that the composition of the top horizontal maps is an orientation-reversing automorphism of $\mathrm{T}_{(0,0)}(\mathbb{R} \times [-1,1])$. The left and right vertical maps are honest-to-G*d the same isomorphism. Therefore it follows from the elementary Lemma 25.7 that the bottom horizontal map is also orientation-reversing.[369]



For the proof of the Orientation-Flipping Proposition 37.4 we will need the following two little lemmas.

Auxiliary Lemma 37.5. Let X be a topological manifold, let $A \subset X$ be a closed subset and let $F\colon X \times [0,1] \to X$ be a homotopy rel A. If F has compact support, then there exists an open neighborhood U of A such that F is a homotopy rel U.

Proof of Lemma 37.5. Let $K \subset X$ be a compact subset such that $F(x,t) = x$ for all $x \notin K$ and all $t \in [0,1]$. Since X is in particular Hausdorff we know by the Compact-Closed

[369] Note that this example gives a solution to Exercise 25.17.

Lemma 1.21 (2) that K is a closed subset of X. Since X is a topological manifold we know by the Manifold Paracompact-Normal Proposition 18.9 (2) that X is normal. This implies in particular that there exists an open neighborhood U of A that is disjoint from K. But then F is evidently a homotopy rel U. ■

Auxiliary Lemma 37.6. Let M be a smooth manifold and let U be an open subset of M. Furthermore let $G\colon U \times [0,1] \to U$ be a diffeotopy with compact support K. Then

$$\begin{array}{rcl} \widetilde{G}\colon M \times [0,1] & \to & M \\ (x,t) & \mapsto & \begin{cases} G(x,t), & \text{if } x \in K, \\ x, & \text{if } x \notin K \end{cases} \end{array}$$

is a diffeotopy of M.

Proof. Since K is a compact subset of U we know by the Compact Image Lemma 2.13 and the Product Topology Properties Proposition 5.6 that $K \times [0,1]$ is a compact, and hence a closed subset of $M \times [0,1]$. Thus we see that $\widetilde{G}$ is smooth on the open subsets $U \times [0,1]$ and $M \times [0,1] \setminus K \times [0,1]$. Since these open subsets cover $M \times [0,1]$ it follows from the Smooth Pasting Proposition 19.23, or to be precise its extension to smooth manifolds with corner, that $\widetilde{G}$ is smooth. Basically the same argument shows that each $\widetilde{G}_t$ is a diffeomorphism. ■

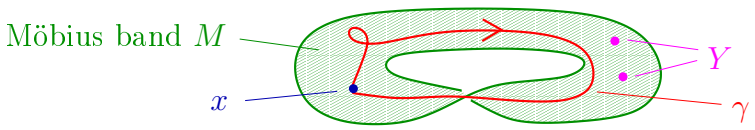
All the work for the proof of the Orientation-Flipping Proposition 37.4 is now done in the proof of the following auxiliary lemma, which deals with the case that $\#X = 1$.

Auxiliary Lemma 37.7. Let M be a connected *non-orientable* smooth manifold. Furthermore let $Y \subset M$ be a finite set and let $x \in M \setminus \partial M$ be a point with $x \notin Y$. There exists a diffeotopy $F\colon M \times [0,1] \to M$ with the following properties:

(1) $F_0 = \mathrm{id}$.
(2) F has compact support.
(3) F is a diffeotopy rel $\partial M \cup Y$.
(4) $\Phi := F_1$ is the identity on $\{x\} \cup Y$.
(5) The map $\Phi := F_1$ has the following properties:
  (a) The map $\mathrm{D}\Phi_x\colon \mathrm{T}_x M \to \mathrm{T}_x M$ is orientation-reversing.
  (b) For each $y \in Y$ the map $\mathrm{D}\Phi_y\colon \mathrm{T}_y M \to \mathrm{T}_y M$ is orientation-preserving.

Proof of Lemma 37.7. Let M be a connected *non-orientable* smooth manifold. Note that by the Smooth 1-Dimensional Manifold Classification Theorem 22.7 we know that $\dim(M) \geq 2$. Let $Y \subset M$ be a finite set and let $x \in M \setminus \partial M$ with $x \notin Y$.

(a) We pick an orientation $\mathcal{O}_x$ for $\mathrm{T}_x M$.
(b) We set $W := M \setminus Y$. Since Y is finite we see that Y is a closed subset. It follows from Lemmas 19.19 and 19.20 that W is a smooth manifold with $\partial W = \partial M$. Furthermore it follows from the Complement-of-Boundary Orientation Lemma 25.16 (2) that W is still non-orientable.
(c) Since Y is a finite set and since $\dim(M) \geq 2$ we know by the elementary Exercise 18.16 that W is also path-connected.
(d) It follows from the above together with the Orientability-via-Loops Proposition 37.3 that there exists an orientation-reversing loop $\gamma\colon [0,1] \to W \setminus \partial W$ that is trivial near the ends with $\gamma(0) = \gamma(1) = x$.



By the Isotopy Extension Theorem 36.1 there exists a diffeotopy $F\colon W \times [0,1] \to W$ with the following properties:

(0) For each $t \in [0,1]$ we have $F_t(x) = \gamma(t)$.
(1) $F_0 = \mathrm{id}$.
(2) F has compact support.
(3) F is a diffeotopy rel ∂W.

Note that it follows from (2), together with Lemma 37.6, that we can extend F to a diffeotopy $F\colon M \times [0,1] \to M$ rel Y on all of M by setting $F(x,t) = x$ for $x \notin W$. Finally we set $\Phi := F_1\colon M \to M$.

By design F has properties (1), (2), (3) and (4). Furthermore it follows almost immediately from (0) and the fact that γ is orientation-reversing that (5a) is satisfied. Furthermore note that by Lemma 37.5 we know that F is actually a diffeotopy rel an open neighborhood of Y. This implies that for each $y \in Y$ we have $\mathrm{D}\Phi_y = \mathrm{id}_{\mathrm{T}_y M}$. Thus (5b) is also satisfied. ■

Proof of the Orientation-Flipping Proposition 37.4. Let $x_1, \dots, x_k$ be the elements of X. It follows from the ever useful Smooth Isotopy Transitivity Proposition 33.3 that it suffices to flip the signs of the points $x_1, \dots, x_k$ one-by-one. But this can be done by applying Lemma 37.7 altogether k times, namely to the points x_i and to the sets $Y_i := Y \cup \{x_1, \dots, \widehat{x_i}, \dots, x_k\}$. ■

37.3. **The Smooth Ball Embedding Theorem.** Our goal of this section is to prove the following theorem which was originally proved by Richard Palais [**Pal1960a**] in 1960.

Theorem 37.8. (Smooth Ball Embedding Theorem) Let M be an n-dimensional smooth manifold. (If M is orientable, then we pick an orientation for M.) In the following let $\varphi_1, \dots, \varphi_k\colon \overline{B}^n \to M \setminus \partial M$ and $\psi_1, \dots, \psi_k\colon \overline{B}^n \to M \setminus \partial M$ be two sets of k smooth embeddings of $\overline{B}^n$ with disjoint images. (If M is orientable, then we demand that all φ_i and ψ_i are orientation-preserving or that all are orientation-reversing.) If $n \geq 2$ or if $k = 1$, then there exists a diffeotopy[370]

$$G\colon M \times [0,1] \;\to\; M$$

with compact support, rel some neighborhood of ∂M, from the identity $G_0 = \mathrm{id}_M$ to a diffeomorphism $G_1\colon M \to M$ such that for each $i \in \{1, \dots, k\}$ the following diagram commutes:

$$\begin{array}{ccccc} & & \overline{B}^n & & \\ & \overset{\varphi_i}{\swarrow} & & \overset{\psi_i}{\searrow} & \\ M & & \xrightarrow[\cong]{G_1} & & M. \end{array}$$

Finally, if $\varphi_i(0) = \psi_i(0)$, $i = 1, \dots, k$ and if M is orientable, then we can find a diffeotopy rel $\{\varphi_1(0), \dots, \varphi_k(0)\}$.

Remark.

(1) The Smooth Ball Embedding Theorem 37.8 has many applications, for example we will see in the Connected Sum-Proposition **??** that it can be used to show that the "connected sum" of connected orientable smooth manifolds is well-defined.
(2) On page **??** we will give a somewhat different proof of the Smooth Ball Embedding Theorem 37.8 using the Tubular Neighborhood Theorem **??**.
(3) In the Topological Ball Embedding Theorem 132.10 we will state an analogue of the Smooth Ball Embedding Theorem 37.8 in the setting of topological manifolds.

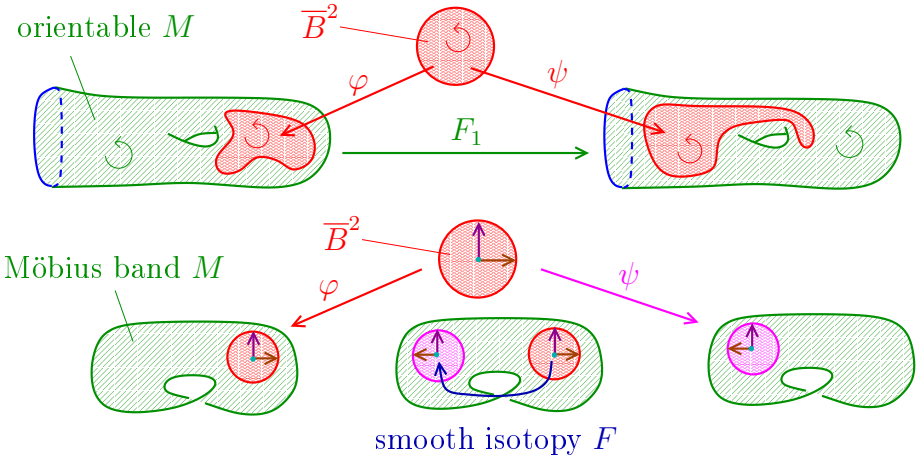
The key to proving the Smooth Ball Embedding Theorem 37.8 is the following proposition.

Proposition 37.9. Let M be a connected n-dimensional smooth manifold. (If M is orientable, then we pick an orientation for M.) In the following let $\varphi_1, \dots, \varphi_k \colon \overline{B}^n \to M \setminus \partial M$ and $\psi_1, \dots, \psi_k \colon \overline{B}^n \to M \setminus \partial M$ be two sets of k smooth embeddings of $\overline{B}^n$ with disjoint images. (If M is orientable, then we demand that all φ_i and all ψ_i are orientation-preserving or that all are orientation-reversing.) If $n \geq 2$ or if $k = 1$, then the smooth embeddings[371]

$$\bigsqcup_{i=1}^{k} \varphi_i \colon \bigsqcup_{i=1}^{k} \overline{B}^n \to M \setminus \partial M \qquad \text{and} \qquad \bigsqcup_{i=1}^{k} \psi_i \colon \bigsqcup_{i=1}^{k} \overline{B}^n \to M \setminus \partial M$$

are smoothly isotopic.

Remark. In the language of page 830 we can say that Proposition 37.9 calculates $\operatorname{Emb}(\overline{B}^n, M^n)$.



Proof of the Smooth Ball Embedding Theorem 37.8 assuming Proposition 37.9. By Proposition 37.9 there exists a smooth isotopy

$$F \colon \left(\bigsqcup_{i=1}^{k} \overline{B}_i^n \right) \times [0,1] \to M$$

with $F_0 = \varphi_1 \sqcup \dots \sqcup \varphi_k$ and $F_1 = \psi_1 \sqcup \dots \sqcup \psi_k$. Furthermore a quick look at the proof of Proposition 37.9 shows that if $\varphi_i(0) = \psi_i(0)$, $i = 1, \dots, k$ and if M is orientable, then

[370] Again note that if M is orientable, then the diffeomorphism G_1 is in fact orientation-preserving by the Smooth Isotopy-Orientation Proposition 34.2.
[371] Since the $\overline{B}^n$ is compact we obtain from Lemma 24.11 that k smooth embeddings of $\overline{B}^n$ with disjoint images give rise to a smooth embedding of the disjoint union of k closed balls.

we can find a smooth isotopy rel the origins of the k balls. Finally, using the Trivial-Near-The-Ends-Lemma 33.4 we can arrange that F is actually trivial near the ends.

We now apply the Isotopy Extension Theorem 36.1 to the smooth isotopy F and we obtain a diffeotopy $G\colon M \times [0,1] \to M$ with all the desired properties. ■

Proof of Proposition 37.9 if M is orientable. Note that after possibly precomposing all the smooth embeddings $\overline{B}^n \to M$ with the reflection in a hyperplane we only need to consider the case that all smooth embeddings are orientation-preserving.

We denote by $\iota\colon \overline{B}^n \to \overline{B}^n_2(0)$ the inclusion map. We pick orientation-preserving smooth embeddings $\{\Theta_i\colon \overline{B}^n_2(0) \to M \setminus \partial M\}_{i=1,\dots,k}$ with disjoint images. By the Smooth Isotopy Transitivity Proposition 33.3 it suffices to show that any orientation-preserving smooth embedding

$$\bigsqcup_{i=1}^k \alpha_i\colon \bigsqcup_{i=1}^k \overline{B}^n \to M \qquad \text{is smoothly isotopic to the smooth embedding} \qquad \bigsqcup_{i=1}^k \Theta_i \circ \iota\colon \bigsqcup_{i=1}^k \overline{B}^n \to M.$$

This statement follows from the subsequent three claims together with the Smooth Isotopy Transitivity Proposition 33.3.

Claim 1. The smooth embedding

$$\bigsqcup_{i=1}^k \alpha_i\colon \bigsqcup_{i=1}^k \overline{B}^n \to M \quad \text{is smoothly isotopic to a smooth embedding} \quad \bigsqcup_{i=1}^k \beta_i\colon \bigsqcup_{i=1}^k \overline{B}^n \to M$$

such that for each $i \in \{1,\dots,k\}$ we have $\beta_i(0) = \Theta_i(0)$.

Proof. Recall that we assume that $n \geq 2$ or that $k = 1$. Also recall that the α_i and Θ_i take values in $M \setminus \partial M$, which implies in particular that the points $\alpha_i(0)$ and $\Theta_i(0)$ lie in $M \setminus \partial M$. Since M is by hypothesis connected it follows from the Homogeneous-via-Diffeotopies Proposition 33.14 and Exercise 33.8[372] that there exists a diffeotopy $F\colon M \times [0,1] \to M$ with $F_0 = \mathrm{id}$ and such that $F_1(\alpha_i(0)) = \Theta_i(0)$ for $i = 1,\dots,k$. For $i = 1,\dots,k$ we set $\beta_i := F_1 \circ \alpha_i$ and we have found the desired smooth embeddings. ⊞

Claim 2. The smooth embedding

$$\bigsqcup_{i=1}^k \beta_i\colon \bigsqcup_{i=1}^k \overline{B}^n \to M \quad \text{is smoothly isotopic to a smooth embedding} \quad \bigsqcup_{i=1}^k \gamma_i\colon \bigsqcup_{i=1}^k \overline{B}^n \to M$$

such that for each $i \in \{1,\dots,k\}$ we have $\gamma_i(\overline{B}^n) \subset \Theta_i(B^n_2)$.

Proof. Let $i \in \{1,\dots,k\}$. Since Θ_i is a smooth embedding of an n-dimensional ball into an n-dimensional smooth manifold we obtain from the Codimension-Zero Smooth Embedding Proposition 24.16 that $\Theta_i(B^n_2(0))$ is an open subset of M. Since $\beta_i(0) = \Theta_i(0)$ we see that there exists an $\epsilon_i > 0$ such that $\beta_i(\overline{B}^n_{\epsilon_i}) \subset \Theta_i(B^n_2)$. We consider the smooth isotopy

$$\begin{array}{rcl} B_i\colon \overline{B}^n \times [0,1] & \to & M \\ (x,t) & \mapsto & \beta_i(x \cdot ((1-t) + t \cdot \epsilon_i)). \end{array}$$

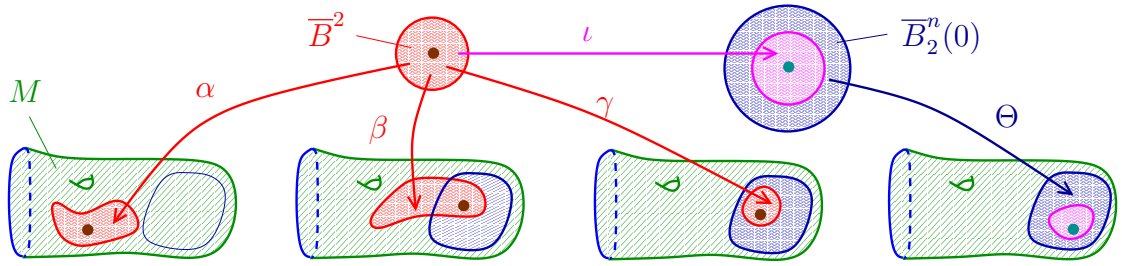
This map is a smooth isotopy from β_i to a map $\gamma_i\colon \overline{B}^n \to M$ with $\gamma_i(\overline{B}^n) \subset \Theta_i(B^n_2(0))$. Since these smooth isotopies it follows from Lemma 24.11 that we have disjoint images we have found the desired smooth isotopy. ⊞

[372] Alternatively one could also use Proposition 36.4

Claim 3. The smooth embedding

$$\bigsqcup_{i=1}^{k} \gamma_i \colon \bigsqcup_{i=1}^{k} \overline{B}^n \to M \quad \text{is smoothly isotopic to the smooth embedding} \quad \bigsqcup_{i=1}^{k} \Theta_i \circ \iota \colon \bigsqcup_{i=1}^{k} \overline{B}^n \to M.$$

Proof. Let $i \in \{1, \dots, k\}$. Note that by the Smooth Embedding Theorem 24.10 together with the Codimension-Zero Smooth Embedding Proposition 24.16 we know that $\Theta_i(B_2^n(0))$ is an open subset of M and we know that $\Theta_i \colon B_2^n(0) \to \Theta_i(B_2^n(0))$ is a diffeomorphism. Therefore we can consider the inverse $\Theta_i^{-1} \colon \Theta_i(B_2^n(0)) \to B_2^n(0)$. Since we arranged that $\gamma_i(\overline{B}^n) \subset \Theta_i(B_2^n(0))$ we can consider the corresponding orientation-preserving smooth embedding $\Theta_i^{-1} \circ \gamma_i \colon \overline{B}^n \to B_2^n(0)$. By the Diffeotopies-of-$\mathbb{R}^n$-Classification Proposition 34.3 (2) we know that $\Theta_i^{-1} \circ \gamma_i \colon \overline{B}^n \to B_2^n(0)$ is smoothly isotopic in $B_2^n(0)$ to the orientation-preserving smooth embedding $\iota \colon \overline{B}^n \to B_2^n(0)$. Therefore γ_i is smoothly isotopic in $\Theta_i(B_2^n(0))$ to $\Theta_i \circ \iota$. Since the $\Theta_i(\overline{B}_2^n(0))$ are disjoint we are done. ∎



Proof of Proposition 37.9 if M is non-orientable. Suppose that M is non-orientable. As before we denote by $\iota \colon \overline{B}^n \to \overline{B}_2^n(0)$ the inclusion map. We pick smooth embeddings $\{\Theta_i \colon \overline{B}_2^n(0) \to M \setminus \partial M\}_{i=1,\dots,k}$ with disjoint images. By the Smooth Isotopy Transitivity Proposition 33.3 it suffices to show that any smooth embedding

$$\bigsqcup_{i=1}^{k} \alpha_i \colon \bigsqcup_{i=1}^{k} \overline{B}^n \to M \qquad \text{is smoothly isotopic to the map} \qquad \bigsqcup_{i=1}^{k} \Theta \circ \iota \colon \bigsqcup_{i=1}^{k} \overline{B}^n \to M.$$

This statement follows again from three claims together with the Smooth Isotopy Transitivity Proposition 33.3.

Before we can state the first claim we need to introduce one new notion. We say that two smooth embeddings $\mu, \nu \colon \overline{B}^n \to M$ are *compatible* if $\mu(0) = \nu(0)$ and if the automorphism $(\mathrm{D}\nu_0)^{-1} \circ \mathrm{D}\mu_0$ of $\mathrm{T}_0\overline{B}^n$ is orientation-preserving.

Now we can move to the modified three claims.

Claim 1. The smooth embedding

$$\bigsqcup_{i=1}^{k} \alpha_i \colon \bigsqcup_{i=1}^{k} \overline{B}^n \to M \quad \text{is smoothly isotopic to a smooth embedding} \quad \bigsqcup_{i=1}^{k} \beta_i \colon \bigsqcup_{i=1}^{k} \overline{B}^n \to M$$

such that for $i = 1, \dots, k$ the smooth embeddings β_i and $\Theta_i \circ \iota$ are compatible.

Proof. As in the oriented case we obtain from Proposition 36.4 that there exists a diffeotopy $F \colon M \times [0,1] \to M$ with $F_0 = \mathrm{id}$ and such that $(F_1 \circ \alpha_i)(0) = (\Theta_i \circ \iota)(0)$ for $i = 1, \dots, k$. For $i = 1, \dots, k$ we set $P_i := (\Theta_i \circ \iota)(0)$. Next we define

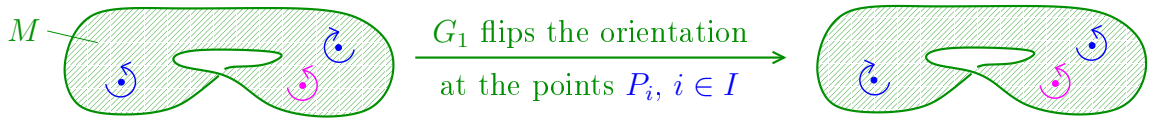
$$I := \{i \in \{1, \dots, k\} \mid \text{the smooth embeddings } F_1 \circ \alpha_i \text{ and } \Theta_i \circ \iota \text{ are not compatible}\}.$$

[372] On page 674 we defined what it means for an *automorphism* of a finite dimensional real vector space to be orientation-preserving.

From the Orientation-Flipping Proposition 37.4 we obtain a diffeotopy $G\colon M \times [0,1] \to M$ with $G_0 = \mathrm{id}$ with the following two properties:

(1) for all $i \in \{1,\dots,k\}$ we have $G_1(P_i) = P_i$,
(2) the map $\mathrm{D}(G_1)_{P_i}\colon \mathrm{T}_{P_i}M \to \mathrm{T}_{P_i}M$ is orientation-preserving if and only if $i \in I$.

For $i = 1,\dots,k$ we now set $\beta_i := G_1 \circ (F_1 \circ \alpha_i)$. It follows easily from (1) and (2), together with the ever-useful Smooth Isotopy Transitivity Proposition 33.3, that these smooth embeddings $\beta_1,\dots,\beta_k$ have the desired properties. ⊞



Claim 2. The smooth embedding

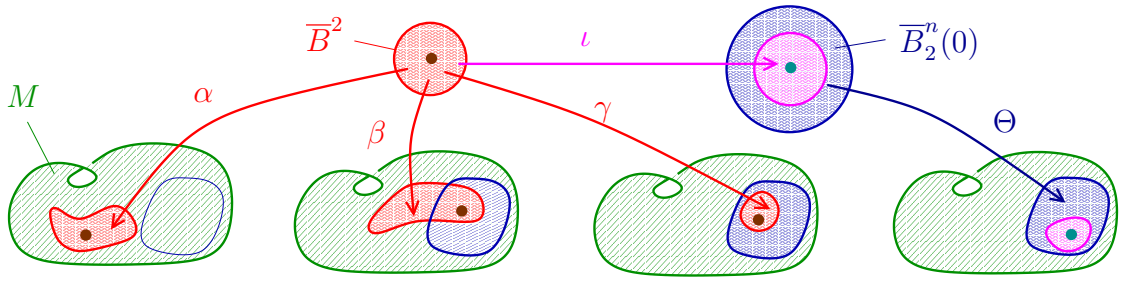
$$\bigsqcup_{i=1}^{k} \beta_i\colon \bigsqcup_{i=1}^{k} \overline{B}^n \to M \quad \text{is smoothly isotopic to a smooth embedding} \quad \bigsqcup_{i=1}^{k} \gamma_i\colon \bigsqcup_{i=1}^{k} \overline{B}^n \to M$$

such that for each $i \in \{1,\dots,k\}$ we have $\gamma_i(\overline{B}^n) \subset \Theta_i(B_2^n)$ and such that for $i = 1,\dots,k$ the smooth embeddings γ_i and $\Theta_i \circ \iota$ are compatible.

Proof. The construction of the γ_i follows by the same recipe as in the oriented case. It is basically clear from the construction that for $i = 1,\dots,k$ the smooth embeddings γ_i and $\Theta_i \circ \iota$ are indeed compatible. ⊞

Claim 3. The map

$$\bigsqcup_{i=1}^{k} \gamma_i\colon \bigsqcup_{i=1}^{k} \overline{B}^n \to M \quad \text{is smoothly isotopic to the smooth embedding} \quad \bigsqcup_{i=1}^{k} \Theta_i \circ \iota\colon \bigsqcup_{i=1}^{k} \overline{B}^n \to M.$$



Proof. Let $i \in \{1,\dots,k\}$. Since $\gamma_i(\overline{B}^n) \subset \Theta_i(B_2^n(0))$ we can consider the corresponding smooth embedding $\iota \circ \Theta_i^{-1} \circ \gamma_i\colon \overline{B}^n \to B_2^n(0)$. We consider the following diagram:

$$\begin{array}{ccccc} & & \mathrm{T}_0 B_2^n(0) & & \\ & \overset{\mathrm{D}(\Theta^{-1}\circ\gamma_i)_0}{\nearrow} & \cong & \overset{\mathrm{D}\iota_0}{\nwarrow} & \\ \mathrm{T}_0\overline{B}^n & & \cong & & \mathrm{T}_0\overline{B}^n. \\ & \underset{(\mathrm{D}(\Theta_i\circ\iota)_0)^{-1}\circ\mathrm{D}(\gamma_i)_0}{\longrightarrow} & & & \end{array}$$

Note that the diagram commutes by default. Since γ_i and $\Theta_i \circ \iota$ are compatible the bottom map is orientation-preserving. By definition of the orientations on $\overline{B}^n$ and $B_2^n(0)$ the right diagonal map is orientation-preserving. It follows that the left diagonal map is orientation-preserving. By the Basics-of-Orientations Lemma 25.13 (3) this implies that the smooth

embedding $\Theta_i^{-1} \circ \gamma_i$ is orientation-preserving. At this point the argument proceeds as in the oriented case.

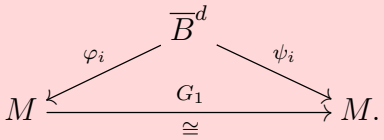
Indeed, by the Diffeotopies-of-$\mathbb{R}^n$-Classification Proposition 34.3 (2) we now know that $\Theta_i^{-1} \circ \gamma_i \colon \overline{B}^n \to B_2^n(0)$ is smoothly isotopic in $B_2^n(0)$ to the orientation-preserving smooth embedding $\iota \colon \overline{B}^n \to B_2^n(0)$. Therefore γ_i is smoothly isotopic in $\Theta_i(B_2^n(0))$ to $\Theta_i \circ \iota$. Since the $\Theta_i(\overline{B}_2^n(0))$ are disjoint we are done. ■

37.4. **The Lower-Dimensional Smooth Ball Embedding Theorem.** We also have a uniqueness theorem for smooth embeddings of lower dimensional balls. In fact it is slightly easier to state than the Smooth Ball Embedding Theorem 37.8 since we will not care about orientations.

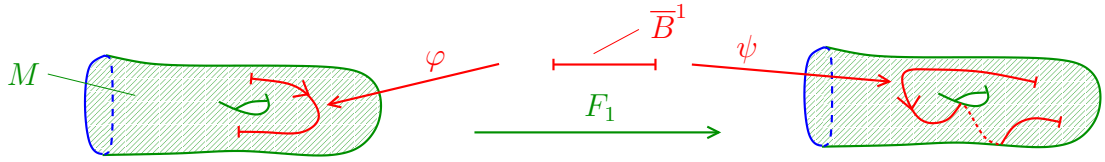
Theorem 37.10. (Lower-Dimensional Smooth Ball Embedding Theorem) Let M be an n-dimensional smooth manifold and let $d \in \{0, \dots, n-1\}$. In the following let $\varphi_1, \dots, \varphi_k \colon \overline{B}^d \to M \setminus \partial M$ and $\psi_1, \dots, \psi_k \colon \overline{B}^d \to M \setminus \partial M$ be two sets of k smooth embeddings of $\overline{B}^d$ with disjoint images. If $n \geq 2$ or if $k = 1$, then there exists a diffeotopy

$$G \colon M \times [0,1] \to M$$

with compact support, rel some neighborhood of ∂M, from the identity $G_0 = \mathrm{id}_M$ to a diffeomorphism $G_1 \colon M \to M$ such that for each $i \in \{1, \dots, k\}$ the following diagram commutes:



Finally, if $\varphi_i(0) = \psi_i(0)$, $i = 1, \dots, k$ and if M is orientable, then we can find a diffeotopy rel $\{\varphi_1(0), \dots, \varphi_k(0)\}$.



Remark.

(1) In the language of page 830 we can say that the Lower-Dimensional Smooth Ball Embedding Theorem 37.10 calculates $\mathrm{Emb}(\overline{B}^d, M^n)$ for $d < n$.
(2) The case $d = 0$ is precisely the content of the Homogeneous-via-Diffeotopies Proposition 36.4.

Proof. The same that we deduced the Smooth Ball Embedding Theorem 37.8 from Proposition 37.9 we can now deduce the Lower-Dimensional Smooth Ball Embedding Theorem 37.10 from the following claim.

Claim. If $n \geq 2$ or if $k = 1$, then the smooth embeddings

$$\bigsqcup_{i=1}^k \varphi_i \colon \bigsqcup_{i=1}^k \overline{B}^d \to M \setminus \partial M \qquad \text{and} \qquad \bigsqcup_{i=1}^k \psi_i \colon \bigsqcup_{i=1}^k \overline{B}^d \to M \setminus \partial M$$

are smoothly isotopic.

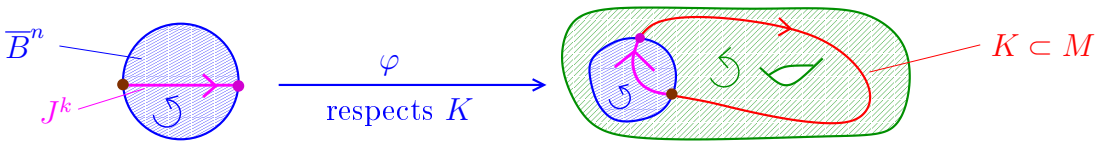
Proof. The proof of the claim is basically identical to the proof of Proposition 37.9 in the orientable case. All we have to do is to replace the Diffeotopies-of-$\mathbb{R}^n$-Classification Proposition 34.3 (2) by the Diffeotopies-of-$\mathbb{R}^n$-Classification Proposition 34.3 (3). We leave it to the reader to fill in the details. ■

Finally we want to consider a variation on the Smooth Ball Embedding Theorem 37.8 and the Lower-Dimensional Smooth Ball Embedding Theorem 37.10 where we look at smooth embeddings of balls that "respect" a submanifold. This discussion is perhaps a little peripheral, but it plays a role when later on we introduce the connected sum of submanifolds and knots.

Definition. Let $n \in \mathbb{N}$ and let $k \in \{0, \dots, n\}$.

(1) We set $J^k := \{(x,0) \in \overline{B}^n \mid x \in \overline{B}^k\} = \overline{B}^n \cap (\mathbb{R}^k \times \{0\})$ and we view equip J^k with the obvious orientation.

(2) Let M be an oriented n-dimensional smooth manifold and let $K \subset M$ be an oriented k-dimensional submanifold. Furthermore let $\varphi\colon \overline{B}^n \to M$ be a smooth embedding. We say φ respects K if $\varphi(\overline{B}^n) \subset M \setminus \partial M$, if $\varphi(J^k) = \varphi(\overline{B}^n) \cap K$, if $\varphi\colon \overline{B}^n \to M$ is orientation-preserving and if $\varphi|_{J^k}\colon J^k \to \varphi(\overline{B}^n) \cap K$ is orientation-preserving.

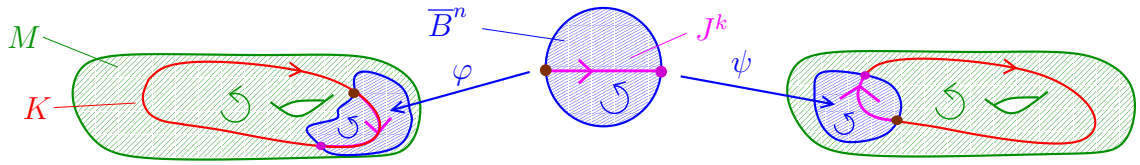


The following theorem gives a uniqueness statement for smooth embeddings of balls that respect a given submanifold.[373]

Theorem 37.11. Let M be an oriented n-dimensional smooth manifold. Furthermore let $K \subset M$ be a connected oriented k-dimensional submanifold. Finally let $\varphi\colon \overline{B}^n \to M$ and $\psi\colon \overline{B}^n \to M$ be two smooth embeddings that respect K. There exists a diffeotopy

$$G\colon M \times [0,1] \to M$$

with compact support, rel some neighborhood of ∂M, from the identity $G_0 = \mathrm{id}_M$ to a diffeomorphism $G_1\colon M \to M$ such that $G_1 \circ \varphi = \psi$ and such that each $G_t \circ \varphi\colon \overline{B}^n \to M$ respects K.



Sketch of proof. As in the proofs of the Smooth Ball Embedding Theorem 37.8 and of the Lower-Dimensional Smooth Ball Embedding Theorem 37.10 it suffices to prove the following claim.

[373] The case $K = M$ is in fact the statement of Smooth Ball Embedding Theorem 37.8 in the orientation-preserving case.

Claim. For any two smooth embeddings $\varphi\colon \overline{B}^n \to M$ and $\psi\colon \overline{B}^n \to M$ that respect K there exists a smooth isotopy $F\colon \overline{B}^n \times [0,1] \to M$ such that $F_0 = \varphi$, such that $F_1 = \psi$ and such that each $F_t \circ \varphi$ respects K.

Proof. The proof of the theorem is a modification of the proof of Proposition 37.9. Evidently now one has to go through the proof of Proposition 37.9, and of all the sources it uses, to make the necessary changes. It seems to this author that most modifications are straightforward, except for possibly one step. Let us elaborate.

Let $f\colon \overline{B}^n \to \mathbb{R}^n$ be a map that respects $K := \mathbb{R}^k \times \{0\}$ and such $f(0) = 0$. We write $A := \mathrm{D}f_0$ and we denote by $\mu(A)\colon \overline{B}^n \to \mathbb{R}^n$ the map that is given by multiplication by A. The fact that f respects K implies that A is of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, where the top left corner is a $(k \times k)$-matrix. It follows fairly easily that $\mu(A)$ also respects $K = \mathbb{R}^k \times \{0\}$. Similar to the discussion on page 834 we need to prove the following claim.

Claim. There exists a smooth isotopy $F\colon \overline{B}^n \to \mathbb{R}^n$ with $F_0 = f$, $F_1 = \mu(A)$ and such that each F_t respects $K = \mathbb{R}^k \times \{0\}$

Proof. Since $\overline{B}^n$ is a convex subset of $\mathbb{R}^n$ that contains the origin, since $f(0) = 0$ and since f respects $K = \mathbb{R}^k \times \{0\}$ we know by the Map-as-Matrix Multiplication Proposition 19.14 (1) and (2) that there exists a smooth map $\Theta\colon \mathbb{R}^n \to \mathrm{M}(n \times n, \mathbb{R})$ with the following properties:

(1) we have $\Theta(0) = A = \mathrm{D}f_0$,
(2) for all $x \in \mathbb{R}^n$ we have $f(x) = \Theta(x) \cdot x$, and
(3) for all $x \in J^k$ we have $\Theta(x) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, where the top left square is a $k \times k$-matrix.

As on page 834 we consider the map

$$\begin{array}{rcl} F\colon \overline{B}^n \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & \underbrace{\Theta(t \cdot x)}_{\in \mathrm{M}(n \times n, \mathbb{R})} \cdot x. \end{array}$$

As on page 834 we see that F is an isotopy with $F_0 = f$ and $F_1 = \mu(A)$. But now the above property (3) implies that each F_t respects $K = \mathbb{R}^k \times \{0\}$. ⊞

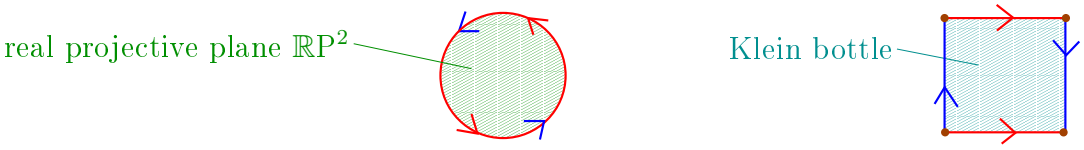
We leave all other details of the proof of Theorem 37.11 to the unquenchable thirst of the reader for technical details. ■

Exercises for Chapter 37.

Exercise 37.1. We consider the

$$\begin{array}{rcll} \text{real projective plane } \mathbb{R}\mathrm{P}^2 & = & \overline{B}^2/\sim & \text{where } x \sim -x \text{ for } x \in S^1 \\ \text{Klein bottle } K & = & ([0,1] \times [0,1])/\sim & \text{where } (x,0) \sim (x,1) \text{ and } (0,y) \sim (1,1-y). \end{array}$$

In both cases give examples of orientation-reversing loops.



Exercise 37.2. Let M be a smooth manifold and let $\gamma, \delta\colon [0,1] \to M \setminus \partial M$ be smooth loops that are trivial near the ends with $\gamma(0) = \gamma(1) = \delta(0) = \delta(1)$. We consider the concatenation

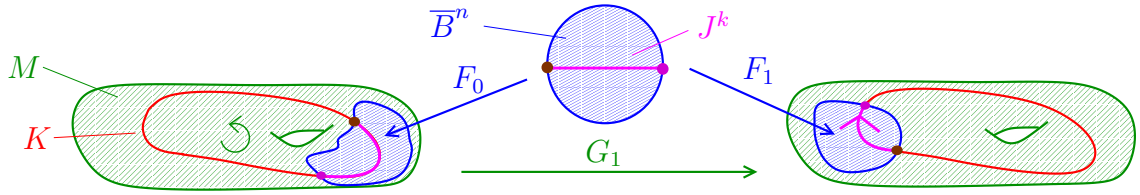
$$\begin{array}{rcl} \gamma * \delta\colon [0,1] & \to & M \\ t & \mapsto & \begin{cases} \gamma(2t), & \text{if } t \in [0, \frac{1}{2}], \\ \delta(2t-1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

Show that $\gamma * \delta$ is orientation-preserving if and only if either both γ and δ are orientation-preserving or both are orientation-reversing.

Exercise 37.3. Let $n \in \mathbb{N}$ and let $k \in \{0, \dots, n\}$. We set $J^k := \{(x,0) \in \overline{B}^n \mid x \in \overline{B}^k\}$. Let N be an n-dimensional smooth manifold and let $K \subset N$ be an k-dimensional submanifold. As on page 894 we say that a smooth embedding $\varphi\colon \overline{B}^n \to N$ *respects* K if $\varphi(\overline{B}^n) \subset N \setminus \partial N$ and if $\varphi(J^k) = \varphi(\overline{B}^n) \cap K$. Let $F\colon \overline{B}^n \times [0,1] \to N$ be a smooth isotopy such that each F_t respects K. Show that there exists a diffeotopy $G\colon N \times [0,1] \to N$ with the following properties:

(1) $G_0 = \mathrm{id}$.
(2) For all $x \in \overline{B}^n$ and all $t \in [0,1]$ we have $G(F_0(x), t) = F(x,t)$.
(3) For all $t \in [0,1]$ the map G_t restricts to a diffeomorphism of K.

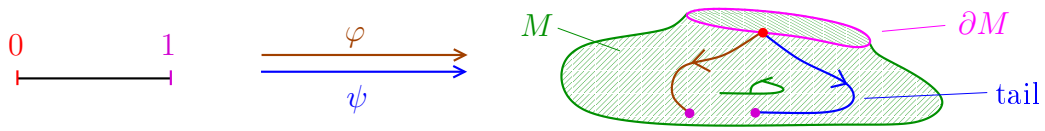
Remark. Evidently you are supposed to at least outline how to modify the proof of the Isotopy Extension Theorem 36.1 to obtain the desired diffeotopy.



Exercise 37.4. Let M be a smooth manifold and let $P \in \partial M$. A *tail at* P is a smooth embedding $\varphi\colon [0,1] \to M$ with the following properties:

(1) $\varphi(0) = P$ and $\varphi^{-1}(\partial M) = \{0\}$,
(2) $\varphi'(0) \notin \mathrm{T}_{\varphi(0)}(\partial M)$.

Show that if $\varphi, \psi\colon [0,1] \to M$ are two tails at P, then they are "smoothly isotopic as tails", i.e. show that there exists a smooth map $F\colon [0,1] \times [0,1] \to M$ with $F_0 = \varphi$, $F_1 = \psi$ and such that each F_t is a tail at P.



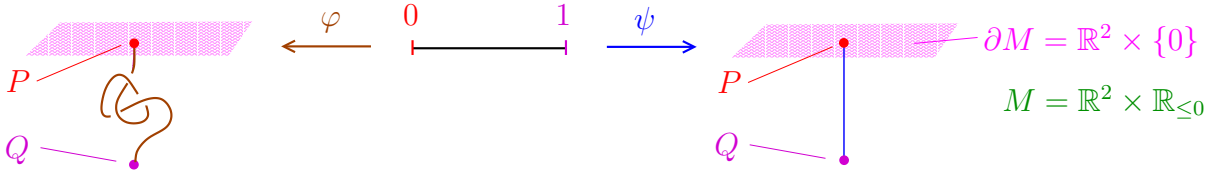
Exercise 37.5. Let M be a smooth manifold and let $P \in \partial M$. As in Exercise 37.4 we define a *tail at* P to be a smooth embedding $\varphi\colon [0,1] \to M$ with the following properties:

(1) $\varphi(0) = P$ and $\varphi^{-1}(\partial M) = \{0\}$,
(2) $\varphi'(0) \notin \mathrm{T}_{\varphi(0)}(\partial M)$.

Solve the following two problems:

(a) Show that if $\varphi, \psi\colon [0,1] \to M$ are two tails at P with the same endpoint, i.e. such that $\varphi(1) = \psi(1)$, then one tail can be turned into the other while keeping the endpoints fixed. More precisely, show that there exists a smooth homotopy $F\colon [0,1]\times[0,1] \to M$ rel $\{0,1\}$ with $F_0 = \varphi$, $F_1 = \psi$ and such that each F_t is a tail at P.

(b) As an explicit example consider the two tails shown in the figure below. We consider a "room" $M = \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ with "ceiling" $\partial M = \mathbb{R}^2 \times \{0\}$. Furthermore we consider two cables φ and ψ from P to Q. Show that you can turn the "twisted cable φ" into the "untwisted cable ψ" while keeping P and Q fixed.



38. The degree of a map between oriented smooth manifolds I

38.1. The definition of the degree of a map between closed smooth manifolds.

Definition. Let $f\colon M \to N$ be a smooth map between two oriented n-dimensional smooth manifolds and let $x \in M \setminus \partial M$. If f is a local diffeomorphism at x, then we define the local degree of f at the point x as[374]

$$\deg(f,x) \;:=\; \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } x \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$

Remark. On page 1694 and page 2379 we will generalize the notion of local degree to maps that are not necessarily local diffeomorphisms and to suitable maps between topological manifolds.

We continue with the following simplified version of the Stack of Records Lemma 26.2.

Lemma 26.2. Let $f\colon M \to N$ be a smooth map between two closed smooth manifolds of the same dimension.

(1) For every regular value $y \in N$ the preimage $f^{-1}(y)$ is a finite subset of M.
(2) Given any regular point $x \in M$ the map $f\colon M \to N$ is a local diffeomorphism at x.

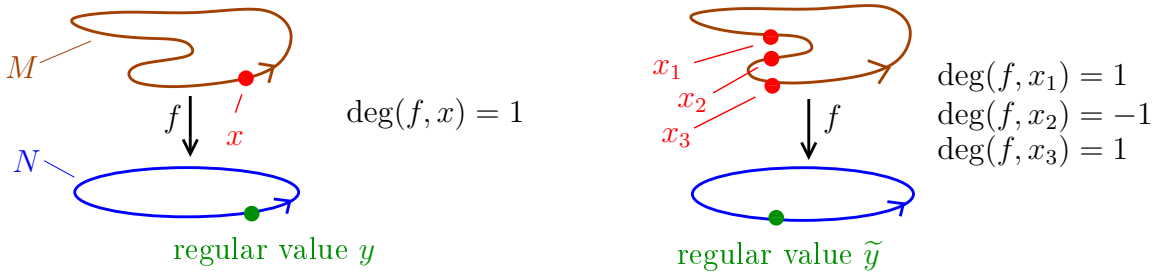
Proof.

(1) By the Regular Value Theorem 26.4 we know that $f^{-1}(y)$ is a proper 0-dimensional submanifold of the closed smooth manifold M. It follows from the discussion on page 514 that $f^{-1}(y)$ is a discrete subset of M. Since M is in particular compact it follows from the Compact-Discrete Lemma 1.22 that $f^{-1}(y)$ is actually a finite set.
(2) Let $x \in M$ be a regular point of f. This means that $\mathrm{D}f_x\colon \mathrm{T}_{f(x)}M \to \mathrm{T}_{f(x)}N$ is an epimorphism. Since M and N have the same dimension this implies that $\mathrm{D}f_x$ is an isomorphism. It follows from the Inverse Mapping Theorem 24.1 that f is a local diffeomorphism around x. ∎

Theorem 38.1. Let $f\colon M \to N$ be a smooth map between two closed oriented connected smooth manifolds of the same dimension. Let $y \in N$ be a regular value. The sum[375]

$$\sum_{x \in f^{-1}(y)} \deg(f,x) \;\in\; \mathbb{Z}$$

does not depend on the choice of y.



[374] Note that by the discussion on page 675 this definition also makes sense for 0-dimensional manifolds.
[375] We know by Lemma 26.2 that this is indeed a finite sum.

We postpone the proof of Theorem 38.1 to the next section. The theorem allows us to introduce the following definition.

Definition. Let $f\colon M \to N$ be a smooth map between two closed oriented connected non-empty smooth manifolds of the same dimension. By Sard's Theorem 31.1 there exists a regular value y. By Theorem 38.1 we can unambiguously define the degree of the smooth map f as

$$\deg(f) \;:=\; \sum_{x\in f^{-1}(y)} \deg(f,x) \;\in\; \mathbb{Z}.$$

Let $f\colon M \to N$ be a map (not necessarily smooth) between two closed oriented connected non-empty smooth manifolds of the same dimension. By the Whitney Approximation Theorem 29.1 we know that f is homotopic to a smooth map $g\colon M \to N$. The idea now is to define the degree of f as the degree of the smooth map g. For this approach to make sense we need the following proposition.

Proposition 38.2. Let $g\colon M \to N$ be a map between two closed oriented connected non-empty smooth manifolds of the same dimension. If $f_0, f_1\colon M \to N$ are smooth maps that are both homotopic to g, then $\deg(f_0) = \deg(f_1)$.

Proof. We postpone the proof of Proposition 38.2 to the next section. ■

Proposition 38.2 allows us to make the following definition.

Definition. Let $f\colon M \to N$ be a map between two closed oriented connected non-empty smooth manifolds of the same dimension. By the Whitney Approximation Theorem 29.1 we know that f is homotopic to a smooth map $g\colon M \to N$. By Proposition 38.2 it makes sense to define

$$\deg(f) \;:=\; \deg(g).$$

Remark.

(1) The notion of the degree was first introduced by Luitzen Brouwer [**Brou1912a**, p. 103], the degree is therefore sometimes also called the *Brouwer degree*.
(2) In Section 39.1 we will extend the definition of a degree to maps between smooth manifolds with non-empty boundary.
(3) On page 2222 we will use homology groups to give a different definition of the degree of a smooth map between closed oriented smooth manifolds. In the Local-Global Degree Proposition 101.6 we will see that the two definitions agree.
(4) On page 2379 we will again use homology groups to extend the notion of a degree from smooth manifolds to topological manifolds.

In the following two theorems we state two interesting properties of the degree.

Theorem 38.3. (Degree Homotopy Invariance Theorem) Let $f_0, f_1\colon M \to N$ be maps between two closed oriented connected non-empty smooth manifolds of the same dimension. If f_0 and f_1 are homotopic, then

$$\deg(f_0) \;=\; \deg(f_1).$$

Theorem 38.4. (Degree Multiplicativity Theorem) If $f\colon L \to M$ and $g\colon M \to N$ are maps between closed oriented connected non-empty smooth manifolds of the same

dimension, then $$\deg(g \circ f) \;=\; \deg(g) \cdot \deg(f).$$

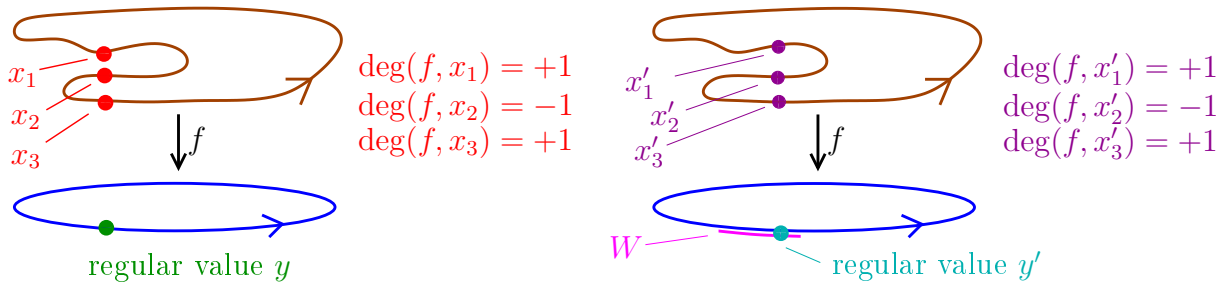
We provide the proofs of these two theorems in the next section.

38.2. **Proof of Theorem 38.1, 38.3 and 38.4.** For the purpose of this section we introduce the following notation.

Notation. Let $f\colon M \to N$ be a smooth map between closed oriented connected non-empty smooth manifolds of the same dimension. Let $y \in N$ be a regular value. We define $$\deg^y(f) \;:=\; \sum_{x \in f^{-1}(y)} \deg(f, x) \;\in\; \mathbb{Z}.$$

The proofs of Theorem 38.1 and 38.3 rely on three lemmas which we will prove in this section. The first lemma is a slight refinement of the Stack of Records Lemma 26.2.

Lemma 38.5. Let $f\colon M \to N$ be a smooth map between two closed smooth manifolds of the same dimension. Every regular value $y \in N$ admits an open neighborhood W such that each $y' \in W$ is a regular value and such that for each $y' \in W$ we have $\deg^y(f) = \deg^{y'}(f)$.



Proof. Let $f\colon M \to N$ be a smooth map between two closed smooth manifolds of the same dimension and let $y \in N$ be a regular value. We denote the points in the preimage $f^{-1}(y)$ by $x_1, \ldots, x_k \in M$. Since $x_1, \ldots, x_k$ are regular points we obtain from the Inverse Mapping Theorem 24.1 connected open neighborhoods $U_1, \ldots, U_k$ of $x_1, \ldots, x_k$ such that each $f_k\colon U_k \to V_k := f(U_k)$ is a diffeomorphism. Since M is Hausdorff and since the $x_1, \ldots, x_k$ are distinct we can arrange, see the elementary Exercise 1.15, that the open neighborhood $U_1, \ldots, U_k$ are disjoint. We set

$$W \;:=\; \Big(\bigcap_{i=1}^{k} V_i\Big) \setminus \underbrace{f\Big(M \setminus \bigcup_{i=1}^{k} U_i\Big)}_{\substack{\text{closed subset of } N \\ \text{since } M \text{ is compact}}}.$$

Note that W is open and that it contains y. Now let $y' \in W$. Note that the preimage of $y' \in W$ consists of precisely k points, namely precisely one x'_i in each U_i.[376] Since the U_i are connected it follows immediately from the Basics-of-Orientations Lemma 25.13 that for each $i \in \{1, \ldots, k\}$ we have $\deg(f, x_i) = \deg(f, x'_i)$. This shows that $\deg^y(f) = \deg^{y'}(f)$. ∎

[376] This can be seen as follows. Let $y' \in W$. Let $i \in \{1, \ldots, k\}$. By definition $y' \in V_i$, thus there exists a unique preimage of y' in U_i. Since $U_1, \ldots, U_k$ are disjoint we see that y' has k preimages in $U_1 \cup \cdots \cup U_k$. By choice of W the point y' has no preimage outside of $U_1 \cup \cdots \cup U_k$.

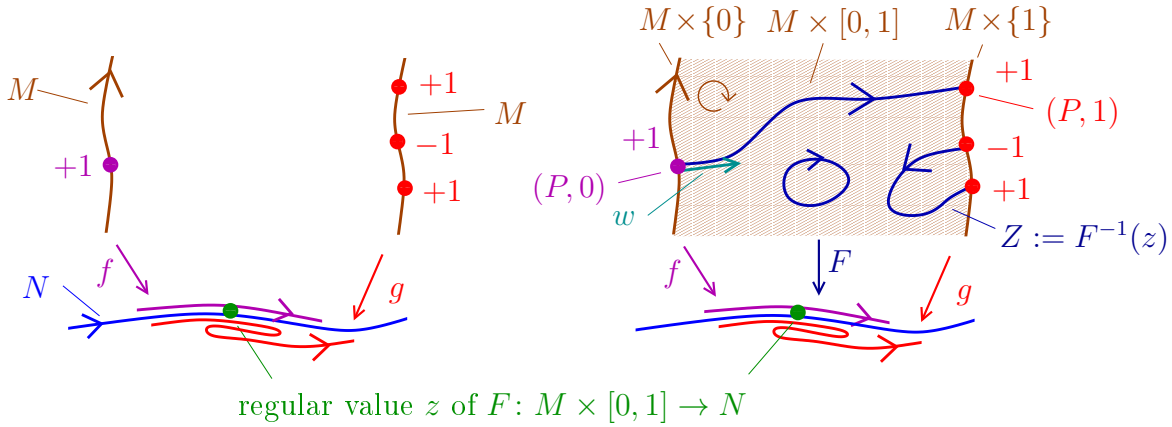
Lemma 38.6. Let M and N be two closed oriented n-dimensional smooth manifolds. Furthermore let $f, g\colon M \to N$ be two smooth maps and let $y \in N$ be a regular value for f and for g. If f and g are smoothly homotopic, then $\deg^y(f) = \deg^y(g)$.

Proof. Let $y \in N$ be a regular value for f and for g. It follows immediately from applying Lemma 38.5 to f and g and taking the intersection of the resulting open neighborhoods of $y \in N$ that there exists an open neighborhood W of $y \in N$ such that for every $y' \in W$ we have $\deg^y(f) = \deg^{y'}(f)$ and $\deg^y(g) = \deg^{y'}(g)$.

Since W is open and non-empty we obtain from Sard's Theorem 31.1 that there exists a regular value $z \in W$ of $F\colon M \times [0,1] \to N$. Note that by the Regular Value Theorem 26.4 we know that $F^{-1}(z)$ is a compact proper 1-dimensional submanifold of $M \times [0,1]$. Furthermore we saw in the proof of the Regular Value Theorem 26.4 on page 707 that we can equip the 1-dimensional smooth manifold $Z := F^{-1}(z) \subset M \times [0,1]$ with an orientation as follows.

(1) We equip $M \times [0,1]$ with the product orientation that we introduced on page 695.
(2) We pick a positive basis $v_1, \dots, v_n$ for $\mathrm{T}_z N$.
(3) Given a point $Q \in Z = F^{-1}(z)$ we pick vectors[377] $\widetilde{v}_1, \dots, \widetilde{v}_n \in \mathrm{T}_Q(M \times [0,1])$ such that $\mathrm{D}F_Q(\widetilde{v}_i) = v_i$ for $i = i, \dots, n$. Now we say that a non-zero vector $w \in \mathrm{T}_Q Z$ is positive if $\widetilde{v}_1, \dots, \widetilde{v}_n, w$ is a positive basis for $\mathrm{T}_Q(M \times [0,1])$.

We equip ∂Z with the corresponding boundary orientation $\epsilon\colon \partial Z \to \{\pm 1\}$ given by the definition on page 688.



Claim. Let $P \in M$.[378]
(1) If $(P,0) \in \partial Z$, then $\epsilon((P,0)) = -\deg(f, P)$.
(2) If $(P,1) \in \partial Z$, then $\epsilon((P,1)) = \deg(g, P)$.

Proof. First let $(P,0) \in \partial Z$. We make the following preparations:

(i) We denote by $q\colon M \times [0,1] \to [0,1]$ the projection. Note that it follows from the fact that Z is a proper submanifold of $M \times [0,1]$ and the fact that $\mathrm{T}_{(P,0)}\partial M = \ker(\mathrm{D}q_{(P,0)})$ that the image of $\mathrm{D}q_{(P,0)}\colon \mathrm{T}_{(P,0)} Z \to \mathrm{T}_{f(P)}\mathbb{R} = \mathbb{R}$ is non-trivial.

[377] Since z is a regular value of F we know that $\mathrm{D}F_Q$ is an epimorphism.
[378] Recall that our definition of a regular value, see page 701, entails that z is also a regular value of the restriction of F to $\partial(M \times [0,1])$, in other words, it is a regular value of $F_0 = f\colon M \to N$ and $F_1 = g\colon M \to N$. In particular $\deg(f, P)$ and $\deg(g, P)$ are actually defined.

(ii) By (i) we can pick $w \in \mathrm{T}_{(P,0)}Z$ with $\mathrm{D}q_{(P,0)}(w) > 0$.
(iii) Recall that our definition of a regular value, see page 701, entails that the differential $\mathrm{D}F_{(P,0)}\colon \mathrm{T}_{(P,0)}(M \times \{0\}) \to \mathrm{T}_{F(P,0)}N$ is also an epimorphism. Therefore we can actually pick vectors $\widetilde{v}_1, \dots, \widetilde{v}_n \in \mathrm{T}_{(P,0)}(M \times \{0\})$ such that $\mathrm{D}F_{(P,0)}(\widetilde{v}_i) = v_i$ for $i = i, \dots, n$.
(iv) We denote by $\iota\colon M \to M \times [0,1]$ the inclusion given by $x \mapsto (x, 0)$.
(v) For $i = 1, \dots, n$ we pick $\widetilde{u}_i \in \mathrm{T}_P M$ such that $\mathrm{D}\iota_P(\widetilde{u}_i) = v_i$.

Now we see that the following statements are equivalent:

we apply the definition of the boundary orientation from page 688 to $\gamma := q\colon Z \to \mathbb{R}_{\geq 0}$
note that $\gamma((P,0)) = 0$ and $\mathrm{D}\gamma_{(P,0)}(w) = \mathrm{D}q_{(P,0)}(w) > 0$

$$\epsilon((P,0)) = -1 \overset{\downarrow}{\iff} w \in \mathrm{T}_{(P,0)}Z \text{ is a positive vector}$$
$$\iff \widetilde{v}_1, \dots, \widetilde{v}_n, w \text{ is a positive basis for } \mathrm{T}_{(P,0)}(M \times [0,1])$$
$$\overset{\uparrow}{\iff} \widetilde{u}_1, \dots, \widetilde{u}_n \text{ is a positive basis for } \mathrm{T}_P M \overset{\uparrow}{\iff} \mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}N \text{ is orientation-preserving} \overset{\uparrow}{\iff} \deg(f,P) = +1.$$

follows from $\mathrm{D}q_{(P,0)}(w) > 0$ and the definition of the product orientation, see the Product Orientation Proposition 25.22

since $F_0 \circ \iota = f$ and since $v_1, \dots, v_n$ is a positive basis for $\mathrm{T}_{f(P)}N$

by definition of $\deg(f,P)$

Since ϵ and deg take values in $\{\pm 1\}$ we are actually done with the proof of (1).

Now suppose that we are given a point $(P,1) \in \partial Z$. The argument is very similar to the above. Again we can and will pick $w \in \mathrm{T}_{(P,1)}W$ with $\mathrm{D}q_{(P,1)}(w) > 0$. Now we see that the following statements are equivalent:

we apply the definition of the boundary orientation from page 688 to $\gamma(t) := q(1-t)\colon Z \to \mathbb{R}_{\geq 0}$
note that $\gamma((P,1)) = 0$ and $\mathrm{D}\gamma_{(P,1)}(w) = -\mathrm{D}q_{(P,0)}(w) < 0$

$$\epsilon((P,1)) = +1 \overset{\downarrow}{\iff} w \in \mathrm{T}_{(P,1)}W \text{ is a positive vector} \overset{\uparrow}{\iff} \dots \overset{\uparrow}{\iff} \deg(g,P) = +1.$$

precisely as above

This concludes the proof of (2). ⊞

Now we can easily complete the argument, namely we perform the following calculation:

$$\deg^y(f) - \deg^y(g) \overset{\uparrow}{=} \deg^z(f) - \deg^z(g) \overset{\uparrow}{=} \sum_{P \in \partial Z \cap (M \times \{0\})} \epsilon(P) + \sum_{P \in \partial Z \cap (M \times \{1\})} \epsilon(P) \overset{\uparrow}{=} 0.$$

since $z \in W$

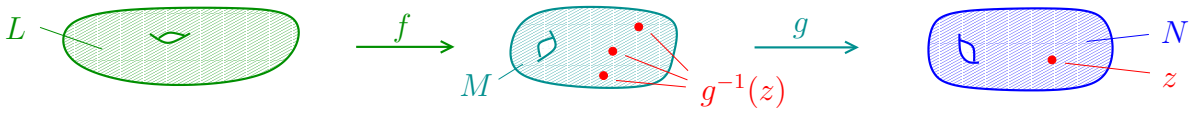
by the claim

by the Boundary-of-Compact 1-Dimensional Smooth Manifold Proposition 25.18 we know that the sum over the orientations of ∂Z is zero, the equality thus follows from the fact that $\partial Z \subset M \times \{0,1\}$

We have thus shown that $\deg^y(f) = \deg^y(g)$. ■

Lemma 38.7. Let $f\colon L \to M$ and $g\colon M \to N$ be two smooth maps between closed oriented connected non-empty smooth manifolds of the same dimension. If $z \in N$ is a regular value of $g\colon M \to N$ such that each point in $g^{-1}(z) \subset M$ is a regular value of $f\colon L \to M$, then

$$\deg^z(g \circ f) = \sum_{y \in g^{-1}(z)} \deg(g,y) \cdot \deg^y(f).$$



Proof. We perform the following calculation:

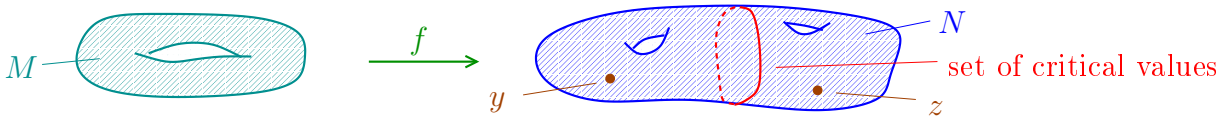
$$\deg^z(g\circ f) = \sum_{x\in(g\circ f)^{-1}(z)} \deg(g\circ f,x) \overset{\text{by the Orientation-Composition Lemma 25.3 we have } \deg(g\circ f,x)=\deg(g,f(x))\cdot\deg(f,x)}{\downarrow =} \sum_{y\in g^{-1}(z)}\sum_{x\in f^{-1}(y)} \deg(g,y)\cdot\deg(f,x)$$
$$= \sum_{y\in g^{-1}(z)} \deg(g,y)\cdot \underbrace{\sum_{x\in f^{-1}(y)} \deg(f,x)}_{=\deg^y(f)}.$$

■

After these preparations we can finally provide the proofs of Theorem 38.1, 38.3 and 38.4.

Proof of Theorem 38.1. Let $f\colon M\to N$ be a smooth map between two closed oriented connected non-empty smooth manifolds of the same dimension. Let $y\in N$ and $z\in N$ be regular values of f. We need to show that $\deg^y(f)=\deg^z(f)$.

The first idea might be to try to connect y and z via a path and to use this path to show that the two degrees are the same. But this approach is doomed to fail since in general the set of regular values is *not* path-connected. Thus we need to use a different approach.



By the Homogeneous-via-Diffeotopies Proposition 33.14 we know that there exists a diffeotopy $F\colon N\times[0,1]\to N$ with $F_0=\mathrm{id}$ and such that $F_1(y)=z$. We now obtain the following equalities:

$$\deg^y(f) = \underbrace{\deg(F_1,y)}_{\substack{=1 \text{ since it follows from the Smooth Isotopy-Orientation Proposition 34.2 that } F_1 \text{ is an orientation-preserving diffeomorphism}}} \cdot \deg^y(f) \overset{\substack{\text{by Lemma 38.7, since } F_1 \text{ is a diffeomorphism and since } y \text{ is a regular value of } f}}{\downarrow =} \deg^{F_1(y)}(F_1\circ f) \overset{\text{since } F_1(y)=z}{\downarrow =} \deg^z(F_1\circ f) \underset{\substack{\uparrow \\ \text{follows from Lemma 38.6, since } F_1\circ f \text{ and } F_0\circ f=f \text{ are smoothly homotopic and since } z \text{ is a regular value of } F_1\circ f \text{ and of } f}}{=} \deg^z(f).$$

■

Proof of Proposition 38.2. Let $g\colon M\to N$ be a map between two closed oriented connected non-empty smooth manifolds of the same dimension and let $f_0,f_1\colon M\to N$ be smooth maps that are both homotopic to g. We need to show that $\deg(f_0)=\deg(f_1)$.

It follows from the transitivity of being homotopic, see the Homotopy-Equivalence Basics Lemma 15.1, that f_0 and f_1 are homotopic. Furthermore it follows from the Replace-by-Smooth Homotopy Proposition 29.5 that f_0 and f_1 are actually smoothly homotopic.

Next note that it follows from Sard's Theorem 31.1 together with the Full Measure-Density Proposition 30.3 that we can find a $z\in N$ which is a regular value for f_0 and also for f_1. Thus it follows from Lemma 38.6 that $\deg(f_0)=\deg(f_1)$. ■

Proof of the Degree Homotopy Invariance Theorem 38.3. Let $f_0, f_1 \colon M \to N$ be maps between two closed oriented connected non-empty smooth manifolds of the same dimension. We assume that f_0 and f_1 are homotopic. Recall that we need to show that $\deg(f_0) = \deg(f_1)$. By the Whitney Approximation Theorem 29.1 we know that for $i = 0, 1$ the map f_i is homotopic to a smooth map $\widetilde{f}_i \colon M \to N$. Now we see that

$$\deg(f_0) \underset{\substack{\uparrow \\ \text{by definition}}}{=} \deg(\widetilde{f}_0) \underset{\substack{\uparrow \\ \text{by Proposition 38.2} \\ \text{applied to } g = f_0}}{=} \deg(\widetilde{f}_1) \underset{\substack{\uparrow \\ \text{by definition}}}{=} \deg(f_1).$$

■

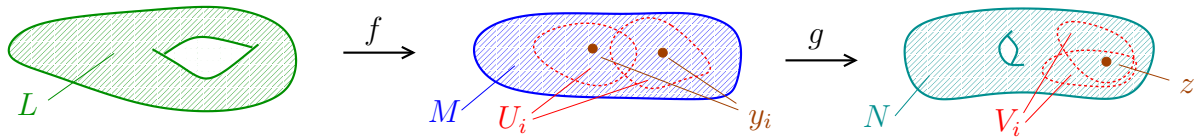
Proof of the Degree Multiplicativity Theorem 38.4 if f and g are smooth. Let $f \colon L \to M$ and $g \colon M \to N$ be smooth maps between closed oriented connected non-empty smooth manifolds of the same dimension. We need to show that $\deg(g \circ f) = \deg(g) \cdot \deg(f)$. Evidently the plan is to apply Lemma 38.7, but to carry out this idea we need the following claim.

Claim. There exists a regular value z of $g \colon M \to N$ such that each point in $g^{-1}(z) \subset M$ is a regular value of $f \colon L \to M$.

Proof. We start out with some regular value z of $g \colon M \to N$. We denote the points in $g^{-1}(z)$ by $y_1, \dots, y_k \in M$. As in the proof of Lemma 38.5 we obtain from the Inverse Mapping Theorem 24.1 open neighborhoods $U_1, \dots, U_k$ of $y_1, \dots, y_k \in M$ such that each $g \colon U_k \to V_k := g(U_k)$ is a diffeomorphism. We set

$$W \;:=\; \Big(\bigcap_{i=1}^{k} V_i\Big) \setminus g\Big(M \setminus \bigcup_{i=1}^{k} U_i\Big) \;\subset\; N.$$

As in the proof of Lemma 38.5 we see that W is an open subset of N and that W contains z. For $i = 1, \dots, k$ we set $\widetilde{U}_i = U_i \cap g^{-1}(W)$ and we denote by C_i the set of regular values of f in $\widetilde{U}_i$. By Sard's Theorem 31.1 we know that the set of critical values of f has measure zero in M. It follows almost immediately, that each C_i is a set of full measure in the open subset $\widetilde{U}_i$ of M.



Since $g \colon \widetilde{U}_i \to W$ is a diffeomorphism we see that each $g(C_i)$ is also a set of full measure in W. It follows from the Full Measure-Density Proposition 30.3 that the set

$$\{\text{set of regular values of } g\} \;\cap W \;\cap \bigcap_{i=1}^{k} g(C_i)$$

is non-empty. Any point in this set has the desired property. ⊞

Now let $z \in N$ be a regular value of g such that each point in $g^{-1}(z)$ is a regular value of f. Then

$$\deg^z(g \circ f) \underset{\substack{\uparrow \\ \text{by Lemma 38.7}}}{=} \sum_{y \in g^{-1}(z)} \deg(g, y) \cdot \underbrace{\deg^y(f)}_{=\deg(f)} = \deg(f) \cdot \underbrace{\sum_{y \in g^{-1}(z)} \deg(g, y)}_{=\deg(g)} = \deg(f) \cdot \deg(g).$$

■

Proof of the Degree Multiplicativity Theorem 38.4. Let $f\colon L \to M$ and $g\colon M \to N$ be maps between closed oriented connected non-empty smooth manifolds of the same dimension. By the Whitney Approximation Theorem 29.1 we know that there exists a homotopy $F\colon L \times [0,1] \to M$ with $F_0 = f$ and such that $\widetilde{f} := F_1$ is smooth. Furthermore we know that there exists a homotopy $G\colon M \times [0,1] \to N$ with $G_0 = g$ and such that $\widetilde{g} := G_1$ is smooth. Note that the map

$$\begin{array}{rccl} G \circ (F \times \mathrm{id}_{[0,1]})\colon L \times [0,1] & \to & N \\ (x,t) & \mapsto & G(F(x,t),t) \end{array}$$

is a homotopy from $g \circ f$ to the smooth map $\widetilde{g} \circ \widetilde{f}$. Thus we see that

$$\deg(g \circ f) \underset{\substack{\uparrow \\ \text{by the above and the definition} \\ \text{of the degree of } g \circ f}}{=} \deg(\widetilde{g} \circ \widetilde{f}) \underset{\substack{\uparrow \\ \text{by the previous discussion} \\ \text{of the smooth case}}}{=} \deg(\widetilde{g}) \cdot \deg(\widetilde{f}) \underset{\substack{\uparrow \\ \text{by definition of the} \\ \text{degrees of } g \text{ and } f}}{=} \deg(g) \cdot \deg(f).$$

■

38.3. **Calculations of the degree.** First we state a few basic properties of the degree.

Lemma 38.8. (Basics of Degree Lemma) Let $f\colon M \to N$ be a smooth map between two closed oriented connected non-empty smooth manifolds of the same dimension.

(1) If f is not surjective, then $\deg(f) = 0$.
(2) If f is an orientation-preserving diffeomorphism, then $\deg(f) = 1$.
(3) If f is an orientation-reversing diffeomorphism, then $\deg(f) = -1$.

Proof.

(1) If $f\colon M \to N$ is not surjective, then we pick $z \in N \setminus f(M)$. Note that z is a regular value, and if we use this regular value for the calculation of $\deg(f)$, then we obtain immediately that $\deg(f) = 0$.
(2),(3) If $f\colon M \to N$ is a diffeomorphism, then every point in N is a regular value. The desired statements now follow from the Basics-of-Orientations Lemma 25.13. ■

Example. Let $A \in \mathrm{O}(n+1)$. In Exercise 25.5 we showed that the map $S^n \to S^n$ given by $x \mapsto A \cdot x$ is orientation-preserving if and only if $\det(A) = 1$. Combining this piece of information with the Basics of Degree Lemma 38.8 we see that

$$\deg\begin{pmatrix} S^n & \to & S^n \\ v & \mapsto & A \cdot v \end{pmatrix} = \det(A).$$

In particular, for $A = -\mathrm{id}_{n+1}$ we obtain that

$$\deg\left(\text{antipodal map } \begin{array}{ccc} S^n & \to & S^n \\ v & \mapsto & -v \end{array}\right) = \det(-\mathrm{id}_{n+1}) = (-1)^{n+1}.$$

The following lemma doubles as an interesting example of a calculation of the degree.

Lemma 38.9.

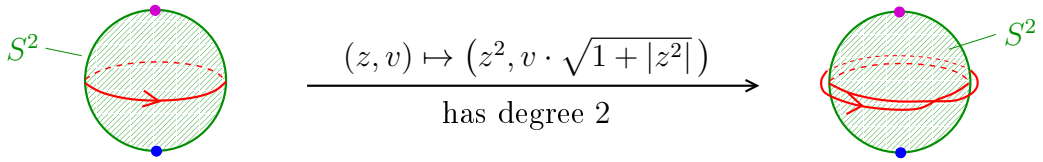
(1) Given any $d \in \mathbb{Z}$ we have

$$\deg\begin{pmatrix} S^1 & \to & S^1 \\ z & \mapsto & z^d \end{pmatrix} = d.$$

(2) Let $n \in \mathbb{N}$. We think of S^n as a subspace of $\mathbb{C} \times \mathbb{R}^{n-1}$. Given any $d \in \mathbb{N}$ we have[379]

$$\deg \begin{pmatrix} S^n & \to & S^n \\ (z,v) & \mapsto & \left(z^d, v \cdot \sqrt{1+|z^2|+\cdots+|z^2|^{d-1}}\right) \end{pmatrix} \;=\; d.$$

(3) Given any $n \in \mathbb{N}$ and given any $d \in \mathbb{Z}$ there exists a map $f\colon S^n \to S^n$ with degree $\deg(f) = d$.



Proof.

(1) The case $d = 0$ follows immediately from the Basics of Degree Lemma 38.8. Now assume that $d \neq 0$. In this case a straightforward calculation shows that each $x \in S^1$ is a regular point, thus every $y \in S^n$ is a regular value. We take $y = 1 \in S^1 \subset \mathbb{C}$ and we calculate

$$\deg(f) \;=\; \sum_{k=1}^{|d|} \deg(f, \exp(2\pi \mathrm{i} k/|d|)) \;=\; \sum_{k=1}^{|d|} \begin{cases} +1, & \text{if } d \geq 1, \\ -1, & \text{if } d < 1 \end{cases} \;=\; d.$$

using the Orientation Checking-via-Differentials Lemma 25.12 the orientation behavior can be deduced easily for the obvious extension of f to $\overline{B}^2$, the statement regarding S^1 then follows from the Boundary Orientation Lemma 25.17

(2) The proof of (2) is almost identical to the proof of (1). We leave it to the reader to fill in the details.

(3) Let $n \in \mathbb{N}$. The case $d > 0$ follows from (2). The case $d = -1$ follows from the example preceding the lemma which says in particular that any reflection in a hyperplane has degree -1. The case $d < 0$ follows from (2), the case $d = -1$ and the Degree Multiplicativity Theorem 38.4. Finally, by the Basics of Degree Lemma 38.8, the case $d = 0$ follows from taking a constant map $S^n \to S^n$. ∎

The following lemma often can be used to determine the degree of a self-map of S^1. The lemma will come in handy in the proof of the Whitney-Graustein Theorem 41.7.

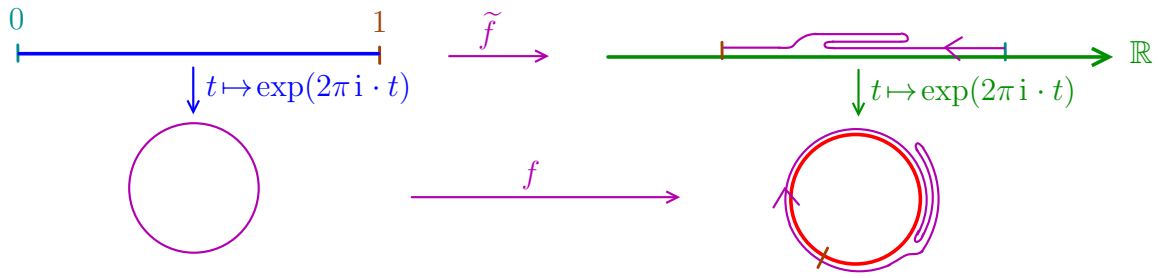
Lemma 38.10. (Degree-via-Lifts Lemma) Let $f\colon S^1 \to S^1$ be a smooth map. If $\widetilde{f}\colon [0,1] \to \mathbb{R}$ is a continuous map such that for every $t \in [0,1]$ we have the equality[380] $f(\exp(2\pi \mathrm{i} \cdot t)) = \exp(2\pi \mathrm{i} \cdot \widetilde{f}(t))$, then

$$\deg(f\colon S^1 \to S^1) \;=\; \widetilde{f}(1) - \widetilde{f}(0).$$

[379] Let $d \in \mathbb{N}$ and let $(z,v) \in S^n$. Note that $\left(z^d, v \cdot \sqrt{1+|z^2|+\cdots+|z^2|^{d-1}}\right)$ does indeed lie in S^n. If $|z| = 1$, then $v = 0$, and in this case clearly $(z^d, 0) \in S^n$. Now assume that $|z| \neq 1$. From $(z,v) \in S^n$ it follows that $|z|^2 + \|v\|^2 = 1$, i.e. $1 - |z^2| = 1 - |z|^2 = \|v\|^2$. Thus we see that

$$|z^d|^2 + \left(\|v\| \cdot \sqrt{1+|z^2|+\cdots+|z^2|^{d-1}}\right)^2 \;=\; |z^2|^d + \|v\|^2 \cdot \frac{1-|z^2|^d}{1-|z^2|} \;=\; |z^2|^d + \|v\|^2 \cdot \frac{1-|z^2|^d}{\|v\|^2} \;=\; 1.$$

[380] In Lemma 34.8 we showed that such a map $\widetilde{f}$ always exists.



Example. Let $n \in \mathbb{Z}$. We consider the map $f\colon S^1 \to S^1$ that is given by $f(z) = z^n$. In this case $\widetilde{f}(t) = n \cdot t$ satisfies the conditions of the Degree-via-Lifts Lemma 38.10. Thus we see, once again, that $\deg(f) = \widetilde{f}(1) - \widetilde{f}(0) = n$.

Proof. We consider the maps

$$\begin{array}{rcl} \Phi\colon [0,1] & \to & S^1 \\ t & \mapsto & \exp(2\pi\mathrm{i}\cdot t) \end{array} \quad \text{and} \quad \begin{array}{rcl} \Theta\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & \exp(2\pi\mathrm{i}\cdot t). \end{array}$$

Let $f\colon S^1 \to S^1$ be a smooth map and let $\widetilde{f}\colon [0,1] \to \mathbb{R}$ be a map such that for any $t \in [0,1]$ we have $f(\exp(2\pi\mathrm{i}\cdot t)) = \exp(2\pi\mathrm{i}\cdot \widetilde{f}(t))$. In other words, we have $f \circ \Phi = \Theta \circ \widetilde{f}$, i.e. we have the following commutative diagram:

$$\begin{array}{ccc} [0,1] & \xrightarrow{\widetilde{f}} & \mathbb{R} \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Theta} \\ S^1 & \xrightarrow{f} & S^1. \end{array}$$

Recall that by Lemma 34.7 we know that Θ is a local diffeomorphism. It follows, as in Lemma 34.8, that $\widetilde{f}$ is also smooth.

By Sard's Theorem 31.1 there exists a regular value $y \in S^1 \setminus \{(\Phi \circ f)(0)\}$ of $f\colon S^1 \to S^1$. We denote the preimages of y by $x_1, \dots, x_m \in S^1$. We denote by $t_1, \dots, t_m$ the unique real numbers in $(0,1)$ with $x_i = \Phi(t_i)$. After reordering them we can assume that $t_1 < \dots < t_m$. We pick $\widetilde{y} \in \mathbb{R}$ with $\Theta(\widetilde{y}) = y$.

We make the following observations:

(1) We equip $[0,1]$, S^1 and $\mathbb{R}$ with the standard orientations. Let $i \in \{1, \dots, m\}$. Note that $\Phi\colon [0,1] \to S^1$ and $\Theta\colon \mathbb{R} \to S^1$ are orientation-preserving. It follows from $f \circ \Phi = \Theta \circ \widetilde{f}$ that $\widetilde{f}$ is orientation-preserving at t_i if and only if f is orientation-preserving at x_i. This implies that

$$\text{sign of } \underbrace{\widetilde{f}'(t_i)}_{\text{derivative of } \widetilde{f}} = \deg(f, x_i) \in \{-1, 1\}.$$

(2)
- As $\widetilde{f}([0,t_1)) \in S^1 \setminus \{y\}$ there is a unique $k_0 \in \mathbb{Z}$ with $\widetilde{f}([0,t_1)) \subset (\widetilde{y}+k_0, \widetilde{y}+k_0+1)$.
- Let $i \in \{1, \dots, m\}$. Since $\widetilde{f}((t_i, t_{i+1})) \in S^1 \setminus \{y\}$ there exists a unique $k_i \in \mathbb{Z}$ such that $\widetilde{f}((t_i, t_{i+1})) \subset (\widetilde{y}+k_i, \widetilde{y}+k_i+1)$.
- Finally, since $\widetilde{f}((t_{m-1}, 1]) \in S^1 \setminus \{y\}$ there exists a unique $k_m \in \mathbb{Z}$ such that $\widetilde{f}((t_m, 1]) \subset (\widetilde{y}+k_m, \widetilde{y}+k_m+1)$.

(3) It follows from (1) that for $i = 0, \dots, m-1$ we have $k_{i+1} = k_i + \deg(f, x_i)$.

Now we see that

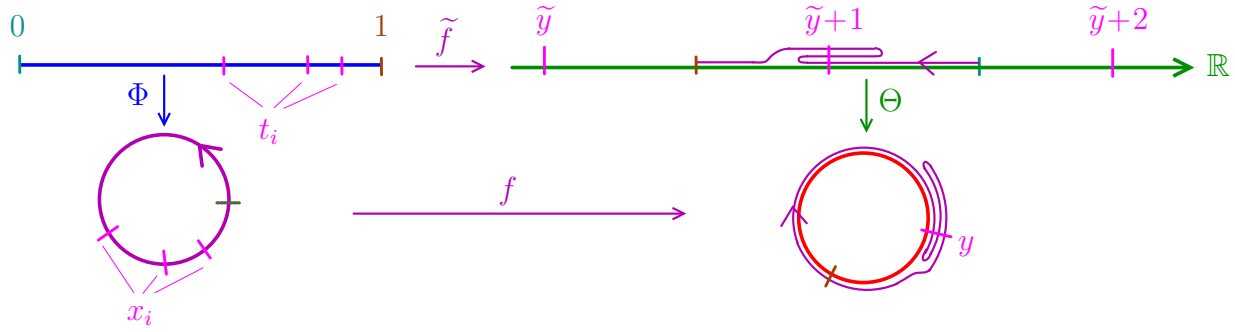
$$\widetilde{f}(1) - \widetilde{f}(0) \underset{\uparrow}{=} k_m - k_0 = \sum_{i=0}^{m-1} (k_{i+1} - k_i) \underset{\uparrow}{=} \sum_{i=0}^{m-1} \deg(f, x_i) \underset{\uparrow}{=} \deg(f\colon S^1 \to S^1).$$

since $\widetilde{f}(1) \in (\widetilde{y} + k_m, \widetilde{y} + k_m + 1)$ and since $\widetilde{f}(0) \in (\widetilde{y} + k_0, \widetilde{y} + k_0 + 1)$ we see that $\widetilde{f}(1) - \widetilde{f}(0) \in (k_m - k_0 - 1, k_m - k_0 + 1)$, the equality then follows from the fact that $\widetilde{f}(1) - \widetilde{f}(0) \in \mathbb{Z}$

by (3)

by the definition of $\deg(f\colon S^1 \to S^1)$ on page 899

We heave thus proved the desired equality. ■



38.4. **Applications of the degree.**

Theorem 38.11. (Sphere-Degree Fixed Point Theorem) Let $n \in \mathbb{N}$. Every map $f\colon S^n \to S^n$ with $\deg(f) \neq (-1)^{n+1}$ has a fixed point.

Example. The Sphere-Degree Fixed Point Theorem Theorem 38.11 says in particular that every map $f\colon S^2 \to S^2$ of degree one has a fixed point. Together with the Degree Homotopy Invariance Theorem 38.3 this implies that every map $f\colon S^2 \to S^2$ that is homotopic to the identity has a fixed point. This means that we cannot "deform" the identity to a map that is fixed point free. This is in contrast to the case $n = 1$ where we can "deform" the identity into the map that is given by rotation by a small angle, which is of course a fixed point free map.

Proof. Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a map without fixed points. We need to show that $\deg(f) = (-1)^{n+1}$. The fact that f has no fixed points means by definition that for all $x \in S^n$ we have $f(x) \neq x$. It follows implies that for all $t \in [0,1]$ and all $x \in S^n$ we have[381] $(1-t) \cdot f(x) \neq t \cdot x$. Therefore we can consider the map

$$\begin{aligned} F\colon S^n \times [0,1] &\to S^n \\ (x,t) &\mapsto \frac{(1-t)\cdot f(x) - t \cdot x}{\|(1-t)\cdot f(x) - t \cdot x\|}. \end{aligned}$$

This map is a homotopy between the map $x \mapsto f(x)$ and the antipodal map $x \mapsto -x$. We obtain that

$$\deg(f) \underset{\uparrow}{=} \deg(\text{antipodal map } x \mapsto -x) \underset{\uparrow}{=} (-1)^{n+1}.$$

by the Degree Homotopy Invariance Theorem 38.3

see page 905

[381] we know that the degree is preserved under homotopies

This case be seen as follows: if $t \neq \frac{1}{2}$ we have $\|(1-t)\cdot f(x)\| = 1 - t \neq t = \|tx\|$. On the other hand, for $t = \frac{1}{2}$ we have $(1-t)\cdot f(x) = \frac{1}{2} f(x) \neq \frac{1}{2} x = tx$.

■

Next we turn to the study of vector fields on spheres. On page 638 we saw that every odd-dimensional sphere admits a nowhere-vanishing vector field. In Question 23.17 we asked whether it is possible to find a nowhere-vanishing vector field on even-dimensional spheres. The following proposition says that this is not possible.

Theorem 38.12. (Hairy Ball Theorem) Every vector field on an even-dimensional sphere S^{2k} vanishes on at least one point.[382]

Remark.

(1) We will discuss the Hairy Ball Theorem 38.12 again in somewhat greater detail on page 1690. At that point we will also give a different proof using homology groups.
(2) We will generalize the Hairy Ball Theorem 38.12 with the Smooth Poincaré–Hopf Theorem 45.17 and the Poincaré–Hopf Theorem 129.7.
(3) The literature contains many proofs of the Hairy Ball Theorem 38.12. Some, as our proofs, require some technology, some, see e.g. [**Miln1978**] get away with less technology. For the case of S^2 one can find especially short proofs, see e.g. [**EG1979, McG2016**].

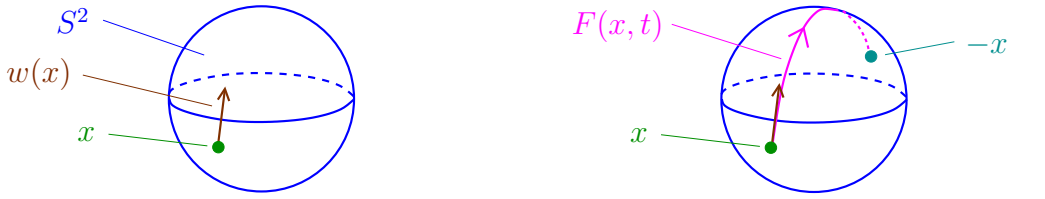
Proof. Suppose S^n admits a nowhere-vanishing vector field v. We have to show that n is odd. We consider the vector field w on S^n that is given by rescaling v to length one, i.e. that is given by

$$\begin{array}{rcl} S^n & \to & \mathbb{R}^{n+1} \\ x & \mapsto & w(x) := \frac{1}{\|v(x)\|} \cdot v(x). \end{array}$$

Furthermore we consider the map

$$\begin{array}{rcl} F\colon S^n \times [0,1] & \to & S^n \\ (x,t) & \mapsto & \cos(\pi t) \cdot x + \sin(\pi t) \cdot w(x). \end{array}$$

The map does indeed take values in S^n, since for any $x \in S^n$ and $t \in [0,1]$ we have



the vector field w gives rise to the homotopy F from id to $-$id

$$\begin{array}{rcl} \langle F(x,t), F(x,t)\rangle & = & \langle \cos(\pi t) \cdot x + \sin(\pi t) \cdot w(x), \cos(\pi t) \cdot x + \sin(\pi t) \cdot w(x)\rangle \\ & = & \cos(\pi t)^2 \cdot \|x\|^2 + \sin(\pi t)^2 \cdot \|w(x)\|^2 \;=\; 1. \\ & & \uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\ & & \text{since } \langle x, w(x)\rangle = 0 \qquad\qquad\qquad\qquad \text{since } \|x\| = \|w(x)\| = 1 \end{array}$$

[382] On S^{2k} we had so far found vector fields which vanish at *two* points. Can we find a vector field on S^{2k} which vanishes on only *one* point? We will discuss this question in Exercise 76.6.

Note that the map F is a homotopy between the two maps $F_0 = \mathrm{id}$ and the antipodal map $F_1 = -\mathrm{id}$. It follows that

$$\begin{array}{ccccccc} 1 & \underset{\uparrow}{=} & \deg(\mathrm{id}_{S^n}) & \underset{\uparrow}{=} & \deg(-\mathrm{id}_{S^n}) & \underset{\uparrow}{=} & (-1)^{n+1}. \\ \text{Basics of Degree Lemma 38.8 (1)} & & & \text{by the Degree Homotopy Invariance Theorem 38.3 applied to the above homotopy} & & \text{see page 905} & \end{array}$$

But this means that n is odd. ■

38.5. **The Hopf Theorem.** Before we turn to the Hopf Theorem let us first show that given any closed oriented smooth manifold any integer can be realized as the degree of a map $M \to S^n$.

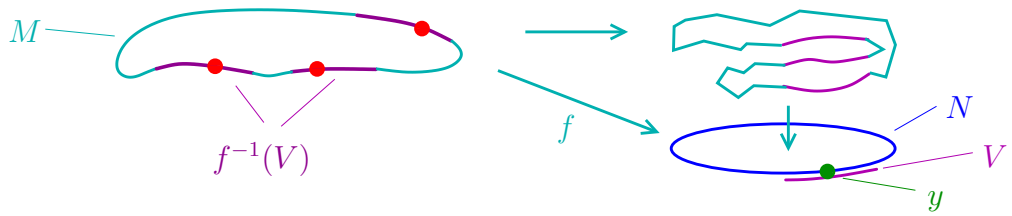
Proposition 38.13. (Degree Realization Proposition) Let $n \in \mathbb{N}$ and let M be a closed oriented connected non-empty n-dimensional smooth manifold. Given any $k \in \mathbb{Z}$ there exists a map $f\colon M \to S^n$ with $\deg(f) = k$.

Remark. We will reprove the Degree Realization Proposition 38.13 on page 2225.

The proof of the Degree Realization Proposition 38.13 rests on the following lemma which can be useful for calculating degrees of non-smooth maps.

Lemma 38.14. (Degree of Partially Smooth Map Lemma) Let $f\colon M \to N$ be a map between two closed oriented connected non-empty n-dimensional smooth manifolds. Let $y \in N$. Suppose that there exists an open neighborhood V of y such that f is smooth on $f^{-1}(V) \subset M$, such that $f^{-1}(y)$ consists of finitely many points and such that y is a regular value of the smooth map $f\colon f^{-1}(V) \to V$. Then

$$\deg(f) \;=\; \sum_{x \in f^{-1}(y)} \deg(f, x).$$



Proof of the Degree of Partially Smooth Map Lemma 38.14. The following claim is the key to proving the lemma.

Claim. The map f is homotopic to a smooth map g such that $f^{-1}(y) = g^{-1}(y)$ and such that for every $x \in f^{-1}(y)$ there exists an open neighborhood of x on which f and g agree.

Proof. We start out with a few preparations:

(a) We denote by $x_1, \ldots, x_m$ the points in $f^{-1}(y)$. For $i = 1, \ldots, m$ we pick a neighborhood U_i of x_i such that the closure $\overline{U}_i$ is contained in $f^{-1}(V)$.

(b) By the Manifold Metrization Proposition 18.10 or alternatively by the Manifold Metrization Proposition 27.4, we know that we can equip N with a metric d.

(c) Since M is compact and since $U_1, \dots, U_m$ are open we know by the Compact-Closed Lemma 1.21 and the Compact Image Lemma 2.13 that $f(M \setminus (U_1 \cup \dots \cup U_m))$ is a closed subset of N. Evidently $y \in f(M \setminus (U_1 \cup \dots \cup U_m))$.

(d) It follows from (b) and (c) together with the Distance-to-Subspace the Distance-to-Subspace Lemma 6.2 that there exists an $\epsilon > 0$ such that $d(f(x), y) > \epsilon$ for all $x \in M \setminus (U_1 \cup \dots \cup U_m)$.

It now follows from the Whitney Approximation Theorem 29.1 that there exists a homotopy $F\colon M \times [0,1] \to N$ with the following properties:

(1) We have $F_0 = f$.
(2) The map $F_1\colon M \to N$ is smooth.
(3) The homotopy is a homotopy rel $\overline{U}_1 \cup \dots \cup \overline{U}_m$.
(4) For every $(x, y) \in M \times [0,1]$ we have $d(F(x,t), f(x)) \leq \frac{\epsilon}{2}$.

We set $g = F_1$. By (2) the map g is smooth and by construction the map g is homotopic to f. It follows easily from (3) and (4) that $f^{-1}(y) = g^{-1}(y)$. Furthermore it follows from (3) and (4) that f and g agree on $(U_1 \cup \dots \cup U_m) \cap f^{-1}(B_{\frac{\epsilon}{2}}(y))$. Thus we see that for any $x \in f^{-1}(y)$ the maps f and g agree on the open neighborhood $(U_1 \cup \dots \cup U_m) \cap f^{-1}(B_{\frac{\epsilon}{2}}(y))$ of x. ⊞

Now we see that

$$\begin{array}{ccccccc} \deg(f) & = & \deg(g) & = & \displaystyle\sum_{x \in g^{-1}(y)} \deg(g, x) & = & \displaystyle\sum_{x \in f^{-1}(y)} \deg(f, x). \\ \uparrow & & & \uparrow & & \uparrow & \\ \text{by the Degree Homotopy} & & & \text{by design we know that } f^{-1}(y) = g^{-1}(y) \text{ and we know that} & & & \\ \text{Invariance Theorem 38.3} & & & \text{maps } f \text{ and } g \text{ agree on neighborhoods around each } x_i, & & & \\ & & & \text{in particular } y \text{ is also a regular value for } f \text{ and } g \text{ and} & & & \\ & & & \text{for each } x \in f^{-1}(y) \text{ we have } \deg(f,x) = \deg(g,x) & & & \end{array}$$

■

Proof of the Degree Realization Proposition 38.13. Let M be a closed oriented connected non-empty n-dimensional smooth manifold and let $k \in \mathbb{Z}$. By Lemma 38.9 there exists a map $g\colon S^n \to S^n$ of degree k. It follows from this fact and the Degree Multiplicativity Theorem 38.4 that it suffices to prove the following claim.

Claim. There exists a map $f\colon M \to S^n$ of degree $+1$.

Proof. Since M is non-empty we can pick an orientation-preserving smooth embedding $\Phi\colon \overline{B}^n \to M$. Next, similar to the setting of the Ball-Quotient Sphere Lemma 5.20 we consider the map

$$\begin{array}{rcl} \Omega\colon \overline{B}^n & \to & S^n \\ r \cdot v & \mapsto & \underbrace{\begin{pmatrix} 0 & (-1)^{n-1} & \dots \\ 0 & 0 & \mathrm{id}_{n-1} \\ -1 & \dots & 0 \end{pmatrix}}_{\in \mathrm{SO}(n+1)} \cdot \underbrace{\begin{pmatrix} \cos(\pi r) \\ \sin(\pi r) \cdot v \end{pmatrix}}_{\in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}} \quad \text{with } r \in [0,1] \text{ and } v \in S^{n-1}. \end{array}$$

By the Ball-Quotient Sphere Lemma 5.20 this map has the following properties:

(1) The map Ω sends every point in S^{n-1} to the North Pole $N := (0, \dots, 0, 1)$.
(2) The map Ω sends the origin to the South Pole $S := (0, \dots, 0, -1)$.
(3) The map $B^n \to \overline{B}^n \xrightarrow{\Omega} S^n$ is a smooth embedding and, if we equip B^n and S^n with the standard orientations, the map is orientation-preserving.

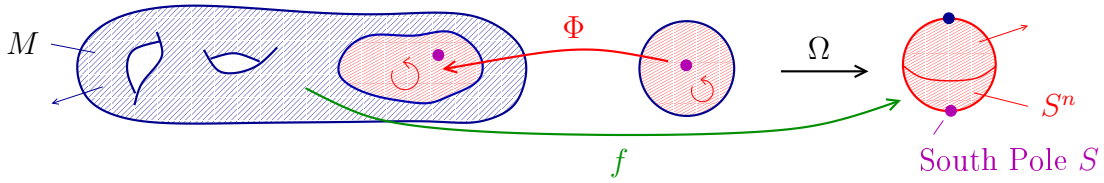
Next we consider the map

$$\begin{array}{rcl} f\colon M & \to & S^n \\ x & \mapsto & \begin{cases} \Omega(\Phi^{-1}(x)), & \text{if } x \in \Phi(\overline{B}^n), \\ N, & \text{if } x \notin \Phi(\overline{B}^n). \end{cases} \end{array}$$

We make the following observations:

(1) It follows from the Compact Image Lemma 2.13, the Compact-Closed Lemma 1.21 (2) and the Pasting Pasting Proposition 2.6 (2) that f is continuous.
(2) The map f is smooth on the open subset $\Phi(B^n)$.
(3) Since Φ and $\Omega\colon B^n \to S^n$ are orientation-preserving smooth embeddings we see that the map $f|_{\Phi(B^n)}\colon \Phi(B^n) \to S^n$ is an orientation-preserving smooth embedding.

It follows from the Partially Smooth Map Lemma 38.14 applied to $y = S$ and $V = S^n \setminus \{N\}$ that $\deg(f) = 1$. ■



In the Degree Realization Proposition 38.13 we just showed that every closed orientable connected non-empty n-dimensional smooth manifold M admits a smooth map $f\colon M \to S^n$ of degree 1. In particular the torus $S^1 \times S^1$ admits a smooth map $f\colon S^1 \times S^1 \to S^2$ of degree 1. The question arises whether we can swap the roles of the torus and the sphere. In other words we have the following question.

Question 38.15. Does there exist a smooth map $f\colon S^2 \to S^1 \times S^1$ of degree one?

We will answer this question in Proposition **??** or alternatively Corollary 101.9, once we have developed many more tools.
Now we can finally state the Hopf Degree Theorem, which is the main theorem of this section. It gives a very satisfying classification of maps to S^n up to homotopy.

Theorem 38.16. (Hopf Degree Theorem) Let $n \in \mathbb{N}$, let M be a closed oriented connected non-empty n-dimensional smooth manifold and let $f, g\colon M \to S^n$ be two maps. Then

$$\deg(f) = \deg(g) \iff \text{the maps } f \text{ and } g \text{ are homotopic.}$$

Remark.

(1) For $n = 2$ the theorem was first proved by Luitzen Brouwer [**Brou1921**, p. 281], the general case was first proved by Heinz Hopf [**Hopf1927a**, Satz IIb].
(2) We will give an alternative proof of the Hopf Degree Theorem 38.16 much later later on page **??**, when we deduce it from the much more general Hopf Theorem **??**.
(3) The approach we use to prove the Hopf Degree Theorem 38.16 will also be used later in the proof of Proposition 157.10 where we calculate the cobordism group $\Omega_0^{\text{fat}}(\mathbb{R}^n)$.
(4) The reader who already knows about higher homotopy groups will have noticed that the Degree Realization Proposition 38.13 and the Hopf Degree Theorem 38.16, both applied to $M = S^n$, imply that $\pi_n(S^n) \cong \mathbb{Z}$.

Proof. We will prove the Hopf Degree Theorem 38.16 in the next and final section of this chapter. ■

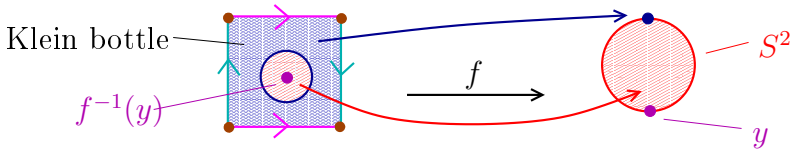
We conclude this section with the non-orientable analogue of the Hopf Degree Theorem 38.16. To formulate this analogue we need to introduce the $\mathbb{Z}_2$-degree.

Definition. Let $f\colon M \to N$ be a map between two closed (possibly non-orientable) connected non-empty n-dimensional smooth manifolds. We do not assume that M or N are orientable. By the Whitney Approximation Theorem 29.1 we know that f is homotopic to a smooth map $g\colon M \to N$. Let $y \in N$ be a regular value of g. We define the $\mathbb{Z}_2$-degree of f as follows:

$$\deg_{\mathbb{Z}_2}(f) \;:=\; \#g^{-1}(y) \;\in\; \mathbb{Z}_2.$$

Using a variation on Proposition 38.2 one can show that this notion is well-defined.

One can easily show (see also Exercise 38.7) that the obvious analogues of Proposition 38.2, Theorem 38.1, the Degree Homotopy Invariance Theorem 38.3, the Degree Multiplicativity Theorem 38.4 and the Degree Realization Proposition 38.13 hold in this context. As an example we show in the figure a map $f\colon K \to S^2$ from the Klein bottle K to the sphere S^2 with $\deg_{\mathbb{Z}_2}(f) = 1 \in \mathbb{Z}_2$.



Now we can formulate the analogue of the Hopf Degree Theorem 38.16 for non-orientable manifolds.

Theorem 38.17. (Non-Orientable Hopf Degree Theorem) Let $n \in \mathbb{N}$ and let M be a closed *non-orientable* connected non-empty n-dimensional smooth manifold and let $f, g\colon M \to S^n$ be two maps. Then

$$\deg_{\mathbb{Z}_2}(f) \;=\; \deg_{\mathbb{Z}_2}(g) \quad\Longleftrightarrow\quad \text{the maps } f \text{ and } g \text{ are homotopic.}$$

Proof. At the very end of this chapter we will explain how to modify the proof of the Hopf Degree Theorem 38.16 to obtain a proof of the Non-Orientable Hopf Degree Theorem 38.17. ■

38.6. **Proof of the Hopf Degree Theorem 38.16.** This final section is totally dedicated to the proofs of the Hopf Degree Theorem 38.16 and of the Non-Orientable Hopf Degree Theorem 38.17. First we recall and introduce some notation.

Notation. As in the Ball-Quotient Sphere Lemma 5.20 we consider the map

$$\begin{aligned} \Omega\colon \overline{B}^n/S^{n-1} &\to S^n \\ [r\cdot v] &\mapsto \underbrace{\begin{pmatrix} 0 & (-1)^{n-1} & \dots \\ 0 & 0 & \mathrm{id}_{n-1} \\ -1 & \dots & 0 \end{pmatrix}}_{\in \mathrm{SO}(n+1)} \cdot \underbrace{\begin{pmatrix} \cos(\pi r) \\ \sin(\pi r)\cdot v \end{pmatrix}}_{\in \mathbb{R}\times\mathbb{R}^n = \mathbb{R}^{n+1}} \quad \text{with } r \in [0,1] \text{ and } v \in S^{n-1}. \end{aligned}$$

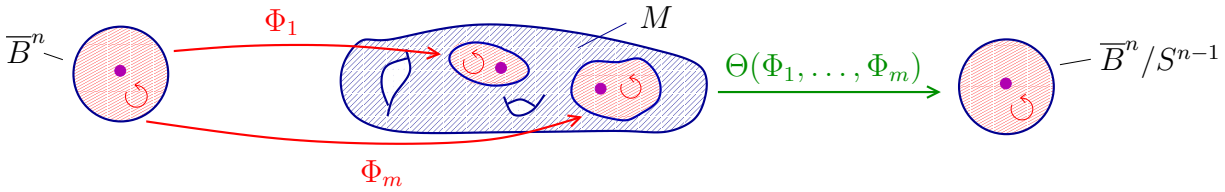
By the Ball-Quotient Sphere Lemma 5.20 this map has the following properties:

(1) The map $\Omega\colon \overline{B}^n/S^{n-1} \to S^n$ is a homeomorphism. We use this homeomorphism and the usual smooth structure on S^n to equip $\overline{B}^n/S^{n-1}$ with a smooth structure and with an orientation.
(2) The map Ω sends the point $[S^{n-1}] \in \overline{B}^n/S^{n-1}$ to the North Pole $N := (0,\dots,0,1)$.
(3) The map Ω sends the origin to the South Pole $S := (0,\dots,0,-1)$.
(4) The map $B^n \to \overline{B}^n/S^{n-1} \xrightarrow{\Omega} S^n$ is an orientation-preserving smooth embedding.

Remark. We use the diffeomorphism $\Omega\colon \overline{B}^n/S^{n-1} \to S^n$ to replace in the following the smooth manifold S^n by $\overline{B}^n/S^{n-1}$. This approach greatly simplifies the notation throughout the following arguments.

Notation. Let M be a closed n-dimensional smooth manifold. Given smooth embeddings $\Phi_1,\dots,\Phi_m\colon \overline{B}^n \to M$ with disjoint images we define, similar to the construction in the proof of the Degree Realization Proposition 38.13, the map

$$\begin{array}{rccl} \Theta(\Phi_1,\dots,\Phi_m)\colon M & \to & \overline{B}^n/S^{n-1} & \\ x & \mapsto & \begin{cases} [\Phi_i^{-1}(x)], & \text{if } x \in \Phi_i(\overline{B}^n), \\ [S^{n-1}], & \text{otherwise.} \end{cases} \end{array}$$



The proof of the Hopf Degree Theorem 38.16 rests on three lemmas.

Lemma 38.18. Let M be a closed n-dimensional smooth manifold. Furthermore let $\Phi_1,\dots,\Phi_m\colon \overline{B}^n \to M$ be smooth embeddings with disjoint images.

(1) If M is oriented, then the degree of $\Theta(\Phi_1,\dots,\Phi_m)\colon M \to \overline{B}^n/S^{n-1}$ can be calculated as follows:

$$\Theta(\Phi_1,\dots,\Phi_m) \;=\; \sum_{i=1}^{m} \begin{cases} +1, & \text{if } \Phi_i \text{ is orientation-preserving,} \\ -1, & \text{if } \Phi_i \text{ is orientation-reversing.} \end{cases}$$

(2) Let $\Psi_1,\dots,\Psi_m\colon \overline{B}^n \to M$ be another set of smooth embeddings with disjoint images. If the smooth embedding[383] $\Phi_1 \sqcup \dots \sqcup \Phi_m\colon \overline{B}^n \sqcup \dots \sqcup \overline{B}^n \to M$ is smoothly isotopic to the smooth embedding $\Psi_1 \sqcup \dots \sqcup \Psi_m\colon \overline{B}^n \sqcup \dots \sqcup \overline{B}^n \to M$, then the maps $\Theta(\Phi_1,\dots,\Phi_m)\colon M \to \overline{B}^n/S^{n-1}$ and $\Theta(\Psi_1,\dots,\Psi_m)\colon M \to \overline{B}^n/S^{n-1}$ are homotopic.

Proof.

(1) The calculation of the degree follows from a slight variation on the argument of Proposition 38.13, using the general case of the Degree of Partially Smooth Map Lemma 38.14.

[383]Since the $\overline{B}^n$ is compact we obtain from Lemma 24.11 that m smooth embeddings of $\overline{B}^n$ with disjoint images give rise to a smooth embedding of the disjoint union of m closed balls.

(2) We start out with a few preparations:
   (a) By hypothesis we can pick a smooth isotopy $F\colon (\overline{B}^n \sqcup \cdots \sqcup \overline{B}^n) \times [0,1] \to M$ from the smooth embedding $\Phi_1 \sqcup \cdots \sqcup \Phi_m\colon \overline{B}^n \sqcup \cdots \sqcup \overline{B}^n \to M$ to the smooth embedding $\Psi_1 \sqcup \cdots \sqcup \Psi_m\colon \overline{B}^n \sqcup \cdots \sqcup \overline{B}^n \to M$.
   (b) We denote by $p\colon \overline{B}^n \times [0,1] \to \overline{B}^n$ and $q\colon \overline{B}^n \times [0,1] \to \overline{B}^n$ the projections.
   (c) Given $i = 1, \ldots, m$ we consider the map

$$\begin{aligned} \widetilde{F}_i = (F|_{\overline{B}_i \times [0,1]} \times q)\colon \overline{B}^n \times [0,1] &\to M \times [0,1] \\ (x,t) &\mapsto (F(x,t),t) \end{aligned}$$

with x in the i-th copy of $\overline{B}^n$

   This map is injective and continuous. Since $\overline{B}^n \times [0,1]$ is compact and since $M \times [0,1]$ is Hausdorff it follows from the Compact-Hausdorff Proposition 2.17 (3) that it is an embedding. We denote by $\widetilde{F}_i^{-1}\colon \widetilde{F}_i(\overline{B}^n \times [0,1]) \to \overline{B}^n \times [0,1]$ the inverse.

Finally we consider the map

$$\begin{aligned} G\colon M \times [0,1] &\to \overline{B}^n/S^{n-1} \\ z &\mapsto \begin{cases} [p(\widetilde{F}_i^{-1}(z))], & \text{if } z \in \widetilde{F}_i(\overline{B}^n \times [0,1]), \\ [S^{n-1}], & \text{otherwise.} \end{cases} \end{aligned}$$

Basically by definition we have $G_0 = \Theta(\Phi_1, \ldots, \Phi_m)$ and $G_1 = \Theta(\Psi_1, \ldots, \Psi_m)$. Thus it remains to show that the map $G\colon M \times [0,1] \to \overline{B}^n/S^{n-1}$ is continuous. First note that it follows from the Codimension-Zero Smooth Embedding Proposition 24.16 that $\widetilde{F}_1(B^n \times [0,1]) \cup \cdots \cup \widetilde{F}_m(B^n \times [0,1])$ is an open subset of $M \times [0,1]$. Furthermore it follows from the Product Topology Properties Proposition 5.6, the Compact Image Lemma 2.13 and the Compact-Closed Lemma 1.21 (2) that $\widetilde{F}_1(\overline{B}^n \times [0,1]) \cup \cdots \cup \widetilde{F}_1(\overline{B}^n \times [0,1])$ is a closed subset of $M \times [0,1]$. It follows from this lengthy discussion and the Pasting Pasting Proposition 2.6 (2) that this map $M \times [0,1] \to \overline{B}^n/S^{n-1}$ is indeed continuous. ■

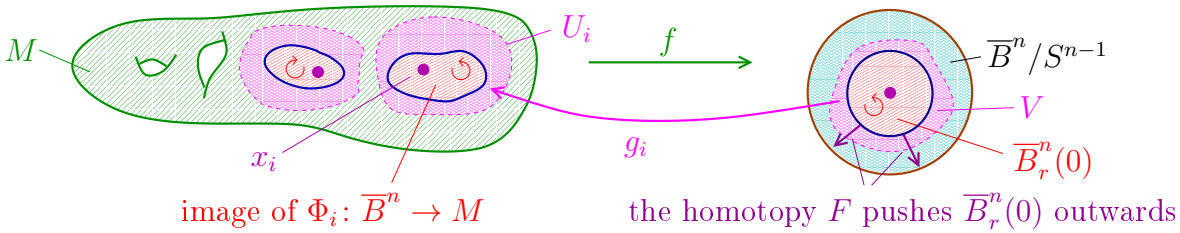
Lemma 38.19. Let M be a closed n-dimensional smooth manifold. Given any map $f\colon M \to \overline{B}^n/S^{n-1}$ there exist smooth embeddings $\Phi_1, \ldots, \Phi_m\colon \overline{B}^n \to M$ with disjoint images such that f is homotopic to $\Theta(\Phi_1, \ldots, \Phi_m)\colon M \to \overline{B}^n/S^{n-1}$.

Proof of Lemma 38.19. We start out with the following straightforward steps:

(1) By the Whitney Approximation Theorem 29.1 and the Degree Homotopy Invariance Theorem 38.3 we can assume that f is smooth.
(2) By Sard's Theorem 31.1 we know that the smooth map $f\colon M \to \overline{B}^n/S^{n-1}$ has a regular value. By applying the Homogeneous-via-Diffeotopies Proposition 33.14 to the smooth manifold $\overline{B}^n/S^{n-1}$ we can assume that the "origin" $0 \in \overline{B}^n/S^{n-1}$ is a regular value of f.
(3) We denote the preimages of 0 by $x_1, \ldots, x_m \in M$.
(4) Since each $\mathrm{D}f_{x_i}\colon \mathrm{T}_{x_i}M \to \mathrm{T}_0\overline{B}^n/S^{n-1}$ is an isomorphism we know by the Inverse Mapping Theorem 24.1 that there exist neighborhoods $U_1, \ldots, U_m$ of $x_1, \ldots, x_m$ and V of $0 \in \overline{B}^n/S^{n-1}$ such that for each $i \in \{1, \ldots, m\}$ the map f restricts to a diffeomorphism $f\colon U_i \to V$.

(5) Since M is Hausdorff we can arrange that the neighborhoods $U_1, \dots, U_m$ of $x_1, \dots, x_m$ are disjoint.
(6) For $i = 1, \dots, m$ we denote by $g_i \colon V \to U_i$ the inverse of $f \colon U_i \to V$.
(7) We pick an $r \in (0,1)$ such that $\overline{B}^n_r(0) \subset V$.
(8) For $i = 1, \dots, m$ we consider the smooth embedding

$$\begin{array}{rcl} \Phi_i \colon \overline{B}^n & \to & M \\ x & \mapsto & g_i(x \cdot r). \end{array}$$



It now remains to prove the following claim.

Claim. The map $f \colon M \to \overline{B}^n/S^{n-1}$ is homotopic to $\Theta(\Phi_1, \dots, \Phi_m) \colon M \to \overline{B}^n/S^{n-1}$.

Proof. We consider the homotopy[384]

$$\begin{array}{rcl} F \colon \overline{B}^n/S^{n-1} \times [0,1] & \to & \overline{B}^n/S^{n-1} \\ ([x],t) & \mapsto & \begin{cases} [x \cdot ((1-t) + t \cdot \frac{1}{r})], & \text{if } \|x\| \le r, \\ [x \cdot ((1-t) + t \cdot \frac{1}{\|x\|})], & \text{if } \|x\| \ge r \end{cases} \end{array}$$

from $F_0 = \mathrm{id}$ to a map $F_1 \colon \overline{B}^n/S^{n-1} \to \overline{B}^n/S^{n-1}$ whose restriction to $\overline{B}^n_r(0)$ is induced by multiplication by $\frac{1}{r}$. Intuitively this homotopy F "expands" the r-ball till it covers all of the sphere $\overline{B}^n/S^{n-1}$ and everything outside of the r-ball gets pushed to the point $[S^{n-1}]$.

Finally note that it follows immediately from the definitions that the map

$$\begin{array}{rcl} G = F \circ (f \times (\text{projection to } [0,1])) \colon M \times [0,1] & \to & \overline{B}^n/S^{n-1} \\ (x,t) & \mapsto & F(f(x),t) \end{array}$$

is a homotopy from f to $\Theta(\Phi_1, \dots, \Phi_m)$. ∎

Our next lemma is a refinement of the previous Lemma 38.19. The proof of the next lemma contains perhaps the most clever idea in the context of the proof of the Hopf Degree Theorem 38.16.

Lemma 38.20. Let M be a closed oriented n-dimensional smooth manifold. Furthermore let $f \colon M \to \overline{B}^n/S^{n-1}$ be a map. We set $k := |\deg(f)|$. There exist k smooth embeddings $\Phi_1, \dots, \Phi_k \colon \overline{B}^n \to M$ with disjoint images such that the map $f \colon M \to \overline{B}^n/S^{n-1}$ is homotopic to $\Theta(\Phi_1, \dots, \Phi_k) \colon M \to \overline{B}^n/S^{n-1}$.

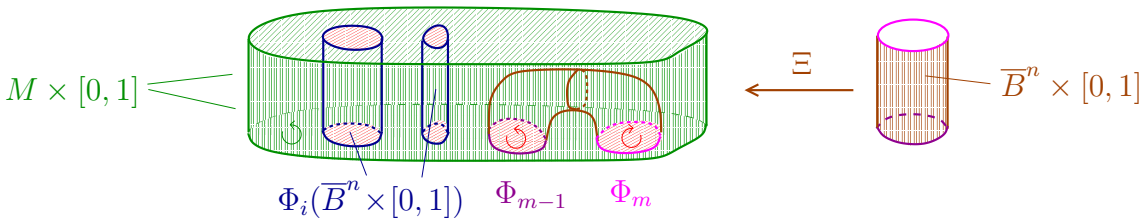
[384] To show that the map is continuous actually requires a bit of footwork. First note that it follows from the Pasting Proposition 2.6 that the obvious corresponding map $\overline{B}^n \times [0,1] \to \overline{B}^n$ is continuous. By the Twice Quotient Lemma 5.23 this map descends to a continuous map $(\overline{B}^n \times [0,1])/\sim \to \overline{B}^n/S^{n-1}$. Finally it follows from the subtle the Product-Quotient Mixing Lemma 13.6 that the given map $\overline{B}^n/S^{n-1} \times [0,1] \to \overline{B}^n/S^{n-1}$ is also continuous.

Proof of Lemma 38.20. Let M be a closed oriented n-dimensional smooth manifold and let $f\colon M \to \overline{B}^n/S^{n-1}$ be a map of degree k. By Lemma 38.19 we know that there exist smooth embeddings $\Phi_1, \dots, \Phi_m\colon \overline{B}^n \to M$ with disjoint images such that f is homotopic to $\Theta(\Phi_1, \dots, \Phi_m)\colon M \to \overline{B}^n/S^{n-1}$.

It follows from Lemma 38.18 (1) that $m \geq k$. If $m = k$, then we are done. Now assume that $m > k$. It follows again from Lemma 38.18 (1) that, after possibly reordering the Φ_i, we can assume that the smooth embedding $\Phi_{m-1}\colon \overline{B}^n \to M$ is orientation-preserving and that the smooth embedding $\Phi_m\colon \overline{B}^n \to M$ is orientation-reversing. By induction on m it suffices to show that $\Theta(\Phi_1, \dots, \Phi_m)$ is homotopic to $\Theta(\Phi_1, \dots, \Phi_{m-2})$.

Claim. There exists a smooth embedding $\Xi\colon \overline{B}^n \times [0,1] \to M \times [0,1)$ with the following two properties:

(1) For every $x \in \overline{B}^n$ we have $\Xi(x,0) = (\Psi_{m-1}(x), 0)$ and $\Xi(x,1) = (\Psi_m(x), 0)$.
(2) The image of Ξ is disjoint from $\Phi_i(\overline{B}^n) \times [0,1]$, $i = 1, \dots, m-2$.



Proof. It follows fairly easily from the Smooth Ball Embedding Theorem 37.8 that there exists an open subset $U \subset M$ and an orientation-preserving chart $\Gamma\colon U \to B^n_6(0)$ with the following properties:

(1) $\Phi_1(\overline{B}^n) \cup \dots \cup \Phi_m(\overline{B}^n)$ are contained in U,
(2) $\Gamma(\Phi_1(\overline{B}^n)) \cup \dots \cup \Gamma(\Phi_{m-2}(\overline{B}^n))$ is contained in $B^n_6(0) \setminus B^n_4(0)$,
(3) $\Gamma \circ \Phi_{m-1}\colon \overline{B}^n \to B^n_6(0)$ is given by $(x_1, \dots, x_n) \mapsto (-2 + x_1, x_2, \dots, x_n)$.
(4) $\Gamma \circ \Phi_m\colon \overline{B}^n \to B^n_6(0)$ is given by $(x_1, \dots, x_n) \mapsto (2 - x_1, x_2, \dots, x_n)$.



The desired smooth embedding is now given by

$$\begin{array}{rcl} \Xi\colon \overline{B}^n \times [0,1] & \to & M \times [0,1] \\ (x_1, x_2, \dots, x_n, t) & \mapsto & \Gamma^{-1}(\cos(\pi \cdot t) \cdot (-2 + x_1), x_2, \dots, x_n, \underbrace{\tfrac{1}{2}\sin(\pi \cdot t) \cdot x_1}_{\in [0,1)}). \end{array}$$

⊞

In the following we will use the smooth embedding $\Xi\colon \overline{B}^n \times [0,1] \to M \times [0,1]$ to construct a homotopy from $\Theta(\Phi_1, \dots, \Phi_m)$ to $\Theta(\Phi_1, \dots, \Phi_{m-2})$. To do so we need to make a few preparations:

(1) For $i = 1, \dots, m-2$ we denote by $\widetilde{F}_i\colon \overline{B}^n \times [0,1] \to M \times [0,1]$ the embedding that is given by $\widetilde{F}_i(x,t) = (\Phi_i(x), t)$.
(2) We denote by $p\colon \overline{B}^n \times [0,1] \to \overline{B}^n$ the projection.

Similar to the proof of Lemma 38.18 we now consider the map

$$\begin{array}{rcl} G\colon M\times[0,1] & \to & \overline{B}^n/S^{n-1} \\ z & \mapsto & \begin{cases} p(\widetilde{F}_i^{-1}(z)), & \text{if } z\in \widetilde{F}_i(\overline{B}^n\times[0,1]), \\ p(\Xi^{-1}(z)), & \text{if } z\in \Xi(\overline{B}^n\times[0,1]), \\ [S^{n-1}], & \text{otherwise.} \end{cases} \end{array}$$

The argument in the proof of Lemma 38.18 shows that $G\colon M\times[0,1]\to\overline{B}^n/S^{n-1}$ is continuous, i.e. G is a homotopy. Finally note that it follows immediately from the definitions that $G_0=\Theta(\Phi_1,\dots,\Phi_m)$ and $G_1=\Theta(\Phi_1,\dots,\Phi_{m-2})$. ■

Proof of the Hopf Degree Theorem 38.16. Let $n\in\mathbb{N}$ and let M be a closed oriented connected non-empty n-dimensional smooth manifold. As we discussed on page 913, the smooth manifold M admits an orientation-preserving diffeomorphism to $\overline{B}^n/S^{n-1}$. Thus it suffices to prove the following claim.

Claim. Given any two maps $f,g\colon M\to\overline{B}^n/S^{n-1}$ we have

$$\deg(f)=\deg(g) \quad\Longleftrightarrow\quad \text{the maps } f \text{ and } g \text{ are homotopic.}$$

Proof. The "$\Leftarrow$"-direction is just a special case of the Degree Homotopy Invariance Theorem 38.3. We turn to the proof of the "$\Rightarrow$"-direction. We set $k:=|\deg(f)|=|\deg(g)|$.

First we point out that by Lemma 38.20 we know that there exist smooth embeddings $\Phi_1,\dots,\Phi_k\colon\overline{B}^n\to M$ with disjoint images such that $f\colon M\to\overline{B}^n/S^{n-1}$ is homotopic to $\Theta(\Phi_1,\dots,\Phi_k)\colon M\to\overline{B}^n/S^{n-1}$. The same way we also see that there exist smooth embeddings $\Psi_1,\dots,\Psi_k\colon\overline{B}^n\to M$ with disjoint images such that $g\colon M\to\overline{B}^n/S^{n-1}$ is homotopic to $\Theta(\Psi_1,\dots,\Psi_k)\colon M\to\overline{B}^n/S^{n-1}$.

We now assume that $\deg(f)=\deg(g)\geq 0$. Note that in this case it follows immediately from $k=\deg(f)=\deg(g)$ and Lemma 38.18 (1) that all of the smooth embeddings $\Phi_1,\dots,\Phi_k\colon\overline{B}^n\to M$ and $\Psi_1,\dots,\Psi_k\colon\overline{B}^n\to M$ are orientation-preserving. Thus it follows from the Smooth Ball Embedding Theorem 37.8 that there exists a smooth isotopy between $\Phi_1\sqcup\dots\sqcup\Phi_k\colon\overline{B}^n\sqcup\dots\sqcup\overline{B}^n\to M$ and $\Psi_1\sqcup\dots\sqcup\Psi_k\colon\overline{B}^n\sqcup\dots\sqcup\overline{B}^n\to M$. By Lemma 38.18 (2) this implies that $\Theta(\Phi_1,\dots,\Phi_k)\colon M\to\overline{B}^n/S^{n-1}$ and $\Theta(\Psi_1,\dots,\Psi_k)\colon M\to\overline{B}^n/S^{n-1}$ are homotopic.

Finally we assume that $\deg(f)=\deg(g)<0$. In this case it follows immediately from $k=-\deg(f)=-\deg(g)$ and Lemma 38.18 (1) that all of the smooth embeddings $\Phi_1,\dots,\Phi_k\colon\overline{B}^n\to M$ and $\Psi_1,\dots,\Psi_k\colon\overline{B}^n\to M$ are orientation-reversing. The remainder of the argument is identical to the above. ■

Proof of the Non-Orientable Hopf Degree Theorem 38.17. The proof is almost entirely the same as the above proof of the Hopf Degree Theorem 38.17. The only real difference is that we have to reformulate Lemma 38.20 in the obvious way and in the proof of Lemma 38.20 we now apply the Smooth Ball Embedding Theorem 37.8 for non-orientable smooth manifolds. Note that in the non-orientable setting we do not have to distinguish between orientation-preserving and orientation-reversing smooth embeddings of balls. We leave it to the reader to fill in the details. ■

Exercises for Chapter 38.

Exercise 38.1. Let M be a connected orientable n-dimensional smooth manifold with *non-empty* boundary. We want to show that every map $f\colon M \to S^n$ is null-homotopic. There are various approaches to proving this result. One way is as follows:

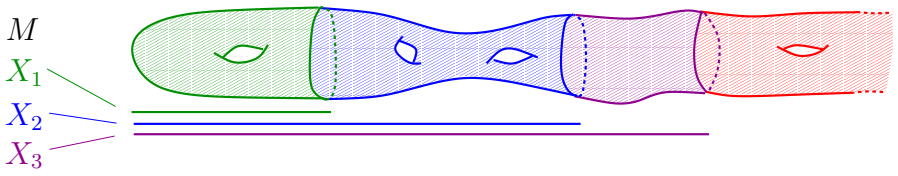
(a) Pick a collar neighborhood $[0,1] \times \partial M$ and show that f is homotopic to a map $g\colon M \to S^n$ which has a regular value $y \in S^n$ such that $f^{-1}(y) \subset [0,1] \times \partial M$.
(b) Use (a) to show that f is homotopic to a map $M \to S^n$ that is not surjective.

Remark. A different approach is sketched in [**OR2009**, p. 193].

Exercise 38.2. Let M be a connected *non-compact* n-dimensional smooth manifold without boundary. We want to show that any map $f\colon M \to S^n$ is null-homotopic. There are various approaches to proving this result. One way is as follows:

(a) By the Compact Submanifold Exhaustion Proposition 31.2 we can pick an exhaustion of M by a sequence of $\{X_k\}_{k\in\mathbb{N}}$ of compact submanifolds.
(b) Use Exercise 38.1 to show that for each $k \in \mathbb{N}$ there exists a homotopy F^k rel $M \setminus X_{k+1}$ such that $F^k_1(X_k) = \{S\}$, where $S = (0,\dots,0,-1)$ is the "South Pole".
(c) Use the homotopies from (b) to define a homotopy $G\colon M \times [0,1] \to M$, where G is defined separately on the subsets $M \times [1 - \frac{1}{2^{k-1}}, 1 - \frac{1}{2^k}]$.

Remark. A different approach is sketched in [**OR2009**, p. 193].



Exercise 38.3. Let $f, g\colon S^n \to S^n$ be two maps. Show that $f \circ g$ and $g \circ f$ are homotopic.
Remark. In Exercise 38.5 we will see that the analogous statement is not true for the torus.

Exercise 38.4. Let $f, g\colon S^1 \to S^1$ be two maps. We consider the map

$$\begin{array}{rccc} f \cdot g\colon & S^1 & \to & S^1 \\ & z & \mapsto & f(z) \cdot g(z). \end{array}$$

Show that $\deg(f \cdot g) = \deg(f) + \deg(g)$.

Exercise 38.5. We consider the following maps

$$\begin{array}{cccc} i_1\colon S^1 \to S^1 \times S^1 & i_2\colon S^1 \to S^1 \times S^1 & p_1\colon S^1 \times S^1 \to S^1 & p_2\colon S^1 \times S^1 \to S^1 \\ z \mapsto (z,1) & z \mapsto (1,z) & (z_1,z_2) \mapsto z_1 & (z_1,z_2) \mapsto z_2. \end{array}$$

(a) Given a map $f\colon S^1 \times S^1 \to S^1 \times S^1$ we define

$$\Theta(f) \;:=\; \begin{pmatrix} \deg(p_1 \circ f \circ i_1) & \deg(p_1 \circ f \circ i_2) \\ \deg(p_2 \circ f \circ i_1) & \deg(p_2 \circ f \circ i_2) \end{pmatrix} \;\in\; \mathrm{M}(2 \times 2, \mathbb{Z}).$$

Show that for any two maps $f, g\colon S^1 \times S^1 \to S^1 \times S^1$ we have $\Theta(f \circ g) = \Theta(f) \cdot \Theta(g)$.
Hint. Use Exercise 38.4 and use that $\mathrm{id} = (p_1 \circ i_1) \cdot (p_2 \circ i_2)\colon S^1 \times S^1 \to S^1 \times S^1$.

(b) Show that there exist maps $f, g\colon S^1 \times S^1 \to S^1 \times S^1$ such that $f \circ g$ and $g \circ f$ are not homotopic.
Remark. This exercise shows that the conclusion of Exercise 38.3 does not hold for the torus.

Exercise 38.6. We defined the degree for a map $f\colon M \to N$ between two closed oriented connected smooth manifolds of the same dimension. We used that M is compact to argue that the degree is actually defined, i.e. given by a finite sum of ± 1. We did not really make use of the hypothesis that N is closed. In fact the same definition makes perfect sense if N is any smooth manifold with empty boundary. Show that if M is closed and N is non-compact, then for any smooth map $f\colon M \to N$ we have $\deg(f) = 0$.

Exercise 38.7. Let $f\colon M \to N$ be a map between two closed connected non-empty n-dimensional smooth manifolds. We do not assume that M or N are orientable. By the Whitney Approximation Theorem 29.1 we know that f is homotopic to a smooth map $g\colon M \to N$. Let $y \in N$ be a regular value of g. We define the $\mathbb{Z}_2$-*degree of* f as follows:

$$\deg_{\mathbb{Z}_2}(f) \;:=\; \#g^{-1}(y) \;\in\; \mathbb{Z}_2.$$

(a) Convince yourself that the obvious analogues of Proposition 38.2, Theorem 38.1, the Degree Homotopy Invariance Theorem 38.3 and the Degree Multiplicativity Theorem 38.4 hold in this context. Put differently, reread the proofs and make mentally the necessary adjustments for the present case.
(b) Does there exist a map f from the Klein bottle to the real projective plane $\mathbb{R}\mathrm{P}^2$ with $\deg_{\mathbb{Z}_2}(f) = 1$?

Exercise 38.8. What is the degree of the map

$$\begin{array}{rcl} f\colon \mathbb{C}\mathrm{P}^n & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \cdots : z_n] & \mapsto & [z_0^2 : \cdots : z_n^2]\,? \end{array}$$

Exercise 38.9.

(1) Let $g\colon S^n \to S^n$ be a smooth map with $g(-x) = g(x)$ for all $x \in S^n$. Show that $\deg(g)$ is even.
(2) Let $f\colon S^n \to S^n$ be a smooth map of odd degree. Show that there exists antipodal points $x, -x$ in S^n which get sent to antipodal points, i.e. $f(x) = -f(-x)$.
Hint. Let $f\colon S^n \to S^n$ be a map such that for each $x \in S^n$ we have $f(x) \neq -f(-x)$. Consider the homotopy

$$\begin{array}{rcl} F\colon S^n \times [0,1] & \to & S^n \\ (x,t) & \mapsto & \dfrac{(1-\frac{1}{2}t)\cdot f(x) + \frac{1}{2}t \cdot f(-x)}{\|(1-\frac{1}{2}t)\cdot f(x) + \frac{1}{2}t \cdot f(-x)\|}. \end{array}$$

Exercise 38.10. Let $n \in \mathbb{N}$ be odd. Recall that by the Real Projective Space Orientability Lemma 25.21 we know that $\mathbb{R}\mathrm{P}^n$ is a closed orientable smooth manifold. Determine the degree of the orientation-preserving map $p\colon S^n \to \mathbb{R}\mathrm{P}^n$ that is given by $x \mapsto [x]$.

Exercise 38.11.

(a) Let $f\colon M \to N$ be a smooth map between two closed oriented connected non-empty n-dimensional smooth manifolds. Suppose that there exists a compact smooth manifold W with $\partial W = M$ and a smooth map $g\colon W \to N$ with $g|_M = f$. Show that

$\deg(f) = 0$.

Hint. Pick up some ideas from the proof of the Degree Homotopy Invariance Theorem 38.3.

(b) Use (a) to give a new proof of the Degree Homotopy Invariance Theorem 38.3.

Exercise 38.12. As in Lemma 2.18 we consider the stereographic projection from the North Pole $N := (0,0,1)$:

$$\begin{array}{rcl} \Phi\colon S^2 & \to & \mathbb{R}^2 \cup \{\infty\} = \mathbb{C} \cup \{\infty\} \\ (x_1,x_2,x_3) & \mapsto & \begin{cases} \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\right), & \text{if } x_3 \neq 1, \\ \infty, & \text{if } x_3 = 1. \end{cases} \end{array}$$

Now let $q(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial with complex coefficients. We consider the map

$$\begin{array}{rcl} \Theta(q)\colon S^2 & \to & S^2 \\ P & \mapsto & \begin{cases} \Phi^{-1}(q(\Phi(P))), & \text{if } P \neq N, \\ N, & \text{if } P = N. \end{cases} \end{array}$$

(For the record we mention that on page 705 we showed that this map is smooth.)

(a) Suppose that $a_n \neq 0$. Show that the maps $\Theta(q)$ and $\Theta(z^n)$ are homotopic.

Remark. This exercise is almost identical to Exercise 14.9.

(b) Show that $\deg(\Theta(z^n)) = n$.

(c) Show that if $n \geq 1$ and $a_n \neq 0$, then $q(z)$ has at least one zero.

Remark. We have just provided another proof of the Fundamental Theorem of Algebra 26.3.

Exercise 38.13. Let $p(z)$ be a complex polynomial.

(a) Let $z_0 \in \mathbb{C}$ be a zero and let $r > 0$ such that z_0 is the only zero of $p(z)$ on $\overline{B}_r(z_0) \subset \mathbb{C}$. We consider the smooth map

$$\begin{array}{rcl} \varphi_{z_0,r}\colon S^1 & \to & S^1 \\ w & \mapsto & \dfrac{p(z_0 + r \cdot w)}{\|p(z_0 + r \cdot w)\|}. \end{array}$$

Show that

$$\deg(\varphi_{z_0,r}) \;=\; \text{order of the zero } z_0 \text{ of the polynomial } p(z).$$

(b) Let $D \subset \mathbb{C}$ be a compact codimension-zero submanifold such that there exists an orientation-preserving diffeomorphism $\gamma\colon S^1 \to \partial D$. We suppose that $p(z)$ does not have a zero on ∂D. Show that

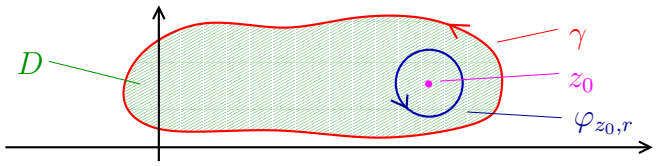
$$\deg\left(\begin{array}{rcl} S^1 & \to & S^1 \\ w & \mapsto & \frac{p(\gamma(w))}{\|p(\gamma(w))\|} \end{array}\right) \;=\; \text{number of zeros of } p(z) \text{ in } D.$$

Hint. Pick disjoint disks around the zeros of $p(z)$ in D.

Remark. This statement is known as the *Argument Principle*.

(c) Use (b) to give one more proof of the Fundamental Theorem of Algebra.

Exercise 38.14. It is easy to come up with a self-map of S^2 with a self-homeomorphism of S^2 with two fixed points. Does there exist a self-homeomorphism of S^2 with precisely one fixed point?



Exercise 38.15. Given a smooth map $\gamma\colon S^1 \to \mathbb{C}\setminus\{0\}$ we define the *winding number*

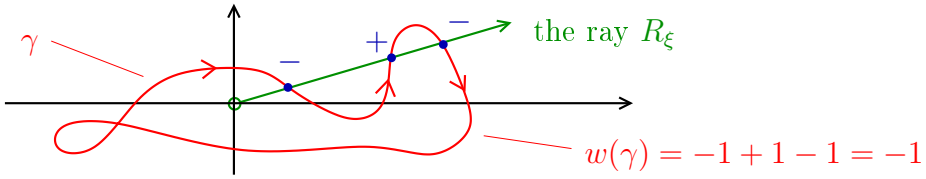
$$\mathrm{w}(\gamma) \;:=\; \deg\begin{pmatrix} S^1 & \to & S^1 \\ z & \mapsto & \frac{\gamma(z)}{\|\gamma(z)\|} \end{pmatrix}.$$

Let $\gamma\colon S^1 \to \mathbb{C}\setminus\{0\}$ be a smooth map. Given $\xi \in S^1$ we consider the corresponding ray $R_\xi := \{r\cdot\xi \,|\, r \in (0,\infty)\}$. We say γ intersects R_z transversally if for every $z \in S^1$ with $\gamma(z) \in R_\xi$ the vectors $\gamma'(z)$ and ξ are linearly independent. Given such a $z \in S^1$ we define

$$\mathrm{sign}(\gamma, z) \;=\; \begin{cases} +1, & \text{if } \xi, \gamma'(z) \text{ is a positive basis of } \mathbb{C} = \mathbb{R}^2, \\ -1, & \text{otherwise.} \end{cases}$$

Show that if γ intersects R_z transversally, then

$$\mathrm{w}(\gamma) \;=\; \sum_{z\in S^1 \text{ with } \gamma(z)\in R_\xi} \mathrm{sign}(\gamma, z).$$



Exercise 38.16. Given a smooth map $\gamma\colon S^1 \to \mathbb{C}\setminus\{0\}$ we denote by $\mathrm{w}(\gamma)$ the winding number that we introduced in Exercise 38.15. Suppose that at a time $t = 0$ a person starts walking around a tree and the person keeps a dog on a leash. At $t = 1$ both the person and the dog are precisely at the starting position. We suppose that at any moment the distance of the person to the tree is longer than the leash. Show that the person and the dog have the same winding number around the tree.
Remark. This statement is colloquially known as the *Dog Walking Theorem*.

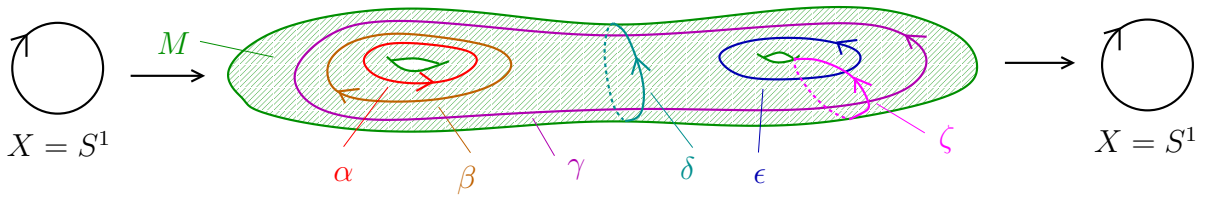
Exercise 38.17. Let $f\colon A \to M$ and $g\colon B \to N$ be smooth maps between closed oriented connected non-empty smooth manifolds with $\dim(A) = \dim(M)$ and $\dim(B) = \dim(N)$. Show that

$$\deg(f\times g\colon A\times B \to M\times N) \;=\; \deg(f\colon A\to M)\cdot\deg(g\colon B\to N).$$

Exercise 38.18.

(a) Let M be a smooth manifold and let X be a closed oriented smooth manifold. Furthermore let $f, g\colon X \to M$ be two smooth maps. Show that if there exists a smooth map $\pi\colon M \to X$ such that $\deg(\pi\circ f) \neq \deg(\pi\circ g)$, then f and g are not homotopic.

(b) Let M be the surface of genus 2. We consider the six maps $S^1 \to M$ shown in the figure below. Show that these six maps are pairwise not homotopic.



Exercise 38.19. In the Diffeotopies-of-$\mathbb{R}^n$-Classification Proposition 34.3 we saw that every orientation-preserving diffeomorphism of $\mathbb{R}^n$ is diffeotopic to the identity. Give examples of orientation-preserving diffeomorphisms of suitable closed connected smooth manifolds that are not diffeotopic to the identity.
Hint. Consider products of spheres.

Exercise 38.20. Let $k \in \mathbb{N}_0$. Show that for every $g \geq k$ there exists a degree-one map from the surface Σ_g of genus g to the surface Σ_k of genus k.

Exercise 38.21. Let $n \in \mathbb{N}_{\geq 2}$. In the Matrix Group-Smooth Manifold Proposition 26.7 we showed that $\mathrm{SU}(n)$ is a closed orientable smooth manifold of dimension $\frac{1}{2}n \cdot (n-1)$. Show that for any $d \in \mathbb{Z}$ we have

$$\deg \begin{pmatrix} \mathrm{SU}(n) & \to & \mathrm{SU}(n) \\ A & \mapsto & A^d \end{pmatrix} = d^{n-1}.$$

Remark. This exercise is a special case of a much more general calculation of the degree of a power map on a Lie group in terms of the dimension of a maximal torus. We refer to [**HoM2020**, Lemma 6.92] for details.

Exercise 38.22. Let $f, g \colon S^n \to S^n$ be two maps. Show that if $\deg(f) \neq \deg(g)$, then there exists a point $x \in S^n$ with $\langle f(x), g(x) \rangle = 0$.
Hint. Consider $\|f(x) \cdot (1-t) + g(x) \cdot t\|$.

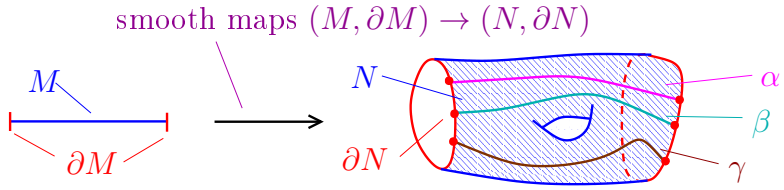
39. Degrees for smooth manifolds with boundary and the Borsuk-Ulam Theorem

In this chapter we will extend some of the definitions and the results from the previous chapter to maps between compact smooth manifolds, possibly with non-empty boundary. This extension is per se not particularly exciting. But we will use it to give our first of many proofs of the celebrated Borsuk-Ulam Theorem 39.8. In the next chapter we will then put the Borsuk-Ulam Theorem 39.8 to good use, namely we will use it to prove four theorems which at first and second glance are far removed from topology.

39.1. **Extension of degrees to smooth manifolds with boundary.** In the following we also intend to study suitable maps between compact oriented smooth manifolds with *non-empty* boundary. At this point it is convenient to introduce the following definitions.

Definition.

(1) A pair of topological spaces is a pair (X, A) where X is a topological space and $A \subset X$.
(2) A map $f\colon (X, A) \to (Y, B)$ between pairs of topological spaces is a map[385] $f\colon X \to Y$ such that $f(A) \subset B$.
(3) We say two maps $f_0, f_1\colon (X, A) \to (Y, B)$ between pairs of topological spaces are homotopic if there exists a homotopy $F\colon X \times [0, 1] \to Y$ with $F_0 = f_0$ and $F_1 = f_1$ and such that for each $t \in [0, 1]$ we have $F_t(A) \subset B$.



$\alpha, \beta\colon (M, \partial M) \to (N, \partial N)$ are homotopic as maps between pairs of topological spaces,
$\alpha, \gamma\colon (M, \partial M) \to (N, \partial N)$ are *not* homotopic as maps of pairs of topological spaces

Now we can formulate the following extension of Theorem 38.1.

Theorem 39.1. Let M and N be two compact oriented connected non-empty smooth manifolds of the same dimension and let $f\colon (M, \partial M) \to (N, \partial N)$ be a smooth map. Let $y \in N \setminus \partial N$ be a regular value. The sum[386]

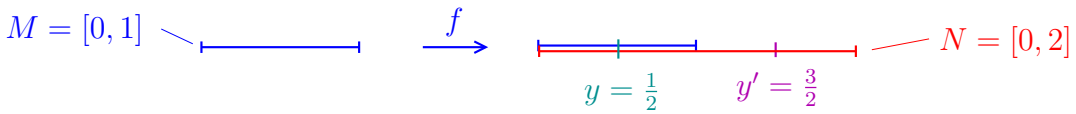
$$\sum_{x \in f^{-1}(y)} \deg(f, x) \ \in \ \mathbb{Z}$$

does not depend on the choice of y.

Proof. We postpone the proof of Theorem 39.1 to the end of the section. ∎

[385] As always, f is understood to be continuous.
[386] It follows from the Regular Value Theorem 26.4 that $f^{-1}(y)$ is a finite set. Furthermore it follows from the hypothesis that $f(\partial M) \subset \partial N$ and $y \in N \setminus \partial N$ that $f^{-1}(y) \subset M \setminus \partial M$. Thus it follows from the Inverse Mapping Theorem 24.1 that for any $x \in f^{-1}(y)$ the map f is a local diffeomorphism at x. Therefore the local degree $\deg(f, x)$ is defined as on page 898.

Non-Example. We consider the inclusion $f\colon [0,1] \to [0,2]$. Both $y = \frac{1}{2}$ and $y' = \frac{3}{2}$ are regular values, but $f^{-1}(y)$ consists of a single point whereas $f^{-1}(\frac{3}{2}) = \varnothing$. This shows that in Theorem 39.1 we cannot drop the hypothesis that $f(\partial M) \subset \partial N$.



Theorem 39.1 leads us to the following generalization of the definition on page 899.

Definition. Let M and N be two compact oriented connected non-empty smooth manifolds of the same dimension and let $f\colon (M, \partial M) \to (N, \partial N)$ be a smooth map. By Sard's Theorem 31.1 there exists a regular value $y \in N \setminus \partial N$. By Theorem 39.1 we can unambiguously define the degree of the smooth map f as

$$\deg(f) \;:=\; \sum_{x \in f^{-1}(y)} \deg(f, x) \;\in\; \mathbb{Z}.$$

Example. Let $n \in \mathbb{Z}$. We view $\overline{B}^2$ as a subspace of $\mathbb{C}$ and we consider the map

$$\begin{array}{rcl} f\colon \overline{B}^2 & \to & \overline{B}^2 \\ z & \mapsto & z^n. \end{array}$$

One can easily show that each $y \in B^2 \setminus \{0\}$ is a regular value. With $y = \frac{1}{2}$ we see that

$$\deg(f) \;=\; \sum_{k=1}^{|n|} \deg\Big(f, \underbrace{\exp\Big(\frac{-\ln(2)}{n} + 2\pi\,\mathrm{i}\cdot\frac{k}{|n|}\Big)}_{n\text{-th root of } \frac{1}{2}}\Big) \;=\; \sum_{k=1}^{|n|} \begin{cases} +1, & \text{if } n \geq 1, \\ -1, & \text{if } n < 1 \end{cases} \;=\; n.$$

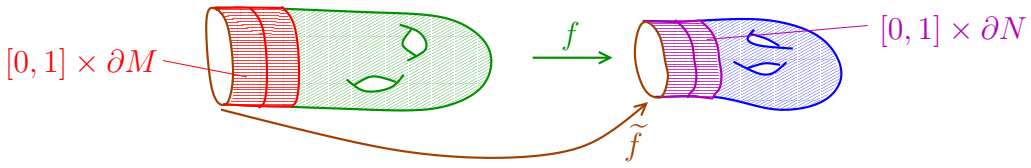
↑ can be deduced using the Orientation Checking-via-Differentials Lemma 25.12

As before we would like to extend the definition of the degree to maps that are continuous but not necessarily smooth. To do so we need the following variation on the Whitney Approximation Theorem 29.1.

Theorem 39.2. (Relative Whitney Approximation Theorem) Let M and N be compact smooth manifolds. Every map $(M, \partial M) \to (N, \partial N)$ is homotopic, as a map of pairs of topological spaces, to a smooth map $(M, \partial M) \to (N, \partial N)$.

Proof. Let $f\colon (M, \partial M) \to (N, \partial N)$ be a map between compact smooth manifolds. By the Collar Neighborhood Theorem 28.3 there exist collars $[0,1] \times \partial M$ and $[0,1] \times \partial N$. We denote by $\widetilde{f}\colon \partial M \to \partial N$ the restriction of f. By the Whitney Approximation Theorem 29.1 we know that there exists a homotopy $\widetilde{F}\colon \partial M \times [0,1] \to \partial N$ with $\widetilde{F}_0 = \widetilde{f}$ and such that $\widetilde{F}_1$ is a smooth map. In fact, by the Trivial-Near-The-Ends-Lemma 33.4, we can assume that $\widetilde{F}_t = \widetilde{F}_1$ for all $t \in [\frac{1}{2}, 1]$. Now we consider the map

$$\begin{array}{rcl} F\colon M \times [0,1] & \to & N \\ (P,t) & \mapsto & \begin{cases} \widetilde{F}(Q, t - 2s), & \text{if } P = (s,Q) \in [0, \frac{1}{2}t] \times \partial M, \\ f((s - \frac{1}{2}t)\cdot\frac{1}{1-\frac{1}{2}t}, Q), & \text{if } P = (s,Q) \in [\frac{1}{2}t, 1] \times \partial M, \\ f(P), & \text{if} P \notin M \setminus ([0,1) \times M) \end{cases} \end{array}$$



It follows from the Pasting Proposition 2.6 together with the Basics-of-Collars Lemma 28.4 (1) that this map $F\colon M \times [0,1] \to N$ is continuous, thus it is indeed a homotopy. Note that for any $t \in [0,1]$ we have $F_t(\partial M) \subset \partial N$. Furthermore note that the restriction of $F_1\colon M \to N$ to $[0,\frac{1}{4}] \times \partial M$ is given by the composition of the projection $[0,\frac{1}{4}] \times \partial M$ to ∂M followed by the smooth map $\widetilde{F}_1$. Thus we see that the restriction of $F_1\colon M \to N$ to $[0,\frac{1}{4}] \times \partial M$ is smooth. In summary we have shown that the map $f\colon (M,\partial M) \to (N,\partial N)$ is homotopic to a map $F_1\colon (M,\partial M) \to (N,\partial N)$ that is smooth on $[0,\frac{1}{4}] \times \partial M$.

Now we can apply the Whitney Approximation Theorem 29.1 to the map $F_1\colon M \to N$ and the closed subset $A := [0,\frac{1}{4}] \times \partial M$. We see that there exists a homotopy rel A from F_1 to a smooth map $M \to N$. In summary we have now finally shown that the map $f\colon (M,\partial M) \to (N,\partial N)$ is indeed homotopic, as a map of pairs of topological spaces, to a smooth map $(M,\partial M) \to (N,\partial N)$. ■

To extend the definition of the degree to non-smooth maps we will also need the following analogue of Proposition 38.2.

Proposition 39.3. Let $g\colon M \to N$ be a map between two compact oriented connected non-empty smooth manifolds of the same dimension. If $f_0, f_1\colon (M,\partial M) \to (N,\partial N)$ are smooth maps that are both homotopic to g, then $\deg(f_0) = \deg(f_1)$.

Proof. We postpone the proof of Proposition 39.3 to the end of this section. ■

The Relative Whitney Approximation Theorem 39.2 and Proposition 38.2 allow us to make the following definition.

Definition. Let $f\colon (M,\partial M) \to (N,\partial N)$ be a map between two compact oriented connected non-empty smooth manifolds of the same dimension. By the Relative Whitney Approximation Theorem 39.2 we know that f is homotopic, as a map of pairs of topological spaces, to a smooth map $g\colon (M,\partial M) \to (N,\partial N)$. By Proposition 39.3 it makes sense to define

$$\deg(f) \;:=\; \deg(g).$$

Remark. On page 2222 we will give an alternative definition of the degree using the "fundamental class". Fortunately we will see in the Local-Global Degree Proposition 101.6 that the two definitions of the degree agree. Furthermore on page 2379 we will extend the notion of a degree to topological manifolds.

The following two theorems are the obvious extensions of the Degree Homotopy Invariance Theorem 38.3 and the Degree Multiplicativity Theorem 38.4.

Theorem 39.4. (Degree Homotopy Invariance Theorem) Let M and N be two compact oriented connected non-empty smooth manifolds of the same dimension and let $f_0, f_1\colon (M,\partial M) \to (N,\partial N)$ be maps. If f_0 and f_1 are homotopic as maps of pairs of topological spaces, then

$$\deg(f_0) \;=\; \deg(f_1).$$

Theorem 39.5. (Degree Multiplicativity Theorem) Let L, M and N be three compact oriented connected non-empty smooth manifolds of the same dimension. Given any maps $f\colon (L, \partial L) \to (M, \partial M)$ and $g\colon (M, \partial M) \to (N, \partial N)$ we have the following equality:

$$\deg(g \circ f) \;=\; \deg(g) \cdot \deg(f).$$

Proof of Theorems 39.1, 39.4 and 39.5 and the proof of Proposition 39.3. The proofs of Theorems 39.1, 39.4 and 39.5 and the proof of Proposition 39.3 for compact smooth manifolds with possibly non-empty boundary are, not surprisingly, very very similar to the proofs of Theorems 38.1, 38.3 and 38.4 and of Proposition 38.2 of the corresponding statements for closed smooth manifolds. To generalize the earlier proofs we need to make three observations. So suppose that M and N are compact connected smooth manifolds.

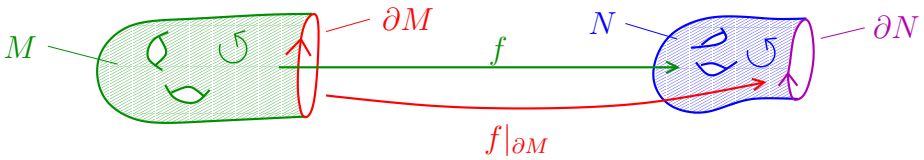
(1) If $y, z \in N \setminus \partial N$, then it follows from the Homogeneous-via-Diffeotopies Proposition 33.14 that there exists a diffeotopy $F\colon N \times [0,1] \to N$ with $F_0 = \mathrm{id}$ and $F_1(y) = z$.
(2) Let $f, g\colon (M, \partial M) \to (N, \partial N)$ be maps. If f and g are homotopic as maps of pairs of topological spaces, then they are also smoothly homotopic as maps of pairs of smooth manifolds. We will prove this slight generalization of the Replace-by-Smooth Homotopy Proposition 29.5 in Exercise 39.1.
(3) Let $f, g\colon (M, \partial M) \to (N, \partial N)$ be smooth maps and let $F\colon M \times [0,1] \to N$ be a smooth homotopy between the maps f and g of pairs of topological spaces. Furthermore let $z \notin \partial N$ be a regular value of $F\colon M \times [0,1] \to N$. Since $f(\partial M \times [0,1]) \subset \partial N$ we see that $F^{-1}(z) \subset (M \setminus \partial M) \times [0,1]$. In fact $Z := F^{-1}(z)$ is a submanifold of $(M \setminus \partial M) \times [0,1]$ such that $\partial Z \subset (M \setminus \partial M) \times \{0,1\}$. Since $M \times [0,1]$ is compact we see that Z is compact. In summary we see that $Z = F^{-1}(z)$ is a proper compact submanifold of $(M \setminus \partial M) \times [0,1]$.

With these three observations it is now straightforward to modify the earlier proofs. We leave it to the reader to fill in the details. ■

39.2. **The Boundary Degree Proposition.** Now we turn to the key result on our new more general notion of degree.

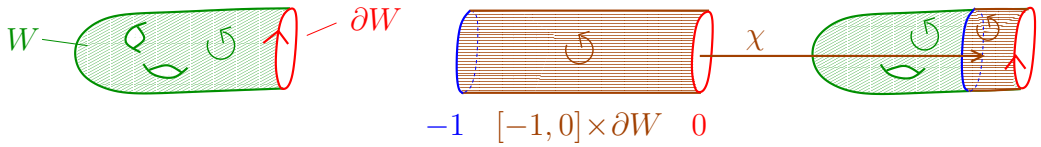
Proposition 39.6. (Boundary Degree Proposition) Let M and N be two compact oriented connected smooth manifolds of the same dimension, such that the boundary ∂M and ∂N are both connected and non-empty. We equip ∂M and ∂N with the boundary orientation that we had introduced on page 689. If $f\colon (M, \partial M) \to (N, \partial N)$ is a map, then

$$\deg(f\colon M \to N) \;=\; \deg(f|_{\partial M}\colon \partial M \to \partial N).$$



For the proof of the Boundary Degree Proposition 39.6 we will need the following slight variation on the Smooth Collar Neighborhood Theorem 28.3.

Lemma 39.7. Let W be a compact oriented smooth manifold. We equip ∂W with the boundary orientation, we equip $[-1,0] \subset \mathbb{R}$ with the standard orientation and we equip $[-1,0] \times \partial W$ with the product orientation. There exists an *orientation-preserving* smooth embedding $\chi\colon [-1,0] \times \partial W \to W$ such that the restriction of χ to $\{0\} \times \partial W$ is the obvious map $(0,P) \mapsto P$.



Proof of Lemma 39.7. It follows immediately from the Smooth Collar Neighborhood Theorem 28.3 that we can find a smooth embedding $\chi\colon [-1,0] \times \partial W \to W$ such that the restriction of χ to $\{0\} \times \partial W$ is the obvious map $(0,P) \mapsto P$. It follows from the Product Orientation Proposition 25.22 (4d) that, as an oriented smooth manifold, we have $\partial([-1,0] \times \partial W) = \partial([-1,0]) \times W = -(\{-1\} \times \partial W) \cup (\{0\} \times \partial W)$. It follows from the Boundary Orientation Lemma 25.17 (3) and (4) that χ is orientation-preserving. ■

Proof of the Boundary Degree Proposition 39.6. Let $f\colon (M,\partial M) \to (N,\partial N)$ be a map. It follows from the Relative Whitney Approximation Theorem 39.2 and the Degree Homotopy Invariance Theorem 39.4 that it suffices to consider the case that f is smooth.

It follows from Lemma 39.7 that we can find *orientation-preserving* smooth embeddings $\Phi\colon [-1,0] \times \partial M \to M$ and $\Psi\colon [-1,0] \times \partial N \to N$ such that for all $P \in \partial M$ we have $\Phi(0,P) = P$ and such that for all $Q \in \partial N$ we have $\Psi(0,Q) = Q$. To simplify the notation we identify $[-1,0] \times \partial M$ with its image under Φ and the same way we proceed with $[-1,0] \times \partial N$ and Ψ. Note that it follows from the Codimension-Zero Smooth Embedding Proposition 24.16 that $(-1,0] \times \partial M \subset M$ and $(-1,0] \times \partial N \subset N$ are open subsets.

By the Homotopic-to-Product-Near-Boundary Lemma 28.6 we know that the map $f\colon (M,\partial M) \to (N,\partial N)$ is homotopic to a map $g\colon (M,\partial M) \to (N,\partial N)$ with the following two properties:

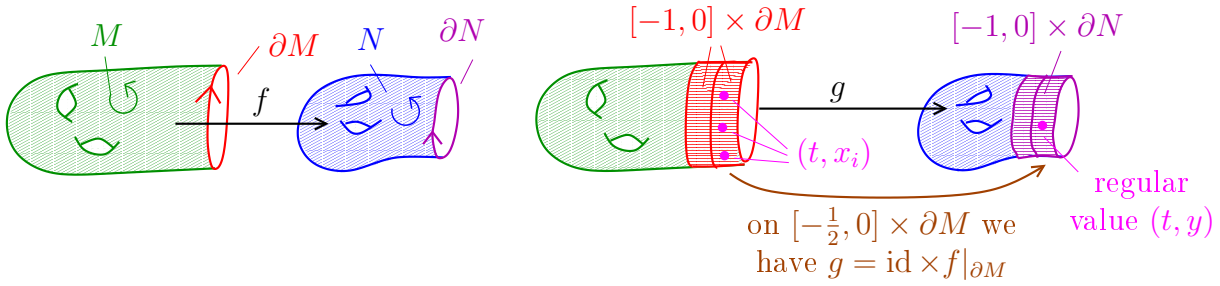
(1) For every $(t,P) \in [-\frac{1}{2},0] \times \partial M$ we have $g(t,P) = (t,f(P)) \in [-\frac{1}{2},0] \times \partial N$.
(2) We have $g^{-1}([-\frac{1}{2},0] \times \partial N) = [-\frac{1}{2},0] \times \partial M$.

Note that a priori we do not know whether $g\colon M \to N$ is smooth. But note that the map $g\colon [-\frac{1}{2},0] \times \partial M \to [-\frac{1}{2},0] \times \partial N$ is smooth. By Sard's Theorem 31.1, together with Proposition 30.3, we know that there exists a regular value $(t,y) \in (-\frac{1}{2},0) \times \partial N$ of $g\colon [-\frac{1}{2},0] \times \partial M \to [-\frac{1}{2},0] \times \partial N$. It follows from (1) and (2) that the preimages of (t,y)

are of the form $(t, x_1), \dots, (t, x_k)$ with $x_1, \dots, x_k \in \partial M$. We see that

$$\begin{aligned}
\deg(f\colon M \to N) &\overset{\substack{\text{by Theorem 38.3, since}\\ f \text{ and } g \text{ are homotopic}\\ \downarrow}}{=} \deg(g\colon M \to N) \overset{\substack{\text{follows from (1) and (2) and the Degree}\\ \text{of Partially Smooth Map Lemma 38.14}\\ \downarrow}}{=} \sum_{i=1}^{k} \deg(g, (t, x_i)) \\
&\underset{\substack{\uparrow\\ \text{by (1) the map } g\colon [-\frac{1}{2},0]\times\partial M \to [-\frac{1}{2},0]\times\partial N \text{ is given by } \mathrm{id}\times f|_{\partial M}, \text{ here we implicitly}\\ \text{use that } [-1,0]\times\partial M \to M \text{ and } [-1,0]\times\partial N \to N \text{ are orientation-preserving}}}{=} \sum_{i=1}^{k} \deg(\mathrm{id} \times f|_{\partial M}\colon [-\tfrac{1}{2}, 0] \times \partial M \to [-\tfrac{1}{2}, 0] \times \partial N, (t, x_i)) \\
&\underset{\substack{\uparrow\\ \text{it follows from the Product Orientation Preserving Lemma 25.23}\\ \text{that the corresponding local degrees are the same}}}{=} \sum_{i=1}^{k} \deg(f|_{\partial M}\colon \partial M \to \partial N, x_i) \underset{\substack{\uparrow\\ \text{by definition of the degree}}}{=} \deg(f|_{\partial M}\colon \partial M \to \partial N).
\end{aligned}$$

■



39.3. **The Borsuk-Ulam Theorem.** We start out with the following simple definition.

Definition. Let $A \subset S^n$ be a subset. We say two points $x, y \in A$ are antipodal if $y = -x$.

The following theorem is arguably the most interesting result one can prove with the methods and tools of this chapter.

Theorem 39.8. (Borsuk-Ulam Theorem) Let $n \in \mathbb{N}_0$. For every map $f\colon S^n \to \mathbb{R}^n$ there exists a pair of antipodal points x and $-x$ on S^n with $f(x) = f(-x)$.

Example. If we think of the surface of earth as S^2, then it follows from the Borsuk-Ulam 39.8 that there exists a pair of antipodal points where the temperature and air pressure agree.

Remark.

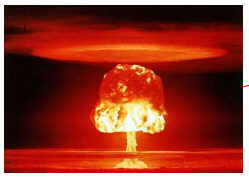
(1) The statement of the Borsuk-Ulam Theorem was conjectured by Stanislaw Ulam, see [**Bor1933**, p. 178]. The theorem was first proved by Karol Borsuk [**Bor1933**] in 1933.
   (a) Karol Borsuk (1905-1982) was a Polish mathematician. He lost his job at the University of Warsaw after the German invasion in 1939. To survive financially he developed a board game and produced and sold it during the war. Almost all copies of the game were lost during the Warsaw uprising in 1944, but in the

1990s one surviving copy was found and the game was republished. We refer to

`https://en.wikipedia.org/wiki/Animal_Husbandry_(game)`

for more details.

(b) Stanislaw Ulam (1909-1984) was a Polish mathematician and nuclear scientist. He was one of the fathers of the hydrogen bomb.



Ulam's other contribution to mankind

(2) In Sections 91.3, 97.6 and 121.3 we will provide three other proofs of the Borsuk-Ulam Theorem 39.8. Furthermore in Theorem 48.21 we will provide a different proof for the special case $n = 2$.

(3) The Borsuk-Ulam Theorem 39.8 has many beautiful applications, in particular the Lusternik-Schnirelmann Theorem 40.1, the Ham-Sandwich Theorem 40.2, Pfister's Level Theorem 40.9 and, my favorite, the Necklace Theorem 40.11. Beyond these theorems the Borsuk-Ulam Theorem has an almost endless string of consequences, many of which lie in combinatorics and graph theory, we refer to [**Mato2008, Steinl1985, Steinl1993**] for an extensive discussion of the Borsuk-Ulam theorem and its corollaries.

The key ingredient in our proof of the Borsuk-Ulam Theorem 39.8 is the Smooth Odd Degree Proposition 39.9. To formulate this proposition we need the following straightforward definition.

Definition. A map $f\colon S^n \to S^n$ is called odd, if $f(-x) = -f(x)$ for all $x \in S^n$.

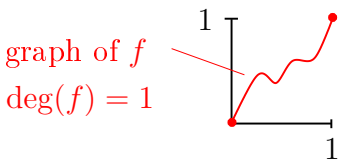
Proposition 39.9. (Smooth Odd Degree Proposition) Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a smooth map. If f is odd, then $\deg(f)$ is odd.

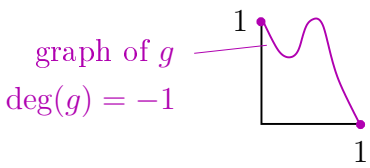
Remark. In the Odd Degree Proposition 91.3 we will use homology theory to provide an alternative proof of the Smooth Odd Degree Proposition 39.9. In fact at that point we will only need that f is continuous.

Eventually we will prove the Smooth Odd Degree Proposition 39.9 by induction on n. For the starting point we will need the following elementary lemma.

Auxiliary Lemma 39.10. Let M and N be two compact oriented connected 1-dimensional smooth manifolds of the same dimension with non-empty boundary. Furthermore let $f\colon (M, \partial M) \to (N, \partial N)$ be a smooth map. If $f\colon \partial M \to \partial N$ is a diffeomorphism, then

$$\deg(f\colon M \to N) \;=\; \begin{cases} +1, & \text{if } f\colon \partial M \to \partial N \text{ is orientation-preserving,} \\ -1, & \text{if } f\colon \partial M \to \partial N \text{ is orientation-reversing.} \end{cases}$$





Proof of Lemma 39.10. By the Smooth 1-Dimensional Manifold Classification Theorem 22.7 we can assume that $M = N = [0,1]$. Note that

$$\begin{array}{rcl} F\colon [0,1]\times[0,1] & \to & [0,1] \\ (x,t) & \mapsto & f(x)\cdot(1-t) + ((1-x)\cdot f(0) + x\cdot f(1))\cdot t \end{array}$$

is a homotopy of maps $([0,1],\{0,1\}) \to ([0,1],\{0,1\})$. If $f(0) = 0$ and $f(1) = 1$, then F is a homotopy from f to the map $t \mapsto t$ and $f(0) = 0$ and $f(1) = 1$, then F is a homotopy from f to the map $t \mapsto 1-t$. Either way it follows from the Degree Homotopy Invariance Theorem 38.3 that $\deg(f) \pm 1$. ∎

Proof of the Smooth Odd Degree Proposition 39.9. We want to prove the proposition by induction on n. In the following we will prove the case $n = 1$ and the induction step $n-1 \rightsquigarrow n$ by almost the same argument. Only at the very end will the two arguments diverge slightly.

In the following let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a smooth map that is odd. We introduce the following notation which we will use throughout this proof:

- We denote by $N := (0,\dots,0,1)$ the North Pole of S^n.
- As on page 152 we denote by $S^n_{\geq 0}$ the upper hemisphere and we denote by $S^n_{=0}$ the equator of S^n. Given a map $g\colon S^n \to M$ we denote by $g_{\geq 0}$ and $g_{=0}$ the obvious restrictions.
- We consider the smooth embedding

$$\begin{array}{rcl} \Phi\colon (-1,1)\times S^{n-1} & \to & S^n \\ (t,x) & \mapsto & \dfrac{(0,x)+t\cdot N}{\|(0,x)+t\cdot N\|}. \end{array}$$

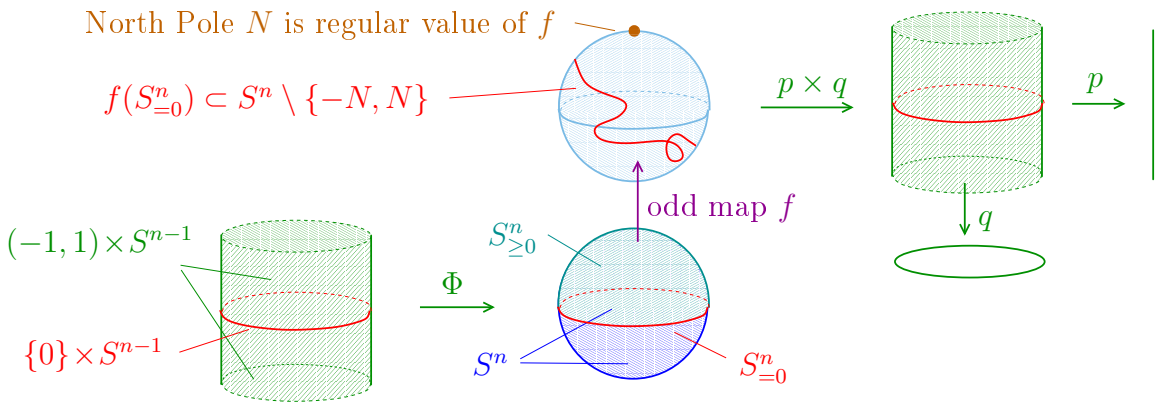
- We consider the maps

$$\begin{array}{rcl} p\colon S^n & \to & [-1,1] \\ (x_1,\dots,x_n,x_{n+1}) & \mapsto & x_{n+1} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} q\colon S^n\setminus\{\pm N\} & \to & S^{n-1} \\ (x_1,\dots,x_n,x_{n+1}) & \mapsto & \dfrac{(x_1,\dots,x_n)}{\|(x_1,\dots,x_n)\|}. \end{array}$$

  Evidently $(p\circ q)\circ\Phi = \mathrm{id}_{(-1,1)\times S^{n-1}}$.

It follows from Sard's Theorem 31.1, together with the Measure-Zero Properties Proposition 30.2, that there exists a $v \in S^n$ which is a regular value of the smooth map $f\colon S^n \to S^n$ and also of the smooth map $f_{=0}\colon S^n_{=0} \to S^n$.

Note that by the Orthogonal Action Lemma 5.26 there exists an orthogonal matrix $A \in \mathrm{O}(n+1)$ with $A\cdot v = N$. Next note that $x \mapsto A\cdot f(x)$ is still odd and note that by the Degree Multiplicativity Theorem 38.4 and the discussion on page 905 we know that $\deg(x \mapsto A\cdot f(x)) = \deg(f)$. This shows that, after composing f with the map that is given by multiplication by A, we might as well assume that $v = N$. Note that since f is odd we see that in fact both N and $-N$ are regular values of $f\colon S^n \to S^n$ and of $f_{=0}\colon S^n_{=0} \to S^n$.



Claim 1. The map f is homotopic to an odd smooth map $g\colon S^n \to S^n$ such that $\pm N$ are again regular values of g and such that g satisfies $g(S^n_{=0}) \subset S^n_{=0}$.[387]

Proof. Since $S^n_{n=0} = \Phi(\{0\} \times S^{n-1})$ is an $(n-1)$-dimensional smooth manifold we see that the fact that $\pm N$ is a regular value of $f\colon S^n_{n=0} \to S^n$ implies that $\pm N \notin (f \circ \Phi)(\{0\} \times S^{n-1})$. An elementary compactness argument, see Exercise 5.9, shows that there exists an $\epsilon \in (0,1)$ such that

$$(f \circ \Phi)([-\epsilon, \epsilon] \times S^{n-1}) \ \subset \ S^n \setminus \{\pm N\}.$$

It follows easily from the Smooth Transition Function Lemma 19.4 that can pick a smooth map $\eta\colon [-\epsilon, \epsilon] \to [0,1]$ such that $\eta(x) = 1$ for $|x| \geq \frac{\epsilon}{2}$, $\eta(0) = 0$ and which is even in the sense that $\eta(-x) = \eta(x)$. Next we consider the map

$$\begin{array}{rcl} F\colon S^n \times [0,1] & \to & S^n \\ (z,s) & \mapsto & \begin{cases} f(z), & \text{if } |p(z)| \geq \epsilon, \\ \Phi(\underbrace{p(f(z)) \cdot (1 - s + s \cdot \eta(p(z)))}_{\in [-\epsilon, \epsilon]}, q(f(z))), & \text{if } |p(z)| \leq \epsilon. \end{cases} \end{array}$$

Note that $F_0 = f$. Since η is smooth we see that each F_s is smooth. Furthermore, since η is symmetric one can easily verify that each F_s is odd. Furthermore note that it follows from $\eta(0) = 0$ that $F_1(S^n_{=0}) \subset S^n_{=0}$. We set $g := F_1$. It remains to show that $\pm N$ are also regular values for g. This can be seen as follows. First recall that $\pm N$ are regular values of f. Next note that by construction of g there exists an open neighborhood U of $\{-N, N\} \subset S^n$, namely $U := S^n \setminus F([-\epsilon, \epsilon] \times S^{n-1} \times [0,1])$, such that $f^{-1}(U) = g^{-1}(U)$ and such that f and g agree on this subset. But this implies that $\pm N$ are also regular values for g. ⊞

Before we can state the next claim we need to add one more piece of notation to the above list.

- We denote by $\pi\colon S^n \to \overline{B}^n$ the projection onto the first n coordinates.

Claim 2. We equip $S^n_{\geq 0}$ with the submanifold orientation coming from S^n and we equip $\overline{B}^n$ with the standard orientation. With these orientations we have the following equality:

$$\deg(g\colon S^n \to S^n) \ \equiv \ \deg(\ \underbrace{\pi \circ g_{\geq 0}\colon S^n_{\geq 0} \to \overline{B}^n}_{\text{defined since } g_{=0}(S^n_{=0}) \subset S^n_{=0}}\ ) \mod 2.$$

[387] Note that we do not claim that one can arrange that $g(S^n_{\geq 0}) \subset S^n_{\geq 0}$.

Proof. We calculate that in $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, or $\mathbb{R}/2\mathbb{Z}$, we have following equalities:

$$\begin{array}{rcl} \deg(g\colon S^n \to S^n) & \overset{\text{since } \pm N \text{ are both regular values of } g \text{ and since mod } \mathbb{Z} \text{ we ignore signs}}{\downarrow\equiv} & \tfrac{1}{2}\cdot\#g^{-1}(N) + \tfrac{1}{2}\cdot\#g^{-1}(-N) \\ & = & \#(\pi\circ g_{\geq 0})^{-1}(0) \\ & \underset{\text{since } \pi^{-1}(0)=\{\pm N\}}{\uparrow=} & \end{array} \qquad \begin{array}{rcl} & \overset{\text{since } g \text{ is odd and since } g(S^n_{=0}) \subset S^n \setminus \{\pm N\}}{\downarrow=} & \#g_{\geq 0}^{-1}(N) + \#g_{\geq 0}^{-1}(-N) \\ & \equiv & \deg(\pi\circ g\colon S^n_{\geq 0} \to \overline{B}^n). \\ & \underset{\text{since } \mathrm{D}\pi_N \text{ is an isomorphism we see that } 0 \text{ is a regular value of } \pi\circ g_{\geq 0}}{\uparrow} & \end{array}$$

⊞

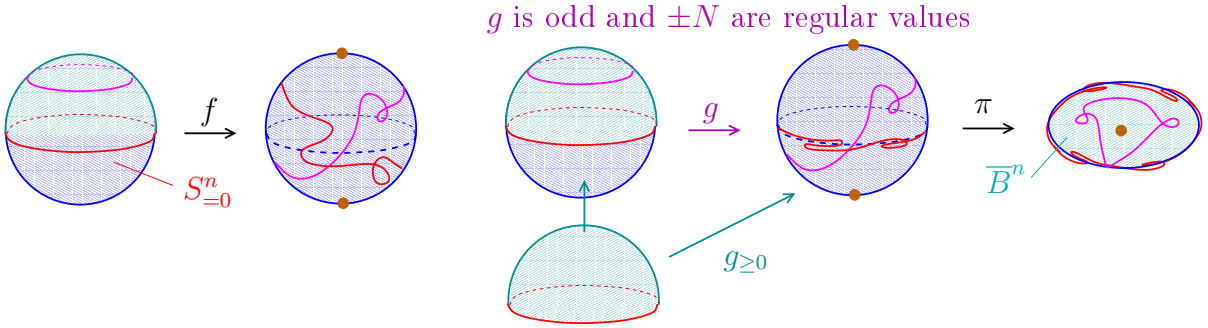
At this point we have to distinguish the cases $n = 1$ and $n \geq 2$. First we deal with the case $n = 1$. We perform the following calculation in $\mathbb{Z}_2$:

$$\deg(f\colon S^1 \to S^1) \overset{\text{by the Degree Homotopy Invariance Theorem 38.3 since } f \text{ and } g \text{ are homotopic}}{\downarrow=} \deg(g\colon S^1 \to S^1) \overset{\text{by Claim 2}}{\downarrow=} \deg(\pi\circ g_{\geq 0}\colon S^1_{\geq 0} \to \overline{B}^1) \underset{\text{follows from Lemma 39.10, since the fact that } g_{=0} \text{ is odd implies that } \pi\circ g_{\geq 0}\colon S^1_{=0} \to S^0 \text{ is a diffeomorphism}}{\uparrow=} \pm 1.$$

Finally let us do the induction step. So now we assume that the conclusion holds for some $n - 1 \in \mathbb{N}$ and we want to show that it holds for n. We perform a very similar calculation in $\mathbb{Z}_2$ as above:

$$\begin{array}{rcl} \deg(f\colon S^n \to S^n) & \overset{\text{by the Degree Homotopy Invariance Theorem 38.3 since } f \text{ and } g \text{ are homotopic}}{\downarrow=} & \deg(g\colon S^n \to S^n) \\ & \underset{\text{by the Boundary Degree Proposition 39.6}}{\uparrow=} & \deg(\pi\circ g_{=0}\colon S^n_{=0} \to S^{n-1}) \end{array} \qquad \begin{array}{rcl} & \overset{\text{by Claim 2}}{\downarrow=} & \deg(\pi\circ g_{\geq 0}\colon S^n_{\geq 0} \to \overline{B}^n) \\ & \underset{\text{since } x \mapsto (x,0) \text{ is a diffeomorphism } S^n_{=0} \to S^{n-1}}{\uparrow=} & \deg\left(S^{n-1} \xrightarrow{x \mapsto (\pi\circ g)(x,0)} S^{n-1}\right) \underset{\text{by our hypothesis on } n-1}{\uparrow=} \text{odd}. \end{array}$$

■



Now we can finally provide the proof of the Borsuk-Ulam Theorem 39.8. We split the proof into two parts. First we consider the case that f is smooth, afterwards we consider the general case.

Proof of the Borsuk-Ulam Theorem 39.8 for smooth maps. Let $n \in \mathbb{N}_0$ and let $f\colon S^n \to \mathbb{R}^n$ be a smooth map. We need to show that there exists an $x \in S^n$ with $f(-x) = f(x)$. The case $n = 0$ is trivial so we only need to deal with the case $n \in \mathbb{N}$.

Suppose there exists no pair of antipodal points x and $-x$ in S^n with $f(x) = f(-x)$. Put differently, suppose that for every $x \in S^n$ we have $f(x) \neq f(-x)$. We consider the

composition of the following two maps

$$\begin{array}{rcccl} \Phi\colon S^{n-1}\times[0,1] & \to & S^n_{\geq 0} & \to & S^{n-1}. \\ (y,t) & \mapsto & (y\cdot\sqrt{1-t^2},t) & & \\ & & x & \mapsto & \underbrace{\frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}}_{\text{defined since } f(x)\neq f(-x)} \end{array}$$

A quick look at the definitions shows that Φ_0 is odd and it shows that Φ_1 is a constant map. Thus we see that

$$\begin{array}{ccccccc} & & & & \overset{\text{by the Degree Homotopy Invariance Theorem 38.3, since } \Phi_0 \text{ and } \Phi_1 \text{ are homotopic}}{\downarrow} & & \\ \text{odd number} & = & \deg(\Phi_0) & = & \deg(\Phi_1) & = & \deg(\text{constant map}) & = & 0. \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & \text{since } \Phi_0 \text{ is odd we obtain from the Smooth Odd Degree Proposition 39.9 that } \deg(\Phi_0) \text{ is odd} & & \text{by definition of } \Phi_1 & & \text{by the Basics of Degree Lemma 38.8 (1) since constant maps are not surjective} & & \end{array}$$

We have thus obtained a contradiction. ■

Proof of the Borsuk-Ulam Theorem 39.8 for continuous maps. Let $f\colon S^n \to \mathbb{R}^n$ be a continuous map. Suppose there exists no pair of antipodal points x and $-x$ in S^n with $f(x) = f(-x)$. It follows that the map

$$\begin{array}{rcl} \varphi\colon S^n & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & \|f(x)-f(-x)\| \end{array}$$

is never zero. Since S^n is compact we see that there exists an $\epsilon > 0$ such for all $x \in S^n$ we have $\|f(x)-f(-x)\| \geq \epsilon$.

By the Smooth Approximation Proposition 21.10 there exists a *smooth* map $\widetilde{f}\colon S^n \to \mathbb{R}^n$ such that for every $x \in S^n$ we have $\|f(x)-\widetilde{f}(x)\| < \frac{\epsilon}{2}$. It follows from the triangle inequality that for every $x \in S^n$ we have $\|\widetilde{f}(x)-\widetilde{f}(-x)\| > 0$. But this contradicts the Smooth Borsuk Ulam Theorem that we had just proved above. ■

Exercises for Chapter 39.

Exercise 39.1. Let M and N be compact smooth manifolds. Let $f, g\colon (M,\partial M) \to (N,\partial N)$ be two smooth maps. We suppose that there exists a homotopy $F\colon M\times[0,1] \to N$ of maps of pairs of topological spaces from f to g.

(a) Show that there exists a homotopy $G\colon M\times[0,1] \to N$ of maps of pairs of topological spaces from f to g such that G is smooth on a neighborhood of $\partial M \times [0,1]$.

(b) Show that there exists a smooth homotopy $G\colon M\times[0,1] \to N$ of maps of pairs of topological spaces from f to g.

Hint. For (a) use the Smooth Collar Neighborhood Theorem 28.3 and the Replace-by-Smooth Homotopy Proposition 29.5. For (b) modify the proof of the Replace-by-Smooth Homotopy Proposition 29.5.

Exercise 39.2. Let M and N be compact smooth manifolds. Let $f, g\colon M \to N$ be two smooth maps with $f|_M = g|_M$. We suppose that there exists a homotopy $F\colon M\times[0,1] \to N$ rel ∂M from f to g.

(a) Show that there exists a homotopy $G\colon M \times [0,1] \to N$ rel ∂M from f to g such that G is smooth on a neighborhood of $\partial M \times [0,1]$.
(b) Show that there exists a smooth homotopy $G\colon M \times [0,1] \to N$ rel ∂M from f to g.

Hint. For (a) use the Smooth Collar Neighborhood Theorem 28.3 and the Replace-by-Smooth Homotopy Proposition 29.5. For (b) modify the proof of the Replace-by-Smooth Homotopy Proposition 29.5.

Exercise 39.3. Let $n \in \mathbb{N}$ and let $f\colon \overline{B}^n \to \overline{B}^n$ be a smooth map such that the restriction $f\colon S^{n-1} \to S^{n-1}$ is a diffeomorphism. Show that $f\colon \overline{B}^n \to \overline{B}^n$ is surjective.
Remark. The case $n = 1$ is equivalent to the Intermediate Value Theorem from real analysis.

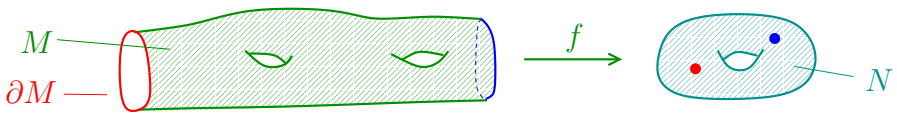
Exercise 39.4.

(a) Generalize the $\mathbb{Z}_2$-degree from page 913 to the setting of maps $(M, \partial M) \to (N, \partial N)$ between compact connected smooth manifolds.
(b) Show that the analogue of Boundary Degree Proposition 39.6 also holds for $\mathbb{Z}_2$-degrees.
(c) Use (b) to give a new proof of the Boundary Non-Retraction Proposition 31.3.

Exercise 39.5. Let M be a compact smooth manifold with non-empty boundary and let N be a closed smooth manifold. Let $f\colon M \to N$ be a smooth map. Given a regular value $y \in N$ we set

$$\deg_y(f) \;:=\; \sum_{x \in f^{-1}(y)} \deg(f,x) \;\in\; \mathbb{Z}.$$

(a) Show that in general the definition of $\deg_y(f)$ depends on the choice of the regular value y.
(b) We suppose that $f\colon M \to N$ is a smooth map which is locally constant on ∂M. Show that in this setting the definition of $\deg_y(f)$ does not depend on the choice of the regular value y.



Exercise 39.6. Let M be a compact connected oriented n-dimensional smooth manifold such that ∂M is connected and non-empty. Show that there exists a map $f\colon M \to \overline{B}^n$ of degree $+1$.

Exercise 39.7. Prove the Borsuk-Ulam Theorem 39.8 for 1-dimensional spheres using methods from real analysis. In other words, show, just using methods from real analysis, that given any map $f\colon S^1 \to \mathbb{R}$ there exists an $x \in S^1$ with $f(-x) = f(x)$.

Exercise 39.8. Let $n \in \mathbb{N}$. Show that the following are equivalent statements:

(a) For every map $f\colon S^n \to \mathbb{R}^n$ there exists an $x \in S^n$ with $f(x) = f(-x)$.
(b) For every odd map $f\colon S^n \to \mathbb{R}^n$ there exists an $x \in S^n$ with $f(x) = 0$.
(c) There is no odd map $f\colon S^n \to S^{n-1}$.
(d) There is no map $f\colon \overline{B}^n \to S^{n-1}$ such that the restriction of f to $S^{n-1} \to S^{n-1}$ is odd.

Exercise 39.9. Let $n \in \mathbb{N}$. Show that the Borsuk-Ulam Theorem 39.8 implies the Brouwer Fixed Point Theorem 31.4.
Remark. From Exercise 39.8 you can pick your favorite formulation of the Borsuk-Ulam Theorem 39.8.

Exercise 39.10. Let $n \in \mathbb{N}$.

(a) Let $f\colon S^n \to S^n$ be a map. Show that if the map f factors through the projection $p\colon S^n \to \mathbb{R}\mathrm{P}^n$, then $\deg(f)$ is even.

(b) Let $f\colon S^n \to S^n$ be a map of odd degree. Show that there exists an $x \in S^n$ with $f(-x) = -f(x)$.
Hint. Prove the statement by contradiction and make use of (a).
Remark. This statement can be viewed as a partial converse to the Smooth Odd Degree Proposition 39.9.

40. Applications of the Borsuk-Ulam Theorem

In this chapter we will state and prove several particularly interesting consequences of the Borsuk-Ulam Theorem 39.8.

40.1. **The Ham-Sandwich Theorem.** Before we can state our first application of the Borsuk-Ulam Theorem 39.8 we need to recall the following harmless definition from page 929.

Definition. Let $A \subset S^n$ be a subset. We say two points $x, y \in A$ are antipodal if $y = -x$.

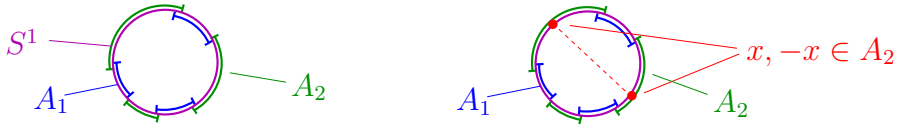
The following application of the Borsuk-Ulam Theorem 39.8 was first proved by the Soviet mathematicians Lazar Lusternik and Lev Schnirelmann [**LuS1930**, p. 26-31] in 1930 using different methods.

Theorem 40.1. (Lusternik-Schnirelmann Theorem) Let $A_1, \dots, A_{n+1}$ be closed subsets of S^n with $A_1 \cup \dots \cup A_{n+1} = S^n$. Then there exists an $i \in \{1, \dots, n+1\}$ such that A_i contains a pair of antipodal points.

Remark. In the Generalized Lusternik-Schnirelmann Theorem 40.7 we will prove a slight generalization of the Lusternik-Schnirelmann Theorem 40.1.

Examples.

(1) For $n = 1$ we obtain from the Lusternik-Schnirelmann Theorem 40.1 that if we have a decomposition $S^1 = A_1 \cup A_2$ where A_1 and A_2 are two closed subsets (which do not need to be connected), then the subset A_1 or the subset A_2 contains a pair of antipodal points x and $-x$. In Exercise 40.1 we will give a naive proof for the special case $n = 1$.



(2) We consider the surface of our earth, we take A_1 to be the closure of the land mass, we take A_2 to be the closure of the Pacific ocean and we take A_3 to be the closure of all seas except for the Pacific. Then the Lusternik-Schnirelmann Theorem 40.1 says that at least one of A_1, A_2 or A_3 contains a pair of antipodal points. Which of A_1, A_2 and A_3 contains a pair of antipodal points?

(3) The Lusternik-Schnirelmann Theorem 40.1 is optimal in the following sense: in Exercise 40.2 we will see that given any $n \in \mathbb{N}_0$ we can find closed subsets $A_1, \dots, A_{n+2}$ that cover all of S^n, but such that none of the A_i contains a pair of antipodal points.

Proof. Let $A_1, \dots, A_{n+1}$ be closed subsets of S^n with $A_1 \cup \dots \cup A_{n+1} = S^n$. Without loss of generality we can assume that all $A_1, \dots, A_{n+1}$ are non-empty.[388] Given a non-empty subset A of S^n and $x \in S^n$ we consider, as on page 283, the distance from x to A:

$$d(x, A) := \inf\{\|x - a\| \mid a \in A\},$$

where $\|x - a\|$ denotes the usual Euclidean distance on $S^n \subset \mathbb{R}^{n+1}$. By the Distance-to-Subspace the Distance-to-Subspace Lemma 6.2 together with Exercise 1.3 we know that the following two statements hold:

[388] Evidently at least one A_i is non-empty. So if one of the A_i is empty we just replace it with one of the non-empty ones.

(a) If A is closed, then $d(x, A) = 0$ if and only if $x \in A$.
(b) The map $x \mapsto d(x, A)$ is continuous.

We consider the map

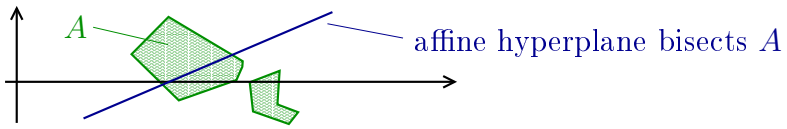
$$\begin{array}{rccc} f\colon & S^n & \to & \mathbb{R}^n \\ & x & \mapsto & (d(x, A_1), \ldots, d(x, A_n)). \end{array}$$

It follows from (b) that this map is indeed continuous. By the Borsuk-Ulam Theorem 39.8 there exists an $x \in S^n$ with $f(x) = f(-x)$. We distinguish three cases:

(1) If there exists an $i \in \{1, \ldots, n\}$ such that $x \in A_i$, then it follows from $f(x) = f(-x)$ that $d(-x, A_i) = d(x, A_i) = 0$. Since A_i is closed it follows from (a) that $-x \in A_i$.
(2) If there exists an $i \in \{1, \ldots, n\}$ such that $-x \in A_i$, then the argument of (1) shows that we also have $x \in A_i$.
(3) If neither x nor $-x$ lies in one of the A_i, then $x, -x$ both lie in A_{n+1}. ■

Before we can state the next consequence of the Borsuk-Ulam Theorem 39.8 we need to introduce one more definition.

Definition. Let $A \subset \mathbb{R}^n$ be a measurable subset[389] of finite volume. We say an affine hyperplane $H \subset \mathbb{R}^n$ bisects A if for the two half-spaces V, V' determined by H we have the equality $\mathrm{vol}(A \cap V) = \mathrm{vol}(A \cap V')$.

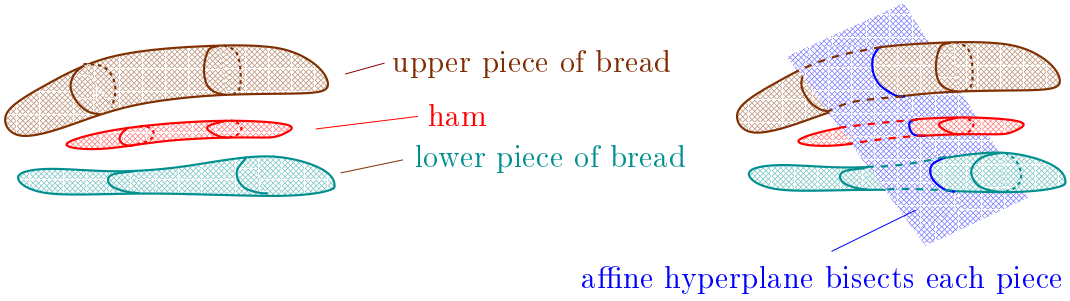


We continue with the following pretty theorem.

Theorem 40.2. (Ham-Sandwich Theorem) Let $A_1, \ldots, A_n$ be measurable finite-volume subsets of $\mathbb{R}^n$. Then there exists an $(n-1)$-dimensional affine hyperplane in $\mathbb{R}^n$ that bisects $A_1, \ldots, A_n$.

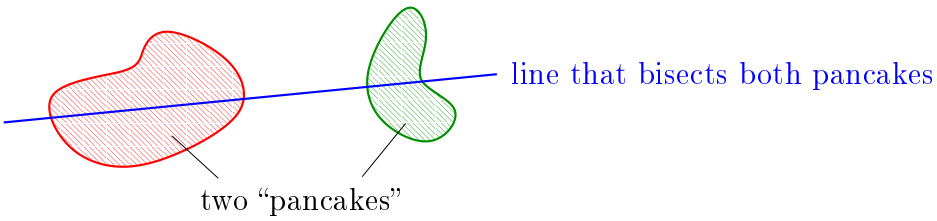
Remark.

(1) The Ham-Sandwich Theorem gets its name from the following "application". Suppose we are given a sandwich with an upper piece of bread A_1, some ham A_2 and a lower piece of bread A_3. Then using a knife one can cut the sandwich, along an affine hyperplane, into two pieces such that each piece of bread and the ham are bisected. This situation is, more or less convincingly, illustrated in the figure below.



[389] Here "measurable" means Lebesgue-measurable in the sense of say [**Caro2000**] or [**RudiW1987**, Chapter 2].

(2) The 2-dimensional version of the Ham-Sandwich Theorem is sometimes referred to as the "Pancake Theorem", since one can bisect the two pancakes along a line into two equal sized halves.



(3) The Ham-Sandwich Theorem is sometimes also called, more prosaically, the Stone-Tukey theorem after the mathematicians Arthur Stone and John Tukey [**StT1942**].

In the proof of the Ham-Sandwich Theorem we will need the following result from measure theory.

Proposition 40.3. Suppose that for each $t \in \mathbb{R}$ we have a measurable subset X_t of $\mathbb{R}^n$ such that for each $s \leq t \in \mathbb{R}$ we have $X_s \subset X_t$. Then $\bigcup_{t\in\mathbb{R}} X_t$ is also measurable and

$$\lim_{t\to\infty} \operatorname{vol}(X_t) \;=\; \operatorname{vol}\Big(\bigcup_{t\in\mathbb{R}} X_t\Big).$$

Proof of Proposition 40.3. The proposition can be deduced easily from basic facts of the Lebesgue measure, see [**Caro2000**, Chapter 16]. ■

Proof of the Ham-Sandwich Theorem 40.2. Let $A_1, \ldots, A_n$ be measurable finite-volume subsets of $\mathbb{R}^n$. If all A_i have volume zero, then there is nothing to prove. Thus, without loss of generality we can assume that the volume of A_n is positive. Given a point $P \in S^{n-1}$ and $t \in \mathbb{R}$ we write

$$\begin{aligned} H_{P,t} &:= \{t \cdot P + w \mid \langle P, w\rangle = 0\} \\ V^+_{P,t} &:= \{t \cdot P + w \mid \langle P, w\rangle \geq 0\} \\ V^-_{P,t} &:= \{t \cdot P + w \mid \langle P, w\rangle \leq 0\}. \end{aligned}$$

(We refer to the figure below on the left for an illustration.) Put differently, $H_{P,t}$ is the affine hyperplane through $t \cdot P$ orthogonal to P, and $V^{\pm}_{P,t}$ are the two half-spaces determined by $H_{P,t}$. We make the following two observations:

(i) by definition we have $V^{\pm}_{-P,-t} = V^{\mp}_{P,t}$,

(ii) for a subset $A \subset \mathbb{R}^n$ of finite volume we have the equality

$$\operatorname{vol}(A \cap V^{\pm}_{P,t}) \underset{\uparrow}{=} \operatorname{vol}(A \cap (\mathbb{R}^n \setminus V^{\mp}_{P,t})) \;=\; \operatorname{vol}(A) - \operatorname{vol}(A \cap V^{\mp}_{P,t}).$$

since $H_{P,t} = V^+_{P,t} \cap V^-_{P,t}$ is a set of measure zero

We continue with the following claim.

Claim. There exists a function

$$t\colon S^{n-1} \;\to\; \mathbb{R}$$

with the following three properties:

(1) for every $P \in S^{n-1}$ we have $\operatorname{vol}(A_n \cap V^+_{P,t(P)}) = \frac{1}{2}\operatorname{vol}(A_n)$, i.e. the affine hyperplane $H_{P,t(P)}$ bisects A_n,
(2) for every $P \in S^{n-1}$ we have $t(-P) = -t(P)$,
(3) the function t is continuous.

Proof. We first fix $P \in S^{n-1}$. We consider the function

$$\begin{array}{rcl} \mathbb{R} & \to & \mathbb{R}_{\geq 0} \\ t & \mapsto & \operatorname{vol}(A_n \cap V^+_{P,t}). \end{array}$$

This function is continuous[390] and we have

$$\lim_{t\to\infty} \operatorname{vol}(A_n \cap V^+_{P,t}) \underset{\substack{\uparrow \\ \text{Proposition 40.3}}}{=} \operatorname{vol}\Big(\bigcup_{t\in\mathbb{R}} (A_n \cap V^+_{P,t})\Big) = \operatorname{vol}(A_n).$$

and

$$\lim_{t\to-\infty} \operatorname{vol}(A_n \cap V^+_{P,t}) \underset{\substack{\uparrow \\ \text{by (ii)}}}{=} \operatorname{vol}(A_n) - \underbrace{\lim_{t\to-\infty} \operatorname{vol}(A_n \cap V^-_{P,t})}_{=\operatorname{vol}(A_n) \text{ by Proposition 40.3}} = 0.$$

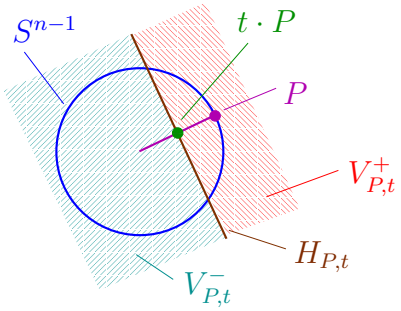
We deduce from the Intermediate Value Theorem that

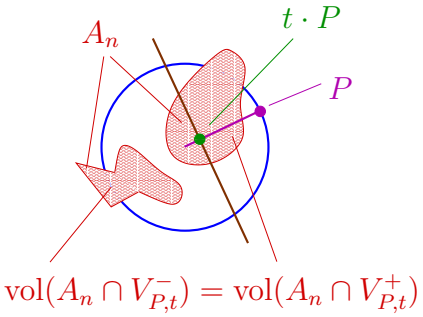
$$T := \big\{t \in \mathbb{R} \,\big|\, \operatorname{vol}(A_n \cap V^+_{P,t}) = \tfrac{1}{2}\operatorname{vol}(A_n)\big\}$$

is a non-empty set.[391] It is straightforward to see that it is a connected subset of $\mathbb{R}$, i.e. it is an interval. Since $\operatorname{vol}(A_n) > 0$ we know that T is a finite interval. We define

$$t(P) := \tfrac{1}{2}(\inf(T) + \sup(T)).$$

Note that it follows immediately from the definitions that $t(-P) = -t(P)$. Furthermore with some elementary measure theory one can show that the function $S^{n-1} \to \mathbb{R}$ given by $P \mapsto t(P)$ is continuous. We leave the verification of the details to the reader. ⊞





Now we consider the map

$$\begin{array}{rcl} f\colon S^{n-1} & \to & \mathbb{R}^{n-1} \\ P & \mapsto & \big(\operatorname{vol}\big(A_1 \cap V^+_{P,t(P)}\big), \ldots, \operatorname{vol}\big(A_{n-1} \cap V^+_{P,t(P)}\big)\big). \end{array}$$

[390] This requires at least a little bit of thought since we do not assume that A is bounded, we only assume it has finite volume. We leave it as a not entirely trivial exercise to prove this statement using basics facts of Lebesgue measures.

[391] One might think that T consists of just a point, but for example if A_n is disconnected, then it can happen that T is in fact an interval that is not just a point.

From the continuity of $t\colon S^{n-1} \to \mathbb{R}$ it follows that f itself is continuous. By the Borsuk-Ulam Theorem 39.8, there exists a point $P \in S^{n-1}$ with $f(P) = f(-P)$. We claim that the affine hyperplane $H_{P,t(P)}$ has the desired property.

By definition of $t(P)$ the affine hyperplane $H_{P,t(P)}$ bisects A_n. Furthermore note that for $i = 1, \ldots, n-1$ we have

$$\operatorname{vol}\left(A_i \cap V^+_{P,t(P)}\right) \underset{\substack{\uparrow \\ \text{since } f(P)=f(-P)}}{=} \operatorname{vol}\left(A_i \cap V^+_{-P,t(-P)}\right) \underset{\substack{\uparrow \\ \text{since } t(-P)=-t(P)}}{=} \operatorname{vol}\left(A_i \cap V^+_{-P,-t(P)}\right) \underset{\substack{\uparrow \\ \text{since } V^+_{P,s}=V^-_{-P,-s}}}{=} \operatorname{vol}\left(A_i \cap V^-_{P,t(P)}\right).$$

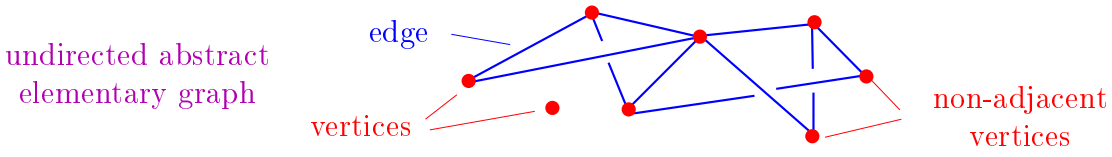
This shows that the affine hyperplane $H_{P,t(P)}$ also bisects $A_1, \ldots, A_{n-1}$. ■

Remark. The proof for the existence of the affine hyperplane that bisects the given sets $A_1, \ldots, A_n$ is non-constructive, in the sense that even with some flexibility in mathematical rigor one cannot turn it into an algorithm that given $A_1, \ldots, A_n$ finds the affine hyperplane that bisects all of the subsets.

40.2. **The chromatic number of Kneser graphs.** In the following we give a definition of an undirected abstract elementary graph.

Definition.

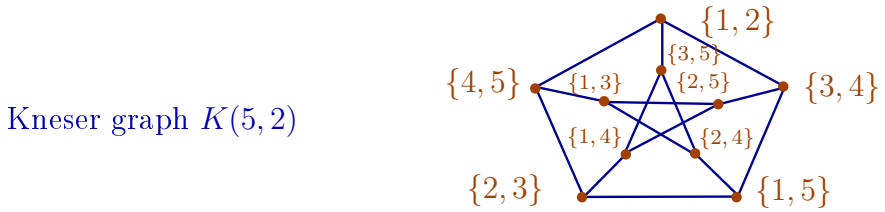
(1) An undirected abstract elementary graph is a pair (V, E) where V is a set, called the vertex set, and a set $E \subset \mathcal{P}(V)$, called the edge set, where each $e \in E$ contains precisely two elements of the vertex set.[392]
(2) Elements of the vertex set are called vertices and elements of the edge set are called edges.
(3) Two vertices v and w are called adjacent if $\{v, w\}$ is an edge.



In this section we only care about the following type of undirected abstract elementary graph.

Definition. Let $n \in \mathbb{N}_0$. The Kneser graph $K(n, k)$ is defined as the undirected abstract elementary graph that is given by the following data:

(1) The vertex set $V(n, k)$ consists of all the k-sets in $\{1, \ldots, n\}$. (A k-set is just a subset with precisely k elements.)
(2) We have an edge between $v, w \in V(n, k)$ precisely if $v \cap w = \varnothing$.



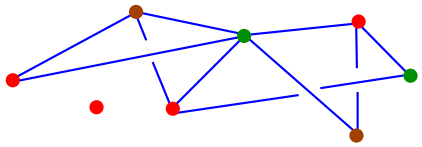
[392] On page 313 we introduced the notion of an undirected abstract graph. The difference is that in an abstract elementary graph we no longer allow loops, i.e. edges where the two endpoints are the same, and we no longer allow multiple edges between two vertices.

Definition. Let $G = (V, E)$ be an undirected abstract elementary graph in the above sense such that V is finite.

(1) Let $c \in \mathbb{N}_0$. A c-vertex coloring of $G = (V, E)$ is a map $\varphi\colon V \to \{1, \dots, c\}$ such that adjacent vertices are assigned different values (colloquially called "colors"). Note that $G = (V, E)$ admits a c-coloring if and only if there exists a decomposition $V = V_1 \sqcup \dots \sqcup V_c$ such that no V_i contains an edge.

(2) The chromatic number of $G = (V, E)$ is the minimal $c \in \mathbb{N}_0$ such that (V, E) admits a c-vertex coloring.

3-coloring



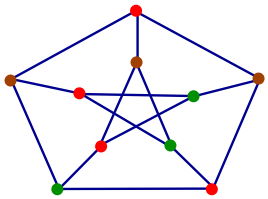
Lemma 40.4. Let $k, d \in \mathbb{N}_0$. The Kneser graph $K(2k + d, k)$ has chromatic number $\chi(K(2k + d, k)) \leq d + 2$.

Proof. As we mentioned above, if we want to show that $\chi(K(n, k)) \leq d+2$ we need to find a decomposition $V(n, k) = V_1 \sqcup \dots \sqcup V_{d+2}$ such that no V_i contains an edge. In other words, we need to find families $V_1, \dots, V_{d+2}$ of disjoint k-sets in $V(n, k)$ with $V(n, k) = V_1 \sqcup \dots \sqcup V_{d+2}$. We do it as follows:

(1) Given $i = 1, \dots, d + 1$ we denote by V_i the k-sets in $\{1, \dots, 2k + d\}$ such that the minimal element is precisely i.

(2) We denote by V_{d+2} all the remaining k-sets. Note that V_{d+2} are precisely the k-sets that are contained in $\{d+2, \dots, 2k+d\}$. Since $\{d+2, \dots, 2k+d\}$ has $2k-1$ elements we see that there are no two disjoint k-sets in $\{d + 2, \dots, 2k + d\}$.

We have thus found the desired decomposition. ■



3-vertex coloring of the Kneser graph $K(5, 2)$:

$V_1 = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}\}$

$V_2 = \{\{2, 3\}, \{2, 4\}, \{2, 5\}\}$

$V_3 = \{\{3, 4\}, \{3, 5\}, \{4, 5\}\}$

In 1955 Martin Kneser [**KnesM1955**] conjectured that the inequality in Lemma 40.4 should actually be an equality:

Conjecture 40.5. (Kneser Conjecture 1955[393]**)** For any choice of $k, d \in \mathbb{N}_0$ the Kneser graph $K(2k + d, k)$ has chromatic number $\chi(K(2k + d, k)) = d + 2$.

This purely combinatorial defeated all attempts to prove it using purely combinatorial methods. The conjecture was finally proved in 1978 by László Lovász[394] [**Lov1978**] using the Borsuk-Ulam Theorem 39.8.

[393] Here the naming gets a bit confusing since in topology there is another Kneser Conjecture, see Theorem **??**, which is named after Hellmuth Kneser, the father of Martin Kneser.

[394] László Lovász (*1948) is a Hungarian mathematician. He was awarded the Abels prize in 2021.

Theorem 40.6. (Lovász Theorem 1978) For any $k, d \in \mathbb{N}_0$ the Kneser graph $K(2k+d, k)$ has chromatic number $\chi(K(2k+d,k)) = d+2$.

The proof of the Kneser Conjecture 40.5 we provide is a simplified version of the proof by László Lovász. This simplified proof was found by Josh Greene [**Greene2002**] in 2002. The proof is so charming that it made it in to "the book of proofs" [**AZ2014**].

The proof of the Lovász Theorem 40.6 rests on the following generalization of the Lusternik-Schnirelmann 40.1.

Theorem 40.7. (Generalized Lusternik-Schnirelmann Theorem) Let $A_1, \ldots, A_{n+1}$ be subsets of S^n with $A_1 \cup \cdots \cup A_{n+1} = S^n$. We assume that each of the first n sets $A_1, \ldots, A_n$ is either an open or a closed subset of S^n. Then there exists an $i \in \{1, \ldots, n+1\}$ such that A_i contains a pair of antipodal points.

Proof of the Generalized Lusternik-Schnirelmann Theorem 40.7. Let $A_1, \ldots, A_{n+1}$ be subsets of S^n with $A_1 \cup \cdots \cup A_{n+1} = S^n$. We assume that each of $A_1, \ldots, A_n$ is closed or open in S^n. Without loss of generality we can assume that all $A_1, \ldots, A_n$ are non-empty. [395] In the following we now assume that none of $A_1, \ldots, A_{n+1}$ contains a pair of antipodal points.

Given a non-empty subset A of S^n and $x \in S^n$ we consider again the distance from x to A:

$$d(x, A) \ := \ \inf\{\|x - a\| \mid a \in A\},$$

where $\| - \|$ denotes the usual Euclidean norm on $S^n \subset \mathbb{R}^{n+1}$. By the Distance-to-Subspace the Distance-to-Subspace Lemma 6.2 together with Exercise 1.3 we know that the following two statements hold:

(a) If A is closed, then $d(x, A) = 0$ if and only if $x \in A$.
(b) The map $x \mapsto d(x, A)$ is continuous.

We consider the map

$$\begin{array}{rccl} f\colon & S^n & \to & \mathbb{R}^n \\ & x & \mapsto & (d(x, A_1), \ldots, d(x, A_n)). \end{array}$$

It follows from (b) that this map is indeed continuous. By the Borsuk-Ulam Theorem 39.8 there exists an $x \in S^n$ with $f(x) = f(-x)$. Since A_{n+1} does not contain a pair of antipodal points we see that x or $-x$ is not contained in A_{n+1}. By symmetry we can assume that x is not contained in A_{n+1}. It follows that there exists an $i \in \{1, \ldots, n\}$ such that x lies in A_i. We now distinguish the cases that A_i is closed or open.

(1) First we assume that A_i is closed. It follows immediately from $f(x) = f(-x)$ that $d(-x, A_i) = d(x, A_i) = 0$. Since A_i is closed it follows from (a) that $-x$ lies in A_i. Thus both x and $-x$ lie in A_i. We have obtained a contradiction.
(2) Now we assume that A_i is open. The proof is given by a sequence of little arguments and observations:
   (i) It follows from $f(-x) = f(x)$ and $x \in A_i$ that $d(-x, A_i) = d(x, A_i) = 0$.
   (ii) It follows from (i) and (a) that $-x$ is contained in the closure $\overline{A_i}$ of A_i.
   (iii) Since A_i is open we see that $S^n \setminus (-A_i)$ is a closed subset.

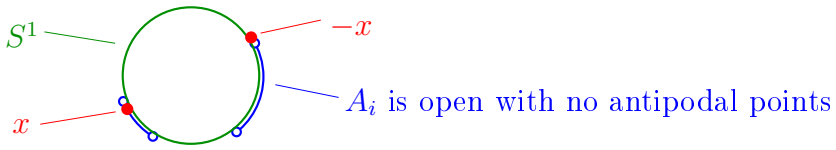
[395] If $A_{n+1} = S^n$ then we are done. Otherwise at least one of the $A_1, \ldots, A_n$ is non-empty, if one of A_i is empty we just replace it with one of the non-empty ones among $A_1, \ldots, A_n$.

(iv) Recall that we assume that A_i has no pair of antipodal points. It follows that $A_i \cap -A_i = \varnothing$, i.e. A_i is contained in $S^n \setminus (-A_i)$.

(v) It follows from (iii) and (iv), together with the definition of the closure, that $\overline{A}_i \subset S^n \setminus (-A_i)$.

(vi) It follows from (ii)-(v) that $-x \in S^n \setminus (-A_i)$.

(vii) It follows from (vi) that $-x \notin -A_i$. This implies that $x \notin A_i$. We have obtained the desired contradiction. ■



We will also need the following elementary lemma.

Lemma 40.8. Given any $d \in \mathbb{N}_0$ and $n \in \mathbb{N}$ there exists a subset $X \subset S^{d+1}$ with n points that is in general position, in the sense that no $d+2$ points of X lie on a hyperplane through the origin.

Proof. [396] Let $d \in \mathbb{N}_0$. We prove the lemma by induction on $n \in \mathbb{N}_0$. The case $n = 0$ is clear. Now suppose that for some $n \in \mathbb{N}_0$ there exists an n-set in S^{d+1} that is in general position. Let V be the union of all hyperplanes in $\mathbb{R}^{d+2}$ that are formed by $(d+1)$-subsets in X. It follows from the Baire Category Theorem 6.16, or alternatively from Propositions 30.2 and 30.3, that $V \subsetneq S^{d+1}$. If we add any point in $S^{d+1} \setminus V$ to X we obtain an $(n+1)$-set in S^{d+1} that is in general position. ■

Proof of the Kneser Conjecture 40.5 aka Lovász Theorem 40.6. Let $k, d \in \mathbb{N}_0$. By Lemma 40.4 it remains to show that $\chi(K(2k+d,k)) \geq d+2$. By Lemma 40.8 there exists a $(2k+d)$-set $X \subset S^{d+1}$ in general position. We now think of $V(2k+d,k)$ as the family of all k-sets in X.

Now let us assume that $\chi(K(2k+d,k)) \leq d+1$. This implies that $K(2k+d,k)$ admits a $(d+1)$-coloring. As we already mentioned in the proof of Lemma 40.4, this means that there exists a decomposition

$$V(2k+d,k) \;=\; \{\text{all } k\text{-sets in } X \;=\; V_1 \sqcup \cdots \sqcup V_{d+1}$$

with the following property: for any V_i any two k-sets $A, B \in V_i$ have non-empty intersection. Our goal is to reach a contradiction. Thus we need to show that there exist disjoint k-sets A and B of X that belong to the same V_i.

For $x \in S^{d+1}$ we consider the corresponding open hemisphere

$$H(x) \;:=\; \{y \in S^{d+1} \mid \underbrace{\langle x, y \rangle}_{\text{scalar product}} > 0\}.$$

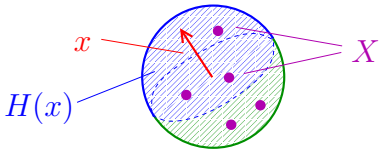
For $i = 1, \ldots, d+1$ we set

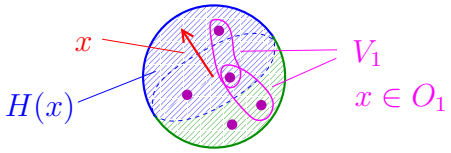
$$O_i \;:=\; \{x \in S^{d+1} \mid H(x) \text{ contains a } k\text{-set from } V_i\}.$$

[396] The proof is quite similar to the proof of Lemma 11.18.

It follows easily from the Neighborhood Openness Criterion 1.5 that each O_i is an open subset of S^{d+1}. Evidently the open subsets $O_1, \ldots, O_{d+1}$ together with the closed subset $C := S^n \setminus (O_1 \cup \cdots \cup O_{d+1})$ form a cover of S^{d+1}.

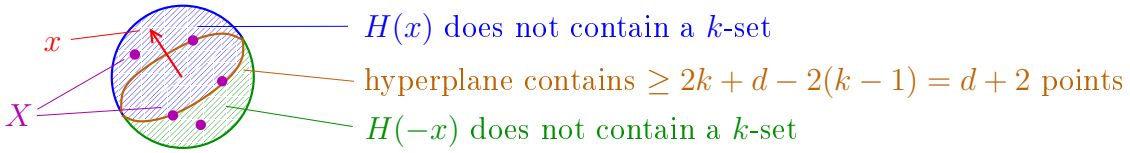
$d = 1$ and $k = 2$





Claim. The subset C does not contain a pair of antipodal points.

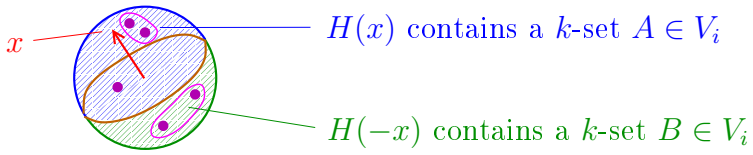
Proof. Suppose that $C = S^n \setminus (O_1 \cup \cdots \cup O_{d+1})$ contains a pair of antipodal points x and $-x$. Note that each k-set of X lies in one of the V_i. It follows from this observation and the definition of the O_i's that both open hemispheres $H(x)$ and $H(-x)$ contain less than k points. But this means that at least $d+2$ points lie on $S^{d+1} \setminus (H(x) \cup H(-x))$. But $S^{d+1} \setminus (H(x) \cup H(-x))$ is the intersection of S^{d+1} with a hyperplane. Thus we have obtained a contradiction to the fact that the points on X are in general position. ⊞



Since the O_i are open and since by the claim we know that $C = S^n \setminus (O_1 \cup \cdots \cup O_{d+1})$ does not pair of antipodal points we deduce from the Generalized Lusternik-Schnirelmann Theorem 40.7 that there exists an $i \in \{1, \ldots, d+1\}$ such that O_i contains a pair of antipodal points x and $-x$. We make the following two simple observations:

(1) By definition of $x \in O_i$ there exists a k-set A from V_i with $A \subset H(x)$.
(2) By definition of $x \in O_i$ there exists a k-set B from V_i with $B \subset H(-x)$.

In summary we see that there exist k-sets A and B, both of which lie in V_i, with $A \subset H(x)$ and $B \subset H(-x)$. Since the open hemispheres $H(x)$ and $H(-x)$ are disjoint we see that A and B are disjoint. ■



40.3. **The level of a commutative ring.** In this section we provide an application of the Borsuk-Ulam Theorem 39.8 to algebra. First we have the following definition.

Definition. Let R be a commutative ring. The level $s(R)$[397] of R is defined as the smallest number $n \in \mathbb{N}_0$ such that -1 can be written as the sum of n squares in R. If -1 is not the sum of squares in R, then we define $s(R) = \infty$.

Examples.

[397] The letter "s" stands for the German word "Stufe", which is the original name of "level".

(1) Evidently $s(\mathbb{C}) = 1$ and $s(\mathbb{R}) = \infty$. In fact a commutative ring R with $s(R) = \infty$ is sometimes called *formally real*.
(2) In Exercise 40.4 we will see that $s(\mathbb{Z}_4) = 3$ and $s(\mathbb{Z}[\sqrt{5}\mathrm{i}]) = 2$.

In 1966 Albrecht Pfister [**Pf1965**] proved the following theorem.

Theorem 40.9. (Pfister's Level Theorem) The level of a field is either ∞ or a power of 2.

Proof. The theorem is of course proved in [**Pf1965**]. There are many accounts of the proof, see e.g. also [**Pf1995**, Theorem 3.1.3] and [**Lam2005**, Theorem XI.2.2]. ■

Albrecht Pfister [**Pf1965**] also showed that any power of 2 appears as the level of a field. In 1977 Manfred Knebusch [**Kneb1977**] asked whether every $n \in \mathbb{N}_0$ is the level of some commutative ring. The following theorem, proved by Dai–Lang–Peng [**DLP1980**] gives an affirmative answer.

Theorem 40.10. Given any $n \in \mathbb{N}_0$ there exists a commutative ring R with $s(R) = n$.

Proof. We consider the ring[398] $R = \mathbb{R}[x_1, \ldots, x_n]/(1 + x_1^2 + \cdots + x_n^2)$. Evidently $s(R) \leq n$. We need to show that $s(R) \geq n$. So suppose that $s(R) < n$. This implies that there exist $f_0(x), f_1(x), \ldots, f_{n-1}(x) \in \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$ such that

$$(1) \qquad -1 \;=\; f_1(x)^2 + \cdots + f_{n-1}(x)^2 + f_0(x) \cdot (1 + x_1^2 + \cdots + x_n^2) \;\in\; \mathbb{R}[x_1, \ldots, x_n].$$

Let $j \in \{0, \ldots, n-1\}$. Since $f_j(x)$ is a real polynomial we can write $f_j(\mathrm{i}x) = p_j(x) + \mathrm{i}q_j(x)$ where $p_j(x)$ is an even real polynomial and $q_j(x)$ is an odd real polynomial. Thus if we plug $\mathrm{i}x$ into Equality (1) and if we compare the real parts we obtain the following equality:

$$(2) \qquad -1 \;=\; \sum_{j=1}^{n-1} (p_j(x)^2 - q_j(x)^2) + p_0(x) \cdot (1 - x_1^2 - \cdots - x_n^2).$$

Next we consider the map $Q := (q_1, \ldots, q_{n-1})\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$. Since the polynomials $q_1, \ldots, q_{n-1}$ are odd we see that for every $a \in S^{n-1}$ we have $Q(-a) = -Q(a)$. Furthermore, by the Borsuk-Ulam Theorem 39.8 there exists an $a = (a_1, \ldots, a_n) \in S^{n-1}$ with $Q(-a) = Q(a)$. Note that it follows from the above symmetry of Q that we have $Q(a) = 0$. Now we see that

$$-1 \;\underset{\text{by Equality (2)}}{\uparrow}\; \sum_{j=1}^{n-1} (p_j(a)^2 - q_j(a)^2) + p_0(a) \cdot \overbrace{(1 - a_1^2 - \cdots - a_n^2)}^{=\,1 \text{ since } a \in S^{n-1}} \;\underset{\text{since } Q(a) = 0 \text{ and } a \in S^{n-1}}{\uparrow}\; \sum_{j=1}^{n-1} p_j(a)^2.$$

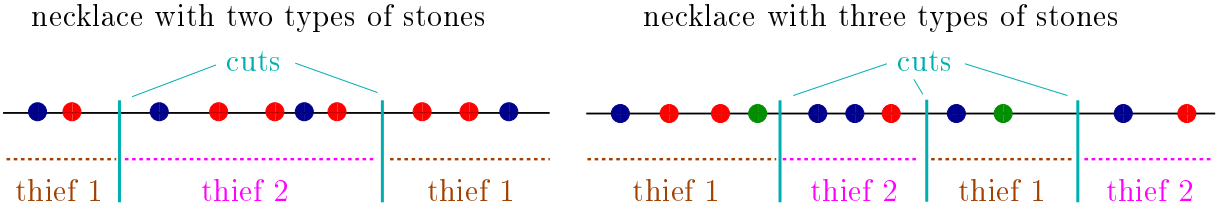
Since the right-hand side is non-negative we have thus obtained a contradiction. ■

Remark.

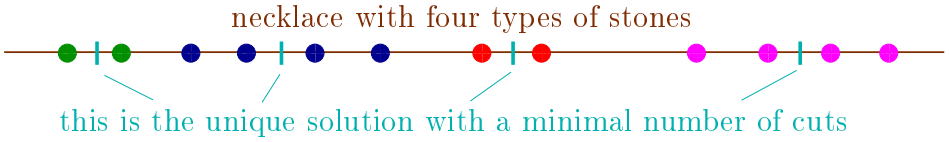
(1) The proof of Theorem 40.10 that we provide is rather ad hoc. One can also introduce a general notion of a level $s(X)$ of a topological space and use it to prove Theorem 40.10. We refer to [**Pf1995**, Chapter 3] for details.
(2) In the proof of Theorem 40.10 we only need to apply the Borsuk-Ulam Theorem 39.8 to maps that are given by polynomials. In this setting a purely algebraic proof of the Borsuk-Ulam Theorem 39.8 was given by Manfred Knebusch [**Kneb1982**].

[398] Note that standard algebra, see e.g. [**Pf1995**, p. 45], shows that R is actually even a domain.

40.4. **The Necklace Theorem.** We conclude this chapter with another, fun consequence of the Borsuk-Ulam Theorem. As in some of the earlier sections this application is again far-removed from what one would usually consider to be the realm of topology.
The setting is the following: Two thieves steal an open necklace with precious stones of k different types. Conveniently enough, of each type of stone there is an even number. The thieves want to cut the string of the open necklace such that they can distribute the resulting pieces in such a way that both thieves obtain the same number of stones of each type. The question is, what is the minimal number of cuts necessary? In the figure below we show two necklaces, one with two types of stones and one with three types of stones. Furthermore we show possible cuts to evenly distribute the stones.



Another open necklace is shown in the figure below. In this particular example the k different types of stones are grouped together. It is pretty clear that one will need at least k cuts to distribute the stones. It turns out that this is the worst case, more precisely, it



turns out that any string of k different types of stones can be distributed evenly using at most k cuts. The following theorem is a formalization of that statement.

Theorem 40.11. (Necklace Theorem) Let $S_1, \dots, S_k$ be disjoint finite subsets of the interval $[0,1)$ such that each subset S_i has an even number of elements. There exists a $d \in \{0, \dots, k\}$ and $0 = z_0 \leq z_1 \leq \cdots \leq z_d \leq z_{d+1} = 1$ such that for each "color" $c \in \{1, \dots, k\}$ we have

$$\underbrace{\sum_{i \text{ even}} \#([z_i, z_{i+1}) \cap S_c)}_{\substack{\text{number of } c\text{-colored pieces} \\ \text{in the even intervals}}} = \underbrace{\sum_{i \text{ odd}} \#([z_i, z_{i+1}) \cap S_c)}_{\substack{\text{number of } c\text{-colored pieces} \\ \text{in the odd intervals}}} .$$

Remark.

(1) The Necklace Theorem 40.11 was first formulated and proved by Charles Goldberg and Douglas West [**GW1985**] in 1985 using topological methods. The proof we give was found shortly afterwards by Noga Alon and Douglas West [**AW1986**].
(2) A topology-free proof of the Necklace Theorem 40.11 was found in 2008 by Frédéric Meunier [**Meu2008**].
(3) The Necklace Theorem 40.11 has been generalized by Noga Alon [**Nog1987**], again using topological methods, to the case of more than two thieves. It seems like there is no known proof for that more general statement, that does not rely on topology.

(4) The proofs of the Necklace Theorem 40.11 and of the Continuous Necklace Theorem 40.12 are both nicely animated in the following video:

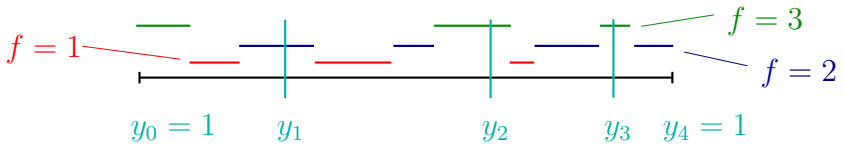
`https://www.youtube.com/watch?v=yuVqxCSsE7c&t=868s`

The Necklace Theorem 40.11 is a priori a discrete problem, far removed from the world of topological spaces. In fact the proof consists of two steps, first we will use the Borsuk-Ulam Theorem 39.8 to prove a "continuous analogue" of the Necklace Theorem 40.11. Afterwards we will show how we can reduce the "discrete Necklace Theorem" from the "continuous Necklace Theorem". Since we will not make use of any of the results of this section we will feel free to use slightly more informal language than usual.
Without further ado, here is the promised continuous analogue of the Necklace Theorem 40.11.

Theorem 40.12. (Continuous Necklace Theorem) Let $f\colon [0,1] \to \{1,\dots,k\}$ be a function[399] such that each preimage $f^{-1}(c)$ is a measurable set (e.g. a union of finitely many intervals). There exists a $d \in \{0,\dots,k\}$ and $0 = y_0 \leq y_1 \leq \dots < y_d \leq y_{d+1} = 1$ such that for each "color" $c \in \{1,\dots,k\}$ we have

$$\underbrace{\sum_{i \text{ even}} \operatorname{vol}\big([y_i, y_{i+1}) \cap f^{-1}(c)\big)}_{\substack{\text{total length of } c\text{-colored} \\ \text{segments in the even intervals}}} = \underbrace{\sum_{i \text{ odd}} \operatorname{vol}\big([y_i, y_{i+1}) \cap f^{-1}(c)\big)}_{\substack{\text{total length of } c\text{-colored} \\ \text{segments in the odd intervals}}}.$$

The same statement also holds for functions f which are defined on $[0,1)$.



Proof. We adopt the following notation: given $(w_0,\dots,w_k) \in S^k$ we set $x_0 = 0$ and for $i = 1,\dots,k$ we write $x_i := w_0^2 + \dots + w_{i-1}^2$. We make the following observation which is the key to the subsequent proof:

> Let $(w_0,\dots,w_k) \in S^k$. We note that $x_0 = 0 \leq x_1 \leq \dots \leq x_k \leq x_{k+1} = 1$. Thus any point in S^k corresponds to a splitting of the interval $[0,1)$ into $k+1$ (possibly empty) intervals. In fact, if we remember the signs of the w_i, then we obtain a way to assign each interval to one of the two thieves, say we award $[x_i, x_{i+1})$ to thief 1 if $w_i < 0$ and to thief 2 if $w_i > 0$. Thus the set of splittings of the interval corresponds to the topological space S^k.

For $c = 1,\dots,k$ we consider the following map:

$$\begin{aligned} g_c\colon S^k &\to \mathbb{R} \\ (w_0,\dots,w_k) &\mapsto \sum_{i \text{ with } w_i > 0} \operatorname{vol}\Big(\big[\underbrace{x_i, x_{i+1}}_{\substack{\uparrow \\ \text{here we use the above definition of } x_0,\dots,x_{k+1} \\ \text{as determined by } w_0,\dots,w_k}}\big) \cap f^{-1}(c)\Big). \end{aligned}$$

[399] Evidently for once the convention that all maps are continuous does not apply.

We leave it to the meticulous reader to show, using standard properties of the Lebesgue-measure, that the map g_c is actually continuous.[400]

Next we combine all these maps and we obtain a map $g = (g_1, \dots, g_k)\colon S^k \to \mathbb{R}^k$. By the Borsuk-Ulam Theorem 39.8 there exists a point $(w_0, \dots, w_k) \in S^k$ such that $g(-w_0, \dots, -w_k) = g(w_0, \dots, w_k)$. This just means that for each $c \in \{1, \dots, k\}$ we have

$$\underbrace{\sum_{i \text{ with } w_i < 0} \operatorname{vol}\big([x_i, x_{i+1}) \cap f^{-1}(c)\big)}_{=g_c(-w_0,\dots,-w_k)} = \underbrace{\sum_{i \text{ with } w_i > 0} \operatorname{vol}\big([x_i, x_{i+1}) \cap f^{-1}(c)\big)}_{=g_c(w_0,\dots,w_k)}.$$

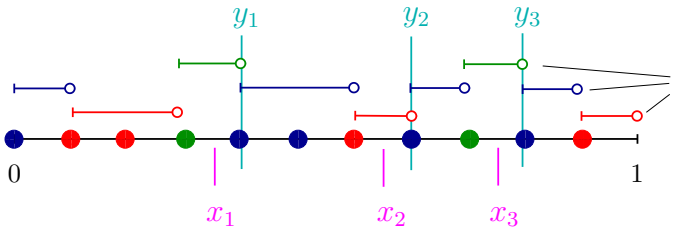
Finally, we consider the numbers $x_0 \le x_1 \le \dots \le x_{k+1} = 1$. Now we just need to combine adjacent intervals for which the w_i's have the same sign. More precisely, iteratively we remove each x_i for which $w_i \ge 0$ and $w_{i+1} \ge 0$ and we remove each x_i for which $w_i \le 0$ and $w_{i+1} \le 0$. It is now fairly elementary to verify that the remaining numbers, let us call them $0 = y_0 < y_1 < \dots < y_d < y_{d+1} = 1$, have the desired properties. ∎

Proof of the Necklace Theorem 40.11. Let $S_1, \dots, S_k$ be disjoint finite subsets of the interval $[0, 1]$ such that each S_i has an even number of elements. Evidently we can assume that $S_1 \cup \dots \cup S_k \neq \varnothing$. We set $s := \#(S_1 \cup \dots \cup S_k)$. It is straightforward to see that without loss of equality we can assume that the points break $[0, 1]$ into intervals of the same lengths $\frac{1}{s}$. More precisely, we can assume that $\bigcup_{i=1}^{k} S_i = \{\frac{i}{s} \mid i = 0, \dots, s-1\}$. Now let $f\colon [0, 1) \to \{1, \dots, k\}$ be the function which is uniquely determined by the property that for each $c \in S_i$ we have $f(x) = i$ for $x \in [c, c + \frac{1}{s})$. By the Continuous Necklace Theorem 40.12 there exists a $d \in \{0, \dots, k\}$ and $0 = y_0 \le y_1 \le \dots \le y_d \le y_{d+1} = 1$ such that for each $c \in \{1, \dots, k\}$ we have

$$(*) \qquad \sum_{i \text{ even}} \operatorname{vol}\big([y_i, y_{i+1}) \cap f^{-1}(c)\big) = \sum_{i \text{ odd}} \operatorname{vol}\big([y_i, y_{i+1}) \cap f^{-1}(c)\big).$$

If each of the y_i lies in the "lattice" $L := \{\frac{i}{s} \mid i = 0, \dots, s\}$, then we can set $z_i := y_i - \frac{1}{2s}$, $i = 1, \dots, d$ and we are done. (This setting is shown in the figure below.) Thus it remains to prove the following claim.

necklace with three types of stones



Claim. We can find $0 = y_0 \le y_1 \le \dots \le y_d \le y_{d+1} = 1$ which satisfy $(*)$ and such that each y_i lies in $L = \{\frac{i}{s} \mid i = 0, \dots, s\}$.

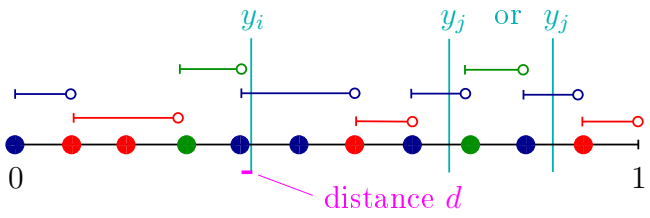
Proof. We prove the claim by induction on the number of y_i that do not lie in the subset $L = \{\frac{i}{s} \mid i = 0, \dots, s\}$. Throughout the proof of the claim we use the following slightly

[400] The fact that we take the sum over all i with $w_i > 0$ might make the reader slightly nervous. But note that for $w_i = 0$ we have $x_{i+1} = x_i$, thus $\operatorname{vol}([x_i, x_{i+1}) \cap f^{-1}(c)) = 0$.

non-standard notation: given $y \in [0,1]$ we define $\lfloor y \rfloor$ and $\lceil y \rceil$ as the result of rounding y down and up to the closest point in L. More precisely, we set $\lfloor y \rfloor := \max\{l \in L \,|\, l \leq y\}$ and we set $\lceil y \rceil := \min\{l \in L \,|\, y \leq l\}$. So suppose there does exist some y_i that does not lie in L. In this case we perform the following three steps:

(1) Among all y_i which do not lie in L we pick a y_i which is closest to a point in L. Without loss of generality we can assume that i is odd.
(2) We define d to be the distance of y_i to some point in L. In other words, if the point on L that is closest to y_i is to the left, then we set $d := y_i - \lfloor y_i \rfloor$. Otherwise we set $d := \lceil y_i \rceil - y_i$.
(3) It follows fairly easily from $(*)$ and the hypothesis that S_c has an even number of elements that there exists a $j \neq i$ such that $f(y_j) = c$ and such that $y_j \notin L$. We distinguish two cases:
   (a) If j is even, then among those we pick the one with a lowest value for $y_j - \lfloor y_j \rfloor$. We replace y_i by $y_i - d \in L$ and we replace y_j by $y_j - d$.
   (b) If j is odd, then among those we pick the one with a lowest value for $\lceil y_j \rceil - y_j$. We replace y_i by $y_i - d \in L$ and we replace y_j by $y_j + d$.

One can easily verify that these new numbers still have all the required properties, except that now y_i also lies in L. By induction we are now done. ■



Exercises for Chapter 40.

Exercise 40.1. Give a proof of the Lusternik-Schnirelmann Theorem 40.1 for $n = 1$ just using methods from real analysis. More precisely, let A, B be two closed subsets of S^1 with $A \cup B = S^1$. Show, just using methods from your real analysis course, that A or B contains a pair of antipodal points.

Exercise 40.2. Let $n \in \mathbb{N}$ and let $P_1, \ldots, P_{n+1}$ be $n+1$ points in $\mathbb{R}^n$ that are in general position, i.e. that are not contained in a hyperplane of $\mathbb{R}^n$. We consider the corresponding "simplex"

$$S(P_1, \ldots, P_{n+1}) \;:=\; \left\{ \sum_{i=1}^{n+1} t_i \cdot P_i \;\middle|\; \sum_{i=1}^{n+1} t_i = 1 \text{ and } t_1, \ldots, t_{n+1} \in \mathbb{R}_{\geq 0} \right\}.$$

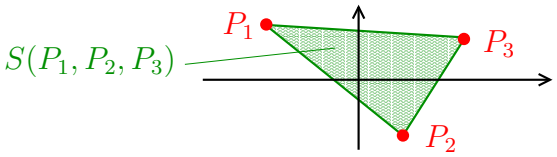
For $i = 1, \ldots, n+1$ we denote by F_i the "i-th face" of $S(P_1, \ldots, P_{n+1})$, i.e. the subspace that is given by setting $t_i = 0$.

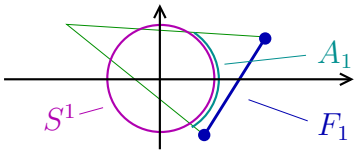
(1) Show that $\underbrace{\partial S(P_1, \ldots, P_{n+1})}_{\text{subspace of } \mathbb{R}^n} = \bigcup_{i=1}^{n+1} F_i$.

(2) Now assume that we have picked $P_1, \dots, P_{n+1}$ such that the origin 0 lies in the interior of $S(P_1, \dots, P_{n+1})$. For $i = 1, \dots, n+1$ we set

$$A_i \;:=\; S^{n-1} \cap \{\lambda \cdot v \,|\, v \in F_i \text{ and } \lambda \in \mathbb{R}_{\geq 0}\}.$$

Show that each A_i is closed, show that $S^{n-1} = A_1 \cup \dots \cup A_{n+1}$ and show that no A_i contains a pair of antipodal points.





Exercise 40.3. Let $k \in \mathbb{N}$ and let $l \in \{1, \dots, k-1\}$. Furthermore let $f\colon S^k \to \mathbb{R}^l$ be a map with $f(x) = -f(x)$ for all $x \in S^k$. Show that there exists an $x \in S^k$ with $f(x) = 0$.

Exercise 40.4. Given a commutative ring R we denote by $s(R)$ the level of R as defined on page 945.

(a) Show that $s(\mathbb{Z}_4) = 3$.
(b) Show that $s(\mathbb{Z}[\sqrt{5}\mathrm{i}]) = 2$.

Exercise 40.5. We consider the string of jewels shown in the figure below. Use four cuts to distribute the jewels evenly.



Exercise 40.6. We consider again the setting of the Continuous Necklace Theorem 40.12. More precisely we are given a map $f\colon [0,1] \to \{1, \dots, k\}$ such that each preimage $f^{-1}(c)$ is a measurable set. For each "color" $c \in \{1, \dots, k\}$ let $y_c \in [0,1]$ be such that we have the equality $\mathrm{vol}\left(f^{-1}(c) \cap [0, y_c]\right) = \mathrm{vol}\left(f^{-1}(c) \cap [y_c, 1]\right)$. Do these points $y_1, \dots, y_k$, after ordering them, necessarily satisfy the conclusion of the Continuous Necklace Theorem 40.12?

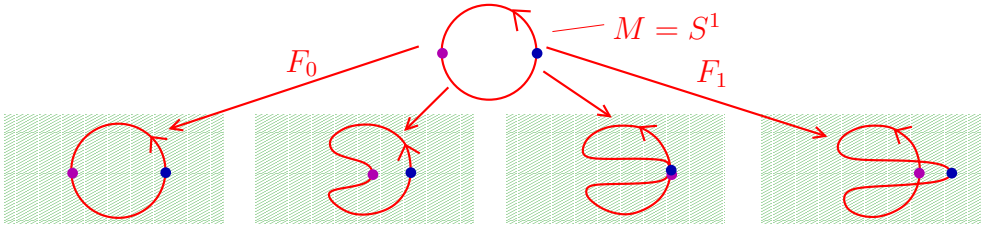
41. Regular homotopies and the Whitney-Graustein Theorem

In this chapter we will take a short detour from explaining the key concepts of differential topology. Namely we will introduce and study the concept of regular homotopies. We will use the notion of the degree of a self-map of S^1 to prove the Whitney-Graustein Theorem 41.7 which gives a complete classification of immersions of S^1 into $\mathbb{R}^2$ up to regular homotopies. We will not make use of the results of this chapter.

41.1. **Regular homotopies.** On page 651 we introduced the notion of an immersion. A smooth homotopy of immersions is called a regular homotopy. More precisely, we have the following definition.

Definition. Let M and N be smooth manifolds. We assume that $\partial N = \varnothing$.

(1) A map $f\colon M \to N$ is called an immersion if f is smooth and if for each $P \in M$ the differential $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}N$ is a monomorphism.
(2) A regular homotopy is a smooth map $F\colon M \times [0,1] \to N$ such that each $F_t\colon M \to N$ is an immersion.
(3) We say that two immersions $f_0, f_1\colon M \to N$ are regularly homotopic if there exists a regular homotopy $F\colon M \times [0,1] \to N$ with $F_0 = f_0$ and $F_1 = f_1$.



regular homotopy $F\colon S^1 \times [0,1] \to \mathbb{R}^2$

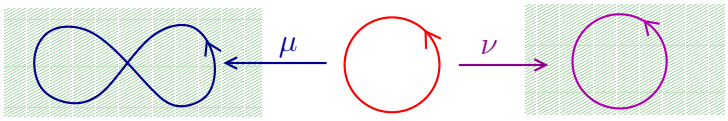
Lemma 41.1. Let M and N be smooth manifolds. We assume that $\partial N = \varnothing$. Being regularly homotopic is an equivalence relation on the set of immersions $M \to N$.

Proof. The proof of the lemma is basically identical to the proof of the Smooth Isotopy Transitivity Proposition 33.3. ■

In the following we restrict ourselves to immersions $S^1 \to \mathbb{R}^n$. First note that any two smooth maps $\gamma, \delta\colon S^1 \to \mathbb{R}^n$ (in particular any two immersions) are smoothly homotopic. For example such a smooth homotopy is given by the map

$$\begin{array}{rcl} F\colon S^1 \times [0,1] & \to & \mathbb{R}^n \\ (z,t) & \mapsto & \gamma(z)\cdot t + \delta(z)\cdot(1-t). \end{array}$$

In Exercise 41.1 we will see that if $n \geq 3$, then any two immersions $S^1 \to \mathbb{R}^n$ are regularly homotopic. This raises the natural question, whether any two immersions $S^1 \to \mathbb{R}^2$ are also regularly homotopic.



are these two immersions regularly homotopic?

In the following we will address this question. Throughout the following discussion we will adopt the following convention and notation.

Convention. We will use the natural isomorphism from the Tangent Space-Isomorphism Proposition 23.16 as an excuse not to distinguish in the notation between the physical tangent spaces and the abstract tangent spaces of S^1 and $\mathbb{R}^n$. Furthermore we will use the Differential-equals-Differential Lemma 23.4 to go back and forth freely between the different notions of differentials.

Definition.

(1) Given a smooth map $f\colon \mathbb{R} \to \mathbb{R}^n$ and $t \in \mathbb{R}$ we write, following the discussion on page 536, $f'(t) := \mathrm{D}\, f_t \cdot 1 \in \mathbb{R}^n$.[401]

(2) Let $r \in \mathbb{R}$. We say that a map $f\colon \mathbb{R} \to X$ is r-periodic if for every $x \in \mathbb{R}$ we have $f(x+r) = r$.

(3) We consider the 2π-periodic map

$$\begin{array}{rccc} \Xi\colon & \mathbb{R} & \to & S^1 \\ & s & \mapsto & \exp(\mathrm{i}s). \end{array}$$

(4) Let $z \in S^1$. We write

$$v_z := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot z \underset{\uparrow}{=} \mathrm{i} \cdot z.$$

under the usual identification $\mathbb{R}^2 = \mathbb{C}$

We make the following two observations:

(a) The vector v_z is the unique vector of length one in $\mathrm{T}_z S^1$ that defines a positive basis of $\mathrm{T}_z S^1$ with respect to the standard orientation of S^1 that we introduced on page 681.

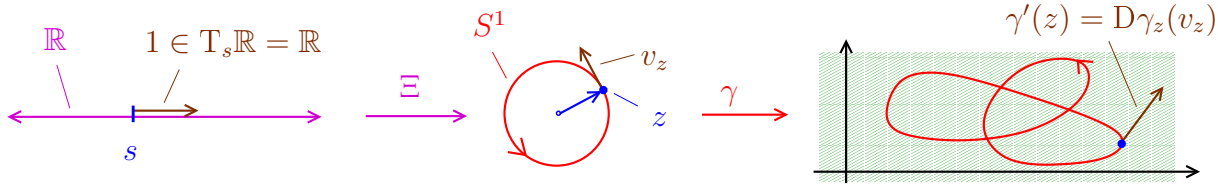
(b) For any $s \in \mathbb{R}$ we have $\mathrm{D}\Xi_s(1) = v_{\Xi(s)}$.

(5) Given a smooth map $\gamma\colon S^1 \to \mathbb{R}^n$ and given $z \in S^1$ we define

$$\gamma'(z) := \mathrm{D}\gamma_z(v_z) \in \mathrm{T}_z S^1,$$

where $\mathrm{D}\gamma_z\colon \mathrm{T}_z S^1_z \to \mathrm{T}_{\gamma(z)}\mathbb{R}^n = \mathbb{R}^n$ denotes the differential of γ at z.

(6) We say that a smooth map $\gamma\colon S^1 \to \mathbb{R}^n$ has unit speed if $\|\gamma'(z)\| = 1$ for all $z \in S^1$.



When studying maps $\gamma\colon S^1 \to X$ it is often convenient to replace such a map by the map $\gamma \circ \Xi\colon \mathbb{R} \to X$, since $\mathbb{R}$ is a more familiar and much easier domain to work with. The following lemma summarizes in our context the relationship between the maps γ and $\gamma \circ \Xi$.

Lemma 41.2.

(1) Let $\gamma\colon S^1 \to \mathbb{R}^n$ be a map.

(a) The map $\gamma \circ \Xi\colon \mathbb{R} \to \mathbb{R}^n$ is 2π-periodic.

[401] Here $1 \in \mathrm{T}_t\mathbb{R} = \mathbb{R}$.

(b) The map $\gamma\colon S^1 \to \mathbb{R}^n$ is smooth if and only if $\gamma \circ \Xi\colon \mathbb{R} \to \mathbb{R}^n$ is smooth.

We now assume that γ is smooth. Then the following statements hold:

(c) For any $t \in \mathbb{R}$ we have $\gamma'(\Xi(t)) = (\gamma \circ \Xi)'(t)$.

(d) The map $\gamma\colon S^1 \to \mathbb{R}^n$ is an immersion if and only if $(\gamma \circ \Xi)'(t) \neq 0$ for all $t \in \mathbb{R}$.

(2) (a) If $\widetilde{\rho}\colon \mathbb{R} \to \mathbb{R}^n$ is a (smooth) 2π-periodic map, then there exists a unique (smooth) map $\rho\colon S^1 \to \mathbb{R}^n$ with $\widetilde{\rho} = \rho \circ \Xi$.

(b) If $\widetilde{F}\colon \mathbb{R} \times [0,1] \to \mathbb{R}^n$ is a smooth homotopy such that each $\widetilde{F}_t$ is 2π-periodic map, then there exists a unique smooth homotopy $F\colon S^1 \times [0,1] \to \mathbb{R}^n$ with $\widetilde{F} = F \circ (\Xi \times \mathrm{id}_{[0,1]})$.

Proof.

(1) (a) This statement is clear.

(b) This statement follows from the Smooth Pasting Proposition 19.23 (3) since Ξ is a local diffeomorphism.

(c) Given $t \in \mathbb{R}$ we have

$$\gamma'(\Xi(t)) \underset{\uparrow}{=} (\mathrm{D}\gamma_{\Xi(t)})(v_{\Xi(t)}) \underset{\uparrow}{=} \overbrace{\mathrm{D}(\gamma \circ \Xi)_t(1)}^{\text{differential}} \underset{\uparrow}{=} \overbrace{\mathrm{D}(\gamma \circ \Xi)_t \cdot 1}^{2 \times 1\text{–matrix}} \underset{\uparrow}{=} (\gamma \circ \Xi)'(t).$$

by definition and since $v_{\Xi(t)} = (\mathrm{D}\Xi_t)(1)$ — Tangent Space-Functor Proposition 23.10 (2b) — Lemma 23.4 — by definition

(d) This statement is an immediate consequence of (1c).

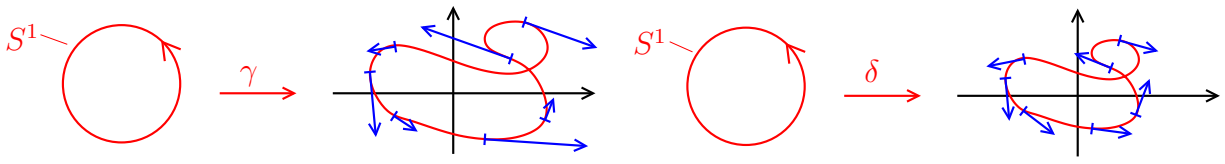
(2) (a) By the Smooth Map-Descends-to-Quotient Corollary 19.33 we know that $\widetilde{\rho}$ descends to a smooth map $\mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^n$. The statement now follows from the fact, which we basically proved in Torus-as-Quotient Lemma 19.34, that the following map is a diffeomorphism:

$$\begin{array}{ccc} \mathbb{R}/2\pi\mathbb{Z} & \to & S^1 \\ [t] & \mapsto & \Xi(t) \end{array}$$

(b) The proof of Statement (b) is basically the same as the proof of Statement (a). ■

We start out our discussions of immersions of S^1 into $\mathbb{R}^n$ with the following proposition.

Proposition 41.3. Every immersion $\gamma\colon S^1 \to \mathbb{R}^n$ is regularly homotopic to an immersion $\delta\colon S^1 \to \mathbb{R}^n$ of unit speed.



Proof. The proposition is proved via a suitable reparametrization and rescaling of the immersion γ. We postpone the full proof of Proposition 41.3, with all the technical details, to Section 41.2. ■

Lemma 41.4. Given any immersion $\gamma\colon S^1 \to \mathbb{R}^2$ the following map is smooth:

$$\begin{array}{ccc} S^1 & \to & S^1 \\ z & \mapsto & \dfrac{\gamma'(z)}{\|\gamma'(z)\|} \end{array}$$

Proof. Given $z \in S^1$ we set $g(z) := \frac{\gamma'(z)}{\|\gamma'(z)\|}$. We need to show that the map $g\colon S^1 \to S^1$ is smooth. As usual let $\Xi\colon \mathbb{R} \to S^1$ be the map that is given by $\Xi(s) = \exp(\mathrm{i}s)$. By Lemma 41.2 (1b) it suffices to show that $g \circ \Xi\colon \mathbb{R} \to S^1$ is smooth. In fact it suffices to show that the map $s \mapsto \gamma'(\Xi(s))$ is smooth. Note that for any $s \in \mathbb{R}$ we have

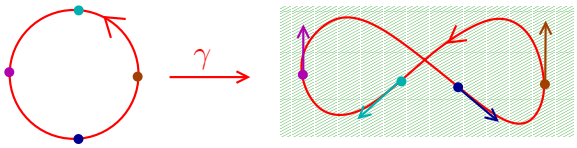
$$\gamma'(\Xi(s)) \overset{\text{Lemma 41.2 (1c)}}{\underset{\downarrow}{=}} (\gamma \circ \Xi)'(s) \overset{\text{by definition}}{\underset{\downarrow}{=}} \underbrace{\mathrm{D}(\gamma \circ \Xi)_s}_{2 \times 1\text{–matrix}} \cdot 1.$$

Since γ and Ξ are smooth we know that $\gamma \circ \Xi\colon \mathbb{R} \to \mathbb{R}^2$ is smooth, which implies, by definition, that the entries of the differential $\mathrm{D}(\gamma \circ \Xi)_s \in \mathrm{M}(2 \times 1, \mathbb{R})$ are smooth in s. This implies the desired statement. ∎

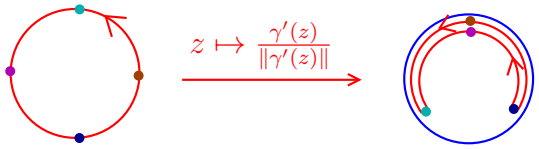
Lemma 41.4 allows us to introduce the following definition.

Definition. Given an immersion $\gamma\colon S^1 \to \mathbb{R}^2$ we define its rotation number as follows:[402][403]

$$\rho(\gamma) := \deg \begin{pmatrix} S^1 & \to & S^1 \\ z & \mapsto & \frac{\gamma'(z)}{\|\gamma'(z)\|} \end{pmatrix}.$$



immersion γ



the corresponding map $z \mapsto \frac{\gamma'(z)}{\|\gamma'(z)\|}$ is null-homotopic, hence $\rho(\gamma) = 0$

Remark. The rotation number of an immersion $\gamma\colon S^1 \to \mathbb{R}^2$ is sometimes also called the *turning number*, see [**BG1988**, Chapter 9.4] or *rotation index*, see [**doC2016**, p. 399] and[**MoR2009**, p. 329].

Lemma 41.5. Let $n \in \mathbb{Z} \setminus \{0\}$. With the usual identification $\mathbb{R}^2 = \mathbb{C}$ we have

$$\rho \begin{pmatrix} S^1 & \to & \mathbb{C} \\ z & \mapsto & z^n \end{pmatrix} = n.$$

Proof. Let $n \in \mathbb{Z} \setminus \{0\}$. We denote by $\theta\colon S^1 \to \mathbb{C}$ the map that is given by $z \mapsto z^n$. Furthermore, very carefully, we denote by $\Theta\colon \mathbb{C} \to \mathbb{C}$ the map that is given by $z \mapsto z^n$.

[402] Since γ is an immersion we know that for every $z \in S^1$ we have $\gamma'(z) \neq 0$. This implies that it actually makes sense to normalize $\gamma'(z)$ to length one.

[403] The map $z \mapsto \frac{\gamma'(z)}{\|\gamma'(z)\|}$ is closely related to the Gauß map that we will introduce on page 1036.

Note that

$$\theta'(z) \overset{\text{definition of } \theta'(z)}{=} \mathrm{D}\theta_z(v_z) \overset{\text{since } \Theta|_{S^1}=\theta}{=} \underbrace{\mathrm{D}\Theta_z}_{\text{Differential}}(v_z) \overset{\text{Differential-equals-Differential Lemma 23.4}}{=} \underbrace{\mathrm{D}\Theta_z}_{2\times 2\text{-matrix}} \cdot v_z$$

$$\overset{\text{by viewing } \Theta \text{ as a holomorphic map, see the Smooth-Complex Differentiability Proposition 32.3, and using the definition of } v_z}{=} \Theta'(z)\cdot \mathrm{i}z = n\cdot z^{n-1}\cdot \mathrm{i}z = n\cdot z^n\cdot \mathrm{i} = n\cdot z^n\cdot \mathrm{i}.$$

Thus we see that

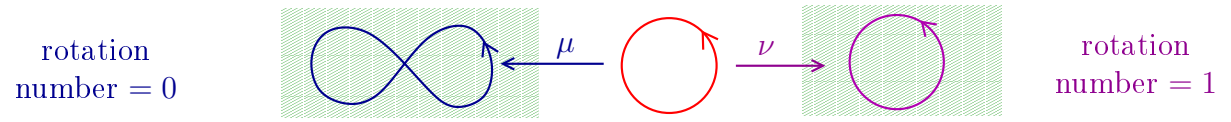
$$\rho(\theta) \overset{\text{by the above, since } |n\cdot z^n\cdot \mathrm{i}| = n}{=} \deg \begin{pmatrix} S^1 & \to & S^1 \\ z & \mapsto & \mathrm{i}\cdot z^n \end{pmatrix} \overset{\text{same calculation as in Lemma 38.9}}{=} n.$$

■

The following proposition explains why we are interested in the rotation number.

Proposition 41.6. Let $\alpha, \beta\colon S^1 \to \mathbb{R}^2$ be two immersions. If α and β are regularly homotopic, then the rotation numbers agree.

Example. We return to the two immersions μ and ν that we encountered in the figure on page 952. By Lemma 41.5 and direct inspection we see that the rotation numbers are 0 and 1. It follows from Proposition 41.6 that these two immersions are not regularly homotopic.



the two immersions have different rotation numbers, thus they are not regularly homotopic

Proof. Let $F\colon S^1 \times [0,1] \to S^1$ be a regular homotopy with $F_0 = \alpha$ and $F_1 = \beta$. It follows immediately from the Degree Homotopy Invariance Theorem 38.3, together with the definition of the winding number, that it suffices to prove the following claim.

Claim. The map

$$\begin{array}{rcl} G\colon S^1 \times [0,1] & \to & S^1 \\ (z,t) & \mapsto & \dfrac{F_t'(z)}{\|F_t'(z)\|} \end{array}$$

is a smooth map, i.e. it is a smooth homotopy.

Proof. Not surprisingly the proof of the claim shares many features with the proof of Lemma 41.4. As before we consider the map $\Xi\colon \mathbb{R} \to S^1$ that is given by $s \mapsto \exp(\mathrm{i}s)$. Note that $\Xi \times \mathrm{id}_{[0,1]}\colon \mathbb{R}\times[0,1] \to S^1 \times [0,1]$ is a local diffeomorphism. Thus it follows from the Smooth Pasting Proposition (3) that it suffices to show that $G \circ \Xi\colon \mathbb{R}\times[0,1] \to S^1$ is smooth. In fact it suffices to show that the map

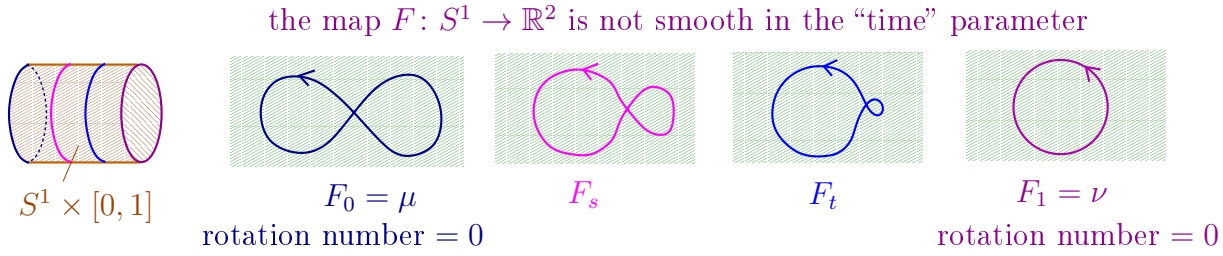
$$\begin{array}{rcl} \mathbb{R}\times[0,1] & \to & \mathbb{R}^2 \setminus \{(0,0)\} \\ (s,t) & \mapsto & F_t'(\Xi(s)) \end{array}$$

is smooth. Note that it follows from Lemma 41.2 that for any $s \in \mathbb{R}$ and $t \in [0,1]$ we have

$$\begin{array}{lcll} & \text{Lemma 41.2 (1c)} & \text{since } F_t \circ \Xi = (F \circ (\Xi \times \mathrm{id}_{[0,1]}))_t \text{ as maps } \mathbb{R} \to \mathbb{R}^2 & \\ F_t'(\Xi(s)) & \overset{\downarrow}{=} & \mathrm{D}(F_t \circ \Xi)_s \cdot 1 & \overset{\downarrow}{=} \mathrm{D}((F \circ (\Xi \times \mathrm{id}_{[0,1]}))_t)_s \cdot 1 \\ & & & = \text{ vector in } \mathbb{R}^2 \text{ given by the two partial derivatives of} \\ & & & \uparrow \quad F \circ (\Xi \times \mathrm{id}_{[0,1]}) \text{ at } (s,t) \text{ in the first coordinate.} \\ & & & \text{by definition of the differential} \end{array}$$

Since $F\colon S^1 \times [0,1] \to S^1$ is smooth and since $\Xi \times \mathrm{id}_{[0,1]}\colon \mathbb{R} \times [0,1] \to S^1 \times [0,1]$ are smooth we know that $F \circ (\Xi \times \mathrm{id}_{[0,1]})\colon \mathbb{R} \times [0,1] \to \mathbb{R}^2$ is smooth. This implies, by the definition on page 531, that the partial derivatives are smooth. By the above calculation this shows that $(s,t) \mapsto F_t'(\Xi(s))$ is smooth. ■

Remark. Note that in the proof of Proposition 41.6 we made very much use of the fact a regular homotopy $F\colon S^1 \times [0,1] \to \mathbb{R}^2$ is by definition a smooth map. In the figure below we show a continuous map $F\colon S^1 \times [0,1] \to \mathbb{R}^2$ with $F_0 = \mu$ and $F_1 = \nu$, such that each $F_t\colon S^1 \to \mathbb{R}^2$ is an immersion. This shows that we cannot easily replace the condition that the map $F\colon S^1 \times [0,1] \to \mathbb{R}^2$ is smooth by a weaker condition.



There is nothing a topologist likes better than a complete characterization of a property. In this setting we are in this lucky situation, since the following theorem says that the converse to Proposition 41.6 also holds.

Theorem 41.7. (Whitney-Graustein Theorem) Two immersions $S^1 \to \mathbb{R}^2$ are regularly homotopic if and only if their rotation numbers agree.

Proof. The "only if"-statement is of course Proposition 41.6. The "if"-statement was first proved by Hassler Whitney [**Why1937**, p. 279] in 1937.[404] We will give the "standard proof" for the Whitney-Graustein Theorem 41.7, which relies on Proposition 41.3, whose proof we still owe to the reader. Thus the plan is as follows:

(1) In Section 41.2 we provide the overdue proof of Proposition 41.3.
(2) In Section 41.3 we will then give the "standard proof" for the Whitney-Graustein Theorem 41.7.

Note that there are several other proofs in the literature, see for example [**Gei2008**, Theorem 6.3.11], [**Gei2009**] and [**Pra2006**, p. 68], finally see also see the discussion in [**Kau1987**, p. 172]. ■

[404] Whitney [**Why1937**, p. 279] writes "this theorem, together with a straightforward proof, was suggested to me by W. C. Graustein".

41.2. **Proof of Proposition 41.3.** In this section we will finally prove Proposition 41.3, i.e. we will show that any immersion is regularly homotopic to an immersion of unit speed. Before we get into the nitty-gritty of the proof let us start out with a little preparation.

Definition. Given a smooth map $\mu\colon S^1 \to \mathbb{R}^n$ we define its length as

$$\text{length of } \mu \;:=\; \int_{r=0}^{r=2\pi} \|(\mu \circ \Xi)'(r)\|\, dr.$$

Lemma 41.8. Every immersion $\mu\colon S^1 \to \mathbb{R}^n$ is regularly homotopic to an immersion of length 2π.

Proof. Let $\mu\colon S^1 \to \mathbb{R}^n$ be an immersion. We denote its length by L. By Lemma 41.2 (1c) we know that $(\mu \circ \Xi)'(r) = \mu'(\Xi(r))$. Since μ is an immersion we see that $L > 0$. One can easily verify that the map

$$\begin{array}{rcl} F\colon \mathbb{R} \times [0,1] & \to & \mathbb{R}^n \\ (s,t) & \mapsto & \dfrac{1}{(1-t) + \frac{L}{2\pi} \cdot t} \cdot \mu(s) \end{array}$$

is a regular homotopy from μ to an immersion of length 2π. ■

Now we are ready to provide the proof of Proposition 41.3.

Proof of Proposition 41.3. Let $\gamma\colon S^1 \to \mathbb{R}^n$ be an immersion. We need to show that γ is regularly homotopic to an immersion $\delta\colon S^1 \to \mathbb{R}^n$ of unit speed, i.e. such that $\|\delta'(z)\| = 1$ for all $z \in S^1$. It follows from Lemmas 41.1 and 41.8 that it suffices to consider the case that the length of γ is equal to 2π.

Claim 1. We set $\widetilde{\gamma} := \gamma \circ \Xi\colon \mathbb{R} \to \mathbb{R}^n$. The map

$$\begin{array}{rcl} \lambda\colon \mathbb{R} & \to & \mathbb{R} \\ s & \mapsto & \displaystyle\int_{r=0}^{r=s} \|\widetilde{\gamma}'(r)\|\, dr \end{array}$$

is a diffeomorphism.

Proof. Recall that it follows from Lemma 41.2 (1c) that $r \mapsto \|\widetilde{\gamma}'(r)\|$ is non-zero for every $r \in \mathbb{R}$. It follows from the Fundamental Theorem of Calculus that $\lambda'(s) = \|\gamma'(s)\| \neq 0$ for all $s \in \mathbb{R}$. Furthermore note that it follows from Lemma 41.2 (1a) that $r \mapsto \|\widetilde{\gamma}'(r)\|$ is 2π-periodic. It follows easily from these two observations that $\lambda\colon \mathbb{R} \to \mathbb{R}$ is a bijection. Since the derivative is non-zero everywhere it follows from the Inverse Function Theorem 19.12 that λ is indeed a diffeomorphism. ⊞

Recall that the assume that γ has length one. This means that $\lambda(2\pi) = 2\pi$. Next we consider the map

$$\begin{array}{rcl} \widetilde{\delta}\colon \mathbb{R} & \to & \mathbb{R}^n \\ s & \mapsto & \widetilde{\gamma}(\lambda^{-1}(s)). \end{array}$$

Claim 2.

(1) The map $\widetilde{\delta}$ is 2π-periodic.

(2) For every $s \in \mathbb{R}$ we have $\|\widetilde{\delta}'(s)\| = 1$.

Proof. First we prove that $\widetilde{\delta}$ is 2π-periodic. We start out this proof with a preliminary calculation. Let $x \in \mathbb{R}$. We pick $n \in \mathbb{Z}$ with $n \cdot 2\pi \in [x, x+2\pi]$. We calculate that

$$\begin{aligned}
\lambda(x+2\pi) &= \int_0^{x+2\pi} \|\widetilde{\gamma}'(r)\|\,dr = \int_0^{x} \|\widetilde{\gamma}'(r)\|\,dr + \int_x^{x+2\pi} \|\widetilde{\gamma}'(r)\|\,dr \\
&= \int_0^{x} \|\widetilde{\gamma}'(r)\|\,dr + \int_x^{n\cdot 2\pi} \|\widetilde{\gamma}'(r)\|\,dr + \int_{n\cdot 2\pi}^{x+2\pi} \|\widetilde{\gamma}'(r)\|\,dr \\
&\underset{\substack{\uparrow \\ \text{since } \widetilde{\gamma}' \text{ is } 2\pi\text{-periodic}}}{=} \int_0^{x} \|\widetilde{\gamma}'(r)\|\,dr + \underbrace{\int_{x-(n-1)\cdot 2\pi}^{2\pi} \|\widetilde{\gamma}'(r)\|\,dr + \int_0^{x-(n-1)\cdot 2\pi} \|\widetilde{\gamma}'(r)\|\,dr}_{=\lambda(2\pi)=2\pi} = \lambda(x) + 2\pi.
\end{aligned}$$

In other words, we have shown that $x + 2\pi = \lambda^{-1}(\lambda(x) + 2\pi)$. It follows that for any $s \in \mathbb{R}$ we have

$$\widetilde{\delta}(s+2\pi) = \widetilde{\gamma}(\lambda^{-1}(s+2\pi)) \underset{\substack{\uparrow \\ \text{by the above applied to } x=\lambda^{-1}(s)}}{=} \widetilde{\gamma}(\lambda^{-1}(s)+2\pi) \underset{\substack{\uparrow \\ \text{since } \widetilde{\gamma} \text{ is } 2\pi\text{-periodic}}}{=} \widetilde{\gamma}(\lambda^{-1}(s)) = \widetilde{\delta}(s).$$

Thus we have shown that $\widetilde{\delta}$ is indeed 2π-periodic.

Now we turn to the proof that for every $s \in \mathbb{R}$ we have $\|\widetilde{\delta}'(s)\| = 1$. This is achieved by the following calculation:

$$\begin{aligned}
\|\widetilde{\delta}'(s)\| &\overset{\substack{\text{by definition of } \widetilde{\delta} \\ \downarrow}}{=} \|(\widetilde{\gamma} \circ \lambda^{-1})'(s)\| \overset{\substack{\text{Chain Rule 19.10} \\ \downarrow}}{=} \|\widetilde{\gamma}'(\lambda^{-1}(s)) \cdot (\lambda^{-1})'(s)\| \overset{\substack{\text{Smooth Invariance of Domain Proposition 19.11 (1)} \\ \downarrow}}{=} \left\| \widetilde{\gamma}'(\lambda^{-1}(s)) \cdot \frac{1}{(\lambda')(\lambda^{-1}(s))} \right\| \\
&\underset{\substack{\uparrow \\ \text{Fundamental Theorem of Calculus}}}{=} \left\| \widetilde{\gamma}'(\lambda^{-1}(s)) \cdot \frac{1}{\|\widetilde{\gamma}'(\lambda^{-1}(s))\|} \right\| = 1.
\end{aligned}$$

⊞

Claim 3. We consider the homotopy

$$\begin{aligned}
\widetilde{F}\colon \mathbb{R} \times [0,1] &\to \mathbb{R}^n \\
(s,t) &\mapsto \widetilde{\gamma}(s \cdot (1-t) + \lambda^{-1}(s) \cdot t)
\end{aligned}$$

from $\widetilde{\gamma}$ to $\widetilde{\delta}$. This homotopy has the property that each $\widetilde{F}_t$ is 2π-periodic and that each $\widetilde{F}_t$ is an immersion.

Proof. Let $t \in [0,1]$. It follows easily from Claim 2 (1) that $\widetilde{F}_t$ is indeed 2π-periodic. Furthermore note that for each $s \in \mathbb{R}$ we have

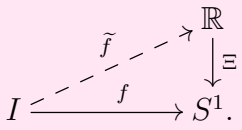
$$\widetilde{F}_t'(s) \underset{\substack{\uparrow \\ \text{chain rule}}}{=} \overbrace{\widetilde{\gamma}'(s \cdot (1-t) + \lambda^{-1}(s) \cdot t)}^{\neq 0} \cdot \overbrace{(1 - t + \underbrace{(\lambda^{-1})'(s)}_{=\|\widetilde{\gamma}'(s)\| \text{ by FTC}} \cdot t)}^{>0} \neq 0.$$

⊞

Since $\widetilde{F}$ is evidently smooth and since by Claim 3 we know that each $\widetilde{F}_t$ is 2π-periodic it follows from Lemma 41.2 (2b) that there exists a smooth homotopy $F\colon S^1 \times [0,1] \to \mathbb{R}^n$ such that $F \circ (\Xi \times \mathrm{id}_{[0,1]}) = \widetilde{F}$. By design we have $F_0 = \gamma$. Note that it follows from Claim 2 (2) and Lemma 41.2 (1c) that F_1 has unit speed. Furthermore it follows from Claim 3 and Lemma 41.2 (1d) that each F_t is an immersion. In summary $F\colon S^1 \times [0,1] \to S^1$ is the desired regular homotopy. ■

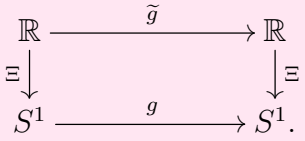
41.3. **Proof of the "if"-statement of the Whitney-Graustein Theorem 41.7.** For the reader's convenience we recall two earlier statements.

Lemma 34.9. Let $I \subset \mathbb{R}$ be an interval and let $x_0 \in I$. Given any map $f\colon I \to S^1$ and given any $\widetilde{y}_0 \in \mathbb{R}$ with $f(x_0) = \widetilde{y}_0$ there exists a unique map $\widetilde{f}\colon I \to \mathbb{R}$ with $\widetilde{f}(x_0) = \widetilde{y}_0$ and such that $f = \Xi \circ \widetilde{f}$, i.e. such that the following diagram commutes;



We refer to $\widetilde{f}$ as a *lift* of f.

Lemma 41.9. Let $g\colon S^1 \to S^1$ be a smooth map and let $\widetilde{g}\colon \mathbb{R} \to \mathbb{R}$ be a map such that $\Xi \circ \widetilde{g} = g \circ \Xi$, i.e. such that the following diagram commutes:



Then for any $s \in \mathbb{R}$ we have

$$\deg(g\colon S^1 \to S^1) \;=\; \tfrac{1}{2\pi} \cdot \big(\widetilde{g}(s + 2\pi) - \widetilde{g}(s)\big).$$

Proof. This lemma follows basically immediately from the Degree-via-Lifts Lemma 38.10 applied to the map $f\colon [0,1] \to \mathbb{R}$ that is given by $f(t) = g(s + 2\pi \cdot t)$. ■

We will also need the following lemma.

Lemma 41.10. If $\Theta\colon [0,1] \to S^n$ is a non-constant map, then

$$\left\| \int_{r=0}^{r=1} \Theta(r)\, dr \right\| \;<\; \int_{r=0}^{r=1} \|\Theta(r)\|\, dr.$$

Proof. We view Θ as a map $[0,1] \to \mathbb{R}^{n+1}$. the Integral Inequality Lemma 0.16 says that the following inequality holds:

$$\left\| \int_{r=0}^{r=1} \Theta(r)\, dr \right\| \;\leq\; \int_{r=0}^{r=1} \|\Theta(r)\|\, dr \qquad \in \mathbb{R}.$$

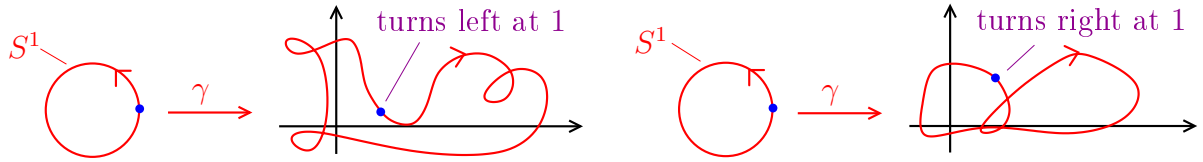
Furthermore the Integral Inequality Lemma 0.16 says that equality holds if and only if there exists a $w \in \mathbb{R}^{n+1}$ and a map $\lambda\colon [a,b] \to \mathbb{R}_{\geq 0}$ such that for every $t \in [a,b]$ we have $\Theta(t) = \lambda(t) \cdot w$. Since Θ takes values in S^n we see that equality only holds if Θ is a constant map. ■

In the remainder of this section we will need the following geometric definition.

Definition. Let $\gamma\colon S^1 \to \mathbb{R}^2 = \mathbb{C}$ be an immersion of unit speed, i.e. such that $\gamma'(z) \in S^1$ for all $z \in S^1$. By the Lifting from S^1-to-$\mathbb{R}$ Lemma 34.9 there exists a smooth map $\Lambda_\gamma\colon \mathbb{R} \to \mathbb{R}$ such that for all $s \in \mathbb{R}$ we have $\Xi(\Lambda_\gamma(s)) = (\gamma \circ \Xi)'(s) \in S^1$.

(1) We say $\gamma\colon S^1 \to \mathbb{R}^2$ turns left at $1 \in S^1$ if $\Lambda'_\gamma(0) > 0$.[405]

(2) We say $\gamma\colon S^1 \to \mathbb{R}^2$ turns right at $1 \in S^1$ if $\Lambda'_\gamma(0) > 0$.



Lemma 41.11. Let $\gamma\colon S^1 \to \mathbb{R}^2$ be an immersion of unit speed.

(1) If the rotation number $\rho(\gamma) \geq 0$, then γ is regularly homotopic to an immersion of unit speed that turns left at $1 \in S^1$.
(2) If the rotation number $\rho(\gamma) \leq 0$, then γ is regularly homotopic to an immersion of unit speed that turns right at $1 \in S^1$.

Proof. Let $\gamma\colon S^1 \to \mathbb{R}^2$ be an immersion of unit speed. Evidently we only deal with one of the two cases of the lemma. For no particular reason we deal with the case that $\rho(\gamma) \geq 0$. Recall that by the Lifting from S^1-to-$\mathbb{R}$ Lemma 34.9 there exists a map $\Lambda\colon \mathbb{R} \to \mathbb{R}$ such that for all $s \in \mathbb{R}$ we have $\Xi(\Lambda(s)) = (\gamma \circ \Xi)'(s)$.

Claim. There exists a $w \in \mathbb{R}$ with $\Lambda'(w) > 0$.

Proof. Note that

$$\text{(A)} \qquad \int_{t=0}^{t=2\pi} \Lambda'(t)\,dt \underset{\text{Fundamental Theorem of Calculus}}{=} \Lambda(2\pi) - \Lambda(0) \underset{\text{by Lemma 41.9}}{=} \deg(\gamma') \underset{\text{by definition of } \rho(\gamma)}{=} \rho(\gamma).$$

If $\rho(\gamma) > 0$ then it is clear from this calculation that there exists a $w \in [0, 2\pi]$ with $\Lambda'(w) > 0$. It remains to consider the case $\rho(\gamma) = 0$. Note that we have

$$\text{(B)} \qquad 0 \underset{\text{since } \gamma\circ\Xi \text{ is } 2\pi\text{-periodic}}{=} (\gamma \circ \Xi)(2\pi) - (\gamma \circ \Xi)(0) \underset{\text{Fundamental Theorem of Calculus}}{=} \int_{t=0}^{t=2\pi} \overbrace{(\gamma \circ \Xi)'(t)}^{\in S^1}\,dt.$$

Since γ has unit speed we know that each $(\gamma \circ \Xi)'(t) \in S^1$. Thus the integral in (B) can only be zero if $(\gamma \circ \Xi)'(t)$ is not constant. This implies that Λ is not constant. But this implies that Λ' cannot be identically zero. But the only way how the integral in (A) can be zero is now that Λ' assumes negative and positive values. ⊞

We have now shown that γ turns left at some point in S^1. It remains to shift this point to $1 \in S^1$. To do so we consider the following map

$$\begin{aligned} F\colon S^1 \times [0,1] &\to \mathbb{R}^2 \\ (z,t) &\mapsto \gamma(z \cdot \exp(\mathrm{i} \cdot w \cdot t)). \end{aligned}$$

One can easily verify that this map is smooth and that each F_t is an immersion of unit speed. In particular F is a regular homotopy from $F_0 = \gamma$ to $F_1 =: \delta$. Finally note that by construction we see that δ turns left at 0. ■

[405] The map Λ_γ is well-defined up to addition of a multiple of 2π. We see that the notion of turning left and right is independent of the choice of Λ_γ.

Now we can finally provide the proof of the "if"-statement of the Whitney-Graustein Theorem 41.7.

Proof of the "if"-statement of the Whitney-Graustein Theorem 41.7. In the following let $\alpha, \beta\colon S^1 \to \mathbb{R}^2$ be two immersions with $\rho(\alpha) = \rho(\beta)$. We need to show that α and β are regularly homotopic. In the following we will deal with the case that $\rho(\alpha) = \rho(\beta) \geq 0$. The case that $\rho(\alpha) = \rho(\beta) < 0$ is proved by an almost identical argument, namely, in the discussion below we just need to replace Lemma 41.11 (1) by Lemma 41.11 (2).

We start out with a few initial observations. In Lemma 41.1 we showed that being regularly homotopic is an equivalence relation. It follows from this observation, together with Proposition 41.3, Lemma 41.11 (1) and Proposition 41.6 that we can assume that the two given immersions $\alpha, \beta\colon S^1 \to \mathbb{R}^2$ have the following properties:

(I) Both α and β are of unit speed.
(II) Both α and β turn left at $1 \in S^1$.

Furthermore, after a translation we can as well assume that the following condition is satisfied:

(III) We have $\alpha(1) = \beta(1) = 0 \in \mathbb{C}$.

Since $\alpha, \beta\colon S^1 \to \mathbb{R}^2 = \mathbb{C}$ are immersions of unit speed we get maps $\alpha', \beta'\colon S^1 \to S^1$. Thus we obtain once again from the Lifting from S^1-to-$\mathbb{R}$ Lemma 34.9 that there exist smooth maps $\Lambda_{\alpha'}, \Lambda_{\beta'}\colon \mathbb{R} \to \mathbb{R}$ such that $\Xi \circ \widetilde{\alpha'} = \alpha' \circ \Xi$ and such that $\Xi \circ \Lambda_{\beta'} = \beta' \circ \Xi$.

For orientation we summarize the various maps in the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\Lambda_{\alpha'}} & \mathbb{R} \\ {\scriptstyle s \mapsto \exp(\mathrm{i}s)}\downarrow{\scriptstyle \Xi} & & {\scriptstyle \Xi}\downarrow{\scriptstyle s \mapsto \exp(\mathrm{i}s)} \\ S^1 & \xrightarrow[z \mapsto \alpha'(z)]{} & S^1. \end{array}$$

On page 952 we saw that the standard homotopy $(z,t) \mapsto \alpha(z)\cdot(1-t)+\beta(z)\cdot t$ shows that α and β are homotopic. In general this homotopy is not a regular homotopy. To obtain a regular homotopy, under the hypothesis that $\rho(\alpha) = \rho(\beta)$ we will use a variation on the standard homotopy. More precisely, the idea is now as follows:

(a) we consider the lifts $\widetilde{\alpha'}$ and $\Lambda_{\beta'}$ of the *derivatives* of $\Xi \circ \alpha$ and $\Xi \circ \beta$,
(b) we consider the standard homotopy of $\widetilde{\alpha'}$ and $\Lambda_{\beta'}$,
(c) we apply Ξ to the homotopy from (b) and we integrate, to "undo" taking the derivative,
(d) we obtain a regular homotopy $\widetilde{F}\colon \mathbb{R} \times [0,1] \to \mathbb{R}^n$ between $\alpha \circ \Xi$ and $\beta \circ \Xi$,
(e) finally, and that is the most delicate step, we need to modify $\widetilde{F}$ to obtain a regular homotopy $\widetilde{G}$ that descends to a regular homotopy on S^1.

Next we consider the homotopy

$$\begin{array}{rcl} \Theta\colon \mathbb{R} \times [0,1] & \to & S^1 \\ (s,t) & \mapsto & \Xi\big(\Lambda_{\alpha'}(s)\cdot(1-t) + \Lambda_{\beta'}(s)\cdot t\big). \end{array}$$

Claim 1.

(1) Each Θ_t is 2π-periodic.
(2) Each $\Theta_t\colon \mathbb{R} \to \mathbb{R}$ is non-constant.

Proof. Let $t \in [0,1]$.

(1) Let $s \in \mathbb{R}$. We need to show that $\Theta_t(s+2\pi) = \Theta_t(s)$. The statement follows immediately from the fact that Ξ is 2π-periodic and the following calculation:

$$\begin{aligned}\Lambda_{\alpha'}(s+2\pi)\cdot(1-t)+\Lambda_{\beta'}(s+2\pi)\cdot t &\overset{\substack{\text{by Lemma 41.9 we have } \Lambda_{\alpha'}(s+2\pi)-\Lambda_{\alpha'}(s)=2\pi\cdot\deg(\alpha')=2\pi\cdot\rho(\alpha)\\ \text{and we have } \Lambda_{\beta'}(s+2\pi)-\Lambda_{\beta'}(s)=2\pi\cdot\deg(\beta')=2\pi\cdot\rho(\beta)}}{\downarrow}{=} (\Lambda_{\alpha'}(s)+2\pi\cdot\rho(\alpha))\cdot(1-t)+(\Lambda_{\beta'}(s)+2\pi\cdot\rho(\beta))\cdot t\\ &\underset{\substack{\uparrow\\ \text{follows from the hypothesis that } \rho(\alpha)=\rho(\beta)}}{=} \Lambda_{\alpha'}(s)\cdot(1-t)+\Lambda_{\beta'}(s)\cdot t+\underbrace{2\pi\cdot\rho(\alpha)}_{\in 2\pi\cdot\mathbb{Z}}.\end{aligned}$$

(2) Let $t \in [0,1]$. Note that by (II) we know that $\Lambda_{\alpha'}(0) > 0$ and that $\Lambda'_{\beta'}(0) > 0$. It follows that the derivative of $s \mapsto \widetilde{\alpha'}(s)\cdot(1-t)+\Lambda_{\beta'}(s)\cdot t$ is positive at $s=0$. It follows that Θ_t is non-constant. ⊞

We use this homotopy $\Theta\colon \mathbb{R}\times[0,1] \to S^1$ to define the following map:

$$\begin{aligned}\widetilde{F}\colon \mathbb{R}\times[0,1] &\to \mathbb{R}^2\\ (s,t) &\mapsto \int_{r=0}^{r=s} \Theta_t(r)\,dr.\end{aligned}$$

Claim 2. The map $\widetilde{F}$ has the following properties:
(1) $\widetilde{F}$ is smooth.
(2) $\widetilde{F}_0 = \alpha\circ\Xi$ and $\widetilde{F}_1 = \beta\circ\Xi$.

Proof.

(1) Note that α and β are smooth. As we mentioned in Lemma 34.8, it follows from the fact that $q\colon \mathbb{R}\to S^1$ is a local diffeomorphism that $\Lambda_{\alpha'}$ and $\Lambda_{\beta'}$ are smooth. It now follows from standard facts in real analysis, e.g. the Leibniz Integral Rule [**PedS2015**, Theorem 9.2.11], that $\widetilde{F}$ is smooth.
(2) Given any $s \in \mathbb{R}$ we perform the following calculation:

$$\widetilde{F}(s,0) \underset{\substack{\uparrow\\ \text{by definition of } \widetilde{F}}}{=} \int_{r=0}^{r=s}\Theta(r,0)\,dr \underset{\substack{\uparrow\\ \text{by definition}\\ \text{of } \Theta(r,0)}}{=} \int_{r=0}^{r=s}(\Xi\circ\Lambda_{\alpha'})(r)\,dr \underset{\substack{\uparrow\\ \text{since we have}\\ \Xi\circ\Lambda_{\alpha'}=\alpha'\circ\Xi=(\alpha\circ\Xi)'}}{=} \int_{r=0}^{r=s}(\alpha\circ\Xi)'(r)\,dr \underset{\substack{\uparrow\\ \text{follows from the Fundamental}\\ \text{Theorem of Calculus and}\\ \text{fact that } (\alpha\circ\Xi)(0)=0}}{=} (\alpha\circ\Xi)(s).$$

The same argument shows that $\widetilde{F}_1 = \beta\circ\Xi$. ⊞

All this sounds very promising. But unfortunately there is a problem with the homotopy $\widetilde{F}\colon \mathbb{R}\times[0,1]\to\mathbb{R}^2$: there is no reason why the maps $\widetilde{F}_t$ should be 2π-periodic, i.e. why

they should descend to a map $S^1 \to \mathbb{R}^2$. To fix this issue we consider the following map:[406]

$$\begin{array}{rccl} \widetilde{G}\colon \mathbb{R}\times[0,1] & \to & \mathbb{R}^2 & \\ (s,t) & \mapsto & \widetilde{F}(s,t) - \frac{s}{2\pi}\cdot\widetilde{F}(2\pi,t) & = \int\limits_{r=0}^{r=s} \Theta_t(r)\,dr \;-\; \frac{s}{2\pi}\cdot\int\limits_{r=0}^{r=2\pi}\Theta_t(r)\,dr. \end{array}$$

Claim 3. The map $\widetilde{G}$ has the following properties:
(1) Each $\widetilde{G}_t$ is 2π-periodic.
(2) $\widetilde{G}$ is smooth.
(3) $\widetilde{G}_0 = \alpha\circ\Xi$ and $\widetilde{G}_1 = \beta\circ\Xi$.
(4) For each $t\in[0,1]$ the map $\widetilde{G}_t\colon\mathbb{R}\to\mathbb{R}^2$ is an immersion.

Proof.

(1) Let $t\in[0,1]$. We need to show that for each $s\in\mathbb{R}$ we have $\widetilde{G}(s+2\pi,t) = \widetilde{G}(s,t)$. Evidently it suffices to show that for every $s\in[0,2\pi)$ and every $n\in\mathbb{Z}$ we have $\widetilde{G}(s+2\pi\cdot n,t) = \widetilde{G}(s,t)$. We perform the following elementary calculation:

$$\begin{aligned} \widetilde{G}(s+2\pi n,t) &= \int_0^{2\pi n+s}\Theta_t(r)\,dr \;-\; \tfrac{2\pi n+s}{2\pi}\cdot\int_0^{2\pi}\Theta_t(r)\,dr \;= \\ &\quad \text{(by definition and by the additivity of the integral)} \\ &= \sum_{k=0}^{n-1}\int_{2\pi k}^{2\pi(k+1)}\Theta_t(r)\,dr \;-\; n\cdot\int_0^{2\pi}\Theta_t(r)\,dr + \int_{2\pi n}^{2\pi n+s}\Theta_t(r)\,dr - \tfrac{s}{2\pi}\cdot\int_0^{2\pi}\Theta_t(r)\,dr \\ &= \underbrace{\sum_{k=0}^{n-1}\int_0^{2\pi}\Theta_t(r)\,dr \;-\; n\cdot\int_0^{2\pi}\Theta_t(r)\,dr}_{=0} + \underbrace{\int_0^{s}\Theta_t(r)\,dr - \tfrac{s}{2\pi}\cdot\int_0^{2\pi}\Theta_t(r)\,dr}_{=\widetilde{G}(s,t)} \;=\; \widetilde{G}(s,t) \end{aligned}$$

(since by Claim 1 (1) we know that Θ_t is 2π-periodic)

(2) It follows easily from Claim 2 (1) that $\widetilde{G}$ is smooth.
(3) Let $s\in\mathbb{R}$. We see that

$$\begin{aligned} \widetilde{G}_0(s) &\overset{\text{by definition}}{=} \widetilde{F}_0(s) - \tfrac{s}{2\pi}\cdot\widetilde{F}_0(2\pi) \overset{\text{by Claim 2 (2)}}{=} (\alpha\circ\Xi)(s) - \tfrac{s}{2\pi}\cdot(\alpha\circ\Xi)(2\pi) \\ &= (\alpha\circ\Xi)(s) - \tfrac{s}{2\pi}\cdot\underbrace{(\alpha\circ\Xi)(0)}_{=0 \text{ by (III)}} \;=\; (\alpha\circ\Xi)(s). \end{aligned}$$

(since $\alpha\circ\Xi$ is 2π-periodic)

The same argument shows that $\widetilde{G}_1 = \beta\circ\Xi$.

[406] While writing this chapter the author regretted on numerous occasions the fact that 1 is not equal to 2π.

(4) Let $t \in [0,1]$. We need to show that for every $s \in \mathbb{R}$ we have $\widetilde{G}_t{}'(s) \neq 0$. Thus let $s \in \mathbb{R}$. Note that

$$\|\widetilde{G}_t{}'(s)\| \;=\; \left\| \Theta_t(s) - \frac{1}{2\pi} \int_{r=0}^{r=2\pi} \Theta_t(r)\, dr \right\| \;\geq\; \|\Theta_t(s)\| - \frac{1}{2\pi} \cdot \left\| \int_{r=0}^{r=2\pi} \Theta_t(r)\, dr \right\|$$

$=$: by the Fundamental Theorem of Calculus applied to the first term of $\widetilde{G}_t(s)$ and by linearity of the second term of $\widetilde{G}_t(s)$

$\geq$: triangle inequality

$$> \; \underbrace{\|\Theta_t(s)\|}_{=1} - \frac{1}{2\pi} \cdot \int_{r=0}^{r=2\pi} \underbrace{\|\Theta_t(r)\|}_{=1}\, dr \; > \; 0.$$

$>$: follows from Lemma 41.10, since we know by Claim 1 (2) that the map Θ_t is not constant ⊞

It follows from Claim 3 (1) and (3) together with Lemma 41.2 (2b) that there exists a smooth map

$$G\colon S^1 \times [0,1] \;\to\; \mathbb{R}^2$$

such that for all $(s,t) \in \mathbb{R} \times [0,1]$ we have $G(\Xi(s),t)) = \widetilde{G}(s,t)$. It follows easily from the above properties of $\widetilde{G}$ that G is the desired regular homotopy between α and β. ■

Exercises for Chapter 41.

Exercise 41.1. Let $n \geq 3$. Show that any two immersions $S^1 \to \mathbb{R}^n$ are regularly homotopic.
Hint. Use Propositions 30.2 and 30.3.

Exercise 41.2. Let $n \in \mathbb{Z} \setminus \{0\}$. Use Lemma 41.9 to give a new proof of Lemma 41.5, i.e. show that

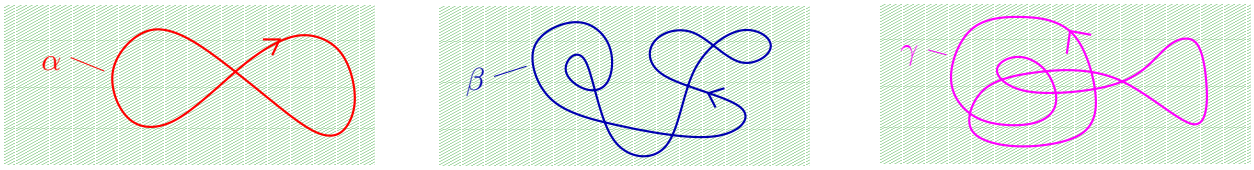
$$\rho \begin{pmatrix} S^1 & \to & \mathbb{C} \\ z & \mapsto & z^n \end{pmatrix} \;=\; n.$$

Exercise 41.3. Let $\mu\colon S^1 \to \mathbb{R}^n$ be a smooth map. As on page 958 we define its length as

$$\text{length of } \mu \;:=\; \int_{r=0}^{r=2\pi} \|(\mu \circ \Xi)'(r)\|\, dr.$$

(Here we use the notation from page 953.) Show that if $\Phi\colon S^1 \to S^1$ is a diffeomorphism, then the lengths of μ and $\mu \circ \Phi$ agree.

Exercise 41.4. Determine the rotation numbers of the immersions indicated in the figure below.



Exercise 41.5. Let $\alpha, \beta\colon S^1 \to \mathbb{R}^2$ be immersions. We suppose that there exists a continuous map $F\colon S^1 \times [0,1] \to \mathbb{R}^2$ with the following properties:

(1) Each F_t is an immersion.
(2) $F_0 = \alpha$ and $F_1 = \beta$.

(3) For each $z \in S^1$ the map $[0,1] \to \mathbb{R}^2$ given by $t \mapsto F'_t(z)$ is continuous.

Does this imply that the rotation numbers of α and β agree?

42. The transversality theorem

In this chapter we will do the following:

(1) We will define what it means for two submanifolds of a given smooth manifold to be transverse.
(2) We will see that if two submanifolds are transverse, then the intersection is again a submanifold.
(3) We will see that given two submanifolds we can always perform an isotopy of one of the two submanifolds to obtain a transverse intersection.

Throughout this chapter we will work with two already familiar conventions which will make our life much easier. First we recall the following convention from page 633

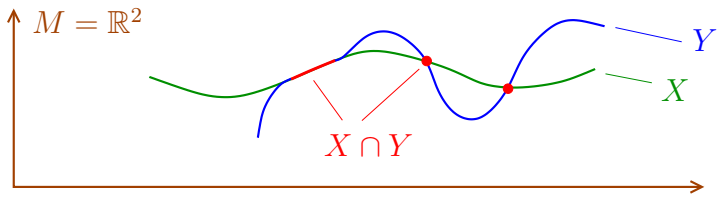
Convention. Let M be a smooth manifold.

(1) Given $P \in \partial M$ we view $\mathrm{T}_P \partial M$ as a vector subspace of $\mathrm{T}_P M$.
(2) Given a submanifold W and given $P \in W$ we view $\mathrm{T}_P W$ as a vector subspace of $\mathrm{T}_P M$.

Secondly we adopt the following convention.

Convention. Given a smooth submanifold M of some $\mathbb{R}^n$ and given $P \in M$ we use the natural isomorphism from the Tangent Space-Isomorphism Proposition 23.16 as an excuse for not distinguishing in our notation between the abstract tangent space $\mathrm{T}_P M$ and the physical tangent space $\widetilde{\mathrm{T}}_P M$. We will always just write $\mathrm{T}_P M$.

42.1. **The Transversal Intersection Theorem.** In this section we want to study the following question: given a smooth manifold M and two submanifolds X and Y, is the intersection $X \cap Y$ again a submanifold of M? It is not difficult to show that in general the answer is negative. For example consider $M = \mathbb{R}^2$ with the two submanifolds X and Y that are shown in the below. The intersection $X \cap Y$ has three components, one of which is a submanifold of dimension one, and the other two components consist of a single point. We see that $X \cap Y$ is not a smooth manifold (after all, what would its dimension be?), in particular it is not a submanifold of M.



If one spends some time pondering the question why in the previous example the intersection $X \cap Y$ is not a submanifold, then one is inexorably led towards the following definition.

Definition. Let M be a smooth manifold. Let X and Y be proper submanifolds of M.

(1) Let $P \in X \cap Y$. We say P is a transverse intersection point if X and Y intersect transversally in P, i.e. if the equality[407]

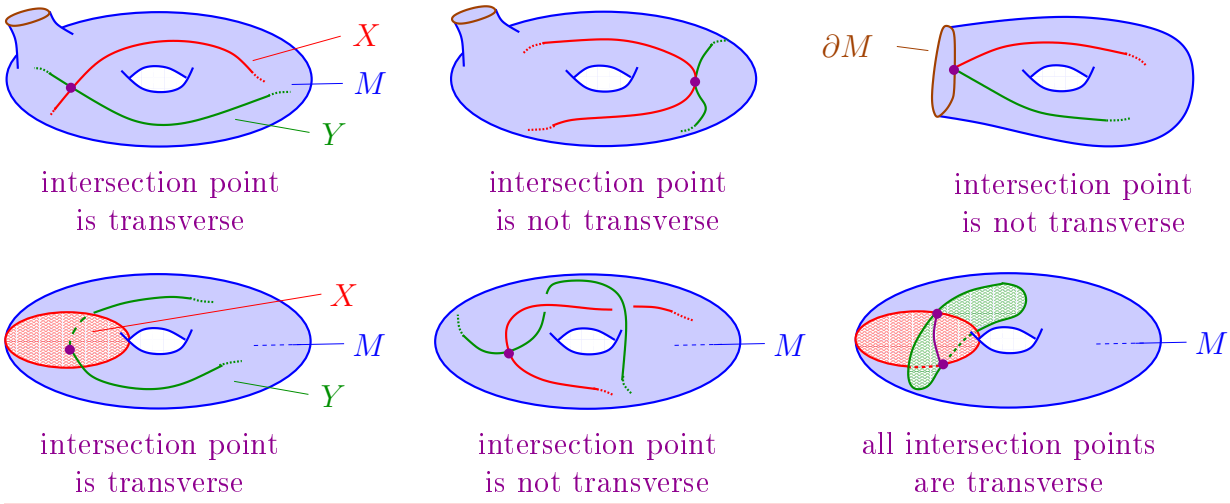
$$\mathrm{T}_P X + \mathrm{T}_P Y \;=\; \mathrm{T}_P M$$

holds. If $P \in \partial M$, then we also demand that[408]

$$\mathrm{T}_P(X \cap \partial M) + \mathrm{T}_P(Y \cap \partial M) \;=\; \mathrm{T}_P\,\partial M.$$

(2) We say that X and Y intersect transversally if each intersection point is a transverse intersection point.

(3) We abbreviate the expression "X and Y intersect transversally" by $X \pitchfork Y$.



intersection point is transverse | intersection point is not transverse | intersection point is not transverse

intersection point is transverse | intersection point is not transverse | all intersection points are transverse

Theorem 42.1. (Submanifold Transversal Intersection Theorem) Let M be an m-dimensional smooth manifold, let $X \subset M$ be a proper submanifold of codimension k and let Y be a proper submanifold of M of codimension l.

(1) Let $P \in M$ be a transverse intersection point of X and Y.

(a) If $P \notin \partial M$, then there exists a chart $\Phi\colon U \to V$ of type (i) around P with

$$\Phi(U \cap X) = V \cap (\mathbb{R}^{m-k} \times \{0\}) \quad \& \quad \Phi(U \cap Y) = V \cap (\{0\} \times \mathbb{R}^{m-l}).$$

(b) If $P \in \partial M$, then there exists a chart $\Phi\colon U \to V$ of type (ii) around P with

$$\Phi(U \cap X) = V \cap (\mathbb{R}^{m-k-1} \times \{0\} \times \mathbb{R}_{\geq 0}) \quad \& \quad \Phi(U \cap Y) = V \cap (\{0\} \times \mathbb{R}^{m-l-1} \times \mathbb{R}_{\geq 0}).$$

(2) If the submanifolds X and Y intersect transversally, i.e. if $X \pitchfork Y$, then the following statements hold:

(a) The intersection $X \cap Y$ is a proper submanifold of M of codimension $k + l$. Furthermore $X \cap Y$ is proper submanifold of X and of Y.

(b) For every $P \in X \cap Y$ we have

$$\mathrm{T}_P(X \cap Y) \;=\; \mathrm{T}_P X \cap \mathrm{T}_P Y \;\subset\; \mathrm{T}_P M.$$

(c) Let M, X and Y be oriented.

(i) By hypothesis for each $P \in X \cap Y$ the vector spaces $\mathrm{T}_P M$, $\mathrm{T}_P X$ and $\mathrm{T}_P Y$ are oriented. By (2b) we can equip $\mathrm{T}_P(X \cap Y) = \mathrm{T}_P X \cap \mathrm{T}_P Y$ with the

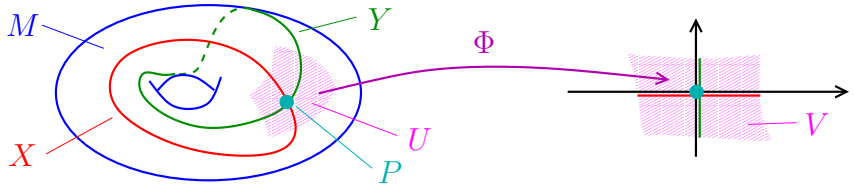
[407] Since X is a submanifold of M we can use the above convention to view the tangent space $\mathrm{T}_P X$ as a vector subspace of $\mathrm{T}_P M$.

[408] Note that in Exercise 19.31 (a) we showed that $X \cap \partial M$ and $Y \cap \partial M$ are submanifolds of ∂M, thus it makes sense to consider their tangent spaces.

intersection orientation that we introduced on page 673. These orientations define an orientation on $X \cap Y$.

(ii) The orientations on the smooth manifolds $\partial X \cap \partial Y$ and $\partial(X \cap Y)$ agree.[409]

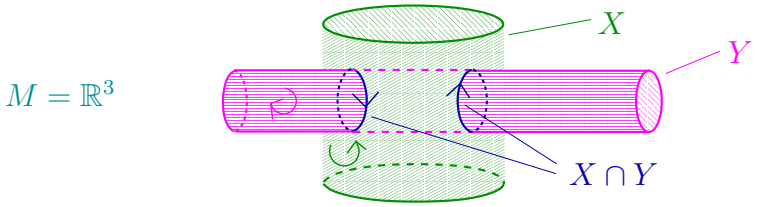
(d) If one of X or Y is compact, then so is $X \cap Y$.



Remark. In the next section we will prove the Transversal Preimage Theorem 42.3 which is a generalization of the Submanifold Transversal Intersection Theorem 42.1 (2).

Definition. Let M be a smooth manifold and let X and Y be proper oriented submanifolds of M that intersect transversally. We refer to the orientation on $X \cap Y$ that we introduced in the Submanifold Transversal Intersection Theorem 42.1 (2) as the intersection orientation.

Example. In the figure below we see two oriented infinite cylinders X and Y in the oriented smooth manifold $\mathbb{R}^3$. The cylinders are submanifolds of codimension one. They intersect transversally and the intersection consists of two circles, i.e. $X \cap Y$ is a one-dimensional submanifold of $\mathbb{R}^3$. Put differently, $X \cap Y$ is a submanifold of codimension two.



Auxiliary Lemma 42.2. Let W be an m-dimensional real vector space, let U be an $(m-k)$-dimensional vector subspace of W and let V be an $(m-l)$-dimensional vector subspace of W. We assume that $U + V = W$. Furthermore let $\Theta\colon W \to \mathbb{R}^l \oplus \mathbb{R}^{m-k-l} \oplus \mathbb{R}^k$ be an isomorphism. If $\Theta(U) = \mathbb{R}^l \oplus \mathbb{R}^{m-k-l} \oplus \{0\}$ and if $\Theta(U \cap V) = \{0\} \oplus \mathbb{R}^{m-k-l} \oplus \{0\}$, then the map

$$V \xrightarrow{\Theta} \mathbb{R}^l \oplus \mathbb{R}^{m-k-l} \oplus \mathbb{R}^k \xrightarrow{\text{projection}} \mathbb{R}^{m-k-l} \oplus \mathbb{R}^k$$

is an isomorphism.

Proof of Lemma 42.2. For dimension reasons we only have to show that the given map $V \to \mathbb{R}^{m-k-l} \oplus \mathbb{R}^k$ is an epimorphism. Thus let $y \oplus z \in \mathbb{R}^{m-k-l} \oplus \mathbb{R}^k$. Since Θ is an isomorphism there exists a $w \in W$ with $\Theta(w) = 0 \oplus y \oplus z$. Since $U + V = W$ there exist $u \in U$ and $v \in V$ with $u + v = w$. By hypothesis $\Theta(u) = x' \oplus y' \oplus 0$ for some $x' \in \mathbb{R}^l$ and $y' \in \mathbb{R}^{m-k-l}$. Furthermore by hypothesis we have $\Theta(U \cap V) = \{0\} \oplus \mathbb{R}^{m-k-l} \oplus \{0\}$. From $U + V = W$ it follows from elementary linear algebra that $U \cap V$ is also of dimension

[409] Here we give $X \cap Y$ the orientation from (i) and we give $\partial(X \cap Y)$ the boundary orientation from page 689. Furthermore we equip ∂X, ∂Y and ∂M with the boundary orientations. By definition of transverse intersection we also have $\partial X \pitchfork \partial Y$. Thus we can use (i) to equip $\partial X \cap \partial Y$ with an orientation.

$m-k-l$. Thus the map $\Theta\colon U\cap V\to\{0\}\oplus\mathbb{R}^{m-k-l}\oplus\{0\}$ is actually an isomorphism. Thus there exists a $\widetilde{v}\in V$ with $\Theta(\widetilde{v})=0\oplus y'\oplus 0$. Now we see that

$$\Theta(v-\widetilde{v}) \;=\; \Theta(w)-\Theta(u)-\Theta(\widetilde{v}) = 0\oplus y\oplus z - x'\oplus y'\oplus 0 - 0\oplus y'\oplus 0 \;=\; (-x')\oplus y\oplus z.$$

Thus we have shown that $y\oplus z$ lies in the image of the given map $V\to\mathbb{R}^{m-k-l}\oplus\mathbb{R}^k$. ■

Proof of the Submanifold Transversal Intersection Theorem 42.1. Let M be an m-dimensional smooth manifold, let $X\subset M$ be a proper submanifold of codimension k and let Y be a proper submanifold of M of codimension l.

(1) Let $P\in M$ be a transverse intersection point of X and Y. In the following we will deal with the case that $P\notin\partial M$. The case $P\in\partial M$ is proved in a similar fashion, we leave it to the reader to fill in the details for the case $P\in\partial M$.

Since X is a proper submanifold of M and since $P\notin\partial M$ we can pick a submanifold chart $\Theta\colon A\to B$ of type (α) for X around P. We can arrange that $\Theta(P)=0$. Note that $\mathrm{D}\Theta_P(\mathrm{T}_PX)=\mathbb{R}^{m-k}\times\{0\}$. Since P is a transverse intersection point we see that $\mathrm{D}\Theta_P(\mathrm{T}_PX\cap\mathrm{T}_PY)$ is a vector subspace of $\mathbb{R}^{m-k}\times\{0\}\subset\mathbb{R}^m$ of dimension $m-k-l$. Elementary linear algebra shows that there exists an $A\in\mathrm{GL}(m,R)$ with $A\cdot(\mathbb{R}^{m-k}\times\{0\})=\mathbb{R}^{m-k}\times\{0\}$ and $A\cdot\mathrm{D}\Theta_P(\mathrm{T}_PX\cap\mathrm{T}_PY)=\{0\}\times\mathbb{R}^{m-k-l}\times\{0\}\subset\mathbb{R}^m$. After composing Θ with multiplication by the matrix A we might as well assume that $\mathrm{D}\Theta_P(\mathrm{T}_PX\cap\mathrm{T}_PY)=\{0\}\times\mathbb{R}^{m-k-l}\times\{0\}\subset\mathbb{R}^m$.

Next we consider the projections

$$\begin{array}{rcl} p_1\colon \mathbb{R}^m=\mathbb{R}^l\times\mathbb{R}^{m-l} & \to & \mathbb{R}^l \\ (x_1,x_2) & \mapsto & x_1 \end{array} \qquad\text{and}\qquad \begin{array}{rcl} p_2\colon \mathbb{R}^m=\mathbb{R}^l\times\mathbb{R}^{m-l} & \to & \mathbb{R}^{m-l} \\ (x_1,x_2) & \mapsto & x_2. \end{array}$$

Since P is a transverse intersection point we obtain from the elementary Lemma 42.2 that $p_2\colon \mathrm{T}_P(\Theta(Y\cap A))\to\mathbb{R}^{m-l}$ is an isomorphism. It follows from the Inverse Mapping Theorem 24.1 that $p_2\colon\Theta(Y\cap A)\to\mathbb{R}^{m-l}$ is a local diffeomorphism at 0. This implies that there exists an open neighborhood B_2 of $0\in\mathbb{R}^{m-l}$ such that the restriction of p_2 to $\Theta(Y\cap A)\cap(\mathbb{R}^l\times B_2)\to B_2$ is a diffeomorphism. We denote by q_2 the inverse. We pick an open subset $B_1\subset\mathbb{R}^l$ such that $B_1\times B_2\subset B$ and we consider the map

$$\begin{array}{rcl} \Omega\colon B_1\times B_2 & \to & \mathbb{R}^l\times\mathbb{R}^{m-l} \\ (x_1,x_2) & \mapsto & (x_1-(p_1\circ q_2)(x_2),x_2). \end{array}$$

Claim. The map

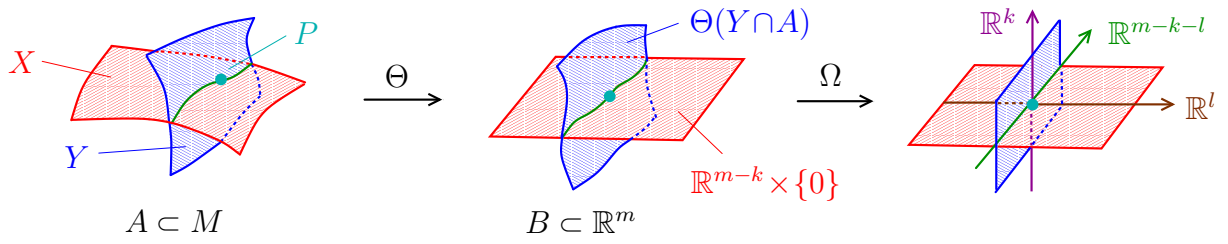
$$\Phi \;:=\; \Omega\circ\Theta\colon \Theta^{-1}(B_1\times B_2) \;\to\; \Omega(B_1\times B_2)$$

has all the desired properties.

Proof. One can easily verify that an inverse to Ψ is given by

$$\begin{array}{rcl} \Psi\colon \Omega(B_1\times B_2) & \to & \Theta^{-1}(B_1\times B_2) \\ (x_1,x_2) & \mapsto & \Theta^{-1}(x_1+(p_1\circ q_2)(x_2),x_2). \end{array}$$

This shows that Φ is a bijection. Furthermore one can easily verify that the differential of Φ at each point is invertible. Thus it follows from the Inverse Mapping Theorem 24.1 together with the Smooth Pasting Proposition 19.23 (2) that Φ is actually a diffeomorphism. This shows that Φ is a chart of type (i). Finally note that by construction we have $\Phi(U\cap X)=V\cap(\mathbb{R}^{m-k}\times\{0\})$ and $\Phi(U\cap Y)=V\cap(\{0\}\times\mathbb{R}^{m-l})$. ⊞



(2) (a) The charts constructed in (1) define, after possibly reordering the coordinates, submanifold charts of type (α) and (β) for $X \cap Y \subset M$. Since X and Y are closed the intersection $X \cap Y$ is also closed. This shows that $X \cap Y$ is a proper submanifold of codimension $k+l$ of M.
A slight modification of the above argument shows that $X \cap Y$ is also a proper submanifold of X and of Y.

(b) Let $P \in X \cap Y$. By (2a) we know that $X \cap Y$ is a proper submanifold of M of codimension $k+l$. Thus $\mathrm{T}_P(X \cap Y)$ is a vector space of dimension $m-k-l$. Since P is a transverse intersection we obtain from an elementary linear algebra argument that $\mathrm{T}_P X \cap \mathrm{T}_P Y$ is also of dimension $m-k-l$. As discussed in (2a), $X \cap Y$ is a proper submanifold of X and Y, thus $\mathrm{T}_P(X \cap Y) \subset \mathrm{T}_P X \cap \mathrm{T}_P Y$. Since the dimensions are equal we see that the vector spaces are the same.

(c) (i) We leave it to the reader to go through the dull task of verifying that the given recipe defines an orientation on $X \cap Y$.

(ii) In the Transversal Preimage Theorem 42.3 (3b) we will prove a generalization of this statement. The proof is a little delicate. To avoid lengthy repetitions we refer the reader to the later argument.

(d) By symmetry we can assume that Y is compact. Since X is a proper submanifold of M it is in particular a closed subset of M. By definition of the subspace topology on Y we see that $X \cap Y$ is a closed subset of the compact space Y. By the Compact-Closed Lemma 1.21 (1) this implies that $X \cap Y$ itself is compact. ■

42.2. **The Transversal Preimage Theorem.** The following definition can be viewed as generalization of the definition from page 968.

Definition. Let M be a smooth manifold and let X be a proper submanifold of M. Let Y be some other smooth manifold and let $f\colon Y \to M$ be a smooth map such that $f^{-1}(\partial M)$ is a union of components of ∂Y.

(1) Let $Q \in Y$ with $f(Q) \in X$. We say X intersects f transversally in Q if

$$\mathrm{D}f_Q(\mathrm{T}_Q Y) + \mathrm{T}_{f(Q)} X \;=\; \mathrm{T}_{f(Q)} M.$$

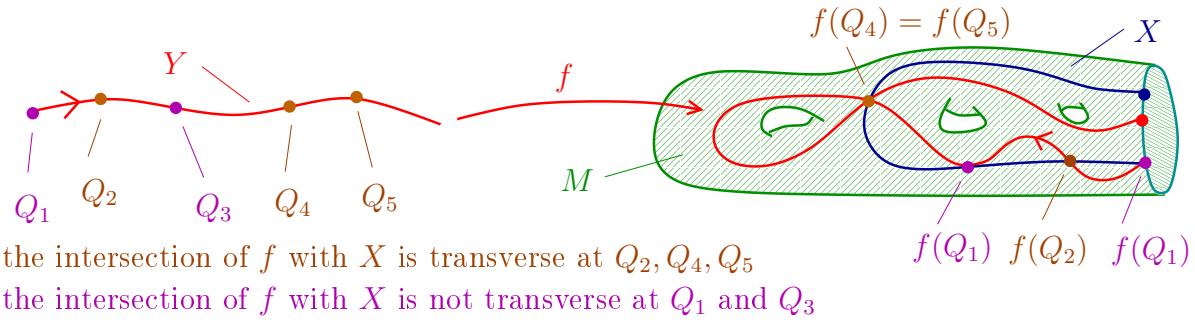
If $Q \in \partial Y$ then we also demand that $f(Q) \in \partial M$ and that [410]

$$\mathrm{D}f_Q(\mathrm{T}_Q(\partial Y)) + \mathrm{T}_{f(Q)}(\partial X) \;=\; \mathrm{T}_{f(Q)}(\partial M).$$

(2) Let C be a subset of Y. We say X intersects f transversally on C if for every $Q \in C$ with $f(Q) \in X$ the submanifold X intersects the map f transversally in Q.

(3) We say X intersects f transversally if for every $Q \in Y$ with $f(Q) \in X$ the submanifold X intersects the map f transversally in Q.

(4) We abbreviate the expression "X intersects f transversally" by $X \pitchfork f$.



Examples.

(1) Let M be a smooth manifold and let X and Y be two proper submanifolds of M. We denote by $i\colon Y \to M$ the inclusion map. It follows immediately from the definitions that i intersects X transversally if and only if X intersects Y transversally in the sense of the definition on page 968.

(2) Let Y and M be smooth manifolds and let $f\colon Y \to M$ be a smooth map. Furthermore let $x \in M \setminus \partial M$. By definition the map f is transverse to the 0-dimensional submanifold $\{x\}$ if and only if x is a regular value of f, as defined on page 701.

The following theorem might ring a bell. Considering the above examples we see that it is quite close to the statement of the Submanifold Transversal Intersection Theorem 42.1 (2) and to the statement of the Regular Value Theorem 26.4.

Theorem 42.3. (Transversal Preimage Theorem) Let M be a smooth manifold and let X be a proper submanifold of M of codimension l. Furthermore let Y be some other smooth manifold and let $f\colon Y \to M$ be a smooth map such that $f^{-1}(X \setminus \partial X) \subset Y \setminus \partial Y$ and such that $f^{-1}(\partial M)$ is a union of components of ∂Y. We denote by $f_\partial\colon \partial Y \to \partial M$ the restriction of f to ∂Y. If X intersects f transversely, i.e. if $X \pitchfork f$, then the following statements hold:

(1) The preimage $f^{-1}(X)$ is a proper submanifold of Y of codimension l where the boundary $\partial(f^{-1}(X))$ is given by $(f_\partial)^{-1}(X) = (f_\partial)^{-1}(\partial X)$.

(2) For every $P \in f^{-1}(X)$ we have

$$\mathrm{T}_P(f^{-1}(X)) \;=\; (\mathrm{D}f_P)^{-1}(\mathrm{T}_{f(P)}X).$$

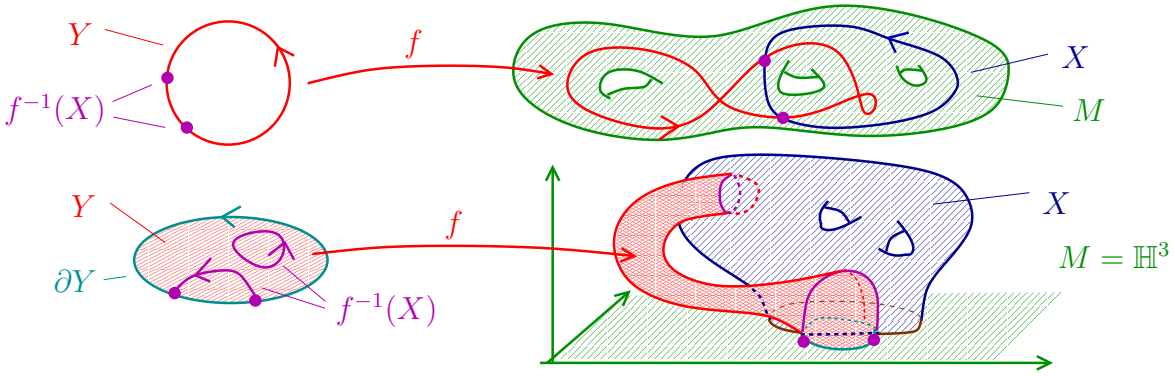
(3) Let M, X and Y be oriented. Furthermore let $P \in Y$.

(a) By hypothesis the homomorphism $\mathrm{D}f_P\colon \mathrm{T}_PY \to \mathrm{T}_{f(P)}M$ intersects T_PX transversally. Thus we can equip $\mathrm{T}_P(f^{-1}(X)) \;=\; (\mathrm{D}f_P)^{-1}(\mathrm{T}_{f(P)}X)$ with the preimage orientation that we introduced on page 673. These orientations define an orientation on the smooth manifold $f^{-1}(X)$.

(b) The equality $\partial(f^{-1}(X)) = (f_\partial)^{-1}(\partial X)$ is an equality of oriented smooth manifolds.[411]

[410] If $Q \in \partial Y$ with $f(Q) \in \partial X$, then by hypothesis the component C of ∂Y that contains Q satisfies $f(C) \subset \partial M$. It follows from the hypothesis that f is smooth together with the Smooth Manifold Boundary Proposition 19.26 (2a) and (2c) that f restricts to a smooth map $f\colon C \to \partial M$. Thus we obtain that $\mathrm{D}f_Q(\mathrm{T}_Q(\partial Y)) = \mathrm{D}f_Q(\mathrm{T}_Q C)$ is a vector subspace of $\mathrm{T}_{f(Q)}(\partial M)$.

(4) If Y is compact, then so is $f^{-1}(X)$.



Remark.

(1) The preimage orientation that we introduced on page 673 is defined via the following diagram

$$\begin{array}{ccccc} \mathrm{T}_{f(P)}X & \hookrightarrow & \mathrm{T}_{f(P)}M & \twoheadrightarrow & \mathrm{T}_{f(P)}M/\mathrm{T}_{f(P)}X \\ & & & & \cong\uparrow{\scriptstyle [v]\mapsto[\mathrm{D}f_P(v)]} \\ \mathrm{T}_P(f^{-1}(X)) & \hookrightarrow & \mathrm{T}_PY & \twoheadrightarrow & \mathrm{T}_PY/\mathrm{T}_P(f^{-1}(X)). \end{array}$$

The orientation on $\mathrm{T}_P(f^{-1}(X))$ is the unique orientation that makes the orientations in the short exact sequences compatible[412] and the right vertical map orientation-preserving.

(2) Let X and Y be proper submanifolds of a smooth submanifold M. If we apply the Transversal Preimage Theorem 42.3 to the inclusion $i\colon Y \to M$, then we see that $i^{-1}(X) = X \cap Y$ is a proper submanifold of Y. It follows from an elementary argument, see Exercise 19.31 (a), that $i^{-1}(X)$ is also a proper submanifold of M. This discussion shows that we can recover the statement of the Submanifold Transversal Intersection Theorem 42.1 (2) from the Transversal Preimage Theorem 42.3.

(3) Suppose that in the setting of the Transversal Preimage Theorem 42.3 the submanifold X consists of a single point $\{x\}$. We equip this 0-dimensional manifold with the orientation given by $+1$. Inspecting the definitions one sees that the preimage orientation on $f^{-1}(x)$ that we just introduced agrees with the orientation on $f^{-1}(x)$ that we introduced in the Regular Value Theorem 26.4 (3).

Proof. Let M be an m-dimensional smooth manifold and let X be a proper submanifold of M of codimension l. Let Y be some other smooth manifold and let $f\colon Y \to M$ be a smooth map such that $f^{-1}(X \setminus \partial X) \subset Y \setminus \partial Y$ and such that $f^{-1}(\partial M)$ is a union of components of ∂Y.

(1) We need to show that the preimage $f^{-1}(X)$ is a proper submanifold of Y of codimension l with boundary $\partial(f^{-1}(X)) = (f_\partial)^{-1}(\partial X)$. First note that X is a closed subset

[411] It takes a while to figure out what orientations we are talking about. We give $f^{-1}(X)$ the orientation from (3a) and we give $\partial(f^{-1}(X))$ the boundary orientation from page 689. Furthermore we equip ∂X, ∂Y and ∂M with the boundary orientations. By definition of transverse intersection we also have $\partial X \pitchfork f_\partial$. Thus $(f_\partial)^{-1}(\partial(X))$ is a proper submanifold of ∂Y and we use (3a) to equip $(f_\partial)^{-1}(\partial X)$ with an orientation.

[412] We refer to page 671 for the definition of compatible orientations.

of M since X is a proper submanifold. Therefore $f^{-1}(X)$ is a closed subset of Y. Furthermore note that it follows from our hypothesis that X is a proper submanifold of M and our hypotheses on $f^{-1}(X\backslash\partial X)$ and on $f^{-1}(\partial M)$ that $(f_\partial)^{-1}(\partial X) = f^{-1}(X)\cap\partial Y$. Thus it remains to show that every point Q in $f^{-1}(X)$ admits a submanifold chart of type (α) or of type (β). To prove this statement it suffices to show that every point Q in $f^{-1}(X)$ admits an open neighborhood $\widetilde{Y} \subset Y$ such that $f^{-1}(X)\cap\widetilde{Y}$ is a proper submanifold of $\widetilde{Y}$.

Let $Q \in f^{-1}(X)$. We set $P := f(Q) \in X$. We need to distinguish the two cases that $P \in X \setminus \partial X$ and that $P \in \partial X$.

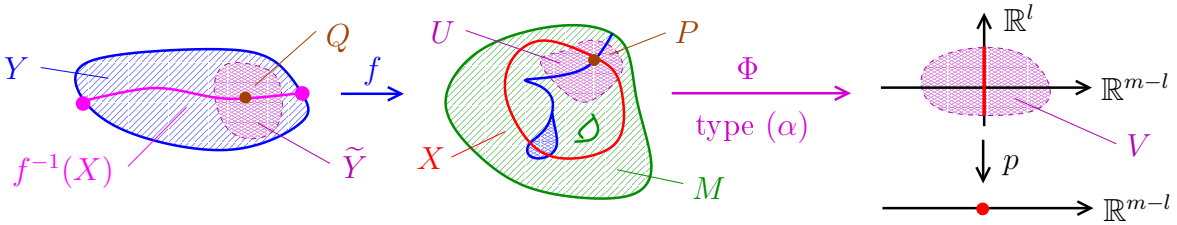
(a) First we consider the case that $P \in X \setminus \partial X$. In this case it follows from the hypothesis that X is a proper submanifold of M that there exists a submanifold chart $\Phi\colon U \to V$ of type (α) around P such that $\Phi(P) = 0$.
Since we assume that $f^{-1}(X \setminus \partial X) \subset Y \setminus \partial Y$ we see that $\widetilde{Y} := f^{-1}(U)\cap(Y \setminus \partial Y)$ is an open neighborhood of Q. Now it suffices to prove the following claim.

Claim. The subset $f^{-1}(X)\cap\widetilde{Y}$ is a proper submanifold of $\widetilde{Y}$.

Proof. We write $\widetilde{f} := f|_{\widetilde{Y}}$. Since Φ is a submanifold chart of type (α) we know that we have $\Phi(X\cap U) = \{0\}\times\mathbb{R}^l$. We denote by $p\colon \mathbb{R}^m = \mathbb{R}^{m-l}\times\mathbb{R}^l \to \mathbb{R}^{m-l}$ the natural projection. Note that

$$f^{-1}(X)\cap\widetilde{Y} \;=\; \widetilde{f}^{-1}(X)\cap\widetilde{Y} \;=\; \widetilde{f}^{-1}(X\cap U) \;=\; (p\circ\Phi\circ\widetilde{f})^{-1}(0).$$

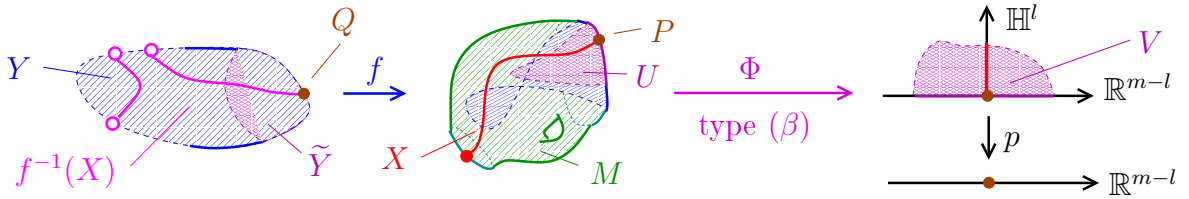
Note that the Regular Value Theorem 26.4 says that $(p\circ\Phi\circ\widetilde{f})^{-1}(0)$ is a proper submanifold of $\widetilde{Y}$ if 0 is a regular value of $p\circ\Phi\circ\widetilde{f}\colon \widetilde{Y} \to \mathbb{R}^{m-l}$. So it remains to show that 0 is a regular value of $p\circ\Phi\circ\widetilde{f}\colon \widetilde{Y} \to \mathbb{R}^{m-l}$. By definition of a regular value we need to show that each $Q' \in \widetilde{Y}$ is a regular point of $p\circ\Phi\circ\widetilde{f}$. By our clever choice of $\widetilde{Y}$ we know that $Q' \notin \partial\widetilde{Y}$. Therefore we only need to show that $\mathrm{D}(p\circ\Phi\circ\widetilde{f})^{-1}_{Q'}\colon \mathrm{T}_{Q'}\widetilde{Y} \to \mathrm{T}_0\mathbb{R}^{m-l}$ is an epimorphism. The fact that this map is indeed an epimorphism follows easily from our hypothesis that f is transverse to X together with Lemma 42.2. ⊞



(b) Now we consider the case that $P \in \partial X$. In this case it follows from the hypothesis that X is a proper submanifold of M that there exists a submanifold chart $\Phi\colon U \to V$ of type (β) around P such that $\Phi(P) = 0$. By our hypothesis on $f^{-1}(\partial M)$ we know that there exists a component C of ∂Y with $C \in \partial Y$ and with $f(C) \subset \partial M$. By the Smooth Manifold Boundary Proposition 19.26 we know that $\partial M \setminus C$ is a closed subset. Thus we see that $\widetilde{Y} := (f^{-1}(U)) \cap (M \setminus (\partial M\backslash))$ is an open neighborhood of Q. Now it suffices to prove the following claim.

Claim. The subset $f^{-1}(X) \cap \widetilde{Y}$ is a proper submanifold of $\widetilde{Y}$ with boundary $(f_\partial)^{-1}(\partial X) \cap \widetilde{Y}$.

Proof. We write $\widetilde{f} := f|_{\widetilde{Y}}$. Since Φ is a submanifold chart of type (β) we know that V is an open subset of $\mathbb{H}^m$ and that $\Phi(X \cap U) = \{0\} \times \mathbb{H}^l$. As in the previous claim we need to show that each $Q' \in \widetilde{Y}$ is a regular point of $p \circ \Phi \circ \widetilde{f}$.
Let $Q' \in \widetilde{Y}$. It follows as before that $\mathrm{D}(p \circ \Phi \circ \widetilde{f})^{-1}_{Q'} \colon \mathrm{T}_{Q'}\widetilde{Y} \to \mathrm{T}_0\mathbb{R}^{m-l}$ is an epimorphism. Now assume that $Q' \in \partial\widetilde{Y}$. From the definition of $\widetilde{Y}$ we see that $Q' \in C$, in particular $Q' \in \partial Y$ and $f(Q') \in \partial M$. It follows from the definition on page 971, that $\mathrm{D}(p \circ \Phi \circ \widetilde{f})^{-1}_{Q'} \colon \mathrm{T}_{Q'}(\partial\widetilde{Y}) \to \mathrm{T}_0\mathbb{R}^{m-l}$ is also an epimorphism. Thus we see that Q' satisfies all the conditions of a regular point as laid out on page 701. ⊞



(2) Let $Q \in f^{-1}(X)$. Evidently we have $(\mathrm{D}f_{f(Q)})^{-1}(\mathrm{T}_Q(f^{-1}(X))) \subset \mathrm{T}_{f(Q)}(X)$, which implies that $\mathrm{T}_Q(f^{-1}(X)) \subset (\mathrm{D}f_{f(Q)})^{-1}(\mathrm{T}_{f(Q)}(X))$. Both sides are vector spaces of dimension $m-k-l$, which implies that equality holds.

(3) (a) We leave the verification of this statement to the reader.

(b) We need to show that the two orientations on $\partial(f^{-1}(X)) = (f_\partial)^{-1}(\partial X)$ agree. Thus let $P \in \partial f^{-1}(X)$ we write $Q = f(P)$. For orientation it might help to consider the following diagram

$$\begin{array}{ccccc}
\mathrm{T}_Q X & \longrightarrow & \mathrm{T}_Q M & \longrightarrow & \mathrm{T}_Q M/\mathrm{T}_Q X \\
 & & & & \uparrow{\scriptstyle \mathrm{D}f_P} \\
\mathrm{T}_P(f^{-1}(X)) & \longrightarrow & \mathrm{T}_P Y & \longrightarrow & \mathrm{T}_P Y/\mathrm{T}_P(f^{-1}(X)) \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle \cong} \\
\mathrm{T}_P((f_\partial)^{-1}(\partial X)) & \longrightarrow & \mathrm{T}_P(\partial Y) & \longrightarrow & \mathrm{T}_P(\partial Y)/\mathrm{T}_P((f_\partial)^{-1}(\partial X))
\end{array}$$

In the following we make a few preparations.

(i) The orientations on $\mathrm{T}_Q X$ and $\mathrm{T}_Q M$ define an orientation on $\mathrm{T}_Q M/\mathrm{T}_Q X$. We pick vectors $v_1, \dots, v_l \in \mathrm{T}_Q M/\mathrm{T}_Q X$ which represent this orientation.

(ii) We pick $\widetilde{v}_1, \dots, \widetilde{v}_l \in \mathrm{T}_P Y$ such that $\mathrm{D}f_P(\widetilde{v}_i) = v_i$ for $i = 1, \dots, l$.

We use the following orientations:

$\mathrm{T}_P(f^{-1}(X))$	with the orientation from (3a) applied to $f \colon Y \to M$
$\mathrm{T}_P(\partial(f^{-1}(X)))$	with the boundary orientation from page 689
$\mathrm{T}_P((f_\partial)^{-1}(X))$	with the orientation from (3a) applied to $f_\partial \colon \partial Y \to M$
$\mathrm{T}_P Y$	with the given orientation
$\mathrm{T}_P(\partial Y)$	with the boundary orientation from page 689.

To simplify the notation we view $\mathrm{T}_P(f^{-1}(X))$ as a vector subspace of $\mathrm{T}_P Y$. We pick an outward pointing vector $o \in \mathrm{T}_P(f^{-1}(X))$. It follows from Exercise 23.8 that o, viewed as a vector in $\mathrm{T}_P Y$, is also outward pointing. Now let $u_1, \dots, u_d$ be a basis of $\mathrm{T}_P(\partial(f^{-1}(X))) = \mathrm{T}_P(f_\partial)^{-1}(\partial X)$. We see:

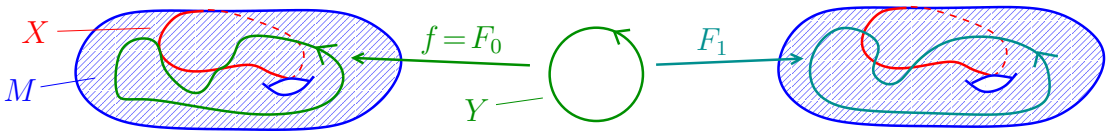
$$\begin{array}{lll} & u_1, \dots, u_d & \text{is a positive basis of } \mathrm{T}_P\left(\partial(f^{-1}(X))\right) \\ \iff & o, u_1, \dots, u_d & \text{is a positive basis of } \mathrm{T}_P(f^{-1}(X)) \\ \iff & o, u_1, \dots, u_d, \widetilde{v}_1, \dots, \widetilde{v}_l & \text{is a positive basis of } \mathrm{T}_P Y \\ \iff & u_1, \dots, u_d, \widetilde{v}_1, \dots, \widetilde{v}_l & \text{is a positive basis of } \mathrm{T}_P\left(\partial Y\right) \\ \iff & u_1, \dots, u_d & \text{is a positive basis of } \mathrm{T}_P((f_\partial)^{-1}(\partial X)). \end{array}$$

(4) Suppose that Y is compact. Since X is a proper submanifold of M it is in particular a closed subset of M. Therefore $f^{-1}(X)$ is a closed subset of the compact space Y. By the Compact-Closed Lemma 1.21 (1) this implies that $f^{-1}(X)$ itself is compact. ■

42.3. **The Transversality Theorem.** Our goal of this section is to state the Transversality Theorem. For the reader's convenience we first state a simplified version before we state the bells-and-whistles version.

Theorem 42.4. (Transversality Theorem) Let M be a smooth manifold with $\partial M = \varnothing$ and let X be a proper submanifold of M. Furthermore let Y be a compact smooth manifold and let $f\colon Y \to M$ be a smooth map.

(1) There exists a smooth homotopy F from $F_0 = f\colon Y \to M$ to a map $F_1\colon Y \to M$ that is transverse to X.
(2) If f is a smooth embedding, then we can choose F to be a smooth isotopy.



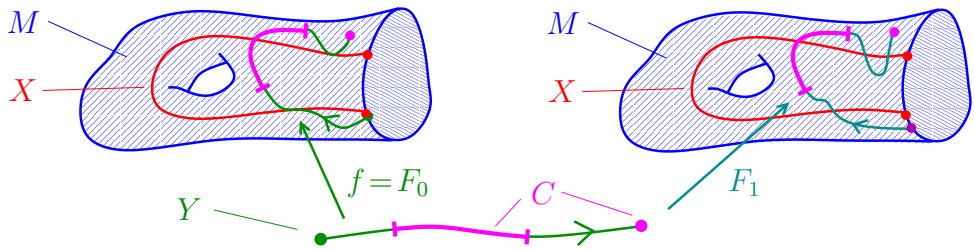
We postpone the proof of the Transversality Theorem 42.4 to Section 42.4. We move straight on to a more general statement.

Theorem 42.5. (Transversality Theorem II) Let M be a smooth manifold and let X be a proper submanifold of M. Furthermore let Y be a compact smooth manifold and let $f\colon Y \to M$ be a smooth map such that $f^{-1}(\partial M)$ is a union of components of ∂Y. We assume that $X \pitchfork f$ on $\partial Y \setminus f^{-1}(\partial M)$. Finally let C be a closed subset of Y such that $X \pitchfork f$ on C.

(1) There exists a smooth homotopy $F\colon Y \times [0,1] \to M$ with the following properties:
  (a) F is a smooth homotopy *rel* C from $F_0 = f$ to a map F_1 with $X \pitchfork F_1$.
  (b) The set of $t \in [0,1]$ such that $X \pitchfork F_t$ has full measure.
  (c) For each $t \in [0,1]$ we have $F_t^{-1}(\partial M) = f^{-1}(\partial M)$.[413]
(2) If f is a smooth embedding, then we can choose F to be a smooth isotopy.

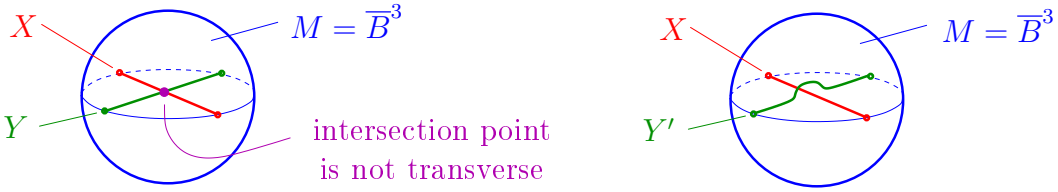
Remark.

[413] In particular if f is proper, i.e. if $f^{-1}(\partial M) = \partial Y$, then for each t we have $F_t^{-1}(\partial M) = \partial Y$, i.e. each F_t is proper.



(1) Similar statements are also proved in [**Lee2002**, Theorem 6.36] and [**GP1974**, p. 70]. Most statements of the theorem also follow from Proposition 4.4.4, Theorem 4.5.6 and Proposition 4.5.7 of [**Wall2016**].[414]
(2) For many purposes the Avoid-Tubular Neighborhoods Proposition **??** can be a useful variation on the Transversality Theorem 42.4.

Example. Let M be a smooth manifold. Furthermore let X and Y be two proper submanifolds of M. If $\dim(X)+\dim(Y) < \dim(M)$, then one deduces easily from the definitions that X and Y intersect transversally if and only if X and Y are disjoint. With this observation the Transversality Theorem 42.4 (2) says in particular that any two proper 1-dimensional submanifolds X and Y of the 3-dimensional ball $\overline{B}^3$ can be made disjoint after a smooth isotopy. If the boundaries are already disjoint, then we can apply the Transversality Theorem 42.4 (2) with $C = \partial Y$ and we see that can make X and Y disjoint while keeping X and the boundary of Y fixed.



The following is a typical and often used consequence of the above transversality theorems.

Corollary 42.6. Let M be a connected n-dimensional smooth manifold and let X be a proper k-dimensional smooth submanifold of M. If $k \leq n-2$, then $M \setminus X$ is path-connected.

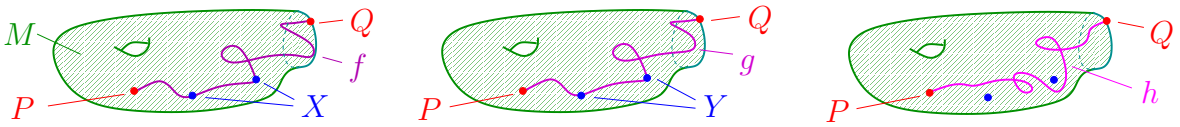
Proof. Let $P, Q \in M \setminus X$. Since M is connected we know by the Smooth Path-Connectivity Proposition 19.29 that there exists a smooth map $f\colon [0,1] \to M$ with $f(0) = P$ and $f(1) = Q$.

Using the Smooth Collar Neighborhood Theorem 28.3 one can show, see Exercise 28.5, that there exists in fact a smooth path $g\colon [0,1] \to M$ with $g(0) = P$ and $g(1) = Q$ and with $g^{-1}(\partial M) \subset \{0,1\}$.

Note that the fact that $P, Q \in M \setminus X$ implies that $X \pitchfork f$ on $\partial[0,1] = \{0,1\}$. Thus we can apply the Transversality Theorem 42.5 to the closed subset $C = \{0,1\}$ and we obtain a smooth map $h\colon [0,1] \to M$ with $h(0) = g(0) = f(0) = P$ and $h(1) = g(1) = f(1) = Q$ and

[414] A first look at [**Wall2016**, Theorem 4.5.6] might be slightly confusing since it demands that X is a "submanifold of $J^r(M,N)$" for some $r \in \mathbb{N}_0$. But as is explained on page 101 of [**Wall2016**], we have $J^0(M,Y) = M \times Y$, and $X \times \{*\}$ is clearly a submanifold of $M \times Y$ for any $* \in Y \setminus \partial Y$.

such that $X \pitchfork h$. But since $\dim(X) + \dim([0,1]) < \dim(M)$ the last condition just says that $X \cap h([0,1]) = \varnothing$. In other words, h is a path in $M \setminus X$ that connects P and Q. ■



We now turn to the proofs of the Transversality Theorems. In Section 42.4 we provide the proof of the Transversality Theorem 42.4. This proof has a neat idea and is quite readable. Afterwards in Section 42.5 we provide the proof of the Transversality Theorem 42.5. This proof is much more technical and arguably much less fun for the author and the reader.

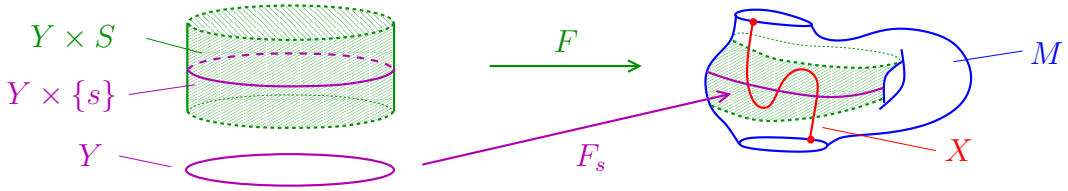
42.4. **Proof of the Transversality Theorem 42.4.** First we introduce or recall the following easy-to-digest notation.

Notation. Let $Y \times S \to M$ be a map. For each $s \in S$ we set

$$\begin{array}{rccc} F_s\colon & Y & \to & M \\ & y & \mapsto & F(y,s). \end{array}$$

The proofs of the Transversality Theorems 42.4 and 42.5 rest on the following proposition.

Proposition 42.7. (Parametric Transversality Proposition) Let Y and M be smooth manifolds and let $X \subset M$ be a proper smooth submanifold. Furthermore let S be a smooth manifold with empty boundary. Finally let $F\colon Y \times S \to M$ be a proper smooth map. If $F\colon Y \times S \to M$ is transverse to X, then for almost every[415] $s \in S$ the map $F_s\colon Y \to M$ is also transverse to X.



Proof of the Parametric Transversality Proposition 42.7. Since $F\colon Y \times S \to M$ is transverse to the proper submanifold $X \subset M$ we obtain from the Transversal Preimage Theorem 42.3 that $W := F^{-1}(X)$ is a submanifold of $Y \times S$. Let $\pi_S\colon Y \times S \to S$ be the obvious projection. By Sard's Theorem 31.1 we know that the set of regular values of the smooth maps $\pi_S\colon W \to S$ and $\pi_S\colon \partial W = \partial Y \times S \to S$ have full measure. Thus, according to the definition on page 971, it suffices to show the following claim.

Claim.

(1) If $s \in S$ is a regular value of $\pi_S|_W\colon W \to S$, then for any $Q \in Y$ with $F_s(Q) \in X$ we have $\mathrm{T}_{F_s(Q)}X + \mathrm{D}(F_s)_Q(\mathrm{T}_Q Y) = \mathrm{T}_{F_s(Q)}M$.

(2) If $s \in S$ is a regular value of $\pi_S|_{\partial W}\colon \partial W \to S$, then for any $Q \in \partial Y$ with $F_s(Q) \in \partial X$ we have $\mathrm{T}_{F_s(Q)}\partial X + \mathrm{D}(F_s)_Q(\mathrm{T}_Q \partial Y) = \mathrm{T}_{F_s(Q)}\partial M$.

[415] Recall that, according to the definition on page 762, the expression "almost every" means that the set of points $s \in S$ for which F_s is transverse to X has full measure.

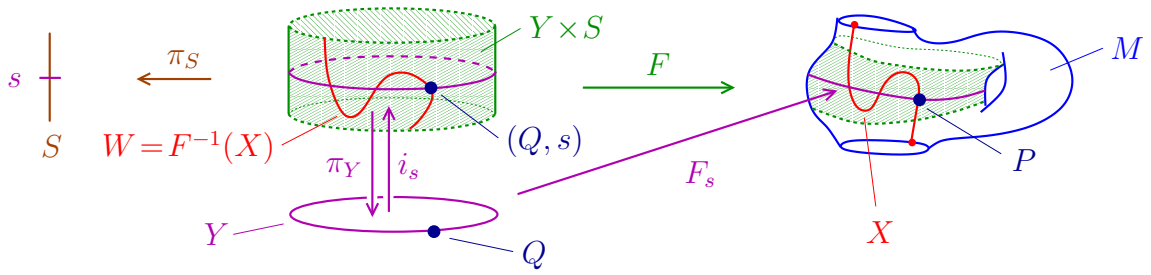
In other words, if $s \in S$ is a regular value of $\pi_S|_W \colon W \to S$ and if $s \in S$ is a regular value of $\pi_S|_{\partial W} \colon \partial W \to S$, then $F_s \colon Y \to M$ is transverse to X.

Proof. We prove Statement (1). The proof of Statement (2) is basically identical. Thus let $s \in S$ be a regular value of $\pi_S|_W \colon W \to S$ and let $Q \in F_s^{-1}(X)$. We set $P := F_s(Q) \in X$. We need to show that $\mathrm{T}_P X + \mathrm{D}(F_s)_Q(\mathrm{T}_Q Y) = \mathrm{T}_P M$. We consider the following diagram

$$\begin{array}{ccccc}
\mathrm{T}_{(Q,s)}W \oplus \mathrm{T}_{(Q,s)}(Y \times \{s\}) & \xrightarrow{\cong} & \mathrm{T}_{(Q,s)}(Y \times S) & \xrightarrow{\mathrm{D}(\pi_Y)_{(Q,s)} \oplus \mathrm{D}(\pi_S)_{(Q,s)}} & \mathrm{T}_Q Y \oplus \mathrm{T}_s S \\
\downarrow{\scriptstyle \mathrm{D}F_{(Q,s)}} & & \downarrow{\scriptstyle \mathrm{D}F_{(Q,s)}} & & \downarrow{\scriptstyle \mathrm{D}F_{(Q,s)}} \\
\mathrm{T}_P X \oplus \mathrm{D}(F_s)_Q(\mathrm{T}_Q Y) & & \longrightarrow & & \mathrm{T}_P M \\
\uparrow{\scriptstyle \mathrm{D}(F_s)_Q} & & & & \downarrow \\
\mathrm{T}_Q Y & & \xrightarrow{\mathrm{D}(F_s)_Q} & & \mathrm{T}_P M / \mathrm{T}_P X.
\end{array}$$

We make the following clarifications and observations:

(1) The top right horizontal map is induced by the projection $\pi_Y \colon Y \times S \to Y$ and the projection $\pi_S \colon Y \times S \to S$.
(2) The undecorated maps are either given by inclusion maps or the natural projections.
(3) It is clear that the diagram commutes.
(4) Since s is a regular value of $\pi_S|_W \colon W \to S$ one sees easily that the composition of the top two horizontal maps is an epimorphisms. Since all vector spaces on the top row are of the same dimension we see that the horizontal map to the top left is actually an isomorphism.
(5) Recall that by hypothesis the map $F \colon Y \times S \to M$ is transverse to X. Thus we know that $\mathrm{T}_P X + \mathrm{D}F_{(Q,s)}(\mathrm{T}_{(Q,s)}(Y \times S)) = \mathrm{T}_P M$. This implies that the composition of the middle two vertical maps is an epimorphism.
(6) By the Transversal Preimage Theorem 42.3 we have $\mathrm{T}_{(Q,s)}W = (\mathrm{D}F_{(Q,s)})^{-1}(\mathrm{T}_P X)$, which implies that $\mathrm{D}F_{(Q,s)}(\mathrm{T}_{(Q,s)}W) \subset \mathrm{T}_P X$.
(7) Evidently we have $\mathrm{D}F_{(Q,s)}(\mathrm{T}_{(Q,s)}(Y \times \{s\})) = \mathrm{D}(F_s)_Q(\mathrm{T}_Q Y)$.
(8) It follows from the above that the map $\mathrm{D}(F_s) \colon \mathrm{T}_Q Y \to \mathrm{T}_P M / \mathrm{T}_P X$ is an epimorphism. But that implies of course that $\mathrm{T}_P X + \mathrm{D}(F_s)_Q(\mathrm{T}_Q Y) = \mathrm{T}_P M$, which is exactly what we needed to show. ∎



Proof of the Transversality Theorem 42.4. Let M be a smooth manifold such that $\partial M = \varnothing$ and let X be a proper submanifold of M. Furthermore let Y be a compact smooth manifold and let $f \colon Y \to M$ be a smooth map. First we need to show that there

exists a smooth homotopy F from $F_0 = f\colon Y \to M$ to a map $F_1\colon Y \to M$ that is transverse to X. To do so, we make the following preparations:

(1) By the Euclidean Embedding Theorem 27.1 and the Smooth Embedding Theorem 24.10 we can assume that M is a proper smooth submanifold of some $\mathbb{R}^n$.
(2) By the Neighborhood Projection Theorem 28.1 there exists an open neighborhood $N(M)$ of $M \subset \mathbb{R}^n$ and a smooth map $\pi\colon N(M) \to M$ which has the following two properties:
   (a) π is a retraction,
   (b) π is a submersion, i.e. for each $Q \in N(M)$ the differential $\mathrm{D}\pi_Q$ is an epimorphism.

Since Y is compact we obtain from the Compact Image Lemma 2.13 together with the elementary Exercise 6.1 that there exists an $r > 0$ such that for each $y \in Y$ we have $B^n_r(f(y)) \subset N(M)$. Therefore we can consider the map

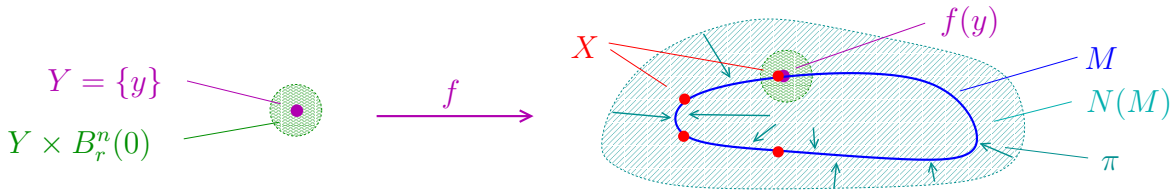
$$\begin{array}{rcl} F\colon Y \times B^n_r(0) & \to & M \\ (y, v) & \mapsto & \pi(f(y) + v). \end{array}$$

Evidently the map $Y \times B^n_r(0) \to N(M)$ given by $(y, v) \mapsto f(y) + v$ is a submersion. Furthermore by (2b) we know that $\pi\colon N(M) \to M$ is a submersion. This discussion shows that for each $(y, v) \in Y \times B^n_r(0)$ the map $\mathrm{D}F_{(y,v)}$ is an epimorphism. This implies that $F\colon Y \times B^n_r(0) \to M$ is transverse to X.

By the Parametric Transversality Proposition 42.7 there exists a $v \in B^n_r(0)$ such that $F_v\colon Y \to M$ is transverse to X. The desired smooth homotopy is now given by the map

$$\begin{array}{rcl} H\colon Y \times [0,1] & \to & M \\ (y, t) & \mapsto & F_{t\cdot v}(y) = F(y, t \cdot v) = \pi(f(y) + t \cdot v). \end{array}$$

Now assume that f is a smooth embedding. Since $F_0 = f$ it follows from the Deform Smooth Embedding Proposition 33.6 (1) that, after possibly shrinking r, we can assume that for each $v \in B^n_r(0)$ the map $F_v\colon Y \to M$ is a smooth embedding. In this setting the above map H is now actually a smooth isotopy. ∎



42.5. **Proof of the Transversality Theorem 42.5.** In hindsight the proof of the Transversality Theorems 42.4 was not all that bad. The proof of the considerably more elaborate Transversality Theorem 42.5 will be, well, considerably more elaborate.

We start out with the following lemma, which is of independent interest.

Lemma 42.8. Given any $m, n, r \in \mathbb{N}_0$ the set

$$\{A \in \mathrm{M}(m \times n, \mathbb{R}) \mid \mathrm{rank}(A) \geq r\}$$

is an open subset of $\mathrm{M}(m \times n, \mathbb{R})$.

Remark. Lemma 42.8 is a slight generalization of Exercise 2.41.

Proof. Given $I \subset \{1,\dots,m\}$ and $J \subset \{1,\dots,n\}$ with $\#I = r$ and $\#J = r$ we consider the map

$$\begin{array}{rccl} \det_{I,J}\colon & \mathrm{M}(m\times n,\mathbb{R}) & \to & \mathbb{R} \\ & A=(a_{ij}) & \mapsto & \text{determinant of the } r\times r\text{-matrix } (a_{ij})_{i\in I, j\in J}. \end{array}$$

It follows from the Leibniz formula that $\det_{I,J}$ is continuous. Using this notation we can formulate the following claim.

Claim. Let $A = (a_{ij}) \in \mathrm{M}(m \times n, \mathbb{R})$. The following two statements are equivalent:

(1) $\mathrm{rank}(A) \geq r$.
(2) There exist subsets $I \subset \{1,\dots,m\}$ and $J \subset \{1,\dots,n\}$ with $\#I = r$ and $\#J = r$ such that $\det_{I,J}(A) \neq 0$.

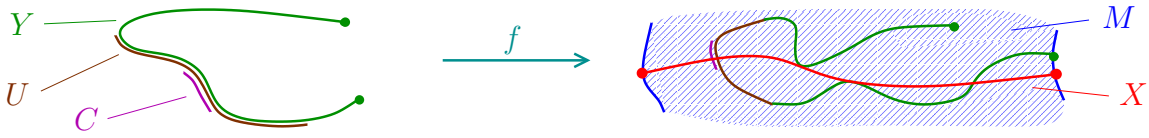
Proof. We leave it to the reader to pull out their favorite linear algebra book and to prove the claim. ⊞

It follows from the claim that

$$\{A \in \mathrm{M}(m\times n,\mathbb{R}) \mid \mathrm{rank}(A) \geq r\} \;=\; \bigcup_{\substack{I,J \text{ with} \\ \#I=r \text{ and } \#J=r}} (\det_{I,J})^{-1}(\mathbb{R}\setminus\{0\}).$$

Since each $\det_{I,J}$ is continuous we see that the set to the left is indeed open. ■

Lemma 42.9. (Transversality-Extends-to-Neighborhood Lemma) Let M be a smooth manifold and let X be a proper submanifold of M. Furthermore let Y be a smooth manifold and let $f\colon Y \to M$ be a smooth map such that $f^{-1}(\partial M) = \partial Y$. If C is a closed subset of Y such that $X \pitchfork f$ on C, then there exists an open neighborhood U of C such that $X \pitchfork f$ on U.



Proof. It suffices to show that for each $P \in C \cap f^{-1}(X)$ there exists an open neighborhood V of $P \in Y$ such that $X \pitchfork f$ on V.

First we consider the case that $P \notin \partial Y$. After using charts of type (i) for $P \in Y$ and for $f(P) \in X$ we can assume that Y is an open subset of $\mathbb{R}^l$, that M is an open subset of $\mathbb{R}^m$ and that $X = \{0\} \times \mathbb{R}^k$. The fact that $X \pitchfork f$ on P implies that the $(m \times l + k)$-matrix

$$(\mathrm{D}f_P \; e_{m-k+1} \; \dots \; e_m)$$

has rank m. It follows from Lemma 42.8 that this condition is satisfied in an open neighborhood of P.

The case that $P \in \partial Y$ is dealt with in almost the same way. In this case we use a chart of type (ii) for $P \in \partial Y$ and for $f(P) \in \partial X$. With the same argument as above one can find an open neighborhood V_1 of $P \in Y$ that satisfies the first condition for transversality and one can find an open neighborhood V_2 of $P \in \partial Y$ that satisfies the second condition for transversality of points on the boundary. On the open neighborhood $V = V_1 \cap (M \setminus (\partial M \setminus V_2))$ we now have $X \pitchfork f$. ■

Now we turn to actual proof of the Transversality Theorem 42.5. The following lemma implies in particular the Transversality Theorem 42.5 if $\partial Y = \varnothing$.

Lemma 42.10. Let M be a smooth manifold and let X be a proper submanifold of M. Furthermore let Y be a compact smooth manifold, let $f\colon Y \to M$ be a smooth map and let C be a closed subset of Y such that $X \pitchfork f$ on C. We assume that $f^{-1}(\partial M) \subset C$.

(1) There exists a smooth homotopy F *rel* C from $F_0 = f$ to a map F_1 with $F_1 \pitchfork X$ with the following properties:
   (a) F is a smooth homotopy *rel* C from $F_0 = f$ to a map F_1 with $X \pitchfork F_1$.
   (b) The set of $t \in [0,1]$ such that $X \pitchfork F_t$ has full measure.
   (c) For each $t \in [0,1]$ we have $F_t^{-1}(\partial M) = f^{-1}(\partial M)$.[416]

(2) If f is a smooth embedding, then we can choose F to be a smooth isotopy.

Proof. Let M be a smooth manifold and let X be a proper submanifold of M. Furthermore let Y be a compact smooth manifold, let $f\colon Y \to M$ be a smooth map and let C be a closed subset of Y such that $X \pitchfork f$ on C and such that $f^{-1}(\partial M) \subset C$.

By the Transversality-Extends-to-Neighborhood Lemma 42.9 there exists an open neighborhood U of $C \subset Y$ such that $X \pitchfork f$ on U. By the Manifold Paracompact-Normal Proposition 18.9 (2) we know that Y is normal. Using the Normal-Open-Closed-Neighborhood Lemma 8.2 twice we see that there exist open subsets V, W of Y with $C \subset W \subset \overline{W} \subset V \subset \overline{V} \subset U$.

Since $\overline{V}$ and $Y \setminus U$ are disjoint closed subsets of Y we obtain from Urysohn's Lemma 21.7 for Manifolds that there exists a smooth map $\tau\colon Y \to \mathbb{R}$ with $\tau|_{\overline{V}} \equiv 0$ and $\tau|_{Y\setminus U} \equiv 1$. In the following we proceed similar to the proof on page 979.

(1) By the Euclidean Embedding Theorem 27.1 and the Smooth Embedding Theorem 24.10 we can assume that M is a proper smooth submanifold of some $\mathbb{H}^n$.

(2) By hypothesis we have $f^{-1}(\partial M) \subset C$. It follows that $f(Y \setminus W) \subset M \setminus \partial M$. Since Y is compact we know by the Compact Image Lemma 2.13 that $f(Y \setminus W)$ is a closed subset of M that is disjoint from ∂M. We now know by Exercise 28.9, which is a mild generalization of the Neighborhood Projection Theorem 28.1, that there exists an open neighborhood N of $f(Y \setminus W) \subset \mathbb{R}^n$ with $N \cap \partial M = \varnothing$ and a smooth map $\pi\colon N \to M$ which has the following two properties:
   (a) for each $P \in f(Y \setminus W)$ we have $\pi(P) = P$,
   (b) π is a submersion, i.e. for each $Q \in N(M)$ the differential $\mathrm{D}\pi_Q$ is an epimorphism.

Since $Y \setminus W$ is compact we obtain from the Compact Image Lemma 2.13 together with the elementary Exercise 6.1 that there exists an $r > 0$ such that for each $y \in f(Y \setminus W)$ we have $B_r^n(f(y)) \subset N(M)$. Therefore we can consider the map

$$\begin{array}{rcl} F\colon Y \times B_r^n(0) & \to & M \setminus \partial M \\ (y, v) & \mapsto & \begin{cases} \pi(f(y) + \tau(y) \cdot v), & \text{if } y \in Y \setminus \overline{W}, \\ y, & \text{if } y \in \overline{W}. \end{cases} \end{array}$$

Since $\tau|_{\overline{V}} \equiv 0$ we see that on the open set $V \times B_r^n(0)$ the map F is just given by the smooth map $(y, v) \mapsto y$. Furthermore by construction F is smooth on $(Y \setminus \overline{W}) \times B_r^n(0)$. Since F is smooth on open sets that cover $Y \times B_r^n(0)$ we see that F is indeed smooth.

Claim. We have $X \pitchfork F$.

Proof. Let $y \in Y$.

- If $\tau(y) = 0$, then by construction of τ we have $y \in U$. By the choice of U this implies that $X \pitchfork f$ at y. But since $\tau(y) = 0$ this implies that $X \pitchfork F$ at every (y, v).
- If $\tau(y) \neq 0$, then it follows from the fact that $f(y) + \tau(y) \cdot B^n_r(0) = B^n_{\tau(y)}(f(y))$ is an open neighborhood of $f(y) \in M \subset N(M) \subset \mathbb{R}^n$ and the fact that π is a submersion that for each $(y, v) \in Y \times B^n_r(0)$ the map $\mathrm{D}F_{(y,v)}$ is an epimorphism, i.e. $X \pitchfork F$ at (y, v). ⊞

The remainder of the argument is similar to the proof on page 979. By the Parametric Transversality Proposition 42.7, together with Exercise 30.2 (which is a consequence of the Mini-Fubini Lemma 30.11), there exists a $v \in B^n_r(0)$ such that $X \pitchfork F_v$, i.e. such that $F_v \colon Y \to M$ is transverse to X, and such that the following condition is satisfied:

$(*)$ The set of $t \in [0, 1]$ such that $X \pitchfork F_{t \cdot v}$ has full measure in $[0, 1]$.

Next we consider the map

$$\begin{array}{rcl} H \colon Y \times [0,1] & \to & M \\ (y, t) & \mapsto & F_{t \cdot v}(y) = F(y, t \cdot v). \end{array}$$

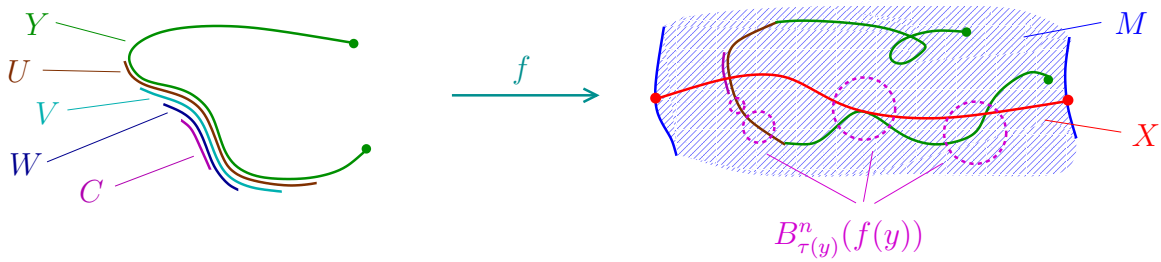
It remains to to verify that the map H has all the desired properties:

(a) Since F is smooth we see that H is smooth. By construction H is a homotopy from $H_0 = f$ to a map H_1 with $X \pitchfork H_1$. Note that for $y \in C$ we have, by the choice of τ, that $H(y, t) = y$ for all $t \in [0, 1]$. In other words, H is indeed a homotopy rel C.

(b) It follows from $(*)$ that the set of $t \in [0, 1]$ such that $X \pitchfork H_t$ has full measure.

(c) Since $N \cap \partial M = \varnothing$ and since $\partial M \subset C$ we see that for each $t \in [0, 1]$ we have $H_t^{-1}(\partial M) = f^{-1}(\partial M)$.

Finally assume that f is a smooth embedding. Since $F_0 = f$ it follows from the Deform Smooth Embedding Proposition 33.6 (1) that, after possibly shrinking r, we can assume that for each $s \in B^n_r(0)$ the map $F_s \colon Y \to M$ is a smooth embedding. In this setting the above map H is now actually a smooth isotopy. ■



Proof of the Transversality Theorem 42.5. Let M be a smooth manifold and let X be a proper submanifold of M. Furthermore let Y be a compact smooth manifold and let $f \colon Y \to M$ be a smooth map such that $f^{-1}(\partial M)$ is a union of components of ∂Y. To simplify the language and notation we actually assume that $f^{-1}(\partial M) = \partial Y$. Finally let C be a closed subset of Y such that $X \pitchfork f$ on C. Recall that we need to show the following:

(1) There exists a smooth homotopy $F \colon Y \times [0, 1] \to M$ with the following properties:
  (a) F is a smooth homotopy *rel* C from $F_0 = f$ to a map F_1 with $X \pitchfork F_1$.
  (b) The set of $t \in [0, 1]$ such that $X \pitchfork F_t$ has full measure.

(c) For each $t \in [0,1]$ we have $F_t^{-1}(\partial M) = \partial M$, i.e. F is a proper smooth homotopy.

(2) If f is a smooth embedding, then we can choose F to be a smooth isotopy.

We start out with the following claim.

Claim.

(1) There exists a proper smooth homotopy G rel C from f to a map $g\colon Y \to M$ such that $X \pitchfork g$ on $C \cup \partial Y$.

(2) If f is a smooth embedding, then we can choose G to be a smooth isotopy.

Proof. [417] We make the following preparations:

(a) We write $f_\partial := f|_{\partial Y}\colon \partial Y \to \partial M$.

(b) By the definition of transversality we know that the submanifold $\partial X \subset \partial M$ is transversal to the map $f_\partial\colon \partial Y \to \partial M$ on $\partial Y \cap C$.

(c) By the Transversality-Extends-to-Neighborhood Lemma 42.9 there exists an open neighborhood U of $\partial Y \cap C \subset \partial Y$ such that $f|_\partial \pitchfork f|_{\partial Y}$ on U.

(d) By the Manifold Paracompact-Normal Proposition 18.9 (2) we know that ∂Y is normal. By the Normal-Open-Closed-Neighborhood Lemma 8.2 this implies that there exists an open subset $V \subset \partial Y$ with $\partial Y \cap C \subset V \subset \overline{V} \subset U$.

(e) By the Smooth Collar Neighborhood Theorem 28.3 we can pick a collar $[0,1] \times \partial Y$ for Y and we can pick a collar $[0,1] \times \partial M$ for M.

(f) Since Y is compact, since $C \subset Y$ is closed and since $[0,1] \times \partial Y$ is closed we see that $C \cap ([0,1] \times \partial Y)$ is compact. Furthermore since $[0,1] \times \partial Y$ is a neighborhood of ∂Y and since V is a neighborhood of $C \cap \partial Y \subset \partial Y$ it follows from Exercise 5.7 that we can pick an $\epsilon > 0$ such that $C \cap ([0,\epsilon] \times \partial Y) = C \cap ([0,\epsilon] \times V)$.

(g) It follows from the Basics-of-Collars Lemma 28.4, and the compactness of ∂Y, that after possibly shrinking ϵ, we can also assume that $f([0,\epsilon]) \subset [0,1) \times \partial M$.

(h) We denote by $\pi\colon [0,1] \times \partial M \to [0,1]$ the projection onto the first factor.

(i) By the Smooth Transition Function Lemma 19.4 we can pick a smooth function $\eta\colon [0,\epsilon] \to \mathbb{R}$ such that $\eta(0) = 0$ and $\eta(t) = 1$ for $t \in [\frac{\epsilon}{2}, \epsilon]$ and we can pick a smooth function $\tau\colon [0,\epsilon] \to \mathbb{R}$ such that $\tau(0) = 1$ and $\tau(t) = 0$ for $t \in [\frac{\epsilon}{2}, \epsilon]$ and

Now we can provide the actual proof of the claim. Since $\partial\partial Y = \varnothing$ we know by Lemma 42.10 that there exists a smooth homotopy $F\colon \partial Y \times [0,1] \to \partial M$ rel $\partial Y \cap \overline{V}$ such that $\partial X \pitchfork F_1$ and such that the following statement holds:

($*$) The set of $t \in [0,1]$ with $\partial X \pitchfork F_t$ has full measure in $[0,1]$.

Now we consider the map

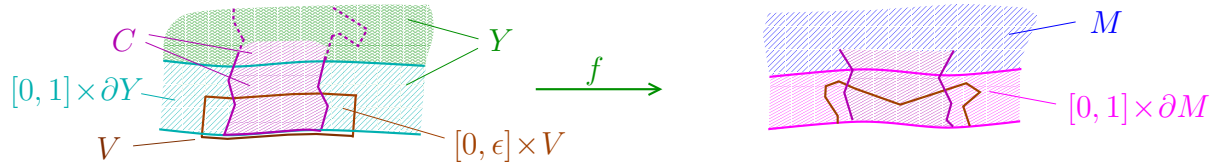
$$\begin{aligned} G\colon Y \times [0,1] &\to M \\ (y,t) &\mapsto \begin{cases} f(y), & \text{if } y \notin [0,\epsilon] \times \partial Y, \\ (\underbrace{F(\widetilde{y}, \tau(s))}_{\in \partial M}, \underbrace{\eta(f(y))}_{\in [0,1]}), & \text{if } y = (s,\widetilde{y}) \in [0,\epsilon] \times \partial Y. \end{cases} \end{aligned}$$

First note that by design the map $G\colon Y \times [0,1] \to M$ is smooth on the open subsets $(Y \setminus [0, \frac{\epsilon}{2}]) \times [0,1]$ and $([0,1) \times \partial Y) \times [0,1]$. It follows from the Smooth Pasting Proposition 19.23

[417] In many situations this claim can also be proved fairly easily using the Homotopic-to-Product-Near-Boundary Lemma 28.6 and Lemma 42.10 applied to ∂Y. But for the most general statements we have to get our hands dirty again.

that G is smooth. Note that this is a smooth homotopy with $G_0 = f$. Furthermore for each $t \in [0,1]$ we have $G_t^{-1}(\partial M) = \partial Y$. Next note that G is constant outside of $[0,\epsilon] \times Y$. Furthermore it is constant on $[0,\epsilon] \times \overline{V}$. By the choice of ϵ this means that G is constant on C. In other words, the above is the desired smooth homotopy.

If f is a smooth embedding then we can arrange, using $(*)$ and the Deform Smooth Embedding Proposition 33.6 (2), by rescaling the original map G that G is actually a smooth isotopy. ⊞



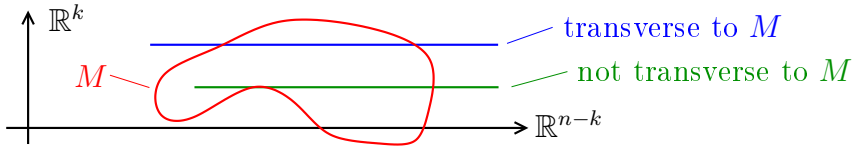
It follows from the claim and the Smooth Isotopy Transitivity Proposition 33.3 that it remains to consider the case that $f\colon Y \to M$ is a smooth map with $f^{-1}(\partial M) = \partial Y$ such that $X \pitchfork f$ on $C \cup \partial Y$, where C is a closed subset of Y. But this case we have dealt with in Lemma 42.10. ■

Exercises for Chapter 42.

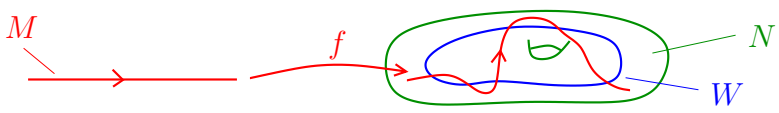
Exercise 42.1. Let $M \subset \mathbb{R}^n$ be a smooth submanifold. Show that the set

$$\{P \in \mathbb{R}^k \mid M \text{ is not transverse to the submanifold } \mathbb{R}^{n-k} \times \{P\}\}$$

has measure zero in $\mathbb{R}^k$.

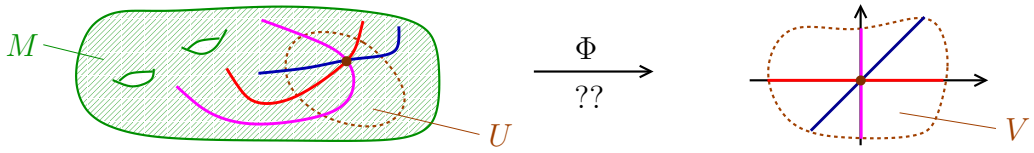


Exercise 42.2. Let M and N be two smooth manifolds with empty boundary. We suppose that $\dim(M) < \dim(N)$. Furthermore let W be a submanifold of N without boundary. Finally we suppose that for every point $P \in M$ with $f(P) \in W$ we have the equality $\mathrm{D}f_P(\mathrm{T}_P M) + \mathrm{T}_{f(P)}W = \mathrm{T}_{f(P)}N$. Show that $W \setminus f(M)$ has full measure in W.
Hint. Use the Local Smooth Embedding Theorem 24.8, the Submanifold Transversal Intersection Theorem 42.1 (2) and the Measure-Zero Properties Proposition 30.2.



Exercise 42.3. Let M be an oriented smooth manifold and let X and Y be two proper smooth submanifolds that intersect transversally. By the Submanifold Transversal Intersection Theorem 42.1 we obtain oriented submanifolds $X \cap Y$ and $Y \cap X$. What is the relationship between the orientations of $X \cap Y$ and $Y \cap X$?
Remark. This exercise is just a reminder for the reader to do Exercise 25.2.

Exercise 42.4. Let M be a 2-dimensional smooth manifold with $\partial M = \varnothing$, let A, B, C be three 1-dimensional smooth submanifolds of M and let $P \in A \cap B \cap C$ be a point such that all pairs among A, B and C intersect transversally. Does there exist a chart $\Phi\colon U \to V$ for $P \in M$ with $\Phi(P) = (0,0)$, $\Phi(A) \subset \mathbb{R} \times \{0\}$, $\Phi(B) \subset \{0\} \times \mathbb{R}$ and with $\Phi(C) \subset \{(x,x) \in \mathbb{R}^2 \,|\, x \in \mathbb{R}\}$?

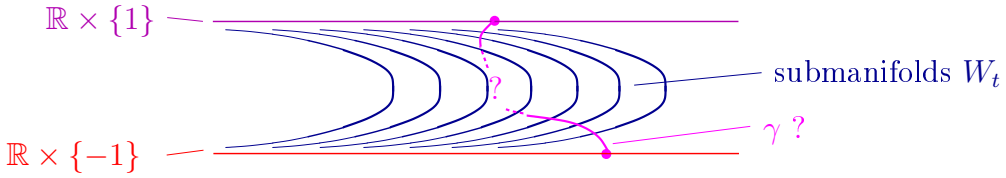


Exercise 42.5. Let $L \subset S^1 \times S^2$ be a 1-dimensional smooth submanifold. Show that L is smoothly isotopic to a 1-dimensional smooth submanifold that is contained in $S^1 \times S^2_{\geq 0}$.

Exercise 42.6. We consider the smooth manifold $M = \mathbb{R} \times [-1,1]$ together with the submanifolds

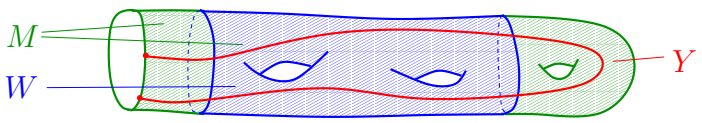
$$W_t \;:=\; \{(t + \tfrac{1}{y^2-1}, y) \,|\, y \in (-1,1)\} \qquad \text{where } t \in \mathbb{R}.$$

Show that there is no smooth path $\gamma\colon [-1,1] \to M$ that connects the bottom $\mathbb{R} \times \{-1\}$ with the top $\mathbb{R} \times \{1\}$ such that γ is transverse to every W_t, i.e. such that for any $s \in (-1,1)$ with $\gamma(s) \in W_t$ we have $\gamma'(s) \notin V_{\gamma(s)} W_t$.
Remark. This decomposition of the interior of $M = \mathbb{R} \times [-1,1]$ into submanifolds is sometimes called a *Reeb foliation*.



Exercise 42.7. Let M be a closed smooth manifold and let $\alpha\colon S^k \to M$ be a map. Show that there exists an $r \in \mathbb{N}_0$ and a smooth embedding $\varphi\colon S^k \to \partial(M \times \overline{B}^r)$ such that the maps $S^k \xrightarrow{\alpha} M \xrightarrow{x \mapsto (x,0)} M \times \overline{B}^r$ and $\varphi\colon S^k \to M \times \overline{B}^r$ are homotopic.

Exercise 42.8. Let M be an m-dimensional smooth manifold, let W be compact codimension-zero submanifold of M and let Y be a proper submanifold of M that intersects ∂W transversally. Show that $Y \cap W$ is a proper submanifold of W.



Exercise 42.9. Let M be an n-dimensional smooth manifold with $\partial M = \varnothing$ and let X and Y be proper submanifolds of M of dimensions k and l. We assume that there exists an $m \in \mathbb{N}_0$ such that for any $P \in X \cap Y$ we have $\dim(\mathrm{T}_P X + \mathrm{T}_P Y) = m$. Show that $X \cap Y$ is a proper submanifold of M of dimension $k + l - m$.
Hint. Given any $P \in M$ you can assume, after applying a chart, that you are working in an open subset of $\mathbb{R}^n$. Apply the Submanifold Transversal Intersection Theorem 42.1 to a suitable setting.

Project 42.1. Formulate and prove a version of the Submanifold Transversal Intersection Theorem 42.1 for three, or more generally $n \in \mathbb{N}$, submanifolds.

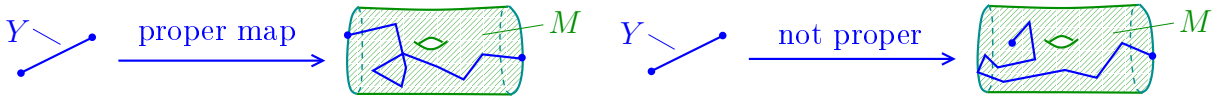
43. The algebraic intersection number

In this section we will use the results from the last chapter to introduce the algebraic intersection number between transverse submanifolds of complementary dimension. The algebraic intersection number is one of the most popular notions of geometric topology. In this chapter we will use this concept to prove the Generalized Smooth Jordan-Brouwer Separation Theorem 43.10 which implies, as we will see in Corollary 43.13, that there is no smooth embedding of $\mathbb{R}\mathrm{P}^2$ or the Klein bottle into $\mathbb{R}^3$.

43.1. The Algebraic intersection number between a submanifold and a map.

We start out with the following definition, which is a special case of the definition on page 650.

Definition. Let M be a topological manifold and let Y be a compact topological manifold. We say that a map $f\colon Y \to M$ is proper if $f^{-1}(\partial M) = \partial Y$.[418]



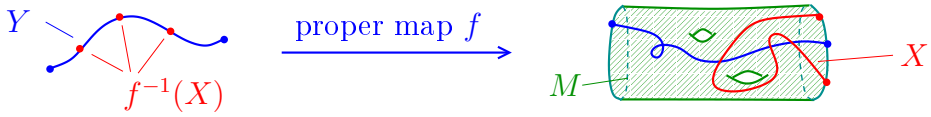
Definition. Let M be an m-dimensional smooth manifold and let $X \subset M$ be a compact proper submanifold of dimension k.

(1) Let Y be a compact smooth manifold of dimension $m-k$ and let $f\colon Y \to M$ be a proper smooth map that intersects X transversally in the sense of the definition on page 971, i.e. such that $X \pitchfork f$. By the Transversal Preimage Theorem 42.3 we know that $f^{-1}(X)$ is a compact 0-dimensional manifold. As we remarked on page 514 this means that $f^{-1}(X)$ is a finite set. Thus we can define

$$\text{geometric intersection number of } X \text{ and } f \;:=\; \#f^{-1}(X).$$

(2) If Y is a compact proper $(m-k)$-dimensional submanifold of M with $X \pitchfork Y$, then we define

$$\text{geometric intersection number of } X \text{ and } Y \;:=\; \#(X \cap Y).$$



The geometric intersection number (except perhaps for its parity) is frankly not a very interesting number. In the presence of orientations we can introduce a significantly more interesting number.

Definition. Let M be an oriented m-dimensional smooth manifold and let $X \subset M$ be a compact oriented proper submanifold of dimension k.

(1) Let Y be a compact oriented smooth manifold of dimension $m-k$ and let $f\colon Y \to M$ be a proper smooth map with $X \pitchfork f$. By the Transversal Preimage Theorem 42.3 we know that $f^{-1}(X)$ is a compact *oriented* 0-dimensional submanifold. Thus we

[418] It follows from the Compact Image Lemma 2.13 that for a compact topological manifold this definition is equivalent to the definition on page 650.

can define

$$\text{algebraic intersection number of } X \text{ and } f \;:=\; X \cdot f \;:=\; \sum_{P \in f^{-1}(X)} \underbrace{\text{orientation of } P}_{\in\{-1,1\}}.$$

(2) If Y is a compact proper $(m-k)$-dimensional submanifold of M with $X \pitchfork Y$, then we define[419]

$$\text{algebraic intersection number of } X \text{ and } Y \;:=\; X \cdot Y \;:=\; \sum_{P \in X \cap Y} \underbrace{\text{orientation of } P}_{\in\{-1,1\}}.$$

Example. Let M and Y be compact oriented smooth manifolds of the same dimension and let $f\colon Y \to M$ be a smooth map with $f^{-1}(\partial M) \subset \partial Y$. By Sard's Theorem 31.1 there exists a regular value $P \in M \setminus \partial M$. Basically by definition the fact that P is regular means that the map $f\colon Y \to M$ is transverse to the 0-dimensional submanifold $\{P\}$. As usual we give $\{P\}$ the orientation given by $+1$. If one goes through the definitions one sees that

$$\{P\} \cdot f \;=\; \deg(f\colon Y \to M),$$

where $\deg(f\colon Y \to M)$ denotes the degree that we introduced on pages 899 and 925.

The above definition of the algebraic intersection number is very elegant, but it is arguably not very intuitive. Therefore we now give an alternative description of the algebraic intersection number.

Definition. Let M be an oriented m-dimensional smooth manifold and let $X \subset M$ be a compact oriented proper submanifold of dimension $k \in \{1, \dots, m-1\}$.

(1) Let Y be a compact oriented smooth manifold of dimension $m-k$ and let $f\colon Y \to M$ be a proper smooth map with $X \pitchfork f$. Let $P \in f^{-1}(X)$. We set $\operatorname{sign}(P) := 1$, if

$$(\text{positive basis of } \mathrm{T}_{f(P)}X, \mathrm{D}f_P(\text{positive basis of } \mathrm{T}_P Y)) \;=\; \text{positive basis of } \mathrm{T}_{f(P)}M.$$

Otherwise we set $\operatorname{sign}(P) := -1$.[420]

(2) If Y is a compact oriented proper submanifold of M of dimension $m-k$, then for any $P \in X \cap Y$ we define $\operatorname{sign}(P)$ as in (1).

Lemma 43.1. We suppose that we are in the setting of the above definition. We have

$$X \cdot f \;=\; \sum_{P \in f^{-1}(X)} \operatorname{sign}(P).$$

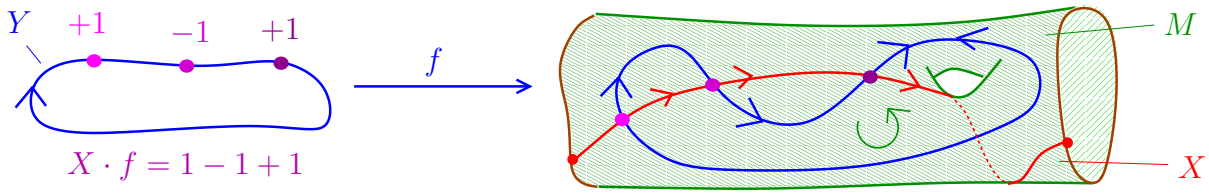
In particular, if Y is a compact proper submanifold of M with $X \pitchfork Y$, then

$$X \cdot Y \;=\; \sum_{P \in X \cap Y} \operatorname{sign}(P).$$

Proof. The lemma follows almost immediately from the definitions. But, for peace of mind, let us provide at least some details, to make sure that the sign conventions really do work out. It suffices to show the following claim.

[419] If $\iota\colon Y \to M$ denotes the inclusion map, then it follows immediately from the remark on page 674 that $X \cdot Y = X \cdot \iota$.

[420] It is a straightforward exercise in linear algebra to show that the definition of $\operatorname{sign}(P)$ does not depend on the choice of the positive bases of $\mathrm{T}_P X$ and $\mathrm{T}_P Y$.



geometric intersection number of X and f $= 3$
algebraic intersection number of X and f $= 1$

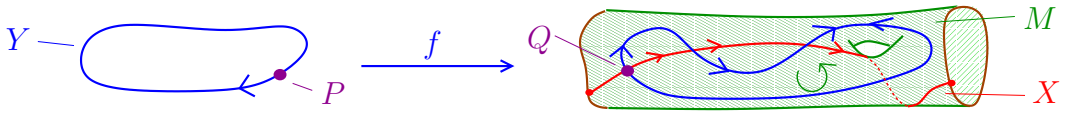
Claim. For each $P \in f^{-1}(X)$ the preimage orientation on P equals $\operatorname{sign}(P)$.

Proof. Let $P \in f^{-1}(X)$. We write $Q = f(P)$. Note that by the Transversal Preimage Theorem 42.3 (2) we have $(\mathrm{D}f_P)^{-1}(\mathrm{T}_Q X) = \mathrm{T}_P\{P\}$. Using this observation, we consider, as in the definition of the preimage orientation on page 673, the following diagram:

$$\begin{array}{ccccc} \mathrm{T}_Q X & \hookrightarrow & \mathrm{T}_Q M & \twoheadrightarrow & \mathrm{T}_Q M/\mathrm{T}_Q X \\ & \nearrow^{\mathrm{D}f_P}_{\cong} & & & \cong\uparrow{\scriptstyle [v]\mapsto[\mathrm{D}f_P(v)]} \\ \mathrm{T}_P\{P\} & \hookrightarrow & \mathrm{T}_P Y & \xrightarrow[\cong]{} & \mathrm{T}_P Y/\mathrm{T}_P\{P\}. \end{array}$$

The diagonal map exists since $\mathrm{T}_P\{P\}$ is 0-dimensional. We equip $\mathrm{T}_Q M/\mathrm{T}_Q X$ with the orientation given by the orientation on $\mathrm{T}_P Y$ and the isomorphism $\mathrm{D}f_P$. It follows immediately from this diagram that the following three statements are equivalent:

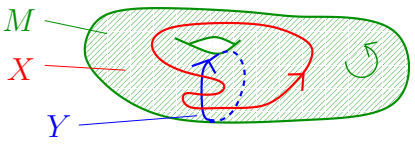
(1) The preimage orientation of $P \in f^{-1}(X)$ equals $+1$.
(2) The orientations on $\mathrm{T}_Q X, \mathrm{T}_Q M$ and $\mathrm{T}_Q M/\mathrm{T}_Q X$ are compatible in the sense of the definition on page 671.
(3) $\operatorname{sign}(P) = 1$. ∎



If we are given two oriented submanifolds X and Y of M, then of course it makes sense to consider $X \cdot Y$ as well as $Y \cdot X$. The following lemma elucidates the relationship between these algebraic intersection numbers.[421]

Lemma 43.2. Let M be an oriented smooth manifold and let $X, Y \subset M$ be compact oriented proper submanifolds of M with $\dim(X) + \dim(Y) = \dim(M)$. If $X \pitchfork Y$, then also $Y \pitchfork X$ and

$$Y \cdot X \;=\; (-1)^{\dim(X)\cdot\dim(Y)} \;\cdot\; X \cdot Y.$$



$$\begin{aligned} X \cdot Y &= +1 - 1 + 1 = +1 \\ Y \cdot X &= -1 + 1 - 1 = -1 \end{aligned}$$

Proof. We set $k = \dim(X)$, $l = \dim(Y)$ and $m = \dim(M)$. The cases $k = 0$ and $l = 0$ follow easily from the definitions. Thus let us now suppose that $k, l \in \{1, \dots, m-1\}$. In this

[421] The next lemma deals with a special case of Exercise 42.3.

case it follows from the claim in the proof of Lemma 43.1 that at each $P \in X \cap Y = Y \cap X$ the orientations differ by the sign of the determinant of the following matrix:

$$\begin{pmatrix} 0_{l\times k} & \mathrm{id}_l \\ \mathrm{id}_k & 0_{k\times l} \end{pmatrix}.$$

We leave it to the reader to bring out their favorite tricks for computing determinants to show that the determinant of this matrix equals $(-1)^{k\cdot l}$. ■

The following elementary lemma shows how the algebraic intersection number is affected by several basic operations.

Lemma 43.3. Let M be an oriented m-dimensional smooth manifold, let $X \subset M$ be a compact oriented proper submanifold of dimension k, let Y be a compact oriented smooth manifold of dimension $m-k$ and let $f\colon Y \to M$ be a proper smooth map with $X \pitchfork f$.

(1) As usual we denote by $-X$ the submanifold with the opposite orientation. We have

$$(-X)\cdot f \;=\; -X\cdot f.$$

(2) Let $\widetilde{M}$ be an oriented smooth manifold with $\dim(\widetilde{M}) = \dim(M)$. If $\Phi\colon M \to \widetilde{M}$ is a smooth embedding, then

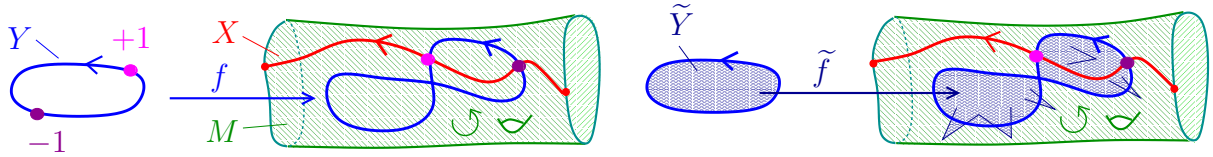
$$\Phi(X)\cdot(\Phi\circ f) \;=\; \begin{cases} X\cdot f, & \text{if } \Phi \text{ is orientation-preserving} \\ -X\cdot f, & \text{if } \Phi \text{ is orientation-reversing.} \end{cases}$$

(3) Let $\widetilde{Y}$ be an oriented smooth manifold. If $\Psi\colon \widetilde{Y} \to Y$ is a diffeomorphism, then

$$X\cdot(f\circ\Psi) \;=\; \begin{cases} X\cdot f, & \text{if } \Psi \text{ is orientation-preserving} \\ -X\cdot f, & \text{if } \Psi \text{ is orientation-reversing.} \end{cases}$$

Proof. All statements follow easily from the definitions together with the Compatible Orientations Lemma 25.5 (2). ■

Proposition 43.4. (Intersection-Number Zero-if-Extends Proposition) Let M be an oriented m-dimensional smooth manifold, let $X \subset M$ be a compact oriented proper submanifold of dimension k, let Y be a *closed* oriented smooth manifold of dimension $m-k$ and let $f\colon Y \to M$ be a proper[422] smooth map with $X \pitchfork f$. If there exists a compact oriented smooth manifold $\widetilde{Y}$ with[423] $\partial\widetilde{Y} = Y$ and if there exists a map[424] $\widetilde{f}\colon \widetilde{Y} \to M$ with $\widetilde{f}|_{\partial\widetilde{Y}} = f$, then $X\cdot f = 0$.



Remark. The Intersection-Number Zero-if-Extends Proposition 43.4 gives in particular a solution to Exercise 38.11.

[422] Since Y is closed this just means that $f(Y) \subset M \setminus \partial M$.

[423] On page 689 we introduced the natural orientation of the boundary of an oriented smooth manifold. With this definition the equality $\partial\widetilde{Y} = Y$ is supposed to be an equality of oriented smooth manifolds.

[424] Note that we do not assume that $\widetilde{f}$ is smooth.

Proof. Let $\widetilde{Y}$ be a compact oriented smooth manifold of dimension $m-k+1$ with $\partial\widetilde{Y} = Y$ and let $\widetilde{f}\colon \widetilde{Y} \to M$ be a map with $\widetilde{f}|_{\partial\widetilde{Y}} = f$. We need to show that $X \cdot f = 0$.

Claim. There exists a smooth map $\gamma\colon \widetilde{Y} \to M \setminus \partial M$ with $\gamma|_{\partial\widetilde{Y}} = f$ such that $X \pitchfork \gamma$.

Proof. We will perform three upgrades to $\widetilde{f}$ to get the desired map.

(α) Since $\widetilde{f}|_{\partial\widetilde{Y}} = f$ is smooth we obtain from the Smooth-on-Boundary-Submanifold Lemma 29.4 together with the Whitney Approximation Theorem 29.1[425] that we can replace $\widetilde{f}$ by a *smooth* map $\alpha\colon \widetilde{Y} \to M$ with $\alpha|_{\partial\widetilde{Y}} = f$.

(β) By hypothesis the map $f\colon Y \to M$ is proper, which in this setting just means that $f(Y) = \alpha(Y) \subset M \setminus \partial M$. It follows from the Push-Away-from-Boundary Corollary 28.5 that we can replace α by a smooth map $\beta\colon \widetilde{Y} \to M \setminus \partial M$ with $\beta|_{\partial\widetilde{Y}} = \alpha|_{\partial\widetilde{Y}} = f$.

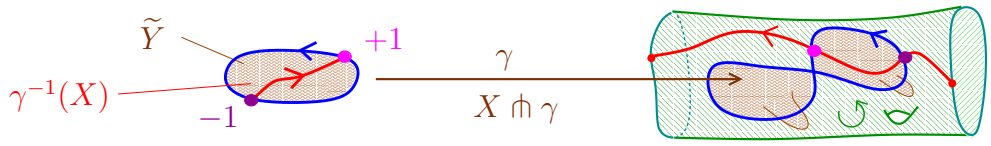
(γ) It follows from $X \pitchfork f$ that $X \pitchfork \beta$ on $Y = \partial\widetilde{Y}$. Since β takes values in $M \setminus \partial M$ we can apply the Transversality Theorem 42.5 to the map $\beta\colon \widetilde{Y} \to M \setminus \partial M$ and the closed subset $C := Y = \partial\widetilde{Y}$ of $\widetilde{Y}$. We obtain a smooth map $\gamma\colon \widetilde{Y} \to M \setminus \partial M$ with $\gamma|_{\partial\widetilde{Y}} = \widetilde{f}$ and with $X \pitchfork \gamma$. ⊞

Given a compact oriented 0-dimensional smooth manifold A we denote by $\epsilon(A) \in \mathbb{Z}$ the sum of the orientations of the points in M. We now see that

$$X \cdot f \underset{\uparrow}{=} \epsilon(f^{-1}(X)) \underset{\uparrow}{=} \epsilon(\partial\gamma^{-1}(X)) \underset{\uparrow}{=} \epsilon\Big(\begin{smallmatrix}\text{compact oriented 1-dimensional}\\ \text{smooth manifold}\end{smallmatrix}\Big) \underset{\uparrow}{=} 0.$$

by definition — by the Transversal Preimage Theorem 42.3 (3) and (4), here we use that $X \pitchfork \gamma$, that $\gamma|_{\partial\widetilde{Y}} = f$ and that $\widetilde{Y}$ is compact — Boundary-of-Compact 1-Dimensional Smooth Manifold Proposition 25.18

We have thus proved that $X \cdot f = 0$. ■



43.2. **The Intersection Number Homotopy Invariance Theorem.** The following theorem can be viewed as an analogue of the Degree Homotopy Invariance Theorem 38.3.

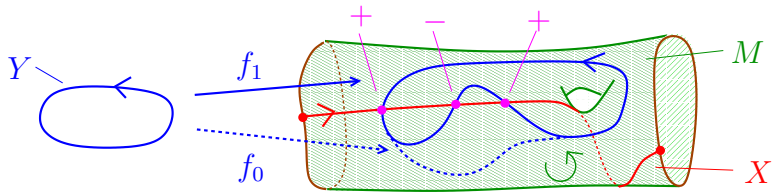
Theorem 43.5. (Intersection Number Homotopy Invariance Theorem) Let M be an oriented m-dimensional smooth manifold, let $X \subset M$ be a compact oriented proper submanifold of dimension k, let Y be a *closed* oriented smooth manifold of dimension $m-k$ and let $f_0, f_1\colon Y \to M$ be proper smooth maps with $X \pitchfork f_0$ and $X \pitchfork f_1$. If there exists a homotopy[426] $F\colon Y \times [0,1] \to M$ between f_0 and f_1, then

$$X \cdot f_0 \;=\; X \cdot f_1.$$

Example. Note that in the statement of the Intersection Number Homotopy Invariance Theorem 43.5 we cannot just drop the hypothesis that Y is closed. More precisely, in the

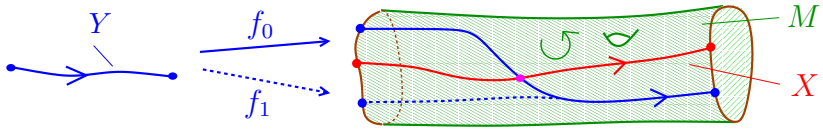
[425] Note that the hypothesis that $f\colon Y \to M$ is proper implies that $f(\partial\widetilde{Y}) \subset M \setminus \partial M$, so we can indeed apply the Whitney Approximation Theorem 29.1.

[426] Note that we do not demand that the homotopy is proper or smooth.



$$X \cdot f_1 = -1+1-1 = +1$$
$$X \cdot f_0 = +1$$

figure below we show a smooth manifold M, a proper submanifold X with $\partial X \neq \varnothing$ and two properly smoothly homotopic maps $Y \to M$ with $\partial Y \neq \varnothing$, which are both transverse to X, such that $X \cdot f_0 \neq X \cdot f_1$. As always in mathematics, under extra hypotheses one can in fact allow Y to have non-empty boundary. We will see one such instance with the Intersection Number Homotopy Invariance Theorem II 43.6 and we will see more examples in Chapter 126.



$$X \cdot f_0 = -1$$
$$X \cdot f_1 = 0$$

Proof. Let M be an oriented m-dimensional smooth manifold, let $X \subset M$ be a compact oriented proper submanifold of dimension k, let Y be a closed oriented smooth manifold of dimension $m-k$ and let $f_0, f_1 \colon Y \to M$ be smooth maps with $X \pitchfork f_0$ and $X \pitchfork f_1$ such that there exists a homotopy between f_0 and f_1. We need to show that $X \cdot f_1 - X \cdot f_0 = 0$.

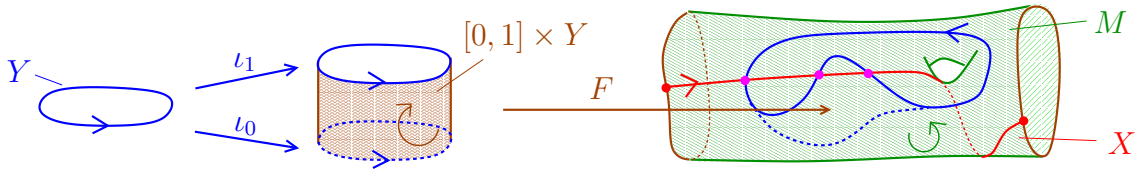
Given $t \in [0,1]$ we equip $\{t\} \times Y$ with the product orientation that we introduced on page 695. Furthermore we denote by

$$\begin{array}{rcl} \iota_t \colon Y & \to & \{t\} \times Y \\ y & \mapsto & (t,y) \end{array}$$

the obvious diffeomorphism. The hypothesis that f_0 and f_1 are homotopic implies that there exists a map $F \colon [0,1] \times Y \to M$ with $F \circ \iota_0 = f_0$ and $F \circ \iota_1 = f_1$. Now we perform the following calculation:

$$X \cdot f_1 - X \cdot f_0 = X \cdot (F \circ \iota_1) - X \cdot (F \circ \iota_0) \overset{\text{by Lemma 43.3 (1) since by the discussion on page 696 we know that } \iota_0 \text{ and } \iota_1 \text{ are orientation-preserving diffeomorphisms}}{=} X \cdot F|_{\{1\}\times Y} - X \cdot F|_{\{0\}\times Y}$$
$$\overset{\text{by Lemma 43.3 (2)}}{=} X \cdot F|_{\{1\}\times Y} + X \cdot F|_{-(\{0\}\times Y)} \overset{\text{since the algebraic intersection number is additive under disjoint unions}}{=} X \cdot F|_{\{1\}\times Y \cup -(\{0\}\times Y)} \overset{\text{see page 696}}{=} X \cdot F|_{\partial([0,1]\times Y)} \overset{\text{Proposition 43.4}}{=} 0.$$

■



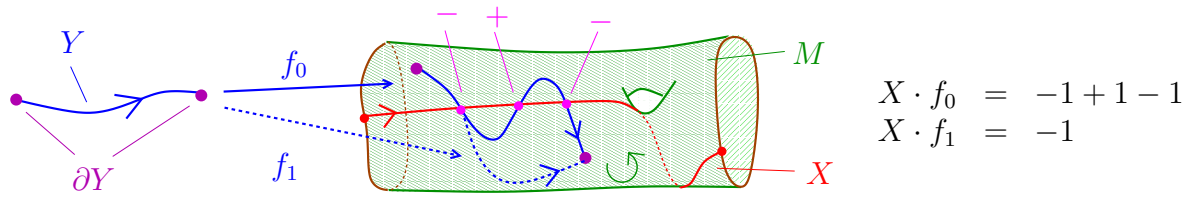
Under suitable restrictions one can generalize the Intersection Number Homotopy Invariance Theorem 43.5 from the case of a closed smooth manifold Y to the case of non-empty boundary. In the next section we will make use of the following theorem.

Theorem 43.6. (Intersection Number Homotopy Invariance Theorem II) Let M be an oriented m-dimensional smooth manifold, let $X \subset M$ be a compact oriented proper submanifold of dimension k, let Y be a compact oriented smooth manifold of dimension $m-k$ and let $f_0, f_1 \colon Y \to M \setminus \partial M$ be smooth maps with $X \pitchfork f_0$ and $X \pitchfork f_1$. We make the following hypothesis on the behavior of the maps on the boundary:

(1) $f_0|_{\partial Y} = f_1|_{\partial Y}$.
(2) $f_0(\partial Y) \cap X = f_1(\partial Y) \cap X = \varnothing$.

If f_0 and f_1 are homotopic rel ∂Y, then

$$X \cdot f_0 \;=\; X \cdot f_1.$$



Proof. The proof can be found in the long version of the lecture notes. ■

Proof. Once again we need to show that $X \cdot f_1 - X \cdot f_0 = 0$. Since $[0,1] \times Y$ is a smooth manifold *with corner* $\partial Y \times \{0,1\}$ we cannot use the off-the-shelf approach that we took in the proof of the Intersection Number Homotopy Invariance Theorem 43.5. Therefore in the following we also need to look under the hood of the proof of the Intersection-Number Zero-if-Extends Proposition 43.4 to make the argument work.

We will outline the changes that need to be made. As before, given $t \in [0,1]$ we equip $\{t\} \times Y$ with the product orientation and we denote by

$$\begin{array}{rccc} \iota_t \colon & Y & \to & \{t\} \times Y \\ & y & \mapsto & (x,y) \end{array}$$

the obvious diffeomorphism.

Claim 1. There exists a map $\delta \colon [0,1] \times Y \to M$ with the following properties:

(1) We have $\delta \circ \iota_0 = f_0$ and $\delta \circ \iota_1 = f_1$.
(2) For each $t \in [0,1]$ and $P \in \partial Y$ we have $\delta(t,P) = f_0(P)$.
(3) We have $\delta([0,1] \times \partial Y) \cap \partial M = \varnothing$.
(4) The map δ is smooth.

Proof. The proof of the claim uses an approach similar to the one taken in the proof of the Intersection-Number Zero-if-Extends Proposition 43.4. Again we proceed in several steps:

(1) Since f_0 and f_1 are homotopic rel ∂Y there exists a map $\alpha \colon [0,1] \times Y \to M$ which satisfies (1) and (2).
(2) the Push-Away-from-Boundary Corollary 28.5 implies that we can replace α by a map β which now satisfies (1), (2) and (3).
(3) Next note that by the Trivial-Near-The-Ends-Lemma 33.4 we can replace β by a homotopy $\gamma \colon [0,1] \times Y \to M$ which has the following properties:
   (a) For each $t \in [0, \frac{1}{4}]$ we have $\gamma \circ \iota_t = f_0$.

[426] Again we do not demand that the homotopy is smooth.

(b) For each $t \in [\frac{3}{4}, 1]$ we have $\gamma \circ \iota_t = f_1$.

(4) We apply the Whitney Approximation Theorem 29.1 to $\gamma\colon (0,1)\times Y \to M \setminus \partial M$ and the closed subset $A = ((0, \frac{1}{8}]\times Y) \cup ((\frac{7}{8}, 1]\times Y) \cup ([0,1]\times \partial Y)$ to obtain a smooth map $\delta\colon (0,1)\times Y \to M\setminus\partial M$. We extend δ to $[0,1]\times M\setminus\partial M$ via f_0 and f_1. It follows from the obvious extension of the Smooth Pasting Proposition 19.23 to smooth manifolds with corner that this map $\delta\colon [0,1]\times Y \to M\setminus\partial M$ is smooth. By construction it has all the desired properties. ⊞

Claim 2. We have $X \cdot \delta_{\partial([0,1]\times Y)} = 0$.

Proof. If $\partial Y = \varnothing$, then this equality follows immediately from the Intersection-Number Zero-if-Extends Proposition 43.4. Now consider the case that $\partial Y \neq \varnothing$. We make the following two observations:

(1) By construction we have $\delta^{-1}(X) \subset M \setminus \partial M$.

(2) Since X is closed and since Y is compact we see that the preimage $\delta^{-1}(X)$ is compact.

Note that it follows from (1) and (2), and the argument of the proof of the Intersection-Number Zero-if-Extends Proposition 43.4, that $\delta^{-1}(X)$ is a compact submanifold of the smooth manifold $[0,1]\times(Y\setminus\partial Y)$. Therefore, by exactly the same argument as in the proof of the Intersection-Number Zero-if-Extends Proposition 43.4 we see that $X \cdot \delta_{\partial([0,1]\times Y)} = 0$. ⊞

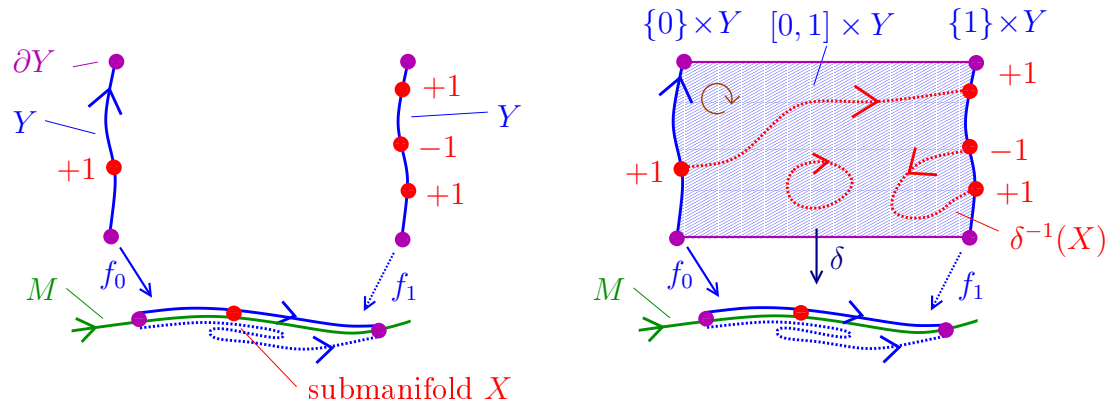
Now we perform the following calculation:

$$X\cdot f_1 - X\cdot f_0 \;=\; X\cdot(\delta\circ\iota_1) - X\cdot(\delta\circ\iota_0) \;\underset{\uparrow}{=}\; X\cdot\delta|_{\{1\}\times Y} - X\cdot\delta|_{\{0\}\times Y}$$

by Lemma 43.3 (1) since by the discussion on page 696 we know that ι_0 and ι_1 are orientation-preserving diffeomorphisms

$$\underset{\uparrow}{=} X\cdot\delta|_{\{1\}\times Y} + X\cdot\delta|_{-(\{0\}\times Y)} \underset{\uparrow}{=} X\cdot\delta|_{\{1\}\times Y \cup -(\{0\}\times Y)} \underset{\uparrow}{=} X\cdot\delta|_{\partial([0,1]\times Y)} \underset{\uparrow}{=} 0.$$

by Lemma 43.3 (2); since the algebraic intersection number is additive under disjoint unions; see page 696; Claim 2 ■



In the Intersection Number Homotopy Invariance Theorem 43.5 we showed that under suitable hypotheses the algebraic intersection number $X \cdot f$ stays constant if we "deform" f. The question arises, what happens if we "deform" the submanifold X by a smooth isotopy. If we are dealing with the algebraic intersection number between two submanifolds, then we

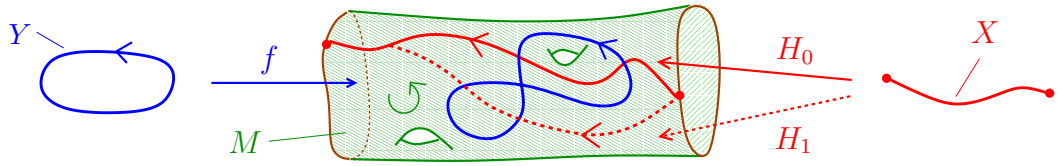
[426] Note that $[0,1]\times Y$ could be a smooth manifold with non-empty corner, but $(0,1)\times Y$ is fortunately smooth manifold without corner. This is the reason why we want to work with $(0,1)\times Y$.

can use the (anti-) symmetry from Lemma 43.2 to swap the roles of the two inputs and we can reduce the problem to the Intersection Number Homotopy Invariance Theorem 43.5.

In the general setting we have the following theorem:

Proposition 43.7. (Intersection Number Homotopy Invariance Theorem III) Let M be an oriented m-dimensional smooth manifold, let Y be a closed oriented smooth manifold of dimension $m-k$, let X be a compact oriented smooth manifold of dimension k and let $H\colon X\times[0,1]\to M$ be a proper smooth isotopy such that[427] $H_0(X)\pitchfork f$ and $H_1(X)\pitchfork f$. Then

$$H_0(X)\cdot f \;=\; H_1(X)\cdot f.$$



Proof. The natural idea would be to try to modify the argument of the Intersection Number Homotopy Invariance Theorem 43.5. We will do it differently though, namely we will employ the Isotopy Extension Theorem 36.1, to sidestep all difficulties.

By the Isotopy Extension Theorem 36.1 there exists a diffeotopy $\Phi\colon M\times[0,1]\to M$ with the following properties:

(1) $\Phi_0=\mathrm{id}$.
(2) $\Phi_1\circ H_0=H_1\colon X\to M$.

We now see that

$$H_0(X)\cdot f \;\underset{\uparrow}{=}\; \Phi_1(H_0(X))\cdot(\Phi_1\circ f) \;=\; H_1(X)\cdot(\Phi_1\circ f) \;\underset{\uparrow}{=}\; H_1(X)\cdot f.$$

since $\Phi_0=\mathrm{id}$ we know by the Smooth Isotopy-Orientation Proposition 34.2 that Φ_1 is an orientation-preserving diffeomorphism, the statement now follows from Lemma 43.3

by the Intersection Number Homotopy Invariance Theorem 43.5, since $(x,t)\mapsto\Phi_t\circ f$ is a homotopy between $f=\mathrm{id}\circ f=\Phi_0\circ f$ and $\Phi_1\circ f$

■

43.3. **The $\mathbb{Z}_2$-algebraic intersection number.** If we do not have orientations at hand, then we cannot define the $\mathbb{Z}$-valued algebraic intersection number. We can of course still consider the geometric intersection number, but the geometric intersection number is clearly not invariant under homotopies. But, playing around with pictures shows that the parity of the geometric intersection number has a chance of being invariant under homotopies. This hope leads us to the following definition.

Definition. Let M be an m-dimensional smooth manifold and let $X\subset M$ be a compact proper submanifold of dimension k.

(1) Let Y be a compact smooth manifold of dimension $m-k$ and let $f\colon Y\to M$ be a proper smooth map with $X\pitchfork f$. We define the

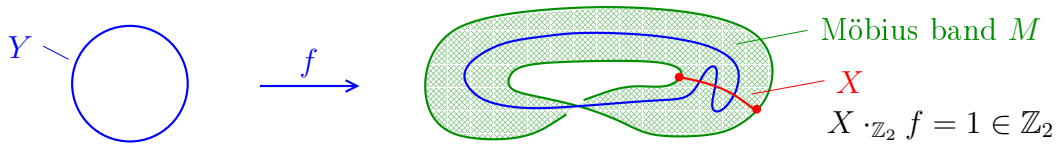
$$\mathbb{Z}_2\text{-valued intersection number of } X \text{ and } f \;:=\; X\cdot_{\mathbb{Z}_2} f \;:=\; [\#f^{-1}(X)]\in\mathbb{Z}_2.$$

[427] Note that it follows from the Smooth Embedding Theorem 24.10 that each $H_t(X)$ is indeed a proper smooth submanifold of M.

(2) If Y is a compact proper $(m-k)$-dimensional submanifold of M with $X \pitchfork Y$, then we define

$$\mathbb{Z}_2\text{-valued intersection number of } X \text{ and } Y \;:=\; X \cdot_{\mathbb{Z}_2} Y \;:=\; [\#(X \cap Y)] \in \mathbb{Z}_2.$$

Remark. In the setting of the above definition, if X, Y and M are actually oriented, then we evidently have $X \cdot_{\mathbb{Z}_2} f = X \cdot f \mod 2$.



Our previous results on the algebraic intersection numbers in the oriented setting also have obvious analogues in the unoriented "mod 2"-setting. We just need to delete the adjective "oriented" and replace $\mathbb{Z}$ by $\mathbb{Z}_2$. For example we have the following analogue the Intersection-Number Zero-if-Extends Proposition 43.4.

Proposition 43.8. (Intersection-Number Zero-if-Extends Proposition) Let M be an m-dimensional smooth manifold, let $X \subset M$ be a compact proper submanifold of dimension k, let Y be a closed smooth manifold of dimension $m-k$ and let $f\colon Y \to M$ be a proper smooth map with $X \pitchfork f$. If there exists a compact smooth manifold $\widetilde{Y}$ with $\partial\widetilde{Y} = Y$ and a map $\widetilde{f}\colon \widetilde{Y} \to M$ with $\widetilde{f}|_{\partial\widetilde{Y}} = f$, then $X \cdot_{\mathbb{Z}_2} f = 0$.

Proof. The proof is almost identical to the proof of the Intersection-Number Zero-if-Extends Proposition 43.4. ■

At the end of the day, the only result about $\mathbb{Z}_2$-value intersection numbers we really care about is the following analogue of the Intersection Number Homotopy Invariance Theorem 43.6. In fact we will put it to good use in the next section in our proof of the Generalized Smooth Jordan-Brouwer Separation Theorem 43.10.

Theorem 43.9. (Intersection Number Homotopy Invariance Theorem) Let M be an m-dimensional smooth manifold, let $X \subset M$ be a compact proper submanifold of dimension k and let Y be compact oriented smooth manifold of dimension $m-k$. Furthermore let $f_0, f_1\colon Y \to M \setminus \partial M$ be smooth maps with $X \pitchfork f_0$ and $X \pitchfork f_1$. We make the following hypotheses on the behavior of the maps on the boundary:

(1) $f_0|_{\partial Y} = f_1|_{\partial Y}$.
(2) $f_0(\partial Y) \cap X = f_1(\partial Y) \cap X = \varnothing$.

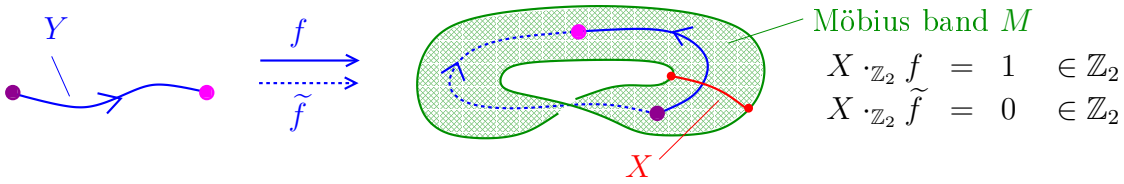
If f_0 and f_1 are homotopic rel ∂Y, then

$$X \cdot_{\mathbb{Z}_2} f_0 \;=\; X \cdot_{\mathbb{Z}_2} f_1.$$

Proof. Again the proof is a worry-free analogue of the proof of the original Intersection Number Homotopy Invariance Theorem 43.6. Along the way we just need to replace the Intersection-Number Zero-if-Extends Proposition 43.4 by the above Intersection-Number Zero-if-Extends Proposition 43.8. ■

Example. Let M be the Möbius band. We consider the proper 1-dimensional submanifold X and the two maps $f, \widetilde{f}\colon [0,1] \to M$ shown in the figure below. Since $X \cdot_{\mathbb{Z}_2} f = 0 \in \mathbb{Z}_2$

and $X \cdot_{\mathbb{Z}_2} \widetilde{f} = 0 \in \mathbb{Z}_2$ we obtain from the Intersection Number Homotopy Invariance Theorem 43.9 that the two maps are not homotopic rel $\{0,1\}$.



43.4. The Smooth Jordan-Brouwer Separation Theorem.

Theorem 43.10. (Generalized Smooth Jordan-Brouwer Separation Theorem) Let M be a connected non-empty smooth codimension-one submanifold of $\mathbb{R}^n$ that is closed if viewed as a smooth manifold in its own right.

(1) The complement $\mathbb{R}^n \setminus M$ has precisely two path-components[428], one of which is bounded and one which is unbounded.
(2) Let A and B be the two path-components of $\mathbb{R}^n \setminus M$. The following statements hold:
  (a) The closures $\overline{A}$ and $\overline{B}$ are codimension-zero submanifolds of $\mathbb{R}^n$.
  (b) $\partial A = \partial \overline{A} = \partial B = \partial \overline{B} = \overline{A} \cap \overline{B} = M$.[429]



As an almost immediate corollary we obtain the following result:

Corollary 43.11. (Smooth Jordan-Brouwer Separation Theorem) Let $n \in \mathbb{N}$.

(1) Let M be a closed connected non-empty n-dimensional smooth manifold. Given any smooth embedding $f\colon M \to \mathbb{R}^{n+1}$ the complement $\mathbb{R}^{n+1} \setminus f(M)$ has precisely two path-components.
(2) If $f\colon S^n \to \mathbb{R}^{n+1}$ is a smooth embedding, then $\mathbb{R}^{n+1} \setminus f(S^n)$ has precisely two path-components.



Proof of Corollary 43.11. The first statement follows from the Smooth Embedding Theorem 24.10 together with the Generalized Smooth Jordan-Brouwer Separation Theorem 43.10. The second statement is evidently an immediate consequence of the first statement. ■

Remark.

(1) The case $n = 1$ is sometimes referred to as the *Smooth Jordan Curve Theorem*.

[428] It follows from the Compact Image Lemma 2.13 and the Compact-Closed Lemma 1.21 (2) that $\mathbb{R}^n \setminus M$ is an open subset of $\mathbb{R}^n$. Thus it follows from the Topological Manifolds-Local Properties Lemma 18.8 that the path-components of $\mathbb{R}^n \setminus M$ are also the components of $\mathbb{R}^n \setminus M$.

[429] Since $\overline{A}$ and $\overline{B}$ are closed subsets and since they are codimension-zero submanifolds it follows from the Codimension-Zero Submanifold Proposition 19.38 that the boundaries of $\overline{A}$ and $\overline{B}$ viewed as smooth manifolds agree with the boundaries of $\overline{A}$ and $\overline{B}$ viewed as subsets of $\mathbb{R}^n$.

(2) (a) In the Jordan–Brouwer Separation Theorem 82.2 we will generalize the Smooth Jordan-Brouwer Separation Theorem 43.11 from smooth embeddings $S^n \to \mathbb{R}^{n+1}$ to injective continuous maps $S^n \to \mathbb{R}^{n+1}$.
(b) In the Generalized Jordan–Brouwer Separation Theorem 124.4 we will generalize the Smooth Jordan-Brouwer Separation Theorem 43.11 (1) from smooth embeddings $M \to \mathbb{R}^{n+1}$ of closed connected non-empty smooth manifolds to embeddings $M \to \mathbb{R}^{n+1}$ of closed connected non-empty topological manifolds.

Proof of the Generalized Smooth Jordan-Brouwer Separation Theorem 43.10. Let M be a connected non-empty smooth codimension-one submanifold of $\mathbb{R}^n$ that is closed if viewed as a smooth manifold in its own right. On several occasions we will use that it follows from the Compact Image Lemma 2.13 and the Compact-Closed Lemma 1.21 (2) that M is a closed subset of $\mathbb{R}^n$. The cases $n = 0$ and $n = 1$ are trivial, hence we now assume that $n \geq 2$.

Since M is closed we obtain from the Neighborhood Projection Theorem 28.1 that there exists an $\epsilon > 0$ such that

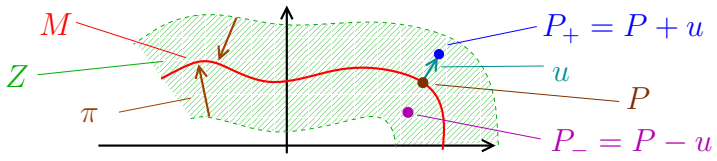
$$Z := \left\{P + w \,|\, P \in M \text{ and } w \in \left(\widetilde{\mathrm{T}}_P M\right)^\perp \text{ with } \|w\| < \epsilon\right\}$$

is an open neighborhood of $M \subset \mathbb{R}^n$ and such that the map

$$\begin{array}{rcll} \pi\colon Z & \to & M & \\ P + w & \mapsto & P & \text{where } P \in M \text{ and } w \in \left(\widetilde{\mathrm{T}}_P M\right)^\perp \end{array}$$

is well-defined. We also saw that the map π is smooth and that π is a retraction $Z \to M$.

Since M is non-empty we can pick a point $P \in M$. Since M has codimension 1 in $\mathbb{R}^n$ there exist precisely two vectors $u \in (\widetilde{\mathrm{T}}_P M)^\perp$ with $\|u\| = \frac{\epsilon}{2}$. Let u be one of the two vectors. We set $P_\pm = P \pm u$.



Claim 0. The topological space $Z \setminus M$ contains at most two path-components.

Proof. Let $Q \in Z \setminus M$. By Lemma 2.30 we know that being connected via a path is an equivalence relation. Thus it suffices to show that there exists a path from Q to P_- or P_+. We pick $z \in M$ and $v \in (\widetilde{\mathrm{T}}_P M)^\perp$ with $\|v\| < \epsilon$ and with $z + v = Q$. Since M is connected there exists a path $\gamma\colon [0,1] \to M$ with $\gamma(0) = z$ and $\gamma(1) = P$. Since M is a codimension-one submanifold one can show easily, either by hand[430] or using the Path-Thickening Proposition 37.1 together with the cross product from page 678, that there is a smooth map $\widetilde{\gamma}\colon [0,1] \to \mathbb{R}^n$ with the following properties:

(a) $\widetilde{\gamma}(0) = \gamma(0) + v$,
(b) for each $t \in [0,1]$ we have $\widetilde{\gamma}(t) - \gamma(t) \in (\widetilde{\mathrm{T}}_{\gamma(t)} M)^\perp$,
(c) for each $t \in [0,1]$ we have $\|\widetilde{\gamma}(t) - \gamma(t)\| = \|v\|$.

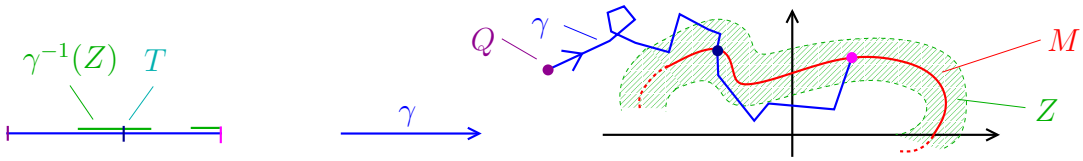
[430] The "by hand" approach is the content of Exercise 25.16.

Note that $\widetilde{\gamma}(1) = P + \lambda \cdot u$ for some $\lambda \in (-2, 0) \cup (0, 2)$. It follows that $\widetilde{\gamma}(1)$ is connected via a path in $Z \setminus M$ to $P + \lambda \cdot u$, thus $\widetilde{\gamma}(1)$ is also connected via a path in $Z \setminus M$ to P_- or to P_+. ⊞



Claim 1. The topological space $\mathbb{R}^n \setminus M$ contains at most two path-components.

Proof. Let $Q \in \mathbb{R}^n \setminus M$. Since $\mathbb{R}^n$ is path-connected we know that there exists a path $\gamma\colon [0,1] \to \mathbb{R}^n$ with $\gamma(0) = Q$ and $\gamma(1) \in M$. Since M is a closed subset of $\mathbb{R}^n$ we know that $\gamma^{-1}(M) \subset [0,1]$ is a closed subset of the compact space $[0,1]$. It follows from the Compact-Closed Lemma 1.21 that $\gamma^{-1}(M)$ is compact. Since $0 \notin \gamma^{-1}(M)$ we deduce from the Compact Image Lemma 2.13 (2) that $\gamma^{-1}(M)$ has a minimal element, i.e. there exists a $T \in \gamma^{-1}(M)$ such that $[0,T) \cap \gamma^{-1}(M) = \varnothing$. Since Z is an open neighborhood of $M \subset \mathbb{R}^n$ there exists an $\epsilon > 0$ such that $(T-\epsilon, T) \subset \gamma^{-1}(Z)$. Thus the restriction of γ to the interval $[0, T - \frac{\epsilon}{2}]$ gives us a path within $\mathbb{R}^n \setminus M$ from P to a point in $Z \setminus M$. It follows from Claim 0 that $\mathbb{R}^n \setminus M$ has indeed at most two path-components. ⊞



Claim 2.

(1) Let $P \in M$ and let $u \in (\widetilde{\mathrm{T}}_P M)^\perp$ with $\|u\| < \epsilon$. The points $P_\pm = P \pm u$ lie in two different path-components of $\mathbb{R}^n \setminus M$.

(2) The topological space $\mathbb{R}^n \setminus M$ contains at least two path-components.

Proof. Since M is non-empty we see that (2) is an immediate consequence of (1). Therefore it suffices to prove (1). Thus let $P \in M$ and let $u \in (\widetilde{\mathrm{T}}_P u)^\perp$ with $\|u\| < \epsilon$. We set $P_\pm = P \pm u$. We need to show that the points $P_-, P_+ \in \mathbb{R}^n \setminus M$ lie in different path-components of $\mathbb{R}^n \setminus M$. Let $\alpha\colon [0,1] \to \mathbb{R}^n$ the "straight path" from P_- to P_+ that is given by $\alpha(t) = (1-t) \cdot P_- + t \cdot P$. It follows from the properties of Z and the choice of u that the path α from $P_- = P - u$ to $P_+ = P + u$ intersects M precisely at $t = \frac{1}{2}$ and that this intersection point is transverse. Suppose that there exists also a path $\beta\colon [0,1] \to \mathbb{R}^n \setminus M$ from P_- to P_+. Since M is a closed subset of $\mathbb{R}^n$ we know that $\mathbb{R}^n \setminus M$ is an open subset, in particular it is a smooth submanifold of $\mathbb{R}^n$. Thus it follows from the Smooth Path-Connectivity Proposition 19.29 that there exists in fact a smooth path $\widetilde{\beta}\colon [0,1] \to \mathbb{R}^n \setminus M$ from P_- to P_+. Note that

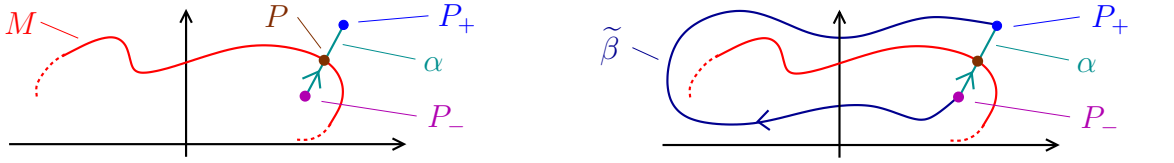
$$\begin{array}{rcl} F\colon [0,1] \times [0,1] & \to & \mathbb{R}^n \\ (t,s) & \mapsto & \alpha(t) \cdot (1-s) + \widetilde{\beta}(t) \cdot s \end{array}$$

is a homotopy rel $\{0,1\}$ from α to $\widetilde{\beta}$. We obtain the following equality of intersection numbers of submanifolds of $\mathbb{R}^n$:

Intersection Number Homotopy Invariance Theorem 43.9

$$0 \underset{\uparrow}{=} M \cdot_{\mathbb{Z}_2} \widetilde{\beta} \overset{\downarrow}{=} M \cdot_{\mathbb{Z}_2} \alpha \underset{\uparrow}{=} 1 \in \mathbb{Z}_2.$$

since $\widetilde{\beta}$ takes values in $\mathbb{R}^n \setminus M$ since α intersects M transversally precisely in $t = 0$ ⊞



Claims 1 and 2 show that $\mathbb{R}^n \setminus M$ contains precisely two path-components. Since M is compact there exists an $r > 0$ such that $M \subset B_r(0)$. It follows from our hypothesis that $n \geq 2$ and the Complement Connected Lemma 2.29 that $\mathbb{R}^n \setminus B_r^n(0)$ is path-connected. Therefore $\mathbb{R}^n \setminus B_r^n(0)$ is contained in a path-component B of $\mathbb{R}^n \setminus M$. We denote the other path-component of $\mathbb{R}^n \setminus M$ by A. We set $\widetilde{A} := A \cup M$ and we set $\widetilde{B} := B \cup M$. By the Codimension-Zero Submanifold Proposition 19.38 it remains to prove the following claim.

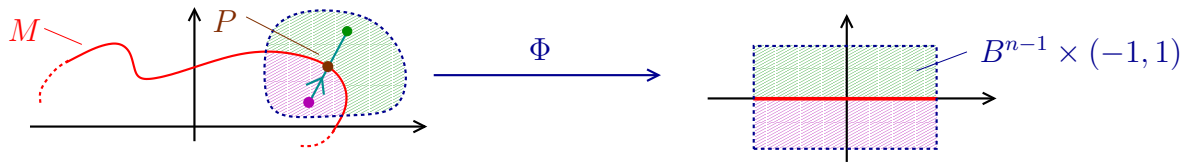
Claim 3.
(1) $\widetilde{A}$ and $\widetilde{B}$ are closed subsets of $\mathbb{R}^n$.
(2) $\widetilde{A}$ is compact.
(3) $\widetilde{A}$ and $\widetilde{B}$ are codimension-zero submanifolds of $\mathbb{R}^n$ with $\partial\widetilde{A} = \partial\widetilde{B} = \widetilde{A} \cap \widetilde{B} = M$.

Proof.

(1) As we remarked above $\mathbb{R}^n \setminus M$ is an open subset of $\mathbb{R}^n$. In particular, as we remarked on page 176, $\mathbb{R}^n \setminus M$ is locally path-connected. It follows from Locally Path-Connected Lemma 2.40 and the Components-are-Closed Lemma 2.34 that A and B are open subsets of $\mathbb{R}^n \setminus M$. Since M is a closed subset of $\mathbb{R}^n$ we see that A and B are both open subsets of $\mathbb{R}^n$. It follows that $\widetilde{A} = \mathbb{R}^n \setminus B$ and $\widetilde{B} = \mathbb{R}^n \setminus A$ are closed.
(2) By design we have $\widetilde{A} \subset B_r^n$. Since $\widetilde{A}$ is closed it follows from the Heine-Borel Theorem 1.24 that $\widetilde{A}$ is in fact compact.
(3) We show that $\widetilde{A}$ is a codimension-zero submanifold of $\mathbb{R}^n$ with $\partial\widetilde{A} = M$. For the points in A the identity defines a submanifold chart. Now let $P \in M$. Since M is a codimension-zero submanifold there exists a chart $\Phi\colon U \to B^{n-1} \times (-1,1)$ around P such that $\Phi(U \cap M) = B^{n-1} \times \{0\}$. Note that $B^{n-1} \times (0,1)$ and $B^{n-1} \times (-1,0)$ are path-connected. Furthermore note that it follows almost immediately from Claim (3a) that $\Phi^{-1}(B^{n-1} \times (0,1))$ and $\Phi^{-1}(B^{n-1} \times (-1,0))$ lie in two different path-components of $\mathbb{R}^n \setminus M$. These two observations imply that either we have $\Phi(U \cap \widetilde{A}) = B^{n-1} \times [0,1)$ or we have $\Phi(U \cap \widetilde{A}) = B^{n-1} \times (-1,0]$. Either way we have found (possibly after composing Φ with a reflection) a submanifold chart of type (γ) for P.

The same argument works for $\widetilde{B}$. We have thus shown the desired statements. ■

Using the Smooth Jordan-Brouwer Separation Theorem 43.11 we can easily prove the following pretty proposition.



Proposition 43.12. (Smooth Codimension-One Orientability Proposition) Let M be some closed n-dimensional smooth manifold. If there exists a smooth embedding $M \to \mathbb{R}^{n+1}$, then M is orientable.

Remark.

(1) On page 2705 we will give an alternative proof of the Smooth Codimension-One Orientability Proposition 43.12.
(2) In the Topological Codimension-One Orientability Proposition 124.5 we will strengthen the result.

Proof. Let M be a closed n-dimensional smooth manifold. that admits a smooth embedding $M \to \mathbb{R}^{n+1}$. It is clear that we only need to deal with the case that M is non-empty and connected. Now let us show that M is orientable. The proof proceeds in three easy steps:

(1) By the Smooth Embedding Theorem 24.10 we know that $f(M)$ is a closed codimension-one smooth submanifold of $\mathbb{R}^{n+1}$ such that $f\colon M \to f(M)$ is a diffeomorphism.
(2) It follows from the Smooth Jordan-Brouwer Separation Theorem 43.11 that there exists a codimension-zero submanifold A of $\mathbb{R}^{n+1}$ such that $\partial A = f(M)$.
(3) By the Basics-of-Orientations Lemma 25.13 we know that A is orientable. It follows from the Boundary Orientation Lemma 25.17 that $\partial A = f(M)$ is orientable. Since f is a diffeomorphism we obtain that M itself is orientable. ■

In Question 24.13 (2) we asked whether $\mathbb{R}\mathrm{P}^2$ or the Klein bottle admit a smooth embedding into $\mathbb{R}^3$. We can now give a negative answer.

Corollary 43.13. Neither $\mathbb{R}\mathrm{P}^2$ nor the Klein bottle admit a smooth embedding into $\mathbb{R}^3$.

Remark. In Corollary 124.6 we will see $\mathbb{R}\mathrm{P}^2$ and the Klein bottle do not even admit a (possibly non-smooth) embedding into $\mathbb{R}^3$.

Proof. By the Surface Non-Orientability Lemma 25.20 we know that $\mathbb{R}\mathrm{P}^2$ and the Klein bottle are both non-orientable. Since both are closed 2-dimensional smooth manifolds it follows from the Smooth Codimension-One Orientability Proposition 43.12 that neither $\mathbb{R}\mathrm{P}^2$ nor the Klein bottle admit a smooth embedding into $\mathbb{R}^3$. ■

Proposition 43.14. If $n \in \mathbb{N}_{\geq 2}$ is even, then there is no smooth embedding from $\mathbb{R}\mathrm{P}^n$ into $\mathbb{R}^{n+1}$.

Proof. Let $n \in \mathbb{N}_{\geq 2}$ be even. It follows from the Real Projective Space Orientability Lemma 25.21 that $\mathbb{R}\mathrm{P}^n$ is non-orientable. Since $\mathbb{R}\mathrm{P}^n$ is a closed n-dimensional smooth manifold it follows from the Smooth Codimension-One Orientability Proposition 43.12 that there is no smooth embedding from $\mathbb{R}\mathrm{P}^n$ into $\mathbb{R}^{n+1}$. ■

Remark. In Proposition 123.11 we will see that the conclusion of Proposition 43.14 holds for any $n \in \mathbb{N}_{\geq 2}$, i.e. regardless of whether or not $\mathbb{R}\mathrm{P}^n$ is orientable, it cannot be smoothly embedded into $\mathbb{R}^{n+1}$.

Exercises for Chapter 43.

Exercise 43.1. Let $f\colon A \to M$ and $g\colon M \to N$ be smooth maps between compact oriented smooth manifolds. Let $Y \subset N$ be a compact oriented proper submanifold such that $Y \pitchfork g$. By the Transversal Preimage Theorem 42.3 we know that $g^{-1}(Y)$ is a compact oriented proper submanifold of M. We suppose that $\dim(A) + \dim(Y) = \dim(M)$ and we suppose that $g^{-1}(Y) \pitchfork f$. Show that

$$X \cdot (g \circ f) \;=\; g^{-1}(X) \cdot f.$$

Exercise 43.2. Let X be a closed oriented smooth manifold, let $f\colon X \to M$ be a smooth map to a compact oriented smooth manifolds and let $Y \subset N$ be a compact oriented proper submanifold with $\dim(X) + \dim(Y) = \dim(M)$. Finally let $\widetilde{X}$ be a closed oriented smooth manifold with $\dim(\widetilde{X}) = \dim(X)$ and let $\varphi\colon \widetilde{X} \to X$ be a smooth map. Show that

$$X \cdot (f \circ \varphi\colon \widetilde{X} \to M) \;=\; (X \cdot (f\colon X \to M)) \;\cdot\; \deg(\varphi\colon \widetilde{X} \to X).$$

Exercise 43.3. In the example on page 993 we saw that the conclusion of Theorem 43.5 does not hold if we do not assume that Y is closed. Go through the proof of Theorem 43.5 and find out where the proof fails if Y is not closed.

Exercise 43.4. Let M be a connected oriented smooth manifold and let X, Y be two closed oriented submanifolds with $\dim(X) = \dim(Y) = \dim(M)$. We suppose that $X \cdot Y = 0$. Does it follow that there exists a smooth homotopy $F\colon Y \times [0,1] \to M$ such that $F_0 = \mathrm{id}$ and $X \cap F_1(Y) = \varnothing$?

Exercise 43.5. Is the curve that goes twice around the Möbius band null-homotopic, i.e. is it homotopic to the constant curve?

Exercise 43.6.

(a) Show that every smooth map $f\colon S^1 \to S^2$ is null-homotopic, i.e. homotopic to a constant map.
(b) Show that S^2 not diffeomorphic to $S^1 \times S^1$.
(c) Can you upgrade the argument to show that S^2 is not homeomorphic to $S^1 \times S^1$?

Exercise 43.7. In the Intersection Number Homotopy Invariance Theorem 43.6 we studied the homotopy invariance of the algebraic intersection number under the following fairly strong hypotheses:

(1) $f_0|_{\partial Y} = f_1|_{\partial Y}$.
(2) $f_0(\partial Y) \cap X = f_1(\partial Y) \cap X = \varnothing$.
(3) There exists a homotopy F rel ∂Y.

Formulate the weakest hypothesis on the behavior of f_0, f_1 and F on ∂Y and prove the corresponding theorem.

Exercise 43.8. Let M be a codimension-zero smooth submanifold of $\mathbb{R}^n$ which is a closed smooth manifold. Determine the number of components of $\mathbb{R}^n \setminus M$ in terms of the number of components of M.

Exercise 43.9. Let M be a smooth manifold. For $i = 1, \dots, k$ let Y_i be a compact smooth manifold and let $f_i \colon Y_i \to M$ be a proper smooth embedding. Show that there exist proper smooth embeddings $\widetilde{f}_i \colon Y_i \to M$, $i = 1, \dots, k$ with the following properties:

(1) Each $\widetilde{f}_i$ is smoothly isotopic to f_i.
(2) For any $i \neq j \in \{1, \dots, k\}$ we have $\widetilde{f}_i(Y_i) \pitchfork \widetilde{f}_j(Y_j)$.

Exercise 43.10. Let $\varphi \colon S^1 \to \mathbb{R}^2$ be a smooth embedding. Show that there exist distinct three points $x, y, z \in S^1$ such that $\varphi(x), \varphi(y), \varphi(z)$ from an equilateral triangle.
Hint. Use the Smooth Jordan-Brouwer Separation Theorem 43.11 for $n = 1$. Let $n \in \mathbb{N}$.

Exercise 43.11. Let

$$S \;:=\; \big\{(0, y) \,\big|\, y \in [-1, 1]\big\} \cup \big\{(x, \sin(\tfrac{1}{x})) \,\big|\, x \in (0, \tfrac{1}{\pi}]\big\} \;\subset\; \mathbb{R}^2$$

be the "Topologist's Sine Curve" from page 164. As on page 276 we turn the Topologist's Sine curve into the *Quasi-Circle* Q by connecting the point $(0, 0)$ with $(\frac{1}{\pi}, 0)$ via the green curve shown in the figure below. We consider the points $A = (\frac{1}{\pi}, -1)$ and $B = (2, -1)$.

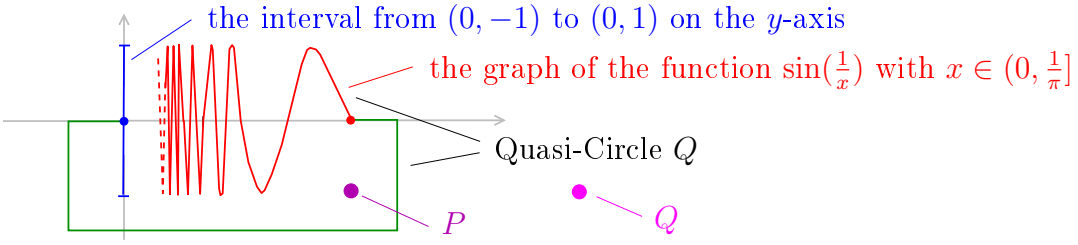
(a) Given $\epsilon > 0$ we consider the thickened up topological space

$$Q_\epsilon \;:=\; \{P \in \mathbb{R}^2 \,|\, \text{there exists } x \in Q \text{ with } d(x, P) < \epsilon\}.$$

Show that for every $\epsilon > 0$ there exists a smooth curve $\varphi \colon S^1 \to Q_\epsilon$ such that A and B lie in different path-components of $\mathbb{R}^2 \setminus \varphi(S^1)$.
Hint. Use the Smooth Jordan-Brouwer Separation Theorem 43.11.

(b) Show that there is no path $f \colon [0, 1] \to \mathbb{R}^2 \setminus Q$ from A to B.
Hint. By Exercise 5.30 we know that Q is compact. Lemma 6.3 might come in handy.

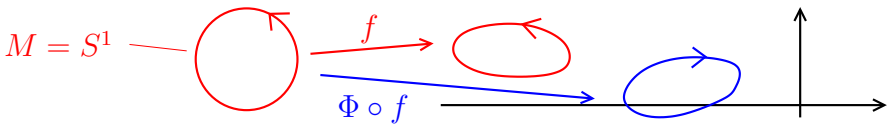
(c) Show that $\mathbb{R}^2 \setminus Q$ has precisely two path components.



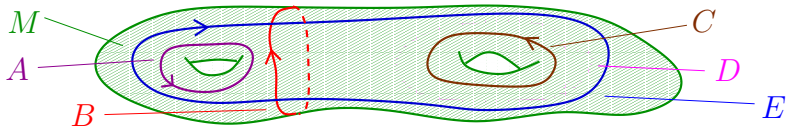
Exercise 43.12. Let M be the smooth manifold that is obtained from the open disk B^2 by removing the three points $(0, -\frac{1}{2}), (0, 0)$ and $(0, \frac{1}{2})$. Show that there is no *proper* smooth embedding $f \colon M \to \mathbb{R}^2$.
Remark. This exercise can be viewed as the complement of Exercise 11.13.

Exercise 43.13. Let M be a closed connected non-empty n-dimensional smooth manifold and let $f \colon M \to \mathbb{R}^{n+1}$ be a smooth embedding. Furthermore let $\Phi \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be an orientation-preserving diffeomorphism. Show that the smooth embeddings $f \colon M \to \mathbb{R}^{n+1}$ and $\Phi \circ f \colon M \to \mathbb{R}^{n+1}$ are not smoothly isotopic.

Hint. You could try to get the Generalized Smooth Jordan-Brouwer Separation Theorem 43.10 into the game.



Exercise 43.14. We consider the smooth manifold M together with the oriented submanifolds A, B, C, D and E are that are shown in the figure below. Show that the smooth submanifolds a pairwise non isotopic.



Exercise 43.15. Let $g \in \mathbb{N}_0$ and let Σ_g be the orientable surface of genus g. We equip Σ_g with some orientation. For the purpose of this exercise, given a closed oriented 1-dimensional submanifold $C \subset \Sigma_g$ we define the self-intersection number $C \cdot C := 0$. What is the maximal $k \in \mathbb{N}_0$ such that there exist closed oriented 1-dimensional submanifolds $A_1, \dots, A_k \subset \Sigma_g$ such that any two A_i, A_j with $i \neq j$ are transverse and such that the matrix $(A_i \cdot A_j)_{i,j=1,\dots,k}$ has non-zero determinant?
Remark. With the current techniques it is perhaps difficult to show that you have achieved the maximal k, but at least you can try to aim for the highest k possible.

Exercise 43.16. We set

$$\begin{aligned} H &:= \{(x,y) \in \mathbb{R}^2 \,|\, y > 0\} && \text{the open half-plane} \\ A_{-1} &:= \{(x,0) \,|\, x < 0\} \\ \text{and for } k \in \mathbb{N}_0 \text{ we set } A_k &:= \{(x,0) \,|\, x \in (2k+1, 2k+2)\}, \\ \text{finally we set} M &:= H \cup (A_{-1} \cup A_0 \cup A_1 \cup \dots). \end{aligned}$$

We need to introduce some more language:

- Let $k \neq l$. A *track* from A_k to A_l is a connected oriented proper 1-dimensional smooth manifold T of M such that $\partial T = \{-P\} \cup \{Q\}$ where $P \in A_k$ and $Q \in A_l$.
- We say that two components A_k and A_l of ∂M are *adjacent* if there exists a track T from A_k to A_l such that all other components of ∂M are contained in one component of $M \setminus T$.

Now we turn to the actual problems:

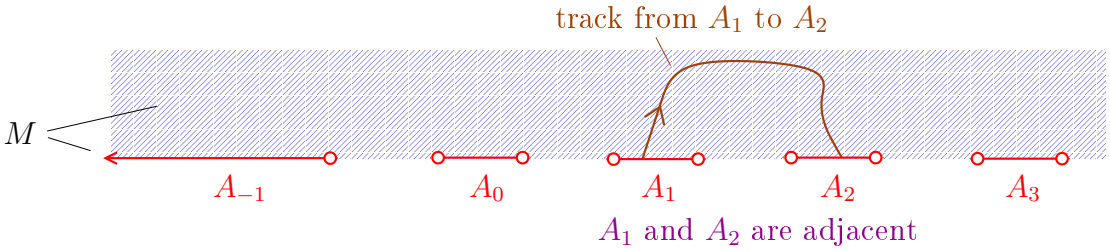
(a) Show that $M := H \cup (A_{-1} \cup A_0 \cup A_1 \cup \dots)$ is a 2-dimensional smooth submanifold with $\partial M = A_{-1} \cup A_0 \cup A_1 \cup \dots$.

(b) Show that for any $k \neq l$ in $\mathbb{Z}_{\geq -1}$ the corresponding components A_k and A_l are adjacent if and only if $k - l \in \{-1, 1\}$.

(c) Let $f \colon M \to M$ be a self-diffeomorphism. Show that for each $k \in \mathbb{Z}_{\geq -1}$ we have $f(A_k) = A_k$.
Hint. Note that adjacency is preserved by diffeomorphisms.

(d) Show that every diffeomorphism of M is orientation-preserving.
Hint. Look at intersection numbers of tracks between components.

Remark. You can use the fact, which we will show later in Exercise 55.19, that any two tracks from A_k to A_l are properly isotopic.



44. Linking numbers

In this short chapter we will use the techniques that we have developed over the last chapters to introduce and discuss linking numbers of suitable disjoint submanifolds in $\mathbb{R}^n$. To define the linking numbers it is convenient to generalize slightly the notion of the degree of a map between closed oriented connected smooth manifolds.

Definition. Let M be a closed oriented smooth manifold with components $M_1, \dots, M_k$ and let N be a closed oriented connected non-empty smooth manifold of the same dimension. Given any map $f\colon M \to N$ we define

$$\deg(f) \;:=\; \sum_{i=1}^{k} \underbrace{\deg(f|_{M_i}\colon M_i \to N)}_{\text{defined on page 899}}.$$

Now we can turn to the key definition of this section.

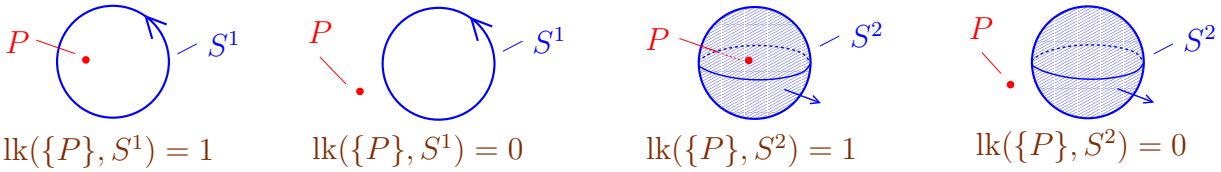
Definition. Let $n \in \mathbb{N}$. Let K and L be disjoint closed oriented smooth submanifolds of $\mathbb{R}^n$ with $\dim(K) + \dim(L) = n - 1$. We define the linking number $\operatorname{lk}(K, L)$ as[431]

$$\operatorname{lk}(K,L) \;:=\; (-1)^{\dim(L)} \cdot \deg \begin{pmatrix} K \times L & \to & S^{n-1} \\ (x,y) & \mapsto & \frac{x-y}{\|x-y\|} \end{pmatrix}$$

Remark. There are many ways think about the linking number. In particular on page 2865 we will also give an alternative definition of the linking number which we will discuss in detail in Section 131.4. In the Linking Number Reformulation Proposition 131.16 we will see that our two definitions of the linking number agree. Our sign convention is chosen in a way such that the latter definition becomes pretty, which unfortunately forces us to add the slightly ugly "$(-1)^{\dim(L)}$"-factor to the above definition.

Example. Let $n \in \mathbb{N}$, let $L = S^{n-1} \subset \mathbb{R}^n$ be the sphere equipped with the standard orientation from page 681 and let $K = \{P\}$ be the 0-dimensional submanifold that consists of a single point $P \in \mathbb{R}^n \setminus S^{n-1}$ and that is equipped with the orientation $+1$. In Exercise 44.3 we will see that

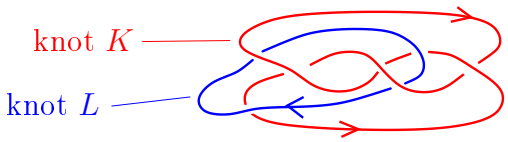
$$\operatorname{lk}(\{P\}, S^{n-1}) \;=\; \begin{cases} 1, & \text{if } P \in B^n, \\ 0, & \text{otherwise.} \end{cases}$$



Our main interest will be in the case $n = 3$ and $\dim(K) = \dim(L) = 1$. It is thus convenient to introduce the following definition.

[431] Here we endow $K \times L$ with the product orientation that we defined on page 695 and we equip S^{n-1} with the standard orientation that we introduced on page 681. Thus we see that $K \times L$ is a closed oriented smooth manifold of dimension $n - 1$. Therefore it makes sense to talk of the degree of a map to S^{n-1}.

Definition. An oriented knot in $\mathbb{R}^3$ is an oriented submanifold of $\mathbb{R}^3$ that is diffeomorphic to S^1.



what is $\mathrm{lk}(K,L)$?

To formulate our next proposition we will need the following notation and definition.

Definition.

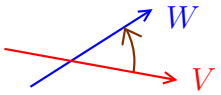
(1) Let $n \in \mathbb{N}$. We denote by

$$\begin{array}{rcl} \pi\colon \mathbb{R}^n & \to & \mathbb{R}^{n-1} \\ (x_1,\dots,x_n) & \mapsto & (x_1,\dots,x_{n-1}) \end{array} \quad \text{and} \quad \begin{array}{rcl} z\colon \mathbb{R}^n & \to & \mathbb{R} \\ (x_1,\dots,x_n) & \mapsto & x_n \end{array}$$

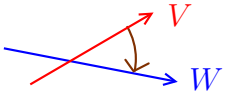
the projections onto the first $n-1$ coordinates and onto the last coordinate.

(2) Given two oriented vector subspaces V, W of $\mathbb{R}^m$ with $\dim(V)+\dim(W)=m$ and with $V+W=\mathbb{R}^m$ we set

$$\mathrm{sign}(V,W) := \begin{cases} +1, & \text{if } V\times W \to \mathbb{R}^m,\ (v,w)\mapsto v+w \text{ is orientation-preserving,} \\ -1, & \text{otherwise.} \end{cases}$$



$\mathrm{sign}(V,W)=+1$

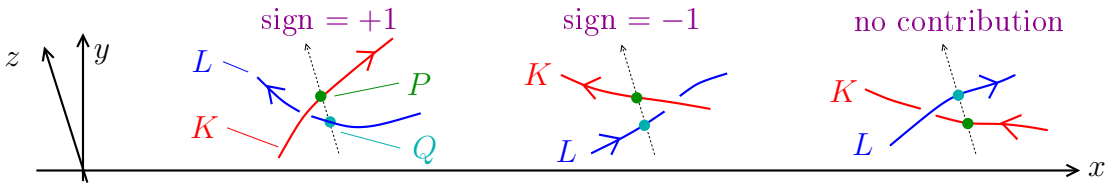


$\mathrm{sign}(V,W)=-1$

The next proposition presents one way to calculate linking numbers. In "real life" it gives a handy way to calculate the linking number of two disjoint oriented knots in $\mathbb{R}^3$. We will also make use of it for a more theoretical purpose.

Proposition 44.1. Let K and L be two disjoint closed oriented smooth submanifolds of $\mathbb{R}^n$ with $\dim(K)+\dim(L)=n-1$. We assume that for each $P\in K$ and $Q\in L$ with $\pi(P)=\pi(Q)$ we have $\pi(\widetilde{\mathrm{T}}_P K)+\pi(\widetilde{\mathrm{T}}_Q L)=\mathbb{R}^{n-1}$. Then there are only finitely many $(P,Q)\in K\times L$ with $\pi(P)=\pi(Q)$ and with $z(P)>z(Q)$. Furthermore[432]

$$\mathrm{lk}(K,L) \;=\; \sum_{\substack{P\in K \text{ and } Q\in L \\ \text{with } \pi(P)=\pi(Q) \text{ and} \\ z(P)>z(Q)}} \mathrm{sign}(\pi(\widetilde{\mathrm{T}}_P K), \pi(\widetilde{\mathrm{T}}_Q L)).$$

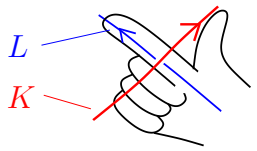


Example. We want to explain in slightly more detail how Proposition 44.1 can be used to calculate the linking number between two disjoint oriented knots K and L in $\mathbb{R}^3$. In

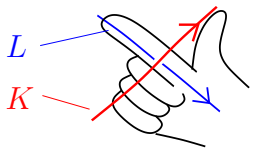
[432] Note that $\dim(K)+\dim(L)=n-1$ and $\pi(\widetilde{\mathrm{T}}_P K)+\pi(\widetilde{\mathrm{T}}_Q L)=\mathbb{R}^{n-1}$ imply that the restriction of π to $\widetilde{\mathrm{T}}_P K$ and to $\widetilde{\mathrm{T}}_Q L$ are both monomorphisms. Thus we can view $\pi(\widetilde{\mathrm{T}}_P K)$ and $\pi(\widetilde{\mathrm{T}}_Q L)$ as oriented vector subspaces of $\mathbb{R}^{n-1}$.

practice the knots K and L are given by a diagram, i.e. the images of the knots in the xy-plane and one remembers which z-coordinate is larger. In this setting Proposition 44.1 says the following:

(1) To each point where K crosses over L we assign $+1$ if the following "right hand rule" is satisfied: if the thumb of the right hand points into the direction of the knot K, then the fingers point in the direction of the knot L. Otherwise we assign -1 to such a crossing point.
(2) We sum up over all signs defined in (1) and we obtain the linking number $\mathrm{lk}(K, L)$ of K and L.

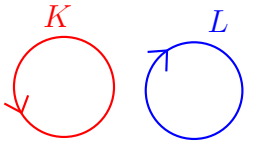


contributes $+1$ to $\mathrm{lk}(K, L)$

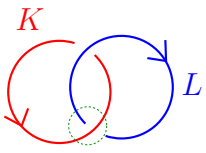


contributes -1 to $\mathrm{lk}(K, L)$

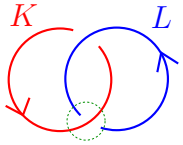
In practice it is now straightforward to calculate linking numbers of disjoint oriented knots in $\mathbb{R}^3$. Some calculations are shown in the following figure.



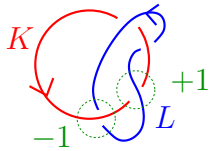
linking number $= 0$



linking number $= +1$



linking number $= -1$



linking number $= 0$

Proof of Proposition 44.1. We set $k := \dim(K)$ and $l := \dim(L)$. We consider the map

$$\begin{array}{rcl} \Theta\colon K \times L & \to & S^{n-1} \\ (P, Q) & \mapsto & \frac{P-Q}{\|P-Q\|}. \end{array}$$

Claim.

(1) For any $(P, Q) \in K \times L$ we have $\widetilde{\mathrm{T}}_{(P,Q)}(K \times L) = \widetilde{\mathrm{T}}_P K \times \widetilde{\mathrm{T}}_Q L \subset \mathbb{R}^n \times \mathbb{R}^n$.
(2) For any $(P, Q) \in K \times L$ with $P - Q = \lambda \cdot e_n$ for some $\lambda > 0$ the map

$$\mathrm{D}(\pi \circ \Theta)_{(P,Q)}\colon \widetilde{\mathrm{T}}_P K \times \widetilde{\mathrm{T}}_Q L \ \to \ \widetilde{\mathrm{T}}_{\pi(\Theta(P,Q))}\mathbb{R}^{n-1} = \mathbb{R}^{n-1}$$

is given by

$$(v, w) \ \mapsto \ \lambda \cdot (\pi(v) - \pi(w)).$$

(3) Given $P \in K$ and $Q \in L$ the following two statements are equivalent:
   (a) $\pi(P) = \pi(Q)$ and $z(P) > z(Q)$,
   (b) $\Theta(P, Q) = e_n$.
(4) $e_n \in S^{n-1}$ is a regular value of Θ.
(5) There are only finitely many $(P, Q) \in K \times L$ with $\pi(P) = \pi(Q)$ and with $z(P) > z(Q)$.

Proof.

(1) We know by Exercise 23.4 that $\widetilde{\mathrm{T}}_{(P,Q)}(K \times L) = \widetilde{\mathrm{T}}_P K \times \widetilde{\mathrm{T}}_Q L$.

(2) Let $(P,Q) \in K \times L$ with $P - Q = \lambda \cdot e_n$ where $\lambda > 0$. We introduce the following two maps:

$$\begin{array}{rcl} \tau\colon \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}^n \\ (v,w) & \mapsto & v - w \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \nu\colon \mathbb{R}^n \setminus \{0\} & \to & S^{n-1} \\ x & \mapsto & \frac{x}{\|x\|}. \end{array}$$

Let $v \in \widetilde{\mathrm{T}}_P K$ and $w \in \widetilde{\mathrm{T}}_Q L$. We perform the following calculation:

$$\begin{array}{rcl} \mathrm{D}(\pi \circ \Theta)_{(P,Q)}(v,w) & \overset{\text{by definition of } \Theta}{\underset{\uparrow}{=}} & \mathrm{D}(\pi \circ \nu \circ \tau)_{(P,Q)}(v,w) \\ & = & \lambda \cdot (\pi \circ \tau)(v,w) \end{array} \quad \begin{array}{rcl} \overset{\text{Chain rule and since } \pi \text{ is linear}}{\underset{\downarrow}{=}} & \pi \circ (\mathrm{D}\nu_{\lambda \cdot e_n} \circ \mathrm{D}\tau_{(P,Q)})(v,w) \\ = & \lambda \cdot (\pi(v) - \pi(w)). \end{array}$$

explicit calculation of $\mathrm{D}\nu_{\lambda \cdot e_n}$ and since τ is linear

(3) This statement follows easily from the definitions.

(4) Note that $\mathrm{D}(\pi|_{S^{n-1}})_{e_n}\colon \widetilde{\mathrm{T}}_{e_n} S^{n-1} \to \mathbb{R}^{n-1}$ is an isomorphism. It follows from this observation, together with (2) and (3) and our hypothesis in the proposition, that e_n is a regular value of Θ.

(5) It follows from (3) and (4) and the Stack of Records Lemma 26.2, that there are only finitely $(P,Q) \in K \times L$ with $\pi(P) = \pi(Q)$ and with $z(P) > z(Q)$. ⊞

Before we move on to the actual calculation of the linking number let us introduce one more piece of notation: given any isomorphism $\varphi\colon V \to W$ between oriented vector spaces we write

$$\deg(\varphi) \;:=\; \begin{cases} +1, & \text{if } \varphi\colon V \to W \text{ is orientation-preserving,} \\ -1, & \text{if } \varphi\colon V \to W \text{ is orientation-reversing.} \end{cases}$$

Using this notation we calculate

$$\begin{array}{rcl} \mathrm{lk}(K,L) & \overset{\text{see above definition}}{\underset{\downarrow}{=}} & (-1)^l \cdot \deg \begin{pmatrix} K \times L \overset{\Theta}{\to} S^{n-1} \\ (x,y) \mapsto \frac{x-y}{\|x-y\|} \end{pmatrix} \overset{\text{by the definition of deg on page 899, since } e_n \text{ is a regular value}}{\underset{\downarrow}{=}} (-1)^l \cdot \sum_{(P,Q) \in \Theta^{-1}(e_n)} \deg(\widetilde{\mathrm{T}}_P K \times \widetilde{\mathrm{T}}_Q L \xrightarrow{\mathrm{D}\Theta_{(P,Q)}} \widetilde{\mathrm{T}}_{e_n} S^{n-1}) \\ & \overset{\text{by the chain rule, and since by the definition of the standard orientation on } S^{n-1}\text{, see page 681, we see that the map } \mathrm{D}(\pi|_{S^{n-1}})_{e_n}\colon \widetilde{\mathrm{T}}_{e_n} S^{n-1} \to \mathbb{R}^{n-1} \text{ is an orientation-preserving isomorphism}}{\underset{\uparrow}{=}} & (-1)^l \cdot \sum_{(P,Q) \in \Theta^{-1}(e_n)} \deg(\mathrm{D}(\pi \circ \Theta)_{(P,Q)}\colon \widetilde{\mathrm{T}}_P K \times \widetilde{\mathrm{T}}_Q L \to \mathbb{R}^{n-1}) \\ & \overset{\text{follows from Statement (2) of the claim, the fact that } z(P-Q) > 0 \text{ and the fact that } -\pi(w) = \pi(-w)}{\underset{\uparrow}{=}} & (-1)^l \cdot \sum_{(P,Q) \in \Theta^{-1}(e_n)} \deg \begin{pmatrix} \widetilde{\mathrm{T}}_P K \times \widetilde{\mathrm{T}}_Q L \to \mathbb{R}^{n-1} \\ (v,w) \mapsto \pi(v) + \pi(-w) \end{pmatrix} \\ & \overset{\text{for the } l\text{-dimensional vector space } W = \widetilde{\mathrm{T}}_Q L \text{ the degree of the map } w \mapsto w \text{ equals } (-1)^l}{\underset{\uparrow}{=}} & \sum_{(P,Q) \in \Theta^{-1}(e_n)} \deg \begin{pmatrix} \widetilde{\mathrm{T}}_P K \times \widetilde{\mathrm{T}}_Q L \to \mathbb{R}^{n-1} \\ (v,w) \mapsto \pi(v) + \pi(w) \end{pmatrix} \overset{\text{by the definition of the sign}}{\underset{\uparrow}{=}} \sum_{(P,Q) \in \Theta^{-1}(e_n)} \mathrm{sign}(\pi(\widetilde{\mathrm{T}}_P K), \pi(\widetilde{\mathrm{T}}_Q L)). \end{array}$$

It follows from Statement (3) of the claim that the sum on the right hand side is precisely the sum on the right hand side of the proposition. ■

We continue with the development of the theory of linking numbers.

Proposition 44.2. Let K and L be disjoint closed oriented smooth submanifolds of $\mathbb{R}^n$ with $\dim(K)+\dim(L)=n-1$.

(1) If $F\colon (K\sqcup L)\times[0,1]\to\mathbb{R}^n$ is a smooth isotopy, then,

$$\operatorname{lk}(F_0(K),F_0(L)) \;=\; \operatorname{lk}(F_1(K),F_1(L)).$$

(2) We have the following equality

$$\operatorname{lk}(L,K) \;=\; (-1)^{\dim(K)\cdot\dim(L)+1}\cdot\operatorname{lk}(K,L).$$

Proof. We write $k=\dim(K)$ and $l=\dim(L)$.

(1) We consider the map

$$\begin{array}{rcl}\Xi\colon K\times L\times[0,1] & \to & S^2\\ (x,y) & \mapsto & \frac{F_t(x)-F_t(y)}{\|F_t(x)-F_t(y)\|}.\end{array}$$

Now we see that

$$\begin{array}{rcl}\operatorname{lk}(F_0(K),F_0(L)) &=& (-1)^l\cdot\deg(K\times L\xrightarrow{\Xi_0}S^2)\\ &\overset{\text{by the Degree Homotopy Invariance Theorem 38.3}}{\downarrow}&\\ &=& (-1)^l\cdot\deg(K\times L\xrightarrow{\Xi_1}S^2)\\ &=& \operatorname{lk}(F_1(K),F_1(L)).\end{array}$$

(2) We consider the three maps

$$\begin{array}{rcl}\Theta\colon K\times L & \to & S^{n-1}\\ (x,y) & \mapsto & \frac{x-y}{\|x-y\|}\end{array}\qquad\begin{array}{rcl}\tau\colon K\times L & \to & L\times K\\ (x,y) & \mapsto & (y,x)\end{array}\qquad\text{and}\qquad\begin{array}{rcl}\sigma\colon S^{n-1} & \to & S^{n-1}\\ z & \mapsto & -z.\end{array}$$

Then

$$\begin{array}{rcl}\operatorname{lk}(L,K) &\overset{\text{by definition}}{=}& (-1)^k\cdot\deg(\sigma\circ\Theta\circ\tau)\\ &=& (-1)^k\cdot(-1)^{k+l+1}\cdot\det(\Theta)\cdot(-1)^{k\cdot l}\\ &\overset{\text{by the Basics of Degree Lemma 38.8 and since}}{\uparrow}&\\ &&\sigma\text{ is orientation-preserving if and only if }n=k+l+1\text{ is even (see Exercise 25.5)}\\ &&\tau\text{ is orientation-preserving if and only if }k\cdot l\text{ is even (see Exercise 25.14)}\end{array}$$

$$\begin{array}{rcl}&\overset{\text{Degree Multiplicativity Theorem 38.4}}{=}&(-1)^k\cdot\deg(\sigma)\circ\deg(\Theta)\cdot\deg(\tau)\\ &=&(-1)^{k\cdot l+1}\cdot\operatorname{lk}(K,L).\end{array}$$

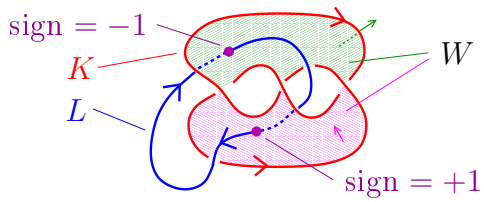
■

We conclude this chapter with the following proposition which can be very useful for calculating linking numbers of knots in $\mathbb{R}^3$.

Proposition 44.3. (Linking Numbers-via-Intersection Numbers Proposition) Let $n\in\mathbb{N}$ and let K and L be two disjoint closed oriented smooth submanifolds of $\mathbb{R}^n$ such that $\dim(K)+\dim(L)=n-1$. Suppose that there exists a compact orientable smooth submanifold W of $\mathbb{R}^n$ with $\partial W=K$ (as oriented smooth manifolds). If W intersects L transversally, then

$$\operatorname{lk}(K,L) \;=\; (-1)^{n+1}\cdot(W\cdot L)$$

where $W\cdot L$ denotes the algebraic intersection number that we introduced on page 989.

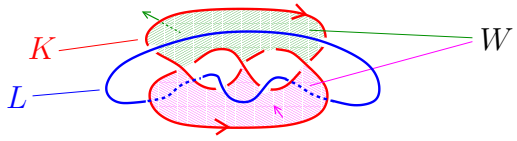


$$\operatorname{lk}(K,L)=W\cdot L=-1+1=0$$

Remark. In Proposition 131.13 we will give an alternative proof of the Linking Numbers-via-Intersection Numbers Proposition 44.3 in the special case that $n=3$.

We will prove the Linking Numbers-via-Intersection Numbers Proposition 44.3 by first considering two special cases.

Auxiliary Lemma 44.4. Let $n \in \mathbb{N}$ and let K and L be closed oriented smooth manifolds of $\mathbb{R}^n$ with $\dim(K) + \dim(L) = n - 1$. If there exists a compact orientable smooth submanifold W of $\mathbb{R}^n$ such that $\partial W = K$ (as oriented smooth manifolds) and which is disjoint from L, then $\mathrm{lk}(K, L) = 0$.



Proof of Lemma 44.4. Since W is disjoint from L we can consider the map

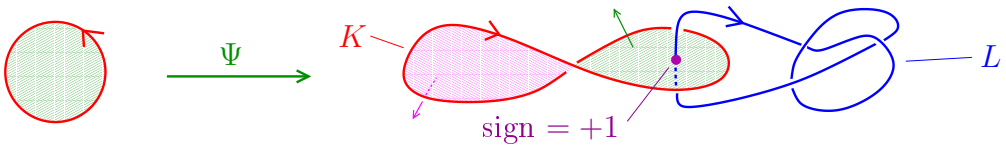
$$\begin{array}{rcl} \Theta\colon W \times L & \to & S^{n-1} \\ (x, y) & \mapsto & \frac{x-y}{\|x-y\|}. \end{array}$$

By Sard's Theorem 31.1 we know that $\Theta|_{K\times L}\colon K \times L \to S^{n-1}$ has a regular value $P \in S^{n-1}$. Now we see that[433]

$$\mathrm{lk}(K, L) \underset{\text{by definition}}{=} (-1)^{\dim(L)} \cdot \deg(\Theta|_{K\times L} \to S^{n-1}) \overset{\text{see page 989}}{=} (-1)^{\dim(L)} \cdot (\{P\} \cdot \Theta|_{K\times L}) \underset{\text{by the Intersection-Number Zero-if-Extends Proposition 43.4}}{=} 0. \qquad \blacksquare$$

Auxiliary Lemma 44.5. Let $n \in \mathbb{N}$ and let K and L be two disjoint closed oriented smooth submanifolds of $\mathbb{R}^n$ with $\dim(K) + \dim(L) = n - 1$. We suppose that there exists a smooth embedding $\Psi\colon \overline{B}^{\dim(K)+1} \to \mathbb{R}^n$ such that $\Psi\colon S^{\dim(K)} \to K$ is a diffeomorphism of oriented smooth manifolds. If $\Psi(\overline{B}^{k+1})$ intersects L transversally, then

$$\mathrm{lk}(K, L) = (-1)^{n+1} \cdot (\Psi(\overline{B}^{k+1}) \cdot L).$$



Proof of Lemma 44.5. First we consider a special case. Afterwards we will reduce the general case to the special case.

Claim 1. Let $\iota\colon \overline{B}^{k+1} \to \mathbb{R}^n$ be the standard smooth embedding that is given by $x \mapsto (x, 0)$. Furthermore let $\widetilde{L}$ be a closed oriented smooth submanifold of $\mathbb{R}^n$ that is disjoint from $\widetilde{K} := \iota(S^k)$ and with $\dim(\widetilde{L}) = n - k - 1$. If $\widetilde{L}$ is transverse to $\overline{B}^{k+1} \times \{0\}$ and to $S^k \times \{0\} \times \mathbb{R}$, then

$$\mathrm{lk}(\widetilde{K}, \widetilde{L}) = (-1)^{n+1} \cdot (\iota(\overline{B}^{k+1}) \cdot \widetilde{L}).$$

Proof. Since $\widetilde{L}$ is compact there exists an $r < 0$ such that $\widetilde{L} \subset \mathbb{R}^{n-1} \times (r, \infty)$. Using the Smooth Transition Function Lemma 19.4 one can easily show that there exists a smooth embedding

$$\begin{array}{rcl} \gamma\colon [0, 1] & \to & \mathbb{R}^2 \\ t & \mapsto & \gamma(t) = (\alpha(t), \beta(t)) \end{array}$$

[433] The argument becomes even more direct if one has solved Exercise 38.11.

with the following properties:

(1) For $t \in [0, \frac{1}{4}]$ we have $\gamma(t) = (t, -r-1)$.
(2) For $t \in [\frac{1}{4}, \frac{3}{4}]$ we have $\gamma(t) \in [\frac{1}{4}, 1] \times [-r-1, -r]$.
(3) For $t \in [\frac{3}{4}, 1]$ we have $\gamma(t) = (1, -r \cdot (1 - (t - \frac{3}{4})))$.

Next we consider the smooth map

$$\begin{array}{rcl} \overline{B}^{k+1} \times [0,1] & \to & \mathbb{R}^n \\ (v,t) & \mapsto & \begin{cases} (v, 0, \ldots, 0, t \cdot (r+1)), & \text{if } \|v\| < \frac{1}{4}, \\ (\frac{v}{\|v\|} \cdot (\|v\| \cdot (1-t) + t \cdot \alpha(\|v\|)), 0, \ldots, 0, t \cdot \beta(\|v\|)), & \text{if } \|v\| > 0. \end{cases} \end{array}$$

We make the following observations and preparations:

(1) Note that $F_1(\overline{B}^{k+1}) \cap (\mathbb{R}^{n-1} \times (-\infty, 0]) = S^k \times \{0\} \times (-r, 0]$.
(2) As above we denote by $\pi \colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ the projection onto the first $n-1$ coordinates.
(3) Since $\widetilde{L}$ is transverse to $S^k \times \{0\} \times \mathbb{R}$ we obtain from the Submanifold Transversal Intersection Theorem 42.1 that $(S^k \times \{0\} \times \mathbb{R}) \cap \widetilde{L}$ is a compact 0-dimensional submanifold, i.e. it is a finite set. We denote the intersection points of $F_1(\overline{B}^{k+1})$ with $\widetilde{L}$ by $(x_i, 0, t_i) \in S^k \times \{0\} \times (-r, 0)$, $1, \ldots, m$.
(4) Note that the points $(x_i, 0, t_i) \in \widetilde{L}$, $i = 1, \ldots, m$ are precisely the points on $\widetilde{L}$ such that $\pi(\widetilde{K}) \cap \pi(\widetilde{L}) \neq \varnothing$ and such that the last coordinate of the point on $\widetilde{K}$ is greater than the last coordinate of the point on $\widetilde{L}$.
(5) It follows from the Boundary Orientation Lemma 25.17 (4) together with the Product Orientation Proposition 25.22 (5) that the map $\Theta \colon (\frac{3}{4}, 1] \times S^k \to \overline{B}^{k+1}$ given by $(t, v) \mapsto t \cdot v$ is orientation-preserving.
(6) It follows easily from (5) that the map

$$\begin{array}{rcl} F_1 \colon \{v \in \overline{B}^{k+1} \mid \|v\| > \frac{3}{4}\} & \to & S^k \times \{0\} \times [0, r) \\ v & \mapsto & (\frac{v}{\|v\|}, 0, \ldots, 0, -r \cdot (1 - (\|v\| - \frac{3}{4}))) \end{array}$$

is *orientation-reversing*.
(7) We have $\det \begin{pmatrix} 0 & \mathrm{id}_{n-1} \\ 1 & 0 \end{pmatrix} = (-1)^n$.
(8) Let $i \in \{1, \ldots, m\}$. It follows easily from (6) and (7) and the various definitions that

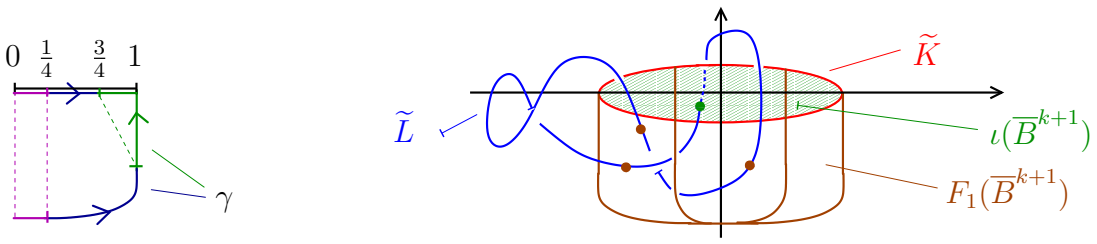
$$\underbrace{\begin{array}{c}\text{sign of the intersection point} \\ (x_i, 0, t_i) \text{ of } F_1(\overline{B}^{k+1}) \text{ and } L\end{array}}_{\text{as defined on page 989}} = (-1)^{n+1} \cdot \underbrace{\mathrm{sign}(\pi(\widetilde{\mathrm{T}}_{(x_i,0,t)}\widetilde{K}), \pi(\widetilde{\mathrm{T}}_{(x_i,0,t)}\widetilde{L}))}_{\text{as defined on page 1008}}$$

Then we see that

$$\iota(\overline{B}^{k+1}) \cdot \widetilde{L} = F_0(\overline{B}^{k+1}) \cdot \widetilde{L} \underset{\uparrow}{=} F_1(\overline{B}^{k+1}) \cdot \widetilde{L} \underset{\uparrow}{=} \sum_{i=1}^{m} \mathrm{sign}(x_i) \underset{\uparrow}{=} (-1)^{n+1} \cdot \mathrm{lk}(\widetilde{K}, \widetilde{L}).$$

Intersection Number Homotopy Invariance Theorem 43.6 — Lemma 43.1 — follows from (8) and Proposition 44.1 ⊞

The statement of the following Claim 2 is identical to the statement of Claim 1, except that we will manage to drop the hypothesis that $\widetilde{L}$ is transverse to $S^k \times \{0\} \times \mathbb{R}$.



Claim 2. Let $\widehat{L}$ be a closed oriented smooth submanifold of $\mathbb{R}^n$ that is disjoint from the submanifold $\widetilde{K} := \iota(S^k)$ and with $\dim(\widehat{L}) = n-k-1$. If $\widehat{L}$ is transverse to $\overline{B}^{k+1} \times \{0\}$, then
$$\mathrm{lk}(\widetilde{K}, \widehat{L}) \;=\; (-1)^{n+1} \cdot (\iota(\overline{B}^{k+1}) \cdot \widehat{L}).$$

Proof. It follows easily from the Transversality-Extends-to-Neighborhood Lemma 42.9 together with the Transversality Theorem 42.5, applied to the smooth manifold $M = \mathbb{R}^n \setminus (S^k \times \{0\})$, that there exists a smooth isotopy $G\colon \widehat{L} \times [0,1] \to \mathbb{R}^n$ rel an open neighborhood of $\widehat{L} \cap (\overline{B}^{k+1} \times \{0\})$, such that $G_0 = \mathrm{id}_{\widehat{L}}$ and such that $(S^k \times \{0\} \times \mathbb{R}) \pitchfork G_1(\widehat{L})$. We set $\widetilde{L} := G_1(\widehat{L})$. Now we see that
$$\mathrm{lk}(\widetilde{K}, \widehat{L}) \underset{\text{Proposition 44.2}}{\uparrow=} \mathrm{lk}(\widetilde{K}, \widetilde{L}) \underset{\text{Claim 1}}{\uparrow=} (-1)^{n+1} \cdot (\iota(\overline{B}^{k+1}) \cdot \widetilde{L}) \underset{\text{since } \widetilde{L} = \widehat{L} \text{ in a neighborhood of } \iota(\overline{B}^{k+1})}{\uparrow=} (-1)^{n+1} \cdot (\iota(\overline{B}^{k+1}) \cdot \widehat{L}).$$
⊞

Now we finally turn to the general setting of the proposition. Thus let K and L be two disjoint closed oriented smooth submanifolds of $\mathbb{R}^n$ with $\dim(K) + \dim(L) = n-1$. We set $k := \dim(K)$. We suppose that there exists a smooth embedding $\Psi\colon \overline{B}^{k+1} \to \mathbb{R}^n$ such that $\Psi\colon S^k \to K$ is a diffeomorphism of oriented smooth manifolds. It follows from the Lower-Dimensional Smooth Ball Embedding Theorem 37.10 that there exists a diffeotopy $F\colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ such that $F_0 = \mathrm{id}$ and such that $F_1 \circ \Psi\colon \overline{B}^{k+1} \to \mathbb{R}^n$ is the standard embedding $\iota\colon \overline{B}^{k+1} \to \mathbb{R}^n$ given by $x \mapsto (x, 0)$. Note that it follows from $F_0 = \mathrm{id}$ and the Smooth Isotopy-Orientation Proposition 34.2 that F_1 is orientation-preserving. Now we see that
$$\mathrm{lk}(K, L) \underset{\text{Proposition 44.2}}{\uparrow=} \mathrm{lk}(F_1(K), F_1(L)) \underset{\substack{\text{by Claim 2 since} \\ F_1(K) = \widetilde{K}}}{\uparrow=} \iota(\overline{B}^{k+1}) \cdot F_1(L) \underset{\substack{\text{Lemma 43.3 applied to the} \\ \text{orientation-preserving diffeomorphism } F_1^{-1}}}{\uparrow=} \Psi(\overline{B}^{k+1}) \cdot L.$$
■

Proof of the Linking Numbers-via-Intersection Numbers Proposition 44.3. Let K and L be disjoint closed oriented smooth submanifolds of $\mathbb{R}^n$ with $\dim(K)+\dim(L) = n-1$. We suppose that there exists a compact orientable smooth submanifold W of $\mathbb{R}^n$ with $\partial W = K$ and with $W \pitchfork L$.

It follows from the Submanifold Transversal Intersection Theorem 42.1 that $W \cap L$ is a compact 0-dimensional submanifold of W. This means that $W \cap L$ consists of finitely many points $P_1, \ldots, P_m \in W \setminus \partial W$.

We set $k := \dim(K)$. Evidently we can pick orientation-preserving smooth embeddings $\Psi_i\colon \overline{B}^{k+1} \to W \setminus \partial W$, $i = 1, \ldots, m$, with disjoint images, and such that $\Psi_i(0) = P_i$ for

$i = 1, \dots, m$. We set $V := W \setminus \bigcup_{i=1}^{m} \Psi_i(B^{k+1})$ and for $i = 1, \dots, m$ we set $J_i := \Psi_i(S^k)$. We equip J_i with the orientation coming from the standard orientation of S^k.

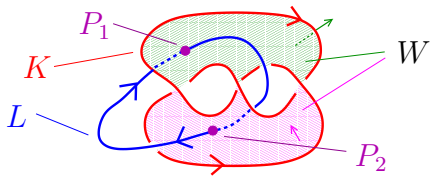
By the Codimension-Zero Smooth Embedding Proposition 24.16 we know that V is a compact codimension-zero submanifold of W. If we equip V with the orientation coming from W, then it follows from Exercise 25.19 that $\partial V = K \cup \bigcup_{i=1}^{m} (-J_i)$. Finally we see that

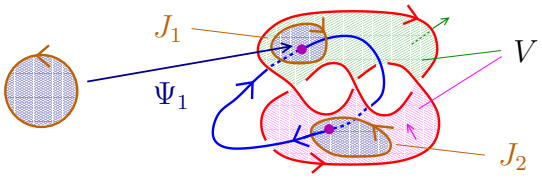
$$\operatorname{lk}(K, L) \underset{\uparrow}{=} \operatorname{lk}\Big(K \cup \bigcup_{i=1}^{m}(-J_i), L\Big) + \sum_{i=1}^{m} \operatorname{lk}(J_i, L) \underset{\uparrow}{=} 0 + \sum_{i=1}^{m} \Psi_i(\overline{B}^{k+1}) \cdot L \underset{\uparrow}{=} W \cdot L$$

follows easily from the definition of the linking number and basic properties of the degree

by Lemma 44.4 since $\partial V = K \cup \bigcup_{i=1}^{m}(-J_i)$ and by Lemma 44.5, since $J_i = \partial \Psi_i(\overline{B}^{k+1})$

since $V \cap L = \varnothing$ ■





Exercises for Chapter 44.

Exercise 44.1. An *oriented link in* $\mathbb{R}^3$ is a closed oriented smooth submanifold of $\mathbb{R}^3$. We say an m-component link $L \subset \mathbb{R}^3$ is a *trivial link* if it bounds m disjoint smoothly embedded disks in $\mathbb{R}^3$. It follows immediately from the Lower-Dimensional Smooth Ball Embedding Theorem 37.10 that all m-component trivial links are in fact smoothly isotopic.

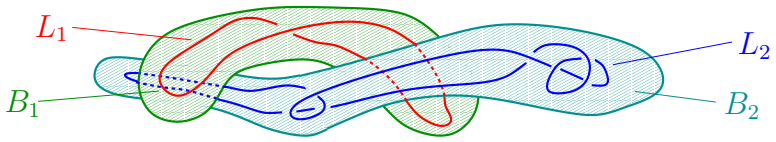
(a) Give an example of two disjoint knots K, L in $\mathbb{R}^3$ such that K and L are smoothly isotopic to the trivial knot, such that $\operatorname{lk}(K, L) = 0$ but such that $K \sqcup L$ is not smoothly isotopic to a trivial link.

(b) Give an example of a 3-component link $L = L_1 \cup L_2 \cup L_3$ in $\mathbb{R}^3$ such that any two components are isotopic to a trivial link but such that L itself is not smoothly isotopic to a trivial link.

(c) Generalize (b) from $m = 3$ to any $m \in \mathbb{N}_{\geq 3}$.

Remark. We lack the tools right now to show that a link with zero linking numbers is not smoothly isotopic to a trivial link. Thus with the methods we have developed so far we cannot really give a complete answer to the challenges. But this should not prevent the reader from trying to find examples.

Exercise 44.2. An *oriented link in* $\mathbb{R}^3$ is a closed oriented smooth submanifold of $\mathbb{R}^3$. An m-component link $L = L_1 \sqcup \dots \sqcup L_m \subset \mathbb{R}^3$ is called *split link* if there exist disjoint smoothly embedded closed 3-dimensional balls $B_1, \dots, B_m \subset \mathbb{R}^3$ such that each L_i is contained in B_i. Show that if $L = L_1 \sqcup \dots \sqcup L_m \subset \mathbb{R}^3$ is a split link, then for any $i \neq j$ we have $\operatorname{lk}(L_i, L_j) = 0$.



Exercise 44.3. Let $n \in \mathbb{N}$, let $L = S^{n-1} \subset \mathbb{R}^n$ be the sphere equipped with the standard orientation from page 681 and let $K = \{P\}$ be the 0-dimensional submanifold that consists of a single point $P \in \mathbb{R}^n \setminus S^{n-1}$ and that is equipped with the orientation $+1$. Show that

$$\mathrm{lk}(\{P\}, S^{n-1}) \;=\; \begin{cases} 1, & \text{if } P \in B^n, \\ 0, & \text{otherwise.} \end{cases}$$

45. The smooth Euler characteristic and the Smooth Poincaré-Hopf Theorem

45.1. **The smooth Euler characteristic.** The following definition generalizes the definition from page 989 of the algebraic intersection number of two transverse submanifolds to the non-transverse case.

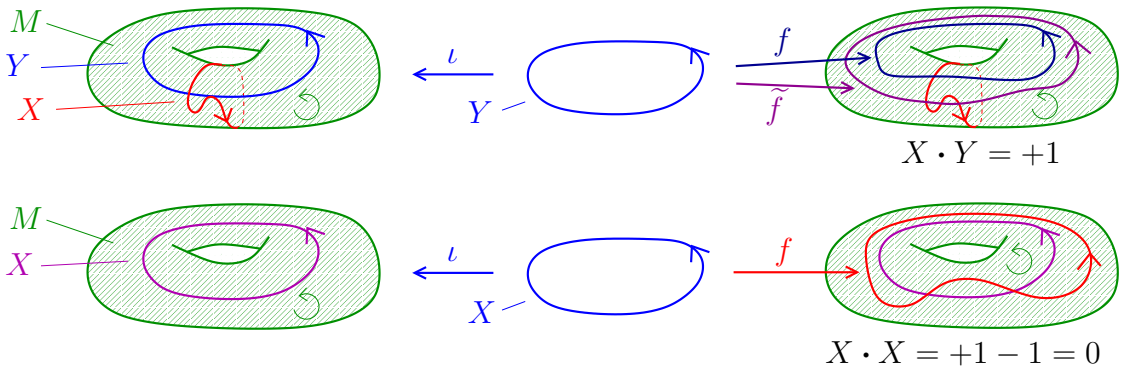
Definition. Let M be an oriented smooth manifold.

(1) Let X and Y be two closed oriented submanifolds of M of complementary dimension, i.e. we have $\dim(X) + \dim(Y) = \dim(M)$. Let $\iota\colon Y \to M$ be the inclusion map. By the Transversality Theorem 42.5 the map ι is smoothly isotopic to a smooth embedding $f\colon Y \to M$ with $X \pitchfork f$. It follows from the Intersection Number Homotopy Invariance Theorem 43.5 that

$$X \cdot Y \;:=\; X \cdot f \;\in\; \mathbb{Z}$$

does not depend on the choice of f. We refer to $X \cdot Y$ as the algebraic intersection number of X and Y.

(2) If M is even-dimensional and if X is a closed oriented submanifold of M such that $2 \cdot \dim(X) = \dim(M)$, then in (1) we can take $Y = X$ and we refer to $X \cdot X \in \mathbb{Z}$ as the algebraic self-intersection number of X



We continue with an analogue of Lemmas 43.2 and 43.3

Lemma 45.1. Let M be an oriented m-dimensional smooth manifold and let $X, Y \subset M$ be two closed oriented submanifolds of M with $\dim(X) + \dim(Y) = \dim(M)$.

(1) The following equality holds:

$$Y \cdot X \;=\; (-1)^{\dim(X)\cdot\dim(Y)} \;\cdot\; X \cdot Y.$$

(2) If we reverse the orientation of precisely one of X, Y or M, then the sign of the algebraic intersection number flips.

Proof.

(1) One might think that this statement is just a special case of Lemma 43.2. But that is not the case: in the definition of the algebraic intersection number in the non-transverse case we apply a smooth isotopy only to the second submanifold. Thus the definition of the algebraic intersection number is not symmetric in X and Y.

We write $k = \dim(X)$ and $l = \dim(Y)$. We denote by $i\colon X \to M$ and $j\colon Y \to M$ the inclusion maps. By the Transversality Theorem 42.4 there exists a smooth isotopy

$G\colon X \times [0,1] \to M$ with $X_0 = i\colon X \to M$ and such that $Y \pitchfork G_1$. By the Isotopy Extension Theorem 36.1 there exists a diffeotopy $H\colon M \times [0,1] \to M$ with $H_0 = \mathrm{id}$ such that for each $(y,t) \in Y \times [0,1]$ we have $H(y,t) = G(y,t)$. Now we see that

$$\begin{array}{lllll}
Y \cdot X & \overset{\text{by definition}}{=} & Y \cdot G_1 & \overset{\text{by Lemma 43.3 and the Smooth Isotopy-Orientation Proposition 34.2}}{=} & H_1^{-1}(Y) \cdot (H_1^{-1} \circ G_1) \\
& & & & = (-1)^{k\cdot l} \cdot X \cdot H_1^{-1}(Y) \\
& & & & \overset{\text{Lemma 43.2}}{=} (-1)^{k\cdot l} \cdot X \cdot H_1^{-1}(Y)
\end{array}$$

$$\begin{array}{lll}
H_1^{-1}(Y) \cdot (H_1^{-1} \circ G_1) & \overset{\text{since } H_1^{-1} \circ G_1 = j}{=} & H_1^{-1}(Y) \cdot X \\
(-1)^{k\cdot l} \cdot X \cdot H_1^{-1}(Y) & = & (-1)^{k\cdot l} \cdot X \cdot H_1^{-1} \\
\overset{\text{by definition}}{=} (-1)^{k\cdot l} \cdot X \cdot H_1^{-1} & = & (-1)^{k\cdot l} \cdot X \cdot Y.
\end{array}$$

by definition of $X \cdot Y$ since the map $Y \times [0,1] \to M,\ (y,t) \mapsto H_t^{-1}(y)$ is a smooth isotopy

(2) This statement follows from the Compatible Orientations Lemma 25.5 (2). ■

Before we can introduce the key definition of this chapter we need to state and prove the following elementary lemma.

Lemma 45.2. Let M be a smooth manifold with $\partial M = \varnothing$.

(1) The diagonal $\Delta_M := \{(x,x) \,|\, x \in M\}$ is a proper submanifold of the smooth manifold $M \times M$.

(2) The diagonal Δ_M is diffeomorphic to M.

Proof. [434] Let M be an n-dimensional smooth manifold with $\partial M = \varnothing$.

(1) Let $(x,x) \in \Delta_M$. We pick a chart $\Phi\colon U \to V$ for $x \in M$. It is straightforward to verify that the map

$$\begin{array}{ccccc}
U \times U & \xrightarrow{\Phi\times\Phi} & V \times V & \to & \{(c,d) \in \mathbb{R}^{2n} \,|\, c+d \in V \text{ and } d \in V\} \\
& & (a,b) & \mapsto & (a-b,b)
\end{array}$$

is a submanifold chart for Δ_M. Since M is by definition Hausdorff we know by Exercise 5.23 that the diagonal Δ_M is a closed subset of $M \times M$. This shows that Δ_M is a proper submanifold of $M \times M$.

(2) We consider the following two maps:

$$\begin{array}{rcl} p\colon \Delta_M & \to & M \\ (x,x) & \mapsto & x \end{array} \qquad \text{and} \qquad \begin{array}{rcl} d\colon M & \to & \Delta_M \\ x & \mapsto & (x,x). \end{array}$$

Evidently these two maps are inverses of one another. Furthermore it follows from the Smooth Manifold Product Proposition 19.31 and the Submanifold Smooth Structure Lemma 19.24 that p and d are smooth. This shows that both maps are diffeomorphisms. ■

Definition. Let M be a closed orientable smooth manifold. We pick an orientation and we consider the product $M \times M$, equipped with the product orientation introduced on page 695, and we consider the submanifold $\Delta_M := \{(x,x) \,|\, x \in M\}$, called the diagonal of M. By Lemma 45.2 (2) we know that Δ_M is orientable. We equip Δ_M with an orientation.

[434] Note that this proof solves Exercise 19.32.

We define the smooth Euler characteristic $\chi^{\mathrm{sm}}(M)$ of M as follows:

$$\chi^{\mathrm{sm}}(M) := \Delta_M \cdot \Delta_M.$$

Remark. On page 1519 and page 1882 we will define the Euler characteristic $\chi(X)$ of suitable topological spaces X. Furthermore we will see in the Smooth Manifolds-Invariants Proposition 96.6 and alternatively on page 2997 that this more general Euler characteristic is defined for every compact smooth manifold. We will see in the Intersection Number-via-Cup Product Theorem 126.5 and Proposition 128.1, or alternatively in Corollary 129.5 (which we prove twice, once on page 2821 and once on page 3016) that the smooth Euler characteristic $\chi^{\mathrm{sm}}(M)$ of a closed orientable smooth manifold M agrees with the more general Euler characteristic $\chi(M)$.

Lemma 45.3.

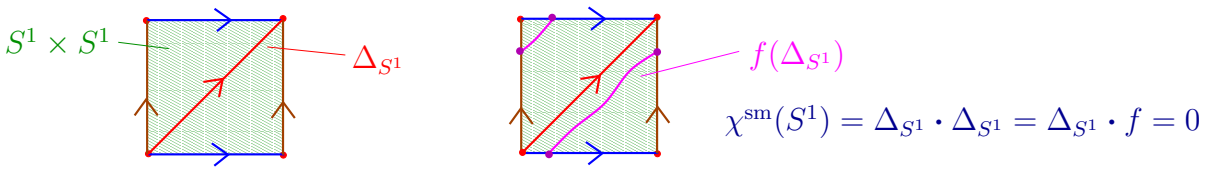
(1) Let M be a closed orientable smooth manifold. The smooth Euler characteristic $\chi^{\mathrm{sm}}(M)$ is independent of the choice of the orientation of M and independent of the choice of the orientation of Δ_M.
(2) Let M and N be two closed orientable smooth manifolds. If M and N are diffeomorphic, then $\chi^{\mathrm{sm}}(M) = \chi^{\mathrm{sm}}(N)$.

Proof.

(1) Let M be a closed orientable smooth manifold. The desired statement follows immediately from the following two facts:
- By the Product Orientation Proposition 25.22 (3) we have $(-M)\times(-M) = M\times M$.
- By Lemma 45.1 (2) we have $(-\Delta_M)\cdot(-\Delta_M) = -\Delta_M\cdot(-\Delta_M) = \Delta_M\cdot\Delta_M$.

(2) This statement follows easily from the definitions. ∎

Example. In the figure below we consider $M = S^1$. The diagonal $\Delta_{S^1} \subset S^1 \times S^1$ is the curve shown in the figure. We denote by $\iota\colon \Delta_{S^1} \to S^1 \times S^1$ the inclusion and as in the figure we pick a map $f\colon \Delta_{S^1} \to S^1 \times S^1$ that is smoothly isotopic to ι and which satisfies $\Delta_{S^1} \pitchfork f$. We see that $\Delta_{S^1} \cdot \Delta_{S^1} = \Delta_{S^1} \cdot f = 0$. In fact the next lemma shows that this is not a coincidence.



Lemma 45.4. (Odd-Dimensional Manifold-Euler Characteristic Lemma) The smooth Euler characteristic of any closed orientable *odd-dimensional* smooth manifold is zero.

Remark. In the Odd-Dimensional Manifold-Euler Characteristic Proposition 119.7 and the Odd-Dimensional Manifold-Euler Characteristic Corollary 140.5 we will reprove and generalize the Odd-Dimensional Manifold-Euler Characteristic Lemma 45.4.

Proof. Let M be a closed orientable $(2n+1)$-dimensional smooth manifold. By Lemma 45.2 we know that the diagonal Δ_M is also an $(2n+1)$-dimensional smooth manifold. Now we see that

$$\chi^{\mathrm{sm}}(M) \underset{\text{by definition}}{=} \Delta_M \cdot \Delta_M \underset{\text{by Lemma 45.1 (1)}}{=} (-1)^{(2n+1)\cdot(2n+1)} \cdot \Delta_M \cdot \Delta_M = -\Delta_M \cdot \Delta_M \underset{\text{by definition}}{=} -\chi^{\mathrm{sm}}(M).$$

It follows that $\chi^{\mathrm{sm}}(M) = 0$. ∎

It is now natural to try to compute the smooth Euler characteristic of our favorite orientable smooth manifolds. We pose the following question.

Question 45.5.

(1) Let $n \in \mathbb{N}$ be even. What is the smooth Euler characteristic of S^n?
(2) Let $g \in \mathbb{N}$. What is the Euler characteristic of the surface Σ_g of genus g?

To answer Question 45.5 we will have to develop tools for calculating smooth Euler characteristics. In the following sections we will formulate and prove the Smooth Poincaré–Hopf Theorem 45.17 which allows us to compute the smooth Euler characteristic of closed orientable smooth manifolds in terms of "indices" of "non-degenerate" zeros of vector fields. We will then be able to answer Question 45.5 (1) in the Sphere-Euler Characteristic-Proposition 45.19 and we will be able to answer Question 45.5 (2) in the Surface-Euler Characteristic Proposition 45.20.

45.2. **Zeros of vector fields.** As on several occasions we adopt for the remainder of this chapter the following popular convention.

Convention. We will use the natural isomorphism from the Tangent Space-Isomorphism Proposition 23.16 to go back and forth between the physical tangent space $\widetilde{\mathrm{T}}_P M$ and the abstract tangent space $\mathrm{T}_P M$ of a smooth submanifold M of some $\mathbb{R}^n$.

Recall that on page 639 we introduced the notion of a smooth vector field on a smooth manifold and recall the following definition from the Vector Field-Pullback Lemma 23.21.

Notation. Let $f\colon M \to N$ be a local diffeomorphism between two smooth manifolds M and N and let w be a vector field on N. We denote by f^*w the vector field on M which assigns to each $P \in M$ the tangent vector $(\mathrm{D}f_P)^{-1}(w(f(P)))$.[435]

We continue with a new aspect of vector fields.

Definition. Let M be a smooth manifold and let v be a smooth vector field on M. Let $P \in M \setminus \partial M$ be a zero of v. We pick a chart $\Phi\colon U \to V$. We set $Q := \Phi(P)$ and we write $\Psi := \Phi^{-1}$. We consider the map

$$\begin{aligned} \Psi^*v\colon V &\to \mathbb{R}^n \\ x &\mapsto (\Psi^*v)(x) = \underbrace{\mathrm{D}\Phi_{\Psi(x)}(v(\Psi(x)))}_{\in \mathrm{T}_x\mathbb{R}^n = \widetilde{\mathrm{T}}_x\mathbb{R}^n = \mathbb{R}^n}. \end{aligned}$$

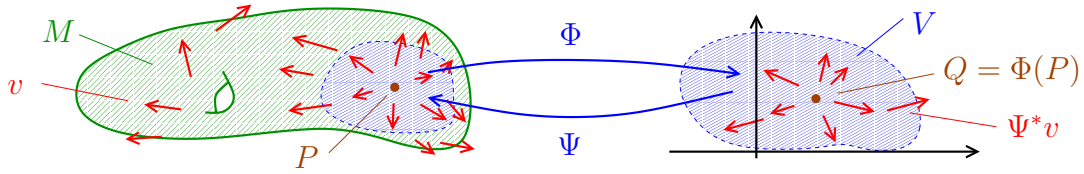
(1) If $\mathrm{D}(\Psi^*v)_Q$ is an invertible matrix, then we say that the zero is non-degenerate. Otherwise we call the zero degenerate.

[435] We will often use that by the Vector Field-Pullback Lemma 23.21 we know that if w is a smooth vector field, then f^*w is also smooth.

(2) If the zero P is non-degenerate, then we define

$$\operatorname{index}(v,P) := \begin{cases} +1, & \text{if } \det(\mathrm{D}(\Psi^* v)_Q) > 0, \\ -1, & \text{if } \det(\mathrm{D}(\Psi^* v)_Q) < 0. \end{cases}$$

In Lemma 45.6 we will see that these notions do not depend on the choice of the chart Φ.



Lemma 45.6. The notions introduced in the previous definition do not depend on the choice of the chart Φ.

In the proof of Lemma 45.6 we will need the following, potentially slightly confusing, little lemma.

Auxiliary Lemma 45.7. Let $A \subset \mathbb{R}^m$ be an open subset and let $\Xi\colon A \to \mathrm{M}(n \times k, \mathbb{R})$ and $u\colon A \to \mathbb{R}^k$ be smooth maps. Let $S \in A$. If $u(S) = 0$, then

$$\mathrm{D}(\underbrace{x \mapsto \Xi(x) \cdot u(x)}_{\text{map } A \to \mathbb{R}^n})_S \;=\; \Xi(S) \cdot \mathrm{D}u_S \quad \in \mathrm{M}(n \times m, \mathbb{R}).$$

Proof of Lemma 45.7. The lemma follows immediately from the Product Rule applied to each entry and the fact that $u(S) = 0$. ∎

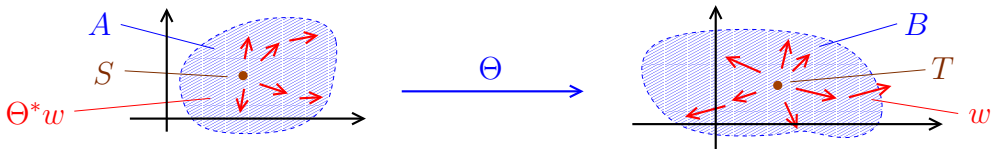
Proof of Lemma 45.6. Let M be an n-dimensional smooth manifold, let v be a smooth vector field on M and let $P \in M \setminus \partial M$ be a zero of v. We need to show that if we have two different charts $\Phi\colon U \to V$ and $\widetilde{\Phi}\colon \widetilde{U} \to \widetilde{V}$ around P, then we end up with the same notion of degeneracy and index.

We set $A := \Phi(U \cap \widetilde{U})$, $B = \widetilde{\Phi}(U \cap \widetilde{U})$ and we denote by $\Theta := \widetilde{\Phi} \circ \Phi^{-1}\colon A \to B$ the transition map between the two charts. It follows easily from Lemma 35.5 that we only have to prove the following claim.

Claim. Let $\Theta\colon A \to B$ be a diffeomorphism between two open subsets of some $\mathbb{R}^n$. Let $w\colon B \to \mathbb{R}^n$ be a smooth vector field on B with a zero at some point $T \in B$. We set $S := \Theta^{-1}(T) \in A$. We consider the maps

$$\begin{array}{rccc} \Theta^* w\colon & A & \to & \mathbb{R}^n \\ & x & \mapsto & (\Theta^* w)(x) \end{array} \qquad \text{and} \qquad \begin{array}{rccc} w\colon & B & \to & \mathbb{R}^n \\ & y & \mapsto & w(y). \end{array}$$

Then $\mathrm{D}(\Theta^* w)_S$ is invertible if and only if $\mathrm{D}w_T$ is invertible. Furthermore $\det(\mathrm{D}(\Theta^* w)_S) > 0$ if and only if $\det(\mathrm{D}w_T) > 0$.



Proof. Note that

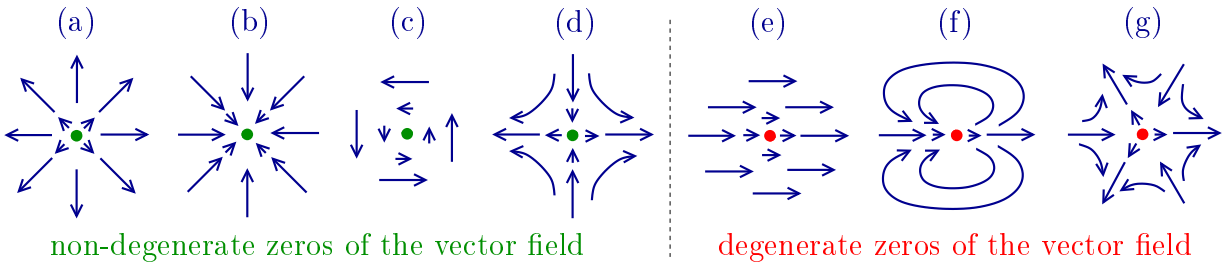
$$\begin{aligned}
\mathrm{D}(x \mapsto (\Theta^* w(x)))_S &\overset{\substack{\text{by definition and the Differential-equals-Differential Lemma 23.4}\\ \downarrow}}{=} \mathrm{D}\big(x \mapsto (\mathrm{D}\Theta_x)^{-1} \cdot (w \circ \Theta)(x)\big)_S \\
&\underset{\substack{\uparrow\\ \text{by Lemma 45.7 and since } (w\circ\Theta)(S)=0}}{=} (\mathrm{D}\Theta_S)^{-1} \cdot \mathrm{D}(w \circ \Theta)_S \underset{\substack{\uparrow\\ \text{by the Chain Rule and since } \Theta(S)=T}}{=} (\mathrm{D}\Theta_S)^{-1} \cdot \mathrm{D}w_T \circ \mathrm{D}\Theta_S.
\end{aligned}$$

This shows that $\mathrm{D}(\Theta^* w)_S$ is invertible if and only if $\mathrm{D}w_T$ is invertible. Furthermore we see that $\det(\mathrm{D}(\Theta^* w)_S) > 0$ if and only if $\det(\mathrm{D}w_T) > 0$. ■

Remark. On page 2818 we will introduce a more general notion of an index of a zero of a vector field which is defined also for degenerate zeros of vector fields. It follows from Lemma 76.15 that the two definitions of the index agree, if both are defined.

Example. We consider the vector fields on $\mathbb{R}^2 = \mathbb{C}$ with isolated zero at the origin that are shown in the figure below. In the following we use the identity chart $\Phi = \mathrm{id}$ to determine the various indices. We obtain the following table:

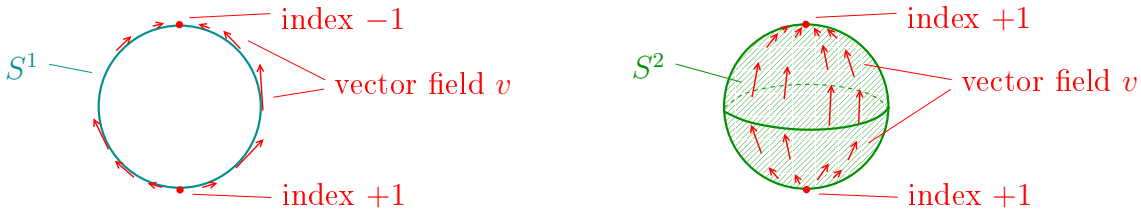
	(a)	(b)	(c)	(d)	(e)	(f)	(g)
vector field v at $(x,y)=z$	(x,y)	$(-x,-y)$	$(-y,x)$	$(x,-y)$	$(x^2+y^2,0)$	z^2	z^{-2}
$\mathrm{D}v_{(0,0)}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$
	non-degenerate zero				degenerate zero		
index	1	1	1	-1	not defined		



We state our last example of indices of zeros of vector fields as a lemma.

Lemma 45.8. Given any $n \in \mathbb{N}_0$ there exists a smooth vector field v on S^n with the following properties:

(1) The vector field has precisely two zeros, namely at the South Pole $S = (0, \dots, 0, -1)$ and at the North Pole $N = (0, \dots, 0, 1)$.
(2) The two zeros are non-degenerate and the indices are given by $\mathrm{index}(v, S) = 1$ and $\mathrm{index}(v, N) = (-1)^n$.



Proof. Statement (1) is taken care of by the following claim.

Claim. We think of S^n as a subspace of $\mathbb{R}^n \times \mathbb{R}$. The map

$$\begin{array}{rcl} v\colon S^n & \to & \mathbb{R}^{n+1} \\ \underbrace{(x,y)}_{x\in\mathbb{R}^n,\, y\in\mathbb{R}} & \mapsto & (x\cdot(-y), \|x\|^2) \end{array}$$

defines a smooth vector field on S^n with precisely two zeros, namely at S and N.

Proof. It follows from $\|x\|^2 = x_1^2 + \cdots + x_n^2$ that v is a smooth map. It is clear that for $(x,y) \in S^n \subset \mathbb{R}^n \times \mathbb{R}$ we have $v(x,y) = 0$ if and only if $x = 0$. In particular N and S are the only two zeros. Next note that for each $(x,y) \in S^n$ we have

$$\langle (x,y), v(x,y)\rangle \;=\; \langle (x,y), (x\cdot(-y), \|x\|^2\rangle \;=\; -\langle x,x\rangle \cdot y + y\cdot\|x\|^2 \;=\; 0.$$

Thus it follows from Lemma 23.1 that $v(x,y) \in \widetilde{\mathrm{T}}_{(x,y)}S^n$. This shows that v defines indeed a smooth vector field on the smooth manifold S^n. ⊞

It remains to prove Statement (2). For $\epsilon = \pm 1$ we consider the zero $P_\epsilon = (0,\dots,0,\epsilon)$ of our vector field v. We consider the chart

$$\begin{array}{rcl} \Phi_\epsilon\colon U := S^n_{>0} = \{(x,y) \in S^n \subset \mathbb{R}^n \times \mathbb{R} \,|\, y\cdot\epsilon > 0\} & \to & B^n \\ (x,y) & \mapsto & x. \end{array}$$

The inverse of the chart is given by $\Psi_\epsilon(x) = (x, \epsilon\cdot\sqrt{1-\|x\|^2})$. A straightforward calculation shows that $\Psi_\epsilon^* v$ is given by

$$\begin{array}{rcl} \Psi_\epsilon^* v\colon B^n & \to & \mathbb{R}^n \\ x & \mapsto & -\epsilon\cdot x\cdot\sqrt{1-\|x\|^2}. \end{array}$$

An elementary calculation shows that $\mathrm{D}(\Psi_\epsilon^* v)_0 = (-\epsilon)\cdot \mathrm{id}$. This shows that the zero of v at P_ϵ is non-degenerate and that $\mathrm{index}(v, P_\epsilon) = (-\epsilon)^n$. ■

Lemma 45.9. Let M be a smooth manifold and let v be a smooth vector field on M.

(1) Every non-degenerate zero of v is an isolated point in the set of zeros of v.
(2) If M is compact and if all zeros of v are non-degenerate, then v has only finitely many zeros.

Proof.

(1) Using a chart it is clear that we only need to deal with the case that M is an open subset of some $\mathbb{R}^n$ and that $v\colon M \to \mathbb{R}^n$ is a smooth vector field with a zero at $x = 0$. In this setting the statement follows easily from the Inverse Function Theorem 19.12.
(2) This statement follows from (1) together with the Compact-Closed Lemma 1.21, the Compact-Discrete Lemma 1.22 and the fact that smooth manifolds are by definition Hausdorff. ■

The above discussion motivates the following question.

Question 45.10. Does every closed smooth manifold admit a smooth vector field such that all zeros are non-degenerate?

To answer Question 45.10 we will develop some more general theory in the following section. We will then be able to answer the question in Proposition 45.15.

45.3. **The total tangent space and vector fields.** In this section we will introduce the total tangent space and we will prove several statements that relate the total tangent space with vector fields. This discussion might initially appear tangential to the topics so far. But in fact we will be so successful in this section, that afterwards the proofs of the main result of this chapter, namely the Smooth Poincaré–Hopf Theorem 45.17, will be a breeze.

We start out with the following new notation.

Notation.

(1) Let M be a smooth submanifold of $\mathbb{R}^n$. We define the total tangent space

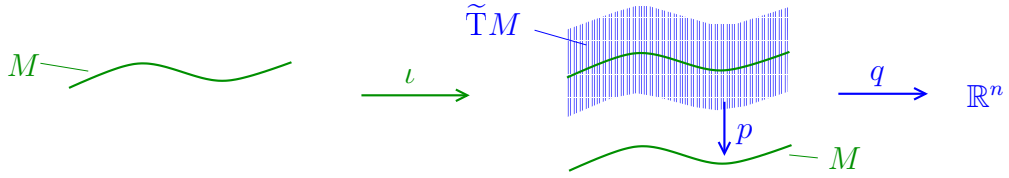
$$\widetilde{\mathrm{T}}M \;:=\; \{(x,v) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, x \in M \text{ and } v \in \widetilde{\mathrm{T}}_x M\}.$$

(2) Let M be a smooth submanifold of $\mathbb{R}^m$ and let N be an smooth submanifold of $\mathbb{R}^n$. Given any smooth map $f\colon M \to N$ we obtain an induced map

$$\begin{array}{rccc} f_*\colon & \widetilde{\mathrm{T}}M & \to & \widetilde{\mathrm{T}}N \\ & (x,v) & \mapsto & (f(x), \mathrm{D}f_x(v)). \end{array}$$

(3) We consider the maps

$$\begin{array}{rccc} p\colon & \widetilde{\mathrm{T}}M & \to & M \\ & (x,y) & \mapsto & x \end{array} \qquad \begin{array}{rccc} q\colon & \widetilde{\mathrm{T}}M & \to & \mathbb{R}^n \\ & (x,y) & \mapsto & y \end{array} \qquad \text{and} \qquad \begin{array}{rccc} \iota\colon & M & \to & \widetilde{\mathrm{T}}M \\ & x & \mapsto & (x,0). \end{array}$$



Remark. Let M be a smooth submanifold of $\mathbb{R}^n$. The total tangent space $\widetilde{\mathrm{T}}M$ *together* with the map $p\colon \widetilde{\mathrm{T}}M \to M$ is called the *tangent bundle* of M. The tangent bundle will play a significant role later on.

Lemma 45.11. Let $k \in \mathbb{N}$.

(1) Let M be a k-dimensional smooth submanifold of $\mathbb{R}^n$.
  (a) The subset $\widetilde{\mathrm{T}}M$ is a $2k$-dimensional smooth submanifold of $\mathbb{R}^n \times \mathbb{R}^n$.
  (b) For each $(x,v) \in \widetilde{\mathrm{T}}M$ we have $\widetilde{\mathrm{T}}_{(x,v)}\widetilde{\mathrm{T}}M = \widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M \subset \mathbb{R}^n \times \mathbb{R}^n$.
  (c) If M is oriented, then there exists a unique orientation on $\widetilde{\mathrm{T}}M$ such that for each $(x,v) \in \widetilde{\mathrm{T}}M$ the equality $\widetilde{\mathrm{T}}_{(x,v)}\widetilde{\mathrm{T}}M = \widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M \subset \mathbb{R}^n \times \mathbb{R}^n$ is an equality of oriented vector spaces.

(2) Let M be a smooth submanifold of $\mathbb{R}^m$ and let N be a smooth submanifold of $\mathbb{R}^n$. Furthermore let $f\colon M \to N$ be a smooth map.
  (a) The map $f_*\colon \widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}N$ is smooth.
  (b) If f is an orientation-preserving local diffeomorphism, then so is $f_*\colon \widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}N$.

(3) The construction in (1a) and the maps in (2a) define a covariant functor from the category $\mathcal{SmMfd}_{\mathbb{R}^n}$ of smooth submanifolds of some $\mathbb{R}^n$, which we introduced on page 626, to itself.

Proof.

(1) (a) Recall that we denote by $p\colon \widetilde{\mathrm{T}}M \to M$ the projection. Let $\Phi\colon U \to V$ be a submanifold chart for M. One can easily verify that

$$\begin{aligned} p^{-1}(U) = U \times \mathbb{R}^n &\to V \times \mathbb{R}^n \\ (x, v) &\mapsto (\Phi(x), \mathrm{D}\Phi_x(v)) \end{aligned}$$

is, after a permutation of coordinates, a submanifold chart for $\widetilde{\mathrm{T}}M$. Thus we have shown that $\widetilde{\mathrm{T}}M \to M$ is a $2k$-dimensional smooth submanifold of $\mathbb{R}^n \times \mathbb{R}^n$.

(b) Let $(x, v) \in \widetilde{\mathrm{T}}M$. One can easily verify that $\widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M \subset \widetilde{\mathrm{T}}_{(x,v)}\widetilde{\mathrm{T}}M$. But since both are $2k$-dimensional vector spaces we actually have equality.

(c) We leave it to the reader to verify that this way we get an orientation for $\widetilde{\mathrm{T}}M$.

(2) We leave the verification of this statement to the reader.

(3) This statement follows easily from the definitions. ■

Notation. Let M be a k-dimensional smooth submanifold of $\mathbb{R}^n$. Given a smooth vector field $v\colon M \to \mathbb{R}^n$ on M we consider the graph

$$\operatorname{graph}(v) \;:=\; \{(x, v(x)) \,|\, x \in M\} \;\subset \widetilde{\mathrm{T}}M.$$

As on page 526 one can easily verify that $\operatorname{graph}(v)$ is a k-dimensional smooth submanifold of $\widetilde{\mathrm{T}}M$ such that the map $p\colon \operatorname{graph}(v) \to M$, $(x, y) \mapsto x$, is a diffeomorphism. If M is oriented, then we equip $\operatorname{graph}(v)$ with the corresponding orientation.[436]

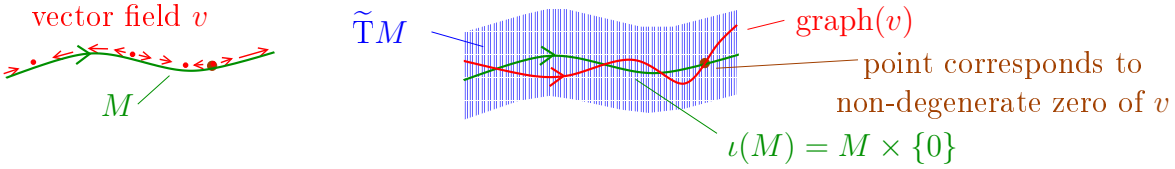
Proposition 45.12. Let M be a smooth submanifold of $\mathbb{R}^n$ with empty boundary and let v be a smooth vector field on M. The following statements hold:

(0) We have $(x, y) \in \iota(M) \cap \operatorname{graph}(v)$ if and only if $x \in M$, $y = 0$ and $v(x) = 0$.

(1) All zeros of the vector field v are non-degenerate if and only if the two submanifolds $\iota(M) = M \times \{0\}$ and $\operatorname{graph}(v)$ intersect transversally in $\widetilde{\mathrm{T}}M$, i.e. if $\iota(M) \pitchfork \operatorname{graph}(v)$.

(2) If x is non-degenerate zero of v, then

$$\operatorname{index}(v, x) \;=\; \text{sign of the intersection point } (x, 0) \text{ of } \iota(M) \text{ and } \operatorname{graph}(v).$$



We first prove Proposition 45.12 for open subsets of $\mathbb{R}^k$. This is almost precisely the content of the following lemma.

Lemma 45.13. Let W be an open subset of $\mathbb{R}^k$ and let $w\colon W \to \mathbb{R}^k$ be a smooth vector field on W. The following statements hold:

(1) Let $y \in W$ be a zero of the vector field w. The zero y is non-degenerate if and only if the submanifolds $\iota(W) = W \times \{0\}$ and $\operatorname{graph}(w)$ of in $\widetilde{\mathrm{T}}W = W \times \mathbb{R}^k$ intersect transversally in $(y, 0)$.

[436] If v is the zero vector field, then $\operatorname{graph}(v) = M \times \{0\} = \iota(M)$, and we see that $\iota(M)$ is also an (oriented) k-dimensional smooth submanifold of $\widetilde{\mathrm{T}}M$.

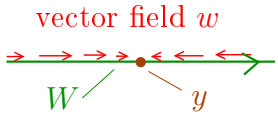
(2) If $y \in W$ is a non-degenerate zero of w, then

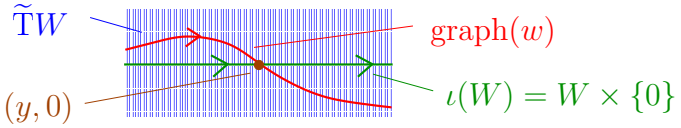
$$\text{index}(w, y) \;=\; \text{sign of the intersection point } (y,0) \text{ of } \iota(W) \text{ and } \text{graph}(w).$$

Proof of Lemma 45.13. Let W be an open subset of $\mathbb{R}^k$, let $w\colon W \to \mathbb{R}^k$ be a smooth vector field on W and let $y \in W$ be a point with $w(y) = 0$. Evidently $(y, 0)$ is an intersection point of $\iota(W) = W \times \{0\}$ and $\text{graph}(w)$. For the definition of a non-degenerate zero and the index of a non-degenerate zero we need to pick a chart. But since W is already an open subset of $\mathbb{R}^k$ we might as well use the identity chart. Next note that

$$\begin{aligned}\widetilde{\mathrm{T}}_{(y,0)}\iota(W) + \widetilde{\mathrm{T}}_{(y,0)}\,\text{graph}(w) \;&=\; \text{im}\begin{pmatrix} \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^k \\ a \mapsto \begin{pmatrix} a \\ 0 \end{pmatrix} \end{pmatrix} + \text{im}\begin{pmatrix} \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^k \\ a \mapsto \begin{pmatrix} a \\ \mathrm{D}w_{(y,0)} \cdot a \end{pmatrix} \end{pmatrix} \\ &=\; \begin{pmatrix} \mathrm{id} & \mathrm{id} \\ 0 & \mathrm{D}w_{(y,0)} \end{pmatrix} \begin{pmatrix} \mathbb{R}^k \\ \mathbb{R}^k \end{pmatrix}.\end{aligned}$$

We see that $\widetilde{\mathrm{T}}_{(y,0)}\iota(W) + \widetilde{\mathrm{T}}_{(y,0)}\,\text{graph}(w) = \mathbb{R}^{2k} = \widetilde{\mathrm{T}}_{(y,0)}\widetilde{\mathrm{T}}W$ if and only if $\mathrm{D}w_{(y,0)}\colon \mathbb{R}^k \to \mathbb{R}^k$ is an isomorphism. This observation, together with the definitions on pages 968 and 1021 concludes the proof of Statement (1).





Finally Statement (2) follows from the following little calculation:

$$\begin{array}{ccccccc} \text{sign of the intersection point} \\ (y,0) \text{ of } \iota(W) \text{ and } \text{graph}(w) \end{array} \;\underset{\uparrow}{=}\; \det\begin{pmatrix} \mathrm{id} & \mathrm{id} \\ 0 & \mathrm{D}w_{(y,0)} \end{pmatrix} \;=\; \det(\mathrm{D}w_{(y,0)}) \;\underset{\uparrow}{=}\; \text{index}(w, y).$$

by the above discussion and the definition on page 989 — see definition on page 1021 ■

Proof of Proposition 45.12. Let M be a k-dimensional smooth submanifold of $\mathbb{R}^n$ with empty boundary and let v be a smooth vector field on M. Statement (0) follows immediately from the definitions.

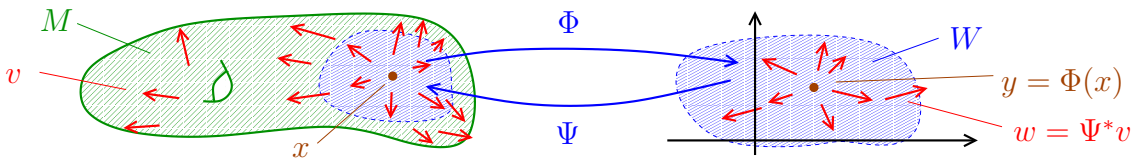
Let $x \in M$ be a zero of the vector field. We pick a chart $\Phi\colon V \to W$ around x. We set $y := \Phi(x)$ and we set $\Psi := \Phi^{-1}$. We consider the vector field $w := \Psi^* v$. The desired statements follow from Lemma 45.13 and the following claim.

Claim.

(1) The point $y \in W$ is a zero of the vector field w.
(2) $x \in M$ is non-degenerate if and only if $y \in W$ is non-degenerate.
(3) $\text{index}(v, x) = \text{index}(w, y)$.
(4) $(x, 0)$ is a transverse intersection point of $\iota(M)$ and $\text{graph}(v)$ if and only if $(y, 0)$ is a transverse intersection point of $\iota(W)$ and $\text{graph}(w)$.
(5) The sign of the intersection point $(x, 0)$ equals the sign of the intersection point $(y, 0)$.

Proof.

(1) This statement is trivial.
(2),(3) These two statements hold by the definitions of non-degeneracy and of the index on page 1021.



(4),(5) We denote by $i\colon V \to M$ the inclusion. We consider the following diagram:

$$\begin{array}{ccccc}
M & \xleftarrow{\ i\ } & V & \longrightarrow & W \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} \\
\widetilde{\mathrm{T}}M & \xleftarrow{\ i_*\ } & \widetilde{\mathrm{T}}V & \xrightarrow{\ \Phi_*\ } & \widetilde{\mathrm{T}}W \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{graph}(v\colon M \to \mathbb{R}^n) & \xleftarrow{\ i_*\ } & \mathrm{graph}(v\colon V \to \mathbb{R}^n) & \xrightarrow{\ \Phi_*\ } & \mathrm{graph}(w\colon W \to \mathbb{R}^k)
\end{array}$$

We make the following observations:

(a) It follows easily from the definitions that $i_*\colon \widetilde{\mathrm{T}}V \to \widetilde{\mathrm{T}}M$ and $\Phi_*\colon \widetilde{\mathrm{T}}V \to \widetilde{\mathrm{T}}W$ restrict to local diffeomorphisms $\mathrm{graph}(v\colon V \to \mathbb{R}^n) \to \mathrm{graph}(v\colon M \to \mathbb{R}^n)$ and $\mathrm{graph}(v\colon V \to \mathbb{R}^n) \to \mathrm{graph}(w\colon W \to \mathbb{R}^k)$.

(b) It follows immediately from the definitions that the diagram commutes and that the horizontal maps are orientation-preserving local diffeomorphisms.

The desired Statements (4) and (5) follow immediately from this discussion. ■

Lemma 45.14. Let M be a closed smooth submanifold of $\mathbb{R}^n$. There exists a smooth vector field v on M such that $\iota(M) \pitchfork \mathrm{graph}(v)$.

Proof. Let M be a closed smooth submanifold of $\mathbb{R}^n$. As before we consider the natural projection $p\colon \widetilde{\mathrm{T}}M \to M$.

Claim. There exists a smooth embedding $g\colon M \to \widetilde{\mathrm{T}}M$ such that $\iota(M) \pitchfork g$ and such that $p \circ g = \mathrm{id}_M$ in $\widetilde{\mathrm{T}}M$.

Proof. Note that by the Transversality Theorem 42.4 there exists a smooth homotopy $F\colon M \times [0,1] \to \widetilde{\mathrm{T}}M$ with $F_0 = \iota$ such that

$$T \ := \ \{t \in [0,1] \,|\, \text{the map } F_t\colon M \to \widetilde{\mathrm{T}}M \text{ is transverse to } \iota(M) \text{ in } \widetilde{\mathrm{T}}M\}$$

has full measure in $[0,1]$. By the Measure-Zero Properties Proposition 30.2 this means that T is dense in $[0,1]$. Next we consider the smooth homotopy

$$\begin{array}{rcl} H\colon M \times [0,1] & \to & M \\ (x,t) & \mapsto & p(F(x,t)). \end{array}$$

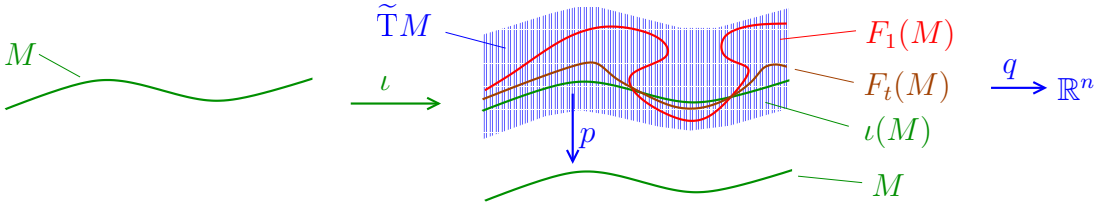
It follows from $H_0 = \mathrm{id}$, the fact that M is closed, the Deform Smooth Embedding Proposition 33.6 (2) and the fact that T is dense in $[0,1]$ that there exists a $t \in T$ such that H_t is a diffeomorphism. By Lemma 45.11 the diffeomorphism $H_t^{-1}\colon M \to M$ induces a diffeomorphism $(H_t^{-1})_*\colon \widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}M$. Finally we set $g := (H_t^{-1})_* \circ F_t\colon M \to \widetilde{\mathrm{T}}M$. We claim that g has the desired properties. This claim follows from the following two observations:

- It follows immediately from the definitions that $(H_t^{-1})_*(\iota(M)) = \iota(M)$. Thus the fact that $\iota(M) \pitchfork F_t$ implies that $\iota(M) \pitchfork (H_t^{-1})_* \circ F_t$.

- We have the following equalities of maps $M \to M$:

$$p \circ g \;=\; p \circ ((H_t^{-1})_* \circ F_t) \;=\; (p \circ (H_t^{-1})_*) \circ F_t \;\overset{\text{by definition of } (H_t^{-1})_* \colon \widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}M}{\downarrow =}\; (H_t^{-1} \circ p) \circ F_t$$
$$= \; H_t^{-1} \circ \underbrace{(p \circ F_t)}_{=H_t} \;=\; \mathrm{id}\,.$$

⊞



Finally we study the vector field

$$\begin{aligned} v \colon M &\to \mathbb{R}^n \\ x &\mapsto \underbrace{q(g(x))}_{\in \widetilde{\mathrm{T}}_x M}. \end{aligned}$$

Since $p \circ g = \mathrm{id}_M$ we see that $\mathrm{graph}(v) = g(M)$. Since $\iota(M) \pitchfork g$ and since $g \colon M \to \widetilde{\mathrm{T}}M$ is a smooth embedding it follows from the remark on page 968 that $\iota(M) \pitchfork \mathrm{graph}(v)$. ■

Now we can prove the following proposition, which gives an affirmative answer to Question 45.10.

Proposition 45.15. (Non-Degenerate Vector Field Existence Proposition) Every closed smooth manifold admits a vector field v such that each zero of v is non-degenerate.

Remark. In Proposition 129.4, which we will prove twice on page 2821 and 2983, we will give an alternative proof of the Non-Degenerate Vector Field Existence Proposition 45.15.

Proof. Let M be a closed smooth manifold. By the Euclidean Embedding Theorem 27.1 it suffices to deal with the case that M is a k-dimensional smooth submanifold of some $\mathbb{R}^n$ and vector fields that are defined in terms of the physical tangent spaces. It follows from Lemma 45.14 that there exists a smooth vector field v on M such that $\iota(M) \pitchfork \mathrm{graph}(v)$ in $\widetilde{\mathrm{T}}M$. By Proposition 45.12 this implies that v is a vector field such that each zero of v is non-degenerate. ■

45.4. The total tangent space and the diagonal.

Proposition 45.16. (Total Tangent Space-Diagonal Embedding Proposition) Let M be a closed smooth submanifold of $\mathbb{R}^n$. There exists a smooth embedding

$$f \colon \widetilde{\mathrm{T}}M \;\to\; M \times M$$

such that for each $x \in M$ we have $f(x,0) = (x,x)$. If M is oriented, then this smooth embedding can be chosen to be orientation-preserving.

The proof of Proposition 45.16 rests on the following tricky theorem from an earlier chapter.

Theorem 21.13. (Neighborhood Smooth Embedding Theorem) Let $b \colon X \to Y$ be a map between smooth manifolds. Furthermore let $C \subset X$ be a compact subset such that the map $b \colon C \to Y$ is an embedding. If b is a local diffeomorphism at each $P \in C$, then there exists an open neighborhood U of C such that the restriction of b to U is a smooth embedding.



Proof of Proposition 45.16. Let M be a closed k-dimensional smooth submanifold of $\mathbb{R}^n$. Given $\sigma > 0$ we write

$$\widetilde{\mathrm{T}}M(\sigma) \;:=\; \{(x,v) \in \widetilde{\mathrm{T}}M \,|\, \|v\| < \sigma\}.$$

We prove a few basic facts about $\widetilde{\mathrm{T}}M(\sigma)$.

Claim 1.

(1) For every $\sigma > 0$ the above set $\widetilde{\mathrm{T}}M(\sigma)$ is an open subset of $\widetilde{\mathrm{T}}M$. Thus we can view $\widetilde{\mathrm{T}}M(\sigma)$ as an (oriented) smooth manifold.
(2) Given any open neighborhood W of $M \times \{0\} \subset \widetilde{\mathrm{T}}M$ there exists a $\sigma > 0$ such that $\widetilde{\mathrm{T}}M(\sigma) \subset W$.
(3) For every $\sigma > 0$ there exists a diffeomorphism $\widetilde{\mathrm{T}}M(\sigma) \to \widetilde{\mathrm{T}}M$ which is the identity on the common subset $M \times \{0\}$. If M is oriented, then the diffeomorphism can be chosen to be orientation-preserving.

Proof.

(1) This statement follows easily from the definition of the product topology on $\mathbb{R}^n \times \mathbb{R}^n$ and the definition of the subspace topology on $\widetilde{\mathrm{T}}M$.
(2) Since W is an open neighborhood of $M \times \{0\} \subset \widetilde{\mathrm{T}}M(\sigma)$ we know, by the definition of the product topology on $\mathbb{R}^n \times \mathbb{R}^n$ and by definition of the subspace topology on $\widetilde{\mathrm{T}}M$, that for each $P \in M$ there exists an open neighborhood U_P of $P \in M$ and a $\sigma_P > 0$ such that $\{(x,v) \in \widetilde{\mathrm{T}}M \,|\, x \in U_P \text{ and } \|v\| < \sigma_P\} \subset W$. Since M is compact there exist $P_1, \ldots, P_m \in M$ with $M = U_{P_1} \cup \cdots \cup U_{P_m}$. It is clear that $\sigma := \min\{\sigma_{P_1}, \ldots, \sigma_{P_m}\}$ has the desired properties.
(3) Using the Smooth Transition Function Lemma 19.4 one can easily show that there exists an orientation-preserving diffeomorphism $\chi\colon [0,\sigma) \to \mathbb{R}_{\geq 0}$ that is the identity on a neighborhood of 0. We consider the map

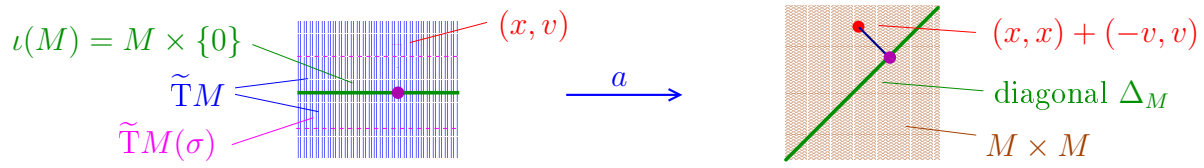
$$\begin{array}{rcl} g\colon \widetilde{\mathrm{T}}M(\sigma) & \to & \widetilde{\mathrm{T}}M \\ (x,v) & \mapsto & \begin{cases} (x,0), & \text{if } v = 0, \\ (x, \frac{v}{\|v\|} \cdot \chi(\|v\|)), & \text{if } v \neq 0. \end{cases} \end{array}$$

Note that g is the identity on a neighborhood of $M \times \{0\}$. Using the Smooth Pasting Proposition 19.3 one can now show that g is smooth. Using the inverse of the diffeomorphism χ one can easily write down a smooth inverse to the map g, thus we see that g is a diffeomorphism. If M is oriented, then g is evidently orientation-preserving. ⊞

We consider the map[437]

$$\begin{array}{rcl} a\colon \widetilde{\mathrm{T}}M & \to & \mathbb{R}^n \times \mathbb{R}^n \\ (x,v) & \mapsto & (x,x) + (-v,v). \end{array}$$

[437]There are many choices for a, for example $a(x,v) = (x,x) + (0,v)$ would also work. We picked the given map for subjective aesthetic reasons.



Claim 2. For each $x \in M$ the map

$$\mathrm{D}a_{(x,0)} \colon \widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M \to \mathbb{R}^n \times \mathbb{R}^n$$

restricts to an isomorphism

$$\mathrm{D}a_{(x,0)} \colon \widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M.$$

If M is oriented, then this isomorphism is orientation-preserving.

Proof. Recall that by Lemma 45.11 we know that $\widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M = \widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M$. Since a is a linear map it follows almost immediately that the map $\mathrm{D}a_{(x,0)} \colon \widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M \to \mathbb{R}^n \times \mathbb{R}^n$ is given by the map

$$\begin{array}{rcl} \mathrm{D}a_{(x,0)} \colon \overbrace{\widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M}^{\subset \mathbb{R}^n \times \mathbb{R}^n} & \to & \mathbb{R}^n \times \mathbb{R}^n \\ \begin{pmatrix} u \\ v \end{pmatrix} & \to & \begin{pmatrix} u \\ u \end{pmatrix} + \begin{pmatrix} -v \\ v \end{pmatrix}. \end{array}$$

Thus we see that the image of $\mathrm{D}a_{(x,0)} \colon \widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M \to \mathbb{R}^n \times \mathbb{R}^n$ lies in $\widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M$. Next we pick a basis for $\widetilde{\mathrm{T}}_x M$ and we equip $\widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M$ with the obvious corresponding basis. It follows from the above discussion, that with this choice of a basis, the map $\mathrm{D}a_{(x,0)} \colon \widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M$ is represented by the matrix $\begin{pmatrix} \mathrm{id} & -\mathrm{id} \\ \mathrm{id} & \mathrm{id} \end{pmatrix}$. Therefore we see that the map $\mathrm{D}a_{(x,0)} \colon \widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M$ is a monomorphism. Since both vector spaces are $2k$-dimensional we see that $\mathrm{D}a_{(x,0)} \colon \widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M$ is in fact an isomorphism. Furthermore, if M is oriented, then we see that, with the appropriate orientations, the map $\mathrm{D}a_{(x,0)} \colon \widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M$ is also orientation-preserving. ⊞

By the Neighborhood Projection Theorem 28.1 there exists an open neighborhood Z of $M \times M \subset \mathbb{R}^n \times \mathbb{R}^n$ and a smooth retraction $\pi \colon Z \to M \times M$. Note that $a^{-1}(Z)$ is an open neighborhood of $M \times \{0\} \subset \widetilde{\mathrm{T}}M$. It follows from Claim 1 (b) that there exists a $\mu > 0$ such that $\widetilde{\mathrm{T}}M(\mu) \subset a^{-1}(Z)$. By Claim 1 (c) and Claim 2 it now suffices to prove the following claim.

Claim 3. There exists a $\nu \in (0, \mu)$ such that the following map is a smooth embedding:

$$\begin{array}{rcl} b \colon \widetilde{\mathrm{T}}M(\nu) & \to & M \times M \\ (x, v) & \mapsto & \pi(a(x, v)). \end{array}$$

Proof. By the choice of μ it makes sense to consider the map

$$\begin{array}{rcl} b \colon \widetilde{\mathrm{T}}M(\mu) & \to & M \times M \\ (x, v) & \mapsto & \pi(a(x, v)). \end{array}$$

We make the following two observations:

(1) Since $\pi\colon Z \to M$ is a retraction we see that the restriction of b to the closed subset $M \times \{0\}$ of $\widetilde{\mathrm{T}}M(\mu)$ is given by the map $x \mapsto (x, x)$. It follows from the Compact-Hausdorff Proposition 2.17 that the map $b\colon M \times \{0\} \to \widetilde{\mathrm{T}}M(\mu)$ is an embedding.

(2) Let $(x, 0) \in M \times \{0\}$ be arbitrary. By Claim 2 we know that we get an isomorphism $\mathrm{D}a_{(x,0)}\colon \widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}_xM \times \widetilde{\mathrm{T}}_xM$. Since $\pi\colon Z \to M \times M$ is a retraction we know that the restriction of $\mathrm{D}\pi_{(x,x)}$ to $\widetilde{\mathrm{T}}_{(x,x)}(M \times M) = \widetilde{\mathrm{T}}_xM \times \widetilde{\mathrm{T}}_xM$ is the identity. In summary we see that the map $\mathrm{D}b_{(x,0)} = \mathrm{D}\pi_{(x,x)} \circ \mathrm{D}a_{(x,0)}\colon \widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}_xM \times \widetilde{\mathrm{T}}_xM$ is an isomorphism. It follows from the Inverse Mapping Theorem 24.1 that b is a local diffeomorphism at $(x, 0) \in M \times \{0\}$.

It follows from the above and the Neighborhood Smooth Embedding Theorem 21.13 that there exists an open neighborhood U of $M \times \{0\} \subset \widetilde{\mathrm{T}}M(\mu)$ such that the restriction of b to U is a smooth embedding. We obtain the desired $\nu \in (0, \mu)$ from Claim 1 (b). ■

45.5. **The Smooth Poincaré–Hopf Theorem.** We move on to the most interesting result of this chapter.

Theorem 45.17. (Smooth Poincaré–Hopf Theorem) Let M be a closed orientable smooth manifold and let v be a smooth vector field on M such that all zeros of v are non-degenerate. Then we have the equality[438]

$$\sum_{x \text{ a zero of } v} \mathrm{index}(v, x) \;=\; \chi^{\mathrm{sm}}(M).$$

Remark.

(1) In case the reader is wondering, as we mentioned on page 1019, the smooth Euler characteristic $\chi^{\mathrm{sm}}(M)$ equals the "usual" Euler characteristic $\chi(M)$, that we will define on page 1519 and on page 1882. In fact in Proposition 129.4, which we will prove twice on page 2821 and on page 2983, we will construct a smooth vector field on M such that all zeros of v are non-degenerate and such that the sum of the indices equals $\chi(M)$. As we point out in Corollary 129.5, this construction together with the Smooth Poincaré–Hopf Theorem 45.17 implies that $\chi(M) = \chi^{\mathrm{sm}}(M)$.

(2) In the Poincaré–Hopf Theorem 129.7 we will reprove the Smooth Poincaré–Hopf Theorem 45.17 and generalize it in two directions:
   (a) We will have a more general notion of the index of a zero of a vector field. Thus we no longer demand that the zeros are non-degenerate, we only demand, that there are only finitely many.
   (b) We will allow compact smooth manifolds with non-empty boundary. If M has boundary, then we will demand that along the boundary ∂M the vector field v is pointing outward.

Our proof of the Smooth Poincaré–Hopf Theorem 45.17 rests on the following proposition, which is interesting in its own right.

[438] Note that by Lemma 45.9 we know that v has only finitely many zeros, so the sum on the left hand side is actually defined.

Proposition 45.18. Let M be a closed orientable k-dimensional smooth submanifold of $\mathbb{R}^n$. As on page 1024 we consider the oriented $2k$-dimensional smooth manifold $\widetilde{\mathrm{T}}M$ and the smooth embedding $\iota\colon M \to \widetilde{\mathrm{T}}M$ that is given by $x \mapsto (x,0)$. Let v be a smooth vector field on M such that all zeros of v are non-degenerate. Then[439]

$$\sum_{x \text{ a zero of } v} \operatorname{index}(v,x) \;=\; \underbrace{\iota(M)\cdot\iota(M)}_{\substack{\text{self-intersection}\\ \text{number in } \widetilde{\mathrm{T}}M}}.$$

Proof. We denote by $x_1,\ldots,x_k$ the zeros of v. We consider the smooth homotopy

$$\begin{array}{rcl} F\colon M\times[0,1] & \to & \widetilde{\mathrm{T}}M \\ (x,t) & \mapsto & x + t\cdot v(x). \end{array}$$

Note that $F_0 = \iota\colon M \to \widetilde{\mathrm{T}}M$ and that $F_1(M) = \operatorname{graph}(v)$. Note that by our hypothesis on v and by Proposition 45.12 we know that $\iota(M) \pitchfork F_1$. The proposition now follows from the following calculation:

$$\begin{array}{ccccc} \displaystyle\sum_{x \text{ a zero of } v} \operatorname{index}(v,x) & = & \displaystyle\sum_{i=1}^{k} \operatorname{index}(v,x_i) & = & \displaystyle\sum_{i=1}^{k} \text{sign of the intersection point} \\ & & & & \qquad (x_i,0) \text{ of } \iota(M) \text{ and } F_1(M) \\ & = & \iota(M)\cdot_{\widetilde{\mathrm{T}}M} F_1(M) & = & \iota(M)\cdot_{\widetilde{\mathrm{T}}M}\iota(M). \\ & & \uparrow & & \uparrow \\ & & \text{by definition} & & \text{by definition of the self-intersection number, here} \\ & & & & \text{we use that } F_1 \text{ is smoothly homotopic to } \iota \end{array}$$

Proposition 45.12 ↓

■

Proof of the Smooth Poincaré–Hopf Theorem 45.17. Let M be a closed orientable smooth manifold and let v be a smooth vector field on M such that all zeros of v are non-degenerate. By the Euclidean Embedding Theorem 27.1 we can assume that M is a smooth submanifold of some $\mathbb{R}^n$.

By the Total Tangent Space-Diagonal Embedding Proposition 45.16 there exists an orientation-preserving smooth embedding

$$f\colon \widetilde{\mathrm{T}}M \;\to\; M\times M$$

such that for each $x \in M$ we have $f(x,0) = (x,x)$. Now we see that

$$\begin{array}{ccccc} \displaystyle\sum_{x \text{ a zero of } v} \operatorname{index}(v,x) & = & \iota(M) \underset{\widetilde{\mathrm{T}}M}{\cdot} \iota(M) & = & \Delta_M \underset{M\times M}{\cdot} \Delta_M \;=\; \chi^{\mathrm{sm}}(M). \\ \uparrow & & \uparrow & & \uparrow \\ \text{by Proposition 45.18} & & \text{follows from } f(\iota(M)) = \Delta_M & & \text{by definition} \\ & & \text{and Lemma 43.3 (2)} & & \end{array}$$

■

Remark. Let M be a closed orientable smooth manifold. In the proof of the Smooth Poincaré–Hopf Theorem 45.17 we showed that $\iota(M)\cdot_{\widetilde{\mathrm{T}}M}\iota(M) = \Delta_M \cdot_{M\times M} \Delta_M = \chi^{\mathrm{sm}}(M)$. In the literature this statement sometimes gets shortened to the catchy and mysterious looking "$M\cdot M = \chi^{\mathrm{sm}}(M)$".

[439] To make sense of the individual terms on the right hand side we have to pick an orientation of M. This orientation then endows $\iota(M)$ and $\widetilde{\mathrm{T}}M$ with an orientation and the self-intersection number $\iota(M)\cdot\iota(M)$ is defined.

45.6. **Applications of the Smooth Poincaré–Hopf Theorem 45.17.** The beauty of the Smooth Poincaré–Hopf Theorem 45.17 is that it can be used for two different purposes:

(1) By constructing suitable vector fields we can use the Smooth Poincaré–Hopf Theorem 45.17 to calculate smooth Euler characteristics.
(2) Once we have calculated the smooth Euler characteristic of a smooth manifold M the Smooth Poincaré–Hopf Theorem 45.17 gives us information on the existence of certain vector fields on M.

In the following we will see both principles in action. First, recall that in Question 45.5 we were challenged to calculate the smooth Euler characteristics of spheres and surfaces of genus g. Fortunately we can now rise to the challenge.

Proposition 45.19. (Sphere-Euler Characteristic Proposition) Given $n \in \mathbb{N}_0$ we have $\chi^{\mathrm{sm}}(S^n) = 1 + (-1)^n$.

Proof. Let $n \in \mathbb{N}_0$. By Lemma 45.8 we know that there exists a smooth vector field v on S^n with the following properties:

(1) The vector field has precisely two zeros, namely at the South Pole $S = (0, \ldots, 0, -1)$ and at the North Pole $N = (0, \ldots, 0, 1)$.
(2) The two zeros are non-degenerate and the indices are given by $\mathrm{index}(v, S) = 1$ and $\mathrm{index}(v, N) = (-1)^n$.

It now follows that

$$\chi^{\mathrm{sm}}(S^n) \underset{\text{Smooth Poincaré–Hopf Theorem 45.17}}{=} \sum_{x \text{ a zero of } v} \mathrm{index}(v, x) \underset{\text{by the above}}{=} \mathrm{index}(v, S) + \mathrm{index}(v, N) \underset{\text{by the above}}{=} 1 + (-1)^n.$$

■

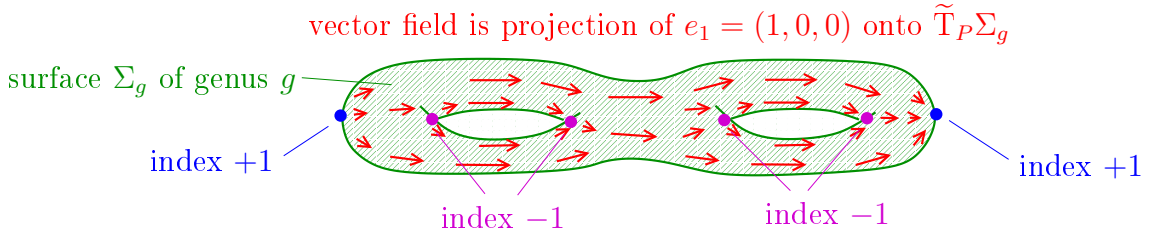
Proposition 45.20. (Surface-Euler Characteristic Proposition) The smooth Euler characteristic of the surface of genus g equals $2 - 2g$.

Proof. Let Σ_g be the surface of genus g. We will prove the proposition using the Smooth Poincaré–Hopf Theorem 45.17 and an explicit smooth vector field. In fact we will provide two constructions of smooth vector fields on Σ_g. The first one is slightly informal, the second one is more formal.

(1) First we view Σ_g as a smooth submanifold of $\mathbb{R}^3$ in the "usual way", as shown in the figure below. We denote by v the smooth vector field on Σ_g that is given at each $x \in \Sigma_g$ by projecting the vector $e_1 = (1, 0, 0)$ onto the tangent plane $\widetilde{\mathrm{T}}_x\Sigma_g$. This vector field has non-degenerate zeros of index $+1$ to the "very left" and it has a non-degenerate zero of index $+1$ to the "very right". Furthermore, for each "hole" there exist two non-degenerate zeros of index -1. The are no other zeros. It follows from this discussion that

$$\chi^{\mathrm{sm}}(\Sigma_g) \underset{\text{Smooth Poincaré–Hopf Theorem 45.17}}{=} \sum_{x \text{ a zero of } v} \mathrm{index}(v, x) \underset{\text{by the above}}{=} +1 + 2g \cdot (-1) + 1 = 2 - 2g.$$

(2) We consider the surface $\Sigma_g = E_{4g}/\sim$ as defined on page 266. We construct a smooth vector field v on $E_{4g}/\sim$ as follows:
   (a) We break $E_{4g}/\sim$ into $4g$ triangles, as shown in the figure below.



(b) The vector field v is zero on each vertex, on each midpoint of an edge and on the midpoint of each triangle.
(c) On the edges and on the triangles the vector field v points towards the midpoint.

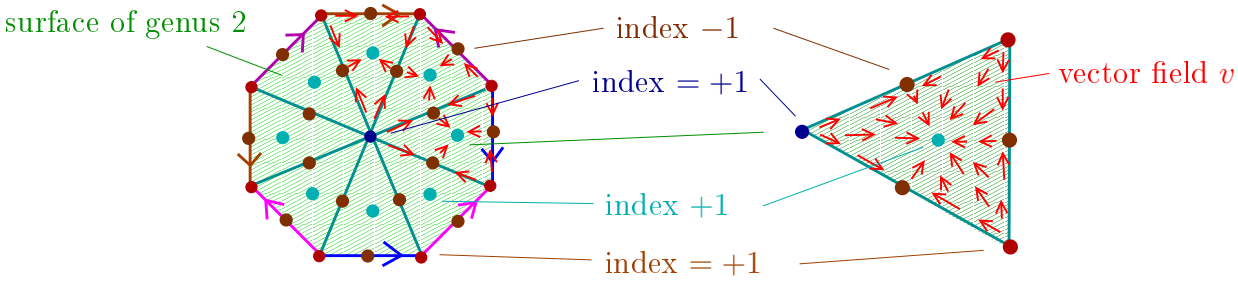
We make the following observations regarding the number of zeros and the indices of the zeros:[440]

(a) at the two vertices the vectors point outward in all directions, thus the index equals $+1$,
(b) at the midpoints of the $2g + 4g$ edges the vectors point inward in one direction and outward in another direction, thus the index equals -1,
(c) at the midpoints of the $4g$ triangles the vectors point inward from all directions, thus the index equals $+1$.

The vector field is non-zero at all other points. In summary we see that

Smooth Poincaré–Hopf Theorem 45.17

$$\chi^{\mathrm{sm}}(\Sigma_g) \overset{\downarrow}{=} \sum_{x \text{ a zero of } v} \mathrm{index}(v,x) = (+1)\cdot\underbrace{\#\text{vertices}}_{=2} + (-1)\cdot\underbrace{\#\text{edges}}_{=2g+4g} + (+1)\cdot\underbrace{\#\text{triangles}}_{=4g}$$
$$= 2 - 2g - 4g + 4g \qquad = 2 - 2g. \qquad \blacksquare$$



Corollary 45.21. (Surfaces-Non Diffeomorphic Corollary) If $g \neq h$ then the surface of genus g and the surface of genus h are not diffeomorphic.

Remark.

(1) In the Diffeomorphism-to-Standard Surfaces Theorem 142.7 we will show that any closed orientable connected non-empty 2-dimensional smooth manifold is diffeomorphic to some surface of genus g. This theorem together with the above Surfaces-Non Diffeomorphic Corollary 45.21 gives us a diffeomorphism classification of closed orientable connected non-empty 2-dimensional smooth manifolds.

[440] On page 2821 we will generalize this construction and calculation to smooth manifolds that are equipped with a "smooth triangulation".

(2) In the Surfaces-Non Homeomorphic Theorem 55.6 we will see that for $g \neq h$ the surface of genus g and the surface of genus h are in fact not even homeomorphic.
(3) We refer to Chapters 55 and 142 for more information on the classification of 2-dimensional smooth and topological manifolds.

Proof. This corollary follows immediately from the fact, see Lemma 45.3 (2), that the smooth Euler characteristic is a diffeomorphism invariant together with the calculation of the smooth Euler characteristic of surfaces in the Surface-Euler Characteristic Proposition 45.20. ■

The following theorem is an almost immediate consequence of the Smooth Poincaré–Hopf Theorem 45.17

Theorem 45.22. (Smooth Euler Characteristic-Vector Field Theorem) Let M be a closed orientable smooth manifold. If M admits a nowhere-vanishing vector field, then $\chi^{\mathrm{sm}}(M) = 0$.

Remark. In the Euler Characteristic-Vector Field Theorem 97.15 we will reprove the Smooth Euler Characteristic-Vector Field Theorem 45.22 and we will extend the statement to non-orientable smooth manifolds.

Proof. Let M be a closed smooth manifold that admits a nowhere-vanishing vector field. By the Replace-by-Smooth Vector Field Proposition 29.6 we know that M also admits a *smooth* nowhere-vanishing vector field. It follows immediately from the Smooth Poincaré–Hopf Theorem 45.17 that $\chi^{\mathrm{sm}}(M) = 0$. ■

The Smooth Euler Characteristic-Vector Field Theorem 45.22 together with the above calculations of smooth Euler characteristics gives us the following pleasant corollary.

Corollary 45.23.
(1) If $n \in \mathbb{N}_0$ is even, then there is no nowhere vanishing vector field on S^n.
(2) If $g \neq 1$, then there is no nowhere vanishing vector field on the surface Σ_g of genus g.

Remark.
(1) The statement of Corollary 45.23 (1) is of course precisely the content of the Hairy Ball Theorem 38.12.
(2) Corollary 45.23 (2) answers Question 23.19 in the orientable case.

Proof.
(1) Let $n \in \mathbb{N}_0$ be even. By the Sphere-Euler Characteristic Proposition 45.19 we know that $\chi^{\mathrm{sm}}(S^n) = 1 + (-1)^n = 2 \neq 0$. It follows from the Smooth Euler Characteristic-Vector Field Theorem 45.22 that S^n cannot admit a nowhere-vanishing vector field.
(2) Let $g \neq 1$. By the Surface-Euler Characteristic Proposition 45.20 we know that $\chi^{\mathrm{sm}}(\Sigma_g) = 2 - 2g \neq 0$. Thus we see, as in (1), that Σ_g does not admit a nowhere-vanishing vector field. ■

We conclude this section with a sneak preview of a theorem which we will prove later on. This proposition is basically the converse to the Smooth Euler Characteristic-Vector Field Theorem 97.15.

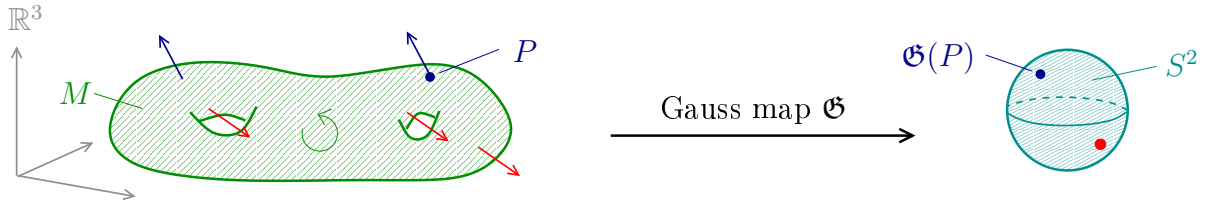
Theorem 129.6. (Nowhere Vanishing Vector Field Existence Theorem) Let M be a closed connected smooth manifold. If $\chi^{\mathrm{sm}}(M) = 0$, then M admits a nowhere-vanishing smooth vector field.

45.7. **The Gauß map I.** In this section we will introduce the Gauß map and we will determine its degree. We will not make use of the results of this section.
The following definition introduces the protagonist of this section.

Definition. Let $n \in \mathbb{N}_0$ and let $M \subset \mathbb{R}^{n+1}$ be a closed oriented smooth submanifold of codimension one. We refer to the map

$$\begin{aligned} \mathfrak{G}\colon M &\to S^n \\ P &\mapsto \begin{array}{l} \text{the unique } \mathfrak{G}(P) \in \mathbb{R}^{n+1} \text{ that has the following properties:} \\ (1)\ \mathfrak{G}(P) \text{ is normal to the physical tangent space } \widetilde{\mathrm{T}}_P M, \\ (2)\ \mathfrak{G}(P) \text{ has length one, i.e. } \mathfrak{G}(P) \in S^n, \\ (3)\ \mathfrak{G}(P) \text{ followed by a positive basis for } \widetilde{\mathrm{T}}_P M \text{ is a positive basis for } \mathbb{R}^{n+1} \end{array} \end{aligned}$$

as the Gauß map of M.[441] [442]



Example. Let $n \in \mathbb{N}$. We consider the submanifold $S^n \subset \mathbb{R}^{n+1}$ with the standard orientation introduced on page 681. It follows almost immediately from the definitions and conventions that the corresponding Gauß map $\mathfrak{G}\colon S^n \to S^n$ is the identity.

The following theorem was first proved by Heinz Hopf [**Hopf1925**] in 1925.

Theorem 45.24. (Gauß Map-Degree Theorem) Let $M \subset \mathbb{R}^{n+1}$ be a closed oriented connected smooth submanifold of codimension one. If n is even, then

$$\deg(\text{Gauß map } \mathfrak{G}\colon M \to \underset{\substack{\uparrow \\ \text{equipped with the standard orientation from page 681}}}{S^n}) \;=\; \tfrac{1}{2} \cdot \chi^{\mathrm{sm}}(M).$$

Remark.

(1) Alternative proofs of the Gauß Map-Degree Theorem 45.24 are given in [**Bre1993**, Theorem VI.12.11], [**Cib2016**], [**Sakk1990**], [**Tu2011**, p. 174] and [**GP1974**, p. 198]. Furthermore we will give a somewhat different proof in Section 129.3.

(2) Note that in general the conclusion of the Gauß Map-Degree Theorem 45.24 does not hold if n is odd. For example if we consider $M = S^1 \subset \mathbb{R}^2$, then we know by the above that the Gauß map $\mathfrak{G}$ is the identity, hence its degree equals one, whereas we saw in the Sphere-Euler Characteristic Proposition 45.19 that $\chi^{\mathrm{sm}}(S^1) = 0$.

[441] A variation on the Gauß map appeared also in the definition of the rotation number of an immersion $S^1 \to \mathbb{R}^2$ on page 955.
[442] As discussed in the Orientations-via-Normal Vector Fields Proposition 25.11, this map is smooth.

Before we provide a proof of the Gauß Map-Degree Theorem 45.24, let us formulate and prove a neat corollary.

Corollary 45.25. If M is a closed orientable even-dimensional smooth manifold that admits a smooth embedding into $\mathbb{R}^{\dim(M)+1}$, then $\chi^{\mathrm{sm}}(M)$ is even.

Example. Let $n \in \mathbb{N}$. In Exercise 45.5 we will see that $\chi^{\mathrm{sm}}(\mathbb{C}\mathrm{P}^n) = n+1$. Thus, if n is in fact even, then it follows from Corollary 45.25 that the $2n$-dimensional smooth manifold $\mathbb{C}\mathrm{P}^n$ does not admit a smooth embedding into $\mathbb{R}^{2n+1}$. No wonder you never saw a copy of $\mathbb{C}\mathrm{P}^2$ in $\mathbb{R}^5$.

Proof of Corollary 45.25. We write $n = \dim(M)$. By the Smooth Embedding Theorem 24.10 we know that M is diffeomorphic to a smooth submanifold $N \subset \mathbb{R}^{n+1}$. It follows that

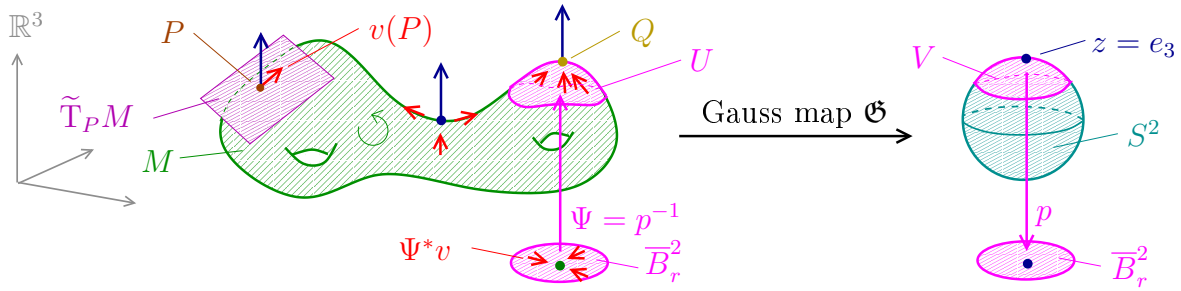
$$\chi^{\mathrm{sm}}(M) \underset{\text{Lemma 45.3}}{\uparrow} = \chi^{\mathrm{sm}}(N) \underset{\text{Gauß Map-Degree Theorem 45.24, since } n \text{ is even}}{\uparrow} = 2 \cdot \deg(\text{Gauß map } \mathfrak{G}\colon N \to S^n) \in 2 \cdot \mathbb{Z}.$$

■

Proof of the Gauß Map-Degree Theorem 45.24. Let $n \in \mathbb{N}_0$ and let $M \subset \mathbb{R}^{n+1}$ be a closed oriented connected smooth submanifold of codimension one. Recall that above we observed that the Gauß map $\mathfrak{G}\colon M \to S^n$ is smooth. Thus we can appeal to Sard's Theorem 31.1 which, together with the Measure-Zero Properties Proposition 30.2, implies that there exists some $z \in S^n$ such that z and $-z$ are both regular values of the Gauß map $\mathfrak{G}$. By the Orthogonal Action Lemma 5.26 we know that $\mathrm{SO}(n+1)$ acts transitively on S^n. It follows easily from this observation, together with Lemma 33.8 and the Degree Homotopy Invariance Theorem 38.3 that, after a rotation, we can assume that $z = e_{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. Next we consider the vector field

$$\begin{aligned} v\colon M &\to \mathbb{R}^{n+1} \\ Q &\mapsto \underbrace{z - \langle z, \mathfrak{G}(Q)\rangle \cdot \mathfrak{G}(Q)}_{\text{orthogonal projection of } z \text{ to } \widetilde{\mathrm{T}}_Q M}. \end{aligned}$$

Clearly we have $v(Q) = 0$ if and only if $\mathfrak{G}(Q) = z$ or $\mathfrak{G}(Q) = -z$.



Let $Q \in M$ with $\mathfrak{G}(Q) = \pm z$. By design we know that Q is a regular value of $\mathfrak{G}$. By the Stack of Records Lemma 26.2 (1) we know that $\mathfrak{G}\colon M \to S^n$ is a local diffeomorphism at Q, thus, as on page 898, it makes sense to introduce the following definition:

$$\deg(\mathfrak{G}, Q) = \begin{cases} +1, & \text{if } \mathfrak{G} \text{ is orientation-preserving at } Q \\ -1, & \text{if } \mathfrak{G} \text{ is orientation-reversing at } Q. \end{cases}$$

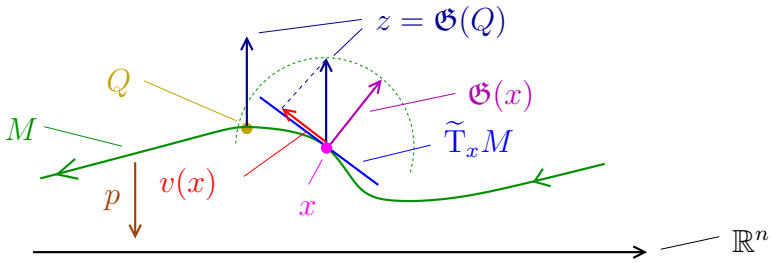
Claim. Let $Q \in M$.
(1) If $\mathfrak{G}(Q) = z$, then $\operatorname{index}(v, Q) = (-1)^n \cdot \deg(\mathfrak{G}, Q)$.
(2) If $\mathfrak{G}(Q) = -z$, then $\operatorname{index}(v, Q) = \deg(\mathfrak{G}, Q)$.

Proof. First we consider the case that we are given $Q \in M$ with $\mathfrak{G}(Q) = z$. We make the following observations and preparations.

(a) We denote by $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ the projection onto the first n coordinates.
(b) It follows fairly easily from the Inverse Mapping Theorem 24.1, see also Exercise 23.6, that there exists a neighborhood U of $Q \in M$, a neighborhood V of $z \in S^n$ and an $r \in \mathbb{R}_{>0}$ with the following properties:
  (i) We have $\mathfrak{G}(U) \subset V$.
  (ii) The restriction of the projection $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ to $U \subset M$ and $V \subset S^n$ defines diffeomorphisms $p\colon U \to \overline{B}^n_r$ and $p\colon V \to \overline{B}^n_r$. It follows immediately from the orientation conventions and some elementary linear algebra that both maps are orientation-preserving if n is even and both maps are orientation-reversing if n is odd.
(c) We set $\Psi := p^{-1}\colon \overline{B}^n_r \to U$.
(d) Recall that by definition we have $v(x) = z - \langle z, \mathfrak{G}(x)\rangle \cdot \mathfrak{G}(x)$. Using the fact that $\mathfrak{G}(Q) = e$, which implies in particular that $p(\mathfrak{G}(Q)) = p(z) = 0$, one can easily show, using the Product Rule for partial differentials, that[443]

$$\mathrm{D}(p \circ (x \mapsto -v(x)) \circ \Psi)_{p(Q)} \;=\; \mathrm{D}(p \circ \mathfrak{G} \circ \Psi)_{p(Q)}.$$

This fact is also illustrated, more or less convincingly, in the figure below.



We now see that

$$\begin{array}{lcll} \deg(\mathfrak{G}, Q) = +1 & \overset{\text{we know that } p \text{ and } \Psi \text{ are either both orientation-preserving or they are both orientation-reversing}}{\Longleftrightarrow} & \det(\mathrm{D}(p \circ \mathfrak{G} \circ \Psi)_{p(Q)}) > 0 & \overset{\text{by (d)}}{\Longleftrightarrow} \det(\mathrm{D}(p \circ (x \mapsto -v(x)) \circ \Psi)_{p(Q)} > 0 \\ & \Longleftrightarrow & \det(\mathrm{D}(\underbrace{p^*(-v)}_{\text{pullback of } -v})_{p(Q)} > 0 & \Longleftrightarrow \quad (-1)^n \cdot \det(\mathrm{D}(p^* v)_{p(Q)} > 0 \\ & & & \Longleftrightarrow \quad \operatorname{index}(v, Q) = (-1)^n. \end{array}$$

Now we consider the case that $\mathfrak{G}(Q) = -z$. The calculation is basically identical to the above. The only difference is that in this setting the map p is orientation-preserving if and

[443] Note that if W is an open subset of $\mathbb{R}^n$ and if $f\colon W \to \mathbb{R}$ and $g\colon W \to \mathbb{R}^n$ are smooth maps, then it follows immediately from the Product Rule for partial differentials that for any $w \in W$ we have

$$\mathrm{D}(f \cdot g)_w \;=\; (\tfrac{\partial f}{\partial x_1}(w) \cdot g(w) \ldots \tfrac{\partial f}{\partial x_n}(w) \cdot g(w)) + f(w) \cdot \mathrm{D}g_w.$$

We now apply this scheme to $f(y) = \langle z, (\mathfrak{G} \circ \Psi)(y)\rangle$ and $g = p \circ \mathfrak{G} \circ \Psi$ and we obtain the desired equality.

only if n is odd, whereas before p was orientation-preserving if and only if n is even. On the other hand the behavior of the orientation of Ψ does not change, i.e. Ψ is orientation-preserving if and only if n is even. Thus we obtain the same result as above, except that by Lemma 76.16 we pick up a sign $(-1)^n$. ⊞

We turn to the actual proof of the promised equality. Thus we now assume that n is even. We perform the following calculation:

$$\begin{array}{rcl} 2\cdot\deg(\text{Gauß map } \mathfrak{G}\colon M\to S^n) & \overset{\text{since } z \text{ and } -z \text{ are regular values we can apply the Local-Global Degree Proposition 101.6 to } z \text{ and to } -z}{\downarrow} & \\ & = \sum_{\substack{Q\in M \text{ with} \\ \mathfrak{G}(Q)=z}} \deg(\mathfrak{G},Q) + \sum_{\substack{Q\in M \text{ with} \\ \mathfrak{G}(Q)=-z}} \deg(\mathfrak{G},Q) & \\ & = \sum_{\substack{Q\in M \text{ with} \\ v(Q)=0}} \deg(\mathfrak{G},Q) & = \sum_{\substack{Q\in M \text{ with} \\ v(Q)=0}} \operatorname{index}(v,Q) = \chi^{\mathrm{sm}}(M). \\ & \uparrow & \uparrow \qquad\qquad\qquad \uparrow \\ & \text{since } v(Q)=0 \text{ if and only if } \mathfrak{G}(Q)=z \text{ or } \mathfrak{G}(Q)=-z & \text{by the claim and since } n \text{ is even} \qquad \text{Smooth Poincaré-Hopf Theorem 45.1} \blacksquare \end{array}$$

Remark. This concludes our discussion of the algebraic (self-) intersection number for the time being. We will return to the study of algebraic (self-) intersection numbers much later in Chapter 126 when we will relate it to the cup product on cohomology.

Exercises for Chapter 45.

Exercise 45.1.

(a) Let M and N be two closed oriented smooth manifold. Show that

$$\chi^{\mathrm{sm}}(M\times N) \;=\; \chi^{\mathrm{sm}}(M)\cdot\chi^{\mathrm{sm}}(N).$$

(b) Show that given any $k\in\mathbb{Z}_{\geq 2}$ there exist closed orientable connected $2k$-dimensional smooth manifolds of arbitrarily large and arbitrarily low values for the smooth Euler characteristic.

Exercise 45.2.

(a) Let M be a closed $2n$-dimensional smooth manifold and let $X\subset M$ be a closed submanifold. We do not make any assumptions on orientability. Show that one can still introduce a $\mathbb{Z}_2$-algebraic self-intersection number $X\cdot_{\mathbb{Z}_2} X\in\mathbb{Z}_2$ in a reasonable way.

(b) Let $\iota\colon \mathbb{R}\mathrm{P}^1\to\mathbb{R}\mathrm{P}^2$ be the smooth embedding given by $[x_0:x_1]\mapsto[x_0:x_1:x_2]$. Determine the $\mathbb{Z}_2$-algebraic self-intersection number of $\iota(\mathbb{R}\mathrm{P}^1)\subset\mathbb{R}\mathrm{P}^2$.

Exercise 45.3. We consider the smooth embedding

$$\begin{array}{rcl} \iota\colon \mathbb{C}\mathrm{P}^1 & \to & \mathbb{C}\mathrm{P}^2 \\ [z_0:z_1] & \mapsto & [z_0:z_1:z_2]. \end{array}$$

Determine the algebraic self-intersection number of $\iota(\mathbb{C}\mathrm{P}^1)\subset\mathbb{C}\mathrm{P}^2$.

Remark. We know by the Complex Projective Space-is-Complex Manifold Lemma 32.16 that each $\mathbb{C}\mathrm{P}^n$ is a closed oriented $2n$-dimensional smooth manifold.

Exercise 45.4. Let M be a smooth submanifold of some $\mathbb{R}^n$ and let $f\colon M \to \mathbb{R}$ be a smooth function. Let $P \in M \setminus \partial M$ be a critical point. We pick a chart $\Phi\colon U \to V$ for P. We write $Q := \Phi(P)$. In Lemma 137.1 we will see that the congruence class of the Hessian matrix

$$\operatorname{Hess}(f \circ \Phi^{-1})_Q \;:=\; \left(\frac{\partial^2}{\partial x_i\, \partial x_j}\Big|_Q f \circ \Phi^{-1} \right)_{i,j=1,\dots,n} \;\in\; \mathrm{M}(n \times n, \mathbb{R})$$

is independent of the choice of the chart. We say the critical point P is *non-degenerate* if $\det(\operatorname{Hess}(f \circ \Phi^{-1})_Q) \neq 0$. We refer to

$$\operatorname{index}(f, P) \;:=\; \text{number of negative eigenvalues of } \operatorname{Hess}(f \circ \Phi^{-1})_Q$$

as the *index of the critical point* P. Next we consider the vector field that is given by

$$\begin{array}{rcl} \nabla f\colon M & \to & \mathbb{R}^n \\ x & \mapsto & \begin{array}{c}\text{the unique } v(x) \in \widetilde{\mathrm{T}}_x M \text{ such that for} \\ \text{all } w \in \widetilde{\mathrm{T}}_x M \text{ we have } \langle v(x), w\rangle = \mathrm{D}f_x(w).\end{array} \end{array}$$

We refer to it as the *gradient vector field of* f. Show that for every $x \in M$ the following statements hold:

(a) x is a critical point of f if and only if x is a zero of ∇f.
(b) If x is a critical point of f, then x is a non-degenerate critical point of f if and only if x is a non-degenerate zero of ∇f.
(c) If x is a non-degenerate critical point of f, then $\operatorname{index}(\nabla f, x) = (-1)^{\operatorname{index}(f,x)}$.

Exercise 45.5. Let $n \in \mathbb{N}_0$. We pick real numbers $c_0 < c_1 < \cdots < c_n$. We consider the following functions on $\mathbb{R}\mathrm{P}^n = S^n/\{\pm\mathrm{id}\}$ and $\mathbb{C}\mathrm{P}^n = S^{2n+1}/S^1$:

$$\begin{array}{rcl} f\colon \mathbb{R}\mathrm{P}^n & \to & \mathbb{R} \\ [x_0 : \cdots : x_n] & \mapsto & \sum_{j=0}^{n} c_j \cdot x_j^2 \end{array} \qquad \text{and} \qquad \begin{array}{rcl} g\colon \mathbb{C}\mathrm{P}^n & \to & \mathbb{R} \\ [z_0 : \cdots : z_n] & \mapsto & \sum_{j=0}^{n} c_j \cdot |z_j|^2. \end{array}$$

By the Smooth Embedding Theorem 24.10 we can think of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ as smooth submanifolds of some $\mathbb{R}^N$. This allows us to consider the corresponding gradient vector fields ∇f and ∇g that we introduced in Exercise 45.4.

(a) Show that ∇f has precisely $n+1$ zeros, that these are non-degenerate and that the indices are $1, -1, \dots, (-1)^{n+1}$.
(b) Show that ∇g has precisely $n+1$ zeros, that these are non-degenerate and that the index is always $+1$.

Remark. You might want to use the explicit charts given in Exercise 19.16 and the Complex Projective Space-is-Complex Manifold Lemma 32.16.
Remark. Note that it follows from this exercise together with the Smooth Poincaré–Hopf Theorem 45.17 that $\chi^{\mathrm{sm}}(\mathbb{R}\mathrm{P}^n) = \frac{1}{2}(1 + (-1)^n)$ and that $\chi^{\mathrm{sm}}(\mathbb{C}\mathrm{P}^n) = n + 1$.

Exercise 45.6. Let M be a closed orientable smooth manifold and let $p \in M$. Does there necessarily exist a smooth embedding $f\colon M \to M \times M$ such that for every $x \in M$ we have $f(x) = (x, p)$?

Exercise 45.7. Let $p\colon \widetilde{M} \to M$ be a local diffeomorphism between two closed oriented connected non-empty smooth manifolds.

(a) Show that for any $x, y \in M$ we have $\#p^{-1}(x) = \#p^{-1}(y)$ and show that the cardinality of this set is finite.
(b) Show that the cardinality of the set in (a) equals $|\deg(p)|$.
(c) Show that $\chi^{\mathrm{sm}}(\widetilde{M}) = \deg(p) \cdot \chi^{\mathrm{sm}}(M)$.
(d) Given $k \in \mathbb{N}$ we denote by N_k the non-orientable surface of genus k as defined on page 266. Show that N_k admits a nowhere vanishing smooth vector field if and only if $k = 2$.
Hint. Make use of Exercise 19.37.

Part V

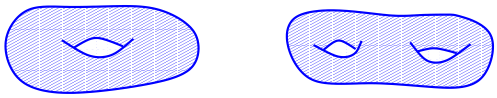
Fundamental Groups

46. How can we show that two topological spaces are (not) homeomorphic?

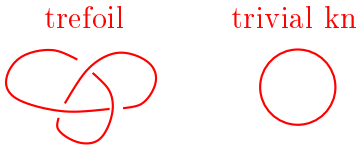
In the previous two sections we collected many examples of topological spaces. In many cases it was relatively straightforward to see that two topological spaces are homeomorphic. For example we wrote down an explicit homeomorphism from $\mathbb{R}^n/\mathbb{Z}^n$ to $(S^1)^n$. Sometimes it is rather painful to explicitly write down a homeomorphism, for example it was not particularly easy to find a homeomorphism from the open cube $(0,1)^n$ to the open ball $B^n = \{x \in \mathbb{R}^n \,|\, \|x\| < 1\}$. Nonetheless, if two topological spaces are homeomorphic, then usually one can write down an explicit homeomorphism.
On the other hand, how can we tell that two topological spaces are not homeomorphic? There are a couple of trivial criteria: being compact, connected, Hausdorff are properties of topological spaces that are shared by homeomorphic topological spaces. This implies for example that the closed cube $[0,1]^n$ is not homeomorphic to the open cube $(0,1)^n$, since the former topological space is compact and the latter is not.
But these criteria are of no use when it comes to studying more subtle examples. For example they do not suffice to address the following questions:

(1) Can surfaces with different genera be homeomorphic?
(2) Given $k \in \mathbb{Z}_{\geq 0}$ we denote by U the complement of k points in $\mathbb{R}^2$. If $k \neq l$, is it possible that U_k and U_l are homeomorphic?
(3) Is the complement of the trefoil homeomorphic to the complement of the trivial knot?

In each case we suspect that the answer is no. Certainly we do not succeed in finding a homeomorphism, but how can we show that this is not due to lack of imagination, but that there is indeed no homeomorphism?



are these surfaces homeomorphic?



are the complements homeomorphic?

The reader might have encountered some ideas for tackling these questions in one or both of the following topics:

(a) Differential forms on smooth manifolds, in particular de Rham cohomology, as developed say in [**BoT1982, Lee2002**].
(b) Complex analysis, see e.g. [**Lang1999, Jä2011**].

For the reader who has seen differential forms and de Rham cohomology we recall that each smooth manifold M and each $k \in \mathbb{Z}_{\geq 0}$ we can assign the de Rham cohomology group $H^k_{\mathrm{dR}}(M)$. It follows more or less from the definition that diffeomorphic smooth manifolds have isomorphic de Rham cohomology groups. This approach has several disadvantages:

(1) the de Rham cohomology groups are only defined for smooth manifolds,
(2) the de Rham cohomology groups can only be used to show that two smooth manifolds are not *diffeomorphic*, a priori they cannot be used to show that two smooth manifolds are not *homeomorphic*,

(3) at least just starting from the definition it is not clear how one can calculate the de Rham cohomology groups of a given smooth manifold.

Later on, in Chapter 108, we will introduce "singular cohomology" which will take care of all of these three issues.

Next we give some definitions that are familiar to the reader who has taken a course on complex analysis.

Definition.

(1) Given a subset U of $\mathbb{C}$, we define a path in U to be a map $\gamma\colon [a,b] \to U$. If $\gamma(a) = \gamma(b)$, then we say that γ is a loop.

(2) Two paths $\gamma_0, \gamma_1\colon [a,b] \to U$ with the same starting point $P := \gamma_0(a) = \gamma_1(a)$ and the same endpoint $Q := \gamma_0(b) = \gamma_1(b)$ are called path-homotopic in U, if there exists a homotopy rel $\{a,b\}$ from γ_0 to γ_1. In other words, if there exists a map

$$\begin{array}{rcl} \Gamma\colon [a,b]\times[0,1] & \to & U \\ (t,s) & \mapsto & \Gamma(t,s), \end{array}$$

with the following properties[444]

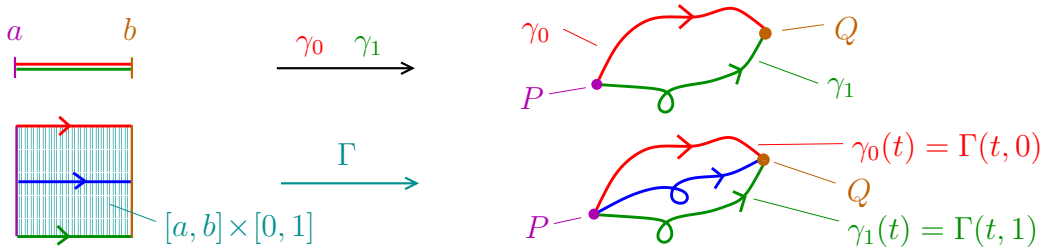
(a) for every $t \in [a,b]$ we have $\Gamma(t,0) = \gamma_0(t)$ and $\Gamma(t,1) = \gamma_1(t)$,

(b) for every $s \in [0,1]$ we have $\Gamma(a,s) = P$ and $\Gamma(b,s) = Q$.

In the figure below we illustrate a path-homotopy between two paths.

(3) A loop $\gamma\colon [a,b] \to U$ is called null-homotopic if it is path-homotopic to the constant path given by $\delta(t) := \gamma(a)$ for all $t \in [a,b]$.

(4) We say U is simply connected if U is non-empty, if U is path-connected and if each loop is null-homotopic.



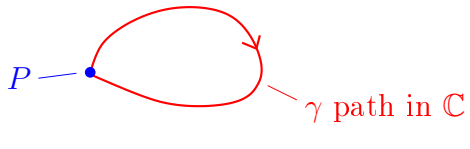
Example. Every loop in $\mathbb{R}^2 = \mathbb{C}$ is null-homotopic. Indeed, we let $\gamma\colon [0,1] \to \mathbb{C}$ be a loop with starting and endpoint P. Then

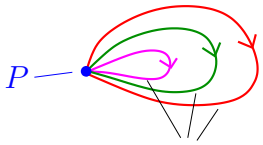
$$\begin{array}{rcl} \Gamma\colon [a,b]\times[0,1] & \to & \mathbb{R}^n \\ (t,s) & \mapsto & \gamma(t)\cdot(1-s) + P\cdot s \end{array}$$

is a path-homotopy between the loop γ and the constant path at P.

Next we recall the following definition from page 791.

[444] Loosely speaking a path-homotopy between two paths consists of a "continuous" family of paths $\{\Gamma(-,s)\}_{s\in[0,1]}$ from P to Q which interpolates between the paths γ_0 and γ_1.





path-homotopy from γ to a constant path

Definition. Let $U \subset \mathbb{C}$ be an open subset, let $f\colon U \to \mathbb{C}$ be a map and let $\gamma\colon [a,b] \to U$ be a smooth path. We define the path integral of f along γ as follows:

$$\int_\gamma f(z)\,dz \;:=\; \int_{t=a}^{t=b} f(\gamma(t)) \cdot \gamma'(t)\,dt \;\in \mathbb{C}.$$

We also recall the following examples from page 791.

Examples.

(1) Let $U = \mathbb{C} \setminus \{0\}$, let $f(z) = \frac{1}{z}$ and let $\gamma\colon [0,1] \to \mathbb{C} \setminus \{0\}$ be the loop given by $\gamma(t) = \exp(2\pi\,\mathrm{i}t)$, i.e. γ is a loop that "goes once around the origin". Then

$$\int_\gamma f(z)\,dz \;=\; \int_{t=0}^{t=1} f(\gamma(t)) \cdot \gamma'(t)\,dt \;=\; \int_{t=0}^{t=1} \frac{1}{\exp(2\pi\,\mathrm{i}t)} \cdot \exp(2\pi\,\mathrm{i}t) \cdot 2\pi\,\mathrm{i}\,dt \;=\; 2\pi\,\mathrm{i}.$$

(2) If γ is a constant path, then $\gamma'(t) \equiv 0$. This shows that the path-integral along a constant path is zero.

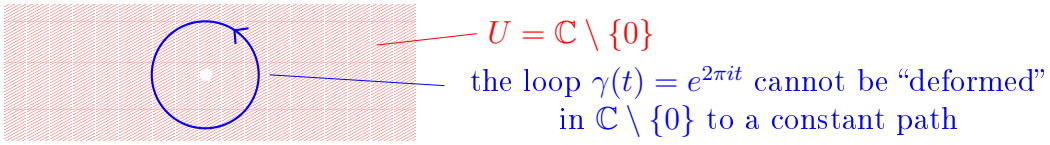
We also recall the following theorem.

Theorem ??. (Cauchy Theorem) Let $U \subset \mathbb{C}$ be an open subset, let $f\colon U \to \mathbb{C}$ be a holomorphic function and let $\gamma, \delta\colon [a,b] \to U$ be two smooth paths. If γ and δ are path-homotopic, then

$$\int_\gamma f(z)\,dz \;=\; \int_\delta f(z)\,dz.$$

The following corollary is a consequence of the Cauchy Theorem **??** and of the above calculations.

Corollary 46.1. The loop $\gamma\colon [0,1] \to \mathbb{C} \setminus \{0\}$ given by $\gamma(t) = \exp(2\pi\,\mathrm{i}t)$ is *not* path-homotopic to a constant path, i.e. γ is not null-homotopic.



Remark. In the following Chapter 46.1 we will use Corollary 46.1 so that we have at least one non-trivial topological statement in our toolbox. Somewhat later, after somewhat lengthy preparations, we will prove a general result, namely Corollary 48.18, which in particular will provide us with a new proof of Corollary 46.1.

Example. If two subsets of $\mathbb{C}$ are homeomorphic, then evidently either both are simply connected or none is. From Corollary 46.1 it now follows immediately that $\mathbb{R}^2 = \mathbb{C}$ and $\mathbb{R}^2 \setminus \{(0,0)\} = \mathbb{C} \setminus \{0\}$ are not homeomorphic.

This second approach to showing that two topological spaces are different also has several disadvantages over the first approach:

(1) A priori these notions are defined only for subsets of $\mathbb{C}$,
(2) this approach does not allow us to show that the two subsets $U = \mathbb{C} \setminus \{0\}$ and $V = \mathbb{C} \setminus \{0, 1\}$ are not homeomorphic, since neither is simply connected.

The first objection is of course easy to rectify, since the definitions make sense for any topological space. But if we work with more general situations a new problem arises: how can we show, without using complex analysis, that a topological space is not simply connected? In the following chapters we will address all these issues.

In particular, when it comes to distinguishing $U = \mathbb{C} \setminus \{0\}$ and $V = \mathbb{C} \setminus \{0, 1\}$, then naively we would say that V has "more" loops that are not null-homotopic than U. In the Chapter 47 we will in particular make precise what "more" means.

47. The fundamental group

47.1. **Path-homotopy classes of paths.** In this section we extend some of the ideas and notions about paths that we introduced in the previous section to the more general setting of paths in topological spaces. The following definition is a slight generalization of the concept of a path that we initially introduced on page 161.

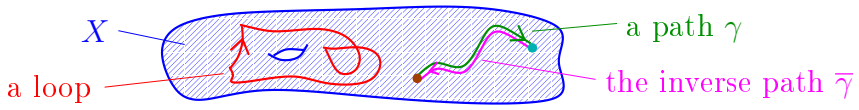
Definition. Let X be a topological space.

(1) We define a path in X as a[445] map $\gamma\colon [a,b] \to X$. We call $\gamma(a)$ the starting point of γ and we call $\gamma(b)$ the endpoint of γ. Often we say that γ is a path from $\gamma(a)$ to $\gamma(b)$.

(2) A loop is a path for which the starting point and the endpoint coincide.

Convention. In the following, if we do not specify the domain of a path or a loop, then it is understood to be $[0,1]$.

Definition. Given a path $\gamma\colon [a,b] \to X$ we refer to the path $\overline{\gamma}\colon [a,b] \to X$ that is given by $\overline{\gamma}(t) = \gamma(b+a-t)$ as the inverse path of γ.



Definition. Let X be a topological space.

(1) Let $\gamma_0, \gamma_1\colon [a,b] \to X$ be two paths with the same starting point P and the same endpoint Q. A path-homotopy between the paths γ_0 and γ_1 is a homotopy rel $\{a,b\}$ between γ_0 and γ_1. In other words, spelling out the definition on page 419, a path-homotopy is a map

$$\begin{array}{rcl} \Gamma\colon [a,b]\times[0,1] & \to & X \\ (t,s) & \mapsto & \Gamma(t,s), \end{array}$$

with the following properties:

(a) for every $t \in [a,b]$ we have $\Gamma(t,0) = \gamma_0(t)$ and $\Gamma(t,1) = \gamma_1(t)$,

(b) for every $s \in [0,1]$ we have $\Gamma(a,s) = P$ and $\Gamma(b,s) = Q$.

(2) If there exists a path-homotopy between two paths γ_0 and γ_1, then we say that γ_0 and γ_1 are path-homotopic, and we write $\gamma_0 \simeq \gamma_1$.

(3) We say a loop $\gamma\colon [a,b] \to X$ is null-homotopic, if it is path-homotopic to the constant path δ given by $\delta(t) = \gamma(a)$ for all $t \in [a,b]$.

Examples.

(A) Let C be a convex subset of $\mathbb{R}^n$ and let $\gamma_0, \gamma_1\colon [a,b] \to C$ be two paths in C with same starting point and the same endpoint. As in the Convex Homotopy Lemma 14.1 we see that

$$\begin{array}{rcl} \Gamma\colon [a,b]\times[0,1] & \to & \mathbb{R}^n \\ (t,s) & \mapsto & \gamma_0(t)\cdot(1-s) + \gamma_1(t)\cdot s \end{array}$$

is a path-homotopy between γ_0 and γ_1.[446]

[445] Recall that all maps are understood to be continuous, unless we say something else.

[446] We had already illustrated this fact in the figure on page 1046.

(B) Let $X = \mathbb{C} \setminus \{0\} = \mathbb{R}^2 \setminus \{(0,0)\}$. As we had just seen in Corollary 46.1, the path

$$\begin{array}{rcl} \gamma\colon [0,1] & \to & X \\ t & \mapsto & \exp(2\pi\,\mathrm{i}t) = (\cos(2\pi t), \sin(2\pi t)) \end{array}$$

is not null-homotopic in X.

(C) Let $Y = S^1 \times (0,2)$ and consider the loop

$$\begin{array}{rcl} \widetilde{\gamma}\colon [0,1] & \to & Y = S^1 \times (0,2) \\ t & \mapsto & (\exp(2\pi\,\mathrm{i}t), 1). \end{array}$$

This loop is not null-homotopic. This can be seen as follows: we consider the map

$$\begin{array}{rcl} \Phi\colon Y = S^1 \times (0,2) & \to & X = \mathbb{C} \setminus \{0\} \\ (z,t) & \mapsto & z \cdot t. \end{array}$$

If $\widetilde{\gamma}$ was null-homotopic in Y, then there would exist a path-homotopy

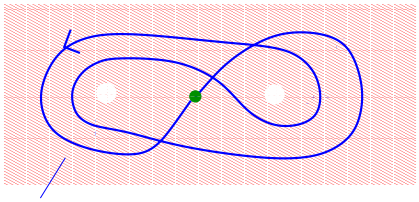
$$\widetilde{F}\colon [0,1] \times [0,1] \to Y$$

from $\widetilde{\gamma}$ to the constant path $\widetilde{\delta}(t) := (1,1)$, $t \in [0,1]$. But then

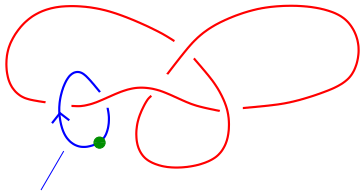
$$F := \Phi \circ \widetilde{F}\colon [0,1] \times [0,1] \to X$$

would be a path-homotopy in $X = \mathbb{C} \setminus \{0\}$ from the path $\gamma := \Phi \circ \widetilde{\gamma}$ to the constant path $\delta := \Phi \circ \widetilde{\delta}$. But $\gamma = \Phi \circ \widetilde{\gamma}$ is precisely the path γ from example (B) which we showed not to be null-homotopic in $X = \mathbb{C} \setminus \{0\}$.

(D) We consider the two loops in the figure below. In both cases it is hard to see how these loops could possibly be path-homotopic to a constant path, but it is also difficult to find an argument why that should not be possible.[447]



loop in $\mathbb{C} \setminus \{0,1\}$



loop in the complement of a knot

(E) We consider the real projective space $\mathbb{R}\mathrm{P}^2$. Recall that in Lemma 5.32 we gave an identification $\mathbb{R}\mathrm{P}^2 = \overline{B}^2/\!\sim$ where $\sim$ is the equivalence relation on $\overline{B}^2$ generated by $P \sim -P$ for $P \in S^1 = \partial\overline{B}^2$. Then

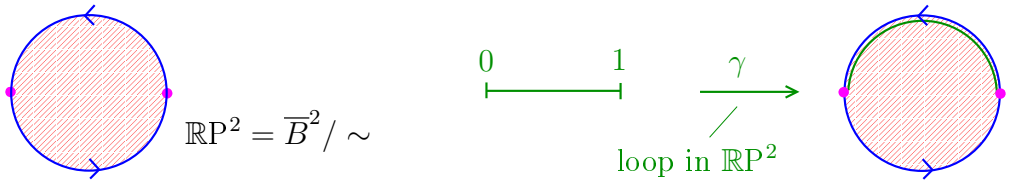
$$\begin{array}{rcl} \gamma\colon [0,1] & \to & \overline{B}^2/\!\sim \\ t & \mapsto & [(\cos(\pi t), \sin(\pi t))] \end{array}$$

is a loop in $\mathbb{R}\mathrm{P}^2 = \overline{B}^2/\!\sim$.[448] But is it null-homotopic?

(F) On page 141 we considered the pseudocircle, namely the set $X = \{A, B, C, D\}$ with a slightly unusual topology. Furthermore we gave an example of a continuous non-constant loop $f\colon S^1 \to X$. The question arises, is this loop null-homotopic?

[447] For example the given loop in $\mathbb{C} \setminus \{0,1\}$ is actually null-homotopic in $\mathbb{C} \setminus \{0\}$ and it is null-homotopic in $\mathbb{C} \setminus \{1\}$. In particular our trick of using Corollary 46.1 does not work in this case.

[448] Why is it a loop?



Remark. Let X be a topological space and let $x_0 \in X$. It follows from Lemma 5.18 that the maps

$$\begin{array}{rcl} \{\text{loops } \gamma\colon [0,1] \to X \text{ with } \gamma(0) = \gamma(1) = x_0\} & \Leftrightarrow & \{\text{maps } f\colon S^1 \to X \text{ with } f(1) = x_0\} \\ (\gamma\colon [0,1] \to X) & \mapsto & (z = \exp(2\pi\,\mathrm{i}t) \mapsto \gamma(t)) \\ (t \mapsto f(\exp(2\pi\,\mathrm{i}t))) & \leftarrow\!\shortmid & (f\colon S^1 \to X) \end{array}$$

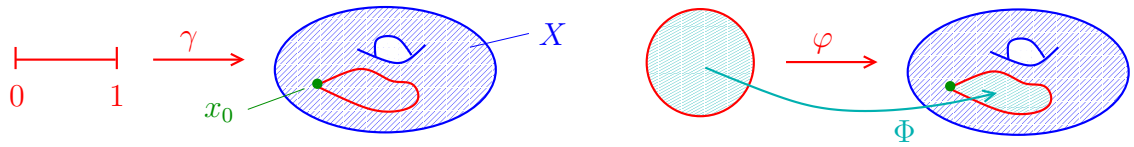
are evidently bijections. Thus we can think of loops in X where the starting and endpoint is x_0 as maps from S^1 to X that send 1 to x_0. We will use this bijection to go back and forth between these two points of view.

The following lemma gives a useful criterion for a loop to be null-homotopic.

Lemma 47.1. (Extension-to-Disk Lemma) Let X be a topological space and let $x_0 \in X$. Let $\gamma\colon [0,1] \to X$ be a loop with $\gamma(0) = \gamma(1) = x_0$. We denote by

$$\begin{array}{rcl} \varphi\colon S^1 & \to & X \\ z = \exp(2\pi\,\mathrm{i}t) & \mapsto & \gamma(t) \end{array}$$

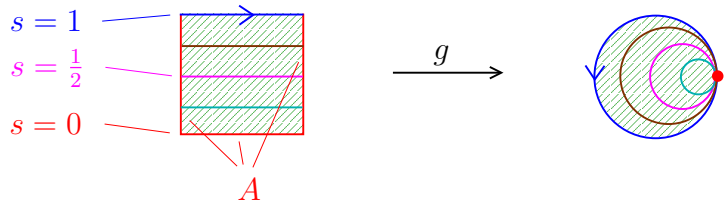
the corresponding map from S^1 to X. Then γ is null-homotopic if and only if there exists a map $\Phi\colon \overline{B}^2 \to X$ so that $\Phi|_{S^1} = \varphi$.



Proof. We start out this proof with a few rather elementary preparations. We write $A := ([0,1] \times \{0\}) \cup (\{0,1\} \times [0,1])$ and we consider the map

$$\begin{array}{rcl} g\colon [0,1] \times [0,1] & \to & \overline{B}^2 \\ (t,s) & \mapsto & s\exp(2\pi\,\mathrm{i}t) + 1 - s. \end{array}$$

Note that this map has the property that all points in A get sent to $1 \in S^1$. Using this



observation, the Topological-Quotient Proposition 5.15 (1b) and the Compact-Hausdorff Proposition 2.17 (3) we see that the map

$$\begin{array}{rcl} h\colon ([0,1] \times [0,1])/A & \to & \overline{B}^2 \\ [P] & \mapsto & g(P) \end{array}$$

is a homeomorphism.

Now we turn to the actual proof of the lemma. Thus let X be a topological space, let $x_0 \in X$ and let $\gamma\colon [0,1] \to X$ be a loop with $\gamma(0) = \gamma(1) = x_0$. We define $\varphi\colon S^1 \to X$ via $\varphi(\exp(2\pi\,\mathrm{i}t)) = \gamma(t)$.

First we suppose that γ is null-homotopic. This means that there exists a path-homotopy $F\colon [0,1] \times [0,1] \to X$ such that $F(t,0) = x_0$ and $F(t,1) = \gamma(t)$ for all $t \in [0,1]$ and such that $F(0,s) = F(1,s) = x_0$ for all $s \in [0,1]$. In other words, F is constant on A. We consider the map[449]

$$\begin{array}{rccc} \Phi\colon \overline{B}^2 & \underset{\cong}{\overset{h}{\leftarrow}} & ([0,1]\times[0,1])/A & \to & X \\ & & [P] & \mapsto & F(P). \end{array}$$

One can easily verify that we have $\Phi|_{S^1} = \varphi$.

Conversely, if there exists a map $\Phi\colon \overline{B}^2 \to X$ so that $\Phi|_{S^1} = \varphi$, then one can easily verify that $\Phi \circ g\colon [0,1] \times [0,1] \to X$ is a path-homotopy from the loop γ to the constant loop c_{x_0}.[450] ■

Before we continue with the examples we need to gather a few more elementary facts about path-homotopies of paths. The following lemma is a special case of Lemma 14.5.

Lemma 47.2. Let X be a topological space and let $x, y \in X$. Then "path-homotopy" is an equivalence relation on the set of paths $\gamma\colon [a,b] \to X$ from x to y.

Proof. This lemma is just the special case Lemma 14.5 with $X = [a,b]$ and $A = \{a,b\}$. For the reader's convenience we provide the straightforward proof also in this context. We have to show that "path-homotopy" satisfies the three properties of an equivalence relation. Let $\alpha, \beta, \gamma\colon [a,b] \to X$ be three paths from x to y.

(1) The "constant path-homotopy" given by $\Gamma(t,s) := \gamma(t)$ defines a path-homotopy between γ and γ.
(2) Let Γ be a path-homotopy from α to β. It is basically clear that the map Γ' defined by $\Gamma'(t,s) := \Gamma(t, 1-s)$ is a path-homotopy from β to α.
(3) Let Γ be a path-homotopy from α to β and let Δ be a path-homotopy from β to γ. We consider the map

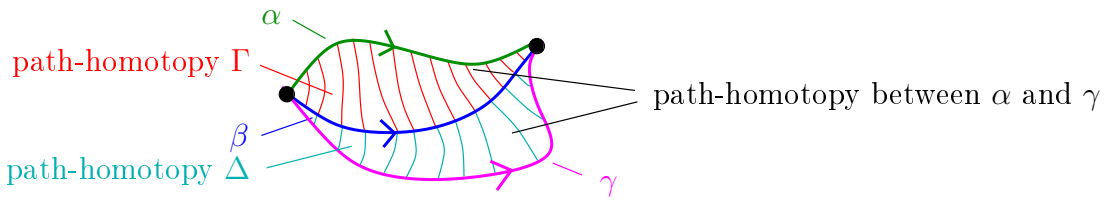
$$\begin{array}{rcl} [a,b]\times[0,1] & \to & X \\ (t,s) & \mapsto & \begin{cases} \Gamma(t,2s), & \text{if } s \in \left[0, \frac{1}{2}\right], \\ \Delta(t,2s-1), & \text{if } s \in \left(\frac{1}{2}, 1\right] \end{cases} \end{array}$$

It follows from the Homotopy Stacking Lemma 14.3 that this map is continuous. It is now clear that this map is a path-homotopy from α to γ. ■

Definition. Given a path $f\colon [0,1] \to X$ in a topological space X we denote by $[f]$ the equivalence class of f with respect to the equivalence relation which is given by path-homotopies. We call $[f]$ the path-homotopy class of f.

[449] Note that we implicitly use the Topological-Quotient Proposition 5.15 (1b) to argue that the map $([0,1]\times[0,1])/A \to X$ is well-defined and continuous.

[450] Why did we not just define g to be the map that sends the horizontal lines to concentric circles where each of them is centered around the origin?



Definition. Let $\gamma\colon [a,b] \to X$ be a path in a topological space and let $\varphi\colon [a,b] \to [a,b]$ be a map[451] with $\varphi(a) = a$ and $\varphi(b) = b$. We refer to $\gamma \circ \varphi\colon [a,b] \to X$ as a reparametrization of γ.

The following lemma says that reparametrizing does not change the path-homotopy class of a path.

Lemma 47.3. Let $\gamma\colon [a,b] \to X$ be a path in a topological space. If a path $\delta\colon [a,b] \to X$ is obtained from γ by a reparametrization, then γ and δ are path-homotopic.

Proof. Let $\gamma\colon [a,b] \to X$ be a path in a topological space and let $\varphi\colon [a,b] \to [a,b]$ be a map with $\varphi(a) = a$, $\varphi(b) = b$. A path-homotopy between γ and $\gamma \circ \varphi$ is given by[452]

$$\begin{array}{rccl} P\colon [a,b] \times [0,1] & \to & X \\ (t,s) & \mapsto & \gamma(t \cdot s + \varphi(t) \cdot (1-s)). \end{array}$$

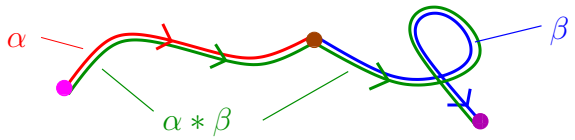
■

We now recall the following definition from page 165.

Definition. Let X be a topological space and let $\alpha\colon [a,b] \to X$ and $\beta\colon [c,d] \to X$ be two paths with $\alpha(b) = \beta(c)$. We define the concatenation of α and β as the path $\alpha * \beta$ which is given by

$$\begin{array}{rccl} \alpha * \beta\colon [0,1] & \to & X \\ t & \mapsto & \begin{cases} \alpha(a + 2t(b-a)), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta(c + (2t-1)(d-c)), & \text{if } t \in \left(\frac{1}{2}, 1\right]. \end{cases} \end{array}$$

The concatenation of two paths is thus given by "first running along α" and then "running along β". Here we reparametrized the path so that the domain of $\alpha * \beta$ is $[0,1]$. This convention might look slightly odd, but later on we will mostly work with paths that are defined on the interval $[0,1]$.



Lemma 47.4. Let $\alpha, \alpha'\colon [a,b] \to X$ and $\beta, \beta'\colon [c,d] \to X$ be two pairs of paths in a topological space X such that $\alpha(b) = \alpha'(b) = \beta(c) = \beta'(c)$. Then

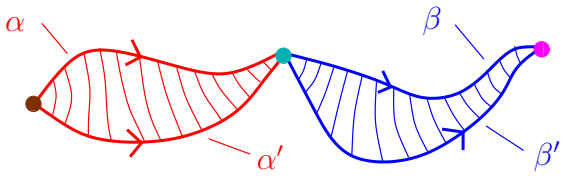
$$\alpha \simeq \alpha' \text{ and } \beta \simeq \beta' \quad \Longrightarrow \quad \alpha * \beta \simeq \alpha' * \beta'.$$

Put differently, if $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$, then $[\alpha * \beta] = [\alpha' * \beta']$.

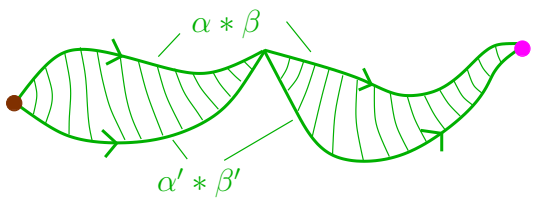
Proof. We leave it to the reader to provide the fairly elementary proof. ■

[451] Note that φ is not required to be a homeomorphism.

[452] Where do we actually use the hypothesis that $\varphi(a) = a$ and $\varphi(b) = b$?



α is path-homotopic to α'
and β is path-homotopic to β'



then $\alpha * \beta$ is path-homotopic to $\alpha' * \beta'$

Definition. Let $[\alpha]$ and $[\beta]$ be two equivalence classes of paths in a topological space X with $\alpha(1) = \beta(0)$. We define the[453] product of the equivalence classes $[\alpha]$ and $[\beta]$ as

$$[\alpha] \cdot [\beta] \;:=\; [\alpha * \beta].$$

The following proposition says that the product of equivalence classes of paths shares many properties of the product structure of a group.

Proposition 47.5. (Path-Homotopy Concatenation Proposition) Let $\alpha, \beta, \gamma\colon [0,1] \to X$ be three paths in a topological space X. Then the following hold:

(1) If $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$, then[454]

$$[\alpha] \cdot ([\beta] \cdot [\gamma]) \;=\; ([\alpha] \cdot [\beta]) \cdot [\gamma].$$

(2) For $x \in X$ we denote by e_x the constant path that is given by $e_x(t) := x$, $t \in [0,1]$. Then the following equality holds

$$[e_{\alpha(0)}] \cdot [\alpha] \;=\; [\alpha] \;=\; [\alpha] \cdot [e_{\alpha(1)}].$$

(3) We denote by $\overline{\alpha}$ the inverse path. Then the following equalities hold

$$[\alpha] \cdot [\overline{\alpha}] \;=\; [e_{\alpha(0)}] \quad \text{and} \quad [\overline{\alpha}] \cdot [\alpha] \;=\; [e_{\alpha(1)}].$$

Proof. Let $\alpha, \beta, \gamma\colon [0,1] \to X$ be three paths in a topological space X.

(1) Suppose that $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$. We need to show that

$$\alpha * (\beta * \gamma) \;\simeq\; (\alpha * \beta) * \gamma.$$

We consider the map

$$\begin{aligned} \Phi\colon [0,1] &\to X \\ t &\mapsto \begin{cases} \alpha(3t), & \text{if } t \in [0, \tfrac{1}{3}], \\ \beta(3t-1), & \text{if } t \in [\tfrac{1}{3}, \tfrac{2}{3}], \\ \gamma(3t-2), & \text{if } t \in [\tfrac{2}{3}, 1]. \end{cases} \end{aligned}$$

It follows from our hypothesis and Pasting Proposition 2.6 (2) that this map is indeed continuous. We consider the maps

$$\begin{aligned} p\colon [0,1] &\to [0,1] \\ t &\mapsto \begin{cases} \tfrac{2}{3}t, & \text{if } t \in [0, \tfrac{1}{2}], \\ \tfrac{4}{3}t - \tfrac{1}{3}, & \text{if } t \in (\tfrac{1}{2}, 1] \end{cases} \end{aligned} \quad \text{and} \quad \begin{aligned} q\colon [0,1] &\to [0,1] \\ t &\mapsto \begin{cases} \tfrac{4}{3}t, & \text{if } t \in [0, \tfrac{1}{2}], \\ \tfrac{2}{3}t + \tfrac{1}{3}, & \text{if } t \in (\tfrac{1}{2}, 1] \end{cases} \end{aligned}$$

[453] According to Lemma 47.4 this definition does not depend on the choice of the representatives α and β of the equivalence classes.

[454] Put differently, the paths $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$ are path-homotopic. Note that these paths are similar in the sense that both paths "run through" the points on α, β and γ, but at different speeds. In general the paths are not equal, for example we have $(\alpha * (\beta * \gamma))(\frac{1}{2}) = \alpha(1)$ and $((\alpha * \beta) * \gamma)(\frac{1}{2}) = \beta(1)$.

Then
$$\alpha * (\beta * \gamma) \underset{\text{by definition}}{=} \Phi \circ p \underset{\text{by Lemma 47.3}}{\simeq} \Phi \simeq \Phi \circ q \underset{\text{by definition}}{=} (\alpha * \beta) * \gamma$$

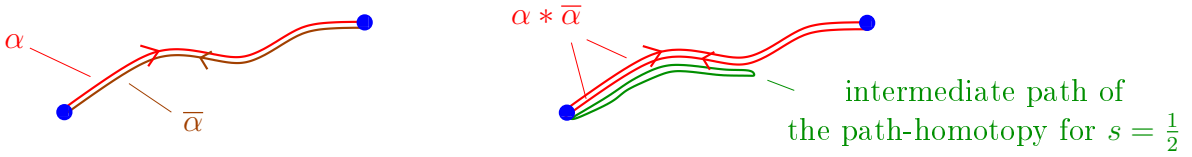
(2) We consider the map
$$\begin{aligned} q\colon [0,1] &\to [0,1] \\ t &\mapsto \begin{cases} 0, & \text{if } t \in [0, \frac{1}{2}], \\ 2t-1, & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{aligned}$$

Then
$$e_{\alpha(0)} * \alpha \underset{\text{by definition}}{=} \alpha \circ q \underset{\text{by Lemma 47.3}}{\simeq} \alpha.$$

This shows that $[e_{\alpha(0)}] \cdot [\alpha] = [e_{\alpha(0)} * \alpha] = [\alpha]$. Almost the same argument also implies that $[\alpha] = [\alpha] \cdot [e_{\alpha(1)}]$.

(3) We write $P = \alpha(0)$. Note that $\alpha * \overline{\alpha}$ is a loop in P. We consider the map
$$\begin{aligned} F\colon [0,1] \times [0,1] &\to X \\ (t,s) &\mapsto \begin{cases} \alpha(2t \cdot s), & \text{if } t \in [0, \frac{1}{2}], \\ \alpha(2s - 2t \cdot s), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{aligned}$$

It follows from the Homotopy Stacking Lemma 14.3 that this map is indeed continuous. For $s = 0$ we obtain the constant path at P and for $s = 1$ we obtain the path $\alpha * \overline{\alpha}$. For any $s \in [0,1]$ we furthermore have $F(0,s) = F(1,s) = P$. Thus F is a path-homotopy from the constant path to $\alpha * \overline{\alpha}$ which implies that $[e_{\alpha(0)}] = [\alpha * \overline{\alpha}] = [\alpha] \cdot [\overline{\alpha}]$. Almost the same argument shows that $[\overline{\alpha} * \alpha] = [e_{\alpha(1)}]$. ■



47.2. **The fundamental group of a pointed topological space.** In the Path-Homotopy Concatenation Proposition 47.5 we saw that the concatenation of path-homotopy classes of paths shares many properties of the multiplication in a group. Nonetheless the path-homotopy classes of paths do not form a group, since the concatenation of two paths is not defined, unless the endpoint of the first path agrees with the starting point of the second path.
We will resolve this problem by restricting our attention to loops.

Definition. Let X be a topological space and let $x_0 \in X$ be a point. A loop in (X, x_0) is a path $f\colon [0,1] \to X$[455] with $f(0) = f(1) = x_0$, i.e. x_0 is the starting as well as the endpoint of f.

Proposition 47.6. Let X be a topological space and $x_0 \in X$. The set
$$\pi_1(X, x_0) := \{\text{path-homotopy classes of loops in } (X, x_0)\}$$
together with the product map $[\alpha] \cdot [\beta] := [\alpha * \beta]$ forms a group where the neutral element is represented by the constant path at x_0 and where the inverse of $[\gamma]$ is given by $[\overline{\gamma}]$.

[455] Note that part of the definition of "loop in (X, x_0)" is that the domain is the interval $[0,1]$.

Proof. It is clear that the concatenation of any two loops in (X, x_0) is defined. According to Lemma 47.4 the concatenation of paths descends to a well-defined map

$$\begin{array}{rcl} \pi_1(X,x_0) \times \pi_1(X,x_0) & \to & \pi_1(X,x_0) \\ ([\alpha],[\beta]) & \mapsto & [\alpha]\cdot[\beta] := [\alpha * \beta]. \end{array}$$

According to the Path-Homotopy Concatenation Proposition 47.5 (1) this product map satisfies the associativity axiom. The trivial element is according to the Path-Homotopy Concatenation Proposition 47.5 (2) given by the path-homotopy class of the constant loop $[e_{x_0}]$. Furthermore according to the Path-Homotopy Concatenation Proposition 47.5 (3) the inverse of the path-homotopy class of a loop $\gamma\colon [0,1] \to X$ in x_0 is given by the path-homotopy class of the loop defined by $\overline{\gamma}(t) := \gamma(1-t)$, $t \in [0,1]$. ■

Definition. We call $\pi_1(X, x_0)$ the fundamental group of X with respect to the base point x_0.

One of our main goals in the intermediate future will be to determine the fundamental group of the topological spaces that we introduced in the earlier sections.

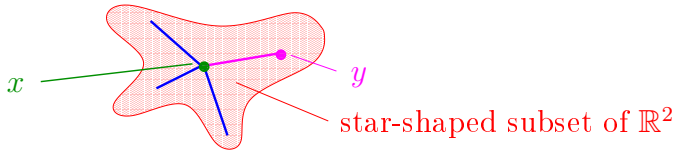
Examples.

(1) Let X be a convex subset of $\mathbb{R}^n$ and let $x_0 \in \mathbb{R}^n$ be an arbitrary point. As we pointed out on page 1048, all loops in (X, x_0) are null-homotopic. More precisely, if $\gamma\colon [0,1] \to \mathbb{R}^n$ is a loop in $(\mathbb{R}^n, x_0)$, then

$$\begin{array}{rcl} F\colon [0,1]\times[0,1] & \to & \mathbb{R}^n \\ (t,s) & \mapsto & \gamma(t)\cdot(1-s) + x_0 \cdot s \end{array}$$

is a path-homotopy between the loop γ and the constant loop e_{x_0}. Thus it follows that $\pi_1(X, x_0) = 0$.[456]

(2) We say that a subset $X \subset \mathbb{R}^n$ is *star-shaped* if there exists an $x \in X$ such that for all $y \in X$ the segment $\{xt + y(1-t) \,|\, t \in [0,1]\}$ lies in X. For example open and closed balls and open and closed rectangles are star-shaped. Furthermore all convex subsets are star-shaped. Precisely the same proof as in (1) shows that the $\pi_1(X, x) = 0$.



Next we state the following theorem from complex analysis.

[456] If $\pi_1(X, x_0)$ is the trivial group, then we usually write $\pi_1(X, x_0) = 0$. This notation is commonly used, even though it is not entirely logical: we use the multiplicative notation for the product structure on the group $\pi_1(X, x_0)$, and it would therefore make more sense to write $\pi_1(X, x_0) = 1$ if the group $\pi_1(X, x_0)$ is trivial.

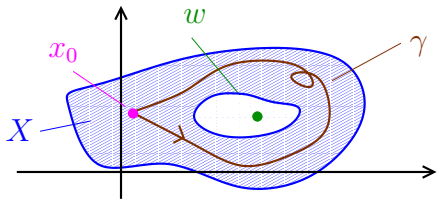
Theorem 47.7. Let X be a proper subset[457] of $\mathbb{C}$, let $x_0 \in X$ be a point and let $w \in \mathbb{C}\setminus X$ be a point in the complement of X. The map

$$\begin{array}{rcl} \pi_1(X, x_0) & \to & \mathbb{Z} \\ [\gamma] & \mapsto & \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{1}{z-w}\, dz \end{array}$$

is a well-defined homomorphism.

Remark. As in the case of Corollary 46.1, we do not really need Theorem 47.7 and its subsequent Corollary 47.8. If one has seen complex analysis, or if one is willing to take Theorem 47.7 on face value, it allows us to show that some fundamental groups are non-trivial, before we have developed the full machinery for calculating fundamental groups. But eventually we will supplant Corollary 47.8 by a much stronger statement proved using purely topological methods.

Proof. A vigilant reader might first have noticed that the statement of Theorem 47.7 contains a slight cheat. We use the path integral over a continuous path γ, whereas so far on page 791 we gave only the definition for the integral over a smooth path. The more general definition of a path integral is given in [**Lang1999**, Chapter III.4]. The fact that the given integral takes values in $\mathbb{Z}$ is proved in [**Lang1999**, Lemma IV.4.1]. Furthermore the fact that this integral defines a well-defined map on the fundamental group is a consequence of [**Lang1999**, Theorem III.5.1]. Finally it follows almost immediately from the definitions that the given map is a homomorphism. ■



the map

$$\begin{array}{rcl} \pi_1(X, x_0) & \to & \mathbb{Z} \\ [\gamma] & \mapsto & \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{1}{z-w}\, dz \end{array}$$

is a homomorphism

The previous example together with Theorem 47.7 give us the following neat corollary.

Corollary 47.8. There exists an epimorphism $\pi_1(S^1, 1) \to \mathbb{Z}$.

Proof. The corollary follows from Theorem 47.7 and the calculation on page 791. ■

This raises the following question.

Question 47.9. Is the above epimorphism $\pi_1(S^1, 1) \to \mathbb{Z}$ in fact an isomorphism?

The definition of the fundamental group of a topological space X relies on the choice of a base point x_0. The following proposition shows that, at least for path-connected topological spaces, the choice of the base point does not affect the isomorphism type of the fundamental group.

Proposition 47.10. (Change-of-Base Point Proposition) Let X be a topological space, let x_0 and x_1 be two points in X and let $p\colon [0,1] \to X$ be a path from x_0 to x_1. Then the following five statements hold:

(1) The map

[457] We say A is a proper subset of B if $A \subset B$ and if $A \neq B$.

$$\begin{array}{rcl} p_*\colon \pi_1(X,x_1) & \to & \pi_1(X,x_0) \\ [\gamma] & \mapsto & [p*\gamma*\overline{p}] \end{array}$$

is well-defined and it is a group isomorphism.

(2) If $x_0 = x_1$, then the map p_* is just conjugation by $[p] \in \pi_1(X,x_0)$, i.e. for any $g \in \pi_1(X,x_0) = \pi_1(X,x_1)$ we have

$$p_*(g) \;=\; [p]\cdot g\cdot [p]^{-1} \;\in\; \pi_1(X,x_0).$$

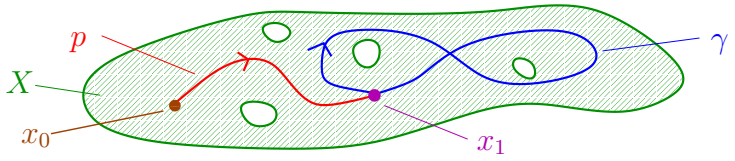
(3) If $q\colon [0,1] \to X$ is a path from x_1 to another point x_2, then

$$(p*q)_* \;=\; p_*\circ q_*\colon \pi_1(X,x_2) \;\to\; \pi_1(X,x_0).$$

(4) If $\pi_1(X,x_0)$ is abelian, then for any two paths $p,q\colon [0,1]\to X$ from x_0 to x_1 we have $p_* = q_*\colon \pi_1(X,x_1)\to\pi_1(X,x_0)$.

(5) If $f\colon X \to Y$ is a map to another topological space, then the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X,x_0) & \xrightarrow{\;p_*\;} & \pi_1(X,x_1) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \pi_1(Y,f(x_0)) & \xrightarrow{\;(f\circ p)_*\;} & \pi_1(Y,f(x_1)). \end{array}$$



Proof. Let X be a topological space and let x_0 and x_1 be two points in X that are connected via a path $p\colon [0,1]\to X$ from x_0 to x_1.

(1) Let γ be a loop in (X,x_1). Then $p*\gamma*\overline{p}$ is evidently a loop in (X,x_0). From Lemma 47.4 and the Path-Homotopy Concatenation Proposition 47.5 it follows that the path-homotopy class of $p*\gamma*\overline{p}$ depends only on the choice of the path-homotopy class of γ, i.e. the map

$$\begin{array}{rcl} \Phi\colon \pi_1(X,x_1) & \to & \pi_1(X,x_0) \\ [\gamma] & \mapsto & [p*\gamma*\overline{p}] \end{array}$$

is well-defined. Furthermore for two loops γ,δ in (X,x_1) we have

$$\begin{array}{rcl} \Phi([\gamma])*\Phi([\delta]) & = & [p*\gamma*\overline{p}]*[p*\delta*\overline{p}] \\ & = & [p*\gamma*e_{x_1}*\delta*\overline{p}] \\ & & \uparrow \\ & & \text{Path-Homotopy Concatenation Proposition 47.5 (3) and Lemma 47.4} \end{array} \qquad \begin{array}{rcl} & = & [p*\gamma*\overline{p}*p*\delta*\overline{p}] \\ & = & [p*(\gamma*\delta)*\overline{p}] \;=\; \Phi([\gamma*\delta]). \\ & & \uparrow \\ & & \text{Path-Homotopy Concatenation Proposition 47.5 (} \end{array}$$

Thus we have shown that Φ is a group homomorphism. Furthermore Φ is even a group isomorphism, since an inverse map is given by[458]

$$\begin{array}{rcl} \Psi\colon \pi_1(X,x_0) & \to & \pi_1(X,x_1) \\ [\delta] & \mapsto & [\overline{p}*\delta*p]. \end{array}$$

(2),(3) These two statements follow basically immediately from the definitions.

(4) This statement is a straightforward consequence of (1), (2) and (3). We will provide the proof in Exercise 47.3.

[458] Why is $\Phi\circ\Psi$ the identity on $\pi_1(X,x_0)$ and why is $\Psi\circ\Phi$ the identity on $\pi_1(X,x_1)$?

(5) This statement again follows immediately from the definitions. ■

We obtain the following corollary:

Corollary 47.11. (Base Point Independence Corollary) Let X be a path-connected topological space and let $x_0, x_1 \in X$. Then the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.

Example. Let X be a star-shaped subset of $\mathbb{R}^n$. Above we saw that there exists an $x \in X$ such that $\pi_1(X, x) = 0$. It now follows from the Base Point Independence Corollary 47.11 that $\pi_1(X, y) = 0$ for any point $y \in X$. Put differently, the fundamental group of every star-shaped subset of $\mathbb{R}^n$ with respect to any base point is trivial.

If X is a path-connected topological space, then the Base Point Independence Corollary 47.11 states that the isomorphism type of the fundamental group does not depend on the choice of the base point. Sometimes we suppress the base point from the notation and we denote by $\pi_1(X)$ the isomorphism type of the fundamental group.

Definition. Let X be a non-empty path-connected topological space. If $\pi_1(X) = 0$[459], then we say that X is simply connected.[460]

Lemma 47.12. Let X be a topological space and let P, Q be two points in X. If X is simply connected, then any two paths from P to Q are path-homotopic.

Proof. The proof is precisely the content of Exercise 47.2. ■

We already saw on page 1055 that $\mathbb{R}^n$ and more generally, all star-shaped subsets of $\mathbb{R}^n$ are simply connected. The following statement is much more interesting:

Proposition 47.13. (Sphere-π_1-Proposition) For any $n \geq 2$ the sphere S^n is simply connected.

In the proof of the Sphere-π_1-Proposition 47.13 we will need the following lemma.

Lemma 47.14. For any $P \in S^n$ the topological space $S^n \setminus \{P\}$ is simply connected.

Proof of Lemma 47.14. Recall that in Lemma 2.18 we showed that there exists a homeomorphism $f\colon S^n \to \mathbb{R}^n \cup \{\infty\}$. We set $Q := f^{-1}(\infty)$. Thus f restricts to homeomorphism $f\colon S^n \setminus \{Q\} \to \mathbb{R}^n$. Now let P be any other point on S^n. There exists a homeomorphism[461] $g\colon S^n \to S^n$ with $g(P) = Q$. Then $f \circ g\colon S^n \to \mathbb{R}^n \cup \{\infty\}$ restricts to a homeomorphism $S^n \setminus \{P\} \to \mathbb{R}^n$. But $\mathbb{R}^n$ is simply connected, hence $S^n \setminus \{P\}$ is also simply connected. ■

Now we can turn to the proof of the Sphere-π_1-Proposition 47.13.

Here is a very short "proof": Let $x_0 \in S^n$ and let $\gamma\colon [0,1] \to S^n$ be a loop in x_0. We have to show that γ is null-homotopic. We pick a point P that does not lie in the image of γ. By Lemma 47.14 the topological space $S^n \setminus \{P\}$ is simply connected, hence γ is null-homotopic in $S^n \setminus \{P\}$, in particular it is null-homotopic in S^n.

[459] Since X is a path-connected topological space it is irrelevant which base point we consider.

[460] It is straightforward to see that for subsets of $\mathbb{C}$ we obtain the same notion of "simply connectedness" as on page 1045.

[461] The existence of the homeomorphism follows from Proposition 36.4. But one can also easily give an elementary proof of that fact. More precisely, using basic facts from linear algebra one can show that there exists an orthogonal matrix A with $A \cdot P = Q$. Then the map $g\colon S^n \to S^n$ that is given by multiplication by A has the desired property.

This sounds convincing, except for one crucial gap: we did not justify why there exists a point P that does not lie in the image of γ. In fact, as we saw in the Space-Filling Curve Theorem 6.13, such a point does not need to exist since there are surjective maps from an interval to any S^n.

We provide two approaches to fixing this problem. The first approach uses the Whitney Approximation Theorem 29.1. The second approach is completely self-contained and considerably longer.

First proof of the Sphere-π_1-Proposition 47.13. Let $n \geq 2$, let $x_0 \in S^n$ and let $\gamma\colon [0,1] \to S^n$ be a loop in x_0. It follows immediately from the Whitney Approximation Theorem 29.1, applied to the closed set[462] $A = \{0,1\} \subset [0,1]$, that there exists a homotopy rel $\{0,1\}$ from $\gamma\colon [0,1] \to S^n$ to a smooth map $\delta\colon [0,1] \to S^n$ with $\delta(0) = \delta(1) = x_0$. It follows from the Measure-Zero Properties Proposition 30.2 and 30.3 and the hypothesis that $n \geq 2$ that the smooth map $\delta\colon [0,1] \to S^n$ is not surjective. We pick $P \in S^n \setminus \{P\}$. By Lemma 47.14 we know that δ is homotopic rel $\{0,1\}$ to the constant map. Thus γ is also homotopic rel $\{0,1\}$ to the constant map. We have thus shown that δ represents the trivial element in $\pi_1(S^n, x_0)$. ■

Second proof of the Sphere-π_1-Proposition 47.13. Let $n \geq 2$. We write

$$\begin{array}{lllllllll} & N & := & (0,\dots,0,1) & \text{“North Pole”} & \text{and} & U & := & S^n \setminus \{N\}, \\ \text{and also} & S & := & (0,\dots,0,-1) & \text{South Pole} & \text{and} & V & := & S^n \setminus \{S\}. \end{array}$$

We fix a base point x_0 on S^n that is neither N nor S. By Lemma 47.14 it suffices to show that any loop in (S^n, x_0) is path-homotopic to a loop in $U = S^n \setminus \{N\}$.

So let γ be a loop in (S^n, x_0).

Claim. There exists a subdivision

$$0 = t_0 < t_1 < \ldots < t_{k-1} < t_k = 1,$$

such that the following hold:

(1) for each $i \in \{0,\dots,k-1\}$ we have $\gamma([t_i, t_{i+1}]) \subset U$ or $\gamma([t_i, t_{i+1}]) \subset V$, and
(2) for each $i \in \{1,\dots,k-1\}$ we have $\gamma(t_i) \neq N$ and $\gamma(t_i) \neq S$.

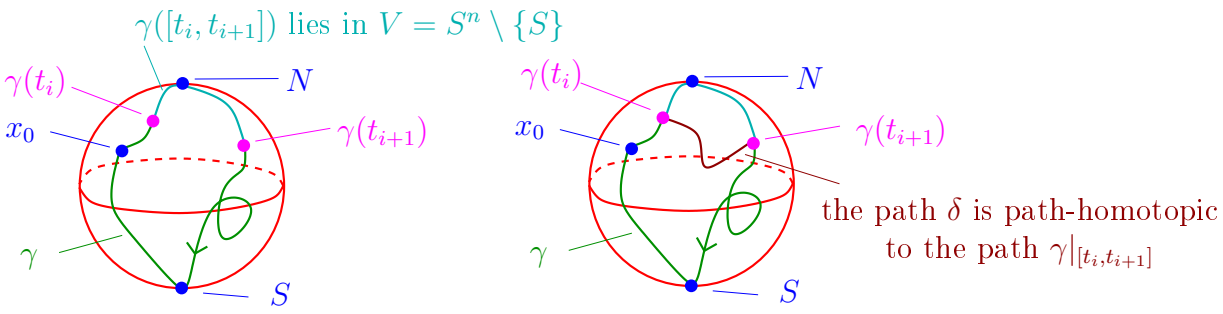
Proof. By the Lebesgue Corollary 6.6 applied to $\gamma\colon [0,1] \to S^n$ and $S^n = U \cup V$ there exists a k such that for any $i \in \{0,\dots,k-1\}$ the image $\gamma([\frac{i}{k}, \frac{i+1}{k}])$ lies in U or in V. In particular the subdivision $t_i = \frac{i}{k}$, $i = 0,\dots,k$ satisfies condition (1).

Now suppose that there exists an $i \in \{1,\dots,k-1\}$ with $\gamma(t_i) = N$. Then neither $\gamma([t_{i-1}, t_i])$ nor $\gamma([t_i, t_{i+1}])$ can be contained in $U = S^n \setminus \{N\}$. So by (1) both have to be contained in V, in particular we have $\gamma([t_{i-1}, t_{i+1}]) \subset V$. But then we can remove t_i from the subdivision. Iterating this process we can assume that for each every i we have $\gamma(t_i) \neq N$. Similarly we can arrange for every i we have $\gamma(t_i) \neq S$. ⊞

As mentioned before, by Lemma 47.14 it now suffices to prove the following claim.

Claim. The loop $\gamma\colon [0,1] \to S^n$ in (S^n, x_0) is path-homotopic to a path that does not hit the North Pole N.

[462] We use the observation that it follows immediately from the definition on page 549 that the restriction of any map $[0,1] \to M$ to a smooth manifold is smooth on the subset $\{0,1\}$.



Proof. We pick a subdivision as in the previous claim. Suppose there exists an $i \in \{0, \ldots, k-1\}$ such that $\gamma([t_i, t_{i+1}])$ hits N. By the choice of our subdivision we know that $\gamma([t_i, t_{i+1}])$ lies entirely in $V = S^n \setminus \{S\}$.[463] Since $S^n \setminus \{N, S\}$ is path-connected and since $\gamma(t_i) \neq N, S$, $\gamma(t_{i+1}) \neq N, S$ we can pick a path $\delta\colon [t_i, t_{i+1}] \to U \cap V = S^n \setminus \{N, S\}$ that connects $\gamma(t_i)$ and $\gamma(t_{i+1})$. Since the paths $\gamma|_{[t_i,t_{i+1}]}$ and δ both lie in the simply connected set $V = S^n \setminus \{S\}$ they are path-homotopic by Lemma 47.12. In the loop γ we now replace $\gamma([t_i, t_{i+1}])$ by δ. We obtain a new loop that is path-homotopic to the original loop and that does not hit N on the interval $[t_i, t_{i+1}]$. Iterating this procedure gives us the desired path-homotopy to a loop that does not hit N.[464] ∎

47.3. The fundamental group as functor.

Definition. Let $f\colon X \to Y$ be a map between topological spaces and let $x_0 \in X$ be a base point. If $\gamma\colon [0,1] \to X$ is a loop in the point x_0, then $f \circ \gamma\colon [0,1] \to Y$ is a loop in the point $f(x_0)$. It is straightforward to see that

$$\begin{array}{rcl} f_*\colon \pi_1(X, x_0) & \to & \pi_1(Y, f(x_0)) \\ [\gamma] & \mapsto & [f \circ \gamma] \end{array}$$

is a well-defined map, i.e. $f_*([\gamma])$ is independent of the choice of the representative of the path-homotopy class.[465] Furthermore it follows easily from the definitions[466] that f_* is a group homomorphism. We call f_* the induced map. It also follows easily from the definitions that

$$\begin{array}{rcll} (\mathrm{id}_X)_* & = & \mathrm{id}_{\pi_1(X,x_0)}, & \text{for all pointed pairs } (X, x_0), \\ (g \circ f)_* & = & g_* \circ f_*, & \text{for all maps } f\colon (X, x_0) \to (Y, y_0) \text{ and } g\colon (Y, y_0) \to (Z, z_0). \end{array}$$

The above description of the properties of the induced maps look very much like the properties of a covariant functor. We just need to figure out what the categories are supposed to be. This leads us to the following definition.

[463] We know it lies in U or V, but since it hits N it cannot lie in U.

[464] Where did we use in the proof that $n \geq 2$?

[465] Indeed, suppose that γ and δ are two loops in (X, x_0) that are path-homotopic. Let F be a path-homotopy between γ and δ. Then $f \circ F$ is a path-homotopy between $f \circ \gamma$ and $f \circ \delta$. We had used almost the same argument on page 1049.

[466] If γ and δ are loops in x_0, then it follows immediately from the definitions that $(f \circ \gamma) * (f \circ \delta) = f \circ (\gamma * \delta)$.

Definition. A pointed topological space is a pair (X, x_0), where X is a topological space and x_0 is a point in X. We refer to the category $\mathcal{PTop}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{PTop}) &:= \text{all pointed topological spaces,} \\ \mathrm{Mor}((X,x_0),(Y,y_0)) &:= \text{all continuous maps } f \text{ from } X \text{ to } Y \text{ with } f(x_0) = y_0 \end{aligned}$$

with the usual composition of maps as the category of pointed topological spaces.

Summarizing we just convinced ourselves of the validity of the following proposition.

Proposition 47.15. Let $\mathcal{PTop}$ be the category of pointed topological spaces and let $\mathcal{Group}$ be the category of groups. Then the map

$$\begin{aligned} \mathrm{Ob}(\mathcal{PTop}) &\to \mathrm{Ob}(\mathcal{Group}) \\ (X,x_0) &\mapsto \pi_1(X,x_0) \end{aligned}$$

together with the maps

$$\begin{aligned} \mathrm{Mor}((X,x_0),(Y,y_0)) &\to \mathrm{Mor}(\pi_1(X,x_0),\pi_1(Y,y_0)) \\ (f\colon (X,x_0)\to(Y,y_0)) &\mapsto \left(\begin{array}{rcl} f_*\colon \pi_1(X,x_0) & \to & \pi_1(Y,y_0) \\ [\gamma] & \mapsto & [f\circ\gamma] \end{array} \right) \end{aligned}$$

is a covariant functor.

Next we will see that the functoriality of the fundamental groups can be helpful for studying certain questions. Let us recall the following definition from page 433.

Definition. Let X be a topological space and let $A \subset X$ be a subspace. We say A is a retract of X if there exists a retraction $r\colon X \to A$, i.e. a map with $r(a) = a$ for all $a \in A$.

Examples.

(1) Let X be a topological space. Then any point $x_0 \in X$ is a retract of X, indeed, the map $r(x) = x_0$ for $x \in X$ is a retraction.

(2) The circle $S^1 \times \{1\}$ is a retract of the torus $S^1 \times S^1$, in fact a retraction is given by $r(z,w) = (z,1)$.

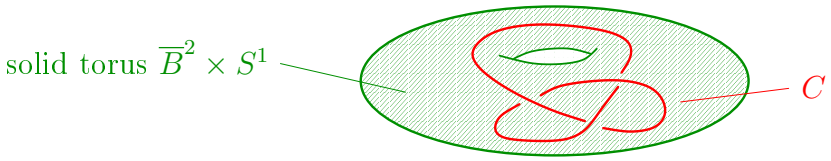
(3) The circle S^1 is a retract of $\overline{B}^2 \setminus \{(0,0)\}$, indeed, the map

$$\begin{aligned} \overline{B}^2 \setminus \{(0,0)\} &\to S^1 \\ z &\mapsto \tfrac{z}{|z|} \end{aligned}$$

is a retraction. (Here we consider $\overline{B}^2$ as subset of $\mathbb{C}$.)

(4) The set $S^0 = \{-1,1\}$ is not a retract of $\overline{B}^1 = [-1,1]$. Indeed, since $[-1,1]$ is connected and since S^0 is discrete it follows from the Connected-to-Discrete Lemma 2.27 that any map from $[-1,1]$ to S^0 has to be constant. Thus there does not exist a map $f\colon [-1,1] \to S^0$ with $f(-1) = -1$ and $f(1) = 1$.

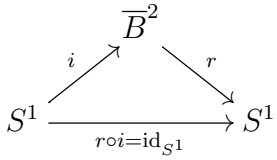
(5) Finally we consider the knotted curve C in the solid torus $\overline{B}^2 \times S^1$ that is shown in the figure below. We invite the reader to think about the question whether or not C is a retract of $\overline{B}^2 \times S^1$. We will answer the question in Section 162.1.



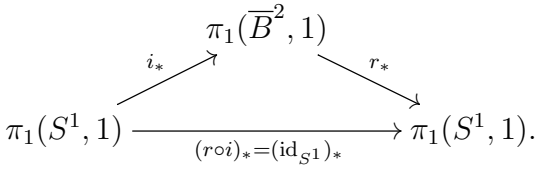
Now we can push the second-to-last example up by one dimension.

Lemma 47.16. The circle S^1 is not a retract of $\overline{B}^2$.

Proof. We denote by $i\colon S^1 \to \overline{B}^2$ the inclusion map. Suppose there exists a retraction $r\colon \overline{B}^2 \to S^1$. By definition this means that $r \circ i = \mathrm{id}_{S^1}$. We obtain the following commutative diagram of maps between topological spaces



which induces



It follows from the functoriality of fundamental groups that $r_* \circ i_* = (r \circ i)_*$, i.e. the diagram on the right-hand side commutes. By the functoriality we know that $(\mathrm{id}_{S^1})_*$ is the identity of $\pi_1(S^1, 1)$.

We saw on page 1058 that $\pi_1(\overline{B}^2, 1)$ is the trivial group whereas we saw in Corollary 47.8 that $\pi_1(S^1, 1)$ is not the trivial group. The lower horizontal map in the second diagram is thus an isomorphism of a non-trivial group, but the upper map factors through the trivial group, i.e. the composition of i_* and r_* cannot be an isomorphism of a non-trivial group.[467] Thus we have obtained a contradiction. ■

The last example and the last lemma now raise the following question.

Question 47.17. Let $n \geq 2$. Is S^n a retract of $\overline{B}^{n+1}$?

At the moment we cannot answer this question. We will first need to develop new invariants before we can return to this question. We will not be able to answer this question for many months.

Lemma 47.18. Let X be a topological space and let $A \subset X$ be a retract of X. Let $a \in A$. We denote by $i\colon A \to X$ the inclusion map.

(1) There exists an epimorphism $\varphi\colon \pi_1(X, a) \to \pi_1(A, a)$ such that $\varphi \circ i_*$ is the identity on $\pi_1(A, a)$.
(2) The inclusion induced map $\pi_1(A, a) \to \pi_1(X, a)$ is a monomorphism.

Proof. Almost exactly the same argument as in the proof of Lemma 47.16 also provides a proof of the lemma. We will fill in the details in Exercise 47.8. ■

Example. We had just seen on page 1061 that the circle $S^1 \times \{1\}$ is a retract of the torus $S^1 \times S^1$. Furthermore we saw in Corollary 47.8 that $\pi_1(S^1, 1)$ is not the trivial group. It follows from Lemma 47.18 that the fundamental group of the torus is also non-trivial. In the Sphere-π_1-Proposition 47.13 we saw that $\pi_1(S^2) = 0$. Thus we see that the torus $S^1 \times S^1$ and the sphere S^2 are not homeomorphic.

[467] Why not?

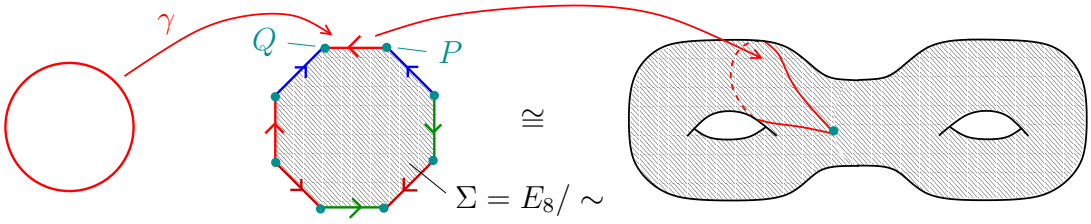
Lemma 47.19. Let E_8 be the regular octagon from the definition of the surface of genus 2, see page 265 for details, and let $\Sigma = E_8/\sim$ be the surface of genus 2. Let P and Q be two adjacent vertices of E_8. Then the following hold:

(1) The map
$$\begin{array}{rcl} \gamma\colon S^1 & \to & E_8/\sim \\ \exp(2\pi i t) & \mapsto & [t\cdot P + (1-t)\cdot Q] \end{array}$$
is an embedding.

(2) There exists a map $\Phi\colon E_8/\sim \to S^1 \times S^1$ such that the following diagram commutes
$$\begin{array}{ccc} & S^1 & \\ {\scriptstyle\gamma}\swarrow & & \searrow{\scriptstyle z\mapsto(z,1)} \\ E_8/\sim & \xrightarrow{\Phi} & S^1\times S^1. \end{array}$$

(3) The subset $\gamma(S^1)$ is a retract of $\Sigma = E_8/\sim$.

(4) There exists an epimorphism $\pi_1(\Sigma, P) \to \mathbb{Z}$ that sends $[\gamma]$ onto 1.[468]



Remark. The lemma shows in particular that the four loops that we drew in the figure on 266 on the surface of genus 2 represent non-trivial elements in the fundamental group. Almost the same way one can show that in the surface of genus g, $g \geq 2$, defined as a quotient of a regular $4g$-gon, each edge gives rise to a non-trivial element of the fundamental group.

Proof.

(1) It is clear that γ is continuous and injective. By the Surface-is-Topological Manifold Proposition 18.7 we know that $\Sigma = E_8/\sim$ is Hausdorff. It follows from the Compact-Hausdorff Proposition 2.17 (2) that γ is a closed embedding.

(2) The map Φ is sketched in the figure below on the top. It is given by sending all points in the lower pentagon to a single point, such a map can be extended continuously to the upper pentagon in such a way that the remaining four sides of the upper

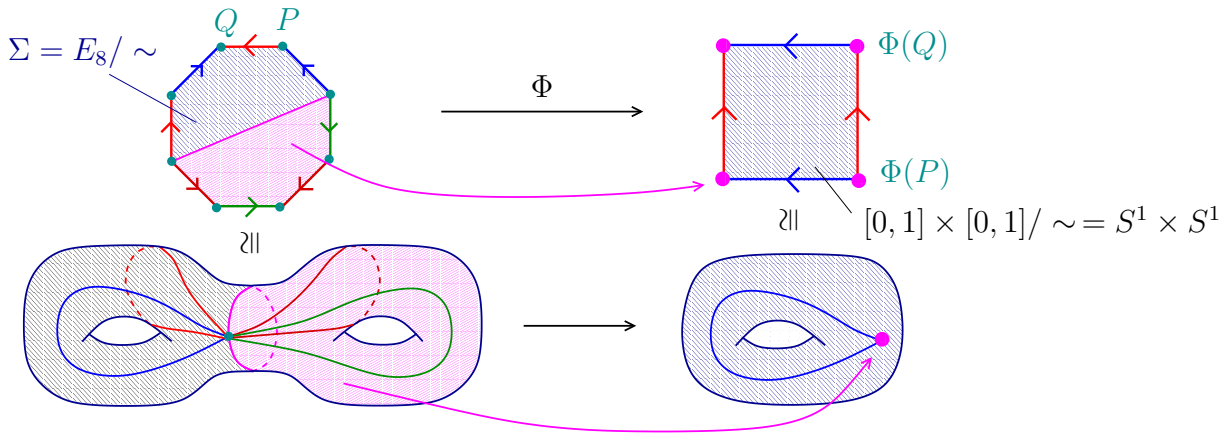
[468] Recall that by the bijection on page 1050 we view maps $f\colon S^1 \to \Sigma$ with $f(1) = P$ as elements in $\pi_1(\Sigma, P)$.

pentagon get sent homeomorphically to the four sides of the square.[469] [470] This map Φ has the desired property.

(3) The retraction is now given by the concatenation of the maps

$$\Sigma \xrightarrow{\Phi} S^1 \times S^1 \xrightarrow{(z,w)\mapsto z} S^1.$$

(4) This statement is a consequence of (3), Lemma 47.18 and Corollary 47.8. ■



In many applications we will want to know whether a given map between topological spaces induces an isomorphism of fundamental groups. The following lemma says that in many settings the property of being an isomorphism is independent of the choice of base point.

Proposition 47.20. Let $\varphi\colon X \to Y$ be a map between topological spaces and let $x_0 \in X$.

(1) If $\gamma\colon [0,1] \to X$ is a path from x_0 to some point x_1, then the following diagram commutes[471]

$$\begin{array}{ccc} \pi_1(X,x_1) & \xrightarrow[\cong]{\gamma_*} & \pi_1(X,x_0) \\ {\scriptstyle\varphi_*}\downarrow & & \downarrow{\scriptstyle\varphi_*} \\ \pi_1(Y,\varphi(x_1)) & \xrightarrow[\cong]{(\varphi\circ\gamma)_*} & \pi_1(Y,\varphi(x_0)). \end{array}$$

(2) If $\varphi_*\colon \pi_1(X,x_0) \to \pi_1(Y,\varphi(x_0))$ is an isomorphism (epimorphism, monomorphism) for some base point $x_0 \in X$, then for any other base point x_1 in the same path component of X the map $\varphi_*\colon \pi_1(X,x_1) \to \pi_1(Y,\varphi(x_1))$ is also an isomorphism (epimorphism, monomorphism).

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[469] It is clearly slightly painful to write down this map explicitly. It is easier to write down a similar map, which maps a square X onto a triangle Y in such a way that one side of the square gets sent to a vertex of the triangle and such that the remaining three sides of the square get sent to the three sides of the triangle. For example the map

$$\begin{array}{ccc} X := [0,1]\times[0,1] & \to & Y := \{(x,y) \in \mathbb{R}^2 \,|\, x \in [0,1], y \in [0,x]\} \\ (x,y) & \mapsto & (x,xy) \end{array}$$

is of that form.

[470] In the lower part of the above figure we illustrate the map for the surfaces viewed as subsets of $\mathbb{R}^3$. There the map Φ is given by crushing the right-hand half of the surface of genus 2 to a single point.

Proof.

(1) This statement follows immediately from the definitions.
(2) This statement follows from (1) and the fact that the horizontal maps are isomorphisms. ■

Exercises for Chapter 47.

Exercise 47.1. Consider the torus $T = S^1 \times S^1$ and consider the loop

$$\begin{array}{rcl} [0,1]/0 \sim 1 & \to & T = S^1 \times S^1 \\ t & \mapsto & (\exp(2\pi \mathrm{i} t), 1). \end{array}$$

Is the loop null-homotopic? If yes, provide a path-homotopy to a constant path. If not, show that the path is not null-homotopic.

Exercise 47.2. Let X be a topological space and let P, Q be two points in X. We suppose that X is simply connected. Show that any two paths from P to Q are path-homotopic.

Exercise 47.3. Let X be a topological space and let x_0 and x_1 be two points in X. Furthermore let $p, q\colon [0,1] \to X$ be two paths from x_0 to x_1. We suppose that $\pi_1(X, x_0)$ is abelian. Show that $p_* = q_*\colon \pi_1(X, x_1) \to \pi_1(X, x_0)$.

Exercise 47.4.

(a) Let $f\colon X \to Y$ be a surjective map of topological spaces and let $x \in X$. Is the induced map $f_*\colon \pi_1(X, x) \to \pi_1(Y, f(x))$ necessarily an epimorphism?
(b) Let $f\colon X \to Y$ be an injective map of topological spaces and let $x \in X$. Is the induced map $f_*\colon \pi_1(X, x) \to \pi_1(Y, f(x))$ necessarily a monomorphism?

Exercise 47.5. Given a group π we define its *automorphism group* $\mathrm{Aut}(\pi)$ as the group of all self-isomorphisms of π. We define the *inner automorphism group* as

$$\mathrm{Inn}(\pi) \;:=\; \{\varphi \in \mathrm{Aut}(\pi) \mid \text{ there exists an } h \in \pi \text{ such that } \varphi(g) = hgh^{-1} \text{ or all } g \in G\}.$$

One can easily verify that $\mathrm{Inn}(\pi)$ is a normal subgroup of $\mathrm{Aut}(\pi)$. Thus we can define the *outer automorphism group* to be the group

$$\mathrm{Out}(\pi) \;:=\; \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi).$$

Now let X be a path-connected topological space, let $x_0 \in X$ and let $n \in \mathbb{N}$.

(a) Show that the map

$$\begin{array}{rcl} \Theta\colon \mathrm{Homeo}(X) & \to & \mathrm{Out}(\pi_1(X, x_0)) \\ (f\colon X \to X) & \mapsto & (\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(X, f(x_0)) \xrightarrow[\uparrow]{\gamma_*} \pi_1(X, x_0)) \end{array}$$

pick any path γ from x_0 to $f(x_0)$ and consider the corresponding isomorphism given by the Change-of-Base Point Proposition 47.10

is well-defined, i.e. independent of the choice of the paths.
(b) Show that $\Theta\colon \mathrm{Homeo}(X) \to \mathrm{Out}(\pi_1(X, x_0))$ is a group homomorphism.

[472] Here the horizontal maps are the isomorphisms that we introduced in the Change-of-Base Point Proposition 47.10 (1).

Exercise 47.6. Show that $\pi_1(S^1 \times S^1)$ admits an epimorphism onto $\mathbb{Z}^2$.

Exercise 47.7. Show that there exists a homeomorphism of the torus $S^1 \times S^1$ that does not extend to a diffeomorphism of the solid torus $S^1 \times \overline{B}^2$.

Exercise 47.8. Let $a \in A$. We denote by $i\colon A \to X$ the inclusion map.

(a) Show that there exists an epimorphism $\varphi\colon \pi_1(X, a) \to \pi_1(A, a)$ such that $\varphi \circ i_*$ is the identity on $\pi_1(A, a)$.

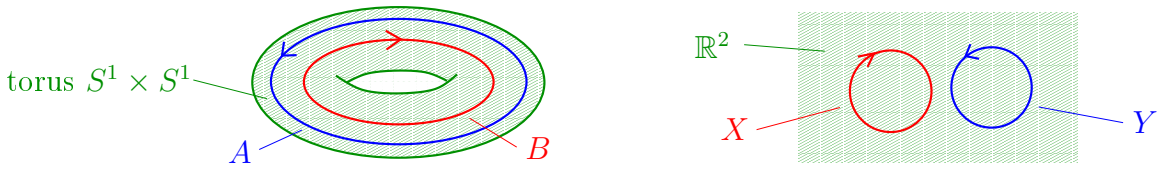
(b) Show that the inclusion induced map $\pi_1(A, a) \to \pi_1(X, a)$ is a monomorphism.

Exercise 47.9.

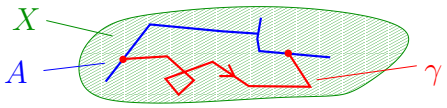
(a) Show that the two oriented 1-dimensional submanifolds A and B of $S^1 \times S^1$ that are shown in the figure on the left are not smoothly isotopic.

(b) We consider $\mathbb{R}^2$ with the two oriented 1-dimensional submanifolds X and Y that are shown in the figure on the right. Are these X and Y smoothly isotopic?

Remark. Beware of base points.



Exercise 47.10. Let X be a topological space, let $A \subset X$ be a path-connected subspace and let $x_0 \in A$. We denote by $i\colon A \to X$ the inclusion map. We suppose that the map $i_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ is an epimorphism. Finally let $\gamma\colon [0, 1] \to X$ be a path with $\gamma(0) \in A$ and $\gamma(1) \in A$. Show that there exists a homotopy $F\colon [0, 1] \times [0, 1] \to X$ rel $\{0, 1\}$ such that $F_0 = \gamma$ and such that F_1 takes values in A.



48. Fundamental groups and coverings

We have now introduced the fundamental group of a (pointed) topological space and we have proved several basic facts. But the only proof that fundamental groups are non-trivial relied on a fact from complex analysis. In particular, so far we have not yet managed to compute a single fundamental group that is not the trivial group. For example, we know that $\pi_1(S^1)$ is non-trivial, but we do not know what group it is.
In this chapter we will compute many non-trivial fundamental groups using the theory of covering spaces.

48.1. **Coverings of topological spaces.** Now we return to the study of topological spaces.

Definition. Let $p\colon X \to B$ be a map between topological spaces.

(1) We say an open subset $U \subset B$ is uniformly covered, if $p^{-1}(U)$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ with the property, that the restriction of p to each subset V_i is a homeomorphism.
(2) We say the map $p\colon X \to B$ is a[473] covering, if it is surjective and if for every $b \in B$ there exists an open neighborhood U of b which is uniformly covered.

Remark. If U is a uniformly covered subset of B, then clearly any open subset of U is also uniformly covered. For example it follows that if $p\colon X \to B$ is a covering and if B is locally (path-) connected, then any point $b \in B$ admits also a (*path*)- *connected* uniformly covered neighborhood.

Examples.

(a) The map

$$\begin{array}{rccc} p\colon \mathbb{R} & \to & S^1 \\ \varphi & \mapsto & \exp(\mathrm{i}\varphi) \end{array}$$

is a covering. Indeed, let $P = \exp(\mathrm{i}\alpha)$ be a point in S^1. We pick the open neighborhood

$$U := \left\{ \exp(\mathrm{i}\varphi) \,\middle|\, \varphi \in \left(\alpha - \tfrac{\pi}{4}, \alpha + \tfrac{\pi}{4}\right)\right\}.$$

Then

$$p^{-1}(U) := \bigsqcup_{j\in\mathbb{Z}} \underbrace{\left(\alpha - \tfrac{\pi}{4} + 2\pi j, \alpha + \tfrac{\pi}{4} + 2\pi j\right)}_{=:V_j},$$

and for each $j \in \mathbb{Z}$ the restriction of p to $V_j \to U$ is a homeomorphism. This example is illustrated in the figure below.

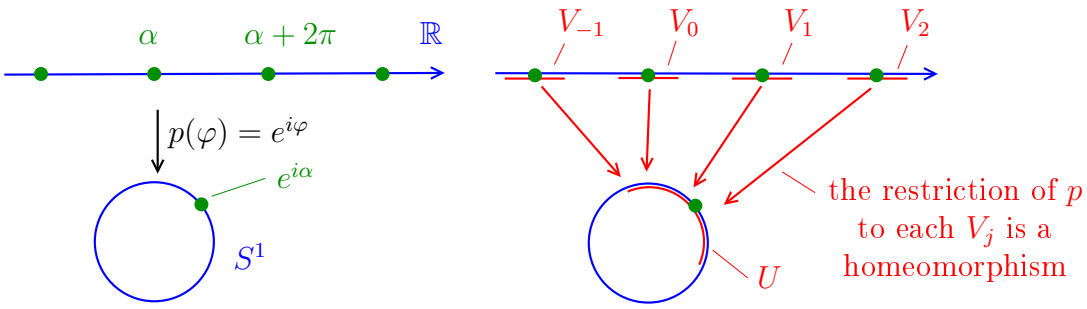
(b) We consider the "infinite helix"

$$H := \left\{ (r \cdot \exp(\mathrm{i}t), t) \,\middle|\, r \in (1,2) \text{ and } t \in \mathbb{R}\right\}$$

together with the "annulus"

$$A := \left\{ r \cdot \exp(\mathrm{i}t) \,\middle|\, r \in (1,2) \text{ and } t \in \mathbb{R}\right\}.$$

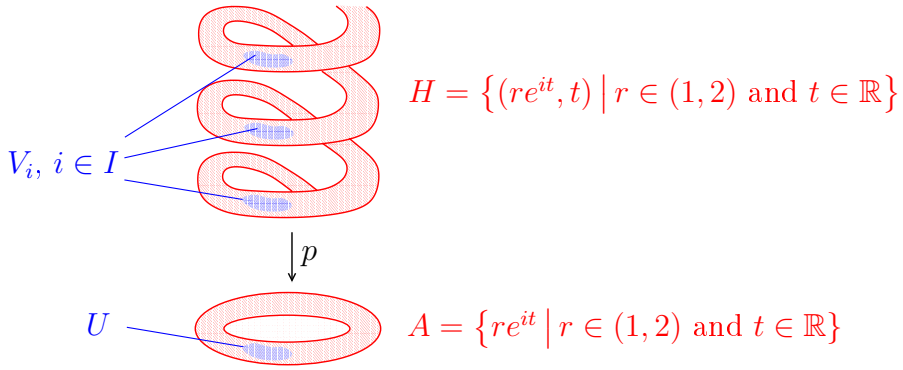
[473] On page 114 we introduced the notion of a "cover". As we mentioned on page 114, such a "cover" is sometimes also called a "covering". Now we introduce the notion of a "covering". Rather maddeningly, in the literature this notion often also gets called a "cover". Fortunately, from the context, it is usually clear what is meant by "cover" and "covering". These issues do not arise in other languages, e.g. in German and French what we call "cover" is referred to as "Überdeckung" and "recouvremement" and what we call "covering" is referred to as "Überlagerung" and "revêtement".



The map

$$\begin{array}{rcl} p\colon H & \to & A \\ (r\cdot\exp(\mathrm{i}t),t) & \mapsto & r\cdot\exp(\mathrm{i}t) \end{array}$$

is a covering. This statement is illustrated in the figure below. The proof of the statement is similar to the proof in the previous example.



(c) The map

$$\begin{array}{rcl} p\colon \mathbb{R}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ x & \mapsto & [x] = x+\mathbb{Z}^n \end{array}$$

is a covering. Indeed, let $[x] = [(x_1,\ldots,x_n)]$ be a point in $\mathbb{R}^n/\mathbb{Z}^n$. We consider

$$W \;:=\; \left\{(y_1,\ldots,y_n)\,\middle|\,|y_i - x_i| < \tfrac{1}{2}\right\} \subset \mathbb{R}^n$$

and we put $U := p(W)$. Then U is an open[474] neighborhood of $[x]$. Furthermore

$$p^{-1}(U) \;=\; \bigcup_{i\in\mathbb{Z}^n} (W+i),$$

and for each i the restriction of p to $W+i \subset \mathbb{R}^n$ is a homeomorphism.[475]

[474]The set U is open since the projection $p\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is open by the Group-Action Projection Lemma 5.24 (2a).

[475]It is straightforward to verify that the map $p\colon V_i := W+i \to U$ is a bijection. It follows from the Group-Action Projection Lemma 5.24 (1) that the projection p is continuous and as we had just mentioned in the previous footnote, the projection is also open. It follows that the map $p\colon V_i = W+i \to U$ is a homeomorphism.

(d) For every $n \in \mathbb{N}$ the map

$$\begin{array}{rcl} p\colon S^n & \to & \mathbb{R}\mathrm{P}^n = S^n/x \sim -x \\ x & \mapsto & [x] = \{x, -x\} \end{array}$$

is a covering. Indeed, for $[x] \in \mathbb{R}\mathrm{P}^n$ we set

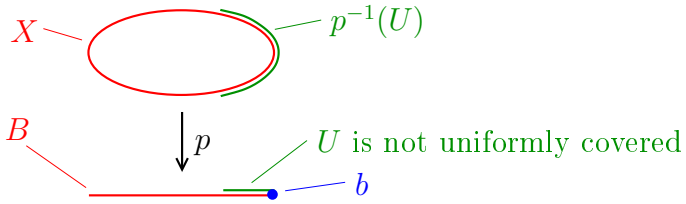
$$W \;=\; \{y \in S^n \mid \underbrace{x \cdot y}_{\text{scalar product}} > 0\}.$$

Then $p(W)$ is an open neighborhood of $[x]$ and it is uniformly covered by the two open sets W and $-W$.

(e) The projection

$$\begin{array}{rcl} p\colon S^1 & \to & [-1,1] \\ (x,y) & \mapsto & x \end{array}$$

is *not* a covering. Indeed, the point $b = 1 \in [-1,1]$ has no open neighborhood that is uniformly covered. This example is illustrated in the figure below.



Lemma 48.1. Let $p\colon X \to B$ be a covering of topological spaces. If B is a path-connected topological space, then for any two points P and Q in B we have

$$\#p^{-1}(P) \;=\; \#p^{-1}(Q).$$

Proof. The proof of the lemma is the content of Exercise 48.1. ■

Example. For a general map $p\colon X \to B$ between topological spaces the number of preimages of a point depends of course on the point chosen in B. For instance in the above example (e) we have $\#p^{-1}(0) = 2$ but $\#p^{-1}(1) = 1$.

Definition. Let $p\colon X \to B$ be a covering of a path-connected non-empty topological space.

(1) We define the degree $[X : B]$ of p as the cardinality of $\#p^{-1}(b)$ for some $b \in B$.[476]
(2) If the degree is a finite number n, then we say that p is an n-fold covering of B. Otherwise we say that p is an infinite covering. If the cardinality is countable, then sometimes we call it a countable covering.

For completeness sake we extend this definition to the case that B, and thus also X, are the empty space. In this case we define the degree to be 1.

Examples.

(1) We had just seen that the map $p\colon S^n \to S^n/\sim$ is a 2-fold covering of the real projective space $S^n/\sim = \mathbb{R}\mathrm{P}^n$.

[476] It follows from Lemma 48.1 that this definition does not depend on the choice of b.

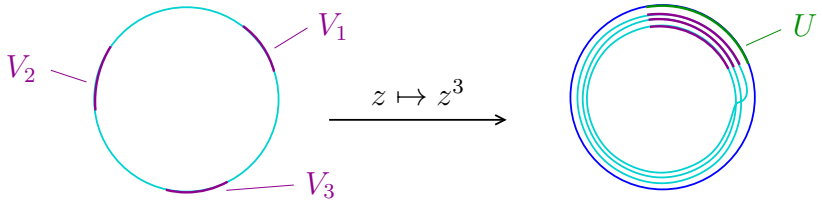
(2) Let $n \in \mathbb{N}$. As usual we consider $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ as a subset of $\mathbb{C}$. Then the map $p\colon S^1 \to S^1$ given by $p(z) := z^n$ is a covering. Indeed, let $z = \exp(2\pi \mathrm{i}\varphi)$ be a point in S^1. Then

$$U \;:=\; \big\{ \exp(2\pi \mathrm{i}\psi) \,\big|\, \psi \in \big(\varphi - \tfrac{1}{4}, \varphi + \tfrac{1}{4}\big)\big\}$$

has the desired property. Indeed, we have

$$p^{-1}(U) \;=\; \bigsqcup_{j=1}^{n} \underbrace{\big\{ \exp(2\pi \mathrm{i}\psi) \,\big|\, \psi \in \big(\tfrac{1}{n}(\varphi - \tfrac{1}{4}) + \tfrac{j}{n}, \tfrac{1}{n}(\varphi + \tfrac{1}{4}) + \tfrac{j}{n}\big)\big\}}_{=:V_j},$$

and the restriction of p to each set V_j is easily seen to be a homeomorphism. This shows that p is a covering and it is clear that p is an n-fold covering.

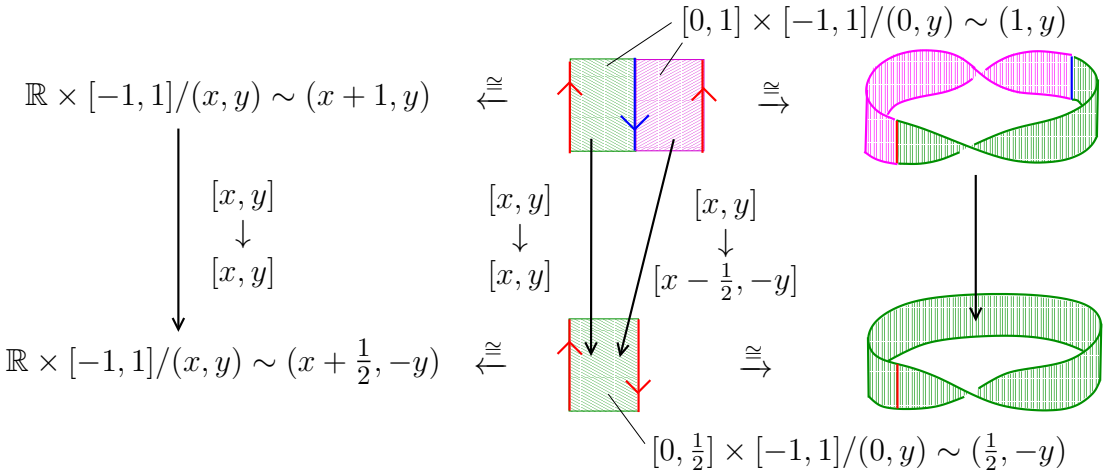


(3) Let $n \in \mathbb{N}$. Note that basically the same argument as before shows that the map $p\colon \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ given by $p(z) = z^n$ is an n-fold covering.

(4) We consider the map

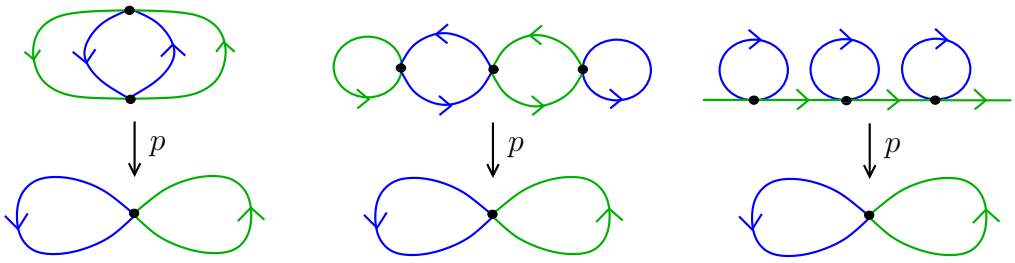
$$\begin{array}{rcl} p\colon \overbrace{\mathbb{R} \times [-1,1]/(x,y) \sim (x+1,y)}^{\text{annulus}} & \to & \overbrace{\mathbb{R} \times [-1,1]/(x,y) \sim (x+\frac{1}{2},-y)}^{\text{Möbius band}} \\ [(x,y)] & \mapsto & [(x,y)]. \end{array}$$

It is straightforward to show that p is a 2-fold covering. Thus we can view the annulus as a 2-fold covering of the Möbius band. This covering is illustrated in the figure below.



(5) If in the previous example we replace the interval $[-1,1]$ by the circle $[-1,1]/-1 \sim 1$, then on the left-hand side we obtain the torus and on the right-hand side we obtain the Klein bottle. This shows that there exists a degree 2 covering map from the torus to the Klein bottle.

(6) In the figure below we sketch three coverings of topological graphs. The coverings are hereby given by sending edges of a certain color, with the indicated direction, to the edge of the same color with the indicated direction.[477]



2-fold covering 3-fold covering infinite covering

Question 48.2. We had just seen that there exists a covering map $S^2 \to \mathbb{R}\mathrm{P}^2$ of degree two and that there exists a covering map from the torus to the Klein bottle of degree two. This raises the question: between which surfaces do there exist covering maps? More precisely we can ask the following:

(1) Does there exist a covering map from the torus $S^1 \times S^1$ to the sphere S^2?
(2) Does there exist a covering map from the sphere S^2 to the torus $S^1 \times S^1$?

The following lemma summarizes some basic facts about coverings.

Lemma 48.3. Let $p\colon X \to B$ be a covering of topological spaces. Then the following hold:

(1) The map $p\colon X \to B$ is open.[478]
(2) For any $b \in B$ the preimage $p^{-1}(b)$ is a discrete subset of X.
(3) If B is Hausdorff, then X is also Hausdorff.
(4) If p is a finite covering and if B is compact, then X is also compact.
(5) If p is an infinite covering, then X is non-compact.

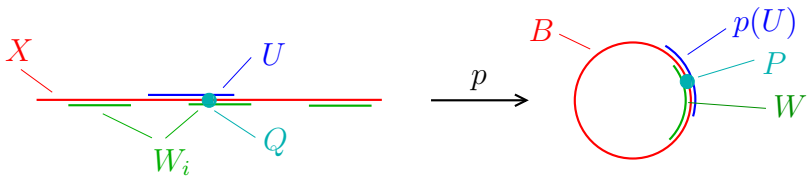
Proof. Let $p\colon X \to B$ be a covering of topological spaces.

(1) Let U be an open subset of X. We need to show that $p(U)$ is open. It suffices to show that for each $P \in p(U)$ there exists an open neighborhood V such that $P \in V \subset U$. Since p is a covering there exists an open neighborhood W of P such that $p^{-1}(W)$ is the union of disjoint open subsets $\{W_i\}_{i\in I}$ with the property that the restriction of p to each subset W_i is a homeomorphism.

Since $P \in p(U)$ there exists a $Q \in U$ with $p(Q) = U$. There exists an $i \in I$ with $Q \in W_i$. Since W_i is open and since U is open the set $U \cap W_i$ is also open in X. Since $p\colon W_i \to W$ is a homeomorphism and since $U \cap W_i$ is open in W_i the set $p(U \cap W_i)$ is also open in $p(W_i) = W$. But since W is open the set $V := p(U \cap W_i)$ is also an open subset of B. It is the desired open neighborhood of P with $P \in V \subset U$.

(2) Let $b \in B$. Since $p\colon X \to B$ is a covering there exists an open neighborhood U of b such that $p^{-1}(U)$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ with the property that the restriction of p to each subset V_i is a homeomorphism. For each $i \in I$ there

[477] It is a good exercise to convince oneself, that the indicated maps are in fact coverings.
[478] Recall that this means that the image of each open set in X is open in B.



exists a unique point $x_i \in V_i$ with $p(x_i) = b$ and the preimage $p^{-1}(b)$ consists of precisely the x_i, $i \in I$. Since each V_i is open in X it follows from the definition of the subspace topology that each $x_i \in p^{-1}(b)$ is an open subset of $p^{-1}(b)$. Put differently, the subspace topology on $p^{-1}(b)$ is the discrete topology. But this is equivalent to saying that $p^{-1}(b)$ is a discrete subset of X.

(3) This statement is relatively easy to prove.We will provide the details in Exercise 48.4.

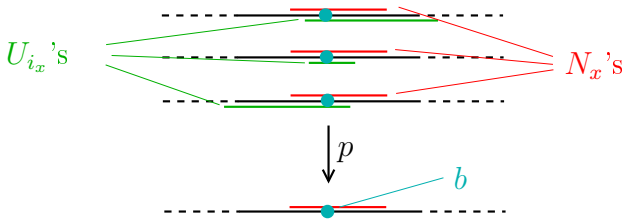
(4) Let $p\colon X \to B$ be an n-fold covering of a compact space B. Let $\{U_i\}_{i\in I}$ be an open cover of X. Let $b \in B$. We pick an open connected neighborhood M of b that is uniformly covered. For each $x \in p^{-1}(b)$ we denote by N_x the component of $p^{-1}(M)$ that contains x. For each point in $x \in p^{-1}(b)$ we pick an $i_x \in I$ with $x \in U_{i_x}$. We write

$$W_b := \bigcap_{x\in p^{-1}(b)} p(U_{i_x} \cap N_x).$$

Claim. The preimage $p^{-1}(W_b)$ is contained in the union of $n = \#p^{-1}(b)$ subsets of our open cover $\{U_i\}_{i\in I}$.

Proof. Since M is uniformly covered the map $p\colon N_x \to M$ is a homeomorphism for any $x \in p^{-1}(y)$. We denote by $p_x\colon N_x \to M$ this homeomorphism. We write $F := p^{-1}(b)$. We then have

$$p^{-1}(W_b) = \bigcup_{y\in F} p_y^{-1}\Big(\bigcap_{x\in F} p_x(U_{i_x} \cap N_x)\Big) = \bigcup_{y\in F}\overbrace{\bigcap_{x\in F} \underbrace{p_y^{-1}(p_x(U_{i_x}))}_{=U_{iy} \text{ if } y=x} \cap \underbrace{p_y^{-1}(p_x(N_x))}_{=N_y}}^{\subset U_{iy}\cap N_y} \subset \bigcup_{y\in F} U_{iy}.$$ ⊞



By (1) each $p(U_{i_x} \cap N_x)$ is an open subset of B and since $p^{-1}(b)$ is a finite set we see that W_b is an open subset of B. Furthermore b is contained in each $p(U_{i_x} \cap N_x)$, so W_b is in fact an open neighborhood of $b \in B$. Since B is compact we can cover B by finitely many of these open sets, i.e. there exist $b_1, \ldots, b_k \in B$ with $B = W_{b_1} \cup \cdots \cup W_{b_k}$. The preimages $p^{-1}(W_{b_1}), \ldots, p^{-1}(W_{b_k})$ cover X. By the above claim each preimage $p^{-1}(W_{b_i})$ is contained in the union of $n = \#p^{-1}(b_i)$ subsets of our open cover $\{U_i\}_{i\in I}$. Summarizing we have shown that we can cover X by $n \cdot k$ sets out of the collection $\{U_i\}_{i\in I}$.

(5) Let $p\colon X \to B$ be an infinite covering. If B is non-compact, then it follows from the Compact Image Lemma 2.13 that X is also non-compact. So now suppose that B is compact. By definition we can cover B by open sets that are uniformly covered.

Since B is compact we can cover B by finitely many open sets $U_1, \ldots, U_k$ that are uniformly covered. Without loss of generality we can assume that U_k is not contained in $U_1 \cup \cdots \cup U_{k-1}$.[479] Now let P be a point in $U_k \setminus (U_1 \cup \cdots \cup U_{k-1})$. Furthermore let V_i, $i \in I$ be the disjoint open subsets of X such that $p^{-1}(U_k)$ is the union of the V_i's and such that the restriction of p to each V_i is a homeomorphism. For $i \in I$ we denote by $Q_i \in V_i$ the unique point with $p(Q_i) = P$. Now we consider the open cover

$$X \;=\; p^{-1}(U_1) \cup \ldots \cup p^{-1}(U_{k-1}) \cup \bigcup_{i \in I} V_i.$$

Note that for $i \in I$ the point Q_i is contained only in V_i. This means that we cannot remove any of the open sets V_i, $i \in I$ from the open cover. Since I is infinite it follows that X does not satisfy the definition of a compact set, i.e. X is not compact. ■

We conclude this section with the following lemma.

Lemma 48.4. Let $p\colon \widetilde{X} \to X$ be a 2-fold covering of a topological space X.

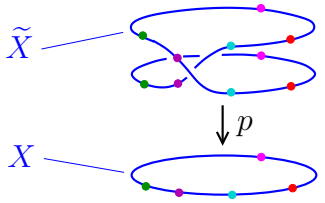
(1) The map

$$\begin{array}{rccl} \Xi\colon & \widetilde{X} & \to & \widetilde{X} \\ & x & \mapsto & \text{the unique element in } p^{-1}(p(\{x\})) \setminus \{x\} \end{array}$$

is continuous.

(2) Let A be a topological space and let $f\colon A \to X$ be a map. We say a map $\widetilde{f}\colon A \to \widetilde{X}$ is a *lift of* f, if $p \circ \widetilde{f} = f$. If $\widetilde{f}\colon A \to \widetilde{X}$ is a lift of f, then $\Xi \circ \widetilde{f}$ is the unique other lift of f.

Proof. We will prove the lemma in Exercise 48.21. ■



$\Xi\colon \widetilde{X} \to \widetilde{X}$ swaps the points of the same color

48.2. **Group actions and covering spaces.** We recall two definitions about actions from pages 248 and 253 and we introduce one new definition.

Definition. Let G be a group which acts continuously on a topological space X.

(1) We say G acts freely if $g \cdot x = x$ for $g \in G$ and $x \in X$ implies that $g = e$.

(2) We say G acts properly if for every two points x and y in X there exist open neighborhoods U of x and V of y such that the set $\{g \in G \,|\, gU \cap V \neq \varnothing\}$ is finite.

(3) We say G acts discretely, if for each $x \in X$ there exists an open neighborhood U so that

$$U \cap g \cdot U \;=\; \varnothing \quad \text{for all } g \neq e.$$

The following lemma summarizes the relationship between these different definitions.

Lemma 48.5. Let G be a group which acts continuously on a topological space X.

(1) If G is finite, then the action is proper.

[479] Indeed, if U_k was contained in $U_1 \cup \cdots \cup U_{k-1}$, then we could remove U_k from the collection of open sets and we continue with the collection $U_1, \ldots, U_{k-1}$.

(2) If G acts discretely, then it also acts freely.
(3) If X is a Hausdorff space and if the action by G is free and proper, then the action is also discrete.

Proof. The first two statements follow immediately from the definitions. The third statement is the content of Lemma 19.35. ■

Examples.

(A) The group $G = \mathbb{Z}^n$ acts on $X = \mathbb{R}^n$ by addition. This action is discrete. Indeed, for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ the open neighborhood

$$U \;=\; \left\{ y = (y_1, \ldots, y_n) \in \mathbb{R}^n \;\middle|\; \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\} < \tfrac{1}{2} \right\}$$

has the desired property.

(B) Let $X = \mathbb{R} \times [-1, 1]$ and $G = \mathbb{Z}$. The action

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n, (x, y)) & \mapsto & (x + n, (-1)^n y) \end{array}$$

is discrete. Indeed, for any $(x, y) \in X$ the open neighborhood

$$U \;=\; \left\{ (a, b) \;\middle|\; |a - x| < \tfrac{1}{2} \text{ and } b \in [-1, 1] \right\}$$

has the desired property.

(C) Let $X = S^n$ and $G = \{\pm 1\}$. The map

$$\begin{array}{rcl} \{\pm 1\} \times S^n & \to & S^n \\ (\epsilon, P) & \mapsto & \epsilon \cdot P \end{array}$$

defines an action that is discrete. Indeed, for any $P \in S^n$ the open hemisphere

$$U \;=\; \{ Q \in S^n \mid \underbrace{P \cdot Q}_{\text{scalar product}} > 0 \}$$

has the desired property.[480]

(D) Every discrete action is free, but as we will see in this example, the converse does not hold. Let $X = \mathbb{R}^{n+1} \setminus \{0\}$ and $G = \mathbb{R} \setminus \{0\}$. The map

$$\begin{array}{rcl} (\mathbb{R} \setminus \{0\}) \times (\mathbb{R}^{n+1} \setminus \{0\}) & \to & \mathbb{R}^{n+1} \setminus \{0\} \\ (r, P) & \mapsto & r \cdot P \end{array}$$

defines an action. It is straightforward to see that the action is free. But the action is not discrete. Indeed, it is easy to see that for each open non-empty set U there exist uncountably many real numbers $r \neq 1$ with $U \cap rU \neq \varnothing$.

(E) As in Exercise 25.11 we consider the following two self-homeomorphisms of $\mathbb{R}^2$:

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x, y) & \mapsto & (x + 1, 1 - y) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x, y) & \mapsto & (x, y + 1). \end{array}$$

[480] Alternatively one can argue, that the action is obviously free and S^n is Hausdorff, hence the action is discrete by Lemma 48.5 (1) and (3).

We denote by G the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by A and B. This means

$$G = \begin{array}{c}\text{all self-homeomorphisms of } \mathbb{R}^2 \text{ that can be written} \\ \text{as a finite concatenation of the maps } A, B, A^{-1} \text{ and } B^{-1}.\end{array}$$

The group G acts by definition on $\mathbb{R}^2$. In Exercise 48.8 we will see that this action of G on $\mathbb{R}^2$ is discrete. Note that in Exercise 25.11 we already saw that the obvious map $[0,1] \times [0,1]/\sim \to \mathbb{R}^2/G$ defines a homeomorphism from the Klein bottle, as defined on page 265, to the quotient space $\mathbb{R}^2/G$.

(F) Let

$$X = S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\}$$

and let $G = \mathbb{Z}_p$ for some $p \in \mathbb{Z} \setminus \{0\}$. Furthermore let $q \in \mathbb{Z}$ with $\gcd(p,q) = 1$. Then the map

$$\begin{array}{rcl} \mathbb{Z}_p \times S^3 & \to & S^3 \\ (k + p\mathbb{Z}, (z_1, z_2)) & \mapsto & (\exp(2\pi \mathrm{i} k/p) \cdot z_1, \exp(2\pi \mathrm{i} kq/p) \cdot z_2) \end{array}$$

defines a smooth, orientation-preserving and discrete action of $\mathbb{Z}_p$ on S^3.[481] We denote by $L(p,q) = S^3/\mathbb{Z}_p$ the quotient space. It follows from the Manifold Quotient-by-Group Action Proposition 19.32 that $L(p,q)$ is again a closed 3-dimensional smooth manifold. Furthermore it follows from the Orientation-Action Proposition 25.19 that $L(p,q)$ is orientable, in fact it has an orientation such that the projection $p\colon S^3 \to L(p,q)$ is orientation-preserving. We will always view $L(p,q)$ as a smooth manifold equipped with this orientation. We refer to the smooth manifolds $L(p,q)$ as *lens spaces*.[482] We consider two special cases:

(a) For $p = 1$ we only have the action of the trivial group on S^3 and the quotient space $L(1,q)$ is therefore just the original sphere S^3.

(b) For $p = 2$ we have q is odd, so for $k = 0$ we have $\exp(2\pi \mathrm{i} k/2) = \exp(2\pi \mathrm{i} kq/2) = 1$ and for $k = 1$ we have $\exp(2\pi \mathrm{i} k/2) = \exp(2\pi \mathrm{i} kq/2) = -1$. Put differently, the action of $\mathbb{Z}_2$ on S^3 is the same as the action of $\mathbb{Z}_2 = \{\pm 1\}$ that we had used in the definition of the real projective space $\mathbb{R}\mathrm{P}^3$. This shows that $L(2,q) = \mathbb{R}\mathrm{P}^3$.

We discuss the last type of examples in slightly more detail. For $p \in \mathbb{N}$ and $q \in \mathbb{N}$ with $\gcd(p,q) = 1$ we just introduced a new topological space, namely the lens space $L(p,q)$. It is natural to ask, for which parameters (p,q) are these topological spaces homeomorphic?

Lemma 48.6. (Lens Spaces-Diffeomorphisms Lemma) Let $p \in \mathbb{N}$ and furthermore let $q, r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$. If $q \equiv \pm r^{\pm 1} \bmod p$,[483] then $L(p,q)$ and $L(p,r)$ are diffeomorphic.[484]

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[481] It is straightforward to verify that the map does indeed define an action. It is clear that the action is smooth and a direct calculation shows that the action is orientation-preserving. One can also easily verify that the action is free. It follows from Lemma 48.5 (1) and (3) that the action is discrete.

[482] The name "lens space" comes from an alternative description of these smooth manifolds.

[483] By our hypothesis we have $\gcd(p,r) = 1$. This implies that there exists a multiplicative inverse of r modulo p, i.e. there exists an s with $r \cdot s \bmod p$. We write $r^{-1} := s$.

[484] In particular any lens space is of the form $L(p,q)$ with $q \in \{0, \ldots, p-1\}$.

Proof. Let $p \in \mathbb{N}$ and let $q, r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$. We suppose that $q \equiv \pm r^{\pm 1} \bmod p$. We define a map

as follows:

$$\begin{array}{lrcl} & L(p,q) & \to & L(p,r) \\ \text{Case 1: } q \equiv r \bmod p. & [(z_1,z_2)] & \mapsto & [(z_1,z_2)] \\ \text{Case 2: } q \equiv -r \bmod p. & [(z_1,z_2)] & \mapsto & [(z_1,\overline{z_2})] \\ \text{Case 3: } q \equiv r^{-1} \bmod p. & [(z_1,z_2)] & \mapsto & [(z_2,z_1)] \\ \text{Case 4: } q \equiv -r^{-1} \bmod p. & [(z_1,z_2)] & \mapsto & [(z_2,\overline{z_1})] \end{array}$$

In all four cases it is straightforward to verify that the given map is well-defined and that it is in fact a diffeomorphism. ■

In the proof of the Lens Spaces-Diffeomorphisms Lemma 48.6 it was quite straightforward to write down the homeomorphism. This raises the question whether there are homeomorphisms between more lens spaces that are harder to find, or whether all other pairs of lens spaces are non-homeomorphic. More precisely, we have the following question.

Question 48.7. Let $p, \widetilde{p} \in \mathbb{N}$ and let $q, \widetilde{q} \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(\widetilde{p},\widetilde{q}) = 1$. If $L(p,q)$ and $L(\widetilde{p},\widetilde{q})$ are homeomorphic, does it follow that $p = \widetilde{p}$ and $q \equiv \pm\widetilde{q}^{\pm 1} \bmod p$?

This question will keep us busy for a very long time. We will eventually provide the answer in the Lens Space-Homeomorphism Classification Theorem **??**.

We had just seen on page 1074 that the group $\{\pm 1\}$ acts freely and continuously on any sphere. We had also seen that any finite cyclic group $\mathbb{Z}_p$ acts freely and continuously on the 3-sphere. This raises the question, whether such an action by $\mathbb{Z}_p$ also exists on the 2-sphere. More generally we can ask the following question.

Question 48.8. Given $n \in \mathbb{N}$, which finite cyclic groups can act freely and continuously on the n-dimensional sphere S^n?[485]

We will give a complete answer to this question once we have introduced homology groups, see Exercise 76.21 and Proposition 87.10.

The following proposition connects the notion of discrete actions to the theory of covering spaces.

Proposition 48.9. (Actions-Covering Proposition) Let X be a topological space together with a discrete and continuous action by a group G. Then the natural projection $p\colon X \to X/G$ is a covering.

Example. This proposition, combined with examples (A) and (C), gives in particular new proofs that the maps $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ and $S^n \to S^n/\!\sim = \mathbb{R}\mathrm{P}^n$ are covering maps.

Proof. Let G be a group which acts discretely and continuously on a topological space X. We want to show that the projection $p\colon X \to X/G$ is a covering. Let $[x] \in X/G$. Since the group acts discretely there exists an open neighborhood V of $x \in X$ such that

$$V \cap gV \;=\; \varnothing \text{ for all } g \neq e.$$

[485] Any cyclic group $\mathbb{Z}_k$, $k \neq 0$ admits a non-trivial action on S^n where $j \in \mathbb{Z}_k$ acts by rotation around the x-axis by the angle $2\pi\frac{j}{k}$. But this action is not free, since the points $(-1,0,\ldots,0)$ and $(1,0,\ldots,0)$ are fixed.

We put $U := p(V)$. It follows from the Group-Action Projection Lemma 5.24 that the set U is an open neighborhood of $[x] = p(x)$. The preimage $p^{-1}(U)$ is the disjoint union of the open subsets[486] $gV, g \in G$. The restriction of the map p to any gV is a continuous, bijective[487] map $gV \to U$ and according to Lemma 5.24 this map is furthermore open. It follows that the restriction of p to any gV is in fact a homeomorphism. Thus we see that U is uniformly covered. Since this holds for any point in X/G we have shown that the projection $p\colon X \to X/G$ is a covering. ■

The following proposition is a slight generalization of the previous proposition.

Proposition 48.10. Let X be a topological space together with a discrete and continuous action by a group G. Furthermore let H be a subgroup of G. Then the natural projection

$$\begin{array}{rcl} p\colon X/H & \to & X/G \\ [x] & \mapsto & [x] \end{array}$$

is a covering.

For H the trivial group we recover the statement of the Actions-Covering Proposition 48.9. In fact the proof of Proposition 48.10 is almost identical to the proof of the Actions-Covering Proposition 48.9. We leave the details to the reader.

Examples.

(1) Let $X = \mathbb{R} \times [-1,1]$ and $G = \mathbb{Z}$. Once again we consider the discrete action

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n,(x,y)) & \mapsto & \big(x + \frac{1}{2}n, (-1)^n y\big). \end{array}$$

Let $H = 2\mathbb{Z}$. According to Proposition 48.10 the projection $p\colon X/H \to X/G$ is a covering. Put differently, all the horizontal maps in the following commutative diagram are coverings:

$$\begin{array}{ccc} (\mathbb{R}\times[-1,1])/2\mathbb{Z} & \longrightarrow & (\mathbb{R}\times[-1,1])/\mathbb{Z} \\ \downarrow \text{id} & & \downarrow \text{id} \\ (\mathbb{R}\times[-1,1])/(x,y) \sim (x+\frac{1}{2}2k,(-1)^{2k}y) & \longrightarrow & (\mathbb{R}\times[-1,1])/(x,y) \sim (x+\frac{1}{2}k,(-1)^k y) \\ \downarrow \text{id} & & \downarrow \text{id} \\ \underbrace{(\mathbb{R}\times[-1,1])/(x,y) \sim (x+k,y)}_{=\text{the annulus}} & \longrightarrow & \underbrace{(\mathbb{R}\times[-1,1])/(x,y) \sim (x+\frac{1}{2}k,(-1)^k y).}_{=\text{Möbius band}} \end{array}$$

Each point on X/G has precisely two preimages, so the covering is in fact a 2-fold covering. Thus we recover the result from page 1070 that the annulus is a 2-fold covering of the Möbius band.

(2) We consider again the group G of self-homeomorphisms of $\mathbb{R}^2$ that is generated by

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1). \end{array}$$

[486] Why is $p^{-1}(U) = \bigcup_{g\in G} gV$ and why are the sets gV disjoint?

[487] Why is the map bijective?

We let H be the subgroup of G generated by A^2 and B. Note that A^2 is the map given by

$$\begin{array}{rcccc} A^2\colon \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1,1-y) & \mapsto & \underbrace{((x+1)+1,(1-(1-y))}_{=(x+2,y)}. \end{array}$$

Note that A^2 and B are two homeomorphisms that commute, so any element in H is in fact of the form

$$\begin{array}{rcl} A^{2k}\circ B^l\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+2k, y+l) \end{array}$$

for some $k,l\in\mathbb{Z}$. By Proposition 48.10 the projection $p\colon \mathbb{R}^2/H \to \mathbb{R}^2/G$ is a covering map. We already know that $\mathbb{R}^2/G$ is the Klein bottle. It follows immediately from the above description of H, $\mathbb{R}^2/H = \mathbb{R}^2/(2\mathbb{Z}\oplus\mathbb{Z})$, so $\mathbb{R}^2/H$ is just the 2-dimensional torus. As on page 1070 we see that we can view the 2-dimensional torus as a 2-fold covering of the Klein bottle.

48.3. Lifting of paths.

Definition. Let $p\colon X\to B$ be a map between topological spaces and let $f\colon [a,b]\to B$ be a path. Let $x\in X$ be a point with $p(x)=f(a)$. A path $g\colon [a,b]\to X$ is called a lift of f to the starting point x, if $g(a)=x$ and if $p\circ g = f$, i.e. if the following diagram of maps commutes:

$$\begin{array}{ccc} & & X \\ & \overset{g}{\nearrow} & \big\downarrow p \\ [a,b] & \xrightarrow[f]{} & B. \end{array}$$

Examples.

(1) We consider the map

$$\begin{array}{rcl} p\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & \exp(\mathrm{i}t) \end{array}$$

and the path

$$\begin{array}{rcl} f\colon [0,b] & \to & S^1 \\ t & \mapsto & \exp(2\mathrm{i}t) \end{array}$$

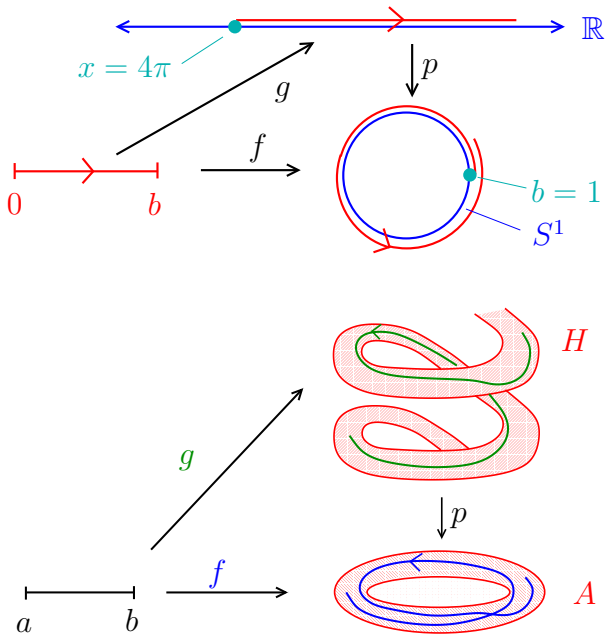
with starting point $f(0)=1$. Let $x=4\pi$. Then $p(x)=p(4\pi)=1$ and

$$\begin{array}{rcl} g\colon [0,b] & \to & \mathbb{R} \\ t & \mapsto & 4\pi+2t \end{array}$$

is a lift of f to the starting point 4π. Note that if $b=\pi$, then f is a loop in $(S^1,1)$, but the lift is not a loop in $(\mathbb{R},4\pi)$. Put differently, the lift of a loop is not necessarily again a loop. This example is illustrated in the figure below.

(2) In the figure below we illustrate the lifting of a path in the covering $p\colon H\to A$ of an infinite helix over an annulus that we had considered on page 1068.

Proposition 48.11. Let $p\colon X\to B$ be a covering and let $f\colon [0,1]\to B$ be a path. Then given any $x\in X$ with $p(x)=f(0)$ there exists a unique lift $g\colon [0,1]\to X$ of f to the starting point x.



Proof. Let $p\colon X \to B$ be a covering, let $f\colon [0,1] \to B$ be a path and let x be a point in X with $p(x) = f(0)$. We first show the existence of a lift and then we prove the uniqueness of the lift.

(a) We consider

$$S \;:=\; \big\{ s \in [0,1] \,\big|\, \text{there exists a lift of } f|_{[0,s]} \text{ to the starting point } x \big\}.$$

Note that S is non-empty since 0 has the desired property. We put $t := \sup(S)$. We want to show that $t = 1$ and that $t \in S$. It suffices to show that there exists an $\epsilon > 0$ such that $[t, t+\epsilon) \cap [0,1]$ still lies in S. We set $b := f(t)$. Since p is a covering there exists an open neighborhood U of $b \in B$, which is uniformly covered. As a reminder, this means that $p^{-1}(U)$ is the union of disjoint open subsets $\{V_i\}_{i \in I}$ with the property that the restriction of p to each subset V_i is a homeomorphism. Since U is open and since f is continuous there exists an $\epsilon > 0$ such that $f((t-\epsilon, t+\epsilon) \cap [0,1]) \subset U$.

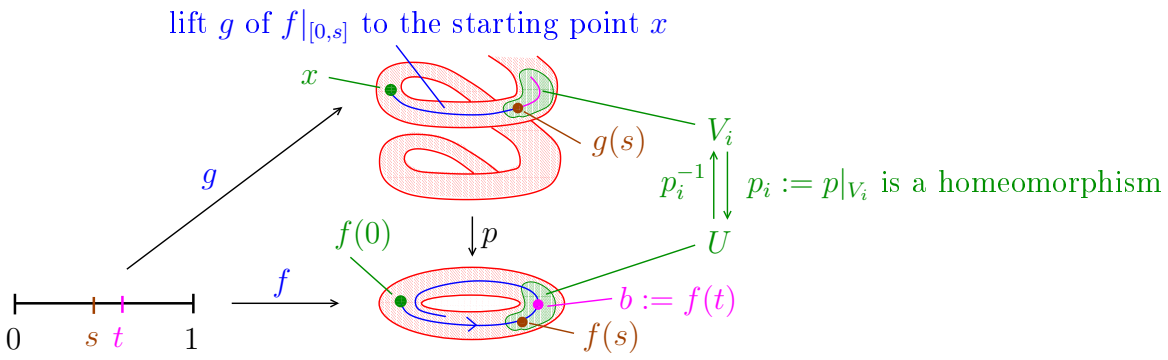
By the definition of t there exists an $s \in (t-\epsilon, t] \cap S$. We denote by g the lift of $f|_{[0,s]}$ the starting point x. There exists a unique $i \in I$ such that $g(s) \in V_i$. We denote by $p_i\colon V_i \to U$ the homeomorphism which is given by the restriction of p to V_i.

We consider the map

$$\begin{array}{rcl} h\colon [0, t+\epsilon) \cap [0,1] & \to & X \\ z & \mapsto & \begin{cases} g(z), & \text{if } z \in [0,s], \\ p_i^{-1}(f(z)), & \text{if } z \in (s, t+\epsilon) \cap [0,1]. \end{cases} \end{array}$$

Since $g(s) = p_i^{-1}(f(s))$ we see that this map is indeed continuous. It is clear that this defines a lift of f to the starting point x.

(b) Now we show the uniqueness of the lift. The proof of that statement is similar to the proof of the existence of a lift. So suppose that $g, h\colon [0,1] \to X$ are two lifts of f to



the starting point x. We want to show that $g = h$. We consider

$$S \;:=\; \{s \in [0,1] \,|\, g(t) = h(t) \text{ for all } t \in [0,s]\}\,.$$

Note that $0 \in S$, i.e. S is non-empty. We put $t := \sup(S)$. We want to show that $t = 1$ and that $t \in S$. Again it suffices to show that there exists an $\epsilon > 0$ such that $[t, t+\epsilon) \cap [0,1]$ still lies in S.

We put $b := f(t)$. Since p is a covering there exists an open neighborhood U of $b \in B$ which is uniformly covered. Since U is open and since f is continuous there exists an $\epsilon > 0$ such that $f((t-\epsilon, t+\epsilon) \cap [0,1]) \subset U$. We pick an $s \in (t-\epsilon, t] \cap [0,1]$. There exists a unique $i \in I$ such that $g(s) \in V_i$. We denote by $p_i \colon V_i \to U$ the homeomorphism which is given by the restriction of p to V_i.

Since $g(s) = h(s)$ it follows that $h(s) \in V_i$. Then it follows that $g(z) \in V_i$ and $h(z) \in V_i$ for all $z \in (t-\epsilon, t+\epsilon) \cap [0,1]$. It follows from the definitions that

$$g(z) \;=\; p_i^{-1}(f(z)) \quad \text{and} \quad h(z) \;=\; p_i^{-1}(f(z))$$

for all $z \in (t-\epsilon, t+\epsilon) \cap [0,1]$. In particular $[t, t+\epsilon) \cap [0,1]$ lies in S. ∎

48.4. **Lifting of path-homotopies.** In the last chapter we saw that given a covering map $p \colon X \to B$ any path $f \colon [0,1] \to B$ can be lifted to a path $g \colon [0,1] \to X$. In this chapter we want to study the question whether the lifts of path-homotopic paths are still path-homotopic.

Before we start this discussion we first generalize the notion of a lift.

Definition. Let $p \colon X \to B$ be a covering and let $f \colon Y \to B$ be a map between topological spaces. A lift of f to X is a map[488] $\widetilde{f} \colon Y \to X$ such that $p \circ \widetilde{f} = f$, i.e. such that the following diagram of maps commutes:

$$\begin{array}{ccc} & & X \\ & \overset{\widetilde{f}}{\nearrow} & \big\downarrow p \\ Y & \xrightarrow[f]{} & B. \end{array}$$

Proposition 48.12. (Path-Homotopy Lifting Proposition) Let $p \colon X \to B$ be a covering, let B be a topological space, let Y be a topological manifold[489] and let $f \colon Y \times$

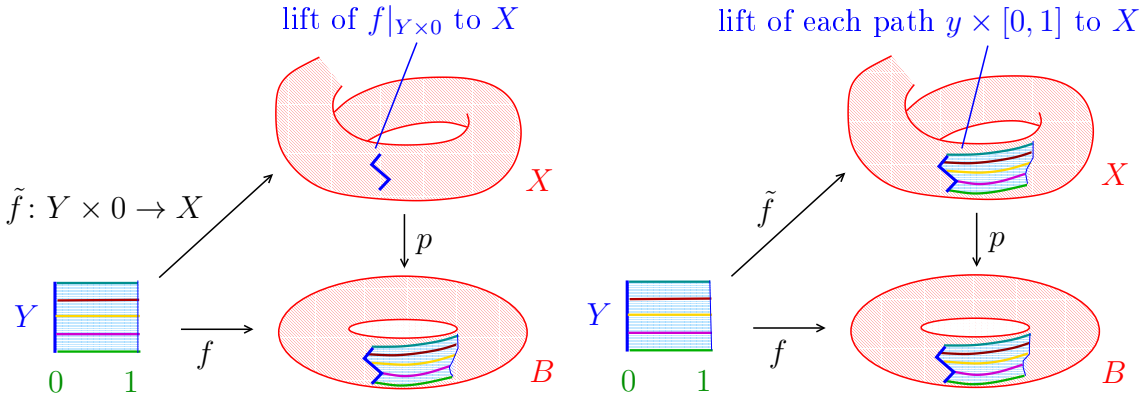
[488] Once again, all maps are understood to be continuous.

$[0,1] \to B$ be a map. Furthermore let $\widetilde{f}\colon Y \times 0 \to X$ be a lift of $f|_{Y\times 0}$. Let

$$\widetilde{f}\colon Y \times [0,1] \ \to \ X$$

be the map that is uniquely determined[490] by the property that for each $y \in Y$ the path $t \mapsto \widetilde{f}(y,t)$ is the lift of the path $t \mapsto f(y,t)$ to the starting point $\widetilde{f}(y,0)$. Then the map $\widetilde{f}\colon Y \times [0,1] \to X$ is continuous.

One way of formulating the Path-Homotopy Lifting Proposition 48.12 is that a "continuous" family of paths in B lifts again to a "continuous" family of paths in X.



Proof. Let $p\colon X \to B$ be a covering, let $f\colon Y \times [0,1] \to B$ be a continuous map and let $\widetilde{f}\colon Y \times 0 \to X$ be a continuous lift of $f|_{Y\times 0}$. (Here and throughout this proof we do not suppose that all maps are continuous.) We begin with the following definitions:

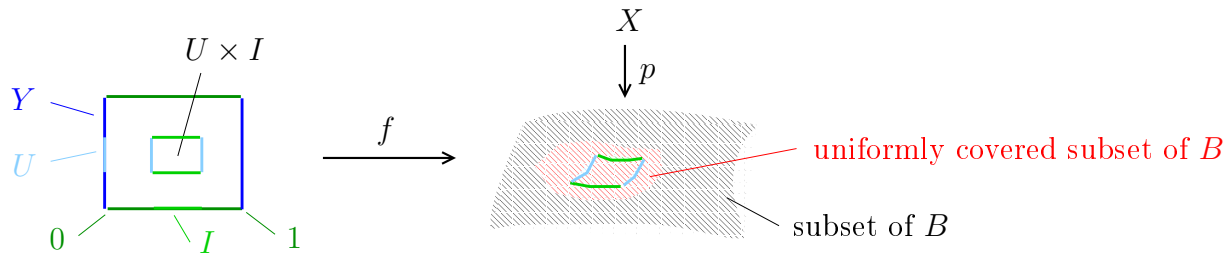
(1) A *good set* $U \times I$ consists of a connected subset $U \subset Y$ and an open interval[491] $I \subset [0,1]$ such that $f(U \times I)$ lies in a uniformly covered subset of B.
(2) We say a good set $U \times I$ is *very good*, if there exists a $t \in I$ such that the restriction of $\widetilde{f}$ to $U \times \{t\}$ is continuous.
(3) Let $y \in Y$ and $t \in [0,1]$. A (*very*) *good neighborhood of* (y,t) is a (very) good set $U \times I$ with $y \in U$ and $t \in I$.

It follows easily[492] from the continuity of f, the hypothesis that Y is a topological manifold and the fact that $p\colon X \to B$ is a covering that each point (y,t) admits a good neighborhood.[493]

Claim. Let $U \subset Y$ open and let $I \subset [0,1]$ be an open interval such that $U \times I$ is very good. Then $\widetilde{f}$ is continuous on $U \times I$.

---

[489] Later on we will apply the proposition to the case that $Y = [0,1]$ and that $Y = \overline{B}^n$ for some $n \in \mathbb{N}_0$.
[490] The existence and the uniqueness of this map is an immediate consequence of Proposition 48.11.
[491] We say $A \subset [0,1]$ is an open interval, if A is an interval and if A is open with respect to the subspace topology of $[0,1]$. Put differently, $A = (a,b)$, or $A = [0,a)$ or $A = (b,1]$ or $A = [0,1]$ for $0 \le a \le b \le 1$.
[492] Why does this follow?
[493] Here we use the fact, established in the Path-Connected Implies Connected Corollary 2.23 and the Topological Manifolds-Local Properties Lemma 18.8 (2), that the topological manifold Y is locally connected.



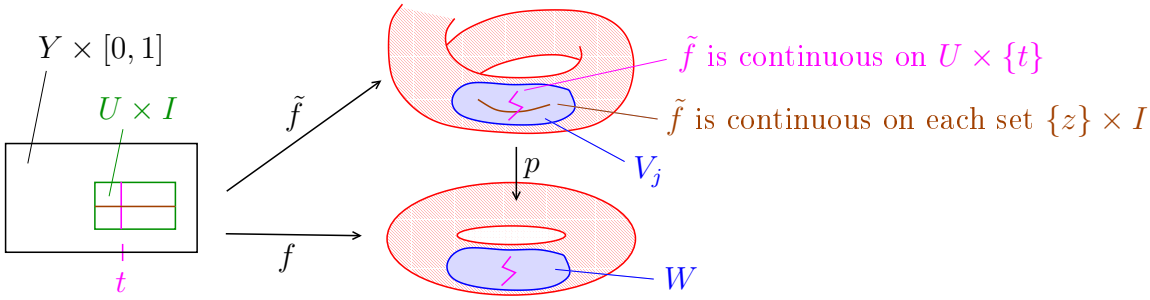
Proof. Let $U \subset Y$ be an open connected subset and let $I \subset [0,1]$ be an interval such that $U \times I$ is very good. This means in particular that there exists a $t \in I$ such that

$$\begin{array}{ccc} U & \to & X \\ y & \mapsto & \widetilde{f}(y,t) \end{array}$$

is continuous.

According to our hypothesis there exists a uniformly covered subset $W \subset B$ such that $f(U \times I) \subset W$. Since $U \times I$ is connected we can without loss of generality suppose that W is connected. Since W is uniformly covered the preimage $p^{-1}(W)$ is the union of disjoint open subsets $\{V_j\}_{j \in J}$, with the property, that the restriction of p to each subset V_j is a homeomorphism.

Now we want to show that there exists a $j \in J$ with $\widetilde{f}(U \times I) \subset V_j$. We choose a $y \in U$. Then there exists a unique $j \in J$ such that $\widetilde{f}(y,t) \in V_j$. Since $\widetilde{f}$ is continuous $U \times \{t\}$ and since U is connected it follows from Lemma 2.35[494] that $\widetilde{f}(U \times \{t\}) \subset V_j$. Now let $z \in U$. We just showed that $\widetilde{f}(z,t) \in V_j$. By the construction of $\widetilde{f}$ the map $\widetilde{f}$ is continuous on $\{z\} \times I$. Since I is connected it follows again from Lemma 2.35 that $\widetilde{f}(\{z\} \times I) \subset V_j$. We have thus shown that $\widetilde{f}(U \times I) \subset V_i$. This part of the argument is sketched in the figure below.



We denote by $p_j \colon V_j \to W$ the homeomorphism which is given by the restriction of p to V_j. Since $\widetilde{f}(U \times I) \subset V_j$ it follows that on $U \times I$ the map $\widetilde{f}$ is given by $\widetilde{f} = p_j^{-1} \circ f$. In particular the map $\widetilde{f}$ is a composition of continuous maps, so it is continuous on $U \times I$. ⊞

Claim. Let $y \in Y$. We consider

$$T \;:=\; \{t \in [0,1] \,|\, \text{there exists a very good neighborhood of } (y,t)\}.$$

[494] We apply Lemma 2.35 to the continuous map $U \times \{t\} \to p^{-1}(W)$ from the connected set U to the set $p^{-1}(W)$ with components V_j, $j \in J$.

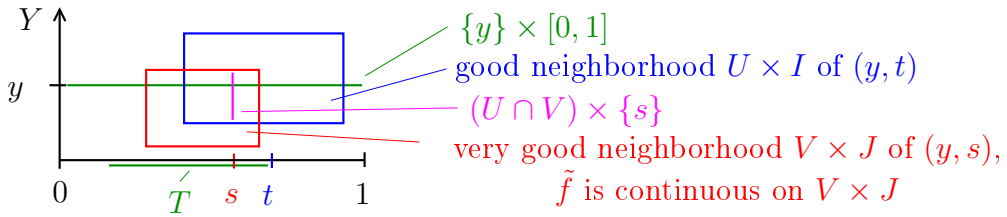
We claim that $T = [0,1]$.

Proof. Thus let $y \in Y$. Since $[0,1]$ is connected it suffices to show the following three statements:

(1) $T \neq \varnothing$,
(2) T is open, and
(3) T is closed.

Now we will prove these three statements:

(1) As we pointed out above, every point in $Y \times I$, in particular the point $(y,0)$, admits a good neighborhood. By our hypothesis $\widetilde{f}$ is continuous on $Y \times \{0\}$. It follows that the good neighborhood is in fact very good.
(2) It is easy to see that T is open. Indeed, suppose that we are given a very good neighborhood $U \times I$ of (y,t). Then $U \times I$ is also a very good neighborhood for any $(y,s) \in U \times I$. Therefore the open neighborhood I of t also lies in T.
(3) It remains to show that T is closed. Put differently, we need to show that T equals its closure $\overline{T}$. Thus let $t \in \overline{T}$. As we already pointed out, there exists a good neighborhood $U \times I$ of (y,t). Since $t \in \overline{T}$ there exists an $s \in T \cap I$. Since $s \in T$ it follows from the above claim that there exists a very good open neighborhood $V \times J$ of (y,s). By the previous claim the restriction of $\widetilde{f}$ to $V \times J$ is continuous, in particular it is continuous on $(U \cap V) \times \{s\}$. We denote by W the component of $U \cap V$ that contains y. Then $W \times I$ is a very good neighborhood of (y,t). This step is illustrated in the figure below. ⊞

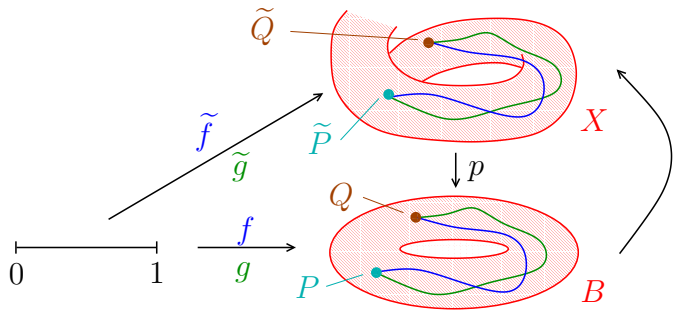


We have just shown that given any $y \in Y$ and any $t \in [0,1]$ there exists a very good neighborhood $U_{y,t}$ of (y,t). By the first claim the map $\widetilde{f}$ is continuous on each $U_{y,t}$. But the open subsets $U_{y,t}$ cover all of $Y \times [0,1]$. It follows from the Pasting Proposition 2.6 that $\widetilde{f}$ is continuous on $Y \times [0,1]$. ■

Corollary 48.13. Let $p\colon X \to B$ be a covering and let $f, g\colon [0,1] \to B$ be two paths. Let $\widetilde{f}, \widetilde{g}\colon [0,1] \to X$ be two lifts with the same starting point. If f and g are path-homotopic, then the endpoints of $\widetilde{f}$ and $\widetilde{g}$ agree and the paths $\widetilde{f}$ and $\widetilde{g}$ are path-homotopic.

Proof. We write $P := f(0) = g(0)$ and $Q := f(1) = g(1)$. Furthermore we denote by $\widetilde{P} := \widetilde{f}(0) = \widetilde{g}(0)$ the common starting point of the lifts and we write $\widetilde{Q} := \widetilde{f}(1)$. Let

$$\begin{array}{rccc} H\colon [0,1] \times [0,1] & \to & B \\ (t,s) & \mapsto & H(t,s) \end{array}$$



path-homotopic paths in B
lift to path-homotopic paths in X

be a path-homotopy between the paths f and g. As a reminder, H is a map such that

$$\begin{array}{lllllllll} & H(t,0) & = & f(t) & \text{and} & H(t,1) & = & g(t) & \text{for all } t \in [0,1], \\ \text{and such that} & H(0,s) & = & P & \text{and} & H(1,s) & = & Q & \text{for all } s \in [0,1]. \end{array}$$

On $\{0\} \times [0,1]$ we consider the lift $\widetilde{H}$ of H which is given by $\widetilde{H}(0,s) = \widetilde{P}$. Furthermore we denote by $\widetilde{H}\colon [0,1] \times [0,1] \to X$ the map which is defined as follows: for each $s \in [0,1]$ the path $t \mapsto \widetilde{H}(t,s)$ is the unique lift of the path $t \mapsto H(t,s)$ to the starting point $\widetilde{P}$ lift. According to the Path-Homotopy Lifting Proposition 48.12 the map $\widetilde{H}$ is in fact continuous. By definition of $\widetilde{H}$ we have

$$\widetilde{H}(t,0) = \widetilde{f}(t) \quad \text{and} \quad \widetilde{H}(t,1) = \widetilde{g}(t) \text{ for all } t \in [0,1],$$

For each $s \in [0,1]$ we have $p(\widetilde{H}(1,s)) = Q$. Since $\widetilde{H}$ is continuous we obtain a map

$$\begin{array}{ccc} [0,1] & \to & p^{-1}(Q) \subset X \\ s & \mapsto & \widetilde{H}(1,s). \end{array}$$

Since $[0,1]$ is connected it follows from the Connected-to-Discrete Lemma 2.27 and Lemma 48.3 that $\widetilde{H}(1,s) = \widetilde{Q}$ for all $s \in [0,1]$. In particular we obtain that

$$\widetilde{f}(1) = \widetilde{H}(1,0) = \widetilde{Q} = \widetilde{H}(1,1) = \widetilde{g}(1).$$

Thus it follows that $\widetilde{H}$ is indeed a path-homotopy between the paths $\widetilde{f}$ and $\widetilde{g}$. ■

Definition. Let $p\colon X \to B$ be a map between topological spaces and let $x_0 \in X$ and $b_0 \in B$. We say a map $p\colon X \to B$ is a covering of pointed topological spaces, if $p\colon X \to B$ is a covering and if $p(x_0) = b_0$.

Corollary 48.14. For any covering $p\colon (X,x_0) \to (B,b_0)$ of pointed topological spaces the induced map $p_*\colon \pi_1(X,x_0) \to \pi_1(B,b_0)$ is a monomorphism.

Proof. We have[495]

$$\begin{array}{l} [f] \in \ker\left(\pi_1(X,x_0) \xrightarrow{p_*} \pi_1(B,b_0)\right) \implies [p \circ f] = e \in \pi_1(B,b_0) \implies p \circ f \simeq e_{b_0} \\ \qquad\qquad \implies f \simeq e_{x_0} \implies [f] = e \in \pi_1(X,x_0). \end{array}$$

by Corollary 48.13, since f is the lift of $p \circ f$ and e_{x_0} is the lift of e_{b_0} to the starting point x_0

■

[495] We use the notation from page 1053: for $y \in Y$ we denote by e_y the constant path $e_y(t) := x$, $t \in [0,1]$.

Example. In Question 48.2 (1) we had asked whether there exists a covering map from the torus $S^1 \times S^1$ to the sphere S^2. By the Sphere-π_1-Proposition 47.13 we know that $\pi_1(S^2) = 0$ whereas we saw on page 1062 that $\pi_1(S^1 \times S^1)$ is non-trivial. Using Corollary 48.14 we can now answer Question 48.2 (1) in the negative: there is no covering map from $S^1 \times S^1$ to S^2.

We conclude the section with the following lemma.

Lemma 48.15. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces.

(1) Let f be a loop in (B, b_0). We denote by $\widetilde{f}$ the lift of f to the starting point x_0. Then the following holds:

$$\widetilde{f} \text{ is a loop in } (X, x_0) \iff [f] \text{ lies in } p_*(\pi_1(X, x_0)) \subset \pi_1(B, b_0).$$

In particular, if $\widetilde{f}$ is not a loop in (X, x_0), then $[f]$ is a non-trivial element of $\pi_1(B, b_0)$.

(2) Now we suppose that X is simply connected. Let f and g loops in (B, b_0) and let $\widetilde{f}$ and $\widetilde{g}$ be the lifts of f and g to the starting point x_0. Then the following holds:

$$f \text{ and } g \text{ are path-homotopic loops} \iff \text{the endpoints of } \widetilde{f} \text{ and } \widetilde{g} \text{ agree.}$$

(3) If X is path-connected, then $[X : B] = [\pi_1(B, b_0) : p_*(\pi_1(X, x_0))]$.

Remark. In Exercise 48.11 we will use Lemma 48.15 (3) to answer Question 48.2 (2) in the negative.

Proof.

(1) We prove the "$\Rightarrow$"-direction. So let f be a loop in (B, b_0). We denote by $\widetilde{f}$ the lift of f to the starting point x_0. If $\widetilde{f}$ is a loop in (X, x_0), then it defines an element in $\pi_1(X, x_0)$ and we have $[f] = [p \circ \widetilde{f}] = p_*([\widetilde{f}]) \in p_*(\pi_1(X, x_0))$. The proof of the "$\Leftarrow$"-direction is Exercise 48.10.[496]

(2) The "$\Rightarrow$"-direction is an immediate consequence of Corollary 48.13. The proof of the "$\Leftarrow$"-direction is done once again in Exercise 48.10. For that direction we need that X is simply connected.

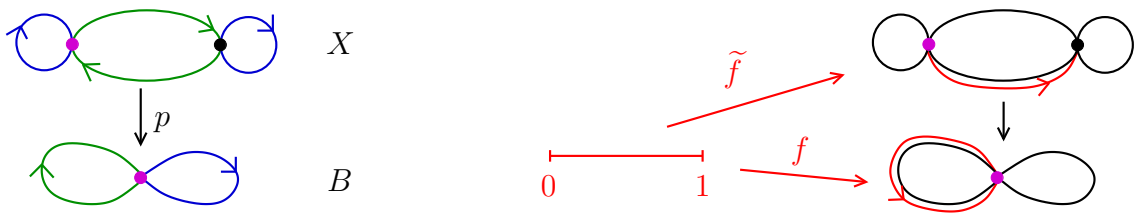
(3) This statement is proved in Exercise 48.10. ∎

Example. The last sentence of Lemma 48.15 (1) can at times be surprisingly useful for showing that a given element in a fundamental group. As proof of concept, we consider the pointed topological space (B, b_0) and we show a loop $f\colon [0, 1] \to B$ shown in the figure below. We also show a covering $p\colon (X, x_0) \to (B, b_0)$ of pointed topological spaces together with the lift $\widetilde{f}$ of f to the starting point x_0. Since $\widetilde{f}$ is not a loop we see that $[f] \in \pi_1(B, b_0)$ is non-trivial. In Exercise 48.12 we will use the same approach to show that $\pi_1(B, b_0)$ is actually non-abelian.

48.5. Group actions and fundamental groups.

The following proposition finally allows us to determine many fundamental groups. This proposition is the reward for the many rather theoretical results of the previous sections.

[496] Note that $[f]$ lies in $p_*(\pi_1(X, x_0)) \subset \pi_1(B, b_0)$ means that there exists a loop g in (X, x_0) such that $p \circ g$ is *path-homotopic* to f. A priori it does not mean that there exists a loop g with $p \circ g = f$.



Theorem 48.16. (Fundamental Group-via-Actions Theorem) Let X be a topological space and let G be a group which acts continuously and discretely on X. We choose an $x \in X$. We denote by $p\colon X \to X/G$ the natural projection .

(1) If X is simply connected, then the map

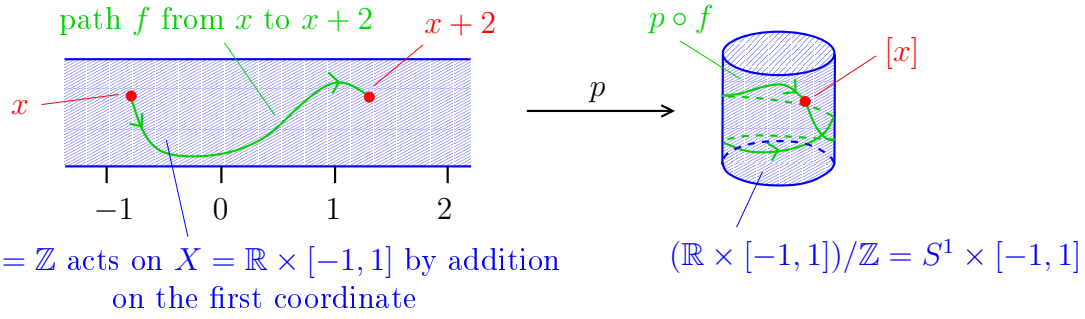
$$\begin{array}{rcl} G & \to & \pi_1(X/G,[x]) \\ g & \mapsto & [p \circ (\text{path in } X \text{ from } x \text{ to } g \cdot x)] \end{array}$$

is a well-defined isomorphism of groups. In particular we have $\pi_1(X/G) \cong G$.

(2) Regardless of whether or not X is simply connected, the subgroup $p_*(\pi_1(X,x))$ is normal in $\pi_1(X/G,[x])$ and the map

$$\begin{array}{rcl} G & \to & \pi_1(X/G,[x])/p_*(\pi_1(X,x)) \\ g & \mapsto & [p \circ (\text{path in } X \text{ from } x \text{ to } g \cdot x)] \end{array}$$

is a well-defined isomorphism of groups.



Before we provide the proof of the Fundamental Group-via-Actions Theorem 48.16, let us impatiently use it to finally confirm our suspicion that the fundamental group of S^1 is isomorphic to $\mathbb{Z}$.

Proposition 48.17.

(1) The maps

$$\Phi\colon \begin{array}{rcl} \mathbb{Z} & \to & \pi_1(\mathbb{R}/\mathbb{Z},[0]) \\ n & \mapsto & \underbrace{\left[\begin{array}{rcl} [0,1] & \to & \mathbb{R}/\mathbb{Z} \\ t & \mapsto & p(nt) = [nt] \end{array}\right]}_{\substack{\text{the map } t \mapsto nt \text{ is a} \\ \text{path from } 0 \text{ to } 0+n}} \end{array}$$

and

$$\Psi\colon \begin{array}{rcl} \mathbb{Z} & \to & \pi_1(S^1,1) \\ n & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & S^1 \\ t & \mapsto & \exp(2\pi i n t) \end{array}\right] \end{array}$$

are isomorphisms of groups.

(2) Let $n \in \mathbb{N}$. We consider the map $f\colon S^1 \to S^1$ that is given by $z \mapsto z^n$. The induced map on $\pi_1(S^1, 1)$ is given by multiplication by n.
(3) The map
$$\begin{array}{rccc} \Theta\colon \pi_1(S^1,1) & \to & \mathbb{Z} \\ [\gamma] & \mapsto & \frac{1}{2\pi \mathrm{i}} \int\limits_\gamma \frac{1}{z}\, dz \end{array}$$
is an isomorphism.

Proof.

(1) We consider the action of $G = \mathbb{Z}$ on the simply connected space $X = \mathbb{R}$. We denote by $p\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ the natural projection . It follows immediately from the Fundamental Group-via-Actions Theorem 48.16 that the given map $\Phi\colon \mathbb{Z} \to \pi_1(\mathbb{R}/\mathbb{Z}, [0])$ is an isomorphism. We denote by $\phi\colon \mathbb{R}/\mathbb{Z} \to S^1$ the homeomorphism given by $\phi([t]) = \exp(2\pi \mathrm{i} t)$. If we compose the above isomorphism with the isomorphism $\phi_*\colon \pi_1(\mathbb{R}/\mathbb{Z}, [0]) \to \pi_1(S^1, 1)$, then we see that the second map $\Psi\colon \mathbb{Z} \to \pi_1(S^1, 1)$ is also an isomorphism.
(2) This statement follows easily from the definitions. To practice the definitions we will prove this statement in Exercise 48.16.
(3) It is straightforward to see that $\Theta \circ \Psi\colon \mathbb{Z} \to \pi_1(S^1, 1) \to \mathbb{Z}$ is the identity. By (1) the first map Ψ is an isomorphism. Thus $\Theta = \Phi^{-1}$ is also an isomorphism. ■

We will provide many more examples for the Fundamental Group-via-Actions Theorem 48.16 immediately after the proof of the Fundamental Group-via-Actions Theorem 48.16.

Proof of the Fundamental Group-via-Actions Theorem 48.16. To simplify the discussion we only provide the proof of the Fundamental Group-via-Actions Theorem 48.16 (1). The proof of the Fundamental Group-via-Actions Theorem 48.16 (2) is almost the same. We leave it to the reader to make the necessary modifications.

Let X be a simply connected space and let G be a group which acts continuously and discretely on X.[497] We choose an $x \in X$. We denote by $p\colon X \to X/G$ the natural projection . We consider the map
$$\begin{array}{rccc} \Phi\colon G & \to & \pi_1(X/G, [x]) \\ g & \mapsto & [p \circ (\text{path in } X \text{ from } x \text{ to } gx)]. \end{array}$$

We need to prove the following four statements:

(1) Φ is well-defined,
(2) Φ is a group homomorphism,
(3) Φ is surjective,
(4) Φ is injective.

Now we provide the proofs for these four claims.

(1) We first show that the map Φ is well-defined. Let $g \in G$. Since X is in particular path-connected there exists a path from x to gx. Now let f be an arbitrary path in X from x to gx. From $p(x) = [x] = [gx] = p(gx)$ it follows that $p \circ f$ is a loop in $(X/G, [x])$. In particular $p \circ f$ does indeed define an element in $\pi_1(X/G, [x])$.

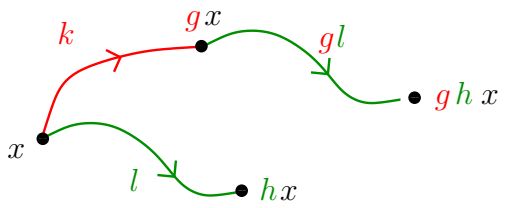
Now we need to show that this element does not depend on the choice of f. So let f' be another path in X from x to gx. Since X is simply connected it follows from Lemma 47.12 that the paths f and f' are path-homotopic in X, i.e. there exists

[497] It is perhaps helpful to read the proof with the example of the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ in mind.

a path-homotopy $H\colon [0,1]\times[0,1]\to X$ from f to f'. But then $p\circ H\colon [0,1]\times[0,1]\to X/G$ is a path-homotopy from the loop $p\circ f$ to the loop $p\circ f'$. This shows that the map Φ is well-defined.

(2) Now we show that Φ is a group homomorphism. Let $g,h\in G$ and pick be a path k from x to gx and pick a path l from x to hx. Then gl[498] is a path from gx to ghx and $k*gl$ is a path from x to ghx. It follows that

$$\Phi(gh) \underset{\substack{\uparrow\\ \text{since } k*gl \text{ is a path from } x \text{ to } ghx}}{=} [p\circ(k*gl)] = [(p\circ k)*(p\circ gl)] \underset{\substack{\downarrow\\ \text{since } p(gx)=p(x)\in X/G \text{ for all } x\in X}}{=} [(p\circ k)*(p\circ l)]$$
$$= [p\circ k]\cdot[p\circ l] = \Phi(g)\cdot\Phi(h).$$
$$\text{definition of } \cdot \text{ in } \pi_1(X/G,[x])\uparrow$$



Before we continue let us recall that according to the Actions-Covering Proposition 48.9 the projection $p\colon X\to X/G$ is a covering.

(3) Now we prove that Φ is surjective. We choose an arbitrary element $c\in\pi_1(X/G,[x])$ and represent it by a loop f in $(X/G,[x])$. According to Proposition 48.11 there exists a lift $\widetilde{f}\colon [0,1]\to X$ of the path f to the starting point x. By definition we have $p\circ\widetilde{f}=f$. In particular we obtain for the endpoint of $\widetilde{f}$ that $p(\widetilde{f}(1))=[x]$, therefore it follows that $\widetilde{f}(1)=gx$ for some $g\in G$. Thus we have proved that

$$\Phi(g) = \big[p\circ(\text{path from } x \text{ to } gx)\big] = [p\circ\widetilde{f}] = [f] = c.$$

(4) We conclude the proof of the proposition by showing that Φ is injective. Let $g\in G$. Then the following holds:

$$\Phi(g)=e \implies [p\circ(\text{path } f \text{ from } x \text{ to } gx)]=e\in\pi_1(X/G,[x])$$
$$\implies p\circ f\simeq e_{[x]} \text{ in } X/G$$
$$\underset{\substack{\uparrow\\ \text{by Corollary 48.13, since } f \text{ is a lift of } p\circ f \text{ and } e_x\\ \text{is a lift of } e_{[x]} \text{ to the starting point } x}}{\implies} f(1)=e_x(1) \implies gx=x \underset{\substack{\uparrow\\ \text{since the action is discrete,}\\ \text{hence free}}}{\implies} g=e.$$

This shows that Φ is indeed injective.

We have thus shown that Φ is a well-defined isomorphism. ■

Now we can determine the fundamental groups of many of the topological spaces that we have encountered so far.

[498] Here gl denotes the path $[0,1]\to X$ that is given by $t\mapsto g\cdot l(t)$.

Corollary 48.18. We have the following isomorphisms of fundamental groups

$$\begin{array}{lrcll}
\text{(A)} & \pi_1(\text{n-dimensional torus}) & = & \pi_1(\mathbb{R}^n/\mathbb{Z}^n) & \cong \mathbb{Z}^n \\
& \text{in particular} & & & \\
& \pi_1(S^1) & = & \pi_1(\mathbb{R}/\mathbb{Z}) & \cong \mathbb{Z}, \\
\text{(B)} & \text{for any } n \neq 0,1 & & & \\
& \pi_1(\text{real projective space } \mathbb{R}\mathrm{P}^n) & = & \pi_1(S^n/\{\pm 1\}) & \cong \mathbb{Z}_2, \\
\text{(C)} & \pi_1(\text{Möbius band }) & = & \pi_1((\mathbb{R}\times[-1,1])/\mathbb{Z}) & \cong \mathbb{Z}, \\
\text{(D)} & \pi_1(\text{lens space } L(p,q)) & = & \pi_1(S^3/\mathbb{Z}_p) & \cong \mathbb{Z}_p.
\end{array}$$

Proof. By the Sphere-π_1-Proposition 47.13 we know that for any $n \geq 2$ the sphere S^n is simply connected. Furthermore we know by the discussion on page 1055 that $\mathbb{R}^n$, $n \geq 1$, and that the strip $\mathbb{R}\times[-1,1]$ are simply connected. We saw on page 1074 that all the actions are discrete. The actions are of course also continuous. The corollary is now an immediate consequence of the Fundamental Group-via-Actions Theorem 48.16. ■

Thus we have now finally determined the fundamental groups of many spaces. In the Sphere-π_1-Proposition 47.13 we saw that $\pi_1(S^2) = 0$ whereas we have now seen that $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$. The fundamental groups are not isomorphic, hence the sphere S^2 and the torus $S^1 \times S^1$ are not homeomorphic.

We also obtained a partial answer to Question 48.7: if two lens spaces $L(p,q)$ and $L(\widetilde{p},\widetilde{q})$ are homeomorphic, then $p = \widetilde{p}$. But as of right now we cannot say anything about the parameters q and $\widetilde{q}$.

Examples. the Fundamental Group-via-Actions Theorem 48.16 not only gives us the fact that $\pi_1(X/G)$ is isomorphic to the group G, but it also gives us an explicit isomorphism. Now we will study this isomorphism in two examples to explicitly determine non-trivial elements in the fundamental group of the torus and the real projective space.

(1) As a reminder, the torus is defined as[499]

$$T \;=\; \mathbb{R}^2/\mathbb{Z}^2 \;=\; [0,1]\times[0,1]/(x,0)\sim(x,1) \text{ and } (0,y)\sim(1,y).$$

The explicit isomorphism of the Fundamental Group-via-Actions Theorem 48.16

$$\Phi\colon \mathbb{Z}^2 \;\to\; \pi_1(T,(0,0))$$

assumes in particular the following values

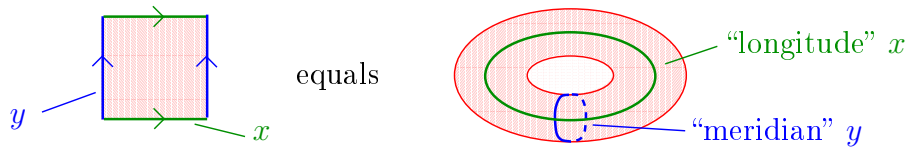
$$\Phi((1,0)) \;=\; x \;:=\; \underbrace{\left[\begin{array}{ccc} [0,1] & \to & ([0,1]\times[0,1])/\!\sim \\ t & \mapsto & [(t,0)] \end{array}\right]}_{t\mapsto(t,0) \text{ is a path from } (0,0) \text{ to } (1,0)}$$

and

$$\Phi((0,1)) \;=\; y \;:=\; \underbrace{\left[\begin{array}{ccc} [0,1] & \to & ([0,1]\times[0,1])/\!\sim \\ t & \mapsto & [(0,t)] \end{array}\right]}_{t\mapsto(0,t) \text{ is a path from } (0,0) \text{ to } (0,1)}.$$

In the description of the torus as a subset of $\mathbb{R}^3$, see the figure below, these two loops correspond to the "longitude" and the "meridian".

[499] Here we are slightly generous in our usage of the equality sign. More precisely, strictly speaking $\mathbb{R}^2/\mathbb{Z}^2$ is not the same as $([0,1]\times[0,1])/\!\sim$. But there exists a canonical homeomorphism between these two spaces that we use to identify these spaces.



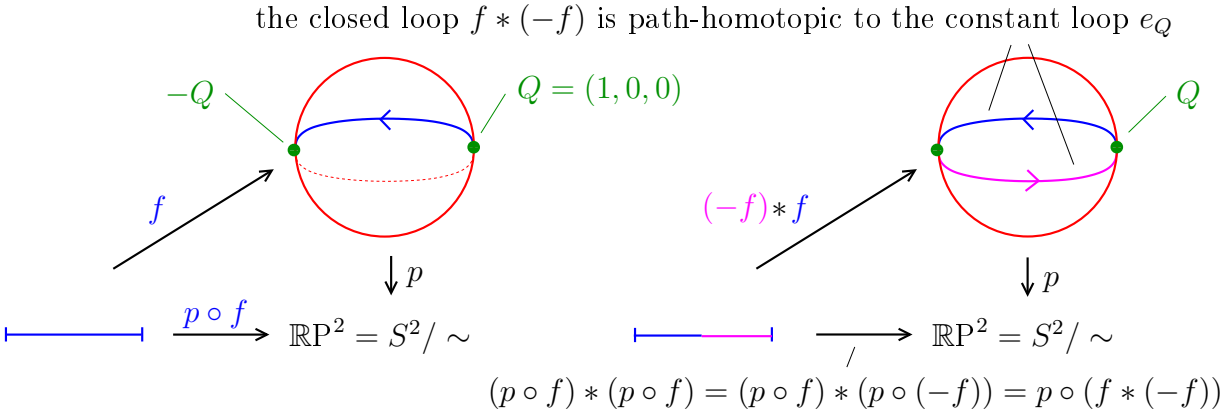
(2) Now we consider the real projective space $\mathbb{R}\mathrm{P}^2 = S^2/\{\pm 1\} = S^2/x \sim -x$ and as usual we denote by $p\colon S^2 \to \mathbb{R}\mathrm{P}^2$ the projection. We put $Q := (1,0,0)$. We consider the following path in S^2:

$$\begin{array}{rcl} f\colon [0,1] & \to & S^2 \\ t & \mapsto & (\cos(\pi t), \sin(\pi t), 0). \end{array}$$

This is a path from Q to $-Q = (-1)\cdot Q$. Since -1 is the one non-trivial element of the group $G = \{\pm 1\}$ it follows from the explicit isomorphism of the Fundamental Group-via-Actions Theorem 48.16 that the equivalence class of $p \circ f$ is precisely the one non-trivial element of the fundamental group $\pi_1(\mathbb{R}\mathrm{P}^2, p(Q)) \cong \{\pm 1\}$.[500] Since $\pi_1(\mathbb{R}\mathrm{P}^2, p(Q))$ is isomorphic to $\mathbb{Z}_2$ we see that $p \circ f$ represents an element of order 2, i.e. $(p \circ f) * (p \circ f)$ represents the trivial element in the fundamental group. Put differently, $(p \circ f) * (p \circ f)$ is null-homotopic. This one can see quite explicitly. More precisely, we have

$$\begin{aligned} [p \circ f] \cdot [p \circ f] &= [(p \circ f) * (p \circ f)] = [(p \circ (-f)) * (p \circ f)] = [p \circ (f * (-f))] \\ &= \left[p \circ \underbrace{\left(\begin{array}{rcl} f\colon [0,1] & \to & S^2 \\ t & \mapsto & (\cos(2\pi t), \sin(2\pi t), 0) \end{array} \right)}_{\text{loop } g \text{ in } S^2} \right]. \end{aligned}$$

The loop g in (S^2, Q) is clearly null-homotopic.[501] But then the loop $p \circ g$ is also null-homotopic. This example is illustrated in the figure below.



[500] In particular we have now shown that the loop in $\mathbb{R}\mathrm{P}^2$ given on page 1049 is not null-homotopic.

[501] More precisely, it is straightforward to write down an explicit path-homotopy from g to the constant path e_Q. Alternatively one can also deduce that g is null-homotopic from our calculation $\pi_1(S^2, Q) = 0$.

Example. We consider the Klein bottle. Recall that on page 1074 we introduced the group G that is given by all self-homeomorphisms of $\mathbb{R}^2$ that can be written as concatenations of the two self-homeomorphisms

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1) \end{array}$$

and their inverses. In Exercise 48.8 we will see G acts discretely on $\mathbb{R}^2$. Also recall we saw in Exercise 25.11 that $\mathbb{R}^2/G$ is homeomorphic to the Klein bottle. It follows from the Fundamental Group-via-Actions Theorem 48.16 that

$$\pi_1(\text{Klein bottle}) \;\cong\; G = \text{ group that is generated by the homeomorphisms } A \text{ and } B.$$

The maps A and B do not commute. Indeed, the map $A \circ B\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by

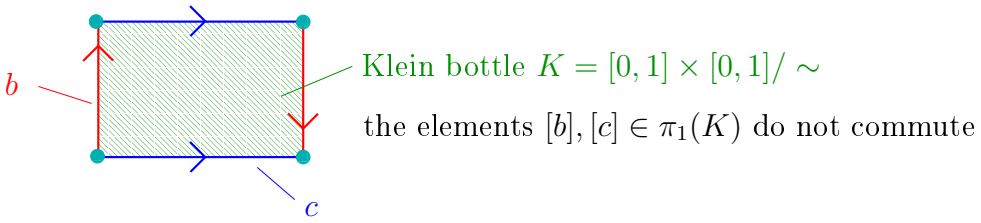
$$(x,y) \;\mapsto\; (x+1, (1-(y+1))) \;=\; (x+1, -y)$$

whereas the map $B \circ A\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$(x,y) \;\mapsto\; (x+1, (1-y)+1) \;=\; (x+1, 2-y).$$

Thus we see that the fundamental group of the Klein bottle is a non-abelian group.

In the figure below we show the two elements b and c of the fundamental group of the Klein bottle $K = ([0,1] \times [0,1])/\sim$ which correspond to B and $C = B^{-1} \circ A$ under the isomorphism of the Fundamental Group-via-Actions Theorem 48.16 and the homeomorphism from Exercise 25.11. Since the two homeomorphisms $B(x,y) = (x, y+1)$ and $C(x,y) = (x+1, -y)$ do not commute, the corresponding elements $[b]$ and $[c]$ of $\pi_1(K)$ do not commute either.



The previous example showed that fundamental groups can be non-abelian groups. In fact, as we will see later in Chapter 52, many, actually arguably "most" fundamental groups are non-abelian. This will pose several major problems. For example, what is a good way to describe a non-abelian group? How can we decide whether or not two non-abelian groups are isomorphic?

We conclude this section with a more unusual topological space. Namely we remark in Exercise 48.15 we will use the Fundamental Group-via-Actions Theorem 48.16 to show that the fundamental group of the pseudocircle, i.e. of the topological space $X = \{A, B, C, D\}$ from page 141, is isomorphic to $\mathbb{Z}$. In particular we show that the loop given on page 141 is not null-homotopic. This answers the question we had posed on page 1049.

48.6. **The fundamental group of the product of two topological spaces.** The next proposition says that the fundamental groups of the product $A \times B$ of two topological spaces A and B is the direct product of the fundamental groups of A and B. Before we state the proposition explicitly we recall the definition of the direct product of two groups.

Definition. Let G and H two groups. We can form the cartesian product

$$G \times H \;:=\; \{(g,h) \,|\, g \in G \text{ and } h \in H\}.$$

This is again a group with the group structure given by

$$(g_1,h_1)\cdot(g_2,h_2) \;:=\; (g_1g_2, h_1h_2)$$

for $g_1, g_2 \in G$ and $h_1, h_2 \in H$. We call the group $G \times H$ the direct product of the groups G and H.

We start out with the following lemma, which strictly speaking we will not need, but which is worth knowing.

Lemma 48.19. Let $\varphi_1 \colon G_1 \to \pi$ and $\varphi_2 \colon G_2 \to \pi$ be two group homomorphisms. If for every $g_1 \in G_1$ and $g_2 \in G_2$ the images $\varphi_1(g_1) \in \pi$ and $\varphi_2(g_2) \in \pi$ commute, i.e. if we have $\varphi_1(g_1)\cdot\varphi_2(g_2) = \varphi_2(g_2)\cdot\varphi_1(g_1)$, then the map

$$\begin{array}{rcl} \varphi_1 \times \varphi_2 \colon G_1 \times G_2 & \to & \pi \\ (g_1,g_2) & \mapsto & \varphi_1(g_1)\cdot\varphi_2(g_2) \end{array}$$

is a homomorphism.[502]

Proof. We write $\Phi = \varphi_1 \times \varphi_2$. Let $g_1, g_1' \in G_1$ and let $g_2, g_2' \in G_2$. We have

$$\begin{array}{rcl} \Phi(g_1,g_2)\cdot\Phi(g_1',g_2') & = & \varphi_1(g_1)\cdot\varphi_2(g_2)\cdot\varphi_1(g_1')\cdot\varphi_2(g_2') \\ & = & \varphi_1(g_1)\cdot\varphi_1(g_1')\cdot\varphi_2(g_2)\cdot\varphi_2(g_2') \quad = \quad \varphi_1(g_1g_1')\cdot\varphi_2(g_2g_2') \\ & \uparrow & \quad = \quad \Phi(g_1g_1', g_2g_2'). \\ & \text{by our hypothesis} & \end{array}$$

■

Now we can formulate the promised proposition.

Proposition 48.20. (Fundamental Group-of-Product Proposition) Let A and B be two topological spaces and let $a_0 \in A$ and $b_0 \in B$. We consider the inclusion maps

$$\begin{array}{rcl} i \colon A & \to & A \times B \\ a & \mapsto & (a,b_0) \end{array} \quad \text{and} \quad \begin{array}{rcl} j \colon B & \to & A \times B \\ b & \mapsto & (a_0,b) \end{array}$$

and we consider the projection s

$$\begin{array}{rcl} p \colon A \times B & \to & A \\ (a,b) & \mapsto & a \end{array} \quad \text{and} \quad \begin{array}{rcl} q \colon A \times B & \to & B \\ (a,b) & \mapsto & b. \end{array}$$

The maps

$$\begin{array}{rcl} \Phi \colon \pi_1(A,a_0) \times \pi_1(B,b_0) & \to & \pi_1(A\times B,(a_0,b_0)) \\ (x,y) & \mapsto & i_*(x)\cdot j_*(y) \end{array}$$

and

$$\begin{array}{rcl} \Psi \colon \pi_1(A\times B,(a_0,b_0)) & \to & \pi_1(A,a_0)\times\pi_1(B,b_0) \\ z & \mapsto & (p_*(z),q_*(z)) \end{array}$$

are both homomorphisms[503] and they are inverses of one another, in particular both are isomorphisms.

---

[502] In fact the converse also holds, if $\varphi_1 \times \varphi_2$ is a homomorphism, then for every $g_1 \in G_1$ and $g_2 \in G_2$ the images $\varphi_1(g_1) \in \pi$ and $\varphi_2(g_2) \in \pi$ commute. Why is that?

[503] Note that a priori it is not clear that Φ is a homomorphism since it is not clear whether the hypothesis of Lemma 48.19 is satisfied.

Example. It follows from the Fundamental Group-of-Product Proposition 48.20 and Corollary 48.18 that

$$\pi_1(S^1 \times L(p,q)) \;\cong\; \pi_1(S^1) \times \pi_1(L(p,q)) \;\cong\; \mathbb{Z} \times \mathbb{Z}_p.$$

Similarly, we see, using the Sphere-π_1-Proposition 47.13, that for any $k \geq 2$ we have

$$\pi_1(S^1 \times S^k) \;\cong\; \pi_1(S^1) \times \pi_1(S^k) \;\cong\; \mathbb{Z} \times 0 \;\cong\; \mathbb{Z}.$$

Of course we can also apply the Fundamental Group-of-Product Proposition 48.20 several times and we see once again that

$$\pi_1\big(\underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}\big) \;\cong\; \underbrace{\pi_1(S^1) \times \ldots \times \pi_1(S^1)}_{n \text{ times}} \;\cong\; \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{n \text{ times}} \;\cong\; \mathbb{Z}^n.$$

Finally it follows from Propositions 47.13 and 48.20 that for any $k, l \geq 2$ we have

$$\pi_1(S^k \times S^l) \;\cong\; \pi_1(S^k) \times \pi_1(S^l) \;\cong\; 0 \times 0 \;=\; 0.$$

In particular we see that the fundamental group cannot distinguish the products of spheres of dimension ≥ 2. For example at the moment we cannot show that the two 6-dimensional smooth manifolds $S^3 \times S^3$ and $S^2 \times S^2 \times S^2$ are not homeomorphic.

In the proof of the Fundamental Group-of-Product Proposition 48.20 we will need the following purely group-theoretic lemma.

Proof of the Fundamental Group-of-Product Proposition 48.20. It is clear that

$$\Psi = (p_*, q_*)\colon \pi_1(A \times B, (a_0, b_0)) \;\to\; \pi_1(A, a_0) \times \pi_1(B, b_0)$$

is a homomorphism. Thus it suffices to show that Φ and Ψ are inverses of one another. Once we have shown that, it follows for free that Φ is also a homomorphism.

To show that Φ and Ψ are inverses of one another we introduce another map, namely we consider[504]

$$\begin{array}{rcl} \Theta\colon \pi_1(A, a_0) \times \pi_1(B, b_0) & \to & \pi_1(A \times B, (a_0, b_0)) \\ ([f\colon (S^1, 1) \to (A, a_0)], [g\colon (S^1, 1) \to (B, b_0)]) & \mapsto & \left[\begin{array}{rcl} (S^1, 1) & \to & (A \times B, (a_0, b_0)) \\ z & \mapsto & (f(z), g(z)) \end{array}\right]. \end{array}$$

It is almost a tautology that Θ and Ψ are inverses of one another. Thus it remains to prove the following claim.

Claim. The two maps $\Phi, \Theta\colon \pi_1(A, a_0) \times \pi_1(B, b_0) \to \pi_1(A \times B, (a_0, b_0))$ agree.

Proof. We use the convention from page 1050 to change our view of fundamental groups. Thus let $f\colon [0,1] \to A$ be a loop in (A, a_0) and $g\colon [0,1] \to B$ be a loop in (B, b_0). We need to show that

$$i_*([f]) \cdot j_*([g]) \;=\; [(t \mapsto (f(t), g(t))] \;\in\; \pi_1(A \times B, (a_0, b_0).$$

In other words, we need to show that the two loops

$$\begin{array}{rcl} [0,1] & \to & A \times B \\ t & \mapsto & \begin{cases} (f(2t), b_0), & \text{if } t \in [0, \frac{1}{2}] \\ (a_0, g(2t-1)), & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} [0,1] & \to & A \times B \\ t & \mapsto & (f(t), g(t)) \end{array}$$

[504] One can easily verify that Θ is well-defined.

are path-homotopic. In fact an explicit path-homotopy is given by the map[505]

$$\begin{array}{rcl} F\colon [0,1]\times[0,1] & \to & A\times B \\ (t,s) & \mapsto & \begin{cases} \big(f\big(\frac{2t}{1+s}\big), g(st)\big), & \text{if } t\in[0,\frac{1}{2}] \\ \big(f(st), g\big(\frac{2t-1+s}{1+s}\big)\big), & \text{if } t\in[\frac{1}{2},1]. \end{cases} \end{array}$$ ■

Summarizing we have now finally managed to calculate many non-trivial fundamental groups. Nonetheless, the list of topological spaces for which we cannot yet calculate the fundamental group is still pretty long, it includes

(1) topological graphs,
(2) knot complements,
(3) the complement of more than one point in $\mathbb{C}$, e.g. $\mathbb{C}\setminus\{0,1\}$,
(4) surfaces of genus ≥ 2.

In the following chapters we will introduce more methods for determining the fundamental group of a given topological space. But first we give in the next section two applications of our results.

48.7. The Fundamental Theorem of Algebra and the Borsuk-Ulam Theorem ❀.

As a first application of our methods we give another proof of the Fundamental Theorem of Algebra. Somewhat surprisingly, contrary to the standard proof [**Lang1999**, Corollary III.7.6], this proof does not make use of complex analysis.

Theorem 26.3. (Fundamental Theorem of Algebra) ❀ Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a zero in $\mathbb{C}$.

Remark. Other proofs are provided on page 704 and page 1700.

Proof. Let $q(z)$ be a polynomial with complex coefficients with $q(z)\neq 0$ for all $z\in\mathbb{C}$ with $|z|=r$. We consider the path

$$\begin{array}{rcl} f_r^{q(z)}\colon [0,1] & \to & S^1 \\ s & \mapsto & \dfrac{q(r\cdot\exp(2\pi\mathrm{i}s))/q(r)}{|q(r\cdot\exp(2\pi\mathrm{i}s))/q(r)|} \end{array}$$

Note that $f_r^{q(z)}(s=0)=f_r^{q(z)}(s=1)=1$, i.e. $f_r^{q(z)}$ is a loop in $(S^1,1)$. Now let

$$p(z) \;=\; a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$$

be a polynomial of degree $n\geq 1$. We assume that $p(z)$ has no zeros in $\mathbb{C}$. For $t\in[0,1]$ we also consider the polynomial

$$p_t(z) \;:=\; a_nz^n+t(a_{n-1}z^{n-1}+\cdots+a_1z+a_0).$$

Note that $p_1(z)=p(z)$ and $p_0(z)=a_nz^n$. Since $a_n\neq 0$ there exists an $s>0$ such that[506]

$$|a_nz^n| \;>\; |a_{n-1}z^{n-1}+\cdots+a_1z+a_0|$$

for any z with $|z|=s$. This inequality implies that for any $t\in[0,1]$ and any z with $|z|=s$ we have $p_t(z)\neq 0$, i.e. $p_t(z)$ has no zeros on the circle $|z|=s$.

[505] Note that this is indeed a path-homotopy of loops.

[506] In fact we can write it down explicitly: it follows easily from the triangle inequality that any $s\in\mathbb{R}$ such that $s\cdot|a_n|>|a_{n-1}|+\cdots+|a_0|$ will work.

We obtain the following sequence of path-homotopic loops:

$$\underset{\text{follows from the definition}}{\text{constant loop at } 1} \overset{\downarrow}{=} f_0^{p(z)} \underset{\uparrow}{\simeq} f_s^{p(z)} = f_s^{p_1(z)} \underset{\uparrow}{\simeq} f_s^{p_0(z)} \overset{\downarrow}{=} \underset{\text{follows from } p_0(z)=a_nz^n}{\text{the loop } \begin{array}{ccc}[0,1] & \to & S^1\\ t & \mapsto & \exp(2\pi \mathrm{i} n t).\end{array}}$$

maps $f_r^{p(z)}$ with $r \in [0,s]$ are defined, since the polynomials $p(z)$ has no zeros they form a path-homotopy from $f_0^{p(z)}$ to $f_s^{p(z)}$

maps $f_s^{p_t(z)}$ with $t \in [0,1]$ are defined, since polynomials $p_t(z)$ have no zeros on $|z|=s$, and and they form a path-homotopy from $f_r^{p_1(z)}$ to $f_r^{p_0(z)}$

By Proposition 48.17 we had know that the loop on the right defines a non-trivial element in the fundamental group $\pi_1(S^1,1)$, but we had just shown that it is path-homotopic to the constant path. Thus we have obtained a contradiction. ■

The following theorem goes back to the Polish mathematicians Karol Borsuk (1905-1982) and Stanislaw Ulam (1909-1984).[507] It was first proved in 1932.

Theorem 48.21. (Borsuk-Ulam Theorem) For every map $f\colon S^2 \to \mathbb{R}^2$ there exists a pair of antipodal points x and $-x$ in S^2 with $f(x) = f(-x)$.

Examples.

(1) It follows from the Borsuk-Ulam Theorem that there exists pair of antipodal points on earth with the same temperature and the same barometric pressure.
(2) The Borsuk-Ulam Theorem implies in particular that there is no injective map from $S^2 \to \mathbb{R}^2$.
(3) The statement of the Borsuk-Ulam Theorem also makes sense for 2 replaced by any $n \in \mathbb{N}$. For $n = 1$ one can prove the statement using elementary methods.[508] We will address the problem for $n \geq 3$ once we have introduced homology groups.

Proof. We prove the theorem by contradiction. So suppose there exists a map $f\colon S^2 \to \mathbb{R}^2$ such that $f(x) \neq f(-x)$ for all $x \in S^2$. This allows to define the map

$$\begin{array}{rcl} g\colon S^2 & \to & S^1 \\ x & \mapsto & \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}. \end{array}$$

Note that $g(-x) = -g(x)$ for all $x \in S^2$. Now let η be the loop that is defined by

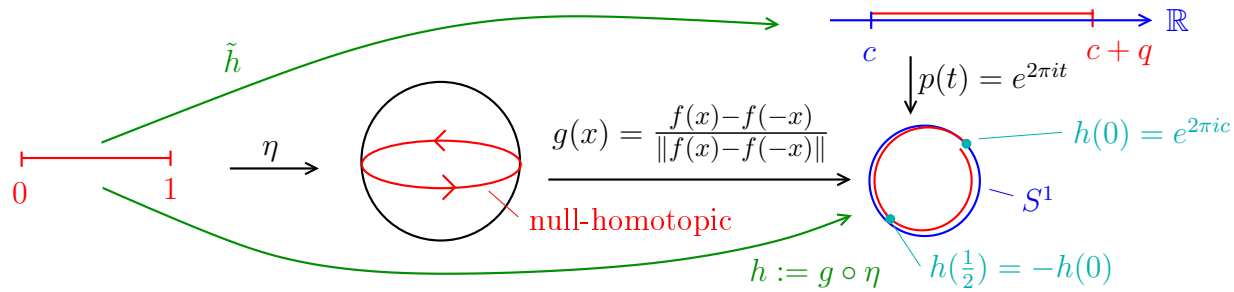
$$\begin{array}{rcl} \eta\colon [0,1] & \to & S^2 \\ s & \mapsto & \eta(s) = (\cos(2\pi s), \sin(2\pi s), 0) \end{array}$$

and let $h = g \circ \eta\colon [0,1] \to S^1$ be the composition of η with g. We pick a $c \in \mathbb{R}$ such that $h(0) = \exp(2\pi \mathrm{i} c)$. We denote by $p\colon \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi \mathrm{i} t)$, the usual covering map. We note that for $z, w \in \mathbb{R}$ we have

$$p(z) = p(w) \iff z - w \in \mathbb{Z} \quad \text{and} \quad p(z) = -p(w) \iff z - w \in \tfrac{1}{2} + \mathbb{Z}.$$

By Proposition 48.11 we can lift the path $h\colon [0,1] \to S^1$ to a path $\widetilde{h}\colon [0,1] \to \mathbb{R}$ with $\widetilde{h}(0) = c$.

[507] Ulam became much more famous through his work on the hydrogen bomb.
[508] How would you prove the statement?



Given any $s \in [0, \frac{1}{2}]$ we have

$$p\big(\widetilde{h}\big(s+\tfrac{1}{2}\big)\big) \;=\; h\big(s+\tfrac{1}{2}\big) \;=\; g\big(\eta\big(s+\tfrac{1}{2}\big)\big) \underset{\text{by the definition of } \eta}{\overset{\uparrow}{=}} g(-\eta(s)) \underset{\text{since } g(-x)=-g(x) \text{ for all } x \in S^2}{\overset{\uparrow}{=}} -g(\eta(s)) \;=\; -h(s) \;=\; -p\big(\widetilde{h}(s)\big).$$

By the above discussion of p this implies that

$$\widetilde{h}\big(s+\tfrac{1}{2}\big) - \widetilde{h}(s) \;=:\; \frac{q(s)}{2}$$

with $q(s)$ an odd integer. The map $q\colon [0, \frac{1}{2}] \to \mathbb{Z}$ is the difference of two continuous maps, hence continuous. It follows from the Connected-to-Discrete Lemma 2.27 that it is constant. We denote this constant value by q. Since q is odd it is in particular non-zero.

It follows from the above that

$$\widetilde{h}(1) \;=\; \widetilde{h}\big(\tfrac{1}{2}\big) + \tfrac{q}{2} \;=\; \widetilde{h}(0) + \tfrac{q}{2} + \tfrac{q}{2} \;=\; \widetilde{h}(0) + q.$$

Since q is non-zero we obtain from Lemma 48.15 that $h = g \circ \eta$ is not null-homotopic. But h was the composition $h = g \circ \eta\colon [0,1] \to S^2 \to S^1$. By the Sphere-$\pi_1$-Proposition 47.13 the loop η is null-homotopic in S^2 which implies by Proposition 47.15 that $h = g \circ \eta$ is null-homotopic in S^1. Thus we have obtained a contradiction. ■

Exercises for Chapter 48.

Exercise 48.1. Let $p\colon X \to B$ be a covering of topological spaces. We suppose that B is a path-connected topological space. Show that for any two points P and Q in B we have

$$\#p^{-1}(P) \;=\; \#p^{-1}(Q).$$

Exercise 48.2.

(a) Let $p\colon X \to B$ be a map between topological spaces that is two-to-one, i.e. such that for any $b \in B$ the preimage $p^{-1}(b)$ consists of precisely two points. Is p necessarily a covering map?

(b) Is the map

$$\begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \setminus \{0\} \\ z & \mapsto & \exp(z) \end{array}$$

a covering map? *Hint.* $z = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$.

(c) Let $p\colon X \to B$ be a surjective map between topological spaces. Let $f\colon [0,1] \to B$ be a map. Does there always exist a lift from f to X, i.e. does there exist a map $g\colon [0,1] \to X$ such that $p \circ g = f$? Give a short justification for your answer.

Exercise 48.3.

(a) Show that there exists a map $p\colon X \to B$ be a map between topological spaces that is a local homeomorphism and that is surjective but which is not a covering map.
(b) Let $p\colon X \to B$ be a map between topological spaces that is a local homeomorphism and that is surjective. Furthermore we suppose that p is proper, i.e. we assume that the preimage of each compact set is compact. Is p necessarily a covering map?

Exercise 48.4. Let $p\colon X \to B$ be a covering of topological spaces. Show that if B is Hausdorff, then X is also Hausdorff.

Exercise 48.5. Let X, Y and Z be topological spaces and let $p\colon X \to Y$ and $q\colon Y \to Z$ be maps. Which of the following two statements are true?

(a) If $q \circ p$ is a covering map, then p is also a covering map.
(b) If $q \circ p$ is a covering map, then q is also a covering map.

Exercise 48.6. Consider the 5-fold covering map

$$\begin{array}{rcl} p\colon X = S^1 & \to & B = S^1 \\ z & \mapsto & z^5. \end{array}$$

Let $f\colon [0,1] \to B = S^1$ be the loop given by $t \mapsto \exp(2\pi \mathrm{i} kt)$. By Proposition 6.11 this loop lifts to a path $g\colon [0,1] \to X = S^1$. For which k is the lift g again a loop?

Exercise 48.7. Let $p\colon \widetilde{X} \to X$ be a covering of topological spaces.

(a) Show that for $A \subset X$ we have $\quad$ closure of $p^{-1}(A) = p^{-1}(\text{closure of } A)$
(b) Show that for $A \subset X$ we have $\quad$ interior of $p^{-1}(A) = p^{-1}(\text{interior of } A)$.

Exercise 48.8. We consider the following two self-homeomorphisms of $\mathbb{R}^2$:

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1). \end{array}$$

We denote by G the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by A and B. This means

$$G = \begin{array}{c} \text{all self-homeomorphisms of } \mathbb{R}^2 \text{ that can be written} \\ \text{as a finite concatenation of the maps } A, B, A^{-1} \text{ and } B^{-1}. \end{array}$$

(a) For $n \in \mathbb{Z}$, what is $A^{-n}BA^n$? (Perhaps start out with $n = 1$.)
(b) We write $C = A^2$. We denote by H the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by C and B. Does A lie in H?
Hint. Consider which self-diffeomorphisms of $\mathbb{R}^2$ are orientation-preserving.
(c) Show that any element X of G lies in H or it is of the form $G = AY$ with $Y \in H$.
(d) Show that the group G acts discretely on $\mathbb{R}^2$.

Exercise 48.9. Let $n \in \mathbb{N}$ and let $A \in \mathrm{M}(n \times n, \mathbb{Z})$ be a matrix with non-zero determinant. Show that the map

$$\begin{array}{rcl} f(A)\colon \mathbb{R}^n/\mathbb{Z}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ [v] & \mapsto & [A \cdot v] \end{array}$$

is a covering map of degree $|\det(A)|$.
Remark. This can be viewed as a generalization of Exercise 19.14.

Exercise 48.10. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces.

(a) Let f be a loop in (B, b_0). We denote by $\widetilde{f}$ the lift of f to the starting point x_0. Show that if $[f]$ lies in $p_*(\pi_1(X, x_0)) \subset \pi_1(B, b_0)$, then $\widetilde{f}$ is a loop in (X, x_0).

(b) Now we suppose that X is simply connected. Let f and g loops in (B, b_0) and let $\widetilde{f}$ and $\widetilde{g}$ be the lifts of f and g to the starting point x_0. Show that if the endpoints of $\widetilde{f}$ and $\widetilde{g}$ agree, then f and g are path-homotopic loops.

(c) Show that if X is path-connected, then $[X : B] = [\pi_1(B, b_0) : p_*(\pi_1(X, x_0))]$.

Exercise 48.11. Show that there is no covering map from the sphere S^2 to the torus $S^1 \times S^1$.
Hint. Use Lemma 48.15.

Exercise 48.12. We consider the pointed topological space (B, b_0) and we consider the loops $\alpha, \beta\colon [0, 1] \to B$ shown in the figure below.

(a) Show that there exists a covering $p\colon (X, x_0) \to (B, b_0)$ of pointed topological spaces such that the loop $\alpha * \beta * \overline{\alpha} * \overline{\beta}$ does not lift to a loop in (X, x_0).

(b) Use Lemma 48.15 (1) to show that $\pi_1(B, b_0)$ is non-abelian.



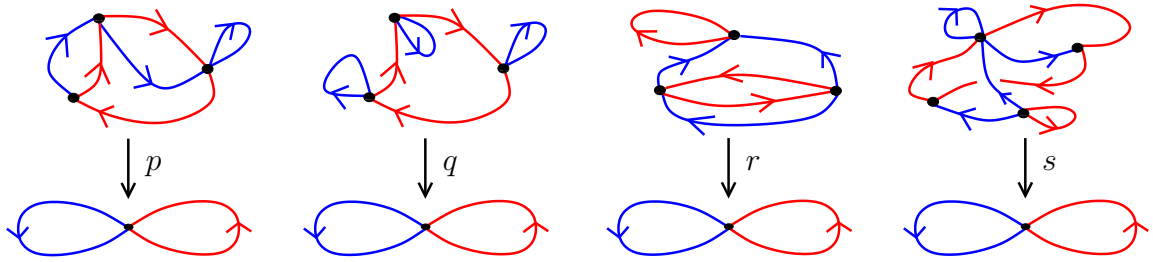
Exercise 48.13.

(a) Does there exist a covering map $p\colon K \to T$ from the Klein bottle to the 2-dimensional torus?

(b) Let X be a topological space and let $b \in \pi_1(X, x_0)$. Does there necessarily exist a map $f\colon (S^1, 1) \to (X, x_0)$ with $b \in f_*(\pi_1(S^1, 1))$?

(c) Let X be a topological space and let $b, c \in \pi_1(X, x_0)$. Does there necessarily exist a map $f\colon (S^1 \times S^1, (1, 1)) \to (X, x_0)$ with $b, c \in f_*(\pi_1(S^1 \times S^1, (1, 1)))$?

Exercise 48.14. Which of the maps between topological graphs that are shown in the figure below are covering maps?



Exercise 48.15.

(a) Use the Fundamental Group-via-Actions Theorem 48.16 to show that the fundamental group of the pseudocircle, i.e. of the topological space $X = \{A, B, C, D\}$ from page 141, is isomorphic to $\mathbb{Z}$.

(b) Let $f\colon S^1 \to X$ be the map that we introduced on page 141. Show that the induced map $f_*\colon \pi_1(S^1, 1) \to \pi_1(X, A)$ is an isomorphism.

Exercise 48.16. Let $n \in \mathbb{N}$. We consider the map $f\colon S^1 \to S^1$ that is given by $z \mapsto z^n$. Show that the induced map on $\pi_1(S^1, 1)$ is given by multiplication by n.

Exercise 48.17. Let A and B be topological spaces and let $a_0 \in A$ and $b_0 \in B$. We consider the maps

$$\begin{array}{rcl} f\colon A & \to & A \times B \\ a & \mapsto & (a, b_0) \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon B & \to & A \times B \\ b & \mapsto & (a_0, b) \end{array}$$

and the map

$$\begin{array}{rcl} \Phi\colon \pi_1(A, a_0) \times \pi_1(B, b_0) & \to & \pi_1(A \times B, (a_0, b_0)) \\ (x, y) & \to & f_*(x) \cdot g_*(y) \end{array}$$

(a) Show that Φ is a monomorphism, i.e. that it is injective.
Hint. Consider the maps

$$\begin{array}{rcl} p\colon A \times B & \to & A \\ (a, b) & \mapsto & a \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon A \times B & \to & B \\ (a, b) & \mapsto & b. \end{array}$$

(b) Show that Φ is an epimorphism, i.e. that it is surjective.
Hint. Let $h(t) = (\alpha(t), \beta(t))$ be a loop in $(A \times B, (a_0, b_0))$. Consider the map

$$\begin{array}{rcl} \Psi\colon [0,1] \times [0,1] & \to & A \times B \\ (s, t) & \mapsto & (\alpha(s), \beta(t)). \end{array}$$

Hint. You might want to use ideas similar to the proof that the map Φ is a homomorphism.

Exercise 48.18. Let $A \in \mathrm{GL}(n, \mathbb{Z})$ be a matrix. In the Maps-on-Torus Lemma 19.37 we saw that the map

$$\begin{array}{rcl} f(A)\colon \mathbb{R}^n/\mathbb{Z}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ v & \mapsto & Av \end{array}$$

is a homeomorphism. We identify $\pi_1(\mathbb{R}^n/\mathbb{Z}^n, 0)$ with $\mathbb{Z}^n$ as in the Fundamental Group-via-Actions Theorem 48.16. Show that the diagram

$$\begin{array}{ccc} \pi_1(\mathbb{R}^n/\mathbb{Z}^n, 0) & \xrightarrow{f(A)_*} & \pi_1(\mathbb{R}^n/\mathbb{Z}^n, 0) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle =} \\ \mathbb{Z}^n & \xrightarrow{v \mapsto Av} & \mathbb{Z}^n \end{array}$$

commutes.

Exercise 48.19. Let (X, d) be a metric space and let $p\colon \widetilde{X} \to X$ be a covering. Given $x, y \in \widetilde{X}$ we define

$$\widetilde{d}(x, y) := \inf \left\{ \sum_{i=0}^{n-1} d(w_i, w_{i+1}) \;\middle|\; \begin{array}{c} \text{with } w_0, \ldots, w_n \in X,\ p(w_0) = x,\ p(w_n) = y \text{ and such} \\ \text{that for each } i \in \{0, \ldots, n-1\} \text{ there exists a uniformly} \\ \text{covered open subset that contains } w_i \text{ and } w_{i+1} \end{array} \right\}.$$

(a) Show that $\widetilde{d}$ defines on $\widetilde{X}$ and that the corresponding topology equals the topology of $\widetilde{X}$.
(b) Show that $p\colon \widetilde{X} \to X$ is a local isometry.

Remark Recall that a topological space X is called *metrizable* if there exists a metric d on X such that the given topology agrees with the topology coming from d. The exercise shows in particular that every covering of a metrizable topological space is again metrizable.

Exercise 48.20.

(a) Consider the following maps:

$$\begin{array}{ccc} & & (S^1,1) \\ & \overset{???\ \text{lift}\ \widetilde{f}\ ???}{\nearrow} & \Big\downarrow{\scriptstyle p\ z\mapsto z^2} \\ (S^1,1) & \xrightarrow[f]{z\mapsto z} & (S^1,1). \end{array}$$

Show that the map f does not lift to a map $\widetilde{f}$.

(b) Let $p\colon \widetilde{M} \to M$ be a finite covering and let $f\colon M \to M$ be map. Does there necessarily exist a map $\widetilde{f}\colon \widetilde{M} \to \widetilde{M}$ that makes the following diagram commute:

$$\begin{array}{ccc} \widetilde{M} & \overset{\widetilde{f}}{\dashrightarrow} & \widetilde{M} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ M & \xrightarrow{\quad f \quad} & M\,? \end{array}$$

Exercise 48.21. Let $p\colon \widetilde{X} \to X$ be a 2-fold covering of a topological space X.

(a) Show that the map

$$\begin{array}{rcl} \Xi\colon \widetilde{X} & \to & \widetilde{X} \\ x & \mapsto & \text{the unique element in } p^{-1}(p(\{x\})) \setminus \{x\} \end{array}$$

is continuous.

(b) Let A be a topological space and let $f\colon A \to X$ be a map. Show that if $\widetilde{f}\colon A \to \widetilde{X}$ is a lift of f, then $\Xi \circ \widetilde{f}$ is the unique other lift of f.

Exercise 48.22. Let $p\colon \widetilde{X} \to X$ be a covering of a connected space X. Show that for each component Y of $\widetilde{X}$ the map $p\colon Y \to X$ is also a covering.

Exercise 48.23. Let X be a topological space and let G be a group that acts continuously, properly and freely on X.

(a) Show that the natural projection $X \to X/G$ is a local homeomorphism.

(b) We denote by $\sim$ the equivalence relation on the topological space $X \times [0,1]$ that is generated by $(x,t) \sim (g \cdot x, t)$ for $g \in G$ and $(x,t) \in X \times [0,1]$. Show that the map

$$\begin{array}{rcl} (X \times [0,1])/\sim & \to & X/G \times [0,1] \\ [(x,t)] & \mapsto & ([x],t) \end{array}$$

is a homeomorphism.

Remark. This is a special case of the Product-Quotient Mixing Lemma 13.6, which relies on the tricky Whitehead Quotient Theorem 13.4. But in our setting, using (a), one can give an elementary proof.

(c) Suppose that $F\colon X \times [0,1] \to X$ is a homotopy that is G-equivariant in the sense that for any $(x,t) \in X \times [0,1]$ and any $g \in G$ we have $F(g \cdot x, t) = g \cdot F(x,t)$. Show that the map

$$\begin{array}{rcl} X/G \times [0,1] & \to & X/G \\ ([x],t) & \mapsto & [F(x,t)] \end{array}$$

is continuous.

Remark. This is a special case of the Homotopy Quotient Theorem 14.7. But in our setting one can deduce it from (b).

Exercise 48.24. We denote by $p\colon \mathbb{R} \to S^1$ the covering map that is given by $t \mapsto \exp(2\pi \mathrm{i} t)$. Let $\epsilon \in \{-1, 1\}$ We say that a map $f\colon \mathbb{R} \to \mathbb{R}$ is ϵ-equivariant if for any $t \in \mathbb{R}$ and $n \in \mathbb{Z}$ we have $f(n + t) = \epsilon \cdot n + f(t)$.

(a) Show that any self-homeomorphism of S^1 lifts to a self-homeomorphism of $\mathbb{R}$ that is ϵ-equivariant for some $\epsilon \in \{-1, 1\}$.

(b) Show that any self-homeomorphism of $\mathbb{R}$ is either monotonously increasing or monotonously decreasing.

(c) Let $g\colon \mathbb{R} \to \mathbb{R}$ be an ϵ-equivariant self-homeomorphism. Show that there exists an isotopy
$$G\colon \mathbb{R} \times [0, 1] \to \mathbb{R}$$
from g to the map that is given by $x \mapsto \epsilon \cdot gx$ and such that each G_t is ϵ-equivariant.

(d) Show that any self-homeomorphism of S^1 is isotopic to the map $f(z) = z$ or to the map $g(z) = \overline{z}$.

Hint. Use Exercise 48.23.

Remark. This gives a new proof of Exercise 14.7 (b).

Exercise 48.25. Let Y be a topological space and let $q\colon \widetilde{Y} \to Y$ be a covering. Furthermore let X be a topological space and let $f\colon X \to Y$ be a map. We define the pullback
$$f^*\widetilde{Y} := \{(x, \widetilde{y}) \in X \times \widetilde{Y} \mid f(x) = q(\widetilde{y})\}.$$
We equip $X \times \widetilde{Y}$ with the product topology and we equip $f^*\widetilde{Y}$ with the subspace topology. Finally we consider the maps
$$\begin{array}{rcl} p\colon f^*\widetilde{Y} & \to & X \\ (x, \widetilde{y}) & \mapsto & x \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \widetilde{f}\colon f^*\widetilde{Y} & \to & \widetilde{Y} \\ (x, \widetilde{y}) & \mapsto & \widetilde{y} \end{array}$$

(a) Show that $p\colon f^*\widetilde{Y} \to X$ is a covering with $[f^*\widetilde{Y} : X] = [\widetilde{Y} : Y]$.

Remark. We refer to $p\colon f^*\widetilde{Y} \to X$ as the *pullback* of the covering $q\colon \widetilde{Y} \to Y$.

(b) Recall that a map between topological spaces is *proper* if the preimage of each compact set is open. Show that if $f\colon X \to Y$ is proper, then $\widetilde{f}\colon f^*\widetilde{Y} \to \widetilde{Y}$ is also proper.

Exercise 48.26.

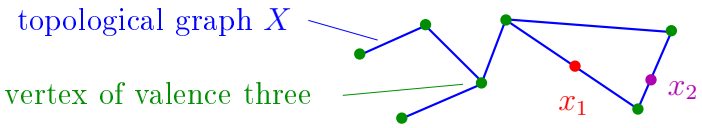
(a) Give an example of a map $f\colon S^1 \times S^1 \to \mathbb{R}^2$ such that there is no $(x, y) \in S^1 \times S^1$ with $f(-x, -y) = f(x, y)$.

(b) Show that given any map $f\colon S^1 \times S^1 \to \mathbb{R}$ there exists an $(x, y) \in S^1 \times S^1$ with $f(-x, -y) = f(x, y)$.

Exercise 48.27. Let $k \in \mathbb{N}$ and let X be a topological manifold. We refer to
$$\mathrm{Conf}_k(X) := \{(x_1, \ldots, x_k) \in X^k \mid \text{for all } i \neq j \text{ we have } x_i \neq x_j\}$$
as the *configuration space of k points in X*. Let $X \subset \mathbb{R}^2$ be a connected topological graph that has at least one vertex of valence three. Show that $\pi_1(\mathrm{Conf}_2(X))$ admits an

epimorphism onto $\mathbb{Z}$.
Hint. Consider the map

$$\begin{array}{rcl} \mathrm{Conf}_2(X) & \to & S^1 \\ (x_1, x_2) & \mapsto & \frac{x_2 - x_1}{\|x_2 - x_1\|}. \end{array}$$



49. Coverings of smooth and topological manifolds

49.1. **Fundamental groups of topological manifolds.** Our only goal in this section is to prove a proposition, which will help us with dealing with a technical issue in the following section. But the proposition is also interesting in its own right and it provides a good way to practice what we learned in the last chapters.

Proposition 49.1. (Countable Fundamental Group Proposition)

(1) Let X be a topological space and let $x_0 \in X$. If X is second-countable and locally simply connected, then $\pi_1(X, x_0)$ is countable.
(2) If X is a topological manifold and if $x_0 \in X$, then $\pi_1(X, x_0)$ is countable.

Remark.

(1) In Proposition 96.7 and 104.15 we will give an alternative proof of the Countable Fundamental Group Proposition 49.1 for smooth respectively topological manifolds.
(2) In Propositions 96.6, 104.14 and on page 2997 we will see that fundamental groups of compact topological manifolds are in fact "finitely generated".

Proof. The second statement follows almost immediately from the first statement. Indeed, first note that topological manifolds are by definition second-countable. Furthermore note that in the Topological Manifolds-Local Properties Lemma 18.8 we showed that an n-dimensional topological manifold is locally homeomorphic to non-empty convex subsets of $\mathbb{R}^n$. It follows immediately from the Convex Homotopy Lemma 14.1, or from the discussion on page 1055, that non-empty convex subsets of $\mathbb{R}^n$ are simply connected. This shows that topological manifolds are locally simply connected.

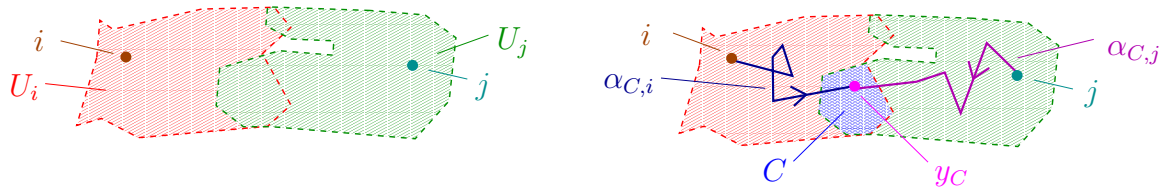
Now we turn to the proof of the first statement. Thus let X be a topological space that is second-countable and locally simply connected. Furthermore let $x_0 \in X$. Since X is locally simply connected we know that for each $x \in X$ there exists an open neighborhood $U_x \subset X$ of x that is simply connected. Since X is second-countable we know by the Countable Covering Lemma 9.2 that there exists a countable subset $I \subset X$ such that $X = \bigcup_{i \in I} U_i$. We pick $i_0 \in I$ in the path-component of x_0. By the Change-of-Base Point Proposition 47.10 it suffices to show that $\pi_1(X, i_0)$ is countable.

Claim 1. For each $i, j \in I$ the intersection $U_i \cap U_j$ has only countably many path-components, i.e. $\pi_0(U_i \cap U_j)$ is countable.

Proof. Let $\{W_k\}_{k \in K}$ be the path-components of $U_i \cap U_j$. For each $k \in K$ we pick a point z_k. Since U_i and U_j are open we know that $U_i \cap U_j$ is also open. Since X is locally simply connected we see that there exists a simply connected neighborhood C_k of z_k with $C_k \subset U_i \cap U_j$. Recall that "simply connected" means in particular that C_k is path-connected. Therefore it follows from the Path-Component Lemma 2.31 that $C_k \subset W_k$. Thus we see that the open sets C_k, $k \in K$ are disjoint. Now we obtain from our hypothesis that X is second-countable, together with say the Countable Covering Lemma 9.2, that K is countable. ⊞

Let $i, j \in I$. Given any $C \in \pi_0(U_i \cap U_j)$ we pick a point $y_C \in C_{i,j}$. Since U_i and U_j are in particular path-connected we can pick a path $\alpha_{C,i}$ from i to y_C and we can pick a path $\alpha_{C,j}$

from j to y_C. Note that it follows from Claim 1 together with the Countability Lemma 0.8 that the set $\bigcup_{i\in I}\bigcup_{j\in J}\pi_0(U_i\cap U_j)$ is countable. Thus it remains to prove the following claim.



Claim 2. Every loop in i_0 is path-homotopic to a composition of the paths $\alpha_{C,i}$ and their inverses.

Proof. Let $\gamma\colon[0,1]\to X$ be a loop in i_0. We start out with a few preparations:

- Since the sets $\{U_i\}_{i\in I}$ form an open cover and since $[0,1]$ is compact we obtain easily from the Lebesgue Corollary 6.6 that there exists an $N\in\mathbb{N}$ and $i_1,\dots,i_{N-1}\in I$ such that for each $k\in\{0,\dots,N-1\}$ we have $\gamma([\frac{k}{N},\frac{k+1}{N}])\subset U_{i_k}$.
- Given $k\in\{0,\dots,N-1\}$ we define γ_k to be the restriction of γ to the interval $[\frac{k}{N},\frac{k+1}{N}]$.
- For $k=1,\dots,N-1$ we have $\gamma(\frac{k}{N})\in U_{i_{k-1}}\cap U_{i_k}$. We denote by C_k the path-component of $U_{i_{k-1}}\cap U_{i_k}$ that contains $\gamma(\frac{k}{N})$. Since C_k is path-connected there exists a path $\mu_k\colon[0,1]\to C_k$ from $\gamma(\frac{k}{N})$ to y_{C_k}.

Next we reuse some notation that we had introduced earlier on.

- Given a path $\delta\colon[a,b]\to X$ we denote, as on page 1051, by $[\delta]$ the path-homotopy class of δ.
- Given two paths $\delta\colon[a,b]\to X$ and $\epsilon\colon[c,d]\to X$ with $\delta(b)=\epsilon(c)$ we denote by $\delta*\epsilon\colon[0,1]\to X$ the concatenation as defined on page 1052.
- Given a path $\delta\colon[0,1]\to X$ we denote by $\overline{\delta}\colon[0,1]\to X$ the inverse path that is given by $\overline{\delta}(t)=\delta(1-t)$.

Now we can provide the actual proof of the claim. By constantly using the Path-Homotopy Concatenation Proposition 47.5 and Lemmas 47.4 we obtain that

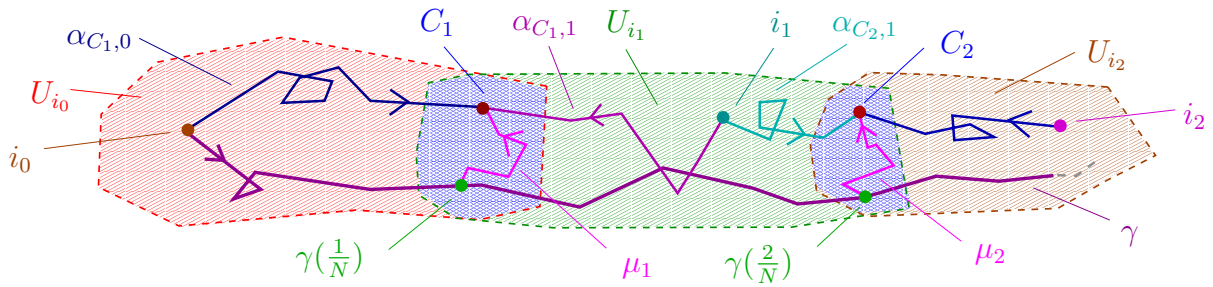
$$\begin{aligned}
[\gamma] &= [\gamma_0 * \gamma_1 * \ldots * \gamma_{N-1}] \\
&= [\underbrace{\gamma_0 * \mu_1}_{\substack{\text{same endpoints}\\ \text{as } \alpha_{C_1,0}}} * \underbrace{\overline{\mu_1} * \gamma_1 * \mu_2}_{\substack{\text{same endpoints}\\ \text{as } \overline{\alpha_{C_1,1}} * \alpha_{C_2,1}}} * \overline{\mu_2} * \ldots * \mu_{N-1} * \underbrace{\overline{\mu_{N-1}} * \gamma_{N-1}}_{\substack{\text{same endpoints}\\ \text{as } \overline{\alpha_{C_{N-1},N}}}}] \\
&\overset{\uparrow}{=} [\alpha_{C_1,0} * \overline{\alpha_{C_1,1}} * \alpha_{C_1,1} * \ldots * \overline{\alpha_{C_{N-1},N}}]
\end{aligned}$$

here we use Lemma 47.12 and that the U_{i_k} are simply connected

We have thus proved the desired statement. ∎

49.2. **Manifold structures on coverings of manifolds.** We recall the following definition from page 1067.

Definition. Let $p\colon X\to B$ be a map between two topological spaces. We say the map $p\colon X\to B$ is a covering, if it is surjective and if for every $b\in B$ there exists an open neighborhood U of b which is uniformly covered, i.e. $p^{-1}(U)$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ with the property that the restriction of p to each subset V_i is a homeomorphism.



In the context of smooth and complex manifolds it is appropriate to introduce the following variation.

Definition.

(1) Let $p\colon X \to B$ be a map between smooth manifolds. We say the map $p\colon X \to B$ is a covering of smooth manifolds, if it is surjective and if for every $b \in B$ there exists an open neighborhood U of b such that $p^{-1}(U)$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ with the property, that the restriction of p to each subset V_i is a diffeomorphism.

(2) We have the analogous definition of a covering between complex manifolds, we just need to replace "diffeomorphism" by "biholomorphism".

Remark. Note that a covering map between smooth manifolds is in particular a local diffeomorphism, which implies by the Smooth Pasting Proposition 19.23 that it is smooth.

Example. We consider the smooth manifold $M = \mathbb{R}$ and the map

$$\begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3. \end{array}$$

This map is a homeomorphism, in particular it is a 1-fold covering between topological spaces. On the other hand this map is not a (local) diffeomorphism, hence it is not a covering between smooth manifolds.

Proposition 49.2. (Manifold Covering Proposition) Let X be an n-dimensional topological manifold and let $p\colon \widetilde{X} \to X$ be a covering.

(0) If X and $\widetilde{X}$ are connected, then $p\colon \widetilde{X} \to X$ is a countable covering.[509]

Now suppose that $p\colon \widetilde{X} \to X$ is a countable covering.

(1) The covering space $\widetilde{X}$ is also an n-dimensional topological manifold.

(2) (a) If X admits a smooth atlas, then $\widetilde{X}$ admits a smooth atlas such that $p\colon \widetilde{X} \to X$ is a covering of smooth manifolds.

(b) If X is equipped with a smooth structure[510] then $\widetilde{X}$ admits a *unique* smooth structure such that $p\colon \widetilde{X} \to X$ is a covering of smooth manifolds.

(3) (a) Regardless of whether we work with smooth or topological manifolds we have the equality $\partial\widetilde{X} = p^{-1}(\partial X)$.

(b) If $p\colon \widetilde{X} \to X$ is a finite covering and if X is a *closed* smooth or topological manifold, then so is $\widetilde{X}$.

(4) If X is an oriented smooth manifold, then $\widetilde{X}$ admits a unique orientation such that $p\colon \widetilde{X} \to X$ is orientation-preserving.
(5) If X is equipped with a complex structure, then $\widetilde{X}$ admits a unique complex structure such that $p\colon \widetilde{X} \to X$ is a covering of complex manifolds.

Convention. If M is a smooth manifold, i.e. if M is a topological manifold equipped with a smooth structure, and if we are given a covering $p\colon \widetilde{M} \to M$, then usually we equip $\widetilde{M}$ with the unique smooth structure given by the Manifold Covering Proposition 49.2 (2b). The same convention also applies to coverings of oriented smooth manifolds and to coverings of complex manifolds.

In the proof of the Manifold Covering Proposition 49.2 we will need the following technical lemma.

Auxiliary Lemma 49.3. Let $p\colon \widetilde{X} \to X$ be a countable covering of a topological space X. If X is second-countable, then $\widetilde{X}$ is also second-countable.

Proof of Lemma 49.3. Let $p\colon \widetilde{X} \to X$ be a countable covering of a topological space X. Recall that we say $U \subset X$ is uniformly covered, if $p^{-1}(U)$ is the union of disjoint open subsets $\{\widetilde{U}_i\}_{i\in I}$ with the property that the restriction of p to each subset $\widetilde{U}_i$ is a homeomorphism. If U is uniformly covered, then it follows from the remark on page 1067 that every open subset of U is also uniformly covered.

Let $\mathcal{B}$ be a countable basis for the topology of X. We denote by $\mathcal{U} = \{U_j\}_{j\in J}$ the family of all open subsets of X that are uniformly covered. Since p is a covering we have $\bigcup_{j\in J} U_j = X$. It follows from Lemma 1.30 that

$$\mathcal{B}' := \{B \in \mathcal{B} \mid \text{ there exists a } j \in J \text{ with } B \subset U_j\}$$

is also a basis for the topology of X.

Now let $B \in \mathcal{B}'$. Since open subsets of uniformly covered subsets are uniformly covered we know that $p^{-1}(B)$ is the union of disjoint open sets B_i, $i \in I_B$ such that the restriction of p to each B_i is a homeomorphism. Since p is a countable covering the index set I_B is countable.

Claim. $$\widetilde{\mathcal{B}} := \{B_i \mid B \in \mathcal{B}' \text{ and } i \in I_B\}$$ is a basis for the topology of $\widetilde{X}$.

Proof. It is straightforward to see that $\widetilde{\mathcal{B}}$ has the basis property. Every set in $\widetilde{\mathcal{B}}$ is of course open in $\widetilde{X}$. By Lemma 1.29 it suffices to show that given any open set W of $\widetilde{X}$ and any point $x \in W$ there exists a set B_i in $\widetilde{\mathcal{B}}$ with $x \in B \subset W$. So let $x \in W$. Since $\mathcal{B}'$ is a basis for the topology of X there exists a $B \in \mathcal{B}'$ with $p(x) \in B \subset p(W)$. But then there exists an $i \in I_B$ with $x \in B_i \subset W$. ⊞

Since $\mathcal{B}'$ is countable and since each I_B is countable we know by the Countability Lemma 0.8 that $\widetilde{\mathcal{B}}$ consists of countably many open sets. This shows that $\widetilde{X}$ is second-countable. ■

[509] Recall that a covering $p\colon \widetilde{X} \to X$ of a connected space is called *countable* if the degree, i.e. the cardinality of the preimage of a point in X, is countable.
[510] We refer to page 543 for the definition of a smooth structure on a smooth manifold.

Proof of the Manifold Covering Proposition 49.2. Let X be an n-dimensional topological manifold and let $p\colon \widetilde{X} \to X$ be a covering.

(0) Suppose that X and $\widetilde{X}$ are path-connected. If X is the empty set there is nothing to show. If X is non-empty, we pick $x_0 \in X$. Furthermore we pick $\widetilde{x}_0 \in \widetilde{X}$ with $p(\widetilde{x}_0) = x_0$. By the Countable Fundamental Group Proposition 49.1 we know that $\pi_1(X, x_0)$ is countable. By Lemma 48.15 we know that $[\widetilde{X} : X] = |\pi_1(X, x_0) : p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))|$. It follows from the fact that $\pi_1(X, x_0)$ is countable and the Countability Lemma 0.8 that $[\widetilde{X} : X]$ is countable.

Now we assume that $p\colon \widetilde{X} \to X$ is a countable covering.

(1) First we note that it follows from the hypothesis that X is a topological manifold that X is Hausdorff. It follows from Lemma 48.3 (3) that $\widetilde{X}$ is also Hausdorff. Next note that it follows from our hypothesis that X is a topological manifold and that p is a countable covering, together with Lemma 49.3, that $\widetilde{X}$ is second-countable.

Since $p\colon \widetilde{X} \to X$ is a covering there exists a family of open subsets $\{U_i\}_{i\in I}$ of X such that each U_i is uniformly covered. Furthermore, since X is an n-dimensional topological manifold there exists an atlas $\{\Phi_j\colon V_j \to W_j\}_{j\in J}$ for X. After possibly replacing each V_j by the family $\{V_j \cap U_i\}_{i\in I}$ we can without loss of generality assume that each V_j is uniformly covered. Thus for each j the preimage $p^{-1}(V_j)$ is the union of disjoint open subsets $\{\widetilde{V}_{ji}\}_{i\in I_j}$ with the property, that the restriction of p to each subset $\widetilde{V}_{ji}$ is a homeomorphism. It is clear that $\{\Phi_j \circ p\colon \widetilde{V}_{ji} \to W_j \mid j \in J \text{ and } i \in I_j\}$ is an n-dimensional atlas for $\widetilde{X}$.

(2) (a) The transition maps of the atlas from (1) are precisely given by restrictions to open subsets of the transition maps of the atlas $\{\Phi_j\colon V_j \to W_j\}_{j\in J}$ for X. Thus if we start out with a smooth atlas for X, then the atlas we just constructed for $\widetilde{X}$ is also smooth.

Since $p\colon \widetilde{X} \to X$ is a covering we know that each $x \in X$ admits an open neighborhood U such that $p^{-1}(U)$ is the union of a family $\{V_i\}_{i\in I}$ of disjoint open subsets such that each $p\colon V_i \to U$ is a homeomorphism. By the definition of the smooth structure on $\widetilde{X}$ these maps are now diffeomorphisms. We have thus shown that $p\colon \widetilde{X} \to X$ is a covering of smooth manifolds.

(b) This statement follows immediately from (a), together with the rather straightforward Exercise 19.33.

(3) (a) Let X be a topological manifold. Since $p\colon \widetilde{X} \to X$ is a local homeomorphism we obtain easily from Lemma 18.5 that $\partial\widetilde{X} = p^{-1}(\partial X)$. If X is in fact a smooth manifold, then the statement follows from the fact that now $p\colon \widetilde{X} \to X$ is a local diffeomorphism together with Lemma 19.27.

(b) This statement follows from (a) together with Lemma 48.3.

(4) This statement follows immediately from the fact that p is a local diffeomorphism together with the Basics-of-Orientations Lemma 25.13 (6).

(5) The proof of this statement is almost identical to the proof of (2). We leave it to the reader to fill in the details. ∎

49.3. **The orientation covering of a non-orientable smooth manifold.** We saw that many of our examples of non-orientable smooth manifolds admit a 2-fold covering such that the covering space is an orientable smooth manifold:

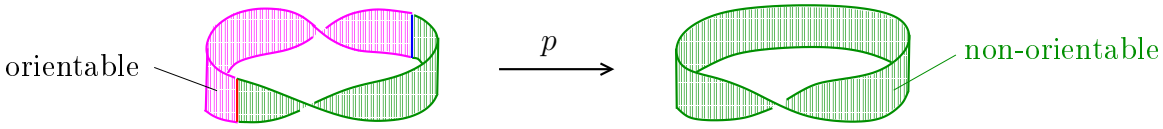
Examples.

(1) By the discussion on page 1069 the projection $p\colon S^2 \to \mathbb{R}\mathrm{P}^2 = S^2/\{\pm 1\}$ is a 2-fold covering. This shows that the real projective plane is covered by the sphere.

(2) On page 1070 we had considered the map

$$\begin{array}{rcl} p\colon \overbrace{(\mathbb{R}\times[-1,1])/(x,y)\sim(x+1,y)}^{\text{annulus}} & \to & \overbrace{(\mathbb{R}\times[-1,1])/(x,y)\sim(x+\frac{1}{2},1-y)}^{\text{Möbius band}} \\ [(x,y)] & \to & [(x,y)] \end{array}$$

and we remarked that this is a 2-fold covering. Thus the non-orientable Möbius band admits a 2-fold covering such that the covering space is orientable.



(3) Almost the same map as in (2) also shows that there exists a 2-fold covering map from the torus to the Klein bottle.

In this section we will see that every non-orientable smooth manifold admits a 2-fold covering such that the covering space is orientable. Furthermore we will see that this 2-fold covering is in a suitable sense unique.
First let us introduce and recall some notation which we will use throughout this section.

Notation.

(1) Given an orientation $\mathcal{O}$ for a real vector space we denote by $-\mathcal{O}$ the unique other orientation.

(2) Given an isomorphism $\varphi\colon V \to W$ between vector spaces and given an orientation $\mathcal{O}$ on V we denote by $\varphi(\mathcal{O})$ the corresponding orientation on W.

Proposition 49.4. Let M be a smooth manifold. We consider the set

$$\widetilde{M} \;=\; \{(Q,\mathcal{O}) \mid Q\in M \text{ and } \mathcal{O} \text{ is an orientation of } \mathrm{T}_Q M\}$$

together with the map

$$\begin{array}{rcl} p = p_M\colon \widetilde{M} & \to & M \\ (Q,\mathcal{O}) & \mapsto & Q. \end{array}$$

(1) There exists a unique topology on $\widetilde{M}$ such that $p_M\colon \widetilde{M} \to M$ is a 2-fold covering.

(2) There exists a unique smooth structure on the topological space $\widetilde{M}$ such that the projection $p_M\colon \widetilde{M} \to M$ is a covering of smooth manifolds.

(3) Let $\widetilde{x} = (x,\mathcal{O}) \in \widetilde{M}$. By (2) the map $p\colon \widetilde{M} \to M$ is a local diffeomorphism, which implies that $\mathrm{D}p_{\widetilde{x}}\colon \mathrm{T}_{\widetilde{x}}\widetilde{M} \to \mathrm{T}_x M$ is an isomorphism. Now we equip $\mathrm{T}_{\widetilde{x}}\widetilde{M}$ with the orientation $(\mathrm{D}p_{\widetilde{x}})^{-1}(\mathcal{O})$. These orientations of the tangent spaces of $\widetilde{M}$ define an orientation of the smooth manifold $\widetilde{M}$.

Definition. Let M be a smooth manifold. The covering $p_M\colon \widetilde{M} \to M$ from Proposition 49.4 is called the orientation covering of M. We always equip $\widetilde{M}$ with the topology, smooth structure and orientation from Proposition 49.4.

Remark. In the Orientation Covering-Proposition 105.15 we will prove an analogue of Proposition 49.4 for topological manifolds.

Proof. Let M be an n-dimensional smooth manifold.

(1) Let $\Phi\colon U \to V$ be a chart of the smooth atlas of M and let $\epsilon \in \{-,+\}$. We introduce the following notation:
- We denote by $i\colon U \to M$ the inclusion map.
- For $x \in V \subset \mathbb{R}^n$ we denote by $\mathcal{O}_{V,x}$ the standard orientation of $\mathrm{T}_x V$ that we introduced on page 677.

We set
$$B_\epsilon(\Phi) \;:=\; \{(Q, \underbrace{\epsilon \cdot (\mathrm{D}i_Q \circ (\mathrm{D}\Phi_Q)^{-1})(\mathcal{O}_{V,\Phi(Q)})}_{\text{orientation of } \mathrm{T}_Q M}) \mid Q \in U\} \;\subset\; \widetilde{M}.$$

It is straightforward to show that the collection of these sets $B_\pm(\Phi)$ satisfies the basis properties (B1) and (B2) from page 132. Now we endow $\widetilde{M}$ with the topology generated by the sets $B_\pm(\Phi)$.

Claim 1. The projection $p\colon \widetilde{M} \to M$ is continuous.

Proof. Note that for each chart $\Phi\colon U \to V$ of M we have $p^{-1}(U) = B_-(\Phi) \cup B_+(\Phi)$. This shows that $p^{-1}(U)$ is in fact an open subset of $\widetilde{M}$. Next note that it follows from the Basis Characterization Lemma 1.32 that the domains of the charts for M have the basis property for the topological space M. Combining these two observations with the Subbasis-Continuity Proposition 2.9 we see that p is indeed continuous. ⊞

Claim 2. The projection $p\colon \widetilde{M} \to M$ is a 2-fold covering.

Proof. Let $Q \in M$. We pick a chart $\Phi\colon U \to V$ around Q such that U is connected. Note that $p^{-1}(U) = B_-(\Phi) \cup B_+(\Phi)$. It follows from the Basics-of-Orientations Lemma 25.13 (1), applied to the connected smooth manifold U, that $B_-(\Phi)$ and $B_+(\Phi)$ are disjoint. Evidently the maps $p\colon B_\pm(\Phi) \to U$ are bijections.

Let $\epsilon \in \{-,+\}$. Using Lemma 2.11 one sees easily that the map $p\colon B_\epsilon(\Phi) \to U$ is open. We already know that the map $p\colon B_\epsilon(\Phi) \to U$ is continuous. Thus it follows from the Open -Injective Map Lemma 2.16 that the map $p\colon B_\epsilon(\Phi) \to U$ is a homeomorphism.

In summary we have shown that U is uniformly covered by the two open sets $B_\pm(\Phi)$. This implies that p is a 2-fold covering. ⊞

Finally note that the uniqueness of the topological structure on $\widetilde{M}$ with the desired properties is an immediate consequence of the fairly elementary Exercise 2.3.

(2) This statement is an immediate consequence of (1) and the Manifold Covering Proposition 49.2 (2).

(3) It follows immediately from the definition of the smooth structure on $\widetilde{M}$, see the Manifold Covering Proposition 49.2 (2), that a smooth atlas for $\widetilde{M}$ is given by the maps $\Phi \circ p\colon B_\epsilon(\Phi) \to V$, where Φ runs over all smooth charts $\Phi\colon U \to V$ of M with

U connected, and where $\epsilon \in \{-,+\}$. By the Orientations-via-Atlas Lemma 25.8 it suffices to consider the continuity of the orientation with respect to the above atlas. So let $\Phi\colon U \to V$ be a chart of M such that U is connected and let $\epsilon \in \{-,+\}$. We denote by $\Phi_\epsilon = \Phi \circ p\colon B_\pm(\Phi) \to V$ the corresponding chart of $\widetilde{M}$. But then by construction we know that Φ_ϵ is orientation-preserving for all points if $\epsilon = +$ and it is orientation-reversing for all points if $\epsilon = -$. ■

The following proposition summarizes the key properties of the orientation coverings.

Proposition 49.5. (Orientation Covering-Proposition)

(1) Let $f\colon M \to N$ be a local diffeomorphism between smooth manifolds. The map

$$\begin{array}{rcl} \widetilde{f}\colon \widetilde{M} & \to & \widetilde{N} \\ (P, \underset{\uparrow}{\mathcal{O}}) & \mapsto & (f(P), \mathrm{D}f_P(\underset{\uparrow}{\mathcal{O}})). \\ \text{orientation of } \mathrm{T}_P M & & \text{orientation of } \mathrm{T}_{f(P)} N \end{array}$$

is the unique orientation-preserving local diffeomorphism $\widetilde{M} \to \widetilde{N}$ that has the property that the following diagram commutes:[511]

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{f}} & \widetilde{N} \\ {\scriptstyle p_M}\downarrow & & \downarrow{\scriptstyle p_N} \\ M & \xrightarrow{f} & N. \end{array}$$

(2) Let M be a connected smooth manifold.

(a) If M is oriented, then there exists a unique orientation-preserving diffeomorphism $\Theta\colon \widetilde{M} \to M \sqcup -M$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow[\cong]{\Theta} & M \sqcup -M \\ & {\scriptstyle p_M}\searrow \quad \swarrow{\scriptstyle q} & \\ & M & \end{array}$$

where q is the obvious map given by the identity on each component of $M \sqcup -M$.

(b) If M is non-orientable, then $\widetilde{M}$ is connected.

(3) Let M be a smooth manifold. The map

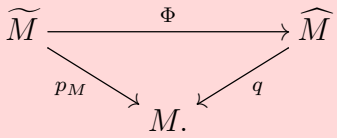
$$\begin{array}{rcl} \Xi_M\colon \widetilde{M} & \to & \widetilde{M} \\ (Q, \mathcal{O}) & \mapsto & (Q, -\mathcal{O}) \end{array}$$

is the unique orientation-reversing self-diffeomorphism $\Xi_M\colon \widetilde{M} \to \widetilde{M}$ which has the property that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow[\cong]{\Xi_M} & \widetilde{M} \\ & {\scriptstyle p_M}\searrow \quad \swarrow{\scriptstyle p_M} & \\ & M. & \end{array}$$

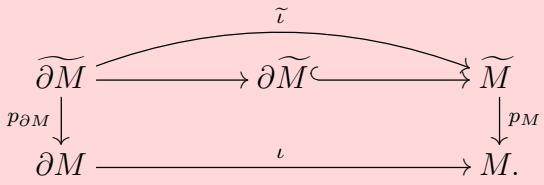
Furthermore, given a local diffeomorphism $f\colon M \to N$ we have $\widetilde{f} \circ \Xi_M = \Xi_N \circ \widetilde{f}$.

(4) Let M be a connected non-orientable smooth manifold. Suppose $q\colon \widehat{M} \to M$ is a 2-fold covering of smooth manifolds such that $\widehat{M}$ is a connected orientable smooth manifold. There exists a unique diffeomorphism $\Phi\colon \widetilde{M} \to \widehat{M}$ such that the following diagram commutes:



If $\widehat{M}$ is oriented, then there exists a unique orientation-preserving diffeomorphism $\Phi\colon \widetilde{M} \to \widehat{M}$ that makes the above diagram commute.

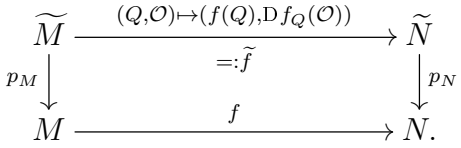
(5) Let M be a smooth manifold. We denote by $\iota\colon \partial M \to M$ the inclusion. There exists a unique natural diffeomorphism $\partial\widetilde{M} \to \widetilde{\partial M}$ such that the following diagram commutes:



Remark. In the Orientation Covering-Proposition 105.15 we will prove an analogue of the Orientation Covering-Proposition 49.5 for topological manifolds.

Sketch of proof.

(1) It is clear that the following diagram commutes:



From the fact that p_M, p_N and f are local diffeomorphisms it follows easily that the top map $\widetilde{f}\colon \widetilde{M} \to \widetilde{N}$ is also a local diffeomorphism. It follows immediately from the definitions of the orientations on $\widetilde{M}$ and $\widetilde{N}$ that $\widetilde{f}$ is indeed orientation-preserving. The proof of the fact that $\widetilde{f}$ is the unique map with the stated properties follows easily from Lemma 48.4 (2).

(2) Let M be a connected smooth manifold.

(a) We suppose that M is oriented. One can easily verify that the map

$$\begin{array}{rcl} \Theta\colon \widetilde{M} & \to & M \sqcup -M \\ (P,\mathcal{O}) & \mapsto & \begin{cases} P \in M, & \text{if } \mathcal{O} \text{ is a positive orientation,} \\ P \in -M, & \text{if } \mathcal{O} \text{ is a negative orientation} \end{cases} \end{array}$$

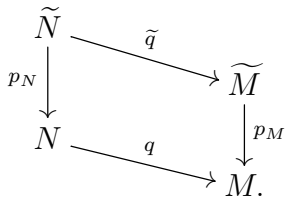
is an orientation-preserving diffeomorphism $\Theta\colon \widetilde{M} \to M \sqcup -M$ that has the desired properties. Uniqueness can once again be deduced from Lemma 48.4 (2).

[511] We leave it to the reader to express this statement in categorical language.

(b) Now we suppose that M is non-orientable. We need to show that $\widetilde{M}$ is connected. If $\widetilde{M}$ was not connected, then $\widetilde{M}$ would consist of at least two components $\{\widetilde{M}_i\}_{i\in I}$. Since M is connected it follows from Exercise 48.22 that the restriction of each $p\colon \widetilde{M} \to M$ to $p_i\colon \widetilde{M}_i \to M$ is also a covering of smooth manifold. Clearly $2 = [\widetilde{M} : M] = \sum_{i\in I}[\widetilde{M}_i : M]$. It follows that $\#I = 2$ and that each $p_i\colon \widetilde{M}_i \to M$ is a covering of degree one, i.e. each p_i is a diffeomorphism. But that is not possible since M was assumed to be non-orientable whereas $\widetilde{M}_i$ is orientable.

(3) It is straightforward to show that Ξ_M is a local diffeomorphism. It is basically clear that it is orientation-preserving. The uniqueness follows easily from Lemma 48.4 (2). Finally, if we are given given a local diffeomorphism $f\colon M \to N$, then it follows immediately from the definitions that we have $\widetilde{f} \circ \Xi_M = \Xi_N \circ \widetilde{f}$.

(4) Let M be a connected non-orientable smooth manifold. Suppose we are given a 2-fold covering $q\colon N \to M$ of smooth manifolds where N is a connected orientable smooth manifold and where q is a local diffeomorphism. Since q is a local diffeomorphism we can consider the corresponding local diffeomorphism $\widetilde{q}\colon \widetilde{N} \to \widetilde{M}$ from (1) which makes the following diagram commute



It follows easily from the fact that q, p_M and p_N are 2-fold coverings of smooth manifolds that $\widetilde{q}\colon \widetilde{N} \to \widetilde{M}$ is also a 2-fold covering of smooth manifolds. From the fact that N is connected and orientable together with (2) we obtain that $\widetilde{N}$ consists of two components $\widetilde{N}_1$ and $\widetilde{N}_2$. As in (2) we see that the restriction of p_N to $\widetilde{N}_1$ and the restriction of $\widetilde{q}$ to $\widetilde{N}_2$ are 1-fold coverings of smooth manifolds. In other words $p_N\colon \widetilde{N}_1 \to N$ is in fact a diffeomorphism. One can now easily verify that $\widetilde{q} \circ (p_N|_{\widetilde{N}_1})^{-1}\colon N \to \widetilde{M}$ has the desired properties.

If N is oriented, then, after possibly composing the diffeomorphism $N \to \widetilde{M}$ with the orientation-reversing diffeomorphism $\Xi\colon \widetilde{M} \to \widetilde{M}$ from (3), we can arrange that the diffeomorphism $N \to \widetilde{M}$ is orientation-preserving. Once again one can deduce from Lemma 48.4 (2) that the orientation-preserving diffeomorphism $\Phi\colon N \to \widetilde{M}$, that commutes with the projections to M, is unique.

(5) We will not make use of this statement. Thus we have no qualms with delegating the proof of this statement to Exercise 49.10. ■

49.4. **The orientation character.** We start out with an elementary lemma in group theory.

Lemma 49.6.

(1) Let π be a group. Any subgroup of index 2 is normal.

(2) If G is a group with two elements, then there exists a unique isomorphism $G \to \mathbb{Z}_2$.

Proof. We outsource the proof to Exercise 49.1. ■

Convention. Let G be a group with two elements. It follows from Lemma 49.6 (2) that we can safely make the identification $G = \mathbb{Z}_2$.

Proposition 49.7. Let M be a non-orientable connected smooth manifold and let $x_0 \in M$. We denote by $p\colon \widetilde{M} \to M$ the orientation cover. Finally we pick a point $\widetilde{x}_0 \in \widetilde{M}$ with $p(\widetilde{x}_0) = x_0$.

(1) The subgroup $p_*(\pi_1(\widetilde{M}, \widetilde{x}_0)) \subset \pi_1(M, x_0)$ is a normal subgroup of index 2.
(2) There exists a unique isomorphism $\pi_1(M, x_0)/p_*(\pi_1(\widetilde{M}, \widetilde{x}_0)) \to \mathbb{Z}_2$.
(3) The projection

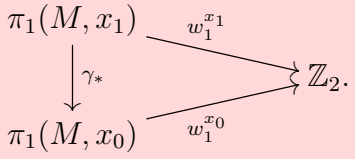
$$w_1^{x_0}\colon \pi_1(M, x_0) \;\to\; \pi_1(M, x_0)/p_*(\pi_1(\widetilde{M}, \widetilde{x}_0)) \;\underset{\uparrow}{=}\; \mathbb{Z}_2$$

(see above convention)

is an epimorphism that does not depend on the choice of $\widetilde{x}_0$.
(4) Let $x_1 \in M$ be some other point and let $\gamma\colon [0,1] \to M$ be a path from x_0 to x_1. We denote by

$$\begin{array}{rcl} \gamma_*\colon \pi_1(M, x_1) & \to & \pi_1(M, x_0) \\ [\varphi] & \mapsto & [\gamma * \varphi * \overline{\gamma}] \end{array}$$

the isomorphism that we introduced in the Change-of-Base Point Proposition 47.10. The following diagram commutes:



Proof. Let M be a non-orientable connected smooth manifold, let $x_0 \in M$ and let $\widetilde{x}_0 \in \widetilde{M}$ with $p(\widetilde{x}_0) = x_0$.

(1) By Proposition 49.4 the orientation covering $p\colon \widetilde{M} \to M$ has degree 2. Furthermore, since M is non-orientable we know by the Orientation Covering-Proposition 49.5 that $\widetilde{M}$ is connected Therefore it follows from Lemma 48.15 that $p_*(\pi_1(\widetilde{M}))$ is an index 2 subgroup of $\pi_1(M)$. By Lemma 49.6 (1) we know that every index 2 subgroup of a group is normal. In particular $p_*(\pi_1(\widetilde{M}))$ is a normal subgroup of $\pi_1(M)$ of index 2.
(2) This statement is an immediate consequence of Lemma 49.6 (2).
(3) By the above it remains to show that the epimorphism $w_1^{x_0}$ does not depend on the choice of $\widetilde{x}_0$. So let $\widehat{x}_0$ be the unique other point in $p^{-1}(x_0)$. By the Orientation Covering-Proposition 49.5 (3) there exists a homeomorphism $\Xi\colon \widetilde{M} \to \widetilde{M}$ with $\Xi(\widetilde{x}_0) = \widehat{x}_0$ and with $p = p \circ \Xi$. We see that

$$\begin{array}{ccccccc} & & \overset{\text{since } p = p\circ\Xi}{\downarrow} & & \overset{\text{functoriality of } \pi_1}{\downarrow} & & \\ p_*(\pi_1(\widetilde{M}, \widetilde{x}_0)) & \overset{}{=} & (p \circ \Xi)_*(\pi_1(\widetilde{M}, \widetilde{x}_0)) & \overset{}{=} & p_*(\Xi_*(\pi_1(\widetilde{M}, \widetilde{x}_0))) & & \\ & & & \underset{\uparrow}{=} & p_*(\pi_1(\widetilde{M}, \Xi(\widetilde{x}_0))) & \underset{\uparrow}{=} & p_*(\pi_1(\widetilde{M}, \widehat{x}_0)). \\ & & & \text{since } \Xi \text{ homeomorphism and since } \pi_1 \text{ is functorial} & & \text{since } \Xi(\widetilde{x}_0) = \widehat{x}_0 & \end{array}$$

(4) Let $\gamma\colon [0,1] \to M$ be a path from x_0 to x_1. By Proposition 48.11 there exists a unique lift $\widetilde{\gamma}\colon [0,1] \to M$ of γ to the starting point $\widetilde{x}_0$. By the Change-of-Base Point

Proposition 47.10 (5) the following diagram commutes:

$$\begin{array}{ccc} \pi_1(\widetilde{M}, \gamma(1)) & \xrightarrow{\quad p_* \quad} & \pi_1(M, x_1) \\ \big\downarrow{\scriptstyle \gamma_*} & & \big\downarrow{\scriptstyle \gamma_*} \\ \pi_1(\widetilde{M}, \widetilde{x}_0) & \xrightarrow{\quad p_* \quad} & \pi_1(M, x_0). \end{array}$$

But this implies immediately the promised statement. ■

Definition. Let M be a connected smooth manifold and let $x_0 \in M$.

(1) If M is non-orientable, then we refer to the epimorphism

$$w_1 \colon \pi_1(M, x_0) \ \to \ \pi_1(M, x_0)/p_*(\pi_1(\widetilde{M}, \widetilde{x}_0)) = \mathbb{Z}_2$$

from Proposition 49.7 as the orientation character of M.

(2) If M is orientable, then we define the orientation character of M to be the trivial homomorphism $w_1 \colon \pi_1(M, x_0) \to \mathbb{Z}_2$.

It is worth formulating the following immediate corollary to Proposition 49.7.

Corollary 49.8. (Non-Orientable $\mathbb{Z}_2$-Epimorphism Corollary)

(1) If M is a non-orientable connected non-empty smooth manifold, then there exists an epimorphism $\pi_1(M) \to \mathbb{Z}_2$.

(2) Every simply connected smooth manifold is orientable.

Definition. Let M be a connected smooth manifold and let $\gamma \colon [0,1] \to M$ be a loop in a point x_0. We say γ is orientation-preserving if $w_1^{x_0}(\gamma) = 0$ otherwise we say that γ is orientation-reversing.

Remark. On page 881 we already introduced the notion of orientation-preserving and orientation-reversing loops in a more restrictive setup. It follows immediately from Proposition 49.10 below that the two definitions agree on the type of loops that are covered by both definitions.

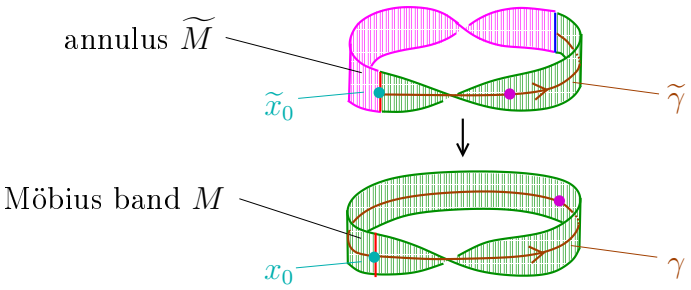
The following lemma gives a practical criterion for determining whether or not a loop is orientation-preserving.

Lemma 49.9. Let M be a connected smooth manifold. As before we denote by $p \colon \widetilde{M} \to M$ the orientation covering provided by Proposition 49.4. Let $x_0 \in M$ be a point and let $\gamma \colon [0,1] \to M$ be a loop in x_0. We pick $\widetilde{x}_0 \in p^{-1}(x_0)$ and we denote by $\widetilde{\gamma} \colon [0,1] \to \widetilde{M}$ the unique lift of γ, provided by Proposition 48.11, to the starting point x_0. Then

$$\begin{array}{rcl} \gamma \text{ is } \textit{orientation-preserving} & \iff & \widetilde{\gamma}(1) = \widetilde{\gamma}(0) \\ \gamma \text{ is } \textit{orientation-reversing} & \iff & \widetilde{\gamma}(1) \neq \widetilde{\gamma}(0). \end{array}$$

Proof. This lemma follows immediately from Lemma 48.15. ■

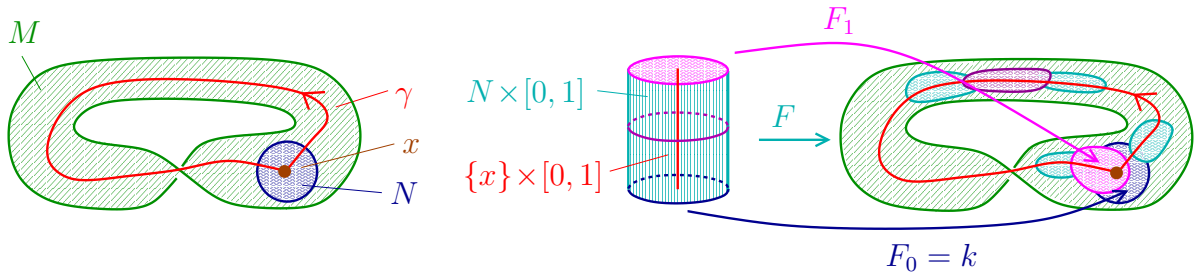
Example. Let M be the Möbius band and let $\widetilde{M}$ be the annulus. We consider the 2-fold covering $p \colon \widetilde{M} \to M$ that is defined on page 1070. Note that by the Orientation Covering-Proposition 49.5 we know that this covering is just an incarnation of the orientation covering. We pick a point in M and we denote by γ a loop that goes "once around M". The lift of γ is not a loop, hence it follows from Lemma 49.9 that γ is orientation-reversing.



The following proposition hopefully justifies the names "orientation-preserving" and "orientation-reversing" for loops.

Proposition 49.10. Let M be a smooth manifold, let $\gamma\colon [0,1] \to M$ be a smooth loop at a point $x \in M$ and let N be a codimension-zero submanifold of M with $x \in N \setminus \partial N$. We denote by $k\colon N \to M$ the inclusion map. Finally let $F\colon N \times [0,1] \to M$ be a smooth isotopy with $F_0 = k$ such that for each $t \in [0,1]$ we have $F_t(x) = \gamma(t)$. Then[512]

$$(\mathrm{D}k_x)^{-1} \circ \mathrm{D}(F_1)_x\colon \mathrm{T}_xN \to \mathrm{T}_xN \text{ is orientation-preserving} \quad \Longleftrightarrow \quad \gamma \text{ is orientation-preserving.}$$



The proof of Proposition 49.10 rests squarely on the shoulders of the following rather technical lemma.

Auxiliary Lemma 49.11. Let M be a smooth manifold. We denote by $p\colon \widetilde{M} \to M$ the orientation cover. Let $\gamma\colon [0,1] \to M$ be a smooth loop at some point $x \in M$ and let N be a codimension-zero submanifold of M with $x \in N \setminus \partial N$. We denote by $k\colon N \to M$ the inclusion map. Let $F\colon N \times [0,1] \to M$ be a smooth isotopy with $F_0 = k$ such that for each $t \in [0,1]$ we have $F_t(x) = \gamma(t)$. We pick an orientation $\mathcal{O}_x$ of T_xN and we denote by $\widetilde{\gamma}\colon [0,1] \to \widetilde{M}$ the unique lift of $\gamma\colon [0,1] \to M$ to the starting point $(x, \mathrm{D}\,k_x(\mathcal{O}_x))$. Then for each $t \in [0,1]$ the map $(\mathrm{D}p_{\widetilde{\gamma}(t)})^{-1} \circ \mathrm{D}(F_t)_x\colon \mathrm{T}_xN \to \mathrm{T}_{\widetilde{\gamma}(t)}\widetilde{M}$ is orientation-preserving.

Proof of Auxiliary Lemma 49.11. We consider the following two subsets of $[0,1]$:

$$\begin{aligned} A &:= \{t \in [0,1] \,|\, (\mathrm{D}p_{\widetilde{\gamma}(t)})^{-1} \circ \mathrm{D}(F_t)_x\colon \mathrm{T}_xN \to \mathrm{T}_{\widetilde{\gamma}(t)}\widetilde{M} \text{ is orientation-preserving}\} \\ B &:= \{t \in [0,1] \,|\, (\mathrm{D}p_{\widetilde{\gamma}(t)})^{-1} \circ \mathrm{D}(F_t)_x\colon \mathrm{T}_xN \to \mathrm{T}_{\widetilde{\gamma}(t)}\widetilde{M} \text{ is orientation-reversing}\}. \end{aligned}$$

[512] On page 674 we defined what it means for an *automorphism* of a finite dimensional real vector space to be orientation-preserving.

We want to show that $A = [0,1]$. By the choice of the lift $\widetilde{\gamma}$ and since $F_0 = k$ we know that $0 \in A$. By the Interval-Connectedness Proposition 2.22 we know that the interval $[0,1]$ is connected. Thus it remains to show that A and B are both open subsets of $[0,1]$.

In the following we will show that A is open. The proof that B is open is basically the same. Thus let $t \in A$. Since $p\colon \widetilde{M} \to M$ is a local diffeomorphism we can pick an open neighborhood $\widetilde{U}$ of $\widetilde{\gamma}(t) \in \widetilde{M}$ such that $p(U)$ is an open neighborhood of $\gamma(t)$ and such that the projection $p\colon \widetilde{U} \to U := p(\widetilde{U})$ is a diffeomorphism.

(1) We denote by $q\colon U \to \widetilde{U}$ the inverse of $p\colon \widetilde{U} \to U$.
(2) We denote by $j\colon U \to M$ and $\widetilde{j}\colon \widetilde{U} \to \widetilde{M}$ the inclusion maps.
(3) We equip $\widetilde{U}$ with the standard codimension-zero submanifold orientation from page 684. In other words, we equip $\widetilde{U}$ with the unique orientation such that for each $y \in \widetilde{U}$ the map $\mathrm{D}\widetilde{j}_y\colon \mathrm{T}_y\widetilde{U} \to \mathrm{T}_y\widetilde{M}$ is orientation-preserving.

Since $F\colon N \times [0,1] \to M$ is continuous and since U is an open subset of M we know that $F^{-1}(U)$ is an open neighborhood of (x,t). Using the fact that $x \notin \partial N$ we see that we can pick a product neighborhood $W \times [a,b]$ of (x,t) with the following properties:

(1) We have $(x,t) \in W \times [a,b] \subset F^{-1}(U)$.
(2) $W \subset N \setminus \partial N$ is an open subset that is diffeomorphic to an open ball.
(3) $[a,b] \subset [0,1]$ is a neighborhood of $t \in [0,1]$.

Next we introduce a few more players:

- We denote by $\iota\colon W \to N$ the inclusion map.
- Since W is diffeomorphic to an open ball we know that W is orientable. We equip it with the unique orientation such that $\mathrm{D}\iota_x\colon \mathrm{T}_xW \to \mathrm{T}_xN$ is orientation-preserving.
- We denote by $F'\colon W \times [a,b] \to U$ the restriction of $F\colon N \times [0,1] \to M$ to $W \times [a,b]$.
- We consider the map

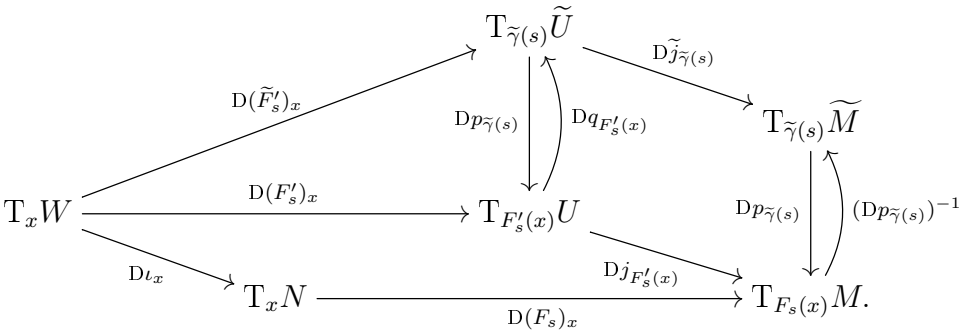
$$\begin{aligned} \widetilde{F}'\colon W \times [a,b] &\to \widetilde{U} \\ (w,s) &\mapsto q(F'(w,s)) = (q \circ F'_s)(w). \end{aligned}$$

Since q is a diffeomorphism we see that $\widetilde{F}'$ is a smooth isotopy.

Claim. For $s \in [a,b]$ we have

$$\text{the map } \mathrm{D}\widetilde{F}'_s\colon \mathrm{T}_xW \to \mathrm{T}_{\widetilde{\gamma}(s)}\widetilde{U} \text{ is orientation-preserving} \quad \Longleftrightarrow \quad s \in A.$$

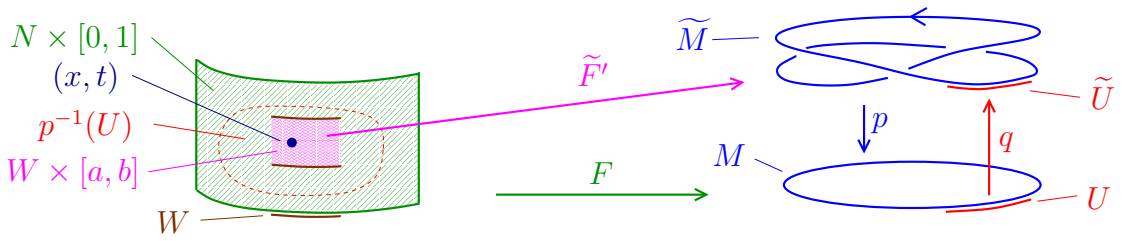
Proof. To prove the claim we consider the following diagram:



Note that the above diagram commutes by the functoriality of the differential and the fact that $\widetilde{F}'_s = q \circ F'_s \colon W \to \widetilde{U}$, $j \circ F'_s = F_s \circ \iota$, $p \circ \widetilde{j} = j \circ p \colon \widetilde{U} \to M$ and $q = p^{-1} \colon \widetilde{U} \to U$. It follows from the commutativity of the diagram that

$$\underbrace{\mathrm{D}\widetilde{j}_{\widetilde{\gamma}(s)}}_{\substack{\text{orientation-preserving}\\ \text{by definition of the}\\ \text{orientation on } \widetilde{U}}} \circ \mathrm{D}\widetilde{F}'_s \circ \underbrace{(\mathrm{D}\iota_x)^{-1}}_{\substack{\text{orientation-preserving}\\ \text{by definition of the}\\ \text{orientation of } \mathrm{T}_x W}} = \underbrace{(\mathrm{D}p_{\widetilde{\gamma}(s)})^{-1} \circ \mathrm{D}(F_s)_x}_{\substack{\text{orientation-preserving}\\ \text{precisely if } s \in A}} \colon \mathrm{T}_x N \to \mathrm{T}_{\widetilde{\gamma}(t)}\widetilde{M}.$$

The claim follows immediately from this observation. ⊞



Now we can easily conclude the proof of the lemma.

(1) Since $t \in A$ we obtain from the claim that $\widetilde{F}'_t \colon W \to \widetilde{U}$ is orientation-preserving at x.
(2) It follows from the Basics-of-Orientations Lemma 25.13 that $\widetilde{F}'_t \colon W \to \widetilde{U}$ is orientation-preserving.
(3) It follows from the Smooth Isotopy-Orientation Proposition 34.2 that $\widetilde{F}'_s \colon W \to \widetilde{U}$ is orientation-preserving for all $s \in [a,b]$.
(4) It follows from (3) that the differential $\mathrm{D}\widetilde{F}'_s \colon \mathrm{T}_x W \to \mathrm{T}_{\widetilde{\gamma}(s)}\widetilde{U}$ is orientation-preserving for all $s \in [a,b]$.
(5) It follows from (4) and the claim that $[a,b] \subset A$.
(6) Since $[a,b]$ is a neighborhood of $t \in [0,1]$ we now deduce from the JH-Lemma 1.7 that A is indeed an open subset of $[0,1]$. ■

Proof of Proposition 49.10. Let M be a smooth manifold. Furthermore let $\gamma \colon [0,1] \to M$ be a smooth loop at a point $x \in M$ and let N be a codimension-zero submanifold of M with $x \in N \setminus \partial N$. We denote by $k \colon N \to M$ the inclusion map. Finally let $F \colon N \times [0,1] \to M$ be a smooth isotopy with $F_0 = k$ such that for each $t \in [0,1]$ we have $F_t(x) = \gamma(t)$. We set $\Phi = F_1 \colon N \to M$ and we set

$$\epsilon := \begin{cases} +1, & \text{if } \gamma \text{ is orientation-preserving,} \\ -1, & \text{if } \gamma \text{ is orientation-reversing.} \end{cases}$$

We need to show that

$$(\mathrm{D}k_x)^{-1} \circ \mathrm{D}\Phi_x \colon \mathrm{T}_x N \to \mathrm{T}_x N \text{ is orientation-preserving} \iff \epsilon = 1.$$

We pick an orientation $\mathcal{O}_x$ of $\mathrm{T}_x N$ and we denote by $\widetilde{\gamma} \colon [0,1] \to \widetilde{M}$ the unique lift of $\gamma \colon [0,1] \to M$ to the starting point $(x, \mathrm{D}\,k_x(\mathcal{O}_x))$. Now we see that

$$\begin{aligned} \mathrm{D}\Phi_x(\mathcal{O}_x) &= \mathrm{D}p_{(x,\epsilon\cdot \mathrm{D}k_x(\mathcal{O}_x))} \circ (\mathrm{D}p_{(x,\epsilon\cdot \mathrm{D}k_x(\mathcal{O}_x))})^{-1} \circ \mathrm{D}\Phi_x(\mathcal{O}_x) \\ &\underset{\uparrow}{=} \mathrm{D}p_{(x,\epsilon\cdot \mathrm{D}k_x(\mathcal{O}_x))}(\text{orientation of } \widetilde{M} \text{ at } (x, \epsilon \cdot \mathrm{D}k_x(\mathcal{O}_x))) \underset{\uparrow}{=} \epsilon \cdot \mathrm{D}k_x(\mathcal{O}_x). \end{aligned}$$

by Lemma 49.11 (1) applied to $t = 1$, note that $\gamma(1) = (x, \epsilon \cdot \mathrm{D}k_x(\mathcal{O}_x))$ and $\Phi = F_1$

by Proposition 49.4

But this is precisely what we needed to show. ■

49.5. **Orientation flipping diffeotopies II.** In this section we will give a new proof of the Orientation-Flipping Proposition 37.4. Before we can restate and reprove said proposition let us recall the following definition from page 822.

Definition. Let X be a topological space. We say that a homotopy $F\colon X \times [0,1] \to X$ has compact support if there exists a compact subset $K \subset X$ such that $F(x,t) = x$ for all $x \notin K$ and all $t \in [0,1]$.

Now we can recall the Orientation-Flipping Proposition 37.4.

Proposition 37.4. (Orientation-Flipping Proposition) Let M be a non-orientable connected smooth manifold and let $X, Y \subset M \setminus \partial M$ be two disjoint finite subsets. There exists a diffeotopy $F\colon M \times [0,1] \to M$ with the following properties:

(1) $F_0 = \mathrm{id}$.
(2) F has compact support.
(3) F is a diffeotopy rel $\partial M \cup Y$.
(4) $\Phi := F_1$ is the identity on $X \cup Y$.
(5) The map $\Phi := F_1$ has the following properties:
  (a) For each $x \in X$ the map $\mathrm{D}\Phi_x\colon \mathrm{T}_x M \to \mathrm{T}_x M$ is orientation-reversing.
  (b) For each $y \in Y$ the map $\mathrm{D}\Phi_y\colon \mathrm{T}_y M \to \mathrm{T}_y M$ is orientation-preserving.

Remark. Our first proof of the Orientation-Flipping Proposition 37.4 rested on the Isotopy Extension Theorem 36.1. Our new proof only requires the existence of the orientation cover. Our new proof also has the big advantage that it can be generalized to topological manifolds. We refer to page 2342 for a few more details.

Our new proof of the Orientation-Flipping Proposition 37.4 is in many ways similar to the previous proof. In particular it rests on elementary two auxiliary lemmas which we now recall.

Auxiliary Lemma 37.5. Let X be a topological manifold, let $A \subset X$ be a closed subset and let $F\colon X \times [0,1] \to X$ be a homotopy rel A. If F has compact support, then there exists an open neighborhood U of A such that F is a homotopy rel U.

Auxiliary Lemma 37.6. Let M be a smooth manifold and let U be an open subset of M. Furthermore let $G\colon U \times [0,1] \to U$ be a diffeotopy with compact support K. Then

$$\begin{array}{rcl} \widetilde{G}\colon M \times [0,1] & \to & M \\ (x,t) & \mapsto & \begin{cases} G(x,t), & \text{if } x \in K, \\ x, & \text{if } x \notin K \end{cases} \end{array}$$

is a diffeotopy of M.

Again all the work for the proof of the Orientation-Flipping Proposition 37.4 is now done in the proof of the following auxiliary lemma, which deals with the case that $\#X = 1$.

Auxiliary Lemma 37.7. Let M be a connected *non-orientable* smooth manifold. Furthermore let $Y \subset M$ be a finite set and let $x \in M \setminus \partial M$ be a point with $x \notin Y$. There exists a diffeotopy $F\colon M \times [0,1] \to M$ with the following properties:

(1) $F_0 = \mathrm{id}$.

(2) F has compact support.
(3) F is a diffeotopy rel $\partial M \cup Y$.
(4) $\Phi := F_1$ is the identity on $\{x\} \cup Y$.
(5) The map $\Phi := F_1$ has the following properties:
  (a) The map $\mathrm{D}\Phi_x \colon \mathrm{T}_x M \to \mathrm{T}_x M$ is orientation-reversing.
  (b) For each $y \in Y$ the map $\mathrm{D}\Phi_y \colon \mathrm{T}_y M \to \mathrm{T}_y M$ is orientation-preserving.

Proof. New proof of Lemma 37.7] Let M be a connected *non-orientable* smooth manifold. Note that by the Smooth 1-Dimensional Manifold Classification Theorem 22.7 we know that $\dim(M) \geq 2$. We denote by $p \colon \widetilde{M} \to M$ the orientation covering from page 1109. We recall that by Propositions 49.4 and 49.5 this covering has the following properties:

(O1) The map p is a local diffeomorphism. In particular for each $a \in \widetilde{M}$ the differential $\mathrm{D}p_a \colon \mathrm{T}_a\widetilde{M} \to \mathrm{T}_{p(a)}M$ is an isomorphism.
(O2) Our hypothesis that M is non-orientable implies that $\widetilde{M}$ is connected.

Now let $Y \subset M$ be a finite set and let $x \in M \setminus \partial M$ with $x \notin Y$.

(a) We pick an orientation $\mathcal{O}_x$ for $\mathrm{T}_x M$.
(b) We set $W := M \setminus Y$ and $\widetilde{W} := p^{-1}(M \setminus Y)$. Since Y is finite we see that Y and $p^{-1}(Y)$ are closed subsets. It follows from Lemmas 19.19 and 19.20 that W and $\widetilde{W}$ are smooth manifolds with $\partial W = \partial M$ and $\partial\widetilde{W} = \partial\widetilde{M}$.
(c) Since $p^{-1}(Y)$ is a finite set and since $\dim(\widetilde{W}) = \dim(\widetilde{M}) = \dim(M) \geq 2$ we know by the elementary Exercise 18.16 that $\widetilde{W}$ is also path-connected.
(d) It follows from the Smooth Path-Connectivity Proposition 19.29 together with the Push Away From Boundary Lemma 18.13 that there exists in fact a smooth path $\widetilde{\gamma} \colon [0,1] \to \widetilde{W} \setminus \partial\widetilde{W}$ from $(x, \mathcal{O}_x)$ to $(x, -\mathcal{O}_x)$.

By the Isotopy Extension Theorem 36.1 there exists a diffeotopy $F \colon W \times [0,1] \to W$ with the following properties:

(0) For each $t \in [0,1]$ we have $F_t(x) = (p \circ \widetilde{\gamma})(t)$.
(1) $F_0 = \mathrm{id}$.
(2) F has compact support.
(3) F is a diffeotopy rel ∂W.

Note that it follows from (2), together with Lemma 37.6, that we can extend F to a diffeotopy $F \colon M \times [0,1] \to M$ rel Y on all of M by setting $F(x,t) = x$ for $x \notin W$. Finally we set $\Phi := F_1 \colon M \to M$.

By design F has properties (1), (2) and (3). Furthermore note that by Lemma 37.5 we know that F is actually a diffeotopy rel an open neighborhood of Y. This implies that for each $y \in Y$ we have $\mathrm{D}\Phi_y = \mathrm{id}_{\mathrm{T}_y M}$. Thus (5b) is also satisfied. Since $(p \circ \widetilde{\gamma})(1) = x$ we also see that $\Phi(x) = x$. Together with the above discussion this shows that Φ is the identity on $\{x\} \cup Y$. We thus verified that (4) holds.

It remains to show that (5a) holds. This statement is fortunately taken care of by Proposition 49.10. Indeed, we just need to apply Proposition 49.10 to $N = M$ and the smooth loop $p \circ \widetilde{\gamma} \colon [0,1] \to M$, which is orientation-reversing by Lemma 49.9. ∎

Proof. New proof of the Orientation-Flipping Proposition 37.4] Let $x_1, \ldots, x_k$ be the elements of X. It follows from the ever useful Smooth Isotopy Transitivity Proposition 33.3

that it suffices to flip the signs of the points $x_1, \dots, x_k$ one-by-one. But this can be done by applying Lemma 37.7 altogether k times, namely to the points x_i and the sets $Y_i := Y \cup \{x_1, \dots, \widehat{x_i}, \dots, x_k\}$. ■

Exercises for Chapter 49.

Exercise 49.1.

(a) Let π be a group. Show that every index 2 subgroup of π is normal.
(b) Give an example of a group π together with a subgroup of index 3 that is not normal in π.
(c) Show that if G is a group with two elements, then there exists a *unique* isomorphism $G \to \mathbb{Z}_2$.
(d) What is the smallest $n \in \mathbb{N}$ such that there exists a group with n elements which does not admit a *unique* isomorphism $G \to \mathbb{Z}_n$?
(e) What is the smallest $n \in \mathbb{N}$ such that there exists a group G with n elements which does not admit an isomorphism $G \to \mathbb{Z}_n$?

Exercise 49.2. Let $f\colon S^1 \to S^1$ be a map with $f(1) = 1$. We denote by $p\colon \mathbb{R} \to S^1$ the covering that is given by $t \mapsto \exp(\mathrm{i}t)$.

(a) Show that there exists a unique map $\widetilde{f}\colon \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\widetilde{f}} & \mathbb{R} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ S^1 & \xrightarrow{f} & S^1. \end{array}$$

(b) Suppose that f is a diffeomorphism. Show that the map $\widetilde{f}$ from (a) is also a diffeomorphism.

Exercise 49.3. Let $f\colon M \to N$ be a smooth map between two smooth manifolds. We assume that f is a covering of topological spaces and we assume that f is a local diffeomorphism. Does it follow that f is a covering of smooth manifolds?

Exercise 49.4.

(a) We consider the following commutative diagram

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{f}} & \widetilde{N} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ M & \xrightarrow{f} & N \end{array}$$

of maps between smooth manifolds $\widetilde{M}$, M, $\widetilde{N}$ and N. We assume that p and q are covering maps of smooth manifolds and we assume that p and q have the same finite degree. Show that f is a diffeomorphism if and only if $\widetilde{f}$ is a diffeomorphism.
(b) Let M and N be smooth manifolds and let $f\colon M \to N$ be a diffeomorphism. We denote by $p\colon \widetilde{M} \to M$ and $q\colon \widetilde{M} \to M$ the orientation coverings of M and N. Show

that there exists an orientation-preserving diffeomorphism $\widetilde{f}\colon \widetilde{M} \to \widetilde{N}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{f}} & \widetilde{N} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ M & \xrightarrow{f} & N. \end{array}$$

Exercise 49.5. Let K be the Klein bottle and let $x \in K$. Describe an explicit self-diffeomorphism $f\colon K \to K$ with $f(x) = x$ such that $\mathrm{D}f_x\colon \mathrm{T}K_x \to \mathrm{T}K_x$ is orientation-preserving.

Exercise 49.6. Let $p\colon T \to K$ be the 2-fold covering of the Klein bottle by the torus that we discussed on page 1070. Give an example of a smooth map $f\colon K \to K$ such that there is no map $\widetilde{f}\colon T \to T$ which makes the following diagram commute:

$$\begin{array}{ccc} T & \xrightarrow{\widetilde{f}} & T \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ K & \xrightarrow{f} & K. \end{array}$$

Remark. This exercise shows that not every self-map of a smooth manifold gives rise to a self-map of its orientation cover.

Exercise 49.7. Let M be an oriented n-dimensional smooth manifold, let N be a non-orientable connected n-dimensional smooth manifold and let $f\colon M \to N$ be a smooth embedding. We denote by $p\colon \widetilde{N} \to N$ the orientation-covering from the Orientation Covering-Proposition 49.5. Show that there exists an orientation-preserving smooth embedding $g\colon M \to \widetilde{N}$ with $p \circ g = f$.

Exercise 49.8. Let $f\colon M \to N$ be an immersion between two smooth manifolds without boundary and of the same dimension.

(a) We suppose that M and N are compact, non-empty and connected. Show that f is a covering map.
Hint. Use the Inverse Mapping Theorem 24.1.

(b) Does the conclusion of (a) still hold if instead we only assume that M and N are connected and that f is surjective?

Exercise 49.9. Let M and $\widetilde{M}$ be closed oriented connected non-empty smooth manifolds and let $p\colon \widetilde{M} \to M$ be a finite covering. We denote by $\deg(p) \in \mathbb{Z}$ the degree as defined on page 899. Show that $\deg(p) = [\widetilde{M} : M]$.

Exercise 49.10. Let M be a smooth manifold. We denote by $\iota\colon \partial M \to M$ the inclusion. Show that there exists a unique natural diffeomorphism $\partial\widetilde{M} \to \widetilde{\partial M}$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & \overset{\widetilde{\iota}}{\frown} & & \\ \widetilde{\partial M} & \longrightarrow & \partial\widetilde{M} & \hookrightarrow & \widetilde{M} \\ {\scriptstyle p_{\partial M}}\downarrow & & & & \downarrow{\scriptstyle p_M} \\ \partial M & & \xrightarrow{\iota} & & M. \end{array}$$

Exercise 49.11. Let $k \in \mathbb{N}$. On page 266 we introduced the non-orientable surface N_k of genus k. The orientation cover of N_k is diffeomorphic to a surface of some genus g. Try to determine g in terms of k.

Exercise 49.12. Let M be a smooth manifold and let $p\colon \widetilde{M} \to M$ be the orientation covering. Show that

$$\begin{aligned}\#\,\text{boundary components of } \widetilde{M} \;\;=\;\; & \#\,\text{non-orientable boundary components of } M \\ & +2\cdot \#\,\text{orientable boundary components of } M.\end{aligned}$$

Exercise 49.13. Let M and N be two connected smooth manifolds, let $f\colon M \to N$ be a local diffeomorphism and finally let $x_0 \in M$. We denote by $w_1^{x_0}(M) \in \operatorname{Hom}(\pi_1(M,x_0),\mathbb{Z}_2)$ and $w_1^{f(x_0)}(N) \in \operatorname{Hom}(\pi_1(N,f(x_0)),\mathbb{Z}_2)$ the corresponding orientation characters. Show that

$$w_1^{x_0}(M) \;=\; w_1^{f(x_0)}(N) \circ (f_*\colon \pi_1(M,x_0) \to \pi_1(N,f(x_0))) \;\in\; \operatorname{Hom}(\pi_1(M,x_0) \to \mathbb{Z}_2).$$

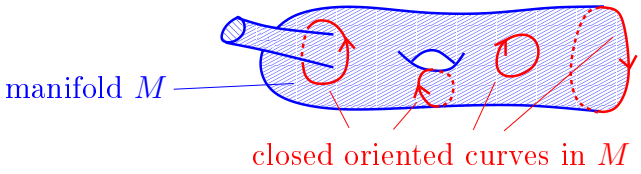
50. Homotopies and fundamental groups

In this chapter we will see that homotopy equivalent topological spaces have isomorphic fundamental groups. In some instances this fact will allow us to reduce the calculation of fundamental groups to some of our earlier calculations.

50.1. Closed oriented curves and fundamental groups.

The following definition gets used frequently throughout the notes.

Definition. Let M be a smooth manifold. A closed (oriented) curve in M is a closed (oriented) connected 1-dimensional submanifold either of ∂M or of M.



Convention. Often we just write "curve" instead of "closed curve".

Remark.

(1) Note that the boundary ∂M a smooth manifold M is *not* a submanifold of M. In particular, if M is a 2-dimensional smooth manifold and C is a closed boundary component, then C is not a submanifold of M, but it is a closed curve in M in the above sense.
(2) On occasions, when we deal with explicit examples, we will use gentle generalizations of the above notion of a closed oriented curve, which in practice should not cause any confusion.

Closed (oriented) curves in a given smooth manifold M have the nice feature that they are easy to draw. But strictly speaking they do not represent elements in $\pi_1(M)$ since elements in $\pi_1(M)$ are represented by *maps* $S^1 \to M$ and not just the images of such maps. Thus we need to clarify the connection between curves and homotopy classes of maps.

The key to this issue is the following lemma, which is also of independent interest.

Lemma 50.1.

(1) Let $f\colon [0,1] \to [0,1]$ be a self-homeomorphism.[513]
   (a) If f is monotonously increasing, then f is isotopic to the map $f(x) = x$.
   (b) If f is monotonously decreasing, then f is isotopic to the map $f(x) = -x$.
(2) Every self-homeomorphism of S^1 is isotopic to the map $f(z) = z$ or to the map $g(z) = \overline{z}$.
(3) Let $f\colon S^1 \to S^1$ be a diffeomorphism. If f is orientation-preserving, then f is diffeotopic to the identity, otherwise f is diffeotopic to the reflection $r\colon S^1 \to S^1$ given by $(x, y) \mapsto (x, -y)$.

Proof. We proved Statements (1) and (2) in Exercise 14.7 or alternatively in Exercise 48.24. We proved Statement (3) in the Diffeotopies-of-S^1-Classification Proposition 34.5. The

[513] It is an elementary real analysis exercise to show that every self-homeomorphism $f\colon [0,1] \to [0,1]$ is either monotonously increasing or monotonously decreasing.

proof of the Diffeotopies-of-S^1-Classification Proposition 34.5 can also be modified easily to take care of Statement (2). ■

Corollary 50.2. Let M be a smooth manifold and let $C \subset M$ be a closed oriented curve in M. Furthermore let $x_0 \in C$. For any two orientation-preserving diffeomorphisms $\alpha, \beta\colon S^1 \to C$ with $\alpha(1) = \beta(1) = x_0$ we have $[\alpha] = [\beta] \in \pi_1(M, x_0)$.

Proof. It is fairly straightforward to prove the corollary using Lemmas 50.1 (1) and 47.3. We leave the task of filling in the details to the reader. ■

Convention. Let M be a smooth manifold and let $C \subset M$ be a closed oriented curve in M. Throughout these notes we use that Corollary 50.2 allows us to view C unambiguously as an element in $\pi_1(M, x_0)$ for any base point $x_0 \in C$.

50.2. **The fundamental groups of homotopy equivalent topological spaces.** Now we will see that homotopy equivalent topological spaces have isomorphic fundamental groups. On numerous occasions this fact will often greatly simplify the calculation of fundamental groups.

Proposition 50.3. (Homotopy-π_1-Proposition)

(1) Let $f, g\colon X \to Y$ be two maps between topological spaces and let $x_0 \in X$. If f and g are homotopic, then there exists a path $\alpha\colon [0,1] \to Y$ from $f(x_0)$ to $g(x_0)$ such that the following diagram commutes:

$$\begin{array}{ccc} & \overset{g_*}{\nearrow} & \pi_1(Y, g(x_0)) \\ \pi_1(X, x_0) & & \Big\downarrow{\scriptstyle \alpha_*}\ {\scriptstyle [s] \mapsto [\alpha * s * \overline{\alpha}]} \\ & \underset{f_*}{\searrow} & \pi_1(Y, f(x_0)). \end{array}$$

Here the vertical map is the isomorphism given by the Change-of-Base Point Proposition 47.10. In particular

$$\begin{array}{c} f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0)) \\ \text{is an isomorphism} \end{array} \iff \begin{array}{c} g_*\colon \pi_1(X, x_0) \to \pi_1(Y, g(x_0)) \\ \text{is an isomorphism.} \end{array}$$

The analogous statement holds if we replace "isomorphism" by "epimorphism" or "monomorphism". Furthermore, the following two refinements hold:

(a) If f and g are homotopic rel x_0, then $f_* = g_*$.[514]

(b) If $f(x_0) = g(x_0)$ and if the maps f and g are homotopic, then there exists an $\alpha \in \pi_1(Y, f(x_0))$ such that $f_*(z) = \alpha \cdot g_*(z) \cdot \alpha^{-1}$ for every $z \in \pi_1(X, x_0)$.

(2) Let $f\colon X \to Y$ be a map between topological spaces and let $x_0 \in X$. If f is a homotopy equivalence, then the induced map

$$f_*\colon \pi_1(X, x_0) \ \to \ \pi_1(Y, f(x_0))$$

is an isomorphism.

(3) If X is contractible topological space, then for any $x_0 \in X$ we have $\pi_1(X, x_0) = 0$.

[514] This statement can be viewed as a special case of Lemma 14.6.

Example. For any $n \geq 2$ and any $Q \in \mathbb{R}^n$ we have

$$\begin{array}{ccccccc} & & & & \text{by the Deformation Retract-Homotopy Equivalence Lemma 15.5, Exercise 15.13 and the Homotopy-}\pi_1\text{-Proposition 50.3 (2)} & & \\ & & & & \downarrow & & \\ \pi_1(\mathbb{R}^n \setminus \{Q\}) & \cong & \pi_1(\mathbb{R}^n \setminus \{0\}) & \cong & \pi_1(S^{n-1}) & \cong & \begin{cases} \mathbb{Z}, & \text{if } n = 2, \\ 0, & \text{if } n \geq 3. \end{cases} \\ & \uparrow & & & & \uparrow & \\ & \text{since } \mathbb{R}^n \setminus \{Q\} \text{ is homeomorphic to } \mathbb{R}^n \setminus \{0\} & & & & \text{by Corollary 48.18 and the Sphere-}\pi_1\text{-Proposition 47.13} & \end{array}$$

Proof.

(1) Let $f, g\colon X \to Y$ be two maps between topological spaces and let $x_0 \in X$. Let $H\colon X \times [0,1] \to Y$ be a homotopy between f and g. We denote by

$$\begin{array}{rcl} \alpha\colon [0,1] & \to & Y \\ t & \mapsto & H(x_0, t) \end{array}$$

the path with which the homotopy H connects the points $H(x_0, 0) = f(x_0)$ and $H(x_0, 1) = g(x_0)$.

Claim. The maps

$$\begin{array}{rcl} f_*\colon \pi_1(X, x_0) & \to & \pi_1(Y, f(x_0)) \\ [s] & \mapsto & [f \circ s] \end{array} \quad \text{and} \quad \begin{array}{rcl} \pi_1(X, x_0) & \to & \pi_1(Y, f(x_0)) \\ [s] & \mapsto & [\alpha * (g \circ s) * \overline{\alpha}] \end{array}$$

are identical.

Proof. Let $s\colon [0,1] \to X$ be a loop in (X, x_0). We need to show that the loops $f \circ s$ and $\overline{\alpha} * (g \circ s) * \alpha$ are path-homotopic. We consider the map

$$\begin{array}{rcl} G\colon [0,1] \times [0,1] & \to & X \\ (r,t) & \mapsto & \begin{cases} \alpha(4r), & \text{if } r \in \left[0, \frac{1}{4}t\right], \\ H\left(s\left(\frac{r - \frac{1}{4}t}{1 - \frac{1}{2}t}\right), 1 - t\right), & \text{if } r \in \left(\frac{1}{4}t, 1 - \frac{1}{4}t\right), \\ \overline{\alpha}(4r - 3), & \text{if } r \in \left[1 - \frac{1}{4}t, 1\right]. \end{cases} \end{array}$$

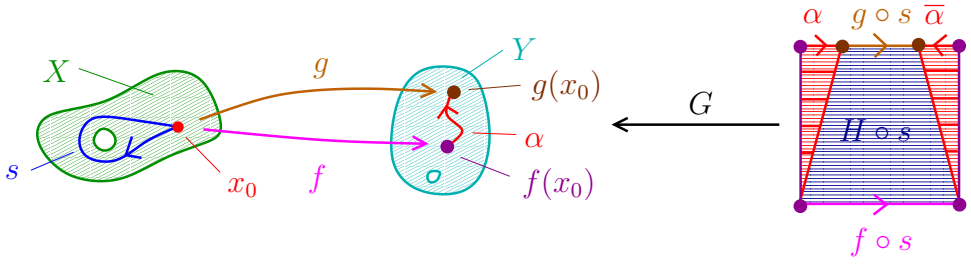
Using the Homotopy Stacking Lemma 14.3 it is straightforward to verify that G is indeed a continuous map. Furthermore it follows easily from the definition that the map G satisfies

$$G(0,t) \;=\; G(1,t) \;=\; f(x_0) \quad \text{for all } t \in [0,1]$$

and that it satisfies

$$G(r,0) \;=\; (f \circ s)(r) \quad \text{and} \quad G(r,1) \;=\; (\overline{\alpha} * (g \circ s) * \alpha)(r) \quad \text{for all } r \in [0,1].$$

This observation shows that G is in fact a path-homotopy between the loops $f \circ s$



and $\alpha * (g \circ s) * \overline{\alpha}$[515] in $(Y, f(x_0))$. ⊞

As in the statement of the proposition we consider the following diagram:

$$\begin{array}{ccc} & \xrightarrow{g_*} & \pi_1(Y, g(x_0)) \\ \pi_1(X, x_0) & & \Big\downarrow{\scriptstyle \alpha_*} \; [s] \mapsto [\alpha * s * \overline{\alpha}] \\ & \xrightarrow[f_*]{} & \pi_1(Y, f(x_0)). \end{array}$$

We make the following observations:

(a) It follows from the claim that the diagram commutes.

(b) By the Change-of-Base Point Proposition 47.10 we know that the vertical map is an isomorphism.

(c) It follows from (a) and (b) that f_* is an isomorphism if and only if g_* is an isomorphism.

(d) Now suppose that f and g are homotopic rel x_0. By definition this means that there exists a homotopy H with $H(x_0, t) = f(x_0)$ for all $t \in [0, 1]$. In other words, $\alpha(t) := H(x_0, t)$ is a constant path. It follows from the above that $f_* = g_*$.

(e) If $f(x_0) = g(x_0)$, then α is a loop in $(Y, f(x_0))$, i.e. it defines an element in $\pi_1(Y, y_0)$. Thus we have $[\alpha * (g \circ s) * \overline{\alpha}] = [\alpha] \cdot g_*(s) \circ [\alpha]^{-1}$. By the above commutative diagram we obtain the desired equality.

The analogous statement holds if we replace "isomorphism" by "epimorphism" or "monomorphism".

(2) Let $f\colon X \to Y$ be a map between topological spaces which is a homotopy equivalence. Let $x_0 \in X$ be a point. We pick a homotopy inverse $g\colon Y \to X$ of f. According to our hypothesis we have $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. It follows from (1) and the functoriality of fundamental groups that $g_* \circ f_*$ is an isomorphism of $\pi_1(X, x_0)$ and that $f_* \circ g_*$ is an isomorphism of $\pi_1(Y, f(x_0))$. Thus it remains to prove the following purely group theoretic claim.

Claim. Let $\alpha\colon A \to B$ and $\beta\colon B \to A$ be group homomorphisms. If $\alpha \circ \beta$ and $\beta \circ \alpha$ are isomorphisms, then α is an isomorphism.

Proof. First note that the fact that $\beta \circ \alpha\colon A \to A$ is an isomorphism implies that α is a monomorphism. Furthermore, the fact that $\alpha \circ \beta\colon B \to B$ is an isomorphism implies that α is an epimorphism. Thus we have shown that α is a monomorphism and an epimorphism, therefore it is an isomorphism. ⊞

(3) This statement is just a frequently used special case of the second statement. ■

In Lemma 2.26 we already saw that the topological spaces $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. Now we can prove a similar statement in one dimension higher.

Lemma 50.4. The topological spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic.

Unfortunately we are still not able to settle the question whether for $k \neq l \geq 3$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ can be homeomorphic. Before we can settle this question we will have to develop a completely new set of tools.

[515] To be precise, G is a homotopy between $f \circ s$ and a suitable parametrization of $\alpha * (g \circ s) * \overline{\alpha}$.

Proof. Let us suppose that there exists a homeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^3$. Let $P \in \mathbb{R}^2$ be a point. We put $Q := f(P)$. Then f restricts to a homeomorphism from $\mathbb{R}^2 \setminus \{P\}$ to $\mathbb{R}^3 \setminus \{Q\}$. But then we obtain that

$$\begin{array}{ccccccc} \mathbb{Z} & \cong & \pi_1(\mathbb{R}^2 \setminus \{P\}) & \cong & \pi_1(\mathbb{R}^3 \setminus \{Q\}) & = & 0. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{page 1125} & & \text{since } f \text{ is a homeomorphism} & & \text{page 1125} & \end{array}$$

Thus we obtained a contradiction. ■

50.3. **The fundamental group of mapping tori.** Let X be a topological space and let $f\colon X \to X$ be a map. On page 467 we introduced the corresponding mapping torus

$$\operatorname{Tor}(X, f) \;:=\; (X \times [0,1])\,/\!\sim \qquad \text{where } (x,0) \sim (f(x),1) \text{ for all } x \in X.$$

Our goal in this section is to determine the fundamental groups of mapping tori. Before we can do so we need to introduce a new concept from group theory.

Definition. Let N be a group, let $k \in \mathbb{N}_0$ and let $\varphi\colon N \to N$ be an isomorphism such that $\varphi^k = \mathrm{id}$.[516] We define the semidirect product of N and $\mathbb{Z}_k$ with respect to φ as the group $N \rtimes_\varphi \mathbb{Z}_k$ where the underlying set is given by the direct product $N \times \mathbb{Z}_k$,[517] but where the group multiplication[518] is given by

$$(h,m) \cdot (h',m') \;:=\; (h \cdot \varphi^m(h'), m + m').$$

If φ is understood from the context, then we drop it from the notation, i.e. we write $N \rtimes \mathbb{Z}_k$ instead of $N \rtimes_\varphi \mathbb{Z}_k$. If $\varphi \neq \mathrm{id}$, then we say that the semidirect product as non-trivial.

Examples.

(1) Let N be a group and $\varphi = \mathrm{id}_N$, then $N \rtimes_{\mathrm{id}} \mathbb{Z} = N \times \mathbb{Z}$ is the direct product of N and $\mathbb{Z}$.

(2) Let $G = \mathbb{Z}_2$ and let $N = \mathbb{Z}_n$ for some $n \in \mathbb{N}_0$. We denote by $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ the isomorphism that is given by multiplication by -1. Clearly this isomorphism satisfies $\varphi^2 = \mathrm{id}$, thus we can form the semidirect product $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$, which is called the *$n$-th dihedral group* D_n. It has $2n$ elements. As an example, the third dihedral group D_3 is isomorphic to the permutation group S_3 via the isomorphism [519]

$$\begin{array}{rcl} \Psi\colon \mathbb{Z}_3 \rtimes \mathbb{Z}_2 & \to & S_3 \\ (h,m) & \mapsto & (1\,2\,3)^h \cdot (1\,2)^m. \end{array}$$

[516] Note that this condition is automatically satisfied if $k = 0$, i.e. for any isomorphism φ of any group we always have $\varphi^0 = \mathrm{id}$.

[517] In particular, if N is finite and $k \in \mathbb{N}$, then the cardinality of $N \rtimes \mathbb{Z}_k$ is given by $|N| \cdot k$.

[518] It is an elementary argument to show that $N \rtimes_\varphi \mathbb{Z}_k$ with the given multiplication is indeed a group.

[519] First note that a straightforward calculation shows that $(1\,2)^{-1} \cdot (1\,2\,3) \cdot (1\,2) = (1\,3\,2) = (1\,2\,3)^{-1}$. This implies that also for any $m \in \mathbb{Z}_2$ and $s \in \mathbb{Z}$ we have $(1\,2)^{-m} \cdot (1\,2\,3)^s \cdot (1\,2)^m = (1\,2\,3)^{s(-1)^m}$. Now for any $(h,m),(h',m') \in \mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$ we have

$$\begin{aligned} \Psi(h,m)\Psi(h',m') &= (1\,2\,3)^h \cdot (1\,2)^m \cdot (1\,2\,3)^{h'} \cdot (1\,2)^{m'} = (1\,2\,3)^h \cdot (1\,2)^m \cdot (1\,2)^{-m} \cdot (1\,2\,3)^{(-1)^m h'} \cdot (1\,2)^m \cdot (1\,2)^{m'} \\ &= (1\,2\,3)^{h+(-1)^m h'} \cdot (1\,2)^{m+m'} \;=\; \Psi(h + (-1)^m h', m + m') \;=\; \Psi((h,m) \cdot (h',m')). \end{aligned}$$

We have thus shown that Ψ is a homomorphism.

Remark. In Exercise 50.16 we will see that $N = \{(h,0) \mid h \in N\} \subset N \rtimes \mathbb{Z}_k$ is a normal subgroup of $N \rtimes \mathbb{Z}_k$. Put differently, we have $N \triangleleft N \rtimes \mathbb{Z}_k$. This is the reason for the symbol "$\rtimes$" in the notation of the semidirect product.

We have the following elementary lemma.

Lemma 50.5. (Semidirect Product Lemma) Let G be a group and let $\gamma\colon G \to \mathbb{Z}$ be an epimorphism. We pick an element $t \in G$ with $\gamma(t) = 1$ and we write $N := \ker(\gamma)$. We denote by φ the isomorphism

$$\begin{array}{rccl} \varphi\colon N & \to & N \\ h & \mapsto & tht^{-1}. \end{array}$$

Then the map[520]

$$\begin{array}{rccl} \Psi\colon G & \to & N \rtimes_\varphi \mathbb{Z} \\ g & \mapsto & \big(gt^{-\gamma(g)}, \gamma(g)\big) \end{array}$$

is an isomorphism of groups.

Proof. It is straightforward to verify that Ψ is injective and surjective. It remains to show that Ψ is a homomorphism: given $g, h \in G$ we have indeed

$$\begin{aligned} \Psi(g \cdot h) \;=\; (ght^{-\gamma(gh)}, \gamma(gh)) \;&=\; (gt^{-\gamma(g)} \cdot \underbrace{t^{\gamma(g)} \cdot ht^{-\gamma(h)} \cdot t^{-\gamma(g)}}_{=\varphi^{\gamma(g)}(ht^{-\gamma(h)})}, \gamma(g) + \gamma(h)) \\ &=\; (gt^{-\gamma(g)}, \gamma(g)) \cdot (ht^{-\gamma(h)}, \gamma(h)) \;=\; \Psi(g) \cdot \Psi(h). \end{aligned}$$

■

The following proposition shows that the fundamental group of a mapping torus $\mathrm{Tor}(X, f)$ is isomorphic to a semidirect product of $\pi_1(X)$ with $\mathbb{Z}$.

Proposition 50.6. (Mapping Torus-π_1-Proposition) Let X be some non-empty path-connected topological space and furthermore let $f\colon X \to X$ be some homeomorphism. Then there exists an isomorphism $\psi\colon \pi_1(\mathrm{Tor}(X,f)) \cong \pi_1(X) \rtimes \mathbb{Z}$ such that the following diagram commutes

$$\begin{array}{ccccc} \pi_1(X) & \longrightarrow & \pi_1(\mathrm{Tor}(X,f)) & \longrightarrow & \pi_1(S^1) \\ \downarrow & & \cong\big\downarrow \psi & & \big\downarrow \cong \\ \pi_1(X) & \longrightarrow & \pi_1(X) \rtimes \mathbb{Z} & \longrightarrow & \mathbb{Z}. \end{array}$$

Here the top-left horizontal map is induced by the inclusion $X = X \times \{0\} \to \mathrm{Tor}(X,f)$ and the top-right horizontal map is induced by the natural projection $\mathrm{Tor}(X,f) \to S^1$. The bottom-left horizontal map is given by the natural inclusion into the first term of the semidirect product and the bottom-right horizontal map is given by the projection onto the second term of the semidirect product.[521]

Proof. Note that

$$\pi_1(\mathrm{Tor}(X,f)) \underset{\uparrow}{=} \pi_1((X \times \mathbb{R})/\mathbb{Z}_f) \overset{\downarrow}{\cong} \pi_1(X \times \mathbb{R}) \rtimes \mathbb{Z} \underset{\uparrow}{\cong} \pi_1(X) \rtimes \mathbb{Z}.$$

Mapping Torus-of-Homeomorphism Lemma 16.28 — Fundamental Group-via-Actions Theorem 48.16 (2) and Semidirect Product Lemma 50.5 — Homotopy-π_1-Proposition 50.3

[520] Note that $\gamma(gt^{-\gamma(g)}) = \gamma(g) + \gamma(t)(-\gamma(g)) = 0$, i.e. $gt^{-\gamma(g)}$ does indeed lie in $N = \ker(\gamma)$.

[521] Note that both maps on the bottom are group homomorphisms.

Thus we have obtained an isomorphism $\psi\colon \pi_1(\mathrm{Tor}(X,f)) \xrightarrow{\cong} \pi_1(X) \rtimes \mathbb{Z}$. We leave the verification that the given diagram commutes to the reader. ■

Remark. A careful reading of the proof of the proposition shows that we can identify the automorphism in the semidirect product. More precisely, let X be a path-connected topological space and let $f\colon X \to X$ be a homeomorphism. Let x be a base point and let γ be a path from $f(x)$ to x. We denote by φ the automorphism of $\pi_1(X,x)$ that is given by

$$\pi_1(X,x) \xrightarrow{f_*} \pi_1(X,f(x)) \xrightarrow[\text{Prop. 47.10}]{\cong} \pi_1(X,x). \qquad =:\varphi$$

Then $\pi_1(\mathrm{Tor}(X,f)) \cong \pi_1(X,x) \rtimes_\varphi \mathbb{Z}$.

We conclude this section with a discussion of several examples.

Examples.

(1) Let $K = \mathrm{Tor}(S^1, f(z) := \overline{z})$ be the Klein bottle. Then the automorphism $f_* =: \varphi$ of $\pi_1(S^1) = \mathbb{Z}$ is multiplication by -1. It follows from the previous remark[522] that

$$\pi_1(\text{Klein bottle}) \;\cong\; \mathbb{Z} \rtimes_\varphi \mathbb{Z}.$$

(2) Let $A \in \mathrm{GL}(n,\mathbb{Z})$ be a matrix. Recall that in Exercise 48.18 we showed that the map $f(A)\colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ given by $[v] \mapsto [Av]$ is a homeomorphism. Furthermore we know that, under the usual identification $\pi_1(\mathbb{R}^n/\mathbb{Z}^n, 0) = \mathbb{Z}^n$ as in the Fundamental Group-via-Actions Theorem 48.16, the induced map $f(A)_*$ on $\pi_1(\mathbb{R}^n/\mathbb{Z}^n, 0) = \mathbb{Z}^n$ is given by multiplication by A. It follows from the above remark that $\pi_1(\mathrm{Tor}(\mathbb{R}^n/\mathbb{Z}^n, f(A))) \cong \mathbb{Z}^n \rtimes_A \mathbb{Z}$. Note that this implies that $\pi_1(\mathrm{Tor}(\mathbb{R}^n/\mathbb{Z}^n, f(A)))$ is solvable.[523]

(3) Let X be a connected topological space. We calculate that

$$\pi_1(X \times S^1) \underset{\text{Mapping Torus-of-id-Lemma 16.24}}{=} \pi_1(\mathrm{Tor}(X, \mathrm{id}_X)) \underset{\text{by the above remark}}{\cong} \pi_1(X) \rtimes_{\mathrm{id}} \mathbb{Z} = \pi_1(X) \times \mathbb{Z} = \pi_1(X) \times \pi_1(S^1).$$

So in this special case we obtain the same result as in the Fundamental Group-of-Product Proposition 48.20.

50.4. **The boundary of 2-dimensional topological manifolds.** The goal of this section is to prove the subtle statement that the boundary of a 2-dimensional topological manifold is what you think it should be. To make sense of this it is perhaps worth reminding us of the definitions from page 511.

[522] On page 1222 we saw that $\pi_1(K) \cong \langle x, y \,|\, yxyx^{-1}\rangle$. What is an explicit isomorphism between the groups $\langle x, y \,|\, yxyx^{-1}\rangle$ and $\mathbb{Z} \rtimes_\varphi \mathbb{Z}$?

[523] Recall that a group G is solvable if there exist subgroups $G_0 = \{1\} \subset G_1 \subset G_2 \subset \cdots \subset G_k = G$ such that for $i = 0, \ldots, k-1$ the group G_i is normal in G_{i+1} and such that G_{i+1}/G_i is abelian. In our case the filtration is given by $\{0\} \subset \mathbb{Z}^n \subset \mathbb{Z}^n \rtimes_A \mathbb{Z}$.

Definition. Let F be an n-dimensional topological manifold.

(a) An n-dimensional chart for F at a point $x \in X$ is a homeomorphism $\Phi\colon U \to V$ where U is an open neighborhood of x and V is one of the following:
  (i) V is an open subset of $\mathbb{R}^n$ or
  (ii) V is an open subset of the upper half-space $H := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ and $\Phi(x)$ lies on $\partial H = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$.

  In the former case we say that Φ is a chart of type (i) in the latter case we say that Φ is a chart of type (ii).

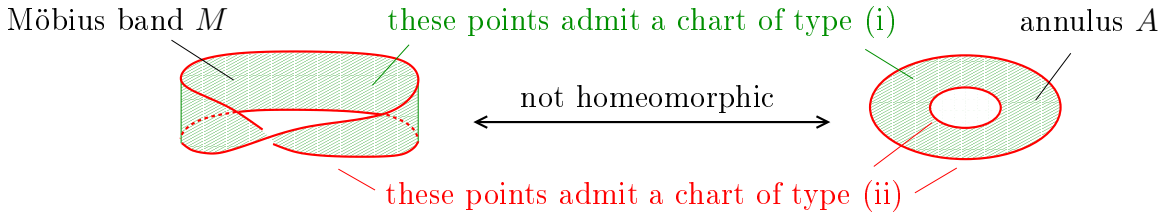
(b) We say that a point x on a topological manifold is a boundary point if x does not admit a chart of type (i). We denote by ∂X the set of all boundary points of X.

Proposition 50.7. Let F be an 2-dimensional topological manifold. The following statements hold:

(1) Every point on F admits either a chart of type (i) or it admits a chart of type (ii).
(2) If F is equipped with a smooth atlas, i.e. if we view F as a smooth manifold, then the boundary of F as a topological manifold agrees with the boundary of F viewed as a topological manifold.
(3) The boundary ∂F is an 1-dimensional topological manifold with empty boundary, i.e. we have $\partial(\partial F) = \varnothing$.

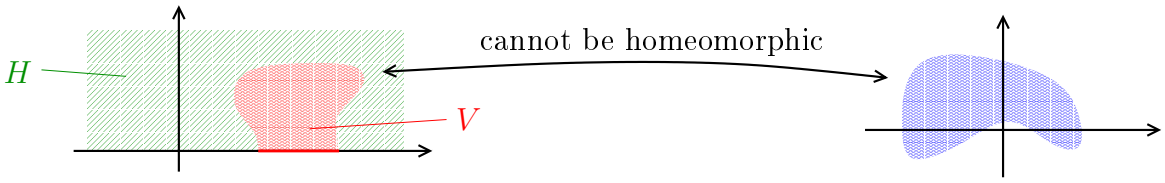
Remark. It might come as a surprise that we do not yet know the statement of the proposition. In fact in the Smooth Manifold Boundary Proposition 19.26 we saw that the boundary of any n-dimensional smooth manifold is an $(n-1)$-dimensional smooth manifold. But we did not show the corresponding statement for topological manifolds. In fact at the moment we can prove the statement only for topological manifolds of dimension *two*. We will deal with the case of higher-dimensional topological manifolds in the Topological Manifold Boundary Proposition 75.2 once we introduced and studied homology groups.

Example. In the figure below we show once again the Möbius band M and the annulus A. Proposition 50.7, together with the Smooth Manifold Boundary Proposition 19.26, says that the boundary of the Möbius band M and the annulus A, *as a topological manifold*, is exactly what we think it should be. In particular we see that ∂M consists of one component whereas ∂A consists of two components. Since homeomorphisms of topological manifolds induced homeomorphisms of their boundaries and since ∂M is not homeomorphic to ∂A we see that the Möbius band M is not homeomorphic to the annulus A.



The key ingredient in the proof of Proposition 50.7 is the following lemma.

Lemma 50.8. Let $H = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ be the upper half-space in $\mathbb{R}^2$. If V is an open subset of H with $V \cap \partial H \neq \varnothing$, then V is not homeomorphic to an open subset of $\mathbb{R}^2$.



Proof. Let V be an open subset of the upper half-space H which contains at least one point $Q \in \partial H = \{(x,0) \mid x \in \mathbb{R}\}$. Furthermore let U be an open subset of $\mathbb{R}^2$. Suppose there exists a homeomorphism $\Phi\colon U \to V$. We write $\Psi := \Phi^{-1}$ and we write $P = \Psi(Q)$.

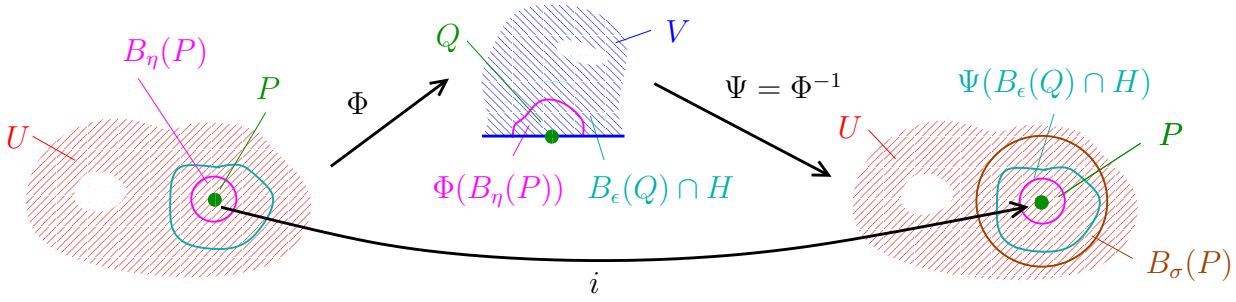
Since U is open in $\mathbb{R}^2$ we can pick a $\sigma > 0$ with $B_\sigma(P) \subset U$. Furthermore, since V is open in H and since Ψ is continuous there exists an $\epsilon > 0$ with $B_\epsilon(Q) \cap H \subset V$ and with $\Psi(B_\epsilon(Q)) \subset B_\sigma(P)$. Finally, by the continuity of Φ there exists an $\eta > 0$ with $\Phi(B_\eta(P)) \subset B_\epsilon(Q))$. We refer to the figure below for an illustration. We consider the following commutative diagram of maps

$$\begin{array}{ccc} & (B_\epsilon(Q) \setminus \{Q\}) \cap H & \\ {\scriptstyle \Phi}\nearrow & & \searrow{\scriptstyle \Psi = \Phi^{-1}} \\ B_\eta(P) \setminus \{P\} & \xrightarrow[i]{} & B_\sigma(P) \setminus \{P\}, \end{array}$$

where $i\colon B_\eta(P)\setminus\{P\} \to B_\sigma(P)\setminus\{P\}$ denotes the inclusion map. Now we pick a base point $x \in B_\eta(P) \setminus \{P\}$. By the functoriality of fundamental groups, the above commutative diagram of maps gives rise to the following commutative diagram

$$\begin{array}{ccc} & \pi_1((B_\epsilon(Q) \setminus \{Q\}) \cap H, \Phi(x)) & \\ {\scriptstyle \Phi_*}\nearrow & & \searrow{\scriptstyle \Psi_* = \Phi_*^{-1}} \\ \pi_1(B_\eta(P) \setminus \{P\}, x) & \xrightarrow[i_*]{} & \pi_1(B_\sigma(P) \setminus \{P\}, x). \end{array}$$

We make the following observations:



(1) The topological spaces on the left and on the right are homotopy equivalent to S^1, in particular the fundamental groups are non-trivial.
(2) By Exercise 15.4 we know that the inclusion map $B_\eta(P) \setminus \{P\} \to B_\sigma(P) \setminus \{P\}$ is a homotopy equivalence, therefore it follows from the Homotopy-π_1-Proposition 50.3 (2) that the inclusion induced map $i_*\colon \pi_1(B_\eta(P) \setminus \{P\}, x) \to \pi_1(B_\sigma(P) \setminus \{P\}, x)$ is an isomorphism.

(3) The topological space $(B_\epsilon(Q) \setminus \{Q\}) \cap H$ is star-shaped[524], which implies by the discussion on page 1058 that $\pi_1((B_\epsilon(Q) \setminus \{Q\}) \cap H, \Phi(x)) = 0$.

Summarizing we just showed that an isomorphism of non-trivial groups factors through the trivial group. This is not possible and hence we have obtained the desired contradiction. ■

Now we can turn to the proof of Proposition 50.7.

Proof of Proposition 50.7. Let F be an 2-dimensional topological manifold.

(1) Let x be a point on F. We need to show that x admits either a chart of type (i) or it admits a chart of type (ii). Suppose that x admits a chart $\Phi\colon U \to V$ of type (i) and also a chart $\Psi\colon X \to Y$ of type (ii). Then $\Phi \circ \Psi^{-1}\colon \Psi(U \cap X) \to \Phi(U \cap X)$ would give rise to a homeomorphism from an open set of H containing a point on $\partial H = \{(x, 0) \mid x \in \mathbb{R}\}$ to an open set in $\mathbb{R}^2$. But by Lemma 50.8 such a homeomorphism cannot exist.

(2) Now suppose that F is equipped with a smooth atlas. For the purpose of this proof we write $\partial^{\text{top}}F$ and $\partial^{\text{sm}}F$. By the definitions on pages 511 and 545 we have

$$\begin{aligned} \partial^{\text{top}}F &= \text{all points that do not admit a chart of type (i)} \\ \partial^{\text{sm}}F &= \text{all points that do not admit a chart of type (i) from the smooth atlas.} \end{aligned}$$

By definition we have $\partial^{\text{top}}F \subset \partial^{\text{sm}}F$. Now let $P \in \partial^{\text{sm}}F$. This means that P does not admit a chart of type (i) from the smooth atlas. By definition of a smooth manifold it admits a chart of type (ii). But by (1) this means that it cannot admit any chart of type (i). Thus we see that $P \in \partial^{\text{sm}}F$.

(3) We need to show that the boundary ∂F is a 1-dimensional topological manifold which has no boundary, i.e. such that $\partial Y = \varnothing$. First let $P \in F$ be a point on the boundary. Let $\Phi\colon U \to V$ be a chart of type (ii) for P. It follows from the (1) and the definitions that $\partial F \cap U = \Phi^{-1}(\partial H \cap V)$. Thus we see that Φ restricts to a chart $\Phi\colon \partial F \cap U \to \partial H \cap V$ of type (i), here we view $\partial H \cap V$ as an open subset of $\partial H = \mathbb{R}$. This shows that ∂F admits a 1-dimensional atlas where all charts are charts of type (i). Since F is second-countable and Hausdorff we obtain from the Second Countability Lemma 9.1 (2) and Lemma 1.14 that ∂F is also second-countable and Hausdorff. ■

50.5. The wedge of pointed topological spaces.

Definition. Let $\{(X_i, x_i)\}_{i \in I}$ be a family of pointed topological spaces.

(1) We refer to

$$\bigvee_{i \in I} X_i \;:=\; \bigvee_{i \in I} (X_i, x_i) \;:=\; \Big(\bigsqcup_{i \in I} X_i\Big)/\!\sim \qquad \text{where } x_i \sim x_j \text{ for all } i, j \in I$$

as the wedge of the pointed topological spaces (X_i, x_i). We refer to the point given by the identifications as the wedge point. If we do not say anything else it will be denoted by $*$.

(2) Let Y be a topological space and suppose that we are given a family of maps $\{f_i\colon X_i \to Z\}_{i \in I}$ such that for every $i, j \in I$ we have $f_i(x_i) = f_j(x_j)$. We define the map

[524] Why is that?

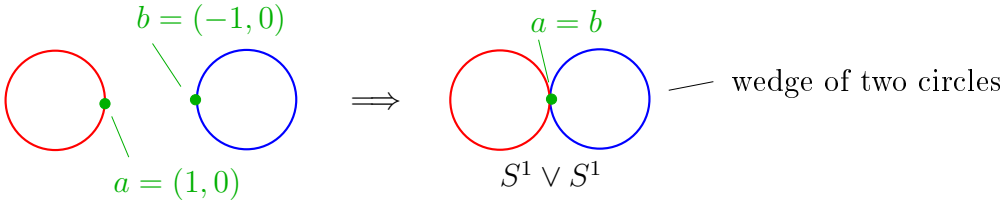
$$\begin{array}{rcl} \bigvee_{i\in I} f_i \colon \bigvee_{i\in I}(X_i, x_i) & \to & Z \\ [y] & \mapsto & f_i([y]) \text{ if } y \in X_i. \end{array}$$

It follows from the Topological-Quotient Proposition 5.15 (1b) that this map is indeed continuous.

(3) If $i = \{i_1, \dots, i_k\}$ is a finite family, then often we just write $(X_{i_1}, x_{i_1}) \vee \cdots \vee (X_{i_k}, x_{i_k})$ and similarly we just write $f_{i_1} \vee \cdots \vee f_{i_k}$.

Examples.

(1) The wedge $(A, a) \vee (B, b)$ of two pointed topological spaces (A, a) and (B, b) should be viewed as the result of gluing A and B along the points a and b.

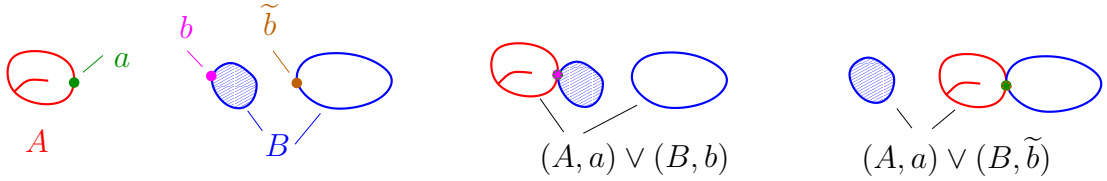


(2) Let $n \in \mathbb{N}_0$ and let $G = (V = \{v\}, E, i, t)$ be an abstract graph with a *single* vertex. It follows quite easily from the Twice Quotient Lemma 5.23 (1) together with the fact that $[0, 1]/0 \sim 1$ is homeomorphic to S^1 that the map

$$\begin{array}{rcl} |G| = (V \sqcup (E \times [0,1]))/\sim & \to & \bigvee_{e\in E} S^1_e \\ [x] & \mapsto & \begin{cases} *, & \text{if } x = v, \\ \exp(2\pi \mathrm{i} t) \in S^1_e, & \text{if } x = (e,t) \in E \times [0,1] \end{cases} \end{array}$$

is a homeomorphism. Evidently this also shows that every topological graph with one vertex and n edges is homeomorphic to the wedge of n circles.

(3) In general the homeomorphism type of the wedge depends on the choice of base points. This becomes particularly clear if we consider disconnected spaces. For example it is pretty clear that the wedges $(A, a) \vee (B, b)$ and $(A, a) \vee (B, \widetilde{b})$ considered in the figure below are not homeomorphic.



Even though we just saw that in general the homeomorphism type of the wedge of two topological spaces depends on the choice of the base points, there are also many other situations where the choice is basically irrelevant. For example we have the following lemma.

Lemma 50.9. Let $\{X_i\}_{i\in I}$ be a family of smooth manifolds. We suppose that for each $i \in I$ we are given points $x_i, y_i \in X_i \setminus \partial X_i$. If each X_i is connected, then there exists a homeomorphism

$$\bigvee_{i\in I}(X_i, x_i) \xrightarrow{\cong} \bigvee_{i\in I}(X_i, y_i).$$

Proof. The lemma is an almost immediate consequence of Proposition 36.4. We leave it to the reader to fill in the details. ■

This leads us to the following convention which is often safe, but which can also be ambiguous.

Convention. Let $\{X_i\}_{i\in I}$ be a family of non-empty topological spaces. For each $i \in I$ we pick a point $x_i \in X_i$ and we write

$$\bigvee_{i\in I} X_i \;:=\; \bigvee_{i\in I} (X_i, x_i).$$

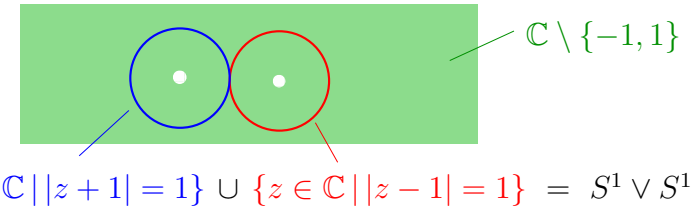
Examples.

(1) On numerous occasions we will talk about the wedge $S^1 \vee S^1$ of two circles. By Lemma 50.9 there is usually no need to specify which points in the two copies of S^1 get identified.

(2) In Exercise 50.2 we will see that the wedge of two circles is homotopy equivalent to $\mathbb{C} \setminus \{\pm 1\}$. More precisely, we will show that

$$S^1 \vee S^1 \;=\; \underbrace{\{z \in \mathbb{C} \,|\, |z+1| = 1\} \cup \{z \in \mathbb{C} \,|\, |z-1| = 1\}}_{\text{two circles that meet at the origin}}$$

is a deformation retract of $\mathbb{C} \setminus \{\pm 1\}$.



Definition. Let $\{(X_j, x_j)\}_{j\in J}$ be a family of pointed topological spaces. Given $k \in J$ we refer to the maps

$$\begin{aligned} i_k \colon X_k &\to \bigvee_{j\in J} X_j \\ x &\mapsto [x] \end{aligned} \qquad \text{and} \qquad \begin{aligned} p_k \colon \bigvee_{j\in J} X_j &\to X_k \\ [x] &\mapsto \begin{cases} x, & \text{if } x \in X_k, \\ x_k, & \text{otherwise} \end{cases} \end{aligned}$$

as the natural inclusion respectively the natural projection.

Lemma 50.10. Let $\{X_j\}_{j\in J}$ be a family of topological spaces and suppose that for each $j \in J$ we are given a point $x_j \in X_j$. Let $k \in J$.

(1) The natural inclusion i_k is an embedding.

(2) A map $f \colon \bigvee_{j\in J} X_j \to Y$ to a topological space is continuous if and only if for each $k \in J$ the map $f \circ i_k \colon X_k \to Y$ is continuous.

(3) The natural projection p_k is continuous.

Remark. In most cases Lemma 50.10 allows us to safely view each X_k as a subset of the wedge $\bigvee_{j\in J} X_j$.

Proof.

(2) This statement follows easily from the Topological-Coproduct Proposition 5.11 (1) and the Topological-Quotient Proposition 5.15 (1b).
(3) This is an immediate consequence of (2).
(1) It follows easily from the Topological-Coproduct Proposition 5.11 (1) and the Topological-Quotient Proposition 5.15 (1a) that the natural inclusion i_k is continuous. It is clear that the natural inclusion i_k is an injection. Evidently we have $p_k \circ i_k = \mathrm{id}_{X_k}$. Since the natural projection p_k is continuous we see that the natural inclusion i_k is indeed an embedding. ■

The following lemma collects a few basic properties of the wedge of topological spaces. It is entirely possible that later on we use the lemma without giving it credit.

Lemma 50.11. Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces.

(1) If I is finite and if each X_i is compact, then $\bigvee_{i\in I} X_i$ is compact.
(2) If each X_i is Hausdorff, then $\bigvee_{i\in I} X_i$ is Hausdorff.
(3) If each X_i is (path)-connected, then $\bigvee_{i\in I} X_i$ is (path)-connected

Proof.

(1) It follows immediately from Lemma 50.10 (3) and the Topological-Coproduct Proposition 5.11 (1) that the map $\bigsqcup_{i\in I} p_i \colon \bigsqcup_{i\in I} X_i \to \bigvee_{i\in I} X_i$ is continuous. Thus the desired statement follows from the Disjoint Union Topology-Properties Lemma 5.13 (2) together with the Compact Image Lemma 2.13.
(2) We will prove this statement in Exercise 50.3.
(3) We will prove this statement in Exercise 50.4. ■

We conclude this section with two technical lemmas which can safely be skipped at a first reading.

Lemma 50.12. Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces and let (Z, z_0) be a pointed topological space. Furthermore let $\{f_i \colon (X_i, x_i) \to (Z, z_0)\}_{i\in I}$ and $\{g_i \colon (X_i, x_i) \to (Z, z_0)\}_{i\in I}$ be two families of maps. The following statement holds:

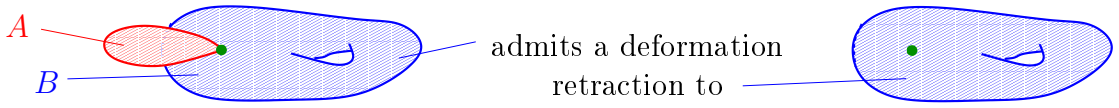
for every $i \in I$ there exists a homotopy rel x_i from f_i to g_i $\iff$ the maps $\bigvee_{i\in I} f_i$ and $\bigvee_{i\in I} g_i$ from $\bigvee_{i\in I} X_i$ to Z are homotopic rel the wedge point.

Proof. The "$\Leftarrow$"-direction of the lemma follows immediately from Lemma 50.10 (1). If the index set I has two elements, then the "$\Rightarrow$"-direction lemma is a straightforward consequence of the Homotopy Pushout Lemma 14.9 which eventually reduces the problem to the Product-Quotient Mixing Lemma 13.6. For a general family the "$\Rightarrow$"-direction lemma can also be reduced easily to the Product-Quotient Mixing Lemma 13.6.

It is worth pointing out that the proof of the "$\Rightarrow$"-direction is not obvious. As we just pointed out, it all boils down to the Product-Quotient Mixing Lemma 13.6. This lemma in turn relies on the delicate, frequently overlooked, underappreciated Whitehead Quotient Theorem 13.4. ■

The following lemma gives in particular a nice little application of Lemma 50.12.

Lemma 50.13. Let (A, a_0) and (B, b_0) be two pointed topological spaces. We consider the wedge $A \vee B = (A, a_0) \vee (B, b_0)$. If $\{a_0\}$ is a deformation retract of A, then $B = \{a_0\} \vee B$ is a deformation retract of $A \vee B$.



Proof. Let $F\colon A \times [0,1] \to A$ be a deformation retraction from A to $\{a_0\}$. Let G be the trivial homotopy on B. It follows again from the Homotopy Pushout Lemma 14.9 that we can combine the two homotopies to get the desired deformation retraction. ■

50.6. **Topological graphs.** In this section we have another look at abstract graphs and their topological incarnations. First we recall the following definitions from page 315.

Definition. Let $G = (V, E, i, t)$ be an abstract graph.

(1) If G is finite, i.e. if V and E are finite, then we refer to

$$\chi(G) \;=\; \#V \,-\, \#E \;=\; \text{number of vertices} \,-\, \text{number of edges}$$

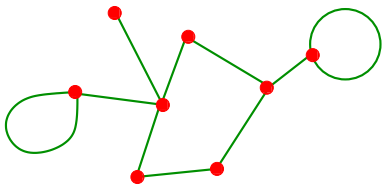
as the Euler characteristic of G.

(2) A tree is a finite connected abstract graph with Euler characteristic 1.

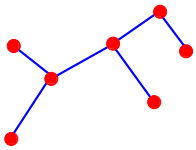
(3) The valence of a vertex v is defined as

$$\text{valence}(v) \;:=\; \#\{e \in E \,|\, i(e) = v\} + \#\{e \in E \,|\, t(e) = v\}.$$

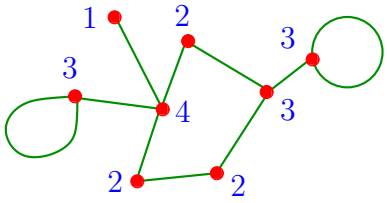
Given a topological graph $(X, G = (V, E, i, t), \Theta\colon |G| \to X)$ we define the properties of X to be the properties of G, e.g. we define $\chi(X) := \chi(G)$ and we say that X is a tree if G is a tree.[525]



Euler characteristic of G
equals $\chi(G) = 8 - 10 = -2$



example of a tree



valences of the vertices

Proposition 50.14. Let T be a topological graph that is a tree. Then the following hold:

(1) If T has at least one edge, then it admits at least two vertices of valence one.

(2) The tree T admits a deformation retraction to a vertex, in particular it is contractible.

(3) The fundamental group $\pi_1(T)$ is trivial.

Remark. Proposition 50.14 is basically the constant of Exercise 15.6.

[525] On many occasions we will use the fact, established in the Abstract Graph-Topological Realization Lemma 7.1 (5), that a topological graph is connected if and only if the underlying abstract graph is connected.

Proof. Basically by definition of a topological graph it suffices to deal throughout with the topological realizations of a tree. Thus let $T = (V, E, i, t)$ be a tree.

(1) This statement is precisely the content of Lemma 7.7.

(2) We will prove that any tree admits a deformation retraction to a vertex. We will prove this statement by induction on the number of edges. If a tree has zero edges, then it follows from the connectedness of T that T consists of precisely one vertex. So we are done.

Now suppose that we know that all trees with n edges admit a deformation retraction to a vertex. Let $T = (V, E, i, t)$ be a tree with $n+1$ edges. By (1) we know that T admits a vertex v of valence one. We denote by e the unique edge with v as an endpoint. We set $T' = (V \setminus \{v\}, E \setminus \{e\}, i|_{E\setminus\{e\}}, j|_{E\setminus\{e\}})$. Note that T' is again an abstract graph. Furthermore note that we reduced the number of vertices and of edges by one, so the Euler characteristic is unchanged, i.e. $\chi(T') = \chi(T) = 1$, so T' is again a tree.

We will now show that $|T'|$ is a deformation retract of $|T|$. We denote by u the other endpoint of e. We consider the maps

$$f\colon |T| \to |T'| \qquad x \mapsto \begin{cases} u, & \text{if } x \in |e|, \\ x, & \text{otherwise} \end{cases} \qquad \text{and} \qquad g\colon |T'| \to |T| \qquad x \mapsto x.$$

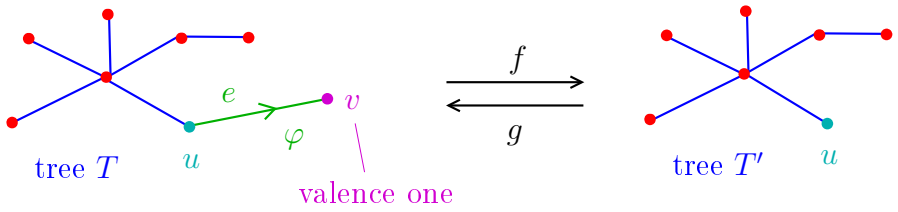
It follows quite easily from the Twice Quotient Lemma 5.23 (2) together with the Topological-Coproduct Proposition 5.11 that both maps are continuous.

We claim that g is a homotopy inverse to f. Evidently $f \circ g = \mathrm{id}_{|T'|}$. We need to show that $g \circ f \simeq \mathrm{id}_{|T|}$. By definition and by the Abstract Graph-Topological Realization Lemma 7.1 (3) we can pick a characteristic map $\Phi_e\colon [0,1] \to |e|$ with $\Phi_e(0) = u$ and $\Phi_e(1) = v$. Next we consider the map

$$H\colon |T| \times [0,1] \to |T| \qquad (x,t) \mapsto \begin{cases} \Phi_e(\Phi_e^{-1}(x) \cdot t), & \text{if } x \in |e|, \\ x, & \text{otherwise.} \end{cases}$$

We leave it to the reader to verify, say using the Homotopy Quotient Theorem 14.7, Lemma 5.23 (2) and the Topological-Coproduct Proposition 5.11 that H is continuous. Now it is clear that H is a homotopy from $g \circ f$ to id_T.

We have thus shown that $|T'|$ is a deformation retract of $|T'|$. By our induction hypothesis $|T'|$ admits a deformation retraction to a vertex $v \in T'$. So it follows from the Homotopy Combination the Homotopy Combination Lemma 14.4 that $|T|$ also admits a deformation retraction to the vertex v. Finally it follows from this fact together with the Point Deformation Retract Lemma 15.6 (1) that T is contractible.



(3) It follows from (2) and the Homotopy-π_1-Proposition 50.3 (3) that the fundamental group of T is trivial. ■

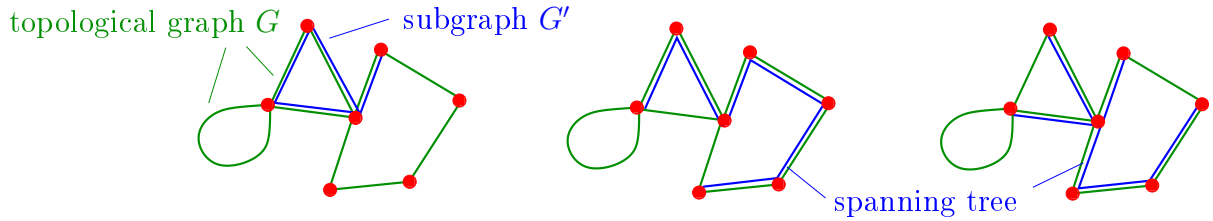
Next we recall a few more definitions from page 317.

Definition. Let $G = (V, E, i, t)$ be an abstract graph.

(1) A subgraph of $G = (V, E, i, t)$ is an abstract graph $G' = (V', E', i', t')$ such that $V' \subset V$ and $E' \subset E$ are subsets such that $i' = i|_{E'}$ and $t' = t|_{E'}$.

(2) A spanning tree for G is a subgraph that is a tree and which does not admit an adjacent edge.

We extend the above definitions in an obvious way to topological graphs.



We conclude this discussion of abstract and topological graphs with the following proposition.

Proposition 50.15. Let $(X, G = (V, E, i, t), \Theta\colon |G| \to X)$ be a finite topological graph.

(1) For every spanning tree T of X the projection $X \to X/T$ is a homotopy equivalence.

(2) If X is connected and non-empty, then the topological graph X is homotopy equivalent to the wedge of $1 - \chi(X)$ circles.

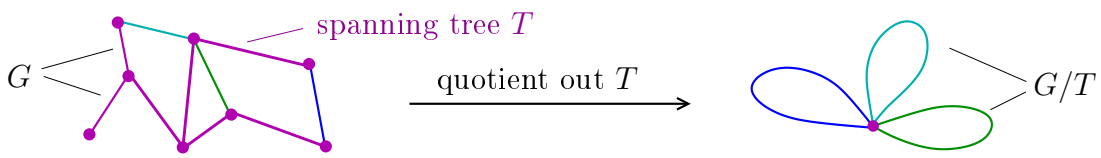
(3) Two finite connected topological graphs with the same Euler characteristic are homotopy equivalent.

Remark.

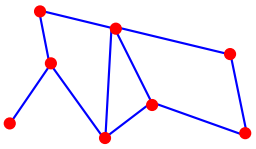
(1) The content of Proposition 50.15 is basically identical to the content of the Graph-mod-Tree Proposition 17.10. WE will now give a somewhat different self-contained proof.

(2) In the Quotient-by-Tree Proposition 69.8 and the Quotient-by-Contractible Corollary 70.3 we will provide generalizations of Proposition 50.15.

(3) Later, in the Fundamental Group-of-Graph Proposition 52.5 we will study the question, whether the converse to Statement (3) holds: if two finite connected topological graphs are homotopy equivalent, does it follow that the Euler characteristics are the same?

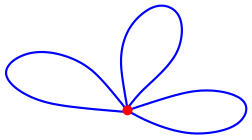


Proof. As we mentioned above, the content of Proposition 50.15 is basically identical to the content of the Graph-mod-Tree Proposition 17.10. The proof of the Graph-mod-Tree Proposition 17.10 relied on the language of cofibrations and, more seriously, on the Homotopy Extension Theorem 70.1. We will now give a self-contained proof.



topological graph of Euler characteristic $8 - 10 = -2$

is homotopy equivalent to



wedge of three circles

First note that basically by definition we only need to deal with the topological realization of an abstract graph. Thus let $G = (V, E, i, t)$ be a finite connected abstract graph.

(1) Let $T = (W, F)$ be a spanning tree of G. Recall that on page 321 we introduced the notion of the quotient abstract graph G/T. Next we consider the following little commutative diagram

$$\begin{array}{ccc} |G| & \xrightarrow{\quad p \quad} & |G/T| \\ & {\scriptstyle x\mapsto[x]}\searrow \quad \nearrow & \\ & |G|/|T| & \end{array}$$

where p is the map introduced in the Quotient Abstract Graph Lemma 7.14 (2). We need to show that the projection $|G| \to |G|/|T|$ to the left is a homotopy equivalence. By the Quotient Abstract Graph Lemma 7.14 (2b) we know that the diagonal map to the right is a homeomorphism. Thus it suffices to show that the horizontal map $p\colon |G| \to |G/T|$ is a homotopy equivalence.

Note that by Proposition 50.14 (2) we know that $|T|$ admits a deformation retraction $R\colon |T| \times [0,1] \to |T|$ to some vertex $t_0 \in T$. Put differently, there exists a homotopy $R\colon |T| \times [0,1] \to |T|$ such that $R_1 = \mathrm{id}_{|T|}$ and such that $R_0 = c_{t_0}$ is the constant map.

Let f be an edge of G that is not contained in T. If $i(f) = t(f)$, then it follows from the Abstract Graph-Topological Realization Lemma 7.1 that there exists a map $\Phi_f\colon [-2,2] \to |G|$ which induces a homeomorphism $[-2,2]/-2 \sim 2 \to |f|$. Furthermore, if $i(f) \neq t(f)$, then it follows from the Abstract Graph-Topological Realization Lemma 7.1 that there exists a map $\Phi_f\colon [-2,2] \to |G|$ which restricts to a homeomorphism $[-2,2] \to |f|$.[526] We write $a_f = \Phi_f(-2)$ and $b_f = \Phi_f(2)$. Note that it follows from our hypothesis that T is a spanning tree together with the Spanning Tree Existence Proposition 7.10 that $a_f, b_f \in |T|$.

Next we consider the quotient graph G/T. Let $p\colon |G| \to |G/T|$ be the projection. For every edge f of G/T, i.e. for every $f \in E \setminus F$, we denote by $\Psi_f\colon [-2,2] \to |G/T|$ the map $p \circ \Phi_f$. We consider the map

$$\begin{array}{rcl} q\colon |G/T| & \to & |G| \\ x & \mapsto & \begin{cases} R(a_f, -s-1), & \text{if } x = \Psi_f(s) \text{ for some } s \in [-2,-1], \\ \Psi_f(2 \cdot s), & \text{if } x = \Psi_f(s) \text{ for some } s \in [-1,1], \\ R(b_f, s-1), & \text{if } x = \Psi_f(s) \text{ for some } s \in [1,2]. \end{cases} \end{array}$$

[526] In other words, the map Φ_f is just a rescaling of characteristic maps of f.

It follows fairly easily from the Twice Quotient Lemma 5.23 that q is continuous. (The maps p and q are illustrated, to the best of the author's abilities, in the figure below.) It remains to prove the following claim.

Claim. The maps p and q are homotopy inverses of one another.

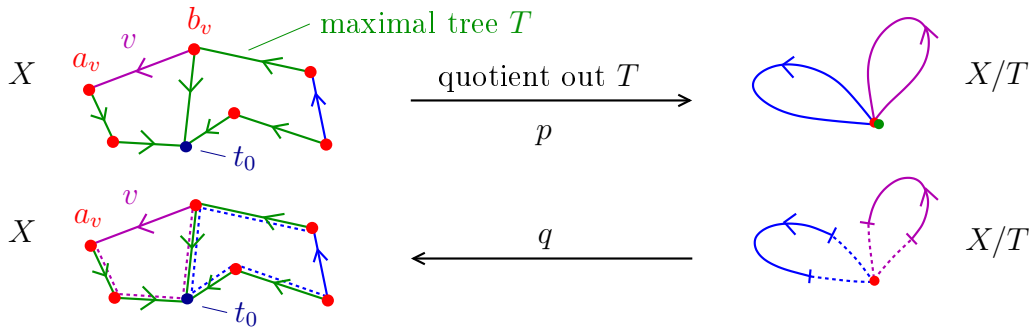
Proof. The proof of the claim is fairly elementary, just notationally a little messy. First we consider the following map

$$H\colon |G/T| \times [0,1] \to |G/T|$$
$$(x,t) \mapsto \begin{cases} *, & \text{if } x = \Psi_f(s) \text{ for some } s \in [-2,-1-t], \\ \Psi_f(\frac{2s}{1+t}), & \text{if } x = \Psi_f(s) \text{ for some } s \in [-1-t,1+t], \\ *, & \text{if } x = \Psi_f(s) \text{ for some } s \in [1+t,2]. \end{cases}$$

It follows, reasonably easily, from the combination of the Twice Quotient Lemma 5.23 and the Homotopy Quotient Theorem 14.7 that H is continuous. It follows immediately from the definitions that $H_0 = p \circ q$ and $H_1 = \mathrm{id}_{|G/T|}$. Next we consider the following map

$$\widetilde{H}\colon |G| \times [0,1] \to |G|$$
$$(x,t) \mapsto \begin{cases} R(x,t), & \text{if } x \in T, \\ R(a_f, s+2+t), & \text{if } x = \Phi_f(s) \text{ for some } s \in [-2,-1-t], \\ \Phi_f(\frac{2s}{1+t}), & \text{if } x = \Phi_f(s) \text{ for some } s \in [-1-t,1+t], \\ R(b_f, -s+2+t), & \text{if } x = \Phi_f(s) \text{ for some } s \in [1+t,2]. \end{cases}$$

Again, using the Twice Quotient Lemma 5.23 and the Homotopy Quotient Theorem 14.7 one can show that $\widetilde{H}$ is continuous. Furthermore it follows again immediately from the definitions that $\widetilde{H}_0 = q \circ p$ and $\widetilde{H}_1 = \mathrm{id}_{|G|}$. ⊞



(2) Note that by the Spanning Tree Existence Proposition 7.10 (1) we know that G admits a spanning tree T. Furthermore note that in (1) we showed that the map $p\colon |G| \to |G/T|$ is a homotopy equivalence. Thus it remains to show that $|G/T|$ is homeomorphic to the wedge of $1-\chi(G)$ circles. We make the following observation:
   (a) By the Spanning Tree Existence Proposition 7.10 the spanning tree T contains all vertices of G. In other words, the abstract graph G/T contains only a single vertex.
   (b) It follows from the Quotient Abstract Graph Lemma 7.14 (2) and the definition of a tree in terms of the Euler characteristic that $\chi(G/T) = \chi(G) - \chi(T) + 1 = \chi(G)$.
   (c) By (1) and (2) we know that G/T has $1-\chi(G)$ edges.

(d) It follows from (1) and (3), together with the discussion on page 1133 that $|G/T|$ is indeed homeomorphic to the wedge of $1-\chi(G)$ circles.

(3) The last statement is an immediate consequence of (2) and the Homotopy-Equivalence Basics Lemma 15.1. ■

50.7. **Homotopy classes of loops.** In the last section of this chapter we want to study the question, when are two maps $S^1 \to X$ to a topological space homotopic? In fact the following proposition gives in some sense a complete answer to the above question.

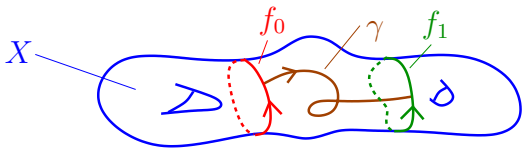
Proposition 50.16. (Homotopies-of-Loops Proposition) Let X be a topological space and let $f_0, f_1 \colon S^1 \to X$ be two maps. We write $x_0 = f_0(1)$ and $x_1 = f_1(1)$.

(1) The following statement holds:[527]

$$f_0 \text{ and } f_1 \text{ are homotopic} \iff \begin{array}{c}\text{there exists a path } \gamma \text{ from } x_0 \text{ to } x_1 \\ \text{such that } [\gamma * f_1 * \overline{\gamma}] = [f_0] \in \pi_1(X, x_0).\end{array}$$

(2) If $x_0 = x_1$, then the following statement holds:

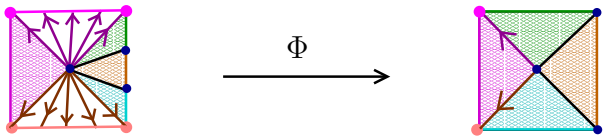
$$f_0 \text{ and } f_1 \text{ are homotopic} \iff [f_0] \text{ and } [f_1] \text{ are conjugate in } \pi_1(X, x_0).$$



Proof. Note that the second statement is an immediate consequence of the first statement and the observation that if $x_0 = x_1$ any path from x_0 to $x_1 = x_0$ is in fact a loop and thus defines an element in $\pi_1(X, x_0)$. Therefore it suffices to prove the first statement.

Now let X be a topological space. We take the view that maps $S^1 \to X$ are the same as loops, i.e. maps $[0,1] \to X$ such that 0 and 1 get sent to the same point. Thus let $f_0, f_1 \colon [0,1] \to X$ be two loops. We write $x_0 = f_0(1)$ and $x_1 = f_1(1)$.

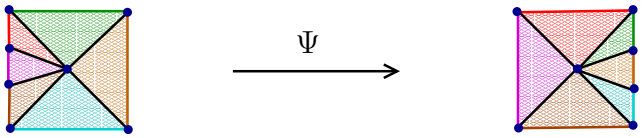
"$\Rightarrow$" Suppose there exists a homotopy $F \colon [0,1] \times [0,1] \to X$ with $F(0,t) = F(1,t)$ for all $t \in [0,1]$ and such that $F_0 = f_0$ and $F_1 = f_1$. We denote by $\gamma \colon [0,1] \to X$ the path that is given by $\gamma(t) = F(0,t)$ for $t \in [0,1]$. We consider the surjective map $\Phi \colon [0,1] \times [0,1] \to [0,1] \times [0,1]$ that is shown in the figure below. The map $\Phi \circ F$ is easily seen to be a homotopy between the loops f_0 and $\gamma * f_1 * \overline{\gamma}$. But that is exactly what we needed to show.



"$\Leftarrow$" In the following we now suppose that there exists a path γ from x_0 to x_1 such that $[\gamma * f_1 * \overline{\gamma}] = [f_0] \in \pi_1(X, x_0)$. By definition this means that there exists a homotopy $F \colon [0,1] \times [0,1] \to X$ with $F(0,t) = F(1,t) = x_0$ for all $t \in [0,1]$ and such that $F_0 = f_0$ and $F_1 = \gamma * f_1 * \overline{\gamma}$. We consider the homeomorphism $\Psi \colon [0,1] \times [0,1] \to [0,1] \times [0,1]$ that is shown in the figure below. The map $\Psi \circ F$ is easily seen to be a homotopy

[527] Sometimes two maps $S^1 \to X$ that are homotopic are called *freely homotopic* to stress the fact that no base points are fixed.

between $c_{x_0} * f_0 * c_{x_0}$ and f_1, viewed as maps $S^1 \to X$. But the map $c_{x_0} * f_0 * c_{x_0}$ is easily seen to be homotopic to the map f_0. Since being homotopic is transitive we have indeed shown that f_0 and f_1 are homotopic.



■

Exercises for Chapter 50.

Exercise 50.1. Let $n \in \mathbb{N}$.

(a) Show that $\mathrm{SL}(n, \mathbb{R})$ is a retract of $\mathrm{GL}(n, \mathbb{R})$ and show that it is not a deformation retract of $\mathrm{GL}(n, \mathbb{R})$.

(b) Solve (a) with $\mathbb{C}$ instead of $\mathbb{R}$.

Exercise 50.2. Show that

$$S^1 \vee S^1 \;=\; \underbrace{\{z \in \mathbb{C} \,|\, |z+1| = 1\} \cup \{z \in \mathbb{C} \,|\, |z-1| = 1\}}_{\text{two circles that meet at the origin}}$$

is a deformation retract of $\mathbb{C} \setminus \{\pm 1\}$.

Exercise 50.3. Let $\{(X_i, x_i)\}_{i \in I}$ be a family of pointed topological spaces. We suppose that each X_i is Hausdorff. Show that $\bigvee\limits_{i \in I} X_i$ is also Hausdorff.

Exercise 50.4. Let $\{(X_i, x_i)\}_{i \in I}$ be a family of pointed topological spaces.

(a) We suppose that each X_i is path-connected. Show that $\bigvee\limits_{i \in I} X_i$ is path-connected.

(b) We suppose that each X_i is connected. Show that $\bigvee\limits_{i \in I} X_i$ is connected.

Exercise 50.5. Let $X := \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}$. We denote by $\| - \|$ the usual Euclidean norm on X. We denote by $P = (0, 0)$ the center of X. The *Paris metric* on X is given by

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} \\ (A, B) & \mapsto & \|A - P\| + \|B - P\|. \end{array}$$

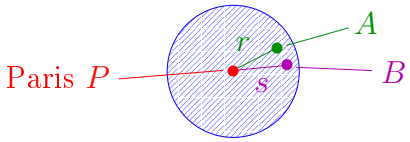
This metric takes its name from the fact that all (OK, many) train lines in France go towards Paris. So if you want to go from A to B you invariably (OK, often) have to go through Paris.

(a) Show that the map

$$\begin{array}{rcccl} \varphi\colon \bigvee\limits_{z \in S^1} [0,1]_z & \to & X & & \\ [r] & \mapsto & r \cdot z & \text{if } r \in [0,1]_z. & \end{array}$$

is not a homeomorphism.

(b) Does there exist a homeomorphism between $\bigvee\limits_{z \in S^1} [0,1]_z$ and X?



in the Paris metric the distance between P and Q equals $r+s$

Exercise 50.6. Let (X,x_0) and (Y,y_0) be two pointed topological spaces. We use the base points to define the wedge $X \vee Y$.

(a) Show that the map

$$\begin{array}{rcl} f\colon X \vee Y & \to & X \times Y \\ [P] & \mapsto & \begin{cases} (P,y_0), & \text{if } P \in X, \\ (x_0,P), & \text{if } P \in Y \end{cases} \end{array}$$

is an embedding.

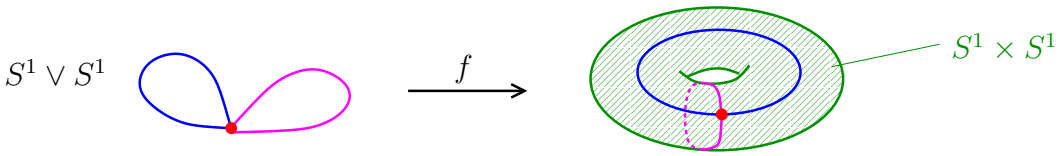
Remark. We do not assume that X and Y have any nice properties.

(b) Using (a) we can view $X \vee Y$ as a subspace of $X \times Y$. We refer to the quotient

$$X \wedge Y \;:=\; (X,x_0) \wedge (Y,y_0) \;:=\; (X \times Y)/(X \vee Y)$$

as the *smash product of* (X,x_0) *and* (Y,y_0). Show that for any $m,n \in \mathbb{N}_0$ the smash product $S^m \wedge S^n$ is homeomorphic to S^{m+n}.

We refer to the figure below for an illustration.



Exercise 50.7. We consider the following two subsets of $\mathbb{C}$:

$$\begin{aligned} X &:= \underbrace{\{z \in \mathbb{C} \,|\, |z-2|=1\}}_{=:A} \cup [-1,1] \cup \underbrace{\{z \in \mathbb{C} \,|\, |z+2|=1\}}_{=:B} \quad \text{and} \\ Y &:= \underbrace{\{z \in \mathbb{C} \,|\, |z-1|=1\}}_{=:C} \cup \underbrace{\{z \in \mathbb{C} \,|\, |z+1|=1\}}_{=:D}. \end{aligned}$$

Show that X and Y are homotopy equivalent.

Remark. It follows basically from Proposition 50.15 that X and Y are homotopy equivalent. The exercise thus consists of showing "by hand" that X and Y are homotopy equivalent.



Exercise 50.8. Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Suppose that for each $i \in I$ we are given a point $x_i \in X_i$ with the property that $\{x_i\}$ is a closed subset of X_i. We use these points to form the wedge $\bigvee\limits_{i \in I} X_i$. We denote by $*$ the wedge point. Suppose that for each $i \in I$ we are given an open neighborhood U_i of x_i in X_i such that x_i is a deformation retract of U_i. We set $U := \bigcup\limits_{i \in I} U_i$.

(a) Show that U is an open neighborhood of $*$.

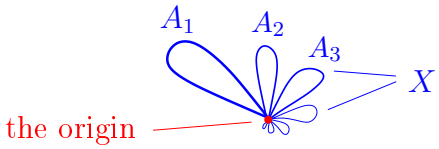
(b) Let $J \subset I$ be a subset. Show that $\bigvee_{i \in J} X_i$ is a deformation retract of $U \cup \bigvee_{i \in J} X_i$. (In particular, with $J = \varnothing$ this implies that $*$ is a deformation retract of U.)
Remark. You need to show that the wannabe deformation retraction which you wrote down is continuous. You might want to use a result along the lines of the Product-Quotient Mixing Lemma 13.6.

Exercise 50.9. Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Suppose that for each $i \in I$ we are given a point $x_i \in X_i$. We use these points to form the wedge $\bigvee_{i \in I} X_i$. Let $K \subset \bigvee_{i \in I} X_i$ be a compact subset.

(a) We suppose that each X_i is Hausdorff. Show that there exists a finite subset $J \subset I$ such that $K \subset \bigvee_{i \in J} X_i$.
Hint. For any $i \in I$ with $K \cap (X_i \setminus \{x_i\}) \neq \varnothing$ pick some point $a_i \in K \cap (X_i \setminus \{x_i\})$. We denote by A the union of these points. Show that A is a discrete subset of K.
(b) Show that in general the conclusion of (a) does not hold if we do not assume that the X_i are Hausdorff.

Exercise 50.10. We consider the subset X of $\mathbb{R}^2$ that is shown in the figure below. It is the union of infinitely many circles $A_i \subset \mathbb{R}^2$, $i \in \mathbb{N}$ that meet in the origin. Furthermore let $Y = \bigvee_{i \in \mathbb{N}} S^1$. There exists a fairly obvious bijection $f\colon X \to Y$ which sends each A_i to the i-th factor of Y.

(a) Is the map f continuous?
(b) Is the map f^{-1} continuous?



Exercise 50.11. In the following we consider the wedge $X := \bigvee_{i \in \mathbb{N}_0} [0,1]_i$ given by "wedging" the points $0 \in [0,1]_i$, $i \in \mathbb{N}_0$.

(a) Show that the sequences of points $x_i := \frac{1}{2^i} \in [0,1]_i \subset X$, $i \in \mathbb{N}_0$ does not converge to the wedge point.
(b) Show that X is not metrizable, i.e. show that it does not admit a metric such that the topology given by the metric agrees with the topology of X.

Exercise 50.12. Let $f, g\colon (X, x_0) \to (Y, y_0)$ be two maps between pointed topological spaces.

(a) Show that if f and g are homotopic rel $\{x_0\}$, then $f_* = g_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$.
(b) Does the conclusion of (a) hold if we only demand that f and g are homotopic, but we no longer demand that they are homotopic rel the base point?

Exercise 50.13. Let $f\colon X \to Y$ be a map between two topological spaces. Let $x_0 \in X$ be a base point. We write $y_0 = f(x_0)$ and we denote by $i\colon f(X) \to Y$ the inclusion map. Suppose that $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism. Does it follow that $i_*\colon \pi_1(f(X), y_0) \to \pi_1(Y, y_0)$ is also an isomorphism?

Exercise 50.14.

(a) Show that every topological graph is locally contractible.
(b) Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces. We suppose that each X_i is locally contractible. Show that the wedge $\bigvee_{i\in I} X_i$ is also locally contractible.

Remark. If you get away without using the Homotopy Quotient Theorem 14.7, then you should go carefully over your continuity arguments.

Exercise 50.15. Let T be a tree and let $v \in T$ be any vertex. Show that T admits a deformation retraction to v.
Remark. This is a slight refinement of the statement of Proposition 50.14 (2).

Exercise 50.16. Let N be a group, let $k \in \mathbb{N}_0$ and let $\varphi\colon N \to N$ be an isomorphism such that $\varphi^k = \mathrm{id}$. As on page 1127 we define the *semidirect product of N and $\mathbb{Z}_k$ with respect to φ* as the group $N \rtimes_\varphi \mathbb{Z}_k$ where the underlying set is given by the direct product $N \times \mathbb{Z}_k$, and where the multiplication is given by

$$(h, m) \cdot (h', m') := (h \cdot \varphi^m(h'), m + m').$$

(a) Show that $N \rtimes_\varphi \mathbb{Z}_k$ is indeed a group.
(b) Show that $N = \{(h, 0) \in N \rtimes_\varphi \mathbb{Z}_k \,|\, h \in N\}$ is a normal subgroup of $N \rtimes \mathbb{Z}_k$.
(c) Is $\mathbb{Z}_k = \{(0, n) \in N \rtimes_\varphi \mathbb{Z}_k \,|\, n \in \mathbb{Z}_k\} \subset N \rtimes_\varphi \mathbb{Z}_k$ necessarily a normal subgroup?
(d) Suppose that $k = 0$. Let $n \in \mathbb{N}$ and let K be a subgroup of N of some finite index m. Show that if K is preserved by φ, then $K \rtimes n\mathbb{Z}$ is a subgroup of $N \rtimes \mathbb{Z}$ of index $m \cdot n$.

Exercise 50.17. Let N be a group and let $\varphi\colon N \to N$ be an isomorphism.

(a) Let $k \in \mathbb{N}$ and let $G \subset N$ be a subgroup with $\varphi^k(G) = G$. Show that $k\mathbb{Z} \rtimes_\varphi G$ is a subgroup of $\mathbb{Z} \rtimes_\varphi N$.
(b) Show that in general not every subgroup of $\mathbb{Z} \rtimes_\varphi N$ is of the form $k\mathbb{Z} \rtimes_\varphi G$.
(c) Let π be a finite-index subgroup of $\mathbb{Z} \rtimes_\varphi N$.
  (i) Show that there exists a $k \in \mathbb{N}$ and a normal subgroup G of N with $\varphi^k(G) = G$ such that $[\mathbb{Z} \rtimes_\varphi N : \pi] = k \cdot [G : \pi]$ and such that π is *isomorphic* to $\mathbb{Z} \rtimes_{\varphi^k} G$.
  (ii) We suppose that N is finitely generated. Show that there exists a $k \in \mathbb{N}$ and a normal finite-index subgroup G of N such that $k\mathbb{Z} \rtimes_\varphi G \subset \pi$.

Exercise 50.18. Let

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K & \xrightarrow{\varphi} & \pi & \xrightarrow{\psi} & G & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
1 & \longrightarrow & \widetilde{K} & \xrightarrow{\widetilde{\varphi}} & \widetilde{\pi} & \xrightarrow{\widetilde{\psi}} & \widetilde{G} & \longrightarrow & 1
\end{array}$$

be a commutative diagram of group homomorphisms with the following properties:

(1) φ and $\widetilde{\varphi}$ are monomorphisms.
(2) ψ and $\widetilde{\psi}$ are epimorphisms.
(3) $\mathrm{im}(\varphi) = \ker(\psi)$ and $\mathrm{im}(\widetilde{\varphi}) = \ker(\widetilde{\psi})$.

Show that if two out of the three vertical maps α, β and γ are isomorphisms, then so is the third one.
Remark. We will generalize this statement later on in the Five Lemma 74.10.

Exercise 50.19. Let π be a group and let $\phi\colon \pi \times \langle t\rangle \to \mathbb{Z}$ be an epimorphism. We suppose that $k := \phi(t) \neq 0$. We set $\Gamma := \ker(\pi \xrightarrow{\phi} \mathbb{Z} \to \mathbb{Z}_k)$. Since ϕ is an epimorphism there exists an $x \in \pi$ such that $\gcd(\phi(x), k) = 1$. Let $\mu\colon \Gamma \to \Gamma$ be the automorphism that is given by $g \mapsto xgx^{-1}$. Show that the map

$$\begin{array}{rcl} \Gamma \rtimes_\mu \mathbb{Z} & \to & \pi \times \mathbb{Z} \\ (g, n) & \mapsto & (gx^n, n + \phi(g)) \end{array}$$

is an isomorphism.
Hint. You could use Exercise 50.18 together with Exercise 50.16 (d).
Remark.

(1) By Exercise 53.2 and Proposition 69.15 we know that if π is finitely generated (presented), then so is Γ. The exercise thus says that if a finitely presented group π admits an epimorphism onto $\mathbb{Z}$, then we can write a product $\pi \times \mathbb{Z}$ in many interesting and perhaps surprising ways as a semidirect product.
(2) We will prove a manifold analogue in Proposition 118.15.

Exercise 50.20. Let G and π be groups and let $\varphi\colon \pi \to \mathrm{Aut}(G)$ be a homomorphism. We consider the set $G \times \pi$ with the multiplication

$$(g, a) \cdot (h, b) \;:=\; (g \cdot \varphi(a)(h), a \cdot b).$$

(a) Show that $G \times \pi$ with the above multiplication is a group. This group is denoted by $G \rtimes_\varphi \pi$ and it is called the *semidirect product of G and π with respect to φ*.
(b) In what sense does this definition of the semidirect product of two groups generalize the concept introduced on page 1127?
(c) Let K be a group and let $\alpha\colon K \to \pi$ be an epimorphism onto a group π. We suppose that α admits a right-inverse, i.e. we suppose that there exists a homomorphism $\varphi\colon \pi \to K$ with $\alpha \circ \varphi = \mathrm{id}_\pi$. Show that K is isomorphic to a semidirect product of the form $\ker(\alpha) \rtimes \pi$.
(d) (i) Let $n \in \mathbb{N}$. Show that $\mathrm{GL}(n, \mathbb{R})$ is isomorphic to a semidirect product of the form $\mathrm{SL}(n, \mathbb{R}) \rtimes (\mathbb{R} \setminus \{0\})$.
(ii) For which $n \in \mathbb{N}$ is $\mathrm{GL}(n, \mathbb{R})$ isomorphic to a direct product $\mathrm{SL}(n, \mathbb{R}) \rtimes (\mathbb{R} \setminus \{0\})$?
(iii) Repeat (i) and (ii) with $\mathbb{C}$ instead of $\mathbb{R}$.

51. Basics of group theory

Before we continue with developing the theory of fundamental groups of topological spaces we want to introduce several basic constructions in group theory and we want state several facts which we will use throughout the remainder of this course.

51.1. **Free abelian groups and finitely generated abelian groups.** The following definition will stay with us throughout these notes.

Definition.

(1) Let S be a non-empty set. We write

$$\mathbb{Z}^S \;:=\; \text{all maps from } S \text{ to } \mathbb{Z}.$$

Furthermore, for S the empty set we write $\mathbb{Z}^S := \{0\}$. We consider $\mathbb{Z}^S$ with the obvious group structure, i.e. the group structure that is given by

$$\begin{array}{ccc} \mathbb{Z}^S \times \mathbb{Z}^S & \to & \mathbb{Z}^S \\ (f,g) & \mapsto & \left(\begin{array}{ccc} S & \to & \mathbb{Z} \\ s & \mapsto & f(s)+g(s) \end{array} \right). \end{array}$$

(2) Given any set S, we are mostly interested in the subgroup

$$\mathbb{Z}^{(S)} \;:=\; \text{all maps from } S \text{ to } \mathbb{Z} \text{ which are non-zero for only finitely many } s \in S.$$

We refer to $\mathbb{Z}^{(S)}$ as the free abelian group generated by S or as the free abelian group on the generating set S.

(3) We refer to the cardinality of S as the rank of $\mathbb{Z}^{(S)}$.

When working with free abelian groups it is usually helpful to use the following notation.

Notation. Let S be a set.

(1) Given $s \in S$ we denote by s also the map

$$\begin{array}{ccc} S & \to & \mathbb{Z} \\ t & \mapsto & \begin{cases} 1, & \text{if } t = s, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

This map lies of course in $\mathbb{Z}^{(S)}$. Thus we can view S as a subset of $\mathbb{Z}^{(S)}$.

(2) Given elements $s_1, \ldots, s_k \in S$ and $n_1, \ldots, n_k \in \mathbb{Z}$ we obtain the corresponding element $n_1 s_1 + \cdots + n_k s_k$ of $\mathbb{Z}^{(S)}$. We refer to $n_1 s_1 + \cdots + n_k s_k$ as a formal linear combination of elements in S. Every element of $\mathbb{Z}^{(S)}$ can be written as such a formal linear combination of finitely many elements of S. Indeed, for $f \in \mathbb{Z}^{(S)}$ we can write[528]

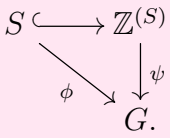
$$f \;=\; \sum_{s \in S \text{ with } f(s) \neq 0} f(s) \cdot s.$$

The following lemma says that the free abelian group generated by a set S has a "universal property". The lemma also says that possessing such a universal property classifies free abelian groups up to isomorphism.

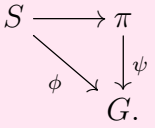
[528] It is straightforward to see that both sides define the same map $S \to \mathbb{Z}$, so they are the same element of $\mathbb{Z}^{(S)}$.

Lemma 51.1.

(1) Let S be a set and let $\phi\colon S \to G$ be a map to an abelian group G. Then there exists a unique homomorphism $\psi\colon \mathbb{Z}^{(S)} \to G$ that makes the following diagram commute



(2) Conversely let π be a group that admits a subset S with the following universal property: given any map $\phi\colon S \to G$ to an abelian group G there exists a unique homomorphism $\psi\colon \pi \to G$ that makes the following diagram commute



Then π is isomorphic to $\mathbb{Z}^{(S)}$.

Lemma 51.1 suggests the following definition.

Definition.

(1) We say that a group π is a free abelian group on a generating set S, if π has the universal property with respect to S. In other words, by Lemma 51.1, G needs to be isomorphic to $\mathbb{Z}^{(S)}$.

(2) We say that a group π is free abelian if it is a free abelian group on some generating set S.

Proof.

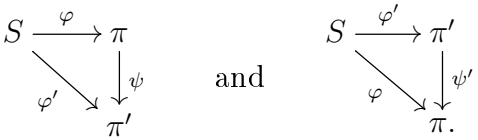
(1) We consider the map[529]

$$\begin{array}{rcl} \psi\colon \mathbb{Z}^{(S)} & \to & G \\ f = \sum_{i=1}^{n} n_i s_i & \mapsto & \psi(f) := \sum_{i=1}^{n} n_i \cdot \phi(s_i). \end{array}$$

It is straightforward to verify that ψ has the desired property and that it is the unique such homomorphism.

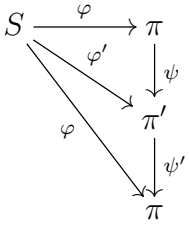
(2) Let π be a group that has the universal property with respect to S. We need to show that π is isomorphic to $\mathbb{Z}^{(S)}$. It is more elegant to show that it is isomorphic to any other group π' that has the universal property with respect to S.

This argument is similar to many other "universal property" proofs in algebra. We consider the inclusion maps $\varphi\colon S \to \pi$ and $\varphi'\colon S \to \pi'$. By the universal property of φ and φ' there exist homomorphisms $\psi\colon \pi \to \pi'$ and $\psi'\colon \pi' \to \pi$ that make the following diagrams commute:

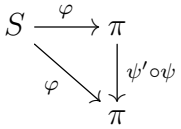


[529]Here we assume that G is written as an additive group. If G is a multiplicative group we would need to replace $n_i \cdot \phi(s_i)$ by $\phi(s_i)^{n_i}$ and we would need to replace the sum by a product.

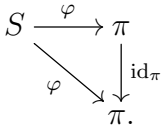
In particular we get the commutative diagrams



which simplifies to



and of course we also have the commutative diagram



It follows from the uniqueness part of the universal property applied to the two diagrams on the right that $\psi' \circ \psi = \mathrm{id}_\pi$. The same way one shows that $\psi \circ \psi' = \mathrm{id}_{\pi'}$. In particular ψ and ψ' are isomorphisms. ■

Lemma 51.2. (Subgroup-of-Free Abelian Group Lemma)

(1) Any subgroup G of a free abelian group F is again a free abelian group.
(2) If $\varphi\colon \mathbb{Z}^{(S)} \to \mathbb{Z}^{(T)}$ is a monomorphism, then[530] $\#S \leq \#T$.
(3) If $\varphi\colon \mathbb{Z}^{(S)} \to \mathbb{Z}^{(T)}$ is an epimorphism, then $\#S \geq \#T$.

Proof. The first two statements are proved in [**Rot1988**, Theorem 9.3]. It remains to prove the third statement. Now let $\varphi\colon \mathbb{Z}^{(S)} \to \mathbb{Z}^{(T)}$ be an epimorphism. For each $t \in T$ there exists therefore some $\phi(t) \in S$ with $\varphi(\phi(t)) = t$. By Lemma 51.1 there exists a unique homomorphism $\psi\colon \mathbb{Z}^{(T)} \to \mathbb{Z}^{(S)}$ with $\psi(t) = \phi(t)$ for all $t \in T$. Therefore we have $\varphi(\psi(t)) = \varphi(\phi(t)) = t$ for all t, so $\varphi \circ \psi = \mathrm{id}$, in particular ψ is a monomorphism. It follows from (2) that $\#T \leq \#S$. ■

Definition.

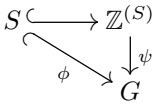
(1) If F is a free abelian group, then we define the rank of F as the cardinality of any set S with $F \cong \mathbb{Z}^{(S)}$.[531]
(2) Given an abelian group G we say that $S \subset G$ is a generating set if the canonical homomorphism $\mathbb{Z}^{(S)} \to G$ is an epimorphism.[532]
(3) Given an abelian group G we say that $S \subset G$ is a basis if the canonical homomorphism $\mathbb{Z}^{(S)} \to G$ is an isomorphism.
(4) We say that an abelian group G is finitely generated if it admits a finite generating set.

Example. Let S be a finite set with m elements $s_1, \ldots, s_m$. Let $\phi\colon S \to \mathbb{Z}^m$ be the map given by $\phi(s_i) = e_i$. Then the resulting homomorphism $\Psi\colon \mathbb{Z}^{(S)} \to \mathbb{Z}^m$ is an isomorphism. Put differently, $\mathbb{Z}^m$ is a free abelian group of rank m.

[530] Here we denote by $\#S$ and $\#T$ the cardinalities of the sets S and T, as defined on page 94.

[531] It follows from the Subgroup-of-Free Abelian Group Lemma 51.2 and the Bernstein-Schröder Theorem 0.11 that the rank of a free abelian group is well-defined.

[532] Here we mean by "canonical map $\mathbb{Z}^{(S)} \to G$" the unique homomorphism $\psi\colon \mathbb{Z}^{(S)} \to G$ from Lemma 51.1 that makes the following diagram commute



where $S \to G$ is the inclusion map.

We leave it to the reader to provide the proof of the following useful characterizations of generating sets.

Lemma 51.3. Let G be an abelian group and let $S \subset G$ be a subset. Then the following two statements are equivalent:

(1) S is a generating set of G.
(2) Any subgroup H of G that contains S is already equal to G.

We use this opportunity to recall the classification of finitely generated abelian groups which had been proved in the algebra course. A proof is for example also provided in [**Hun1980**, Chapter II.2].

Theorem 51.4. (Finitely Generated Abelian Group Classification Theorem) Let A be a finitely generated abelian group. There exist non-zero natural numbers $a_1, \dots, a_k$ with $a_i | a_{i+1}$ for $i = 1, \dots, k-1$ and an $r \in \mathbb{N}_0$ such that

$$A \;\cong\; \mathbb{Z}^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}_{a_i}.$$

Furthermore the numbers $a_1, \dots, a_k$ and r are uniquely determined by A.

Definition. Given a finitely generated abelian group A we denote the direct sum of the cyclic groups on the right-hand side of the Finitely Generated Abelian Group Classification Theorem 51.4 as the canonical form of A.

Example. The Chinese Remainder Theorem says that for any coprime m and n we have an isomorphism

$$\begin{array}{rcl} \mathbb{Z}_{mn} & \xrightarrow{\cong} & \mathbb{Z}_m \times \mathbb{Z}_n = \mathbb{Z}_m \oplus \mathbb{Z}_n \\ k + mn\mathbb{Z} & \mapsto & (k + m\mathbb{Z}, k + n\mathbb{Z}). \end{array}$$

For example consider $A = \mathbb{Z}_3 \oplus \mathbb{Z}_5$. It is not in the form of the Finitely Generated Abelian Group Classification Theorem 51.4, but we have the isomorphism $\mathbb{Z}_3 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{15}$, which is of the desired form. Thus $\mathbb{Z}_{15}$ is the canonical form of $\mathbb{Z}_3 \oplus \mathbb{Z}_5$.

We will not provide a proof for the Finitely Generated Abelian Group Classification Theorem 51.4. But we will sketch the proof of a weaker version of the Classification Theorem 51.4. Before we can get to this point we need to prove the following theorem.

Theorem 51.5. (Smith Normal Form Theorem) Let V be an $m \times n$–matrix over $\mathbb{Z}$. There exist two matrices $A \in \mathrm{GL}(m, \mathbb{Z})$ and $B \in \mathrm{GL}(n, \mathbb{Z})$[533] such that[534]

$$A \cdot V \cdot B \;=\; \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & p_t & 0 \\ 0 & 0 & 0 & 0_{(m-t)\times(n-t)} \end{pmatrix} \;\in\; \mathrm{M}(m \times n, \mathbb{Z})$$

for some $p_1, \dots, p_t \in \mathbb{N}$. The matrix on the right is sometimes called a Smith normal form of V.

[533] The matrices A and B are invertible over $\mathbb{Z}$, i.e. their determinants are ± 1.
[534] Here the last row and column of zeros can be arbitrarily large.

Remark. In Exercise 51.19 we will discuss a more refiend version of the Smiith Normal Form Theorem 51.5, which then also gives us a uniqueness statement for the Smith normal form. Alternativel we refer to [**NewM1972**, Chapter 15] and [**Nor2012**, Chapter 1] for more information.

Proof. First note that if an $m \times n$–matrix P is obtained from an $m \times n$–matrix Q by swapping two rows, then P is obtained by left-multiplying Q by an $m \times m$–matrix of the form[535]

$$\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}.$$

Similarly if P is obtained from an $m \times n$–matrix Q by adding the k-th row to the l-th row, then P is obtained by left-multiplying Q by an $m \times m$–matrix of the form

$$\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}$$

where the extra 1 sits in the $(1,k)$-entry. Both these $m \times m$-matrices lie in $\mathrm{GL}(m,\mathbb{Z})$. If we do the same operations for columns, then this corresponds to right multiplication by the same type of matrices in $\mathrm{GL}(n,\mathbb{Z})$ as above.

Now let V be an $m \times n$–matrix over $\mathbb{Z}$. If V is the zero matrix, then there is nothing to show. So suppose that V is not the zero matrix. We perform the following steps:

(1) After swapping rows and columns we can arrange that the absolute value of the $(1,1)$-entry is less or equal than the absolute value of all other entries. After possibly multiplying the first column by -1 we can assume that the $(1,1)$-entry v_{11} is positive.
(2) By adding and subtracting the first column from the other columns we can arrange that all entries in the first row, except for the $(1,1)$-entry, lie between 0 and $v_{11} - 1$.
(3) The same way as in (2) we can arrange that all entries in the first column, except for the $(1,1)$-entry, lie between 0 and $v_{11} - 1$.

Now we have to distinguish two cases:

(1) If there exists outside of the $(1,1)$-entry another entry in the first row or column that is non-zero, then we start again with (1). Note that the new entry $(1,1)$-entry will be less than before. So this process will come to a halt after finitely many steps.
(2) If all entries in the first row and column, except for the $(1,1)$-entry are zero, then

$$V = \begin{pmatrix} v_{11} & 0 \\ 0 & V' \end{pmatrix}$$

for some $(m-1) \times (n-1)$–matrix V' and we perform the same operations on the smaller matrix V'. ∎

[535] Here is the reality check whether that is really correct: we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}.$$

The second statement of the following lemma is a somewhat weaker version of Theorem 51.4.

Lemma 51.6. (Matrix Quotient Lemma)

(1) Let V be an $m \times n$–matrix over $\mathbb{Z}$. For matrices $A \in \mathrm{GL}(m, \mathbb{Z})$ and $B \in \mathrm{GL}(n, \mathbb{Z})$ we have an isomorphism

$$\mathbb{Z}^m / AVB\mathbb{Z}^n \;\cong\; \mathbb{Z}^m / V\mathbb{Z}^n.$$

(2) Let A be a finitely generated abelian group. Then there exists an $r \geq 0$ and there exist $p_1, \ldots, p_t \in \mathbb{N}$ such that

$$A \;\cong\; \mathbb{Z}^r \oplus \bigoplus_{i=1}^{t} \mathbb{Z}_{p_i}.$$

Proof. (1) The proof of the first statement is elementary. We leave it to the reader to fill in the details.

(2) Let A be a finitely generated abelian group. By the definition of a finitely generated abelian group there exists an epimorphism $\psi\colon \mathbb{Z}^m \to A$. Since $\mathbb{Z}$ is a Noetherian ring we know that $\ker(\psi)$ is again a finitely generated free abelian group of rank $n \leq m$. In particular there exists an isomorphism $v\colon \mathbb{Z}^n \to \ker(\psi)$. We denote by V the $m \times n$–matrix that represents the resulting map $\mathbb{Z}^n \to \ker(\psi) \subset \mathbb{Z}^m$ with respect to the standard bases. Then $A \cong \mathbb{Z}^m / V\mathbb{Z}^n$. We pick matrices A and B as in (2). From (1) it follows that $\mathbb{Z}^m / V\mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^m / AVB\mathbb{Z}^n$. But the latter is isomorphic to $\mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_t} \oplus \mathbb{Z}^{m-n}$. ■

Definition. Let A be an abelian group.

(1) If A is finitely generated, then we refer to the r in the statement of the Finitely Generated Abelian Group Classification Theorem 51.4 as the rank of the finitely generated abelian group A. We denote it by $\mathrm{rank}(A)$.[536]

(2) We refer to

$$\mathrm{Tor}(A) \;:=\; \{a \in A \mid \text{ there exists an } n \in \mathbb{N} \text{ with } na = 0\}$$

as the torsion subgroup of A.[537] If $\mathrm{Tor}(A)$ is the trivial subgroup, then we say that A is torsion-free.

In the following three lemmas we collect several basic facts about finitely generated abelian groups. Most of the statements follow more or less directly from the Finitely Generated Abelian Group Classification Theorem 51.4. We will not prove the lemmas. We leave it to the reader to deduce the statements from the Finitely Generated Abelian Group Classification Theorem 51.4 and the algebra course the reader took.

Lemma 51.7. (Rank Additivity Lemma) Let

$$0 \;\to\; A \;\to\; B \;\to\; C \;\to\; 0$$

[536] If A is a finitely generated free abelian group, then this definition of rank agrees with the definition given for free abelian groups on page 1147.

[537] In Exercise 51.16 we will see that A is indeed a subgroup.

be a short exact sequence of abelian groups[538] such that at least two of these groups are finitely generated. Then the following holds:

(1) all three groups are finitely generated abelian groups,
(2) $\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C)$,
(3) if A and C are torsion-free, then B is also torsion-free.

Remark. Let B be a finitely generated group and let $\varphi\colon B \to C$ be an epimorphism. Note that C is evidently also finitely generated. If B is abelian it thus follows from the Rank Additivity Lemma 51.7 (1) that the kernel $\ker(\varphi)$ is also finitely generated. Note that if B is not abelian, then this conclusion does not necessarily hold. For example in Exercise 51.17 we will give an example of a group epimorphism $\varphi\colon B \to C$ from a non-commutative group B on two generators onto an abelian group C, such that the kernel is infinitely generated.

Lemma 51.8. Let $A \in \mathrm{M}(n \times n, \mathbb{Z})$ be a matrix. Then

$$\mathbb{Z}^n/A\mathbb{Z}^n \text{ is finite} \quad \Longleftrightarrow \quad \det(A) \neq 0.$$

Furthermore, if $\det(A) \neq 0$, then

$$0 \;\to\; \mathbb{Z}^n \;\xrightarrow{A\cdot}\; \mathbb{Z}^n \;\to\; \mathbb{Z}^n/A\mathbb{Z}^n \;\to\; 0$$

is a short exact sequence and $\mathbb{Z}^n/A\mathbb{Z}^n$ is a finite group with $|\det(A)|$ elements.

We also have the following lemma.

Lemma 51.9.

(1) If A, B and C are finitely generated abelian groups, then

$$A \oplus C \cong B \oplus C \quad \Longrightarrow \quad A \cong B.$$

(2) Let A be a subgroup of an abelian group B.
   (a) $\operatorname{Tor}(A)$ is a subgroup of $\operatorname{Tor}(B)$.
   (b) If B is finitely generated, then A is also finitely generated, and we have the inequality $\operatorname{rank}(A) \leq \operatorname{rank}(B)$.
(3) If A is a subgroup of a free abelian group of rank r, then A is a free abelian group with rank less or equal than r.[539]
(4) If A is a subgroup of a free abelian group B and if $\operatorname{rank}(A) < \operatorname{rank}(B)$, then A is a proper subgroup, i.e. $A \neq B$.
(5) Let $\varphi\colon A \to B$ be a homomorphism between two finitely generated abelian groups that are abstractly isomorphic.[540] If φ is an epimorphism, then φ is already an isomorphism.
(6) If A is a finitely generated abelian group and $\varphi\colon A \to B$ is an epimorphism, then B is also finitely generated and $\operatorname{rank}(A) \geq \operatorname{rank}(B)$.

Abelian groups that are not finitely generated are not classified and they tend to be significantly harder to deal with. For example the following lemma shows that not every torsion-free abelian group is a free abelian group.

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[538] Recall that this means that the map $A \to B$ is injective, that $B \to C$ is surjective and that the image of $A \to B$ agrees with the kernel of $B \to C$.

[539] In fact this statement is already contained in the Subgroup-of-Free Abelian Group Lemma 51.2.

[540] For example A and B could be two free abelian groups of the same finite rank.

Lemma 51.10. The abelian group $(\mathbb{Q}, +)$ is torsion-free but not free abelian.

Proof. Clearly $(\mathbb{Q}, +)$ is torsion-free. If it was free abelian, then there would exist an epimorphism to $\mathbb{Z}$. In Exercise 51.1 we will show that such an epimorphism cannot exist. ■

In Exercise 51.3 (1) we will see that the conclusion of Lemma 51.9 (1) does not hold for general abelian groups. But the following statement actually does hold.

Proposition 51.11. Let A and B be any two abelian groups. If C is a finitely generated abelian group, then

$$A \oplus C \cong B \oplus C \quad \Longrightarrow \quad A \cong B.$$

Proof. We will consider a special case in Exercise 51.3 (b). The general case is proved in [**Cohn1956, WalkE1956**]. ■

51.2. **The free product of groups.** We first recall the definition of the direct product of two groups G and H that we already gave on page 1092.

Definition. The direct product of two groups G and H is defined as

$$G \times H \ := \ \{(g, h) \mid g \in G \text{ and } h \in H\}$$

together with the group structure that is given by

$$(g_1, h_1) \cdot (g_2, h_2) \ := \ (g_1 g_2, h_1 h_2)$$

for $g_1, g_2 \in G$ and $h_1, h_2 \in H$. We call the group $G \times H$ the direct product of the groups G and H. Via the monomorphisms

$$\begin{array}{ccc} G & \to & G \times H \\ g & \mapsto & (g, e) \end{array} \quad \text{and} \quad \begin{array}{ccc} H & \to & G \times H \\ h & \mapsto & (e, h) \end{array}$$

we can view G and H as subgroups of $G \times H$. Note that these subgroups commute, i.e. for every $g \in G$ and $h \in H$ we have the equality $(g, e) \cdot (e, h) = (g, h) = (e, h) \cdot (g, e)$.

Now we will introduce the "free product" $G * H$ of two groups G and H. The definition is perhaps initially harder than the definition of the "direct product" $G \times H$. But for many, perhaps even most applications, the free product is more useful.

Definition. Let G be and H be two groups. We consider the set $G * H$ of all finite sequences $(x_1, \ldots, x_m)$ such that the following conditions are satisfied:

(a) each x_i lies in one of the groups G or H,
(b) no x_j is the neutral element of G or of H,
(c) any two consecutive x_j's lie in two different groups.

Here we also allow the "empty sequence" (). Such sequences are sometimes called reduced words in G and H.

Let $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$ be two elements in $G * H$. We define the product of these two elements as follows:

(1) First suppose that x_m and y_1 lie in different groups. Then we define

$$(x_1, \ldots, x_m) \cdot (y_1, \ldots, y_n) \ := \ (x_1, \ldots, x_m, y_1, \ldots, y_n).$$

(2) Now suppose that x_m and y_1 lie in the same group. Let $j \in \{1, \dots, \min\{m,n\}\}$ be the maximal element such that $x_{m+1-i} = y_i^{-1}$ for $i = 1, \dots, j$. If $j < \min\{m,n\}$, then we define[541]

$$(x_1, \dots, x_m) \cdot (y_1, \dots, y_n) \;:=\; (x_1, \dots, \underbrace{x_{m-j} \cdot y_{j+1}}_{\neq e}, \dots, y_n).$$

If $j = m$, then we define

$$(x_1, \dots, x_m) \cdot (y_1, \dots, y_n) \;:=\; (y_{m+1}, \dots, y_n).$$

We proceed similarly if $j = n$.

Another way of stating the product formula is as follows: given two sequences $(x_1, \dots, x_m)$ and $(y_1, \dots, y_n)$ we stack them together $(x_1, \dots, x_m, y_1, \dots, y_n)$ and then we delete any occurrence of a subsequence of the form a, a^{-1} for $a \in G$ or $a \in H$ and if a subsequence is of the form a, b with $a, b \in G$ or $a, b \in H$, then we replace it by ab.

We henceforth refer to $G * H$ together with this product structure as the free product of G and H.

We have the following important lemma.

Lemma 51.12. The free product of two groups G and H is again a group. The neutral element of $G * H$ is hereby given by the empty sequence $()$ and the inverse of an element $(x_1, \dots, x_m) \in G * H$ is given by

$$(x_1, \dots, x_m)^{-1} \;=\; (x_m^{-1}, \dots, x_1^{-1}) \;\in\; G * H.$$

Proof. It is clear that the empty sequence is a neutral element. It follows immediately from the definition of the product that the inverse of an element $(x_1, \dots, x_m) \in G * H$ is given by $(x_m^{-1}, \dots, x_1^{-1})$. It remains to show that the multiplication satisfies associativity.

In principle one can show associativity naively "by hand", but there are surprisingly many cases one needs to distinguish, so that the proof becomes rather painful.

We will therefore follow a different approach. We write $W = G * H$ and we denote by $(P(W), \circ)$ the group of all permutations of W, i.e. $P(W)$ is the set of all bijections from W to W and the group structure is given by composition of maps. For $g \in G$ we consider the map[542]

$$\begin{array}{rcl} \lambda_g \colon W & \to & W \\ (x_1, \dots, x_m) & \mapsto & \begin{cases} (g, x_1, x_2, \dots, x_m), & \text{if } x_1 \in H, \\ (gx_1, x_2, \dots, x_m), & \text{if } x_1 \in G \text{ and } g \neq x_1^{-1}, \\ (x_2, \dots, x_m), & \text{if } g = x_1^{-1}. \end{cases} \end{array}$$

It is now straightforward to see that for any $g, g' \in G$ and any $(x_1, \dots, x_m) \in W$ we have $g(g' \cdot (x_1, \dots, x_m)) = (gg') \cdot (x_1, \dots, x_m)$. This means that $\lambda_g \circ \lambda_{g'} = \lambda_{gg'}$. It is now straightforward to see that the map

$$\begin{array}{rcl} \lambda \colon G & \to & P(W) \\ g & \mapsto & \lambda_g \end{array}$$

[541] Since x_m and y_1 lie in the same group, the elements x_{m-j} and y_{j+1} also lie in the same group, so it makes sense to consider $x_{m-j} \cdot y_{j+1}$.

[542] It is straightforward to see that the sequences on the right do indeed lie in $G * H$.

is a group homomorphism. Similarly we define $\lambda\colon H \to P(W)$. Now we consider the map

$$\begin{array}{rcl} \lambda\colon W & \to & P(W) \\ (x_1, \ldots, x_n) & \mapsto & \lambda_{x_1} \circ \cdots \circ \lambda_{x_n}. \end{array}$$

This map is injective since $\lambda(x_1, \ldots, x_n)$ applied to the empty word $()$ gives the word $(x_1, \ldots, x_n)$. Hence $\lambda(x_1, \ldots, x_n)$ is not the identity map from $W \to W$, i.e. $\lambda(x_1, \ldots, x_n)$ is non-trivial in $P(W)$.

Furthermore the map $\lambda\colon W \to P(W)$ satisfies $\lambda(x \cdot y) = \lambda_x \circ \lambda_y$. Put differently, λ defines an injective map $\lambda\colon (W, \cdot\,) \to (P(W), \circ)$ that preserves the product structure. But since $(P(W), \circ)$ satisfies the associativity law we now see that $(W, \cdot\,)$ also satisfies the associativity law. ∎

The following lemma summarizes two main properties of the free product of two groups.

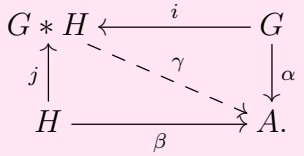
Lemma 51.13. Let G and H be two groups. The following two statements hold:

(1) The maps

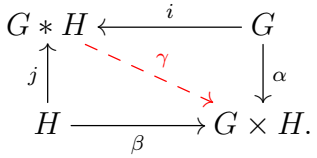
$$\begin{array}{rcl} i\colon G & \to & G * H \\ g & \mapsto & \begin{cases} (g), & \text{if } g \neq e, \\ (), & \text{if } g = e \end{cases} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} j\colon H & \to & G * H \\ h & \mapsto & \begin{cases} (h), & \text{if } h \neq e, \\ (), & \text{if } h = e \end{cases} \end{array}$$

are monomorphisms.

(2) For any two group homomorphisms $\alpha\colon G \to A$ and $\beta\colon H \to A$ there exists a unique group homomorphism $\gamma\colon G * H \to A$ such that the following diagram commutes

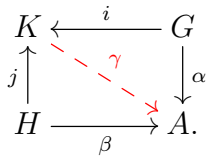


Example. Let G and H be groups. We denote by $\alpha\colon G \to G \times H$ and $\beta\colon H \to G \times H$ the obvious inclusion maps. It follows from Lemma 51.13 there exists a unique homomorphism $\gamma\colon G * H \to G \times H$ which makes the following diagram commute:



Remark. In the literature the direct product of two groups often gets introduced in a slightly different way, namely by a "universal property". More precisely, let G be and H be two groups. The *direct product of G and H* is then defined as a group K together with two homomorphisms $i\colon G \to K$ and $j\colon G \to K$ which satisfy the following universal property: for every two group homomorphisms $\alpha\colon G \to A$ and $\beta\colon H \to A$ there exists a *unique* group

homomorphism $\gamma\colon K \to A$ such that the following diagram commutes



Lemma 51.13 can now be reinterpreted as saying that the direct product of two groups always exists. An argument that is very similar to the one we gave on page 1149 shows that the direct product is unique in the following sense: If $(K, i\colon G \to K, j\colon H \to K)$ and $(K', i'\colon G \to K', j'\colon H \to K')$ both satisfy the above universal property, then there exists a unique isomorphism $\Phi\colon K \to K'$ such that $i' = \Phi \circ i$ and $j' = \Phi \circ j$. [543]

Proof. It is clear that the maps i and j are injective, furthermore it follows immediately from the definition of the group structure on $G*H$ that the maps are group homomorphisms.

Now let $\alpha\colon G \to A$ and $\beta\colon H \to A$ be two group homomorphisms. For an element $(g_1, h_1, \dots, g_k, h_k) \in G * H$ with $g_i \in G, h_i \in H$, $i = 1, \dots, n$ we define

$$\gamma(g_1, h_1, \dots, g_k, h_k) \;:=\; \alpha(g_1) \cdot \beta(h_1) \cdot \dots \cdot \alpha(g_k) \cdot \beta(h_k).$$

The same way we define γ for all other elements in $G * H$. Now one can show easily that γ is a group homomorphism.

It remains to show the uniqueness of γ. Any element in $G * H$ is a product of elements of the form $i(g)$, $g \in G$ and $j(h)$, $h \in H$. It follows that any homomorphism $\delta\colon G * H \to A$ is already uniquely determined by the values of δ on all elements of the form $i(g)$, $g \in G$ and $j(h)$, $h \in H$. Put differently, there is at most one homomorphism $\delta\colon G * H \to A$ with $\delta(i(g)) = \alpha(g)$ for all $g \in G$ and $\delta(i(h)) = \beta(h)$ for all $h \in H$. ■

Remark.

(1) We will use the monomorphisms from Lemma 51.13 to view G and H as subgroups of $G * H$.

(2) If there is no danger of confusion, then we often shorten $(x_1, \dots, x_m) \in G * H$ to the notation $x_1 \dots x_m$. This notation is coherent with the convention (1). Note that if $g \in G$ and $h \in H$ are non-trivial elements, then the sequences (g, h) and (h, g) both lie in $G * H$ and they are different. With the convention that we just introduced that means that $g \cdot h \neq h \cdot g$. Put differently, a non-trivial element of G never commutes with a non-trivial element of H.

(3) The construction of the free product of two groups generalizes in an obvious way to the free product of finitely many, in fact even to infinitely many groups.

Let t be a symbol. We define

$$\langle t \rangle \;:=\; \{\dots, t^{-2}, t^{-1}, e, t, t^2, \dots\}.$$

The set $\langle t \rangle$ admits an obvious group structure that is given by $t^i \cdot t^j := t^{i+j}$. This group is called *the infinite cyclic group generated by* t. The map $\mathbb{Z} \to \langle t \rangle$, $n \mapsto t^n$ is evidently an isomorphism. Now we come to one of the main definitions in group theory.

[543] Why does $K = G \times H$ with the obvious maps $i\colon G \to K$ and $j\colon H \to K$ not satisfy this universal property? Or does it?

Definition. For a set S we refer to

$$\langle S \rangle \ := \ \text{free product of the groups } \langle s \rangle,\ s \in S$$

as the free group on the (generating) set S. Via the map $s \mapsto (s)$ we can and will view S as a subset of $\langle S \rangle$.

Example. If $S = \{t_1, \ldots, t_k\}$, then we obtain the group

$$\langle t_1, \ldots, t_k \rangle \ := \ \langle t_1 \rangle * \cdots * \langle t_k \rangle$$

which we also refer to as the *free group on the generators* $t_1, \ldots, t_k$. Each element of the free group $\langle t_1, \ldots, t_k \rangle$ is thus of the form

$$t_{r_1}^{s_1} t_{r_2}^{s_2} \ldots t_{r_l}^{s_l}$$

with $r_i \neq r_{i+1}$ for $i = 1, \ldots, l-1$ and $s_i \neq 0$ for $i = 1, \ldots, l$.

Convention. Sometimes we denote the free group on k generators just by F_k.

Example. We consider the free group $\langle x, y \rangle$ on the two generators x and y. Elements in this group are of the form $xy^3x^{-2}y$ or $y^{-2}xy$. The multiplication is the "obvious one", for example

$$(xy^3x^{-1}y^2) \cdot (y^{-2}x^3yxy^{-1}) \ = \ xy^3x^2yxy^{-1}.$$

The group is not abelian since $xy \neq yx$.[544]

Lemma 51.14. Let S be a set. The free group $\langle S \rangle$ is torsion-free. In other words, every non-trivial element in the free group $\langle S \rangle$ has infinite order.

Proof. We will prove the lemma in Exercise 51.4. ■

The following innocent looking lemma summarizes the key property of free groups.

Lemma 51.15. Let S be a set, let G be an arbitrary group and let $g\colon S \to G$ be a map. Then there exists a unique homomorphism $\varphi\colon \langle S \rangle \to G$ with $\varphi(s) = g(s)$ for every $s \in S$.

Proof. For $s \in S$ the map

$$\begin{array}{ccc} \langle s \rangle & \to & G \\ s^n & \mapsto & g(s)^n \end{array}$$

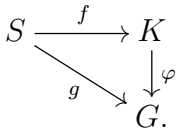
is evidently the unique homomorphism $\varphi_s\colon \langle s \rangle \to G$ with $\varphi_s(s) = g(s)$. It now follows from the obvious generalization of Lemma 51.13 to the case of the free product of arbitrarily many groups that there exists a unique homomorphism $\varphi\colon \langle S \rangle \to G$ with $\varphi(s) = \varphi_s(s) = g(s)$ for $s \in S$. ■

Remark. Sometimes in the literature free groups are defined via a universal property. More precisely, let S be a set. A *free group on* S is sometimes defined as a group K together with a map $f\colon S \to K$ which satisfies the following universal property: for any map $g\colon S \to G$ from the set S to a group G[545] there exists a unique group homomorphism

[544] Why is $xy \neq yx$?

[545] Note that in contrast to Lemma 51.1, the group G is now not necessarily abelian.

$\varphi\colon K \to G$ such that the following diagram commutes



In this language our above Lemma 51.15 says that the obvious map $S \to \langle S\rangle$ is a free group on S in the above sense. As before, a slight variation on the argument on page 1149 shows that the free group on S is unique in an appropriate sense.

Definition.

(1) A group G is called free, if it is isomorphic to the free group $\langle S\rangle$ on some set S. We say a subset $S \subset G$ is a basis, if the natural map[546] $\langle S\rangle \to G$ is an isomorphism.
(2) Given group G we say that $S \subset G$ is a generating setif the natural homomorphism $\langle S\rangle \to G$ is an epimorphism.Sometimes we also say that the elements in S are generators of G.
(3) We say that a group G is finitely generated if it admits a finite generating set.

Examples.

(1) As on page 1074 we consider the following two self-homeomorphisms of $\mathbb{R}^2$:

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1) \end{array}$$

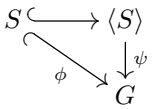
and we denote by G the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by A and B. This means that

$$G = \begin{array}{c} \text{all self-homeomorphisms of } \mathbb{R}^2 \text{ that can be written} \\ \text{as a finite concatenation of the maps } A, B, A^{-1} \text{ and } B^{-1}. \end{array}$$

We saw in Exercise 25.11 that $\mathbb{R}^2/G$ equals the Klein bottle K and using the Fundamental Group-via-Actions Theorem 48.16 we saw on page 1091 that $\pi_1(\text{Klein bottle}) \cong G$. The group G is finitely generated in the above sense. More precisely, it follows immediately from the definition of G that $S = \{A, B\} \subset G$ is a generating set for G.
(2) For any set S there exists by Lemma 51.15 a unique homomorphism $\langle S\rangle \to \mathbb{Z}^{(S)}$ that sends $s \in S$ to s viewed as an element in $\mathbb{Z}^{(S)}$. Since the set $S \subset \mathbb{Z}^{(S)}$ generates $\mathbb{Z}^{(S)}$ we see that this homomorphism is an epimorphism. It follows that any finitely generated abelian group in the sense of page 1149 is also a finitely generated group in the above sense.

The subgroup of a free abelian group is again a free abelian group. Does the same hold for free groups? We record this as a question for future reference.

[546]Here we mean by "canonical map $\langle S\rangle \to G$" the unique homomorphism $\psi\colon \langle S\rangle \to G$ from Lemma 51.15 that makes the following diagram commute



where $S \to G$ is the inclusion map.

Question 51.16. Is every subgroup of a free group again a free group?

The next lemma says in particular that free groups on two sets are isomorphic if and only if the cardinalities of the sets are the same.

Lemma 51.17. Let S and T be two sets.

(1) If there exists an epimorphism $\langle S\rangle \to \langle T\rangle$, then $\#S \geq \#T$.
(2) If $\langle S\rangle$ is isomorphic to $\langle T\rangle$, then $\#S = \#T$.
(3) If given $n \in \mathbb{N}$ we denote by F_n the free group on n generators, then for $k \neq l$ the groups F_k and F_l are not isomorphic.

Proof.

(1) We first recall that we had just seen in the previous example that for any set U there exists a unique epimorphism $\psi_U\colon \langle U\rangle \to \mathbb{Z}^{(U)}$ that sends $u \in U$ to u viewed as an element in $\mathbb{Z}^{(U)}$.

Now let S and T be two sets and let $\varphi\colon \langle S\rangle \to \langle T\rangle$ be an epimorphism. By Lemma 51.1 there exists a unique homomorphism $\widetilde{\varphi}\colon \mathbb{Z}^{(S)} \to \mathbb{Z}^{(T)}$ that sends $s \in S$ to $\psi_T(\varphi(s)) \in \mathbb{Z}^{(T)}$. We obtain the following diagram of homomorphisms

$$\begin{array}{ccc} \langle S\rangle & \xrightarrow{\ \varphi\ } & \langle T\rangle \\ {\scriptstyle\psi_S}\downarrow & & \downarrow{\scriptstyle\psi_T} \\ \mathbb{Z}^{(S)} & \xrightarrow{\ \widetilde{\varphi}\ } & \mathbb{Z}^{(T)}. \end{array}$$

It follows immediately from the definition of $\widetilde{\varphi}$ that this diagram commutes.[547] Since the top horizontal is an epimorphism and since the vertical maps are epimorphisms we see that the bottom horizontal map is also an epimorphism. It follows from the Subgroup-of-Free Abelian Group Lemma 51.2 that $\#S \geq \#T$.

(2) Let $f\colon \langle S\rangle \to \langle T\rangle$ be an isomorphism. We apply (1) to f and to f^{-1} and we obtain that $\#S \geq \#T$ and $\#T \geq \#S$. It follows from the Bernstein-Schröder Theorem 0.11 that $\#S = \#T$.

(3) This statement is an immediate consequence of (2).[548] ∎

51.3. **An alternative definition of the free product of groups.** For some purposes it is sometimes useful to work with a different model for the free product of two groups.

Definition. Let A and B be two groups. We set[549]

$$S(A,B) \ := \ \{\text{all finite sequences } (g_1,\ldots,g_k) \text{ with each } g_i \text{ in } A \text{ or } B\}$$

[547] Here we use that a homomorphism $\gamma\colon \langle S\rangle \to G$ is uniquely determined by the values of γ on S. Put differently, if $\alpha\colon \langle S\rangle \to G$ and $\beta\colon \langle S\rangle \to G$ are two homomorphisms with $\alpha(s) = \beta(s)$ for all $s \in S$, then it follows immediately from the uniqueness statement of Lemma 51.15 that $\alpha = \beta$.

[548] Note that if we just want to prove Statement (3), then in the proof of (2) we could replace the slightly esoteric the Subgroup-of-Free Abelian Group Lemma 51.2 by the more conventional the Rank Additivity Lemma 51.7 that was proved in the Algebra course.

and we denote by $\sim$ the equivalence relation on $S(A,B)$ that is generated by the relations

$$(g_1,\dots,g_l,e,g_{l+2},\dots,g_k) \sim (g_1,\dots,g_l,g_{l+2},\dots,g_k)$$

and by the relations

$$(g_1,\dots,g_l,g_{l+1},\dots,g_k) \sim (g_1,\dots,g_lg_{l+1},\dots,g_k)$$

if both g_l and g_{l+1} lie in A or both lie in B. It is now straightforward to verify that the map

$$\begin{array}{rcl} S(A,B)/\sim \times S(A,B)/\sim & \to & S(A,B)/\sim \\ ([(g_1,\dots,g_k)],[h_1,\dots,h_l]) & \mapsto & [(g_1,\dots,g_k,h_1,\dots,h_l)] \end{array}$$

is well-defined and that it defines a group structure on $S(A,B)/\sim$.[550]

Proposition 51.18. Let A and B be two groups. Then the map[551]

$$\begin{array}{rcl} \Phi\colon S(A,B)/\sim & \to & A*B \\ [(g_1,\dots,g_k)] & \mapsto & g_1\cdot\dots\cdot g_k \end{array}$$

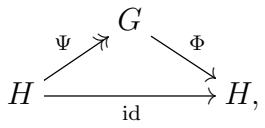
is well-defined and it is an isomorphism. The inverse map is given by

$$\begin{array}{rcl} \Psi\colon A*B & \to & S(A,B)/\sim \\ (g_1,\dots,g_k) & \mapsto & [(g_1,\dots,g_k)]. \end{array}$$

In the proof of Proposition 51.18 we will need the following little lemma.

Lemma 51.19. Let $\Phi\colon G\to H$ and $\Psi\colon H\to G$ be two group homomorphisms such that $\Phi\circ\Psi=\mathrm{id}_H$ and such that Ψ is an epimorphism. Then Φ is an isomorphism.

The lemma can be summarized as follows, if we are given a commutative diagram of the form



then Φ is an isomorphism.[552]

Proof. It follows from $\Phi\circ\Psi=\mathrm{id}_H$ that Φ is an epimorphism. It remains to show that Φ is a monomorphism. So let $g\in\ker(\Phi)$. Since Ψ is an epimorphism there exists an $h\in H$ with $\Psi(h)=g$. Since $\Phi\circ\Psi=\mathrm{id}_H$ we see that $h=\Phi(\Psi(h))=\Phi(g)=e$. Thus $h=e$ and thus $g=\Psi(h)=\Psi(e)=e$. ∎

Proof. It is straightforward to verify that the map

$$\begin{array}{rcl} \Phi\colon S(A,B)/\sim & \to & A*B \\ [(g_1,\dots,g_k)] & \mapsto & g_1\cdot\dots\cdot g_k \end{array}$$

[549]This definition is similar to the definition of $A*B$, but now we allow that elements of the sequence are trivial and we allow that consecutive elements lie in the same group.
[550]The neutral element is again represented by the empty sequence ( ) and the inverse of $[(g_1,\dots,g_k)]$ is given by $[(g_k^{-1},\dots,g_1^{-1})]$. This time associativity is not a problem. Why not?
[551]Recall that we use the canonical monomorphisms from Lemma 51.13 to view A and B as subgroups of $A*B$. If we want to be more precise, then we could say that Φ is given by $[(g_1,\dots,g_k)]\mapsto(g_1)\cdot\dots\cdot(g_k)$.
[552]Is Ψ also necessarily an isomorphism?

is well-defined and that it is a homomorphism. This homomorphism is surjective since $A * B$ is a subset of $S(A, B)$. Next we consider the map[553]

$$\begin{array}{rcl} \Psi\colon A * B & \to & S(A,B)/\sim \\ (g_1, \ldots, g_k) & \mapsto & [(g_1, \ldots, g_k)]. \end{array}$$

It is straightforward to see that Ψ is a homomorphism. It is clear that $\Phi \circ \Psi = \mathrm{id}_{A*B}$. It is not so obvious that $\Psi \circ \Phi = \mathrm{id}_{S(A,B)/\sim}$. In fact we will proceed in a slightly different fashion: to show that Φ is an isomorphism it suffices, according to Lemma 51.19, to show that Ψ is surjective. This in turn follows from the following claim.

Claim. Any element in $S(A, B)$ is equivalent to an element in $A * B$.

Proof. We prove the claim by induction on the length of a sequence in $S(A, B)$. Clearly the claim holds for all sequences of length 0. Suppose we already know the claim for all sequences of length $\leq k-1$. Let $(g_1, \ldots, g_k)$ be a sequence of length k. If $(g_1, \ldots, g_k) \in A*B$, then we are done. Now suppose that this is not the case. Then one of the following occurs:

(1) either there exists an $i \in \{1, \ldots, k\}$ with $g_i = e$, or
(2) there exists an $i \in \{1, \ldots, k-1\}$ with $g_i, g_{i+1} \in A$, or
(3) there exists an $i \in \{1, \ldots, k-1\}$ with $g_i, g_{i+1} \in B$.

In the first case the sequence is equivalent to $(g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_k)$. In the second and third case the sequence is equivalent to $(g_1, \ldots, g_{i-1} \cdot g_{i+1}, \ldots, g_k)$. In all three cases we have thus shown that our original sequence is equivalent to a sequence of length $k-1$. So we are done by our induction hypothesis. ∎

Let A and B be two groups. In the following we will use the isomorphism of Proposition 51.18 to identify the group $S(A, B)/\sim$ with the free product $A * B$. Both groups have their advantages:

(1) In the group $S(A, B)/\sim$ it is easier to write down the product of two elements, we just need to juxtapose sequences of elements in A and B.
(2) On the other hand in the group $A*B$ it is trivial to check whether two elements are the same. For example for two non-trivial elements $a \in A$ and $b \in B$ the elements (a, b) and (b, a) of $A * B$ are by definition different. But from the definition of $S(A, B)/\sim$ it is not immediately clear that $[(a, b)]$ and $[(b, a)]$ are indeed different.

51.4. The Grushko-Neumann Theorem.

We start out with the following natural definition.

Definition. Let G be a finitely generated group. We denote by $d(G) \in \mathbb{N}_0$ the minimal number of elements in a generating set for G.

Remark. It is an unfortunate fact of life that in the literature the number $d(G)$ is usually referred to as the rank of the group G. But for abelian groups the invariants $\mathrm{rank}(G)$ and $d(G)$ can differ quite dramatically. For example for $G = (\mathbb{Z}_2)^n$ we have $\mathrm{rank}(G) = 0$ but $d(G) = n$.

The following theorem was proved by Igor Grushko [**Gru1940**] in 1940 and independently by Bernhard Neumann [**Neum1943**] in 1940.

[553]Put differently, $A * B \to S(A, B)/\sim$ is the homomorphism that is given by the universal property of $A * B$.

Theorem 51.20. (Grushko-Neumann Theorem) Given any two finitely generated groups A and B we have $$d(A * B) \;=\; d(A) + d(B).$$

Proof. The proof is evidently given in the above two references, but it can also be found in any self-respecting book on combinatorial group theory, see e.g. [**LyS1977**, Corollay IV.1.9] and [**MKS1976**, p. 192]. ■

We already saw that some groups can be written as a direct product of two groups in different ways. For example it follows from the Chinese Remainder Theorem that the groups $\mathbb{Z}_3 \times \mathbb{Z}_{35}$ and $\mathbb{Z}_5 \times \mathbb{Z}_{21}$ are isomorphic. The following theorem says that the situation is much more rigid for free products. Before we can state the theorem we have to introduce the following definition.

Definition. A group G is indecomposable if it is not isomorphic to the free product of two non-trivial groups.

Lemma 51.21. Abelian groups and finite groups are indecomposable.

Proof. The lemma will be proved in Exercise 51.6. ■

Now we can state the Grushko[554] Decomposition Theorem.

Theorem 51.22. (Grushko Decomposition Theorem) Let G be a finitely generated group.

(1) The group G is isomorphic to the free product $G = A_1 * \cdots * A_k$ of non-trivial indecomposable groups.[555]
(2) The isomorphism types of $A_1, \ldots, A_k$ are unique up to permutation. More precisely, if $A_1, \ldots, A_k$ and $B_1, \ldots, B_l$ are indecomposable finitely generated groups such that
$$A_1 * \cdots * A_k \;\cong\; B_1 * \cdots * B_l,$$
then $k = l$ and furthermore there exists a permutation $\sigma \in S_k$ such that $A_i \cong B_{\sigma(i)}$ for $i = 1, \ldots, k$.

Proof.

(1) This statement follows easily from induction on $d(G)$ and the Grushko-Neumann Theorem 51.20. Indeed, if $d(G) = 0$ then there is nothing to prove. Next suppose that the statement is known for all groups G with $d(G) < k$. Let G be a group with $d(G) = k$. If G is indecomposable, we are done. If G is not indecomposable, then we can write G as the free product of two non-trivial groups A and B. By the Grushko-Neumann Theorem 51.20 we have $d(G) = d(A) + d(B)$. Since A and B are non-trivial we have $d(A) \geq 1$ and $d(B) \geq 1$. Thus we have $d(A) < k$ and $d(B) < k$ and by induction we can find decompositions for A and B. These two decompositions put together give the desired decomposition for $G = A * B$.
(2) The uniqueness statement was proved by Aleksandr Kurosh in 1934, see [**Kuro1934**, Isomorphismsatz] and [**Kuro1960**, p. 27]. Alternative proofs are given in [**Stal1975**, p. 168f] and [**Nel**, Theorem 4.5]. Also see [**CM1982**, Chapter II.4] for more context. ■

[554] Igor Grushko was a Russian mathematician. The theorem was proved in 1940.
[555] If G is trivial then $k = 0$ and the free product of zero groups is by convention the trivial group.

Example. It follows from Lemma 51.21 and from the Grushko Decomposition Theorem 51.22 that the free product $\mathbb{Z}_3 * \mathbb{Z}_{35}$ is not isomorphic to the free product $\mathbb{Z}_5 * \mathbb{Z}_{21}$.

Exercises for Chapter 51.

Exercise 51.1. Show that $\mathbb{Q}$ does not admit an epimorphism onto $\mathbb{Z}$.

Exercise 51.2. Given $n \in \mathbb{Z}$ we consider the matrix $B(n) = \begin{pmatrix} 0 & 3 \\ 3 & n \end{pmatrix}$ and we consider the abelian group $A_n := \mathbb{Z}^2/B(n)\mathbb{Z}^2$. Complete the following sentence: given $m, n \in \mathbb{Z}$ the abelian groups A_m and A_n are isomorphic if and only if ....

Exercise 51.3.

(a) Show that there exists an abelian group G with $G \oplus G \cong G$.

(b) Let A be a finitely generated abelian group. Show that if B is an abelian group with $B \oplus A \cong A$, then $B \cong 0$.
Hint. Arguably it is easiest to start out with the special case that $A = \mathbb{Z}$.

Exercise 51.4. . Let S be a set. Show that every non-trivial element in the free group $\langle S \rangle$ has infinite order.

Exercise 51.5. Let M be a connected smooth manifold, let $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ be points on M. Does there exist an action by a free group F on M such that for any $i \in \{1, \ldots, n\}$ there exists a $g \in F$ with $g \cdot P_i = Q_i$?

Exercise 51.6. Show that abelian groups and finite groups are indecomposable.

Exercise 51.7. Let G be an additive abelian group. We say that an element $g \in G$ is *primitive* if there does not exist an $h \in G$ and an $n \geq 2$ with $n \cdot h = g$.

(1) Suppose G is finitely generated. Let $g \in G$. Show that the following statements are equivalent:
   (a) g is a primitive element.
   (b) We can write $G = \mathbb{Z} \cdot g \oplus H$ for some abelian group G.
   (c) There exists a homomorphism $\varphi\colon G \to \mathbb{Z}$ with $\varphi(g) = 1$.

(2) Which of the implications in (1) also hold for infinitely generated abelian groups?

Exercise 51.8. Let $p, q, r \in \mathbb{N}_{\geq 2}$ be coprime. By the Chinese Remainder Theorem we know that $\mathbb{Z}_{pq} \times \mathbb{Z}_r \cong \mathbb{Z}_p \times \mathbb{Z}_{qr}$. Show, without making use of the Grushko Decomposition Theorem, that $\mathbb{Z}_{pq} * \mathbb{Z}_r$ is not isomorphic to $\mathbb{Z}_p * \mathbb{Z}_{qr}$.
Perhaps it is notationally easier to write $A = \{e, a, \ldots, a^{pq-1}\}$, $B = \{e, b, \ldots, b^{r-1}\}$, $C = \{e, c, \ldots, c^{p-1}\}$ and $D = \{e, d, \ldots, d^{qr-1}\}$, where A, B, C and D have the obvious multiplicative group structure. Then the problem is to show that $A * B$ is not isomorphic to $C * D$.

Exercise 51.9. Let S and T be sets. If there exists a monomorphism $\varphi\colon \langle S \rangle \to \langle T \rangle$, does it follow that $\#S \leq \#T$?

Exercise 51.10. Let P be a property of groups. A group G is called *locally* P if every finitely generated subgroup has property P.

(1) Is every locally abelian group in fact an abelian group?

(2) Is every locally free group in fact a free group?
(3) Is every locally finite group in fact a finite group?

Exercise 51.11. Let A and B be two non-trivial groups. Show that the free product $A * B$ is infinite.

Exercise 51.12. Let G be a group and let $g \in G$. The *centralizer* of $g \in G$ is defined as

$$C_G(g) \;:=\; \{h \in G \,|\, gh = hg\}.$$

(a) Let A and B be groups and let $g \in A * B$. We view A and B as subgroups of $A * B$ in the obvious way. Let $g \in A * B$. Show that

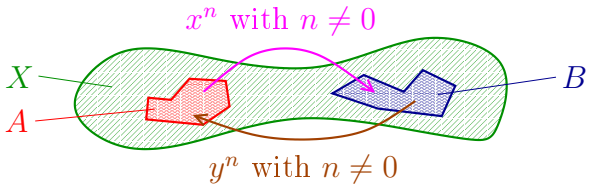
$$C_G(g) \;=\; \begin{cases} C_A(g), & \text{if } g \in A, \\ C_B(g), & \text{if } g \in B, \\ \langle h \rangle, \quad h \in A * B \text{ of infinite order} & \text{otherwise.} \end{cases}$$

(b) Let S be a set. Show that every abelian subgroup of the free group $\langle S \rangle$ is isomorphic to $\mathbb{Z}$.
Remark. Note that S is not necessarily finite.

Exercise 51.13. Let G be a group and let $x, y \in G$. Suppose there exists an action of G on some set X and two disjoint subsets A and B of X such that the following two conditions are satisfied:

(1) for every $n \in \mathbb{Z} \setminus \{0\}$ we have $x^n \cdot A \subset B$, and
(2) for every $n \in \mathbb{Z} \setminus \{0\}$ we have $y^n \cdot B \subset A$.

Show that x and y generate a free subgroup of G, i.e. show that the map $\langle x \rangle * \langle y \rangle \to G$ is a monomorphism.
Hint. Show that any non-trivial element $(g_1, \ldots, g_k) \in \langle x \rangle * \langle y \rangle$ acts non-trivially on X.
Remark. This statement is known as the *Ping-Pong Lemma*.



Exercise 51.14. We consider the group $G = \mathrm{SL}(2, \mathbb{Z})$ and we consider the two matrices $x = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Use the Ping-Pong Lemma from Exercise 51.13 to show that x and y generate a free subgroup of G.
Hint. Consider $X = \mathbb{R}^2$ together with the subsets $A = \{(x, y) \in \mathbb{R}^2 \,|\, |y| > |x|\}$ and $B = \{(x, y) \in \mathbb{R}^2 \,|\, |x| > |y|\}$.

Exercise 51.15. A group π is called *residually finite* if given any non-trivial element $g \in \pi$ there exists a homomorphism $\alpha \colon \pi \to G$ to a finite group G such that $\alpha(g)$ is non-trivial.

(a) Show that for any $n \in \mathbb{N}$ the group $\mathrm{GL}(n, \mathbb{Z})$ is residually finite.
(b) Show that the free group on two generators is residually finite.
Remark. Use Exercise 51.14.

Exercise 51.16. Let G be a group. We consider the subset

$$\mathrm{Tor}(G) \;:=\; \{g \in G \,|\, \text{ there exists an } n \in \mathbb{N} \text{ with } g^n = e\}.$$

(a) Suppose that G is abelian. Show that $\mathrm{Tor}(G)$ is a subgroup of G. We refer to it as the *torsion subgroup of* G.
(b) Give an example of a group G such that $\mathrm{Tor}(G)$ is not a subgroup of G.

Exercise 51.17.

(a) Let $\phi\colon \langle x, y\rangle \to \mathbb{Z}$ be the homomorphism given by $\phi(x) = 1$ and $\phi(y) = 0$. Show that $\ker(\phi)$ is infinitely generated.
Remark. In Proposition 53.14 we will see that any non-trivial *normal* subgroup of infinite index of a free group is infinitely generated.
(b) Let $\psi\colon \langle s, t\rangle \times \langle x, y\rangle \to \mathbb{Z}$ be the homomorphism given by $\psi(s) = 1$, $\psi(t) = 0$, $\psi(x) = 1$ and $\psi(y) = 0$. Show that $\ker(\psi)$ is finitely generated.

Exercise 51.18. We consider the free group $F = \langle x, y\rangle$. Let $a = (a_1, a_2, \ldots, a_m)$ be a reduced word in $\langle x\rangle$ and $\langle y\rangle$. We denote by $\overline{a} = (a_m, \ldots, a_1)$ the reverse word. Given any reduced word $z = (z_1, \ldots, z_n)$ we write

$$|z|_a \;=\; \text{number of non-overlapping subsequences of } z_1, \ldots, z_n \text{ that agree with } a_1, \ldots, a_m.$$

Now we consider the following map

$$\begin{array}{rcl} \varphi_a\colon \langle x, y\rangle & \to & \mathbb{Z}_{\geq 0} \\ z & \mapsto & |z|_a - |z|_{\overline{a}}. \end{array}$$

(a) Show that in general φ_a is not a homomorphism.
(b) Show that there exists a $D > 0$ such that for every $g, h \in \langle x, y\rangle$ we have

$$|\varphi_a(g \cdot h) - \varphi_a(g) - \varphi_a(h)| \;\leq\; D.$$

Remark. Any map $\varphi\colon G \to \mathbb{R}$ from a group G to $\mathbb{R}$ that admits a D as above is called a *quasi-morphism*. The above quasi-morphism is called the *Brooks quasi-morphism* [**Brooks1981**].

Exercise 51.19.

(a) Let $p, q \in \mathbb{N}$. Show that if p and q are coprime, then there exist $X, Y \in \mathrm{GL}(2, \mathbb{Z})$ with

$$X \cdot \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \cdot Y \;=\; \begin{pmatrix} x \cdot y & 0 \\ 0 & 1 \end{pmatrix}.$$

(b) Let $A \in \mathrm{M}(m \times n; \mathbb{Z})$. By the Smith Normal Form Theorem 51.5 we know that there exist matrices $P \in \mathrm{GL}(m, \mathbb{Z})$ and $Q \in \mathrm{GL}(n, \mathbb{Z})$ such that $P \cdot A \cdot Q$ is of the form

$$P \cdot A \cdot Q \;=\; \begin{pmatrix} D_{k\times k} & 0_{k\times(n-k)} \\ 0_{k\times(m-k)} & 0_{(n-k)\times(m-k)} \end{pmatrix}$$

where $D = \mathrm{diag}(d_1, \ldots, d_k)$ is a diagonal $(k \times k)$-matrix such that the diagonal entries $d_1, \ldots, d_k$ lie in $\mathbb{N}$. Show that we can find P and Q such that for each $i \in \{1, \ldots, k-1\}$ we have $d_i | d_{i+1}$.
(c) Show that $d_1, \ldots, d_k$ with $d_i | d_{i+1}$ as in (b) are unique.

Exercise 51.20. Show that every matrix in $\mathrm{GL}(n,\mathbb{Z})$ can be written as a product of elementary matrices, i.e. of matrices of the following form:

$$\underbrace{\begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}}_{\text{(I)}}, \quad \underbrace{\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}}_{\text{(II)}} \quad \text{and} \quad \underbrace{\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & r & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}}_{\text{(III)}}, \quad \text{with } r \in \{-1, 1\}.$$

Exercise 51.21. Let G and H be two groups.

(a) Suppose there exist monomorphisms $\varphi\colon G \to H$ and $\psi\colon H \to G$. Does it follow that G and H are isomorphic?

(b) Suppose there exist epimorphisms $\varphi\colon G \to H$ and $\psi\colon H \to G$. Does it follow that G and H are isomorphic?

52. The basic Seifert–van Kampen Theorem

We are still lacking the computation of fundamental groups of even fairly simple topological spaces. Now we will take a new approach to calculating fundamental groups. Namely in the following two chapters we will formulate and prove results which allow us to calculate the fundamental group of a given topological space from the fundamental groups of suitable subspaces.

52.1. **The formulation of the basic Seifert–van Kampen Theorem.** The idea of the Seifert–van Kampen theorem is to reduce the calculation of the fundamental group of a topological space X to the determination of fundamental groups of "simpler subsets" of X. Let X be a topological space and let $U, V \subset X$ be two subsets with $X = U \cup V$ and with $U \cap V \neq \varnothing$. We choose a base point $x_0 \in U \cap V$ and we consider the following inclusion maps and the corresponding induced homomorphisms of the fundamental groups:

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle i} \\ V & \xrightarrow[j]{} & X = U \cup V \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \pi_1(U \cap V, x_0) & \longrightarrow & \pi_1(U, x_0) \\ \downarrow & & \downarrow{\scriptstyle i_*} \\ \pi_1(V, x_0) & \xrightarrow[j_*]{} & \pi_1(X, x_0) = \pi_1(U \cup V, x_0). \end{array}$$

Our goal is to determine the fundamental group of the total space $X = U \cup V$ in terms of the fundamental groups of U, V, $U \cap V$ and the induced maps $\pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ and $\pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$.

We start out with the following lemma:

Lemma 52.1. Let X be a topological space and let $U, V \subset X$ be two open subsets with $X = U \cup V$ and such that $U \cap V \neq \varnothing$. We choose a base point $x_0 \in U \cap V$. If $U \cap V$ is path-connected, then the homomorphism

$$\pi_1(U, x_0) * \pi_1(V, x_0) \;\to\; \pi_1(X, x_0),$$

which is induced by the homomorphisms $\pi_1(U, x_0) \to \pi_1(X, x_0)$ and $\pi_1(V, x_0) \to \pi_1(X, x_0)$ and by Lemma 51.13 (2), is surjective.

Examples.

(1) We consider the sphere

$$X \;=\; S^2 \;=\; \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\},$$

with the covering $X = U \cup V$ given by the two open subsets

$$\begin{aligned} U &= \{(x, y, z) \mid x^2 + y^2 + z^2 = 1 \text{ and } z > -\tfrac{1}{2}\}, \\ V &= \{(x, y, z) \mid x^2 + y^2 + z^2 = 1 \text{ and } z < \tfrac{1}{2}\}. \end{aligned}$$

We pick the base point $x_0 = (1, 0, 0) \subset U \cap V$. Here the intersection $U \cap V$ is path-connected, therefore we can apply Lemma 52.1. It says that we have an epimorphism

$$\pi_1(U, x_0) * \pi_1(V, x_0) \;\to\; \pi_1(S^2, x_0).$$

The subspaces U and V are homeomorphic to an open disk,[556] thus the fundamental groups are trivial. It follows that the trivial group[557] surjects onto $\pi_1(S^2, x_0)$, thus $\pi_1(S^2, x_0)$ itself is trivial. Precisely the same argument shows that $\pi_1(S^n) = 0$ for $n \geq 3$.

We had already obtained this result in the Sphere-π_1-Proposition 47.13. In fact, as we will see shortly, the proof of Lemma 52.1 has clear parallels to the proof of the Sphere-π_1-Proposition 47.13.

(2) We consider the sphere

$$X = S^1 = \{\exp(\mathrm{i}t) \mid t \in \mathbb{R}\},$$

with the covering $X = U \cup V$ given by the two open subsets

$$\begin{aligned} U &:= \{\exp(\mathrm{i}t) \mid t \in (-\tfrac{\pi}{4}, \tfrac{5\pi}{4})\} \\ V &:= \{\exp(\mathrm{i}t) \mid t \in (\tfrac{3\pi}{4}, \tfrac{9\pi}{4})\}. \end{aligned}$$

We pick the base point $x_0 = 1$. The subsets U and V are both homeomorphic to an open interval, hence $\pi_1(U, x_0) = \pi_1(V, x_0) = 0$, but of course we know from Corollary 48.18 that $\pi_1(X, x_0) \cong \mathbb{Z}$. Thus we see that $\pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ is not an epimorphism. This is not a contradiction to Lemma 52.1, since in this example the intersection $U \cap V$ is not path-connected.

Now we turn to the proof of Lemma 52.1.

Proof of Lemma 52.1. Let X be a topological space and let $U, V \subset X$ be two open subsets with $X = U \cup V$ and such that $U \cap V \neq \varnothing$. We choose a base point $x_0 \in U \cap V$. It suffices to prove the following claim.[558]

Claim. Every loop $s\colon [0,1] \to X = U \cup V$ in x_0 is path-homotopic to a loop which is the product of finitely many loops in (U, x_0) or in (V, x_0).

Proof. So let $s\colon [0,1] \to X = U \cup V$ be a loop in x_0. Since U and V are open we can use the Lebesgue Corollary 6.6 to find a subdivision

$$0 = t_0 < t_1 < t_2 < \ldots < t_k < t_{k+1} = 1$$

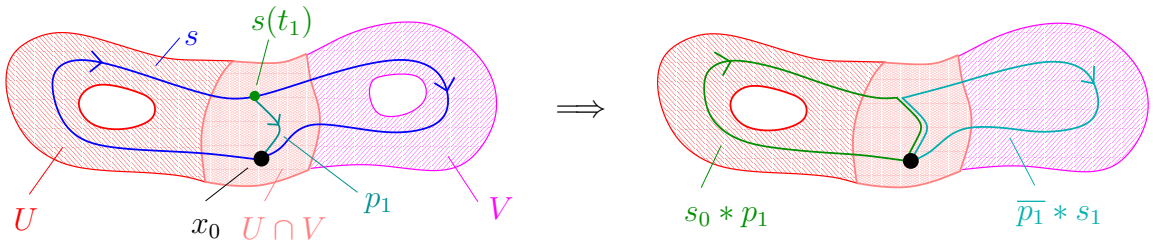
of the interval $[0,1]$ such that the image of each $s_i := s|_{[t_i, t_{i+1}]}$ lies in U or in V. The same argument as in the proof of the Sphere-π_1-Proposition 47.13 shows that we can assume that each $s(t_i)$ lies in $U \cap V$.[559] The problem is that even though the paths s_i lie in U respectively in V, they are not necessarily loops in x_0. But since $U \cap V$ is path-connected we can find for each $i \in \{1, \ldots, k\}$ a path p_i in $U \cap V$ which connects $s(t_i) \in U \cap V$ to the

[556] A homeomorphism is for example given by stereographic projection.

[557] Here we use the obvious fact that the free product of two copies of the trivial group is again the trivial group.

[558] Why does it suffice to prove this claim?

[559] Indeed, if $s(t_i)$ lies in $U \setminus V$, then $s([t_{i-1}, t_i])$ and $s([t_i, t_{i+1}])$ cannot lie in V, hence they both lie in U. This implies that $s_{([t_{i-1}, t_{i+1}])}$ lies in U, and therefore we can remove t_i from the subdivision.



base point $x_0 \in U \cap V$. Now we have the following path-homotopies:

$$\begin{aligned} s \overset{\substack{\text{by Lemma 47.3, since } s \\ \text{is a reparametrization} \\ \text{of } s_0 * \cdots * s_k}}{\simeq} s_0 * \cdots * s_k &\overset{\substack{\text{since each } p_i * \overline{p_i} \\ \text{is null-homotopic}}}{\simeq} s_0 * p_1 * \overline{p_1} * s_1 * p_2 * \overline{p_2} * s_2 * p_3 * \overline{p_3} * \ldots * \overline{p_k} * s_k \\ &\simeq (\underbrace{s_0 * p_1}_{\substack{\text{loop in } (U,x_0) \\ \text{or in } (V,x_0)}}) * (\underbrace{\overline{p_1} * s_1 * p_2}_{\substack{\text{loop in } (U,x_0) \\ \text{or in } (V,x_0)}}) * (\underbrace{\overline{p_2} * s_2 * p_3}_{\substack{\text{loop in } (U,x_0) \\ \text{or in } (V,x_0)}}) * (\overline{p_3} * \ldots * (\overline{p_k} * s_k). \end{aligned}$$

Thus we have shown that s is path-homotopic to a product of loops in (U, x_0) and (V, x_0).[560] ■

Theorem 52.2. (Seifert–van Kampen Theorem)[561] Let X be a topological space and let $U, V \subset X$ be two open subsets with $X = U \cup V$ and with $U \cap V \neq \varnothing$. We choose a base point $x_0 \in U \cap V$. If $U \cap V$ is simply connected, then the map

$$\pi_1(U, x_0) * \pi_1(V, x_0) \;\to\; \pi_1(X, x_0)$$

that is induced by $\pi_1(U, x_0) \to \pi_1(X, x_0)$ and $\pi_1(V, x_0) \to \pi_1(X, x_0)$ and Lemma 51.13 (2), is an isomorphism.

Remark. One might wonder whether the hypothesis that $U \cap V$ is simply connected is really necessary. But this is indeed the case. For example we could take $X = S^1 \times [-1, 1]$ with $U = S^1 \times [-1, \frac{1}{2})$ and $V = S^1 \times (-\frac{1}{2}, 1]$. In this case we have $\pi_1(X) \cong \mathbb{Z}$ which is not isomorphic to the non-abelian group $\pi_1(U) * \pi_1(V) \cong \mathbb{Z} * \mathbb{Z}$.

We postpone the proof of the Seifert–van Kampen Theorem 52.2 to the next section and we first discuss several examples and applications of the theorem.
Let A and B be two topological spaces and let $a \in A$ and $b \in B$ be two points. Recall that on page 1132 we defined the wedge $A \vee B$ of A and B as

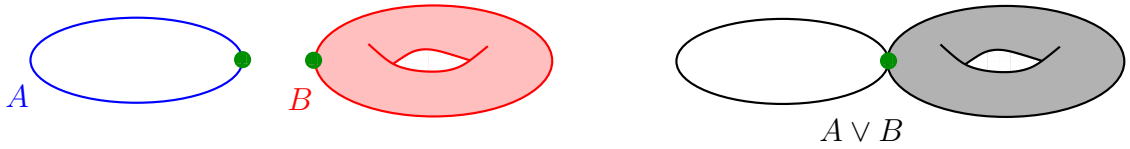
$$A \vee B \;:=\; A \sqcup B \;/\; a = b.$$

We illustrate this definition in the figure below. In the following we will often identify A and B with their respective images in $A \vee B$. Another way of thinking about the wedge of

[560] The proof of Lemma 52.1 is arguably easier to follow than the proof of the Sphere-π_1-Proposition 47.13. The reason is that in the special case the Sphere-π_1-Proposition 47.13 we introduced more notation, e.g. North Pole and South Pole, than was perhaps strictly necessary. All we really needed was that we could cover S^2 by two open simply connected subsets U and V such that $U \cap V$ is path-connected. The precise choice of U and V is irrelevant.

[561] The theorem was proved independently by Herbert Seifert (1907-1996) and Egbert van Kampen (1908-1942).

A and B is to say that the wedge $X = A \vee B$ can be decomposed into two subsets A and B such that $A \cap B$ is a point. Since $A \cap B$ is a point it is in particular simply connected. Thus it sounds like one should be able to apply the Seifert–van Kampen Theorem 52.2. This is almost correct, except that in the Seifert–van Kampen Theorem we have to deal with decompositions of a topological space X into *open* subsets. We therefore need to introduce a somewhat technical definition.



Definition. We say a point x in a topological space X is good, if the following two conditions are satisfied:

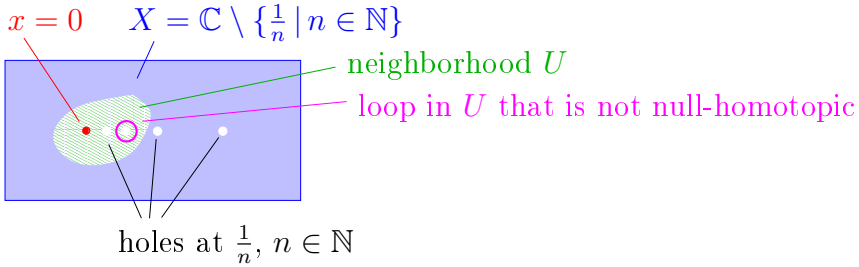
(1) $\{x\}$ is a closed subset of X and
(2) there exists an open neighborhood U of x such that x is a deformation retract of U.

Remark.

(1) If X is Hausdorff, then it we know by the Points-in-Hausdorff Space Lemma 1.15 (1) that any subset consisting of a single point is closed. Thus the first condition in the definition of a good point is satisfied "for free".
(2) What we call a "good point" is sometimes called a "non-degenerate point", see e.g. [**Lee2000**, p. 212]. On the other hand, the term "non-degenerate point" is defined in a subtly different way, see e.g. [**Spa1995**, p. 380] or [**Bre1993**, p. 435]. Also note that a point $x \in X$ is good if the pair $(X, \{x\})$ is good in the sense of [**Hat2002**, p. 114].

Examples.

(1) Let X be an n-dimensional topological manifold (e.g. a sphere). We claim that every point is good. Since X is by definition Hausdorff we only need to show that Condition (2) is always satisfied. Thus let $x \in X$. Using a chart around x one easily sees that x admits an open neighborhood that is homeomorphic to the open ball B^n or to the "open half-ball" $\{(x_1, \ldots, x_n) \in B^n \mid x_n \geq 0\}$. In both cases x is evidently a deformation retract of that open neighborhood.
(2) We consider $X = \mathbb{C} \setminus \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Morally speaking the point $x = 0$ is not good, "since any open neighborhood of 0 has non-trivial fundamental group". In Exercise 52.1 we will provide a proper proof that $x = 0$ is not a good point of X.



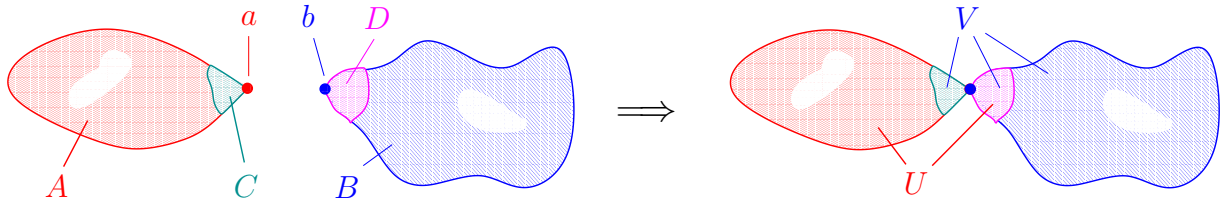
Proposition 52.3. (Wedge-π_1-Proposition) Let A and B be two path-connected topological spaces and let $a \in A$ and $b \in B$ be good points. Then the inclusion maps induce an isomorphism[562]

$$\pi_1(A,a) * \pi_1(B,b) \xrightarrow{\cong} \pi_1(A \vee B, a = b).$$

Remark. In the definition of a "good point" we demanded that the one-point subset given by the point is closed. Strictly speaking we do not need this fact in the proof of the Wedge-π_1-Proposition 52.3.

Proof. Let A and B be two path-connected topological spaces and let $a \in A$ and $b \in B$ be good points. We write $X = A \vee B$ and $x_0 = \{a, b\} \in A \vee B$. By Lemma 50.10 we can view A and B as subsets of $X = A \vee B$ in the obvious way.

We pick an open neighborhood C in A of a that deformation retracts to a. Furthermore we pick an open neighborhood D of b in B that deformation retracts to b. We consider $U := A \vee D$ and $V := B \vee C$, each viewed as subsets of $X = A \vee B$.



Now we consider the following diagram

$$\begin{array}{ccc} \pi_1(A,a) * \pi_1(B,b) & \longrightarrow & \pi_1(X,x_0) \\ \downarrow & & \downarrow \mathrm{id} \\ \pi_1(U,x_0) * \pi_1(V,x_0) & \longrightarrow & \pi_1(X,x_0). \end{array}$$

We make the following observations:

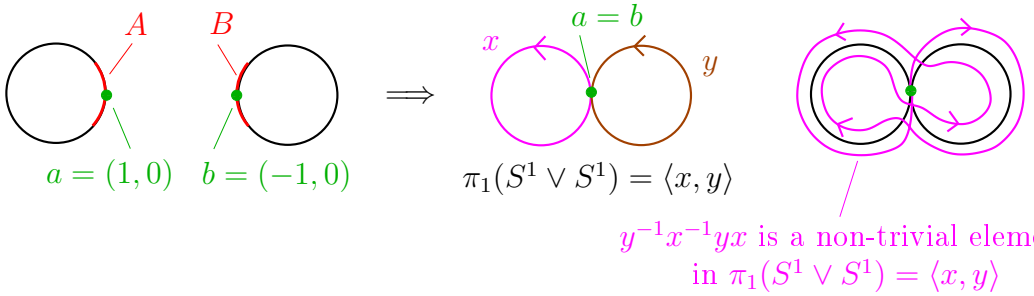
(1) The vertical and horizontal maps are all induced by the various inclusions of topological spaces. In particular the diagram commutes.
(2) It follows from the definition of the quotient topology on $A \vee B = (A \sqcup B)/\sim$ that U and V are open subsets of X.
(3) It follows from Lemma 50.13 that A is a deformation retract of $U = A \vee C$.
(4) As in (3) we see that B is a deformation retract of $V = B \vee D$.
(5) Furthermore, very similar to the argument in (3), we see that $A \cap B$ is a deformation retract of $U \cap V = C \vee D$. It follows from the Deformation Retract-Homotopy Equivalence Lemma 15.5 and the Homotopy-π_1-Proposition 50.3 (2) that $U \cap V$ is simply connected.
(6) By (2) and (5) we can apply the Seifert–van Kampen Theorem 52.2 to the decomposition $A \vee B = U \cup V$ and we obtain that the bottom horizontal map of the above diagram is an isomorphism.

[562] More precisely, the two natural inclusions $A \to A \vee B$ and $B \to A \vee B$ induce homomorphisms $\pi_1(A,a) \to \pi_1(A \vee B, a = b)$ and $\pi_1(B,b) \to \pi_1(A \vee B, a = b)$. Combined with Lemma 51.13 (2) we obtain the homomorphism $\pi_1(A,a) * \pi_1(B,b) \to \pi_1(A \vee B, a = b)$.

(7) It follows from (3), (4) and the Deformation Retract-Homotopy Equivalence Lemma 15.5 that the inclusions $A \to U = A \cup D$ and $B \to V = B \cup C$ are homotopy equivalences.
(8) It follows from (6) together with the Homotopy-π_1-Proposition 50.3 (2) that the two inclusion induced maps $\pi_1(A, a) \to \pi_1(U, a)$ and $\pi_1(B, b) \to \pi_1(V, b)$ are isomorphisms, which implies that the left vertical map in the above commutative diagram is an isomorphism.
(9) We deduce that the top horizontal map is, as desired, also an isomorphism. ■

Examples.

(1) We consider again the wedge $S^1 \vee S^1$ with $a = (1, 0)$ and $b = (-1, 0)$. These points are simple.[563] By Proposition 48.17 we know that $\pi_1(S^1, a) = \langle x \rangle$[564] and $\pi_1(S^1, b) = \langle y \rangle$



where x and y are represented by loops that go "once around the circle", as illustrated in the figure above. the Wedge-π_1-Proposition 52.3 says that the inclusion maps induce an isomorphism

$$\langle x, y \rangle \;=\; \langle x \rangle * \langle y \rangle \;=\; \pi_1(S^1, a) * \pi_1(S^1, b) \;\xrightarrow{\cong}\; \pi_1(S^1 \vee S^1, a = b).$$

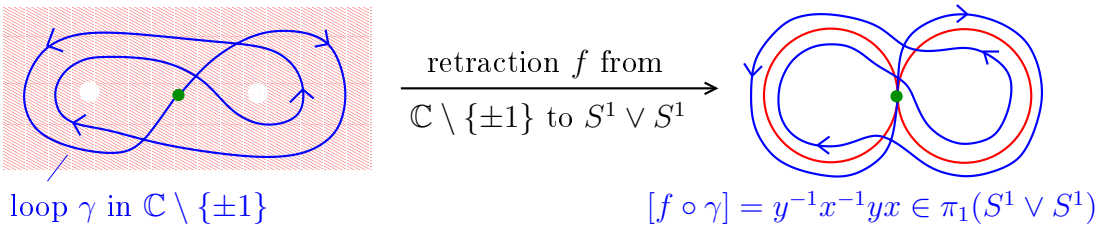
(2) We consider the loop γ from page 1049 in $\mathbb{C} \setminus \{\pm 1\}$ that is also illustrated in the figure below. We had already pointed out on page 1134 that there exists a deformation retraction f from $\mathbb{C} \setminus \{\pm 1\}$ to the wedge of two circles that is given by

$$S^1 \vee S^1 \;=\; \underbrace{\{z \in \mathbb{C} \mid |z + 1| = 1\} \cup \{z \in \mathbb{C} \mid |z - 1| = 1\}}_{\text{two circles that meet at the origin}}.$$

We then have $f_*([\gamma]) = [f \circ \gamma] = \langle y^{-1}x^{-1}yx \rangle \in \langle x, y \rangle = \pi_1(S^1 \vee S^1)$, which is non-trivial. Thus we see that $f \circ \gamma$ is not null-homotopic in $S^1 \vee S^1$ which implies that γ is not null-homotopic in $\mathbb{C} \setminus \{\pm 1\}$.[565]

Before we continue we condense the above example into a lemma.

[563] This follows either from the general statement that any point on a topological manifold is simple, or more simply, by taking A and B to be small "intervals" around a and b.
[564] Recall that $\langle x \rangle$ denotes the free group generated by x, i.e. $\langle x \rangle = \{\dots, x^{-1}, e = x^0, x, x^2, \dots\}$.
[565] We could also have applied Seifert–van Kampen Theorem 52.2 to the covering $\mathbb{C} \setminus \{\pm 1\} = U \cup V$ where $U = \{z \in \mathbb{C} \setminus \{-1\} \mid \mathrm{Re}(z) < \frac{1}{2}\}$ and $V = \{z \in \mathbb{C} \setminus \{1\} \mid \mathrm{Re}(z) > -\frac{1}{2}\}$. Both sets U and V are open and they are homotopy equivalent to a circle.



Lemma 52.4. (Fundamental Group-Wedge-of-Circles Lemma) Let $k \in \mathbb{N}_0$. The map

$$\begin{array}{rcl} \langle x_1, \ldots, x_k \rangle & \to & \pi_1\Big(\bigvee_{i=1}^{k} S^1, *\Big) \\ x_i & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & S^1_i \\ t & \mapsto & \exp(2\pi \mathrm{i} t) \end{array}\right] \end{array}$$

that is given by

is an isomorphism.

Convention. We refer to the images of $x_1, \ldots, x_k$ under the isomorphism of the Fundamental Group-Wedge-of-Circles Lemma 52.4 as the standard generating set of $\pi_1\Big(\bigvee_{i=1}^{k} S^1, *\Big)$.

Proof. Similar to the above example the lemma follows from applying the Wedge-π_1-Proposition 52.3 altogether $k-1$ times. ■

Proposition 52.5. (Fundamental Group-of-Graph Proposition)

(1) Let $G = (V, E, i, t)$ be a finite connected non-empty abstract graph and let T be a maximal tree of G. Let $e_1, \ldots, e_n$ be the edges not contained in T. We pick some $v \in V$. For $j = 1, \ldots, n$ we pick a path $\alpha_j \colon [0,1] \to |T|$ from v to $i(e_j)$, we denote by $\beta_j \colon [0,1] \to [(e_j, t)] \in |G|$ the obvious path from $i(e_j)$ to $t(e_j)$, and we pick a path $\gamma_j \colon [0,1] \to |T|$ from $t(e_j)$ to v. Then the map

$$\begin{array}{rcl} \Theta_G \colon \langle x_1, \ldots, x_n \rangle & \to & \pi_1(|G|, v) \\ x_j & \mapsto & [\alpha_j * \beta_j * \gamma_j] \end{array}$$

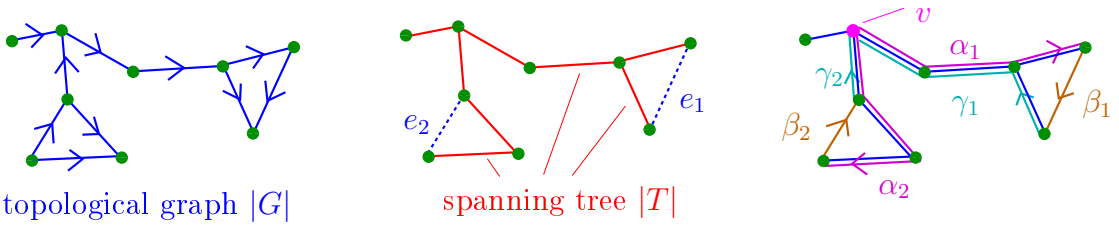
that is given by

is an isomorphism.

(2) Let X be a finite connected non-empty topological graph with v vertices and e edges. Then

$$\pi_1(X) \;\cong\; \text{free group on } 1 - \chi(X) = 1 + e - v \text{ generators.}$$

(3) Two finite connected topological graphs are homotopy equivalent if and only if they have the same Euler characteristic. ■



Proof.

(1) We consider the maps

$$\langle x_1,\dots,x_n\rangle \xrightarrow{\Theta_G} \pi_1(|G|,v) \underset{\text{isomorphism by Propositions 50.15 and 50.3 (2)}}{\to} \pi_1(|G|/|T|,v) \overset{\text{isomorphism induced by the homeomorphism on page 1133}}{\overset{\cong}{\leftarrow}} \pi_1\Big(\bigvee_{j=1}^{n} S^1_j\Big) \underset{\text{isomorphism given by the Fundamental Group-Wedge-of-Circles}}{\overset{\cong}{\leftarrow}} \langle y_1,\dots,y_n\rangle.$$

It follows fairly easily from the explicit descriptions of the various maps that for each $j\in\{1,\dots,n\}$ the images of x_j and y_j in $\pi_1(|G|/|T|,v)$ are represented by homotopic loops, in particular the images of x_j and y_j in $\pi_1(|G|/|T|,v)$ are the same. But now it follows from the above and the fact that all other maps are isomorphisms that Θ_G itself is indeed an isomorphism.

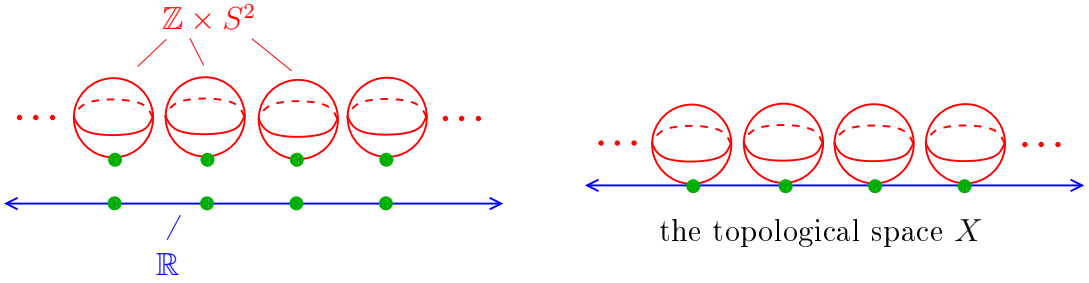
(2) Basically by definition we only need to prove the statement if $X=|G|$ is the topological realization of a finite connected non-empty abstract graph $G=(V,E,i,t)$. By the Spanning Tree Existence Proposition 7.10 (1) we know that G admits a spanning tree $T=(W,F)$. By the Spanning Tree Existence Proposition 7.10 we know that T contains all vertices of V. Furthermore, since T is a tree we know by definition of a tree that $\#W-\#F=1$. Thus it follows that there are precisely $\#E-\#F=e-(\#W-1)=e-v+1$ edges of G that are not contained in T. The desired statement now follows from (1).

(3) Let X and X' be two finite connected topological graphs. If $\chi(X)=\chi(X')$, then we already saw in Proposition 50.15 (3) that X and X' are homotopy equivalent. On the other hand, if $\chi(X)\neq\chi(X')$, then it follows from (1) together with Lemma 51.17 that X and X' are not homotopy equivalent. ■

As a last example in this section we would like to determine the fundamental group of

$$X \;=\; \big(\mathbb{R}\sqcup(\mathbb{Z}\times S^2)\big)\,/\,n\sim(n,(0,0,-1)).$$

The topological space X consists of an infinite line where to each $n\in\mathbb{Z}$ we attach a 2-dimensional sphere. The topological space X is illustrated in the figure below. Given



$k\in\mathbb{N}$ we denote by X_k the topological space that is obtained by X by restricting oneself to the interval $(-k-\frac{1}{2},k+\frac{1}{2})$, i.e.

$$X_k \;=\; \Big(\big(-k-\tfrac{1}{2},k+\tfrac{1}{2}\big)\sqcup(\{-k,\dots,k-1,k\}\times S^2)\Big)\,/\,n\sim(n,(0,0,-1)).$$

Note that X_k can be viewed as the wedge of an open interval with $2k+1$ spheres. It follows immediately from the Sphere-π_1-Proposition 47.13, the Wedge-π_1-Proposition 52.3 and a straightforward induction argument that $\pi_1(X_k)=0$ for all k. Furthermore one can easily convince oneself that the sets X_k are open. It now follows from the following lemma that $\pi_1(X)$ is also trivial.

Lemma 52.6. Let X be a path-connected topological space and let $x_0 \in X$. Furthermore let $X_k, k \in \mathbb{N}$ be a sequence of subsets of X such that the following hold:

(1) each X_k is open,
(2) each X_k is simply connected,
(3) the sequence X_k is nested, i.e. for each k we have $X_k \subset X_{k+1}$,
(4) we have $\bigcup_{k\in\mathbb{N}} X_k = X$.

Then $\pi_1(X, x_0) = 0$.

Proof. The proof of the lemma, which will make essential use of Lemma 2.12, is the content of Exercise 52.2. ∎

We conclude this section with the following generalization of the Wedge-π_1-Proposition 52.3.

Proposition 52.7. Let $\{A_k\}_{k\in K}$ be a family of topological spaces. For each $k \in K$ suppose that we are given a good point $a_k \in A_k$. Given $j \in K$ we denote by

$$i_j \colon A_j \to \bigvee_{k\in K} A_k \quad \text{respectively} \quad p_j \colon \bigvee_{k\in K} A_k \to A_j$$

the natural inclusion map respectively the natural projection . We denote by $*$ the wedge point of $\bigvee_{i\in I} X_i$. For every $n \in \mathbb{N}_0$ the projections p_k induces an isomorphism

$$\ast_{k\in K} p_{k*} \colon \pi_1\Big(\bigvee_{k\in K} A_k, *\Big) \to \text{ free product of the groups } \{\pi_1(A_k, a_k)\}_{k\in K}.$$

The inverse is induced by the inclusions i_k.

Proof. Let $\{A_k\}_{k\in K}$ be a family of topological spaces. For each $k \in K$ suppose that we are given a good point $a_k \in A_k$. We use these points to form the wedge $\bigvee_{k\in K} A_k$. Recall that the fact that a_k is good means that

(1) $\{a_k\}$ is a closed subset of A_k and
(2) that there exists an open neighborhood U_k of a_k in A_k such that a_k is a deformation retract of U_k.

The trick in the proof of Proposition 52.7 is to reduce the statement to the case of Proposition 52.7. We will show that the map

$$\ast_{k\in K} i_k \colon \ast_{k\in K} \pi_1(A_k) \to \pi_1\Big(\bigvee_{k\in K} A_k\Big)$$

is an isomorphism. (Here and throughout out we work with the obvious base points, which we suppress from the notation.) Once we will have established this fact it is straightforward to show that the inverse is given by $\ast_{k\in K} p_{k*}$.

The idea of the proof is to show that the map is a monomorphism and an epimorphism.

Claim. The map

$$\ast_{k\in K} i_k \colon \ast_{k\in K} \pi_1(A_k) \to \pi_1\Big(\bigvee_{k\in K} A_k\Big)$$

is an epimorphism.

Proof. We introduce the following notation:

(1) We write $B := \bigvee_{k\in K} A_k$ and we set $U := \bigcup_{k\in K} U_k \subset B$.

(2) Given a subset $J \subset K$ we write $B_J := \bigvee_{j\in J} A_j$ and we write $\widehat{B}_J := U \cup B_J$.

Given a finite subset $J \subset K$ we have the following commutative diagram

$$\begin{array}{ccccc}
\underset{j\in J}{*}\pi_1(A_j) & & \xrightarrow{\cong} & & \pi_1(B_J) \\
\downarrow & & & \searrow^{\cong} & \\
 & & & & \pi_1(\widehat{B}_J) \\
 & & & \swarrow & \\
\underset{k\in K}{*}\pi_1(A_k) & & \longrightarrow & & \pi_1(B).
\end{array}$$

We make the following observations and comments:

(1) The maps are either the obvious algebraic inclusion maps or they are the obvious maps induced by inclusions of topological spaces. It follows almost immediately from the definitions that the diagram commutes.
(2) Since J is finite we obtain, from applying the Wedge-π_1-Proposition 52.3 finitely many times, that the top horizontal map is an isomorphism.
(3) By Exercise 50.8 we know that B_J is a deformation retract of $\widehat{B}_J$. It follows from the Homotopy-π_1-Proposition 50.3 (2) that the top diagonal map is an isomorphism.
(4) By Exercise 50.9 we know that given any compact subset $L \subset B$ there exists a finite subset $J \subset K$ with $L \subset \widehat{B}_J$.

Now we show that the above map $\underset{k\in K}{*} i_{k*}$ is an epimorphism. So suppose we are given some $[\sigma\colon S^1 \to B] \in \pi_1(B)$. It follows from (4) there exists a finite subset $J \subset K$ such that $\sigma \in C_n(\widehat{B}_J)$. But considering the above diagram, and using the fact that the top horizontal map is an isomorphism we see that $[\sigma]$ actually lies in the image of $\underset{k\in K}{*} i_k$. ⊞

Claim. The map

$$\underset{k\in K}{*} i_k\colon \underset{k\in K}{*}\pi_1(A_k) \to \pi_1\Big(\bigvee_{k\in K} A_k\Big)$$

is a monomorphism.

Proof. We continue with the notation that we introduced in the proof of the previous claim. Given any two finite subsets $J \subset \widetilde{J} \subset K$ we have the following commutative diagram

$$\begin{array}{ccccccc}
\underset{j\in J}{*}\pi_1(A_j) & & \xrightarrow{\cong} & & \pi_1(B_J) & & \\
 & \searrow & & \swarrow & \downarrow & \searrow & \\
 & & \underset{j\in \widetilde{J}}{*}\pi_1(A_j) & \xrightarrow{\cong} & \pi_1(B_{\widetilde{J}}) & \xrightarrow{\cong} & \pi_1(\widehat{B}_{\widetilde{J}}). \\
 & \swarrow & & \searrow & & \swarrow & \\
\underset{k\in K}{*}\pi_1(A_k) & & \longrightarrow & & \pi_1(B) & &
\end{array}$$

In Exercise 52.7 we will use this diagram to show that the map $\underset{k\in K}{*} i_{k*}$ is also a monomorphism. ■

52.2. **Proof of the Seifert–van Kampen Theorem 52.2.** After this long list of applications we finally provide the proof of the Seifert–van Kampen Theorem 52.2.

Proof. Let X be a topological space and let $U, V \subset X$ be two open subsets with $X = U \cup V$ and with $U \cap V \neq \varnothing$ such that $U \cap V$ is simply connected. We choose a base point $x_0 \in U \cap V$. We write $A := \pi_1(U, x_0)$ and $B := \pi_1(V, x_0)$. We need to show that the map

$$\Phi\colon A * B \;=\; \pi_1(U, x_0) * \pi_1(V, x_0) \;\to\; \pi_1(X, x_0)$$

that is induced by the inclusions $U \to X$ and $V \to X$ is an isomorphism. Taking into account Lemma 52.1 it suffices to show that the map is injective. To prove injectivity it is enough to show that there exists a map $\Psi\colon \pi_1(X, x_0) \to A * B$ such that $\Psi \circ \Phi = \mathrm{id}_{A*B}$. We will prove the existence of such a map Ψ in three steps:

(1) we first construct a map $\overline{\Psi}\colon \{\text{loops in } (X, x_0)\} \to A * B$,
(2) then we show that $\overline{\Psi}$ descends to a well-defined map

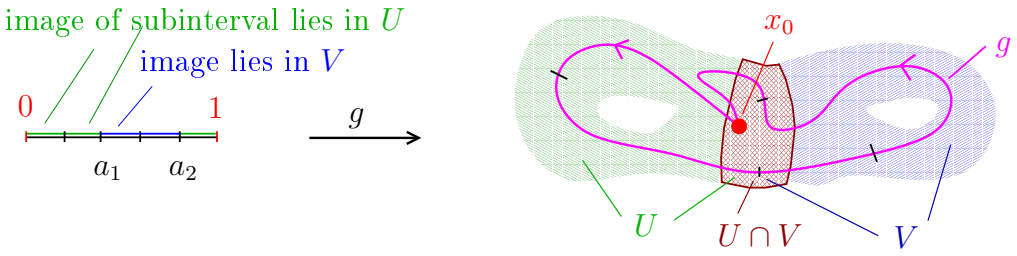
$$\begin{aligned} \Psi\colon \pi_1(X, x_0) &\to A * B \\ [g] &\mapsto \overline{\Psi}(g) \end{aligned}$$

(3) and finally we show that $\Psi \circ \Phi = \mathrm{id}_{A*B}$.

For the remainder of the proof we will identify $A*B$ with $S(A, B)/\!\sim$ using the isomorphism given by Proposition 51.18.

We start out with step (1). Let $g\colon [0,1] \to X$ be a loop in (X, x_0). We apply the Lebesgue Corollary 6.6 to the map $g\colon [0,1] \to X$ and the open cover $X = U \cup V$. We obtain an N such that for every $i \in \{0, \ldots, N-1\}$ the image of the interval $[\frac{i}{N}, \frac{i+1}{N}]$ under the map g lies in U or in V. We introduce a few definitions.

(a) We say a subinterval of $[0,1]$ can be *colored green* if the image of the subinterval under the map g lies in U and we say an interval can be *colored blue* if the image of the subinterval lies in V. Note that the image of a subinterval lies in $U \cap V$ if and only if the subinterval can be colored green as well as blue.
(b) A *coloring* is a map $\alpha\colon \{0, \frac{1}{N}, \ldots, 1 - \frac{1}{N}\} \to \{\text{blue}, \text{green}\}$ such that each subinterval $[i, i + \frac{1}{N}]$ can be colored by $\alpha(i)$. This definition is illustrated in the figure below.

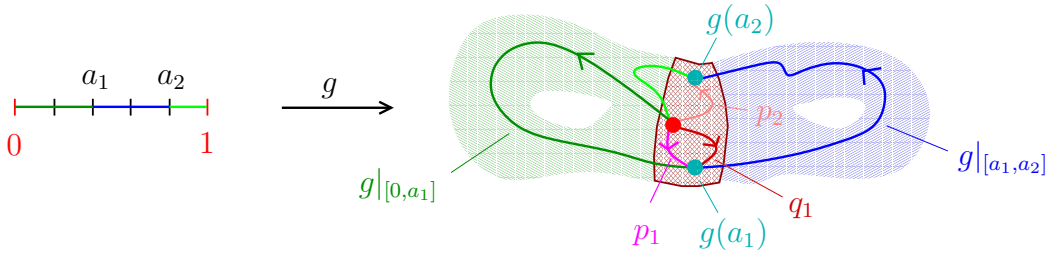


(c) Given a coloring α we denote by $0 < a_1 < \cdots < a_{k-1} < 1$ the numbers where the coloring of the subintervals changes from one color to another. This definition is also illustrated in the figure above. Note that for each a_i the intervals $[a_i - \frac{1}{N}, a_i]$ and $[a_i, a_i + \frac{1}{N}]$ can be colored in two different colors, which implies that $g(a_i) \in U \cap V$.
(d) A *basing* for a coloring α is a collection of paths $p_1, \ldots, p_{k-1}$ in $U \cap V$ such that each path p_i goes from x_0 to $g(a_i)$.[566]

[566] Such basing exists since each $g(a_i)$ lies in $U \cap V$ and since $U \cap V$ is path-connected.

Now given a coloring α and a basing $p_1, \dots, p_{k-1}$ we define[567] [568]

$$\overline{\Psi}(g) := \big[\big[\underbrace{g|_{[0,a_1]} * \overline{p}_1}_{\substack{\text{loop in } (U,x_0) \text{ or} \\ \text{loop in } (V,x_0)}}\big], \big[\underbrace{p_1 * g|_{[a_1,a_2]} * \overline{p}_2}_{\substack{\text{loop in } (U,x_0) \text{ or} \\ \text{loop in } (V,x_0)}}\big], \dots, \big[\underbrace{p_{k-1} * g|_{[a_{k-1},1]}}_{\substack{\text{loop in } (U,x_0) \text{ or} \\ \text{loop in } (V,x_0)}}\big]\big] \in S(A,B)/{\sim} = A * B.$$



Our next goal is to prove the following claim.

Claim. The definition of $\overline{\Psi}(g)$ does not depend on any of the choices made.

Proof. We need to prove the following three statements:

(a) given N and a coloring the definition does not depend on the choice of the basings,
(b) given N the definition does not depend on the choice of the coloring,
(c) the definition does not depend on the choice of N.

We first fix a coloring α and we show that the definition of $\overline{\Psi}(g)$ does not depend on the choice of the basing. First we pick a different path q_1 from x_0 to $g(a_1)$ instead of the original path p_1. (See the above figure for an illustration.) In this case we have

$$\begin{aligned}
&\text{definition of } \overline{\Psi}(g) \text{ using the basing } p_1, p_2, \dots, p_{k-1} \;=\\
&\quad= \big[[g|_{[0,a_1]} * \overline{p}_1],\ [p_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big]\\
&\quad= \big[[g|_{[0,a_1]} * \overline{p}_1 * q_1 * \overline{q}_1],\ [q_1 * \overline{q}_1 * p_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big]\\
&\qquad\uparrow \text{ since } q_1 * \overline{q}_1 \text{ is null-homotopic}\\
&\quad= \big[[g|_{[0,a_1]} * \overline{q}_1],\ [q_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big]\\
&\qquad\uparrow \overline{p}_1 * q_1 \text{ and } \overline{q}_1 * p_1 \text{ are null-homotopic since } U \cap V \text{ is simply connected}[569]\\
&\quad= \text{definition of } \overline{\Psi}(g) \text{ using the basing } q_1, p_2, \dots, p_{k-1}.
\end{aligned}$$

Precisely the same argument shows that the choices of the other paths p_i are also irrelevant. This concludes the proof of (a).

Now we show that the definition does not depend on the choice of the coloring. So suppose we have two colorings α and β. We can turn one coloring into the other by a finite sequence of color swaps of a single subinterval. Therefore it suffices to consider the case that there exists a single i where the values for $\alpha(i)$ and $\beta(i)$ differ.

[567] Each path $p_m * g|_{[a_m,a_{m+1}]} * \overline{p}_{m+1}$ is a loop in (U,x_0) or (V,x_0), so each path represents an element in $\pi_1(U,x_0)$ or $\pi_1(V,x_0)$. Thus we obtain a sequence of elements in $\pi_1(U,x_0)$ and $\pi_1(V,x_0)$. The equivalence class of this sequence defines an element in $S(A,B)/{\sim} = A * B$.

[568] This definition is almost the same as in the proof of Lemma 52.1.

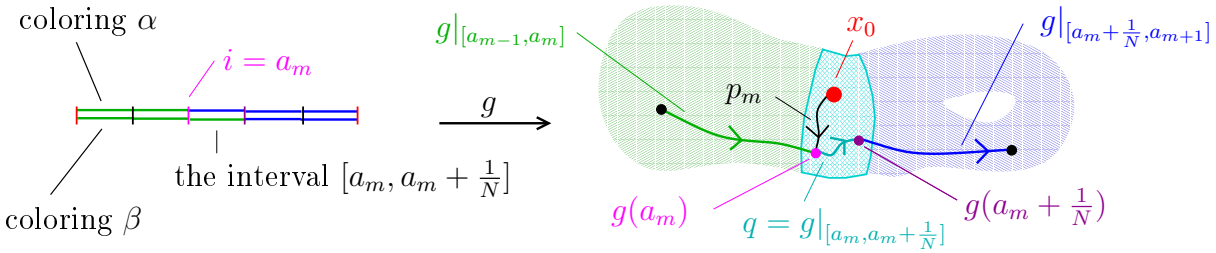
[569] This is the only time that in the proof we use the hypothesis that $U \cap V$ is simply connected. In fact if one studies the argument carefully one sees that we only used that the maps $\pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ and $\pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$ are trivial.

We consider the case, sketched in the figure below, with $\alpha(i) =$ blue and $\beta(i) =$ green such that the common coloring to the left is green and the common coloring to the right is blue. All other cases are dealt with in a similar way.[570]

Let $0 = a_0 < a_1 < \cdots < a_k = 1$ be the decomposition for α. We pick a basing $p_1, \ldots, p_{k-1}$ for α. There exists a unique m with $a_m = i$. The corresponding decomposition for the coloring β is given by $0 = a_0 < a_1 < \cdots < a_m + \frac{1}{N} < a_{m+1} < a_k = 1$. We write $q = g_{[a_m, a_m + \frac{1}{N}]}$. Since $[a_m, a_m + \frac{1}{N}]$ can be colored green and blue we see that q lies actually in $U \cap V$. We have

$$\begin{aligned}
&\text{definition of } \overline{\Psi}(g) \text{ using the coloring } \alpha \text{ and the basing } p_1, \ldots, p_{k-1} \;= \\
&\quad = \big[[g|_{[0,a_1]} * \overline{p}_1], \ldots, [p_{m-1} * g|_{[a_{m-1}, a_m]} * \overline{p}_m], [p_m * g|_{[a_m, a_{m+1}]} * \overline{p}_{m+1}], \ldots\big] \\
&\quad = \big[[g|_{[0,a_1]} * \overline{p}_1], \ldots, [p_{m-1} * g|_{[a_{m-1}, a_m + \frac{1}{N}]} * \overline{q} * \overline{p}_m], [p_m * q * g|_{[a_m + \frac{1}{N}, a_{m+1}]} * \overline{p}_{m+1}], \ldots\big] \\
&\qquad \uparrow \\
&\text{since } g_{[a_{m-1}, a_m]} \simeq g_{[a_{m-1}, a_m + \frac{1}{N}]} * \overline{q} \text{ and } g_{[a_m, a_m + \frac{1}{N}]} \simeq q * g_{[a_m + \frac{1}{N}, a_{m+1}]} \\
&\quad = \text{definition of } \overline{\Psi}(g) \text{ using the coloring } \beta \text{ and basing } p_1, \ldots, p_{m-1}, p_m * q, p_{m+1}, \ldots, p_{k-1}.
\end{aligned}$$

(We refer to the figure below for an illustration of the proof.) This concludes the proof of (b).



It remains to prove (c). First suppose that instead of dividing $[0,1]$ into N intervals we divide it into $m \cdot N$ intervals for some $m \in \mathbb{N}$. A coloring for the subdivision into N subintervals then also gives rise to a coloring into $m \cdot N$ subintervals, and it follows immediately from the definitions that we end up with the same element of $A * B$. Now if had chosen a different N' instead, then we can replace N and N' by $N \cdot N'$ and by the argument we just made we obtain the same element of $A * B$. This concludes the proof of (c). Summarizing, we are now done with the proof of the claim.

We turn to step (2) of the above program. We need to prove the following claim.

Claim. If $g, g' \colon [0,1] \to X$ are two path-homotopic loops in (X, x_0), then $\overline{\Psi}(g) = \overline{\Psi}(g')$.

Proof. Let $g, g' \colon [0,1] \to X$ be two path-homotopic loops in (X, x_0). We pick a path-homotopy

$$\begin{aligned} H \colon [0,1] \times [0,1] &\to X \\ (t, s) &\mapsto H^s(t). \end{aligned}$$

from the loop g to the loop g'. We apply the Lebesgue Corollary 6.6 to the map $H \colon [0,1] \times [0,1] \to X$ and the open cover $X = U \cup V$. We obtain an $N \in \mathbb{N}$ such that for every choice of $i, j \in \{0, \frac{1}{N}, \ldots, 1 - \frac{1}{N}\}$ the image of the square $[i, i + \frac{1}{N}] \times [j, j + \frac{1}{N}]$ under the map H

[570] In fact the only case which requires a slightly modified argument is that the coloring to the left and right are the same, i.e. either both are blue or both are green. We leave it to the reader to perform that argument.

lies in U or in V. Given $j \in \{0, \frac{1}{N}, \dots, 1\}$ we now define

$$\begin{array}{rcl} g(j)\colon [0,1] & \to & X \\ t & \mapsto & H^j(t) \end{array}$$

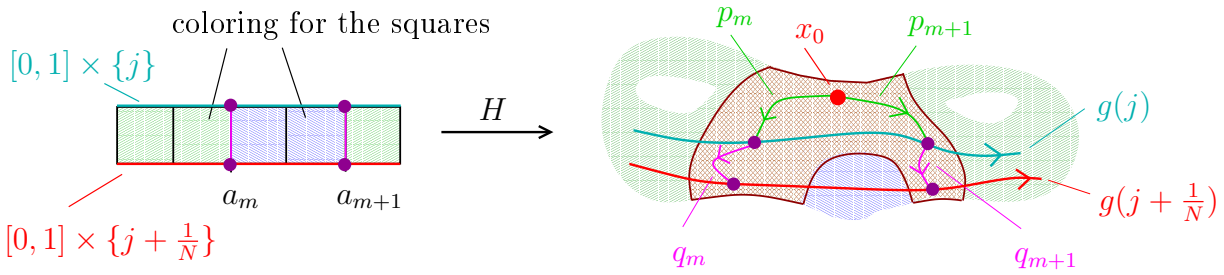
Note that each $g(j)$ is a loop in x_0 and that $g(0) = g$ and $g(1) = g'$. Therefore it suffices to show that for any $j \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$ we have $\overline{\Psi}(g(j)) = \overline{\Psi}(g(j + \frac{1}{N}))$. So let $j \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$.

As above we say that a subset of $[0,1] \times [0,1]$ can be *colored green* if the image of the subset under the map H lies in U and we say the subset can be *colored blue* if the image lies in V. We pick a map $\alpha\colon \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\} \to \{\text{blue}, \text{green}\}$ such that for each i the value $\alpha(i)$ is a color with which the square $[i, i + \frac{1}{N}] \times [\frac{j}{N}, \frac{j+1}{N}]$ can be colored. Note that α also defines a coloring for the loops $g(j)$ and $g(j + \frac{1}{N})$.

We define $0 = a_0 < a_1 < \cdots < \dots a_k = 1$ as above for the coloring α. Furthermore we pick a basing p_i, $i = 1, \dots, k-1$ for the loop $g(j)$. For $i = 0, \dots, k$ we write $q_i = H^{[j, j+\frac{1}{N}]}|_{a_i}$. Put differently, q_j connects a point on the loop $g(j)$ to the corresponding point on the loop $g(j + \frac{1}{N})$. Now we have

$$\begin{aligned} \overline{\Psi}(g(j)) &= \overline{\Psi}(g(j)) \text{ defined using the coloring } \alpha \text{ and the basing } p_1, \dots, p_{k-1} \\ &= \big[\cdots * \big[p_m * \underbrace{H^j|_{[a_m, a_{m+1}]}}_{= g(j)|_{[a_m, a_{m+1}]}} * \overline{p}_{m+1}\big] * \dots\big] \\ &= \big[\cdots * \big[p_m * q_m * \overline{q}_m * H^j|_{[a_m, a_{m+1}]} * q_{m+1} * \overline{q}_{m+1} * \overline{p}_{m+1}\big] * \dots\big] \\ &\quad \uparrow \\ &\quad \text{since each } q_m * \overline{q}_m \text{ is null-homotopic} \\ &= \big[\cdots * \big[p_m * q_m * \underbrace{H^{j+\frac{1}{N}}|_{[a_m, a_{m+1}]}}_{= g(j+\frac{1}{N})|_{[a_m, a_{m+1}]}} * \overline{q}_{m+1} * \overline{p}_{m+1}\big] * \dots\big] \\ &\quad \uparrow \\ &\quad \text{since } H \text{ restricts to path-homotopies from } \overline{q}_m * H^j|_{[a_m, a_{m+1}]} * q_{m+1} \\ &\quad \text{to } H^{j+\frac{1}{N}}|_{[a_m, a_{m+1}]} \text{ for } m = 0, \dots, k-1 \\ &= \overline{\Psi}(g(j + \tfrac{1}{N})) \text{ defined using } \alpha \text{ and the basing } p_1 * q_1, \dots, p_{k-1} * q_{k-1}. \end{aligned}$$

This last part of the proof is illustrated in the figure below. This concludes the proof of the claim and thus of part (2) of our three step program.



Summarizing we now showed that the map $\overline{\Psi}$ descends to a well-defined map

$$\begin{array}{rcl} \Psi\colon \pi_1(X, x_0) & \to & A * B \\ [g] & \mapsto & \big[\overline{\Psi}(g)\big]. \end{array}$$

It remains to prove the following claim.

Claim. We have $\Psi \circ \Phi = \mathrm{id}_{A*B}$.

Proof. So let $[y_1] \cdot \dots \cdot [y_k] \in A * B$ where $y_1, \dots, y_k$ are loops in (U, x_0) and (V, x_0). Then

$$(\Psi \circ \Phi)([y_1] \cdot \dots \cdot [y_k]) \overset{\text{definition of } \Phi\colon A*B = \pi_1(U,x_0) * \pi_1(V,x_0) \to \pi_1(X,x_0)}{=} \Psi([y_1 * \dots * y_k]) = \big[\overline{\Psi}(y_1 * \dots * y_k)\big] = [y_1 * \dots * y_k]$$
$$= [y_1] \cdot \dots \cdot [y_k].$$

by definition of $\overline{\Psi}$, where we use $N = k$ and where we can use the constant paths as basings since $y_1, \dots, y_k$ are already loops in (U, x_0) and (V, x_0)

This concludes the proof of the claim and thus also the proof of the Seifert–van Kampen Theorem 52.2 ∎

52.3. More examples: surfaces and the connected sum of smooth manifolds.

Given $g \in \mathbb{N}_0$ we denote by Σ_g the surface of genus g. In this section we first want to prove the following proposition.

Proposition 52.8. For $g \neq h$ the surface Σ_g of genus g is not homeomorphic to the surface Σ_h of genus h.

As on several earlier occasions we want to prove this result by considering what happens if we remove a point from the topological spaces.

Lemma 52.9. Let $g \in \mathbb{N}_0$. We denote by Σ_g the surface of genus g.

(1) For any point P the complement $\Sigma_g \setminus \{P\}$ is homotopy equivalent to the wedge of $2g$ circles. In particular

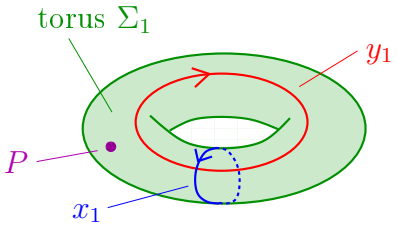
$$\pi_1(\Sigma_g \setminus \{P\}) \cong \langle x_1, y_1, \dots, x_g, y_g \rangle.$$

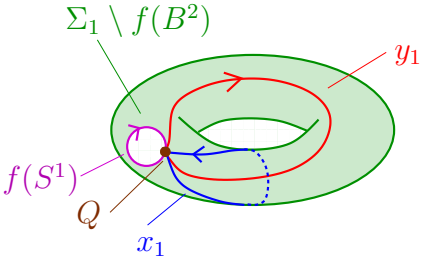
(2) Let $f\colon \overline{B}^2 \to \Sigma_g$ be a smooth embedding. The complement $\Sigma_g \setminus f(B^2)$ is homotopy equivalent to the wedge of $2g$ circles. Furthermore, if $Q \in f(S^1)$ is a point, then there exists an isomorphism

$$\begin{array}{rcl} \pi_1(\Sigma_g \setminus f(B^2), Q) & \overset{\cong}{\to} & \langle x_1, y_1, \dots, x_g, y_g \rangle \\ [f\colon S^1 \to \Sigma_g \setminus f(B^2)] & \mapsto & [x_1, y_1] \cdot \dots \cdot [x_g, y_g] \end{array}$$

with

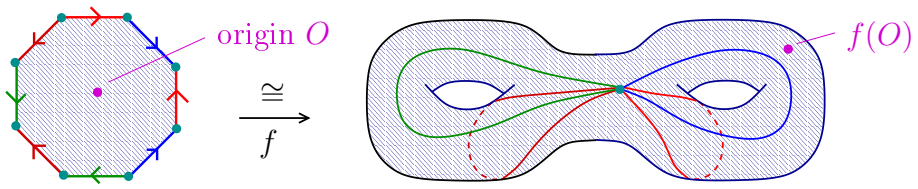
where $[x_i, y_i] := x_i y_i x_i^{-1} y_i^{-1}$ is the commutator of x_i and y_i. In particular the map $f\colon S^1 \to \Sigma_g \setminus f(B^2)$ represents a non-trivial element in $\pi_1(\Sigma_g \setminus f(B^2), Q)$.





Proof of Lemma 52.9. We prove the lemma for the surface Σ of genus 2. The proof for $g \geq 3$ is almost identical. So let $\Sigma = E_8/\sim$ as sketched in the figure below.

(1) We denote by O the origin in $E_8 \subset \mathbb{C}$. We first show that $\Sigma \setminus \{O\}$ is homotopy equivalent to the wedge of 4 circles.



Claim. There exists a deformation retraction from $\Sigma \setminus \{O\} = (E_8 \setminus \{O\})/\sim$ to the subset $\partial E_8/\sim$.

Proof. First we consider the map[571][572]

$$\begin{array}{rcl} r\colon E_8 \setminus \{O\} & \to & \partial E_8 \\ r \cdot \exp(\mathrm{i}\varphi) & \mapsto & \text{unique point } s\exp(\mathrm{i}\varphi) \text{ on } \partial E_8. \end{array}$$

This map "projects points radially outward towards the boundary". The deformation retraction is now given by "pushing outward radially", i.e. it is given by the map

$$\begin{array}{rcl} ((E_8 \setminus \{O\})/\sim) \times [0,1] & \to & (E_8 \setminus \{O\})/\sim \\ ([P], t) & \mapsto & [P \cdot (1-t) + r(P) \cdot t]. \end{array}$$ ⊞

Claim. The topological space $\partial E_8/\sim$ is homeomorphic to the wedge of 4 circles.

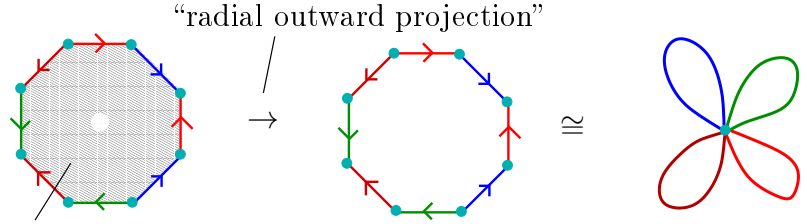
Proof. A homeomorphism is given by sending the point on $\partial E_8/\sim$ that corresponds to the vertices to the common point of the four circles, and by sending the edges that get identified to the same circle. We refer to the figure below for an illustration. We leave it to the reader to write down the proof in detail.

The two claims put together show that $\Sigma \setminus \{O\}$ is homotopy equivalent to the wedge of 4 circles. Now we obtain that

$$\pi_1(\Sigma_2 \setminus \{P\}) \underset{\uparrow}{\cong} \pi_1(\text{wedge of 4 circles}) \underset{\uparrow}{\cong} \langle x_1, y_1, x_2, y_2 \rangle.$$

Fundamental Group-Homotopy Proposition 50.3 (2) — by Lemma 52.4

Now let P be some other point on Σ. By Proposition 36.4 there exists a homeomorphism $\Phi\colon \Sigma \to \Sigma$ with $\Phi(P) = O$.[573] This shows that $\Sigma \setminus \{P\}$ is homeomorphic to $\Sigma \setminus \{O\}$. We can thus appeal to the above calculation to get the desired result.



(2) The proof of this statement is very similar to the proof of the first statement. First remove an open disk centered at O and we perform almost the same calculation as above. One easily sees that the boundary curve corresponds to the product of

[571] Note that this map cannot be extended continuously to O.

[572] Here we use polar coordinates, i.e. r, s are positive real numbers.

[573] Note that here we use Proposition 19.18 which says that Σ is a smooth manifold with empty boundary.

commutators. It follows from the Smooth Ball Embedding Theorem 37.8 that the same calculation also works for any other disk. We leave the details to the reader. ■

Now we can turn to the proof of Proposition 52.8.

Proof of Proposition 52.8. Let S and T be two surfaces of genus g and h. Suppose that there exists a homeomorphism $f\colon S \to T$. Pick a point $P \in S$. We write $Q = f(P)$. The homeomorphism f restricts to a homeomorphism $f\colon S \setminus \{P\} \to T \setminus \{Q\}$. It now follows that

$$\langle x_1, \dots, x_{2g} \rangle \underset{\text{Lemma 52.9 (1)}}{\cong} \pi_1(S \setminus \{P\}) \xrightarrow[f_*]{\cong} \pi_1(T \setminus \{Q\}) \underset{\text{Lemma 52.9 (1)}}{\cong} \langle y_1, \dots, y_{2h} \rangle.$$

We obtain from Lemma 51.17 that $g = h$. ■

For many applications the following theorem, which is a variation on the Seifert–van Kampen Theorem 52.2, is very useful.

Theorem 52.10. (Seifert–van Kampen Theorem for topological manifolds) Let M be an m-dimensional topological manifold and let $R, S \subset M$ be two m-dimensional submanifolds such that the following hold:

(1) $M = R \cup S$,
(2) $R \cap S$ is a component of ∂R and it is a component of ∂S,
(3) $R \cap S$ is non-empty and simply connected,
(4) R and S are closed subsets of M. (Note that by the Compact-Closed Lemma 1.21 (2) this condition is satisfied if R and S are compact.)

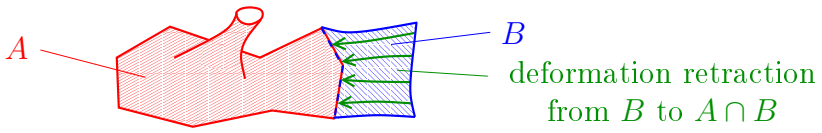
Then for any base point $x_0 \in R \cap S$ the inclusion induced map

$$\pi_1(R, x_0) * \pi_1(S, x_0) \;\to\; \pi_1(M, x_0)$$

is an isomorphism.

We will of course prove Theorem 52.10 by reducing it to Theorem 52.2. As we will see, the proof is very similar to the proof of the Wedge-π_1-Proposition 52.3 where we determined the fundamental group of a wedge of two spaces.
The proof of Theorem 52.10 relies on the following lemma which we will use on several occasions.

Lemma 52.11. Let X be a topological space. Furthermore let A and B be two subsets with $X = A \cup B$. If $A \cap B$ is a deformation retract of B and if A and B are closed subsets of X, then A is a deformation retract of X.



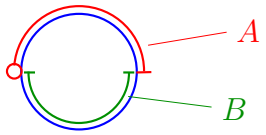
Proof. Lemma 52.11 can also be viewed as a consequence of Lemma 5.36 and the slightly scary the Homotopy Pushout Lemma 14.9. Since our situation is fairly straightforward we also give a direct, fairly elementary proof.

We pick a deformation retraction $F\colon B \times [0,1] \to B$ from B to $A \cap B$. We consider the map

$$\begin{array}{rcl} G\colon X\times[0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} x, & \text{if } x\in A, \\ F(x,t), & \text{if } x\in B. \end{cases} \end{array}$$

Since A and B are closed subsets of X we obtain from the Homotopy Stacking Lemma 14.3 that G is continuous. It is now clear that G is the desired deformation retraction from X to A. ■

Remark. On two occasions we are going to struggle a lit bit to verify the hypothesis of Lemma 52.11 that A and B need to be closed. Thus it is good to remind ourselves of the fact that hypothesis are usually added not to vex the reader but since they are necessary. For example, let us consider the situation shown in the figure below. Here $A\cap B$ is a deformation retract of B, but it should be pretty clear that A is not a deformation retract of $A\cup B = S^1$.

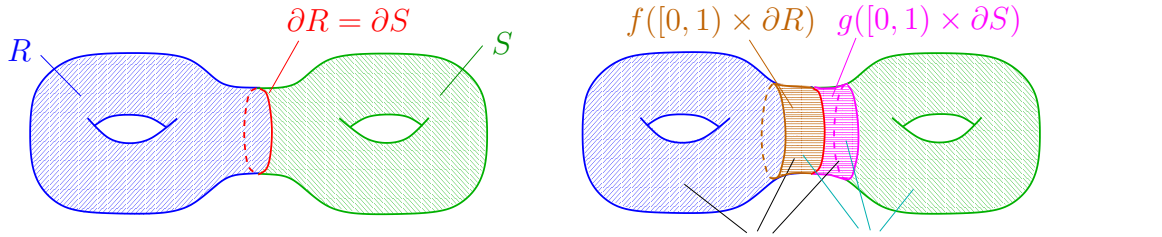


Proof of Theorem 52.10. Let M be an m-dimensional topological manifold and let $R, S\subset M$ be two m-dimensional submanifolds such that the following hold:

(1) $M = R\cup S$,
(2) $R\cap S$ is a union of components of ∂R and it is a union of components of ∂S,
(3) $R\cap S$ is simply connected,
(4) R and S are closed subsets of M.

We pick a base point[574] $x_0\in R\cap S$. Furthermore we pick maps $f\colon [0,1]\times\partial R\to R$ and $g\colon [0,1]\times\partial S\to S$ that are provided by the Smooth Collar Neighborhood Theorem 75.5[575] It follows from (2) that we can set

$$U := R\cup\underbrace{[0,1)\times g((R\cap S))}_{\subset[0,1)\times S} \quad\text{and}\quad V := S\cup\underbrace{[0,1)\times f((R\cap S))}_{\subset[0,1)\times R}.$$



We consider the following diagram

$$\begin{array}{ccc} \pi_1(R,x_0)*\pi_1(S,x_0) & \longrightarrow & \pi_1(M,x_0) \\ \downarrow & & \downarrow\mathrm{id} \\ \pi_1(U,x_0)*\pi_1(V,x_0) & \longrightarrow & \pi_1(M,x_0). \end{array}$$

[574]Recall that being simply connected entails being non-empty, hence such a point exists.
[575]If we are dealing with a smooth manifold we can also use the Smooth Collar Neighborhood Theorem 28.3.

The following steps are similar to the ones we performed in the proof of the Wedge-π_1-Proposition 52.3.

(1) Since the vertical maps are induced by inclusions we see that the diagram commutes.
(2) We claim that U is an open subset of $M = R \cup S$. Since R and S are by our hypothesis (4) closed subsets of M we obtain from Lemma 1.8 (2a) it suffices to show that $U \cap R$ is open in R and that $U \cap S$ is open in S. Evidently $U \cap R = R$ is open in R. Furthermore, $U \cap S = g([0,1] \times (R \cap S))$ is an open subset of S by (2) and the Basics-of-Collars Lemma 28.4.
(3) We want to use Lemma 52.11 to show that R is a deformation retract of U. We need to verify that the hypotheses of Lemma 52.11 are satisfied, i.e. we need to show that R and $g([0,1) \times (R \cap S))$ are closed subsets of U. By our hypothesis R and S are closed subsets of M. It follows from the Neighborhood Openness Criterion 1.5 (2) that $R = R \cap U$ and $g([0,1] \times (R \cap S)) = S \cap U$ are closed subsets of U.
(4) The obvious analogues of (2) and (3) hold for V and S.
(5) We obtain from (2), (3), (4) and the Homotopy-π_1-Proposition 50.3 (2) that the vertical map on the left is an isomorphism.
(6) Similar to (3) one can show that $R \cap S$ is a deformation retract of $U \cap V$. It follows from (3), the Deformation Retract-Homotopy Equivalence Lemma 15.5 and the Homotopy-π_1-Proposition 50.3 (2) that $U \cap V$ is simply connected.
(7) It follows from the Seifert–van Kampen Theorem 52.2 that the bottom horizontal map of the above commutative diagram is an isomorphism.
(8) Since the vertical maps are isomorphisms we see that the top horizontal map is also an isomorphism. ■

We conclude this section by considering the connected sum of two smooth manifolds of dimension ≥ 3. More precisely, we have the following proposition:

Proposition 52.12. (Connected Sum-π_1-Proposition) Let $n \geq 3$ and let M and M' be two oriented connected non-empty n-dimensional smooth manifolds. Then[576]

$$\pi_1(M \# M') \ \cong \ \pi_1(M) * \pi_1(M').$$

Examples.

(1) Let p, q be two coprime integers and let r, s be another two coprime integers. We can then consider the lens spaces $L(p,q)$ and $L(r,s)$. It follows from the Connected Sum-π_1-Proposition 52.12 and Corollary 48.18 that

$$\pi_1(L(p,q) \# L(r,s)) \ \cong \ \mathbb{Z}_p * \mathbb{Z}_r.$$

It follows for example from this calculation of fundamental groups, from Lemma 51.21 and from the Grushko Decomposition Theorem 51.22 that the two topological spaces $L(3,1) \# L(35,1)$ and $L(5,1) \# L(21,1)$ are not homeomorphic.
(2) Let $n \geq 3$ and let M and N be two oriented connected n-dimensional smooth manifolds. We denote by $-N$ the same smooth manifold as N but with the opposite orientation. It follows from the Connected Sum-π_1-Proposition 52.12 that $\pi_1(M \# N) \cong \pi_1(M \# (-N))$.

[576]Note that by the Connected Sum-Proposition **??** (3) we know that $M \# M'$ is also connected, thus we do not need to worry about base points.

Proof of the Connected Sum-π_1-Proposition 52.12. Let M and M' be two oriented connected non-empty n-dimensional smooth manifolds and let $\varphi\colon \overline{B}^n \to M$ and $\varphi'\colon \overline{B}^n \to M'$ be two smooth embeddings where φ is orientation-preserving and φ' is orientation-reversing. Recall that the connected sum of M and M' is defined as

$$M\#M' := \big(\underbrace{(M \setminus \varphi(B^n))}_{=:X} \sqcup \underbrace{(M' \setminus \varphi'(B^n))}_{=:X'}\big) / \varphi(P) = \varphi'(P) \text{ for } P \in S^{n-1}.$$

We pick a base point x_0 that lies in the intersection of the images of X and X' in $M\#M'$.

Claim. The inclusion induced maps $\pi_1(X, x_0) \to \pi_1(M, x_0)$ and $\pi_1(X', x_0) \to \pi_1(M', x_0)$ are isomorphisms.

Proof. We show that the inclusion induced map $\pi_1(X, x_0) \to \pi_1(M, x_0)$ is an isomorphism. We write $Y = \varphi(\overline{B}^n)$. It follows from the Smooth Embedding Theorem 24.10 that X and Y are n-dimensional submanifolds of M. Note that $X \cup Y = M$ and that $X \cap Y = \partial Y \cong S^{n-1}$. Since $n \geq 3$ we obtain from the Sphere-π_1-Proposition 47.13 that $X \cap Y$ is simply connected. We obtain that

$$\pi_1(X, x_0) \xrightarrow[\uparrow]{\cong} \pi_1(X, x_0) * \pi_1\big(\overline{B}^n, x_0\big) \xrightarrow[\uparrow]{\cong} \pi_1(M, x_0).$$

since $\pi_1\big(\overline{B}^n, x_0\big) = \{e\}$ — Seifert–van Kampen Theorem 52.10

Since the horizontal maps are induced by the inclusion maps we see that the inclusion induced map $\pi_1(X, x_0) \to \pi_1(M, x_0)$ is an isomorphism. The same way we can show that the inclusion induced map $\pi_1(X', x_0) \to \pi_1(M', x_0)$ is an isomorphism. ⊞

Now we see that[577]

$$\pi_1(M\#M', x_0) \xleftarrow[\uparrow]{\cong} \pi_1(X, x_0) * \pi_1(X', x_0) \xrightarrow[\uparrow]{\cong} \pi_1(M, x_0) * \pi_1(M', x_0).$$

by Theorem 52.10, since $X \cap X' \cong S^{n-1}$ is simply connected — above claim

■

Remark. We cannot apply the Seifert–van Kampen Theorem 52.2 to the connected sum of two surfaces, since in this case the fundamental group of the overlap $R \cap R'$ in the proof of the Connected Sum-π_1-Proposition 52.12 is not simply connected. In particular we cannot use this approach to determine the fundamental group of the surface of genus 2, which by the figure on page **??** is the connected sum of two tori. Our next goal is to formulate and prove a generalization of the Seifert–van Kampen Theorem 52.2 which will allow us to determine the fundamental group of the surface of genus g for $g \geq 2$.

Exercises for Chapter 52.

Exercise 52.1. We consider $X = \mathbb{C} \setminus \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Show that the point $x = 0$ is not good, i.e. show that there does not exist an open neighborhood U of $x = 0$ such that $x = 0$ is a deformation retract of U.

Exercise 52.2.

[577] Here the directions of the arrow indicate in which direction the obvious inclusion induced maps point.

(1) Let X be a path-connected topological space and let $x_0 \in X$. Furthermore let $X_k, k \in \mathbb{N}$ be a sequence of subsets of X such that the following hold:
   (a) each X_k is open,
   (b) each X_k is simply connected,
   (c) the sequence X_k is nested, i.e. for each k we have $X_k \subset X_{k+1}$,
   (d) we have $\bigcup_{k\in\mathbb{N}} X_k = X$.

   Show that $\pi_1(X, x_0) = 0$.
   Hint. Use Lemma 2.12.
(2) Does the conclusion of (1) also hold if the sets X_k are not open?

Exercise 52.3. Let M be a *topological* manifold of dimension $n \geq 3$. Let P be a point and let $x_0 \in M \setminus \{P\}$ be a base point. Show that the inclusion induced map

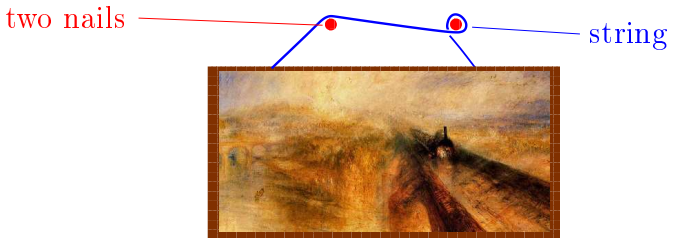
$$\pi_1(M \setminus \{P\}, x_0) \;\to\; \pi_1(M, x_0)$$

is an isomorphism.

Exercise 52.4.

(a) How can one hang a painting with two nails provided, such that if one removes either of the two nails, the painting falls down?
(b) What is a solution to (a) if we use three nails instead of two nails?

Hint. What does this have to do with free groups?



Exercise 52.5. Show that condition (4) in the statement of the Seifert–van Kampen Theorem 52.10 for smooth manifolds cannot be dropped. More precisely, give an example of an m-dimensional smooth manifold and m-dimensional submanifolds R and S such that conditions (1), (2) and (3) are satisfied, but such that $\pi_1(R, x_0) * \pi_1(S, x_0)$ is not isomorphic to $\pi_1(M, x_0)$.
Hint. The dimension m does not have to be taken particularly large.

Exercise 52.6. We consider the two topological manifolds X and Y that are shown in the figure below. Show that X and Y are not homeomorphic.
Remark. As a smooth manifold X is orientable and Y is non-orientable. But as of right now we do not know whether orientability is preserved under homeomorphisms.
Hint. You could use Proposition 50.7 and consider the interaction between the topological manifolds and their boundaries.

Exercise 52.7. In the proof of Proposition 52.7, given on page 1176, show that the map

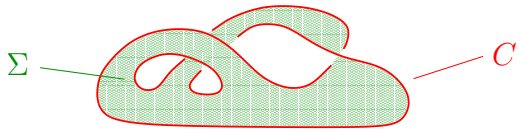
$$\ast_{k\in K} i_{k*} \colon \ast_{k\in K} \pi_1(A_k) \;\to\; \pi_1\Big(\bigvee_{k\in K} A_k\Big)$$



is a monomorphism.
Hint. Consider the commutative diagram shown on page 1177.

Exercise 52.8. Let Σ be the 2-dimensional smooth manifold that is shown in the figure below. We denote by C the boundary of Σ.

(a) What is the homeomorphism type of the topological space that we obtain from gluing a disk $\overline{B}^2$ to Σ along the boundary C?
(b) Is C a retract of Σ?



Exercise 52.9.

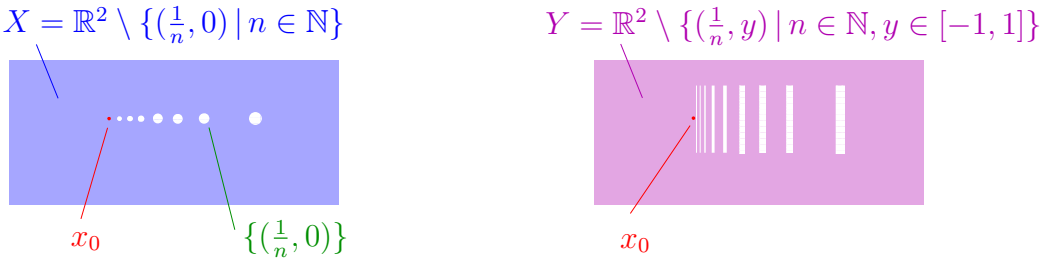
(a) Let $n \geq 3$ and furthermore let $K \subset \mathbb{R}^n$ be a compact subset. We view K also as a subset of $S^n = \mathbb{R}^n \cup \{\infty\}$. Let $x_0 \in \mathbb{R}^n \setminus K$. Show that the inclusion induced map $\pi_1(\mathbb{R}^n \setminus K, x_0) \to \pi_1(S^n \setminus K, x_0)$ is an isomorphism.
(b) Does the conclusion of (a) also hold for $n = 2$?
(c) Does the conclusion of (c) also hold for non-compact subsets of $\mathbb{R}^n$?

Exercise 52.10. We consider the two topological spaces

$$X = \mathbb{R}^2 \setminus \{(\tfrac{1}{n}, 0) \,|\, n \in \mathbb{N}\} \text{ and } \quad \text{and } Y = \mathbb{R}^2 \setminus \{(\tfrac{1}{n}, y) \,|\, n \in \mathbb{N}, y \in [-1, 1]\}.$$

We write $x_0 = (0, 0)$. We refer to the figure below for an illustration.

(a) Show that $\pi_1(X, x_0)$ is uncountable.
Hint. For $n \in \mathbb{N}$ we write $Z_n := \mathbb{R}^2 \setminus \{\frac{1}{n}, 0\}$ and we denote by $f_n \colon X \to Z_n$ the inclusion map. Use the induced maps $f_{n*} \colon \pi_1(X, x_0) \to \pi_1(Z_n, x_0) \cong \mathbb{Z}$ to find a homomorphism from $\pi_1(X, x_0)$ to a group such that the image is uncountable.
(b) Show that $\pi_1(Y, x_0)$ is countable.
Hint. Use Exercise 2.44 to find a monomorphism from $\pi_1(Y, x_0)$ into a countable group.



Exercise 52.11. Show that given any $k \in \mathbb{N}$ and given any $n \geq 3$ there exists a closed orientable connected n-dimensional smooth manifold M such that $\pi_1(M) \cong \langle x_1, \ldots, x_k \rangle$.

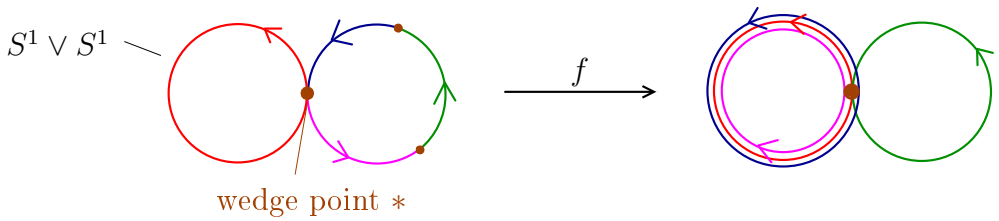
Exercise 52.12. We consider the wedge $S^1 \vee S^1$ of two circles and we denote by $*$ the wedge point. We consider the map $f\colon S^1 \vee S^1 \to S^1 \vee S^1$ that is shown in the figure below.

(a) Show that f is a homotopy equivalence.
(b) Show that every homeomorphism of $S^1 \vee S^1$ preserves the wedge point $*$.
(c) Determine

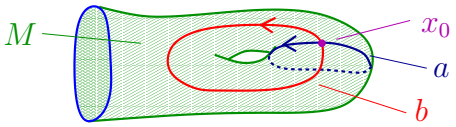
$$\{\varphi \in \mathrm{Aut}(\pi_1(S^1 \vee S^1, *)) \mid \text{ there exists a homeomorphism } g \text{ with } \varphi = g_*\}.$$

(d) Show that f is not homotopic to a self-homeomorphism of $S^1 \vee S^1$.

Remark. In Proposition **??** we will show that there exist self-homotopy equivalences of smooth manifolds that are not homotopic to homeomorphisms.



Exercise 52.13. We consider 2-dimensional smooth manifold M and the two oriented loops a and b in the point $x_0 \in M$ that are shown in the figure below. Sketch an *embedded* loop $g\colon S^1 \to M$ that represents the commutator $[a, b] \in \pi_1(M, x_0)$.



Exercise 52.14. Let $g \in \mathbb{N}$. We denote by Σ_g the surface of genus g.

(a) Does there exist a smooth map $\Sigma_g \to S^2$ that is a local diffeomorphism?
(b) Let M be the smooth manifold that we obtain from removing a single point from the surface of genus g. Does there exist a smooth map $M \to S^2$ that is a local diffeomorphism?

53. Presentations of groups and amalgamated products

We have now made some progress, for example we have shown that surfaces of different genus are not homeomorphic. But this proof is perhaps slightly unsatisfactory in so far as we used the trick of removing a point, instead of dealing directly with the fundamental groups of surfaces.

In the next section we will cover the generalized Seifert–van Kampen Theorem. It will allow us to compute many more fundamental groups. In particular we will determine the fundamental groups of surfaces and knot complements.
Before we can state the generalized Seifert–van Kampen Theorem we need to introduce more concepts from group theory. We will do so in this chapter.

53.1. **Basic definitions in group theory.** We recall a few basic definitions from group theory.

Definition. Let G be a group.

(1) A subgroup H is called normal if for every $g \in G$ we have $g^{-1}Hg = H$. If $H \subset G$ is normal, then we often write $H \lhd G$. If $\{H_i\}_{i\in I}$ is a family of normal subgroups of G, then it is straightforward to see that the intersection $\bigcap_{i\in I} H_i$ is again a normal subgroup of G.

(2) Let H be a subgroup of G. The subgroup H defines an equivalence relation on G by defining $g \sim \widetilde{g}$ if there exists an $h \in H$ with $gh = \widetilde{g}$. Put differently, we have $g \sim \widetilde{g}$ if $\widetilde{g}^{-1}g \in H$. (Note that the equivalence classes are precisely the sets of the form $gH = \{gh \mid h \in H\}$ for some $g \in G$.) We denote the set of equivalence classes by G/H.

Remark. Let H be a normal subgroup of G.

(1) For any $h \in H$ and $a, b \in G$ we have

$$ahbH \;=\; ab\cdot \underbrace{b^{-1}hb}_{\text{in } H \text{ since } H \text{ is normal}} H \;=\; abH.$$

(2) If $aH = bH$, then for any $c, d \in G$ we have

$$cadH \;=\; cb\underbrace{(b^{-1}a)}_{\in H}dH \;\underset{\text{above remark (1)}}{=}\; cbdH.$$

Now we will see that the group structure on G descends to a group structure on G/H. More precisely, given aH and bH we set

$$aH \cdot bH \;:=\; abH.$$

It follows immediately from the above remark (1) that this definition does not depend on the representatives a and b. It is now straightforward to see that this defines indeed a group structure on G/H. The following lemma plays the same role as the Topological-Quotient Proposition 5.15 (1b) does for topological spaces.

Lemma 53.1. Let G be a group and let H be a normal subgroup. Let $\varphi\colon G \to \Gamma$ be a group homomorphism with the property that $H \subset \ker(\varphi)$. Then there exists a unique

homomorphism $\psi\colon G/H \to \Gamma$ such that the following diagram commutes

$$\begin{array}{ccc} G & \longrightarrow & G/H \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \psi} \\ & & \Gamma. \end{array}$$

Proof. For $gH \in G/H$ we put $\psi(gH) := \varphi(g)$. Since $H \subset \ker(\varphi)$ this definition is independent of the choice of the representatives g.[578] It is straightforward to verify that ψ is a homomorphism and that it is uniquely determined by the desired property. ∎

53.2. Presentations of groups.

Definition. Let G be a group and let $A \subset G$ be a subset. We define

$$\text{subgroup of } G \text{ generated by } A \;:=\; \langle A \rangle \;:=\; \bigcap_{\substack{A \subset H \subset G \\ H \text{ subgroup}}} H,$$

i.e. $\langle A \rangle$ is the intersection of all subgroups H of G which contain A. This is again a subgroup of G, namely the smallest[579] subgroup of G which contains A. Similarly we define

$$\text{subgroup of } G \text{ normally generated by } A \;:=\; \langle\langle A \rangle\rangle \;:=\; \bigcap_{A \subset H \lhd G} H,$$

i.e. $\langle\langle A \rangle\rangle$ is the intersection of all normal subgroups H of G which contain A. This is again a normal subgroup of G, namely the smallest normal subgroup of G which contains A.

The following lemma follows immediately from the definitions and from the fact that in an abelian group every subgroup is normal.

Lemma 53.2. Let G be a group and let $A \subset G$ be a subset.

(1) If A is a subgroup, then $\langle A \rangle = A$.
(2) If A is a normal subgroup, then $\langle\langle A \rangle\rangle = A$.
(3) If G is abelian, then $\langle\langle A \rangle\rangle = \langle A \rangle$.
(4) For any $B \subset G$ with $A \subset B \subset \langle\langle A \rangle\rangle$ we have $\langle\langle B \rangle\rangle = \langle\langle A \rangle\rangle$.
(5) Suppose that for each $a \in A$ we are given $g_a \in A$. Then $\langle\langle \{g_a a g_a^{-1}\}_{a \in A} \rangle\rangle = \langle\langle A \rangle\rangle$.

The following lemma gives a more explicit description of $\langle A \rangle$ and $\langle\langle A \rangle\rangle$.

Lemma 53.3. Let G be a group and let $A \subset G$ be a subset.

(1) Then $\langle A \rangle$ equals the set of all elements of G which can be written as products of elements in A and their inverses, i.e.

$$\langle A \rangle \;=\; \left\{ a_1^{\epsilon_1} \cdot \dots \cdot a_k^{\epsilon_k} \mid a_1, \dots, a_k \in A \text{ and } \epsilon_1, \dots, \epsilon_k \in \{-1, 1\} \right\}.$$

(2) The subset $\langle\langle A \rangle\rangle$ equals the set of all elements of G which can be written as products of conjugates of elements in A and their inverses, i.e.

$$\langle\langle A \rangle\rangle = \left\{ g_1 a_1^{\epsilon_1} g_1^{-1} \dots g_k a_k^{\epsilon_k} g_k^{-1} \mid a_1, \dots, a_k \in A, g_1 \dots, g_k \in G \text{ and } \epsilon_1, \dots, \epsilon_k \in \{-1, 1\} \right\}.$$

[578] Indeed, if $gH = \widetilde{g}H$, then $g = \widetilde{g}h$ for some $h \in H$, therefore $\varphi(g) = \varphi(\widetilde{g}h) = \varphi(\widetilde{g}) \cdot \varphi(h) = \varphi(\widetilde{g})$.

[579] Here "smallest" means that if B is another subgroup that contains A, then $\langle A \rangle \subset B$. This statement is obvious, since B appears in the definition of $\langle A \rangle$ as one of the H's.

Example. On page 1074 we introduced a subgroup of the group H of all self-homeomorphisms of $\mathbb{R}^2$ that was generated by two given self-homeomorphisms A and B. The lemma shows that the definition of "being generated" by a subset used on page 1074 agrees with the above more formal definition.

Proof. We only prove the second statement. The proof of the first statement is very similar. So let G be a group and let $A \subset G$ be a subset. The subgroup $\langle\langle A\rangle\rangle$ contains all elements of A and their inverses. The normal subgroup $\langle\langle A\rangle\rangle$ then contains also all elements of the form $ga^{\pm 1}g^{-1}, g \in G, a \in A$. Since $\langle\langle A\rangle\rangle$ is a subgroup it is also closed under multiplication, i.e. it contains products of such elements. It follows that $\langle\langle A\rangle\rangle$ contains the right-hand side of (2). Conversely it is straightforward to see that the right-hand side is indeed a normal[580] subgroup of G that contains A, hence it is contained in the left-hand side of (2). ■

Lemma 53.4. Let G be a group and let $A \subset G$ be a subset. Let $\alpha\colon G \to \Gamma$ be a group homomorphism such that $\alpha(a) = e \in \Gamma$ for all $a \in A$. Then there exists a unique group homomorphism $\beta\colon G/\langle\langle A\rangle\rangle \to \Gamma$ such that the following diagram commutes:

$$\begin{array}{ccc} G & \longrightarrow & G/\langle\langle A\rangle\rangle \\ & \searrow^{\alpha} & \downarrow{\scriptstyle \beta} \\ & & \Gamma. \end{array}$$

Proof. Let G be a group and let $A \subset G$ be a subset. Furthermore let $\alpha\colon G \to \Gamma$ be a group homomorphism such that $\alpha(a) = e \in \Gamma$ for all $a \in A$. This means that $A \subset \ker(\alpha)$. Since $\ker(\alpha)$ is a normal subgroup it follows from the definition of $\langle\langle A\rangle\rangle$ that $\langle\langle A\rangle\rangle \subset \ker(\alpha)$. The proposition now follows immediately from Lemma 53.1. ■

The following definition will subsequently play an important role:

Definition. Let X be a set and let R be a subset of the free group $\langle X\rangle$ generated by the set X. We define

$$\langle X \mid R\rangle \;:=\; \langle X\rangle \,/\, \langle\langle R\rangle\rangle.$$

If $X = \{x_1, \ldots, x_k\}$ and $R = \{r_1, \ldots, r_l\}$, then we use the convenient shorthand

$$\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l\rangle \;:=\; \langle \{x_1, \ldots, x_k\} \,|\, \{r_1, \ldots, r_l\}\rangle.$$

Examples.

(1) For any $k \in \mathbb{N}_0$ we have

$$\langle g \mid g^k\rangle \;=\; \langle g\rangle/\langle\langle g^k\rangle\rangle \;\underset{\uparrow}{=}\; \langle g\rangle/\langle g^k\rangle \;\cong \mathbb{Z}_k.$$

since $\langle g\rangle$ is abelian it follows from Lemma 53.2 that $\langle\langle g^k\rangle\rangle = \langle g^k\rangle$

(2) The group $\langle x_1, \ldots, x_k \,|\,\rangle$ with $R = \varnothing$ is just the free group on $x_1, \ldots, x_k$.

Before we continue we formulate an elementary but useful lemma.

Lemma 53.5. Let $\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l\rangle$ be a presentation.

(1) Given any $a, b \in \langle x_1, \ldots, x_k\rangle$, given any $i \in \{1, \ldots, l\}$ and given any $\epsilon \in \{-1, 1\}$ we have

[580] Why is the right-hand side normal?

$$a \cdot r_i^{\epsilon} \cdot b \;=\; a \cdot b \;\in\; \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle.$$

(2) Let $a, b \in \langle x_1, \ldots, x_k \rangle$ and $c, d \in \langle x_1, \ldots, x_k \rangle$. If $a = b \in \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$, then $c \cdot a \cdot d \;=\; c \cdot b \cdot d \;\in\; \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$.

Proof. The first statement is just a special case of Remark (1) on page 1191 and the second statement is just a special case of Remark (2) on page 1191. ■

This lemma allows us to prove the following lemma.

Lemma 53.6.

(1) There exists a unique homomorphism

$$\beta\colon \pi = \langle x, y \rangle / \langle\langle [x, y] \rangle\rangle = \langle x, y \mid [x, y] \rangle \;\to\; \mathbb{Z}^2$$

with $\beta(x) = (1, 0)$ and $\beta(y) = (0, 1)$. Furthermore this homomorphism is an isomorphism.

(2) Let $m \in \mathbb{N}$. There exists a unique isomorphism

$$\beta\colon \langle x_1, \ldots, x_m \mid [x_i, x_j] = e \text{ for all } i, j \in \{1, \ldots, m\} \rangle \;\xrightarrow{\cong}\; \mathbb{Z}^m$$

which sends each x_i to the i-th standard basis vector e_i of $\mathbb{Z}^m$.

Proof. By Lemma 51.15 there exists a unique homomorphism $\alpha\colon \langle x, y \rangle \to \mathbb{Z}^2$ which satisfies $\alpha(x) = (1, 0)$ and $\alpha(y) = (0, 1)$. Note that

$$\alpha([x, y]) \;=\; \alpha(xyx^{-1}y^{-1}) \;=\; \alpha(x) + \alpha(y) - \alpha(x) - \alpha(y) \;=\; 0.$$

Thus by Lemma 53.4 there exists a unique group homomorphism

$$\beta\colon \pi = \langle x, y \rangle / \langle\langle [x, y] \rangle\rangle = \langle x, y \mid [x, y] \rangle \;\to\; \mathbb{Z}^2$$

with $\beta(x) = (1, 0)$ and $\beta(y) = (0, 1)$. It is clear that β is an epimorphism. Thus it remains to show that α is a monomorphism. This follows immediately from the following claim.

Claim. Given any $w \in \langle x, y \rangle$ there exist $m, n \in \mathbb{Z}$ with $w = x^m y^n \in \langle x, y \mid [x, y] \rangle$.

Proof. First we note that we have the following equalities in $\langle x, y \mid [x, y] \rangle$

$$\underset{\text{Lemma 53.5 (1)}}{yx \;=\; xyx^{-1}y^{-1} \cdot yx} \;=\; xy \qquad \text{and} \qquad xy^{-1} \;=\; y^{-1} \cdot yx \cdot y^{-1} \;\underset{\text{calculation on the left and Lemma 53.5 (2)}}{=\; y^{-1} \cdot xy \cdot y^{-1}} \;=\; y^{-1}x.$$

In Exercise 53.4 we will show that the equalities $yx^{-1} = x^{-1}y$ and $x^{-1}y^{-1} = y^{-1}x^{-1}$ hold in $\pi = \langle x, y \mid [x, y] \rangle$. This means that in π an arbitrary product of powers of x and y can be reordered to equal a product of the $x^m y^n$. For example in π we have the equality

$$x^3y^2x^{-1}y \;=\; x^3y \cdot yx^{-1} \cdot y \;\underset{\text{since } yx^{-1} = x^{-1}y}{=}\; x^3y \cdot x^{-1}y \cdot y \;=\; x^3 \cdot yx^{-1} \cdot y^2 \;\underset{\text{since } yx^{-1} = x^{-1}y}{=}\; x^3 \cdot x^{-1}y \cdot y^2 \;=\; x^2y^3.$$

■

We consider one more example.

Example. We consider the following three self-homeomorphisms of $\mathbb{R}^2$:

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & (s+1, 1-t) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & (s, t+1) \end{array} \quad \text{and} \quad \begin{array}{rcl} \widetilde{A}\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & (s+1, -t). \end{array}$$

We already know the first two maps from page 1074. We denote by G the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by A and B. We are interested in the group G

since $\mathbb{R}^2/G$ is the Klein bottle K and by the Fundamental Group-via-Actions Theorem 48.16 we have $\pi_1(K) \cong G$.

Note that $\widetilde{A} = B^{-1}A$. Let $X = \{x, y\}$ be the set that consists of two elements. By Lemma 51.15 there exists a unique homomorphism $\alpha\colon \langle x, y\rangle \to G$ with $\alpha(x) = \widetilde{A}$ and $\alpha(y) = B$. Now let R be the subset of $\langle X\rangle = \langle x, y\rangle$ that consists of the single element $yxyx^{-1}$. We want to consider $\alpha(yxyx^{-1}) \in G$, which is by definition a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. For any point $(s, t) \in \mathbb{R}^2$ this map is given by

$$\begin{aligned}\alpha(yxyx^{-1})(s,t) &= B\widetilde{A}B\widetilde{A}^{-1}(s,t) &&= B\widetilde{A}B(s-1,-t)\\ &&&= B\widetilde{A}(s-1,-t+1) = B(s,t-1) = (s,t).\end{aligned}$$

This shows that $\alpha(yxyx^{-1})$ is the identity map of $\mathbb{R}^2$, i.e. it is the trivial element in G. It follows from Lemma 53.4 that α descends to a homomorphism

$$\beta\colon \langle x, y\rangle/\langle\langle yxyx^{-1}\rangle\rangle = \langle x, y \mid yxyx^{-1}\rangle \to G.$$

This homomorphism is an epimorphism, since the image of β is the group generated by $\widetilde{A} = B^{-1}A$ and B, but this group is the same as the group generated by A and B, i.e. the image of β is G. Is the homomorphism β also a monomorphism? We will come back to this question later on page 1223.

We continue with the following definitions.

Definition.

(1) A presentation is a set X together with a subset $R \subset \langle X\rangle$. We call the elements of X the generators and we call the elements of R the relators[581] of the presentation.

(2) Let π be a group. A presentation for π is a presentation (X, R) together with an isomorphism
$$\langle X \mid R\rangle \xrightarrow{\cong} \pi.$$
Very often one just says $\langle X \mid R\rangle$ is a presentation for π and the explicit isomorphism is suppressed from the notation.

(3) We say that a group π is finitely generated, if π admits a presentation with finitely many generators.[582]

(4) We say a group π is finitely presented, if π admits a presentation with finitely many generators and finitely many relators.

Now we give some examples of finitely generated and finitely presented groups.

Examples.

(1) Every group admits a presentation. Indeed, let π be a group. Then we set $X := \pi$, i.e. X is the set of elements of π. By Lemma 51.15 there exists a unique homomorphism $\alpha\colon \langle X\rangle \to \pi$ with $\alpha(g) = g$ for all $g \in X = \pi$. We set $R := \ker(\alpha)$. Then

$$\langle X \mid R\rangle = \langle X\rangle/\langle\langle R\rangle\rangle = \langle X\rangle/\langle\langle\ker(\alpha)\rangle\rangle \underset{\uparrow}{=} \langle X\rangle/\ker(\alpha) \xrightarrow[\alpha]{\cong} G.$$
by Lemma 53.2 since $R = \ker(\alpha)$ is a normal subgroup

[581] The relators are often also called relations.

[582] This definition of a finitely generated group is equivalent to the one we gave on page 1159. Why is that?

(2) We had just see in Lemma 53.6 (2) that

$$\mathbb{Z}^n \;\cong\; \langle x_1, \dots, x_n \mid [x_i, x_j] = e \text{ with } i, j \in \{1, \dots, n\}\rangle$$

which shows that $\mathbb{Z}^n$ is a finitely presented group.

(3) In fact every finitely generated abelian group is also finitely presented. Indeed, let A be a finitely generated abelian group. Then

$$\begin{array}{ccc} A & \cong & \bigoplus_{i=1}^{n} \mathbb{Z}_{a_i} \\ \uparrow & & \\ \text{Theorem 51.4} & & \\ \text{for some } a_i \in \mathbb{N}_0 & & \end{array} \quad \begin{array}{c} = \\ \uparrow \\ \text{as in (2)} \end{array} \quad \langle x_1, \dots, x_n \mid [x_i, x_j] = e \text{ for } i, j = 1, \dots, n, x_i^{a_i} = e \text{ for } i = 1, \dots, n\rangle.$$

(4) There exist groups which are finitely generated but not finitely presented. In fact there are countably many isomorphism types of finitely presented groups,[583] whereas there are uncountably many isomorphism types of finitely generated groups.[584] A proof for both statements is given in Chapter 14 of [**Rob1995**].

(5) We saw on page 1159 that the fundamental group of the Klein bottle is finitely generated. But is it also finitely presented?

53.3. Tietze transformations.

Definition. Let π be a group and let

$$\phi\colon \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l\rangle \;\xrightarrow{\cong}\; \pi$$

be a presentation. We obtain new presentations as follows:

(1) For any element s of $\langle\langle r_1, \dots, r_l\rangle\rangle \subset \langle X\rangle$ we obtain the presentation

$$\langle x_1, \dots, x_k \mid r_1, \dots, r_l, s\rangle \;=\; \langle x_1, \dots, x_k \mid r_1, \dots, r_l\rangle \;\xrightarrow{\phi}\; \pi.$$

(2) For any $x \neq x_1, \dots, x_k$ and any two words s, t in $x_1, \dots, x_k$ we obtain the presentation[585]

$$\begin{array}{rcl} \langle x_1, \dots, x_k, x \mid r_1, \dots, r_l, s \cdot x \cdot t\rangle & \to & \langle x_1, \dots, x_k \mid r_1, \dots, r_l\rangle \xrightarrow{\phi} \pi. \\ x_i & \mapsto & x_i \\ x & \mapsto & s^{-1} \cdot t \end{array}$$

The above two methods, and their inverses, of obtaining new presentations out of a given one are called Tietze transformations.[586]

The above Tietze transformations show that from a given finite presentation we can obtain infinitely many new finite presentations. This fact is not that surprising. What is much more interesting is the following theorem which was first proved by Heinrich Tietze [**Tie1908**] in 1908.

Theorem 53.7. (Tietze Theorem) Any two finite presentations for a given group are related by a finite sequence of Tietze transformations.

---

[583] That statement is pretty obvious.

[584] This statement is much less obvious. The idea is that in a free group $\langle x_1, \dots, x_k\rangle$ there are uncountably many infinite sequences of elements $\{r_1, r_2, \dots\}$. The difficulty is to show that there exist uncountably many sequences that give raise to non-isomorphic groups.

[586] The map to the left is an isomorphism, in fact an inverse is given by $x_i \mapsto x_i$.

In the proof of the Tietze Theorem 53.7 we will need the following four pretty elementary preparatory lemmas.

Lemma 53.8. Let $\theta\colon G \to H$ be a group homomorphism and let X be a generating set for G. The subgroup of $G * H$ that is generated by $\{g \cdot \theta(g^{-1}) \,|\, g \in G\}$ equals that the subgroup of $G * H$ that is normally generated by $\{x \cdot \theta(x^{-1}) \,|\, x \in X\}$.

Proof of Lemma 53.8. We set $N := \{x \cdot \theta(x^{-1}) \,|\, x \in X\}$. We only need to show that given any $g \in G$ we have $g \cdot \theta(g^{-1}) \in \langle\langle N \rangle\rangle$. This statement follows easily by induction on the length of g as a word in the generating set X and the observation that for $x, y \in X$ we have the following equality:

$$xy \cdot \theta((xy)^{-1}) \;=\; xy \cdot \theta(y^{-1}) \cdot \theta(x^{-1}) \;=\; x \cdot \underbrace{y \cdot \theta(y)^{-1}}_{\in N} \cdot x^{-1} \cdot \underbrace{x \cdot \theta(x^{-1})}_{\in N}.$$

■

Lemma 53.9. Let G and H be two groups, let $\theta\colon G \to H$ be an isomorphism and let $\alpha\colon G \to \pi$ be an isomorphism. We consider the homomorphism

$$\Psi\colon G * H \;\to\; \pi$$

defined by α on G and by $\alpha \circ \theta^{-1}$ on H. The kernel of the homomorphism Ψ is generated by $\{g \cdot \theta(g^{-1}) \,|\, g \in G\}$.

Proof of Lemma 53.9. We set $M := \{g \cdot \theta(g^{-1}) \,|\, g \in G\}$. We need to show that the kernel of the homomorphism Ψ is generated by M. Let $g_1 \cdot h_1 \cdot \dots \cdot g_k \cdot h_k \in \ker(\Psi) \subset G * H$, where we use self-explanatory notation.

If $k = 1$, then it follows from $e = \Psi(g_1 \cdot h_1) = \alpha(g_1) \cdot \alpha(\theta^{-1}(g_1)) = \alpha(g_1 \cdot \theta^{-1}(g_1))$ and the fact that α and θ are isomorphisms that $g_1 \cdot h_1 = g_1 \cdot \theta({g_1}^{-1})$.

Now suppose that $k > 1$. If $g_1 \neq e$, then we can write

$$g_1 \cdot h_1 \cdot g_2 \cdot h_2 \cdot \dots \cdot g_k \cdot h_k \;=\; \underbrace{g_1 \cdot \theta({g_1}^{-1})}_{\in M} \cdot \underbrace{\theta(g_1) \cdot h_1}_{\in H} \cdot g_2 \cdot h_2 \cdot \dots g_k \cdot h_k.$$

If $g_1 = e$, then we write

$$h_1 \cdot g_2 \cdot h_2 \cdot \dots \cdot g_k \cdot h_k \;=\; \underbrace{(\theta^{-1}(h_1) \cdot \theta(\theta^{-1}(h_1)^{-1}))^{-1}}_{\in M} \cdot \underbrace{\theta^{-1}(h_1) \cdot g_2}_{\in G} \cdot h_2 \cdot \dots g_k \cdot h_k.$$

Iterating this procedure we obtain the desired statement. ■

Lemma 53.10. Let

$$\phi\colon \langle x_1, \dots, x_k \,|\, r_1(x_1, \dots, x_k), \dots, r_l(x_1, \dots, x_k)\rangle \;\to\; \pi$$

be a presentation. For pairwise different $y_1, \dots, y_k$ the presentation

$$\begin{aligned} \langle y_1, \dots, y_k \,|\, r_1(y_1, \dots, y_k), \dots, r_l(y_1, \dots, y_k)\rangle &\;\to\; \pi \\ y_i &\;\mapsto\; \phi(x_i) \end{aligned}$$

can be obtained from the original one by Tietze transformations.

Proof of Lemma 53.10. We consider the Tietze transformations

$$\begin{aligned}
\langle x_1,\dots,x_k \,|\, r_1(x_1,\dots,x_k),\dots\rangle &\overset{(2)}{\rightsquigarrow} \langle x_1,\dots,x_k,y_1 \,|\, r_1(x_1,\dots,x_k),\dots,y_1\cdot x_1^{-1}\rangle\\
&\overset{(1)}{\rightsquigarrow} \langle x_1,\dots,x_k,y_1 \,|\, r_1(x_1,\dots,x_k),\dots,y_1\cdot x_1^{-1},r_1(y_1,\dots,x_k),\dots\rangle\\
&\overset{(1)}{\rightsquigarrow} \langle x_1,\dots,x_k,y_1 \,|\, y_1\cdot x_1^{-1},r_1(y_1,\dots,x_k),\dots\rangle\\
&\overset{(1)}{\rightsquigarrow} \langle x_1,\dots,x_k,y_1 \,|\, x_1\cdot {y_1}^{-1},y_1\cdot x_1^{-1},r_1(y_1,\dots,x_k),\dots\rangle\\
&\overset{(1)}{\rightsquigarrow} \langle x_1,\dots,x_k,y_1 \,|\, x_1\cdot {y_1}^{-1},r_1(y_1,\dots,x_k),\dots\rangle\\
&\overset{(2)}{\rightsquigarrow} \langle x_2,\dots,x_k,y_1 \,|\, r_1(y_1,\dots,x_k),\dots\rangle.
\end{aligned}$$

Iterating this process leads to the desired sequence of Tietze transformations. ∎

Lemma 53.11. Let $\alpha\colon A\to B$ and $\beta\colon B\to C$ be group epimorphisms. Let $X\subset A$ be a subset such that $\ker(\alpha)=\langle\langle X\rangle\rangle$ and let $Y\subset B$ be a subset such that $\ker(\beta)=\langle\langle\alpha(Y)\rangle\rangle$. Then $\ker(\beta\circ\alpha\colon A\to C)=\langle\langle X\cup Y\rangle\rangle$.

Proof of Lemma 53.11. It is clear that $X\cup Y\subset\ker(\beta\circ\alpha\colon A\to C)$. To prove the reverse inclusion let $g\in\ker(\beta\circ\alpha\colon A\to C)$. We have $\alpha(g)\in\ker(\beta)=\langle\langle\alpha(Y)\rangle\rangle$. Since α is an epimorphism the map $\alpha\colon\langle\langle Y\rangle\rangle\to\langle\langle\alpha(Y)\rangle\rangle$ is an epimorphism. Thus there exists an $h\in\langle\langle Y\rangle\rangle$ with $\alpha(g)=\alpha(h)$. But this implies that $g\cdot h^{-1}\in\ker(\alpha)=\langle\langle X\rangle\rangle$. It follows that $g\in\langle\langle X\cup Y\rangle\rangle$. ∎

Proof of the Tietze Theorem 53.7. Let π be a group and let $\phi_1\colon\langle X_1\,|\,R_1\rangle\xrightarrow{\cong}\pi$ and $\phi_2\colon\langle X_2\,|\,R_2\rangle\xrightarrow{\cong}\pi$ be two finite presentations for π. We need to show that they are related by Tietze transformations. First note that it follows from Lemma 53.10 that by applying a sequence of Tietze transformations we can arrange that $X_1\cap X_2=\varnothing$.

We denote by $p_1\colon\langle X_1\rangle\to\langle X_1\,|\,R_1\rangle$ and $p_2\colon\langle X_2\rangle\to\langle X_2\,|\,R_2\rangle$ the obvious epimorphisms. We set $X:=X_1\sqcup X_2$. We consider the homomorphism

$$\Psi\colon\langle X_1\rangle*\langle X_2\rangle=\langle X\rangle\ \to\ \pi$$

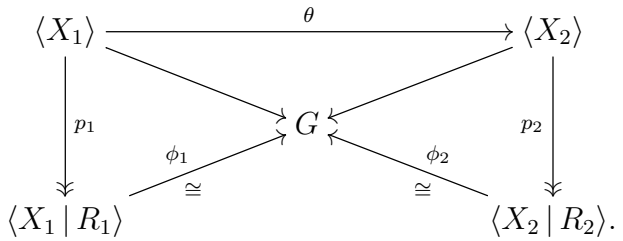
that is induced by $\phi_1\circ p_1$ on $\langle X_1\rangle$ and $\phi_2\circ p_2$ on $\langle X_2\rangle$.

Note that the restriction of Ψ to $\langle X_2\rangle$ is an epimorphism. Therefore we can pick for each $x\in X_1$ a $w_x\in\langle X_2\rangle$ with $\Psi(x)=\Psi(w_x)$. We write $S_1:=\{x\cdot w_x^{-1}\,|\,x\in X_1\}\subset\langle X\rangle$. The same way we define $S_2\subset\langle X\rangle$.

Claim. We have $\ker(\Psi)=\langle\langle R_2\cup S_1\rangle\rangle$.

Proof. First note that it follows almost immediately from Lemmas 53.8, 53.9 and 53.11 that $\ker(\Psi)=\langle\langle R_1\cup R_2\cup S_1\rangle\rangle$. Thus it remains to show that $\langle\langle R_1\cup R_2\cup S_1\rangle\rangle=\langle\langle R_2\cup S_1\rangle\rangle$. Put differently, given $r_1\in R_1$ we need to show that $r_1\in\langle\langle R_2\cup S_1\rangle\rangle$. Let $\theta\colon\langle\langle X_1\rangle\rangle\to\langle\langle X_2\rangle\rangle$ be the homomorphism that is given by $x\mapsto w_x$ for $x\in X_1$. Note that we have the following

commutative diagram:



In particular we have $(\phi_2 \circ p_2)(\theta(r_1)) = (\phi_1 \circ p_1)(r_1) = 1$. This implies that $\theta(r_1) \in \langle\langle R_2 \rangle\rangle$. On the other hand it follows from Lemma 53.8 that $r_1 \cdot \theta(r_1)^{-1} \in \langle\langle S_1 \rangle\rangle$. Thus we see that $r_1 = r_1 \cdot \theta(r_1)^{-1} \cdot \theta(r_1) \in \langle\langle S_1 \cup R_2 \rangle\rangle$. ⊞

After these preparations we turn to the actual proof of the theorem.

(a) First note that we can turn the presentation $\langle X_2 \,|\, R_2 \rangle$ into $\langle X \,|\, R_2 \cup S_1 \rangle$ by a sequence of Tietze transformations (2), where we add one by one the elements of X_1 and for each element of X_1 we add the corresponding relation in S_1.

(b) Note that $R_1 \cup S_2 \subset \ker(\Psi)$. By the claim this implies that $R_1 \cup S_2 \subset \langle\langle R_2 \cup S_1 \rangle\rangle$. In particular, using Tietze transformations (1) we can turn the presentation $\langle X \,|\, R_2 \cup S_1 \rangle$ into the presentation $\langle X \,|\, R_1 \cup S_2 \cup R_2 \cup S_1 \rangle$.

(c) By exactly the same reasoning as in (a) and (b) we can also turn the presentation $\langle X_1 \,|\, R_1 \rangle$ into the presentation $\langle X \,|\, R_1 \cup S_2 \cup R_2 \cup S_1 \rangle$ via a sequence of Tietze transformations.

Since $\langle X_1 \,|\, R_1 \rangle$ and $\langle X_2 \,|\, R_2 \rangle$ can be turned into precisely the same presentation using Tietze transformations, the two presentations are in fact related by Tietze transformations. ■

Remark. Alternative proofs of the Tietze Theorem 53.7 can be found in [**Tie1908**] and also [**LyS1977**, Proposition II.2.2] and [**MKS1976**, Theorem 1.5].

The following lemma often works well in practice to show that two presentations correspond to isomorphic groups.

Lemma 53.12. Let $X = \{x_1, \ldots, x_k\}$ be a finite set and furthermore let $r = \{r_1, \ldots, r_l\}$ be a finite subset of $\langle X \rangle = \langle x_1, \ldots, x_k \rangle$. Suppose that we can write each $r_i(x_1, \ldots, x_k)$ as a word $s_i(x_1, \ldots, x_{k-1}, x_k \cdot w)$ where w is a word in $x_1, \ldots, x_{k-1}$, then the homomorphism

$$\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle \;\to\; \langle x_1, \ldots, x_{k-1}, t \,|\, s_1, \ldots, s_l \rangle$$

that is induced by $x_i \mapsto x_i$, $i = 1, \ldots, k-1$, and $x_k \mapsto t w^{-1}$ is an isomorphism.

Definition. We refer to the isomorphism of Lemma 53.12 as the isomorphism given by the substitution $t = x_k \cdot w$.

Example. A picky reader might decide that the formulation of Lemma 53.12 is not completely precise. But the following example should make the meaning clear:

$$\langle a, b \,|\, a^{-1}ba^{-1}bab^{-1} \rangle \;=\; \langle a, b \,|\, \underbrace{a^{-1}b \cdot a^{-1}b \cdot a \cdot (a^{-1}b)^{-1} \cdot a^{-1}}_{=a^{-1}ba^{-1}bab^{-1}} \rangle \;\underset{\text{substitution } c = a^{-1}b}{\overset{\uparrow}{=}}\; \langle a, c \,|\, c \cdot c \cdot a \cdot c^{-1} \cdot a^{-1} \rangle.$$

Proof. It follows from Lemma 53.13 that there exists in fact a unique homomorphism

$$\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle \;\to\; \langle x_1, \ldots, x_{k-1}, t \,|\, s_1, \ldots, s_l \rangle$$

that has the property that $x_i \mapsto x_i$, $i = 1, \dots, k-1$ and that $x_k \mapsto tw^{-1}$. Furthermore, we have a map from the right to left by the reverse substitution $t \mapsto x_k \cdot w$. It is straightforward to verify that these homomorphisms are inverses of one another, hence both are isomorphisms. ■

As we will now see, finite presentations of groups are useful for finding group homomorphisms. To formulate the corresponding lemma we need the following definition.

Definition. Let $X = \{x_1, \dots, x_k\}$ be a set and let $r \in \langle X \rangle$. Furthermore let G be a group and let $g_1, \dots, g_k$ be elements of G. We say that $g_1, \dots, g_k$ satisfy the relation r if the homomorphism[587] $\langle x_1, \dots, x_k \rangle \to G$ that is given by $x_i \mapsto g_i$, $i = 1, \dots, k$ sends r to the trivial element.

Examples.

(1) We consider $\{x_1, x_2\}$ and the word $r = x_1 x_2 x_1^{-1} x_2^{-1}$. If a group is abelian, then any two elements g_1, g_2 satisfy the relation $x_1 x_2 x_1^{-1} x_2^{-1}$.
(2) Let G be a finite group with d elements. We consider the set $\{x\}$. It follows from Lagrange's Theorem, see e.g. [**KuS2004**, p. 8], that every element in G satisfies the relation x^d.

Lemma 53.13. Let π be a group with a finite presentation $\pi = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$. For any group G the map

$$\begin{array}{rcl} \mathrm{Hom}(\pi, G) & \to & \{\text{all } (g_1, \dots, g_k) \in G^k \text{ that satisfy } r_1, \dots, r_l\} \\ f & \mapsto & (f(x_1), \dots, f(x_k)) \end{array}$$

is a bijection.

Remark. Note that if G is a finite group, then the right-hand side of Lemma 53.13 is a finite set. In practice it is usually straightforward, albeit tedious, to determine the right-hand side.

Proof. This lemma follows immediately from Lemma 53.4 and Lemma 51.15. ■

Next, recall that in Exercise 51.17 we gave an explicit example of a group epimorphism $\varphi\colon \langle x, y \rangle \to \mathbb{Z}$ such that the kernel $\ker(\varphi)$ is infinitely generated. The next proposition shows that this was not a fluke.

Proposition 53.14. Every non-trivial normal subgroup of a free group is infinitely generated.

Proof. A purely group theoretic proof is given in [**LyS1977**, Proposition I.3.12]. Alternatively a proof using topological methods is provided in [**dlH2000**, p. 45]. ■

Fortunately the following proposition, which we obtain from [**DrK2018**, Lemma 7.29], gives us at least some control over the "size" of a kernel.

Proposition 53.15. Let $\varphi\colon \pi \to G$ be a group epimorphism. If π is finitely generated and if G is finitely presented, then $\ker(\varphi)$ is finitely normally generated. In other words, there exists a finite subset $S \subset \ker(\varphi)$ such that $\langle\langle S \rangle\rangle = \ker(\varphi)$.

[587] The fact that there exists a unique such homomorphism is a consequence of Lemma 51.15.

Example. Let $\varphi\colon \langle x, y\rangle \to \mathbb{Z}^2$ be the epimorphism that is given by $\varphi(x) = (1, 0)$ and $\varphi(y) = (0, 1)$. In Exercise 53.10 we will see that $\ker(\varphi)$ is normally generated by a single element, namely the commutator $xyx^{-1}y^{-1}$.

We conclude this section with an interesting little lemma. For the formulation we need the following harmless definition.

Definition. Let G and H be groups. We say H is a retract of G if there exist a group homomorphisms $\varphi\colon H \to G$ and $\rho\colon G \to H$ such that $\rho \circ \varphi = \mathrm{id}_H$.

Lemma 53.16. (Finite Presentation-Retract Lemma) Every retract of a finitely presented group is also finitely presented.

Proof. Let H be a group, let G be a finitely presented group and let $\varphi\colon H \to G$ and $\rho\colon G \to H$ be group homomorphisms such that $\rho \circ \varphi = \mathrm{id}_H$. We need to show that H is also finitely presented.

To do so we pick a finite presentation $G = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l\rangle$ for the group G. We write $F = \langle g_1, \dots, g_k\rangle$. In the following we denote by $p\colon F \to G$ the obvious epimorphism. For each $i \in \{1, \dots, k\}$ we pick $w_i \in F$ with $p(w_i) = (\varphi \circ \rho \circ p)(g_i) \in G$. It suffices to prove the following claim.

Claim. A finite presentation for H is given by $\langle g_1, \dots, g_k \,|\, r_1, \dots, r_l, g_1^{-1}w_1, \dots, g_k^{-1}w_k\rangle$.

Proof. Let L be the group that is defined by the finite presentation of the claim. We denote by $u\colon G \to L$ and $q\colon F \to L$ the obvious projections. Note that for each $j \in \{1, \dots, l\}$ we have $\rho(p(r_i)) = \rho(e) = e$ and for each $i \in \{1, \dots, k\}$ we have

$$\rho(p(g_i^{-1}w_i)) \;=\; \rho(p(g_i^{-1})) \cdot \rho(p(w_i)) \;\underset{\substack{\uparrow\\ \text{since } p(w_i) = (\varphi\circ\rho\circ p)(g_i) \text{ and since } \rho\circ\varphi = \mathrm{id}_H}}{=}\; \rho(p(g_i^{-1})) \cdot \rho(p(g_i)) \;=\; e.$$

It follows from Lemma 53.4 that the homomorphism $p \circ \varphi\colon F \to H$ factors through a homomorphism $v\colon L \to H$.

We want to show that $u \circ \varphi$ is an isomorphism. First note that

$$v \circ (u \circ \varphi) \;=\; (v \circ u) \circ \varphi \;\underset{\substack{\uparrow\\ \text{since } \rho = v\circ u}}{=}\; \rho \circ \varphi \;=\; \mathrm{id}_H\,.$$

By Lemma 51.19 it remains to show that $u\circ\varphi$ is an epimorphism. Note that for $i = 1, \dots, k$ we have

$$(u \circ \varphi)(\underbrace{(\rho \circ p)(w_i)}_{\in H}) \;\underset{\substack{\uparrow\\ \text{since } \varphi\circ\rho = \mathrm{id}_H}}{=}\; u(p(w_i)) \;=\; q(w_i) \;\underset{\substack{\uparrow\\ \text{since } q(w_i) = g_i \in L}}{=}\; g_i \;\in\; L.$$

Since $g_1, \dots, g_k$ is a generating set for L we see that $u \circ \varphi$ is indeed an epimorphism.

For convenience we summarize all objects in the following diagram

$$\begin{array}{ccccc} & \overset{\rho}{\curvearrowleft} & & & \\ H & \underset{v}{\overset{\varphi}{\rightleftarrows}} & G = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l\rangle & \xleftarrow{\;p\;} & F = \langle g_1, \dots, g_k\rangle \\ & & \downarrow u & \swarrow q & \\ & & L := \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l, g_1^{-1}w_1, \dots, g_k^{-1}w_k\rangle. & & \end{array}$$

■

53.4. **The Andrews-Curtis Conjecture.** In this section we want to introduce Nielsen transformations and we want to present the Andrews-Curtis Conjecture. Both are mostly peripheral to the content of these notes. But for general culture it is perhaps interesting to know that in this supposedly fairly elementary context of finite presentations there are difficult conjectures lurking in the background.
First we introduce the following notation.

Notation. Given a set S and $n \in \mathbb{N}_0$ we refer to an element $(g_1, \dots, g_n) \in S^n$ as an ordered set of elements in S.[588]

Now we can introduce the definition we are really interested in.

Definition. Let π be a group and let $(g_1, \dots, g_n)$ be an ordered set of elements in π. A Nielsen operation is one of the following three ways of obtaining a new ordered set of elements in π:

$$\begin{array}{lrcl} (1) & (g_1, g_2, g_3, \dots, g_n) & \rightsquigarrow & (g_1^{-1}, g_2, g_3, \dots, g_n) \\ (2) & (g_1, g_2, g_3, \dots, g_n) & \rightsquigarrow & (g_2, g_1, g_3, \dots, g_n) \\ (3) & (g_1, g_2, g_3, \dots, g_n) & \rightsquigarrow & (g_1 \cdot g_2, g_2, g_3, \dots, g_n). \end{array}$$

The following theorem that was proved by Jakob Nielsen [**Nie1924**] (see also [**LyS1977**, Proposition I.4.1] and [**MKS1976**, Theorem 3.2]) in 1924.

Theorem 53.17. (Nielsen 1924) Let $F = \langle x_1, \dots, x_n \rangle$ be a free group. If $y_1, \dots, y_n \in F$ form a basis[589] for the free group F, then $(y_1, \dots, y_n)$ can be obtained from $(x_1, \dots, x_n)$ through a finite sequence of Nielsen operations (1), (2) and (3).

Remark.

(1) Let $F = \langle g_1, \dots, g_n \rangle$ be a finitely generated free group. It follows from Lemma 51.15 that any Nielsen operation, applied to $(g_1, \dots, g_n)$, gives rise to an endomorphism of the free group F. One can easily show that these endomorphisms are in fact automorphisms of the free group. We refer to any such automorphism as a *Nielsen automorphism*. Nielsen's Theorem 53.17 implies almost immediately that any automorphism of the free group is in fact a composition of finitely many Nielsen automorphisms. In other words, the group $\mathrm{Aut}(F)$ is generated by the Nielsen transformations.
(2) The fact that the group $\mathrm{Aut}(F)$ is generated by the Nielsen operations is similar in spirit to the fact, established in any self-respecting linear algebra course, that the group $\mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}(n, \mathbb{Z})$ is generated by elementary matrices.

The following conjecture was formulated by James Andrews and Morton Curtis [**AC1965**] in 1965.

Conjecture 53.18. (Andrews-Curtis Conjecture) Let $\langle g_1, \dots, g_n \,|\, r_1, \dots, r_n \rangle$ be a finite presentation. If the corresponding group is the trivial group, then $(r_1, \dots, r_n)$ can be obtained from $(g_1, \dots, g_n)$ through a finite sequence of operations where we allow the Nielsen operations (1), (2), (3) and the following fourth operation:

$$(4) \qquad (g_1, g_2, g_3, \dots, g_n) \rightsquigarrow (hg_1h^{-1}, g_2, g_3, \dots, g_n) \quad \text{for some } h \in \pi.$$

[588] Note that we do not assume that $g_1, \dots, g_n$ are pairwise different.
[589] Recall that by the definition on page 1159 the elements $y_1, \dots, y_n$ form a basis if the natural map $\langle y_1, \dots, y_n \rangle \to F$ is an isomorphism.

The Andrews-Curtis Conjecture is notoriously difficult. It seems like most experts believe the conjecture to be false, but not much progress has been made over the last decades. Finally note that the Andrews-Curtis Conjecture will raise its head once more in Section 141.2.

53.5. **The abelianization of a group.** The following definition should already be familiar from the algebra course.

Definition. Let π be a group. We refer to

$$[\pi,\pi] \;:=\; \langle\langle \{ \underbrace{[x,y]}_{=xyx^{-1}y^{-1}} \mid x,y\in\pi\}\rangle\rangle$$

as the commutator subgroup of π.

We start out with the following elementary lemma.

Lemma 53.19. Let π be a group. Then the following hold:

(1) The group $\pi/[\pi,\pi]$ is abelian.

(2) We have

$$[\pi,\pi] \;:=\; \left\{ \prod_{i=1}^{n} [x_i,y_i] \,\middle|\, x_1,y_1,\ldots,x_n,y_n\in\pi \right\}.$$

Remark. Note that in general not every element of the commutator subgroup is a commutator. For example, if $\pi=\langle a,b,c,d\rangle$ is the free group on four generators a,b,c,d, then we have $[a,b]\cdot[c,d]\in[\pi,\pi]$, but it is not a commutator.[590]

Proof. Let π be a group.

(1) For any two elements $x,y\in\pi$ we have

$$xy\,[\pi,\pi] \;=\; yx\cdot x^{-1}y^{-1}xy\,[\pi,\pi] \;=\; yx\cdot \underbrace{[x^{-1},y^{-1}]}_{\in[\pi,\pi]}\,[\pi,\pi] \;=\; yx\,[\pi,\pi].$$

This shows that $\pi/[\pi,\pi]$ is indeed abelian.

(2) Note that the inverse of a commutator is again a commutator, i.e. given $x,y\in\pi$ we have

$$[x,y]^{-1} \;=\; (xyx^{-1}y^{-1})^{-1} \;=\; yxy^{-1}x^{-1} \;=\; [y,x].$$

Furthermore a conjugate of a commutator is again a commutator. Indeed, given $x,y,g\in\pi$ we have

$$\begin{aligned} g[x,y]g^{-1} = gxyx^{-1}y^{-1}g^{-1} &= gxg^{-1}gyg^{-1}gx^{-1}g^{-1}gy^{-1}g^{-1} \\ &= (gxg^{-1})(gyg^{-1})(gxg^{-1})^{-1}(gyg^{-1})^{-1} = [gxg^{-1},gyg^{-1}]. \end{aligned}$$

It follows from this discussion and from Lemma 53.3 that each element of the commutator subgroup can be written as a product of commutators, i.e.

$$[\pi,\pi] \;:=\; \left\{ \prod_{i=1}^{n} [x_i,y_i] \,\middle|\, x_1,y_1,\ldots,x_n,y_n\in\pi \right\}.$$

■

Definition. Let π be a group. We call

$$\pi_{\mathrm{ab}} \;:=\; \pi/[\pi,\pi]$$

[590] Why not?

the abelianization of π.[591]

The following proposition summarizes some of the key properties of the abelianization of groups.

Proposition 53.20. (Abelianization Proposition)

(1) If π is abelian, then $\pi_{\text{ab}} = \pi$.

(2) Let π be a group and let $\alpha\colon \pi \to H$ be a homomorphism to an abelian group. Then there exists a unique homomorphism $\beta\colon \pi_{\text{ab}} = \pi/[\pi,\pi] \to H$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi & \longrightarrow & \pi_{\text{ab}} = \pi/[\pi,\pi] \\ & \searrow^{\alpha} & \downarrow{\scriptstyle \beta} \\ & & H. \end{array}$$

In particular for any group π and any abelian group G the map

$$\begin{array}{rcl} \text{Hom}(\pi_{\text{ab}}, G) & \to & \text{Hom}(\pi, G) \\ (\alpha\colon \pi_{\text{ab}} \to G) & \mapsto & \big(\pi \to \pi_{\text{ab}} \xrightarrow{\alpha} G\big) \end{array}$$

is a bijection.

(3) Let G and H be two groups. There exists a unique isomorphism

$$(G * H)_{\text{ab}} \xrightarrow{\cong} G_{\text{ab}} \times H_{\text{ab}}$$

which makes the following diagram commute:

$$\begin{array}{ccc} G * H & \twoheadrightarrow & G \times H \\ \downarrow\!\downarrow & & \downarrow\!\downarrow \\ (G * H)_{\text{ab}} & \xrightarrow{\cong} & G_{\text{ab}} \times H_{\text{ab}}, \end{array}$$

where the other three maps are the natural projections.

(4) Let $F = \langle x_1, \ldots, x_m\rangle$ be the free group on m generators $x_1, \ldots, x_m$. Then the map[592]

$$\begin{array}{rcl} \Phi\colon F_{\text{ab}} = F/[F,F] & \to & \mathbb{Z}^m \\ [x_i] & \mapsto & e_i := \text{the } i\text{-th standard basis vector} \end{array}$$

is an isomorphism.

(5) Let $\pi = \langle x_1, \ldots, x_m \mid r_1, \ldots, r_n\rangle$ be a finitely presented group. We denote by

$$\Phi\colon \langle x_1, \ldots, x_m\rangle \to \mathbb{Z}^m$$

the homomorphism that is given by $\Phi(x_i) = e_i$. There exists a unique isomorphism

$$\Psi\colon \pi/[\pi,\pi] \xrightarrow{\cong} \mathbb{Z}^m/\langle \Phi(r_1), \ldots, \Phi(r_n)\rangle$$

[591] The abelianization of a group π is sometimes also written as $\text{H}_1(\pi)$. The reason for that somewhat strange looking notation will become clear later on when we formulate the Hurewicz Theorem 84.5 and when we consider the homology groups, see Chapter **??**.

such that the following diagram commutes

$$\begin{array}{ccc} \langle x_1,\dots,x_m\rangle & \xrightarrow[\Phi]{x_i\mapsto e_i} & \twoheadrightarrow \mathbb{Z}^m \\ \downarrow & & \\ \pi = \langle x_1,\dots,x_m\rangle/\langle\langle r_1,\dots,r_n\rangle\rangle & & \downarrow \\ \downarrow & & \\ \pi_{\mathrm{ab}} = \pi/[\pi,\pi] & \xrightarrow[\cong]{\Psi} & \mathbb{Z}^m/\langle\Phi(r_1),\dots,\Phi(r_n)\rangle. \end{array}$$

(6) The maps

$$G \;\mapsto\; G_{\mathrm{ab}} = G/[G,G]$$

and

$$(\varphi\colon G\to H) \;\mapsto\; \left(\begin{array}{rcl} \varphi_*\colon G_{\mathrm{ab}} = G/[G,G] & \to & H_{\mathrm{ab}} = H/[H,H] \\ [g] & \mapsto & [\varphi(g)] \end{array}\right)$$

define a covariant functor from the category *Group* of groups to the category *AbGroup* of abelian groups.

(7) Given a group G we denote by

$$\begin{array}{rcl} \Phi_G\colon G & \to & G_{\mathrm{ab}} = G/[G,G] \\ g & \mapsto & [g] \end{array}$$

the projection . These maps Φ_G define a natural transformation from the identity functor *Group* $\to$ *Group* to the abelianization functor *Group* $\to$ *Group* that we had introduced in (6).[593]

(8) Isomorphic groups have isomorphic abelianizations.

(9) If $\varphi\colon A\to B$ is an epimorphism, then the induced map $A_{\mathrm{ab}}\to B_{\mathrm{ab}}$ is also an epimorphism.

Examples.

(1) By the Abelianization Proposition 53.20 (1) and (3) we have

$$(\mathbb{Z}_3 * \mathbb{Z}_{35})_{\mathrm{ab}} \;\cong\; \mathbb{Z}_3\times\mathbb{Z}_{35} \quad\text{and}\quad (\mathbb{Z}_5 * \mathbb{Z}_{21})_{\mathrm{ab}} \;\cong\; \mathbb{Z}_5\times\mathbb{Z}_{21}.$$

It follows from the Chinese Remainder Theorem that the abelianizations are isomorphic. On the other hand on page 1164 we had already observed that the Grushko Decomposition Theorem 51.22 implies that the groups $\mathbb{Z}_3 * \mathbb{Z}_{35}$ and $\mathbb{Z}_5 * \mathbb{Z}_{21}$ are not isomorphic. Thus we have seen that non-isomorphic groups can have isomorphic abelianizations.

(2) We give an explicit example for Statement (5). Let

$$\pi \;=\; \langle x, y \mid x^2y^{-2}x, y^{-1}x^3yx\rangle.$$

[592]By Lemma 51.15 there exists a unique homomorphism $F\to\mathbb{Z}^m$ with $x_i\mapsto e_i$. By (1) this map descends to a homomorphism $F/[F,F]\to\mathbb{Z}^m$. The way we write down the map is a little bit sloppy, since we only specify the map on the generators of $F/[F,F]$.

[593]Super-picky readers might argue that in (6) we introduced a functor from *Group* to *AbGroup*, but of course we can also view it as a functor from $\mathcal{G}$ to $\mathcal{G}$.

Let $\Phi\colon \langle x, y\rangle \to \mathbb{Z}^2$ be the epimorphism given by $\Phi(x) = (1, 0)$ and $\Phi(y) = (0, 1)$. Then the abelianization of π is given by[594]

$$\pi_{\text{ab}} \;=\; \mathbb{Z}^2/\langle \underbrace{\Phi(x^3y^{-2})}_{=\binom{3}{-2}}, \underbrace{\Phi(y^{-1}x^3yx)}_{=\binom{4}{0}} \rangle \;=\; \mathbb{Z}^2/\begin{pmatrix} 3 & 4 \\ -2 & 0 \end{pmatrix}\mathbb{Z}^2 \;\cong\; \mathbb{Z}_8.$$

(3) Let S be a set. It is straightforward to show that the abelianization of the free group $\langle S\rangle$ generated by S is isomorphic to the free abelian group $\mathbb{Z}^{(S)}$ generated by S.
(4) Given $k \in \mathbb{N}$ we denote by S_k the permutation group on k elements and we denote by $A_k \subset S_k$ the alternating group, i.e. the group of all permutations with positive sign. In the algebra course it was shown that for $k \geq 5$ the group A_k is simple, i.e. the only normal subgroups of A_k are the trivial group and the group A_k itself.[595] The commutator subgroup $[A_k, A_k]$ is normal. Since A_k is not abelian we cannot have $[A_k, A_k] = \{e\}$. Thus we see that $[A_k, A_k] = A_k$, which implies that the abelianization of A_k is the trivial group.

Now we provide the proof of the Abelianization Proposition 53.20.

Proof of the Abelianization Proposition 53.20.

(1) The first statement is obvious.[596]
(2) Let π be a group and let $\alpha\colon \pi \to H$ be a homomorphism to an abelian group. Then $[\pi, \pi]$ lies in the kernel of α. The desired statement now follows immediately from Lemma 53.1.
(3) The maps

$$\begin{array}{rcl} \Phi\colon (G * H)_{\text{ab}} & \to & G_{\text{ab}} \times H_{\text{ab}} \\ [(x_1, \ldots, x_k)] & \mapsto & \left(\left[\prod_{x_i \in G} x_i\right], \left[\prod_{x_i \in H} x_i\right]\right) \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi\colon G_{\text{ab}} \times H_{\text{ab}} & \to & (G * H)_{\text{ab}} \\ ([g], [h]) & \mapsto & [g \cdot h] \end{array}$$

are easily seen to be well-defined, to be homomorphisms and to be inverses of one another.
(4) This statement follows immediately from the statements (1) and (3) together with the observation that $\langle x_1, \ldots, x_m\rangle = \langle x_1\rangle * \cdots * \langle x_m\rangle$, that $\langle x_i\rangle_{\text{ab}} \cong \mathbb{Z}$ and that $\mathbb{Z}^m = \mathbb{Z} \times \cdots \times \mathbb{Z}$.
(5) It is straightforward to see that there exists a unique homomorphism

$$\Psi\colon \pi/[\pi, \pi] \;\to\; \mathbb{Z}^m/\langle \Phi(r_1), \ldots, \Phi(r_n)\rangle$$

that makes the diagram of Statement (5) commute. It remains to show that Ψ is an isomorphism. We write $F = \langle x_1, \ldots, x_m\rangle$ and we write $G = \langle y_1, \ldots, y_n\rangle$. We

[594] Why is the group isomorphic to $\mathbb{Z}_8$?
[595] A proof of this statement is also given in any self-respecting algebra book, see e.g. [**Rob1995**, p. 69].
[596] Why?

consider the following diagram

$$\begin{array}{ccc} G_{\mathrm{ab}} = G/[G,G] & \xrightarrow[\cong]{[y_i]\mapsto e_i} & \mathbb{Z}^n \\ \downarrow{\scriptstyle [y_i]\mapsto [r_i]} & & \downarrow{\scriptstyle e_i\mapsto \Phi(r_i)} \\ F_{\mathrm{ab}} = F/[F,F] & \xrightarrow[\cong]{\Phi([x_i])=e_i} & \mathbb{Z}^m \\ \downarrow & & \downarrow \\ \pi_{\mathrm{ab}} = \pi/[\pi,\pi] & \xrightarrow{\Psi} & \dfrac{\mathbb{Z}^m}{\langle \Phi(r_1),\dots,\Phi(r_n)\rangle}. \end{array}$$

Note that the diagram commutes. By (4) the top two horizontal maps are isomorphisms. The bottom vertical maps are clearly epimorphisms and it is clear that the images of the top vertical maps lie in the kernel of the bottom vertical maps. It now follows from an elementary "diagram chase" that the bottom horizontal map is also an isomorphism.[597] We leave the details to the reader.

(6) This statement follows easily from the definitions and the observation that the image of a commutator under a group homomorphism is once again a commutator.
(7) This statement is basically obvious once one has internalized the definition of a natural transformation.
(8) This statement is an immediate consequence of (6).
(9) We will prove this statement in Exercise 53.6. ■

53.6. **The amalgamated product of groups I.** In the formulation of the generalized version of the Seifert–van Kampen theorem we will need to replace the free product of two groups by the "amalgamated product" of two groups.

In Section 51.2 we had first defined the free product via an explicit construction. We had then explained that one can also characterize the free product via a universal property. In this section we will proceed the other way round, we will first state the desired universal property and then we will explicitly construct a group which satisfies the universal property.

Proposition 53.21. Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. Then there exists a triple

$$(A *_G B, \varphi\colon A \to A *_G B, \psi\colon B \to A *_G B)$$

[597] More precisely, we use the following fact: suppose we are given a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\cong} & A' \\ \downarrow & & \downarrow \\ B & \xrightarrow{\cong} & B' \\ \downarrow & & \downarrow \\ C & \longrightarrow & C', \end{array}$$

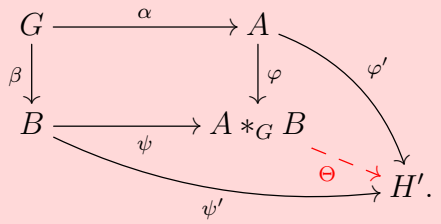
of group homomorphisms where the bottom vertical maps are epimorphisms, where the top two horizontal maps are isomorphisms and where the kernels of the bottom vertical maps are the images of the top vertical maps. Then the bottom horizontal map is also an isomorphism. This general statement can be proved by a "diagram chase". This statement can also be viewed as a special case of the "five lemma".

where $A *_G B$ is a group and φ and ψ are homomorphisms with the following properties:

(a) the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & A \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\varphi} \\ B & \xrightarrow[\psi]{} & A *_G B, \end{array}$$

(b) if $(H', \varphi' \colon A \to H', \psi' \colon B \to H')$ is another triple which satisfies property (a), then there exists a unique homomorphism $\Theta \colon A *_G B \to H'$ such that $\varphi' = \Theta \circ \varphi$ and $\psi' = \Theta \circ \psi$, i.e. such that the following diagram commutes



Proof. Let $\alpha \colon G \to A$ and $\beta \colon G \to B$ be two group homomorphisms. As usual we view A and B as subgroups of $A * B$. We put

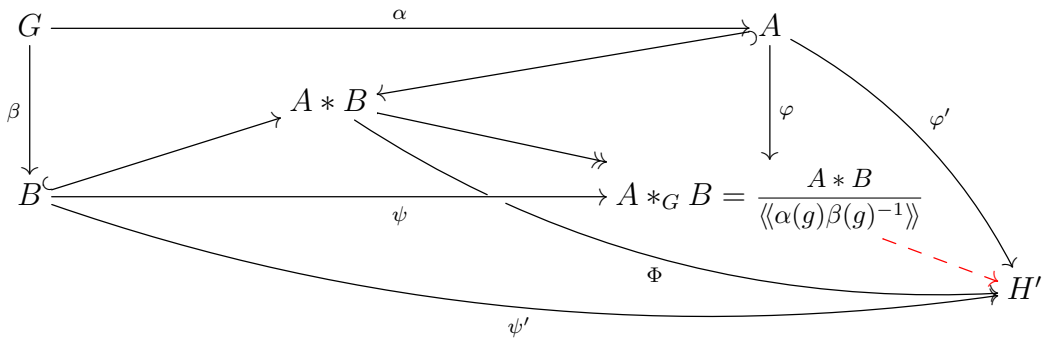
$$A *_G B \;:=\; A * B / \langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g \in G} \rangle\rangle$$

and we define $\varphi \colon A \to A *_G B$ as the composition of the obvious homomorphisms

$$A \;\hookrightarrow\; A * B \;\twoheadrightarrow\; A * B / \langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g \in G} \rangle\rangle,$$

and precisely the same way we define the homomorphism $\psi \colon B \to A *_G B$. It follows from the construction of $A *_G B$ that $\varphi(\alpha(g)) = \psi(\beta(g))$ for all $g \in G$. Therefore the triple $(A *_G B, \varphi \colon A \to A *_G B, \psi \colon B \to A *_G B)$ satisfies property (a).

Now let $(H', \varphi' \colon A \to H', \psi' \colon B \to H')$ be another triple which satisfies property (a). We consider the following diagram



where $\Phi \colon A * B \to H'$ is the unique homomorphism determined by ψ' and φ'. Since $\varphi' \circ \alpha = \psi' \circ \beta$ it follows that for any $g \in G$ we have

$$\Phi(\alpha(g) \cdot \beta(g)^{-1}) \;=\; \Phi(\alpha(g)) \cdot \Phi(\beta(g))^{-1} \;=\; \varphi'(\alpha(g)) \cdot \psi'(\beta(g))^{-1} \;=\; e.$$

Put differently, all elements of the form $\alpha(g)\beta(g)^{-1}$ in $A * B$ lie in the kernel of Φ. By Lemma 53.4 there exists a unique homomorphism $\Theta \colon A * B / \langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g \in G} \rangle\rangle \to H'$

which makes the following diagram commute

$$\begin{array}{ccc} A * B & \longrightarrow & A *_G B = A * B / \langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g\in G} \rangle\rangle \\ & \searrow^{\Phi} & \downarrow \Theta \\ & & H'. \end{array}$$

This proves the existence of Θ. The uniqueness follows from the observation that another choice of Θ' would give another choice for Φ, but this homomorphism is unique. ■

Definition. Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. We write

$$A *_G B \;:=\; A * B / \langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g\in G} \rangle\rangle$$

and we refer to $A *_G B$ as the amalgamated product of A and B with amalgam G.[598] We refer to the obvious homomorphisms $A \to A *_G B$ and $B \to A *_G B$ as the natural homomorphisms.[599]

Remark. A variation on the argument on page 1149 shows that if

$$(H, \varphi\colon A \to H, \psi\colon B \to H)$$

is a triple that has both the properties (a) and (b) of Proposition 53.21, then there exists a unique isomorphism $\Theta\colon A *_G B \xrightarrow{\cong} H$ such that $\varphi(a) = \Theta(a)$ for all $a \in A$ and $\psi(b) = \Theta(b)$ for all $b \in B$. Often we will use this uniquely defined isomorphism to identify H with the amalgamated product $A *_G B$.

Example. Let $G = \mathbb{Z}_2, A = \mathbb{Z}_4$ and $B = \mathbb{Z}_6$ and consider the homomorphisms

$$\begin{array}{rcl} \alpha\colon G & \to & A \\ [k] & \mapsto & [2k] \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta\colon G & \to & B \\ [k] & \mapsto & [3k]. \end{array}$$

Furthermore we consider the homomorphisms[600]

$$\begin{array}{rcl} \varphi\colon A = \mathbb{Z}_4 & \to & \mathrm{SL}(2,\mathbb{Z}) \\ k & \mapsto & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^k \end{array} \quad \text{and} \quad \begin{array}{rcl} \varphi\colon B = \mathbb{Z}_6 & \to & \mathrm{SL}(2,\mathbb{Z}) \\ k & \mapsto & \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^k. \end{array}$$

It is straightforward to verify that[601] $\varphi \circ \alpha = \psi \circ \beta$, therefore there exists by Proposition 53.21 a homomorphism

$$A *_G B \;=\; \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6 \;\to\; \mathrm{SL}(2,\mathbb{Z}).$$

More surprisingly, this homomorphism is in fact an isomorphism. We refer to [**Rosb1994**, p. 218] and [**Ser1980**, p. 35] for a proof.

The following lemma summarizes some important special cases of the amalgamated product of groups.

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[598] The notation is a little misleading since the homomorphisms α and β do not get mentioned in the notation, even though they play of course an important role. In particular different choices of α and β can lead to non-isomorphic amalgamated products.

[599] We leave it again to the reader to figure out in what sense these homomorphisms are "natural".

[600] Are these really homomorphisms?

[601] Indeed, the one non-trivial element of $G = \mathbb{Z}_2$ gets sent under both maps to $-\mathrm{id}$.

Lemma 53.22. (Amalgamated Product Lemma)

(1) For two groups A and B and $G = \{e\}$ the trivial group, the corresponding amalgamated product is the usual free product, i.e.

$$A *_{\{e\}} B \ = A * B.$$

(2) If $\alpha\colon G \to A$ is a group homomorphism and if $\beta\colon G \to \{e\}$ is the trivial homomorphism, then

$$A *_G \{e\} \ = \ A/\langle\langle \alpha(G) \rangle\rangle.$$

(3) Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be group homomorphisms. We consider the diagram

$$\begin{array}{ccc} G & \xrightarrow{\quad\alpha\quad} & A \\ {\scriptstyle\beta}\downarrow & & \downarrow \\ B & \longrightarrow & A *_G B. \end{array}$$

The following two statements hold:[602]

(a) If the left vertical map β is an isomorphism, then the right vertical map is also an isomorphism and the inverse $A *_G B \to A$ is given by the identity on A and $\beta^{-1}\colon B \to G$.

(b) If the left vertical map β is an epimorphism, then the right vertical map is also an epimorphism.

(4) Let A, B, C and G, H be groups and let $\alpha\colon G \to A$, $\beta\colon G \to B$, $\widetilde{\beta}\colon H \to B$ and $\gamma\colon H \to C$ be group homomorphisms. Then the universal properties of the amalgamated products define an isomorphism[603]

$$(A *_G B) *_H C \ \xrightarrow{\cong}\ A *_G (B *_H C).$$

(5) Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be group homomorphisms and let $\varphi\colon H \to G$ be an epimorphism. The natural map $A *_G B \to A *_H B$ is an isomorphism.

(6) Let $\alpha\colon G * H \to A$ and $\beta\colon G * H$ to be group homomorphisms. If the restriction of β to H is trivial, then the natural map $A *_{G*H} B \to A/\langle\langle H \rangle\rangle *_G B$ is an isomorphism.

Remark. The Amalgamated Product Lemma 53.22 (4) is actually quite useful since it allows us to form iterated amalgamated products without having to worry about parentheses.

Sketch of proof. The first two statements follow immediately from the definition

$$A *_G B \ := \ A * B/\langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g\in G} \rangle\rangle.$$

The remaining statement follow either directly from the universal property of the amalgamated product or from the actual description. For some statements we will fill in the details in Exercise 53.14. ■

The following lemma gives us a convenient description of the amalgamated product of two finitely presented groups.

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[602] The statements are similar in spirit to the statement of the Topological Pushout-Maps Lemma 5.34 for the pushout of topological spaces.

[603] In an attempt to lessen the burden on the reader we refrained from writing down the super-precise statement.

Lemma 53.23. Let

$$A = \langle x_1, \dots, x_k \mid r_1, \dots, r_p \rangle$$

and

$$B = \langle y_1, \dots, y_l \mid s_1, \dots, s_q \rangle$$

be two finitely presented groups. Furthermore let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms and let $g_1, \dots, g_m$ be generators of G. Then

$$A *_G B = \langle x_1, \dots, x_k, y_1, \dots, y_l \mid r_1, \dots, r_p, s_1, \dots, s_q \text{ and } \alpha(g_i) = \beta(g_i), i = 1, \dots, m \rangle.$$

Examples.

(1) Let $A_1 = \langle x_1, y_1 \rangle$ and $A_2 = \langle x_2, y_2 \rangle$ be two free groups. Furthermore for $i = 1, 2$ let $\varphi_i\colon \langle t \rangle \to A_i$ be the homomorphism that is given by $\varphi_i(t^n) := [x_i, y_i]^n$. Then we obtain from Lemma 53.23 that

$$A_1 *_{\langle t \rangle} A_2 = \langle x_1, y_1, x_2, y_2 \mid [x_1, y_1] = [x_2, y_2] \rangle.$$

(2) Let $A = \langle a \rangle$ and let $B = \langle b \rangle$ be two infinite cyclic groups and let $G = \langle x, y \mid [x, y] \rangle$ be a group that is isomorphic to $\mathbb{Z}^2$. We consider the homomorphism $\alpha\colon G \to A$ that is given by $\alpha(x) = a$ and $\alpha(y) = e$ and we consider the homomorphism $\beta\colon G \to B$ that is given by $\beta(x) = e$ and $\beta(y) = b$. Then

$$A *_G B \underset{\text{Lemma 53.23}}{=} \langle a, b \mid \alpha(x) = \beta(x), \alpha(y) = \beta(y) \rangle = \langle a, b \mid a = e, e = b \rangle \underset{\text{Tietze transformation (2), see page 1196}}{=} \langle | \rangle = \{e\}.$$

This shows in particular that the amalgamated product of two non-trivial groups can in fact be trivial.

Proof of Lemma 53.23. We need to show that

$$\underbrace{\frac{A*B}{\langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g \in G} \rangle\rangle}}_{\text{definition of } A*_G B} = \underbrace{\frac{A*B}{\langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\dots,m} \rangle\rangle}}_{\text{right-hand side of the proposition}}.$$

Put differently, we need to prove the following claim.

Claim. In the free product $A * B$ we have

$$\langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g \in G} \rangle\rangle = \langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\dots,m} \rangle\rangle.$$

Proof. It is clear that the right-hand side is contained in the left-hand side. To show the reverse inclusion we need to show that for each $g \in G$ we have $\alpha(g)\beta(g)^{-1} \in \langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\dots,m} \rangle\rangle$. We start out with two observations:

(1) We first consider the case $g = g_i^{-1}$ for some i. Then

$$\alpha(g)\beta(g)^{-1} = \alpha(g_i^{-1})\beta(g_i^{-1})^{-1} = \alpha(g_i)^{-1}\big(\alpha(g_i)\beta(g_i^{-1})\big)^{-1}\alpha(g_i),$$

thus $\alpha(g)\beta(g)^{-1}$ is a conjugate of the inverse of $\alpha(g_i)\beta(g_i^{-1})$, in particular it lies in $\langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\dots,m} \rangle\rangle$.

(2) Note that if a, b and c are elements in a group π such that ac and b lie in a normal subgroup Γ, then $abc = ac \cdot c^{-1}bc$ also lies in Γ.

Now we turn to the proof of the claim. We define the *length* of $g \in G$ as

$$\ell(g) = \min\{n \mid \text{ we can write } g = g_{i_1}^{\epsilon_1} \cdot \dots \cdot g_{i_n}^{\epsilon_n} \text{ for } \epsilon_j \in \{-1, 1\} \text{ and } i_j \in \{1, \dots, m\}\}.$$

Since $g_1, \dots, g_m$ are generators of G any $g \in G$ has a length.

We prove the claim by induction on the length of $g \in G$. For $\ell(g) = 0$ there is nothing to prove. Now suppose we have shown the claim for all elements of length $n-1$. Now let $g \in G$ be an element of length n. Thus we can write $g = g_{i_1}^{\epsilon_1} \cdot \cdots \cdot g_{i_n}^{\epsilon_n}$ for $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$ and $i_1, \ldots, i_n \in \{1, \ldots, m\}$. We then have

$$\begin{aligned} \alpha(g)\beta(g)^{-1} = \alpha(g)\beta(g^{-1}) &= \alpha\big(g_{i_1}^{\epsilon_1} \cdots g_{i_n}^{\epsilon_n}\big) \cdot \beta\big(g_{i_n}^{-\epsilon_n} \cdots g_{i_1}^{-\epsilon_1}\big) \\ &= \underbrace{\alpha\big(g_{i_1}^{\epsilon_1}\big) \cdots \alpha\big(g_{i_{n-1}}^{\epsilon_{n-1}}\big)}_{=:a} \cdot \underbrace{\alpha\big(g_{i_n}^{\epsilon_n}\big) \cdot \beta\big(g_{i_n}^{-\epsilon_n}\big)}_{=:b} \cdot \underbrace{\beta\big(g_{i_{n-1}}^{-\epsilon_{n-1}}\big) \cdots \beta\big(g_{i_1}^{-\epsilon_1}\big)}_{=:c}. \end{aligned}$$

Note that b lies in the normal subgroup $\langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\ldots,m} \rangle\rangle$ by (1) and ac lies in the normal subgroup $\langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\ldots,m} \rangle\rangle$ by induction. Thus $abc = \alpha(g)\beta(g)^{-1}$ lies in $\langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\ldots,m} \rangle\rangle$ by (2). ■

53.7. **The amalgamated product of groups II.** In this last section we will state and outline proofs of a few more advanced properties of amalgamated products.
Recall that on page 1211 we saw that the amalgamated product of two non-trivial groups can be trivial. The following proposition gives a criterion for the amalgamated product to be non-trivial.

Proposition 53.24. (Amalgamated Product-Monomorphism Proposition) Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. If α and β are both monomorphisms, then the natural homomorphisms $A \to A *_G B$ and $B \to A *_G B$ are also both monomorphisms.

Convention. If we are in the setting of the Amalgamated Product-Monomorphism Proposition 53.24, then we will use the natural monomorphisms $A \to A *_G B$ and $B \to A *_G B$ to identify A, B and G with the images in $A *_G B$ under these monomorphisms.

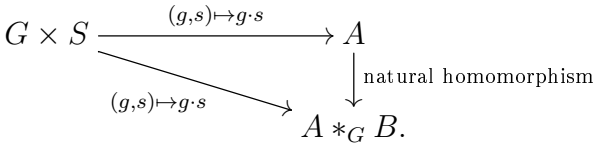
Remark. In Exercise 53.12 we will see that there exists an injective group homomorphism $\alpha\colon G \to A$ and a non-injective group homomorphism $\beta\colon G \to B$ such that the amalgamated product $A *_G B$ is trivial. This shows that there is no obvious way how one can weaken the hypothesis in the Amalgamated Product-Monomorphism Proposition 53.24 that both maps are monomorphisms.

The proof of the Amalgamated Product-Monomorphism Proposition 53.24 relies on the following proposition which is proved in [**Ser1980**, p. 3], or alternatively [**vdW1948**], [**LyS1977**, Theorem IV.2.6] and [**CgRR2008**, Satz 5.2].

Theorem 53.25. (Normal Form Theorem) Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group monomorphisms. We identify G with its image in A and B. Let $S \subset A$ be a set of coset representatives for $G\backslash A$ and let $T \subset B$ be a set of coset representatives for $G\backslash B$. Then any element h in $A *_G B$ we can be written *uniquely* as a product $h = g \cdot c_1 \cdots c_k$ where $g \in G$ and $c_1, \ldots, c_k \in S \sqcup T$.

Proof of the Amalgamated Product-Monomorphism Proposition 53.24. Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group monomorphisms. Let S be a set of coset representatives for $G\backslash A$ and let T be a set of coset representatives for $G\backslash B$. We have the following modest

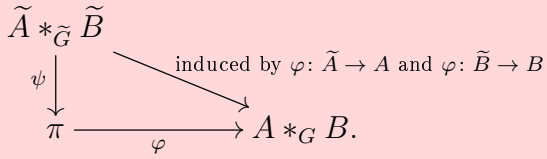
commutative diagram



The top map is a bijection by definition of S. Furthermore the diagonal map is an injection by the Normal Form Theorem 53.25. Thus the vertical map is a monomorphism. ■

We conclude this chapter with another consequence of the Normal Form Theorem 53.25.

Proposition 53.26. Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group monomorphisms. Furthermore let $\varphi\colon \pi \to A *_G B$ be a group epimorphism. We write $\widetilde{A} := \varphi^{-1}(A)$, $\widetilde{B} := \varphi^{-1}(B)$ and $\widetilde{G} := \varphi^{-1}(G)$. Then there exists a unique isomorphism $\psi\colon \widetilde{A} *_{\widetilde{G}} \widetilde{B} \to \pi$ which makes the following diagram commute:



Proof. Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group monomorphisms. It follows from the universal property of the amalgamated product that there exists a unique homomorphism $\psi\colon \widetilde{A} *_{\widetilde{G}} \widetilde{B} \to \pi$ which makes the above diagram commute. It remains to show that ψ is an isomorphism.

Let S be a set of coset representatives for $G\backslash A$ and let T be a set of coset representatives for $G\backslash B$. By the Normal Form Theorem 53.25 we know that any element h in $A *_G B$ we can be written *uniquely* as a product $h = g \cdot c_1 \cdots c_k$ where $g \in G$ and $c_1, \ldots, c_k \in S \sqcup T$. We write $\ell(h) := k$.

Now we turn to the proof that ψ is an epimorphism. We need to show that every $g \in \pi$ lies in the image of ψ. We prove this statement by induction on $\ell(\varphi(g))$. If $\ell(\varphi(g)) \leq 1$, then $\varphi(g) \in A$ or $\varphi(g) \in B$. But this means that $g \in \psi(\widetilde{A})$ or $g \in \psi(\widetilde{B})$. Suppose we know that we have an $n \in \mathbb{N}$ such that the desired statement holds whenever $\ell(\varphi(g)) \leq n$. Let $g \in \pi$ with $\ell(\varphi(g)) = n + 1$. Since φ is an epimorphism we see that there exist $x, y \in \pi$ with $g = x \cdot y$ such that $\ell(\varphi(x)) = n$ and $\ell(\varphi(y)) = 1$. By induction we know that x and y lie in the image of ψ. But then so does $g = x \cdot y$.

It remains to show that ψ is also a monomorphism. Thus let $h \in \ker(\psi)$. Since φ restricts to a surjective map $\widetilde{G}\backslash\widetilde{A} \to G\backslash A$ we see that there exists a set $\widetilde{S}$ of coset representatives for $\widetilde{G}\backslash\widetilde{A}$ with $\varphi(\widetilde{S}) \subset S$. Similarly there exists a set $\widetilde{T}$ of coset representatives for $\widetilde{G}\backslash\widetilde{B}$ with $\varphi(\widetilde{T}) \subset T$. By the Normal Form Theorem 53.25 we can write $h = \widetilde{g} \cdot \widetilde{c}_1 \cdot \cdots \cdot \widetilde{c}_k$ with $\widetilde{g} \in \widetilde{G}$ and $\widetilde{c}_1, \ldots, \widetilde{c}_k \in \widetilde{S} \sqcup \widetilde{T}$. Note that

$$e \;=\; \varphi(e) \;=\; \varphi(\psi(h)) \underset{\text{since the diagram commutes}}{=} \varphi(\widetilde{g}) \cdot \underbrace{\varphi(\widetilde{c}_1)}_{\in S \sqcup T} \cdot \cdots \cdot \underbrace{\varphi(\widetilde{c}_k)}_{\in S \sqcup T} \;\in\; A *_G B.$$

It follows immediately from the uniqueness statement of the Normal Form Theorem 53.25 that $k = 0$, i.e. we see that $h = \widetilde{g} \in \widetilde{G}$. But the restriction of $\psi\colon \widetilde{A} *_{\widetilde{G}} \widetilde{B} \to \pi$ to $\widetilde{G}$ is of

course just the inclusion $\widetilde{G} \to \pi$, in particular the restriction is a monomorphism. Thus we see that $h = e$. ■

Exercises for Chapter 53.

Exercise 53.1. Let π be a group and let $A, B \subset \pi$ be two subsets of π. Show that there exists a natural isomorphism between $(\pi/\langle\langle A \rangle\rangle)/\langle\langle B \rangle\rangle$ and $\pi/\langle\langle A \cup B \rangle\rangle$.

Exercise 53.2. Let π be a finitely generated group. Show that every finite-index subgroup is also finitely generated.
Remark. In Proposition 69.15 we will prove the more subtle result that every finite-index subgroup of a finitely presented group is finitely presented.

Exercise 53.3. Let π be a finitely generated group and let G be a finite group. Show that the set $\mathrm{Hom}(\pi, G)$ of homomorphisms from π to G is a finite set.

Exercise 53.4.

(a) Show that
$$\langle x_1, \ldots, x_k \,|\, x_k \rangle \;\cong\; \langle x_1, \ldots, x_{k-1} \rangle.$$
(b) Show that in the group $\langle x, y \,|\, xyx^{-1}y^{-1} \rangle$ we have the equalities $yx^{-1} = x^{-1}y$ and $x^{-1}y^{-1} = y^{-1}x^{-1}$.

Exercise 53.5.

(a) Let $k \geq 5$. Determine the abelianization of the permutation group S_k.
(b) Let G be an abelian group, let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be epimorphisms. Show that $A *_G B$ is also abelian.

Exercise 53.6. Given a group G we denote by G_{ab} its abelianization.

(a) Let $\varphi\colon A \to B$ be an epimorphism. Show that the induced map $A_{\mathrm{ab}} \to B_{\mathrm{ab}}$ is also an epimorphism.
(b) Let $\varphi\colon A \to B$ be a monomorphism. Is the induced map $A_{\mathrm{ab}} \to B_{\mathrm{ab}}$ also a monomorphism?

Exercise 53.7. Let (X, x_0) be a path-connected topological space and let $f\colon X \to X$ be a homeomorphism with $f(x_0) = x_0$. Show that there exists an isomorphism of the following form:
$$\pi_1(\mathrm{Tor}(X, f))_{\mathrm{ab}} \;\cong\; \mathbb{Z} \oplus \mathrm{coker}\left(\pi_1(X, x_0)_{\mathrm{ab}} \xrightarrow{f_* - \mathrm{id}} \pi_1(X, x_0)_{\mathrm{ab}}\right).$$
Remark. Make use of the Mapping Torus-π_1-Proposition 50.6.

Exercise 53.8. Let $\varphi\colon G \to H$ be a group epimorphism. We write $K := \ker(\varphi)$. We consider its abelianization K_{ab}.

(a) Let $h \in G$. We pick $\widetilde{h} \in G$ with $\varphi(\widetilde{h}) = h$. Show that the map
$$\begin{array}{ccc} K_{\mathrm{ab}} & \to & K_{\mathrm{ab}} \\ [k] & \mapsto & [\widetilde{h} \cdot k \cdot \widetilde{h}^{-1}] \end{array}$$
is a group isomorphism and show that this map does not depend on the choice of $\widetilde{h}$.
(b) What goes wrong in (a) if we work with K instead of K_{ab}?

Exercise 53.9. Given any $l, m, n \in \mathbb{N}$. We consider the von Dyck group

$$T(l,m,n) \;:=\; \langle a, b \,|\, a^l, b^m, (ab)^n \rangle.$$

Show that $T(l,m,n)_{\mathrm{ab}} \cong \mathbb{Z}_{\gcd(l,m,n)}$.
Remark. It follows in particular that if l, m, n are coprime, then the abelianization is trivial. If $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$, then it follows from Corollary 66.24 together with Lemma 66.23 that the von Dyck group $T(l,m,n)$ is nonetheless infinite.

Exercise 53.10. Let $\varphi\colon \langle x, y\rangle \to \mathbb{Z}^2$ be the epimorphism that is given by $\varphi(x) = (1,0)$ and $\varphi(y) = (0,1)$. Show that the kernel $\ker(\varphi)$ is normally generated by the commutator $[x,y] = xyx^{-1}y^{-1}$.

Exercise 53.11. Let $\varphi\colon G \to A$ and $\psi\colon G \to B$ group monomorphisms which we use to define the amalgamated product $A *_G B$. Suppose that the map $A \to A *_G B$ is an isomorphism. Show that ψ is an isomorphism.

Exercise 53.12. Let G be a group. Let $\alpha\colon G \to A$ be a monomorphism to a simple group A and let $\beta\colon G \to B$ be an epimorphism that is not an isomorphism. Show that the amalgamated product $A *_G B$ is trivial.

Exercise 53.13. Let A and G be groups and let $\varphi\colon A \to G$ be an epimorphism. Furthermore let G' be a copy of G. We consider the amalgamated product $G *_A G'$ given by using twice the epimorphism $\varphi\colon A \to G$ and $\varphi\colon A \to G' = G$. Show that the natural homomorphism $G \to G *_A G'$ is an isomorphism.

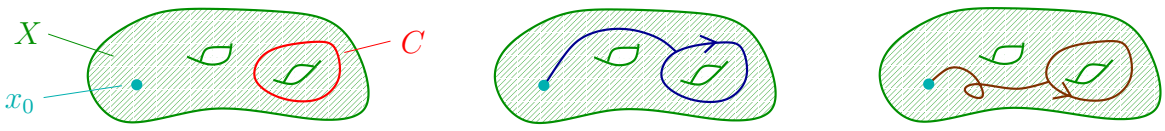
Exercise 53.14.

(a) Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be group homomorphisms and let $\varphi\colon H \to G$ be an epimorphism. Show that the natural map $A *_G B \to A *_H B$ is an isomorphism.
(b) Let $\alpha\colon G * H \to A$ and $\beta\colon G * H$ to be group homomorphisms. We assume that the restriction of β to H is trivial. Show that the natural map $A *_{G*H} B \to A/\langle\langle H \rangle\rangle *_G B$ is an isomorphism.

Exercise 53.15. Let X be a path-connected topological space and let $x_0 \in X$. Furthermore let $C \subset X$ be a subset that is homeomorphic to S^1. We pick a homeomorphism $\varphi\colon S^1 \to C$ and a path $\gamma\colon [0,1] \to X$ from x_0 to $\varphi(1)$. We denote by $\widetilde{\varphi}\colon [0,1] \to X$ the map that is given by $t \mapsto \varphi(\exp(2\pi\,\mathrm{i}\,t))$. Show that the subgroup

$$\langle\langle C \rangle\rangle \;:=\; \langle\langle \gamma * \varphi * \widetilde{\gamma} \rangle\rangle \;\subset\; \pi_1(X, x_0)$$

neither depends on the choice of φ nor on the choice of γ.



Exercise 53.16. Let π be a group and let S be a generating set for π. We set $V := \pi$, we set $E := V \times S$ and we define the initial map $i\colon E = V \times S \to V$ by $i((v,s)) = v$ and we define the terminal map $t\colon E = V \times S \to V$ by $t((v,s)) = v \cdot s$. Note that (V, E, i, t) is an abstract graph. It is called the *Cayley graph* $\Gamma(\pi, S)$ of the group π with respect to the generating set S.

(a) We consider $\pi = \langle x, y \,|\, xyx^{-1}y^{-1} \rangle$ with the generating set $S = \{x, y\}$. Sketch $|\Gamma(\pi, S)|$ as a subset of $\mathbb{R}^2$.
(b) We consider $\pi = \langle x, y \,|\, \varnothing \rangle$ with the generating set $S = \{x, y\}$. Sketch $|\Gamma(\pi, S)|$ as a subset of $\mathbb{R}^2$.
(c) We consider $\pi = \langle x, y \,|\, x^2, xyx^{-1}y^{-1} \rangle$ with the generating set $S = \{x, y\}$. Sketch $|\Gamma(\pi, S|$ as a subset of $\mathbb{R}^3$.

Exercise 53.17. Let $K = \mathbb{R}$ or $K = \mathbb{C}$. Let $n \in \mathbb{N}$.

(a) Show that the abelianization of $\mathrm{SL}(n, \mathbb{K})$ is the trivial group.
Hint. Try to write the elementary matrices as commutators.
(b) Determine the abelianization of $\mathrm{GL}(n, \mathbb{K})$.

Remark. A group with trivial abelianization is sometimes called *perfect*. We have thus shown that the groups $\mathrm{SL}(n, \mathbb{K})$ are perfect.

54. The general Seifert–van Kampen Theorem

In this chapter we will generalize the "basic" Seifert–van Kampen Theorem 52.2 to the setting where we no longer need to assume that the intersection $U \cap V$ is simply connected. We will use the resulting Seifert–van Kampen Theorem 54.1 to determine the fundamental groups of the (non-) orientable surfaces Σ_g and N_g. In Chapter 55 we will use these calculations to give the classification of compact 2-dimensional smooth and topological manifolds.

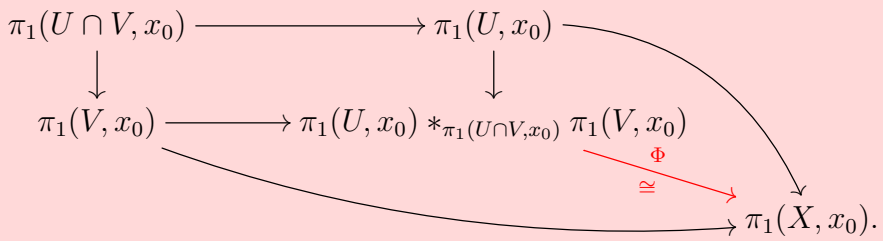
Towards the end of this chapter we will use the Seifert–van Kampen Theorem 54.1 to prove the Fundamental Group-Manifold Realization Proposition 54.12 which says, in a nutshell, that every finitely presented group occurs as the fundamental group of a closed smooth manifold of dimension ≥ 4.

54.1. **The formulation of the general Seifert–van Kampen Theorem.** With the group theoretic preparations from Chapter 53 we can now formulate and prove the general Seifert–van Kampen Theorem.

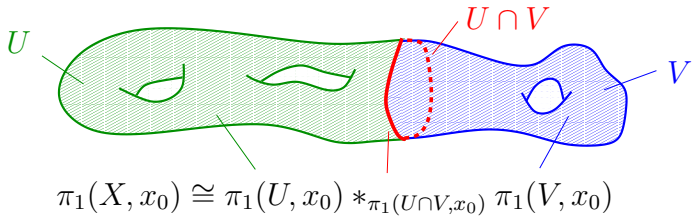
Theorem 54.1. (Seifert–van Kampen Theorem) Let X be a topological space and let $X = U \cup V$ be a decomposition of X in two open subsets U and V such that $U \cap V$ is non-empty and path-connected. Let $x_0 \in U \cap V$. Then there exists an isomorphism

$$\Phi\colon \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$$

such that the following diagram commutes:

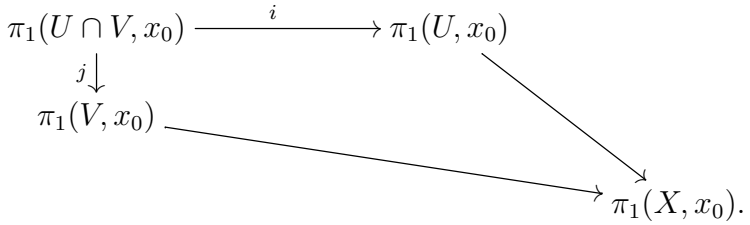


Here all the "undecorated" maps are the obvious inclusion induced homomorphisms.



The proof of the general Seifert–van Kampen Theorem is a modification of the proof of the more "basic" Seifert–van Kampen Theorem 52.2 where we had assumed that $U \cap V$ is in fact simply connected. We only sketch the proof of the general Seifert–van Kampen Theorem 54.1. The full details are for example given in Section 1.2 of [**Hat2002**].

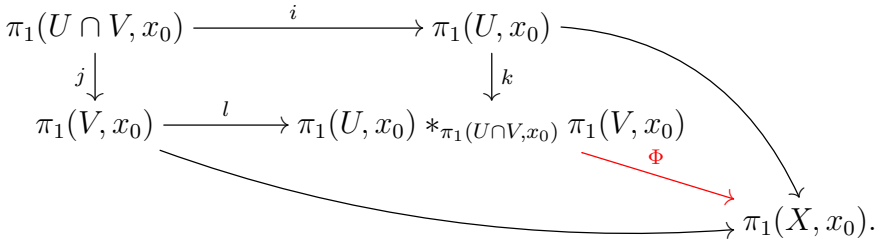
Sketch of the proof. The various inclusions of topological spaces give rise to the following commutative diagram:



By Proposition 53.21 there exists a unique homomorphism

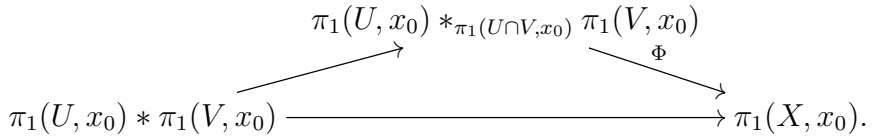
$$\Phi\colon \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \;\to\; \pi_1(X, x_0)$$

such that the following diagram commutes:



It remains to show that Φ is surjective and that Φ is injective.

We first show surjectivity. We consider the following commutative diagram[604] of homomorphisms



By Lemma 52.1 the horizontal map is surjective. It follows that the diagonal map Φ on the right is also surjective.

Now we turn to the proof of the injectivity of Φ. We write $A = \pi_1(U, x_0), B = \pi_1(V, x_0)$ and $G = \pi_1(U \cap V, x_0)$. As on page 1160 we consider the set

$$S(A, B) \;:=\; \{\text{all finite sequences } (x_1, \ldots, x_k) \text{ with each } x_i \text{ in } A \text{ or } B\}.$$

Furthermore we denote by $\sim_G$ the equivalence relation on $S(A, B)$ generated by the relation

$$(x_1, \ldots, x_l, e, x_{l+2}, \ldots, x_k) \;\sim\; (x_1, \ldots, x_l, x_{l+2}, \ldots, x_k)$$

where e is trivial in A or B, by the relation

$$(x_1, \ldots, x_l, x_{l+1}, \ldots, x_k) \;\sim\; (x_1, \ldots, x_l x_{l+1}, \ldots, x_k)$$

if both x_l and x_{l+1} lie in A or both lie in B, and also by the relation

$$(x_1, \ldots, x_l, i(g), x_{l+2}, \ldots, x_k) \;\sim\; (x_1, \ldots, x_l, j(g), x_{l+2}, \ldots, x_k)$$

for $g \in G$. Similar to Proposition 51.18 one can now show that the map

$$\begin{array}{rcl} S(A, B)/\!\sim_G & \to & A *_G B \\ [(x_1, \ldots, x_k)] & \mapsto & x_1 \cdot \cdots \cdot x_k \end{array}$$

[604] What are the maps, and why does the diagram commute?

is a well-defined bijection. We will use this isomorphism to identify $S(A,B)/\sim_G$ with $A *_G B$.

The proof of the present theorem is now almost identical to the proof of Theorem 52.2, one just needs to replace the discussion involving $S(A,B)/\sim$ by the corresponding objects in $S(A,B)/\sim_G$. More precisely, in Footnote 569 we pointed out where in the proof of Theorem 52.2 we used the hypothesis that $U \cap V$ is simply connected. At that point in the proof of Theorem 52.2 we need to replace $A * B = S(A,B)/\sim$ with $A *_G B = S(A,B)/\sim_G$. Full details for the proof can be found for example in Section 1.2 of [**Hat2002**]. ■

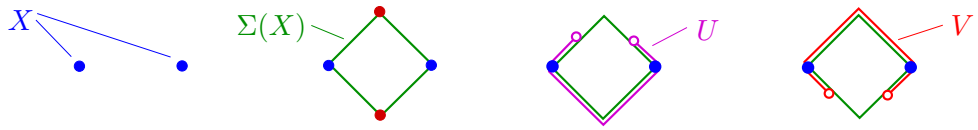
As a first example we can now prove the following lemma.

Lemma 54.2. (Suspension-π_1-Lemma) If X is a path-connected[605] non-empty topological space, then its suspension[606] $\Sigma(X) = (X \times [-1,1])/\sim$ is simply connected.

Proof. Let X be a path-connected non-empty topological space. In the following we denote by $p\colon X \times [-1,1] \to \Sigma(X)$ the natural projection. We consider the subsets $U = p(X \times [-1, \frac{1}{2}))$ and $V = p(X \times (-\frac{1}{2}, 1])$. It follows from the definition of the quotient topology on $\Sigma(X)$ that U and V are open subsets of $\Sigma(X)$. We pick a base point x on $X = X \times \{0\} \subset \Sigma(X)$. We calculate that

$$\pi_1(\Sigma(X),x) \underset{\text{Seifert–van Kampen Theorem 54.1}}{\overset{\uparrow}{\cong}} \pi_1(U,x) *_{\pi_1(U\cap V,x)} \pi_1(V,x) \underset{\text{as in the Cone Contractibility Lemma 16.7 we see that } U \text{ and } V \text{ are contractible, hence their fundamental groups are trivial by Homotopy-}\pi_1\text{-Proposition 50.3 (3)}}{\overset{\uparrow}{=}} \underset{\text{of trivial groups}}{\text{amalgamated product}} = \text{trivial group.}$$

We have thus shown that $\Sigma(X)$ is indeed simply connected. ■



The following lemma calculates the fundamental groups of mapping cones, which we introduced on page 457.

Lemma 54.3. (Mapping Cone-π_1-Lemma) Let $f\colon A \to X$ be a map between topological spaces and let $a_0 \in A$. We write $x_0 := f(a_0)$. If A is path-connected, then the inclusion induces an isomorphism

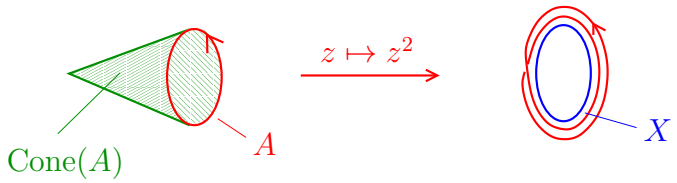
$$\pi_1(X,x_0)/f_*(\pi_1(A,a_0)) \;\to\; \pi_1(\text{Cone}(f\colon A \to X),[x_0]).$$

Example. Let $n \in \mathbb{N}$ and let $A = X = S^1$. We consider the map $f\colon A = S^1 \to X = S^1$ that is given by $f(z) = z^n$. It follows from Lemma 54.3 and Proposition 48.17, perhaps counter-intuitively, that $\pi_1(\text{Cone}(f\colon A \to X)) \cong \mathbb{Z}_n$. We refer to the figure below for an attempt at an illustration of this example.

Proof. We will provide the proof in Exercise 54.1. ■

[605] In Exercise 57.5 we will consider the fundamental group of topological spaces that are not path-connected.

[606] We refer to page 451 for the definition of the suspension of a topological space.



The same way that we deduced Theorem 52.10 from Theorem 52.2 we can now obtain the following theorem from Theorem 54.1.

Theorem 54.4. (Seifert–van Kampen Theorem for Topological Manifolds) Let M be an m-dimensional topological manifold and let $R, S \subset M$ be two m-dimensional submanifolds such that the following hold:

(1) $M = R \cup S$,
(2) $R \cap S$ is a component of ∂R and it is a component of ∂S,
(3) R and S are closed subsets of M.

We pick a base point $x_0 \in R \cap S$. Then the inclusion induced map

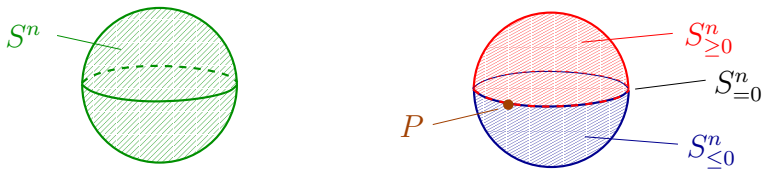
$$\pi_1(R, x_0) *_{\pi_1(R\cap S, x_0)} \pi_1(S, x_0) \;\to\; \pi_1(M, x_0)$$

is an isomorphism.

Example. Given $n \in \mathbb{N}_0$ we denote as usual by $S^n_{\geq 0}$ the upper hemisphere of S^n and we denote by $S^n_{\leq 0}$ the lower hemisphere of S^n. We denote by $S^n_{=0} = S^n_{\geq 0} \cap S^n_{\leq 0}$ the "equator" of S^n, which is of course homeomorphic to S^{n-1}. Given any $n \in \mathbb{N}_{\geq 2}$ and any $P \in S^n_{=0}$ we can perform the following calculation:

$$\pi_1(S^n, P) \;\underset{\text{by the Seifert–van Kampen Theorem 54.4, note that we use that our hypothesis } n\in\mathbb{N}_{\geq 2} \text{ implies that } S^{n-1} \text{ is path-connected}}{\cong}\; \pi_1(S^n_{\leq 0}, P) *_{\pi_1(S^n_{=0},P)} \pi_1(S^n_{\geq 0}, P) \;\underset{\text{by page 152 the hemispheres are homeomorphic to } \overline{B}^n, \text{ hence the fundamental groups are trivial}}{\cong}\; \{e\} *_{\pi_1(S^{n-1},P)} \{e\} \;\underset{\text{by definition or by the Amalgamated Product Lemma 53.22}}{\cong}\; \{e\}.$$

This calculation gives a new, and arguably more conceptual proof of the Sphere-π_1-Proposition 47.13.



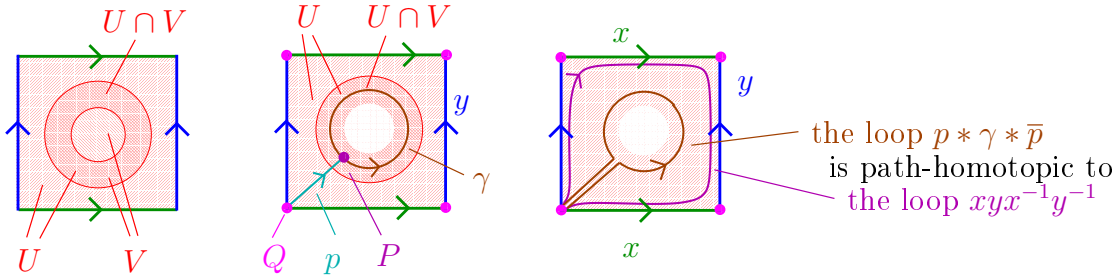
54.2. **The fundamental groups of surfaces.** Now we want to use the Seifert–van Kampen Theorem 54.1 to determine the fundamental groups of surfaces of genus $g \geq 2$. Before

we do so we consider the torus, the Klein bottle and the real projective plane to gain some confidence in our newly acquired methods.[607]

(1) We first consider the torus $T = ([0,1] \times [0,1])/\sim$. In the following we will use the following notation, see also the figure below for an illustration:
   (a) We denote by Q the point on T corresponding to the four vertices of the square.
   (b) We write $T = U \cup V$ where

$$U = T \setminus \overline{B_{\frac{1}{8}}(\tfrac{1}{2},\tfrac{1}{2})} \subset T \quad \text{and } V = B_{\frac{1}{4}}(\tfrac{1}{2},\tfrac{1}{2}) \subset T,$$

   i.e. U is the complement of a closed disk and V is an open disk and $U \cap V$ is an open annulus.
   (c) We let $P = (r,r) \in U \cap V$ and we denote by p the direct path from Q to P.
   (d) We pick a loop γ in $(U \cap V, P)$ that "goes counterclockwise around the annulus" and that represents a generator of the infinite cyclic group $\pi_1(U \cap V, P) \cong \mathbb{Z}$.[608]
   (e) We denote by x and y the two loops in (T,Q) corresponding to the horizontal and the vertical edge of the square. By a slight abuse of notation we denote the corresponding elements in $\pi_1(T,Q)$ by x and y as well.



Now we have

$$\begin{array}{lll}
\pi_1(T,P) & \cong & \pi_1(U,P) *_{\pi_1(U\cap V,P)} \pi_1(V,P) \\
\uparrow & & \\
\text{Seifert–van Kampen Theorem 54.1} & & \\
& = & \pi_1(U,P)/\langle\langle i_*(\pi_1(U\cap V,P))\rangle\rangle \\
& \uparrow & \\
& \text{by the Amalgamated Product Lemma 53.22} & \\
& \text{we have } A *_G \{e\} = A/\langle\langle \alpha(G)\rangle\rangle & \\
& \xrightarrow[\uparrow]{\cong} & \pi_1(U,Q)/\langle\langle [p*\gamma*\overline{p}]\rangle\rangle \\
& & \text{by the Change-of-Base Point Proposition 47.10} \\
& & \text{path } p \text{ gives rise to an isomorphism } p_*
\end{array}$$

$$\begin{array}{lll}
= \pi_1(U,P) *_{\pi_1(U\cap V,P)} \{e\} & & \\
\uparrow & & \\
V \text{ is a disk, hence } \pi_1(V,P) = \{e\} & & \\
= \pi_1(U,P)/\langle\langle [\gamma]\rangle\rangle & & \\
\uparrow & & \\
\text{since } \pi_1(U\cap V,P) = \langle[\gamma]\rangle & & \\
= \pi_1(U,Q)/\langle\langle xyx^{-1}y^{-1}\rangle\rangle = (*). & & \\
\uparrow & & \\
\text{since } p*\gamma*\overline{p} \text{ is path-homotopic to } xyx^{-1}y^{-1} & &
\end{array}$$

As in the proof of Lemma 52.9 we can show that $W := \partial([0,1] \times [0,1])/\sim$ is a deformation retract of U and that W is homeomorphic to the wedge of two circles. By the Fundamental Group-Wedge-of-Circles Lemma 52.4 we can therefore make the identification $\pi_1(W,P) = \langle x, y\rangle$, see also the figure below. Now we continue with the

[607] The following examples can also be handled using Theorem 54.4 instead of Theorem 54.1. The advantage of using Theorem 54.1 at this point is that in the illustrations one can draw $U \cap V$ better if U and V are open subsets.

[608] Since $U \cap V$ is an open annulus its fundamental group is isomorphic to $\mathbb{Z}$.

above calculation of $\pi_1(T,P)$. We have

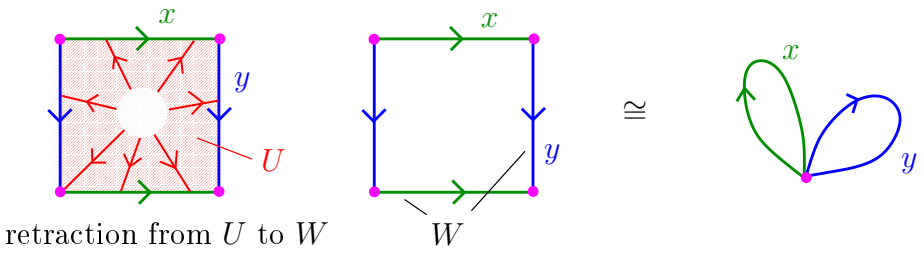
$$(*) \;=\; \pi_1(U,Q)/\langle\langle xyx^{-1}y^{-1}\rangle\rangle \;\overset{\cong}{\underset{\uparrow}{\to}}\; \pi_1(W,P)/\langle\langle xyx^{-1}y^{-1}\rangle\rangle$$

by the Deformation Retract-Homotopy Equivalence Lemma 15.5 and the Homotopy-π_1-Proposition 50.3 (2)

$$\underset{\uparrow}{=}\; \langle x,y\rangle/\langle\langle xyx^{-1}y^{-1}\rangle\rangle \;=\; \langle x,y \mid [x,y]\rangle \;\underset{\uparrow}{\cong}\; \mathbb{Z}^2.$$

since W is the wedge of the circles x and y — see Lemma 53.6 (1)

Thus we have reproved that the fundamental group of the torus is isomorphic to the free abelian group $\mathbb{Z}^2$.



retraction from U to W

(2) We turn to the Klein bottle K. The approach to calculating the fundamental group of the Klein bottle is almost identical to the above approach to calculating the fundamental group of the torus. We just replace the situation illustrated in the figure above by the situation illustrated in the figure below. The same argument as above shows that[609]

$$\pi_1(K) \;\cong\; \langle x,y \mid yxyx^{-1}\rangle.$$

Using this presentation and using the Abelianization Proposition 53.20 (5) we obtain in particular that the abelianization of $\pi_1(K)$ is given by

$$\pi_1(K)_{\text{ab}} \;\cong\; \langle x,y \mid yxyx^{-1}\rangle_{\text{ab}} \;\cong\; \mathbb{Z}^2/\tbinom{0}{2}\mathbb{Z} \;\cong\; \mathbb{Z}\oplus\mathbb{Z}_2.$$

On page 1091 we already gave a very different calculation of $\pi_1(K)$. More precisely, we introduced the group G that is given by all self-homeomorphisms of $\mathbb{R}^2$ that can be written as concatenations of the two self-homeomorphisms

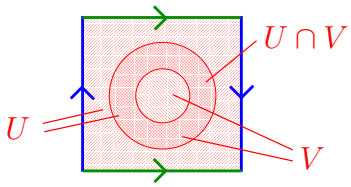
$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & (s+1,1-t) \end{array} \quad\text{and}\quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & (s,t+1) \end{array}$$

and their inverses and we showed that $\pi_1(K)\cong G$. On page 1195 we already showed that there exists a homomorphism $\Phi\colon \langle x,y \mid yxyx^{-1}\rangle \to G$ with $\Phi(x)=B^{-1}A$ and $\Phi(y)=B$. This homomorphism is in fact an isomorphism. This can be seen as follows. Denote by H the group of all self-homeomorphisms of $\mathbb{R}^2$. We consider the diagram

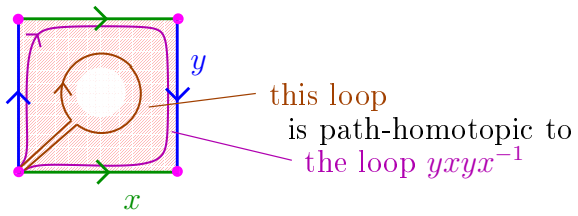
$$\langle x,y \mid yxyx^{-1}\rangle \xrightarrow{\;\cong\;} \pi_1(K) \xleftarrow{\;\cong\;} \text{subgroup of } H \text{ generated by } A,B.$$
$$\langle x,y \mid yxyx^{-1}\rangle \xrightarrow[x\mapsto B^{-1}A,\,y\mapsto B]{\;\Phi\;} \text{subgroup of } H \text{ generated by } A,B.$$

[609] In particular this shows that $\pi_1(K)$ is finitely presented. This answers the question we had raised on page 1196.

Here the isomorphisms to $\pi_1(K)$ are the ones given by the two calculations of the fundamental group. If one goes carefully through the definitions of the two horizontal maps, then one can see that the diagram commutes. Since two of the three maps are isomorphisms and since the diagram commutes it follows that Φ is also an isomorphism.



the Klein bottle



(3) Next we consider again the real projective space $\mathbb{R}\mathrm{P}^2$. Here we use the description of $\mathbb{R}\mathrm{P}^2$ from Lemma 5.32 as

$$\mathbb{R}\mathrm{P}^2 \;=\; \overline{B}^2/\!\sim \qquad \text{where for } z \in S^1 \text{ we have } z \sim -z.$$

As in the previous two examples we decompose $\mathbb{R}\mathrm{P}^2$ into an open disk V and the complement of a closed disk U such that $U \cap V$ is an annulus. We consider the base point $Q = [1] = [-1] \in \mathbb{R}\mathrm{P}^2$ and we denote by α the loop

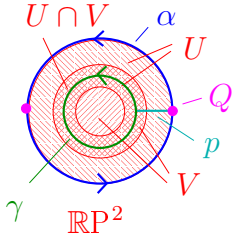
$$\begin{array}{rcl} \alpha\colon [0,1] & \to & \mathbb{R}\mathrm{P}^2 \\ t & \mapsto & [\exp(\mathrm{i}\pi t)]. \end{array}$$

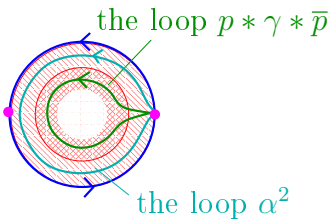
By a slight abuse of notation we now denote by α also the corresponding element in $\pi_1(\mathbb{R}\mathrm{P}^2, Q)$. Furthermore we denote by W the image of α in $\mathbb{R}\mathrm{P}^2$. Evidently α is a generator of $\pi_1(W, Q)$. Then, with similar arguments as in the torus case, we see that

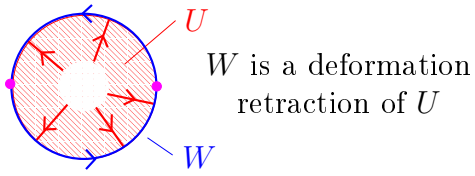
$$\pi_1(\mathbb{R}\mathrm{P}^2) \;\underset{\uparrow}{\cong}\; \pi_1(U,Q)/\langle\langle p*\gamma*\overline{p}\rangle\rangle \;\underset{\uparrow}{\cong}\; \pi_1(W,Q)/\langle\langle p*\gamma*\overline{p}\rangle\rangle \;\underset{\uparrow}{\cong}\; \langle\alpha\rangle/\langle\alpha^2\rangle \;\cong\; \mathbb{Z}_2.$$

by the Seifert–van Kampen Theorem 54.1 and Amalgamated Product Lemma 53.22 (2) — since W is a deformation retract of U — since $\pi_1(W,Q) = \langle\alpha\rangle$ and $[p*\gamma*\overline{p}] = \alpha^2$

In particular we obtain the same result, as we should, as in Corollary 48.18.



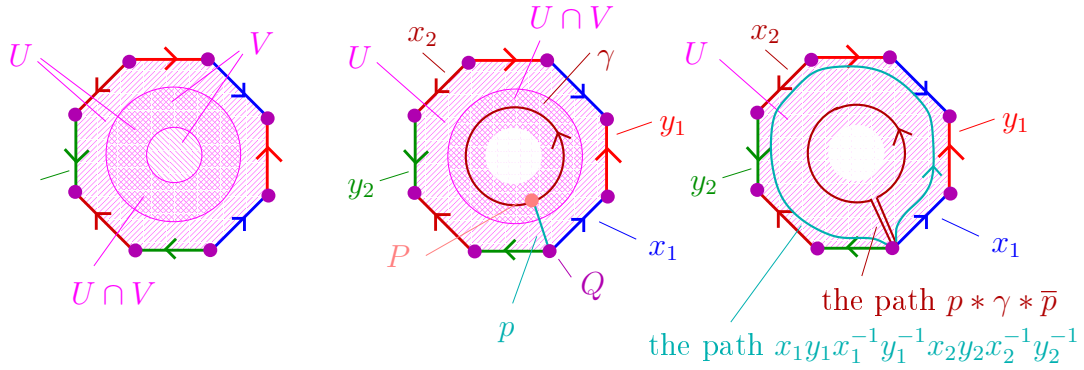




(4) Now that we have gained some confidence that the approach of determining fundamental groups using Seifert–van Kampen Theorem 54.1 does indeed give the right fundamental groups we turn to the surfaces of higher genus that we could not handle with our previous methods.

So let us consider the surface $F = E_8/\!\sim$ of genus 2. As in the torus case we write $F = U \cup V$ where V is an open disk and where U is the complement of a closed disk such that $U \cap V$ is an annulus. Similar to the torus case we write $W = \partial E_8/\!\sim$. By the proof of Proposition 52.8 we know that W is a deformation retraction of U and

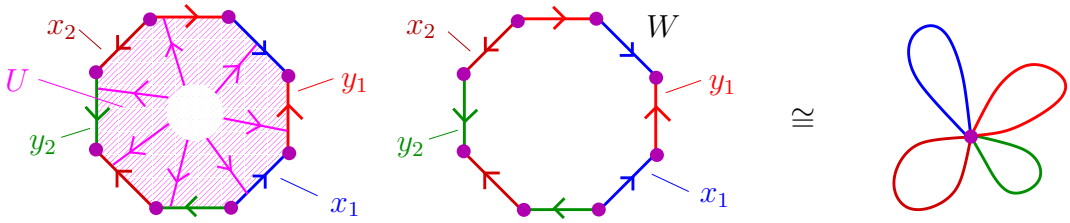
that W is homeomorphic to the wedge of four circles corresponding to x_1, y_1, x_2, y_2. See also the figures below for an illustration. Basically exactly the same argument



as in the torus case shows that

$$\begin{aligned}\pi_1(E_8/\!\sim, P) &\cong \pi_1(U,Q)/\langle\langle \overbrace{x_1y_1x_1^{-1}y_1^{-1}x_2y_2x_2^{-1}y_2^{-1}}^{\text{path-homotopic to } p*\gamma*\overline{p}} \rangle\rangle \\ &\qquad\qquad \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{=[x_1,y_1][x_2,y_2]} \\ &\cong \pi_1(W,Q)/\langle\langle [x_1,y_1][x_2,y_2]\rangle\rangle \\ &\cong \langle x_1,y_1,x_2,y_2\rangle/\langle\langle [x_1,y_1][x_2,y_2]\rangle\rangle \;=\; \langle x_1,y_1,x_2,y_2 \mid [x_1,y_1][x_2,y_2]\rangle.\end{aligned}$$

Thus we have finally succeeded in calculating the fundamental group of the surface of genus 2.



The following gives in particular a new proof for Proposition 52.8.

Proposition 54.5. (Fundamental Group-of-Surfaces Proposition)

(1) For any $g \in \mathbb{N}_0$ we have

$$\pi_1(\text{surface } \Sigma_g \text{ of genus } g) \;\cong\; \langle x_1, y_1, \ldots, x_g, y_g \mid [x_1,y_1]\cdot\cdots\cdot[x_g,y_g]\rangle$$

and we have $\pi_1(\text{surface of genus } g)_{\text{ab}} \;\cong\; \mathbb{Z}^{2g}$.

(2) For $g \neq h$ the surfaces of genus g and h are not homeomorphic.

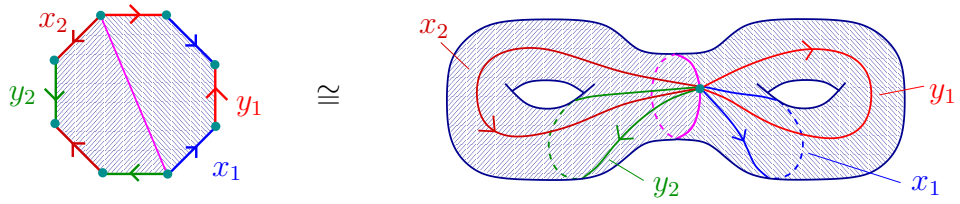
Proof.

(1) Let $g \in \mathbb{N}_0$. For $g = 0$ we obtain the fundamental group from the Sphere-π_1-Proposition 47.13 and for $g = 1$ we obtain the presentation from the discussion on page 1221. For $g \geq 2$ the presentation follows from a straightforward generalization of the calculation preceding the proposition. Finally the stated abelianization of the fundamental group is an immediate consequence of the Abelianization Proposition 53.20 (5).

(2) For $g \neq h$ we obtain from the Finitely Generated Abelian Group Classification Theorem 51.4 that the abelianizations of the fundamental groups of the surfaces of genus g and h are not isomorphic. Hence the surfaces are not homeomorphic. ■

Remark.

(1) In the figure below we show loops that represent the generators x_1, y_1, x_2, y_2 of the fundamental group for the surface of genus 2.



(2) As explained in the figure on page **??** we can view the surface Σ of genus 2 as the connected sum of two copies of the torus T. We can also use this point of view to determine the fundamental group of the surface Σ of genus 2. More precisely, we have $\Sigma = F_1 \cup F_1$ with $F_1 = T \setminus B^2$ and $F_2 = T \setminus B^2$ where $F_1 \cap F_2 = \partial F_1 = \partial F_2 \cong S^1$ and we obtain that[610]

$$\begin{array}{l} \text{Seifert–van Kampen Theorem 54.4} \qquad\qquad \text{by Lemma 52.9 (2) where } t \mapsto [x_1, y_1] \text{ and } t \mapsto [x_2, y_2] \\ \pi_1(\Sigma) \overset{\downarrow}{\cong} \pi_1(F_1) *_{\pi_1(F_1 \cap F_2)} \pi_1(F_2) \overset{\downarrow}{\cong} \langle x_1, y_1 \rangle *_{\langle t \rangle} \langle x_2, y_2 \rangle \\ \quad \cong \langle x_1, x_2, y_1, y_2 \mid [x_1, y_1] = [x_2, y_2] \rangle. \\ \quad \uparrow \\ \quad \text{see page 1211} \end{array}$$

Let $g \geq 2$ and let π be the fundamental group of the surface of genus g. We have just shown that $\pi_{\text{ab}} \cong \mathbb{Z}^{2g}$. This raises the question, whether perhaps π is isomorphic to $\mathbb{Z}^{2g}$ or to the free group F_{2g} on $2g$ generators. The proof of the following lemma is Exercise 54.2.

Lemma 54.6. The group $\pi = \langle x_1, y_1, \ldots, x_g, y_g \mid [x_1, y_1] \cdot \cdots \cdot [x_g, y_g] \rangle$ is non-abelian.

The lemma says in particular that π is not isomorphic to $\mathbb{Z}^{2g}$. It is much harder to determine whether or not π is a free group. We record this as a question for later.

Question 54.7. Let π be the fundamental group of the surface of genus g. Is π isomorphic to the free group F_{2g} on $2g$ generators?

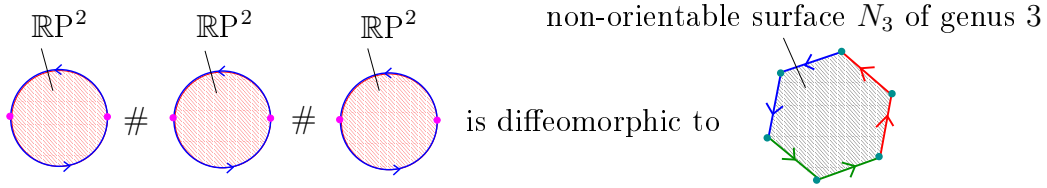
We will give an answer to the question in Proposition 64.17 and also on page 1887.
Let π be the fundamental group of the surface of genus g. In Exercise 54.4 we will see that if $g \geq 2$, then π contains a free group on two generators as a subgroup. This raises the question, what other groups can occur as subgroups of π? Let us record this as a question.

Question 54.8. Let π be the fundamental group of the surface of genus g. What isomorphism types of subgroups can appear? Is every subgroup again a free group or the fundamental group of a surface?

We will provide the answer to Question 54.8 in the Subgroups-of-Surface Groups Proposition 64.6.

[610] Why is this group isomorphic to the above group $\langle x_1, y_1, x_2, y_2 \mid [x_1, y_1][x_2, y_2] \rangle$?

54.3. **Non-orientable surfaces.** In this section we will discuss the fundamental groups of the non-orientable surfaces N_g of some genus $g \in \mathbb{N}$. It is also worth recalling that in Exercise **??** we showed that the connected sum of g copies of $\mathbb{R}\mathrm{P}^2$ is diffeomorphic to the non-orientable surface of genus g that we introduced on page 266.



The following proposition is the analogue of the Fundamental Group-of-Surfaces Proposition 54.5.

Proposition 54.9. (Fundamental Group-of-Surfaces Proposition II)

(1) For any $g \in \mathbb{N}$ we have

$$\pi_1(\text{non-orientable surface } N_g \text{ of genus } g) \;\cong\; \langle x_1, \ldots, x_g \mid x_1^2 \cdot \cdots \cdot x_g^2 \rangle$$

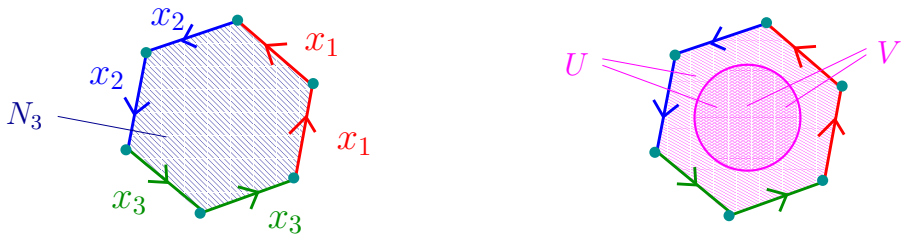
and we have

$$\pi_1(\text{non-orientable surface } N_g \text{ of genus } g)_{\mathrm{ab}} \;\cong\; \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2.$$

(2) For $g \neq h$ the non-orientable surfaces of genus g and h are not homeomorphic.

Proof.

(1) Let $g \in \mathbb{N}$. Recall that the non-orientable surface of genus one is by definition the real projective plane. In Corollary 48.18 we saw that $\pi_1(\mathbb{R}\mathrm{P}^2) \cong \mathbb{Z}_2 \cong \langle x \,|\, x^2 \rangle$. The case $g \geq 2$ follows from a straightforward modification of the calculation of the fundamental group of the surface of genus 2 that we gave on page 1224. Using the figure below the reader should have no troubles with assembling a full argument. Finally note that the stated abelianization of the fundamental group is an immediate consequence of the Abelianization Proposition 53.20 (5).



(2) For $g \neq h$ we obtain from the above calculation and from the Finitely Generated Abelian Group Classification Theorem 51.4 that the abelianizations of the fundamental groups of the non-orientable surfaces of genus g and h are not isomorphic. Hence the non-orientable surfaces of genus g and h are not homeomorphic. ■

As mentioned above, in Exercise **??** we showed that for any $g \in \mathbb{N}$ the non-orientable surface of genus g is diffeomorphic to the connected sum of g copies of $\mathbb{R}\mathrm{P}^2$. In particular the Klein bottle is diffeomorphic to $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$. Just for fun and for practice we provide the direct calculation of the fundamental group of $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ using the Seifert–van Kampen

Theorem 54.4 for Smooth Manifolds. The calculation is somewhat interesting since it involves some interesting maps between fundamental groups.

Lemma 54.10. $\pi_1(\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2) \cong \langle x, y \mid x^2 = y^2\rangle$.

Proof. In order to distinguish the two copies of $\mathbb{R}\mathrm{P}^2 = \overline{B}^2/\sim$ we write $P = \mathbb{R}\mathrm{P}^2$ and $Q = \mathbb{R}\mathrm{P}^2$. The proof is of course a variation on the proof of the Connected Sum-π_1-Proposition 52.12. We have

$$\pi_1(\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2) = \pi_1(P\#Q) \underset{\uparrow}{\cong} \pi_1\big(P \setminus \tfrac{1}{2}B^2\big) *_{\pi_1(\frac{1}{2}S^1)} \pi_1\big(Q \setminus \tfrac{1}{2}B^2\big).$$

by the definition of $P\#Q$ and the Seifert–van Kampen Theorem 54.4

We pick a generator g for $\pi_1(\frac{1}{2}S^1)$. In the calculation of the fundamental group of the real projective plane on page 1223 we saw that $\pi_1(P \setminus \frac{1}{2}B^2) \cong \langle x\rangle$ and that the inclusion induced map $\pi_1(\frac{1}{2}S^1) \to \pi_1(P \setminus \frac{1}{2}B^2)$ is given by $g \mapsto x^2$. Evidently the same holds for Q with $\pi_1(Q \setminus \frac{1}{2}B^2) \cong \langle y\rangle$. Summarizing we see that

$$\begin{aligned}
\pi_1(\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2) &\cong \pi_1\big(P \setminus \tfrac{1}{2}B^2\big) *_{\pi_1(\frac{1}{2}S^1)} \pi_1\big(Q \setminus \tfrac{1}{2}B^2\big) \\
&\cong \text{amalgamated product of } \begin{pmatrix} \langle g\rangle & \to & \langle x\rangle \\ g & \mapsto & x^2 \end{pmatrix} \text{ and } \begin{pmatrix} \langle g\rangle & \to & \langle y\rangle \\ g & \mapsto & y^2 \end{pmatrix} \\
&\underset{\uparrow}{\cong} \langle x, y \mid x^2 = y^2\rangle = \langle x, y \mid x^2y^{-2}\rangle.
\end{aligned}$$

Lemma 53.23

We have thus shown that $\pi_1(\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2) \cong \langle x, y \mid x^2 = y^2\rangle$. ∎

We conclude this section with a short discussion of the properties of fundamental groups of compact 2-dimensional smooth manifolds. We recall that the fundamental group of the real projective space $\mathbb{R}\mathrm{P}^2$ is isomorphic to $\mathbb{Z}_2$, therefore it is torsion and in particular it is not torsion-free. It is less clear whether or not the fundamental groups of the other 2-dimensional smooth manifolds are torsion-free. More precisely, we have the following question.

Question 54.11.

(1) Let $g \in \mathbb{N}$. Is the fundamental group

$$\pi_1(\text{surface of genus } g) \cong \langle x_1, y_1, \dots, x_g, y_g \mid [x_1, y_1] \cdot \dots \cdot [x_g, y_g]\rangle$$

torsion-free?

(2) Let $k \geq 2$. Is the fundamental group

$$\pi_1(k \cdot \mathbb{R}\mathrm{P}^2) \cong \langle x_1, \dots, x_k \mid x_1^2 \cdot \dots \cdot x_k^2\rangle$$

torsion-free?

We will come back to this question in Exercise 56.12, Corollary 64.7 and Proposition **??**.

54.4. **Fundamental groups of high-dimensional smooth manifolds.** Later on, in the Smooth Manifolds-Invariants Proposition 96.6 we will see that the fundamental group of a compact connected smooth manifold is necessarily finitely presented. In this section we want to study the converse: which finitely presented groups can occur as fundamental groups of smooth manifolds?

First note that it follows from the Smooth 1-Dimensional Manifold Classification Theorem 22.7 that only the trivial group and $\mathbb{Z}$ occur as the fundamental group of a compact connected 1-dimensional topological manifold. Similarly, we will see shortly in Theorem 55.4 that only "few" finitely presented groups occur as the fundamental group of compact connected 2-dimensional topological manifold. In contrast we will now see that once we go to dimension ≥ 4 any finitely presented group occurs as the fundamental group of a closed orientable smooth manifold

Proposition 54.12. (Fundamental Group-Manifold Realization Proposition) Let π be a finitely presented group. Given any $n \geq 4$ there exists a closed orientable connected non-empty n-dimensional smooth manifold M with $\pi_1(M) \cong \pi$.

Remark.

(1) We will give an alternative proof of the Fundamental Group-Manifold Realization Proposition 54.12 on page 3031.
(2) the Fundamental Group-Manifold Realization Proposition 54.12 is a key ingredient of the proof of the fact, stated in Theorem 60.4, that closed smooth manifolds of a fixed dimension ≥ 4 cannot be classified.

The proof of the Fundamental Group-Manifold Realization Proposition 54.12 relies on two lemmas and one theorem. The first lemma is actually quite interesting in its own right.

Lemma 54.13. Let W be a connected n-dimensional smooth manifold. Suppose we are given $k \in \mathbb{N}_0$ and a submanifold of the form $\overline{B}^{n-k} \times K$ where K is diffeomorphic to S^k.

(1) If $n \geq k+3$, then for every base point $w_0 \in W \setminus (B^{n-1} \times K)$ the inclusion induced map $\pi_1(W \setminus (B^{n-1} \times K), w_0) \to \pi_1(W, w_0)$ is an isomorphism.
(2) If $n = k+2$, then for every base point $w_0 \in W \setminus (B^{n-1} \times K)$ the inclusion induced map $\pi_1(W \setminus (B^{n-1} \times K), w_0) \to \pi_1(W, w_0)$ is an epimorphism.

Proof of Lemma 54.13. We write $X = W \setminus (B^{n-k} \times K)$. Note that it follows from Proposition 47.20 that it suffices to prove the statement for some base point $w_0 \in S^{n-k-1} \times K$.

First we consider the case that $n \geq k+3$. Next note that we have the following two isomorphisms:

$$\pi_1(X, w_0) \underset{\uparrow}{\xrightarrow{\cong}} \pi_1(X, w_0) *_{\pi_1(S^{n-k-1} \times K, w_0)} \pi_1\big(\overline{B}^{n-k} \times K, w_0\big) \underset{\uparrow}{\xrightarrow{\cong}} \pi_1(W, w_0).$$

map is an isomorphism by Lemma 53.22 (3a) and since map $\pi_1(S^{n-k-1} \times K, w_0) \to \pi_1(\overline{B}^{n-k} \times K, w_0)$ is an isomorphism, note that for the latter statement we used that $n \geq k+3$

by the Seifert–van Kampen Theorem 54.4, which we can apply by Proposition 24.16

Since the horizontal maps are induced by the inclusions we see that the inclusion induced map $\pi_1(X, w_0) \to \pi_1(W, w_0)$ is indeed an isomorphism.

If $n = k+2$, then the inclusion induced map $\pi_1(S^{n-k-1} \times K) \to \pi_1(\overline{B}^{n-k} \times K)$ is an epimorphism but no longer an isomorphism. If we now use the Amalgamated Product Lemma 53.22 (3b) instead of the Amalgamated Product Lemma 53.22 (3a) we see that $\pi_1(X, w_0) \to \pi_1(W, w_0)$ is at the very least an epimorphism. ■

Lemma 54.14. Let W be an n-dimensional smooth manifold and suppose we are given a boundary component of the form $S^{n-2} \times C$ where C admits a diffeomorphism $\varphi\colon S^1 \to C$. We set

$$X \;:=\; (W \sqcup (S^{n-2} \times \overline{B}^2))/\!\sim \qquad \text{where } (x, \varphi(y)) \sim (x, y) \text{ for } (x, y) \in S^{n-2} \times S^1.$$

If $n \geq 4$, then for any $w_0 \in W$ the inclusion induced map $\pi_1(W, w_0) \to \pi_1(X, w_0)$ induces an isomorphism[611]

$$\pi_1(W, w_0)/\langle\!\langle C \rangle\!\rangle \xrightarrow{\cong} \pi_1(X, w_0).$$

Proof. Note that it follows from Proposition 47.20 that it suffices to prove the statement for some base point $w_0 \in S^{n-2} \times C$. Next note that we have the following two isomorphisms:

$$\pi_1(W, w_0)/\langle\!\langle C \rangle\!\rangle \xrightarrow[\uparrow]{\cong} \pi_1(W, w_0) *_{\pi_1(S^{n-2}\times C, w_0)} \pi_1\big(S^{n-2} \times \overline{B}^2, w_0\big) \xrightarrow[\uparrow]{\cong} \pi_1(X, w_0).$$

map is an isomorphism by the Amalgamated Product Lemma 53.22 (2), hereby we use that $\pi_1(S^{n-2} \times \overline{B}^2, w_0) = 0$ and that $\pi_1(S^{n-2} \times C, w_0) = \langle C \rangle$, note that for the latter statement we used that $n \geq 4$

by the Seifert–van Kampen Theorem 54.4 which we can apply by the Gluing-Smooth Manifolds-Proposition

Since the horizontal maps are induced by the inclusions we see that the inclusion induced map $\pi_1(W, w_0)/\langle\!\langle C \rangle\!\rangle \to \pi_1(X, w_0)$ is indeed an isomorphism. ∎

Now we are fully equipped for proving the Fundamental Group-Manifold Realization Proposition 54.12.

Proof of the Fundamental Group-Manifold Realization Proposition 54.12. Throughout the proof we will mostly ignore base points. But at the end of the proof we will argue, why being a little sloppy with base points in between does not affect the outcome.

Let $\pi = \langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_l \rangle$ be a finitely presented group. Furthermore let $n \geq 4$. We consider

$$X \;:=\; \underbrace{(S^1 \times S^{n-1}) \# \ldots \# (S^1 \times S^{n-1})}_{k \text{ copies of } S^1 \times S^{n-1}}.$$

It follows basically from the Connected Sum-Proposition **??** that X is a closed orientable connected n-dimensional smooth manifold.

We see that

$$\pi_1(\overbrace{(S^1 \times S^{n-1}) \# \ldots \# (S^1 \times S^{n-1})}^{=X}) \xrightarrow[\uparrow]{\cong} \pi_1(S^1 \times S^{n-1}) * \ldots * \pi_1(S^1 \times S^{n-1}) \xrightarrow[\uparrow]{\cong} \overbrace{\langle g_1 \rangle * \ldots * \langle g_k \rangle}^{=\langle g_1, \ldots, g_k \rangle}.$$

by the Connected Sum-π_1-Proposition 52.12 since $n \geq 3$

by the discussion on page 1093 we know that $\pi_1(S^1 \times S^{n-1}) \cong \mathbb{Z}$

We use this isomorphism to make the identification $\pi_1(X, x_0) = \langle g_1, \ldots, g_k \rangle$. Next we pick loops $b_1, \ldots, b_l\colon (S^1, *) \to (X, x_0)$ that represent $r_1, \ldots, r_l \in \pi_1(X, x_0) = \langle g_1, \ldots, g_k \rangle$.

By hypothesis we have $\dim(X) = n \geq 4$. Thus we can apply Theorem **??** and we obtain that the map $b_1 \sqcup \cdots \sqcup b_l\colon S^1 \sqcup \cdots \sqcup S^1 \to X$ is in fact homotopic to a smooth embedding $c_1 \sqcup \cdots \sqcup c_l\colon S^1 \sqcup \cdots \sqcup S^1 \to X$. For $i = 1, \ldots, l$ we write $C_i := c_i(S^1)$. It follows from the Smooth Embedding Theorem 24.10 and the Tubular Neighborhood Theorem **??** that we

[611] Here we use that Exercise 53.15 tells us that $C = \{0\} \times C$ gives us a well-defined normal subgroup $\langle\!\langle C \rangle\!\rangle$.

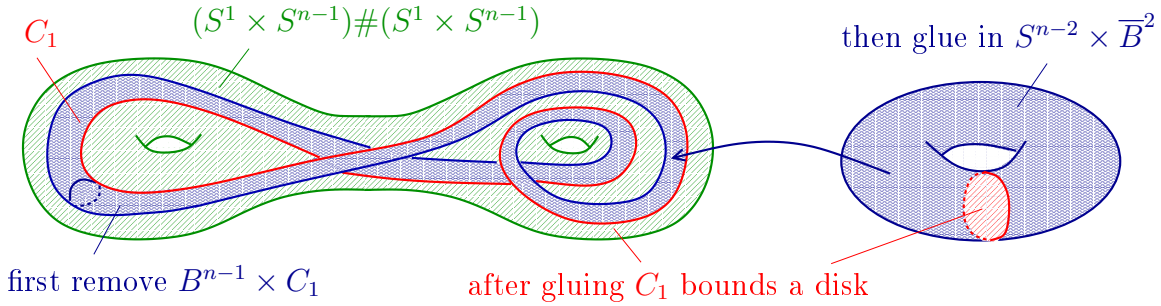
can thicken up the C_i to disjoint submanifolds $\overline{B}^{n-1} \times C_1, \ldots, \overline{B}^{n-1} \times C_l$. We consider[612]

$$Y := X \setminus \bigcup_{i=1}^{l} (B^{n-1} \times C_i) \quad \text{and} \quad Z := \Big(Y \sqcup \bigsqcup_{i=1}^{l} (S^{n-2} \times \overline{B}^2_i) \Big) / \sim \qquad \begin{array}{l} \text{where } (x, c_i(y)) \sim (x, y) \text{ for} \\ x \in S^{n-2} \text{ and } y \in S^1_i = S^1. \end{array}$$

It follows from the Codimension-Zero Smooth Embedding Proposition 24.16 and the Gluing-Smooth Manifolds-Proposition **??** that Z is indeed a closed oriented connected n-dimensional smooth manifold. Finally we see that

$$\begin{array}{ccccc} & \text{Lemma 54.14 applied } l \text{ times} & & \text{Lemma 54.13 applied } k \text{ times} & \\ & \downarrow & & \downarrow & \\ \pi_1(Z) & \cong & \pi_1(Y)/\langle\langle C_1, \ldots, C_l \rangle\rangle & \cong & \pi_1(X)/\langle\langle C_1, \ldots, C_l \rangle\rangle \\ & \cong & \langle g_1, \ldots, g_k \rangle / \langle\langle x_1 r_1 x_1^{-1}, \ldots, x_l r_l x_l^{-1} \rangle\rangle & \cong & \underbrace{\langle g_1, \ldots, g_k \rangle / \langle\langle r_1, \ldots, r_l \rangle\rangle}_{= \langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_l \rangle}. \\ & \uparrow & & \uparrow & \\ & \text{by the above, the } x_i \text{ are necessary since} & & \text{by Lemma 53.2 (5)} & \\ & \text{we lost track of the base points} & & & \end{array}$$

We have thus found the promised smooth manifold. ■



In the proof of the Fundamental Group-Manifold Realization Proposition 54.12 we used on several occasions that we are dealing with dimensions ≥ 4. The following question arises naturally.

Question 54.15. Is every finitely presented group the fundamental group of a closed (or at least compact) orientable connected 3-dimensional smooth manifold?

We will answer Question 54.15 in Theorem 146.5.

Exercises for Chapter 54.

Exercise 54.1. Let $f\colon A \to X$ be a map between topological spaces and let $a_0 \in A$. We write $x_0 := f(a_0)$. We suppose that A is path-connected. Show that the natural inclusion map $X \to \mathrm{Cone}(f\colon A \to X))$ induces an isomorphism

$$\pi_1(X, x_0)/f_*(\pi_1(A, a_0)) \;\to\; \pi_1(\mathrm{Cone}(f\colon A \to X)).$$

Exercise 54.2. Let $g \in \mathbb{N}_{\geq 2}$. We consider the group

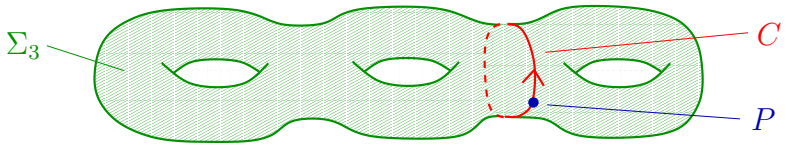
$$\pi \;:=\; \pi_1(\text{surface of genus } g) \;\cong\; \langle x_1, y_1, \ldots, x_g, y_g \,|\, [x_1, y_1] \cdot \cdots \cdot [x_g, y_g] \rangle.$$

[612] The smooth manifold Y is effectively obtained from $(S^1 \times S^{n-1}) \# \ldots \# (S^2 1 \times S^{n-2})$ by removing copies of $\overline{B}^{n-1} \times S^1$ and gluing in copies of $S^{n-2} \times \overline{B}^2$. This operation is frequently called a surgery. We will discuss surgeries in greater detail in Chapter **??**.

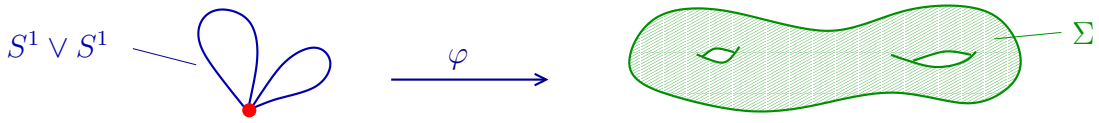
(a) Show that π admits an epimorphism onto a free group of rank ≥ 2.
(b) What is the largest $k \in \mathbb{N}$ for which you manage to find an epimorphism onto a free group of rank k?

Exercise 54.3. We consider the curve C on the surface Σ_3 of genus three that is shown in the figure below.

(a) Show that C represents the trivial element in $\pi_1(\Sigma_3, P)_{\text{ab}}$.
(b) Show that C represents a non-trivial element in $\pi_1(\Sigma_3, P)$.



Exercise 54.4. Let π be the fundamental group of the surface Σ of genus g. Show that π contains a free subgroup of rank 2.
Hint. You could make use of your solution to Exercise 54.2 or you could show that there exists an embedding $\varphi\colon S^1 \vee S^1 \to \Sigma$ such that $\varphi(S^1 \vee S^1)$ is a retract of Σ. We refer to the figure below for an illustration.



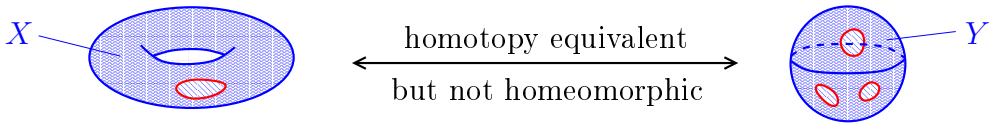
Exercise 54.5. Let K be the Klein bottle. Recall that on page 1222 we showed that $\pi_1(K) \cong \langle x, y \mid yxyx^{-1}\rangle$ and in the Fundamental Group-of-Surfaces Proposition 54.5 we showed that $\pi_1(K) \cong \langle x, y \,|\, x^2y^2\rangle$. Give a purely algebraic proof that the groups $\langle x, y \mid yxyx^{-1}\rangle$ and $\langle x, y \,|\, x^2y^2\rangle$ are isomorphic.

Exercise 54.6.

(a) Let M be an oriented connected m-dimensional smooth manifold and let N be a 1-dimensional submanifold *that is diffeomorphic to* S^1. Let $x_0 \in M \setminus N$. Suppose that $m \geq 4$. Show that the inclusion induced map $\pi_1(M \setminus N, x_0) \to \pi_1(M, x_0)$ is an isomorphism.
Hint. Use the Tubular Neighborhood Theorem **??**.
(b) Show that the conclusion (a) does not hold for $m = 3$, i.e. give an example of a connected 3-dimensional smooth manifold M and a connected 1-dimensional submanifold N such that the inclusion induced map $\pi_1(M \setminus N, x_0) \to \pi_1(M, x_0)$ is not an isomorphism.
Remark. The smooth manifold M in the example does not have to be closed.

Exercise 54.7. Let X be the result of removing the interior of a smoothly embedded disk from a torus. Furthermore let Y be the result of removing the interiors of three disjointly smoothly embedded disks from the sphere S^2.

(a) Show that X and Y are homotopy equivalent.
(b) Show that X and Y are not homeomorphic.



Exercise 54.8. Let $g \in \mathbb{N}$ and let Σ be the surface of genus g minus one open disk. We pick a diffeomorphism $\varphi\colon S^1 \to \partial\Sigma$. Let $\psi\colon S^1 \times \partial\Sigma \to \partial\Sigma \times S^1$ be the diffeomorphism given by $(z,w) \mapsto (\varphi(z), \varphi^{-1}(w))$. We consider the inclusion map

$$\begin{array}{rcl} f\colon (S^1 \times \partial\Sigma) \cup_\psi (\partial\Sigma \times S^1) & \to & \Sigma \times \Sigma \\ [(z,w)] & \mapsto & \begin{cases} (z, \varphi(w)), & \text{if } (z,w) \in S^1 \times \partial\Sigma, \\ (\varphi(z), w), & \text{if } (z,w) \in \partial\Sigma \times S^1. \end{cases} \end{array}$$

Show that the induced map

$$f_*\colon \pi_1((S^1 \times \partial\Sigma) \cup_\varphi (\partial\Sigma \times S^1)) \to \pi_1(\Sigma \times \Sigma)$$

has non-trivial kernel.

Remark. You might want to use the Normal Form Theorem 53.25.

55. The classification of compact 2-dimensional manifolds I

One of the goals of mathematicians is to classify certain objects. For example one goal would be, given a dimension $n \in \mathbb{N}$, to "classify" all compact connected non-empty n-dimensional topological manifolds. But what does "classify" mean? One interpretation would be the following:

(I) Provide a collection $\{X_i\}_{i\in I}$ of compact connected non-empty n-dimensional topological manifolds with the following two properties:
  (a) Any compact connected non-empty n-dimensional topological manifold is homeomorphic to some X_i.
  (b) If $i \neq j$, then X_i and X_j are not homeomorphic.

(II) Furthermore provide an algorithm which determines, given a compact connected n-dimensional topological manifold M, to which X_i it is homeomorphic to.

With obvious modifications the same goal can also be formulated in the smooth setting. For $n = 1$ we have achieved this goal with the Classification Theorems 22.1 and 22.7. In this setting a complete list is given by $\{S^1, [0,1]\}$ and given a compact connected non-empty 1-dimensional topological or smooth manifold it is trivial to determine to which of the two it corresponds.
In this chapter we turn our gaze to the 2-dimensional setting. We will state the classification theorem for topological and smooth manifolds and we will prove (I b) and (II) in the above setting. For the proof of (I a) we will need some patience though, we will only provide it in Chapter 142.

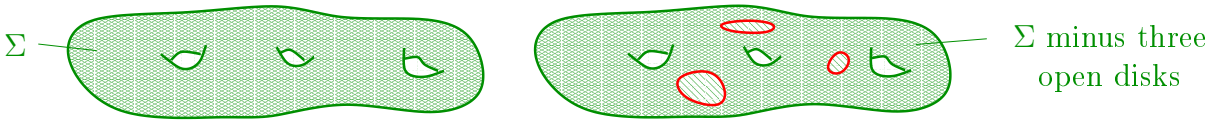
55.1. **Surfaces with boundary.** As mentioned above, one goal of any classification theorem is to provide a "complete" list of objects. We already have a good list of closed surfaces. Now we need to add surfaces with non-empty boundary to our arsenal. We start out with reminding the reader of the following subtle fact.[613]

Proposition 50.7.

(2) Let M be a 2-dimensional smooth manifold. The boundary of M, viewed as a smooth manifold agrees with the boundary of M viewed as a topological manifold.

Now let us move on to the first definition of this chapter,

Definition. Let Σ be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$. We pick smooth embeddings $\varphi_1, \dots, \varphi_n \colon \overline{B}^2 \to \Sigma$ with disjoint images. We refer to $\Sigma \setminus (\varphi_1(B^2) \cup \dots \cup \varphi_n(B^2))$ as Σ minus n open disks.



The following lemma captures some of the main features of the above definition.

[613] The statement is indeed subtle, for example we will only be able to prove the higher-dimensional analogue in the Topological Manifold Boundary Proposition 75.2 after a thorough study of homology groups.

Lemma 55.1. Let Σ be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$.

(1) The topological space Σ minus n open disks is a compact connected topological submanifold of Σ with n boundary components, which are given by the boundaries of the closed disks whose interiors were removed.
(2) Σ minus n open disks is a submanifold of the smooth manifold Σ, in particular it is a smooth manifold in its own right.
(3) The diffeomorphism type of the smooth manifold Σ minus n open disks does not depend on the choice of the disks.

Sketch of proof.

(1),(2) Let $\varphi_1, \dots, \varphi_n \colon \overline{B}^2 \to \Sigma$ be smooth embeddings with disjoint images. Note that we obtain from applying the Codimension-Zero Smooth Embedding Proposition 24.16 (2) iteratively altogether n times that $\Sigma \setminus (\varphi_1(B^2) \cup \dots \cup \varphi_n(B^2))$ is a compact connected submanifold of Σ. In particular we see that it is a smooth and topological manifold in its own right. The fact that the boundary, as a topological manifold, is precisely given by $\varphi_1(S^1), \dots, \varphi_n(S^1)$ follows from Propositions 19.26 and 50.7.
(3) This statement follows easily from the Smooth Ball Embedding Theorem 37.8. ■

This leads us to the following notation:

Notation. Given $g, n \in \mathbb{N}_0$ we write

$$\Sigma_{g,n} \quad := \quad \text{the surface of genus } g \text{ minus } n \text{ open disks.}$$

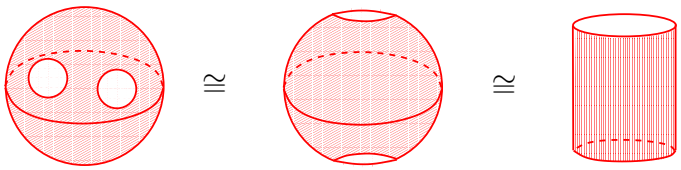
Furthermore for $g \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we write

$$N_{g,n} \quad := \quad \text{the non-orientable surface of genug } g \text{ minus } n \text{ open disks.}$$

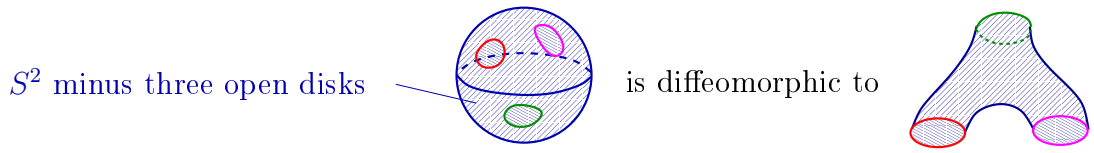
Sometimes we refer to g as the genus of $\Sigma_{g,n}$ and also of $N_{g,n}$.

Examples.

(1) The surface $\Sigma_{0,2}$ is the result of removing two open disjoint disks from the 2-sphere. It is pretty elementary to show that the resulting smooth manifold is diffeomorphic to the "cylinder" $[0,1] \times S^1$. We refer to the figure below for an illustration of this statement.



(2) A smooth manifold that is diffeomorphic to S^2 minus three open disks is often called a *pair of pants*. We refer to the figure below for an illustration which hopefully justifies the name. As we will see in Exercise 55.8, pairs of pants can be viewed as a "building block" for almost all compact connected 2-dimensional smooth manifolds.



(3) We saw in the figure on page 571 that the result of removing an open disk from $\mathbb{RP}^2$ is the Möbius band, i.e. $N_{1,1}$ is the Möbius band.

The following lemma shows that the fundamental groups of our new friends are rather dull and uninformative.

Lemma 55.2.

(1) Given $g \in \mathbb{N}_0$ and $n \in \mathbb{N}$ the surface $\Sigma_{g,n}$ admits a deformation retract to a topological graph of Euler characteristic $2-2g-n$. In particular $\pi_1(\Sigma_{g,n})$ is a free group on $2g+n-1$ generators.

(2) Given $k \in \mathbb{N}$ and $n \in \mathbb{N}$ the surface $N_{k,n}$ admits a deformation retract to a topological graph of Euler characteristic $2-k-n$. In particular $\pi_1(N_{k,n})$ is a free group on $k+n-1$ generators.

Remark. Lemma 55.2 can be viewed as a generalization of the fact, essentially shown in Lemma 52.9, that $\Sigma_{g,1}$ admits a deformation retraction to a wedge Γ of $2g$ circles, which is of course a topological graph of Euler characteristic $1-2g$.

Sketch of proof.

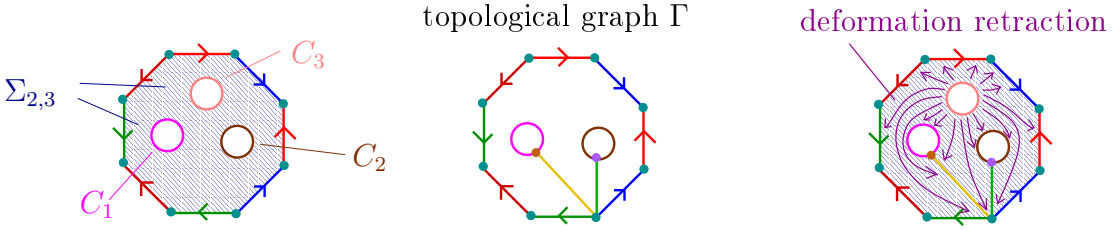
(1) We denote by $C_1, \dots, C_n$ the boundary components of $\Sigma_{g,n}$. Let Γ be the topological space that is given by the wedge of $2g$-circles (which is obtained from identifying as usual the edges of the $4g$-gon pairwise), together with the boundary components $C_1, \dots, C_{n-1}$ and together with edges $e_1, \dots, e_{n-1}$ that connect the wedge point to $C_1, \dots, C_{n-1}$. The deformation retraction from $\Sigma_{g,n}$ to Γ is given by pushing the remaining boundary component C_n "inward". We illustrate this in the figure below. (This is one of the rare occasions where we provide a "proof by picture". It is of course possible to give a completely rigorous proof, but one does not gain anything from doing so.) We equip Γ with the obvious structure of a topological graph which has $n = 1 + (n-1)$ vertices and $2g + 2(n-1)$ edges. Note that $\chi(\Gamma) = 2-2g-n$. Finally we see that

$$\pi_1(\Sigma_{g,n}) \underset{\uparrow}{\cong} \pi_1(\Gamma) \cong \pi_1\left(\begin{array}{c}\text{connected topological graph } \Gamma \\ \text{with } \chi(\Gamma) = 2-2g-n.\end{array}\right) \underset{\uparrow}{\cong} \text{free group on } 2g+n-1 \text{ genera}$$

by the above together with the Deformation Retract-Homotopy Equivalence Lemma 15.5 and the Homotopy-π_1-Proposition 50.3

by the Fundamental Grou

(2) The proof of this statement is almost identical to the proof of the first statement.



■

In the following we will see that we can "recover" the closed surfaces Σ_g and N_g from any $\Sigma_{g,n}$ respectively any $N_{g,n}$. The key to doing so is the following definition.

Definition. Let M be a topological manifold. Let "$\sim$" be the equivalence relation on M that is generated by $x \sim y$ whenever x and y lie on the same component of ∂M. We write $\widehat{M} := M/\sim$.

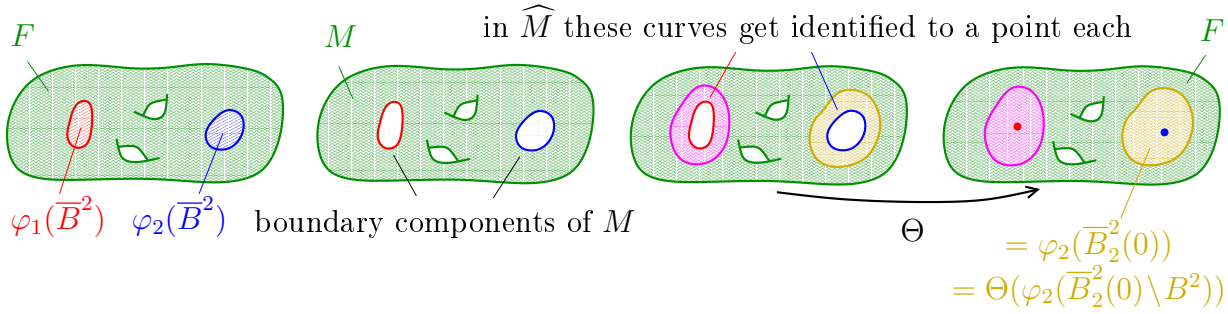
The following lemma shows in particular that $\widehat{\Sigma_{g,n}}$ is homeomorphic to Σ_g and that $\widehat{N_{g,n}}$ is homeomorphic to N_g.

Lemma 55.3. Let F be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$. If we set M to be F minus n open disks, then $\widehat{M}$ is homeomorphic to F.

Proof. Let $\varphi_1, \dots, \varphi_n \colon \overline{B}^2 \to F$ be smooth embeddings with disjoint images. We need to show that for $M := F \setminus (\varphi_1(B^2) \cup \dots \cup \varphi_n(B^2))$ there exists a homeomorphism $\widehat{M} \to F$. By the Embedding Extension Theorem **??** we can extend the smooth embeddings $\varphi_1, \dots, \varphi_n$ to smooth embeddings $\overline{B}^2_2(0) \to F$ with disjoint images. We consider the following map:[614]

$$\begin{aligned} \Theta \colon \widehat{M} &\to F \\ P &\mapsto \begin{cases} \varphi_i(Q \cdot 2 \cdot (\|Q\| - 1)), & \text{if } P = [\varphi_i(Q)] \text{ for some } i \in \{1, \dots, n\} \text{ and } Q \in \overline{B}^2_2(0) \setminus B^2, \\ P, & \text{otherwise.} \end{cases} \end{aligned}$$

It follows from the Topological-Quotient Proposition 5.15 (1b) and the Smooth Proposition 2.6 (2) that Θ is continuous. Note that by the Compact Image Lemma 2.13 we know that $\widehat{M}$ is compact. Now it follows almost immediately from the Compact-Hausdorff Proposition 2.17 (3) that the given map $\Theta \colon \widehat{M} \to F$ is indeed a homeomorphism. ■



55.2. **The Surface Classification Theorem.** The following theorem achieves Goal (I) that we set out on page 1233.

Theorem 55.4. (Surface Classification Theorem) Every compact connected non-empty 2-dimensional topological manifold is homeomorphic to either

(1) the surface $\Sigma_{g,n}$ for *unique* $g, n \in \mathbb{N}_0$ or

(2) to the surface $N_{k,n}$ for *unique* $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.

Furthermore, if we are given a smooth manifold, then we can upgrade "homeomorphic" to diffeomorphic.

Remark. The surfaces Σ_g, and thus also the surfaces $\Sigma_{g,n}$, come with a canonical orientation and each of these surfaces admits an orientation-reversing self-diffeomorphism given by a reflection. Thus, if we are given a compact *oriented* connected 2-dimensional smooth

[614] Note that discretely we are using Lemma 55.1 (1) which says that the boundary of M, viewed as a topological manifold, is indeed what we think it should be, namely it is given by $\varphi_1(S^1) \sqcup \dots \sqcup \varphi_n(S^1)$.

manifold, then we can in fact find an *orientation-preserving* diffeomorphism to the corresponding $\Sigma_{g,n}$.

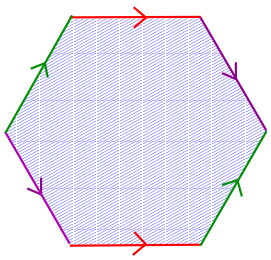
Theorem 55.4 leads us to the following definition.

Definition. Let M be a compact connected non-empty 2-dimensional topological manifold. By the Surface Classification Theorem 55.4 there exists a unique $g \in \mathbb{N}_0$ such that M is homeomorphic to some $\Sigma_{g,n}$ or to some $N_{g,n}$. We refer to g as the genus(M) of M.

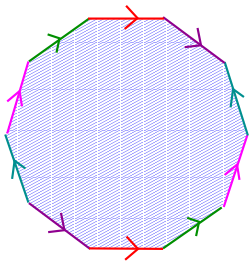
At the end of the section we will state a slightly more precise version of the Surface Classification Theorem 55.4 that gives us Goal (II) from page 1233. Before we talk about the proof of the Surface Classification Theorem 55.4 let us discuss a few examples.

Examples.

(1) At least for closed orientable connected 2-dimensional smooth manifolds it is perhaps tempting to dismiss the Surface Classification Theorem 55.4 (1) as "obvious". After all, what other closed orientable connected 2-dimensional smooth manifold could there possibly be? It is thus instructive to consider the figure below. It shows a hexagon and a decagon where *opposite sides* are identified. The quotient spaces are closed orientable 2-dimensional smooth manifolds.[615] To which of the standard surfaces are they diffeomorphic to? We will come back to this question on page 1880.



hexagon with opposite sides identified



dodecagon with opposite sides identified

(2) Let $g \in \mathbb{N}$. We consider the surface Σ_g of genus g. The connected sum $\Sigma_g \# \mathbb{R}\mathrm{P}^2$ contains a Möbius band as a submanifold of codimension zero,[616] hence $\Sigma_g \# \mathbb{R}\mathrm{P}^2$ is non-orientable. It follows from the Surface Classification Theorem 55.4 that $\Sigma_g \# \mathbb{R}\mathrm{P}^2$ is diffeomorphic to $k \cdot \mathbb{R}\mathrm{P}^2$ for some unique $k \in \mathbb{N}$. In Exercise 55.2 we will determine k as a function of g.

The Surface Classification Theorem 55.4 is an immediate consequence of the following three theorems.

Theorem 142.7. Let Σ be a compact connected non-empty 2-dimensional smooth manifold.

(1) If Σ is orientable, then Σ is diffeomorphic to the surface $\Sigma_{g,n}$ for some $g, n \in \mathbb{N}_0$.

[615] Indeed, basically the same proof that showed that the $4g$-gon, with appropriate sides identified, is a closed orientable 2-dimensional smooth manifold also shows that the topological spaces shown in the figure above are closed orientable 2-dimensional smooth manifolds.

[616] This follows from the observation that $\Sigma_g \# \mathbb{R}\mathrm{P}^2$ contains $\mathbb{R}\mathrm{P}^2 \setminus \frac{1}{2}B^2$, which by the figure on page 571 is a Möbius band.

(2) If Σ is non-orientable, then Σ is diffeomorphic to the surface $N_{k,n}$ for some $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.

Theorem 55.5. (Surface-Smooth Structure Existence Theorem) Every compact 2-dimensional topological manifold admits a smooth structure.

Theorem 55.6. (Surfaces-Non Homeomorphic Theorem) The topological spaces $\Sigma_{g,m}$ and $N_{h,n}$ are pairwise non-homeomorphic.[617]

We now turn to the discussion of the proofs of the three theorems.

Proof of the Diffeomorphism-to-Standard Surfaces Theorem 142.7. We provide the proof of the Diffeomorphism-to-Standard Surfaces Theorem 142.7 in Chapter 142. Actually the proof is not as advanced as it may sound. The only ingredient one needs is the notion and existence of a handle decomposition which are given in Chapter 138. We will also sketch an alternative approach to proving the Diffeomorphism-to-Standard Surfaces Theorem 142.7 in Exercise **??**. Also note that there are several other approaches to proving the Diffeomorphism-to-Standard Surfaces Theorem 142.7. For example [**Mata2002**, Chapter 5.2] and [**Hirs1976**, Chapter 9.3] give a proof using "Morse theory".[618] ■

Proof of the Surface-Smooth Structure Existence Theorem 55.5. Let M be a compact 2-dimensional topological manifold. There are two approaches to showing that M admits a smooth structure.

(1) In 1922 Tibor Radó [**Rad1926**] showed that M is homeomorphic to a finite 2-dimensional simplicial complex. Alternative proofs are given in [**Moi1977**, p. 60], [**DM1968**, Corollary 1], [**Tn1992**, Theorem 4.1], [**MTh2001**, Theorem 3.1.1], [**AS1960**, p. 107] and [**GX2013**, Appendix E]. A sketch of the proof is provided in [**Lee2000**, Theorem 5.12].[619]

We will give the definition of a simplicial complex on page 1994. For the purpose of this proof it suffices to know that this means that M is homeomorphic to a topological space that is obtained from gluing finitely many triangles linearly along their sides.

Recall that in Proposition 19.18 we showed that the surface $E_{2g}/\sim$ of genus g admits a smooth structure. The proof can be modified without too many problems to show that any compact 2-dimensional topological manifold that admits a simplicial structure, also admits a smooth structure.

(2) In [**Hat2013**, Theorem A] a direct proof is given for the fact that M admits a smooth structure. Note that this approach does not rely on the Schönflies Theorem 82.4. ■

Since we did not really prove the first two theorems we should at least provide full details for the third theorem. The idea is to find a complete set of invariants to distinguish the $\Sigma_{g,m}$ and $N_{h,n}$ pairwise. The first invariant we want to use is of course the (abelianization) of

[617] In the Surfaces-Non Diffeomorphic Corollary 45.21 we showed the weaker statement that the surfaces Σ_g are pairwise non-diffeomorphic.

[618] We will provide an introduction to Morse Theory in Chapter 138 but we will not discuss how Morse theory can be used directly to prove the Diffeomorphism-to-Standard Surfaces Theorem 142.7. We will only use Morse theory indirectly, since it allows us to study smooth manifolds via handle decompositions, which is the key to our proof of the Diffeomorphism-to-Standard Surfaces Theorem 142.7.

[619] The key ingredient to the proof is the Schönflies Theorem 82.4 which we will formulate later, but which we will not prove.

the fundamental group. But since this invariant is not enough we have to look for further invariants. One idea would be to use the number of boundary components. But as we saw in the figure on page 1189, even this extra information is not enough. To resolve this conundrum let us recall the following notation from page 1236.

Notation. Let M be a topological manifold. Let "$\sim$" be the equivalence relation on M that is generated by $x \sim y$ whenever x and y lie on the same component of ∂M. We write $\widehat{M} := M/\sim$.

The following proposition contains all invariants that we will need.

Proposition 55.7. For $g \in \mathbb{N}_0$ and $k \in \mathbb{N}$ we have the following table of invariants:

X	Σ_g	$\Sigma_{g,m}$ with $n \geq 1$	N_k	$N_{k,n}$ with $n \geq 1$
isomorphism type of $\pi_1(X)_{\text{ab}}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{2g+m-1}$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{k+n-1}$
number of boundary components	0	n	0	n
isomorphism type of $\pi_1(\widehat{X})_{\text{ab}}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}_2$.

Proof. The proposition is an immediate consequence of Propositions 54.5 and 54.9, together with Lemmas 55.1, 55.2 and 55.3. ■

Now we can prove the Surfaces-Non Homeomorphic Theorem 55.6.

Proof of the Surfaces-Non Homeomorphic Theorem 55.6. It follows immediately from Proposition 55.7 that the $\Sigma_{g,m}$ and $N_{h,n}$ are pairwise non-homeomorphic. For completeness we point out that in Exercise 80.15 we will use (relative) homology groups to give an alternative proof of the Surfaces-Non Homeomorphic Theorem 55.6. ■

Proposition 55.7 also gives us a practical way to determine whether two surfaces are homeomorphic respectively diffeomorphic. This proposition also gives a hopefully satisfying answer to Goal (II) from page 1233.

Proposition 55.8. Let M and N be two compact connected non-empty 2-dimensional topological (respectively smooth) manifolds. The following are equivalent:

(1) M and N are homeomorphic (respectively diffeomorphic),
(2) (a) $\pi_1(M)_{\text{ab}} \cong \pi_1(N)_{\text{ab}}$,
(b) M and N have the same number of boundary components,
(c) $\pi_1(\widehat{M})_{\text{ab}} \cong \pi_1(\widehat{N})_{\text{ab}}$.
In the smooth setting we can replace (c) by
(c') M and N have the same orientability.

Remark.

(1) Only much later, in the Smooth-Topological Orientation Proposition 105.11, will we see that homeomorphisms between smooth manifolds preserve orientability. If we factor this proposition in, then we can replace (c) by (c') also in the topological setting.
(2) In Proposition 64.17 we will see that the fundamental group of the surface Σ_g is not isomorphic to the free group on $2g$ generators. Once we know this fact, one can replace (a) and (b) by $\pi_1(M) \cong \pi_1(N)$.

Proof. In the topological setting this proposition follows immediately from Proposition 55.7 and the Surface Classification Theorem 55.4. In the smooth setting the proposition follows from the above, together with the fact that on page 686 we showed that the $\Sigma_{g,m}$ are orientable and that in the Surface Non-Orientability Lemma 25.20 we showed that the $N_{k,n}$ are non-orientable. ■

We conclude this discussion of the classification theorem with a long remark on various other approaches to giving the topological classification of 2-dimensional topological manifolds.

Remark. The Surface Classification Theorem 55.4 says in particular that every compact connected non-empty 2-dimensional topological (respectively smooth) manifold is homeomorphic (respectively diffeomorphic) to one of the $\Sigma_{g,n}$ or one of the $N_{k,n}$. Basically all proofs in the literature that deal with the purely topological setting use the fact, mentioned on page 1238, that a 2-dimensional topological manifold admits a simplicial structure. The idea then is to modify the simplicial structure to turn it into a simplicial structure of one of $\Sigma_{g,n}$ or $N_{k,m}$. The traditional account of this proof is given in [**ST1934, ST1980**] or alternatively in [**Mass1991**, Theorem 5.1]. A variation on this proof, which is usually referred to as Conway's Zip Proof, is outlined in [**FW1999, Wild2010**], with full details given in [**Gle**] and [**SchubJ2017**]. Other proofs are given in [**Tn1992**], [**Meie**] and [**AKM2022**]. There is also a proof due to Eric Zeeman, see [**Zee1966, Put2**]. The proof given in [**Arm1988**, Chapter 7] is somewhat different, its basic idea is related to the approach taken in Exercise **??**. Finally we point to [**GX2013**] for a book that is completely dedicated to the proof of the Surface Classification Theorem 55.4 in the topological setting.

By now we might have the impression that we have a solid understanding of surfaces. The following question provides a good test for our true understanding.

Question 55.9. Let M be the surface of genus $g \in \mathbb{N}_0$ and let $p\colon \widetilde{M} \to M$ be a covering of finite degree d such that $\widetilde{M}$ itself is connected. By the Manifold Covering Proposition 49.2 and Lemma 48.3 we know that $\widetilde{M}$ is a closed orientable connected 2-dimensional smooth manifold. Can we determine the genus of $\widetilde{M}$ in terms of g and d?

We will answer this question later on in the Subgroups-of-Surface Groups Proposition 64.6. It is worth trying to come up with an answer with our current knowledge.

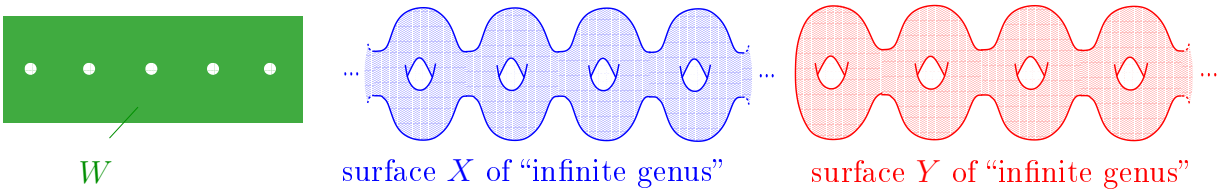
55.3. **Non-compact 2-dimensional smooth manifolds.** Just for kicks we also want to consider *non-compact* 2-dimensional smooth manifolds. A short moment's thought there is a plethora of examples:

(1) Given any connected 2-dimensional smooth manifold Σ and given any non-empty closed discrete subset $A \subset \Sigma$ one can easily show that the complement $\Sigma \setminus A$ is in fact a non-compact connected 2-dimensional smooth manifold. For example, this shows that $W = \mathbb{R}^2 \setminus \{(n,0) \,|\, n \in \mathbb{Z}\}$ is a non-compact connected 2-dimensional smooth manifold.

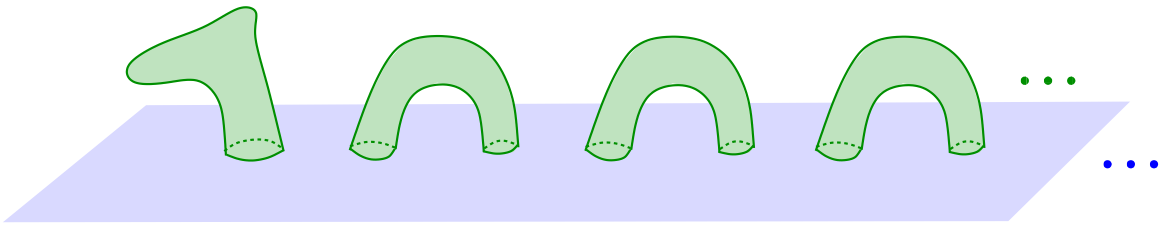
(2) In the figure below we show two non-compact 2-dimensional smooth manifolds X and Y for which the name "infinite genus" seems appropriate.

In Exercises 55.14 and Exercise 55.15 we will show, with some effort, that the non-compact 2-dimensional smooth manifolds W, X and Y are pairwise not diffeomorphic. This already shows that the classification of non-compact connected 2-dimensional smooth manifolds

can get a little tricky. In fact the classification of non-compact orientable connected 2-dimensional smooth manifolds without boundary was first given by Ian Richards [**Ric1963**] in 1963, building on earlier work of Béla Kerékjártó [**Kere1923**] from 1923. In [**PM2007**] this result was extended to the case of 2-dimensional smooth manifolds with non-empty boundary.



Remark. As an aside we point out that the surface Y of infinite genus shown in the figure above is sometimes called the *Loch Ness monster*. The reason is shown in the figure below: the visible part of an infinite Loch Ness monster, together with the surface of the infinite Loch Ness form a 2-dimensional smooth manifold that is diffeomorphic to Y. More information on the mathematical Loch Ness monster and an explicit rigorous construction are given in [**ARM2017**].



55.4. **Retractions onto boundary components of 2-dimensional smooth manifolds.** In this section we will deal with the following question: for which 2-dimensional smooth manifolds is a given boundary component a retract? This discussion is useful but not essential and one can safely skip this section.

Without further apologies let us recall the following definition from page 1061.

Definition. Let X be a topological space. We say that a subset A of X is a retract of X if there exists a retraction $r\colon X \to A$, i.e. a map with $r(a) = a$ for all $a \in A$.

For the reader's convenience we also recall the statement of Lemma 47.16.

Lemma 47.16. The circle S^1 is not a retract of $\overline{B}^2$.

We quickly recall the proof of Lemma 47.16.

Proof. We denote by $i\colon S^1 \to \overline{B}^2$ the inclusion map. Suppose there exists a retraction $r\colon \overline{B}^2 \to S^1$, i.e. a map with $r \circ i = \mathrm{id}_{S^1}$. We obtain the commutative diagram

$$\begin{array}{ccc} & \overline{B}^2 & \\ {\scriptstyle i}\nearrow & & \searrow{\scriptstyle r} \\ S^1 & \xrightarrow[r\circ i = \mathrm{id}_{S^1}]{} & S^1 \end{array}$$

which induces the commutative diagram

$$\begin{array}{ccccc} & & \underbrace{\pi_1(\overline{B}^2, 1)}_{=0} & & \\ & {\scriptstyle i_*}\nearrow & & \searrow{\scriptstyle r_*} & \\ \underbrace{\pi_1(S^1, 1)}_{\cong \mathbb{Z}} & & \xrightarrow[(r\circ i)_* = (\mathrm{id}_{S^1})_*]{} & & \underbrace{\pi_1(S^1, 1)}_{\cong \mathbb{Z}}. \end{array}$$

But a commutative diagram as given on the right-hand side cannot exist since an isomorphism of a non-trivial group cannot factor through the trivial group. ■

In light of Lemma 47.16 the following question arises:

Question 55.10. Let Σ be a 2-dimensional smooth manifold and let C be a boundary component of Σ. Is C a retract of Σ?

By the Surface Classification Theorem 55.4 it suffices to address Question 55.10 for the smooth manifolds $\Sigma_{g,m}$ and $N_{h,n}$. For notational convenience we will work throughout with $\Sigma = \Sigma_{g,m}$. We will deal with $\Sigma = N_{h,n}$ in Exercise 55.13.

So let us consider $\Sigma = \Sigma_{g,n}$. Lemma 47.16 says that the answer to Question 55.10 is no if $g = 0$ and $n = 1$. The proof of Lemma 47.16 suggests that if we want to answer Question 55.10, then given a boundary component C of $\Sigma_{g,n}$ we should try to understand the inclusion induced map $\pi_1(C) \to \pi_1(\Sigma_{g,n})$. Of course by Lemma 55.2 we know that $\pi_1(\Sigma_{g,n})$ is a free group on $2g + n - 1$ generators. But back then we mostly cared about the isomorphism type of $\pi_1(\Sigma_{g,n})$ and we did not worry about the relationship of $\Sigma_{g,n}$ and its boundary components.

The following lemma gives a new description of $\pi_1(\Sigma_{g,n})$. This description is more complicated in the sense that it is not totally immediate that $\pi_1(\Sigma_{g,n})$ is a free group. On the other hand it makes it clearer how the boundary components of $\Sigma_{g,n}$ interact with $\Sigma_{g,n}$.

Lemma 55.11. Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}$. We denote the boundary components of $\Sigma_{g,n}$ by $C_1, \dots, C_n$. The following statements hold:

(1) Let $* \in \Sigma_{g,n}$ be the standard base point given by the "vertex" of $E_{2g}/\sim$. We pick an orientation for $\Sigma_{g,n}$. There exists a presentation

$$\pi_1(\Sigma_{g,n}, *) \;=\; \langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_n \mid [x_1, y_1] \cdot \dots \cdot [x_g, y_g] \cdot z_1 \cdot \dots \cdot z_n \rangle$$

with the following properties:

(a) The elements $x_1, y_1, \dots, x_g, y_g$ are represented by the edges of the $2g$-gon in the "usual way".

(b) For any $i \in \{1, \dots, n\}$ the element z_i is represented by a curve of the form $\alpha * \gamma * \overline{\alpha}$ where α is a path from $*$ to a point on C_i and γ is a path that goes once around C_i in an orientation-preserving way.[620]

We refer to the figure below for an illustration.

(2) For every $i \in \{1, \dots, n\}$ the obvious map

$$\underbrace{\langle x_1, y_1, .., x_g, y_g, z_1, .., \widehat{z_i}, .., z_n \rangle}_{\text{free group}} \to \langle x_1, y_1, .., x_g, y_g, z_1, .., z_n \mid \underbrace{[x_1, y_1] \dots [x_g, y_g] \cdot z_1 .. z_n}_{\text{relation}} \rangle$$

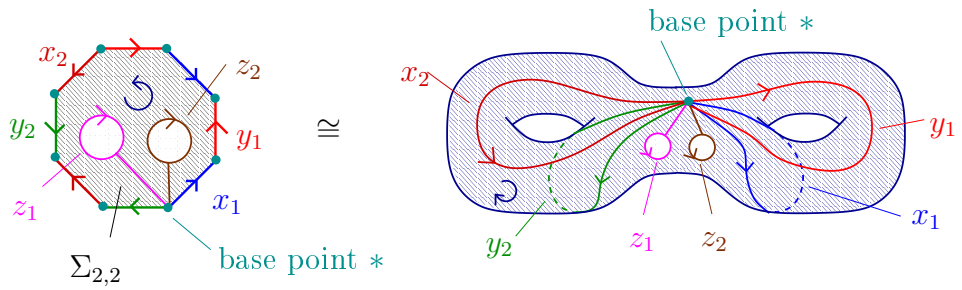
is an isomorphism. The inverse is induced by the obvious assignments together with

$$z_i \;\mapsto\; {z_{i-1}}^{-1} \cdot \dots \cdot {z_1}^{-1} \cdot [y_g, x_g] \cdot \dots \cdot [y_1, x_g] \cdot {z_n}^{-1} \cdot \dots {z_{i+1}}^{-1}.$$

In particular $\pi_1(\Sigma_{g,n})$ is a free group on $2g + n - 1$ generators.

(3) If $g \geq 1$ or if $n \geq 2$, then for every $i \in \{1, \dots, n\}$ the inclusion induced map $\pi_1(C_i) \to \pi_1(\Sigma_{g,n})$ is a monomorphism.

[620] Since we picked an orientation for $\Sigma_{g,n}$ each boundary component also inherits an orientation.



Sketch of the proof. Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}$.

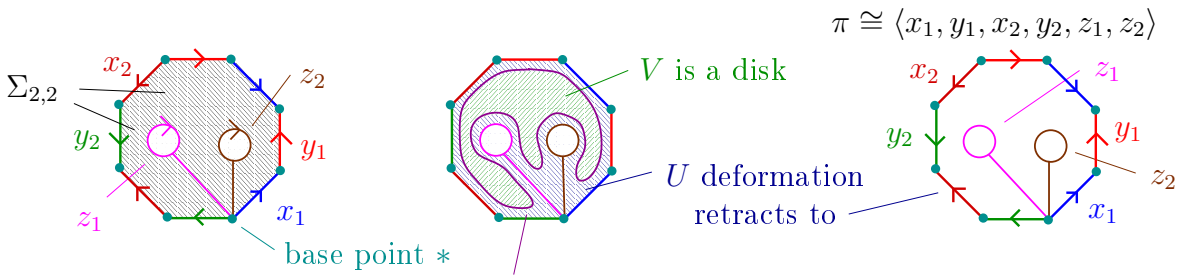
(1) First note that by Proposition 36.4 we can without loss of generality assume that the base point corresponds to any of the vertices of the $4g$-gon as shown in the figure below. The following argument is very similar to the argument on the pages 1221 and 1224 and it relies heavily on the notation from the figure below. We have

Seifert–van Kampen Theorem 54.1 Amalgamated Product Lemma 53.22

$$\pi_1(\Sigma_{g,n}) \overset{\downarrow}{=} \pi_1(U) *_{\pi_1(U\cap V)} \underbrace{\pi_1(V)}_{=\{e\}} \overset{\downarrow}{\cong} \pi_1(U)/\langle\langle \pi_1(U\cap V)\rangle\rangle$$

$$\overset{\uparrow}{=} \langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_n\rangle / \langle\langle [x_1,y_1]\cdot\dots\cdot[x_g,y_g]\cdot z_1\cdot\dots\cdot z_n\rangle\rangle.$$

followings from the deformation retraction shown in the figure below together with the Deformation Retract-Homotopy Equivalence Lemma 15.5 and the Homotopy-π_1-Proposition 50.3 (2)



(2) It follows immediately from the discussion on page 1196 that for any $i \in \{1,\dots,n\}$ the obvious map

$$\langle x_1, y_1, .., x_g, y_g, z_1, .., \widehat{z_i}, .., z_n\rangle \to \langle x_1, y_1, .., x_g, y_g, z_1, .., z_n \mid [x_1,y_1]\dots[x_g,y_g]\cdot z_1..z_n\rangle$$

is an isomorphism. The map that we give in the inverse direction is evidently a converse.

(3) Now suppose that $g \geq 1$ or that $n \geq 2$. Let $i \in \{1,\dots,n\}$. The image of $\pi_1(C_i)$ in $\pi_1(\Sigma_{g,n})$ is the subgroup generated by z_i. It suffices to show that z_i is an element of infinite order in $\pi_1(\Sigma_{g,n})$. We apply (2) to i. We obtain an isomorphism

$$\begin{aligned} \pi_1(\Sigma_{g,n}) &\to \langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, \widehat{z_i}, \dots, z_n\rangle \\ z_i &\mapsto z_{i-1}{}^{-1}\cdot\dots\cdot z_1{}^{-1}\cdot[y_g,x_g]\cdot\dots\cdot[y_1,x_1]\cdot\dots z_n{}^{-1}\cdot\dots\cdot z_{i+1}{}^{-1}. \end{aligned}$$

Since $g \geq 1$ or $n \geq 2$ we see that the element on the right-hand side is represented by a non-empty reduced word. Thus by definition it represents a non-trivial element

in a free group. By Lemma 51.14 we know that every non-trivial element of a free group has infinite order. Thus we have shown that z_i is indeed an element of infinite order of $\pi_1(\Sigma_{g,n})$. ■

Finally the following technical lemma can be viewed as a refinement of Lemma 55.2 (1).

Lemma 55.12. Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}$.

(1) We denote the boundary components of $\Sigma_{g,n}$ by $C_1, \dots, C_n$. Given any $i \in \{1, \dots, n\}$ there exists a topological graph $G \subset \Sigma_{g,n}$ which contains all C_j with $i \neq j$ and which contains a point on C_j, a deformation retraction $r\colon \Sigma_{g,n} \to G$, a spanning tree $T \subset G$ and a homeomorphism[621]

$$\Theta\colon G/T \;\to\; C_1 \vee \dots \vee \widehat{C_i} \vee \dots \vee C_n \vee \underbrace{S^1 \vee \dots \vee S^1}_{2g \text{ copies}}$$

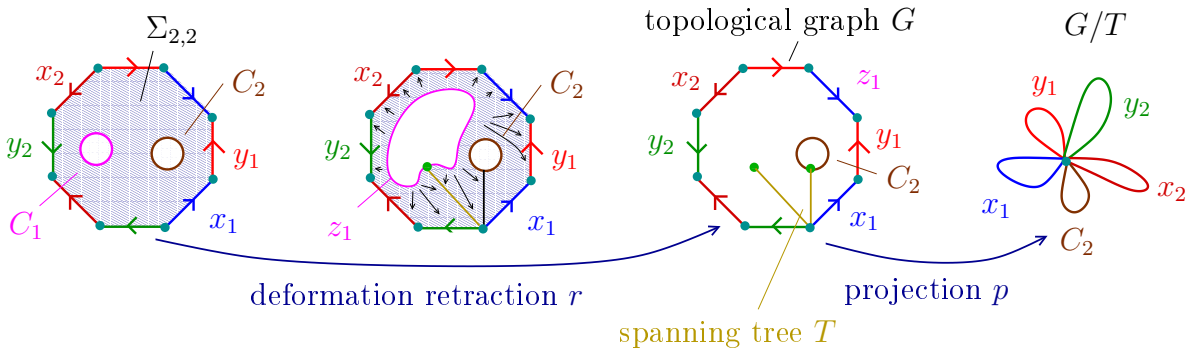
such that the following diagram commutes:

$$\begin{array}{ccccc} & & C_j & \longrightarrow & C_1 \vee \dots \vee \widehat{C_i} \vee \dots \vee C_n \vee S^1 \vee \dots \vee S^1 \\ & \swarrow & \downarrow & & \Theta \downarrow \cong \\ \Sigma_{g,n} & \xrightarrow{r} & G & \xrightarrow{\quad p \quad} & G/T. \end{array}$$

Note that $(p \circ r)_*\colon \pi_1(\Sigma_{g,n}) \to \pi_1(G/T)$ is an isomorphism by Propositions 50.3 and 50.15 (1).

(2) The fundamental group $\pi_1(\Sigma_{g,n})$ is a free group on $2g + n - 1$ generators.

Given $k \in \mathbb{N}$ and $n \in \mathbb{N}$ completely analogous statements also hold for $N_{k,n}$.



Sketch of proof.

(1) In the figure we show that, starting from any boundary component C_i of $\Sigma_{g,n}$, we can push "inward" to obtain a deformation retraction[622][623] to a subset that is a topological graph G that can be described as

$$G \;=\; \big(C_1 \sqcup \dots \sqcup \widehat{C_i} \sqcup \dots \sqcup C_n \sqcup \underbrace{S^1 \vee \dots \vee S^1}_{2g \text{ copies}}\big) \text{ together with a spanning tree } T.$$

[621] Here $\widehat{C_i}$ means that we leave out C_i.

[622] We leave it to others with more time and energy to write down a rigorous argument. One possibility would be to apply several times the deformation retractions that we found in Exercise 15.13.

[623] This argument is evidently closely related to the proof of Lemma 52.9.

The vertical map to the right is the homeomorphism given by the discussion on page 1133. It follows immediately from the construction that the diagram commutes.

(2) This statement follows from (1) together with the Fundamental Group-Wedge-of-Circles Lemma 52.4. ■

The proof for $N_{k,n}$ is almost identical to the above proof for $\Sigma_{g,n}$. We leave it to the reader to figure out the precise statements and to write down the proof.

The following lemma gives an answer to Question 55.10.

Lemma 55.13. Let $g \in \mathbb{N}_0$.

(1) The unique boundary component of $\Sigma_{g,1}$ is not a retract of $\Sigma_{g,1}$.
(2) For $k \geq 2$ every boundary component of $\Sigma_{g,k}$ is a retract of $\Sigma_{g,k}$.

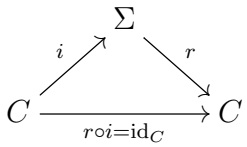
Proof. Let $g \in \mathbb{N}_0$.

(1) The case $g = 0$ was dealt with in Lemma 47.16. Now we assume that $g \geq 1$. We denote by C the unique boundary component of $\Sigma := \Sigma_{g,1}$. We had just seen in Lemma 55.11 that $\pi_1(C) \to \pi_1(\Sigma)$ is a monomorphism if $g \geq 1$. So we cannot apply the same argument as in Lemma 47.16 where we studied the induced map on the level of fundamental groups. The idea now is to study instead the induced map on the *abelianizations* of the fundamental groups.

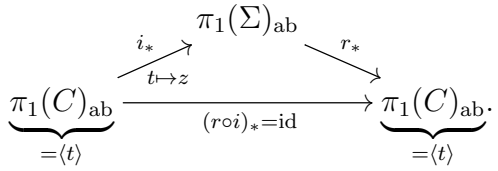
We denote by $i\colon C \to \Sigma$ the inclusion map and we suppose that there exists a retraction $r\colon \Sigma \to C$, i.e. a map with $r \circ i = \mathrm{id}_C$. We write $\pi_1(C) = \langle t \rangle$ and we use the isomorphism from Lemma 55.11 (1) to make the identification

$$\pi_1(\Sigma) \;=\; \langle x_1, y_1, \ldots, x_g, y_g, z \mid [x_1, y_1] \cdot \cdots \cdot [x_g, y_g] \cdot z \rangle$$

where z corresponds to t under the inclusion $C = \partial\Sigma \to \Sigma$. We obtain the commutative diagram



which induces by Proposition 53.20 (6) the commutative diagram



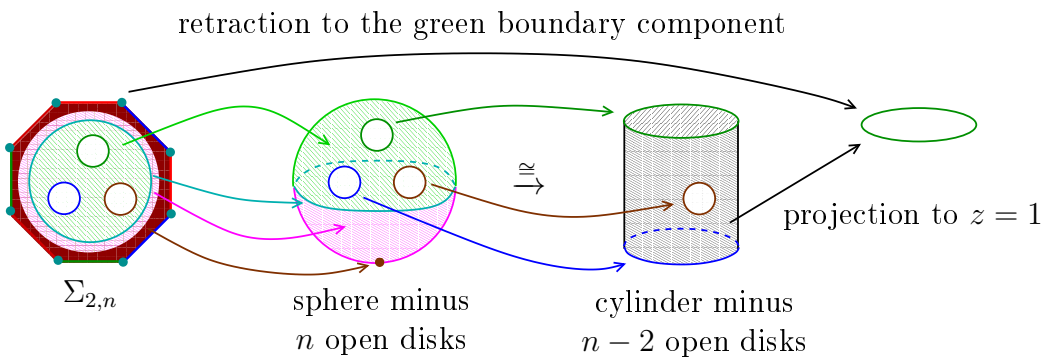
In $\pi_1(\Sigma)$ we have

$$z \;=\; ([x_1, y_1] \cdot \cdots \cdot [x_g, y_g])^{-1} \;\in\; \pi_1(\Sigma).$$

This shows that z lies in the commutator subgroup of $\pi_1(\Sigma)$, i.e. z represents the trivial element in $\pi_1(\Sigma)_{\mathrm{ab}}$. Thus we have shown that the left diagonal map is the zero map. But the bottom horizontal map is an isomorphism of a non-trivial group and the diagram commutes. We have thus obtained a contradiction.

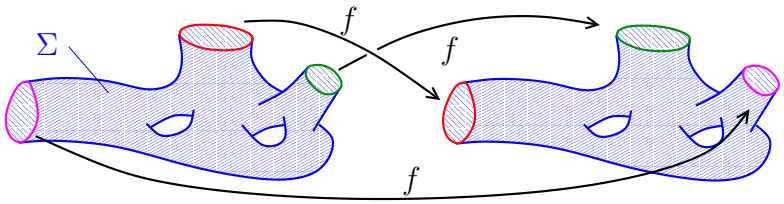
(2) This statement follows immediately from Lemma 55.12 since we can project the wedge of circles onto any circle. Alternatively, a retraction from $\Sigma_{2,n}$ to a boundary component is sketched in the figure below. The general case is treated the same way.[624] ■

[624] The proof is somewhat similar to the proof of Lemma 47.19.



55.5. **Curves on surfaces.** In this section we will use the Surface Classification Theorem 55.4 to draw several hopefully interesting consequences on curves on surfaces. We start out with the following lemma.

Lemma 55.14. (Boundary Curve-Permutation Lemma) Let Σ be a compact connected 2-dimensional smooth manifold. We denote the boundary components of Σ by $C_1, \ldots, C_n$. Given any permutation $\sigma \in S_n$ there exists a diffeomorphism f of Σ with $f(C_i) = C_{\sigma(i)}$, $i = 1, \ldots, n$. Furthermore, if Σ is oriented, then we can choose f to be orientation-preserving.



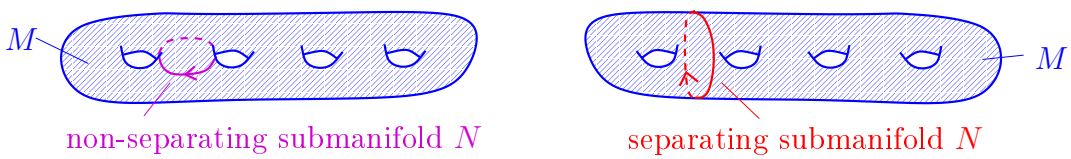
Proof. In the proof we deal with the setting that Σ is orientable. The case that Σ is non-orientable is dealt with in almost the same way. We pick an orientation for Σ and we equip the boundary components $C_1, \ldots, C_n$ with the corresponding orientations. Furthermore we pick orientation-reversing diffeomorphisms $\varphi_1, \ldots, \varphi_n \colon S^1 \to C_i$ and we consider

$$M \;:=\; (\Sigma \sqcup (\overline{B}^2_1 \sqcup \cdots \sqcup \overline{B}^2_n))/\varphi_i(x) \sim x \qquad \text{with } x \in S^1 \text{ and } i \in \{1, \ldots, n\}.$$

It follows from the Codimension-Zero Submanifold Proposition 19.38 that M is a closed oriented connected 2-dimensional smooth manifold with obvious orientation-preserving smooth embeddings $\psi_i \colon \overline{B}^2 \to M$. Now let $\sigma \in S_n$ be a permutation. Note that it follows from the Smooth Ball Embedding Theorem 37.8 that there exists a diffeotopy $F \colon M \times [0,1] \to M$ with $F_0 = \mathrm{id}$ such that for each $i \in \{1, \ldots, n\}$ we have $F_1 \circ \varphi_i = \varphi_{\sigma(i)} \colon \overline{B}^2 \to M$. Note that F_1 is orientation-preserving by the Smooth Isotopy-Orientation Proposition 34.2. The desired diffeomorphism is now given by the restriction of F_1 to Σ. ■

We move on to the following general definition.

Definition. Let M be a smooth manifold and furthermore let $N \subset M$ be a closed submanifold of codimension one. We say N is non-separating if $M \setminus N$ is connected, otherwise we see that N is separating.

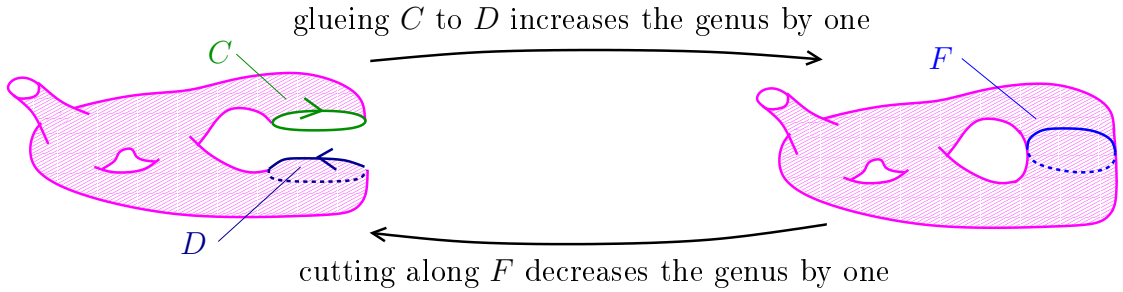


Next we recall that on page **??** we introduced the notion of cutting an orientable smooth manifold along a proper codimension-one submanifold. In Proposition **??** (4) and Proposition **??** we saw that this is operation is the inverse of the operation of gluing two boundary components that we introduced in the Gluing-Smooth Manifolds-Proposition **??**. We will not recall the precise definitions but we hope that the figure below will bring back fond memories.

The following lemma explains how gluing and cutting affects surfaces.

Lemma 55.15. (Surface-Gluing-Cutting Lemma) Let Σ be a compact oriented connected 2-dimensional smooth manifold.

(1) Let C and D be two distinct boundary components of Σ. Gluing C to D via an orientation-reversing diffeomorphism increases the genus by one and it decreases the number of boundary components by two.
(2) Let F be a closed non-separating curve on Σ. Cutting along F decreases the genus by one and it increases the number of boundary components by two. (In particular this implies that the genus of Σ is ≥ 1.)



Sketch of proof.

(1) By the Surface Classification Theorem 55.4 we can assume that $\Sigma = \Sigma_{g,n}$ for some $g \in \mathbb{N}_0$ and some $n \in \mathbb{N}_{\geq 2}$. We denote by M the result of gluing the boundary components C and D via an orientation-reversing diffeomorphism. Note that it follows from the Gluing-Smooth Manifolds-Proposition **??** that M is a compact oriented connected 2-dimensional smooth manifold. In particular it has some genus h. Also note that it follows from the Boundary Curve-Permutation Lemma 55.14 that the diffeomorphism type of M does not depend on the choice of the two boundary components C and D.

It is now elementary, albeit painful, to show clearly that gluing C to D via an orientation-reversing diffeomorphism increases the genus by one. We will deal with a special case in Exercise 55.12. The general case is not much harder. Finally note that evidently the number of boundary components goes down by two.

For readers who are slightly unhappy about the above proof of the statement that the genus goes up by one we also give an alternative approach. Namely, the genus of M can be read off directly from suitable invariants. For example later on we will

use the HNN-Seifert–van Kampen Theorem 57.3 to calculate $\pi_1(M)$ which, together with Proposition 55.7 implies that $h = g + 1$. Alternatively one can use the Euler characteristic, which we will introduce in Chapter 87, together with Lemmas 87.5 and 87.6 to see once again that $h = g + 1$.

(2) Note that if we cut along a non-separating curve, then the resulting 2-dimensional smooth manifold is again connected. Thus we see that Statement (2) is an immediate consequence of Statement (1) together with Proposition **??**. ■

We move on to the following definition.

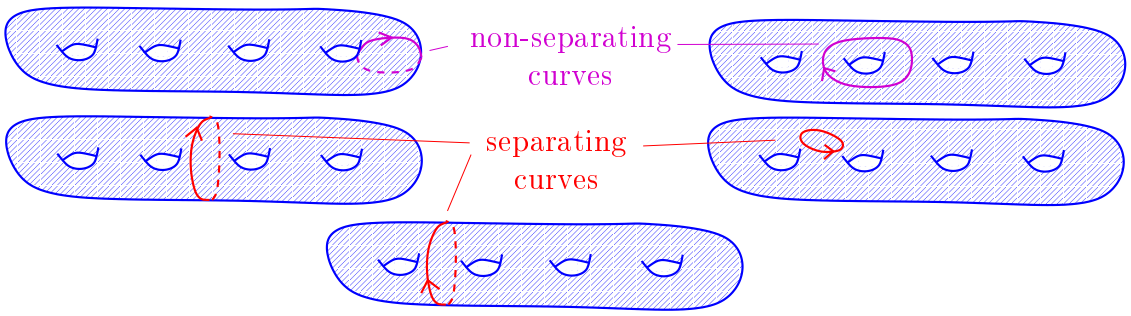
Definition. Let Σ be an oriented 2-dimensional smooth manifold and let C and D be two closed oriented curves in Σ. We say C and D are equivalent if there exists an orientation-preserving self-diffeomorphism f of Σ that restricts to an orientation-preserving diffeomorphism $f\colon C \to D$.

It is clear that if two curves are equivalent, then either both are separating or both are non-separating. Furthermore if they are both separating, then the results of cutting the 2-dimensional smooth manifold along the curves are diffeomorphic. The following theorem gives us the converse to this observation.

Theorem 55.16. (Curves-on-Surfaces-Classification Theorem) Let Σ be a compact oriented connected 2-dimensional smooth manifold and let C and D be two closed oriented curves in $\Sigma \setminus \partial\Sigma$.

(1) If C and D are non-separating, then C and D are equivalent.
(2) Suppose C and D are separating. If Σ cut along C is diffeomorphic to Σ cut along D, then C and D are equivalent.

Example. It follows from the Curves-on-Surfaces-Classification Theorem 55.16 (1) that the two non-separating curves shown in the figure below are actually equivalent. Furthermore it follows easily from the Curves-on-Surfaces-Classification Theorem 55.16 (2) and the future Exercise 87.5 that any separating curve on the surface of genus four is equivalent to one of the separating curves shown in the figure below.



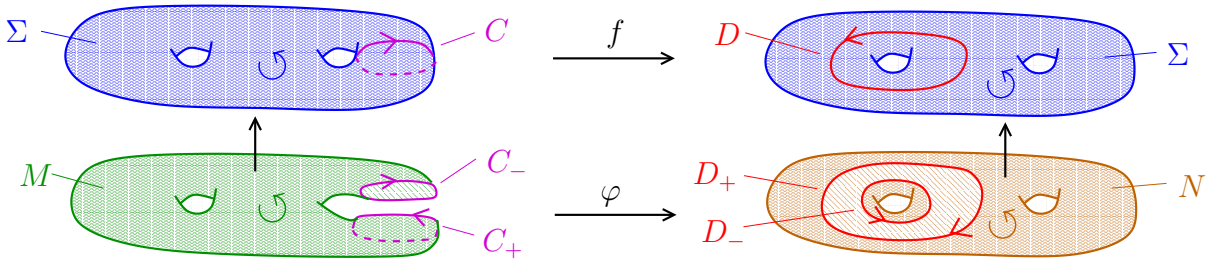
Remark. In Exercise 55.17 we will generalize the Curves-on-Surfaces-Classification Theorem 55.16 to certain pairs of curves.

Sketch of proof. Let Σ be a compact oriented connected 2-dimensional smooth manifold.

(1) Let C and D be two closed oriented non-separating curves on Σ. We denote by M respectively N the smooth manifold obtained from Σ by cutting along C respectively D. We denote by $C_\pm$ and $D_\pm$ the boundary components arising from

C and D. Since C and D are non-separating it follows from the Surface-Gluing-Cutting Lemma 55.15 (2) and the Boundary Curve-Permutation Lemma 55.14 that there exists an orientation-preserving diffeomorphism $\varphi\colon M \to N$ that restricts to orientation-preserving diffeomorphisms $\varphi\colon C_\pm \to D_\pm$. Using Lemma 50.1 and using a collar neighborhood one can arrange that under the obvious identifications $C_- = C_+$ and $D_- = D_+$ the maps $\varphi\colon C_- \to D_-$ and $\varphi\colon C_+ \to D_+$ agree. As in the Gluing-Smooth Manifolds-Proposition **??** (7) we see that $\varphi\colon M \to N$ descends to an orientation-preserving diffeomorphism $\overline{\varphi}\colon M/C_- = C_+ \to N/D_- = D_+$. If we combine this diffeomorphism with the diffeomorphisms $\Sigma \to M/C_- = C_+$ and $\Sigma \to N/D_- = D_+$ from Proposition **??** we obtain the desired diffeomorphism.

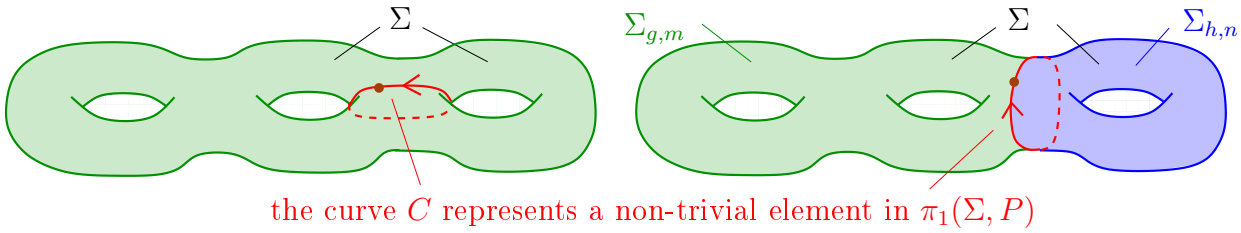
(2) The argument is similar to the proof of Statement (1), except that we do not need to appeal to the Surface-Gluing-Cutting Lemma 55.15 (2), instead we can use our hypothesis directly. We leave it to the reader to fill in the details. ■



In the remainder of this section we will state a few consequences of the Curves-on-Surfaces-Classification Theorem 55.16.

Proposition 55.17. Let Σ be a compact orientable connected 2-dimensional smooth manifold and let C be a closed oriented curve on Σ.

(1) If C is non-separating, then C is not null-homotopic.
(2) If C is separating, and if the two components of Σ cut along C are not disks, then C is not null-homotopic.



Proof. Let Σ be a compact 2-dimensional smooth manifold and let C be a closed oriented curve on Σ.

(1) First assume that C is non-separating. By the Surface Classification Theorem 55.4 we can assume that $\Sigma = \Sigma_{g,n}$ for some $g \in \mathbb{N}_0$ and some $n \in \mathbb{N}_0$. Since C is non-separating we know by the Surface-Gluing-Cutting Lemma 55.15 (2) that $g \geq 1$. Next note that by the Curves-on-Surfaces-Classification Theorem 55.16 we only need to show that there exists a single non-separating curve on $\Sigma_{g,n}$ that is not null-homotopic. By Lemma 55.11 we have a presentation

$$\pi_1(\Sigma_{g,n}, *) \;=\; \langle x_1, y_1, \ldots, x_g, y_g, z_1, \ldots, z_n \mid [x_1, y_1] \cdot \cdots \cdot [x_g, y_g] \cdot z_1 \cdot \cdots \cdot z_n \rangle$$

where each x_i is represented by a non-separating curve. But the x_i represent non-trivial elements in the abelianization of $\pi_1(\Sigma_{g,n})$, in particular the corresponding curves are not null-homotopic.

(2) Now we suppose that C is separating. We cut Σ along C and we obtain two connected surfaces. By the Surface Classification Theorem 55.4 we can assume that the two surfaces are of the form $\Sigma_{g,m}$ and $\Sigma_{h,n}$. By hypothesis neither surface is a disk, so we have $g > 0$ or $m > 1$ and we have $h > 0$ or $n > 1$. To simplify the notation we now assume that $g = 1$ and $h = 1$ and $m = 2$ and $n = 2$. We perform the following calculation:

$$\begin{array}{l} \text{Seifert–van Kampen Theorem 54.4} \qquad \text{by Lemma 55.11, here } C \text{ corresponds to } c_1 \text{ and to } z_1^{-1} \\ \pi_1(\Sigma) \overset{\downarrow}{\cong} \pi_1(\Sigma_{1,2}) *_{\pi_1(C)} \pi_1(\Sigma_{1,2}) \overset{\downarrow}{=} \langle a,b,c_1,c_2 | [a,b]\cdot c_1\cdot c_2\rangle *_{\pi_1(C)} \langle x,y,z_1,z_2 | [x,y]\cdot z_1\cdot z_2\rangle \\ \qquad = \langle a,b,c_1,c_2,x,y,z_1,z_2 \,|\, [a,b]\cdot c_1\cdot c_2, [x,y]\cdot z_1\cdot z_2, c_1 = z_1^{-1}\rangle \\ \qquad\quad \overset{\uparrow}{} \\ \qquad \text{by Lemma 53.23} \end{array}$$

where the curve C corresponds to $c_1 = z_1^{-1}$. We need to show that c_1 represents a non-trivial element in $\pi_1(\Sigma)$. In our present setting we see that c_1 represents a non-trivial element in the abelianization, but this is *not* the case if $m = 1$. So we need to come up with an alternative argument to show that c_1 is non-trivial. Note that it follows from Lemma 53.13 that there exists a homomorphism

$$\varphi\colon \langle a,b,c_1,c_2,x,y,z_1,z_2 \mid [a,b]\cdot c_1\cdot c_2, [x,y]\cdot z_1\cdot z_2, c_1 = z_1^{-1}\rangle \;\to\; \langle u,v\rangle$$

with[625]

$$a \mapsto u, \quad b \mapsto v, \quad c_1 \mapsto [v,u], \quad c_2 \mapsto e, \quad x \mapsto v, \quad y \mapsto u, \quad z_1 \mapsto [u,v], \quad z_2 \mapsto e.$$

Finally note that $\varphi(c) = [u,v]$ is a non-trivial element of the free group $\langle u,v\rangle$. It follows that C is not null-homotopic. ■

Remark. What is the "right" proof for a statement is often a matter of taste. For example an alternative, arguably more conceptual, proof for Proposition 55.17 (1) can be obtained by the combination of the Surface Classification Theorem 55.4 together with the future HNN-Seifert–van Kampen Theorem 57.3, Lemma 55.11 (3) and the HNN-Extension Monomorphism Proposition 57.2. Similarly an alternative proof for Proposition 55.17 (2) can be obtained by the combination of the Surface Classification Theorem 55.4 with Seifert–van Kampen Theorem 54.4, Lemma 55.11 (3) and the Amalgamated Product-Monomorphism Proposition 53.24.

We conclude this section with the following consequence of Proposition 55.17.

Proposition 55.18. Let Σ be a compact orientable 2-dimensional smooth manifold. Furthermore let $f\colon S^1 \to \Sigma$ be a smooth embedding. If f is null-homotopic, then there exists a smooth embedding $F\colon \overline{B}^2 \to \Sigma$ with $F|_{S^1} = f$.

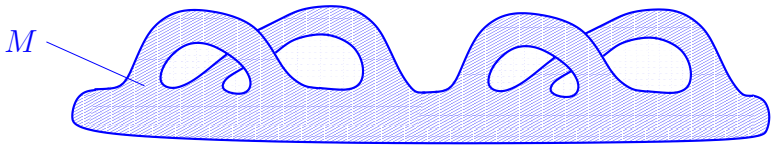
Proof. Let $f\colon S^1 \to \Sigma$ be a smooth embedding that is null-homotopic. By Proposition 55.17 (1) we know that $f(S^1)$ is separating. Let A and B be the two components of Σ cut along $f(S^1)$. By Proposition 55.17 (2) we know that at least one of A and B is

[625] Recall that $[s,t]$ denotes the commutator of s and t, i.e. we have $[s,t] = sts^{-1}t^{-1}$. The verification that the relations are indeed satisfied relies on the trivial but useful observation that $[s,t]^{-1} = [t,s]$.

diffeomorphic to a disk. Let us assume that A is diffeomorphic to $\overline{B}^2$. It follows easily from Lemma 50.1 (3), see also the Diffeomorphisms-of-S^1-Extension Corollary 34.6, that there exists such a diffeomorphism $F\colon \overline{B}^2 \to A$ with $F|_{S^1} = f$. ■

Exercises for Chapter 55.

Exercise 55.1. Determine the genus of the surface M shown in the figure below.

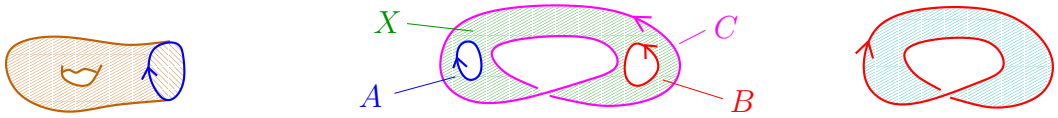


Exercise 55.2. Let $g \in \mathbb{N}$. By the Surface Classification Theorem 55.4 we know that $\Sigma_g \# \mathbb{R}\mathrm{P}^2$ is diffeomorphic to $k \cdot \mathbb{R}\mathrm{P}^2$ for some unique $k \in \mathbb{N}$. Determine k as a function of g.

Exercise 55.3. Let X be the Möbius band minus two disks. We denote by A, B, C the three boundary components of X. We glue a torus minus one open disk and a Möbius band to the boundary components of X. By the Gluing-Smooth Manifolds-Proposition **??** we obtain a compact 2-dimensional smooth manifold.

(a) Do the resulting diffeomorphism types depend on the gluing?
(b) How do the resulting smooth manifolds fit into the classification scheme of the Surface Classification Theorem 55.4?

We refer to the figure below for an illustration.

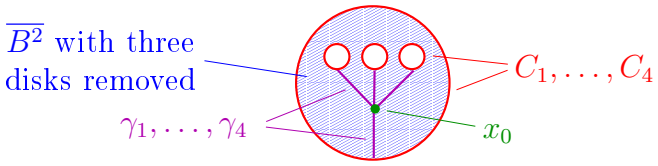


torus minus one disk | Möbius band minus two disks | Möbius band

Exercise 55.4. Recall that in the figure on page 571 we saw that $N_{1,1}$ is a Möbius band. In particular we can view $N_{0,1}$ as a submanifold of $\mathbb{R}^3$.

(a) Is $N_{2,1}$ diffeomorphic to a submanifold of $\mathbb{R}^3$?
(b) Show that $N_{3,1}$ is diffeomorphic to a submanifold of $\mathbb{R}^3$.
(c) For which $k \in \mathbb{N}$ is $N_{k,1}$ diffeomorphic to a submanifold of $\mathbb{R}^3$?

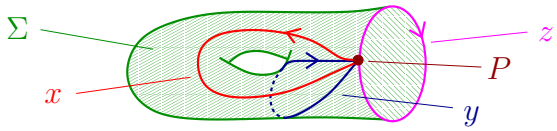
Exercise 55.5. Let F be the sphere minus k open disks. (Alternatively, let F be the disk $\overline{B}^2$ minus $k-1$ open disks.) We denote by $C_1, \dots, C_k$ the boundary components of F. We pick a base point $x_0 \in F$. Furthermore for each $i \in \{1, \dots, k\}$ we pick a path γ_i from a point on C_i to x_0. Finally let $\Gamma \subsetneq \pi_1(F, x_0)$ be a proper normal subgroup. Show that there exists an $i \in \{1, \dots, k\}$ such that C_i, viewed as an element in $\pi_1(F, x_0)$ via the path γ_i, does not lie in Γ.
Hint. You might want to start out with your preferred base point x_0 and your preferred paths $\gamma_1, \dots, \gamma_k$, e.g. as shown in the figure below.



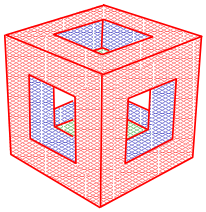
Exercise 55.6. Let Σ be the torus minus one disk. We denote by $P \in \partial\Sigma$ a base point. Furthermore let x, y, z be the curves shown in the figure below. We denote by x, y, z also the elements in $\pi_1(\Sigma, P)$ represented by these curves.

(a) Convince yourself of the fact that $\pi_1(\Sigma, P) = \langle x, y, z \,|\, [x, y] = z \rangle$.
Remark. This statement is closely related to the statement of Lemma 55.11, but it takes some effort to get the orientations of the curves right.

(b) Show that $z \in \pi_1(\Sigma, P)$ is not a proper power, i.e. show that there is no $w \in \pi_1(\Sigma, P)$ such that $z = w^d$ for some $d \in \mathbb{N}_{\geq 2}$.



Exercise 55.7. What is the genus of the surface shown in the figure below.



Exercise 55.8. We say a 2-dimensional smooth manifold M admits a *pair of pants decomposition* if there exist submanifolds $P_1, \dots, P_k$ such that each P_i is a pair of pants and such that for each $i \neq j$ the intersection $P_i \cap P_j$ is a (possibly empty) union of boundary components of P_i and also of P_j.

(a) Determine all $g, n \in \mathbb{N}_0$ such that $\Sigma_{g,n}$ admits a pair of pants decomposition.

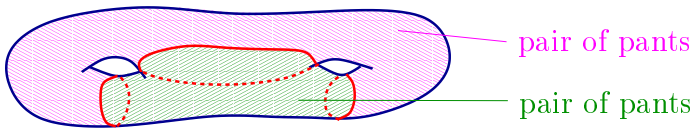
(b) Determine all $g \in \mathbb{N}$ and $n \in \mathbb{N}_0$ for which $N_{k,n}$ admits a pair of pants decomposition.

Exercise 55.9. Let Σ be a compact connected 2-dimensional smooth manifold and let C and D be two closed oriented non-separating curves in $\Sigma \setminus \partial\Sigma$. In the Curves-on-Surfaces-Classification Theorem 55.16 we showed that if Σ is orientable, then there exists a self-diffeomorphism φ of Σ with $\varphi(C) = D$.

(a) Show that this conclusion does not necessarily hold if we drop the hypothesis that Σ is orientable.

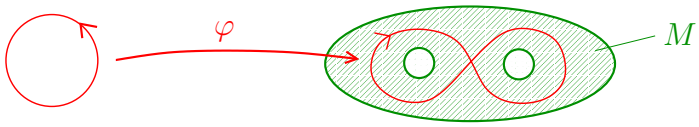
(b) Where in the proof of the Curves-on-Surfaces-Classification Theorem 55.16 did we really make use of the hypothesis that Σ is orientable?

Exercise 55.10. We consider the 2-dimensional smooth manifold M together with the map $\varphi\colon S^1 \to M$ that is shown in the figure below. Show that φ is not homotopic to a



smooth embedding.

Remark. At this stage it is a challenging, but by now means impossible, task to provide a completely rigorous proof.



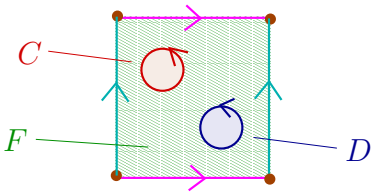
Exercise 55.11.

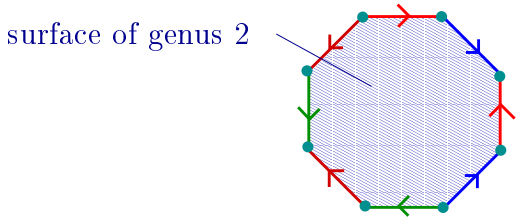
(a) Let $k \in \mathbb{N}$. By Proposition 49.4 we know that the non-orientable surface N_k admits an essentially unique 2-fold covering $\Sigma \to N_k$ that is orientable. What is the genus of Σ?
Hint. First deal with the cases $N_1 \cong \mathbb{R}\mathrm{P}^2$ and $N_2 \cong \mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$. Given $k \in \mathbb{N}_{\geq 2}$ you can write N_k as the connected sum of an orientable smooth manifold with $\mathbb{R}\mathrm{P}^2$ or with $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$.

(b) Let $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$. By Proposition 49.4 we now know that the non-orientable surface $N_{k,m}$ admits an essentially unique 2-fold covering $\Sigma \to N_{k,m}$ that is orientable. What is the genus of Σ and what is the number of boundary components of Σ?

Hint. In Exercise 87.2 we will give a different, perhaps less ad hoc approach to dealing with these two questions.

Exercise 55.12. Let F be the torus minus two open disks. We pick an orientation-reversing diffeomorphism $\varphi\colon C \to D$ between the two boundary components. Show that $F/c \sim \varphi(c)$ is diffeomorphic to the surface of genus two.



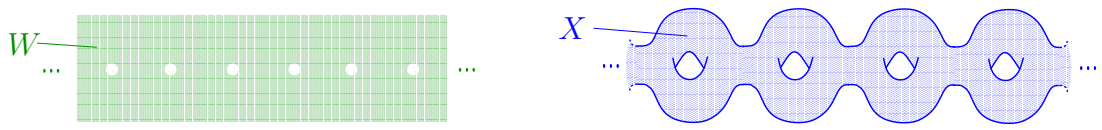


Exercise 55.13. For which values of $g \in \mathbb{N}$ and $n \in \mathbb{N}$ is every boundary component of $N_{g,n}$ a retract of $N_{g,n}$?



Exercise 55.14. Let $W = \mathbb{R}^2 \setminus \{(n, 0) \,|\, n \in \mathbb{Z}\}$ and let X be the surface of infinite genus that is shown in the figure below. Show that W and X are not diffeomorphic.

Hint. What compact submanifolds can occur?
Remark. The same argument also shows that W is not diffeomorphic to the other surfaces of "infinite genus" that are shown in the figure below.
Remark. Later in Exercise 115.5 we will see that W and X are not homeomorphic. With the present knowledge it is at best quite challenging to give a watertight proof that W and X are not homeomorphic.

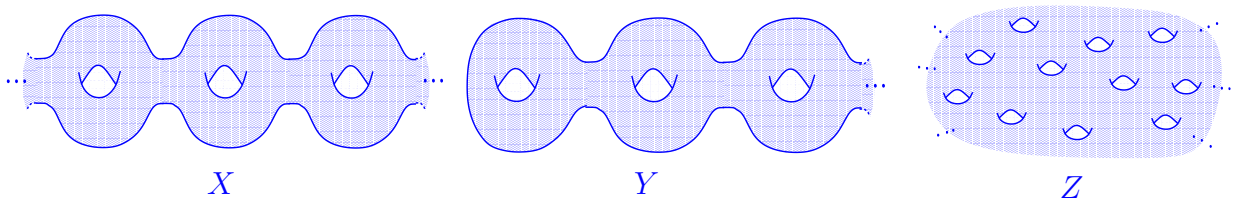


Exercise 55.15. Let X be a topological space. Given a compact subset $K \subset X$ we define $m(K)$ to be the number of non-compact path-components of $X \setminus K$ that are *not* contained in a compact subset of X. As in Exercise 2.27 we refer to

$$\mathrm{End}(X) \;:=\; \max\{m(K) \,|\, K \text{ is a compact subset of } X\} \in \mathbb{N}_0 \cup \{\infty\}$$

as the *number of ends of* X.

(a) Determine the number of ends for each of the three non-compact surfaces shown in the figure below.
(b) Which (if any) of the three "surfaces of infinite genus" shown in the figure below are homeomorphic?
(c) For each $n \in \mathbb{N}$ give an example of a non-compact connected 2-dimensional smooth manifolds such that the number of ends equals n.



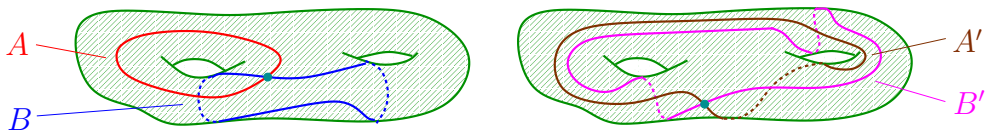
Exercise 55.16. Let $n \in \mathbb{N}$, let M_n be S^2 minus n open disks and let C be a closed curve in $M_n \setminus \partial M_n$.

(a) Show that C is separating.
(b) Let $n = 3$. Show that C either bounds a disk or that it is isotopic to a boundary component of M_n.
(c) Show that the conclusion of (b) is in general not true for $n \geq 4$.

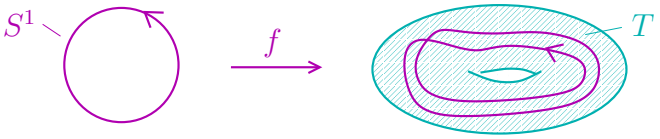
Exercise 55.17. Let Σ be a closed orientable connected 2-dimensional smooth manifold and let $\{A, B\}$ and $\{A', B'\}$ be pairs of closed curves on Σ such that A intersects B transversally in a single point and such that A' intersects B' transversally in a single point.

(a) Show that there exists a diffeomorphism $f\colon \Sigma \to \Sigma$ with $f(A) = A'$ and $f(B) = B'$.
Remark. The statement of this exercise, and of the closely related Curves-on-Surfaces-Classification Theorem 55.16, are proved and extended, in [**FaM2011**, Chapter 1.3]. They refer to statements of this type as the *change of coordinates principle*.

(b) Suppose that all the curves are oriented. Can we find f such that the maps $f\colon A \to A'$ and $f\colon B \to B'$ are both orientation-preserving?



Exercise 55.18. Show that the map $f\colon S^1 \to T = S^1 \times S^1$ that is shown in the figure below is not homotopic to a smooth embedding.
Remark. This settles an issue that arose on page **??**.



Exercise 55.19. We set

$$\begin{array}{rcll} H & := & \{(x,y) \in \mathbb{R}^2 \,|\, y > 0\} & \text{the open half-plane} \\ A & := & \{(x,0) \,|\, x < -1\} & \\ B & := & \{(x,0) \,|\, x > 1\}. & \end{array}$$

We consider the smooth manifold $M = H \cup A \cup B$. A *track* from A to B is a connected oriented proper 1-dimensional smooth manifold T of M such that $\partial T = \{-P\} \cup \{Q\}$ where $P \in A$ and $Q \in B$.

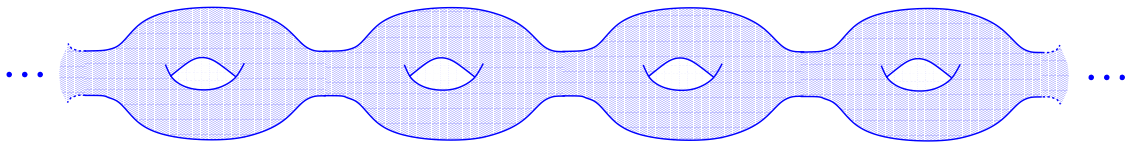
(a) Show that if S and T are two tracks from A to B, then there exists a proper smooth isotopy from S to a track S' and from T to a track T' such that S' and T' are disjoint.

(b) Show that all tracks from A to B are smoothly isotopic.



56. Limits

The goal of this section is to determine the fundamental group of "unbounded spaces", e.g. we would like to determine the fundamental group of the surface of "infinite genus" illustrated in the figure below.[626] One idea might be to view the surface of "infinite genus" as the limit of a sequence given by compact surfaces. But then we need to make precise what we mean by a "limit of topological spaces" and "limit of the corresponding fundamental groups". We will do so in the following sections.



surface of "infinite genus"

56.1. **Preordered and directed sets.** The following definition partly recalls some notions that we introduced on page 89.

Definition. Let X be a set and let $\leq$ be a relation on X.

(1) The relation is called reflexive if for every $x \in X$ we have $x \leq x$.
(2) The relation is called transitive if for every $x, y, z \in X$ we have
$$x \leq y \text{ and } y \leq z \quad \Longrightarrow \quad x \leq z.$$
(3) A reflexive and transitive relation is called a preorder.
(4) A set together with a preorder is called a preordered set.
(5) A directed set is a preordered set such that for any $x, y \in X$ there exists a $z \in X$ with $x \leq z$ and $y \leq z$.

Examples.

(1) Let X be a subset of $\mathbb{R}$ and denote by $\leq$ the usual "less or equal" relation on the real numbers. Then $(X, \leq)$ is a directed set. In particular $(\mathbb{N}, \leq)$ is a directed set.
(2) Given any set X the relation "$\leq$" where $x \leq y$ only if $x = y$ is a preorder, which we call the *trivial preorder* on X.
(3) Let M be a set. We denote by $\mathcal{P}(M)$ the power set of M, i.e. the set of all subsets of M. Then "being a subset", i.e. the relation given by "$\subseteq$", defines a preorder on $\mathcal{P}(M)$. Thus $(\mathcal{P}(M), \subseteq)$ is a preordered set, in fact it is straightforward to see that it is even a directed set.
(4) The set $X = \{0, x, y\}$ with the trivial relations $0 \leq 0, x \leq x, y \leq y$ and the non-trivial relations $0 \leq x$ and $0 \leq y$ is a preordered set. We refer to $(\{0, x, y\}, \leq)$ as the *push-pull set*. Note that this is not a directed set, since there is no $z \in X$ with $x \leq z$ and $y \leq z$.
(5) We consider the set $\mathbb{N}$. The divisibility relation $k|l$ for $k, l \in \mathbb{N}$ is a preorder on $\mathbb{N}$ and with this preorder $\mathbb{N}$ is in fact a directed set.[627]

[626] Actually we also need to make precise what we mean by the surface of "infinite genus".
[627] Indeed, given $k, l \in \mathbb{N}$ we have $k|kl$ and $l|kl$.

56.2. The direct limit of a direct system.

Definition. Let $(I, \leq)$ be a preordered set[628] and let $\mathcal{C}$ be a category. A direct system in the category $\mathcal{C}$ over I is a family of objects $\{X_i\}_{i\in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ij}\colon X_i \to X_j\}$ for all $i, j \in I$ with $i \leq j$ such that the following two conditions are satisfied:

$$\begin{array}{lrcll} (1) & f_{ii} & = & \mathrm{id}_{X_i} & \text{for all } i \in I, \\ (2) & f_{ik} & = & f_{jk} \circ f_{ij} & \text{for all } i, j, k \in I \text{ with } i \leq j \leq k. \end{array}$$

Examples.

(A) We consider the directed set $I = (\mathbb{N}, \leq)$ and the category of groups. For $i \in \mathbb{N}$ we consider the group $\mathbb{Z}^i$ and for $i \leq j$ we denote by

$$\begin{array}{rcl} f_{ij}\colon \mathbb{Z}^i & \to & \mathbb{Z}^j \\ (x_1, \ldots, x_i) & \mapsto & (x_1, \ldots, x_i, 0, \ldots, 0) \end{array}$$

the inclusion map. Then $(\{\mathbb{Z}^i\}_{i\in\mathbb{N}}, \{f_{ij}\}_{i\leq j})$ forms clearly a direct system.

(B) Again we consider $I = (\mathbb{N}, \leq)$ and the category of groups. Given $i \in \mathbb{N}$ we define

$$F_i \;:=\; \langle x_1, \ldots, x_i \rangle \;=\; \text{the free group on } i \text{ generators}$$

and given $i \leq j$ we set

$$\begin{array}{rcl} f_{ij}\colon F_i & \to & F_j \\ x_k & \mapsto & x_k, \text{ for } k = 1, \ldots, i. \end{array}$$

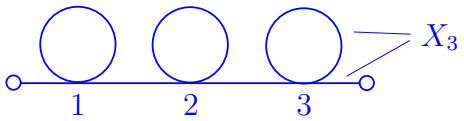
This is clearly a direct system of groups.

(C) Given $r \in \mathbb{N} \cup \{\infty\}$ we set

$$X_r \;:=\; \big(0, r + \tfrac{1}{2}\big) \;\cup\; \bigcup_{j=1}^{r} \underbrace{\big\{z \in \mathbb{C} \,\big|\, \big|z - (j + \tfrac{1}{4}i)\big| = \tfrac{1}{4}\big\}}_{\text{circle of radius } \frac{1}{4} \text{ around } j + \frac{1}{4}i \in \mathbb{C}} \;\subset\; \mathbb{C}.$$

For example, for $r \in \mathbb{N}$ the set X_r is the subset of $\mathbb{C} = \mathbb{R}^2$ that is given by the interval $(0, r + \frac{1}{2})$ with r circles attached. We refer to the figure below for an illustration.

Now we consider the directed set $(\mathbb{N}, \leq)$ and for $r \leq s$ we denote by $f_{rs}\colon X_r \to X_s$ the obvious inclusion map. This is clearly a direct system of topological spaces.



(D) We can generalize example (C) as follows. Let $(I, \leq)$ be a preordered set and let $\{X_i\}_{i\in I}$ be a family of topological spaces such that for any $i \leq j$ we have $X_i \subset X_j$. Given $i \leq j$ we denote by $\iota_{ij}\colon X_i \to X_j$ the inclusion map. Then $(\{X_i\}_{i\in I}, \{\iota_{ij}\}_{i\leq j})$ is a direct system in the category of topological spaces.

(E) Let $(\{0, x, y\}, \leq)$ be the push-pull set and let $\mathcal{C}$ be a category. A direct system in the category $\mathcal{C}$ over the preordered set $(\{0, x, y\}, \leq)$ consists of three objects A_0, A_x, A_y, the identity morphisms $\mathrm{id}_{A_0}, \mathrm{id}_{A_x}, \mathrm{id}_{A_y}$ and morphisms $f_x\colon A_0 \to A_x$ and

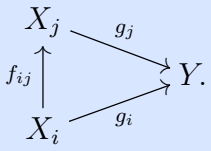
[628] In the literature one often demands that I is a directed set, see e.g. [**Hat2002**, p. 243], but this is not necessary.

$f_y\colon A_0 \to A_y$. We refer to such a system as a *pushout system* and often we arrange the morphisms in a diagram as follows

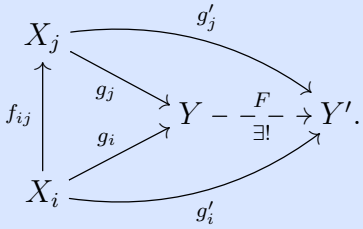
$$\begin{array}{ccc} A_0 & \xrightarrow{f_x} & A_x \\ {\scriptstyle f_y}\downarrow & & \\ A_y. & & \end{array}$$

Definition. Let $(I, \leq)$ be a preordered set and let $\mathcal{C}$ be a category. Suppose we are given a direct system $(\{X_i\}_{i\in I}, \{f_{ij}\}_{i\leq j})$ in the category $\mathcal{C}$. A direct limit[629] of the direct system is an object Y in $\mathcal{C}$ together with morphisms $g_i\colon X_i \to Y, i \in I$ in the category $\mathcal{C}$ such that the following two conditions are satisfied:

(1) For all $i \leq j$ we have $g_j \circ f_{ij} = g_i\colon X_i \to Y$, i.e. for all $i \leq j$ the following diagram commutes



(2) If we are given another object Y' and morphisms $g_i'\colon X_i \to Y', i \in I$ that satisfy (1), then there exists a unique morphism $F\colon Y \to Y'$ such that for all $i \leq j$ the following diagram commutes[630]



The "usual argument" for objects that satisfy a universal property[631] shows that if the direct limit exists, then it is unique in an appropriate sense. We denote the direct limit by $\varinjlim X_i$ or sometimes by $\varinjlim_I X_i$.[632]

Examples. Now we discuss the direct limits of the direct systems we had discussed above.

(A) We consider the above direct system in the category of groups that is given by the free abelian groups $\mathbb{Z}^i$, $i \in \mathbb{N}$, with the obvious inclusion homomorphisms $\mathbb{Z}^i \to \mathbb{Z}^j$ for $i \leq j$. We claim that[633]

$$\varinjlim \mathbb{Z}^i \;=\; \mathbb{Z}^{(\mathbb{N})} \;=\; \{(x_1, x_2, \dots) \mid x_i \in \mathbb{Z} \text{ but only finitely many } x_i\text{'s are non-zero}\},$$

[629] Somewhat confusingly a "direct limit" is often also called a "projective limit" or "colimit of a direct system".

[630] The notation $\exists!$ means that there exists a uniquely defined morphism that makes the diagram commute.

[631] See for instance the discussion on page 1149 for an example of such an argument.

[632] Here we ignore the maps f_{ij} and g_i in the notation. Usually these maps are clear from the context.

[633] Why is the direct limit not given by $\mathbb{Z}^{\mathbb{N}}$, i.e. the group of all infinite sequences?

where the maps $\mathbb{Z}^j \to \varinjlim \mathbb{Z}^i = \mathbb{Z}^{(\mathbb{N})}$, $j \in \mathbb{N}$ are the obvious inclusion maps. Indeed, given a group Y' and homomorphisms $g'_j \colon \mathbb{Z}^j \to Y'$ that satisfy condition (1) we define

$$\begin{array}{rcl} F \colon \varinjlim \mathbb{Z}^i \ = \ \mathbb{Z}^{(\mathbb{N})} & \to & Y' \\ (x_1, \dots, x_j, 0, \dots) & \mapsto & g'_j(x_1, \dots, x_j). \end{array}$$

It is straightforward to see that this definition is well-defined, i.e. independent of the choice of x_j.[634] It is clear that the diagram (2) commutes. Furthermore, since $(x_1, \dots, x_j, 0, \dots) = g_j(x_1, \dots, x_j)$ it is clear that F is the unique homomorphism that makes the diagram (2) commute.

Thus we have now shown that the direct limit is the free abelian group on an infinite countable generating set.

(B) We consider the above direct system in the category of groups that is given by the free groups $F_i := \langle x_1, \dots, x_i \rangle$, $i \in \mathbb{N}$, with the obvious inclusion homomorphisms $F_i \to F_j$ for $i \leq j$. Similar to the proof of (A) one can show that

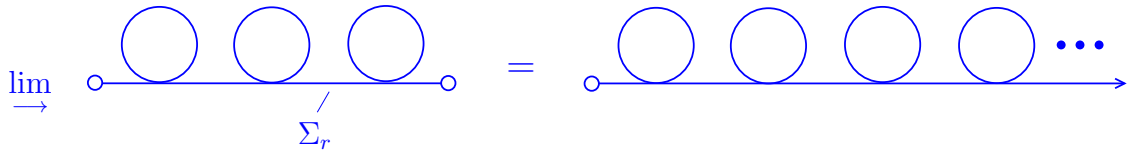
$$\varinjlim F_i \ = \ \langle x_1, x_2, x_3, \dots \rangle \ = \ \text{free group on the generators } x_1, x_2, x_3, \dots$$

where the maps $F_j \to \varinjlim F_i = \langle x_1, x_2, \dots \rangle$, $j \in \mathbb{N}$ are the obvious maps. Put differently, the direct limit is the free group on an infinite countable generating set.

(C) We continue with the direct system $(X_r, r \in \mathbb{N})$ in the category of topological spaces from above. Perhaps not surprisingly the direct limit is the interval $(0, \infty)$ with infinitely many circles attached, i.e.

$$\varinjlim X_r \ = \ X_\infty \ = \ (0, \infty) \ \cup \ \bigcup_{j=1}^{\infty} \left\{ z \in \mathbb{C} \,\middle|\, \left| z - (j + \tfrac{1}{4} i) \right| = \tfrac{1}{4} \right\} \ \subset \ \mathbb{C},$$

where the maps $X_s \to \varinjlim X_r = X_\infty$, $s \in \mathbb{N}$ are the obvious inclusion maps. We leave the verification of this statement to the reader.



(D) Let $(I, \leq)$ be a preordered set and let $\{X_i\}_{i \in I}$ be a family of topological spaces such that for any $i \leq j$ we have $X_i \subset X_j$. In the category of topological spaces we have

$$\varinjlim X_i \ = \ X \ := \ \bigcup_{i \in I} X_i,$$

where the topology on $X = \bigcup_{i \in I} X_i$ is given by the rule that $U \subset X$ is open if and only if $U \cap X_i$ is open for all $i \in I$. Furthermore the maps $g_j \colon X_j \to X = \bigcup_{i \in I} X_i$, $j \in \mathbb{N}$ are the obvious inclusion maps.[635] The verification of this statement is Exercise 56.5.

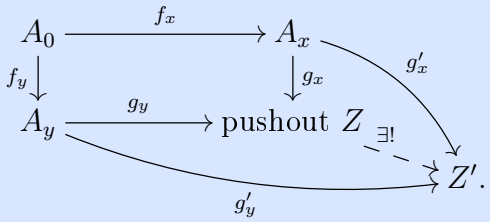
[634] Here we did not demand that $x_j \neq 0$.

[635] Why are the maps g_j continuous, i.e. why are they morphisms in the category of topological spaces.

Definition. If the direct limit of a pushout system

$$\begin{array}{ccc} A_0 & \xrightarrow{f_x} & A_x \\ {\scriptstyle f_y}\downarrow & & \\ A_y & & \end{array}$$

exists, then we refer to it as the pushout of the pushout system. The defining property of the pushout Z can be summarized in the following commutative diagram[636]



Examples.

(1) We first consider the pushout in the category of groups. By Proposition 53.21 the pushout of a pushout system in the category of groups

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & A \\ {\scriptstyle \beta}\downarrow & & \\ B & & \end{array} \qquad \text{is given by} \qquad \begin{array}{ccc} G & \xrightarrow{\quad\alpha\quad} & A \\ {\scriptstyle \beta}\downarrow & & \downarrow \\ B & \longrightarrow & A *_G B = \dfrac{A * B}{\langle\langle \{\alpha(g) \cdot \beta(g)^{-1}\}_{g \in G} \rangle\rangle}. \end{array}$$

(2) Now we consider the pushout in the category of topological spaces. In the Topological Pushout Proposition 5.33 we already showed, in ever so slightly different words, that the pushout of the pushout system of topological spaces

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \\ Z & & \end{array} \qquad \text{is given by} \qquad \begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle g}\downarrow & & \downarrow \\ Z & \longrightarrow & Y \cup_X Z := (Y \sqcup Z)/f(x) \sim g(x). \end{array}$$

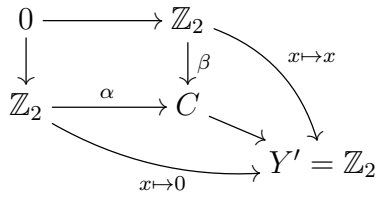
Here we equip $Y \cup_X Z$ with the quotient topology induced from the obvious topology on $Y \sqcup Z$.

(3) The direct limit of a direct system does not always exist. For example, denote by $\mathcal{C}$ the category with two objects, namely the trivial group and the group $\mathbb{Z}_2$, and where the morphisms are the usual group homomorphisms. Then the pushout system

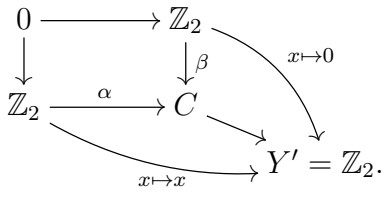
$$\begin{array}{ccc} 0 & \longrightarrow & \mathbb{Z}_2 \\ \downarrow & & \\ \mathbb{Z}_2 & & \end{array} \qquad \begin{array}{c} \text{does not have a direct limit} \\ \text{since there is no commutative} \\ \text{diagram of the form} \end{array} \qquad \begin{array}{ccc} 0 & \longrightarrow & \mathbb{Z}_2 \\ \downarrow & & \downarrow{\scriptstyle \beta} \\ \mathbb{Z}_2 & \xrightarrow{\alpha} & C \end{array}$$

[636] Here we suppress the morphism $g_0 \colon A_0 \to Z$ from the diagram since it is given by $g_x \circ f_x = g_y \circ f_y$.

with $C = 0$ or $C = \mathbb{Z}_2$ such that in the following two diagrams there exists the desired diagonal maps from C to $Y' = \mathbb{Z}_2$:[637]



and



Proposition 56.1. (Direct Limit Existence Proposition) The direct limit of any direct system exists in the following categories:

(0) the category of sets,
(1) the category *Top* of topological spaces,
(2) the category *Group* of groups,
(3) the category *AbGroup* of abelian groups,
(4) the category *Ring* of rings,
(5) the category R-*Mod* of R-modules, where R is a commutative ring.

Sketch of proof. Let $(I, \leq)$ be a preordered set and let $\mathcal{C}$ be one of the six given categories. Furthermore let $(\{X_i\}_{i\in I}, \{f_{ij}\colon X_i \to X_j\}_{i\leq j})$ be a direct system in $\mathcal{C}$.

(0),(1) If we work in the category of sets, then we define the direct limit as the set

$$\varinjlim X_i \ := \ \Big(\bigsqcup_{i\in I} X_i\Big)/\sim \quad \text{where } x \sim f_{ij}(x) \text{ for all } i \leq j,\ x \in X_i.$$

If we work in the category of topological spaces, then $\bigsqcup_{i\in I} X_i$ is equipped with the topology where a subset U is open if and only if $U \cap X_i$ is open for all i.[638]

(2) If we work in the category of groups, then we cannot use the definition of (0) and (1), since the union of groups is not a group. Therefore we define the direct limit via the "smallest group that contains all X_i", i.e. we define it as the group

$$\varinjlim X_i \ := \ \Big(\mathop{*}_{i\in I} X_i\Big)/\langle\langle\{f_{ij}(x)\cdot x^{-1} \mid i \leq j \text{ and } x \in X_i\}\rangle\rangle.$$

(3),(4),(5) If we work in the category of abelian groups, or in the category of rings or in the category of R-modules, then we define[639]

$$\varinjlim X_i \ := \Big(\bigoplus_{i\in I} X_i\Big)/\{f_{ij}(x) - x \mid \ i \leq j \text{ and } x \in X_i\}.$$

Furthermore in all six cases we denote by $g_j\colon X_j \to \varinjlim X_i$, $j \in I$ the obvious maps. It is now straightforward to verify that in all five categories we provided a direct limit. We leave the details to the reader. ∎

[637] Why is there no such diagram with $C = 0$ or $C = \mathbb{Z}_2$ and homomorphisms α and β?

[638] Note that for the above example of the pushout we do *not* get the same topological space (why not?) but we get a topological space that is homeomorphic (why?) to the above pushout. Since we only care about the direct limit up to homeomorphism this makes no difference.

[639] Here we really mod out by the subgroup generated by the differences $f_{ij}(x) - x$.

Remark.

(1) Suppose we are given a direct system $(\{X_i\}_{i\in I}, \{f_{ij}\colon X_i \to X_j\}_{i\leq j})$ in any of the four algebraic categories mentioned in the previous proposition. Now we assume that $(I, \leq)$ is not only a preordered set, but that it is in fact a directed set. Then, as for direct limits of sets and topological spaces, we can also define

$$\varinjlim X_i \;:=\; \Big(\bigsqcup_{i\in I} X_i\Big)/\sim \quad \text{where } x \sim f_{ij}(x) \text{ for all } i \leq j,\ x \in X_i$$

and we denote by $g_j\colon X_j \to \varinjlim X_i$, $j \in I$, the obvious maps. If we work in the category of groups, then $\varinjlim X_i$ is naturally a group as follows: if we are given $a \in X_i$ and $b \in X_j$, by definition of a directed set there exists $k \in I$ with $i \leq k$ and $j \leq k$, and we define

$$[a] \cdot [b] \;:=\; \big[\underbrace{f_{ik}(a)}_{\in X_k} \cdot \underbrace{f_{jk}(b)}_{\in X_k}\big] \;\in\; \varinjlim X_i.$$

It follows easily from the definition of a direct system that this definition of the product structure is independent of the choice of k and that this defines indeed a group structure on $\varinjlim X_i$. Similarly we deal with the case that we work in the other three algebraic categories.

(2) An important special case is given by $I = \mathbb{N}_0$ and where the maps $f_{ij}\colon X_i \to X_j$ are inclusion maps. In this case one can in fact take

$$\varinjlim X_i \;:=\; \bigcup_{i\in\mathbb{N}_0} X_i.$$

The algebraic structure is defined as in the previous remark.

We continue our discussion with the following lemma that gives a particularly simple but commonly used calculation of a direct limit.

Lemma 56.2. (Direct Limit Stabilization Lemma) Let $f_n\colon X_n \to X_{n+1}$ be a sequence of morphisms in any category. Suppose there exists an $N \in \mathbb{N}$ such that all f_n for $n \geq N$ are isomorphisms. Then $\varinjlim X_n$ exists and it is naturally isomorphic to X_N, more precisely, a direct limit is given by $\varinjlim X_n := X_N$ and the maps

$$\begin{aligned} f_{N-1} \circ \cdots \circ f_{n+1} \circ f_n\colon X_n &\to X_N \quad \text{if } n \leq N, \\ (f_n \circ \cdots \circ f_{N+1} \circ f_N)^{-1}\colon X_n &\to X_n \quad \text{if } n > N \end{aligned}$$

define a direct limit.

Proof. We will provide the proof in Exercise 56.4. ∎

56.3. **The direct limit in the topological category.** In this short section we discuss a few results about direct limits that are specific to the category of topological spaces. We start out with the discussion of an important example of direct limits in the topological category.

Lemma 56.3. (Direct Limit-of-Topological Spaces Lemma) Let $(\{X_i\}_{i\in\mathbb{N}_0}, \{f_{ij}\}_{i\leq j})$ be a direct system in the category of topological spaces. We assume that all the maps $f_{ij}\colon X_i \to X_j$ are inclusion maps.

(1) In this case one can in fact take

$$\varinjlim X_i \;:=\; \bigcup_{i\in\mathbb{N}_0} X_i$$

where the topology is given, as in the proof of the Direct Limit Existence Proposition 56.1, by the condition that $U \subset X$ is open if and only if each $U \cap X_i$ is an open subset of X_i.

(2) For each $i \in \mathbb{N}$ the obvious map $X_i \to \varinjlim X_i$ is an embedding.

Proof.

(1) The argument is the same as in the proof of the Direct Limit Existence Proposition 56.1 (1).

(2) Let $i \in \mathbb{N}$. The map $X_i \to \varinjlim X_i$ is evidently injective and continuous. By the Open -Injective Map Lemma 2.16 (1) it remains to show that the map is an open map. Thus let $U_i \subset X_i$ be an open subset. By definition of the subspace topology there exists an open subset $U_{i+1} \subset X_{i+1}$ such that $U_{i+1} \cap X_i = U_i$. We iterate this procedure to obtain subsets $U_i, U_{i+1}, U_{i+2}, \ldots$. We set $U := \bigcup_{j\geq n} U_j$. Evidently $U \cap X_i = U_i$. ■

Finally we conclude this section with two lemmas which show that in the category of topological spaces direct limits commute, under favorable circumstances, with direct products and quotients.

Lemma 56.4. (Direct Limits-and-Products Lemma) Let $(I, \leq)$ be a preordered set and let $(\{X_i\}_{i\in I}, \{f_{ij}\colon X_i \to X_j\}_{i\leq j})$ be a direct system in the category of topological spaces. We denote by $g_i\colon X_i \to \varinjlim X_i$ the canonical maps. Furthermore let Y be a topological space. By definition of the direct limit the maps

$$g_i \times \mathrm{id}_Y\colon X_i \times Y \;\to\; (\varinjlim X_i) \times Y$$

induce a map

$$\varinjlim (X_i \times Y) \;\to\; \big(\varinjlim X_i\big) \times Y.$$

If Y is regionally compact, then this map is a homeomorphism.

Example. Let $f\colon X \to Y$ and $g\colon X \to Z$ be maps between topological spaces. These maps define a pushout system, and taking the product with $[0,1]$ also defines a pushout system. We get the following two diagrams:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i} \\ Z & \xrightarrow{j} & X \cup_Y Z \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X\times[0,1] & \xrightarrow{f\times\mathrm{id}} & Y\times[0,1] \\ {\scriptstyle g\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle I} \\ Z\times[0,1] & \xrightarrow{J} & X\times[0,1]\cup_{Y\times[0,1]} Z\times[0,1] \end{array}$$

$i\times\mathrm{id}$: $Y\times[0,1] \to (X\cup_Y Z)\times[0,1]$; $j\times\mathrm{id}$: $Z\times[0,1] \to (X\cup_Y Z)\times[0,1]$.

By the property of the pushout we have the diagonal map. Since $[0,1]$ is regionally compact we obtain from the Direct Limits-and-Products Lemma 56.4, or more directly from the Product-Quotient Mixing Lemma 13.6, that the diagonal map is a homeomorphism.

Sketch of proof. This lemma follows easily from the explicit description of the direct limit in the category of topological spaces that we gave in the Direct Limit Existence Proposition 56.1 together with the Product-Quotient Mixing Lemma 13.6. ■

Lemma 56.5. Let $(I, \leq)$ be a preordered set and let $(\{X_i\}_{i\in I}, \{f_{ij}\colon X_i \to X_j\}_{i\leq j})$ be a direct system in the category of topological spaces. Let G be a group. Suppose that for each $i \in I$ we are given an action of G on X_i such that given any $i \leq j$ the following diagram commutes:

$$\begin{array}{ccc} G \times X_i & \longrightarrow & X_i \\ \downarrow{\scriptstyle \mathrm{id}_G \times f_{ij}} & & \downarrow{\scriptstyle f_{ij}} \\ G \times X_j & \longrightarrow & X_j. \end{array}$$

It follows almost immediately from this data that G also acts on $\varinjlim X_i$. We denote by $g_i\colon X_i \to \varinjlim X_i$ the canonical maps. By definition of the direct limit the corresponding maps

$$X_i/G \;\to\; (\varinjlim X_i)/G$$

induce a map

$$\varinjlim (X_i/G) \;\to\; \big(\varinjlim X_i\big)/G.$$

This map is a homeomorphism.

Sketch of proof. This lemma follows easily from the explicit description of the direct limit in the category of topological spaces that we gave in the Direct Limit Existence Proposition 56.1. ■

56.4. **Fundamental groups and direct limits.** In this section we want to address the following question:

Question 56.6. Let $(\{X_i\}_{i\in I}, \{f_{ij}\}_{i\leq j})$ be a direct system of topological spaces. Now we can form the direct limit $\varinjlim X_i$ of the topological spaces. The maps f_{ij} also induce homomorphisms on fundamental groups. The question is, when does "taking limits commute with taking fundamental groups", i.e. under what circumstances do we have[640]

$$\pi_1\big(\varinjlim X_i\big) \;=\; \varinjlim \pi_1(X_i)\ ?$$

Example. We have already seen two instances where the equality of the question holds.

(1) Suppose X is a topological space that is the union of two open subsets U and V such that U, V and $U \cap V$ are path-connected. We fix a base point in $U \cap V$. Then

$$\pi_1\left(\varinjlim \begin{array}{c} U\cap V \to U \\ \downarrow \\ V \end{array}\right) \;=\; \pi_1(X) \;=\; \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) \;=\; \varinjlim \begin{array}{c} \pi_1(U\cap V) \to \pi_1(U) \\ \downarrow \\ \pi_1(V) \end{array}$$

see page 1260 — Seifert–van Kampen Theorem 54.1 — see page 1260

[640] As it is this question is not well-phrased, since fundamental groups depend on base points and we ignored them in the above discussion.

(2) Let X be a path-connected topological space and let X_k, $k \in \mathbb{N}$ be a sequence of subsets of X such that the following hold:
   (a) each X_k is open,
   (b) each X_k is simply connected,
   (c) the sequence X_k is nested, i.e. for each k we have $X_k \subset X_{k+1}$,
   (d) we have $\bigcup\limits_{k\in\mathbb{N}} X_k = X$.

   Then for any $w \in X_1$ we have[641]

$$\pi_1\big(\varinjlim X_i, w\big) \underset{\uparrow}{=} \pi_1\Big(\bigcup_{i\in\mathbb{N}} X_i, x_0\Big) \underset{\uparrow}{=} \{e\} = \varinjlim \{e\} = \varinjlim \pi_1(X_i, w).$$

see the Direct Limit-of-Topological Spaces Lemma 56.1 Lemma 52.6

The following proposition generalizes the second example.

Proposition 56.7. (Direct Limit-π_1-Proposition) Let $X_i, i \in \mathbb{N}$ be a sequence of topological spaces such that the following holds:

(1) the sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $X_i \subset X_{i+1}$,

and such that one of the following two conditions holds:

(2) every compact subset of $\varinjlim X_i$ is already contained in one of the X_i, or

(3) each X_i is open in X_{i+1}.

Then for any $w \in X_1$ we have

$$\pi_1\big(\varinjlim X_i, w\big) = \varinjlim \pi_1(X_i, w).$$

Remark. We will see that the proof of the Direct Limit-π_1-Proposition 56.7 is a generalization of the ideas behind the proof of Lemma 52.6.

The following lemma shows that the hypothesis (3) in the Direct Limit-π_1-Proposition 56.7 implies in fact hypothesis (2).

Lemma 56.8. Let $X_i, i \in \mathbb{N}$ be a sequence of topological spaces such that each X_i is open in X_{i+1}. Then every compact subset of $\varinjlim X_i$ is already contained in one of the X_i.

Proof of Lemma 56.8. Let $X_i, i \in \mathbb{N}$ be a nested sequence of topological spaces such that each X_i is open in X_{i+1}.

Claim. For every $i \in \mathbb{N}$ the subset X_i is open in X.

Proof. By the definition of the topology on X we have to show that for each $j \in \mathbb{N}$ the intersection $X_i \cap X_j$ is open in X_j. Since the sequence is nested it follows that for $j \leq i$ we have $X_i \cap X_j = X_j$, in particular $X_i \cap X_j$ is open in X_j. Now we need to show that X_i is open in any X_{i+k} with $k \in \mathbb{N}_0$. We do so by induction on k. For $k = 0$ the statement is clear. Suppose we already know that X_i is open in X_{i+k}. Then X_i is an open subset in

[641] Recall that by the discussion on page 1257 the topological spaces $\{X_i\}_{i\in\mathbb{N}}$ together with the inclusion maps form a direct system in the category of topological spaces. The direct limit is given by the union of all X_i's, where the union is given a suitable topology. Furthermore note that the inclusion induced maps $\pi_1(X_i, w) \to \pi_1(X_j, w)$ turn the groups $\{\pi_1(X_i, w)\}_{i\in\mathbb{N}}$ into a direct system in the category of groups.

X_{i+k} and X_{i+k} is in turn an open subset of X_{i+k+1} by our hypothesis. It follows that X_i is open in X_{i+k+1}.[642] ⊞
Now let $K \subset \varinjlim X_i$ be a compact subset. From the claim we know that each X_i is open in $X = \varinjlim X_i$ we obtain immediately from Lemma 2.12 and the hypothesis that the X_i's are nested that there exists a $k \in \mathbb{N}$ such that $K \subset X_i$. ■

The proof of the Direct Limit-π_1-Proposition 56.7 also relies on the following lemma that we will use implicitly and explicitly on numerous occasions.

Lemma 56.9. Let (X, x_0) be a pointed topological space.

(1) Given any $g \in \pi_1(X, x_0)$ there exists a compact subset K of X with $x_0 \in K$ such that g lies in the image of the inclusion induced map $\pi_1(K, x_0) \to \pi_1(X, x_0)$.
(2) Let Y be a subset of X with x_0. If $g \in \pi_1(Y, x_0)$ lies in the kernel of the inclusion induced map $\pi_1(Y, x_0) \to \pi_1(X, x_0)$, then there exists a compact subset K of X such that g lies in the kernel of the inclusion induced map $\pi_1(Y, x_0) \to \pi_1(Y \cup K, x_0)$.

Proof.

(1) Let $g \in \pi_1(X, x_0)$. We represent g by a loop $\gamma\colon [0,1] \to X$. By the Compact Image Lemma 2.13 the image $\gamma([0,1])$ is compact. Thus $K := \gamma([0,1])$ has the desired property.
(2) Now suppose that we are given $g = [\gamma] \in \pi_1(Y, x_0)$ that lies in the kernel of the inclusion induced map $\pi_1(Y, x_0) \to \pi_1(X, x_0)$. By definition this means that there exists a path-homotopy $H\colon [0,1] \times [0,1] \to X$ from the path γ to the constant path. We appeal again to the Compact Image Lemma 2.13 to deduce that $K := H([0,1] \times [0,1])$ is a compact subset of X. Evidently K has the desired property. ■



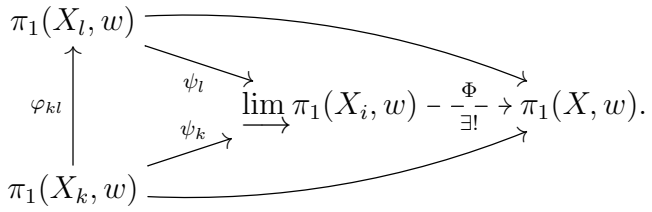
Now we can provide the proof of the Direct Limit-π_1-Proposition 56.7.

Proof of the Direct Limit-π_1-Proposition 56.7. By Lemma 56.8 we only have to prove the proposition under the hypothesis (2). Thus let $X_i, i \in \mathbb{N}$ be a nested sequence of topological spaces such that every compact subset of $\varinjlim X_i$ is already contained in one of the X_i. Let $w \in X_1$. We write $X = \bigcup_{i \in I} X_i = \varinjlim X_i$.

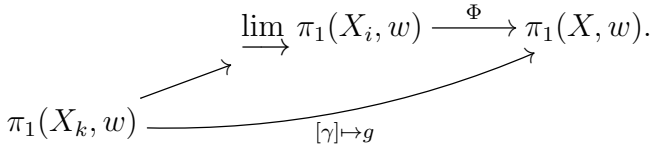
Given $k \leq l$ we denote by φ_{kl} the inclusion induced map $\pi_1(X_k, w) \to \pi_1(X_l, w)$. We denote by $\Phi\colon \varinjlim \pi_1(X_i, w) \to \pi_1(X, w)$ the unique homomorphism that makes the

[642] Here we use the following general fact: if U is an open subset in V and V is an open subset of W, then U is an open subset of W. Why is that true?

following diagram commute for any $k \leq l$:



We need to show that Φ is an isomorphism. First we show that the map Φ is surjective. Let $g \in \pi_1(X, w)$. By Lemma 56.9 (1) and our hypothesis (2) there exists a $k \in \mathbb{N}$ such that g lies in the image of the inclusion induced map $\pi_1(X_k, w) \to \pi_1(X, w)$. We consider the following commutative diagram

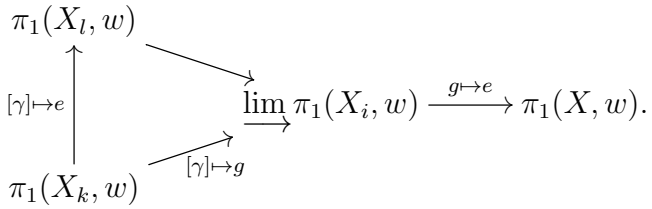


By our choice of k we see that g lies in the image of the curved map, hence it also lies in the image of Φ. This concludes the proof that Φ is surjective.

Now we show that the map Φ is injective. So let $g \in \varinjlim \pi_1(X_i, w)$. Since $(\mathbb{N}, \leq)$ is a directed set we can use the description of the direct limit from page 1262, i.e. we can write

$$\varinjlim \pi_1(X_i, w) \;=\; \Big(\bigsqcup_{i \in \mathbb{N}} \pi_1(X_i, w) \Big) / \sim \quad \text{where } g \sim \varphi_{kl}(g) \text{ for all } k \leq l,\ g \in \pi_1(X_k, w).$$

In particular we see that there exists a k such that g lies in the image of the map $\psi_k \colon \pi_1(X_k, w) \to \varinjlim \pi_1(X_k, w)$. It follows from Lemma 56.9 (1) and our hypothesis (2) that there exists an $l \geq k$ such that $\varphi_{kl}([\gamma])$ is trivial in $\pi_1(X_l, w)$. We obtain the following diagram



It follows immediately that g itself is trivial. This concludes the proof that Φ is injective. ■

Examples.

(1) As on page 1257 we denote by X_∞ the interval $(0,\infty)$ with infinitely many circles attached. We pick the base point $w = \frac{1}{4}$. We then have [643]

$$\pi_1\big(X_\infty, w\big) \underset{\uparrow}{=} \pi_1\big(\varinjlim X_r, w\big) \underset{\uparrow}{=} \varinjlim \pi_1\big(X_r, w\big) \underset{\uparrow}{=} \varinjlim \langle x_1, \dots, x_r\rangle \underset{\uparrow}{=} \langle x_1, x_2, \dots\rangle.$$

see page 1259 — Direct Limit-π_1-Proposition 56.7 — Proposition 52.5 — see page 1259

Furthermore the proof of the Fundamental Group-of-Graph Proposition 52.5 and the above argument show that the generators $x_1, x_2, \dots$ of $\pi_1(X_\infty)$ can be chosen such that x_n, $n \in \mathbb{N}$, corresponds to a loop which goes "once around the n-th circle".

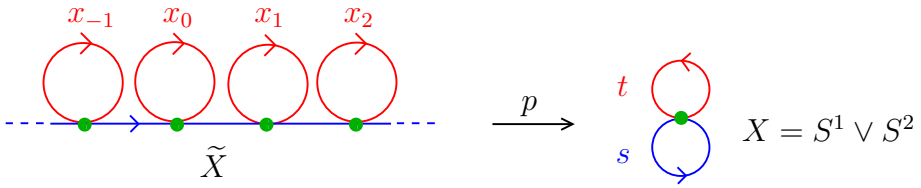
(2) Now we consider

$$\widetilde{X} \;:=\; \mathbb{R} \;\cup\; \bigcup_{j\in\mathbb{Z}} \underbrace{\big\{z \in \mathbb{C} \,\big|\, \big|z - (j + \tfrac{1}{4}i)\big| = \tfrac{1}{4}\big\}}_{\text{circle of radius } \frac{1}{4} \text{ around } j+\frac{1}{4}i \in \mathbb{C}} \;\subset\; \mathbb{C}.$$

Put differently, $\widetilde{X}$ is the real axis with infinitely many circles attached. The same argument as in (1) shows that $\pi_1(\widetilde{X})$ is the free group on infinitely many generators x_n, $n \in \mathbb{Z}$. In the figure below we illustrate a covering map $p\colon \widetilde{X} \to X = S^1 \vee S^1$. By Corollary 48.14 the map $\pi_1(\widetilde{X}) \to \pi_1(X)$ is a monomorphism. We had just seen that $\pi_1(\widetilde{X})$ is a free group on infinitely generators and by the discussion on page 1173 we know that $\pi_1(X)$ is a free group on two generators. In fact, with appropriate orientations and conventions this shows that the homomorphism

$$\begin{array}{rcl} \pi_1(\widetilde{X}) = \langle \dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\rangle & \to & \langle s, t\rangle = \pi_1(X) \\ x_n & \mapsto & s^n t s^{-n} \end{array}$$

is a monomorphism. In particular we see that the free group $\langle s, t\rangle$ contains a subgroup that is "much larger" than the original free group itself.



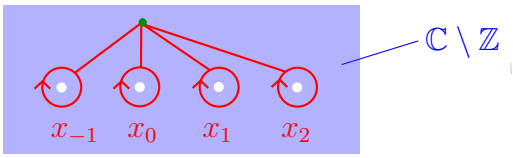
(3) An argument similar to (1) shows that $\pi_1(\mathbb{C}\backslash\mathbb{Z})$ is a free group generated by $x_n, n \in \mathbb{Z}$ where x_n is a loop which goes "once around the n-th hole". We leave the task of providing the details to the reader. We refer to the figure below for an illustration.

---

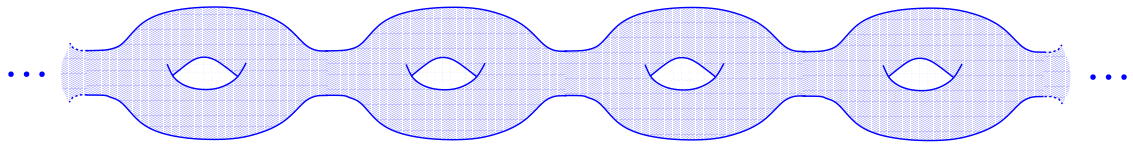
[643] The third equality follows from the Fundamental Group-of-Graph Proposition 52.5, to be more precise, it follows from the fact that for any $r \le s$ the inclusion map $\iota_{rs}\colon X_r \to X_s$ and the isomorphisms of the Fundamental Group-of-Graph Proposition 52.5 give rise to a commutative diagram

$$\begin{array}{ccc} \pi_1(X_r, w) & \xrightarrow{\cong} & \langle x_1, \dots, x_r\rangle \\ {\scriptstyle (\iota_{rs})_*}\downarrow & & \downarrow{\scriptstyle x_i \mapsto x_i} \\ \pi_1(X_s, w) & \xrightarrow{\cong} & \langle x_1, \dots, x_s\rangle. \end{array}$$

The fact that this diagram commutes follows easily from the proof of the Fundamental Group-of-Graph Proposition 52.5. We leave the details to the reader.



56.5. **The surface of "infinite genus".** In this section we return to the initial goal of this chapter, namely we want to determine the fundamental group of the surface of "infinite genus" that we illustrate in the figure below. Before we can calculate its fundamental group we have to give a precise description of the surface of infinite genus.



surface of "infinite genus"

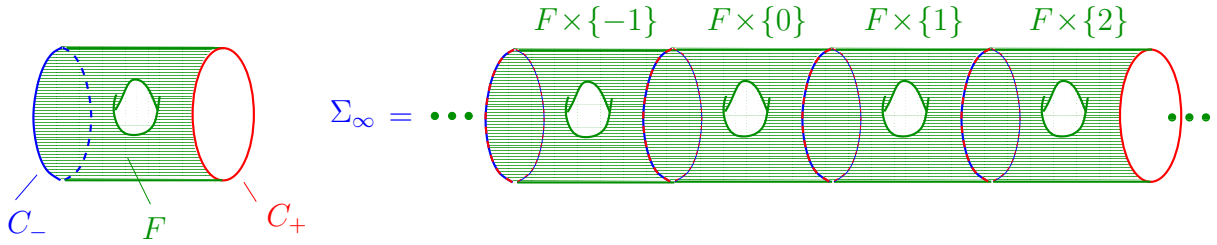
Definition. We write

$$F \;:=\; \text{the torus minus two open disks.}$$

We label the two boundary components by C_- and C_+. We pick orientation-preserving diffeomorphisms $\varphi_\pm \colon S^1 \to C_\pm$. We define

$$\Sigma_\infty \;:=\; \Big(\bigcup_{i\in\mathbb{Z}} (F \times \{i\})\Big)\Big/ \sim \qquad \text{where } (\varphi_+(x), i) \sim (\varphi_-(x), i+1) \text{ for } x \in S^1 \text{ and } i \in \mathbb{Z}.$$

We refer to Σ_∞ as the surface of infinite genus. It should be clear that this definition gives a rigorous description of the surface that we sketched in the figure on page 1269.[644]



Lemma 56.10.

(1) The surface Σ_∞ is a 2-dimensional orientable smooth manifold.

(2) Given $k \in \mathbb{N}$ we write

$$X_k \;:=\; \Big(\bigcup_{i=-k+1}^{k} (F \times \{i\})\Big)\Big/ \sim \qquad \begin{array}{l}\text{where } (\varphi_+(x), i) \sim (\varphi_-(x), i+1) \text{ for } x \in S^1 \\ \text{and } i \in \{-k+1, \ldots, k-1\}.\end{array}$$

The obvious maps $X_k \to \Sigma_\infty$ define a homeomorphism

$$\varinjlim X_i \;\xrightarrow{\cong}\; \Sigma_\infty.$$

Sketch of proof.

[644] One might be tempted to say that the surface of infinite genus is the connected sum of infinitely many tori. But it is not clear that "connected sum of infinitely many tori" is actually a well-defined concept.

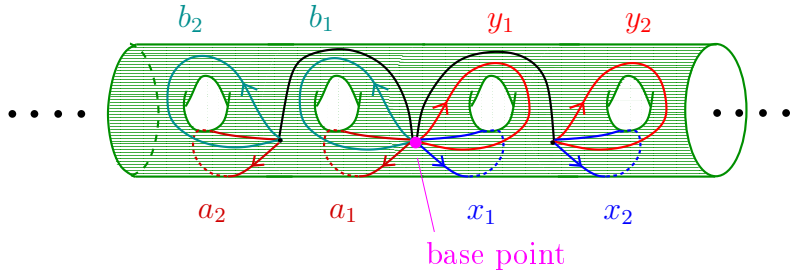
(1) This statement follows fairly easily from the Gluing-Smooth Manifolds-Proposition **??**.
(2) It follows almost immediately from the Compact-Hausdorff Proposition 2.17 (2) that for each $k \in \mathbb{Z}$ the two obvious maps $Y_k := F \times \{k\} \to \varinjlim X_i$ and $F \times \{k\} \to \Sigma_\infty$ are embeddings. Thus we can view each Y_k as a subset of $\varinjlim X_i$ and as a subset of Σ_∞. By unraveling the definitions of the topology one sees that in both cases a subset U is open if and only if the intersection with each Y_k is open. It now follows easily that the map $\varinjlim X_i \to \Sigma_\infty$ is a homeomorphism.

In both cases we leave it to the reader to fill in all the details. ■

In the following lemma we compute the fundamental group of the surface of infinite genus.

Lemma 56.11. The fundamental group of the surface of infinite genus is the free group on infinitely many generators.

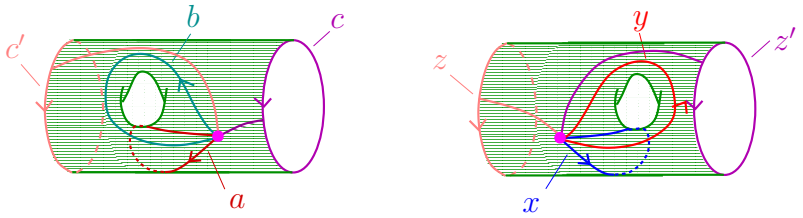
Remark. In Proposition 64.5 we will see that the fundamental group of any non-compact orientable 2-dimensional smooth manifold is a free group.



Proof. We continue with the notation from Lemma 56.10. By Lemma 56.10 (2) it suffices to show that the fundamental group of $\varinjlim X_i$ is a free group on infinitely many generators. By Lemma 55.11 we have[645]

$$\pi_1(F) \;=\; \langle a, b, c, c' \mid c = [a, b] \cdot c' \rangle \;=\; \langle x, y, z, z' \mid z = [x, y] \cdot z' \rangle$$

where the generators z, z' and c, c' correspond to the two boundary components. An induc-



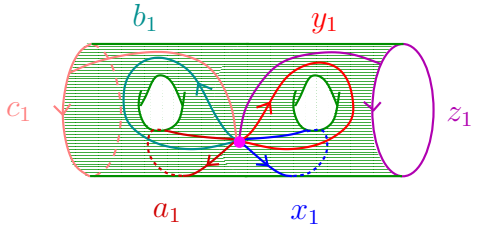
tion argument, using the Seifert–van Kampen Theorem 54.4 for smooth manifolds together with Lemma 53.23 and the above presentations of $\pi_1(F)$ shows that for any $l \geq k$ we have

[645]This is indeed a consequence of Lemma 55.11, the only change is that we reversed the orientation of one of the boundary curves.

a commutative diagram

$$\begin{array}{ccc}
\pi_1(X_k) & \xrightarrow{\cong} & \left\langle \begin{matrix} x_1,y_1,\ldots,x_k,y_k \\ a_1,b_1,\ldots,a_k,b_k \end{matrix} z_k,c_k \;\middle|\; \prod_{j=1}^{k}[a_j,b_j]c_k = \prod_{j=1}^{k}[x_j,y_j]z_k \right\rangle \\
\downarrow & & \downarrow {\scriptstyle c_k\mapsto \prod_{j=k+1}^{l}[a_j,b_j]c_l,\ z_k\mapsto \prod_{j=k+1}^{l}[x_j,y_j]z_l} \\
\pi_1(X_l) & \xrightarrow{\cong} & \left\langle \begin{matrix} x_1,y_1,\ldots,x_l,y_l \\ a_1,b_1,\ldots,a_l,b_l \end{matrix} z_l,c_l \;\middle|\; \prod_{j=1}^{l}[a_j,b_j]c_l = \prod_{j=1}^{l}[x_j,y_j]z_l \right\rangle
\end{array}$$

where the left vertical map is induced by the inclusion $X_k \to X_l$. We leave the details to the energetic reader. For any $l \geq k$ we also have the following commutative diagram of



homomorphisms:

$$\begin{array}{ccc}
\left\langle \begin{matrix} x_1,y_1,\ldots,x_k,y_k \\ a_1,b_1,\ldots,a_k,b_k \end{matrix} p \right\rangle & \xrightarrow[p\mapsto \prod_{j=1}^{k}[a_j,b_j]c_k]{\cong} & \left\langle \begin{matrix} x_1,y_1,\ldots,x_k,y_k \\ a_1,b_1,\ldots,a_k,b_k \end{matrix} z_k,c_k \;\middle|\; \prod_{j=1}^{k}[a_j,b_j]c_k = \prod_{j=1}^{k}[x_j,y_j]z_k \right\rangle \\
\downarrow & & \downarrow {\scriptstyle c_k\mapsto \prod_{j=k+1}^{l}[a_j,b_j]c_l,\ z_k\mapsto \prod_{j=k+1}^{l}[x_j,y_j]z_l} \\
\left\langle \begin{matrix} x_1,y_1,\ldots,x_l,y_l \\ a_1,b_1,\ldots,a_l,b_l \end{matrix} p \right\rangle & \xrightarrow[p\mapsto \prod_{j=1}^{l}[a_j,b_j]c_l]{\cong} & \left\langle \begin{matrix} x_1,y_1,\ldots,x_l,y_l \\ a_1,b_1,\ldots,a_l,b_l \end{matrix} z_l,c_l \;\middle|\; \prod_{j=1}^{l}[a_j,b_j]c_l = \prod_{j=1}^{l}[x_j,y_j]z_l \right\rangle
\end{array}$$

The horizontal homomorphisms are in fact isomorphisms, this is for example a consequence of the Tietze transformations from page 1196.[646]

We denote by Σ_∞ the surface of infinite genus. Furthermore given any $i \in \mathbb{N}_0$ we write $U_i = X_i \setminus \partial X_i$. Note that each U_i is in fact an open subset of Σ_∞. We have

$$\pi_1(\Sigma_\infty) \underset{\text{since } \Sigma_\infty = \bigcup_{i\in\mathbb{N}_0} U_i}{=} \pi_1\big(\varinjlim U_i\big) \overset{\text{Direct Limit-}\pi_1\text{-Proposition 56.7}}{=} \varinjlim \pi_1(U_i) \underset{\text{since } \pi_1(U_i)\xrightarrow{\cong}\pi_1(X_i)}{=} \varinjlim \pi_1(X_i) \overset{\text{by the above two commutative diagrams}}{=} \varinjlim \left\langle \begin{matrix} x_1,y_1,..,x_i,y_i \\ a_1,b_1,..,a_i,b_i \end{matrix} p \right\rangle \underset{\text{see example on page 1259}}{=} \left\langle \begin{matrix} x_1,y_1,.. \\ a_1,b_1,.. \end{matrix} p \right\rangle.$$

[646] Indeed, we write $X = \{x_1,y_1,\ldots,x_m,y_m,a_1,b_1,\ldots,a_m,b_m\}$, $u = \prod_{j=1}^{m}[a_j,b_j]$, $v = \prod_{j=1}^{m}[x_j,y_j]$, $c = c_m$ and $z = z_m$, then

$$\langle X,z,c \mid uc = vz\rangle = \langle X,z,c \mid z = v^{-1}uc\rangle \xleftarrow{\cong} \langle X,c\rangle \xrightarrow{\cong} \langle X,c,p \mid p = uc\rangle = \langle X,c,p \mid X,c = u^{-1}p\rangle \xleftarrow{\cong} \langle X,p\rangle$$

where all three isomorphisms are the isomorphisms of the Tietze transformation (2). It is straightforward to see that the concatenation equals for $m = k,l$ precisely the horizontal maps in the diagram.

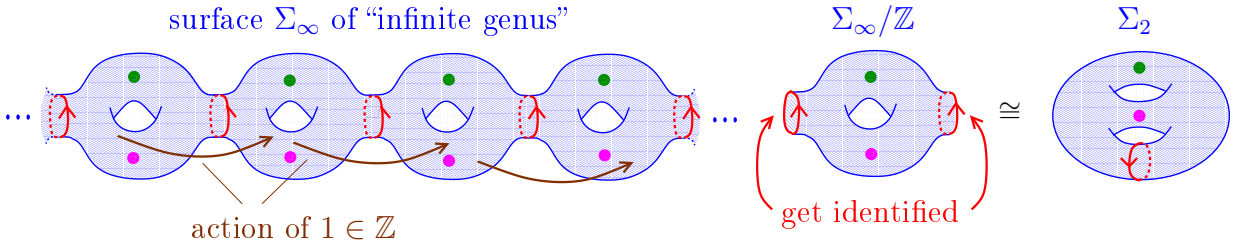
This concludes the proof of the lemma. ■

In the final part of this section we will see that we can actually use Lemma 56.11 to draw an interesting conclusion on fundamental groups of surfaces of *finite* genus. First we have the following lemma.

Lemma 56.12. Let $g \geq 2$. We denote by Σ_∞ the surface of infinite genus and we denote by Σ_g the surface of genus g.

(1) There exists an action of $\mathbb{Z}$ on Σ_∞ that is discrete and continuous and such that the quotient $\Sigma_\infty/\mathbb{Z}$ is homeomorphic to Σ_g.

(2) There exists a covering $p\colon \Sigma_\infty \to \Sigma_g$ of infinite degree.



Sketch of proof.

(1) We continue with the notation from page 1269 and Lemma 56.10. Let $g \in \mathbb{N}_{\geq 2}$. As before we consider

$$\Sigma_\infty := \Big(\bigcup_{i\in\mathbb{Z}} (F \times \{i\}) \Big) / \sim \quad \text{where } (\varphi_+(x), i) \sim (\varphi_-(x), i+1) \text{ for } x \in S^1 \text{ and } i \in \mathbb{Z}.$$

Given $n \in \mathbb{Z}$ and $[(x,k)] \in \Sigma_\infty$ we define $n \cdot [(x,k)] := [(x, k + (g-1)\cdot n)]$. It follows from Exercise 56.14[647] that this action is continuous and proper. Furthermore we obtain from Exercise 56.14 that the quotient $\Sigma_\infty/\mathbb{Z}$ is homeomorphic to the topological space that is given by identifying the two boundary components of

$$\Big(\bigcup_{i=0}^{g-1} (F \times \{i\}) \Big) / \sim \qquad \begin{array}{c} \text{where } (\varphi_+(x), i) \sim (\varphi_-(x), i+1) \text{ for } x \in S^1 \\ \text{and } i \in \{0, \dots, g-1\}. \end{array}$$

The experienced reader will surely be convinced that this topological space is homeomorphic to the surface of genus g. We leave it to the scrupulous reader to fill in the details.

(2) This statement follows immediately from (1) together with the Actions-Covering Proposition 48.9. ■

Using the results of this section we can prove the following proposition.

[647] It follows from the Smooth Collar Neighborhood Theorem 28.3 together with the Basics-of-Collars Lemma 28.4 that there exist open disjoint neighborhoods of the two boundary components of F, thus the extra hypothesis for Exercise 56.14 (c) is in fact satisfied.

Proposition 56.13. Let $g \geq 2$. We consider the automorphism

$$\varphi\colon \underbrace{\langle \{x_{1,i}, y_{1,i}, \ldots, x_{g-1,i}, y_{g-1,i}\}_{i\in\mathbb{Z}} \rangle}_{\text{free group of infinite rank}} \to \underbrace{\langle \{x_{1,i}, y_{1,i}, \ldots, x_{g-1,i}, y_{g-1,i}\}_{i\in\mathbb{Z}} \rangle}_{\text{same free group of infinite rank}}$$

that is induced by

$$\begin{aligned} x_{j,i} &\mapsto x_{j,i+1} \\ y_{j,i} &\mapsto y_{j,i+1}. \end{aligned}$$

There exists an isomorphism

$$\pi_1(\Sigma_g) \cong \langle \{x_{1,i}, y_{1,i}, \ldots, x_{g-1,i}, y_{g-1,i}\}_{i\in\mathbb{Z}} \rangle \rtimes_\varphi \mathbb{Z}.$$

Sketch of proof. We consider the surface Σ_∞ of infinite genus together with the $\mathbb{Z}$-action that is provided by Lemma 56.12 (1). By the Actions-Covering Proposition 48.9 (2) we know that the projection $p\colon \Sigma_\infty \to \Sigma_\infty/\mathbb{Z} = \Sigma_g$ is a covering.

We pick a base point $a_0 \in \Sigma_g$. Next let $\widetilde{a}_0 \in \Sigma_\infty$ be a point with $p(\widetilde{a}_0) = a_0$. It follows from Corollary 48.14 that $p_*\colon \pi_1(\Sigma_\infty, \widetilde{a}_0) \to \pi_1(\Sigma_g, a_0)$ is a monomorphism. We will use this monomorphism to view $\pi_1(\Sigma_\infty, \widetilde{a}_0)$ as a subgroup of $\pi_1(\Sigma_g, a_0)$. We pick a path $\widetilde{\mu} \in \Sigma_\infty$ from $a_0 = 0 \cdot a_0$ to $1 \cdot a_0$. We write $\mu := p \circ \widetilde{\mu}$ and we consider the isomorphism

$$\begin{aligned} \psi\colon \pi_1(\Sigma_\infty, \widetilde{a}_0) &\to \pi_1(\Sigma_\infty, \widetilde{a}_0) \\ g &\mapsto [\mu] \cdot g \cdot [\mu]^{-1}. \end{aligned}$$

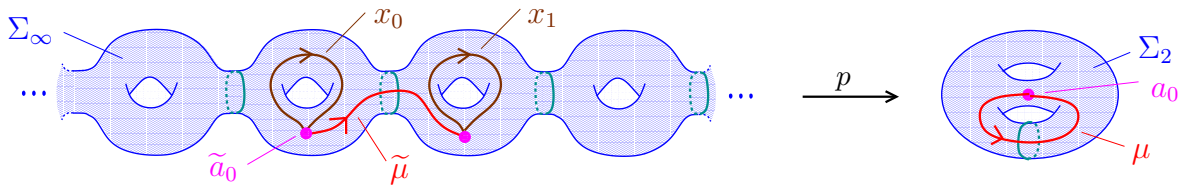
Now we have the following isomorphisms:

by Lemma 56.11 we have an explicit isomorphism $\pi_1(\Sigma_\infty, \widetilde{a}_0) \cong \langle \{x_{1,i}, y_{1,i}, \ldots, x_{g-1,i}, y_{g-1,i}\}_{i\in\mathbb{Z}} \rangle$
we leave it to the reader to verify that ψ and φ match up

$$\pi_1(\Sigma_g, a_0) \underset{\uparrow}{\overset{\cong}{\to}} \pi_1(\Sigma_\infty, \widetilde{a}_0) \rtimes_\psi \mathbb{Z} \underset{\cong}{\overset{\downarrow}{\to}} \langle \{x_{1,i}, y_{1,i}, \ldots, x_{g-1,i}, y_{g-1,i}\}_{i\in\mathbb{Z}} \rangle \rtimes_\varphi \mathbb{Z}.$$

by the Fundamental Group-via-Actions Theorem 48.16 (2) we have an epimorphism $\gamma\colon \pi_1(\Sigma_g, a_0) \to \mathbb{Z}$ with $\gamma([\mu]) = 1$ and with kernel $\pi_1(\Sigma_\infty, \widetilde{a}_0)$
thus we obtain the isomorphism from the Semidirect Product Lemma 50.5

We have thus found the promised isomorphism. ∎



Using Proposition 56.13 we can prove the following proposition.

Proposition 56.14. Let $g \geq 2$. Every abelian subgroup of $\pi_1(\Sigma_g)$ is isomorphic to $\mathbb{Z}$.

Proof. This proposition is a pleasant application of Proposition 56.13 and Exercise 51.12. We will provide the details in Exercise 56.11. ∎

56.6. **The inverse limit of an inverse system.** For completeness' sake we also introduce the inverse limit of an inverse system, even though it will not play a role in the intermediate future.

Definition. Let $(I, \leq)$ be a preordered set and let $\mathcal{C}$ be a category. An inverse system in the category $\mathcal{C}$ over I is a family of objects $\{X_i\}_{i\in I}$ in $\mathcal{C}$, together with a family of

morphisms $\{f_{ji}\colon X_j \to X_i\}$ for all $i, j \in I$ with $i \leq j$[648] such that the following two conditions are satisfied:

(1) $f_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$,
(2) $f_{ki} = f_{ji} \circ f_{kj}$ for all $i, j, k \in I$ with $i \leq j \leq k$.

Examples.

(A) For each $i \in \mathbb{N}$ we define $X_i := \mathbb{Z}^i$ and for $i \leq j$ we denote by

$$\begin{array}{rcl} f_{ji}\colon \mathbb{Z}^j & \to & \mathbb{Z}^i \\ (x_1, \ldots, x_j) & \mapsto & (x_1, \ldots, x_i) \end{array}$$

the projection onto the first i coordinates. Then $(\{\mathbb{Z}^i\}_{i\in\mathbb{N}}, \{f_{ji}\}_{i\leq j})$ forms an inverse system in the category of groups.

(B) Let p be a prime. For each $i \in \mathbb{N}$ we define $X_i := \mathbb{Z}_{p^i}$ and for $i \leq j$ we denote by

$$\begin{array}{rcl} f_{ji}\colon \mathbb{Z}_{p^j} & \to & \mathbb{Z}_{p^i} \\ n + p^j\mathbb{Z} & \mapsto & n + p^i\mathbb{Z} \end{array}$$

the natural projection. Then $(\{\mathbb{Z}_{p^i}\}_{i\in\mathbb{N}}, \{f_{ji}\}_{i\leq j})$ forms an inverse system in the category of rings.

(C) We let $I = \mathbb{N}$, but this time we use the preorder given by divisibility, i.e. we write $k \leq l$ if and only if $k|l$. For each $k \in \mathbb{N}$ we define $X_k := \mathbb{Z}_k$ and for $k|l$ we denote by

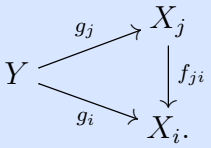
$$\begin{array}{rcl} f_{lk}\colon \mathbb{Z}_l & \to & \mathbb{Z}_k \\ n + l\mathbb{Z} & \mapsto & n + k\mathbb{Z} \end{array}$$

the natural projection. Then $(\{\mathbb{Z}_k\}_{k\in\mathbb{N}}, \{f_{lk}\}_{k|l})$ forms an inverse system in the category of rings.

The following definition is the "mirror" image of the definition of the direct limit of a direct system.

Definition. Let $(I, \leq)$ be a preordered set and let $\mathcal{C}$ be a category. Furthermore suppose we are given an inverse system $(\{X_i\}_{i\in I}, \{f_{ji}\}_{i\leq j})$ in the category $\mathcal{C}$. An inverse limit[649] of the inverse system is an object Y in $\mathcal{C}$ together with morphisms $g_i\colon Y \to X_i$, $i \in I$, in the category $\mathcal{C}$ such that the following two conditions are satisfied:

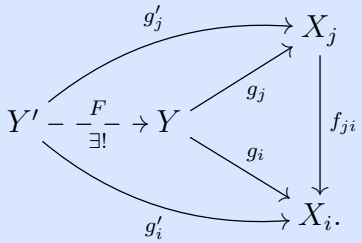
(1) For all $i \leq j$ we have $f_{ji} \circ g_j = g_i\colon Y \to X_i$, i.e. for all $i \leq j$ the following diagram commutes



(2) If we are given another object Y' and morphisms $g'_i\colon Y' \to X_i$, $i \in I$, that satisfy (1), then there exists a unique morphism $F\colon Y' \to Y$ such that for all $i \leq j$ the

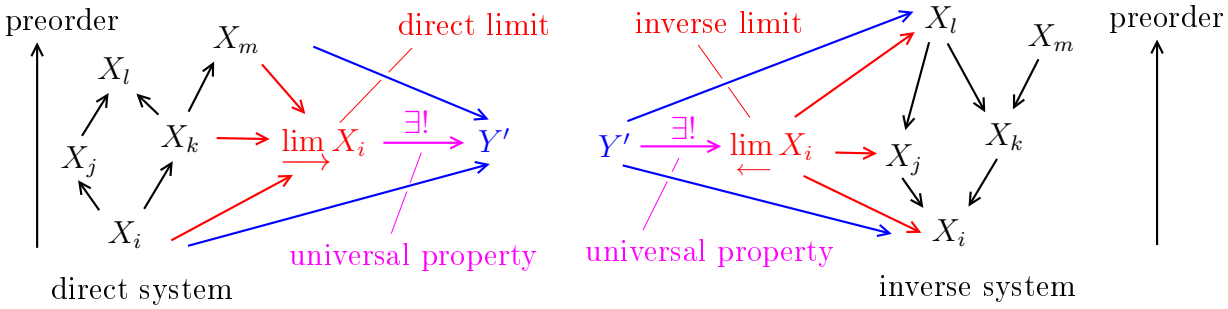
[648] So here for $i \leq j$ we have a morphism $X_j \to X_i$ instead of $X_i \to X_j$ in a direct system.

following diagram commutes



The "usual argument" for objects that satisfy a universal property shows that if the inverse limit exists, then it is unique in an appropriate sense. We denote the inverse limit by $\varprojlim X_i$ or sometimes by $\varprojlim_I X_i$.

In the following diagram we compare schematically the definition of the direct limit of a direct system to the inverse limit of an inverse system:[650]



Examples. In order to get an idea of how to construct inverse limits we first consider the special case that the preorder on I is trivial, in the sense that we have $i \leq j$ if and only if $i = j$.

(D) Let I be a set with the trivial preorder. Suppose that for every $i \in I$ we are given a set X_i. Together with the identity maps of the X_i this defines an inverse system

[649] Somewhat confusingly an "inverse limit" is often also called an "inductive limit" or "limit of an inverse system".

[650] The difference between a direct limit and an inverse limit is not the direction of the arrows between the X_i. In fact given a preorder "$\leq$" one could define a preorder "$\widetilde{\leq}$" via $a \widetilde{\leq} b$ if and only if $b \leq a$, hence a direct system over $(I, \leq)$ is an inverse system over $(I, \widetilde{\leq})$. The real difference is whether the X_i map to the limit, or whether the limit maps to the X_i.

over I in the category of sets. We claim that the inverse limit is given by[651]

$$\varprojlim X_i \;=\; \prod_{i\in I} X_i$$

together with the maps g_i that are given by the projection onto the i-component. Now we verify this statement. Since we only have $i \leq j$ if $i = j$ there is nothing to verify in part (1) of the definition of the inverse limit. Now suppose we are given a set Y' and maps $g_i' \colon Y' \to X_i$. Then we define

$$\begin{array}{rcl} F \colon Y' & \to & \varprojlim X_i \\ y & \mapsto & \left(\begin{array}{ccc} I & \to & \prod_{i\in I} X_i \\ i & \mapsto & g_i'(y) \end{array} \right). \end{array}$$

It is straightforward to verify that F is uniquely determined and that it has the desired property. We have thus determined the inverse limit $\varprojlim X_i$.

(E) Let I be a set with the trivial preorder and suppose that for any $i \in I$ we are given a topological space X_i. Together with the identity maps of the X_i this defines an inverse system over I in the category of topological spaces. As in (D) we define

$$\varprojlim X_i \;=\; \prod_{i\in I} X_i$$

and each map g_i is again given by the projection onto the i-component.

To conclude that we have found an inverse limit in the category of topological spaces we still have to equip $\prod_{i\in I} X_i$ with a topology which has the following two properties:

(i) the maps $g_i \colon \prod_{i\in I} X_i \to X_i$ are continuous,

(ii) given a topological space Y' and continuous maps $g_i' \colon Y' \to X_i$ the map F, defined as in (D), is continuous.[652]

We set

$$\mathcal{B} \;:=\; \left\{ \prod_{i\in I} U_i \;\subset\; \prod_{i\in I} X_i \;\middle|\; \begin{array}{ll} (1) & \text{each } U_i \text{ is open in } X_i \text{ and} \\ (2) & \text{there exist only finitely many } i\text{'s with } U_i \neq X_i \end{array} \right\}.$$

---

[651] Recall that the product $\prod_{i\in I} X_i$ is defined as the set of all maps $f \colon I \to \bigsqcup_{i\in I} X_i$ with the property that $f(i) \in X_i$ for all $i \in I$. Such a map f is often written as $(f(i))_{i\in I}$ and often we refer to $f(i)$ as the *i-component* of f. Furthermore we refer to the map

$$\begin{array}{rcl} \prod_{i\in I} X_i & \to & X_i \\ f & \mapsto & f(i) \end{array}$$

as the *projection onto the i-component*. Note that if the X_i's are groups, rings or modules, then the product inherits this algebraic structure by component-wise multiplication and addition.

[652] On the product $\prod_{i\in I} X_i$ we could also consider the topology $\mathcal{T}$ that is generated by all sets of the form $\prod_{i\in I} U_i$ with U_i open in X_i. This might be a more "natural" topology on $\prod_{i\in I} X_i$. But with this topology $\mathcal{T}$ the map F does not necessarily have to be continuous. The problem is that $\mathcal{T}$ has "too many" open sets.

It is straightforward to verify that $\mathcal{B}$ has the basis property that we introduced on page 132. We equip $\prod_{i\in I} X_i$ with the topology generated by this basis and we refer to this topology as the *product topology*.[653] Now we need to verify that we found the correct topology on the direct limit.

(i) It follows immediately from the definitions that the maps g_i are continuous with respect to the product topology.

(ii) Suppose we are given Y and g_i' as above. We denote by F the map defined as above. We claim that F is continuous. By the Subbasis-Continuity Proposition 2.9 it suffices to verify that the preimage of any set in $\mathcal{B}$ is open. So let $\prod_{i\in I} U_i$ be a set in $\mathcal{B}$. Then

$$F^{-1}\Big(\prod_{i\in I} U_i\Big) = \bigcap_{i\in I} (g_i')^{-1}(U_i) = \bigcap_{U_i\neq X_i} \underbrace{(g_i')^{-1}(U_i)}_{\text{open in } Y\text{'}} = \begin{array}{c}\text{intersection of finitely}\\ \text{many open subsets of } Y'.\end{array}$$

This concludes the proof that F is continuous.

Now we can prove fairly easily that inverse limits exist in many categories that are of interest to us.

Proposition 56.15. (Inverse Limit Existence Proposition) The inverse limit of any inverse system exists in the following categories:

(0) the category *Set* of sets,
(1) the category *Top* of topological spaces,
(2) the category *Group* of groups,
(3) the category *TopGr* of topological groups,
(4) the category *AbGroup* of abelian groups,
(5) the category *Ring* of rings,
(6) the category of R-modules, where R is a commutative ring.

Furthermore, the direct limits of (0),...,(6) are "coherent". For example if $\{X_i\}_{i\in I}$ is a direct system of topological groups, then the direct limit is the direct system of the corresponding topological spaces together together with an appropriate group structure.

We will see that the proof of the Inverse Limit Existence Proposition 56.15 is a relatively straightforward generalization of the examples (D) and (E) above.

Proof. Let $(I,\leq)$ be a preordered set and let $\mathcal{C}$ be one of the seven given categories. Furthermore let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_j \to X_i\}_{i\leq j})$ be an inverse system in the category $\mathcal{C}$. We define

$$\varprojlim X_i := \big\{(x_i)_{i\in I} \in \prod_{i\in I} X_i \,\big|\, f_{ji}(x_j) = x_i \text{ for all } i \leq j\big\}.$$

Furthermore we denote by $g_i\colon \varprojlim X_i \to X_i$ the natural projections. It follows immediately from the definitions that for any $i \leq j$ we have $f_{ji}\circ g_j = g_i$. Furthermore, given an object Y' and morphisms $g_i'\colon Y' \to X_i$, $i\in I$ which satisfy the condition $f_{ji}\circ g_j' = g_i'$ for all $i\in I$

[653] If the index set I is finite, i.e. if we consider the product of finitely many topological spaces, then this definition of the product topology agrees with the definition given on page 224.

we define

$$\begin{array}{rcl} F\colon Y' & \to & \varprojlim X_i \\ y & \mapsto & \left(\begin{array}{ccc} I & \to & \prod\limits_{i\in I} X_i \\ i & \mapsto & g_i'(y) \end{array} \right). \end{array}$$

It is straightforward to verify that $F(Y')$ lies in $\varprojlim X_i$ and that F is the unique map with the desired property. This shows that the inverse limit exists in the category of sets.[654]

For the remaining six categories we will use the same definition of the direct limit as a set and the same maps, but we now have to verify that we have objects and morphisms in the given category.

(1) Suppose that we work in the category of topological spaces. We equip $\prod\limits_{i\in I} X_i$ with the product topology that we defined in Example (E) above. We equip $\varprojlim X_i$ with the subspace topology. The same argument as in Example (E) shows that the maps g_i and F are continuous.

(2) In the category of groups we view $\prod\limits_{i\in I} X_i$ as a group via component-wise multiplication. Since the f_{ij} are group homomorphisms we see that $\varprojlim X_i$ is a subgroup of $\prod\limits_{i\in I} X_i$. It is straightforward to verify that the above maps are all morphisms in the category of groups.

(3) We equip the set $\varprojlim X_i$ with the topological structure and the group structure as in the above statements (1) and (2). It follows from Lemma 5.39 that this turns $\varprojlim X_i$ into a topological group, which turns out to be the inverse limit of the inverse system of topological groups.

(4),(5),(6) These categories are dealt with the same way as (2).

For all seven categories we have now provided an inverse limit. Since we always equipped the same set with different structures we see that the direct limits are coherent. ■

Remark. We consider the important special case that the preordered set $(I, \leq) = (\mathbb{N}, \leq)$. Furthermore let

$$(\{X_n\}_{n\in\mathbb{N}}, \{f_{nm}\colon X_n \to X_m\}_{m\leq n})$$

[654]Now we will see that we can use the "same" inverse limit for all the categories, this is very different from the construction of the direct limit for the same categories. There, in the proof of the Direct Limit Existence Proposition 56.1, we had to use several different construction.

be an inverse system over $(\mathbb{N}, \leq)$ in any of the above seven categories. Given $n \in \mathbb{N}_{\geq 2}$ we write $g_n := f_{n,n-1} \colon X_n \to X_{n-1}$. We have

$$\begin{aligned} \varprojlim X_i &= \big\{(x_1, x_2, x_3, \dots) \mid x_n \in X_n \text{ for all } n \in \mathbb{N} \text{ and } f_{nm}(x_n) = x_m \text{ for all } m \leq n\big\} \\ &= \big\{(x_1, x_2, x_3, \dots) \mid x_n \in X_n \text{ for all } n \in \mathbb{N} \text{ and } g_n(x_n) = x_{n-1} \text{ for all } n \in \mathbb{N}_{\geq 2}\big\} \end{aligned}$$

$\uparrow$ follows from $f_{nm} = f_{n,n-1} \circ \cdots \circ f_{m+1,m} = g_n \circ \cdots \circ g_{m+1}$

$$= \text{all infinite sequences of the form } \begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ x_3 & \in & X_3 \\ \downarrow & & \downarrow g_3 \\ x_2 & \in & X_2 \\ \downarrow & & \downarrow g_2 \\ x_1 & \in & X_1 \end{array}$$

This discussion shows that we can view of elements in the inverse limit as "upward going sequences".

Now we return to our initial examples of inverse systems and their inverse limits.

Examples.

(A) It follows quite easily from the previous remark that

$$\varprojlim_{n \in \mathbb{N}} \mathbb{Z}^n = \{(x_1, x_2, x_3, \dots) \mid x_n \in \mathbb{Z}\} \cong \mathbb{Z}^{\mathbb{N}}.$$

Note that this is an uncountable group. Recall that on page 1258 we had determined that

$$\varinjlim \mathbb{Z}^i = \mathbb{Z}^{(\mathbb{N})} = \{(x_1, x_2, \dots) \mid x_i \in \mathbb{Z} \text{ but only finitely many } x_i \text{ are non-zero}\}$$

which is a countable group.[655] Thus we see that in this particular case the inverse limit is "much larger" than the direct limit.[656]

(B) Let p be a prime. The inverse limit $\widehat{\mathbb{Z}}_p := \varprojlim \mathbb{Z}_{p^i}$ of the rings $\mathbb{Z}_{p^i} = \mathbb{Z}/p^i\mathbb{Z}$ is called the *ring of the p-adic numbers*.[657] The p-adic numbers play a very prominent role in many branches of pure mathematics.

(C) The inverse limit $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}_n$ of the rings $\mathbb{Z}_n$, $n \in \mathbb{N}$, with the preorder on $\mathbb{N}$ defined by the divisibility relation $k|l$, is called the *ring of the profinite integers*. Using the Chinese Remainder Theorem one can show that

$$\widehat{\mathbb{Z}} \cong \prod_{\text{prime } p} \widehat{\mathbb{Z}}_p.$$

[655] Why is $\mathbb{Z}^{(\mathbb{N})}$ countable?

[656] Note though that we are comparing limits of different system, first of a direct system and then of an inverse system.

[657] In number theory one often uses a different convention, namely one writes explicitly $\mathbb{Z}/p\mathbb{Z}$ for the quotient ring and one denotes the p-adic numbers by $\mathbb{Z}_p$. If the expression "$\mathbb{Z}_p$" appears in a paper or a book it is a good idea to check the convention used by the author.

Now we continue with three examples that we had not considered yet.

(D) We let $(I, \leq) = (\mathbb{N}, \leq)$. Given $i \in \mathbb{N}$ we set $X_i := \mathbb{N}$ and for $i \leq j$ we define

$$\begin{aligned} f_{ji} \colon X_j = \mathbb{N} &\to X_i = \mathbb{N} \\ n &\mapsto n + (j - i). \end{aligned}$$

Then $(\{X_i = \mathbb{N}\}_{i \in \mathbb{N}}, \{f_{ji}\}_{i \leq j})$ forms an inverse system in the category of sets. In Exercise 56.10 we will see that in this case the inverse limit is the empty set.

(E) Let X be a topological space. We denote by K the set of all compact subsets of X. We consider K with the preorder given by inclusion, i.e. we have $A \leq B$ if $A \subseteq B$. Given $A \in K$ we define $X_A := X \setminus A$. Furthermore given $A \leq B$ we denote by[658] $f_{AB} \colon \pi_0(X_B) \to \pi_0(X_A)$ the map induced by the inclusion $X_B \to X_A$. It is straightforward to verify that the sets $\{\pi_0(X_A)\}_{A \in K}$, together with the inclusion induced maps f_{AB} for $A \leq B$ define an inverse system in the category of sets. The elements of $\text{ends}(X) := \varprojlim \pi_0(X_A)$ are called the *ends of* X. In Exercise 56.13 we will determine the number of ends of $X = \mathbb{R}^n$.

(F) Given a topological space Y we denote by $C(Y, \mathbb{R})$ the abelian group of all continuous functions on Y. Note that for each subset $W \subset Y$ we have a homomorphism $C(Y, \mathbb{R}) \to C(W, \mathbb{R})$ given by $f \mapsto f|_W$, i.e. given by restricting a function on Y to a function on W.

Let X be a topological space and let $\{U_i\}_{i \in \mathbb{N}}$ be a sequence of open subsets such that for each $i \in \mathbb{N}$ we have $U_i \subset U_{i+1}$ and such that $\bigcup_{i \in \mathbb{N}} U_i = X$. Then the groups $\{C(U_i, \mathbb{R})\}_{i \in \mathbb{N}}$ together with the restriction maps form an inverse system. We leave it to the reader to verify that the restriction maps from X to the U_i gives rise to an isomorphism

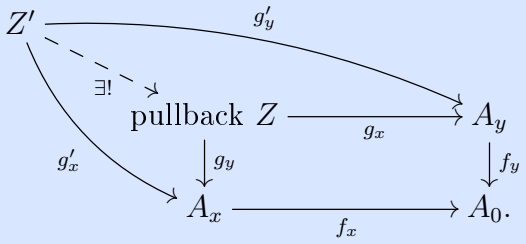
$$C(X, \mathbb{R}) \to \varprojlim C(U_i, \mathbb{R}).$$

Definition. Let $(\{0, x, y\}, \leq)$ be the push-pull set as defined on page 1256, i.e. the preordered set with the non-trivial relations $0 \leq x$ and $0 \leq y$, and let $\mathcal{C}$ be a category. An inverse system in the category $\mathcal{C}$ over $(\{0, x, y\}, \leq)$ consists of three objects A_0, A_x, A_y, the identity morphisms $\text{id}_{A_0}, \text{id}_{A_x}, \text{id}_{A_y}$ and morphisms $f_x \colon A_x \to A_0$ and $f_y \colon A_y \to A_0$. We refer to such a system as a pullback system and often we arrange the morphisms in a diagram as follows

$$\begin{array}{ccc} & & A_y \\ & & \downarrow{\scriptstyle f_y} \\ A_x & \xrightarrow[f_x]{} & A_0. \end{array}$$

If the inverse limit of such a pullback system exists, then we refer to it as the pullback of the pullback system. The defining property of the pullback Z can be summarized in the following diagram

[658] Recall that on page 167, given a topological space Y we defined $\pi_0(Y)$ as the set of path-components of Y. A map $f \colon Y \to Z$ between two topological spaces induces a map $f_* \colon \pi_0(Y) \to \pi_0(Z)$ in an obvious way.



The following lemma deals with the pullback in the category of topological spaces.

Lemma 56.16. (Topological-Pullback Lemma) Let $p\colon X \to B$ be a map of topological spaces, let C be a topological space and let $f\colon C \to B$ be a map.

(1) In the category of topological spaces the pullback of the pullback system

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle p} \\ C & \xrightarrow[f]{} & B \end{array} \quad \text{is given by} \quad \begin{array}{ccc} f^*X := \{(c,x) \in C \times X \mid f(c) = p(x)\} & \xrightarrow{(c,x)\mapsto x} & X \\ {\scriptstyle (c,x)\mapsto c}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow[f]{} & B. \end{array}$$

Often the pullback is denoted by f^*X.

(2) If X and C are connected, then f^*X is also connected.

(3) Let $p\colon X \to B$ be a covering.

(a) The pullback $q\colon f^*X \to C$ is also a covering.

(b) If B is connected, then

$$\deg(q\colon f^*X \to C) \;=\; \deg(p\colon X \to B).$$

(c) Let $c \in C$. We write $b = f(c)$ and we pick[659] $x \in X$ such that $p(x) = b$. The following equality holds:

$$q_*(\pi_1(f^*X, (c,x))) \;=\; f_*^{-1}(p_*(\pi_1(X,x))) \;\subset\; \pi_1(C,c).$$

Proof.

(1) This statement follows quite easily from the definitions or alternatively from the proof of the Inverse Limit Existence Proposition 56.15.

(2) We will prove this statement in Exercise 56.17.

(3) Let $p\colon X \to B$ be a covering. We write $Z := f^*X$ and $z := (c,x)$. Finally we denote by $q\colon f^*X \to C$ the map given by $(c,x) \mapsto c$.

(a) Suppose $V \subset B$ is uniformly covered. In Exercise 56.16 we will verify that $f^{-1}(V)$ is uniformly covered. Since $f^{-1}(V)$ is open we see that $q\colon Z \to C$ is a covering.

(b) Since B is connected it makes sense to speak of the degree of p. For any $c \in C$ we have a bijection

$$q^{-1}(c) \;=\; \{(c,x) \in C \times X \mid f(c) = p(x)\} \xrightarrow{\text{bijection}} p^{-1}(f(c)) \;=\; \deg(p).$$

This shows that the degree of q equals the degree of p.

[659] Recall that covering maps are in particular surjective, in particular such x exists.

(c) We need to show that $q_*(\pi_1(Z,z)) = f_*^{-1}(\pi_1(X,x))$. We show the two inclusions "$\subset$" and "$\supset$" separately.

"$\subset$" Let $\widetilde{f}\colon Z \to X$ be the map given by $(c,x) \to x$. By construction we have $f \circ q = \widetilde{f} \circ p$. But this implies immediately that $f_*(q_*(\pi_1(Z,z))$ is contained in $p_*(\pi_1(X,x))$.

"$\supset$" Let $\gamma\colon [0,1] \to C$ be a loop in (C,c) with $[f \circ \gamma] = f_*([\gamma]) \in p_*(\pi_1(X,x))$. We write $g := f \circ \gamma$. Let $\widetilde{g}\colon [0,1] \to X$ be the lift with $\widetilde{g}(0) = x$. Since $[g] \in p_*(\pi_1(X,x))$ we obtain from Lemma 48.15 (1) that $\widetilde{g}$ is actually a loop. Thus the map $\alpha\colon [0,1] \to C \times X$ that is given by $t \mapsto (\gamma(t), \widetilde{g}(t))$ is a loop. By construction the loop takes values in Z and evidently we have $q_*([\alpha]) = [\gamma]$. This shows that $[\gamma] \in q_*(\pi_1(Z,z))$. ∎

We conclude this section on inverse limits with the following lemma.

Lemma 56.17. Let $(I,\leq)$ be a preordered set and let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_j \to X_i\}_{i\leq j})$ be an inverse system of topological spaces.

(1) If all the X_i's are Hausdorff, then the inverse limit $\varprojlim X_i$ is a closed subset of $\prod_{i\in I} X_i$.

(2) If all the X_i's are compact and Hausdorff, then the inverse limit $\varprojlim X_i$ is also compact.[660]

Example. Let p be a prime. The abelian group $\widehat{\mathbb{Z}}_p$ is the inverse limit of finite, hence compact groups. It follows from Proposition 56.18 that $\widehat{\mathbb{Z}}_p$ is also a compact topological group. On the other hand $\widehat{\mathbb{Z}}_p$ is infinite,[661] which shows that the topology on $\widehat{\mathbb{Z}}_p$ is *not* the discrete topology.

Proof. Let $(I,\leq)$ be a preordered set and let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_j \to X_i\}_{i\leq j})$ be an inverse system of topological spaces.

(1) We suppose that all the X_i are Hausdorff. Given $i \in I$ we denote by $p_i\colon \prod_{j\in J} X_j \to X_i$ the projection and we denote by $D_i = \{(x,x) \mid x \in X_i\} \subset X_i \times X_i$ the diagonal. By Exercise 5.23 the hypothesis that X_i is Hausdorff implies that the diagonal D_i is a closed subset of $X_i \times X_i$. By the proof of the Inverse Limit Existence Proposition 56.15 we have

$$\begin{aligned}\varprojlim X_i &:= \Big\{(x_i)_{i\in I} \in \prod_{i\in I} X_i \,\Big|\, f_{lk}(x_l) = x_k \text{ for all } k \leq l\Big\} \\ &= \bigcap_{k\leq l} \Big\{x \in \prod_{i\in I} X_i \,\Big|\, \underbrace{((f_{lk}\circ p_l) \times p_k)}_{\substack{(f_{lk}\circ p_l)\times p_k \text{ is a map} \\ \prod_{i\in I} X_i \to X_k \times X_k}}(x) \in D_k\Big\} = \bigcap_{k\leq l} \underbrace{((f_{lk}\circ p_l)\times p_k)^{-1}(D_k)}_{\text{closed since } D_k \text{ closed}}.\end{aligned}$$

Thus we see that $\varprojlim X_i$ is the intersection of closed subsets of $\prod_{i\in I} X_i$, hence $\varprojlim X_i$ itself is a closed subset.

[660] Note that this is very different from the situation of the direct limit. The direct limit of topological spaces is not necessarily compact, for example the direct limit of the direct system $[0,n]$, $n \in \mathbb{N}$ is given by the non-compact space $\bigcup_{n\in\mathbb{N}} [0,n] = [0,\infty)$.

[661] In fact $\widehat{\mathbb{Z}}_p$ is uncountable (why?).

(2) Now we suppose that all the X_i are Hausdorff and compact. We obtain from Tychonoff's Theorem 5.9 that $\prod_{i\in I} X_i$ is compact. Together with (1) we see that $\varprojlim X_i$ is a closed subset of a compact space. Thus it follows from the Compact-Closed Lemma 1.21 (1) that $\varprojlim X_i$ is also compact. ■

56.7. **The profinite completion of a group.** Note that we can view any group as a topological group by equipping it with the discrete topology. With this topology a group is compact if and only if it is finite. Furthermore any homomorphism of groups equipped with the discrete topology is also a morphism in the category $\mathcal{TopGr}$ of topological groups.

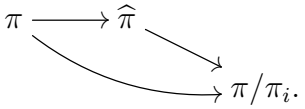
Proposition 56.18. Let $(I, \leq)$ be a preordered set and let $(\{G_i\}_{i\in I}, \{f_{ji}\colon G_j \to G_i\}_{i\leq j})$ be an inverse system of finite groups. Then the inverse limit $\varprojlim G_i$ exists in the category of topological groups and it is a compact topological group.

Proof. It follows from the Inverse Limit Existence Proposition 56.15 that the inverse limit $\varprojlim G_i$ exists in the category of topological groups. Since all the G_i's are finite, in particular compact, it follows from Lemma 56.17 (2) that the inverse limit $\varprojlim G_i$ is also compact. ■

Now let π be a group. We consider the set of all finite index normal subgroups $\{\pi_i\}_{i\in I}$. We equip the index set I with the preorder defined by the convention that $i \leq j$ if and only if $\pi_j \subset \pi_i$.[662] Given $i \leq j$ we denote by $\varphi_{ji}\colon \pi/\pi_j \to \pi/\pi_i$ the natural projection. Then $(\{\pi/\pi_i\}_{i\in I}, \{\varphi_{ji}\colon \pi/\pi_j \to \pi/\pi_i\}_{i\leq j})$ forms an inverse system. We refer to

$$\widehat{\pi} := \varprojlim \pi/\pi_i$$

as the *profinite completion of* π. By Proposition 56.18 the profinite completion $\widehat{\pi}$ is a compact topological group. It follows immediately from the definition of an inverse limit that there exists a unique homomorphism $\pi \to \widehat{\pi}$ such that for any i the following diagram commutes:



Examples.

(1) The profinite completion $\widehat{\mathbb{Z}}$ of the group $\mathbb{Z}$ equals precisely the profinite integers that we had encountered on page 1279.
(2) The profinite completion of a finite group G is the group itself, i.e. $\widehat{G} = G$.[663]
(3) Let $G = \mathbb{Q}$. This group has no proper subgroups of finite index.[664] It follows immediately from the definitions that $\widehat{\mathbb{Q}} = \{0\}$.

Now let $\varphi\colon A \to B$ be a group homomorphism. We denote by $\{B_i\}_{i\in I}$ the finite index normal subgroups of B. For $B_j \subset B_i$ we denote again by $p_{ji}\colon B/B_j \to B/B_i$ the projection . For every $i \in I$ the preimage $\varphi^{-1}(B_i)$ is a finite index normal subgroup of A.[665] Now we define $\psi_i\colon \widehat{A} \to A/\varphi^{-1}(B_i) \xrightarrow{\varphi} B/B_i$. Note that ψ_i is continuous since the first map is

[662] Note that we wrote $\pi_j \subset \pi_i$ instead of $\pi_i \subset \pi_j$.
[663] Why is that?
[664] Why not?
[665] Why is it a finite index subgroup?

continuous by definition and the second map is a map between two topological spaces that are equipped with the discrete topology. It follows immediately from the definitions that for $B_j \subset B_i$ we have $p_{ji} \circ \psi_j = \psi_i$. It now follows from the definition of the inverse limit in the category of topological groups that there exists a unique continuous homomorphism $\widehat{\varphi}\colon \widehat{A} \to \widehat{B}$ such that for any i the following diagram commutes

$$\begin{array}{ccc} \widehat{A} & \xrightarrow{\widehat{\varphi}} & \widehat{B} \\ & \searrow^{\psi_i} & \downarrow \\ & & B/B_i. \end{array}$$

It follows easily from the definitions and Proposition 56.18 that the maps

together with the maps

$$\begin{array}{rcl} A & \mapsto & \widehat{A} \\ \mathrm{Mor}(A,B) & \to & \mathrm{Mor}(\widehat{A},\widehat{B}) \\ (\varphi\colon A \to B) & \mapsto & (\widehat{\varphi}\colon \widehat{A} \to \widehat{B}) \end{array}$$

define a functor from the category of groups to the category of compact topological groups.

Exercises for Chapter 56.

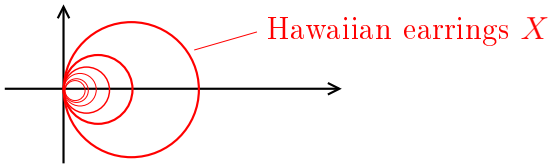
Exercise 56.1. We consider the following subset of $\mathbb{R}^2$:

$$X := \bigcup_{n\in\mathbb{N}} \text{circle of radius } \tfrac{1}{n} \text{ around } (\tfrac{1}{n},0) = \bigcup_{n\in\mathbb{N}} \{(x,y)\in\mathbb{R}^2 \mid (x-\tfrac{1}{n})^2 + y^2 = \tfrac{1}{n^2}\}.$$

This subset, equipped with the subspace topology is colloquially referred to as the *Hawaiian earrings*, see the figure below for an illustration.

(a) Let $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Show that X is homeomorphic to $[0,1]/A$.
Remark. We encountered the topological space $[0,1]/A$ in Exercise 5.37.
(b) Show that X is compact.
(c) Is X homeomorphic to the wedge of countably infinitely many circles?
(d) Show that $\pi_1(X)$ is infinitely generated.
(e) Even better than (d): Show that $\pi_1(X)$ is uncountably generated, i.e. show that there is no countable generating set for $\pi_1(X)$.

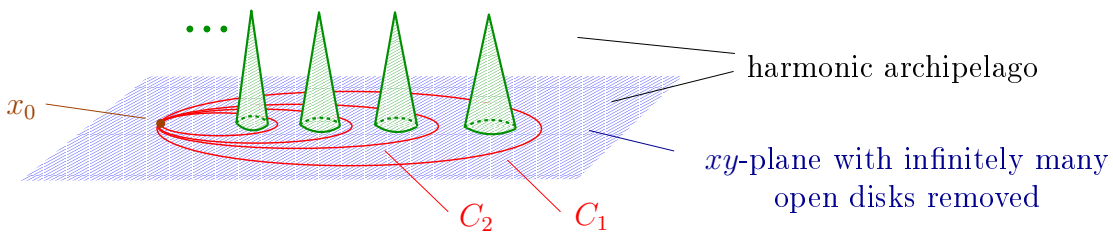
The combination of (b) and (d) implies, perhaps somewhat surprisingly, that there exist compact spaces whose fundamental groups are infinitely generated.
Remark. The precise determination of $\pi_1(X)$ is actually rather delicate, we refer to **[MoM1986]** and **[Smit1992]** for details. In particular, despite appearances, it is shown in **[Smit1992]** that $\pi_1(X)$ is not a free group.



Exercise 56.2. For $n \in \mathbb{N}$ let C_n be the circle in $\mathbb{R}^2 \times \{0\}$ of radius $\frac{1}{2n}$ with center $(\frac{1}{n},0)$. The *harmonic archipelago* H is the topological space that obtained from $\mathbb{R}^2$ as follows: For

each $n \in \mathbb{N}$ we delete an open disk between C_n and C_{n+1} and we glue on a cone such that the z-coordinate of the cone equals 1. We refer to the figure below for an illustration.

(a) Give a precise description of the harmonic archipelago H.
Remark. You might want to skip this exercise, unless you are inclined to super-rigorous mathematics.
(b) Show that the harmonic archipelago H is non-compact.
(c) Let $x_0 = (0,0)$ be the origin. Show that for each n the inclusion induced map $\pi_1(C_n, x_0) \to \pi_1(H, x_0)$ is a monomorphism.
(d) As in Exercise 56.1 we refer to $X := \bigcup_{n\in\mathbb{N}} C_n$ as the Hawaiian earrings. Show that the inclusion induced map $\pi_1(X, x_0) \to \pi_1(H, x_0)$ is not a monomorphism.
(e) Is $\pi_1(H, x_0)$ countable or uncountable?



Exercise 56.3.

(a) Given $k \leq l \in \mathbb{N}$ we consider the homomorphism

$$\begin{array}{rccc} \varphi_{kl}\colon & \mathbb{Z} & \to & \mathbb{Z} \\ & n & \mapsto & \frac{l!}{k!} \cdot n. \end{array}$$

These homomorphisms define a direct system $(\{\mathbb{Z}\}_{k\in\mathbb{Z}}, \{\varphi_{kl}\}_{k\leq l})$ in the category of abelian groups. Show that the corresponding direct limit $\varinjlim \mathbb{Z}$ is isomorphic to $(\mathbb{Q}, +)$.
(b) Provide a direct system $(\{G\}_{i\in\mathbb{N}}, \{\varphi_i\colon G_i \to G_j\}_{i\leq j})$ consisting of finitely generated abelian groups such that the direct limit is isomorphic to $\mathbb{Z}[\frac{1}{2}]$.
(c) Provide a direct system $(\{G\}_{i\in\mathbb{N}}, \{\varphi_i\colon G_i \to G_j\}_{i\leq j})$ consisting of finitely generated abelian groups such that the direct limit is isomorphic to $\mathbb{Q}^2$.

Exercise 56.4. Let $f_n\colon X_n \to X_{n+1}$ be a sequence of morphisms in any category. Suppose there exists an $N \in \mathbb{N}$ such that all f_n for $n \geq N$ are isomorphisms. Show that $\varinjlim X_n$ exists and show that it is naturally isomorphic to X_N, more precisely, show that a direct limit is given by $\varinjlim X_n := X_N$ and show that the maps

$$\begin{array}{rccll} f_{N-1} \circ \cdots \circ f_{n+1} \circ f_n\colon & X_n & \to & X_N & \text{if } n \leq N, \\ (f_n \circ \cdots \circ f_{N+1} \circ f_N)^{-1}\colon & X_n & \to & X_n & \text{if } n > N \end{array}$$

define a direct limit.

Exercise 56.5. Let $(I, \leq)$ be a preordered set and let $\{X_i\}_{i\in I}$ be a family of topological spaces such that for any $i \leq j$ we have $X_i \subset X_j$. We define

$$\varinjlim X_i \;=\; X \;:=\; \bigcup_{i\in I} X_i,$$

where the topology on $X = \bigcup_{i\in I} X_i$ is given by the rule that $U \subset X$ is open if and only if $U \cap X_i$ is open for all $i \in I$. Furthermore we denote by $g_j \colon X_j \to X = \bigcup_{i\in I} X_i, j \in \mathbb{N}$ the obvious inclusion maps. Show that $\varinjlim X_i$ together with the maps g_j is the direct limit of the direct system in the category of topological spaces.

Exercise 56.6. Let $\{X_i\}_{i\in\mathbb{N}}$ be a family of topological spaces. Suppose that for each $i \in \mathbb{N}$ we are given an injective map $f_i \colon X_i \to X_{i+1}$.

(a) Show the map $X_i \to \varinjlim X_i$ is not necessarily an embedding.

(b) Suppose that each $f_i \colon X_i \to X_{i+1}$ is an embedding. Show that in this case each map $X_i \to \varinjlim X_i$ is in fact an embedding.

Exercise 56.7.

(a) For $i \in \mathbb{N}$ we define $F_i = \langle x_i, x_{i+1}\rangle$. For $i \in \mathbb{N}$ we denote by $g_i \colon F_i \to F_{i+1}$ the homomorphism that is given by $g_i(x_i) = e, g_i(x_{i+1}) = x_{i+1}$. Furthermore for $i < j$ we define $f_i \colon F_i \to F_j$ to be the homomorphism $g_{j-1} \circ \cdots \circ g_{i+1} \circ g_i$. What is the direct limit $\varinjlim F_i$ in the category of groups?

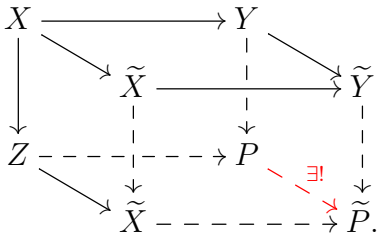
(b) Given $k \in \mathbb{N}$ we define $X_k = \mathbb{N}$ to be the topological space where the open sets are $\{1, \dots, k\}, \{k+1\}, \{k+2\}, \dots$. For $k \le l$ we denote by $f_{kl} \colon X_k \to X_l$ the identity map. (Note that with the choice of our topologies on X_k and X_l this map is in fact continuous). What is the direct limit $\varinjlim X_k$ in the category of topological spaces?

Remark. as a set the direct limit is again $\mathbb{N}$, but what is the topology?

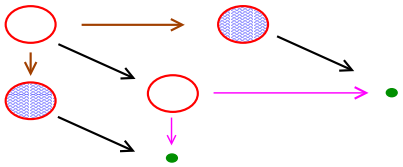
Exercise 56.8. Suppose we are given two pushouts

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \dashrightarrow & P \end{array} \quad \text{and} \quad \begin{array}{ccc} \widetilde{X} & \longrightarrow & \widetilde{Y} \\ \downarrow & & \downarrow \\ \widetilde{Z} & \dashrightarrow & \widetilde{P}. \end{array}$$

Furthermore suppose we are given a commutative diagram



An example for such a situation is sketched in the figure below.



(a) Show that there exists a unique map $P \to \widetilde{P}$ which makes the cube commute.

(b) Suppose that the maps $X \to \widetilde{X}$, $Y \to \widetilde{Y}$ and $Z \to \widetilde{Z}$ are homotopy equivalences. Does it follows that the map $P \to \widetilde{P}$ is a homotopy equivalence?

Exercise 56.9. Proof the Direct Limit Stabilization Lemma 56.2. It is best to prove it using the definition of direct limits instead of using any of the above explicit descriptions of direct limits.

Exercise 56.10. We let $(I, \leq) = (\mathbb{N}, \leq)$. Given $i \in \mathbb{N}$ we set $X_i := \mathbb{N}$ and for $i \leq j$ we define

$$\begin{array}{rccc} f_{ji}\colon X_j = \mathbb{N} & \to & X_i = \mathbb{N} \\ n & \mapsto & n + (j - i). \end{array}$$

Then $(\{X_i = \mathbb{N}\}_{i\in\mathbb{N}}, \{f_{ji}\}_{i\leq j})$ forms an inverse system in the category of sets. Show that the inverse limit is the empty set.

Exercise 56.11. Let $g \geq 2$. We denote by Σ_g the surface of genus g. Show that every abelian subgroup of $\pi_1(\Sigma_g)$ is isomorphic to $\mathbb{Z}$.
Hint. Make use of Proposition 56.13 and Exercise 51.12.

Exercise 56.12. Let $g \in \mathbb{N}_{\geq 2}$. Use Proposition 56.13 to show that the fundamental group of the surface of genus g is torsion-free.

Exercise 56.13. Let X be a topological space. We denote by $\mathcal{K}$ the set of all compact subsets of X. We consider $\mathcal{K}$ with the preorder given by inclusion, i.e. given $A, B \in \mathcal{K}$ we have $A \leq B$ if $A \subseteq B$. Given $A \in \mathcal{K}$ we define[666] $X_A := \pi_0(X \setminus A)$. Furthermore given $A \leq B$ we denote by $f_{AB}\colon \pi_0(X_B) \to \pi_0(X_A)$ the map induced by the inclusion $X_B \to X_A$. It follows immediately from the discussion on page 200 that the sets $\{\pi_0(X_A)\}_{A\in\mathcal{K}}$, together with the inclusion induced maps f_{AB} for $A \leq B$ define an inverse system in the category of sets. The elements of $\varprojlim \pi_0(X_A)$ are called the *ends of* X.

(a) Let *Top* be the category of topological spaces and let *Set* be the category of sets. Show that

$$X \mapsto \varprojlim \pi_0(X_A)$$

gives rise to a covariant functor from *Top* to *Set*.
(b) Show that for a topological space X the cardinality of $\varprojlim \pi_0(X_A)$, viewed as an element in $\mathbb{N}_0 \cup \{\infty\}$, equals the number of ends of M as defined in Exercise 2.27.
(c) Let $n \in \mathbb{N}_0$. Determine the number of ends of $X = \mathbb{R}^n$.
(d) For each $k \in \mathbb{N}_0$ give an example of a connected 1-dimensional CW-complex X with precisely k ends.

Exercise 56.14. Let X be a topological space, let A and B be two subsets of X and let $f\colon A \to B$ be a homeomorphism. We define

$$\widetilde{X} := \Big(\bigcup_{i\in\mathbb{Z}} X \times \{i\}\Big) / (a, i) \sim (f(a), i - 1) \quad \text{for } a \in A \text{ and } i \in \mathbb{Z}.$$

We consider the action of $\mathbb{Z}$ on $\widetilde{X}$ that is given by $n\cdot[(x, i)] := [(x, i+n)]$ for $(x, i) \in X \times \{i\}$.

(a) Show that this action is continuous.
(b) We write $X(f) = X/\!\sim$ where $a \sim f(a)$ for $a \in A$. Show that the maps

$$\begin{array}{rccc} \Phi\colon X(f) & \to & \widetilde{X}/\mathbb{Z} \\ [x] & \mapsto & [(x, 0)] \end{array} \quad \text{and} \quad \begin{array}{rccc} \Psi\colon \widetilde{X}/\mathbb{Z} & \to & X(f) \\ [(x, i)] & \mapsto & [x] \end{array}$$

are continuous and inverses of one another. (This implies of course that both maps are homeomorphisms.)

[666] Recall that by the definition on page 167, given a topological space Y we had denoted $\pi_0(Y)$ by the set of path-components of X.

(c) Suppose that there exist disjoint open neighborhoods U of A and V of B. Show that the action is discrete.

Exercise 56.15. We consider the profinite integers, i.e. the inverse limit $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}_n$ of the rings $\mathbb{Z}_n$, $n \in \mathbb{N}$, with the preorder on $\mathbb{N}$ defined by the divisibility relation $k|l$. Does there exist an epimorphism $\widehat{\mathbb{Z}} \to \mathbb{Z}$ of groups?

Exercise 56.16. Show that the pullback $f^*X \to C$ of a covering $p\colon X \to B$ under a map $f\colon C \to B$ is again a covering.

Exercise 56.17. Let $f^*X \to C$ be the pullback of a map $p\colon X \to B$ and a map $f\colon C \to B$. Show that if X and C are connected, then f^*X is also connected.

Exercise 56.18. Let $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ be a short exact sequence of groups. Furthermore let $g\colon Z \to C$ be a group homomorphism. Show that there exists a commutative diagram of group homomorphisms of the form below such that the top horizontal sequence is also exact:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \xrightarrow{\widetilde{\alpha}} & Y & \xrightarrow{\widetilde{\beta}} & Z & \longrightarrow & 1 \\ & & \downarrow{=} & & \downarrow{f} & & \downarrow{g} & & \\ 1 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 1. \end{array}$$

Exercise 56.19.

(a) We consider the inverse system $(X_i)_{i\in\mathbb{N}}$ where $X_i = \mathbb{Z}$ and where for each $i \geq j$ the map $X_i \to X_j$ is given by multiplication by 2^{i+j}. Determine $\varprojlim X_i$ in the category of abelian groups.

(b) Now we work in the category of sets. Show that there exists an inverse system $(\{X_i\}_{i\in\mathbb{N}}, \{f_{ji}\colon X_j \to X_i\}_{i\leq j})$ of non-empty sets such that $\varprojlim X_i = \varnothing$.

Remark. In Exercise 56.20 we will see that the inverse limit of an inverse system of *finite non-empty sets* is in fact non-empty.

Exercise 56.20. In this exercise we consider the category of topological space. Let $(\{X_i\}_{i\in\mathbb{N}}, \{f_{ji}\colon X_j \to X_i\}_{i\leq j})$ be an inverse system of topological spaces that are non-empty, Hausdorff and compact.

(a) For each $j \in \mathbb{N}$ we set

$$Z_j := \{(x_i)_{i\in\mathbb{N}} \mid \text{ for all } i \leq j \text{ we have } x_i = f_{ji}(x_j)\} \subset \prod_{i\in\mathbb{N}} X_i.$$

Show that Z_j is closed and non-empty.
Hint. You might want to consider Lemma 56.17 (1).

(b) Show that for any $i, j \in \mathbb{N}$ we have $Z_i \cap Z_j \neq \varnothing$.

(c) Apply Tychonoff's Theorem 5.9, together with Exercise 1.29, to conclude that $\varprojlim X_i \neq \varnothing$.

Remark. We can apply this statement in particular to an inverse system of finite non-empty sets, viewed as topological spaces by equipping each of the finite sets with the discrete topology. Evidently a finite discrete space is Hausdorff and compact.

Exercise 56.21. Let I be preordered set, let $\{f_i \colon A_i \to A_j\}_{i \leq j}$ be a direct system in the category of abelian groups and let $\{g_i \colon B_i \to B_j\}_{j \leq i}$ be an inverse system in the category of abelian groups. Suppose that for each $i \in I$ we are given a bilinear map $\varphi_i \colon A_i \times B_i \to \mathbb{Z}$ such that for any $i \leq j$, and $a \in A_i$ and $b \in B_j$ we have $\varphi_i(a_i, g_j(b_j)) = \varphi_j(f_i(a_i), b_j)$. Define an "interesting" bilinear map

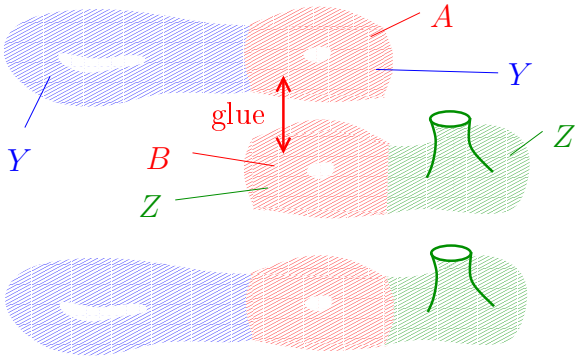
$$\varinjlim A_i \times \varprojlim B_i \to \mathbb{Z}.$$

57. The HNN-Seifert–van Kampen Theorem

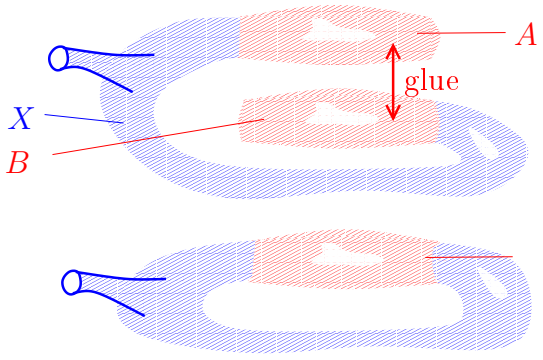
One of our main goals at the moment is to calculate fundamental groups of interesting topological spaces. As we have discovered, one of the most powerful tools is the Seifert–van Kampen Theorem 54.1.

In light of Exercise 5.38 (c) we can say, modulo technical details, that the Seifert–van Kampen Theorem 54.1 allows us to calculate the fundamental group of a topological space $X = Y \cup_f Z$ that is obtained from gluing two path-connected topological spaces Y and Z along path-connected open subsets $A \subset Y$ and $B \subset Z$. This situation is illustrated in the figure below, to the left.

Sometimes one is given the situation that is shown in the figure below, to the right. Namely we are given a path-connected topological space X and we glue two disjoint path-connected open subsets A and B together. In this chapter we will prove the HNN-Seifert–van Kampen Theorem which will allow us to calculate the fundamental group in the latter situation.



$\pi_1(Y \cup_f Z) = \pi_1(Y) *_{\pi_1(A)} \pi_1(Z)$



$\pi_1 = ???$

We start out with the following notation.

Notation. Let X be a topological space, let A and B be two subsets of X and let $f\colon A \to B$ be a homeomorphism. We write

$$X(f) \;:=\; X(f\colon A \to B) \;:=\; X/\!\sim \qquad \text{where } f(a) \sim a \text{ for } a \in A.$$

Examples.

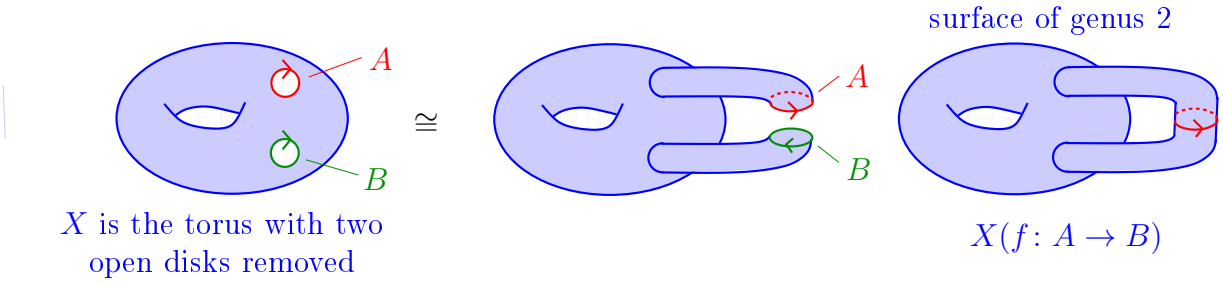
(1) Let X be the torus with two open disks removed. We denote by A and B the two boundary components and we endow them with the orientation coming from a fixed orientation of the torus. Let $f\colon A \to B$ be an orientation-reversing diffeomorphism. Then one sees that $X(f\colon A \to B)$ is diffeomorphic[667] to the surface of genus 2. This statement is illustrated in the figure below.[668]

(2) Let Y be a topological space and let $f\colon Y \to Y$ be a homeomorphism. We consider $X = Y \times [0,1]$. Furthermore, by an abuse of notation, we denote by $f\colon Y \times \{0\} \to Y \times \{1\}$ also the map defined by $f(y,0) = (f(y),1)$. It follows immediately from the definitions that

$$X(f) \;=\; (Y \times [0,1])(f) \;=\; (Y \times [0,1])/\!\sim \;=\; \mathrm{Tor}(X,f),$$

[667] Note that by Lemma **??** we know that $X(f\colon A \to B)$ is indeed a smooth manifold.

[668] What topological space do we obtain if f is chosen to be orientation-preserving?



where $\mathrm{Tor}(X, f)$ denotes the mapping torus that we introduced on page 467.

Our goal now is to determine $\pi_1(X(f))$ in terms of the fundamental groups of X, A and the induced map $f_*\colon \pi_1(A) \to \pi_1(B)$. To formulate the Seifert–van Kampen Theorem 54.1 we needed to introduce the notion of an amalgamated product of groups. So it does perhaps now come as a surprise that we need a new construction from group theory to formulate the desired result.

Definition. Let π and Γ be two groups and let $\alpha, \beta\colon \Gamma \to \pi$ be two homomorphisms. We refer to

$$\langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle \;:=\; (\pi * \langle t \rangle) \,/\, \langle\langle \{\alpha(g)t\beta(g)^{-1}t^{-1}\}_{g\in\Gamma} \rangle\rangle$$

as the HNN-extension corresponding to (π, α, β).[669]

Example. Let π be a group and let Γ be the trivial group. It follows immediately from the definitions that the corresponding HNN-extension is the free product $\pi * \langle t \rangle$.

The following lemma summarizes some basic properties of HNN-extensions. It says in particular that a semidirect product is a special case of an HNN-extension.

Lemma 57.1.

(1) Let $\pi = \Gamma$. Furthermore let $\alpha\colon \pi \to \pi$ and $\beta\colon \pi \to \pi$ be isomorphisms. We write $\varphi = \beta \circ \alpha^{-1}\colon \pi \to \pi$. In this setting the map

$$\begin{array}{rcl} \Phi\colon \pi \rtimes_\varphi \mathbb{Z} & \to & \langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle \\ (g, n) & \mapsto & g \cdot t^{-n} \end{array}$$

is an isomorphism.

(2) Let $\pi = \langle x_1, \dots, x_k \mid r_1, \dots, r_l \rangle$ be a finitely presented group and let $g_1, \dots, g_m$ is a generating set for Γ. A presentation for the HNN-extension of (π, α, β) is given by

$$\langle x_1, \dots, x_k, t \mid r_1, \dots, r_l, \alpha(g_i) = t\beta(g_i)t^{-1} \text{ for } i = 1, \dots, m \rangle.$$

(3) Let π and Γ be two groups and let $\alpha, \beta\colon \Gamma \to \pi$ be two homomorphisms. Then there exists an epimorphism

$$\langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle \;\to\; \mathbb{Z}$$

that is uniquely determined by $g \mapsto 0$ if $g \in \pi$ and $t^n \mapsto n$.

Sketch of proof.

[669] The concept of such a group was introduced by Graham Higman, Bernhard Neumann and Hannah Neumann [**HNN1950**] in 1950, whence the name.

(1) We first verify that the given map is indeed a homomorphism. Let (g, m) and (h, n) be two elements in $\pi \rtimes_\varphi \mathbb{Z}$. First we assume that $m \geq 0$. Then the following equalities hold in $\langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle$:

$$\begin{array}{c} \text{since } \alpha \text{ is an isomorphism we can write} \\ h = \alpha(\alpha^{-1}(h)) = t\beta(\alpha^{-1}(h))t^{-1} = t\varphi(h)t^{-1} = t \cdot t\varphi(\varphi(h))t^{-1} \cdot t^{-1} = \cdots = t^m\varphi^m(h)t^{-m} \end{array}$$

$$\begin{array}{rcll} \Phi(g,m) \cdot \Phi(h,n) = gt^{-m} \cdot ht^{-n} & \overset{\downarrow}{=} & gt^{-m} \cdot t^m\varphi^m(h)t^{-m} \cdot t^{-n} & = g\varphi^m(h)t^{-m-n} \\ & = & \Phi(g\varphi^m(h), m+n) & = \Phi((g,m)(h,n)). \\ & & & \uparrow \\ & & & \text{definition of product structure on } \pi \rtimes_\varphi \mathbb{Z} \end{array}$$

If $m < 0$ then a very similar calculation, using that β is an isomorphism, shows again that $\Phi(g, m) \cdot \Phi(h, n) = \Phi((g, m)(h, n))$. This shows that Φ is indeed a homomorphism. By Lemma 51.13 there exists a unique homomorphism

$$\Psi \colon \pi * \langle t \rangle \ \to \ \pi \rtimes \mathbb{Z}$$

with $\Psi(g) = (g, 0)$ for every $g \in \pi$ and with $\Psi(t) = (0, 1)$. Using Lemma 53.4 and using an elementary calculation one can easily show that Ψ descends to a map $\langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle \to \pi \rtimes \mathbb{Z}$. Since $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identity on the generating sets π and t we see that both homomorphisms are the identity. In particular Φ is an isomorphism.

(2) This time we apply Lemma 53.4 twice to obtain the desired isomorphism and its inverse.

(3) This statement is once again an immediate consequence of Lemma 53.4. ■

The following proposition is an analogue of the Amalgamated Product-Monomorphism Proposition 53.24.

Proposition 57.2. (HNN-Extension Monomorphism Proposition) Let π and Γ be two groups and let $\alpha, \beta \colon \Gamma \to \pi$ be two homomorphisms. If α and β are monomorphisms, then the obvious inclusion map

$$\begin{array}{ccl} \pi & \to & \langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle \ := \ (\pi * \langle t \rangle) \,/\, \langle\langle \{\alpha(g)t\beta(g)^{-1}t^{-1}\}_{g \in \Gamma} \rangle\rangle \\ h & \mapsto & h \end{array}$$

is also a monomorphism.

Proof. The proposition is proved in [**Ser1980**, p. 45]. Alternatively the proposition follows from the Normal Form Theorem for HNN-extensions which is formulated and proved in [**LyS1977**, Theorem IV.2.1] or alternatively in [**CgRR2008**, Satz 6.3] and [**Sti1993**, Chapter 9.2]. ■

Now we return to topological spaces. The following theorem can be considered a sibling of the Seifert–van Kampen Theorem 54.1.

Theorem 57.3. (HNN-Seifert–van Kampen Theorem) Suppose we are in one of the following two settings:

(1) We are given a path-connected topological space X, two open disjoint subsets A and B of X and a homeomorphism $f \colon A \to B$.

(2) We are given a path-connected smooth manifold X, two distinct components A and B of ∂X and a homeomorphism $f \colon A \to B$.

We pick a base point $x_0 \in A$ and we pick a path γ in X from x_0 to $f(x_0)$. Furthermore we write $\Gamma = \pi_1(A, x_0)$. Finally we denote by

$$\alpha\colon \Gamma = \pi_1(A, x_0) \to \pi_1(X, x_0)$$

the inclusion induced map and we denote by

$$\beta\colon \Gamma = \pi_1(A, x_0) \xrightarrow{f_*} \pi_1(B, f(x_0)) \to \pi_1(X, f(x_0)) \xrightarrow{\gamma_*} \pi_1(X, x_0)$$

the concatenation of the maps induced by f, the inclusion and the base point change isomorphism from the Change-of-Base Point Proposition 47.10, using the path γ. Then there exists an isomorphism

$$\Phi\colon \pi_1(X(f), x_0) \xrightarrow{\cong} \langle \pi_1(X, x_0), t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle$$

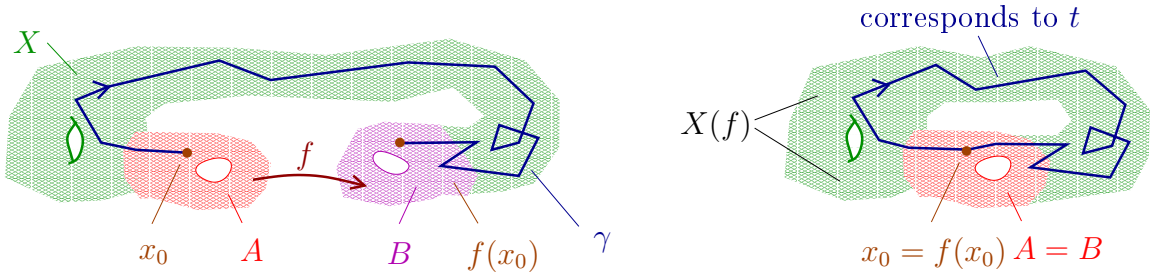
which has the following two properties:

(1) The diagram

$$\begin{array}{ccc} & \pi_1(X, x_0) & \\ \swarrow & & \searrow \\ \pi_1(X(f), x_0) & \xrightarrow{\Phi} & \langle \pi_1(X, x_0), t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle \end{array}$$

with the obvious diagonal maps commutes.

(2) The closed loop in $X(f)$ defined by γ corresponds under the isomorphism Φ to t.



Remark. The HNN-Seifert–van Kampen Theorem 57.3 is a frequently used tool in the calculation of fundamental groups of topological spaces. The popularity of the result is in contrast to the sparsity of proofs in the literature. A sketch of an outline of a proof is given in [**ScW1979**]. A much more general statement is proved in [**Bou2007**, Proposition IV.5.5 on p. 431].

Example. As on page 1290 we consider the surface X that is given by removing two open disks from a torus. We refer to the two boundary components as A and B. We equip A and B with the orientations given in the figure below. We pick a diffeomorphism $f\colon A \to B$ that preserves the orientation. As discussed on page 1290, the resulting topological space $X(f)$ is homeomorphic to the surface of genus two.

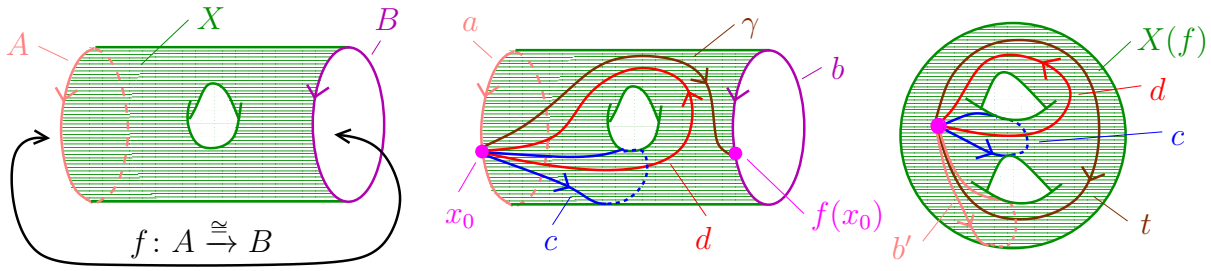
We pick a base point $x_0 \in A$. We pick a loop a in (A, x_0), we pick loops c, d in (X, x_0), a path γ in X from x to $f(x)$ and a loop b in $(B, f(x))$ as indicated in the figure below. Finally we write $\widetilde{b} = \gamma * b * \overline{\gamma}$. By Lemma 55.11 and as in the proof of Lemma 56.11 we have an isomorphism

$$\pi_1(X, x_0) \xrightarrow{\cong} \langle c, d, a, \widetilde{b} \mid a = [c, d] \cdot \widetilde{b} \rangle.$$

We write $\Gamma = \pi_1(A, x_0)$. Note that $\Gamma = \langle a \rangle$ and $\alpha(a) = a$ and $\beta(a) = \widetilde{b}$. We see that

$$\begin{array}{ccccc} \pi_1(X(f), x_0) & \underset{\text{Theorem 57.3 (2)}}{\cong} & \langle \pi_1(X), t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle & \underset{\text{Lemma 57.1}}{=} & \langle c, d, a, \widetilde{b}, t \mid a = [c,d] \cdot \widetilde{b}, a = t\widetilde{b}t^{-1} \rangle \\ & \underset{\text{Tietze transformations}}{=} & \langle c, d, \widetilde{b}, t \mid [c,d] \cdot \widetilde{b} = t\widetilde{b}t^{-1} \rangle & = & \langle c, d, \widetilde{b}, t \mid \underbrace{[c,d] \cdot \widetilde{b}t\widetilde{b}^{-1}t^{-1}}_{=[c,d][\widetilde{b},t]} \rangle. \end{array}$$

Thus we have now obtained the same presentation that we had already obtained in the Fundamental Group-of-Surfaces Proposition 54.5.



In the proof of Theorem 57.3 (1) we will need one extra piece of notation and one more lemma.

Notation. Let π and Γ be groups and let $\alpha, \beta\colon \Gamma \to \pi$ be homomorphisms. Given $i \in \mathbb{Z}$ we write $\pi_i := \pi$ and $\Gamma_i := \Gamma$. We write[670]

$$\ldots \pi_{-2} \underset{\Gamma_{-1}}{*} \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \underset{\Gamma_2}{*} \pi_2 \ldots := \varinjlim \Big(\underbrace{\pi_{-i} \underset{\Gamma_{-i+1}}{*} \ldots \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \ldots \underset{\Gamma_i}{*} \pi_i}_{\text{iterated amalgamated product defined by } \alpha \text{ and } \beta} \Big).$$

By a slight abuse of language we refer to this group as an infinite amalgamated product. By the universal property of the direct limit there exists an automorphism Ξ of this infinite amalgamated product that shifts all indices by one to the right.

Lemma 57.4. With the above notation there exists a well-defined map

$$\begin{array}{rcl} \Big(\ldots \pi_{-2} \underset{\Gamma_{-1}}{*} \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \underset{\Gamma_2}{*} \pi_2 \ldots \Big) \rtimes_\Xi \mathbb{Z} & \xrightarrow{\cong} & \langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle \\ (g_i, n) & \mapsto & gt^{-i} \cdot t^n, \quad \text{where } g_i \in \pi_i \end{array}$$

which is an isomorphism.

Example. If α and β are isomorphisms, then it follows easily from the Amalgamated Product Lemma 53.22 (3) that the "infinite amalgamated product" is isomorphic to π. We see that in this special case the statement of Lemma 57.4 is basically the same as the statement of Lemma 57.1 (1).

Sketch of proof of Lemma 57.4. It follows easily from the universal property of the direct limit and of the amalgamated product that there exists a well-defined map

$$\begin{array}{rcl} \ldots \pi_{-2} \underset{\Gamma_{-1}}{*} \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \underset{\Gamma_2}{*} \pi_2 \ldots & \to & \langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle \\ g_i & \mapsto & gt^{-i}, \quad \text{where } g_i = g \in \pi_i = \pi. \end{array}$$

[670] By the Amalgamated Product Lemma 53.22 (4) we can form the iterated amalgamated product without having to worry about parentheses.

A straightforward calculation as in the proof of Lemma 57.1 (1) shows that we get an induced map on the semidirect product. This shows that the map given in the lemma actually makes sense. It remains to show that the map is an isomorphism. Note that using Lemma 53.4 one can easily show that there exists a well-defined homomorphism from right to left which is an inverse to our given map. We leave it to the reader to fill in the details. ■

Proof of Theorem 57.3 (1). [671] Let X be a path-connected topological space, let A and B be two open disjoint path-connected subsets of X and let $f\colon A \to B$ be a homeomorphism. We pick a base point $x \in A$ and we pick a path γ in X from x to $f(x)$. Furthermore we define

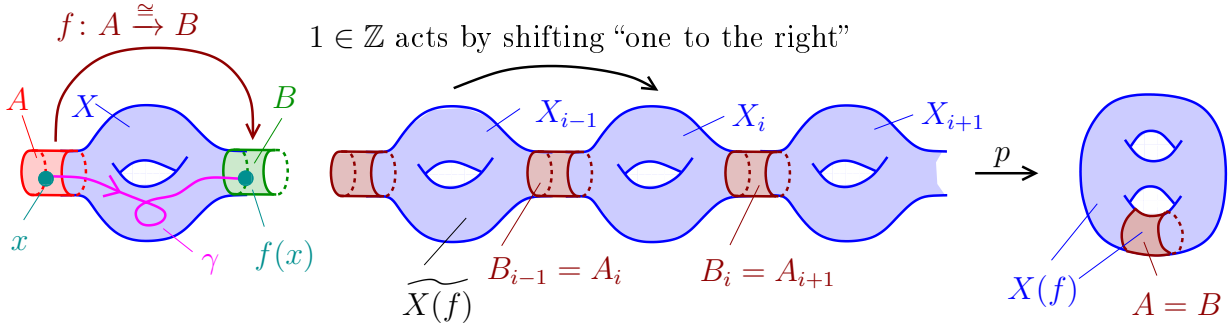
$$\widetilde{X(f)} \;:=\; \Big(\bigcup_{i\in\mathbb{Z}} X \times \{i\}\Big) \,/\, (a,i) \sim (f(a), i-1) \qquad \text{for } a \in A \text{ and } i \in \mathbb{Z}.$$

We equip $\widetilde{X(f)}$ with the base point $\widetilde{x}$ which is represented by $(x,0)$. There is an obvious action of $\mathbb{Z}$ on $\widetilde{X(f)}$ given by $n \cdot [(P,i)] := [(P,i+n)]$ for $(P,i) \in X \times \{i\}$. Since A and B are open and disjoint we can apply Exercise 56.14. We obtain that this action is discrete and continuous and that the map

$$\begin{array}{rccc} \Phi\colon & X(f) & \to & \widetilde{X(f)}/\mathbb{Z} \\ & [x] & \mapsto & [(x,0)] \end{array}$$

is a homeomorphism.

For $i \in \mathbb{Z}$ we now write $X_i := X \times \{i\}$, $A_i := A \times \{i\}$, $B_i := B \times \{i\}$ and $x_i := \{x\} \times \{i\}$. Furthermore we write $\pi_i := \pi_1(X_i, x_i)$ and $\Gamma_i = \pi_1(A_i, x_i)$. We refer to the figure below for an illustration.



Claim. Given any $i \in \mathbb{Z}$ we consider the homomorphism

$$\pi_i \;=\; \pi_1(X_i, x_i) \;\underset{\uparrow}{\longrightarrow}\; \pi_1(\widetilde{X(f)}, x_i) \;\underset{\uparrow}{\longrightarrow}\; \pi_1(\widetilde{X(f)}, \widetilde{x}).$$

induced by the inclusion $X_i \to \widetilde{X(f)}$

induced by the Change-of-Base Point Proposition 47.10 (1) where we use i translates of the path γ to connect x_i to $\widetilde{x} = x_0$

[671] As we saw on pages 1291 and in Lemma 57.1 (1), we can view Theorem 57.3 as a generalization of the Mapping Torus-π_1-Proposition 50.6. In fact the subsequent proof is partly modelled on the proof of the Mapping Torus-π_1-Proposition 50.6.

These homomorphisms induce an isomorphism

$$\dots \pi_{-2} \underset{\Gamma_{-1}}{*} \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \underset{\Gamma_2}{*} \pi_2 \dots \xrightarrow{\cong} \pi_1(\widetilde{X(f)}, \widetilde{x}).$$

Here given $i \in \mathbb{Z}$ the maps in the amalgamated product are defined as follows:

$$\overbrace{\pi_1(X_{i-1}, x_{i-1})}^{=\pi_{i-1}} \xleftarrow[\uparrow]{\gamma_*} \pi_1(X_{i-1}, f(x_{i-1})) \xleftarrow[\uparrow]{} \pi_1(B_{i-1}, f(x_i)) \xleftarrow{f_*} \overbrace{\pi_1(A_i, x_i)}^{=\Gamma_i} \xrightarrow[\uparrow]{} \overbrace{\pi_i(X_i, x_i)}^{=\pi_i}.$$

given by the Change-of-Base Point

induced by the inclusion $B_{i-1} \to X_{i-1}$

Proposition 47.10 (1)

induced by the inclusion $A_i \to X_i$

Proof. To prove the claim we need to introduce one more gadget. Namely, given $n \in \mathbb{N}$ we write

$$\widetilde{Y}_n := \Big(\bigcup_{i=-n}^{n} X_i \Big) / (a, i) \sim (f(a), i-1) \quad \text{for } a \in A, i \in \{-n+1, \dots, n\}.$$

It follows from the Seifert–van Kampen Theorem 54.1, applied iteratively $2n$ times, that

$$\pi_1\big(\widetilde{Y}_n, \widetilde{x}\big) \xleftarrow{\cong} \pi_{-n} \underset{\Gamma_{-n+1}}{*} \pi_{-n+1} \dots \pi_{n-1} \underset{\Gamma_n}{*} \pi_n.$$

Now we see that

$$\pi_1\big(\widetilde{X(f)}, \widetilde{x}\big) = \pi_1\big(\varinjlim \widetilde{Y}_n, \widetilde{x}\big) \xleftarrow[\uparrow]{\cong} \varinjlim \pi_1\big(\widetilde{Y}_n, \widetilde{x}\big) \xleftarrow[\uparrow]{\cong} \dots \pi_{-2} \underset{\Gamma_{-1}}{*} \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \underset{\Gamma_2}{*} \pi_2 \dots$$

since A, B are open in X we can apply the Direct Limit-π_1-Proposition 56.7 (3)

by the above calculation and the definition of the right hand side ⊞

In summary we obtain isomorphisms

$$\begin{array}{ccc} \pi_1(X(f), x) & \cong & \pi_1\big(\widetilde{X(f)}/\mathbb{Z}, [\widetilde{x}]\big) \\ & \cong & \pi_1(\widetilde{X(f)}, \widetilde{x}) \rtimes \mathbb{Z} \\ & \xrightarrow{\cong} & \langle \pi_1(X(f), x), t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle. \end{array}$$

above homeomorphism

by the Fundamental Group-via-Actions Theorem 48.16 (2) and and the Semidirect Product Lemma 50.5, here the automorphism Θ is given by conjugation by $[\gamma] \in \pi_1(X(f), x)$

Corollary 48.14

here, as on page 1294, we denote by Ξ the automorphism of the infinite amalgamated product that "shifts every term by one to the right", isomorphism is then given by the above claim and the verification that under this isomorphism the automorphisms Θ and Ξ match up

Lemma 57.4

We leave it to the reader to verify that the given isomorphism has all the properties that were promised. ■

Sketch of the proof of Theorem 57.3 (2). The proof of Theorem 57.3 (2) is basically the same as the proof of Theorem 57.3 (1), we just need to replace the Seifert–van Kampen Theorem 54.1 by the Seifert–van Kampen Theorem 54.4 for smooth manifolds. We leave it to the reader to fill in the details and to make the necessary changes to the proof. ■

In many situations the following corollary to the HNN-Seifert–van Kampen Theorem 57.3 is quite useful.

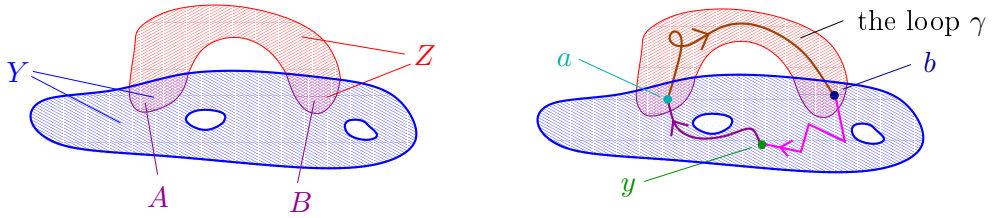
Corollary 57.5. Let X be a topological space. We suppose that it can be written as a union $X = Y \cup Z$ such that the following conditions are satisfied:

(1) Y and Z are open subsets,
(2) Y is path-connected,
(3) Z is simply connected,
(4) $Y \cap Z$ consists of two simply connected path components A and B, each of which is open in X.

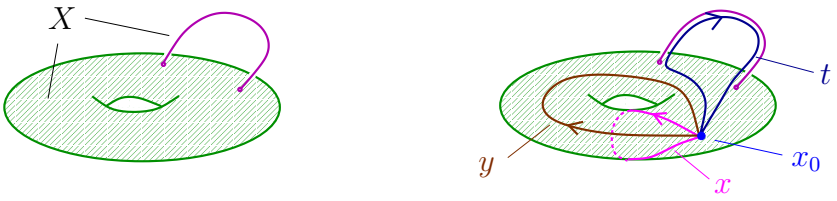
Let y be a point in Y and let γ be the concatenation of a path in Y from y to a point $a \in A$, a path in Z from a to a point $b \in B$ and a path in Y from b to y. Then the map

$$\pi_1(Y, y) * \langle t \rangle \ \to \ \pi_1(X, y)$$

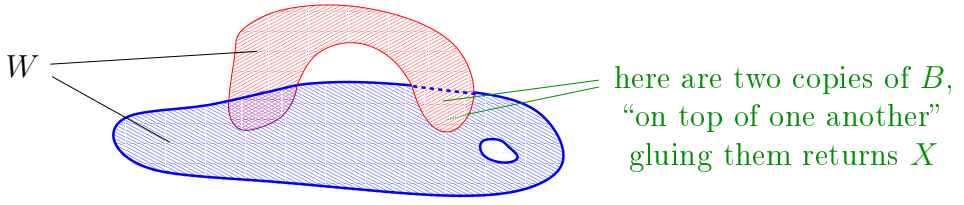
that is given by the inclusion induced map $\pi_1(Y, y) \to \pi_1(X, y)$ and given by $t \mapsto [\gamma]$ is an isomorphism.



Example. We consider the topological space X that is obtained from "gluing an interval" to the torus. Using the notation from the figure below one can show quite easily, using Corollary 57.5 that $\pi_1(X, x_0) \cong \langle x, y, t \,|\, [x, y] \rangle \cong \mathbb{Z}^2 * \langle t \rangle$. We will work out the details of this example in Exercise 57.1.



Proof. Let Z' be a disjoint copy of Z (e.g. we could set $Z' = Z \times \{0\}$). We consider $W = (Y \sqcup Z')/a \sim a'$ where we identify each point $a \in A$ with the corresponding point in Z'. We denote by B' the copy of B in Z' and thus also the copy in W. Thus we see



that B and its doppelgänger B' are contained in W. We denote by $f \colon B' \to B$ the obvious

homeomorphism. We obtain the following isomorphisms:[672]

$$\pi_1(X,y) \underset{\text{by Lemma 5.36 the obvious map } X \to W(f) \text{ is a homeomorphism}}{\uparrow\cong} \pi_1(W(f),y) \underset{\text{by the HNN-Seifert–van Kampen Theorem 57.3 and since } B \text{ is simply connected}}{\downarrow\cong} \pi_1(W,y)*\langle t\rangle \underset{\text{Seifert–van Kampen Theorem 52.2}}{\uparrow\cong} (\pi_1(Y,y)*_{\pi_1(A,y)}\pi_1(Z',y))*\langle t\rangle \underset{\text{since } A \text{ and } Z' \text{ are simply connected}}{\uparrow=} \pi_1(Y,y)*\langle t\rangle.$$

One can easily verify that this is precisely the map in the statement of the corollary. ■

We conclude this chapter with the following technical lemma which we will need once at a later occasion.

Lemma 57.6. Let $n \in \mathbb{N}$ and let M be the 2-sphere with n open disks removed. We enumerate the boundary components by $C_1, \dots, C_n$. For $i = 1, \dots, n$ we pick a path $\beta_i \colon [0,1] \to C_i$ that goes once around C_i. We pick a base point $* \in M \setminus \partial M$. We pick paths $\alpha_1, \dots, \alpha_n \colon [0,1] \to M$ with the following properties:

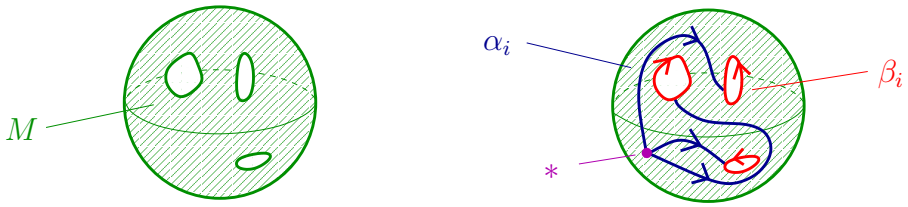
(1) each α_i is a smooth embedding,
(2) each α_i is a path from $*$ to the point $\beta_i(0)$,
(3) for each i we have $\alpha_i([0,1)) \subset M \setminus \partial M$,
(4) the images of the paths only intersect at $*$.

For any $j \in \{1, \dots, n\}$ the homomorphism

$$\begin{array}{rcl} \langle y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n\rangle & \to & \pi_1(M,*) \\ y_i & \mapsto & [\alpha_i * \beta_i * \overline{\alpha_i}] \end{array}$$

that is uniquely determined by

is an isomorphism.



Remark.

(1) This proposition should ring a bell, it sounds quite like Lemma 55.11. The difference is that in Lemma 55.11 we only said that there exist certain paths β_i such that the given map is an isomorphism. The above lemma gives a more precise statement what type of paths work.
(2) One might think that perhaps one can use any old paths α_i to obtain a basis for the free group $\pi_1(M,*)$. But as we will see in Exercise 57.6, this is not the case.

We conclude this chapter with a sketch of a proof of Lemma 57.6. Perhaps the most friendly words that can be uttered regarding the proof are that the proof shows that it can be quite hard to write down rigorous arguments once one leaves the friendly word of submanifolds and instead tries to deal with more general subsets.

[672] It follows from our hypothesis that A, Y, Z are open in X that in the settings of the HNN-Seifert–van Kampen Theorem 57.3 and the Seifert–van Kampen Theorem 52.2 we do in fact deal with open subsets.

Sketch of a proof. Without loss of generality we can assume that $j = n$. We consider $N := M \setminus \alpha_n([0,1])$. This is an open subset of the smooth manifold M, thus N itself is a smooth manifold. Furthermore by Exercise **??** we know that the inclusion $M \setminus \alpha_n([0,1]) \to M$ is a homotopy equivalence.[673] For $i = 1, \dots, n-1$ we write $X_i := \alpha_i((0,1]) \subset N$. It follows from (1) and the Smooth Embedding Theorem 24.10 (1) that $X = X_1 \cup \dots \cup X_{n-1}$ is a submanifold of N. By the Tubular Neighborhood Theorem **??** we can pick a tubular map $[-1,1] \times X$. We write $W := N \setminus (-1,1) \times X$, we pick a base point $\star \in W$ and we pick path p in M from $*$ to $\star$.

Using Corollary 57.5 one can prove[674] the following:

(1) there exist loops $\eta_1, \dots, \eta_{n-1}$ in $(N, \star)$ such that the obvious map

$$\pi_1(W, \star) * \langle [\eta_1], \dots, [\eta_{n-1}] \rangle \ \to \ \pi_1(N, \star)$$

is an isomorphism,

(2) for each $i \in \{1, \dots, n-1\}$ the loop $p * \eta_i * \overline{p}$ is path-homotopic to $\alpha_i * \beta_i * \overline{\alpha_i}$.

Putting everything together we see that we have an isomorphism

$$\pi_1(W) * \langle [\alpha_1 * \beta_1 * \overline{\alpha_1}], \dots, [\alpha_{n-1} * \beta_{n-1} * \overline{\alpha_{n-1}}] \rangle \ \xrightarrow{\cong} \ \pi_1(M).$$

By Lemma 55.11 we know that the right-hand side is a free group of rank $n-1$. It follows from the Grushko-Neumann Theorem 51.20 that $\pi_1(W)$ is trivial. We have thus obtained the desired isomorphism.[675] ■

Exercises for Chapter 57.

Exercise 57.1. Give full details for the calculation of the fundamental group of the topological space shown in the figure on page 1297.

Exercise 57.2. Let $\pi = \langle a \rangle$, let $\Gamma = \langle t \rangle$ and let $\alpha, \beta \colon \Gamma \to \pi$ be the homomorphisms given by $\alpha(t) = a$ and $\beta(t) = a^2$. Show that the corresponding HNN-extension is isomorphic to the semidirect product $\mathbb{Z}[\frac{1}{2}]_\varphi \rtimes \mathbb{Z}$ where $\varphi \colon \mathbb{Z}[\frac{1}{2}] \to \mathbb{Z}[\frac{1}{2}]$ is the group homomorphism given by multiplication by 2.

Exercise 57.3. Let X be a path-connected topological space. Furthermore let $f \colon X \to X$ be a homeomorphism. In the Mapping Torus-π_1-Proposition 50.6 we saw that there exists an isomorphism $\psi \colon \pi_1(\mathrm{Tor}(X, f)) \cong \pi_1(X) \rtimes \mathbb{Z}$. Give an alternative proof of this statement using the HNN-Seifert–van Kampen Theorem 57.3.

---

[673] The alert reader will not have failed to notice that we are referring to an exercise that is dealt with later in the notes. Fortunately we do not need to be afraid of circular logic. The exercise builds on an extension of the Tubular Neighborhood Theorem **??** to submanifolds with boundary. This formulation, see Theorem **??**, is just a little awkward and was thus postponed to later on. But if one uses a naive extension of the Tubular Neighborhood Theorem **??** to the case of submanifolds with boundary, then one can easily prove Exercise **??** with our present knowledge.

[674] Here is the point where the demanding reader might feel that not enough details are provided.

[675] The esthetically-inclined reader might object to the use of the Grushko-Neumann Theorem 51.20. But any other approach also opens up a can of worms. For example one might be tempted to show that W is contractible.

Exercise 57.4. Let X be the result of removing two open disks from the surface of genus g. We denote by C and D the resulting boundary components. We pick orientation-preserving diffeomorphisms $\varphi\colon S^1 \to C$ and $\psi\colon S^1 \to D$.

(a) We consider the homeomorphism

$$\begin{array}{rcl} f\colon C \times S^1 & \to & D \times S^1 \\ (c, z) & \mapsto & (\psi(z), \varphi^{-1}(c)). \end{array}$$

What is the abelianization of the fundamental group of $(X \times S^1)(f)$?

(b) Does the answer to (a) change if we pick φ to be orientation-preserving and ψ to be orientation-reversing?

Exercise 57.5. Let X be a topological space. In the Suspension-π_1-Lemma 54.2 we showed that if X is non-empty and path-connected, then the fundamental group of the suspension $\Sigma(X)$ is trivial.

(a) Suppose that X has two path-components. Show that $\pi_1(\Sigma(X)) \cong \mathbb{Z}$.

(b) Suppose that X has k path-components with $k \geq 1$. Show that $\pi_1(\Sigma(X))$ is isomorphic to a free group on $k-1$ generators.

Exercise 57.6. We consider the permutation group S_4 where given $\alpha, \beta \in S_4$ we write $\alpha \cdot \beta = \alpha \circ \beta$. We consider the cycles[676]

$$\sigma = (1\,2\,3\,4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \mu = (1\,2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \quad \text{and } \nu = (1\,3) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}.$$

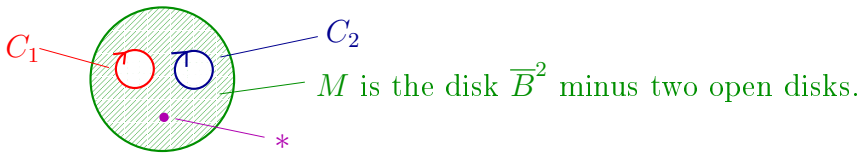
(a) Show that σ and μ form a generating set for S_4.

(b) Show that σ and ν do not form a generating set for S_4.

(c) Show that $(\mu \cdot \sigma) \cdot \mu \cdot (\mu \cdot \sigma)^{-1} = \nu$.

(d) Show that the endomorphism $\langle x, y\rangle \to \langle x, y\rangle$ that is given by $x \mapsto (yx)^{-1}x(yx)$ and $y \mapsto y$ is not an isomorphism.

(e) We consider M which is defined as the disk $\overline{B}^2$ minus two open disks[677] We consider the base point $*$ and the curves β_1 and β_2 as shown in the figure below. Show that for $i = 1, 2$ there exist paths α_i from $*$ to the starting point of β_i such that $[\alpha_1 * \beta_1 * \overline{\alpha_1}]$ and $[\alpha_2 * \beta_2 * \overline{\alpha_2}]$ do *not* form a generating set for $\pi_1(M, *)$.



[676] Recall that for pairwise disjoint $a_1, \ldots, a_k \in \{1, \ldots, n\}$ we denote by $(a_1\ a_2\ \ldots\ a_k) \in S_n$ the permutation that is given by

$$\begin{array}{rcl} \{1, \ldots, n\} & \to & \{1, \ldots, n\} \\ m & \mapsto & \begin{cases} a_{i+1}, & \text{if } m = a_i \text{ for some } i \in \{1, \ldots, k-1\}, \\ a_1 & \text{if } m = a_k, \\ m, & \text{otherwise.} \end{cases} \end{array}$$

Put differently, the numbers $(a_1, \ldots, a_k)$ get cyclically permuted and all other numbers stay the same.

[677] It follows easily from stereographic projection that M is diffeomorphic to S^2 minus three open disks.

58. Introduction to knot theory

Before we introduce knots we recall several different ways how we can describe the 3-sphere S^3. More precisely, by definition and Lemma 2.18 we have

$$S^3 = \{(z,w)\in\mathbb{C}^2 \,|\, |w|^2+|z|^2=1\} = \{(w,x,y,z)\in\mathbb{R}^4 \,|\, w^2+x^2+y^2+z^2=1\} \underset{\text{via the stereographic projection from Lemma 2.18}}{\cong} \mathbb{R}^3\cup\{\infty\}.$$

We will go back and forth between these models without mentioning these maps. In particular, as discussed on page 659, we will always view $\mathbb{R}^3$ as a submanifold of the smooth manifold $S^3 = \mathbb{R}^3 \cup \{\infty\}$.

Definition. A knot is a submanifold of $S^3 = \mathbb{R}^3 \cup \{\infty\}$ that is diffeomorphic to S^1.[678] We say that the knot is oriented if it is oriented as a 1-dimensional smooth manifold.

Remark.

(1) It follows from the Smooth Embedding Theorem 24.10 that a subset of S^3 is a knot if and only if it is the image of a smooth embedding $S^1 \to S^3$.[679]
(2) The definition of a knot is supposed to model the "physical objects" that we have in mind and that are sketched in the figure below. It is therefore perhaps at first not clear why we consider knots in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ instead of knots in $\mathbb{R}^3$. The reason is that topologists prefer, if possible, to work with compact spaces. In particular the compact space $S^3 = \mathbb{R}^3 \cup \{\infty\}$ is often strongly preferable to the non-compact space $\mathbb{R}^3$.

We start out with the most boring but also the most important knot.

Definition. The trivial knot is defined as $\{(x,y,0) \in \mathbb{R}^3 \,|\, x^2+y^2 = 1\} \subset \mathbb{R}^3 \cup \{\infty\} = S^3$.

In the figure below we show the trivial knot together with two other examples of knots. More precisely, we show 1-dimensional submanifolds of $\mathbb{R}^3$ that we view as 1-dimensional submanifolds of $\mathbb{R}^3 \cup \{\infty\} = S^3$.[680]
Now we want to say that two knots are "the same" if one can be "deformed" into the other. Fortunately we already have the language to make this picture precise: The notion of a smooth isotopy between (oriented) submanifolds that we introduced on page 817 makes this notion precise. For the reader's convenience we repeat the definition of "smoothly isotopic" in our context.

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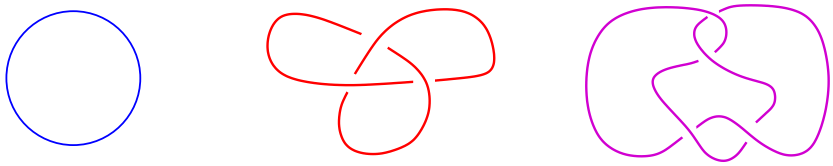
[678] Sometimes, to distinguish this knot from the higher dimensional siblings that we will introduce on pages 1338 and **??** we also refer to such a knot as a *classical knot*.

[679] To be more precise, the "if"-direction is a consequence of Theorem 24.10. On the other hand, if $K \subset S^3$ is a knot, then by definition there exists a diffeomorphism $\varphi\colon S^1 \to K \subset S^3$ and it follows easily from the definitions that the map φ viewed as a map from $S^1 \to S^3$ is a smooth embedding whose image is precisely K.

[680] In principle it is possible to give a precise description of all these three knots. For example we just gave a definition of the trivial knot as $\{(x,y,0) \,|\, x^2+y^2=1\} \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$. Later, on page 1312, we will give a precise definition of the trefoil. It is also clear that one can give a precise description of the figure-8 knot, but this description would be painful to write down and it would not add to our understanding. We therefore stick with the picture, with the understanding, that if somebody was challenging us, we could write down a precise description in coordinates. But it is considered very impolite to challenge a topologist to give a rigorous description.



trivial knot trefoil figure-8 knot

Definition. We say that two knots K and J are smoothly isotopic if there exists a smooth isotopy from K to J, i.e. a smooth map

$$\begin{aligned} F\colon S^1 \times [0,1] &\to S^3 \\ (z,t) &\mapsto F(z,t) \end{aligned}$$

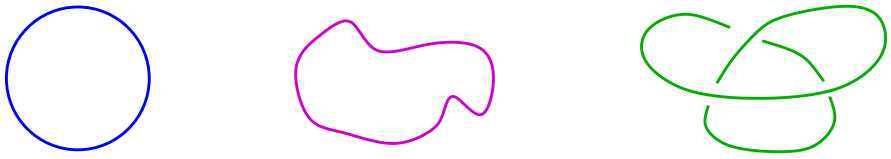
such that the following hold:

(1) we have $F(S^1 \times \{0\}) = K$ and $F(S^1 \times \{1\}) = J$, and
(2) for each $t \in [0,1]$ the map $S^1 \to S^3$ given by $z \mapsto F(z,t)$ is a smooth embedding.

If K and J are oriented, then we demand that the map $K \to J$ given by $x \mapsto F_1(F_0^{-1}(x))$ is orientation-preserving.

Remark. It follows from the Smooth Isotopy Transitivity Proposition 33.3 that the property of being smoothly isotopic defines an equivalence relation on the set of knots.

Example. The three knots shown in the figure below are smoothly isotopic. Usually we do not distinguish two knots if they are smoothly isotopic. For example, any knot that is smoothly isotopic to a trivial knot is called *trivial knot*. Sometimes we also say that a knot that is smoothly isotopic to a trivial knot is *trivial*.



Playing around with pictures for some time shows that it might be quite difficult to show that the trefoil is smoothly isotopic to the trivial knot. This arouses the suspicion, that the trefoil is in fact not smoothly isotopic to the trivial knot. This raises the following question.

Question 58.1. How can we show that the trefoil is not smoothly isotopic to the trivial knot?

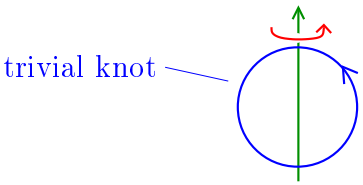
It is also fun to consider orientations. This leads us to the following definition.

Definition. A knot K is called reversible if the two orientations on K give smoothly isotopic oriented knots.[681]

Example. As we see in the figure below it is basically clear that the trivial knot is reversible. In Exercise 34.7 we showed that the trefoil and the figure-8 knot are also reversible.

This leads us to the following question which we will answer on page 2859.

[681] In the literature what we call a reversible knot is often called an invertible knot. We reserve the adjective invertible for a related, but different notion, see page 2857.



rotation around the z-axis
by the angle π turns the trivial knot
into itself but flips the orientation

Question 58.2. Is every knot reversible?

In the remainder of this section we will ignore orientations and instead we will discuss the following generalization of Question 58.1.

Question 58.3. Given a knot K[682], how can we determine whether or not K is smoothly isotopic to the trivial knot?

The following proposition allows us to turn these two questions into the question, whether two topological spaces are diffeomorphic.

Proposition 58.4. Let K and J be two knots. If they are smoothly isotopic, then the following two statements hold:

(1) There exists an orientation-preserving diffeomorphism $\Phi\colon S^3 \to S^3$ with $\Phi(K) = J$. If K and J are oriented, then Φ restricts to an orientation-preserving diffeomorphism $K \to J$.
(2) There exists an orientation-preserving diffeomorphism between the knot complements $S^3 \setminus K$ and $S^3 \setminus J$.

The proposition is basically a special case of the Smooth Isotopies-Complements Proposition 36.3. But for the reader's convenience let us recall the argument. The key input is the following (simplified) version of the Isotopy Extension Theorem 36.1.

Theorem (Isotopy Extension Theorem 36.1) Let M and K be two closed smooth manifolds. Given any smooth isotopy $F\colon K \times [0,1] \to M$ there exists a diffeotopy $G\colon M \times [0,1] \to M$ with the following properties:

(1) $G_0 = \mathrm{id}$.
(2) We have $G \circ (F_0 \times \mathrm{id}) = F\colon N \times [0,1] \to M$, in other words, we have

$$G(F_0(x), t) \;=\; F(x,t) \quad \text{for all } x \in N \text{ and } t \in [0,1].$$

Proof of Proposition 58.4. Let

$$\begin{array}{rcl} F\colon S^1 \times [0,1] & \to & S^3 \\ (z,t) & \mapsto & F(z,t) \end{array}$$

be a smooth isotopy from a knot K to a knot J. By the Isotopy Extension Theorem 36.1 we can extend the smooth isotopy F to a diffeotopy of S^3. This means in particular that there exists a diffeotopy

$$\begin{array}{rcl} G\colon S^3 \times [0,1] & \to & S^3 \\ (z,t) & \mapsto & G(z,t) \end{array}$$

from the identity to a diffeomorphism $\Phi = G_1\colon S^3 \to S^3$ with $\Phi|_K = F_1$, in particular with $\Phi(K) = J$. By the Smooth Isotopy-Orientation Proposition 34.2 the diffeomorphism Φ is

[682]The phrase "given a knot K" is consciously slightly vague.

orientation-preserving. The map Φ restricts to an orientation-preserving diffeomorphism $\Phi\colon S^3 \setminus K \to S^3 \setminus J$. ■

It is natural to ask to what degree the converse of Proposition 58.4 holds. It turns out that in both cases the converse holds, but in both cases these are difficult theorems. First we have the following theorem that was proved in 1969 by Jean Cerf [**Cer1968**].

Theorem 58.5. (Cerf Theorem) Every orientation-preserving diffeomorphism of S^3 is diffeotopic to the identity.

Now we obtain the promised converse to Proposition 58.4 (1).

Corollary 58.6. Let K and J be two knots. If there exists an orientation-preserving diffeomorphism $\Phi\colon S^3 \to S^3$ with $\Phi(K) = J$, then K and J are smoothly isotopic.

Proof. Let $\Phi\colon S^3 \to S^3$ be an orientation-preserving diffeomorphism with $\Phi(K) = J$. By the Cerf Theorem 58.5 there exists a diffeotopy $F\colon S^3 \times [0,1] \to S^3$ with $F_0 = \mathrm{id}$ and $F_1 = \Phi$. The restriction of F to $[0,1] \times K$ is the desired smooth isotopy from K to J. ■

The following theorem also gives us the converse to Proposition 58.4 (2).

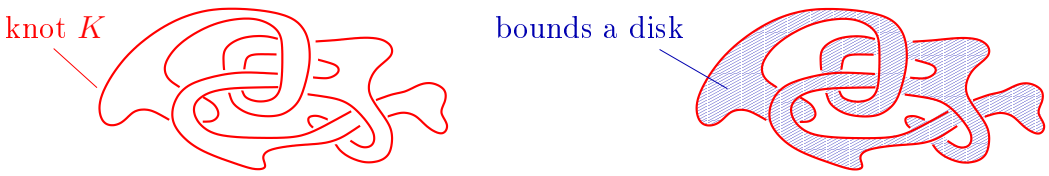
Theorem 58.7. (Gordon-Luecke) Let K and J be two knots in S^3. If there exists an orientation-preserving diffeomorphism between the knot complements $S^3 \setminus K$ and $S^3 \setminus J$, then K and J are smoothly isotopic.

Proof. This theorem is significantly more difficult to prove than Proposition 58.4. In fact a proof was given only in the 1980s by Cameron Gordon and John Luecke [**GL1989**, Theorem 1]. The proof builds on the work of William Thurston [**Thu1982**] for which he was awarded the Fields medal in 1982. ■

The following proposition gives a neat characterization of trivial knots.

Proposition 58.8. (Genus-Zero Knots Proposition) Let K be a knot. The following two statements are equivalent:

(1) The knot K bounds a smoothly embedded disk, i.e. there exists a smooth embedding $\varphi\colon \overline{B}^2 \to S^3$ with $\varphi(S^1) = K$.
(2) The knot K is smoothly isotopic to the trivial knot.



Proof. Let K be a knot. We need to prove the equivalence of (1) and (2).

First we prove the almost trivial "(2)⇒(1)"-direction. It is clear that the trivial knot $\{(x,y,0) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1\} \subset \mathbb{R}^3 \cup \{\infty\} = S^3$ bounds a disk. It follows immediately from the Isotopy Extension Theorem 36.1 that any knot that is smoothly isotopic to the trivial knot also bounds a smoothly embedded disk.

Now we turn to the proof of the much more interesting "(1)⇒(2)"-direction. In fact this is a special case of the Lower-Dimensional Smooth Ball Embedding Theorem 37.10. But to make this section slightly more self-contained we show how one can deduce this statement directly from the Diffeotopies-of-$\mathbb{R}^n$-Classification Proposition 34.3 (2).

Thus let $\varphi\colon \overline{B}^2 \to S^3 = \mathbb{R}^3 \cup \{\infty\}$ be a smooth embedding with $\varphi(S^1) = K$. We write $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ and we make the identification $\mathbb{C} = \mathbb{C} \times \{0\} \subset \mathbb{R}^3$. It suffices to show that there exists a smooth isotopy from φ to a smooth embedding $\psi\colon \overline{B}^2 \to \mathbb{R}^3$ such that $\psi(S^1) = S^1 \subset \mathbb{C}$. We perform the following steps:

(0) By the Matrix Group Path-Component Proposition 2.37 (3) we know that $\mathrm{SO}(4)$ is path-connected. Together with the Smooth Path-Connectivity Proposition 19.29 we see that we can find a smooth isotopy such that afterwards we have $\infty \notin \varphi(\overline{B}^2)$.

(1) Using that $\mathrm{SL}(3,\mathbb{R})$ is path-connected, see again the Matrix Group Path-Component Proposition 2.37 (2), and using again the Smooth Path-Connectivity Proposition 19.29 we can find a smooth isotopy such that afterwards we have $\varphi(0) = 0$ and $(\mathrm{D}\varphi_0)(\mathrm{T}_0\overline{B}^2) = \mathbb{C} \times \{0\}$.

(2) We can shrink the disk, i.e. we can precompose the map $\varphi\colon \overline{B}^2 \to \mathbb{C} \times \mathbb{R}$ with the smooth isotopy $\overline{B}^2 \times [\epsilon, 1] \to \overline{B}^2$ that is given by $(p,t) \mapsto p \cdot t$ for some $\epsilon \in (0,1]$, such that composition $p \circ \varphi$ of φ with the projection $p\colon \mathbb{R}^3 = \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ is an embedding.

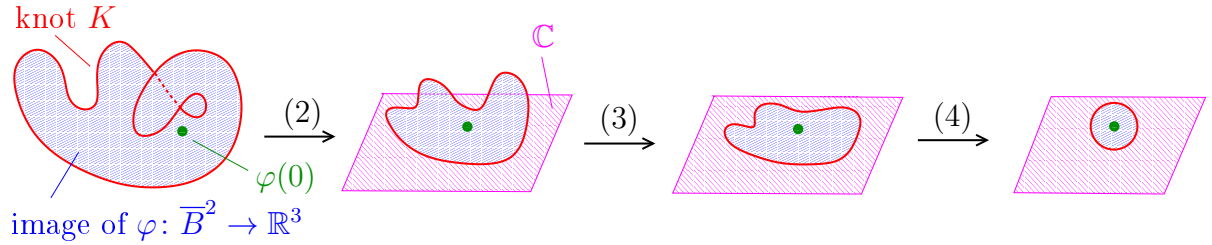
(3) We write $\varphi = \varphi_{\mathbb{C}} \times \varphi_{\mathbb{R}}\colon \overline{B}^2 \to \mathbb{C} \times \mathbb{R}$. It follows from (2) that the map

$$\begin{array}{rcl} \overline{B}^2 \times [0,1] & \to & \mathbb{C} \times \mathbb{R} \\ (x,t) & \mapsto & (\varphi_{\mathbb{C}}(x), \varphi_{\mathbb{R}}(x) \cdot t) \end{array}$$

is a smooth isotopy. This shows that we only need to deal with a smooth embedding $\varphi\colon \overline{B}^2 \to \mathbb{C}$ with $\varphi(0) = 0$.

(4) Now we can apply the Diffeotopies-of-$\mathbb{R}^n$-Classification Proposition 34.3 (2).

We illustrate the steps (2) to (4) in the figure below. We leave it to the reader to fill in all the technical details and to make all steps rigorous. ■



We continue our discussion of knots with a definition.

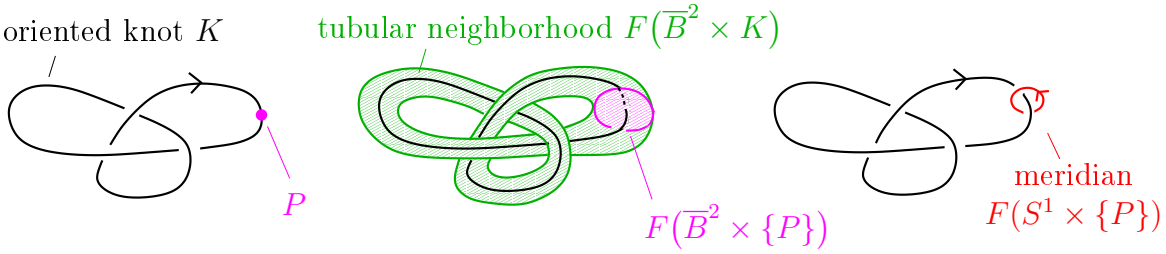
Definition. Let K be an oriented knot in S^3. By the Tubular Neighborhood Theorem **??** there exists an orientation-preserving smooth embedding

$$F\colon \overline{B}^2 \times K \to S^3$$

such that $F(0,P) = P$ for all $P \in K$. For any $P \in K$ we refer to the oriented submanifold[683] $F(S^1 \times \{P\})$ as a meridian of K.

A priori the definition of the meridian depends on the choice of the tubular neighborhood. The following lemma, which is an immediate consequence of Proposition **??**, says that the meridian is well-defined up to smooth isotopy.

[683]It follows immediately from the Smooth Embedding Theorem 24.10 that $F(S^1 \times \{P\})$ is indeed a submanifold of $S^3 \setminus K$.



Lemma 58.9. The meridian of an oriented knot in S^3 is well-defined up to smooth isotopy in $S^3 \setminus K$.

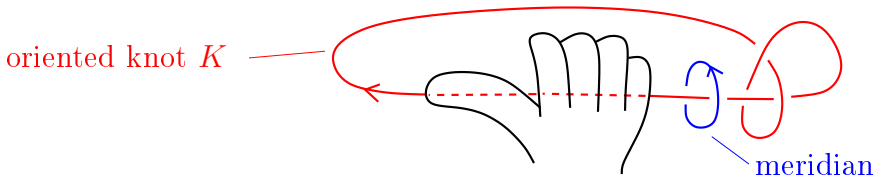
The following proposition gives a convenient criterion for showing that a given closed oriented curve is indeed a meridian.

Proposition 58.10. Let K be an oriented knot in S^3. Let $\mu \subset S^3 \setminus K$ be a closed oriented curve. Suppose there exists a smooth embedding $\varphi\colon \overline{B}^2 \subset S^3$ with the following two properties:

(1) the map $\varphi|_{S^1}$ defines an orientation-preserving diffeomorphism $S^1 \to \mu$,
(2) the image $\varphi(\overline{B}^2)$ intersects K transversally in a single point P,
(3) a positive basis for $\mathrm{T}_P(\varphi(\overline{B}^2))$ together with a positive basis for $\mathrm{T}_P K$ gives a positive basis for $\mathrm{T}_P S^3$,[684]

then μ is a meridian of K.

Remark. More casually speaking, Proposition 58.10 says that a meridian is any closed curve which circles once around a knot where the orientation is given by the right-hand-rule, i.e. if the thumb points into the direction of the knot, then the fingers point into the direction of the meridian. This rule is illustrated in the figure below.



Sketch of proof. By the Tubular Neighborhood Theorem **??** there exists an orientation-preserving smooth embedding $F\colon \overline{B}^2 \times K \to S^3$ such that $F(0, P) = P$ for all $P \in K$. For convenience we denote the image of F by $\overline{B}^2 \times K$. We perform the following steps which are quite similar to the steps in the proof of the Genus-Zero Knots Proposition 58.8.

(a) We pick a point $Q \neq P$ on K. We can shrink the disk such that $\varphi(\overline{B}^2)$ is contained in $\overline{B}^2 \times (K \setminus \{Q\})$.
(b) It follows from our hypothesis (1) and the definition of differentiability that we can shrink the disk even further, such that the map

$$\overline{B}^2 \xrightarrow{\varphi} \overline{B}^2 \times (K \setminus \{Q\}) \xrightarrow{(v,z)\mapsto(v,P)} \overline{B}^2 \times \{P\}$$

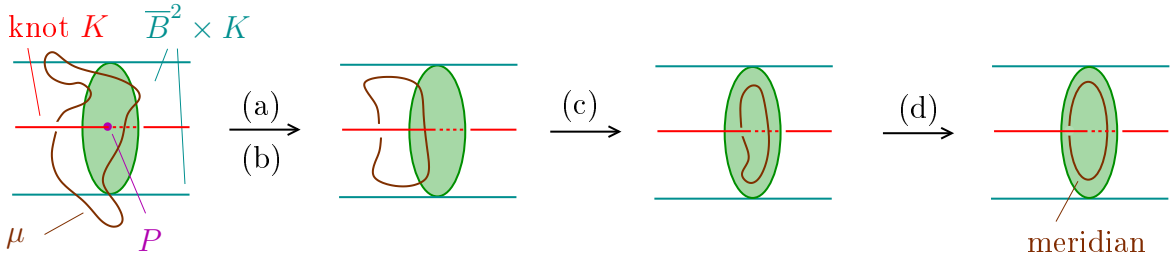
[684] Later, on page 2748, we will say that P is a positive intersection point of $\varphi(\overline{B}^2)$ and K.

is an embedding. It follows from the hypothesis (2) that this map is actually orientation-preserving.

(c) Similar to Step (3) of the proof of the Genus-Zero Knots Proposition 58.8 we see that φ is smoothly isotopic, rel $(0, P)$, to an orientation-preserving embedding into $\overline{B}^2 \times \{P\}$.

(d) The statement now follows from the Diffeotopies-of-$\mathbb{R}^n$-Classification Proposition 34.3 (2).

As in the proof of the Genus-Zero Knots Proposition 58.8 we leave it to the reader to fill in the details. ■



test test

Later on, in Chapter 59, we will study the fundamental group of a knot complement. Implicitly we will make use of the following lemma.

Lemma 58.11. Given any knot $K \subset S^3$ the complement $S^3 \setminus K$ is path-connected.

Proof. We will provide the proof in Exercise **??**. ■

59. An excursion into knot theory

In this chapter we will study the fundamental groups of the complements of knots. In particular we will use these calculations to show that the trivial knot, the trefoil and the figure-8 knot are pairwise different, i.e. we will show that they are pairwise not smoothly isotopic.

59.1. **One more avatar of spheres.** The spheres $S^n = \{x \in \mathbb{R}^{n+1} \,|\, \|x\| = 1\}$ are some of the most important topological spaces. We already saw that there are different ways to think about spheres, in particular on pages 156 and 261 and in the Ball-Quotient Sphere Lemma 5.20 we made the following identifications

$$S^n = \mathbb{R}^n \cup \{\infty\}, \quad S^n = \overline{B}^n/S^{n-1}, \quad \text{and} \quad S^n = \overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_-.$$

Furthermore, on page 263 we had given a canonical homeomorphism

$$(S^{m-1} \times \overline{B}^n) \cup_{S^{m-1}\times S^{n-1}} (\overline{B}^m \times S^{n-1}) \xrightarrow{\cong} S^{m+n-1}.$$

Even though the homeomorphism is canonical, it is not particularly explicit. Furthermore it has the disadvantage that, if one looks up the definition, one sees that the restriction to the two solid tori $S^{m-1} \times \overline{B}^n$ and $\overline{B}^m \times S^{n-1}$ is not smooth. The following lemma rectifies these two problems.

Lemma 59.1. Let $m, n \in \mathbb{N}_0$. We consider the following two maps:

$$\begin{array}{rcl} \Phi\colon S^{m-1} \times \overline{B}^n & \to & A := \left\{(z,w) \in \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n} \,\middle|\, \|w\|^2 + \|z\|^2 = 1 \text{ and } \|z\|^2 \geq \frac{1}{2}\right\} \\ (a,b) & \mapsto & \left(\frac{a}{\sqrt{\|a\|^2+\|b\|^2}}, \frac{b}{\sqrt{\|a\|^2+\|b\|^2}}\right) \end{array}$$

and

$$\begin{array}{rcl} \Psi\colon \overline{B}^m \times S^{n-1} & \to & B := \left\{(z,w) \in \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n} \,\middle|\, \|w\|^2 + \|z\|^2 = 1 \text{ and } \|w\|^2 \geq \frac{1}{2}\right\} \\ (a,b) & \mapsto & \left(\frac{a}{\sqrt{\|a\|^2+\|b\|^2}}, \frac{b}{\sqrt{\|a\|^2+\|b\|^2}}\right). \end{array}$$

These maps have the following properties:

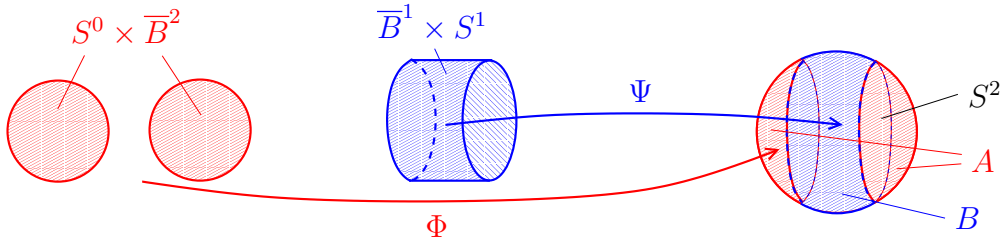
(1) The map $\Phi\colon S^{m-1} \times \overline{B}^n \to A$ is an orientation-preserving diffeomorphism.
(2) The map $\Psi\colon \overline{B}^m \times S^{n-1} \to B$ is a diffeomorphism, it is orientation-preserving if and only if m is even.
(3) The map[685]

$$\begin{array}{rcl} \Theta\colon (S^{m-1} \times \overline{B}^n) \cup_{S^{m-1}\times S^{n-1}} (\overline{B}^m \times S^{n-1}) & \to & S^{m+n-1} \\ [P] & \mapsto & \begin{cases} \Phi(P), & \text{if } P \in S^{m-1} \times \overline{B}^n, \\ \Psi(P), & \text{if } P \in \overline{B}^m \times S^{n-1} \end{cases} \end{array}$$

is well-defined, it is a diffeomorphism, and it is orientation-preserving if and only if m is even.

[685] We equip the boundaries of $S^{m-1} \times S^n$ and $\overline{B}^m \times S^{n-1}$ with the "obvious" collars. We use these collars and the Gluing-Smooth Manifolds-Proposition ?? (1b) to equip the left-hand side with the structure of a smooth manifold. Furthermore we use the Gluing-Smooth Manifolds-Proposition ?? (6) to equip the left-hand side with the orientation, where the orientation matches the orientation of $S^{m-1} \times \overline{B}^n$ and which matches the orientation of $\overline{B}^m \times S^{n-1}$ if and only if m is even.

Convention. We use the diffeomorphism Θ from Lemma 59.1 (2) to add the smooth manifold $(S^{m-1}\times\overline{B}^n)\cup_{S^{m-1}\times S^{n-1}}(\overline{B}^m\times S^{n-1})$ to our list of avatars of S^{m+n-1}.



Sketch of proof.

(2) Since the orientations are a little more interesting in the second case let us treat the second case in detail. We consider the map

$$\begin{aligned}\widetilde{\Psi}\colon B=\big\{(z,w)\in\mathbb{R}^m\times\mathbb{R}^n=\mathbb{R}^{m+n}\;\big|\;\|w\|^2+\|z\|^2=1\;\&\;\|w\|^2\geq\tfrac{1}{2}\big\}\;&\to\;\overline{B}^m\times S^{n-1}\\(z,w)\;&\mapsto\;\Big(\tfrac{z}{\|w\|},\tfrac{w}{\|w\|}\Big).\end{aligned}$$

One can easily verify that $\widetilde{\Psi}$ and Ψ are inverses of one another and that Ψ and $\widetilde{\Psi}$ are smooth. This observation implies that $\Psi\colon\overline{B}^m\times S^{n-1}\to B$ is a diffeomorphism. It remains to show that Ψ is orientation-preserving. By the Basics-of-Orientations Lemma 25.13 it suffices to prove the following claim.

Claim. For $e_{m+1}=((0,\dots,0),(1,0,\dots,0))\in\overline{B}^m\times S^{n-1}$ the corresponding differential $\mathrm{D}\Psi_{e_{m+1}}\colon\widetilde{\mathrm{T}}_{e_{m+1}}(\overline{B}^m\times S^{n-1})\to\widetilde{\mathrm{T}}_{\Psi(e_{m+1})}B$ is orientation-preserving if and only if m is even.

Proof. By the orientation conventions from page 681 and the Smooth Manifold Product Proposition 19.31 we know that $\{e_1,\dots,e_m,e_{m+2},\dots,e_{m+n}\}$ is a positive basis for $\widetilde{\mathrm{T}}_{e_{m+1}}(\overline{B}^m\times S^{n-1})$. First note that a straightforward calculation shows that the differential of Ψ at the point e_{m+1} is actually the identity. Next note that $\Psi(e_{m+1})=e_{m+1}$. Finally note that by definition $\{v_1,\dots,v_{m+n-1}\}$ is a positive basis for $\widetilde{\mathrm{T}}_{\Psi(e_{m+1})}B=\widetilde{\mathrm{T}}_{e_{m+1}}S^{m+n-1}$ if and only if $\det(e_{m+1},v_1,\dots,v_{m+n-1})>0$. The claim follows almost immediately from these observations. ⊞

(1) The proof of this statement is almost identical to the above proof of (2).

(3) It is clear that Φ and Ψ agree on $S^{m-1}\times S^{n-1}$. Therefore it follows from the Topological-Quotient Proposition 5.15 (1b) that the given map $\Theta\colon(S^{m-1}\times\overline{B}^n)\cup_{S^{m-1}\times S^{n-1}}(\overline{B}^m\times S^{n-1})\to S^{m+n-1}$ is well-defined and continuous. Using (1) one easily verifies that the map is a bijection. It follows from the Compact-Hausdorff Proposition 2.17 (3) that the map Θ is a homeomorphism. Going through the definition of the smooth structure of the smooth manifold on the left, as defined in the Gluing-Smooth Manifolds-Proposition **??** (1b), shows that the map is actually a diffeomorphism. Finally it follows from (1) and (2) that the diffeomorphism is actually orientation-preserving. ■

Throughout this chapter we are mostly interested in the case $m = n = 2$. In this setting we have the following identifications which are given by definition and b Lemma 2.18:

$$S^3 = \big\{(w,x,y,z)\,\big|\,w^2+x^2+y^2+z^2=1\big\} = \big\{(z,w)\in\mathbb{C}^2\,\big|\,|w|^2+|z|^2=1\big\} \underset{\uparrow}{=} \mathbb{R}^3\cup\{\infty\}.$$

via the stereographic projection as defined in Lemma 2.18
note that in Exercise 25.9 we saw that this map
is (lucky us!) orientation-preserving

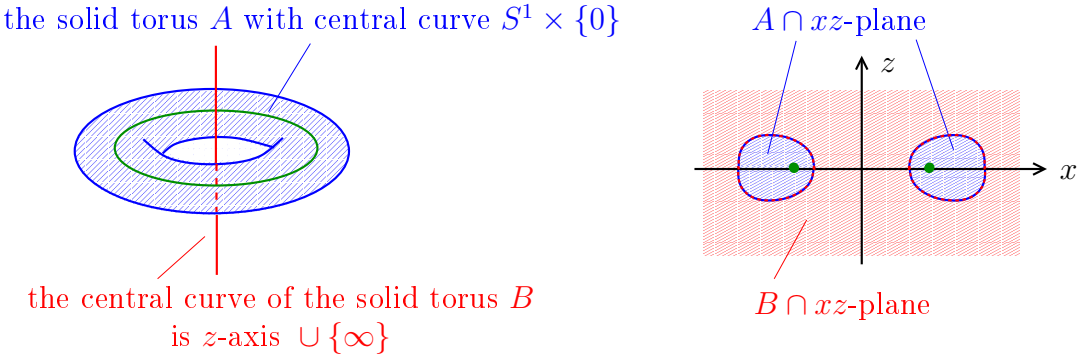
We will go back and forth between these models without mentioning these maps. In particular, as discussed on page 659, we will view $\mathbb{R}^3$ as a submanifold of the smooth manifold $S^3 = \mathbb{R}^3\cup\{\infty\}$. Furthermore we now view A and B as subsets of $S^3 = \mathbb{R}^3\cup\{\infty\}$. In this setting we have

$$\text{“central curve of } A\text{”} = \Phi(S^1\times\{0\}) = S^1\times\{0\} = \text{the trivial knot}$$

and we have

$$\text{“central curve of } B\text{”} = \Psi(\{0\}\times S^1) = \text{the } z\text{-axis}\cup\{\infty\}.$$

We visualize the solid torus A and the central curve of the solid torus B in the figure below. The solid torus B is more difficult to illustrate. It consists of a closed disk attached to each point on the central curve. For example the closed disk attached to the origin is just the “obvious” closed disk in the xy-plane that touches the torus $\Phi(S^1\times S^1)$.[686]



Lemma 5.7 and Lemma 59.1 give us two smooth embeddings of the 2-dimensional torus $S^1\times S^1$ into $\mathbb{R}^3$. The following lemma says that these two smooth embeddings are essentially the same.

Lemma 59.2. We denote by $\Omega\colon S^3\setminus\{(0,1)\}\to\mathbb{R}^3$ the stereographic projection as defined in Lemma 2.18. We consider the two smooth embeddings[687][688]

$$\begin{aligned}\overline{B}^2\times S^1 &\xrightarrow{\Theta} \mathbb{R}^3 \\ ((x,y),\exp(\mathrm{i}\varphi)) &\mapsto \begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1+\frac{x}{2}\\ 0\\ -\frac{y}{2}\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\overline{B}^2\times S^1 &\xrightarrow{\Xi} S^3\setminus\{(0,1)\} \xrightarrow{\Omega} \mathbb{R}^3 \\ (z,w) &\mapsto \Big(\tfrac{w}{\sqrt{|z|^2+|w|^2}}, \tfrac{z}{\sqrt{|z|^2+|w|^2}}\Big)\end{aligned}$$

provided by Lemma 5.7 and provided by Lemma 59.1. There exists a diffeotopy G of S^3 rel $\{(0,0)\}\times S^1$ with $G_0 = \mathrm{id}$ and $G_1\circ\Theta = \Omega\circ\Xi$.

---

[686] What is the disk attached to the point ∞?

[687] Note that $\Xi\colon \overline{B}^2\times S^1\to S^3$ is just the composition of the swap $\overline{B}^2\times S^1\to S^1\times\overline{B}^2$ and the smooth embedding $\Phi\colon S^1\times\overline{B}^2\to S^3$.

[688] As a reality check note that it follows from Lemma 2.18 that Θ and $\Omega\circ\Xi$ agree on $\{(0,0)\}\times S^1$.

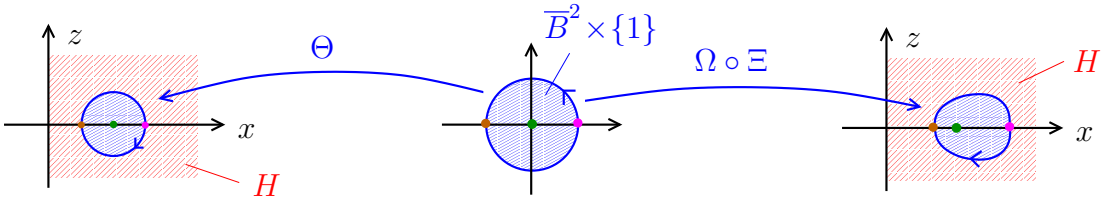
Convention. In all the future examples we will not distinguish between the two smooth embeddings of the solid torus $\overline{B}^2 \times S^1 \to \mathbb{R}^3$ that are given in Lemma 59.2.

Sketch of proof. By the Isotopy Extension Theorem 36.1 it suffices to show that there exists a smooth isotopy rel $\{(0,0)\} \times S^1$ from Θ to $\Omega \circ \Xi$.

We write $H = \{(x,0,z) \in \mathbb{R}^3 \,|\, x > 0\}$. Let us first consider the restrictions of Θ and $\Omega \circ \Xi$ to the disk $\overline{B}^2 \times \{1\}$. In both cases the image lies in H. These maps are illustrated in the figure below. We leave it to the dear reader to show that there exists a smooth isotopy $F\colon \overline{B}^2 \times \{1\} \times [0,1] \to H$ rel $\{(0,0,1)\}$ from Θ to $\Omega \circ \Xi$.

The desired isotopy is given by "rotating F" around the z-axis. Slightly more precisely it is given by the map

$$\begin{array}{rcl} \overline{B}^2 \times S^1 \times [0,1] & \to & \mathbb{R}^3 \\ ((x,y), \exp(\mathrm{i}\varphi), t) & \mapsto & \text{rotation of } F((x,y),t) \in \mathbb{R}^3 \text{ by } \varphi \text{ around the } z\text{-axis.} \end{array}$$ ■



59.2. **The fundamental group of the complement of the trefoil.** Now we want to finally turn to the study of knots. For the reader's convenience we recall the relevant definitions from page 1301 and 1302.

Definition.

(1) A knot is a submanifold of $S^3 = \mathbb{R}^3 \cup \{\infty\}$ that is diffeomorphic to S^1.[689] We say that the knot is oriented if it is oriented as a 1-dimensional smooth manifold.

(2) We say that two knots K and J are smoothly isotopic if there exists a smooth isotopy from K to J, i.e. a smooth map

$$\begin{array}{rcl} F\colon S^1 \times [0,1] & \to & S^3 \\ (z,t) & \mapsto & F(z,t) \end{array}$$

such that the following hold:

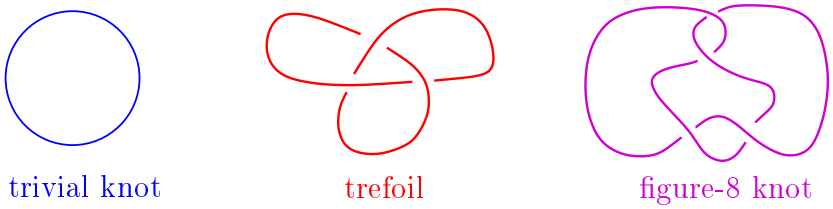
(a) we have $F(S^1 \times \{0\}) = K$ and $F(S^1 \times \{1\}) = J$, and

(b) for each $t \in [0,1]$ the map $S^1 \to S^3$ given by $z \mapsto F(z,t)$ is a smooth embedding.

If K and J are oriented, then we demand that the map $K \to J$ given by $x \mapsto F_1(F_0^{-1}(x))$ is orientation-preserving.

In the following we would like to study the knots that are illustrated in the figure below. In Question 58.1 we had raised the question, how can one show that the trefoil is not trivial, i.e. not smoothly isotopic to the trivial knot?

Now we have some tools to attack this question. We saw in Proposition 58.4 that if K and J are smoothly isotopic knots, then the knot complements $S^3 \setminus K$ and $S^3 \setminus J$ are diffeomorphic. In particular if $\pi_1(S^3 \setminus K)$ and $\pi_1(S^3 \setminus J)$ are not isomorphic, then K and J are not smoothly isotopic. Thus we arrive at the following variation on Question 58.1.



Question 59.3. Can the fundamental group be used to distinguish the trefoil from the trivial knot?

Before we continue we want to give a precise description of the trivial knot and the trefoil. In fact, we will use the occasion to introduce an interesting family of knots which contains the trefoil knot.

Definition.

(1) As on page 1301 we define the trivial knot as the knot[690]

$$U := \{(\exp(it), 0) \mid t \in \mathbb{R}\} \subset \left\{(z,w) \in \mathbb{C}^2 \,\middle|\, |w|^2 + |z|^2 = 1\right\}$$

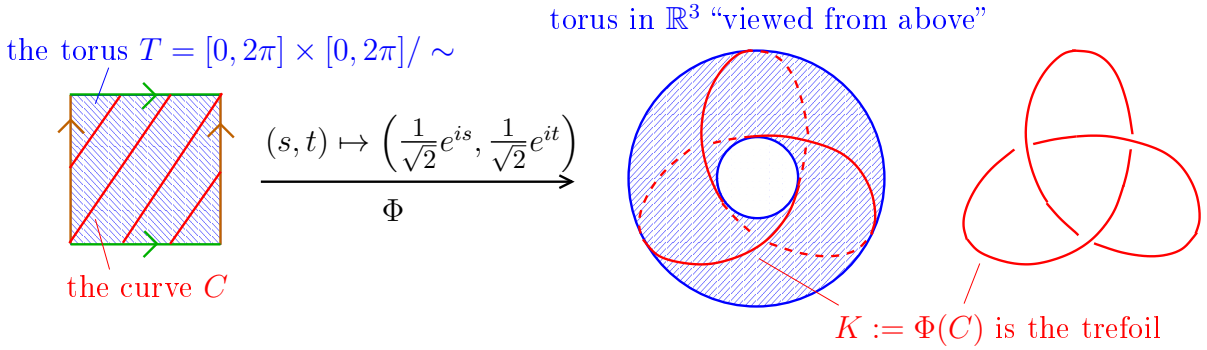
or equivalently

$$U := \{(\cos t, \sin t, 0) \mid t \in \mathbb{R}\} \subset \mathbb{R}^3 \cup \{\infty\}.$$

(2) Let $p, q \in \mathbb{N}$ be coprime. We consider the curve $C = \{(p \cdot t, q \cdot t) \mid t \in \mathbb{R}\}$ on the torus $T = [0, 2\pi] \times [0, 2\pi]/\sim$. We define the (p,q)-torus knot $T(p,q)$ as the image of C under the map

$$\begin{aligned} \Phi\colon T = [0,2\pi] \times [0,2\pi]/\sim &\to S^3 \\ (s,t) &\mapsto \left(\tfrac{1}{\sqrt{2}} \exp(is), \tfrac{1}{\sqrt{2}} \exp(it)\right). \end{aligned}$$

(3) We define the trefoil to be the $(2,3)$-torus. This definition of the trefoil is illustrated in the figure below.



Now we turn to the calculation of the fundamental groups of the complements of the trivial knot and of the trefoil.

Proposition 59.4. If U is the trivial knot, then $S^3 \setminus U$ is diffeomorphic to $S^1 \times \mathbb{C}$, in particular $\pi_1(S^3 \setminus U) \cong \mathbb{Z}$.

[690] The name U comes from the fact that the trivial knot is often also called the *unknot*.

Proof. Note that

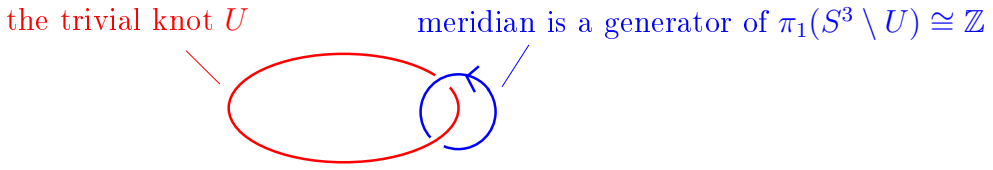
$$\begin{aligned} S^3 \setminus U &= \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} \setminus \{(z,0) \mid |z| = 1\} \\ &= \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1 \text{ and } |w|^2 > 0\}. \end{aligned}$$

Now basically the same argument as in the proof of Lemma 59.1 shows that the map

$$\begin{aligned} \Phi\colon S^1 \times \mathbb{C} &\to S^3 \setminus U = \left\{(z,w) \in \mathbb{C}^2 \,\middle|\, |w|^2 + |z|^2 = 1 \text{ and } |w|^2 > 0\right\} \\ (a,b) &\mapsto \left(\frac{b}{\sqrt{|a|^2+|b|^2}}, \frac{a}{\sqrt{|a|^2+|b|^2}}\right) \end{aligned}$$

is a diffeomorphism. Since $S^1 \times \mathbb{C}$ is homotopy equivalent to S^1 we obtain from the above and from the Homotopy-π_1-Proposition 50.3 (2) that $\pi_1(S^3 \setminus U) = \pi_1(S^1 \times \mathbb{C}) = \pi_1(S^1) \cong \mathbb{Z}$. ∎

Remark. Let U be the trivial knot. A careful reading of the proof of Proposition 59.4 shows that a generator of $\pi_1(S^3 \setminus U)$ is given by a meridian that we defined on page 1305. We refer to the figure below for an illustration of the trivial knot and a meridian.



Proposition 59.5.
(1) Let $p, q \in \mathbb{N}$ be coprime. Then $\pi_1(S^3 \setminus T(p,q)) \cong \langle x, y \mid x^p = y^q \rangle$.
(2) If K is the trefoil, then $\pi_1(S^3 \setminus K) \cong \langle x, y \mid x^2 = y^3 \rangle$.

Proof. By definition of the trefoil Statement (2) is just a special case of Statement (1). Thus we only need to prove Statement (1). Let $p, q \in \mathbb{N}$ be coprime. By Lemma 59.1 there exists a diffeomorphism

$$\Phi\colon S^3 \xrightarrow{\cong} \underbrace{\left(S^1 \times \overline{B}^2\right)}_{=:A} \cup_{S^1 \times S^1 = S^1 \times S^1} \underbrace{\left(\overline{B}^2 \times S^1\right)}_{=:B}$$

which is, up to scaling by a factor of $\sqrt{2}$, the identity on $T := S^1 \times S^1$. We identify the (p,q)-torus knot with its image under Φ, i.e. we have

$$K \;=\; \{(\exp(ps\mathrm{i}), \exp(qs\mathrm{i})) \mid s \in \mathbb{R}\} \;\subset\; T.$$

We write $A_K = A \setminus K, B_K = B \setminus K$ and $T_K = T \setminus K$. Note that A_K and B_K are submanifolds of $X_K := S^3 \setminus K$ and with[691] $A_K \cap B_K = \partial A_k = \partial B_K = T_K$ and with $A_K \cup B_K = X_K$. Note that $x := S^1 \times \{0\}$ is a deformation retract of A_K and that $y := \{0\} \times S^1$ is a deformation retract of B_K. Finally note that T_K retracts onto a parallel copy of K. More precisely, for $\epsilon > 0$ sufficiently small, the subset T_K retracts onto the loop

$$z \;=\; \{(\exp(ps\mathrm{i} + \epsilon), \exp(qs\mathrm{i})) \mid s \in \mathbb{R}\} \;\subset\; T_K.$$

[691] Here we mean by ∂A_K and ∂B_K the boundary of A_K and B_K viewed as submanifolds of S^3.

By an abuse of notation we denote by x, y and z also the loops corresponding to the circles x, y and z with the obvious orientation. Then the inclusion induced maps

$$\begin{array}{ccccc} \pi_1(A_K) & \leftarrow & \pi_1(T_K) & \rightarrow & \pi_1(B_K) \\ i_*(g) & \mapsfrom & g & \mapsto & j_*(g) \end{array} \quad \text{become} \quad \begin{array}{ccccc} \langle x\rangle & \leftarrow & \langle z\rangle & \rightarrow & \langle y\rangle \\ x^p & \mapsfrom & z & \mapsto & y^q. \end{array}$$

Thus we see that

$$\begin{array}{ccccccc} \pi_1(X_K) & = & \pi_1(A_K) *_{\pi_1(T_K)} \pi_1(B_K) & \cong & \langle x\rangle *_{x^p=y^q} \langle y\rangle & = & \langle x, y \mid x^p = y^q\rangle. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Seifert–van Kampen Theorem 54.4} & & \text{above discussion} & & \text{Lemma 53.23} & \end{array}$$

■

Now the question is whether or not $\pi_1(S^3 \setminus K) = \langle x, y \mid x^2 = y^3\rangle$ is isomorphic to $\pi_1(S^3 \setminus U) = \mathbb{Z}$. As usual we first consider the abelianization of $\pi_1(S^3 \setminus K)$, but it is straightforward to see, using the Abelianization Proposition 53.20 that the abelianization is isomorphic to $\mathbb{Z}$.[692][693] Despite this slight setback the following lemma does fortunately show that the group $\pi_1(S^3 \setminus K) = \langle x, y \mid x^2 = y^3\rangle$ is not isomorphic to $\pi_1(S^3 \setminus U) = \mathbb{Z}$.

Proposition 59.6.

(1) The group $\langle x, y \mid x^2 = y^3\rangle$ admits an epimorphism onto the permutation group S_3, in particular the group $\langle x, y \mid x^2 = y^3\rangle$ is non-abelian and it is not isomorphic to $\mathbb{Z}$.
(2) The trefoil is not smoothly isotopic to the trivial knot.

Proof.

(1) It follows from Lemma 53.4 and a straightforward calculation that there exists a unique homomorphism

$$\begin{array}{rcl} \Phi\colon \langle x, y \mid x^2 = y^3\rangle & \rightarrow & S_3 = \text{permutation group on three elements} \\ \text{with} \quad \Phi(x) & = & \sigma := (1\,2) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \\ \text{and} \quad \Phi(y) & = & \tau := (1\,2\,3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}. \end{array}$$

The permutations σ and τ do not commute, hence the image of Φ is a non-abelian subgroup of S_3.[694] On the other hand the abelian group $\mathbb{Z}$ cannot admit an epimorphism onto a non-abelian group[695].

(2) It follows from (1) and above discussion that the trefoil is non-trivial. ■

The following question arises naturally.

Question 59.7. Are the (p, q)-torus knots pairwise different, i.e. if (p, q) and (r, s) for coprime pairs of natural numbers, does it follow that $T(p, q)$ and $T(r, s)$ are not smoothly isotopic?

The approach of the proof of Proposition 59.6 is most likely too pedestrian to deal with Question 59.7. We will return to Question 59.7 at a much later stage.

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[692] In fact that is not a coincidence, using homology groups we will soon see that for any knot J the abelianization of $\pi_1(S^3 \setminus J)$ is isomorphic to $\mathbb{Z}$.

[693] If one goes very carefully through the definitions and calculations then one can see that a meridian once again becomes a generator of the abelianization.

[694] In fact it is easy to see that the only non-abelian subgroup of S_3 is S_3 itself, i.e. Φ is an epimorphism onto S_3.

[695] Why not?

Remark. Let $p, q \in \mathbb{N}$ be coprime. We consider the map

$$\begin{aligned} f\colon S^3 = \{(z,w) \in \mathbb{C}^2 \,|\, |z|^2 + |w|^2 = 1\} &\to \mathbb{C} \\ (z,w) &\mapsto z^q - w^p. \end{aligned}$$

Note that by definition the (p,q)-torus knot $T(p,q)$ equals $f^{-1}(0)$. It follows from work of Milnor, see [**Miln1968**, Theorem 4.8] and [**Sav2012**, Chapter 8] (see also [**EN1985**, p. 125] and [**Rolf1990**, Chapter 10.I]), that there exists a surface and a self-diffeomorphism of order pq together with a diffeomorphism $\Theta\colon S^3 \setminus T(p,q) \to \mathrm{Tor}(\Sigma, \varphi)$ such that the following diagram commutes

$$\begin{array}{ccc} S^3 \setminus T(p,q) & \xrightarrow[\cong]{\Theta} & \mathrm{Tor}(\Sigma,\varphi) \\ & {\scriptstyle P \mapsto \frac{f(P)}{|f(P)|}} \searrow \quad \swarrow {\scriptstyle [(z,t)] \mapsto \exp(2\pi i t)} & \\ & S^1. & \end{array}$$

This generalizes the observation from 470 that the complement of the trefoil is diffeomorphic to a mapping torus.

The question arises, whether the fundamental group is strong enough to detect the smooth isotopy class of a knot. In the section we will see that this is perhaps a trifle to optimistic.

59.3. **The mirror image of a knot.** We continue with the following nicely geometric definition.

Definition. Given a knot $K \subset S^3$ the reflection of K in any hyperplane of $\mathbb{R}^4$ is called mirror of K and denoted by K^{mir}.

Lemma 59.8.

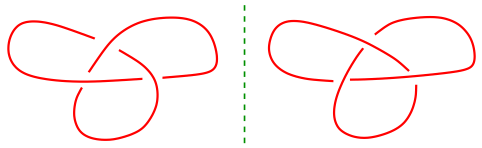
(1) Let $K \subset S^3$ be a knot. The reflections in any two hyperplanes of $\mathbb{R}^4$ give rise to smoothly isotopic knots.
(2) Let $K \subset \mathbb{R}^3$ be a knot. We use the diffeomorphism $\mathbb{R}^3 \cup \{\infty\} \to S^3$ from Lemma 2.18 to view K as a knot in S^3. The image of K under a reflection of an affine hyperplane in $\mathbb{R}^3$ corresponds under this embedding to a mirror of K.

Sketch of proof.

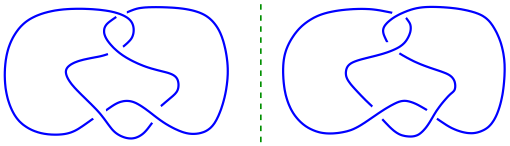
(1) We will prove this statement in Exercise 59.1, making use of the Matrix Group Path-Component Proposition 2.37 and the Smooth Path-Connectivity Proposition 19.29.
(2) Let $K \subset \mathbb{R}^3$ be a knot. First note that the same argument as in (1) shows that reflection in any two affine hyperplanes of $\mathbb{R}^3$ lead to smoothly isotopic knots in $\mathbb{R}^3$. Thus it remains to show that there exists a single hyperplane H of $\mathbb{R}^3$ such that the reflection of K in H corresponds to the mirror image of $K \subset \mathbb{R}^3 \subset S^3$ as defined above. Next note that the stereographic projection, that we used in Lemma 2.18, commutes with the reflection in the $(x_1 = 0)$-hyperplane of $\mathbb{R}^4$ respectively $\mathbb{R}^3$. Thus we see that the hyperplane $(x_1 = 0)$ has the desired property. ∎

Convention. Let $K \subset S^3$. Since usually we only care about knots up to smooth isotopy we use Lemma 59.8 to talk of "the mirror image" of K instead of the slightly more correct "a mirror image of K".

Example. Let $K \subset \mathbb{R}^3 \subset \mathbb{R}^3 \cup \{\infty\} = S^3$ be a knot. Keeping Lemma 59.8 (2) we see in the figure below the trefoil and its mirror and we also see the figure-8 knot and its mirror.



the trefoil K and its mirror K^{mir}



the figure-8 knot J and its mirror J^{mir}

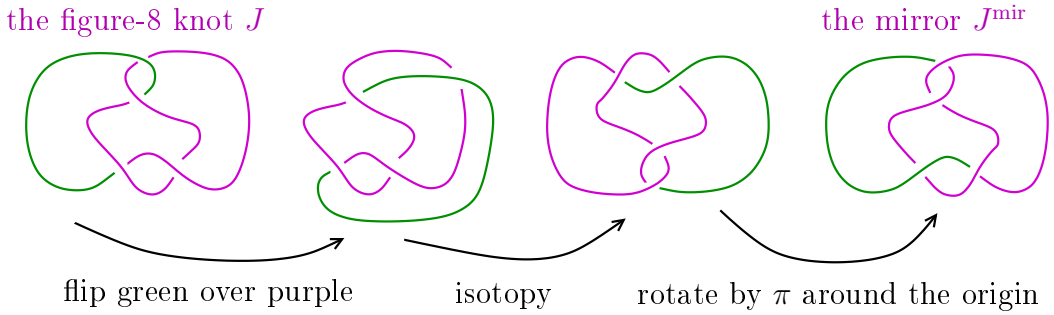
The question arises, whether the mirror image of a knot is "truly different" from the original knot. We are now naturally led to the following definition.

Definition. We say that a knot K in $\mathbb{R}^3$ is amphichiral if it is smoothly isotopic to its mirror image, otherwise we call the knot K chiral.

Examples.

(1) The trivial knot equals of course its mirror, in particular the trivial knot is amphichiral.

(2) In the figure below we show that the figure-8 knot J is smoothly isotopic to its mirror image J^*, i.e. the figure-8 knot is also amphichiral.



It is now natural to ask whether the trefoil is amphichiral. But endless hours of playing around with the trefoil do not lead to any success. So the suspicion arises that the following question should be answered in the negative.

Question 59.9. Is the trefoil chiral?

Much later, in Proposition 131.5 and 149.10 we will use two different approaches to give negative answer to this question. In the first approach we use the fundamental group together with the "longitude", in the latter approach we use linking pairings of the "2-fold branched covering of K".

We return to the study of fundamental groups. Let $K \subset S^3$ be a knot and let K^{mir} be its mirror image. Evidently $S^3 \setminus K$ is diffeomorphic to $S^3 \setminus K^{\text{mir}}$, so we cannot hope to distinguish K and K^{mir} using the fundamental groups of their complements. This leads us to the following question:

Question 59.10. Let K and J be two knots such that the fundamental groups of $S^3 \setminus K$ and $S^3 \setminus J$ are isomorphic. Does this imply that K is smoothly isotopic to J or to its mirror J^*?

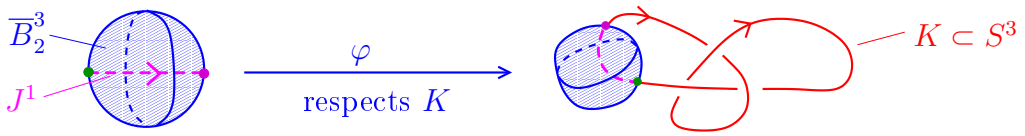
59.4. **The connected sum of oriented submanifolds and knots.** In this section, given two submanifolds $K \subset M$ and $L \subset N$ we introduce the corresponding connected sum $K\#L \subset M\#N$. Eventually our main interest will be in the connected sum of knots $K, L \subset S^3$.

To introduce the connected sum of submanifolds and knots we need to make a few preparations. We recall and expand on some definitions from page 894.

Notation. Let $n \in \mathbb{N}$ and let $k \in \{1, \dots, n-1\}$.

(1) We denote by $\overline{B}_2^n$ and $\overline{B}_2^k$ the closed balls of radius 2 in $\mathbb{R}^n$ and $\mathbb{R}^k$.
(2) We set $J^k := \{(x,0) \in \overline{B}_2^n \,|\, x \in \overline{B}_2^k\}$ and we view equip J^k with the obvious orientation.
(3) Let M be an oriented n-dimensional smooth manifold and let $K \subset M$ be an oriented k-dimensional submanifold. Furthermore let $\varphi\colon \overline{B}_2^n \to M$ be a smooth embedding.
   (a) We say φ respects K if $\varphi(J^k) = \varphi(\overline{B}_2^n) \cap K$, if $\varphi\colon \overline{B}_2^n \to M$ is orientation-preserving and if $\varphi|_{J^k}\colon J^k \to \varphi(\overline{B}_2^n) \cap K$ is orientation-preserving.
   (b) We say φ anti-respects K if the same conditions as in (a) hold, except that we now demand that both maps are orientation-reversing.

In the figure below we illustrate the definition of a map that respects K in the special case that $M = S^3$ and that K is in fact a knot.



The following proposition says that maps that respect a submanifold K always exist and that in a sense they are unique.

Proposition 59.11. Let M be an oriented n-dimensional smooth manifold. Furthermore let $K \subset M$ be an oriented k-dimensional submanifold.

(1) If K is non-empty, then there exists a map $\varphi\colon \overline{B}_2^n \to M$ that respects K.
(2) If K is connected, then for any two maps $\varphi\colon \overline{B}_2^n \to M$ and $\psi\colon \overline{B}_2^n \to M$ that respect K there exists a diffeotopy $F\colon M \times [0,1] \to M$ such that $F_0 = \mathrm{id}$, such that $F_1 \circ \varphi = \psi$ and such that each $F_t \circ \varphi$ respects K.

The obvious analogues of (1) and (2) also hold for "anti-respect" instead of "respect".

Remark. An analogue of Proposition 59.11 for topological manifolds is proved in [**Liv2021**].

Sketch of proof. The first statement follows easily from the definition of a submanifold. Now let us turn to the proof of the second statement. Thus we assume that K is connected. Let $\varphi\colon \overline{B}_2^n \to M$ and $\psi\colon \overline{B}_2^n \to M$ be two maps that respect K. By Theorem 37.11 there exists a smooth isotopy $I\colon \overline{B}^n \times [0,1] \to M$ such that $I_0 = \varphi$, such that $I_1 = \psi$ and such that each $I_t \circ \varphi$ respects K.

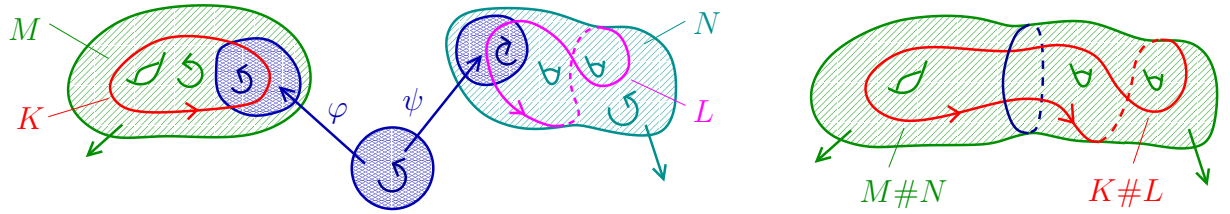
It follows from a slight generalization of the Isotopy Extension Theorem 36.1, see Exercise 37.3, that there exists a diffeotopy $F\colon M \times [0,1] \to M$ with the following properties:

(1) $F_0 = \mathrm{id}$.
(2) For all $x \in \overline{B}^n$ and all $t \in [0,1]$ we have $F(I_0(x), t) = F(x,t)$.

This diffeotopy has all the desired properties. ■

Let us move on to the key definition of this section which can be viewed as a variation on the definition of the connected sum of two oriented smooth manifolds that we gave on page **??**.

Definition. Let M be a closed oriented n-dimensional smooth manifold and let $K \subset M$ be a closed oriented k-dimensional submanifold. Furthermore let N be a closed oriented n-dimensional smooth manifold and let $L \subset N$ be a closed oriented k-dimensional submanifold. We pick a map $\varphi\colon \overline{B}^n_2 \to M$ that respects K and we pick a map $\psi\colon \overline{B}^n_2 \to N$ that anti-respects L. We use $\varphi\colon \overline{B}^n \to M$ and $\psi\colon \overline{B}^n \to N$ to define $M\#N$. Furthermore we use[696] $\overline{B}^k = \overline{B}^k_1 \xrightarrow{x \mapsto (x,0)} J^n_k \xrightarrow{\varphi} K$ and $\overline{B}^k = \overline{B}^k_1 \xrightarrow{x \mapsto (x,0)} J^n_k \xrightarrow{\psi} L$ to define the connected sum $K\#L$ as we did on page **??**. We refer to the resulting pair $(M\#N, K\#L)$ as the connected sum of (M,K) and (N,L).



The following proposition can be viewed as an analogue of the Connected Sum-Proposition **??**.

Proposition 59.12. Let M be a closed oriented n-dimensional smooth manifold and let K be a closed oriented k-dimensional submanifold of M. Furthermore let N be a closed oriented n-dimensional smooth manifold and let L be a closed oriented k-dimensional submanifold of N.

(1) The subset $K\#L$ is a closed k-dimensional smooth submanifold of $M\#N$.
(2) If M and N are both connected, then the oriented diffeomorphism type of the pair $(M\#N, K\#L)$ does not depend on the choice of the map that respects K and the map that anti-respects L.

Sketch of proof.

(1) We leave it to the reader to verify this statement. Note that here one needs to make use of the fact that the maps that (anti-) respect the submanifold give us control on $\overline{B}^n_2$ and not just on $\overline{B}^n$.
(2) This statement follows fairly easily from Proposition 59.11. ■

Now we can finally specialize to our beloved setting of knots in S^3. Following most books in knot theory, see e.g. [**Rolf1990**, p. 40] and [**Kaw1996**, p. 30] we introduce the connected sum of two oriented knots as follows:

Definition.

(1) Let K and L be two oriented knots in S^3. We pick a map $\varphi\colon \overline{B}^3_2 \to S^3$ that respects K and we pick a map $\psi\colon \overline{B}^3_2 \to S^3$ that anti-respects L. We perform the connected

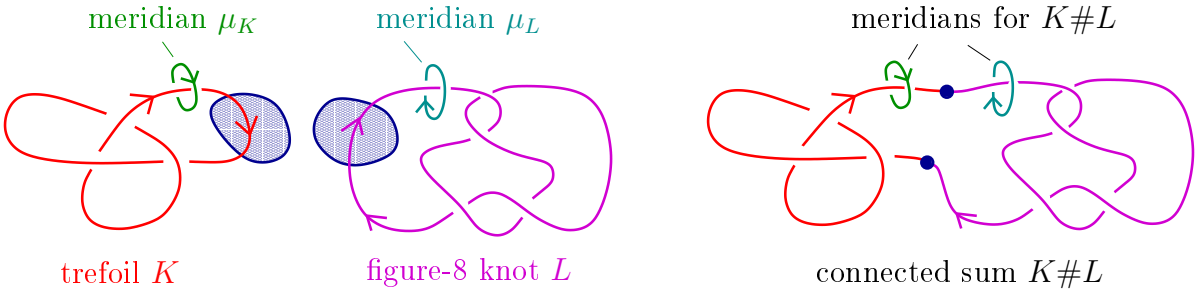
[696] Note that here we deliberately restrict ourselves from balls of radius 2 to balls of radius 1.

sum $(S^3\#S^3, K\#L)$ and we use the explicit orientation-preserving diffeomorphism $S^3\#S^3 \to S^3$ from Lemma **??** to view $K\#L$ as an oriented knot in S^3.

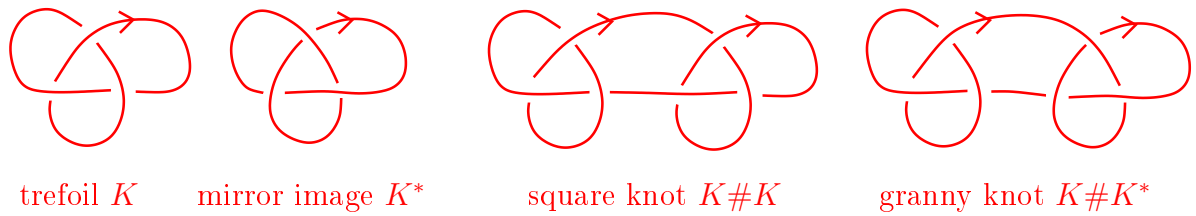
(2) We say a knot K is prime if it is not smoothly isotopic to the connected sum of two non-trivial knots.

Examples.

(1) In the figure below we show the connected sum of the trefoil K and the figure-8 knot L. Admittedly it can take a minute to connect the picture to the actual definition. Just for fun we also show meridians of K, L and $K\#L$.



(2) In the figure below we show the trefoil and its "mirror image" K^{mir}, i.e. the reflection of K in the xy-hyperplane. The connected sum $K\#K$ is called the *square knot* and the connected sum $K\#K^{\text{mir}}$ is called the *granny knot*. It is notoriously tricky to distinguish the square knot and the granny knot, even up to taking "mirror images".



Proposition 59.13. On the set of smooth isotopy classes of oriented knots the connected sum operation is well-defined and commutative.

Sketch of proof. First note that it follows almost immediately from the Isotopy Extension Theorem 36.1 together with Proposition 59.12 that the connected sum operation is indeed well-defined.[697] A slight variation on the proof of the Connected Sum-Proposition **??** (5) shows that the connected sum of knots is commutative. ■

But now let us return to our main hobby these days, namely the calculation of fundamental groups. We have the following proposition.

[697] Note though that the statement does not follow just from Proposition 59.12 since we claim that $K\#L$ is well-defined up to isotopy in S^3. This is a priori not the same statement as saying that the oriented diffeomorphism type of $(S^3\#S^3, K\#L)$ is well-defined.

Proposition 59.14. Let K and L be two oriented knots in S^3. We pick meridians μ_K and μ_L and base points $x_0 \in \mu_K$ and $y_0 \in \mu_L$. There exists an isomorphism

$$\pi_1(S^3 \setminus (K\#L)) \;\cong\; \pi_1(S^3 \setminus K, x_0) *_{\langle \mu_K \rangle = \langle \mu_L \rangle} \pi_1(S^3 \setminus L, y_0).$$

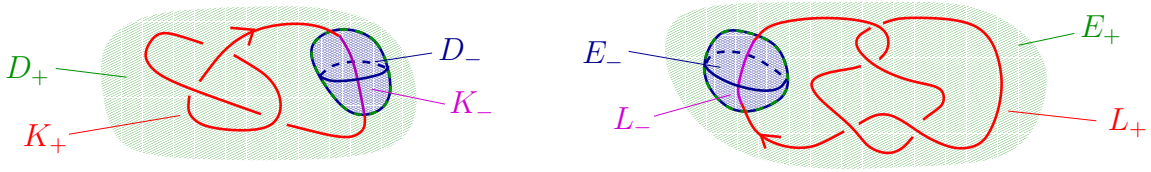
given by making the identification $\mu_K = \mu_J$

Sketch of proof. To preserve our sanity we only provide a sketch of the proof. We pick a map $\varphi\colon \overline{B}^3_2 \to S^3$ that respects K and we pick a map $\psi\colon \overline{B}^3_2 \to S^3$ that anti-respects L. We write $D^3_+ := S^3 \setminus \varphi(B^3)$, $D^3_- := \varphi(\overline{B}^3)$ and $K^\pm := D^3_\pm \cap K$. Similarly we write $E^3_+ := S^3 \setminus \psi(B^3)$, $E^3_- := \psi(\overline{B}^3)$ and $L^\pm := E^3_\pm \cap K$. Since all meridians are equivalent we can arrange that $\mu_K \subset \varphi(S^2)$ and that $\mu_L \subset \psi(S^2)$.[698] Now we consider the following diagram

isomorphism by the Seifert–van Kampen Theorem 54.4 for Smooth Manifolds

$$\begin{array}{ccccc} \pi_1(S^3 \setminus K\#L) & = & \pi_1(S^3\#S^3 \setminus K\#L) & \overset{\cong}{\downarrow} & \pi_1(D^3_+ \setminus K_+) *_{\langle \mu_K \rangle = \langle \mu_L \rangle} \pi_1(E^3_+ \setminus L_+) \\ & & & & \downarrow\cong \qquad\qquad \downarrow\cong \\ & & & & \pi_1(S^3 \setminus K) *_{\langle \mu_K \rangle = \langle \mu_L \rangle} \pi_1(S^3 \setminus L). \end{array}$$

Here the vertical maps are isomorphisms by another two applications of the Seifert–van Kampen Theorem 54.4, applied this time to $S^3 \setminus K = (D^3_+ \setminus K_+) \cup (D^3_- \setminus K_-)$ and to $S^3 \setminus L = (E^3_+ \setminus L_+) \cup (E^3_- \setminus L_-)$. Note that here we secretly use the Amalgamated Product Lemma 53.22 (3) twice. ■



Remark. In Section 102.7 we will introduce the "internal connected sum" of two oriented submanifolds. In particular we will introduce the "internal connected sum" of two oriented knots which, for most intents and purposes agrees with the connected sum of two oriented knots.

In Exercise 59.6 we will prove the following corollary to Proposition 59.14.

Corollary 59.15. Let K and L be oriented knots. The fundamental groups of $S^3 \setminus (K\#L)$ and $S^3 \setminus (K\#L^*)$ are isomorphic.

In particular we see that the square knot and the granny knot cannot be distinguished by the fundamental group of their complements. On the other hand we will see in Exercise 149.3 that the granny knot is neither smoothly isotopic to the square knot nor to its mirror image. In particular we see that in general the answer to Question 59.10 is negative.
Fortunately the following deep theorem shows that the situation is much better once we restrict to prime knots. Namely, under this extra hypothesis we actually do get a positive answer to Question 59.10.

[698] Readers who go through the proof with a fine logical comb will notice that the last sentence requires some extra thought.

Theorem 59.16. (Whitten Theorem) Let K and L be two prime knots in $\mathbb{R}^3 \subset \mathbb{R}^3 \cup \{\infty\} = S^3$. If $\pi_1(S^3 \setminus K)$ is isomorphic to $\pi_1(S^3 \setminus L)$, then K is smoothly isotopic to L or to its mirror image L^*.

Proof. Let K and L be two prime knots such that the fundamental groups $\pi_1(S^3 \setminus K)$ and $\pi_1(S^3 \setminus L)$ are isomorphic. Wilbur Whitten [**Whn1987**, Corollary 2.1], building on the Gordon-Luecke Theorem 58.7, showed that there exists a diffeomorphism $F\colon S^3 \to S^3$ with $F(K) = L$. By possibly composing F with the reflection in a hyperplane we see that there exists an orientation-preserving diffeomorphism $F\colon S^3 \to S^3$ with $F(K) = L$ or with $F(K) = L^*$. The claimed statement now follows from Corollary 58.6. ■

59.5. **Knot diagrams.** In the previous section we gave a completely rigorous calculation of the isomorphism type of the fundamental group of the complement of a trefoil and we used this to show that the trefoil is not smoothly isotopic to the trivial knot. In the next section we will give a practical algorithm for determining the isomorphism type of the fundamental group of the complement of a given knot. The key to formulating the algorithm is the notion of a knot diagram that we will study in this section.
Before we start with knot theory we introduce one bit of notation.

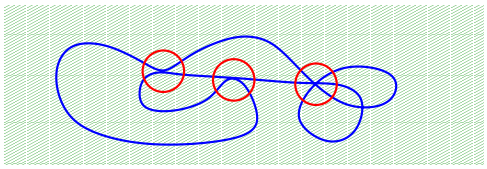
Notation. Given a smooth map $f\colon S^1 \to \mathbb{R}^n$ and $z = \exp(\mathrm{i}s) \in S^1$ we write

$f'(z)$:= derivative at the point s of the function $\mathbb{R} \to \mathbb{R}^n$ given by $t \mapsto f(\exp(\mathrm{i}t))$.

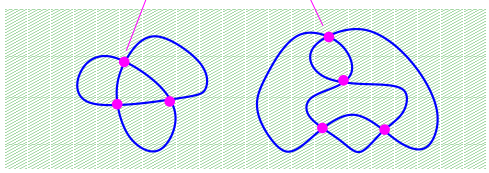
Definition. A map $\gamma\colon S^1 \to \mathbb{R}^2$ is called diagrammatic if the following conditions are satisfied:

(a) the map is an immersion,
(b) if $z \neq w \in S^1$ satisfy $\gamma(z) = \gamma(w)$, then $\gamma'(z)$ and $\gamma'(w)$ are linearly independent,
(c) given any $P \in \gamma(S^1)$ the preimage $\gamma^{-1}(P)$ consists of either one or two points.

Furthermore any $z \in S^1$ for which there exists a $w \neq z \in S^1$ with $\gamma(z) = \gamma(w)$ is called a double point of γ.



not allowed in a diagrammatic map



images of diagrammatic maps

Lemma 59.17. Every diagrammatic map has only finitely many double points.

Proof. The lemma will be proved in Exercise 59.4. ■

Definition.

(1) A knot diagram is a diagrammatic map $\gamma\colon S^1 \to \mathbb{R}^2$ together with a map

$$c\colon \{\text{double points of } \gamma\} \to \{\pm 1\}$$

which has the property that for $z \neq w \in S^1$ with $\gamma(z) = \gamma(w)$ we have $c(z) \neq c(w)$.

(2) Let (γ, c) be a knot diagram. Let z be a double point. If $c(z) = +1$, then we say that the double point is an overcrossing, otherwise we call it an undercrossing.

The next lemma shows that knot diagrams give rise to knots.

Lemma 59.18. Let $(\gamma\colon S^1 \to \mathbb{R}^2, c)$ be a knot diagram. We can pick disjoint compact segments $I_1, \ldots, I_k$ of S^1 and a smooth function $\eta\colon S^1 \to [-1, 0]$ with the following properties:

(a) Each segment contains a unique undercrossing and each undercrossing is contained in some segment.
(b) No segment contains an overcrossing.
(c) The value of η outside the segments is equal to 0.
(d) On each segment the function has a unique local minimum, namely at the corresponding undercrossing where the value of η is equal to -1.

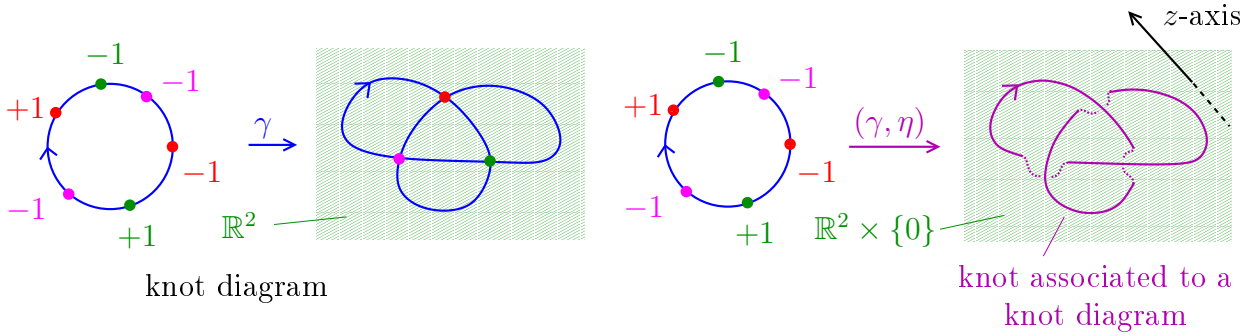
The following two statements hold:

(1) The image of S^1 under the map $S^1 \to \mathbb{R}^3$ given by $z \mapsto (\gamma(z), \eta(z))$ is a knot.
(2) Any two choices of segments and η as above give rise to smoothly isotopic knots.

If we equip S^1 with the standard orientation, then in the above statements we can also replace "knot" by "oriented knot".

Lemma 59.18 shows that a knot diagram gives rise to an essentially unique (oriented) knot. We refer to this knot as the *knot associated to the knot diagram*. At times we will rather blur the difference between a knot diagram and the associated knot.

Example. In the figure below we show to the left a knot diagram. To the right we show the associated knot. It is pretty clear that the resulting knot is smoothly isotopic to the trefoil.



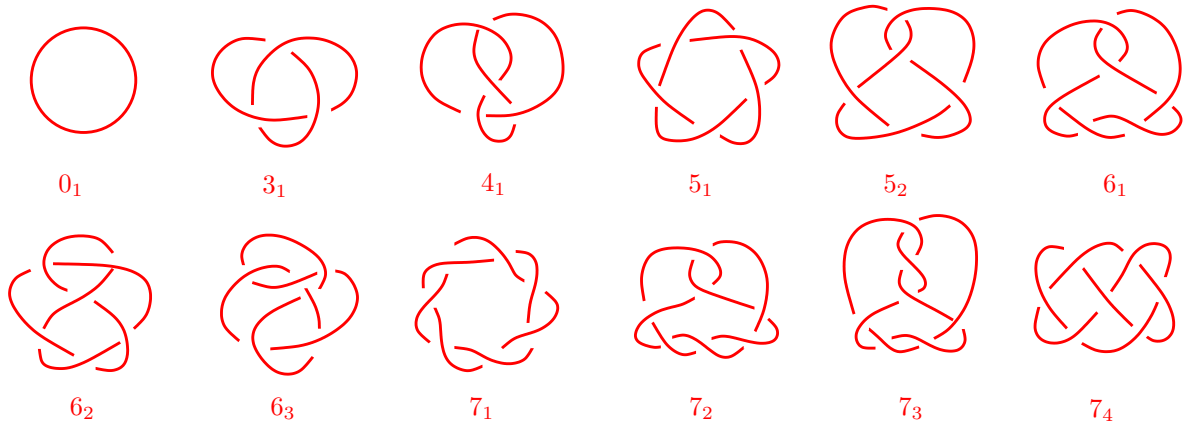
Proof. It follows quite easily from the Smooth Transition Function Lemma 19.4 that such segments $I_1, \ldots, I_k$ and that such a function η exist. Given such a function it is clear that the map $z \mapsto (\gamma(z), \eta(z))$ is injective. Since γ is an immersion it follows that the map $z \mapsto (\gamma(z), \eta(z))$ is in fact a smooth embedding. Thus it follows from the Smooth Embedding Theorem 24.10 (1) that the image is a one-dimensional submanifold of $\mathbb{R}^3 \cup \{\infty\} = S^3$ that is diffeomorphic to S^1. In other words, the image is a knot.

Finally suppose we are given functions η and η' with the properties stated as above. We consider the map

$$\begin{aligned} H\colon S^1 \times [0,1] &\to \mathbb{R}^3 \cup \{\infty\} = S^3 \\ (z, t) &\mapsto (\gamma(z), \eta(z) \cdot (1-t) + \eta'(z) \cdot t). \end{aligned}$$

This map is smooth. By construction we know that for any two distinct points $z, w \in S^1$ with $\gamma(z) = \gamma(w)$ we have $\eta(z) < \eta(w)$ if and only if $c(z) = -1$ if and only if $\eta'(z) < \eta'(w)$. Using this observation one can easily verify that each H_t is a smooth embedding. In other words, the map H_t is a smooth isotopy between the two (oriented) knots corresponding to η and η'. ■

In practice, i.e. in future examples, we do what every other sane topologist does, namely we draw a suggestive picture and we interpret it as a knot diagram and a corresponding knot in the obvious way. For example in the figure below we show the first knots in the standard table of knots. We refer to the standard textbooks on knot theory [**Rolf1990, BZH2014**] for a continuation of the table. But it should already be pretty clear from this table that knots form a straightforward way to produce an almost endless list of interesting topological problems.



In Lemma 59.18 we saw that any knot diagram gives rise to a knot. The following proposition gives us the converse:

Proposition 59.19. (Knot Diagram Existence Proposition) Given any knot $K \subset S^3$ there exists a knot diagram such that K is smoothly isotopic to the knot associated to the knot diagram.

The proof of the Knot Diagram Existence Proposition 59.19 mostly rests on the next lemma. The formulation of this lemma requires the following notation.

Notation. Given $v \in S^2$ we denote by

$$\begin{array}{rccl} \pi_v \colon & \mathbb{R}^3 & \to & v^\perp := \{w \in \mathbb{R}^3 \,|\, \langle v, w\rangle = 0\} \\ & w & \mapsto & w - \langle v, w\rangle v \end{array}$$

the projection onto the orthogonal complement of v.

Now we can formulate the lemma.

Lemma 59.20. Let $K \subset \mathbb{R}^3$ be a knot. We pick a diffeomorphism $f \colon S^1 \to K$. We consider the following three properties of a vector $v \in S^2$:

(a) the map $\pi_v \circ f \colon S^1 \to v^\perp = \{w \in \mathbb{R}^3 \,|\, \langle v, w\rangle = 0\}$ is an immersion,
(b) whenever $z \neq w \in S^1$ satisfy $(\pi_v \circ f)(z) = (\pi_v \circ f)(w)$, then $(\pi_v \circ f)'(z)$ and $(\pi_v \circ f)'(w)$ are linearly independent,

(c) for every $P \in (\pi_v \circ f)(S^1)$ the preimage $(\pi_v \circ f)^{-1}(P)$ consists of either one or two points.

The set
$$\{v \in S^2 \mid v \text{ satisfies (a), (b) and (c)}\}$$
has full measure in S^2.[699]

Remark. Recall that in the Full Measure-Density Proposition 30.3 we showed that a subset of full measure is dense. We leave it to the chronically bored reader to show that the set $\{v \in S^2 \mid v \text{ satisfies (a), (b) and (c)}\}$ considered in Lemma 59.20 is open.

Proof. Let $K \subset \mathbb{R}^3$ be a knot. We pick a diffeomorphism $f\colon S^1 \to K$. We say that a subset of a smooth manifold W is *large* if it is open and if it has full measure in W. It follows from the Measure-Zero Properties Proposition 30.2 that the intersection of two large subsets is again large. Furthermore, it follows from the Measure-Zero Properties Proposition 30.2 that any self-diffeomorphism of S^2 sends large sets to large sets.

It is convenient to consider an extra property of vectors in S^2 which is somewhat weaker than the above property (c):

(c′) given any $P \in (\pi_v \circ f)(S^1)$ the preimage $(\pi_v \circ f)^{-1}(P)$ consists of finitely many points.

We write
$$V(a) \;:=\; \{v \in S^2 \mid v \text{ satisfies (a)}\}.$$
Similarly we define $V(b), V(c), V(a,b), V(a,b,c')$ and so on. Note that with this language we need to show that $V(a,b,c)$ has full measure.

Let $v \in S^2$. We start out with three elementary observations regarding the projections π_v.

(1) Given $u \in \mathbb{R}^3 \setminus \{0\}$ we have
$$\pi_v(u) = 0 \quad\Longleftrightarrow\quad u \in \mathbb{R} \cdot v \quad\Longleftrightarrow\quad \tfrac{u}{\|u\|} \in \{v, -v\}.$$

(2) The projection π_v is linear, hence for any $P \in \mathbb{R}^3$ we have $(\mathrm{D}\pi_v)_P = \pi_v$.

(3) It follows from (2) and the Chain Rule 19.10 that given any $z \in S^1$ we have the equality $(\pi_v \circ f)'(z) = \pi_v(f'(z))$.

Next we consider the three maps[700] [701]
$$\varphi\colon S^1 \to S^2,\quad z \mapsto \tfrac{f'(z)}{\|f'(z)\|} \qquad \psi\colon \overbrace{\{(z,w) \in S^1 \times S^1 \mid z \neq w\}}^{=:M} \to S^2,\quad (z,w) \mapsto \tfrac{f(z)-f(w)}{\|f(z)-f(w)\|} \qquad \text{and} \qquad \rho\colon S^2 \to S^2,\quad v \mapsto -v.$$

[699] We refer to page 762 for the definition of a subset of full measure.

[700] Since f is an immersion we see that the map φ is well-defined, i.e. we do not divide by zero. Similarly, since f is in particular injective we see that the map ψ is well-defined, i.e. once again we do not divide by zero.

[701] The astute reader will notice that throughout the argument it might be more reasonable to work with maps φ and ψ that take values in $\mathbb{R}\mathrm{P}^2 = S^2/x \sim -x$ instead of taking values in S^2. We stick with the maps to S^2 since S^2 has the advantage that its tangent spaces can be described easily as vector subspaces of $\mathbb{R}^3$, which makes it easy to write down differentials. Also, the approach of working with maps to $\mathbb{R}\mathrm{P}^2$ instead creates an extra layer of notation which is as annoying as our notation which requires the extra map ρ.

We make the following observations:

(4) If X is a subset of full measure (respectively large subset) of S^2, then it follows from the above discussion that $X \cap \rho(X)$ is also a subset of full measure (respectively large).
(5) Since f is smooth we see that the map φ is also smooth.
(6) Given $v \in S^2$ it follows easily from (1) and (3) that $\pi_v \circ f \colon S^1 \to \mathbb{R}^2$ is an immersion if and only if neither v nor $-v$ lies in $\varphi(S^1)$. In other words, we have the equality $V(a) = (S^2 \setminus \varphi(S^1)) \cap \rho(S^2 \setminus \varphi(S^1))$.
(7) Note that $M = \{(z,w) \in S^1 \times S^1 \,|\, z \neq w\}$ is an open subset of the smooth manifold $S^1 \times S^1$, thus it is a smooth manifold in an obvious way. With this smooth manifold structure the map $\psi \colon M \to S^2$ is easily seen to be smooth.
(8) For each $(x,y) \in S^1$ we make the identification[702] $\mathrm{T}_{(x,y)}S^1 = \mathbb{R}$ via the basis vector $(-y,x)$. Given $(z,w) \in M$ we use the above identification to make the identification $\mathrm{T}_{(z,w)}M = \mathrm{T}_{(z,w)}(S^1 \times S^1) = \mathbb{R}^2$. Furthermore given $v \in S^2$ we already saw in Lemma 23.1 that we have the equality $\mathrm{T}_v S^2 = v^\perp = \{w \in \mathbb{R}^3 \,|\, \langle v, w\rangle = 0\} \subset \mathbb{R}^3$.
(9) Using the identifications from (8) an elementary calculation[703] shows that for a point $(z,w) \in M$ we can write the differential $\mathrm{D}\psi_{(z,w)}$, viewed as a homomorphism $\mathrm{T}_{(z,w)}M = \mathbb{R}^2 \to \mathrm{T}_{\psi(z,w)}S^2 \subset \mathbb{R}^3$, in the following way as a (3×2)–matrix:

$$\mathrm{D}\psi_{(z,w)} \;=\; \tfrac{1}{\|f(z)-f(w)\|} \cdot \underbrace{\big(\pi_{\psi(z,w)}(f'(z)) \quad -\pi_{\psi(z,w)}(f'(w))\big)}_{(3\times 2)\text{–matrix, in particular two columns}}.$$

(10) It follows from (9) that $(z,w) \in M$ is a regular point of ψ if and only if the vectors $\pi_{\psi(z,w)}(f'(z))$ and $\pi_{\psi(z,w)}(f'(w))$ are linearly independent.
(11) Let $v \in S^2$. It follows immediately from (1) and the linearity of π_v that we have $\pi_v(f(z)) = \pi_v(f(w))$ if and only if $\psi(z,w) = v$ or $\psi(z,w) = -v$.
(12) We denote by $\sigma \colon S^1 \times S^1$ the diffeomorphism given by $(z,w) \mapsto (w,z)$. It follows immediately from the definitions that $\psi \circ \sigma = \rho \circ \psi$.
(13) It follows easily from (3), (10), (11) and (12) that

$$V(b) \;=\; \{\text{regular values of } \psi\}.$$

After these initial remarks we turn to our first claim.

Claim.
(α) The sets $V(a)$ and $V(a,b)$ are large.
(β) We have $V(a) = V(a,c')$ and $V(a,b) = V(a,b,c')$.

Proof. We prove the two statements of the claim in several steps.

(i) Since S^1 is compact we obtain from the Compact Image Lemma 2.13 and the Compact-Closed Lemma 1.21 that $S^2 \setminus \varphi(S^1)$ is an open subset of S^2. Furthermore, since φ is smooth and since $\dim(S^1) < \dim(S^2)$ we obtain from the Full Measure-Density Proposition 30.3 (5) that $S^2 \setminus \varphi(S^1)$ is a subset of full measure. In summary, we

[702] As on many other occasions we use the Tangent Space-Isomorphism Proposition 23.16 to view the tangent space of a submanifold of some $\mathbb{R}^n$ as a vector subspace of $\mathbb{R}^n$.
[703] This calculation can be performed easily using the Chain Rule 19.10 and by writing ψ as the composition of the map $(z,w) \mapsto f(z) - f(w)$ followed by the map $P \mapsto \frac{P}{\|P\|}$.

have shown that $S^2 \setminus \varphi(S^1)$ is large. It follows from (5) and (6) that $V(a) = (S^2 \setminus \varphi(S^1)) \cap \rho(S^2 \setminus (\varphi(S^1)))$ is large.

(ii) It follows from Sard's Theorem 31.1 that the set of regular values of ψ is a subset of full measure. It follows from (i) and (13), together with the Full Measure-Density Proposition 30.3 (1) that $V(a,b) = V(a) \cap V(b)$ has full measure.

(iii) Let $v \in V(a)$. It follows from (11), from the fact that S^1 is compact and from Exercise 24.4 that $\psi^{-1}(v)$ is finite. In particular we see that $V(a,c') = V(a)$ and thus also $V(a,b,c') = V(a,b)$.

(iv) It remains to show that $V(a,b)$ is open. Let $v \in V(a,b) = V(a) \cap V(b)$. From (13) and (iii), together with Lemma 26.2 (2), we obtain an open neighborhood U of $v \in S^2$ that is still contained in $V(b)$. Since $V(a)$ is open we see that $U \cap V(a)$ is an open neighborhood of $v \in S^2$ that is still contained in $V(a,b)$. It follows from the JH-Lemma 1.7 that $V(a,b)$ is indeed open. ⊞

Before we continue it is perhaps helpful to summarize what we have shown so far: In the claim we have seen that $V(a,b,c') = V(a,b)$ is large, in particular that it has full measure. But our actual goal is to show that $V(a,b,c)$ has full measure.

To do so we consider the map

$$\begin{array}{rccc} \Xi\colon & \overbrace{\{(x,y,z) \in S^1 \times S^1 \times S^1 \,|\, x \neq y \text{ and } x \neq z\}}^{=:N} & \to & S^2 \times S^2 \\ & (x,y,z) & \mapsto & (\psi(x,y), \psi(x,z)). \end{array}$$

We make the following simple observations:

(14) We view N as a smooth manifold in an obvious way. Evidently the map Ξ is smooth.

(15) It follows easily from (1) that we have $v \in V(c)$ if and only if none of the four vectors $(\pm v, \pm v)$ lies in $\Xi(N)$.

(16) If one of the four vectors $(\pm v, \pm v)$ lies in $\Xi(N)$, then so does (v,v). This can be seen as follows: If $\Xi(x,y,z) \in \{(\pm v, \pm v)\}$, then $f(x), f(y), f(z)$ lie on the line $\mathbb{R} \cdot v$. After possibly permuting x, y, z we can assume that $f(x) = f(y) + r \cdot v$ and that $f(x) = f(z) + s \cdot v$ with $r, s > 0$. But with this permutation we have $\Xi(x,y,z) = (v,v)$.

Claim. We consider the "partial diagonal"

$$\Delta \;:=\; \{(v,v) \in S^2 \times S^2 \,|\, v \in V(a,b)\}.$$

The complement of $\Xi(N) \cap \Delta$ has full measure in Δ.

Proof. In the previous claim we saw that $V(a,b)$ is an open subset of S^2. Using this fact it is straightforward to see that Δ is a 2-dimensional submanifold of $S^2 \times S^2$. Thus, by Exercise 42.2 (b) it suffices to prove that if $(x,y,z) \in N$ satisfies $\Xi(x,y,z) = (v,v)$ for some $v \in V(a,b)$, then Ξ intersects Δ transversally, i.e. we have the equality

$$(\mathrm{D}\Xi_{(x,y,z)})(\mathrm{T}_{(x,y,z)}N) + \mathrm{T}_{(v,v)}\Delta \;=\; \mathrm{T}_{(v,v)}(S^2 \times S^2) \;\subset\; \mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6.$$

We write $p = \pi_v(f'(x))$, $q = \pi_v(f'(y))$ and $r = \pi_v(f'(z))$ and we write $\mu = \frac{1}{\|f(x) - f(y)\|}$ and $\nu = \frac{1}{\|f(x) - f(z)\|}$. By (10) and (13), and by the fact that $v \in V(b)$, we know that p, q and r are pairwise linearly independent. In particular q and r form a basis for $\mathrm{T}_v S^2$. Note that this implies that $\begin{pmatrix} q \\ q \end{pmatrix}$ and $\begin{pmatrix} r \\ r \end{pmatrix}$ form a basis for $\mathrm{T}_{(v,v)}(S^2 \times S^2)$. Thus, using (9) we see

that

$$(\mathrm{D}\Xi_{(x,y,z)})(\mathrm{T}_{(x,y,z)}N) + \mathrm{T}_{(v,v)}\Delta \;=\; \text{span of the columns of } \begin{pmatrix} \mu\cdot p & -\mu\cdot q & 0 & q & r \\ \nu\cdot p & 0 & -\nu\cdot r & q & r \end{pmatrix}.$$

Using that p, q and r are pairwise linearly independent it is not hard to see that the five columns of the matrix on the right span a 4-dimensional subspace of $\mathbb{R}^6$. Since this subspace is contained in the 4-dimensional subspace $\mathrm{T}_{(v,v)}(S^2 \times S^2)$ we see that it equals $\mathrm{T}_{(v,v)}(S^2 \times S^2)$. ⊞

We consider the "diagonal map" $d\colon V(a,b) \to \Delta \subset S^2 \times S^2$ that is given by $d(v) = (v, v)$. One can easily show that $d\colon V(a,b) \to \Delta$ is a diffeomorphism. It follows from (16) that $V(a,b,c) = d^{-1}(\Delta \setminus \Xi(N))$. By the claim $\Delta \setminus \Xi(N)$ is a subset of full measure of Δ. Since d is a diffeomorphism we obtain from the Measure-Zero Properties Proposition 30.2 that $V(a,b,c) = d^{-1}(\Delta \setminus \Xi(N))$ has full measure in $V(a,b)$. We saw above that $V(a,b)$ has full measure in S^2. It follows easily from the Measure-Zero Properties Proposition 30.2 that $\Delta \setminus \Xi(N)$ has full measure in S^2. ■

Proof of the Knot Diagram Existence Proposition 59.19. Let $K \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ be a knot. As we saw in Exercise **??** (a), it follows easily from the Orthogonal Action Lemma 5.26 (4) and the Diffeotopies-via-Matrices Lemma 33.8 (2) that, possibly after applying a smooth isotopy, we can assume that $K \subset \mathbb{R}^3$.

Next we pick $v \in S^2$ as in Lemma 59.20. It follows from the Orthogonal Action Lemma 5.26 (4) and the Diffeotopies-via-Matrices Lemma 33.8 (2) that, possibly after applying a smooth isotopy, we can assume that $v = e_3$. We define $\gamma := \pi_{e_3} \circ f\colon S^1 \to \mathbb{R}^2 \times \{0\} = \mathbb{R}^2$. It follows immediately from the fact that $v = e_3$ has the three properties (a), (b) and (c) stated in Lemma 59.20 that γ is diagrammatic. Next we consider the function

$$\begin{array}{rcl} \epsilon\colon S^1 & \to & \mathbb{R} \\ w & \mapsto & z\text{-coordinate of } f(w). \end{array}$$

Furthermore, given $z_1 \neq z_2 \in S^1$ with $\gamma(z_1) = \gamma(z_2)$ we define

$$c(z_i) \;:=\; \begin{cases} +1, & \text{if } \epsilon(z_i) > \epsilon(z_{3-i}), \\ -1, & \text{if } \epsilon(z_i) < \epsilon(z_{3-i}). \end{cases}$$

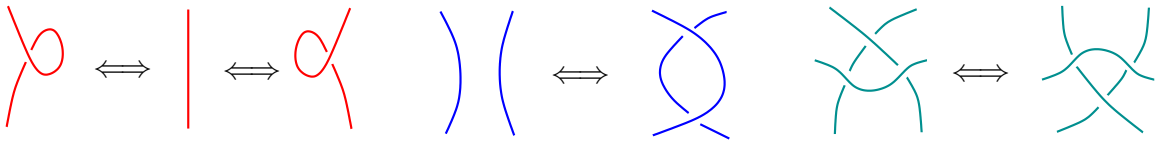
It is clear that (γ, c) is a knot diagram. We pick a function $\eta\colon S^1 \to [-1, 0]$ as in Lemma 59.18. We consider the map

$$\begin{array}{rcl} H\colon S^1 \times [0,1] & \to & \mathbb{R}^3 \\ (z,t) & \mapsto & (\gamma(z), (1-t)\cdot\epsilon(t) + t\cdot\eta(t)). \end{array}$$

Clearly this map is smooth. Given any $z, w \in S^1$ with $\gamma(z) = \gamma(w)$ we have $\eta(z) < \eta(w)$ if and only if $c(z) = -1$ if and only if $\eta(z) < \eta(w)$. From this observation we deduce that each H_t is actually a smooth embedding. Thus we see that H is a smooth isotopy from K to the knot associated to the knot diagram (γ, η). ■

We have now seen that every knot, up to smooth isotopy, arises from a knot diagram. The question arises, when do two knot diagrams give rise to smoothly isotopic knots. The following theorem gives a complete answer. We formulate the theorem in a slightly informal way since we will not make use of it.

Theorem 59.21. (Reidemeister Moves Theorem) Two knot diagrams give rise to smoothly isotopic knots if and only if the two diagrams are related by a finite sequence of smooth isotopies of $\mathbb{R}^2$ and Reidemeister moves. The three Reidemeister moves are illustrated in the figure below.



Reidemeister move 1 Reidemeister move 2 Reidemeister move 3

Proof. The theorem, not surprisingly, goes back to work of Kurt Reidemeister [**Rei1932**, Chapter 3] in 1932. The formulation and proof in [**Rei1932**] is in terms of "polygonal knots". The statement of the theorem is well known in our setting, where knots are defined as submanifolds of S^3 and it appears in any textbook on knot theory. A satisfactory proof can be found in [**OSS2015**, Theorem B.1.1]. ■

Remark. Throughout these lecture notes we will see on several occasions that the tools of algebraic topology, e.g. fundamental groups, linking pairings and Reidemeister torsion can be used fruitfully to study knots. the Knot Diagram Existence Proposition 59.19 and the Reidemeister Moves Theorem 59.21 open up an alternative route to studying knots. More precisely, these two results show that there exists a bijection

$$\begin{array}{c}\text{knot diagrams up to smooth isotopy}\\ \text{and Reidemeister moves}\end{array} \xrightarrow{\cong} \text{knots up to smooth isotopy}$$

This bijection lies at the heart of the definition of several knot invariants, like the Jones polynomial [**JonV1985**] that was introduced in 1984 and the HOMFLY-PT polynomial [**FYHLMO1985, PT1987**] that was discovered in 1985. These polynomial invariants are quite different from any of the invariants that are obtained through algebraic-topological methods. We refer to [**Lic1997a**, Chapters 3 and 15] and [**BZH2014**, Chapter 17] for more information on these invariants.

59.6. **The Wirtinger presentation.** In the previous section we saw that we can associate a knot diagram to any given knot. In this section we will see that we give an explicit algorithm that, given a knot diagram, produces a presentation for the fundamental group of the complement $S^3 \setminus K$ of a knot K.

In this section we will take several liberties and we will not attempt to give full details in the proofs. The issue is that any attempt to make the proofs completely rigorous renders them totally unreadable.

Before we can state the promised algorithm we need to introduce a few more definitions.

Definition. Let $K \subset \mathbb{R}^3$ be an oriented knot that is associated to some knot diagram $(f\colon S^1 \to \mathbb{R}^2, c)$.

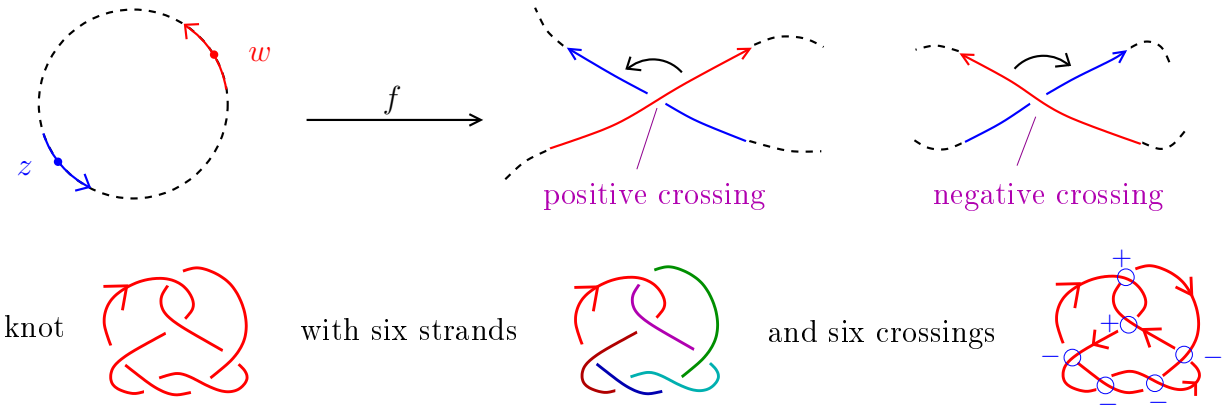
(1) We refer to the images of the double points as the crossings of K.
(2) Let $x \in f(S^1)$ be a crossing. Let $z, w \in S^1$ with $f(z) = f(w) = x$ and such that z is an undercrossing and w is an overcrossing. We say x is a positive crossing if the

ordered basis $\{f'(w), f'(z)\}$ is a positive basis for $\mathbb{R}^2$. Otherwise we say that x is a negative crossing.

(3) We refer to the components of $K \cap (\mathbb{R}^2 \times \{0\})$ as the strands of K.

These definitions are illustrated in the figures below.

Remark. In a picture of a diagram the strands are understood to correspond precisely to the segments drawn.



The following lemma will be proved in Exercise 59.8.

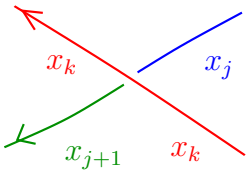
Lemma 59.22. If a knot diagram has at least one crossing, then the number of crossings equals the number of strands.

The following proposition shows how to determine the fundamental group of a knot that is associated to a knot diagram. We will sacrifice some rigor in the formulation in an attempt to make it more readable. We leave it to the reader to make the proposition totally rigorous.

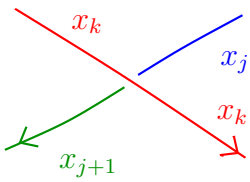
Proposition 59.23. (Wirtinger Presentation Proposition) Let K be a knot that is associated to a given knot diagram that has at least one crossing. We enumerate the strands cyclically by $x_1, \ldots, x_n$ where we make the identifications $x_0 = x_n$ and $x_{n+1} = x_1$. For the crossing between the strands x_j and x_{j+1} we define a relation r_j as shown in the figure below. Then we have an isomorphism

$$\pi_1(S^3 \setminus K) \;\cong\; \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_n \rangle.$$

Furthermore we can drop any one of the relations and we still obtain an isomorphism.



the positive crossing gives rise to the relation $r_j = x_k \cdot x_{j+1} \cdot x_k^{-1} \cdot x_j^{-1}$

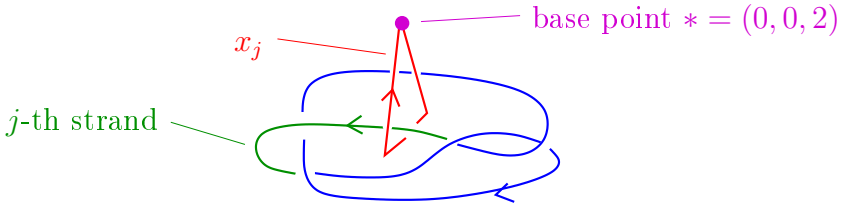


the negative crossing gives rise to the relation $r_j = x_k^{-1} \cdot x_{j+1} \cdot x_k \cdot x_j^{-1}$

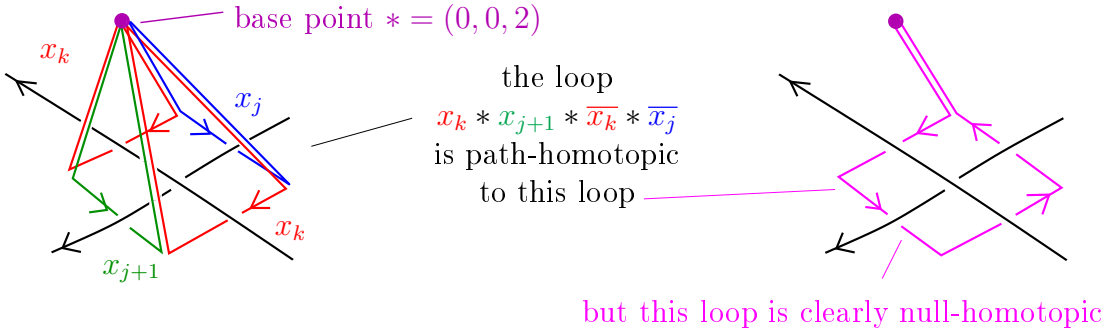
Remark. Given a knot K, any presentation with n generators and $n-1$ relators as obtained from the Wirtinger Presentation Proposition 59.23 is called a *Wirtinger presentation of* $\pi_1(S^3 \setminus K)$.

In the following we first explain why the Wirtinger Presentation Proposition 59.23 is plausible. Afterwards we will provide the technical proof of the Wirtinger Presentation Proposition 59.23.

Proof. "Plausibility argument for the Wirtinger Presentation Proposition 59.23"] Suppose we are given a knot diagram that has n crossings. We denote by K the corresponding knot. We work with the base point $* = (0, 0, 2) \in \mathbb{R}^3$ which lies "above" the knot K. We enumerate the strands cyclically. For the j-th strand we denote by x_j the oriented triangle that starts at $*$ and "circles once around the j-th strand" according to the "right-hand rule". We illustrate the definition in the figure below.



Next we consider the situation at a positive crossing. By "sliding" various x_j along the strands and by some further path-homotopies we can assume that we are in the situation in the figure below on the left. If one follows the loop $x_k * x_{j+1} * \overline{x_k} * \overline{x_j}$ on its journey one realizes that on three occasions one goes back and forth to the base point. Thus the loop is actually path-homotopic to the loop shown in the figure below on the right. But that loop is clearly null-homotopic. Almost the same logic applies to negative crossings.



This discussion, together with Lemmas 51.15 and 53.4, shows that there exists a unique group homomorphism

$$\begin{array}{rcl} \langle x_1, \ldots, x_n \,|\, \text{relations corresponding to crossings} \rangle & \to & \pi_1(S^3 \setminus K, *) \\ x_i & \mapsto & [x_i]. \end{array}$$

with

It remains to show that this group homomorphism is in fact an isomorphism. We will do so next in the proper proof of the Wirtinger Presentation Proposition 59.23. ■

Sketch of a proof for the Wirtinger Presentation Proposition 59.23. Suppose once again that we are given a knot diagram $(\gamma\colon S^1 \to \mathbb{R}^2, c)$ that has $n \geq 1$ crossings. We denote by $P_1, \ldots, P_n$ the double points of γ. We pick a smooth function $\eta\colon S^1 \to [-1, 0]$ as

in the statement of Lemma 59.18. We denote by K the image of the map $S^1 \to \mathbb{R}^3$ given by $z \mapsto (\gamma(z), \eta(z))$. It follows from property (d) of Lemma 59.18 that

$$K \cap (\mathbb{R}^2 \times \mathbb{R}_{\leq -1}) \;=\; K \cap (\mathbb{R}^2 \times \{-1\}) \;=\; \{(P_1, -1), \ldots, (P_n, -1)\}.$$

In contrast to the above plausibility argument we now actually work with the base point $\infty \in S^3 = \mathbb{R}^3 \cup \{\infty\}$. We set

$$A \;:=\; (\{(x, y, z) \in \mathbb{R}^3 \,|\, z \geq -1\} \cup \{\infty\}) \setminus K,$$

and

$$B \;:=\; (\{(x, y, z) \in \mathbb{R}^3 \,|\, z \leq -1\} \cup \{\infty\}) \setminus K.$$

We start out with the following claim.

Claim 1. The inclusion maps induce an isomorphism

$$\pi_1(A, \infty) *_{\pi_1(A \cap B, \infty)} \pi_1(B, \infty) \;\cong\; \pi_1(S^3 \setminus K).$$

Proof. Evidently the statement of the claim smells like it should be a consequence of the Seifert–van Kampen 54.1. The problem is that A and B are not open subsets of $S^3 \setminus K$. We rectify this problem by replacing A and B by suitable open subsets. We define

$$\begin{aligned} U \;&:=\; A \;\cup\; (\mathbb{R}^2 \setminus \{P_1, \ldots, P_n\}) \times (-2, -1) \\ \text{and} \qquad V \;&:=\; B \;\cup\; (\mathbb{R}^2 \setminus \gamma(S^1)) \times [-1, 0) \\ &\qquad\; \cup\; \{(z, t) \,|\, \text{ there exists } w \in S^1 \text{ with } z = \gamma(w) \text{ and } t \in [-1, \eta(z))\}. \end{aligned}$$

One can easily verify that U and V are open subsets of $S^3 \setminus K$. Furthermore the "obvious retractions" of the blue sets to the corresponding subsets of $\mathbb{R}^2 \times \{-1\}$ show that A is a deformation retract of U, that B is a deformation retract of V and that $A \cap B$ is a deformation retract of $U \cap V$. Now we see that the inclusion maps induce isomorphisms

$$\pi_1(S^3 \setminus K) \;\overset{\cong}{\underset{\uparrow}{\leftarrow}}\; \pi_1(U, \infty) *_{\pi_1(U \cap V, \infty)} \pi_1(V, \infty) \;\overset{\cong}{\underset{\uparrow}{\rightarrow}}\; \pi_1(A, \infty) *_{\pi_1(A \cap B, \infty)} \pi_1(B, \infty).$$

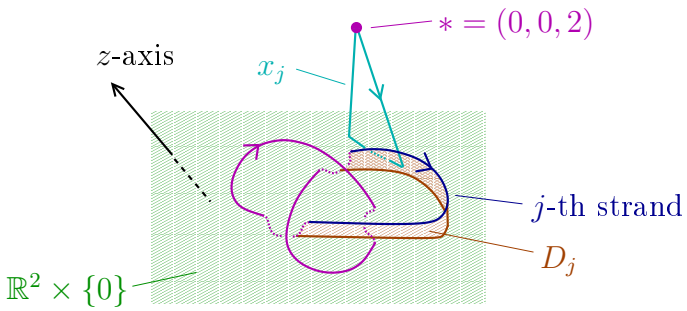
by the Seifert–van Kampen Theorem 54.1

by the above, the Deformation Retract-Homotopy Equivalence Lemma 15.5 and the Homotopy-π_1-Proposition 50.3 (2) ⊞

Our next goal is to understand the groups $\pi_1(A, \infty)$, $\pi_1(B, \infty)$ and $\pi_1(A \cap B, \infty)$. We handle these three groups consecutively. To better understand $\pi_1(A, \infty)$ we introduce the following objects.

(1) As in the plausibility argument we set $* = (0, 0, 2)$. We fix once and for all a path-connecting $*$ to ∞.
(2) We write $D := \{(\gamma(z), w) \,|\, z \in S^1 \text{ and } w \in [-1, \eta(z))\}$.
(3) We cyclically enumerate the strands of the knot.
(4) For $j \in \{1, \ldots, n\}$ we denote by D_j the unique component of D that contains the j-th strand in its closure.
(5) For $j = 1, \ldots, n$ we denote by x_j a loop in $*$ that is the boundary of a triangle that intersects D_j precisely once in a positive[704] direction. We use the path from (1) to view each x_j as a loop in ∞.

[704]Here by "positive direction" we mean that the orientation vectors of the triangle followed by the orientation vector of the knot should form a positive basis for $\mathbb{R}^3$.



Claim 2. The obvious map $\langle x_1, \dots, x_n \rangle \to \pi_1(A, \infty)$ is an isomorphism.

Proof. Note that each D_j is contractible and hence simply connected. Furthermore one can easily show that $A \setminus D$ is contractible. We use Corollary 57.5 iteratively n times to prove the claim. We leave it to the assiduous reader to fill in the details.[705] ⊞

Fortunately the next group $\pi_1(B, \infty)$ on our list is easy to handle.

Claim 3. The group $\pi_1(B, \infty)$ is trivial.

Proof. Using the stereographic projection from Lemma 2.18 and using the map defined on page 152 it is not hard to see that there exists a homeomorphism

$$f \colon (\{(x,y,z) \in \mathbb{R}^3 \,|\, z \leq -1\} \cup \{\infty\} \;\to\; \overline{B}^3$$

which sends the points in $\mathbb{R}^2 \times \{-1\}$ to S^2. Since $K \cap (\mathbb{R}^2 \times \mathbb{R}_{\leq -1}) = \{(P_1, -1), \dots, (P_n, -1)\}$ we see that B is homeomorphic to $\overline{B}^3$ with n points on its boundary removed. But this does not change the fact that the origin is a deformation retract. We have thus shown that $f(B)$, and hence also B, is contractible. But this implies by the Homotopy-π_1-Proposition 50.3 (3) that $\pi_1(B, \infty)$ is indeed trivial. ⊞

Finally we turn to the group $\pi_1(A \cap B, \infty)$. We pick orientation-preserving smooth embeddings $\varphi_i \colon \overline{B}^2 \to \mathbb{R}^2 \times \{-1\}$, $i = 1, \dots, n$ with disjoint images and such that for each i we have $\varphi_i(0) = (P_i, -1)$. It follows almost immediately[706] from Lemma 57.6 that we have an equality

$$\langle z_1, \dots, z_n \mid z_1 \cdot \dots \cdot z_n \rangle \;=\; \pi_1(A \cap B, \infty)$$

where for any $i \in \{1, \dots, n\}$ the element z_i is represented by a curve of the form $\alpha_i * \gamma_i * \overline{\alpha}_i$ where α_i is a suitable path from ∞ to a point on $\varphi_i(S^1)$ and γ_i is a path that goes once around $\varphi_i(S^1)$ in an orientation-preserving way.

[705] The details involve replacing the set D_j by a suitable open subset to be able to apply Corollary 57.5.

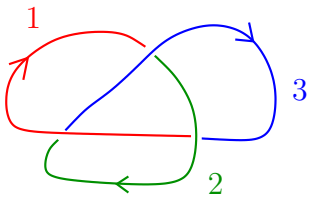
[706] The slightly cautious "almost" refers to the fact that we also make use of the fairly obvious fact that $((\mathbb{R}^2 \times \{-1\}) \cup \{\infty\}) \setminus \bigcup_{i=1}^{n} \varphi_i(B^2)$ is a deformation retract of $A \cap B = ((\mathbb{R}^2 \times \{-1\}) \cup \{\infty\}) \setminus \{P_1, \dots, P_n\}$.

Now we can finally prove the desired statement. Indeed, we have

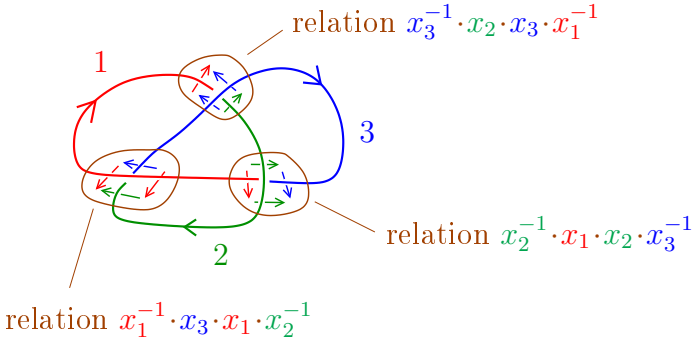
$$\begin{array}{ccccc}
\pi_1(S^3 \setminus K) & \cong & \pi_1(A,\infty) *_{\pi_1(A\cap B,\infty)} \pi_1(B,\infty) & \cong & \pi_1(A,\infty)/\langle\langle \pi_1(A\cap B,\infty)\rangle\rangle \\
& \uparrow & & \uparrow & \\
& \text{by Claim 1} & & \text{by Claim 3 and the Amalgamated Product Lemma 53.22 (2)} & \\
& \cong & \langle x_1,\dots,x_n\rangle/\langle\langle z_1,\dots,z_n\rangle\rangle & \cong & \langle x_1,\dots,x_n\rangle/\langle\langle r_1,\dots,r_n\rangle\rangle. \\
& \uparrow & & \uparrow & \\
& \text{by Claim 2, Lemma 53.2 (4) and the above fact that } \{z_1,\dots,z_n\} \text{ is a generating set for } \pi_1(A\cap B,\infty) & & \text{“plausibility argument shows that each } z_i \text{ is conjugate to } r_i, \text{ thus it follows from Lemma 53.2 (5) that } \langle\langle z_1,\dots,z_n\rangle\rangle = \langle\langle r_1,\dots,r_n\rangle\rangle &
\end{array}$$

It remains to show that we can drop any one of the relations. Thus let $j \in \{1,\dots,n\}$. It follows easily from $z_1 \cdot \dots \cdot z_n = e$ that $z_j \in \langle z_1,\dots,z_{j-1},z_{j+1},\dots,z_n\rangle$. Now note that by Lemma 53.2 (4) this implies that $\langle\langle z_1,\dots,z_n\rangle\rangle = \langle\langle z_1,\dots,z_{j-1},z_{j+1},\dots,z_n\rangle\rangle$. Therefore we see that in the above equality we can drop the relation r_j. ■

As mentioned above, we continue with two examples, namely the trefoil and the figure-8 knot. First we consider the knot diagram for the trefoil K that is shown in the figure below to the left. We number the strands as shown. To the right we show how the three crossings give rise to three relations. As pointed out in the Wirtinger Presentation Proposition 59.23, we can drop any one of the relations.



knot diagram for the trefoil
with three negative crossings



This discussion shows that we have proved the following lemma.

Lemma 59.24. Let K be the trefoil. There exists an isomorphism

$$\pi_1(S^3 \setminus K) \cong \langle x_1,x_2,x_3 \,|\, x_3^{-1}x_2x_3x_1^{-1}, x_2^{-1}x_1x_2x_3^{-1}\rangle.$$

Remark. Let K be the trefoil. In Proposition 59.5 we had also shown that there exists an isomorphism $\pi_1(S^3 \setminus K) \cong \langle x,y \mid x^2 = y^3\rangle$. In particular we have now shown that the two groups $\langle x_1,x_2,x_3 \,|\, x_3^{-1}x_2x_3x_1^{-1}, x_2^{-1}x_1x_2x_3^{-1}\rangle$ and $\langle x,y \mid x^2 = y^3\rangle$ are isomorphisms. In Exercise 59.9 we will give a purely algebraic proof of this statement.

The following corollary to the Wirtinger Presentation Proposition 59.23 gives us the disappointing news that our old trick of using abelianizations of fundamental groups cannot be used to distinguish knots.

Corollary 59.25. (Knot Group-Abelianization Corollary) Given any knot K the abelianization of the group $\pi_1(S^3 \setminus K)$ is isomorphic to $\mathbb{Z}$.

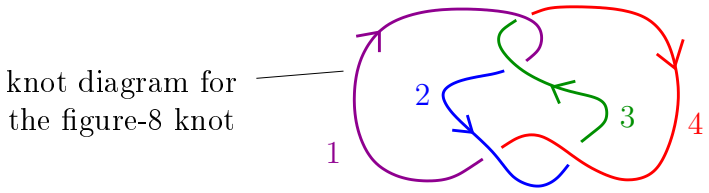
Proof. We will prove the corollary in Exercise 59.5. Later, on page 1836, once we have introduced homology groups we will give a different, arguably more conceptual proof of the lemma. ■

After this sobering corollary we move on to the next example.

Lemma 59.26. Let J be the figure-8 knot. There exists an isomorphism

$$\pi_1(S^3 \setminus J) \;\cong\; \langle x_1, x_2, x_3, x_4 \,|\, x_3 x_2 x_3^{-1} x_1^{-1}, x_4^{-1} x_3 x_4 x_2^{-1}, x_1 x_4 x_1^{-1} x_3^{-1} \rangle.$$

Proof. We leave it to the reader to use the knot diagram shown in the figure below to prove the lemma. ■



On its own Lemma 59.26 is not particularly enlightening. It is initially not clear whether this group is isomorphic to $\mathbb{Z}$ or to the fundamental group of the complement of the trefoil. By the Knot Group-Abelianization Corollary 59.25 we know that there is no hope that the abelianization will tell us something interesting.

So we need a different approach. Recall that in Proposition 59.6 we used the (non-) existence of an epimorphism onto the symmetric group S_3 to distinguish the trefoil from the trivial knot. The idea now is to repeat this trick with a different finite group. Recall that on page 1127 we introduced the dihedral groups D_n which have $2n$ elements. It is encouraging that $D_3 \cong S_3$ thus we try our luck with D_5.[707]

Using Lemma 53.13 and enough time on our hands we obtain the following table:

	trivial knot	trefoil	figure-8 knot
epimorphism onto S_3	✗	✓	✗
epimorphism onto the dihedral group D_5	✗	✗	✓

This table shows that we have proved the following corollary.

Corollary 59.27. The trivial knot, the trefoil and the figure-8 knot are pairwise not smoothly isotopic.

This approach of trying to distinguish knots by the (non-) existence of epimorphisms onto finite groups is rather ad hoc, but not without its merits. For example the following proposition is proved in [**BF2020, BR2020**].

Proposition 59.28. Let K be either the trivial knot, the trefoil or the figure-8 knot. Furthermore let J be another knot. If for all finite groups G the group $\pi_1(S^3 \setminus K)$ admits an epimorphism onto G if and only if $\pi_1(S^3 \setminus J)$ does, then J is smoothly isotopic to K or to the mirror image of K, i.e. the reflection of K in the xy-plane.

We conclude this section with the following question which is still open.

Question 59.29. Let K and J be two knots in $\mathbb{R}^3 \subset \mathbb{R}^3 \cup \{\infty\} = S^3$. Suppose that the finite groups onto which $\pi_1(S^3 \setminus K)$ and $\pi_1(S^3 \setminus J)$ admit epimorphisms are the same. Does this imply that K and J are smoothly isotopic, up to possibly taking a reflection in the xy-plane?

[707] The reader might ask, why we do not try our luck with D_4. But for reasons we cannot get into right now, given a knot K and an even number $2n$ there cannot be an epimorphism $\pi_1(S^3 \setminus K) \to D_{2n}$.

59.7. **The Reidemeister-Schreier process.** Even though the Knot Group-Abelianization Corollary 59.25 was quite disappointing, using a clever jiu-jitsu move we will see in this section that we can actually turn the Knot Group-Abelianization Corollary 59.25 to our advantage. The key to doing so is contained in the following definition and the subsequent lemma.

Definition. Let π be a group and let $n \in \mathbb{N}$. We say that a subgroup $\Gamma \subset \pi$ is cocyclic of order n if Γ is a normal subgroup such that π/Γ is isomorphic to $\mathbb{Z}_n$.

Lemma 59.30. If π is a group whose abelianization is isomorphic to $\mathbb{Z}$, then given any $n \in \mathbb{N}$ there exists a unique subgroup that is cocyclic of order n. It is given by the kernel of any epimorphism $\pi \to \mathbb{Z}_n$.

Proof. Let π be a group whose abelianization is isomorphic to $\mathbb{Z}$ and let $n \in \mathbb{N}$. We denote by $\psi\colon \pi \to \pi_{\text{ab}}$ the natural projection and we pick an isomorphism $\varphi\colon \pi_{\text{ab}} \xrightarrow{\cong} \mathbb{Z}$. It is clear that $(\varphi \circ \psi)^{-1}(n\mathbb{Z})$ is a subgroup of π that is cocyclic of order n. On the other hand, suppose that Γ is a subgroup of π that is cocyclic of order n. We pick an isomorphism $\theta\colon \pi/\Gamma \cong \mathbb{Z}_n$. It follows from the Abelianization Proposition 53.20 that there exists a unique epimorphism $\alpha\colon \mathbb{Z} \to \mathbb{Z}_n$ such that the following diagram commutes:

$$\begin{array}{ccccc} \pi & \xrightarrow[\cong]{\psi} & \pi_{\text{ab}} & \xrightarrow[\cong]{\varphi} & \mathbb{Z} \\ {\scriptstyle g\mapsto g\Gamma}\big\downarrow & & & & \big\downarrow{\scriptstyle \alpha} \\ \pi/\Gamma & & \xrightarrow[\cong]{\theta} & & \mathbb{Z}_n. \end{array}$$

It follows immediately that $\Gamma = (\varphi \circ \psi)^{-1}(n\mathbb{Z})$. ∎

The Knot Group-Abelianization Corollary 59.25 and Lemma 59.30 allow us to make the following definition.

Definition. Given a knot K and $n \in \mathbb{N}$ we denote by $\pi(K, n)$ the unique subgroup of $\pi_1(S^3 \setminus K)$ that is cocyclic of order n.

Example. If K is the trivial knot, then we saw in Proposition 59.4 that we can make an identification $\pi_1(S^3 \setminus K) = \mathbb{Z}$. It follows easily that for any $n \in \mathbb{N}$ we have $\pi(K, n) = n \cdot \mathbb{Z}$, in particular $\pi(K, n)$ is isomorphic to $\mathbb{Z}$.

We have now a new tool for trying to distinguish knots:

Lemma 59.31. Let K and J be two knots. If K and J are smoothly isotopic, then given any $n \in \mathbb{N}$ the groups $\pi(K, n)$ and $\pi(J, n)$ are isomorphic. In particular the abelianizations of these groups are isomorphic.

The question arises, given a knot K and $n \in \mathbb{N}$, how can we determine $\pi(K, n)$ from a given presentation for $\pi_1(S^3 \setminus K)$? Fortunately the following proposition gives us an explicit algorithm. To facilitate the uptake we formulate the following proposition in a slightly informal way.

Proposition 59.32. (Reidemeister-Schreier Process) Let $\pi = \langle x_1, \dots, x_m, t \,|\, r_1, \dots, r_l \rangle$ be a group and let $\phi\colon \pi \to \mathbb{Z}_n$ be an epimorphism such that $\phi(x_1) = \cdots = \phi(x_m) = 0$ and

$\phi(t) = 1$. Then a presentation for the subgroup $\ker(\phi\colon \pi \to \mathbb{Z}_n)$ is given by

$$\langle y_{1,0}, \dots, y_{1,n-1}, \dots, y_{m,0}, \dots, y_{m,n-1}, u \,|\, s_1^0, \dots, s_1^{n-1}, \dots, s_l^0, \dots, s_l^{n-1} \rangle$$

where each s_i^j is obtained as follows:

(1) For $i = 1, \dots, l$ and $j = 0, \dots, n-1$ we write $t^j r_i t^{-j}$ as an expression in

$$x_1, tx_1t^{-1}, \dots, t^{n-1}x_1t^{-n+1}, \dots, x_m, tx_mt^{-1}, \dots, t^{n-1}x_mt^{-n+1} \text{ and } t^n.$$

(2) Then we replace each term $t^j x_i t^{-j}$ by $y_{i,j}$ and we replace t^n by u. We obtain a word in the new generators and denote it by s_i^j.

The inclusion map $\ker(\phi\colon \pi \to \mathbb{Z}_n) \to \pi$ is then given by sending each $y_{i,j}$ to $t^j x_i t^{-j}$ and by sending u to t^n.

Proof. As we had mentioned above, the proposition follows fairly easily from the more general Reidemeister-Schreier process that is explained in [**LyS1977**, Chapter II.4], [**MKS1976**, Chapter 2.3] or [**Bog2008**, Chapter 2.9]. ■

Instead of studying the proof of Proposition 59.32 our time is better spent trying to understand two explicit examples.

Example. Let $K \subset S^3$ be the trefoil. We write $\pi = \pi_1(S^3 \setminus K)$. In Proposition 59.5 we showed that we can make the identification $\pi = \langle x, t \mid \underbrace{x^3 \cdot t^{-2}}_{=r} \rangle$. By Lemma 53.13 there exists a unique epimorphism $\phi\colon \pi \to \mathbb{Z}_2$ with $\phi(x) = 0$ and $\phi(t) = 1$. Evidently $\pi(K, 2) = \ker(\phi\colon \pi \to \mathbb{Z}_2)$. Now we will use Proposition 59.32 to find a presentation $\langle y_0, y_1, u \,|\, s_0, s_1 \rangle$ for $\ker(\phi\colon \pi \to \mathbb{Z}_2)$. We write

$t^0 r t^{-0} = \underbrace{x^3}_{=y_0^3} \cdot \underbrace{t^{-2}}_{=u^{-1}}$ and more interestingly $t^1 r t^{-1} = t x^3 t^{-3} = (txt^{-1})^3 \cdot t^{-2} = \underbrace{(txt^{-1})}_{y_1}^3 \cdot \underbrace{t^{-2}}_{=u^{-1}}$.

Thus we see that

$$\pi(K,2) \;=\; \ker(\phi\colon \pi \to \mathbb{Z}_2) \;\underset{\text{Proposition 59.32}}{=}\; \langle y_0, y_1, u \,|\, y_0^3 u^{-1}, y_1^3 u^{-1} \rangle \;\underset{\text{by the Tietze transformations from page 1196}}{=}\; \langle y_0, y_1 \,|\, y_0^3 = y_1^3 \rangle.$$

Using the Abelianization Proposition 53.20 we can now easily compute that the abelianization of the above group $\pi(K, 2) = \ker(\phi\colon \pi \to \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_3$.

This calculation, together with Lemma 59.31 and the above discussion of the trivial knot, gives a new proof of Proposition 59.6.

The previous example was fairly straightforward since we had started out with a presentation where one generator got sent to one, and the other ones got sent to zero. Unfortunately this is not the case for Wirtinger presentations, so we will need to work somewhat harder. In the following example we consider the figure-8 knot K and we carry out the first steps needed to calculate the isomorphism type of $\pi(K, 2)$.

Example. Let J be the figure-8 knot. In Lemma 59.26 we saw that we can make the identification

$$\pi_1(S^3 \setminus J) \;=\; \langle y_1, y_2, y_3, y_4 \,|\, y_1^{-1} y_4 y_1 y_3^{-1}, y_3^{-1} y_2 y_3 y_1^{-1}, y_2^{-1} y_1 y_2 y_4^{-1} \rangle.$$

We denote by $\phi\colon \pi_1(S^3 \setminus J) \to \mathbb{Z}_2$ the epimorphism that is given by sending each y_i to 1. We set $t := y_1$. The goal is to replace the generators y_1, y_2, y_3 by generators x_1, x_2, x_3 which

satisfy $\phi(x_i) = 0$. In fact we have

$$\begin{aligned}\pi_1(S^3 \setminus J) &= \langle t, y_2, y_3, y_4 \,|\, t^{-1}y_4 t y_3^{-1}, y_3^{-1}y_2y_3t^{-1}, y_2^{-1}ty_2y_4^{-1}\rangle \\ &= \langle t, y_2, y_3, y_4 \,|\, \underbrace{t^{-1}y_4t^{-1}tty_3^{-1}}_{=t^{-1}y_4ty_3^{-1}}, \underbrace{t^{-1}ty_3^{-1}y_2t^{-1}ty_3t^{-1}}_{=y_3^{-1}y_2y_3t^{-1}}, \underbrace{t^{-1}ty_2^{-1}ty_2t^{-1}ty_4^{-1}}_{=y_2^{-1}ty_2y_4^{-1}}\rangle \\ &= \langle t, x_2, x_3, x_4 \,|\, t^{-1}x_4tx_3^{-1}, t^{-1}x_3^{-1}x_2tx_3, t^{-1}x_2^{-1}tx_2x_4^{-1}\rangle. \\ &\quad\uparrow \\ &\quad\text{substitution } x_i = y_it^{-1},\ x_i^{-1} = ty_i^{-1}, \text{ see Lemma 53.12}\end{aligned}$$

At this point we have a presentation to which we can apply Proposition 59.32. A heroic calculation shows that $\pi(J,2)_{\text{ab}} \cong \mathbb{Z} \oplus \mathbb{Z}_5$. Thus we obtain a new proof that the figure-8 knot and the trefoil are not smoothly isotopic.

The above procedure for distinguishing knots works fairly well in practice. For example, with enough time at one's hand, or alternatively a convenient computer implementation, one can use the above methods to show that all the knots shown in the figure on page 1323 are in fact pairwise different.

Nonetheless, the above approach is not overly satisfying. It would be much better to have invariants of knots that are much quicker to calculate. We will come back to this challenge at a much later point.

The above discussion, in particular Proposition 59.32 raises the following purely algebraic question.

Question 59.33. Let π be a group with a given finite presentation and let $\phi\colon \pi \to G$ be an epimorphism onto a finite group.

(1) Is the kernel $\ker(\phi\colon \pi \to G)$ a finitely presented group?
(2) If the answer to (1) is yes, how can we determine a finite presentation for the kernel in terms of the given data?

It turns out that the answer to the first question is yes.

Proposition 59.34. If π is a finitely presented group and $\phi\colon \pi \to G$ is an epimorphism onto a finite group, then $\ker(\phi\colon \pi \to G)$ is also finitely presented.

The proof of the proposition is given in most books on combinatorial group theory, see e.g. [**LyS1977**, Chapter II.4], [**MKS1976**, Chapter 2.3] or [**Bog2008**, Chapter 2.9]. In fact the proof is algorithmic, in particular it also gives an affirmative answer to the second question. The algorithm is called the *Reidemeister-Schreier process*. The general Reidemeister-Schreier process is somewhat messy to formulate. Thus we refer to the above literature for details.

Remark. In the proof of Proposition 59.6 we showed that the group $\langle x, t \,|\, x^3 \cdot t^{-2}\rangle$ admits an epimorphism α onto the permutation group S_3. In [**Bog2008**, Example 2.9.3] the general Reidemeister-Schreier process is used to determine a presentation for the kernel of $\alpha\colon \langle x, t \,|\, x^3 \cdot t^{-2}\rangle \to S_3$.

59.8. **High-dimensional knots.** Since the study of knots in S^3 was so much fun, it would be a shame not to consider high-dimensional knots.

Definition. An n-dimensional knot in S^k is a smooth submanifold of S^k that is diffeomorphic to S^n. For $n = 1$ and $k = 3$ we say that such a knot is classical, when we allow any dimension then sometimes we say the knot is high-dimensional.

We start out with a boring example.

Example. Let $n, k \in \mathbb{N}$ with $k \geq n$. We refer to $U := \{(x, 0) \in S^{n+2} \subset \mathbb{R}^{k+1} \,|\, x \in S^n\}$ as the *trivial n-dimensional knot in S^k*.

Definition. We say an n-dimensional knot in S^k is trivial if it is smoothly isotopic to the trivial n-dimensional knot in S^k.

In Exercise **??** we showed, building on Proposition **??**, that every n-dimensional knot in a sphere of dimension $\geq 2n + 3$ is trivial. The following theorem is a refinement of this result.

Theorem 59.35. (Haefliger's (Un-)knotting Theorem)

(1) Let $n, k \in \mathbb{N}$. If $2k > 3(n + 1)$, then every n-dimensional knot in S^k is trivial.
(2) Given any $m \in \mathbb{N}$ there exists a $(4m-1)$-dimensional knot in S^{6m} that is non-trivial.

Example. It follows immediately from Haefliger's Unknotting Theorem 59.35, and an elementary calculation, that for any $n \geq 2$ every n-dimensional knot in S^{2n+1} is trivial. As we saw in Proposition 59.6, this conclusion does not hold for $n = 1$.

Proof. The first statement was proved by André Haefliger [**Hae1961a**, p. 47] in 1961. (An alternative proof is given in [**Wall2016**, p. 192].) Shortly afterwards André Haefliger [**Hae1962b**] also proved the second statement. ■

Remark. In the remainder of this section we will only consider knots of codimension two, i.e. n-dimensional knots in S^{n+2}. In principle one could also study embedded spheres of other codimensions. But first note that the Generalized Smooth Schönflies Theorem 82.5 implies that, at least for $n \neq 4$, codimension one knots in S^n are trivial. Also note that André Haefliger [**Hae1967, Hae1962a, Hae1961b, Hae1962b**] showed that in codimensions ≥ 3 there can be knots that are non-trivial. But Eric Zeeman [**Zee1963c**, p. 505] and John Stallings [**Stal1963**, Corollary 9.3] showed that knots of codimension ≥ 3 are "unknotted" in the "PL-category" and in the "topological category". Thus usually topologists tend to stick to knots of codimension two. Nonetheless, we will make use of knots of larger codimension in page **??**.

We continue with the following very unsurprising lemma.

Lemma 59.36. Let $n \in \mathbb{N}$.

(1) Let K and L be two n-dimensional knots in S^{n+2}. If K and L are smoothly isotopic, then $\pi_1(S^{n+2} \setminus K) \cong \pi_1(S^{n+2} \setminus L)$.
(2) If K is a trivial n-dimensional knot in S^{n+2}, then $\pi_1(S^{n+2} \setminus K) \cong \mathbb{Z}$.

Proof.

(1) The proof of this statement is verbatim the same as the proof of Proposition 58.4.
(2) Let K be a trivial n-dimensional knot in S^{n+2}. We see that

$$\pi_1(S^{n+2} \setminus K) \underset{\text{by (1)}}{\overset{\uparrow}{\cong}} \pi_1(S^{n+2} \setminus U) \overset{\downarrow}{\underset{\text{follows from Exercise 59.15 together with the Homotopy-}\pi_1\text{-Proposition 50.3}}{\cong}} \pi_1(S^1) \underset{\text{by Corollary 48.18}}{\overset{\uparrow}{\cong}} \mathbb{Z}.$$

■

Evidently the question now is whether there exist non-trivial high-dimensional knots. Or perhaps better, the question is whether we will manage to explicitly construct such examples. The good news is, we can indeed do so.

In the following we will present a construction, which has its origins in the work of Emil Artin [**Art1925**] and Eric Zeeman [**Zee1965**] that turns an n-dimensional knot into a family of $(n+1)$-dimensional knots. We make the following preparations:

(1) As on page 261 we make the identification $S^{n+2} = \overline{B}^{n+2}_+ \cup_{S^{n+1}} \overline{B}^{n+2}_-$.

(2) By Lemma 59.1 we can make the identification $S^{n+3} = (S^1 \times \overline{B}^{n+2}) \cup_{S^1 \times S^{n+1}} (\overline{B}^2 \times S^{n+1})$.

(3) We make the identification $\mathbb{R}^{n+2} = \mathbb{C} \times \mathbb{R}^n$. In particular we view $\overline{B}^{n+2}$ as a subset of $\mathbb{C} \times \mathbb{R}^n$.

(4) Given $z \in S^1$ let

$$\begin{array}{rccc} \rho_z \colon & \overline{B}^{n+2} & \to & \overline{B}^{n+2} \\ & (w,x) & \mapsto & (z \cdot w, x) \end{array}$$

the rotation of $\overline{B}^{n+2}$ by the "angle" $z \in S^1$ around "the $0 \times \overline{B}^n$-axis".

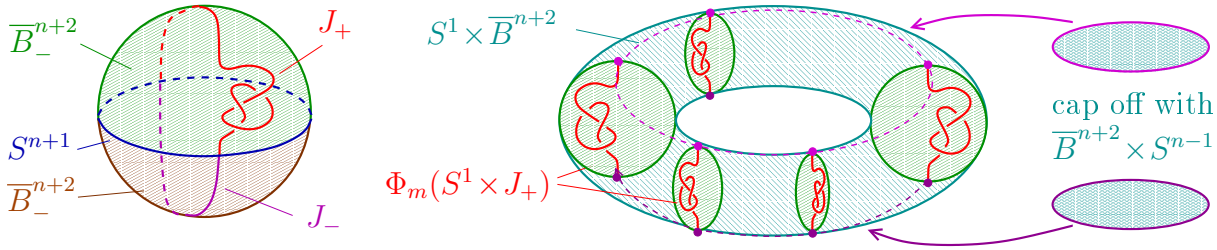
Now let $K \subset S^{n+2}$ be a knot. It follows fairly easily from the Smooth Ball Embedding Theorem 37.8 and the fact that K is a smooth submanifold that, after possibly applying a diffeotopy of S^{n+2}, we can assume that $J_- := K \cap \overline{B}^{n+2}_- = \{(0,w) \,|\, w \in \overline{B}^n\}$ is the "trivial disk knot" in $\overline{B}^{n+2}$. We denote by $J_+ := K \cap \overline{B}^{n+2}_+$ the "other disk knot". Next, given $m \in \mathbb{Z}$ we denote by Φ_m the diffeomorphism

$$\begin{array}{rccc} \Phi_m \colon & S^1 \times \overline{B}^{n+2} & \to & S^1 \times \overline{B}^{n+2} \\ & (z,x) & \mapsto & (z, \rho_{z^m}(x)). \end{array}$$

Informally speaking Φ_m spins $\overline{B}^{n+2}$ altogether m times as we go around the S^1-direction. Note that Φ_0 is the identity and also note that each Φ_m leaves the subset $S^1 \times S^{n-1}$ invariant as a subset. Now we define the *m-twist spin* $S_m(K)$ as follows:

$$S_m(K) := \underbrace{\Phi_m(S^1 \times J_+)}_{\subset S^1 \times \overline{B}^{n+2}} \cup_{S^1 \times S^{n-1}} \underbrace{(\overline{B}^2 \times S^{n-1})}_{\subset \overline{B}^2 \times S^{n+1}} \subset \underbrace{(S^1 \times \overline{B}^{n+2}) \cup_{S^1 \times S^{n+1}} (\overline{B}^n \times S^{n+1})}_{= S^{n+3}}.$$

We leave it to the reader to verify that $S_m(K)$ is indeed an $(n+1)$-dimensional knot. In the figure below we do our best to illustrate this construction for $n = 1$. Again informally speaking, $S_m(K)$ is given by spinning the disk knot J_+ around the S^1-direction, performing m spins around J_+ as you go around S^1, and then capping off the result by $\overline{B}^2 \times S^{n-1}$.



The following proposition will show shortly, perhaps unsurprisingly, that some twist spins are non-trivial. Perhaps more surprisingly the proposition also shows that (± 1)-twist spins are always trivial.

Proposition 59.37. Let $K \subset S^{n+2}$ be an n-dimensional knot.
(1) For the 0-twist spin there exists an isomorphism $\pi_1(S^{n+3} \setminus S_0(K)) \cong \pi_1(S^{n+2} \setminus K)$.
(2) For every $\epsilon \in \{-1, 1\}$ the ϵ-twist spin knot $S_\epsilon(K)$ is trivial.

Sketch of proof.

(1) We have the following isomorphisms:

$$\begin{array}{c}
\text{by the Seifert–van Kampen Theorem 54.4, we also use that } \Phi_0(S^1 \times J_+) = S^1 \times J_+ \\
\pi_1(S^{n+3} \setminus S_0(K)) \overset{\downarrow}{\cong} \pi_1(S^1 \times (\overline{B}^{n+2} \setminus J_+)) *_{\pi_1(S^1 \times (S^{n+1} \setminus S^{n-1}))} \pi_1(\overline{B}^2 \times (S^{n+1} \setminus S^{n-1})) \\
\cong \big\downarrow \quad \text{isomorphisms given by the Fundamental Group-of-Product Proposition 48.20} \\
(\langle t \rangle \times \pi_1(\overline{B}^{n+2} \setminus J_+)) *_{\langle s \rangle \times \langle t \rangle} \langle s \rangle \\
\cong \big\uparrow \quad \text{isomorphism by the group theoretic fact that } \Gamma \xrightarrow{\cong} (\langle t \rangle \times \Gamma) *_{\langle s \rangle \times \langle t \rangle} \langle s \rangle \\
\pi_1(\overline{B}^{n+2} \setminus J_+) \\
\cong \big\downarrow \quad \text{isomorphism by Theorem 54.4} \\
\pi_1(S^{n+2} \setminus K).
\end{array}$$

By now the reader will be fully aware of the fact that it is supremely painful to completely keep track of all identifications, base points, induced maps and so on. Thus we also took some liberties to improve readability.

(2) This statement is proved implicitly in [**Zee1965**, p. 486] and explicitly in [**FO2015**, Corollary 2.1]. The reason for writing [**FO2015**] was that it felt easier to prove the statement, than to try to read the proofs written down by others. So we warmly recommend to try to prove the statement on your own. ■

Corollary 59.38. Given any $n \in \mathbb{N}$ there exists an n-dimensional knot in S^{n+2} that is non-trivial.

Proof. Let $K \subset S^3$ be the trefoil and let $n \in \mathbb{N}$. We see that

$$\pi_1(S^{n+2} \setminus (n-1)\text{-st iterated 0-twist spin of } K) \underset{\text{Proposition 59.37}}{\overset{\uparrow}{\cong}} \pi_1(S^3 \setminus K) \underset{\text{Proposition 59.5 and Proposition 59.6}}{\overset{\uparrow}{\ncong}} \mathbb{Z}.$$

It now follows from Lemma 59.36 (2) that the $(n-1)$-st iterated 0-twist spin of the trefoil is a non-trivial knot. ■

Exercises for Chapter 59.

Exercise 59.1. Let $K \subset S^3$ be a knot and let H_1 and H_2 be two hyperplanes of $\mathbb{R}^4$. For $i = 1, 2$ we denote by K_i^* the reflection of K in the hyperplane H_i. Show that K_1^* and K_2^* are smoothly isotopic.
Hint. Use the Matrix Group Path-Component Proposition 2.37 and the Smooth Path-Connectivity Proposition 19.29.

Exercise 59.2. Given $k \in \mathbb{Z}$ we consider the map

$$\begin{array}{rcl} S^1 \times S^1 & \to & S^1 \times S^1 \\ (w,z) & \mapsto & (wz^k, z). \end{array}$$

Show that $(S^1 \times \overline{B}^2) \cup_{\varphi_k} (S^1 \times \overline{B}^2)$ is homeomorphic to S^3 if and only if $k \in \{-1, 1\}$.
Hint. It might be helpful to compute the fundamental group or the first homology group of this topological space.

Exercise 59.3. We consider the two knot diagrams shown in the figure below. Use Reidemeister moves to show that the associated knots are smoothly isotopic.



Exercise 59.4. Show that every diagrammatic map has only finitely many double points.

Exercise 59.5. Let K be a knot. Use the Wirtinger Presentation Proposition 59.23 to show that the abelianization of the group $\pi_1(S^3 \setminus K)$ is isomorphic to $\mathbb{Z}$.

Exercise 59.6. Let K and L be oriented knots. We denote by L^* the mirror image of L. Show that the fundamental groups of $S^3 \setminus (K \# L)$ and $S^3 \setminus (K \# L^*)$ are isomorphic.

Exercise 59.7. We defined the sign of a crossing of a diagram of an oriented knot. How does the sign change if we reverse the orientation of the knot?

Exercise 59.8. Show that if a knot diagram has at least one crossing, then the number of crossings equals the number of strands.

Exercise 59.9. Give a purely algebraic proof that the two groups $\langle x, y \mid x^2 = y^3 \rangle$ and $\langle a, b, c \,|\, c^{-1}bca^{-1}, b^{-1}abc^{-1} \rangle$ are isomorphic.
Remark. You could use the Tietze transformations introduced on page 1196 and you could use Lemma 53.12.

Exercise 59.10. Let $K \subset S^3$ be a knot. We consider the "cone"

$$C \;:=\; \{r \cdot P \,|\, P \in K \text{ and } r \in [0,1]\} \;\subset\; \overline{B}^4.$$

(a) Show that C is homeomorphic to the disk $\overline{B}^2$.
(b) We suppose that K is smoothly isotopic to the unknot. Show that C is a topological submanifold of the topological manifold $\overline{B}^4$.

Exercise 59.11. Let $K \subset S^3$ be the trefoil. As in Exercise 59.10 we consider the "cone"

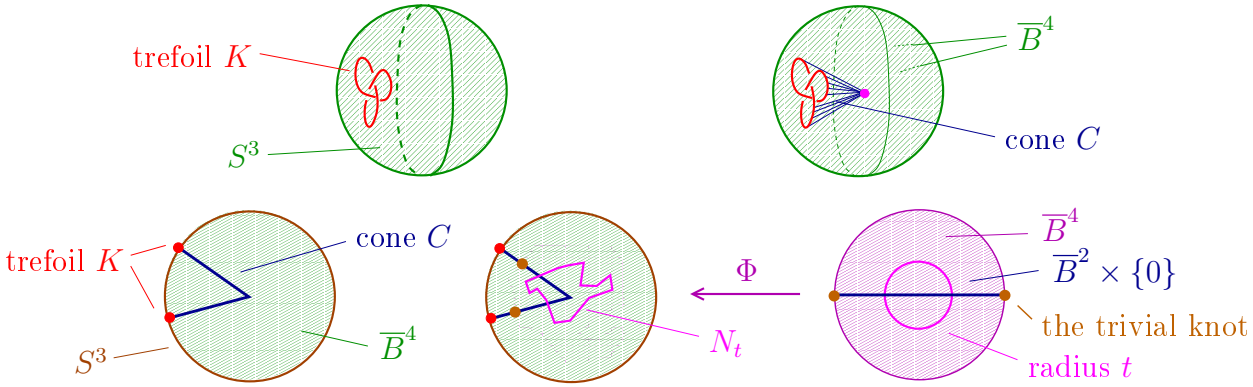
$$C \;:=\; \{r \cdot P \,|\, P \in K \text{ and } r \in [0,1]\} \;\subset\; \overline{B}^4.$$

In Exercise 59.10 we shewed that C is homeomorphic to the disk $\overline{B}^2$. In the following we will show that C is not a submanifold of the topological manifold $\overline{B}^4$.

(a) Let $\alpha\colon \pi_1(S^3 \setminus K) \to \mathbb{Z}$ and $\beta\colon \mathbb{Z} \to \pi_1(S^3 \setminus K)$ be two maps. Show that the composition $\beta \circ \alpha\colon \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K)$ is not an isomorphism.

(b) Suppose that F is a proper 2-dimensional topological submanifold of $\overline{B}^4$. Show that given any point $P \in F \cap B^4$ there exists a map $\Phi\colon \overline{B}^4 \to \overline{B}^4$ that is an embedding such that $\Phi(0) = P$ and such that $\Phi(\overline{B}^2 \times \{0\}) = \Phi(\overline{B}^4) \cap C$.

(c) Show that C is not a *topological submanifold* of $\overline{B}^4$.
Hint. Evidently the idea is to apply (c) to the point $P = 0 \in C$. The difficulty is that the map Φ from (c) is not just a rescaling of the identity but it is potentially a completely different chart. To deal with the problem we introduce some notation.
(i) Given $I \subset [0,1]$ we write $D_I := \{v \in \overline{B}^4 \mid \|v\| \in I\}$.
(ii) Given $I \subset [0,1]$ we write $N_I := \Phi(D_I)$.
(iii) Given $t \in (0,1]$ we write $D_t := D_{\{t\}}$ and $N_t := N_{\{t\}}$.
Show that the following statements hold:
(1) Given any $t \in (0,1]$ there exists an $s \in (0,1)$ with $D_{[0,s]} \subset N_{[0,t)}$.
(2) Given any $s \in (0,1]$ there exists a $t \in (0,1)$ with $N_{[0,t]} \subset D_{[0,s)}$.
(3) Given any choice of $r < s$ in $(0,1]$ the two inclusion maps $D_r \backslash C \to D_{[r,s]} \backslash C$ and $D_s \backslash C \to D_{[r,s]} \backslash C$ are homotopy equivalences, in particular they induce isomorphism of fundamental groups. Each of the fundamental groups is isomorphic to $\pi_1(S^3 \setminus K)$.
(4) Given any choice of $r < s$ in $(0,1]$ the two inclusion maps $N_r \backslash C \to N_{[r,s]} \backslash C$ and $N_s \backslash C \to N_{[r,s]} \backslash C$ are homotopy equivalences, in particular they induce isomorphism of fundamental groups. Each of the fundamental groups is isomorphic to $\mathbb{Z}$.
(5) Use the above discussion together with (b) to show that such Φ cannot exist.



Exercise 59.12. Let K be the trefoil. Compute the abelianization of the group $\pi(K, 3)$.

Exercise 59.13. Let π be a group. We say that a subset $S \subset \pi$ *normally generates* π if $\langle\langle S \rangle\rangle = \pi$. We define the *weight of* π as

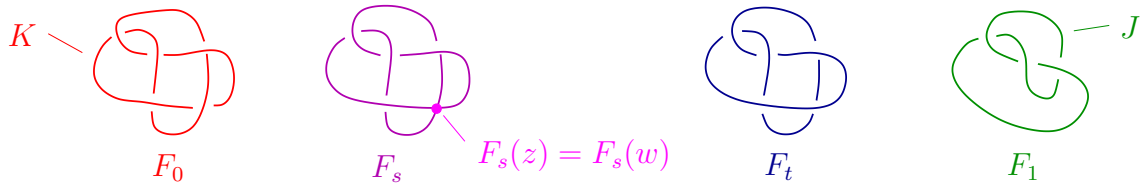
$$w(\pi) := \min\{\#S \mid S \text{ is a subset of } \pi \text{ that normally generates } \pi\} \in \mathbb{N}_0 \cup \{\infty\}.$$

(a) Determine the weight of $\mathbb{Z}^m$.
(b) Determine the weight of the free group on m generators.
(c) Let $K \subset S^3$ be a knot. Show that $w(\pi_1(S^3 \setminus K)) = 1$.

Exercise 59.14. We say two knots K and J in $\mathbb{R}^3$ are related by a *crossing change* if there exists a smooth map $F\colon S^1 \times [0,1] \to \mathbb{R}^3$ and some $s \in (0,1)$ with the following properties:
(1) We have $F_0(S^1) = K$ and $F_1(S^1) = J$.

(2) For every $t \neq s$ the map $F_t\colon S^1 \to S^3$ is an embedding.
(3) The map $F_s\colon S^1 \to S^3$ is an immersion and there exist $w, z \in S^1$ with the following properties:
   (i) the restriction of F_s to $S^1 \setminus \{w, z\}$ is an injection,
   (ii) $F_s(w) = F_s(z)$ and the vectors $F_s'(w)$ and $F_s'(z)$ are linearly independent.

In the figure below we illustrate the definition of a crossing change.



Given two knots K and J the minimal number of crossing changes needed to turn K into J is called the *Gordian distance between K and J*. The Gordian distance of a knot K to the trivial knot is called the *unknotting number of K*. Given explicit knots these numbers are notoriously difficult to calculate.

(a) Show that any two knots in $\mathbb{R}^3$ are related by a sequence of finitely many crossing changes. In other words, show that the Gordian distance between two knots is finite. *Hint.* You could use the fact that every knot admits a knot diagram.
(b) What are the best lower and upper bounds you can find on the Gordian distance between the trefoil and the figure-8 knot?

Remark. For readers who prefer completely formal proofs it can be a fun challenge to tweak the argument of the proof of Proposition **??** to obtain a proof of (a).

Exercise 59.15. Let $n \in \mathbb{N}$ and let $k \in \{1, \dots, n-1\}$. We consider the submanifold $K := \{(x, 0) \in \mathbb{R}^{n+1} \mid x \in S^k\}$. Show that $S^n \setminus K$ is homotopy equivalent to S^{n-k-1}.

Exercise 59.16. Let $K \subset S^3$ be an oriented knot and let $m \in \mathbb{Z}$. As on page 1339 we consider the m-twist spin $S_m(K) \subset S^4$. We view S^3 as a subset of S^4 via the embedding $z \mapsto (z, 0)$. The intersection $S_m(K) \cap S^3$ is a knot. What is the smooth isotopy type of this knot? A possible answer is that this knot is a connected sum of K with some suitable other knot.

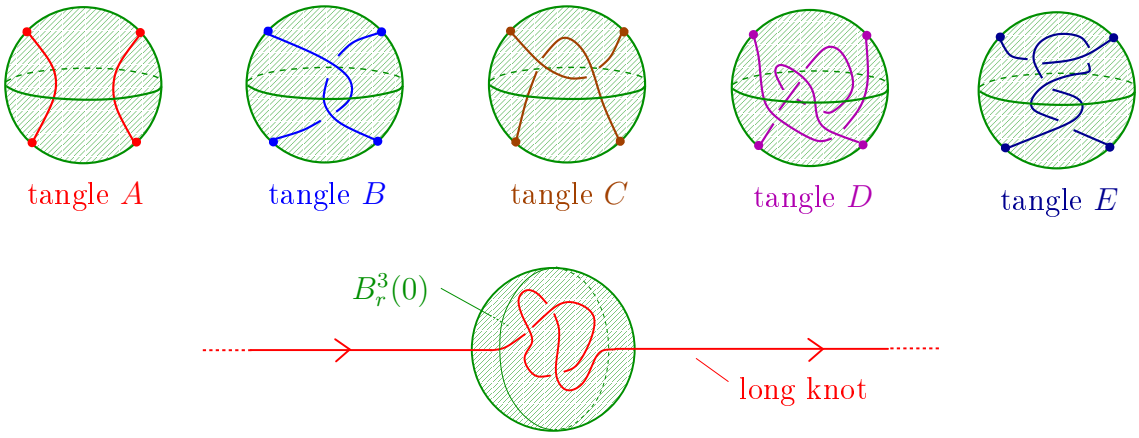
Exercise 59.17. A *tangle* is a proper 1-dimensional submanifold of $\overline{B}^3$. We consider the five tangles shown in the figure below. Note that the boundary of each of the four tangles is the same.

(a) For which pairs S, T of the five tangles does there exist an orientation-preserving diffeomorphism $\varphi\colon \overline{B}^3 \to \overline{B}^3$ with $\varphi(S) = T$?
(b) For which pairs S, T of the five tangles does there exist an orientation-preserving diffeomorphism $\varphi\colon \overline{B}^3 \to \overline{B}^3$ with $\varphi(S) = T$ and with $\varphi|_{S^2} = \mathrm{id}_{S^2}$.

Exercise 59.18. We start out with a few definitions.

(1) A *long knot* is a submanifold K of $\mathbb{R}^3$ that is diffeomorphic to $\mathbb{R}$ and which has the property that there exists an $r > 0$ such that

$$K \cap (\mathbb{R}^3 \setminus B_r^3(0)) \;=\; (\mathbb{R} \times \{(0,0)\}) \cap (\mathbb{R}^3 \setminus B_r^3(0)).$$



A long knot comes with a natural orientation given by "going from $-\infty$ to ∞".

(2) We say two long knots K and J are *equivalent* if there exists a *proper*[708] smooth isotopy $F\colon \mathbb{R}\times[0,1]\to\mathbb{R}^3$, in the sense of the definition on page 812, such that $F_0(\mathbb{R})=K$, $F_1(\mathbb{R})=J$ and such that each $F_t(\mathbb{R})$ is a long knot.

(3) Let L be a long knot. We pick an $r>0$ such that

$$L\cap B^3_r(0) \;\subset\; \mathbb{R}\times\{(0,0)\}.$$

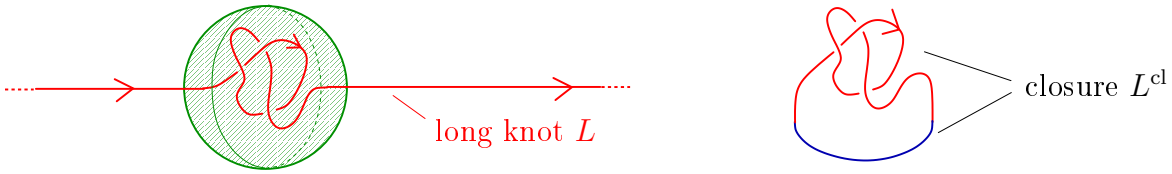
We denote by

$$\begin{array}{rcl}\Phi\colon \overline{B}^3_{2r}(0) & \to & S^3\\ x & \mapsto & \left(\frac{x}{2r}, \sqrt{1-|\frac{x}{2r}|^2}\right)\end{array}$$

the obvious smooth embedding whose image is precisely the upper hemisphere. We refer to

$$L^{\mathrm{cl}} \;:=\; \Phi\big(L\cap\overline{B}^3_{2r}(0)\big)\cup\big\{\big(x,0,0,\sqrt{1-x^2}\,\big)\,|\,x\in[-1,1]\big\}$$

as the *closure* L^{cl} *of the long knot* L. Note that the closure inherits an orientation from L.



Now we turn to the actual exercises:

(a) There is a pretty obvious definition of a composition $L\#L'$ of long knots L and L'. Turn this idea into a proper definition. Are $L\#L'$ and $L\#L'$ equivalent?

(b) Show that the fundamental groups of $\mathbb{R}^3\setminus L$ and $S^3\setminus L^{\mathrm{cl}}$ are isomorphic.

(c) Show that the closure L^{cl} is well-defined up to smooth isotopy and show that the map

$$\begin{array}{rcl}\{\text{long knots up to equivalence}\} & \to & \{\text{oriented knots up to smooth isotopy}\}\\ [L] & \mapsto & [L^{\mathrm{cl}}]\end{array}$$

is a bijection.

[708] The discussion in Exercise 36.9 shows that we cannot drop the requirement that the smooth isotopy is proper.

Exercise 59.19. We consider the epimorphism

$$\phi\colon \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \,|\, a^2, b^2 \rangle \;\to\; \mathbb{Z}_2$$

that is given by $\phi(a) = \phi(b)$. Show that $\ker(\phi)$ is isomorphic to $\mathbb{Z}$.
Hint. Use the Reidemeister-Schreier Process 59.32.

60. Decision problems

In the last chapters we had developed several techniques for determining the fundamental groups of topological spaces. In particular we managed to find finite presentations for the fundamental groups of large classes of topological spaces, e.g. topological graphs, surfaces, knot complements and mapping tori. In fact we now have enough tools to determine, with enough effort and perseverance, a presentation for the fundamental groups of most "nice" spaces. Furthermore we will see later in Propositions 96.6 and 104.14 on that the fundamental group of every compact topological manifold admits a *finite* presentation.
The question that now arises is, what information can we extract from a finite presentation of a group? Here are a couple of natural questions.

Question 60.1.

(1) Does there exist an algorithm that can decide whether or not a given finite presentation $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ represents the trivial group?
(2) Does there exist an algorithm that can decide whether or not two given finite presentations $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ and $\Gamma = \langle h_1, \dots, h_m \mid s_1, \dots, s_n \rangle$ represent isomorphic groups?[709]
(3) Does there exist an algorithm that can determine, given
   (a) a finite presentation $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ and
   (b) a word w in $g_1, \dots, g_k$, i.e. an element $w \in \langle g_1, \dots, g_k \rangle$,
   whether or not w represents the trivial element in the group π?[710]
(4) Does there exist an algorithm that can determine, given
   (a) a finite presentation $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ and
   (b) and two words v and w in $g_1, \dots, g_k$,
   whether or not v and w represent conjugate elements in the group π?[711]

Remark.

(1) This question was first posed by Max Dehn [**Deh1911**] in 1911. In fact shortly afterwards, in 1912 Max Dehn [**Deh1912**] gave positive answers to the word problem and the conjugacy problem for the fundamental groups of compact 2-dimensional smooth manifolds.
(2) The above question is formulated in a slightly informal way. After all, what is an "algorithm" supposed to be? The questions are made more precise in [**CZ1993**, Chapter 7], [**Stil1982**] and [**MillC1992**].

On page 1196 we introduced the Tietze transformations on presentations. We pointed out that two finite presentations for a given group are in fact related by a finite sequence of Tietze transformations. In particular, if $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ is a presentation for the trivial group, then we can turn the presentation after finitely many Tietze transformations into the trivial presentation $\langle | \rangle$. The problem is that we do not know how many Tietze transformations are required to turn a presentation of a trivial group into a trivial presentation. In particular at times it can be very hard to find the correct Tietze transformations.

[709] The question whether such an algorithm exists is often called the *isomorphism problem*.
[710] The question whether such an algorithm exists is often called the *word problem*.
[711] The question whether such an algorithm exists is often called the *conjugacy problem*.

Example. We consider the presentation

$$\pi \;=\; \langle a, b, c \mid a^3, b^3, c^4, ac = ca^{-1}, aba^{-1} = bcb^{-1}\rangle.$$

This is a presentation of the trivial group as can be seen as follows:[712]

(1) from $aba^{-1} \sim bcb^{-1}$ we get $(aba^{-1})^3 \sim (bcb^{-1})^3$,
(2) since $a^{-1}a \sim 1$ and $b^{-1}b \sim 1$ it follows that $(aba^{-1})^3 \sim ab^3a^{-1}$ and $(bcb^{-1})^3 \sim bc^3b^{-1}$,
(3) we have $b^3 \sim 1$, hence $ab^3a^{-1} \sim 1$, therefore we get from (1) and (2) that $bc^3b^{-1} \sim 1$ hence $c^3 \sim 1$,
(4) we have $c^3 \sim 1$ but we also have the relation $c^4 \sim 1$, so $c \sim 1$,
(5) since $aba^{-1} \sim bcb^{-1}$ it follows that $aba^{-1} \sim 1$ and so $b \sim 1$,
(6) from $ac \sim ca^{-1}$ and $c \sim 1$ it follows that $a \sim a^{-1}$, hence $a^2 \sim 1$,
(7) together with $a^3 \sim 1$ we obtain $a \sim 1$.
(8) we now showed that a, b, c are all equivalent to 1, thus we see that the group defined by the presentation is the trivial group.

This example shows that if a presentation corresponds to the trivial group, then it can be surprisingly complicated to verify this statement.

Now suppose we are given a finite presentation and we have to decide whether or not it is a presentation of the trivial group. We can modify the presentation using Tietze transformations. If at some point we end up with the trivial presentation we know that the given presentation corresponds to the trivial group. But what happens if after many attempts at modifying the presentation using Tietze transformations we still do not have the trivial presentation. Does that mean that the presentation corresponds to a non-trivial group? Or does it just mean that we did not try long enough?
So the real question is, does there exist an algorithm, that given a presentation which defines a non-trivial group, actually verifies that the group is non-trivial?
Unfortunately the answer is quite sobering.

Theorem 60.2. The answer to all four questions raised in Question 60.1 is no.

Remark. Here "no" means that one can show that no such algorithm exists. This is a much stronger statement than saying that we do not know of an example of such an algorithm.

Proof.

(1) Adyan [**Ady1955**] and Michael Rabin [**Rabi1958**] showed in 1955 that the answer to the first question is no. Alternatively see [**MillC1992**, Theorem 3.3] for a proof.
(2) An algorithm that performs (2) would also give an algorithm for determining whether or not a given presentation is isomorphic to the trivial group with the empty presentation $\langle|\rangle$. But since the answer to (1) is no, it follows that the answer to (2) is also no. Historically, it was first proved by Sergey Novikov [**Nov1965**] and William Boone [**Boo1958**] that (2) has a negative answer before a negative answer to (1) was given.
(3) If there did exist an algorithm that dealt with the third problem, then we could apply it in particular to the generators. Since a presentation presents the trivial group if and only if all generators are trivial, it follows that an algorithm for (3) would also

[712] For elements g, h of $\langle a, b, c\rangle$ we write $g \sim h$ if they represent the same element in π.

give an algorithm for (1). Since the latter cannot exist, the former cannot exist either.

(4) An element in a group is trivial if and only if it is conjugate to the trivial element. Hence the fact that (3) has a negative answer leads to a negative answer to (4). ■

Remark.

(1) There are many other non-existence results in the vein of Theorem 60.2. For example, [**MillC1992**, Corollary 3.4], see also Exercise 60.2 says the following:
   (a) it is undecidable whether a finitely presented group is free,
   (b) it is undecidable whether a finitely presented group is abelian,
   (c) it is undecidable whether a finitely presented group is nilpotent,
   (d) it is undecidable whether a finitely presented group is solvable,
   (e) it is undecidable whether a finitely presented group is simple,
   (f) it is undecidable whether a finitely presented group is torsion-free.

(2) The statement of Theorem 60.2 that the answer to all four questions of Question 60.1 is no, is of course somewhat disappointing. But this raises the question whether there are interesting classes of groups for which the answer is actually yes. For example Mikhail Gromov [**Grom1987**][713] introduced the notion of a "word hyperbolic group" for which the answer to all four questions turns out to be yes. Examples of word hyperbolic groups are given by
   (a) finite groups,
   (b) free groups, fundamental groups of surfaces of genus ≥ 2,
   (c) fundamental groups of hyperbolic smooth manifolds of any dimension,
   (d) fundamental groups of "Riemannian manifolds with negative sectional curvature".

   In fact in a precise sense a "random" finitely presented group is word hyperbolic, see e.g. [**Ol2005**] for more details.

   The topic of word hyperbolic groups is a very important topic of mathematics in its own right. See e.g. [**Löh2015**] for an introduction to word hyperbolic groups.

Of course our main goal is not to classify groups, but our goal is to understand "nice" topological spaces. For example ideally we would like to classify compact topological and smooth manifolds. The following theorem gives us some hope that this task is not completely hopeless.

Theorem 60.3. Let $n \in \mathbb{N}_0$.

(1) There exist only countably many homeomorphism classes of compact n-dimensional topological manifolds.

(2) There exist only countably many diffeomorphism classes of compact n-dimensional smooth manifolds.

Proof. The first statement is proved in [**CK1970**]. The second statement follows from [**Pets1984**, p. 77] and the fact that every smooth manifold admits a Riemannian structure, see e.g. [**Wall2016**, Theorem 1.3.1]. For $n \geq 6$ the second statement can also be

[713] Mikhail Gromov (1943-) is a French-Russian mathematician famous for his many contributions to geometry and group theory. He was the first person to view groups as "geometric objects", giving rise to the subject of "geometric group theory".

deduced from the fact that every compact n-dimensional smooth manifold admits a finite simplicial structure, see the Smooth Manifold-Simplicial Structure Theorem 96.2, the fact that there are only countably many isomorphism classes of finite simplicial complexes, see Exercise 93.4, and the (very difficult) Theorem **??** which says that for $n \geq 6$ every compact topological manifold admits at most finitely many smooth structures. ■

Given $n \in \mathbb{N}$ we would like to classify closed orientable connected n-dimensional smooth manifolds up to diffeomorphism. For $n = 1$ we did so in the Smooth 1-Dimensional Manifold Classification Theorem 22.7 and for $n = 2$ we saw the classification in the Surface Classification Theorem 55.4. In particular we saw that in these dimensions only "few groups" appear as fundamental groups.

We also proved the following result which shows that in dimension ≥ 4 the situation is dramatically different.

Proposition 54.12. Given any finitely presented group π and any $n \geq 4$ there exists a closed orientable connected n-dimensional smooth manifold with fundamental group π.

Theorem 60.2 and the Fundamental Group-Manifold Realization Proposition 54.12 are at the heart of the proof of the following theorem.

Theorem 60.4. Let $n \geq 4$.

(1) Then there is no algorithm that can decide whether or not two closed orientable n-dimensional smooth manifolds are homeomorphic.
(2) Then there is no algorithm that can decide whether or not two closed orientable n-dimensional smooth manifolds are diffeomorphic.

Proof. The first statement was first proved by Markov [**Mark1958**]. We refer to [**BHP1968**, Theorem 1] and [**Hak1973**] for a detailed discussion of the proofs of both statements. ■

This leaves us with the case of 3-dimensional smooth manifolds. It turns out that in this dimension there does in fact exist such an algorithm.

Theorem 60.5. There exists an algorithm that can decide whether or not two closed orientable connected 3-manifolds are diffeomorphic.

In this case the algorithm is considerably more complicated than in the 1-dimensional and the 2-dimensional case. The proof that such an algorithm exists is due to the work of many mathematicians, first and foremost William Thurston [**Thu1982**] and Grigori Perelman. The proof of the theorem was completed about 2003. It is impossible to mention here all the work that goes into proving Theorem 60.5. We refer to the survey paper [**AFW2015**, Theorem 4.27] for details and precise references.

Exercises for Chapter 60.

Exercise 60.1.

(a) Show that every finite group is isomorphic to some permutation group S_n.
(b) Let $\pi = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$ be a finitely presented group. Suppose that we know that π is finite. Show that there exists an algorithm which determines an injective homomorphism $\varphi\colon \pi \to S_n$ for some $n \in \mathbb{N}$.

(c) Let $\pi = \langle g_1, \ldots, g_k \mid r_1, \ldots, r_l \rangle$ and $\Gamma = \langle h_1, \ldots, h_m \mid s_1, \ldots, s_n \rangle$ be two finitely presented groups. Suppose we know that π and Γ are finite. Show that there exists an algorithm that can determine whether or not π and Γ are isomorphic.

Exercise 60.2.

(a) Show that it is undecidable whether a finitely presented group is free.
(b) Show that it is undecidable whether a finitely presented group is abelian.
(c) Show that it is undecidable whether a finitely presented group is nilpotent.
(d) Show that it is undecidable whether a finitely presented group is solvable.

Somewhat more precisely, for each group property show that there is no algorithm that can decide whether or not a given finite presentation $\pi = \langle g_1, \ldots, g_k \mid r_1, \ldots, r_l \rangle$ represents a group that has said property.

Exercise 60.3. A group π is called *residually finite* if given any non-trivial element $g \in \pi$ there exists a homomorphism $\alpha \colon \pi \to G$ to a finite group G such that $\alpha(g)$ is non-trivial. Show that the word problem is solvable in a finitely generated group that is residually finite.
Remark. A solution to this exercise can be found in [**MillC1992**, Theorem 5.2].

Part VI

The universal covering

61. The existence of universal coverings

After spending a lot of time on determining the fundamental groups of topological spaces we now want to return to the study of covering maps. One key goal of this section is to address the following question.

Question 61.1. Does every path-connected topological space B admit a path-connected covering $p\colon X \to B$ such that X is simply connected?

We know of course that the answer is yes for the torus, the Klein bottle and the Möbius band. But does such a covering exist for the surface of genus 2? Or does it exist for the complement $S^3 \setminus K$ of a knot $K \subset S^3$? How about the wedge $S^1 \vee S^1$ of two circles?

61.1. Local properties of topological spaces.

Before we can address Question 61.1 we need to recall the definition of a local property of topological spaces that we already gave in Section 2.8.

Definition. Let P be a property of topological spaces. We say a topological space X is locally P if given any $Q \in X$ and any neighborhood U of Q there exists an open neighborhood V of Q that is contained in U that has the property P.

Examples.

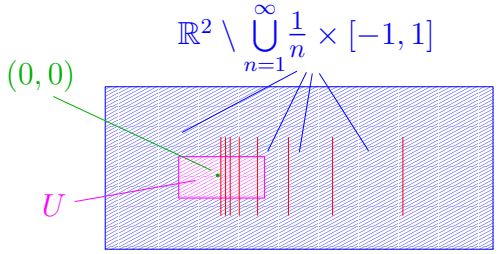
(1) By the Homotopy-π_1-Proposition 50.3 we know that every contractible topological space is simply connected. Thus every locally contractible topological space is also locally simply connected. Furthermore every simply connected space is by definition path-connected, thus every locally simply connected space is also locally path-connected.

(2) On page 176 we saw that every open subset of $\mathbb{R}^n$ is locally simply connected.

(3) By the Topological Manifolds-Local Properties Lemma 18.8 (1) every topological manifold is locally homeomorphic to some non-empty convex subset of $\mathbb{R}^n$. Since non-empty convex subsets are contractible we see that every topological manifold is locally contractible.

(4) In Exercise 61.1 we will show that every topological graph is locally contractible and we will show that the wedge of locally contractible topological spaces is again locally contractible.

(5) We consider
$$X := \mathbb{R}^2 \setminus \bigcup_{n=1}^{\infty} \left\{\tfrac{1}{n}\right\} \times [-1,1],$$
with the usual subspace topology coming from $\mathbb{R}^2$. Then X is path-connected but not locally path-connected. Indeed, we consider the point $(0,0) \in X$ and the neighborhood $U = (-\frac{1}{2},\frac{1}{2}) \times (-\frac{1}{2},\frac{1}{2}) \cap X$. It is straightforward to show that there does not exist a path-connected neighborhood V of $(0,0)$ which is contained in U.



(6) We consider

$$X \;:=\; \mathbb{R}^2 \;\setminus\; \bigcup_{n=1}^{\infty} \big(\tfrac{1}{n}, 0\big),$$

with the subspace topology coming from $\mathbb{R}^2$. Then X is path-connected, locally path-connected but it is not locally simply connected. Indeed the origin admits no neighborhood of the desired form, since any neighborhood of the origin contains "infinitely many holes". We made this statement precise in Exercise 52.1.

61.2. **Lifting maps to coverings.** Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Furthermore let $f\colon (Z, z_0) \to (B, b_0)$ be a map. Recall that on page 1080 we said that f *lifts*, if there exists a *lift* $\widetilde{f}\colon (Z, z_0) \to (X, x_0)$, i.e. a map $\widetilde{f}\colon Z \to X$ with $\widetilde{f}(z_0) = x_0$ and which makes the following diagram commute

$$\begin{array}{ccc} & & (X, x_0) \\ & \overset{\widetilde{f}}{\nearrow} & \big\downarrow p \\ (Z, z_0) & \xrightarrow[f]{} & (B, b_0). \end{array}$$

For example, if $f\colon ([0,1], 0) \to (B, b_0)$ is a map from the interval $[0,1]$ to B, i.e. if f is a path with starting point b_0, then we saw in Proposition 48.11 that there exists a unique lift of f to a map $\widetilde{f}\colon ([0,1], 0) \to (X, x_0)$.

Example. Consider the following maps:

$$\begin{array}{ccc} & & (S^1, 1) \\ & \overset{???\ \text{lift}\ \widetilde{f}\ ???}{\nearrow} & \big\downarrow p \;z\mapsto z^3 \\ (S^1, 1) & \xrightarrow[f]{z\mapsto z^2} & (S^1, 1). \end{array}$$

Here the map p is a 3-fold covering. Does the map f lift to a map $\widetilde{f}$?

To answer the question we need to study lifts of maps more carefully. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces and let $f\colon (Z, z_0) \to (B, b_0)$ be a map. Suppose that there exists a lift $\widetilde{f}\colon (Z, z_0) \to (X, x_0)$. We then obtain the commutative diagram

$$\begin{array}{ccc} & & (X, x_0) \\ & \overset{\widetilde{f}}{\nearrow} & \big\downarrow p \\ (Z, z_0) & \xrightarrow[f]{} & (B, b_0) \end{array}$$

from the functoriality of the fundamental group it follows that $f_* = \big(p \circ \widetilde{f}\big)_* = p_* \circ \widetilde{f}_*$, i.e. we get the commutative diagram

$$\begin{array}{ccc} & & \pi_1(X, x_0) \\ & \overset{\widetilde{f}_*}{\nearrow} & \big\downarrow p_* \\ \pi_1(Z, z_0) & \xrightarrow[f_*]{} & \pi_1(B, b_0). \end{array}$$

Thus it follows that if f lifts, then

$$\operatorname{im}(f_*) \;=\; \operatorname{im}\big(\big(p \circ \widetilde{f}\big)_*\big) \;=\; \operatorname{im}\big(p_* \circ \widetilde{f}_*\big) \;\subset\; \operatorname{im}(p_*).$$

So summarizing, if there exists a lift of f, then

$$\operatorname{im}(f_*\colon \pi_1(Z, z_0) \to \pi_1(B, b_0)) \;\subset\; \operatorname{im}(p_*\colon \pi_1(X, x_0) \to \pi_1(B, b_0)).$$

Example. In the above example we have $\pi_1(S^1, 1) = \mathbb{Z}$ and $\operatorname{im}(f_*) = 2\mathbb{Z}$ whereas we have $\operatorname{im}(p_*) = 3\mathbb{Z}$. So we see that f does not lift.

The following proposition says, that the above conclusion has a converse, for "reasonably" topological spaces. More precisely, we have the following proposition which, under a modest technical hypothesis, gives a complete criterion for when a map lifts.

Proposition 61.2. (Map Lifting Criterion) Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected topological space and let $f\colon (Z, z_0) \to (B, b_0)$ be a map.

(1) If Z is locally path-connected, then we have the following equivalence of statements:

$$\begin{array}{ccc} & & (X, x_0) \\ & \overset{\text{there exists a lift } \widetilde{f}}{\nearrow} & \downarrow p \\ (Z, z_0) & \xrightarrow[f]{} & (B, b_0) \end{array} \qquad \Longleftrightarrow \qquad \operatorname{im}(f_*) \subset \operatorname{im}(p_*).$$

(2) There exist at most one such lift, in other words, any two such lifts agree.[714]

Examples. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces.

(1) The condition on the fundamental groups in the Map Lifting Criterion 61.2 is automatically satisfied if Z is simply connected.
(2) Now let $f\colon (S^n, s_0) \to (B, b_0)$ be a map. The topological manifold S^n is path-connected and locally path-connected. By the Sphere-π_1-Proposition 47.13 the sphere S^n is simply connected if $n \geq 2$. Thus it follows from (1) that there exists a uniquely determined lift $\widetilde{f}\colon (S^n, s_0) \to (X, x_0)$ of the map $f\colon (S^n, s_0) \to (B, b_0)$.

Remark. In Exercise 61.4 we will see that we cannot drop the hypothesis that Z is locally path-connected.

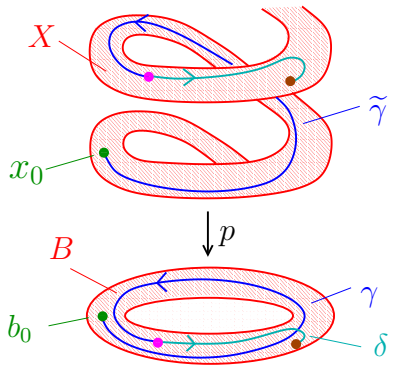
In the proof of the Map Lifting Criterion 61.2 we will need the following lemma.

Lemma 61.3. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed spaces, let γ be a path in B with starting point b_0 and let δ be another path in B with $\delta(0) = \gamma(1)$. We denote by $\widetilde{\gamma}$ the lift of γ with starting point x_0. Then

$$\begin{array}{c}\text{endpoint of the lift of } \gamma * \delta \\ \text{to the starting point } x_0\end{array} = \begin{array}{c}\text{endpoint of the lift of } \delta \\ \text{to the starting point } \widetilde{\gamma}(1).\end{array}$$

Proof. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed spaces, let γ be a path in B with starting point b_0 and let δ be another path in B with $\delta(0) = \gamma(1)$. We denote by $\widetilde{\gamma}$ the lift of γ with starting point x_0 and we denote by $\widetilde{\delta}$ the lift of δ to the starting point $\widetilde{\gamma}(1)$. Finally we denote by $\widetilde{\gamma * \delta}$ the lift of $\gamma * \delta$ to the starting point x_0. For $t \in [0, 1]$ we write $\beta(t) = (\gamma * \delta)(\frac{1}{2}(t+1))$ and $\widetilde{\beta}(t) = (\widetilde{\gamma * \delta})(\frac{1}{2}(t+1))$. Note that by definition we have $\beta(t) = \delta(t)$ for $t \in [0, 1]$ and that $\widetilde{\beta}$ and $\widetilde{\delta}$ are both lifts of $\beta = \delta$ to the starting point

[714] Note that for (2) we do not need the annoying hypothesis that Z is locally path-connected.



$\beta(0) = \gamma(1) = \delta(0)$. Thus it follows that

$$\text{endpoint of } \widetilde{\gamma * \delta} \;=\; \widetilde{(\gamma * \delta)}(1) \;=\; \widetilde{\beta}(1) \;\underset{\uparrow}{=}\; \widetilde{\delta}(1) \;=\; \text{endpoint of } \widetilde{\delta}.$$

by Proposition 48.11 ■

Now we can give the proof of the Map Lifting Criterion 61.2. To make the proof slightly more readable we prove the two statements (1) and (2) separately.

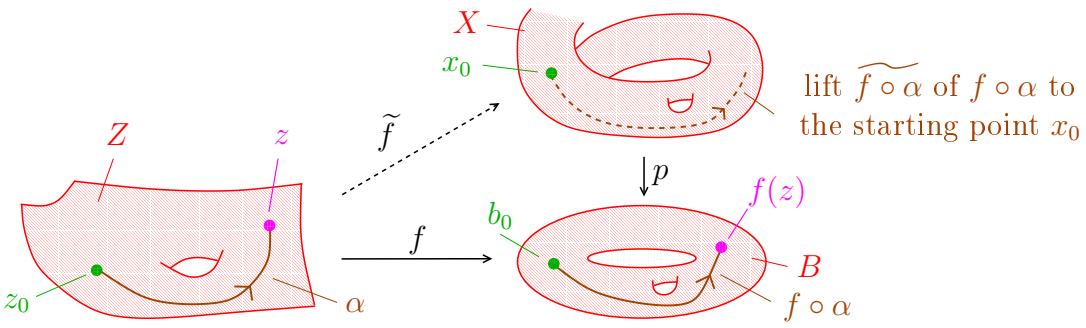
Proof of the Map Lifting Criterion 61.2 (1). Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected and locally path-connected topological space and let $f\colon (Z, z_0) \to (B, b_0)$ be a map.

By the discussion on page 1353 we only have to prove the "$\Longleftarrow$"-direction of the proposition. So suppose that we have

$$\operatorname{im}(f_*\colon \pi_1(Z, z_0) \to \pi_1(B, b_0)) \;\subset\; \operatorname{im}(p_*\colon \pi_1(X, x_0) \to \pi_1(B, b_0)).$$

We have to construct a lift $\widetilde{f}\colon (Z, z_0) \to (X, x_0)$. We will do so as follows. Let $z \in Z$ be a point. Since Z is path-connected we can find a path $\alpha\colon [0,1] \to Z$ with $\alpha(0) = z_0$ and $\alpha(1) = z$. According to Proposition 48.11 we can lift the path $f \circ \alpha\colon [0,1] \to B$ to a path $\widetilde{(f \circ \alpha)}\colon [0,1] \to X$ with starting point x_0. We define

$$\widetilde{f}(z) \;:=\; \widetilde{(f \circ \alpha)}(1).$$



We have to verify the following statements:

(1) the map $\widetilde{f}\colon Z \to X$ is well-defined,

(2) the map $\widetilde{f}$ is continuous,

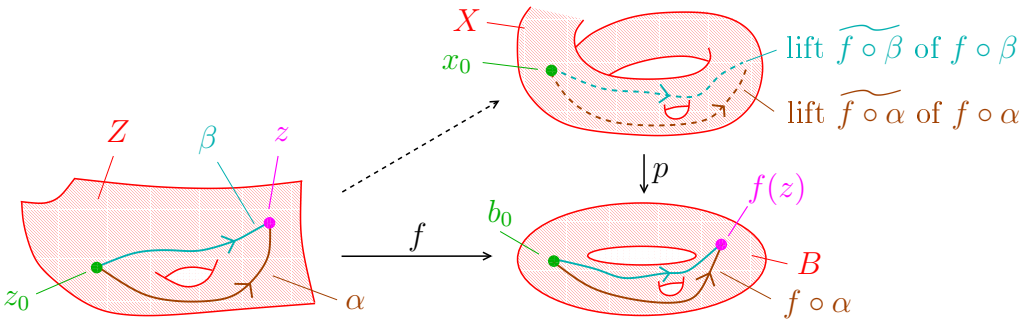
(3) the map $\widetilde{f}$ is a lift.
(4) and the map $\widetilde{f}$ is the unique lift.

We will do so in the following:

(1) We have to show that for $z \in Z$ the point $\widetilde{f}(z) \in X$ is well-defined, i.e. independent of the choice of the path α. Thus let $\beta\colon [0,1] \to Z$ be another path with $\beta(0) = z_0$ and $\beta(1) = z$. (This situation is illustrated in the figure below.) It follows that

$$\begin{aligned}
\widetilde{(f \circ \alpha)}(t=1) &= (\text{lift of } f \circ \alpha \text{ to the starting point } x_0)(t=1) \\
&= (\text{lift of } (f \circ \alpha) * \overline{(f \circ \beta)} * (f \circ \beta) \text{ to the starting point } x_0)(t=1) \\
&\quad \uparrow \\
&\quad \text{by Corollary 48.13 since } f \circ \alpha \text{ and } (f \circ \alpha) * \overline{(f \circ \beta)} * (f \circ \beta) \text{ are path-homotopic} \\
&= (\text{lift of } (f \circ (\underbrace{\alpha * \overline{\beta}}_{=:\gamma,\ \text{a loop in } (Z, z_0)})) * (f \circ \beta) \text{ to the starting point } x_0)(t=1) \\
&= (\text{lift of } f \circ \beta \text{ to the starting point } (\widetilde{f \circ \gamma})(1))(t=1) \\
&\quad \uparrow \\
&\quad \text{Lemma 61.3, where } \widetilde{f \circ \gamma} \text{ denotes the lift of } f \circ \gamma \text{ to the starting point } x_0 \\
&= (\text{lift of } f \circ \beta \text{ to starting point } x_0)(t=1) \;=\; \widetilde{(f \circ \beta)}(t=1). \\
&\quad \uparrow \\
&\quad \text{by our hypothesis we have } [f \circ \gamma] = f_*([\gamma]) \in \operatorname{im}(f_*) \subset p_*(\pi_1(X, x_0)), \\
&\quad \text{hence } \widetilde{f \circ \gamma} \text{ is a loop in } x_0 \text{ by Lemma 48.15}^{715}
\end{aligned}$$

Thus we have shown that the map $\widetilde{f}$ is well-defined.



(2) Now we want to show that $\widetilde{f}$ is continuous. So let $z \in Z$. We pick a path α from z_0 to z. Since p is a covering there exists an open neighborhood V of $\widetilde{f}(z)$ such that the restriction of p to V is an embedding. We denote by $q\colon p(V) \to V$ the inverse map. Since Z is locally path-connected there exists an open neighborhood U of z that is contained in $f^{-1}(p(V))$ and that is path-connected. Since f and q are continuous it suffices to prove the following claim.

Claim. On the open neighborhood U of z the map $\widetilde{f}$ agrees with $q \circ f$.

---

[715] Lemma 48.15 (1) says the following: if δ is a loop in (B, b_0) and if $\widetilde{\delta}$ denotes the lift of δ to the starting point x_0, then the following holds:

$$\widetilde{\delta} \text{ is a loop in } (X, x_0) \iff [\delta] \text{ lies in } p_*(\pi_1(X, x_0)) \subset \pi_1(B, b_0).$$

We apply the lemma to $\delta = f \circ \gamma$.

Proof. So let $w \in U$. Since U is path-connected there exists a path δ from z to w that lies in U. Then

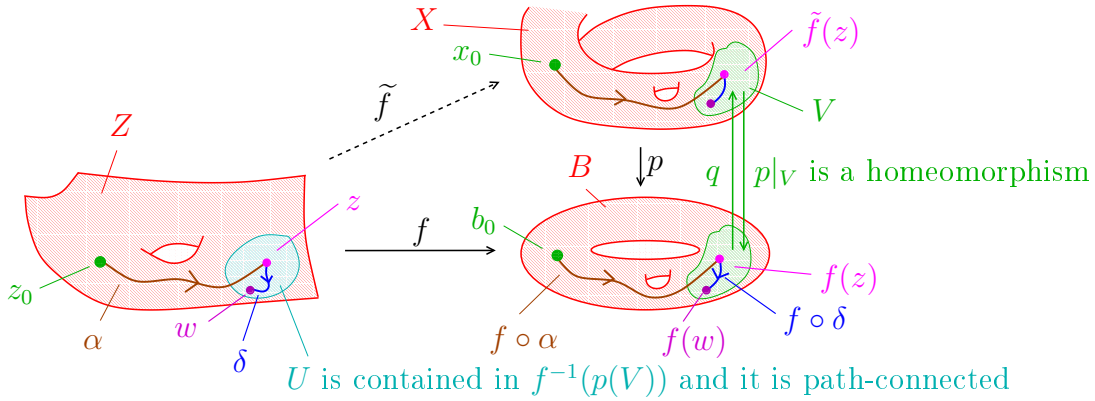
$$\begin{aligned}\widetilde{f}(w) &= \text{endpoint of the lift of } f\circ(\alpha*\delta) = (f\circ\alpha)*(f\circ\delta) \text{ to the starting point } x_0\\ &= \text{endpoint of the lift of } f\circ\delta \text{ to the starting point } \widetilde{f}(z)\\ &\quad\uparrow \text{ by Lemma 61.3 and the definition of } \widetilde{f}(z) \text{ as the endpoint of } \widetilde{f\circ\alpha}\\ &= \text{endpoint of } q\circ(f\circ\delta) \;=\; (q\circ f)(\delta(1)) \;=\; (q\circ f)(w).\\ &\quad\uparrow \text{ since } q = p^{-1} \text{ we know that } q\circ(f\circ\delta) \text{ is the lift of } f\circ\delta \text{ to the starting point } f(z)\end{aligned}$$

(3) We need to show that $\widetilde{f}$ is a lift, i.e. we need to show that $p\circ\widetilde{f} = f$. So let $z \in Z$. Then

$$p\big(\widetilde{f}(z)\big) \;=\; p\big(\big(\widetilde{f\circ\alpha}\big)(1)\big) \;=\; (f\circ\alpha)(1) \;=\; f(\alpha(1)) \;=\; z.$$

$\uparrow$ pick path α from z_0 to z and denote by $\widetilde{f\circ\alpha}$ the lift of $f\circ\alpha$

■



Proof of the Map Lifting Criterion 61.2 (2). Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected topological space. Finally suppose we are given a map $f\colon (Z, z_0) \to (B, b_0)$.

We will now see, using the uniqueness of the lift of paths established in Proposition 48.11, that any two lifts $\widetilde{f}, \widetilde{f}'\colon (Z, z_0) \to (X, x_0)$ agree. Thus let $z \in Z$. Since Z is path-connected there exists a path $\alpha\colon [0,1] \to Z$ from z_0 to z. Then $f\circ\alpha\colon [0,1] \to B$ is a path and $\widetilde{f}\circ\alpha$ and $\widetilde{f}'\circ\alpha$ are both lifts of $f\circ\alpha$ to the starting point x_0. By Proposition 48.11 we have $\widetilde{f}\circ\alpha = \widetilde{f}'\circ\alpha$. In particular the endpoints agree. But this implies the desired statement that $\widetilde{f}(z) = (\widetilde{f}\circ\alpha)(1) = (\widetilde{f}'\circ\alpha)(1) = \widetilde{f}'(z)$. ■

In Exercise 61.3 we will prove the following corollary.

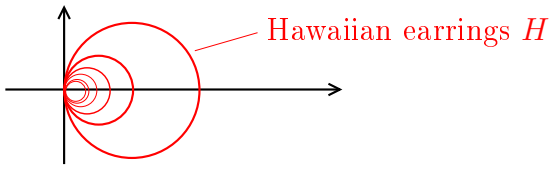
Corollary 61.4. Let $\pi\colon X \to B$ be a covering of topological spaces. Then every open simply connected and locally path-connected subset $U \subset B$ is uniformly covered.

61.3. **Semi-locally simply connected spaces.** Our next goal will be to show that every "reasonable" path-connected non-empty topological spaces admits a covering that is simply connected. To make precise what we mean by "reasonable" we need to introduce the following definition.

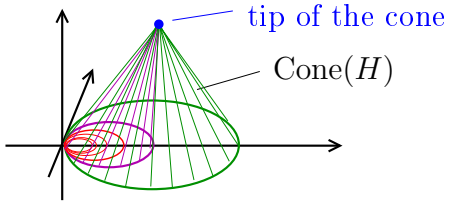
Definition. A topological space X is called semi-locally simply connected[716] if for every point $x \in X$ there exists an open[717] neighborhood U of x such that the inclusion induced map $\pi_1(U,x) \to \pi_1(X,x)$ is the trivial map.

Examples.

(1) Every simply connected space is semi-locally simply connected.
(2) Let H be the Hawaiian earring that we introduced on page 1284. In Exercise 61.2 we will see that H is not semi-locally simply connected.
(3) Evidently every topological space that is locally simply connected is also semi-locally simply connected. But the converse does not hold. For example let us consider the cone(H) on the Hawaiian earring.
   (a) By the Cone Contractibility Lemma 16.7 we know that the tip of cone(H) is a deformation retract. This implies that cone(H) is simply connected, in particular it is semi-locally simply connected.
   (b) On the other hand we will see in Exercise 61.2 that cone(H) is not locally simply connected.



not semi-locally simply connected



semi-locally simply connected
but not locally simply connected

61.4. **Existence of covering spaces.** Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed topological spaces. In Corollary 48.14 we saw that

$$p_*\colon \pi_1(X,x_0) \;\to\; \pi_1(B,b_0)$$

is an injective map. Thus we can view $\pi_1(X,x_0)$ as a subgroup of $\pi_1(B,b_0)$.
Generalizing Question 61.1 one can now ask, whether to each subgroup Γ of $\pi_1(B,b_0)$ there exists a corresponding covering with $p_*(\pi_1(X,x_0)) = \Gamma$. We will see that this is indeed the case, at least under the rather technical assumption that the topological space B is locally path-connected and semi-locally simply connected.
The following theorem gives in particular an affirmative answer to Question 61.1 under rather modest hypotheses.

[716] Note that in contrast to "locally simply connected" there is no general notion of what "semi-locally" means. In other words, we define "semi-locally simply connected" as a single object.
[717] If there exists a neighborhood W of x such that the inclusion induced map $\pi_1(W,x) \to \pi_1(X,x)$ is the trivial map, then the interior U of W is an open neighborhood of x and the inclusion induced map $\pi_1(U,x) \to \pi_1(X,x)$ is trivial, since it factors through the trivial map $\pi_1(W,x) \to \pi_1(X,x)$. Thus in the definition it makes no real difference whether we demand a "neighborhood" or an "open neighborhood".

Theorem 61.5. (Covering Existence Theorem) Let Y be a topological space that is path-connected, locally path-connected and semi-locally simply connected, let $y_0 \in Y$ and let $\Gamma \subset \pi = \pi_1(Y, y_0)$ be a subgroup.

(1) There exists a path-connected covering[718] $p\colon (X, x_0) \to (Y, y_0)$ of pointed topological spaces such that $p_*(\pi_1(X, x_0)) = \Gamma$.
(2) For the covering p from (1) we have

$$[X : Y] \ = [\pi_1(Y, y_0) : \Gamma].$$

We will provide the proof of the Covering Existence Theorem 61.5 in the next section. In the following we discuss a few examples and an important variation on the Covering Existence Theorem 61.5.

Examples.

(1) We consider $(Y, y_0) = (S^1, 1)$. We pick an identification[719] $\pi_1(S^1, 1) = \mathbb{Z}$. Given $n \in \mathbb{N}$ we consider $\Gamma = n\mathbb{Z} \subset \mathbb{Z} = \pi_1(S^1, 1)$. Then the covering

$$\begin{array}{rcl} p\colon S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}$$

has the property that $p_*(\pi_1(S^1, 1)) = n\mathbb{Z}$. Another example is given by the covering

$$\begin{array}{rcl} q\colon \mathbb{R}/n\mathbb{Z} & \to & S^1 \\ t + n\mathbb{Z} & \mapsto & \exp(2\pi i t). \end{array}$$

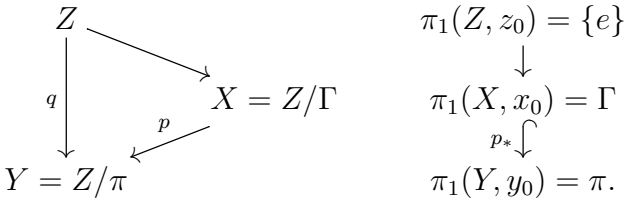
Here we also have $\mathrm{im}(q_*) = n\mathbb{Z}$.

(2) Let Z be a simply connected space and let $z_0 \in Z$. Furthermore let π be a group that acts on Z continuously and discretely. We consider $Y = Z/\pi$, we denote by $q\colon Z \to Y = Z/\pi$ the projection and we write $y_0 = q(z_0)$. Using the isomorphism of the Fundamental Group-via-Actions Theorem 48.16 we can identify π with $\pi_1(Y, y_0)$.

Now let $\Gamma \subset \pi = \pi_1(Y, y_0)$ be a subgroup. Then the group Γ also acts on Z and we can consider the projection

$$p\colon X := Z/\Gamma \ \to \ Y := Z/\pi$$

We denote by x_0 the image of z_0 under the projection $Z \to X = Z/\Gamma$. One can show easily that $p\colon X = Z/\Gamma \to Y = Z/\pi$ is a covering map with $p_*(\pi_1(X, x_0)) = \Gamma$. We summarize the situation in the following diagram



In the following we apply the Covering Existence Theorem 61.5 to any flavor of a manifold.

[718] Here we say that a covering $p\colon X \to Y$ is path-connected if X is path-connected.

[719] This means that we pick an isomorphism $\pi_1(S^1, 1) \xrightarrow{\cong} \mathbb{Z}$ and then ignore it in the notation.

Proposition 61.6. (Coverings-of-Manifolds Existence Proposition) Let Y be a path-connected topological manifold, let $y_0 \in Y$ and let $\Gamma \subset \pi_1(Y, y_0)$ be a subgroup.

(1) There exists a path-connected covering $p\colon (X, x_0) \to (Y, y_0)$ of pointed topological spaces such that $p_*(\pi_1(X, x_0)) = \Gamma$. Any such X is a topological manifold.
(2) If Y is a smooth manifold, then any cover as in (1) can be equipped with a unique smooth structure such that $p\colon X \to Y$ is a covering of smooth manifolds in the sense of the definition on page 1105.
(3) If Y is a complex manifold, then any cover as in (1) can be equipped with a unique complex structure such that $p\colon X \to Y$ is a covering of complex manifolds.

Proof. First of all note that by the Topological Manifolds-Local Properties Lemma 18.8 we know that topological manifolds are locally contractible. In particular they are locally path-connected and semi-locally simply connected. Thus we obtain from the Covering Existence Theorem 61.5 that there exists a path-connected covering $p\colon (X, x_0) \to (Y, y_0)$ of pointed topological spaces such that $p_*(\pi_1(X, x_0)) = \Gamma$ and such that $[X : Y] = [\pi_1(Y, y_0) : \Gamma]$.

The statements regarding smooth and complex manifolds follow immediately from the Manifold Covering Proposition 49.2. ■

61.5. **Proof of the Covering Existence Theorem 61.5.** Now we turn to the proof of the Covering Existence Theorem 61.5. Note that we only need show the existence of such a covering, the statement about the degree of the covering is an immediate consequence of Lemma 48.15 (3).

We start out with a general discussion of the construction of the covering. Only afterwards we will turn to the nitty-gritty of the proof. In particular for the time being let Y be any path-connected topological space, without any other adjectives. Furthermore let $y_0 \in Y$ and let $\Gamma \subset \pi_1(Y, y_0)$ be a subgroup.

To get an idea for the construction, let us first consider the case that Γ is the trivial group. In this case we want to construct a covering $p\colon X \to Y$ such that X is simply connected. Let us assume for a second, that we have such an X. We pick $x_0 \in X$ with $p(x_0) = y_0$. Then, somewhat similar to the statement of the Fundamental Group-via-Actions Theorem 48.16, we have a map

$$\begin{array}{rcl} \Phi\colon X & \to & \{\text{path-homotopy classes of paths starting in } y_0\} \\ x & \mapsto & p \circ (\text{any path from } x_0 \text{ to } x). \end{array}$$

In fact the proof of the Fundamental Group-via-Actions Theorem 48.16 comes close to showing that the map Φ is a bijection. Hence Φ is in fact a homeomorphism, provided we equip the right-hand side with an appropriate topology.

Now that we are not given X, the idea is, that given (Y, y_0) we can define the right-hand side of the above bijection, and with a suitable topology we will have found our X.

As on page 392 we denote by $Y^{[0,1]}$ the set of all maps $[0, 1] \to Y$. Next we consider the path space[720]

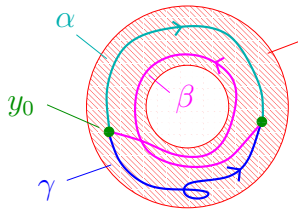
$$P_{y_0} \;:=\; \{\gamma \in Y^{[0,1]} \,|\, \gamma(0) = y_0\} \;=\; \{\text{all paths in } Y \text{ with starting point } y_0\}.$$

[720] On page 3280 the path space will be called $P(Y, y_0)$. Later on it will play an important role, but for the purpose of this discussion it is easier to work with a shorter notation.

For $u, v \in P_{y_0}$ we write

$$u \sim v \quad :\Longleftrightarrow \quad u \text{ and } v \text{ have the same endpoint and } [u * \overline{v}] \in \Gamma.$$

Using the fact that $\Gamma \subset \pi_1(Y, y_0)$ is a subgroup one can show easily that $\sim$ is indeed an equivalence relation on P_{y_0}. The definitions of P_{y_0} and $\sim$ are illustrated in the figure below. We write $X_{y_0,\Gamma} := P_{y_0}/\sim$ and $x_0 := [e_{y_0}]$, i.e. x_0 is the equivalence class of the constant path.



Y is an annulus

we consider $\Gamma = 2\mathbb{Z} \subset \pi_1(Y, y_0) = \mathbb{Z}$,
then α and β are equivalent,
but α and γ are inequivalent

Now we consider the evaluation map

$$\begin{array}{rccc} p\colon X_{y_0,\Gamma} = P_{y_0}/\sim & \to & Y \\ [v] & \mapsto & v(1). \end{array}$$

Note that this map is well-defined, i.e. it is independent of the choice of the representative v. It remains to equip $X_{y_0,\Gamma} = P_{y_0}/\sim$ with a sensible topology and to show that the map $p\colon X_{y_0,\Gamma} = P_{y_0}/\sim \to Y$ has all the desired properties.
In fact in the following we provide two approaches to defining such a topology:

(1) The first one is is "by hand", namely we construct a topology on $X_{y_0,\Gamma} = P_{y_0}/\sim$ from scratch.
(2) The second approach is "off-the-shelves", meaning we make use of the compact-open topology that we introduced on page 392.

Put differently, both Lemma 61.7 and Lemma 61.8 can be used to complete the proof of The Covering Existence Theorem 61.5.
We start out with the "by hand" approach.

Lemma 61.7. Let Y be a path-connected topological space, let $y_0 \in Y$ and let $\Gamma \subset \pi_1(Y, y_0)$ be a subgroup. We write $P := P_{y_0}$ and we write $X := X_{y_0,\Gamma}$. Given $x = [f] \in X = P/\sim$ and given an open neighborhood V of $p(x) = f(1) \in Y$ we consider

$$U(x, V) \;:=\; \{[f * u] \mid u \text{ is a path in } V \text{ with starting point } f(1)\} \subset X = P/\sim .$$

With this notation the following statement holds:

(0) The family of all such subsets $U(x, V)$ of X satisfy the "basis property" that we introduced on page 132.

The topology that is generated on $X = X_{Y_0,\Gamma}$ via the sets $U(x, V)$ is called the *whisker topology*. If Y is locally path-connected and semi-locally simply connected, and if we equip $X = X_{Y_0,\Gamma}$ with the whisker topology, then the following statements hold:

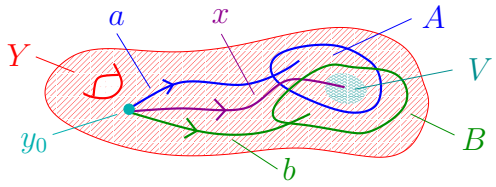
(1) The above evaluation map $p\colon X \to Y$ is continuous and open.
(2) The map $p\colon X \to Y$ is a covering.
(3) The topological space X is path-connected,
(4) We have $p_*(\pi_1(X, x_0)) = \Gamma$.

Proof. Let Y be a path-connected, let $y_0 \in Y$ and let $\Gamma \subset \pi_1(Y, y_0)$ be a subgroup. We write $P := P_{y_0}$ and we write $X := X_{y_0,\Gamma}$. Given $x = [f] \in X = P/\!\sim$ and given an open path-connected neighborhood V of $p(x) = f(1) \in Y$ we consider

$$U(x, V) \;:=\; \{[f * u] \mid u \text{ is a path in } V \text{ with starting point } f(1)\} \subset X = P/\!\sim .$$

We denote by $\mathcal{B}$ the family of all such subsets of X.

(0) We need to show that $\mathcal{B}$ has the basis property.
  (B1) For each $x \in X$ we have $x \in U(x, X)$. This shows that $\mathcal{B}$ satisfies (B1).
  (B2) Now let $x = [f]$ be a point in the intersection of $U(a, A)$ and $U(b, B)$. Let V be any open neighborhood V of $f(1)$ with $V \subset A \cap B$, e.g. we could take $V = A \cap B$. It is straightforward to see that $U(x, V) \subset U(a, A) \cap U(b, B)$.



In the following we assume that Y is locally path-connected and that Y is semi-locally simply connected. We equip $X = X_{Y_0,\Gamma}$ with the *whisker topology*, i.e. we equip it with the topology generated by $\mathcal{B}$. Since Y is locally path-connected it follows easily that the whisker topology is also generated by

$$\mathcal{C} \;:=\; \{U(x, V) \in \mathcal{B} \,|\, x \in X \text{ and } V \text{ is a } \textit{path-connected} \text{ open neighborhood of } p(x)\}.$$

(1) We need to show that the map $p\colon X = P/\!\sim \to Y$ is continuous and open. We start with the following observation: for every $x \in X$ and every open path-connected neighborhood V of $p(x)$ we have $p(U(x, V)) = V$.[721]

Now we turn to the actual proof of continuity. So let $A \subset Y$ be an open subset and let $x = [f] \in p^{-1}(A)$. It suffices to show that there exists an open subset B of Y such that $U(x, B) \subset p^{-1}(A)$. Since Y is locally path-connected, there exists an open neighborhood B of x that is contained in A and that is path-connected. By the above observation we have $p(U(x, B)) = B \subset A$, i.e. $U(x, B) \subset p^{-1}(A)$.

Finally we show that p is also open. By Lemma 2.11 it suffices to verify that images of the basis $\mathcal{C}$ of the topology of X are open. But this statement we had just verified in the above observation.

(2) Now we want to prove that $p\colon X = P/\!\sim \to Y$ is a covering map. So let $y \in Y$ be a point. We need to show that there exists a connected open neighborhood V of y such that the restriction of p to each component of $p^{-1}(V)$ is a homeomorphism.

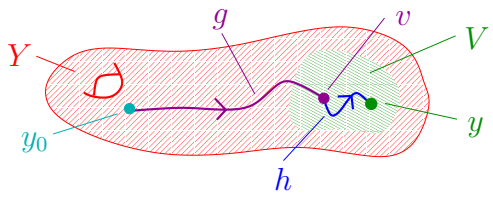
Since Y is semi-locally simply connected, there exists an open neighborhood U of y such that the inclusion induced map $\pi_1(U, x) \to \pi_1(Y, y)$ is the trivial map. We claim that V already has the desired property.

Claim. $$p^{-1}(V) \;=\; \bigsqcup_{\widetilde{y} \in p^{-1}(y)} U(\widetilde{y}, V).$$

[721] By definition we have $p(U(x, V)) \subset V$. But since V is path-connected we also have $V \subset p(U(x, V))$.

Proof. It is clear that the right-hand side is contained in the left-hand side. Next we want to show that the left-hand side is contained in the right-hand side. So let $[g] \in p^{-1}(V)$. Here g is a path in Y from y_0 to a point $v \in V$. Since V is in particular path-connected there exists a path h in V from v to y. We then have

$$[g] \;=\; [g * h * \overline{h}] \;\in\; U(\underbrace{[g * h]}_{\in p^{-1}(y)}, V).$$

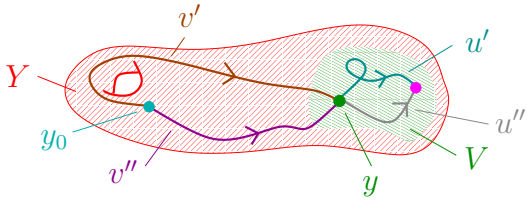


It remains to show that the sets on the right-hand side of the claim are disjoint. So let $[v'], [v''] \in p^{-1}(y)$ and let $[g] \in U([v'], V) \cap U([v''], V)$. Then there exist paths u' and u'' in V with $[v' * u'] = [v'' * u'']$. It follows that

$$[v'] \;=\; [v' * u' * \overline{u}'] \;=\; [v'' * u'' * \overline{u}'] \;\underset{\uparrow}{=}\; [v''].$$

$u'' * \overline{u}'$ is null-homotopic since the map $\pi_1(V, y) \to \pi_1(Y, y)$ is the trivial map

Thus $U([v'], V) \cap U([v''], V)$ intersect only if $[v'] = [v'']$. We refer to the figure below for an illustration. ⊞



The sets $U(\widetilde{y}, V)$ are by definition open. By the above claim the sets are also disjoint. Thus it suffices to prove the following claim.

Claim. For each $\widetilde{y} \in p^{-1}(y)$ the restriction of the projection $p\colon X = P/\!\sim \to Y$ to the map $p\colon U(\widetilde{y}, V) \to V$ is a homeomorphism.

Proof. So let $\widetilde{y} \in p^{-1}(y)$. As we had just pointed out in (1), the map $p\colon U(\widetilde{y}, V) \to V$ is surjective. From the fact that the inclusion induced map $\pi_1(V, y) \to \pi_1(Y, y)$ is trivial it also follows immediately that p is injective. Furthermore we had just verified in (1) that p is continuous and open. Therefore $p\colon U(\widetilde{y}, V) \to V$ is a homeomorphism. ⊞

(3) Now we want to show that $X = P/\!\sim$ is path-connected. We start out with the following claim.

Claim. Let $\alpha\colon [0,1] \to Y$ be a path. Then

$$\begin{array}{rcl} \widetilde{\alpha}\colon [0,1] & \to & X = P/\!\sim \\ t & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & Y \\ s & \mapsto & \alpha(ts) \end{array}\right]. \end{array}$$

is a continuous map.

Proof. By the Subbasis-Continuity Proposition 2.9 it suffices to show that for each $U(x,V)$ in the basis $\mathcal{C}$ of the topology on X the preimage $\widetilde{\alpha}^{-1}(U(x,V))$ is open in $[0,1]$. So suppose we are given $U(x,V)$. We denote by $\{I_j\}_{j\in J}$ the path-components of $\alpha^{-1}(V)$. Since α is continuous each I_j is an open subset of $[0,1]$. Note that for each $j\in J$ we either have[722] $\widetilde{\alpha}(I_j)\subset U(x,V)$ or $\widetilde{\alpha}(I_j)\cap U(x,V)=\varnothing$. It follows that

$$\widetilde{\alpha}^{-1}(U(x,V)) \;=\; \text{union of all the } I_j \text{ with the property that } \widetilde{\alpha}(I_j)\subset U(x,V).$$

Thus we see that $\widetilde{\alpha}^{-1}(U(x,V))$ is a union of open subsets of $[0,1]$, hence the preimage itself is open. ⊞

To show that X is path-connected it suffices to show that for every point $x\in X$ there exists a path in X from $x_0=[e_{y_0}]$ to x. So let $x=[\alpha]\in X$. Then such a path is given by $\widetilde{\alpha}\colon[0,1]\to X$, where $\widetilde{\alpha}$ is defined as in the claim. By the claim $\widetilde{\alpha}$ is indeed continuous.

(4) It remains to show that $p_*(\pi_1(X,x_0))=\Gamma$. Let $g\in\pi_1(Y,y_0)$. We choose a representatives $\alpha\colon[0,1]\to Y$ of g. We consider the following path[723] in $X=P/\!\sim$

$$\begin{array}{rcl} \widetilde{\alpha}\colon [0,1] & \to & X=P/\!\sim \\ t & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & Y \\ s & \mapsto & \alpha(ts) \end{array}\right]. \end{array}$$

Note that $\widetilde{\alpha}$ is evidently the lift of α to the starting point x_0. Then we have

$$g\in p_*(\pi_1(X,x_0)) \underset{\text{Lemma 48.15 (1)}}{\Leftrightarrow} \widetilde{\alpha} \text{ is a loop} \Leftrightarrow \text{in } X=P/\!\sim \text{ we have } \widetilde{\alpha}(1)=x_0=[e_{y_0}] \underset{\text{definition of } \sim \text{ on } P}{\Leftrightarrow} \alpha\in\Gamma.$$

■

Next we use the "off-the-shelves"-topology, namely we will make use of the compact-open topology. For the reader's convenience we recall the definition from page 392.

Definition. Let Y and A be topological spaces. The compact-open topology on the set Y^A, i.e. on the set of all maps from A to Y, is the topology generated, in the sense of the definition on page 135, by all the sets of the form

$$M(K,U) \;:=\; \{f\in Y^A \mid f(K)\subset U\}$$

where $K\subset A$ is compact and where $U\subset Y$ is open.

Lemma 61.8. Let Y be a path-connected topological space and let $y_0\in Y$. Furthermore let $\Gamma\subset\pi_1(Y,y_0)$ be a subgroup.

(i) We equip $Y^{[0,1]}=$ set of all maps $[0,1]\to Y$ with the compact-open topology.
(ii) We equip $P_{y_0}=\{\gamma\in Y^{[0,1]}\mid\gamma(0)=y_0\}$ with the subspace topology.
(iii) We equip $X_{y_0,\Gamma}=P_{y_0}/\!\sim$ with the quotient topology.

We refer to this topology on $X_{y_0,\Gamma}=P_{y_0}/\sim$ as the *CO-topology* on $X_{y_0,\Gamma}$. We write $X:=X_{y_0,\Gamma}$. If Y is locally path-connected and semi-locally simply connected, then the following statements hold:

(1) The above evaluation map $p\colon X\to Y$ is continuous and open.

[722] This might require a few minutes of thought.
[723] Note that this map is continuous by the claim in (3).

(2) The map $p\colon X \to Y$ is a covering.
(3) The topological space X is path-connected,
(4) We have $p_*(\pi_1(X, x_0)) = \Gamma$.

Proof. Let Y be a topological space that is path-connected, locally path-connected and semi-locally simply connected, let $y_0 \in Y$ and let $\Gamma \subset \pi_1(Y, y_0)$ be a subgroup. We write $X := X_{y_0,\Gamma}$ and we equip $X := P/\sim$ with the CO-topology. We make a few preparations:

(i) We consider the maps

$$\begin{array}{ccc} P & \hookrightarrow & Y^{[0,1]} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \mathrm{ev},\ f\mapsto f(1)} \\ X = P/\sim & \xrightarrow{\quad p \quad} & Y \end{array}$$

where q is the projection and $\mathrm{ev}\colon Y^{[0,1]} \to Y$ is given by the evaluation at 1. By the Topological-Quotient Proposition 5.15 (1a) we know that q is continuous. Furthermore by the Evaluation Map Proposition 12.4 we know that the map $\mathrm{ev}\colon Y^{[0,1]} \to Y$ is continuous. Thus it follows from Lemma 2.1 (3) and the Topological-Quotient Proposition 5.15 (1b) (2) that the induced map $p\colon X = P/\sim \to Y$ is continuous.

(ii) Since Y is locally path-connected it follows from Lemma 2.39 that the open path-connected subsets of Y form a basis for the topology of Y. Since $[0,1]$ is Hausdorff we obtain from Lemma 12.8 that the sets $M(K,U)$, with $K \subset [0,1]$ compact and U open and path-connected form a basis for the topology of the compact-open topology on $Y^{[0,1]}$.

(iii) Let $x = [f] \in X = P/\sim$ and let V be an open path-connected neighborhood of $p(x) = f(1) \in Y$. As in Lemma 61.7 we consider

$$U(x,V) \ := \ \{[f * u] \mid u \text{ is a path in } V \text{ with starting point } f(1)\} \ \subset \ X = P/\sim .$$

We want to show that $U(x,V)$ is an open subset of X. By definition of the quotient topology we need to show that $q^{-1}(U(x,V))$ is an open subset of P. This statement is easily seen to be a consequence of the following claim.

Claim. Let $g\colon [0,1] \to Y$ be a path with $g(0) = y_0$ and $g(1) \in V$. The set

$$\left\{ h \in P \,\middle|\, \begin{array}{c} \text{there exists a path } u\colon [0,1] \to V \text{ with } u(0) = g(1) \\ \text{such that } h \text{ is path-homotopic to } g * u \end{array} \right\}$$

is an open subset of P.

Proof. Let $h \in P$ be a path that is path-homotopic to $g * u$ for some path $u\colon [0,1] \to V$. Note that it follows from the fact that Y is semi-locally simply connected, together with the Lebesgue Corollary 6.6, that there exists an $n \in \mathbb{N}$ and open subsets $D_1, \ldots, D_n$ of Y with the following properties:
(a) For each $i \in \{0, \ldots, n-1\}$ we have $h([\frac{i}{n}, \frac{i+1}{n}]) \subset D_i$.
(b) For each $i \in \{0, \ldots, n-1\}$ the inclusion induced map $\pi_1(D_i, h(\frac{i}{n})) \to \pi_1(Y, h(\frac{i}{n}))$ is the trivial map.

Since Y is locally path-connected we can furthermore find open path-connected neighborhood $E_1, \dots, E_{n-1}$ of $h(\frac{1}{n}), \dots, h(\frac{n-1}{n})$. Now let

$$Z := \bigcap_{i=0}^{n-1} M([\tfrac{i}{n}, \tfrac{i+1}{n}], D_i) \;\cap\; \bigcap_{i=1}^{n-1} M(\{\tfrac{i}{n}\}, E_i) \;\cap\; M(\{1\}, V).$$

We make the following observations:

(α) By definition of the compact-open topology we know that Z is an open subset of $Y^{[0,1]}$. Furthermore it follows from (a) and the choice of $E_1, \dots, E_{n-1}$ that $h \in Z$.

(β) By the JH-Lemma 1.7 it remains to show that $Z \cap P$ is contained in the set that we described in the statement of the claim.

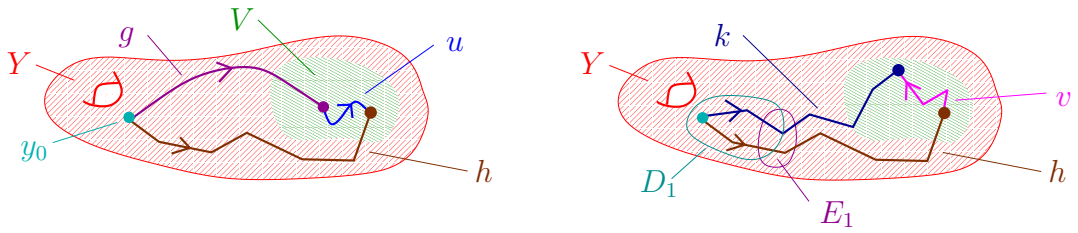
Now let $k \in Z \cap P$.

(γ) Since $k(1) \in V$ and since V is path-connected there exists a path $v\colon [0,1] \to V$ from $h(1)$ to $k(1)$.

(δ) Since the $E_1, \dots, E_{n-1}$ are path-connected we can arrange, after a path-homotopy, that $k(\frac{i}{n}) = (h * v)(\frac{i}{n})$ for $i = 0, \dots, n-1$.

(η) It follows from (b) together with a slight variation on Lemma 47.12 that we can show that k is path-homotopic to $h * v$.

Since k is path-homotopic to $h * v$ and since h is path-homotopic to $g * u$ we have now shown that k is path-homotopic to $g * (u * v)$. But this means that k lies again in the set defined in the statement of the claim. ⊞



(iv) Let $\alpha\colon [0,1] \to Y$ be a path with $\alpha(0) = y_0$. It follows easily from the Compact-Open Exponential Proposition 12.6 and Lemma 2.1 (4) that the map

$$\begin{array}{ccc} [0,1] & \to & P \subset Y^{[0,1]} \\ t & \mapsto & \left(\begin{array}{ccc} [0,1] & \to & Y \\ s & \mapsto & \alpha(ts) \end{array} \right). \end{array}$$

is continuous.

Now we turn to the actual proofs of the four statements.

(1) In (i) we already showed that $p\colon X = P/\!\sim \to Y$ is continuous. We still need to show that p is open. It follows from an elementary argument, see Exercise 5.26, that it suffices to show that the evaluation map $\mathrm{ev} = p \circ q\colon P \to Y$ is open. In fact it follows from (ii), together with Lemmas 1.33 and Lemma 2.11 that it suffices to prove the following claim.

Claim. For each compact subset $K \subset [0,1]$ and each path-connected open subset $V \subset Y$ we have $\mathrm{ev}(M(K,V) \cap P) = V$.

Proof. A straightforward argument, that uses that Y and V are path-connected, shows that

$$\mathrm{ev}(M(K,V)\cap P) \;=\; \begin{cases} \varnothing, & \text{if } M(K,V)=\varnothing,\\ Y, & \text{if } 1\notin K \text{ and } M(K,V)\neq\varnothing,\\ V, & \text{if } 1\in K. \end{cases}$$

Thus we see that in any case the image $\mathrm{ev}(M(K,V))$ is an open subset of Y. ⊞

(2) Now we want to prove that $p\colon X=P/\!\sim\to Y$ is a covering map. So let $y\in Y$ be a point. We need to show that there exists a connected open neighborhood V of y such that the restriction of p to each component of $p^{-1}(V)$ is a homeomorphism.

Since Y is semi-locally simply connected, there exists an open neighborhood V of y such that the inclusion induced map $\pi_1(V,y)\to\pi_1(Y,y)$ is the trivial map. We claim that V already has the desired property.

Claim. $$p^{-1}(V) \;=\; \bigsqcup_{\widetilde{y}\in p^{-1}(y)} U(\widetilde{y},V).$$

Proof. For the reader's convenience we repeat word-by-word the proof from page 1363. First note that it is clear that the right-hand side is contained in the left-hand side. Next we want to show that the left-hand side is contained in the right-hand side. So let $[g]\in p^{-1}(V)$. Here g is a path in Y from y_0 to a point $v\in V$. Since V is in particular path-connected there exists a path h in V from v to y. We then have

$$[g] \;=\; [g*h*\overline{h}] \;\in\; U(\underbrace{[g*h]}_{\in p^{-1}(y)},V).$$

It remains to show that the sets on the right-hand side of the claim are disjoint. So let $[v'],[v'']\in p^{-1}(y)$ and let $[g]\in U([v'],V)\cap U([v''],V)$. Then there exist paths u' and u'' in V with $[v'*u']=[v''*u'']$. It follows that

$$[v'] \;=\; [v'*u'*\overline{u}'] \;=\; [v''*u''*\overline{u}'] \underset{\uparrow}{=} [v''].$$

$u''*\overline{u}'$ is null-homotopic since the map $\pi_1(V,y)\to\pi_1(Y,y)$ is the trivial map

Thus $U([v'],V)\cap U([v''],V)$ intersect only if $[v']=[v'']$. ⊞

By (iii) we know that the sets $U(\widetilde{y},V)$ are open. By the above claim the sets are also disjoint. Thus it suffices to prove the following claim.

Claim. For each $\widetilde{y}\in p^{-1}(y)$ the restriction of the projection $p\colon X=P/\!\sim\to Y$ to the map $p\colon U(\widetilde{y},V)\to V$ is a homeomorphism.

Proof. The proof of this claim is again almost verbatim the same as on page 1363. So let $\widetilde{y}\in p^{-1}(y)$. It follows easily from the fact that V that the map $p\colon U(\widetilde{y},V)\to V$ is surjective. From the fact that the inclusion induced map $\pi_1(V,y)\to\pi_1(Y,y)$ is trivial it also follows immediately that p is injective. Furthermore we had just verified in (1) that p is continuous and open. Therefore $p\colon U(\widetilde{y},V)\to V$ is a homeomorphism. ⊞

(3) It follows from the fact that Y is path-connected together with (iv) that P is path-connected. By the Quotient-Compact Connected Lemma 5.16 (2) this implies that $X=P/\!\sim$ is path-connected.

(4) It remains to show that $p_*(\pi_1(X,x_0))=\Gamma$. The proof we now give is basically identical to the proof on page 1364. Thus let $g\in\pi_1(Y,y_0)$. We choose a representatives

$\alpha\colon [0,1] \to Y$ of g. We consider the following map:

$$\begin{array}{rcl} \widetilde{\alpha}\colon [0,1] & \to & X = P/\!\sim \\ t & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & Y \\ s & \mapsto & \alpha(ts) \end{array}\right]. \end{array}$$

By (iv) we know that $\widetilde{\alpha}$ is in fact continuous. It is now clear that $\widetilde{\alpha}$ is the lift of α to the starting point x_0. Then we have

$$g \in p_*(\pi_1(X,x_0)) \underset{\uparrow}{\Leftrightarrow} \widetilde{\alpha} \text{ is a loop} \Leftrightarrow \text{in } X = P/\!\sim \text{ we have } \widetilde{\alpha}(1) = x_0 = [e_{y_0}] \underset{\uparrow}{\Leftrightarrow} \alpha \in \Gamma.$$

Lemma 48.15 (1) — definition of $\sim$ on P ■

Remark.

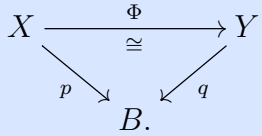
(1) The construction of the universal covering of a topological space has its origin in the work of Hermann Weyl from 1913, see [**Wey1913**][**Wey1964**, p. 58].

(2) Let Y be a path-connected topological space, let $y_0 \in Y$ and let $\Gamma \subset \pi = \pi_1(Y, y_0)$ be a subgroup. Recall that in Lemma 61.7 we introduced the whisker topology and that in Lemma 61.8 we introduced the CO-topology on $X_{y_0,\Gamma} := P_{y_0}/\!\sim$.
   (a) In [**FZ2007**, Lemma 2.1] it is shown that the CO-topology is contained in the whisker topology.
   (b) In [**FZ2007**, Example 2.2] it is shown that the two topologies are in general not equal, for example they disagree for the Hawaiian earrings that we introduced on page 1284.
   (c) It follows from the Covering Uniqueness Theorem 61.9 (see also [**FZ2007**, Lemma 2.1]) that the two topologies agree for every topological space that is locally path-connected and semi-locally simply connected.

(3) The various choices of topologies on $X_{y_0,\Gamma} = P_{y_0}/\!\sim$ are discussed in greater detail in [**FZ2007, VZ2014**] and [**Pav2020**]. The history of the construction of covering spaces is also outlined [**Pav2020**].

61.6. **Uniqueness of covering spaces.** Now the question arises, to what degree is the covering corresponding to the subgroup Γ unique. To answer that question we need an appropriate notion of two coverings being equivalent.

Definition. Let $p\colon X \to B$ and $q\colon Y \to B$ be two coverings of a topological space B. We say p and q are equivalent, if there exists a homeomorphism $\Phi\colon X \to Y$ such that the following diagram commutes:



Similarly we define the equivalence of coverings of pointed topological spaces.

Example. Given $n \in \mathbb{N}$ the two coverings

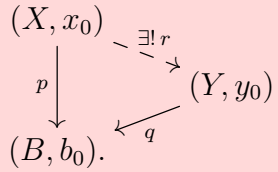
$$\begin{array}{rcl} p\colon S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon \mathbb{R}/n\mathbb{Z} & \to & S^1 \\ t + n\mathbb{Z} & \mapsto & \exp(\mathrm{i}t) \end{array}$$

are equivalent. In fact the desired map $\Phi\colon S^1 \to \mathbb{R}/n\mathbb{Z}$ is given by $\exp(2\pi\,\mathrm{i}t) \mapsto [nt]$.

Now we can formulate the following theorem:

Theorem 61.9. (Covering Uniqueness Theorem) Let B be a topological space, let $b_0 \in B$ and let $\Gamma \subset \pi_1(B, b_0)$ be a subgroup.

(1) Let $p\colon (X, x_0) \to (B, b_0)$ be a path-connected and locally path-connected covering such that $p_*(\pi_1(X, x_0)) = \Gamma$. This covering has the following universal property: for any other path-connected covering $q\colon (Y, y_0) \to (B, b_0)$ with $\Gamma \subset q_*(\pi_1(Y, y_0))$ there exists a unique covering $r\colon (X, x_0) \to (Y, y_0)$ such that the following diagram commutes:

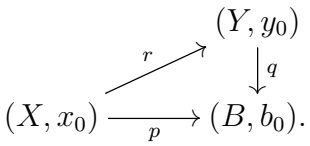


(2) If B is path-connected, locally path-connected and semi-locally simply connected, then there exists, up to equivalence, a unique path-connected covering $p\colon (X, x_0) \to (B, b_0)$ of pointed topological spaces such that $p_*(\pi_1(X, x_0)) = \Gamma$.

Remark. Let B be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $b_0 \in B$ and let $\Gamma \subset \pi_1(B, b_0)$ be a subgroup. It follows from the Covering Uniqueness Theorem 61.9 that the whisker topology and the CO-topology on the set $X = \{\gamma \in B^{[0,1]} \mid \gamma(0) = b_0\}/\sim$, which we introduced in Lemmas 61.7 and 61.8, are actually the same.

Proof. Let B be a topological space and let $b_0 \in B$.

(1) Let $p\colon (X, x_0) \to (B, b_0)$ and $q\colon (Y, y_0) \to (B, b_0)$ be two path-connected coverings such that $p_*(\pi_1(X, x_0)) \subset q_*(\pi_1(Y, y_0))$. We assume furthermore that X is locally path-connected. It follows from the Map Lifting Criterion 61.2 that there exists a unique map $r\colon (X, x_0) \to (Y, y_0)$ which makes the following diagram commute



It remains to show that r is a covering. This will be done in Exercise 61.5.

(2) The existence of such a covering is just the statement of the Covering Existence Theorem 61.5. The uniqueness statement is an immediate consequence of (1)[724] and the usual proof of uniqueness for an object satisfying a universal property. We leave the details to the reader. ■

The Covering Existence Theorem 61.5 and the Covering Uniqueness Theorem 61.9 (2) lead us to the following definition.

[724] If one does the argument carefully, then one sees, that here we use the following fact: if B is locally path-connected, then also every covering space of B is locally locally path-connected.

Definition. Let B be a topological space that is path-connected, locally path-connected and semi-locally simply connected, let $b_0 \in B$ and let $\Gamma \subset \pi_1(B, b_0)$ be a subgroup.

(1) As in the proof of the Covering Existence Theorem 61.5 we consider the set

$$\widetilde{B}^{\Gamma}_{b_0} \;:=\; \{\text{all paths in } B \text{ with starting point } b_0\}/\!\sim$$

where $u \sim v$ if and only u and v have the same endpoint and if $[u * \overline{v}] \in \Gamma$. We equip the set $\widetilde{B}^{\Gamma}_{b_0}$ with the topology introduced in the proof of the Covering Existence Theorem 61.5. We refer to $\widetilde{B}^{\Gamma}_{b_0}$ together with map $p\colon \widetilde{B}^{\Gamma}_{b_0} \to B$ given by $[u] \mapsto u(1)$ as the natural covering corresponding to (B, b_0, Γ). Furthermore we refer to the point in $\widetilde{B}^{\Gamma}_{b_0}$ that is represented by the constant path as the natural base point of $\widetilde{B}^{\Gamma}_{b_0}$.

(2) If $\Gamma = \{e\}$ is the trivial subgroup of $\pi_1(B, b_0)$, then we write $\widetilde{B}_{b_0} := \widetilde{B}^{\Gamma}_{b_0}$ and we refer to $p\colon \widetilde{B}_{b_0} \to B$ as the natural universal covering of (B, b_0).

(3) We refer to any path-connected covering $p\colon (X, x_0) \to (B, b_0)$ of pointed topological spaces with $p_*(\pi_1(X, x_0)) = \Gamma$ as a covering of B corresponding to Γ. By the Covering Uniqueness Theorem 61.9 (2) any two such coverings are equivalent, in particular any such covering is equivalent to the above natural covering. In practice usually we do not care about this indeterminacy.

The Covering Uniqueness Theorem 61.9 can be viewed as a "machine" that constructs covering spaces of a given topological space X. In the next section we discuss the case that X is a surface in somewhat greater detail to get a better feeling for what is happening.

Example. Let M be a connected smooth manifold and let $x_0 \in M$. On page 1114 we introduced the orientation character $w_1\colon \pi_1(M, x_0) \to \mathbb{Z}_2$. It follows from the Orientation Covering-Proposition 49.5 that the covering of M corresponding to $\ker(w_1)$ is equivalent to the orientation covering that we introduced on page 1109.

Let B be a topological space and let $\rho\colon [0,1] \to B$ be a path from a point $b_0 = \rho(0)$ to a point $b_1 = \rho(1)$. In the Change-of-Base Point Proposition 47.10 we saw that ρ induces an isomorphism

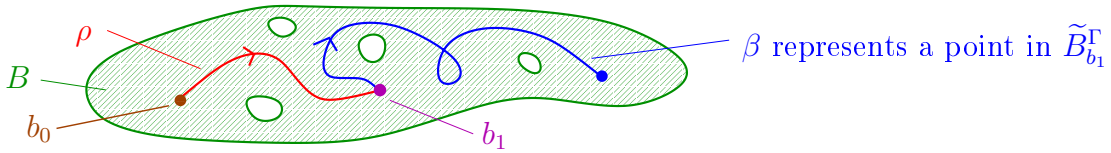
$$\begin{array}{rcl} \rho_*\colon \pi_1(B, b_1) & \to & \pi_1(B, b_0) \\ [\alpha] & \mapsto & [\rho * \alpha * \overline{\rho}] \end{array}$$

The following lemma now shows that ρ also induce a homeomorphism of natural coverings.

Lemma 61.10. Let B be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $\rho\colon [0,1] \to B$ be a path from a point $b_0 = \rho(0)$ to a point $b_1 = \rho(1)$. Finally let $\Gamma \subset \pi_1(B, b_1)$ be a subgroup. The map

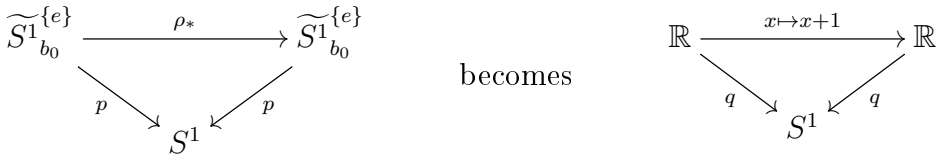
$$\begin{array}{rcl} \rho_*\colon \widetilde{B}^{\Gamma}_{b_1} & \to & \widetilde{B}^{\rho_*(\Gamma)}_{b_0} \\ [\beta] & \mapsto & [\rho * \beta] \end{array}$$

is a homeomorphism.



Proof. One can easily verify that the map $\rho_*\colon \widetilde{B}^{\Gamma}_{b_1} \to \widetilde{B}^{\rho_*(\Gamma)}_{b_0}$ is well-defined and that it is continuous. An inverse is given by the map $\overline{\rho}_*$, which shows that the map ρ_* is indeed a homeomorphism. ■

Example. The map defined in Lemma 61.10 is also interesting if $b_0 = b_1$ and if ρ is a loop in (B, b_0). Let us consider $B = S^1 \subset \mathbb{C}$ with base point $b_0 = 1 = \exp(0\mathrm{i})$ and the trivial subgroup $\Gamma = \{1\} \subset \pi_1(S^1, 1)$. It follows from the Covering Uniqueness Theorem 61.9 that the natural covering $p\colon \widetilde{S^1}^{\{e\}}_{b_0} \to S^1$ is equivalent to the usual covering $q\colon \mathbb{R} \to S^1$ that is given by $q(t) = \exp(2\pi\,\mathrm{i}t)$. In fact they can become equivalent via the homeomorphism $\mathbb{R} \to \widetilde{S^1}^{\{e\}}_{b_0}$ that sends $x \in \mathbb{R}$ to the path $[0,1] \to S^1$ that is given by $t \mapsto \exp(2\pi\,\mathrm{i}tx)$. Now let $\rho\colon [0,1] \to S^1$ be the loop that is given by $t \mapsto \exp(2\pi\,\mathrm{i}t)$. One can now easily show that under the homeomorphism Φ the map



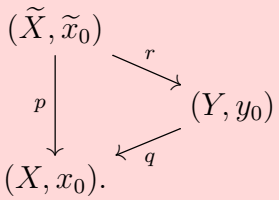
In summary we see that ρ_* acts non-trivially on $\widetilde{S^1}^{\{e\}}_{b_0}$. We will explore this action of loops on natural coverings in much greater detail in the subsequent Chapter 62 on the deck transformation group of coverings.

61.7. **The universal covering.** In the Covering Uniqueness Theorem 61.9 we saw that subgroups of fundamental groups give rise to covering spaces. An obvious choice of a subgroup is the trivial group and we immediately obtain the following corollary to the Covering Uniqueness Theorem 61.9.

Corollary 61.11. Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $x_0 \in X$. Then the following holds:

(1) There exists, up to equivalence, a unique path-connected covering
$$p\colon (\widetilde{X}, \widetilde{x}_0) \;\to\; (X, x_0)$$
such that $\widetilde{X}$ is simply connected.

(2) Let $q\colon (Y, y_0) \to (X, x_0)$ be another path-connected covering. Then there exists a unique covering $r\colon (\widetilde{X}, \widetilde{x}_0) \to (Y, y_0)$ such that the following diagram commutes:



Proof. Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected and let $x_0 \in X$.

(1) We just apply the Covering Uniqueness Theorem 61.9 to the trivial subgroup of $\pi_1(X, x_0)$ and we obtain, up to equivalence, a unique path-connected covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ is the trivial group. But then it follows from Corollary 48.14 that $\widetilde{X}$ is simply connected.

(2) The second statement is also an immediate consequence of the Covering Uniqueness Theorem 61.9. ■

Definition. Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected and let $x_0 \in X$.

(1) We refer to the natural covering $p\colon \widetilde{X}_{x_0} := \widetilde{X}_{x_0}^{\{e\}} \to X$, as defined on page 1370, as the natural universal covering of (X, x_0).

(2) We refer to any covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that $\widetilde{X}$ is simply connected as a universal covering of (X, x_0). As we remarked in the definition on page 1370, very often we suppress the fact that the universal covering is well-defined only up to equivalence.[725]

(3) Sometimes we ignore base points and we say that a covering $p\colon \widetilde{X} \to X$ is the universal covering if $\widetilde{X}$ is simply connected.

Remark. Note that most "reasonable" topological spaces are locally path-connected and semi-locally simply connected. For example we saw on page 1352 that every topological manifold is locally simply connected, which implies in particular that it is locally path-connected and semi-locally simply connected. Thus, given a path-connected topological manifold we can talk of is universal covering.

Examples.

(1) The map

$$\begin{array}{rcl} p\colon \mathbb{R} & \to & \mathbb{R}/\mathbb{Z} = S^1 \\ t & \mapsto & \exp(2\pi\,\mathrm{i}t) \end{array}$$

is the universal covering of S^1.

(2) More generally, the map

$$\begin{array}{rcl} p\colon \mathbb{R}^n & \to & (S^1)^n \\ (t_1, \ldots, t_n) & \mapsto & \big(\exp(2\pi\,\mathrm{i}t_1), \ldots, \exp(2\pi\,\mathrm{i}t_n)\big) \end{array}$$

is the universal covering of the n-dimensional torus $(S^1)^n$.

(3) The projection $p\colon S^3 \to S^3/\mathbb{Z}_p = L(p, q)$ is the universal covering of the lens space $L(p, q)$.

Remark. The universal covering is often also just called the *universal cover*. Also very often one suppresses the actual map p from the notation. For example, often one says that "$\mathbb{R}^n$ is the universal covering of $(S^1)^n$". The name "universal cover(ing)" stems from the fact, proved in Corollary 61.11, that the universal cover(ing) of a topological space X "covers" every other covering of X.

[725] In the literature the universal covering of a topological space is usually indicated by the same symbol but decorated with a tilde, e.g. the universal covering of X is usually denoted by $\widetilde{X}$.

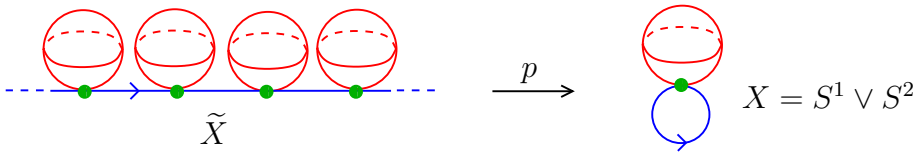
We have now shown that for every topological space X that is locally path-connected and semi-locally simply connected there exists a universal covering $p\colon \widetilde{X} \to X$. So far we have given an abstract construction, but if possible we would like to visualize it. We have just given an explicit description of the universal covering of the n-dimensional torus $(S^1)^n$ and the lens spaces. Now we will describe the universal covers of $X = S^1 \vee S^2$ and $X = S^1 \vee S^1$. First we consider $X = S^1 \vee S^2$ where we identify $i \in S^1$ with $(0,0,-1) \in S^2$. As on page 1175 we consider the topological space

$$\widetilde{X} \;=\; \big(\mathbb{R} \cup (\mathbb{Z} \times S^2)\big) \,/\, n \sim (n,(0,0,-1)), \quad n \in \mathbb{Z}.$$

It is straightforward to see that the map

$$p\colon \big(\mathbb{R} \cup (\mathbb{Z} \times S^2)\big) \,/\, n \sim (n,(0,0,-1)) \to S^1 \vee S^2$$
$$P \;\mapsto \begin{cases} \exp(\mathrm{i}(2\pi P + \frac{\pi}{2})), & \text{if } P \in \mathbb{R} \\ Q, & \text{if } P = (n,Q) \text{ with } n \in \mathbb{Z}, Q \in S^2 \end{cases}$$

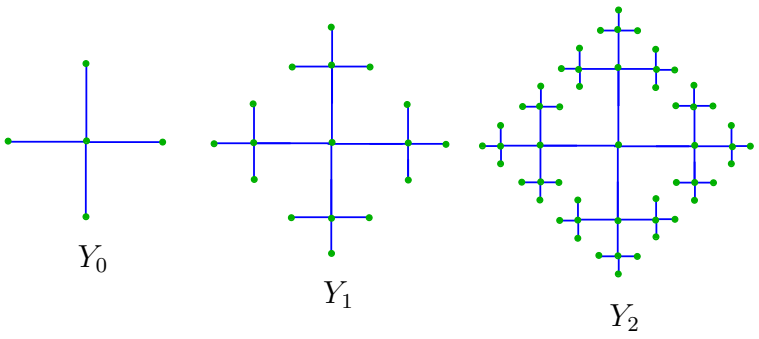
is well-defined and in fact a covering. We saw on page 1175 that $\widetilde{X}$ is simply connected. Thus $p\colon \widetilde{X} \to X$ is the universal covering of X.



Now we turn to $X = S^1 \vee S^1$. In the following we give a semi-rigorous description of the universal covering of $S^1 \vee S^1$ as a subset of $\mathbb{R}^2$. Iteratively we build a sequence of topological spaces Y_k, $k \in \mathbb{N}_0$ as follows:

(1) The topological space Y_0 is the cross centered at the origin where each segment has length 1.
(2) Given Y_k we attach to each endpoint of Y_k three segments of length $\frac{1}{4^k}$[726] such that each endpoint of Y_k now has a neighborhood that is a cross. We denote the resulting space by Y_{k+1}.

Finally we put $Y = \bigcup_{k \in \mathbb{N}_0} Y_k$.



We summarize some of the key properties of the construction in the following claim.

Claim.

[726] In the figures the lengths of the segments is $\frac{1}{2^k}$ since these are easier to draw.

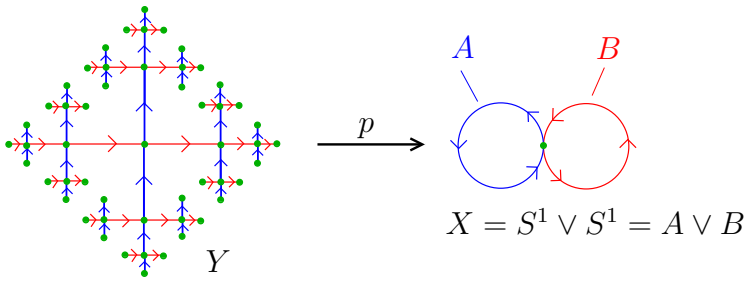
(1) Each Y_k is a tree.[727]
(2) Y is simply connected.

Sketch of proof. First note that $\chi(Y_0) = 5 - 4 = 1$. For $k \geq 1$ we obtain Y_k from Y_{k-1} by adding three vertices and three edges, hence $\chi(Y_k) = \chi(Y_{k-1}) = \cdots = \chi(Y_0) = 1$. This shows that each Y_k is a tree. It follows from Proposition 50.14 that each Y_k is simply connected.

Given $k \in \mathbb{N}$ we denote by U_k the result of removing the 4^{k+1} endpoints of Y_k. Note that each U_k is open in Y_k, hence each U_k is open in Y. Furthermore each U_k deformation retracts to Y_{k-1}. Hence it follows from the fact that each Y_k is simply connected that also each U_k is simply connected.

It follows as in Lemma 52.6 that $Y = \bigcup_{k\in\mathbb{N}} Y_k = \bigcup_{k\in\mathbb{N}} U_k$ itself is simply connected. ■

Now we construct a covering $p\colon Y \to S^1 \vee S^1$. To distinguish the two copies of S^1 we name them A and B, which are wedged together along $a \in A$ and $b \in B$. We give both A and B the usual counterclockwise orientation. Furthermore we orient each edge of the infinite topological graph Y as indicated in the figure below. We consider the map $p\colon Y \to A \vee B$



that is given as follows:

(1) each vertex of the infinite topological graph Y gets sent to the point $a = b \in A \vee B$,
(2) each horizontal edge of Y gets sent in an orientation-preserving way to $A \setminus \{a\}$,
(3) each vertical edge of Y gets sent in an orientation-preserving way to $B \setminus \{b\}$.

This map is illustrated in the figure above. It is relatively straightforward to convince oneself that $p\colon Y \to X$ is in fact a covering.[728] Since Y is simply connected, this is the universal covering of $X = A \vee B = S^1 \vee S^1$.

We have now determined explicitly the universal covers of several topological spaces. The surfaces of genus $g \geq 2$ are arguably one of the most interesting types of topological manifolds. Therefore it is natural to ask the following question.

Question 61.12. What does the universal covering of the surface of genus $g \geq 2$ look like?

We continue with the following proposition which will prove helpful at times.

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[727] Recall that by definition a tree is a topological graph with finitely many vertices V and edges E, such that the Euler characteristic $\chi(G) = \#V - \#E = 1$.

[728] We have to check that each point in $X = A \vee B$ admits an open neighborhood that is uniformly covered. It is clear that such a point exists for all points that are not the wedge point. For the wedge point itself the desired open neighborhood is given by small open intervals on A and B. Then each component of the preimage consists of a little cross with center P, with one horizontal edge going into P, one horizontal edge going out of P, one vertical edge going into P and one vertical edge going out of P.

Proposition 61.13. (Lift-to-Universal Coverings Proposition) Let X and Y be topological spaces that are path-connected, locally path-connected and semi-locally simply connected. Let $p\colon \widetilde{X} \to X$ and $q\colon \widetilde{Y} \to Y$ be universal coverings. Finally suppose we are given a map $f\colon X \to Y$ and points $x_0 \in X$, $\widetilde{x}_0 \in \widetilde{X}$ and $\widetilde{y}_0 \in \widetilde{Y}$ with $p(\widetilde{y}_0) = f(x_0)$.

(1) There exists a unique map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ with $\widetilde{f}(\widetilde{x}_0) = \widetilde{y}_0$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y. \end{array}$$

(2) If f is a homeomorphism, then $\widetilde{f}$ is also a homeomorphism.

(3) Suppose that X and Y are smooth manifolds and that p and q are coverings of smooth manifolds.[729]

(a) If f is smooth (respectively a diffeomorphism), then f is also smooth (respectively a diffeomorphism). The analogous statement also holds for complex manifolds.

(b) If X and Y are oriented, and if p and q are orientation-preserving, then f is orientation-preserving if and only if $\widetilde{f}$ are orientation-preserving.

Remark.

(1) the Lift-to-Universal Coverings Proposition 61.13 is a generalization of Exercise 49.2 (a).

(2) In Proposition 62.13 we will prove that the map $\widetilde{f}$ has an interesting extra property.

Proof.

(1) We consider the map $f \circ g := p\colon \widetilde{X} \to Y$. Since $\widetilde{Y}$ is the universal covering, since $\pi_1(\widetilde{X}, \widetilde{x}_0)$ and since X is locally path-connected and semi-locally simply connected we obtain from the Map Lifting Criterion 61.2 that there exists a unique lift $\widetilde{g}\colon \widetilde{X} \to \widetilde{Y}$ of g with $\widetilde{g}(\widetilde{x}_0) = \widetilde{y}_0$. This map is precisely the map we were trying to find.

(2) This statement follows easily from the uniqueness statement of (1). We will fill in the details in Exercise 61.11.

(3) Note that coverings of smooth manifolds are in particular local diffeomorphisms. The statement now follows easily from (1) and (2), together with the Smooth Pasting Proposition 19.23. The case of complex manifolds is treated in almost entirely the same way. ■

We conclude this section with a more general statement. Recall that in the Fundamental Group-via-Actions Theorem 48.16 we saw that if X is a simply connected space and if G is a group that acts continuously and discretely on X, then the fundamental group of the quotient space X/G is isomorphic to G. Now we will show that, under modest extra hypotheses, the converse holds. More precisely, we have the following proposition.

Proposition 61.14. p Let Y be a topological space that is path-connected, locally path-connected and semi-locally simply connected and let $y_0 \in Y$. We write $\pi = \pi_1(Y, y_0)$. Then there exists a simply connected space X, namely the universal covering of Y, and a continuous and discrete action of π on X such that Y is homeomorphic to X/π.

[729] We refer to page 1105 for the definition of a covering between smooth manifolds.

Proof. Let Y be a topological space that is path-connected, locally path-connected and semi-locally simply connected and let $y_0 \in Y$ be a base point. In the Covering Existence Theorem 61.5 we already showed that Y admits a simply connected covering. We recall the explicit construction given in the proof of the Covering Existence Theorem 61.5. Thus we consider

$$P \;:=\; \{f \in Y^{[0,1]} \,|\, f(0) = y_0\} \;=\; \{\text{all paths in } Y \text{ with starting point } y_0\}.$$

For $u, v \in P$ we write

$$u \sim v \quad :\Leftrightarrow \quad u \text{ and } v \text{ have the same endpoint and } [u * \overline{v}] = e \in \pi_1(Y, y_0)$$

and we put $X = P/\sim$. As in Lemma 61.7 given $x = [f] \in X = P/\sim$ and an open path-connected neighborhood V of $f(1)$ we define

$$U(x, V) \;:=\; \{[f * u] \,|\, u \text{ is a path in } V \text{ with starting point } f(1)\} \;\subset\; X = P/\sim .$$

Furthermore, as in Lemma 61.7 we endow X with the whisker topology, i.e. with the topology generated by these sets $U(x, V)$.

Claim. The map

$$\begin{array}{rcl} \pi_1(Y, y_0) \times X & \to & X \\ ([g], [u]) & \mapsto & [g * u] \end{array}$$

is a well-defined action.

Proof. We first show that the map is independent of the choice of u. So suppose that $u, v \in P$ are equivalent. Then $g * u$ and $g * v$ still have the same endpoint. Now we need to consider the second condition in the definition of $\sim$. We have

$$[(g * u) * (\overline{g * v})] \;=\; [g * u * \overline{v} * \overline{g}] \;=\; [g] * \underbrace{[u * \overline{v}]}_{= e \in \pi_1(Y, y_0)} * [g]^{-1} \;=\; e \;\in \pi_1(Y, y_0).$$

It is straightforward to see that the map does not depend on the choice of representative g. Finally we want to show that it is indeed an action. So let $[g], [h] \in \pi_1(Y, y_0)$. Then

$$[g] \cdot ([h] \cdot [u]) \;=\; [g] \cdot [h * u] \;=\; [g * (h * u)] \;=\; [(g * h) * u] \;=\; [g * h] \cdot [u] \;=\; ([g] \cdot [h]) \cdot [u].$$

⊞

Now we need to show the following:

(1) the group $\pi_1(Y, y_0)$ acts continuously and discretely on $X = P/\sim$,
(2) there exists a homeomorphism $f\colon X/\pi_1(Y, y_0) \to Y$.

We do this in the following two steps:

(1) It follows immediately from the definition of the whisker topology on $X = P/\sim$ and from the Subbasis-Continuity Proposition 2.9 that the action is continuous.

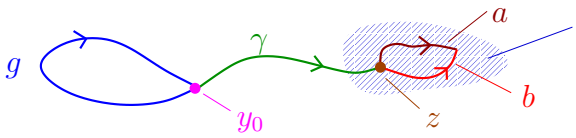
Now we need to show that the action is discrete. So let $x = [\gamma] \in P/\sim$. We set $z := \gamma(1) \in Y$. Since Y is semi-locally simply connected there exists an open neighborhood V of $z = \gamma(1)$ such that the inclusion induced map $\pi_1(V, z) \to \pi_1(Y, z)$ is trivial. It suffices to show that for $[g] \in \pi_1(Y, y_0)$ we have

$$[g] \cdot U(x, V) \cap U(x, V) \neq \varnothing \quad \Longrightarrow \quad [g] = e \in \pi_1(Y, y_0).$$

So suppose that $[g] \cdot U(x, V) \cap U(x, V) \neq \varnothing$. This means that there exist paths a and b in U, starting from P such that $g * \gamma * a$ and $\gamma * b$ have the same endpoint and

they are path-homotopic in Y. But then

$$[g] \;=\; [g]\cdot\underbrace{[\gamma]\cdot\overbrace{[a*\bar{b}]}^{=e\in\pi_1(Y,y_0)=\{e\}}*[\gamma]^{-1}}_{=e\in\pi_1(Y,y_0)} \;=\; [(g*\gamma*a)*\overline{(\gamma*b)}] \;=\; e\;\in\;\pi_1(Y,y_0).$$



open neighborhood V of z such that $\pi_1(V,z)\to\pi_1(Y,z)$ is trivial

(2) We consider the map

$$\begin{array}{rcl} f\colon X/\pi_1(Y,y_0) & \to & Y \\ [\gamma] & \mapsto & \gamma(1). \end{array}$$

It is straightforward to see that this map is a bijection. It follows immediately from the Topological-Quotient Proposition 5.15 (1b) that the map is continuous and it follows from Lemma 2.11 that the map is open. Thus it is a homeomorphism. ■

Exercises for Chapter 61.

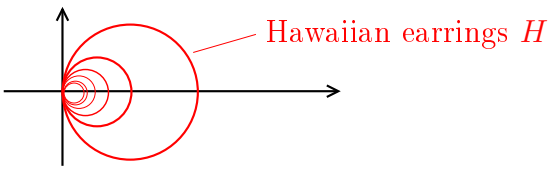
Exercise 61.1.

(a) Show that every topological graph is locally contractible.
(b) Let $\{X_i\}_{i\in I}$ be a family of locally contractible topological spaces. Show that any wedge $\bigvee_{i\in I} X_i$ is also locally contractible.

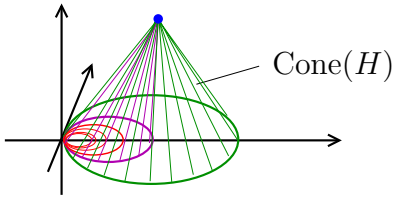
Remark. For a rigorous argument you will want to use the Homotopy Quotient Theorem 14.7.

Exercise 61.2. Let H be the Hawaiian earring that we introduced on page 1284 and that we show in the figure below to the left.

(a) Show that H is not semi-locally simply connected.
(b) Show that cone(H) is not locally simply connected.



not semi-locally simply connected



not locally simply connected

Exercise 61.3. Let $\pi\colon X\to B$ be a covering of topological spaces. Show that every simply connected and locally path-connected subset $U\subset B$ is uniformly covered.

Exercise 61.4. On page 164 we introduced the Topologist's Sine Curve

$$X \;:=\; \big\{(0,y)\,\big|\,y\in[-1,1]\big\}\cup\big\{(x,\sin(\tfrac{1}{x}))\,\big|\,x\in(0,\tfrac{1}{\pi}]\big\}\;\subset\;\mathbb{R}^2.$$

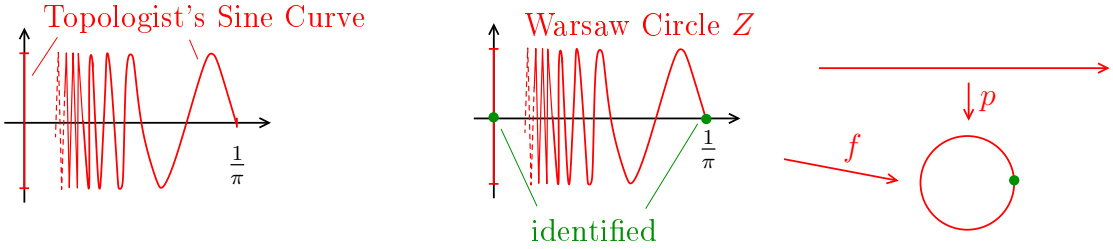
As on page 276 we refer to quotient space $Z := X/(0,0) \sim (\frac{1}{\pi}, 0)$ as the *Warsaw Circle*. We consider the map

$$\begin{array}{rcl} f\colon Z & \to & S^1 \\ [(x,y)] & \mapsto & \exp(2\pi^2 \cdot x \cdot \mathrm{i}) \end{array} \qquad \text{and the covering map} \qquad \begin{array}{rcl} p\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & \exp(t \cdot \mathrm{i}). \end{array}$$

(a) Show that $f\colon Z \to S^1$ is continuous.

(b) Show that there is no continuous map $\widetilde{f}\colon Z \to \mathbb{R}$ with $p \circ \widetilde{f} = f\colon Z \to S^1$.

Remark. This Exercise shows that in the Map Lifting Criterion 61.2 we cannot drop the hypothesis that Z is locally path-connected.



Exercise 61.5. Let $p\colon (X, x_0) \to (B, b_0)$ and $q\colon (Y, y_0) \to (B, b_0)$ be two path-connected coverings of topological spaces such that $p_*(\pi_1(X, x_0)) \subset q_*(\pi_1(Y, y_0))$. It follows from the Map Lifting Criterion 61.2 that there exists a unique map $r\colon (X, x_0) \to (Y, y_0)$ which makes the following diagram commute

$$\begin{array}{ccc} & & (Y, y_0) \\ & \overset{r}{\nearrow} & \downarrow q \\ (X, x_0) & \xrightarrow[p]{} & (B, b_0). \end{array}$$

Show that r is a covering.

Exercise 61.6. We consider the figure below.

(a) Let X be the wedge of two circles and let $p\colon Y \to X$ be the universal covering that is illustrated in the figure. Denote the usual generators of $\pi_1(X)$ by a and b. By an abuse of notation we denote by a and b also representing loops. What does the lift of the loop $a * b * \overline{a} * \overline{b}$ to Y to the starting point y_0 look like?

(b) We continue with the notation from (1). Now consider the loop

$$\gamma := a * a * b * a * a * \overline{b} * \overline{a} * b * b * \overline{a} * \overline{a} * \overline{b} * \overline{a} * \overline{b}.$$

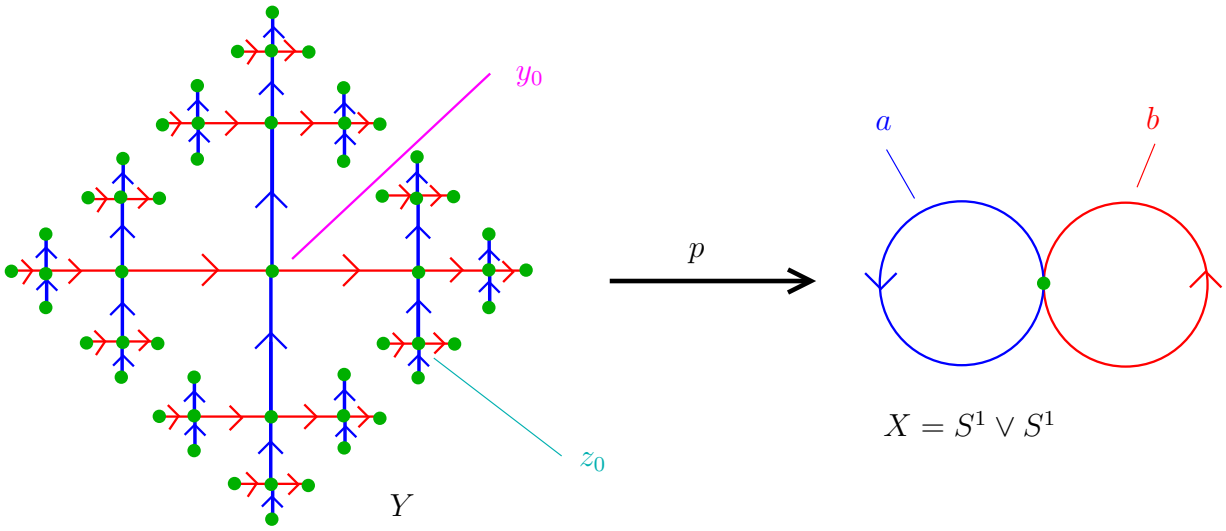
We denote by $\widetilde{\gamma}$ the lift of γ to the starting point y_0. Is y_0 the endpoint of $\widetilde{\gamma}$?

(c) Does there exist a homeomorphism $f\colon Y \to Y$ with $f(y_0) = z_0$?

Exercise 61.7. Let $X = S^1 \cup \{*\}$ be $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ with two "1", where the topology is defined as for the line with two zeros, see page 122. Does X admit a universal cover? If yes, can you describe it explicitly?

Exercise 61.8.

(a) Let M be the Möbius band. We pick an identification $\pi_1(M) = \mathbb{Z}$. Given $k \in \mathbb{N}$ we denote by Γ_k the kernel of the epimorphism $\pi_1(M) = \mathbb{Z} \to \mathbb{Z}_k$. Let $M_k \to M$ be the



covering corresponding to Γ_k. What well-known topological space is M_k? You do not have to justify your answer.

(b) Draw the universal covering of $\mathbb{R}\mathrm{P}^2 \vee S^1$ as a subset of $\mathbb{R}^3$.

(c) Let $X = A \vee B$ be the wedge of two circles. Let $\Gamma \subset \pi_1(X) = \langle a, b \rangle$ be the commutator subgroup. What does the covering $p\colon \widetilde{X} \to X$ corresponding to Γ look like? You can draw it as a subset of $\mathbb{R}^2$.
Hint. You have seen $\widetilde{X}$ many times in your life.

Exercise 61.9. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. We denote by $M := \mathrm{Tor}(X, f) := (X \times [0,1])/(x,0) \sim (f(x),1)$ the corresponding mapping torus (see page 467.) Let $p\colon \widetilde{M} \to M$ be a finite covering. Show that $\widetilde{M}$ is homeomorphic to a mapping torus.

Exercise 61.10. We say two smooth manifolds M and N are *commensurable* if M and N admit finite covers $\widetilde{M}$ and $\widetilde{N}$ that are diffeomorphic.

(a) Show that the torus and the surface of genus two are not commensurable.

(b) Let $g, h \in \mathbb{N}_{n\geq 2}$. Let M be the surface of genus g and let N be the surface of genus h. Show that M and N are commensurable.

(c) Determine which compact connected 2-dimensional smooth manifolds are commensurable.

Exercise 61.11. Let X be a topological space and let $p\colon \widetilde{X} \to X$ be a covering. Let $f\colon X \to X$ be a homeomorphism and let $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ be a map such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & X. \end{array}$$

Show that $\widetilde{f}$ is also a homeomorphism.
Remark. This is a generalization of Exercise 49.2 (b).

Exercise 61.12. Give an example of a topological space X, a finite covering $p\colon \widetilde{X} \to X$ and a homeomorphism $f\colon X \to X$ which does *not* lift to a map $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$, i.e. such there is no map $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & X. \end{array}$$

Exercise 61.13. Let X be a path-connected topological space that is locally path-connected and let $p\colon \widetilde{X} \to X$ be a finite covering of degree d.

(a) Let $f\colon X \to X$ be a homeomorphism. Show that there exists a map $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ such that $p \circ \widetilde{f} = f^d \circ p$.

(b) Does the conclusion of (a) also hold, if we do not assume that f is a homeomorphism?

Exercise 61.14. Let X be a path-connected topological space that is locally path-connected and semi-locally simply connected. Furthermore let $x_0 \in X$. By the Covering Uniqueness Theorem 61.9 there exists an, essentially unique, covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. We refer this covering as the *universal abelian covering of* (X, x_0).

(a) Let X be torus minus one open disk. Give an explicit description of the universal abelian cover of X.

(b) Let X be the sphere minus three open disks. Give an explicit description of the universal abelian cover of X.

Evidently the expression "explicit" is a little vague, you could interpret it as an explicit description as a submanifold of say $\mathbb{R}^3$.

Exercise 61.15. Let $f\colon (X, x_0) \to (Y, y_0)$ be a map between pointed topological spaces such that $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a monomorphism. We assume that both X and Y are locally path-connected and semi-locally simply connected. Let Γ be a subgroup of $\pi_1(Y, y_0)$ Let $q\colon \widetilde{Y} \to Y$ be the covering corresponding to Γ. Show that the pullback $p\colon f^*\widetilde{Y} \to X$, as defined in the Topological-Pullback Lemma 56.16, is the covering corresponding to the subgroup $f_*^{-1}(\Gamma)$ of $\pi_1(X, x_0)$.

Exercise 61.16. Let X be a path-connected topological space that is locally path-connected and that is semi-locally simply connected. Furthermore let $p\colon \widetilde{X} \to X$ be the universal covering. Let $x_0 \in X$ be a base point and let $\varphi\colon \pi_1(X, x_0) \to \Gamma$ be a group homomorphism. We consider

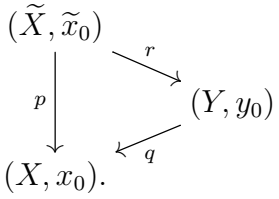
$$\begin{array}{rcl} q\colon \widetilde{X}_\varphi := (\widetilde{X} \times \Gamma)/\overbrace{(x,g) \sim (h \cdot x, \varphi(h) \cdot g)}^{x \in X, g \in \Gamma \text{ and } h \in \pi_1(X,x_0)} & \to & X \\ [(x,g)] & \mapsto & p(x) \end{array}$$

where we equip Γ with the discrete topology, $\widetilde{X} \times \Gamma$ with the product topology and $\widetilde{X}_\varphi$ with the quotient topology.

(a) Suppose that φ is an epimorphism. Show that $q\colon \widetilde{X}_\varphi \to X$ is the covering corresponding to $\ker(\varphi)$.

(b) In general, show that the number of components of $\widetilde{X}_\varphi$ has the same cardinality as $\operatorname{coker}(\varphi)$ and show that the restriction of $q\colon \widetilde{X}_\varphi \to X$ to each component of $\widetilde{X}_\varphi$ is the covering corresponding to $\ker(\varphi)$.

Exercise 61.17. Let X be a topological space and let $x_0 \in X$. We say that a covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is *universal* if given any other covering $q\colon (Y, y_0) \to (X, x_0)$ (we do not demand that Y is path-connected) there exists a map $r\colon (\widetilde{X}, \widetilde{x}_0) \to (Y, y_0)$ such that the following diagram commutes:



Show that if X is path-connected, locally path-connected and semi-locally simply connected, then a covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is universal in the above sense if and only if $\widetilde{X}$ is simply connected.

Remark 1. This exercise shows that the above definition of a universal covering is compatible with the definition on page 1372.

Remark 2. In Exercise 61.18 we will see that in general a simply connected covering is not a universal covering in the above sense.

Exercise 61.18. On page 164 we introduced the Topologist's Sine Curve

$$S \;:=\; \big\{(0,y)\,\big|\,y\in[-1,1]\big\} \cup \big\{(x,\sin(\tfrac{1}{x}))\,\big|\,x\in(0,\tfrac{1}{\pi}]\big\} \;\subset\; \mathbb{R}^2.$$

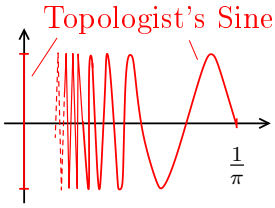
As on page 276 we refer to quotient space $W := S/(0,0)\sim(\frac{1}{\pi},0)$ as the *Warsaw Circle*.

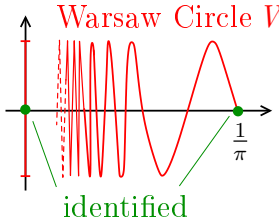
(a) Show that the Warsaw Circle is path-connected and simply connected.
Remark. This exercise is similar to Exercise 2.48 where we showed that the Topologist's Sine Curve S is connected but not path-connected.

(b) Show that the Warsaw Circle is not locally path-connected.

(c) We set $w_0 = [(0,0)]$. Provide an example of a 2-fold covering $q\colon (Y, y_0) \to (W, w_0)$.

Remark. By (a) we know that (W, w_0) is simply connected. The example in (c) shows that the identity map $(W, w_0) \to (W, w_0)$ does not have the universal property stated in Exercise 61.17 respectively Corollary 61.11, even though (W, w_0) is simply connected.





Exercise 61.19. We define the *Modified Warsaw Circle* to be the topological space

$$C \;:=\; X/(0,0)\sim(\tfrac{1}{\pi},0) \qquad \text{where } S \;:=\; \{(0,0)\} \cup \big\{(x,\sin(\tfrac{1}{x}))\,\big|\,x\in(0,\tfrac{1}{\pi}]\big\} \;\subset\; \mathbb{R}^2.$$

As in Exercise 5.30 and 61.18 one can show that C is compact and simply connected.

(a) We define the *Topologist's Electrocardiogram* to be the topological space

$$E \;:=\; (S \times \mathbb{Z})/((\tfrac{1}{\pi}, 0), k) \sim ((0,0), k+1) \quad \text{for } k \in \mathbb{Z}.$$

Show that the map

$$\begin{array}{rcl} p_E \colon E & \to & C \\ [(P,k)] & \mapsto & [P] \end{array}$$

is a covering.

(b) We consider the following subspace of $\mathbb{R}^2$:

$$L \;:=\; \Big(0 \times \mathbb{Z} \;\sqcup\; \bigcup_{k \in \mathbb{Z}} \{(x, \tfrac{1}{x} + k) \,|\, x \in (0, \tfrac{1}{\pi}]\}\Big) \setminus (0,k) \sim (\tfrac{1}{\pi}, k) \text{ for } k \in \mathbb{Z}.$$

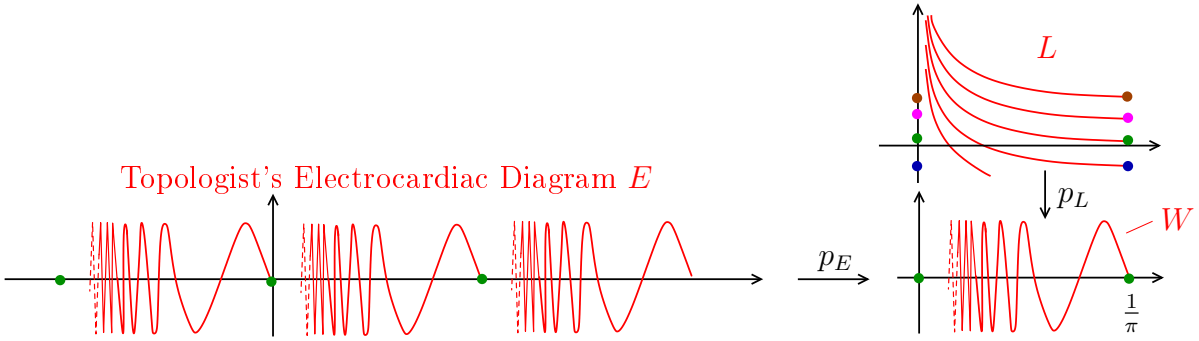
Show that the map

$$\begin{array}{rcl} p_L \colon L & \to & C \\ [(x,y)] & \mapsto & \begin{cases} [(0,0)], & \text{if } (x,y) \in \{0\} \times \mathbb{Z}, \\ [(x, \sin(\frac{1}{x}))], & \text{if } x > 0. \end{cases} \end{array}$$

is a covering. This covering is called the *Löhlein covering*.

(c) Show that there is no map $f \colon E \to L$ such that $p_L \circ f = p_E$.

Remark. This shows that the covering $p \colon E \to C$ is not a universal covering in any sense.



62. The deck transformation group

The following definition introduces the main actor of this chapter.

Definition. Let $p\colon X \to B$ be a covering.

(1) A deck transformation is a homeomorphism $d\colon X \to X$ such that $p(d(x)) = p(x)$ for all $x \in X$, i.e. such that the following diagram commutes:

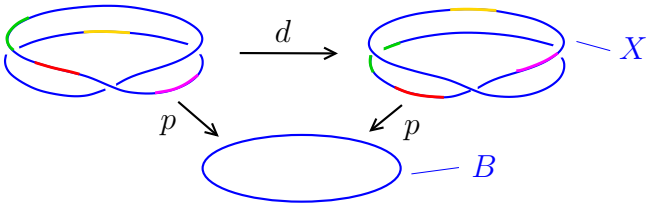
$$\begin{array}{ccc} X & \xrightarrow[\cong]{d} & X \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & B. & \end{array}$$

The group of deck transformations[730] is called the deck transformation group of the covering $p\colon X \to B$ and we denote it by $\mathrm{Deck}(p\colon X \to B)$.

(2) We say that the covering is regular if for every $b \in B$ the deck transformation group $\mathrm{Deck}(p\colon X \to B)$ acts transitively on $p^{-1}(b)$, i.e. if given any two points x, x' with $p(x) = p(x') = b$ there exists a deck transformation d with $d(x) = x'$. Otherwise we say that the covering is irregular.

For completeness' sake we also mention the following definition even though we will barely make use of it:

(3) Let P be a property of isomorphism types of groups, e.g. cyclic, free, nilpotent, solvable etc. We say the covering $p\colon X \to B$ is a P-covering if the covering is regular and if the deck transformation group has the property P.



Example.

(1) We consider the covering

$$\begin{array}{rccc} p\colon & S^1 & \to & S^1 \\ & z & \mapsto & z^2. \end{array}$$

The map $d\colon S^1 \to S^1$ given by $d(z) := -z$ is a deck transformation. It is now clear that the covering is regular.

(2) Let $p\colon \widetilde{X} \to X$ be a 2-fold covering. We define

$$\begin{array}{rccc} f\colon & \widetilde{X} & \to & \widetilde{X} \\ & Q & \mapsto & \text{the unique other point in } p^{-1}(p(Q)). \end{array}$$

In Lemma 48.4 we saw that f is a deck transformation. It follows immediately that any 2-fold covering is regular.

[730] Why do deck transformations form a group?

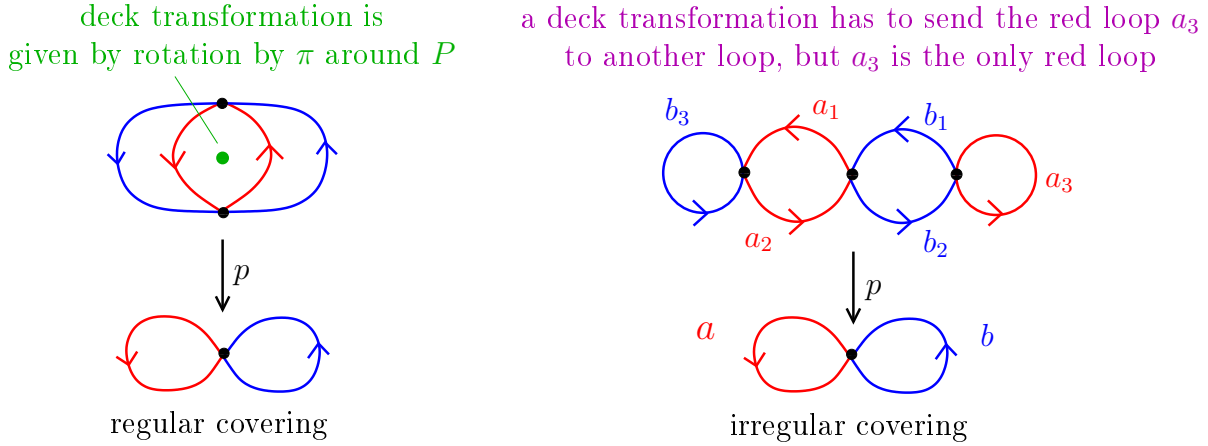
(3) As an important special case of (2) consider a non-orientable path-connected smooth manifold M. We consider the orientation covering

$$\begin{array}{rcl} \widetilde{M} := \{(Q,O) \mid Q \in M \text{ and } O \text{ an orientation of } \mathrm{T}_Q M\} & \to & M \\ (Q,O) & \mapsto & Q \end{array}$$

from page 1109. It is straightforward to show that the map $d\colon \widetilde{M} \to \widetilde{M}$ given by $d((Q,O)) := (Q,-O)$ is a deck transformation of p.

(4) We consider again the first two coverings of topological graphs from the figure on page 1071 which are also shown again in the figure below. The covering maps are the ones suggested by the colors and the directions. We first make the following observations. A deck transformation d does the following:
   (a) it sends a vertex again to a vertex,
   (b) it sends a blue edge to a blue edge, preserving the orientation,
   (c) it sends a red edge to a red edge, preserving the orientation,
   (d) if e is an edge with endpoints P and Q, then $d(e)$ is an edge with endpoints $d(P)$ and $d(Q)$, in particular if e is a loop, then $d(e)$ is also a loop.

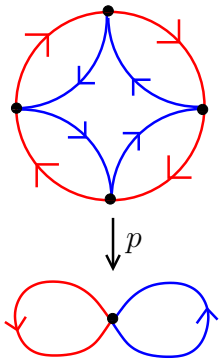
   Now we consider the two examples.
   (a) The example on the left admits one non-trivial deck transformation, namely the map that is given by "rotation by π" around the center of the topological graph. The covering is now easily seen to be regular.
   (b) The loop a_3 in the topological graph on the right is the only red loop, so any deck transformation has to send it to itself. It follows that there can be no deck transformation that sends the vertex to the right to any of the other vertices.[731] Thus the covering on the right is irregular.



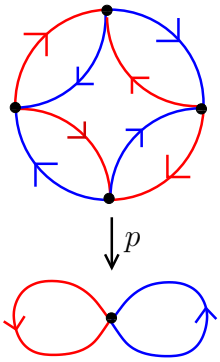
regular covering irregular covering

(5) Finally we consider the 4-fold coverings illustrated in the figure below. They are both regular coverings. For the topological graph on the left the deck transformation group is isomorphic to $\mathbb{Z}_4$, a generator is given by "rotation by $\frac{\pi}{2}$". The deck transformation group on the right is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, here one $\mathbb{Z}_2$ summand is given by "rotation by π" and the other one corresponds to "swapping the inner and the outer circle".

[731] Does this covering admit any deck transformation?



deck transformation group $\mathbb{Z}_4$



deck transformation group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

Lemma 62.1. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed path-connected topological spaces. The deck group $\mathrm{Deck}(p\colon X \to B)$ acts discretely and continuously on X.

Proof. By definition of a deck transformation the action is continuous. We will show in Exercise 62.1 that the action is discrete. ■

Lemma 62.2. (Deck Group-Free Action Lemma) Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed path-connected topological spaces and let d_1 and d_2 be two deck transformations of p. Then the following holds

$$d_1 = d_2 \iff d_1(x_0) = d_2(x_0).$$

In other words, given any $b_0 \in B$ the deck transformation group $\mathrm{Deck}(p\colon X \to B)$ acts freely on $p^{-1}(b_0)$.

Proof. Evidently we only need to show the "$\Leftarrow$"-direction. Thus let d be a deck transformation of p. One way of stating that d is a deck transformation is to state that the following diagram commutes:

$$\begin{array}{ccc} & & (X, d(x_0)) \\ & \overset{d}{\nearrow} & \downarrow p \\ (X, x_0) & \xrightarrow[p]{} & (B, b_0). \end{array}$$

Put differently, the map d can be viewed as a lift of the map $(X, x_0) \to (B, b_0)$ to the covering $(X, d(x_0)) \to (B, b_0)$. By the uniqueness of lifts[732], that we showed in the Map Lifting Criterion 61.2 (2), we now see that two deck transformations d_1 and d_2 of p with $d_1(x_0) = d_2(x_0)$ agree. ■

Example. We let $B = (S^1 \times \{a\} \sqcup S^1 \times \{b\})/(1, a) \sim (1, b)$, i.e. B is the wedge of two circles and we consider

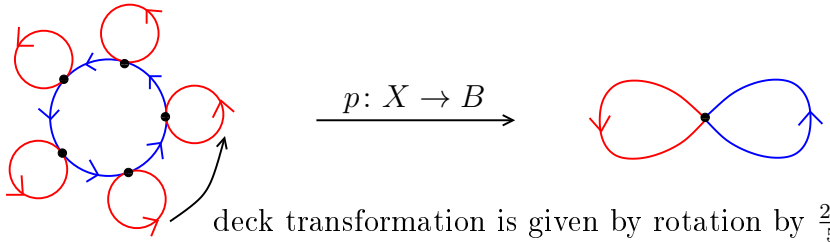
$$X = (S^1 \sqcup (S^1 \times \mathbb{Z}_5))/(\exp(2\pi \mathrm{i} n/5) \sim (1, n)) \text{ for } n \in \mathbb{Z}_5,$$

i.e. X is a circle with five circles attached at the points $\exp(2\pi \mathrm{i} k/5)$, $k = 0, \ldots, 4$. Then the map

$$p\colon (S^1 \sqcup (S^1 \times \mathbb{Z}_5))/\sim \to \ G = (S^1 \times \{a\} \sqcup S^1 \times \{b\})/(1, a) \sim (1, b)$$

[732] Here we use that X is path-connected.

that is given by $p(z) = (z^5, a)$ for $z \in S^1$ and $p((z,n)) = (z,b)$ for $(z,n) \in S^1 \times \mathbb{Z}_5$, is a covering map of degree five.



We consider the map

$$\begin{array}{rcl} \mathbb{Z}_5 & \to & \mathrm{Deck}(p\colon X \to B) \\ m & \mapsto & \left(\begin{array}{rcll} X & \to & X & \\ z & \mapsto & \exp(2\pi \mathrm{i} m/5) \cdot z, & \text{for } z \in S^1, \\ (z,n) & \mapsto & (z, m+n) & \text{for } (z,n) \in S^1 \times \mathbb{Z}_5 \end{array} \right) \end{array}$$

In the figure above this corresponds to "rotation of X by the angle $\frac{2\pi m}{5}$". It is straightforward to show that this map is well-defined, i.e. the maps on the right are deck transformations, and using the Deck Group-Free Action Lemma 62.2 one can easily see that this map is an isomorphism. Also one can now convince oneself easily that the covering $p\colon X \to B$ is regular.

Lemma 62.3. Let X be a path-connected topological space on which a group G acts continuously and discretely. We consider the corresponding covering $p\colon X \to X/G$. Then $p\colon X \to X/G$ is regular and the map

$$\begin{array}{rcl} \Phi\colon G & \to & \mathrm{Deck}(p\colon X \to X/G) \\ g & \mapsto & \begin{pmatrix} X & \to & X \\ x & \mapsto & g \cdot x \end{pmatrix} \end{array}$$

is an isomorphism of groups.

Proof. It is straightforward to see that the maps $x \mapsto gx$ are indeed deck transformations. It follows almost immediately that the covering $p\colon X \to X/G$ is regular. Also it follows easily from the axioms of a group action that Φ is a homomorphism of groups. Furthermore it is clear that Φ is a monomorphism.

It remains to show that Φ is an epimorphism. So let $d \in \mathrm{Deck}(p\colon X \to X/G)$ be a deck transformation. We pick $x \in X$. Since $p(d(x)) = p(x)$ we have $d(x) = gx$ for some $g \in G$. Since X is path-connected it follows from the Deck Group-Free Action Lemma 62.2 that $d(y) = gy$ for all $y \in X$. This shows that Φ is an epimorphism. ■

Lemma 62.4. Any 2-fold covering of path-connected topological spaces is regular and the deck transformation group contains precisely two elements.

Proof. Let $p\colon X \to B$ be a 2-fold covering of path-connected topological spaces. By Lemma 48.4 we know that

$$\begin{array}{rcl} d\colon X & \to & X \\ x & \mapsto & \text{the unique other element of } p^{-1}(p(x)) \end{array}$$

is a deck transformation. Evidently it is regular. It follows easily from the Deck Group-Free Action Lemma 62.2 and our hypothesis that X, B are path-connected that any deck transformation of p is either the identity or it equals the above deck transformation d. ■

We recall the following important, albeit somewhat technical, definition.

Definition. Let G be a group and let $H \subset G$ be a subgroup. The normalizer $N(H)$ of the subgroup $H \subset G$ is defined as

$$N(H) \ := \ \{g \in G \mid g^{-1}Hg = H\}.$$

The following lemma summarizes a few elementary properties of the normalizer of a subgroup.

Lemma 62.5. Let G be a group and let $H \subset G$ be a subgroup.

(1) The subgroup H of G is normal if and only if $N(H) = G$.
(2) The normalizer $N(H)$ of $H \subset G$ is a subgroup of G.
(3) The normalizer $N(H)$ of $H \subset G$ is the largest subgroup of G which contains H as a normal subgroup, in the sense that if A is a subgroup of G which contains H as a normal subgroup, then $A \subset N(H)$.

Proof. Both statements are easily verified. Alternatively see [**Isa1994**, p. 26] for the elementary proof. ■

We return to topology and we consider one more example in detail.

Example. Let Y be a topological space that is path-connected, locally path-connected and semi-locally simply connected and let $y_0 \in Y$. Furthermore let $\Gamma \subset \pi = \pi_1(Y, y_0)$ be a subgroup. By the Covering Existence Theorem 61.5 there exists a path-connected covering $p\colon (X, x_0) \to (Y, y_0)$ of pointed topological spaces such that $p_*(\pi_1(X, x_0)) = \Gamma$. We recall the construction of X. We first considered

$$P \ := \ \{\text{all paths in } Y \text{ with starting point } y_0\}$$

and for $u, v \in W$ we wrote

$$u \sim v \quad :\Leftrightarrow \quad u \text{ and } v \text{ have the same endpoint and } [u * \overline{v}] \in \Gamma.$$

We defined $X = P/\sim$ and the projection $p\colon P/\sim \to Y$ that is given by $[w] \mapsto w(1)$. We then showed that, with an appropriate topology on $P/\sim$, the map $p\colon P/\sim \to Y$ is a path-connected covering with $p_*(\pi_1(P/\sim, x_0)) = \Gamma$.

As in the proof of Proposition 61.14 we now would like to construct deck transformations by precomposing elements in P by a loop in y_0. More precisely, let g be a loop in (Y, y_0). We consider the map

$$\begin{array}{ccc} P & \to & P \\ w & \mapsto & g * w \end{array}$$

We need to check that the map respects our equivalence relation $\sim$ on P. So suppose that $u, v \in P$ are equivalent. Then $g * u$ and $g * v$ still have the same endpoint. Now we need to consider the second condition. We have

$$[(g * u) * (\overline{g * v})] \ = \ [g * u * \overline{v} * \overline{g}] \ = \ [g] * \underbrace{[u * \overline{v}]}_{\in \Gamma} * [g]^{-1}.$$

In general it is not the case that the right-hand side lies again in Γ. But it is true if $[g]$ lies in the normalizer of Γ. As in the proof of Proposition 61.14 it is now straightforward to see that the map

$$\begin{array}{rcl} \Psi\colon N(\Gamma)/\Gamma & \to & \mathrm{Deck}(p\colon X \to Y) \\ [g] & \mapsto & \begin{pmatrix} P/\!\sim & \to & P/\!\sim \\ [u] & \mapsto & [g*u] \end{pmatrix} \end{array}$$

is a well-defined homomorphism. We will see in Proposition 62.6 that this in fact an isomorphism of groups.

Now we want to consider the general case of a covering. So let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed path-connected topological spaces. We write $\pi := \pi_1(B,b_0)$ and $\Gamma := p_*(\pi_1(X,x_0))$.[733] Now the question arises whether there is a connection between the groups π, Γ and $\mathrm{Deck}(p\colon X \to B)$.

Definition. Similar to the proof of the Fundamental Group-via-Actions Theorem 48.16 we can consider the following map[734]

$$\begin{array}{rcl} \Xi_{(X,x_0)}\colon \mathrm{Deck}(p\colon X \to B) & \to & \pi/\Gamma \\ d & \mapsto & \big[p \circ (\text{path in } X \text{ from } x_0 \text{ to } d(x_0))\big]. \end{array}$$

It is straightforward to see that this map is well-defined, i.e. it does not depend on the choice of the path.

The question now arises, whether this map Φ is a group isomorphism. But in general this is not the case, after all, if Γ is not normal in π, then π/Γ is not even a group. But we have the following proposition:

Proposition 62.6. Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed topological spaces. We suppose that X and B are both path-connected and locally path-connected. We write $\pi := \pi_1(B,b_0)$ and $\Gamma := p_*(\pi_1(X,x_0))$. The above map

$$\begin{array}{rcl} \Xi_{(X,x_0)}\colon \mathrm{Deck}(p\colon X \to B) & \to & \pi/\Gamma \\ d & \mapsto & \big[p \circ (\text{path in } X \text{ from } x_0 \text{ to } d(x_0))\big]. \end{array}$$

restricts to an isomorphism of groups

$$\Xi_{(X,x_0)}\colon \mathrm{Deck}(p\colon X \to B) \xrightarrow{\cong} N(\Gamma)/\Gamma.$$

Proof. Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed path-connected topological spaces. We write $\pi := \pi_1(B,b_0)$, $\Gamma := p_*(\pi_1(X,x_0))$, $D := \mathrm{Deck}(p\colon X \to B)$ and $\Phi := \Xi_{(X,x_0)}$.

(A) First we have to show that $\Phi(D)$ lies indeed in $N(\Gamma)/\Gamma$. Thus let $d \in D$. We need to show that $\Phi(d)$ lies in $N(\Gamma)/\Gamma$. Thus let f be a loop in (X,x_0) and and let g be a path in X from x_0 to $d(x_0)$. We have to verify that $[\overline{p\circ f}]\cdot[p\circ g]\cdot[p\circ f] \in \Gamma$. Indeed the following holds

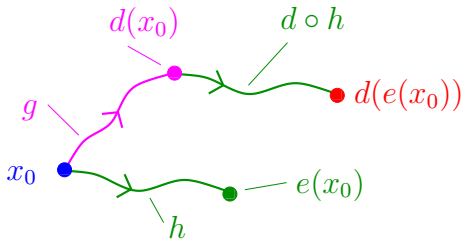
$$[\overline{p\circ f}]\cdot[p\circ g]\cdot[p\circ f] \;=\; \big[\overline{p\circ f} * p\circ g * p\circ f\big] \;=\; \big[p\circ \big(\underbrace{\overline{f} * g * f}_{\text{loop in } (X,x_0)}\big)\big] \;\in\; p_*(\pi_1(X,x_0)) = \Gamma.$$

[733] Recall that by Corollary 48.14 the induced map $p_*\colon \Gamma \to \pi$ is injective, thus $\Gamma \cong \pi_1(X,x_0)$.

[734] We denote hereby with $[p\circ f]$ not only the equivalence class of the loop $p\circ f$ in π, but also its equivalence class π/Γ, since the notation $[p\circ f]\Gamma$ is too cumbersome.

(B) Now we have to show that Φ is a group homomorphism. So let d and e be two deck transformations. We pick a path g from x_0 to $d(x_0)$ and we pick a path h from x_0 to $e(x_0)$. Then

$$\Phi(d)\cdot\Phi(e) \underset{\text{definition of } \Phi(e) \text{ and } \Phi(d)}{=} [p\circ g]\cdot[p\circ h] \underset{\text{since } p\circ d = p}{=} [p\circ g]\cdot[p\circ(d\circ h)] \overset{\text{endpoint of } g \text{ equals starting point of } d\circ h}{=} [g*(d\circ h)] \underset{\substack{\text{since } g*(d\circ h) \text{ is a path} \\ \text{from } x_0 \text{ to } d(e(x_0)), \text{ see the figure below}}}{=} \Phi(e\circ d).$$



(C) Now we will construct a map $\Psi\colon N(\Gamma)/\Gamma \to D$ in the inverse direction. Thus let $[f]\in N(\Gamma)\subset\pi=\pi_1(B,b_0)$ where f is a loop in (B,b_0). Let x_1 be the endpoint of the lift $\widetilde{f}$ of f to the starting point x_0.[735] We have $p(x_1)=p\big(\widetilde{f}(1)\big)=f(1)=x_0$. Thus we obtain the following diagram of maps of pointed topological spaces:

$$\begin{array}{ccc} & & (X,x_1) \\ & & \big\downarrow p \\ (X,x_0) & \xrightarrow{\ p\ } & (B,b_0). \end{array}$$

Then

$$p_*(\pi_1(X,x_1)) \underset{\text{by Proposition 47.10 since } \widetilde{f} \text{ is a path from } x_0 \text{ to } x_1}{=} \big\{\big[p\circ\big(\overline{\widetilde{f}}*g*\widetilde{f}\big)\big]\,\big|\,[g]\in\pi_1(X,x_0)\big\} \underset{\text{since } [f]\in N(\Gamma)}{\substack{=\ [f]\,p_*(\pi_1(X,x_0))\,[f]^{-1} \\ =\ p_*(\pi_1(X,x_0)).}}$$

Note that it follows from the Map Lifting Criterion 61.2 that there exists a lift $\widetilde{p}\colon(X,x_0)\to(X,x_1)$ of p, i.e. there exists a map $\widetilde{p}$ such that the following diagram commutes:

$$\begin{array}{ccc} & & (X,x_1) \\ & \overset{\widetilde{p}}{\nearrow} & \big\downarrow p \\ (X,x_0) & \xrightarrow{\ p\ } & (B,b_0). \end{array}$$

The map is $\widetilde{p}$ is now by definition a deck transformation.

We have thus constructed a map $N(\Gamma)\to D$. We still need to show that this map descends to a map $N(\Gamma)/\Gamma\to D$. So let $[f]\in N(\Gamma)$ and let $[g]\in\Gamma$. We denote by $\widetilde{f}$ the lift of f to the starting point x_0. Similarly we define $\widetilde{g*f}$. By the above we only

[735] According to Corollary 48.13 the definition of x_1 does not depend on the choice of the representative of $[f]\in N(\Gamma)$.

need to show that the endpoints of $\widetilde{f}$ and $\widetilde{g * f}$ agree. In fact we have

$$\widetilde{g * f}(1) \underset{\text{Lemma 61.3}}{=} \begin{array}{c}\text{endpoint of the lift of } f\\ \text{to the starting point } \widetilde{g}(1)\end{array} \underset{\text{by Lemma 48.15 (2) (1) since } [g] \in p_*(\pi_1(X,x_0))}{=} \begin{array}{c}\text{endpoint of the lift of } f\\ \text{to the starting point } x_0\end{array} = \widetilde{f}(1)$$

(D) Now we want to show that Φ is a bijection. It suffices to show that $\Psi \circ \Phi = \mathrm{id}$ and $\Phi \circ \Psi = \mathrm{id}$.

First let $d \in D$. Then

$$\begin{aligned}\Psi(\Phi(d)) &= \Psi(\text{loop } f \text{ in } (B,b_0) \text{ that lifts to a path from } x_0 \text{ to } d(x_0))\\ &= \text{deck transformation that sends } x_0 \text{ to } d(x_0) \underset{\text{Deck Group-Free Action Lemma 62.2}}{=} d\end{aligned}$$

This shows that $\Psi \circ \Phi$ is the identity.

Now let $g \in N(\Gamma)/\Gamma$. We pick a representative loop s in b_0 and we denote by $\widetilde{s}$ the lift of s to the starting point x_0. We denote by x_1 the endpoint of $\widetilde{s}$. Then

$$\Phi(\Psi(g)) = \Big[p \circ \big(\text{path in } X \text{ from } x_0 \text{ to } \overbrace{\Psi(g)(x_0)}^{=x_1}\big)\Big] \underset{\widetilde{s} \text{ is a path from } x_0 \text{ to } x_1}{=} \big[p \circ \widetilde{s}\big] = [s] = g.$$

This shows that $\Phi \circ \Psi$ is the identity. ∎

Remark. Let X be a path-connected topological space on which a group G acts continuously and discretely. We denote by $p\colon X \to X/G$ the natural projection . Finally let $x_0 \in X$. We now have three isomorphisms relating the groups G, $\mathrm{Deck}(p\colon X \to X/G)$ and $\pi_1(X/G,[x_0])$. More precisely we consider the isomorphisms

$$\begin{array}{rcl}\Psi\colon G &\to& \mathrm{Deck}(p\colon X \to X/G)\\ g &\mapsto& \begin{pmatrix} X & \to & X\\ x & \mapsto & g \cdot x_0\end{pmatrix}\end{array} \quad \text{and} \quad \begin{array}{rcl}\Theta\colon G &\to& \pi_1(X/G,[x])\\ g &\mapsto& [p \circ (\text{path in } X \text{ from } x_0 \text{ to } g \cdot x_0)]\end{array}$$

from Lemma 62.3 and the Fundamental Group-via-Actions Theorem 48.16. Furthermore we consider isomorphism

$$\begin{array}{rcl}\Xi_{(X,x_0)}\colon \mathrm{Deck}(p\colon X \to B) &\to& \pi_1(X,x_0)\\ d &\mapsto& \big[p \circ (\text{path in } X \text{ from } x_0 \text{ to } d(x_0))\big]\end{array}$$

from Proposition 62.6. Fortunately these three isomorphisms are compatible in the sense that $\Xi_{(X,x_0)} \circ \Psi = \Theta$.

Proof. This lemma follows immediately from the definitions. ∎

In most applications we will use the following special case of Proposition 62.6.

Proposition 62.7. Let (X,x_0) be a pointed topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $\gamma\colon \pi_1(X,x_0) \to G$ be an epimorphism onto a group G. We denote by $p\colon (\widetilde{X},\widetilde{x}_0) \to (X,x_0)$ the covering corresponding to $\ker(\gamma)$ that is given by the Covering Existence Theorem 61.5 and that is unique up to covering equivalence by the Covering Uniqueness Theorem 61.9. Then the

map

$$\begin{array}{rccccc} \Phi\colon \mathrm{Deck}(p\colon \widetilde{X} \to X) & \to & \pi_1(X,x_0)/p_*(\pi_1(\widetilde{X},\widetilde{x}_0)) & \xrightarrow{\gamma} & G. \\ d & \mapsto & \big[p \circ (\text{path from } x_0 \text{ to } d(x_0))\big] & & \end{array}$$

is an isomorphism.

Proof. First note that $p_*(\pi_1(\widetilde{X},\widetilde{x}_0)) = \ker(\gamma)$ is a normal subgroup of $\pi_1(X,x_0)$. Therefore the normalizer of $\Gamma := p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$ equals $\pi_1(X,x_0)$. It now follows immediately from Proposition 62.6 that the homomorphism on the left is an isomorphism. The homomorphism on the right is an isomorphism since $p_*(\pi_1(\widetilde{X},\widetilde{x}_0)) = \ker(\gamma)$. ■

Convention. Let (X,x_0) be a pointed topological space that is path-connected, locally path-connected and semi-locally simply connected.

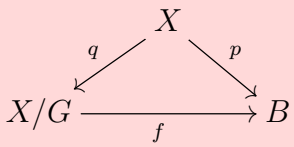
(1) Let $\gamma\colon \pi_1(X,x_0) \to G$ be an epimorphism onto some group G. Furthermore let $p\colon (\widetilde{X},\widetilde{x}_0) \to (X,x_0)$ be a covering corresponding to $\ker(\gamma)$. We use the inverse of the isomorphism $\mathrm{Deck}(p\colon \widetilde{X} \to X) \xrightarrow{\cong} G$ from Proposition 62.7 to equip $\widetilde{X}$ with a left G-action.

(2) Let $p\colon (\widetilde{X},\widetilde{x}_0) \to (X,x_0)$ be a universal covering. We use the inverse of the isomorphism $\mathrm{Deck}(p\colon \widetilde{X} \to X) \xrightarrow{\cong} \pi_1(X,x_0)$ from Proposition 62.6 to equip $\widetilde{X}$ with a left $\pi_1(X,x_0)$-action.

The following corollary to Proposition 62.6 gives a complete characterization of regular coverings.

Corollary 62.8. (Regular Covering-Characterization Corollary) Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed topological spaces that are path-connected and locally path-connected. Then the following statements are equivalent:[736]

(1) The covering is regular.
(2) The deck transformation group $\mathrm{Deck}(p\colon X \to B)$ acts transitively on $p^{-1}(b_0) \subset X$.[737]
(3) The subgroup $p_*(\pi_1(X,x_0))$ is a normal subgroup of $\pi_1(B,b_0)$.
(4) There exists a group G which acts discretely on X and there exists a homeomorphism $f\colon X/G \to B$ such that the following diagram commutes:



where $q\colon X \to X/G$ is the natural projection .

If p is a finite covering then the above statements are furthermore equivalent to

(5) $\#\,\mathrm{Deck}(p\colon X \to B) = [X:B]$.

Remark. Let $p\colon (X,x_0) \to (B,b_0)$ be a 2-fold covering of path-connected pointed topological spaces. In Lemma 62.4 we saw that p is regular, which by the Regular Covering-Characterization Corollary 62.8 means that the index two subgroup $p_*(\pi_1(X,x_0))$ of $\pi_1(B,b_0)$

[736] Note that being regular does not depend on the choice of $b_0 \in B$. Thus, by the equivalence of the statements, we see that if Statements (2) and (3) hold for a single b_0, they hold for any point in B.
[737] Note that by the Deck Group-Free Action Lemma 62.2 we already know that the deck transformation group acts freely on $p^{-1}(b_0)$.

is normal. In fact we already showed in Lemma 49.6 that any index 2 subgroup Γ of a group π is normal.

Proof. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed path-connected topological spaces. We write $\pi := \pi_1(B, b_0)$, $\Gamma := p_*(\pi_1(X, x_0))$ and $D = \mathrm{Deck}(p\colon X \to B)$. The following four steps prove the equivalence of (1), (2) (3) and (4).

(1) $\Leftrightarrow$ (2) Recall that a covering is regular if for every $b \in B$ the group D acts transitively on $p^{-1}(b)$. Thus (1) $\Rightarrow$ (2) is trivial. Now suppose that d acts transitively on $p^{-1}(b_0)$. Let b be some other point in B. Since B is path-connected there exists a path s from b to b_0.

Now let x, x' be two points in $p^{-1}(b)$. We denote by $\widetilde{s}$ the lift of s to the starting point x and similarly we define $\widetilde{s}'$. We denote by x_0 and x_0' the endpoints of $\widetilde{s}$ and $\widetilde{s}'$. Since D acts transitively on $p^{-1}(b_0)$ there exists a deck transformation d with $d(x_0) = x_0'$. By the uniqueness of lifts to a starting point we then also have $d \circ \widetilde{s} = \widetilde{s}'$. But then we also have that d turns the starting point of $\widetilde{s}$ into the starting point of $\widetilde{s}'$, i.e. we have $d(x) = x'$.

(2) $\Leftrightarrow$ (3) We consider again the map

$$\begin{aligned} \Phi\colon \mathrm{Deck}(p\colon X \to B) &\mapsto \pi/\Gamma \\ d &\mapsto \big[p \circ (\text{path in } X \text{ from } x_0 \text{ to } d(x_0))\big]. \end{aligned}$$

We first prove the following claim.

Claim. We have the following equivalence of statements:

$$D \text{ acts transitively on } p^{-1}(b_0) \subset X \quad \Leftrightarrow \quad \Phi\colon D \to \pi/\Gamma \text{ is surjective.}$$

Proof. First suppose that D acts transitively on $p^{-1}(b_0) \subset X$. Let $g \in \pi/\Gamma$ and pick a loop f that represents g. We denote by $\widetilde{f}$ the unique lift of f to the starting point x_0 and we denote by $\widetilde{x}_0$ the endpoint of $\widetilde{f}$. Since D acts transitively there exists a deck transformation d with $d(x_0) = \widetilde{x}_0$. Now

$$\Phi(d) = [p \circ (\text{path in } X \text{ from } x_0 \text{ to } d(x_0))] \underset{\substack{\uparrow \\ \text{since } \widetilde{f} \text{ is a path from } x_0 \text{ to } d(x_0)}}{=} [p \circ (d \circ \widetilde{f})] \underset{\substack{\uparrow \\ \text{since } p \circ d = p}}{=} [p \circ \widetilde{f}] = [f] = g.$$

Now suppose that $\Phi\colon D \to \pi/\Gamma$ is surjective. Let $\widetilde{x}_0 \in p^{-1}(b_0)$. We pick a path $\widetilde{f}$ from x_0 to $\widetilde{x}_0$ and we denote by f the loop $p \circ \widetilde{f}$. Since Φ is surjective there exists a deck transformation d with $\Phi(d) = [f]$. But then $d(x_0) = \widetilde{x}_0$. ⊞

Now we have the following equivalences

$$\begin{aligned} D \text{ acts transitively on } p^{-1}(b_0) \subset X &\iff \Phi\colon D \to \pi/\Gamma \text{ is surjective} \\ &\underset{\substack{\uparrow \\ \text{Proposition 62.6}}}{\iff} \pi = N(\Gamma) \underset{\substack{\uparrow \\ \text{by the definitions}}}{\iff} \Gamma \text{ is normal in } \pi. \end{aligned}$$

(1) $\Rightarrow$ (4) We will show that the deck transformation group D has the desired properties.

We first show that the deck transformation group acts discretely on X. So let $x \in X$. We write $b = p(x)$. Since $p\colon X \to B$ is a covering there exists a connected open neighborhood U of b that is uniformly covered. This means that the restriction of p to each component of $p^{-1}(U)$ is a homeomorphism. Now

let V be the component of $p^{-1}(U)$ that contains x. Then $gV \cap V = \varnothing$ for each non-trivial g. This shows that the action of D is discrete.

The map $p\colon X \to B$ has the property that $p(gx) = p(x)$ for all $g \in D$. Therefore the map p factors through a map $f\colon X/D \to B$. The map f is clearly surjective. Furthermore it is injective since we suppose that D acts transitively on each $p^{-1}(b)$. Finally the map f is not only continuous but, as follows easily from Lemma 48.3 (1), it is also open. Thus f is a homeomorphism.

(4) $\Rightarrow$ (1) Now suppose that such a homeomorphism $f\colon X/G \to B$ exists. First note that for each $g \in G$ the map $x \mapsto gx$ not only defines a deck transformation of the covering $q\colon X \to X/G$ but also of the covering $p\colon X \to B$. Now let $b \in B$ and let $x, x' \in X$ with $p(x) = p(x')$. Since f is a homeomorphism we also have $q(x) = q(x')$. Hence there exists a $g \in G$ with $gx = x'$.

Now we suppose that p is a finite covering. We show the following equivalence:

(5) $\Leftrightarrow$ (3) We have the following equivalences:[738]

$$\begin{array}{ccccc} & \overset{\text{Lemma 48.15 (3)}}{\downarrow} & & \overset{\text{Proposition 62.6}}{\downarrow} & \\ \#D = [X : B] & \Longleftrightarrow & \#D = [\pi : \Gamma] & \Longleftrightarrow & [N(\Gamma) : \Gamma] = [\pi : \Gamma] \\ & \Longleftrightarrow & N(\Gamma) = \pi & \Longleftrightarrow & \Gamma \subset \pi \text{ is a normal subgroup.} \end{array}$$

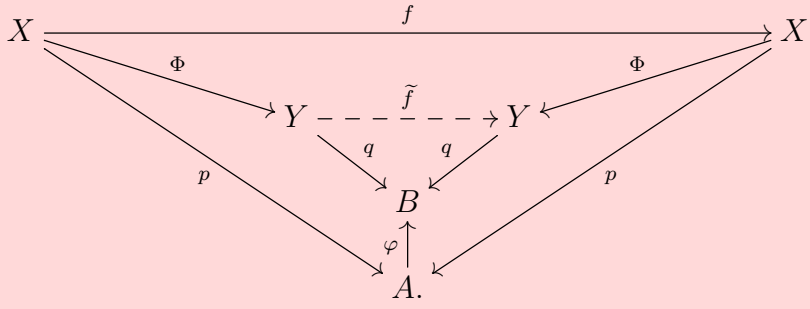
■

Proposition 62.9. (Induced-Deck Transformation Proposition) Let X, Y, A and B be topological spaces that are path-connected and locally path-connected. Furthermore let $p\colon X \to A$ be a covering and let $q\colon Y \to B$ be a *regular* covering. Finally let $\varphi\colon A \to B$ and let $\Phi\colon X \to Y$ be maps such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{\varphi} & B. \end{array}$$

In this setting the following two statements hold:

(1) Given any $f \in \mathrm{Deck}(p\colon X \to B)$ there exists a unique $\widetilde{f} \in \mathrm{Deck}(q\colon Y \to B)$ such that the following diagram commutes:



(2) The map $f \mapsto \widetilde{f}$ from (1) defines a group homomorphism

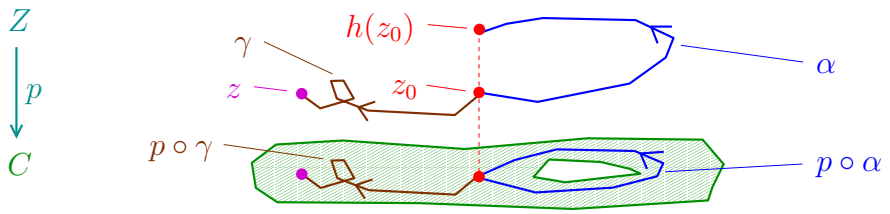
$$\mathrm{Deck}(p\colon X \to A) \;\to\; \mathrm{Deck}(q\colon Y \to B).$$

[738] Where do we actually use in the argument that the covering is finite?

In the proof of the Induced-Deck Transformation Proposition 62.9 we will use the following lemma which gives a sometimes useful description of a deck transformation.

Lemma 62.10. Let $r\colon (Z, z_0) \to (C, c_0)$ be a covering and let $h\colon Z \to Z$ be a deck transformation. Let $z \in Z$. We pick a path $\alpha\colon [0,1] \to Z$ from z_0 to $h(z_0)$ and we pick a path $\gamma\colon [0,1] \to Z$ from z_0 to z. Then

$h(z)$ = endpoint of the unique lift of $(p \circ \alpha) * (p \circ \gamma)\colon [0,1] \to C$ to the starting point z_0.



Proof of Lemma 62.10. Note that

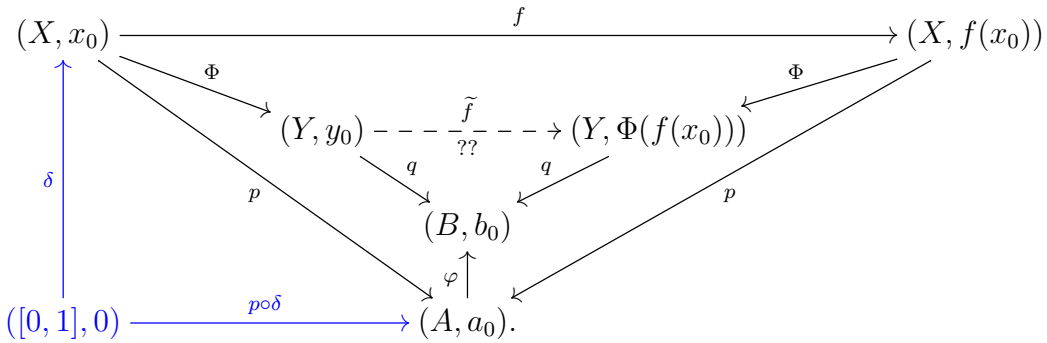
endpoint of the unique lift of $(p \circ \alpha) * (p \circ \gamma)\colon [0,1] \to C$ to the starting point z_0
$= \uparrow$ endpoint of $\alpha * h \circ \gamma = (h \circ \gamma)(1) = h(\gamma(1)) = h(z)$.

since h is a deck transformation we see that $h \circ \gamma$ is the lift of $p \circ \gamma$ to starting point $\alpha(1) = h(z_0)$

■

Proof of the Induced-Deck Transformation Proposition 62.9.

(1) We pick $x_0 \in X$. We set $a_0 = p(x_0)$, $y_0 := \Phi(x_0)$ and $b_0 := q(y_0)$. We consider the following diagram of maps between pointed topological spaces:



(The maps in blue will only be explained a little later on.)

Claim 1. We have $q_*(\pi_1(Y, y_0)) = q_*(\pi_1(Y, \Phi(f(x_0))))$.

Proof. Since Y is path-connected we can pick a path $\gamma\colon [0,1] \to Y$ with $\gamma(0) = y_0$ and $\gamma(1) = \Phi(f(x_0))$. Then

$$q_*(\pi_1(Y, y_0)) \underset{\uparrow}{=} \overbrace{[q \circ \gamma]}^{\in \pi_1(B, b_0)} \cdot q_*(\pi_1(Y, \Phi(f(x_0)))) \cdot [q \circ \gamma]^{-1} \underset{\uparrow}{=} q_*(\pi_1(Y, \Phi(f(x_0)))).$$

by the Change-of-Base Point Proposition 47.10

by the Regular Covering-Characterization Corollary 62.8, since $q\colon Y \to B$ is a regular covering

⊞

Since $q_*(\pi_1(Y, y_0)) = q_*(\pi_1(Y, \Phi(f(x_0))))$ and since Y is locally path-connected we obtain from the Map Lifting Criterion 61.2 a unique map $\widetilde{f}\colon (Y, y_0) \to (Y, \Phi(f(x_0)))$

such that $q = q \circ \widetilde{f}$, i.e. such that the small triangle commutes. It remains to prove the following claim.

Claim 2. The quadrilateral on top also commutes.

Proof. Let $x \in X$. We need to show that $(\widetilde{f} \circ \Phi)(x) = (\Phi \circ f)(x)$.

(a) We pick a path $\alpha\colon [0,1] \to X$ from x_0 to $f(x_0)$ and we pick a path $\gamma\colon [0,1] \to X$ from x_0 to x.

(b) Evidently $\Phi \circ \alpha\colon [0,1] \to Y$ is a path from $y_0 = \Phi(x_0)$ to $\Phi(f(x_0))$ and evidently $\Phi \circ \gamma\colon [0,1] \to Y$ is a path from $y_0 = \Phi(x_0)$ to $\Phi(x)$.

(c) Let $\delta\colon [0,1] \to X$ be the unique lift of

$$(p \circ \alpha) * (p \circ \gamma) = p \circ (\alpha * \gamma)\colon [0,1] \to A$$

to the starting point x_0.

(d) Note that

$$q \circ \Phi \circ \delta \underset{\substack{\uparrow \\ \text{since } q \circ \Phi = \varphi \circ p}}{=} \varphi \circ p \circ \delta \underset{\substack{\uparrow \\ \text{since } \delta \text{ lift of } \alpha * \gamma}}{=} \varphi \circ p \circ (\alpha * \gamma) \underset{\substack{\uparrow \\ \text{since } q \circ \Phi = \varphi \circ p}}{=} (q \circ \Phi) \circ (\alpha * \gamma) = q \circ (\Phi \circ \alpha * \Phi \circ \gamma).$$

(e) It follows from (d) that $\Phi \circ \delta$ is the (necessarily unique) lift of $q \circ (\Phi \circ \alpha) * q \circ (\Phi \circ \gamma)$ to the starting point y_0.

In summary we see that

$$\widetilde{f}(\Phi(x)) \underset{\substack{\uparrow \\ \text{by Lemma 62.10 with (b) and (e)}}}{=} (\Phi \circ \delta)(1) = \Phi(\delta(1)) \underset{\substack{\uparrow \\ \text{by Lemma 62.10 and choice of } \delta}}{=} \Phi(f(x)) = (\Phi \circ f)(x).$$

⊞

(2) This statement follows easily from Statement (1). We leave it to the reader to fill in details. ■

Lemma 62.11. Let X, Y, A and B be topological spaces that are path-connected and locally path-connected. Furthermore let $p\colon X \to A$ be a covering and let $q\colon Y \to B$ be a *regular* covering. Finally let $\varphi\colon A \to B$ and let $\Phi\colon X \to Y$ be maps such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\quad\Phi\quad} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{\quad\varphi\quad} & B. \end{array}$$

Let $x_0 \in X$. We write $a_0 = p(x_0)$, $y_0 = \Phi(x_0)$ and $b_0 = q(\Phi(x_0))$. In this setting the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Deck}(p\colon X \to A) & \xrightarrow{\quad\Xi_{(X,x_0)}\quad} & \pi_1(A,a_0)/p_*(\pi_1(X,x_0)) \\ \downarrow & & \downarrow{\scriptstyle \varphi_*} \\ \mathrm{Deck}(q\colon Y \to B) & \xrightarrow{\quad\Xi_{(Y,y_0)}\quad} & \pi_1(B,b_0)/q_*(\pi_1(Y,y_0)), \end{array}$$

where the horizontal maps are the maps we introduced on page 1388, where the left vertical map is the map given by the Induced-Deck Transformation Proposition 62.9 and the right vertical map is induced by $\Phi_*\colon \pi_1(A,a_0) \to \pi_1(B,b_0)$.[739]

[739] It follows from $q \circ \Phi = \varphi \circ p$ that $\varphi_*(p_*(\pi_1(X,x_0))) \subset q_*(\pi_1(Y,y_0))$.

Proof. The lemma follows immediately from the definitions. ■

The following proposition tells us how one can obtain in a relatively explicit way "all" coverings from the universal covering. It can be viewed as a refinement of Proposition 61.14.

Proposition 62.12. Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected, let $x_0 \in X$ and let $p\colon \widetilde{X} \to X$ be a universal covering. We pick $\widetilde{x}_0 \in \widetilde{X}$ with $p(\widetilde{x}) = x_0$. We endow $\widetilde{X}$ with the $\pi_1(X, x_0)$-action that we introduced on page 1391. Let $\Gamma\colon \pi_1(X, x_0)$ be a subgroup. The map

$$\begin{array}{rcl} q\colon \widetilde{X}/\Gamma & \to & X \\ [\widetilde{x}] & \mapsto & p(x) \end{array}$$

is a covering with $q_*(\pi_1(\widetilde{X}/\Gamma, [\widetilde{x}_0])) = \Gamma$.

Proof. By Lemma 62.1 we know that the action of Γ on $\widetilde{X}$ is discrete and continuous. Thus it follows from Proposition 48.10 that the map $q\colon \widetilde{X}/\Gamma \to X$ is indeed a covering. In the Fundamental Group-via-Actions Theorem 48.16 we gave an explicit isomorphism $\pi_1(\widetilde{X}/\Gamma, [\widetilde{x}_0]) \cong \Gamma$. One can easily verify that this map is precisely q_*. ■

The following proposition is a refinement of the Lift-to-Universal Coverings Proposition 61.13.

Proposition 62.13. Let X and Y be topological spaces that are path-connected, locally path-connected and semi-locally simply connected. Let $p\colon \widetilde{X} \to X$ and $q\colon \widetilde{Y} \to Y$ be universal coverings. Furthermore suppose we are given a map $f\colon X \to Y$ and points $x_0 \in X$, $\widetilde{x}_0 \in \widetilde{X}$ and $\widetilde{y}_0 \in \widetilde{Y}$ with $p(\widetilde{y}_0) = f(x_0)$. By the Lift-to-Universal Coverings Proposition 61.13 there exists a unique map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ with $\widetilde{f}(\widetilde{x}_0) = \widetilde{y}_0$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \overset{\widetilde{f}}{\dashrightarrow} & \widetilde{Y} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{\quad f \quad} & Y. \end{array}$$

We equip $\widetilde{X}$ with the action by $\pi_1(X, x_0)$ and we equip $\widetilde{Y}$ with the action by $\pi_1(Y, y_0)$ that we introduced on page 1391. The map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ has the following property: given any $\widetilde{x} \in \widetilde{X}$ and given any $g \in \pi_1(X, x_0)$ we have

$$\widetilde{f}(g \cdot \widetilde{x}) \;=\; \underbrace{f_*(g)}_{\in \pi_1(Y, y_0)} \cdot \widetilde{f}(\widetilde{x}).$$

Proof. It is worth recalling how the action of fundamental groups on coverings are actually defined. In Proposition 167.1 we saw that if $r\colon (\widehat{Z}, \widehat{z}_0) \to (Z, z_0)$ is a regular covering of pointed topological spaces that are both path-connected and locally path-connected, then the map

$$\begin{array}{rcl} \Phi_{\widehat{Z}, \widehat{z}_0}\colon \mathrm{Deck}(p\colon \widehat{Z} \to Z) & \to & \pi_1(Z, z_0)/p_*(\pi_1(\widehat{Z}, \widehat{z}_0)) \\ d & \mapsto & \big[p \circ (\text{path in } \widehat{Z} \text{ from } \widehat{z}_0 \text{ to } d(\widehat{z}_0))\big] \end{array}$$

is an isomorphism, and we used the inverse of this isomorphism to define the action of $\pi_1(Z, z_0)$ on $\widehat{Z}$.

Now let $\widetilde{x} \in \widetilde{X}$. First we consider the case that $p(\widetilde{x}) = x_0$.

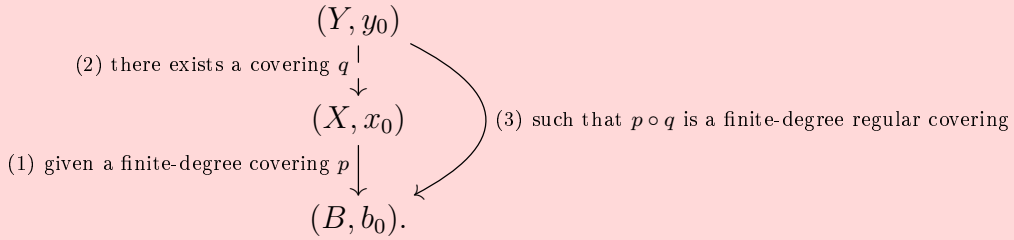
Now let $\widetilde{x}$ be an arbitrary point. Since X is path-connected we can pick a path $\gamma\colon [0,1] \to X$ from x_0 to $p(\widetilde{x}_0)$.

■

Proof. Once one has digested the definition this statement follows immediately from the definitions. We leave it to the reader to fill in the details. ■

We conclude this chapter with the following proposition which often, rather conveniently, allows us to replace an irregular covering by a regular covering.

Proposition 62.14. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces that are path-connected, locally path-connected and semi-locally simply connected. If p is a finite-degree covering, then there exists a covering $q\colon (Y, y_0) \to (X, x_0)$ such that $p \circ q\colon (Y, y_0) \to (B, b_0)$ is a finite-degree *regular* covering. For convenience this statement is summarized in the following diagram:



Proof. Let $p\colon (X, x_0) \to (B, b_0)$ be a finite-degree covering of pointed topological spaces that are path-connected, that are locally path-connected and that are semi-locally simply connected. In the following we write $G := \pi_1(X, x_0)$ and $H := p_*(\pi_1(B, b_0))$. By Lemma 48.15 we know that H is a finite-index subgroup of G. It follows from the Normal Core Lemma 64.13 that the normal core H^c of H in G is a finite-index normal subgroup of G that is contained in H. Since B is locally path-connected and semi-locally simply connected we obtain from the Covering Existence Theorem 61.5 a finite-degree path-connected covering $r\colon (Y, y_0) \to (B, b_0)$ of pointed topological spaces such that $r_*(\pi_1(Y, y_0)) = H^c$.

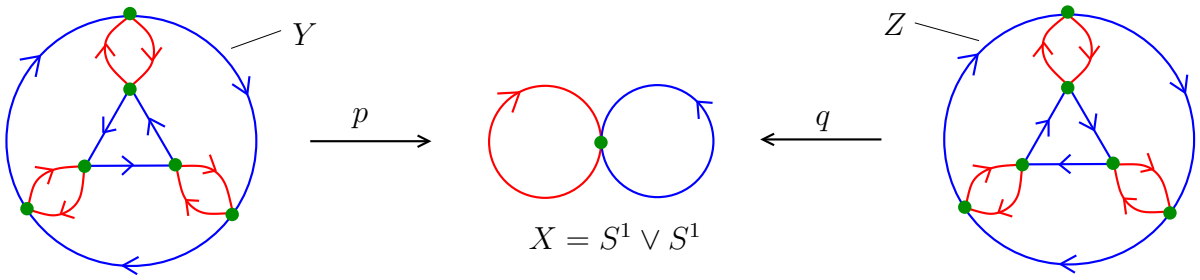
The existence of the desired covering map $q\colon (Y, y_0) \to (X, x_0)$ with $r = p \circ q$ is an immediate consequence of the Covering Uniqueness Theorem 61.9 and the fact that H^c is contained in H. ■

Exercises for Chapter 62.

Exercise 62.1. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed path-connected topological spaces. Show that the action of the deck group $\mathrm{Deck}(p\colon X \to B)$ on X is discrete.

Exercise 62.2. In the figure below we show two coverings $p\colon Y \to X = S^1 \vee S^1$ and $q\colon Z \to X = S^1 \vee S^1$.

(a) Determine which coverings are regular.
(b) For the regular covering(s) in (a), determine the isomorphism type of the deck transformation group.



Exercise 62.3. Let Σ be the surface of genus 2. Does there exist a regular covering $p\colon \widetilde{\Sigma} \to \Sigma$ such that the deck transformation group is isomorphic to the permutation group S_3?

Exercise 62.4. Let G be a finite group. Show that there exists a closed orientable 2-dimensional smooth manifold that admits a free and transitive action by G.

Exercise 62.5. Let $p\colon \widetilde{X} \to X$ be a covering with deck transformation group G and let $K \subset \widetilde{X}$ be a compact subset. We suppose that X is Hausdorff and regionally compact. Show that the set $\{g \in G \,|\, g \cdot K \cap K \neq \varnothing\}$ is finite.
Remark. In [**Scot1978**, Lemma 1.2] claims that the same conclusion holds without the hypothesis that X is regionally compact. Try to prove this more general statement.

Exercise 62.6. Verify Proposition 62.13.

Exercise 62.7. Let X be a topological space that is locally contractible and connected and let $x_0 \in X$. As in the proof of the Covering Existence Theorem 61.5 we consider the topological space

$$\widetilde{X}_{x_0} \;=\; \{\gamma\colon [0,1] \to X \,|\, \gamma(0) = x_0\}/\!\sim \qquad \begin{array}{l}\text{where } \alpha \sim \beta \text{ if } \alpha(1) = \beta(1) \text{ and} \\ \alpha * \beta \text{ is null-homotopic.}\end{array}$$

We denote by $p\colon \widetilde{X}_{x_0} \to X$ the covering map that is given by the evaluation map $[\gamma] \mapsto \gamma(1)$ In Proposition 62.6 we can an explicit isomorphism $\mathrm{Deck}(p\colon \widetilde{X}_{x_0} \to X) \cong \pi_1(X, x_0)$. Show that the corresponding action of the fundamental group $\pi_1(X, x_0)$ on $\widetilde{X}_{x_0}$ is given by

$$\begin{array}{rcl}\pi_1(X, x_0) \times \widetilde{X}_{x_0} & \to & \widetilde{X}_{x_0} \\ ([\alpha], [\gamma]) & \mapsto & [\alpha * \gamma].\end{array}$$

63. Topics in covering theory

In this chapter we will discuss several applications and examples of the existence of (universal) coverings of suitable topological spaces.

63.1. **Self-diffeomorphisms of S^1.** Using covering theorem we can now provide a new prove of the following proposition.

Proposition 34.5 Let $f\colon S^1 \to S^1$ be a diffeomorphism. If f is orientation-preserving, then f is diffeotopic to the identity, otherwise f is diffeotopic to the reflection $r\colon S^1 \to S^1$ given by $(x, y) \mapsto (x, -y)$.

Remark. The proof of the Diffeotopies-of-S^1-Classification Proposition 34.5 that we now provide is very similar to the original proof. The key difference is that we replace the ad hoc the Lifting from S^1-to-$\mathbb{R}$ Lemma 34.9 by the more general the Map Lifting Criterion 61.2.

Proof. In Torus-as-Quotient Lemma 19.34 we showed that the smooth manifolds S^1 and $\mathbb{R}/\mathbb{Z}$ are diffeomorphic. Thus we might as well work with $\mathbb{R}/\mathbb{Z}$. First let $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an orientation-preserving diffeomorphism. We denote by $p\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ the usual covering map. By the Lift-to-Universal Coverings Proposition 61.13 there exists an orientation-preserving diffeomorphism $\widetilde{f}\colon \mathbb{R} \to \mathbb{R}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\widetilde{f}} & \mathbb{R} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{R}/\mathbb{Z} & \xrightarrow{f} & \mathbb{R}/\mathbb{Z}. \end{array}$$

We make the following observations:

(a) The map $\widetilde{f}$ is in particular smooth.
(b) Since $\widetilde{f}$ is orientation-preserving we know, by Exercise 25.18, that $\widetilde{f}$ is a strictly monotonously increasing function.
(c) Since $\widetilde{f}$ is a diffeomorphism the restriction of $\widetilde{f}$ to the interval $[x, x+1)$ defines a bijection with $[f(x), f(x+1))$.
(d) Given $x \in \mathbb{R}$ we obtain from the commutative diagram that $\widetilde{f}(x+1) - \widetilde{f}(x) \in \mathbb{Z}$.
(e) It follows from (a), (b), (c) and (d) that for any $x \in \mathbb{R}$ we have $\widetilde{f}(x+1) = \widetilde{f}(x) + 1$.

Now we consider the map

$$\begin{array}{rcl} \widetilde{F}\colon \mathbb{R} \times [0,1] & \to & \mathbb{R} \\ (x, t) & \mapsto & \widetilde{f}(x) \cdot (1-t) + x \cdot t. \end{array}$$

This map is evidently smooth. Since $x \mapsto x$ is a strictly monotonously increasing diffeomorphism of $\mathbb{R}$ one can see easily that for any $t \in [0,1]$ the map $\widetilde{F}_t\colon \mathbb{R} \to \mathbb{R}$ has the above Properties (a) to (e). This implies in particular that the map $\widetilde{F}$ descends to a map

$$\begin{array}{rcl} F\colon (\mathbb{R}/\mathbb{Z}) \times [0,1] & \to & \mathbb{R}/\mathbb{Z} \\ ([x], t) & \mapsto & \widetilde{f}(x) \cdot (1-t) + x \cdot t \end{array}$$

where each $F_t\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a homeomorphism. Since each $\widetilde{F}_t$ is a local diffeomorphism and since p is a local diffeomorphism we obtain from the Smooth Pasting Proposition 19.23

(2) that each F_t is in fact a diffeomorphism. We have thus shown that F is indeed a diffeotopy from f to the identity.

Finally we consider the case that f is orientation-reversing. In this case $r \circ g$ is orientation-preserving. We apply the above result to $r \circ g$ and we easily obtain the desired diffeotopy from f to r. ■

63.2. **Complex one-dimensional tori.** The following definition is inspired by Lemma 32.18 and Exercise 19.34.

Notation.

(1) A lattice in $\mathbb{C}$ is a subgroup of $\mathbb{C}$ that is of the form $\mathbb{Z} \cdot a + \mathbb{Z} \cdot b$ where $a, b \in \mathbb{C}$ are two complex numbers that are linearly independent over $\mathbb{R}$.
(2) Given a lattice Λ we write $T(\Lambda) = \mathbb{C}/\Lambda$. The holomorphic analogue of the Manifold Quotient-by-Group Action Proposition 19.32 shows that $T(\Lambda)$ is naturally a 1-dimensional complex manifold such that the projection $p\colon \mathbb{C} \to T(\Lambda)$ is a covering map of complex manifolds, in particular it is a local biholomorphism.
(3) We say that two lattices Λ and $\widetilde{\Lambda}$ in $\mathbb{C}$ are homothetic if there exists an $r \in \mathbb{C} \setminus \{0\}$ such that $r \cdot \Lambda = \widetilde{\Lambda}$.

The following proposition gives a complete and satisfactory characterization for when two lattices define biholomorphic complex manifolds.

Proposition 63.1. Let Λ and $\widetilde{\Lambda}$ be two lattices in $\mathbb{C}$. The complex manifolds $T(\Lambda)$ and $T(\widetilde{\Lambda})$ are biholomorphic if and only if Λ and $\widetilde{\Lambda}$ are homothetic.

Proof. Let Λ and $\widetilde{\Lambda}$ be two lattices in $\mathbb{C}$. We denote by $p\colon \mathbb{C} \to T(\Lambda)$ and $\widetilde{p}\colon \mathbb{C} \to T(\widetilde{\Lambda})$ the natural projections. Note that there local holomorphisms.

First assume that Λ and $\widetilde{\Lambda}$ are homothetic, i.e. we assume that there exists an $r \in \mathbb{C}\setminus\{0\}$ such that $r \cdot \Lambda = \widetilde{\Lambda}$. In this case we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{z \mapsto r \cdot z} & \mathbb{C} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \widetilde{p}} \\ T(\Lambda) & \xrightarrow{[z] \mapsto [r \cdot z]} & T(\widetilde{\Lambda}). \end{array}$$

Note that our hypothesis shows that bottom horizontal map is defined and a bijection. The top horizontal map is of course a biholomorphism. Since the vertical maps are local biholomorphisms we see that the bottom horizontal map is a bijection and a local biholomorphism. In other words, it is a biholomorphism.

Now we assume that there exists a biholomorphism $f\colon T(\Lambda) \to T(\widetilde{\Lambda})$. By the Lift-to-Universal Coverings Proposition 61.13 there exists a biholomorphism $\widetilde{f}\colon \mathbb{C} \to \mathbb{C}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\widetilde{f}} & \mathbb{C} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \widetilde{p}} \\ T(\Lambda) & \xrightarrow{f} & T(\widetilde{\Lambda}). \end{array}$$

Next we make the following observations and preparations:

(a) We equip the universal covering $\mathbb{C}$ with the actions of $\pi_1(T(\Lambda),[0])$ and $\pi_1(T(\widetilde{\Lambda}),[s])$ that we introduced on page 1391.

(b) By Proposition 62.13 we know that for all $z \in \mathbb{C}$ and all $g \in \pi_1(T(\Lambda), x_0)$ we have

$$\widetilde{f}(g \cdot z) \;=\; \underbrace{f_*(g) \cdot \widetilde{f}(z)}_{\in \pi_1(T(\widetilde{\Lambda}),[s])}.$$

(c) Note that the Fundamental Group-via-Actions Theorem 48.16 provides us with explicit isomorphisms $\Theta\colon \Lambda \to \pi_1(T(\Lambda), x_0)$ and $\widetilde{\Theta}\colon \widetilde{\Lambda} \to \pi_1(T(\widetilde{\Lambda}), x_0)$.

(d) It follows from the discussion on page 1390 that (c) translates into the statement that for all $z \in \mathbb{C}$ and all $\lambda \in \Lambda$ we have

$$\widetilde{f}(\lambda + z) \;=\; (\widetilde{\Theta} \circ f_* \circ \Theta^{-1})(\lambda) + \widetilde{f}(z).$$

(e) Since $\widetilde{f}\colon \mathbb{C} \to \mathbb{C}$ is a biholomorphism we know by Theorem 32.11 that there exist $r, s \in \mathbb{C}$ such that $f(z) = r \cdot z + s$ for all $z \in \mathbb{C}$.

(f) The combination of (d), applied to $z = 0$, and (e) shows that for all $\lambda \in \Lambda$ we have

$$r \cdot \lambda + s \;=\; (\widetilde{\Theta} \circ f_* \circ \Theta^{-1})(\lambda) + s.$$

(g) The map $f_*\colon \pi_1(T(\Lambda),[0]) \to \pi_1(T(\widetilde{\Lambda}),[s])$ is an isomorphism since f is in particular a homeomorphism. In particular $\widetilde{\Theta} \circ f_* \circ \Theta^{-1}\colon \Lambda \to \widetilde{\Lambda}$ is an isomorphism.

The combination of (f) and (g) shows that $r \cdot \Lambda = \widetilde{\Lambda}$. We have thus shown that the lattices Λ and $\widetilde{\Lambda}$ are homothetic. ■

Corollary 63.2. There are closed complex manifolds that are diffeomorphic but that are not biholomorphic.

Proof. Let Λ and $\widetilde{\Lambda}$ be two lattices in $\mathbb{C}$. It follows from Exercise 19.34 that the tori $T(\Lambda)$ and $T(\widetilde{\Lambda})$, viewed as smooth manifolds, are diffeomorphic. In Exercise 63.1 we will see that there exist lattices Λ and $\widetilde{\Lambda}$ in $\mathbb{C}$ that are not homothetic. It follows from Proposition 63.1 that $T(\Lambda)$ and $T(\widetilde{\Lambda})$ are not biholomorphic. ■

63.3. **Complex logarithms.** In this short section we will discuss complex logarithms. This discussion does not play a role in the remainder of these notes, but it is a fun application of covering theory.

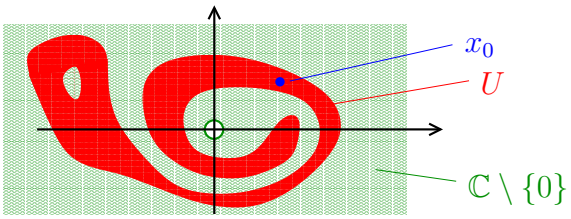
Definition. Let $U \subset \mathbb{C} \setminus \{0\}$ be a subset. We say the complex logarithm exists on U if there exists a map[740] $\ln\colon U \to \mathbb{C}$ such that $\exp(\ln(z)) = z$ for all $z \in U$.

The following proposition gives a very pleasing sufficient and necessary criterion for complex logarithms to exist.

Proposition 63.3. Let $U \subset \mathbb{C} \setminus \{0\}$ be an open path-connected subset. The following statement holds:[741]

$$\text{the complex logarithm exists on } U \quad\Longleftrightarrow\quad \begin{array}{c}\text{for some } x_0 \in U \text{ the inclusion induced map}\\ \pi_1(U, x_0) \to \pi_1(\mathbb{C} \setminus \{0\}, x_0)\\ \text{is the trivial map.}\end{array}$$

[740] As always, all maps are understood to be continuous.



the map $\pi_1(U, x_0) \to \pi_1(\mathbb{C} \setminus \{0\}, x_0)$
is the trivial map, hence
the complex logarithm exists on U

Proof. Let $U \subset \mathbb{C} \setminus \{0\}$ be an open path-connected subset and let $x_0 \in U$. We consider the following rather simple minded diagram

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "z \mapsto \exp(z)"] \\ U \arrow[ur, dashed] \arrow[r, hook, "i"'] & \mathbb{C} \setminus \{0\}. \end{tikzcd}$$

We make the following observations:

(1) As on page 1067 one can show that the vertical map exp is actually a covering map.
(2) The question whether the complex logarithm exists on U is equivalent to the question whether the inclusion map $i\colon U \to \mathbb{C} \setminus \{0\}$ lifts to a map $U \to \mathbb{C}$.
(3) Since $\exp\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is surjective we can pick a $y_0 \in \mathbb{C}$ with $\exp(y_0) = x_0$.
(4) Note that $\mathbb{C}$ is simply connected. The image $\operatorname{im}(\exp_*\colon \pi_1(\mathbb{C}, y_0) \to \pi_1(\mathbb{C} \setminus \{0\}, x_0))$ is therefore the trivial group.
(5) Since U is an open subset of $\mathbb{C} \setminus \{0\}$ it is in particular locally path-connected. Thus we can apply the Map Lifting Criterion 61.2. Together with (4) we obtain from the Map Lifting Criterion 61.2 the statement

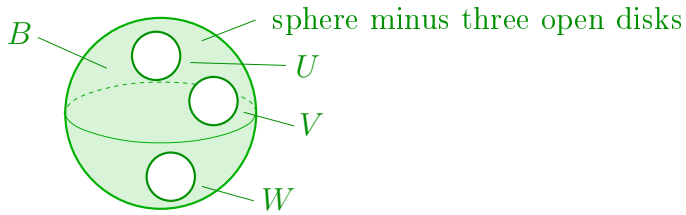
$$\begin{array}{c}\text{there exists a lift of}\\ \text{the map } i\colon U \to \mathbb{C} \setminus \{0\}\end{array} \iff \begin{array}{c}\text{the map } i_*\colon \pi_1(U, x_0) \to \pi_1(\mathbb{C} \setminus \{0\}, x_0)\\ \text{is the trivial map.}\end{array}$$

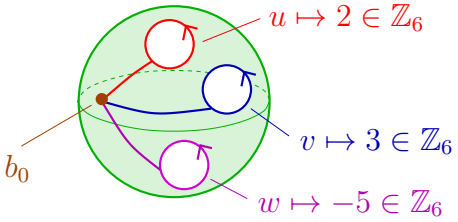
(6) The combination of (2) and (5) is precisely the statement we had set out to prove. ■

63.4. **Preimages of subsets under coverings.** Let (B, b_0) be a pointed topological space that is path-connected, locally path-connected and semi-locally simply connected. In the Covering Existence Theorem 61.5 (2) we saw that subgroups of $\pi_1(B, b_0)$ give rise to coverings of B. Arguably the most common source of subgroups is given by considering kernels of group homomorphisms $\pi_1(B, b_0) \to G$.

Example. Let B be the 2-dimensional sphere S^2 minus three open disks. We pick a base point b_0 for B and we pick loops u, v, w as shown in the figure below. From Lemma 55.11 we know that $\pi_1(B, b_0) = \langle u, v, w \mid uvw \rangle$. By Lemma 53.4 there exists a unique homomorphism $\varphi\colon \pi_1(B, b_0) \to \mathbb{Z}_6$ with $\varphi(u) = 2$, $\varphi(v) = 3$ and $\varphi(w) = -5$. Let $p\colon X \to B$ be the covering corresponding to $\ker(\varphi)$. By the Manifold Covering Proposition 49.2 we know that X is again a compact orientable 2-dimensional smooth manifold, where the boundary is given by $\partial X = p^{-1}(\partial B)$. Thus we know from the Surface Classification Theorem 55.4 that there exist unique $g \in \mathbb{N}_0$ and $n \in \mathbb{N}$ such that X is homeomorphic to $\Sigma_{g,n}$, i.e. such that X is homeomorphic to the surface of genus g minus n open disks.

[741] It follows from Proposition 47.20 that the condition on the right-hand side does not depend on the choice of the base point.





Therefore the following question arises:

Question 63.4. In the previous example, what are g and n?

The final proposition of this chapter will allow us to answer one half of Question 63.4, namely we can determine n. We will determine g in Exercise 64.3.

Proposition 63.5. (Preimage-of-Covering Proposition) Let (B, b_0) be a pointed topological space that is path-connected, locally path-connected and semi-locally simply connected. Let $\varphi\colon \pi_1(B, b_0) \to G$ be a group epimorphism and let $p\colon (X, x_0) \to (B, b_0)$ be the covering with $p_*(\pi_1(X, x_0)) = \ker(\varphi)$. Finally let A be a path-connected subset of B and let $a_0 \in A$. We denote by $i\colon A \to B$ the inclusion map and we pick a path γ from b_0 to a_0. We consider the homomorphism

$$\psi := \varphi \circ \gamma_* \circ i_*\colon \pi_1(A, a_0) \xrightarrow{i_*} \pi_1(B, a_0) \xrightarrow{\gamma_*} \pi_1(B, b_0) \xrightarrow{\varphi} G.$$

With these preparations the following two statements hold:

(1) Each path-component of $p^{-1}(A)$ is a covering of A corresponding to the subgroup $\ker(\psi\colon \pi_1(A, a_0) \to G)$.

(2) If G is a finite group, then

$$\text{number of path-components of } p^{-1}(A) \;=\; [G : \operatorname{im}(\psi\colon \pi_1(A, a_0) \to G)].$$

(3) Let $\widetilde{A}$ be a path-component of $p^{-1}(A)$. The following diagram commutes:

$$\begin{array}{ccc} \operatorname{Deck}(p\colon p^{-1}(A) \to A) & \xrightarrow{\Xi_{(A,a_0)}} & \pi_1(A, a_0)/\ker(\psi) \\ \downarrow & & \downarrow{\scriptstyle \gamma_* \circ i_*} \\ \operatorname{Deck}(p\colon X \to B) & \xrightarrow{\Xi_{(B,a_0)}} & \pi_1(B, a_0)/\ker(\varphi \circ \gamma_*). \end{array}$$

Here the horizontal maps are the isomorphisms that we introduced on page 1388 and the left vertical map is the map that we introduced in the Induced-Deck Transformation Proposition 62.9.

Example. As promised we can now determine the number n of boundary components of the covering X in the previous example. More precisely, denote by U, V, W the three boundary components of B. We make the following three observations:

(1) The image of $\pi_1(V)$ in $\mathbb{Z}_6$ is the subgroup generated by $\varphi(u) = 2$, thus the subgroup has index two, which by the Preimage-of-Covering Proposition 63.5 implies that $p^{-1}(U)$ consists of two components.

(2) The image of $\pi_1(W)$ in $\mathbb{Z}_6$ is the subgroup generated by $\varphi(v) = 3$, thus $p^{-1}(V)$ consists of three components.

(3) The image of $\pi_1(U)$ in $\mathbb{Z}_6$ is the subgroup generated by $\varphi(w) = -5$. Since -5 is coprime to 6 we see that this subgroup is all of $\mathbb{Z}_6$, thus $p^{-1}(W)$ consists of precisely one component.

Altogether we see that X has $2 + 3 + 1 = 6$ boundary components.

Remark. The reader who is in need of a brain teaser might want to come up with an appropriate generalization of the Preimage-of-Covering Proposition 63.5 to the case where we consider coverings of B coming from a non-normal subgroup.

Proof. To simplify the notation a little bit we assume that $a_0 = b_0$, in particular we ignore the role of γ. Throughout the proof we refer to the figure below for an illustration.

(1) Let W be a path-component of $p^{-1}(A)$. We pick a point $w_0 \in W$ with $p(w_0) = a_0$. We start out with the following slightly subtle claim.

Claim. We have the following equality:

$$p_*(\pi_1(X, w_0)) \;=\; \ker(\varphi\colon \pi_1(B, a_0) \to G).$$

Proof. We pick a path γ in X from x_0 to w_0. Since $p \circ \gamma$ is a loop in a_0 we can write $h := [p \circ \gamma] \in \pi_1(B, a_0)$. We consider the following diagram

$$\begin{array}{ccc}
\pi_1(X, w_0) & \xrightarrow[\cong]{\gamma_*} & \pi_1(X, x_0) \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\
\pi_1(B, a_0) & \xrightarrow[\cong]{g \mapsto hgh^{-1}} & \pi_1(B, a_0) \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\
G & \xrightarrow[\cong]{g \mapsto \varphi(h) g \varphi(h)^{-1}} & G.
\end{array}$$

The top square commutes by the Change-of-Base Point Proposition 47.10 (2) and (5). Evidently the bottom square also commutes. By definition of (X, x_0) we have

$$p_*(\pi_1(X, x_0)) \;=\; \ker(\varphi\colon \pi_1(B, a_0) \to G).$$

The claim now follows from the fact that the diagram commutes and that the horizontal maps are isomorphisms.

In the following we denote by $j\colon W \to X$ the inclusion map. We consider the following diagram

$$\begin{array}{ccc}
\pi_1(W, w_0) & \xrightarrow{j_*} & \pi_1(X, w_0) \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\
\pi_1(A, a_0) & \xrightarrow{i_*} & \pi_1(B, a_0) \\
& {\scriptstyle \psi}\searrow & \downarrow{\scriptstyle \varphi} \\
& & G.
\end{array}$$

Recall that we need to show the equality $p_*(\pi_1(W, w_0)) = \ker(\psi)$. We show the two inclusions separately.

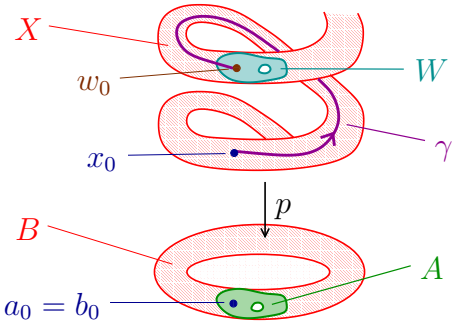
"$\subset$" This inclusion follows immediately from the definitions, from the commutativity of the above diagram and from the claim.

"$\supset$" Thus let $g \in \ker(\psi)$. We represent it by a loop γ in (A, a_0). By the above claim and Lemma 48.15 the loop $i \circ \gamma$ in (B, a_0) lifts to a loop $\widetilde{\gamma}$ in (X, w_0). By definition

the image of $\widetilde{\gamma}$ lies in $p^{-1}(A)$ and it contains w_0. In particular the image of $\widetilde{\gamma}$ lies in the path-component of $p^{-1}(A)$ that contains w_0, in other words, $\widetilde{\gamma}$ is actually a loop in (W, w_0). Since $g = p_*([\widetilde{\gamma}])$ we obtain that $g \in p_*(\pi_1(W, w_0))$.

(2) We write $n = |G|$ and $k = |\operatorname{im}(\psi\colon \pi_1(A, a_0) \to G)|$. It follows from Lemma 48.15 and (1) that each path-component of $p^{-1}(A)$ is a k-fold covering of A. By Lemma 48.15 the map $p\colon X \to B$ is an n-fold covering. Evidently $p\colon p^{-1}(A) \to A$ is also an n-fold covering. It follows almost immediately that $p^{-1}(A)$ has necessarily n/k path-components. But $n/k = |G|/|\operatorname{im}(\psi)|$ is by definition the degree of $\operatorname{im}(\psi)$ in G.

(3) This statement follows easily from the definition of the horizontal maps. We leave the details to the reader. ∎



63.5. **Covering spaces of surfaces.** Suppose we start out with a surface F of genus g and a finite index subgroup Γ of $\pi_1(F)$, then by the Covering Existence Theorem 61.5 we obtain a corresponding finite covering $p\colon \widetilde{F} \to F$. By the Manifold Covering Proposition 49.2 and Lemma 48.3 (4) the topological space $\widetilde{F}$ is again a closed orientable connected 2-dimensional smooth manifold. From the Surface Classification Theorem 55.4 we know that $\widetilde{F}$ is again a surface of some genus h. How can we determine h from g and Γ?
For future reference we record the question.

Question 63.6. Given a surface F of genus g and a finite index subgroup Γ of $\pi_1(F)$, what is the genus of the corresponding finite cover?

It will take a while till we can give a satisfactory answer. In fact we will obtain a complete answer only once we have introduced the Euler characteristic.

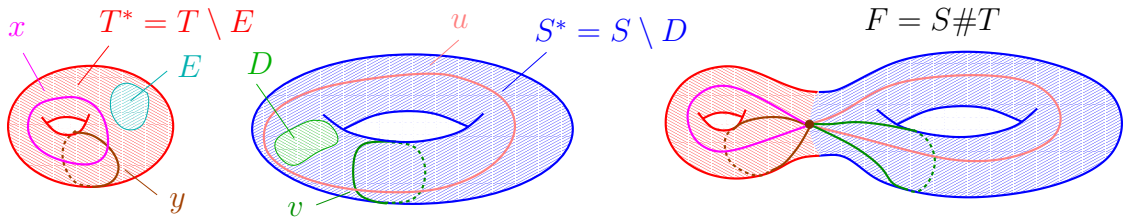
But we want to discuss one example in detail. We consider the surface F of genus two that we view as the connected sum of two tori $S = S^1 \times S^1$ and $T = S^1 \times S^1$. We denote by u, v the standard generators for $\pi_1(S)$ given by the two copies of S^1 in each torus. Similarly we denote by x, y the standard generators for $\pi_1(T)$. We pick an embedded closed disk in S and an embedded closed disk in T which both lie in the complement of the standard generators. We denote the corresponding open disks by D and E and we write $S^* = S \setminus D$ and $T^* \setminus E$. We build the connected sum $S\#T$ by gluing S^* to T^* along the boundary.[742] Furthermore, by a slight abuse of notation, we denote by u, v and x, y also the generators of $\pi_1(S^*) = \langle u, v\rangle$ and $\pi_1(T^*) = \langle x, y\rangle$.

[742]It follows from Corollary **??** that we can be a bit casual in our language, i.e. we can talk about "gluing surfaces" without specifying explicit diffeomorphisms of boundary components, as long as we take into account orientations.

By the discussion on page 1225 the abelianization of $\pi_1(S\#T) = \pi_1(S^* \cup_{\partial S^* = \partial T^*} T^*)$ is the free abelian group on u, v, x, y. Now we consider the epimorphism

$$\varphi\colon \pi_1(S\#T) \;\to\; \pi_1(S\#T)/[\pi_1(S\#T), \pi_1(S\#T)] \;\to\; \mathbb{Z}_3$$

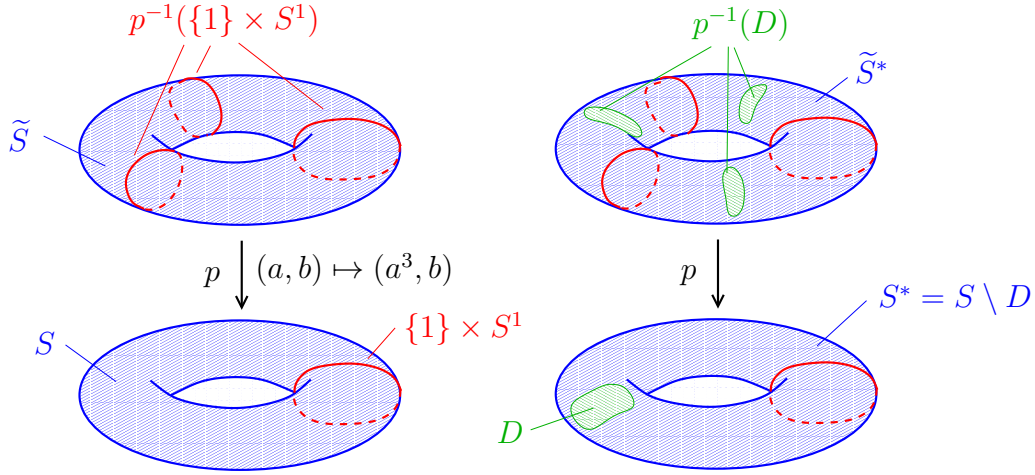
given by $u \mapsto 1, v \mapsto 0, x \mapsto 0$ and $y \mapsto 0$. Our goal in the following discussion is to visualize the covering $p\colon \widetilde{F} \to F = S\#T$ corresponding to $\ker(\varphi)$. Since φ is an epimorphism onto a group of order 3 we see that $\ker(\varphi)$ is a subgroup of index 3. By Lemma 48.15 this implies that p is a 3-fold covering.



We first consider the torus $S = S^1 \times S^1$ and furthermore we consider the epimorphism $\psi\colon \pi_1(S) = \langle u, v \mid [u,v]\rangle \to \mathbb{Z}_3$ given by $\psi(u) = 1$ and $\psi(v) = 0$. The covering corresponding to $\ker(\psi) \subset \pi_1(S)$ is equivalent to the covering[743]

$$\begin{array}{ccc} \widetilde{S} = S^1 \times S^1 & \to & S = S^1 \times S^1 \\ (a,b) & \mapsto & (a^3, b). \end{array}$$

Now we consider the epimorphism $\psi^*\colon \pi_1(S^*) \to \mathbb{Z}_3$ given by $\psi^*(u) = 1$ and $\psi^*(v) = 0$. By the argument of Footnote 743 the covering corresponding to $\ker(\psi^*)$ is then given by $p\colon \widetilde{S}^* := \widetilde{S} \setminus p^{-1}(D) \to S^* = S \setminus D$. Recall that we perform the connected sum operation



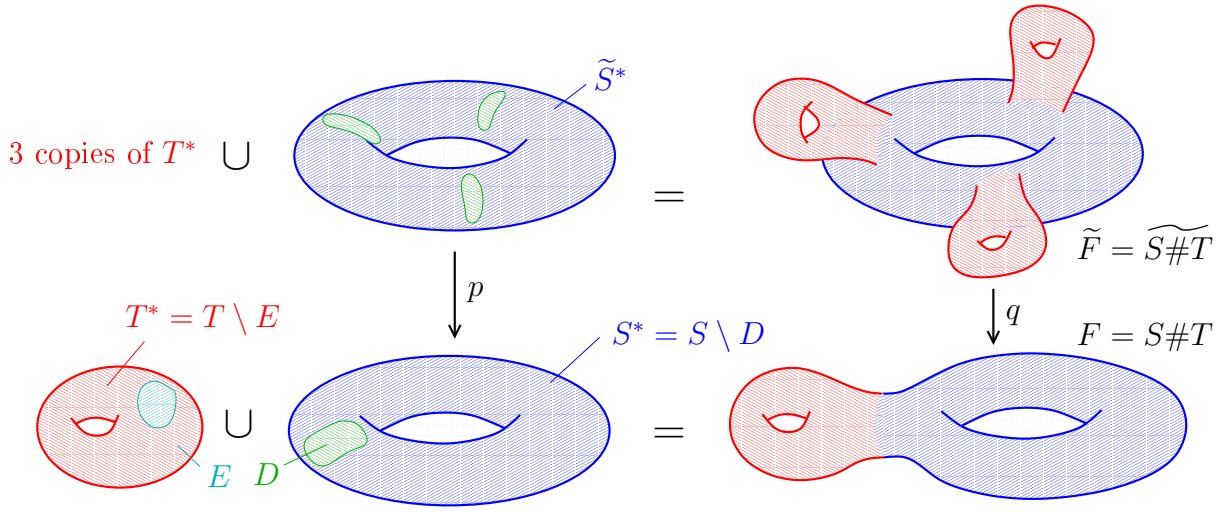
$S\#T$ by gluing $T^* := T \setminus E$ to $S^* = S \setminus D$ to obtain $F = S\#T$. The same way we can also glue three copies of T^*, i.e. $T^* \times \{1,2,3\}$, to the three boundary components of $\widetilde{S}^*$ to obtain a closed smooth manifold $\widetilde{F}$. It follows from the Pasting Proposition 2.6 $(2')$ that

[743] Indeed, since $\widetilde{S} \to S$ is a connected 3-fold covering we know from Lemma 48.15 that $p_*(\pi_1(\widetilde{S}))$ is a subgroup of $\pi_1(S)$ of index three. Furthermore it is contained in the subgroup $\ker\big(\pi_1(S) \xrightarrow{\psi} \mathbb{Z}_3\big)$ which also has index three, thus $p_*(\pi_1(\widetilde{S})) = \ker(\psi\colon \pi_1(S) \to \mathbb{Z}_3)$. But by the Covering Uniqueness Theorem 61.9 all path-connected coverings that correspond to the same subgroup are equivalent.

the map

$$\begin{array}{rcl} q\colon \widetilde{F} & \to & F \\ P & \mapsto & \begin{cases} p(P), & \text{if } P \in \widetilde{S}^*, \\ Q, & \text{if } P = Q \times \{n\} \text{ with } Q \in T^* \text{ and } n \in \{1,2,3\} \end{cases} \end{array}$$

is continuous. Similar to Footnote 743 one can now see that $q\colon \widetilde{F} \to F$ is indeed the covering of F corresponding to $\ker(\varphi)$. This construction is illustrated in the figure below which also shows that $\widetilde{F}$ is a surface of genus 4.



Exercise for Chapter 63.

Exercise 63.1. Show that there exist lattices Λ and $\widetilde{\Lambda}$ in $\mathbb{C}$ that are not homothetic, i.e. such that there exists no $r \in \mathbb{C} \setminus \{0\}$ such that $r \cdot \Lambda = \widetilde{\Lambda}$.

Exercise 63.2. We write $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

(a) Let $m \in \mathbb{N}$. Show that the map

$$\begin{array}{rcl} \mathbb{C}^* & \to & \mathbb{C}^* \\ z & \mapsto & z^m \end{array}$$

is a holomorphic map between complex manifolds and show that this map is a covering map with degree $= m$.

(b) Show that the exponential map

$$\begin{array}{rcl} \exp\colon \mathbb{C} & \to & \mathbb{C}^* := \mathbb{C} \setminus \{0\} \\ z & \mapsto & \exp(z) \end{array}$$

is a holomorphic map between complex manifolds and show that this map is a covering map of infinite degree.

(c) A *complex logarithm* on an open subset $U \subset \mathbb{C}^*$ is a map $f\colon U \to \mathbb{C}$ such that for all $z \in U$ we have $f(\exp(z)) = z$.

(i) Show that every complex logarithm is holomorphic.

(ii) Show that on any simply connected open subset $U \subset \mathbb{C}^*$ one can define a complex logarithm.

(iii) Show that it is not possible to define a complex logarithm on all of $\mathbb{C}^*$.

Exercise 63.3. Let Σ be a compact oriented connected 2-dimensional smooth manifold and let $f\colon \partial\Sigma \to \partial\Sigma$ be a diffeomorphism. Show that there exists a diffeomorphism $\Sigma \to \Sigma$ with $F|_{\partial\Sigma} = f$ if and only if f is either orientation-preserving or orientation-reversing.
Hint. Use the Boundary Curve-Permutation Lemma 55.14 , the Diffeotopies-of-S^1-Classification Proposition 34.5 and the Smooth Collar Neighborhood Theorem 28.3.

Exercise 63.4. Let $n \in \mathbb{N}_{\geq 3}$.

(a) We consider the connected sum $M = \mathbb{R}\mathrm{P}^n \# \mathbb{R}\mathrm{P}^n$. We consider the following homomorphism

$$\begin{array}{lcccccc} \varphi\colon \pi_1(\mathbb{R}\mathrm{P}^n \# \mathbb{R}\mathrm{P}^n) & \xrightarrow{\cong} & \pi_1(\mathbb{R}\mathrm{P}^n) * \pi_1(\mathbb{R}\mathrm{P}^n) & \xrightarrow{\cong} & \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a \,|\, a^2\rangle * \langle b \,|\, b^2\rangle & \to & \mathbb{Z}_2. \\ & & \uparrow & & \uparrow & & a \mapsto 1 \\ & & \text{Connected Sum-}\pi_1\text{-Proposition 52.12} & & \text{Corollary 48.18} & & b \mapsto 1 \end{array}$$

We denote by $\widetilde{M}$ the cover corresponding to $\ker(\varphi\colon \pi_1(M) \to \mathbb{Z}_2)$. Show that $\widetilde{M}$ is diffeomorphic to $S^1 \times S^{n-1}$.
Hint. Use the Preimage-of-Covering Proposition 63.5.

(b) Let $k \in \mathbb{N}$. We consider the connected sum M of k copies of $\mathbb{R}\mathrm{P}^n$. As in (a) we have an obvious epimorphism $\pi_1(M) \to \mathbb{Z}_2$. It turns out that the corresponding covering space $\widetilde{M}$ is diffeomorphic to the connected sum of m copies of $S^1 \times S^{n-1}$. Determine m in terms of k.

64. Coverings of topological graphs and applications to group theory

In this chapter we will apply the results obtained in Chapter 61 to prove several results in group theory.

64.1. **Coverings of topological graphs.** In our proofs of the promised group-theoretic results we will need the following proposition which is also of interest in its own right.

Proposition 64.1. (Covering-of-Graph Proposition) Let $G = (V, E, i, t)$ be a connected abstract graph and let $p\colon \widetilde{X} \to |G|$ be a covering of the topological realization $X := |G|$ of G. Then there exists a natural[744] abstract graph $\widetilde{G} = (\widetilde{V}, \widetilde{E}, i, t)$, a natural homeomorphism $f\colon |\widetilde{G}| \to \widetilde{X}$ and a natural map $q\colon \widetilde{G} \to G$ of abstract graphs with the following properties:

(1) We have $\#\widetilde{V} = [\widetilde{X} : X] \cdot \#V$ and $\#\widetilde{E} = [\widetilde{X} : X] \cdot \#E$.
(2) The following diagram commutes:

$$\begin{array}{ccc} |\widetilde{G}| & \xrightarrow{\ f\ } & \widetilde{X} \\ {\scriptstyle |q|}\downarrow & & \downarrow{\scriptstyle p} \\ |G| & \xrightarrow{\ \mathrm{id}\ } & X. \end{array}$$

Here $|q|$ denotes the topological realization of the map q, as defined in Lemma 7.2.

Remark. In the CW-complex Covering Proposition 68.37 we will generalize the above proposition to the setting of "CW-complexes".

Proof. Let $G = (V, E, i, t)$ be a connected abstract graph. Recall that on page 308 we defined the topological realization of G as follows:

$$|G| \ := \ \big(V \ \sqcup \ (E \times [0,1])\big)/\!\sim \quad \text{where for } e \in E \text{ we have } (e,0) \sim i(e) \text{ and } (e,1) \sim t(e).$$

Note that by the Abstract Graph-Topological Realization Lemma 7.1 we know that $|G|$ is connected. Given $e \in E$ we set $\mu(e) := [(e, \frac{1}{2})]$, i.e. $\mu(e)$ is the "midpoint" of e. Furthermore, given $e \in E$ we denote by $\gamma_e\colon [0,1] \to X$ the path given by $t \mapsto [(e,t)]$.

Note that in Exercise 50.14 we showed that $X = |G|$ is locally contractible, in particular it is locally path-connected. Thus in the following discussion we will have no troubles with applying the results from Chapter 61.

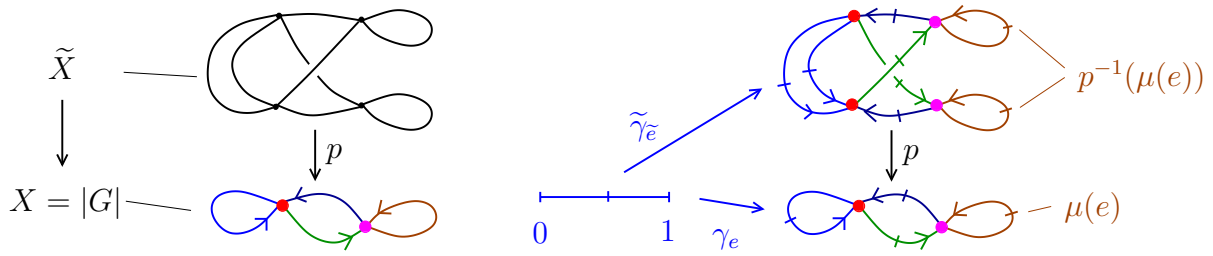
Now let $p\colon \widetilde{X} \to |G|$ be a covering of the topological realization $X = |G|$ of G. We set $\widetilde{V} := p^{-1}(V) \subset \widetilde{X}$ and we set $\widetilde{E} := p^{-1}(\{\mu(e) \,|\, e \in E\}) \subset \widetilde{X}$. Since $X = |G|$ is connected we know that the degree $[\widetilde{X} : X]$ is defined. By definition, or if you want, by Lemma 48.1, we have $\#\widetilde{V} = [\widetilde{X} : X] \cdot \#V$ and $\#\widetilde{E} = [\widetilde{X} : X] \cdot \#E$.

Now let $\widetilde{e} \in \widetilde{E}$. We denote by $\varphi(\widetilde{e})$ the unique edge in E with $p(\widetilde{e}) = \mu(\varphi(\widetilde{e}))$. By the Map Lifting Criterion 61.2 there exists a unique lift $\widetilde{\gamma}_{\widetilde{e}}\colon [0,1] \to \widetilde{X}$ of $\gamma_{\varphi(\widetilde{e})}\colon [0,1] \to X$ with $\widetilde{\gamma}(\frac{1}{2}) = \widetilde{e}$. We set $i(\widetilde{e}) = \widetilde{\gamma}(0)$ and $t(\widetilde{e}) = \widetilde{\gamma}(1)$. Since $\widetilde{\gamma}_{\widetilde{e}}$ is a lift of γ_e we see that $i(\widetilde{e})$ and $t(\widetilde{e})$ actually lie in $\widetilde{V}$. In other words, $(\widetilde{V}, \widetilde{E}, i, t)$ is actually an abstract graph. Furthermore note that the two maps $p\colon \widetilde{V} \to V$ and $\varphi\colon \widetilde{E} \to E$ define by, basically by construction, a map $q\colon \widetilde{G} = (\widetilde{V}, \widetilde{E}, i, t) \to (V, E, i, t) = G$ of abstract graphs.

[744] We leave the pleasure of figuring out what "natural" means to the reader.

Next we define $f\colon |\widetilde{G}| \to \widetilde{X}$ via the inclusion $\widetilde{V} \to \widetilde{X}$ and by sending $(\widetilde{e}, t)$ to $\widetilde{\gamma}_{\widetilde{e}}(t)$. It follows immediately from the Topological-Quotient Proposition 5.15 (1b) that f is continuous. Furthermore, using the existence and uniqueness statement of the Map Lifting Criterion 61.2 it is not difficult to see that the map f is actually a bijection. Next note that, with a little bit of an effort, one can also show that f is an open map. Thus we obtain from the Open -Injective Map Lemma 2.16 that f is actually a homeomorphism.

Finally note that it is straightforward to see that the given diagram in the proposition does indeed commute. ■



64.2. **Fundamental group of countable topological graphs.** In the Fundamental Group-of-Graph Proposition 52.5 we showed that the fundamental group of any finite topological graph is a free group. We generalize this result to countable topological graphs.

Proposition 64.2. If X is a countable connected non-empty topological graph, then $\pi_1(X)$ is a free group.

Remark. In the Fundamental Group-of-1-Complex Theorem 69.9 we will show that the fundamental of any (not necessarily countable) connected topological graph is a free group. Since the proof of this more general statement is technically rather demanding we feel that it is worth to prove this intermediate statement.

The proof of Proposition 64.2 builds on the following reasonably straightforward lemma.

Lemma 64.3. Let H be a finite connected graph and let G be a connected non-empty subgraph of H. Let v be a vertex of G. We denote by $i\colon |G| \to |H|$ the inclusion. We set $s := \chi(H) - \chi(G)$. Then there exists a homomorphism $\varphi\colon \langle x_1, \ldots, x_s\rangle \to \pi_1(|H|, v)$ such that the map

$$i_* * \varphi\colon \pi_1(|G|, v) * \langle x_1, \ldots, x_s\rangle \;\to\; \pi_1(|H|, v)$$

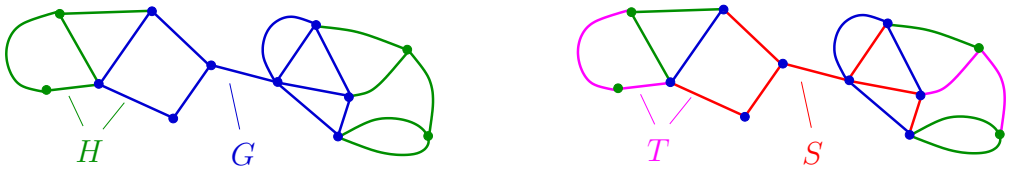
is an isomorphism.

Proof of Lemma 64.3. Let $H = (W, F, i, t)$ be a finite connected graph. Furthermore let $G = (V, E, i, t)$ be a connected non-empty subgraph of H. By the Spanning Tree Existence Proposition 7.10 (1) there exists a spanning tree S for G. By a rather elementary argument, which we outsourced to Exercise 64.1, there exists a spanning tree T of H such that $T \cap G = S$. Let $e_1, \ldots, e_m$ be the edges of G not contained in S and let $e_{m+1}, \ldots, e_n$ be the edges of H not contained in T. For $j = 1, \ldots, m$ we pick a path α_j in $|S|$ from v to $i(e_j)$, we denote by β_j the obvious path form $i(e_j)$ to $t(e_j)$ and we pick a path γ_j in $|S|$ from $t(e_j)$ to v. We proceed analogously for $j = m+1, \ldots, n$, except that now can no longer demand that the paths α_j and γ_j lie in $|S|$, but we will demand that they lie in $|T|$.

Finally we consider the following diagram:

$$\begin{array}{ccc} & & \pi_1(|G|,v) * \langle y_{m+1},\dots,y_{m+n}\rangle \\ & \overset{\Theta_G * \mathrm{id}}{\nearrow} & \searrow \\ \langle y_1,\dots,y_m\rangle * \langle y_{m+1},\dots,y_{m+n}\rangle & \xrightarrow{\qquad\qquad\Theta_H\qquad\qquad} & \pi_1(|H|,v). \end{array}$$

Here the homomorphisms Θ_G and Θ_H are the maps induced by $y_j \mapsto [\alpha_j * \beta_j * \gamma_j]$. By the Fundamental Group-of-Graph Proposition 52.5 we know that both Θ_G and Θ_H are isomorphisms. This shows that the homomorphism $\langle x_1,\dots,x_{n-m}\rangle \to \pi_1(|H|,v)$ given by $x_j \mapsto \Theta_H(y_{m+j})$ has the desired properties. Finally one can easily verify that $\chi(G) = 1+m$ and $\chi(H) = 1+n$. Thus we see that $n-m = \chi(H) - \chi(G)$. ■



Proof of Proposition 64.2. Let $G = (V,E,i,t)$ be a countable connected non-empty abstract graph. We pick a vertex $v \in V$. We need to show that $\pi_1(G,v)$ is a free group. By Exercise 7.8 we know that there exists a sequence $\{G_j\}_{j\in\mathbb{N}}$ of subgraphs with the following properties:

(1) Each G_j is finite connected.
(2) Each G_j contains v.
(3) We have $G = \bigcup\limits_{i\in\mathbb{N}} G_j$ in the sense that $V = \bigcup\limits_{j\in\mathbb{N}} V_j$ and $E = \bigcup\limits_{j\in\mathbb{N}} G_j$.

We see that

$$\pi_1(|G|,v) \;=\; \pi_1\Big(\bigcup_{j\in\mathbb{N}} |G_j|, v\Big) \;\underset{\uparrow}{=}\; \varinjlim \pi_1(|G_j|,v) \;\underset{\uparrow}{=}\; \text{free group.}$$

by Exercise 7.5 and the Direct Limit-π_1-Proposition 56.7

follows from Lemma 64.3 and discussion on page 1259

■

64.3. **Subgroups of free groups.** In this section we will apply our topological results to the study of free groups. First we introduce the following definition.

Definition. Given a free group F, i.e. given a group F that is isomorphic to the free group $\langle S\rangle$ generated by some set S, we refer to the cardinality of S as the rank of F.[745]

The following proposition is arguably the most interesting result of this chapter. The statement is purely group-theoretical, but as we will see, a topological point of view leads to a very neat proof.

Proposition 64.4. Let F be a free group of countable rank.

(1) Every subgroup of F is again a free group.
(2) If F is finitely generated and if G is a finite-index subgroup of F, then[746]

$$\mathrm{rank}(G) - 1 \;=\; [F:G]\cdot(\mathrm{rank}(F)-1).$$

[745] It follows from the discussions on the pages 1206 and 1149 that the rank of F is well-defined.

Examples.

(1) If $F = \mathbb{Z}$ then the formula from Proposition 64.4 does indeed show that every finite-index subgroup of $\mathbb{Z}$ is again a free group of rank one, i.e. every finite-index subgroup of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$.

(2) Let $\alpha\colon F := \langle a, b\rangle \to \mathbb{Z}_3$ be the epimorphism defined given by $\alpha(a) = 1$ and $\alpha(b) = 0$. It follows from Proposition 64.4 that the kernel $G := \ker(\alpha)$ is a free group of rank $[F : G] \cdot (\mathrm{rank}(F) - 1) + 1 = 3 \cdot 1 + 1 = 4$. In Exercise 64.2 we will determine four elements of $G = \ker(\alpha)$ that are generators for this free group.

Remark.

(1) In the Subgroup-of-Free Group Corollary 69.10 we will extend the statement to all free groups, i.e. we will drop the condition that the rank should be countable. The idea of the proof is fairly similar to the one used below, but the proof of the more general statement is overall "technologically" somewhat more fancy.

(2) There are purely group theoretic proofs of Proposition 64.4 and of the Subgroup-of-Free Group Corollary 69.10, see e.g. [**Schre1927**], [**HallM1976**, Theorem 7.2.1] or [**Bau1993**, Chapter III.3], but the topological proof that we had just provided is arguably conceptually the clearest proof.

(3) the Subgroup-of-Free Group Corollary 69.10 and Proposition 64.4 are a special case of the *Kurosh Subgroup Theorem*. This theorem says that if G is the product of groups $A_i, i \in I$ and if H is a subgroup of G, then H is a free product of a free group and groups $B_i, i \in I$ where each B_i is a (potentially trivial) group that is conjugate to a subgroup of A_i. We refer to [**LyS1977**, Theorem IV.1.10] for details and a proof.

Proof. Let $F = \langle E\rangle$ be a free group on a countable generating set E and let G be a subgroup of F. Note that it follows easily from the Countability Lemma 0.8 that G is a countable set. Let $\Gamma = (\{x_0\}, E, i, t)$ be the abstract graph with one vertex and one edge for each element of E. By the discussion on page 1133 we know that the topological realization $X := |G|$ is homeomorphic to the wedge $\bigvee_{e\in E} S^1_e$. It follows from the Fundamental Group-Wedge-of-Circles Lemma 52.4 that there exists an isomorphism $\varphi\colon F = \langle E\rangle \to \pi_1(X, x_0)$. By the CW-Complex Properties Proposition 68.10 (6) and the Covering Existence Theorem 61.5 there exists a path-connected covering $p\colon (Y, y_0) \to (X, x_0)$ of degree $[F : G]$ such that $p_*(\pi_1(Y, y_0)) = G$. By Corollary 48.14 the map p_* is injective. Thus we see that $\pi_1(Y, y_0)$ itself is isomorphic to G. By the Covering-of-Graph Proposition 64.1 we can equip Y with the structure of a topological graph. Since E and G are countable we see that Y is a countable topological graph. We deduce from Proposition 64.2 that $G \cong \pi_1(Y, y_0)$ is a free group. This completes the proof of (1).

Now suppose that F is finitely generated and that G is a finite-index subgroup of F. We see that we have the following equalities:

$$\underbrace{\mathrm{rank}(\pi_1(Y, y_0))}_{=\mathrm{rank}(G)} - 1 \overset{\text{Fundamental Group-of-Graph Proposition 52.5 (2)}}{\overset{\downarrow}{=}} \chi(Y) \overset{\text{Covering-of-Graph Proposition 64.1}}{=\!\!\uparrow} [Y : X] \cdot \chi(X) \overset{\downarrow}{=} [F : G] \cdot (\underbrace{\mathrm{rank}(\pi_1(X, x_0))}_{=\mathrm{rank}(F)} - 1).$$

[746] Does the equality hold if G is infinite-index and if we interpret $[F : G] \cdot (\mathrm{rank}(F) - 1)$, as usual, as $\infty \cdot (\mathrm{rank}(F) - 1) = \infty$ for $F \ncong \mathbb{Z}$?

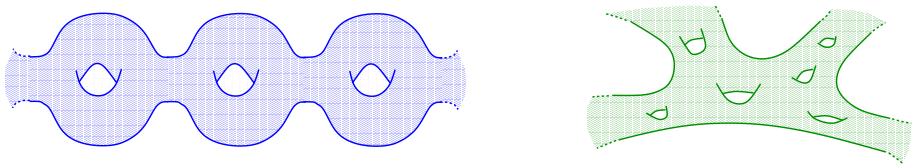
This completes the proof of (2). ■

64.4. **Subgroups of surface groups.** We start out with the following proposition that is similar in spirit to Proposition 64.2, but potentially it is more surprising.

Proposition 64.5. If M is a connected 2-dimensional smooth manifold that is non-compact, then $\pi_1(M)$ is a free group.

Remark. A somewhat different proof of Proposition 64.5 is given in [**Sti1993**, p. 142].

Example. The 2-dimensional smooth manifolds shown in the figure below are non-compact and thus their fundamental groups are free. For the example on the left we saw this already in Lemma 56.11.



Proof. Let M be a connected 2-dimensional smooth manifold that is non-compact. In Corollary 75.7 we will show, using the Smooth Collar Neighborhood Theorem 28.3, that M is homotopy equivalent to a smooth manifold with empty boundary. So we might as well assume that M has no boundary.

By the Compact Submanifold Exhaustion Proposition 31.2 we know that there exists a sequence $X_1, X_2, \ldots$ of n-dimensional smooth submanifolds of M with the following four properties:

(1) The sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $X_i \subset X_{i+1}$.
(2) Each X_i is compact and connected.
(3) The X_i are submanifolds in the sense of the definition on page 524.
(4) We have $\bigcup_{i\in\mathbb{N}} \overset{\circ}{X}_i = M$.

We pick an embedding $\varphi\colon \overline{B}^2 \to X_1 \backslash \partial X_1$ and given any $i \in \mathbb{N}$ we set $X_i' := X_i \backslash \varphi(B^2)$. With a moderate effort we see, by iteratively applying the Surface Classification Theorem 55.4 and Lemma 55.12, that there exists a sequence of finite topological graphs $\{G_i\}_{i\in I}$ with the following properties:

(5) G_1 contains a point $v \in \varphi(S^1)$.
(6) For each $i \in \mathbb{N}$ we have $G_i \subset G_{i+1}$.
(7) Each G_i is a deformation retract of X_i'.

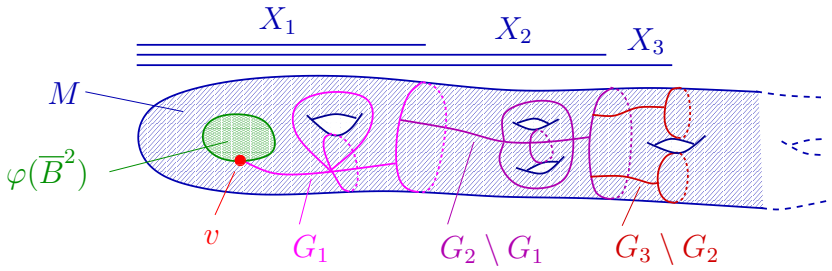
We see that

$$\begin{array}{ccccc} \pi_1(M \setminus \varphi(B^2), v) & = & \pi_1\Big(\bigcup_{j\in\mathbb{N}} X_j', v\Big) & \overset{\text{follows from (4) and the Direct Limit-}\pi_1\text{-Proposition 56.7}}{=} & \varinjlim \pi_1(X_j', v) \\ & = & \varinjlim \pi_1(|G_j|, v) & = & \pi_1(G_1, v) * \text{free group.} \\ & \overset{\text{by (7) and the Homotopy-}\pi_1\text{-Proposition 50.3}}{\uparrow} & & \overset{\text{follows from Lemma 64.3 and the discussion on page 1259}}{\uparrow} & \end{array}$$

Now we can perform the following calculation:

$$\begin{array}{rcl} \pi_1(M,v) & = & \pi_1((M \setminus \varphi(B^2)) \cup_C \varphi(\overline{B}^2), v) \overset{\text{Seifert–van Kampen Theorem 54.4}}{\cong} \pi_1(M \setminus \varphi(B^2), v) *_{\pi_1(C,v)} \overbrace{\pi_1(\varphi(\overline{B}^2), v)}^{=\{e\}} \\ & \cong & (\pi_1(G_1, v) * \text{free group})/\langle\langle \pi_1(C,v) \rangle\rangle \overset{\text{by Lemma 55.12 we know that we can write } \pi_1(G_1,v) = \pi_1(C,v) * \text{free group}}{=} \text{free group}. \\ & & \uparrow \text{by the above calculation of } \pi_1(M \setminus \varphi(B^2), v) \text{ together with the Amalgamated Product Lemma 53.22 (2)} \end{array}$$

■



Next we address the following question.

Question 54.8. Let π be the fundamental group of the surface of genus $g \geq 2$. What isomorphism types of subgroups can appear? Is every subgroup again a free group or the fundamental group of a surface?

The following proposition gives an affirmative answer.

Proposition 64.6. (Subgroups-of-Surface Groups Proposition) Let M be the surface of genus g and let $\Gamma \subset \pi_1(M)$ be a subgroup.

(1) If the index $[\pi : \Gamma]$ is finite, then Γ is the fundamental group of the surface of genus $[\pi : \Gamma] \cdot (g-1) + 1$.
(2) If the index $[\pi : \Gamma]$ is infinite, then Γ is a free group.

Remark. In Exercise 64.6 we will consider the case of subgroups of the fundamental group of a *non-orientable* surface of some genus $g \in \mathbb{N}$.

Proof. Let $g \in \mathbb{N}_{\geq 2}$ and let Σ be the surface of genus g. We pick a base point $x_0 \in \Sigma$. Let $\Gamma \subset \pi_1(\Sigma, x_0)$ be a subgroup. We obtain from the CW-Complex Properties Proposition 68.10 (6) and the Covering Existence Theorem 61.5 that there exists a path-connected covering $p\colon (\widetilde{\Sigma}, \widetilde{x}_0) \to (\Sigma, x_0)$ of index $[\pi_1(M, x_0) : \Gamma]$ with $p_*(\pi_1(\Sigma, \widetilde{x}_0)) = \Gamma$. By the Manifold Covering Proposition 49.2 we know that $\widetilde{\Sigma}$ is an orientable 2-dimensional smooth manifold with empty boundary.

(1) If the index $[\pi : \Gamma]$ is finite, then we know by Lemma 48.3 (4) that $\widetilde{\Sigma}$ is again compact. This implies that we now know that $\widetilde{\Sigma}$ is a closed 2-dimensional smooth manifold. In Exercise 64.6 we will show that the genus is given by the slightly weird formula
(2) By Lemma 48.3 (5) we know that $\widetilde{\Sigma}$ is non-compact. Thus it follows from Proposition 64.5 that $\Gamma \cong \pi_1(\widetilde{\Sigma})$ is a free group. ■

Note that the following lemma gives an affirmative answer to Question 54.11 (1).

Corollary 64.7. For every $g \in \mathbb{N}$ the fundamental group of the surface of genus g is torsion-free.

Proof. We will prove the corollary to the Subgroups-of-Surface Groups Proposition 64.6 in Exercise 64.4. ■

We conclude this section with the following proposition which is closely related to Proposition 53.14. We will not prove the proposition, but we will it use at one occasion later on.

Proposition 64.8. Let M be a surface of genus ≥ 2. If $\Gamma \subset \pi_1(M)$ is a *normal* non-trivial subgroup of infinite-index, then $\pi_1(M)$ is a free group of *infinite* rank.

Proof. Let M be a surface of genus ≥ 2 and let $\Gamma \subset \pi_1(M)$ be a *normal* non-trivial subgroup of infinite-index. By the Subgroups-of-Surface Groups Proposition 64.6 it remains to show that Γ is infinitely generated. A proof for this statement is given in [**Cat2003**, Lemma 3.4], alternatively one can prove the second statement using [**Lü1997**, Theorem 0.7]. A significantly stronger statement is proved in [**BrH2007**, Theorem 1]. ■

64.5. **Residually finite groups and Hopfian groups.** Let (X, x_0) be a pointed topological space that is path-connected, locally path-connected and semi-locally simply connected. In the Covering Uniqueness Theorem 61.9 we saw that there exists a bijection between the set of subgroups of $\pi_1(X, x_0)$ and the set of equivalence classes of path-connected coverings of (X, x_0).
Let (X, x_0) be a pointed topological space that is path-connected, locally path-connected and semi-locally simply connected that is compact. It follows from the above, that given a (finite-index) subgroup of $\pi_1(X, x_0)$ we can associate to (X, x_0) a (finite) covering $p\colon \widetilde{X} \to X$. For the most part we like to stick to the world of compact spaces X. By Lemma 48.3 this means that we are interested in finite-index subgroups of $\pi_1(X, x_0)$. This leads us to the following, purposefully vague question.

Question 64.9. Which (finitely generated) groups have lots of finite-index subgroups?

In this section we will try to address Question 64.9. The following definition, which we already stated in Exercise 51.15, gives one popular way to turn Question 64.9 into a proper mathematical question.

Definition. A group π is called residually finite if given any non-trivial element $g \in \pi$ there exists a homomorphism $\alpha\colon \pi \to G$ to a finite group G such that $\alpha(g)$ is non-trivial.

Examples.

(1) The group $\mathbb{Z}$ is residually finite, indeed given $g \in \mathbb{Z}$ we can consider the epimorphism $\alpha\colon \mathbb{Z} \to \mathbb{Z}/(|g|+1)\cdot\mathbb{Z}$ onto a finite group, which evidently satisfies $\alpha(g) \neq 0$.
(2) It follows easily from the Finitely Generated Abelian Group Classification Theorem 51.4, a slight generalization of (1), that every finitely generated abelian group is residually finite.
(3) In Exercise 64.12 we will see that the group $(\mathbb{Q}, +)$ is not residually finite.

The following is now a meaningful interpretation of Question 64.9.

Question 64.10. Which (finitely generated) groups are residually finite?

Before we address Question 64.10 let us heighten the interest in Question 64.10 by stating a surprising property of residually finite groups.

Definition. A group π is called Hopfian[747] if every epimorphism $\varphi\colon \pi \to \pi$ is in fact a monomorphism.

Example.

(1) It follows immediately from Lemma 51.9 (5) that every finitely generated abelian group is Hopfian.
(2) Note that in contrast to (1), the *infinitely generated* free abelian group $\mathbb{Z}^{(\mathbb{N})}$ is actually not Hopfian. Indeed, the homomorphism $\varphi\colon \mathbb{Z}^{(\mathbb{N})} \to \mathbb{Z}^{(\mathbb{N})}$ which is given by $\varphi(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$ is an epimorphism, but evidently it is not a monomorphism.

It is self-evident that being Hopfian is a convenient property. As we will see later in the Hopf Conjecture 101.13, the notion of a Hopfian group also naturally arises in topology. The following proposition relates the two concepts that we just introduced.

Proposition 64.11. (Residually Finite-Hopfian Proposition) Let π be a finitely generated group. If π is residually finite, then it is Hopfian.

Remark. Shortly we will see that the converse to the Residually Finite-Hopfian Proposition 64.11 does not hold.

Proof. Let π be a finitely generated group that is residually finite. Let $\varphi\colon \pi \to \pi$ be an epimorphism. Now we assume that φ is not a monomorphism. We pick a non-trivial element $g \neq e \in \ker(\varphi)$. Since π is residually finite there exists a homomorphism $\alpha\colon \pi \to G$ onto a finite group G such that $\alpha(g) \neq e$.

Since π is finitely generated we know by the rather elementary Exercise 53.3 that the set $\mathrm{Hom}(\pi, G)$ of homomorphisms from π to G is finite. We set $n := \#\,\mathrm{Hom}(\pi, G)$ and we denote by $\beta_1, \dots, \beta_n\colon \pi \to G$ the distinct homomorphisms.

Next we consider the homomorphisms $\beta_1 \circ \varphi, \dots, \beta_n \circ \varphi\colon \pi \to G$. Since φ is an epimorphism we see that these homomorphisms are also distinct. But note they all satisfy $(\beta_i \circ \varphi)(g) = \beta_i(\varphi(g)) = \beta_i(e) = e$. So these n homomorphisms are different from our initial homomorphism $\alpha\colon \pi \to G$. This means that we have found $n+1$ distinct homomorphisms from $\pi \to G$. This is a contradiction to the definition of n. ∎

Now that we are fully convinced of the value of residually finite groups, let us explore this class of groups in greater detail. The following proposition gives a convenient characterization of residually finite groups.

Proposition 64.12. Let π be a group. The following two statements are equivalent.

(1) The group is residually finite.
(2) Given any non-trivial element $g \in \pi$ there exists a finite-index subgroup Γ with $g \notin \Gamma$.

If π is countable, then the above are also equivalent to the following statement:

(3) There exists a sequence $\pi \supset \pi_0 \supset \pi_1 \supset \pi_2 \supset \dots$ of finite index normal subgroups of π such that $\bigcap_{i \in \mathbb{N}} \pi_i = \{e\}$.

The proof of Proposition 64.12 rests on the following group-theoretic concept.

[747] This notion is once again named after Heinz Hopf.

Definition. Let G be a group and let H be a subgroup. We refer to

$$H^c := \bigcap_{g\in G} gHg^{-1}$$

as the normal core of G.

The following lemma is the key ingredient for the proof of Proposition 64.12. But note that the lemma is also interesting in its own right.

Lemma 64.13. (Normal Core Lemma) Let G be a group and let H be a subgroup.

(1) The normal core of H is a normal subgroup of G.
(2) If X is a subgroup of G that is contained in H and that is normal in G, then $X \subset H^c$.
(3) If H is of finite-index d in G, then H^c is a subgroup of G of index at most $d!$, in particular H^c is also of finite-index in G.

Proof of the Normal Core Lemma 64.13.

(1) Let $k \in G$. We have

$$kH^ck^{-1} = \bigcap_{g\in G} k\cdot gHg^{-1}\cdot k^{-1} = \bigcap_{g\in G}(kg)H(kg)^{-1} = \bigcap_{g\in G} gHg^{-1} = H^c.$$

(2) Let X be a subgroup of G that is contained in H and that is normal in G. Then

$$\begin{array}{ccccc} X & = & \bigcap_{g\in G} gXg^{-1} & \subset & \bigcap_{g\in G} gHg^{-1} = H^c. \\ & & \uparrow & & \uparrow \\ & & \text{since } X \text{ normal in } G & & \text{since } X\subset H \end{array}$$

(3) The proof of the last statement is arguably more interesting. Thus let H be a subgroup of index $d \in \mathbb{N}$ of G. We consider the set G/H of right coset of G with respect to H. We denote by $\mathrm{Perm}(G/H)$ the group of permutations of G/H, i.e. it is the group of all bijections of the set G/H. Note that G/H and thus also $\mathrm{Perm}(G/H)$ are finite. We consider the map

$$\begin{array}{rcl} \varphi\colon G & \mapsto & \mathrm{Perm}(G/H) \\ g & \mapsto & \begin{pmatrix} G/H & \to & G/H \\ xH & \mapsto & gxH \end{pmatrix}. \end{array}$$

This map is easily seen to be a homomorphism. Since $\mathrm{Perm}(G/H)$ is finite we see that $\ker(\varphi)$ is a finite-index normal subgroup of G. We claim that $\ker(\varphi)\subset H$. Thus let $g\in\ker(\varphi)$. This implies in particular that $\varphi(g)(H)=H$, but that means that $gH=H$, i.e. that $g\in H$. We have thus shown that indeed $\ker(\varphi)\subset H$. From (2) and the above observation that $\ker(\varphi)$ is normal in G we obtain that $\ker(\varphi)\subset H^c$. But since the index of $\ker(\varphi)$ in G is at most $d!$ we see that the index of the normal H^c in G is also at most $d!$. ∎

Proof of Proposition 64.12. Let π be a group.

(1)⇒(2) Let $g\in\pi$ be a non-trivial element. Since π is residually finite there exists a homomorphism $\alpha\colon \pi\to G$ to some finite group G such that $\alpha(g)$ is non-trivial. The kernel $\Gamma := \ker(\alpha)$ is a finite-index subgroup of π with $g\notin\Gamma$.

(2)⇒(1) Let $g\in\pi$ be a non-trivial element and let Γ be a finite-index subgroup of π with $g\notin\Gamma$. By the Normal Core Lemma 64.13 we know that the normal core

Γ^c is a finite-index normal subgroup of π. Thus we see that the homomorphism $\alpha\colon \pi \to \pi/\Gamma^c$ is an epimorphism onto a finite group with $\gamma(g) \neq e$.

(3)⇒(1) This statement is trivial. Note that here we do not need that π is countable.

(1)⇒(3) We outsource the straightforward proof to Exercise 64.9. ■

The following lemma gives a topological criterion for a (fundamental) group to be residually finite.

Lemma 64.14. Let (X, x_0) be a pointed topological space that is path-connected, locally path-connected and semi-locally simply connected. The following two statements are equivalent:

(1) The group $\pi_1(X, x_0)$ is residually finite.

(2) Given any loop $\gamma\colon [0,1] \to X$ in (X, x_0) that represents a non-trivial element in $\pi_1(X, x_0)$ there exists a finite covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that the corresponding lift $\widetilde{\gamma}\colon [0,1] \to \widetilde{X}$ is not a loop.

Proof. Let (X, x_0) be a topological space that is path-connected, locally path-connected and semi-locally simply connected. First we prove the (1)⇒(2)-implication. Thus we assume that the group $\pi_1(X, x_0)$ is residually finite. Let $\gamma\colon [0,1] \to X$ be a loop in (X, x_0) that represents a non-trivial element in $\pi_1(X, x_0)$. By Proposition 64.12 there exists a finite-index subgroup Γ of $\pi_1(X, x_0)$ such that $[\gamma] \notin \Gamma$. Since X is locally path-connected and semi-locally simply connected we know by the Covering Existence Theorem 61.5 that there exists a covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ of X of degree $[\pi : \Gamma]$ with $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = \Gamma$. By Lemma 48.15 we know that the corresponding lift $\widetilde{\gamma}\colon [0,1] \to \widetilde{X}$ is not a loop.

Now we prove the (2)⇒(1)-implication. Let $g \in \pi_1(X, x_0)$ be a non-trivial element. We pick a representative loop $\gamma\colon [0,1] \to (X, x_0)$. By hypothesis there exists a finite covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that the corresponding lift $\widetilde{\gamma}\colon [0,1] \to \widetilde{X}$ is not a loop. By Lemma 48.15 we know that $g = [\gamma]$ does not lie in $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ and that this is a finite-index subgroup of $\pi_1(X, x_0)$. It follows from Proposition 64.12 that $\pi_1(X, x_0)$ is residually finite. ■

Proposition 64.15. (Free Groups-are-Residually Finite Proposition) Every free group is residually finite.

In the following we will provide two proofs of the Free Groups-are-Residually Finite Proposition 64.15. The first is algebraic, very short and elegant. The other proof is topological in nature and great fun. The eagle-eyed reader will notice that the basic ideas of the two proofs are actually not that dissimilar.

Remark. In Exercise 51.15 we gave a third proof for the Free Groups-are-Residually Finite Proposition 64.15. Furthermore, a completely different proof of the Free Groups-are-Residually Finite Proposition 64.15, based again on topology, is given in [**Hem1972**].

First proof of the Free Groups-are-Residually Finite Proposition 64.15. Let X be a set and let $g \in \langle X \rangle$ be a non-trivial element in the free group generated by the set X. We can and will write g as a product $g = y_1^{\epsilon_1} \cdot \cdots \cdot y_k^{\epsilon_k}$ with $y_1, \ldots, y_k \in X$ and $\epsilon_1, \ldots, \epsilon_k \in \{-1, 1\}$ such that the following condition is satisfied:

(∗) For every $i \in \{1, \ldots, k-1\}$ we have $(y_i, \epsilon_i) \neq (y_{i+1}, -\epsilon_{i+1})$.

It follows easily from $(*)$ that for every $x \in X$ we can pick a permutation $\sigma_x \in S_{k+1}$ with the following two properties:

(1) For every $i \in \{1, \ldots, k\}$ with $x = y_i$ and $\epsilon_i = 1$ we have $\sigma_x(i+1) = i$.
(2) For every $i \in \{1, \ldots, k\}$ with $x = y_i$ and $\epsilon_i = -1$ we have $\sigma_x(i) = i+1$. In other words, we have $\sigma_x^{-1}(i+1) = i$.

By Lemma 51.15 there exists a unique homomorphism $\varphi\colon \langle X \rangle \to S_{k+1}$ such that $\varphi(x) = \sigma_x$ for every $x \in X$. We claim that $\varphi(g) \in S_{k+1}$ is a non-trivial permutation. In fact we have

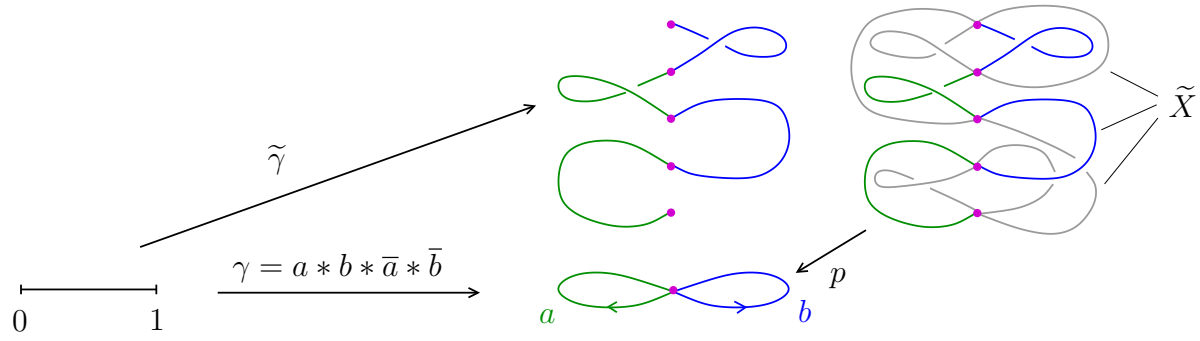
$$\varphi(g)(k+1) = \varphi(y_1^{\epsilon_1} \cdot \ldots \cdot y_k^{\epsilon_k})(k+1) = \varphi(y_1)^{\epsilon_1}(\ldots(\underbrace{\varphi(y_k)^{\epsilon_k}(k+1)}_{=k})\ldots) = \ldots = \varphi(y_1)^{\epsilon_1}(2) = 1.$$

This shows that the permutation $\varphi(g) \in S_{k+1}$ sends $k+1$ to 1, in other words, it is a non-trivial permutation. ■

Sketch of second proof of the Free Groups-are-Residually Finite Proposition 64.15. For simplicity we only provide the proof that the free group on two generators is residually finite. Recall that by the Fundamental Group-Wedge-of-Circles Lemma 52.4 we can make an identification $\langle a, b \rangle = \pi_1(S_a^1 \vee S_b^1, *)$ where a and b correspond to loops around S_a^1 and S_b^1. It follows fairly easily from Lemma 48.15 and Lemma 64.14 that it remains to show that for any loop $\gamma = a_1^{m_1} * b_1^{n_1} * \cdots * a_k^{m_k} * b_k^{n_k}\colon [0,1] \to X = S_a^1 \vee S_b^1$ there exists a finite-covering $p\colon \widetilde{X} \to X$ such that the lift of γ is not a loop. In the figure below we show the following two-step construction of such a covering for the loop $\gamma = a * b * \overline{a} * \overline{b}$.[748]

(1) First we build an "upward turning lift $\widetilde{\gamma}$ of γ".
(2) Afterward we complete the "image of $\widetilde{\gamma}$" to a finite covering $\widetilde{X}$.

It is an extraordinary charming exercise to turn the above idea into a rigorous argument for the general case. We would not dream of robbing the reader from this experience. ■



Basically for free we now get the following result.

Corollary 64.16. Every finitely generated free group is Hopfian.

Proof. This statement follows immediately from the Free Groups-are-Residually Finite Proposition 64.15 together with the Residually Finite-Hopfian Proposition 64.11. An alternative proof of the corollary is given in [**MKS1976**, p. 109]. ■

[748] The reader will notice that this is basically the solution to Exercise 48.12.

In Question 54.7 we had asked whether the fundamental group of the surface of genus g is isomorphic to the free group F_{2g} on $2g$ generators. Now we can answer this question in the negative.

Proposition 64.17. If $g \geq 1$, then the group

$$\pi_1(\text{surface of genus } g) \;\cong\; \langle x_1, y_1, \ldots, x_g, y_g \,|\, [x_1, y_1] \cdot \cdots \cdot [x_g, y_g] \rangle$$

is not isomorphic to the free group F_{2g} on $2g$ generators.[749]

Remark. We will give alternative proofs of Proposition 64.17 on page 1887 and in Corollary ??.

In fact Proposition 64.17 is an immediate consequence of the following more general statement.

Proposition 64.18. Let $X = \{x_1, \ldots, x_k\}$ be a finite set with k elements. If r is a non-trivial element in the free group $\langle x_1, \ldots, x_k \rangle$, then the group $\langle x_1, \ldots, x_k \mid r \rangle$ is not isomorphic to $\langle x_1, \ldots, x_k \rangle$.

Proof. Let X is a finite set and let r be an element in $\langle X \rangle$. Suppose there exists an isomorphism $\varphi\colon \langle X \,|\, r \rangle \to \langle X \rangle$. We denote by $\psi\colon \langle X \rangle \to \langle X \,|\, r \,\rangle$ the obvious epimorphism. Then $\varphi \circ \psi\colon \langle X \rangle \to \langle X \rangle$ is an epimorphism. It follows from Corollary 64.16 that $\varphi \circ \psi$ is actually an isomorphism. This implies in particular that ψ is a monomorphism. But r lies in the kernel of ψ, this is only possible if $r = e \in \langle X \rangle$. ■

Our next goal is to give a useful criterion for a group to be residually finite. To do so we need to introduce the following definition.

Definition. Let R be a ring. We say that a group G is linear over R if there exists an $n \in \mathbb{N}$ and a monomorphism $G \to \mathrm{GL}(n, R)$.

Examples.

(1) In Exercise 51.14 we showed that the free group on two generators is linear over $\mathbb{Z}$.
(2) Let G be a group and let R be a ring. In Exercise 64.15 we will show that if G admits a finite-index subgroup which is linear over R, then G itself is linear over R.

Now we can formulate the promised proposition.

Proposition 64.19. Let G be a finitely generated group. If G is linear over $\mathbb{C}$, then G is residually finite.

Sketch of proof. We only provide a full proof of the following easier claim which is the content of Exercise 51.15 (a).

Claim. The group $\mathrm{GL}(n, \mathbb{Z})$ is residually finite.

Proof. Let $A = (a_{ij}) \in \mathrm{GL}(n, \mathbb{Z})$ be a matrix with $A \neq \mathrm{id}$. We pick some $m \in \mathbb{N}$ such that $m > \max\{|a_{ij} - \delta_{ij}| \,|\, i, j = 1, \ldots, n\}$. The ring homomorphism $\mathbb{Z} \to \mathbb{Z}_m$ induces a group homomorphism $\alpha\colon \mathrm{GL}(n, \mathbb{Z}) \to \mathrm{GL}(n, \mathbb{Z}_m)$. Basically by construction we have $\alpha(A) \neq \mathrm{id}$. Since $\mathrm{GL}(n, \mathbb{Z}_m)$ is a finite group we are done. ⊞

[749] It follows from considering the abelianizations, see the Fundamental Group-of-Surfaces Proposition 54.5, that $\pi_1(\text{surface of genus } g)$ cannot be isomorphic to any other free group either.

Now let G be a finitely generated subgroup of $\mathrm{GL}(n,\mathbb{C})$. The proof that G is residually finite consists of two steps:

(1) First, since G is finitely generated one sees, almost immediately, that there exists a finitely generated subring S of $\mathbb{C}$ such that $G \subset \mathrm{GL}(n,S)$.
(2) Next one needs to show that S is a "residually finite ring", i.e. one needs to show that given any non-zero $s \in S$ there exists a ring homomorphism $\alpha\colon S \to T$ to some finite ring T with $\alpha(s) \neq 0$. This part requires some input from ring theory.

Once one has proved (1) and (2) it is straightforward to modify the proof of the claim to conclude that G is residually finite.

The full details for the above proof are given in [**Mal1940, Mal1965**] and also in [**Weh1973**, Theorems 4.1, 4.3] and in [**Hem87**]. ■

We already mentioned that $(\mathbb{Q},+)$ is not residually finite and that $\mathbb{Z}^{(\mathbb{N})}$ is not Hopfian. These are amusing observations, but most group theorists and geometric topologists restrict their interest to finitely generated groups. Thus the question arises, whether perhaps every finitely generated group is residually finite or Hopfian.

Proposition 64.20. Given $m,n \in \mathbb{N}$ we consider the Baumslag-Solitar group

$$\mathrm{BS}(m,n) \;=\; \langle x,y \,|\, x^{-1}y^m x \,=\, y^n \rangle.$$

The following statements hold:

(1) The group $\mathrm{BS}(m,n)$ is Hopfian if and only if m and n are "meshed", i.e. if and only if m or n divides the other or if m and n have precisely the same prime divisors.
(2) The group $\mathrm{BS}(m,n)$ is residually finite if and only if $|m| = 1$, or if $|n| = 1$ or if $|m| = |n|$.

Example. By Proposition 64.20 the group $\mathrm{BS}(2,6)$ is Hopfian but not residually finite. This shows, as promised, that the converse to the Residually Finite-Hopfian Proposition 64.11 does not hold.

Proof. The first statement was proved in 1962 by Gilbert Baumslag and Solitar [**BaS1962**]. The second statement is proved in [**Mes1972**, Theorem C] and also [**Cam1990**, p. 61]. ■

By Proposition 64.20 we now know that there exist finitely generated groups which do not have "many" finite quotients. The question arises, does every non-trivial group admit at least some finite index subgroups? It is now time to recall the following standard definition from group theory.

Definition. A group G is called simple if the only normal subgroups are the trivial group $\{e\}$ and the whole group G.

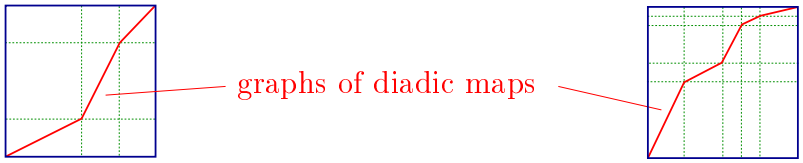
Example. One of the most basic statements in the theory of finite groups is the fact that for $n \geq 5$ the alternating group A_n is actually simple, we refer to [**Bog2008**, Chapter 1.12] for a proof.

It is now natural to ask whether there exist infinite simple groups. The answer is yes, but one needs to be rather creative when it comes to producing such groups. This discussion leads us to the following groups which were introduced by Richard Thompson in 1965.

Definition.

(1) A continuous self-homeomorphism $f\colon [0,1] \to [0,1]$ is called dyadic if there exists an $m \in \mathbb{N}$ such that the restriction of f to each interval $[\frac{i}{2^m}, \frac{i+1}{2^m}]$ is linear and the slope on each such interval is a (possibly negative) power of 2.

(2) We say a map $g\colon S^1 \to S^1$ is dyadic if there exists a dyadic map $f\colon [0,1] \to [0,1]$ such that $g(\exp(2\pi \mathrm{i} t)) = \exp(2\pi \mathrm{i} f(t))$.



We leave it to the reader to verify that the set of dyadic self-homeomorphisms of $[0,1]$ and of S^1 actually form groups. This observation leads us to the key definition.

Definition.

(1) Thompson's group F is defined as the group of dyadic self-homeomorphisms of $[0,1]$.

(2) Thompson's group T is defined as the subgroup of the self-homeomorphisms of S^1 that is generated by the dyadic self-homeomorphisms together with the homeomorphism given by $z \mapsto -z$.

The following theorem summarizes some of the many interesting properties of Thompson's groups.

Theorem 64.21.

(1) Thompson's group F and Thompson's group T are both finitely presented.

(2) The commutator subgroup of Thompson's group F is simple, in particular Thompson's group F is not residually finite.

(3) Thompson's group T is simple, in particular it is not residually finite.

Proof.

(1) The statements are proved in [**CFP1996**] and [**Geo2008**, Chapters 9.2].

(2) By [**CFP1996**] and [**Geo2008**, Chapters 9.2] the commutator subgroup of Thompson's group F is simple. One can easily verify that Thompson's group F is non-abelian and torsion-free. It now follows from an elementary argument that it is not residually finite.

(3) It is shown in [**CFP1996**] and [**Geo2008**, Theorem 9.4.3] that Thompson's group T is simple. Clearly T is infinite, hence it cannot be residually finite. ■

In this section we have shown, among many other statements, that free groups are residually finite. It is natural to ask which of the other friendly groups which we have encountered are actually residually finite. Let us pose two questions:

Question 64.22.

(1) Are the fundamental groups of closed 2-dimensional smooth manifolds residually finite?

(2) Is the fundamental group of every knot complement residually finite?

We will return to these questions later on in Proposition 66.19.

Exercises for Chapter 64.

Exercise 64.1. Let H be a finite connected graph and let G be a connected non-empty subgraph of H. Let S be a spanning tree of G. Show that there exists a spanning tree T of H such that $T \cap G = S$.
Remark. This exercise is a rather straightforward generalization of the Spanning Tree Existence Proposition 7.10 (1).

Exercise 64.2. Let $\alpha\colon \langle a, b\rangle \to \mathbb{Z}_3$ be the epimorphism defined given by $\alpha(a) = 1$ and $\alpha(b) = 0$. Furthermore let X be the wedge of two circles A and B. As usual we equip X with the CW-structure with one 0-cell P and two 1-cells a and b. Furthermore we make the identification $\pi_1(X) = \langle a, b\rangle$.

(a) "Construct" the covering $p\colon \widetilde{X} \to X$ corresponding to $\ker(\alpha)$, i.e. construct the covering p such that $p_*(\pi_1(\widetilde{X})) = \ker(\alpha)$. More precisely, give the CW-structure on $\widetilde{X}$ and indicate, e.g. using colors, the cellular map p.
(b) We know from Proposition 64.4 (2) that $\ker(\alpha)$ is a free group on four generators. Determine four elements of $\ker(\alpha)$ that generate $\ker(\alpha)$ as a free group. You do not have to justify your answer.

Exercise 64.3. In Question 63.4 we had considered an explicit 6-fold covering $X \to B$ where B is the 2-dimensional sphere minus three open disks. On page 1404 we saw that X has six boundary components. Determine the genus of X.

Exercise 64.4. Let $g \in \mathbb{N}_{\geq 2}$. Use Proposition 64.5 to show that the fundamental group of the surface of genus g is torsion-free.
Remark. We posed this already as Exercise 56.12 but with different instructions.

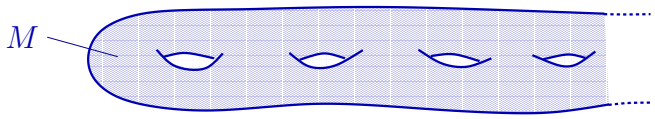
Exercise 64.5. Let P be a property of groups. We say a group G is *virtually* P if G admits a finite-index subgroup that has property P. Let G be a finitely generated group. Show that if G is virtually abelian, then there exists a short exact sequence of the form $0 \to \mathbb{Z}^d \to G \to A \to 1$ where $d \in \mathbb{N}_0$ and A is a finite group.

Exercise 64.6. Let $p\colon \widetilde{\Sigma} \to \Sigma$ be a covering map of degree d between two closed *non-orientable* connected 2-dimensional smooth manifolds. Show that the following equality holds:

$$\operatorname{genus}(\widetilde{\Sigma}) \;=\; d \cdot (\operatorname{genus}(\Sigma) - 1) + 1.$$

Hint. Let $\varphi\colon \overline{B}^2 \to \Sigma$ be an embedding. Consider the compact connected 2-dimensional smooth manifolds $M := \Sigma \setminus \varphi(B^2)$ and $\widetilde{M} := p^{-1}(M)$. What can you say about the fundamental group and the number of boundary components of $\widetilde{M}$?
Remark. This exercise is meant as the non-orientable analogue of the Subgroups-of-Surface Groups Proposition 64.6. In Lemma 87.9 we will use "Euler characteristics" to give an alternative proof of this equality.

Exercise 64.7. We consider the non-compact 2-dimensional smooth manifold M shown in the figure below. Does there exist a closed 2-dimensional smooth manifold N such that M is a covering space of N?



Exercise 64.8. In Exercise 51.15 we introduced the notion of a residually finite group and we showed that the free group on two generators is residually finite.

(a) Show that every finitely generated free group is residually finite.
Remark. Use Proposition 64.4 (2).
(b) Show that every free group is residually finite.

Exercise 64.9. Let π be a countable residually finite group. Show that there exists a descending sequence $\pi \supset \pi_0 \supset \pi_1 \supset \pi_2 \supset \dots$ of finite index normal subgroups of π such that we have $\bigcap_{i \in \mathbb{N}} \pi_i = \{e\}$.

Exercise 64.10. Let G be a finitely generated group. Show that given any $n \in \mathbb{N}$ there exist only finitely many subgroups of G of index n.

Exercise 64.11. Let G be a group. A subgroup Γ of G is called *characteristic* if for every automorphism $\varphi\colon G \to G$ we have $\varphi(\Gamma) = \Gamma$.

(a) Show that if subgroup is characteristic, then it is also normal.
(b) Let H be a subgroup of G. Suppose that G is finitely generated. Show that there exists a characteristic subgroup $K \subset G$ with $K \subset H$ and such that K has finite index in G.
(c) Show that in (b) we cannot drop the hypothesis that G is finitely generated.

Exercise 64.12. Show that the group $(\mathbb{Q}, +)$ is not residually finite.

Exercise 64.13. Let $\varphi\colon A \to B$ be a homomorphism between free groups of the same finite rank. Suppose that $\varphi_*\colon A_{\mathrm{ab}} \to B_{\mathrm{ab}}$ is an isomorphism. Show that φ is a monomorphism.
Hint. Use Proposition 64.4 and Corollary 64.16.

Exercise 64.14. Let G be a group and let $\varphi\colon G \to G$ be an automorphism. We suppose that G is residually finite and finitely generated. Show that the semidirect product $G \rtimes_\varphi \mathbb{Z}$, that we defined on page 1127, is also residually finite.
Hint. Make good use of Exercise 64.11.

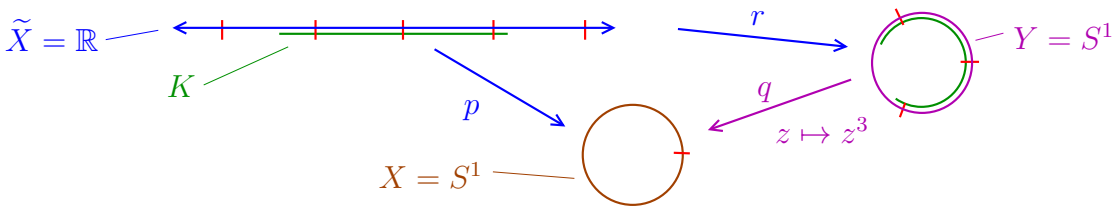
Exercise 64.15. Let R be a ring, let G be a group and let H be a subgroup of finite index $d \in \mathbb{N}$. Suppose there exists a monomorphism $\varphi\colon H \to \mathrm{GL}(n, R)$. Show that there exists a monomorphism $\varphi\colon G \to \mathrm{GL}(n \cdot d, R)$.

Exercise 64.16.

(a) Provide an example of an infinitely generated group that is Hopfian.
(b) Show that every infinite simple group is Hopfian.

Exercise 64.17. Let $k \in \mathbb{N}_{\geq 2}$. We consider the direct product $\langle x_1, \dots, x_k \rangle \times \mathbb{Z}$. Show that given any $N \in \mathbb{N}$ there exists a finitely generated free group F of rank $\geq N$ and an automorphism φ of F such that $\langle x_1, \dots, x_k \rangle \times \mathbb{Z}$ is isomorphic to the semidirect product $F \rtimes_\varphi \mathbb{Z}$.
Remark. Most of the hard work was already done in Exercise 50.19.

Exercise 64.18. Let X be a path-connected non-empty topological space. We denote by $p\colon \widetilde{X} \to X$ the universal covering. Let $K \subset \widetilde{X}$ be a compact subset. Show that if $\pi_1(X)$ is residually finite, then there exists a finite covering $q\colon Y \to X$ such that for the induced covering $r\colon \widetilde{X} \to Y$, that is given by Corollary 61.11, the restriction of r to $r|_K\colon K \to Y$ is an embedding.



65. Hyperbolic geometry

In this chapter we will switch gears and we will introduce some basics on hyperbolic geometry. The reason for doing so is that in Chapter 66 we will see that every surface of genus ≥ 2 admits a hyperbolic structure, and this fact will be extremely helpful in getting a better understanding of surfaces.

65.1. Hyperbolic space.

We start out with the following definition.

Definition. Let V be a real vector space. A symmetric form on V is a map[750]

$$g\colon V \times V \to \mathbb{R}$$

that is bilinear, i.e. for all $u, v, w \in V$ and $\mu, \lambda \in \mathbb{R}$ we have

$$\begin{aligned} g(\lambda \cdot u + \mu v, w) &= \lambda \cdot g(u, w) + \mu g(v, w) \\ g(u, \lambda \cdot v + \mu \cdot w) &= \lambda \cdot g(u, v) + \mu \cdot g(u, w) \end{aligned}$$

and that is symmetric, i.e. for all $v, w \in V$ we have

$$g(v, w) = g(w, v).$$

We say g is positive-definite if for all $v \in V \setminus \{0\}$ we have

$$g(v, v) > 0.$$

Now we can recall the following key definition from the theory of smooth manifolds, see e.g. [**Lee2002**, Chapter 17].

Definition.

(1) A Riemannian structure on a smooth manifold M is a smooth[751] map g which assigns to each point P a positive-definite symmetric form g_P on the tangent space $\mathrm{T}_P M$.
(2) A Riemannian manifold is a pair (M, g) consisting of a smooth manifold M together with a Riemannian structure g.

As so often in mathematics, given some structure one is very much interested in maps that preserve the given structure. This leads us to the following definition.

Definition.

(1) Given a smooth map $f\colon M \to N$ between smooth manifolds and given $P \in M$ we denote by $\mathrm{D}f_P = f_*\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ the differential, i.e. the induced homomorphism from the tangent space $\mathrm{T}_P M$ to the tangent space $\mathrm{T}_{f(P)} N$.
(2) Let $f\colon M \to N$ be a local diffeomorphism. Given a Riemannian structure h on N we denote by f^*h the Riemannian structure on M that at each point $P \in M$ is given by $(f^*h)(v, w) = h(f_* v, f_* w)$.[752]
(3) We say that a map $f\colon (M, g) \to (N, h)$ between Riemannian manifolds is a (local) isometry if f is a (local) diffeomorphism and if $g = f^*h$.
(4) Given a Riemannian manifold (M, g) we denote by $\mathrm{Isom}(M, g)$ the group of isometries of (M, g). If M is orientable, then we denote by $\mathrm{Isom}^+(M, g)$ the group of orientation-preserving isometries of (M, g).

[750] Sometimes such a structure is called a *bilinear symmetric form*, but we abbreviate this to *symmetric form*.
[751] The notion of "smooth" in this context is made precise in [**Lee2002**, Chapter 17]. Once one has digested the definition of a "smooth map" it is straightforward to see that all examples below satisfy this condition.

(5) We say that Riemannian manifolds (M, g) and (N, h) are isometric if there exists an isometry from (M, g) to (N, h).

In this chapter and the following chapter we are mostly interested in Riemannian structures on surfaces, and more specifically, in hyperbolic structures. Now we give the definition of hyperbolic space.

Definition. The hyperbolic half-plane is the smooth manifold

$$\mathbb{H} \;=\; \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \;=\; \{z = x + \mathrm{i}y \in \mathbb{C} \mid y > 0\}$$

together with the Riemannian structure that assigns to the tangent space $\mathrm{T}_z\mathbb{H} = \mathbb{R}^2$ at a point $z = x + \mathrm{i}y \in \mathbb{H}$ the positive-definite symmetric form

$$\begin{array}{rcl} g\colon \mathbb{R}^2 \times \mathbb{R}^2 & \to & \mathbb{R} \\ (v, w) & \mapsto & \frac{1}{y^2} \cdot \langle v, w \rangle \end{array}$$

where $\langle v, w \rangle$ denotes the usual scalar product on $\mathbb{R}^2$.

Example. In the figure below we see a decomposition of $\mathbb{H}$ into angels and devils. With respect to the hyperbolic metric all the angels have the same size and all the devils have the same size, even though they differ in the usual Euclidean sense.



As we mentioned before, we are very much interested in the group $\mathrm{Isom}^+(\mathbb{H})$ of orientation-preserving isometries of the Riemannian manifold $\mathbb{H}$. This leads us to the following definition.

Definition. A Möbius transformation of $\mathbb{H}$ is a map of the form[753]

$$\begin{array}{rcl} \mathbb{H} & \to & \mathbb{H} \\ z & \mapsto & \dfrac{az + b}{cz + d} \end{array}$$

where $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$.

Examples. We give three special types of Möbius transformations of $\mathbb{H}$:

(1) given $r > 0$ the scalar multiplication $z \mapsto rz = \frac{\sqrt{r}\cdot z + 0}{0 \cdot z + \frac{1}{\sqrt{r}}}$ is a Möbius transformation,

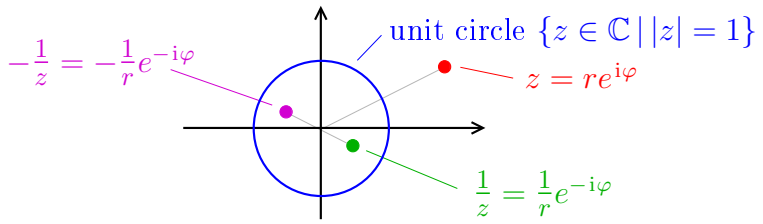
(2) given $d \in \mathbb{R}$ the horizontal translation $z \mapsto z + d = \frac{1 \cdot z + d}{0 \cdot z + 1}$ is a Möbius transformation,

(3) the map $z \mapsto -\frac{1}{z} = \frac{0 \cdot z + 1}{(-1) \cdot z + 0}$ is a Möbius transformation.

[752] It follows from [**Lee2002**, Proposition 13.9] that f^*h is indeed a Riemannian structure on M. Note that in the proof, to show that f^*h is indeed non-degenerate, one needs that f is a local diffeomorphism.

[753] An elementary calculation shows that the image of a Möbius transformation $\mathbb{H} \to \mathbb{C}$ does indeed lie again in $\mathbb{H}$.

The third Möbius transformation is the composition of taking the inverse $z \mapsto \frac{1}{z}$ and the reflection $z \mapsto -z$ in the origin. We illustrate this map in the figure below. For simplicity we refer to the map $z \mapsto -\frac{1}{z}$ as an inversion.



Rather amazingly we are already done with listing all elements in $\mathrm{Isom}^+(\mathbb{H})$. More precisely, we have the following proposition.

Proposition 65.1. The map

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbb{R}) & \to & \mathrm{Isom}^+(\mathbb{H}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & \begin{pmatrix} \mathbb{H} & \to & \mathbb{H} \\ z & \mapsto & \frac{az+b}{cz+d} \end{pmatrix} \end{array}$$

is a group isomorphism.

Sketch of proof. One can easily calculate by hand that each Möbius transformation is an orientation-preserving isometry of $\mathbb{H}$. Furthermore, it is an amusing calculation to verify that the given map is indeed a group homomorphism. It is elementary to see that the map is a monomorphism. The only bit of the proposition which requires some thought is the statement that every orientation-preserving isometry is already a Möbius transformation. This statement is for example shown in [**And2005**, Theorem 3.19]. ■

Now that we have seen that every orientation-preserving isometry of $\mathbb{H}$ is a Möbius transformation it makes sense to study Möbius transformations in greater detail.

Proposition 65.2. Every Möbius transformation of $\mathbb{H}$ is the composition of scalar multiplications, horizontal translations and inversions.

Proof. Let $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$ and denote by

$$\Phi(z) \;=\; \frac{az+b}{cz+d}$$

the corresponding Möbius transformation. If $c = 0$, then

$$\Phi(z) \;=\; \frac{a}{d}z + \frac{b}{d}$$

is the composition of the scalar multiplication[754] $z \mapsto \frac{a}{d}z$ with the horizontal translation $z \mapsto \frac{b}{d} + z$. If $c \neq 0$, then

$$\Phi(z) \;=\; \frac{az+b}{cz+d} \;=\; \frac{acz+bc}{c^2z+dc} \underset{\substack{\uparrow \\ \text{since } ad-bc=1}}{=} \frac{acz+ad-1}{c^2z+cd} \;=\; \frac{-1+a(cz+d)}{c^2z+cd} \;=\; \frac{-1}{c^2z+dc} + \frac{a}{c}.$$

Thus we see that Φ is the composition of the following maps

$$z \mapsto c^2z, \quad z \mapsto z+dc, \quad z \mapsto \frac{-1}{z} \quad \text{and } z \mapsto z + \frac{a}{c}.$$

■

[754] Note that $c = 0$ implies that $1 = ad - bc = ad$, in particular $ad > 0$, hence $\frac{a}{d} > 0$.

Definition.

(1) The hyperbolic disk[755] is the smooth manifold

$$\mathbb{D} \;=\; \{z \in \mathbb{C} \mid |z|^2 < 1\}$$

together with the Riemannian structure which assigns to the tangent space $\mathrm{T}_z\mathbb{D} = \mathbb{R}^2$ at a point $z \in \mathbb{D}$ the positive-definite symmetric form

$$\begin{array}{rcl} g_z \colon \mathbb{R}^2 \times \mathbb{R}^2 & \to & \mathbb{R} \\ (v,w) & \mapsto & \dfrac{2}{(1-|z|^2)^2}\langle v, w\rangle \end{array}$$

where, as above, $\langle v, w\rangle$ denotes the usual scalar product on $\mathbb{R}^2$.

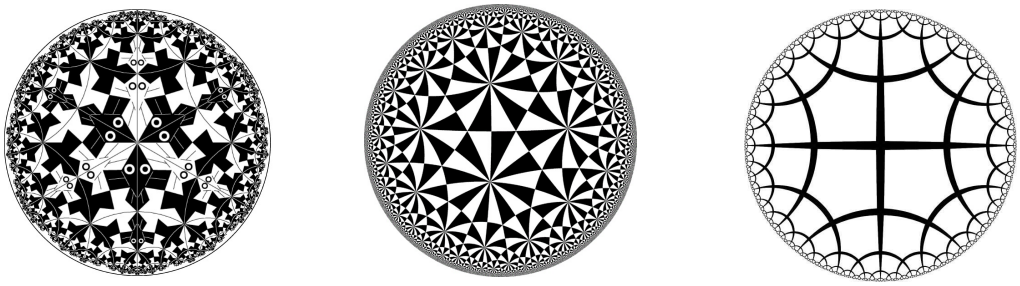
(2) A Möbius transformation of $\mathbb{D}$ is a map of the form[756]

$$\begin{array}{rcl} \mathbb{D} & \to & \mathbb{D} \\ z & \mapsto & \exp(\mathrm{i}\theta) \cdot \frac{z-a}{1-\bar{a}z} \end{array}$$

for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$.

Examples.

(1) In the figure below we see three different decompositions of $\mathbb{D}$ into subsets. In each of the three cases these subsets have the same hyperbolic size, even though they differ in the usual Euclidean sense.



(2) If we set $a = 0$ in the definition of a Möbius transformation of $\mathbb{D}$, then we obtain precisely the rotation by the angle θ. Put differently, the Möbius transformations preserving 0 are precisely given by rotations around the origin.

The following proposition will allow us to go back and forth between the hyperbolic half-plane and the hyperbolic disk.

Proposition 65.3. The maps

$$\begin{array}{rcl} \Phi \colon \mathbb{H} & \to & \mathbb{D} \\ z & \mapsto & \frac{z-\mathrm{i}}{z+\mathrm{i}} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \Psi \colon \mathbb{D} & \to & \mathbb{H} \\ w & \mapsto & \frac{\mathrm{i}+\mathrm{i}w}{1-w} \end{array}$$

have the following properties:

(1) Φ and Ψ are inverses to one another, in particular they are biholomorphisms,

[755] The hyperbolic disk is often also called the *Poincaré model for hyperbolic space*.

[756] An elementary calculation shows that the image of a Möbius transformation $\mathbb{D} \to \mathbb{C}$ does indeed lie again in $\mathbb{D}$.

(2) they extend to homeomorphisms between $\overline{\mathbb{H}} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0\} \cup \{\infty\}$[757] and $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$,
(3) they are orientation-preserving,[758]
(4) they are isometries,
(5) they give a bijection between the Möbius transformations of $\mathbb{D}$ and $\mathbb{H}$.[759]

Proof. All five statements follow from a straightforward calculation. The statements can also be deduced, more or less directly, from any book that deals with Möbius transformations, see e.g. [**Schwe1979**, Chapter 6] and [**And2005**, Chapter 2]. ■

Remark. Proposition 65.3 says that $\mathbb{D}$ and $\mathbb{H}$ are two equivalent ways to describe hyperbolic space. In fact there are several other ways to describe hyperbolic space, for example we could also use the Beltrami-Klein model

`https://en.wikipedia.org/wiki/Beltrami-Klein_model`

or the hyperboloid model

`https://en.wikipedia.org/wiki/Hyperboloid_model`.

Some of the connections between these models are explained in the following video of Henry Segerman and Saul Schleimer:

`https://www.youtube.com/watch?v=eGEQ_UuQtYs`.

The following proposition summarizes some of the key properties of Möbius transformations.

Proposition 65.4. [760]

(1) Compositions of Möbius transformations are again Möbius transformations and the inverse of a Möbius transformation is again a Möbius transformation.
(2) Möbius transformations are biholomorphisms, they are isometries and they are orientation-preserving.
(3) Möbius transformations act transitively, i.e. given any two points P and Q there exists a Möbius transformation which sends P to Q.
(4) Given $P, Q \in \mathbb{D}$ and non-zero vectors $v \in \mathrm{T}_P\mathbb{D}$ and $w \in \mathrm{T}_Q\mathbb{D}$ with $\|v\|_g = \|w\|_g$ there exists a Möbius transformation ϕ with $\phi(P) = Q$ and $d\phi_P(v) = w$.
(5) Let ϕ and ψ be two Möbius transformations. Suppose there exists a $P \in \mathbb{D}$ such that $\phi(P) = \psi(P)$ and such that $d\phi_P = d\psi_P$. Then $\phi = \psi$.

Proof. By Proposition 65.3 it suffices to prove the statements for $\mathbb{D}$ or for $\mathbb{H}$, whichever is more convenient.

(1),(2) The first two statements follow from a straightforward calculation, see e.g. [**Schwe1979**, p. 43].

[757] Here $\overline{\mathbb{H}}$ has the same topology as defined on page 122.
[758] The hyperbolic half-plane $\mathbb{H}$ and the hyperbolic disk $\mathbb{D}$ are both subsets of $\mathbb{C} = \mathbb{R}^2$ and inherit from $\mathbb{C} = \mathbb{R}^2$ the natural orientation.
[759] More precisely, if α is a Möbius transformation of $\mathbb{H}$, then $\Phi \circ \alpha \circ \Psi$ is a Möbius transformation of $\mathbb{D}$ and conversely, if α is a Möbius transformation of $\mathbb{D}$, then $\Psi \circ \alpha \circ \Phi$ is a Möbius transformation of $\mathbb{H}$.
[760] The statement of the proposition applies to Möbius transformations of $\mathbb{D}$ and to Möbius transformations of $\mathbb{H}$.

(3) We prove this statement for $\mathbb{D}$. Note that by (1) it suffices to show that given any $P \in \mathbb{D}$ there exists a Möbius transformation ψ with $\psi(P) = 0$. But such a Möbius transformation is given by

$$\psi(z) \;=\; \tfrac{z-P}{1-\overline{P}z}.$$

(4) Using (3) we see that we only have to deal with the case that $P = Q = 0$. But then the desired Möbius transformation is given by a rotation.

(5) By (3) there exists a Möbius transformation θ such that $\theta(0) = P$ and there exists a Möbius transformation ω such that $\omega(\phi(P)) = \omega(\psi(P)) = 0$. We consider the Möbius transformations $\phi' = \omega \circ \phi \circ \theta$ and $\psi' = \omega \circ \psi \circ \theta$. It follows from our hypothesis that both ϕ' and ψ' fix 0. As we pointed out above, the only Möbius transformations of $\mathbb{D}$ that fix 0 are rotations.

By our hypothesis we have $d\phi'_0 = d\psi'_0$, which means that the rotations ϕ' and ψ' agree. But this implies that ϕ and ψ also agree. ■

Note that Möbius transformations extend in an obvious way uniquely to homeomorphisms of $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and $\overline{\mathbb{H}} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0\} \cup \{\infty\}$. We will denote the extension of a Möbius transformation with the same symbol.

Lemma 65.5. Let $\{P, Q, R\}$ and let $\{S, T, U\}$ be two sets of three distinct points on the boundary $\partial\overline{\mathbb{H}} = \mathbb{R} \cup \{\infty\}$. Then there exists a Möbius transformation ψ of $\mathbb{H}$ with $\psi(P) = S$ and $\psi(\{Q, R\}) = \{T, U\}$.

Proof. Let P, Q, R be three distinct points on $\partial\overline{\mathbb{H}} = \mathbb{R} \cup \{\infty\}$. It suffices to show that there exists a Möbius transformation ψ with $\psi(P) = \infty$ and with $\psi(\{Q, R\}) = \{0, 1\}$. We proceed as follows:

(1) We first claim that there exists a Möbius transformation α with $\alpha(P) = \infty$. Indeed, if $P = \infty$ then we just take $\alpha = \mathrm{id}$. Otherwise we can combine a horizontal translation with the inversion to obtain a Möbius transformation α with $\alpha(P) = \infty$. Given such α we write $Q' = \alpha(Q)$ and $R' = \alpha(R)$. Note that the points Q', R' lie on $\mathbb{R} = \overline{\mathbb{H}} \backslash \{\infty\}$. Without loss of generality we can assume that $Q' < R'$.

(2) Next we pick a horizontal translation such that $\beta(Q') = 0$. Note that we still have $(\beta \circ \alpha)(P) = \beta(\infty) = \infty$. Also note that it follows from $Q' < R'$ that $\beta(R') > 0$.

(3) Finally we pick a scalar multiplication γ such that $(\gamma \circ \beta)(R') = 1$. Note that γ fixes 0 and ∞. Thus we see that $\psi = \gamma \circ \beta \circ \alpha$ has the desired property. ■

Definition. A hyperbolic line is a subset of $\mathbb{D}$ of one of the following two types:

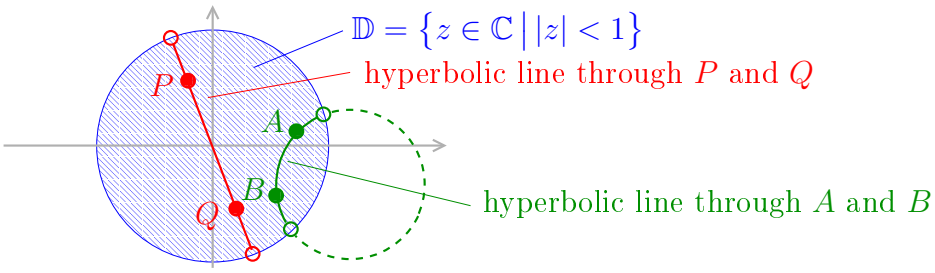
(1) it is the intersection of $\mathbb{D}$ with a Euclidean line through the origin,

(2) it is the intersection of $\mathbb{D}$ with a Euclidean circle that intersects the circle $S^1 = \partial\overline{\mathbb{D}}$ orthogonally.

In each of the two cases we refer to the intersection of the Euclidean object with $S^1 = \partial\overline{\mathbb{D}}$ as the endpoints of the hyperbolic line.

Lemma 65.6.

(1) For any two distinct points P and Q in $\mathbb{D}$ there exists a unique hyperbolic line that contains P and Q.



(2) For any two distinct points P and Q in $\partial\overline{\mathbb{D}}$ there exists a unique hyperbolic line with P and Q as endpoints.
(3) Let P and Q be two distinct points in $\partial\overline{\mathbb{D}}$. Every Möbius transformation ϕ sends the hyperbolic line with endpoints P and Q to the hyperbolic line with endpoints $\phi(P)$ and $\phi(Q)$.
(4) Möbius transformations act transitively on hyperbolic lines, i.e. given any two hyperbolic lines g and h there exists a Möbius transformation ϕ with $\phi(g) = h$.

Proof. In principle one can prove the lemma directly in $\mathbb{D}$. But it is easier to prove it using $\mathbb{H}$. In $\mathbb{H}$ the hyperbolic lines are defined to be vertical lines $\{x + \mathrm{i}y \mid y > 0\}$ and Euclidean half-circles such that the origin lies on $\mathbb{R}$. It is straightforward to see that the diffeomorphisms of Proposition 65.3 define a bijection between the hyperbolic lines in $\mathbb{D}$ and in $\mathbb{H}$.

The corresponding statements for hyperbolic lines in $\mathbb{H}$ are straightforward to prove. They can also be obtained from any textbook on hyperbolic geometry, see e.g. [**And2005**] and [**Bon2009**, Chapter 2]. ■

65.2. **Angles in Riemannian manifolds.** We continue our quick introduction to Riemannian manifolds and hyperbolic space with a short discussion of angles.

Definition.

(1) Let g be a positive-definite symmetric form on a real vector space V. Given two non-zero vectors $v, w \in V \setminus \{0\}$ we define[761]

$$\text{angle between } v \text{ and } w \;:=\; \text{unique } \alpha \in [0, \pi] \text{ with } g(v, w) = \cos(\alpha) \cdot \|v\| \cdot \|w\|.$$

(2) Given a Riemannian manifold (M, g) and non-zero vectors $v, w \in \mathrm{T}_P M$ we use the form g_P to define the angle between v and w.

Examples.

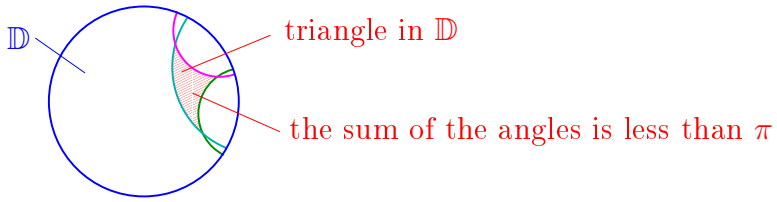
(1) Let U be an open subset of $\mathbb{R}^n$ and let $f\colon U \to \mathbb{R}_{>0}$ be a smooth function. For any $P \in U$ we can define a positive-definite symmetric form g_P on $\mathrm{T}_P U = \mathbb{R}^n$ via

$$g_P(v, w) \;=\; f(P) \cdot \langle v, w \rangle,$$

where $\langle\,,\,\rangle$ denotes the usual scalar product. A straightforward calculation shows that in this Riemannian structure the angle at any point is the usual Euclidean angle in $\mathbb{R}^n$. This argument implies in particular to the Riemannian manifolds $\mathbb{D}$ and $\mathbb{H}$, in both cases the hyperbolic angle, defined by the hyperbolic Riemannian structure, is the same as the Euclidean angle.

[761] We have $|g(v, w)| \leq \big|\|v\| \cdot \|w\|\big|$ by the Cauchy-Schwarz Inequality 0.12.

(2) A hyperbolic triangle is defined to be a compact subset of $\mathbb{H}$ whose boundary is given by segments of three hyperbolic lines. One of the many elegant theorems in hyperbolic geometry, see [**And2005**, Theorem 5.16], says that given such a hyperbolic triangle its angle sum equals π minus its "area". Even without the definition of "area" it is pretty clear that this result implies that the angle sum of a hyperbolic triangle is always less than π.



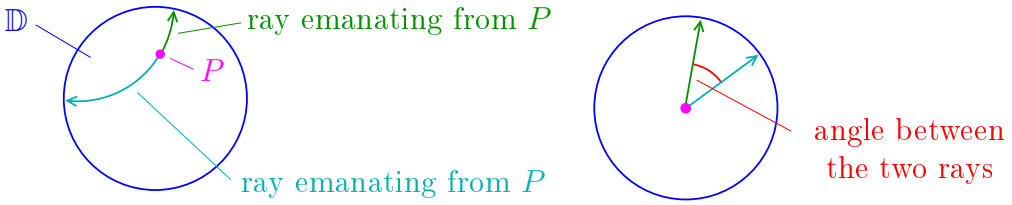
The following lemma follows easily from the definitions and from Proposition 65.4 (2).

Lemma 65.7. Local isometries, in particular Möbius transformations, preserve angles.

Definition.

(1) Let $P \in \mathbb{D}$ and let g be a hyperbolic line through P. Let Q be one of the two endpoints of g. We refer to the set of points between P and Q as a ray emanating from P.

(2) Let $P \in \mathbb{D}$ and let r, s be two rays emanating from P. We refer to the angle between the tangent vectors of r and s at P as the angle between the rays r and s.



Examples.

(1) Given $\varphi \in [0, 2\pi)$ the set $S_\varphi := \{r \cdot \exp(\mathrm{i}\varphi) \mid r \in [0, 1)\}$ is a ray emanating from 0.

(2) It follows from the above discussion of angles in $\mathbb{D}$, that given $\varphi, \theta \in [0, 2\pi)$ the angle between the two rays S_φ and S_θ in $\mathbb{D}$ is the usual Euclidean angle.

Lemma 65.8. Let r, s be two rays emanating from $P \in \mathbb{D}$ and let r', s' be two rays emanating from $P' \in \mathbb{D}$. Then the following holds:

$$\text{there is a Möbius transformation } \phi \text{ with } \phi(r \cup s) = r' \cup s' \iff \text{angle between } r, s = \text{angle between } r', s'.$$

Furthermore, if such a Möbius transformation exists, then it is unique.

Proof. By Lemma 65.7 Möbius transformations preserve angles and by Proposition 65.4 Möbius transformations act transitively on $\mathbb{D}$. Therefore we can without loss of generality assume that $P = P' = 0$. The lemma follows from the observation, made on page 1429, that Möbius transformations preserving 0 are precisely given by rotations around the origin. ■

65.3. The distance metric of a Riemannian manifold.

Definition. Let (M, g) be a connected Riemannian manifold.

(1) Given $v \in \mathrm{T}_P M$ we write $\|v\|_g := \sqrt{g_P(v,v)}$. If g is understood from the context, then we just write $\|v\|$.

(2) Let $\gamma\colon [a,b] \to M$ be a piecewise smooth path. By definition we can pick a subdivision $a = s_0 < s_1 < \cdots < s_m = b$ such that for $i = 0, \ldots, m-1$ the map $\gamma|_{[s_i, s_{i+1}]}$ is smooth. We define the length of γ as[762]

$$\ell_{(M,g)}(\gamma) := \sum_{i=0}^{m-1} \int_{t=s_i}^{t=s_{i+1}} \|\gamma'(t)\|_g \, dt.$$

If (M, g) is understood from the context, then we write $\ell(\gamma)$ instead of $\ell_{(M,g)}(\gamma)$.

(3) Let P and Q be two points on M.

$$d_{(M,g)}(P,Q) := \inf\{\ell(\gamma) \mid \underbrace{\gamma \text{ is a smooth path in } M \text{ from } P \text{ to } Q}_{\text{exists by the Smooth Path-Connectivity Proposition 19.29}} \}.$$

If (M, g) is understood from the context, then we just write $d(P, Q)$.

We start our discussion of the distance function d on Riemannian manifolds with the following somewhat technical proposition.

Proposition 65.9. Let (M, g) be a connected Riemannian manifold. The following statements hold:

(1) The pair $(M, d_{(M,g)})$ is a metric space.

(2) The topology defined by the metric $d_{(M,g)}$ agrees with the topology of M.

(3) Given any $P \in M$ and given any open neighborhood U of P there exists an $r > 0$ such that

$$B_r^{(M,g)}(P) := \{Q \in M \mid d_{(M,g)}(P,Q) < r\} \subset U.$$

Proof. Let (M, g) be a connected n-dimensional Riemannian manifold. We first show that $(M, d_{(M,g)})$ is a metric space.

It is clear that $d = d_{(M,g)}$ is symmetric. It follows easily from the definitions and the fact that the concatenation of two piecewise smooth paths is again piecewise smooth that d satisfies the triangle inequality.

Now let $P \neq Q$ be two distinct points in M. We need to show that $d(P,Q) > 0$. Since M is Hausdorff and since it is locally diffeomorphic to open subsets of $\mathbb{R}^n$ we can find a chart $\Phi\colon U \to B_{2r}(0)$ for P with $\Phi(P) = 0$ and such that $Q \notin U$. We write $\Psi = \Phi^{-1}$. Before we continue we introduce the following definitions:

(1) We write $U_r = \Phi^{-1}(\{x \in \mathbb{R}^n \mid \|x\| = r\})$ and given a subset $I \subset [0, 2r)$ we write $U_I = \Phi^{-1}(\{x \in \mathbb{R}^n \mid \|x\| \in I\})$.

(2) For a positive-definite bilinear form h on $\mathbb{R}^n$ we set

$$\|h\| := \min\left\{\sqrt{h(v,v)} \,\middle|\, v \in S^{n-1}\right\}.$$

Note that h is positive-definite, that $h\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuous and that S^{n-1} is compact. It follows from the Compact Image Lemma 2.13 that the minimum exists

[762] Recall that given a smooth path $\delta\colon [a,b] \to M$ and given any $t \in [a,b]$ the path defines, basically by definition of the tangent space $\mathrm{T}_{\delta(t)}M$, an element in $\mathrm{T}_{\delta(t)}M$. We denote this vector by $\delta'(t)$.

and that $\|h\| > 0$. It is straightforward to see that for any $v \in \mathbb{R}^n$ we have

$$h(v,v) \;\geq\; \|h\| \cdot \|v\|_{\text{eucl}}.$$

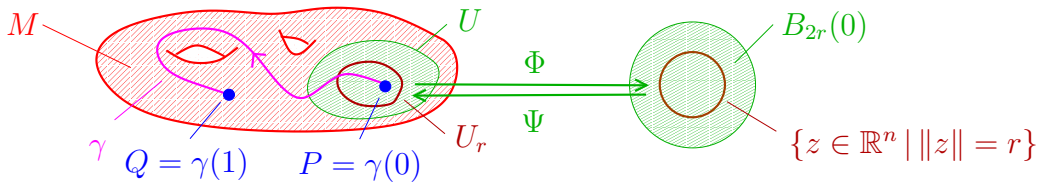
(3) We set

$$C \;:=\; \min\left\{ \|(\Psi^* g)_P\| \;\middle|\; P \in \overline{B_r(0)} \right\}.$$

Since $P \mapsto \|(\Psi^* g)_P\|$ is continuous and since $\overline{B_r(0)}$ is compact it follows from the Compact Image Lemma 2.13 that $C > 0$.

Claim. Let $\gamma\colon [0,1] \to M$ be a piecewise smooth path from P to Q. There exists a $t \in [0,1]$ with $\gamma(t) \in U_r$.

Proof. Suppose there is no $t \in (0,1)$ with $\gamma(t) \in U_r$. We consider the sets $A = U_{[0,r)}$ and $B = M \setminus U_{[0,r]}$. The set A is clearly open. Furthermore it follows from Lemma 18.11 that $U_{[0,r]}$ is closed, hence B is open. Since $\gamma([0,1]) \cap U_r = \varnothing$ we have $[0,1] = \gamma^{-1}(A) \sqcup \gamma^{-1}(B)$. But both sets $\gamma^{-1}(A)$ and $\gamma^{-1}(B)$ are open by the continuity of γ and both are non-empty since $0 \in \gamma^{-1}(A)$ and $1 \in \gamma^{-1}(B)$. But this contradicts the fact, established with some effort in the Interval-Connectedness Proposition 2.22, that $[0,1]$ is connected. ⊞



Now it suffices to prove the following claim.

Claim. Let $\gamma\colon [0,1] \to M$ be a piecewise smooth path from P to Q. Then $\ell_g(\gamma) \geq C \cdot r$.

Proof. Let $T := \inf\{t \in (0,1) \mid \gamma(t) \in U_r\}$. By the above claim this definition makes sense since we take the infimum of a non-empty set. By Lemma 18.11 the set U_r is a closed subset of M. Since γ is continuous we also have $\gamma(T) \in U_r$. Now we have

$$\begin{aligned}
\ell_g(\gamma) \;\geq\; \ell_g(\gamma|_{[0,T]}) \;&= \ell_{\Psi^* g}(\Phi \circ \gamma|_{[0,T]}) && = \int_{t=0}^{t=T} \|(\Phi\circ\gamma)'(t)\|_{\Psi^* g}\, dt \\
&\geq \int_{t=0}^{t=T} C \cdot \|(\Phi\circ\gamma)'(t)\|_{\text{eucl}}\, dt && \geq C \cdot \left\| \int_{t=0}^{t=T} (\Phi\circ\gamma)'(t)\, dt \right\|_{\text{eucl}} \\
&\uparrow && \uparrow \\
&\text{for any } v \in \mathrm{T}_P\overline{B_r(0)} = \mathbb{R}^n \text{ we have} && \text{Integral Inequality Lemma 0.16} \\
&\|v\|_{(\Psi^* g)_P} \;\geq\; \|(\Psi^* g)_P\| \cdot \|v\|_{\text{eucl}} \\
&= C \cdot \|\underbrace{(\Phi\circ\gamma)(T)}_{\|-\|_{\text{eucl}} = r} - \underbrace{(\Phi\circ\gamma(0))}_{=0}\|_{\text{eucl}} \;=\; C \cdot r.
\end{aligned}$$

This concludes the proof of the claim and therefore the proof of Statement (1) of the proposition.

Next we turn to the proof of Statement (2). This means that we need to show that the topology defined by d is the same as the original topology of M. The proof is again somewhat technical and uses ideas similar to the proof of (1). We refer to [**Lee1997**, Lemma 6.2] for the proof.

Finally Statement (3) is just an immediate consequence of the definitions and of Statement (2). ■

Lemma 65.10. Let (M, g) and (N, h) be two connected Riemannian manifolds. Furthermore let $f\colon (M, g) \to (N, h)$ be a local isometry. Then the following hold:

(1) For any piecewise smooth path $\gamma\colon [a, b] \to M$ we have
$$\ell_{(N,h)}(f \circ \gamma) \;=\; \ell_{(M,g)}(\gamma).$$
(2) For any $P, Q \in M$ we have
$$d_{(N,h)}(f(P), f(Q)) \;\leq\; d_{(M,g)}(P, Q).$$

Proof. Let $f\colon (M, g) \to (N, h)$ be a local isometry. We start out with the following claim.

Claim.

(1) For a smooth path $\gamma\colon [c, d] \to M$ and $t \in (c, d)$ we have $\|(f\circ\gamma)'(t)\|_{(N,h)} = \|\gamma'(t)\|_{(M,g)}$.
(2) For any piecewise smooth path $\gamma\colon [a, b] \to M$ we have $\ell_{(N,h)}(f \circ \gamma) = \ell_{(M,g)}(\gamma)$.

Proof.

(1) The first statement is an immediate consequence of the definitions and the hypothesis that f is a local isometry.
(2) By the additivity of the lengths of piecewise smooth paths it suffices to consider the case that $\gamma\colon [a, b] \to M$ is a smooth path. But in that case the equality follows immediately from the definitions and from (1). ⊞

Now let P and Q be two points on M. For any piecewise smooth path p in M from P to Q we have
$$d_{(N,h)}(f(P), f(Q)) \underset{\substack{\uparrow \\ \text{by definition of } d_{(N,h)}}}{\leq} \ell_{(N,h)}(f \circ p) \underset{\substack{\uparrow \\ \text{by the claim}}}{=} \ell_{(M,g)}(p).$$
It follows that the left-hand side is also less or equal than the infimum over all paths on the right-hand side. Thus $d_{(N,h)}(f(P), f(Q)) \leq d_{(M,g)}(P, Q)$. ■

65.4. The hyperbolic distance function.

We start out with the following definition.

Definition. Let (M, g) be a Riemannian manifold. A path $\gamma\colon [0, a] \to M$ is called a geodesic if the following conditions are satisfied:

(1) γ is smooth,
(2) $\|\gamma'(t)\|_g = 1$ for every $t \in [0, a]$ and
(3) $\ell(\gamma) = d_{(M,g)}(\gamma(0), \gamma(a))$.

The following proposition summarizes the key facts about the distance metric on $\mathbb{D}$.

Proposition 65.11.

(1) Let $r \in (0, 1)$ and $\varphi \in \mathbb{R}$. In the Riemannian manifold $\mathbb{D}$ the path
$$\begin{array}{rcl} \gamma\colon [0, r] & \to & \mathbb{D} \\ t & \mapsto & t \exp(\mathrm{i}\varphi) \end{array}$$
is, up to reparametrization, the unique shortest path from 0 to $r \cdot \exp(\mathrm{i}\varphi)$.
(2)

$$d_{(\mathbb{D},g)}\big(0, r\cdot\exp(\mathrm{i}\varphi)\big) \;=\; \ell(\gamma) \;=\; \int_{t=0}^{t=r} \frac{\sqrt{2}}{1-|\gamma(t)|^2}\|\gamma'(t)\|_{\mathrm{eucl}}\,dt \;=\; \int_{t=0}^{t=r} \frac{\sqrt{2}}{1-t^2}\,dt.$$

(3) Given $\varphi \in \mathbb{R}$ we have

$$\lim_{r\to 1} d_{(\mathbb{D},g)}\big(0, r\cdot\exp(\mathrm{i}\varphi)\big) \;=\; \infty.$$

(4) Given any P and Q in $\mathbb{D}$ there is a unique geodesic from P to Q.

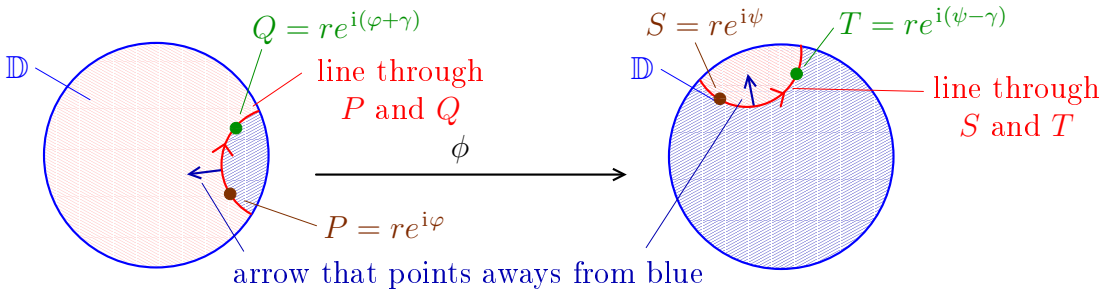
Proof. The first statement can be deduced easily from [**Bon2009**, Lemma 2.4 and Proposition 2.22]. The second statement follows immediately from the first statement, and the third statement is a straightforward consequence of the second statement. For the fourth statement, note that it suffices to prove it for $P = 0$. In that case it is a straightforward consequence of (1). ■

Lemma 65.12. Let (P, Q) and (S, T) be two pairs of distinct points on $\mathbb{D}$ such that $d(P, Q) = d(S, T)$. Then there exists a unique Möbius transformation ϕ with $\phi(P) = S$ and $\phi(Q) = T$.

Example. Let $r \in (0, 1)$, $\varphi \in [0, 2\pi]$, $\psi \in [0, 2\pi]$ and $\gamma \in (0, \pi)$. We consider the pairs of points $P = r\cdot\exp(\mathrm{i}\varphi), Q = r\cdot\exp(\mathrm{i}(\varphi+\gamma))$ and $S = r\cdot\exp(\mathrm{i}\psi), T = r\cdot\exp(\mathrm{i}(\psi-\gamma))$. We then have

$$d(S,T) \;=\; d(T,S) \;=\; d(r\cdot\exp(\mathrm{i}(\psi-\gamma)), r\cdot\exp(\mathrm{i}\psi)) \underset{\substack{\uparrow\\ \text{since rotation by } \varphi+\gamma-\psi \text{ is an isometry}}}{=} d(r\cdot\exp(\mathrm{i}\varphi), r\cdot\exp(\mathrm{i}(\varphi+\gamma))).$$

By Lemma 65.12 there exists a Möbius transformation ϕ with $\phi(P) = S$ and $\phi(Q) = T$. This Möbius transformation is illustrated in the figure below. Recall that from Proposition 65.4 and Lemma 65.6 we know that Möbius transformations preserve hyperbolic lines and that they are orientation-preserving. In particular ϕ sends the hyperbolic line through P and Q to the hyperbolic line through S and T. Since ϕ is orientation-preserving it "swaps" the two sides of the hyperbolic line. We refer to the figure below for an illustration.



Proof of Lemma 65.12. It follows from Proposition 65.4 (3) that, without loss of generality, we can assume that $P = 0$ and $S = 0$. We write $Q = r\cdot\exp(\mathrm{i}\varphi)$ and $T = \exp(\mathrm{i}\psi)$. It is a direct consequence of Proposition 65.11 (1) and our hypothesis that

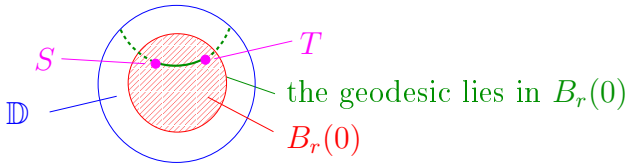
$$d(0, r\cdot\exp(\mathrm{i}\varphi)) \;=\; d(P,Q) \;=\; d(S,T) \;=\; d(0, s\cdot\exp(\mathrm{i}\psi))$$

that we have $r = s$. But this means that we can apply the rotation by the angle $\varphi - \psi$, which is a Möbius transformation, to turn (P, Q) into (S, T). This proves the existence of

ϕ. The uniqueness follows from the observation, that we had made already on page 1429, that the only Möbius transformations that fix 0 are precisely rotations. ■

Lemma 65.13. Let $P \in \mathbb{D}$ and $r > 0$. Let $S, T \in B_r(P)$. Then the image of the unique geodesic in $\mathbb{D}$ from S to T lies in $B_r(P)$.

Proof. As usual, this time by Propositions 65.4, 65.6 and 65.11, it suffices to prove the lemma for $P = 0$. In this special case the reader will have no troubles providing the elementary argument. We also refer to the figure below for an illustration. ■



Next we recall the following definition from page 289.

Definition. We say that a metric space (X, d) is complete if every Cauchy sequence in (X, d) converges.

The present setting motivates the following definition.

Definition. We say that a connected Riemannian manifold (M, g) is complete if the corresponding metric space $(M, d_{(M,g)})$ is complete.

In the Compact-Complete Proposition 6.10 we showed that every compact metric space is complete. The Riemannian manifolds $\mathbb{D}$ and $\mathbb{H}$ are of course not compact, but nonetheless, as we will now see, we can use the Compact-Complete Proposition 6.10 to show that $\mathbb{D}$ and $\mathbb{H}$ are complete.

Lemma 65.14. The Riemannian manifolds $\mathbb{D}$ and $\mathbb{H}$ are complete.

Proof. By Proposition 65.3 the two Riemannian manifolds $\mathbb{D}$ and $\mathbb{H}$ are isometric. Therefore it suffices to show that $\mathbb{D}$ is complete. We denote by $d = d_{(\mathbb{D},g)}$ the metric on $\mathbb{D}$ corresponding to the Riemannian structure. Let $\{a_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to d. An elementary argument using the triangle inequality and the definition of a Cauchy sequence shows that there exists a $C \in \mathbb{R}$ such that $d(0, a_n) \leq C$ for all $n \in \mathbb{N}$.

It follows from Proposition 65.11 that there exists an $r \in (0, 1)$ such that all a_n are contained in the closed Euclidean ball $\overline{B}_r^{eucl}(0)$. By the Heine-Borel Theorem 1.24 this ball is a compact subset of $\mathbb{R}^2$, viewed with the usual topology. It follows from Proposition 65.9 that $\overline{B}_r^{eucl}(0)$ is also compact in the topology defined by d. Now we see that $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the compact metric space $(\overline{B}_r^{eucl}(0), d)$. It follows from the Compact-Complete Proposition 6.10 that the sequence converges. ■

Exercises for Chapter 65.

Exercise 65.1. Does the smooth manifold $B^2 \setminus \{0\}$ admit a complete Riemannian metric?

66. The universal covering of surfaces

Throughout this chapter we mean by a *surface* a connected 2-dimensional smooth manifold. For example the surface of genus g, $g \geq 0$, but also $\mathbb{C}$, $\mathbb{C} \setminus \{0,1\}$ and the result of removing finitely many points from Σ_g are surfaces.

Definition. Let M be a surface. We say that a Riemannian structure g on M is hyperbolic if the following two conditions are satisfied:

(1) (M,g) is locally isometric to $\mathbb{D}$, i.e. if given any $P \in M$ there exists an isometry $\Phi\colon U \to V$ from an open neighborhood U of P to an open subset V of $\mathbb{D}$,
(2) (M,g) is complete.

If g is hyperbolic, then we refer to (M,g) as a hyperbolic surface.

Remark. By the Compact-Complete Proposition 6.10 the condition (2) of the definition is automatically satisfied if M is a closed surface.

In this chapter we will pursue the following three goals:

(1) we want to show that for any $g \geq 2$ the surface of genus g is a complex manifold,
(2) we intend to prove that for any $g \geq 2$ the surface of genus g is hyperbolic,
(3) given $g \geq 2$ we want to determine the universal covering of the surface of genus g.

66.1. Hyperbolic surfaces.

In this section we will show that the surfaces of genus ≥ 2 are complex manifolds and that they are hyperbolic. In the subsequent subsection we will then consider more general types of surfaces.

Definition. Let M be a 2-dimensional topological manifold without boundary. A Möbius structure for M is a family of homeomorphisms $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ from open subsets of M to open subsets of $\mathbb{D}$ such that $\bigcup_{i\in I} U_i = M$ and such that for any $i,j \in I$ the transition map

$$\underbrace{\Phi_i(U_i \cap U_j)}_{\subset \mathbb{D}} \xrightarrow{\left(\Phi_i|_{U_i\cap U_j}\right)^{-1}} U_i \cap U_j \xrightarrow{\Phi_j|_{U_i\cap U_j}} \underbrace{\Phi_j(U_i \cap U_j)}_{\subset \mathbb{D}}$$

is given by a Möbius transformation. Sometimes we refer to a smooth manifold together with a Möbius structure as a Möbius manifold.

Example.

(1) We will always view $\mathbb{D}$ as equipped with the Möbius structure that is given by the chart $\mathrm{id}\colon \mathbb{D} \to \mathbb{D}$.
(2) Any open subset of a Möbius manifold is also a Möbius manifold.

The following lemma says that Möbius structures are useful for solving two problems at once: we can use them to show that a smooth manifold is a complex manifold and we can use them to show that a smooth manifold has a hyperbolic structure.

Lemma 66.1. Let M be a surface and let $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ be a Möbius structure for M. Then the following hold:

(1) The charts form a holomorphic atlas for M, in particular M is a complex 1-dimensional smooth manifold.

(2) (a) The smooth manifold M admits a unique Riemannian structure g such that all the charts in the atlas $\{\Phi_i \colon U_i \to V_i\}_{i\in I}$ are isometries.
(b) If the Riemannian structure from (a) is complete, then (M, g) is hyperbolic.

In the following we will always view a Möbius manifold as a 1-dimensional complex manifold and as a Riemannian manifold with the structures coming from Lemma 66.1.

Proof. Let M be a surface and let $\{\Phi_i \colon U_i \to V_i\}_{i\in I}$ be a Möbius structure for M.

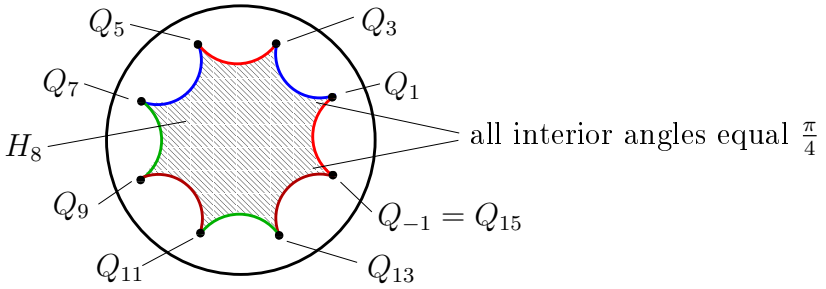
(1) The first statement is an immediate consequence of the definitions and of the fact, proved in Proposition 65.4 (2), that Möbius transformations are biholomorphisms.
(2) (a) We denote by g the Riemannian structure on $\mathbb{D}$. We define a Riemannian structure on M as follows. Let $P \in M$. We pick a chart $\Phi_i \colon U_i \to V_i$ from our Möbius structure and we define $h_P := \Phi_i^*(g_{\Phi_i(P)})$. Now suppose that we had picked a different chart Φ_j from the Möbius structure. The transition map from Φ_i to Φ_j is by definition a Möbius transformation, which is an isometry by Proposition 65.4 (2). It follows easily that Φ_j gives rise to the same definition of h_P. It is now straightforward to verify that this defines a Riemannian structure on M and that it has the desired properties.
(b) This statement is an immediate consequence of (a) and of the definition of a hyperbolic surface. ■

The following proposition gives in particular an affirmative answer to Question 32.25.

Proposition 66.2. Let $g \geq 2$. Then the following hold:
(1) The surface of genus g admits a Möbius structure.
(2) The surface of genus g admits the structure of a 1-dimensional complex manifold.
(3) The surface of genus g admits a hyperbolic Riemannian structure.

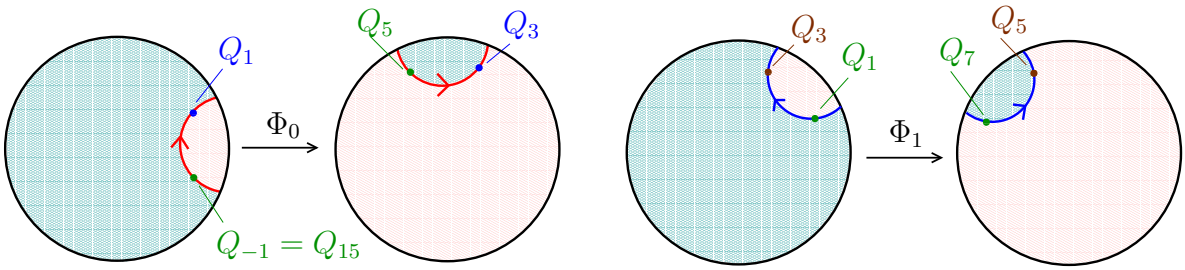
Proof. We prove the proposition for the case that $g = 2$. The general case is proved by almost the same argument.

We first show that the surface of genus 2 admits a Möbius structure. By [**Bon2009**, Lemma 5.3] (see also [**Frie2016a**, Lemma 21.5]) there exists an $r > 0$ such that the interior angle at any vertex of the hyperbolic octagon H_8 in $\mathbb{D}$, defined by $Q_k = r \cdot \exp(2\pi \mathrm{i} k/16)$, $k = 1, 3, \ldots, 15$, equals $\frac{\pi}{4}$.[763]

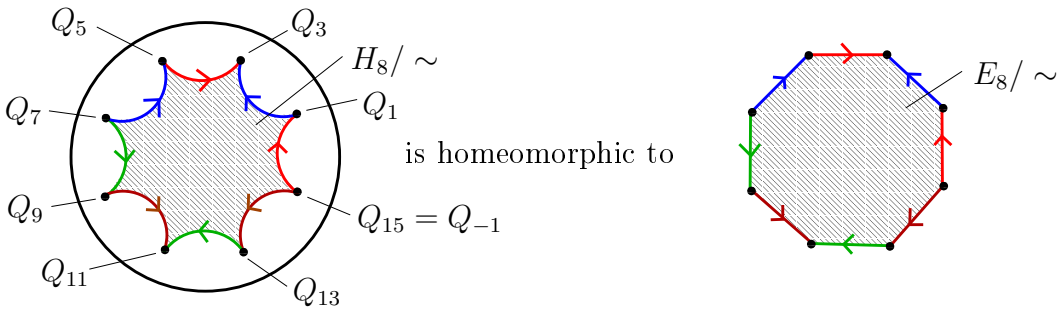


[763]Here is a sketch for the proof: For $r \to 0$ the interior angle converges to the interior angle of a Euclidean regular octagon, namely $\frac{3\pi}{4}$, whereas for $r \to \infty$ the interior angle converges to 0. The interior angle changes continuously with r (that is sort of clear, except that it is somewhat painful to provide a rigorous proof), so by the Intermediate Value Theorem there exists an r such that the interior angle equals $\frac{\pi}{4}$.

Let $k \in \{0,1,4,5\}$. By Lemma 65.12 and the example on page 1437 there exists a unique Möbius transformation Φ_k with $\Phi_k(Q_{2k-1}) = Q_{2k+5}$ and $\Phi_k(Q_{2k+1}) = Q_{2k+3}$. The Möbius transformations Φ_0 and Φ_1 are illustrated in the figure below.



Given $k \in \{0,1,4,5\}$ we declare any point P on the edge connecting Q_{2k-1} and Q_{2k+1} to be equivalent to $\Phi_k(P)$. We denote by $\sim$ the equivalence relation that is generated by these equivalences. It is straightforward to see that $H_8/\sim$ is homeomorphic to the surface of genus 2 that we defined on page 265 as $E_8/\sim$. In the following we denote by $p\colon H_8 \to H_8/\sim$ the natural projection.



Now we will give an explicit Möbius structure for $H_8/\sim$. More precisely, given any $P \in H_8/\sim$ we will give a chart ψ_P such that the transitions maps are given by Möbius transformations. So let $P \in H_8/\sim$.

(1) If $P = p(z)$ for some $z \in \mathring{H}_8$, then the map

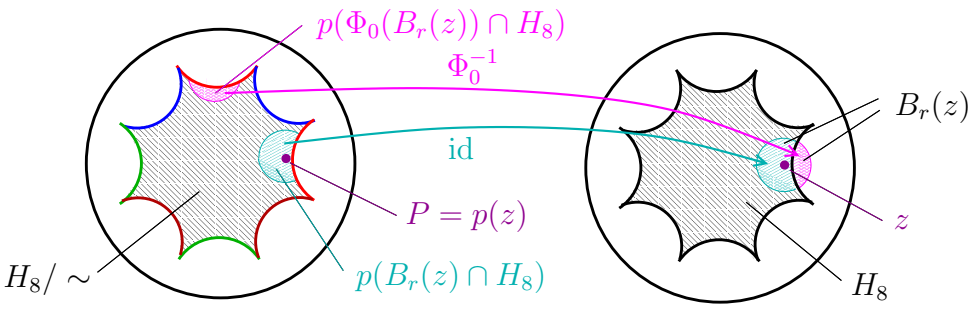
$$\begin{array}{rcl} \psi_P\colon p\big(\mathring{H}_8\big) & \to & \mathring{H}_8 \\ p(w) & \mapsto & w \end{array}$$

is a chart around $P = p(z)$.

(2) Now suppose that $P = p(z)$ where z lies in the interior of one of the eight edges of ∂H_8. In the following we deal with the case that z lies on the edge from Q_{-1} to Q_1. We pick an $r > 0$ such that $B_{2r}(z)$ does not contain any of the vertices $Q_1, Q_3, \ldots, Q_{15}$ and does not hit any of the other edges. Then we consider the map

$$\begin{array}{rcl} \psi_P\colon p(B_r(z) \cap H_8) \cup p(\Phi_0(B_r(z)) \cap H_8) & \to & B_r(z) \\ p(w) & \mapsto & \begin{cases} w, & \text{if } w \in B_r(z) \cap H_8 \\ \Phi_0^{-1}(w), & \text{if } w \in \Phi_0(B_r(z)) \cap H_8. \end{cases} \end{array}$$

This map is a chart around $P = p(z)$. We refer to the figure below for an illustration of the definition of the map ψ_P.



(3) Finally suppose that $P = p(Q_1) = \cdots = p(Q_{15})$. We choose $r \in (0, \frac{1}{2}d(Q_{-1}, Q_1))$. Given $k \in \{1, 3, \ldots, 15\}$ we define $\gamma(k)$ via the following table:

k	1	3	5	7	9	11	13	15
$\gamma(k)$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{1}{2}\pi$	$\frac{7}{4}\pi$	0	$\frac{1}{4}\pi$	$\frac{3}{2}\pi$.

Recall that we had chosen the octagon H_8 such that the angle at any vertex equals $\frac{\pi}{4}$. For $k = 1, 3, \ldots, 15$ we can therefore apply Lemma 65.8 to find a unique Möbius transformation θ_k with the following two properties:

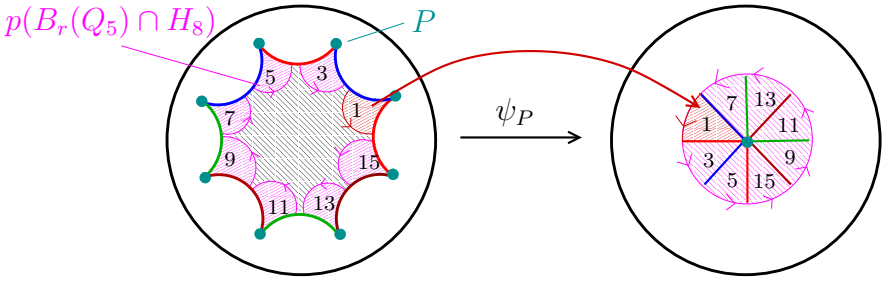
(a) $\theta_k(Q_k) = 0$,

(b) The map θ_k applied to the two rays emanating from Q_k equals the two rays $\{s \cdot \exp(\mathrm{i}\gamma(k)) \mid s \geq 0\}$ and $\{s \cdot \exp(\mathrm{i}(\gamma(k) + \frac{\pi}{4})) \mid s \geq 0\}$ emanating from 0.

We consider the map

$$\begin{aligned} \psi_P \colon p(B_r(Q_1) \cap H_8) \cup \cdots \cup p(B_r(Q_{15}) \cap H_8) &\to B_r(0) \\ p(z) &\mapsto \theta_k(z), \quad \text{if } z \in B_r(Q_k) \cap H_8. \end{aligned}$$

We leave it to the reader to verify that ψ_P is well-defined and that it is indeed a homeomorphism. This map is a chart around $P = p(z)$. We refer to the figure below for an illustration of the definition of the map ψ_P.



It is straightforward to verify that all the transition maps between the above maps are given by Möbius transformations. We have thus shown that the maps $\{\psi_P\}_{P \in H_8}$ form a Möbius structure for $H_8/\sim$. This concludes the proof of (1).

Now we turn to the proof of (2). The existence of a Möbius structure together with Lemma 66.1 (2) shows immediately that the surface of genus 2 is a 1-dimensional complex manifold.

Finally we provide the proof of (3). In light of (1) and Lemma 66.1 (2) it suffices to show that the Riemannian manifold $(H_8/\sim, g)$ is complete. By Proposition 65.9 the topology on M defined by the metric $d_{(M,g)}$ agrees with the given topology of M. Since M is compact it now follows from the Compact-Complete Proposition 6.10 that (M, g) is complete. ∎

66.2. **More hyperbolic structures on the surfaces of genus $g \geq 2$p.** Now we want to study the question, how many hyperbolic metrics does a surface of genus g admit. To make this question more precise, we introduce the following definition.

Definition. Let M be a surface of genus $g \geq 2$ and let h, h' be two hyperbolic metrics on M. We say h and h' are equivalent if there exists an isometry from (M, h) to (M, h').

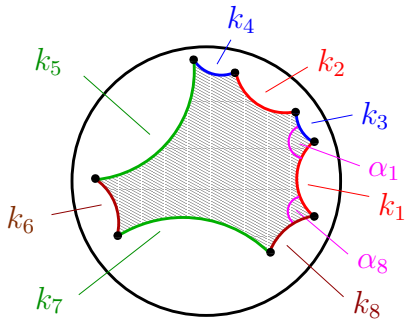
This raises the question, whether a surface of genus $g \geq 2$ admits non-equivalent hyperbolic metrics. We return to the construction of a hyperbolic metric in Proposition 66.2. In our construction we started out with a regular $4g$-gon with interior angle $\frac{2\pi}{4g}$ and we identified pairs of sides. Two facts were crucial to obtain a hyperbolic metric:

(1) the sides were identified via an isometry,
(2) the sum of the interior angles equals 2π.

Or put differently, any choice of a $4g$-gon with side lengths $k_1, \dots, k_{4g}$ and interior angles $\alpha_1, \dots, \alpha_{4g}$ that satisfy the equations

$$k_1 = k_3,\ k_2 = k_4,\ \dots,\ k_{4g-3} = k_{4g-1},\ k_{4g-2} = k_{4g} \quad \text{and} \quad \alpha_1 + \dots + \alpha_{4g} = 2\pi$$

gives rise to a hyperbolic metric on a topological space that is homeomorphic to the surface of genus g. It is of course not clear whether different parameters give rise to non-equivalent hyperbolic structures. One way to distinguish them, in principle at least, is to look at the shortest length of a closed curve that is not null-homotopic. We will not elaborate on this procedure.



How many such $4g$-gons are there? On the one hand we have parameters $k_1, k_3, \dots, k_{4g-1}$ and $\alpha_1, \dots, \alpha_{4g}$. On the other hand we have the condition that $\alpha_1 + \dots + \alpha_{4g} = 2\pi$. Furthermore we want a $4g$-gon, so the last angle and the length of the last edge are determined by the others. Thus we obtain two other conditions. Summarizing we have $2g + 4g = 6g$ parameters that need to satisfy 3 conditions. Of course it is not clear which of these Riemannian structures are isometric. This naive discussion suggests that the space of hyperbolic metrics is $(6g-3)$-dimensional. Surprisingly this is not too far off the correct answer, in fact in [**BP1992**, Chapter B.4] the following proposition is proved.

Proposition 66.3. The topological space of hyperbolic metrics on a surface of genus $g \geq 2$ can be viewed as a smooth manifold of dimension $6g - 6$.

We will not make use of this proposition. The topological space of hyperbolic metric is called the *Teichmüller space*, more information can be found at

`https://en.wikipedia.org/wiki/Teichmueller_space`.

The theory is named after Oswald Teichmüller (1913-1943) who was a brilliant mathematician and an ardent Nazi, see

`https://en.wikipedia.org/wiki/Oswald_Teichmueller`

`http://www-history.mcs.st-andrews.ac.uk/Biographies/Teichmuller.html`

and

`https://de.wikipedia.org/wiki/Oswald_Teichmueller`.

66.3. **More examples of hyperbolic surfaces.** In this section we will generalize the discussion from the previous section to surfaces that are non-compact.
In the following let $Q_1 = 1, Q_2 = \mathrm{i}, Q_3 = -1$ and $Q_0 = Q_4 = -\mathrm{i}$. We denote by H_4 the closed, non-compact subset of $\mathbb{D}$ that is bounded by the four hyperbolic lines with endpoints Q_k, Q_{k+1} where $k = 0, 1, 2, 3$. We refer to the figure on page 1445 for an illustration.
We consider the Möbius transformations[764]

$$\Phi_1(z) := (-\mathrm{i}) \cdot \frac{z - \frac{-1+\mathrm{i}}{2}}{1 - \frac{-1-\mathrm{i}}{2} z} \quad \text{and} \quad \Phi_2(z) := (-\mathrm{i}) \cdot \frac{z + \frac{-1-\mathrm{i}}{2}}{1 + \frac{-1+\mathrm{i}}{2} z}.$$

We record a few key properties of Φ_1 and Φ_2 in a lemma.

Lemma 66.4.

(1) We have $\Phi_1(Q_2) = Q_4$ and $\Phi_1(Q_3) = Q_3$ and we also have $\Phi_2(Q_2) = Q_4$ and $\Phi_1(Q_1) = Q_1$.
(2) The restriction of Φ_1 to the hyperbolic line with endpoints Q_3 and Q_2 is given by reflection in the x-axis.[765]
(3) The restriction of Φ_2 to the hyperbolic line with endpoints Q_1 and Q_2 is given by reflection in the x-axis.

Proof. The first statement follows immediately from plugging in the points. A straightforward, albeit slightly painful calculation shows that for any $\alpha \in \mathbb{R}$ we have

$$\Phi_1\big(-1 + \mathrm{i} + e^{-\mathrm{i}\alpha}\big) \;=\; -1 - \mathrm{i} + \exp(\mathrm{i}\alpha).$$

This proves the second statement. The third statement is proved by a similarly tedious calculation. ∎

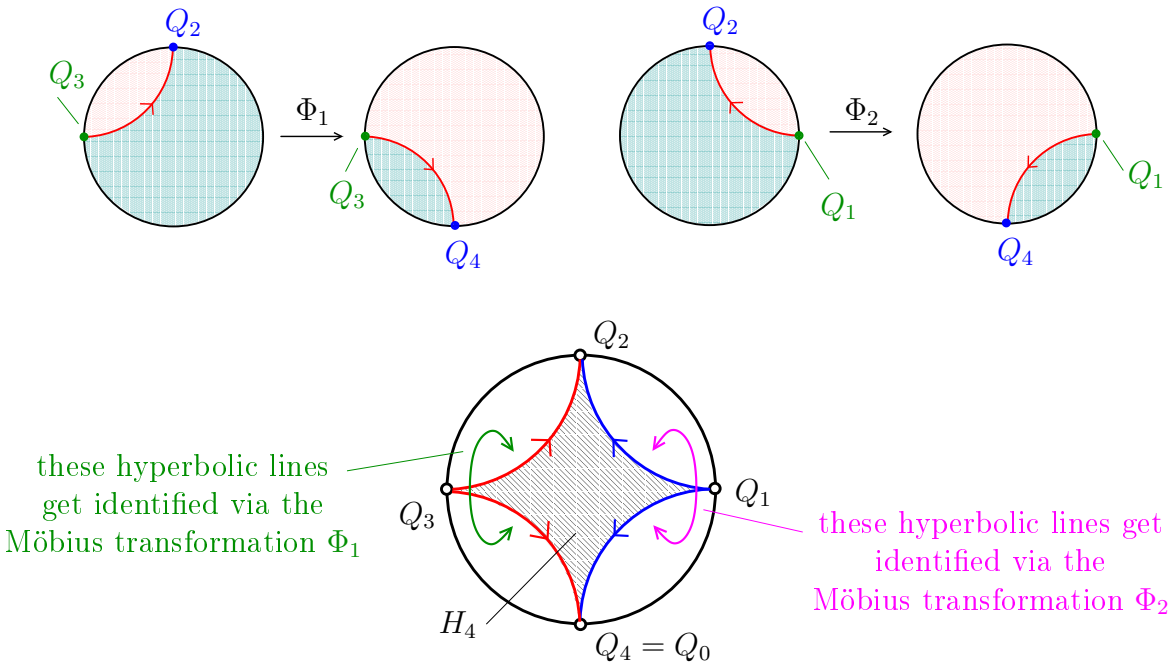
Remark. We consider the Möbius transformation Φ_1 in more detail. It sends the hyperbolic line with endpoints Q_3 and Q_2 to the hyperbolic line with endpoints Q_3 and Q_4. In particular it sends the two halves determined by the hyperbolic lines to one another. Which half gets sent to which half is determined by the fact that Φ_1 is orientation-preserving. We refer to the figure below for an illustration of the Möbius transformations Φ_1 and Φ_2.

Given $k \in \{1, 2\}$ we declare any point P on the line with the endpoints Q_k and Q_{k+1} to be equivalent to $\Phi_{3-k}(P)$. We denote by $\sim$ the equivalence relation that is generated by these equivalences.

---

[764] Recall that a Möbius transformation of $\mathbb{D}$ is a map of the form $\psi(z) = e^{\mathrm{i}\theta} \cdot \frac{z-a}{1-\bar{a}z}$ for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$.

[765] The Möbius transformation Φ_1 sends the hyperbolic line with endpoints Q_3 and Q_2 to the hyperbolic line with endpoints Q_3 and Q_4. There are many such Möbius transformations, but the Möbius transformation we picked has the extra property, that on the hyperbolic line it is the "obvious map".



Proposition 66.5.

(1) The topological space $H_4/\sim$ admits a Möbius structure. In particular it is a 1-dimensional complex manifold.
(2) The Riemannian structure on $H_4/\sim$ coming from (1) and Lemma 66.1 is complete.
(3) The topological space $H_4/\sim$ admits a hyperbolic metric.

Proof. We start out with the proof of (1). We have to show that $H_4/\sim$ admits a Möbius structure. The following argument is almost identical to the proof of Proposition 66.2 (1).

We denote by $p\colon H_4 \to H_4/\sim$ the projection . In the following let $P \in H_4/\sim$.

(1) If $P = p(z)$ for some $z \in \mathring{H}_4$, then

$$\begin{array}{rcl} \psi_P\colon p\big(\mathring{H}_4\big) & \to & \mathring{H}_4 \\ p(w) & \mapsto & w \end{array}$$

is a chart around $P = p(z)$.

(2) Now suppose that $P = p(z)$ where z lies on ∂H_4. In the following we deal with the case that z lies on the edge from Q_1 to Q_2. All other cases are dealt with almost the same way. We pick an $r > 0$ such that $B_{2r}(z)$ intersects no other component of ∂H_4. We consider the map

$$\begin{array}{rcl} \psi_P\colon p(B_r(z) \cap H_4) \cup p(\Phi_2(B_r(z)) \cap H_4) & \to & B_r(z) \\ p(w) & \mapsto & \begin{cases} w, & \text{if } w \in B_r(z) \cap H_4 \\ \Phi_2^{-1}(w), & \text{if } w \in \Phi_1^{-1}(B_r(z)) \cap H_4. \end{cases} \end{array}$$

This map is a chart around $P = p(z)$.

The charts that we just constructed form an atlas for $H_4/\sim$. It follows immediately from the definitions that all transition maps are given by Möbius transformations. It follows that the maps ψ_P form a Möbius structure for $H_4/\sim$. As in the proof of Proposition 66.2

it follows that $H_4/\!\sim$ is a 1-dimensional complex manifold and that these charts satisfy the conditions of Lemma 66.1. This concludes the proof of (1).

We continue with the proof of (2). We have to show that the Riemannian structure on $H_4/\!\sim$ coming from (1) and Lemma 66.1 is complete. The subsequent argument is a variation on the proof of the Compact-Complete Proposition 6.10.

We denote by d the metric corresponding to the Riemannian structure on $H_4/\!\sim$. Let $\{a_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $H_4/\!\sim$ with respect to d. As in the proof of the Compact-Complete Proposition 6.10 we see that there exists a $C \in \mathbb{R}$ such that $d(0, a_n) \leq C$ for all $n \in \mathbb{N}$.

Claim. There exists an $r \in [0, 1)$ such that for any $z \in H_4$ with $d(0, p(z)) \leq C$ we have $|z| \leq r$.

Proof. Given $s \in [0, 1)$ we set

$$g(s) \;:=\; \int_{t=0}^{t=s} \frac{\sqrt{2}}{1-t^2}\, dt.$$

Clearly we have $\lim_{s\to 1} g(s) = \infty$. In particular there exists an $r \in [0, 1)$ such that $g(s) > C+1$ for all $s \geq \frac{1}{2}r$.

We claim that r has the desired property. So let $z = x + \mathrm{i}y \in H_4$ with $|z| > r$. Since H_4 is a subset of the square spanned by ± 1 and $\pm i$ we see that we have $|z| \leq |x| + |y|$. Put differently, we have $|x| \geq \frac{1}{2}r$ or $|y| \geq \frac{1}{2}r$. Without loss of generality we can assume that $|x| \geq \frac{1}{2}r$.

We need to show that $d(0, p(z)) \geq C + 1$. It suffices to show that for any piecewise smooth path $\gamma\colon [0, a] \to H_4/\!\sim$ with $\gamma(0) = 0$ and $\gamma(a) = p(z)$ we have $\ell(\gamma) \geq C + 1$.

We denote by

$$\begin{array}{rcl} q\colon H_4/\!\sim & \to & (-1, 1) \\ [x + \mathrm{i}y] & \mapsto & x \end{array}$$

the projection onto the real part. It follows from Lemma 66.4 that this map is indeed well-defined. A straightforward calculation shows that for all $t \in [0, a]$ we have

$$\big\|\gamma'(t)\big\|_{H_4/\!\sim} \;\geq\; \big\|(q\circ\gamma)'(t)\big\|_{\mathbb{D}}.$$

It follows that

$$\ell(\gamma) \;=\; \int_0^a \big\|\gamma'(t)\big\|_{H_4/\!\sim} dt \;\underset{\text{above inequality}}{\geq} \int_0^a \big\|(q\circ\gamma)'(t)\big\|_{\mathbb{D}}\, dt \;\underset{\text{definition of } d_{(\mathbb{D},g)}}{\geq} d_{(\mathbb{D},g)}(0, \overbrace{q(\gamma(a))}^{=x}) \;=\; g(|x|) \underset{\text{Proposition 65.11}}{\overset{\text{since } |x|\geq \frac{1}{2}r}{\geq}} C+1.$$

⊞

By the above claim the sequence $\{a_n\}_{n\in\mathbb{N}}$ lies in the compact set $p(\{z \in H_4 \mid |z| \leq r\})$. But by the Compact-Complete Proposition 6.10 the metric space $p(\{z \in H_4 \mid |z| \leq r\})$ is complete, so the sequence $\{a_n\}_{n\in\mathbb{N}}$ converges in $p(\{z \in H_4 \mid |z| \leq r\})$, in particular it converges in $p(H_4) = H_4/\!\sim$. Thus we have proved that $H_4/\!\sim$ is complete. This concludes the proof of (2).

Finally Statement (3) is an immediate consequence of the definition of a hyperbolic metric and of Statement (2). ■

We had just seen in Proposition 66.5 that $H_4/\!\sim$ admits a Möbius structure, in particular it is a 1-dimensional complex manifold. It is relatively straightforward to show that $H_4/\!\sim$ is homeomorphic to the sphere minus three points, or equivalently, to $\mathbb{C}$ minus two points.

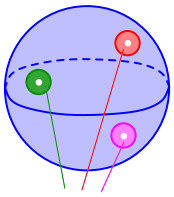
Now we show the much stronger statement that $H_4/\sim$ is actually biholomorphic to $\mathbb{C} \setminus \{0,1\}$.

Proposition 66.6. There exists a biholomorphism $H_4/\sim \to \mathbb{C} \setminus \{0,1\}$.

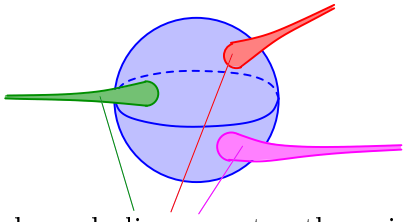
Remark. We equip $H_4/\sim$ with the Riemannian metric g coming from the Möbius structure and we equip $\mathbb{C} \setminus \{0,1\}$ with the usual Euclidean Riemannian metric that we denote by h.

By Proposition 66.5 the metric space $(H_4/\sim, g)$ is complete whereas $(\mathbb{C} \setminus \{0,1\}, h)$ is evidently not complete. Thus we see that the biholomorphism $H_4/\sim \to \mathbb{C} \setminus \{0,1\}$ from Proposition 66.6 is *not* an isometry.

sphere minus three points



neighborhoods of the points we removed



in hyperbolic geometry the neighborhoods become infinitely long open annuli, called cusps

Proof of Proposition 66.6. We denote by Δ the open subset of $\mathbb{D}$ bounded by the three hyperbolic lines with endpoints $-1, 1$ and i. (We refer to the figure below for an illustration of Δ.) It follows from applying the Riemann Mapping Theorem 32.12 together with Carathéodory's Theorem 32.13 that there exists a biholomorphism $\Psi\colon \Delta \to \mathbb{D}$ that extends to a homeomorphism $\Psi\colon \overline{\Delta} \to \overline{\mathbb{D}}$. We combine Ψ with the biholomorphism from $\mathbb{D}$ to $\mathbb{H}$ coming from Proposition 65.3. This way we obtain a biholomorphism $\Phi\colon \Delta \to \mathbb{H}$ that extends to a homeomorphism $\Phi\colon \overline{\Delta} \to \overline{\mathbb{H}}$.

Using Lemma 65.5 we can, if necessary, compose Φ with a Möbius transformation of $\mathbb{H}$, to arrange that $\Phi(i) = \infty$ and that $\Phi(\{-1,1\}) = \{0,1\}$.

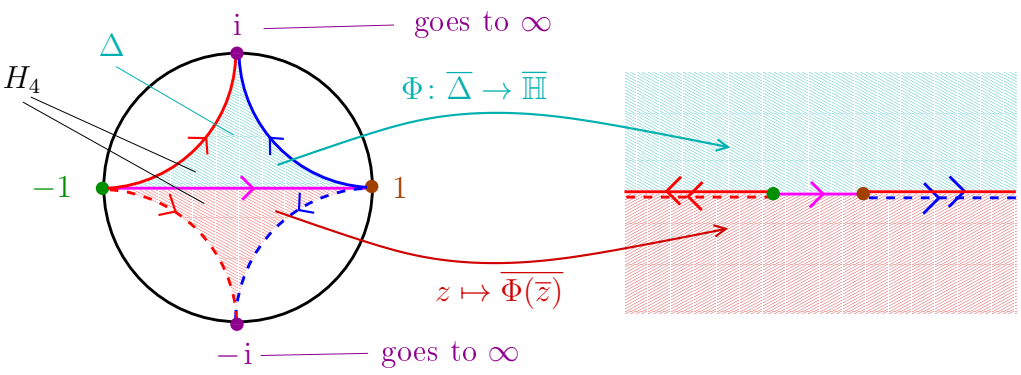
Note that Φ restricts to a homeomorphism $\Phi\colon \partial\overline{\Delta} \to \partial\overline{\mathbb{H}} = \mathbb{R} \cup \{\infty\}$. By the above we have $\Phi(\partial\overline{\Delta}) = \mathbb{R} \setminus \{0,1\}$. Now we consider the map

$$\begin{aligned} \Psi\colon H_4 &\to \mathbb{C} \setminus \{0,1\} \\ z &\mapsto \begin{cases} \Phi(z), & \text{if } z \in \overline{\Delta}, \\ \overline{\Phi(\overline{z})}, & \text{if } \overline{z} \in \overline{\Delta}, \end{cases} \end{aligned}$$

Note that by the above we have $f((-1,1)) = (0,1)$. Thus it follows from the Schwarz Reflection Principle, see Proposition 32.7, that the restriction of Ψ to the interior of H_4 is holomorphic. It follows from Lemma 66.4 that if P and Q are two equivalent points on ∂H_4, then $\Psi(P) = \Psi(Q)$. Therefore the map Ψ factors through a map $H_4/\sim \to \mathbb{C} \setminus \{0,1\}$. Using once again the Schwarz Reflection Principle one can show that this map is in fact a biholomorphism. We leave the verification of this step to the reader. ∎

Definition. Given $g \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$ we now denote by $F_{g,n}$ the surface that is obtained from the surface of genus g by removing n points. We refer to $F_{g,n}$ as the n-punctured surface of genus g.

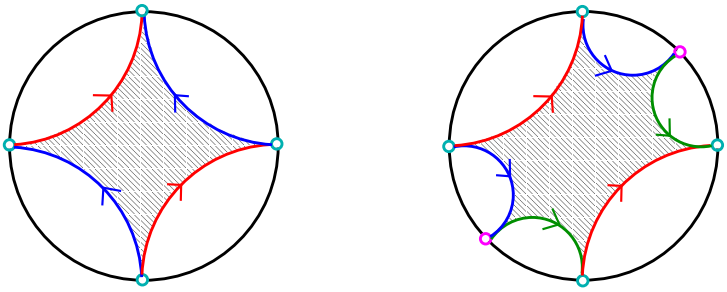
For example the three-punctured sphere $F_{0,3}$ is the sphere with three points removed. As we pointed out above, $F_{0,3}$ is diffeomorphic to $\mathbb{C} \setminus \{0,1\}$. In particular, as we have just



seen, the three-punctured sphere admits a hyperbolic structure. The following theorem says that "most" surfaces admit a hyperbolic structure.

Theorem 66.7. Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}_0$. If $2g + n - 2 > 0$, then $F_{g,n}$ admits a hyperbolic structure.

Sketch of a proof. Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}_0$. If $2g + n - 2 > 0$ then variations on the argument of Propositions 66.2 and 66.5 show that $F_{g,n}$ admits a hyperbolic metric. In the figure below we sketch the constructions needed to show that the once-punctured and the twice-punctured torus have a Möbius structure. The proof that the corresponding Riemannian manifolds are complete is similar to the proof of Proposition 66.5. ■



once punctured torus twice punctured torus

66.4. The universal covering of surfaces.

Definition. Let $f\colon M \to N$ be a map between two Möbius manifolds.

(1) We say that f is a Möbius map if for every two chart $\Phi\colon U \to V$ in the Möbius structure of M and any chart $\Psi\colon X \to Y$ in the Möbius structure of N the restriction of the map

$$\underbrace{\Phi(U \cap f^{-1}(X))}_{\subset \mathbb{D}} \xrightarrow{\Psi \circ f \circ \Phi^{-1}} \underbrace{\Psi(X \cap f(U))}_{\subset \mathbb{D}}$$

to any component of $\Phi(U \cap f^{-1}(X))$ is given by a Möbius transformation.

(2) We say that f is a Möbius isomorphism, if f is a Möbius map, if f is a bijection, and if f^{-1} is also a Möbius map.

The following lemma is an immediate consequence of the definitions and the fact, proved in Proposition 65.4, that Möbius transformations are biholomorphisms and isometries.

Lemma 66.8. Let M and N be two Möbius manifolds and let $f\colon M \to N$ be a Möbius map. Then the following two statements hold:

(1) the map f is a local biholomorphism,
(2) the map f is a local isometry.

The following theorem is the main result of this chapter.

Theorem 66.9. (Hadamard's Theorem) Let $g \geq 2$, and let $M = H_{4g}/\sim$ be the surface of genus g with the Möbius structure constructed in Proposition 66.2. Then there exists a covering map $p\colon \mathbb{D} \to M$ with the following properties:

(1) p is a Möbius map,
(2) p is a local isometry,
(3) p is a local biholomorphism.

The hyperbolic disk $\mathbb{D}$ is of course simply connected. The theorem thus says that for any $g \geq 2$ the hyperbolic disk $\mathbb{D}$ is the universal covering of the surface of genus g. Thus we have now answered Question 61.12.
The proof of Theorem 66.9 will require the following two sections.

66.5. **Proof of Theorem 66.9 I.** In this section we will lay the technical foundations for the proof of Theorem 66.9.
In the following lemma we first summarize a few properties of Möbius maps.

Lemma 66.10.

(1) If $U \subset \mathbb{D}$ is a connected subset and if $f\colon U \to \mathbb{D}$ is a Möbius map, then f is given by a Möbius transformation.
(2) Let $f\colon M \to N$ be a map between two Möbius manifolds. If f is locally a Möbius map, then f is a Möbius map.
(3) The composition of two Möbius maps is again a Möbius map.
(4) A bijective Möbius map is a Möbius isomorphism.

Proof.

(1) Let $U \subset \mathbb{D}$ be a connected subset and let $f\colon U \to \mathbb{D}$ be a Möbius map. It follows from Proposition 65.4 (4) that given $P \in U$ there exists a unique Möbius transformation ϕ_P such that $f = \phi_P$ in an open neighborhood of P. We consider the map

$$\begin{array}{rcl} \phi\colon U & \to & \text{set of all Möbius transformations} \\ P & \mapsto & \phi_P. \end{array}$$

We equip the right-hand side of the map ϕ with the discrete topology. It follows from the definitions that this map is continuous. But then it follows from the Connected-to-Discrete Lemma 2.27 that this map is constant. But that is exactly what we had tried to show.
(2) This statement follows immediately from (1) and the definitions.
(3) This statement is an immediate consequence of definitions and the fact, proved in Proposition 65.4, that the compositions of two Möbius transformations is again a Möbius transformation.
(4) Let $f\colon M \to N$ be a bijective Möbius map. In Proposition 65.4 we saw that the inverse of a Möbius transformation is again a Möbius transformation. It follows that

f^{-1} is locally a Möbius map. But then it follows from (2), that f^{-1} is in fact a Möbius map. ■

Lemma 66.11. Let U be an open connected subset of $\mathbb{D}$ and let M be a Möbius manifold. Furthermore let $\Phi, \Psi \colon U \to M$ be two Möbius maps.[766] Suppose one of the following two conditions is satisfied:

(1) There exists an open non-empty subset of U on which Φ and Ψ agree, or
(2) there exists a point $Q \in U$ with $\Phi(Q) = \Psi(Q)$ and a non-zero vector $v \in \mathrm{T}_Q\mathbb{D}$ such that $\mathrm{D}\Phi_Q(v) = \mathrm{D}\Psi_Q(v)$,

then the Möbius maps Φ and Ψ agree on all of U.

Proof. Let U be an open connected subset of $\mathbb{D}$, let M be a Möbius manifold and let $\Phi, \Psi \colon U \to M$ be two Möbius maps. We consider one more statement:

(3) There exists a point $Q \in U$ such that $\Phi(Q) = \Psi(Q)$ and such that $\mathrm{D}\Phi_Q = \mathrm{D}\Psi_Q$.

Clearly we have (1) $\Rightarrow$ (3). We also have (2) $\Rightarrow$ (3). Indeed, the differentials $\mathrm{D}\Phi_Q$ and $\mathrm{D}\Psi_Q$ are homomorphisms between two 1-dimensional complex vector spaces. So if they are agree on one non-zero vector, then they agree on all of the vector space $\mathrm{T}_Q\mathbb{D}$.

So it suffices to show that if (3) is satisfied, then the maps Φ and Ψ agree on all of U. So suppose that there exists a point $Q \in U$ such that $\Phi(Q) = \Psi(Q)$ and such that $\mathrm{D}\Phi_Q = \mathrm{D}\Psi_Q$. We consider

$$Y \;:=\; \{P \in U \mid \Phi(P) = \Psi(P) \text{ and } \mathrm{D}\Phi_P = \mathrm{D}\Psi_P\}.$$

By our hypothesis we know that Y contains Q, in particular Y is non-empty. Since Φ and Ψ are smooth it follows that Y is closed. It remains to show that Y is open.

So let $P \in Y$. We pick a map $\Theta \colon W \to \mathbb{D}$ from the Möbius structure of M such that W is an open neighborhood of $\Phi(P) = \Psi(P)$. We denote by V the component of $\Phi^{-1}(W) \cap \Psi^{-1}(W)$ that contains P. Since Φ and Ψ are Möbius maps it follows from Lemma 66.10 (3) that the maps $\Theta \circ \Phi \colon V \to \mathbb{D}$ and $\Theta \circ \Psi \colon V \to \mathbb{D}$ are also Möbius maps. Since V is connected it follows from Lemma 66.10 (1) that both maps are given by Möbius transformations. These two Möbius transformations agree at P and they have the same differential at P. It follows from Proposition 65.4 (5) that Φ and Ψ agree on an open neighborhood of P. This shows that Y is also open. ■

Definition. Let M be a Möbius manifold and let $Q \in M$ and $r > 0$. We say $B_r^M(Q)$ is small if there exists a Möbius isomorphism $\Omega \colon B_r^{\mathbb{D}}(0) \to B_r^M(Q)$ with $\Omega(0) = Q$.

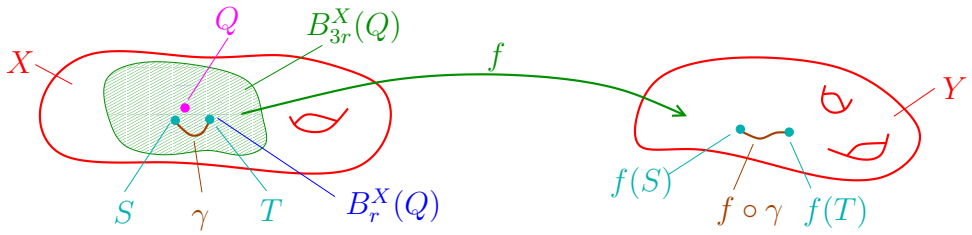
Proposition 66.12. Let M be a Möbius manifold and let $Q \in M$. Then there exists an $r > 0$ such that $B_r^M(Q)$ is small.

In the proof of Proposition 66.12 we will make use of the following lemma.

Lemma 66.13. Let (X, g) and (Y, h) be Riemannian manifolds. Let $Q \in X$ and $r > 0$. Let $f \colon B_{3r}^X(Q) \to (Y, h)$ be a local isometry and let $S, T \in B_r^X(Q)$. Then we have

$$d_Y(f(S), f(T)) \;\leq\; d_X(S, T).$$

[766] Here we view $U \subset \mathbb{D}$ as equipped with the obvious Möbius structure.



Proof. Let $S, T \in B^X_r(Q)$. Furthermore let $\epsilon > 0$. By the triangle inequality we have $d_X(S,T) < 2r$. In particular there exists a piecewise smooth path γ in X from S to T such that $\ell_X(\gamma) < 2r$ and with $\ell_X(\gamma) < d_X(S,T) + \epsilon$. Given any t we have

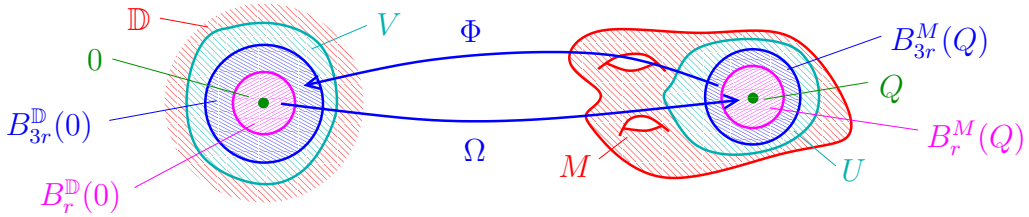
$$d_X(Q, \gamma(t)) \;\leq\; d_X(Q,S) + d_X(S, \gamma(t)) \;<\; r + \ell_X(\gamma) \;<\; 3r.$$

Thus we have shown that the image of γ lies in $B^X_{3r}(Q)$. In particular the image of γ lies in the domain of f. Thus we see that

$$d_Y(f(S), f(T)) \underset{\substack{\uparrow \\ \text{since } f\circ\gamma \text{ is a path from} \\ f(S) \text{ to } f(T)}}{\leq} \ell_Y(f \circ \gamma) \underset{\substack{\uparrow \\ \text{by Lemma 65.10, since} \\ f \text{ is a local isometry}}}{=} \ell_X(\gamma) \;<\; d_X(S,T) + \epsilon.$$

Since this inequality holds for all $\epsilon > 0$ we see that $d_Y(f(S), f(T)) \leq d_X(S,T)$. ∎

Proof of Proposition 66.12. Let M be a Möbius manifold and let $Q \in M$. We pick a map $\Phi\colon U \to V$ from the Möbius structure of M such that U is an open neighborhood of Q. By Proposition 65.4 (3), after possibly composing Φ with a Möbius transformation, we can furthermore assume that $\Phi(Q) = 0$. We write $\Omega := \Phi^{-1}$. By Proposition 65.9 (3) and by the openness of V there exists an $r > 0$ such that $B^M_{3r}(Q) \subset U$ and such that $B^{\mathbb{D}}_{3r}(0) \subset V$.[767]



It follows immediately from Lemmas 66.8 and 66.13 and from our choice of r that the following holds:

(1) For any $S, T \in B^M_r(Q)$ we have $d_{\mathbb{D}}(\Phi(S), \Phi(T)) \leq d_M(S,T)$.
(2) For any $S, T \in B^{\mathbb{D}}_r(Q)$ we have $d_M(\Omega(S), \Omega(T)) \leq d_{\mathbb{D}}(S,T)$.

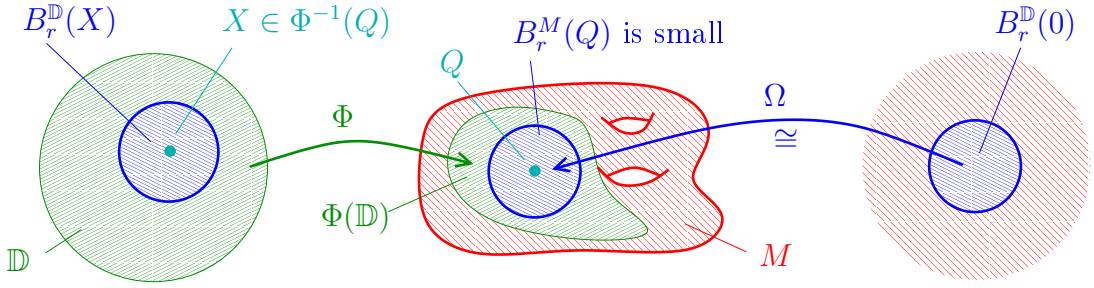
It follows immediately from these two inequalities, from $\Phi(Q) = 0$ and from the fact that $\Phi = \Omega^{-1}$ that Ω restricts to an isometry $\Omega\colon (B^{\mathbb{D}}_r(0), d_{\mathbb{D}}) \to (B^M_r(Q), d_M)$ of metric spaces which is furthermore a bijection. Furthermore, since Φ is a Möbius isomorphism it follows that the inverse map $\Omega = \Phi^{-1}\colon (B^{\mathbb{D}}_r(0), d_{\mathbb{D}}) \to (B^M_r(Q), d_M)$ is also a Möbius isomorphism. ∎

[767] Note the discrete factor 3 in the radius of the ball.

Lemma 66.14. Let M be a Möbius manifold. Let $Q \in M$ and let $r > 0$ such that $B_r^M(Q)$ is small. Let $\Phi\colon \mathbb{D} \to M$ be a Möbius map such that $B_r^M(Q) \subset \Phi(\mathbb{D})$. Then the following hold:

(1) For any $X \in \Phi^{-1}(Q)$ the map Φ restricts to a Möbius isomorphism $B_r^{\mathbb{D}}(X) \to B_r^M(Q)$.
(2) Let $s \leq \frac{r}{2}$ and let $Y \in \mathbb{D}$ such that $\Phi(Y) \in B_s^M(Q)$. Then there exists an $X \in \Phi^{-1}(Q)$ with $Y \in B_s^{\mathbb{D}}(X)$.

Proof. Let M be a Möbius manifold. Let $Q \in M$ and let $r > 0$ such that $B_r^M(Q)$ is small. We pick a Möbius isomorphism $\Omega\colon B_r^{\mathbb{D}}(0) \to B_r^M(Q)$ as in the definition of "small". Let $\Phi\colon \mathbb{D} \to M$ be a Möbius map such that $B_r^M(Q) \subset \Phi(\mathbb{D})$.

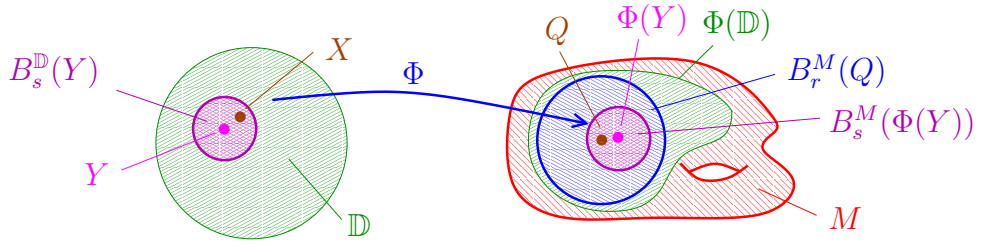


(1) Let $X \in \Phi^{-1}(Q)$. It follows immediately from the fact that Möbius maps are local isometries and from Lemma 66.13 that $\Phi(B_r^{\mathbb{D}}(X)) \subset B_r^M(Q)$. Therefore we can consider the Möbius map $\Omega^{-1} \circ \Phi\colon B_r^{\mathbb{D}}(X) \to B_r^{\mathbb{D}}(0)$. By Lemma 66.10 this Möbius map is the restriction of a Möbius transformation. In particular it is a Möbius isomorphism. But since Ω is a Möbius isomorphism as well we obtain that $\Phi\colon B_r^{\mathbb{D}}(X) \to B_r^M(Q)$ is also a Möbius isomorphism.

(2) Let $s \leq \frac{r}{2}$ and let $Y \in \mathbb{D}$ such that $\Phi(Y) \in B_s^M(Q)$, i.e. such that $d_M(Q, \Phi(Y)) < s$. By the triangle inequality we have

$$B_s^M(\Phi(Y)) \subset B_{2s}^M(Q) \subset B_r^M(Q).$$

The same argument as in (1) shows that the map Φ restricts to a Möbius isomorphism $B_s^{\mathbb{D}}(Y) \to B_s^M(\Phi(Y))$. We have $d_M(Q, \Phi(Y)) < s$, i.e. we have $Q \in B_s^M(\Phi(Y))$. Hence there exists an $X \in B_s^{\mathbb{D}}(Y)$ with $\Phi(X) = Q$. ∎



66.6. **Proof of Theorem 66.9 II.** In this section we will now finally provide the proof of Theorem 66.9. We will only cover the case $g = 2$, the proof of the general case is almost identical.

Throughout this section let $M = H_8/\sim$ be the surface of genus 2 with the Möbius structure constructed in Proposition 66.2. In the following we will show that there exists a Möbius

map $p\colon \mathbb{D} \to M$ that is in fact a covering map. It then follows from Lemma 66.8 that p is also a local isometry and a local biholomorphism.
For the remainder of the proof we adopt the following notation and conventions.

(1) We view $\mathring{H}_8$ as a subset of $\mathbb{D}$ and also of $H_8/\!\sim$.
(2) Given $Z \in \mathbb{D}$ we denote by
$$\begin{array}{rccc} \gamma_Z\colon [0,1] & \to & \mathbb{D} \\ t & \mapsto & t \cdot Z \end{array}$$
the radial path from the origin $0 \in \mathbb{D}$ to Z.
(3) We say that $U \subset \mathbb{D}$ is *convex* if it is convex in the usual Euclidean sense, i.e. if given any $P, Q \in U$ the points $t \cdot P + (1-t) \cdot Q$, $t \in [0,1]$ also lie in U. On several occasions throughout the proof we will make use of the following elementary observations:
(a) convex sets are path-connected,
(b) the intersection of two convex sets is also convex,
in particular, combining (a) and (b) we get
(c) the intersection of two convex sets is path-connected.

Our first goal is to construct a suitable map $p\colon \mathbb{D} \to M = H_8/\!\sim$. So let $Z \in \mathbb{D}$. We write $\gamma = \gamma_Z$. We say $t \in [0,1]$ is *good* if there exists a convex open neighborhood U of $\gamma([0,t])$ and a map $\Psi\colon U \to M$ which has the following properties:

(i) Ψ is a Möbius map,
(ii) it restricts to the identity on $\mathring{H}_8 \cap U \subset M$.

We start out with the following lemma.

Lemma 66.15. The point $t = 1$ is good.

Proof. We set
$$T \;:=\; \{t \in [0,1] \mid t \text{ is good}\}.$$
We start out with the following observations:

(1) We have that $0 \in T$ since the identity on the open neighborhood $\mathring{H}_8$ of 0 has the required properties.[768]
(2) Being good is clearly an open condition, hence $T \subset [0,1]$ is open.

We want to show that $T = [0,1]$. By the above, and since $[0,1]$ is connected, it suffices to show that $s := \sup(T)$ is good.

Since $0 \in S$ and since T is open we see that $s > 0$. Therefore there exists an increasing sequence $\{t_n\}_{n\in\mathbb{N}}$ of good numbers $t_n \in [0,t)$ that converges to s. Since each t_n is good we can pick for each n a convex open neighborhood U_n of $\gamma([0,t_n])$ and a map $\Psi_n\colon U_n \to M$ that has the desired properties (i) and (ii).

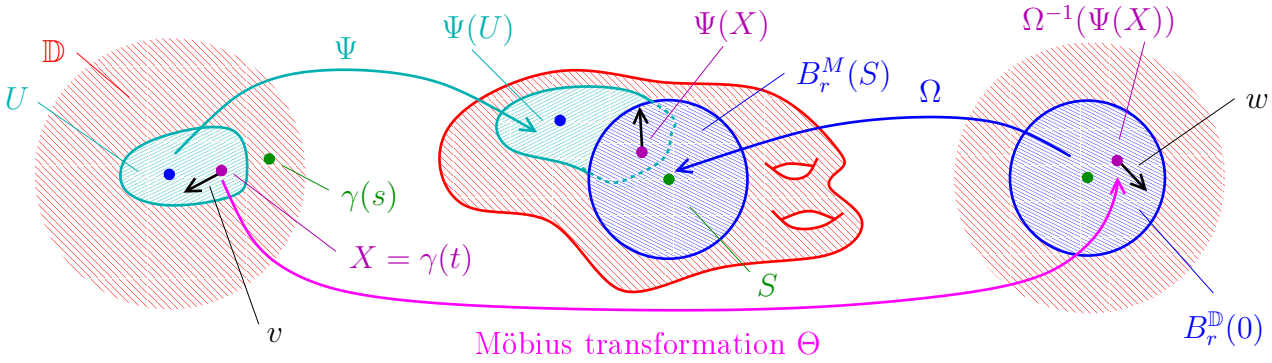
Let $m, n \in \mathbb{N}$. The Möbius maps Ψ_m and Ψ_n agree on the open non-empty connected subset $\mathring{H}_8 \cap U_n \cap U_m$. It follows from Lemma 66.11 that the maps Ψ_m and Ψ_n agree on $U_n \cap U_m$.

By the continuity of γ the sequence $\{\gamma(t_n)\}_{n\in\mathbb{N}}$ converges to $\gamma(t)$, in particular it is a Cauchy sequence with respect to $d_{\mathbb{D}}$. It follows from Lemma 65.10 that $\{\Psi_n(\gamma(t_n))\}_{n\in\mathbb{N}}$

[768]Here we use that for each point $Z \in \mathring{H}_8$ the identity map $\mathrm{id}\colon \mathring{H}_8 \to \mathring{H}_8$ is part of the Möbius structure of M.

is a Cauchy sequence with respect to d_M.[769] It follows from Proposition 66.2 that the metric space $(M = H_8/\!\sim, d_M)$ is complete. Therefore the Cauchy sequence $\{\Psi_n(\gamma(t_n))\}_{n\in\mathbb{N}}$ converges to a point $S \in M$.

It follows from Proposition 66.12 that there exists an $r > 0$ and a Möbius isomorphism $\Omega\colon B_r^{\mathbb{D}}(0) \to B_r^M(S)$ such that $\Omega(0) = S$. We pick an n such that $\Psi_n(\gamma(t_n)) \in B_r^M(S)$ and such that $d_{\mathbb{D}}(\gamma(t), \gamma(s)) < r$. We write $t = t_n, \Psi = \Psi_n, U = U_n$ and we write $X = \gamma(t)$. Let $v \in \mathrm{T}_X\mathbb{D}$ be a non-zero vector. We let $w := d(\Omega^{-1} \circ \Psi)_X(v)$. The Möbius maps Ω and Ψ are in particular local isometries, therefore we have $\|w\| = \|v\|$. It follows from Proposition 65.4 (4) that there exists a Möbius transformation Θ with $\Theta(X) = \Omega^{-1}(\Psi(X))$ and such that $d\Theta_X(v) = w$.



The Möbius maps $\Omega \circ \Theta\colon B_r^{\mathbb{D}}(X) \to M$ and $\Psi\colon U \to M$ agree at $X = \gamma(t)$ and we have $\mathrm{D}(\Omega \circ \Theta)_X(v) = \mathrm{D}\Psi_X(v)$. It follows from Lemma 66.11 that $\Omega \circ \Theta$ and Ψ agree on the path-connected set $B_r^{\mathbb{D}}(X) \cap U$.

Now we consider the map

$$\begin{array}{rcl} \Phi\colon U \cup B_r^{\mathbb{D}}(\gamma(t)) & \to & M \\ Q & \mapsto & \begin{cases} \Psi(Q), & \text{if } Q \in U, \\ \Omega^{-1}(\Theta(Q)), & \text{if } Q \in B_r^{\mathbb{D}}(\gamma(t)). \end{cases} \end{array}$$

By the above discussion this map is well-defined and it is locally a Möbius map. It follows from Lemma 66.10 (3) that this map is a Möbius map. It follows from $d_{\mathbb{D}}(\gamma(t), \gamma(s)) < r$ and the fact that γ is a geodesic in $\mathbb{D}$ that $\gamma([t, s]) \subset B_r^{\mathbb{D}}(\gamma(t))$, in particular $U \cup B_r^{\mathbb{D}}(\gamma(t))$ is an open subset of $\mathbb{D}$ that contains $\gamma([0, t]) \cup \gamma([t, s]) = \gamma([0, s])$. It is now straightforward to see that we can find a *convex* open subset W of $U \cup B_r^{\mathbb{D}}(\gamma(t))$ that contains $\gamma([0, s])$. The Möbius map $\Phi\colon W \to M$ now certifies that s is also good. ∎

Thus we have now shown that there exists a convex open neighborhood U of $\gamma([0, 1])$ and a local isometry $\Psi\colon U \to H_8/\!\sim$ which agrees with the identity map on $U \cap \mathring{H}_8$. We define $p(Z) := \Psi(Z)$. By Lemma 66.11 this definition does not depend on the choice of U and Ψ.

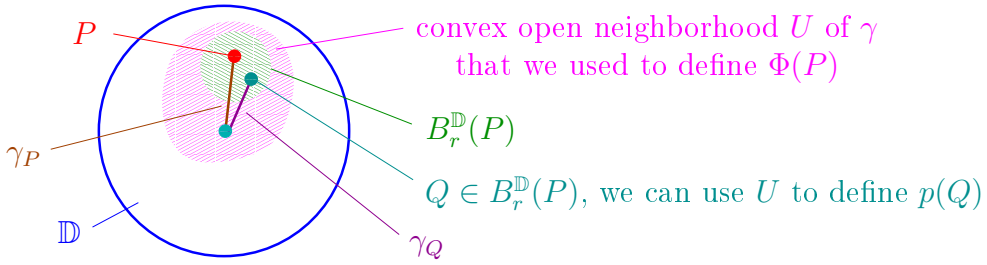
---

[769] Indeed, suppose we have $\epsilon > 0$. Since $\{\gamma(t_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence there exists an N such that $d_{\mathbb{D}}(\gamma(t_m), \gamma(t_n)) < \epsilon$ for all $m, n \geq N$. Now let $m, n \geq N$. Without loss of generality we have $m > n$. Then

$$d_M(\Psi_m(\gamma(t_m)), \Psi_n(\gamma(t_n))) \underset{\uparrow}{=} d_M(\Psi_m(\gamma(t_m)), \Psi_m(\gamma(t_n))) \underset{\uparrow}{\leq} d_{\mathbb{D}}(\gamma(t_m), \gamma(t_n)) < \epsilon.$$

since Ψ_m and Ψ_n agree on $\gamma|_{[0,t_n]}$ — Lemma 65.10

Lemma 66.16. The map $p\colon \mathbb{D} \to M$ is a Möbius map.



Proof. By Lemma 66.10 (2) it suffices to show that p is locally a Möbius map. This means that it suffices to show that every point $P \in \mathbb{D}$ admits an open neighborhood W such that the restriction of p to W is a Möbius map. Thus let $P \in \mathbb{D}$. By the above claim there exists a convex open neighborhood U of $\gamma_P([0,1])$ and a Möbius map $\Psi\colon U \to H_8/\!\sim$ which restricts to the identity on $U \cap \mathring{H}_8$. Since U is open we can find an $r > 0$ such that for all $Q \in B_r^{\mathbb{D}}(P)$ the image of the radial path γ_Q from 0 to Q also lies in U. (We refer to the above figure for an illustration.) This implies that we can use the Möbius map $\Psi\colon U \to M$ to define p for any $Q \in B_r^{\mathbb{D}}(P)$. Put differently, $p = \Psi$ on $B_r^{\mathbb{D}}(P)$. We have thus shown that the restriction of p to the open neighborhood $W = B_r^{\mathbb{D}}(P)$ of P is a Möbius map. ∎

The following lemma now concludes the proof of Theorem 66.9

Lemma 66.17. The map $p\colon \mathbb{D} \to M = H_8/\!\sim$ is a covering map.

Proof. First we show that the map $p\colon \mathbb{D} \to M = H_8/\!\sim$ is surjective. Recall that by construction p is the identity on $\mathring{H}_8$. Since p is continuous[770] it follows that the restriction of p to $H_8 \to M = H_8/\!\sim$ is surjective.[771]

It remains to show that every $Q \in M$ admits a uniformly covered neighborhood. Let $Q \in M$. By Proposition 66.12 there exists an $r > 0$ such that $B_{2r}^M(Q)$ is small.[772] We claim that $B_r^M(Q)$ is uniformly covered. More precisely, we make the following claims:

(1) $p^{-1}(B_r^M(Q)) = \bigcup_{X \in p^{-1}(Q)} B_r^{\mathbb{D}}(X)$,
(2) for $X \neq X' \in p^{-1}(Q)$ we have $B_r^{\mathbb{D}}(X) \cap B_r^{\mathbb{D}}(X') = \varnothing$,
(3) given any $X \in p^{-1}(Q)$ the restriction of p to $B_r^{\mathbb{D}}(X) \to B_r^M(Q)$ is a homeomorphism.

We apply Lemma 66.14 to the map $p\colon \mathbb{D} \to M$. It gives us the following three facts:

(a) If $Y \in \mathbb{D}$ satisfies $p(Y) \in B_r^M(Q)$, then there exists an $X \in p^{-1}(Q)$ with $Y \in B_r^{\mathbb{D}}(X)$.
(b) For any $X \in p^{-1}(Q)$ the restriction of p to $B_r^{\mathbb{D}}(X) \to B_r^M(Q)$ is a Möbius isomorphism, in particular it is a homeomorphism.
(c) For any $X \in p^{-1}(Q)$ the restriction of p to $B_{2r}^{\mathbb{D}}(X)$ is injective.

Now the desired properties (1) and (3) follow immediately from (a) and (b). It remains to prove (2). Suppose that we have $X \neq X' \in p^{-1}(Q)$ with $B_r^{\mathbb{D}}(X) \cap B_r^{\mathbb{D}}(X') \neq \varnothing$. We

[770] The map p is a Möbius map, in particular it is a local biholomorphism which implies that it is a local homeomorphism. Thus it follows from the Pasting Proposition 2.6 that p itself is continuous.
[771] This is the only part of the proof where we actually use that we work with $H_8/\!\sim$, the remainder of the proof works for any smooth manifold M with a Möbius structure such that the corresponding Riemannian manifold is complete.
[772] Note the factor 2.

need to show $X = X'$. We pick $Y \in B_r^{\mathbb{D}}(X) \cap B_r^{\mathbb{D}}(X')$. The triangle inequality implies that $X' \in B_{2r}^{\mathbb{D}}(X)$. But p restricted to $B_{2r}^{\mathbb{D}}(X)$ is injective by (c). Hence $X = X'$. ■

66.7. **Fundamental groups of surfaces.** In this section we will combine Hadamard's Theorem 66.9 and the discussion of the deck transformation group in Chapter 62 to draw some interesting consequences on the structure of fundamental groups of surfaces. First we have the following proposition.

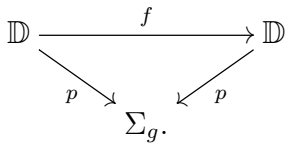
Proposition 66.18. Let $g \in \mathbb{N}_{\geq 2}$. We denote by Σ_g the surface of genus g. We pick a base point $x_0 \in \Sigma_g$. There exists a monomorphism $\pi_1(\Sigma_g, x_0) \to \mathrm{SL}(2, \mathbb{R})$.

Proof. Throughout the proof we will make use of the notion of a Möbius structure, of a Möbius manifold and of a Möbius map between Möbius manifolds. We refer to pages 1439 and 1448 for the definitions, in case the reader needs a refresher.

Now let $g \geq 2$. As in the proof of Proposition 66.2 we can and will make the identification $\Sigma_g = H_{4g}/\sim$ where H_{4g} is a regular hyperbolic $4g$-gon in $\mathbb{D}$ with interior angle $\frac{\pi}{2g}$. Furthermore, as in the proof of Proposition 66.2 we equip $\Sigma_g = H_{4g}/\sim$ with the canonical Möbius structure. By Hadamard's Theorem 66.9 there exists a covering map $p\colon \mathbb{D} \to \Sigma_g$ such that p is a Möbius map.

Claim. Every deck transformation of the covering map $p\colon \mathbb{D} \to \Sigma_g$ is a Möbius transformation.

Proof. Let $f\colon \mathbb{D} \to \mathbb{D}$ be a deck transformation. Recall that this means that the following diagram commutes:



Since p is a Möbius map and since p is a local diffeomorphism we see that f is locally a Möbius map. By Lemma 66.11 (2) this implies that f is in fact a Möbius map. It now follows from Lemma 66.11 (1) that f is in fact a Möbius transformation. ⊞

Now we can easily conclude the proof of the proposition, indeed we have the following isomorphisms:

$$\pi_1(\Sigma_g, x_0) \underset{\text{by Proposition 62.6}}{\overset{\cong}{\uparrow}} \mathrm{Deck}(p\colon \mathbb{D} \to \Sigma_g) \underset{\text{by the claim}}{\overset{\subset}{\uparrow}} \text{group of Möbius transformations of } \mathbb{D} \underset{\text{by Proposition 65.1}}{\overset{\cong}{\uparrow}} \mathrm{SL}(2, \mathbb{R}).$$

■

The following proposition gives in particular a positive answer to Question 64.22.

Proposition 66.19. Let Σ be a closed connected 2-dimensional smooth manifold. We pick a base point $x_0 \in \Sigma$. The following two statements hold:

(1) The group $\pi_1(\Sigma, x_0)$ is linear over $\mathbb{R}$.
(2) The group $\pi_1(\Sigma, x_0)$ is residually finite.

Remark.

(1) In [**NewM1985**, Lemma 1], see also [**Scot1978**, Section 3] together with [**Bou1981**, Chapitre V, §4, Section 4], it shown that $\pi_1(\Sigma, x_0)$ is in fact linear over $\mathbb{Z}$.
(2) There are many other proofs that $\pi_1(\Sigma, x_0)$ is residually finite, see e.g. [**Chis1999**, Corollary 4.9] and [**Hem1972**] for alternative arguments.

Proof. Let Σ be a closed connected 2-dimensional smooth manifold. By Proposition 49.4 we know that there exists a finite covering $p\colon (\widetilde{\Sigma}, \widetilde{x}_0) \to (\Sigma, x_0)$ such that $\widetilde{\Sigma}$ is a closed orientable connected 2-dimensional smooth manifold. By Proposition 66.18 we know that $\pi_1(\widetilde{\Sigma}, \widetilde{x}_0)$ is isomorphic to a subgroup of $\mathrm{SL}(2, \mathbb{R})$. In particular we see that $\pi_1(\widetilde{\Sigma}, \widetilde{x}_0)$ is linear over $\mathbb{R}$.

By Corollary 48.14 and Lemma 48.15 we know that $\pi_1(\widetilde{\Sigma}, \widetilde{x}_0)$ is isomorphic to a finite-index subgroup of $\pi_1(\Sigma, x_0)$. Thus it follows from the above together with Exercise 64.15 that $\pi_1(\Sigma, x_0)$ is also linear over $\mathbb{R}$. By Proposition 64.19 this implies that $\pi_1(\Sigma, x_0)$ is residually finite. ■

66.8. **Picard's Theorem ❀.** We also have the following variation on Theorem 66.9.

Proposition 66.20. Let $H_4/\!\sim$ be the three-punctured sphere with the Möbius structure constructed in Proposition 66.5. Then there exists a covering map $p\colon \mathbb{D} \to H_4/\!\sim$ with the following properties:

(1) p is a Möbius map,
(2) p is a local isometry,
(3) p is a local biholomorphism.

Proof. The proof of Proposition 66.20 is almost the same as the proof of Theorem 66.9. In the proof of Theorem 66.9 we used that $H_8/\!\sim$ is complete. In the present case it follows from Proposition 66.5 (3) that $H_4/\!\sim$ is complete. The construction of the Möbius map $\Phi\colon \mathbb{D} \to H_4/\!\sim$ is verbatim the same as the construction of the Möbius map $\Phi\colon \mathbb{D} \to H_8/\!\sim$. The same argument as before shows that $\Phi\colon \mathbb{D} \to H_4/\!\sim$ is surjective and that it is a covering map. ■

Remark. We consider again the covering $p\colon \mathbb{D} \to H_8/\!\sim$ that we had just constructed. It follows from Corollary 61.4 that the interior of H_8 is uniformly covered. In particular we see that

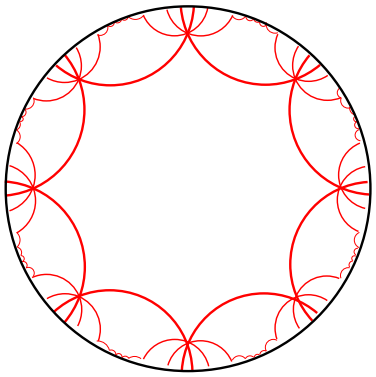
$$\mathbb{D} \setminus \underbrace{p^{-1}(\partial H_8)}_{\text{“1-dimensional”}} \;=\; \bigsqcup \text{copies of } \mathring{H}_8.$$

Put differently, up to the “one-dimensional” subset $p^{-1}(\partial H_8)$ we can cover $\mathbb{D}$ by infinitely many disjoint copies of the open hyperbolic octagon. Such a decomposition is often called a *tessellation*. This tessellation is shown in the figure below on the left. The same argument also applies to the covering $\mathbb{D} \to H_4/\!\sim$ that we also had just constructed. The corresponding tessellation of $\mathbb{D}$ is shown in the figure below on the right.

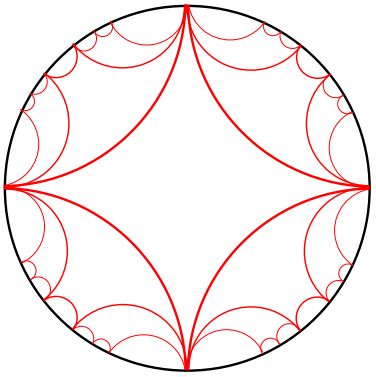
At the end of this chapter we return to complex analysis. There are different types of holomorphic functions defined on $\mathbb{C}$.

(1) There are the (rather dull) constant functions.
(2) There are holomorphic functions $f\colon \mathbb{C} \to \mathbb{C}$ that are surjective. For example it is an easy consequence of the fundamental theorem of algebra that all maps defined by non-constant polynomials are surjective.
(3) There are also non-constant maps that are not surjective, for example the exponential function $z \mapsto \exp(z)$ is never zero[773], in fact its image is $\mathbb{C} \setminus \{0\}$.

[773] Why not?



preimages of $\mathring{H}_8$ under the universal cover $p\colon \mathbb{D} \to H_8/\sim$



preimages of $\mathring{H}_4$ under the universal cover $p\colon \mathbb{D} \to H_4/\sim$

On the other hand it is fairly easy prove, using Liouville's Theorem 32.6, that the image of a non-constant holomorphic function on $\mathbb{C}$ is dense.
Now we can prove Picard's Theorem[774] which is a significantly stronger statement than the above statement that the image is dense.

Theorem 66.21. (Picard's Theorem) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a non-constant holomorphic function. Then there exists at most one $z \in \mathbb{C}$ which does not lie in the image of f.

Proof. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that there exist two different complex numbers a, b that do not lie in the image of f. We need to show that f is constant. We consider the biholomorphism

$$\begin{array}{rcl} \alpha\colon \mathbb{C} \setminus \{a,b\} & \to & \mathbb{C} \setminus \{0,1\} \\ z & \mapsto & \dfrac{z-a}{b-a}. \end{array}$$

Furthermore we denote by $\beta\colon \mathbb{C}\setminus\{0,1\} \to H_4/\sim$ the biholomorphism from Proposition 66.6. Now we consider the following diagram of maps

$$\begin{array}{ccccccc} & & & & & & \mathbb{D} \\ & & & & & & \downarrow \Phi \\ \mathbb{C} & \xrightarrow{f} & \mathbb{C} \setminus \{a,b\} & \xrightarrow[\cong]{\alpha} & \mathbb{C} \setminus \{0,1\} & \xrightarrow[\cong]{\beta} & H_4/\sim \end{array}$$

where $\Phi\colon \mathbb{D} \to H_4/\sim$ is the covering map from Proposition 66.20. Recall that Φ is a Möbius map which implies by Lemma 66.8 that Φ is a local biholomorphism.

Since $\mathbb{C}$ is simply connected we can appeal to the Map Lifting Criterion 61.2 to obtain a lift of the map $\beta \circ \alpha \circ f\colon \mathbb{C} \to H_4/\sim$ to the universal covering. More precisely, there

[774] Emile Picard (1856-1941) was a French mathematician. Many readers will have encountered him already in real analysis, as one of the two namesakes of the Picard-Lindelöf Theorem.

exists a map $\widetilde{\beta \circ \alpha \circ f} \colon \mathbb{C} \to \mathbb{D}$ such that the following diagram commutes

$$\begin{array}{ccccccc} \mathbb{C} & \xrightarrow[f]{} & \mathbb{C} \setminus \{a,b\} & \xrightarrow[\alpha]{\cong} & \mathbb{C} \setminus \{0,1\} & \xrightarrow[\beta]{\cong} & H_4/\!\sim \\ & \searrow^{\widetilde{\beta\circ\alpha\circ f}} & & & & & \downarrow \Phi \\ & & & & & & \mathbb{D} \end{array}.$$

Note that it follows from the fact that $\beta \circ \alpha \circ f$ is holomorphic and the fact that Φ is a local biholomorphism, that the map $\widetilde{\beta \circ \alpha \circ f} \colon \mathbb{C} \to \mathbb{D}$ is also holomorphic. But $\mathbb{D}$ is of course bounded. Therefore it follows from Liouville's Theorem 32.6 that $\widetilde{\beta \circ \alpha \circ f}$ is constant. But then $\beta \circ \alpha \circ f$ is also constant. Since α and β are biholomorphisms this implies that f itself is already constant. ∎

Remark. In the literature Theorem 66.21 is sometimes referred to as Picard's Little Theorem, to distinguish it from Picard's Great Theorem which says that if z_0 is an essential singularity of a holomorphic function, then in any neighborhood of z_0 the function f assumes all, but possibly one, value *infinitely* many times. We refer to [**Conw78**, Section 12] for details.

66.9. **Triangle groups.** In this short bonus section we will discuss the triangle groups. But before we can do so we need to introduce the notion of the reflection in a hyperbolic plane.

Definition. We define the reflection in the hyperbolic line $l := \{x + 0\mathrm{i} \mid x \in (-1,1)\} \subset \mathbb{D}$ to be the orientation-reversing isometry

$$\begin{array}{rcl} \rho_l \colon \mathbb{D} & \to & \mathbb{D} \\ z = x + \mathrm{i}y & \mapsto & \overline{z} = x - \mathrm{i}y. \end{array}$$

Now let k be any hyperbolic line in $\mathbb{D}$. By Lemma 65.6 there exists a Möbius transformation Φ with $\Phi(k) = l$. We define the reflection in k to be the map

$$\begin{array}{rcl} \rho_k \colon \mathbb{D} & \to & \mathbb{D} \\ x & \mapsto & \Phi^{-1} \circ \rho_l \circ \Phi \end{array}$$

In Exercise 66.1 we will see that the definition of ρ_k does not depend on the choice of Φ.

Next recall that given any two distinct points $P, Q \in \mathbb{D}$ there exists by Lemma 65.6 a unique hyperbolic line $g(P,Q)$ through P and Q. Now we can formulate the key proposition of this section.

Proposition 66.22. Let $l, m, n \in \mathbb{N}$ with $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$.

(1) There exists a hyperbolic triangle Δ_{ABC} with interior angles $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$ at the vertices A, B and C.

Let G be the subgroup of that is generated by the reflections in the hyperbolic lines $g(C,A)$, $g(A,B)$ and $g(B,C)$. The following statements hold:

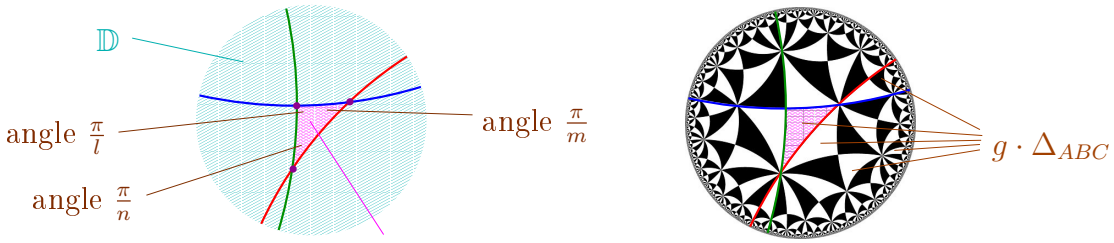
(2) (a) Given any $g \neq h \in G$ the intersection $g \cdot \Delta_{ABC} \cap h \cdot \Delta_{ABC}$ is either empty or a vertex or a side.

(b) $\bigcup_{g \in G} g \cdot \Delta_{ABC} = \mathbb{D}$.

(3) The map

$$\begin{array}{rcl} \langle x, y, z \,|\, x^2, y^2, z^2, (xy)^l = (yz)^m = (zx)^n \rangle & \to & G \\ x & \mapsto & \text{reflection in } g(C, A) \\ y & \mapsto & \text{reflection in } g(A, B) \\ z & \mapsto & \text{reflection in } g(B, C). \end{array}$$

is an isomorphism.



Proof.

(1) This statement is shown in [**Rat2019**, Theorem 3.5.6]. In fact in that reference it is also shown that the triangle is unique up to an isometry.

(2) The statements (2a) and (2b) are both shown in [**Cara1954**, p. 177-182] or alternatively in [**Mask1971**].

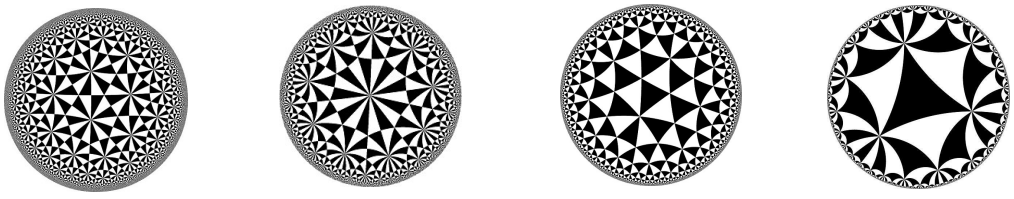
(3) (a) First note that the map is actually well-defined. In fact this follows from the observation that every reflection has order 2 and that xy, yz and zx get sent to a rotation around a vertex with rotation angles $\frac{2\pi}{l}$, $\frac{2\pi}{m}$ and $\frac{2\pi}{n}$.

(b) The fact that the map is in fact an isomorphism is shown in [**Mag1974**, Theorem 2.8]. ■

Remark.

(1) A decomposition of the euclidean plane $\mathbb{R}^2$ or the hyperbolic disk $\mathbb{D}$ into isometric geometric shapes that only overlap on the boundary is called a *tessellation*. Proposition 66.22 thus shows that the hyperbolic disk $\mathbb{D}$ can be tessellated by any hyperbolic triangle with angles $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$ for $l, m, n \in \mathbb{N}$ with $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. Tessellations for various values of (l, m, n) are shown in the figure below.

(2) Note that, as illustrated in the figure on page 1458, Hadamard's Theorem 66.9 and Proposition 66.20 also provide tessellations of $\mathbb{D}^2$.



$(2, 3, 7)$ $(2, 3, 9)$ $(3, 3, 4)$ $(6, 6, 6)$

Proposition 66.22 partly motivates the following definition.

Definition. Let $l, m, n \in \mathbb{N}$.

(1) The group

$$T^*(l,m,n) := \langle x, y, z \,|\, x^2, y^2, z^2, (xy)^l = (yz)^m = (zx)^n \rangle$$

is called the (l,m,n)-triangle group.

(2) The group

$$T(l,m,n) := \ker \left\{ \begin{array}{ccc} T^*(l,m,n) & \to & \mathbb{Z}_2 \\ x,y,z & \mapsto & 1 \end{array} \right\}$$

is called the (l,m,n)$-von Dyck group$.

Remark. Let $l, m, n \in \mathbb{N}$.

(1) (a) If $\frac{1}{l}+\frac{1}{m}+\frac{1}{n} < 1$, then we saw in Proposition 66.22 that $T^*(l,m,n)$ is isomorphic to the subgroup of $\mathrm{Isom}(\mathbb{D})$ generated by reflections in the hyperbolic lines defined by a hyperbolic triangle with interior angles $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$.

(b) If $\frac{1}{l}+\frac{1}{m}+\frac{1}{n} = 1$, then it is shown in [**Mag1974**, Section II.4] that $T^*(l,m,n)$ is isomorphic to the subgroup of $\mathrm{Isom}(\mathbb{R}^2)$ generated by reflections in the euclidean lines defined by a *euclidean* triangle with interior angles $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$.

(c) Finally, if $\frac{1}{l}+\frac{1}{m}+\frac{1}{n} > 1$, then it is shown in [**Mag1974**, Section II.4] that $T^*(l,m,n)$ is isomorphic to the subgroup of $\mathrm{Isom}(S^2)$ generated by reflections in the great circles defined by a *spherical* triangle with interior angles $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$.

(2) In each of the three cases above the (l,m,n)-von Dyck group is the intersection of the (l,m,n)-triangle group with the subgroup of orientation-preserving isometries.

The following lemma gives a particularly simple presentation of the von Dyck groups.

Lemma 66.23. Given any $l, m, n \in \mathbb{N}$ the map

$$\begin{array}{ccc} \langle a, b \,|\, a^l, b^m, (ab)^n \rangle & \to & T(l,m,n) \\ a & \mapsto & xy \\ b & \mapsto & yz \end{array}$$

is an isomorphism.

Proof. We will prove Lemma 66.23 in Exercise 66.3. ∎

We conclude this chapter with the following corollary to Proposition 66.22.

Corollary 66.24. Let $l, m, n \in \mathbb{N}$. If $\frac{1}{l}+\frac{1}{m}+\frac{1}{n} < 1$, then the triangle group $T^*(l,m,n)$ and the von Dyck group $T(l,m,n)$ are infinite.

Proof. Let $l, m, n \in \mathbb{N}$ with $\frac{1}{l}+\frac{1}{m}+\frac{1}{n} < 1$. We denote by G the group introduced in Proposition 66.22. Note that it follows from Proposition 66.22 (2), the fact that Δ_{ABC} is compact and the fact that $\mathbb{D}$ is non-compact that G is infinite. By Proposition 66.22 (3) this implies that the triangle group $T^*(l,m,n)$ is infinite. The index two subgroup $T(l,m,n)$ is evidently also infinite. ∎

Exercises for Chapter 66.

Exercise 66.1. We define reflection in the hyperbolic line $l := \{x \,|\, x \in (-1,1)\} \subset \mathbb{D}$ to be the map

$$\begin{array}{rcl} \rho_l\colon \mathbb{D} & \to & \mathbb{D} \\ z = x + \mathrm{i}y & \mapsto & \overline{z} = x - \mathrm{i}y. \end{array}$$

Now let k be any hyperbolic line in $\mathbb{D}$. By Lemma 65.6 there exists a Möbius transformation Φ with $\Phi(k) = l$. We define the *reflection in* k to be the map

$$\begin{array}{rcl} \rho_k\colon \mathbb{D} & \to & \mathbb{D} \\ x & \mapsto & \Phi^{-1} \circ \rho_l \circ \Phi \end{array}$$

Show that the definition of ρ_k does not depend on the choice of Φ.

Exercise 66.2. Show that the map

$$\begin{array}{rcl} \mathrm{SL}(2,\mathbb{R}) \times \mathbb{H} & \to & \mathbb{H} \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) & \mapsto & \dfrac{az+b}{cz+d} \end{array}$$

defines an action of the group $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}$ and show that this action is transitive.

Exercise 66.3. Let $l, m, n \in \mathbb{N}$. Show that the map

$$\begin{array}{rcl} \langle a, b \,|\, a^l, b^m, (ab)^n \rangle & \to & T(l,m,n) \\ a & \mapsto & xy \\ b & \mapsto & yz \end{array}$$

is an isomorphism.
Hint. First use Proposition 59.32 to find an explicit presentation for $T(l,m,n)$.

67. Related constructions in algebraic geometry and Galois theory

In this short chapter we indicate some connections between the topics that we have covered so far to algebraic geometry and Galois theory. Since the author of these notes does not know much about either field the following discussion should be taken with a grain of salt. In particular for an expert the subsequent discussion will certainly look too simplistic.

67.1. The fundamental group of an algebraic variety.

Definition. Let $\mathbb{K}$ be a field.[775]

(1) Given a set of multivariable polynomials S, i.e. given a subset $S \subset \mathbb{K}[t_1, \ldots, t_n]$ we denote by

$$V(S) \;=\; \{(x_1, \ldots, x_n) \in \mathbb{K}^n \mid f(x_1, \ldots, x_n) = 0 \text{ for all } f \in S\}$$

the vanishing set of S, i.e. the set of all points in $\mathbb{K}^n$ where all the polynomials in S, viewed as maps $\mathbb{K}^n \to \mathbb{K}$, vanish.

(2) We say $U \subset \mathbb{K}^n$ is Zariski open,[776] if there exists a subset S of $\mathbb{K}[t_1, \ldots, t_n]$ such that $U = \mathbb{K}^n \setminus V(S)$.

Remark. Let S be a subset of $\mathbb{K}[t_1, \ldots, t_n]$. Since the ring $\mathbb{K}[t_1, \ldots, t_n]$ is Noetherian there exists a finite set $\{f_1, \ldots, f_k\} \subset \mathbb{K}[t_1, \ldots, t_n]$ such that $V(S) = V(f_1, \ldots, f_k)$.

Lemma 67.1. Let $\mathbb{K}$ be a field. The Zariski open subsets form a topology on $\mathbb{K}^n$.

Remark.

(1) The topology defined by the Zariski open subsets is of course called the *Zariski topology on* $\mathbb{K}^n$.

(2) If $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, then every Zariski open subset is also open with respect to the usual topology on $\mathbb{R}^n$ or $\mathbb{C}^n = \mathbb{R}^{2n}$. But for $n \geq 1$ the converse does not hold, in fact "most" subsets that are open with respect to the usual topology are not open with respect to the Zariski topology.

Proof. Let $\mathbb{K}$ be a field.

(1) The vanishing set of the empty set, i.e. $V(\varnothing)$ is all of $\mathbb{K}^n$, hence the empty set $\varnothing = \mathbb{K}^n \setminus V(\varnothing)$ is open.

(2) The vanishing set of the constant polynomial $f = 1$ is the empty set, therefore $\mathbb{K}^n = \mathbb{K}^n \setminus V(1)$ is open.

(3) We need to show that the intersection of finitely many Zariski open sets is again Zariski open. By induction it suffices to consider the case of two Zariski open sets. So suppose we are given Zariski open sets $X = \mathbb{K}^n \setminus V(S)$ and $Y = \mathbb{K}^n \setminus V(T)$. Then

$$\mathbb{K}^n \setminus (X \cap Y) \;=\; (\mathbb{K}^n \setminus X) \cup (\mathbb{K}^n \setminus Y) \;=\; V(S) \cup V(T) \;=\; V\big(\{s \cdot t\}_{s \in S, t \in T}\big).$$

[775] For example we could take $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ but $\mathbb{K}$ could also be a field of non-zero characteristic, e.g. for a prime p we could take $\mathbb{K} = \mathbb{F}_p$ or $\mathbb{K}$ could be the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$.

[776] Oscar Zariski (1899-1986) was a Russian-American mathematician working in algebraic geometry.

(4) We need to show that the union of Zariski open sets is again Zariski open. So suppose we are given a family of Zariski open sets $X_i = \mathbb{K}^n \setminus V(S_i)$, $i \in I$. Then

$$\mathbb{K}^n \setminus \bigcup_{i\in I} X_i \;=\; \bigcap_{i\in I}(\mathbb{K}^n \setminus X_i) \;=\; \bigcap_{i\in I} V(S_i) \;=\; V\Big(\bigcup_{i\in I} S_i\Big).$$

Thus we have verified that the Zariski open sets have all the properties of a topology. ■

Definition. Let $\mathbb{K}$ be a field.

(1) A variety over $\mathbb{K}$ is defined as the vanishing set of a set of multivariable polynomials.
(2) The Zariski topology on a variety $V \subset \mathbb{K}^n$ is defined as the subspace topology on V induced from the Zariski topology on $\mathbb{K}^n$.

Example. Let $\mathbb{K}$ be a field and let and $f(s,t) = s^2 + t^2 - 1$. We define

$$S^1_{\mathbb{K}} \;:=\; V(f) \;=\; \{(x,y) \in \mathbb{K}^2 \mid f(x,y) = x^2 + y^2 - 1 = 0\} \;=\; \{(x,y) \in \mathbb{K}^2 \mid x^2 + y^2 = 1\}.$$

For $\mathbb{K} = \mathbb{R}$ we obtain the usual circle S^1.

Any variety V is again a topological space, so in principle one could study the usual fundamental group. But in many cases, e.g. if the characteristic of $\mathbb{K}$ is non-zero, one will see that all continuous maps $S^1 \to V$ are constant, hence the usual fundamental group is trivial.
The correct analogue of a finite covering $p\colon X \to B$ between topological spaces for varieties is a finite étale morphism. The precise definition of a finite étale morphism is irrelevant for our purpose.[777] A heuristic fact is that varieties tend to have many finite étale morphisms, but there is no well-behaved analogue of an infinite covering in the context of varieties. In particular usually there is no concept of a "universal" étale morphism. The question is now, whether one can recover the notion of a fundamental group from the knowledge of finite covers.
Given a variety V we can consider all finite regular étale morphisms $p_i\colon W_i \to V$, $i \in I$. The corresponding deck transformations groups $\mathrm{Deck}(p_i\colon W_i \to V)$, $i \in I$ are finite groups that form an inverse system. Now we can define the étale fundamental group of V as

$$\pi_1^{\text{ét}}(V) \;:=\; \varprojlim_i \mathrm{Deck}(p_i\colon W_i \to V).$$

By Proposition 56.18 the étale fundamental group $\pi_1^{\text{ét}}(V)$ is naturally a compact topological group. In algebraic geometry the étale fundamental group of a variety plays a role similar to the role of the fundamental group of a topological space in topology.
If V is a complex variety without singularities, then the étale fundamental group is isomorphic to the profinite completion[778] of the usual topological fundamental group $\pi_1(V)$, i.e. we have

$$\pi_1^{\text{ét}}(V) \;\cong\; \widehat{\pi_1(V)}.$$

We refer to [**Groth1971**] for a proof of this statement.
More information on this topic can be found at

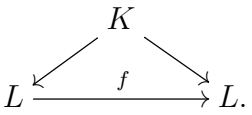
`https://en.wikipedia.org/wiki/etale_fundamental_group`

[777] Furthermore it is unknown to the author of these lecture notes.
[778] Recall that on page 1283 we introduced the profinite-completion $\widehat{\pi}$ of a group π.

67.2. **Galois theory.** The theory of coverings of topological spaces has some similarities to the Galois theory of field extensions.
Throughout this section let K be a field of characteristic zero[779] and let L/K be a field extension of K, i.e. L is a field which contains K as a subfield. The dimension of L as a K vector space is denoted by $[L:K]$. We consider the group

$$\mathrm{Gal}(L/K) = \text{ all field automorphisms of } L \text{ which are the identity on } K.$$

Put differently, a field automorphism $f\colon L \to L$ lies in $\mathrm{Gal}(L/K)$ if and only if the following diagram commutes:



We consider two examples:

(1) $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is a field extension with $[\mathbb{Q}(\sqrt{2})\colon \mathbb{Q}] = 2$ and $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ consists of the identity and the homomorphism $a + b\sqrt{2} \mapsto a - b\sqrt{2}$.
(2) $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is a field extension with $[\mathbb{Q}(\sqrt[3]{2})\colon \mathbb{Q}] = 3$ and the group $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$ consists only of the identity.

Given a field K we denote by $\overline{K}$ the algebraic closure of K. A field extension L/K is called *Galois*, if for each $f \in \mathrm{Gal}(\overline{K}/K)$ we have $f(L) = L$.[780] In the usual course on Galois theory one shows that L/K is Galois if and only if each polynomial in $K[x]$ which admits a zero in L splits over L, i.e. all other zeros of f lies also in L. For a Galois extension L/K we refer to $\mathrm{Gal}(L/K)$ as the *Galois group* of the field extension.
We return to the above two examples:

(1) $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is a Galois field extension.
(2) $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not a Galois field extension, since $\sqrt[3]{2}$ is a zero of x^3-2, but the complex zero $\sqrt[3]{2}\cdot\exp(2\pi\mathrm{i}/3)$ of f does not lie in $\mathbb{Q}(\sqrt[3]{2})$.[781]

If L/K is a finite extension, then the main theorem of Galois theory says that the following two statements are equivalent:

(1) The field extension L/K is Galois.
(2) $\#\,\mathrm{Gal}(L/K) = [L:K]$.

The statement is formally very similar to the statement of the Regular Covering-Characterization Corollary 62.8 which says that for some finite covering $p\colon X \to B$ of topological spaces the following statements are equivalent:

(1) The covering $p\colon X \to B$ is regular.
(2) $\#\,\mathrm{Deck}(p\colon X \to B) = [X:B]$.

The analogy between Galois theory and covering theory goes considerably further. Many statements about the Galois group of a Galois extension have a corresponding statement for the deck transformation group of a normal covering. For example, given a finite extension L/K the (normal) subgroups of Galois group $\mathrm{Gal}(L/K)$ correspond to (normal)

[779] This restriction to characteristic zero is not necessary, but it simplifies the discussion.
[780] Strictly speaking this is the definition of a normal extension, but since K is a field of characteristic zero, a normal extension is already Galois.
[781] We put $z = \sqrt[3]{2}$ and $w = \exp(2\pi\mathrm{i}/3)$. It is a good exercise to show that $\mathbb{Q}(z,w)/\mathbb{Q}$ is a normal field extension with $[\mathbb{Q}(z,w)\colon \mathbb{Q}] = 6$. The group $\mathrm{Gal}(\mathbb{Q}(z,w)/\mathbb{Q})$ is isomorphic to the permutation group S_3.

intermediate fields and (normal) subgroups of the deck transformation group correspond to (regular) intermediate coverings.
Now let L/K be a Galois extension of infinite degree. All finite intermediate Galois extensions of K, i.e. all finite Galois extensions F/K with $F \subset L$, form a preordered set via the inclusion. The corresponding Galois groups form an inverse set. Indeed, given two finite Galois extensions F/K and G/K with $F \subseteq G$ the restriction of an automorphism of G to an automorphism of F defines a group homomorphisms $\mathrm{Gal}(G/K) \to \mathrm{Gal}(F/K)$. Therefore we can form the inverse limit of these Galois groups. By [**FJ2008**, Section 1.3] there exists a canonical isomorphism of groups

$$\mathrm{Gal}(L/K) \;=\; \varprojlim_{\substack{F/K \text{ finite} \\ \text{Galois extension}}} \mathrm{Gal}(F/K).$$

On the right-hand side we consider the inverse limit of an inverse system of finite groups. It follows from Proposition 56.18 that the right-hand side, and thus also the Galois group $\mathrm{Gal}(L/K)$, is naturally a compact topological group. This topology on $\mathrm{Gal}(L/K)$ is called the *Krull topology*.

Example. Let p be a prime. We denote by $\mathbb{F}_p$ the finite field with p elements and we denote by $\overline{\mathbb{F}}_p$ its algebraic closure. It is shown in [**FJ2008**, Section 1.5] that there exists a canonical isomorphism

$$\mathrm{Gal}\left(\overline{\mathbb{F}}_p/\mathbb{F}_p\right) \;\cong\; \widehat{\mathbb{Z}}$$

where $\widehat{\mathbb{Z}}$ denotes the profinite completion of $\mathbb{Z}$, or equivalently the profinite integers that we introduced on page 1279.

Part VII

CW-complexes

68. CW-complexes I: Definitions, basic properties and constructions

In this chapter we will introduce CW-complexes. Initially the definition of a CW-complex might look rather confusing and artificial. But in this and in Chapters 69 and 70 we will see that the notion of a CW-complex strikes the right balance:

(1) CW-complexes are topological space that have many desirable properties,
(2) the notion is flexible enough so that most of the topological spaces that we are interested in are actually CW-complexes.

68.1. **Definition of finite-dimensional CW-complexes and examples.** We start out with the definition of a finite-dimensional CW-complex and its topological realization.

Definition.

(a) A CW-complex X of dimension ≤ -1 is defined to be the empty set. Its topological realization $|X|$ is also defined to be the empty set.
(b) We define inductively higher dimensional CW-complexes. Let $n \in \mathbb{N}_0$. A CW-complex X^n of dimension $\leq n$ is a pair $(X^{n-1}, \{\{\varphi_i \colon S^{n-1} \to |X^{n-1}|\}_{i \in I_n}\})$ where the following holds:
   (i) X^{n-1} is a CW-complex X^{n-1} of dimension $\leq n-1$, with topological realization denoted by $|X^{n-1}|$,
   (ii) $\{\varphi_i \colon S^{n-1} \to |X^{n-1}|\}_{i \in I_n}$ is a set of maps.
   The topological realization $|X^n|$ of X^n is defined as the topological space
$$|X^n| := \Big(|X^{n-1}| \sqcup \bigsqcup_{i \in I_n} \overline{B}^n\Big)/\sim$$
   where for each $i \in I_n$ we use the map φ_i to identify the points on the corresponding copy of $S^{n-1} = \partial \overline{B}^n$ with the corresponding points in $|X^{n-1}|$.
(c) A CW-structure for a topological space Y is a CW-complex X together with a homeomorphism $f \colon |X| \to Y$.

Remark.

(1) Let X be a CW-complex as above and let $n \in \mathbb{N}_0$. It follows from the the Topological Pushout-Maps Lemma 5.34 (3)that the obvious map $|X^{n-1}| \to |X^n|$ is an embedding. Thus we can and will view $|X^{n-1}|$ as a subspace of $|X^n|$.
(2) Note that $\overline{B}^0$ consists of a point and that $S^{-1} = \varnothing$. It follows from the above definition that a CW-complex of dimension ≤ 0 is exactly the same as set of points with the discrete topology.
(3) CW-complexes were introduced by John H. C. Whitehead[782]. The name comes from "C" for "closure-finite", and the "W" for "weak topology". We will explain on pages 1479 and 1491 what "closure-finite" and "weak topology" mean. Some people also say it goes back to the initials C. W. of Whitehead. Some authors, see e.g. [**Lee2000**, p. 132] and [**Mun1984**, p. 214] require "closure-finiteness" as part of the definition, but as we will see, this property follows from the definition as given.

[782] John Henry Constantine Whitehead (1904-1960) was a British mathematician.

(4) In the notation we usually do not distinguish between the CW-complex and its topological realization, i.e. we usually denote the topological realization of a CW-complex X again by X.

Before we discuss a long list of examples of CW-complexes we want to introduce a few more definitions.

Definition. Let X be a CW-complex as above and let $k \in \mathbb{N}_0$. We continue with the notation used in the previous definition

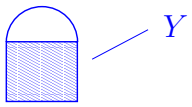
(1) The images of the $\overline{B}_i^k$, $i \in I_k$, in X are called the k-dimensional cells, or often short k-cells, of X.
(2) The images of the B_i^k, $i \in I_k$, in X are called the open k-cells of X.
(3) The maps $\varphi_i \colon S_i^{k-1} \to X$[783] are called the attaching maps of the k-cells and the obvious maps $\overline{B}_i^k \to X$ are referred to as the characteristic maps of the k-cells.
(4) We refer to the CW-complex X^k as the k-skeleton of X.
(5) If X is empty, then we define its dimension to be -1. If X is non-empty, then we define the dimension $\dim(X) \in \mathbb{N}_0 \cup \{\infty\}$ as the maximal dimension of a cell.

Remark. At times, when we say that "a CW-complex X is k-dimensional", we allow the possibility, that the highest dimension of a cell is actually less than k. This slightly fuzzy language should not pose a problem in practice.

Example. We consider the topological space which is sketched in the figure below, namely

$$Y := ([-1,1] \times [-1,1]) \cup \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + (y-1)^2 = 1 \text{ and } y \geq 1\}.$$

The topological space X thus consists of a square together together with a semicircle.

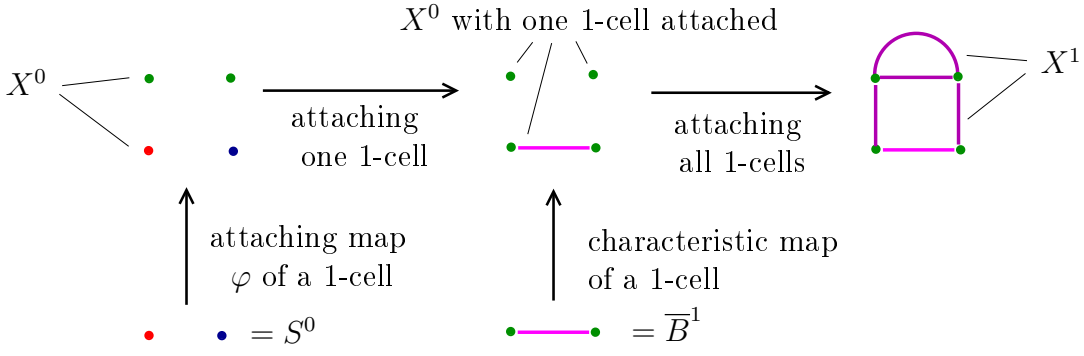


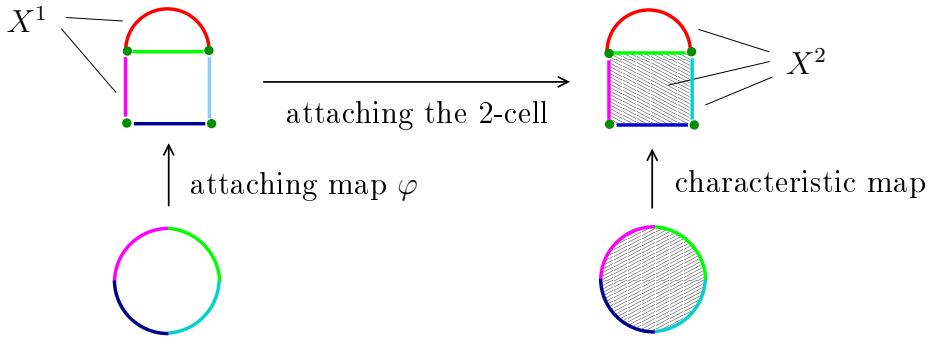
Our goal is to find a CW-structure for X. First we consider the 0-dimensional CW-complex which consists of the 4 vertices of the square. Next we add a 1-cell. To do so we consider the map φ which sends $\partial\overline{B}^1 = \{\pm 1\}$ to the two points $\pm 1 \times -1$. The topological space $X^0 \cup_\varphi \overline{B}^1$ consists of the four points $\{\pm 1, \pm 1\}$ together with an edge which connects the lower two points. This topological space is sketched in the center of the figure below. The same way we attach four more 1-cells and we obtain the 1-dimensional CW-complex which is sketched on the right of the figure below. Finally we attach a 2-cell to X^1 with the attaching map shown in the figure below and we obtain a CW-complex X^2 whose topological realization is homeomorphic to Y.

Now we will see that most of the topological spaces that we have encountered so far can be described as CW-complexes.

(1) It follows almost immediately from the definitions that every topological graph, as defined on page 310, admits naturally the structure of a one-dimensional CW-complex where the vertices correspond precisely to the 0-cells and the edges correspond precisely to the 1-cells. Conversely, every 1-dimensional CW-complex admits naturally

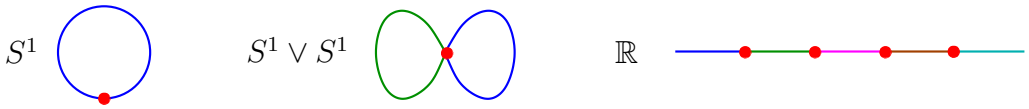
[783] Here we already use the convention that in our notation we do not distinguish between a CW-complex and its topological realization.





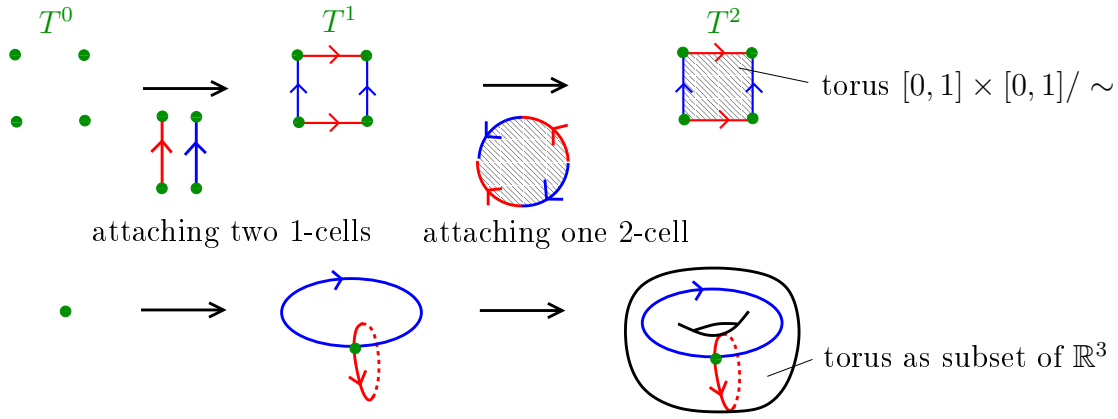
the structure of a topological graph. Therefore in the following we will freely go back and forth between these objects.

(2) As a special case of (1) we see that S^1 can be viewed as a CW-complex with one 0-cell and one 1-cell. Furthermore we can view the wedge $S^1 \vee S^1$ as a CW-complex with one 0-cell and two 1-cells. Finally $\mathbb{R}$ can be viewed as a CW-complex where the 0-cells are given by the integers $\mathbb{Z}$ and the 1-cells are given by the intervals $[k, k+1]$ with $k \in \mathbb{Z}$. We refer to the figure below for an illustration.



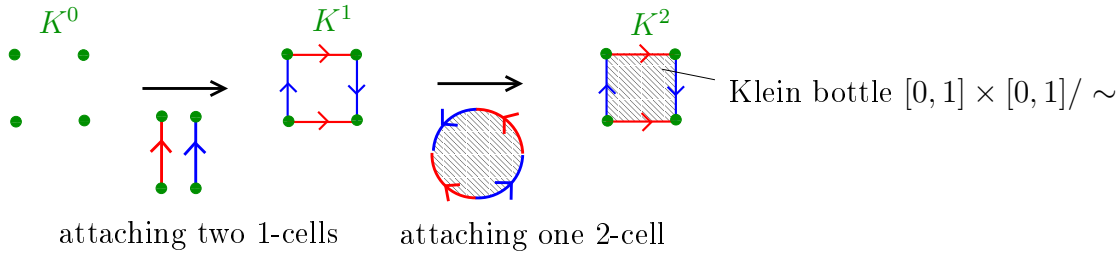
(3) Now we consider the torus $T = ([0,1] \times [0,1])/\sim$. We denote by $p\colon [0,1] \times [0,1] \to T$ the natural projection. We can view T as a CW-complex with one 0-cell corresponding to the point $p(0,0) = p(0,1) = p(1,0) = p(1,1)$, two 1-cells corresponding to $p(0 \times [0,1]) = p(1 \times [0,1])$ and to $p([0,1] \times 0) = p([0,1] \times 1)$ and one 2-cell corresponding to $p([0,1] \times [0,1])$. Note that since we have only one 0-cell there is a unique choice of attaching map for the 1-cells. Finally the attaching map for the 2-cell is illustrated in

the figure below.[784] In the figure below we sketch the CW-structure of the torus and also the corresponding CW-structure of the torus viewed as a subset of $\mathbb{R}^3$. We refer to it as the *standard* CW-*structure* of the torus.



(4) Next we consider the Klein bottle $K = ([0,1]\times[0,1])/\sim$. As usual we denote by $p\colon [0,1]\times[0,1]\to K$ the projection . Similar to the torus we can view K as CW-complex with one 0-cell corresponding to $p(0,0) = p(0,1) = p(1,0) = p(1,1)$, two 1-cells corresponding to $p(0\times[0,1]) = p(1\times[0,1])$ and to $p([0,1]\times 0) = p([0,1]\times 1)$ and one 2-cell corresponding to $p([0,1]\times[0,1])$ with the attaching map sketched in the figure below.[785]

The 1-skeleta of the torus and the Klein bottle are both wedges of two circles. Furthermore both CW-complexes have precisely one 2-cell. The difference between the torus and the Klein bottle is that the attaching maps for the 2-cells are different.



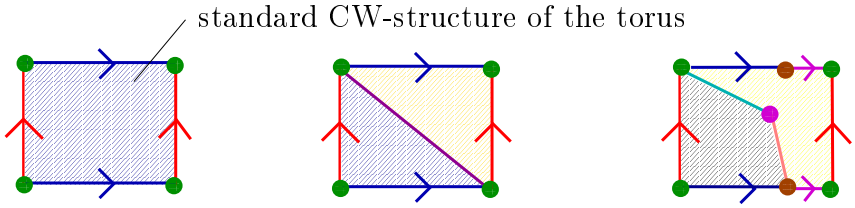
(5) In the figure below we show that we can endow the torus with different CW-structures.

[784]Strictly speaking the attaching map is given by

$$\begin{array}{rcl} S^1 & \to & T^1 \\ \exp(it) & \mapsto & \begin{cases} p(\frac{2}{\pi}t, 0), & \text{if } t\in[0,\frac{\pi}{2}] \\ p(1, \frac{2}{\pi}(t-\frac{\pi}{2})), & \text{if } t\in[\frac{\pi}{2},\pi] \\ p(1-\frac{2}{\pi}(t-\pi), 1), & \text{if } t\in[\pi,\frac{3\pi}{2}] \\ p(0, 1-\frac{2}{\pi}(t-\frac{3}{2}\pi)), & \text{if } t\in[\frac{3\pi}{2},2\pi]. \end{cases} \end{array}$$

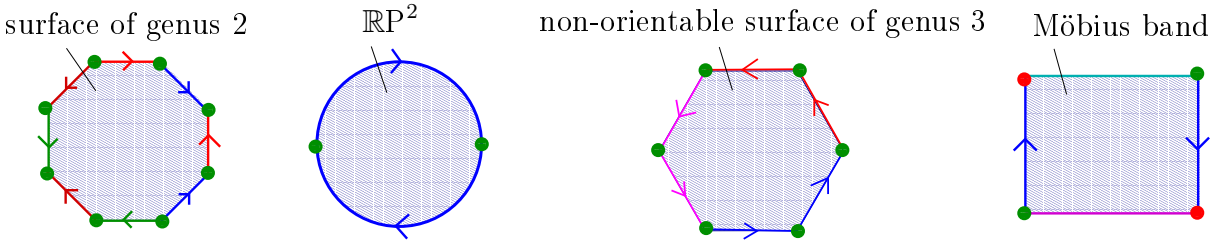
But writing out the attaching map with explicit formulas is clearly not very instructive.

[785]Evidently, as in Footnote 784, one could spell out the attaching map with precise formulas, but there is no point in doing it.



one	0-cell	one	0-cell	three	0-cells
two	1-cells	three	1-cells	five	1-cells
one	2-cell	two	2-cells	two	2-cells

(6) In the figure below we sketch how we can view the surface of genus 2, the projective plane $\mathbb{R}\mathrm{P}^2$, the non-orientable surface of genus 3 and the Möbius band as CW-complexes.[786]



one	0-cell	one	0-cell	one	0-cell	two	0-cells
four	1-cells	one	1-cell	three	1-cells	three	1-cells
one	2-cell	one	2-cell	one	2-cell	one	2-cell

(7) We conclude this discussion of surfaces with the 2-sphere. Since it makes no difference, we will immediately discuss the case of any sphere. We can describe the n-dimensional sphere S^n as a CW-complex X as follows: We take a single 0-cell given by a point P and one n-cell where the attaching map $\varphi\colon S^{n-1} \to X^0 = \{P\}$ sends the whole boundary S^{n-1} to the point P. The topological realization of X is thus given by

$$X^0 \cup_\varphi \overline{B}^n \;=\; \{P\} \cup_{P=\partial\overline{B}^n} \overline{B}^n \;=\; \overline{B}^n/S^{n-1} \;\underset{\uparrow}{\overset{\cong}{\Rightarrow}}\; S^n.$$

identification from the Ball-Quotient Sphere Lemma 5.20

We refer to this CW-structure on S^n as the *canonical* CW-*structure on* S^n. Note that the identification from the Ball-Quotient Sphere Lemma 5.20 identifies the unique 0-cell to the left with the North Pole $(0,\dots,0,1)$. Unless we say something else we will equip S^n with the above CW-structure.

We continue the discussion of CW-complexes with two more explicit examples, namely the real projective spaces and the complex projective spaces which we had already defined on page 255. In the following let $\mathbb{A} = \mathbb{R}$ or $\mathbb{A} = \mathbb{C}$. We denote by $\sim$ the equivalence relation on $\mathbb{A}^{n+1} \setminus \{0\}$ given by multiplication by an element in $\mathbb{A} \setminus \{0\}$ and we define

$$\mathbb{A}\mathrm{P}^n \;:=\; (\mathbb{A}^{n+1} \setminus \{0\})/\sim .$$

[786] What does the 1-skeleton of the given CW-structure for the surface of genus two look like, if we view the surface Σ as a "physical object" in $\mathbb{R}^3$?

On page 564 and the Complex Projective Space-is-Complex Manifold Lemma 32.16 we saw that $\mathbb{A}\mathrm{P}^n$ is a smooth manifold. Given a point $(v_0, \ldots, v_n) \in \mathbb{A}^{n+1} \setminus \{0\}$ we denote by $[v_0 : \ldots : v_n]$ its equivalence class. For $k \leq n$ the map

$$\begin{array}{rcl} \mathbb{A}\mathrm{P}^k & \to & \mathbb{A}\mathrm{P}^n \\ [v_0 : \ldots : v_k] & \mapsto & [v_0 : \ldots : v_k : 0 : \ldots : 0] \end{array}$$

is easily seen to be a smooth embedding and we use it to view $\mathbb{A}\mathrm{P}^k$ as a submanifold of $\mathbb{A}\mathrm{P}^n$.

We can give an explicit description of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ as CW-complexes.

Lemma 68.1. Let $n \in \mathbb{N}_0$.

(1) We can view $\mathbb{R}\mathrm{P}^n$ as a CW-complex with the following properties:
   (a) the CW-structure has exactly one cell in the dimensions $0, 1, \ldots, n$ and no other cells,
   (b) for each $k \leq n$ the k-skeleton of $\mathbb{R}\mathrm{P}^n$ is given by $\mathbb{R}\mathrm{P}^k$.

(2) We can view $\mathbb{C}\mathrm{P}^n$ as a CW-complex with the following properties:
   (a) the CW-structure has exactly one cell in the dimensions $0, 2, \ldots, 2n$ and no other cells,
   (b) for each $k \leq n$ the $2k$-skeleton of $\mathbb{C}\mathrm{P}^n$ is given by $\mathbb{C}\mathrm{P}^k$.

Proof.

(1) We prove the statement by induction on n. For $n = 0$ the topological space $\mathbb{R}\mathrm{P}^0$ consists of a single point which means that it is a CW-complex with a single 0-cell. Now suppose that we have equipped $\mathbb{R}\mathrm{P}^{n-1}$ with a CW-structure with the properties stated in (a) and (b). We view $\mathbb{R}\mathrm{P}^{n-1}$ as the subspace of $\mathbb{R}\mathrm{P}^n$ that is given by setting the last coordinate to zero. We consider the attaching map

$$\begin{array}{rcl} \varphi\colon S^{n-1} & \to & \mathbb{R}\mathrm{P}^{n-1} \\ (x_1, \ldots, x_n) & \mapsto & [x_1 : \ldots : x_n]. \end{array}$$

It remains to show that there exists a homeomorphism $\mathbb{R}\mathrm{P}^{n-1} \cup_\varphi \overline{B}^n \to \mathbb{R}\mathrm{P}^n$ that restricts to the identity on $\mathbb{R}\mathrm{P}^{n-1}$. We consider the map

$$\begin{array}{rcl} \Theta\colon \mathbb{R}\mathrm{P}^{n-1} \cup_\varphi \overline{B}^n & \to & \mathbb{R}\mathrm{P}^n \\ [P] & \mapsto & \begin{cases} [x_1 : \cdots : x_n : 0], & \text{if } P = [x_1 : \cdots : x_n] \in \mathbb{R}\mathrm{P}^{n-1}, \\ [P : 1 - \|P\|], & \text{if } P \in \overline{B}^n. \end{cases} \end{array}$$

It follows immediately from the definition of φ that the map Θ is well-defined. Furthermore it follows easily from the Topological-Quotient Proposition 5.15 (1b) that the map Θ is continuous. It is basically clear that the map Θ is a surjection and it is straightforward to verify that the map Θ is an injection. Using the Projective Spaces-Compact+Hausdorff Proposition 5.30, the Topological-Quotient Proposition 5.15 (1b) and the Compact-Hausdorff Proposition 2.17 (3) one can easily verify that the map is a homeomorphism. Finally it is clear that the restriction of the map Θ to $\mathbb{R}\mathrm{P}^{n-1}$ is the identity.

(2) The proof for $\mathbb{C}\mathrm{P}^n$ is basically identical to the proof of (1). We just need to view S^{2n-1} and $\overline{B}^{2n}$ as subsets of $\mathbb{C}^n$. We leave it to the reader to make the necessary minuscule modifications. ■

To enhance the reader's interest in CW-complexes we now give a sneak preview of a later result.

Theorem 96.5. Every compact connected n-dimensional smooth manifold M admits a finite CW-structure with the following properties:

(1) the maximal dimension of a cell is n,
(2) the boundary ∂M is a subcomplex.

Furthermore, if $M = A \cup B$ is a decomposition into two n-dimensional submanifolds such that $A \cap B$ is a union of boundary components of A and a union of boundary components of B, then we can find a CW-structure on M which has all the above properties and such that A and B are subcomplexes.

In the above discussion we have seen that many of the topological spaces we are interested in can be viewed as CW-complexes. But note that not every topological space can be described as a CW-complex. For example consider the topological space X which consists of precisely two points P and Q and which is endowed with the trivial topology. (Recall that this means that the only open sets are the empty set and all of X.) Then we cannot view X as CW-complex since any CW-complex with finitely many points consists only of 0-cells together with the *discrete* topology. For the same reason $\{0\} \cup \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$ is not a CW-complex.

68.2. **Two topologies on $\mathbb{R}^\infty$.** In the next section we want to introduce infinite-dimensional CW-complexes. Before we do so we want to get more acquainted with $\mathbb{R}^\infty$ and topologies on the set $\mathbb{R}^\infty$. First of all we write

$$\mathbb{R}^\infty \;:=\; \mathbb{R}^{(\mathbb{N})} \;:=\; \{(x_1, x_2, \dots) \,|\, \text{only finitely many } x_i \text{ are non-zero}\}.$$

As in the finite-dimensional case we can equip $\mathbb{R}^\infty$ with the Euclidean metric[787]

$$d((x_1, x_2, \dots), (y_1, y_2, \dots)) \;:=\; \sqrt{\sum_{i=1}^{\infty} |x_i - y_i|^2}.$$

By Metric-Topology Lemma 1.1 the metric d defines a topology on $\mathbb{R}^\infty$, more precisely

$$\mathcal{S} \;:=\; \{U \subset \mathbb{R}^\infty \,|\, \text{ for every } P \in U \text{ there exists an } \epsilon > 0 \text{ such that } B_\epsilon(P) \subset U\}$$

is a topology on $\mathbb{R}^\infty$.
This topology might feel like it is the "right topology" on $\mathbb{R}^\infty$. But the next two examples might sow some doubts.

Examples.

(1) We claim that the map

$$\begin{aligned} \alpha \colon \mathbb{R}^\infty &\to \mathbb{R} \\ (x_1, x_2, \dots) &\mapsto \sum_{i=1}^{\infty} 2^i \cdot x_i \end{aligned}$$

is not continuous with respect to $\mathcal{S}$. To show this claim we consider the open subset $(-1, 1) \subset \mathbb{R}$, then $\alpha^{-1}((-1, 1))$ contains the origin 0. We claim that there is no $\epsilon > 0$

[787] Note that the series on the right is in fact a finite sum, since all but finitely many terms are zero. It is again straightforward to verify that d is a metric.

with $B_\epsilon(0) \subset \alpha^{-1}((-1,1))$. Indeed, suppose such $\epsilon > 0$ exists. We pick $i \in \mathbb{N}$ with $\frac{1}{2^i} < \epsilon$ and we consider the point

$$Q \;=\; (0, \dots, 0, \tfrac{1}{2^i}, 0, \dots).$$

Then $Q \in B_\epsilon(0)$ but $\alpha(Q) = 1$, i.e. $Q \notin \alpha^{-1}((-1,1))$.

(2) Let B be the infinite diagonal matrix where the diagonal entries are given by 2^i for $i = 1, 2, \dots$, i.e. let

$$B \;:=\; \begin{pmatrix} 2 & 0 & 0 & \dots \\ 0 & 4 & 0 & \dots \\ 0 & 0 & 8 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We claim that the map

$$\begin{array}{rcl} \beta\colon \mathbb{R}^\infty & \to & \mathbb{R}^\infty \\ x & \mapsto & B \cdot x \end{array}$$

is not continuous with respect to $\mathcal{S}$. Indeed, an argument similar to (1) shows that the preimage of the open set $U = B_1(0)$ is not open.

The maps α and β feel quite natural, so we might want to look around for a different topology on $\mathbb{R}^\infty$ for which these two maps are in fact continuous. Of course that would be achieved by equipping $\mathbb{R}^\infty$ with the discrete topology, but clearly that's not what we want. Another reasonable condition on the desired topology is that for any $n \in \mathbb{N}_0$ the inclusion map

$$\begin{array}{rcl} \iota_n\colon \mathbb{R}^n & \hookrightarrow & \mathbb{R}^\infty \\ (x_1, \dots, x_n) & \mapsto & (x_1, \dots, x_n, 0, \dots) \end{array}$$

should also be continuous. This immediately eliminates the discrete topology as a contender for the "right topology".

In fact this last condition leads us almost immediately to an alternative topology on $\mathbb{R}^\infty$. We saw in the proof of the Direct Limit Existence Proposition 56.1 that in the category of topological spaces we have

$$\varinjlim \mathbb{R}^n \;=\; \mathbb{R}^\infty,$$

where $\mathbb{R}^\infty$ is equipped with the "weak topology", i.e. with the topology

$$\mathcal{T} \;:=\; \{U \subset \mathbb{R}^\infty \mid \text{for every } n \in \mathbb{N} \text{ the set } U \cap \mathbb{R}^n \text{ is open in } \mathbb{R}^n\}.$$

Put differently, a map $f\colon \mathbb{R}^\infty \to X$ to a topological space X is continuous with respect to $\mathcal{T}$ if and only if for each $n \in \mathbb{N}$ the map $\mathbb{R}^n \hookrightarrow \mathbb{R}^\infty \xrightarrow{f} X$ is continuous.

Examples.

(1) By definition of $\mathcal{T}$ the inclusion map $\mathbb{R}^n \to \mathbb{R}^\infty$ is continuous with respect to $\mathcal{T}$.
(2) The above map $\alpha\colon \mathbb{R}^\infty \to \mathbb{R}$ is continuous with respect to the topology $\mathcal{T}$. Indeed, for any $n \in \mathbb{N}$ the restriction $\mathbb{R}^n \to \mathbb{R}^\infty \to \mathbb{R}$ is a linear map, hence continuous.
(3) In Exercise 68.2 we will see that $\mathcal{S} \subset \mathcal{T}$. This implies in particular that the identity map defines a continuous map $(\mathbb{R}^\infty, \mathcal{T}) \to (\mathbb{R}^\infty, \mathcal{S})$.

(4) Let a_i, b_i, $i \in \mathbb{N}$ be real numbers. The infinite hyperrectangle

$$(a_1, b_1) \times (a_2, b_2) \times (a_3, b_3) \times \dots$$

is easily seen to be an open subset of $(\mathbb{R}^\infty, \mathcal{T})$.[788]

The following lemma shows in particular that the above map $\beta\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ is continuous with respect to the topology $\mathcal{T}$.

Lemma 68.2. Let $C = (c_{ij})_{i,j\in\mathbb{N}}$ be an infinite diagonal matrix. Then the map[789]

$$\begin{aligned} \gamma\colon \mathbb{R}^\infty &\to \mathbb{R}^\infty \\ x &\mapsto C \cdot x \end{aligned}$$

is continuous with respect to the topology $\mathcal{T}$.

Proof. Let $U \in \mathcal{T}$. We need to show that $\gamma^{-1}(U)$ lies in $\mathcal{T}$. So given any $n \in \mathbb{N}$ we need to show that $\gamma^{-1}(U) \cap \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$. So let $n \in \mathbb{N}$. Given $m \in \mathbb{N}$ we denote by $C_{m,n}$ the submatrix of C given by the top left $m \times n$-submatrix of C and we denote by $\gamma_{m,n}\colon \mathbb{R}^n \to \mathbb{R}^m$ the map given by $x \mapsto C_{m,n} \cdot x$. We have[790]

$$\begin{aligned} \gamma^{-1}(U) \cap \mathbb{R}^n &= \big\{x \in \mathbb{R}^n \,\big|\, C \cdot x \in U\big\} = \big\{x \in \mathbb{R}^n \,\big|\, C \cdot x \in \bigcup_{m\in\mathbb{N}} (U \cap \mathbb{R}^m)\big\} \\ &= \bigcup_{m\in\mathbb{N}} \big\{x \in \mathbb{R}^n \,\big|\, C \cdot x \in U \cap \mathbb{R}^m\big\} = \bigcup_{m\in\mathbb{N}} \big\{x \in \mathbb{R}^n \,\big|\, C_{m,n} \cdot x \in U \cap \mathbb{R}^m\big\}. \\ &= \bigcup_{m\in\mathbb{N}} \underbrace{\gamma_{m,n}^{-1}\big(\overbrace{U \cap \mathbb{R}^m}^{\text{open in } \mathbb{R}^m}\big)}_{\substack{\text{open in } \mathbb{R}^n \text{ since} \\ \gamma_{m,n} \text{ is continuous}}}. \end{aligned}$$

This shows that $\gamma^{-1}(U) \cap \mathbb{R}^n$ is a union of open subsets of $\mathbb{R}^n$, hence itself is open. ∎

This discussion shows hopefully that the following convention is reasonable.

Convention. Unless we say something else we will view $\mathbb{R}^\infty$ as a topological space with the topology $\mathcal{T}$. Put differently, we think of $\mathbb{R}^\infty$ as $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$. Similarly we define $\mathbb{C}^\infty = \varinjlim \mathbb{C}^n$ as topological spaces.

Remark. In Exercise 68.3 we will verify that the map

$$\begin{aligned} \mathbb{C}^\infty &\to \mathbb{R}^\infty \\ (x_1 + \mathrm{i}y_1, x_2 + \mathrm{i}y_2, \dots) &\mapsto (x_1, y_1, x_2, y_2, \dots) \end{aligned}$$

is a homeomorphism. At times we will use this homeomorphism to make the identification $\mathbb{C}^\infty = \mathbb{R}^\infty$.

In the future, the more we work with $\mathbb{R}^\infty$ and $\mathbb{C}^\infty$, the more we will see that the "direct limit topology" is in fact the "right" topology to work with.
To get acquainted with this new topological space we want to prove the following lemma.

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[788] This example is less obvious. The topology $\mathcal{T}$ has more open sets than $\mathcal{S}$ which made it easier in (2) to argue that the map α is continuous. But here in (3) it also makes it harder, since we have to check the continuity for more open sets on the right-hand side.

[789] Does this also work for any infinite-square matrix?

[790] For each $k \in \mathbb{N}$ we view $\mathbb{R}^k$ as a subset of $\mathbb{R}^\infty$ via the inclusion ι_k.

Lemma 68.3. Given any compact subset K of $\mathbb{R}^\infty$ there exists an $n \in \mathbb{N}$ with $K \subset \mathbb{R}^n$.

Proof. So let $K \subset \mathbb{R}^\infty$ be a compact set. Suppose that there is no $n \in \mathbb{N}$ such that $K \subset \mathbb{R}^n$. Then there exists an infinite increasing sequence $n_1, n_2, n_3, \dots$ of natural numbers and points $x_1, x_2, x_3, \dots$ in K such that $x_i \in \mathbb{R}^{n_i} \setminus \mathbb{R}^{n_i - 1}$. We denote by X the set whose elements are precisely the x_i.

Claim 1. The set X is a discrete subset of $\mathbb{R}^\infty$.

Proof. To simplify the notation we assume that $n_i = i$ for all $i \in \mathbb{N}$. For each i we denote by z_i the i-coefficient of x_i. Put differently, z_i is the highest non-zero coefficient of x_i. Furthermore for each $i \in \mathbb{N}$ we pick $\epsilon_i \in (0, \frac{\|z_i\|}{2})$. Finally, given $x \in \mathbb{R}$ and $\epsilon > 0$ we write $I_\epsilon(x) = (x - \epsilon, x + \epsilon)$. We define

$$\begin{array}{rcccccccccc} U_1 & = & I_{\epsilon_1}(z_1) & \times & I_{\epsilon_2}(0) & \times & I_{\epsilon_3}(0) & \times & I_{\epsilon_4}(0) & \times & \dots \\ U_2 & = & \mathbb{R} & \times & I_{\epsilon_2}(z_2) & \times & I_{\epsilon_3}(0) & \times & I_{\epsilon_4}(z_4) & \times & \dots \\ U_3 & = & \mathbb{R} & \times & \mathbb{R} & \times & I_{\epsilon_3}(z_3) & \times & I_{\epsilon_4}(0) & \times & \dots \\ \vdots & & & & & & & & & & \end{array}$$

As we remarked on page 1476, each U_i is an open subset of $\mathbb{R}^\infty$. Evidently U_i contains x_i and thus each U_i is an open neighborhood of x_i. Furthermore the sets U_i, $i \in \mathbb{N}$ are easily seen to be disjoint. But this shows, by definition of the subspace topology, that X is discrete. ⊞

Claim 2. The set X is a closed subset of K.

Proof. We need to show that $X \setminus K$ is open. By definition of the subspace topology it suffices to show that $\mathbb{R}^\infty \setminus X$ is open. In the following we continue with the notation from the previous claim. Thus let $y \in \mathbb{R}^\infty$. We pick $n \in \mathbb{N}$ such that $y \in \mathbb{R}^n$. In $\mathbb{R}^n$ we can and will pick an open neighborhood U of $(y_1, \dots, y_n)$ that does not contain $x_1, \dots, x_n \in \mathbb{R}^n$. One can now verify easily that

$$V \ := \ U \times I_{\epsilon_{n+1}}(0) \times I_{\epsilon_{n+2}}(0) \times \dots$$

is an open neighborhood of $y \in \mathbb{R}^\infty$ that does not intersect X. ⊞

It follows from Claim 1, Claim 2 and the Compact-Discrete Lemma 1.22 (2) that X is indeed finite. But that is a contradiction to our assumption that there is no $n \in \mathbb{N}$ such that K is contained in $\mathbb{R}^n$. ■

We conclude this section with a short discussion of the infinite-dimensional sphere S^∞.

Definition. We refer to

$$S^\infty \ := \ \{x \in \mathbb{R}^\infty \,|\, d(x, \text{origin}) = 1\} \ = \ \Big\{(x_1, x_2, \dots) \in \mathbb{R}^\infty \,\Big|\, \sum_{i=1}^{\infty} x_i^2 = 1\Big\},$$

equip with the subspace topology coming from $\mathbb{R}^\infty$, as the infinite-dimensional sphere.

Even though we did not show it yet, we strongly expect that the finite-dimensional spheres S^n are not contractible. In contrast to this expectation we have the following mind-bending lemma.

Lemma 68.4. The infinite-dimensional sphere S^∞ is contractible.

Proof. We want to show that S^∞ is contractible. We recall that by the π_0-Homotopy Equivalence Lemma 15.3 it suffices to show that there exists a homotopy H from id_{S^∞} to

a constant map. Our goal is to find such H. We set

$$X \;:=\; \{(0, p_2, p_3, \dots) \in \mathbb{R}^\infty \mid p_i \in \mathbb{R}\}.$$

Next we consider the maps

$$\begin{aligned} F\colon \mathbb{R}^\infty \times [0,1] &\to \mathbb{R}^\infty \\ (P,t) &\mapsto \underbrace{P\cdot(1-t) + (0,P)\cdot t}_{\neq 0 \text{ if } P \neq 0} \end{aligned} \quad \text{and} \quad \begin{aligned} G\colon X \times [0,1] &\to \mathbb{R}^\infty \\ (P,t) &\mapsto \underbrace{P\cdot(1-t) + (1,0,\dots)\cdot t}_{\neq 0 \text{ if } P \neq 0}. \end{aligned}$$

For non-zero $P \in \mathbb{R}^\infty$ and $P \in X$ we define $\widetilde{F}(P,t)$ and $\widetilde{G}(P,t)$ to be the normalization of $F(P,t)$ and $G(P,t)$ to length one. Next we consider the map

$$\begin{aligned} H\colon S^\infty \times [0,1] &\to S^\infty \\ (P,t) &\mapsto \begin{cases} \widetilde{F}(P, 2t), & \text{if } t \in [0, \frac{1}{2}], \\ \widetilde{G}(\widetilde{F}(P,1), 2t-1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{aligned}$$

Clearly $H_0 = \mathrm{id}$ and it is straightforward to see that H_1 is a constant map. It is tempting to conclude that we are done. But in fact we still need to show the following claim.

Claim. The map H is continuous.

Proof. By the Homotopy Stacking Lemma 14.3 it suffices to show that the restriction of H to the subsets $S^\infty \times [0, \frac{1}{2}]$ and $S^\infty \times [\frac{1}{2}, 1]$ are continuous. Since intervals are regionally compact it follows from the Direct Limits-and-Products Lemma 56.4 that it suffices to show for each $n \in \mathbb{N}$ the restrictions of H to the subsets $S^n \times [0, \frac{1}{2}]$ and $S^n \times [\frac{1}{2}, 1]$ are continuous. But now we are in familiar territory and the continuity is a consequence of elementary arguments. ∎

68.3. **Infinite-dimensional CW-complexes.** The following definition now generalizes the notion of a CW-complex.

Definition.

(a) A CW-complex X is defined as a sequence

$$X^{-1} \;:=\; \varnothing \subset X^0 \subset X^1 \subset X^2 \subset \dots$$

of CW-complexes (in the sense of the definition of page 1468) such that for each $m \leq n$ the CW-complex X^m is the m-skeleton of X^n. Let $n \in \mathbb{N}_0$. We adopt the following language:

(i) The (open) n-cells of X^n are called the (open) n-cells of X.

(ii) We refer to X^n as the *n-skeleton of X*

(b) We define the topological realization $|X|$ of such a CW-complex X as

$$|X| \;:=\; \varinjlim |X^n|.$$

Recall that on page 1468 we argued that for each $n \in \mathbb{N}_0$ we can view $|X^n|$ as a subset of $|X^{n+1}|$. Thus it follows from the Direct Limit-of-Topological Spaces Lemma 56.3 that we can and will write

$$|X| \;:=\; \bigcup_{n \in \mathbb{N}_0} |X^n|$$

where a subset $U \subset |X|$ is open if and only if for every $n \in \mathbb{N}_0$ the intersection $U \cap |X^n|$ is an open subset of $|X^n|$. As before, in the notation we usually do not distinguish between the notation for a CW-complex and its topological realization.

(c) We define the dimension $\dim(X) \in \mathbb{N}_0 \cup \{\infty\}$ of a non-empty CW-complex X as the maximal dimension of a cell.[791] We define the dimension of the empty CW-complex to be -1.

(d) We say a CW-complex is finite if it has only finitely many cells and we say it is countable if it has only countably many cells.

(e) A CW-structure for a topological space Y is a CW-complex X together with a homeomorphism $f\colon |X| \to Y$.

Remark.

(1) Let X be a CW-complex and let $n \in \mathbb{N}_0$. By the Direct Limit-of-Topological Spaces Lemma 56.3 we know that the inclusion map $|X^n| \to |X|$ is an embedding. Thus we can and will view the n-skeleton as a subset of X.

(2) The topology given in (b) on a union of topological spaces is sometimes called the the *weak topology*. This explains the "W" of the name CW-complex.

(3) It follows almost immediately from the definitions that the topological realization $|X|$ of a CW-complex is, *as a set*, the disjoint union of all the open cells.

(4) On a few occasions we will refer to $X^\infty := X$ as the ∞-skeleton.

Examples.

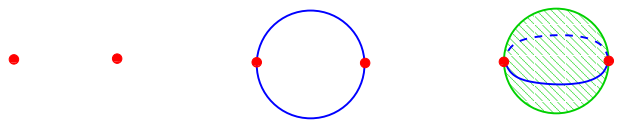
(1) We consider the infinite sphere

$$S^\infty \;=\; \{x \in \mathbb{R}^\infty \,|\, d(x, \text{origin}) = 1\} \;=\; \left\{(x_1, x_2, \dots) \in \mathbb{R}^\infty \,\middle|\, \sum_{i=1}^{\infty} x_i^2 = 1\right\}$$

which equip with the subspace topology coming from $\mathbb{R}^\infty$. It follows easily from the fact that $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$ that we have the equality $S^\infty = \varinjlim S^n$. We consider iteratively the spheres S^n as CW-complexes, where S^0 consists of two 0-cells and where S^n is built starting from S^{n-1} by attaching two n-cells ("the upper hemisphere and the lower hemisphere") to S^{n-1}. We then obtain an ascending sequence

$$S^0 \;\subset\; S^1 \;\subset\; S^2 \;\subset\; \dots$$

of CW-complexes which satisfies precisely the required properties of a CW-complex as defined above. This shows that the sphere S^∞ is an infinite-dimensional CW-complex. We refer to the figure below for an illustration.[792]



S^0 S^1 S^2

[791] Here we define $\dim(X) := \infty$ if X admits cells of arbitrarily large dimensions.

[792] If we equip S^∞ with the topology $\mathcal{S}$ coming from the Euclidean metric on $\mathbb{R}^\infty$, then this argument does not work and it is not clear (and presumably wrong), that $(S^\infty, \mathcal{S})$ is a CW-complex.

(2) Let I be a set and for each $i \in I$ let $n_i \in \mathbb{N}$. In Exercise 68.10 we will see that the wedge $\bigvee_{i\in I} S_i^{n_i}$ has a natural CW-structure with one 0-cell and where for each $n \in \mathbb{N}_0$ the n-cells correspond precisely to the n-dimensional spheres in our family of spheres $\{S_i^{n_i}\}_{i\in I}$.

The following definition and notation is the obvious infinite-dimensional analogue of the finite-dimensional definitions introduced on page 255.

Definition. Let $\mathbb{A} = \mathbb{R}$ or $\mathbb{A} = \mathbb{C}$.

(1) We define the infinite-dimensional projective space $\mathbb{A}\mathrm{P}^\infty$ as the set
$$\mathbb{A}\mathrm{P}^\infty \;=\; (\mathbb{A}^\infty \setminus \{0\})/(\mathbb{A} \setminus \{0\})$$
together with the quotient topology.

(2) Given a point $(x_1, x_2, \dots) \in \mathbb{A}^\infty \setminus \{0\}$ we denote its equivalence class in $\mathbb{A}\mathrm{P}^\infty$ by $[x_1 : x_2 : \dots]$.

(3) Given $n \in \mathbb{N}$ we view $\mathbb{A}\mathrm{P}^n$ as a subset of $\mathbb{A}\mathrm{P}^\infty$ via the inclusion map
$$\begin{array}{rcl} \mathbb{A}\mathrm{P}^n & \to & \mathbb{A}\mathrm{P}^\infty \\ [x_0 : \cdots : x_n] & \mapsto & [x_0 : \cdots : x_n : 0 : \dots]. \end{array}$$

The following lemma shows in particular that we can view the real infinite projective space $\mathbb{R}\mathrm{P}^\infty$ as an infinite-dimensional CW-complex.

Lemma 68.5.

(1) For each $n \in \mathbb{N}$ the obvious map $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^\infty$ is an embedding. Furthermore these maps define a homeomorphism $\varinjlim \mathbb{R}\mathrm{P}^n \xrightarrow{\cong} \mathbb{R}\mathrm{P}^\infty$.

(2) The topological space $\mathbb{R}\mathrm{P}^\infty$ admits a CW-structure such that for every $n \in \mathbb{N}$ the n-skeleton is given by $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^\infty$ where $\mathbb{R}\mathrm{P}^n$ is equipped with the CW-structure coming from Lemma 68.1.

(3) The infinite real projective space $\mathbb{R}\mathrm{P}^\infty$ admits a CW-structure which has precisely one n-cell for every $n \in \mathbb{N}_0$.

(4) (a) The action $\{\pm 1\} \times S^\infty \to S^\infty$ given by $(\epsilon, v) \mapsto \epsilon \cdot v$ is continuous and discrete.
(b) The natural projection $S^\infty \to S^\infty/x \sim -x$ is a 2-fold covering.
(c) The map
$$\begin{array}{rcl} S^\infty/x \sim -x & \to & \mathbb{R}\mathrm{P}^\infty \\ [(x_1, x_2, \dots)] & \mapsto & [x_1 : x_2 : \dots] \end{array}$$
is a homeomorphism.

Proof.

(1) We leave it as a very mildly interesting exercise to the reader to show that for each $n \in \mathbb{N}$ the obvious map $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^\infty$ is an embedding. Next we note that we have the following equalities:
$$\begin{aligned} \varinjlim \mathbb{R}\mathrm{P}^n \;=\; \varinjlim (\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\}) &\xrightarrow[\cong]{\text{homeomorphism by Lemma 56.5}} \left(\varinjlim (\mathbb{R}^{n+1} \setminus \{0\})\right)/(\mathbb{R} \setminus \{0\}) \\ &= (\mathbb{R}^\infty \setminus \{0\})/(\mathbb{R} \setminus \{0\}) \;=\; \mathbb{R}\mathrm{P}^\infty. \end{aligned}$$

It follows easily from the definitions that this homeomorphism is the inclusion induced map.

(2) This statement follows immediately from (1) and the definition of a CW-complex on page 1478.

(3) This statement is an immediate consequence of (2) and the definition of the CW-structure on each $\mathbb{R}\mathrm{P}^n$.

(4) (a) The statement is not difficult to prove, but one needs to argue carefully with the given topology on S^∞. We will fill in the details in Exercise 68.4.

(b) This statement follows from (a) and the Actions-Covering Proposition 48.9.

(c) It is straightforward to verify that the given map is a bijection. We leave it to the reader to verify that the map is continuous and that its inverse is continuous. ■

The following lemma is the complex sibling of Lemma 68.5.

Lemma 68.6.

(1) For each $n \in \mathbb{N}$ the obvious map $\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^\infty$ is an embedding. Furthermore the maps $\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^\infty$ define a homeomorphism $\varinjlim \mathbb{C}\mathrm{P}^n \xrightarrow{\cong} \mathbb{C}\mathrm{P}^\infty$.

(2) The topological space $\mathbb{C}\mathrm{P}^\infty$ admits a CW-structure such that for every $n \in \mathbb{N}$ the $2n$-skeleton is given by $\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^\infty$ where $\mathbb{C}\mathrm{P}^n$ is equipped with the CW-structure coming from Lemma 68.1.

(3) The infinite complex projective space $\mathbb{C}\mathrm{P}^\infty$ admits a CW-structure which has precisely one n-cell for every even $n \in \mathbb{N}_0$.

Proof. The proof of this lemma is basically verbatim the same as the proof of the corresponding parts of Lemma 68.5. ■

Convention. We will use Lemmas 68.5 (1) and 68.6 as an excuse to view each $\mathbb{R}\mathrm{P}^n$ as a subset of $\mathbb{R}\mathrm{P}^\infty$ and to view each $\mathbb{C}\mathrm{P}^n$ as a subset of $\mathbb{C}\mathrm{P}^\infty$.

The following lemma says that characteristic maps are continuous and it gives a convenient approach to showing that a map starting at a CW-complex is continuous.

Lemma 68.7. (Characteristic Maps Lemma) Let X be a CW-complex. We denote by $\{\Phi_i \colon \overline{B}^{n_i} \to X\}_{i \in I}$ the characteristic maps of all the cells of X. The following four statements hold:

(1) Every characteristic map Φ_i is continuous.

(2) A subset $U \subset X$ is open if and only if for each $i \in I$ the preimage of U under the characteristic map Φ_i is open, i.e. if each $\Phi_i^{-1}(U) \subset \overline{B}^{n_i}$ is an open subset of $\overline{B}^{n_i}$.

(3) The statement in (2) also holds if we replace "open" by "closed".

(4) A map $f \colon X \to Y$ to a topological space Y is continuous if and only if each map $f \circ \Phi_i \colon \overline{B}^{n_i} \to Y$ is continuous.

Proof. Given $n \in \mathbb{N}_0$ we denote by $I_n \subset I$ the indices corresponding to the characteristic maps of the n-dimensional cells.

(1) Let $i \in I_n$. The characteristic map $\Phi \colon \overline{B}^n \to X$ is the composition of the maps

$$\overline{B}^n = \overline{B}^n_i \to |X^{n-1}| \sqcup \bigsqcup_{j \in I_n} \overline{B}^n_j \to |X^n| = \Big(|X^{n-1}| \sqcup \bigsqcup_{j \in I_n} \overline{B}^n_j\Big)/\sim \to |X| = \varinjlim |X^k|.$$

All these maps are obviously continuous, hence the composition is also continuous.

(2) Once one has unraveled the definition of the topology on a CW-complex one sees that the first statement of the lemma is obvious. In the following we will try to convince the reader that this glib statement is actually correct.

Thus let $U \subset X$ be a subset. By the definition on page 1479 we know that U is open if and only if for each $n \in \mathbb{N}_0$ the intersection $U \cap |X_n|$ is an open subset of $|X_n|$. The statement follows from this observation together with the following claim.

Claim. Let $n \in \mathbb{N}$ and let U be a subset of $|X_n|$. The following two statements are equivalent:
(a) U is an open subset of $|X_n|$.
(b) $U \cap |X_{n-1}|$ is an open subset of $|X_{n-1}|$ and for every $i \in I_n$ the preimage $\Phi_i^{-1}(U)$ is an open subset of $\overline{B}^n$.

Proof. By the definition on page 1468 we have

$$|X^n| \;=\; \Big(|X^{n-1}| \;\sqcup\; \bigsqcup_{i\in I_n} \overline{B}_i^n\Big)/\!\sim \;\text{ where } \Phi_i(x) \sim x \text{ for } x \in \partial\overline{B}_i^n.$$

The statement now follows easily from the definition of the quotient topology and the disjoint union topology. We leave it to the reader to weave these slogans into a coherent argument. ⊞

(3) The "only if"-direction follows from the Characteristic Maps Lemma 68.7 (1). The "if"-direction follows from (1) and a slight generalization of Lemma 2.7 which one can easily prove using the Image-Preimage Lemma 0.2 (8). We leave it to the reader to make sense of this cryptic remark.

(4) Let $f\colon X \to Y$ be a map to a topological space. If f is continuous, then it follows from the Characteristic Maps Lemma 68.7 (1) that each map $f \circ \Phi_i\colon \overline{B}^{n_i} \to Y$ is continuous. Now suppose conversely that each map $f \circ \Phi_i\colon \overline{B}^{n_i} \to Y$ is continuous. Let $U \subset Y$ be an open subset. We need to show that $f^{-1}(U)$ is an open subset of X. By (1) it suffices to show that each preimage $\Phi_i^{-1}(f^{-1}(U))$ is an open subset of $\overline{B}^{n_i}$. But this follows from the observation that $\Phi_i^{-1}(f^{-1}(U)) = (f \circ \Phi_i)^{-1}(U)$ and the hypothesis that each $f \circ \Phi_i$ is continuous. ■

68.4. **Homotopies on infinite CW-complexes.** On several occasions throughout these notes, in particular in the next section, we will want to construct a homotopy on an infinite-dimensional CW-complex by working our way up through the skeleta. In the end one needs to combine infinitely many homotopies which obviously leads to continuity issues. This rather technical section deals with these issues. It is best to initially skip this section and to go back to it if later on one really wants to know how an argument works.

For the undaunted reader we start out with the following subtle lemma.

Lemma 68.8. (Intervall CW-Product Lemma) Let X be a CW-complex. The following three topologies on $X \times [0,1]$ agree:
(1) The product topology on the product $X \times [0,1]$.
(2) The topology on $X \times [0,1]$ viewed as the direct limit $\varinjlim (X^k \times [0,1])$.[793]

(3) The topology which is defined on the set $X \times [0,1]$ by the condition that a given subset $U \subset X \times [0,1]$ is open if and only if for each characteristic map $\Phi\colon \overline{B}^n \to X$ the preimage $(\Phi \times \mathrm{id}_{[0,1]})^{-1}(U)$ is an open subset of $\overline{B}^n \times [0,1]$.

Proof. Let X be a CW-complex. First note that it follows immediately from the fact that $X = \varinjlim X_k$ together with the Product Topology-Basics Lemma 5.4 (4) that any subset that is open with respect to (1) is also open with respect to (2). Furthermore it follows from the Characteristic Maps Lemma 68.7 (1) together with the Product Topology-Basics Lemma 5.4 (3) that any subset that is open with respect to (2) is also open with respect to (3). Thus it suffices to show that any set that is open with respect to (3) is also open with respect to (1).

Let $W \subset X \times [0,1]$ be a subset that is open with respect to (3). We need to show that W is also open with respect to the product topology on $X \times [0,1]$. Let $(x_0, t_0) \in W$ be a point. By the Product Topology-Basics Lemma 5.4 it suffices to show the following statement:

(∗) There exists a neighborhood V of $x_0 \in X$ and a neighborhood K of $t_0 \in [0,1]$ such that $V \times K$ is contained in W.

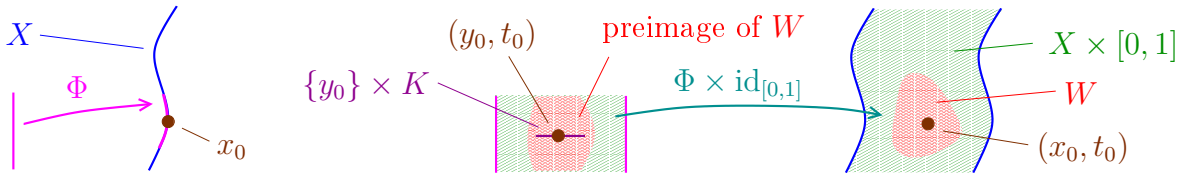
To prove (∗) we start out with the following claim.

Claim. There exists a compact neighborhood K of $t_0 \in [0,1]$ such that $\{x_0\} \times K \subset W$.

Proof. Let $\Phi\colon \overline{B}^n \to X$ be the characteristic map of the open cell that contains x_0. Note that $\Phi\colon B^n \to X$ is injective. Thus there exists a unique $y_0 \in B^n$ with $\Phi(y_0) = x_0$. We write $\Psi := \Phi \times \mathrm{id}_{[0,1]}$. Since W is open with respect to (3) we know that $\Psi^{-1}(W)$ is an open subset of $\overline{B}^n \times [0,1]$. Since $(y_0, t_0) \in \Psi^{-1}(W)$ we obtain from the definition of the product topology on $\overline{B}^n \times [0,1]$ there exists in particular an open neighborhood J of $t_0 \in [0,1]$ such that $\{y_0\} \times J \subset \Psi^{-1}(W)$. Since $[0,1]$ is regionally compact we see that there exists a compact neighborhood K of t_0 that is contained in J. Now we see that

$$\{x_0\} \times K \;=\; \{\Phi(y_0)\} \times K \;\underset{\text{definition of } \Psi}{=}\; \Psi(\{y_0\} \times K) \;\subset\; \Psi(\{y_0\} \times J) \;\underset{\text{since } \{y_0\} \times J \subset \Psi^{-1}(W)}{\subset}\; \Psi(\Psi^{-1}(W)) \;\underset{\text{Image-Preimage Lemma 0.2 (2)}}{\subset}\; W.$$

Thus K has the desired property. ⊞



Next we consider $V := \{x \in X \mid \{x\} \times K \subset W\}$. By the choice of K we have $x_0 \in V$. Thus to prove (∗) it remains to prove the following claim.

Claim. The subset V is open in X.

[793] Recall that we showed in the Direct Limit-of-Topological Spaces Lemma 56.3 that this is the topology where a given subset W of $X \times [0,1] = \varinjlim (X^k \times [0,1])$ is open if and only each intersection $W \cap (X^k \times [0,1])$ is an open subset of $X^k \times [0,1]$.

Proof. We use the Characteristic Maps Lemma 68.7 (2) to show that V is open in X. Thus let $\Phi\colon \overline{B}^n \to X$ be the characteristic map of an n-cell of X. We need to show that the preimage $\Phi^{-1}(V)$ is open in $\overline{B}^n$. We write $U := (\Phi \times \mathrm{id}_{[0,1]})^{-1}(W)$. Since W is open with respect to (3) we see that U is an open subset of $\overline{B}^n \times [0,1]$. Basically by definition we have the equality

$$\Phi^{-1}(V) \;=\; \{y \in \overline{B}^n \mid \{y\} \times K \subset U\}.$$

It follows from elementary point-set topology, see Exercise 5.10, that the set on the right-hand side is indeed open. This concludes the proof of the second claim and thus of the lemma. ■

Later on we will frequently need the following corollary.

Corollary 68.9. (Interval CW-Product Corollary) Let X be a CW-complex and let $\Phi\colon X \times [0,1] \to Y$ be a map to a topological space Y. The following statements are equivalent:

(1) The map f is continuous.
(2) For every $k \in \mathbb{N}_0$ the restriction of f to $X^k \times [0,1]$ is continuous.
(3) For every characteristic map $\Phi\colon \overline{B}^n \to X$ the map

$$\overline{B}^n \times [0,1] \xrightarrow{(v,t)\mapsto(\Phi(v),t)} X \times [0,1] \xrightarrow{f} Y$$

is continuous.

Proof. The corollary follows immediately from the equivalence of the three topologies in the Intervall CW-Product Lemma 68.8. ■

68.5. **Properties of CW-complexes.** We have seen, as promised, that many topological spaces that we are interested in are indeed CW-complexes. In this section we will prove a somewhat technical proposition that shows that CW-complexes have, again as promised, many nice topological properties.
To formulate the key result of this section we need the following definition which we will explore in much greater detail in the subsequent section.

Definition. A subcomplex of a CW-complex X is a subset of X that is the union of arbitrarily many cells of X. We say a subcomplex is finite it contains only finitely many cells of X.

Now we can state the main result of this section.

Proposition 68.10. (CW-Complex Properties Proposition)

(1) Every point on a CW-complex is closed.[794]
(2) (a) CW-complexes are Hausdorff.
    (b) CW-complexes are normal.[795]
(3) (a) Every cell of a CW-complex X is compact and it is a closed subset of X.
    (b) Every finite CW-complex is compact.
    (c) The closure of an open cell is the corresponding cell. More precisely, given a characteristic map $\Phi\colon \overline{B}^n \to X$ the closure of $\Phi(B^n)$ is indeed $\Phi(\overline{B}^n)$.
    (d) If a closed subset of X contains an open cell, then it also contains the corresponding cell.

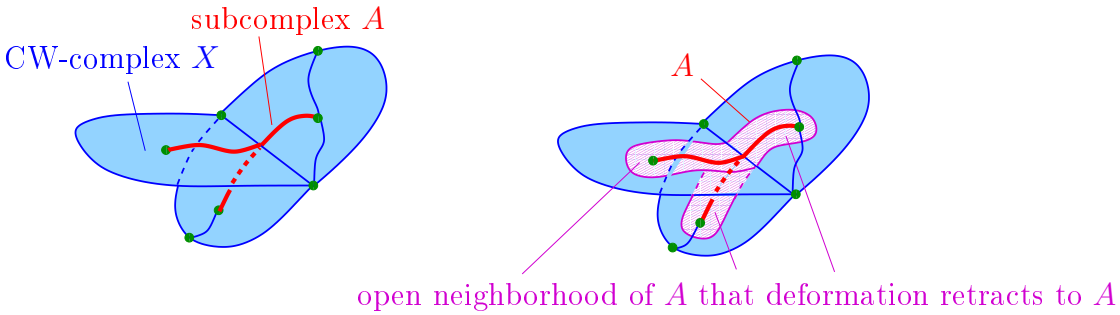
(4) Let X be a CW-complex, let $x \in X$ and let U be an open neighborhood U of x.
   (a) There exists an open neighborhood V of x that is contained in U and that admits a deformation retraction to $\{x\}$.
   (b) There exists a closed neighborhood V of x that is contained in U and that admits a deformation retraction to $\{x\}$.

(5) Every point on a CW-complex is good in the sense of the definition on page 1171.

(6) CW-complexes are locally contractible, locally path-connected, locally simply-connected and semi-locally simply connected.

(7) Let X be a CW-complex. The following statements are equivalent:
   (a) X is connected.
   (b) X is path-connected.
   (c) The 1-skeleton X^1 is connected.
   (d) The 1-skeleton X^1 is path-connected.
   (e) Given any two distinct points P, Q in the 0-skeleton of X there exists an embedding $\gamma\colon [0,1] \to X^1$ with $\gamma(0) = P$, $\gamma(1) = Q$ and such that $\gamma([0,1])$ is a subcomplex of X^1.

(8) There exists a map Φ which assigns to each subcomplex A a subset $\Phi(A) \subset X$ such that the following statements hold:
   (a) for each subcomplex A the set $\Phi(A)$ is an open neighborhood of A such that A is a deformation retract of $\Phi(A)$,
   (b) for any two subcomplexes A_1 and A_2 we have
   $$\Phi(A_1 \cap A_2) = \Phi(A_1) \cap \Phi(A_2) \quad \text{and} \quad \Phi(A_1 \cup A_2) = \Phi(A_1) \cup \Phi(A_2),$$
   (c) for any two subcomplexes A and B with $A \subset B$ we have $\Phi(A) \subset \Phi(B)$.



Remark.

(1) In a slightly simplified form the CW-Complex Properties Proposition 68.10 (8) says that we can assign to each subcomplex A in a "systematic way" an open neighborhood $\Phi(A)$ that deformation retracts to A.

[794]Note that we need (1) in the proof of (2), thus we cannot just use the Points-in-Hausdorff Space Lemma 1.15 (1) to deduce (2) from (1).

[795]Recall that on page 123 we introduced the definition that a topological space X is called *normal* if any two disjoint closed subsets A and B of X are separated by neighborhoods, i.e. if there exist disjoint neighborhoods U of A and V of B.

(2) In the CW-Complex Properties Proposition 68.10 (8) it is essential that A is a subcomplex. For example, if we consider $X = \mathbb{R}$ and $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$, then A does not admit an open neighborhood U such that A is a deformation retract of U.[796]
(3) the CW-Complex Properties Proposition 68.10 (6) sounds somewhat technical, but it ensures that later on we can apply the Covering Existence Theorem 61.5 to obtain coverings of CW-complexes.
(4) Many other useful properties of CW-complexes are proved in [**LW1969**, Section II.3].
(5) There is a limit to what degree CW-complexes are nice, for example in Exercise 50.11 we saw that the wedge $\bigvee_{i \in \mathbb{N}_0} [0,1]_i$, which is easily seen to be a CW-complex, is not metrizable.

Proof. We start out with the following notation. Given a subset A of $\overline{B}^n$ and $\epsilon > 0$ we denote by

$$N_\epsilon(A) \;:=\; \big\{x \in \overline{B}^n \;\big|\; \text{ there exists an } y \in A \text{ with } \|x - y\| < \epsilon\big\}$$

the ϵ-neighborhood around A. One easily verifies that $N_\epsilon(A)$ is an open subset of $\overline{B}^n$. Furthermore given two non-empty subsets X and Y of $\overline{B}^n$ we follow the notation introduced on page 283 and we write

$$d(X,Y) \;:=\; \inf\{\|x - y\| \,|\, x \in X \text{ and } y \in Y\}.$$

Furthermore, if X or Y is empty, then we define $d(X,Y) = \infty$ and evidently we write $\infty > 0$.

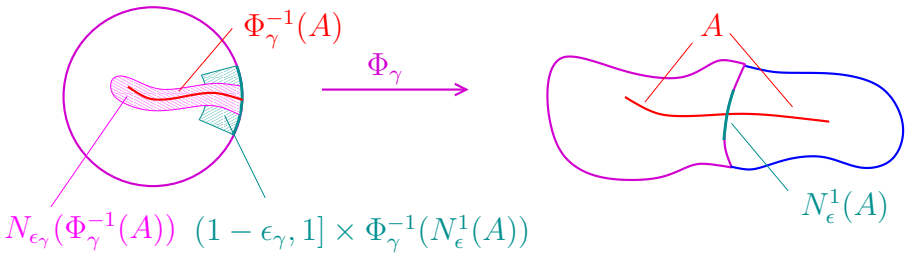
Now let X be a CW-complex. We denote by Γ the set of cells of X and given $n \in \mathbb{N}_0$ we denote by Γ_n the set of n-cells. Given a cell γ we denote by Φ_γ the corresponding characteristic map.

Let $A \subset X$ be a subset and let ϵ be a map that assigns to each $\gamma \in \Gamma$ a positive number ϵ_γ. Given $n \in \mathbb{N}_0$ we define a subset $N^n_\epsilon(A)$ inductively as follows:

(i) For $n = 0$ we define $N^0_\epsilon(A) := A \cap X^0$.
(ii) Suppose that $N_\epsilon(A)^n$ has already been defined. We define

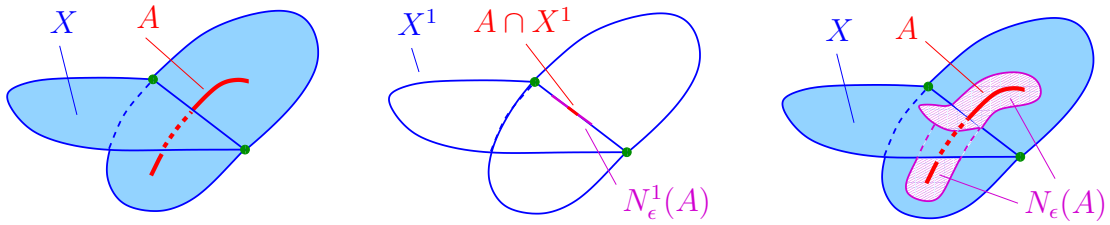
$$N^{n+1}_\epsilon(A) \;:=\; N^n_\epsilon(A) \cup \bigcup_{\gamma \in \Gamma_{n+1}} \Phi_\gamma\Big(N_{\epsilon_\gamma}(\overbrace{\Phi^{-1}_\gamma(A)}^{\subset \overline{B}^{n+1}}) \cup \underbrace{(1-\epsilon_\gamma, 1] \times \overbrace{\Phi^{-1}_\gamma(N^n_\epsilon(A))}^{\subset S^n}}_{\text{radial coordinates on } \overline{B}^{n+1}}\Big).$$

It is straightforward to verify that each $N^n_\epsilon(A)$ is an open neighborhood of $A \cap X^n$ in X^n. The definition of $N^{n+1}_\epsilon(A)$ is illustrated in the figure below. Finally we set $N_\epsilon(A) :=$



[796] Why not?

$\bigcup_{n\in\mathbb{N}} N_\epsilon^n(A)$. It follows easily from the definition of the topology of a CW-complex that $N_\epsilon(A)$ is an open neighborhood of A in X. The definition of $N_\epsilon(A)$ is illustrated in the figure below.



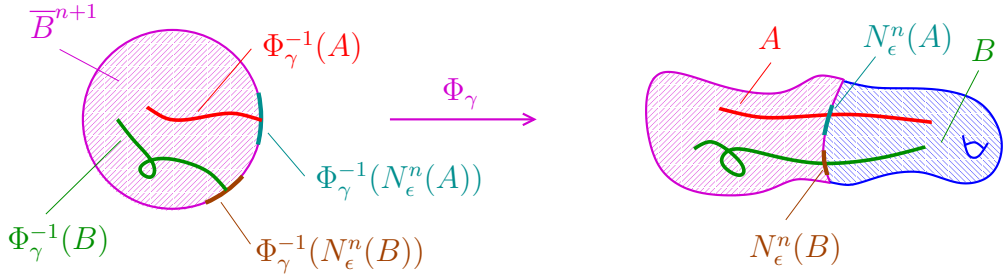
Now we turn to the proofs of the eight statements:

(1) This statement follows fairly easily from the the Topological Pushout-Maps Lemma 5.34 (4). We will fill in the details in Exercise 68.6.

(2) Since by (1) we know that points are closed it suffices to show that CW-complexes are normal. Thus let A and B of X be disjoint closed subsets of X. We need to show that there exist open disjoint neighborhoods of A and B.
So let A and B be two disjoint closed subsets of X. We will find ϵ as above such that $N_\epsilon(A)$ and $N_\epsilon(B)$ are disjoint. We will define the values of ϵ inductively on the dimension of the cells. For 0-cells we take $\epsilon = 1$. Now suppose that we have already found disjoint open sets $N_\epsilon^n(A)$ and $N_\epsilon^n(B)$. Let γ be an $(n+1)$-cell with characteristic map $\Phi_\gamma\colon \overline{B}^{n+1} \to X$.

Claim. We have the following inequalities:

$$\begin{array}{lrcl} \text{(a)} & \mu_1 := d(\Phi_\gamma^{-1}(N_\epsilon^n(A)), \Phi_\gamma^{-1}(B)) & > & 0, \\ \text{(b)} & \mu_2 := d(\Phi_\gamma^{-1}(N_\epsilon^n(B)), \Phi_\gamma^{-1}(A)) & > & 0, \\ \text{(c)} & \mu_3 := d(\Phi_\gamma^{-1}(B), \Phi_\gamma^{-1}(A)) & > & 0. \end{array}$$



Proof. We deal with the three inequalities separately:

(a) We first consider μ_1. Suppose that $\mu_1 = 0$. If this was the case then there would exist a sequence of points in $\Phi_\gamma^{-1}(B)$ that converges to a point of $\Phi_\gamma^{-1}(B)$ in $\partial\overline{B}^{n+1}$ of distance zero from $\Phi_\gamma^{-1}(N_\epsilon^n(A))$. But this is impossible since $\Phi_\gamma^{-1}(N_\epsilon^n(B))$ is a neighborhood of $\Phi_\gamma^{-1}(B) \cap \partial\overline{B}^{n+1}$ in $\partial\overline{B}^{n+1}$ which is disjoint from $\Phi_\gamma^{-1}(N_\epsilon^n(A))$.

(b) The proof of the second inequality $\mu_2 > 0$ is proved exactly the same as the proof of the inequality $\mu_1 > 0$.

(c) It follows from the Characteristic Maps Lemma 68.7 (1) and our hypothesis that A and B are disjoint closed subsets that the preimages $\Phi_\gamma^{-1}(A)$ and $\Phi_\gamma^{-1}(B)$ are disjoint closed subsets of $\overline{B}^{n+1}$. By the Compact-Closed Lemma 1.21 (1) we know that they are in fact disjoint compact subsets of $\overline{B}^n$. Thus it follows from Lemma 6.3 that $\mu_3 = d(\Phi_\gamma^{-1}(B), \Phi_\gamma^{-1}(A)) > 0$. ⊞

We set $\epsilon_\gamma := \frac{1}{2}\min\{\mu_1, \mu_2, \mu_3\}$. We do this for all $(n+1)$-cells γ. It then follows immediately from the definitions that $N_\epsilon^{n+1}(A)$ and $N_\epsilon^{n+1}(B)$ are disjoint.

(3) (a) Every k-cell is the image of the compact set $\overline{B}^k$, hence it is compact by the Compact Image Lemma 2.13. It follows from the Compact-Closed Lemma 1.21 (2) and the fact, proved in (2), that X is Hausdorff that every k-cell is also a closed subset of X.

(b) A finite CW-complex is the union of finitely many cells. Thus it follows from (3a) together with the Compact-Union-Intersection Lemma 1.20 that a finite CW-complex is compact.

(c) Let $\Phi\colon \overline{B}^n \to X$ be a characteristic map. We have the following inclusions

$$\begin{array}{ccccccc} \Phi(\overline{B}^n) & \subset & \overline{\Phi(B^n)} & = & \bigcap \text{ closed subsets containing } \Phi(B^n) & \subset & \Phi(\overline{B}^n). \\ \uparrow & & \uparrow & & & & \uparrow \\ \text{Closure-Interior} & & \text{definition of the closure} & & & & \text{by (3a)} \\ \text{Mapping Lemma 2.5} & & & & & & \end{array}$$

We have thus shown that we indeed have $\overline{\Phi(B^n)} = \Phi(\overline{B}^n)$.

(d) This statement follows immediately from (3b) and the definition of the closure of a set.

(4) Let $x \in X$ and let U be an open neighborhood of x. We need to prove the following two statements;

(a) There exists an open neighborhood V of x that is contained in U and that admits a deformation retraction to $\{x\}$.

(b) There exists a closed neighborhood V of x that is contained in U and that admits a deformation retraction to $\{x\}$.

In the following we will prove Statement (a). We leave it to the reader to make the slight modifications to obtain Statement (b).

So let us start with the proof of Statement (a). We leave it to the long-suffering reader to show that there exists an ϵ such that $N_\epsilon(x)$ is contained in U.

It remains to show that $N_\epsilon(x)$ admits a deformation retraction to $\{x\}$. We denote by m the unique natural number with $x \in X^m \setminus X^{m-1}$. It follows easily from the definitions that $N_\epsilon^m(x)$ is an open ball around x, in particular $N_\epsilon^m(x)$ deformation retracts onto $\{x\}$. Therefore it suffices to find a deformation retraction from $N_\epsilon(x)$ to $N_\epsilon^m(x)$.

First note that for $n > m$ "sliding radially outward" we obtain a deformation retraction

$$R(n)\colon N_\epsilon^n(x) \times [0,1] \;\to\; N_\epsilon^n(x)$$

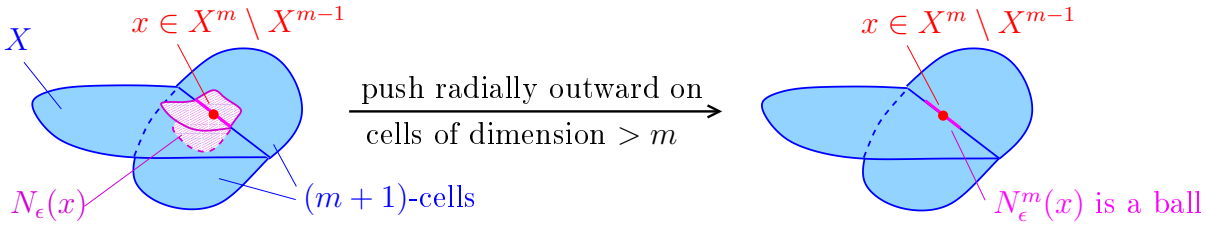
from $N_\epsilon^n(x)$ to $N_\epsilon^{n-1}(x)$. We denote by $r_n := R(n)_1 \colon N_\epsilon^n(x) \to N_\epsilon^{n-1}(x)$ the corresponding retraction. For each $k \in \mathbb{N}_{\geq m+1}$ we consider the map

$$\begin{array}{rcl} f_k \colon N_\epsilon(x) & \to & N_\epsilon^{k-1}(x) \\ y & \mapsto & \begin{cases} (r_k \circ \cdots \circ r_l)(y), & \text{if } y \in N_\epsilon^l(x) \text{ and } l \geq k, \\ y, & \text{if } y \in N_\epsilon^{k-1}(x). \end{cases} \end{array}$$

It follows from the Interval CW-Product Corollary 68.9 that this map f_k is continuous. Now we consider the following map

$$\begin{array}{rcl} F \colon N_\epsilon(x) \times [0,1] & \to & N_\epsilon(x) \\ (y,s) & \mapsto & \begin{cases} y, & \text{if } x \in N_\epsilon^k(x) \text{ and } s \in [0, \frac{1}{2^{k+1}}], \\ r_k\big(y, 2^{k+1} \cdot (s - \frac{1}{2^{k+1}})\big), & \text{if } x \in N_\epsilon^k(x) \text{ and } s \in [\frac{1}{2^{k+1}}, \frac{1}{2^k}], \\ f_k(y), & \text{if } x \in N_\epsilon^k(x) \text{ and } s \in [\frac{1}{2^k}, 1]. \end{cases} \end{array}$$

We refer to the figure below for an illustration. It follows from the Homotopy Stacking Lemma 14.3 that the restriction of F to each $N_\epsilon^k(x) \times [0,1]$ is continuous. Thus it follows from the Interval CW-Product Corollary 68.9 that $F \colon N_\epsilon(x) \times [0,1] \to N_\epsilon(x)$ is continuous. This map is the desired deformation retraction.



(5) This statement follows immediately from (2) and (4).

(6) Note that this statement follows immediately from (4) since every topological space that admits a deformation retraction to a point is contractible, and since every contractible topological space is path-connected and simply connected. Finally note that a topological space that is locally simply connected is in particular semi-locally simply connected.

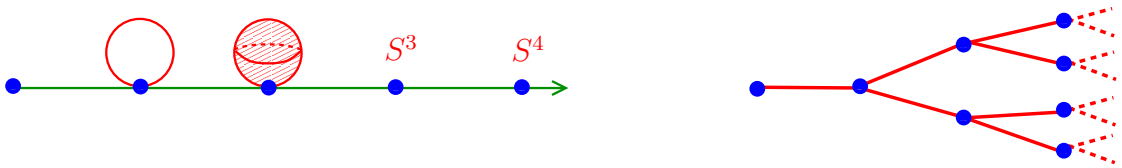
(7) By Locally Path-Connected Lemma 2.40 an locally path-connected topological space is connected if and only if it is path-connected. This shows the equivalence (a) $\Leftrightarrow$ (b) and (c) $\Leftrightarrow$ (d). It follows from the Abstract Graph-Topological Realization Lemma 7.1 (5) and the discussion on page 1470 that we also have (d) $\Leftrightarrow$ (e). We will prove (d) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) in Exercise 68.7.

(8) We pick an $\epsilon \in (0,1)$. Given a subcomplex A of X we set $\Phi(A) := N_\epsilon(A)$. In each cell of $X \setminus A$ the set $N_\epsilon(A)$ is a product neighborhood of the boundary of this cell. So a deformation retraction of $N_\epsilon(A)$ onto A can be constructed as in (4). We leave the details to the reader. It is clear that Φ has all the other desired properties. ■

Our next goal is to formulate an addendum to the CW-Complex Properties Proposition 68.10. To do so we need to introduce the following definition.

Definition. We say that a CW-complex is locally finite if every cell is disjoint from all but finitely many cells of X.

Examples. The CW-complex $\mathbb{C}\mathrm{P}^\infty$ is not locally finite, whereas the CW-complexes illustrated in the figure below are locally finite.



The following proposition is the promised addendum to the CW-Complex Properties Proposition 68.10.

Proposition 68.11. A CW-complex is regionally compact if and only if it is locally finite.

Proof. We will not make use of Proposition 68.11, except that in the negative sense, that now we know that many CW-complexes we are interested in are unfortunately not regionally compact. The proposition is proved in [**Geo2008**, Proposition 10.1.8]. ■

One of the curses of the working topologist is the unfortunate fact that a bijection continuous map between topological spaces does not need to be a homeomorphism. The following lemma gives a nice criterion for a map between CW-complexes to be a homeomorphism.

Lemma 68.12. (CW-Complex Homeomorphism Lemma) Let $f\colon X \to Y$ be a map between CW-complexes. Suppose the following three conditions are satisfied:

(1) f is a bijection,
(2) f is continuous,
(3) the images of the cells of X are precisely the cells of Y.

Then $f\colon X \to Y$ is a homeomorphism.

Proof. We denote by $g = f^{-1}\colon Y \to X$ the inverse of f. It remains to show that g is continuous. Let $\{\Phi_i\colon \overline{B}^{m_i} \to X\}_{i\in I}$ be the characteristic maps of the CW-complex X and let $\{\Phi_j\colon \overline{B}^{n_j} \to Y\}_{j\in J}$ be the characteristic maps of the CW-complex Y. By the Characteristic Maps Lemma 68.7 it suffices to show that for each $j \in J$ the map $g \circ \Psi_j\colon \overline{B}^{n_j} \to X$ is continuous. We consider the cell $\Psi_j(\overline{B}^{n_j})$. By hypothesis (3) there exists an $i \in I$ such that $f(\Phi_i(\overline{B}^{m_i})) = \Psi_j(\overline{B}^{n_j})$. Since $\Phi_i(\overline{B}^{m_i})$ is compact and since we know by the CW-Complex Properties Proposition 68.10 that $\Psi_j(\overline{B}^{n_j}) \subset Y$ is Hausdorff it follows from the Compact-Hausdorff Proposition 2.17 (3) that $f\colon \Phi_i(\overline{B}^{m_i}) \to \Psi_j(\overline{B}^{n_j})$ is a homeomorphism. But this implies that $f^{-1} = g$ is continuous on $\Psi_j(\overline{B}^{n_j})$. It follows that, as desired, the map $g \circ \Psi_j$ is continuous. ■

We conclude this section with the following proposition.

Proposition 68.13. (CW-Complex Paracompactness Proposition) Every CW-complex is paracompact.

Proof. As always, when it comes to CW-complexes, the proof is slightly delicate. Since we will not really make use of the proposition we refer instead to [**FrPi1990a**, Theorem 1.3.5], or alternatively to [**Hat2**, Proposition 1.20], [**Lee2000**, Theorem 5.22], [**LW1969**, Theorem II.4.2] and [**Miy1952**], for a proof. ■

68.6. **The Finiteness Theorem for CW-complexes.** The following proposition is one of the key technical results about infinite CW-complexes.

Theorem 68.14. (CW-Complex-Finiteness Theorem) Every compact (connected) subset of a CW-complex X is contained in a finite (connected) subcomplex.

Remark. The statement of the proposition and also its proof have more than a passing resemblance with Lemma 68.3.

Example. We consider the wedge $X := \bigvee_{i\in\mathbb{N}_0} S^1_i$. As discussed on page 1480, we can view X as a CW-complex with one 0-cell, given by the wedge point, and 1-cells indexed by $i \in \mathbb{N}_0$. It follows from the CW-Complex-Finiteness Theorem 68.14 that each compact subset of X is contained in the union of finitely many S^1_i. In fact this statement already follows from Exercise 50.9.

Before we start out with the proof of the CW-Complex-Finiteness Theorem 68.14 we formulate the following pretty corollary.

Corollary 68.15. (CW-Complex Compactness Corollary) A CW-complex is compact if and only if it is finite.

Proof. In the CW-Complex Properties Proposition 68.10 (3) we showed that every finite CW-complex is compact. Now we want to prove the converse. Thus let X be a compact CW-complex. It follows from the CW-Complex-Finiteness Theorem 68.14 that X is contained in a finite subcomplex Y of X. But from $X \subset Y \subset X$ it follows that $X = Y$, i.e. X is a finite CW-complex. ■

The proof of the CW-Complex-Finiteness Theorem 68.14 will be a fairly straightforward consequence of the following rather technical lemma.

Lemma 68.16. Let X be a CW-complex.

(1) Let $A \subset X$ be a subset. If every open cell of X contains at most one point of A, then A is closed and discrete.
(2) If $C \subset X$ is a compact subset, then C is contained in a finite union of open cells.
(3) Every cell of X is contained in a finite subcomplex of X.[797]

Remark. Lemma 68.16 (3) together with the CW-Complex Properties Proposition 68.10 (3a) and (3c) shows in particular that the *closure* of each open cell is contained in the union of *finitely* many open cells. This property is sometimes referred to as X being "closure finite" which explains the "C" in the name CW-complex.

Proof of Lemma 68.16. We start out with the following claim.

Claim. Let Y be a CW-complex. Then the statements (1), (2) and (3) are equivalent.

Proof. Let Y be a CW-complex. First we show (1) $\Rightarrow$ (2). Let $C \subset Y$ be a compact subset. For each cell that intersects C we pick a single point. We denote the resulting set of points by A. By (1) the set A is closed and discrete. Since A is a closed subset of compact set we obtain from the Compact-Closed Lemma 1.21 (1) that A is compact. Since A is compact and discrete we obtain from the Compact-Discrete Lemma 1.22 (1) that A is in fact finite. But that means that C intersects only finitely many cells, i.e. C is contained in a finite union of open cells as claimed.

[797] Recall that by definition, see page 1484, a subcomplex is *finite* if it contains only finitely many cells of X. So even a subcomplex that is given by a single cell might a priori contain many other cells.

Now we will prove (2) $\Rightarrow$ (3). So let $C_\sigma \subset Y$ be an n-cell. We denote by U_σ the corresponding open cell. By the CW-Complex Properties Proposition 68.10 (3) and by definition of a cell we know that $C_\sigma \setminus U_\sigma$ is a compact subset of the $(n-1)$-skeleton Y^{n-1}. It follows from (2) that the subset $C_\sigma \setminus U_\sigma$ is contained in a finite union of open cells of dimension $\leq n-1$. By the same argument each of these lower-dimensional closed cells is again contained in a finite union of open cells of smaller dimension, except again for the cell itself. A straightforward induction argument now shows that we end up with finitely many cells which form a subcomplex of Y.

Next we will show (3) $\Rightarrow$ (1). Let $A \subset Y$ be a subset such that no two points lie in the same open cell. We consider the intersection of A with a closed cell. By (3) this intersection is contained in a finite subcomplex Y'. Since A has at most one point in common with any open cell this intersection $A \cap Y'$ must be finite and hence closed in $A \cap Y'$ by the CW-Complex Properties Proposition 68.10 (2). By the CW-Complex Properties Proposition 68.10 (3) the finite subcomplex Y' is a closed subset of X. Hence $A \cap Y'$ is a closed subset of X. For any point $x \in A$ the set $A \setminus \{x\}$ also satisfies the hypothesis for A, and so it must also be a closed subset of X. Hence $\{x\}$ is open in A, so A is discrete. ⊞

Now let X be a CW-complex. First note that Statement (1) holds for X^0 by definition of the topology on X^0. Now suppose we know (1) for the n-skeleton X^n for some $n \in \mathbb{N}_0$. By claim we know that (2) and (3) also for X^n. But the proof of (2) $\Rightarrow$ (3) for a given n-cell only used (2) for subsets of X^{n-1}. Thus we actually have (3) for X^{n+1}. Thus we also have (1) for X^{n+1}. Summarizing we have all three statements for X^n for all n. But any cell lies in some X^n, and so we know (3) for X itself, and by the claim this implies that (1) and (2) also hold for X. ■

Proof of the CW-Complex-Finiteness Theorem 68.14. Let X be a CW-complex and let $C \subset X$ be a compact subset. By Lemma 68.16 (2) the subset C is contained in a union of finitely many open cells. By Lemma 68.16 (3) each of these open cells is contained in a finite subcomplex. The union of these finitely many finite subcomplexes is a finite subcomplex Z which contains C.
If C is connected, then we take the component of Z that contains C. By Lemma 68.21 (3) this is again a subcomplex. ■

68.7. **Cellular maps.** Before we move on to considering subcomplexes and constructions of CW-complexes let us quickly introduce the type of maps between CW-complexes that are reasonable.

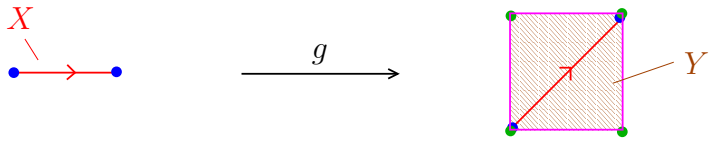
Definition.

(1) A map $g\colon X \to Y$ between two CW-complexes is called cellular if for each $n \in \mathbb{N}_0$ we have $g(X^n) \subset Y^n$.
(2) A map $g\colon X \to Y$ between two CW-complexes is called a cellular isomorphism if f is cellular, if f is a bijection and continuous, and if f^{-1} is also cellular.

Example. In general maps between CW-complexes are not cellular. For example consider the CW-complex $X = [0,1]$ with two 0-cells and one 1-cell and we consider the CW-complex $Y = [0,1] \times [0,1]$ with four 0-cells, two 1-cells and one 2-cell. Then the diagonal map

$$\begin{array}{rccc} g\colon X = [0,1] & \to & Y = [0,1] \times [0,1] \\ t & \mapsto & (t,t) \end{array},$$

which is illustrated in the figure below is not a cellular map.



It is clear that the composition of two cellular maps is again cellular. This leads us to the following definition.

Definition. We call the category $\mathcal{CW}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{CW}) &:= \text{all CW-complexes,} \\ \mathrm{Mor}(X,Y) &:= \text{all } \textit{cellular} \text{ maps from } X \text{ to } Y, \end{aligned}$$

together with the usual composition of maps the category of CW-complexes. This category should not be confused with the *full category of* CW*-complexes* that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{FullCW}) &:= \text{all CW-complexes,} \\ \mathrm{Mor}(X,Y) &:= \text{all } \textit{continuous} \text{ maps from } X \text{ to } Y. \end{aligned}$$

Now is as good as any other moment to introduce a little more categorical language.

Definition. A covariant functor $F\colon \mathcal{C} \to \mathcal{D}$ between two categories is called faithful if for every two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ the map $\mathrm{Mor}(X,Y) \to \mathrm{Mor}(F(X),F(Y))$ is an injection. If the map is in fact a bijection, then we call the functor fully faithful. The same definition applies, with the obvious modifications, to contravariant functors.

To practice the definitions we formulate the following lemma.

Lemma 68.17. The maps

$$\begin{aligned} \text{CW-complex } X &\mapsto |X| \\ (f\colon X \to Y) &\mapsto (f\colon |X| \to |Y|) \end{aligned}$$

define a covariant functor from the category $\mathcal{CW}$ of CW-complexes to the category of topological spaces. The functor is faithful but not fully faithful. The maps also define a covariant functor from the full category $\mathcal{FullCW}$ of CW-complexes to the category of topological spaces, this time the functor is fully faithful.

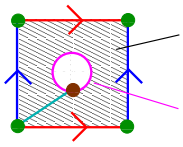
Proof. The statements are basically obvious. The fact that we just found a continuous map which is not cellular gives us that the functor $\mathcal{CW} \to \mathcal{Top}$ is not fully faithful. ■

68.8. **Subcomplexes.** In this section we will discuss in great deal the concept of a subcomplex. We defined the notion on a subcomplex in a perfunctory way on page 1484, but to give the concept its due prominence let us repeat the definition.

Definition. A subcomplex of a CW-complex X is a subset of X that is the union of arbitrarily many cells of X.

Examples.

(1) In the figure below we consider the smooth manifold X that is given by removing an open disk from a torus. We see that X has the structure of a CW-structure such that the boundary ∂X is a CW-subcomplex.



(2) Let X be a CW-complex. For each $n \in \mathbb{N}_0$ the n-skeleton X^n is a subcomplex.

The name "subcomplex" suggests that a subcomplex should again be a CW-complex. That is indeed the case.

Lemma 68.18. (Subcomplex Lemma) Let X be a CW-complex and let A be a subcomplex of X.

(1) The subset A admits a CW-structure such that for each $n \in \mathbb{N}_0$ we have the equality $|A^n| = A \cap |X^n|$ and such that the characteristic maps of A correspond precisely to the characteristic maps of X that take values in A.
(2) The inclusion map $A \to X$ is a closed embedding, in particular A is a closed subset of X.

Proof. We start out with the proof of Statement (1). We set $Y^{-1} = |Y^{-1}| := \varnothing$. For $n = 0, 1, 2, 3, \ldots$ we iteratively do the following:

(1) Let $\{\varphi_i \colon S^{n-1} \to |Y^{n-1}|\}_{i \in I}$ be the attaching maps from the CW-structure of X that correspond to n-cells that lie in A. We define $Y^n = (Y^{n-1}, \{\varphi_i \colon S^{n-1} \to |Y^{n-1}|\}_{i \in I})$.
(2) We set $|Y^n| := \left(|Y^{n-1}| \sqcup \bigsqcup_{i \in I} \overline{B}^n\right)/\sim$.

Finally we define Y to be the CW-complex that is given by the sequence of CW-complexes $Y_{-1} \subset Y_0 \subset Y_1 \subset \ldots$.

Since A is by definition the union of cells we see that the obvious map $\varphi \colon |Y| \to A$ is a bijection. It remains to show that the map φ is actually a homeomorphism. Note that by definition, if $\Phi \colon \overline{B}^n \to |Y|$ is a characteristic map of the CW-complex Y, then $\varphi \circ \Phi \colon \overline{B}^n \to |X|$ is a characteristic map of the CW-complex X. Thus it follows from the Characteristic Maps Lemma 68.7 (1) and (4) that φ is continuous.

By the Open -Injective Map Lemma 2.16 (1) it remains to show that the map $\varphi \colon |Y| \to X$ is a closed map.

Claim. The map $\varphi \colon |Y| \to X$ is a closed map.

Proof. Lt $C \subset |Y|$ be a closed subset. We need to show that $\varphi(C)$ is also a closed subset of A. By the Characteristic Maps Lemma 68.7 (1) and (3) it suffices to show that for each cell $e \subset X$ the set $\varphi(C) \cap e$ is closed. By Lemma 68.16 we know that e intersects only finitely many cells. In particular there are only finitely many cells contained in $A \cap e$. We denote these by $a_1, \ldots, a_m$. Now we see that

$$\varphi(C) \cap e \underset{\substack{\uparrow \\ \text{since } \varphi(C) \subset A}}{=} \varphi(C) \cap (A \cap e) \underset{\substack{\uparrow \\ \text{since } A \text{ is a subcomplex}}}{=} \varphi(C) \cap \bigcup_{i=1}^m a_i = \bigcup_{i=1}^m (\varphi(C) \cap a_i)$$

$$\underset{\substack{\uparrow \\ \text{since } C \text{ is closed in } |Y| \text{ and since} \\ a_1, \ldots, a_m \in S \text{ are cells of } Y}}{=} \bigcup_{i=1}^m \text{closed subset of } a_i \underset{\substack{\uparrow \\ \text{by the Open -Closed Inclusion Lemma 2.10 (2) since the } a_i \\ \text{are closed subsets of } e}}{=} \bigcup_{i=1}^m \text{closed subset of } e = \text{closed subset of } e.$$

⊞

Now we turn to the proof of Statement (2). In fact this follows immediately from the claim. Indeed, evidently $|Y|$ is a closed subset of $|Y|$ and from the claim we obtain that $\varphi(|Y|)$ is a closed subset of X. But $\varphi(|Y|) = A$. Thus we have shown that A is a closed subset of X. ■

Definition. Let A and X be CW-complex. A map $f\colon A \to X$ is called a cellular embedding if f is injective and if the image of a cell of A is a cell of X.

Lemma 68.19. (Cellular Embedding Lemma) Let $f\colon A \to X$ be a cellular embedding between two CW-complexes. The following statements hold:

(1) The image $f(A)$ is a subcomplex of A and the map $f\colon A \to f(A)$ is a cellular isomorphism.
(2) The map $f\colon A \to X$ is a closed embedding.

Proof. It follows immediately from the above definition of a subcomplex and of a cellular embedding that the image of a cellular embedding $f\colon A \to X$ is a subcomplex. Next we consider the map $f^{-1}\colon f(A) \to A$. We deduce from the Characteristic Maps Lemma 68.7 together with the Compact-Hausdorff Proposition 2.17 (3) that the map $f^{-1}\colon f(A) \to A$ is continuous. It follows almost immediately that $f\colon A \to f(A)$ is a cellular isomorphism. Finally we obtain from the above together with the Subcomplex Lemma 68.18 that $f\colon A \to X$ is a closed embedding. ■

Lemma 68.20. (CW-Complex Pasting Lemma) Let X be a CW-complex and let $\{X_i\}_{i\in I}$ be a family of subcomplexes such that $\bigcup_{i\in I} X_i = X$. The following three statements hold:

(1) A subset $U \subset X$ is open if and only if each $X_i \cap U$ is an open subset of X_i.
(2) A subset $A \subset X$ is closed if and only if each $X_i \cap A$ is a closed subset of X_i.
(3) A map $f\colon X \to Y$ to a topological space Y is continuous if and only if each map $f|_{X_i}\colon X_i \to Y$ is continuous.

Proof.

(1) This statement is an easy consequence of the Characteristic Maps Lemma 68.7 (2).
(2) The "only if"-direction is an immediate consequence of the Neighborhood Openness Criterion 1.5 (2). The "if"-direction follows from (1) and Lemma 2.7 (1) ⇒ (2).
(3) The "only if"-direction is an immediate consequence of Lemma 2.1 (3). The "if"-direction follows from (1) and Lemma 2.7 (1) ⇒ (3). ■

The following lemma shows in particular that some straightforward constructions provide examples of subcomplexes.

Lemma 68.21. Let X be a CW-complex.

(1) For any $n \in \mathbb{N}_0$ the n-skeleton X^n is a subcomplex of X.
(2) The union and also the intersection of arbitrarily many subcomplexes is again a subcomplex.
(3) Every path-component of a complex is again a subcomplex.

Proof.

(1) This statement is clear.

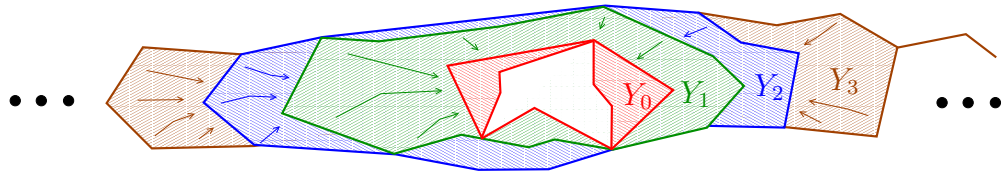
(2) This statement is also clear.
(3) This statement is basically obvious since each closed ball is path-connected. ■

We conclude this section with the following proposition.

Proposition 68.22. (Iterated Deformation Retracts Proposition) Let X be a CW-complex and let $Y = Y_0 \subset Y_1 \subset Y_2 \subset \dots$ be a sequence of subcomplexes such that the following two conditions are satisfied:

(1) we have $X = \bigcup_{i \in \mathbb{N}} Y_i$,
(2) each Y_i is a deformation retract of Y_{i+1}.

Then Y_0 is a deformation retract of X.



Proof. By hypothesis there exists for each $k \in \mathbb{N}_0$ a deformation retraction

$$R(k)\colon Y_k \times [0,1] \;\to\; Y_{k-1}.$$

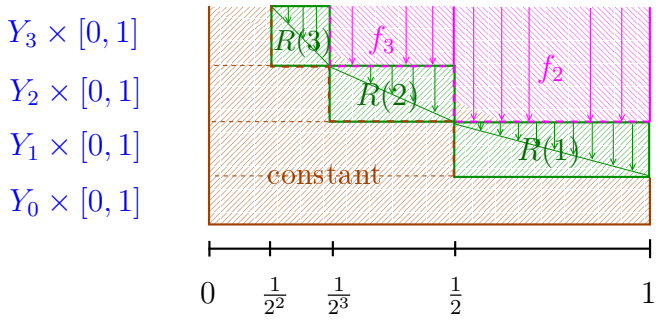
We denote by $r_k := R(k)_1\colon Y_k \to Y_{k-1}$ the corresponding retraction. For each $k \in \mathbb{N}_0$ we consider the map

$$\begin{array}{rcl} f_k\colon X \times [0,1] & \to & Y_{k-1} \times [0,1] \\ (x,t) & \mapsto & \begin{cases} (r_k \circ \dots \circ r_l)(x), & \text{if } x \in Y_l \text{ and } l \geq k, \\ x, & \text{if } x \in Y_{k-1}. \end{cases} \end{array}$$

It follows from the Interval CW-Product Corollary 68.9 that this map f_k is continuous. Now we consider the following map

$$\begin{array}{rcl} F\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} x, & \text{if } x \in Y_k \text{ and } t \in [0, \frac{1}{2^{k+1}}], \\ r_k\big(x, 2^{k+1} \cdot (t - \frac{1}{2^{k+1}})\big), & \text{if } x \in Y_k \text{ and } t \in [\frac{1}{2^{k+1}}, \frac{1}{2^k}], \\ f_k(x), & \text{if } x \in Y_k \text{ and } t \in [\frac{1}{2^k}, 1]. \end{cases} \end{array}$$

We refer to the figure below for an illustration. It follows from the Homotopy Stacking Lemma 14.3 that the restriction of F to each $Y_k \times [0,1]$ is continuous. Thus it follows from the Interval CW-Product Corollary 68.9 that $F\colon X \times [0,1] \to X$ is continuous. This map is the desired deformation retraction. ■



68.9. **Products of CW-complexes I.** Let X and Y be two CW-complexes. We can form the product topological space $X \times Y$. Ideally one would like to equip $X \times Y$ with a CW-structure. Throughout this section we use the homeomorphism from the Convex-to-Ball Proposition 2.20 (2) to identify the n-dimensional closed ball $\overline{B}^n$ with the n-dimensional cube $I^n = [0,1]^n$ and we use it to make the identification $S^{n-1} = \partial I^n$. For $p+q=n$ we evidently have $I^n = I^p \times I^q$ and we have

$$\partial(I^n) \;=\; \partial I^p \times I^q \;\cup\; I^p \times \partial I^q.$$

The idea is now to define a CW-structure on $X \times Y$ such that the product of a p-cell in X with a q-cell in Y corresponds to a $(p+q)$-cell in $X \times Y$.
In the following we want to execute this idea rigorously. First, to be on the safe side, we consider the case of two finite complexes.

Definition. Let X and Y be two CW-complexes. Given $p \in \mathbb{N}_0$ let $\{\varphi_j \colon S^{p-1} \to X^{p-1}\}_{j \in J_p}$ and $K_p := \{\psi_k \colon S^{p-1} \to Y^{p-1}\}_{k \in K_p}$ be the set of attaching maps of p-dimensional cells of X and Y. Furthermore we denote by $\{\Phi_j \colon \overline{B}^p \to X^p\}_{j \in J_p}$ and $\{\Psi_k \colon \overline{B}^p \to Y^p\}_{k \in K_p}$ the corresponding characteristic maps of the cells. We define a new CW-complex Z as follows:

(1) We put $Z^{-1} = \varnothing$.
(2) Suppose Z^{n-1} is already defined for some $n \in \mathbb{N}_0$. Given $j \in J_p$ and $k \in K_{n-p}$ we define[798]
$$\begin{aligned} \theta_{(j,k)} \colon \partial I^n = \partial I^p \times I^{n-p} \;\cup\; I^p \times \partial I^{n-p} &\to Z^{n-1} \\ (x,y) &\mapsto (\Phi_j(x), \Psi_k(y)). \end{aligned}$$
We set
$$Z^n \;:=\; \Big(Z^{n-1} \;\sqcup \bigsqcup_{p+q=n} \bigsqcup_{j \in J_p, k \in K_{n-p}} \underbrace{\big(I^p \times I^{n-p}\big)}_{=\overline{B}^n}{}_{(j,k)} \Big) / \sim$$
where for each $j \in J_p$, $k \in K_{n-p}$ we define $(x,y) \in (I^p \times I^{n-p})_{(j,k)}$ with the point $\theta_{(i,j)}(x,y) \in Z^{n-1}$.
(3) Given $j \in J_p$ and $k \in K_{n-p}$ we denote by $\Theta_{(j,k)} \colon I^n = I^p \times I^{n-p} \to Z^n$ the corresponding characteristic map.
(4) We set $Z := \varinjlim Z^n$.

As the discussion suggests, the above construction does indeed (often) define a CW-structure on the topological space $X \times Y$. To simplify the initial discussion we first consider the case that X and Y are two finite CW-complexes.

Proposition 68.23. (CW-Complex Product Proposition) Let X and Y be two CW-complexes. We use the notation from the above definition.

(1) The above data turns Z into a CW-complex.
(2) The characteristic maps $\Theta_{(i,j)}$ define a continuous bijection $\Theta \colon Z \to X \times Y$.
(3) Suppose we are in one of the following two settings:
   (a) X and X are both finite CW-complexes, or
   (b) $Y = [0,1]$ where Y is equipped with the usual CW-structure with two 0-cells and one 1-cell,

[798] Why does the image of this attaching map lie in $(X \times Y)^{n-1}$?

then the map $\Theta\colon Z \to X \times Y$ from (2) is a homeomorphism. In particular the topological space $X \times Y$ has a CW-structure such that for each $n \in \mathbb{N}_0$ we have

$$\underbrace{(X \times Y)^n}_{n\text{-skeleton of } X \times Y} = \bigcup_{p+q=n} X^p \times Y^q.$$

Definition. Let X and Y be finite CW-complexes. Suppose we are in one of the two settings of the CW-Complex Product Proposition 68.23 (3).

(1) We refer to the above CW-structure on $X \times Y$ as the product CW-structure.
(2) Given a p-cell e for X and given a q-cell for Y we denote by $e \times f$ the corresponding $(p+q)$-cell for $X \times Y$.

Remark. The second case of the CW-Complex Product Proposition 68.23 (3) might appear somewhat random, but it will actually prove important for showing that several constructions, like mapping cylinders and mapping cones, give rise to CW-complexes.

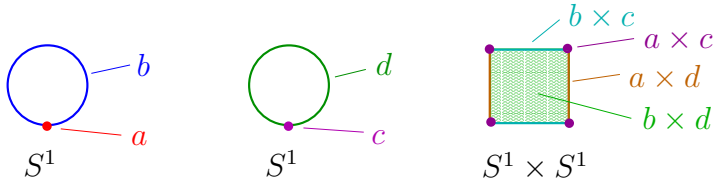
Proof. Let X and Y be two CW-complexes.

(1) This statement follows basically immediately from the definitions.
(2) It follows from the Characteristic Maps Lemma 68.7 (4) that the map $\Theta\colon Z \to X \times Y$, which is defined via the characteristic maps $\Theta_{(i,j)}$, is continuous.
(3) (a) Suppose that X and X are both finite CW-complexes. In this case Z has only finitely many cells, hence Z is compact by the CW-Complex Compactness Corollary 68.15. Furthermore note that $X \times Y$ is Hausdorff by the CW-Complex Properties Proposition 68.10 (2) and the Product Topology Properties Proposition 5.6. It follows from the Compact-Hausdorff Proposition 2.17 (3) that $\Theta\colon Z \to X \times Y$ is indeed a homeomorphism.
(b) Suppose we have $Y = [0,1]$ with the obvious CW-structure. It follows basically immediately from the equivalence of the topologies (1) and (3) of the Intervall CW-Product Lemma 68.8 that the map $\Theta\colon Z \to X \times Y$ is a homeomorphism.

Since $\Theta\colon Z \to X \times Y$ is a homeomorphism and since Z is a CW-complex we have endowed $X \times Y$ with a CW-structure. The statement about the skeleta follows immediately from the definition of the CW-structure on Z. ■

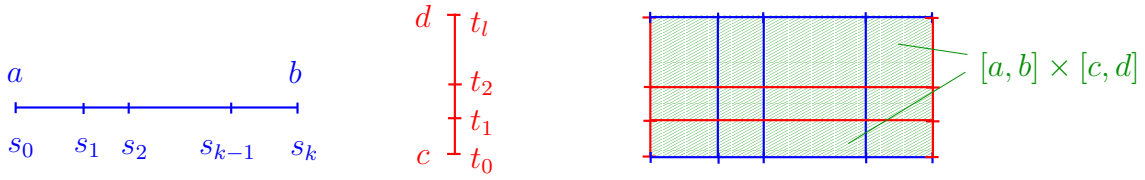
Examples.

(1) If we equip $S^1 = [0,1]/0 \sim 1$ with the usual CW-structure with one 0-cell and one 1-cell, then the product CW-structure on the torus $S^1 \times S^1 = ([0,1] \times [0,1])/\sim$ is precisely the CW-structure that we had already encountered on page 1471. We refer to the figure below for an illustration.



(2) Given $a = s_0 < s_1 < \cdots < s_k = b$ and $c = t_0 < t_1 < \cdots < t_l = d$ we can view the intervals $[a,b]$ and $[c,d]$ as CW-complexes where the 1-cells are given precisely by the intervals $[s_i, s_{i+1}]$, $i = 0, \ldots, k-1$ and $[t_j, t_{j+1}]$, $j = 0, \ldots, l-1$. The product

CW-complex is the topological space $[a,b]\times[c,d]$ where the 2-cells are given by the rectangles $[s_i,s_{i+1}]\times[t_j,t_{j+1}]$ with $i\in\{0,\dots,k-1\}$ and $j\in\{0,\dots,l-1\}$. The analogous statement holds also for higher-dimensional hyperrectangles. For example, given $k\in\mathbb{N}$ and $n\in\mathbb{N}$ we can equip the cube $[0,1]^n$ with a CW-structure where the n-cells are precisely given by the cubes $[\frac{s_1}{k},\frac{s_1+1}{k}]\times\cdots\times[\frac{s_n}{k},\frac{s_n+1}{k}]$ with $s_1,\dots,s_n\in\{0,\dots,k-1\}$.



The formulation of Proposition 68.23 already suggests that the conclusion does not hold without some assumptions on X and Y. In fact, the following lemma, which goes back to **[Dow1952]**, shows that there are indeed examples of CW-complexes X and Y such that the map $\Theta\colon Z\to X\times Y$ is *not* a homeomorphism. In other words, in general the "naive" attempt to endow a product of CW-complexes with a CW-structure fails.

Lemma 68.24. Let $S:=\{(s_1,s_2,\dots)\,|\,s_i\in\mathbb{N}\}$ be the set of all infinite sequences of positive integers. Let $X=\bigvee_{i\in[0,1]}[0,1]_s$ be the wedge of uncountably many copies of $[0,1]$, obtained by wedging all the $0\in[0,1]_s$. Similarly we form the wedge $Y=\bigvee_{j\in\mathbb{N}}[0,1]_j$ of countably many copies of $[0,1]$. We equip X and Y with the "obvious CW-structure", see e.g. the CW-Complex Construction Lemma 68.32 (4). In this setting the map $\Theta\colon Z\to X\times Y$ from the CW-Complex Product Proposition 68.23 is *not* a homeomorphism.

Proof. For $s\in S$ and $j\in\mathbb{N}$ we consider $(\frac{1}{s_j},\frac{1}{s_j})\in[0,1]_s\times[0,1]_j\in X\times Y$. We denote by $P\subset X\times Y$ the subset defined by all these points. Since each 2-cell of the CW-complex Z contains a single point of P we obtain from the Characteristic Maps Lemma 68.7 (3) that P is a closed subset of Z. To show that Θ is not a homeomorphism it remains to prove the following claim.

Claim. The set P is not closed with respect to the product topology on $X\times Y$.

Proof. We denote by $x_0\in X$ and $y_0\in Y$ the wedge points. Note that $(x_0,y_0)\notin P$. Thus it suffices to show that every neighborhood of $(x_0,y_0)\in X\times Y$ contains a point of P. In fact, by definition of the product topology, see page 224, it suffices to show that any subset $U\times V$ of $X\times Y$ of the form $U=\bigvee_{s\in S}[0,a_s]\subset X$ and $V=\bigvee_{j\in\mathbb{N}}[0,b_j]\subset Y$ contains a point of P. We pick a sequence $t=(t_1,t_2,\dots)\in S$ such that for every $j\in\mathbb{N}$ we have $t_j>j$ and such that $t_j>\frac{1}{b_j}$. Furthermore we pick a $k\in\mathbb{N}$ with $k>\frac{1}{a_t}$. It follows that $t_k>k>\frac{1}{a_t}$, which implies that $\frac{1}{t_k}<a_t$. Recall that by construction we also have $\frac{1}{t_k}<b_k$. Thus we see that $(\frac{1}{t_k},\frac{1}{t_k})$ is a point of P that lies in $[0,a_t]\times[0,b_k)$. Hence it also lies in $U\times V$. ∎

68.10. **Products of CW-complexes II.** In this section we will discuss the product of two infinite CW-complexes in somewhat greater detail. We will not make use of the results of this and the coming section, thus these two sections can be skipped safely.

First we mention the following proposition which can be viewed as a generalization of the CW-Complex Product Proposition 68.23.

Proposition 68.25. Let X and Y be two CW-complexes. If at least one of the two CW-complexes has only countably many cells, then the map $\Theta\colon Z \to X \times Y$ from the CW-Complex Product Proposition 68.23 (2) is a homeomorphism, in particular $X \times Y$ has a natural CW-structure.

Proof. This statement is proved in [**LW1969**, Theorem II.5.2]. If one of X or Y is a finite CW-complex, then the statement is also proved in [**Hat2002**, Theorem A.6]. If both CW-complexes have countably many cells, then the statement is also proved in [**FrPi1990a**, Proposition 2.2.3] or [**Miln1956b**, Lemma 2.1]. ■

Definition. Let X and Y be two CW-complexes.

(1) If at least one of the two CW-complexes has only countably many cells, then we refer to the above CW-structure on $X \times Y$ as the product CW-structure.
(2) Following [**LW1969**, p. 56] we denote by $X \otimes Y$ the topological space whose underlying set is given by $X \times Y$ but whose topology is the unique topology that turns the map $\Theta\colon Z \to X \otimes Y$ from Proposition 68.23 into a homeomorphism.

Remark. Note that [**Brooke2017**, Theorem 1] gives a necessary and sufficient condition for the product of two CW-complexes to be a CW-complex.

We conclude this section with a short discussion on the relationship between $X \otimes Y$ and $X \times Y$.

Proposition 68.26. Let X and Y be two CW-complexes.

(1) The identity map $\mathrm{id}\colon X \otimes Y \to X \times Y$ is a continuous bijection.
(2) A subset $K \subset X \otimes Y$ is compact if and only if it is compact in $X \times Y$.
(3) Given any $(x, y) \in X \times Y$ the induced map $\pi_1(X \otimes Y, (x, y)) \to \pi_1(X \times Y, (x, y))$ is an isomorphism.
(4) If at least one of the two CW-complexes has only countably many cells, then the map $X \otimes Y \to X \times Y$ is a homeomorphism.

The proof of Proposition 68.26 (3) is mostly outsourced to the following lemma.

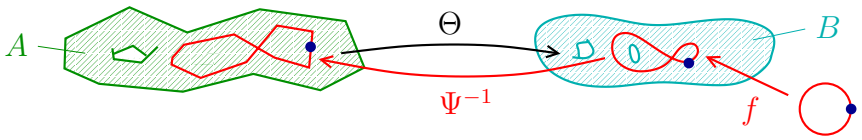
Lemma 68.27. Let $\Theta\colon A \to B$ be a continuous bijection between two topological spaces which has the property that a subset $K \subset A$ is compact if and only if $\Theta(K) \subset B$ is compact. If A is Hausdorff, then for any $a_0 \in A$ the map $\Theta_*\colon \pi_1(A, a_0) \to \pi_1(B, f(a_0))$ is an isomorphism.

Proof. We write $b_0 := f(a_0)$. The proof of the lemma naturally breaks up into showing that $\Theta_*\colon \pi_1(A, a_0) \to \pi_1(B, b_0)$ is an epimorphism and that it is a monomorphism.

Let us first show that the map Θ_* is an epimorphism. Suppose we are given an element $[f\colon (S^1, *) \to (B, b_0)] \in \pi_1(B, b_0)$. By the Compact Image Lemma 2.13 we know that $L := f(S^1)$ is a compact subset of B. We set $K := \Theta^{-1}(L)$. Since Θ is a bijection we have $\Theta(K) = L$. It follows from our hypothesis that K is a compact subset of A. Since Θ is a bijection, since K is compact and since L is Hausdorff we obtain from the Compact-Hausdorff Proposition 2.17 (3) that $\Psi := \Theta|_K\colon K \to L$ is a homeomorphism. In particular the inverse $\Psi^{-1}\colon L \to K$ is continuous. Next we set $g := \Psi^{-1} \circ f\colon (S^1, *) \to (A, a_0)$.

We have $\Theta_*([g]) = [\Theta \circ g] = [\Theta|_K \circ \Psi^{-1} \circ f] = [f]$. Thus we have shown that Θ_* is an epimorphism.

The proof that Θ_* is a monomorphism is quite similar. The key observation is that the image of a homotopy in B between two loops is a compact subset. We leave it to the reader to fill in the few remaining details. ■



Proof of Proposition 68.26. Let X and Y be two CW-complexes.

(1) It follows immediately from Proposition 68.23 (2) and the definition of the topological space $X \otimes Y$ that the identity map $\mathrm{id}\colon X \otimes Y \to X \times Y$ is a continuous bijection.

(2) Let $K \subset X \otimes Y$. If K is compact in $X \otimes Y$. It follows from (1) and the Compact Image Lemma 2.13 that K is also compact in $X \times Y$. Now suppose conversely that K is compact in the product topological space $X \times Y$.
   (a) Note that $X \times Y$ is Hausdorff by the CW-Complex Properties Proposition 68.10 (2) and the Product Topology Properties Proposition 5.6. Thus it follows from the Compact-Closed Lemma 1.21 (2) that K is a closed subset of $X \times Y$. It follows from (1) that K is closed in $X \otimes Y$.
   (b) We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the obvious projections. It follows from the Compact Image Lemma 2.13 that $p(K)$ and $q(K)$ are compact. By the CW-Complex-Finiteness Theorem 68.14 we know that there exist finite subcomplexes $X' \subset X$ and $Y' \subset Y$ such that $p(K) \subset X'$ and $q(K) \subset Y'$. But this implies that K is contained in the finite subcomplex $X' \otimes Y'$ of $X \otimes Y$. By the CW-Complex-Finiteness Theorem 68.14 we know that $X' \otimes Y'$ is compact. Together with (a) we see that K is a closed subset of $X \otimes Y$ that is contained in a compact subset $X' \otimes Y'$. It follows from the Compact-Closed Lemma 1.21 (1) that K itself is compact.

(3) This statement follows immediately from (1) and (2), the fact that $X \times Y$ is Hausdorff and Lemma 68.27.

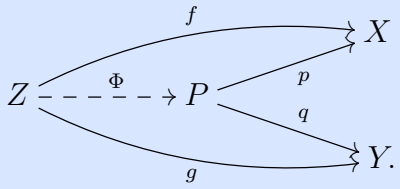
(4) This statement is just a reformulation of Proposition 68.23 (3). ■

68.11. **A convenient category of topological spaces.** Let X and Y be two CW-complexes. We can equip the set $X \times Y := \{(x, y) \,|\, x \in X \text{ and } y \in Y\}$ with two different topologies, namely the familiar product topology from the Topological-Product Proposition 5.1 and the less familiar topology $X \otimes Y$ from the previous section. As we saw in Lemma 68.24, in general these two topologies are different.

This is quite a nuisance. In this section we will discuss one approach to addressing this issue. We will not make much use of the ideas discussed in this section, thus the reader can safely move on to the next section.

Let us first reflect about what "product" is supposed mean in the first place. After contemplating on the concept of "product" for a while one will surely arrive at the following definition that we already gave on page 211.

Definition. Let $\mathcal{C}$ be a category and let $X, Y \in \mathrm{Ob}(\mathcal{C})$ be two objects. A product of X and Y in the category $\mathcal{C}$ is some object $P \in \mathrm{Ob}(\mathcal{C})$ together with two morphisms $p\colon P \to X$ and $q\colon P \to Y$ such that the following property is satisfied: Whenever we are given two morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ there exists a unique morphism $\Phi\colon Z \to P$ such that $f = p \circ \Phi$ and $g = q \circ \Phi$. The situation is summarized in the following diagram



Sometimes we refer to P also as the categorical product of X and Y if we want to distinguish it from the usual set-theoretic product.

Remark. The definition of the product can be formulated in terms of inverse limits. More precisely, we consider the set $I = \{*, \star\}$ with the trivial preorder. An inverse system over I in a category $\mathcal{C}$ is just the data of two objects X and Y in $\mathcal{C}$. The corresponding inverse limit, if it exists, is *by definition*[799] precisely a product of X and Y in $\mathcal{C}$.

The usual "formal nonsense" shows that the product, if it exists, is unique up to a canonical isomorphism. Thus we will use usual parlance and we will speak of *the product*, if it exists.

Examples.

(1) Let $\mathcal{Set}$ be the category of sets and let $X, Y \in \mathrm{Ob}(\mathcal{Set})$. As we mentioned already on page 212, one can easily verify that the product is given by the set $X \times Y := \{(x, y) \,|\, x \in X \text{ and } y \in Y\}$ together with the obvious maps $X \times Y \to X$ and $X \times Y \to Y$.

(2) Let $\mathcal{Top}$ be the category of topological spaces and let $(X, \mathcal{A}), (Y, \mathcal{B}) \in \mathrm{Ob}(\mathcal{Top})$. Let $\mathcal{P}$ be the product topology on the set $X \times Y$ that we defined on page 224. Recall that in the Topological-Product Proposition 5.1 we showed that in the category of topological space the product is given by the topological space $(X \times Y, \mathcal{P})$ together with the obvious maps $X \times Y \to X$ and $X \times Y \to Y$.

(3) Let $\mathcal{PTop}$ be the category of pointed topological spaces that we introduced on page 1061. Furthermore let $(X, x_0), (Y, y_0) \in \mathrm{Ob}(\mathcal{PTop})$ together with the obvious maps. In Exercise 68.12 we will see that the product in $\mathcal{PTop}$ is given by the smash product $(X, x_0) \wedge (Y, y_0)$ that we introduced on page 1143, together with the obvious morphisms $(X, x_0) \wedge (Y, y_0) \to (X, x_0)$ and $(X, x_0) \wedge (Y, y_0) \to (Y, y_0)$.

(4) Let $\mathcal{SmMfd}_{\partial=\varnothing}$ be the category of smooth manifolds with empty boundary that we introduced on page 637 In the Smooth Manifold Product Proposition 19.31 we saw that "the obvious construction" implies that this category admits a product. Note though that this does not apply to the category $\mathcal{SmMfd}$ of all smooth manifolds. In Exercise 68.19 we will see that this category does not have products.[800]

[799] We refer to page 1273 for the definition of an inverse limit.

[800] Note that in Propositions 20.6 and ?? we saw two hacks which at times allow us to get around this annoying issue.

The following theorem shows that if we move from considering all topological spaces to a slightly more constrained setup, then we can combine the two concepts of products that we introduced for CW-complexes.

Theorem 68.28. (Convenient Category Theorem) There exists a full subcategory $\mathcal{C}$ of the category of topological spaces with the following properties:

(1) Every regionally compact Hausdorff space is an object of $\mathcal{C}$.
(2) Every CW-complex is an object of $\mathcal{C}$.
(3) Every direct system in $\mathcal{C}$ has a direct limit and every inverse system in $\mathcal{C}$ has an inverse limit.
(4) If X and Y are two CW-complexes, then the CW-complex $X \otimes Y$ is in fact the product in $\mathcal{C}$.

Remark. A category as in the Convenient Category Theorem 68.28 is often called a *convenient category*. Note though that the notion of a convenient category might be used somewhat differently by different authors. In fact there are many "convenient categories", possibly with somewhat different properties from the ones we stated. We refer to [**Stee1967, Miln1959a, Spa1963, VogR1971, McCor1967, Stri, Pre2002, BrownR1963**] and [**Stro2011**, Chapter 3.4] for more examples and for the discussion of various notions of a "convenient category".

We will outline the proof of the Convenient Category Theorem 68.28 by giving an explicit example of a convenient category. We will need the following definitions.

Definition.

(1) A subset A of a topological space X is called compactly closed if for every map $\varphi\colon K \to X$, where K is a compact Hausdorff space, the preimage $\varphi^{-1}(A)$ is a closed subset of K.[801]
(2) A topological space X is called compactly generated if every compactly closed subset of X is actually closed.[802]
(3) A topological space X is called weakly Hausdorff if for every map $\varphi\colon K \to X$, where K is a compact Hausdorff space, the image $\varphi(K)$ is closed.[803]
(4) A topological space X is called a CGWH-space[804] if it is compactly generated and if it is weakly Hausdorff.

First, for the readers who like weird topological spaces we give some non-examples.

Examples.

(1) We equip the set $X = (\mathbb{N} \times \mathbb{N}) \cup \{*\}$ with the topology where $U \subset X$ is open if one of the following holds:
   (a) the point $*$ is contained in U, or
   (b) for all but finitely many $n \in \mathbb{N}$ the set U contains all but finitely many points in $\{n\} \times \mathbb{N}$.

[801] It follows from the Continuity-via-Closed Subsets Lemma 2.4 that every closed subset of X is actually compactly closed.
[802] Sometimes in the literature a compactly generated topological space is also called a *k-space*.
[803] By the Compact-Closed Lemma 1.21 (2) we know that a topological space that is Hausdorff is also, as it should be, weakly Hausdorff.
[804] Evidently the acronym CGWH stands for Compactly Generated and Weakly Hausdorff.

In Exercise 68.13 we will see that X is not compactly generated.

(2) We consider $X = \mathbb{R} \setminus \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$ and we consider $Y = \mathbb{R}/Z$ where $Z \subset \mathbb{R}$ is the subset given by integers.[805] In Exercise 68.16 we will see that the product topological space $X \times Y$ is not compactly generated.

(3) In Exercise 68.17 we will give an example of a topological space that is weakly Hausdorff but that is not Hausdorff.

We proceed with an actual example of a CGWH-space.

Proposition 68.29. Every CW-complex is a CGWH-space.

Proof. Let X be a CW-complex.

(1) Let A be a compactly closed subset of X. It follows easily from the Characteristic Maps Lemma 68.7 (3) that A is in fact a closed subset of X.

(2) By the CW-Complex Properties Proposition 68.10 (2) we know that X is Hausdorff. It follows from the Compact-Closed Lemma 1.21 (2) that X is in fact weakly Hausdorff. ∎

We continue with the following simple-minded definition.

Definition. The category $\mathcal{CGWH}$ of CGWH-spaces is the full subcategory of the category of topological spaces whose objects are precisely the CGWH-spaces.

Remark. The category of CGWH-spaces was introduced by Michael McCord [**McCor1969**], building on work by Norman Steenrod [**Stee1967**].

The following theorem shows that the category of CGWH-spaces has many charming properties. The theorem provides us in particular with a proof of the Convenient Category Theorem 68.28.

Theorem 68.30. (CGWH-Category Theorem) The category $\mathcal{CGWH}$ of CGWH-spaces has the following properties:

(0) The category $\mathcal{CGWH}$ is a full subcategory of the category of topological spaces.

(1) Every regionally compact Hausdorff space is an object of $\mathcal{CGWH}$.

(2) Every CW-complex is an object of $\mathcal{CGWH}$.

(3) Every direct system in $\mathcal{CGWH}$ has a direct limit and every inverse system in $\mathcal{CGWH}$ has an inverse limit.

(4) Given any two objects X and Y in $\mathcal{CGWH}$ there exists a product in the category $\mathcal{CGWH}$.

(5) Let X and Y be two objects in $\mathcal{CGWH}$. The following statements hold:

(a) The product of X and Y in $\mathcal{CGWH}$ is given by the set $X \times Y$ together with a topology $\mathcal{T}$ that contains the usual product topology $\mathcal{P}$. In particular the identity map
$$(X \times Y, \mathcal{T}) \;\to\; (X \times Y, \mathcal{P})$$
is a bijection and continuous.

(b) If X and Y are compact and Hausdorff, then $\mathcal{T} = \mathcal{P}$, i.e. the map in (a) is a homeomorphism.

[805] We use this cumbersome notation to make sure that we do not confuse the quotient space $\mathbb{R}/Z$ with the quotient of $\mathbb{R}$ by the obvious action of the group $(\mathbb{Z}, +)$ on the topological space $\mathbb{R}$.

(c) If X and Y are two CW-complexes, then the identity map $(X \times Y, \mathcal{T}) \to X \otimes Y$ is a homeomorphism.

Remark. Let X and Y be two CGWH-spaces. We denote by $X \widetilde{\times} Y$ the categorical product of X and Y in the category of CGWH-spaces. the CGWH-Category Theorem 68.30 (4) can be reformulated as saying that we have a continuous bijection $X \widetilde{\times} Y \to X \times Y$. Furthermore, if X and Y are two CW-complexes, then the identity map $X \widetilde{\times} Y \to X \otimes Y$ is a homeomorphism.

Sketch of proof. Most of the statements below were first proved by Michael McCord [**McCor1969**], building on work by Norman Steenrod [**Stee1967**]. Arguably the best account of the proof is given in [**Stri**]. In the following we show that our theorem, as formulated, can be deduced from the statements in [**Stri**].

(0) This statement holds by definition.
(1) [**Stri**, Proposition 1.7] says that every regionally compact Hausdorff space is in fact a CGWH-space.
(2) In Proposition 68.29 we just saw that every CW-complex is a CGWH-space.
(3) In [**Stri**, Propositions 2.23 and 2.30] it is shown that every direct system and every inverse system in the category of CGWH-spaces actually has a limit in the category of CGWH-spaces.
(4) As we discussed on page 1502, this is just a special case of (3).
(5) Let X and Y be two topological spaces. Let $\mathcal{T}$ be the usual product topology on $X \times Y$. Furthermore let $\mathcal{P}$ be the topology on $X \times Y$ that is generated by all compactly closed subsets of $(X \times Y, \mathcal{T})$. By the Continuity-via-Closed Subsets Lemma 2.4 we know that every closed subset is compactly closed, thus we see that $\mathcal{T} \subset \mathcal{P}$. It follows from [**Stri**, Proposition 2.4 and Corollary 2.16] that $(X \times Y, \mathcal{P})$ is the product in the category of CGWH-spaces.
   (a) This statement follows immediately from the above discussion.
   (b) Now suppose that X and Y are compact and Hausdorff. By the Product Topology Properties Proposition 5.6 we know that the product topological space $X \times Y$ is also compact and Hausdorff. But it follows almost immediately from the definitions that in a compact Hausdorff space a subset is compactly closed if and only if it is closed. Thus we see that $\mathcal{P} = \mathcal{T}$.
   (c) Now let X and Y be CW-complexes. We denote by $\mathcal{X}$ the topology of $X \widetilde{\times} Y$ that we introduced on page 1500. We need to show that $\mathcal{P} = \mathcal{X}$. We make the following two observations:
      (i) By definition a subset $A \subset X \times Y$ is closed in $\mathcal{P}$ if and only if the intersection of A with each compact subset of $(X \times Y, \mathcal{T})$ is closed.
      (ii) By definition a subset $A \subset X \times Y$ is closed in $\mathcal{X}$ if and only if the intersection of A with each cell of the CW-complex $X \otimes Y$ is closed.

   Using the Characteristic Maps Lemma 68.7 (3), the CW-Complex-Finiteness Theorem 68.14 and Exercise 5.17 one can show that $\mathcal{P}$ and $\mathcal{X}$ have the same closed subsets. We leave it to the reader to fill in the details. ■

Remark. Given CGWH-spaces X and Y it is also convenient to equip X^Y with a slight modification of the usual compact-open topology. We refer to [**Stri**, Proposition 2.24] for details.

We conclude this section with the following proposition that gives us three other convenient properties of the category of CGWH-spaces. The reader who has struggled with the intricacies of the Whitehead Quotient Theorem 13.4 and Proposition 13.11 will surely appreciate these statements.

Proposition 68.31.

(1) Let $f\colon X \to Y$ be a bijection between two topological spaces. If f is proper in the sense of the definition on page 180, i.e. if the preimage of every compact subset of Y is a compact subset of X, and if X and Y are CGWH-spaces, then f is actually a homeomorphism.

(2) Let $f\colon W \to X$ and $g\colon Y \to Z$ be quotient maps. If W, X, Y and Z are CGWH-spaces, then $f \times g\colon W \times Y \to X \times Z$ is also a quotient map.

(3) Let X and Y be topological spaces and let $B \subset Y$ be a closed subset. If X and Y are CGWH-spaces, then the obvious map $(X \times Y)/(X \times B) \to X \times (Y/B)$ is a homeomorphism.

Proof. This proposition is proved in [**Stri**, Propositions 2.20, 2.40 and 3.17]. ■

Remark. The category of CGWH-spaces admits a fully faithfully embedding into the category of "quasiseparated condensed sets". The category of "quasiseparated condensed sets" is particularly well-suited for dealing with technical problems that arise in topology. We refer to [**Scholz2019**, Theorem 2.16] and [**Scholz2019b**, Proposition 1.2] for details.

68.12. **Constructions of more CW-complexes.** In this section we will see how one can build new CW-complexes out of given CW-complexes. We start out with the following lemma.

Lemma 68.32. (CW-Complex Construction Lemma)

(1) Let $\{X_i\}_{i\in I}$ be a family of CW-complexes. The disjoint union $\bigsqcup_{i\in I} X_i$ admits a natural CW-structure such that each X_i is a subcomplex.

(2) Let X and Y be CW-complexes, let A be a subcomplex of X and let $f\colon A \to Y$ be a cellular map. As on page 259 we consider the topological pushout:

$$X \cup_A Y \;:=\; (X \sqcup Y)/\!\sim \qquad \text{where } a \sim f(a) \text{ for } a \in A$$

Furthermore we consider the following pushout diagram:

$$\begin{array}{ccc} A & \hookrightarrow & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle x \mapsto [x]} \\ Y & \xrightarrow{y \mapsto [y]} & X \cup_A Y. \end{array}$$

The following statements hold:[806]

(a) The map pushout $X \cup_A Y$ admits a natural CW-structure.

(b) The map $Y \to X \cup_A Y$ is a cellular embedding and for any subcomplex B of X that is disjoint from A, the map $B \to X \cup_A Y$ is also a cellular embedding.

(c) If $f\colon A \to Y$ is cellular embedding, then the opposite map $X \to X \cup_A Y$ is also a cellular embedding.

(3) Let X be a CW-complex and let A be a subcomplex. The quotient X/A admits a natural CW-structure such that the projection $X \to X/A$ is cellular and where the cells of X/A are in one-to-one correspondence to the cells of X that do not entirely lie in A together with one 0-cell given by the point $[A]$.[807]

(4) Let $\{X_i\}_{i\in I}$ be a family of CW-complexes and suppose that for each $i \in I$ we are given a point $a_i \in X_i^0$. If we use these points to form the wedge $\bigvee_{i\in I} X_i$, then the wedge admits a natural CW-complex structure such that each X_i is a subcomplex.

(5) Let (X, x_0) and (Y, y_0) be pointed CW-complexes. If at least one of X or Y has only countably many cells, then the smash product

$$X \wedge Y \;=\; (X \times Y)/((X \times \{y_0\}) \cup (\{x_0\} \times Y)),$$

as defined in Exercise 50.6, has a natural CW-structure.

(6) Let X be a CW-complex, let A be a complex of X and let $f\colon A \to X$ be a cellular map with the following properties:
   (a) For each n-cell σ of A we have either $f(\sigma) \subset X^{n-1}$ or $f|_\sigma$ is a homeomorphism.
   (b) The image of f is contained in a subcomplex of X that is disjoint from A.

   Then $X/\!\sim$, where $a \sim f(a)$ for all $a \in A$, admits a natural CW-structure such that for any subcomplex B that is disjoint from $f(A)$ the natural map $B \to X/\!\sim$ is a cellular embedding.

(We point out that by the Cellular Embedding Lemma 68.19 we know that the image of a cellular embedding is in fact a cellular homeomorphism onto a subcomplex and we will often use this cellular isomorphism to identify the domain with its image.)

Convention. When we form the wedge of a family of a CW-complexes, then we always use a point that lies in the 0-skeleton, so that we can view the wedge in a natural way as a CW-complex.

Proof.

(1) The first statement is basically obvious.

(2) First note that it is pretty clear what the CW-structure on $X \cup_A Y$ should be. The problem that arises is that one needs to show that the topology of the new CW-complex agrees in fact with the intended topology. If X (and Y) are finite CW-complexes then this follows easily from the Compact-Hausdorff Proposition 2.17 (3). The general case gets more technical. We refer to [**LW1969**, p. 60 and Theorem 5.11] or alternatively to [**FrPi1990a**, Theorem 2.31 on p. 62] for complete proofs. The remaining two statements follow easily from the actual definition of the CW-structure on $X \cup_A Y$.

(3) Note that this statement is the special case of (2) where we consider the constant map $f\colon A \to Y = \{*\}$ where $Y = \{*\}$ is a topological space consisting of a single point.

[806] The statements below should be viewed as an extension of the discussion of the properties of the pushout that we gave in the Topological Pushout-Maps Lemma 5.34.

[807] It is worth considering the case that $A = \varnothing$ is the empty subcomplex. By the definition on page 242 we know that $X/A = X/\varnothing$ equals the disjoint union of X with the topological space $\{\varnothing\}$ consisting of a single point. The above definition of the CW-structure on X/A still makes, namely it is given by the disjoint union of the CW-complex with $\{\varnothing\}$ viewed as a CW-complex in the only possible way.

(4) This follows easily from (1) and (2).
(5) This statement follows immediately from Proposition 68.23 (3) and (2).
(6) This statement follows from [**LW1969**, Proposition 5.8]. We can apply [**LW1969**, Proposition 5.8] since our hypothesis on f ensures that f defines a "cellular equivalence relation" in the sense of [**LW1969**, p. 32]. ■

Later on we will make good use of the following lemma.

Lemma 68.33. (Skeleton-Quotient Lemma) Let X be a CW-complex and let $n \in \mathbb{N}$. Let $\{\varphi_i \colon \overline{B}^{n+1} \to X^n\}_{i \in I}$ be the characteristic maps of the $(n+1)$-cells of X. For each $i \in I$ let $\overline{B}_i^{n+1}$ be a copy of $\overline{B}^{n+1}$ and let S_i^n be a copy of S^n. We write $S_i^n = \partial \overline{B}_i^{n+1}$.

(1) The map

$$\begin{aligned} f \colon \bigvee_{i \in I} \overbrace{(\overline{B}_i^{n+1}/S_i^n)}^{\text{homeomorphic to } S^{n+1}\text{, see Lemma 5.20}} &\to X^{n+1}/X^n \\ [x] &\mapsto [\varphi_i(x)] \quad \text{if } x \in \overline{B}_i^{n+1} \end{aligned}$$

is a homeomorphism.

(2) If X has only one 0-cell $\{x_0\}$ and no cells in dimensions $1, \dots, n$, then the homeomorphism f, together with the homeomorphism $\overline{B}^{n+1}/S^n \to S^{n+1}$ from the Ball-Quotient Sphere Lemma 5.20 induces a homeomorphism

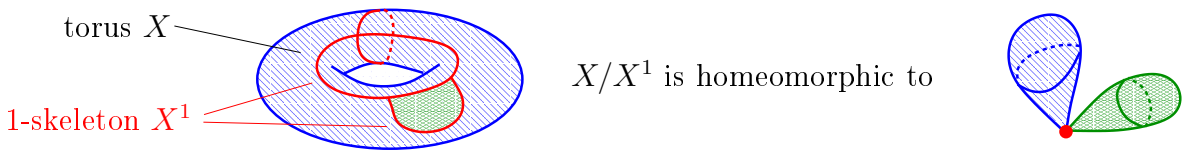
$$\bigvee_{i \in I} S_i^{n+1} \to X^{n+1}.$$

(3) For every $i \in I$ the subset $X^{n+1} \setminus \Phi_i(B_i^{n+1})$ is a subcomplex of X^{n+1} and the map

$$\begin{aligned} \overline{B}_i^{n+1}/S_i^n &\to X^{n+1}/(X^{n+1} \setminus \Phi_i(B_i^{n+1})) \\ [x] &\mapsto [\Phi_i(x)] \end{aligned}$$

is a homeomorphism.

Example. In the figure below we see the torus together with a CW-structure X that has two 2-dimensional cells. It follows from the Skeleton-Quotient Lemma 68.33 (1) that $X/X^1 = X^2/X^1$ is homeomorphic to the wedge of two spheres.



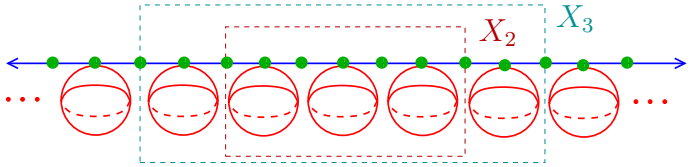
Proof.

(1) First note that it is basically clear that the given map f is a bijection. Furthermore it follows immediately from Lemma 50.10 that f is continuous. We use the CW-Complex Construction Lemma 68.32 to view both sides of the Skeleton-Quotient Lemma 68.33 as CW-complexes. The map f gives a bijection between the cells of the two sides. Thus it follows from the CW-Complex Homeomorphism Lemma 68.12 that f is a homeomorphism.
(2) This statement follows immediately from (1).

(3) Since we remove a "top-dimensional" open cell from X^{n+1} it follows immediately from the definitions that $X^{n+1} \setminus \Phi_i(B_i^{n+1})$ is a subcomplex of X^{n+1}. Similar to the argument in (1) we now see that the map is a homeomorphism. ■

Lemma 68.34. (Ascending Union-CW Complex Lemma) Let $f_i \colon X_i \to X_{i+1}$, $i \in \mathbb{N}$ be a sequence of cellular inclusion maps between CW-complexes. Then the direct limit[808] $\varinjlim X_i = \bigcup_{i \in I} X_i$ admits a CW-structure such that each X_i is a subcomplex of $\varinjlim X_i = \bigcup_{i \in I} X_i$.

Example. The Ascending Union-CW Complex Lemma 68.34 allows us to view the line with infinitely many spheres attached as a CW-complex.[809] We refer to the figure below for an illustration.



the topological space
$X = \bigcup_{i \in \mathbb{N}} X_i$

Proof. We leave it to the reader to write down the, notationally messy but mathematically elementary, proof. ■

We conclude this section with the following corollary that shows that several constructions that we are familiar with can also be performed in the category of CW-complexes.

Corollary 68.35. (CW-Complex Cone Corollary) Let X be a CW-complex.[810]

(1) The cone

$$\mathrm{Cone}(X) \;=\; (X \times [0,1])/\!\sim \qquad \text{where } (x,0) \sim (y,0) \text{ for every } x, y \in X$$

admits a natural CW-structure such that the inclusion map $X \to \mathrm{Cone}(X)$ given by $x \mapsto [(x,1)]$ is a cellular embedding.

(2) The suspension

$$\Sigma(X) \;=\; (X \times [-1,1])/\!\sim \qquad \text{where for every } x, y \in X \text{ we have } (x,-1) \sim (y,-1) \text{ and } (x,1) \sim (y,1)$$

admits a natural CW-structure such that the map $X \to \Sigma(X)$ given by $x \mapsto [(x,0)]$ is a cellular embedding.

(3) If A is a CW-complex and if $f \colon A \to X$ is a cellular map, then the mapping cone

$$\mathrm{Cone}(f \colon A \to X) \;=\; (\;)/\!\sim \;\text{ where } (a,1) \sim f(a) \text{ for all } a \in A$$

and the mapping cylinder

$$\mathrm{Cyl}(f \colon A \to X) \;=\; (A \times [0,1] \sqcup X)/\!\sim \;\text{ where } (a,1) \sim f(a) \text{ for all } a \in A$$

---

[808] We write explicitly "direct limit" to remind the reader that, according to the discussion on page 1259, the topology on $\varinjlim X_i = \bigcup_{i \in I} X_i$ is given by the condition that a subset U is open if and only if each intersection $U \cap X_i$ is an open subset of X_i.

[809] It follows from the Closed Embedding-to-$\mathbb{R}^n$ Proposition 2.19 that the topology on the "line with infinitely many spheres" coming from $\mathbb{R}^3$ agrees with the topology coming from the CW-structure.

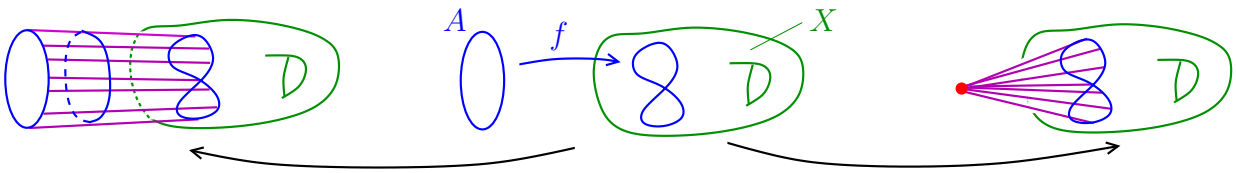
admit natural CW-structures. Furthermore the obvious maps $X \to \mathrm{Cone}(f\colon A \to X)$ and $\mathrm{Cyl}(f\colon A \to X)$ are cellular embeddings and the map $A \mapsto \mathrm{Cyl}(f\colon A \to X)$ given by $a \mapsto [(a,0)]$ is a cellular embedding.

(4) If $f\colon X \to X$ is a cellular map, then the mapping torus

$$\mathrm{Tor}(X,f) \;:=\; (X \times [0,1])\,/\!\sim \qquad \text{where } (x,0) \sim (f(x),1) \text{ for all } x \in X$$

admits a natural CW-structure such that the natural embedding $X \to \mathrm{Tor}(X,f)$ given by $x \mapsto (x,0)$ is in fact a cellular embedding.

(As in the statement of the CW-Complex Construction Lemma 68.32 we point out that often we will use a cellular embedding to identify the domain with its image.)



mapping cylinder $\mathrm{Cyl}(f\colon X \to Y)$ mapping cone $\mathrm{Cone}(f\colon A \to X)$

Proof. All four statements can be proved fairly easily using Proposition 68.23 (3b) and the CW-Complex Construction Lemma 68.32. We only point out that the CW-structure on the mapping torus $\mathrm{Tor}(X,f)$ is obtained in a 2-stage process, first we equip the mapping cylinder $\mathrm{Cyl}(f\colon X \to X)$ with the natural CW-structure given by (3). Afterwards we apply the CW-Complex Construction Lemma 68.32 (6). We leave it to the reader to fill all the other details. ■

We conclude this discussion of constructions of CW-structures with the following slightly sobering proposition.

Proposition 68.36. There exists a finite 3-dimensional CW-complex that admits an open subset that does not admit a CW-structure.

Proof. This statement is proved in [**Cau1992**, Exemple 2]. ■

68.13. **Coverings of CW-complexes.** In this section we will see that a covering of a CW-complex inherits a natural CW-structure.

Proposition 68.37. (CW-complex Covering Proposition) Let $p\colon \widetilde{X} \to X$ be a covering of a connected CW-complex X. We can equip $\widetilde{X}$ with a natural structure of a CW-complex such that the following statements hold:

(1) The map $p\colon \widetilde{X} \to X$ is a cellular map.

(2) For any $k \in \mathbb{N}_0$ we have[811]

$$\#k\text{-cells of } \widetilde{X} \;=\; \big[\widetilde{X} : X\big] \cdot \#k\text{-cells of } X.$$

(3) The dimension of $\widetilde{X}$ is the same as the dimension of X.

(4) If A is a subcomplex of X, then $p^{-1}(A)$ is a subcomplex of $\widetilde{X}$.

[810] Strictly speaking, or Bourbakily speaking if that's an adverb, we should really specify in each case what the "natural CW-structure" is. But this should hopefully be clear in each case, and we do not encumber the reader with unnecessary extra notation.

Remark. The CW-complex Covering Proposition 68.37 is a generalization of the Covering-of-Graph Proposition 64.1 where we showed that any covering of a topological graph is again a topology graph. In fact, as we will see, the proofs are structurally very similar.

Proof. Let $p\colon \widetilde{X} \to X$ be a covering of a connected CW-complex X of degree $n := [\widetilde{X} : X]$. Let $\psi\colon \overline{B}^k \to X$ be a characteristic map of a k-cell of X. We write $x = \psi(0)$. We consider the following diagram

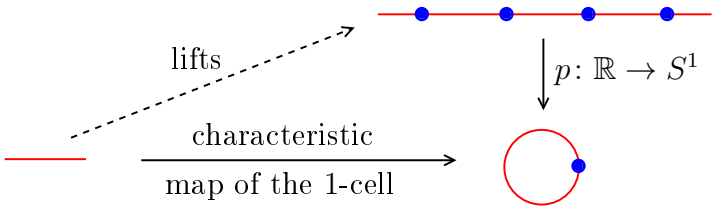
$$\begin{array}{ccc} & & \widetilde{X} \\ & \overset{\widetilde{\psi}}{\nearrow} & \big\downarrow p \\ \overline{B}^k & \xrightarrow[\psi]{} & X. \end{array}$$

Since $\overline{B}^k$ is simply connected and locally path-connected we can apply the Map Lifting Criterion 61.2 which says in this context that for each $\widetilde{x} \in p^{-1}(x)$ there exists a unique lift $\overline{B}^k \to \widetilde{X}$ with $\widetilde{\psi}(0) = \widetilde{x}$. By the definition on page 1069, the degree $n = [\widetilde{X} : X]$ of the covering equals the cardinality of $p^{-1}(x)$. Thus we see that ψ gives rise to n maps $\overline{B}^k \to \widetilde{X}$. It is straightforward to verify that all these lifts of the characteristic maps for the cells of X define a CW-structure for $\widetilde{X}$. We leave the joyful task of verifying the details to the reader. Alternatively we refer to [**Bre1993**, Theorem IV.8.10] for details.

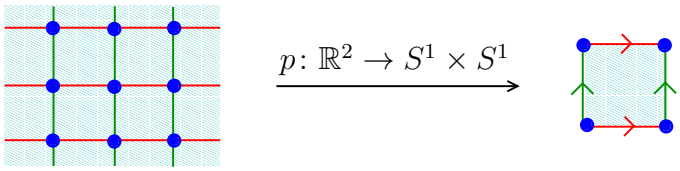
It follows immediately from the above construction and the definitions that the CW-structure we had just defined on $\widetilde{X}$ has the desired properties (1) to (4). ■

Examples.

(1) We equip S^1 with the CW-structure consisting of one 0-cell and one 1-cell. We denote by $p\colon \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi i t)$ the universal covering of S^1. If we apply the construction of the CW-structure on $\mathbb{R}$ given in the proof of the CW-complex Covering Proposition 68.37, then we end up equipping $\mathbb{R}$ with the CW-structure where the 0-cells correspond to $\mathbb{Z}$ and the 1-cells are precisely the intervals $[k, k+1]$, with $k \in \mathbb{Z}$. We illustrate the CW-structure of $\mathbb{R}$ in the figure below.

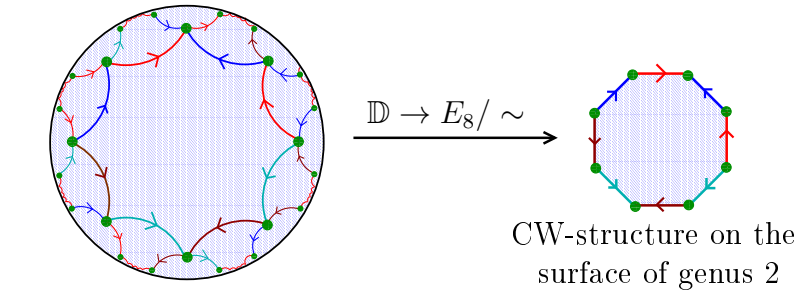


(2) Almost the same approach as in (1), applied to the universal covering $\mathbb{R}^2 \to S^1 \times S^1$, endows $\mathbb{R}^2$ with a CW-structure that is illustrated in the figure below.



[811] Note that this is an equality of cardinalities. For example, if $[\widetilde{X} : X]$ is infinite countable and if X has finitely many k-cells, then $\widetilde{X}$ has infinitely many, but countably many, k-cells.

(3) In the proof of Hadamard's Theorem 66.9 we constructed an explicit covering map $\mathbb{D} \to H_8/\sim \cong E_8/\sim$. If we equip $E_8/\sim$ with the "usual" CW-structure then we obtain from the CW-complex Covering Proposition 68.37 the CW-structure on $\mathbb{D}$ that is shown in the figure below.



induced CW-structure on $\mathbb{D}$

CW-structure on the surface of genus 2

Exercises for Chapter 68.

Exercise 68.1.

(a) Does every compact subset of $\mathbb{R}^2$ admit a CW-structure?
(b) Let $n \in \mathbb{N}$. Does the open ball $B_1^n(0)$ admit a CW-structure?
(c) Let C be the Cantor set as defined on page 130. Does C admit a CW-structure?

Exercise 68.2.

(a) We equip $\mathbb{R}^\infty$ with the metric that is given by

$$d((x_1, x_2, \dots), (y_1, y_2, \dots)) \;:=\; \sqrt{\sum_{i=1}^{\infty} |x_i - y_i|^2}.$$

We consider the following two topologies on $\mathbb{R}^\infty$:

$$\mathcal{S} \;:=\; \{U \subset \mathbb{R}^\infty \mid \text{for every } P \in U \text{ there exists } \epsilon > 0 \text{ such that } B_\epsilon(P) \subset U\}$$

and

$$\mathcal{T} \;:=\; \{U \subset \mathbb{R}^\infty \mid \text{for every } n \in \mathbb{N} \text{ the set } U \cap \mathbb{R}^n \text{ is open in } \mathbb{R}^n\}.$$

Show that $\mathcal{S} \subset \mathcal{T}$.

(b) We view $\mathbb{R}^\infty$ as equipped with the topology $\mathcal{S}$. Does the Heine-Borel Theorem hold in $\mathbb{R}^\infty$? More precisely, is every bounded closed subset of $\mathbb{R}^\infty$ compact?

(c) We view $\mathbb{R}^\infty$ as equipped with the topology $\mathcal{T}$. Show that the projection $p\colon \mathbb{R}^\infty \to \mathbb{R}^n$ is continuous and open.

Exercise 68.3. Show that the map

$$\begin{array}{rcl} \mathbb{C}^\infty & \to & \mathbb{R}^\infty \\ (x_1 + \mathrm{i}y_1, x_2 + \mathrm{i}y_2, \dots) & \mapsto & (x_1, y_1, x_2, y_2, \dots) \end{array}$$

is a homeomorphism.

Exercise 68.4. Show that the action

$$\begin{array}{rcl} \{\pm 1\} \times S^\infty & \to & S^\infty \\ (\epsilon, v) & \mapsto & \epsilon \cdot v \end{array}$$

of the group $\{\pm 1\}$ on the topological space S^∞ is continuous and discrete.

Exercise 68.5. Let $P, Q \in S^\infty$. Show that there exists a homeomorphism h of S^∞ with $h(P) = Q$.

Exercise 68.6. Let X be a CW-complex and let $x \in X$. Show that $\{x\}$ is a closed subset of $|X|$.
Hint. Make good use of the Topological Pushout-Maps Lemma 5.34 (4).

Exercise 68.7. Let X be a CW-complex.

(a) Show that if the 1-skeleton X^1 is path-connected, then X itself is path-connected.
(b) Show that if X is connected, then X^1 is also connected.

Hint. Make good use of Lemma 2.24, the Ball-Sphere-Connected Lemma 2.25 and Lemma 2.28.

Exercise 68.8. Show that a CW-complex is locally finite if and only if it is regionally compact.

Exercise 68.9. We equip $\mathbb{R}\mathrm{P}^2$ with the CW-structure with one cell in dimensions $0, 1, 2$. Let $p\colon S^2 \to \mathbb{R}\mathrm{P}^2 = S^2/\pm 1$ be the usual 2-fold covering. The CW-structure on $\mathbb{R}\mathrm{P}^2$ induces a CW-structure on S^2. What does this CW-structure on S^2 look like?

Exercise 68.10. Let I be a set and for each $i \in I$ let $n_i \in \mathbb{N}$. Show that the wedge $\bigvee_{i\in I} S_i^{n_i}$ has a natural CW-structure with one 0-cell and where for each $n \in \mathbb{N}_0$ the n-cells correspond precisely to the n-dimensional spheres in the family of spheres $\{S_i^{n_i}\}_{i\in I}$.
Hint. Use the Twice Quotient Lemma 5.23 and use the fact, shown in the Ball-Quotient Sphere Lemma 5.20, that for each $n \in \mathbb{N}$ the sphere S^n is homeomorphic to $\overline{B}^n/S^{n-1}$.

Exercise 68.11. Let X be a finite CW-complex. Show that there exists an $n \in \mathbb{N}_0$ and a map $\varphi\colon |X| \to \mathbb{R}^n$ that is an embedding.

Exercise 68.12. Recall that we denote by $\mathcal{PT}op$ the category of pointed topological spaces. Suppose we are given $(X, x_0), (Y, y_0) \in \mathrm{Ob}(\mathcal{PT}op)$. Show that the direct product of (X, x_0) and (Y, y_0) in $\mathcal{PT}op$ is given by the smash product $(X, x_0) \wedge (Y, y_0)$ that we introduced on page 1143.
Hint. This exercise is very similar to Exercise 4.6.

Exercise 68.13. We consider the set $X = (\mathbb{N} \times \mathbb{N}) \cup \{*\}$. We say $U \subset X$ is open if $* \notin U$ or if for all but finitely many $n \in \mathbb{N}$ the set U contains all but finitely many points in $\{n\} \times \mathbb{N}$.

(a) Show that these open sets do indeed define a topology on X.
(b) Show that X is Hausdorff.
(c) Show that every infinite subset of X contains an infinite closed and discrete subset.
(d) Show that every compact subset of X is finite.
(e) Show that the compactly generated topology on X is the discrete topology.

These steps show that X is not compactly generated.

Exercise 68.14. Let $\mathcal{C}$ be a convenient category of topological spaces, i.e. let $\mathcal{C}$ be a full subcategory $\mathcal{C}$ of the category of topological spaces that has the properties stated in the Convenient Category Theorem 68.28. Let X and Y be CW-complexes. Show that if the

usual product $X \times Y$ lies in $\mathcal{C}$, then $X \times Y$ is homeomorphic to the product CW-complex $X \otimes Y$.

Exercise 68.15. Let X and Y be CGWH-spaces. We denote by $X \widetilde{\times} Y$ the product in the category of CGWH-spaces, which exists by the CGWH-Category Theorem 68.30 (4). Show that for each $n \in \mathbb{N}$, each $x_0 \in X$ and $y_0 \in Y$ the natural maps $p\colon X \widetilde{\times} Y \to X$ and $q\colon X \widetilde{\times} Y \to Y$ induce an isomorphism

$$p_* \times q_*\colon \pi_n(X \widetilde{\times} Y, (x_0, y_0)) \xrightarrow{\cong} \pi_n(X, x_0) \times \pi_n(Y, y_0).$$

Remark. The exercise is to verify that the proofs of Propositions 48.20 and 71.8 also apply to $X \widetilde{\times} Y$ instead of the usual product topological space $X \times Y$.

Exercise 68.16. We consider $X = \mathbb{R} \setminus \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and we consider $Y = \mathbb{R}/Z$ where $Z \subset \mathbb{R}$ is the subset given by integers. We consider the subset

$$A \;=\; \big\{\big(\tfrac{1}{i} + \tfrac{a_i}{j}, i + \tfrac{1}{2j}\big) \in X \times Y \,\big|\, i, j \in \mathbb{N}\big\} \qquad \text{where } a_i = \tfrac{1}{2}\big(\tfrac{1}{i} - \tfrac{1}{i+1}\big).$$

(a) Show that the closure of A contains $(0, [0])$.
(b) Show that for any compact subset $K \subseteq X \times Y$, the set $K \cap A$ has only finitely many points.
(c) Show that X is not compactly generated.

Exercise 68.17.

(a) We consider

$$X \;:=\; \{0\} \cup \bigcup_{n,m \in \mathbb{N}} \{\tfrac{1}{n} + \tfrac{1}{m+n}\} \;\subset\; \mathbb{R}.$$

We equip X with the usual metric coming from $\mathbb{R}$. We set $x := 0$.
(i) Show that each point in $X \setminus \{x\}$ is open.
(ii) Show that X is not regionally compact at x.

(b) Let X be a countable metric space which admits a point $x \in X$ such that the following conditions are satisfied:
(i) Each point in $X \setminus \{x\}$ is open.
(ii) The topological space X is not regionally compact at x.
Let $Y := X \cup \{\infty\}$ be the one-point compactification as defined in Exercise 2.59.
(α) Show that compact sets of Y are closed. (This implies that Y is weakly Hausdorff.)
(β) Show, by considering the points x and ∞, that Y is not Hausdorff.

Remark. The combination of (a) and (b) shows that there exist topological spaces that are weakly Hausdorff but not Hausdorff.

Exercise 68.18. Let X be a CW-complex with countably many cells.

(a) Show that there exists a sequence $A_1 \subset A_2 \subset \ldots$ of *finite* subcomplexes A_i of X such that $X = \bigcup_{i \in \mathbb{N}_0} A_i$.
(b) If X is connected, then show that we can arrange in (a) that every A_i is connected.

Exercise 68.19. Let $\mathcal{SmMfd}$ be the category of all smooth manifolds that we introduced on page 637 (in particular there are no restrictions on the boundary). Show that this category does not have a product in the sense of the definition on page 211.

69. CW-complexes II: Fundamental groups

In this chapter we will consider fundamental groups of CW-complexes. First we collect a few general statements. Then we will show that fundamental groups of 1-dimensional CW-complexes are free groups. Finally we will see how can calculate fundamental groups of more general CW-complexes in practice.

69.1. **Fundamental groups of CW-complexes.** Using the CW-Complex Properties Proposition 68.10 we can now prove the following very useful version of the Seifert–van Kampen Theorem 54.1. In view of the Smooth Manifold-CW Structure Theorem 96.5 , which we mentioned already on page 1474, this can be viewed as a generalization of the Seifert–van Kampen Theorem 54.4 for smooth manifolds.

Theorem 69.1. (Seifert–van Kampen Theorem for CW-complexes) Let X be a CW-complex and let $X = A \cup B$ be a decomposition of X in two subcomplexes such that $A \cap B$ is path-connected. Let $x_0 \in A \cap B$. Then

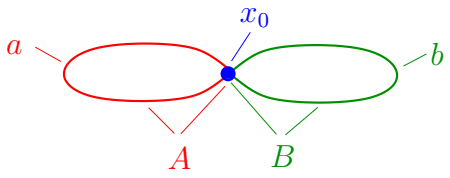
$$\pi_1(X, x_0) \;\cong\; \pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0).$$

Examples.

(1) Let X be the wedge of two circles. We can view X as a CW-complex with one 0-cell x_0 and two 1-cells a and b. Then $A = \{x_0, a\}$ and $B = \{x_0, b\}$ are subcomplexes, each of which is homeomorphic to S^1. We refer to the figure below for an illustration. It follows, as on page 1173, that

$$\pi_1(X, x_0) \underset{\uparrow}{\cong} \pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0) \underset{\uparrow}{\cong} \langle a\rangle * \langle b\rangle \;=\; \langle a, b\rangle.$$

Seifert-van Kampen Theorem 69.1 — since A and B are homeomorphic to S^1



(2) We consider

$$K \;:=\; \underbrace{(S^1 \times S^1)}_{=:A} \cup \underbrace{(\overline{B}^2 \times \{1\})}_{=:B},$$

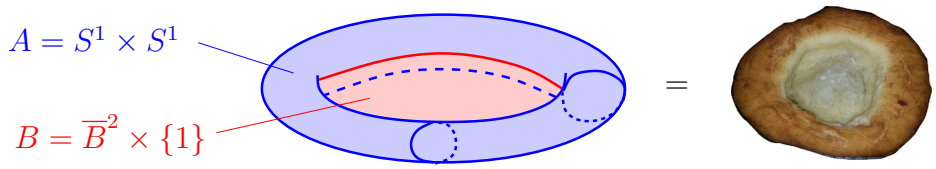
i.e. K is given by a torus where the "central hole" is filled in with a disk. We refer to the figure below for an illustration.[812] In the above definition of K we already wrote K as the union of two subsets. It is quite straightforward to see that K can be given a CW-structure such that A and B are subcomplexes.[813] We pick the base point $x_0 = (1, 1)$. We obtain that

Theorem 69.1 — since $A \cap B = S^1 \times \{1\}$ — Amalgamated Product Lemma 53.22

$$\pi_1(K, x_0) \underset{\downarrow}{\cong} \pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0) \underset{\downarrow}{\cong} \underbrace{\langle x, y | [x, y]\rangle *_{\langle s\rangle} \{e\}}_{\text{with } x \leftrightarrow s} \underset{\downarrow}{\cong} \langle y\rangle.$$

[812] Here the name K stands for "kiacherl".

[813] How would you define the CW-structures?



Kiacherl

Proof of Theorem 69.1. [814] Let X be a CW-complex and let $X = A \cup B$ be a decomposition of X in two subcomplexes such that $A \cap B$ is path-connected. Note that $A \cap B$ is also a subcomplex. Let $x_0 \in A \cap B$.

We use the notation of the CW-Complex Properties Proposition 68.10 (8). We write $U = \Phi(A)$ and $V = \Phi(B)$. Note that by the CW-Complex Properties Proposition 68.10 (8b) we have $U \cap V = \Phi(A \cap B)$. Since $A \cap B$ is path-connected and since $A \cap B$ is a deformation retract of $\Phi(A \cap B)$ we see that $U \cap V = \Phi(A \cap B)$ is also path-connected. Therefore we can apply the Seifert–van Kampen Theorem 54.1 to the decomposition $X = U \cup V$. We obtain the following commutative diagram

$$\begin{array}{ccc} \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) & \xrightarrow{\cong} & \pi_1(X, x_0) \\ \uparrow & & =\uparrow \\ \pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0) & \longrightarrow & \pi_1(X, x_0), \end{array}$$

where the two vertical maps are induced by the inclusions. By the CW-Complex Properties Proposition 68.10 (8a) and the Homotopy-π_1-Proposition 50.3 (2) the three inclusion induced maps $\pi_1(A, x_0) \to \pi_1(U, x_0)$, $\pi_1(B, x_0) \to \pi_1(V, x_0)$ and $\pi_1(A \cap B, x_0) \to \pi_1(U \cap V, x_0)$ are isomorphisms. The top horizontal map is an isomorphism by the Seifert–van Kampen Theorem 54.1. It follows that the bottom horizontal map is also an isomorphism. ■

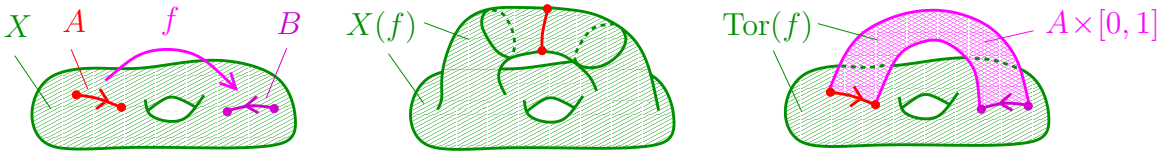
Next we want to compute the fundamental group of a CW-complex that is obtained by gluing two subcomplexes of a connected CW-complex. We recall and introduce the following two definitions.

Definition. Let X be a topological space, let A be a subset of X and let $f\colon A \to X$ be an embedding. Following the discussion on page 1290 we define

$$X(f) \;:=\; X(f\colon A \to X) \;:=\; X/\!\sim \qquad \text{where } f(a) \sim a \text{ for every } a \in A.$$

Furthermore we define the partial mapping torus[815]

$$\mathrm{Tor}(f) \;:=\; \mathrm{Tor}(f\colon A \to X) \;:=\; (X \sqcup (A \times [0,1]))/\!\sim \qquad \text{where } (a,0) \sim a \text{ and } (a,1) \sim f(a) \text{ for every } a \in A.$$



Continuing with the above notation, note that the projection $A \times [0,1] \to A$ induces a map $p\colon \mathrm{Tor}(f) \to X(f)$. In Proposition 70.5 we will see that under reasonably mild hypotheses

[814] The proof is very similar to the proof of the Seifert–van Kampen Theorem for Smooth Manifolds 54.4.

[815] If $A = X$, then one can easily show that the partial mapping torus is naturally homeomorphic to the mapping torus that we introduced on page 467.

the map $p\colon \mathrm{Tor}(f) \to X(f)$ is a homotopy equivalence. In particular by the Homotopy-π_1-Proposition 50.3 we see that p induces an isomorphism of fundamental groups.
Now we can formulate, prove and discuss the following theorem.

Theorem 69.2. (HNN-Seifert–van Kampen Theorem for CW-complexes) Suppose we are given a path-connected CW-complex X and two path-connected disjoint subcomplexes A and B. Let $f\colon A \to B$ be a cellular isomorphism. We pick a base point $x_0 \in A$. Let $\gamma\colon [0,1] \to X$ be an embedding with $\gamma(0) = x_0$ and with $\gamma(1) = f(x_0)$ such that $\gamma([0,1])$ is a subcomplex of the 1-skeleton X^1.[816] We write $\Gamma = \pi_1(A, x_0)$. Finally we denote by

$$\alpha\colon \Gamma = \pi_1(A, x_0) \to \pi_1(X, x_0)$$

the inclusion induced map and we denote by

$$\beta\colon \Gamma = \pi_1(A, x_0) \xrightarrow{f_*} \pi_1(B, f(x_0)) \to \pi_1(X, f(x_0)) \xrightarrow{\gamma_*} \pi_1(X, x_0)$$

the concatenation of the maps induced by f, the inclusion and the base point change isomorphism from the Change-of-Base Point Proposition 47.10, using the path γ. Then there exists an isomorphism

$$\Phi\colon \pi_1(\mathrm{Tor}(f), x_0) \xrightarrow{\cong} \langle \pi_1(X, x_0), t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle$$

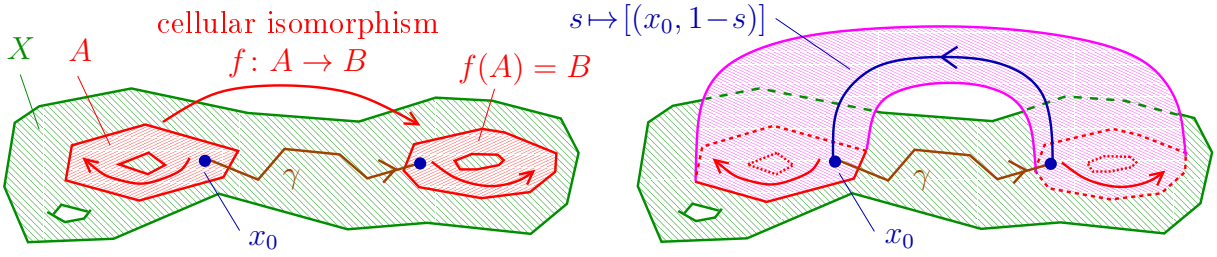
which has the following two properties:

(1) The diagram

$$\begin{array}{ccc} & \pi_1(X, x_0) & \\ \swarrow & & \searrow \\ \pi_1(\mathrm{Tor}(f), x_0) & \xrightarrow{\Phi} & \langle \pi_1(X, x_0), t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle \end{array}$$

with the obvious diagonal maps commutes.

(2) The closed loop in $\mathrm{Tor}(f)$ given by the composition of γ with the path $s \mapsto [(x_0, 1-s)]$ corresponds under the isomorphism Φ to t.



Remark.

(1) Basically the same way that we obtained the HNN-Seifert–van Kampen Theorem 57.3 from the Seifert–van Kampen Theorem 54.1 one can also obtain the above HNN-Seifert–van Kampen Theorem 69.2 for CW-complexes from the Seifert–van Kampen Theorem 69.1 for CW-complexes. Below we will give a different proof of the HNN-Seifert–van Kampen Theorem 57.3 for CW-complexes which is easier than the argument provided in the proof of the HNN-Seifert–van Kampen Theorem 69.2.

[816] Note that it follows from our hypothesis that X is path-connected together with the CW-Complex Properties Proposition 68.10 (7) that such γ exists.

(2) The original HNN-Seifert–van Kampen Theorem 57.3 is more general than the above HNN-Seifert–van Kampen Theorem 57.3 for CW-complexes, but for most applications the latter result will be just fine.
(3) Using Proposition 70.5 one can also deal with certain situations where the map $f\colon A \to B$ is not a cellular isomorphism. We leave it to the reader to figure out what these generalized statements might be.

Sketch of a proof of Theorem 69.2. Let X be a path-connected CW-complex X and let $f\colon A \to B$ be a cellular isomorphism between two path-connected disjoint subcomplexes of X. Let $x_0 \in A$ and let $\gamma\colon [0,1] \to X$ be an embedding with $\gamma(0) = x_0$ and with $\gamma(1) = f(x_0)$ such that $C := \gamma([0,1])$ is a subcomplex.

It follows from the CW-Complex Product Proposition 68.23 (3b) and the CW-Complex Construction Lemma 68.32 that $\mathrm{Tor}(f)$ is equipped with a natural CW-structure such that we can view $A \times [0,1]$ and X as subcomplexes of $\mathrm{Tor}(f)$.

We consider $Y := (X \sqcup (\{x_0\} \times [0,1])/\sim$ where $(x_0, 0) \sim x_0$ and $(x_0, 1) \sim f(x_0)$. We view Y as a CW-complex given by attaching a single 1-cell to X. We denote by D the image of $\{x_0\} \times [0,1]$ in Y. Furthermore we denote by $\delta\colon [0,1] \to D$ the path given by $s \mapsto (x_0, 1-s)$. Now note that we have the following two isomorphisms:

$$\pi_1(Y, x_0) \underset{\uparrow}{\cong} \pi_1(X, x_0) *_{\pi_1(C, x_0)} \pi_1(C \cup D, x_0) \underset{\uparrow}{\cong} \pi_1(X, x_0) * \langle t = [\gamma * \delta] \rangle$$

Seifert–van Kampen Theorem 69.1 for CW-complexes

since C contractible and since $C \cup D$ is homeomorphic to S^1

and

$$\pi_1(A, x_0) * \pi_1(B, f(x_0)) \underset{\uparrow}{\cong} \pi_1(A, x_0) * \pi_1(D \cup B, x_0) \underset{\uparrow}{\cong} \pi_1(A \cup D \cup B, x_0).$$

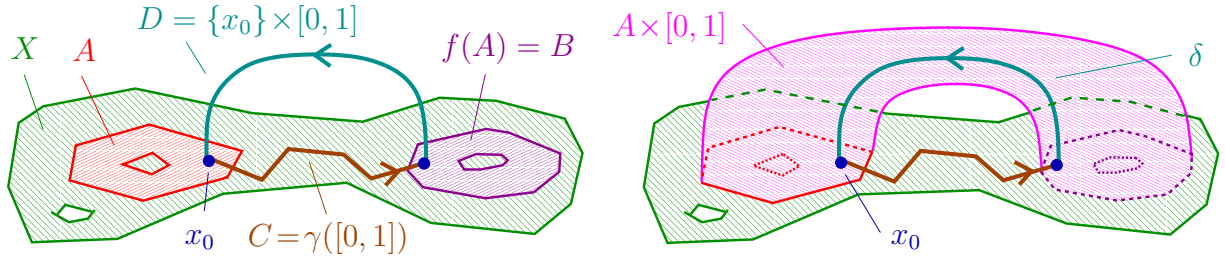
by the Seifert–van Kampen Theorem 69.1 and the base point change $f(x_0) \mapsto x_0$ induced by δ

Finally we also have the isomorphism:

$$\pi_1(\mathrm{Tor}(f), x_0) \underset{\uparrow}{\cong} \pi_1(Y, x_0) *_{\pi_1(A \cup D \cup B, x_0)} \underbrace{*\pi_1(A \times [0,1], x_0)}_{\cong \pi_1(A, x_0)}.$$

by the Seifert–van Kampen Theorem 69.1, here we use that $Y \cap (A \times [0,1]) = A \cup D \cup B$ is connected

Note that in the above discussion basically all maps are induced by the obvious inclusion maps We leave it to the intrepid reader to go through the explicit descriptions of the various homomorphisms and the description of the amalgamated product in Proposition 53.21 to show that we do indeed end up with the promised description of $\pi_1(\mathrm{Tor}(f), x_0)$. ■



We conclude this section with the following proposition that is an immediate consequence of the Direct Limit-π_1-Proposition 56.7 and the Finiteness the CW-Complex-Finiteness Theorem 68.14

Proposition 69.3. Let X be a CW-complex and let $X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subcomplexes such that $X = \bigcup_{i\in\mathbb{N}} X_i$. Then for any $w \in X_1$ we have

$$\pi_1(X, w) \;=\; \varinjlim \pi_1(X_i, w).$$

Examples.

(1) We have[817]

$$\pi_1(\mathbb{R}\mathrm{P}^\infty) \;\underset{\text{see page 1480}}{=}\; \pi_1\Big(\bigcup_{n\in\mathbb{N}} \mathbb{R}\mathrm{P}^n\Big) \;\underset{\text{Proposition 69.3}}{=}\; \varinjlim \begin{array}{c} \uparrow \\ \pi_1(\mathbb{R}\mathrm{P}^3) \\ \uparrow \\ \pi_1(\mathbb{R}\mathrm{P}^2) \\ \uparrow \\ \pi_1(\mathbb{R}\mathrm{P}^1) \end{array} = \varinjlim \begin{array}{c} \uparrow\cong \\ \mathbb{Z}_2 \\ \uparrow\cong \\ \mathbb{Z}_2 \\ \uparrow \\ \mathbb{Z} \end{array} \;\cong\; \mathbb{Z}_2.$$

(2) In Exercise 69.1 we will use Proposition 69.3 to show that there exists a CW-complex X with $\pi_1(X) \cong (\mathbb{Q}, +)$.

69.2. **The Euler characteristic of a CW-complex.** In the following sections we will give ea complete calculation of the fundamental group of a 1-dimensional CW-complex. In the subsequent discussion it will be convenient to have the following definition at our disposal.

Definition. Given a finite CW-complex X we define its Euler characteristic $\chi(X)$ as follows:

$$\chi(X) \;:=\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \text{number of } n\text{-cells}.$$

Examples.

(1) If G is a finite topological graph, then, according to the discussion on page 1470, we can view G as a 1-dimensional CW-complex where the number of 0-cells equals the number of vertices and the number of 1-cells equals the number of edges of G. Thus it follows immediately from the definitions that in this case the above Euler characteristic agrees with the definition of the Euler characteristic of a topological graph that we gave on page 315.

(2) For the CW-structures provided by Lemma 68.1 we have $\chi(\mathbb{R}\mathrm{P}^n) = \frac{1}{2}(1+(-1)^n)$ and $\chi(\mathbb{C}\mathrm{P}^n) = n+1$.

(3) A short glance at the three examples in the figure below shows that these three different CW-structures for the torus have the same Euler characteristic. Later on, in Chapter 87, we will find out whether or not that is a coincidence.

The following three propositions give us several convenient statements about the Euler characteristic of a finite CW-complex.

Proposition 69.4. (Covering-χ-Proposition) Let $p\colon \widetilde{X} \to X$ be a finite covering of a finite CW-complex X. We equip $\widetilde{X}$ with the structure of a CW-complex given by the

[817] By Corollary 48.18 we know that $\pi_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}_2$ for $n \geq 2$. But why is $\pi_1(\mathbb{R}\mathrm{P}^1) \cong \mathbb{Z}$?

CW-complex Covering Proposition 68.37. Then

$$\chi(\widetilde{X}) \;=\; [\widetilde{X} : X] \cdot \chi(X).$$

Proof. Let $d := [\widetilde{X} : X]$. As we mentioned in the CW-complex Covering Proposition 68.37 (2), it follows immediately from the construction of the CW-structure on $\widetilde{X}$ that for each k we have

$$\#k\text{-cells of } \widetilde{X} \;=\; \big[\widetilde{X} : X\big] \cdot \#k\text{-cells of } X.$$

The proposition is an immediate consequence of this equality. ■

The following lemma can be viewed as a generalization of the Quotient Abstract Graph Lemma 7.14.

Lemma 69.5. (Quotient-χ-Lemma) Let X be a finite CW-complex and let A be a subcomplex. We equip the quotient X/A with the CW-structure given by the CW-Complex Construction Lemma 68.32 (3). Then

$$\chi(X/A) \;=\; \chi(X) - \chi(A) + 1.$$

Proof. It follows immediately from the construction of the CW-structure on X/A given in the CW-Complex Construction Lemma 68.32 (3) that for each $k \in \mathbb{N}_0$ we have the following equalities:

$$\#k\text{-cells of } X/A \;=\; \begin{cases} \#k\text{-cells of } X \text{ not in } A, & \text{if } k > 0, \\ \#k\text{-cells of } X \text{ not in } A \;+\; 1, & \text{if } k = 0. \end{cases}$$

Evidently these equalities implies the equality $\chi(X/A) = \chi(X) - \chi(A) + 1$. ■

Proposition 69.6. Let X be a non-empty finite connected 1-dimensional CW-complex. Then $\chi(X) = 0$ if and only if X is contractible.

Proof. This statement is an immediate consequence of the Fundamental Group-of-Graph Proposition 52.5 and the fact that we can view any 1-dimensional CW-complex as a topological graph of the same Euler characteristic. ■

We have now assembled enough information to get us through this chapter. Later on in Chapter 87 we will give a much more elaborate discussion of the Euler characteristic of a CW-complex.

69.3. **The fundamental group of 1-dimensional CW-complexes.** The goal in this section is to show that the fundamental group of a connected non-empty 1-dimensional CW-complex X is a free group. The idea is quite simple:

(1) If the CW-complex X has a single 0-cell, then it follows basically from Proposition 52.7 that $\pi_1(X)$ is a free group.
(2) In the general case we show that X is homotopy equivalent to a 1-dimensional CW-complex with a single 0-cell.

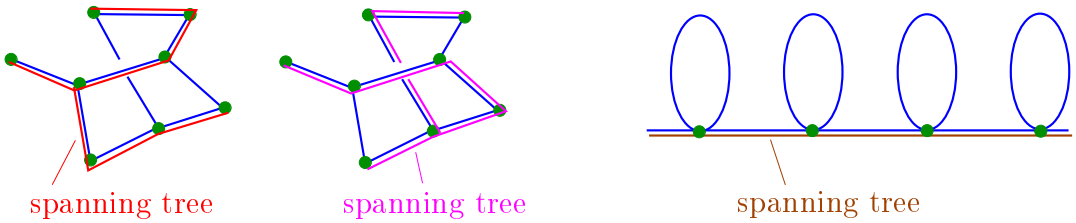
The following definition is the key to proving Statement (2).

Definition.

(1) A tree is a 1-dimensional CW-complex that admits a deformation retraction to a 0-cell.
(2) Let G be a 1-dimensional CW-complex. A spanning tree is a subcomplex T that is a tree and that contains all vertices of G.

Remark.

(1) On page 315 we defined a *tree* to be a finite connected non-empty topological graph with Euler characteristic 1. It follows from Proposition 69.6 that the above definition of a tree is consistent with the definition that we had given on page 315.
(2) We leave it to the reader to show that for a finite topological graph the above definition of a spanning tree is equivalent to the one we had given on page 317.
(3) A spanning tree is often also called a *maximal tree*.



In the figure below we give examples of spanning trees in topological graphs. We see in particular that in general there is no unique spanning tree.

Proposition 69.7. (Spanning Tree Existence Proposition)

(1) Given any connected 1-dimensional CW-complex G and given any tree $S \subset G$ there exists a spanning tree T that contains S and such that S is a deformation retract of T.
(2) Every connected non-empty 1-dimensional CW-complex admits a spanning tree.

Remark.

(1) the Spanning Tree Existence Proposition 69.7 can be viewed as a generalization of the Spanning Tree Existence Proposition 7.10.
(2) A somewhat different proof of the Spanning Tree Existence Proposition 69.7 is given in [**FrPi1990a**, Lemma 2.6.3].

Proof. Statement (2) follows from Statement (1) by applying Statement (1) to a tree that is given by a single vertex. Thus it remains to prove Statement (2). Now let G be a connected 1-dimensional CW-complex and let $S \subset G$ be a tree.

Claim. There exists a sequence $S = Y_0 \subset Y_1 \subset Y_2 \subset \dots$ of subcomplexes of G with the following properties:

(1) The union $Y := \bigcup_{i \in \mathbb{N}} Y_i$ contains all 0-simplices of G.
(2) Each Y_i is a deformation retract of Y_{i+1}.

Proof. To prove the claim we first construct a sequence $S = X_0 \subset X_1 \subset X_2 \subset \dots$ of subcomplexes of G as follows.

(a) We set $X_0 := S$.
(b) Given X_i we define X_{i+1} to be the union of X_i with all 1-dimensional cells which contain at least one 0-cell of X_i.

We make the following observations:

(i) It follows from Lemma 68.21 (2) that X_{i+1} is a subcomplex of G.

(ii) We claim that $G = \bigcup_{i\in\mathbb{N}} X_i =: X$. This can be seen as follows. Basically by construction we have for each characteristic map[818] $\Phi\colon \overline{B}^n \to G$, $n = 0, 1$ that $\Phi^{-1}(X) = \varnothing$ or $\Phi^{-1}(X) = \overline{B}^n$. Since G is 1-dimensional it follows from the Characteristic Maps Lemma 68.7 that X is a subset of G that is closed and that is open. Since G is connected and since X is non-empty we see that $X = G$.

Next we do the following iterative construction:

(a) We set $Y_0 := X_0 = S$.

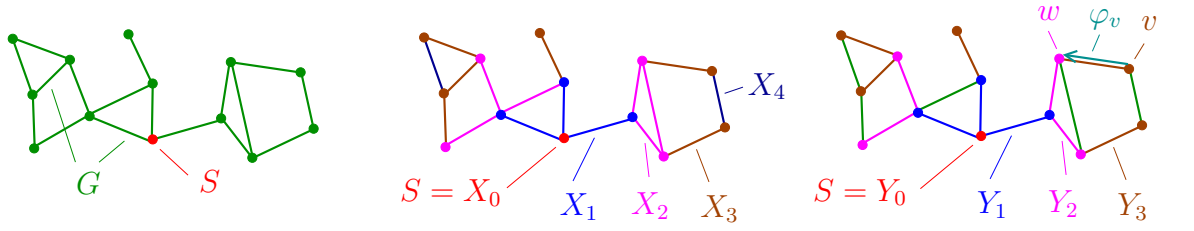
(b) Suppose we are given Y_i. For each 0-cell v in Y_{i+1} that is not contained in Y_i we pick precisely one 1-cell e_v of Y_{i+1} that contains v. We define Y_{i+1} to be the union of Y_i with all these 1-cells. By Lemma 68.21 (2) we know that Y_{i+1} is a subcomplex of Y_i.

By construction the 0-cells of each Y_i are precisely the 0-cells of X_i. It follows from (ii) that $Y := \bigcup_{i\in\mathbb{N}} Y_i$ contains all 0-cells of G. It remains to show that each Y_i is a deformation retract of Y_{i+1}. As we will see, this is almost obvious, except that we need to worry a little bit about continuity issues. Let V be the set of 0-cells in Y_{i+1} that are not contained in Y_i. Note that the corresponding 1-cell e_v contains two 0-cells, namely v and one 0-cell w in Y_i. Let $\varphi_v\colon e_v \times [0,1] \to e_v$ be the obvious deformation retraction to w. Finally we consider the map

$$\begin{array}{rcl} F\colon \overbrace{\left(Y_i \cup \bigsqcup_{v\in V} e_v\right)}^{=Y_{i+1}} \times [0,1] & \to & Y_{i+1} = Y_i \cup \bigsqcup_{v\in V} e_v \\ ([P], t) & \mapsto & \begin{cases} P, & \text{if } P \in Y_i, \\ \varphi_v(P,t), & \text{if } P \in e_v. \end{cases} \end{array}$$

It follows almost immediately from the Interval CW-Product Corollary 68.9 that the map is continuous. It is now basically clear that F defines a deformation retraction from Y_{i+1} to Y_i. ⊞

We set $Y := \bigcup_{i\in\mathbb{N}} Y_i$ By Lemma 68.21 (2) we know that Y is a subcomplex of G. We need to show that Y is contractible. Since S is contractible it follows from the Homotopy-Equivalence Basics Lemma 15.1 (2) and the Deformation Retract-Homotopy Equivalence Lemma 15.5 that it suffices to that S is a deformation retract of Y. But this is an immediate consequence of Statement (2) of the claim together with the Iterated Deformation Retracts Proposition 68.22. ■



[818] For $n = 0$ the statement is a tautology. For $n = 1$ note that if $\Phi^{-1}(X) \neq \varnothing$, then there exists an i with $\Phi^{-1}(X_i) \neq \varnothing$. Since X_i is a subcomplex we see that $\Phi^{-1}(X_i)$ contains at least one 0-cell. But by construction this means that $\Phi^{-1}(X_{i+1}) = \overline{B}^1$.

Proposition 69.8. (Quotient-by-Tree Proposition) Let X be a 1-dimensional CW-complex.

(1) For every spanning tree T of X the following holds:
   (a) The projection $p\colon X \to X/T$ is a homotopy equivalence.
   (b) The quotient X/T X is homeomorphic to a wedge of circles. If X is finite, then X/T is homeomorphic to the wedge of $1-\chi(X)$ circles.

(2) If X is connected and non-empty, then X is homotopy equivalent to the wedge of circles. Furthermore if X is finite, then X is homotopy equivalent to the wedge of $1-\chi(X)$ circles.

Remark.

(1) the Quotient-by-Tree Proposition 69.8 is a fairly straightforward generalization of Proposition 50.15. In fact the proof below is at times verbatim the same as the proof of Proposition 50.15.

(2) Later on, in the Quotient-by-Contractible Corollary 70.3, we will generalize the Quotient-by-Tree Proposition 69.8 in a suitable way to the case of arbitrary CW-complexes.

Proof. Let X be a 1-dimensional CW-complex.

(1a) Let T be a spanning tree of X. By hypothesis and by definition there exists a 0-cell t_0 of T such that there exists a deformation retraction from T to $\{t_0\}$, i.e. there exists a homotopy $R\colon T\times[0,1]\to T$ such that $R_1 = \mathrm{id}$ and such that $R_0 = c_{t_0}$ is the constant map. For notational convenience we make the obvious identification $\overline{B}^1 = [-2,2]$. Now let $\{\Phi_i\colon [-2,2]\to X\}_{i\in I}$ be the characteristic maps of the 1-cells of X that are not contained in T. We write $a_i = \Phi_i(-2)$ and $b_i = \Phi_i(2)$.

By the CW-Complex Construction Lemma 68.32 (3) we can view X/T as a 1-dimensional CW-complex with a single 0-cell $*$ and one 1-cell for each $i\in I$. We denote by $\{\Psi_i\colon [-2,2]\to X/T\}_{i\in I}$ the characteristic maps of these 1-cells. We consider the map

$$\begin{array}{rcl} q\colon X/T & \to & X \\ x & \mapsto & \begin{cases} R(a_i, -y-1), & \text{if } x=\Psi_i(y) \text{ for some } y\in[-2,-1], \\ \Psi_i(2\cdot y), & \text{if } x=\Psi_i(y) \text{ for some } y\in[-1,1], \\ R(b_i, y-1), & \text{if } x=\Psi_i(y) \text{ for some } y\in[1,2]. \end{cases} \end{array}$$

It follows from the Characteristic Maps Lemma 68.7 (1) and (4) that q is continuous. (We refer to the figure below from a long time ago for an illustration of q.) It remains to prove the following claim.

Claim. The maps p and q are homotopy inverses of one another.

Proof. The proof of the claim is fairly elementary, just notationally a little messy. First we consider the following map

$$\begin{array}{rcl} F\colon (X/T)\times[0,1] & \to & X/T \\ (x,t) & \mapsto & \begin{cases} *, & \text{if } x=\Psi_i(y) \text{ for some } y\in[-2,-1-t], \\ \Psi_i(\frac{2y}{1+t}), & \text{if } x=\Psi_i(y) \text{ for some } y\in[-1-t,1+t], \\ *, & \text{if } x=\Psi_i(y) \text{ for some } y\in[1+t,2]. \end{cases} \end{array}$$

Using the Interval CW-Product Corollary 68.9 (3) one can easily show that F is continuous. It follows immediately from the definitions that $F_0 = p \circ q$ and $F_1 = \mathrm{id}_{X/T}$. Next we consider the following map

$$G\colon X \times [0,1] \to X$$
$$(x,t) \mapsto \begin{cases} R(x,t), & \text{if } x \in T, \\ R(a_i, y+2+t), & \text{if } x = \Phi_i(y) \text{ for some } y \in [-2,-1-t], \\ \Phi_i(\frac{2y}{1+t}), & \text{if } x = \Phi_i(y) \text{ for some } y \in [-1-t, 1+t], \\ R(b_i, -y+2+t), & \text{if } x = \Phi_i(y) \text{ for some } y \in [1+t, 2]. \end{cases}$$

Again, using the Interval CW-Product Corollary 68.9 (3) one can easily show that G is continuous. Furthermore it follows again immediately from the definitions that $G_0 = q \circ p$ and $G_1 = \mathrm{id}_X$. ⊞

(1b) Let T be a spanning tree of X. We denote by I the set of 1-cells of X that are not contained in T. As mentioned in (1), it follows from the CW-Complex Construction Lemma 68.32 (3) that we can view X/T as a 1-dimensional CW-complex with a single 0-cell $*$ and one 1-cell for each $i \in I$. By the Skeleton-Quotient Lemma 68.33 we know that X/T is homeomorphic to the wedge $\bigvee_{i \in I} S^1_i$.

Finally we suppose that X is finite. At this point it suffices to show that X/T is homeomorphic to the wedge of $1 - \chi(X)$ circles, in other words, we want to show that $|I| = 1 - \chi(X)$. This in turn follows from the following calculation:

$$1 - |I| \underset{\uparrow}{=} \chi\Big(\bigvee_{i \in I} S^1_i\Big) = \chi(X/T) \underset{\uparrow}{=} \chi(X) - \chi(T) \underset{\uparrow}{=} \chi(X).$$

see page 1480 — by the Quotient-χ-Lemma 69.3 — Proposition 69.6

(2) Now we assume that X is connected and non-empty. By the Spanning Tree Existence Proposition 69.7 we know that X admits a spanning tree T. The desired conclusion now follows from (1) together with (2). ■

Now we no longer have any troubles with computing the fundamental group of 1-dimensional CW-complexes. More precisely, we have the following theorem which generalizes the Fundamental Group-of-Graph Proposition 52.5 and 64.2

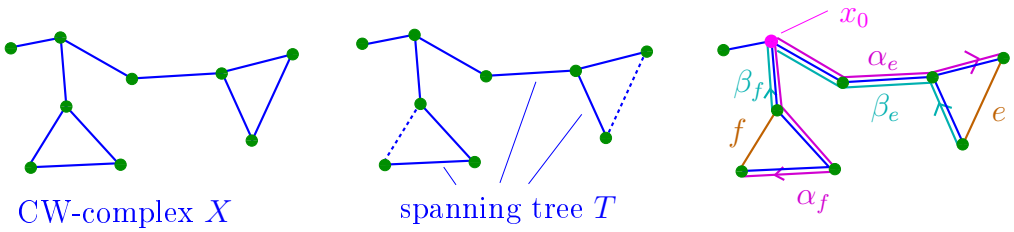
Theorem 69.9. (Fundamental Group-of-1-Complex Theorem) Let X be a connected non-empty 1-dimensional CW-complex. Furthermore let $x_0 \in X$. Then the following holds:

(1) The group $\pi_1(X, x_0)$ is isomorphic to a free group. In fact, if T is a spanning tree for X, then the cardinality of the generating set of the free group equals the cardinality of the set of edges in $X \setminus T$.
(2) If X is a finite CW-complex, then $\pi_1(X, x_0)$ is a free group with $1 - \chi(X)$ generators.
(3) An explicit description of the isomorphism in (1) is given as follows. Pick a base point x_0 in the spanning tree and for each 1-cell e that is not contained in T we pick a path α_e from p to one of the endpoints and pick a path β_e from the other endpoint back to p[819] and let t_e be the loop given by first going along α_e, then going through

e and then going back to the base point via β_e. Then the map

$$\begin{array}{rcl} \langle \{x_e\}_{e \text{ cell not in } T} \rangle & \to & \pi_1(X, x_0) \\ x_e & \mapsto & [t_e] \end{array}$$

is an isomorphism.



Proof. Let X be a connected non-empty 1-dimensional CW-complex. By the Spanning Tree Existence Proposition 69.7 we know that X admits a spanning tree T. We denote by E the set of 1-cells of X that are not contained in T. Let x_0 be a point in T. We denote by $p\colon X \to X/T$ the projection . We write $* = p(x_0)$. We have the following isomorphisms:

$$\pi_1(X, x_0) \xrightarrow{p_*} \pi_1(X/T, *) \xrightarrow{\cong} \pi_1\Big(\bigvee_{e \in E} S^1_e, *\Big) \xleftarrow{\cong} \langle \{x_e\}_{e \in E} \rangle.$$

by the Quotient-by-Tree Proposition 69.8 we know that the map $p\colon X \to X/T$ is a homotopy equivalence, from the Fundamental Group-Homotopy Proposition 50.3 (1a) we obtain that p_* is an isomorphism

Fundamental Group-Homotopy Proposition 50.3 (1b)

isomorphism given by Proposition 52.7

If one goes through the explicit isomorphism, then one sees that the map given in (3) is indeed an isomorphism.

Finally suppose that X is a finite 1-dimensional CW-complex. It is not terribly hard to show directly that E has cardinality $1-\chi(X)$. But since we proved the desired statement regarding the rank of $\pi_1(X, x_0)$ already in the Fundamental Group-of-Graph Proposition 52.5 we will not work out the details of the argument. ■

We conclude this discussion of CW-complexes with an application to group theory, namely to the study of free groups. Recall that in Proposition 64.4 we used topological graphs to show that if F is a free group of countable rank, then every subgroup is also a free group. Now that we have developed the theory of CW-complexes we can remove the slightly artificial restriction on the cardinality of the generating set. More precisely, we have the following corollary to our previous results.

Corollary 69.10. (Subgroup-of-Free Group Corollary) Every subgroup of a free group (of any rank) is again a free group.

Remark. Even for free groups of countable rank the proof of the Subgroup-of-Free Group Corollary 69.10 might be of interest, since it differs somewhat from the proof of Proposition 64.4.

[819] The two paths exist since a tree is in particular connected.

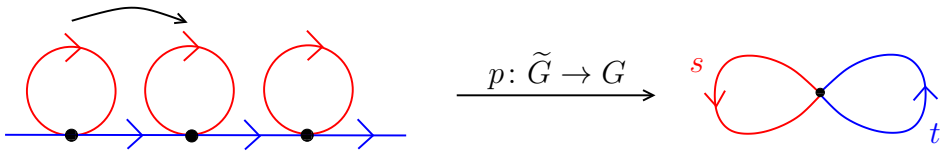
Proof. Let $F = \langle S \rangle$ be a free group on a set S and let G be a subgroup of F.

(1) By attaching one 1-cell for each $s \in S$ to a one-point set $\{P\}$ we obtain a 1-dimensional CW-complex X with one 0-cell P and with one 1-cell for each $s \in S$. By the Fundamental Group-of-1-Complex Theorem 69.9 we have $\pi_1(X, P) \cong F$. By the CW-Complex Properties Proposition 68.10 (6) and the Covering Existence Theorem 61.5 there exists a covering $p\colon (Y, Q) \to (X, P)$ of degree $[F : G]$ such that $p_*(\pi_1(Y, Q)) = G$. By Corollary 48.14 the map p_* is injective. Thus we see that $\pi_1(Y, Q)$ itself is isomorphic to G. By the CW-complex Covering Proposition 68.37 we can equip Y with the structure of a 1-dimensional CW-structure. We deduce from the Fundamental Group-of-1-Complex Theorem 69.9 that $G \cong \pi_1(Y, Q)$ is a free group.

(2) Now suppose that F is a free group of finite rank m and suppose that G is a finite-index subgroup of F. By Proposition 69.4 we know that $\chi(Y) = [F : G] \cdot \chi(X)$. It is now a consequence of the Fundamental Group-of-1-Complex Theorem 69.9 that $\operatorname{rank}(G) - 1 = [F : G] \cdot (\operatorname{rank}(F) - 1)$. ■

Example. Let G be the CW-complex consisting of one 0-cell v and two 1-cells which we denote by s and t. By a slight abuse of notation we denote the corresponding elements in $\pi_1(G, v)$ by s and t as well. By the Fundamental Group-of-Graph Proposition 52.5 we can identify $\pi_1(G, v)$ with the free group $\langle s, t \rangle$ on the generators s and t. Now let $\phi\colon \pi_1(G, v) = \langle s, t \rangle \to \mathbb{Z}$ be the homomorphism that is given by $\phi(s) = 0$ and $\phi(t) = 1$. By the Covering Existence Theorem 61.5 there exists a covering $q\colon (\widehat{G}, \widehat{v}) \to (G, v)$ with $q_*(\pi_1(\widehat{G}, \widehat{v})) = \ker(\phi)$, furthermore by the Covering Uniqueness Theorem 61.9 this covering is unique up to equivalence, in the sense of the definition on page 1368.

We consider the infinite covering illustrated in the figure below, which is very similar to the finite covering studied on pages 1385.[820] For $i \in \mathbb{Z}$ we denote by x_i the loop in $(\widetilde{G}, \widetilde{v})$ as



illustrated in the figure below. We denote by x_i also the corresponding element in $\pi_1(\widetilde{G}, \widetilde{v})$. It follows from the Fundamental Group-of-1-Complex Theorem 69.9 that $\pi_1(\widetilde{G}, \widetilde{v})$ equals the free group $\langle \dots x_{-2}, x_{-1}, x_0, x_1, x_2 \dots \rangle$. It is straightforward to see that

$$p_*(x_i) = t^i s t^{-i}.$$

[820] This covering can also be described rigorously. More precisely, we view G as the union of two copies A and B of S^1 glued along the common point 1. We define

$$\widetilde{G} = (\mathbb{R} \sqcup (S^1 \times \mathbb{Z}))/(n \sim (1, n)) \text{ for } n \in \mathbb{Z}.$$

Then the projection

$$p\colon (\mathbb{R} \sqcup (S^1 \times \mathbb{Z}))/\sim \to G = (S^1 \times \{A\} \sqcup S^1 \times \{B\})/(1, A) \sim (1, B)$$

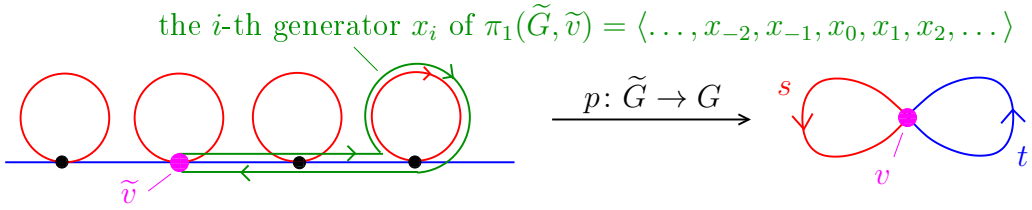
is given by $p(x) = (\exp(2\pi \mathrm{i} x), A)$ for $x \in \mathbb{R}$ and $p((z, n)) = (z, B)$ for $(z, n) \in S^1 \times \mathbb{Z}$.

By Corollary 48.14 the homomorphism p_* is a monomorphism. Thus we see that $t^i s t^{-i}$, $i \in \mathbb{Z}$, generate an infinitely generated free subgroup of the finitely generated generated free group $\langle s, t\rangle$. Furthermore we see that $p_*(\pi_1(\widetilde{G}, \widetilde{v})) \subset \ker(\phi\colon \pi_1(G, v) \to \mathbb{Z})$. It is easy to see that the deck transformation group is isomorphic to $\mathbb{Z}$.

Now we claim that the coverings $p\colon \widetilde{G} \to G$ and $q\colon \widehat{G} \to G$ are equivalent. By the Covering Uniqueness Theorem 61.9 it suffices to show that $p_*(\pi_1(\widetilde{G}, \widetilde{v})) = q_*(\pi_1(\widehat{G}, \widehat{v}))$. We consider the following sequence of maps

$$\begin{array}{ccccccccc} \mathbb{Z} & \cong & \mathrm{Deck}(p\colon \widetilde{G} \to G) & \cong & \pi_1(G,v)/p_*(\pi_1(\widetilde{G},\widetilde{v})) & \twoheadrightarrow & \pi_1(G,v)/\ker(\phi\colon \pi_1(G,v) \to \mathbb{Z}) & \xrightarrow[\cong]{\phi} & \mathbb{Z} \\ & \uparrow & & & \uparrow & & \uparrow & & \\ & \text{see above} & & & \text{isomorphism of Proposition 62.7} & & \text{since } p_*(\pi_1(\widetilde{G},\widetilde{v})) \subset \ker(\phi\colon \pi_1(G,v) \to \mathbb{Z}) & & \end{array}$$

Thus, reading from left to right, we get an epimorphism from $\mathbb{Z}$ onto itself. But any such epimorphism has to be an isomorphism. It follows that the third map is also an isomorphism, i.e. $p_*(\pi_1(\widetilde{G}, \widetilde{v})) = \pi_1(G, v)/\ker(\phi\colon \pi_1(G, v) \to \mathbb{Z})$, but the latter group is precisely $q_*(\pi_1(\widehat{G}, \widehat{v}))$. So we get the desired equality of subgroups.



69.4. **Fundamental groups of CW-complexes.** In this section we will discuss fundamental groups of CW-complexes. For the statement of our first proposition it is useful to introduce the following definition.
The following definition is evidently inspired by the definition of a CW-complex.

Definition. Let X be a topological space and let $\varphi\colon S^{n-1} \to X$ be a map. We refer to $X \cup_\varphi \overline{B}^n$ as the topological space X with an n-cell attached via the attaching map φ. We refer to the image of $\overline{B}^n$ in $X \cup_\varphi \overline{B}^n$ as an n-cell.

Remark. On page 450 we gave an explicit homeomorphism $\mathrm{Cone}(S^{n-1}) \xrightarrow{\cong} \overline{B}^n$. Using this homeomorphism one sees almost immediately that, using the notation from the above definition, we have a homeomorphism $\mathrm{Cone}(\varphi\colon S^{n-1} \to X) \xrightarrow{\cong} X \cup_\varphi \overline{B}^n$ which is the identity on X. In other words, attaching a cell is just a special case of taking a mapping cone.

The following proposition now tells us how attaching a cell affects the fundamental group of a given topological space.

Proposition 69.11. (Cell Attachment-π_1-Proposition) Let X be a topological space, let $n \geq 1$, let $\varphi\colon S^{n-1} \to X$ be a map and let x_0 be a point in $\varphi(S^{n-1})$. Then the following hold:

($n > 2$) The inclusion induced map $\pi_1(X, x_0) \to \pi_1(X \cup_\varphi \overline{B}^n, x_0)$ is an isomorphism.
($n = 2$) The map $\varphi\colon S^1 \to X$ defines an element in $\pi_1(X, x_0)$ that we denote by $[\varphi]$, and the inclusion induced map $\pi_1(X, x_0) \to \pi_1(X \cup_\varphi \overline{B}^n, x_0)$ descends to an

isomorphism

$$\pi_1(X, x_0)/\langle\langle[\varphi]\rangle\rangle \xrightarrow{\cong} \pi_1\big(X \cup_\varphi \overline{B}^2, x_0\big).$$

($n = 1$) Let $\epsilon \in \{-1, 1\}$ such that $x_0 = \varphi(\epsilon)$. We set $y_0 := \varphi(-\epsilon)$.

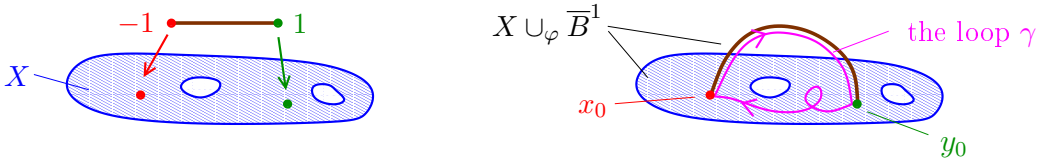
(a) Suppose that x_0 and y_0 lie in the same path-component of X. Let γ be a loop in $X \cup_\varphi \overline{B}^1$ that is given by the concatenation of the obvious path in the 1-cell from $x_0 = \varphi(\epsilon)$ to $y_0 = \varphi(-\epsilon)$ and some path in X from $y_0 = \varphi(-\epsilon)$ to $x_0 = \varphi(\epsilon)$. With this notation the map

$$\pi_1(X, x_0) * \langle t\rangle \;\to\; \pi_1(X \cup_\varphi \overline{B}^1, x_0)$$

that is given by the inclusion induced map $\pi_1(X, x_0) \to \pi_1(X \cup_\varphi \overline{B}^1, x_0)$ and given by $t \mapsto [\gamma]$ is an isomorphism.

(b) Suppose that x_0 and y_0 lie in two different path components of X. Then we get an isomorphism

$$\pi_1(X, x_0) * \pi_1(X, y_0) \;\to\; \pi_1(X \cup_\varphi \overline{B}^1, x_0).$$



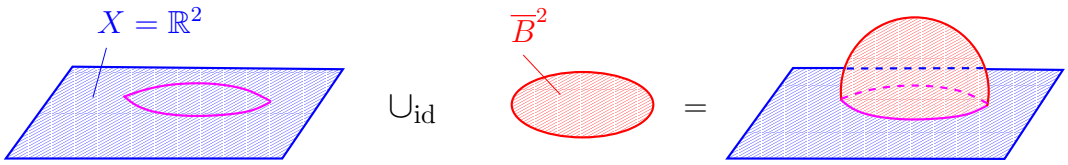
Examples.

(1) Let $X = S^1$ and let $n \in \mathbb{N}$. We attach to S^1 a 2-cell via the attaching map $\varphi\colon S^1 \to S^1$ that is given by $\varphi(z) = z^n$. Then[821]

$$\pi_1\big(S^1 \cup_\varphi \overline{B}^2, 1\big) \underset{\uparrow}{=} \pi_1(S^1, 1)/\langle[\varphi]\rangle \underset{\uparrow}{=} \langle x\rangle/\langle x^n\rangle \cong \mathbb{Z}_n.$$

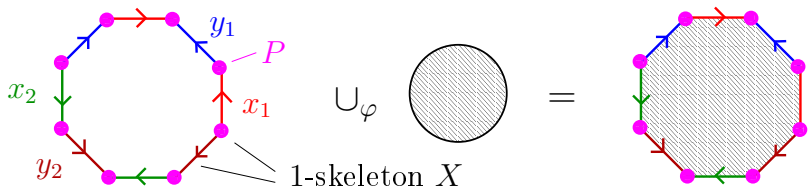
Cell Attachment-π_1-Proposition 69.11 — see Proposition 48.17

(2) We attach a 2-cell to $X = \mathbb{R}^2$ via the attaching map $\mathrm{id} = \varphi\colon S^1 \to S^1$ that is given by the identity. The resulting topological space is illustrated in the figure below. This topological space is again simply connected by the Cell Attachment-π_1-Proposition 69.11.



(3) Using the Fundamental Group-of-1-Complex Theorem 69.9 and the Cell Attachment-π_1-Proposition 69.11 we can now easily compute fundamental groups of finite CW-complexes. For example, as in the figure below we view the surface Σ of genus 2 as a CW-complex with one 0-cell P, four 1-cells x_1, y_1, x_2, y_2 and one 2-cell. We denote by X the 1-skeleton of Σ and we denote by $\varphi\colon S^1 \to X$ the attaching map of the

[821] We leave it to the reader to show that for $n = 2$ the topological space $S^1 \cup_\varphi \overline{B}^2$ is actually homeomorphic to $\mathbb{R}\mathrm{P}^2$.



2-cell. We have

$$\pi_1(\Sigma, P) \;=\; \pi_1\big(X \cup_\varphi \overline{B}^2, P\big) \underset{\uparrow}{=} \pi_1(X, P)/\langle\langle [\varphi] \rangle\rangle \underset{\uparrow}{=} \langle x_1, y_1, x_2, y_2 \rangle / \langle\langle [x_1, y_1][x_2, y_2] \rangle\rangle.$$

Cell Attachment-π_1-Proposition 69.11 — by the Fundamental Group-of-1-Complex Theorem 69.9 and since $[\varphi] = [x_1, y_1][x_2, y_2]$

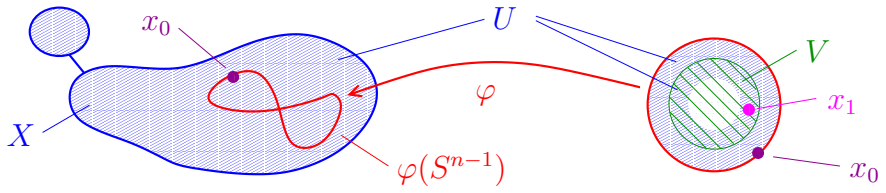
Of course we had obtained the same result much earlier. The advantage of CW-complexes is that the Cell Attachment-π_1-Proposition 69.11 absorbs many of the technical arguments we had to deal with in the proof of the Fundamental Group-of-Surfaces Proposition 54.5, i.e. in our initial calculation of $\pi_1(\Sigma, P)$.

Proof of the Cell Attachment-π_1-Proposition 69.11 for $n \geq 2$. Let X be a topological space, let $n \geq 2$, let $\varphi\colon S^{n-1} \to X$ be a map and let x_0 be a point in $\varphi(S^{n-1})$. The key idea of the proof is to apply the Seifert–van Kampen Theorem 54.1 to the following two open[822] subsets of $X \cup_\varphi \overline{B}^n$

$$U \;=\; X \cup_\varphi \big\{ z \in \overline{B}^n \,\big|\, \|z\| > \tfrac{1}{4} \big\} \quad \text{and} \quad V \;=\; \big\{ z \in \overline{B}^n \,\big|\, \|z\| < \tfrac{3}{4} \big\}.$$

The remainder of the proof consists in showing that the resulting isomorphism implies the desired, a priori slightly different, isomorphism.

We pick a point $x_1 \in U \cap V$ and we pick a path in U from x_0 to x_1. We refer to the figure below for an illustration. Note that V is a ball, hence $\pi_1(V, x_1)$ is the trivial group.



[822] Why are they open subsets?

We consider the following diagram of maps

$$\begin{array}{ccc}
\pi_1(X,x_0) & \longrightarrow & \pi_1(X\cup_\varphi \overline{B}^n,x_0)\\
\cong\downarrow & & =\downarrow\\
\pi_1(U,x_0) & \longrightarrow & \pi_1(X\cup_\varphi \overline{B}^n,x_0)\\
\cong\downarrow & & \downarrow\cong\\
\pi_1(U,x_1) & \longrightarrow & \pi_1(X\cup_\varphi \overline{B}^n,x_1)\\
\downarrow & & \downarrow=\\
\pi_1(U,x_1)/\langle\langle\pi_1(U\cap V,x_1)\rangle\rangle & \longrightarrow & \pi_1(X\cup_\varphi \overline{B}^n,x_1)\\
\cong\big\downarrow \text{theAmalgamatedProductLemma 53.22} & & \big\downarrow=\\
\pi_1(U,x_1)*_{\pi_1(U\cap V,x_1)} \underbrace{\pi_1(V,x_1)}_{=\{e\}} & \xrightarrow[\text{Seifert–van Kampen}]{\cong} & \pi_1(X\cup_\varphi \overline{B}^n,x_1).
\end{array}$$

We want to prove the following three statements:

(i) the diagram commutes,
(ii) all vertical maps except possibly the third vertical map on the left are isomorphisms,
(iii) the bottom horizontal map is an isomorphism.

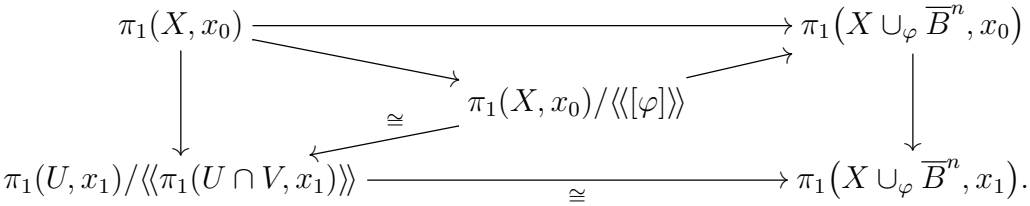
The two vertical maps on the second level are the isomorphisms induced by the choice of base point coming from the chosen path from x_0 to x_1. All the other vertical and horizontal maps are induced by inclusions or they are the obvious algebraic map. It follows easily that the diagram commutes.

Most of the maps are obviously isomorphisms. Using the Homotopy Pushout Lemma 14.9 one can easily show that X is a deformation retract of U. It follows from the Deformation Retract-Homotopy Equivalence Lemma 15.5 and the Homotopy-π_1-Proposition 50.3 (2) that the top left vertical map is an isomorphism. The bottom vertical map on the left is an isomorphism by the Amalgamated Product Lemma 53.22.

Finally the bottom horizontal map is an isomorphism by the Seifert–van Kampen Theorem 54.1.

Now we consider the case $n \geq 3$. In this case $\pi_1(U\cap V,x_1)$ is trivial since $U\cap V$ is homotopy equivalent to S^{n-1} which has trivial fundamental group by the Sphere-π_1-Proposition 47.13. It is now clear that the one missing vertical map on the left is also an isomorphism. Since all vertical maps are now isomorphisms and since the bottom horizontal map is an isomorphism it follows that the top horizontal map is an isomorphism.

Finally we consider the case $n = 2$. In this case $U\cap V$ is homotopy equivalent to S^1 which implies that $\pi_1(U\cap V,x_1)\cong\mathbb{Z}$. Under the isomorphism $\Phi\colon \pi_1(X,x_0)\to\pi_1(U,x_1)$ this infinite cyclic subgroup corresponds to $\langle[\varphi]\rangle$. It is straightforward to see that we get a commutative diagram



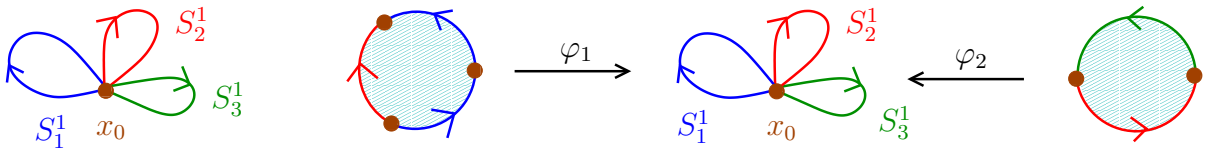
This concludes the proof of the proposition. ■

Proof of the Cell Attachment-π_1-Proposition 69.11 for $n = 1$. The proof of the case $n = 1$ is quite similar to the proof of the case $n \geq 2$. The only serious difference is that we need to replace the Seifert–van Kampen Theorem 54.1 by Corollary 57.5, which in turn is a corollary to the HNN-Seifert–van Kampen Theorem 57.3. We leave it to the reader to fill in the details. ■

In the following corollary we will see that every finitely presented group is the fundamental group of a finite CW-complex. The key to proving the corollary is the following construction, which associates to any finite presentation a 2-dimensional CW-complex. The construction is illustrated in the figure below. If the reader feels comfortable with that illustration it might be best to skip the official definition.

2-dimensional CW-complex for the presentation

$\langle g_1, g_2, g_3 \,|\, g_1 \cdot g_2^{-1} \cdot g_1, g_3 \cdot g_2 \rangle$



CW-complex is given by attaching two 2-cells to $S_1^1 \vee S_2^1 \vee S_3^1$

Construction. Let $\langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_m \rangle$ be a finite presentation. For $i = 1, \ldots, k$ let S_i^1 be a copy of S^1. For $j = 1, \ldots, m$ we do the following:

(1) We write the relation r_j as a reduced word $g_{s(1)}^{\epsilon_1} \cdot \cdots \cdot g_{s(n)}^{\epsilon_n}$ with $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$.

(2) We denote by $\varphi_j \colon S^1 = [0,1]/0 \sim 1 \to \bigvee_{i=1}^{k} S_i^1$ the (hopefully obvious) path which on the interval $[\frac{i-1}{n}, \frac{i}{n}]$ travels through $S_{s(i)}^1$ at constant speed in the direction determined by ϵ_i.

We denote by X the CW-complex that is given by attaching m 2-cells to $\bigvee_{i=1}^{k} S_i^1$ along the attaching maps $\varphi_1, \ldots, \varphi_m$. We refer to X as the CW-complex associated to the presentation.

Now we can state the corollary that we promised above.

Corollary 69.12. Let $\langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_m \rangle$ be a finite presentation for a group π. If Y is the associated 2-dimensional CW-complex, then $\pi_1(Y) \cong \pi$.

Proof. Let $\pi = \langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_m \rangle$ be a finitely presented group. We use the notation introduced in the above construction of the CW-complex associated to π. We denote by X the wedge of the k circles and we denote by x_0 the wedge point. Following the discussion on page 1173 we can make the identification $\pi_1(X, x_0) = \langle g_1, \ldots, g_k \rangle$. This identification has the property that for $j = 1, \ldots, m$ the corresponding loop $\varphi_j \colon S^1 \to X$ represents $r_j \in \pi_1(X, x_0) = \langle g_1, \ldots, g_k \rangle$. We write $Y_0 = X$ and for $j = 1, \ldots, m$ we iteratively define Y_j to be the 2-dimensional CW-complex that is given by attaching a 2-cell to Y_{j-1} with the

attaching map φ_j. We obtain that

$$\begin{array}{ccccc} & \text{Cell Attachment-}\pi_1\text{-Proposition 69.11, applied } m \text{ times} & & \text{since } [\varphi_j] = r_j & \\ & \downarrow & & \downarrow & \\ \pi_1(Y_m, x_0) & \cong & \langle g_1, \ldots, g_k \rangle / \langle\langle [\varphi_1] \rangle\rangle \ldots / \langle\langle [\varphi_m] \rangle\rangle & = & \langle g_1, \ldots, g_k \rangle / \langle\langle r_1 \rangle\rangle \ldots / \langle\langle r_m \rangle\rangle \\ & = & \langle g_1, \ldots, g_k \rangle / \langle\langle r_1, \ldots, r_m \rangle\rangle & = & \langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_m \rangle. \\ & \uparrow & & \uparrow & \\ & \text{by Exercise 53.1} & & \text{by definition} & \end{array}$$

■

The following proposition says in particular that for many CW-complexes the fundamental group of a CW-complex is the same as the fundamental group of its 2-skeleton.

Proposition 69.13. (CW-Skeleton-π_1-Proposition) Let X be a connected CW-complex and let x_0 be a point in the 0-skeleton of X.

(1) If X has finitely many cells in each dimension, then the following statements hold:
   (a) The inclusion induced map[823] $\pi_1(X^1, x_0) \to \pi_1(X, x_0)$ is an epimorphism.
   (b) For any $k \geq 2$ the inclusion induced map $\pi_1(X^k, x_0) \to \pi_1(X, x_0)$ is an isomorphism.

(2) If X has only finitely many cells in dimensions 0 and 1, then $\pi_1(X, x_0)$ is finitely generated.

(3) If X has only finitely many cells in dimensions 0, 1 and 2, then $\pi_1(X, x_0)$ is finitely presented.

Remark. In the CW-Skeleton-π_n-Proposition 71.9 we will generalize the CW-Skeleton-π_1-Proposition 69.13 to all connected CW-complexes.

Example. Let $X = \mathbb{R}\mathrm{P}^\infty$ be the infinite real projective space with the CW-structure from page 1480. For this CW-complex the 1-skeleton is $X^1 = \mathbb{R}\mathrm{P}^1 \cong S^1$ and the 2-skeleton is $X^2 = \mathbb{R}\mathrm{P}^2$. It follows from the CW-Skeleton-π_1-Proposition 69.13 (1) that $\mathbb{Z} \cong \pi_1(S^1) \to \pi_1(\mathbb{R}\mathrm{P}^\infty)$ is an epimorphism and it follows from the CW-Skeleton-π_1-Proposition 69.13 (1) that $\mathbb{Z}_2 \cong \pi_1(\mathbb{R}\mathrm{P}^2) \to \pi_1(\mathbb{R}\mathrm{P}^\infty)$ is an isomorphism. This calculation of $\pi_1(\mathbb{R}\mathrm{P}^\infty)$ is certainly easier than the argument from page 1519.

Proof.

(1) We first suppose that X is a finite CW-complex. Then the two statements follow immediately from iteratively applying the Cell Attachment-π_1-Proposition 69.11 finitely many times. If X is an infinite CW-complex, but with finitely many cells in each dimension, then one can reduce the two statements to the finite case using Propositions 69.3. We leave the details to the reader.

(2) Now we turn to the proof of (2). By (1) it suffices to show that the fundamental group of a finite connected 1-dimensional CW-complex is finitely generated, but we showed this in the Fundamental Group-of-1-Complex Theorem 69.9.

(3) Finally we turn to the proof of (3). By (1) it suffices to show that a finite connected 2-dimensional CW-complex X has a finitely presented fundamental group. We pick $x_0 \in X$. We denote by $\varphi_1, \ldots, \varphi_k$ the attaching maps of the 2-cells. For $i = 1, \ldots, k$ we pick a path p_i from x_0 to a point on $\varphi_i(S^1)$ and we write $\psi_i = p_i * \varphi_i * \overline{p}_i$. We then have

[823] Recall that for $k \in \mathbb{N}$ we denote by X^k the k-skeleton of X, i.e. the union of all cells of dimension $\leq k$.

$$\pi_1(X) \underset{\uparrow}{\cong} \pi_1(X^1)/\langle\langle [\psi_1] \rangle\rangle \dots /\langle\langle [\psi_k] \rangle\rangle \underset{\uparrow}{=} \pi_1(X^1)/\langle\langle [\psi_1], \dots, [\psi_k] \rangle\rangle$$

Propositions 47.10 and 69.11 — by Exercise 53.1

$$= \underset{\uparrow}{} \{\text{finitely generated free group}\}/\langle\langle [\psi_1], \dots, [\psi_k] \rangle\rangle.$$

Statement (2)

We have thus shown that $\pi_1(X)$ is a finitely presented group. ■

Remark. On page 1474 we already mentioned the Smooth Manifold-CW Structure Theorem 96.5 which says in particular that every compact smooth manifold admits the structure of a finite CW-complex. Together with the CW-Skeleton-π_1-Proposition 69.13 (3) we now obtain that the fundamental group of any connected compact smooth manifold M is finitely presented.

The following corollary is an immediate but rather convenient consequence of the CW-Skeleton-π_1-Proposition 69.13 (1).

Corollary 69.14. Let X be a connected CW-complex with countably many cells. If X has no 1-cells, then X is simply connected, i.e. $\pi_1(X) = 0$.

Examples.

(1) On page 1472 we pointed out that for $n \geq 2$ the sphere S^n admits a CW-structure with one 0-cell and one n-cell. From Corollary 69.14 we now recover the result from the Sphere-π_1-Proposition 47.13 that S^n is simply connected.
(2) It follows from Lemma 68.1 and Corollary 69.14 that for any n we have $\pi_1(\mathbb{C}\mathrm{P}^n) = 0$. Similarly it follows, using Lemma 68.6, that $\pi_1(\mathbb{C}\mathrm{P}^\infty) = 0$.

We conclude this chapter with a purely group theoretic result. To put the result in context, first recall that in Exercise 53.2 we saw that every finite-index subgroup of a finitely generated group is finitely generated. Now we can prove the following analogue.

Proposition 69.15. Every finite-index subgroup of a finitely presented group is finitely presented.

Remark. We will provide a proof of Proposition 69.15 that makes heavy use of topological methods. Not surprisingly one can also give a purely group theoretic proof. In fact such a proof can be given by the Reidemeister-Schreier process that we alluded to in Section 59.7. We refer to [**LyS1977**, p. 103] and [**MKS1976**, p. 90] for details.

Proof. Let $\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_m \rangle$ be a finitely presented group. Let Y be the finite 2-complex associated to this presentation that we constructed on page 1531. By Corollary 69.12 we know that $\pi_1(Y) \cong \pi$. Now let Γ be a finite-index subgroup of π. By the CW-Complex Properties Proposition 68.10 together with the Covering Existence Theorem 61.5 there exists a finite covering $p\colon \widetilde{Y} \to Y$ such that $\pi_1(\widetilde{Y}) \cong \Gamma$. By the CW-complex Covering Proposition 68.37 (2) the 2-complex $\widetilde{Y}$ has finitely many 1-cells and 2-cells. Thus it follows from the CW-Skeleton-π_1-Proposition 69.13 (3) that $\Gamma \cong \pi_1(\widetilde{Y})$ is a finitely presented group. ■

Exercises for Chapter 69.

Exercise 69.1. Let $A = S^1 \times [0,1]$ be an annulus.

(a) (i) We consider

$$X := \Big(\bigcup_{k\in\mathbb{N}_0} A \times \{k\} \Big) / (z,0,k) \sim (z^2,1,k-1) \quad \text{for } z \in S^1, k \in \mathbb{N}.$$

Determine $\pi_1(X)$.

(ii) We consider

$$Y := \Big(\bigcup_{k\in\mathbb{N}_0} A \times \{-k\} \Big) / (z,0,-k) \sim (z^2,1,-k-1) \quad \text{for } z \in S^1, k \in \mathbb{N}.$$

Determine $\pi_1(Y)$.

In both cases the group you obtain is an abelian group that is isomorphic to a subgroup of $(\mathbb{R},+)$.

(b) Give an example of topological space Z such that $\pi_1(Z) \cong (\mathbb{Q},+)$.

Exercise 69.2. Next let X and Y be two finite CW-complexes. On page 1498 we introduced the product CW-structure which by Proposition 68.23 is indeed a CW-structure for the topological space $X \times Y$. Is $\chi(X \times Y)$ determined by $\chi(X)$ and $\chi(Y)$?

70. CW-complexes III: Advanced properties

In this chapter we will prove several key technical theorems on CW-complexes, in particular we will prove the Homotopy Extension Theorem 70.1 and the Cellular Approximation Theorem 70.15. The reader who is in a hurry could just have a look at the Cellular Approximation Theorem 70.15 and then move on.

70.1. **The Homotopy Extension Theorem.** We recall the following definition from page 478.

Definition.

(1) A map $i\colon A \to X$ between topological spaces is called a cofibration if it has the homotopy extension property, i.e. if given any topological space Y and any two maps $f\colon X \times \{0\} \to Y$ and $G\colon A \times [0,1] \to Y$ with $f(i(a),0) = G(a,0)$ for all $a \in A$ there exists a map $F\colon X \times [0,1] \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} A \times \{0\} & \hookrightarrow & A \times [0,1] \\ \downarrow{\scriptstyle i} & \overset{f}{\nearrow} \; \underset{F}{\nwarrow} \; \overset{G}{\swarrow} & \downarrow{\scriptstyle i\times \mathrm{id}} \\ X \times \{0\} & \hookrightarrow & X \times [0,1]. \end{array} \qquad \begin{array}{l} \\ Y \\ \\ \end{array}$$

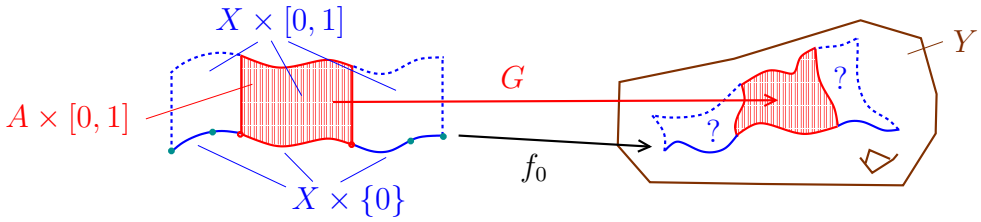
We illustrate the definition in the figure below.

(2) If $i\colon A \to X$ is a cofibration such that $i(A)$ is a closed subset of X, then we refer to i as a closed cofibration.

Theorem 70.1. (Homotopy Extension Theorem) Let X be a CW-complex and let A be a subcomplex of X.

(1) The inclusion map $A \to X$ is a closed cofibration.

(2) In slightly different words: let Y be a topological space and let $f_0\colon X \to Y$ be map. Furthermore, suppose that $A \subset X$ is a subcomplex and suppose we are given a homotopy $G\colon A \times [0,1] \to Y$ from $g_0 := f_0|_A$ to some map $g_1\colon A \to Y$. Then there exists a homotopy $F\colon X \times [0,1] \to Y$ starting from f_0 such that the restriction $F|_{A\times[0,1]}$ equals G.

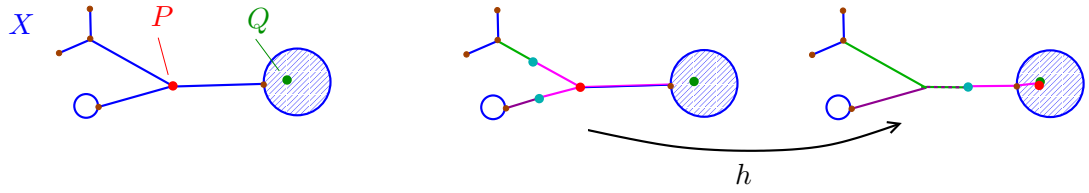


We postpone the long and strenuous proof of the Homotopy Extension Theorem 70.1 to the following section. Let us first revel in some exciting corollaries of the Homotopy Extension Theorem 70.1.

Corollary 70.2. Let X be connected CW-complex, let P be a point in the 0-skeleton of X and let Q be some other point on Y.

(1) Given any path $\gamma\colon [0,1] \to X$ with $\gamma(0) = P$ and $\gamma(1) = Q$ there exists a homotopy $F\colon X \times [0,1] \to X$ with $F(P,t) = \gamma(t)$ for all $t \in [0,1]$ and such that $F_0 = \mathrm{id}$.
(2) There exists a map $h\colon X \to X$ with $h(P) = Q$ that is homotopic to the identity.

Example. We consider the CW-complex X and the two points P and Q shown in the figure below. It should be pretty clear that there is no homeomorphism f of X with $f(P) = Q$. But Corollary 70.2 gives us a decent replacement, namely a map $h\colon X \to X$ with $h(P) = Q$ which is homotopic to the identity.



Proof. Let X be connected CW-complex, let P be a point in the 0-skeleton of X and let Q be some other point on Y.

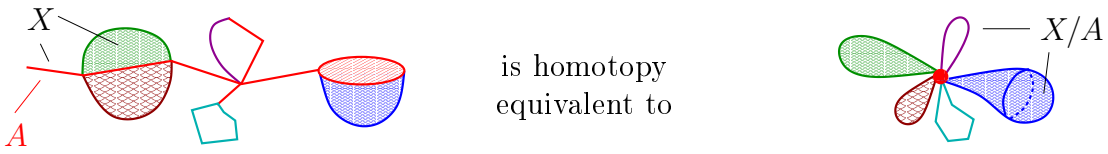
(1) Let $\gamma\colon [0,1] \to X$ be a path with $\gamma(0) = P$ and $\gamma(1) = Q$. We write $A := \{P\}$. Since P is a point in the 0-skeleton of X we see that A is a subcomplex of A. We apply the Homotopy Extension Theorem 70.1 to $f_0 = \mathrm{id}\colon X \to X$, $g_0(P) := P$, $g_1(P) = Q$ and $G(P,t) = \gamma(t)$. The resulting homotopy $F\colon X \times [0,1] \to X$ has all the desired properties.
(2) Since X is connected we obtain from the CW-Complex Properties Proposition 68.10 that there exists a map $\gamma\colon [0,1] \to X$ with $\gamma(0) = P$ and $\gamma(1) = Q$. From (1) we obtain a homotopy $F\colon X \times [0,1] \to X$ with $F_0 = \mathrm{id}$ and $F_1(P) = Q$. The map $h := F_1$ evidently has all the promised properties. ∎

We can now dip into the theory of cofibrations that we developed in Chapter 17. In particular we obtain the following useful corollary.

Corollary 70.3. (Quotient-by-Contractible Corollary) Let X be a CW-complex and let $A \subset X$ be a subcomplex.

(1) If A is contractible, then the projection $X \to X/A$ is a homotopy equivalence.
(2) If $a_0 \in A$ is a point such that $\{a_0\}$ is a deformation retract of A, then the projection $X \to X/A$ is a homotopy equivalence between the pointed topological spaces (X, a_0) and $(X/A, [A])$.

Remark. the Quotient-by-Contractible Corollary 70.3 (1) is a generalization of the Quotient-by-Tree Proposition 69.8.

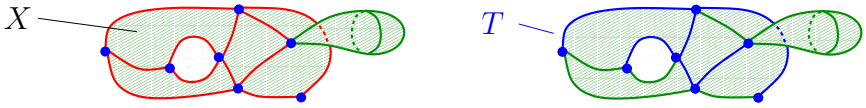


Proof. By the Subcomplex Cofibration Theorem 17.9 we know that the inclusion $A \to X$ is a closed cofibration. The two statements now follow immediately from the corresponding statements in the Cofibration-Quotient Lemma 17.7. ■

The following proposition is a neat corollary to the Quotient-by-Contractible Corollary 70.3.

Proposition 70.4. Let X be a 0-connected CW-complex.

(1) There exists a contractible 1-dimensional subcomplex T which contains all 0-cells.[824]
(2) Given any T as in (1) the quotient X/T is a CW-complex with precisely one 0-cell. Furthermore the projection $X \to X/T$ is a homotopy equivalence.



Remark.

(1) Using the Cell Attachment-π_1-Proposition 69.11 it is very easy to calculate the fundamental group of a finite 0-connected CW-complex with a single 0-cell. Proposition 70.4 says in particular that given a finite 0-connected CW-complex we can always reduce the calculation of the fundamental group to the case of a finite CW-complex with a single 0-cell.
(2) Let X be a 0-connected CW-complex with countably many 1-cells. Let $x_0 \in X$ be a 0-cell. By the CW-Skeleton-π_n-Proposition 71.9 the inclusion $X^1 \to X$ of the 1-skeleton induces an epimorphism $\pi_1(X^1, x_0) \to \pi_1(X, x_0)$. Furthermore it follows from Proposition 70.4 and 52.7 that $\pi_1(X^1, x_0)$ is a free group on a countable generating set. It follows from this discussion, together with the Countability Lemma 0.8 (2), that $\pi_1(X, x_0)$ is countable.

Proof. Let X be a 0-connected CW-complex.

(1) In the Spanning Tree Existence Proposition 69.7 (1) we showed that the 1-skeleton X^1 admits a subcomplex T that contains all 0-cells of X and that admits a deformation retraction to x_0.
(2) By the CW-Complex Construction Lemma 68.32 the quotient X/T admits a CW-structure with one 0-cell. Furthermore note that it follows from the Quotient-by-Contractible Corollary 70.3 (2) that the projection $X \to X/T$ is a homotopy equivalence between. ■

The following proposition gives another typical application of the Homotopy Extension Theorem 70.1.

Proposition 70.5. Let $f\colon A \to B$ be a cellular map between CW-complexes that has the following property:

($*$) For each n-cell σ of A we have either $f(\sigma) \subset B^{n-1}$ or $f|_\sigma$ is a homeomorphism.

We consider the corresponding mapping cylinder

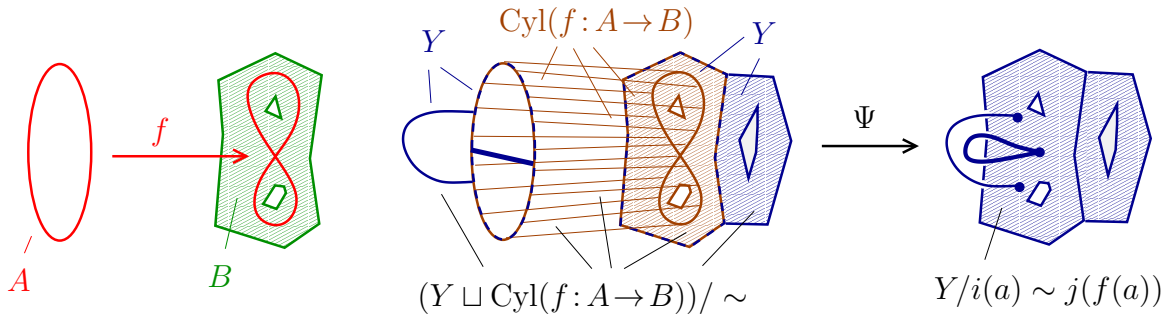
$$\mathrm{Cyl}(f) := \mathrm{Cyl}(f\colon A \to B) := ((A \times [0,1]) \sqcup B)/\sim \text{ where } (a,1) \sim f(a) \text{ for all } a \in A.$$

[824] Such a subcomplex is basically what we called a spanning tree on page 317.

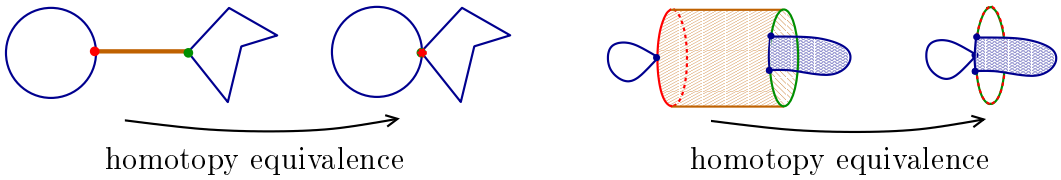
that we defined on page 454. Finally let Y be a CW-complex together with a cellular embedding $i \sqcup j \colon A \sqcup B \to Y$. The map

$$\begin{array}{rcl} \Phi \colon (Y \sqcup \mathrm{Cyl}(f \colon A \to B))/\sim & \to & Y/i(a) \sim j(f(a)) \\ [P] & \mapsto & \begin{cases} [P], & \text{if } P \in Y, \\ [i(a)], & \text{if } P = [(a,t)] \text{ with } a \in A \text{ and } t \in [0,1], \\ [j(b)], & \text{if } P = [b] \text{ with } b \in B \end{cases} \end{array}$$

is a homotopy equivalence.



Example. In many applications the map f will actually be a cellular isomorphism. In this special case Proposition 70.5 basically says that one can compress a product $A \times [0,1]$ to a single copy of A without changing the homotopy type. Two such examples are shown in the figure below.

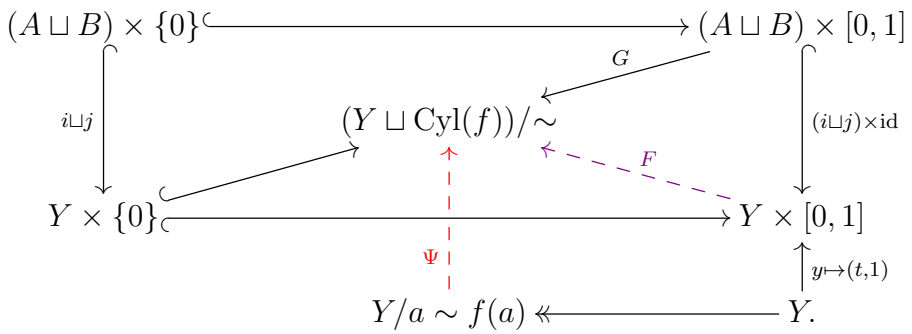


Remark. Note that the statement of Proposition 70.5 is related in spirit to the statement of the Cofibration-Quotient Lemma 17.7 (2).

Proof. First note that by the Cellular Embedding Lemma 68.19 we can and will view $A \sqcup B$ as a subcomplex of Y. In the following let $R \colon \mathrm{Cyl}(f \colon A \to B) \times [0,1] \to B$ be the natural deformation retraction given by the Mapping Cylinder Basics Lemma 16.12. Furthermore note that it follows quite easily from the CW-Complex Cone Corollary 68.35 (3) and the CW-Complex Construction Lemma 68.32 (2) that we can and will equip $\mathrm{Cyl}(f)$ and $(Y \sqcup \mathrm{Cyl}(f))/\sim$ with natural CW-structures. Finally, it follows from our slightly technical hypothesis on f together with the CW-Complex Construction Lemma 68.32 (6) that $Y/\sim$ also admits a natural CW-structure. Next we consider the homotopy

$$\begin{array}{rcl} G \colon (A \sqcup B) \times [0,1] & \to & \mathrm{Cyl}(f) \subset (Y \sqcup \mathrm{Cyl}(f))/\sim \\ [(Q,t)] & \mapsto & \begin{cases} R([(Q,0)],t), & \text{if } Q \in A, \\ Q, & \text{if } Q \in B. \end{cases} \end{array}$$

We obtain the following diagram:



We make the following observations and clarifications:

(1) The map $i \sqcup j \colon A \sqcup B \to Y$ is the inclusion map.
(2) It follows from the Homotopy Extension Theorem 70.1 and the fact that $A \sqcup B$ is a subcomplex of Y that there exists a map $F \colon Y \times [0,1] \to (Y \sqcup \mathrm{Cyl}(f))/\sim$ that makes the two triangles involving F commute.
(3) For any $a \in A$ we have

$$F_1(a) = F(a,1) = G(a,1) = \underbrace{f(a)}_{\in B} = G(f(a),1) = F(f(a),1) = F_1(f(a)).$$

It follows from this calculation that the map $F_1 \colon Y \to (Y \sqcup \mathrm{Cyl}(f))/\sim$ descends to a map $\Psi \colon Y/a \sim f(a) \to (Y \sqcup \mathrm{Cyl}(f))/\sim$.

It remains to show the following claim.

Claim. The maps $\Phi \colon (Y \sqcup \mathrm{Cyl}(f))/\sim \to Y/a \sim f(a)$ and $\Psi \colon Y/a \sim f(a) \to (Y \sqcup \mathrm{Cyl}(f))/\sim$ are homotopy inverses of one another.

Proof. First we consider the map

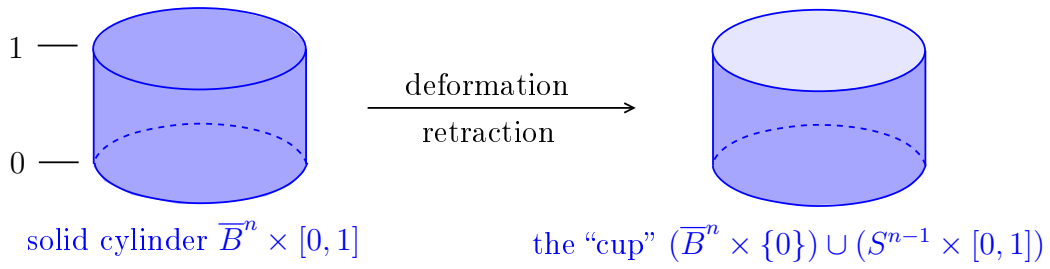
$$\begin{array}{rcl} H \colon (Y/a \sim f(a)) \times [0,1] & \to & Y/a \sim f(a) \\ (\underbrace{[P]}_{\in Y}, t) & \mapsto & \Phi(F(P,t)). \end{array}$$

One can easily verify that the map is, despite first appearances, actually well-defined. Using the fact that $Y/a \sim f(a)$ is a CW-complex and using the Interval CW-Product Corollary 68.9 one can show quite easily that H is continuous. Basically by construction we have $H_0 = \mathrm{id}$ and $H_1 = \Phi \circ \Psi$. Thus we have found our first homotopy. Finally we consider the map

$$\begin{array}{rcl} H \colon (Y \sqcup \mathrm{Cyl}(f))/\sim & \to & (Y \sqcup \mathrm{Cyl}(f))/\sim \\ ([P], t) & \mapsto & \begin{cases} R(P,t), & \text{if } P \in \mathrm{Cyl}(f), \\ F(P,t), & \text{if } P \in Y. \end{cases} \end{array}$$

Similar to the above one sees that H is continuous, that $H_0 = \mathrm{id}$ and that $H_1 = \Psi \circ \Phi$. ∎

70.2. **Proof of the Homotopy Extension Theorem 70.1.** Before we can provide the proof of Theorem 70.1 we will need to formulate and prove several lemmas.



Lemma 70.6. For any $n \in \mathbb{N}_0$ there exists a deformation retraction from the solid cylinder $\overline{B}^n \times [0,1]$ to the "cup" $\overline{B}^n \times \{0\} \cup S^{n-1} \times [0,1]$.

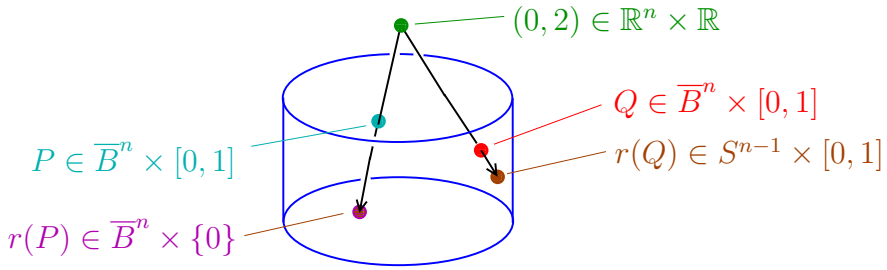
Proof. It is straightforward to verify that the map

$$\begin{array}{rcl} r\colon \overline{B}^n \times [0,1] & \to & (\overline{B}^n \times \{0\}) \cup (S^{n-1} \times [0,1]) \\ P & \mapsto & \text{the unique point on } \overline{B}^n \times \{0\} \cup S^{n-1} \times [0,1] \text{ that} \\ & & \text{gets hit by the ray from } (0,2) \in \mathbb{R}^n \times \mathbb{R} \text{ through } P. \end{array}$$

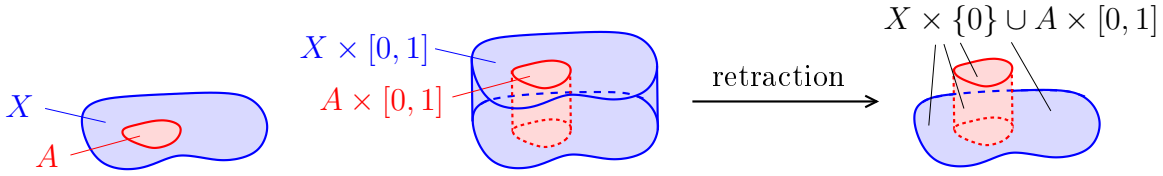
is a retraction. We refer to the figure below for an illustration. Since $\overline{B}^n \times [0,1]$ is convex we can define the corresponding deformation retraction as follows:

$$\begin{array}{rcl} (\overline{B}^n \times [0,1]) \times [0,1] & \to & \overline{B}^n \times [0,1] \\ (P,t) & \mapsto & P \cdot (1-t) + r(P) \cdot t. \end{array}$$

■



Lemma 70.7. If X is a CW-complex and if A is a subcomplex of X, then there exists a deformation retraction from $X \times [0,1]$ to the subcomplex $(X \times \{0\}) \cup (A \times [0,1])$.



Proof. Let X be a CW-complex and let A be a subcomplex. For each $n \in \mathbb{N}_0$ we set

$$Z_n \;:=\; X \times \{0\} \;\cup\; (X^{n-1} \cup A) \times [0,1].$$

Note that each Z_n is a subcomplex of the product CW-complex $X \times [0,1]$. Evidently we have a sequence

$$(X \times \{0\}) \cup (A \times [0,1]) = Z_0 \;\subset\; Z_1 \;\subset\; Z_2 \;\subset\; \ldots \qquad \text{with } X \times [0,1] \;=\; \bigcup_{n \in \mathbb{N}_0} Z_n.$$

Thus it follows from the Iterated Deformation Retracts Proposition 68.22 that it suffices to prove the following claim.

Claim. For every $n \in \mathbb{N}_0$ there exists a deformation retraction from Z_{n+1} to Z_n.

Proof. Let $n \in \mathbb{N}_0$. We denote by $\Phi_i \colon \overline{B}^n \to X$, $i \in I$, the characteristic maps of all the n-cells of X that do not lie in A. By Lemma 70.6 there exists a deformation retraction $r \colon \overline{B}^n \times [0,1] \times [0,1] \to \overline{B}^n \times [0,1]$ from $\overline{B}^n \times [0,1]$ to $\overline{B}^n \times \{0\} \cup S^{n-1} \times [0,1]$. The idea is to apply this retraction to all k-cells of X to obtain the desired deformation retraction. More precisely we consider the map

$$\begin{aligned} r_n \colon \overbrace{(X \times \{0\} \cup (X^n \cup A) \times [0,1])}^{=Z_{n+1}} \times [0,1] &\to \overbrace{X \times \{0\} \cup (X^n \cup A) \times [0,1]}^{=Z_{n+1}} \\ ((P,t),s) &\mapsto \begin{cases} \Phi_i(r((Q,t),s)), & \text{if } P = \Phi_i(Q) \text{ with} \\ & \quad Q \in \overline{B}^n \text{ and } i \in I, \\ (P,t), & \text{otherwise.} \end{cases} \end{aligned}$$

This map is easily seen to be a deformation retraction from Z_{n+1} to Z_n.[825] ∎

Proof of the Homotopy Extension Theorem 70.1. Let X be a CW-complex and let $A \subset X$ be a subcomplex. We need to show that the inclusion map $i \colon A \to X$ is a closed cofibration. Note that by the Subcomplex Lemma 68.18 (2) we know that A is a closed subset of X. It remains to show tht $i \colon A \to X$ is a cofibration.

Thus let Y be a topological space, let $f_0 \colon X \to Y$ be a map and let $G \colon A \times [0,1] \to Y$ be a homotopy from $g_0 := f_0|_A$ to some map $g_1 \colon A \to Y$. We need to show that we can extend G to a homotopy on all of $X \times [0,1]$.

Note that by Lemma 70.7 there exists a retraction

$$H \colon X \times [0,1] \to (X \times \{0\}) \cup (A \times [0,1]).$$

Therefore the theorem follows from the Cofibration Characterization Lemma 17.2 or, equivalently, from the following claim.

Claim. The map

$$\begin{aligned} F \colon X \times [0,1] \xrightarrow{H} (X \times \{0\}) \cup (A \times [0,1]) &\to Y \\ (x,0) &\mapsto f_0(x), \quad \text{for } x \in X, \\ (a,t) &\mapsto G(a,t), \quad \text{for } (a,t) \in A \times [0,1]. \end{aligned}$$

has all the desired properties.

Proof. First we have to verify that the map is continuous. In fact, for continuity we do have to worry a little bit about the right-hand map. But note that the Subcomplex Lemma 68.18 (2) says that the subcomplex A is a closed subset of X. Thus we see that the map on the right-hand side is continuous when restricted to the *closed* subsets $X \times \{0\}$ and $A \times [0,1]$. Thus it is continuous by the Pasting Proposition 2.6 (2). Note it is clear that F has the desired properties, i.e. that it is a homotopy $F \colon X \times [0,1] \to Y$ starting from f_0 such that the restriction $F|_{A \times [0,1]}$ equals G. ∎

[825] It follows from the Characteristic Maps Lemma 68.7 (4) that the map is in fact continuous.

70.3. **Dropping base points.** Even though base points are necessary and valuable, in practice they are frequently a major nuisance. Fortunately, using the Homotopy Extension Theorem 70.1, we can prove a proposition that often allows us to stop worrying about base points.

We start out with a few definitions. Some of them we have encountered before.

Definition.

(1) A pointed topological space is a pair (X, x_0) where X is a topological space and $x_0 \in X$ is a point. We refer to x_0 as the base point of the pointed topological space.
(2) A map $f\colon (X, x_0) \to (Y, y_0)$ between pointed topological spaces is a map $f\colon X \to Y$ with $f(x_0) = y_0$.
(3) We say that two maps $f, g\colon (X, x_0) \to (Y, y_0)$ between pointed topological spaces are homotopic if there exists a homotopy $F\colon X \times [0,1] \to Y$ rel $\{x_0\}$ from f to g.
(4) Given two pointed topological spaces (X, x_0) and (Y, y_0) we write[826]

$$\langle (X, x_0), (Y, y_0) \rangle := \{\text{maps } (X, x_0) \to (Y, y_0)\}/\sim$$

where $f \sim g$ if there exists a homotopy in the above sense between f and g.

Example. Let (X, x_0) be a pointed topological space, let $n \in \mathbb{N}$ and let $*$ be the usual base point of S^n. By definition we have $\langle (S^n, *), (X, x_0) \rangle = \pi_n(X, x_0)$.

We have a natural map $\langle (X, x_0), (Y, y_0) \rangle \to [X, Y]$. It is worth spending a minute on trying to find examples that show that in general this map is neither an injection nor a surjection. In this section we want to give some criteria that give us some control over this map. We start out with the following easy, albeit fairly restrictive, example.

Proposition 70.8. Let (Y, y_0) be a path-connected topological space. If $\pi_1(Y, y_0)$ is abelian, then the map

$$\begin{array}{rcl} \langle (S^1, *), (Y, y_0) \rangle & \to & [S^1, Y] \\ [f] & \mapsto & [f] \end{array}$$

is a bijection.

Proof. It follows from the fact that Y is path-connected together with the Homotopies-of-Loops Proposition 50.16 (1) that the map is a surjection. Furthermore it follows from the hypothesis that $\pi_1(Y, y_0)$ is abelian together with the Homotopies-of-Loops Proposition 50.16 (2) that the map is an injection. ∎

Now we want to move away from the fairly simple topological space S^1. In the proof of the Homotopies-of-Loops Proposition 50.16 we had to construct fairly explicitly maps on S^1 and $S^1 \times [0,1]$. In the following we will now replace S^1 by CW-complexes, we have now developed several tools for building maps *out* of a CW-complex. This leads us to the following definition.

Definition. A pointed CW-complex is a pair (X, x_0) where X is a CW-complex and x_0 is a point in the 0-skeleton X^0 of X.

[826] The notation $\langle X, Y \rangle$ goes back to [**Hat2002**, p. 357], where it says "the notation $\langle \dots \rangle$ is intended to suggest pointed homotopy classes".

Proposition 70.9. Let (X, x_0) be pointed a CW-complex and let (Y, y_0) be a pointed topological space. We consider the map

$$\begin{array}{rcl} \langle (X,x_0),(Y,y_0)\rangle & \to & [X,Y] \\ [f] & \mapsto & [f]. \end{array}$$

The following two statements hold:

(1) If Y is path-connected, then the map $\langle (X, x_0), (Y, y_0)\rangle \to [X, Y]$ is a surjection.
(2) If Y is simply connected, then the map $\langle (X, x_0), (Y, y_0)\rangle \to [X, Y]$ is a bijection.

Remark. In Proposition 164.25 we will formulate a variation on Proposition 70.9 for CW-complexes with abelian fundamental groups.

For the record we state the following corollary to our results.

Corollary 70.10.

(1) Let (X, x_0) be a path-connected pointed topological space and let $n \in \mathbb{N}$. If $n = 1$, then we assume that $\pi_1(X, x_0)$ is abelian and if $n \geq 2$ we assume that $\pi_1(X, x_0)$ is trivial. In this setting the map

$$\underbrace{\langle (S^n, x_0), (X, x_0)\rangle}_{=\pi_n(X,x_0)} \xrightarrow{[f]\mapsto[f]} [S^n, X]$$

is a bijection.
(2) Given any $n \in \mathbb{N}$ and given any base point $x_0 \in S^n$ the map

$$\langle (S^n, x_0), (S^n, x_0)\rangle \xrightarrow{[f]\mapsto[f]} [S^n, S^n]$$

is a bijection.

Proof of Corollary 70.10.

(1) (a) The case $n = 1$ is precisely the content of Proposition 70.8.
(b) Now let $n \in \mathbb{N}_{\geq 2}$. We can and will equip S^n with a CW-structure such that x_0 is a 0-cell. The desired statement now follows from Proposition 70.9 (2).
(2) This statement follows immediately from (1) and the fact, established in Corollary 48.18 and the Sphere-π_1-Proposition 47.13, that for $n \geq 2$ the fundamental group $\pi_1(S^n)$ is trivial. ∎

Now we turn to the proof of Proposition 70.9. This proof relies very much on the following lemma.

Lemma 70.11. Let (X, x_0) and (Y, y_0) be pointed CW-complexes and let $F\colon X \times [0, 1] \to Y$ be a homotopy with $F_0(x_0) = F_1(x_0) = y_0$. We consider the loop $\gamma\colon [0, 1] \to Y$ that is given by $\gamma(t) := F(x_0, t)$. If the loop γ is null-homotopic, then there exists a homotopy $J\colon X \times [0, 1] \to Y$ with $J_0 = F_0$ and $J_1 = F_1$ and such that $J(x_0, t) = y_0$ for all $t \in [0, 1]$.

Proof of Lemma 70.11. We prove the lemma in three easily digestible steps:

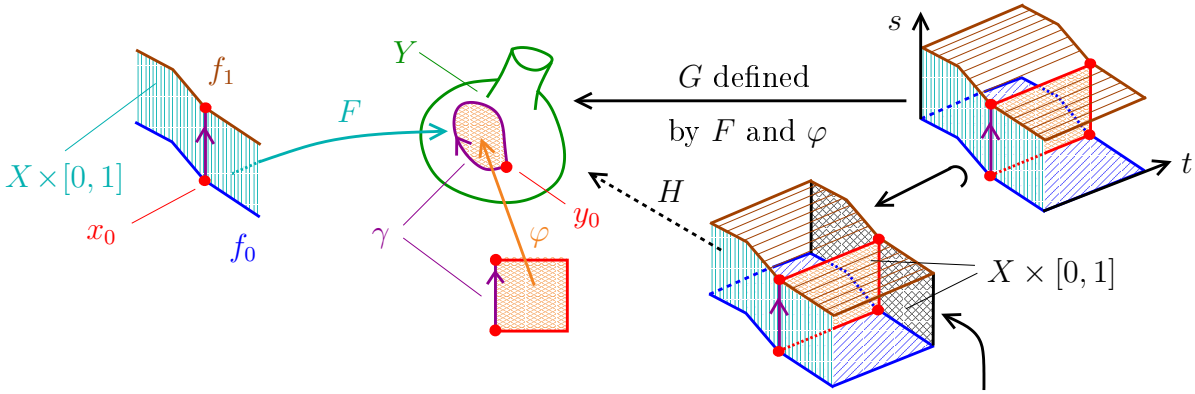
(1) We consider the loop $\gamma\colon [0, 1] \to Y$ that is given by $\gamma(t) := F(x_0, t)$. By our hypothesis the loop γ is null-homotopic. By definition this means that there exists a map $\varphi\colon [0, 1] \times [0, 1] \to Y$ with $\varphi(t, 0) = \gamma(t)$ for all $t \in [0, 1]$ and such that $\varphi(x, y) = y_0$ for all other points in $\partial([0, 1] \times [0, 1])$.

(2) We consider the map[827]

$$G\colon \overbrace{(X\times[0,1])\times\{0\}\cup\underbrace{(X\times\{0,1\}\cup\{x_0\}\times[0,1])\times[0,1]}_{\text{subcomplex of } X\times[0,1]}}^{\text{subcomplex of } (X\times[0,1])\times[0,1]} \to Y$$
$$(x,s,t)\mapsto\begin{cases}F(x,s), & \text{if } t=0,\\ F(x,s), & \text{if } s=0,1,\\ \varphi(s,t), & \text{if } x=x_0.\end{cases}$$

It follows from the CW-Complex Pasting Lemma 68.20 that this map G is continuous.

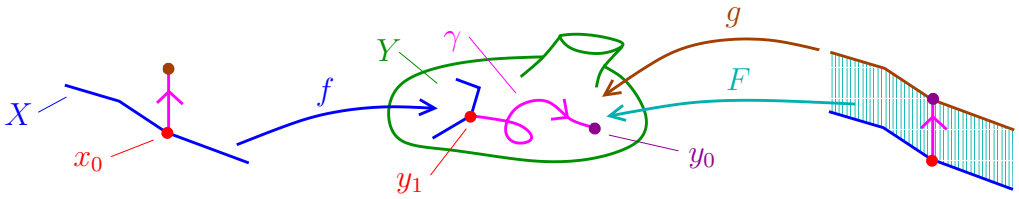
(3) By the Homotopy Extension Theorem 70.1 the map G from (2) extends to a map $H\colon (X\times[0,1])\times[0,1]\to Y$. It follows basically immediately from the definitions that $J:=H_1\colon X\times[0,1]\to Y$ has the desired properties. ■



Now we turn to the actual proof of Proposition 70.9.

Proof of Proposition 70.9. Let (X,x_0) and (Y,y_0) be pointed CW-complexes.

(1) We suppose that Y is path-connected. Let $f\colon X\to Y$ be map. We set $y_1=f(x_0)$. Since Y is path-connected there exists a path $\gamma\colon[0,1]\to Y$ with $\gamma(0)=y_1$ and $\gamma(1)=y_0$. Since $\{x_0\}$ is a subcomplex of X we obtain from Corollary 70.2 that we can extend γ, viewed as a homotopy $\{x_0\}\times[0,1]\to Y$, to a homotopy $F\colon X\times[0,1]\to Y$. This shows that $f=F_0$ is homotopic to a map $g:=F_1$ with $g(x_0)=y_0$. This implies that the map $\langle(X,x_0),(Y,y_0)\rangle\to[X,Y]$ is a surjection.



(2) By (1) it remains to show that the map $\langle(X,x_0),(Y,y_0)\rangle\to[X,Y]$ is injective. Thus let $f_0,f_1\colon(X,x_0)\to(Y,y_0)$ be two maps which represent the same element in $[X,Y]$. This means that there exists a homotopy $F\colon X\times[0,1]\to Y$ with $F_0=f_0$ and $F_1=f_1$. We need to show that there exists a homotopy $J\colon X\times[0,1]\to Y$ with

[827] Secretly we use Proposition 68.23 which tells us in this instance that the product CW-structure $X\times[0,1]\times[0,1]$ is indeed a CW-structure for the topological space $X\times[0,1]\times[0,1]$.

$J_0 = f_0$ and $J_1 = f_1$ and such that $J(x_0, t) = y_0$ for all $t \in [0,1]$. We consider the loop $\gamma\colon [0,1] \to Y$ that is given by $\gamma(t) := F(x_0, t)$. Since Y is simply connected we know that γ is null-homotopic. Thus we can apply Lemma 70.11 and we obtain the desired homotopy J. ■

We conclude this section with the following proposition.

Proposition 70.12. Let X be a CW-complex and let x_0 and x_1 be two points in the 0-skeleton of X. If X is connected, then there exists a map $f\colon X \to X$ with the following properties:

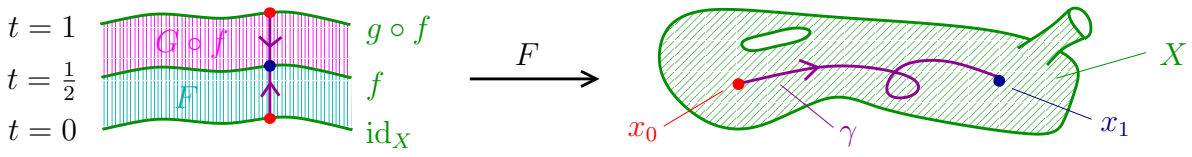
(1) The map f is homotopic to the identity.
(2) We have $f(x_0) = x_1$.
(3) The map f defines a homotopy equivalence $(X, x_0) \to (X, x_1)$ of pointed topological spaces.

Proof. Since X is connected we obtain from the CW-Complex Properties Proposition 68.10 (7) that there exists a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Since x_0, x_1 lie in the 0-skeleton of X we can apply Corollary 70.2 and we obtain a homotopy $F\colon X \times [0,1] \to X$ with $F(x_0, t) = \gamma(t)$ for all $t \in [0,1]$ and such that $F_0 = \mathrm{id}$. We set $f := F_1$. Evidently f is homotopic to the identity and $f(x_0) = x_1$. It remains to show that $f\colon (X, x_0) \to (X, x_1)$ is a homotopy equivalence of pointed topological spaces.

We need to find a homotopy inverse to f. Once again we appeal to Corollary 70.2 to obtain a homotopy $G\colon X \times [0,1] \to X$ with $G(x_1, t) = \overline{\gamma}(t) := \gamma(1-t)$ for all $t \in [0,1]$ and such that $G_0 = \mathrm{id}$. We set $g := G_1$. We need to show that $g \circ f\colon (X, x_0) \to (X, x_0)$ and $f \circ g\colon (X, x_1) \to (X, x_1)$ are homotopic[828] to the identity. We consider the homotopy[829]

$$\begin{array}{rcl} F\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} F(x, 2t), & \text{if } t \in [0, \frac{1}{2}], \\ G(f(x), 2t-1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

Note that this is a homotopy from id_X to $g \circ f\colon X \to X$. Furthermore note that the loop given by $\gamma(t) = F(x_0, t)$ is precisely the loop $\gamma * \overline{\gamma}$. By the Path-Homotopy Concatenation Proposition 47.5 (3) we know that this loop is null-homotopic. Thus we can apply Lemma 70.11. We obtain a homotopy $J\colon X \times [0,1] \to X$ with $J_0 = F_0 = \mathrm{id}_X$, $J_1 = F_1 = g \circ f$ and $J(x_0, t) = x_0$ for all $t \in [0,1]$. But that is precisely what we needed. ■



70.4. **Combining infinitely many homotopies and the telescope construction.** In the Homotopy Combination the Homotopy Combination Lemma 14.4 we saw that one can combine finitely many homotopies. Later on we will want to combine infinitely many homotopies of CW-complexes. This slightly delicate procedure can be performed using the following lemma.

[828] Here, as a reminder, "homotopic" means "homotopic as maps of pointed topological spaces".
[829] It follows from the Homotopy Stacking Lemma 14.3 that F is indeed continuous.

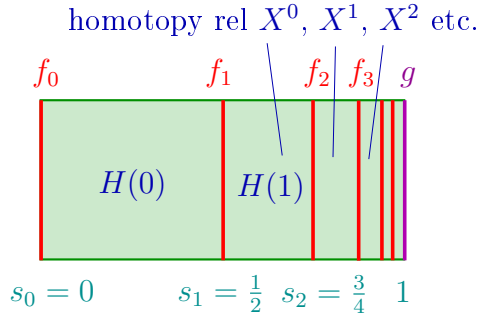
Lemma 70.13. Let X be a CW-complex, let A be a subset of X and let Y be a topological space. Finally let $W_0 \subset W_1 \subset W_2 \subset \ldots$ be a sequence of subcomplexes of X such that for each $k \in \mathbb{N}_0$ there exists an $l \in \mathbb{N}_0$ such that the k-skeleton X^k is contained in W_l. (In many applications we have $W_k = X^k$.) Let $f_0, f_1, \ldots$ be maps from X to Y. Suppose that for each $k \in \mathbb{N}_0$ we are given a homotopy $H(k)\colon X \times [0,1] \to Y$ rel $A \cup W_k$ from f_k to f_{k+1}. (Since it is easy to overlook this bit of the hypothesis we stress that $H(k)$ is in particular a homotopy rel W_k). We define[830]

$$\begin{array}{rcl} g\colon X & \to & Y \\ x & \mapsto & f_k(x) \quad \text{where } k \in \mathbb{N}_0 \text{ such that } x \in W_k. \end{array}$$

For $k = 0, 1, 2, \ldots$ we write $s_k = \sum_{j=1}^{k} \frac{1}{2^j}$. Then the map

$$\begin{array}{rcl} H\colon X \times [0,1] & \to & Y \\ (x,t) & \mapsto & \begin{cases} H(k)(x, 2^{k+1} \cdot (t - s_k)), & \text{if } t \in [s_k, s_{k+1}), \\ g(x), & \text{if } t = 1 \end{cases} \end{array}$$

that is illustrated in the figure below is a homotopy rel A from f_0 to g. We refer to the homotopy H as the *combination of* $\{H(k)\}_{k \in \mathbb{N}_0}$.



Proof. We only need to show that H is continuous. Note that for each $l \in \mathbb{N}_0$ it follows from our choice of maps f_l and the Homotopy Stacking Lemma 14.3 that the restriction of H to $W_l \times [0,1]$ is continuous. Since each X^k is contained in some W_l we see that the restriction of H to each $X^k \times [0,1]$ is continuous. But now it follows from the Interval CW-Product Corollary 68.9 that $H\colon X \times [0,1] \to Y$ is also continuous. ■

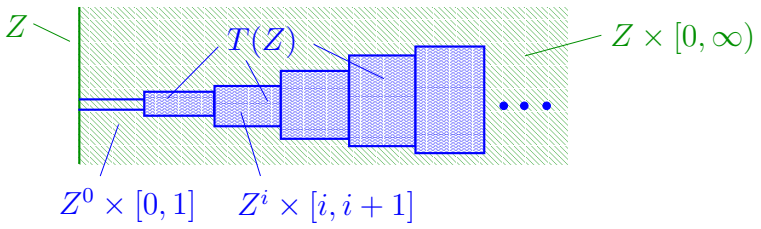
We conclude this technical section with an application that we will need once, namely in the proof Proposition **??**. But the application doubles also as a nice illustration for how to apply the previous lemma.

Definition. Let Z be a CW-complex. We equip $[0,\infty)$ with the CW-structure where the 0-skeleton is given by $\mathbb{N}_0$. We equip $Z \times [0,\infty)$ with the product CW-structure. We refer to the subcomplex

$$T(Z) \;:=\; \bigcup_{i \in \mathbb{N}_0} Z^i \times [i, i+1] \quad \subset \; Z \times [0,\infty)$$

as the telescope of Z. .

[830] It follows easily from the setup that the map g is well-defined, i.e. given $x \in X$ the point $g(x)$ does not depend on the choice of k.



The following lemma shows that the telescope $T(Z)$ is homotopy equivalent to Z.

Lemma 70.14. (Telescope Lemma) Let Z be a CW-complex. The following statements hold:

(1) $T(Z)$ is a deformation retract of $Z \times [0, \infty)$.
(2) The telescope $T(Z)$ is homotopy equivalent to Z.

Remark. One can also show that the Telescope Lemma 70.14 is a consequence of the much more general the Whitehead Deformation Retract Proposition 162.2.

Proof. First note that $Z = Z \times \{0\}$ is clearly a deformation retract of $Z \times [0, \infty)$. Thus it follows from Lemmas 15.5 and 15.1 that (2) is a consequence of (1). Therefore it remains to prove (1).

As in the definition of the telescope $T(Z)$ we view $[0, \infty)$, $X := Y := Z \times [0, \infty)$ and $T(Z)$ as CW-complexes. It follows from Lemma 70.7 that for each $k \in \mathbb{N}_0$ there exists a deformation retraction

$$r_k \colon (Z \times [0,1]) \times [0,1] \;\to\; Z \times [0,1]$$

from $Z \times [0,1]$ to $(Z^k \times [0,1]) \cup (Z \times \{1\})$. We denote by

$$\begin{array}{rcl} \rho_k \colon Z \times [k, k+1] & \to & (Z^{k-1} \times [k, k+1]) \cup (Z^k \times \{k+1\}) \\ (z,t) & \mapsto & (r_k(z, t-k) + (0,k), 1) \end{array}$$

the corresponding retraction, with the intervals shifted by k. We make the following preparations:

(1) Given $k \in \mathbb{N}_0$ we define $W_k := Z^k \times [0, \infty)$. Note that it follows from the definition of the product CW-structure on $Z \times [0, \infty)$ that the k-skeleton of $X = Z \times [0, \infty)$ is contained in W_{k+1}.
(2) For each $k \in \mathbb{N}_0$ we consider the map

$$\begin{array}{rcl} f_k \colon X = Z \times [0,\infty) & \to & Y = Z \times [0,\infty) \\ (z,t) & \mapsto & \begin{cases} (z,t), & \text{if } x \in Z^k \text{ and } t \geq k+1, \\ (\rho_k \circ \cdots \circ \rho_1 \circ \rho_0)(x), & \text{if } z \in Z^k \text{ and } t < k+1. \end{cases} \end{array}$$

(3) We consider the map

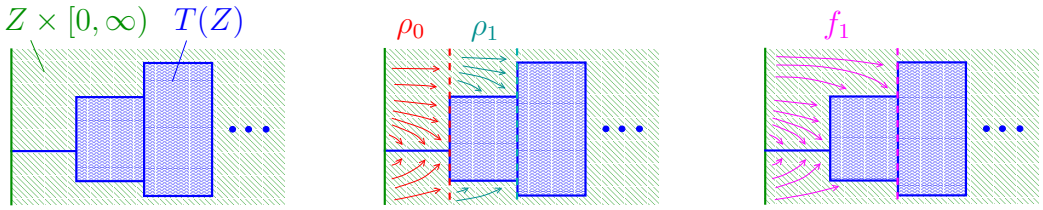
$$\begin{array}{rcl} H(k) \colon X \times [0,1] & \to & Y = Z \times [0,\infty) \\ ((z,t),s) & \to & \begin{cases} (z,t), & \text{if } t \geq k+1, \\ r_{k+1}(\rho_k(z,t),s), & \text{if } t < k+1. \end{cases} \end{array}$$

This defines a homotopy from f_k to f_{k+1} rel W_k. Note that $H(0)_1 = \mathrm{id}_X$.

As in the statement of Lemma 70.13 we consider the map

$$\begin{array}{rcl} g\colon X & \to & Y \\ x & \mapsto & f_k(x) \quad \text{where } k \in \mathbb{N}_0 \text{ such that } x \in W_k. \end{array}$$

One can easily verify that this is a retraction of $X = Z \times [0,\infty)$ to the telescope $T(Z)$. It follows from Lemma 70.13 that there exists a homotopy from $H(0)_1 = \mathrm{id}$ to g, i.e. $T(Z)$ is in fact a deformation retract of $X = Z \times [0,\infty)$. ■



70.5. **The Cellular Approximation Theorem.** We recall the following definition from page 1492.

Definition. A map $g\colon X \to Y$ between CW-complexes is called cellular if for each $n \in \mathbb{N}_0$ we have $g(X^n) \subset Y^n$

It will be instructive to return to the example of a non-cellular map that we gave on page 1492.

Example. In general maps between CW-complexes are not cellular. For example consider the CW-complex $X = [0,1]$ with two 0-cells and one 1-cell and we consider the CW-complex $Y = [0,1] \times [0,1]$ with four 0-cells, two 1-cells and one 2-cell. Then the diagonal map

$$\begin{array}{rcl} g\colon X = [0,1] & \to & Y = [0,1] \times [0,1] \\ t & \mapsto & (t,t) \end{array},$$

which is illustrated in the figure below on the left, is not a cellular map.

The following theorem is the main result of this section.

Theorem 70.15. (Cellular Approximation Theorem) Let $g\colon X \to Y$ be a map between CW-complexes and let A be a subcomplex of X such that the restriction of g to A is a cellular map.[831] The map g is homotopic rel A to a cellular map $\widetilde{g}\colon X \to Y$.

Remark. Let $g\colon X \to Y$ be a *homeomorphism* between CW-complexes. By the Cellular Approximation Theorem 70.15 we know that g is homotopic to a cellular map h. But in general we cannot arrange that h is again a *homeomorphism*. For example consider the torus $T = ([0,1] \times [0,1])/\sim\, = \mathbb{R}^2/\mathbb{R}^2$ with the standard CW-structure from page 1471 which has precisely one 0-cell $*$ that corresponds to the origin. As in the Fundamental Group-via-Actions Theorem 48.16 we make the identification $\pi_1(\mathbb{R}^n/\mathbb{Z}^2, *)$ with $\mathbb{Z}^2$. Let $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$. We consider the map

$$\begin{array}{rcl} f\colon \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ v & \mapsto & P \cdot v. \end{array}$$

[831] For example $f\colon X \to Y$ could be any map between CW-complexes, then the hypothesis is satisfied for A the empty set.

In the Maps-on-Torus Lemma 19.37 we saw that this map is a homeomorphism. Furthermore, in Exercise 48.18 we saw that the induced map on $\pi_1(\mathbb{R}^2/\mathbb{Z}^2, *) = \mathbb{Z}^2$ is given by multiplication by P. On the other hand the map on $\pi_1(\mathbb{R}^2/\mathbb{Z}^2, *) = \mathbb{Z}^2$ induced by any *cellular homeomorphism* $\widetilde{g}$ on $\mathbb{R}^2/\mathbb{Z}^2$ is represented by a matrix of the form[832] $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$. It follows from Exercise 50.12 (a) that g cannot be homotopic, rel $*$, to any such $\widetilde{g}$.

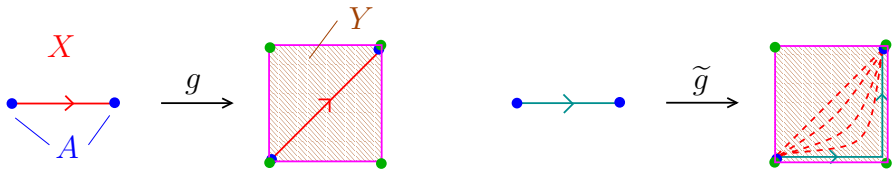
Example. We consider again the above example of a non-cellular map. More precisely, we consider again the above diagonal map

$$\begin{array}{rcl} g\colon X = [0,1] & \to & Y = [0,1] \times [0,1] \\ t & \mapsto & (t,t). \end{array}$$

We want to deform g into a cellular map, while keeping the endpoints fixed. Put differently, we want to find a homotopy rel $A = \{(0,0),(1,1)\}$ from g to a cellular map $\widetilde{g}$. Such a map $\widetilde{g}$ is given by

$$\begin{array}{rcl} \widetilde{g}\colon X = [0,1] & \to & Y = [0,1] \times [0,1] \\ t & \mapsto & \begin{cases} (2t,0), & \text{if } t \in [0,\frac{1}{2}] \\ (1,2t-1), & \text{if } t \in [\frac{1}{2},1] \end{cases} \end{array}$$

One way of stating this example is that we turn g into a cellular map $\widetilde{g}$ by "pushing g off the



g is not a cellular map $\widetilde{g}$ is a cellular map

high-dimensional cell". The proof of Cellular Approximation Theorem 70.15 is built on the same concept. But somewhat similar to the proof of the Sphere-π_1-Proposition 47.13, we have to worry about space-filling curves that exist by the Space-Filling Curve Theorem 6.13.

Before we attack the proof of the Cellular Approximation Theorem 70.15 we want to formulate a relative version thereof. To formulate this relative version we need the following semi-obvious definition.

Definition. A pair of CW-complexes is a pair (X, A) where X is a CW-complex and A is a subcomplex of X.

Now we can formulate the promised theorem.

Theorem 70.16. (Cellular Approximation Theorem for Pairs) Suppose we are given a map $g\colon (X, A) \to (Y, B)$ between pairs of CW-complexes. Let Z be a subcomplex of A such that the restriction of g to Z is a cellular map. Then g is homotopic rel Z to a cellular map $\widetilde{g}\colon (X, A) \to (Y, B)$.[833]

[832] Why is that?

[833] Here we refer to page 440 for the discussion on the notion of homotopies of pairs of topological spaces rel a subset.

Proof assuming the Cellular Approximation Theorem 70.15. The desired homotopy is constructed as follows:

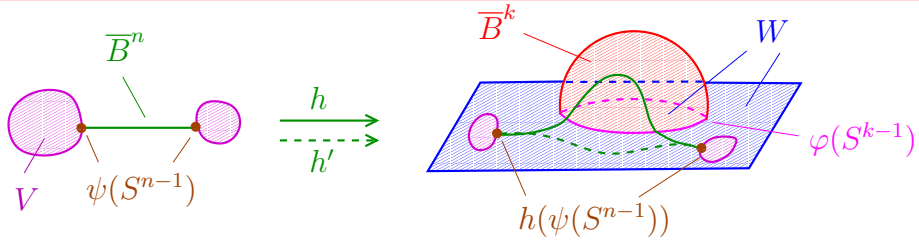
(1) By the Cellular Approximation Theorem 70.15 applied to $g|_A\colon A \to B$ there exists a homotopy F rel Z from $g|_A\colon A \to B$ to a cellular map.
(2) By the Homotopy Extension Theorem 70.1 we can extend the homotopy F from (1) to a homotopy $G\colon X \times [0,1] \to Y$. Note that G is still a homotopy rel Z. We write $h = g_1\colon X \to Y$. Note that $h|_A$ is cellular and that $h(A) \subset B$.
(3) We apply the Cellular Approximation Theorem 70.15 to h and we obtain a homotopy H rel A from h to a cellular map.
(4) The combination of the homotopies G and H has all the desired properties. ■

Now we turn to the proof of Cellular Approximation Theorem 70.15. The following proposition is the key step towards the desired proof.

Proposition 70.17. Let V be a CW-complex and W be a topological space. Furthermore let $\psi\colon S^{n-1} \to V^{n-1}$ be a map to the $(n-1)$-skeleton of V and let $\varphi\colon S^{k-1} \to W$ be another map. Finally let

$$h\colon V \cup_\psi \overline{B}^n \;\to\; W \cup_\varphi \overline{B}^k$$

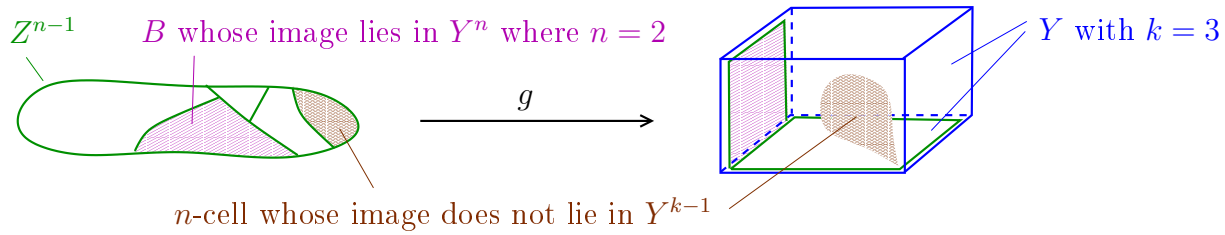
be a map with $h(\psi(S^{n-1})) \subset W$. If $n < k$, then h is homotopic rel V to a map h' such that $h'(B^n)$ does not intersect B^k.



The statement of Proposition 70.17 is fairly obvious, if there exists a point $P \in B^k$ that does not get hit by $h(B^n)$. Indeed, if such a point P exists, then we can use the deformation retraction $\overline{B}^n \setminus \{p\} \to S^{n-1}$ to push h on B^n off the ball B^k. Put in general such a point P does not exist, since even for $n < k$ there exist continuous surjective maps $B^n \to B^k$. To get around this problem requires a fairly hard, not particularly inspiring, argument which we will give later on in Section 70.6.
In the following we will use Proposition 70.17 to prove the Cellular Approximation Theorem 70.15.

Corollary 70.18. Let Z be a finite n-dimensional CW-complex.

(1) Let $g\colon Z \to Y$ be a map to a k-dimensional CW-complex Y with $g(Z^{n-1}) \subset Y^{n-1}$. Let B be a subcomplex of Z with $g(B) \subset Y^n$. If $n < k$, then g is homotopic rel $Z^{n-1} \cup B$ to a map g' with $g'(Z) \subset Y^{k-1}$.
(2) Let $g\colon Z \to Y$ be a map to a CW-complex Y such that $g(Z^{n-1}) \subset Y^{n-1}$. Let B be a subcomplex of Z with $g(B) \subset Y^n$. The map g is homotopic rel $Z^{n-1} \cup B$ to a map g' with $g'(Z) \subset Y^n$.



Proof. Let Z be a finite n-dimensional CW-complex. By the CW-Complex Properties Proposition 68.10 (3) the CW-complex Z is compact. It follows from the CW-Complex-Finiteness Theorem 68.14 that the image of the compact set Z in a CW-complex is contained in a finite subcomplex. Therefore we can assume in both (1) and (2) that Y is in fact a finite CW-complex.

(1) Let $g\colon Z \to Y$ be a map to a finite k-dimensional CW-complex Y with $g(Z^{n-1}) \subset Y^{n-1}$. Let B be a subcomplex of Z with $g(B) \subset Y^n$. We suppose that $n < k$. We denote by $\widehat{Z}$ the union of Z^{n-1} with all the n-cells of Z whose image is contained in Y^{k-1}. In other words we can write $Z = \widehat{Z} \cup e_1 \cup \cdots \cup e_r$ where $e_1, \ldots, e_r$ are the n-cells such that the image is not contained in Y^{k-1}. We denote by $f_1, \ldots, f_s$ the k-cells of Y. Note that by hypothesis on B the interiors of the cells $e_1, \ldots, e_r$ are disjoint from B.

In the following we apply Proposition 70.17 precisely $r \cdot s$ times to get the desired homotopy. More precisely, we do the following:

(a) We apply Proposition 70.17 to the map

$$h\colon (\widehat{Z} \cup e_1 \cup \cdots \cup e_{r-1}) \cup e_r \;\to\; (Y \cup f_1 \cup \cdots \cup f_{s-1}) \cup f_s$$

to obtain a homotopy rel $\widehat{Z} \supset Z^{n-1} \cup B$ to a map h_{rs} such that the image of the interior of e_r under the map h_{rs} does not intersect the interior of f_s. Next we apply Proposition 70.17 to the map

$$h_{rs}\colon (\widehat{Z} \cup e_1 \cup \cdots \cup e_{r-2} \cup e_r) \cup e_{r-1} \;\to\; (Y \cup f_1 \cup \cdots \cup f_{s-1}) \cup f_s.$$

Iterating this process we obtain a homotopy rel $\widehat{Z}$ to a map such that the image of the interiors of the cells $e_1, \ldots, e_r$ does not intersect the interior of f_s.

(b) Again iterating the process above we obtain a homotopy rel $\widehat{Z}$ to a map such that image of the interiors of the cells $e_1, \ldots, e_r$ does not intersect the interior of any of the f_i. But that means that the image of Z lies in Y^{k-1}.

(2) Let $g\colon Z \to Y$ be a map to a CW-complex Y such that $g(Z^{n-1}) \subset Y^{n-1}$ and let B be a subcomplex of Z with $g(B) \subset Y^n$. As we pointed out in the beginning of the proof, we can assume that Y is finite-dimensional, we denote the dimension by k. But then Statement (2) is just an immediate consequence of applying (1) iteratively $k - n$ times. ∎

Now we turn to the proof of the Cellular Approximation Theorem 70.15.

Proof. Let X be a CW-complex and let A be a (possibly empty) subcomplex of X. Furthermore let Y be a CW-complex. We prove the Cellular Approximation Theorem 70.15 only in the special case that X has finitely many cells. The general case is dealt with in [**Hat2002**, Theorem 4.8].

Let $g\colon X \to Y$ be map.

(1) We say an n-cell e of X is *g-bad* if $g(e)$ is not contained in Y^n.
(2) If g is non-cellular, then we define

$$n(g) \;:=\; \text{minimal dimension of a cell that is } g\text{-bad.}$$

If g is cellular, then we define $n(g) := \infty$.

Claim. Let $g\colon X \to Y$ be a non-cellular map such that the restriction of g to A is a cellular map. Then there exists a homotopy from g to a map g' rel A such that $n(g') > n(g)$.

Proof. We write $n = n(g)$. We apply Corollary 70.18 (2) to $g\colon Z := X^n \to Y$ and $B = A^n$.[834] We obtain a homotopy G rel $X^{n-1} \cup A^n$ from $g\colon Z = X^n \to Y$ to a map $g'\colon Z = X^n \to Y$ with $g'(X^n) \subset Y^n$. We can extend this homotopy to a homotopy G on $(X^n \cup A) \times [0,1]$ via the trivial homotopy on $A \times [0,1]$. Now we apply the Homotopy Extension Theorem 70.1 to obtain a homotopy rel $X^n \cup A$ from $g\colon X \to Y$ to a map $g'\colon X \to Y$ such that $g'(X^n) \subset Y^n$. Note that by the choice of our homotopy we have $g'|_{X^{n-1}} = g|_{X^{n-1}}$. By definition of $n(g)$ the map $g'|_{X^{n-1}} = g|_{X^{n-1}}$ is cellular. Thus we see that $n(g') \geq n+1 = n(g)+1$. ⊞

Finally let $g\colon X \to Y$ be a non-cellular map such that the restriction of g to A is a cellular map. We apply the claim iteratively as many times as possible to obtain a sequence of maps $g = g_1, g_2, g_3, \ldots : X \to Y$, each of which is homotopic to g rel A, such that $n(g) = n(g_1) < n(g_2) < \ldots$. Note that n can take on only finitely many values, namely $\{0, \ldots, \dim(X), \infty\}$. This means that the sequence is finite which implies that the last map is cellular. ■

70.6. **Proof of Proposition 70.17 (∗).** [835] Before we can even think of proving Proposition 70.17 we need to introduce a few more definitions and we need to prove a rather technical lemma.

Definition.

(1) A closed half-space in $\mathbb{R}^n$ is a set of the form $\{v \in \mathbb{R}^n \mid \langle v, w\rangle \geq 0\}$, for some fixed non-zero vector $w \in \mathbb{R}^n$.
(2) A convex polyhedron in $\mathbb{R}^n$ is a bounded subset of $\mathbb{R}^n$ that is the intersection of finitely many closed half-spaces.
(3) A polyhedron in $\mathbb{R}^n$ is the union of a finite number of convex polyhedra.[836]
(4) Given a polyhedron P a map $f\colon P \to \mathbb{R}^k$ is called piecewise linear if there exists a subdivision of P into finitely many convex polyhedra $Q_1, \ldots, Q_m$ such that the restriction of f to each Q_i is affine linear.[837]

The following, very technical lemma, will get us around the problem of space-filling curves, i.e. it will get us around the problem that we saw in Space-Filling Curve Theorem 6.13 that there exist surjective continuous maps $\overline{B}^n \to \overline{B}^k$ even if $n < k$.

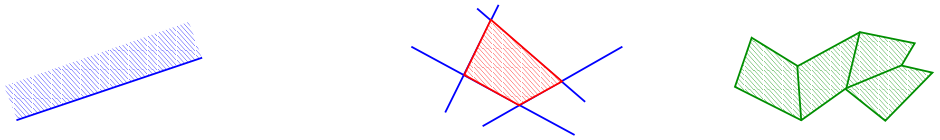
---

[834] Note that by definition of n we have $g(X^{n-1}) \subset Y^{n-1}$.

[835] Here the (∗) indicates once again that we do not provide the proof in the lecture. You are reading this section at your own risk.

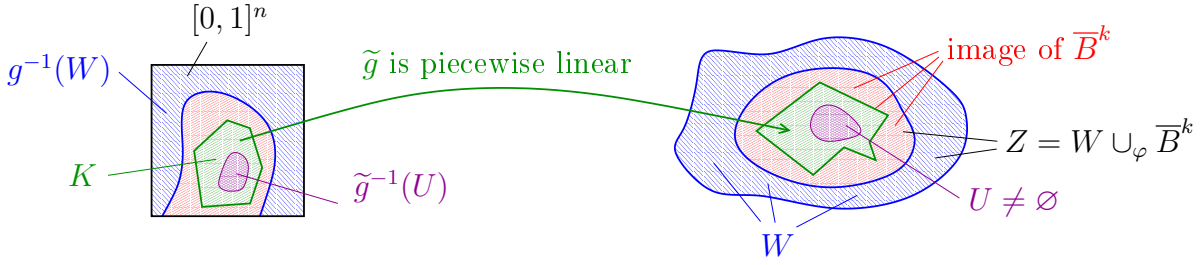
[836] Note that the union of zero convex polyhedra is the empty set.

[837] A map $f\colon V \to \mathbb{R}^k$ from a subset V of $\mathbb{R}^n$ is called *affine linear* if there exists a $P \in \mathbb{R}^k$ and a $k \times n$–matrix A such that $f(v) = P + Av$ for all $v \in V$.



Lemma 70.19. Let W be a topological space and let $\varphi\colon S^{k-1} \to W$ be map. Furthermore let $g\colon [0,1]^n \to Z := W \cup_\varphi \overline{B}^k$ be a map. Then there exists a (possibly empty) polyhedron $K \subset [0,1]^n$ and a map $\widetilde{g}\colon [0,1]^n \to Z$ with the following properties:

(1) The map $\widetilde{g}$ is homotopic to g rel $g^{-1}(W)$.
(2) We have $\widetilde{g}(K) \subset B^k$.
(3) The restriction $\widetilde{g}|_K\colon K \to B^k$ is piecewise linear.
(4) There exists an open non-empty subset $U \subset B^k$ such that $\widetilde{g}^{-1}(U) \subset K$.



Example. Suppose that in the setting of Lemma 70.19 there exists an open subset $U \subset B^k$ such that $g([0,1]^n) \cap U = \varnothing$. Then the empty polyhedron $K = \varnothing$ and the map $\widetilde{g} = g$ already have the desired property.

Proof. Let W be a topological space and let $\varphi\colon S^{k-1} \to W$ be map. Furthermore let $g\colon [0,1]^n \to Z := W \cup_\varphi \overline{B}^k$ be a map. We write $D_1 := \overline{B}_{\frac{1}{4}}(0) \subset B^k$ and $D_2 := \overline{B}_{\frac{1}{2}}(0) \subset B^k$. The set $g^{-1}(D_2) \subset [0,1]^n$ is closed and therefore compact by the Heine-Borel Theorem 1.24. We obtain from Proposition 6.7 that g is uniformly continuous on $g^{-1}(D_2)$.[838] This implies that we can find an $\epsilon > 0$ with the following two properties:

(i) For any $x, y \in g^{-1}(D_2)$ with $\|x - y\| < \epsilon$ we have $\|g(x) - g(y)\| < \frac{1}{8}$.
(ii) We have[839]

$$\epsilon \;<\; \text{distance between } g^{-1}(D_1) \text{ and } [0,1]^n \setminus g^{-1}\big(\mathring{D}_2\big).$$

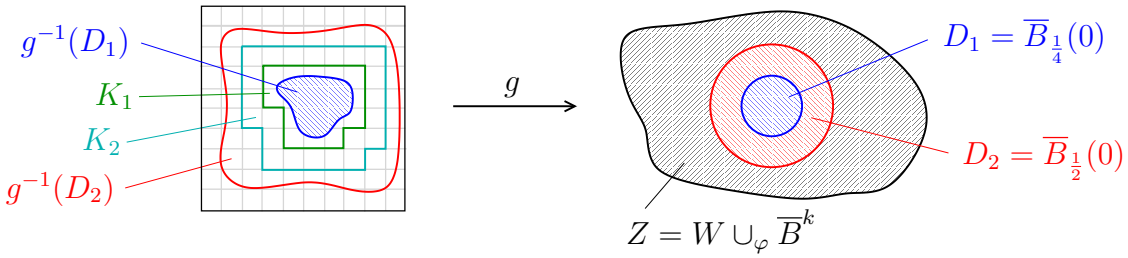
We divide $[0,1]^n$ into cubes such that the diameter of each cube is less than ϵ. We denote by K_1 the union of all the cubes that meet $g^{-1}(D_1)$ and we denote by K_2 the union of all the cubes meeting K_1. By the second property of ϵ we have $K_2 \subset g^{-1}(D_2)$. We refer to the figure below for an illustration.

[838] Why can't we conclude that g is uniformly continuous on all of $[0,1]^n$?

[839] Here recall that, according to the definition on page 283, for two non-empty subsets A and B of $\mathbb{R}^n$ we refer to

$$d(A,B) \;:=\; \inf\{\|P - Q\| \,|\, P \in A \text{ and } Q \in B\}$$

as the distance between A and B. If A and B are compact and disjoint, then it follows from Lemma 6.3 that $d(A,B) > 0$.



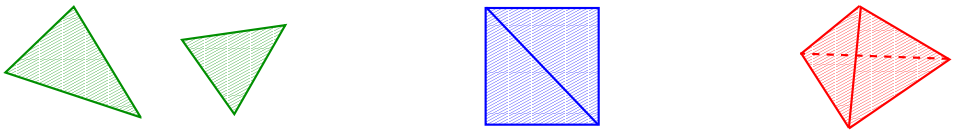
We need one more definition: Suppose we are given points $P_1, \ldots, P_{n+1} \in \mathbb{R}^n$ such that the vectors $P_2 - P_1, \ldots, P_{n+1} - P_1$ are linearly independent. We refer to

$$\text{convex hull of } P_1, \ldots, P_{n+1} \;=\; \Big\{ \sum_{i=1}^{n+1} t_i \cdot P_i \;\Big|\; \sum_{i=1}^{n+1} t_i = 1 \Big\}$$

as a *simplex* in $\mathbb{R}^n$.[840] We will make use of the following trivial observation: if we are given $w_1, \ldots, w_{n+1} \in \mathbb{R}^k$, then there exists a unique affine linear map Φ with $\Phi(P_i) = w_i$, namely the map that is given by

$$\Phi\Big(\sum_{i=1}^{n+1} t_i \cdot P_i \Big) \;:=\; \sum_{i=1}^{n+1} t_i w_i.$$

We leave it to the reader to show that each cube can be subdivided[841] into simplices. We also refer to the figure below for an illustration.



simplices in $\mathbb{R}^2$ | a square can be decomposed into two simplices | simplex in $\mathbb{R}^3$

Now we subdivide all the cubes in K_2 into simplices. By the above discussion there exists a map $h\colon K_2 \to \mathbb{R}^k$ that equals g on all the vertices of the simplices in the subdivision of the cubes.[842] Furthermore we choose a map $\psi\colon K_2 \to [0,1]$ with $\psi|_{\partial K_2} \equiv 0$ and $\varphi|_{K_1} \equiv 1$.[843]

---

[840] Given a subset S of $\mathbb{R}^k$ we defined on page 157 the *convex hull* of S as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S. As we saw in Exercise 2.38, if S is a finite subset of $\mathbb{R}^k$, then the *convex hull* of S is given by

$$\Big\{ \sum_{s \in S} a_s \cdot s \;\Big|\; a_s \in [0,1] \text{ and } \sum_{s \in S} a_s = 1 \Big\}.$$

[841] Here, by "subdivide" we mean that given any cube K there exist simplices $S_1, \ldots, S_k$ such that the interiors of the S_i are disjoint and such that $K = S_1 \cup \cdots \cup S_k$.

[842] Note that it follows from the uniqueness statement above that the map g is in fact continuous.

[843] Such a map can be constructed as follows: we assign the value 1 to all vertices in K_1 and we assign the value 0 to all other vertices in K_2. By the above "trivial observation" this defines a map on K_2 with the desired property.

Now we consider the homotopy[844]

$$\begin{array}{rcl} F\colon [0,1]^n \times [0,1] & \to & Z = W \cup_\varphi \overline{B}^k \\ (P,t) & \mapsto & \begin{cases} (1 - t\cdot\psi(P))\cdot g(P) + t\cdot\psi(P)\cdot h(P), & \text{if } P \in K_2 \\ g(P), & \text{otherwise.} \end{cases} \end{array}$$

This is a homotopy from g to a map $\widetilde{g}\colon [0,1] \to Z = W \cup_\varphi \overline{B}^k$ that restricts to the piecewise linear map h on K_1.

Claim. The point $0 \in D_1$ does not lie in $\widetilde{g}\big(\overline{[0,1]^n \setminus K_1}\big)$.

Proof. The claim is proved in two steps:

(1) On $[0,1]^n \setminus K_2$ we have $\widetilde{g} = g$. By Property (ii) of ϵ we have $g^{-1}(D_1) \subset K_2$. It follows that the image of $[0,1]^n \setminus K_2$ lies outside of D_1.
(2) Now let σ be a simplex of K_2 that does not lie in K_1. Note that $K_2 \subset g^{-1}(D_2)$. It follows from Property (i) of ϵ that $g(\sigma)$ is contained in some ball B_σ of radius $\frac{1}{8}$. Since $g(\sigma) \subset B_\sigma$ and since B_σ is convex we also have $h(\sigma) \subset B_\sigma$. Again by the convexity of B_σ we have $\widetilde{g}(\sigma) \subset B_\sigma$. We know that B_σ is not contained in D_1 since σ contains points outside K_1, hence outside $g^{-1}(D_1)$. The radius of B_σ is half that of D_1, so it follows that 0 is not in B_σ, and hence 0 does not lie in $\widetilde{g}(\sigma)$. ⊞

Since $\widetilde{g}\big(\overline{[0,1]^n \setminus K_1}\big)$ is compact it follows from the claim that there exists in fact an open neighborhood U of 0 that does not intersect the image of $[0,1]^n \setminus K_1$. Put differently, we have $\widetilde{g}^{-1}(U) \subset K_1$. It is now straightforward to verify that $K = K_1$, $\widetilde{g}$ and U have the desired properties. ■

Lemma 70.20. Let W be a topological space and let $\varphi\colon S^{k-1} \to W$ be the attaching map of a k-cell. Furthermore let $h\colon \overline{B}^n \to Z := W \cup_\varphi \overline{B}^k$ be a map with $h(S^{n-1}) \subset W$. If $n < k$, then h is homotopic rel S^{n-1} to a map h' that misses B^k.

Proof. Let W be a topological space and let $\varphi\colon S^{k-1} \to W$ be the attaching map of a k-cell. Furthermore let $h\colon \overline{B}^n \to Z := W \cup_\varphi \overline{B}^k$ be a map with $h(S^{n-1}) \subset W$. We suppose that $n < k$. We have to show that h is homotopic rel S^{n-1} to a map h' that misses B^k.

By the Convex-to-Ball Proposition 2.20 there exists a homeomorphism $\Phi\colon [0,1]^n \to \overline{B}^n$. So after precomposing h with Φ we might as well assume that the domain of h is $[0,1]^n$, i.e. that h is a map $h\colon [0,1]^n \to Z := W \cup_\varphi \overline{B}^k$ with $h(\partial[0,1]^n) \subset W$.

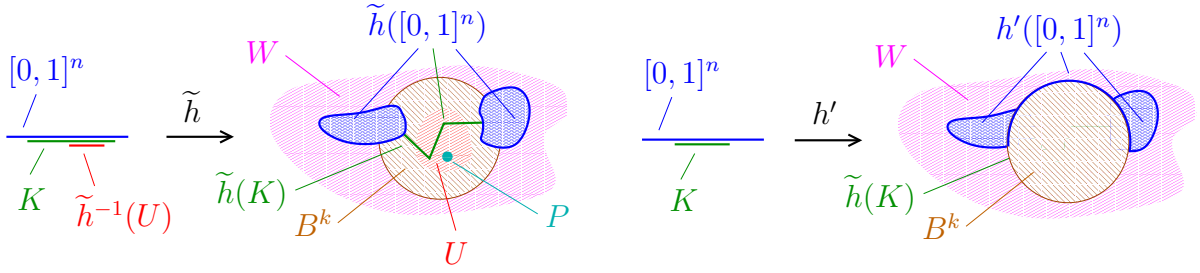
We pick a polyhedron K, a map $\widetilde{h}\colon [0,1]^n \to Z$ and an open non-empty subset $U \subset B^n$ with $\widetilde{h}^{-1}(U) \subset K$ as in the statement of Lemma 70.19. Since $n < k$ and since $\widetilde{h}|_K\colon K \to B^k$ is piecewise linear we deduce that $\widetilde{h}(K)$ cannot contain the open subset U of B^n.[845] We pick a point $P \in U \setminus \widetilde{h}(K)$. By Statement (4) of Lemma 70.19 we know that P does not lie in the image of $\widetilde{h}\colon [0,1]^n \to Z$.

As we pointed out on page 434, there exists a deformation retraction from $\overline{B}^k \setminus \{P\}$ to S^{k-1}. We can extend this deformation retraction via the identity map to a deformation

[844] Since $\varphi|_{\partial K_2} \equiv 0$ we see that F is indeed continuous.

[845] Indeed, since $\widetilde{h}|_K\colon K \to B^k$ is piecewise linear and since $n < k$ we know that the image $\widetilde{h}(K)$ is set of Lebesgue-measure zero in B^n whereas an open non-empty subset has positive Lebesgue-measure.

retraction from $Z = W \cup_\varphi (\overline{B}^k \setminus \{P\})$ to W. If we compose $\widetilde{h}$ with this deformation retraction we obtain a homotopy, rel S^{n-1}, from $\widetilde{h}$ to a map h' that misses B^k. ■



Now we can finally provide the proof of Proposition 70.17.

Proof of Proposition 70.17. Let V be a CW-complex and let W be some topological space. Furthermore let $\psi\colon S^{n-1} \to V^{n-1}$ be a map to the $(n-1)$-skeleton of V and let $\varphi\colon S^{k-1} \to W$ be another map. Finally let

$$h\colon V \cup_\psi \overline{B}^n \;\to\; W \cup_\varphi \overline{B}^k$$

be a map with $h(\psi(S^{n-1})) \subset W$. We suppose that $n < k$.

We denote by θ the obvious map $\overline{B}^n \to V \cup_\psi \overline{B}^n$, furthermore we write $h = g \circ \theta$. By our hypothesis we have $h(S^{n-1}) \subset W$. By Lemma 70.20 there exists a homotopy G rel S^{n-1} from h to a map $h'\colon \overline{B}^n \to W$ that misses B^k. The desired homotopy is now given by[846][847]

$$\begin{array}{rcl} F\colon (V \cup_\psi \overline{B}^n) \times [0,1] & \to & W \cup_\varphi \overline{B}^k \\ (P,t) & \mapsto & \begin{cases} G(Q,t), & \text{if } P = \theta(Q), \text{ for some } Q \in \overline{B}^k, \\ g(P), & \text{otherwise.} \end{cases} \end{array}$$

■

Exercises for Chapter 70.

Exercise 70.1. Let X be a contractible CW-complex and let $x_0 \in X$. Show that X admits a deformation retraction to the point x_0.
Hint. Use the Homotopy Extension Theorem 70.1.

Exercise 70.2. We consider the map

$$\begin{array}{rcl} \varphi\colon S^1 & \to & S^1 \\ \exp(\mathrm{i}t) & \mapsto & \begin{cases} \exp(3\mathrm{i}t), & \text{if } t \in [0, \frac{2\pi}{3}], \\ \exp(-3\mathrm{i}t), & \text{if } t \in [\frac{2\pi}{3}, \frac{4\pi}{3}], \\ \exp(3\mathrm{i}t), & \text{if } t \in [\frac{4\pi}{3}, 2\pi]. \end{cases} \end{array}$$

[846] We view $V \cup_\psi \overline{B}^n$ as a CW-complex where V is a subcomplex and the image of $\overline{B}^n$ is an n-cell. Furthermore we equip $V \cup_\psi \overline{B}^n$ with the product CW-structure. It follows from the Intervall CW-Product Lemma 68.8 and the CW-Complex Pasting Lemma 68.20, applied to the subcomplexes given by $V \times [0,1]$ and $\theta(\overline{B}^n) \times [0,1]$, that the map F is continuous.

[847] One might be tempted to use a different approach to show that F is continuous: Namely one might want to write $(V \cup_\psi \overline{B}^n) \times [0,1] = (V \times [0,1]) \cup_{\psi \times \mathrm{id}_{[0,1]}} (\overline{B}^n \times [0,1])$ and then argue using the Topological Pushout Proposition 5.33 (1b) that the given map is continuous. But this approach is problematic since it is not clear that the two topologies on $(V \cup_\psi \overline{B}^n) \times [0,1] = (V \times [0,1]) \cup_{\psi \times \mathrm{id}_{[0,1]}} (\overline{B}^n \times [0,1])$ agree.

Put differently, φ wraps S^1 first around S^1 clockwise, then counter clockwise and then once again clockwise. We write $Y = S^1$ and we consider

$$X \ := \ Y \cup_\varphi \overline{B^2},$$

where we identify $z \in \partial\overline{B}^2 = S^1$ with $\varphi(z) \in S^1 = Y$.

(a) Show that X is homotopy equivalent to $\overline{B}^2$.

(b) Is X homeomorphic to $\overline{B}^2$?

Exercise 70.3. Let X be a CW-complex which admits a single 0-cell. Show that every subcomplex Y of X is path-connected.

Exercise 70.4. Let X be a CW-complex. Use the Cellular Approximation Theorem 70.15 to give a direct proof that if X is path-connected, then its 1-skeleton X^1 is also path-connected.

Remark. We gave a roundabout proof of this statement in the CW-Complex Properties Proposition 68.10 (7).

Part VIII

Singular Homology

71. Higher homotopy groups: a primer

After these rather technical chapters on CW-complexes we now return to our original goal, namely trying to distinguish topological spaces. Using the fundamental group we had established several non-trivial results, for example we proved that

(1) S^1 and S^k with $k \geq 2$ are not homeomorphic,
(2) $\mathbb{R}^2$ and $\mathbb{R}^n$ with $n \geq 3$ are not homeomorphic,
(3) surfaces of different genus are not homeomorphic,
(4) we showed that S^1 is not a deformation retract of $\overline{B}^2$,
(5) the trefoil is not isotopic to the trivial knot.

On the other hand there are still many even rather elementary questions we cannot answer. For example at the moment we cannot show that for $k \neq l$ with $k, l \geq 2$ the spheres S^k and S^l are not homeomorphic and we cannot show that for $k \neq l$ with $k, l \geq 3$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic. Also, the somewhat mysterious lens spaces $L(p, q)$ still evaded our attempts at distinguishing them for a given p.
The problem with fundamental groups is that we only probe a topological space X with loops. The natural idea is now to go from using maps $S^1 \to X$ to studying maps $S^n \to X$ for $n \geq 2$. As we will see, this idea leads us naturally to the definition of the "higher homotopy groups" of a topological space.

71.1. Definition of the higher homotopy groups.

Notation. Throughout this chapter we write $I = [0, 1]$ for the unit interval. Given $n \in \mathbb{N}$ we denote by $I^n \subset \mathbb{R}^n$ the cube in $\mathbb{R}^n$ and we denote, as usual, by ∂I^n its boundary as a subset of $\mathbb{R}^n$.

Now let X be a topological space and let $x_0 \in X$. With the definitions from page 439 we can write

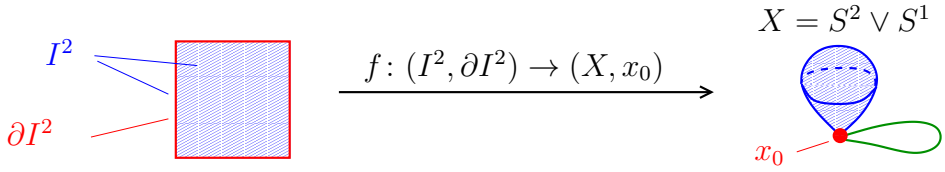
$$\pi_1(X, x_0) \;=\; \text{homotopy classes of maps } (I, \partial I) \to (X, x_0).$$

This definition can be easily generalized to higher dimensions:

Definition. Given a pointed topological space (X, x_0) and $n \in \mathbb{N}_0$ we define

$$\pi_n(X, x_0) \;:=\; \text{homotopy classes of maps } (I^n, \partial I^n) \to (X, x_0).$$

For $n \geq 1$ we refer to $\pi_n(X, x_0)$ as the n-th homotopy group.



Remark. This definition makes in particular sense for $n = 0$, we can thus consider $\pi_0(X, x_0)$. In Section 2.5 we had already defined $\pi_0(X)$ as the set of path-components of X. In fact it is straightforward to see that the map

$$\begin{array}{rcl} \pi_0(X, x_0) & \to & \pi_0(X) \\ [\gamma\colon I^0 = \{0\} \to X] & \mapsto & \text{path-component of } X \text{ that contains } \gamma(0) \end{array}$$

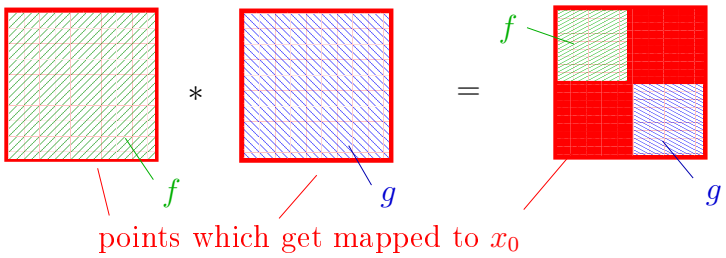
is a bijection.

Note that $\pi_n(X, x_0)$ is a priori only a set. So to justify the name "homotopy group" we need to define a group structure on $\pi_n(X, x_0)$ which for $n = 1$ agrees with our previous definition of the group structure. Here is what we do:

Definition. Let (X, x_0) be a pointed topological space and let $n \geq 1$. Furthermore let $f, g\colon (I^n, \partial I^n) \to (X, x_0)$ be two maps. We define

$$\begin{array}{rcl} f * g\colon (I^n, \partial I^n) & \to & (X, x_0) \\ (x_1, \ldots, x_n) & \mapsto & \begin{cases} f(2x_1, \ldots, 2x_n), & \text{if } (x_1, \ldots, x_n) \in [0, \frac{1}{2}]^n, \\ g(2x_1 - 1, \ldots, 2x_n - 1), & \text{if } (x_1, \ldots, x_n) \in [\frac{1}{2}, 1]^n, \\ x_0, & \text{otherwise.} \end{cases} \end{array}$$

This map is illustrated in the figure below.[848] Applying the Homotopy Stacking Lemma 14.3 several times one sees that this map is indeed continuous.



It is straightforward to see that the homotopy class of $f * g$ only depends on the homotopy classes of f and g. This leads us to the following definition:

Definition. Let (X, x_0) be a pointed topological space and let $n \geq 1$. We define

$$\begin{array}{rcl} \pi_n(X, x_0) \times \pi_n(X, x_0) & \to & \pi_n(X, x_0) \\ ([f], [g]) & \mapsto & [f] \cdot [g] := [f * g]. \end{array}$$

Note that for $n = 1$ this is precisely the same definition of the product structure that we had we already introduced in Chapter 47.2.

Now we can formulate the following proposition:

Proposition 71.1. Let (X, x_0) be a pointed topological space and let $n \geq 1$. The following two statements hold:

(1) The map

$$\begin{array}{rcl} \pi_n(X, x_0) \times \pi_n(X, x_0) & \to & \pi_n(X, x_0) \\ ([f], [g]) & \mapsto & [f] \cdot [g] := [f * g] \end{array}$$

defines a group structure on $\pi_n(X, x_0)$.

(2) Given a map $f\colon (I^n, \partial I^n) \to (X, x_0)$ the inverse of $[f] \in \pi_n(X, x_0)$ is represented by the map $f \circ \Phi\colon (I^n, \partial I^n) \to (X, x_0)$ where $\Phi\colon I^n \to I^n$ is the homeomorphism given by $\Phi(x_1, x_2, \ldots, x_n) = (1 - x_1, x_2, \ldots, x_n)$.

As we will see, the proof of Proposition 71.1 is very similar to the proof of Proposition 47.6 and the proof of the Path-Homotopy Concatenation Proposition 47.5. To deal with the technical differences between the case $n = 1$ and the general case we need a few more definitions and one lemma.

[848] For mysterious reasons in all figures of this chapter the origin $(0, 0)$ is in the top left corner and the point $(1, 1)$ is the vertex at the bottom right.

Definition.

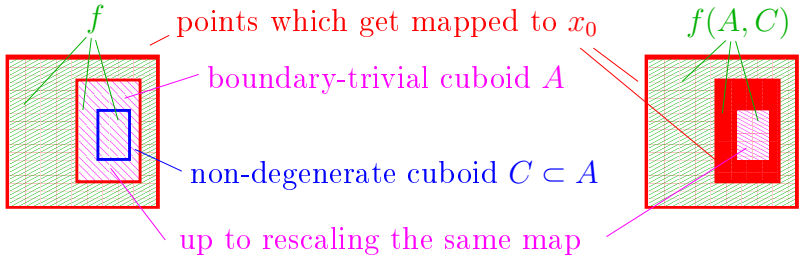
(1) A hyperrectangle is any subset of $\mathbb{R}^n$ of the form $[c_1, d_1] \times \cdots \times [c_n, d_n]$.
(2) We say a hyperrectangle $[c_1, d_1] \times \cdots \times [c_n, d_n]$ is non-degenerate if for all $i = 1, \ldots, n$ we have $d_i > c_i$.

Let X be a topological space, let $x_0 \in X$ and let $f\colon I^n \to X$ be a map.

(3) We say a hyperrectangle $A \subset I^n$ is boundary-trivial if f sends all points on the boundary ∂A of A to x_0.
(4) Let $A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset I^n$ be a boundary-trivial hyperrectangle and furthermore let $C = [c_1, d_1] \times \cdots \times [c_n, d_n]$ be a non-degenerate hyperrectangle that is contained in A. We define

$$\begin{aligned} f(A,C)\colon I^n &\to X \\ x = (x_1, \ldots, x_n) &\mapsto \begin{cases} f(x), & \text{if } x \notin A, \\ x_0, & \text{if } x \in A \setminus C, \\ f\left(a_1 + (x_1 - c_1)\frac{b_1 - a_1}{d_1 - c_1}, \ldots, a_n + (x_n - c_n)\frac{b_n - a_n}{d_n - c_n}\right) & \text{if } x \in C. \end{cases} \end{aligned}$$

By applying the Homotopy Stacking Lemma 14.3 several times one can easily verify that this map is continuous.



Lemma 71.2. Let (X, x_0) be a pointed topological space and let $f\colon (I^n, \partial I^n) \to (X, x_0)$ be a map. Furthermore let $A \subset I^n$ be a boundary-trivial hyperrectangle and let C be a non-degenerate hyperrectangle that is contained in A. Then the maps f and $f(A, C)$ from $(I^n, \partial I^n)$ to (X, x_0) are homotopic.

Proof. Let $f\colon (I^n, \partial I^n) \to (X, x_0)$ be a map, let $A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset I^n$ be a boundary-trivial hyperrectangle and let $C = [c_1, d_1] \times \cdots \times [c_n, d_n]$ be a non-degenerate hyperrectangle that is contained in A. For $t \in [0, 1]$ we write

$$o_t = \big[a_1(1-t) + c_1 t, b_1(1-t) + d_1 t\big] \times \cdots \times \big[a_n(1-t) + c_n t, b_n(1-t) + d_n t\big].$$

Note that $C_0 = A$ and $C_1 = C$. It is now straightforward to see that the map

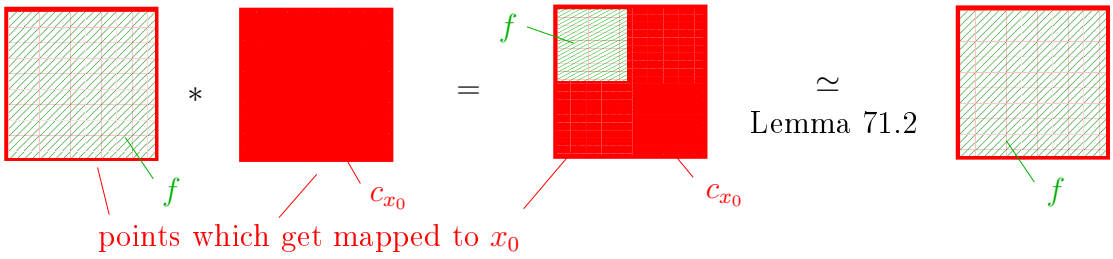
$$\begin{aligned} I^n \times [0,1] &\to X \\ (x, t) &\mapsto f(A, C_t)(x) \end{aligned}$$

is a homotopy between f and $f(A, C)$. ∎

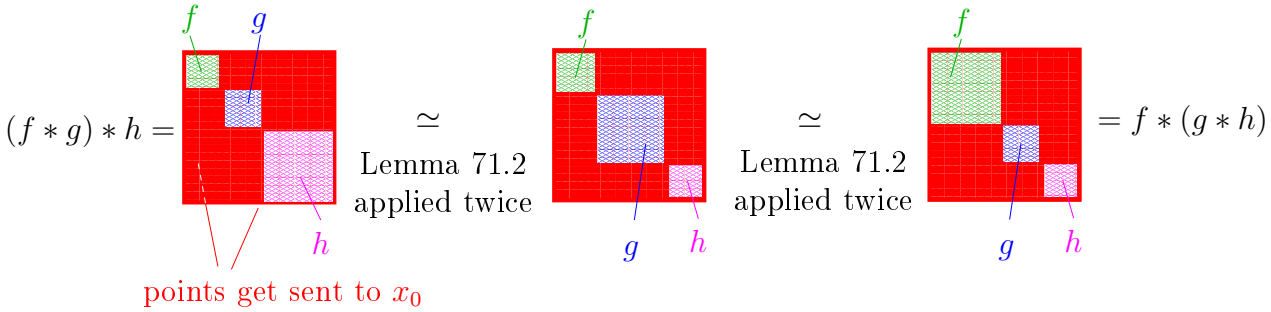
Proof of Proposition 71.1. The neutral element of the product structure is given by the equivalence class of the constant map

$$\begin{aligned} c_{x_0}\colon I^n &\to X \\ x &\mapsto x_0. \end{aligned}$$

Indeed, it follows immediately from Lemma 71.2, applied to $A = [0,1]^n$ and $C = [0,\frac{1}{2}]^n$, that given any map $f\colon (I^n, \partial I^n) \to (X, x_0)$ we have $f * c_{x_0} \simeq f$. Basically the same argument shows also that $c_{x_0} * f \simeq f$.



Now consider three maps $f, g, h\colon (I^n, \partial I^n) \to (X, x_0)$. By applying Lemma 71.2 several times one can easily show that $(f*g)*h \simeq f*(g*h)$, see e.g. the figure below. This proves that the multiplication on $\pi_n(X, x_0)$ is associative.



Now we turn to the most interesting question, namely finding the inverse of a given element in $\pi_n(X, x_0)$. Thus suppose we are given a map $f\colon (I^n, \partial I^n) \to (X, x_0)$. We consider the map

$$\begin{array}{rcl} \overline{f}\colon I^n & \to & X \\ (x_1, \ldots, x_n) & \mapsto & f(1 - x_1, x_2, \ldots, x_n). \end{array}$$

We claim that $[f] \cdot [\overline{f}] = [f * \overline{f}] = [c_{x_0}]$. Thus we need to show that $f * \overline{f}$ is homotopic to c_{x_0}. It follows from Lemma 71.2 that $f * \overline{f}$ is homotopic to the map

$$\begin{array}{rcl} h\colon (I^n, \partial I^n) & \to & (X, x_0) \\ (x_1, \ldots, x_n) & \mapsto & \begin{cases} f(2x_1, x_2, \ldots, x_n), & \text{if } x_1 \in [0, \frac{1}{2}], \\ \underbrace{\overline{f}(2x_1 - 1, x_2, \ldots, x_n)}_{= f(2-2x_1, x_2, \ldots, x_n)}, & \text{if } x_1 \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

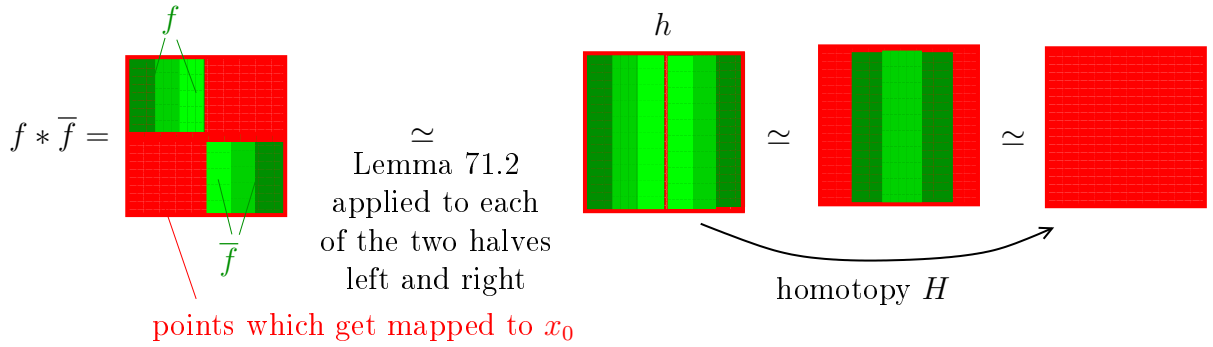
Thus it suffices to show that h is homotopic to c_{x_0}. It is in fact straightforward to see that the map[849][850]

$$\begin{array}{rcl} H\colon I^n \times [0,1] & \to & X \\ (x_1, \ldots, x_n, t) & \mapsto & \begin{cases} f(2(x_1 - \frac{t}{2}), x_2, \ldots, x_n), & \text{if } x_1 \in [\frac{t}{2}, \frac{1}{2}], \\ f(2 - 2(x_1 + \frac{t}{2}), x_2, \ldots, x_n), & \text{if } x_1 \in (\frac{1}{2}, 1 - \frac{t}{2}], \\ x_0, & \text{otherwise} \end{cases} \end{array}$$

[849] This map is continuous by the Homotopy Stacking Lemma 14.3.

[850] Even for $n = 1$ this homotopy is different from the homotopy used in the Path-Homotopy Concatenation Proposition 47.5 (3). Why is that?

is a homotopy from h to c_{x_0}. We refer to the figure below for an illustration of the argument. Basically the same argument also shows that $[\overline{f}]\cdot[f] = [\overline{f} * f] = [c_{x_0}]$. We have thus verified both statements of the proposition. ■



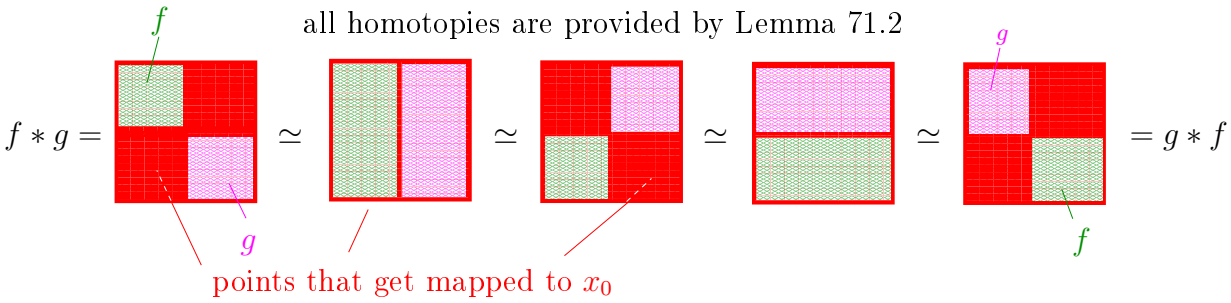
Example. There is one non-empty topological space for which we can immediately compute the higher homotopy groups, namely the topological space consisting of a single point P. In this case the only map $I^n \to \{P\}$ is the constant map, hence we see that $\pi_n(\{P\}) = 0$ for any $n \geq 1$.

The higher homotopy groups $\pi_n(X, x_0)$ seem even more complicated than the fundamental group. Surprisingly at least their algebraic structure is much simpler.

Proposition 71.3. (Homotopy Group-Abelian Proposition) For any pointed topological space (X, x_0) and any $n \geq 2$ the homotopy group $\pi_n(X, x_0)$ is abelian.

Remark. The higher homotopy groups were initially introduced by Eduard Čech in 1932 [**Čec1932**]. Since it was quickly noticed that they are abelian it was initially assumed that higher homotopy groups are of little interest. This perception changed soon afterwards.

Proof. Suppose we are given two maps $f, g\colon (I^n, \partial I^n) \to (X, x_0)$. We have to show that $f * g$ is homotopic to $g * f$. The homotopy is described in the figure below for $n = 2$. It is straightforward to write down an explicit homotopy between $f * g$ and $g * f$ by composing four instances of the homotopy provided by Lemma 71.2. We leave the details to the reader. ■



71.2. **Alternative descriptions of homotopy groups.** In this short section we present several alternative points of views regarding elements in homotopy groups. In the following let $n \in \mathbb{N}$.

(1) We denote by $\Phi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$ the homeomorphism from page 158.

(2) We write $* = (0, \dots, 0, 1) \in S^n$. We denote by $\Psi\colon (\overline{B}^n/S^{n-1}, [S^{n-1}]) \to (S^n, *)$ the homeomorphism from the Ball-Quotient Sphere Lemma 5.20.

Now let X be a topological space and let $x_0 \in X$. The above homeomorphisms Φ and Ψ, together with Lemma 5.18, define bijections

$$\begin{array}{rl} & \text{(homotopy classes) of maps } (I^n, \partial^n) \to (X, x_0) \\ \iff & \text{(homotopy classes) of maps } (\overline{B}^n, S^{n-1}) \to (X, x_0) \\ \iff & \text{(homotopy classes) of maps } (\overline{B}^n/S^{n-1}, [S^{n-1}]) \to (X, x_0) \\ \iff & \text{(homotopy classes) of maps } (S^n, *) \to (X, x_0). \end{array}$$

The four equivalences combined say that we can view elements in $\pi_n(X, x_0)$ as homotopy classes of maps $(S^n, *) \to (X, x_0)$. This point of view is perhaps closest to our visual approach to thinking about topological spaces. But as we have seen above, for technical purposes it is often much easier to work with the maps $(I^n, \partial I^n) \to (X, x_0)$.

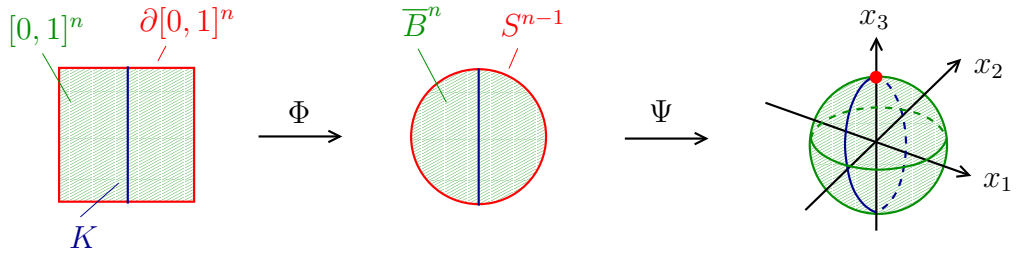
Remark. It is visually most natural to work with the last point of view, namely to think of elements in $\pi_n(X, x_0)$ as homotopy classes of maps $(S^n, *) \to (X, x_0)$. Unfortunately with this definition it is a little less clear how the group structure on $\pi_n(X, x_0)$ is defined. To visualize the group structure we need to work a little:

(1) First note that the homeomorphism $\Phi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$ descends to a homeomorphism $\overline{\Phi}\colon I/\partial I^n \to \overline{B}^n/S^{n-1}$.

(2) We consider the homeomorphism $\Theta := \overline{\Phi}^{-1} \circ \Psi^{-1}\colon S^n \to I^n/\partial I^n$. Note that it identifies the North Pole $(0, \dots, 0, 1)$ with the point $[\partial I^n]$.

(3) We write $K := \{(x_1, \dots, x_n) \in I^n \,|\, x_1 = \frac{1}{2}\}$.

In the figure below we show K and its journey through the maps Φ and Ψ.



(4) We denote by $S^n_1 \vee S^n_2$ the wedge of two copies of S^n along the point $* \in S^n$.

(5) We refer to the map[851]

$$\begin{array}{rcl} S^n \xrightarrow{\Theta} I^n/\partial I^n \to I^n/(\partial I^n \cup K) & \to & S^n_1 \vee S^n_2 \\ [x_1, \dots, x_n] & \mapsto & \begin{cases} \Theta^{-1}([2x_1, x_2, \dots, x_n]) \in S^n_1, & \text{if } x_1 \in [0, \frac{1}{2}), \\ \Theta^{-1}([2x_1 - 1, x_2, \dots, x_n]), & \text{if } x_1 \in [\frac{1}{2}, 1] \end{cases} \end{array}$$

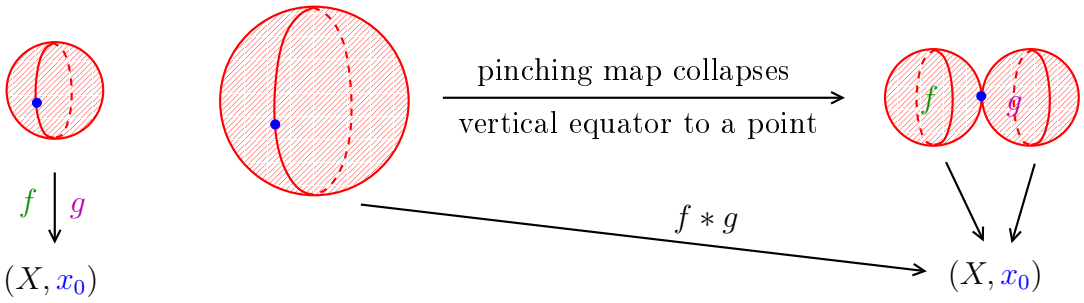
as the *pinching map*.

(6) It follows basically immediately from the definitions that the group structure on $\pi_n(X, x_0)$ is given as follows:

$$\begin{array}{rcl} \pi_n(X, x_0) \times \pi_n(X, x_0) & \to & \pi_n(X, x_0) \\ ([f_1\colon S^n \to X], [f_2\colon S^n \to X]) & \mapsto & \big[S^n \to S^n_1 \vee S^n_2 \xrightarrow{f_1 \vee f_2} X\big]. \end{array}$$

[851] It follows easily from the Pasting Proposition 2.6 (2′) that this map is continuous.

In the figure below we sketch the pinching map and we sketch the corresponding definition of the group structure on $\pi_n(X, x_0)$.[852]



We conclude this section with the following lemma that can be viewed as a generalization of the Extension-to-Disk Lemma 47.1.

Lemma 71.4. (Trivial-in-π_n-Lemma) Let X be a topological space, let $x_0 \in X$ and let $n \geq 1$. Furthermore let $f\colon (S^n, *) \to (X, x_0)$ be a map. Then the following statements are equivalent:

(1) the map $f\colon S^n \to X$ is homotopic to a constant map,
(2) there exists a map $F\colon \overline{B}^{n+1} \to X$ so that $F|_{S^n} = f$,
(3) f represents the trivial element in $\pi_n(X, x_0)$.

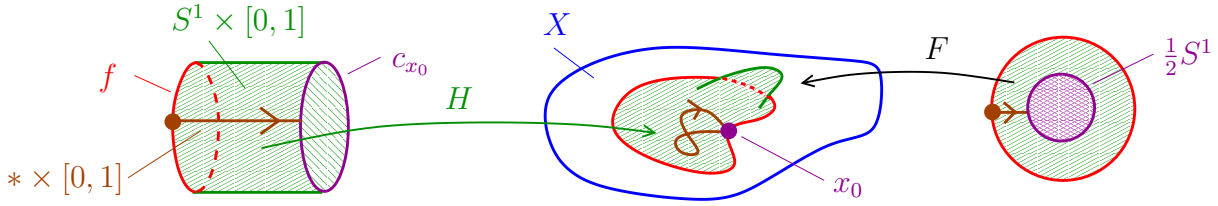
Proof. The equivalence of (2) and (3) is proved in a similar way as the Extension-to-Disk Lemma 47.1. We leave it to the reader to make the fairly straightforward modifications.

We turn to the proof that (3) implies (2). This is basically obvious. Indeed, the fact that f represents the trivial element in $\pi_n(X, x_0)$ means that there exists a homotopy of maps of pairs $(S^n, *) \to (X, x_0)$ from f to the constant map. But this means in particular that f is homotopic to the constant map c_{x_0}.

Finally suppose that (1) holds. This means that we assume that there exists a homotopy $H\colon S^n \times [0,1] \to X$ with $H_0 = f$ and $H_1 = c_{x_0}$. This time we do not assume that $H(*, t) = *$ for all $\in [0,1]$. We consider the map

$$\begin{array}{rcl} F\colon \overline{B}^{n+1} & \to & X \\ P & \mapsto & \begin{cases} H(Q, 2-2t), & \text{if } P = Q \cdot t \text{ with } Q \in S^n \text{ and } t \in [1, \frac{1}{2}], \\ x_0, & \text{otherwise.} \end{cases} \end{array}$$

It follows from the Pasting Proposition 2.6 (2) that this map is indeed continuous. By construction we have $F|_{S^n} = f$. Thus we see that (2) is satisfied. This concludes the proof that all three statements are equivalent. ■



[852] The meticulous reader will notice that it is perhaps not completely clear how the maps f and g are defined on the two smaller spheres.

71.3. **Properties and basic calculations of higher homotopy groups.** Now we want to understand some of the basic properties of higher homotopy groups. Perhaps not surprisingly, many of the basic statements about the fundamental group can be easily generalized to the setup of higher homotopy groups.
For example the following proposition can be viewed as a generalization of the Change-of-Base Point Proposition 47.10.

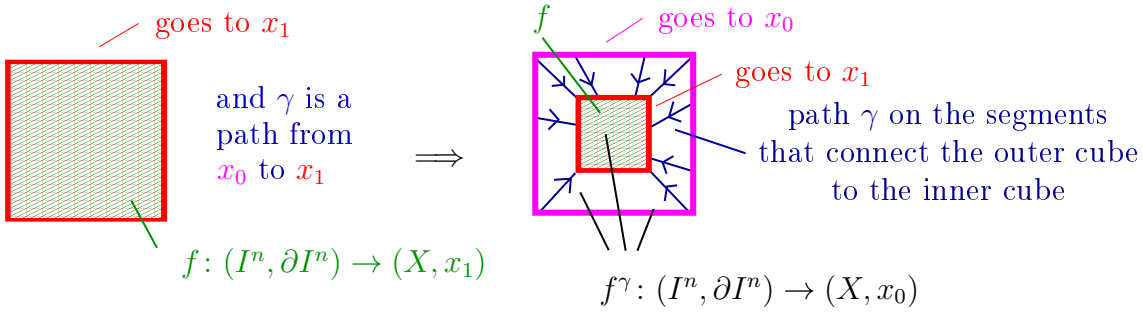
Proposition 71.5. (Change-of-Base Point Proposition) Let X be a topological space, let x_0 and x_1 be two points in X and let $\gamma\colon [0,1] \to X$ be a path from x_0 to x_1. Given a map $f\colon (I^n, \partial I^n) \to (X, x_1)$ we consider the map[853][854]

$$\begin{array}{rcl} f^\gamma\colon (I^n, \partial I^n) & \to & (X, x_0) \\ (t_1, \dots, t_n) & \mapsto & \begin{cases} f\big(2t_1 - \frac{1}{2}, \dots, 2t_n - \frac{1}{2}\big), & \text{if } (t_1, \dots, t_n) \in \big[\frac{1}{4}, \frac{3}{4}\big]^n, \\ \gamma(t), & \text{if } (t_1, \dots, t_n) \in \partial\big(\big[\frac{t}{4}, \frac{4-t}{4}\big]^n\big) \text{ for } t \in [0,1]. \end{cases} \end{array}$$

The definition of f^γ is illustrated in the figure below. The map

$$\begin{array}{rcl} \gamma_*\colon \pi_n(X, x_1) & \to & \pi_n(X, x_0) \\ [f] & \mapsto & [f^\gamma] \end{array}$$

is well-defined and it is a group isomorphism.



Remark. If X is a path-connected topological space and if we are only interested in the isomorphism type of the n-th homotopy group of X, then sometimes we use this proposition as an excuse to just write $\pi_n(X)$, i.e. we suppress the base point from the notation.

Proof. Let X be a topological space, let x_0 and x_1 be two points in X and let $\gamma\colon [0,1] \to X$ be a path from x_0 to x_1. We need to prove the following claim.

Claim.

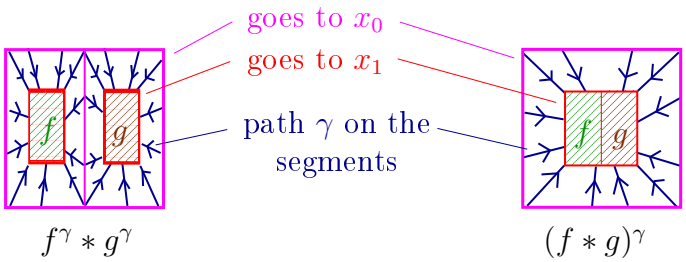
(1) The map $\Phi\colon \pi_n(X, x_1) \to \pi_n(X, x_0)$ given by $[f] \mapsto [f^\gamma]$ is well-defined,
(2) the map Φ is a bijection, and
(3) the map Φ is a group homomorphism.

Proof.

(1) This statement can be verified easily by hand.
(2) As usual we denote by $\overline{\gamma}$ the path given by $\overline{\gamma}(t) = \gamma(1-t)$. It is straightforward to verify that the map $\Psi\colon \pi_n(X, x_0) \to \pi_n(X, x_1)$ given by $[g] \mapsto [f^{\overline{\gamma}}]$ is an inverse to Φ.

[853] If we view elements in $\pi_1(X, x_i)$ as represented by maps $(S^1, *) \to (X, x_i)$, what does the corresponding map $\pi_1(X, x_1) \to \pi_1(X, x_0)$ look like?
[854] Using Pasting Proposition 2.6 (2) one can easily verify that the map is continuous.

(3) Now let $f, g\colon (I^n, \partial I^n) \to (X, x_0)$ be maps. We need to show that $f^\gamma * g^\gamma$ is homotopic to $(f * g)^\gamma$. (These two maps are illustrated in the figure below.) It should be clear that verifying this statement is elementary, in the sense that one can write down an explicit homotopy. On the other hand it should also be clear that writing down an explicit homotopy is supremely painful and highly non-instructive. Thus we will skip this step and leave it to the eager reader to fill in the details. ■



Next we will see that higher homotopy groups are functorial.

Definition. Let $f\colon X \to Y$ be a map between two topological spaces, let $n \in \mathbb{N}$ and let $x_0 \in X$. We write $y_0 = f(x_0)$. The map

$$\begin{array}{rcl} f_*\colon \pi_n(X, x_0) & \to & \pi_n(Y, y_0) \\ [\sigma\colon (I^n, \partial I^n) \to (X, x_0)] & \mapsto & [f \circ \sigma\colon (I^n, \partial I^n) \xrightarrow{\sigma} (X, x_0) \xrightarrow{f} (Y, y_0)] \end{array}$$

is well-defined map and it is a group homomorphism for every $n \geq 1$.[855] As usual we say that f_* is the map induced by f.

The following proposition is a straightforward generalization of Proposition 47.15.

Proposition 71.6. Let $n \geq 1$. Recall that we denote by $\mathcal{PTop}$ the category of pointed topological spaces and that we denote by $\mathcal{Group}$ be the category of groups. Then

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{PTop}) & \to & \mathrm{Ob}(\mathcal{Group}) \\ (X, x_0) & \mapsto & \pi_n(X, x_0) \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}(\mathcal{PTop}) & \to & \mathrm{Mor}(\mathcal{Group}) \\ (f\colon (X, x_0) \to (Y, y_0)) & \mapsto & (f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)) \end{array}$$

is a covariant functor.

Proof. It follows immediately from the definitions that

$$\mathrm{id}_* = \mathrm{id} \qquad \text{for all pointed pairs } (X, x_0),$$

and that

$$(g \circ f)_* = g_* \circ f_* \qquad \text{for all maps } f\colon (X, x_0) \to (Y, y_0) \text{ and } g\colon (Y, y_0) \to (Z, z_0).$$

But this is exactly what we need to show. ■

We continue with the following fairly elementary but useful proposition, which generalizes the Homotopy-π_1-Proposition 50.3.

[855] Both statements follow basically immediately from the definitions.

Proposition 71.7. (Homotopy-π_n-Proposition) Let $n \geq 1$.

(1) Let $f, g\colon X \to Y$ be two maps between topological spaces and let $x_0 \in X$. If f and g are homotopic, then

$$\begin{array}{c} f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0)) \\ \text{is an isomorphism} \end{array} \iff \begin{array}{c} f_*\colon \pi_n(X, x_0) \to \pi_n(Y, g(x_0)) \\ \text{is an isomorphism.} \end{array}$$

The analogous statement holds if we replace "isomorphism" by "epimorphism" or "monomorphism".

(2) Let $f\colon X \to Y$ be a map between topological spaces and let $x_0 \in X$ be a point. If f is a homotopy equivalence, then

$$f_*\colon \pi_n(X, x_0) \ \to \ \pi_n(Y, f(x_0))$$

is an isomorphism.

(3) If X is a contractible topological space, then the following two statements hold:
   (a) for any base point $x_0 \in X$ we have $\pi_n(X, x_0) = 0$,
   (b) any map $f\colon S^n \to X$ can be extended to a map $\overline{B}^{n+1} \to X$.

(4) Let (X, x_0) be a pointed topological space. If we denote by Y the component of X that contains x_0 and if we denote by $i\colon Y \to X$ the inclusion map, then the map

$$i_*\colon \pi_n(Y, x_0) \ \to \ \pi_n(X, x_0)$$

is an isomorphism.

Proof.

(1) This statement is proved basically the same way as the Homotopy-π_1-Proposition 50.3 (1). We leave it to the reader to make the necessary changes to the argument.

(2) As in the case of the Homotopy-π_1-Proposition 50.3 one can easily deduce Statement (2) from Statement (1).

(3) Let X be a contractible topological space.
   (a) By definition of "contractible" the topological space X is homotopy equivalent to a topological space consisting of a single point $\{*\}$. Evidently $\pi_n(\{*\}, *) = 0$, which by (2) implies that $\pi_n(X, x_0) = 0$ for some base point x_0. In the π_0-Homotopy Equivalence Lemma 15.3 we saw that a contractible topological space is path-connected. Thus it follows from the Change-of-Base Point Proposition 71.5 that $\pi_n(X, x_0) = 0$ for any base point x_0.
   (b) This statement is proved basically the same way as the Extension-to-Disk Lemma 47.1. For completeness' sake we carry out the argument. So let $f\colon S^n \to X$ be a map. We pick a base point $*$ for S^n and we set $x_0 := f(*)$. By (a) we know that $\pi_n(X, x_0) = 0$. We take the point of view from page 1565. Thus we can view $[f]$ as an element in $\pi_n(X, x_0) = 0$. By definition $[f] = 0$ means that there exists a homotopy $F\colon S^n \times [0, 1] \to X$ such that $F(p, 0) = f(p)$ for all $p \in S^n$, that $F(p, 1) = x_0$ for all $p \in S^n$ and that $F(*, t) = x_0$ for all $t \in [0, 1]$. We consider the map

$$\begin{array}{rcl} \overline{B}^{n+1} & \to & X \\ r \cdot v & \mapsto & F(v, (1 - r)) \end{array}$$

where we write a point in $\overline{B}^{n+1}$ as $r \cdot v$ with $r \in [0,1]$ and $v \in S^n$. It follows from Lemma 5.19 (2) that this map is indeed continuous. Clearly this is the desired extension.

(c) This statement is an easy consequence of Lemma 2.35 and the fact that I^n is path-connected. ■

Example. The topological space $\mathbb{R}^n$ is homotopy equivalent to a point P. Thus it follows from the Homotopy-π_n-Proposition 71.7 (2) and the discussion on page 1564 that $\pi_k(\mathbb{R}^n) \cong \pi_k(P) \cong 0$. Exactly the same way, using the discussion on page 432, one can show that all the higher homotopy groups of convex subsets of $\mathbb{R}^n$ vanish, e.g. we have $\pi_k\big(\overline{B}^n\big) = \pi_k(B^n) = 0$.

The following proposition is the high-dimensional analogue of the Fundamental Group-of-Product Proposition 48.20.

Proposition 71.8. (Homotopy Groups-of-Product Proposition) Let A and B be two topological spaces and let $a_0 \in A$ and $b_0 \in B$. We consider the inclusion maps

$$\begin{array}{rcl} i\colon A & \to & A \times B \\ a & \mapsto & (a, b_0) \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon B & \to & A \times B \\ b & \mapsto & (a_0, b) \end{array}$$

and we consider the projection s

$$\begin{array}{rcl} p\colon A \times B & \to & A \\ (a,b) & \mapsto & a \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon A \times B & \to & B \\ (a,b) & \mapsto & b. \end{array}$$

(1) The maps

$$\begin{array}{rcl} \Phi\colon \pi_n(A,a_0) \times \pi_n(B,b_0) & \to & \pi_n(A \times B, (a_0,b_0)) \\ (x,y) & \mapsto & i_*(x) \cdot j_*(y) \end{array}$$

and

$$\begin{array}{rcl} \Psi\colon \pi_n(A \times B, (a_0,b_0)) & \to & \pi_n(A,a_0) \times \pi_n(B,b_0) \\ z & \mapsto & (p_*(z), q_*(z)) \end{array}$$

are both homomorphisms[856] and they are inverses of one another, in particular both are isomorphisms.

(2) The map Φ agrees with the map

$$\begin{array}{rcl} \Theta\colon \pi_n(A,a_0) \times \pi_n(B,b_0) & \to & \pi_n(A \times B, (a_0,b_0)) \\ ([f\colon (S^n,*) \to (A,a_0)], [g\colon (S^n,*) \to (B,b_0)]) & \mapsto & \left[\begin{array}{rcl} (S^n,*) & \to & (A \times B, (a_0,b_0)) \\ z & \mapsto & (f(z), g(z)) \end{array}\right]. \end{array}$$

In particular we obtain from (1) that Θ is a homomorphism.

Proof. The proof of this proposition is almost verbatim the same as the proof of the Fundamental Group-of-Product Proposition 48.20. We leave it to the reader to make the minute modifications. ■

We continue with the following very general and actually very convenient proposition regarding homotopy groups of CW-complexes. The proposition greatly generalizes the CW-Skeleton-π_1-Proposition 69.13 (1).

[856] In fact in this case it follows immediately from the Homotopy Group-Abelian Proposition 71.3 and Lemma 48.19 that Φ is a homomorphism.

Proposition 71.9. (CW-Skeleton-π_n-Proposition)

(1) Let X be a CW-complex and let $k \in \mathbb{N}_0$. As usual we denote by X^k the k-skeleton of X. Let $x_0 \in X^k$. The following two statements hold:
   (a) For $j = 1, \ldots, k-1$ the inclusion induced map $\pi_j(X^k, x_0) \to \pi_j(X, x_0)$ is an isomorphism.
   (b) The inclusion induced map $\pi_k(X^k, x_0) \to \pi_k(X, x_0)$ is an epimorphism.

(2) Let X be a CW-complex with a single 0-cell x_0 and let $n \in \mathbb{N}$. If X has no cells in dimensions $1, \ldots, n$, then $\pi_n(X, x_0) = 0$ for $k = 1, \ldots, n$.

Note that the above proposition is a generalization of Corollary 69.14.

Proof.

(1) Let X be a CW-complex and let $x_0 \in X^k$. We denote by $i\colon X^k \to X$ the inclusion. We take the point of view from page 1565, i.e. we view elements of $\pi_j(X, x_0)$ as homotopy classes of maps $(S^j, *) \to (X, x_0)$. As on page 1472 we consider the j-sphere S^j as a CW-complex with one 0-cell $*$ and one j-cell.
   (a) First we show that for $j = 1, \ldots, k$ the map $i_*\colon \pi_j(X^k, x_0) \to \pi_j(X, x_0)$ is an epimorphism.
   Let $[g] \in \pi_j(X, x_0)$ be some element. In other words, we are given a map $g\colon (S^j, *) \to (X, x_0)$ be a map. By the Cellular Approximation Theorem 70.15, applied to $g\colon S^j \to X$ and the subcomplex $\{*\} \subset S^j$, there exists a homotopy F rel $\{*\}$ from the map g to a cellular map $\widetilde{g}$. Since F is a homotopy rel $\{*\}$ we see that F also defines a homotopy of maps $(S^j, *) \to (X, x_0)$ of pairs of topological spaces. Thus we see that $[g] = [\widetilde{g}] \in \pi_j(X, x_0)$.
   Since $\widetilde{g}$ is cellular we know that $\widetilde{g}(S^j)$ lies in the j-skeleton of X. Since $j \leq k$ we see in particular that $\widetilde{g}(S^j)$ is contained in X^k. In other words, $\widetilde{g}$ is a map $(S^k, *) \to (X^k, x_0)$. This shows that $[g] = [\widetilde{g}] = [i \circ \widetilde{g}] = i_*([\widetilde{g}])$ lies in the image of $\pi_j(X^k, x_0)$.
   (b) Now we show that for $j = 1, \ldots, k-1$ the map $i_*\colon \pi_j(X^k, x_0) \to \pi_j(X, x_0)$ is a monomorphism. So suppose that we are given $[f_0], [f_1] \in \pi_j(X^k, x_0)$ with $i_*([f_0]) = i_*([f_1])$. By the Cellular Approximation Theorem 70.15 we can assume that the maps $f_0, f_1\colon (S^j, *) \to (X^k, x_0)$ are cellular. The fact that $i_*([f_0]) = i_*([f_1])$ means that there exists a homotopy $H\colon S^j \times [0,1] \to X$ with $H_0 = f_0$ and $H_1 = f_1$. This time we apply the Cellular Approximation Theorem 70.15 to the map $H\colon S^j \times [0,1] \to X$ and the subcomplex $(S^j \times \{0\}) \cup (S^j \times \{1\})$ and we obtain a cellular map $\widetilde{H}\colon S^j \times [0,1] \to X$ such that $\widetilde{\mathrm{H}}_0 = H_0 = f_0$ and $\widetilde{H}_1 = H_1 = f_1$. Since $\widetilde{H}$ is cellular it takes values in the $(j+1)$-skeleton of $X \times [0,1]$. Since $j < k$ we see in particular that $\widetilde{H}$ takes values in $X^k \times [0,1]$. But this means that we can use $\widetilde{H}$ to show that $[f_0] = [f_1] \in \pi_j(X^k, x_0)$.

(2) Let $n \in \mathbb{N}$. Let X be a CW-complex with a single 0-cell x_0 and which has no cells in dimensions $1, \ldots, n$. Let $k \in \{1, \ldots, n\}$. By hypothesis the k-skeleton of X equals $\{x_0\}$. By (1a) we know that the map $\pi_k(\{x_0\}, x_0) \to \pi_k(X, x_0)$ is an epimorphism. But the former group is evidently trivial, hence $\pi_k(X, x_0)$ is trivial as well. ∎

In the Sphere-π_1-Proposition 47.13 we saw that $\pi_1(S^n) = 0$ for $n \geq 2$. Now we can prove the analogous statement for higher homotopy groups. Now we obtain, basically for free, the following much more general proposition.

Proposition 71.10. Let $n \in \mathbb{N}$. For any $k \in \{1, \dots, n-1\}$ we have

$$\pi_k(S^n) \;=\; 0.$$

Proof.

(1) Again, as on page 1472 we consider the n-sphere S^n as a CW-complex with one 0-cell and one n-cell. With this CW-structure it follows immediately from the CW-Skeleton-π_n-Proposition 71.9 that $\pi_k(S^n) = 0$ for $k = 1, \dots, n-1$.

(2) An alternative approach to proving Proposition 71.10 is given by the point of view taken on page 1059 where we used the Whitney Approximation Theorem 29.1, the Replace-by-Smooth Homotopy Proposition 29.5 and Proposition 30.3 to show that for an $n \geq 2$ we have $\pi_1(S^n) = 0$. ■

The following proposition is also a straightforward generalization from the setting of fundamental groups to the setting of higher homotopy groups. The proof is almost verbatim the same as the proof of Proposition 69.3 and of the Direct Limit-π_1-Proposition 56.7.

Proposition 71.11. (Direct Limit-π_n-Proposition) Let X be a topological space and let $X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subsets such that $X = \bigcup_{i \in \mathbb{N}} X_i$ and such that of one the following three conditions holds:

(1) every compact subset of X is already contained in one of the X_i, or
(2) each X_i is open in X, or
(3) X is a CW-complex and each X_i is a subcomplex.

Then for any $w \in X_1$ and any $n \in \mathbb{N}$ the inclusion induced maps $\pi_n(X_i, w) \to \pi_n(X, w)$ induce an isomorphism

$$\pi_n(X, w) \;=\; \varinjlim \pi_n(X_i, w).$$

In Lemma 68.4 we already saw that S^∞ is contractible, thus we know by that all homotopy groups are trivial. But it is also illuminating to apply the Direct Limit-π_n-Proposition 71.11 to obtain this result.

Corollary 71.12. For every $n \in \mathbb{N}$ we have $\pi_n(S^\infty) = 0$.

Proof. We equip S^∞ and all spheres S^k, $k \in \mathbb{N}$ with the CW-complex structure that we defined on page 1479. We have

$$\pi_n(S^\infty) \;=\; \pi_n\Big(\bigcup_{k \in \mathbb{N}} S^k\Big) \;\overset{\text{by the Direct Limit-}\pi_n\text{-Proposition 71.11 and by definition of the topology of } S^\infty}{=}\; \varinjlim \pi_n(S^k) \;\overset{\text{by Proposition 71.10 we have } \pi_n(S^k) = 0 \text{ for } k > n}{=}\; 0.$$

■

71.4. **Covering spaces and higher homotopy groups.** One of the main tools in studying fundamental groups was to study the relationship between the fundamental group of a topological space and coverings of the given topological space. Perhaps somewhat surprisingly, for $n \geq 2$ we have a completely different relationship between the n-th homotopy group and coverings.

Proposition 71.13. (Coverings-π_n-Proposition) Let $p\colon (X,x_0)\to(B,b_0)$ be a covering of pointed topological spaces. Let $n\geq 2$. Then the following two statements hold:

(1) The map $p_*\colon \pi_n(X,x_0)\to\pi_n(B,b_0)$ is an isomorphism.

(2) The map[857]

$$\begin{array}{rcl}\pi_n(B,b_0) & \to & \pi_n(X,x_0)\\ [f\colon (S^n,*)\to(B,b_0)] & \mapsto & [\text{unique lift of } f \text{ to a map } (S^n,*)\to(X,x_0)]\end{array}$$

is well-defined and it is the inverse to p_*.

Before we provide the proof of the Coverings-π_n-Proposition 71.13 let us first state an immediate, and very useful, corollary to the Coverings-π_n-Proposition 71.13 and let us give a few examples.

Corollary 71.14. Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Given any $n\geq 2$ the universal covering $p\colon \widetilde{X}\to X$ induces an isomorphism[858]

$$p_*\colon \pi_n(\text{universal covering } \widetilde{X} \text{ of } X) \xrightarrow{\cong} \pi_n(X).$$

Now we present some examples for Corollary 71.14. We will consider a few more examples after the proof of the Coverings-π_n-Proposition 71.13.

Examples.

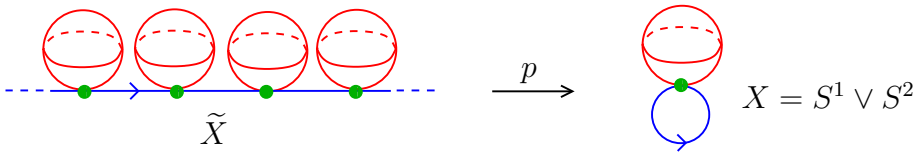
(1) Let $k\in\mathbb{N}$ and $n\geq 2$. We have

$$\pi_n((S^1)^k) \;=\; \pi_n(\mathbb{R}^k/\mathbb{Z}^k) \xleftarrow[\uparrow]{\cong} \pi_n(\mathbb{R}^k) \underset{\uparrow}{=} 0.$$

Coverings-π_n-Proposition 71.13 — page 1570

In particular we see that all of the higher homotopy groups of S^1 and the torus $S^1\times S^1$ vanish.

(2) We consider $X=S^1\vee S^2$. On page 1373 we had determined its universal covering $\widetilde{X}$. For convenience we picture it again in the figure below. It follows from the Coverings-π_n-Proposition 71.13 that $\pi_2(S^1\vee S^2)$ is isomorphic to $\pi_2(\widetilde{X})$. At the moment we cannot determine $\pi_2(\widetilde{X})$, but since $\widetilde{X}$ has "infinitely many spheres" one might expect $\pi_2(\widetilde{X})$ to be pretty big.



(3) One of our goals is still to complete the classification of lens spaces, in particular we would like to answer Question 48.7. So let $L(p,q)$ and $L(p,r)$ be two lens spaces.

[857] Here we take the point of view from page 1565 where we saw that we view the elements of $\pi_n(B,b_0)$ as homotopy classes of maps $(S^n,*)\to(B,b_0)$.

[858] In Corollary 61.11 we showed that every topological space X that is path-connected, locally path-connected and semi-locally simply connected admits a universal covering $p\colon \widetilde{X}\to X$, i.e. a covering such that $\widetilde{X}$ is simply connected.

Then for any $n \geq 2$ we have

$$\pi_n(L(p,q)) \underset{\uparrow}{\overset{\cong}{\longleftarrow}} \pi_n(S^3) \underset{\uparrow}{\overset{\cong}{\longrightarrow}} \pi_n(L(p,r)).$$

Coverings-π_n-Proposition 71.13

We do not know the higher homotopy groups of S^3, but this argument shows that higher homotopy groups are useless for distinguishing lens spaces.

Proof of the Coverings-π_n-Proposition 71.13. Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed topological spaces and furthermore let $n \geq 2$. Our goal is to construct an inverse to the map $p_*\colon \pi_n(X,x_0) \to \pi_n(B,b_0)$.

We saw on page 1565 that we can think of elements in $\pi_n(B,b_0)$ as homotopy classes of maps $(S^n,*) \to (B,b_0)$. Thus let $f\colon (S^n,*) \to (B,b_0)$ be given. We are in the following setting[859]

$$\begin{array}{ccc} & & (X,x_0) \\ & & \downarrow p \\ (S^n,*) & \xrightarrow[f]{} & (B,b_0). \end{array} \qquad \overset{\text{by the Map Lifting Criterion 61.2}}{\underset{\text{since } \pi_1(S^n)=0}{\rightsquigarrow}} \qquad \begin{array}{ccc} & \overset{\text{exists}}{\overset{\text{unique } \widetilde{f}}{\nearrow}} & (X,x_0) \\ & \swarrow & \downarrow p \\ (S^n,*) & \xrightarrow[f]{} & (B,b_0). \end{array}$$

Furthermore it follows again from the Map Lifting Criterion 61.2, applied to $Z = S^n \times [0,1]$, that the lifts of two homotopic maps $(S^n,*) \to (B,b_0)$ are two homotopic maps $(S^n,*) \to (X,x_0)$.[860] Summarizing we just constructed a map

$$\begin{array}{rcl} \lambda\colon \pi_n(B,b_0) & \to & \pi_n(X,x_0) \\ [f] & \mapsto & [\widetilde{f}]. \end{array}$$

Now we have to show that λ is an inverse to p_*. First of all, given any $[f] \in \pi_n(B,b_0)$ we have

$$(p_* \circ \lambda)([f]) \;=\; p_*\big([\widetilde{f}]\big) \;=\; \big[p \circ \widetilde{f}\big] \;=\; [f].$$

Now let $[g] \in \pi_n(X,x_0)$. We then have

$$(\lambda \circ p_*)([g]) \;=\; \lambda([p \circ g]) \underset{\uparrow}{=} [g].$$

since g is the unique lift of $p \circ g$ to x_0.

■

Now we give a few more applications of the Coverings-π_n-Proposition 71.13.

Examples.

(1) Let Σ be a surface of genus ≥ 2. In Theorem 66.9 we have seen that there exists a covering $p\colon \mathbb{D} \to \Sigma$ where $\mathbb{D} = B_1(0) \subset \mathbb{C}$. Since $\mathbb{D}$ is convex it follows from

[859] In the Map Lifting Criterion 61.2 we need that the sphere S^n is of path-connected and locally path-connected, but there are no hypotheses on B or X.

[860] Here there is a little hidden subtlety: if $F\colon S^n \times [0,1] \to B$ is a homotopy between two maps $f,g\colon (S^n,*) \to (B,b_0)$, then we can lift f,g to maps $\widetilde{f},\widetilde{g}\colon (S^n,*) \to (X,x_0)$. Furthermore we can apply the Map Lifting Criterion 61.2 to lift F to a map $\widetilde{F}\colon S^n \times [0,1] \to X$ such that $\widetilde{F}(*,0) = x_0$. We still need to show that $\widetilde{F}(*,t) = x_0$ for all $t \in [0,1]$. We know that for every $t \in [0,1]$ we have $\widetilde{F}(*,t) \in p^{-1}(b_0)$. By Lemma 48.3 the preimage $p^{-1}(b_0)$ is discrete. Since $[0,1]$ is connected it follows that the map $p\colon [0,1] \to p^{-1}(b_0)$ is constant by the Connected-to-Discrete Lemma 2.27. But since it takes the value x_0 for $t=0$ it takes the value x_0 for all $t \in [0,1]$.

the Coverings-π_n-Proposition 71.13 and the discussion on page 1570 that $\pi_n(\Sigma) \cong \pi_n(\mathbb{D}) = 0$ for all $n \geq 2$.

Together with the example on page 1573 we have now shown that the higher homotopy groups of every surface of genus ≥ 1 vanish.

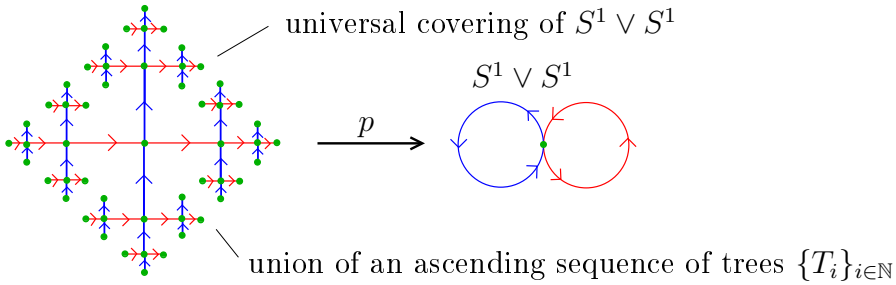
(2) Let Σ be a surface of genus ≥ 1 and let $f\colon \Sigma \to \Sigma$ be a self-homeomorphism. We denote by $\mathrm{Tor}(\Sigma, f)$ the mapping torus that we introduced on page 467. For $n \geq 2$ we have

$$\pi_n(\mathrm{Tor}(\Sigma, f)) \underset{\text{by the Mapping Torus Covering Lemma 11.29 and the Coverings-}\pi_n\text{-Proposition 71.13}}{\cong} \pi_n(\Sigma \times \mathbb{R}) \underset{\text{Homotopy-}\pi_n\text{-Proposition 71.7}}{\cong} \pi_n(\Sigma) \underset{\text{previous example}}{=} 0.$$

Lemma 71.15. Let X be a 1-dimensional CW-complex and let $x_0 \in X$ be base point. Then $\pi_n(X, x_0) = 0$ for every $n \geq 2$.

Sketch of proof. Let X be a 1-dimensional CW-complex, let $x_0 \in X$ and let $n \geq 2$. We first assume that X is finite and connected. We write $k = 1 - \chi(X)$. We have

$$\pi_n(X) \underset{\text{Quotient-by-Tree Proposition 69.8 (2) and Homotopy-}\pi_n\text{-Proposition 71.7}}{\cong} \pi_n\Big(\bigvee_{i=1}^{k} S^1\Big) \underset{\text{by the Coverings-}\pi_n\text{-Proposition 71.13}}{\cong} \pi_n\Big(\text{universal covering of } \bigvee_{i=1}^{k} S^1\Big) \underset{\text{by the Direct Limit-}\pi_n\text{-Proposition 71.11 and the figure below}}{\cong} \varinjlim \pi_n(\text{trees } T_i) \underset{\text{by Proposition 50.14 and Homotopy-}\pi_n\text{-Proposition 71.7}}{=} \varinjlim 0 = 0.$$



Now suppose that X is an arbitrary 1-dimensional CW-complex. Let $[f] \in \pi_n(X, x_0)$. It follows from the CW-Complex-Finiteness Theorem 68.14 that the image of f lies in a finite connected subcomplex Y of X. But by the above discussion $[f] = 0 \in \pi_n(Y, x_0)$, hence, by functoriality of π_n, we also have $[f] = 0 \in \pi_n(X, x_0)$. ■

The discussion of $\pi_2(S^1 \vee S^2)$ on page 1573 strongly suggests that even simple topological spaces can have surprisingly large higher homotopy groups. This raises the following anxious question, which after these examples is no longer totally absurd.

Question 71.16. Are the homotopy groups of a countable CW-complex countable?

Remark. On page 1537 we saw that at least the fundamental of a countable CW-complex is countable.

We will answer this question in Proposition 104.21.

71.5. **Are there any higher homotopy groups that are non-trivial?** We have now computed lots of higher homotopy groups, but so far the results have been rather dull, all the groups we have computed so far are trivial.
So are there are any higher homotopy groups that are non-trivial? A natural candidate for a non-trivial group is surely $\pi_n(S^n)$. The following question naturally arises:

Question 71.17. Let $n \geq 2$. Does the identity map $\mathrm{id}\colon S^n \to S^n$ define a non-trivial element in the homotopy group $\pi_n(S^n, *)$?

We will answer Question 71.17 in Lemma 85.2 and Corollary 85.6. For readers familiar with the degree of a smooth map between smooth manifolds, that we introduced on page 898, we will alternatively answer Question 71.17 in Exercise 71.6. In a similar vein we will use smooth methods to answer the question in Corollary 157.12.
We can also ask, what are the higher homotopy groups of spheres, i.e. what can we say about $\pi_k(S^n)$ for $k > n$? On page 1573 we saw that all these groups are trivial for $n = 1$. But what happens for $n > 1$? Can the higher homotopy groups of spheres be non-trivial?
To construct our first example of a potentially interesting map from a higher-dimensional to a lower-dimensional sphere we recall that on page 258 we gave an identification $\mathbb{C}\mathrm{P}^1 = S^2$. We consider the following map which is usually referred to as the *Hopf*[861] *map*:

$$\begin{array}{rcl} H\colon S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^2 = \mathbb{C}\mathrm{P}^1 \\ (z_1, z_2) & \mapsto & [z_1 : z_2]. \end{array}$$

The homotopy class of H defines an element in $\pi_3(S^2)$.

Question 71.18. Does the Hopf map $H\colon S^3 \to S^2$ represent a non-trivial element in the homotopy group $\pi_3(S^2)$?

Remark. The definition of the Hopf map is very elegant, but unfortunately we pay for the elegance by loosing any intuition for what the map looks like. So how are supposed to think about the Hopf map? A straightforward calculation shows that for each $P \in S^2$ the preimage $H^{-1}(P)$ is a circle. Furthermore, if we use the explicit homeomorphism from Lemma 2.18 to make the identification of S^3 with $\mathbb{R}^3 \cup \{\infty\}$ then we can make the following observations:

(1) the circle $S^1 \times \{0\}$ corresponds to the points $(z, 0) \in S^3$, in particular the circle gets sent to the point $[1 : 0] \in \mathbb{C}\mathrm{P}^1$,
(2) the circle that is given by the z-axis together with ∞ corresponds to the points $(0, z) \in S^3$, in particular the circle gets sent to the point $[0 : 1] \in \mathbb{C}\mathrm{P}^1$.

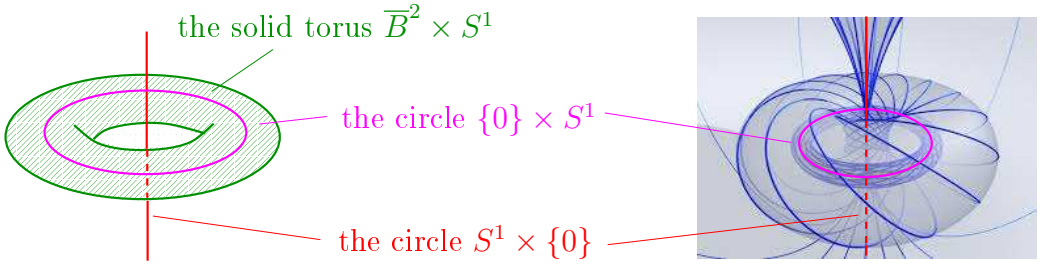
Next recall that Lemma 59.1 we gave an explicit identification[862]

$$\left(S^1 \times \overline{B}^2\right) \cup_{S^1 \times S^1} \left(\overline{B}^2 \times S^1\right) \xrightarrow{\cong} S^3.$$

It follows from the definition of this identification that on the torus $S^1 \times S^1$ the "diagonal circle" $\{(z, z) \,|\, z \in S^1\}$ is the preimage of the point $[1 : 1] \in \mathbb{C}\mathrm{P}^1$. In the figure below we show the preimages of H for some points in $\mathbb{C}\mathrm{P}^2 = S^2$.

[861] Heinz Hopf (1894-1971) was a German mathematician.

[862] Note that in Lemma 59.2 we saw that the corresponding smooth embedding $S^1 \times \overline{B}^2 \to S^3$ is essential given by the standard smooth embedding.



We introduce one more construction of maps that represent potentially non-trivial elements in homotopy groups of spheres.

Definition.

(1) Let X be a topological space. Recall that on page 451 we defined the suspension of X to be the topological space

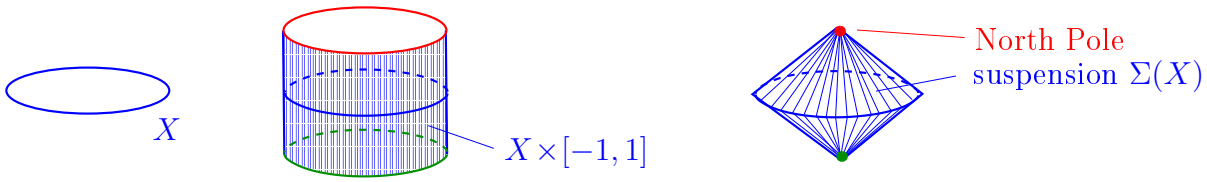
$$\Sigma(X) \;:=\; (X \times [-1,1])/\!\sim$$

where we identify all points in $X \times \{-1\}$ to a single point and we identify all points in $X \times \{1\}$ to a single point. We refer to the point $[X \times \{1\}]$ as the North Pole of $\Sigma(X)$.

(2) Given a map $f\colon X \to Y$ between topological spaces we denote by

$$\begin{array}{rcl} \Sigma(f)\colon \Sigma(X) & \to & \Sigma(Y) \\ [(x,t)] & \mapsto & [(f(x),t)] \end{array}$$

the suspension of f.



Remark. Let $n \in \mathbb{N}_0$. Recall that on page 452 we wrote down an explicit homeomorphism $\Sigma(S^n) \to S^{n+1}$ that sends the North Pole of $\Sigma(S^n)$ to the North Pole $(0,\ldots,0,1)$ of S^{n+1}. We will use this homeomorphism to identify $\Sigma(S^n)$ with S^{n+1}.

Now we have the following elementary lemma.

Lemma 71.19. (Suspension Homomorphism Lemma) Let $k \in \mathbb{N}_0$.

(1) Let (X, x_0) be a pointed topological space. We denote by $*$ the North Pole of $\Sigma(X)$. The map

$$\begin{array}{rcl} \pi_k(X,x_0) & \to & \pi_{k+1}(\Sigma(X),*) \\ [f\colon S^k \to X] & \mapsto & \big[\Sigma(f)\colon \Sigma(S^k) = S^{k+1} \to \Sigma(X)\big] \end{array}$$

is a well-defined group homomorphism. We refer to it as the suspension homomorphism.

(2) Let $n \in \mathbb{N}_0$. We denote by $*$ the North Pole of S^n and S^{n+1}. The map

$$\begin{array}{rcl} \pi_k(S^n,*) & \to & \pi_{k+1}(S^{n+1},*) \\ [f\colon S^k \to S^n] & \mapsto & \big[\Sigma(f)\colon \Sigma(S^k) = S^{k+1} \to \Sigma(S^n) = S^{n+1}\big] \end{array}$$

is a well-defined group homomorphism. We refer to it also as the suspension homomorphism.

Proof.

(1) It follows from the Suspension Functor Lemma 16.9 (4) that this map is well-defined.[863] We leave it the task of verifying that the map is in fact a group homomorphism as a somewhat challenging exercise to the reader. Alternatively we refer to [**Hilt1953**, Theorem 2.1] for a proof.

(2) This statement follows immediately from (1). ■

We conclude this chapter with the following question:

Question 71.20. Let $H\colon S^3 \to S^2$ be the Hopf map. Is the image of $[H] \in \pi_3(S^2)$ under the iterated suspension homomorphisms $\pi_3(S^2, x_0) \to \pi_{3+k}(S^{2+k}, x_1)$ non-trivial?

Unfortunately at the moment we do not have the tools to answer any of the above questions. In general it is very hard to compute higher homotopy groups as there are few computational techniques that can be used to show that higher homotopy groups are non-zero. One of the key problems is that there is no analogue to the Seifert–van Kampen Theorem 54.1. We will get to know a few techniques to compute higher homotopy groups of spheres, but we will cover these only in much later chapter, since the calculations will require the use of singular homology groups. But even then we will only be able to determine some homotopy groups of spheres. The results we obtain are summarized in Chapter 160. But note that the full computation of homotopy groups of spheres is still an open problem.

This sorry state of affairs means that as of right now we cannot use higher homotopy groups to distinguish any of the topological spaces that we could not distinguish already using the fundamental group. In particular we still cannot show that for $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic. We will only be able to do so after developing a new tool, namely the singular homology groups, in the coming sections.

Exercises for Chapter 71.

Exercise 71.1. Let X be a topological space, let $x_0 \in X$ and let $f\colon (I^n, \partial I^n) \to (X, x_0)$ be a map with $n \geq 1$. We consider the map

$$\begin{array}{rcl} \overline{f}\colon I^n & \to & X \\ (x_1, \ldots, x_n) & \mapsto & f(1-x_1, 1-x_2, \ldots, 1-x_n). \end{array}$$

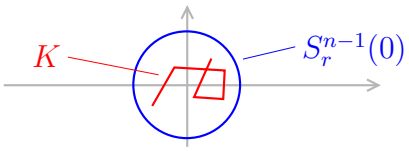
Is $[\overline{f}] \in \pi_n(X, x_0)$ the inverse of $[f] \in \pi_n(X, x_0)$?

Exercise 71.2. Let $k \leq l$ be natural numbers, let $f\colon S^k \to S^l$ be a map and let $x_0 \in S^k$. Show that for any $i \in \mathbb{N}$ the map $f_*\colon \pi_i(S^k, x_0) \to \pi_i(S^l, f(x_0))$ is the trivial map.
Hint. Use Proposition 71.10.

Exercise 71.3. In this exercise you can assume that we already know that $\pi_3(S^2) \cong \mathbb{Z}$. Show that there exists an epimorphism $\pi_3(S^1 \vee S^2) \to \mathbb{Z}^3 \oplus \mathbb{Z}_4$.

[863] In an earlier version of these it was written that "it is straightforward to see that the map is well-defined". If one traces through the ancestry of the Suspension Functor Lemma 16.9 (4) one realizes that this lemma is, despite appearances, by no means obvious.

Exercise 71.4. Let $n \in \mathbb{N}$ and let K be a non-empty compact subset of $\mathbb{R}^n$. Let $r \in \mathbb{R}$ such that $r > \sup\{\|P\| \,|\, P \in K\}$. Show that for any $k \in \mathbb{N}$ the inclusion induced map $\pi_k(S_r^{n-1}(0)) \to \pi_k(\mathbb{R}^n \setminus K)$ is a monomorphism.



Exercise 71.5. We consider the pseudocircle, i.e. we consider the slightly weird topological space $X = \{A, B, C, D\}$ from page 141. Show that $\pi_n(X, A) = 0$ for any $n \geq 2$.
Hint. Find a convenient covering of X.

Exercise 71.6. Let $n \in \mathbb{N}$. Given a smooth map $f\colon S^n \to S^n$ we introduced on page 899 the corresponding degree $\deg(S^n) \in \mathbb{Z}$. Now let $f\colon S^n \to S^n$ be any (not necessarily smooth) map. By the Whitney Approximation Theorem 29.1 we know that f is homotopic to a smooth map $g\colon M \to N$. It follows from the Degree Homotopy Invariance Theorem 38.3 that the map

$$\begin{array}{rcl} \deg\colon \pi_n(S^n, x_0) & \to & \mathbb{Z} \\ [f] & \mapsto & \deg(g\colon S^n \to S^n \text{ a smooth map homotopic to } f) \end{array}$$

is well-defined.

(a) Show that $\deg\colon \pi_n(S^n, x_0) \to \mathbb{Z}$ is well-defined.
(b) Show that $\deg\colon \pi_n(S^n, x_0) \to \mathbb{Z}$ is an epimorphism.
(c) The adventurous reader can try to show that $\deg\colon \pi_n(S^n, x_0) \to \mathbb{Z}$ is actually a monomorphism. For example, suppose that we have a smooth map $f\colon S^n \to S^n$ with $\deg(f\colon S^n \to S^n) = 0$. Pick a regular value $y \in S^n$. If $\#f^{-1}(y) > 0$ there exist two points x_0, x_1 in the preimage with $\deg(f, x_0) = -\deg(f, x_1)$. Can one find a homotopy from f to a map g such that $g^{-1}(y)$ has two preimages less?

Exercise 71.7. Let (X, x_0) be a topological space. We denote by $\Sigma(X)$ the suspension of X and we denote by $*$ the North Pole of $\Sigma(X)$. Is the suspension homomorphism

$$\pi_1(X_0, x_0) \;\to\; \pi_2(\Sigma(X), *),$$

that we introduced in the Suspension Homomorphism Lemma 71.19, necessarily a monomorphism?

Exercise 71.8. Let M be a smooth manifold, let $Y \subset M$ be a compact submanifold and let $P \in M \setminus Y$.

(a) Show that for every $i < \dim(M) - \dim(Y) - 1$ the map $\pi_i(M \setminus Y, P) \to \pi_1(M, P)$ is an isomorphism.
(b) Show that for $i = \dim(M) - \dim(Y) - 1$ the map $\pi_i(M \setminus Y, P) \to \pi_1(M, P)$ is an epimorphism.

Hint. Use the Transversality Theorem 42.4.

Exercise 71.9. On page 164 we introduced the Topologist's Sine Curve

$$S \;:=\; \big\{(0, y) \,\big|\, y \in [-1, 1]\big\} \cup \big\{(x, \sin(\tfrac{1}{x})) \,\big|\, x \in (0, \tfrac{1}{\pi}]\big\} \;\subset\; \mathbb{R}^2.$$

As on page 276 we refer to quotient space $W := S/(0,0) \sim (\frac{1}{\pi}, 0)$ as the *Warsaw Circle*.

(a) Show that the image of every map $S^n \to W$ lies in a contractible subset of W.
Hint. Use Exercise 2.48 and use the fact that each S^n is path-connected.

(b) Show that for every $n \in \mathbb{N}$ we have $\pi_n(W, [(0,0)]) = 0$.

72. The homology groups of a topological space

One of our main goals is to show that for $m \neq n$ the spheres S^m and S^n are not homeomorphic. We have seen that for most cases fundamental groups are of no use. Furthermore, it might be that in principle higher homotopy groups can distinguish such spheres, but since we cannot compute higher homotopy groups, those are of no use either. So we need a new approach to finding invariants that can distinguish spheres.
In this chapter we want to introduce a new functor from the category of topological spaces to an algebraic category, more precisely, we will introduce a functor from the category of topological spaces to the category of abelian groups.

72.1. Singular chains.

Definition. Let $n \in \mathbb{N}_0$.

(1) Given k points $P_1, \dots, P_k$ in $\mathbb{R}^n$ such that $P_2 - P_1, \dots, P_k - P_1$ are linearly independent we refer to the convex hull of $P_1, \dots, P_k$ as a k-simplex.

(2) Given $n \in \mathbb{N}_0$ we refer to the convex hull of the standard basis vectors $e_1, \dots, e_{n+1}$ of $\mathbb{R}^{n+1}$, i.e. to

$$\Delta^n := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \,|\, x_0 + \dots + x_n = 1 \text{ and } x_i \geq 0 \text{ for all } i = 0, \dots, n\}$$

as the standard n-simplex.[864]

(3) Given the standard n-simplex Δ^n we define its boundary as[865]

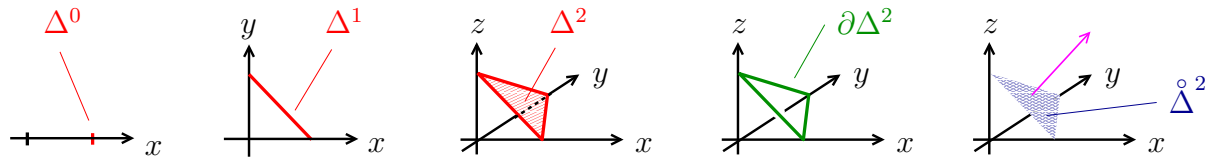
$$\partial\Delta^n := \bigcup_{i=0}^{n} \{(t_0, \dots, t_n) \in \Delta^n \,|\, t_i = 0\}.$$

(4) We write

$$\mathring{\Delta}^n := \Delta^n \setminus \partial\Delta^n.$$

Sometimes it will be helpful to view $\mathring{\Delta}^n$ as a smooth manifold where a smooth atlas for $\mathring{\Delta}^n$ is given by projection to the first n coordinates. We equip the smooth manifold $\mathring{\Delta}^n$ with the orientation where for each $P \in \mathring{\Delta}^n$ a basis $v_1, \dots, v_n$ of $\widetilde{\mathrm{T}}_P\mathring{\Delta}^n$ is positive if the vectors $\{(1, \dots, 1), v_1, \dots, v_n\}$ are a positive basis for $\mathbb{R}^{n+1}$.

Example. The standard 0-simplex Δ^0 is just the point $\{1\} \in \mathbb{R}$, the standard 1-simplex Δ^1 equals the segment in $\mathbb{R}^2$ from $(1, 0)$ to $(0, 1)$ and the standard 2-simplex Δ^2 is the triangle in $\mathbb{R}^3$ spanned by the vertices $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$. We refer to the figure below for an illustration of the standard simplices in dimension 0, 1 and 2 and for an illustration of the boundary of the standard 2-simplex.



[864] In Exercise 2.38 (a) we showed that the standard n-simplex Δ^n is the convex hull of $e_i \in \mathbb{R}^{n+1}$, $i = 1, \dots, n+1$. In particular the standard n-simplex is an n-simplex.

[865] The definition of the "boundary" of the standard n-simplex Δ^n is the "obvious one". Nonetheless note that if we had taken the original definition of the boundary of a subset A in some subspace X as introduced on page 116 and applied it to $\Delta^n \subset \mathbb{R}^{n+1}$, then the boundary of Δ^n in $\mathbb{R}^{n+1}$ would be all of Δ^n.

For the record we state the following elementary lemma.

Lemma 72.1. (Standard Simplex Homeomorphism Lemma) Let $n \in \mathbb{N}$.

(1) Every n-simplex, in particular the standard n-simplex Δ^n, is compact and contractible.

We set

$$\triangle_n := \left\{(x_1, \ldots, x_n) \in [0,1]^n \,\middle|\, \sum_{i=1}^{n} x_i \in [0,1]\right\}$$

and refer to it as the planar n-simplex. The following statements hold:

(2) The maps

$$\begin{array}{rcl} \Phi\colon \Delta^n & \to & \triangle_n \\ (t_0, \ldots, t_n) & \mapsto & (t_0, \ldots, t_{n-1}) \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi\colon \triangle_n & \to & \Delta^n \\ (s_0, \ldots, s_{n-1}) & \mapsto & \left(x_1, \ldots, x_n, 1 - \sum_{i=1}^{n} x_i\right) \end{array}$$

are homeomorphisms that are inverses of one another.

(3) The map

$$\Delta^n \xrightarrow{\Phi} \triangle_n \xrightarrow[\text{applied to } \triangle_n \text{ and } Q = (\frac{1}{n+1}, \ldots, \frac{1}{n+1})]{\text{the homeomorphism } \Xi \text{ obtained from the Convex-to-Ball Proposition 2.20 (2)}} \overline{B}^n$$

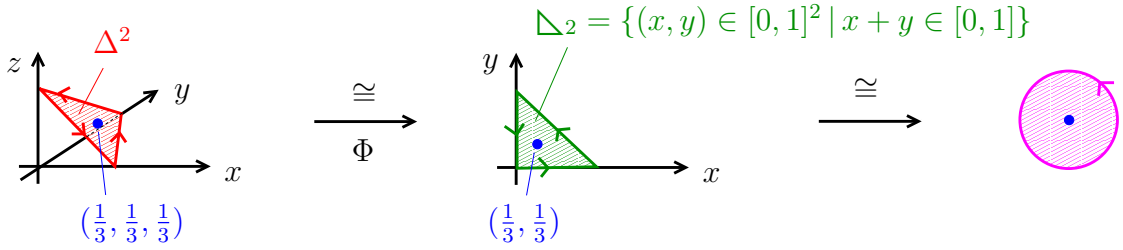
is a homeomorphism that restricts to a homeomorphism from $\partial\Delta^n$ to the sphere S^{n-1}.

Proof.

(1) We had proved in Exercise 2.38 (b) that each n-simplex is compact. (Of course it is not difficult to prove the statement using Heine-Borel Theorem 1.24.) Note that each n-simplex is convex, thus star-shaped, thus it is contractible by the elementary discussion on page 432.

(2) Note that Φ extends "by the same formula" to a map $\mathbb{R}^{n+1} \to \mathbb{R}^n$ which is evidently continuous. Hence the restriction to Δ^n is also continuous. Now that this is settled we see that the remaining statements are basically trivial.

(3) This statement follows easily from (1) and the Convex-to-Ball Proposition 2.20 (2). ■



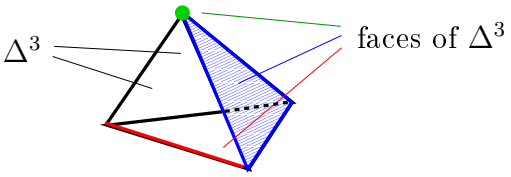
Convention. Let $n \in \mathbb{N}$. We use the homeomorphism from the Standard Simplex Homeomorphism Lemma 72.1 to make the identifications $\Delta^n = \overline{B}^n$ and $\partial\Delta^n = S^{n-1}$.

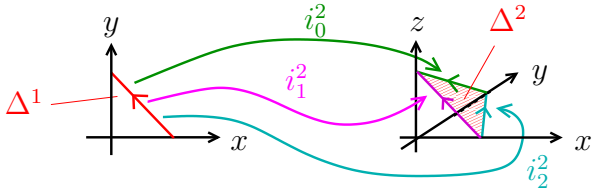
Definition. An r-dimensional face of the standard n-simplex is defined as a subset of the form

$$\{(t_0, \ldots, t_n) \in \Delta^n \,|\, t_{d_1} = t_{d_2} = \cdots = t_{d_{n-r}} = 0\},$$

where $d_1 < d_2 < \cdots < d_{n-r} \in \{0, \ldots, n\}$ are fixed.

Remark.





(1) The boundary $\partial\Delta^n$ of the standard n-simplex is the union of all $(n-1)$-dimensional faces of Δ^n.
(2) Every r-dimensional face of Δ^r is homeomorphic to the standard r-simplex. For later we fix homeomorphism in the case of $(n-1)$-dimensional faces. More precisely, for $j \in \{0, \dots, n\}$ we consider the map

$$\begin{array}{rcl} i_j^n \colon \Delta^{n-1} & \to & \Delta^n \\ (t_0, \dots, t_{n-1}) & \mapsto & (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{n-1}). \end{array}$$

This map is sometimes called the *j-th face map*. It is straightforward to see that these face maps are homeomorphisms. The face maps $i_0^2, i_1^2, i_2^2 \colon \Delta^1 \to \Delta^2$ are illustrated in the figure below on the right.

The first idea for the definition of homology groups is now the following. In the definition of the fundamental group $\pi_1(X, x_0)$ we had considered all maps $f \colon [0,1] \to X$ with the extra condition $f(0) = f(1) = x_0$. Using this extra condition we could concatenate loops to eventually end up with a group structure. Now given $n \in \mathbb{N}_0$ we want to consider all maps from Δ^n to X with no extra condition on the boundary. This time there is no reasonable way how two such maps can be composed to give a new map. Since we nonetheless want the notion of a group we turn the set of all maps $\Delta^n \to X$ into a group by "brute force", i.e. by considering the free abelian group generated by these maps.
This leads us to the following definition.

Definition. Let X be a topological space and let $n \in \mathbb{N}_0$.
(1) A singular n-simplex is a map $\sigma \colon \Delta^n \to X$.
(2) We define

$$\mathrm{C}_n(X) \ := \ \text{free abelian group generated by the set of singular } n\text{-simplices in } X.$$

For $n \in \mathbb{Z}_{<0}$ we set $\mathrm{C}_n(X) = 0$.
(3) We refer to an element in $\mathrm{C}_n(X)$ as a singular n-chain in X.

Remark. In principle, given a map $\sigma \colon \Delta^n \to X$ to some topological space X we need to show that this map is actually continuous. If one really feels compelled to give an argument, then often it is more convenient to show the equivalent statement that the map $\sigma \circ \Psi \colon \triangle_n \to X$ is continuous, where $\Psi \colon \triangle_n \to \Delta^n$ is the homeomorphism given in the Standard Simplex Homeomorphism Lemma 72.1 (1).

Examples.

(1) For $n = 0$ a singular n-simplex is a map from Δ^0 to X. Such a map is uniquely determined by the corresponding point in X. Therefore frequently we will identify singular 0-simplices with points in X.

(2) For points $z_0, \dots, z_k \in \mathbb{R}^n$ we consider the map[866]

$$\begin{aligned} [z_0, \dots, z_k] \colon \Delta^k &\to \mathbb{R}^n \\ (t_0, \dots, t_k) &\mapsto \sum_{j=0}^{k} t_j \cdot z_j. \end{aligned}$$

The map $[z_0, \dots, z_k]$ thus sends the standard k-simplex Δ^k to the simplex spanned by $z_0, \dots, z_k$ in $\mathbb{R}^n$. Put differently, $[z_0, \dots, z_k]$ is a singular k-simplex in $\mathbb{R}^n$.

Remark. Let X be a topological space. By the discussion on page 1147 we can view every element in $\mathrm{C}_n(X)$, i.e. every singular n-chain, as a linear combination $a_1\sigma_1 + \cdots + a_k\sigma_k$ where $a_1, \dots, a_k \in \mathbb{Z}$ and where $\sigma_1, \dots, \sigma_k$ are singular n-simplices.

Now let $f \colon X \to Y$ be a map between topological spaces. For each n we consider

$$\begin{aligned} f_* \colon \mathrm{C}_n(X) &\to \mathrm{C}_n(Y) \\ \sum_{i=1}^{k} a_i \cdot \sigma_i &\mapsto \sum_{i=1}^{k} a_i \cdot (f \circ \sigma_i). \end{aligned}$$

It is easy to verify that $X \mapsto \mathrm{C}_n(X)$ and $f \mapsto f_*$ defines a functor from the category of topological spaces to the category of abelian groups.
Have we made any progress towards our goal of distinguishing topological spaces? Not really. For example, if X is a smooth manifold of dimension ≥ 1, then given any $n \in \mathbb{N}_0$ there exist uncountably many maps $\Delta^n \to X$, i.e. the group $\mathrm{C}_n(X)$ is a free abelian group with uncountably many generators. Such groups are totally useless for distinguishing non-homeomorphic topological spaces.

72.2. **Definition of the homology groups of a topological space.** Before we give the definition of homology groups it is perhaps helpful to recall the definition of de Rham cohomology.
Let M be a smooth manifold. We write

$$\Omega^n(M) \;:=\; \{\text{smooth } n\text{-forms on } M\}.$$

The isomorphism type of this vector space is an invariant of the diffeomorphism type of the smooth manifold M, but it is rather useless, since for $1 \leq n \leq \dim(M)$ this is a vector space of uncountable dimension. To cut $\Omega^n(M)$ into something much smaller we defined linear maps $d_n \colon \Omega^n(M) \to \Omega^{n+1}(M)$ with $d_{n+1} \circ d_n = 0$. Finally we defined the n–th de Rham cohomology group of M as

$$\mathrm{H}^n_{\mathrm{dR}}(M) \;:=\; \ker(d_n \colon \Omega^n(M) \to \Omega^{n+1}(M)) / \operatorname{im}(d_{n-1} \colon \Omega^{n-1}(M) \to \Omega^n(M)).$$

We are now in a similar situation: to a topological space X we associated the free abelian group $\mathrm{C}_n(X)$ which is "too big". The idea now is to introduce a "differential" as for the case of differential forms. But in this case there is no good way to associate to a map $\Delta^n \to X$ a new map $\Delta^{n+1} \to X$. But by restricting the map $\Delta^n \to X$ to each of the $(n-1)$-dimensional faces of Δ^n which are, as we have seen, homeomorphic to the standard $(n-1)$-simplex, we can associate to a singular n-simplex $\sigma \colon \Delta^n \to X$ a sum of singular $(n-1)$-simplices, i.e. a singular $(n-1)$-chain. To do this we will use the maps $i^n_k \colon \Delta^{n-1} \to \Delta^n$, $k = 0, \dots, n$ from

[866] Using the Standard Simplex Homeomorphism Lemma 72.1 (2) one can easily show that this map is continuous.

page 1585 to identify the standard $(n-1)$-simplex with the $(n-1)$-dimensional faces of Δ^n. Now we can define

$$\widetilde{\partial}_n\sigma \;:=\; \sum_{j=0}^{n}\sigma\circ i_j^n.$$

By Lemma 51.1 this defines a map $\widetilde{\partial}_n\colon \mathrm{C}_n(X)\to\mathrm{C}_{n-1}(X)$. But it is quite straightforward to see that for this definition the composition of the maps $\widetilde{\partial}_n\colon \mathrm{C}_n(X)\to\mathrm{C}_{n-1}(X)$ and $\widetilde{\partial}_{n-1}\colon \mathrm{C}_{n-1}(X)\to\mathrm{C}_{n-2}(X)$ is not zero[867]. But we can rectify this problem by cleverly introducing signs.

Now we are ready to define homology groups. For clarity we first repeat the definition of the face maps. Afterwards we will provide the definition of homology groups.

Definition. Let $n\in\mathbb{N}_0$. For $j\in\{0,\dots,n\}$ we define the j-th face map as follows:

$$\begin{array}{rcl} i_j^n\colon \Delta^{n-1} & \to & \Delta^n \\ (t_0,\dots,t_{n-1}) & \mapsto & (t_0,\dots,t_{j-1},0,t_j,\dots,t_{n-1}). \end{array}$$

Definition. Let X be a topological space and let $\sigma\colon\Delta^n\to X$ be a singular n-simplex. The boundary $\partial_n\sigma\in\mathrm{C}_{n-1}(X)$ of σ is defined as

$$\partial_n\sigma \;:=\; \sum_{j=0}^{n}(-1)^j\cdot\sigma\circ i_j^n.$$

By Lemma 51.1 the map $\sigma\mapsto\partial_n\sigma$ extends uniquely to a linear map

$$\begin{array}{rcl} \partial_n\colon \mathrm{C}_n(X) & \to & \mathrm{C}_{n-1}(X) \\ \displaystyle\sum_{r=1}^{k} a_r\cdot\sigma_r & \mapsto & \displaystyle\sum_{r=1}^{k} a_r\cdot\partial\sigma_r, \end{array}$$

which we refer to as the n-th boundary map. If it is clear with what dimension we are working with, then we just write ∂ instead of ∂_n.

Examples.

(1) For any singular 1-simplex $\sigma\colon\Delta^1\to X$ we have[868]

$$\partial\sigma \;=\; (-1)^0\cdot\sigma\circ i_0^1+(-1)^1\cdot\sigma\circ i_1^1 \;=\; \sigma(0,1)-\sigma(1,0).$$

Put differently, a singular 1-simplex can be viewed as a map σ from the "interval" Δ^1 to X. The boundary of σ is thus the "endpoint" $\sigma(0,1)$ "minus" the "starting point" $\sigma(1,0)$. Here we view points in X again as singular 0-simplices. We sketch the definition of a singular 1-simplex in the figure below.

(2) In the figure below we sketch the boundary of a singular 2-simplex.

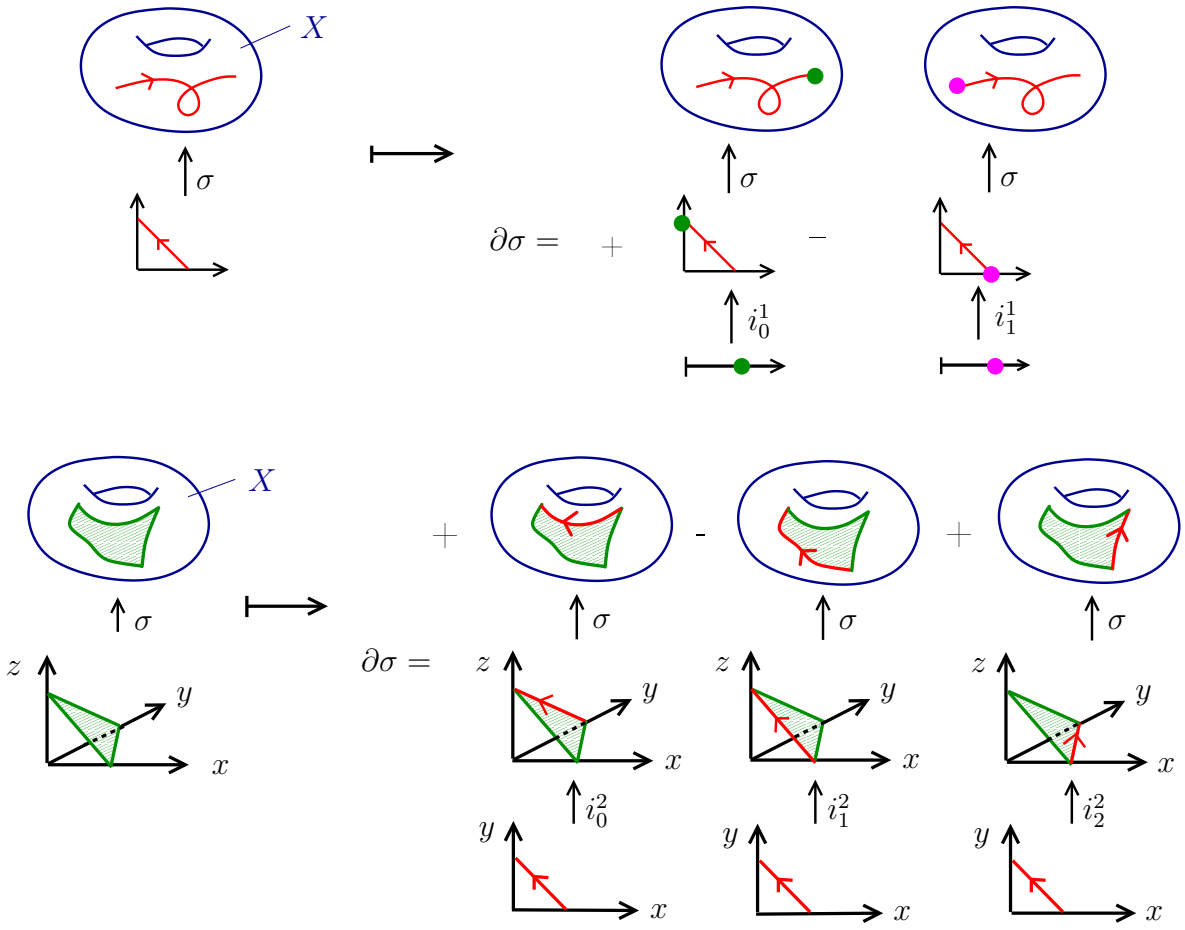
As promised, these boundary maps have the following fundamental property:

Proposition 72.2. Let X be a topological space. For any $n\in\mathbb{N}_0$ the composition

$$\partial_{n-1}\circ\partial_n\colon \mathrm{C}_n(X)\;\to\;\mathrm{C}_{n-2}(X)$$

[867]Indeed, let S be a set and let $\mathbb{Z}^{(S)}$ be the free abelian group generated by S. We define $\epsilon\colon\mathbb{Z}^{(S)}\to\mathbb{Z}$ to be the linear map that assigns to each $s\in S$ the value 1. It follows easily from the definitions that for any $\sigma\colon\Delta^n\to X$ we have $\epsilon(\widetilde{\partial}_n\sigma)=n+1$ and $\epsilon((\widetilde{\partial}_{n-1}\circ\widetilde{\partial}_n)(\sigma))=n\cdot(n+1)$.

[868]Here we again use the convention that given a point $x\in X$ we denote by x also the corresponding singular 0-simplex.



is the zero map.

Proof. It suffices to show that $\partial_{n-1}(\partial_n(\sigma)) = 0$ for every singular n-simplex $\sigma\colon \Delta^n \to X$. Thus let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. Then it follows immediately from the definitions that

$$\begin{aligned}\partial_{n-1}(\partial_n(\sigma)) &= \partial_{n-1}\Big(\sum_{j=0}^{n}(-1)^j\cdot\sigma\circ i_j^n\Big) = \sum_{k=0}^{n-1}(-1)^k\cdot\sum_{j=0}^{n}(-1)^j\cdot(\sigma\circ i_j^n)\circ i_k^{n-1}\\ &= \sum_{0=k<j\le n}(-1)^{k+j}\cdot\sigma\circ(i_j^n\circ i_k^{n-1}) + \sum_{0=j\le k\le n-1}(-1)^{k+j}\cdot\sigma\circ(i_j^n\circ i_k^{n-1}) =: (*).\end{aligned}$$

Now we want to show that each expression $\sigma\circ(i_j^n\circ i_k^{n-1})$ appears precisely twice, but with opposite signs. From the definitions it follows easily that for $j\le k$ we have[869]

$$i_j^n\circ i_k^{n-1} = i_{k+1}^n\circ i_j^{n-1}.$$

[869] Indeed, if $j\le k$, then the zero-entries of $(i_j^n\circ i_k^{n-1})(t_0,\dots,t_{n-1})$ are the entries j and $k+1$.

Now we continue with the above calculation:

$$(*) \underset{\substack{\uparrow \\ \text{above calculation of } i_j^n \circ i_k^{n-1}}}{=} \sum_{0=k<j\leq n} (-1)^{k+j} \cdot \sigma \circ (i_j^n \circ i_k^{n-1}) + \sum_{0=j\leq k\leq n-1} (-1)^{k+j} \cdot \sigma \circ (i_{k+1}^n \circ i_j^{n-1})$$

$$\underset{\substack{\uparrow \\ \text{substitution } l=k+1}}{=} \sum_{0=k<j\leq n} (-1)^{k+j} \cdot \sigma \circ (i_j^n \circ i_k^{n-1}) + \sum_{0=j<l\leq n} (-1)^{l+j-1} \cdot \sigma \circ (i_l^n \circ i_j^{n-1}) \underset{\substack{\uparrow \\ \text{all summands left and right cancel}}}{=} 0.$$

■

The following is arguably the most important definition of algebraic topology.

Definition. Let X be a topological space and let $n \in \mathbb{N}_0$. We define the n-th singular homology group of X as the quotient group[870]

$$\mathrm{H}_n(X) := \frac{\ker(\partial_n \colon \mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X))}{\mathrm{im}(\partial_{n+1} \colon \mathrm{C}_{n+1}(X) \to \mathrm{C}_n(X))}.$$

For $n \in \mathbb{Z}_{<0}$ we extend this definition to $\mathrm{H}_n(X) := 0$. Throughout the lecture notes we will mostly shorten the name "singular homology" to "homology".[871]

One of the key goals of this course is to develop tools for computing homology groups of topological spaces. In particular we want to compute the homology groups for all spheres. In the subsequent discussions it is helpful to introduce a few more definitions.

Definition. Let X be a topological space.

(1) We say $c \in \mathrm{C}_n(X)$ is a singular n-cycle, or short n-cycle, if $\partial c = 0$.
(2) If c is an n-cycle, then we refer to $[c] \in \mathrm{H}_n(X)$ as the homology class of c.
(3) We say two singular n-chains c and d are homologous if there exists a singular $(n+1)$-chain e with $\partial e = c - d$. In particular, if c and d are cycles, then they are homologous if they represent the same homology class.
(4) An n-cycle c is called null-homologous, if $[c] = 0 \in \mathrm{H}_n(X)$.
(5) Given a singular 1-simplex $\sigma \colon \Delta^1 \to X$ we denote by $\overline{\sigma}$ the singular 1-simplex that is given by $\overline{\sigma}(t, 1-t) = \sigma(1-t, t)$. Put differently, $\overline{\sigma}$ is given by reversing the orientation of σ.

Example. In the figure below we show that torus $T = ([0,1] \times [0,1])/\sim$ together with two singular 1-simplices c and d and together with two singular 2-simplices α and β. It follows easily from the definitions that c and d are 1-cycles. It is a good exercise to verify that $\alpha - \beta$ is a 2-cycle. With our present knowledge it is impossible though to determine whether these cycles represent non-trivial homology classes.

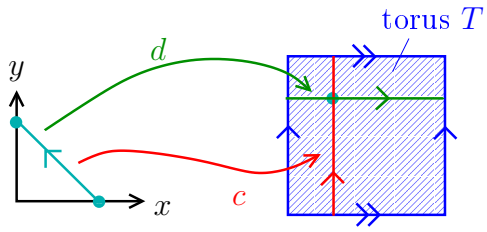
The following lemma will be proved in Exercises 72.3 and 72.6.

[870] It follows from Proposition 72.2 that we have the following inclusion:

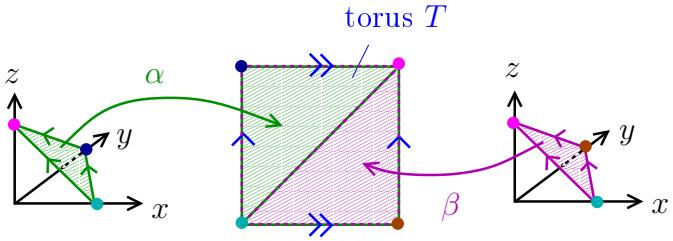
$$\mathrm{im}(\partial_{n+1} \colon \mathrm{C}_{n+1}(X) \to \mathrm{C}_n(X))) \subset \ker(\partial_n \colon \mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X)).$$

Furthermore the groups $\mathrm{C}_*(X)$ are abelian, in particular any subgroup is also abelian and normal. Therefore it does indeed make sense to consider the quotient group.

[871] Later on, in Chapters 80 and 95, we will also introduce "cellular homology" and "simplicial homology". At that point it will be useful to remember that our present homology groups have a more precise name, namely singular homology groups.



c and d are 1-cycles



$\alpha - \beta$ is a 2-cycle

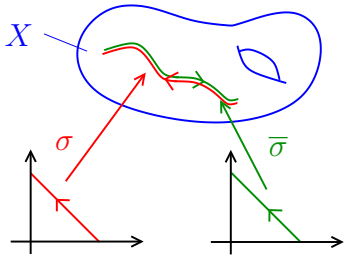
Lemma 72.3. Let X be a topological space and let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex.

(1) If σ is a constant map, then σ is null-homologous.
(2) The singular 1-chains $-\sigma$ and $\overline{\sigma}$ are homologous.
(3) Let $s \in (0,1)$ and consider the singular 1-simplices

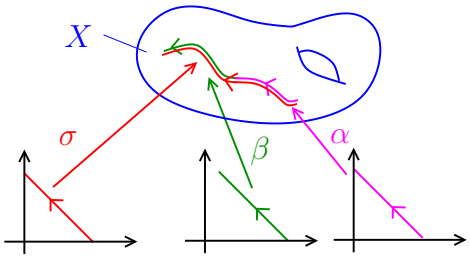
$$\begin{array}{rcl} \alpha\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & \sigma(st, 1-st) \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & \sigma(s+t(1-s), 1-(s+t(1-s))). \end{array}$$

Then the singular 1-chains σ and $\alpha + \beta$ are homologous.

The second part of the lemma says that for a singular 1-simplex the geometric operation of reversing the orientation is (up to the boundary of a simplicial 2-chain) the same as algebraically inverting the sign. Furthermore, the third part of the lemma says that for a singular 1-simplex (again up to the boundary of a simplicial 2-chain) it does not make a difference whether or not we split the singular 1-simplex into two singular 1-simplices.



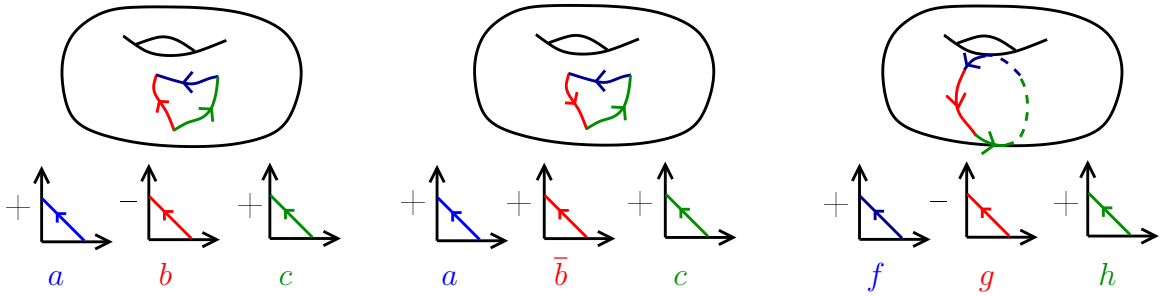
σ and $-\overline{\sigma}$ are homologous



σ and $\alpha + \beta$ are homologous

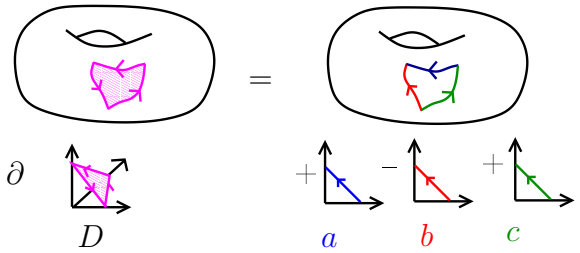
Examples.

(1) We consider the figure below. We see three singular chains on the torus, and it is straightforward to show that all three singular chains are cycles. Now the question



arises, which of these cycles are in fact null-homologous. In the figure below we

see that the first cycle $a - b + c$ in the figure below is the boundary of a singular 2-simplex D. One can easily see that the second cycle, $a + \overline{b} + c$ in the figure below



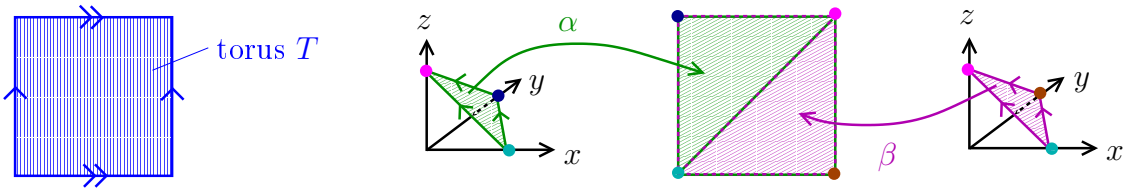
is *not* the boundary of a singular 2-simplex. On the other hand by Lemma 72.3 we know that $b + \overline{b}$ is a null-homologous cycle, i.e. there exists an $E \in \mathrm{C}_2(X)$ such that $b + \overline{b} = \partial E \in \mathrm{C}_1(X)$. Thus it follows that

$$a + \overline{b} + c \;=\; a + \partial E - b + c \;=\; a - b + c + \partial E = \partial D + \partial E \;=\; \partial(D + E),$$

i.e. $a + \overline{b} + c$ is null-homologous.

It is not clear whether the third cycle in the figure below is the boundary of a 2-dimensional singular chain. We will see later that this not the case. But it will take us quite some time to develop the techniques which will allow us to prove this statement.

(2) In the figure below we show the torus $T = ([0,1] \times [0,1])/\sim$ and two singular simplices $\alpha\colon \Delta^2 \to T$ and $\beta\colon \Delta^2 \to T$ that are given by affine linear maps.[872] It is straightforward to verify that $\alpha - \beta$ is a cycle, hence it represents an element in $\mathrm{H}_2(T)$. We will see later on page 1589 whether or not this element is zero in $\mathrm{H}_2(T)$.



(3) In Exercise 72.8 we will try to generalize the approach taken for the torus to other 2-dimensional smooth manifolds, namely the sphere, the Klein bottle and the projective space $\mathbb{R}P^2$.

72.3. **First calculations of homology groups.** Given a topological space the corresponding homology groups are defined as the quotient of two abelian groups which in general will be infinitely generated. Somewhat surprisingly we will see in the coming chapters that the homology groups are nonetheless very frequently finitely generated abelian groups.

We start out our computations of homology groups with the 0-th homology of a path-connected topological space. The key to the calculation is the following definition.

[872] More precisely, α is the unique affine linear map with $\alpha(1,0,0) = [(0,0)]$, $\alpha(0,1,0) = [(1,0)]$ and $\alpha(0,0,1) = [(1,1)]$ whereas β is the unique affine linear map with $\beta(1,0,0) = [(0,0)]$, $\beta(0,1,0) = [(0,1)]$ and $\beta(0,0,1) = [(1,1)]$.

Definition. Given a topological space X we refer to the map

$$\begin{array}{rcl} \epsilon_X \colon \mathrm{C}_0(X) & \to & \mathbb{Z} \\ \sum_{i=1}^{k} a_i \sigma_i & \mapsto & \sum_{i=1}^{k} a_i \end{array}$$

as the augmentation map.

The following lemma shows that the augmentation map is a neat addition to the singular chain complex of a topological space.

Lemma 72.4. Let X be a topological space.

(1) The map $\epsilon_X \circ \partial_1 \colon \mathrm{C}_1(X) \to \mathbb{Z}$ is the zero map.

(2) The map

$$\begin{array}{rcl} \epsilon_X \colon \mathrm{H}_0(X) & \to & \mathbb{Z} \\ [\sigma] & \mapsto & [\epsilon_X(\sigma)] \end{array}$$

is well-defined. In the following we refer to this map also as the *augmentation map*.

Proof. Let X be a topological space.

(1) Let $c = \sum_{i=1}^{n} b_i \sigma_i \in \mathrm{C}_1(X)$. We calculate that

$$\epsilon(\partial_1(c)) = \epsilon\Big(\sum_{i=1}^{n} b_i \cdot \partial_1(\sigma_i)\Big) \underset{\uparrow}{=} \epsilon\Big(\sum_{i=1}^{n} b_i \cdot \sigma_i(0,1) - b_i \cdot \sigma_i(1,0)\Big) \underset{\uparrow}{=} \sum_{i=1}^{n} b_i - \sum_{i=1}^{n} b_i = 0.$$

for every singular 1-simplex σ we have $\partial\sigma = \sigma(0,1) - \sigma(1,0)$, see page 1585

by definition of ϵ

(2) Note that

$$\begin{array}{rcl} \mathrm{H}_0(X) & = & \ker(\partial_0 \colon \mathrm{C}_0(X) \to \mathrm{C}_{-1}(X)) \,/\, \mathrm{im}(\partial_1 \colon \mathrm{C}_1(X) \to \mathrm{C}_0(X)) \\ & \underset{\uparrow}{=} & \mathrm{C}_0(X) \,/\, \mathrm{im}(\partial_1 \colon \mathrm{C}_1(X) \to \mathrm{C}_0(X)). \end{array}$$

since $\mathrm{C}_{-1}(X) = 0$

Thus we see that Statement (2) is an immediate consequence of Statement (1). ∎

Before we can state the key result of this section let us introduce the following handy convention.

Notation. Let X be a topological space. Given a point $P \in X$ we denote by P also the singular 0-simplex $\Delta^0 \to X$ that is given by sending the single point in Δ^0 to P.

Now we can formulate the following proposition.

Proposition 72.5. (H_0-Proposition) Let X be a path-connected non-empty topological space and let P be a point in X. Then the augmentation map

$$\begin{array}{rcl} \epsilon = \epsilon_X \colon \mathrm{H}_0(X) & \to & \mathbb{Z} \end{array}$$

is a natural isomorphism where the inverse is given by the map

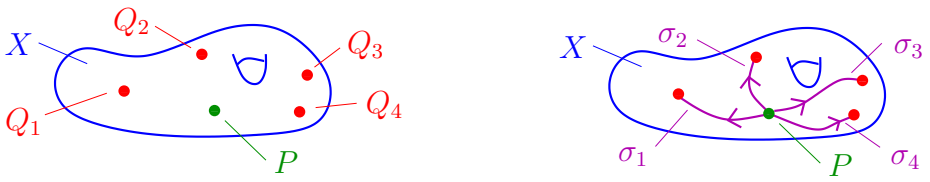
$$\begin{array}{rcl} \iota \colon \mathbb{Z} & \to & \mathrm{H}_0(X) \\ n & \mapsto & n \cdot [P]. \end{array}$$

Proof. Let X be a non-empty path-connected topological space. Let P be a point in X. It is clear that $\epsilon \circ \iota = \mathrm{id}_{\mathbb{Z}}$, in particular the homomorphism ϵ is surjective. It remains to show that ϵ is injective.

Thus let $c \in \mathrm{C}_0(X)$ with $\epsilon(c) = 0$. By definition of $\mathrm{C}_0(X)$ we can find points $Q_1, \ldots, Q_k$ in X and $a_1, \ldots, a_k \in \mathbb{Z}$ such that $c = \sum_{i=1}^{k} a_i \cdot Q_i$. We choose a point P in X. Since X is path-connected we can find, for $i = 1, \ldots, k$, a path from P to Q_i. Put differently, for each $i = 1, \ldots, k$ we can find a singular 1-simplex σ_i with $\partial\sigma_i = Q_i - P$. Then we have

$$\partial\Big(\sum_{i=1}^{k} a_i \cdot \sigma_i\Big) \;=\; \sum_{i=1}^{k} a_i \cdot \partial\sigma_i \;=\; \sum_{i=1}^{k} a_i \cdot (Q_i - P) \;=\; \sum_{i=1}^{k} a_i \cdot Q_i - \underbrace{\Big(\sum_{i=1}^{k} a_i\Big)}_{=\epsilon(c)=0} \cdot P \;=\; \sum_{i=1}^{k} a_i \cdot Q_i.$$

This implies that $[c] = 0 \in \mathrm{H}_0(X)$. Thus we have shown that ϵ is injective. ■



Lemma 72.6. (Single Point-Homology Lemma) Let X be a topological space which consists of a single point. Then the following holds

$$\mathrm{H}_n(X) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

Proof. The proof of the lemma is the content of Exercise 72.5. ■

The calculation of $\mathrm{H}_0(X)$ in the H_0-Proposition 72.5 and the calculation of the homology of a point in the Single Point-Homology Lemma 72.6 are the only calculations of homology groups of path-connected topological spaces that one can do "by hand". For all other topological spaces we will need to work significantly harder.

72.4. **Algebraic chain complexes.** Before we continue with the discussion of homology groups it is convenient to introduce several algebraic definitions. The following definitions are just the obvious algebraic abstractions of the definitions that we had already introduced.

Definition.

(1) An (algebraic) chain complex (C_*, ∂_*) is a sequence

$$\ldots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots C_1 \xrightarrow{\partial_1} C_0 \to 0$$

of maps between abelian groups[873] such that for every $i \in \mathbb{N}$ we have $\partial_{i-1} \circ \partial_i = 0$. As before we introduce the convention that $C_n = 0$ for $n \in \mathbb{Z}_{<0}$.

(2) We define the n-th homology group of the chain complex precisely the same way as we defined the homology of a topological space, namely we set

$$\mathrm{H}_n(C) \;:=\; \ker(\partial_n \colon C_n \to C_{n-1}) \,/\, \mathrm{im}(\partial_{n+1} \colon C_{n+1} \to C_n).$$

(3) As before we refer to elements in the kernel of $\partial_n \colon C_n \to C_{n-1}$ as cycles. Furthermore we say that a cycle $z \in C_n$ is null-homologous if $[z] = 0 \in \mathrm{H}_n(C)$.

(4) An algebraic chain complex (C_*, ∂_*) is called acyclic if all homology groups are zero.

On some occasions we will need the following generalization:

(5) A generalized (algebraic) chain complex (C_*, ∂_*) is a sequence

$$\ldots \to C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \to \ldots$$

of maps between abelian groups such that for every $i \in \mathbb{Z}$ we have $\partial_{i-1} \circ \partial_i = 0$. Basically all present and future definitions and results for chain complexes also make sense for generalized chain complexes. In most cases we will not explicitly state the generalization to generalized chain complex.

Now we introduce a new notion, namely the notion of a chain map between chain complexes.

Definition. A chain map $f\colon C_* \to D_*$ between two chain complexes[874] consists of a family $\{f_n\colon C_n \to D_n\}_{n\in\mathbb{N}_0}$ of maps such that for each n the following equality holds:[875]

$$f_{n-1} \circ \partial_n \;=\; \partial_n \circ f_n.$$

Put differently the maps f_n have the property that the following diagram commutes:

$$\begin{array}{ccccccccc}
\ldots & \xrightarrow{\partial} & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & \ldots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\ldots & \xrightarrow{\partial} & D_{n+1} & \xrightarrow{\partial_{n+1}} & D_n & \xrightarrow{\partial_n} & D_{n-1} & \xrightarrow{\partial_{n-1}} & \ldots
\end{array}$$

Remark. On many occasions we will use the elementary observation that the composition of two chain maps $f\colon C_* \to D_*$ and $g\colon D_* \to E_*$ defines a chain map $g \circ f\colon C_* \to E_*$.

Now we can prove the following lemma:

Lemma 72.7. Let $f\colon C_* \to D_*$ be a chain map between chain complexes C_* and D_*. Then the map

$$\begin{array}{rcl} f_*\colon \mathrm{H}_n(C_*) & \to & \mathrm{H}_n(D_*) \\ [c] & \mapsto & [f_n(c)] \end{array}$$

is well-defined.

Definition. In the setting of Lemma 72.7 we refer to the map $f_*\colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ as the induced map on homology.

Proof. Let $c \in C_n$ be a cycle. First we have to show that $f_n(c)$ is again a cycle. Indeed we have

$$\partial_n(f_n(c)) \underset{\substack{\uparrow \\ \text{definition of a chain map}}}{=} f_{n-1}(\partial_n c) \underset{\substack{\uparrow \\ \text{since } c \text{ is a cycle}}}{=} f_n(0) \;=\; 0.$$

[873] Note that we do not assume that the chain groups C_* are free abelian groups.

[874] As usual we suppress bits from the notation which are understood from the context. In this instance we suppress the notation for the boundary maps in the chain complexes. Put differently, instead of saying "let (C_*, ∂_*) be a chain complex" we say simply "let C_* be a chain complex".

[875] More precisely, for any $c \in C$ we have $f_{n-1}(\partial_n(c)) = \partial_n(f_n(c))$ where on the left-hand side we consider the boundary map $C_n \to C_{n-1}$, and on the right-hand side we consider the boundary map $D_n \to D_{n-1}$ which we also denote by ∂_n. Usually it is obvious from the context which boundary map we consider.

Now let $c, d \in C_n$ be two cycles which represent the same element in $\mathrm{H}_n(C)$. We have to show that $[f_n(c)] = [f_n(d)]$. Indeed we have

$$[f_n(c)] - [f_n(d)] \;=\; [f_n(c-d)] \underset{\substack{\text{since } c,d \text{ are homologous there} \\ \text{exists an } e \in C_{n+1} \text{ with } \partial_{n+1}e = c-d}}{\uparrow =} [f_n(\partial_{n+1}(e))] \underset{\text{definition of a chain map}}{\uparrow =} [\partial_{n+1}(f_{n+1}(e))] \;=\; 0 \;\in \mathrm{H}_n(D).$$

■

We can state the above results more formally. To do so it is convenient to introduce the following definition.

Definition. We refer to category *ChCplx* that is given by

$$\mathrm{Ob}(\mathcal{ChCplx}) \;:=\; \text{all chain complexes,}$$

and with morphisms

$$\mathrm{Mor}(C_*, D_*) \;:=\; \text{all chain maps from } C_* \text{ to } D_*.$$

with the obvious composition of morphisms as the category of chain complexes.

The following lemma follows easily from Lemma 72.7 and from the definitions:

Lemma 72.8. For each $n \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} C_* & \mapsto & \mathrm{H}_n(C_*) \\ (f\colon C_* \to D_*) & \mapsto & (f_*\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)) \end{array}$$

define a functor from the category *ChCplx* of chain complexes to the category *AbGroup* of abelian groups.

72.5. **The functoriality of homology groups.** Let $f\colon X \to Y$ be a map between topological spaces. We already saw on page 1584 that for each $n \in \mathbb{N}_0$ the map f induces a map

$$\begin{array}{rcl} f_*\colon \mathrm{C}_n(X) & \to & \mathrm{C}_n(Y) \\ \sum_{i=1}^{k} a_i \cdot \sigma_i & \mapsto & \sum_{i=1}^{k} a_i \cdot f \circ \sigma_i. \end{array}$$

Now we can formulate the following lemma:

Lemma 72.9.

(1) The boundary maps $\partial_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X)$ are natural transformations. Put differently, in plain English, if $f\colon X \to Y$ is a map between topological spaces, then given any $c \in \mathrm{C}_n(X)$ we have

$$f_*(\partial_n c) \;=\; \partial_n(f_*(c)) \;\in\; \mathrm{C}_n(Y).$$

In particular $f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ is a chain map.

(2) The maps

$$\begin{array}{rcl} \text{topological space } X & \mapsto & \mathrm{C}_*(X) \\ (f\colon X \to Y) & \mapsto & (f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)) \end{array}$$

and

define a functor from the category *Top* of topological spaces to the category *ChCplx* of chain complexes.

Proof.

(1) Let $f\colon X \to Y$ be a map between topological spaces. Given any singular n-simplex $\sigma\colon \Delta^n \to X$ we have

$$f_*(\partial\sigma) = f_*\Big(\sum_{j=0}^{n}(-1)^j\cdot\sigma\circ i_j^n\Big) = \sum_{j=0}^{n}(-1)^j\cdot f\circ(\sigma\circ i_j^n) = \sum_{j=0}^{n}(-1)^j\cdot(\underbrace{f\circ\sigma}_{=f_*\sigma})\circ i_j^n = \partial(f_*\sigma).$$

Since the singular n-simplices generate $\mathrm{C}_n(X)$ we obtain the desired equality for all singular n-chains. By definition this means that $f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ is a chain map.

(2) The second statement is an almost immediate consequence of the first statement. ■

In particular, using Lemma 72.7 we immediately obtain the following lemma.

Lemma 72.10. Let $f\colon X \to Y$ be a map between topological spaces. Then the map

$$\begin{array}{rcl} f_*\colon \mathrm{H}_n(X) & \to & \mathrm{H}_n(Y) \\ \Big[\sum_{i=1}^{k} a_i\cdot\sigma_i\Big] & \mapsto & \Big[\sum_{i=1}^{k} a_i\cdot(f\circ\sigma_i)\Big] \end{array}$$

is well-defined.

Example. Let $f\colon X \to Y$ be a map between two topological spaces. It follows immediately from the definitions that the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}_0(X) & \xrightarrow{\quad f_*\quad} & \mathrm{H}_0(Y). \\ & {\scriptstyle\epsilon_X}\searrow \quad \swarrow{\scriptstyle\epsilon_Y} & \\ & \mathbb{Z} & \end{array}$$

where the maps ϵ_X and ϵ_Y denote the augmentation maps. In particular, if X and Y are path-connected topological spaces, then it follows easily from the H_0-Proposition 72.5 that the map $f_*\colon \mathrm{H}_0(X) \to \mathrm{H}_0(Y)$ is an isomorphism. Furthermore, if $X = Y$, i.e. if f is a self-map on a path-connected topological space X, then f_* is in fact the identity on $\mathrm{H}_0(X)$.

The combination of Lemmas 72.9 and 72.8 now gives us the following lemma.

Lemma 72.11. For each $n \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} X & \mapsto & \mathrm{H}_n(X) \end{array}$$

together with the map

$$\begin{array}{rcl} (f\colon X \to Y) & \mapsto & \left(\begin{array}{rcl} f_*\colon \mathrm{H}_n(X) & \to & \mathrm{H}_n(Y) \\ \Big[\sum_{i=1}^{k} a_i\cdot\sigma_i\Big] & \mapsto & \Big[\sum_{i=1}^{k} a_i\cdot(f\circ\sigma_i)\Big] \end{array}\right) \end{array}$$

defines a functor from the category of topological spaces to the category of abelian groups.

Example. Let X and Y be non-empty topological spaces. Let $f\colon X \to Y$ be a map. We suppose that f is constant, i.e. we suppose that there exists a $y \in Y$ such that $f(X) = \{y\}$. We denote by $i\colon \{y\} \to Y$ the inclusion. Let $n \in \mathbb{N}$. We consider the commutative diagram

$$\begin{array}{ccc} & \{y\} & \\ {\scriptstyle f}\nearrow & & \searrow{\scriptstyle i} \\ X & \xrightarrow{\quad f\quad} & Y \end{array}$$

which gives rise to the commutative diagram

$$\begin{array}{ccc} & \mathrm{H}_n(\{y\}) & \\ {\scriptstyle f_*}\nearrow & & \searrow{\scriptstyle i_*} \\ \mathrm{H}_n(X) & \xrightarrow{\quad f_*\quad} & \mathrm{H}_n(Y). \end{array}$$

By the Single Point-Homology Lemma 72.6 we know that $\mathrm{H}_n(\{y\}) = 0$. Thus we see that $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ factors through the trivial group, i.e. the map is the trivial map. On many occasions we will make use of this observation without referring to this discussion.

72.6. **Direct products and direct sums** (∗). In this short section we recall the definition of the direct product and the direct sum of a family of abelian groups.

Definition. Let $G_i, i \in I$ be a family of abelian groups. We define the direct product as

$$\prod_{i\in I} G_i := \Big\{\text{all maps } g\colon I \to \bigcup_{i\in I} G_i \text{ such that for all } i \in I \text{ we have } g(i) \in G_i\Big\}.$$

Given such a map we usually write $g_i := g(i)$ for $i \in I$ and we denote such a map by $\{g_i\}_{i\in I}$. The direct product is again a group with respect to the group structure

$$\begin{array}{rcl} \prod_{i\in I} G_i \times \prod_{i\in I} G_i & \to & \prod_{i\in I} G_i \\ \big((a_i)_{i\in I}, (b_i)_{i\in I}\big) & \mapsto & (a_i + b_i)_{i\in I}. \end{array}$$

Furthermore we consider the direct sum

$$\bigoplus_{i\in I} G_i := \Big\{(g_i)_{i\in I} \in \prod_{i\in I} G_i \,\Big|\, \text{there exists only finitely many } i \in I \text{ with } g_i \neq e\Big\}.$$

The same way one can also define the direct product respectively the direct sum of rings, algebras, vector spaces, modules and so on.

Example. Let $I = \mathbb{N}$ and for each $i \in I = \mathbb{N}$ we pick $G_i = \mathbb{R}$. Then

$$\prod_{i\in\mathbb{N}} \mathbb{R} = \text{all real-valued sequences } i \mapsto a_i$$

and

$$\bigoplus_{i\in\mathbb{N}} \mathbb{R} = \text{all real-valued sequences with only finitely many non-zero terms.}$$

In general the direct product is "significantly larger" than the direct sum. For example the direct product $\prod_{i\in\mathbb{N}} \mathbb{Z}$ is uncountable while the direct sum $\bigoplus_{i\in\mathbb{N}} \mathbb{Z}$ is countable.[876]

Convention. Let $\{G_i\}_{i\in I}$ be family of abelian groups.

(1) For some element $g_j \in G_j$, following the discussion on page 1147, we sometimes denote by g_j also the element in $\bigoplus_{i\in I} G_i$ that is given by $i \mapsto 0$ if $i \neq j$ and $j \mapsto g_j$. In particular we can view each G_j as a subgroup of $\bigoplus_{i\in I} G_i$.

(2) Given $i_1, \ldots, i_l \in I$ and $g_{i_j} \in G_{i_j}$ we can use (1) to consider the corresponding element

$$g_{i_1} + \cdots + g_{i_l} \in \bigoplus_{i\in I} G_i.$$

It follows immediately from the definitions that any element of $\bigoplus_{i\in I} G_i$ is of that form.

Finally we introduce the following notation.

[876] Why is that the case?

Notation.

(I) Let $\{G_i\}_{i\in I}$ be a family of abelian groups and let H be an abelian group. Furthermore let $\{\varphi_i\colon G_i \to H\}_{i\in I}$ be a family of homomorphisms. We refer to

$$\begin{aligned} \bigoplus_{i\in I}\varphi_i\colon \bigoplus_{i\in I} G_i &\to H \\ g_{i_1}+\cdots+g_{i_l} &\mapsto \underbrace{\varphi_{i_1}(g_{i_1})}_{\in H}+\cdots+\underbrace{\varphi_{i_l}(g_{i_l})}_{\in H} \end{aligned}$$

as the direct sum of the homomorphisms $\{\varphi_i\}_{i\in I}$.

(II) Let G be an abelian group, let $\{H_i\}_{i\in I}$ be a *finite* family of abelian groups and let $\{\varphi_i\colon G \to H_i\}_{i\in I}$ be a family of homomorphisms. We refer to

$$\begin{aligned} \bigoplus_{i\in I}\varphi_i\colon G &\to \bigoplus_{i\in I} H_i \\ g &\mapsto \sum_{i\in I}\underbrace{\varphi_i(g)}_{\in H_i} \end{aligned}$$

also as the direct sum of the family of homomorphisms $\{\varphi\}_{i\in I}$.

(III) Let $\{\varphi_i\colon G_i \to H_i\}_{i\in I}$ be a *finite* family of homomorphisms between abelian groups. We refer to

$$\begin{aligned} \bigoplus_{i\in I}\varphi_i\colon \bigoplus_{i\in I} G_i &\to \bigoplus_{i\in I} H_i \\ \sum_{i\in I} g_i &\mapsto \sum_{i\in I}\underbrace{\varphi_i(g_i)}_{\in H_i} \end{aligned}$$

also as the direct sum of the homomorphisms $\{\varphi_i\}_{i\in I}$.

The notation is arguably somewhat ambiguous. But usually it should be clear from the context what we precisely mean when we write $\bigoplus_{i\in I}\varphi_i$. Sometimes, in desperation, we will refer to $\bigoplus_{i\in I}\varphi_i$ a direct sum of type (I), (II) or (III).

For reference we jot down the following lemma which follows immediately from the definitions.

Lemma 72.12. Let I be a finite set and let $\{\varphi_i\colon G_i \to H_i\}_{i\in I}$ be a family of homomorphisms between abelian groups.

(1) Let F be an abelian group and let $\{\alpha\colon F \to G_i\}_{i\in I}$ be a family of homomorphisms. Then
$$\left(\bigoplus_{i\in I}\varphi_i\right)\circ\left(\bigoplus_{i\in I}\alpha_i\right) = \bigoplus_{i\in I}\alpha_i\circ\varphi_i\colon F \to \bigoplus_{i\in I} G_i.$$

(2) Let $\{\alpha_i\colon F_i \to G_i\}_{i\in I}$ be another family of homomorphisms between abelian groups. Then
$$\left(\bigoplus_{i\in I}\varphi_i\right)\circ\left(\bigoplus_{i\in I}\alpha_i\right) = \bigoplus_{i\in I}\alpha_i\circ\varphi_i\colon \bigoplus_{i\in I} F_i \to \bigoplus_{i\in I} G_i.$$

72.7. **The homology groups of a direct sum.** Now we return to the study of chain complexes. Let C_* and D_* be two chain complexes. The same way as we introduced the direct sum of abelian groups we can also define the direct sum of chain complexes. The following lemma says that homology commutes with direct sums and direct products. The proof is a standard exercise in homological algebra, which oddly enough nobody ever feels like doing, see [**Weib1994**, p. 5].

Lemma 72.13. If $\{C_a\}_{a\in A}$ is a family of chain complexes, then the obvious inclusion maps $C_a \to \bigoplus_{a\in A} C_a$ and $C_a \to \prod_{a\in A} C_a$ induce isomorphisms

$$\bigoplus_{a\in A} \mathrm{H}_n(C_a) \xrightarrow{\cong} \mathrm{H}_n\Big(\bigoplus_{a\in A} C_a\Big) \quad \text{and} \quad \prod_{a\in A} \mathrm{H}_n(C_a) \xrightarrow{\cong} \mathrm{H}_n\Big(\prod_{a\in A} C_a\Big).$$

Now we can formulate and prove the following lemma.

Lemma 72.14. Let X be a topological space with path-components $\{X_a\}_{a\in A}$. Given $a \in A$ we denote by $i_a\colon X_a \to X$ the inclusion map. Then the map

$$\bigoplus_{a\in A} i_a\colon \quad \bigoplus_{a\in A} \mathrm{C}_*(X_a) \quad \to \quad \mathrm{C}_*(X)$$

is an isomorphism of chain complexes. In particular for each $n \in \mathbb{N}_0$ the map

$$\bigoplus_{a\in A} i_{a*}\colon \quad \bigoplus_{a\in A} \mathrm{H}_n(X_a) \quad \to \quad \mathrm{H}_n(X)$$

is an isomorphism.

The lemma says in particular that the homology groups of a topological space X are the direct sum of the homology groups of the path-components of X. Put differently, this shows that in most cases it suffices to consider the homology groups of path-connected topological spaces.

Examples.

(1) One of our goals is to calculate the homology groups of the spheres. Now we can calculate the homology groups of the 0-dimensional sphere $S^0 = \{\pm 1\}$. More precisely, by Lemmas 72.6 and 72.14 we have

$$\mathrm{H}_n(S^0) \cong \begin{cases} \mathbb{Z}^2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

Later in the proof of Proposition 74.4 we will cleverly reduce the computation of the homology groups of all other spheres to this calculation.

(2) Let X be a topological space and let $\{U_a\}_{a\in A}$ be a family of disjoint open subsets of X. It follows fairly easily from Lemma 72.14 together with the Component Lemma 2.33 (4) and (5) that the map

$$\bigoplus_{a\in A} i_{a*}\colon \bigoplus_{a\in A} \mathrm{H}_n(U_a) \to \mathrm{H}_n\Big(\bigcup_{a\in A} U_a\Big)$$

is an isomorphism.

Proof. Evidently the standard simplex Δ^n is path-connected. Thus it follows from Lemma 2.35 that the image of a map $\sigma\colon \Delta^n \to X$ is contained in a path-component X_a. This shows that

$$\bigsqcup_{a\in A} \text{singular } n\text{-simplices in } X_a \quad = \quad \text{singular } n\text{-simplices in } X.$$

It follows that the map

$$\begin{aligned} \bigoplus_{a\in A} i_a\colon \bigoplus_{a\in A} \mathrm{C}_n(X_a) &\to \mathrm{C}_n(X) \\ \Big(\sum_{i=1}^k c_i(\sigma\colon \Delta^n \to X_{a_i})\Big) &\mapsto \Big(\sum_{i=1}^k c_i(\sigma\colon \Delta^n \to X_{a_i})\Big) \end{aligned}$$

is an isomorphism of free abelian groups. From the definitions it follows immediately that this a chain map. The second statement of the lemma now follows from Lemma 72.13. ■

The following corollary is an immediate consequence of the H_0-Proposition 72.5 and Lemma 72.13.

Corollary 72.15. Let X be a topological space. As on page 167 we denote by $\pi_0(X)$ the set of path-components of X. Then

$$H_0(X) \;\cong\; \mathbb{Z}^{(\pi_0(X))}.$$

In particular, if X consists of n path-components with $n \in \mathbb{N}_0$, then

$$H_0(X) \;\cong\; \mathbb{Z}^n.$$

Exercises for Chapter 72.

Exercise 72.1. How many faces does the standard simplex Δ^n have?

Exercise 72.2. Let $n \in \mathbb{N}$. As in the Standard Simplex Homeomorphism Lemma 72.1 we consider

$$\triangleleft_n \;:=\; \left\{(x_1,\ldots,x_n) \in [0,1]^n \;\middle|\; \sum_{i=1}^{n} x_i \in [0,1]\right\} \;\subset\; \mathbb{R}^n.$$

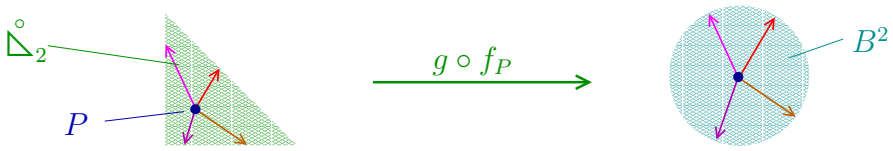
We denote by $\Phi\colon \Delta^n \to \triangleleft_n$ the natural projection. We pick any $P \in \overset{\circ}{\triangleleft}_n$. Next we consider the maps

$$\begin{array}{rcl} f_P\colon \overset{\circ}{\triangleleft}_n \to \mathbb{R}^n & & g\colon \mathbb{R}^n \to B^n \\ (x_1,\ldots,x_n) \mapsto \frac{((x_1,\ldots,x_n)-P)}{x_1\cdot\cdots\cdot x_n\cdot(1-x_1-\cdots-x_n)} & \text{and} & (x_1,\ldots,x_n) \mapsto \frac{x_1^2+\cdots+x_n^2}{1+x_1^2+\cdots+x_n^2}\cdot(x_1,\ldots,x_n) \end{array}$$

Note that f_P is defined in such a way that it sends rays emanating from P to rays emanating from the origin.

(a) Show that $f_P\colon \overset{\circ}{\triangleleft}_n \to \mathbb{R}^n$ is an orientation-preserving diffeomorphism.
(b) Show that $g\colon \mathbb{R}^n \to B^n$ is an orientation-preserving diffeomorphism.
Note that for the most part we had solved this problem on page 645.
(c) Show that $g \circ f_P\colon \overset{\circ}{\triangleleft}_n \to B^n$ extends to a homeomorphism $\Theta\colon \triangleleft_n \to \overline{B}^n$.
(d) Let $\Xi\colon \triangleleft_n \to \overline{B}^n$ be the homeomorphism obtained from the Convex-to-Ball Proposition 2.20 (2) applied to P. Show that Θ and Ξ are homotopic rel P.

Remark. The composition $\Psi \circ \Phi\colon \Delta^n \to \overline{B}^n$ is now a homeomorphism such that the restriction $\overset{\circ}{\Delta}^n \to B^n$ is a diffeomorphism. Furthermore this diffeomorphism is orientation-preserving if and only if n is even. Here we use the definitions on page 1581 to view $\overset{\circ}{\Delta}^n$ as an oriented smooth manifold.



Exercise 72.3.

(a) Let X be a topological space and let $x \in X$. For each $n \in \mathbb{N}_0$ denote by σ_n the singular n-simplex $\sigma_n\colon \Delta^n \to X$ which is defined by $\sigma_n(P) = x$ for all $P \in \Delta^n$. Determine $\partial\sigma_n \in C_{n-1}(X)$.

(b) Let X be a topological space and let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex. Show that if σ is a constant map, then σ is null-homologous.

Exercise 72.4. Let (C_*, c_*) and (D_*, d_*) be a chain complex and let $\varphi\colon C_* \to D_*$ be a chain map. Show that the groups $K_n := \ker(\varphi_n\colon C_n \to D_n)$ form a subcomplex of (C_*, c_*).

Exercise 72.5. Let X be a topological space which consists of a single point. Show that the following holds:

$$\mathrm{H}_n(X) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

Exercise 72.6.

(a) Let X be a topological space and let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex. We consider the singular 1-simplices

$$\begin{array}{rcl} \sigma_1\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & \sigma(\frac{1}{2}t, 1-\frac{1}{2}t) \end{array} \quad \text{and} \quad \begin{array}{rcl} \sigma_2\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & (\frac{1}{2}+\frac{1}{2}t, \frac{1}{2}-\frac{1}{2}t). \end{array}$$

Show that σ and $\sigma_1 + \sigma_2$ are homologous.

(b) Let X be a topological space and let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex. We consider the singular 1-simplex

$$\begin{array}{rcl} \overline{\sigma}\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & \sigma(1-t, t). \end{array}$$

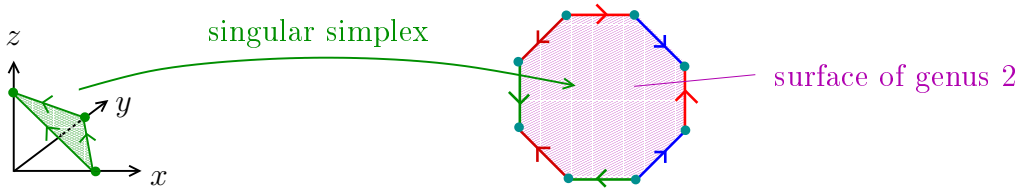
Show that $\sigma + \overline{\sigma}$ is a 1-cycle and that this 1-cycle is null-homologous.

We refer to the figure below for an illustration of the definitions.

Exercise 72.7. Let $\Sigma = E_8/\!\sim$ be the surface of genus 2. Provide an example of a 2-cycle $\sum_{i=1}^{m} a_i \cdot \sigma_i \in \mathrm{C}_2(\Sigma)$ with the following three properties:

(1) At least one a_i is non-zero.
(2) The images of the singular 2-simplices cover the whole surface.
(3) The images of the singular 2-simplices overlap only in "1-dimensional" subsets.

Remark. Note that on page 1589 we provided an example of such a 2-cycle for the torus $\Sigma = ([0,1] \times [0,1])/\!\sim$.



Exercise 72.8.

(a) Try to find "interesting" elements in the following homology groups:
(i) $\mathrm{H}_2(S^2)$,
(ii) H_2(surface Σ of genus 2),
(iii) H_2(Klein bottle K) and
(iv) $\mathrm{H}_2(\mathbb{R}\mathrm{P}^2)$.

More precisely, try to find 2-cycles which have the properties (1), (2) and (3) from Exercise 72.7.

(b) Make a guess for which surfaces you expect H_2 to be zero and for which it is non-zero.

Exercise 72.9. Let $n \in \mathbb{N}$.

(a) Provide an example of an n-cycle $\sum_{i=1}^{m} a_i \cdot \sigma_i \in \mathrm{C}_n(S^n)$ which has the properties (1), (2) and (3) from Exercise 72.7.

(b) Make an educated guess what the homology groups of S^n might be.

Exercise 72.10. Provide an example of a chain complex $(\mathrm{C}_*, \partial_*)$ such that all C_* are finitely generated free abelian groups and such that $\mathrm{H}_i(C_*) = 0$ for all even $i \in \mathbb{N}_0$ and such that $\mathrm{H}_i(C_*) = \mathbb{Z}_2$ for all odd $i \in \mathbb{N}_0$.

Exercise 72.11.

(a) Determine the isomorphism types of the homology groups of the chain complex

$$0 \;\to\; C_1 = \mathbb{Z}^2 \;\xrightarrow{\begin{pmatrix} 3 & 7 \\ 4 & 2 \end{pmatrix}}\; C_0 = \mathbb{Z}^2 \;\to\; 0.$$

(b) Determine the isomorphism types of the homology groups of the chain complex

$$0 \;\to\; C_2 = \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 2 & 5 \\ 4 & 10 \end{pmatrix}} C_1 = \mathbb{Z}^2 \;\xrightarrow{\begin{pmatrix} -2 & 1 \end{pmatrix}}\; C_0 = \mathbb{Z} \;\to\; 0.$$

(Why is this a chain complex?)

Exercise 72.12. Given a topological space X and $n \in \mathbb{N}_0$ we consider the map

$$\begin{array}{rcl} \epsilon \colon \mathrm{C}_n(X) & \to & \mathbb{Z} \\ \sum_{i=1}^{k} a_i \sigma_i & \mapsto & \sum_{i=1}^{k} a_i. \end{array}$$

(a) Show that the following diagram commutes

$$\begin{array}{ccccccccccccc} \ldots & \longrightarrow & \mathrm{C}_3(X) & \xrightarrow{\partial_3} & \mathrm{C}_2(X) & \xrightarrow{\partial_2} & \mathrm{C}_1(X) & \xrightarrow{\partial_1} & \mathrm{C}_2(X) & \xrightarrow{\partial_0} & 0 \\ & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} & & \\ \ldots & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

(b) Show that if n is even, then for every cycle $\sigma \in \mathrm{C}_n(X)$ we have $\epsilon(\sigma) = 0$.

Exercise 72.13. Let $f \colon C \to B$ be a homomorphism between abelian groups and let A be a subgroup of B with the following two properties:

(1) We have $f(C) \cap A = 0$.

(2) The subgroups $f(C)$ and A generate B.

Show that the map

$$\begin{array}{rcl} A & \to & \operatorname{coker}(f) = B/\operatorname{im}(f) \\ a & \mapsto & [a] \end{array}$$

is an isomorphism.

Exercise 72.14. Suppose we are chain complex of the following form

$$D_* \;=\; \ldots \;\to\; C_{k+2} \;\xrightarrow{\partial_{k+2}}\; A_{k+1} \oplus B_{k+1} \;\xrightarrow{\partial_{k+1}}\; A_k \oplus B_k \;\xrightarrow{\partial_k} C_{k-1} \;\to\; \ldots$$

Given $m \in \{k, k+1\}$ we denote by $i_m \colon A_m \to A_m \oplus B_m$ and $j_m \colon B_m \to A_m \oplus B_m$ the natural inclusions and we denote by $p_m \colon A_m \oplus B_m \to A_m$ and $q_m \colon A_m \oplus B_m \to B_m$ the natural projections. We suppose that $p_k \circ \partial_{k+1} \circ i_{k+1} \colon A_{k+1} \to A_k$ is an isomorphism. Show that we can "cancel" A_{k+1} and A_k from the chain complex without changing the isomorphism type of the homology groups. More precisely, consider the following chain complex:

$$\widetilde{D}_* \;=\; \ldots \;\to\; C_{k+2} \;\xrightarrow{p_{k+1} \circ \partial_{k+2}}\; B_{k+1} \;\xrightarrow{q_k \circ \partial_{k+1} \circ j_{k+1}}\; B_k \;\xrightarrow{\partial_k \circ j_k} C_{k-1} \;\to\; \ldots$$

Show that for each $i \in \mathbb{N}_0$ there exists an isomorphism $\mathrm{H}_i(D_*) \to \mathrm{H}_i(\widetilde{D}_*)$.

Remark. This implies in particular that D_* is acyclic if and only if $\widetilde{D}_*$ is acyclic.

Exercise 72.15. We define the direct sum of two chain complexes (C_*, ∂_*) and (D_*, ∂_*) in the obvious way. A chain complex is called *indecomposable* if it cannot be written as the direct sum of two non-zero chain complexes. Show that every chain complex (C_*, ∂_*), where the groups C_i are all finitely generated free abelian groups, can be written as a direct sum of chain complexes of the form

$$0 \;\to\; \underset{\substack{\uparrow \\ \text{degree } i}}{\mathbb{Z}} \;\to\; 0 \qquad \text{and} \qquad 0 \;\to\; \underset{\substack{\uparrow \\ \text{degree } i}}{\mathbb{Z}} \;\xrightarrow{\cdot n}\; \underset{\substack{\uparrow \\ \text{degree } i-1}}{\mathbb{Z}} \;\to\; 0.$$

Remark. Use the Smith Normal Form Theorem 51.5.

Exercise 72.16. Let (C_*, ∂_*) be a chain complex of abelian groups. Show that there exists a chain complex (F_*, ∂_*) of *free abelian groups* together with a chain map $\varphi \colon F_* \to C_*$ such that the induced maps $f_* \colon \mathrm{H}_k(F_*) \to \mathrm{H}_k(C_*)$ are isomorphisms for all $k \in \mathbb{N}_0$.

73. Homology and homotopies

It was not overly difficult to introduce the homology groups of a topological space. We also proved several straightforward properties of homology groups. But so far we are unable to calculate any interesting homology group. In this section we will show in particular that homotopy equivalent topological spaces have isomorphic homology groups. That does not yet lead to the calculation of non-trivial homology groups, but this result will play an important role in the calculation of the homology groups of spheres that we will give in Proposition 74.4.

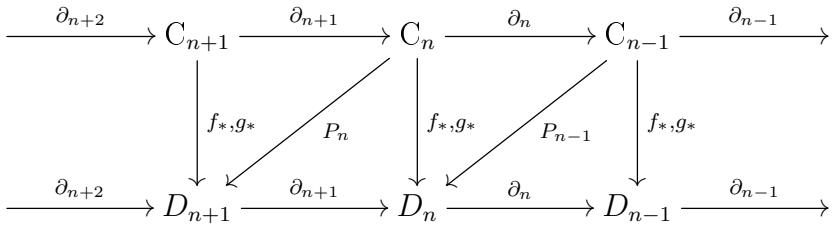
73.1. **Chain homotopies.** In this section we introduce a criterion that ensures that two chain maps $f, g\colon C_* \to D_*$ induce the same maps on the corresponding homology groups.

Definition. Let $f, g\colon C_* \to D_*$ be chain maps between chain complexes C_* and D_*. A chain homotopy $P = \{P_n\}_{n\in\mathbb{N}_0}$ between f and g consists of family $\{P_n\colon \mathrm{C}_n \to D_{n+1}\}_{n\in\mathbb{N}_0}$ such that for each $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; f_n - g_n.$$

If there exists a chain homotopy between chain maps f and g, then we say that f and g are chain homotopic and we write $f \simeq g$.

For the definition of a chain homotopy and the subsequent discussion of chain homotopies it is sometimes helpful to keep in mind the following diagram that combines all objects involved in the definition:



Lemma 73.1. Let C_* and D_* be two chain complexes.

(1) Chain homotopy defines an equivalence relation on the set of all chain maps from C_* to D_*. In particular, if $f, g, h\colon C_* \to D_*$ are chain maps with $f \simeq g$ and $g \simeq h$, then $f \simeq h$.
(2) Let $f, g\colon C_* \to D_*$ be chain maps, and let $e\colon B_* \to \mathrm{C}_*$ and $h\colon D_* \to E_*$ be two other chain maps. Then

$$f \simeq g \quad\Longrightarrow\quad f \circ e \simeq g \circ e \quad\text{and}\quad h \circ f \simeq h \circ g.$$

Proof. The lemma will be proved in Exercise 73.1. ∎

Now we can formulate the following lemma.

Lemma 73.2. Let $f, g\colon C_* \to D_*$ be two chain maps between chain complexes. If f and g are chain homotopic, then $f_* = g_*\colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ for all $n \in \mathbb{N}_0$.

Proof. Let $P = \{P_n\}_{n\in\mathbb{N}_0}$ be a chain homotopy between f and g. Then for any cycle $z \in C_n$ we get the equalities

$$f(z) - g(z) \;=\; (f-g)(z) \;=\; (\partial_{n+1}P_n + P_{n-1}\partial_n)(z) \;=\; (\partial_{n+1}P_n)(z) + P_{n-1}\underbrace{\partial_n z}_{=0} \;=\; \partial P_n z,$$

i.e. $f(z)$ and $g(z)$ are homologous. ■

Definition. Let C_* and D_* be two chain complexes.

(1) A chain map $f\colon \mathrm{C}_* \to D_*$ is called a chain homotopy equivalence between C_* and D_*, if it admits a chain homotopy inverse, i.e. a chain map $g\colon D_* \to \mathrm{C}_*$ with $g\circ f \simeq \mathrm{id}_{C_*}$ and $f\circ g \simeq \mathrm{id}_{D_*}$.

(2) If there exists a chain homotopy equivalence between C_* and D_*, then we say that the chain complexes C_* and D_* are chain homotopy equivalent and we write $C_* \simeq D_*$.

Lemma 73.3. The relation "homotopy equivalence" of chain complexes is indeed an equivalence relation on the class of all chain complexes.

Proof. This statement follows easily from the definitions.We fill in the details in Exercise 73.2. ■

For later we record the following straightforward corollary to Lemma 73.2.

Corollary 73.4. If $f\colon C_* \to D_*$ is a chain homotopy equivalence between chain complexes, then the induced maps $f_*\colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ are isomorphisms.

Proof. Let $g\colon D_* \to C_*$ be a chain homotopy inverse to f. Then we see that

$$g_* \circ f_* \quad = \quad (g\circ f)_* \quad \underset{\uparrow}{=} \quad (\mathrm{id}_{C_*})_* \quad = \quad \mathrm{id}_{\mathrm{H}_n(C)}\,.$$

by Lemma 73.2 since $g\circ f$ is chain homotopic to the identity

The same way we see that $f_* \circ g_* = \mathrm{id}_{\mathrm{H}_n(D)}$. ■

We also record the following useful little lemma.

Lemma 73.5. Let $(\mathrm{C}_*, \partial_*)$ be a chain complex. If there exist "diagonal" maps P_n, $n \in \mathbb{Z}_{\geq -1}$, to obtain a diagram[877]

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & C_3 & \xrightarrow{\partial_3} & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0} & 0 \\
 & \swarrow^{P_3} & \downarrow{\scriptstyle \mathrm{id}} & \swarrow^{P_2} & \downarrow{\scriptstyle \mathrm{id}} & \swarrow^{P_1} & \downarrow{\scriptstyle \mathrm{id}} & \swarrow^{P_0} & \downarrow{\scriptstyle \mathrm{id}} & & \\
\ldots & \longrightarrow & C_3 & \xrightarrow[\partial_3]{} & C_2 & \xrightarrow[\partial_2]{} & C_1 & \xrightarrow[\partial_1]{} & C_0 & \xrightarrow[\partial_0]{} & 0
\end{array}$$

such that for every $n \in \mathbb{N}_0$ we have the equality

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; \mathrm{id}_{\mathrm{C}_n},$$

then all homology groups of $(\mathrm{C}_*, \partial_*)$ vanish.

Proof. Let D_* be the zero chain complex and for each $n \in \mathbb{N}_0$ let $f_n\colon \mathrm{C}_n \to D_n = 0$ be the trivial map. Since the homology groups of D_* it suffices, by Corollary 73.4, to show that f_* is a chain homotopy equivalence. For each $n \in \mathbb{N}_0$ let $g_n\colon D_n \to C_n$ be the trivial map.

[877] The diagram has no particular role, except that it shows the domain and the target of the maps P_n and that it is convenient for keeping track of the various maps.

Evidently $f_n \circ g_n = \mathrm{id}_{D_*}$. Thus it remains to show that $g_* \circ f_* \simeq \mathrm{id}_{C_*}$. In other words we need to show that $0 \simeq \mathrm{id}_{C_*}$. But that is exactly what the maps P_n do. ■

73.2. **Homology and homotopic maps.** The name *chain homotopy* introduced in the previous section suggests that this concept is related to the notion of a homotopy between two topological spaces. As we will see shortly, this is indeed the case. For the reader's convenience we recall that if $f, g\colon X \to Y$ are two maps between topological spaces, then a *homotopy between the maps f and g* is a map

$$F\colon X \times [0,1] \to Y,$$

such that

$$F(x,0) = f(x) \text{ and } F(x,1) = g(x) \text{ for all } x \in X.$$

If there exists a homotopy between f and g, then we say that f and g are *homotopic* and we write $f \simeq g$.

Proposition 73.6. (Homotopic Maps-and-Homology Proposition) Let X and Y be two topological spaces and let $f, g\colon X \to Y$ be two maps. If f and g are homotopic, then the following hold:

(1) the induced maps f_* and g_* from $\mathrm{C}_*(X)$ to $\mathrm{C}_*(Y)$ are chain homotopic,
(2) for each $n \in \mathbb{N}_0$ we have $f_* = g_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$.

Remark. The proposition can be stated more succinctly as follows:

$$f \simeq g\colon X \to Y \quad \Longrightarrow \quad f_* \simeq g_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y) \quad \Longrightarrow \quad f_* = g_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y).$$

We start out with a discussion of the key idea behind the proof of the Homotopic Maps-and-Homology Proposition 73.6. Afterwards we will fill in the details. So let $f, g\colon X \to Y$ be two maps between topological spaces and let $F\colon X \times [0,1] \to Y$ be a homotopy between the maps f and g. Our goal is to find a chain homotopy between f_* and g_*. Thus let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. We need to find $P(\sigma) \in \mathrm{C}_{n+1}(Y)$ and $P(\partial\sigma) \in \mathrm{C}_n(Y)$ such that

$$\partial P(\sigma) \;=\; -P(\partial\sigma) + f_*(\sigma) - g_*(\sigma).$$

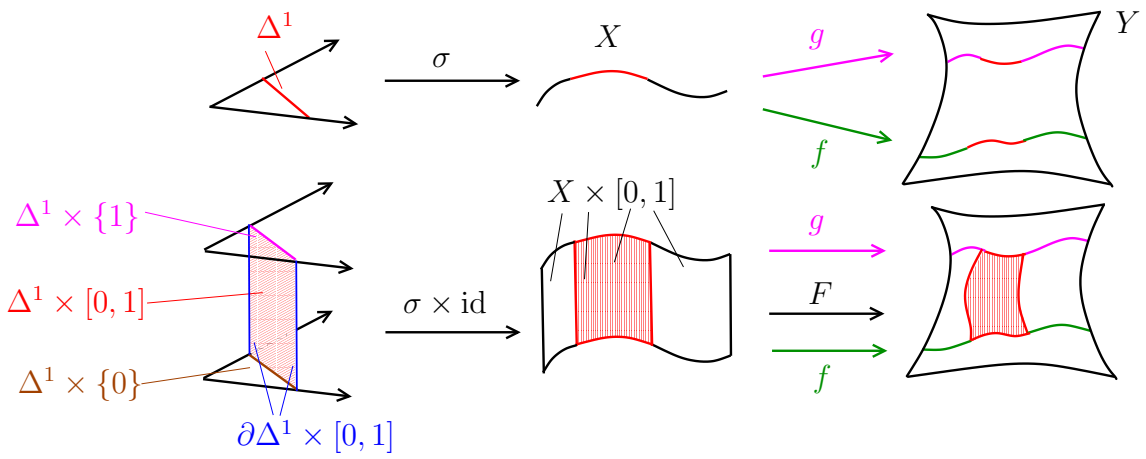
Note that $F \circ (\sigma \times \mathrm{id}_{[0,1]})$ defines a map $\Delta^n \times [0,1] \to Y$. Now we can rewrite the above desired equality as follows:

$$\partial P(\sigma) \;=\; -P(\partial\sigma) + \underbrace{\begin{matrix}\text{map } F \circ (\sigma \times \mathrm{id}_{[0,1]}) \\ \text{restricted to } \Delta^n \times \{0\}\end{matrix}}_{=\Delta^n} - \underbrace{\begin{matrix}\text{map } F \circ (\sigma \times \mathrm{id}_{[0,1]}) \\ \text{restricted to } \Delta^n \times \{1\}\end{matrix}}_{=\Delta^n}.$$

To get an idea we consider the figure below. We make the following trivial observation: we have

$$\partial(\Delta^n \times [0,1]) \;=\; \partial\Delta^n \times [0,1] \;\cup\; \Delta^n \times \{0\} \;\cup\; \Delta^n \times \{1\}.$$

Comparing the desired equality of maps and the equality of sets suggests that each map $P_k\colon \mathrm{C}_k(X) \to \mathrm{C}_{k+1}(Y)$ should take the role of replacing a singular k-simplex $\mu\colon \Delta^k \to X$ by the map $F \circ (\mu \times \mathrm{id}_{[0,1]})\colon \Delta^k \times [0,1] \to Y$. This is a good idea, but the problem is that $\Delta^k \times [0,1]$ is not a $(k+1)$-simplex. The idea now is to write $\Delta^k \times [0,1]$ as the union of $(k+1)$-simplices. In the figure below we see for example that we can view $\Delta^1 \times [0,1]$ as the union of the triangles with the vertices v_0, w_0, w_1 respectively v_0, v_1, w_1.
To generalize this observation we have to introduce some more notation.



Notation. Let $n \in \mathbb{N}_0$.

(1) For $i = 0, \ldots, n$ we write

$$v_i := (0, \ldots, 1, 0, \ldots, 0) \times \{0\} \in \mathbb{R}^{n+1} \times \{0\},$$

and

$$w_i := (0, \ldots, 1, 0, \ldots, 0) \times \{1\} \in \mathbb{R}^{n+1} \times \{1\}.$$

The points $v_0, \ldots, v_n$ thus span the simplex $\Delta^n \times 0$ and the points $w_0, \ldots, w_n$ span the simplex $\Delta^n \times 1$. Furthermore for each i the points $v_0, \ldots, v_i, w_i, \ldots, w_n$ span an $(n+1)$-simplex that is a subset of $\Delta^n \times [0,1]$.[878]

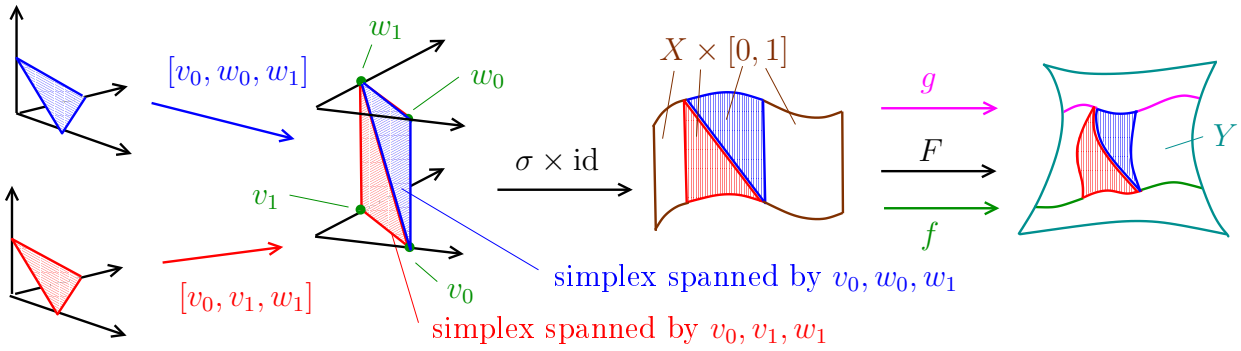
(2) For $a_0, \ldots, a_k \in \Delta^n \times [0,1]$ we consider, as on page 1584, the map

$$\begin{aligned} [a_0, \ldots, a_k] \colon \Delta^k &\to \Delta^n \times [0,1] \\ (\lambda_0, \ldots, \lambda_k) &\mapsto \sum_{j=0}^{k} \lambda_j \cdot a_j. \end{aligned}$$

The map $[a_0, \ldots, a_k]$ thus sends the standard k-simplex Δ^k to the simplex spanned by $a_0, \ldots, a_k$ in $\Delta^n \times [0,1]$.

(3) We set

$$\Omega_n := \sum_{j=0}^{n} (-1)^j \cdot \underbrace{[v_0, \ldots, v_j, w_j, \ldots, w_n]}_{\Delta^{n+1} \to \Delta^n \times [0,1]} \in \mathrm{C}_{n+1}(\Delta^n \times [0,1]).$$



[878] It is a straightforward exercise to show that the union of these $(n+1)$-dimensional simplices is precisely $\Delta^n \times [0,1]$. We will not make use of this fact.

In the following lemma we compute the boundary of $\Omega_n \in \mathrm{C}_{n+1}(\Delta^n \times [0,1])$. The statement might not be particularly visually appealing, but it will turn out to be very useful.

Lemma 73.7. (Prism Boundary Lemma) Let $n \in \mathbb{N}_0$. Given $t \in \{0,1\}$ we write

$$\begin{array}{rccc} \eta_t \colon & \Delta^n & \to & \Delta^n \times [0,1] \\ & x & \mapsto & (x,t). \end{array}$$

With this notation we have

$$\partial_{n+1}(\Omega_n) + \sum_{j=0}^{n} (-1)^j \cdot (i_j^n \times \mathrm{id}_{[0,1]})_*\Big(\underbrace{\Omega_{n-1}}_{\in \mathrm{C}_n(\Delta^{n-1}\times[0,1])} \Big) \;=\; \eta_1 - \eta_0 \;\in\; \mathrm{C}_n(\Delta^n \times [0,1]).$$

Proof. The proof of the lemma is basically an elementary, albeit slightly confusing calculation. First note that

$$\begin{aligned} \partial\Omega_n &= \partial\Big(\sum_{j=0}^{n} (-1)^j \cdot [v_0, \dots, v_j, w_j, \dots, w_n] \Big) \\ &= \sum_{k=0}^{n+1} (-1)^k \cdot \sum_{j=0}^{n} (-1)^j \cdot [v_0, \dots, v_j, w_j, \dots, w_n] \circ i_k^{n+1}. \end{aligned}$$

It is straightforward to see that for any $k \in \{0, \dots, n+1\}$ we have[879]

$$[v_0, \dots, v_j, w_j, \dots, w_n] \circ i_k^{n+1} \;=\; \begin{cases} [v_0, \dots, \widehat{v_k}, \dots, v_j, w_j, \dots, w_n], & \text{if } k \le j, \\ [v_0, \dots, v_j, w_j, \dots, \widehat{w_{k-1}}, \dots, w_n], & \text{if } k > j. \end{cases}$$

Thus it follows that[880]

$$\begin{aligned} \partial\Omega_n \;=\; & \sum_{0 \le k \le j \le n} (-1)^k \cdot (-1)^j \cdot [v_0, \dots, \widehat{v_k}, \dots, v_j, w_j, \dots, w_n] \\ & + \sum_{n+1 \ge k > j \ge 0} (-1)^k \cdot (-1)^j \cdot [v_0, \dots, v_j, w_j, \dots, \widehat{w_{k-1}}, \dots, w_n]. \end{aligned}$$

Now we split the first sum into the cases $k = j$ and $k < j$ and we split the second sum into the cases $k = j+1$ and $k > j+1$. This leads us to the following equality:

$$\begin{aligned} \partial\Omega_n \;=\; & \sum_{k=0}^{n} (-1)^{2k} \cdot [v_0, \dots, v_{k-1}, w_k, \dots, w_n] \\ & + \sum_{k=1}^{n+1} (-1)^{2k+1} \cdot [v_0, \dots, v_{k-1}, w_k, \dots, w_n] \\ & + \sum_{0 \le k < j \le n} (-1)^k \cdot (-1)^j \cdot [v_0, \dots, \widehat{v_k}, \dots, v_j, w_j, \dots, w_n] \\ & + \sum_{n+1 \ge k > j+1 \ge 1} (-1)^k \cdot (-1)^j \cdot [v_0, \dots, v_j, w_j, \dots, \widehat{w_{k-1}}, \dots, w_n]. \end{aligned}$$

[879] The first case, $k \le j$ is slightly easier: one has to figure out, where the h-th vertex e_h of the standard n-simplex gets sent to. If $h < k$, then i_k^{n+1} sends the h-th vertex again to the h-th vertex, and $[v_0, \dots, v_j, w_j, \dots, w_n]$ sends this vertex to v_h. If $h \ge k$, then i_k^{n+1} sends the h-th vertex again to the $(h+1)$-th vertex, and $[v_0, \dots, v_j, w_j, \dots, w_n]$ sends this vertex to v_{h+1}. If $k \le j$ this corresponds just to the map $[v_0, \dots, \widehat{v_k}, \dots, v_j, w_j, \dots, w_n]$.
The second case, $k > j$ is proved precisely the same way.

[880] Here we use the usual notation that $\widehat{v_k}$ means that we do not plug in $\widehat{v_k}$, the same applies to $\widehat{w_{k-1}}$.

Next we consider the first two sums. The summands cancel each other, except for the summands for $k = 0$ and $k = n + 1$. Thus we obtain that

$$
\begin{aligned}
\partial\Omega_n \;&=\; [w_0, \ldots, w_n] - [v_0, \ldots, v_n] \\
(*) \qquad &+ \sum_{0 \le k < j \le n} (-1)^k \cdot (-1)^j \cdot [v_0, \ldots, \widehat{v_k}, \ldots, v_j, w_j, \ldots, w_n] \\
&+ \sum_{n+1 \ge k > j+1 \ge 1} (-1)^k \cdot (-1)^j \cdot [v_0, \ldots, v_j, w_j, \ldots, \widehat{w_k}, \ldots, w_n].
\end{aligned}
$$

The first summand is just η_1 and the second is $-\eta_0$. It remains to show that the remaining summand corresponds to the remaining term in the formula. Thus we continue our proof with the discussion of the one term in the statement of the lemma that we have not studied yet. We have[881]

$$
\sum_{j=0}^{n} (-1)^j \cdot (i_j^n \times \mathrm{id}_{[0,1]})_*(\Omega_{n-1}) = \sum_{r=0}^{n} \sum_{s=0}^{n-1} (-1)^r \cdot (-1)^s \cdot (i_r^n \times \mathrm{id}_{[0,1]}) \circ [v_0, \ldots, v_s, w_s, \ldots, w_{n-1}].
$$

$\uparrow$ definition of $(i_j^n \times \mathrm{id}_{[0,1]})_*$ and substitution $r = j$

It follows from an elementary (but arguably confusing) argument, using only the definitions, that we have the following equalities of maps $\Delta^n \to \Delta^n \times [0,1]$:

$$
(i_r^n \times \mathrm{id}_{[0,1]}) \circ [v_0, \ldots, v_s, w_s, \ldots, w_{n-1}] \;=\; \begin{cases} [v_0, \ldots, \widehat{v_r}, \ldots, v_{s+1}, w_{s+1}, \ldots, w_n], & \text{if } r \le s, \\ [v_0, \ldots, v_s, w_s, \ldots, \widehat{w_{r+1}}, \ldots, w_n], & \text{if } r > s. \end{cases}
$$

Thus we see that

$$
\begin{aligned}
\sum_{j=0}^{n} (-1)^j \cdot (i_j^n \times \mathrm{id}_{[0,1]})_*(\Omega_{n-1}) \;&=\; \sum_{0 \le r \le s \le n-1} (-1)^s \cdot (-1)^r \cdot [v_0, \ldots, \widehat{v_r}, \ldots, v_{s+1}, w_{s+1}, \ldots, w_n] \\
&+ \sum_{n \ge r > s \ge 0} (-1)^s \cdot (-1)^r \cdot [v_0, \ldots, v_s, w_s, \ldots, \widehat{w_{r+1}}, \ldots, w_n].
\end{aligned}
$$

In the first term we perform the substitutions $r = k$ and $s = j - 1$ and in the second term we perform the substitutions $r = k - 1$, $s = j$. We obtain the following equality:

$$
\begin{aligned}
\sum_{j=0}^{n} (-1)^j \cdot (i_j^n \times \mathrm{id}_{[0,1]})_*(\Omega_{n-1}) \;&=\; \sum_{0 \le k \le j-1 \le n-1} (-1)^{j-1} \cdot (-1)^k \cdot [v_0, \ldots, \widehat{v_k}, \ldots, v_j, w_j, \ldots, w_n] \\
&+ \sum_{n \ge k-1 > j \ge 0} (-1)^j \cdot (-1)^{k-1} \cdot [v_0, \ldots, v_j, w_j, \ldots, \widehat{w_k}, \ldots, w_n].
\end{aligned}
$$

We see immediately that these terms are just the negatives of the corresponding terms in the above equation $(*)$. ∎

The following lemma implies in particular the first statement of the Homotopic Maps-and-Homology Proposition 73.6. The second statement of the Homotopic Maps-and-Homology Proposition 73.6 regarding the induced maps on homology follows immediately from the first statement together with Lemma 73.2. In other words, the following lemma completes the proof of the Homotopic Maps-and-Homology Proposition 73.6.

[881] Here $v_0, \ldots, v_{n-1}, w_0, \ldots, w_{n-1}$ are now, unfortunately somewhat confusingly, the corresponding vertices of $\Delta^{n-1} \times [0,1]$.

Lemma 73.8. Let X be a topological space and let $F\colon X \times [0,1] \to Y$ be a homotopy from a map f to a map g. Given any $n \in \mathbb{N}_0$ we define

$$\begin{array}{rcl} P_n\colon \mathrm{C}_n(X) & \to & \mathrm{C}_{n+1}(Y) \end{array}$$

to be the map that is induced by

$$\begin{array}{rcl} (\sigma\colon \Delta^n \to X) & \mapsto & (F \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\Omega_n). \end{array}$$

Then for any $n \in \mathbb{N}_0$ we have

$$\partial P_n + P_{n-1}\partial \;=\; g_* - f_*$$

as maps from $\mathrm{C}_n(X)$ to $\mathrm{C}_n(Y)$.

Proof. Let X be a topological space and let $F\colon X \times [0,1] \to X$ be a homotopy from a map f to a map g. Furthermore let $\sigma\colon \Delta^n \to X$ be a singular k-simplex. Then

$$\begin{aligned} (\partial P_n + P_{n-1}\partial)(\sigma) \;&=\; \partial(P_n(\sigma)) + P_{n-1}\Big(\sum_{j=0}^{n}(-1)^j \cdot \sigma \circ i_j^n\Big) \;=\; \partial(P_n(\sigma)) + \sum_{j=0}^{n}(-1)^j \cdot P_{n-1}(\sigma \circ i_j^n) \\ &=\; \partial((F \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\Omega_n)) + \sum_{j=0}^{n}(-1)^j \cdot (F \circ (\sigma \circ_j^n \times \mathrm{id}_{[0,1]}))_*(\Omega_{n-1}) \\ &=\; (F \circ (\sigma \times \mathrm{id}_{[0,1]}))_*\Big(\partial(\Omega_n)) + \sum_{j=0}^{n}(-1)^j \cdot (i_j^n \times \mathrm{id}_{[0,1]})_*(\Omega_{n-1})\Big) \\ &\quad\uparrow \\ &\quad\text{by Lemma 72.9 (1) and since } (\sigma \times \mathrm{id}) \circ (i_j^n \times \mathrm{id}) = (\sigma \circ i_j^n) \times \mathrm{id} \\ &=\; (F \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\eta_1 - \eta_0) \;\underset{\uparrow}{=}\; f \circ \sigma - g \circ \sigma \;=\; g_*(\sigma) - f_*(\sigma). \end{aligned}$$

Prism Boundary Lemma 73.7 — since F is a homotopy from f to g ∎

We recall the following definition from page 430.

Definition. Let X and Y be topological spaces.

(1) A map $f\colon X \to Y$ is a homotopy equivalence between X and Y, if there exists a homotopy inverse to f, i.e. if there exists a continuous map $g\colon Y \to X$ with $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

(2) If there exists a homotopy equivalence between X and Y, then we say that the topological spaces X and Y are homotopy equivalent and we write $X \simeq Y$.

(3) We say X is contractible, if X is homotopy equivalent to $Y = \{*\}$, i.e. if X is homotopy equivalent to a topological space which consists of precisely one point.

Now we obtain the following corollary to the Homotopic Maps-and-Homology Proposition 73.6. This corollary mirrors the fact, proved in Propositions 50.3 (2) and 71.7 (2), that homotopy equivalences induce isomorphisms of fundamental groups and higher homotopy groups.

Corollary 73.9. (Homotopy Equivalence-H_*-Corollary) Let X and Y be topological spaces and let $n \in \mathbb{N}_0$.

(1) If $f\colon X \to Y$ is a homotopy equivalence (e.g. a homeomorphism), then the induced map $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is an isomorphism.

(2) If X and Y are homotopy equivalent, then $\mathrm{H}_n(X) \cong \mathrm{H}_n(Y)$.

(3) If X is contractible, then $\mathrm{H}_0(X) \cong \mathbb{Z}$ and $\mathrm{H}_n(X) = 0$ for $n \geq 1$.

(4) If X is a deformation retract of Y, then the inclusion induced map $\mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is an isomorphism.

Proof.

(1) Let $f\colon X \to Y$ be a homotopy equivalence and let $g\colon Y \to X$ be a homotopy inverse of f, i.e. g is a map such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. It follows from the Homotopic Maps-and-Homology Proposition 73.6 and the functoriality of homology that $f_* \circ g_* = (f \circ g)_* = \mathrm{id}_{\mathrm{H}_n(Y)}$ and similarly $g_* \circ f_* = \mathrm{id}_{\mathrm{H}_n(X)}$. But that means precisely that $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is an isomorphism.

(2) This statement follows immediately from (1).

(3) This statement follows immediately from (2) and the Single Point-Homology Lemma 72.6.

(4) This statement follows immediately from (1) and the Deformation Retract-Homotopy Equivalence Lemma 15.5. ■

Examples.

(1) On page 432 we saw that every convex subset of $\mathbb{R}^n$, e.g. the open disk B^n, the closed disk $\overline{B}^n$, any simplex or $\mathbb{R}^n$, is contractible. Thus it follows from the Homotopy Equivalence-H_*-Corollary 73.9, that

$$\mathrm{H}_i(\text{convex subset of } \mathbb{R}^n) \;\cong\; \mathrm{H}_i(B^n) \;\cong\; \mathrm{H}_i(\overline{B}^n) \;\cong\; \mathrm{H}_i(\mathbb{R}^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{if } i > 0. \end{cases}$$

(2) In Exercise 73.3 we will use the Homotopy Equivalence-H_*-Corollary 73.9 to show that if A is a compact convex subset of $\mathbb{R}^n$ and if x is a point in the interior of A, then the inclusion $\mathbb{R}^n \setminus A \to \mathbb{R}^n \setminus \{x\}$ induces isomorphisms $\mathrm{H}_i(\mathbb{R}^n \setminus A) \to \mathrm{H}_i(\mathbb{R}^n \setminus \{x\})$ for all i.

Remark. Given a map $f\colon X \to Y$ between topological spaces we denote by $[f]$ the homotopy equivalence class of f, and we denote by $[X, Y]$ the set of equivalence classes of maps from X to Y. As we pointed out on page 441,

$$\begin{aligned} \mathrm{Ob}(\mathcal{HomTop}) &:= \text{ all topological spaces,} \\ \mathrm{Mor}(X, Y) &:= [X, Y] \end{aligned}$$

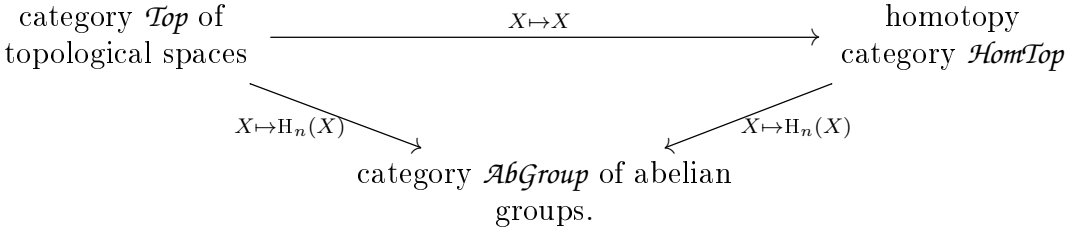
together with the composition

$$\begin{aligned} \mathrm{Mor}(X, Y) \times \mathrm{Mor}(Y, Z) &\to \mathrm{Mor}(X, Z) \\ ([f], [g]) &\mapsto [g \circ f] \end{aligned}$$

defines a category $\mathcal{HomTop}$, called the *homotopy category*. the Homotopic Maps-and-Homology Proposition 73.6 thus says that for each $n \in \mathbb{N}_0$ the maps

$$\begin{aligned} X &\mapsto \mathrm{H}_n(X) \\ \text{and} \qquad [f] &\mapsto (f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)) \end{aligned}$$

define a functor from the category $\mathcal{HomTop}$ to the category $\mathcal{AbGroup}$ of abelian groups. In particular we now have the following commutative diagram of categories and functors:



Exercises for Chapter 73.

Exercise 73.1. Let C_* and D_* be two chain complexes.

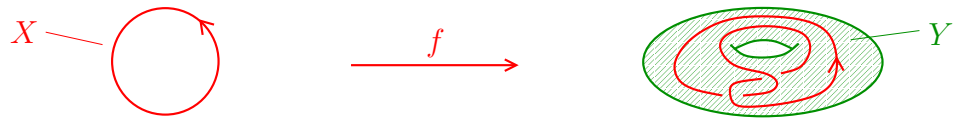
(a) Show that chain homotopy defines an equivalence relation on the set of all chain maps from C_* to D_*.

(b) Let $f, g\colon C_* \to D_*$ be chain maps, and let $e\colon B_* \to \mathrm{C}_*$ and $h\colon D_* \to E_*$ be two other chain maps. Show that if $f \simeq g$, then $f \circ e \simeq g \circ e$ and $h \circ f \simeq h \circ g$.

Exercise 73.2. Show that the relation "homotopy equivalence" of chain complexes is indeed an equivalence relation on the class of all chain complexes.

Exercise 73.3. Let A be a compact convex subset of $\mathbb{R}^n$ and let $x \in A$ be a point. Show that the inclusion $\mathbb{R}^n \setminus A \to \mathbb{R}^n \setminus \{x\}$ induces isomorphisms $\mathrm{H}_i(\mathbb{R}^n \setminus A) \to \mathrm{H}_i(\mathbb{R}^n \setminus \{x\})$ for all i.
Hint. Use Exercise 15.5.

Exercise 73.4. Let $X = S^1$ and let $Y = \overline{B}^2 \times S^1$. We consider the map $f\colon X \to Y$ shown in the figure below. Show that for every $i \in \mathbb{N}$ the induced map $f_*\colon \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$ is the zero map.



74. The Excision Theorem

In this section, given a topological space X and a "good" subset A we will relate the homology groups of X, A and the quotient space X/A via a long exact sequence. We will use this long exact sequence to fulfill our dream of calculating the homology groups of all spheres. Unfortunately it will take a very long time to prove the existence of this long exact sequence.

74.1. **Reduced homology groups.** In this section we introduce the reduced homology groups of a topological space. These are a minute but at times surprisingly useful variation on the usual homology groups.

Definition. Let X be a topological space. By Lemma 72.4 (1) we know that the following is a generalized[882] chain complex

$$\begin{array}{ccccccccccc} & & & & & & \text{degree } 0 & & \text{degree } -1 & & \\ & & & & & & \downarrow & & \downarrow & & \\ \ldots & \xrightarrow{\partial_3} & \mathrm{C}_2(X) & \xrightarrow{\partial_2} & \mathrm{C}_1(X) & \xrightarrow{\partial_1} & \mathrm{C}_0(X) & \xrightarrow{\epsilon} & \mathbb{Z} & \to & 0. \\ & & & & & & \sum_{i=1}^k r_i \cdot \sigma_i & \mapsto & \sum_{i=1}^k r_i & & \end{array}$$

augmentation map from page 1590

We introduce the following three definitions.

(1) We refer to this chain complex as the augmented chain complex $\widetilde{\mathrm{C}}_*(X)$ of X.
(2) Given $n \in \mathbb{Z}_{\geq -1}$ we define the n-th reduced homology group $\widetilde{\mathrm{H}}_n(X)$ of X to be the n-th homology group of the augmented chain complex.
(3) If $f\colon X \to Y$ is a map between topological spaces we denote by $f_*\colon \widetilde{\mathrm{C}}_*(X) \to \widetilde{\mathrm{C}}_*(Y)$ the map that is the usual map on $\widetilde{\mathrm{C}}_n(X) = \mathrm{C}_n(X)$ for $n \geq 0$ and that is the identity on the final $\mathbb{Z}$-term.

The following elementary lemma summarizes a few properties of reduced homology groups.

Lemma 74.1. (Reduced Homology Lemma)

(0) Given a topological space X we have[883]

$$\widetilde{\mathrm{H}}_{-1}(X) = \begin{cases} 0, & \text{if } X \neq \varnothing, \\ \mathbb{Z}, & \text{if } X = \varnothing. \end{cases}$$

(1) If $f\colon X \to Y$ is a map between two topological spaces, then the induced maps $f_*\colon \widetilde{\mathrm{C}}_*(X) \to \widetilde{\mathrm{C}}_*(Y)$ are chain maps. In particular we see that for each $n \in \mathbb{Z}_{\geq -1}$ we get a well-defined induced map $f_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)$.
(2) For every $n \in \mathbb{Z}_{\geq -1}$ the maps

$$\begin{array}{rcl} X & \mapsto & \widetilde{\mathrm{H}}_n(X) \\ (f\colon X \to Y) & \mapsto & (f_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)) \end{array}$$

define a functor from the category of topological spaces to the category of abelian groups.

[882] Recall that a generalized chain complex is like a chain complex, but we also allow non-zero groups in negative degrees.

(3) For each $n \in \mathbb{Z}_{\geq -1}$ the maps

$$\begin{array}{rcl} \text{ж}\colon \widetilde{\mathrm{H}}_n(X) & \to & \mathrm{H}_n(X) \\ [\sigma] & \mapsto & [\sigma] \end{array}$$

are well-defined. Furthermore they define a natural transformation from the functor $X \mapsto \widetilde{\mathrm{H}}_n(X)$ to the functor $X \mapsto \mathrm{H}_n(X)$.

(4) For every topological space X the following statements hold:

(a) Given any point $P \in X$ the map

$$\begin{array}{rcl} \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} & \to & \mathrm{H}_0(X) \\ [\sigma] \oplus n & \mapsto & [\sigma] + [n \cdot P] \end{array}$$

is an isomorphism.

(b) For every $n \geq 1$ the natural map $\text{ж}\colon \widetilde{\mathrm{H}}_n(X) \to \mathrm{H}_n(X)$ is the identity.

(5) If X is a topological space with $n \in \mathbb{N}$ path-components, then $\widetilde{\mathrm{H}}_0(X) \cong \mathbb{Z}^{n-1}$. In particular X is path-connected if and only if $\widetilde{\mathrm{H}}_0(X) = 0$.

(6) The statements of the Homotopy Equivalence-H_*-Corollary 73.9 (1), (2) and (4) also hold the same way for reduced homology groups and the corresponding induced maps. For example, if X is a deformation retract of some topological space Y, then the inclusion induced maps $\widetilde{\mathrm{H}}_k(X) \to \widetilde{\mathrm{H}}_k(Y)$ are isomorphisms for all $k \in \mathbb{Z}_{\geq -1}$.

(7) If X is a contractible topological space, then $\widetilde{\mathrm{H}}_k(X) = 0$ for all $k \in \mathbb{Z}_{\geq -1}$.

The lemma shows in particular that reduced homology groups are almost the same as the usual homology groups. So it might sound slightly silly to introduce them. But we will see that on several occasions reduced homology groups make it easier to formulate some statements.

Remark.

(1) If X is a non-empty topological space, then it follows from the Reduced Homology Lemma 74.1 (4) that we have isomorphisms

$$\mathrm{H}_n(X) \;\cong\; \begin{cases} \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z}, & \text{if } n = 0, \\ \widetilde{\mathrm{H}}_n(X), & \text{if } n \neq 0. \end{cases}$$

Furthermore, by the Reduced Homology Lemma 74.1 (3) we know that for $n \geq 1$ the isomorphism is natural. On other hand for $n = 0$ the isomorphism depends on the choice of a point.

(2) In Lemma 74.2 we will formulate a useful variation on the Reduced Homology Lemma 74.1 (3).

(3) By the Reduced Homology Lemma 74.1 (7) we know for any $k \in \mathbb{N}_0$ and any $n \in \mathbb{Z}_{\geq -1}$ we have

$$\widetilde{\mathrm{H}}_n(\Delta^k) \;=\; \widetilde{\mathrm{H}}_n(B^k) \;=\; \widetilde{\mathrm{H}}_n(\overline{B}^k) \;=\; \widetilde{\mathrm{H}}_n(\mathbb{R}^k) \;=\; 0.$$

(4) We will often use the Reduced Homology Lemma 74.1 without explicitly mentioning it. For example we will refer to earlier results about homology groups and state the corresponding results for reduced homology groups.

[883] This calculation shows that in almost all cases we can ignore $\widetilde{\mathrm{H}}_{-1}(X)$. Nonetheless, some formulas later on become cleaner if we take into account that $\widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$.

Proof.

(0) First let X be a non-empty topological space. It follows immediately from the definitions that the augmentation map ϵ is an epimorphism which entails that $\widetilde{\mathrm{H}}_{n-1}(X) = 0$. The fact that $\widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$ is clear.

(1) Let $f\colon X \to Y$ be a map between two topological spaces. It is a straightforward consequence of the definition of the augmentation maps that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{C}_0(X) & \xrightarrow{\quad f_* \quad} & \mathrm{C}_0(Y) \\ & {\scriptstyle\epsilon}\searrow \quad \swarrow{\scriptstyle\epsilon} & \\ & \mathbb{Z}. & \end{array}$$

It follows almost immediately from the fact that this diagram commutes that the induced maps $f_*\colon \widetilde{\mathrm{C}}_*(X) \to \widetilde{\mathrm{C}}_*(Y)$ are chain maps. Furthermore we obtain from Lemma 72.7 that we get induced maps on reduced homology.

(2) This statement is basically obvious.

(3) This statement follows easily from the definitions.

(4) Let X be a topological space. By the definition of $\widetilde{\mathrm{H}}_n(X)$ we only have to deal with the case $n = 0$. Let $P \in X$ be a point. It is straightforward to show that the maps

$$\begin{array}{rcl} \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} & \to & \mathrm{H}_0(X) \\ [\sigma] \oplus n & \mapsto & [\sigma] + [n \cdot P] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{H}_0(X) & \to & \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} \\ [\sigma] & \mapsto & [\sigma - \epsilon_X(\sigma) \cdot P] + \epsilon_X(\sigma) \end{array}$$

are inverses of one another. In particular both maps are isomorphisms.

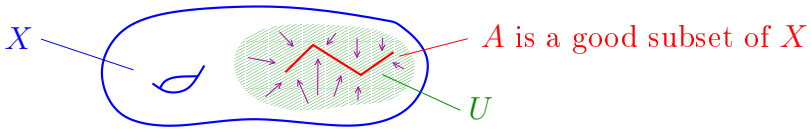
(5) This statement follows immediately from (1), Corollary 72.15 and Lemma 51.9 (4).

(6) This statement is proved exactly the same way as the Homotopy Equivalence-H_*-Corollary 73.9.

(7) This final statement follows immediately from (5) and (6). ∎

74.2. **Long exact sequences.** We need two more definitions before we can formulate one of the key theorems of homology theory.

Definition. Let X be a topological space and let $A \subset X$ be a subset. We say A is good if A is closed and if there exists an open neighborhood U of A in X such that A is a deformation retract of U.



Remark. Our definition of a good subset is the same as on [**Hat2002**, p. 114], except that we do not demand that a good subset is non-empty.

Examples.

(1) Let $x \in X$ be a point. Basically by definition the subset $\{x\} \subset X$ is good in the above sense if and only the point is good in the sense of the definition on page 1171.

(2) The sphere S^{n-1} is a good subset of the closed ball $\overline{B}^n$. Indeed, it is clear that S^{n-1} is a closed subset. Furthermore, if we consider the open neighborhood $U = \overline{B}^n \setminus \{0\}$,

then a deformation retraction is given by the map

$$\begin{array}{rcl}(\overline{B}^n \setminus \{0\}) \times [0,1] & \to & \overline{B}^n \setminus \{0\} \\ (x,t) & \mapsto & \left(1 - t \cdot \frac{1-\|x\|}{\|x\|}\right) \cdot x.\end{array}$$

(3) We can generalize the second example: It follows easily from the Smooth Manifold Boundary Proposition 19.26 (3) and the Smooth Collar Neighborhood Theorem 28.3 that every boundary component of a smooth manifold is good.
(4) Finally it follows from the Subcomplex Lemma 68.18 (2) and the CW-Complex Properties Proposition 68.10 (8a) that every subcomplex A of a CW-complex X is good.

The following definition might be familiar from earlier algebra courses.

Definition. A sequence of group homomorphisms

$$\ldots A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \to \ldots$$

is called exact, if for each n we have

$$\ker(f_n) = \operatorname{im}(f_{n+1}).$$

Example. Let A and B be abelian groups. The sequence

$$0 \to A \xrightarrow{a \mapsto (a,0)} A \oplus B \xrightarrow{(a,b) \mapsto b} B \to 0$$

is easily seen to be exact.

Remark. We collect a few basic statements about exact sequences which we will use frequently.

(1) In any exact sequence the composition of two consecutive maps is necessarily zero.
(2) We have the following equivalences of statements:

$$\begin{array}{rcl}\text{the sequence } 0 \to A \xrightarrow{f} B \text{ is exact} & \iff & f \text{ is injective} \\ \text{the sequence } A \xrightarrow{f} B \to 0 \text{ is exact} & \iff & f \text{ is surjective} \\ \text{the sequence } 0 \to A \xrightarrow{f} B \to 0 \text{ is exact} & \iff & f \text{ is an isomorphism.}\end{array}$$

(3) Sometimes we also use the following observations:

$$\begin{array}{rcl}\text{the sequence } 0 \to A \to B \xrightarrow{f} C \text{ is exact} & \implies & A \cong \ker(f) \\ \text{the sequence } A \xrightarrow{f} B \to C \to 0 \text{ is exact} & \implies & C \cong \operatorname{coker}(f).\end{array}$$

The following lemma gives us the first slightly interesting example of an exact sequence.

Lemma 74.2. If X is a non-empty topological space, then the natural sequence

$$0 \to \widetilde{\mathrm{H}}_0(X) \xrightarrow{\text{ж}} \mathrm{H}_0(X) \xrightarrow[\text{from Lemma 72.4}]{\text{augmentation map } \epsilon_X} \mathbb{Z} \to 0$$

is exact.

Proof. Let $P \in X$ be a point. We consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{\mathrm{H}}_0(X) & \xrightarrow{\text{ж}} & \mathrm{H}_0(X) & \xrightarrow{\epsilon_X} & \mathbb{Z} & \longrightarrow & 0 \\ & & & \searrow^{v\mapsto(v,0)} & \cong\uparrow{\scriptstyle (v,n)\mapsto(\text{ж}(v),[n\cdot P])} & & \nearrow_{(v,n)\mapsto n} & & \\ & & & & \widetilde{\mathrm{H}}_0(X)\oplus\mathbb{Z} & & & & \end{array}$$

We make the following observations:

(1) It follows immediately from the definitions that the diagram commutes.
(2) By the Reduced Homology Lemma 74.1 (4) we know that the vertical map is an isomorphism.
(3) The "bottom sequence" via $\widetilde{\mathrm{H}}_0(X)\oplus\mathbb{Z}$ is precisely of the form of the above example, i.e. this sequence is exact.
(4) It follows easily from the above that the original sequence is also exact. ■

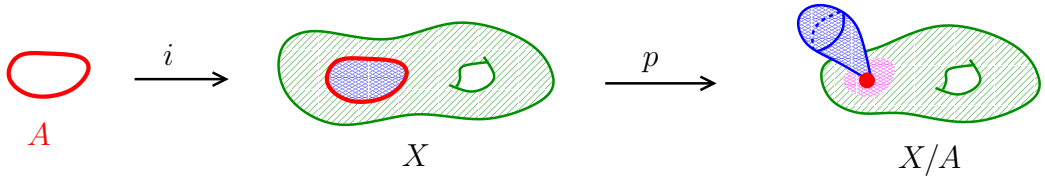
Now we can formulate the following theorem.

Theorem 74.3. Let X be a topological space and let $A \subset X$ be a good subset.[884] We denote by $i\colon A \to X$ the inclusion map and we denote by $p\colon X \to X/A$ the projection . For each $n \in \mathbb{N}_0$ there exists a natural[885] homomorphism $\partial\colon \widetilde{\mathrm{H}}_n(X/A) \to \widetilde{\mathrm{H}}_{n-1}(A)$ such that the sequence

$$\begin{array}{ccccccccc} & & & & \ldots & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_{n+1}(X/A) & \xrightarrow{\partial} & \\ \to & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_n(X) & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_n(X/A) & \xrightarrow{\partial} & & \\ \to & \widetilde{\mathrm{H}}_{n-1}(A) & \xrightarrow{i_*} & \ldots & & & & & \end{array}$$

is exact.

Example. In the figure below we show a topological space X together with a good subset A and we show the inclusion map $i\colon A \to X$ and the projection $p\colon X \to X/A$. Theorem 74.3 relates the (reduced) homology groups of A, X and X/A in a mysterious way.



Remark.

(1) In Exercise 74.19 we will see that in general the conclusion of Theorem 74.3 does not hold, if we do not assume that A is a good subset.
(2) If A is non-empty, then it follows from the Reduced Homology Lemma 74.1 that the last term that is possibly non-zero is $\widetilde{\mathrm{H}}_0(X/A)$.

[884] The most common situation is that X is a CW-complex and that A is a subcomplex.

[885] Here natural means the following: If $f\colon (X,A) \to (Y,B)$ is a map between pairs of topological spaces and if A and B are good subsets, then the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_n(X/A) & \xrightarrow{\partial} & \widetilde{\mathrm{H}}_{n-1}(A) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \widetilde{\mathrm{H}}_n(Y/B) & \xrightarrow{\partial} & \widetilde{\mathrm{H}}_{n-1}(B). \end{array}$$

It will take a while to prove Theorem 74.3. Therefore we will first consider several applications which will convince us that proving Theorem 74.3 is worth the effort.

74.3. **The homology groups of spheres.** Using Theorem 74.3 we can now determine the homology groups of spheres and we get a nice and satisfactory result.

Proposition 74.4. For each $n \in \mathbb{Z}_{\geq -1}$ we have

$$\widetilde{\mathrm{H}}_k(S^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

Remark.

(1) Put differently, the combination of the Reduced Homology Lemma 74.1 and Proposition 74.4 says that for any $n > 0$ we have

$$\mathrm{H}_k(S^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0 \text{ or } k = n, \text{ and} \\ 0, & \text{if } k \neq 0, n. \end{cases}$$

Furthermore we have

$$\mathrm{H}_k(S^0) \;\cong\; \begin{cases} \mathbb{Z}^2, & \text{if } k = 0, \text{ and} \\ 0, & \text{if } k \neq 0. \end{cases}$$

(2) Note that the statement of Proposition 74.4 is in marked contrast to the situation for higher homotopy groups. As formulated in Question 71.18 we have the strong suspicion that $\pi_3(S^2) \neq 0$.

By the Smooth Manifold-Dimension Proposition 19.28 we know that for $k \neq l$ the spheres S^k and S^l are not diffeomorphic. The following corollary is a serious upgrade of this fact.

Corollary 74.5. For $k \neq l$ the spheres S^k and S^l are not homeomorphic.

So we have finally managed to distinguish spheres of different dimensions. Our initial idea had been to use higher homotopy groups to prove that result. Even though the definition of higher homotopy groups is arguably much more natural than the definition of homology groups, it turns out that the latter groups are significantly easier to compute.

Proof of Proposition 74.4 assuming Theorem 74.3. We prove the proposition by induction on n. First note that for $n = -1$ we have $S^n = S^{-1} = \varnothing$, thus the statement follows immediately from the Reduced Homology Lemma 74.1 (0).[886]

Now we suppose that the statement of the proposition holds for $n - 1$. We want to determine the reduced homology of S^n. In the Ball-Quotient Sphere Lemma 5.20 we had shown that S^n is homeomorphic to $\overline{B}^n/S^{n-1}$. We had just seen on page 1614 that S^{n-1} is a good subset of $\overline{B}^n$. Therefore we can apply Theorem 74.3 to $X = \overline{B}^n$ and the closed subset $A = S^{n-1}$ and we obtain the following exact sequence

$$\begin{array}{rcccccc} & \ldots & \longrightarrow & \widetilde{\mathrm{H}}_{k+1}(\overline{B}^n) & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_{k+1}(\overline{B}^n/S^{n-1}) & \xrightarrow{\partial} \\ \to & \widetilde{\mathrm{H}}_k(S^{n-1}) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_k(\overline{B}^n) & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_k(\overline{B}^n/S^{n-1}) & \xrightarrow{\partial} \\ \to & \widetilde{\mathrm{H}}_{k-1}(S^{n-1}) & \xrightarrow{i_*} & \ldots & & & \end{array}$$

From the induction hypothesis, from the homeomorphism $S^n \cong \overline{B}^n/S^{n-1}$ and from the fact that $\widetilde{\mathrm{H}}_i(\overline{B}^n) = 0$ for all i (obtained on page 1612) we now see that the above long exact

[886] To readers who prefer S^0 as a starting point we point out that S^0 consists of precisely two points. Thus the promised isomorphisms follow from Lemmas 72.6, 72.14 and 74.1.

sequence gives rise to the following exact sequence:

$$\begin{array}{rcccccc} & \dots & \longrightarrow & 0 & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_{k+1}(S^n) & \xrightarrow{\partial} \\ \to & \widetilde{\mathrm{H}}_k(S^{n-1}) & \xrightarrow{i_*} & 0 & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_k(S^n) & \xrightarrow{\partial} \\ \to & \widetilde{\mathrm{H}}_{k-1}(S^{n-1}) & \xrightarrow{i_*} & \dots & & & \end{array}$$

In particular we obtain for each k an exact sequence

$$0 \;\to\; \widetilde{\mathrm{H}}_{k+1}(S^n) \;\xrightarrow{\partial}\; \widetilde{\mathrm{H}}_k(S^{n-1}) \;\to\; 0.$$

But this means that the maps ∂ are isomorphisms, and using our induction hypothesis we obtain the desired result that

$$\widetilde{\mathrm{H}}_k(S^n) \;=\; \widetilde{\mathrm{H}}_{k-1}(S^{n-1}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k-1 = n-1, \text{ i.e. if } k = n \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

■

In Lemma 2.26 we had used elementary arguments to show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$. Furthermore, in Lemma 50.4 we had used the fundamental group to prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \geq 3$. Now we can finally show that for any $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic.

Proposition 74.6. For any $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic.

Proof. Clearly we only have to deal with the case $k, l \geq 1$. Let $k, l \in \mathbb{N}$ and suppose there exists a homeomorphism $f\colon \mathbb{R}^k \to \mathbb{R}^l$. Then we obtain that

$$\begin{array}{ccccccc} & & & & \text{follows from the Homotopy Equivalence-H}_*\text{-Corollary 73.9} & & \\ & & & & \downarrow & & \\ \mathbb{Z} & \underset{\uparrow}{\cong} & \widetilde{\mathrm{H}}_{k-1}(S^{k-1}) & \cong & \widetilde{\mathrm{H}}_{k-1}(\mathbb{R}^k \setminus \{0\}) & \xrightarrow[\uparrow]{f_*} & \widetilde{\mathrm{H}}_{k-1}(\mathbb{R}^l \setminus \{f(0)\}) \;\underset{\downarrow}{\cong}\; \widetilde{\mathrm{H}}_{k-1}(S^{l-1}). \\ & \text{Proposition 74.4} & & & & \text{isomorphism since } f \text{ is a homeomorphism} & \end{array}$$

So we have $\widetilde{\mathrm{H}}_{k-1}(S^{l-1}) \cong \mathbb{Z}$. But by Proposition 74.4 that is only possible if $k = l$. ■

74.4. **The Brouwer Fixed Point Theorem II.** In Lemma 47.16 we had used the fundamental group to show that S^1 is not a retract of $\overline{B}^2$. Precisely the same argument, together with Proposition 74.4, now gives us the following proposition.

Proposition 74.7. For every $n \in \mathbb{N}$ the sphere $S^{n-1} = \partial\overline{B}^n$ is not a retract of the closed ball $\overline{B}^n$.

Remark. I found the following amusing description of Proposition 74.7 online:
"Suppose a shark jumps into a shoal of fish (a kind of big ball). The small fishes start escaping in all directions towards the border of the shoal, where the fishes stand still. Yet they escape with a certain disposition to follow a continuous flow, as they usually do, since everybody tends to follow its neighborhoods. But since there is no continuous retraction to the boundary, somebody doesn't know where to go, and stay there for a moment, much to the shark's satisfaction. There is also a 2D version, with a wolf entering into a herd of sheep."

Proof. Suppose there exists a retraction $r\colon \overline{B}^n \to S^{n-1}$. We denote by $i\colon S^{n-1} \to \overline{B}^n$ the inclusion map. We consider the following diagram

$$\begin{array}{ccc} & \widetilde{\mathrm{H}}_{n-1}(\overline{B}^n) & \\ {\scriptstyle i_*}\nearrow & & \searrow{\scriptstyle r_*} \\ \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) & \xrightarrow{\quad (r\circ i)_* \quad} & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}). \end{array}$$

This diagram commutes since it follows from the functoriality of the homology groups that $r_* \circ i_* = (r \circ i)_*$. Since r is a retraction it follows that $r \circ i = \mathrm{id}_{S^{n-1}}$, i.e. $(r \circ i)_* = (\mathrm{id}_{S^{n-1}})_*$ is the identity map on $\widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ which, by Proposition 74.4, is isomorphic to $\mathbb{Z}$. The lower map is thus an isomorphism, but the upper map factorizes through the trivial group, i.e. the composition of i_* and r_* cannot be an isomorphism of $\mathbb{Z}$. Thus we have obtained a contradiction. ■

Now we can again provide a proof of Brouwer Fixed Point Theorem 31.4.

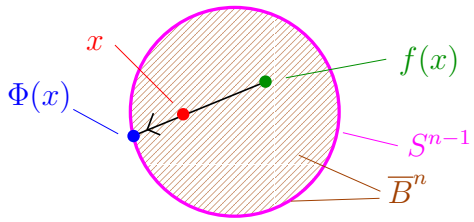
Theorem (Brouwer Fixed Point Theorem 31.4. Every map $f\colon \overline{B}^n \to \overline{B}^n$ admits a fixed point, i.e. for every map $f\colon \overline{B}^n \to \overline{B}^n$ there exists a point $x \in \overline{B}^n$ such that $f(x) = x$.

Remark. We already provided on proof on page 784 and we will provide another proof on page 2118.

Proof. The proof of the Brouwer Fixed Point Theorem 31.4 is basically identical to the one provided on page 1618. We just need to replace the Boundary Non-Retraction Proposition 31.3 by Proposition 74.7. For the reader's convenience we provide the argument, minus the technical details, again.

Suppose there exists a map $f\colon \overline{B}^n \to \overline{B}^n$ without a fixed point. Since f admits no fixed points we saw on page 784 that for each $x \in \overline{B}^n$ the open ray $\{f(x) + t \cdot (x - f(x)) \,|\, t > 0\}$ which starts at $f(x)$ and which goes through x contains precisely one point in S^{n-1}. Thus we can consider the following map:

$$\begin{array}{rcl} \Phi\colon \overline{B}^n & \to & \overline{B}^n \\ x & \mapsto & \text{the intersection point of } S^{n-1} \text{ with the} \\ & & \text{uniquely determined open ray from } f(x) \text{ to } x. \end{array}$$



As we saw on page 785, it is elementary to show that Φ is continuous. It is evident that $\Phi(x) = x$ for all $x \in S^{n-1}$ and that $\Phi(x) \in S^{n-1}$ for all $x \in S^{n-1}$. The map Φ is thus a retraction from $\overline{B}^n$ to S^{n-1}. But according to Proposition 74.7 this not possible. Thus we have obtained a contradiction. ■

As we mentioned already in Section 31.2, the Brouwer Fixed Point Theorem pops up many times in mathematics. For example we can use it to prove the following theorem in linear algebra.

Theorem 31.5. (Perron-Frobenius Theorem) Let $A = (a_{ij})$ be an $n \times n$-real matrix. If all entries are positive, then A has a positive eigenvalue λ with an eigenvector v such that all entries of v are non-negative.

Proof. For the reader's convenience we recall the proof from page 785. We need to introduce two players:

(1) Given $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ we write $|v| := \sum_{i=1}^{n} |v_i|$.
(2) We consider again the standard $(n-1)$-simplex

$$\Delta^{n-1} := \{(t_1, \dots, t_n) \in [0,1]^n \,\big|\, |v| = 1\}.$$

We saw in the Standard Simplex Homeomorphism Lemma 72.1 that Δ^{n-1} is homeomorphic to $\overline{B}^{n-1}$. It follows immediately from this observation and from the Brouwer Fixed Point Theorem 31.4 that any self-map $f\colon \Delta^{n-1} \to \Delta^{n-1}$ has a fixed point.

Now let $A = (a_{ij})$ be an $n \times n$-real matrix such that all entries are positive. We consider the map

$$\begin{array}{rcl} f\colon \Delta^{n-1} & \to & \Delta^{n-1} \\ v = (v_1, \dots, v_n) & \mapsto & \frac{A \cdot v}{|A \cdot v|}. \end{array}$$

Note that this map is defined since our hypothesis that all a_{ij} are positive and the fact that all entries of v are non-negative and that at least one entry of v is positive imply that $A \cdot v \neq 0$. As we had just discussed above, the map $f\colon \Delta^{n-1} \to \Delta^{n-1}$ has a fixed point $v \in \Delta^{n-1}$. But this means that $A \cdot v = |A \cdot v| \cdot v$. Put differently, we have just shown that v is an eigenvector to the positive eigenvalue $|A \cdot v|$. ■

74.5. **Basic homological algebra.** Now that we have made use of Theorem 74.3 we also need to provide a proof of Theorem 74.3. In this chapter we will first develop several algebraic methods which later on will be used in the proof of the existence of the exact sequence of Theorem 74.3.

Definition. A short exact sequence is an exact sequence of the form

$$0 \to A \to B \to C \to 0,$$

which consists of at most three non-trivial groups. A short exact sequence of chain complexes

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} \mathrm{C}_* \to 0$$

consists of chain complexes A_*, B_*, C_* and chain maps $i\colon A_* \to B_*$ and $p\colon B_* \to \mathrm{C}_*$ such that for each n the maps

$$0 \to A_n \xrightarrow{i} B_n \xrightarrow{p} \mathrm{C}_n \to 0$$

form a short exact sequence.

Remark. Throughout the subsequent discussion in the following pages it is helpful to keep in mind the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_{n+1} & \xrightarrow{i} & B_{n+1} & \xrightarrow{p} & \mathrm{C}_{n+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} & & \\
0 & \longrightarrow & A_{n} & \xrightarrow{i} & B_{n} & \xrightarrow{p} & \mathrm{C}_{n} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_{n}} & & \downarrow{\scriptstyle \partial_{n}} & & \downarrow{\scriptstyle \partial_{n}} & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{i} & B_{n-1} & \xrightarrow{p} & \mathrm{C}_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_{n-1}} & & \downarrow{\scriptstyle \partial_{n-1}} & & \downarrow{\scriptstyle \partial_{n-1}} & & \\
0 & \longrightarrow & A_{n-2} & \xrightarrow{i} & B_{n-2} & \xrightarrow{p} & \mathrm{C}_{n-2} & \longrightarrow & 0.
\end{array}$$

Definition. Now suppose that we are given such a short exact sequence of chain complexes. We will define a map

$$\partial_n \colon \mathrm{H}_n(C) \ \to \ \mathrm{H}_{n-1}(A)$$

as follows:

(1) Let $z \in \mathrm{H}_n(C)$.
(2) We pick $c \in C_n$ with $z = [c]$.
(3) By the surjectivity of p we have $c = p(b)$ for some $b \in B_n$.
(4) We have

$$\begin{array}{ccccccc}
p(\partial b) & = & \partial p(b) & = & \partial c & = & 0. \\
 & \uparrow & & & & \uparrow & \\
\text{since the diagram commutes} & & & & & \text{since } c \text{ is a cycle} &
\end{array}$$

(5) Since the second horizontal row in the above diagram is exact it follows immediately from (4) that $\partial b = i(a)$ for some uniquely determined $a \in A_{n-1}$.
(6) We put $\partial_n(z) := [a]$.

We refer to this map $\partial_n \colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$ as the connecting homomorphism.

The following lemma summarizes two of the key properties of the connecting homomorphisms.

Lemma 74.8. (Connecting Homomorphism Lemma)

(1) Let $0 \to A_* \to B_* \to C_* \to 0$ be a short exact sequence of chain complexes. The connecting homomorphism $\partial_n \colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$ is well-defined and it is indeed a homomorphism.
(2) The connecting homomorphism is natural, i.e. if

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_{*} & \xrightarrow{i} & B_{*} & \xrightarrow{p} & C_{*} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\
0 & \longrightarrow & A'_{*} & \xrightarrow{i} & B'_{*} & \xrightarrow{p} & C'_{*} & \longrightarrow & 0
\end{array}$$

is a commutative diagram of chain maps where the horizontal sequences are exact, then the following diagram commutes for every $n \in \mathbb{N}$:

$$\begin{array}{ccc} \mathrm{H}_n(C) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A) \\ {\scriptstyle c_*}\downarrow & & \downarrow{\scriptstyle a_*} \\ \mathrm{H}_n(C') & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A') \end{array}$$

Remark. Let us explain why we say that the connecting homomorphism is "natural". Let $\mathcal{S}$ be the category of short exact sequences of chain complexes where the morphisms are chain maps. the Connecting Homomorphism Lemma 74.8 (2) says given any $n \in \mathbb{N}_0$ the connecting homomorphism $\partial_n \colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$ is a natural transformation from the functor

$$\begin{array}{rcll} \mathcal{S} & \to & \mathit{AbGroup} & \text{to the} \\ (A_* \to B_* \to \mathrm{C}_*) & \mapsto & \mathrm{H}_n(C) & \text{functor} \end{array} \qquad \begin{array}{rcl} \mathcal{S} & \to & \mathit{AbGroup} \\ (A_* \to B_* \to \mathrm{C}_*) & \mapsto & \mathrm{H}_{n-1}(A). \end{array}$$

The proof of this lemma and the following proposition are done by a "diagram chase". This means that one needs to keep the commutative diagram on page 1620 in mind and one has to use on several occasions that the vertical maps are chain maps and that the horizontal sequences are exact. In both cases the proofs are quite elementary.

Proof.

The proof below is elementary but not very readable. It is much easier to prove the lemma on your own than to try to read the proof.

For the undeterred reader we now provide the fully gory details.

(1) We start out with the following claim.

Claim. The map
$$\partial_n \colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$$
is well-defined.

Proof. Let $z \in \mathrm{H}_n(C)$. We denote by c, b and a the elements as in the definition of ∂_n. We need to show that the element a is indeed a cycle and that the definition of $\partial_n([c])$ does not depend on the choices of c and b.

The proofs of these two statements is not particularly difficult and it is done via "diagram chasing". First we note that

$$i(\partial a) \underset{\text{since the diagram commutes}}{\overset{\uparrow}{=}} \partial i(a) = \partial(\partial b) = 0.$$

Since i is injective we obtain that $\partial a = 0$, i.e. a is indeed a cycle. Now we show that the homology class $[a]$ depends only on the homology class $[c]$.

(a) If we chose a different element in B_n, then it follows from the fact that the horizontal sequences are exact that this element would be of the form $b + i(a')$ for some $a' \in A_n$. But then we obtain the following equality:

$$\partial(b + i(a')) = \partial b + \partial(i(a')) \overset{\text{since the diagram commutes}}{\overset{\downarrow}{=}} \underbrace{\partial b}_{=i(a)} + i(\partial a') = i(a + \partial a').$$

Since $[a] = [a + \partial a'] \in \mathrm{H}_{n-1}(A)$ this does not change the homology class.

(b) If we picked an other representative for $[c]$, then this representative would be of the form $c+\partial c'$ for some $c' \in \mathrm{C}_{n+1}$. Since the map $p\colon B_n \to C_n$ is an epimorphism there exists a $b' \in B_{n+1}$ with $p(b') = c'$. It follows that

$$p(b+\partial b') \;=\; p(b) + p(\partial b') \;\underset{\text{since the diagram commutes}}{\overset{\uparrow}{=}}\; c + \partial p(b') \;=\; c + \partial c'.$$

Since $\partial(b+\partial b') = \partial b$ we see that we have ended up with the same element in B_{n-1} as before. ⊞

Claim. The map

$$\partial_n\colon \mathrm{H}_n(C) \;\to\; \mathrm{H}_{n-1}(A)$$

is a homomorphism.

Proof. Thus let $[c_1]$ and $[c_2]$ be given. For $j = 1, 2$ we choose b_j as in this definition of ∂_n, and we denote by a_j the uniquely determined element with $i(a_j) = \partial b_j$. Then we also have $p(b_1+b_2) = c_1+c_2$ and $i(a_1+a_2) = i(a_1)+i(a_2) = \partial b_1+\partial b_2 = \partial(b_1+b_2)$. It follows that

$$\partial_n([c_1+c_2]) \;=\; [a_1+a_2] \;=\; [a_1]+[a_2] \;=\; \partial_n([c_1])+\partial_n([c_2]).$$ ⊞

(2) The second statement follows easily from the definitions. We leave the verification of the details to the reader. ■

Now we can formulate the following proposition which explains the name "connecting homomorphism".

Proposition 74.9. (LES Proposition)

(1) Let

$$0 \;\to\; A_* \;\xrightarrow{i}\; B_* \;\xrightarrow{p}\; C_* \;\to\; 0$$

be a short exact sequence of chain complexes. Then the sequence

$$\ldots \to \mathrm{H}_n(A) \xrightarrow{i_*} \mathrm{H}_n(B) \xrightarrow{p_*} \mathrm{H}_n(C) \xrightarrow{\partial} \mathrm{H}_{n-1}(A) \xrightarrow{i_*} \mathrm{H}_{n-1}(B) \xrightarrow{p_*} \ldots$$

is exact.

(2) The connecting homomorphism is natural. More precisely, let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \xrightarrow{i} & B_* & \xrightarrow{p} & C_* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A'_* & \xrightarrow{i} & B'_* & \xrightarrow{p} & C'_* & \longrightarrow & 0 \end{array}$$

be a commutative diagram of chain maps where the horizontal sequences are exact. Then the following diagram commutes:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \mathrm{H}_n(A) & \xrightarrow{i_*} & \mathrm{H}_n(B) & \xrightarrow{p_*} & \mathrm{H}_n(C) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(A) & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle a_*} & & \downarrow{\scriptstyle b_*} & & \downarrow{\scriptstyle c_*} & & \downarrow{\scriptstyle a_*} & & \\ \ldots & \longrightarrow & \mathrm{H}_n(A') & \xrightarrow{i_*} & \mathrm{H}_n(B') & \xrightarrow{p_*} & \mathrm{H}_n(C') & \xrightarrow{\partial} & \mathrm{H}_{n-1}(A') & \longrightarrow & \ldots \end{array}$$

The fact that the square to the right commutes is just in plain words the statement that the connecting homomorphism is natural.

In the following we refer to the exact sequence of the LES Proposition 74.9 as the as *long exact sequence in homology* corresponding to the short exact sequence of chain complexes.

Proof of the LES Proposition 74.9. First note that Statement (2) is an immediate consequence of the Connecting Homomorphism Lemma 74.8. We included the statement for the convenience of the reader. Thus let us turn to the actual meat, namely Statement (1).

The proof below is elementary but not very readable. It is much easier to prove the proposition on your own than to try to read the proof.

Let
$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$$
be a short exact sequence of chain complexes. Then the sequence
$$\ldots \to \mathrm{H}_n(A) \xrightarrow{i_*} \mathrm{H}_n(B) \xrightarrow{p_*} \mathrm{H}_n(C) \xrightarrow{\partial} \mathrm{H}_{n-1}(A) \xrightarrow{i_*} \mathrm{H}_{n-1}(B) \xrightarrow{p_*} \ldots$$
is exact. We examine the exactness at the three groups $\mathrm{H}_{n-1}(A), \mathrm{H}_n(B)$ and $\mathrm{H}_n(C)$. Put differently, at each of the three groups we want to show the following two statements:

(1) the image of the map to the left is contained in the image of the map to the right, or equivalently, the composition of the map to the left and the map to the right is zero,
(2) the kernel of the map to the right is contained in the image of the map to the left.

Now we perform these steps:

(A) First let $[c] \in \mathrm{H}_n(C)$. We want to show that $i(\partial[c]) = 0$. We choose b with $p(b) = c$ and denote by $a \in A_{n-1}$ the element with $i(a) = \partial b$. Then we have
$$i(\partial[c]) \underset{\substack{\uparrow \\ \text{definition of } \partial\colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)}}{=} i([a]) = [\partial b] = 0 \in \mathrm{H}_{n-1}(B).$$
Now let $[a] \in \ker(i_*)$. Then $i(a) = \partial b$ for some $b \in B_n$. It follows immediately from the definitions that $\partial[p(b)] = [a]$.

(B) From $p \circ i = 0$ it follows that $p_* \circ i_* = 0$, i.e. $\mathrm{im}(i_*) \subset \ker(p_*)$.
Conversely let $b \in \ker(p_*)$. Then $p(b) = \partial c'$ for some $c' \in \mathrm{C}_{n+1}$. Since p is surjective, there exists a $b' \in B_{n+1}$ with $p(b') = c'$. From the commutativity it follows that $p(\partial b') = \partial p(b') = \partial c' = p(b)$. Thus it follows that $p(b - \partial b') = 0$, i.e. $b - \partial b' = i(a)$ for some $a \in A_n$. But then it follows that $[b] = [b - \partial b'] = [i(a)] \in \mathrm{im}(i_*)$.

(C) Let $[b] \in \mathrm{H}_n(B)$. We have to show that $\partial(p_*([b])) = 0$. In the definition of $\partial p_*([b])$ we choose some $b \in B$. Since $\partial b = 0$ it follows immediately that $\partial p_*([b]) = 0$.
Finally let $[c] \in \ker(\partial\colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A))$. We choose $b \in B_n$ with $p(b) = c$. Then there exists a uniquely determined $a \in A_{n-1}$ with $i(a) = \partial b$. Since we have $[a] = \partial[c] = 0 \in \mathrm{H}_{n-1}(A)$ there exists an $a' \in A_n$ with $\partial a' = a$. Then it follows that
$$\partial(b - i(a')) = \partial b - \partial i(a') = \partial b - i(\partial a') = \partial b - i(a) = 0,$$
i.e. $b - i(a')$ is a cycle. On the other hand we have $p(b - i(a')) = p(b) - (p \circ i)(a') = p(b)$. Thus we see that $p_*([b - i(a')]) = [c]$. We have thus shown that $\ker(\partial) \subset \mathrm{im}(p_*)$.

Thus we have proved the claim and therefore we have also completed the proof of the proposition. ∎

Lemma 74.10. (Five Lemma) Let

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \downarrow{\scriptstyle a} & & \cong\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \cong\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E' \end{array}$$

be a diagram of groups (here for once we do not suppose that the groups are abelian) such that the horizontal sequences are exact and such that every squares commute (if all the groups are abelian, then it suffices it the squares commute up to a sign). If a is surjective, if b and d are isomorphisms, and if e is injective, then c is an isomorphism.[887]

Proof. One can prove the lemma via a "diagram chase". Filling in the details is precisely the task of Exercise 74.6. ∎

74.6. **Relative homology groups.** In this section we will introduce the relative homology groups which will play an important role in the proof of Theorem 74.3 and also in the remainder of this course.

Definition. Let (X, A) be a pair of topological spaces.[888] We view each chain group $\mathrm{C}_n(A)$ as a subgroup of $\mathrm{C}_n(X)$. Thus we can define

$$\mathrm{C}_n(X, A) \;:=\; \mathrm{C}_n(X)/\mathrm{C}_n(A)$$

and we consider the boundary map[889]

$$\begin{array}{rcl} \partial_n \colon \mathrm{C}_n(X, A) & \to & \mathrm{C}_{n-1}(X, A) \\ [c] & \mapsto & [\partial_n c]. \end{array}$$

We refer to

$$\ldots \xrightarrow{\partial_{n+2}} \mathrm{C}_{n+1}(X, A) \xrightarrow{\partial_{n+1}} \mathrm{C}_n(X, A) \xrightarrow{\partial_n} \mathrm{C}_{n-1}(X, A) \xrightarrow{\partial_{n-1}} \ldots$$

as the chain complex of the pair (X, A).[890] Given any $n \in \mathbb{N}_0$ we define the n-th relative homology group of (X, A) as follows:

$$\mathrm{H}_n(X, A) \;:=\; \mathrm{H}_n(\mathrm{C}_*(X, A)).$$

Remark. Let (X, A) be a pair of topological spaces and let $n \in \mathbb{N}_0$. We write

$$S \;:=\; \{\sigma \colon \Delta^n \to X \mid \mathrm{im}(\sigma) \cap (X \setminus A) \neq \varnothing\},$$

i.e. S denotes the set of all singular n-simplices of X that are not completely contained in A. Then it is straightforward to show that the map

$$\begin{array}{rcl} \mathbb{Z}^{(S)} & \to & \mathrm{C}_n(X, A) \\ \sum\limits_{i=1}^{n} n_i \sigma_i & \mapsto & \left[\sum\limits_{i=1}^{n} n_i \sigma_i\right] \end{array}$$

---

[887] In many applications it suffices to remember the following slightly weaker statement: if the two vertical maps on the left and right are isomorphisms, then the middle map is also an isomorphism.

[888] Recall that according to the definition on page 439 this means that X is a topological space and $A \subset X$ is a subset.

[889] The map ∂_n is well-defined, since for any $c \in \mathrm{C}_n(A) \subset \mathrm{C}_n(X)$ the boundary ∂c lies obviously in $\mathrm{C}_{n-1}(A) \subset \mathrm{C}_{n-1}(X)$.

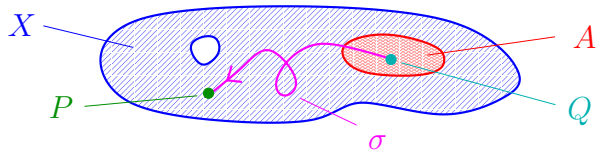
[890] Since $\partial_{n-1} \circ \partial_n \colon \mathrm{C}_n(X) \to \mathrm{C}_{n-2}(X)$ is the zero map it follows immediately that the composition $\partial_{n-1} \circ \partial_n \colon \mathrm{C}_n(X, A) \to \mathrm{C}_{n-2}(X, A)$ is also the zero map.

is an isomorphism. In particular we see that $\mathrm{C}_n(X, A)$ is a free abelian group.

Examples.

(1) If $A = \varnothing$, then for all $n \in \mathbb{N}_0$ we have $\mathrm{C}_n(\varnothing) = 0$ and $\mathrm{C}_n(X, \varnothing) = \mathrm{C}_n(X)$, i.e. we have $\mathrm{H}_n(X, \varnothing) = \mathrm{H}_n(X)$. In the following we will freely go back and forth between the notations $\mathrm{H}_n(X, \varnothing)$ and $\mathrm{H}_n(X)$.

(2) If X is a path-connected topological space and if A is non-empty, then $\mathrm{H}_0(X, A) = 0$. Indeed, let $P \in X$ be a point. We pick a point $Q \in A$. Since X is path-connected we can pick a map $\sigma\colon \Delta^1 \to X$ such that $\sigma(0, 1) = P$ and $\sigma(1, 0) = Q$. As discussed on page 1585, in $\mathrm{C}_*(X, A)$ we have $\partial\sigma = P - Q$, but since $Q = 0 \in \mathrm{C}_0(X, A)$ we have $\partial\sigma = P$, which shows that $P = 0 \in \mathrm{H}_0(X, A)$. But this clearly implies that $\mathrm{H}_0(X, A) = 0$. We refer to the figure below for an illustration.

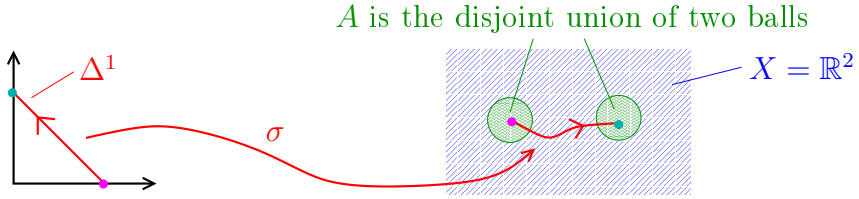


(3) Whereas it is often difficult to write down singular cycles in absolute chain complexes $\mathrm{C}_*(X)$, it can be surprisingly easy to write down singular cycles in a relative chain complex $\mathrm{C}_*(X, A)$. Indeed, given any map $\sigma\colon \Delta^n \to X$ with $\sigma(\partial\Delta^n) \subset A$ we have

$$\partial_n(\sigma) \;=\; \sum_{j=0}^{n} (-1)^j \cdot \sigma \circ i_j^n \;\underset{\uparrow}{=}\; 0 \;\in\; \mathrm{C}_{n-1}(X, A) = \mathrm{C}_{n-1}(X)/\mathrm{C}_{n-1}(A).$$

since $\sigma \circ i_j^n\colon \Delta^{n-1}$ takes values in A we see that $\sigma \circ i_j^n \in \mathrm{C}_{n-1}(A)$

Thus σ defines a cycle in the chain complex $\mathrm{C}_n(X, A)$. As an example we consider $X = \mathbb{R}^2$ and $A = B_1(-2) \cup B_1(2)$. Let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex with $\sigma(0, 1) \in B_1(2)$ and $\sigma(1, 0) \in B_1(-2)$. (We refer to the figure below for an illustration.) It follows from the above discussion that σ is a cycle in $\mathrm{C}_*(X, A)$. Later, on page 1630, we will see whether or not $[\sigma] \in \mathrm{H}_1(X, A)$ is non-trivial.



Definition.

(1) Recall that we refer to the category that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{PairTop}) &:= \text{all pairs of topological spaces,} \\ \mathrm{Mor}((X, A), (Y, B)) &:= \text{all continuous maps } f\colon X \to A \text{ with } f(A) \subset B \end{aligned}$$

with the usual composition of maps as the category of pairs of topological spaces.

(2) Let $f\colon (X,A) \to (Y,B)$ be a map of pairs of topological spaces, i.e. a map $f\colon X \to Y$ with $f(A) \subset B$. Then f induces maps

$$\begin{array}{rcl} f_*\colon \mathrm{H}_n(X,A) & \to & \mathrm{H}_n(Y,B) \\ \Big[\sum\limits_{j=1}^{n} a_j\sigma_j\Big] & \mapsto & \Big[\sum\limits_{j=1}^{n} a_j(f\circ\sigma_j)\Big]. \end{array}$$

It is straightforward to show that these maps are well-defined.

The following lemma follows almost immediately from the definitions.

Lemma 74.11. For each $n \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} (X,A) & \to & \mathrm{H}_n(X,A) \\ (f\colon (X,A) \to (Y,B)) & \mapsto & (f_*\colon \mathrm{H}_n(X,A) \to \mathrm{H}_n(Y,B)) \end{array}$$

define a covariant functor from the category *PairTop* of pairs of topological spaces to the category *AbGroup* of abelian groups.

Remark. If (Y,B) is a pair of topological spaces and if $X \subset Y$ and $A \subset B$, then we have the obvious inclusion map $(X,A) \to (Y,B)$ and we therefore obtain the induced maps $\mathrm{H}_n(X,A) \to \mathrm{H}_n(Y,B)$ in relative homology. If it is clear from the context that we are dealing with such an inclusion induced map, then on many occasions we will not explicitly point out that we are working with the inclusion induced maps.

Next we study the interaction between ordinary homology, relative homology and reduced homology. We introduce the following notation.

Notation. Let X be a topological space, let $A \subset X$ and let $n \in \mathbb{N}_0$. We introduce the following maps:

$$\begin{tikzcd} \mathrm{H}_n(X) \arrow[r, "="] \arrow[rr, bend right, "=:\text{я}=:\text{я}_{X,A}"'] & \mathrm{H}_n(X,\varnothing) \arrow[r] & \mathrm{H}_n(X,A) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \widetilde{\mathrm{H}}_n(X) \arrow[r, "\text{ж}"] \arrow[rr, bend right, "=:\text{ж}_{X,A}"'] & \mathrm{H}_n(X) \arrow[r, "\text{я}"] & \mathrm{H}_n(X,A). \end{tikzcd}$$

Lemma 74.12. Let X be a topological space and let $n \in \mathbb{N}_0$. The following statements hold:

(1) Let $A \subset X$. The maps $\text{я}_{X,A}$ and $\text{ж}_{X,A}$ are natural in the sense that if we are given a map $f\colon (X,A) \to (Y,B)$ of pairs of topological spaces then the following diagrams commute:

$$\begin{tikzcd} \mathrm{H}_n(X) \arrow[r, "f_*"] \arrow[d, "\text{я}_{X,A}"'] & \mathrm{H}_n(Y) \arrow[d, "\text{я}_{Y,B}"] \\ \mathrm{H}_n(X,A) \arrow[r, "f_*"] & \mathrm{H}_n(Y,B) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \widetilde{\mathrm{H}}_n(X) \arrow[r, "f_*"] \arrow[d, "\text{ж}_{X,A}"'] & \widetilde{\mathrm{H}}_n(Y) \arrow[d, "\text{ж}_{Y,B}"] \\ \mathrm{H}_n(X,A) \arrow[r, "f_*"] & \mathrm{H}_n(Y,B). \end{tikzcd}$$

(2) For each $x_0 \in X$ the map

$$\text{ж}_{X,\{x_0\}}\colon \widetilde{\mathrm{H}}_n(X) \to \mathrm{H}_n(X,\{x_0\})$$

is a natural isomorphism.

Proof.

(1) The first statement follows immediately from the definitions.
(2) We will provide the proof of this statement in Exercise 74.14. ■

Before we can state the next proposition we need to introduce the notion of a triple of topological spaces which is defined in almost the same fashion as a pair of topological spaces.

Definition.

(1) A triple of topological spaces is a triple (X, B, A) where X is a topological space and $A \subset B \subset X$ are subsets.
(2) A map $f\colon (X, B, A) \to (X', B', A')$ between triples of topological spaces is a continuous map $f\colon X \to X'$ such that $f(B) \subset B'$ and $f(A) \subset A'$.

Proposition 74.13. (Topological LES-Proposition) Let X be a topological space and let $A \subset B \subset X$ be two subsets. We denote by $i\colon (B, A) \to (X, A)$ and by $p\colon (X, A) \to (X, B)$ the obvious maps.

(1) For each $n \in \mathbb{N}_0$ the map[891][892]
$$\begin{array}{rcl} \partial_n\colon \mathrm{H}_n(X, B) & \to & \mathrm{H}_{n-1}(B, A) \\ \left[\sum_{j=1}^{k} a_j \cdot (\sigma_j\colon \Delta^n \to X)\right] & \mapsto & \left[\sum_{j=1}^{k} a_j \cdot \partial\sigma_j\right] \end{array}$$
is well-defined.
(2) We consider the category of triples of topological spaces.[893] The maps ∂_n from (1) define a natural transformation from the functor $(X, B, A) \to \mathrm{H}_n(X, B)$ to the functor $(X, B, A) \to \mathrm{H}_{n-1}(B, A)$. In down-to-earth language this means that for a map $f\colon (X, A, B) \to (\widetilde{X}, \widetilde{A}, \widetilde{B})$ of triples of topological spaces the following diagram commutes:
$$\begin{array}{ccc} \mathrm{H}_n(X, B) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(B, A) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathrm{H}_n(\widetilde{X}, \widetilde{B}) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(\widetilde{B}, \widetilde{A}). \end{array}$$
In particular the following diagram commutes:
$$\begin{array}{ccccccccc} \ldots & \longrightarrow & \mathrm{H}_n(B, A) & \longrightarrow & \mathrm{H}_n(X, A) & \longrightarrow & \mathrm{H}_n(X, B) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(B, A) & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\ \ldots & \longrightarrow & \mathrm{H}_n(\widetilde{B}, \widetilde{A}) & \longrightarrow & \mathrm{H}_n(\widetilde{X}, \widetilde{A}) & \longrightarrow & \mathrm{H}_n(\widetilde{X}, \widetilde{B}) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(\widetilde{B}, \widetilde{A}) & \longrightarrow & \ldots. \end{array}$$
(3) The maps
$$\ldots \to \mathrm{H}_n(B, A) \xrightarrow{i_*} \mathrm{H}_n(X, A) \xrightarrow{p_*} \mathrm{H}_n(X, B) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(B, A) \to \ldots$$
form an exact sequence.

The long exact sequence of the Topological LES-Proposition 74.13 (2) is sometimes called the *long exact sequence of the triple* (X, B, A) and we refer to ∂_n as the *connecting homomorphism* of the triple (X, B, A).

[891] The fact that here the symbol "∂_n" denotes the boundary map of singular chains and the desired map $\mathrm{H}_n(X, B) \to \mathrm{H}_{n-1}(B, A)$ is an unfortunate problem in our choice of notation.
[892] Here we use that every homology class in $\mathrm{H}_n(X, B)$ admits by definition a representative in the quotient $\mathrm{C}_n(X, B) = \mathrm{C}_n(X)/\mathrm{C}_n(B)$, so it can be represented by a singular n-chain in X.
[893] It should be clear from the definitions preceding the definition what the objects and morphisms of this category are.

Proof. It follows immediately from the definitions and the third isomorphism theorem[894] that

$$0 \to \mathrm{C}_*(B,A) \xrightarrow{i_*} \mathrm{C}_*(X,A) \xrightarrow{p_*} \mathrm{C}_*(X,B) \to 0$$

is a short exact sequence of chain complexes.

Claim. The maps ∂_n introduced in Statement (1) of the propositions are precisely the connecting homomorphisms, as defined on page 1620, of the above short exact sequence of chain complexes. In particular the maps ∂_n are well-defined.

Proof. The claim follows basically immediately from the definitions. But for completeness' sake we give some details. We consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{C}_n(B,A) & \longrightarrow & \mathrm{C}_n(X,A) & \longrightarrow & \mathrm{C}_n(X,B) & \longrightarrow & 0 \\ & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \\ 0 & \longrightarrow & \mathrm{C}_{n-1}(B,A) & \longrightarrow & \mathrm{C}_{n-1}(X,A) & \longrightarrow & \mathrm{C}_{n-1}(X,B) & \longrightarrow & 0. \end{array}$$

Let $z \in \mathrm{H}_n(X,A)$. Now we will determine the image of z under the connecting homomorphism $\mathrm{H}_n(X,B) \to \mathrm{H}_{n-1}(B,A)$ which during the proof of the claim we denote by ∂^c.

We pick a representative $\sum_{j=1}^{k} a_j \cdot (\sigma_j \colon \Delta^n \to X)$ in $\mathrm{C}_n(X)$. Note that $\sum_{j=1}^{k} a_j \cdot (\sigma_j \colon \Delta^n \to X)$ also defines an element in $\mathrm{C}_n(X,A)$. We then take the boundary of the above singular n-chain under the boundary map $\partial_n \colon \mathrm{C}_n(X,A) \to \mathrm{C}_{n-1}(X,A)$ and we end up with the singular n-chain $\sum_{j=1}^{k} a_j \cdot \partial\sigma_j \in \mathrm{C}_{n-1}(X,A)$. As we have seen in the proof of the Connecting Homomorphism Lemma 74.8 this singular n-chain lies in fact in $\mathrm{C}_{n-1}(B,A) \subset \mathrm{C}_{n-1}(X,A)$. The homology class thereof is by definition precisely $\partial^c(z) \in \mathrm{H}_{n-1}(B,A)$. Summarizing we have shown that

$$\partial^c(z) \;=\; \Big[\sum_{j=1}^{k} a_j \cdot \partial\sigma_j\Big].$$

This shows that the map defined in (1) agrees with ∂^c. ⊞

Statement (2) of the lemma is an easy consequence of the Connecting Homomorphism Lemma 74.8. The fact that the stated sequence of homology groups is exact is now an immediate consequence of the LES Proposition 74.9. ■

For convenience we record two important special cases of the Topological LES-Proposition 74.13 in the following corollary.

Corollary 74.14. (Topological LES-Corollary)

(1) Let X be a topological space and let $B \subset X$ be a subset. The connecting homomorphism of the Topological LES-Proposition 74.13 applied to $A = \varnothing$ gives rise to along exact sequence

$$\ldots \longrightarrow \mathrm{H}_n(B) \xrightarrow{i_*} \mathrm{H}_n(X) \xrightarrow{p_*} \mathrm{H}_n(X,B) \xrightarrow{\partial} \mathrm{H}_{n-1}(B) \longrightarrow \ldots$$

[894] The third isomorphism theorem in group theory implies that we have a natural isomorphism $(\mathrm{C}_*(X)/\mathrm{C}_*(A))/(\mathrm{C}_*(B)/\mathrm{C}_*(A)) \cong \mathrm{C}_*(X)/\mathrm{C}_*(B)$.

Furthermore, for $n = 1$ the connecting homomorphism $\partial\colon \mathrm{H}_1(X, B) \to \mathrm{H}_0(B)$ takes values in $\widetilde{\mathrm{H}}_0(B)$ and we obtain a long exact sequence[895]

$$\ldots \longrightarrow \widetilde{\mathrm{H}}_n(B) \xrightarrow{i_*} \widetilde{\mathrm{H}}_n(X) \xrightarrow{\text{ж}_{X,B*}} \mathrm{H}_n(X, B) \xrightarrow{\partial} \widetilde{\mathrm{H}}_{n-1}(B) \longrightarrow \ldots$$

(2) The connecting homomorphisms in (1) are natural. In particular, given a map $f\colon (X, A) \to (Y, B)$ between pairs of topological spaces we obtain a commutative diagram

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & \mathrm{H}_n(A) & \xrightarrow{i_*} & \mathrm{H}_n(X) & \xrightarrow{\text{я}} & \mathrm{H}_n(X, A) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(A) \longrightarrow \ldots \\
 & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\ldots & \longrightarrow & \mathrm{H}_n(B) & \xrightarrow{i_*} & \mathrm{H}_n(Y) & \xrightarrow{\text{я}} & \mathrm{H}_n(Y, B) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(B) \longrightarrow \ldots
\end{array}$$

The analogous statement also holds for reduced homology groups.

The long exact sequences of the Topological LES-Corollary 74.14 (1) are sometimes called the *long exact sequence* (*in reduced homology*) *of the pair* (X, B).

Proof. First note that the statement regarding unreduced homology follows from the Topological LES-Proposition 74.13 applied to $A = \varnothing$. Now we turn to the statement regarding reduced homology. This statement follows basically immediately from the unreduced case together with the Reduced Homology Lemma 74.1 (4) and Lemma 74.2. We leave it to the reader to fill in the dull details for the reduced case. ■

Remark. Let (X, B, A) be a triple of topological spaces. Then $(X, B, \varnothing) \to (X, B, A)$ is a map between triples of topological spaces and by the naturality of the long exact sequence we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & \mathrm{H}_n(B) & \longrightarrow & \mathrm{H}_n(X) & \longrightarrow & \mathrm{H}_n(X, B) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(B) \longrightarrow \ldots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \downarrow \\
\ldots & \longrightarrow & \mathrm{H}_n(B, A) & \longrightarrow & \mathrm{H}_n(X, A) & \longrightarrow & \mathrm{H}_n(X, B) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(B, A) \longrightarrow \ldots.
\end{array}$$

Example. Let X be a topological space. Furthermore let $x_0 \in X$ be a point. We denote by $i\colon \{x_0\} \to X$ the inclusion map. By the Topological LES-Corollary 74.14 we have a long exact sequence

$$\to \underbrace{\mathrm{H}_1(\{x_0\})}_{=0} \xrightarrow{i_*} \mathrm{H}_1(X) \to \mathrm{H}_1(X, \{x_0\}) \xrightarrow{i_*} \mathrm{H}_0(\{x_0\}) \to \mathrm{H}_0(X) \to \mathrm{H}_0(X, \{x_0\}) \to 0.$$

We obtain from this long exact sequence that the map $i_*\colon \mathrm{H}_0(\{x_0\}) \to \mathrm{H}_0(X)$ is a monomorphism. Together with the Single Point-Homology Lemma 72.6 we see that for each $n \in \mathbb{N}$ the map $\mathrm{H}_n(X) \to \mathrm{H}_n(X, \{x_0\})$ is an isomorphism.

We move on to a more elaborate example.

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[895] Note that if B is non-empty, then it follows from the Reduced Homology Lemma 74.1 (0) that $\widetilde{\mathrm{H}}_{-1}(B) = 0$. Thus the last possibly non-zero term is $\mathrm{H}_0(X, B)$. On the other hand, if $B = \varnothing$, then we have $\widetilde{\mathrm{H}}_{-1}(B) = \widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$, and if $X = \varnothing$, then we have $\widetilde{\mathrm{H}}_{-1}(X) = \widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$. If $B = \varnothing$ the sequence is understood to continue with $\mathrm{H}_0(B, \varnothing) \xrightarrow{\epsilon} \mathrm{H}_{-1}(\varnothing) \to \mathrm{H}_{-1}(X)$.

Example. We let $X = \mathbb{R}^2$ and $A = B_1^2(-2) \cup B_1^2(2)$. We want to determine the relative homology groups of (X, A). We denote by $i\colon A \to X$ the inclusion map. By the Topological LES-Corollary 74.14 we have a long exact sequence

$$\mathrm{H}_2(X) \to \mathrm{H}_2(X, A) \to \mathrm{H}_1(A) \xrightarrow{i_*} \mathrm{H}_1(X) \to \mathrm{H}_1(X, A) \to \mathrm{H}_0(A) \to \mathrm{H}_0(X) \to \mathrm{H}_0(X, A) \to 0.$$

By the Homotopy Equivalence-H_*-Corollary 73.9 (3) together with Lemma 72.14 we have $\mathrm{H}_i(X) = \mathrm{H}_i(\mathbb{R}^2) = 0$ and also $\mathrm{H}_i(A) = \mathrm{H}_i(B_1(-2)) \oplus \mathrm{H}_i(B_1(2)) = 0$ for $i \geq 1$. Thus we see from the above long exact sequence that $\mathrm{H}_i(X, A) = 0$ for $i \geq 2$. We denote by $\epsilon\colon \mathrm{H}_0(\mathbb{R}^2) \to \mathbb{Z}$ the augmentation map that is an isomorphism by the H_0-Proposition 72.5. Similarly we denote by $\mu_1\colon \mathrm{H}_0(B_1(-2)) \to \mathbb{Z}$ and $\mu_2\colon \mathrm{H}_0(B_1(2)) \to \mathbb{Z}$ the isomorphisms that are given by the augmentation maps. We obtain the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}_1(X, A) & \xrightarrow{\partial} & \mathrm{H}_0(A) & \xrightarrow{i_*} & \mathrm{H}_0(X) & \longrightarrow & \mathrm{H}_0(X, A) \longrightarrow 0 \\
 & & \Big\downarrow{\scriptstyle =} & & \Big\downarrow{\scriptstyle \mu_1 \oplus \mu_2}^{\cong} & & \Big\downarrow{\scriptstyle \cong}^{\epsilon} & & \Big\downarrow{\scriptstyle =} \\
0 & \longrightarrow & \mathrm{H}_1(X, A) & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathrm{H}_0(X, A) \longrightarrow 0.
\end{array}$$

We want to determine the matrix representing the homomorphism $\mathbb{Z}^2 \to \mathbb{Z}$ at the bottom. We do so in the following claim.

Claim. The homomorphism $\epsilon \circ i_* \circ (\mu_1 \oplus \mu_2)^{-1}\colon \mathbb{Z}^2 \to \mathbb{Z}$ is given by the matrix $(1\ 1)$.

Proof. We pick a point $P \in B_1(-2)$ and we pick a point $Q \in B_1(2)$. It follows from the H_0-Proposition 72.5 and Lemma 72.14 that the map

$$\begin{array}{rcl}
\Psi\colon \mathbb{Z}^2 & \to & \mathrm{H}_0(A) = \mathrm{H}_0(B_1(-2)) \oplus \mathrm{H}_0(B_1(2)) \\
(m, n) & \mapsto & [m \cdot P + n \cdot Q]
\end{array}$$

is the inverse to $\mu_1 \oplus \mu_2$. Now it is easy to verify that $\epsilon \circ i_* \circ (\mu_1 \oplus \mu_2)^{-1} = \epsilon \circ i_* \circ \Psi$ applied to $(1, 0)$ and $(0, 1)$ gives in both cases the value 1. So the map is described by the matrix $(1\ 1)$. ⊞

Since the top sequence in the above commutative diagram is exact and since the vertical maps are isomorphisms we see that the bottom sequence is also exact. It is now straightforward to see that

$$\mathrm{H}_1(X, A) \ \cong\ \ker\left(\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}\right) \ \cong\ \mathbb{Z} \quad \text{and} \quad \mathrm{H}_0(X, A) \ \cong\ \operatorname{coker}\left(\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}\right) \ \cong\ 0.$$

Now we return to the situation of the example on page 1625. Let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex with $\sigma(0, 1) \in B_1(2)$ and $\sigma(1, 0) \in B_1(-2)$. We have

$$((\mu_1 \oplus \mu_2) \circ \partial)[\sigma] \underset{\substack{\uparrow \\ \text{description of } \partial \text{ in the Topological LES-Proposition 74.13}}}{=} ((\mu_1 \oplus \mu_2)([\partial\sigma]) = (\mu_1 \oplus \mu_2)([\underbrace{\sigma(0, 1)}_{\in B_1(2))}] - [\underbrace{\sigma(1, 0)}_{\in B_1(-2))}]) = (1, -1).$$

Since $(1, -1)$ is a generator of the kernel of $\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}$ we see that $[\sigma]$ is in fact a generator of $\mathrm{H}_1(X, A) \cong \mathbb{Z}$. In particular it is non-zero.

Recall that, as defined on page 439, we say that two maps $f, g\colon (X, A) \to (Y, B)$ between pairs of topological spaces are homotopic, if there exists a map $F\colon X \times [0, 1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$, and $F(a, t) \in B$ for all $a \in A$

and all $t \in [0,1]$. The following proposition extends the Homotopic Maps-and-Homology Proposition 73.6 to the relative case.

Proposition 74.15. Let $f, g\colon (X,A) \to (Y,B)$ be two maps between pairs of topological spaces. If f and g are homotopic, then the maps $f_*, g_*\colon \mathrm{C}_*(X,A) \to \mathrm{C}_*(Y,B)$ are chain homotopic, in particular for every $n \in \mathbb{N}_0$ we have

$$f_* = g_*\colon \mathrm{H}_n(X,A) \to \mathrm{H}_n(Y,B).$$

Proof. Let $f, g\colon (X,A) \to (Y,B)$ be two homotopic maps between pairs of topological spaces. To prove the proposition we have to look again at the proof of the Homotopic Maps-and-Homology Proposition 73.6. In that proof we explicitly constructed a chain $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(Y)$, $n \in \mathbb{N}_0$ between the maps f_* and g_* from $\mathrm{C}_*(X)$ to $\mathrm{C}_*(Y)$. It follows immediately from the definition of the chain homotopy that it has the property that $\mathrm{C}_n(A)$ gets sent to $\mathrm{C}_{n+1}(B)$. In particular the chain homotopy descends to maps $Q_n\colon \mathrm{C}_n(X,A) \to \mathrm{C}_{n+1}(Y,B)$ which then form a chain homotopy between the maps f_* and g_* from $\mathrm{C}_*(X,A)$ to $\mathrm{C}_*(Y,B)$. It now follows immediately from Lemma 73.2 that $f_* = g_*\colon \mathrm{H}_n(X,A) \to \mathrm{H}_n(Y,B)$. ∎

We conclude this section with the following result that can be viewed as a generalization of the Homotopy Equivalence-H_*-Corollary 73.9 to the relative setting.

Corollary 74.16.

(1) If $f\colon (X,A) \to (Y,B)$ is a homotopy equivalence of pairs of topological spaces,[896] then for every $n \in \mathbb{N}_0$ the induced map $f_*\colon \mathrm{H}_n(X,A) \to \mathrm{H}_n(Y,B)$ is an isomorphism.

(2) Let X be a topological space.
   (a) Let $A \subset B$ be subsets of X. If A is a deformation retract of B, then for every $n \in \mathbb{N}_0$ the inclusion induced map $\mathrm{H}_n(X,A) \to \mathrm{H}_n(X,B)$ is an isomorphism.
   (b) If A is a deformation retract of X, then $\mathrm{H}_n(X,A) = 0$ for all $n \in \mathbb{N}_0$.
   (c) Let $A \subset Y$ be subsets of X. If Y is a deformation retract of X, then for every $n \in \mathbb{N}_0$ the inclusion induced map $\mathrm{H}_n(Y,A) \to \mathrm{H}_n(X,A)$ is an isomorphism.

Proof. Let X be a topological space.

(1) We deduce this statement from Proposition 74.15 the same way we deduced the Homotopy Equivalence-H_*-Corollary 73.9 from the Homotopic Maps-and-Homology Proposition 73.6.

(2) (a) Let $A \subset B$ be a deformation retract of B. As we mentioned on page 440, the inclusion map $(X,A) \to (X,B)$ is a homotopy equivalence of pairs of topological spaces. Thus the desired statement follows from (1).
   (b) This statement follows from Statement (a) applied to $B = X$ and the trivial observation that $\mathrm{H}_n(X,X) = 0$ for all $n \in \mathbb{N}_0$.
   (c) Let $A \subset Y$ be subsets of X. If Y is a deformation retract of X, then we saw already on page 440 that the inclusion map $(Y,A) \to (X,A)$ is a homotopy equivalence of pairs of topological spaces. Thus the desired statement follows again from (1). ∎

The statement of the Topological LES-Corollary 74.14 looks already very similar to the statement of Theorem 74.3 that we actually want to prove. Now we "only" have to find

[896] We refer to page 440 for the definition.

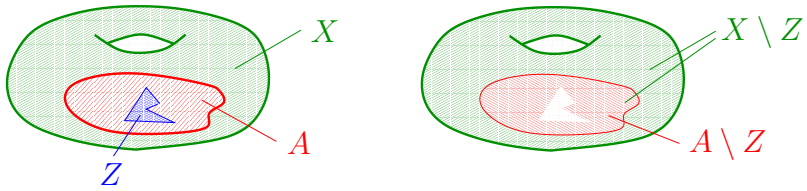
the connection between the groups $\mathrm{H}_n(X, A)$ and $\mathrm{H}_n(X/A)$. The notations already look very similar, and they do so for good reasons, since in both cases "one ignores what lies in A". Nonetheless, proving the precise relationship will still require some substantial piece of work.

74.7. **The Excision Theorem.** The key step in finding the connection between the groups $\mathrm{H}_n(X, A)$ and $\mathrm{H}_n(X/A)$ is given by the following Excision Theorem 74.17.

Theorem 74.17. (Excision Theorem) Let X be a topological space and let $Z \subset A \subset X$ be subsets such that the closure of Z is contained in the interior of A. Then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} \mathrm{H}_n(X, A).$$

The theorem thus says in particular that if we are given a pair of topological spaces (X, A) and if we excise [897] a subset Z from the interior of A, then this does not affect the relative homology groups.



$$\mathrm{H}_n(X, A) \cong \mathrm{H}_n(X \setminus Z, A \setminus Z)$$

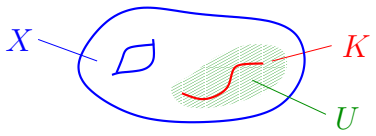
On many occasions we will need the following variation on the Excision Theorem 74.17.

Theorem 74.18. (Excision Theorem) Let X be a topological space.

(1) Let K be a subset of X and let U be a neighborhood of the closure of K. Then the inclusion $(U, U \setminus K) \to (X, X \setminus K)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(U, U \setminus K) \xrightarrow{\cong} \mathrm{H}_n(X, X \setminus K).$$

(2) The hypotheses of (1) are satisfied if X is Hausdorff (e.g. if X is a topological manifold) and if K is a compact subset that is contained in an open set U.



here we have $\mathrm{H}_n(U, U \setminus K) \xrightarrow{\cong} \mathrm{H}_n(X, X \setminus K)$

Proof of Theorem 74.18 assuming Theorem 74.17.

(1) We have

$$\mathrm{H}_n(U, U \setminus K) \underset{\uparrow}{=} \mathrm{H}_n(X \setminus Z, A \setminus Z) \underset{\uparrow}{\xrightarrow{\cong}} \mathrm{H}_n(X, A) = \mathrm{H}_n(X, X \setminus K).$$

set $A := X \setminus K$ and $Z := X \setminus U$

since U is a neighborhood of the closure of K we see that the closure of Z is contained in the interior of A, thus we can apply the Excision Theorem 74.17

[897] The verb "to excise" means to cut out, or in German "ausschneiden".

(2) If X is Hausdorff and if K is a compact subset that is contained in an open set U, then it follows from the Compact-Closed Lemma 1.21 (2) that K is also a closed subset, hence we have $\overline{K} \subset U$. ■

Before we turn to the proof of the Excision Theorem 74.17 we first want to show that we can now relate the relative homology groups of a pair (X, A) to the homology groups of the quotient space X/A, assuming $A \subset X$ is good. To do so we need the following definition.

Definition. Let X be a topological space and let $A \subset X$ be a[898] subset. We denote by $p\colon (X, A) \to (X/A, A/A)$ the natural projection. For each $n \in \mathbb{N}_0$ we define

$$\mathrm{H}_n(X, A) \xrightarrow{p_*} \mathrm{H}_n(X/A, A/A) \xleftarrow[\cong]{\text{ж from Lemma 74.12}} \widetilde{\mathrm{H}}_n(X/A)\ .$$
$$\text{=: ю}$$

The following lemma summarizes two properties of the maps ю.

Lemma 74.19.

(1) Let X be a topological space and let $A \subset X$ be a subset. We denote by $p\colon X \to X/A$ the projection. The following diagram commutes:[899]

$$\begin{array}{ccc} & \mathrm{H}_k(X, A) & \\ {\scriptstyle \text{ж}}\nearrow & & \searrow{\scriptstyle \text{ю}} \\ \widetilde{\mathrm{H}}_k(X) & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_k(X/A). \end{array}$$

(2) We denote by *PairTop* the category of pairs of topological spaces that we introduced on page 439. Furthermore as usual we denote by *AbGroup* the category of abelian groups. The above maps ю define a natural transformation from the functor

$$\begin{array}{ccc} \mathit{PairTop} & \to & \mathit{AbGroup} \\ (X, A) & \mapsto & \mathrm{H}_n(X, A) \end{array} \quad \text{to the functor} \quad \begin{array}{ccc} \mathit{PairTop} & \to & \mathit{AbGroup} \\ (X, A) & \mapsto & \widetilde{\mathrm{H}}_n(X/A). \end{array}$$

Proof.

(1) We consider the following diagram

$$\begin{array}{ccccc} & & \mathrm{H}_k(X, A) & & \\ & {\scriptstyle \text{ж}}\nearrow & \downarrow{\scriptstyle p_*} & \searrow{\scriptstyle \text{ю}} & \\ \widetilde{\mathrm{H}}_k(X) & & \mathrm{H}_k(X/A, A/A) & \xleftarrow[\cong]{\text{Lemma 74.12}} & \widetilde{\mathrm{H}}_k(X/A). \\ & {\scriptstyle [\sigma]\mapsto[\sigma]}\searrow & & \nearrow & \\ & & {\scriptstyle p_*} & & \end{array}$$

It follows immediately from the definitions that all three regions of the diagram commute. This observation gives us the desired statement.

(2) This statement follows basically immediately from the definitions. ■

The following proposition is much more interesting than the previous lemma.

[898] Note that the definition also makes sense if A is the empty set.

[899] The left diagonal map is the composition of the natural map $\widetilde{\mathrm{H}}_k(X) \to \mathrm{H}_k(X)$ from the Reduced Homology Lemma 74.1 with the natural map $\mathrm{H}_k(X) \to \mathrm{H}_k(X, A)$.

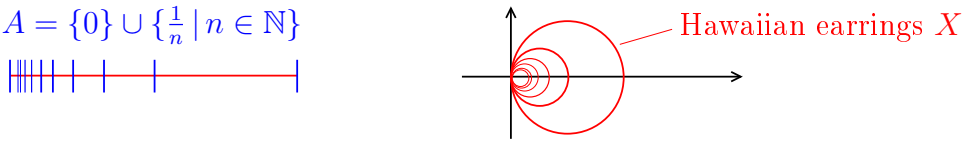
Proposition 74.20. (Homology-of-Quotients Proposition) Let X be a topological space, let $A \subset X$ be a subset and let $n \in \mathbb{N}_0$. If A is a good subset[900], then the natural homomorphism $\mathrm{H}_n(X,A) \xrightarrow{\text{IO}} \widetilde{\mathrm{H}}_n(X/A)$ is an isomorphism.

Remark. In Corollary 78.16 we will show the analogue of the Homology-of-Quotients Proposition 74.20 under the slightly different hypothesis that the inclusion map $i\colon A \to X$ is a closed cofibration.

Example. Let X be a topological space, let $A \subset X$ be a subset and let $n \in \mathbb{N}_0$. For better or worse the natural homomorphism $\mathrm{H}_n(X,A) \to \widetilde{\mathrm{H}}_n(X/A)$ is in general not an isomorphism. In fact in general the two groups $\mathrm{H}_n(X,A)$ and $\widetilde{\mathrm{H}}_n(X/A)$ are not even abstractly isomorphic.

For example consider $X = [0,1]$ and $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$. In Exercise 56.1 we showed that the quotient space X/A is homeomorphic to the Hawaiian earrings. (See the figure below for an illustration.) In Exercise 74.18 we will prove the following two statements:

(1) The group $\mathrm{H}_1(X,A)$ is countable.
(2) The group $\mathrm{H}_1(X/A)$ is uncountable.



We will first provide a proof of the Homology-of-Quotients Proposition 74.20 assuming the Excision Theorem 74.17. This proof relies on the following elementary little lemma, which is precisely the content of Exercise 5.46.

Auxiliary Lemma 74.21. Let X be a topological space and let A be a closed subset of X.

(1) For every open subset $U \subset X \setminus A$ the obvious map $U \to X/A$ is an open embedding.
(2) For every open neighborhood W of A the obvious map $W \setminus A \to (W/A) \setminus (A/A)$ is a homeomorphism.



Proof of Lemma 74.21. Let X be a topological space and let A be a closed subset of X. We denote by $p\colon X \to X/A$ the natural projection.

(1) Let $U \subset X \setminus A$ be an open subset. Recall that we need to show that the map $q\colon U \to X \to X/A$ is an open embedding. It follows from $U \subset X \setminus A$ that q is an injection. Furthermore note that by the Topological-Quotient Proposition 5.15 (1a) we know that q is continuous. By the Open -Injective Map Lemma 2.16 (2) it remains to show that $q\colon U \to X/A$ is an open map. Now let $V \subset U$ be an open

[900] We refer to page 1614 for several examples of good subsets. For example we saw that if X is a CW-complex, then any subcomplex A is a good subset.

subset. We need to show that $q(V) \subset X/A$ is open. We break the argument into three easy steps:

(a) It follows from the Open -Closed Inclusion Lemma 2.10 (1) and the fact that A is by hypothesis closed, that V is also an open subset of X.

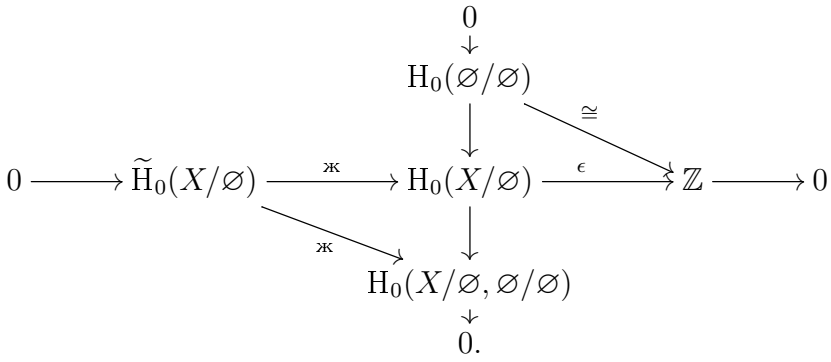
(b) Since $A \cap V = \varnothing$ we see that $p^{-1}(p(V)) = V$, which by (a) is in fact an open subset of X.

(c) It follows from (b) and the definition of the quotient topology that $p(V)$ is an open subset of X/A.

(2) Let W be an open neighborhood of A. We set $U := W \setminus A$. Since A is closed we see that U is an open subset of $W \setminus A$. Recall that we want to show that the obvious map $q\colon W \setminus A \to (W/A) \setminus (A/A)$ is a homeomorphism. It is clear that q is a bijection and it follows from the Twice Quotient Lemma 5.23 (2) that q is continuous. By the Open -Injective Map Lemma 2.16 (2) it remains to show that $q\colon W \setminus A \to (W/A) \setminus (A/A)$ is an open map. Thus let V be an open subset of $W \setminus A$. We have $q(V) = p(V) \cap (W/A \setminus A/A)$. It follows from (1c) and the definition of the subspace topology that $q(V)$ is an open subset of $(W/A) \setminus (A/A)$. ■

Proof of Proposition 74.20 assuming the Excision Theorem 74.17. Let X be a topological space, let $A \subset X$ be a good subset and let $n \in \mathbb{N}_0$.

First we need to get the annoying case that $A = \varnothing$ out of the way. Note that the map $(X, \varnothing) \to (X/\varnothing, \varnothing/\varnothing)$ is easily seen to be a homeomorphism of pairs of topological spaces. Thus it remains to show that the map $\text{ж}\colon \widetilde{\mathrm{H}}_n(X/\varnothing) \to \mathrm{H}_n(X, \varnothing)$ is an isomorphism. It follows immediately from the Reduced Homology Lemma 74.1 (4) and the Topological LES-Corollary 74.14 that the only dimension one needs to worry about for a second is $n = 0$. In this case we have the following diagram:



It follows from Lemma 74.2 and the Topological LES-Corollary 74.14 that the horizontal and the vertical sequences are both exact. The diagonal map on the top right is easily seen to be an isomorphism. Some mild diagram chasing, see Exercise 74.10, now shows that the diagonal map on bottom the is also an isomorphism.

Now we turn to the much more interesting case that $A \neq \varnothing$. We make the following simple preparations:

(1) Since A is good we can pick an open neighborhood U of A in X such that A is a deformation retract of U.

(2) We denote by $p\colon X \to X/A$ the projection .

(3) We denote by $q\colon X \setminus A \to (X/A) \setminus (A/A)$ the restriction of p to $X \setminus A$. For completeness' sake we point out that it follows from Lemma 2.1 that q is continuous.

Now we consider the following diagram

$$\begin{array}{ccccc} \mathrm{H}_n(X,A) & \longrightarrow & \mathrm{H}_n(X,U) & \longleftarrow & \mathrm{H}_n(X \setminus A, U \setminus A) \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle q_*} \\ \mathrm{H}_n(X/A, A/A) & \longrightarrow & \mathrm{H}_n(X/A, U/A) & \longleftarrow & \mathrm{H}_n((X/A) \setminus (A/A), (U/A) \setminus (A/A)), \end{array}$$

here all the horizontal maps are induced by inclusions of pairs of topological spaces. It follows easily from the definitions that the diagram commutes. It follows from our hypothesis that A is closed, together with the above auxiliary Lemma 74.21 (2), that the map $q\colon (X \setminus A, U \setminus A) \to ((X/A) \setminus (A/A), (U/A) \setminus (A/A))$ is a homeomorphism of pairs of topological spaces. We obtain from this observation that the vertical map on the right is an isomorphism. Our actual goal is to show that the vertical map on the left is an isomorphism. This follows from the commutativity of the above diagram and from the following claim:

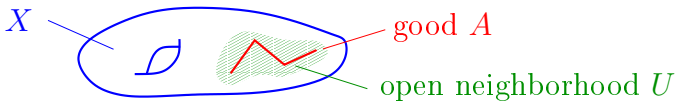
Claim. All horizontal maps are isomorphisms.

Proof. We distinguish three different situations:

(1) It follows from the Excision Theorem 74.17 that both the top horizontal and the bottom horizontal map on the right-hand side are isomorphisms.

(2) Since A is deformation retract of U we obtain from Corollary 74.16 that the top left horizontal map is an isomorphism.

(3) Recall that A is a deformation retract of U. It follows from the Quotient Deformation Retract Proposition 15.9 that A/A is a deformation retract of U/A.[901] As in (2) we now see that the bottom-left horizontal map is an isomorphism. ∎



Proof of Theorem 74.3 assuming the Excision Theorem 74.17. Let X be a topological space and let $A \subset X$ be a good subset. We denote by $i\colon A \to X$ the inclusion and we denote by $q\colon X \to X/A$ the projection . We consider the following diagram:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_n(X) & \xrightarrow{\text{ж}_{X,A*}} & \mathrm{H}_n(X,A) & \xrightarrow{\partial} & \widetilde{\mathrm{H}}_{n-1}(A) & \longrightarrow & \widetilde{\mathrm{H}}_{n-1}(X) \xrightarrow{i_*} \ldots \\ & & & & & \searrow^{q_*} & \downarrow{\scriptstyle \text{ю}} & & & & \\ & & & & & & \widetilde{\mathrm{H}}_n(X/A). & & & & \end{array}$$

We make the following comments and observations:

(1) The horizontal sequence is the long exact sequence coming from the Topological LES-Corollary 74.14.

[901] The diligent reader who checks the statements we refer to will not fail to notice, that the Quotient Deformation Retract Proposition 15.9 rests eventually on the Whitehead Quotient Theorem 13.4. It is all too often overlooked (see see e.g. earlier version of these notes) that the Quotient Deformation Retract Proposition 15.9 is required and that it is non-trivial.

(2) The vertical map is the natural homomorphism that we defined on page 1633. By Lemma 74.19 (1) we know that the triangle commutes.
(3) Since A is good we obtain from the Homology-of-Quotients Proposition 74.20 that the vertical map ю is an isomorphism.

Now we consider the maps $\widetilde{\partial} := \text{ю}^{-1} \circ \partial \colon \widetilde{\mathrm{H}}_n(X/A) \to \widetilde{\mathrm{H}}_{n-1}(A)$. Since ю and ∂ are natural we see that the maps $\text{ю}^{-1} \circ \partial$ are also natural. By the above the sequence

$$\ldots \longrightarrow \widetilde{\mathrm{H}}_n(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_n(X) \xrightarrow{q_*} \widetilde{\mathrm{H}}_n(X/A) \xrightarrow{\widetilde{\partial}} \widetilde{\mathrm{H}}_{n-1}(A) \longrightarrow \widetilde{\mathrm{H}}_{n-1}(X) \xrightarrow{i_*} \ldots$$

is exact. ■

74.8. **The proof of the Excision Theorem 74.17: the idea.** We are still not done with the proof of Theorem 74.3 since we still have to prove the Excision Theorem 74.17. In this section we want to come up with the idea for the proof of the Excision Theorem 74.17. The full details of the proof will then be given in the following section.
Thus let X be a topological space and let $Z \subset A \subset X$ be two subsets such that the closure of Z is contained in the interior of A. Our goal is to show that the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for every $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \;\to\; \mathrm{H}_n(X, A).$$

Let us attempt to prove that the map $\mathrm{H}_n(X \setminus Z, A \setminus Z) \to \mathrm{H}_n(X, A)$ is surjective. We take an element in $\mathrm{H}_n(X, A)$ and we choose a representative $a_1\sigma_1 + \cdots + a_r\sigma_r$ where $\sigma_i \colon \Delta^n \to X$, $i = 1, \ldots, r$ are singular n-simplices in X.
We make the following two basic observations:

(1) If the singular n-simplices $\sigma_1, \ldots, \sigma_r$ lie in $X \setminus Z$,[902] then $[a_1\sigma_1 + \cdots + a_r\sigma_r]$ evidently lies in the image of the inclusion induced map $\mathrm{H}_n(X \setminus Z, A \setminus Z) \to \mathrm{H}_n(X, A)$.
(2) If a singular n-simplex lies entirely in A, then it represents the zero element in the quotient group $\mathrm{C}_n(X, A) = \mathrm{C}_n(X)/\mathrm{C}_n(A)$.

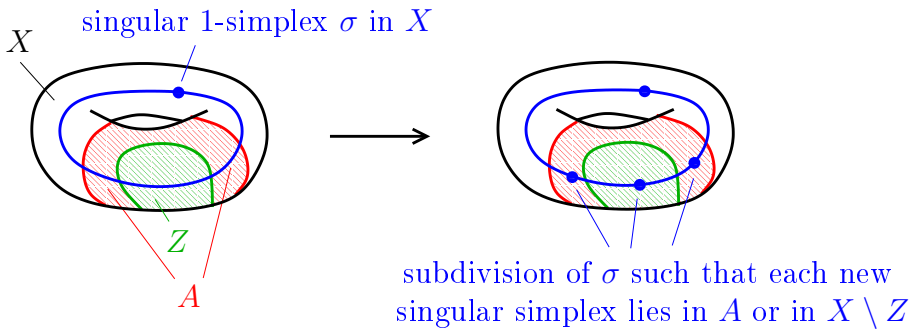
The combination of these two observations proves the following lemma.

Lemma 74.22. Let X be a topological space and let $Z \subset A$ be two subsets of X. Let $c = a_1\sigma_1 + \cdots + a_r\sigma_r$ be a cycle in X such that each σ_i lies entirely in A or it lies entirely in $X \setminus Z$. Then $[c]$ lies in the image of $\mathrm{H}_n(X \setminus Z, A \setminus Z) \to \mathrm{H}_n(X, A)$.

Unfortunately not every cycle in $\mathrm{C}_n(X)$ satisfies the hypothesis of the lemma. But we will see in the next section that we can subdivide each singular simplex into smaller singular simplices that lie entirely in A or lie entirely in $X \setminus Z$.

74.9. **The proof of the Excision Theorem 74.17: the full details.** In this chapter we will now turn the vague idea of the last section into a rigorous proof of the Excision Theorem 74.17. This will take a surprisingly long time.
First we introduce a few definitions.

[902] Given a topological space X and a subset A we say that *a singular simplex* $\sigma \colon \Delta^n \to X$ *lies in* A if the image of σ lies in A.



Definition. Let again X be a topological space.

(1) On page 114 we defined a cover of X to be a family $\{U_i\}_{i\in I}$ of subsets of X such that $X = \bigcup_{i\in I} U_i$.

(2) We say a cover $\{U_i\}_{i\in I}$ is comfortable if $X = \bigcup_{i\in I} \overset{\circ}{U}_i$, i.e. if the interiors of the U_i also form a cover of X.

We move straight to the next definition.

Definition. Let again X be a topological space. Given a comfortable cover $\mathcal{U} = \{U_i\}_{i\in I}$ we consider the following subgroup of $\mathrm{C}_n(X)$:

$$\mathrm{C}_n^{\mathcal{U}}(X) := \left\{ \sum_{j=1}^{k} a_j \cdot \sigma_j \;\middle|\; \text{for each } j \text{ there exists a } U_i \in \mathcal{U} \text{ such that the image of } \sigma_j \text{ lies in } U_i \right\}.$$

It follows quite easily from the definitions that the restriction of the usual boundary map $\partial_n \colon \mathrm{C}_n(X) \xrightarrow{\partial} \mathrm{C}_{n-1}(X)$ to $\mathrm{C}_n^{\mathcal{U}}(X)$ restricts to a map $\partial_n \colon \mathrm{C}_n^{\mathcal{U}}(X) \xrightarrow{\partial} \mathrm{C}_{n-1}^{\mathcal{U}}(X)$. Since this map is the restriction of the usual boundary map it follows in particular that the composition $\partial_n \circ \partial_{n+1} \colon \mathrm{C}_{n+1}^{\mathcal{U}}(X) \xrightarrow{\partial} \mathrm{C}_{n-1}^{\mathcal{U}}(X)$ is the zero map. Thus we can define

$$\begin{aligned} \mathrm{H}_n^{\mathcal{U}}(X) &:= \mathrm{H}_n\big(C_*^{\mathcal{U}}(X), \partial_*\big) \\ &= \ker\big(\partial_n \colon \mathrm{C}_n^{\mathcal{U}}(X) \to \mathrm{C}_{n-1}^{\mathcal{U}}(X)\big) \,/\, \mathrm{im}\,\big(\partial_{n+1} \colon \mathrm{C}_{n+1}^{\mathcal{U}}(X) \to \mathrm{C}_n^{\mathcal{U}}(X)\big). \end{aligned}$$

The homology groups $\mathrm{H}_n^{\mathcal{U}}(X)$ are thus defined by singular chains which are "small" in the sense that the image of each singular simplex lies in some U_i.

The following proposition says that we can determine the homology of a topological space through such "small" singular chains. As we will see later on many occasions, this proposition is at the heart of many methods for computing homology groups.

Proposition 74.23. (Comfortable Covering Proposition) Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a comfortable cover of X. Then the inclusion map

$$\mathrm{C}_*^{\mathcal{U}}(X) \;\to\; \mathrm{C}_*(X)$$

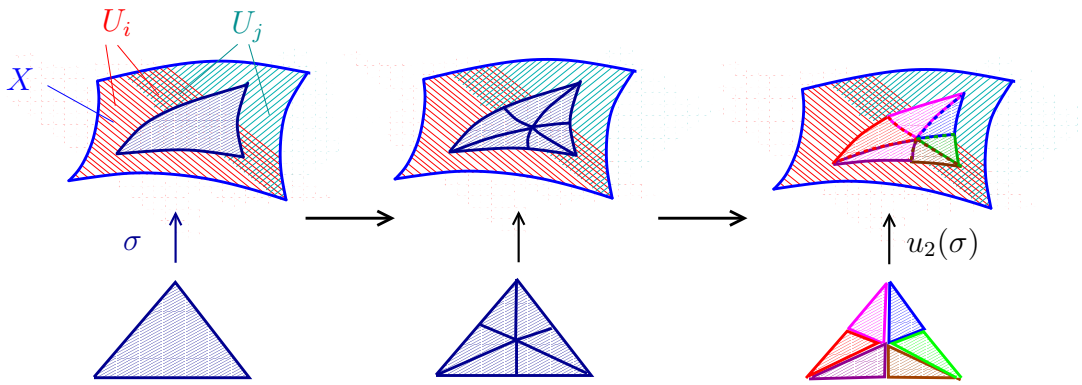
induces for every $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n^{\mathcal{U}}(X) \;\xrightarrow{\cong}\; \mathrm{H}_n(X).$$

Remark. In fact a stronger statement holds: the inclusion map $\mathrm{C}_*^{\mathcal{U}}(X) \to \mathrm{C}_*(X)$ is in fact a chain homotopy equivalence. Later, in Section 81.2, we will show that this stronger statement follows in fact from the Comfortable Covering Proposition 74.23 and a purely

algebraic argument. Alternatively one can fight one's way through the proof of the Comfortable Covering Proposition 74.23 and upgrade the proof to this stronger statement. This approach is for example pursued in [**Hat2002**, Proposition 2.21]. The disadvantage is that it makes the proof even more complicated than it already is.

The idea of the proof of the Comfortable Covering Proposition 74.23 is the one we already described in the previous section. We want to replace systematically each singular n-simplex $\sigma\colon \Delta^n \to X$ by a singular n-chain of the form $\sum_{j=1}^{r} a_j \cdot \sigma_j$ such that the image of each singular simplex σ_j lies in some $U_i \in \mathcal{U}$. As we have seen before in Lemma 72.3 in the 1-dimensional case, an appropriate subdividing procedure does not change the homology class. The systematic approach we are going to use is given by "barycentric subdivision". The key idea is sketched in the figure below for the case $n = 2$.



Before we can execute this idea we have to introduce a few more definitions.

(1) Given $n \in \mathbb{N}_0$ we denote by $B_n := (\frac{1}{n+1}, \ldots, \frac{1}{n+1}) \in \Delta^n$ the *barycenter*[903] of the standard n-simplex Δ^n.

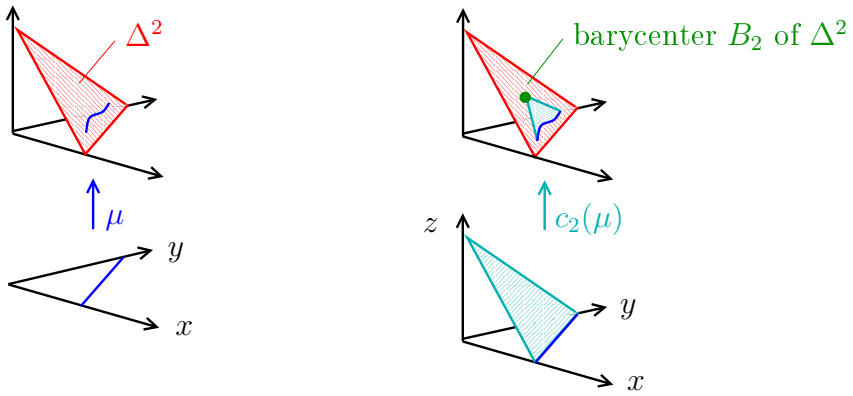
(2) Given a singular l-simplex $\mu\colon \Delta^l \to \Delta^n$ we consider the singular $(l+1)$-simplex $c_{l+1}(\mu)\colon \Delta^{l+1} \to \Delta^n$ which is defined as the composition of the (inverses) of the following two maps:

homeomorphism by the Compact-Hausdorff Proposition 2.17 (3) ; continuous by the Topological-Quotient Proposition 5.15 (1b)

$$\begin{array}{rcccl} c_{l+1}(\mu)\colon \Delta^{l+1} & \overset{\cong}{\leftarrow} & (\Delta^l \times [0,1])/\Delta^l \times \{1\} & \to & \Delta^n \\ ((t_0,\ldots,t_l)\cdot(1-s),s) & \mapsto & [((t_0,\ldots,t_l),s)] & & \\ & & [((t_0,\ldots,t_l),s)] & \mapsto & \underbrace{\mu(t_0,\ldots,t_l)\cdot(1-s)+B\cdot s}_{\text{contained in } \Delta^n \text{ since } \Delta^n \text{ is convex}} \end{array}$$

Pictorially speaking the singular $(l+1)$-simplex $c_{l+1}(\mu)$ thus corresponds to the "cone" which is given by the singular l-simplex μ and the barycenter B_n. By Lemma 51.1 this map $\mu \to c_{l+1}(\mu)$ defines a linear map $\mathrm{C}_l(\Delta^n) \to \mathrm{C}_{l+1}(\Delta^n)$ that we also denote by c_{l+1} and which we call the $(l+1)$-*st cone map*.

[903] The word "barycenter" comes from the Greek word "barys" which means heavy. In physics the barycenter of an object is the center of mass, in our case the barycenter of the standard n-simplex is the center of mass if we view Δ^k as a physical object of uniform density.



(3) For $n = 0, 1, 2, 3, \ldots$ we now inductively do the following:
   (a) we define an element $\chi_n \in \mathrm{C}_n(\Delta^n)$, and
   (b) for each topological space X we define a homomorphism $u_n \colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$.

We do this inductive procedure as follows:
   (i) We set $\chi_0 = \mathrm{id} \colon \Delta^0 \to \Delta^0$ and for each topological space X we define u_0 to be the identity on $\mathrm{C}_0(X)$.
   (ii) Now suppose $\chi_{n-1} \in \mathrm{C}_{n-1}(\Delta^{n-1})$ and that furthermore all the homomorphisms $u_{n-1} \colon \mathrm{C}_{n-1}(X) \to \mathrm{C}_{n-1}(X)$ are already defined. Then we define

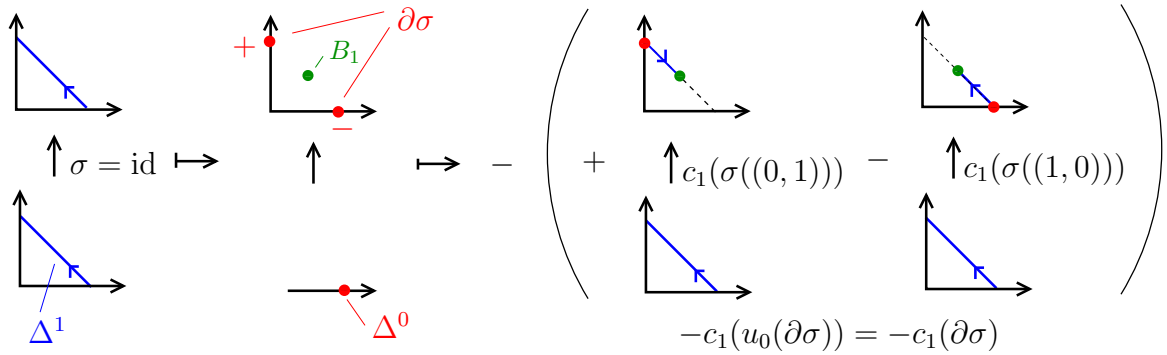
$$\chi_n \;:=\; (-1)^n \cdot c_n(u_{n-1}(\underbrace{\partial\, \mathrm{id}_{\Delta^n}}_{\in \mathrm{C}_{n-1}(\Delta^n)})) \;\in\; \mathrm{C}_n(\Delta^n).$$

Furthermore, for each topological space X we define

$$\begin{array}{rcl} u_n \colon \mathrm{C}_n(X) & \to & \mathrm{C}_n(X) \\ \sum_{l=1}^{r} a_l \cdot (\sigma_l \colon \Delta^n \to X) & \mapsto & \sum_{l=1}^{r} a_l \cdot (\sigma_l)_*(\underbrace{\chi_n}_{\in \mathrm{C}_n(\Delta^n)}). \end{array}$$

map $\sigma_l \colon \Delta^l \to X$ induces a map $(\sigma_l)_* \colon \mathrm{C}_n(\Delta^l) \to \mathrm{C}_n(X)$

Note that with this notation we have $u_n(\mathrm{id}_{\Delta^n}) = \chi_n$. The map u_n is illustrated for $n = 1$ in the figure below. For $n = 2$ we already gave a more schematic sketch in the figure below without considering signs and orientations.



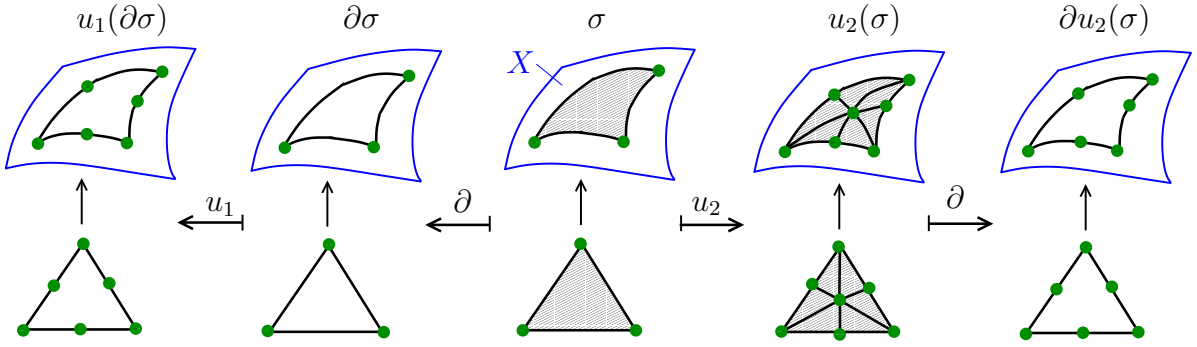
Before we can continue we need to prove several technical lemmas about the subdivision maps.

Lemma 74.24. (Chain Subdivision Lemma) Let X be a topological space. The maps $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$, $n \in \mathbb{N}_0$, form a chain map from the chain complex $(\mathrm{C}_*(X), \partial)$ to itself.

The statement of the Chain Subdivision Lemma 74.24 is equivalent to the statement that for any singular n-simplex σ in X we have

$$u_{n-1}(\partial\sigma) \;=\; \partial(u_n(\sigma)).$$

Put differently, the Chain Subdivision Lemma 74.24 says that it does not make a difference whether we first take the boundary and then subdivide, or whether we first subdivide and then take the boundary. This statement is sketched in the figure below. The picture is quite convincing.



Proof of the Chain Subdivision Lemma 74.24. [904] We adopt the following notation and we make the following remarks:

(1) We denote the barycenter of Δ^n by b.

(2) As on page 1584, given points $x_0, \ldots, x_k \in \Delta^n$ we denote by $[x_0, \ldots, x_k]$ the singular k-chain $\Delta^k \to \Delta^n$ that is given by[905]

$$\begin{aligned} \Delta^k &\to \Delta^n \\ (t_0, \ldots, t_k) &\mapsto \sum_{i=0}^{k} t_i x_i, \end{aligned}$$

we refer to such a singular k-simplex as a *linear* singular simplex.

(3) We denote by $\mathrm{C}_k^{\mathrm{lin}}(\Delta^n)$ the subgroup of $\mathrm{C}_k(\Delta^n)$ generated by all the linear singular k-simplices. Note that the boundary map $\partial\colon \mathrm{C}_k(\Delta^n) \to \mathrm{C}_{k-1}(\Delta^n)$ restricts to a boundary map $\partial\colon \mathrm{C}_k^{\mathrm{lin}}(\Delta^n) \to \mathrm{C}_{k-1}^{\mathrm{lin}}(\Delta^n)$.

(4) For $x_0, \ldots, x_k \in \Delta^n$ we write

$$[x_0, \ldots, \widehat{x_i}, \ldots, x_k] \;:=\; [x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k].$$

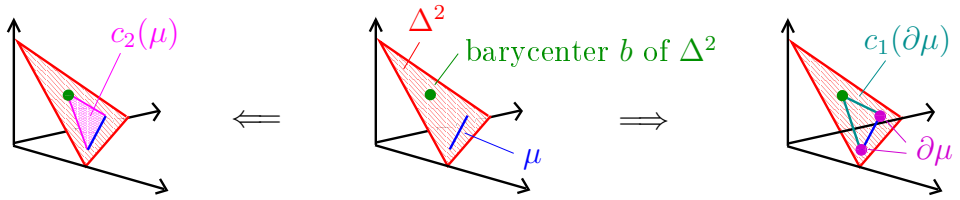
The proof of the lemma will be broken into three claims.

Claim. For any $\mu \in \mathrm{C}_k^{\mathrm{lin}}(\Delta^n)$ we have

$$\partial(c_{k+1}(\mu)) \;=\; c_k(\partial\mu) + (-1)^{k+1} \cdot \mu \;\in\; \mathrm{C}_k(\Delta^n).$$

[904] We will not do this proof in the lectures. The proof is somewhat lengthy but straightforward. The cleverness hereby does not lie in the proof, but in the cleverly chosen definitions, which turn the verification into a simple calculation.

[905] It is straightforward to show that this map does indeed take values in Δ^n.



Proof. To prove the claim we first note that for any choice of points $x_0, \dots, x_{l-1} \in \Delta^n$ we have by definition of c_l that

$$c_l([x_0, \dots, x_{l-1}]) \;=\; [x_0, \dots, x_{l-1}, b].$$

It now follows that for any $w_0, \dots, w_k \in \Delta^n$ we have the following equality in $\mathrm{C}_k(\Delta^n)$:

$$\begin{aligned}
\partial(c_{k+1}([w_0, \dots, w_k])) &\overset{\text{by the above equality}}{\downarrow} \\
&= \partial([w_0, \dots, w_k, b]) \\
&\underset{\text{definition of } \partial \text{ and by the above equality}}{\uparrow} \\
&= \sum_{j=0}^{k} (-1)^j \cdot [w_0, \dots, \widehat{w_j}, \dots, w_k, b] + (-1)^{k+1} \cdot [w_0, \dots, w_k] \\
&\overset{\downarrow}{=} c_k(\partial([w_0, \dots, w_k])) + (-1)^{k+1} \cdot [w_0, \dots, w_k].
\end{aligned}$$

The claim immediately follows from this equality.

Claim. For any $n \in \mathbb{N}_0$ and any $\nu \in \mathrm{C}_n^{\mathrm{lin}}(\Delta^n)$ we have

$$\partial u_n(\nu) \;=\; u_{n-1}(\partial\nu).$$

Proof. We will prove this claim by induction on n. The case $n = 0$ is trivial. So now suppose we have already shown the case $n - 1$. It suffices to show the claim for all linear singular n-simplices. So let $\nu\colon \Delta^n \to \Delta^n$ be a linear singular n-simplex. We then have

$$\begin{aligned}
\partial u_n(\nu) &\overset{\text{def. of } u_n \text{ and } \chi_n, \text{ see page 1640}}{\downarrow} \partial(\nu_*(\chi_n)) \overset{\downarrow}{=} (-1)^n \cdot \partial(\nu_*(c_n(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})))) \overset{\text{since } \partial\circ\nu_* = \nu_*\circ\partial}{\downarrow} (-1)^n \cdot \nu_*\big(\partial(c_n(\underbrace{u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})}_{=:\mu}))\big) \\
&\underset{\text{we can apply the previous claim since } \mu := u_{n-1}(\partial\,\mathrm{id}_{\Delta^n}) \in \mathrm{C}_{n-1}^{\mathrm{lin}}(\Delta^n)}{\uparrow} (-1)^n \cdot \nu_*\big(c_{n-1}(\partial(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})) + (-1)^n \cdot u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})\big) \\
&\underset{\uparrow}{=} (-1)^n \cdot \nu_*\big(c_{n-1}(u_{n-2}(\underbrace{\partial\partial\,\mathrm{id}_{\Delta^n}}_{=0}))\big) \;+\; \nu_*(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})) \\
&\underset{\text{by induction hypothesis we have } \partial u_{n-1}(\rho) = u_{n-2}(\partial\rho) \text{ for } \rho \in \mathrm{C}_{n-1}^{\mathrm{lin}}(\Delta^n), \text{ applied here to } \rho = \partial\,\mathrm{id}_{\Delta^n}}{\uparrow} \\
&\underset{\text{Lemma 74.25 applied to } f=\nu}{\uparrow} u_{n-1}(\nu_*(\partial\,\mathrm{id}_{\Delta^n})) \;\underset{\text{by definition}}{\uparrow}{=}\; u_{n-1}(\partial\nu).
\end{aligned}$$

⊞

The following claim is just a reformulation of the statement of the lemma.

Claim. For any singular n-simplex σ in X we have $u_{n-1}(\partial\sigma) = \partial(u_n(\sigma))$.

Proof. So let $\sigma\colon \Delta^n \to X$ be a singular n-simplex in X. Then we have

$$\begin{array}{rcccccl} & & \text{Lemma 74.25 applied to } f=\sigma & & \text{since } \sigma_* \text{ is a chain map} & & \\ \partial(u_n(\sigma)) &=& \partial(u_n(\sigma_*(\mathrm{id}_{\Delta^n}))) &\overset{\downarrow}{=}& \partial(\sigma_*(u_n(\mathrm{id}_{\Delta^n}))) &\overset{\downarrow}{=}& \sigma_*(\partial(u_n(\mathrm{id}_{\Delta^n}))) \\ &=& \sigma_*(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})) &=& u_{n-1}(\sigma_*(\partial\,\mathrm{id}_{\Delta^n})) &=& u_{n-1}(\partial\sigma_*(\mathrm{id}_{\Delta^n})) \;=\; u_{n-1}(\partial\sigma). \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \text{by the previous claim} & & \text{Lemma 74.25 applied to } f=\sigma & & \text{since } \sigma_* \text{ is a chain map} \end{array}$$

■

We continue with a refreshingly straightforward lemma.

Lemma 74.25. The maps $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$ are natural,[906] in particular for any map $f\colon X \to Y$ between topological spaces and any $n \in \mathbb{N}_0$ we have

$$f_* \circ u_n \;=\; u_n \circ f_*\colon \mathrm{C}_n(X) \;\to\; \mathrm{C}_n(Y).$$

As we will see, Lemma 74.25 follows immediately from the definitions.

Proof. Let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. Then

$$\begin{array}{ccccc} f_*(u_n(\sigma)) & = & f_*(\sigma_*(\chi_n)) & = & (f\circ\sigma)_*(\chi_n) \quad = \quad u_n(f_*(\sigma)). \\ \uparrow & & \uparrow & & \uparrow \\ \text{by definition of } u_n(\sigma) & & \text{since } (f\circ\sigma)_* = f_*\circ\sigma_* & & \text{by definition of } u_n(f_*(\sigma)) \\ & & \text{as maps } \mathrm{C}_n(\Delta^n) \to \mathrm{C}_n(Y) & & \end{array}$$

Since the desired equality holds for all generators of $\mathrm{C}_n(X)$ it also holds for all of $\mathrm{C}_n(X)$. ■

The proof of the next lemma is again rather involved.

Lemma 74.26. Let X be a topological space. The chain maps

$$u_n\colon \mathrm{C}_n(X) \;\to\; \mathrm{C}_n(X)$$

are chain homotopic to the identity.

Proof. Showing that the chain maps $(u_n)_{n\in\mathbb{N}}$ are chain homotopic to the identity requires us to find maps $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X), n \in \mathbb{N}_0$ such that for each $n \in \mathbb{N}_0$ the following equality holds:

$$\partial_{n+1} \circ P_n \;=\; -P_{n-1} \circ \partial_n + u_n - \mathrm{id}\,.$$

First we consider the case that X is a convex subset of some $\mathbb{R}^m$. Recall that we defined $\mathrm{C}_{-1}(X) = 0$, accordingly we define $P_{-1}\colon \mathrm{C}_{-1}(X) \to \mathrm{C}_0(X)$ to be the zero map. Let us suppose that we have already found maps $P_{-1}, \dots, P_{n-1}$ with the desired properties.

We want to define the next map $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X)$. This map has to have the property that for each singular n-simplex σ we have

$$\partial_{n+1}(P_n(\sigma)) \;=\; -(P_{n-1}\circ\partial_n)(\sigma) \,+\, (u_n - \mathrm{id})(\sigma).$$

Put differently, to define $P_n(\sigma)$ we have to find a singular $(n+1)$-chain $E_\sigma \in \mathrm{C}_{n+1}(X)$ with

$$\partial E_\sigma \;=\; -(P_{n-1}\circ\partial)(\sigma) + (u_n - \mathrm{id})(\sigma),$$

---

[906] Let us explain in what sense these maps deserve to be called "natural". Given $n \in \mathbb{N}_0$ we consider the functor C_* from the category $\mathcal{T}op$ of topological spaces to the category $\mathcal{ChCplx}$ of chain complexes which is given by

$$\begin{array}{rcl} \text{topological space } X & \mapsto & \mathrm{C}_*(X) \\ \text{and} \qquad (f\colon X \to Y) & \mapsto & (f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)). \end{array}$$

Given a topological space X we denote by $\Phi_X\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X)$ the map that is given by the n-th subdivision map that we defined on page 1640. Lemmas 74.24 an 74.25 say that these maps define a natural transformation from the functor C_* to itself.

so that we can then define $P_n(\sigma) := E_\sigma$. Thus let $\sigma\colon \Delta^n \to X$ be a singular n-simplex.

(1) For $n = 0$ we have

$$\big(\underbrace{P_{-1}}_{=0} \circ \partial_0 + \underbrace{u_0}_{=\mathrm{id}} - \mathrm{id}\big)(\sigma) \;=\; 0 \in \mathrm{C}_0(X),$$

and we put $E_\sigma := 0 \in \mathrm{C}_1(X)$.

(2) Now suppose that $n > 0$. Since X is convex we know from the discussion on page 1609 that $\mathrm{H}_n(X) = 0$. Therefore to show that an element $c \in \mathrm{C}_n(X)$ is the boundary of some element in $\mathrm{C}_{n+1}(X)$ it suffices to show that $\partial c = 0$. In our specific situation we calculate

by the Chain Subdivision Lemma 74.24 the subdivision maps form a chain map, i.e. $\partial \circ u_n = u_{n-1} \circ \partial$

$$\begin{aligned}\partial\big(\,(-P_{n-1}\partial_n + u_n - \mathrm{id})\,(\sigma)\big) &\overset{\downarrow}{=} -\partial P_{n-1}\partial\sigma + u_{n-1}\partial\sigma - \partial\sigma \\ &= \underbrace{(-\partial P_{n-1} + u_{n-1} - \mathrm{id})}_{=P_{n-2}\partial}(\partial\sigma) \;=\; P_{n-2}(\underbrace{\partial\partial\sigma}_{=0}) \;=\; 0.\end{aligned}$$

As we had just remarked, this implies that there exists an $E_\sigma \in \mathrm{C}_{n+1}(X)$ with $\partial(E_\sigma) = (P_{n-1}\partial_n - u_{n-1} - \mathrm{id})(\sigma)$.

Now we denote by $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X)$ the map which assigns to each singular n-simplex σ the element $E_\sigma \in \mathrm{C}_{n+1}(X)$. Then it follows from the definition of E_σ that

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; u_n - \mathrm{id}$$

as maps $\mathrm{C}_n(X) \to \mathrm{C}_n(X)$. This concludes the proof of Lemma 74.26 if X is a convex subset of some $\mathbb{R}^m$.

Now we deal with the general case. So let X be a topological space. We consider

$$\begin{aligned} P_n\colon \mathrm{C}_n(X) &\to \mathrm{C}_{n+1}(X) \\ \sum_{l=1}^r a_l \cdot (\sigma_l\colon \Delta^n \to X) &\mapsto \sum_{l=1}^r a_l \cdot (\sigma_l)_*(\underbrace{P_n(\mathrm{id}_{\Delta^n})}_{\uparrow}). \end{aligned}$$

defined since Δ^n is convex

It is straightforward to see that this is a chain homotopy from the subdivision map to the identity map. ∎

Remark. Later on, in Chapter 113.7, we will generalize the idea behind the proof of Lemma 74.26, to prove the Acyclic Model Theorems.

Definition. Given a topological space X and $m \in \mathbb{N}_0$ we denote by u_n^m the m-fold composition of $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$.

Lemma 74.27. For each topological space X and every $m \in \mathbb{N}$ the maps $\{u_n^m\}_{n\geq 0}$ form a chain map $\mathrm{C}_*(X) \to \mathrm{C}_*(X)$ that is chain homotopic to the identity.

Proof. Recall that in the Chain Subdivision Lemma 74.24 we already saw that the maps $\{u_n\}_{n\geq 0}$ form a chain map $\mathrm{C}_*(X) \to \mathrm{C}_*(X)$. As we remarked on page 1592, the composition of chain maps is again a chain map. Thus for any $m \in \mathbb{N}_0$ the maps $\{u_n^m\}_{n\geq 0}$ are also chain maps.

By Lemma 74.26 we have $u_* \simeq \mathrm{id}$. By Lemma 73.1 (2) this implies $u_*^2 \simeq u_*$, and by Lemma 73.1 (1) this implies $u_*^2 \simeq \mathrm{id}$. An induction argument now shows that $u_*^m \simeq \mathrm{id}$ for all m. ∎

Definition. Given a singular chain c we can write it uniquely as $c = a_1\sigma_1 + \cdots + a_s\sigma_s$ with $a_1, \ldots, a_s \in \mathbb{Z} \setminus \{0\}$ and where the σ_i are pairwise different. We then say $\sigma_1, \ldots, \sigma_s$ are the pieces of c.

If $\sigma\colon \Delta^n \to X$ is a singular n-simplex, then the pieces of $u_n(\sigma)$ are evidently "smaller" than the pieces of σ, and by iteratively applying u_n one can hope to make the resulting pieces "arbitrary small". In the following two lemmas we will make it precise what we mean by "arbitrary small". Before we formulate the first lemma we need to recall and introduce several definitions.

Definition.

(1) Given a bounded subset A of $\mathbb{R}^{n+1}$ the diameter is defined as

$$\operatorname{diam}(A) \;:=\; \sup\big\{\|a-b\| \,\big|\, a,b \in A\big\} \in \mathbb{R}_{\geq 0}.$$

(2) Given a singular simple $\sigma\colon \Delta^n \to X$ we denote by $|\sigma|$ the image of σ.

Now we can formulate the following lemma.

Lemma 74.28. For any $x_0, \ldots, x_k \in \mathbb{R}^n$ and any piece τ of $u_k([x_0, \ldots, x_k])$ we have[907]

$$\operatorname{diam}(|\tau|) \;\leq\; \tfrac{n}{n+1} \cdot \operatorname{diam}(|[x_0, \ldots, x_k]|).$$



In the proof of Lemma 74.28 we will need the following elementary[908] geometric lemma.

Lemma 74.29. Let $y_0, \ldots, y_k \in \mathbb{R}^n$.

(1) We have the equality

$$\operatorname{diam}(|[y_0, \ldots, y_k]|) \;=\; \max\big\{\|y_i - y_j\| \,\big|\, i,j = 1, \ldots, k\big\}.$$

(2) We denote by $b = \frac{1}{k}\sum_{i=0}^{k} y_i$ the barycenter of $y_0, \ldots, y_k$. Then for any $i = 0, \ldots, k$ we have

$$\|y_i - b\| \;\leq\; \operatorname{diam}(|[y_0, \ldots, y_k]|).$$

Proof of Lemma 74.29.

(1) The inequality "$\geq$" follows immediately from the definitions and the observation that each y_i lies in

$$|[y_0, \ldots, y_k]| \;=\; \Big\{\sum_{i=0}^{k} t_i \cdot y_i \,\Big|\, t_i \in [0,1] \text{ and } \sum_{i=0}^{k} t_i = 1\Big\}.$$

We turn to the proof of the reverse inequality "$\leq$". So we pick $a, b \in |[y_0, \ldots, y_k]|$. By definition we can write $a = \sum_{i=0}^{k} s_i y_i$ and $b = \sum_{i=0}^{k} t_i y_i$ with $s_i, t_i \in [0,1]$ and such that

[907] Recall the definition on page 1584 of the map $[x_0, \ldots, x_k]\colon \Delta^k \to \mathbb{R}^n$.

[908] Here "elementary" means that the statement and the proof are basically high school level mathematics which does not mean that the proofs are entirely obvious.

$\sum_{i=0}^{k} s_i = \sum_{i=0}^{k} t_i = 1$. We then have

$$\begin{array}{rcl} \|a-b\| & \overset{\text{since } \sum_{i=0}^{k} t_i = 1}{=} & \left\|a - \sum_{i=0}^{k} t_i y_i\right\| = \left\|\sum_{i=0}^{k} t_i(a - t_i y_i)\right\| \leq \sum_{i=0}^{k} \|t_i(a-t_i y_i)\| \\ & \leq & \sum_{i=0}^{k} t_i \cdot \max_{i=0,\dots,k}\{\|a-y_i\|\} \overset{\text{since } \sum_{i=0}^{k} t_i = 1}{=} \max_{i=0,\dots,k}\{\|a-y_i\|\} \overset{\text{same argument applied to } a = \sum_{j=0}^{k} s_j y_j}{\leq} \max_{i,j=0,\dots,k}\{\|y_j - y_i\|\} \end{array}$$

(2) We denote by b_i the barycenter of $y_0, \dots, \widehat{y_i}, \dots, y_k$, i.e.

$$b_i = \tfrac{1}{k}\sum_{j\neq i} y_j.$$

We then have
$$b = \tfrac{1}{k+1} y_i + \tfrac{k}{k+1} b_i.$$

It follows that

$$\begin{array}{rcl} \|b-y_i\| = \left\|\frac{1}{k+1}y_i + \frac{k}{k+1}b_i - y_i\right\| & = & \left\|\frac{k}{k+1}b_i - \frac{k}{k+1}y_i\right\| \\ & = & \frac{k}{k+1}\|b_i - y_i\| \leq \frac{k}{k+1}\cdot \operatorname{diam}(|[y_0,\dots,y_k]|). \end{array}$$

Thus we have obtained the desired inequality. ■

Now we are ready to provide the proof of Lemma 74.28.

Proof of Lemma 74.28. We need to show that for any $x_0, \dots, x_k \in \mathbb{R}^n$ and any piece τ of $u_k([x_0,\dots,x_k])$ we have

$$\operatorname{diam}(|\tau|) \leq \tfrac{n}{n+1}\cdot \operatorname{diam}(|[x_0,\dots,x_k]|).$$

We prove the claim by induction on k. For $k=0$ all the diameters are zero, so the inequality holds trivially. Now suppose that the claim holds for $k-1$. Let $x_0,\dots,x_k \in \Delta^n$ and denote by $b = \frac{1}{k}\sum_{i=0}^{k} x_i$ the barycenter of $x_0,\dots,x_k$. We denote by $\sigma_1,\dots,\sigma_r$ the pieces of $\partial[x_0,\dots,x_k]$.[909]

Now let τ be a piece of $u_k([x_0,\dots,x_k])$. By the definition of the subdivision map given on page 1640 we see that $|\tau| = |[w_0,\dots,w_{k-1},b]|$ where $[w_0,\dots,w_{k-1}]$ is a piece of $u_{k-1}(\sigma_s)$ for some $s \in \{1,\dots,r\}$. By Lemma 74.29 (1) it now suffices to prove the following claim.

Claim.
(1) For any $i,j = 0,\dots,k-1$ we have $\|w_i - w_j\| \leq \frac{k}{k+1}\cdot \operatorname{diam}(|[x_0,\dots,x_k]|)$.
(2) For any $i = 0,\dots,k-1$ we have $\|w_i - b\| \leq \frac{k}{k+1}\cdot \operatorname{diam}(|[x_0,\dots,x_k]|)$.

Proof. First let $i,j \in \{0,\dots,k-1\}$. Then

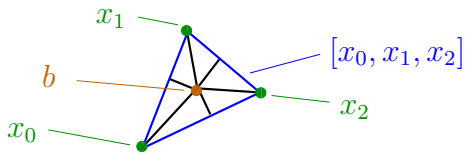
$$\|w_i - w_j\| \leq \operatorname{diam}([w_0,\dots,w_{k-1}]) \overset{\text{induction hypothesis applied to } \sigma_s}{\leq} \tfrac{k-1}{k}\cdot\operatorname{diam}(\sigma_s) \overset{\text{since } |\sigma_s| \subset |[x_0,\dots,x_k]| \text{ and } \frac{k-1}{k} \leq \frac{k}{k+1}}{\leq} \tfrac{k}{k+1}\cdot\operatorname{diam}([x_0,\dots,x_k]).$$

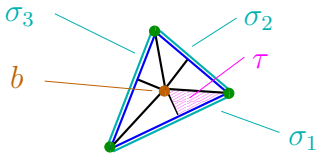
[909] Why does it not say that the number of pieces is $k+1$?

Now let $i \in \{0, \dots, k-1\}$. We then have

$$\|w_i - b\| \underset{\text{Lemma 74.29 (2)}}{\leq} \frac{k}{k+1} \cdot \operatorname{diam}(|[w_0, \dots, w_{k-1}]|) \underset{\text{since } |[w_0, \dots, w_{k-1}]| \subset |[x_0, \dots, x_k]|}{\leq} \frac{k}{k+1} \cdot \operatorname{diam}(|[x_0, \dots, x_k]|)$$

■





Lemma 74.30. Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. For every singular n-simplex $\sigma\colon \Delta^n \to X$ there exists an m such that $u_n^m(\sigma)$ lies in $\mathrm{C}_n^{\mathcal{U}}(X)$.

Proof. Let X be a topological space let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X and let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. By the Lebesgue Lemma 6.5 there exists a $\delta > 0$ such that for any subset A of Δ^n with diameter $\operatorname{diam}(A) < \delta$ there exists an $i \in I$ such that $A \subset \sigma^{-1}(\mathring{U}_i)$.

We pick an $m \in \mathbb{N}$ such that $\left(\frac{n}{n+1}\right)^m < \delta$. We have

$$\begin{aligned} u_n^m(\sigma) &= u_n^m(\sigma_*(\mathrm{id}_{\Delta^n})) \underset{\text{Lemma 74.25}}{=} \sigma_*(u_n^m(\mathrm{id}_{\Delta^n})) \underset{\text{by Lemma 74.28 and } \left(\frac{n}{n+1}\right)^m < \delta}{=} \sigma_*\begin{pmatrix}\text{simplicial chain such that the image}\\ \text{has diameter less than } \delta\end{pmatrix} \\ &= \sigma_*\begin{pmatrix}\text{simplicial chain such that the image}\\ \text{of each piece lies in some } \sigma^{-1}(\mathring{U}_i)\end{pmatrix}. \\ &\qquad \underset{\text{by definition of } \delta}{\uparrow} \end{aligned}$$

But this means precisely that $u_n^m(\sigma) \in \mathrm{C}_n^{\mathcal{U}}(X)$. ■

Now we can prove the Comfortable Covering Proposition 74.23.

Proof of the Comfortable Covering Proposition 74.23. Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. We want to show that the inclusion map $\iota\colon \mathrm{C}_*^{\mathcal{U}}(X) \to \mathrm{C}_*(X)$ induces for each n an isomorphism

$$\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X).$$

We first show that the map $\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is surjective. Let $z \in \mathrm{H}_n(X)$ be given. We choose a cycle $c = \sum_{j=1}^{r} a_j \sigma_j \in \mathrm{C}_n(X)$ which represents $z \in \mathrm{H}_n(X)$. It follows from Lemma 74.30, applied to the singular n-simplices $\sigma_1, \dots, \sigma_r$, that there exists an $m \in \mathbb{N}$ such that $u_n^m(c) \in \mathrm{C}_n^{\mathcal{U}}(X)$. By Lemma 74.27 the maps $\{u_n^m\}_{n \in \mathbb{N}_0}$ form a chain map that is chain homotopic to the identity. By definition this means that there exists a chain homotopy $P_k\colon \mathrm{C}_k(X) \to \mathrm{C}_{k+1}(X)$, $k \in \mathbb{N}_0$. Thus we obtain that

$$z = [c] \underset{\text{since } \partial_{n+1}P_n + P_{n-1}\partial_n \,=\, u_n^m - \mathrm{id}}{=} \left[u_n^m(c) - P_{n-1}\overbrace{\partial_n(c)}^{=0} - \partial_{n+1}P_n(c)\right] = \left[u_n^m(c) - \partial_{n+1}P_n(c)\right] \underset{\text{equality in } \mathrm{H}_n(X)}{=} \overbrace{[u_n^m(c)]}^{\in \iota_*(\mathrm{H}_n^{\mathcal{U}}(X))}.$$

Thus we have shown that $\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is surjective.

The argument that the map $\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is injective is very similar. Indeed, let $[c] \in \mathrm{H}_n^{\mathcal{U}}(X)$ be an element such that $[c] = 0 \in \mathrm{H}_n(X)$. This means that there exists an $e \in \mathrm{C}_{n+1}(X)$ with $\partial e = c$. From Lemma 74.30 it follows, as above, that there exists an $m \in \mathbb{N}$ with $u_{n+1}^m(e) \in \mathrm{C}_{n+1}^{\mathcal{U}}(X)$. We again choose a chain homotopy $\{P_k\}_{k\in\mathbb{N}_0}$ between the chain maps $\{u_k^m\}_{k\in\mathbb{N}_0}$ and id. Then we have

$$c \;=\; \partial e \;\underset{\substack{\uparrow\\ \text{since } \partial P_{n+1} + P_n\partial = u_{n+1}^m - \mathrm{id}}}{=}\; \partial\big(u_{n+1}^m(e) - \partial P_{n+1}(e) - P_n\partial e\big) \;=\; \partial\underbrace{u_{n+1}^m(e)}_{\in \mathrm{C}_{n+1}^{\mathcal{U}}(X)} \;-\; \partial\underbrace{P_n(c)}_{\in \mathrm{C}_{n+1}^{\mathcal{U}}(X)}.$$

Thus have we have shown that c is indeed the boundary of an $(n+1)$-dimensional singular chain, in $C_{n+1}^{\mathcal{U}}(X)$ i.e. $[c] = 0 \in \mathrm{H}_n^{\mathcal{U}}(X)$. Put differently, we have now proved that $\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is injective. ■

Using the Comfortable Covering Proposition 74.23 we can now prove the Excision Theorem 74.17.

Proof of the Excision Theorem 74.17. Let X be a topological space. Furthermore let $Z \subset A \subset X$ be subsets such that the closure of Z is contained in the interior of A. We have to show that the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \;\xrightarrow{\cong}\; \mathrm{H}_n(X, A).$$

We put $B := X \setminus Z$. Note that $A \cap B = A \setminus Z$. It follows from our hypothesis on Z and A that X is the union of the interiors of A and B, put differently, the two sets $\{A, B\}$ is a cover of X.

For each n we consider the following commutative diagram of abelian groups:[910]

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{C}_n(A) & \longrightarrow & \mathrm{C}_n^{\{A,B\}}(X) & \longrightarrow & \mathrm{C}_n^{\{A,B\}}(X)/\mathrm{C}_n(A) & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{C}_n(A) & \longrightarrow & \mathrm{C}_n(X) & \longrightarrow & \mathrm{C}_n(X, A) & \longrightarrow & 0.
\end{array}$$

Note that by definition the horizontal sequences are in fact short exact sequences of chain complexes. We first make the observation that it follows immediately from the definition of $\mathrm{C}_n^{\{A,B\}}(X)$ that the inclusion

$$\begin{array}{rcl} \mathrm{C}_n(B) & \to & \mathrm{C}_n^{\{A,B\}}(X) \\ \mathrm{C}_n(B, A\cap B) & \to & \mathrm{C}_n^{\{A,B\}}(X)/\mathrm{C}_n(A) \end{array}$$

induces a chain map

that is in fact an isomorphism. Thus in the above diagram we can replace the chain complex $\mathrm{C}_n^{\{A,B\}}(X)/\mathrm{C}_n(A)$ by $\mathrm{C}_n(B, A\cap B) = \mathrm{C}_n(X \setminus Z, A \setminus Z)$.

Now we consider the long exact sequence of homology groups arising from the above short exact sequences of chain complexes. By the LES Proposition 74.9 we obtain the

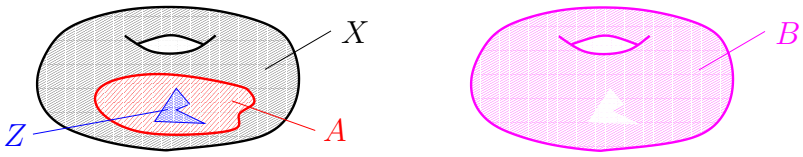
[910] Here we use the definition from page 1638 which in this context says that

$$\mathrm{C}_n^{\{A,B\}}(X) \;:=\; \{\sigma + \tau \,|\, \sigma \in \mathrm{C}_n(A) \text{ and } \tau \in \mathrm{C}_n(B)\}.$$

following commutative diagram where the horizontal sequences are exact:

$$\begin{array}{ccccccccccc}
\ldots \longrightarrow & \mathrm{H}_n(A) & \longrightarrow & \mathrm{H}_n^{\{A,B\}}(X) & \longrightarrow & \overbrace{\mathrm{H}_n(B, A\cap B)}^{=\mathrm{H}_n(X\setminus Z, A\setminus Z)} & \xrightarrow{\partial} & \mathrm{H}_{n-1}(A) & \longrightarrow & \mathrm{H}_{n-1}^{\{A,B\}}(X) & \longrightarrow \ldots \\
& \downarrow{=} & & \downarrow & & \downarrow & & \downarrow{=} & & \downarrow & \\
\ldots \longrightarrow & \mathrm{H}_n(A) & \longrightarrow & \mathrm{H}_n(X) & \longrightarrow & \mathrm{H}_n(X, A) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(A) & \longrightarrow & \mathrm{H}_{n-1}(X) & \longrightarrow \ldots
\end{array}$$

From the Comfortable Covering Proposition 74.23 it follows that the second and fifth vertical map are isomorphisms. Since the first and fourth map are evidently isomorphisms it now follows from the Five Lemma 74.10 that the middle map is also an isomorphism. ■



74.10. **Excision for CW-complexes.** On some occasions the following variation on the Excision Theorem 74.17 is useful.

Theorem 74.31. (Excision Theorem for CW-complexes) Let X be a CW-complex and let A and B be two subcomplexes with $X = A \cup B$. For every $n \in \mathbb{N}_0$ the inclusion induced map

$$\mathrm{H}_n(B, A\cap B) \xrightarrow{\cong} \mathrm{H}_n(X, A)$$

is an isomorphism.

Proof. Let X be a CW-complex. Furthermore let A and B be two subcomplexes of X with $X = A\cup B$. We denote by $i\colon B \to X$ the inclusion. We consider the following diagram

$$\begin{array}{ccc}
\mathrm{H}_n(B, A\cap B) & \xrightarrow{\quad i_* \quad} & \mathrm{H}_n(X, A) \\
{\scriptstyle \text{ю}}\downarrow & & \downarrow{\scriptstyle \text{ю}} \\
\widetilde{\mathrm{H}}_n(B/(A\cap B)) & \xrightarrow{\quad i_* \quad} & \widetilde{\mathrm{H}}_n(X/A).
\end{array}$$

By Lemma 74.19 we know that the diagram commutes. Furthermore, since A and B are subcomplexes we obtain from the Homology-of-Quotients Proposition 74.20 that the vertical maps are isomorphisms. Finally note that by the CW-Complex Construction Lemma 68.32 (3) the quotients $B/(A\cap B)$ and X/A have natural CW-structures. It follows from the CW-Complex Homeomorphism Lemma 68.12 that the induced map $B/(A\cap B)$ and X/A is a homeomorphism. Thus we see that the bottom horizontal map is an isomorphism. It follows from this discussion that the top map is, as promised, an isomorphism. ■

Remark. In the Excision Theorem 75.10 for Topological Manifolds we will get to see another variation on the Excision Theorem 74.17.

Exercises for Chapter 74.

Exercise 74.1. Let X be a non-empty topological space. By the Reduced Homology Lemma 74.1 (4) we know that there exists an isomorphism $\mathrm{H}_0(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_0(X)\oplus\mathbb{Z}$. Show that

there is no *natural* isomorphism $\mathrm{H}_0(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z}$. More precisely, show that it is not possible to assign to each non-empty topological space X an isomorphism $\Phi_X \colon \mathrm{H}_0(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z}$ such that for any map $f \colon X \to Y$ between non-empty topological spaces the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_0(X) & \xrightarrow{\Phi_X} & \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_* \oplus \mathrm{id}_{\mathbb{Z}}} \\ \mathrm{H}_0(Y) & \xrightarrow{\Phi_Y} & \widetilde{\mathrm{H}}_0(Y) \oplus \mathbb{Z}. \end{array}$$

Hint. Let X be the discrete space consisting of two points. If such a natural isomorphism Φ_X did exist, what should the map Φ_X do?

Exercise 74.2. Let

$$\mathbb{Z} \xrightarrow{f} B \xrightarrow{g} \mathbb{Z} \oplus \mathbb{Z}_6 \to 0$$

be an exact sequence, i.e. we know that $g \circ f = 0$, we know that $\mathrm{im}(f) = \ker(g)$ and we know that g is an epimorphism. What are the possible isomorphism types of the group B?

Exercise 74.3. We consider the following commutative diagram of homomorphisms between abelian groups:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow \\ 0 & \longrightarrow & \widetilde{A} & \longrightarrow & \widetilde{B} & \longrightarrow & \widetilde{C} & \longrightarrow & 0. \end{array}$$

We suppose that the horizontal sequences are exact and that the maps a and c to the left and right are the zero map. Does it follow that the middle map b is also the zero map?

Exercise 74.4. Let X be a topological space with $\widetilde{\mathrm{H}}_0(X) \cong \mathbb{Z}$.

(a) Show that given any $\epsilon \in \{-1, 0, 1\}$ there exists a map $f \colon X \to X$ such that the induced map f_* on $\widetilde{\mathrm{H}}_0(X)$ is given by multiplication by ϵ.

(b) Show that given any map $f \colon X \to X$ the induced map f_* on $\widetilde{\mathrm{H}}_0(X)$ is given by multiplication by some $\epsilon \in \{-1, 0, 1\}$.

Exercise 74.5. Determine the homology groups of $S^2 \vee S^1$.

Exercise 74.6.

(a) Let

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E' \end{array}$$

be a commutative diagram such that the horizontal sequences are exact. Show that if a is surjective, if b and d are isomorphisms, and if e is injective, then c is an isomorphism.

(b) Does the statement in (a) also hold, if a injective and if e is surjective ?

Exercise 74.7. Let M be the Möbius band. Determine the relative homology groups of the pair $(M, \partial M)$.

Exercise 74.8. Give examples of two contractible subspaces A and B of $\mathbb{R}^2$ such that $\mathrm{H}_1(A \cup B)$ is infinitely generated.

Exercise 74.9. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \xrightarrow{\varphi} & B_* & \xrightarrow{\psi} & C_* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A'_* & \xrightarrow{\varphi'} & B'_* & \xrightarrow{\psi'} & C'_* & \longrightarrow & 0 \end{array}$$

be a commutative diagram of homomorphisms between abelian groups. Show that there exists an exact sequence

$$\begin{array}{l} 0 \longrightarrow \ker(a) \xrightarrow{\varphi} \ker(b) \xrightarrow{\psi} \ker(c) \\ \qquad\qquad \longrightarrow \operatorname{coker}(a) \xrightarrow{\varphi'} \operatorname{coker}(b) \xrightarrow{\psi'} \operatorname{coker}(c) \longrightarrow 0. \end{array}$$

This statement is sometimes referred to as the "Snake Lemma".
Hint. Try to prove the statement with as little effort as possible.

Exercise 74.10. We consider the following diagram of homomorphisms between abelian groups:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & M & & & & \\ & & & & \downarrow{\scriptstyle \mu} & \searrow{\scriptstyle \beta\circ\mu} & & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & & \searrow{\scriptstyle \nu\circ\alpha} & \downarrow{\scriptstyle \nu} & & & & \\ & & & & N & & & & \\ & & & & \downarrow & & & & \\ & & & & 0. & & & & \end{array}$$

We suppose that the horizontal and the vertical sequences are exact.

(a) Show that $\beta \circ \mu$ is a monomorphism if and only if $\nu \circ \alpha$ is a monomorphism.
(b) Show that $\beta \circ \mu$ is an epimorphism if and only if $\nu \circ \alpha$ is an epimorphism.

Exercise 74.11. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \longrightarrow & B_* & \longrightarrow & C_* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A'_* & \longrightarrow & B'_* & \longrightarrow & C'_* & \longrightarrow & 0 \end{array}$$

be a commutative diagram of short exact sequences of chain complexes. Show that if two of the vertical maps induce isomorphisms on homology, then so does the third.

Exercise 74.12. We consider the following commutative diagram:

$$\begin{array}{ccccccccc} & & & & \ker(\beta) & & & & \\ & & & \swarrow{\scriptstyle \varphi} & \downarrow & & & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0. \end{array}$$

Evidently we assume that the horizontal sequences are exact. Furthermore we assume that $\alpha\colon A \to A'$ and $\beta\colon B \to B'$ are epimorphisms. Show that $\gamma\colon C \to C'$ is an isomorphism.

Exercise 74.13. For readers who really really enjoy diagram chases we have the following problem on offer. Suppose we are given the following commutative diagram of

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_* & \longrightarrow & B_* & \xrightarrow{h_B} & C_* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle v_B} & & \downarrow & & \\
0 & \longrightarrow & D_* & \xrightarrow{h_D} & E_* & \longrightarrow & F_* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle v_D} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_* & \longrightarrow & \mathrm{H}_* & \longrightarrow & J_* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the vertical and horizontal sequences are exact. We suppose that the chain complex H_* is acyclic, i.e. we suppose that homology groups of H_* vanish. Show that for any $k \in \mathbb{N}_0$ the following diagram commutes up to a minus sign:

$$\begin{array}{ccccc}
\mathrm{H}_k(D_*) & \xrightarrow{v_{D*}} & \mathrm{H}_k(G_*) & \xleftarrow[\cong]{\partial} & \mathrm{H}_{k+1}(K_*) \\
\downarrow & & & & \partial\downarrow\cong \\
\mathrm{H}_k(E) & \xrightarrow[\cong]{v_{B*}^{-1}} & \mathrm{H}_k(B) & \xrightarrow{h_{B*}} & \mathrm{H}_k(C).
\end{array}$$

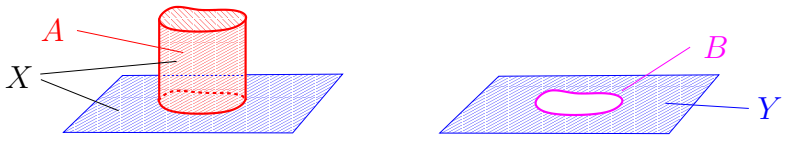
Exercise 74.14. Let X be a topological space, let $x_0 \in X$ be a point and let $n \in \mathbb{N}_0$. Show that the map

$$\widetilde{\mathrm{H}}_n(X) \longrightarrow \mathrm{H}_n(X) \xrightarrow{=} \mathrm{H}_n(X, \varnothing) \longrightarrow \mathrm{H}_n(X, \{x_0\})$$
$$=: \Phi_{X,x_0}$$

is an isomorphism.

Exercise 74.15. We say that a map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ has *bounded displacement* if there exists a $C \in \mathbb{R}$ such that for any $x \in \mathbb{R}^n$ we have $\|x - f(x)\| \leq C$. Show that any map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ of bounded displacement is onto.
Hint. Use the Brouwer Fixed Point Theorem.

Exercise 74.16. Let (X, A) be a pair of topological spaces. Suppose there exists a deformation retraction $r\colon X \times [0,1] \to X$ from X to some subset $Y = r_1(X)$. We write $B = r_1(A)$. Show that the inclusion induced maps $\mathrm{H}_k(Y, B) \to \mathrm{H}_k(X, A)$ are isomorphisms.



Exercise 74.17. Let $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_m$ be a nested sequence of topological spaces.

(a) Suppose that for each $k = 0, \dots, m-1$ we have $\mathrm{H}_n(X_{k+1}, X_k) = 0$ for all $n \in \mathbb{N}_0$. Show that $\mathrm{H}_n(X_m, X_0) = 0$ for all $n \in \mathbb{N}_0$.
(b) Let $n \in \mathbb{N}_0$. Suppose that for each $k = 0, \dots, m-1$ we have $\mathrm{H}_n(X_{k+1}, X_k) = 0$. Does it also follow that $\mathrm{H}_n(X_m, X_0) = 0$?

Exercise 74.18. We consider $X = [0,1]$ and $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Prove the following two statements:

(a) The group $\mathrm{H}_1(X, A)$ is countable.
(b) The group $\mathrm{H}_1(X/A)$ is uncountable. For psychological reasons it might be convenient to recall that in Exercise 56.1 we showed that the quotient space X/A is homeomorphic to the Hawaiian earrings.

Remark. The homology groups of the Hawaiian earrings are computed in [**EKa2000**]. In particular it is shown that the higher homology groups $\mathrm{H}_i(X/A)$ are zero for $i \geq 2$, which is not clear at all, see e.g. [**BaM1962**].

Exercise 74.19. Give an example of a topological space X and some non-empty subset A for which there is no exact sequence of the form

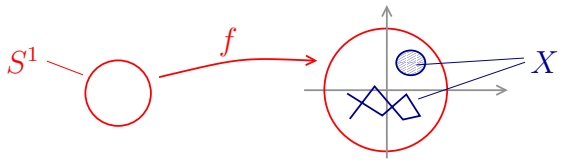
$$\dots \to \widetilde{\mathrm{H}}_{n+1}(X/A) \to \widetilde{\mathrm{H}}_n(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_n(X) \xrightarrow{p_*} \widetilde{\mathrm{H}}_n(X/A) \to \dots$$

where $i\colon A \to X$ is the inclusion and $p\colon X \to X/A$ is the projection.
Remark. You might want to consult your calculations from Exercise 74.18.

Exercise 74.20. Let X be a non-empty bounded subset of $\mathbb{R}^n$. We pick an $r \in \mathbb{R}_{>0}$ such that X is contained in the bounded n-ball $\overline{B}_r^n$ of radius r. We consider the map

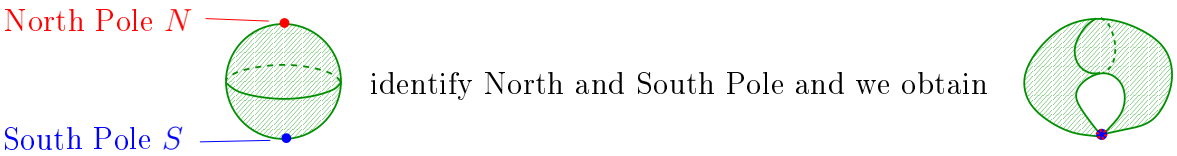
$$\begin{aligned} f\colon S^{n-1} &\to \mathbb{R}^n \setminus X \\ x &\mapsto 2r \cdot x. \end{aligned}$$

Show that the induced map $f_*\colon \mathrm{H}_{n-1}(S^{n-1}) \to \mathrm{H}_{n-1}(\mathbb{R}^n \setminus X)$ is a monomorphism.



Exercise 74.21. Show that for $k \geq 1$ the wedge $S^k \vee S^k$ is not a topological manifold.

Exercise 74.22. Compute the homology groups of the quotient of S^2 obtained by identifying the North Pole N and the South Pole S to a point. We refer to the figure below for an illustration.



Exercise 74.23. Let $T = S^1 \times S^1$ be the torus. We consider the subset $A := S^1 \times \{1\}$.

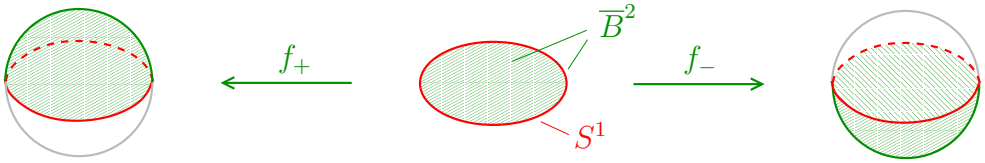
(a) Show that T/A is homeomorphic to the topological space $S^2/(S \cup N)$ considered in Exercise 74.22.
(b) Compute the homology groups of the torus.
(c) Bonus question: How far can you push the above approach if we start out with the Klein bottle instead of the torus?

Remark. We will soon see a more conceptual, less ad hoc, approach to calculating the homology groups of the torus.

Exercise 74.24. We consider the two maps

$$\begin{array}{rcl} f_+\colon \overline{B}^2 & \to & S^2 \\ (x,y) & \mapsto & \left(x, y, \sqrt{1-x^2-y^2}\right) \end{array} \quad \text{and} \quad \begin{array}{rcl} f_-\colon \overline{B}^2 & \to & S^2 \\ (x,y) & \mapsto & \left(x, y, -\sqrt{1-x^2-y^2}\right) \end{array}$$

Show that these two maps $f_-, f_+\colon \overline{B}^2 \to S^2$ are not homotopic rel S^1.



75. Applications and variations of the Excision Theorem 74.17

As the title of this chapter suggests, in this chapter we will consider some applications of the Excision Theorem 74.17 and we will state and prove several variations thereof.

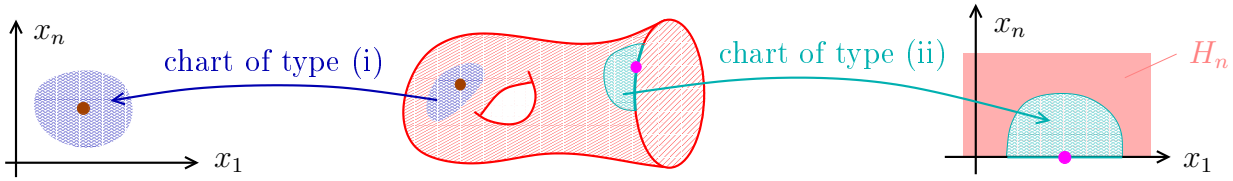
In particular we will prove several statements on the topology of topological manifolds which extend the results obtain in Section 19.8 from the setting of smooth manifolds to the more general context of topological manifolds. We will also use the opportunity to state a few other results on topological manifolds. At the end of the chapter we will introduce the double of a manifold. This construction makes sense, and is of interest, for smooth and for topological manifolds.

The chapter does not contain any real surprises. The overall message is that the statement of the Excision Theorem 74.17 holds in several convenient alternative settings and that many of the statements which we know and love about the topology of smooth manifolds also hold for topological manifolds. These variations and generalizations of earlier results are non-trivial, but most readers will have a hard time building up enthusiasm for these results. Thus it might be best to have a quick glimpse at the results and then move on to the next section.

75.1. **The local homology groups of a topological space.** We start out with the following definition which at first glance might look somewhat artificial.

Definition. Given a topological space X and a point $x_0 \in X$ the relative homology group $\mathrm{H}_n(X, X \setminus \{x_0\})$ is called the n-th local homology group of X at the point x_0.

The main goal of this section is to calculate the local homology groups of topological manifolds and to draw several consequences from this calculation. At this point it is helpful to recall the definition on page 511 of charts of type (i) and (ii) of a topological manifold. For the reader's convenience we illustrate the definitions in the figure below. We also recall that on page 511 we said that a point x on a topological manifold is a *boundary point* if x does not admit a chart of type (i).



The following lemma calculates the local homology groups of topological manifolds.

Lemma 75.1. (Manifold-Local Homology Lemma) Let M be a k-dimensional topological manifold.

(1) Every point on X admits either a chart of type (i) or it admits a chart of type (ii).
(2) Given any point $P \in M$, any $l \in \mathbb{N}_0$ and any neighborhood U of P we have

$$\mathrm{H}_l(U, U\setminus\{P\}) \xrightarrow[\uparrow]{\cong} \mathrm{H}_l(M, M\setminus\{P\}) \cong \begin{cases} \mathbb{Z}, & \text{if } P \in M \setminus \partial M \text{ and if } l = k, \\ 0, & \text{otherwise.} \end{cases}$$

induced by the inclusion $(U, U\setminus\{P\}) \to (M, M\setminus\{P\})$

Proof. Let P be a point on a k-dimensional topological manifold M. We start out with the following claim.

Claim. We have the following isomorphisms:

$$\mathrm{H}_l(U, U\backslash\{P\}) \xrightarrow{\cong} \mathrm{H}_l(M, M\backslash\{P\}) \cong \begin{cases} \mathbb{Z}, & \text{if } P \text{ admits a chart of type (i) and if } l = k, \\ 0, & \text{if } P \text{ admits a chart of type (ii) or if } l \neq k. \end{cases}$$

Proof. First we consider the case that P admits a chart of type (i). It follows easily from the definition of an open subset of $\mathbb{R}^k$ that we can pick a chart $\Phi\colon U \to B^k$ for P with $\Phi(P) = 0$. We then have the following isomorphisms:

$$\mathrm{H}_l(M, M\backslash\{P\}) \xleftarrow[\uparrow]{\cong} \mathrm{H}_l(U, U\backslash\{P\}) \xrightarrow[\downarrow]{\Phi_*} \mathrm{H}_l(B^k, B^k\backslash\{0\}) \xrightarrow[\uparrow]{\partial} \widetilde{\mathrm{H}}_{l-1}(B^k\backslash\{0\}) \xleftarrow[\uparrow]{\cong} \widetilde{\mathrm{H}}_{l-1}(S^{k-1}).$$

isomorphism by the Excision Theorem 74.18

isomorphism since Φ is a homeomorphism

isomorphism by the long exact sequence in reduced homology of the pair $(B^k, B^k \setminus \{0\})$ and the fact that $\widetilde{\mathrm{H}}_*(B^k) = 0$ by page 1612

isomorphism since S^{k-1} is homotopy equivalent to $B^k \setminus \{0\}$

The desired statement now follows from the calculation of the reduced homology groups of spheres given in Proposition 74.4.

Now we consider the case that P admits a chart of type (ii). Similar to the above we can pick a chart $\Phi\colon U \to B^k_{\geq 0} := B^k \cap \mathrm{H}_k$ for P with $\Phi(P) = 0$. By arguments that are very similar to the ones used above we see that we have the following isomorphisms:

$$\mathrm{H}_l(M, M\backslash\{P\}) \xleftarrow{\cong} \mathrm{H}_l(U, U\backslash\{P\}) \xrightarrow[\cong]{\Phi_*} \mathrm{H}_l(B^k_{\geq 0}, B^k_{\geq 0}\backslash\{0\}) \xrightarrow[\cong]{\partial} \widetilde{\mathrm{H}}_{l-1}(B^k_{\geq 0}\backslash\{0\}) \underset{\uparrow}{=} 0.$$

one can easily show that $B^k_{\geq 0}$ is contractible, hence its reduced homology vanishes by the Reduced Homology Lemma 74.1 ⊞

Now it is straightforward to prove the statements of the lemma.

(1) It follows from the above claim that $\mathrm{H}_k(M, M \setminus \{P\}) \cong \mathbb{Z}$ if and only if P admits a chart of type (i). This observation immediately implies the promised statement.
(2) The first map is an isomorphism by the Excision Theorem 74.18. The second isomorphism follows immediately from the above claim. ■

75.2. **The topology of topological manifolds.** With the results from Chapter 74 we can now generalize the Smooth Manifold Boundary Proposition 19.26 from the case of smooth manifolds to the more general setting of topological manifolds.

Proposition 75.2. (Topological Manifold Boundary Proposition) Let X be an n-dimensional topological manifold. The following statements hold:

(1) Every point on X admits either a chart of type (i) or it admits a chart of type (ii).
(2) If F is equipped with a smooth atlas, i.e. if we view F as a smooth manifold, then the boundary of F as a topological manifold agrees with the boundary of F viewed as a topological manifold.
(3) The boundary ∂X is an $(n-1)$-dimensional topological manifold with empty boundary, i.e. we have $\partial(\partial X) = \varnothing$.
(4) (a) Any union of components of ∂X is a closed subset of X. In particular ∂X itself is a closed subset of X.
    (b) If X is compact, then ∂X is also compact.
(5) If ∂X is non-empty, then $X \setminus \partial X$ is non-compact.

(6) Let N be a submanifold of X. The following two statements are equivalent:
   (a) N is a proper submanifold.
   (b) The following two conditions are satisfied:
      (i) N is a closed subset of X.
      (ii) We have the equality[911] $\partial N = N \cap \partial X$.
(7) If $N \subset X$ is a proper k-dimensional submanifold, then we have $N \cap \partial X = \partial N$, and $N \cap \partial X = \partial N$ is a proper $(k-1)$-dimensional submanifold of ∂X.
(8) If X is an m-dimensional topological manifold and if Y is an n-dimensional topological manifold, then $X \times Y$ is an $(m+n)$-dimensional topological manifold with $\partial(X \times Y) = (\partial X \times Y) \cup (X \times \partial Y)$.

Example. It follows from the Topological Manifold Boundary Proposition 75.2 and the examples of charts for $\overline{B}^n$ given on page 514 that the boundary of the n-dimensional topological manifold is indeed S^{n-1}. We had suspected this for a long time, but only recently did we acquire the tools for providing a proof.

Remark.

(1) For 2-dimensional topological manifolds we had proved some of the statements of the Topological Manifold Boundary Proposition 75.2 in Proposition 50.7.
(2) In Propositions 19.31 and 20.6 we showed the analogue of the Topological Manifold Boundary Proposition 75.2 (7) in the setting of smooth manifolds.
(3) Let M and N be two connected n-dimensional topological manifolds with empty boundary. If $M \times [0,1]$ is homeomorphic to $N \times [0,1]$, then evidently the corresponding boundaries are also homeomorphic. Note that it follows from the Topological Manifold Boundary Proposition 75.2 (7) that $\partial(M \times [0,1]) = M \times \{0,1\}$ and $\partial(N \times [0,1]) = N \times \{0,1\}$. Since M and N are connected we obtain easily from the above that M and N are homeomorphic. Even though this sounds quite reasonable, very similar sounding statements can easily be wrong. For example there exists a 3-dimensional topological manifold W (called the *Whitehead manifold*) which is non-compact and has no boundary, which is not homeomorphic to $\mathbb{R}^3$ but such that $W \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$. We refer to [**Kir1989**, p. 87] and [**AnR1965**] for details.

Proof. Let X be an n-dimensional topological manifold.

(1) This statement is precisely the statement of the Manifold-Local Homology Lemma 75.1 (1). The only reason why we wrote down the statement again is to preserve the analogy with the Smooth Manifold Boundary Proposition 19.26.
(2) Verbatim the same argument as in the 2-dimensional setting, see the proof of Proposition 50.7 (2) on page 1132, shows how to deduce this statement from Statement (1).
(3) First note that it follows from Lemma 1.14 and from the Second Countability Lemma 9.1 (2) that ∂X is Hausdorff and second-countable. Now it remains to show that ∂X admits an $(n-1)$-dimensional atlas. Let $P \in \partial X$. By definition of ∂X there exists a chart $\Phi\colon U \to V$ of type (ii) for X with $P \in U$. Note that Φ is a chart of type (ii) precisely for the points in $\Phi^{-1}(\partial\,\mathbb{H}^n)$. Also note that it follows from (1) and

[911] Here ∂N denotes the boundary of N, viewed as a topological manifold in its own right.

the definition of ∂X that $\Phi^{-1}(\partial \mathbb{H}^n) = \partial X \cap U$. In other words, under the obvious identification $\partial \mathbb{H}^n = \mathbb{R}^{n-1}$ we see that Φ restricts to an $(n-1)$-dimensional chart of type (i) for the topological space ∂X. This shows that ∂X is an $(n-1)$-dimensional topological manifold with empty boundary.

(4) This statement is precisely the content of the Manifold Boundary-Closed Subset Lemma 18.6 (1) and (2).

(5) This statement is the content of the Manifold Boundary-Closed Subset Lemma 18.6 (3).

(6) As we already saw in the proof of Statements (5) and (6) of the Smooth Manifold Boundary Proposition 19.26, this statement is straightforward consequence of (1) and (4).

(7) In the Topological Manifold Product Proposition 18.4 we already showed that $X \times Y$ is an $(m+n)$-dimensional topological manifold. Furthermore recall that in the Topological Manifold Product Proposition 18.4 we showed that each point in $(X \times Y) \setminus ((\partial X \times Y) \cup (X \times \partial Y))$ admits a chart of type (i) and we showed that each point in $(\partial X \times Y) \cup (X \times \partial Y)$ admits a chart of type (ii). By (1) and by definition of the boundary of a topological manifold this means that we have indeed $\partial(X \times Y) = (\partial X \times Y) \cup (X \times \partial Y)$. ■

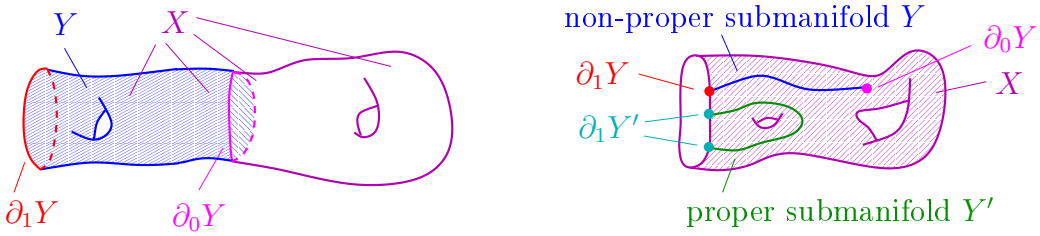
Let X be a topological manifold and let Y be a submanifold. It follows almost immediately from the definitions that Y is a topological manifold in its own right. In particular it makes sense to consider its boundary ∂Y and in particular it makes sense to introduce the following definition which can be viewed as a generalization of the definition on page 568.

Definition. Let X be a topological manifold and let Y be a submanifold. We write

$$\partial_0 Y := \underset{\uparrow}{\partial Y} \setminus \partial X \quad \text{and} \quad \partial_1 Y := \underset{\uparrow}{\partial Y} \cap \partial X.$$

here ∂Y denotes the boundary of Y viewed as a topological manifold

The definition is illustrated in the figure below.



The next proposition gives us some results on the boundary of submanifolds. It can be viewed as an analogue of the Codimension-Zero Submanifold Proposition 19.38, which we had formulated in the smooth context.

Proposition 75.3. Let X be an m-dimensional topological manifold and let Y be an n-dimensional submanifold. We assume that Y is a closed subset of X. (In most applications Y will be compact).

(1) $\partial_0 Y$ and $\partial_1 Y$ are disjoint and each of them is a union of components of ∂Y.

(2) If Y is a proper submanifold, then $\partial Y = \partial_1 Y = Y \cap \partial X$.

(3) If Y is of codimension zero, then the interior $\overset{\circ}{Y}$ of Y as a subset of X agrees with $Y \setminus \partial_0 Y$. In particular $Y \setminus \partial_0 Y$ is an open subset of X.
(4) Suppose that Y is of codimension zero and that Y is compact.
   (a) The set $Z := X \setminus \overset{\circ}{Y}$ is also a codimension-zero submanifold of X and we have $Y \cap Z = \partial_0 Y = \partial_0 Z$.
   (b) If X and ∂Y are connected, then $Z = X \setminus \overset{\circ}{Y}$ is also connected.
   (c) If X is compact, then $Z = X \setminus \overset{\circ}{Y}$ is also compact.

Proof. The proposition follows easily from making the obvious modifications to the proof of the Codimension-Zero Submanifold Proposition 19.38. The biggest change is that we need to replace the Smooth Manifold Boundary Proposition 19.26 by the Topological Manifold Boundary Proposition 75.2. We leave it to the reader to fill in the details. ■

We conclude this section with the following proposition which is the generalization of the Smooth Manifold-Dimension Proposition 19.28 from smooth manifolds to topological manifolds.

Proposition 75.4. (Topological Manifold-Dimension Proposition) Let M be an m-dimensional non-empty topological manifold and let N be an n-dimensional topological manifold. If M is homeomorphic to an open subset of N (e.g. if M is homeomorphic to N), then $m = n$.

It follows in particular from the Topological Manifold-Dimension Proposition 75.4 that the dimension of a non-empty topological manifold is well-defined.

Proof. By the remark on page 526 it suffices to consider the case that we are given a homeomorphism $f\colon M \to N$ between a non-empty m-dimensional topological manifold M and a topological manifold N. By the Topological Manifold Boundary Proposition 75.2 (5) we know that $M \setminus \partial M$ is non-empty. In particular we can find a point $P \in M \setminus \partial M$. Then

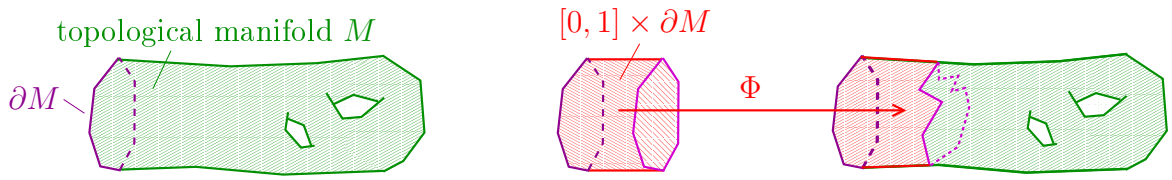
$$\begin{array}{ccccc} \mathbb{Z} & \underset{\uparrow}{\cong} & \mathrm{H}_m(M, M \setminus \{P\}) & \underset{\uparrow}{\overset{\cong}{\to}} & \mathrm{H}_m(N, N \setminus \{f(P)\}). \\ & \text{Manifold-Local Homology Lemma 75.1} & & \text{since } f \text{ is a homeomorphism} & \end{array}$$

It now follows from the Manifold-Local Homology Lemma 75.1 that N is also m-dimensional. ■

75.3. **The Topological Collar Neighborhood Theorem.** In the next section we will consider some useful variations on the Excision Theorem 74.17 for topological manifolds. To prove these results we will make use of the following following theorem which is an analogue of the Smooth Collar Neighborhood Theorem 28.3 for topological manifolds.

Theorem 75.5. (Topological Collar Neighborhood Theorem) Given any topological manifold M there exists a map $\Phi\colon [0,1) \times \partial M \to M$ with the following four properties:
(1) The map Φ is an embedding.
(2) The image of $[0,1) \times \partial M$ is an open subset of M.
(3) For every $P \in \partial M$ we have $\Phi(0, P) = P$.
(4) The image of $[0,1] \times \partial M$ is a closed subset of M.

Remark.



(1) In the compact case the Topological Collar Neighborhood Theorem 75.5 is also proved in [**Vic1994**, Theorem Appendix II.7] and [**Hat2002**, p. 253]. The general case, where M is also allowed to be non-compact, is also proved in [**BrownM1962**, Theorem 3] and [**Arm1970**, Theorem 1]. Note that [**Arm1970**, Theorem 1] has a few refinements which could be useful in some applications. The proof we provide is based on [**Conn1971**]

(2) In Exercise 120.1 we will prove a variation on the Topological Collar Neighborhood Theorem 75.5 for the case that M is a codimension-zero submanifold *with corner* of some other topological manifold.

(3) In Exercise 75.11 we will use the Topological Collar Neighborhood Theorem 75.5 to give a new proof of the Push Away From Boundary Lemma 18.13. In particular we will show again that a topological manifold X is path-connected if and only if $X \setminus \partial X$ is path-connected.

Proof of Theorem 75.5 for compact M. We consider M with an "external collar" attached, i.e. we consider $W := M \cup_{\partial M = \{0\} \times \partial M} ([0,1] \times \partial M)$. Let us assume for a second that we have found a homeomorphism $F\colon M \to W$ with $F(\partial M) = \{1\} \times \partial M$. In this fortunate setting one can verify easily, say using the Topological Pushout-Maps Lemma 5.34 (3) and (4), that the map

$$\begin{array}{rcl} \Phi\colon [0,1] \times \partial M & \to & M \\ (t,x) & \mapsto & F^{-1}([(1-t,x)]) \end{array}$$

has all the desired properties. Thus it remains to show that such an F actually exists.

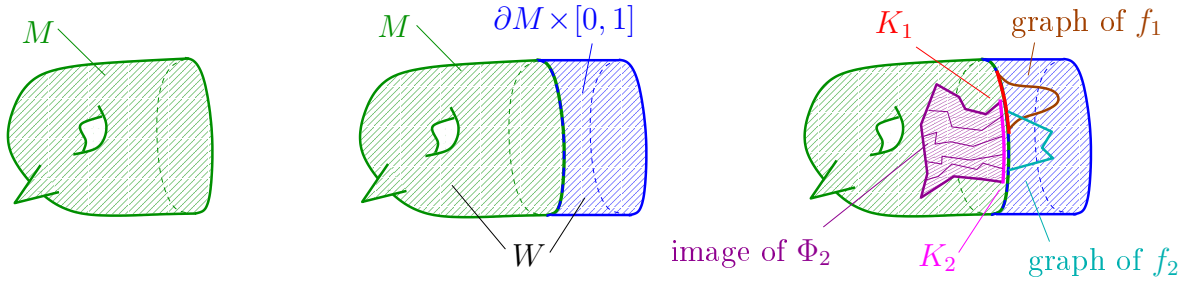
We start out our search for such F with the following claim.

Claim. There exist compact subsets $K_1, \ldots, K_m \subset \partial M$, embeddings $\Phi_i\colon K_i \times [-1,0] \to M$ and maps $f_i\colon \partial M \to [0,1]$ such that the following are satisfied:

(1) For each $i \in \{1, \ldots, m\}$ we have $\Phi_i^{-1}(\partial M) = K_i \times \{0\}$.
(2) For each $i \in \{1, \ldots, m\}$ we have $\operatorname{supp}(f_i) \subset K_i$.
(3) We have $f_1 + \cdots + f_m = 1$ on all of ∂M.

Proof. It follows from the definition of ∂M, together with the Topological Manifold Boundary Proposition 75.2 (1), that for each $x \in \partial M$ there exists an open neighborhood U_x of x and an embedding $\Phi_x\colon [-1,0] \times U_x \to M$ with $\Phi_x^{-1}(\partial M) = \{0\} \times U_x$. For each x we can and will pick a compact neighborhood K_x with $K_x \subset U_x$. Since M is compact we know by the Topological Manifold Boundary Proposition 75.2 (4b) that ∂M is also compact. Evidently the interiors of the K_x cover all of ∂M. Since ∂M is compact we see that there exist $x_1, \ldots, x_m \in \partial M$ such that ∂M is covered by the interiors of the K_{x_i}. Finally note that it follows from the Manifold-Partitions of Unity Existence Theorem 21.1 that there exist maps $f_1, \ldots, f_m\colon \partial M \to [0,1]$ such that for each $i \in \{1, \ldots, m\}$ we have $\operatorname{supp}(f_i) \subset K_{x_i}$

and such that $f_1 + \cdots + f_m = 1$. Renaming K_{x_i} and Φ_{x_i} as K_i and Φ_i we obtain the desired result. ⊞



Next, for $i = 0, \ldots, m$ we set $g_i := f_1 + \cdots + f_i$ and we set

$$W_i := M \cup_{\partial M = \{0\} \times \partial M} \{(t, x) \,|\, x \in \partial M \text{ and } t \in [0, g_i(x)]\}.$$

By the Topological Pushout-Maps Lemma 5.34 (1) we know that W_0 is homeomorphic to M and evidently $W_m = W$.

Claim. For $i = 1, \ldots, m$ there exists a homeomorphism $F_i \colon W_{i-1} \to W_i$.

Proof. Let $i \in \{1, \ldots, m\}$. We consider the map

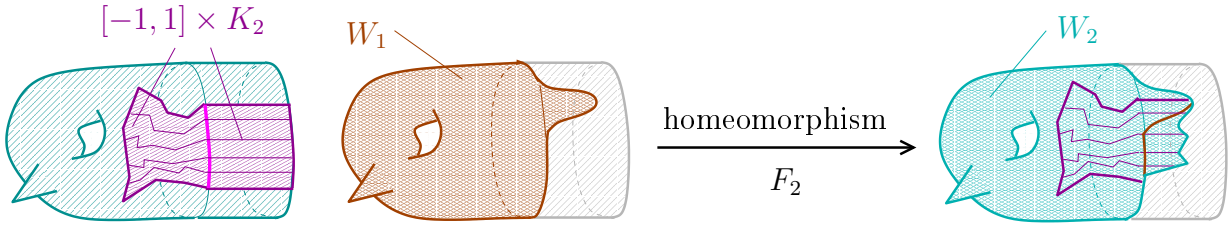
$$\begin{array}{rcl} \Theta \colon [-1,1] \times K_i & \to & W \\ (t, x) & \mapsto & \begin{cases} [\Phi_i((t,x))], & \text{if } t \in [-1, 0], \\ [(t, x)], & \text{if } t \in [0, 1]. \end{cases} \end{array}$$

It follows fairly easily from the Topological Pushout-Maps Lemma 5.34 (3) and (4), Lemma 5.36 and the Compact-Hausdorff Proposition 2.17 (2) that this map is a closed embedding. To simplify the notation we will identify $[-1, 1] \times K_i$ with its image in W. The desired homeomorphism $W_{i-1} \to W_i$ is now given by "stretching horizontally" along $[-1, 1] \times K_i$. More precisely, we consider the map

$$\begin{array}{rcl} F_i \colon W_{i-1} & \to & W_i \\ x & \mapsto & \begin{cases} x, & \text{if } x \notin [-1, 1] \times K_i \\ \big(-1 + \frac{1+g_i(y)}{1+g_{i-1}(y)} \cdot (t+1), y\big), & \text{if } x = (t, y) \in [-1, 1] \times K_i. \end{cases} \end{array}$$

It follows easily from the fact that $\operatorname{supp}(f_i) \subset K_i$, the Product Topology-Basics Lemma 5.4 (3) and Pasting Proposition 2.6 that this map is continuous and it follows from the Compact-Hausdorff Proposition 2.17 (2) that this map is in fact a homeomorphism. ⊞

It is now clear that $F := F_m \circ \cdots \circ F_1 \colon M \to W$ is a homeomorphism which has the feature that $F(\partial M) = \{1\} \times \partial M$. ■



Proof of the Topological Collar Neighborhood Theorem 75.5. Now let M be any topological manifold. The proof of the Topological Collar Neighborhood Theorem 75.5 in

this setting is not that different from the compact case. For space and time reasons we only sketch the argument.

It follows from the fact that ∂M is in particular second-countable together with the Manifold-Partitions of Unity Existence Theorem 21.1 that there exists an infinite sequence $K_1, K_2, \ldots$ of compact subsets, embeddings $\Phi_i \colon K_i \times [-1,0] \to M$ and maps $f_i \colon \partial M \to [0,1]$ such that beyond the aforementioned Statements (1), (2) and (3) the following extra condition is satisfied:

(4) For each $x \in \partial M$ there exists an open neighborhood V such that there exist only finitely many $i \in \mathbb{N}$ with $\operatorname{supp}(f_i) \cap U \neq \varnothing$.

As in the previous argument we now define functions $g_1, g_2, \ldots$, subsets $W_1, W_2, \ldots$ and homeomorphisms $F_i \colon W_{i-1} \to W_i$. It follows from (4) that the map

$$\begin{array}{rcll} F \colon M & \to & W & \\ x & \mapsto & (F_k \circ \cdots \circ F_1)(x) & \text{if } f_i(x) = 0 \text{ for } i \geq k \end{array}$$

is well-defined and continuous. In fact using the Closed Embedding-to-$\mathbb{R}^n$ Proposition 2.19 it is straightforward to see that F is actually a homeomorphism. Furthermore, note that by construction we again have $F(\partial M) = \{1\} \times \partial M$. Thus, as in the proof of the compact case, we are done. ■

The following uniqueness statement for collar neighborhoods is almost verbatim the same as the statement of the Uniqueness of Collar-Proposition **??**.

Proposition 75.6. Let M be a topological manifold and let $f_0, f_1 \colon [0,1] \times \partial M \to M$ be two embeddings that are the identity map on $\{0\} \times \partial M = \partial M$. If ∂M is compact, then there exists an isotopy

$$H \colon M \times [0,1] \ \to \ M$$

rel ∂M from $H_0 = \mathrm{id}_M$ to a homeomorphism $\Phi = H_1 \colon M \to M$ with $\Phi \circ f_0 = f_1$. In particular Φ restricts to a homeomorphism

$$\Phi \colon f_0([0,1] \times \partial M) \ \to \ f_1([0,1] \times \partial M).$$

Remark. In the study of smooth manifold several constructions, in particular the gluing construction, depends on the choice of a smooth collar. The Uniqueness of Collar-Proposition **??** then shows that these constructions are well-defined up to diffeomorphism. In the context of topological manifolds constructions usually do *not* depend on the choice of a collar. Thus the uniqueness statement of Proposition 75.6 is nice to know, but we will never need to make use of it.

Proof. The proof is sketched in [**KSi1969**, Essay I, Theorem A.2]. The proof is also given in [**Arm1970**, Theorem 2].[912] ■

Convention. Suppose we are given a topological manifold M with compact boundary ∂M. We pick a map $\Phi \colon [0,1] \times \partial M \to M$ as in the conclusion of the Topological Collar

[912] The proof in [**Arm1970**] makes me slightly nervous, there is no hypothesis that ∂M is compact, and there is no statement that the image of $[0,1] \times \partial M$ needs to be closed. But that would imply that the "usual" collar neighborhood for $M = \{(x,y) \,|\, y \geq 0\}$ is equivalent to the one shown in the figure below, which is certainly *not* the case.

Neighborhood Theorem 75.5. Usually we identify $[0,1] \times \partial M$ with the image in M under the map Φ and we just refer to $[0,1] \times \partial M$ as a "collar neighborhood of ∂M". Sometimes we also find it convenient to use other intervals in our notation, e.g. we might write "let $[-3,0] \times \partial M$ be a collar neighborhood". It is then understood that $\{0\} \times \partial M = \partial M$.

For later we record the following corollary to the Topological Collar Neighborhood Theorem 75.5 which is the analogue of the Basics-of-Collars Lemma 28.4 and the Push-Away-from-Boundary Corollary 28.5.

Corollary 75.7. Let M be an n-dimensional topological manifold. Let $[0,1] \times \partial M$ be a collar neighborhood.

(1) The subset $W := M \setminus ([0,1) \times \partial M)$ is an n-dimensional topological manifold with boundary given by $\{1\} \times \partial M$.
(2) The subset $W = M \setminus ([0,1) \times \partial M)$ is a deformation retract of M and it is a deformation retract of $M \setminus \partial M$.
(3) If M is non-empty, then M is homotopy equivalent to a non-compact n-dimensional topological manifold without boundary.

Proof. The proof of Corollary 75.7 is basically the same as the proof of the Basics-of-Collars Lemma 28.4 and the Push-Away-from-Boundary Corollary 28.5. We just need to replace the Smooth Collar Neighborhood Theorem 28.3 by the Topological Collar Neighborhood Theorem 75.5 and we need to replace the Smooth Manifold Boundary Proposition 19.26 by the Topological Manifold Boundary Proposition 75.2. ■

We conclude this discussion of the Topological Collar Neighborhood Theorem 75.5 with the following proposition that looks uncannily similar to the Gluing-Smooth Manifolds-Proposition **??**.

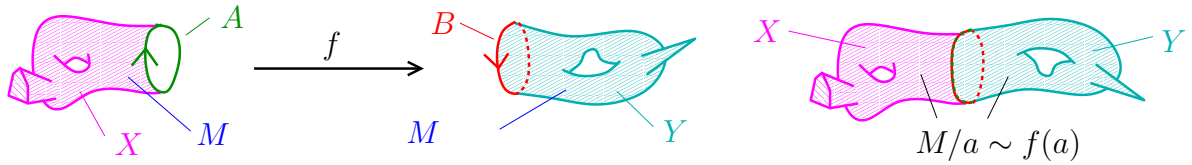
Proposition 75.8. (Gluing-Topological Manifolds-Proposition) Let M be an n-dimensional topological manifold (we do not assume that M is connected). Let A and B be disjoint unions of boundary components of M. Furthermore let $f\colon A \to B$ be a homeomorphism. Then the following two statements hold:

(1) The topological space
$$M/a \sim f(a) \quad \text{for } a \in A$$
is an n-dimensional topological manifold.
(2) If M consists of two components X and Y and if $A \subset X$ and $B \subset Y$, then the images of X and Y in $M/a \sim f(a)$ are codimension-zero submanifolds.
(3) If M is compact, then $M/a \sim f(a)$ is also compact.
(4) The boundary of $M/a \sim f(a)$ is given by $\partial M \setminus (A \cup B)$.
(5) If each component of M contains at least one component of A or B, then $M/a \sim f(a)$ is connected.

In Chapter 25 we will introduce the notion of an orientation of a topological manifold. To preserve the symmetry with the Gluing-Smooth Manifolds-Proposition **??** we add the

following statement, even though with the present knowledge it does not have any meaning yet.

(6) Assume that M is oriented. We equip A and B with the boundary orientations, as defined in the Topological Manifold-Boundary Orientation Proposition 105.19.
  (a) If f is orientation-reversing, then $M/a \sim f(a)$ admits an orientation such that the inclusion $M \setminus (A \cup B) \to M/a \sim f(a)$ is orientation-preserving.
  (b) If M is connected and if f is not orientation-reversing, then $M/a \sim f(a)$ is non-orientable.



Proof. For the most part the proof of the Gluing-Topological Manifolds-Proposition 75.8 is almost the same as the proof of the Gluing-Smooth Manifolds-Proposition **??**. For statements (1) to (5) we just need to make the following replacements:

(1) We need to replace the Smooth Manifold Boundary Proposition 19.26 (3a) by Lemma 18.6.
(2) We need to replace the Smooth Collar Neighborhood Theorem 28.3 by the Topological Collar Neighborhood Theorem 75.5.
(3) We need to replace the Smooth Manifold Boundary Proposition 19.26 (1) by the Topological Manifold Boundary Proposition 75.2 (1).

The alert reader will notice that in the proof of the Gluing-Topological Manifolds-Proposition 75.8 it is not really necessary to use the Topological Collar Neighborhood Theorem 75.5. For example, as we will see in Exercise 75.5, one can define an atlas for $M/a \sim f(a)$ "by hand". Nonetheless we leave the proof as it is, since the Topological Collar Neighborhood Theorem 75.5 greatly simplifies the discussion.

Finally we will prove Statement (6) in Exercise 105.4. ■

75.4. **Variations on the Excision Theorem 74.17.** In this section we will state and prove two variations on the Excision Theorem 74.17. First we have the following rather technical sounding theorem.

Theorem 75.9. Let X be a topological space and let $Z \subset A \subset X$ be two subsets. We suppose that there exists a subset $U \subset Z$ such that the closure of U is contained in the interior of A and such that the following statements hold:

(1) $X \setminus Z$ is a deformation retract of $X \setminus U$
(2) $A \setminus Z$ is a deformation retract of $A \setminus U$.

Then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} \mathrm{H}_n(X, A).$$

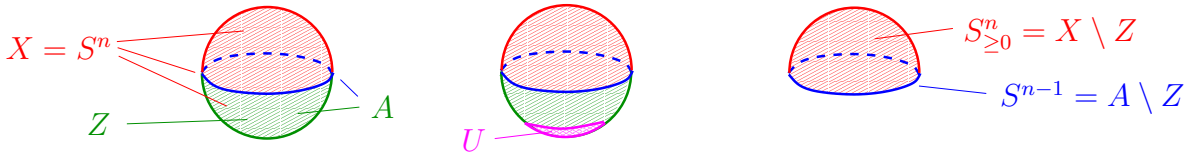
The hypotheses of Theorem 75.9 might sound complicated, but as we will see, in practice these conditions are often satisfied.

Example. Let $n \in \mathbb{N}$. As usual we denote by $S^n_{\geq 0}$ and $S^n_{\leq 0}$ the upper and lower hemisphere of S^n and we make the obvious identification $S^{n-1} = S^n_{\geq 0} \cap S^n_{\leq 0}$. It follows from

Theorem 75.9 and the choices of X, A, Z and U illustrated in the figure below that the inclusion induced map

$$\mathrm{H}_n(S^n_{\geq 0}, S^{n-1}) \;\to\; \mathrm{H}_n(S^n, S^n_{\leq 0})$$

is an isomorphism.

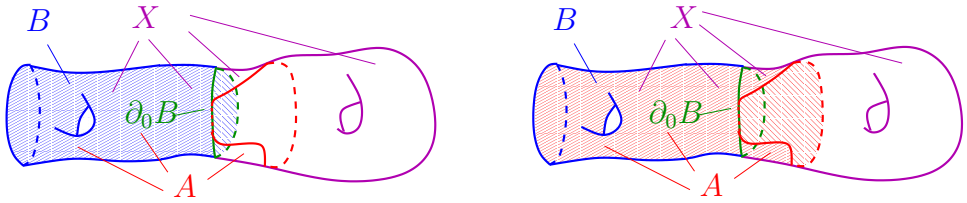


We can easily generalize the above example. More precisely, Theorem 75.9 also allows us to prove the following variation on the Excision Theorem 74.17. It says, under mild technical hypotheses, that in the relative homology of an m-dimensional topological manifold we can excise the interior of a codimension-zero submanifold.

Theorem 75.10. (Excision Theorem for Topological Manifolds) Let X be an m-dimensional topological manifold and let A be a subset of X. Furthermore let B be an m-dimensional submanifold of X that is a closed subset of X. We write $\mathring{B} := B \setminus \partial_0 B$.[913] If $\mathring{B} \subset A$, then for every $n \in \mathbb{N}_0$ the inclusion induced map

$$\mathrm{H}_n\big(X \setminus \mathring{B}, A \setminus \mathring{B}\big) \;\xrightarrow{\cong}\; \mathrm{H}_n(X, A)$$

is an isomorphism.



Example. If we take $X = S^n$, $A = S^n_{\leq 0}$ and $B = S^n_{\leq 0}$, then all the hypothesis of the Excision Theorem 75.10 for Topological Manifolds are satisfied and we obtain, as above, that the inclusion induced map

$$\mathrm{H}_n(S^n_{\geq 0}, S^{n-1}) \;\to\; \mathrm{H}_n(S^n, S^n_{\leq 0})$$

is an isomorphism.

Remark. In Theorem 120.2 we will generalize the statement of the Excision Theorem 75.10 for Topological Manifolds to the setting that B is a submanifold "with corner".

Proof of Theorem 75.10 assuming Theorem 75.9. Note that by Proposition 75.3 we know that $\partial_0 B$ is a union of boundary components of B and that $\mathring{B}$ is indeed the interior of B viewed as a subset of X. In particular we can apply the Topological Collar Neighborhood Theorem 75.5 to $\partial_0 B$. We obtain an embedding $\varphi\colon [-1,0] \times \partial_0 B \to B$ that is the identity map on $\{0\} \times \partial_0 B = \partial_0 B$. To simplify the notation we identify $[-1,0] \times \partial_0 B$ with its image in B. We set $Z = \mathring{B} = B \setminus (\{0\} \times \partial_0 B)$ and we set $U = B \setminus ((-1,0] \times \partial_0 B)$.

[913] We refer to page 1658 for the definition of $\partial_0 B$.

Claim. There exists a deformation retract from $X \setminus U$ to $X \setminus Z$.

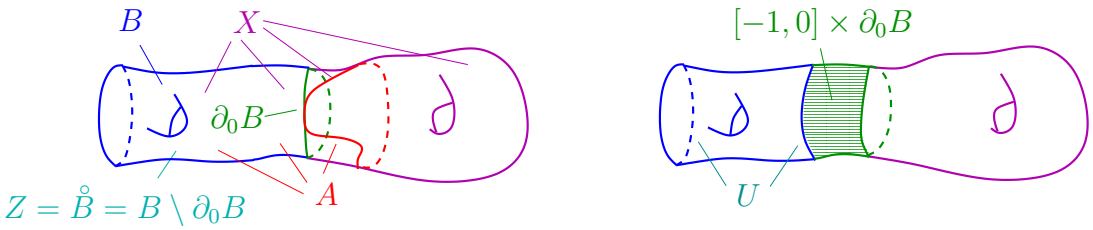
Proof. First note that we can write $X \setminus U = (X \setminus Z) \cup ([-1,0] \times \partial_0 B)$. Furthermore note that $(X \setminus Z) \cap ([-1,0] \times \partial_0 B) = \{0\} \times \partial_0 B$. The idea is to use Lemma 52.11 to conclude that the obvious deformation retraction from $[-1,0] \times \partial_0 B$ to $\{0\} \times \partial_0 B$ gives rise to the desired deformation retraction.

We still have to perform the task of verifying the hypotheses of Lemma 52.11, i.e. we need to argue that $X \setminus Z$ and $[-1,0] \times \partial_0 B$ are closed subsets of $X \setminus U$. First note that by Proposition 75.3 we know that $Z = \mathring{B}$ is an open subset of X. It follows that $X \setminus Z$ is a closed subset of $X \setminus U$. Furthermore by Statement (4) of the Topological Collar Neighborhood Theorem 75.5 we know that $[-1,0] \times \partial_0 B$ is a closed subset of X, in particular it is a closed subset of $X \setminus U$. ⊞

Claim. There exists a deformation retract from $A \setminus U$ to $A \setminus Z$.

Proof. Since $\mathring{B} \subset A$ we have $A \setminus U = (A \setminus Z) \cup ([-1,0] \times \partial_0 B)$. Furthermore note that $(A \setminus Z) \cap ([-1,0] \times \partial_0 B) = \{0\} \times \partial_0 B$. Thus basically the same argument as in the proof of the previous claim shows that such a deformation retraction exists. ⊞

The statement of Excision Theorem 75.10 for Topological Manifolds now follows immediately from Theorem 75.9. ■



We conclude this section with the proof of Theorem 75.9.

Proof of Theorem 75.9. We consider the following commutative diagram of inclusion induced maps

$$\begin{array}{ccc} \mathrm{H}_n(X \setminus Z, A \setminus Z) & \longrightarrow & \mathrm{H}_n(X, A) \\ \downarrow & \nearrow_{\cong} & \\ \mathrm{H}_n(X \setminus U, A \setminus U). & & \end{array}$$

The diagonal map is an isomorphism by the Excision Theorem 74.17. Thus it suffices to show that the vertical map is an isomorphism.

We consider the following diagram:

$$\begin{array}{ccccccccc} \mathrm{H}_n(A \setminus Z) & \longrightarrow & \mathrm{H}_n(X \setminus Z) & \longrightarrow & \mathrm{H}_n(X \setminus Z, A \setminus Z) & \longrightarrow & \mathrm{H}_{n-1}(A \setminus Z) & \longrightarrow & \mathrm{H}_{n-1}(X \setminus Z) \\ \downarrow{=} & & \downarrow & & \downarrow & & \downarrow{=} & & \downarrow \\ \mathrm{H}_n(A \setminus Z) & \longrightarrow & \mathrm{H}_n(X \setminus U) & \longrightarrow & \mathrm{H}_n(X \setminus U, A \setminus Z) & \longrightarrow & \mathrm{H}_{n-1}(A \setminus Z) & \longrightarrow & \mathrm{H}_{n-1}(X \setminus U) \\ \downarrow & & \downarrow{=} & & \downarrow & & \downarrow & & \downarrow{=} \\ \mathrm{H}_n(A \setminus U) & \longrightarrow & \mathrm{H}_n(X \setminus U) & \longrightarrow & \mathrm{H}_n(X \setminus U, A \setminus U) & \longrightarrow & \mathrm{H}_{n-1}(A \setminus U) & \longrightarrow & \mathrm{H}_{n-1}(X \setminus U). \end{array}$$

Here the horizontal maps are pieces of the long exact sequences of the various pairs of topological spaces provided by the Topological LES-Corollary 74.14 (1). Note that by the Topological LES-Corollary 74.14 (2) we know that the diagram commutes.

Furthermore the vertical maps are induced by the various inclusion maps. On the upper level the first and the fourth vertical maps are the identity. Furthermore it follows from the hypothesis that $X \setminus Z$ is a deformation retract of $X \setminus U$ and from Corollary 74.16 that the second and the fifth vertical map are also isomorphisms. It follows from the Five Lemma 74.10 that the upper middle vertical map is also an isomorphism.

Using the hypothesis that $A \setminus Z$ is a deformation retract of $A \setminus U$ one can show exactly the same way that the lower middle vertical map is also an isomorphism. Combining the two middle vertical maps we obtain the desired isomorphism. ■

75.5. **The double of topological and smooth manifolds.** In this short section we introduce the double of a topological manifold. This construction associates to a compact manifold with non-empty boundary a closed manifold. This procedure comes in handy on several occasions. In particular sometimes this procedure allows us to reduce the proof of a certain statement for compact manifolds to the closed case. As the reader will notice, this section has only a moderately convincing justification for fitting into this particular chapter, but the material has to go somewhere, and some of the more technical statements in this section make use of the earlier results of this section.
After this apology, let us give the key definition of this section.

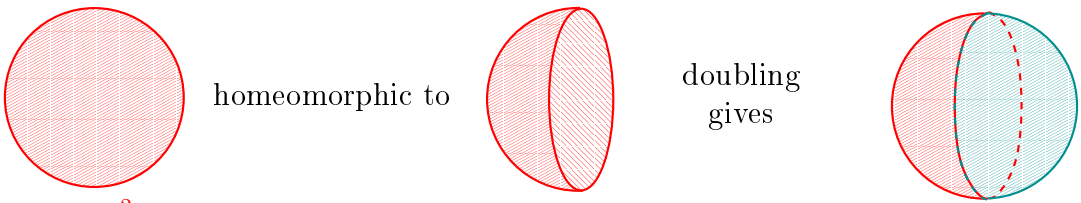
Definition. Let M be a topological manifold and let W be the union of some components of ∂M. We refer to

$$\mathrm{D}_W M \;:=\; (M \times \{1\}) \cup_{W \times \{1\} = W \times \{2\}} (M \times \{2\})$$

as the double of M along W. We refer to $\mathrm{D}M := D_{\partial M} M$ as the double of M.

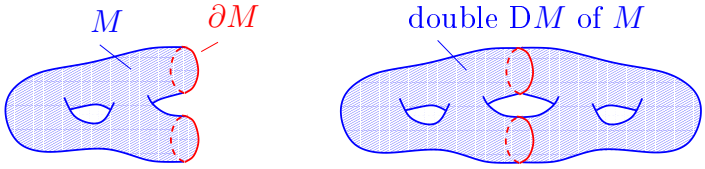
Example. The double of M is given by taking two copies of M and identifying the boundary points "left and right" in the obvious way.

(1) On page 152 we gave an explicit homeomorphism $\overline{B}^n \xrightarrow{\cong} S^n_{\leq 0}$. Using this homeomorphism it is straightforward to see that the double of the n-ball $\overline{B}^n$ is homeomorphic to the n-dimensional sphere S^n. We refer to the figure below for an illustration.



disk $\overline{B}^2$ — lower hemisphere $S^2_{\leq 0}$ — double is homeomorphic to S^2

(2) the figure below makes a convincing case that the double of the torus minus two open disks is homeomorphic to the surface of genus 3. We refer to the figure below for an illustration.



(3) In Exercise 75.7 we will see that the double of the Möbius band and the Klein bottle minus an open disk are homeomorphic to well-known topological spaces.

The following definition allows us to relate a topological manifold to its double.

Definition. Let M be a topological manifold and let W be the union of some components of ∂M. We refer to the map

$$i\colon M \xrightarrow{P\mapsto(P,1)} M\times\{1\} \to (M\times\{1\})\sqcup(M\times\{2\}) \to \mathrm{D}_W M$$

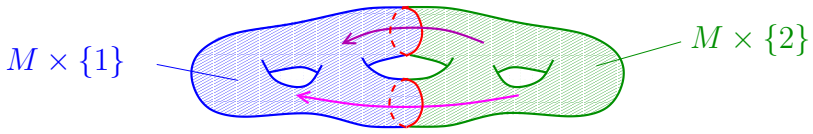
as the natural inclusion. Furthermore we refer to the map

$$\begin{aligned} r\colon \mathrm{D}_W M = ((M\times\{1\})\cup(M\times\{2\}))/\!\sim &\to M \\ [(P,i)] &\mapsto P \end{aligned}$$

as the folding map. Finally we refer to the map

$$\begin{aligned} \rho\colon \mathrm{D}_W M &\to \mathrm{D}_W M \\ [(P,i] &\mapsto [(P,3-i)] \end{aligned}$$

as the natural involution on $\mathrm{D}_W M$.



Lemma 75.11. Let M be a topological manifold and let W be the union of some components of ∂M.

(1) The natural inclusion $i\colon M\to \mathrm{D}_W M$ is an embedding.
(2) The folding map $\mathrm{D}_W M\to M$ is continuous and it satisfies $r\circ i=\mathrm{id}_M$.
(3) The natural involution $\mathrm{D}_W M\to \mathrm{D}_W M$ is continuous and it is an involution, i.e. $\rho\circ\rho=\mathrm{id}$.

Proof. It follows from the Topological-Coproduct Proposition 5.11 and the Topological-Quotient Proposition 5.15 (1a) that the natural inclusion i is continuous and that the involution ρ are continuous. It is clear that $\rho\circ\rho=\mathrm{id}$. Furthermore it follows from the Topological-Coproduct Proposition 5.11 and the Topological-Quotient Proposition 5.15 (1b) that the folding map r is continuous. Clearly we have $r\circ i=\mathrm{id}_M$. This shows in particular that $i\colon M\to i(M)$ is a homeomorphism, i.e. i is an embedding.

Finally it follows from the Topological-Coproduct Proposition 5.11, the Topological-Quotient Proposition 5.15 (1a) and the Topological-Quotient Proposition 5.15 (1b) that the involution ρ is continuous. It is clear that $\rho\circ\rho=\mathrm{id}$. ■

Convention. Sometimes we will use the fact, established in Lemma 75.11, that the natural inclusion $M\to \mathrm{D}_W M$ is an embedding as an excuse to view M as a subspace of $\mathrm{D}_W M$ and to view $r\colon \mathrm{D}_W M\to M$ as a retraction.

The following lemma summarizes a few key properties of the doubling operation.

Lemma 75.12. (Doubling-Manifold Lemma) Let M be an n-dimensional topological (smooth) manifold and let W be the union of some components of ∂M. The following statements hold:

(1) (a) The double $\mathrm{D}_W M$ along W is an n-dimensional topological manifold such that $M \subset \mathrm{D}_W M$ is a submanifold and such that M is a closed subset of $\mathrm{D}_W M$.
   (b) If M is actually a smooth manifold, then the double $\mathrm{D}_W M$ is naturally an n-dimensional smooth manifold such that $M \subset \mathrm{D}_W M$ is a smooth submanifold and such that the natural inclusion $i\colon M \to \mathrm{D}_W M$ is a smooth embedding.[914]

(2) The boundary of the double $\mathrm{D}_W M$ is given by two copies of $\partial M \setminus W$. In particular the boundary of the double $\mathrm{D}M$ is empty.

(3) M is compact if and only if the double $\mathrm{D}_W M$ is compact.

(4) If M is connected and if $W \neq \varnothing$, then the double $\mathrm{D}_W M$ is also connected.

(5) If M is an oriented smooth manifold, then $\mathrm{D}_W M$ admits an orientation such that $M \to \mathrm{D}_W M$ is an orientation-preserving embedding. Conversely, if $\mathrm{D}_W M$ is oriented, then M admits an orientation such that the inclusion $M \to \mathrm{D}_W M$ is orientation-preserving.[915]

(6) For every $k \in \mathbb{N}_0$ the inclusion induced map

$$\mathrm{H}_k(M, W) \;\to\; \mathrm{H}_k(\mathrm{D}_W M, M)$$

is an isomorphism.

Proof. Let M be an n-dimensional topological (smooth) manifold. Statements (1), (2) and (4) follow immediately from applying the Gluing-Smooth Manifolds-Proposition **??** and the Gluing-Topological Manifolds-Proposition 75.8 iteratively to $(W \times \{1\}) \sqcup (W \times \{2\}) \subset (M \times \{1\}) \sqcup (M \times \{2\})$.

We turn to the proof of Statement (3). Note that it follows easily from Lemmas 75.11, 2.13 and 1.20 that M is compact if and only if the double $\mathrm{D}_W M$ is compact.

Next we provide the proof of Statement (5). Thus we suppose that M is an oriented smooth manifold. We equip $M \times \{1\}$ with the obvious orientation. Furthermore we equip $M \times \{2\}$ with the opposite orientation. With these orientations it is straightforward to see that the map $\partial(M \times \{1\}) \xrightarrow{(P,1) \mapsto (P,2)} \partial(M \times \{2\})$ is orientation-reversing. Thus it follows from the Gluing-Smooth Manifolds-Proposition **??** that the double $\mathrm{D}_W M$ admits an orientation such that $M \to \mathrm{D}_W M$ is an orientation-preserving embedding. Conversely, if $\mathrm{D}_W M$ is oriented, then it follows from the Basics-of-Orientations Lemma 25.13 (4) that M inherits an orientation such that the inclusion $M \to \mathrm{D}M$ is orientation-preserving.

Finally note that Statement (6) is an immediate consequence of the Excision Theorem 75.10 for Topological Manifolds, and the fact, established in (1), that M is a closed subset of $\mathrm{D}M$. ■

The following lemma relates some of the algebraic invariants of a manifold M to the invariants of its double $\mathrm{D}M$.

[914] Note though that the folding map $r\colon \mathrm{D}_W M \to M$ is *not* smooth.

[915] In the Double-Orientability Lemma 105.18 we will prove an analogous statement for topological manifolds.

Lemma 75.13. Let M be a topological manifold and let W be the union of some components of ∂M.

(1) For every $j \in \mathbb{N}_0$ the inclusion induced map $\mathrm{H}_j(M) \to \mathrm{H}_j(\mathrm{D}_W M)$ is a monomorphism.
(2) If ∂M is non-empty and if x_0 is a point in ∂M, then for every $j \in \mathbb{N}_0$ the inclusion induced map $\pi_j(M, x_0) \to \pi_j(\mathrm{D}_W M, x_0)$ is a monomorphism.

Remark. In the Homology-of-Double Proposition 141.2 we will calculate the homology groups of $\mathrm{D}M$ in terms of the homology groups of M.

Proof. Let $i\colon M \to \mathrm{D}_W M$ be the natural inclusion map and let $r\colon \mathrm{D}_W M \to M$ be the folding map. In Lemma 75.11 we saw that both maps are continuous and that $r \circ i = \mathrm{id}_M$. The two statements now follow as usual from the functoriality of homology groups and homotopy groups. More precisely, in both cases we have $r_* \circ i_* = \mathrm{id}_*$, i.e. i_* needs to be a monomorphism. ■

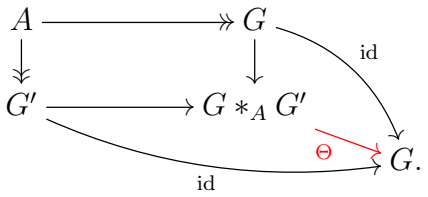
In favorable circumstances we can fully control the fundamental group of the double.

Proposition 75.14. Let M be a compact connected n-dimensional topological manifold, let W be a component of ∂M and let $P \in W$. If the inclusion induced map $\pi_1(W, P) \to \pi_1(M, P)$ is an epimorphism, then the inclusion induced map $\pi_1(M, P) \to \pi_1(\mathrm{D}_W M, P)$ is an isomorphism.

The proof of Proposition 75.14 results on the following group-theoretic lemma, which is precisely the content of Exercise 53.13.

Lemma 75.15. Let A and G be groups and let $\varphi\colon A \to G$ be an epimorphism. Furthermore let G' be a copy of G. We consider the amalgamated product $G *_A G'$ given by using twice the epimorphism $\varphi\colon A \to G$ and $\varphi\colon A \to G' = G$. In this setting the natural homomorphism $G \to G *_A G'$ is an isomorphism.

Proof of Lemma 75.15. By Proposition 53.21 there exists a unique homomorphism $\Theta\colon G *_A G' \to G'$ that makes the following diagram commutes:



In particular we see that the right vertical map is a monomorphism. The left vertical map is by hypothesis an epimorphism. It follows from the Amalgamated Product Lemma 53.22 (3) that the right vertical map is also an epimorphism. Thus we shown that the right vertical map is indeed an isomorphism. ■

Proof of Proposition 75.14. Let $P \in W$. Since W is non-empty and connected we obtain from the Seifert-van Kampen Theorem 54.4 that the inclusions of the two copies of M into the double $\mathrm{D}_W M$ induce an isomorphism $\pi_1(\mathrm{D}M, P) \cong \pi_1(M, P) *_{\pi_1(W,P)} \pi_1(M, P)$. By our hypothesis the map $\pi_1(W, P) \to \pi_1(M, P)$ is an epimorphism. It follows from the above, together with Lemma 75.15, that the map $\pi_1(M, P) \to \pi_1(\mathrm{D}_W M, P)$ is indeed an epimorphism. ■

We conclude this introduction to the double of a topological manifold with the following harmless but at times useful lemma. The lemma follows almost immediately from the definitions, hence we leave it to the conscientious reader to fill in the details.

Lemma 75.16. (Doubling-Homology Lemma) Let M be a topological manifold and let W be the union of some components of ∂M. Given $i \in \{1,2\}$, we denote by $j_i \colon M \to M \times \{i\} \to \mathrm{D}_W M$ the obvious map.

(1) For every $k \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{D} \colon \mathrm{C}_k(M,W) & \to & \mathrm{C}_k(\mathrm{D}_W M) \\ \big[\underbrace{\sigma}_{\in \mathrm{C}_k(M)} \big] & \mapsto & j_{1*}(\sigma) - j_{2*}(\sigma) \end{array}$$

is well-defined.

(2) The maps from (1) define a natural chain map, in particular for every $k \in \mathbb{N}_0$ these maps induce a map

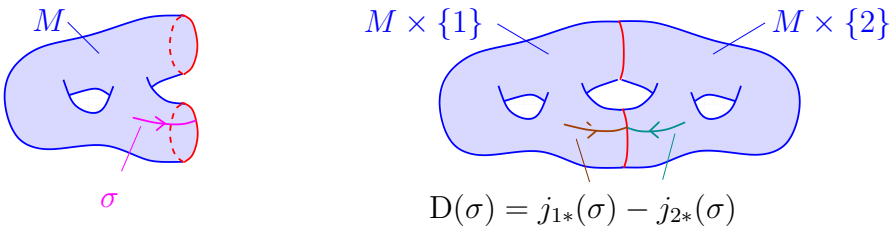
$$\mathrm{D} \colon \mathrm{H}_k(M,W) \to \mathrm{H}_k(\mathrm{D}_W M)$$

which is called the doubling homomorphism.

(3) We make the identification $M = j_1(M \times \{1\})$ and we write $M' = j_2(M \times \{2\})$. The doubling homomorphism has the property that the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}_k(M,W) & \xrightarrow{\quad \mathrm{D} \quad} & \mathrm{H}_k(\mathrm{D}_W M) \\ & \searrow \quad \swarrow & \\ & \mathrm{H}_k(\mathrm{D}_W M, M'), & \end{array}$$

where the diagonal maps are the obvious maps. (As an aside we mention that it follows from the Excision Theorem 75.10 that the left diagonal map is an isomorphism.)



Exercises for Chapter 75.

Exercise 75.1. Let M be a topological manifold. Furthermore let $x_1, \ldots, x_k$ be distinct points in M. Let $n \in \mathbb{N}_0$. For $i \in \{1, \ldots, k\}$ we denote by

$$\nu_i \colon \mathrm{H}_n(M, M \setminus \{x_1, \ldots, x_k\}) \to \mathrm{H}_n(M, M \setminus \{x_i\})$$

the map that is induced by the inclusion $M \setminus \{x_1, \ldots, x_k\} \to M \setminus \{x_i\}$. Show that the map

$$\bigoplus_{i=1}^{k} \nu_i \colon \mathrm{H}_n(M, M \setminus \{x_1, \ldots, x_k\}) \to \bigoplus_{i=1}^{k} \mathrm{H}_n(M, M \setminus \{x_i\})$$

is an isomorphism.

Hint. Use Excision Theorem 74.18.

Exercise 75.2. Let X be a finite topological graph, in other words, let X be a finite 1-dimensional CW-complex. Given any point $x \in X$ compute the local homology groups $\mathrm{H}_*(X, X \setminus \{x\})$.

Exercise 75.3. Let $n_1, \ldots, n_k \in \mathbb{N}$. We form the wedge $X := S^{n_1} \vee \cdots \vee S^{n_k}$. We denote by P the wedge point. Compute the local homology groups $\mathrm{H}_*(X, X \setminus \{P\})$.

Exercise 75.4. Let X be a topological space and let $n \in \mathbb{N}_0$. We consider the following two maps.

(1) Let $p\colon X \times S^1 \to X$ be the projection.
(2) We consider the homomorphism $\psi\colon \mathrm{H}_n(X \times S^1) \to \mathrm{H}_{n-1}(X)$ that is given by

$$\begin{aligned} \mathrm{H}_n(X \times S^1) \;&\to\; \mathrm{H}_n(X \times S^1, X \times S^1_{\leq 0}) \;\underset{\text{isomorphism by the Excision Theorem 75.9}}{\overset{\cong}{\leftarrow}}\; \mathrm{H}_n(X \times S^1_{\geq 0}, X \times \{\pm 1\}) \\ &\underset{\text{connecting homomorphism}}{\to} \mathrm{H}_{n-1}(X \times \{\pm 1\}) \underset{\text{projection induced homomorphism}}{\to} \mathrm{H}_{n-1}(X \times \{1\}) \;=\; \mathrm{H}_{n-1}(X). \end{aligned}$$

Show that the map

$$p_* \oplus \psi\colon \mathrm{H}_n(X \times S^1) \;\to\; \mathrm{H}_n(X) \oplus \mathrm{H}_{n-1}(X)$$

is an isomorphism.
Hint. Consider the long exact sequence of the pair $(X \times S^1, X \times S^1_{\leq 0})$.
Remark. In Lemma 78.21 we will give a different approach to calculating $\mathrm{H}_n(X \times S^1)$.

Exercise 75.5. Let M be an n-dimensional topological manifold and let A and B be disjoint unions of boundary components of M. Furthermore let $f\colon A \to B$ be a homeomorphism. Show, without using the Topological Collar Neighborhood Theorem 75.5, that the topological space

$$M/a \sim f(a) \quad \text{for } a \in A$$

is an n-dimensional topological manifold.

Exercise 75.6. We recall the following definitions:

(1) The suspension of a topological space X is defined to be the topological space

$$\Sigma(X) \;:=\; (X \times [-1, 1])/\!\sim$$

where we identify all points in $X \times \{-1\}$ and we identify all points in $X \times \{1\}$.
(2) A topological space X that has the property that for each $x \in X$ we have

$$\mathrm{H}_k(X, X \setminus \{x\}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise} \end{cases}$$

is called an *n-dimensional homology manifold*.
(3) We say that an n-dimensional topological manifold X is a *topological homology n-sphere* if for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X) \cong \mathrm{H}_k(S^n)$.

Let Y be a homology n-sphere.

(a) Show that $Y \times \mathbb{R}$ is an $(n+1)$-dimensional homology manifold.
(b) Show that the suspension $\Sigma(Y)$ of Y is an $(n+1)$-dimensional homology manifold.
(c) Show that if $\pi_1(Y)$ is non-trivial, then the suspension $\Sigma(Y)$ is *not* a topological manifold.

Remark. In Proposition 98.11 we will see that there exists a closed orientable 3-dimensional smooth manifold Y that is a homology 3-sphere, but such that $\pi_1(Y)$ is a non-trivial group. Together with the above this shows that there exist homology manifolds that are not topological manifolds.

Exercise 75.7. What is a familiar topological space that is homeomorphic to the double of the Möbius band? Furthermore, which one is homeomorphic to the double of the Klein bottle minus an open disk?

Exercise 75.8. Let M be a compact connected n-dimensional topological manifold such that ∂M has $k \in \mathbb{N}$ components.

(a) Show that $\operatorname{rank}(\mathrm{H}_1(\mathrm{D}M)) \geq k-1$.
(b) Show that $\pi_1(M)$ admits an epimorphism onto a free group of rank $k-1$.

Exercise 75.9. Let X be a topological space and let $x \in X$. On page 319 we defined the *topological valence of* x as

$$\operatorname{topval}(x,X) := \min\left\{ n \in \mathbb{N} \cup \{\infty\} \;\middle|\; \begin{array}{c} \text{for every open neighborhood } U \text{ of } x \text{ there} \\ \text{exists an open neighborhood } V \text{ of } x \text{ with } V \subset U \\ \text{such that } V \setminus \{x\} \text{ has at least } n \text{ components} \end{array} \right\}.$$

(a) Show that if X is locally contractible and if x is not an isolated point, then

$$\operatorname{topval}(x,X) \;=\; \operatorname{rank}(\mathrm{H}_1(X, X \setminus \{x\})).$$

(b) Show that the conclusion of (a) does not necessarily hold if X is not locally contractible.

Exercise 75.10. Let M be a compact smooth manifold. Show that there exists a compact smooth manifold W with $\partial W = \mathrm{D}M$.

Exercise 75.11. Let M be a topological manifold, let X be a Rham co, let $A \subset X$ be closed and let $f\colon X \to M$ be a map with $f(A) \subset M \setminus \partial M$. Use the Topological Collar Neighborhood Theorem 75.5 to show that f is homotopic rel A to a map $g\colon X \to M \setminus \partial M$.
Remark. This exercise gives a new proof of the Push Away From Boundary Lemma 18.13. In particular one can use it to show again that if M is path-connected, then $M \setminus \partial M$ is also path-connected.
Remark. The exercise is the topological analogue of Exercise 28.2.

76. Homology and self-maps of spheres

By Proposition 74.4 we know that for any $n \in \mathbb{N}_0$ we have

$$\widetilde{\mathrm{H}}_k(S^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

Thus we have determined the homology groups of all finite-dimensional spheres. In this chapter, given $n \in \mathbb{N}_0$, we will determine an explicit singular n-chain which is a generator of $\widetilde{\mathrm{H}}_n(S^n)$. Furthermore, given a map $f\colon S^n \to S^n$ we will get to know several techniques for determining the induced map $f_*\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n)$. These results will allow us to prove various interesting results, e.g. the mysterious sounding Hair Ball Theorem. The results of this chapter will also be essential later on in Chapter 80 when we will introduce cellular homology.

76.1. **Explicit generators of relative homology groups.** We start out with the following elementary lemma that we will use explicitly and implicitly on many occasions.

Lemma 76.1. Let $n \in \mathbb{N}_0$. We have

$$\widetilde{\mathrm{H}}_k(\overline{B}^n, S^{n-1}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We consider the long exact sequence in reduced homology of the pair $(\overline{B}^n, S^{n-1})$:

$$\ldots \to \underbrace{\widetilde{\mathrm{H}}_k(\overline{B}^n)}_{=0,\ \text{see page 1612}} \to \mathrm{H}_k(\overline{B}^n, S^{n-1}) \xrightarrow{\partial} \underbrace{\widetilde{\mathrm{H}}_{k-1}(S^{n-1})}_{\substack{\text{calculated in} \\ \text{Proposition 74.4}}} \to \underbrace{\widetilde{\mathrm{H}}_{k-1}(\overline{B}^n)}_{=0,\ \text{see page 1612}} \to \ldots$$

The statement of the lemma is now an immediate consequence of Proposition 74.4. ∎

We continue with the following lemma.

Lemma 76.2. Identity-is-Generator Lemma) For any $n \in \mathbb{N}_0$ the following two statements hold:

(1) $\mathrm{H}_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$.
(2) The identity map $\mathrm{id}\colon \Delta^n \to \Delta^n$ represents a generator of $\mathrm{H}_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$.

Proof. The first statement is an immediate consequence of Lemmas 76.1 and 72.1. Now we turn to the proof of the second statement. We prove the second statement by induction on n. It follows easily from the H_0-Proposition 72.5 that the statement holds for $n = 0$. Now we suppose that the desired statement holds for $n-1$ with $n \in \mathbb{N}$.

Our next goal is to find a connection between the two homology groups $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$ and $\mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$. We introduce the following notation.

(1) We consider

$$\Lambda \;:=\; \bigcup_{i=0}^{n-1} \{(t_0, \ldots, t_n) \in \Delta^n \,|\, t_i = 0\},$$

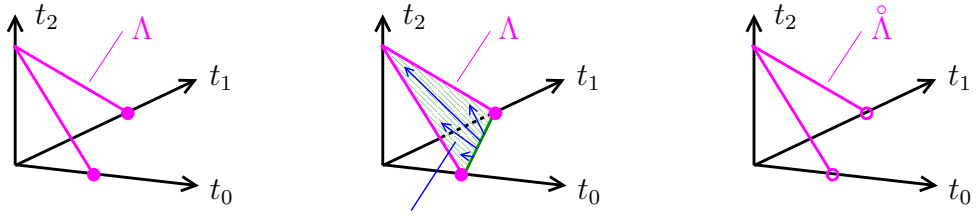
i.e. Λ is the union of all $(n-1)$-dimensional faces of Δ^n except for the face given by $t_n = 0$. A generalization of the argument of Exercise 15.13 (c) shows that Λ is a deformation retract of Δ^n. We refer to the figure below for an illustration.

(2) We write

$$\mathring{\Lambda} := \bigcup_{i=0}^{n-1} \{(t_0, \ldots, t_n) \in \Lambda \,|\, t_i = 0 \text{ and } t_n > 0\},$$

the set $\mathring{\Lambda}$ is thus the "interior" of Λ.

In the following we view $\mathbb{R}^n$ as the subset of $\mathbb{R}^{n+1}$ given by setting the last coordinate equal to zero. This point of view allows us to view Δ^{n-1} as a subset of Δ^n. Note that with this convention we have $\partial\Delta^{n-1} \subset \Lambda$.



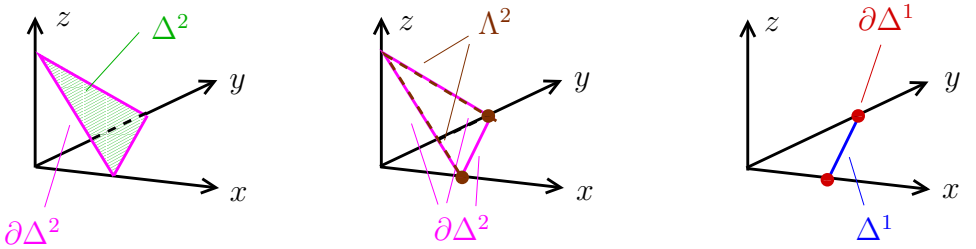
deformation retraction from Δ^2 to Λ

Claim. We consider the maps

$$\mathrm{H}_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial} \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda) \xleftarrow{i_*} \mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}).$$

Here the map on the left is the connecting homomorphism in the long exact sequence of the triple $(\Delta^n, \partial\Delta^n, \Lambda)$ and the right-hand map is induced by the inclusion. Then the following holds:

(1) Both maps are isomorphisms.
(2) $\partial([\mathrm{id}_{\Delta^n}]) = (-1)^n \cdot i_*([\mathrm{id}_{\Delta^{n-1}}])$.



It is clear that a proof of the claim together with the induction hypothesis implies that $[\mathrm{id}_{\Delta^n}]$ is a generator of $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$. In particular a proof of the claim implies the Identity-is-Generator Lemma 76.2.

Proof. Of the two maps in the claim we first consider the map ∂ on the left. It follows from the exact sequence of the triple $(\Delta^n, \partial\Delta^n, \Lambda)$ that the following sequence is exact:

$$\underbrace{\mathrm{H}_n(\Delta^n, \Lambda)}_{\substack{= 0 \text{ by Corollary 74.16 since} \\ \Lambda \text{ is a deformation} \\ \text{retract of } \Delta^n}} \xrightarrow{p} \mathrm{H}_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial} \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda) \xrightarrow{i_*} \underbrace{\mathrm{H}_{n-1}(\Delta^n, \Lambda).}_{\substack{= 0 \text{ by Corollary 74.16 since} \\ \Lambda \text{ is a deformation} \\ \text{retract of } \Delta^n}}$$

This shows that $\mathrm{H}_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial} \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda)$ is an isomorphism. Furthermore we have

$$\underset{\text{Topological LES-Proposition 74.13}}{\partial([\mathrm{id}_{\Delta^n}])} \underset{\uparrow}{=} [\partial\,\mathrm{id}_{\Delta^n}] = \Big[\underbrace{\sum_{j=0}^{n-1}(-1)^j\,\mathrm{id}\circ i_j^n}_{\text{lies in } \mathrm{C}_{n-1}(\Lambda)} + (-1)^n\cdot\underbrace{\mathrm{id}\circ i_n^n}_{=\mathrm{id}_{\Delta^{n-1}}}\Big] \underset{\uparrow}{=} \underset{\text{since we work in } \mathrm{H}_{n-1}(\partial\Delta^n,\Lambda)}{(-1)^n\cdot[\mathrm{id}_{\Delta^{n-1}}]}.$$

Now we turn our attention to the right-hand map in the statement of the claim. It follows from Theorem 75.9, applied to $X = \partial\Delta^n$, $A = \Lambda$, $Z = \mathring{\Lambda}$ and $U = (0,\ldots,0,1)$[916] that the inclusion induced map

$$\mathrm{H}_{n-1}(\Delta^{n-1},\partial\Delta^{n-1}) = \mathrm{H}_{n-1}(\underbrace{\partial\Delta^n\setminus\mathring{\Lambda}}_{=\Delta^{n-1}},\underbrace{\Lambda\setminus\mathring{\Lambda}}_{=\partial\Delta^{n-1}}) \xrightarrow{i_*} \mathrm{H}_{n-1}(\partial\Delta^n,\Lambda)$$

is an isomorphism. Furthermore we evidently have $i_*([\mathrm{id}_{\Delta^{n-1}}]) = [\mathrm{id}_{\Delta^{n-1}}]$.

Thus we have proved that both maps in the claim are isomorphisms and that

$$\partial([\mathrm{id}_{\Delta^n}]) = (-1)^n\cdot[\mathrm{id}_{\Delta^{n-1}}] = (-1)^n\cdot i_*([\mathrm{id}_{\Delta^{n-1}}]).$$ ■

In the remainder of this section we explain how to find explicit generators of the local homology groups of a topological manifold. We recall the following lemma from the previous section.

Lemma 75.1. (Manifold-Local Homology Lemma) Let M be a k-dimensional topological manifold. Then for any $P \in M$, any $l \in \mathbb{N}_0$ and any open neighborhood U of P we have

$$\mathrm{H}_l(M, M\setminus\{P\}) \cong \begin{cases} \mathbb{Z}, & \text{if } P \in M\setminus\partial M \text{ and } l = k,\\ 0, & \text{otherwise.}\end{cases}$$

Examples. Our goal now is to give explicit generators for the relative homology groups $\mathrm{H}_k(M, M\setminus\{x\}) \cong \mathbb{Z}$. We first do so for open subsets of $\mathbb{R}^k$ and then we "transport" those generators to topological manifolds using charts.

(1) Let $U \subset \mathbb{R}^k$ be an open subset that contains 0. Let $\sigma\colon \Delta^k \to U$ be an injective affine linear map. We suppose that 0 lies in the image of the "interior" $\mathring{\Delta}^k := \Delta^k\setminus\partial\Delta^k$ of Δ^k. We claim that σ represents a generator of $\mathrm{H}_k(U, U\setminus\{0\})$. To prove this claim we consider the commutative diagram

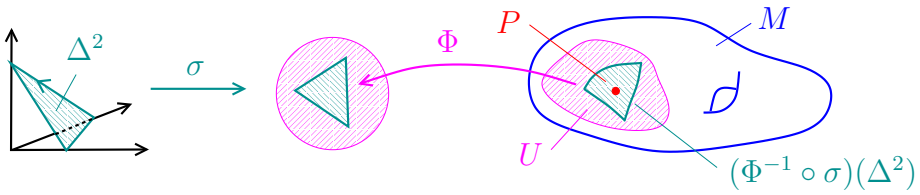
$$\begin{array}{ccccc} \mathrm{H}_k(\Delta^k,\partial\Delta^k) & & \xrightarrow{\quad\sigma_*\quad} & & \mathrm{H}_k(U, U\setminus\{0\}) \\ \downarrow{\scriptstyle\sigma_*} & & & & \downarrow{\scriptstyle\cong} \\ \mathrm{H}_k(\sigma(\Delta^k),\sigma(\partial\Delta^k)) & \longrightarrow & \mathrm{H}_k(\mathbb{R}^k,\sigma(\partial\Delta^k)) & \longrightarrow & \mathrm{H}_k(\mathbb{R}^k,\mathbb{R}^k\setminus\{0\}) \end{array}$$

where the maps that are not induced by σ are induced by the obvious inclusions. The left vertical map is an isomorphism since we know by the Compact-Hausdorff Proposition 2.17 (3) that $\sigma\colon \Delta^k \to \sigma(\Delta^k)$ is a homeomorphism. The bottom horizontal maps are isomorphisms by Corollary 74.16 since one can easily show that $\sigma(\Delta^k)$ is a deformation retract of $\mathbb{R}^k$ and similarly one can show that $\sigma(\partial\Delta^k)$ is deformation retract of $\mathbb{R}^k\setminus\{0\}$. By the Identity-is-Generator Lemma 76.2 we know that id_{Δ^n} represents a generator of $\mathrm{H}_n(\Delta^n,\partial\Delta^n)$. It follows from the above that $\sigma\circ\mathrm{id}_{\Delta^n} = \sigma$ represents a generator of $\mathrm{H}_k(\mathbb{R}^k,\mathbb{R}^k\setminus\{0\})$. But the right vertical map is

[916] One can easily show that U has the property required in Theorem 75.9.

an isomorphism by the Excision Theorem 74.18, hence σ also represents a generator of $\mathrm{H}_k(U, U \setminus \{0\})$.

(2) Let M be a k-dimensional topological manifold and let $P \in M \setminus \partial M$. In the following we want to give an example of an explicit singular k-simplex that represents a generator for $\mathrm{H}_k(M, M \setminus \{P\}) \cong \mathbb{Z}$. We pick a chart $\Phi\colon U \to B^k$ around P with $\Phi(P) = 0$ and we pick an affine linear map $\sigma\colon \Delta^k \to B^k$ as in (1). It follows from (1) together with the proof of the Manifold-Local Homology Lemma 75.1 that $\Phi^{-1} \circ \sigma\colon \Delta^k \to M$ represents a generator of $\mathrm{H}_k(M, M \setminus \{P\})$. We refer to the figure below for an illustration.



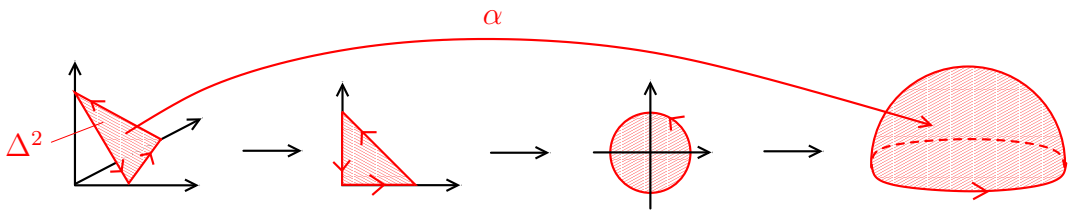
76.2. **The standard generator of $\mathrm{H}_n(S^n)$.** In this section we want to find explicit generators of $\mathrm{H}_n(S^n) \cong \mathbb{Z}$ for $n \geq 1$.

Notation. We consider the map $\alpha\colon \Delta^n \to S^n$ that is given by the concatenation of the following two maps

$$\Delta^n \xrightarrow[\text{by the Standard Simplex Homeomorphism Lemma 72.1}]{\text{homeomorphism given}} \overline{B}^n \subset S^n \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$$
$$x \mapsto \big(x, \sqrt{1 - \|x\|^2}\big).$$

We refer to the figure below for an illustration of the definition of the map α.

It follows easily from the definitions and the Compact-Hausdorff Proposition 2.17 (3) that this map α defines a homeomorphism $\Delta^n \to S^n_{\geq 0} = \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} \geq 0\}$ and that it restricts to a homeomorphism $\partial\Delta^n$ to $S^{n-1} = \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} = 0\}$.

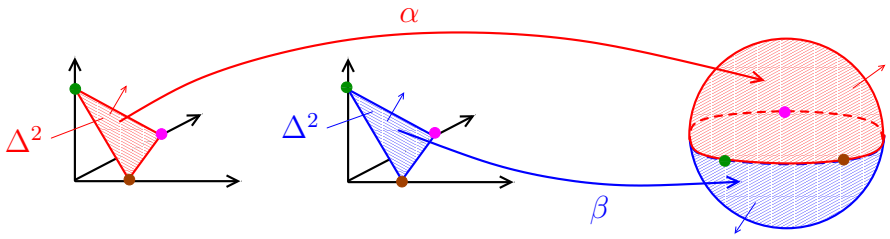


Notation. We denote by $\beta\colon \Delta^n \to S^n$ the map which is the composition of α with reflection in the hyperplane defined by $x_{n+1} = 0$. The maps $\alpha\colon \Delta^n \to S^n$ and $\beta\colon \Delta^n \to S^n$ both define singular n-chains in S^n. Note that α and β agree on $\partial\Delta^n$. Both maps are sketched in the figure below.

Lemma 76.3. Let $n \geq 1$. The above singular n-chain $\alpha - \beta \in \mathrm{C}_n(S^n)$ is a cycle and it represents a generator of $\widetilde{\mathrm{H}}_n(S^n) = \mathrm{H}_n(S^n)$.

Definition.

(1) Let $n \geq 1$. We refer to the generator from Lemma 76.3 as the standard generator of $\mathrm{H}_n(S^n)$.[917] With great foresight we denote it by $[S^n]$.



(2) We define $[c_1 - c_{-1}] \in \widetilde{\mathrm{H}}_0(S^0)$ to be the standard generator of $\widetilde{\mathrm{H}}_0(S^0)$.

Proof. We start out with the following claim.

Claim. The singular n-chain $\alpha - \beta$ is a cycle.

Proof. We denote by ρ the reflection in the hyperplane defined by $x_{n+1} = 0$. We have

$$\partial(\alpha - \beta) \;=\; \partial(\alpha - \rho_*(\alpha)) \;\underset{\text{since } \rho_* \text{ is a chain map}}{=}\; (\mathrm{id} - \rho_*)(\underset{\text{since } \alpha(\partial\Delta^n) \subset \mathbb{R}^n \times 0 \text{ we have } \partial\alpha \in \mathrm{C}_{n-1}(\mathbb{R}^n \times 0)}{\partial\alpha}) \;\underset{\text{since } \rho = \mathrm{id} \text{ on } \mathbb{R}^n \times 0 \subset \mathbb{R}^{n+1}}{=}\; 0.$$

⊞

As usual we denote by

$$\begin{aligned} S^n_{\geq 0} &:= \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} \geq 0\} \\ S^n_{\leq 0} &:= \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} \leq 0\} \end{aligned}$$

the two "hemispheres". By the discussion on page 152 both hemispheres are homeomorphic to the closed ball $\overline{B}^n$, in particular they are contractible which implies by the discussion on page 1612 that their reduced homology groups vanish. We also consider the "equator"

$$S^n_{=0} \;:=\; \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} = 0\}.$$

Next we consider the following sequence of maps

$$\begin{array}{ccccccc} \underset{\text{isomorphism by the long exact sequence in reduced homology of the pair } (S^n, S^n_{\leq 0})}{\downarrow} & & \underset{\text{isomorphism by example on on page 1665}}{\downarrow} & & \underset{\text{isomorphism since } \alpha \text{ is a homeomorphism}}{\downarrow} & & \\ \widetilde{\mathrm{H}}_n(S^n) & \to & \mathrm{H}_n(S^n, S^n_{\leq 0}) & \xleftarrow{i_*} & \mathrm{H}_n(S^n_{\geq 0}, S^n_{=0}) & \xleftarrow{\alpha_*} & \mathrm{H}_n(\Delta^n, \partial\Delta^n) \\ [\alpha - \beta] & \mapsto & [\alpha - \beta] = [\alpha] & \mapsto & [\alpha] & \mapsto & [\mathrm{id}_{\Delta^n}]. \\ & & \underset{\text{since } \beta = 0 \text{ in } \mathrm{C}_n(S^n, S^n_{\leq 0})}{\uparrow} & & & & \end{array}$$

So we have shown that all maps are isomorphisms. Furthermore, by the Identity-is-Generator Lemma 76.2 we know that $[\mathrm{id}_{\Delta^n}]$ is a generator of $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$. But $[\mathrm{id}_{\Delta^n}]$ and $[\alpha - \beta]$ get sent to the same element in $\mathrm{H}_n(S^n, S^n_{\leq 0})$, therefore $[\alpha - \beta]$ must also be generator of $\widetilde{\mathrm{H}}_n(S^n)$. ■

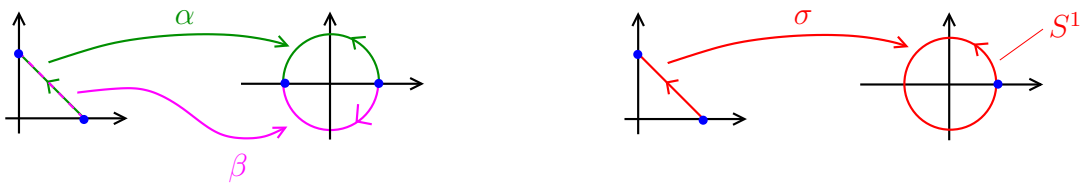
In the following lemma we give an alternative description of the standard generator of the homology group $\mathrm{H}_1(S^1)$.

[917] It follows easily from Lemma 72.3 and the definitions that the two definitions of standard generator of $\mathrm{H}_1(S^1)$ agree.

Lemma 76.4. The singular 1-simplex

$$\begin{array}{rcl} \sigma\colon \Delta^1 & \to & S^1 \\ (1-t,t) & \mapsto & \exp(2\pi\,\mathrm{i}t) \end{array}$$

is a cycle that represents the standard generator $[S^1]$ of $\mathrm{H}_1(S^1) \cong \mathbb{Z}$.



Proof. We denote by $\alpha, \beta\colon \Delta^1 \to S^1$ the singular 1-simplices in the definition of the standard generator of $\mathrm{H}_1(S^1)$. We have

$$[\alpha - \beta] \underset{\uparrow}{=} [\alpha + \overline{\beta}] \underset{\uparrow}{=} [\sigma] \in \mathrm{H}_1(S^1).$$

by Lemma 72.3 (2) and (3)

We refer to the figure below for an illustration. ∎

Remark. By Lemma 76.4 we know that there exists a singular 1-simplex σ that represents a generator of $\mathrm{H}_1(S^1)$. In Exercise 76.3 we will discuss the question, for which $n \in \mathbb{N}$ does there exist a singular n-simplex $\sigma\colon \Delta^n \to S^n$ that represents a generator of $\mathrm{H}_n(S^n)$.

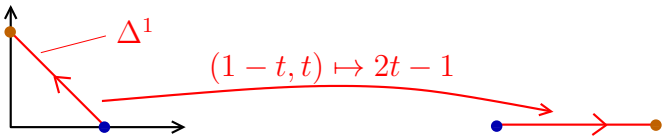
We conclude this section with a short discussion of several related homology groups and their "standard generators".

Definition. Let $n \in \mathbb{N}$. The standard generator of $\mathrm{H}_n(\overline{B}^n, S^{n-1})$ is defined to be the unique element that gets sent to the standard generator of $\widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ under the connecting homomorphism $\mathrm{H}_n(\overline{B}^n, S^{n-1}) \to \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ of the long exact sequence in reduced homology of the pair $(\overline{B}^n, S^{n-1})$. We denote the standard generator by $[\overline{B}^n]$.

Example. Recall that the standard generator of $\widetilde{\mathrm{H}}_0(S^0)$ is defined as $[c_1 - c_{-1}] \in \widetilde{\mathrm{H}}_0(S^0)$. It follows easily from the explicit description of the connecting homomorphism, see the Topological LES-Proposition 74.13, and the explicit description of the boundary of a singular 1-simplex, see page 1585, that the standard generator of $\mathrm{H}_1(\overline{B}^1, S^0)$ is represented by the cycle

$$\begin{array}{rcl} \Delta^1 & \to & \overline{B}^1 \\ (1-t,t) & \mapsto & 2t-1. \end{array}$$

We refer to the figure below for an illustration.



We have now introduced several standard generators of homology groups. The following proposition shows the relationship between these standard generators under various maps. The proposition, let alone the proof, are not much fun to read, but it is an absolutely essential proposition since it settles several rather subtle signs.

Proposition 76.5. (Standard Generator-Relationship Proposition) Let $n \in \mathbb{N}$. We denote by $f\colon (\Delta^n, \partial\Delta^n) \to (\overline{B}^n, S^{n-1})$ the homeomorphism from the Standard Simplex Homeomorphism Lemma 72.1 and we denote by $g\colon \overline{B}^n/S^{n-1} \to S^n$ the homeomorphism from the Ball-Quotient Sphere Lemma 5.20. We set $* := g([S^{n-1}])$.[918] We consider the following diagram:

$$\begin{array}{ccccccccc}
 & & & & \widetilde{\mathrm{H}}_n(\overline{B}^n/S^{n-1}) & \xrightarrow{g_*} & \widetilde{\mathrm{H}}_n(S^n) & & \\
 & & & \overset{\text{ю}}{\nearrow} & \text{ж}\downarrow\cong & & \cong\downarrow\text{ж} & & \\
\mathrm{H}_n(\Delta^n, \partial\Delta^n) & \xrightarrow{f_*} & \mathrm{H}_n(\overline{B}^n, S^{n-1}) & \longrightarrow & \mathrm{H}_n(\overline{B}^n/S^{n-1}, [S^{n-1}]) & \xrightarrow{g_*} & \mathrm{H}_n(S^n, *) & \longleftarrow & \mathrm{H}_n(S^n). \\
\partial\downarrow & & \downarrow\partial & & & & & & \\
\widetilde{\mathrm{H}}_{n-1}(\partial\Delta^{n-1}) & \xrightarrow{f_*} & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) & & & & & &
\end{array}$$

The following statements hold:

(1) The diagram commutes and the top vertical maps are isomorphisms.

(2) We have
$$f_*([\mathrm{id}_{\Delta^n}]) \;=\; (-1)^n \cdot \text{standard generator } [\overline{B}^n] \text{ of } \mathrm{H}_n(\overline{B}^n, S^{n-1}).$$

(3) The standard generator $[\overline{B}^n]$ of $\mathrm{H}_n(\overline{B}^n, S^{n-1})$ has the same image in $\mathrm{H}_n(S^n, *)$ as the standard generator $[S^n]$ of $\mathrm{H}_n(S^n)$.

(4) The image of the standard generator $[\overline{B}^n] \in \mathrm{H}_n(\overline{B}^n, S^{n-1})$ in $\mathrm{H}_{n-1}(S^{n-1})$ equals the standard generator $[S^{n-1}]$ of $\mathrm{H}_{n-1}(S^{n-1})$.

(5) We have
$$f_*(\partial[\mathrm{id}_{\Delta^n}]) \;=\; (-1)^n \cdot \text{standard generator } [S^{n-1}] \text{ of } \mathrm{H}_{n-1}(S^{n-1}).$$

Proof of the Standard Generator-Relationship Proposition 76.5.

(1) The triangle commutes by definition of ю, see page 1633. Furthermore it follows from Lemma 74.12 and the Topological LES-Corollary 74.14 that the two squares commute.

(3),(4) The calculation of these two signs is delicate and tricky. It becomes somewhat easier after we have developed a few more tools. Thus we postpone the proof of these two statements to Section 76.7.

(4) This statement is just a repetition of the definition of the standard generator $[\overline{B}^n]$.

(5) This statement is an immediate consequence of (1), (2) and (4). ■

We conclude this section with the following lemma which gives us another slightly subtle sign calculation.

Lemma 76.6. Let $n \in \mathbb{N}$. We consider the following sequence of maps

$$\mathrm{H}_n(S^n) \xrightarrow{\text{я}} \mathrm{H}_n(S^n, S^n_{\geq 0}) \xleftarrow[\uparrow]{\cong} \mathrm{H}_n(S^n_{\geq 0}, S^{n-1}) \xrightarrow{\partial} \widetilde{\mathrm{H}}_{n-1}(S^{n-1}).$$

inclusion induced map is an isomorphism by the discussion on page 1665

The image of the standard generator $[S^n] \in \mathrm{H}_n(S^n)$ under this map is $(-1)^n \cdot [S^{n-1}]$.

[918] The fact that we use the expression $[S^{n-1}]$ for two very different objects, namely first for the point in $\overline{B}^n/S^{n-1}$ given by the equivalence class S^{n-1} and secondly for the standard generator of $\widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ is unfortunate, but difficult to avoid if one wants to follow the standard conventions.

Proof. We recall and introduce the following notation:

(1) Let $f\colon (\Delta^n, \partial\Delta^n) \to (\overline{B}^n, S^{n-1})$ be the homeomorphism from the Standard Simplex Homeomorphism Lemma 72.1.
(2) Let $p_+\colon S^n_{\geq 0} \to \overline{B}^n$ be the natural projection and let $q_+\colon \overline{B}^n \to S^n_{\geq 0}$ be the map given by $x \mapsto (x, \sqrt{1 - \|x\|^2})$. Evidently p and q are inverses of one another. We define $p_-\colon S^n_{\leq 0} \to \overline{B}^n$ and $q_-\colon \overline{B}^n \to S^n_{\leq 0}$ in an analogous fashion.
(3) Note that $\alpha := q_+ \circ f\colon \Delta^n \to S^n$ and $\beta := q_- \circ f\colon \Delta^n$ are the singular n-simplices that we introduced on page 1677.

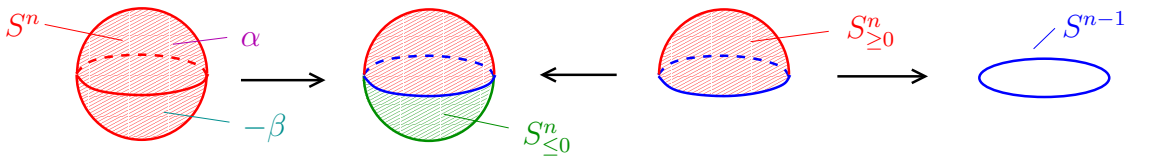
Next we consider the following diagram

$$\begin{array}{ccccccc} \mathrm{H}_n(S^n) & \xrightarrow{\text{я}} & \mathrm{H}_n(S^n, S^n_{\geq 0}) & \xleftarrow{i} & \mathrm{H}_n(S^n_{\geq 0}, S^{n-1}) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(S^{n-1}). \\ & & & & \uparrow{\scriptstyle q_+} \downarrow{\scriptstyle p_+} & \nearrow{\scriptstyle \partial} & \\ & & & & \mathrm{H}_n(\overline{B}^n, S^{n-1}) & & \end{array}$$

Note that the triangle to the right commutes by the naturality of the connecting homomorphism, see the Topological LES-Corollary 74.14. Now we see that

$$\begin{array}{ccccccc} & \text{definition of } [S^n] & & \text{definition of я} & & \text{since } \beta = 0 \in \mathrm{C}_n(S^n, S^n_{\leq 0}) & \\ \partial(i_*^{-1}(\text{я}([S^n]))) & \downarrow & & \downarrow & & \downarrow & \\ & = \partial(i_*^{-1}(\text{я}([\alpha - \beta]))) & = \partial(i_*^{-1}([\alpha - \beta])) & = \partial(i_*^{-1}([\alpha])) & = \partial([\alpha]) & & \\ & = \partial(q_{+*}([p_+ \circ \alpha])) & = \partial(q_{+*}([f])) & = \partial([f]) & = (-1)^n \cdot [S^{n-1}]. & & \\ & \uparrow & \uparrow & \uparrow & \uparrow & & \\ & \text{since } p_+ = q_+^{-1} & \text{since } \alpha = q_+ \circ f & \text{since the triangle commutes} & \text{Proposition 76.5 (5)} & & \end{array}$$

■



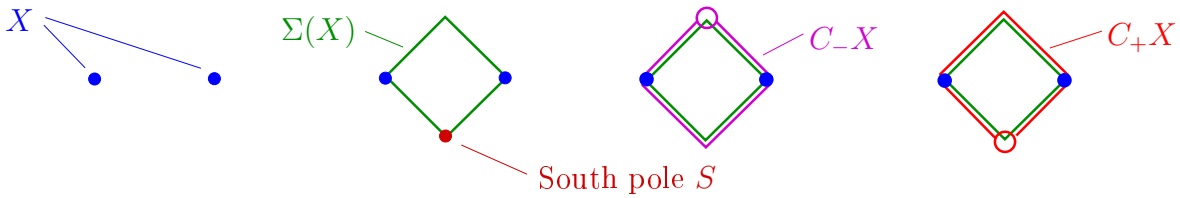
76.3. **Suspensions and homology groups.** In the next section we will study self-maps of spheres. In that section it will be useful to consider the effect of taking suspensions on homology. Since this is going to be interesting for larger classes of topological spaces we dedicate this short section to the study of homology groups of suspensions of topological spaces.

The following definition follows the definition on page 451 and the convention given by the Cone-Suspension Basics Lemma 16.8 (3).

Definition. Let X be a non-empty topological space.

(1) We define the suspension of X to be the topological space
$$\Sigma(X) \;:=\; (X \times [-1, 1])/\!\sim$$
where we identify all points in $X \times \{-1\}$ to a single point and we identify all points in $X \times \{1\}$ to a single point.
(2) We denote by $p\colon X \times [-1, 1] \to \Sigma(X)$ the natural projection. We write
$$C_-X \;:=\; p(X \times [-1, 1)) \quad \text{and} \quad C_+X \;:=\; p(X \times (-1, 1]).$$

By the Cone-Suspension Basics Lemma 16.8 (3) we know that the map $X \xrightarrow{x \mapsto p(x,0)} \Sigma(X)$ is a closed embedding. We use this embedding to view X as a subspace of $\mathrm{C}_{\pm}X$ and of $\Sigma(X)$.



In the Suspension-π_1-Lemma 54.2 and Exercise 57.5 we calculated the fundamental group of the suspension of a non-empty topological space with finitely many path components. The following proposition, which is the main result of this section, determines the homology groups of a suspension.

Proposition 76.7. (Suspension-H_*-Proposition) Let X be a non-empty topological space and let $n \in \mathbb{Z}$.[919]

(1) The following maps are isomorphisms:

$$\widetilde{\mathrm{H}}_n(X) \xleftarrow{\partial} \mathrm{H}_{n+1}(C_+X, X) \xrightarrow{} \mathrm{H}_{n+1}(\Sigma(X), C_-X) \xleftarrow{\text{ж}} \widetilde{\mathrm{H}}_{n+1}(\Sigma(X)).$$

connecting homomorphism of the pair (C_+X, X) from the Topological LES-Corollary 74.14 — inclusion induced — natural map given by the Reduced Homology Lemma

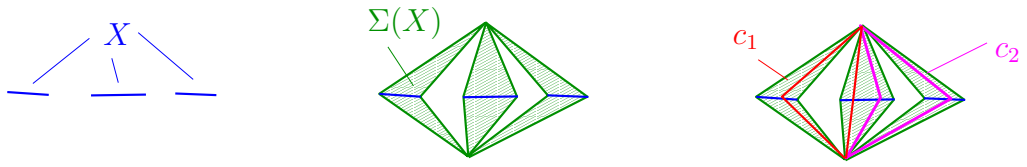
(2) The resulting isomorphism

$$\Sigma_X \colon \widetilde{\mathrm{H}}_n(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_{n+1}(\Sigma(X))$$

is natural. In other words, given any map $f \colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_n(X) & \xrightarrow{\Sigma_X} & \widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \Sigma(f)_*} \\ \widetilde{\mathrm{H}}_n(Y) & \xrightarrow{\Sigma_Y} & \widetilde{\mathrm{H}}_{n+1}(\Sigma(Y)). \end{array}$$

Example. Let $n \in \mathbb{N}$ and let X be the disjoint union of n intervals. It follows from the Suspension-H_*-Proposition 76.7 together with the Reduced Homology Lemma 74.1 that $\mathrm{H}_1(\Sigma(X)) \cong \mathbb{Z}^{n-1}$. In the figure below we show a generating set for $\mathrm{H}_1(\Sigma(X))$ for $n = 3$. We leave it to the reader to deduce from the proof of the Suspension-H_*-Proposition 76.7 that the given cycles do indeed form a generating set for $\mathrm{H}_1(\Sigma(X))$.



[919] Note that the proposition has non-trivial content for $n = -1$.

Proof. Let $n \in \mathbb{Z}$.

(1) (a) By the same argument as in the proof of the Cone Contractibility Lemma 16.7 we see that both C_-X and C_+X admit a deformation retraction to a point. It follows from the Deformation Retract-Homotopy Equivalence Lemma 15.5 that both are contractible, therefore by the Reduced Homology Lemma 74.1 (7) the reduced homologies of C_-X and C_+X vanish. Note that it follows in particular that the connecting homomorphism $\partial\colon \mathrm{H}_{n+1}(C_+X, X) \to \widetilde{\mathrm{H}}_n(X)$ is an isomorphism.

(b) We denote by $S := [X \times \{-1\}] \in \Sigma(X)$ the South Pole of the suspension. Note that $\Sigma(X) \setminus S = C_+X$ and $C_-X \setminus S = C_-X \cap C_+X$. Using this observation we obtain from the Excision Theorem 74.17[920] that the second horizontal map is an isomorphism.

(c) Similar to (a) we obtain from the Topological LES-Corollary 74.14 that the third horizontal map is an isomorphism.

(2) First note that the connecting homomorphism is natural by the Topological LES-Proposition 74.13 (2) and the fact that $\Sigma(f)|_X = f$. It is clear that the second horizontal map is natural. Finally it follows from the Reduced Homology Lemma 74.1 (3) that the map ж is natural. The combination of these observations shows that Σ_X itself is natural. ■

Example. Let $n \in \mathbb{N}$. Let $\Sigma(S^n) \to S^{n+1}$ be the homeomorphism from page 452. It follows quite easily from the definition of the suspension isomorphism $\Sigma_{S^n}\colon \mathrm{H}_n(S^n) \to \mathrm{H}_{n+1}(S^{n+1})$ and Lemma 76.6 that

$$\Sigma_{S^n}([S^n]) \;=\; (-1)^{n+1} \cdot [S^{n+1}] \qquad \in \mathrm{H}_{n+1}(S^{n+1}).$$

We conclude this section with the following corollary.

Corollary 76.8. Let X be a topological space that is compact and Hausdorff. Furthermore let $k \in \mathbb{N}_{\geq -1}$. For every $n \in \mathbb{Z}$ there exists a natural isomorphism

$$\widetilde{\mathrm{H}}_n(X) \;\xrightarrow{\cong}\; \widetilde{\mathrm{H}}_{n+k+1}(S^k * X).$$

($S^k * X$: join of S^k and X, see page 446 for the definition)

Here, recall that if X is the empty topological space, then by the Reduced Homology Lemma 74.1 (0) we have $\widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$.

Proof. First note that the case $X = \varnothing$ follows from the fact that, by definition, $S^k * \varnothing = S^k$, and the calculation of the reduced homology groups of S^k that we gave in Proposition 74.4.

Now assume that $X \neq \varnothing$. If $k = -1$, then we have, again by definition, the equality $S^{-1} * X = \varnothing * X = X$ which implies the desired statement. Thus it remains to deal with case that $k \geq 0$. To simplify the notation we only consider the case $k = 1$. In this case we have

$$\widetilde{\mathrm{H}}_{n+2}(S^1 * X) \overset{\text{Join of Spheres Lemma 16.4}}{\cong} \widetilde{\mathrm{H}}_{n+2}((S^0 * S^0) * X) \underset{\substack{\text{by the Join Associativity Proposition 16.6}\\ \text{see page 448 for the definition}\\ \text{of the homeomorphism}}}{\cong} \widetilde{\mathrm{H}}_{n+2}(S^0 * (S^0 * X)) \overset{\text{Exercise 16.5}}{\cong} \widetilde{\mathrm{H}}_{n+2}(\Sigma(\Sigma(X))) \underset{\substack{\text{Suspension-H}_*\text{-Proposition 76.7}\\ \text{applied twice}}}{\cong} \widetilde{\mathrm{H}}_n(X).$$

[920] One can easily verify that the hypotheses for the Excision Theorem 74.17 are satisfied.

It follows easily from the definitions of the maps that the isomorphism is in fact natural. ■

76.4. **The degree of a self-map of a sphere.** Now that we know the homology groups of spheres, we want to study the maps on homology induced by maps $f\colon S^n \to S^n$. This is interesting in its own right, but it will also play a major role later on in Chapter 80 when we will try to determine the homology groups of CW-complexes.
Throughout the remaining sections of this chapter we will work a lot with the following simple-minded definition.

Definition. Let $(G,+)$ be a group that is isomorphic to $\mathbb{Z}$. Furthermore let $\Phi\colon G \to G$ be a homomorphism. Note that Φ is given by multiplication by a unique integer which we refer to as the degree $\deg(\Phi)$.

For completeness' sake we record two basically obvious statements regarding the degree.

Lemma 76.9. Let $(G,+)$ be a group that is isomorphic to $\mathbb{Z}$.
(1) Let $g \in G$ be any non-trivial element. Furthermore let $\Phi\colon G \to G$ be a homomorphism. The degree $\deg(\Phi)$ is uniquely determined by the equality $\Phi(g) = \deg(\Phi)\cdot g$.
(2) Let $\varphi\colon G \to H$ be an isomorphism and let $f\colon H \to H$ be a homomorphism. Then $\deg(\varphi^{-1}\circ f\circ\varphi\colon G\to G) = \deg(f\colon H\to H)$.

Next we relate the above definition to topology.

Definition. Let $n \in \mathbb{N}_0$ and let $f\colon S^n \to S^n$ be map. Since $\widetilde{\mathrm{H}}_n(S^n)$ is isomorphic to $\mathbb{Z}$ we can define the degree of f as

$$\deg(f) \;:=\; \deg\big(f_*\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n)\big).$$

Remark. Let $n \in \mathbb{N}$. On page 899 we introduced in particular the degree of a *smooth* map $f\colon S^n \to S^n$. On page 1697 we will see that in the setting of a smooth map $f\colon S^n \to S^n$ the two definitions of the degree agree.

Example. We consider the map $f\colon S^1 \to S^1$ that is illustrated in the figure below on the left.[921] We denote by $\tau\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $\tau(1-t,t) = \exp(2\pi i t)$. By Lemma 76.4 we know that $[\tau] \in \mathrm{H}_1(S^1)$ is a generator.

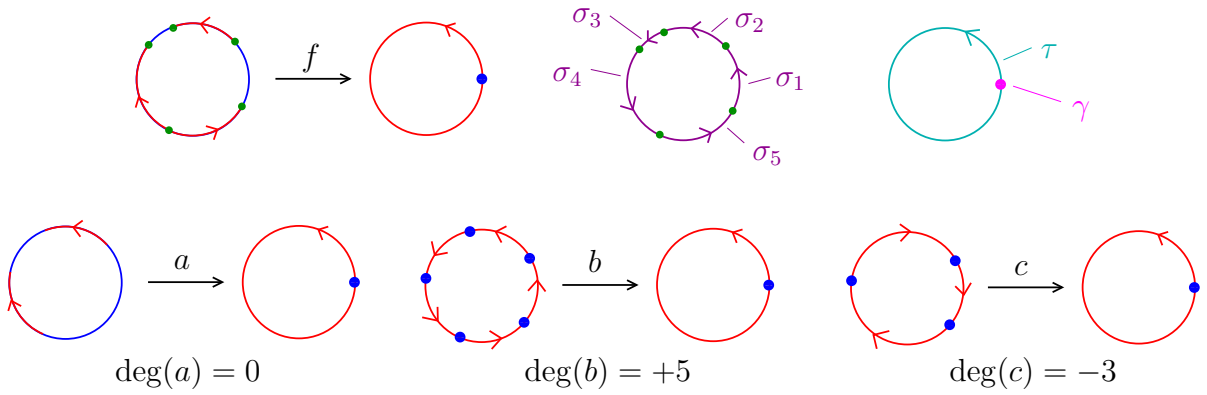
We pick singular 1-simplices $\sigma_1,\dots,\sigma_5$ as in the middle of the figure below and we denote by $\gamma\colon \Delta^1 \to S^1$ the constant singular 1-simplex that sends all points to 1. Then

$$\begin{aligned} f_*([\tau]) \overset{\text{by Lemma 72.3 (3)}}{=} f_*([\sigma_1+\cdots+\sigma_5]) &= [\overbrace{\sigma_1\circ f}^{=\gamma} + \overbrace{\sigma_2\circ f}^{=\tau} + \overbrace{\sigma_3\circ f}^{=\gamma} + \overbrace{\sigma_4\circ f}^{=\overline{\tau}} + \overbrace{\sigma_5\circ f}^{=\tau}] \\ &= [2\cdot\tau + \overline{\tau} + 2\cdot\gamma] \overset{\text{Lemma 72.3 (2)}}{=} [\tau] = \underbrace{(+1)}_{=:\deg(f)}\cdot[\tau]. \end{aligned}$$

Summarizing we have shown that $\deg(f) = +1$. With almost the same approach we can calculate the degrees of the maps that are illustrated in the figure below.

Almost the same arguments that we used in the discussion of the above example gives us the following lemma.

[921] More precisely, given an "interval" on S^1 going from $\exp(i\alpha)$ to $\exp(i\beta)$ we mean the singular 1-simplex that is given by $(1-t,t) \mapsto \exp(i(\alpha\cdot(1-t)+\beta\cdot t))$.



Lemma 76.10. For any $k \in \mathbb{Z}$ the degree of the map

$$\begin{array}{ccc} S^1 & \to & S^1 \\ z & \mapsto & z^k \end{array}$$

equals k.

Proof. We denote by $\tau\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $\tau(1-t,t) = \exp(2\pi i t)$. By Lemma 76.4 we know that $[\tau] \in \mathrm{H}_1(S^1)$ is a generator.

Now we first consider the case $k \geq 1$. For $j = 0, \ldots, k-1$ we denote by $\sigma_j\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $(1-t,t) \mapsto \exp(2\pi \mathrm{i}(\frac{t+j}{k}))$. Then

$$f_*([\tau]) \underset{\substack{\uparrow \\ \text{by Lemma 72.3 (3)}}}{=} f_*([\sigma_0 + \cdots + \sigma_{k-1}]) = [\underbrace{f \circ \sigma_0}_{=\tau} + \cdots + \underbrace{f \circ \sigma_{k-1}}_{=\tau}] = k \cdot [\tau].$$

The case $k < 0$ is proved is proved in a similar fashion using Lemma 72.3 (2). Finally the case $k = 0$ follows from the fact that the map $S^1 \to S^1$, $z \mapsto 1$ is constant, hence, as discussed on page 1595, the induced map on first homology is the zero map. ■

Before we move on to the next lemma it is convenient to introduce the following notation.

Notation. As on page 419, given a matrix $A \in \mathrm{O}(n+1)$ we denote by $\rho(A)\colon S^n \to S^n$ the map given by $x \mapsto A \cdot x$.

The following lemma summarizes several important properties of the degree of the self-map of a sphere.

Lemma 76.11. (Sphere Degree Lemma) Let $f, g\colon S^n \to S^n$ be two maps. Then the following hold:

(1) $\deg(\mathrm{id}_{S^n}) = 1$.
(2) If f is not surjective, then $\deg(f) = 0$.
(3) If f is homotopic to g, then $\deg(f) = \deg(g)$. (In the Degree Isomorphism Proposition 86.8 we will see that the converse to this statement holds.)
(4) $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.
(5) If f is a self-homeomorphism of S^n, then $\deg(f^{-1}) = \deg(f)$.
(6) Given any $A \in \mathrm{O}(n)$ we have $\deg(\rho(A)) = \det(A) \in \{-1, 1\}$.

Examples.

(1) If $f\colon S^n \to S^n$ is the reflection in a hyperplane, then it follows from the Sphere Degree Lemma 76.11 (6) that $\deg(f) = -1$,
(2) If $f\colon S^n \to S^n$ is the reflection in the origin i.e. if $f(x) = -x$ for all $x \in S^n$, then it follows from the Sphere Degree Lemma 76.11 (6) we have $\deg(f) = (-1)^{n+1}$.
(3) Let $A \in \mathrm{O}(n+1)$. Let $f\colon S^n \to S^n$ be the map that is given by $f(P) = A \cdot P$. If $\det(A) = 1$, then we saw on page 419 that f is actually homotopic to the identity. On the other hand, if $\det(A) = -1$, then it now follows from the Sphere Degree Lemma 76.11 (1), (3) together with the Sphere Degree Lemma 76.11 (6) that f is not homotopic to the identity.

Proof.

(1) Since $(\mathrm{id}_{S^n})_* = \mathrm{id}_{\mathrm{H}_n(S^n)}$ we know that $\deg(\mathrm{id}_{S^n}) = 1$.
(2) Suppose that the map $f\colon S^n \to S^n$ misses the point $P \in S^n$. We consider the commutative diagram

$$\begin{array}{ccc} & S^n\backslash\{P\} & \\ {\scriptstyle f}\nearrow & & \searrow{\scriptstyle i} \\ S^n & \xrightarrow[f]{} & S^n \end{array} \qquad \text{which gives rise to the commutative diagram} \qquad \begin{array}{ccc} & \mathrm{H}_n(S^n\backslash\{P\}) & \\ {\scriptstyle f_*}\nearrow & & \searrow{\scriptstyle i_*} \\ \mathrm{H}_n(S^n) & \xrightarrow[f_*]{} & \mathrm{H}_n(S^n). \end{array}$$

By the Orthogonal Action Lemma 5.26 (6) we know $S^n\backslash\{P\}$ is homeomorphic to $\mathbb{R}^n$. Thus we see that $S^n\backslash\{P\}$ is contractible which implies by the Homotopy Equivalence-H_*-Corollary 73.9 that $\mathrm{H}_n(S^n\backslash\{P\}) = 0$. But this means that $f_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$ factors through the zero group, i.e. f_* is the zero map.
(3) If f is homotopic to g, then it follows from the Homotopic Maps-and-Homology Proposition 73.6 that $\deg(f) = \deg(g)$.
(4) We consider the following diagram

$$\mathrm{H}_n(S^n) \xrightarrow[\cdot\deg(f)]{f_*} \mathrm{H}_n(S^n) \xrightarrow[\cdot\deg(g)]{g_*} \mathrm{H}_n(S^n). \qquad \text{(curved arrow: } (f\circ g)_*\text{)}$$

This diagram commutes since $(f \circ g)_* = f_* \circ g_*$. It follows that the curved map is given by multiplication by $\deg(f) \cdot \deg(g)$, i.e. $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.
(5) This statement follows immediately from (1) and (4).
(6) We start out with the following claim.

Claim. Let $\mu\colon S^n \to S^n$ be the reflection in the $(x_{n+1} = 0)$-hyperplane. Then we have $\deg(\mu) = -1$.

Proof. We consider again the singular chain

$$\begin{array}{rcccccc} \alpha\colon \Delta^n & \overset{\cong}{\to} & \left\{(x_1,\dots,x_n) \in [0,1]^n \;\middle|\; \sum_{i=1}^n x_i \in [0,1]\right\} & \overset{\cong}{\to} & \overline{B}^n & \to & S^n. \\ (t_0,\dots,t_n) & \mapsto & (t_0,\dots,t_{n-1}) & & x & \mapsto & \big(x, \sqrt{1-\|x\|^2}\big). \end{array}$$

that we had already considered on page 1677. We write $\beta = \mu \circ \alpha$. In Lemma 76.3 we saw that the singular n-chain $\alpha - \beta$ is a generator of $\mathrm{H}_n(S^n)$. Next we calculate

$$\mu_*([\alpha-\beta]) \;=\; [\mu\circ\alpha] - [\mu\circ\beta] \;=\; [\beta] - [\underbrace{\mu\circ\mu}_{=\mathrm{id}}\circ\alpha] \;=\; [\beta] - [\alpha] \;=\; -[\alpha-\beta].$$

The map μ_* is therefore given by multiplication with -1. ⊞

Now let $A \in \mathrm{O}(n+1)$. First we consider the case $\det(A) = 1$, i.e. we assume that $A \in \mathrm{SO}(n+1)$. By the Matrix Group Path-Component Proposition 2.37 we know that $\mathrm{SO}(n+1)$ is path-connected, so there exists a map $h\colon [0,1] \to \mathrm{SO}(n+1)$ with $h(0) = \mathrm{id}$ and $h(1) = A$. But then $F(x,t) = h(t) \cdot x$ defines a homotopy from id to $\rho(A)$. It follows from the Sphere Degree Lemma 76.11 (1) and (3) that $\deg(\rho(A)) = \deg(\mathrm{id}) = 1$.

Finally we consider the case that $\det(A) \neq 1$. Let $R \in \mathrm{O}(n+1)$ be the matrix such that $\rho(R) = \mu$. Now we see that

$$\deg(\rho(A)) \;=\; \deg(\rho(A \cdot R \cdot R^{-1})) \underset{\text{by (4)}}{\overset{\uparrow}{=}} \deg(\rho(A \cdot R)) \cdot \deg(\rho(R)) \underset{\text{since } \det(A\cdot R) = 1 \text{ and by the claim}}{\overset{\uparrow}{=}} 1 \cdot (-1) \;=\; -1.$$

■

We have now seen that the circle admits self-maps of arbitrary degrees and that every sphere admits self-maps of degree $-1, 0, 1$. It is natural to ask whether there are also maps on higher-dimensional spheres that have other degrees. The combination of Lemma 76.10 and the following lemma shows that in fact every $k \in \mathbb{Z}$ is realized as the degree of every high-dimensional sphere.

Lemma 76.12. (Degree-Suspension Lemma) Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a map. Let $\Sigma(f)\colon \Sigma(S^n) \to \Sigma(S^n)$ be the suspension of f that we defined on page 452. Furthermore let $\Theta\colon \Sigma(S^n) \to S^{n+1}$ be the homeomorphism from page 452. Then

$$\deg(\Theta \circ \Sigma(f) \circ \Theta^{-1}\colon S^{n+1} \to S^{n+1}) \;=\; \deg(f\colon S^n \to S^n).$$

Examples.

(1) Let $k \in \mathbb{Z}$. We view S^2 as a subset of $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. We consider the map

$$\begin{array}{rcl} g\colon S^2 & \to & S^2 \\ (r \cdot \exp(\mathrm{i}\varphi), w) & \mapsto & (r \cdot \exp(\mathrm{i}\varphi k), w). \end{array}$$

One can easily verify that the map g equals $\Theta \circ \Sigma(z \mapsto z^k) \circ \Theta^{-1}$. Thus it follows from Lemma 76.10 and the Degree-Suspension Lemma 76.12 that $\deg(g) = \deg(\Sigma(z \mapsto z^k)) = k$.

(2) Let $n \in \mathbb{N}$ and let $k \in \mathbb{Z}$. It follows from Lemma 76.10 and the Degree-Suspension Lemma 76.12 that there exists a self-map $f\colon S^n \to S^n$ of degree k.

Proof. We consider the following diagram

$$\begin{array}{ccccc} \widetilde{\mathrm{H}}_n(S^n) & \xrightarrow{\Sigma_{S^{n+1}}} & \widetilde{\mathrm{H}}_n(\Sigma(S^n)) & \xrightarrow{\Theta_*} & \widetilde{\mathrm{H}}_{n+1}(S^{n+1}) \\ \downarrow{\scriptstyle \cdot \deg(f)} & & \downarrow{\scriptstyle \Sigma(f)_*} & & \downarrow{\scriptstyle (\Theta\circ\Sigma(f)\circ\Theta^{-1})_*} \\ \widetilde{\mathrm{H}}_n(S^n) & \xrightarrow{\Sigma_{S^{n+1}}} & \widetilde{\mathrm{H}}_{n+1}(\Sigma(S^n)) & \xrightarrow{\Theta_*} & \widetilde{\mathrm{H}}_{n+1}(S^{n+1}). \end{array}$$

First note that it follows from the Suspension-H_*-Proposition 76.7 that the square to the left commutes and that the horizontal maps to the left are isomorphisms. It is clear that the square to the right commutes. Furthermore, since Θ is a homeomorphism we see that the horizontal maps to right are isomorphism. Note that it follows immediately from the above discussion together with Lemma 76.9 (2) that $\deg(\Theta \circ \Sigma(f) \circ \Theta^{-1}) = \deg(f)$. ■

The following theorem is an interesting application of our discussion of degrees of self-maps of spheres.

Theorem 38.11. (Sphere-Degree Fixed Point Theorem) Let $f\colon S^n \to S^n$ be a map. If $\deg(f) \neq (-1)^{n+1}$, then f has a fixed point.

Example. The Sphere-Degree Fixed Point Theorem Theorem 38.11 says in particular that every map $f\colon S^2 \to S^2$ of degree one has a fixed point. Together with the Sphere Degree Lemma 76.11 this implies that every map $f\colon S^2 \to S^2$ that is homotopic to the identity has a fixed point. This means that we cannot "deform" the identity to a map that is fixed point free. This is in contrast to the case $n = 1$ where we can "deform" the identity into the map that is given by rotation by a small angle, which is of course a fixed point free map.

Proof. We had proved the Sphere-Degree Fixed Point Theorem Theorem 38.11 on page 908 using the notion of the degree of a smooth map $f\colon S^n \to S^n$. In a rather blatant case of self-plagiarism the proof we now provide is almost identical to the proof on page 908.

Let $f\colon S^n \to S^n$ be a map which admits no fixed point. This means that for all $x \in S^n$ we have $f(x) \neq x$. We need to show that $\deg(f) = (-1)^{n+1}$. We will do this by showing that f is homotopic to $-\operatorname{id}$.

Our hypothesis implies that $(1-t)f(x) \neq tx$ for all $t \in [0,1]$ and all $x \in S^n$.[922] Therefore we can consider the map

$$\begin{array}{rcl} F\colon S^n \times [0,1] & \to & S^n \\ (x,t) & \mapsto & \dfrac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}. \end{array}$$

This map is a homotopy between the maps $x \mapsto f(x)$ and $x \mapsto -x$. Thus it follows that

$$\deg(f) \underset{\uparrow}{=} \deg(-\operatorname{id}) \underset{\uparrow}{=} (-1)^{n+1}.$$

Sphere Degree Lemma 76.11 (3) Sphere Degree Lemma 76.11 (6) ■

In the following we recall the definition from page 638 of a vector field on a submanifold of $\mathbb{R}^n$.

Definition. Let $M \subset \mathbb{R}^n$ be a submanifold. Given $P \in M \setminus \partial M$ we denote by $\widetilde{\mathrm{T}}_P M \subset \mathbb{R}^n$ the physical tangent vector space as defined on page 620. A vector field on M is a continuous map $v\colon M \to \mathbb{R}^n$ such that for any $P \in M$ we have $v(P) \in \widetilde{\mathrm{T}}_P M$.

Examples.

(1) Let P be a point on S^n. By Lemma 23.1 we have

$$\widetilde{\mathrm{T}}_P S^n = \{v \in \mathbb{R}^{n+1} \mid \langle P, v\rangle = 0\}.$$

(2) For an odd-dimensional sphere S^{2k-1} a vector field is given by[923]

$$\begin{array}{rcl} v\colon S^{2k-1} & \to & \mathbb{R}^{2k} \\ (x_1, x_2, \ldots, x_{2k-1}, x_{2k}) & \mapsto & (-x_2, x_1, \ldots, -x_{2k}, x_{2k-1}). \end{array}$$

[922] This case be seen as follows: if $t \neq \frac{1}{2}$ we have $\|(1-t)f(x)\| = 1 - t \neq t = \|tx\|$. On the other hand, for $t = \frac{1}{2}$ we have $(1-t)f(x) = \frac{1}{2}f(x) \neq \frac{1}{2}x = tx$.

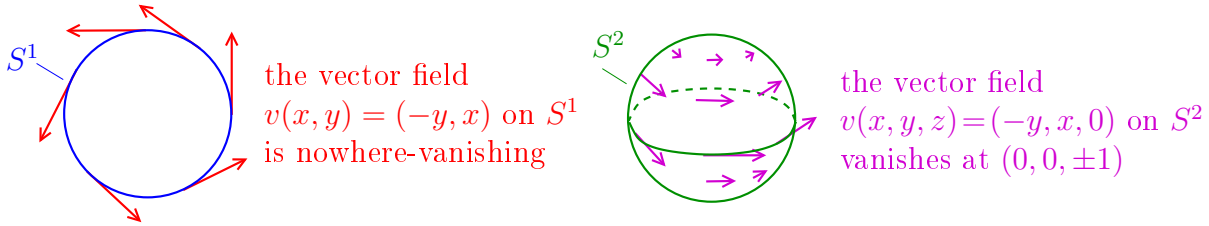
[923] Indeed, for any $P \in S^{2k-1}$ we have $\langle P, v(P)\rangle = 0$, i.e. $v(P)$ does indeed lie in the tangent space $\mathrm{T}_P S^{2k-1}$.

This vector field has the property that it is nowhere-vanishing. In the case of the 1-sphere S^1 this is just the vector field of constant length 1 that points in the "counterclockwise direction".

(3) For an even-dimensional sphere S^{2k} a vector field is similarly given by

$$\begin{array}{rcl} v\colon S^{2k} & \to & \mathbb{R}^{2k+1} \\ (x_1, x_2, \ldots, x_{2k-1}, x_{2k}, x_{2k+1}) & \mapsto & (-x_2, x_1, \ldots, -x_{2k}, 0). \end{array}$$

This vector field vanishes at the "North Pole" $(0, \ldots, 0, 1)$ and it also vanishes at the "South Pole" $(0, \ldots, 0, -1)$. Both types of vector fields are sketched in the figure below.



In Question 23.17 we asked whether it is possible to find a nowhere-vanishing vector field on even-dimensional spheres. The following theorem says that this is not possible.

Theorem 38.12. (Hairy Ball Theorem) Every vector field on an even-dimensional sphere S^{2k} vanishes on at least one point.[924]

Remark.

(1) For $n = 2$ we get the following "real life application": on the surface of earth there always exists at least one point where the wind does not blow at all. This fact is illustrated at the following website

`https://earth.nullschool.net/`

(2) The "Hairy Ball Theorem" gets its observation that for $n = 2$ the theorem says that "you can't comb a hairy ball flat without creating a cowlick".

(3) Contrary to most other languages in German the Hairy Ball Theorem 38.12 is called the "Satz vom (gekämmten) Igel", i.e. it is referred to as the "Combed Hedgehog Theorem".

(4) Evidently the fun never stops. Now that we know that odd-dimensional spheres admit one nowhere-vanishing vector field the question arises, how many linearly independent vector fields can we find on a given odd-dimensional sphere. It turns out that there is complete and highly non-trivial answer, see [**Adam1962**], [**MillH1989**] and [**VogI**] for details.

Proof. We had proved the Hairy Ball Theorem 38.12 on page 909 using the notion of the degree of a smooth map $f\colon S^n \to S^n$. In an another shameless case of self-plagiarism the proof we now provide is again almost identical to our earlier proof.

[924] On S^{2k} we had so far found vector fields which vanish at *two* points. Can we find a vector field on S^{2k} which vanishes on only *one* point? We will discuss this question in Exercise 76.6.

Let v be a vector field on the sphere S^n which is nowhere-vanishing. We have to show that n is odd. We consider the vector field w on S^n that is given by rescaling v to length one, i.e. that is given by

$$\begin{array}{rcl} S^n & \to & \mathbb{R}^{n+1} \\ x & \mapsto & w(x) := \frac{1}{\|v(x)\|} v(x). \end{array}$$

Furthermore we consider the map

$$\begin{array}{rcl} F\colon S^n \times [0,1] & \to & S^n \\ (x,t) & \mapsto & \cos(\pi t)\cdot x + \sin(\pi t)\cdot w(x). \end{array}$$

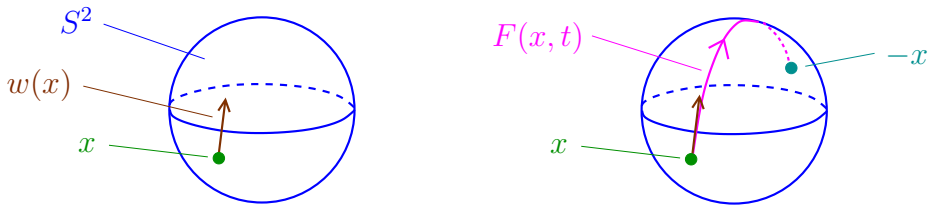
We refer to the figure below for an illustration. The map does indeed take values in S^n, since

$$\langle \cos(\pi t)x + \sin(\pi t)w(x), \cos(\pi t)x + \sin(\pi t)w(x)\rangle \underset{\substack{\uparrow \\ \text{since } \langle x, w(x)\rangle = 0}}{=} \cos(\pi t)^2\|x\|^2 + \sin(\pi t)^2\|w(x)\|^2 \underset{\substack{\uparrow \\ \text{since } \|x\| = \|w(x)\| = 1}}{=} 1.$$

The map $F_0 = F(-,0)$ is the identity and the map $F_1 = F(-,1)$ equals F is $-\mathrm{id}$. Put differently, the map F is a homotopy between the two maps id and $-\mathrm{id}$. It follows that

$$1 \underset{\substack{\uparrow \\ \text{Sphere Degree Lemma 76.11 (1)}}}{=} \deg(\mathrm{id}) \underset{\substack{\uparrow \\ \text{Lemma 76.11 (3)}}}{=} \deg(-\mathrm{id}) \underset{\substack{\uparrow \\ \text{Lemma 76.11 (6)}}}{=} (-1)^{n+1}.$$

But this means that n is odd. ∎



76.5. **The local degree I.** As the reader will have noticed, it is not entirely trivial to compute the isomorphism types of the homology groups of a given topological space. In the following two sections we will get to know a few tricks to address this problem.

We start out with the following harmless lemma which is a variation on the Manifold-Local Homology Lemma 75.1.

Lemma 76.13. Let U be a subset of $\mathbb{R}^n$. For every point P in the interior of U we have

$$\mathrm{H}_k(U, U\setminus\{P\}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since P lies in the interior of U there exists an $r > 0$ such that $B_r^n(P) \subset U$. Now we see that

$$\mathrm{H}_k(U, U\setminus\{P\}) \underset{\substack{\uparrow \\ \text{Excision Theorem 74.18}}}{\leftarrow} \mathrm{H}_l(B_r(P), B_r(P)\setminus\{P\}) \underset{\substack{\uparrow \\ \text{isomorphism by the long exact sequence in reduced homology of the pair } (B_r^n(P), B_r^n(P)\setminus\{P\})}}{\xrightarrow{\partial}} \widetilde{\mathrm{H}}_{k-1}(B_r^n(P)\setminus\{P\}) \underset{\substack{\uparrow \\ \text{isomorphism since } S^{n-1} \text{ is homotopy equivalent to } B_r^n(P)\setminus\{P\}}}{\leftarrow} \widetilde{\mathrm{H}}_{k-1}(S^{n-1}).$$

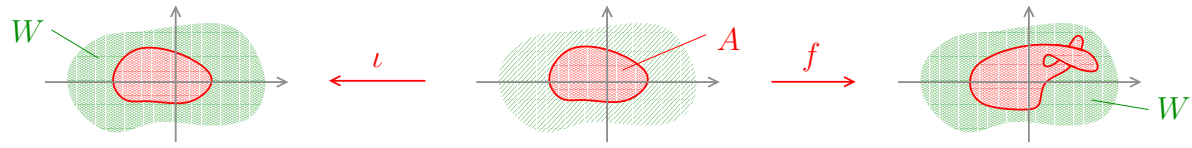
The lemma now follows from the calculation of $\widetilde{\mathrm{H}}_*(S^{n-1})$ in Proposition 74.4. ∎

We move on to the following much more interesting proposition.

Proposition 76.14. (Local Diffeomorphism-Degree Proposition) Let W be a subset of $\mathbb{R}^n$ and let A be a subset of W such the origin 0 is contained in the interior of U. Let $f\colon A \to W$ be an injective map with $f(0) = 0$. We suppose that f is a local diffeomorphism at 0. Then the degree of the map[925]

$$\mathrm{H}_n(W, W \setminus \{0\}) \xleftarrow[\cong]{\iota_*} \mathrm{H}_n(A, A \setminus \{0\}) \xrightarrow{f_*} \mathrm{H}_n(W, W \setminus \{0\})$$

equals $+1$ if f is orientation-preserving at 0 and -1 if f is orientation-reversing at 0.



In the proof of the Local Diffeomorphism-Degree Proposition 76.14 we will make use of the following two lemmas which we will also use on several other occasions later on.

Lemma 76.15. Let $f\colon U \to V$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. Let $P \in U$. We denote by

$$\begin{array}{rcl} \lambda\colon \mathbb{R}^n & \to & \mathbb{R}^n \\ x & \mapsto & (\mathrm{D}f_P)(x) + f(P) \end{array}$$

the linearization of f at P. There exists an open neighborhood $U' = B_r^n(P)$ of P such that the restrictions of f and λ to maps $(U', U' \setminus \{P\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{f(P)\})$ are homotopic.

Remark. Lemma 76.15 is the key to translating information from smooth methods into topology and later on homology groups.

Proof. Let $f\colon U \to V$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. To simplify the notation we assume that $P = 0$ and that $f(P) = 0$. Furthermore, by restriction to an open ball we might as well assume that U is convex. We write $D := \mathrm{D}f_0$. Thus the linearization of f at $P = 0$ is given by the map $\lambda(x) = D \cdot x$. Since f is in particular a local diffeomorphism a $P = 0$ we know by the Chain Rule 19.10 together with the Smooth Invariance of Domain Proposition 19.11, that D is an invertible matrix.

Since $U \subset \mathbb{R}^n$ is convex we obtain from the Map-as-Matrix Multiplication Proposition 19.14 that there exists a smooth map $A\colon U \to \mathrm{M}(n \times n, \mathbb{R})$ such that $A(0) = \mathrm{D}f_0$ and such that $f(x) = A(x) \cdot x$ for all $x \in U$. Since $\det(A(0)) \neq 0$ we see that there exists an $r > 0$ such that $\det(A(x)) \neq 0$ for all $x \in B_r^n(0)$. This implies that the map

$$\begin{array}{rcl} B_r^n(0) \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & A(x \cdot (1-t)) \cdot x \end{array}$$

is a homotopy of the maps f and λ as maps $(B_r^n(0), B_r^n(0) \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ of pairs of topological spaces. ∎

We adopt the following notation:

Notation.

[925] By Lemma 76.13 we know that $\mathrm{H}_n(W, W \setminus \{0\}) \cong \mathbb{Z}$, thus it makes sense to talk of the degree of an automorphism of $\mathrm{H}_n(W, W \setminus \{0\})$.

(1) Given a matrix $P \in \mathrm{GL}(n,\mathbb{R})$ we denote by $\rho(P)\colon \mathbb{R}^n \to \mathbb{R}^n$ the map given by $x \mapsto P \cdot x$.
(2) Given a non-zero real number r we denote its sign by $\mathrm{sign}(r) \in \{-1,+1\}$.

Lemma 76.16. Let $n \in \mathbb{N}$. Given $A \in \mathrm{GL}(n,\mathbb{R})$ we have

$$\deg\big(\rho(A)_*\colon \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) \to \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\})\big) = \mathrm{sign}(\det(A))$$

and given $A \in \mathrm{O}(n)$ we have

$$\deg\big(\rho(A)_*\colon \mathrm{H}_n(\overline{B}^n,\overline{B}^n\setminus\{0\}) \to \mathrm{H}_n(\overline{B}^n,\overline{B}^n\setminus\{0\})\big) = \mathrm{sign}(\det(A)).$$

Proof of Lemma 76.16. For simplicity we only prove the first statement of the lemma. Thus let $A \in \mathrm{GL}(n,\mathbb{R})$. We start out by picking an orthogonal matrix $B \in \mathrm{O}(n)$ such that $\mathrm{sign}(\det(B)) = \mathrm{sign}(\det(A))$. Next we consider the following diagram

$$\begin{array}{ccccccc}
\mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) & \xrightarrow{\mathrm{id}} & \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(\mathbb{R}^n\setminus\{0\}) & \longleftarrow & \mathrm{H}_{n-1}(S^{n-1}) \\
\downarrow{\scriptstyle \rho(A)_*} & & \downarrow{\scriptstyle \rho(B)_*} & & \downarrow{\scriptstyle \rho(B)_*} & & \downarrow{\scriptstyle \rho(B)_*} \\
\mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) & \xrightarrow{\mathrm{id}} & \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(\mathbb{R}^n\setminus\{0\}) & \longleftarrow & \mathrm{H}_{n-1}(S^{n-1}).
\end{array}$$

We make the following observations:

(1) Since $\mathrm{sign}(\det(B)) = \mathrm{sign}(\det(A))$ we obtain from the Matrix Group Path-Component Proposition 2.37 (1) that there exists a path $[0,1] \to \mathrm{GL}(n,\mathbb{R})$ from A to B. Similar to the discussion on page 419 this path gives rise to a homotopy between $\rho(A)$ and $\rho(B)$. We obtain from Proposition 74.15 that $\rho(A)_* = \rho(B)_*$, in other words, the square to the left commutes.
(2) The square in the middle commutes by the naturality of the connecting homomorphism, see the Topological LES-Corollary 74.14.
(3) The horizontal maps to the right are induced by the inclusion $S^{n-1} \to \mathbb{R}^n \setminus \{0\}$. It is clear that the square to the right commutes.

Now we see that

$$\begin{array}{ccccc}
 & & & & \text{by the above commutative diagram together with Lemma 76.9 (2)} \\
 & & & & \downarrow \\
\deg\big(\mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) & \xrightarrow{\rho(A)_*} & \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\})\big) & = & \deg(\rho(B)\colon S^{n-1}\to S^{n-1}) \\
 & & = \det(B) & = \mathrm{sign}(\det(B)) & = \mathrm{sign}(\det(A)). \\
 & & \uparrow & \uparrow & \uparrow \\
 & & \text{Sphere Degree Lemma 76.11 (6)} & \text{since } B\in\mathrm{O}(n) \text{ we have } \det(B)\in\{\pm1\} & \text{choice of } B
\end{array}$$

■

Proof of the Local Diffeomorphism-Degree Proposition 76.14 . Let W be a subset of $\mathbb{R}^n$ and let A be a subset of W such that the origin 0 is contained in the interior. Let $f\colon A \to W$ be an injective map with $f(0) = 0$. We suppose that f is a local diffeomorphism at 0. Recall that this means that there exists an open neighborhood U of 0 that is contained in A and an open neighborhood V of 0 that is contained in B such that the map $f\colon U \to V$ is a diffeomorphism.

We start out with the following preparations.

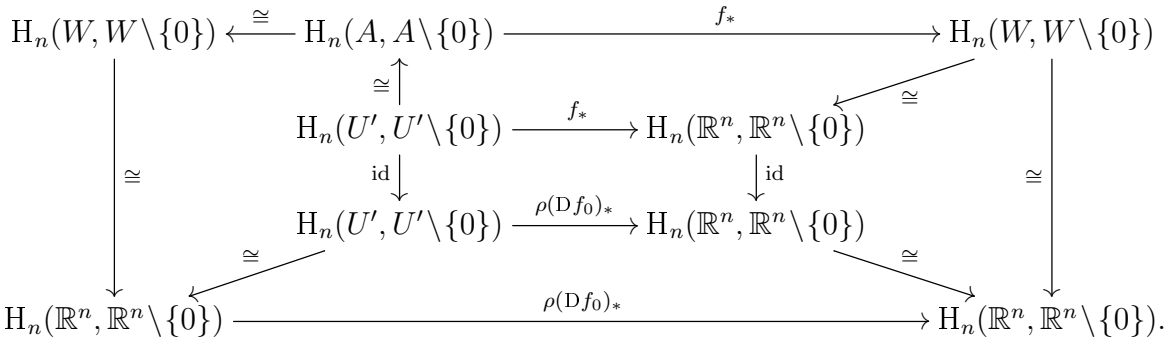
(1) We set

$$\epsilon := \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } 0, \\ -1, & \text{if } f \text{ is orientation-reversing at } 0. \end{cases}$$

Note that it follows from the Orientation Checking-via-Differentials Lemma 25.12 that we have $\epsilon = \operatorname{sign}(\det(\mathrm{D}f_0))$.

(2) By Lemma 76.15 there exists an open neighborhood U' of 0 that is contained in U such that the restrictions of f and $\rho(\mathrm{D}f_0)$ to maps $(U', U' \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ are homotopic.

Now we consider the following diagram



We continue with the following clarifications and observations:

(3) The undecorated maps are induced by inclusions. All of them are isomorphisms by the Excision Theorem 74.18.

(4) It follows from (2) and Proposition 74.15 that the central square commutes. All other regions of the diagram commute by the functoriality of homology groups.

(5) By Lemma 76.16 the bottom horizontal map equals multiplication by $\operatorname{sign}(\det(\mathrm{D}f_0))$.

(6) It follows from all of the above that the top map is given by multiplication by ϵ. But that is precisely what we wanted to show. ■

76.6. **The local degree II.** As mentioned in the previous section, we want to develop some tools for addressing the following problem: given a map $f\colon X \to Y$ between topological spaces, how can we determine the induced map on homology? For example, let us consider the following question.

Question 76.17. Let f be a complex polynomial of degree ≥ 1. By Exercise 2.35 we know that the map

$$\begin{aligned} \Theta(f)\colon S^2 = \mathbb{C} \cup \{\infty\} &\to \mathbb{C} \cup \{\infty\} = S^2 \\ z &\mapsto \begin{cases} f(z), & \text{if } z \in \mathbb{C}, \\ \infty, & \text{if } z = \infty \end{cases} \end{aligned}$$

is continuous. What is the induced map on homology, i.e. what is the degree of $\Theta(f)$?

In this section we will develop a method for determining the degree of a self-map of a sphere which can be applied in many "reasonable" cases. In particular we will be able to answer Question 76.17. The answer will give us an alternative proof of the Fundamental Theorem 26.3 of Algebra.

We start out with the following observation.

Observation. Let $n \geq 1$. Now let $x \in S^n$ be a point and let U be a neighborhood of x in S^n. We have isomorphisms

$$\begin{array}{ccccc} \mathrm{H}_n(S^n) & \xrightarrow{\text{я}} & \mathrm{H}_n(S^n, S^n \setminus \{x\}) & \xleftarrow{\cong} & \mathrm{H}_n(U, U \setminus \{x\}). \\ & \uparrow & & \uparrow & \\ & \text{isomorphism by the long exact of the pair } (S^n, S^n \setminus \{x\}) & & \text{isomorphism by the Excision Theorem 74.18} & \end{array}$$

Throughout this section will use frequently that these maps are isomorphisms.

Definition. We say a map $f\colon X \to Y$ between two topological spaces is nice at a point $x \in X$ if there exists a neighborhood U of x such that for all $x' \neq x$ in U we have $f(x') \neq f(x)$. We refer to such a neighborhood as a nice neighborhood of x.

Examples.

(1) Let $f\colon X \to Y$ be a map between two topological spaces and let $x \in X$. If U is a neighborhood such that $f|_U$ is injective, then U is evidently a nice neighborhood of x.

(2) We consider the functions

$$\begin{array}{rcl} f\colon \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & t^2 \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & \begin{cases} 0, & \text{if } t = 0, \\ t\sin(\frac{1}{t}), & \text{if } t \neq 0. \end{cases} \end{array}$$

For the function f every point on $\mathbb{R}$ is nice, in fact for $x < 0$ we can take $U = (-\infty, 0)$, for $x > 0$ we can take $U = (0, \infty)$ and for $x = 0$ we can take $U = \mathbb{R}$. On the other hand it is straightforward to see that for the function g the point $x = 0$ is not nice.

The following is the key definition of this section.

Definition. Let $n \geq 1$, let $f\colon S^n \to S^n$ be a map and let $x \in S^n$ be a nice point. Let U be a nice neighborhood of x. The local degree of f at the point x is defined as the degree of the homomorphism[926]

$$\mathrm{H}_n(S^n) \xrightarrow{\text{я}} \mathrm{H}_n(S^n, S^n \setminus \{x\}) \xleftarrow{\cong} \mathrm{H}_n(U, U \setminus \{x\}) \xrightarrow{f_*} \mathrm{H}_n(S^n, S^n \setminus \{f(x)\}) \underset{\cong}{\xleftarrow{\text{я}}} \mathrm{H}_n(S^n).$$

We denote this local degree by $\deg(f, x)$.

Remark. We will generalize the notion of a local degree on page 2379 to maps between oriented topological manifolds.

Lemma 76.18. The local degree $\deg(f, x)$ of a map $f\colon S^n \to S^n$ at a nice point $x \in S^n$ is well-defined.

Proof. Let $n \geq 1$, let $f\colon S^n \to S^n$ be a map and let $x \in S^n$ be a nice point. Let U, U' be nice neighborhoods of x. After replacing U' by $U \cap U'$ we can without loss of generality

[926] Note that f induces a map of pairs $(U, U \setminus \{x\}) \to (S^n, S^n \setminus \{f(x)\})$ since U is nice.

assume that $U' \subset U$. We consider the following commutative diagram

$$\begin{array}{ccccccc} & & \mathrm{H}_n(U', U'\backslash\{x\}) & \xrightarrow{f_*} & \mathrm{H}_n(S^n, S^n\backslash\{f(x)\}) & & \\ & \overset{\cong}{\swarrow} & \downarrow \cong & & \downarrow \mathrm{id} & \overset{\cong}{\searrow} & \\ \mathrm{H}_n(S^n) \longrightarrow \mathrm{H}_n(S^n, S^n\backslash\{x\}) & & & & & & \mathrm{H}_n(S^n). \\ & \underset{\cong}{\nwarrow} & & & & \underset{\cong}{\nearrow} & \\ & & \mathrm{H}_n(U, U\backslash\{x\}) & \xrightarrow{f_*} & \mathrm{H}_n(S^n, S^n\backslash\{f(x)\}). & & \end{array}$$

All the non-decorated maps are induced by inclusions. It is clear that the diagram commutes. Also note that the left vertical map is an isomorphism by the Excision Theorem 74.18. It follows immediately from these observations that the local degree defined using U' agrees with the local degree defined using U. ■

Lemma 76.19. Let $n \geq 1$, let $f\colon S^n \to S^n$ be a map and let $x \in S^n$. If f is a homeomorphism, then for every nice point $x \in S^n$ we have

$$\deg(f, x) \;=\; \deg(f).$$

Proof. From the hypothesis that f is a homeomorphism we obtain that $U = S^n$ is a nice neighborhood for x. We take $V = S^n$ and we consider the following diagram

$$\begin{array}{ccc} \mathrm{H}_n(S^n) & \xrightarrow{\qquad f_* \qquad} & \mathrm{H}_n(S^n) \\ \cong \downarrow \text{я} & & \text{я} \downarrow \cong \\ \mathrm{H}_n(S^n, S^n \setminus \{x\}) & \xrightarrow{f_*} & \mathrm{H}_n(S^n, S^n \setminus \{f(x)\}) \end{array}$$

The statement now follows immediately from the definitions. ■

The following proposition can often be used to determine the local degree of a map.

Proposition 76.20. Let $f\colon S^n \to S^n$ be a map with $n \geq 1$ and let $x \in S^n$.

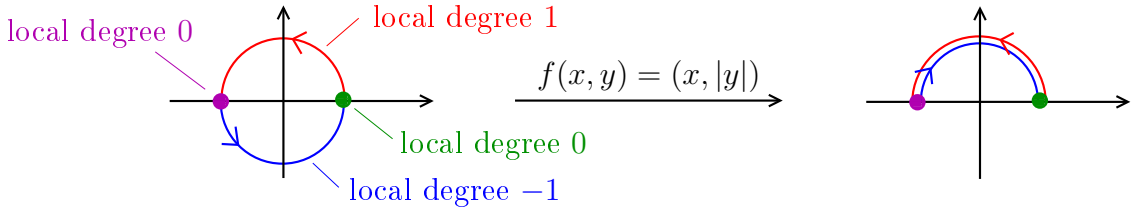
(1) If f is a local homeomorphism around x, then x is nice and

$$\deg(f, x) \;=\; \pm 1.$$

(2) If f is a local diffeomorphism around x, then x is nice and

$$\deg(f, x) \;=\; \begin{cases} 1, & \text{if } f \text{ is orientation-preserving at } x, \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$

Example. We consider the map $f\colon S^1 \to S^1$ given by $(x, y) \mapsto (x, |y|)$. It follows from Proposition 76.20 (2) that for every point on the open upper semicircle the local degree is $+1$ and that for every point on the open lower semicircle the local degree is -1. In Exercise 76.10 we will prove that the local degree at the points ± 1 is in both cases zero.



Remark. Proposition 76.20 shows that for a smooth map $f\colon S^n \to S^n$ and a point $x \in S^n$ at which f is a local diffeomorphism the definition of the local degree on page 898 agrees with the above definition.

Proof of Proposition 76.20. Let $f\colon S^n \to S^n$ be a map with $n \geq 1$ and let $x \in S^n$.

(1) Suppose that f is a local homeomorphism around the point x. Recall that this means that there exists an open neighborhood U of x and an open neighborhood V of $f(x)$ such that $f\colon U \to V$ is a homeomorphism. This implies in particular that U is nice. Now we consider the following rather simple diagram:

$$\begin{array}{ccccccccc} \mathrm{H}_n(S^n) & \xrightarrow{\cong} & \mathrm{H}_n(S^n, S^n\backslash\{x\}) & \xleftarrow{\cong} & \mathrm{H}_n(U, U\backslash\{x\}) & \longrightarrow & \mathrm{H}_n(S^n, S^n\backslash\{f(x)\}) & \xleftarrow{\cong} & \mathrm{H}_n(S^n). \\ & & & & \downarrow{\scriptstyle f_*} & \nearrow{\scriptstyle \cong} & & & \\ & & & & \mathrm{H}_n(V, V\backslash\{f(x)\}) & & & & \end{array}$$

It is clear that the diagram commutes. Since the map f restricts to a homeomorphism $(U, U\backslash\{x\}) \to (V, V\backslash\{f(x)\})$ of pairs we see that the vertical map is an isomorphism. Furthermore the diagonal map is an isomorphism by the Excision Theorem 74.18. Thus we see that all horizontal maps are isomorphism. This implies immediately that $\deg(f, x) = \pm 1$.

(2) Let $f\colon S^n \to S^n$ be a map with $n \geq 1$ and let $x \in S^n$. We assume that f is a local diffeomorphism around x. By the Orthogonal Action Lemma 5.26 we know that there exists an $A \in \mathrm{SO}(n-1)$ with $A \cdot f(x) = x$. By Exercise 25.5 we know that the map $z \mapsto A \cdot z$ is orientation-preserving and by the Sphere Degree Lemma 76.11 (6) we know that $\deg(z \mapsto A \cdot z) = 1$. Thus, after replacing f by the $z \mapsto A \cdot f(z)$, we might as well assume that $f(x) = x$.

We make a few further preparations.

(a) We pick a chart $\Phi\colon V \to W$ around x.

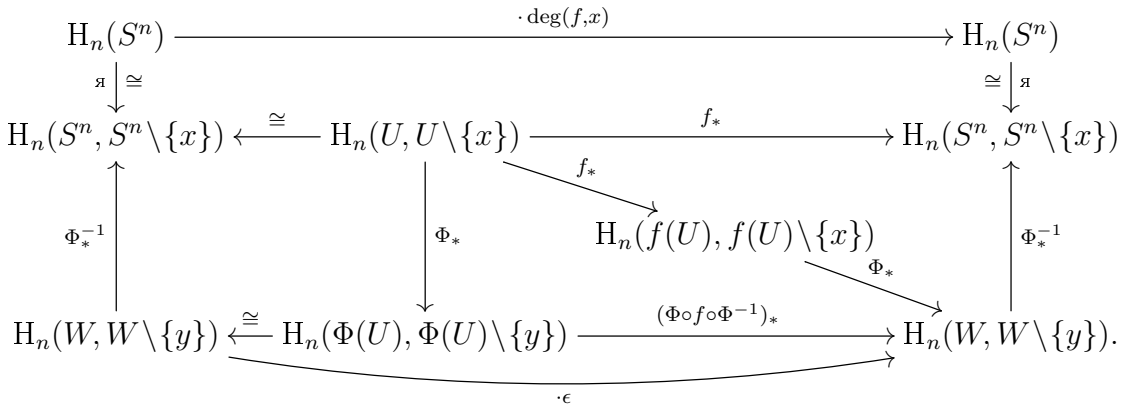
(b) Since f is a local diffeomorphism around x there exists an open neighborhood U of x such that $f\colon U \to f(U)$ is a diffeomorphism. Note that after possibly replacing U by $U \cap V \cap f^{-1}(V)$ we can assume that $U \subset V$ and $f(U) \subset V$.

(c) We set $y := \Phi(x)$.

(d) We set

$$\epsilon := \begin{cases} 1, & \text{if } f \text{ is orientation-preserving at } x, \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$

Next we consider the following diagram



We make the following observations and clarifications.

(a) As usual the undecorated maps are induced by inclusions.

(b) The top rectangle commutes by the definition of $\deg(f, x)$. The middle regions of the diagram commute by the functoriality of relative homology. Finally, note that $\Phi \circ f \circ \Phi^{-1}$ is orientation-preserving at y if and only if f is orientation-preserving at x. Thus we see that the bottom part of the diagram commutes by the Local Diffeomorphism-Degree Proposition 76.14.

(c) It follows from the above together with Lemma 76.9 (2) that $\deg(f, x) = \epsilon$. ■

Using the following proposition we can determine the degree of a map from local degrees.

Proposition 76.21. (Local-Global Degree Proposition) Let $f\colon S^n \to S^n$ be a map where $n > 0$ and let $y \in S^n$ such that $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$.[927] Then[928]

$$\deg(f) \;=\; \sum_{i=1}^{m} \deg(f, x_i).$$

Remark.

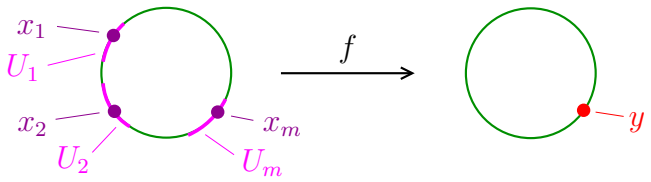
(1) In Proposition 106.32 we will generalize the Local-Global Degree Proposition 76.21 to a more general setting.

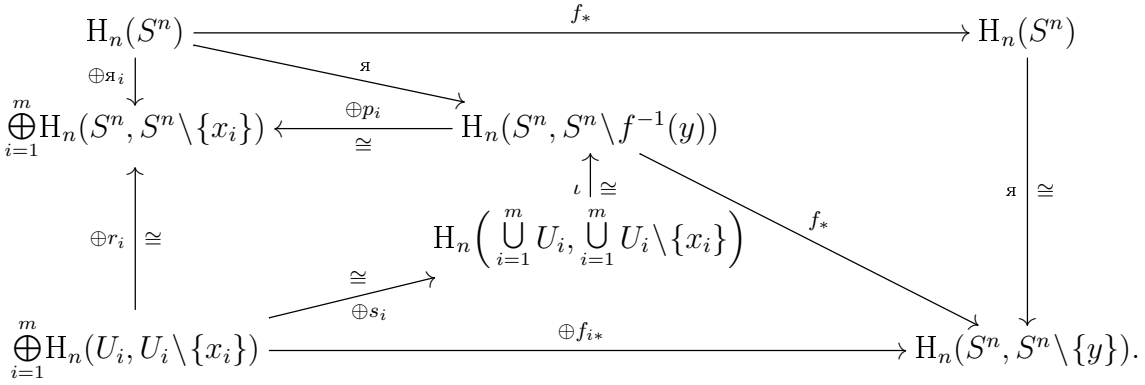
(2) Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a *smooth* map. In this setting we have two definitions of "degree", namely we have the definition from page 1684 and also from page 899. Mercifully it follows from the Local-Global Degree Proposition 76.21 together with Proposition 76.20 that the two definitions agree. We now see that we have some overlap of results, for example the Sphere Degree Lemma 76.11 overlaps with Theorems 38.3 and 38.4 and the Basics of Degree Lemma 38.8.

Proof. It follows easily from the fact that S^n is Hausdorff that we can choose disjoint open neighborhoods $U_1, \ldots, U_m$ for $x_1, \ldots, x_m$ and we choose an open neighborhood V of y such that $f(U_i) \subset V$ for $i = 1, \ldots, m$. We consider the following diagram:

[927] Does there always exist such a y?

[928] The hypothesis implies immediately that all the points $x_1, \ldots, x_m$ are nice, hence the local degrees are defined.





We make the following clarifications and observations:

(1) The maps r_i, s_i, p_i and ι are the maps induced by the obvious inclusions of pairs of topological spaces. Furthermore the maps $f_i \colon (U_i, U_i \setminus \{x_i\}) \to (S^n, S^n \setminus \{y\})$ are given by the restriction of f to U_i.

(2) It follows easily from the functoriality of relative homology groups that the diagram commutes.

(3) The argument on page 1597 shows that the diagonal map $\bigoplus_{i=1}^m s_i$ to the bottom left is an isomorphism.

(4) It follows from the Excision Theorem 74.17 that the vertical map ι and that the maps r_i are isomorphisms.

(5) It follows from the fact that the diagram commutes together with (3) and (4) that the horizontal map $\bigoplus_{i=1}^m p_i$ is also an isomorphism.

Since $я \colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n, S^n \setminus \{y\})$ is an isomorphism it suffices to prove the following claim.

Claim. We have

$$я(f_*([S^n])) \;=\; \sum_{i=1}^m \deg(f, x_i) \cdot я([S^n]) \;\in\; \mathrm{H}_n(S^n, S^n \setminus \{y\}).$$

Proof. We verify the claim by performing the following calculation:

$$я(f_*([S^n])) \overset{\text{since the diagram commutes and since the maps } \oplus p_i \text{ and } \oplus r_i \text{ are isomorphisms}}{=} \Big(\bigoplus_{i=1}^m f_{i*}\Big) \circ \Big(\bigoplus_{i=1}^m r_{i*}\Big)^{-1} \circ \Big(\bigoplus_{i=1}^m я_{i*}\Big)([S^n]) \overset{\text{by Lemma 72.12 (1) and (2) and since each } r_i \text{ is an isomorphism}}{=} \Big(\bigoplus_{i=1}^m f_{i*}\Big) \circ \Big(\bigoplus_{i=1}^m r_{i*}^{-1} \circ я_i\Big)([S^n])$$

$$= \Big(\bigoplus_{i=1}^m f_{i*}\Big)\Big(\sum_{i=1}^m \underbrace{(r_i^{-1} \circ я_i)([S^n])}_{\in \mathrm{H}_n(U_i, U_i \setminus \{x_i\})}\Big) = \sum_{i=1}^m (f_i \circ r_i^{-1} \circ я_i)([S^n]) = \sum_{i=1}^m \deg(f, x_i) \cdot я([S^n]).$$

↑ definition of direct sum of type (II) of homomorphisms, see page 1596

↑ definition of direct sum of type (I) of homomorphisms, see page 1596

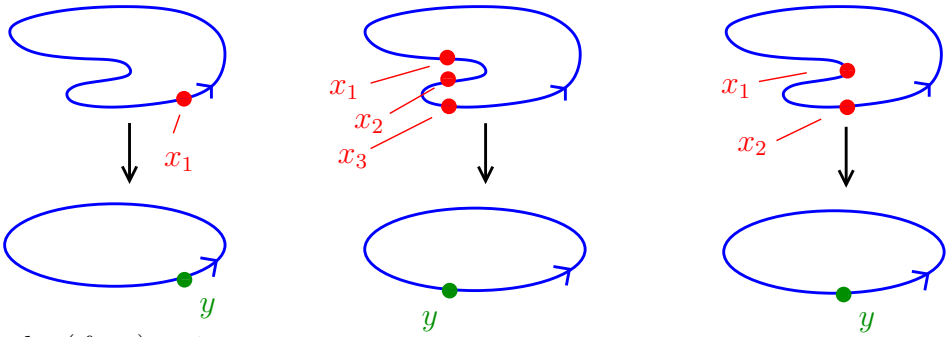
↑ definition of local degree, see page 1694

■

Examples.

(1) The map $f\colon S^1 \to S^1$ given by $f(x,y) = (x, |y|)$ that is illustrated in the figure below has degree zero by the Sphere Degree Lemma 76.11. Taking preimages of points in S^1 one sees that in each case the sum of the local degrees of the preimages is zero.

(2) In the figure below we sketch a map $f\colon S^1 \to S^1$ and we choose several points $y \in S^1$.



$\deg(f, x_1) = 1$

$\deg(f, x_1) = 1$
$\deg(f, x_2) = -1$
$\deg(f, x_3) = 1$

$\deg(f, x_1) = 0$
$\deg(f, x_2) = 1$

We use Proposition 76.20 to determine the various degrees at the given points.[929] We see that the number of preimages and the degrees at the points in the preimages vary, but in all three cases the sum of the degrees at the points in the preimage is the same.

(3) We consider the map

$$\begin{array}{rccc} f\colon & S^1 & \to & S^1 \\ & z & \mapsto & z^m \end{array}$$

We had already determined in Lemma 76.10 that $\deg(f) = m$. Now we will show, just for fun, that the degree of f can also easily be determined using the Local-Global Degree Proposition 76.21.

First let $m \neq 0$. Then we choose $y = 1$. The preimages of y are given by $x_j = \exp(2\pi \mathrm{i} j/m)$ where $j = 1, \ldots, m$. It follows from Proposition 76.20 (2) that for each $j \in \{1, \ldots, m\}$ we have

$$\deg(f, x_j) \;=\; \begin{cases} 1, & \text{if } m > 0, \\ -1, & \text{if } m < 0. \end{cases}$$

Thus it follows from the Local-Global Degree Proposition 76.21 that $\deg(f) = |m| \cdot \operatorname{sign}(m) = m$. If $m = 0$, then $\deg(f) = 0$. The easiest way to see this is to take $y = -1$, since in this case the preimage is the empty set.

As promised, we can now provide an answer to Question 76.17:

[929] Why is in the third example the degree $\deg(f, x_1)$ equal to zero?

Proposition 76.22. Let f be a complex polynomial of degree ≥ 1. The degree of the map

$$\begin{array}{rcl} \Theta(f)\colon S^2 = \mathbb{C}\cup\{\infty\} & \to & \mathbb{C}\cup\{\infty\} = S^2 \\ z & \mapsto & \begin{cases} f(z), & \text{if } z\in\mathbb{C}, \\ \infty, & \text{if } z=\infty \end{cases} \end{array}$$

equals the degree of the polynomial f.

Sketch of proof. Let f be a complex polynomial of degree $n \geq 1$. We also consider the polynomial $g(z) = z^n$. We calculate that

$$\deg(\Theta(f)) \overset{\text{by the Sphere Degree Lemma 76.11 (3), since by Exercise 14.9 we know that } \Theta(f) \text{ and } \Theta(g) \text{ are homotopic}}{\downarrow =} \deg(\Theta(g(z)=z^n)) \overset{\uparrow}{=} n.$$

follows from the Local-Global Degree Proposition 76.21 and Proposition 76.20, see Exercise 76.7 ∎

More excitingly we can provide one more proof for the Fundamental Theorem of Algebra.

Theorem 26.3. (Fundamental Theorem of Algebra) ❀ Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a zero in $\mathbb{C}$.

Remark. We already gave two other proofs on page 704 and page 704.

Proof. Let f be a nonconstant polynomial with complex coefficients. In other words, f is a complex polynomial of degree $n \geq 1$. By Proposition 76.22 we know that the degree of the self-map $\Theta(f)$ of $S^2 = \mathbb{C}\cup\{\infty\}$ equals $n \geq 1$. By the Sphere Degree Lemma 76.11 (2) this implies that $\Theta(f)\colon \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$ is surjective, in particular 0 lies in the image of $\Theta(f)$. Since $\Theta(f)(\infty) = \infty$ we see that 0 lies in the image of $f\colon \mathbb{C} \to \mathbb{C}\cup\{\infty\}$. In other words, there exists a $z \in \mathbb{C}$ with $\underbrace{\Theta(f)(z)}_{=f(z)} = 0$. ∎

76.7. Proof of the Standard Generator-Relationship Proposition 76.5.

For the tenacious reader we now provide the long-delayed proof of Statements (2) and (3) of the Standard Generator-Relationship Proposition 76.5. For the convenience of the reader we recall the relevant two statements.

Proposition 76.5. Let $n \in \mathbb{N}$. We denote by $f\colon (\Delta^n, \partial\Delta^n) \to (\overline{B}^n, S^{n-1})$ the homeomorphism from the Standard Simplex Homeomorphism Lemma 72.1 and we denote by $g\colon \overline{B}^n/S^{n-1} \to S^n$ the homeomorphism from Lemma 5.20. We set $* := g([S^{n-1}])$. We consider the following sequence of maps:

$$\mathrm{H}_n(\Delta^n, \partial\Delta^n) \xrightarrow{f_*} \mathrm{H}_n(\overline{B}^n, S^{n-1}) \longrightarrow \mathrm{H}_n(\overline{B}^n/S^{n-1}, [S^{n-1}]) \xrightarrow{g_*} \mathrm{H}_n(S^n, *) \xleftarrow{\text{я}} \mathrm{H}_n(S^n).$$

The following statements hold:

(2) We have

$$f_*([\mathrm{id}_{\Delta^n}]) \;=\; (-1)^n \cdot \underbrace{\text{standard generator } [\overline{B}^n] \text{ of } \mathrm{H}_n(\overline{B}^n, S^{n-1})}_{\text{defined by the condition } \partial([\overline{B}^n]) = [S^{n-1}], \text{ see page 1679}}.$$

(3) The standard generator $[\overline{B}^n]$ of $\mathrm{H}_n(\overline{B}^n, S^{n-1})$ has the same image in $\mathrm{H}_n(S^n, *)$ as the standard generator $[S^n]$ of $\mathrm{H}_n(S^n)$.

In the following we introduce and recall the following notation:

Notation. Let $n \in \mathbb{N}_0$.

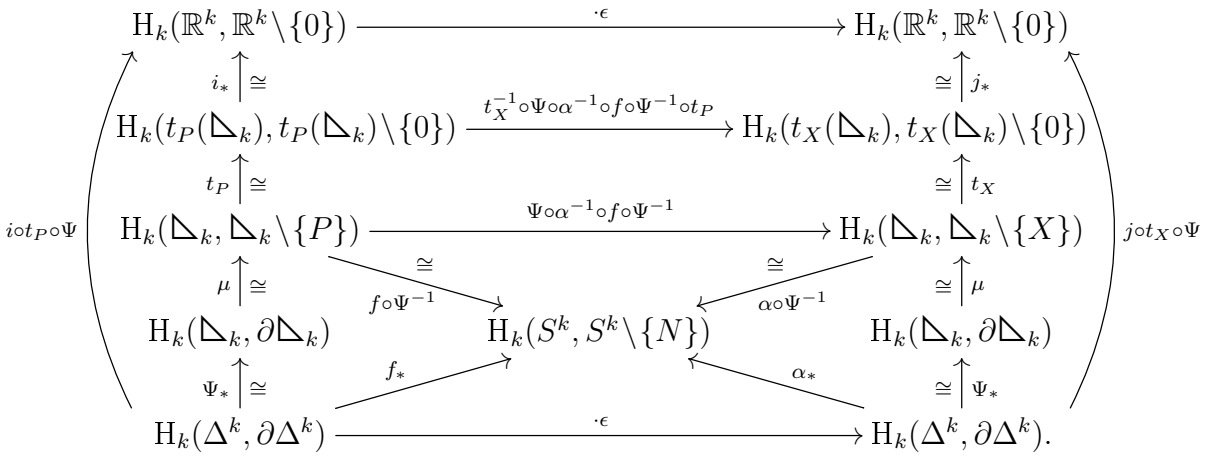
(1) Given $P \in S^n$ we denote by $[S^n]_P$ the image of the standard generators $[S^n]$ under the map $\mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n, S^n \setminus \{P\})$.
(2) Let $\Psi\colon \Delta^n \to \mathbin{\triangle}_n$ be the obvious projection. In the Standard Simplex Homeomorphism Lemma 72.1 (1) we saw that Ψ is a homeomorphism. We write $\Phi := \Psi^{-1}\colon \mathbin{\triangle}_n \to \Delta^n$.
(3) Let $\alpha\colon \Delta^n \to S^n_{\geq 0} \subset S^n$ and $\beta\colon \Delta^n \to S^n_{\leq 0} \subset S^n$ be the two maps defined on page 1677.
(4) We denote by $N = (0, \ldots, 0, 1)$ the North Pole of S^n.

The proof of the Standard Generator-Relationship Proposition 76.5 requires the following lemma.

Lemma 76.23. Let $k \in \mathbb{N}_0$ and let $f\colon \Delta^k \to S^k$ be an injective map such that the map $f \circ \Phi\colon \mathring{\mathbin{\triangle}}_k \to S^k$ is a smooth embedding.[930] Then for every $Q \in f(\mathring{\Delta}^k)$ we have the following equality in $\mathrm{H}_k(S^k, S^k \setminus \{Q\})$:

$$[f_*(\mathrm{id}_{\Delta^k})] \;=\; \epsilon \cdot [S^k]_Q \qquad \text{where } \epsilon = \begin{cases} +1, & \text{if } f \circ \Phi\colon \mathring{\mathbin{\triangle}}^k \to S^k \text{ is orientation-preserving,} \\ -1, & \text{if } f \circ \Phi\colon \mathring{\mathbin{\triangle}}^k \to S^k \text{ is orientation-reversing.} \end{cases}$$

Proof of Lemma 76.23. First we deal with the special case that $Q = N$ and that $f(\Delta^k) \subset S^k_{\geq 0}$. We consider the following diagram:



We make the following clarifications and observations:

(1) We set $P := \Psi(f^{-1}(N)) \in \mathring{\mathbin{\triangle}}_k$ and $X := \Psi(\alpha^{-1}(N)) \in \mathring{\mathbin{\triangle}}_k$. Evidently we have the equality $(f \circ \Psi^{-1})(P) = N = (\alpha \circ \Psi^{-1})(X)$.
(2) Given $v \in \mathbb{R}^k$ we denote by $t_v\colon \mathbb{R}^k \to \mathbb{R}^k$ the map that is given by $x \mapsto x - v$.
(3) The maps $i\colon t_P(\mathbin{\triangle}_k) \to \mathbb{R}^k$ and $j\colon t_X(\mathbin{\triangle}_k) \to \mathbb{R}^k$ are the inclusion maps.
(4) The maps $\mu\colon (\mathbin{\triangle}_k, \partial\mathbin{\triangle}_k) \to (\mathbin{\triangle}_k, \mathbin{\triangle}_k \setminus \{P\})$ and $\mu\colon (\mathbin{\triangle}_k, \partial\mathbin{\triangle}_k) \to (\mathbin{\triangle}_k, \mathbin{\triangle}_k \setminus \{X\})$ are the inclusion maps of pairs of topological spaces.

[930] Note that $\mathring{\mathbin{\triangle}}_k$ is an open subset of $\mathbb{R}^k$, in particular, following the convention established on page 677, we will view it as an oriented k-dimensional smooth manifold.

(5) Since $f(\Delta^k) \subset S^k_{\geq 0}$ we have in particular that $f(\Delta^k) \subset \alpha(\Delta^k)$ which means that the map $\alpha^{-1} \circ f$ is actually defined.
(6) One can easily verify that $\alpha \circ \Psi^{-1}$ is orientation-preserving. As mentioned on page 683, the translation maps t_v are also orientation-preserving. Therefore we see that the map $t_X^{-1} \circ \Psi \circ \alpha^{-1} \circ f \circ \Psi^{-1} \circ t_P$ is orientation-preserving if and only if the map $f \circ \Psi^{-1} = f \circ \Phi$ is orientation-preserving. Thus we obtain from the Local Diffeomorphism-Degree Proposition 76.14 that the top square commutes.
(7) The second square from the top, the top triangle and the two triangles to the left and right commute basically by definition.
(8) It is straightforward to see that the maps $i \circ t_P \circ \Psi$ and $j \circ t_X \circ \Psi$ are homotopic as maps of pairs $(\Delta^k, \partial\Delta^k) \to (\mathbb{R}^k, \mathbb{R}^k \setminus \{0\})$. In particular we obtain from this observation together with Proposition 74.15 that the two sequences of vertical maps on the left and on the right are actually the same map.
(9) Using Corollary 74.16 (2a) and using the Excision Theorem 74.18 one can show fairly easily that all the vertical maps are isomorphisms and we see that the upper diagonal maps are isomorphisms. Since the triangles to the left and right commute we see that all diagonal maps are isomorphisms.
(10) It follows from (6), (7), (8) and (9) that the bottom triangle also commutes.

Now we are basically done with the proof for our special case. Indeed we see that

$$[f_*(\mathrm{id}_{\Delta^k})] \underset{\text{by (10)}}{\overset{\uparrow}{=}} \epsilon \cdot [\alpha_*(\mathrm{id}_{\Delta^k})] = \epsilon \cdot [\alpha] \underset{\text{since } \beta = 0 \in \mathrm{C}_k(S^n, S^n \setminus \{N\})}{\overset{\uparrow}{=}} \epsilon \cdot [\alpha - \beta]_Q \underset{\text{definition of } [S^k]}{\overset{\uparrow}{=}} \epsilon \cdot [S^k]_Q.$$

Now we still need to deal with the general case. In all likelihood few readers will have made it so far. So let us just sketch the ingredients for reducing the general case to the above special case:

(1) By the Orthogonal Action Lemma 5.26 (4) there exists an $A \in \mathrm{SO}(n+1)$ with $A \cdot Q = N$. By the discussion on page 419 the map $\varphi\colon S^n \to S^n$ that is given by $\varphi(x) = A \cdot x$ is homotopic to the identity.
(2) Every map $\varphi\colon \Delta^k \to S^k$ which contains N in its image is homotopic to some map $\psi\colon \Delta^k \to S^k_{\geq 0}$ into the upper hemispheres which has otherwise "the same properties".

We leave it to the reader to fill in the details. ■

Proof of the Standard Generator-Relationship Proposition 76.5 (2) and (3).

(2) We need to show that $f_*([\mathrm{id}_{\Delta^n}]) = (-1)^n \cdot [\overline{B}^n]$. To do so we recall the following maps:
  (a) As on page 1585 we denote by $i = i^n_n\colon \Delta^{n-1} \to \Delta^n$ the n-th face map that is given by $i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{n-1}, 0)$.
  (b) Below we will denote by ж the natural map from reduced homology to relative homology that we introduced in Lemma 74.12.

We consider the following diagram

$$\begin{array}{ccccccc}
\overset{[\mathrm{id}_{\Delta^n}]}{\mathrm{H}_n(\Delta^n,\partial\Delta^n)} & \xrightarrow{\partial} & \overset{\partial[\mathrm{id}_{\Delta^n}]}{\widetilde{\mathrm{H}}_{n-1}(\partial\Delta^n)} & \xrightarrow{\text{ж}} & \overset{\partial[\mathrm{id}_{\Delta^n}]=(-1)^n\cdot i_*([\mathrm{id}_{\Delta^{n-1}}])}{\mathrm{H}_{n-1}(\partial\Delta^n,\Lambda)} & \underset{\cong}{\overset{i_*}{\longleftarrow}} & \overset{(-1)^n\cdot[\mathrm{id}_{\Delta^{n-1}}]}{\mathrm{H}_{n-1}(\Delta^{n-1},\partial\Delta^{n-1})} \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & \swarrow{\scriptstyle (f\circ i)_*} & \\
\underset{[\overline{B}^n]}{\mathrm{H}_n(\overline{B}^n,S^{n-1})} & \xrightarrow{\partial} & \underset{[S^{n-1}]=[\alpha-\beta]}{\widetilde{\mathrm{H}}_{n-1}(S^{n-1})} & \xrightarrow{\text{ж}} & \underset{[\alpha-\beta]=[\alpha]}{\mathrm{H}_{n-1}(S^{n-1},S^{n-1}\setminus\{N\})} & \xleftarrow[\alpha_*]{} & \underset{[\mathrm{id}_{\Delta^{n-1}}]}{\mathrm{H}_i(\Delta^{n-1},\partial\Delta^{n-1})}.
\end{array}$$

(Here we put on top and above of the various groups the elements which will play a role in the subsequent argument.) Note that the diagram commutes by the Topological LES-Corollary 74.14 (2) and Lemma 74.12 (1). Now we see that we have the following equalities in $\mathrm{H}_{n-1}(S^{n-1},S^{n-1}\setminus\{N\})$:

$$\begin{array}{ccccccc}
 & & \text{since the diagram commutes} & & \text{we have } \text{ж}(\partial(\mathrm{id}_{\Delta^n}))=(-1)^n\cdot i_*(\mathrm{id}_{\Delta^{n-1}})\in \mathrm{C}_{n-1}(\partial\Delta^n,\Lambda) \text{ since all other summands of } \partial(\mathrm{id}_{\Delta^n}) \text{ are zero in } \mathrm{C}_{n-1}(\partial\Delta^n,\Lambda) & & \\
\text{ж}(\partial(f_*([\mathrm{id}_{\Delta^n}]))) & \overset{\downarrow}{=} & f_*(\text{ж}(\partial([\mathrm{id}_{\Delta^n}]))) & \overset{\downarrow}{=} & f_*((-1)^n\cdot i_*([\mathrm{id}_{\Delta^{n-1}}])) & = & (-1)^n\cdot[\alpha] \\
 & = & (-1)^n\cdot(f\circ i)_*([\mathrm{id}_{\Delta^{n-1}}]) & = & (-1)^n\cdot\alpha_*([\mathrm{id}_{\Delta^{n-1}}]) & & \\
 & & & & \uparrow & & \\
 & & \multicolumn{3}{c}{\text{we leave it to the reader to verify that } f\circ i\colon \Delta^{n-1}\to S^{n-1} \text{ and } \alpha\colon\Delta^{n-1}\to S^{n-1} \text{ are both orientation-preserving, thus the equality follows from Lemma 76.23}} & & \\
 & = & (-1)^n\cdot([\alpha-\beta]) & = & (-1)^n\cdot\text{ж}([S^{n-1}]) & = & (-1)^n\cdot\text{ж}(\partial([\overline{B}^n])). \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \text{since } \beta=0\in \mathrm{C}_{n-1}(S^n,S^{n-1}\setminus\{N\}) & & \text{definition of } [S^{n-1}] & & \text{definition of } [\overline{B}^n]
\end{array}$$

As discussed earlier, by the various long exact sequences we know that the maps ∂ and ж are both isomorphisms. Thus we obtain from the above the desired equality $f_*([\mathrm{id}_{\Delta^n}])=(-1)^n\cdot[\overline{B}^n]$.

(3) We consider the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{H}_n(\Delta^n,\partial\Delta^n) & \xrightarrow{f_*} & \mathrm{H}_n(\overline{B}^n,S^{n-1}) & \longrightarrow & \mathrm{H}_n(\overline{B}^n/S^{n-1},[S^{n-1}]) & \xrightarrow{g_*} & \mathrm{H}_n(S^n,*) \longleftarrow \mathrm{H}_n(S^n). \\
 & \searrow & & & & & \downarrow\cong \quad \swarrow \\
 & & & & & & \mathrm{H}_n(S^n,S^n\setminus\{-*\})
\end{array}$$

Let $\Phi\colon \triangle_n\to\Delta^n$ be the map that we introduced above. One can easily verify that the map $g\circ f\circ\Phi\colon \overset{\circ}{\triangle}_n\to S^n$ is orientation-preserving if and only if n is even. This observation, together with Lemma 76.23 implies that the image of $[\mathrm{id}_{\Delta^n}]$ in $\mathrm{H}_n(S^n,S^n\setminus\{-*\})$ under the diagonal map agrees with $[S^n]_{-*}$. Since the vertical map is an isomorphism we obtain the desired equality. ■

Exercises for Chapter 76.

Exercise 76.1. Let $n\in\mathbb{N}_0$ and let $P\in\overset{\circ}{\Delta}{}^n$. Show that $\mathrm{H}_n(\Delta^n,\Delta^n\setminus\{P\})=\mathbb{Z}\cdot[\mathrm{id}_{\Delta^n}]$.
Hint. Make use of the Identity-is-Generator Lemma 76.2.

Exercise 76.2. Let $n\in\mathbb{N}$. Let

$$\begin{array}{rcl}
\rho\colon\Delta^n & \to & \Delta^n \\
(t_0,t_1,t_2,\dots,t_n) & \mapsto & (t_1,t_0,t_2,\dots,t_n)
\end{array}$$

be the map that swaps the first two coordinates of Δ^n. Show that the induced map $\rho_*\colon \mathrm{H}_n(\Delta^n, \partial\Delta^n) \to \mathrm{H}_n(\Delta^n, \partial\Delta^n)$ is given by multiplication by -1.
Hint. First prove the statement for $n = 1$. Then use an induction argument similar to the proof of the Identity-is-Generator Lemma 76.2 (2). Hereby you need to use the naturality of the connecting homomorphism of the long exact sequence in homology of a pair of topological spaces.

Exercise 76.3. For which $n \in \mathbb{N}$ does there exist a singular simplex $\sigma\colon \Delta^n \to S^n$ that represents a generator of $\mathrm{H}_n(S^n)$?

Exercise 76.4. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a map with $f(0) = 0$, with $f(\mathbb{R}^n \setminus \{0\}) \subset \mathbb{R}^n \setminus \{0\}$ and with $f(S^{n-1}) \subset S^{n-1}$. Show that

$$\deg(f_*\colon \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})) \;=\; \deg(f_*\colon \mathrm{H}_{n-1}(S^{n-1}) \to \mathrm{H}_n(S^{n-1})).$$

Remark. The analogous statement also holds if we replace $\mathbb{R}^n$ by $\overline{B}^n$.

Exercise 76.5.

(a) Let $n \in \mathbb{N}_{\geq 2}$. Show that given any $k \in \mathbb{N}$ there exists a map $f\colon \overline{B}^n \to \overline{B}^n$ with $f(S^{n-1}) \subset S^{n-1}$ such that

$$\deg\big(f_*\colon \underbrace{\mathrm{H}_n(\overline{B}^n, S^{n-1})}_{\cong \mathbb{Z}} \to \mathrm{H}_n(\overline{B}^n, S^{n-1})\big) \;=\; k.$$

(b) Show that the conclusion of (a) does not hold for $n = 1$.

Exercise 76.6. On S^{2k} we had found a vector field which vanishes at *precisely two* points. Does there exist a vector field on S^2 which vanishes on only *one* point?

Exercise 76.7. Let $n \geq 1$. We consider the complex polynomial $g(z) = z^n$. Show that the degree of the map $\Theta(g)\colon S^2 \to S^2$ equals n.
Hint. Use the Local-Global Degree Proposition 76.21 and Proposition 76.20.

Exercise 76.8. Let X be a topological space. Suppose that the suspension $\Sigma(X)$ is a closed n-dimensional topological manifold. Show that X is a topological homology $(n-1)$-sphere, i.e. show that $\mathrm{H}_i(X) \cong \mathrm{H}_i(S^{n-1})$ for all $i \in \mathbb{N}_0$.
Remark. Also have a look at Exercise 76.9.

Exercise 76.9. Let $n \in \mathbb{N}_{\geq 3}$ and let M be a 0-connected space with non-trivial fundamental group.

(a) Show that the suspension $\Sigma(M)$ is *not* an n-dimensional topological manifold.
Hint. There are many different approaches one can take. One possibility is to study the fundamental group of $\Sigma(M)$ and to study the fundamental group of $\Sigma(M)$ with some points removed, and to use Exercise 52.3.

(b) Show that the suspension $\Sigma(M)$ is not a topological manifold of any dimension.

Remark. In Exercise 76.8 we just showed that if $\Sigma(M)$ is a closed n-dimensional topological manifold, then M is a topological homology $(n-1)$-sphere, i.e. $\mathrm{H}_i(M) \cong \mathrm{H}_i(S^{n-1})$ for all $i \in \mathbb{N}_0$. Together with (a) and the proof of the Poincaré Conjecture in all dimensions, see Chapter **??**, this shows that if $n \in \mathbb{N}_{\geq 2}$ and if M is a 0-connected n-dimensional topological manifold such that $\Sigma(M)$ is a topological manifold, then M is necessarily homeomorphic to S^n.

Exercise 76.10.

(a) We consider the map $f\colon S^1 \to S^1$ given by $(x, y) \mapsto (x, |y|)$. Show that the local degree at the point -1 is zero.

(b) Let $f\colon S^1 \to S^1$ be a map and let $z \in S^1$ be a nice point. Suppose that $\deg(f, z) = 1$. Does it follow that f is a local homeomorphism at z? It suffices to give a short justification for your answer.

Exercise 76.11. Let $f = f(z)$ be a polynomial with complex coefficients of degree ≥ 1. Let $\Theta(f)$ be the corresponding self-map of $S^2 = \mathbb{C} \cup \{\infty\}$ that we introduced on page 1693. Show that the local degree of $\Theta(f)$ at a root of $f(z)$ equals the multiplicity of the root.

Exercise 76.12. Let $n \in \mathbb{N}$ be even. Show that every map $f\colon \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ has a fixed point.

Hint. Apply the Lift-to-Universal Coverings Proposition 61.13 to get a suitable map $\widetilde{f}\colon S^n \to S^n$. If $\deg(\widetilde{f}) \neq -1$, then we are basically done by the Sphere-Degree Fixed Point Theorem Theorem 38.11. What can we do if $\deg(\widetilde{f}) = -1$?

Exercise 76.13. Let $n \in \mathbb{N}_{\geq 2}$. In this exercise we identify S^{n-1} with the equatorial sphere $\{(x, 0) \in S^n \mid x \in S^{n-1}\}$.

(a) Let $f\colon S^n \to S^n$ be a map with $f(S^n_{\geq 0}) \subset S^n_{\geq 0}$ and with $f(S^n_{\leq 0}) \subset S^n_{\leq 0}$. Show that $\deg(f\colon S^n \to S^n) = \deg(f\colon S^{n-1} \to S^{n-1})$.

(b) Let $f\colon S^n \to S^n$ be a map with $f(S^n_{\geq 0}) \subset S^n_{\leq 0}$ and with $f(S^n_{\leq 0}) \subset S^n_{\geq 0}$. Show that $\deg(f\colon S^n \to S^n) = -\deg(f\colon S^{n-1} \to S^{n-1})$.

Exercise 76.14. Let X be a topological space. Given a homology class $\varphi \in \mathrm{H}_k(X)$ we define

$$\|\varphi\|_{\mathbb{Z}} \;:=\; \inf\Big\{ \sum_{i=1}^{n} |a_i| \;\Big|\; \sigma = \underbrace{\sum_{i=1}^{n} a_i \cdot \sigma_i}_{\text{with } \sigma_i \neq \sigma_j \text{ for } i \neq j} \text{ is a cycle with } [\sigma] = \varphi \in \mathrm{H}_k(X) \Big\}.$$

(a) Let X be a path-connected topological space. Show that for any $\varphi \in \mathrm{H}_1(X)$ we have $\|\varphi\|_{\mathbb{Z}} = 1$.

(b) Let $n \in \mathbb{N}$. Determine $\|[S^n]\|_{\mathbb{Z}}$.

Remark. The notation with the subscript "$\mathbb{Z}$" is used to distinguish this definition from the concept that we will introduce in Exercise 101.9.

Exercise 76.15. Let $f\colon S^n \to S^n$ be an orientation-preserving diffeomorphism. Show that f is homotopic to the identity.

Hint. You can easily arrange that f fixes say the North Pole $N = (0, \ldots, 0, 1)$. Now use Lemma 76.15 and push everything that is "awkward" into the South Pole.

Exercise 76.16. Given $z \in S^1$ we denote by $\Phi_z\colon S^2 \to S^2$ the map that is given by rotating S^2 along the z-axis by the angle z. The map

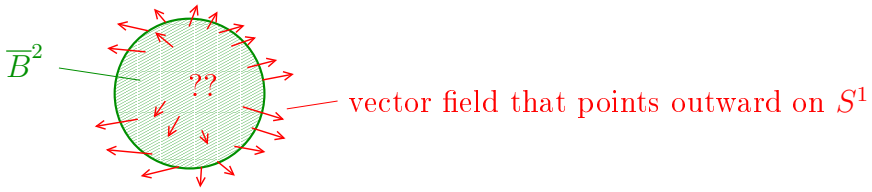
$$\begin{array}{rcl} G\colon S^1 \times S^2 & \to & S^1 \times S^2 \\ (z, v) & \mapsto & (z, \Phi_z(v)) \end{array}$$

is called the *Gluck twist*. Show that the Gluck twist is homotopic to the identity.

Hint. The exercise builds on Exercise 76.15.

Remark. The Gluck twist is in so far remarkable as it is not diffeotopic to the identity. This non-trivial fact is proved in [**Glue1962**, Chapter 5] and [**Wall1970**, p. 232].

Exercise 76.17. Let $n \in \mathbb{N}$. Show that every vector field on $\overline{B}^n$ that points outward on S^{n-1} admits at least one zero in $\overline{B}^n$.



Exercise 76.18. Let $k, l \in \mathbb{Z}$. We consider the sphere $S^3 = \{(z,w) \in \mathbb{C}^2 \,|\, |z|^2 + |w^2| = 1\}$ and the self-map

$$\begin{array}{rcl} f\colon S^3 & \to & S^3 \\ (z,w) & \mapsto & (z^k, w^l). \end{array}$$

Determine the degree of f.

Exercise 76.19. Let X be a topological space with subsets A, B, U, V and suppose that we have the following diagram of inclusion maps

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & U \\ \downarrow{\scriptstyle j} & & \uparrow{\scriptstyle k} \\ V & \xleftarrow{\ l\ } & B. \end{array}$$

Suppose that the inclusion induced maps i_*, j_*, k_*, l_* on the first homology groups are isomorphisms. Does it follow that $i_* \circ j_*^{-1} = k_* \circ l_*^{-1}\colon \mathrm{H}_1(V) \to \mathrm{H}_1(U)$?

Exercise 76.20. Let $n \in \mathbb{N}$ and let $f\colon S^n \times S^n \to S^n$ be a map. We pick $x_0, y_0 \in S^n$. We define the *bidegree* of f as

$$\mathrm{bideg}(f) := (p,q) \text{ where } p := \deg\begin{pmatrix} S^n & \xrightarrow{i} & S^n \\ x & \mapsto & f(x, y_0) \end{pmatrix} \text{ and } q := \deg\begin{pmatrix} S^n & \xrightarrow{j} & S^n \\ y & \mapsto & f(x_0, y) \end{pmatrix}.$$

(a) Show that the definition not depend on the choice of x_0 and y_0.

(b) Let $f\colon S^1 \times S^1 \to S^1$ be the map that is given by complex multiplication. Show that the bidegree of f equals $(1,1)$.
Remark. If you already know quaternions, perform the same exercise for the map $f\colon S^3 \times S^3 \to S^3$ that is given by quaternionic multiplication.

(c) Show that the map

$$\mathrm{H}_n(S^n) \oplus \mathrm{H}_n(S^n) \xrightarrow{i_* \oplus j_*} \mathrm{H}_n(S^n \times S^n) \xrightarrow{f_*} \mathrm{H}_n(S^n)$$

is of the form $\varphi \oplus \psi \mapsto p \cdot \varphi + q \cdot \psi$.

Exercise 76.21. Let $n \in \mathbb{N}$ and let G be a group that acts continuously on S^n. We suppose that the action of G is free. Recall that according to the definition on page 248 this means that for every $g \neq e$ and every $x \in S^n$ we have $g \cdot x \neq x$.

(a) Show that if n is even, then the map

$$\begin{array}{ccc} G & \to & \{\pm 1\} \\ g & \mapsto & \deg\begin{pmatrix} S^n & \to & S^n \\ x & \mapsto & g\cdot x \end{pmatrix} \end{array}$$

is a monomorphism.

Remark. This exercise shows in particular that if a group admits a continuous free action on an even-dimensional sphere, then G has order at most 2. This gives in particular a complete answer to Question 48.8. We will give a different proof of this statement in Proposition 87.10.

(b) Show that the conclusion of (a) does not necessarily hold if n is odd.

Exercise 76.22. Let $n \in \mathbb{N}_0$. We consider

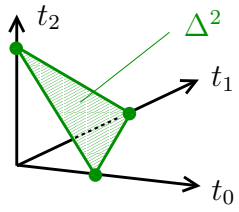
$$\Lambda \;:=\; \bigcup_{i=0}^{n-1} \{(t_0,\ldots,t_n) \in \Delta^n \,|\, t_i = 0\},$$

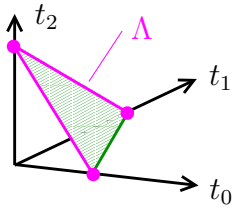
i.e. Λ is the union of all $(n-1)$-dimensional faces of Δ^n except for the face given by $t_n = 0$. We make the identification

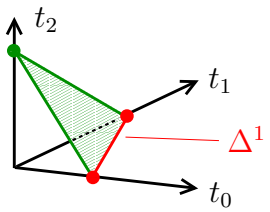
$$\Delta^{n-1} \;:=\; \{(t_0,\ldots,t_n) \in \Delta^n \,|\, t_n = 0\}.$$

Let $f\colon \Delta^n \to \Delta^n$ be a map with $f(\Lambda) \subset \Lambda$, with $f(\Delta^{n-1}) \subset \Delta^{n-1}$ and which also satisfies $f(\partial\Delta^{n-1}) \subset \partial\Delta^{n-1}$. Show that

$$\deg\big(\mathrm{H}_n(\Delta^n,\partial\Delta^n) \xrightarrow{f_*} \mathrm{H}_n(\Delta^n,\partial\Delta^n)\big) \;=\; \deg\big(\mathrm{H}_{n-1}(\Delta^{n-1},\partial\Delta^{n-1}) \xrightarrow{f_*} \mathrm{H}_{n-1}(\Delta^{n-1},\partial\Delta^{n-1})\big).$$







77. The covering dimension of topological spaces II

Recall that in Chapter 11 we introduced the covering dimension of a topological space. Also recall that in Chapter 11 we managed to give upper bounds on the covering dimension for various topological spaces. For example in Proposition 11.11 we showed that for every subspace $X \subset \mathbb{R}^n$ we have $\operatorname{cdim}(X) \leq n$. But we failed to give any meaningful lower bounds on the covering dimension. Now, with the homology groups under our belt, we have the tools to rectify this issue. In particular we now have the tools to answer some questions that had occurred naturally, but that back then we could not answer.
First let us recall the following definition from page 369.

Definition. Let X be a topological space.

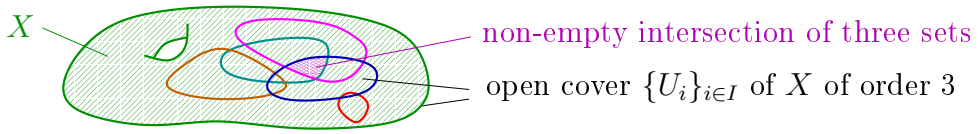
(1) A family $\{U_i\}_{i\in I}$ of subsets of X has order m if the following two statements hold:
   (a) There exists $J \subset I$ with $\#J = m$ such that $\bigcap_{j\in J} U_j \neq \varnothing$.
   (b) For every $J \subset I$ with $\#J > m$ we have $\bigcap_{j\in J} U_j = \varnothing$.

   If no such m exists, then we say that the family has infinite order.

(2) If X is non-empty, then we refer to

$$\operatorname{cdim}(X) \;:=\; \min\left\{ m \in \mathbb{N}_0 \cup \{\infty\} \;\middle|\; \begin{array}{c}\text{every open cover of } X \text{ admits an}\\ \text{open refinement of order} \leq m+1\end{array} \right\}$$

as the covering dimension of X. If X is empty, then we set $\operatorname{cdim}(X) := -1$.



The following is the main technical result of this section.

Theorem 77.1. Let $n \in \mathbb{N}_0$. The covering dimension of the standard n-simplex

$$\Delta^n \;:=\; \{(x_0,\ldots,x_n) \in \mathbb{R}^{n+1} \,|\, x_0 + \cdots + x_n = 1 \text{ and } x_i \geq 0 \text{ for all } i = 0,\ldots,n\}$$

equals n.

Theorem 77.1 has many applications. For example, later in Theorem 77.4 we will use it to nail down the covering dimension of topological manifolds. But before we can revel in the applications, we first have to go through the hard slog of proving Theorem 77.1.
The proof of Theorem 77.1 requires some preparations. First let us recall the following definition from page 1582.

Definition. Let $n \in \mathbb{N}_0$. We denote by $e_1,\ldots,e_{n+1}$ the standard basis vectors of $\mathbb{R}^{n+1}$.

(1) Given a finite subset $K \subset \{1,\ldots,n+1\}$ with $r = \#K$ elements we refer to

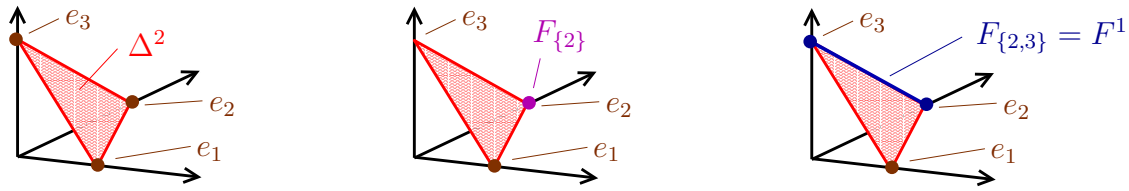
$$\left\{ \sum_{k\in K} \lambda_k \cdot e_k \;\middle|\; \sum_{k\in K} \lambda_k = 1 \text{ and each } \lambda_k \in \mathbb{R}_{\geq 0} \right\} \;\subset\; \Delta^n$$

as an r-dimensional face of the standard n-simplex.

(2) Given $i \in \{1, \dots, n+1\}$ we denote by F^i the $(n-1)$-dimensional face that is given by the set $\{1, \dots, \widehat{i}, \dots, n+1\}$, i.e. we set

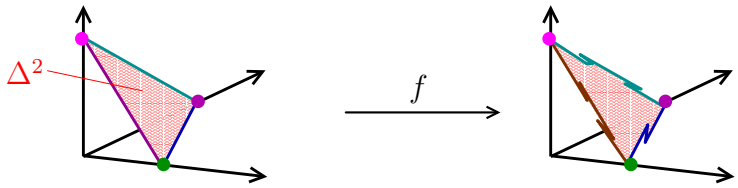
$$F^i := F_{\{1,\dots,\widehat{i},\dots,n+1\}} = \Big\{ \sum_{j=1}^{n+1} \lambda_j \cdot e_j \in \Delta_n \,|\, e_i = 0 \Big\}.$$

(3) As on page 1581 we write $\partial\Delta^n = \bigcup_{i=1}^{n+1} F_i$.



Auxiliary Lemma 77.2. Let $f\colon \Delta^n \to \Delta^n$ be a map. If f sends each face of Δ^n *into* itself, then[931] $f_*\colon \mathrm{H}_n(\Delta^n, \partial\Delta^n) \to \mathrm{H}_n(\Delta^n, \partial\Delta^n)$ is the identity, in particular it is an isomorphism.

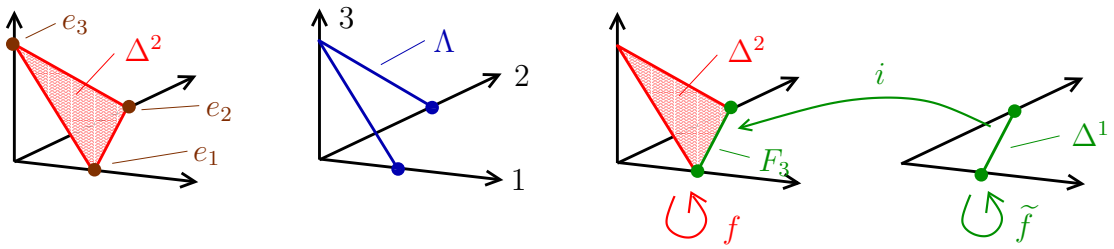
Proof of Lemma 77.2. We prove the lemma by induction on n. The case $n = 0$ is basically trivial, it follows for example from the discussion on page 1594. So now suppose that the statement holds for some $n-1$. Let $f\colon \Delta^n \to \Delta^n$ be a map that sends each face of Δ^n into itself.



We introduce the following notation:

- As on page 1674 we denote by $\Lambda = F^1 \cup \dots \cup F^n$ the union of all $(n-1)$-dimensional faces of Δ^n except for the "last one", i.e. except for the $(n-1)$-dimensional face F^{n+1} that is given by $\{1, \dots, n\}$.
- We denote by $i\colon \Delta^{n-1} \to F^{n+1} \to \Delta^n$ the inclusion of Δ^{n-1} as the "last n-dimensional face", i.e. $i\colon \Delta^{n-1} \to F^{n+1} \to \Delta^n$ is the map given by $i(x) = (x, 0)$.
- The hypothesis that the map $f\colon \Delta^n \to \Delta^n$ sends each face of Δ^n into itself implies that $f(\Lambda) \subset \Lambda$, it implies that $f(\partial\Delta^n) \subset \partial\Delta^n$ and finally it implies that there exists a self-map $\widetilde{f}\colon \Delta^{n-1} \to \Delta^{n-1}$ such that $f \circ i = i \circ \widetilde{f}$. Note that $\widetilde{f}$ again has the property that it sends each face of Δ^{n-1} into itself.
- We denote by $\partial\colon \mathrm{H}_n(\Delta^n, \partial\Delta^n) \to \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda)$ the connecting homomorphism of the long exact sequence of the triple $(\Delta^n, \partial\Delta^n, \Lambda)$, see the Topological LES-Proposition 74.13 for details.

[931] Note that the hypothesis that f sends each face of Δ^n into itself implies that $f(\partial\Delta^n) \subset \partial\Delta^n$. Thus f does indeed induce a self-map of $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$.



We obtain the following commutative diagram

$$\begin{array}{ccccc} \mathrm{H}_n(\Delta^n, \partial\Delta^n) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda) & \xleftarrow{i_*} & \mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \widetilde{f}_*} \\ \mathrm{H}_n(\Delta^n, \partial\Delta^n) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda) & \xleftarrow{i_*} & \mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}). \end{array}$$

We make the following observations:

- The square to the left commutes by the naturality of the connecting homomorphism, see the Topological LES-Corollary 74.14. The square to the right commutes by the functoriality of homology groups and the fact that $f \circ i = ß \circ \widetilde{f}$.
- As on page 1675 we see that all of the horizontal maps are isomorphisms. For the reader's convenience we recall the argument. First note that the long exact sequence of the triple $(\Delta^n, \partial\Delta^n, \Lambda)$ looks as follows:

$$\underbrace{\mathrm{H}_n(\Delta^n, \Lambda)}_{\substack{=0 \text{ by Corollary 74.16 since} \\ \Lambda \text{ is a deformation} \\ \text{retract of } \Delta^n}} \xrightarrow{p} \mathrm{H}_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial} \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda) \xrightarrow{i_*} \underbrace{\mathrm{H}_{n-1}(\Delta^n, \Lambda).}_{\substack{=0 \text{ by Corollary 74.16 since} \\ \Lambda \text{ is a deformation} \\ \text{retract of } \Delta^n}}$$

  Thus we see that the connecting homomorphism is an isomorphism. Now we turn our attention to the inclusion induced maps i_*. It follows from a variation on the Excision Theorem 74.17, namely Theorem 75.9, applied to $X = \partial\Delta^n$, $A = \Lambda$, $Z = \mathring{\Lambda}$ and $U = (0, \ldots, 0, 1)$[932] that the map

$$\mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \;=\; \mathrm{H}_{n-1}(\underbrace{\partial\Delta^n \setminus \mathring{\Lambda}}_{=\Delta^{n-1}}, \underbrace{\Lambda \setminus \mathring{\Lambda}}_{=\partial\Delta^{n-1}}) \xrightarrow{i_*} \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda)$$

  is an isomorphism.
- By induction hypothesis the right hand vertical map is the identity.
- It follows from the above that the left hand vertical map is also the identity. ■

Before we head to the actual proof of Theorem 77.1 we recall, for the reader's convenience, the following lemma.

Lemma 11.2 Let X be a topological space with covering dimension $d := \mathrm{cdim}(X) \in \mathbb{N}_0$. Given any open cover $\{U_i\}_{i \in I}$ there exists an open cover $\{V_i\}_{i \in I}$ of order $\leq d+1$ such that for each $i \in I$ we have $V_i \subset U_i$.

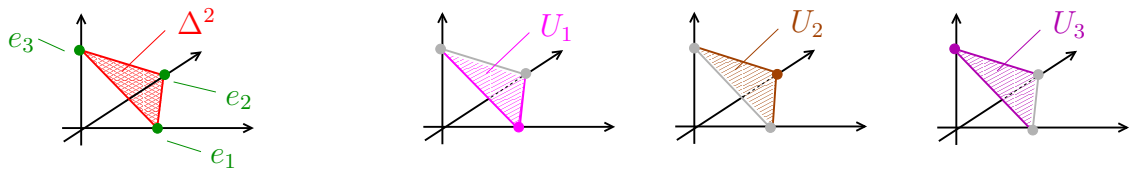
[932] One can easily show that U has the property required in Theorem 75.9.

Proof of Theorem 77.1. Let $n \in \mathbb{N}_0$. We need to show that $\operatorname{cdim}(\Delta^n) = n$. By the Standard Simplex Homeomorphism Lemma 72.1 we know that Δ^n is homeomorphic to a compact subset of $\mathbb{R}^n$. Therefore it follows from Theorem 11.3 that $\operatorname{cdim}(\Delta^n) \leq n$. Thus it remains to show that $\operatorname{cdim}(\Delta^n) \geq n$. We assume by contradiction that $\operatorname{cdim}(\Delta^n) < n$.

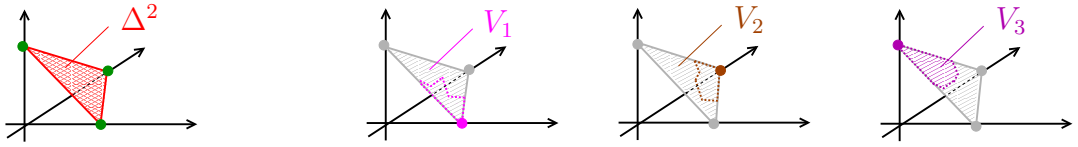
We introduce some notation. Given $i \in \{1, \ldots, n+1\}$ we denote by F^i the $(n-1)$-dimensional face that is given by the set $\{1, \ldots, \widehat{i}, \ldots, n+1\}$, i.e.

$$F^i := F_{\{1,\ldots,\widehat{i},\ldots,n+1\}} = \left\{ \sum_{j=1}^{n+1} \lambda_j \cdot e_j \in \Delta_n \,|\, e_i = 0 \right\}.$$

Furthermore for $i = 1, \ldots, n+1$ we set $U_i := \Delta^n \setminus F^i$. Note that e_i is the only vertex of Δ^n that is contained in U_i. It is rather elementary to verify that each U_i is open in Δ^n. Furthermore, note that $U_1 \cup \cdots \cup U_{n+1} = \Delta^n$. In other words, $U_1, \ldots, U_{n+1}$ is an open cover of Δ^n. Since we assume that $\operatorname{cdim}(\Delta^n) < n$ we know, by Lemma 11.2, that there exists an



open refinement $\{V_1, \ldots, V_{n+1}\}$ of the open cover $\{U_1, \ldots, U_{n+1}\}$ of order $\leq n$ such that for each $i \in \{1, \ldots, n+1\}$ we have $V_i \subset U_i$.



Claim 1. There exists a partition of unity $\{\Phi_i \colon \Delta^n \to [0,1]\}_{i=1,\ldots,n+1}$ on Δ^n such that for each $i \in \{1, \ldots, n+1\}$ we have $\operatorname{supp}(\Phi_i) \subset V_i$.

Proof. In Lemma 10.1 we saw that it follows from the Paracompact-Criterion Lemma 10.2 that $\mathbb{R}^{n+1}$ is paracompact. We deduce from the Closed Subset-Paracompact Lemma 10.3 that the closed subset Δ^n is also paracompact. The statement now follows from the Paracompact-Partition of Unity Theorem 10.9 (1)⇒(3).[933] ⊞

Claim 2. The map

$$\begin{aligned} r \colon \Delta^n &\to \mathbb{R}^{n+1} \\ x &\mapsto \sum_{i \in \{1,\ldots,n+1\}} \Phi_i(x) \cdot e_i \end{aligned}$$

has the following properties:

(1) it sends each face of Δ^n into itself, in particular it sends Δ^n into itself,
(2) we have $r(\Delta^n) \subset \partial\Delta^n$.

Proof.

[933] Alternatively note that by the Standard Simplex Homeomorphism Lemma 72.1 we know that Δ^n is homeomorphic to $\overline{B}^n$, which implies easily that Δ^n is an n-dimensional topological manifold. Thus one can deduce Claim 1 from the Manifold-Partitions of Unity Existence Theorem 21.1

(1) Let $K \subset \{1, \ldots, n+1\}$ be a subset. As above we denote by

$$F_K \ := \ \Big\{ \sum_{k \in K} \lambda_k \cdot e_k \ \Big| \ \sum_{k \in K} \lambda_k = 1 \text{ and each } \lambda_k \in \mathbb{R}_{\geq 0} \Big\}$$

the corresponding face. We need to show that $f(F_K) \subset F_K$. We start out with the three observations:

(a) We have the following description of the face F_K:

$$F_K \ = \Big\{ \sum_{i=1}^{n+1} \lambda_i \cdot e_i \in \Delta^n \ \Big| \ \lambda_i = 0 \text{ if } i \notin K \Big\} = \bigcap_{i \notin K} F^i \ = \bigcap_{i \notin K} (\Delta^n \backslash U_i) \ = \ \Delta^n \backslash \Big(\bigcup_{i \notin K} U_i \Big).$$

(b) It follows from (a) that if $x \in F_K \cap U_i$, then $i \in K$.

(c) Let $x \in \Delta^n$. If $\Phi_i(x) \neq 0$, then $x \in \operatorname{supp}(\Phi_i) \subset V_i \subset U_i$.

Recall that we need to show that $r(F_K) \subset F_K$. Let $x \in F_K$. We calculate

$$r(x) \ \underset{\substack{\uparrow \\ \text{definition of } r}}{=} \ \sum_{i \in \{1,\ldots,n+1\}} \Phi_i(x) \cdot e_i \ \underset{\substack{\uparrow \\ \text{follows from (c)}}}{=} \ \sum_{i \ \in \ \{1,\ldots,n+1\} \text{ with } x \in U_i} \Phi_i(x) \cdot e_i \ \underset{\substack{\uparrow \\ \text{by (b) we know that } x \in F_K \cap U_i \\ \text{implies that } i \in K}}{=} \ \sum_{i \in K} \Phi_i(x) \cdot e_i \ \in \ F_K.$$

(2) Since the cover $\{V_i\}_{i \in \{1,\ldots,n+1\}}$ has order $\leq n$ we see that for each $x \in \Delta^n$ there exist at most n indices $i \in \{1, \ldots, n+1\}$ for which $\Phi_i(x) \neq 0$. But this means that $r(x)$ is the linear combination of at most n vertices, which means that $r(x) \in \bigcup_{i=1}^{n+1} F^i = \partial \Delta^n$. ⊞

We consider the following diagram

$$\begin{array}{ccc} \mathrm{H}_n(\Delta^n, \partial\Delta^n) & \xrightarrow{\quad r_* \quad} & \mathrm{H}_n(\Delta^n, \partial\Delta^n). \\ & {\scriptstyle r_*} \searrow \quad \nearrow & \\ & \mathrm{H}_n(\partial\Delta^n, \partial\Delta^n) & \end{array}$$

We make the following observations:

- By Claim 2 (2) we know that $r(\partial\Delta^n) \subset \partial\Delta^n$. Thus we see that the top horizontal map is actually defined. Next with note that it follows from Claim 2 (2) together with Lemma 77.2 that the top horizontal map is an isomorphism.
- It follows from Claim 2 (3) that the top horizontal map factors through the bottom group.
- The bottom group is evidently zero.
- We know by the Identity-is-Generator Lemma 76.2 that the top groups are non-trivial.

The above four statements are incompatible. Thus we have found a contradiction to our hypothesis that $\operatorname{cdim}(\Delta^n) < n$. ■

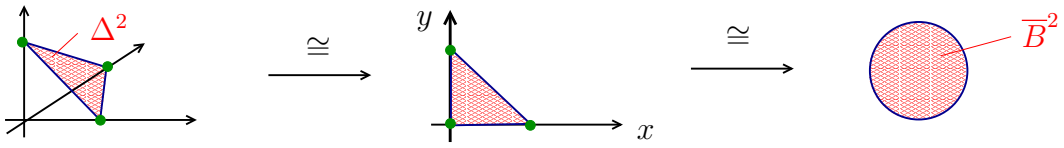
Corollary 77.3. For each $n \in \mathbb{N}_0$ we have $\operatorname{cdim}(\overline{B}^n) = n$.

Proof. Note that

$$\operatorname{cdim}(\overline{B}^n) \ \underset{\substack{\uparrow \\ \text{by Lemmas 11.1 and 72.1}}}{=} \ \operatorname{cdim}(\Delta^n) \ \underset{\substack{\uparrow \\ \text{by Theorem 77.1}}}{=} \ n.$$

■

The name of the game is now to identify topological spaces that contained closed subsets that are homeomorphic to $\overline{B}^n$ or equivalently to Δ^n.



Theorem 77.4. Let $n \in \mathbb{N}_0$. Given any non-empty n-dimensional topological manifold M we have $\operatorname{cdim}(M) = n$.

Proof. Let M be a non-empty n-dimensional topological manifold. In the Covering Dimension-of-Manifolds Proposition 27.7 we already showed that $\operatorname{cdim}(M) \leq n$. Thus it remains to show that $\operatorname{cdim}(M) \geq n$.

Since M is non-empty there exists a chart $\Phi\colon U \to V$ where V is an open non-empty subset of $\mathbb{R}^n$ or $\mathbb{H}^n$. By pick an embedding $f\colon \overline{B}^n \to V$. Note that $\Phi^{-1} \circ f\colon \overline{B}^n \to M$ is an embedding. Now we see that

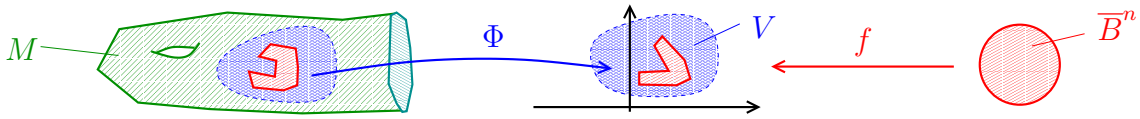
$$\operatorname{cdim}(M) \underset{\uparrow}{\geq} \operatorname{cdim}((\Phi^{-1} \circ f)(\overline{B}^n)) \overset{\downarrow}{=} \operatorname{cdim}(\overline{B}^n) \overset{\downarrow}{\geq} n.$$

this follows from the Closed Subspace Covering Dimension Lemma 11.5, since by the Compact Image Lemma 2.13 and the Compact-Closed Lemma 1.21 we know that $(\Phi^{-1} \circ f)(\overline{B}^n)$ is a closed subset

Lemma 11.1

Corollary 77.3

■



In Proposition 11.11 we showed that for every subset $X \subset \mathbb{R}^n$ we have $\operatorname{cdim}(X) \leq n$. We now have the following complementary result, which gives in particular an affirmative answer to Question 11.10.

Corollary 77.5. Let $X \subset \mathbb{R}^n$ be a subset. If the interior of X is non-empty, then $\operatorname{cdim}(U) = n$.

Proof. Let $X \subset \mathbb{R}^n$ be a subset with $\mathring{X} \neq \varnothing$. Note that by the Interior-Closure Lemma 1.9 we know that $\mathring{X}$ is an open subset of $\mathbb{R}^n$. It follows that there exists an embedding $f\colon \overline{B}^n \to \mathring{X}$. Now we see that

$$n \underset{\uparrow}{\geq} \operatorname{cdim}(X) \underset{\uparrow}{\geq} \operatorname{cdim}(f(\overline{B}^n)) \underset{\uparrow}{=} \operatorname{cdim}(\overline{B}^n) \underset{\uparrow}{\geq} n.$$

Proposition 11.11

by the Closed Subspace Covering Dimension Lemma 11.5 and since it follows from Lemmas 1.21 and 2.13 that $f(\Delta)^n)$ is a closed subset of M

Lemma 11.1

Corollary 77.3

■

Interestingly the converse to Corollary 77.5 holds:

Theorem 77.6. Given any subset $X \subset \mathbb{R}^n$ we have

$$\operatorname{cdim}(X) = n \iff \text{the interior of } X \text{ is non-empty.}$$

Proof. The "$\Leftarrow$"-direction is precisely the content of Corollary 77.5. The proof of the "$\Rightarrow$"-direction is given in [**Enge1978**, Theorem 1.8.10], except that the statement is formulated with the "inductive dimension" instead of the covering dimension. Also note that the

definition of "covering dimension" given in [**Enge1978**, Definition 1.6.7] differs slightly from our. But by [**Enge1978**, Theorem 1.7.7] and [**Enge1978**, Exercise 1.7.E] these three different notions of "dimension" agree for subsets of $\mathbb{R}^n$. ■

For completeness' sake we mention the following proposition which we will not make use of.

Proposition 77.7. Given any CW-complex X we have

$$\dim(X) \;=\; \operatorname{cdim}(X).$$

Proof. Let X be a CW-complex that contains an n-dimensional cell. Let $\Phi\colon \overline{B}^n \to X$ be the characteristic map of an n-cell. By the Characteristic Maps Lemma 68.7 we know that Φ is continuous. Furthermore, by definition, the restriction of Φ to $\overline{B}^n_{\frac{1}{2}} \to X$ is injective. Furthermore note that by the CW-Complex Properties Proposition 68.10 (1) we know that X is Hausdorff.

Since $\overline{B}^n_{\frac{1}{2}}$ is compact, since Φ is continuous and since X is Hausdorff we know by the Compact Image Lemma 2.13, the Compact-Closed Lemma 1.21 (2) and the Compact-Hausdorff Proposition 2.17 (2) that $\Phi\colon \overline{B}^n_{\frac{1}{2}} \to X$ is a closed embedding. Thus we see that

$$\begin{array}{ccccccc} \operatorname{cdim}(X) & \underset{\uparrow}{\geq} & \operatorname{cdim}(\Phi(\overline{B}^n_{\frac{1}{2}})) & \underset{\uparrow}{=} & \operatorname{cdim}(\overline{B}^n_{\frac{1}{2}}) & \underset{\uparrow}{=} & n. \\ & \text{Closed Subspace Covering Dimension Lemma 11.5} & & \text{Lemma 11.1} & & \text{Corollary 77.3} & \end{array}$$

This shows that $\operatorname{cdim}(X) \geq \dim(X)$. The reverse inequality is proved in [**FrPi1990b**, Theorem 4.4]. ■

Exercises for Chapter 77.

Exercise 77.1. We consider the cube $[-1,1]^n$. Given a subset $J = (j_1, \ldots, j_n) \in \{-1, 0, 1\}^n$ we define the corresponding J-face as the subset

$$F_J \;:=\; \{(x_1, \ldots, n_n) \in [-1,1]^n \,|\, \text{for every } i \in \{1, \ldots, n\} \text{ with } j_i \neq 0 \text{ we have } x_i = \epsilon_i\}.$$

Let $f\colon [-1,1]^n \to [-1,1]^n$ be a map that sends every J-face into itself. Show that f is surjective.

Exercise 77.2. Let $n \in \mathbb{N}$.

(a) Provide a finite set $\{M_j\}_{j \in J}$ of submanifolds of S^n such that the following statement holds: if $f\colon S^n \to S^n$ is a map such that for every $j \in J$ we have $f(M_j) \subset M_j$, then $f_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$ is an isomorphism.

(b) What is the minimal cardinality for J that you can achieve?
Remark. The author does not know the answer, but evidently one should strive for minimal cardinality of J. If we allow all submanifolds, in particular open subsets, then the record is $n+2$. If we restrict ourselves to compact submanifolds, then the record is $2n+2$.

Exercise 77.3. Let X be an n-dimensional topological manifold. Show that for every subset $A \subset X$ the covering dimension A is $\leq n$.

78. The Mayer–Vietoris sequence and its applications

In this chapter we will state and prove the Mayer–Vietoris sequence for homology groups. It makes often allows us to compute the homology groups of a topological space from the homology groups of appropriate subspaces together with the knowledge of the nature of inclusion induced maps on homology.
The role of the Mayer–Vietoris sequence in homology theory is similar to the role of the Seifert–van Kampen Theorem 54.1 for fundamental groups. The Mayer–Vietoris sequence will give us the tool to compute the homology groups of many topological spaces that we are interested in. In particular we will determine the homology groups of $\mathbb{R}\mathrm{P}^2$, the torus and the Klein bottle.

78.1. **Split exact sequences.** We already saw in Theorem 74.3 that homology groups of different topological spaces are sometimes related by a long exact sequence of homology groups. The Mayer–Vietoris sequence that we will introduce later on in this chapter is another instance of such a long exact sequence. Very often one ends up in the situation that one knows the homology groups of two out of three topological spaces and hopefully one knows something about maps between these homology groups, and given this data one would like to determine the homology groups of the third topological space from the long exact sequence. Algebraically the question is as follows: suppose we are given an exact sequence

$$\ldots \to A_n \to B_n \to \mathrm{C}_n \to A_{n-1} \to \ldots$$

of abelian groups where we know the groups A_* and C_*, can we determine the groups B_* from this long exact sequence? The same question of course also arises when the roles of A, B and C are switched.
In this section we will study the simplest case, namely the case of a short exact sequence

$$0 \to A \to B \to C \to 0.$$

If we know A and C, can we read "read off" B from this short exact sequence? In general the answer is no. For example the short exact sequence

$$0 \to \mathbb{Z}_2 \longrightarrow B \longrightarrow \mathbb{Z}_2 \to 0$$

could be of the form

$$0 \to \mathbb{Z}_2 \xrightarrow{x\mapsto(x,0)} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{(x,y)\mapsto y} \mathbb{Z}_2 \to 0$$

but it could also be of the form[934]

$$0 \to \mathbb{Z}_2 \xrightarrow{x\mapsto 2x} \mathbb{Z}_4 \xrightarrow{x\mapsto x} \mathbb{Z}_2 \to 0.$$

In this short section we will give a criterion for when the naive guess $B \cong A \oplus C$ is actually the correct answer.

Definition. We say that a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

of (not necessarily abelian) groups splits, if there exists a splitting $s\colon C \to B$, i.e. a homomorphism $s\colon C \to B$ such that $p \circ s = \mathrm{id}_C$.

[934] Are there any other possibilities for the isomorphism type of B?

For example the short exact sequence

$$0 \;\to\; \mathbb{Z} \;\xrightarrow{k\mapsto(k,0)}\; \mathbb{Z}\oplus\mathbb{Z}_n \;\xrightarrow{(k,l)\mapsto l}\; \mathbb{Z}_n \;\to\; 0$$

splits, in fact a splitting is given by the map $b \mapsto (0, b)$. On the other hand the short exact sequence

$$0 \;\to\; \mathbb{Z} \;\xrightarrow{k\mapsto nk}\; \mathbb{Z} \;\to\; \mathbb{Z}_n \;\to\; 0.$$

does not split.[935]

Lemma 78.1. Let

$$0 \;\to\; A \;\xrightarrow{i}\; B \;\xrightarrow{p}\; C \;\to\; 0$$

be a short exact sequence of abelian groups. If C is a free abelian group, then the short exact sequence splits.

Proof. Let $\{c_i\}_{i\in I}$ be a basis of the free abelian group C. For each $i \in I$ we choose a $b_i \in B$ with $p(b_i) = c_i$. By Lemma 51.1 there exists a unique homomorphism $s\colon C \to B$ with $s(c_i) = b_i$ for all i. We have $(p \circ s)(c_i) = c_i$ for each c_i. But then it follows from the uniqueness part of Lemma 51.1 that $p \circ s = \mathrm{id}_C$. The homomorphism s is thus indeed a splitting. ■

The following lemma gives two useful criteria for a short exact sequence to split.

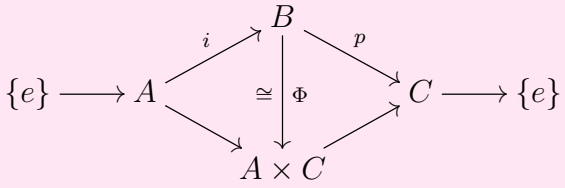
Lemma 78.2. (Splitting Lemma) Let

$$\{e\} \;\to\; A \;\xrightarrow{i}\; B \;\xrightarrow{p}\; C \;\to\; \{e\}$$

be a short exact sequence of groups. The following two statements are equivalent:

(1) There exists a homomorphism $t\colon B \to A$ such that $t \circ i = \mathrm{id}_A$.

(2) There exists an isomorphism $\Phi\colon B \to A \times C$ such that the following diagram commutes:



where the lower left diagonal map is the obvious inclusion and the lower right diagonal map is the natural projection.

Furthermore, if B is abelian, then the above statements are also equivalent to the following statement:

(3) The short exact sequence splits.

Remark. If B is not abelian, then the implication (3) $\Rightarrow$ (1) of Splitting Lemma 78.2 does in general not hold. For example consider the sequence $0 \to \mathbb{Z}_3 \xrightarrow{\varphi} S_3 \xrightarrow{\sigma} \mathbb{Z}_2 \to 0$ where S_3 is as always the permutation group on the three elements, where $\varphi\colon \mathbb{Z}_3 \to S_3$ is the homomorphism given by $\varphi(1) = (1\,2\,3)$ and where $\sigma\colon S_3 \to \mathbb{Z}_2$ is the unique epimorphism from the permutation group S_3 onto $\mathbb{Z}_2$. A splitting of σ is given by $\varphi(1) = (1\,2)$. Since σ is also the only epimorphism from S_3 onto a non-trivial group we see that the sequence does not split, hence (1) is not satisfied.

[935] Why does it not split?

Proof.

(1)⇒(2) Let $t\colon B \to A$ be a homomorphism such that $t \circ i = \mathrm{id}_A$. We define

$$\begin{array}{rccc} \Phi\colon & B & \to & A \times C \\ & b & \mapsto & (t(b), p(b)). \end{array}$$

It follows easily from the exactness of the initial sequence that this map is in fact an isomorphism.[936] It is straightforward to verify that Φ has all the other desired properties.

(2)⇒(1),(3) Now suppose we are given an isomorphism $\Phi\colon B \to A \times C$ such that the given diagram commutes. It follows immediately from the definitions and given properties that the maps

$$B \xrightarrow{\Phi} A \times C \xrightarrow{(a,c)\mapsto a} A \quad \text{and} \quad C \xrightarrow{c\mapsto(0,c)} A \times C \xrightarrow{\Phi^{-1}} B$$

have the required properties.

Finally we assume that B is abelian.

(3)⇒(1) Let $s\colon C \to B$ be a splitting, i.e. let s be a homomorphism such that $p \circ s = \mathrm{id}_C$. We need to find an appropriate homomorphism $t\colon B \to A$. Since the sequence is exact, our task is basically the same as finding an appropriate homomorphism $B \to \ker(p)$. Let $b \in B$. We have

$$p(b - s(p(b))) \;=\; p(b) - \underbrace{(p \circ s)}_{=\mathrm{id}}(p(b)) \;=\; p(b) - p(b) \;= 0.$$

By the exactness of the short exact sequence there exists a unique $a \in A$ with $i(a) = b - s(p(b))$. We define $t(b) := a$. It follows easily from the fact that p and s are homomorphisms that the map $t\colon B \to A$ is a homomorphism. It remains to show that $t \circ i = \mathrm{id}_A$. Let $a \in A$. We have to show that $t(i(a)) - a = 0$. Since i is injective it suffices to show that $i(t(i(a))) - i(a) = 0$. We have

$$i(t(i(a))) - i(a) \underset{\substack{\uparrow \\ \text{since } t(y) \text{ is the unique element} \\ \text{with } i(t(y)) = y - s(p(y))}}{=} (i(a) - s(p(i(a)))) - i(a) \underset{\substack{\uparrow \\ \text{since } p \circ i = 0}}{=} i(a) - i(a) \;=\; 0.$$

■

In many applications we only need the following corollary that is an immediate consequence of Lemmas 78.1 and 78.2.

Corollary 78.3. (Direct Sum Corollary) Let

$$0 \;\to\; A \xrightarrow{i} B \xrightarrow{p} C \;\to\; 0$$

be a short exact sequence of abelian groups. Then the following hold:

(1) If C is a free abelian group, then $B \cong A \oplus C$.
(2) Suppose A and C are free abelian groups. If X is a basis for A and if $Y \subset B$ is a subset such that $p(Y)$ is a basis for C, then $X \sqcup Y$ is a basis for the free abelian group B.

Proof. The first statement is an immediate consequence of Lemmas 78.1 and 78.2. The second statement follows easily from Splitting Lemma 78.2 (1) ⇒ (3). We leave the details to the reader. ■

[936] In fact the argument is basically the same as in the proof of the Reduced Homology Lemma 74.1 (4a).

Example. Let B be an abelian group, let $b \in B$ and let $p\colon B \to \mathbb{Z}$ be a homomorphism with $p(b) = 1$. It follows immediately from the Direct Sum Corollary 78.3, applied to the short exact sequence $0 \to \ker(p) \to B \to \mathbb{Z} \to 0$, that we can write $B = K \oplus \mathbb{Z} \cdot b$ for some subgroup K that is isomorphic to $\ker(p)$.

The above discussion gives us tools for dealing with short exact sequence. The following lemma allows us to break a long exact sequence into shorter exact sequences.

Lemma 78.4. (Break-into-SES Lemma) Let

$$\ldots \to A_{n+2} \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \to \ldots$$

be an exact sequence of abelian groups. Then for any n the sequences

$$0 \to \operatorname{coker}\left(A_{n+2} \xrightarrow{f_{n+2}} A_{n+1}\right) \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \to \ldots$$

and

$$\ldots \to A_{n+2} \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} \ker\left(A_{n-1} \xrightarrow{f_{n-1}} A_{n-2}\right) \to 0$$

are also exact. In particular

$$0 \to \operatorname{coker}\left(A_{n+2} \xrightarrow{f_{n+2}} A_{n+1}\right) \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} \ker\left(A_{n-1} \xrightarrow{f_{n-1}} A_{n-2}\right) \to 0$$

is a short exact sequence.

Proof. The statements follow immediately from the definitions. We leave it to the reader to fill in the details. ■

Example. A typical application is that if we are given an exact sequence

$$\ldots A_{n+2} \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n=0} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \xrightarrow{f_{n-2}} A_{n-3} \ldots$$

of abelian groups such that the map f_n is the zero map, then the exact sequence can be split "into the two halves" along that zero map, i.e.

$$0 \to A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \xrightarrow{f_{n-2}} A_{n-3} \ldots$$

and

$$\ldots A_{n+2} \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \to 0$$

are also exact sequences. In the remainder of this course we will frequently use this fact without explicitly citing the Break-into-SES Lemma 78.4.

78.2. **The Mayer–Vietoris sequence.** In the following theorem and throughout the subsequent discussion we will use the following notation: given a topological space D and a subset $C \subset D$, we denote the corresponding inclusion map $C \to D$ by i_C. To avoid pandemonium with subscripts we often drop denote the inclusion induced maps on homology by i_C instead of the more accurate $(i_C)_*$.

Theorem 78.5. (Mayer–Vietoris Theorem) [937] Let X be a topological space and let $A, B \subset X$ be subsets such that $X = \mathring{A} \cup \mathring{B}$. Then the following two statements hold:

(1) For each $n \in \mathbb{N}$ there exists a unique homomorphism

$$\partial_n\colon \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A \cap B)$$

which has the following property: for every $a \in \mathrm{C}_n(A)$ and $b \in \mathrm{C}_n(B)$ such that $a+b \in \mathrm{C}_n(X)$ is a cycle we have[938]

$$\partial_n([a+b]) \;=\; [\,\partial a\,] \;=\; -\,[\,\partial b\,] \;\in\; \mathrm{H}_{n-1}(A \cap B).$$

We refer to the maps $\partial_n \colon \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A \cap B)$ as the *connecting homomorphisms* corresponding to (X, A, B).

(2) The connecting homomorphism of (1) is natural in the following sense. Let X and Y be two topological spaces and let $A, B \subset X$ and $C, D \subset Y$ be subsets with $X = \mathring{A} \cup \mathring{B}$ and $Y = \mathring{C} \cup \mathring{D}$. If $f \colon X \to Y$ is a map with $f(A) \subset C$ and $f(B) \subset D$, then for every $n \in \mathbb{N}$ the following diagram commutes

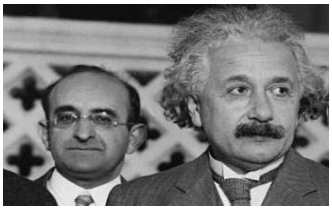
$$\begin{array}{ccc} \mathrm{H}_n(X) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A \cap B) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{H}_n(Y) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(C \cap D) \end{array}$$

where the horizontal maps are the connecting homomorphisms from (1).

(3) The following sequence is exact:

$$\begin{array}{ccccccc} \ldots \to \mathrm{H}_n(A\cap B) & \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} & \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) & \xrightarrow{i_A + i_B} & \mathrm{H}_n(X) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A\cap B) \to \ldots \\ s & \longmapsto & (i_{A\cap B}(s), -i_{A\cap B}(s)) & & & & \\ & & & & s & \longmapsto & i_A(s) + i_B(s) \end{array}$$

Totally analogous statements also hold if we replace the homology groups of $A \cap B, A, B$ and X by the corresponding reduced homology groups.[939]



Walther Mayer (1887-1948) with some other person



Leopold Vietoris (1891-2002) at 110

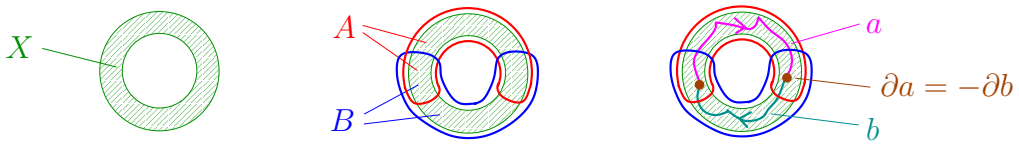
The long exact sequence in Theorem 78.5 is usually called the *Mayer–Vietoris sequence* of the decomposition $X = A \cup B$.

Example. In the figure below we consider the annulus X together with two closed subsets A and B with $X = \mathring{A} \cup \mathring{B}$. Furthermore we show chains $a \in \mathrm{C}_1(A)$ and $b \in \mathrm{C}_1(B)$ such that $a+b$ form a cycle in $\mathrm{C}_1(X)$. The connecting homomorphism of the Mayer–Vietoris sequence sends $[a+b] \in \mathrm{H}_1(X)$ to $[\partial a] = -[\partial b] \in \mathrm{H}_0(A \cap B)$.

[937] Walther Mayer (1887-1948) and Leopold Vietoris (1891-2002) were two Austrian mathematicians. Vietoris was the longest living Austrian man ever.

[938] Here in the expression "∂a" we mean by ∂ the boundary map $\partial \colon \mathrm{C}_n(A) \to \mathrm{C}_{n-1}(A)$. In our case we will see that $\partial a \in \mathrm{C}_{n-1}(A \cap B) \subset \mathrm{C}_{n-1}(A)$. A similar statement holds for ∂b.

[939] Note though that for reduced homology groups we might need to use the fact, obtained in the Reduced Homology Lemma 74.1 (0), that $\widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$.



Remark.

(1) In many applications some of the inclusions appearing in Mayer–Vietoris Theorem 78.5 are homotopy equivalences, in particular, by the Homotopy Equivalence-H_*-Corollary 73.9 many of the inclusion induced maps on homology will be isomorphisms. If that is the case, then it is worth looking up Exercise 72.14 which might greatly simplify the calculations.

(2) Suppose that we are given a topological space X together with an open cover $\{U_i\}_{i\in I}$. If the cover consists of precisely two sets U_1 and U_2, then the Mayer–Vietoris Theorem 78.5 relates the homology groups of X, U_1, U_2 and the intersection $U_1 \cap U_2$. If the open cover consists of more than two sets, then the situation is much more tricky. In this more general setting the homology groups of X are related to the homology groups of the U_i and the homology groups of all finite intersections $U_{i_1} \cap \cdots \cap U_{i_m}$ via the "Mayer–Vietoris spectral sequence". We refer to [**LSVJ2011, Staf2015**], [**Dim2004**, Remark 2.3.9], [**TC2019**, Chapter 3] and [**BoT1982**, Chapter 15] for details. It is also worth having a look at Exercise 78.19.

Proof. Let X be a topological space and let $A, B \subset X$ be subsets such that $X = \mathring{A} \cup \mathring{B}$. There are two approaches to proving the Mayer–Vietoris Theorem 78.5.

The more elegant approach is as follows: In Exercise 78.1 we prove a purely algebraic statement, namely we construct the algebraic Mayer–Vietoris sequence. In Exercise 78.2 we use this exact sequence together with the Excision Theorem 74.17 and the long exact sequences of pairs of topological spaces to prove the desired statements. This approach allows us to prove Statements (2) and (3) of the Mayer–Vietoris Theorem 78.5 without working again on the "chain level".

In the following we will carry out the second approach. It works on the "chain level" and is thus arguably less conceptual. On the other hand it has the advantage that it is clearer "why it works". After this long preamble, let us remind the reader of the following notation from page 1638:

$$\mathrm{C}_n^{\{A,B\}}(X) \;:=\; \{a+b \,|\, a \in \mathrm{C}_n(A) \text{ and } b \in \mathrm{C}_n(B)\}.$$

The inclusion maps induce the following sequence of chain maps:

$$0 \to \mathrm{C}_*(A\cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} \mathrm{C}_*(A) \oplus \mathrm{C}_*(B) \xrightarrow{i_A + i_B} \mathrm{C}_*^{\{A,B\}}(X) \to 0.$$

Claim. The above sequence of chain maps is exact.

Proof. We prove the claim in the following four steps:

(a) It is clear that each of the maps $\mathrm{C}_*(A\cap B) \to \mathrm{C}_*(A)$ and $\mathrm{C}_*(A\cap B) \to \mathrm{C}_*(B)$ is already injective, in particular the map $i_{A\cap B} \oplus -i_{A\cap B}$ is injective.

(b) It is an immediate consequence of the definition of $\mathrm{C}_*^{\{A,B\}}(X)$ that the map $i_A + i_B$ is surjective.

(c) It follows immediately from the definitions that the composition of both maps is the zero map,[940] i.e. $\operatorname{im}(i_{A\cap B} \oplus -i_{A\cap B}) \subset \ker(i_A + i_B)$.

(d) It remains to show that $\ker(i_A + i_B) \subset \operatorname{im}(i_{A\cap B} \oplus -i_{A\cap B})$. Thus let $x \in \mathrm{C}_n(A)$ and $y \in \mathrm{C}_n(B)$ such that $(i_A + i_B)(x + y) = 0$. By definition this means that x and y, viewed as elements in $\mathrm{C}_n(X)$, satisfy $x = -y$. This means that x lies in $\mathrm{C}_n(A)$ and also in $\mathrm{C}_n(B)$. Put differently, $x \in \mathrm{C}_n(A) \cap \mathrm{C}_n(B) = \mathrm{C}_n(A \cap B)$. But then we have $(i_{A\cap B} \oplus -i_{A\cap B})(x) = x + y$. ⊞

Next we consider the following maps:

$$\begin{array}{ccccccccc} \ldots \longrightarrow \mathrm{H}_n(A\cap B) & \xrightarrow{i_{A\cap B}\oplus -i_{A\cap B}} & \mathrm{H}_n(A)\oplus\mathrm{H}_n(B) & \xrightarrow{i_A+i_B} & \mathrm{H}_n^{\{A,B\}}(X) & \xrightarrow{p_n} & \mathrm{H}_{n-1}(A\cap B) & \longrightarrow \ldots \\ & & & \searrow^{i_A+i_B} & \downarrow{\scriptstyle \Phi_n} & \nearrow_{:=\partial_n} & & \\ & & & & \mathrm{H}_n(X) & & & \end{array}$$

We make the following clarifications and observations:

(a) Given $n \in \mathbb{N}$ we denote by $p_n \colon \mathrm{H}_n^{\{A,B\}}(X) \to \mathrm{H}_{n-1}(A \cap B)$ the connecting homomorphism of the above short exact sequence of chain complexes, as defined on page 1620.

(b) By the LES Proposition 74.9 the top horizontal sequence is exact.

(c) It follows from the Comfortable Covering Proposition 74.23, together with our hypothesis that $X = \mathring{A} \cup \mathring{B}$, that for every $n \in \mathbb{N}_0$ the inclusion map

$$\mathrm{C}_*^{\{A,B\}}(X) \;\to\; \mathrm{C}_*(X)$$

induces an isomorphism

$$\Phi_n \colon \mathrm{H}_n^{\{A,B\}}(X) \;\xrightarrow{\cong}\; \mathrm{H}_n(X).$$

For each $n \in \mathbb{N}_0$ we now write $\partial_n := p_n \circ \Phi_n^{-1}$. It remains to show that these maps have the two desired properties stated in the theorem:

(3) It is clear from the above discussion that the sequence given in (3) is indeed exact.

(1) We leave it to the reader to verify that it follows from the definition of the connecting homomorphism p_n on page 1620 that for every $a \in \mathrm{C}_n(A)$ and $b \in \mathrm{C}_n(B)$ such that $a + b \in \mathrm{C}_n(X)$ is a cycle we have

$$\partial_n([a+b]) \;=\; p_n([a+b]) \;=\; [\partial a] \;=\; -\,[\partial b] \;\in\; \mathrm{H}_{n-1}(A \cap B).$$

Furthermore, the isomorphism $\Phi_n \colon \mathrm{H}_n^{\{A,B\}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ shows that every element in $\mathrm{H}_n(X)$ is of the form $[a + b]$ for some $a \in \mathrm{C}_n(A)$ and $b \in \mathrm{C}_n(B)$, which implies that ∂_n is uniquely determined by the above rule.

(2) It follows basically immediately from the existence and uniqueness of the connecting homomorphisms ∂_n that these maps are also natural.

We leave it to the reader to verify, using the Reduced Homology Lemma 74.1, that the analogous statements also hold if we replace the homology groups of $A \cap B, A, B$ and X by the corresponding reduced homology groups. ■

The following lemma gives an alternative description of the connecting homomorphism of the Mayer–Vietoris sequence.

[940] Note that here we use the minus-sign that appears in the first map.

Lemma 78.6. Let X be a topological space and let A, B be two open subsets of X with $X = A \cup B$. The connecting homomorphism $\partial_n \colon \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A \cap B)$ equals the map

$$\mathrm{H}_n(X) \xrightarrow{\text{я}} \mathrm{H}_n(X, B) \xleftarrow[\uparrow]{\cong} \mathrm{H}_n(A, A \cap B) \xrightarrow[\uparrow]{\partial_n} \mathrm{H}_{n-1}(A \cap B).$$

inclusion induced map is an isomorphism by the Excision Theorem 74.17 — connecting homomorphism of pair $(B, A \cap B)$

Proof. This equality follows easily from the precise description of the connecting homomorphisms given in the Mayer–Vietoris Theorem 78.5 and in the Topological LES-Proposition 74.13. ■

We conclude this section with the following variation on the Mayer–Vietoris Theorem 78.5. This theorem might sound rather unnatural, but it will play an important role in the proof of the Non-Compact PD Theorem 119.18.

Theorem 78.7. (Inverted Mayer–Vietoris Theorem) Let X be a topological space and let U and V be open subsets. Then for all $n \in \mathbb{N}_0$ there exists a natural homomorphism $\partial_n \colon \mathrm{H}_n(X, U \cup V) \to \mathrm{H}_{n-1}(X, U \cap V)$ such that

$$\ldots \to \mathrm{H}_n(X, U \cap V) \xrightarrow{i \oplus -i} \mathrm{H}_n(X, U) \oplus \mathrm{H}_n(X, V) \xrightarrow{i+i} \mathrm{H}_n(X, U \cup V) \xrightarrow{\partial} \mathrm{H}_{n-1}(X, U \cap V) \to \ldots$$

is an exact sequence.

Proof. Note that as in the proof of Theorem 78.5 we have a short exact sequence

$$0 \;\to\; \mathrm{C}_n(X, U \cap V) \;\to\; \mathrm{C}_n(X, U) \oplus \mathrm{C}_n(X, V) \;\to\; \mathrm{C}_n(X) / \mathrm{C}_n^{\{U,V\}}(U \cup V) \;\to\; 0.$$

Thus we get a corresponding long exact sequence of homology groups. It suffices to show that the homology groups of the chain complex on the right are precisely the homology groups of the pair $(X, U \cup V)$. To do so we consider the following commutative pair of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{C}_*^{\{U,V\}}(U \cup V) & \longrightarrow & \mathrm{C}_*(X) & \longrightarrow & \mathrm{C}_*(X) / \mathrm{C}_*^{\{U,V\}}(U \cup V) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{C}_*(U \cup V) & \longrightarrow & \mathrm{C}_*(X) & \longrightarrow & \mathrm{C}_*(X, U \cup V) & \longrightarrow & 0. \end{array}$$

We obtain the corresponding long exact sequence of homology groups:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \mathrm{H}_n^{\{U,V\}}(U \cup V) & \longrightarrow & \mathrm{H}_n(X) & \longrightarrow & \mathrm{H}_n\big(\mathrm{C}_*(X) / \mathrm{C}_*^{\{U,V\}}(U \cup V)\big) & \longrightarrow & \mathrm{H}_{n-1}^{\{U,V\}}(U \cup V) & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & \mathrm{H}_n(U \cup V) & \longrightarrow & \mathrm{H}_n(X) & \longrightarrow & \mathrm{H}_n(X, U \cup V) & \longrightarrow & \mathrm{H}_{n-1}(U \cup V) & \longrightarrow & \ldots \end{array}$$

It follows from the Comfortable Covering Proposition 74.23 that the first and fourth vertical map are isomorphisms. The second map and the fifth map (which for space reasons is not shown) are the identity. Hence it follows from the Five Lemma 74.10 that the third vertical map is also an isomorphism. ■

78.3. **Applications of the Mayer–Vietoris sequence.** As a warm-up we first consider the homology groups of the suspension $\Sigma(X)$ of a topological space X. More precisely, we have the following lemma which basically gives a new proof for the Suspension-H_*-Proposition 76.7.

Proposition 76.7. (Suspension-H$_*$-Proposition) Let $k \in \mathbb{Z}$. Given any topological space X we have a natural isomorphism

$$\Sigma_X \colon \widetilde{\mathrm{H}}_k(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)).$$

Example. Let $n \in \mathbb{N}_0$. We perform the following calculation:

$$\begin{array}{ccccccc} \widetilde{\mathrm{H}}_k(S^n) & \cong & \widetilde{\mathrm{H}}_k(\Sigma(\ldots(\Sigma(S^0)))) & \cong & \widetilde{\mathrm{H}}_{k-n}(S^0) & \cong & \begin{cases} \mathbb{Z}, & \text{if } k=n, \\ 0, & \text{else.} \end{cases} \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{see page 452} & & \text{see the Suspension-H}_*\text{-Proposition 76.7} & & \text{see page 1597} & \end{array}$$

This recovers the calculation we made in Proposition 74.4.

Remark. Let X be a topological space. In Exercise 78.11 we will see that the two natural isomorphisms $\widetilde{\mathrm{H}}_k(\Sigma(X)) \xrightarrow{\cong} \widetilde{\mathrm{H}}_{k-1}(X)$ that we constructed in the Suspension-H$_*$-Proposition 76.7 and that we construct in the proof of the Suspension-H$_*$-Proposition 76.7 are actually the same on the nose.

Proof. Let $k \in \mathbb{Z}$.

(1) Let X be a topological space. We denote by $p \colon X \times [-1,1] \to \Sigma(X)$ the natural projection. As on page 1681 we consider the subspaces $C_+ := C_+X = p(X \times (-1,1]))$ and $C_- := C_-X = p(X \times [-1,1))$. Note that C_+X and C_-X are both open subsets of $\Sigma(X)$ and that $\Sigma(X) = C_+X \cup C_-X$. We consider the corresponding long exact sequence in reduced homology from the Mayer–Vietoris Theorem 78.5:

$$\underbrace{\widetilde{\mathrm{H}}_{k+1}(C_+)}_{=0} \oplus \underbrace{\widetilde{\mathrm{H}}_{k+1}(C_-)}_{=0} \longrightarrow \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)) \xrightarrow{\partial_k} \widetilde{\mathrm{H}}_k(C_+ \cap C_-) \longrightarrow \underbrace{\widetilde{\mathrm{H}}_{k+1}(C_+)}_{=0} \oplus \underbrace{\widetilde{\mathrm{H}}_{k+1}(C_-)}_{=0}$$

$$\widetilde{\mathrm{H}}_k(X) \xrightarrow{\Sigma_X} \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)) \qquad \widetilde{\mathrm{H}}_k(X) \xrightarrow{i_*} \widetilde{\mathrm{H}}_k(C_+ \cap C_-)$$

We make the following clarifications and observations:

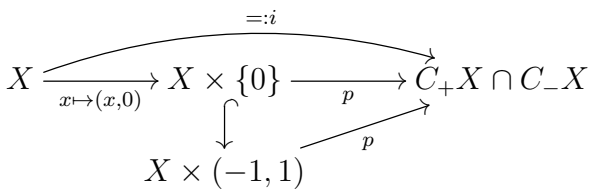
(a) Note that by the same argument as in the proof of the Cone Contractibility Lemma 16.7 we see that both $C_+ = C_+X$ and $C_- = C_-X$ admit a deformation retraction to a point. It follows from the Deformation Retract-Homotopy Equivalence Lemma 15.5 that both C_+ and C_- are contractible, therefore we know by the Reduced Homology Lemma 74.1 (7) the reduced homology groups of C_+X and C_-X vanish.

(b) It follows immediately from (1) that the connecting homomorphism

$$\partial \colon \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)) \to \widetilde{\mathrm{H}}_k(C_+X \cap C_-X)$$

is in fact an isomorphism.

(c) We consider the following little diagram:

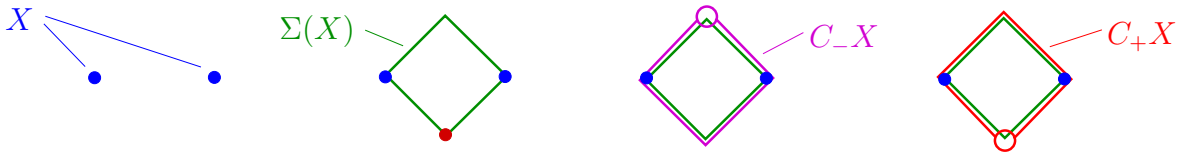


The vertical map is evidently a homotopy equivalence. Furthermore one can easily show, either by hand, or using Lemma 74.21 (1), that the diagonal map is a homeomorphism. We obtain that the map $i\colon X \to C_+X \cap C_-X$ is a homotopy equivalence. Thus we obtain from the Reduced Homology Lemma 74.1 (6) that the induced map $i_*\colon \widetilde{\mathrm{H}}_k(X \times \{0\}) \to \widetilde{\mathrm{H}}_k(C_+X \cap C_-X)$ is an isomorphism.

We define Σ_X to be the map

$$\widetilde{\mathrm{H}}_k(X) \;=\; \widetilde{\mathrm{H}}_k(X \times \{0\}) \xrightarrow{i_*} \widetilde{\mathrm{H}}_k(C_+X \cap C_-X) \xrightarrow{\partial_k^{-1}} \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)).$$

(2) It follows easily from the definition of Σ_X and the naturality of the connecting homomorphisms of the Mayer–Vietoris sequence, see Theorem 78.5 (2), that the isomorphism Σ_X is natural. ∎



As our next example we want to determine the homology groups of the 2-dimensional projective space $\mathbb{R}\mathrm{P}^2$.

Lemma 78.8. Let $n \in \mathbb{N}_0$. We have

$$\mathrm{H}_n(\mathbb{R}\mathrm{P}^2) \;\cong\; \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z}_2, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Remark. In Exercise 78.10 we will determine an explicit cycle that represents the unique non-trivial element of $\mathrm{H}_1(\mathbb{R}\mathrm{P}^2)$.

Proof. In the proof of Lemma 78.8 we use the description from page 258 of $\mathbb{R}\mathrm{P}^2$ as

$$\mathbb{R}\mathrm{P}^2 \;=\; \overline{B}^2/\!\sim \qquad \text{where } x \sim y \text{ if } x, y \in \partial\overline{B}^2 \text{ and } x = -y.$$

In the following we denote by p the projection $\overline{B}^2 \to \overline{B}^2/\!\sim$. We consider the open subsets

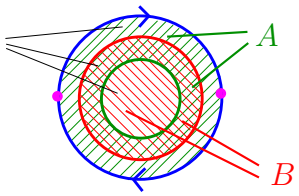
$$A \;:=\; p\big(\big\{z \in \overline{B}^2 \,\big|\, |z| > \tfrac{1}{4}\big\}\big) \qquad \text{and} \qquad B \;:=\; p\big(\big\{z \in \overline{B}^2 \,\big|\, |z| < \tfrac{3}{4}\big\}\big).$$

Evidently we have $\mathring{A} \cup \mathring{B} = A \cup B = \mathbb{R}\mathrm{P}^2$. From the Mayer–Vietoris Theorem 78.5 we obtain the following long exact sequence of reduced homology groups:

$$\cdots \to \widetilde{\mathrm{H}}_2(A) \oplus \widetilde{\mathrm{H}}_2(B) \to \widetilde{\mathrm{H}}_2(\mathbb{R}\mathrm{P}^2) \xrightarrow{\partial} \widetilde{\mathrm{H}}_1(A \cap B) \to \widetilde{\mathrm{H}}_1(A) \oplus \widetilde{\mathrm{H}}_1(B) \to \widetilde{\mathrm{H}}_1(\mathbb{R}\mathrm{P}^2) \to \ldots$$

Now we have to collect all the available information on the reduced homology groups of

$\mathbb{R}\mathrm{P}^2 = \overline{B}^2/\!\sim$



A, B and $A \cap B$. We make the following observations:

(1) We remarked on page 1612 that the reduced homology of a disk vanishes, i.e. we have $\widetilde{\mathrm{H}}_n(B) = 0$ for all $n \in \mathbb{N}_0$.

(2) Note that $p(\frac{1}{2}S^1)$ is a deformation retract of $A \cap B = p(\{z \in \overline{B}^2 \,|\, |z| \in (\frac{1}{4}, \frac{3}{4})\})$. This implies that the map $j\colon S^1 \to A \cap B = p\big(\{z \in \overline{B}^2 \,|\, |z| \in (\frac{1}{4}, \frac{3}{4})\}\big)$ given by $z \mapsto \frac{1}{2}z$ is a homotopy equivalence. It follows from the Homotopy Equivalence-H_*-Corollary 73.9 (2) and Proposition 74.4 that $\widetilde{\mathrm{H}}_n(A \cap B) = 0$ for $n \neq 1$ and that $j_*\colon \widetilde{\mathrm{H}}_1(S^1) \to \widetilde{\mathrm{H}}_1(A \cap B)$ is an isomorphism.

(3) We consider the retraction

$$\begin{array}{rcl} r\colon A & \to & \{z \in \partial\overline{B}^2\}/z \sim -z \\ p(s\exp(\mathrm{i}\varphi)) & \mapsto & \exp(\mathrm{i}\varphi) \end{array}$$

which is easily seen to be a homotopy equivalence and we also consider the homeomorphism

$$\begin{array}{rcl} f\colon \{z \in \partial\overline{B}^2\}/z \sim -z & \to & S^1 \\ z & \mapsto & z^2. \end{array}$$

As in (2) it follows that $\widetilde{\mathrm{H}}_n(A) = 0$ for $n \neq 1$ and that $(f \circ r)_*\colon \widetilde{\mathrm{H}}_1(A) \to \widetilde{\mathrm{H}}_1(S^1)$ is an isomorphism.

The above Mayer–Vietoris sequence is thus in our case of the following form:

$$\begin{array}{l} \ldots \to 0 \to \widetilde{\mathrm{H}}_3(\mathbb{R}\mathrm{P}^2) \to 0 \to 0 \to \widetilde{\mathrm{H}}_2(\mathbb{R}\mathrm{P}^2) \xrightarrow{\partial} \widetilde{\mathrm{H}}_1(A \cap B) \xrightarrow{\quad i_* \quad} \widetilde{\mathrm{H}}_1(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_1(\mathbb{R}\mathrm{P}^2) \to 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad {\scriptstyle j_*}\uparrow{\scriptstyle \cong} \qquad\qquad\qquad\qquad\quad {\scriptstyle \cong}\downarrow{\scriptstyle (f\circ r)_*} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathrm{H}_1(S^1) \xrightarrow[\cdot \deg(f\circ r\circ i\circ j)]{(f\circ r\circ i\circ j)_*} \mathrm{H}_1(S^1). \end{array}$$

We obtain immediately that $\widetilde{\mathrm{H}}_n(\mathbb{R}\mathrm{P}^2) = 0$ for $n \geq 3$. Furthermore it follows immediately from the definitions that the map $f \circ r \circ i \circ j\colon S^1 \to S^1$ is given by $z \mapsto z^2$. By Lemma 76.10 this implies that $\deg(f \circ r \circ i \circ j) = 2$. Thus we obtain the following exact sequence

$$0 \;\to\; \widetilde{\mathrm{H}}_2(\mathbb{R}\mathrm{P}^2) \;\to\; \mathbb{Z} \;\xrightarrow{\cdot 2}\; \mathbb{Z} \;\to\; \widetilde{\mathrm{H}}_1(\mathbb{R}\mathrm{P}^2) \;\to\; 0.$$

It follows from the discussion on page 1614 that

$$\widetilde{\mathrm{H}}_2(\mathbb{R}\mathrm{P}^2) \cong \ker\big(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}\big) = 0 \quad \text{and} \quad \widetilde{\mathrm{H}}_1(\mathbb{R}\mathrm{P}^2) \cong \operatorname{coker}\big(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}\big) = \mathbb{Z}_2. \qquad \blacksquare$$

Remark. It is a non-trivial, but certainly very amusing exercise to generalize the calculation of Lemma 78.8 to compute the homology groups of any real projective space $\mathbb{R}\mathrm{P}^n$. We cordially invite the reader to try to make an educated guess of what the homology groups of $\mathbb{R}\mathrm{P}^n$ might be. We will compute the homology groups later on in Proposition 80.11 using a somewhat different approach.

78.4. **The Mayer–Vietoris Theorem for manifolds and CW-complexes.** Recall that when we studied fundamental groups we first proved the Seifert–van Kampen Theorem 54.1 for the decomposition of a topological space X into two *open* subsets. Afterwards we showed in Theorem 54.4 and Theorem 69.1 that an analogous statement holds if we decompose a smooth manifold into suitable submanifolds and if we decompose a CW-complex into subcomplexes.

In a similar fashion we now state and prove the "manifold version" and the "CW-complex version" of the Mayer–Vietoris Theorem 78.5.

Theorem 78.9. (Mayer–Vietoris Theorem for Topological Manifolds) Let X be an m-dimensional topological manifold and let $A, B \subset X$ be two m-dimensional submanifolds such that the following statements hold:

(a) $X = A \cup B$,
(b) $A \cap B$ is a union of components of ∂A and it is a union of components of ∂B,
(c) A and B are closed subsets of X. (Note that by the Compact-Closed Lemma 1.21 (2) this condition is satisfied if A and B are compact.)

The following three statements hold:

(1) For each $n \in \mathbb{N}_0$ there exists a unique homomorphism $\partial_n \colon \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A \cap B)$ which has the following property: If we are given $a \in \mathrm{C}_n(A)$ and $b \in \mathrm{C}_n(B)$ are such that $a + b \in \mathrm{C}_n(X)$ is a cycle, then

$$\partial_n([a+b]) \;=\; [\partial a] \;=\; -\,[\partial b] \;\in\; \mathrm{H}_{n-1}(A \cap B).$$

(2) The following sequence is exact:

$$\ldots \to \mathrm{H}_n(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \xrightarrow{i_A + i_B} \mathrm{H}_n(X) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A \cap B) \to \ldots$$

(3) The connecting homomorphism $\partial_n \colon \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A \cap B)$ equals the map

$$\mathrm{H}_n(X) \xrightarrow{\text{я}} \mathrm{H}_n(X, B) \xleftarrow[\uparrow]{\cong} \mathrm{H}_n(A, A \cap B) \xrightarrow[\uparrow]{\partial_n} \mathrm{H}_{n-1}(A \cap B).$$

inclusion induced map is an isomorphism by the Excision Theorem 75.10

connecting homomorphism of pair $(B, A \cap B)$

A totally analogous statement also holds if we replace the homology groups of $A \cap B, A, B$ and X by the corresponding reduced homology groups.

Proof. Let X be an m-dimensional topological manifold and let $A, B \subset X$ be two m-dimensional submanifolds such that$A$ and B are closed subset of M, such that $X = A \cup B$ and such that $A \cap B$ is a union of components of ∂A and also a union of components of ∂B.

We pick maps $f \colon [0,1) \times \partial A \to A$ and $g \colon [0,1) \times \partial B \to B$ that are provided by the Topological Collar Neighborhood Theorem 75.5[941] We set

$$U \;:=\; A \cup \underbrace{(g([0,1) \times (A \cap B))}_{\subset [0,1) \times B} \quad \text{and} \quad V \;:=\; B \cup \underbrace{(f([0,1) \times (A \cap B))}_{\subset [0,1) \times A}.$$

As in the proof of Theorems 52.10 and 54.4 we note that the following statements hold:

(i) U and V are open subsets of X,[942]
(ii) A is a deformation retract of U,
(iii) B is a deformation retract of V, and
(iv) $A \cap B$ is a deformation retract of $U \cap V$.

[941] Of course, if X is a smooth manifold, then we could also use the Smooth Collar Neighborhood Theorem 28.3.
[942] At this point we secretly use our hypothesis that A and B are closed subsets of X.

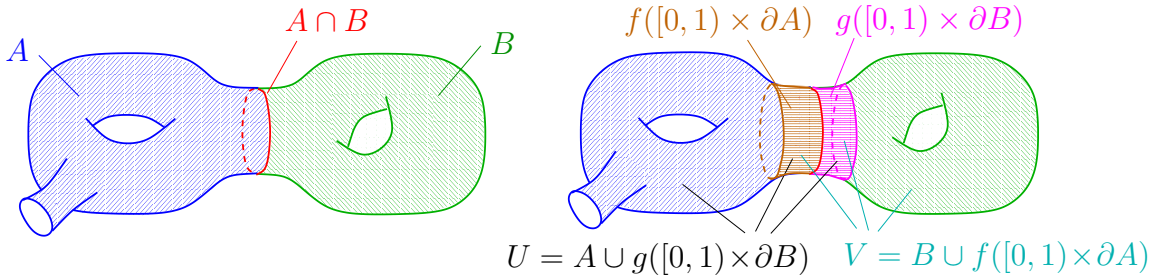
Now we consider the following diagram

$$\begin{array}{ccccccccccc}
\dots & \dashrightarrow & \mathrm{H}_n(A\cap B) & \xrightarrow{i\oplus -i} & \mathrm{H}_n(A)\oplus \mathrm{H}_n(B) & \xrightarrow{i+i} & \mathrm{H}_n(X) & \overset{\partial_n}{\dashrightarrow} & \mathrm{H}_{n-1}(A\cap B) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow \mathrm{id} & & \downarrow & & \\
\dots & \longrightarrow & \mathrm{H}_n(U\cap V) & \xrightarrow{i\oplus -i} & \mathrm{H}_n(U)\oplus \mathrm{H}_n(V) & \xrightarrow{i+i} & \mathrm{H}_n(X) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(U\cap V) & \longrightarrow & \dots
\end{array}$$

where the bottom sequence is given by the Mayer–Vietoris Theorem 78.5. We make the following observations:

(a) The two squares to the left commute since all the relevant maps are given by inclusions.
(b) The vertical maps are isomorphisms by the Homotopy Equivalence-H_*-Corollary 73.9.
(c) By (ii) we can now define the dotted map $\mathrm{H}_n(X) \overset{\partial_n}{\longrightarrow} \mathrm{H}_{n-1}(A\cap B)$ such that the third square also commutes.

It now follows from the Mayer–Vietoris Theorem 78.5 that the sequence in (2) is exact. We leave the somewhat technical task of verifying Statement (1) to the reader. Finally note that, as in the proof of Lemma 78.6, Statement (3) is a straightforward consequence of the descriptions of the various maps. ■



Now we turn to CW-complexes.

Theorem 78.10. (Mayer–Vietoris Theorem for CW-complexes) Let X be a CW-complex. If A and B are two subcomplexes of X such that $X = A \cup B$, then basically exactly the same conclusion as in the Mayer–Vietoris Theorem 78.9 for Manifolds holds.[943]

Proof. Let X be a CW-complex and let A and B be two subcomplexes of X with $X = A \cup B$. By the CW-Complex Properties Proposition 68.10 (8) there exist open neighborhoods U of A and V of B such that the following statements hold:

(1) A is a deformation retract of U,
(2) B is a deformation retract of V, and
(3) $A \cap B$ is a deformation retract of $U \cap V$.

The proof is now verbatim the same as the proof of the Mayer–Vietoris Theorem 78.9 for Topological Manifolds. ■

Later on in Theorem 109.14 we will generalize the above Mayer–Vietoris theorems to an even more general setting, namely to "excisive triads". But for the time being we will be able to handle most situations with the Mayer–Vietoris theorems that we stated above.

[943] In Statement (3) we just need to replace the Excision Theorem 75.10 for Topological Manifolds by the Excision Theorem 74.31 for CW-complexes.

78.5. **The homology groups of the torus and the Klein bottle.** In this section we will determine the homology groups of the torus and the Klein bottle. We do so partly since the results are interesting, but partly also to get to some practice for applying the various Mayer–Vietoris Theorems.

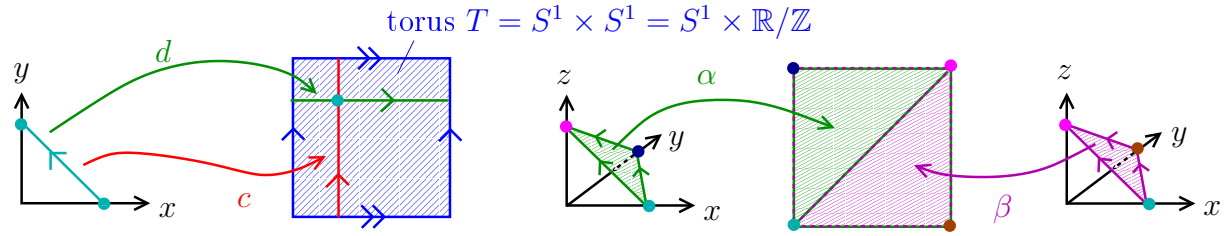
Lemma 78.11. Let $T = S^1 \times S^1$ be the torus. The following two statements hold.

(1) We have the following isomorphisms

$$\mathrm{H}_n(T) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 2, \\ \mathbb{Z}^2, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

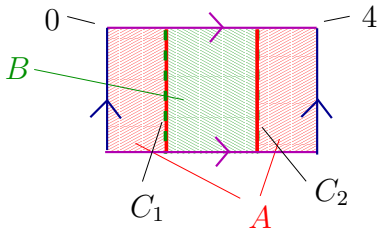
(2) The homology groups are generated by the following cycles.[944]
   (a) The singular 1-simplices c and d shown in the figure below on the left are generators for $\mathrm{H}_1(T) \cong \mathbb{Z}^2$.
   (b) The singular 2-chain $\alpha - \beta$ that is given by the singular 2-simplices α and β shown in the figure below on the right is a generator for $\mathrm{H}_2(T) \cong \mathbb{Z}$. Note that this singular 2-chain is precisely the one that we had already considered in the figure below. In particular this answers our question from page 1589, the cycle $\alpha - \beta$ does indeed represent a non-trivial element in $\mathrm{H}_2(T)$.

Note that Lemma 78.11 (2) shows in particular that the cycles considered in the figure below represent non-trivial homology classes.



To increase readability we break the proof of Lemma 78.11 into two parts.

Proof of Lemma 78.11 (1). First we point out that throughout the proof we identify the torus T with $S^1 \times \mathbb{R}/4\mathbb{Z}$ in the obvious way. Next we introduce some notation. We denote by $p\colon S^1 \times \mathbb{R} \to T = S^1 \times \mathbb{R}/4\mathbb{Z}$ the projection . In the following we write $A := p(S^1 \times ([0,1] \cup [3,4]))$ and $B := p(S^1 \times [1,3])$. Evidently we have $T = A \cup B$. Furthermore we write $\mathrm{C}_1 := p(S^1 \times \{1\})$ and $\mathrm{C}_2 := p(S^1 \times \{3\})$. Evidently we have $A \cap B = \mathrm{C}_1 \cup \mathrm{C}_2$.



[944] Here the singular simplices in the figure below are understood to be given by the obvious affine maps $\Delta^n \to ([0,1] \times [0,1])/\sim$.

It is fairly easy[945] to verify that the hypotheses for the Mayer–Vietoris Theorem 78.9 for Manifolds are satisfied for $T = A \cup B$. Thus we obtain the following long exact sequence in homology[946]

$$\begin{array}{ccccccccc} & & & & \dots & \to & \mathrm{H}_2(T) & \xrightarrow{\partial} & \\ \to & \begin{matrix}\mathrm{H}_1(\mathrm{C}_1)\\ \mathrm{H}_1(\mathrm{C}_2)\end{matrix} & \xrightarrow{\begin{pmatrix} i_{\mathrm{C}_1} & i_{\mathrm{C}_2} \\ -i_{\mathrm{C}_1} & -i_{\mathrm{C}_2}\end{pmatrix}} & \begin{matrix}\mathrm{H}_1(A)\\ \mathrm{H}_1(B)\end{matrix} & \xrightarrow{i_A+i_B} & \mathrm{H}_1(T) & \xrightarrow{\partial} & \\ \to & \begin{matrix}\mathrm{H}_0(\mathrm{C}_1)\\ \mathrm{H}_0(\mathrm{C}_2)\end{matrix} & \xrightarrow{\begin{pmatrix} i_{\mathrm{C}_1} & i_{\mathrm{C}_2} \\ -i_{\mathrm{C}_1} & -i_{\mathrm{C}_2}\end{pmatrix}} & \begin{matrix}\mathrm{H}_0(A)\\ \mathrm{H}_0(B)\end{matrix} & \xrightarrow{i_A+i_B} & \mathrm{H}_0(T) & \longrightarrow & 0. \end{array}$$

Given $x \in [0,4]$ we consider the 0-cycle $Q_x := (1,x)$ and we consider the 1-cycle

$$\begin{array}{rcl} \sigma_x \colon \Delta^1 & \to & S^1 \times \mathbb{R}/4\mathbb{Z} \\ (1-t,t) & \mapsto & \exp(2\pi \mathrm{i} t). \end{array}$$

Let $I \subset \mathbb{R}$ be a compact interval of length less than four. We make the following four observations:

(a) Since the length of I is less than four one obtains almost immediately from the Compact-Hausdorff Proposition 2.17 (3) that $p \colon S^1 \times I \to p(S^1 \times I)$ is a homeomorphism.
(b) It is a straightforward consequence of the Homotopic Maps-and-Homology Proposition 73.6 that for any two points x, x' in I we have $[Q_x] = [Q_{x'}] \in \mathrm{H}_1(p(S^1 \times I))$ and $[\sigma_x] = [\sigma_{x'}] \in \mathrm{H}_1(p(S^1 \times I))$.
(c) It follows from (1) together with the H_0-Proposition 72.5 that for each $x \in I$ we have $\mathrm{H}_0(p(S^1 \times I)) = \mathbb{Z} \cdot [\sigma_x]$.
(d) It follows from (1) together with Lemma 76.4 and the Homotopic Maps-and-Homology Proposition 73.6 that for each $x \in I$ we have $\mathrm{H}_1(p(S^1 \times I)) = \mathbb{Z} \cdot [Q_x]$.

It follows from (c) and (d) that we can rewrite the above Mayer–Vietoris sequence as follows:

$$\begin{array}{ccccccccc} & & & & 0 & \to & \mathrm{H}_2(T) & \xrightarrow{\partial} & \\ \to & \begin{matrix}\mathbb{Z}\cdot[\sigma_1]\\ \mathbb{Z}\cdot[\sigma_3]\end{matrix} & \xrightarrow{\begin{pmatrix} i_{\mathrm{C}_1} & i_{\mathrm{C}_2} \\ -i_{\mathrm{C}_1} & -i_{\mathrm{C}_2}\end{pmatrix}} & \begin{matrix}\mathbb{Z}\cdot[\sigma_0]\\ \mathbb{Z}\cdot[\sigma_2]\end{matrix} & \xrightarrow{i_A+i_B} & \mathrm{H}_1(T) & \xrightarrow{\partial} & \\ \to & \begin{matrix}\mathbb{Z}\cdot[Q_{x_1}]\\ \mathbb{Z}\cdot[Q_{x_1}]\end{matrix} & \xrightarrow{\begin{pmatrix} i_{\mathrm{C}_1} & i_{\mathrm{C}_2} \\ -i_{\mathrm{C}_1} & -i_{\mathrm{C}_2}\end{pmatrix}} & \begin{matrix}\mathbb{Z}\cdot[Q_{x_0}]\\ \mathbb{Z}\cdot[Q_{x_2}]\end{matrix} & \xrightarrow{i_A+i_B} & \mathrm{H}_0(T) & \longrightarrow & 0. \end{array}$$

To determine the homology groups $\mathrm{H}_2(T)$ and $\mathrm{H}_1(T)$ we now need to understand the maps in the above long exact sequence. For example, it follows immediately from (b) that we have $i_{\mathrm{C}_1}([\sigma_1]) = [\sigma_1] = [\sigma_0]$. Basically identical considerations using (b) tell us that we

[945] To be completely fair, it might take a few lines to write down a complete argument why A and B are submanifolds of T.

[946] Here we write

$$\begin{matrix}\mathrm{H}_i(\mathrm{C}_1)\\ \mathrm{H}_i(\mathrm{C}_2)\end{matrix} \text{ instead of } \mathrm{H}_i(\mathrm{C}_1) \oplus \mathrm{H}_i(\mathrm{C}_2) \quad \text{and} \quad \begin{matrix}\mathrm{H}_i(A)\\ \mathrm{H}_i(B)\end{matrix} \text{ instead of } \mathrm{H}_i(A) \oplus \mathrm{H}_i(B).$$

The reason is that we think of direct sums as columns vectors so that we can use the usual matrix notation for describing the maps between the direct sums.

obtain the following long exact sequence:

$$\begin{array}{ccccccccc} & & & & 0 & \to & \mathrm{H}_2(T) & \xrightarrow{\partial} & \\ \to & \begin{matrix}\mathbb{Z}\cdot[\sigma_1]\\ \mathbb{Z}\cdot[\sigma_3]\end{matrix} & \xrightarrow{\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}} & \begin{matrix}\mathbb{Z}\cdot[\sigma_0]\\ \mathbb{Z}\cdot[\sigma_2]\end{matrix} & \xrightarrow{(1\ 1)} & \mathrm{H}_1(T) & \xrightarrow{\partial} & \\ \to & \begin{matrix}\mathbb{Z}\cdot[Q_1]\\ \mathbb{Z}\cdot[Q_3]\end{matrix} & \xrightarrow{\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}} & \begin{matrix}\mathbb{Z}\cdot[Q_0]\\ \mathbb{Z}\cdot[Q_2]\end{matrix} & \xrightarrow{(1\ 1)} & \mathrm{H}_0(T) & \longrightarrow & 0. \end{array}$$

We will now calculate the homology groups by hand. Alternatively one could now also use Exercise 72.14 to quickly calculate the homology groups. Note that it follows from the discussion on page 1614 that

$$\mathrm{H}_2(T) \ \cong\ \ker\left(\begin{matrix}\mathbb{Z}\\ \mathbb{Z}\end{matrix} \xrightarrow{\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}} \begin{matrix}\mathbb{Z}\\ \mathbb{Z}\end{matrix}\right) \ =\ \mathbb{Z}\cdot\begin{pmatrix}1\\ -1\end{pmatrix} \ \cong\ \mathbb{Z}.$$

To determine $\mathrm{H}_1(T)$ we consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \to & \operatorname{coker}\left(\begin{matrix}\mathbb{Z}\cdot[\sigma_1]\\ \mathbb{Z}\cdot[\sigma_3]\end{matrix} \xrightarrow{\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}} \begin{matrix}\mathbb{Z}\cdot[\sigma_0]\\ \mathbb{Z}\cdot[\sigma_2]\end{matrix}\right) & \to & \mathrm{H}_1(T) & \xrightarrow{\partial_1} & \ker\left(\begin{matrix}\mathbb{Z}\cdot[\sigma_0]\\ \mathbb{Z}\cdot[\sigma_2]\end{matrix} \xrightarrow{\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}} \begin{matrix}\mathbb{Z}\cdot[Q_1]\\ \mathbb{Z}\cdot[Q_2]\end{matrix}\right) & \to & 0. \\ & & \cong\uparrow & & \mathrm{id}\uparrow & & \mathrm{id}\uparrow & & \\ 0 & \longrightarrow & \begin{matrix}\mathbb{Z}\cdot[\sigma_1]\\ 0\end{matrix} & \longrightarrow & \mathrm{H}_1(T) & \xrightarrow{\partial} & \mathbb{Z}\cdot\begin{pmatrix}[Q_1]\\ -[Q_2]\end{pmatrix} & \longrightarrow & 0. \end{array}$$

It follows from the above discussion and the Break-into-SES Lemma 78.4 that the top horizontal sequence is exact. Elementary linear algebra shows that the vertical maps are isomorphisms. Finally it is basically clear that the diagram commutes. Since in the bottom the groups to the left and right are both isomorphic to $\mathbb{Z}$ we obtain from the Direct Sum Corollary 78.3 that $\mathrm{H}_1(T) \cong \mathbb{Z}^2$. Finally it follows from the above long exact sequence and the discussion on page 1614 that[947]

$$\mathrm{H}_0(T) \ \cong\ \operatorname{coker}\left(\begin{matrix}\mathbb{Z}\cdot[Q_1]\\ \mathbb{Z}\cdot[Q_3]\end{matrix} \xrightarrow{\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}} \begin{matrix}\mathbb{Z}\cdot[Q_0]\\ \mathbb{Z}\cdot[Q_2]\end{matrix}\right) \ \cong\ \mathbb{Z}.$$

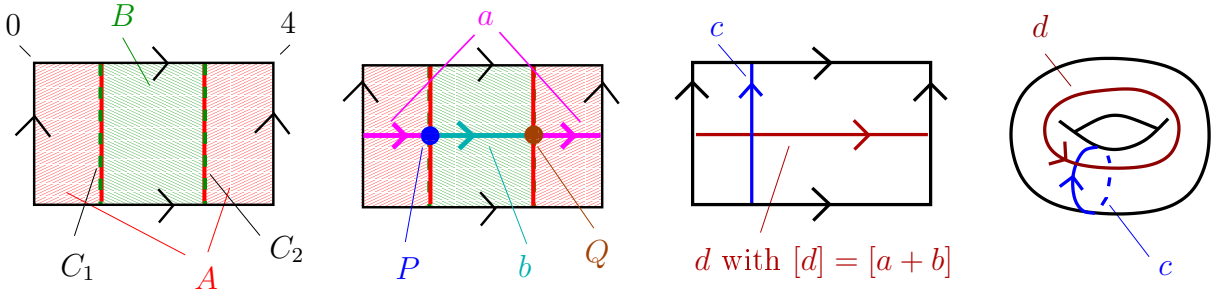
This concludes the proof of Statement (1) of the lemma. ■

Now we turn to the proof of the second statement of Lemma 78.11.

Proof of Lemma 78.11 (2). We continue with the notation that we introduced in the proof of Statement (1) of Lemma 78.11. Next we provide the proof of (2a). Note that $[c] = [\sigma_1] \in \mathrm{H}_1(C)$. Let a and b be the singular simplices in A and B that are indicated in the figure below. Recall that we have the following short exact sequence:

$$0 \longrightarrow \mathbb{Z}\cdot[\sigma_1] \longrightarrow \mathrm{H}_1(T) \xrightarrow{\partial} \mathbb{Z}\cdot\begin{pmatrix}[Q_1]\\ -[Q_2]\end{pmatrix} \longrightarrow 0.$$

[947] Of course we already knew by the H_0-Proposition 72.5 that $\mathrm{H}_0(T) \cong \mathbb{Z}$.



Evidently $[c] = [\sigma_1]$ represents a generator of the group on the left-hand side. Furthermore note that $a + b$ is a cycle in T with $a \in \mathrm{C}_1(A)$ and $b \in \mathrm{C}_1(B)$, thus we have

$$\partial_1([a+b]) \underset{\uparrow}{=} [\partial a] = [P - Q] = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \begin{matrix} \mathrm{H}_1(\mathrm{C}_1) = \mathbb{Z} \\ \mathrm{H}_1(\mathrm{C}_2) = \mathbb{Z} \end{matrix}$$

see the definition of ∂_1 in Theorem 78.9 (1)

Thus we see that $\partial_1([a+b])$ is a generator of the group on the right-hand side of the above short exact sequence. It follows from the Direct Sum Corollary 78.3 (2) that $[c]$ and $[a+b]$ generate the free abelian group $\mathrm{H}_1(T) \cong \mathbb{Z}^2$. The desired statement now follows from the observation that Lemma 72.3 (3) implies that $[a + b] = [d]$. This concludes the proof of (2a).

Now we turn to the proof of (2b). Let α and β be the two singular 2-simplices that are shown in the figure below on the right. It is straightforward to verify that the singular 2-chain $\alpha - \beta$ is indeed a cycle. Note that the proof that $\alpha - \beta$ represents a generator of $\mathrm{H}_2(T)$ is less immediate than the previous argument since the singular simplices α and β do not "fit" into any of the groups of our long exact sequence. In fact, as we will see, the proof that $\alpha - \beta$ represents a generator of $\mathrm{H}_2(T)$ is quite similar to the proof of Lemma 76.3.

We denote by A and B the images of $\alpha\colon \Delta^2 \to T$ and $\beta\colon \Delta^2 \to T$. (We refer to the figure below for an illustration.) We consider the following maps

$$\begin{array}{ccccccc} \mathrm{H}_2(T) & & \xrightarrow{[\alpha-\beta]\mapsto[\alpha-\beta]} & & \xleftarrow{[\mathrm{id}_{\Delta^2}]\mapsto[\alpha]} & & \mathrm{H}_2(\Delta^2, \partial\Delta^2) \\ \downarrow & & & & & & \downarrow \cong \\ \mathrm{H}_2(T/B) & \longrightarrow & \underbrace{\mathrm{H}_2(T/B, B/B)}_{\text{here } [\alpha-\beta]=[\alpha]} & \xleftarrow{\cong} & \mathrm{H}_2(A/\partial A, \partial A/\partial A) & \xleftarrow[\cong]{\alpha_*} & \mathrm{H}_2(\Delta^2/\partial\Delta^2, \partial\Delta^2/\partial\Delta^2). \end{array}$$

We make the following observations:

(i) By the Identity-is-Generator Lemma 76.2 we know that $[\mathrm{id}_{\Delta^2}] \in \mathrm{H}_2(\Delta^2, \partial\Delta^2) \cong \mathbb{Z}$ is a generator.

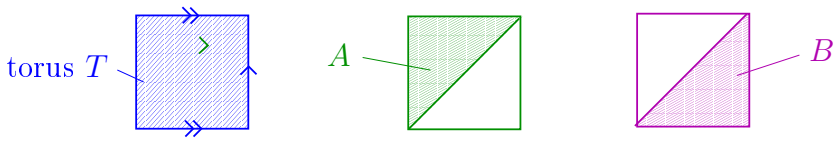
(ii) The two horizontal maps on the right are induced by homeomorphisms of pairs of topological spaces, hence they are isomorphisms.

(iii) The vertical homomorphism on the right is an isomorphism by the Homology-of-Quotients Proposition 74.20 and Lemma 74.12.

(iv) Note that statements (i), (ii) and (iii) imply that $[\alpha]$ represents a generator of the group $\mathrm{H}_2(T/B, B/B) \cong \mathbb{Z}$.

(v) The singular 2-simplex $\beta\colon \Delta^2 \to B$ is zero in $\mathrm{C}_2(T/B, B/B)$. Therefore we have $[\alpha - \beta] = [\alpha] \in \mathrm{H}_2(T/B, B/B)$.

(vi) Next note that statements (iv) and (v) imply that the image of $[\alpha - \beta] \in \mathrm{H}_2(T) \cong \mathbb{Z}$ under the map $\mathrm{H}_2(T) \to \mathrm{H}_2(T/B, B/B)$ is a generator of $\mathrm{H}_2(T/B, B/B) \cong \mathbb{Z}$. But this implies that $[\alpha - \beta]$ is a generator of $\mathrm{H}_2(T) \cong \mathbb{Z}$. ■



Lemma 78.12. Let

$$K \;=\; ([0,1]\times[0,1])/\sim \qquad \text{where } (x,0)\sim(x,1) \text{ and } (0,y)\sim(1,1-y)$$

be the Klein bottle as defined on page 265. We consider the two singular 1-cycles

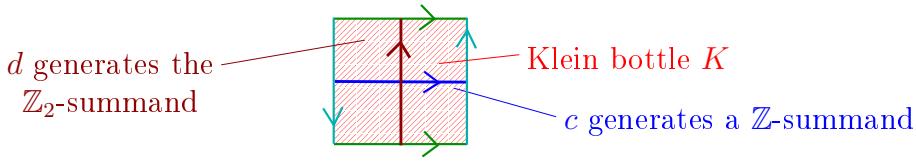
$$\begin{array}{rcl} c\colon \Delta^1 & \to & K \\ (t,1-t) & \mapsto & [(\frac{1}{2},t)] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} d\colon \Delta^1 & \to & K \\ (t,1-t) & \mapsto & [(\frac{1}{2},t)] \end{array}$$

Then

$$\mathrm{H}_n(K) \;=\; \begin{cases} \mathbb{Z}, & \text{if } n=0, \\ \mathbb{Z}\cdot[c]\oplus\mathbb{Z}_2\cdot[d], & \text{if } n=1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The lemma can be proved using the Mayer–Vietoris Theorem 78.9 for Manifolds. We will carry out the details in Exercise 78.3. ■

Furthermore we can determine explicitly two generators of $\mathrm{H}_1(K)$. More precisely, we consider the figure below. There the singular 1-simplex d generates the $\mathbb{Z}_2$-summand and the singular 1-simplex c generates a $\mathbb{Z}$-summand of $\mathrm{H}_1(K)$.[948] We leave the verification of this statement to the reader.



Remark. With some effort one could now use the Mayer–Vietoris Theorem 78.9 for Manifolds to compute the homology groups of the surfaces of genus ≥ 2. We will not carry out the argument in this chapter since in Proposition 80.10 we will perform the calculation using an alternative argument, namely using "cellular homology" which greatly simplifies the calculation. For the impatient reader we state the calculation of the homology groups of surfaces of genus ≥ 2 as Exercise 78.13.

78.6. **The first homology group of a knot complement.** Finally we want to determine the first homology group of the complement of a knot. We recall the relevant definitions from pages 1301 and 1305.

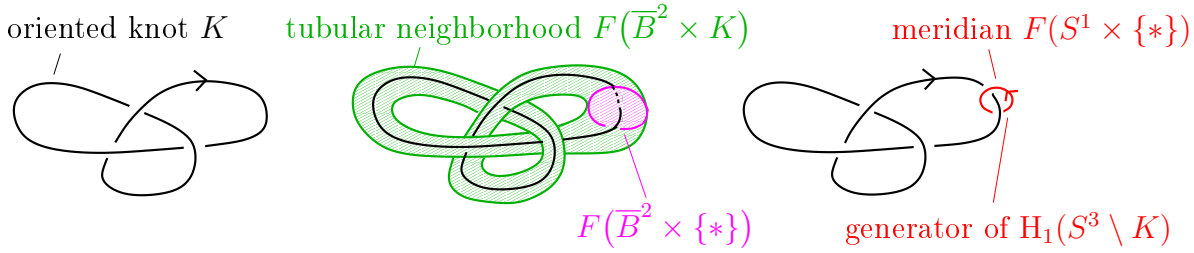
[948] It is straightforward to see that c generates an element of infinite order since the "vertical projection" $K \to [0,1]/\sim$ sends c to a generator of $\mathrm{H}_1([0,1]/\sim) = \mathrm{H}_1(S^1)$.

Definition. (1) A knot is a 1-dimensional submanifold of S^3 that is diffeomorphic to S^1.

(2) Let K be an oriented knot in S^3. By the Tubular Neighborhood Theorem **??** there exists an orientation-preserving smooth embedding $F\colon \overline{B}^2 \times K \to S^3$ such that $\Phi(0,P) = P$ for all $P \in K$. For any $* \in S^1$ we refer to the oriented submanifold $F(S^1 \times \{*\})$ as a meridian of the knot K.

We refer to the figure below for an illustration of an oriented knot and a meridian. We also introduce one new definition:

(3) Let M be a smooth manifold and let C be a closed oriented curve in M, see page 1123 for the definition. Sometimes, by a slight abuse of notation, we denote by C also the element in $\mathrm{H}_1(M)$ that is given by the image of the standard generator $[S^1]$ of $\mathrm{H}_1(S^1)$ under the map $\mathrm{H}_1(S^1) \xrightarrow{\Phi_*} \mathrm{H}_1(C) \to \mathrm{H}_1(M)$.[949]



Lemma 78.13. Let $K \subset S^3$ be a knot. Then[950]

$$\mathrm{H}_0(S^3 \setminus K) \cong \mathbb{Z} \quad \text{and} \quad \mathrm{H}_1(S^3 \setminus K) = \mathbb{Z} \cdot [\mu].$$

here μ is any meridian of K

Later, in Lemma 100.16, we will compute the remaining homology groups of $S^3 \setminus K$.

Remark. The result of Lemma 78.13 is easy to state but also slightly disappointing since it shows that the first homology groups cannot be used to distinguish different knots. In contrast to this fact we saw in Proposition 59.5 and Proposition 59.6 that the fundamental group can be used to distinguish the trivial knot from the trefoil.

Proof. Let $K \subset S^3$ be a knot. By the Tubular Neighborhood Theorem **??** there exists a smooth embedding $\Phi\colon \overline{B}^2 \times K \to S^3$ with $\Phi(0,P) = P$ for all $P \in K$. To simplify the notation we drop Φ from the notation, in particular we write $\overline{B}^2 \times K$ instead of $\Phi(\overline{B}^2 \times K)$. It is basically clear that the interiors of $S^3 \setminus K$ and $\overline{B}^2 \times K$ cover S^3. Therefore we can apply the Mayer–Vietoris Theorem 78.5 to the decomposition $S^3 = (S^3 \setminus K) \cup (\overline{B}^2 \times K)$

[949] It follows from the Diffeotopies-of-S^1-Classification Proposition 34.5 and the Homotopic Maps-and-Homology Proposition 73.6 that, up to a sign, the homology class does not depend on the choice of the orientation-preserving diffeomorphism Φ.

[950] By the H_0-Proposition 72.5 the first statement just says that $S^3 \setminus K$ is path-connected. This seems obvious from the pictures, but it is nonetheless somewhat non-trivial to prove. For example, as discussed in Corollary 42.6, this statement can be viewed as a consequence of the Transversality Theorem 42.4.

and we obtain the following exact sequence

$$\begin{aligned}\overbrace{\mathrm{H}_2(S^3)}^{=0} &\longrightarrow \mathrm{H}_1\big((\overline{B}^2\setminus\{0\})\times K\big) \longrightarrow \mathrm{H}_1(S^3\setminus K)\oplus\mathrm{H}_1(\overline{B}^2\times K) \longrightarrow \overbrace{\mathrm{H}_1(S^3)}^{=0}\longrightarrow\\ &\longrightarrow \widetilde{\mathrm{H}}_0((\overline{B}^2\setminus\{0\})\times K) \longrightarrow \widetilde{\mathrm{H}}_0(S^3\setminus K)\oplus\widetilde{\mathrm{H}}_0\big(\overline{B}^2\times K\big) \longrightarrow \underbrace{\widetilde{\mathrm{H}}_0(S^3)}_{=0}\longrightarrow 0.\end{aligned}$$

Now we have to study the homology groups of the various spaces in more detail.[951]

(1) We denote by $[\mu]=S^1\times\{*\}$ the meridian and we define $[\lambda]=\{*\}\times K$. It follows from Lemma 78.11 (2) that $[\mu]$ and $[\lambda]$ is a basis for $\mathrm{H}_1(S^1\times K)$. The inclusion $S^1\times K\to(\overline{B}^2\setminus\{0\})\times K$ is a homotopy equivalence. It follows from the above, the H_0-Proposition 72.5 and the Homotopy Equivalence-H_*-Corollary 73.9 that $\mathrm{H}_1\big((\overline{B}^2\setminus\{0\})\times K\big)\cong\mathbb{Z}\cdot[\mu]\oplus\mathbb{Z}\cdot[\lambda]$ and that $\widetilde{\mathrm{H}}_0((\overline{B}^2\setminus\{0\})\times K)=0$.

(2) We write $[\gamma]=\{0\}\times K$. The inclusion $\{0\}\times K\to\overline{B}^2\times K$ is a homotopy equivalence. Similar to (1) we see that $\mathrm{H}_1(\overline{B}^2\times K)\cong\mathbb{Z}\cdot[\gamma]$ and that $\widetilde{\mathrm{H}}_0(\overline{B}^2\times K)=0$.

(3) Under the inclusion induced map

$$\widetilde{\mathrm{H}}_1((\overline{B}^2\setminus\{0\})\times K) \;=\; \mathbb{Z}\cdot[\mu]\oplus\mathbb{Z}\cdot[\lambda] \;\to\; \widetilde{\mathrm{H}}_1(\overline{B}^2\times K) \;=\; \mathbb{Z}\cdot[\gamma]$$

we have $[\mu]\mapsto 0$[952] and we have $[\lambda]\mapsto[\gamma]$.[953].

Thus we obtain an exact sequence of the following form

$$\begin{aligned}0 &\longrightarrow \mathbb{Z}\cdot[\mu]\oplus\mathbb{Z}\cdot[\lambda] \xrightarrow[\cong]{\substack{[\mu]\mapsto([\mu],0)\\ [\lambda]\mapsto([\lambda],[\gamma])}} \mathrm{H}_1(S^3\setminus K)\oplus\mathbb{Z}\cdot[\gamma] \longrightarrow 0 \longrightarrow\\ &\longrightarrow 0 \longrightarrow \widetilde{\mathrm{H}}_0(S^3\setminus K)\oplus 0 \longrightarrow 0 \longrightarrow 0.\end{aligned}$$

Now it follows immediately from the Direct Sum Corollary 78.3 and Lemma 51.9 (1) that $\mathrm{H}_1(S^3\setminus K)\cong\mathbb{Z}$ and that $\widetilde{\mathrm{H}}_0(S^3\setminus K)\cong 0$. Furthermore note that it follows easily that $[\mu]\in\mathrm{H}_1(S^3\setminus K)=\mathrm{H}_1(S^3\setminus K)$ is a generator.[954] ■

Remark.

(1) In the proof of Lemma 78.13 we used the Mayer–Vietoris Theorem 78.5 in a different way than in the previous examples. More precisely, instead of decomposing $S^3\setminus K$ into smaller pieces, we used that for a decomposition $X=U\cup V$ the Mayer–Vietoris sequence not only relates the homology groups of U,V and $U\cap V$ to the homology groups of X, but also that it relates the homology groups of U to the homology groups of $V,U\cap V$ and X.

(2) The reader might wonder what the approach of Lemma 78.13 can say about the groups $\mathrm{H}_2(S^3\setminus K)$ and $\mathrm{H}_3(S^3\setminus K)$. In fact the approach shows that the exists an

[951] We will do the calculation by hand. Alternatively this calculation is also a good moment to use Exercise 72.14.

[952] This can be seen as follows: the map $\mathrm{H}_1(S^1\times\{*\})\to\mathrm{H}_1(\overline{B}^2\times K)$ factors through $\mathrm{H}_1(\overline{B}^2\times\{*\})=0$.

[953] Why is that?

[954] Why does this follow?

exact sequence

$$\mathrm{H}_3(S^3 \setminus K) \;\to\; \underbrace{\mathrm{H}_3(S^3)}_{\cong\mathbb{Z}} \xrightarrow[=??]{\partial} \underbrace{\mathrm{H}_2((\overline{B}^2 \setminus \{0\}) \times K)}_{\cong\mathbb{Z}} \;\to\; \mathrm{H}_2(S^3 \setminus K) \;\to\; 0.$$

Since do not know whether the connecting homomorphism is the zero map, injective or perhaps even an isomorphism we cannot determine $\mathrm{H}_2(S^3 \setminus K)$ and $\mathrm{H}_3(S^3 \setminus K)$. Later, in Chapter 100 will acquire the tools necessary to study this map. This will allow us in Lemma 100.16 to completely determine the isomorphism classes of the homology groups of $S^3 \setminus K$.

Lemma 78.14. Let $X \subset \mathbb{R}^n$ be a non-empty compact subset. As usual we denote by $S^n = \mathbb{R}^n \cup \{\infty\}$ the topological space $\mathbb{R}^n$ with a "point at infinity". Then for any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(\mathbb{R}^n \setminus X) \;\cong\; \begin{cases} \mathrm{H}_k((\mathbb{R}^n \cup \{\infty\}) \setminus X) \oplus \mathbb{Z}, & \text{if } k = n-1, \\ \mathrm{H}_k((\mathbb{R}^n \cup \{\infty\}) \setminus X), & \text{otherwise.} \end{cases}$$

Proof. We outsource the proof of the lemma to Exercise 78.12. ∎

78.7. **The homology groups of a mapping cone (∗).** In this section we will discuss the homology groups of a mapping cone. Before we start the discussion we recall the relevant definitions.

Definition. (1) On page 450, given a non-empty topological space A we defined the cone on A as

$$\mathrm{Cone}(A) \;:=\; (A \times [0,1])/\!\sim \qquad \text{where } (a,0) \sim (a',0) \text{ for all } a, a' \in A.$$

If $A = \varnothing$, then we defined $\mathrm{Cone}(\varnothing) = \{*\}$ to be the topological space consisting of a single point $*$.

(2) Given a map $f\colon A \to X$ between topological spaces we defined on page 457 the corresponding mapping cone as

$$\mathrm{Cone}(f\colon A \to X) \;:=\; (\mathrm{Cone}(A) \sqcup X)/\!\sim \;\text{ where } (a,1) \sim f(a) \text{ for all } a \in A.$$

We refer to $[A \times 0]$ respectively $*$ as the cone point of $\mathrm{Cone}(A)$ and of $\mathrm{Cone}(i\colon A \to X)$.

The following lemma, which is an analogue of the Mapping Cone-π_1-Lemma reflem:mapping-cone, gives an elegant calculation of the homology groups of a mapping cone of an inclusion.

Lemma 78.15. (Mapping Cone-H_*-Lemma)

(1) Let X be a topological space and let $A \subset X$ be a subset. We denote by $i\colon A \to X$ the inclusion map. For any $n \in \mathbb{N}_0$ the two maps

$$\mathrm{H}_n(X,A) \underset{\uparrow}{\to} \mathrm{H}_n(\mathrm{Cone}(i\colon A \to X), \mathrm{Cone}(A)) \overset{\text{ex}}{\leftarrow} \widetilde{\mathrm{H}}_n(\mathrm{Cone}(i\colon A \to X))$$

induced by the embedding $X \to \mathrm{Cone}(i\colon A \to X)$

are isomorphisms.

(2) We denote by *PairTop* the category of pairs of topological spaces and we denote by *AbGroup* the category of abelian groups. The composition of the two isomorphisms

in (1) define a natural isomorphism from the functor[955]

$$\begin{array}{ccc} \mathcal{PairTop} & \to & \mathcal{AbGroup} \\ (X,A) & \mapsto & \widetilde{\mathrm{H}}_n(\mathrm{Cone}(i\colon A \to X)) \end{array} \quad \text{to the functor} \quad \begin{array}{ccc} \mathcal{PairTop} & \to & \mathcal{AbGroup} \\ (X,A) & \mapsto & \mathrm{H}_n(X,A). \end{array}$$

Proof.

(1) Just for kicks we throw in one more map:

$$\begin{array}{ccccc} & & \mathrm{H}_n\big(\mathrm{Cone}(A \xrightarrow{i} X)\setminus\{P\}, \mathrm{Cone}(A)\setminus\{P\}\big) & & \\ & \overset{\text{(a)}}{\nearrow} & & \overset{\text{(b)}}{\downarrow} & \\ \mathrm{H}_k(X,A) & \longrightarrow & \mathrm{H}_k(\mathrm{Cone}(i\colon A \to X), \mathrm{Cone}(A)) & \xleftarrow[\text{(c)}]{\text{ж}} & \widetilde{\mathrm{H}}_k(\mathrm{Cone}(i\colon A \to X)). \end{array}$$

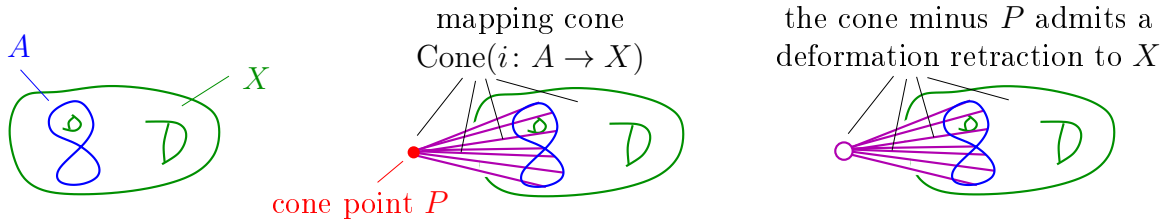
Since the diagram commutes it suffices to show that the maps (a), (b) and (c) are isomorphisms.

(a) From a slight variation on the Mapping Cylinder Basics Lemma 16.12 (1) we obtain a deformation retraction from $\mathrm{Cone}(i\colon A \to X)\setminus\{P\}$ to X which restricts to a deformation retraction from $\mathrm{Cone}(A)$ to A. Thus it follows from Exercise 74.16 that the map (a) is an isomorphism.

(b) It follows from the Excision Theorem 74.17 that the vertical map (b) is an isomorphism.

(c) By the Cone Contractibility Lemma 16.7 we know that the cone $\mathrm{Cone}(A)$ is contractible. It follows from the Topological LES-Corollary 74.14 that the map (c) is an isomorphism.

(2) It is clear that the two maps in (1) are natural. Thus it follows that the two maps together define a natural isomorphism. ■



Remark.

(1) It is worth comparing the Mapping Cone-H_*-Lemma 78.15 to Lemma 74.12 which says that given a pointed topological space (X, x_0) the reduced homology groups $\widetilde{\mathrm{H}}_i(X)$ are naturally isomorphic to the relative homology groups $\mathrm{H}_i(X, \{x_0\})$.

(2) In the Homology-of-Quotients Proposition 74.20 we showed that, under favorable circumstances, we can replace the relative homology group $\mathrm{H}_n(X, A)$ by the homology group $\widetilde{\mathrm{H}}_n(X/A)$. the Mapping Cone-$\mathrm{H}_*$-Lemma 78.15 says that we can replace any relative homology group by the homology group of the mapping cone of the inclusion.

(3) Loosely speaking, the Mapping Cone-H_*-Lemma 78.15 says that for all intents and purposes we can usually replace the relative homology groups of a pair (X, A) of

[955] As in the Mapping Cone Functor Lemma 16.14 (3) we see that the left-hand side does indeed define a functor.

topological spaces by the reduced homology of the mapping cone of the inclusion $A \to X$.

Proof. We had just seen that the map is an isomorphism. It is straightforward to verify that the maps define a natural transformation. ■

We obtain the following convenient corollary which is a very close relative of the Homology-of-Quotients Proposition 74.20.

Corollary 78.16. Let X be a topological space and let $A \subset X$ be a subset. We denote by $i\colon A \to X$ the inclusion map. If i is a closed cofibration, then for any $n \in \mathbb{N}_0$ the natural map $\text{ю}\colon \mathrm{H}_n(X, A) \to \widetilde{\mathrm{H}}_n(X/A)$ from page 1633 is an isomorphism.

Proof. The case that $A = \varnothing$ is basically trivial and is left to the reader. In the following we assume that $A \subset X$ is a non-empty subset. Let $n \in \mathbb{N}_0$. We consider the following diagram

$$\begin{array}{ccc} \mathrm{H}_n(X, A) & \xrightarrow{\quad \text{ю} \quad} & \widetilde{\mathrm{H}}_n(X/A) \\ & {\scriptstyle\cong}\searrow \quad \nearrow{\scriptstyle\varphi} & \\ & \widetilde{\mathrm{H}}_n(\mathrm{Cone}(i\colon A \to X)). & \end{array}$$

We make the following clarifications and observations:

(1) The top horizontal map is the natural homomorphism that we introduced on page 1633.
(2) The left diagonal map is the natural *isomorphism* of the Mapping Cone-H_*-Lemma 78.15.
(3) The right diagonal map is the map

$$\begin{array}{rcl} \varphi\colon \mathrm{Cone}(i\colon A \to X) & \to & X/A \\ P & \mapsto & \begin{cases} [P], & \text{if } P \in X, \\ [A], & \text{if } P = [(a,t)] \text{ with } a \in A \text{ and } t \in [0,1] \end{cases} \end{array}$$

that we introduced in the Cofibration-Quotient Lemma 17.7 (1).
(4) It follows easily from the definitions that the diagram commutes.
(5) Since $i\colon A \to X$ is by hypothesis a closed cofibration we obtain from the Cofibration-Quotient Lemma 17.7 (1) that φ is a homotopy equivalence. Therefore we obtain from the Homotopy Equivalence-H_*-Corollary 73.9 that the right diagonal map is also an isomorphism.
(6) Putting everything together we see that the top map is indeed an isomorphism. ■

Let A be a subset of a topological space X. We denote by $i\colon A \to X$ the corresponding inclusion map. It follows from the Topological LES-Corollary 74.14 together with the Mapping Cone-H_*-Lemma 78.15 that the inclusion induced maps $i_*\colon \mathrm{H}_k(A) \to \mathrm{H}_k(X)$ fit into the following natural long exact sequence

$$\ldots \to \mathrm{H}_k(A) \xrightarrow{i_*} \mathrm{H}_k(X) \to \underbrace{\mathrm{H}_k(X, A)}_{\cong \widetilde{\mathrm{H}}_k(\mathrm{Cone}(i\colon A \to X))} \xrightarrow{\partial} \mathrm{H}_{k-1}(A) \xrightarrow{i_*} \mathrm{H}_{k-1}(X) \to \ldots$$

The following lemma generalizes this neat fact to the setting of any map $f\colon A \to X$ between topological spaces.

Lemma 78.17. (Mapping Cone-H_*-Lemma)[956]

(1) Let $f\colon A \to X$ be a map between topological spaces.
   (a) We denote by $j(f)\colon X \to \mathrm{Cone}(f) = (\mathrm{Cone}(A) \sqcup X)/\sim$ the obvious inclusion map.
   (b) We denote by U the image of $A \times [0, \frac{3}{4})$ in $\mathrm{Cone}(f)$ and we denote by V the complement of the image of $A \times [0, \frac{1}{4}]$. (See also the figure below.) We consider the following diagram[957]

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & \mathrm{H}_k(U) \oplus \mathrm{H}_k(V) & \longrightarrow & \mathrm{H}_k(\mathrm{Cone}(f)) & \xrightarrow{\partial_k} & \mathrm{H}_{k-1}(U \cap V) & \longrightarrow & \ldots \\ & & & & \uparrow \wr & & \uparrow \cong & & \\ & & & & \mathrm{H}_k(\mathrm{Cone}(f)) & \xrightarrow[=:\partial_k(f)]{\partial_k} & \mathrm{H}_{k-1}(A). & & \end{array}$$

   Here the top sequence is the long exact sequence arising from the Mayer–Vietoris Theorem 78.5[958] and the vertical map is induced by the inclusion $a \mapsto (a, \frac{1}{2})$ which is a homotopy equivalence. From the above diagram we obtain a map

$$\partial_k(f)\colon \widetilde{\mathrm{H}}_k(\mathrm{Cone}(f)) \;\to\; \mathrm{H}_{k-1}(A).$$

With these definitions the following sequence is exact

$$\ldots \longrightarrow \mathrm{H}_k(A) \xrightarrow{f_*} \mathrm{H}_k(X) \xrightarrow{j(f)_*} \widetilde{\mathrm{H}}_k(\mathrm{Cone}(f)) \xrightarrow{\partial_k(f)} \mathrm{H}_{k-1}(A) \longrightarrow \ldots$$

(2) The connecting homomorphism in (1) is natural. In particular given any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \widetilde{A} & \xrightarrow{\widetilde{f}} & \widetilde{X} \end{array}$$

of maps between topological spaces the following diagram commutes:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \mathrm{H}_k(A) & \xrightarrow{f_*} & \mathrm{H}_k(X) & \xrightarrow{j(f)_*} & \widetilde{\mathrm{H}}_k(\mathrm{Cone}(f)) & \xrightarrow{\partial_k(f)} & \mathrm{H}_{k-1}(A) & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle\varphi_*} & & \downarrow{\scriptstyle\psi_*} & & \downarrow & & \downarrow{\scriptstyle\varphi_*} & & \\ \ldots & \longrightarrow & \mathrm{H}_k(\widetilde{A}) & \xrightarrow{\widetilde{f}_*} & \mathrm{H}_k(\widetilde{X}) & \xrightarrow{j(\widetilde{f})_*} & \widetilde{\mathrm{H}}_k(\mathrm{Cone}(\widetilde{f})) & \xrightarrow{\partial_k(\widetilde{f})} & \mathrm{H}_{k-1}(\widetilde{A}) & \longrightarrow & \ldots. \end{array}$$

Here the vertical maps between the homology groups of the mapping cones are induced by the maps from the Mapping Cone Functor Lemma 16.14 (3).

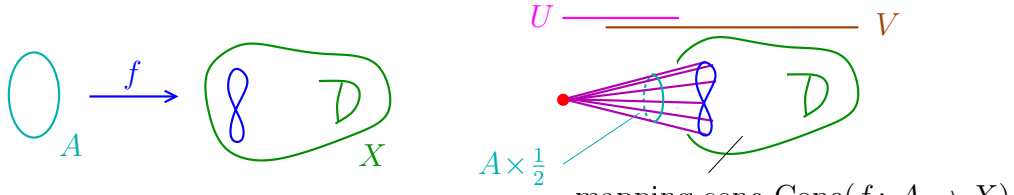
(3) Let X be a topological space and let $A \subset X$ be a subset. We denote by $i\colon A \to X$ the inclusion map. The following diagram

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \mathrm{H}_k(A) & \xrightarrow{i} & \mathrm{H}_k(X) & \longrightarrow & \mathrm{H}_k(X, A) & \xrightarrow{\partial} & \widetilde{\mathrm{H}}_{k-1}(A) & \longrightarrow & \widetilde{\mathrm{H}}_{k-1}(X) & \longrightarrow & \ldots \\ & & \downarrow{=} & & \downarrow{=} & & \downarrow{\cong} & & \downarrow{=} & & \downarrow{=} & & \\ \ldots & \longrightarrow & \mathrm{H}_k(A) & \xrightarrow{i} & \mathrm{H}_k(X) & \longrightarrow & \widetilde{\mathrm{H}}_k(\mathrm{Cone}(i)) & \xrightarrow{\partial} & \widetilde{\mathrm{H}}_{k-1}(A) & \longrightarrow & \widetilde{\mathrm{H}}_{k-1}(X) & \longrightarrow & \ldots \end{array}$$

commutes. Here the middle vertical map is the natural isomorphism from the Mapping Cone-H_*-Lemma 78.15, the top sequence is the long exact sequence arising from

the Topological LES-Corollary 74.14 and the bottom sequence is the exact sequence from (1).

The obvious analogues of the above statements for reduced homology of A and X also hold.



mapping cone $\mathrm{Cone}(f\colon A \to X)$

Proof.

(1) We consider the following diagram:

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & \mathrm{H}_k(U\cap V) & \longrightarrow & \mathrm{H}_k(U)\oplus\mathrm{H}_k(V) & \longrightarrow & \mathrm{H}_k(\mathrm{Cone}(f)) & \xrightarrow{\partial_k} & \mathrm{H}_{k-1}(U\cap V) \longrightarrow \ldots \\
 & & \cong\uparrow & & \cong\uparrow \quad \uparrow\cong & & \mathrm{id}\uparrow & & \uparrow\cong \\
\ldots & \longrightarrow & \mathrm{H}_k(A) & \xrightarrow{c_*\oplus f_*} & \mathrm{H}_k(\{*\})\oplus\mathrm{H}_k(X) & \longrightarrow & \mathrm{H}_k(\mathrm{Cone}(f)) & \longrightarrow & \mathrm{H}_{k-1}(A) \longrightarrow \ldots \\
 & & \cong\uparrow & & \cong\uparrow \quad \uparrow\text{ж} & & \text{ж}\uparrow & & \uparrow\cong \\
\ldots & \longrightarrow & \mathrm{H}_k(A) & \xrightarrow{f_*} & \underbrace{\widetilde{\mathrm{H}}_k(\{*\})}_{=0}\oplus\mathrm{H}_k(X) & \xrightarrow{j(f)_*} & \widetilde{\mathrm{H}}_k(\mathrm{Cone}(f)) & \xrightarrow{\partial_k(f)} & \mathrm{H}_{k-1}(A) \longrightarrow \ldots
\end{array}$$

We make the following clarifications and observations:

(a) The top sequence is the long exact sequence arising from the Mayer–Vietoris Theorem 78.5 applied to the open cover $\mathrm{Cone}(f) = U \cup V$.

(b) We denote by $*$ the cone point of $\mathrm{Cone}(f)$.

(c) The vertical maps are the almost obvious inclusion maps. The map $c\colon A \to \{*\}$ is the constant map. It follows almost immediately from the definitions together with the Homotopic Maps-and-Homology Proposition 73.6 that the upper part of the diagram commutes.

(d) As in the Cone Contractibility Lemma 16.7 we see that the inclusion $\{*\} \to U$ is a homotopy equivalence. Furthermore, by the Reduced Homology Lemma 74.1 (7) we know that $\widetilde{\mathrm{H}}_k(\{*\}) = 0$.

(e) It follows now fairly easily from Lemma 74.2 that the lower part of the diagram also commutes and that the lower sequence is exact. But that is what we wanted to show.

(2) This statement follows from the naturality of the connecting homomorphisms of the Mayer–Vietoris sequence, see Theorem 78.5 (2).

(3) We will prove this statement in Exercise 78.18. ∎

[956] In the statement and the proof of the lemma we implicitly use Lemmas 16.8 and 16.13 (2) several times to show that various maps are open and closed embeddings. For example it follows from the Cone-Suspension Basics Lemma 16.8 that the map $j(f)$ given in (1a) is actually a closed embedding.

[957] We leave it to the reader to give a reasonable interpretation of what we say here in the special case that $A = \varnothing$.

[958] Note that U and V are open subsets of $\mathrm{Cone}(f)$.

In the Cell Attachment-π_1-Proposition 69.11 we saw how attaching an n-cell to a topological space affects its fundamental group. Basically for free we now get a corresponding statement for homology groups.

Lemma 78.18. (Cell Attachment-H_*-Lemma) Let X be a topological space, let $n \geq 2$ and let $\varphi\colon S^{n-1} \to X$ be a map. We denote by $i\colon X \to X \cup_\varphi \overline{B}^n$ the obvious map. (Note that by the Topological Pushout-Maps Lemma 5.34 (3) and (4) we know that i is a closed embedding.) There exists a natural exact sequence of the form

$$0 \to \mathrm{H}_n(X) \xrightarrow{i_*} \mathrm{H}_n\big(X \cup_\varphi \overline{B}^n\big) \to \underbrace{\mathrm{H}_{n-1}(S^{n-1})}_{\cong \mathbb{Z}} \xrightarrow{\varphi_*} \mathrm{H}_{n-1}(X) \xrightarrow{i_*} \mathrm{H}_{n-1}\big(X \cup_\varphi \overline{B}^n\big) \to 0.$$

Furthermore, for any $k \neq n-1, n$ the inclusion induced map

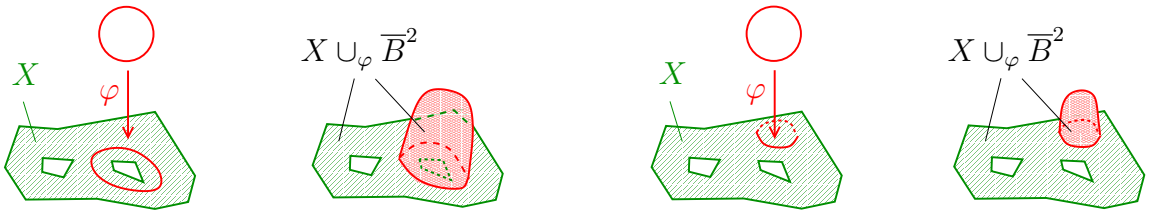
$$i_*\colon \mathrm{H}_k(X) \xrightarrow{\cong} \mathrm{H}_k(X \cup_\varphi \overline{B}^n)$$

is an isomorphism.

Example. Let X be a topological space with finitely generated homology groups. Let $n \geq 2$ and let $\varphi\colon S^{n-1} \to X$ be a map. Loosely speaking the Cell Attachment-H_*-Lemma 78.18 says that gluing on a copy of $\overline{B}^n$ along φ has the following effect on homology:

	H_n	H_{n-1}
$\mathrm{H}_{n-1}(S^{n-1}) \xrightarrow{\varphi_*} \mathrm{H}_{n-1}(X)$ has infinite image	unchanged	rank goes down by 1
$\mathrm{H}_{n-1}(S^{n-1}) \xrightarrow{\varphi_*} \mathrm{H}_{n-1}(X)$ has finite image	rank goes up by 1	unchanged.

We illustrate these two different behaviors in the figure below.



$\varphi_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(X)$ has infinite image

$\varphi_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(X)$ has trivial image

Proof. On page 1527 we made the, almost trivial, observation that there exists a homeomorphism $\mathrm{Cone}(\varphi\colon S^{n-1} \to X) \xrightarrow{\cong} X \cup_\varphi \overline{B}^n$ which is the identity on X. The conclusion of the Cell Attachment-H_*-Lemma 78.18 follows immediately from this observation together with the Mapping Cone-H_*-Lemma 78.17, the calculation of the homology groups of S^{n-1}, see Proposition 74.4 and the observation that our hypothesis that $n \geq 2$ implies that we do not need to worry about reduced homology groups. ■

The last lemma of this section gives a neat reinterpretation of the connecting homomorphism in the long exact sequence of the Mapping Cone-H_*-Lemma 78.17.

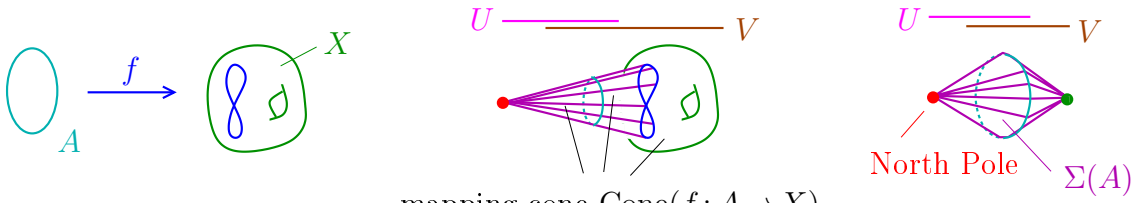
Lemma 78.19. Let $f\colon A \to X$ be a map between topological spaces. For every $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_k(\mathrm{Cone}(f\colon A \to X)) & \xrightarrow{\partial} & \widetilde{\mathrm{H}}_{k-1}(A) \\ \downarrow & & \cong\downarrow \Sigma_A \\ \widetilde{\mathrm{H}}_k(\mathrm{Cone}(f\colon A \to X)/X) & \xrightarrow[\cong]{\varphi_*} & \widetilde{\mathrm{H}}_k(\Sigma(A)). \end{array}$$

Here we used the following notation:

(1) The top horizontal map is the connecting homomorphism of the long exact sequence given in the Mapping Cone-H_*-Lemma 78.17.
(2) The left vertical map is induced by the obvious projection.
(3) The map $\varphi\colon \mathrm{Cone}(f\colon A \to X)/X \to \Sigma(A)$ is the natural homeomorphism from the Mapping Cone Basics Lemma 16.13 (4).
(4) The map $\Sigma_A\colon \widetilde{\mathrm{H}}_{k-1}(A) \to \widetilde{\mathrm{H}}_k(\Sigma(A))$ is the natural isomorphism from the Suspension-H_*-Proposition 76.7.

Proof. The statement follows immediately from the explicit definition of the maps ∂ and Σ_A via Mayer–Vietoris sequences, from the explicit definition of the map φ, and from the naturality of the connecting homomorphism of the Mayer–Vietoris sequence, see Theorem 78.5 (2), that the isomorphism Σ_X is natural. We leave it to the reader to fill in the details. ■



mapping cone $\mathrm{Cone}(f\colon A \to X)$

78.8. **The homology groups of a mapping torus** (∗). On page 467 we defined the mapping torus of a self-homeomorphism and in the Mapping Torus-π_1-Proposition 50.6 we had determined the fundamental group of the mapping torus. In this section we will determine the isomorphism types of the homology groups of a mapping torus.
For convenience we recall from page 467 the definition of the mapping torus. Let X be a topological space and let $f\colon X \to X$ be a map. We refer to

$$\mathrm{Tor}(X, f) \;:=\; (X \times [0,1])/(x,0) \sim (f(x),1)$$

as the *mapping torus of* (X, f). In the following we will use on several occasions Lemma 16.22 which says that for any closed respectively open interval $I \subset \mathbb{R}$ of length less than one, the map

$$\varphi\colon X \times I \xrightarrow{(x,t)\mapsto[(x,t)]} (X \times \mathbb{R})/\!\sim \xrightarrow{[(x,t)]\mapsto[(x,t)]} \mathrm{Tor}(X, f)$$

is a closed respectively open embedding. Now we can state the main result of this section.

Proposition 78.20. (Mapping Torus-Homology Proposition) Let X be a topological space and let $f\colon X \to X$ be a map. For every $n \in \mathbb{N}_0$ we have a natural short exact

sequence[959]

$$0 \to \operatorname{coker}\big(\mathrm{H}_n(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_n(X)\big) \xrightarrow{i} \mathrm{H}_n(\mathrm{Tor}(X,f)) \to \ker\big(\mathrm{H}_{n-1}(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_{n-1}(X)\big) \to 0,$$

where the map i is induced by the embedding $X \xrightarrow{x \mapsto [(x,0)]} \mathrm{Tor}(X,f)$.

Example. First let $X = S^1$ and let $f\colon S^1 \to S^1$ be the self-homeomorphism that is given by $f(z) = \overline{z} = z^{-1}$. On page 469 we saw that the mapping torus $\mathrm{Tor}(S^1, f)$ is homeomorphic to the Klein bottle. It follows from the discussion on page 1594 the induced map $f_*\colon \mathrm{H}_0(S^1) \to \mathrm{H}_0(S^1)$ is the identity map. Furthermore we showed in Lemma 76.10 that the induced homomorphism $f_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(S^1)$ is given by multiplication by -1. Since $\mathrm{H}_0(S^1) \cong \mathbb{Z}$ and $\mathrm{H}_1(S^1) \cong \mathbb{Z}$ we now see that the short exact sequences of the Mapping Torus-Homology Proposition 78.20 are of the form

$$\begin{array}{lccccccccc}
n=2\colon & & & & 0 & \to & \mathrm{H}_2(\mathrm{Tor}(S^1,f)) & \to & \ker(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}) & \to & 0 \\
n=1\colon & 0 & \to & \operatorname{coker}\big(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}\big) & \to & \mathrm{H}_1(\mathrm{Tor}(S^1,f)) & \to & \ker(\mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z}) & \to & 0
\end{array}$$

and

$$n=0\colon \quad 0 \to \operatorname{coker}\big(\mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z}\big) \to \mathrm{H}_0(\mathrm{Tor}(S^1,f)) \to 0.$$

Using the Direct Sum Corollary 78.3 we now deduce that

$$\mathrm{H}_2(\mathrm{Tor}(S^1,f)) = 0, \quad \mathrm{H}_1(\mathrm{Tor}(S^1,f)) \cong \mathbb{Z}_2 \oplus \mathbb{Z} \quad \text{and} \quad \mathrm{H}_0(\mathrm{Tor}(S^1,f)) = \mathbb{Z}.$$

This confirms, as it should, the calculation that we had made in Lemma 78.12.

Proof. The proof of the proposition is quite similar to the calculation of the homology groups of the torus.[960] Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. We compute the homology of the following space that is clearly naturally homeomorphic to the mapping torus:

$$M := (X \times [0,10])/(x,0) \sim (f(x),10).$$

We denote by $p\colon X \times [0,10] \to M$ the natural projection. Furthermore we write $A := p(X \times ([0,4) \cup (6,10])) \subset M$ and $B := p(X \times (2,8)) \subset M$.[961] Finally, given $t \in [0,10]$ we denote by

$$\begin{array}{rccc} i_t\colon & X & \to & p(X \times \{t\}) \\ & x & \mapsto & p(x,t) \end{array}$$

the obvious inclusion map. We write $i_{3,A}$ for the composition of i_3 with the inclusion map $p(X \times \{3\}) \to A$. We denote analogous compositions by the obvious variations in the notation. The proof below crucially relies on the following claim.

[959] Recall that $\mathrm{H}_{-1}(X)$ is defined to be the trivial group.

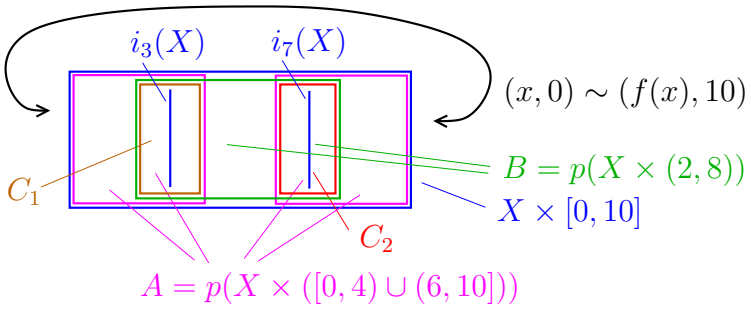
[960] In fact if one reads the subsequent proof with $X = S^1$ and $f = \mathrm{id}$ in mind, then one obtains almost the same argument as for the torus.

[961] Throughout this proof we will freely use Lemma 16.22 which says that $p\colon X \times [0,1] \to M$, restricted to reasonable open subsets, is in fact an open embedding.

Claim. For any $n \in \mathbb{N}_0$ the following two diagrams commute

$$\begin{array}{ccc} \mathrm{H}_n(X) & \xrightarrow{\mathrm{id}} & \mathrm{H}_n(X) \\ & {\scriptstyle i_{3,B}} \searrow \quad \swarrow {\scriptstyle i_{7,B}} & \\ & \mathrm{H}_n(B) & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathrm{H}_n(X) & \xrightarrow{f_*^{-1}} & \mathrm{H}_n(X). \\ & {\scriptstyle i_{3,A}} \searrow \quad \swarrow {\scriptstyle i_{7,A}} & \\ & \mathrm{H}_n(B) & \end{array}$$

Proof. It follows almost immediately from the Homotopic Maps-and-Homology Proposition 73.6 that the two triangles commute. We leave the details to the reader. ⊞



Next we consider the following diagram:[962]

$$\begin{array}{ccccccccccc}
\to & \begin{matrix}\mathrm{H}_n(C_1)\\ \mathrm{H}_n(C_2)\end{matrix} & \xrightarrow{\left(\begin{smallmatrix} i_{C_1} & i_{C_2} \\ -i_{C_1} & -i_{C_2}\end{smallmatrix}\right)} & \begin{matrix}\mathrm{H}_n(A)\\ \mathrm{H}_n(B)\end{matrix} & \xrightarrow{(i_A\ i_B)} & \mathrm{H}_n(M) & \xrightarrow{\partial} & \begin{matrix}\mathrm{H}_{n-1}(C_1)\\ \mathrm{H}_{n-1}(C_2)\end{matrix} & \xrightarrow{\left(\begin{smallmatrix} i_{C_1} & i_{C_2} \\ -i_{C_1} & -i_{C_2}\end{smallmatrix}\right)} & \begin{matrix}\mathrm{H}_{n-1}(A)\\ \mathrm{H}_{n-1}(B)\end{matrix} & \to \\
 & \uparrow{\scriptstyle \left(\begin{smallmatrix} i_3 & 0 \\ 0 & i_7\end{smallmatrix}\right)} & & \uparrow{\scriptstyle \left(\begin{smallmatrix} i_3 & 0 \\ 0 & i_7\end{smallmatrix}\right)} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \left(\begin{smallmatrix} i_3 & 0 \\ 0 & i_7\end{smallmatrix}\right)} & & \uparrow{\scriptstyle \left(\begin{smallmatrix} i_3 & 0 \\ 0 & i_7\end{smallmatrix}\right)} & \\
\to & \begin{matrix}\mathrm{H}_n(X)\\ \mathrm{H}_n(X)\end{matrix} & \xrightarrow{\left(\begin{smallmatrix} \mathrm{id} & f_* \\ -\mathrm{id} & -\mathrm{id}\end{smallmatrix}\right)} & \begin{matrix}\mathrm{H}_n(X)\\ \mathrm{H}_n(X)\end{matrix} & \xrightarrow{(i_A\ i_B)} & \mathrm{H}_n(M) & \xrightarrow{\partial} & \begin{matrix}\mathrm{H}_{n-1}(X)\\ \mathrm{H}_{n-1}(X)\end{matrix} & \xrightarrow{\left(\begin{smallmatrix} \mathrm{id} & f_* \\ -\mathrm{id} & -\mathrm{id}\end{smallmatrix}\right)} & \begin{matrix}\mathrm{H}_{n-1}(X)\\ \mathrm{H}_{n-1}(X)\end{matrix} & \to \\
 & \uparrow{\scriptstyle \left(\begin{smallmatrix} \mathrm{id} \\ -\mathrm{id}\end{smallmatrix}\right)} & & \uparrow{\scriptstyle \left(\begin{smallmatrix} \mathrm{id} \\ 0\end{smallmatrix}\right)} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \left(\begin{smallmatrix} \mathrm{id} \\ -\mathrm{id}\end{smallmatrix}\right)} & & \uparrow{\scriptstyle \left(\begin{smallmatrix} \mathrm{id} \\ 0\end{smallmatrix}\right)} & \\
\longrightarrow & \mathrm{H}_n(X) & \xrightarrow{\mathrm{id}-f_*} & \mathrm{H}_n(X) & \xrightarrow{i} & \mathrm{H}_n(M) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(X) & \xrightarrow{\mathrm{id}-f_*} & \mathrm{H}_{n-1}(X) & \longrightarrow
\end{array}$$

The above diagram arose as follows:

(1) Note that the interiors of $A = p(X \times ([0,4) \cup (6,10])) \subset M$ and $B = p(X \times (2,8)) \subset M$ cover all of M. Furthermore note that we have $A \cap B = C_1 \sqcup C_2$. On the top of the diagram we wrote down the corresponding exact Mayer–Vietoris sequence from Theorem 78.5.

(2) Note that the maps $i_3 \colon X \to A$, $i_3 \colon X \to C_1$, $i_7 \colon X \to C_2$ and $i_7 \colon X \to B$ are homotopy equivalences. It follows from the Homotopy Equivalence-H_*-Corollary 73.9 that the induced maps on homology are isomorphisms. In particular the vertical maps between the first and the second horizontal sequence are isomorphisms.

(3) Next we want to show that the top part of the diagram commutes. The squares involving $\mathrm{id} \colon \mathrm{H}_n(M) \to \mathrm{H}_n(M)$ commute by the functoriality of homology groups and by definition of the connecting homomorphism ∂ in the middle row. It remains to consider the squares which in our excerpt above are to the left and right. The

[962] Here we use the same notation as on page 1729.

discussion is perhaps less confusing if we explain how the maps on the middle row were found. The task was to find maps $a, b, c, d\colon \mathrm{H}_n(X) \to \mathrm{H}_n(X)$ such that the following equality holds:

$$\underbrace{\begin{pmatrix} i_{C_1,A} & i_{C_2,A} \\ -i_{C_1,B} & -i_{C_2,B} \end{pmatrix}\begin{pmatrix} i_{3,C_1} & 0 \\ 0 & i_{7,C_2} \end{pmatrix}}_{=\begin{pmatrix} i_{3,A} & i_{7,A} \\ -i_{3,B} & -i_{7,B} \end{pmatrix}} = \underbrace{\begin{pmatrix} i_{3,A} & 0 \\ 0 & i_{7,B} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{=\begin{pmatrix} i_{3,A}\circ a & i_{3,A}\circ b \\ i_{7,B}\circ c & i_{7,B}\circ d \end{pmatrix}}.$$

It follows from the above claim that a solution is given by $a = \mathrm{id}, b = f_*, c = -\mathrm{id}$ and $d = -\mathrm{id}$.

(4) It is straightforward to see that the bottom squares in the above diagram commute. We leave it to the reader to verify that the bottom sequence is also exact.

As we had just pointed out, the bottom horizontal sequence is exact. The proposition is thus an immediate consequence of the Break-into-SES Lemma 78.4. ∎

Now we can also easily prove the following lemma.

Lemma 78.21. Let X be a topological space. We denote by $p\colon X \times S^1 \to X$ the natural projection. For every $n \in \mathbb{N}_0$ there exists a natural map $\psi\colon \mathrm{H}_n(X \times S^1) \to \mathrm{H}_{n-1}(X)$ such that the map

$$\Xi_X := p_* \circ \psi\colon \mathrm{H}_n(X \times S^1) \xrightarrow{\cong} \mathrm{H}_n(X) \oplus \mathrm{H}_{n-1}(X)$$

is a natural isomorphism. Here natural means that for any map $f\colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_n(X \times S^1) & \xrightarrow[\cong]{\Xi_X} & \mathrm{H}_n(X) \oplus \mathrm{H}_{n-1}(X) \\ \downarrow{\scriptstyle (f\times \mathrm{id})_*} & & \downarrow{\scriptstyle f_* \quad f_*} \\ \mathrm{H}_n(Y \times S^1) & \xrightarrow[\cong]{\Xi_Y} & \mathrm{H}_n(X) \oplus \mathrm{H}_{n-1}(X). \end{array}$$

Example. Using an elementary and amusing induction argument one can show quite easily, using Lemma 78.21, that for $k = 0, \ldots, n$ we have

$$\mathrm{H}_k(n\text{-dimensional torus } (S^1)^n) \;=\; \text{free abelian group of rank } \binom{n}{k}$$

and that all other homology groups vanish.

Remark. We gave a different proof of Lemma 78.21 in Exercise 75.4.

Proof. Let $n \in \mathbb{N}_0$. Note that by the Mapping Torus-of-id Lemma 16.24 the map $\Theta\colon \mathrm{Tor}(X, f := \mathrm{id}) \to X \times S^1$ given by $[(x,t)] \mapsto (x, \exp(2\pi\,\mathrm{i}t))$ is a natural homeomorphism. We denote by $i\colon X \to \mathrm{Tor}(f)$ the natural embedding given by $x \mapsto [(x,0)]$ and we denote by $j\colon X \to X \times S^1$ the natural embedding given by $x \mapsto (x,1)$. Next we consider the following

diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{coker}\big(\mathrm{H}_n(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_n(X)\big) & \xrightarrow{i_*} & \mathrm{H}_n(\mathrm{Tor}(X,f)) & \xrightarrow{\partial} & \ker\big(\mathrm{H}_{n-1}(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_{n-1}(X)\big) & \longrightarrow & 0 \\
 & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \\
0 & \longrightarrow & \mathrm{H}_n(X) & \xrightarrow{i} & \mathrm{H}_n(\mathrm{Tor}(X,f)) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(X) & \longrightarrow & 0 \\
 & & \downarrow \mathrm{id} & & \cong \downarrow \Theta_* & & \downarrow \mathrm{id} & & \\
0 & \longrightarrow & \mathrm{H}_n(X) & \underset{p_*}{\overset{j_*}{\rightleftarrows}} & \mathrm{H}_n(X \times S^1) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(X) & \longrightarrow & 0.
\end{array}$$

We make the following clarifications and observations:

(1) The top sequence is the short exact sequence obtained from the Mapping Torus-Homology Proposition 78.20.
(2) Since $f_* - \mathrm{id}_* = 0$ we obtain the two vertical identities on top.
(3) Since Θ is an isomorphism and since $\Theta \circ i = j$ we see that the bottom sequence also commutes.
(4) The map $p\colon X \times S^1 \to X$ is the natural map that is given by projection onto the first factor. Note that $p_* \circ i_* = (p \circ i)_* = \mathrm{id}$.

As in the proof of Splitting Lemma 78.2 we now see that the map

$$\begin{array}{rcl}
\Xi_X\colon \mathrm{H}_n(X \times S^1) & \to & \mathrm{H}_n(X) \oplus \mathrm{H}_{n-1}(X) \\
\sigma & \mapsto & p_*(\sigma) \oplus \partial(\sigma - (j_* \circ p_*)(\sigma))
\end{array}$$

is an isomorphism. Since all maps involved are natural we see that this isomorphism is in fact natural. ■

We conclude this long chapter with the following lemma which can be viewed as the analogue of the Mapping Torus-π_1-Proposition 50.6.

Lemma 78.22. (Mapping Torus-Homology Lemma) Let X be a path-connected topological space and let $f\colon X \to X$ be a map. We denote by $i\colon X \xrightarrow{x \mapsto [(x,0)]} \mathrm{Tor}(X,f)$ the natural embedding and we denote by $q\colon \mathrm{Tor}(X,f) \to S^1$ the natural projection that we introduced on page 467. With this notation the following sequence is natural and exact:

$$0 \;\to\; \operatorname{coker}\big(\mathrm{H}_1(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_1(X)\big) \;\xrightarrow{i}\; \mathrm{H}_1(\mathrm{Tor}(X,f)) \;\to\; \mathrm{H}_1(S^1) \;\to\; 0.$$

In particular we have an isomorphism

$$\mathrm{H}_1(\mathrm{Tor}(X,f)) \;\cong\; \mathbb{Z} \oplus \operatorname{coker}\big(\mathrm{H}_1(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_1(X)\big).$$

Remark. Later in the Hurewicz Theorem 85.5 we will show that the abelianization of the fundamental group $\pi_1(X,x_0)$ of a pointed path-connected topological space (X,x_0) is naturally isomorphic to the first homology group $\mathrm{H}_1(X)$. Thus we see that the Mapping Torus-Homology Lemma 78.22 is a modest generalization of Exercise 53.7.

Proof. Since X is path-connected we obtain immediately from the discussion on page 1594 that the map $f_* - \mathrm{id}\colon \mathrm{H}_0(X) \to \mathrm{H}_0(X)$ is the zero map. We consider the following maps:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \operatorname{coker}\big(\mathrm{H}_1(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_1(X)\big) & \xrightarrow{i} & \mathrm{H}_1(\mathrm{Tor}(X,f)) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0. \\ & & & & \downarrow{\scriptstyle p_*} & \swarrow{\scriptstyle \Psi} & & & \\ & & & & \underbrace{\mathrm{H}_1(S^1)}_{\cong \mathbb{Z}}. & & & & \end{array}$$

We make the following observations:

(1) The horizontal map is the short exact sequence from the Mapping Torus-Homology Proposition 78.20.
(2) Since $p \circ i$ is a constant map we see that $p_* \circ i_*$ is the trivial map.
(3) It follows from (1) and (2) that there exists a map $\Psi\colon \mathbb{Z} \to \mathrm{H}_1(S^1)$ that makes the diagram commute.
(4) Furthermore, since X is path-connected one can easily construct a left-inverse to $p\colon \mathrm{Tor}(X,f) \to S^1$. This implies that $p_*\colon \mathrm{H}_1(\mathrm{Tor}(X,f)) \to \mathrm{H}_1(S^1)$ is an epimorphism.
(5) Since $\mathrm{H}_1(S^1) \cong \mathbb{Z}$ and since any epimorphism from $\mathbb{Z} \to \mathbb{Z}$ is an isomorphism we obtain from the above that Ψ is an isomorphism.

This concludes the proof that the sequence in the statement of the lemma is indeed exact.

Finally note that since $\mathbb{Z}$ is free abelian it follows from the Direct Sum Corollary 78.3 that we have an isomorphism

$$\mathrm{H}_1(\mathrm{Tor}(X,f)) \;\cong\; \mathbb{Z} \oplus \operatorname{coker}\big(\mathrm{H}_1(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_1(X)\big). \qquad \blacksquare$$

Exercises for Chapter 78.

Exercise 78.1. We consider the following commutative diagram of homomorphisms between abelian groups:

$$\begin{array}{ccccccccccccc} \ldots & \longrightarrow & C''_{n+1} & \xrightarrow{\partial_{n+1}} & C'_n & \xrightarrow{i_n} & C_n & \xrightarrow{p_n} & C''_n & \xrightarrow{\partial_n} & C'_{n-1} & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle f''_{n+1}} & & \downarrow{\scriptstyle f'_n} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f''_n} & & \downarrow{\scriptstyle f'_{n-1}} & & \\ \ldots & \longrightarrow & D''_{n+1} & \xrightarrow{\partial_{n+1}} & D'_n & \xrightarrow{j_n} & D_n & \xrightarrow{q_n} & D''_n & \xrightarrow{\partial_n} & D'_{n-1} & \longrightarrow & \ldots \end{array}$$

We suppose that the horizontal sequences are exact and we suppose that each f''_n is an isomorphism. For each $n \in \mathbb{N}_0$ we set $\Delta_n := \partial_n \circ (f''_n)^{-1} \circ q_n\colon D_n \to C'_{n-1}$. Show that the following sequence is exact

$$\ldots \longrightarrow C'_n \xrightarrow{i_n + f'_n} C_n \oplus D'_n \xrightarrow{f_n + j_n} D_n \xrightarrow{\Delta_n} C'_{n-1} \longrightarrow \ldots$$

Remark. The resulting exact sequence is sometimes called the *algebraic Mayer–Vietoris sequence*. The name will become clear in Exercise 78.2.

Exercise 78.2. Let X be a topological space and let $A, B \subset X$ be subsets such that $X = \mathring{A} \cup \mathring{B}$. Use the algebraic Mayer–Vietoris sequence from Exercise 78.1 together with the Excision Theorem 74.17 and the long exact sequence of the pair of topological spaces to give a new proof of of the Mayer–Vietoris Theorem 78.5 92) and (3). More precisely you need to show that there exists a long exact sequence of the form

$$\ldots \to \mathrm{H}_n(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \xrightarrow{i_A + i_B} \mathrm{H}_n(X) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A \cap B) \to \ldots$$

where ∂_n is natural.
Hint. Consider the commutative diagram

$$\begin{array}{ccc} A \cap B & \xrightarrow{i_{A\cap B}} & A \\ {\scriptstyle i_{A\cap B}}\downarrow & & \downarrow{\scriptstyle i_A} \\ B & \xrightarrow{i_B} & X. \end{array}$$

Exercise 78.3.

(a) We denote by

$$K \;=\; ([0,1] \times [0,1])/\!\sim \;\text{ with } (x,0) \sim (x,1) \text{ and } (0,y) \sim (1,1-y)$$

the Klein bottle as defined on page 265. We consider the two singular 1-cycles

$$\begin{array}{rcl} c\colon \Delta^1 & \to & K \\ (t, 1-t) & \mapsto & [(\frac{1}{2}, t)] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} d\colon \Delta^1 & \to & K \\ (t, 1-t) & \mapsto & [(\frac{1}{2}, t)] \end{array}$$

Show that

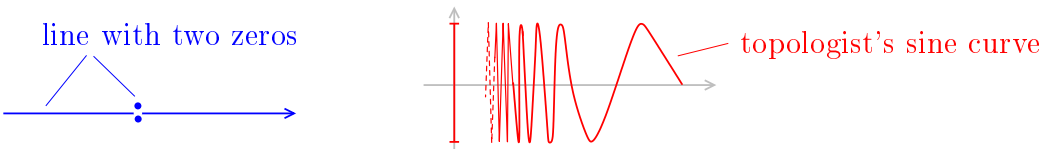
$$\mathrm{H}_n(K) \;=\; \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z} \cdot [c] \oplus \mathbb{Z}_2 \cdot [d], & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Give an example of an explicit map $f\colon S^1 \to K$ such that the inclusion induced map $f_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(K)$ is a monomorphism. Prove that the map f that you provide does indeed have this property.

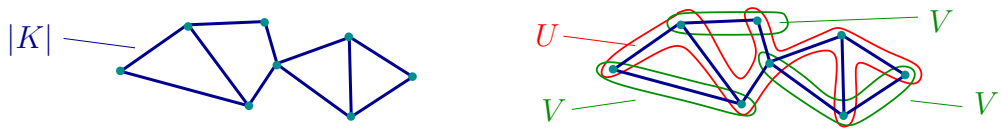
Exercise 78.4. Let X be a topological space and let U and V be two open subset of X such that X and $U \cap V$ are path-connected and such that $U \cap V$ is non-empty. Show that U and V are path-connected.
Remark. This is of course just a special case of Exercise 6.11 (a). But it is great fun to prove this statement using the Mayer–Vietoris Theorem 78.5.

Exercise 78.5. Compute the isomorphism types of the homology groups of the real line with two zeros.

Exercise 78.6. Determine the isomorphism types of the homology groups of the Topologist's Sine Curve that we introduced on page 164.
Hint. Use Exercise 2.48.



Exercise 78.7. Let $K = (V, E, i, t)$ be a connected abstract graph. Determine the isomorphism types of the homology groups of the topological realization $|K|$.
Hint. The figure below shows an example of a convenient covering by open sets.



Exercise 78.8. In Lemma 78.11 we computed the homology groups of the torus $T = S^1 \times S^1$ the Mayer–Vietoris Theorem 78.9 for Manifolds. compute the *reduced* homology of the torus using the Mayer–Vietoris sequence with reduced homology groups.
Remark. Depending on how you go about this calculation you can easily end up with a contradictory calculation.

Exercise 78.9. Let $T = S^1 \times S^1$ be the torus. We denote by $p, q\colon S^1 \times S^1 \to S^1$ the two obvious projections. Show that the map

$$\begin{array}{rcl} p_* \oplus q_*\colon \mathrm{H}_1(S^1 \times S^1) & \to & \mathrm{H}_1(S^1) \oplus \mathrm{H}_1(S^1) \\ \varphi & \mapsto & p_*(\varphi) + q_*(\varphi) \end{array}$$

is an isomorphism.
Hint. You could make use of Lemma 78.11 (2).

Exercise 78.10. We consider the real projective plane

$$\mathbb{R}\mathrm{P}^2 \;=\; \overline{B}^2/\!\sim \qquad \text{where } x \sim y \text{ if } x, y \in \partial\overline{B}^2 \text{ and } x = -y.$$

Determine an explicit cycle in $\mathrm{C}_1(\mathbb{R}\mathrm{P}^2)$ that represents the unique non-trivial element of $\mathrm{H}_1(\mathbb{R}\mathrm{P}^2)$.

Exercise 78.11. Let X be a topological space and let $k \in \mathbb{Z}$. Show that the two natural isomorphisms $\widetilde{\mathrm{H}}_k(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_{k-1}(\Sigma(X))$ that we constructed in the Suspension-H_*-Proposition 76.7 and in the proof of the Suspension-H_*-Proposition 76.7 are actually the same.

Exercise 78.12. Let $X \subset \mathbb{R}^n$ be a non-empty compact subset. As in Lemma 2.18 we make the identification $S^n = \mathbb{R}^n \cup \{\infty\}$ of the sphere with the space $\mathbb{R}^n$ together with a "point at infinity".

(a) Show that for every $k \neq n-1$ the inclusion induced map

$$\mathrm{H}_k(\mathbb{R}^n \setminus X) \;\to\; \mathrm{H}_k(\underbrace{(\mathbb{R}^n \cup \{\infty\})}_{=S^n} \setminus X)$$

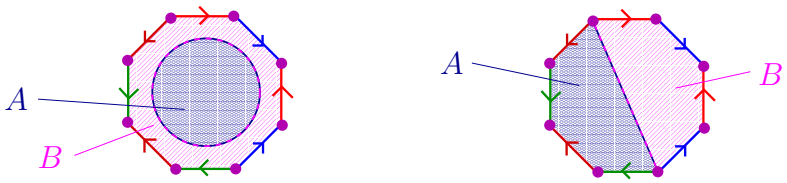
is an isomorphism.

(b) Show that there exists a short exact sequence

$$0 \;\to\; \mathbb{Z} \;\to\; \mathrm{H}_{n-1}(\mathbb{R}^n \setminus X) \;\to\; \mathrm{H}_{n-1}((\mathbb{R}^n \cup \{\infty\}) \setminus X) \;\to\; 0.$$

(c) Show that the short exact sequence in (b) splits.

Exercise 78.13. Let $g \in \mathbb{N}_{\geq 2}$. Use the Mayer–Vietoris Theorem 78.9 for Manifolds, or some other flavor of the Mayer–Vietoris sequence, to determine the isomorphism types of the surface of genus g.
Hint. In the figure below we show two possible approaches to decomposing the surface of genus two.
Remark. We will give the calculation in Proposition 80.10 using "cellular homology".



Exercise 78.14. Let X be a topological space, let $k \in \mathbb{N}$ and let $n \in \mathbb{N}_0$. What is the isomorphism type of $\mathrm{H}_n(X \times S^k)$ in terms of the homology groups of X?

Exercise 78.15. Let $k \in \mathbb{N}$. We consider the topological space

$$X_k \;:=\; ([-k,k] \cup S^2 \times \{-k,\dots,k\})/\!\sim$$

where given $i \in \{-k,\dots,k\}$ we identify $i \in [-k,k]$ with $((0,0,-1),i) \in S^2 \times \{-k,\dots,k\}$. For $i = -k,\dots,k$ we denote by

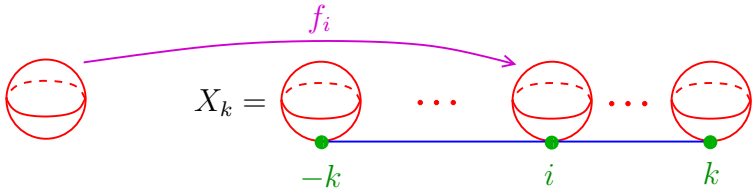
$$\begin{array}{rcl} f_i\colon S^2 & \to & X_k \\ P & \mapsto & (P,i) \end{array}$$

the various "obvious" inclusion maps. (We refer to the figure below for an illustration.)

(a) Show that $\mathrm{H}_0(X_k) \cong \mathbb{Z}$.
(b) Show that $\mathrm{H}_i(X_k) = 0$ for $i \neq 0, 2$.
(c) Let $[S^2] \in \mathrm{H}_2(S^2)$ be the standard generator. Show that the map

$$\begin{array}{rcl} \mathbb{Z}^{2k+1} & \to & 2(X_k) \\ (a_{-k},\dots,a_k) & \to & \sum\limits_{i=-k}^{k} a_i \cdot f_{i*}([S^2]) \end{array}$$

is an isomorphism.



Exercise 78.16. Let X be a topological space. We consider the corresponding suspension $\Sigma(X) = (X \times [-1,1])/\!\sim$ which is given by identifying all points in $X \times \{-1\}$ and by identifying all points in $X \times \{1\}$.

(a) Given a singular k-simplex $\sigma\colon \Delta^k \to X$ we define the two maps

$$\begin{array}{rcl} c_\pm(\sigma)\colon \Delta^{k+1} & \to & \Sigma(X) \\ (t_0,t_1,\dots,t_{k+1}) & \mapsto & \begin{cases} \big([\big(\sigma\big(\frac{1}{1-t_0}(t_1,\dots,t_{k+1})\big),\pm t_0\big)\big], & \text{if } t_0 < 1, \\ [X \times \{\pm 1\}], & \text{if } t_0 = 1. \end{cases} \end{array}$$

Show that the maps $c_-(\sigma)$ and $c_+(\sigma)$ are continuous. (We refer to the figure below for an illustration of the definition of $c_\pm(\sigma)$.)
Hint. Consider $\Delta^k \times [0,1]$ and $(\Delta^k \times [0,1])/(\Delta^k \times \{1\})$.

(b) Note that the maps $\sigma \mapsto c_\pm(\sigma)$ define a homomorphisms $c_\pm \colon \mathrm{C}_k(X) \to \mathrm{C}_{k+1}(\Sigma(X))$. We set $s = (-1)^{k+1} \cdot (c_+ - c_-)$. Show that for any $\sigma \colon \Delta^k \to X$ we have

$$\partial s(\sigma) \;=\; \begin{cases} s(\partial\sigma), & \text{if } k > 0, \\ [X \times \{1\}] - [X \times \{-1\}], & \text{if } k = 0. \end{cases}$$

(c) Let $k \in \mathbb{N}_0$. Show that the map

$$\begin{array}{rcl} \widetilde{\mathrm{H}}_k(X) & \to & \mathrm{H}_{k+1}(\Sigma(X)) \\ \sigma & \mapsto & s(\sigma) \end{array}$$

is well-defined and that it is an isomorphism.

Remark. We refer to c_+, c_- as *cone operators* and we refer to s as the *suspension operator*.
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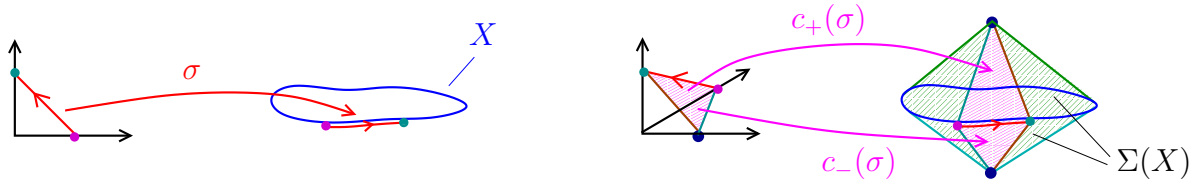


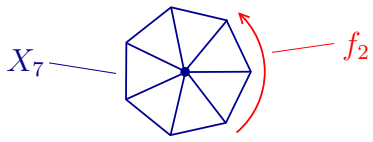
Exercise 78.17. Let $k \in \mathbb{N}_{\geq 3}$ and let $m \in \mathbb{N}$. We consider the topological space

$$X_k \;:=\; S^1 \cup \{r \cdot \exp(2\pi \mathrm{i} j/k) \,|\, r \in [0,1] \text{ and } j \in \{0, \ldots, k-1\}\}$$

and we consider the homeomorphism

$$\begin{array}{rcl} f_m \colon X_k & \to & X_k \\ z & \mapsto & \exp(2\pi \mathrm{i} m/k) \cdot z. \end{array}$$

Determine the homology groups of the mapping torus $\mathrm{Tor}(X_k, f_m)$.



Exercise 78.18. Let X be a topological space and let $A \subset X$ be a non-empty subset. We denote by $i \colon A \to X$ the inclusion map. We consider the following diagram

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i} & \widetilde{\mathrm{H}}_n(X) & \longrightarrow & \mathrm{H}_n(X,A) & \xrightarrow{\partial} & \widetilde{\mathrm{H}}_{n-1}(A) & \longrightarrow & \widetilde{\mathrm{H}}_{n-1}(X) \longrightarrow \ldots \\ & & \downarrow{=} & & \downarrow{=} & & \downarrow{\cong} & & \downarrow{=} & & \downarrow{=} \\ \ldots & \longrightarrow & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i} & \widetilde{\mathrm{H}}_n(X) & \longrightarrow & \widetilde{\mathrm{H}}_n(\mathrm{Cone}(i)) & \xrightarrow{\partial} & \widetilde{\mathrm{H}}_{n-1}(A) & \longrightarrow & \widetilde{\mathrm{H}}_{n-1}(X) \longrightarrow \ldots \end{array}$$

where the middle vertical map is the natural isomorphism from the Mapping Cone-H_*-Lemma 78.15, where the top sequence is the long exact sequence arising from the Topological LES-Corollary 74.14. and where the bottom sequence is the exact sequence from the Mapping Cone-H_*-Lemma 78.17. Show that the diagram commutes.

Exercise 78.19. Let X be a topological space and let U_1, U_2 and U_3 be open subsets of X with $X = U_1 \cup U_2 \cup U_3$.

(a) We suppose that $U_1 \cap U_2 \cap U_3 = \varnothing$. Show that there exists a long exact sequence of the form

$$\dots \to \begin{matrix} \mathrm{H}_n(U_1 \cap U_2) \\ \mathrm{H}_n(U_2 \cap U_3) \\ \mathrm{H}_n(U_1 \cap U_3) \end{matrix} \xrightarrow{\begin{pmatrix} i_{U_1\cap U_2} & 0 & i_{U_1\cap U_3} \\ -i_{U_1\cap U_2} & i_{U_2\cap U_3} & 0 \\ 0 & -i_{U_2\cap U_3} & -i_{U_1\cap U_3} \end{pmatrix}} \begin{matrix} \mathrm{H}_n(U_1) \\ \mathrm{H}_n(U_2) \\ \mathrm{H}_n(U_3) \end{matrix} \xrightarrow{i_{U_1}+i_{U_2}+i_{U_3}} \mathrm{H}_n(X) \xrightarrow{\partial_n} \dots$$

(b) Show that in general the conclusion of (a) does not hold if $U_1 \cap U_2 \cap U_3 \neq \varnothing$.

Remark. This exercise is related to the discussion on page 1720.

Exercise 78.20. Let X and Y be topological spaces. We consider the join $X * Y$ as defined on page 446. Show that for each $k \in \mathbb{N}_0$ there exists a short exact sequence of the following form:

$$0 \;\to\; \widetilde{\mathrm{H}}_k(X * Y) \;\to\; \widetilde{\mathrm{H}}_{k-1}(X) \oplus \widetilde{\mathrm{H}}_{k-1}(Y) \;\to\; \widetilde{\mathrm{H}}_{k-1}(X \times Y) \;\to\; 0.$$

Hint. In Exercise 16.15 we gave an explicit homeomorphism

$$X * Y \;\cong\; (\mathrm{Cone}(X) \times Y) \cup_{(X\times\{1\})\times Y = X\times(Y\times\{1\})} (X \times \mathrm{Cone}(Y)).$$

Note though that you need to argue why you can use this decomposition.
Remark. In the Künneth Theorem 90.8 we will learn how to calculate the homology groups of the product $X \times Y$.

Exercise 78.21. Let $X = \mathbb{Z}_n$ be the topological space whose underlying set is $\mathbb{Z}_n$ and whose topology is given by the discrete topology. Let $f\colon X = \mathbb{Z}_n \to X = \mathbb{Z}_n$ be the homeomorphism given by $f(x) = x + 1 \in \mathbb{Z}_n$. Compute the homology groups of the mapping torus $\mathrm{Tor}(X, f)$.

Exercise 78.22. Let $p, q \in \mathbb{N}$ be coprime. We consider the torus knot $T_{p,q}$ as defined on page 1312. Compute the homology groups of $S^3 \setminus T_{p,q}$.
Hint. Use the decomposition of $S^3 \setminus T_{p,q}$ given in the proof of Proposition 59.5.

Exercise 78.23. Let K be the Klein bottle minus one open disk. Show that ∂K is not a retract of K.

Exercise 78.24. Let $n \in \mathbb{N}$ and let M be S^2 minus n open disks. Show that the inclusion induced map $\mathrm{H}_1(\partial M) \to \mathrm{H}_1(M)$ is an epimorphism.

Exercise 78.25. We consider the two topological manifolds X and Y that are shown in the figure below. Use (relative?) homology groups to show that X and Y are not homeomorphic.
Remark. In Exercise 52.6 we attacked the same problem using fundamental groups. It should be much easier to deal with the problem at hand using (relative) homology groups.



Exercise 78.26. Given an n-dimensional smooth manifold M we define the

$$\text{embedding covering number } N(M) := \begin{array}{c}\text{minimal cardinality of an open cover } \{U_i\}_{i\in I} \\ \text{of } M \text{ such that each } U_i \text{ admits a} \\ \text{smooth embedding into } \mathbb{R}^n.\end{array}$$

In other words, the embedding covering number is precisely the minimal cardinality of a smooth atlas of M. Let M be a non-orientable smooth manifold. Show that if $N(M) = 2$, then $\mathrm{H}_1(M)$ is infinite.

Hint. Assume that there exists a smooth atlas for M consisting of two charts $\Phi_1 \colon U_1 \to V_1$ and $\Phi_2 \colon U_2 \to V_2$. Consider the Mayer–Vietoris sequence corresponding to the open cover $M = U_1 \cup U_2$ and use the Define Orientations-via Transition Maps Lemma 25.15. It helps to first consider the case that U_1 and U_2 are connected.

Remark. This exercise, together with the calculation of $\mathrm{H}_1(\mathbb{R}\mathrm{P}^2)$ in Lemma 78.8, shows that the embedding covering number of the real projective plane $\mathbb{R}\mathrm{P}^2$ equals three.

79. Homology groups and direct limits

In this chapter we will discuss the relationship between homology groups and direct limits. This will allow us to determine the homology groups of suitable nested sequences of topological spaces. In a similar vein we will determine the homology groups of the wedge of arbitrarily many topological spaces. We concludes this chapter with the introduction of the Moore spaces. We will in particular see that all abelian group can appear as homology groups of topological spaces.
For the reader who is in a hurry only few of the results of this chapter are really important. In particular, very soon we will make use of the Wedge-H_*-Proposition 79.9. For this proposition we give two proofs, one building on the theme of this chapter. The other proof can be read independently of the rest of the chapter.

79.1. Direct limits and induced maps.

Notation. Given a directed set $(I, \leq)$ and a direct system $(\{K_i\}_{i\in I}, \{\chi_{ij}\}_{i\leq j})$ of abelian groups we write, as on page 1262

$$\varinjlim K_i \ := \ \Big(\bigsqcup_{i\in I} K_i\Big)/\sim \quad \text{where } k_i \sim \chi_{ij}(k_i) \text{ for all } i \leq j,\ k_i \in K_i$$

We adopt the following notation, given $k_i \in K_i$ we denote by $[k_i] \in \varinjlim K_i$ the element represented by k_i.

We make the following observation that follows immediately from the above definition of the direct limit.

Observation 79.1. Any element in $\varinjlim K_i$ is of the form $[k_i]$ for some $k_i \in K_i$.

Definition. Let $(I, \leq)$ be a directed set and let $(\{K_i\}_{i\in I}, \{\chi_{ij}\}_{i\leq j})$ and $(\{L_i\}_{i\in I}, \{\lambda_{ij}\}_{i\leq j})$ be direct systems of abelian groups.

(1) We define a homomorphism between the above two direct systems to be a family $\{f_i \colon K_i \to L_i\}_{i\in I}$ of homomorphisms that has the property that for any $i \leq j$ the following diagram commutes:

$$\begin{array}{ccc} K_i & \xrightarrow{f_i} & L_i \\ {\scriptstyle \chi_{ij}}\downarrow & & \downarrow{\scriptstyle \lambda_{ij}} \\ K_j & \xrightarrow{f_j} & L_j. \end{array}$$

(2) Given such a homomorphism we refer to

$$\begin{array}{ccc} \varinjlim K_i & \to & \varinjlim L_i \\ [k_i] & \mapsto & [f_i(k_i)] \end{array}$$

as the induced map. Sometimes we denote it by $\varinjlim f_i$.[963964] It is straightforward to verify that the induced maps are functorial.

Proposition 79.2. (Direct Limit-Exactness Proposition) Let $(I, \leq)$ be a directed set. Also let $(\{K_i\}_{i\in I}, \{k_{ij}\}_{i\leq j})$, $(\{L_i\}_{i\in I}, \{l_{ij}\}_{i\leq j})$ and $(\{M_i\}_{i\in I}, \{m_{ij}\}_{i\leq j})$ be direct systems of abelian groups. Furthermore suppose that we are given homomorphisms $f_i\colon K_i \to L_i$ and $g_i\colon L_i \to M_i$ of the direct systems. If for all $i \in I$ the sequence

$$K_i \xrightarrow{f_i} L_i \xrightarrow{g_i} M_i$$

is exact, then the induced sequence

$$\varinjlim K_i \xrightarrow{\varinjlim f_i} \varinjlim L_i \xrightarrow{\varinjlim g_i} \varinjlim M_i$$

is also exact.

Proof. We write $f = \varinjlim f_i$ and $g = \varinjlim g_i$. We have to prove that $g \circ f = 0$ and we need to show that $\ker(g) \subset \mathrm{im}(f)$. Both statements statements follow our hypothesis together with Observation 79.1 and the fact that in the definition of the induced map between direct limits we can take any representative. We leave the few remaining details to the reader. ■

The following corollary says that direct limits commute with kernels and cokernels.

Corollary 79.3. (Direct Limit-(Co-)-Kernel Corollary) Let $(I, \leq)$ be a directed set and let $(\{K_i\}_{i\in I}, \{k_{ij}\}_{i\leq j})$ and $(\{L_i\}_{i\in I}, \{l_{ij}\}_{i\leq j})$ be direct systems of abelian groups. Furthermore suppose that we are given a homomorphism $f_i\colon K_i \to L_i$ of the direct systems. There exist canonical isomorphisms

$$\varinjlim \ker(f_i\colon K_i \to L_i) \xrightarrow{\cong} \ker\left(\varinjlim K_i \to \varinjlim L_i\right)$$

and

$$\varinjlim \mathrm{coker}(f_i\colon K_i \to L_i) \xrightarrow{\cong} \mathrm{coker}\left(\varinjlim K_i \to \varinjlim L_i\right).$$

Proof. For each $i \in I$ the sequence

$$0 \to \ker(f_i) \to K_i \xrightarrow{f_i} L_i \to \mathrm{coker}(f_i) \to 0.$$

is exact. Evidently these homomorphisms form a homomorphism between the various direct systems. It follows from the Direct Limit-Exactness Proposition 79.2 (applied four times!) that the sequence

$$0 \to \varinjlim \ker(f_i) \to \varinjlim K_i \to \varinjlim L_i \to \varinjlim \mathrm{coker}(f_i) \to 0$$

is also exact. But this observation already implies the desired statements. ■

79.2. **The homology groups of a nested sequence of topological spaces.** In Propositions 56.7 and 71.11 we saw that in favorable situations the homotopy groups of a union of a nested sequence of topological spaces is the direct limit of the corresponding fundamental groups. Now we will see that a similar statement holds for homology groups.

[963] It follows immediately from the definitions that this map is well-defined, i.e. the map does not depend on the choice $[k_i]$ of a representative for an element in the direct limit.

[964] More formally one could say that the maps $K_i \to L_i \to \varinjlim L_i$ induce, by the universal property of a direct limit, a map $\varinjlim K_i \to \varinjlim L_i$. It is straightforward to see that this map is exactly the one we wrote down.

Proposition 79.4. (Exhaustions-H$_*$-Proposition) Let X be a topological space and let $X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subsets such that $X = \bigcup_{i\in\mathbb{N}} X_i$ and such that one the following three conditions holds:

(1) every compact subset of X is already contained in one of the X_i, or
(2) each X_i is open in X, or
(3) X is a CW-complex and each X_i is a subcomplex.

Then for any $n \in \mathbb{N}_0$ the inclusion induced maps $\mathrm{H}_n(X_i) \to \mathrm{H}_n(X)$ induce an isomorphism

$$\varinjlim \mathrm{H}_n(X_i) \xrightarrow{\cong} \mathrm{H}_n(X).$$

We remark that it follows easily from the Finiteness the CW-Complex-Finiteness Theorem 68.14 that hypothesis (3) implies hypothesis (1). Furthermore we point out that it follows from Lemma 56.8 that hypothesis (2) implies hypothesis (1). Thus it suffices to prove the Exhaustions-H$_*$-Proposition 79.4 under the hypothesis (1). But in that case the proposition is an immediate consequence of the next two lemmas.

Lemma 79.5. (Exhaustion-Chain Complex Lemma) Let X be a topological space and let $X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subsets such that every compact subset of X is already contained in one of the X_i. Then the chain complexes $\mathrm{C}_*(X_i), i \in \mathbb{N}$ together with the inclusion induced chain maps $\mathrm{C}_*(X_i) \to \mathrm{C}_*(X_j)$ for $i \leq j$ form a direct system of chain complexes.[965] Furthermore the inclusion induced maps $\mathrm{C}_*(X_i) \to \mathrm{C}_*(X)$ induce an isomorphism[966]

$$\varinjlim \mathrm{C}_*(X_i) \xrightarrow{\cong} \mathrm{C}_*(X)$$

of chain complexes.

Proof. Given a topological space Y and $n \in \mathbb{N}_0$ we denote by $\mathrm{S}_n(Y)$ the set of singular n-simplices in Y. It follows immediately from our hypothesis on the X_i, the fact that Δ^n is compact and the Compact Image Lemma 2.13 that

$$\mathrm{S}_n(X) = \bigcup_{i\in\mathbb{N}} \mathrm{S}_n(X_i).$$

As in the example on page 1258 we now see that

$$\varinjlim \mathrm{C}_n(X_i) = \varinjlim \mathbb{Z}^{(\mathrm{S}_n(X_i))} = \mathbb{Z}^{(\mathrm{S}_n(X))} = \mathrm{C}_n(X).$$

We leave it to the reader to verify that the inclusion maps also induce an isomorphism of chain complexes. ■

The second lemma required for the proof of the Exhaustions-H$_*$-Proposition 79.4 is purely algebraic.

Lemma 79.6. (Direct Limit-Homology Lemma) Let $(I, \leq)$ be a directed set and furthermore let $(\{\mathrm{C}_i\}_{i\in I}, \{f_{ij}\}_{i\leq j})$ be a direct system of chain complexes. Then for each $n \in \mathbb{N}_0$ we obtain an induced direct system $(\{\mathrm{H}_n(\mathrm{C}_i)\}_{i\in I}, \{f_{ij*}\}_{i\leq j})$ of abelian groups.

[965] We refer to page 1257 for the definition of a direct system.

[966] The argument of the Direct Limit Existence Proposition 56.1 or respectively of page 1262 shows that the direct limit of a direct system of chain complexes always exists.

Furthermore, given any $n \in \mathbb{N}_0$ the maps $\mathrm{H}_n(\mathrm{C}_i) \to \mathrm{H}_n\big(\varinjlim \mathrm{C}_i\big)$ induce an isomorphism

$$\varinjlim \mathrm{H}_n(\mathrm{C}_i) \ \xrightarrow{\cong}\ \mathrm{H}_n\big(\varinjlim \mathrm{C}_i\big).$$

Proof. We have

$$\begin{aligned} \varinjlim \mathrm{H}_n(\mathrm{C}_i) &= \varinjlim \overbrace{\operatorname{coker}(\mathrm{C}_{i+1} \to \ker(\mathrm{C}_i \to \mathrm{C}_{i-1}))}^{=\mathrm{H}_n(\mathrm{C}_i)} \\ &= \operatorname{coker}\big(\varinjlim \mathrm{C}_{i+1} \to \ker\big(\varinjlim \mathrm{C}_i \to \varinjlim \mathrm{C}_{i-1}\big)\big) \ = \ \mathrm{H}_n\big(\varinjlim \mathrm{C}_i\big). \\ &\uparrow \\ &\text{Direct Limit-(Co-)-Kernel Corollary 79.3 applied twice} \end{aligned}$$

The fact that this isomorphism is induced by the maps $\mathrm{H}_n(\mathrm{C}_i) \to \mathrm{H}_n(\varinjlim \mathrm{C}_i)$ follows easily from the various definitions. ∎

We conclude this short section with a few examples:

Example. For any $n \in \mathbb{N}_0$ we have

$$\begin{aligned} \mathrm{H}_n(S^\infty) &= \varinjlim_{k\in\mathbb{N}} \mathrm{H}_n(S^k) &&= \varinjlim_{k\in\mathbb{N}} \begin{cases} \mathbb{Z}, & \text{if } n = k, 0, \\ 0, & \text{else} \end{cases} &&= \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{else.} \end{cases} \\ &\uparrow &&\uparrow &&\uparrow \\ &\text{by the Homology-via-Exhaustions Proposition 79.4, where we view } S^\infty \text{ as a CW-complex as on page 1479} &&\text{Proposition 74.4} &&\text{Direct Limit Stabilization Lemma 56.2} \end{aligned}$$

The next lemma is basically also just an example. But since we will need it later on we formulate it as a lemma.

Lemma 79.7. We consider the real line with infinitely many 2-dimensional spheres attached, i.e. we consider

$$X \ := \ (\mathbb{R} \cup (S^2 \times \mathbb{Z}))/\!\sim \qquad \text{where } n \sim ((0,0,-1),n) \text{ for } n \in \mathbb{Z}.$$

Then[967]

$$\mathrm{H}_n(X) \ \cong \ \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z}^\infty, & \text{if } n = 2, \\ 0, & \text{else.} \end{cases}$$

In fact we can specify an isomorphism on $\mathrm{H}_2(X)$. Given $i \in \mathbb{Z}$ we denote by $f_i \colon S^2 \to X$ the "obvious" i-th inclusion map. Then an isomorphism is given by

$$\begin{aligned} \mathbb{Z}^\infty \cong \bigoplus_{i\in\mathbb{Z}} \mathbb{Z} \ &\to \ \mathrm{H}_2(X) \\ \bigoplus_{i\in\mathbb{Z}} a_i \ &\mapsto \ \sum_{i\in\mathbb{Z}} a_i \cdot f_i([S^2]). \end{aligned}$$

Proof. We view X as a CW-complex in the obvious way. Furthermore, given $k \in \mathbb{N}$ we consider the subcomplex

$$X_k \ := \ ([-k,k] \cup S^2 \times \{-k,\dots,k\})/\!\sim .$$

[967] Recall that

$$\mathbb{Z}^\infty \ := \ \mathbb{Z}^{(\mathbb{N})} \ = \ \{(x_1,x_2,\dots) \,|\, x_i \in \mathbb{Z} \text{ but only finitely many } x_i\text{'s are non-zero}\}.$$

Note that X_k is the interval $[-k, k]$ with $2k+1$ spheres attached. (We refer to the figure below for an illustration.) We see that

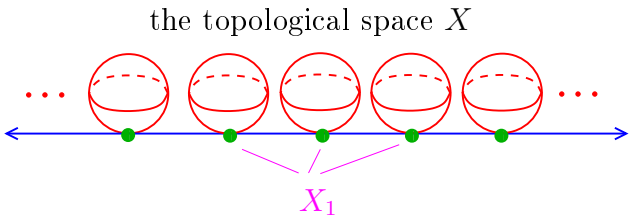
$$\mathrm{H}_n(X) \;=\; \varinjlim \mathrm{H}_n(X_k) \;=\; \varinjlim \begin{cases} \mathbb{Z}, & \text{if } n=0, \\ \mathbb{Z}^{2k+1}, & \text{if } n=2, \\ 0, & \text{else} \end{cases} \;=\; \begin{cases} \mathbb{Z}, & \text{if } n=0, \\ \mathbb{Z}^{\infty}, & \text{if } n=2, \\ 0, & \text{else.} \end{cases}$$

Homology-via-Exhaustions Proposition 79.4

see Exercise 78.15

see page 1258 and the Direct Limit Stabilization Lemma 56.2

The statement about the precise isomorphism of $\mathrm{H}_2(X)$ also follows from the explicit description of the isomorphism Exercise 78.15 and the discussion on page 1258. ■



79.3. **The homology groups of the wedge of topological spaces.** The following proposition says that under a mild hypothesis "homology groups are additive" under the operation of taking the wedge of two topological spaces. The proposition is very similar in vein to the Wedge-π_1-Proposition 52.3.

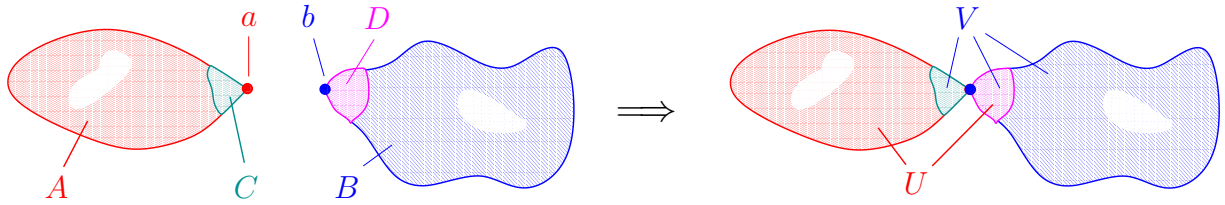
Proposition 79.8. (Wedge-H_*-Proposition) Let A and B be two topological spaces and let $a \in A$ and $b \in B$ be good points. We denote by $A \vee B$ the wedge defined using a and b. Then the natural inclusion maps $i\colon A \to A \vee B$ and $j\colon B \to A \vee B$ that we introduced on page 1134 induce for every $n \in \mathbb{N}_0$ an isomorphism

$$i_* \oplus j_*\colon \widetilde{\mathrm{H}}_n(A) \oplus \widetilde{\mathrm{H}}_n(B) \;\to\; \widetilde{\mathrm{H}}_n(A \vee B).$$

Furthermore, if we denote by $p\colon A \vee B \to A$ and $q\colon A \vee B \to A$ the natural projection s, then an inverse to the above isomorphism is given by

$$p_* + q_*\colon \widetilde{\mathrm{H}}_n(A \vee B) \;\to\; \widetilde{\mathrm{H}}_n(A) \oplus \widetilde{\mathrm{H}}_n(B).$$

Proof. We write $X = A \vee B$ and $x_0 = \{a, b\} \in A \vee B$. We pick an open neighborhood C in A of a such that $\{a\}$ is a deformation retract of C. Furthermore we pick an open neighborhood D of b in B such that b is a deformation retract of D. We consider $U := A \vee D$



and $V := B \vee C$. We consider the following diagram of maps

$$\begin{array}{ccccccccc} & & & & \widetilde{\mathrm{H}}_n(A) \oplus \widetilde{\mathrm{H}}_n(B) & & & & \\ & & & & \cong\downarrow \quad \downarrow\cong & & & & \\ \ldots & \longrightarrow & \widetilde{\mathrm{H}}_n(U \cap V) & \longrightarrow & \widetilde{\mathrm{H}}_n(U) \oplus \widetilde{\mathrm{H}}_n(V) & \longrightarrow & \widetilde{\mathrm{H}}_n(X) & \longrightarrow & \widetilde{\mathrm{H}}_{n-1}(U \cap V) \longrightarrow \ldots \end{array}$$

We make the following observations:

(1) The bottom sequence is exact by the Mayer–Vietoris Theorem 78.5.
(2) As in the proof of the Wedge-π_1-Proposition 52.3 we see that the intersection $U \cap V$ deformation retracts to x_0, in particular we obtain from the Homotopy Equivalence-H_*-Corollary 73.9 that we have the equality $\widetilde{\mathrm{H}}_n(U \cap V) = \widetilde{\mathrm{H}}_{n-1}(U \cap V) = 0$.
(3) By the proof of the Wedge-π_1-Proposition 52.3 we know that A is a deformation retract of U and that B is a deformation retract of V. Therefore the vertical maps are isomorphisms by the Homotopy Equivalence-H_*-Corollary 73.9.

The combination of these three observations shows that

$$i_* \oplus j_* \colon \widetilde{\mathrm{H}}_n(A) \oplus \widetilde{\mathrm{H}}_n(B) \;\to\; \widetilde{\mathrm{H}}_n(A \vee B)$$

is an isomorphism. It is clear that $(p_* + q_*) \circ (i_* + j_*)$ is the identity on $\widetilde{\mathrm{H}}_n(A) \oplus \widetilde{\mathrm{H}}_n(B)$, so we also get the second isomorphism. ■

The Wedge-H_*-Proposition 79.8 has nothing to do with the content of the present chapter, we could easily have proved it in Chapter 78. But odd as it may sound, soon we will be quite interested in the homology groups of the wedge $\bigvee\limits_{i \in I} X_i$ of an arbitrary family of topological spaces. If I is a finite set, i.e. if we take the wedge of finitely many topological spaces, then we can use the Wedge-H_*-Proposition 79.8 and an elementary induction argument to determine the homology groups.

It is a priori less clear how one can deal with the case that I is an infinite set. Fortunately the next proposition allows us to handle the general situation. This proposition is the generalization of the Wedge-H_*-Proposition 79.8 to the wedge of an arbitrary family of topological spaces.

Proposition 79.9. (Wedge-H_*-Proposition) Let $\{A_k\}_{k \in K}$ be a family of topological spaces. For each $k \in K$ suppose that we are given a good point $a_k \in A_k$.[968] We use these points to form the wedge $\bigvee\limits_{k \in K} A_k$. Given $j \in K$ we denote by

$$i_j \colon A_j \;\to\; \bigvee_{k \in K} A_k \quad \text{respectively} \quad p_j \colon \bigvee_{k \in K} A_k \;\to\; A_j$$

the natural inclusion map respectively the natural projection that we introduced on page 1134. Then for every $n \in \mathbb{N}_0$ the map

$$\bigoplus_{k \in K} p_{k*} \colon \widetilde{\mathrm{H}}_n\Big(\bigvee_{k \in K} A_k\Big) \;\to\; \bigoplus_{k \in K} \widetilde{\mathrm{H}}_n(A_k)$$

is an isomorphism where the inverse is given by

$$\bigoplus_{k \in K} i_{k*} \colon \bigoplus_{k \in K} \widetilde{\mathrm{H}}_n(A_k) \;\to\; \widetilde{\mathrm{H}}_n\Big(\bigvee_{k \in K} A_k\Big).$$

We like the Wedge-H_*-Proposition 79.9 so much that we provide two different proofs. The first proof is quick and slick, but depending on your taste the first proof might have feel slightly unsatisfactory. The second proof is somewhat longer, but it fits in better with the discussion of this chapter. In fact, as an encore we will suggest one more proof of the Wedge-H_*-Proposition 79.9 in Exercise 79.2.

First proof of the Wedge-H_*-Proposition 79.9. We have the following isomorphisms:

$$\widetilde{\mathrm{H}}_n\Big(\bigvee_{k\in K} A_k\Big) \underset{\text{definition of the wedge operation}}{=} \widetilde{\mathrm{H}}_n\Big(\bigsqcup_{k\in K} A_k \Big/ \bigsqcup_{k\in K}\{a_k\}\Big) \underset{\text{since each } a_k \in A_k \text{ is good we see that } \bigsqcup_{k\in K}\{a_k\} \text{ is a good subset of } \bigsqcup_{k\in K} A_k, \text{ thus we can apply the Homology-of-Quotients Proposition 74.20}}{\overset{\cong}{\leftarrow}} \mathrm{H}_n\Big(\bigsqcup_{k\in K} A_k, \bigsqcup_{k\in K}\{a_k\}\Big) \underset{\text{by the obvious generalization of Lemma 72.14}}{\overset{\cong}{\leftarrow}} \bigsqcup_{k\in K} \mathrm{H}_k(A_k, \{a_k\})$$

$$\overset{\cong}{\leftarrow} \bigsqcup_{k\in K} \widetilde{\mathrm{H}}_k(A_k).$$

Lemma 74.12

We leave it to the reader to verify that the isomorphism is indeed given by $\bigoplus_{k\in K} p_{k*}$ and that the inverse is the map $\bigoplus_{k\in K} i_{k*}$. ■

The second proof of the Wedge-H_*-Proposition 79.9 is almost embarrassingly similar to the proof of Proposition 52.7.

Second proof of the Wedge-H_*-Proposition 79.9. Let $\{A_k\}_{k\in K}$ be a family of topological spaces. For each $k \in K$ suppose that we are given a good point $a_k \in A_k$. We use these points to form the wedge $\bigvee_{k\in K} A_k$. Recall that the fact that a_k is good means that

(1) $\{a_k\}$ is a closed subset of A_k and
(2) that there exists an open neighborhood U_k of a_k in A_k such that a_k is a deformation retract of U_k.

We will show that the map

$$\bigoplus_{k\in K} i_k \colon \bigoplus_{k\in K} \widetilde{\mathrm{H}}_n(A_k) \;\to\; \widetilde{\mathrm{H}}_n\Big(\bigvee_{k\in K} A_k\Big)$$

is an isomorphism. Once we will have established this fact it is straightforward to show that the inverse is given by $\bigoplus_{k\in K} p_{k*}$.

The idea of the proof is to show that the map is a monomorphism and an epimorphism, in a way which is reminiscent to the approach taken in the proofs of the Exhaustions-H_*-Proposition 79.4.

Claim. The map

$$\bigoplus_{k\in K} i_k \colon \bigoplus_{k\in K} \widetilde{\mathrm{H}}_n(A_k) \;\to\; \widetilde{\mathrm{H}}_n\Big(\bigvee_{k\in K} A_k\Big)$$

is an epimorphism.

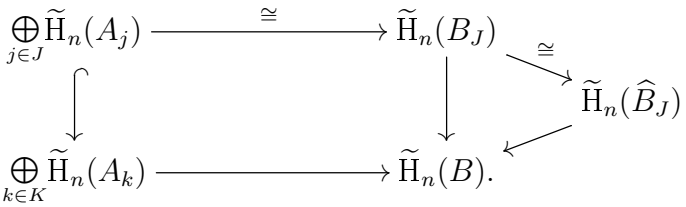
Proof. We introduce the following notation:

(1) We write $B := \bigvee_{k\in K} A_k$ and we set $U := \bigcup_{k\in K} U_k \subset B$.

[968] Recall that by the discussion on page 1171 any point on a topological manifold is good and that by the CW-Complex Properties Proposition 68.10 (5) any point on a CW-complex is good.

(2) Given a subset $J \subset K$ we write $B_J := \bigvee_{j\in J} A_j$ and we write $\widehat{B}_J := U \cup B_J$.

Given a finite subset $J \subset K$ we have the following commutative diagram



We make the following observations and comments:

(1) The maps are either the obvious algebraic inclusion maps or they are the obvious maps induced by inclusions of topological spaces. It follows almost immediately from the definitions that the diagram commutes.
(2) Since J is finite we obtain, as pointed out above, from the Wedge-H_*-Proposition 79.8 that the top horizontal map is an isomorphism.
(3) By Exercise 50.8 we know that B_J is a deformation retract of $\widehat{B}_J$. It follows from the Homotopy Equivalence-H_*-Corollary 73.9 (4) that the top diagonal map is an isomorphism.
(4) By Exercise 50.9 we know that given any compact subset $L \subset B$ there exists a finite subset $J \subset K$ with $L \subset \widehat{B}_J$.

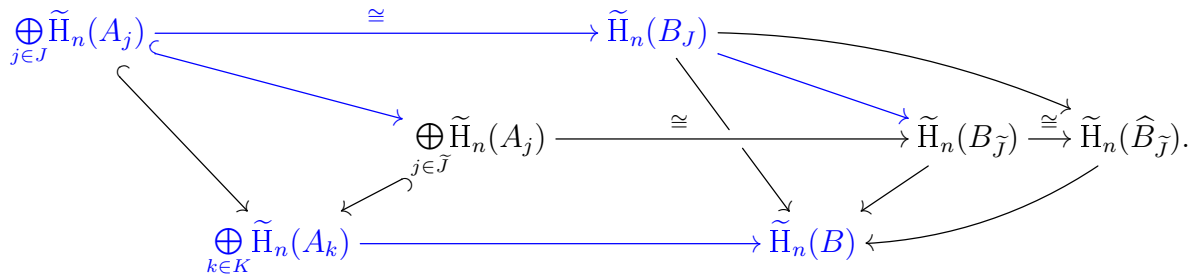
Now we show that the above map $\bigoplus_{k\in K} i_{k*}$ is an epimorphism. So suppose we are given a homology class $[\sigma] \in \widetilde{\mathrm{H}}_n(B)$. As in the proof of the Exhaustion-Chain Complex Lemma 79.5 we see that it follows from (4) there exists a finite subset $J \subset K$ such that $\sigma \in \mathrm{C}_n(\widehat{B}_J)$. But considering the above diagram, and using the fact that the top horizontal map is an isomorphism we see that $[\sigma]$ actually lies in the image of $\bigoplus_{k\in K} i_k$. ⊞

Claim. The map

$$\bigoplus_{k\in K} i_k \colon \bigoplus_{k\in K} \widetilde{\mathrm{H}}_n(A_k) \to \widetilde{\mathrm{H}}_n\Big(\bigvee_{k\in K} A_k\Big)$$

is a monomorphism.

Proof. We continue with the notation that we introduced in the proof of the previous claim. Given any two finite subsets $J \subset \widetilde{J} \subset K$ we have the following commutative diagram



The diagram together with the argument of Exercise 52.7 shows that the map $\bigoplus_{k\in K} i_{k*}$ is also a monomorphism. ■

79.4. **Moore spaces.** Now that we have played with homology groups the following question arises: which sequences of abelian groups can appear as the sequence of homology groups $\mathrm{H}_*(X)$ of a non-empty topological space? It turns out that anything goes, there are basically no restrictions, except that we know by Corollary 72.15 that $\mathrm{H}_0(X)$ is necessarily a non-trivial free abelian group.
In other words, we have the following proposition.

Proposition 79.10. If $\{A_n\}_{n\in\mathbb{N}_0}$ is a sequence of abelian groups. If A_0 is a non-trivial free abelian group, then there exists a CW-complex Z such that for any $n \in \mathbb{N}_0$ we have $\mathrm{H}_n(Z) \cong A_n$.

This proposition is hardly the most exciting result of these notes. But the following definition, which we will need in the proof of Proposition 79.10, will in fact turn out to be convenient.

Definition. Let $n \in \mathbb{N}$ and let π be an abelian group. We say that a non-empty topological space X is a Moore space of type $\mathrm{M}(\pi, n)$ if the following conditions are satisfied:

(1) X admits a CW-structure,
(2) for any $i \neq n$ we have $\widetilde{\mathrm{H}}_i(X) = 0$,
(3) $\mathrm{H}_n(X) \cong \pi$.
(4) If $n \geq 2$, then we also demand that X is simply connected.

Example. Let $n \in \mathbb{N}$. It follows from the Sphere-π_1-Proposition 47.13 and Proposition 74.4 that S^n is a Moore space of type $(\mathbb{Z}, n)$.

Proposition 79.11. (Moore Space Existence Proposition) Given any $n \in \mathbb{N}$ and given any abelian group π there exists an $(n+1)$-dimensional CW-complex that has no cells in dimensions $1, \ldots, n-1$ and that is a Moore space of type $\mathrm{M}(\pi, n)$.

Proof of Proposition 79.10 assuming the Moore Space Existence Proposition 79.11. Let $\{A_n\}_{n\in\mathbb{N}_0}$ be a sequence of abelian groups such that A_0 is a non-trivial free abelian group. In other words, we have $A_0 \cong \mathbb{Z}^{(S)}$ for some non-empty set S.

By the Moore Space Existence Proposition 79.11 we can find for each $n \in \mathbb{N}$ a CW-complex X_n that is a Moore space of type $\mathrm{M}(\pi, n)$. We consider the wedge $Y := \bigvee_{n\in\mathbb{N}} X_n$. By the CW-Complex Construction Lemma 68.32 (4) we can equip Y with a CW-structure. It follows from the Wedge-H_*-Proposition 79.9 that for every $n \in \mathbb{N}$ we have $\mathrm{H}_n(Y) \cong \mathrm{H}_n(X_n) \cong A_n$ and that $\mathrm{H}_0(Y) \cong \mathbb{Z}$. Since S is non-empty we can pick a point $* \in S$. We define $Z := Y \sqcup (S \setminus \{*\})$ where we equip $S \setminus \{*\}$ with the discrete topology. It follows from the CW-Complex Construction Lemma 68.32 that Z is a CW-complex. Finally it follows from the H_0-Proposition 72.5 and Lemma 72.13 that Z has the promised homology groups. ∎

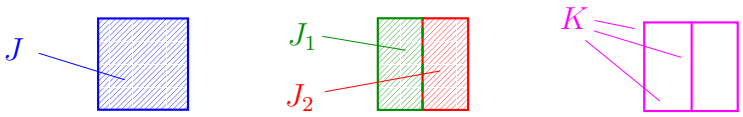
In the proof of the Moore Space Existence Proposition 79.11 we will also need the following little calculation which will be of independent use.

Lemma 79.12. Let (X, x_0) be a pointed topological space, let $n \in \mathbb{N}$, let $\omega \in \mathrm{H}_n(I^n, \partial I^n)$ and let $f, g\colon (I^n, \partial I^n) \to (X, x_0)$ be two maps. Then

$$[f_*(\omega)] + [g_*(\omega)] \underset{\uparrow}{=} [(f * g)_*(\omega)] \in \mathrm{H}_n(X, x_0) = \mathrm{H}_n(X).$$

$\uparrow$ product of f and g as defined on page 1561

Proof. We write $J = I^n$, $J_1 = [0, \frac{1}{2}] \times I^{n-1}$ and $J_2 = [\frac{1}{2}, 1] \times I^{n-1}$. Furthermore we write $K = \partial J \cup (\{\frac{1}{2}\} \times I^{n-1})$. For $k = 1, 2$ we denote by $i_k\colon (J_k, \partial J_k) \to (J, K)$ the inclusion map. Finally we denote by $\varphi_k\colon (J_k, \partial J_k) \to (J, \partial J)$ the obvious homeomorphism given by stretching the last coordinate.



The map $f * g\colon (J, \partial J) \to (X, x_0)$ factors through a map $(J, K) \to (X, x_0)$ that we also denote by $f * g$. Thus we have the following commutative diagram

$$\begin{array}{ccccc} & & (J_1, \partial J_1) & & \\ & \overset{\varphi_1}{\nearrow} & \downarrow i_1 & \overset{f\circ\varphi_1^{-1}}{\searrow} & \\ (J, \partial J) & \xrightarrow{i} & (J, K) & \xrightarrow{f*g} & (X, x_0) \\ & \underset{\varphi_2}{\searrow} & \uparrow i_2 & \underset{g\circ\varphi_2^{-1}}{\nearrow} & \\ & & (J_2, \partial J_2). & & \end{array}$$

For $k = 1, 2$ we write $\omega_k = (\varphi_k)_*(\omega)$. Thus it suffices to show that $i_{1*}(\omega_1) + i_{2*}(\omega_2) = i_*(\omega)$. To prove this equality we consider the following commutative diagram[969]

$$\begin{array}{ccccc} & & \mathrm{H}_n(J_1, \partial J_1) & \xrightarrow{k_1} & \mathrm{H}_n(J, \partial J_1 \cup J_2) \\ & \overset{\varphi_1}{\nearrow} & \downarrow i_1 & \overset{j_1}{\nearrow} & \\ \mathrm{H}_n(J, \partial J) & \xrightarrow{i} & \mathrm{H}_n(J, K) & \overset{j_2}{\searrow} & \\ & \underset{\varphi_2}{\searrow} & \uparrow i_2 & & \\ & & \mathrm{H}_n(J_2, \partial J_2) & \xrightarrow[k_2]{} & \mathrm{H}_n(J, J_1 \cup \partial J_2), \end{array}$$

where all maps, except for φ_1 and φ_2, are induced by the obvious inclusion maps. We have

$$(j_1)(i(\omega)) = (k_1 \circ \varphi_1)(\omega) = j_1(i_1(\omega_1)) \underset{\uparrow}{=} j_1(i_1(\omega_1)) + j_1(i_2(\omega_2))$$

$\uparrow$ since $j_1 \circ i_2 = 0$

and similarly we show that $(j_2)(i(\omega)) = j_2(i_1(\omega_1)) + j_2(i_2(\omega_2))$.

[969] To simplify the notation we denote the induced map by the same symbol as the map between pairs of topological spaces, i.e. we drop the subscript $*$ from the notation.

It remains to show that $\ker(j_1) \cap \ker(j_2) = 0$. We consider the following diagram

$$\begin{array}{ccccc} & & & & \mathrm{H}_n(J, \partial J_1 \cup J_2) \\ & & & \overset{j_1}{\nearrow} & \\ & & \mathrm{H}_n(J, K) & & \\ & \nearrow & \uparrow{\scriptstyle i_2} & \searrow^{j_2} & \\ \mathrm{H}_n(J, \partial J_1 \cup J_2) & \xleftarrow[\cong]{} & \mathrm{H}_n(J_2, \partial J_2) & \xrightarrow[\cong]{k_2} & \mathrm{H}_n(J, J_1 \cup \partial J_2). \end{array}$$

We make the following clarifications and observations:

(1) The diagonal map to the bottom left of the diagram is induced by the inclusion $(J, \partial J_1 \cup J_2) \to (J, K)$.
(2) It follows easily from Corollary 74.16 (2a) and (2c) that the bottom two horizontal maps are isomorphisms.
(3) The diagonal sequence is exact by the Topological LES-Proposition 74.13.
(4) It now follows from some mild diagram chasing that $\ker(j_1) \cap \ker(j_2) = 0$. ■

Lemma 79.13.

(1) Let $f\colon S^n \to S^n_1 \vee S^n_2$ be the pinching map that we introduced on page 1565. Let $i \in \{1, 2\}$. If we denote by $S^n_1 \vee S^n_2 \to S^n_i = S^n$ the natural projection, then the induced map $(p_i \circ f)_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$ is the identity.
(2) Given any $k \in \mathbb{Z}$ there exists a map $g_k\colon S^n \to S^n$ with $g_k(*) = *$ such that $g_{k*}\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$ is given by multiplication by k.

Proof.

(1) The first statement follows easily from Lemma 79.12.
(2) The second statement follows almost immediately from Lemma 76.10 together with the Suspension-H_*-Proposition 76.7. ■

Now we turn to the proof of the Moore Space Existence Proposition 79.11.

Proof of the Moore Space Existence Proposition 79.11. Let $n \in \mathbb{N}$ and let π be an abelian group. Since π is abelian it follows immediately from Lemma 89.14 (1)[970] that there exist sets A and B, a homomorphism $\varphi\colon \mathbb{Z}^{(A)} \to \mathbb{Z}^{(B)}$ and a homomorphism $\rho\colon \mathbb{Z}^{(B)} \to \pi$ such that the following sequence is exact:

$$0 \to \mathbb{Z}^{(A)} \xrightarrow{\varphi} \mathbb{Z}^{(B)} \xrightarrow{\rho} \pi \to 0.$$

We let $X := \bigvee_{a \in A} S^n_a$ and we let $Y := \bigvee_{b \in B} S^n_b$. By the CW-Complex Construction Lemma 68.32 (4) we can equip X with the obvious CW-structure which has precisely one 0-cell and which has precisely one n-cell for each $a \in A$. Similarly we equip Y with the obvious CW-structure. Using the Wedge-H_*-Proposition 79.9 and using the standard generator of $\mathrm{H}_n(S^n)$ we can make the obvious identifications $\mathrm{H}_n(X) = \mathbb{Z}^{(A)}$ and $\mathrm{H}_n(Y) = \mathbb{Z}^{(B)}$.

[970] In this case the fact that we refer to a later lemma need not worry us. As the reader can see, the proof of Lemma 89.14 (1) relies only on information that we already know.

Claim. There exists a map $h\colon X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_n(X) & \xrightarrow{h_*} & \mathrm{H}_n(Y) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\ \mathbb{Z}^{(A)} & \xrightarrow{\varphi} & \mathbb{Z}^{(B)}. \end{array}$$

Proof. This map φ can be constructed explicitly using the following ingredients:

(1) the pinching map $f\colon S^n \to S^n_1 \vee S^n_2$,
(2) the maps $g_k\colon S^n \to S^n$ from Lemma 79.13,
(3) the natural inclusions and projections from page 1134,
(4) and the wedge of maps as introduced on page 1133.

The notation becomes somewhat cumbersome, thus in a short moment of lassitude we decide to leave it to the reader to fill in the details. A baby example is the content of Exercise 79.3. ⊞

Now let $Z := \mathrm{Cone}(h\colon X \to Y)$ be the mapping cone of the map $h\colon X \to Y$. It follows from the above claim and the Mapping Cone-H_*-Lemma 78.17 that we have the following commutative diagram of long exact sequences

$$\begin{array}{ccccccccccccc} \ldots & \longrightarrow & \overbrace{\mathrm{H}_{n+1}(Y)}^{=0} & \longrightarrow & \mathrm{H}_{n+1}(Z) & \longrightarrow & \mathrm{H}_n(X) & \xrightarrow{h_*} & \mathrm{H}_n(Y) & \longrightarrow & \mathrm{H}_n(Z) & \longrightarrow & \overbrace{\mathrm{H}_{n-1}(X)}^{=0} & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & & & \downarrow & & \\ \ldots & \longrightarrow & 0 & \longrightarrow & \mathrm{H}_{n+1}(Z) & \longrightarrow & \mathbb{Z}^{(A)} & \xrightarrow{\varphi} & \mathbb{Z}^{(B)} & \longrightarrow & \mathrm{H}_n(Z) & \longrightarrow & 0 & \longrightarrow & \ldots. \end{array}$$

It follows that $\mathrm{H}_n(Z) \cong \mathrm{coker}(\varphi) \cong \pi$. Furthermore, since φ is injective we see that $\mathrm{H}_{n+1}(Z) = 0$. Furthermore we see that all other reduced homology groups of Z are also zero.

Next we equip Z with the natural CW-structure coming from the CW-Complex Cone Corollary 68.35. There is one small fly in the ointment. The CW-structure has two 0-cells (namely one corresponding to the tip of the cone and the 0-cell of Y), one 1-cell and otherwise indeed only cells in dimensions n and $n+1$. Let A be the subcomplex that consists of the 1-cell that connects the two 0-cells. By the CW-Complex Construction Lemma 68.32 we can view Z/A as a CW-complex which now has only one 0-cell and otherwise only cells in dimension n and $n+1$. By the discussion on page 1614 we know that A is a good subset of Z. It follows from Theorem 74.3 that the reduced homology groups of Z/A coincide with the reduced homology groups of Z. Thus we see that Z/A is the CW-complex of our dreams. ■

Given an abelian group π and $n \in \mathbb{N}$ there are in general infinitely many pairwise non-homeomorphic Moore spaces of type $\mathrm{M}(\pi, n)$. For example the topological spaces $\overline{B}^k$, $k \in \mathbb{N}_0$ are Moore spaces of type $\mathrm{M}(\{e\}, 1)$. It is slightly less absurd to wonder whether Moore spaces are well-defined up to say homotopy equivalence. We formulate this as a question, we will come back to it at a later point.

Question 79.14. Let π be an abelian group and let $n \in \mathbb{N}$. Are any two Moore spaces of type (π, n) homotopy equivalent?

Exercises for Chapter 79.

Exercise 79.1. Let X and Y be topological spaces and let $x_0 \in X$ and $y_0 \in Y$ be good points. We write $\widetilde{x}_0 = [(x_0, 0)] \in \Sigma(X)$ and $\widetilde{y}_0 = [(y_0, 0)] \in \Sigma(Y)$.

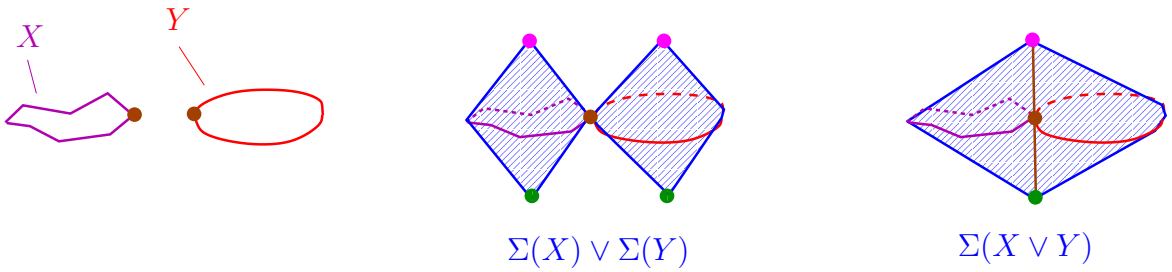
(a) Show that $\widetilde{x}_0 \in \Sigma(X)$ and $\widetilde{y}_0 \in \Sigma(Y)$ are good points. We use these points to define the wedge $\Sigma(X) \vee \Sigma(Y)$.

(b) Let $i\colon X \to X \vee Y$ and $j\colon Y \to X \vee Y$ be the obvious maps. Show that the map

$$(\Sigma(i) \vee \Sigma(j))_*\colon \mathrm{H}_k(\Sigma(X) \vee \Sigma(Y)) \;\to\; \mathrm{H}_k(\Sigma(X \vee Y))$$

is an isomorphism for each k.

Remark. You could make use of the *natural* isomorphism of the Suspension-H_*-Proposition 76.7.



Exercise 79.2. Let $\{A_k\}_{k\in K}$ be a family of topological spaces. For each $k \in K$ suppose that we are given a good point $a_k \in A_k$. By definition, see page 1171, this means that given any $k \in K$ the following statements hold:

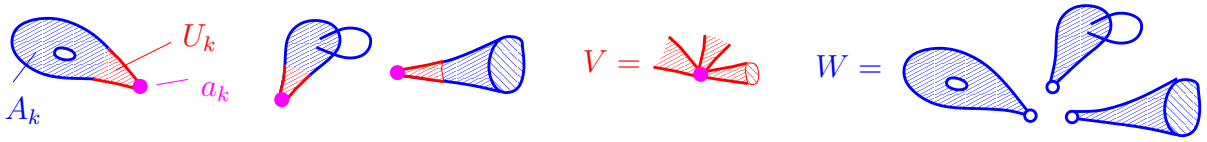
(1) $\{a_k\}$ is a closed subset of A_k and

(2) there exists an open neighborhood U_k of a_k in A_k such that a_k is a deformation retract of U_k.

Provide a third proof of the Wedge-H_*-Proposition 79.9 using the following approach. First we write $W := \bigcup_{k\in K}(A_k \setminus \{a_k\})$. Now apply the (natural) reduced Mayer–Vietoris sequences from Theorem 78.5 to the decompositions

$$\bigvee_{k\in K} A_k \;=\; V \cup W \quad \text{and} \quad \bigsqcup_{k\in K} A_k \;=\; \Big(\bigsqcup_{k\in K}(U_k \setminus \{a_k\})\Big) \cup \Big(\bigsqcup_{k\in K}(A_k \setminus \{a_k\})\Big).$$

Remark. Do not forget to make use of the analogue of Exercise 52.7.



Exercise 79.3. Let $n \in \mathbb{N}$ and let $X = S^n \vee S^n$ and let $Y = S^n \vee S^n \vee S^n$. As in the proof of the Moore Space Existence Proposition 79.11 we can make the identification $\mathrm{H}_n(X) = \mathbb{Z}^2$ and $\mathrm{H}_n(Y) = \mathbb{Z}^3$. Show that there exists a map $\varphi\colon X \to Y$ such that the map

$\varphi_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is represented by the matrix

$$\begin{pmatrix} 2 & 1 \\ -5 & 1 \\ 0 & 7 \end{pmatrix}.$$

80. Cellular homology

We saw in the Fundamental Group-of-1-Complex Theorem 69.9 and the Cell Attachment-π_1-Proposition 69.11 that it is quite easy to compute the fundamental group of a finite CW-complex. In this section we will develop a relatively straightforward method for computing the homology groups of a finite CW-complex. We will use this method to finally determine the homology groups of the surfaces of genus ≥ 2 and all the projective spaces $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$.

80.1. **The definition of cellular homology.** We turn to the systematic study of homology groups of CW-complexes. We need to recall two definitions:

(1) Recall that in the Standard Simplex Homeomorphism Lemma 72.1 we gave a homeomorphism $f\colon (\Delta^n, \partial\Delta^n) \to (\overline{B}^n, \partial\overline{B}^n)$. This homeomorphism together with the Identity-is-Generator Lemma 76.2 gives us the following isomorphism:

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\cong} & \mathrm{H}_n(\Delta^n, \partial\Delta^n) & \underset{\cong}{\xrightarrow{f_*}} & \mathrm{H}_n(\overline{B}^n, \partial\overline{B}^n) \\ 1 & \mapsto & [\mathrm{id}_{\Delta^n}] & & \end{array}$$

In the following we will usually identify $\mathrm{H}_n(\overline{B}^n, \partial\overline{B}^n)$ with $\mathbb{Z}$ using the above isomorphism.

(2) Let X be a topological space and let $A \subset X$ be a subset. As on page 1633 we consider the map

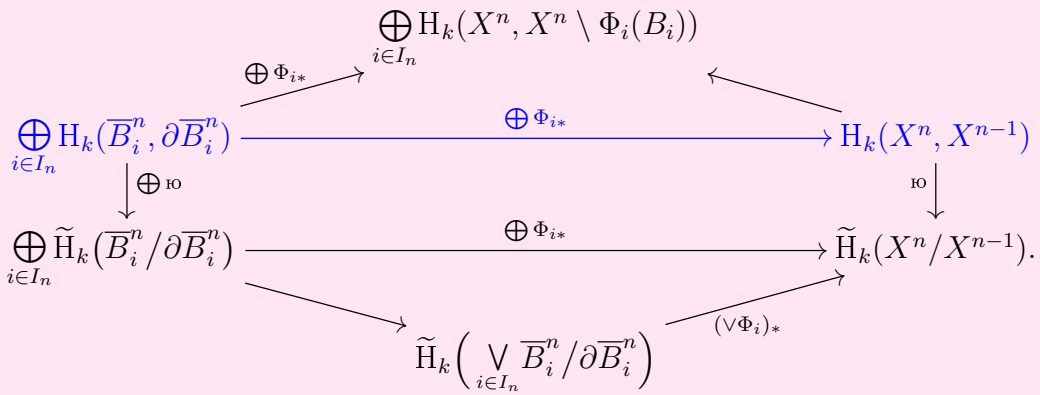
$$\mathrm{H}_n(X, A) \longrightarrow \mathrm{H}_n(X/A, A/A) \xleftarrow[\cong]{\text{Lemma 74.12}} \widetilde{\mathrm{H}}_n(X/A).$$
$$\text{ю}$$

By Lemma 74.19 we know that the map is a natural homomorphism.

Now we can formulate one of the key lemmas regarding homology groups of CW-complexes.

Lemma 80.1. (Skeleton-Homology Lemma) Let X be a CW-complex and let $n \in \mathbb{N}_0$.

(1) Given $n \in \mathbb{N}_0$ we denote by I_n the set of n-cells and given $i \in I_n$ we denote by $\Phi_i\colon \overline{B}_i^n \to X^n$ the corresponding characteristic map. Then for every $k \in \mathbb{N}_0$ we have the following commutative diagram where all maps are isomorphisms:



In particular the following holds:

(a) For $k \neq n$ we have $\mathrm{H}_k(X^n, X^{n-1}) = 0$.

(b) The group $\mathrm{H}_n(X^n, X^{n-1})$ is a free abelian group where the rank equals the cardinality of the set I_n of n-cells in the CW-structure.

(2) For $k > n$ we have $\mathrm{H}_k(X^n) = 0$.

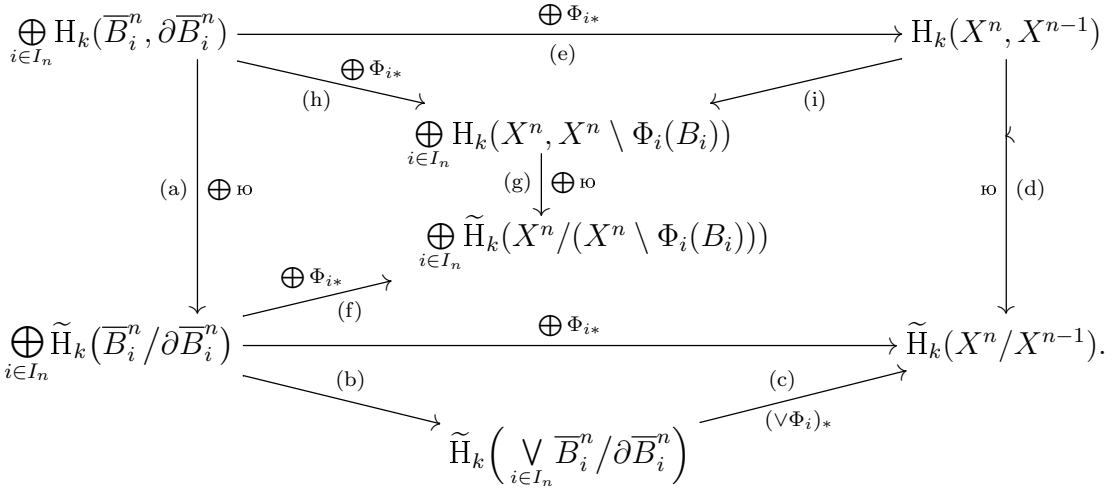
(3) For $k < n$ the inclusion $i\colon X^n \to X$ induces an isomorphism $i_*\colon \mathrm{H}_k(X^n) \xrightarrow{\cong} \mathrm{H}_k(X)$.

(4) The inclusion $i\colon X^n \to X$ induces an epimorphism $i_*\colon \mathrm{H}_n(X^n) \twoheadrightarrow \mathrm{H}_n(X)$.

Remark. The statements (3) and (4) of the Skeleton-Homology Lemma 80.1 can be viewed as "homology analogues" of the CW-Skeleton-π_n-Proposition 71.9.

Proof. Let X be a CW-complex.

(1) We consider the following diagram



We make the following observations:

(α) It follows from the definitions that the two triangles at the top and bottom commute.

(β) It follows almost immediately from the naturality of the maps ю that the big outer rectangle commutes and that the quadrilateral to the left commutes.

It remains to show that the maps numbered with (a) to (h) are isomorphisms.

(a) This map is an isomorphism by the Homology-of-Quotients Proposition 74.20.

(b) This map is an isomorphism by the Wedge-H_*-Proposition 79.9.

(c) This map is an isomorphism by the Skeleton-Quotient Lemma 68.33 (1).

(d) It follows from the Homology-of-Quotients Proposition 74.20 together with the CW-Complex Properties Proposition 68.10 (8) that this map is an isomorphism.

(e) It follows from the commutativity of the diagram and the fact that (a) to (d) are isomorphisms that this map is an isomorphism.

(f) This map is an isomorphism by the Skeleton-Quotient Lemma 68.33 (3).

(g) It follows immediately from the Homology-of-Quotients Proposition 74.20 together with the CW-Complex Properties Proposition 68.10 (8) and the Skeleton-Quotient Lemma 68.33 (3) that this map is an isomorphism.

(h) It follows from the fact that the quadrilateral to the left commutes and the fact that (a), (f) and (g) are isomorphisms that this map is an isomorphism.

(i) It follows from the fact that the top triangle commutes and the fact that (e) and (h) are isomorphisms that this map is an isomorphism.

The remaining statements of (1) are now an immediate consequence of Lemma 76.1.

(2) We fix $k \in \mathbb{N}$. We want to show that $\mathrm{H}_k(X^n) = 0$ for $n = 0, \dots, k-1$. For $r = 0, \dots, k-2$ we consider the long exact sequence of the pair (X^{r+1}, X^r):

$$\dots \to \mathrm{H}_{k+1}(X^{r+1}, X^r) \to \mathrm{H}_k(X^r) \to \mathrm{H}_k(X^{r+1}) \to \underbrace{\mathrm{H}_k(X^{r+1}, X^r)}_{\substack{= 0 \text{ by (1) since } r < k-1 \\ \text{i.e. since } k \neq r+1}} \to \dots$$

It follows that we have epimorphisms

$$\mathrm{H}_k(X^0) \to \mathrm{H}_k(X^1) \to \dots \to \mathrm{H}_k(X^{k-1}).$$

By the H_0-Proposition 72.5 and Lemma 72.14.[971] we know that $\mathrm{H}_k(X^0) = 0$. Thus all of the groups are zero.

(3) We fix $k \in \mathbb{N}_0$. Let $m > k$. This time we consider the following excerpt from the long exact sequence of the pair (X^{m+1}, X^m):

$$\dots \to \underbrace{\mathrm{H}_{k+1}(X^{m+1}, X^m)}_{= 0 \text{ by (1) since } k < m} \to \mathrm{H}_k(X^m) \to \mathrm{H}_k(X^{m+1}) \to \underbrace{\mathrm{H}_k(X^{m+1}, X^m)}_{= 0 \text{ by (1) since } k < m} \to \dots$$

This implies in particular that for each $k < n \leq l$ we have inclusion induced isomorphisms $\mathrm{H}_k(X^n) \xrightarrow{\cong} \mathrm{H}_k(X^{n+1}) \xrightarrow{\cong} \dots \xrightarrow{\cong} \mathrm{H}_k(X^l)$. In particular, if X is finite-dimensional, then claim (3) follows immediately from this observation. The infinite-dimensional case follows from the finite-dimensional case together with the Exhaustions-H_*-Proposition 79.4.

(4) We consider the following excerpt from long exact sequence of the pair (X^{n+1}, X^n):

$$\dots \longrightarrow \mathrm{H}_{n+1}(X^{n+1}, X^n) \longrightarrow \mathrm{H}_n(X^n) \longrightarrow \mathrm{H}_n(X^{n+1}) \longrightarrow \underbrace{\mathrm{H}_n(X^{n+1}, X^n)}_{= 0 \text{ by (1)}} \longrightarrow \dots$$

$$\mathrm{H}_n(X^n) \to \mathrm{H}_n(X) \quad \text{and} \quad \mathrm{H}_n(X^{n+1}) \xrightarrow{\cong \text{ by (3)}} \mathrm{H}_n(X)$$

We see that the inclusion induced map $\mathrm{H}_n(X^n) \to \mathrm{H}_n(X)$ is an epimorphism. ∎

Definition. Given a CW-complex X, we denote by $d = d_n$ the map

$$\mathrm{H}_n(X^n, X^{n-1}) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(X^{n-1}) \xrightarrow{j_{n-1}} \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}), \quad =: d = d_n$$

where ∂_n is the connecting map in the long exact sequence of the pair (X^n, X^{n-1}) and where j_{n-1} is the map induced by the projection $\mathrm{C}_{n-1}(X^{n-1}) \to \mathrm{C}_{n-1}(X^{n-1}, X^{n-2})$. We refer to d_n as the cellular boundary map.[972]

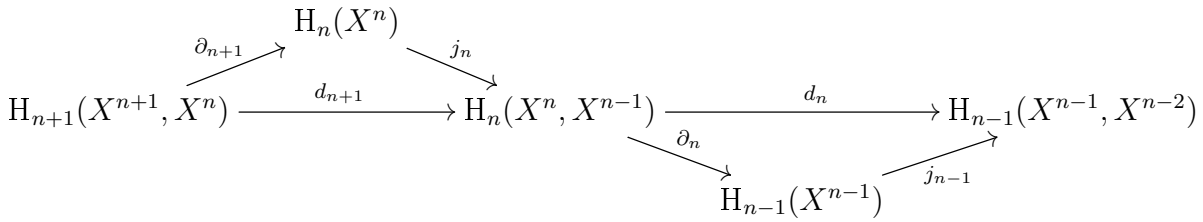
The name "cellular boundary map" already suggests that the maps d_n have the following important property:

[971] Here we use that X^0 has the discrete topology.

[972] It follows easily from the definitions that d_n is precisely the connecting homomorphism of the long exact sequence of the triple (X^n, X^{n-1}, X^{n-2}).

Lemma 80.2. For all $n \in \mathbb{N}_0$ we have $d_n \circ d_{n+1} = 0$.

Proof. We consider the following commutative diagram of maps



The map $\partial_n \circ j_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_{n-1}(X^{n-1})$ is the composition of two successive maps in the long exact sequence of the pair (X^n, X^{n-1}), i.e. the map $\partial_n \circ j_n$ is the zero map. Since the diagram commutes we obtain that $d_n \circ d_{n+1} = 0$. ■

Definition. Let X be a CW-complex.

(1) Given $n \in \mathbb{N}_0$ we write $\mathrm{C}_n^{\mathrm{CW}}(X) := \mathrm{H}_n(X^n, X^{n-1})$.
(2) We refer to $(\mathrm{C}_*^{\mathrm{CW}}(X), d_*)$ as the cellular chain complex of the CW-complex X.[973]
(3) We denote the homology groups of the cellular chain complex by $\mathrm{H}_n^{\mathrm{CW}}(X)$ and we refer to these groups as the cellular homology groups of the CW-complex X.

Remark. Let us summarize the content of the Skeleton-Homology Lemma 80.1 (1) in light of the above definition. Given a CW-complex X and given $n \in \mathbb{N}_0$ the n-th cellular chain group $\mathrm{C}_n^{\mathrm{CW}}(X)$ is a free abelian group and the characteristic maps of the n-cells give naturally rise to a basis of this free abelian group.

Example. We consider the n-sphere S^n for $n \geq 2$. On page 1472 we showed that we can view S^n as a CW-complex with one 0-cell and one n-cell. The cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(X)$ is thus of the form

$$0 \to \underbrace{\mathbb{Z}}_{=\mathrm{C}_n^{\mathrm{CW}}(X)} \to 0 \to \ldots \to 0 \to \underbrace{\mathbb{Z}}_{=\mathrm{C}_0^{\mathrm{CW}}(X)} \to 0.$$

Since $n \geq 2$ both the $\mathbb{Z}$'s are separated by at least one zero group. The cellular boundary maps are therefore necessarily the zero maps and we obtain that

$$\mathrm{H}_i^{\mathrm{CW}}(X) \cong \begin{cases} \mathbb{Z}, & \text{if } i = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

These are of course isomorphic to the "usual" homology groups of S^n. In the next section we will see that this is not a coincidence.

The "usual" homology groups define functors from the category of topological spaces to the category of abelian groups. We will now see that in a more restricted setting cellular homology groups also define a functor.

Definition.

(1) Recall that according to the definition on page 1492 a map $f \colon X \to Y$ between CW-complexes is called cellular if for each $n \in \mathbb{N}_0$ we have $f(X^n) \subset Y^n$.

[973] It follows from Lemma 80.2 that $(\mathrm{C}_*^{\mathrm{CW}}(X), d_*)$ is indeed a chain complex.

(2) As before we denote by $\mathcal{CW}$ the category of CW-complexes where the objects are CW-complexes and where the morphisms are cellular maps.

We conclude this section with the following lemma.

Lemma 80.3.

(1) Every cellular map $f\colon X \to Y$ between two CW-complexes induces a chain map

$$\begin{array}{rcl} f_*\colon \mathrm{C}_*^{\mathrm{CW}}(X) & \to & \mathrm{C}_*^{\mathrm{CW}}(Y) \\ f_*\colon \mathrm{H}_*^{\mathrm{CW}}(X) & \to & \mathrm{H}_*^{\mathrm{CW}}(Y) \end{array}$$

and hence also maps

on cellular homology.

(2) For each $n \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} X & \mapsto & \mathrm{H}_n^{\mathrm{CW}}(X) \\ (f\colon X \to Y) & \mapsto & (f_*\colon \mathrm{H}_n^{\mathrm{CW}}(X) \to \mathrm{H}_n^{\mathrm{CW}}(Y)) \end{array}$$

define a covariant functor from the category $\mathcal{CW}$ of CW-complexes to the category *AbGroup* of abelian groups.

Proof.

(1) Let $f\colon X \to Y$ be a cellular map. Since the map f is cellular we get for every $n \in \mathbb{N}_0$ an induced map $\mathrm{H}_n(X^n, X^{n-1}) \to \mathrm{H}_n(Y^n, Y^{n-1})$, i.e. we get an induced map $f_*\colon \mathrm{C}_n^{\mathrm{CW}}(X) \to \mathrm{C}_n^{\mathrm{CW}}(Y)$. It follows almost immediately from the naturality of the connecting homomorphism, see the Topological LES-Proposition 74.13 (2), that the above maps $f_*\colon \mathrm{C}_n^{\mathrm{CW}}(X) \to \mathrm{C}_n^{\mathrm{CW}}(Y)$, $n \in \mathbb{N}_0$ actually form a chain map.

(2) This statement follows easily from the fact that the usual chain complexes and homology groups are covariantly functorial. ■

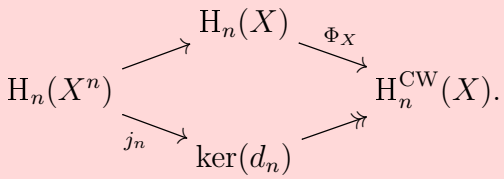
80.2. **The relationship between cellular and singular homology.** Now we can formulate and prove the following proposition.

Proposition 80.4. (Singular-Cellular H_*-Isomorphism Proposition) Let $n \in \mathbb{N}_0$.

(1) For each CW-complex X there exists a uniquely determined isomorphism

$$\Phi_X\colon \mathrm{H}_n(X) \xrightarrow{\cong} \mathrm{H}_n^{\mathrm{CW}}(X)$$

with the property that the following diagram commutes:[974]



(2) Recall that we denote by $\mathcal{CW}$ the category of CW-complexes where the objects are CW-complexes and where the morphisms are cellular maps. The homomorphisms from (1) define a natural isomorphism between the functors $X \mapsto \mathrm{H}_n(X)$ and the functor $X \mapsto \mathrm{H}_n^{\mathrm{CW}}(X)$. In other words, for every cellular map $f\colon X \to Y$ and any

$n \in \mathbb{N}_0$ the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}_n(X) & \xrightarrow{f_*} & \mathrm{H}_n(Y) \\ {\scriptstyle \Phi_X}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \Phi_Y} \\ \mathrm{H}_n^{\mathrm{CW}}(X) & \xrightarrow{f_*} & \mathrm{H}_n^{\mathrm{CW}}(Y). \end{array}$$

Remark. Later on, in the Singular-Cellular Chain Complex Proposition 81.12, we will give an alternative proof that singular and cellular homology groups are naturally isomorphic. The latter proof has the advantage that it shows a stronger statement, namely it shows in fact that the singular and cellular chain complexes are chain homotopy equivalent.

Example. Even though the cellular boundary maps in the cellular chain complex are still somewhat mysterious, we can already use the proposition to do some calculations. As an example, we return to the *n-dimensional complex projective space* $\mathbb{C}\mathrm{P}^n$ which we defined on page 255 as

$$\mathbb{C}\mathrm{P}^n \;:=\; (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\}).$$

In Lemma 68.1 we showed that we can view $\mathbb{C}\mathrm{P}^n$ as a CW-complex which admits exactly one cell in dimensions $0, 2, \ldots, 2n$ and which admits no other cells. In particular the cellular chain complex is of the form

$$0 \to \underbrace{\mathrm{C}_{2n}^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to \underbrace{\mathrm{C}_{2n-1}^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n)}_{=0} \to \cdots \to \underbrace{\mathrm{C}_{2}^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to \underbrace{\mathrm{C}_{1}^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n)}_{=0} \to \underbrace{\mathrm{C}_{0}^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to 0.$$

Since every other group is trivial it follows that all boundary maps are also zero. In particular the homology groups equal the cellular chain groups. Thus we have shown that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \;\underset{\uparrow}{\cong}\; \mathrm{H}_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n) \;\cong\; \mathrm{C}_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \ldots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

Singular-Cellular H_*-Isomorphism Proposition 80.4

It follows immediately from inspecting the cellular chain complex that for any $k \leq l$ and any $i \in \{0, \ldots, 2k\}$ the inclusion induced map

$$\mathrm{H}_i^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^k) \;\xrightarrow{\cong}\; \mathrm{H}_i^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^l)$$

is an isomorphism. By the naturality of the isomorphism of the Singular-Cellular H_*-Isomorphism Proposition 80.4 the same conclusion also holds for the inclusion induced map

$$\mathrm{H}_i(\mathbb{C}\mathrm{P}^k) \;\xrightarrow{\cong}\; \mathrm{H}_i(\mathbb{C}\mathrm{P}^l)$$

on singular homology. It follows from Lemma 68.6 (2) that we can view the infinite-dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty$ as a CW-complex that has precisely one cell in each even dimension and no cells in odd dimensions. From the arguments above it follows that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty) \;\cong\; \mathrm{H}_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^\infty) \;\cong\; \mathrm{C}_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^\infty) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

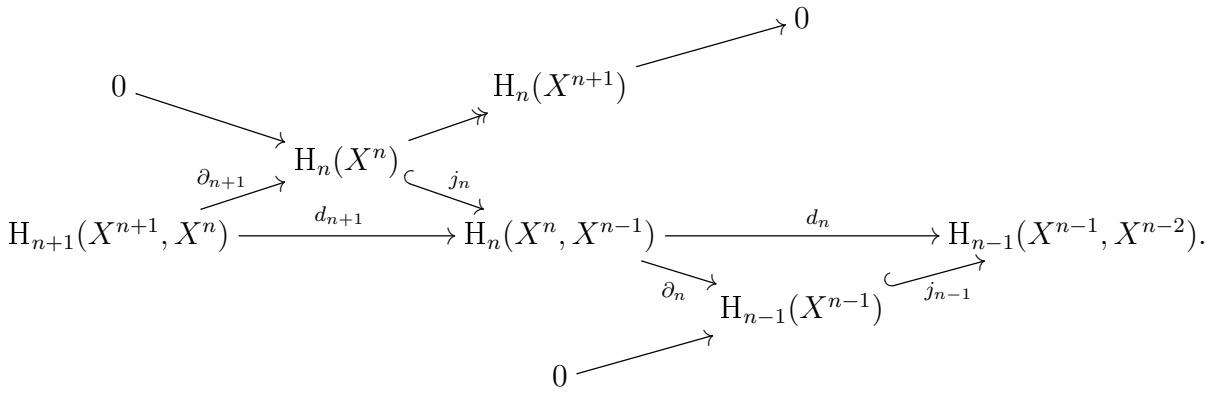
[974] Here we again denote by $j_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ the obvious map. In the long exact sequence of the pair (X^n, X^{n-1}) we see that the image of $\mathrm{H}_n(X^n)$ in $\mathrm{H}_n(X^n, X^{n-1})$ lies in the kernel of the connecting homomorphism ∂_n. This implies that the map $j_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ takes values in $\ker(d_n)$.

Furthermore, as above we see that the inclusion induced map

$$\mathrm{H}_i(\mathbb{C}\mathrm{P}^k) \xrightarrow{\cong} \mathrm{H}_i(\mathbb{C}\mathrm{P}^\infty)$$

is an isomorphism for $i \leq k+1$.

Proof of Proposition 80.4 (1). We add a couple of maps to the commutative diagram that we had already encountered in the proof of Lemma 80.2:



Claim. All the diagonal sequences in the above commutative diagram are exact.

Proof. From the Skeleton-Homology Lemma 80.1 (1) and (2) we know that

$$\mathrm{H}_n(X^{n+1}, X^n) = 0, \quad \mathrm{H}_n(X^{n-1}) = 0 \quad \text{and} \quad \mathrm{H}_{n-1}(X^{n-2}) = 0.$$

It thus follows that the diagonal sequences in the above commutative diagram are just the long exact sequences of the corresponding pairs of topological spaces. In particular all the diagonal sequences are exact. ⊞

Next we prove the following claim.

Claim. The map $j_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ induces an isomorphism

$$\Phi \colon \mathrm{H}_n(X^n)/\operatorname{im}(\partial_{n+1}) \to \ker(d_n)/\operatorname{im}(d_{n+1}) = \mathrm{H}_n^{\mathrm{CW}}(X).$$

Proof. We will prove this claim once again by a "diagram chase".[975]

(1) Since $\partial_n \circ j_n = 0$ it follows that $j_n(\mathrm{H}_n(X^n)) \subset \ker(d_n)$.

(2) The map j_{n-1} is injective, i.e. $\ker(d_n) = \ker(\partial_n) = \operatorname{im}(j_n)$, i.e. the map j_n induces an epimorphism

$$\mathrm{H}_n(X^n) \to \ker(d_n).$$

Since the left triangle commutes it also induces an epimorphism

$$\mathrm{H}_n(X^n)/\operatorname{im}(\partial_{n+1}) \to \ker(d_n)/\operatorname{im}(d_{n+1}).$$

(3) If an element $j_n(c) \in \mathrm{H}_n(X^n, X^{n-1})$ lies in the image of $d_{n+1} = j_n \circ \partial_{n+1}$, then $j_n(c) = j_n(\partial_{n+1}(d))$ for some $d \in \mathrm{H}_{n+1}(X^{n+1}, X^n)$. Since j_n is injective it follows that $c = \partial_{n+1}(d)$. Thus we have now shown that the map

$$\mathrm{H}_n(X^n)/\operatorname{im}(\partial_{n+1}) \to \ker(d_n)/\operatorname{im}(d_{n+1})$$

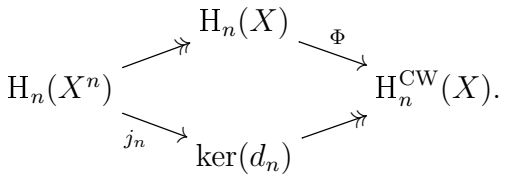
is injective. ⊞

[975] It is arguably best not to read the remainder of the proof but to go on the chase on one's own.

Next we consider the diagram which gives us the definition of the isomorphism Φ:

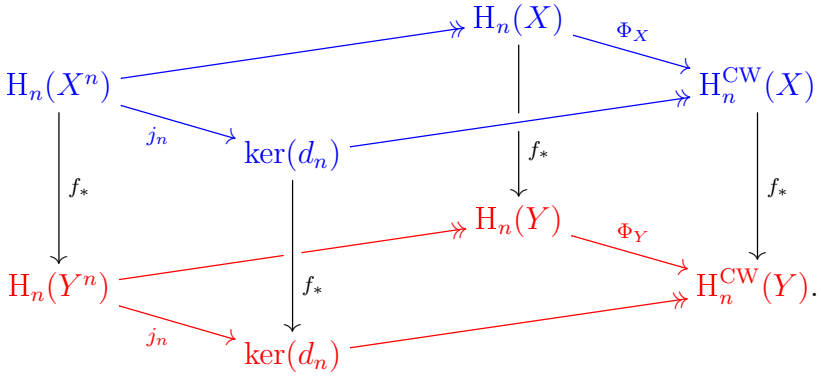
$$\begin{array}{ccccc}
 & \text{isomorphism since in the above diagram left diagonal sequence is exact} & & \text{isomorphism by Skeleton-Homology Lemma 80.1 (3)} & \\
 & \downarrow & & \downarrow & \\
\mathrm{H}_n(X^n)/\operatorname{im}(\partial_{n+1}) & \to & \mathrm{H}_n(X^{n+1}) & \to & \mathrm{H}_n(X). \\
\text{by the claim we know that } j_n \text{ induces an isomorphism} \Big\downarrow & \cong & & & \Big\downarrow =: \Phi \\
\ker(d_n)/\operatorname{im}(d_{n+1}) & & \xrightarrow[\text{id}]{} & & \mathrm{H}_n^{\mathrm{CW}}(X)
\end{array}$$

By construction the isomorphism Φ has the property that the following diagram commutes:



Note that the top-left diagonal map is an epimorphism by the Skeleton-Homology Lemma 80.1 (4). This implies immediately that Φ is uniquely determined. ■

Proof of Proposition 80.4 (2). We need to show that the isomorphisms Φ_X that we just constructed is natural. Thus suppose we are given a cellular map $f\colon X \to Y$ and let $n \in \mathbb{N}_0$. We consider the following diagram:



Note that the sides of the hyperrectangle that do not involve Φ_X and Φ_Y, commute basically by definition. Furthermore the top and bottom parallelogram commute by definition of Φ_X and Φ_Y. But then it follows from the fact that the map $\mathrm{H}_n(X^n) \to \mathrm{H}_n(X)$ is an epimorphism that also the parallelogram involving Φ_X and Φ_Y commutes, and that is exactly, what we had to show. ■

In general the natural isomorphism from the Singular-Cellular H_*-Isomorphism Proposition 80.4 can be hard to pin down. The following lemma gives one instance where we can actually say exactly what is happening.

Lemma 80.5. Let X be a CW-complex and let $\Theta\colon \overline{B}^n \to X$ be the characteristic map of an n-cell such that $\Theta(\overline{B}^n)$ consists of a single point. Note that this means that Θ descends to a map $f\colon \overline{B}^n/S^{n-1} \to X$ and that it defines a cycle in $\mathrm{H}^n_{\mathrm{CW}}(X)$. We denote by $g\colon S^n \to \overline{B}^n/S^{n-1}$ the homeomorphism from the Ball-Quotient Sphere Lemma 5.20.

Under the natural isomorphism

$$\Phi_X \colon \mathrm{H}_n(X) \;\to\; \mathrm{H}_n^{\mathrm{CW}}(X)$$

of the Singular-Cellular H_*-Isomorphism Proposition 80.4 we have

$$\Phi_X((f \circ g)_*([S^n])) \;=\; [\Theta].$$

Proof. The lemma follows almost immediately from the definitions and the Standard Generator-Relationship Proposition 76.5. We leave it to the reader to fill in the details. ■

On page 1772 we already saw that the Singular-Cellular H_*-Isomorphism Proposition 80.4 can be useful, even if we cannot determine the boundary maps d_n. Another, more theoretical application is, that the cellular chain complex is "much smaller" than the singular chain complex. For example, if X is a CW-complex with finitely many cells, then the chain group $\mathrm{C}_*^{\mathrm{CW}}(X)$ in the cellular chain complex are *finitely generated* free abelian groups. This is in contract to the singular chain groups $\mathrm{C}_*(X)$ which are in almost all cases free abelian groups of uncountable rank.[976]

Using this observation we can now easily prove the following proposition. It can be viewed as an analogue to the CW-Skeleton-π_1-Proposition 69.13.

Proposition 80.6. (Homology-of-CW-Complex Proposition) Let X be a CW-complex. The following statements hold:

(1) If X contains precisely $d \in \mathbb{N}_0$ cells of dimension n, then $\mathrm{H}_n(X)$ is generated by at most d elements. In particular, if X has no cells of dimension n, then $\mathrm{H}_n(X) = 0$. Similarly, if X contains only countable many cells of dimension n, then $\mathrm{H}_n(X)$ is countable.

(2) If X consists only of finitely many cells, then all homology groups of X are finitely generated groups.

(3) If X is finite-dimensional, then

$$\mathrm{H}_k(X) \;=\; 0 \quad \text{for all } k > \dim(X).$$

(4) If X is an n-dimensional CW-complex, then $\mathrm{H}_n(X)$ is a free abelian group.

Example. It follows from the Homology-of-CW-Complex Proposition 80.6 that for any topological graph G we have $\mathrm{H}_i(G) = 0$ for every $i \geq 2$ and that $\mathrm{H}_1(G)$ is a free abelian group.

Remark.

(1) Later on, in the Singular-Cellular Chain Complex Proposition 81.12, we will give an alternative proof that singular and cellular homology groups are naturally isomorphic. The latter proof has the advantage that it shows a stronger statement, namely it shows in fact that the singular and cellular chain complexes are chain homotopy equivalent.

(2) In Chapter 140, in particular in Proposition 140.9, we will formulate and prove a pretty elementary algebraic statement that allows us in Proposition 140.11 (2) to prove that the "size" of homology groups give a lower bound on the number of cells

[976] For which topological spaces and which n is the singular chain complex $\mathrm{C}_n(X)$ finitely generated? Can you give a complete answer?

in a CW-structure. This result can be viewed as a refinement of the Homology-of-CW-Complex Proposition 80.6 (1).

Proof. Let X be a CW-complex.

(1) If X contains precisely $d \in \mathbb{N}_0$ cells of dimension n, then it follows from the Skeleton-Homology Lemma 80.1 that $\mathrm{C}_n^{\mathrm{CW}}(X) \cong \mathbb{Z}^d$. By Lemma 51.9 any subgroup U of $\mathbb{Z}^d$ is a free abelian group such that $\mathrm{rank}(U) \leq d$. Thus it follows that

$$\mathrm{H}_n(X) \cong \underbrace{\ker\big(d_n \colon \mathrm{C}_n^{\mathrm{CW}}(X) \to \mathrm{C}_{n-1}^{\mathrm{CW}}(X)\big)}_{\text{free abelian group of rank} \leq d} \big/ \operatorname{im}\big(d_{n+1} \colon \mathrm{C}_{n+1}^{\mathrm{CW}}(X) \to \mathrm{C}_n^{\mathrm{CW}}(X)\big)$$

is generated by at most d elements. The argument in the case that X has countably many cells of dimension n is very similar, we just need to use the Countability Lemma 0.8 several times.

(2) This statement follows immediately from (1).

(3) This statement is in fact already a consequence of the Skeleton-Homology Lemma 80.1 (2).

(4) Now suppose that X is an n-dimensional CW-complex X. We have

$$\mathrm{H}_n(X) \underset{\text{Proposition 80.4}}{\cong} \mathrm{H}_n^{\mathrm{CW}}(X) \underset{\text{we have } \mathrm{C}_{n+1}^{\mathrm{CW}}(X) = 0 \text{ since } X \text{ is } n\text{-dimensional}}{=} \ker\big(d_n \colon \mathrm{C}_n^{\mathrm{CW}}(X) \to \mathrm{C}_{n-1}^{\mathrm{CW}}(X)\big),$$

i.e. $\mathrm{H}_n(X)$ is a subgroup of the free abelian group $\mathrm{C}_n^{\mathrm{CW}}(X)$. It follows from the Subgroup-of-Free Abelian Group Lemma 51.2 that the group $\mathrm{H}_n(X)$ itself is a free abelian group. ∎

By the Smooth Manifold-CW Structure Theorem 96.5 , which we already mentioned in a sneak preview on page 1474 we know that every compact n-dimensional smooth manifold admits a CW-structure with finitely many cells and such that the maximal dimension of a cell is n. If we combine this result with the Homology-of-CW-Complex Proposition 80.6 we obtain immediately the following useful result.

Proposition 96.6. Let M be a compact n-dimensional smooth manifold. Then the following holds:

(1) All homology groups are finitely generated abelian groups.

(2) For any $k > n$ we have $\mathrm{H}_k(M) = 0$.

Remark. We have now shown that for a CW-complex X the singular homology groups $\mathrm{H}_n(X)$ are isomorphic to the cellular homology groups $\mathrm{H}_n^{\mathrm{CW}}(X)$. One might now ask, whether one can generalize this statement, namely whether the singular chain complex $(\mathrm{C}_*(X), \partial_*)$ and the cellular chain complex $(\mathrm{C}_*^{\mathrm{CW}}, d_*)$ are in fact chain homotopic. This is indeed the case. But the proof requires more homological algebra than we have at the moment at our disposal. Therefore we postpone the proof to a later chapter.

80.3. **The cellular boundary maps.** If we want to use the Singular-Cellular H_*-Isomorphism Proposition 80.4 to calculate homology groups of CW-complexes we need to develop tools for computing the cellular boundary maps d_n in the cellular chain complex. Let X be a CW-complex and let $n \in \mathbb{N}_0$. Throughout this section we will use the following notation:

(1) We denote by I_n the set of n-cells of X.

(2) Given an n-cell α we denote by $\varphi_\alpha\colon S^{n-1}_\alpha \to X^{n-1}$ the corresponding attaching map and we denote by $\Phi_\alpha\colon \overline{B}^n_\alpha \to X^n$ the corresponding characteristic map.

(3) We denote by

$$\begin{aligned} \Psi_n\colon \bigvee_{\alpha\in I_n} \overline{B}^n_\alpha/\partial\overline{B}^n_\alpha &\to X^n/X^{n-1} \\ x &\mapsto \Phi_\alpha(x), \quad \text{if } x\in \overline{B}^n_\alpha. \end{aligned}$$

the map that is given by

It is straightforward to verify that Ψ_n is a homeomorphism.

(4) Given $\beta\in I_n$ we denote by

$$i_\beta\colon \overline{B}^n_\beta/\partial\overline{B}^n_\beta \to \bigvee_{\alpha\in I_n} \overline{B}^n_\alpha/\partial\overline{B}^n_\alpha \quad \text{and} \quad p_\beta\colon \bigvee_{\alpha\in I_n} \overline{B}^n_\alpha/\partial\overline{B}^n_\alpha \to \overline{B}^n_\beta/\partial\overline{B}^n_\beta,$$

the natural injection respectively the natural projection. By the Wedge-H_*-Proposition 79.9 the maps

$$\bigoplus_{\beta\in I_n} \mathrm{H}_n\big(\overline{B}^n_\beta/\partial\overline{B}^n_\beta\big) \underset{\oplus i_\beta}{\overset{\oplus p_\beta}{\rightleftarrows}} \mathrm{H}_n\Big(\bigvee_{\beta\in I_n} \overline{B}^n_\beta/\partial\overline{B}^n_\beta\Big).$$

are inverses of one another.

(5) Given $\alpha\in I_n$ we denote, by a slight abuse of notation, the standard generator of $\mathrm{H}_n(\overline{B}^n_\alpha,\partial\overline{B}^n_\alpha)\cong\mathbb{Z}$, as defined on page 1679, by α. Put differently, we have $\mathrm{H}_n(\overline{B}^n_\alpha,\partial\overline{B}^n_\alpha)=\mathbb{Z}\cdot\alpha$.

(6) By the Skeleton-Homology Lemma 80.1 we have the following commutative diagram where all maps are isomorphisms:

$$\begin{array}{ccccc} \bigoplus_{i\in I_n} \mathrm{H}_n(\overline{B}^n_i,\partial\overline{B}^n_i) & & \xrightarrow{\quad\Omega_n:=\oplus\Phi_{i*}\quad} & & \mathrm{H}_n(X^n,X^{n-1}) \\ \downarrow{\scriptstyle \oplus \mathrm{ю}} & & & & \downarrow{\scriptstyle \mathrm{ю}} \\ \bigoplus_{i\in I_n} \widetilde{\mathrm{H}}_n\big(\overline{B}^n_i/\partial\overline{B}^n_i\big) & \longrightarrow & \widetilde{\mathrm{H}}_n\Big(\bigvee_{i\in I_n} \overline{B}^n_i/\partial\overline{B}^n_i\Big) & \xrightarrow{(\vee\Phi_i)_*} & \widetilde{\mathrm{H}}_n(X^n/X^{n-1}). \end{array}$$

(7) By another abuse of notation we often denote $\Omega_n(\alpha)$ also by α. Put differently, we identify $\mathrm{C}^{\mathrm{CW}}_n(X)=\mathrm{H}_n(X^n,X^{n-1})$ with the free abelian group generated by the set I_n of all n-cells.

Our goal now is to determine the matrix of the cellular boundary map

$$d_n\colon \overbrace{\mathrm{H}_n(X^n,X^{n-1})}^{=\mathrm{C}^{\mathrm{CW}}_n(X)} \to \overbrace{\mathrm{H}_{n-1}(X^{n-1},X^{n-2})}^{=\mathrm{C}^{\mathrm{CW}}_{n-1}(X)}.$$

free abelian group generated by I_n — free abelian group generated by I_{n-1}

corresponding to the bases given by the cells. As a warm-up with start out with the case $n=1$.

Lemma 80.7. Let X be a CW-complex. We adopt all the notations introduced above.

(1) For a 1-cell α with attaching map $\varphi_\alpha\colon \overline{B}^1\to X^0$ we have

$$d_1(\alpha) \;=\; \varphi_\alpha(1)-\varphi_\alpha(-1).$$

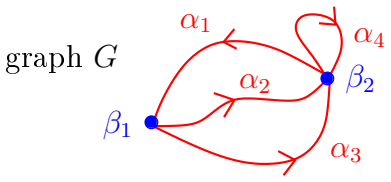
(2) If X has precisely one 0-cell, then the cellular boundary map d_1 is the zero map.

Proof. The second statement of the lemma is evidently a consequence of the first statement. Therefore it suffices to prove the first statement. So let α be a 1-cell with characteristic map $\Phi_\alpha\colon \overline{B}^1 \to X^0$. We have to determine the map

$$\begin{array}{ccccc} \mathbb{Z}\cdot\alpha \;\to\; \bigoplus\limits_{j\in I_1} \mathrm{H}_1(\overline{B}^1_j, \partial\overline{B}^1_j) & \xrightarrow{\Omega_1} & \mathrm{H}_1(X^1, X^0) & & \\ & & \downarrow{\scriptstyle d_1} & & \\ & & \mathrm{H}_0(X^0, \underbrace{X^{-1}}_{=\varnothing}) & \xrightarrow{\Omega_0^{-1}} & \bigoplus\limits_{i\in I_0} \mathrm{H}_0(\overline{B}^0_i, \underbrace{\partial\overline{B}^0_i}_{=\varnothing}). \end{array}$$

Since $X^{-1} = \varnothing$ we see that in the present case we have $d_1 = \partial_1$, i.e. the cellular boundary map d_1 is the connecting homomorphism $\mathrm{H}_1(X^1, X^0) \to \mathrm{H}_0(X^0)$ of the long exact sequence of the pair (X^1, X^0). It follows immediately from the definitions of the connecting homomorphism, as defined in the Topological LES-Proposition 74.13 (1), that $d_1(\alpha) = \varphi_\alpha(1) - \varphi_\alpha(-1)$. ■

Example. We consider the topological graph G that is given in the figure below with two vertices β_1, β_2 and four edges $\alpha_1, \ldots, \alpha_4$. We view it as a CW-complex in the obvious way. According to Lemma 80.7 the cellular chain complex of X is of the following form



$$0 \;\to\; \mathrm{C}_1^{\mathrm{CW}}(G) = \mathbb{Z}^4 \xrightarrow{\begin{array}{c}\alpha_1\;\alpha_2\;\alpha_3\;\alpha_4\\ \beta_1\begin{pmatrix} 1 & -1 & -1 & 0\\ -1 & 1 & 1 & 0\end{pmatrix}\\ \beta_2\end{array}} \mathrm{C}_0^{\mathrm{CW}}(G) = \mathbb{Z}^2 \;\to\; 0.$$

Thus we obtain from the Singular-Cellular H_*-Isomorphism Proposition 80.4 that

$$\mathrm{H}_1(G) \cong \ker\left(\mathbb{Z}^4 \xrightarrow{\begin{pmatrix}1 & -1 & -1 & 0\\ 1 & 1 & 1 & 0\end{pmatrix}} \mathbb{Z}^4\right) \cong \mathbb{Z}^3 \quad \text{and} \quad \mathrm{H}_0(G) \cong \mathrm{coker}\left(\mathbb{Z}^4 \xrightarrow{\begin{pmatrix}1 & -1 & -1 & 0\\ 1 & 1 & 1 & 0\end{pmatrix}} \mathbb{Z}^4\right) \cong \mathbb{Z}.$$

After this example we return to the discussion of the cellular boundary maps. Let X be a CW-complex and let $n > 1$. We want to determine the cellular boundary map

$$d_n\colon \mathrm{H}_n(X^n, X^{n-1}) \;\to\; \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}).$$

Given an n-cell α and an $(n-1)$-cell β we consider the map

$$S^{n-1}_\alpha = \partial\overline{B}^n_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} X^{n-1}/X^{n-2} \xrightarrow[\cong]{\Psi_{n-1}^{-1}} \bigvee_{j\in I_{n-1}} \overline{B}^{n-1}_j/\partial\overline{B}^{n-1}_j \xrightarrow{p_\beta} \overline{B}^{n-1}_\beta/\partial\overline{B}^{n-1}_\beta \xrightarrow[\cong]{\Theta} S^{n-1}_\beta$$

where $q\colon X^{n-1} \to X^{n-1}/X^{n-2}$ is the projection and where $\Theta\colon \overline{B}^{n-1}/\partial\overline{B}^{n-1} \xrightarrow{\cong} S^{n-1}$ denotes the homeomorphism from the Ball-Quotient Sphere Lemma 5.20. Since S^{n-1}_α and S^{n-1}_β are copies of S^{n-1} we can consider the degree of the map as defined on page 1684.

Now we will see that this degree is just the $\beta \times \alpha$–entry of the matrix that represents the cellular boundary map d_n with respect to the bases given by all n-cells and all $(n-1)$-cells. More precisely we have the following proposition:

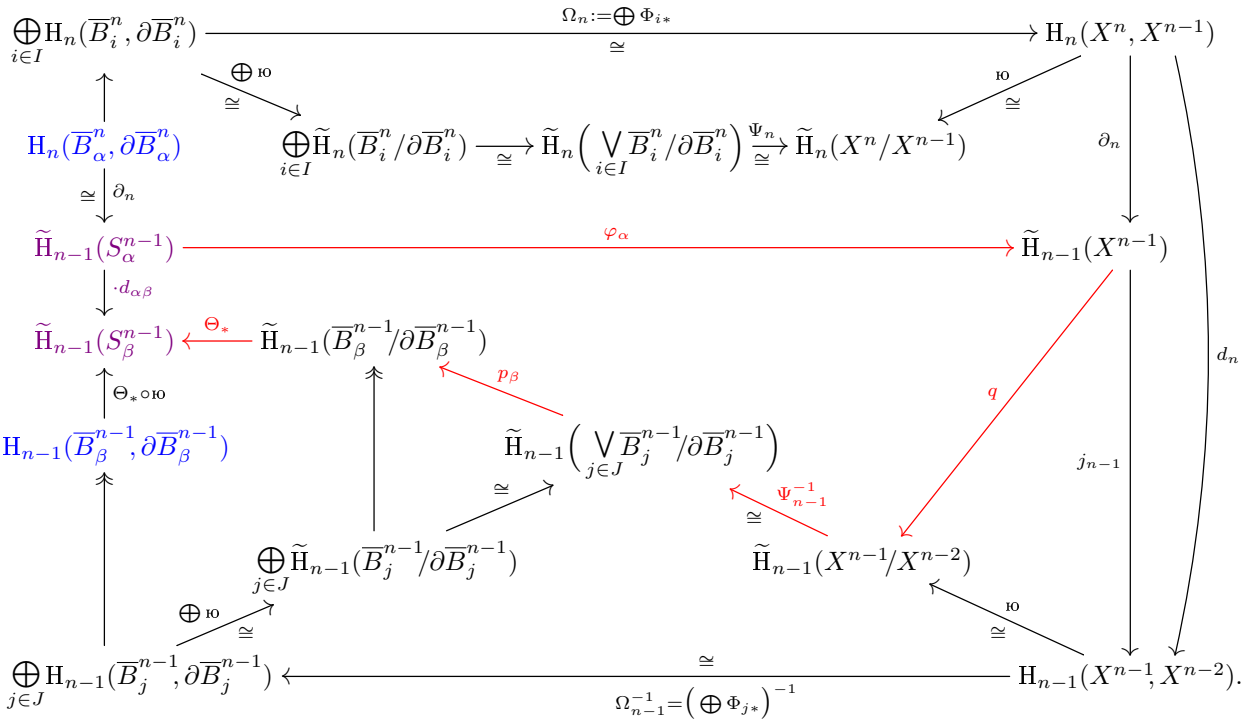
Proposition 80.8. (Cellular Boundary Map-via-Degrees Proposition) Let X be a CW-complex. We use all the notation that we introduced above. Then for each n-cell α we have

$$d_n(\alpha) \;=\; \sum_{\beta \in I_{n-1}} d_{\alpha,\beta} \cdot \beta$$

where

$$d_{\alpha\beta} \;:=\; \text{degree of the map } S^{n-1}_\alpha \xrightarrow{\Theta \circ p_\beta \circ \Psi_{n-1}^{-1} \circ q \circ \varphi_\alpha} S^{n-1}_\beta.$$

Proof. We write $I := I_n$ and $J := I_{n-1}$. We consider the following diagram[977]:



We make the following claim.

Claim. The diagram commutes.

Proof. We consider the various parts of the diagram.

(1) It follows from the Skeleton-Homology Lemma 80.1 (1) that the pentagon at the top and the large triangle at the bottom commute.

[977] The strange shape of the diagram is due to space constraints.

(2) It follows from the naturality of the connecting homomorphism, see the Topological LES-Corollary 74.14 (2), that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_n(\overline{B}^n_\alpha, \partial\overline{B}^n_\alpha) & \xrightarrow{\Phi_\alpha} & \mathrm{H}_n(X^n, X^{n-1}) \\ \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} \\ \mathrm{H}_{n-1}(S^{n-1}_\alpha) & \xrightarrow{\varphi_\alpha} & \mathrm{H}_{n-1}(X^{n-1}). \end{array}$$

From this observation it follows that the upper rectangle in the diagram commutes.

(3) It follows immediately from the definitions that the pentagon to the bottom left commutes.

(4) It follows again immediately from the definitions that the triangle towards the bottom left, involving the map p_β commutes.

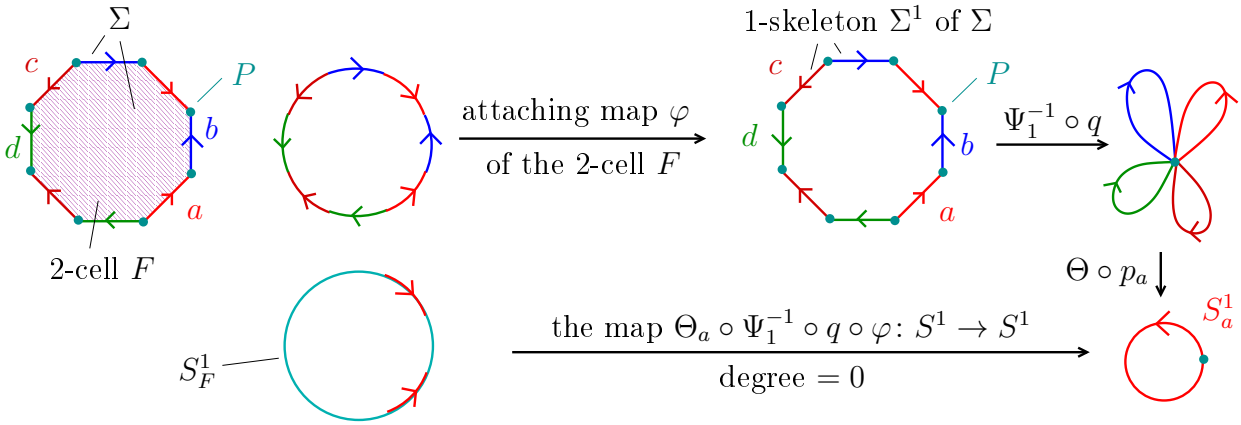
(5) It follows from Lemma 74.19 (1) that the triangle to the bottom right commutes. ⊞

Next note that it follows from the definition of the standard generator of $\mathrm{H}_n(\overline{B}^n, \partial\overline{B}^n)$ on page 1679 and from the Standard Generator-Relationship Proposition 76.5 that the two maps

$$\begin{array}{rcll} \partial_n \colon \mathrm{H}_n(\overline{B}^n_\alpha, \partial\overline{B}^n_\alpha) & \to & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}_\alpha) & \text{and} \\ \Theta_* \circ \text{ю} \colon \mathrm{H}_{n-1}(\overline{B}^{n-1}_\beta, \partial\overline{B}^{n-1}_\beta) & \to & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}_\beta) & \end{array}$$

send the standard generator to the standard generator. The reader who is still awake will notice that the proposition follows immediately from the claim and this observation. ■

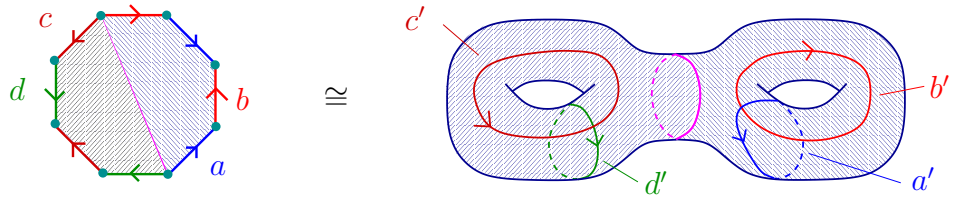
Example. We consider the surface Σ of genus 2. As usual we view Σ as a CW-complex with one 0-cell P, four 1-cells a, b, c, d and one 2-cell F. We denote by $\varphi\colon S^1 \to \Sigma^1$ the attaching map of the 2-cell F. (We refer to the figure below for an illustration.) It follows from the discussion on page 1684 that the degree of the map $p_a \circ \Psi_1^{-1} \circ q \circ \varphi\colon S^1_F \to S^1_a$ is zero. The same conclusion holds for all other 1-cells. It follows from the Cellular Boundary



Map-via-Degrees Proposition 80.8 that the cellular boundary map d_2 in the cellular chain complex of Σ is zero. Since Σ has only one 0-cell we know from Lemma 80.7 that the cellular boundary map d_1 is also zero. Thus we see that

$$\mathrm{H}_n(\Sigma) \underset{\text{Singular-Cellular Homology Isomorphism Proposition 80.4}}{\overset{\uparrow}{\cong}} \mathrm{H}_n^{\mathrm{CW}}(\Sigma) \underset{\text{since all the cellular boundary maps are zero}}{\overset{\uparrow}{=}} \mathrm{C}_n^{\mathrm{CW}}(\Sigma) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}^4, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

In the figure below on the left-hand side we show closed curves a, b, c, d. It follows from the above calculation and Lemma 80.5 that these curves represent a basis for $\mathrm{H}_1(\Sigma) \cong \mathbb{Z}^4$. These curves in turn are isotopic to the curves a', b', c', d' on the right-hand side, which therefore, by the Homotopic Maps-and-Homology Proposition 73.6, also represent a basis for $\mathrm{H}_1(\Sigma) \cong \mathbb{Z}^4$.

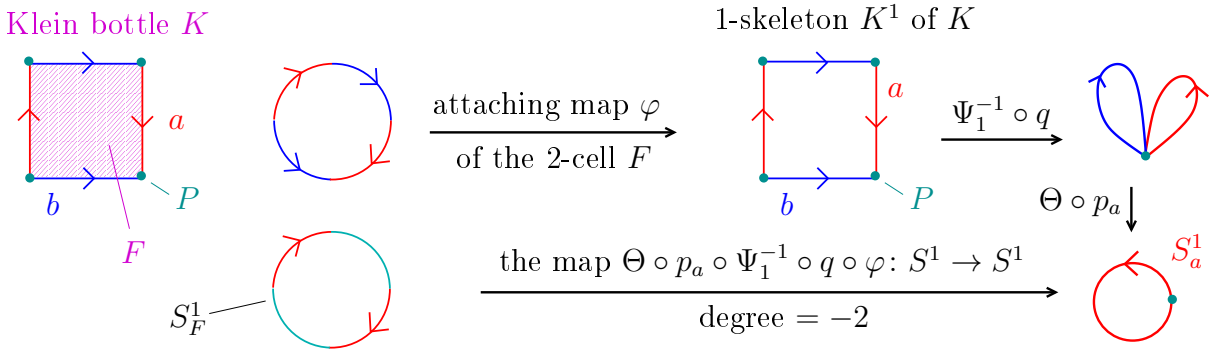


In the previous example we determined that $\mathrm{H}_1(\Sigma) \cong \mathbb{Z}^4$ and we gave an explicit basis for $\mathrm{H}_1(\Sigma)$. One might think that we now fully understand the homology groups of Σ. We will try to challenge this idea in Exercise 80.16.

Example. As a reality check we also consider once again the Klein bottle K. We view K as a CW-complex with one 0-cell P, two 1-cells a, b and one 2-cell F. (We refer to the figure below for an illustration.) It follows from the Cellular Boundary Map-via-Degrees Proposition 80.8, the discussion on page 1684 and Lemma 80.7 that the cellular chain complex is of the form

$$0 \to \mathbb{Z} \xrightarrow{\begin{pmatrix} -2 \\ 0 \end{pmatrix}} \mathbb{Z} \cdot a \oplus \mathbb{Z} \cdot b \xrightarrow{\begin{pmatrix} 0 & 0 \end{pmatrix}} \mathbb{Z} \to 0.$$

Therefore we see that $\mathrm{H}_2(K) = 0$, $\mathrm{H}_1(K) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$ and $\mathrm{H}_0(K) \cong \mathbb{Z}$. This is of course the same result that we had already obtained in Lemma 78.12.



80.4. **The homology groups of 2-dimensional smooth manifolds.** Recall that given any $g, m \in \mathbb{N}_0$ we write[978]

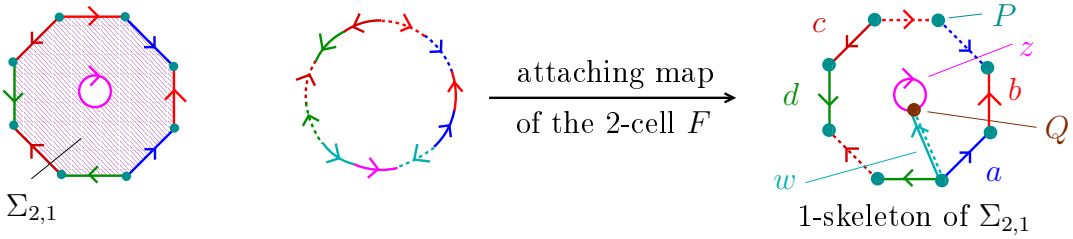
$$\Sigma_{g,m} \quad := \quad \text{the surface of genus } g \text{ minus } m \text{ open disks}$$

and for $k \in \mathbb{N}$ and for $m \in \mathbb{N}_0$ we write

$$N_{k,m} \quad := \quad \text{the non-orientable surface of genus } g \text{ minus } m \text{ open disks.}$$

In the Surface Classification Theorem 55.4 we saw that every connected, compact 2-dimensional topological manifold is homeomorphic to either $\Sigma_{g,m}$ for unique $g, m \in \mathbb{N}_0$ or to $N_{k,m}$ for unique $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$.
Our goal in this section will be to determine the homology groups of $\Sigma_{g,n}$ and $N_{k,n}$. Before we consider the general case we will consider the special case $\Sigma_{2,1}$ in great detail.

Example. We consider $\Sigma_{2,1}$ and we equip it with the CW-structure that is illustrated in the figure below. This CW-structure has two 0-cells P, Q, six 1-cells a, b, c, d, w, z and it has one 2-cell F. Given $i = 1, 2$ we denote by A_i the matrix that represents the cellular



boundary map $d_i \colon \mathrm{C}_i^{\mathrm{CW}}(\Sigma_{2,1}) \to \mathrm{C}_{i-1}^{\mathrm{CW}}(\Sigma_{2,1})$ with respect to the given bases.

The argument on page 1780 shows that $A_2 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ w \\ z \end{array} \begin{array}{c} F \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{array}$

It follows from Lemma 80.7 that we have $A_1 = \begin{array}{c} \\ P \\ Q \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & w & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}.$

Some elementary linear algebra shows that

$$\mathrm{H}_n(\Sigma_{2,1}) \underset{\uparrow}{\cong} \mathrm{H}_n^{\mathrm{CW}}(\Sigma_{2,1}) = \mathrm{H}_n\Big(0 \to \mathbb{Z} \xrightarrow{A_2} \mathbb{Z}^6 \xrightarrow{A_1} \mathbb{Z}^2 \to 0\Big) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^4, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Singular-Cellular H_*-Isomorphism Proposition 80.4

From this discussion we get in fact a more precise statement. Namely, by the choice of the CW-structure on $\Sigma_{2,1}$ we have that the map inclusion map $\partial\Sigma_{2,1} \to \Sigma_{2,1}$ is in fact a cellular map. Therefore it follows from the Singular-Cellular H_*-Isomorphism Proposition 80.4 (2)

[978] Also recall the definition on page 1233 what we mean by a "surface minus m open disks".

that the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}_1(\partial\Sigma_{2,1}) & \xrightarrow{\quad f_* \quad} & \mathrm{H}_1(\Sigma_{2,1}) \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{Z} = \mathrm{H}_1^{\mathrm{CW}}(\partial\Sigma_{2,1}) & \xrightarrow[[z]\mapsto[z]]{f_*} & \mathrm{H}_1^{\mathrm{CW}}(\Sigma_{2,1}) = \mathbb{Z}^4 \end{array}$$

where the vertical maps denote the isomorphisms from the Singular-Cellular H_*-Isomorphism Proposition 80.4. The 1-cell z is null-homologous in $\mathrm{C}_*^{\mathrm{CW}}(\Sigma_{2,1})$ since we have just seen that it is the boundary of the 2-cell F. Thus we see that the bottom horizontal map is the zero map, but that implies that the top horizontal map is also the zero map.

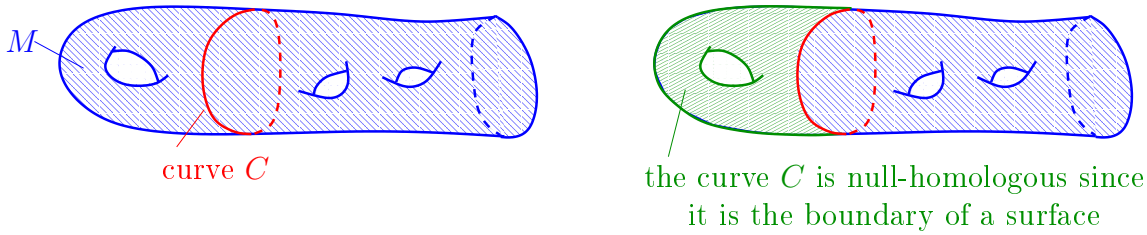
After the calculations that we just did we can now easily prove the following lemma.

Lemma 80.9.

(1) For any $g \in \mathbb{N}_0$ the inclusion induced map $\mathrm{H}_1(\partial\Sigma_{g,1}) \to \mathrm{H}_1(\Sigma_{g,1})$ is the zero map.
(2) Let M be a smooth manifold and let C be an oriented curve in M. If there exists a $g \in \mathbb{N}_0$ and a map $\psi\colon \Sigma_{g,1} \to M$ such that $\psi(\partial\Sigma_{g,1}) = C$, then $C = 0 \in \mathrm{H}_1(M)$.[979]

Remark. In Exercise 80.14 we will prove a generalization of Lemma 80.9 (1).

Example. Loosely speaking Lemma 80.9 (2) says that if a curve C is the boundary of a compact orientable surface in M, then the homology class $[C]$ represented by C is zero. For example in the figure below we see a curve C on a surface M. Since C is the boundary of a compact orientable subsurface contained in M we see that $[C] = 0 \in \mathrm{H}_1(M)$.



Proof.

(1) For $g = 1$ we proved the statement in the previous example. The general case is proved entirely the same way.
(2) Now let M be a smooth manifold and let C be an oriented curve in M. We denote by $i\colon C \to M$ the inclusion map. Suppose there exists a $g \in \mathbb{N}_0$ and a map $\psi\colon \Sigma_{g,1} \to M$ such that $\psi(\partial\Sigma_{g,1}) = C$. We pick a diffeomorphism $f\colon S^1 \to \partial\Sigma_{g,1}$. We get the commutative diagram

$$\begin{array}{ccccc} & & \mathrm{H}_1(C) & & \\ & \overset{(\psi\circ f)_*}{\nearrow} & & \overset{i_*}{\searrow} & \\ \mathrm{H}_1(S^1) & & & & \mathrm{H}_1(M). \\ & \underset{f_*}{\searrow} & & \overset{\psi_*}{\nearrow} & \\ & & \mathrm{H}_1(\partial\Sigma_{g,1}) & \longrightarrow & \mathrm{H}_1(\Sigma_{g,1}) \end{array}$$

[979] Following the convention of page 1733 the curve C defines a homology class in $\mathrm{H}_1(M)$ that we also denote by $[C]$.

By the first statement the bottom horizontal map is the zero map. But that shows that the homology class represented by C is also zero. ■

In the following proposition we give the calculation for the homology groups of all connected, compact 2-dimensional topological manifolds.

Proposition 80.10. For any $g \in \mathbb{N}_0$ we have

$$\mathrm{H}_n(\Sigma_g) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}^{2g}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases} \quad \begin{array}{c} \text{Furthermore} \\ \text{for } m \geq 1 \\ \text{we have} \end{array} \quad \mathrm{H}_n(\Sigma_{g,m}) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^{2g+m-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

For any $k \in \mathbb{N}$ we have

$$\mathrm{H}_n(N_k) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}_2 \oplus \mathbb{Z}^{k-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases} \quad \begin{array}{c} \text{Furthermore} \\ \text{for } m \geq 1 \\ \text{we have} \end{array} \quad \mathrm{H}_n(N_{k,m}) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^{k+m-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

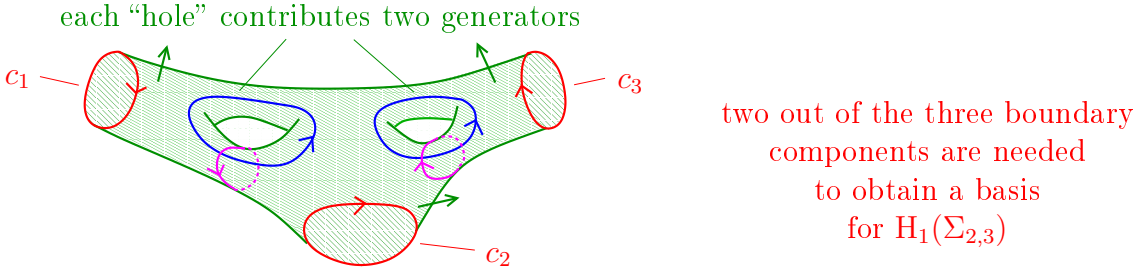
Proof. The calculation of the homology groups $\Sigma_{g,m}$ is a slight generalization of the arguments that we provided above and on page 1780. We leave the details to the reader.
We will deal with the homology of $N_{k,m}$ in Exercise 80.1. The calculation is evidently just a variation on the arguments we provided for Σ_2 and $\Sigma_{2,2}$. ■

Remark. A careful reading of the Proof of Proposition 80.10 gives us some extra information. First of all we endow $\Sigma_{g,m}$ with an orientation. The boundary components of $\Sigma_{g,m}$ then inherit an orientation. By the discussion on page 1733 the boundary components, viewed as oriented curves, then define elements $c_1, \ldots, c_m \in \mathrm{H}_1(\Sigma_{g,m})$. Then there exists an exact sequence

$$0 \to \mathbb{Z} \xrightarrow{1 \mapsto \sum c_i} \bigoplus_{i=1}^{m} \mathbb{Z} \cdot c_i \to \mathrm{H}_1(\Sigma_{g,m}) \to \mathrm{H}_1(\Sigma_g) \to 0.$$

(the map $\mathrm{H}_1(\Sigma_{g,m}) \to \mathrm{H}_1(\Sigma_g)$ is the inclusion induced map)

As we saw in the figure below, a basis for $\mathrm{H}_1(\Sigma_g)$ is given by a pair of curves for each "hole". The above exact sequence shows that a basis for the homology group $\mathrm{H}_1(\Sigma_{g,m}) \cong \mathbb{Z}^{2g+m-1}$ is obtained by a pair of curves for each "hole" together with boundary curves for all except one boundary component. This statement is illustrated in the figure below.

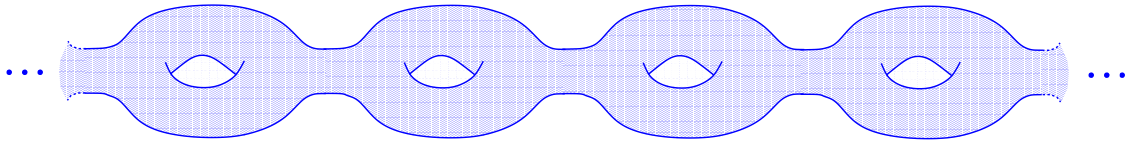


We conclude this section with the surface Σ_∞ of infinite genus that we had already encountered on page 1269. Using a variation on the proof of Lemma 56.11 one can show, using

the above calculations and the Exhaustions-H_*-Proposition 79.4, that

$$H_n(\Sigma_\infty) \cong \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z}^\infty, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

We leave it to the reader to fill in the details.



surface of "infinite genus"

80.5. **The homology groups of real projective spaces.** In this section we will compute the homology groups of real projective spaces $\mathbb{R}P^n$. This calculation is significantly trickier than the previous ones, since the calculation of the cellular boundary maps is not as straightforward as in our previous examples. This time we will need the theory of local degrees that we developed in Section 76.6.

Proposition 80.11. For any $n \in \mathbb{N}_0$ we have[980]

$$H_k(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd and } k < n, \\ 0, & \text{if } k \text{ is even and } 0 < k \neq 0, \\ 0, & \text{if } k > n, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd.} \end{cases}$$

Proof. As a reminder we defined

$$\mathbb{R}P^n = S^n/\!\sim \qquad \text{where } x \sim -x \text{ for all } x \in S^n.$$

In Lemma 68.1 we already saw that we can view $\mathbb{R}P^n$ as a CW-complex with the following two properties:

(1) the CW-structure has exactly one cell in the dimensions $0, 1, \ldots, n$ and no other cells,
(2) for each $k \leq n$ the k-skeleton of $\mathbb{R}P^n$ is given by $\mathbb{R}P^k$.

Now we recall the definition of this CW-structure. For each $k \in \mathbb{N}_0$ we define

$$\begin{array}{rcl} \varphi_k \colon S^{k-1} = \partial \overline{B}^k & \to & \mathbb{R}P^{k-1} \\ (x_1, \ldots, x_k) & \mapsto & [(x_1, \ldots, x_k)]. \end{array}$$

Then we obtain a homeomorphism

$$\begin{array}{rcl} \mathbb{R}P^{k-1} \cup_{\varphi_k} \overline{B}^k & \xrightarrow{\cong} & \mathbb{R}P^k \\ x & \mapsto & \begin{cases} (x, 0), & \text{if } x \in \mathbb{R}P^{k-1} \\ \big[\big(x, \sqrt{1 - \|x\|^2}\big)\big], & \text{if } x \in \overline{B}^k. \end{cases} \end{array}$$

Thus we see inductively that we can view $\mathbb{R}P^n$ as a CW-complex with a single cell in the dimensions $0, \ldots, n$ where the attaching maps are given precisely by the maps $\varphi_1, \ldots, \varphi_n$.

[980] For $n = 2$ we thus obtain, as we should, the same result as in Section 78.3.

As usual we denote the characteristic map of the k-cell, which in this case is the obvious map $\overline{B}^k \to \mathbb{R}\mathrm{P}^k$, by Φ_k.

Claim. For each $k = 1, \dots, n$ the cellular boundary map

$$d_k \colon \mathbb{Z} = \mathrm{C}_k^{\mathrm{CW}}(X) \;\to\; \mathrm{C}_{k-1}^{\mathrm{CW}}(X) = \mathbb{Z}$$

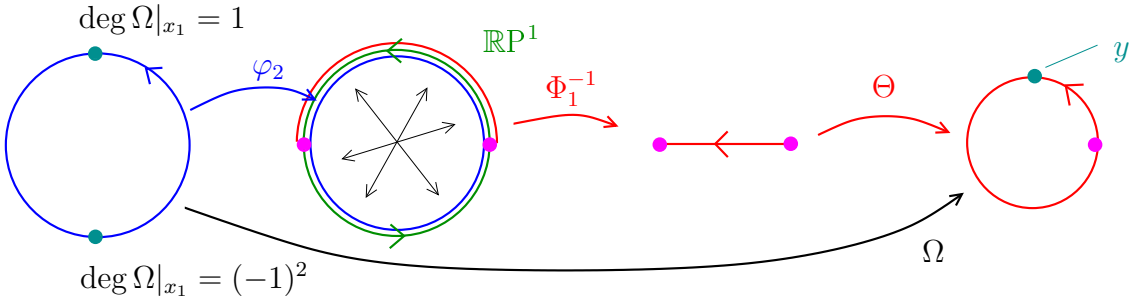
is given by multiplication by $1 + (-1)^k$.

Proof. Let $k \in \{1, \dots, n\}$. We denote by $\Theta \colon \overline{B}^{k-1}/\partial\overline{B}^{k-1} \to S^{k-1}$ the homeomorphism from the Ball-Quotient Sphere Lemma 5.20 and we denote by $q \colon \mathbb{R}\mathrm{P}^{k-1} \to \mathbb{R}\mathrm{P}^{k-1}/\mathbb{R}\mathrm{P}^{k-2}$ the projection . By the Cellular Boundary Map-via-Degrees Proposition 80.8 it suffices to show that the degree of the map

$$S^{k-1} \xrightarrow{\varphi_k} \mathbb{R}\mathrm{P}^{k-1} \xrightarrow{q} \mathbb{R}\mathrm{P}^{k-1}/\mathbb{R}\mathrm{P}^{k-2} \xrightarrow{\Phi_{k-1}^{-1}} \overline{B}^{k-1}/\partial\overline{B}^{k-1} \xrightarrow{\Theta} S^{k-1}$$

$$=:\Omega$$

equals $1 + (-1)^k$. (We refer to the figure below for an illustration of the map.) We pick the point $y = (0, \dots, 0, 1) \in S^{k-1}$. The preimages of y under the map Ω are $x_1 = (0, \dots, 0, 1)$ and $x_2 = (0, \dots, 0, -1)$. We leave it to the reader to verify that[981] $\deg \Omega|_{x_1} = 1$ and that $\deg \Omega|_{x_2} = (-1)^n$. The desired statement now follows from the Local-Global Degree Proposition 76.21. ⊞



It follows from the claim that the cellular chain complex of X is of the form

$$0 \;\to\; \mathbb{Z} \xrightarrow{1+(-1)^n} \mathbb{Z} \;\dots\; \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \;\to\; 0.$$

It is now straightforward to determine the cellular homology groups which, by the Singular-Cellular H_*-Isomorphism Proposition 80.4, are isomorphic to the singular homology groups of $\mathbb{R}\mathrm{P}^n$. ■

Remark. Using the approach of the proof of Proposition 80.11, applied to the infinite CW-complex $\mathbb{R}\mathrm{P}^\infty$, or alternatively using $\mathbb{R}\mathrm{P}^\infty = \bigcup_n \mathbb{R}\mathrm{P}^n$ and the Exhaustions-H_*-Proposition 79.4, we see that

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^\infty) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd}, \\ 0, & \text{if } k \text{ is even and } k > 0. \end{cases}$$

[981] It is not entirely easy to struggle through all the orientation conventions and all the maps, but it can indeed be done.

Remark. We have now seen that the homology groups of CW-complexes can be computed fairly easily using cellular chain complexes. One might ask, why did we introduce singular chain complexes, if it is so much easier to do calculations with cellular chain complexes. There are two related answers:

(1) we also want to study topological spaces that are not CW-complexes,
(2) it is much harder to prove some statements using cellular chain complexes, for example it is very difficult, if not impossible, to show that homeomorphic CW-complexes have isomorphic homology groups, just using the definition of cellular chain complexes.

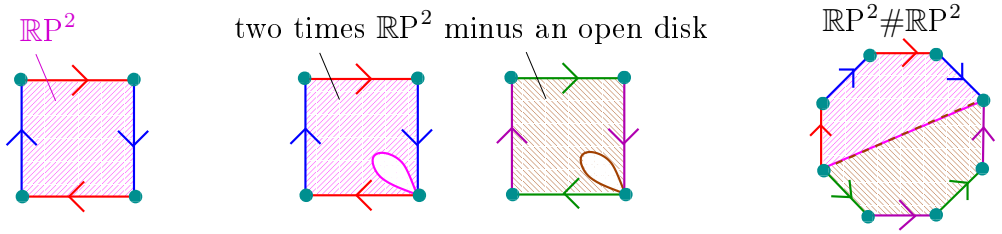
A general slogan in topology is that for calculations it is usually best to work with "small objects", whereas for proving general properties, it is often best to work with "big objects".

Exercises for Chapter 80.

Exercise 80.1.

(a) Determine the homology groups of $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$.
(b) Determine the homology groups of $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ minus an open disk.

Note: In both parts you can make use of the CW-structure for $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ illustrated in the figure below.
Remark. The homology groups are stated as Proposition 80.10. But it is a very good exercise to try to determine the homology groups without looking at the proposition.



Exercise 80.2. Show that given any finitely generated abelian group A there exists a topological space X with $\mathrm{H}_0(X) = \mathbb{Z}$ and $\mathrm{H}_1(X) \cong A$ and with $\mathrm{H}_i(X) = 0$ for $i \geq 2$.

Exercise 80.3. Let A be a finitely generated abelian group. Does there exist a topological space X with $\mathrm{H}_0(X) = \mathbb{Z}$, $\mathrm{H}_3(X) \cong A$ and $\mathrm{H}_i(X) = 0$ for $i \neq 0, 3$?

Exercise 80.4.

(a) Determine $\min\{\#\text{cells of } X \mid X \text{ is a CW-complex with } \mathrm{H}_2(X) \cong \mathbb{Z} \oplus \mathbb{Z}_2\}$.
(b) Determine $\min\{\#\text{cells of } X \mid X \text{ is a CW-complex with } \mathrm{H}_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}_2\}$.
(c) Determine $\min\{\#\text{cells of } X \mid X \text{ is a connected CW-complex with } \pi_1(X) \cong \mathbb{Z} \times \mathbb{Z}_2\}$.

Exercise 80.5.

(a) Let X be the quotient space of S^2 under the identification $x \sim -x$ for all $x \in S^1$. Determine the homology groups of $X = S^2/\!\sim$.
(b) Let $m, n \in \mathbb{N}$ with $m < n$. As usual we view $\mathbb{R}\mathrm{P}^m$ as a subset of $\mathbb{R}\mathrm{P}^n$. Compute the homology groups of the quotient space $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^m$.

Exercise 80.6. Let C be a closed curve in a smooth manifold M. Suppose that C is the boundary of an embedded compact orientable surface in M. Show that the inclusion induced map $\mathrm{H}_1(C) \to \mathrm{H}_1(M)$ is the zero map.
More concisely, if a closed curve bounds a compact orientable surface, then the closed curve represents the trivial element in homology.

Exercise 80.7.

(a) Let Σ be a compact orientable connected 2-dimensional smooth manifold and let $F \subset \Sigma$ be a 2-dimensional submanifold. We denote by $i\colon F \to \Sigma$ the inclusion map. Show that $i_*(\mathrm{H}_1(F)) \subset \mathrm{H}_1(\Sigma)$ is a summand.
Hint. Consider the long exact sequence of the pair (Σ, F).

(b) Does the conclusion of (a) also hold if we do not assume that Σ is orientable?

Exercise 80.8. Let X be a topological space and let $n \in \mathbb{N}_0$. Suppose X is compact. Does it follow that $\mathrm{H}_n(X)$ is a finitely generated group?

Exercise 80.9. Let $n \in \mathbb{N}_0$. In Lemma 68.1 we showed that $\mathbb{R}\mathrm{P}^n$ admits a CW-structure which has exactly one cell in the dimensions $0, 1, \ldots, n$ and no other cells. Show that this is the "smallest" CW-structure on $\mathbb{R}\mathrm{P}^n$. More precisely, show that every CW-structure for $\mathbb{R}\mathrm{P}^n$ has at least one cell in the dimensions $0, 1, \ldots, n$.
Hint. Consider the (cellular) homology groups of $\mathbb{R}\mathrm{P}^n$.

Exercise 80.10. Let $m \leq n$. We denote by $i\colon \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$ the inclusion map. Determine for each $k \in \mathbb{N}_0$ the induced map $i_*\colon \mathrm{H}_k(\mathbb{R}\mathrm{P}^m) \to \mathrm{H}_k(\mathbb{R}\mathrm{P}^n)$.

Exercise 80.11. Let n be odd. By Proposition 80.11 we know that $\mathrm{H}_n(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}$. Show that there exists a singular simplex $\sigma\colon \Delta^n \to \mathbb{R}\mathrm{P}^n$ such that $[\sigma]$ is a generator of $\mathrm{H}_n(\mathbb{R}\mathrm{P}^n)$.
Hint.

(a) First you need to find an "interesting" cycle σ. You could use the identification $\mathbb{R}\mathrm{P}^n = \overline{B}^n/\sim$ from page 258. Or perhaps better, write $\mathbb{R}\mathrm{P}^n = \Delta^n/\sim$ for a suitable equivalence relation $\sim$.

(b) To show that the "interesting" cycle σ is actually a generator you could use the map $\overline{B}^n/\sim \to \overline{B}^n/S^{n-1}$ (which is continuous by the Twice Quotient Lemma 5.23 (2)) and results from Section 76.1.

Exercise 80.12. Let $n \in \mathbb{N}$ be odd. By Proposition 80.11 we know that $\mathrm{H}_n(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}$. Let $p\colon S^n \to \mathbb{R}\mathrm{P}^n = S^n/\{\pm \mathrm{id}\}$ be the projection . Determine the induced map $p_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(\mathbb{R}\mathrm{P}^n)$.
Hint. You could use the fact that the isomorphism between cellular and singular homology is natural.

Exercise 80.13. Let $n \in \mathbb{N}$ be odd. By Proposition 80.11 we know that $\mathrm{H}_n(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}$. Let $A \in \mathrm{GL}(n+1, \mathbb{R})$. We denote by $f\colon \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ the map given by $f(x) := A \cdot x$. Determine the degree of the map $f_*\colon \mathrm{H}_n(\mathbb{R}\mathrm{P}^n) \to \mathrm{H}_n(\mathbb{R}\mathrm{P}^n)$.
Remark. You could make use of the homotopies considered on page 426.

Exercise 80.14. Let $g \in \mathbb{N}_0$ and let $k \in \mathbb{N}_0$. As on page 1234 we consider $\Sigma_{g,n}$, i.e. we consider the surface of genus g minus k open disks.

(a) Show that $\mathrm{H}_2(\Sigma_{g,n}, \partial\Sigma_{g,n}) \cong \mathbb{Z}$.

(b) Let $C_1, \dots, C_n$ be the boundary components. We equip $\Sigma_{g,n}$ with an orientation and we equip its boundary $\partial\Sigma_{g,n} = C_1 \sqcup \dots \sqcup C_n$ with the corresponding orientation given by the Boundary Orientation Lemma 25.17. For $i = 1, \dots, n$ we pick an orientation-preserving diffeomorphism $\gamma_i \colon S^1 \to C_i$. We write $[C_i] := \gamma_{i*}([S^1])$ and by a serious abuse of notation we denote by $[C_i]$ also the image of $[C_i]$ under the inclusion induced map $C_i \to \Sigma_{g,n}$. Show that there exists a generator φ of $\mathrm{H}_2(\Sigma_{g,n}, \partial\Sigma_{g,n}) \cong \mathbb{Z}$ such that

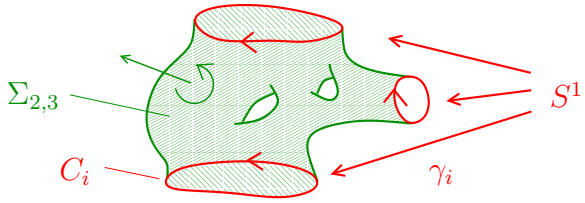
$$\underset{\substack{\uparrow \\ \text{connecting homomorphism of the pair } (\Sigma_{g,n}, \partial\Sigma_{g,n})}}{\partial_2(\varphi)} \;=\; [C_1] + \dots + [C_n] \;\in\; \mathrm{H}_1(\partial\Sigma_{g,n}).$$

(c) Show that $\mathrm{H}_1(\Sigma_{g,n}, \partial\Sigma_{g,n}) \cong \mathbb{Z}^{2g}$.

(d) We denote by $i \colon \partial\Sigma_{g,n} \to \Sigma_{g,n}$ and $j \colon \Sigma_{g,n} \to \Sigma_n$ the inclusion maps. Show that the following sequence is exact:

$$0 \;\to\; \mathrm{H}_2(\Sigma_{g,n}) \;\xrightarrow{\partial_2}\; \mathrm{H}_1(\partial\Sigma_{g,n}) \;\xrightarrow{i_*}\; \mathrm{H}_1(\Sigma_{g,n}) \xrightarrow{j_*} \mathrm{H}_1(\Sigma_g) \to 0.$$

Remark. In the Fundamental Class-of-Boundary Corollary 100.11 we will generalize Statement (a) to compact orientable manifolds in a suitable way.

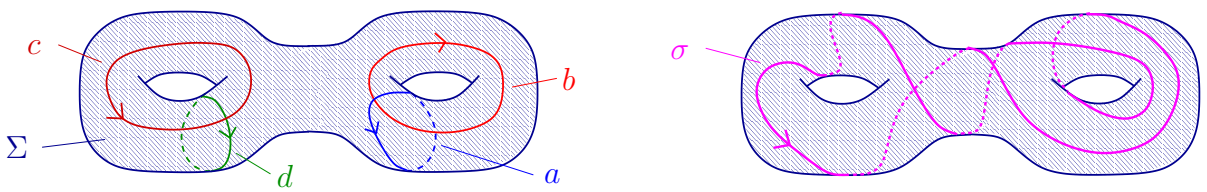


Exercise 80.15. Let $k \in \mathbb{N}$ and let $m \in \mathbb{N}_0$. As on page 1234 we denote by $N_{k,m}$ the non-orientable surface of genus g minus m open disks.

(a) Determine the relative homology groups $\mathrm{H}_*(N_{k,m}, \partial N_{k,m})$.

(b) Use (a) together with Exercise 80.14 and Proposition 80.10 to show that the topological spaces $\Sigma_{g,n}$ and $N_{k,n}$ are pairwise non-homeomorphic.
Remark. This gives a new, perhaps more natural proof for the Surfaces-Non Homeomorphic Theorem 55.6.

Exercise 80.16. In the figure below we see to the left the surface Σ of genus two together with four oriented curves a, b, c, d and to the right we see an oriented curve σ. On page 1781 we showed that $\mathrm{H}_1(\Sigma) \cong \mathbb{Z}^4$ and we showed that the curves a, b, c and d represent a basis for $\mathrm{H}_1(\Sigma)$. In particular we can write $[\sigma] \in \mathrm{H}_1(\Sigma)$ uniquely as a linear combination of $[a]$, $[b]$, $[c]$ and $[d]$. Determine the coefficients of this linear combination.
Remark. In Exercise 126.13 we will get to know a charming approach to dealing with problems of this form.



Exercise 80.17. Let X be a CW-complex and let A be a subcomplex of X. We denote by $i\colon A \to X$ the inclusion map. We define the relative cellular chain complex

$$\mathrm{C}_*^{\mathrm{CW}}(X, A) \;:=\; \operatorname{coker}\big(i_*\colon \mathrm{C}_*^{\mathrm{CW}}(A) \to \mathrm{C}_*^{\mathrm{CW}}(X)\big),$$

and we refer to its homology groups as the relative cellular homology groups $\mathrm{H}_*^{\mathrm{CW}}(X, A)$. Show that there exists a natural isomorphisms

$$\Phi_{(X,A)}\colon \mathrm{H}_n(X, A) \;\to\; \mathrm{H}_n^{\mathrm{CW}}(X, A)$$

such that we obtain a commutative diagram as follows:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & \mathrm{H}_n(A) & \xrightarrow{i_*} & \mathrm{H}_n(X) & \longrightarrow & \mathrm{H}_n(X, A) & \xrightarrow{\partial} & \mathrm{H}_{n-1}^{\mathrm{CW}}(A) \longrightarrow \dots \\
 & & \downarrow{\scriptstyle \Phi_A} & & \downarrow{\scriptstyle \Phi_X} & & \downarrow{\scriptstyle \Phi_{(X,A)}} & & \downarrow{\scriptstyle \Phi_A} \\
\dots & \longrightarrow & \mathrm{H}_n^{\mathrm{CW}}(A) & \xrightarrow{i_*} & \mathrm{H}_n^{\mathrm{CW}}(X) & \longrightarrow & \mathrm{H}_n^{\mathrm{CW}}(X, A) & \xrightarrow{\partial} & \mathrm{H}_{n-1}^{\mathrm{CW}}(A) \longrightarrow \dots
\end{array}$$

Exercise 80.18. Let $k \leq n$ and let $i\colon \mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^n$ be the embedding that is given by $i([x_0 : \cdots : x_k]) = [x_0 : \cdots : x_k : 0 : \cdots : 0]$. Use the relative cellular chain complex of Exercise 80.17 to compute the relative singular homology groups $\mathrm{H}_*(\mathbb{R}\mathrm{P}^n, i(\mathbb{R}\mathrm{P}^k))$.

81. The Fundamental Theorem of Homological Algebra

On several occasions we have now shown that certain homology groups are isomorphic, and it is reasonable to ask whether the underlying chain complexes are in fact chain homotopy equivalent. In this chapter we will see that, for purely algebraic reasons, the question can always be answered in the affirmative.

81.1. **The Fundamental Theorem of Homological Algebra.** We start out with the following definition which will arise time and again.

Definition.

(1) Let S be a non-empty set. We refer to

$$\mathbb{Z}^{(S)} := \text{all maps from } S \text{ to } \mathbb{Z} \text{ which are non-zero for only finitely many } s \in S$$

as the free abelian group generated by S.

(2) A group G is called free abelian if G is isomorphic to $\mathbb{Z}^{(S)}$ for some set S.

(3) A (free) resolution of an abelian group H is an exact sequence

$$\ldots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

where the F_i are (free) abelian groups.

Example.

(1) The following is a free resolution for the abelian group $H = \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5$:

$$0 \to \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 5 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\text{obvious projection}} \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5.$$

(2) Let $(\mathrm{C}_*, \partial_*)$ a chain complexes of *free* abelian groups. For each $n \in \mathbb{N}_0$ we write

$$Z_n := \ker(\partial_n \colon \mathrm{C}_n \to \mathrm{C}_{n-1}), \quad B_n := \operatorname{im}(\partial_{n+1} \colon \mathrm{C}_{n+1} \to \mathrm{C}_n) \quad \text{and} \quad H_n = Z_n / B_n.$$

Note that these groups form the following two types of short exact sequences

$$\text{(a)} \qquad 0 \to Z_n \hookrightarrow \mathrm{C}_n \xrightarrow{\partial_n} B_{n-1} \to 0,$$
$$\text{(b)} \qquad 0 \to B_n \hookrightarrow Z_n \to H_n \to 0.$$

Since all the chain groups are free abelian it follows from the Subgroup-of-Free Abelian Group Lemma 51.2 that the subgroups Z_n and B_n are also free abelian. Thus the above two short exact sequences are free resolutions of B_{n-1} and H_n.

We will give many more examples of free resolutions in Section 89.4.

Definition. Let $\alpha \colon H \to H'$ be a homomorphism between abelian groups. Furthermore let F_* be a resolution of H and let F'_* be a free resolution of H'. An extension of α to the resolutions F_* and F'_* is a sequence of homomorphisms $\alpha_i \colon F_i \to F'_i$, $i \in \mathbb{N}_0$ such that the

following diagram commutes:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\ \ldots & \longrightarrow & F_2' & \xrightarrow{f_2'} & F_1' & \xrightarrow{f_1'} & F_0' & \xrightarrow{f_0'} & H' & \longrightarrow & 0. \end{array}$$

Note that the horizontal exact sequences on top and bottom are in particular chain complexes and the maps α_* define a chain map from one complex to the other.
The following fundamental theorem underlies many arguments in homological algebra.

Theorem 81.1. (Fundamental Theorem of Homological Algebra) Let $\alpha\colon H \to H'$ be a homomorphism between abelian groups. Furthermore let F_* be a free resolution of H and let F_*' be a resolution[982] of H'.

(1) There exists an extension of α to the free resolutions F_* and F_*'.
(2) Any two extensions α_* and α_*', viewed as chain maps, are chain homotopic.

Proof. Let F_* be a free resolution of H and let F_*' be a resolution of H'. Statements (1) and (2) are both a straightforward consequence of some mild diagram chasing and the flexibility which Lemma 51.1 provides for defining homomorphisms starting from a free abelian group. In the following we fill in all the details.

(1) We have to show that there exists a sequence $\{\alpha_i\}_{i\in\mathbb{N}_0}$ of homomorphisms such that the following diagram commutes:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\ \ldots & \longrightarrow & F_2' & \xrightarrow{f_2'} & F_1' & \xrightarrow{f_1'} & F_0' & \xrightarrow{f_0'} & H' & \longrightarrow & 0. \end{array}$$

First we define the homomorphism α_0. We choose a basis $\mathcal{B}_0$ of the free abelian group F_0. For each $x_i \in \mathcal{B}_0$ we choose an $x_i' \in F_0'$ with $f_0'(x_i') = \alpha(f_0(x_i))$. This is possible, since f_0' is surjective. By Lemma 51.1 there exists a unique homomorphism $\alpha_0\colon F_0 \to F_0'$ with $\alpha_0(x_i) = x_i'$ for all $i \in I$. It is now evident that the right square commutes.

Now we turn to the definition of the map α_1. We choose a basis $\mathcal{B}_1$ of the free abelian group F_1. Let $x_i \in \mathcal{B}_1$. It follows from the commutativity of the square on the right that

$$f_0'((\alpha_0 \circ f_1)(x_i)) \;=\; \alpha(\underbrace{(f_0 \circ f_1)}_{=0}(x_i)) \;=\; 0.$$

Since the lower sequence is exact there exists an $x_i' \in F_1'$ with $f_1'(x_i') = (\alpha_0 \circ f_1)(x_i)$. Again by Lemma 51.1 there exists a unique homomorphism $\alpha_1\colon F_1 \to F_1'$ such that $\alpha_1(x_i) = x_i'$ for all $i \in I$. It is now evident that the "second square from the right" also commutes. It is now clear how inductively we can construct the homomorphisms $\alpha_2, \alpha_3, \ldots$.

(2) Now let $\alpha_i, \alpha_i'\colon F_i \to F_i', i \in \mathbb{N}_0$ be two extensions of α to the free resolutions F_* and F_*'. We have to find a chain homotopy from the chain map α_i to the chain map α_i'. The maps $\beta_i = \alpha_i - \alpha_i'$ are an extension of the zero map $\beta = 0\colon H \to H'$. Therefore

[982] Note that we do not demand that F_*' is a *free* resolution.

it suffices to find a chain homotopy λ from $\beta_i = \alpha_i - \alpha'_i$ to the zero chain map. The construction of the chain homotopy is very similar to the construction of the maps α_i in the first part of the proof.

Recall that we have to find a sequence of homomorphisms λ_i, $i \in \mathbb{N}_0$ such that for every $i \in \mathbb{N}_0$ we have

$$f'_{i+1} \circ \lambda_i + \lambda_{i-1} \circ f_i \;=\; \beta_i.$$

In the following discussion it is convenient to keep an eye on the following diagram that contains all the homomorphisms that we are given and that we need to construct:

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\
 & \swarrow^{\lambda_2} & \downarrow{\scriptstyle \beta_2} & \swarrow^{\lambda_1} & \downarrow{\scriptstyle \beta_1} & \swarrow^{\lambda_0} & \downarrow{\scriptstyle \beta_0} & \swarrow^{\lambda_{-1}} & \downarrow{\scriptstyle \beta=0} & & \\
\ldots & \longrightarrow & F'_2 & \xrightarrow[f'_2]{} & F'_1 & \xrightarrow[f'_1]{} & F'_0 & \xrightarrow[f'_0]{} & H' & \longrightarrow & 0.
\end{array}$$

We put $\lambda_{-1} = 0$. Next we need to find a map $\lambda_0 \colon F_0 \to F'_1$ such that $\beta_0 = f'_1 \circ \lambda_0$. For each $x \in \mathcal{B}_0$ we choose an $x' \in F'_1$ with $f'_1(x') = \beta_0(x)$. The existence of such x' follows from the observation that $f'_0(\beta_0(x)) = \beta(f_0(x)) = 0$ and from the exactness of the lower sequence.

Now we suppose that we have already found maps $\lambda_0, \ldots, \lambda_{i-1}$ which have the desired property that

$$(*) \qquad f'_j \circ \lambda_{j-1} + \lambda_{j-2} \circ f_{j-1} \;=\; \beta_{j-1} \quad \text{for } j = 0, \ldots, i.$$

We choose again a basis $\mathcal{B}_i$ of the free abelian group F_i. We have to assign to x an element $x' \in F'_{i+1}$ which satisfies the equation

$$f'_{i+1}(x') \;=\; \beta_i(x) - \lambda_{i-1}(f_i(x)).$$

From the exactness of the lower sequence it follows that $\operatorname{im}(f'_{i+1}) = \ker(f'_i)$. Thus it suffices to show that $f'_i \circ (\beta_i - \lambda_{i-1} \circ f_i) = 0$. It turns out that indeed we have

$$\begin{aligned}
f'_i \circ (\beta_i - \lambda_{i-1} \circ f_i) \;&=\; f'_i \circ \beta_i - f'_i \circ \lambda_{i-1} \circ f_i \;\underset{\uparrow}{=}\; \beta_{i-1} \circ f_i - f'_i \circ \lambda_{i-1} \circ f_i \\
&\qquad \text{since the } \beta_i\text{'s are a chain map, i.e. } f'_i \circ \beta_i = \beta_{i-1} \circ f_i \\
&=\; (\beta_{i-1} - f'_i \circ \lambda_{i-1}) \circ f_i \;\underset{\uparrow}{=}\; \lambda_{i-2} \circ f_{i-1} \circ f_i \;\underset{\uparrow}{=}\; 0. \\
&\qquad\qquad \text{by } (*) \qquad\qquad \text{since } f_{i-1} \circ f_i = 0
\end{aligned}$$

■

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We conclude this section with two applications to the Fundamental Theorem 81.1 of Homological Algebra.

Corollary 81.2. Let

$$\ldots \;\to\; F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \;\to\; 0$$

and

$$\ldots \;\to\; F'_2 \xrightarrow{f'_2} F'_1 \xrightarrow{f'_1} F'_0 \xrightarrow{f'_0} H \;\to\; 0$$

be two free resolutions of an abelian group H. Let $\alpha_i, i \in \mathbb{N}$ be an extension of the identity map $\mathrm{id} \colon H \to H$ to the free resolutions F_* and F'_*. The maps α_* form a chain homotopy equivalence between the two chain complexes F_* and F'_*.

Proof. We pick an extension $\alpha_i, i \in \mathbb{N}$ of the identity map $\mathrm{id}\colon H \to H$. We apply the Fundamental Theorem 81.1 of Homological Algebra to the free resolutions F'_* and F_* (i.e. we swap the roles of the free resolutions) and we obtain maps $\{\beta_i\}_{i\in\mathbb{N}_0}$ that give us the lower half of the following commutative diagram.

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\ldots & \longrightarrow & F'_2 & \xrightarrow{f'_2} & F'_1 & \xrightarrow{f'_1} & F'_0 & \xrightarrow{f'_0} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0.
\end{array}$$

The maps $\beta_i \circ \alpha_i$ now provide us with the following commutative diagram:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta_2\circ\alpha_2} & & \downarrow{\scriptstyle \beta_1\circ\alpha_1} & & \downarrow{\scriptstyle \beta_0\circ\alpha_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0.
\end{array}$$

This shows that the maps $\beta_i \circ \alpha_i$ define an extension of $\mathrm{id}\colon H \to H$ to the free resolutions F_i and F_i. But the identity maps also define an extension. By the Fundamental Theorem 81.1 of Homological Algebra the extensions $\{\beta_i \circ \alpha_i\}_{i\in\mathbb{N}_0}$ and $\{\mathrm{id}\}_{i\in\mathbb{N}_0}$ are chain homotopic. The same way one shows that the extensions $\{\alpha_i \circ \beta_i\}_{i\in\mathbb{N}_0}$ and $\{\mathrm{id}\}_{i\in\mathbb{N}_0}$ are chain homotopic. ■

Corollary 81.3. Let $(\mathrm{C}_*, \partial_*)$ be a chain complex of free abelian groups. If $(\mathrm{C}_*, \partial_*)$ is acyclic, i.e. if $\mathrm{H}_n(C_*) = 0$ for all $n \in \mathbb{N}_0$, then the identity map and the zero map of C_* to itself are chain homotopic.

Proof. We consider the following pretty dull diagram:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{0} & 0 & \longrightarrow & 0 \\
 & & & & & & & & \downarrow{\scriptstyle 0} & & \\
\ldots & \longrightarrow & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{0} & 0 & \longrightarrow & 0.
\end{array}$$

The hypothesis that $\mathrm{H}_n(C_*) = 0$ for all $n \in \mathbb{N}_0$ means that C_* is a free resolution of the trivial group, i.e. the top and bottom are free resolutions. The zero map on the right now has two extensions:

(1) The identity maps $\mathrm{id}\colon C_i \to C_i$ define an extension.
(2) The zero maps $\mathrm{id}\colon C_i \to C_i$ define an extension.

By Statement (2) of the Fundamental Theorem 81.1 of Homological Algebra these two extensions are chain homotopic. But that is precisely what we wanted to show. ■

81.2. **Quasi-isomorphisms are chain homotopy equivalences.** In this section we will state and discuss more consequences of the Fundamental Theorem 81.1 of Homological Algebra. We start out with the following definition.

Definition. Let $f\colon C_* \to D_*$ be a chain map between chain complexes. We say f is a quasi-isomorphism if for each $k \in \mathbb{N}_0$ the induced map $f_*\colon \mathrm{H}_k(C_*) \to \mathrm{H}_k(D_*)$ is an isomorphism.

In Lemma 73.2 we saw that chain homotopy equivalences are quasi-isomorphisms. If we are dealing with chain complexes of free abelian groups, the following proposition gives us now the converse statement.

Proposition 81.4. (Quasi Isomorphisms-are-Homotopy Equivalences Proposition) Let $f\colon C_* \to D_*$ be a chain map between chain complexes of *free abelian groups*. If f is a quasi-isomorphism, then f is in fact a chain homotopy equivalence.

Remark. The hypothesis that C and D_* are free abelian groups is essential. For example in Exercise 81.3 we will see that there exists a quasi-isomorphism between the chain complex

$$C_* = 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$$

and the zero chain complex D_*, but we will see that the two chain complexes are not chain homotopy equivalent.

The following proposition is a convenient variation on the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 81.4.

Proposition 81.5. Let C_* and D_* be two chain complexes of free abelian groups and let $\iota\colon C_* \to D_*$ be a chain map with the following two properties:

(a) The chain map $\iota\colon C_* \to D_*$ induces an isomorphism of homology groups.
(b) For each $k \in \mathbb{N}_0$ the map $\iota\colon \mathrm{C}_k \to D_k$ is a monomorphism and the image is a subsummand of D_k.

Then there exists a chain map $\pi\colon D_* \to \mathrm{C}_*$ with the following two properties:

(1) For each $k \in \mathbb{N}_0$ we have $\pi_k \circ \iota_k = \mathrm{id}_{\mathrm{C}_k}$.
(2) The maps ι and π are chain homotopy homotopy equivalences.

We provide the proofs of these two propositions in the next section. For the remainder of this section we want to discuss some consequences. For example we have have the following interesting corollary to the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 81.4.

Corollary 81.6. (Two-Out-Of-Three-Homotopy Equivalence Corollary)[983] Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \longrightarrow & B_* & \longrightarrow & C_* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A'_* & \longrightarrow & B'_* & \longrightarrow & C'_* & \longrightarrow & 0 \end{array}$$

be a commutative diagram of short exact sequences of chain complexes of free abelian groups. If two of the vertical maps are chain homotopy equivalences, then the third is also a chain homotopy equivalence.

Proof. By the LES Proposition 74.9 we get a commutative diagram of long exact sequences

$$\begin{array}{ccccccccccccc} \ldots & \longrightarrow & \mathrm{H}_{n+1}(C_*) & \xrightarrow{\partial} & \mathrm{H}_n(A_*) & \longrightarrow & \mathrm{H}_n(B_*) & \longrightarrow & \mathrm{H}_n(C_*) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(A_*) & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & \mathrm{H}_{n+1}(C'_*) & \xrightarrow{\partial} & \mathrm{H}_n(A'_*) & \longrightarrow & \mathrm{H}_n(B'_*) & \longrightarrow & \mathrm{H}_n(C'_*) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(A'_*) & \longrightarrow & \ldots \end{array}$$

[983] This corollary is very close to the content of Exercise 74.11.

By our hypothesis, together with Corollary 73.4, we know that always two out of three vertical maps are isomorphisms. But then it follows from the Five Lemma 74.10 that the third vertical map is also an isomorphism. Now the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 81.4 implies that the corresponding map of chain complexes is also a chain homotopy equivalence. ■

As we pointed out above, the first time we had mentioned the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 81.4 was in the remark after the Comfortable Covering Proposition 74.23. In fact we can now prove the following strengthened version of the Comfortable Covering Proposition 74.23.

Proposition 81.7. Let X be a topological space. For every comfortable cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X the inclusion map[984]
$$\mathrm{C}_*^{\mathcal{U}}(X) \;\to\; \mathrm{C}_*(X)$$
is a chain homotopy equivalence.

Proof. Let $k \in \mathbb{N}_0$. Note that $\mathrm{C}_k^{\mathcal{U}}(X)$ is the free abelian group generated by the set
$$\{\sigma\colon \Delta^k \to X \,|\, \text{there exists an } i \in I \text{ with } \sigma(\Delta^k) \subset U_i\}.$$
Therefore we can apply the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 81.4 and we obtain the desired result. ■

81.3. **Proof of Propositions 81.4 and 81.5.** This section will be occupied with the proofs of the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 81.4 and 81.5. We start out with the following definition.

Definition. Let (C_n, c_n) and (D_n, d_n) be two chain complexes and let $\varphi\colon C_* \to D_*$ be a chain map. The algebraic mapping cone of φ is the chain complex $(\mathrm{M}(\varphi), \partial_*)$ that is given by the chain groups
$$\mathrm{M}(\varphi)_n \;:=\; \begin{matrix}\mathrm{C}_{n-1}\\ D_n\end{matrix} \;:=\; \mathrm{C}_{n-1} \oplus D_n$$
and the boundary maps[985]
$$\partial\colon \begin{matrix}\overbrace{}^{\mathrm{M}(\varphi)_n}\\ \mathrm{C}_{n-1}\\ D_n\\ x\\ y\end{matrix} \;\begin{matrix}\to\\ \\ \\ \mapsto\\ \\ \end{matrix}\; \begin{matrix}\overbrace{}^{\mathrm{M}(\varphi)_{n-1}}\\ \mathrm{C}_{n-2}\\ D_{n-1}\\ -c_{n-1}(x)\\ \varphi_{n-1}(x) + d_n(y)\end{matrix} \qquad \text{which we write as block matrices} \qquad \begin{matrix}\mathrm{C}_{n-1} & D_n & \\ \end{matrix} \begin{pmatrix} -c_{n-1} & 0 \\ \varphi_{n-1} & d_n \end{pmatrix} \begin{matrix}\mathrm{C}_{n-2}\\ D_{n-1}\end{matrix}$$

[984] Recall that a *comfortable cover of* X is a family $\mathcal{U} = \{U_i\}_{i\in I}$ of subsets of X such that the union of the interiors of the U_i equals X. On page 1638 we defined
$$\mathrm{C}_n^{\mathcal{U}}(X) \;=\; \Big\{ \sum_{j=1}^{k} a_j \sigma_j \,\Big|\, \text{for each } j \text{ there exists a } U_i \in \mathcal{U} \text{ such that the image of } \sigma_j \text{ lies in } U_i \Big\}$$
and we had observed that these groups form a subcomplex of $\mathrm{C}_*(X)$.

[985] Note that
$$\partial_{n-1} \circ \partial_n \;=\; \begin{pmatrix} -c_{n-2} & 0 \\ \varphi_{n-2} & d_{n-1} \end{pmatrix} \begin{pmatrix} -c_{n-1} & 0 \\ \varphi_{n-1} & d_n \end{pmatrix} \;=\; \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
since φ is a chain map and $c_{n-2} \circ c_{n-1} = 0$ and $d_{n-1} \circ d_n = 0$

This shows that we have indeed defined a chain complex.

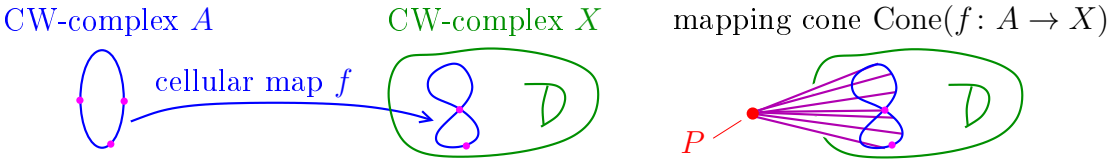
Remark. Let $f\colon A \to X$ be a map between topological spaces. Recall that on page 457 we defined the corresponding *mapping cone* as

$$\mathrm{Cone}(f\colon A \to X) \;:=\; (A \times [0,1] \sqcup X)/\!\sim$$

where $(a,1) \sim f(a)$ for all $a \in A$ and $(a,0) \sim (b,0)$ for all $a,b \in A$. As an application we showed on page 457 that for an inclusion map $i\colon A \to X$ there exists a natural isomorphism

$$\widetilde{\mathrm{H}}_n(\mathrm{Cone}(i\colon A \to X)) \;\xrightarrow{\cong}\; \mathrm{H}_n(X,A),$$

i.e. we can reinterpret relative homology groups as reduced homology groups of the corresponding mapping cone. Now suppose that A and X are finite CW-complexes and that f is a cellular map. By the CW-Complex Cone Corollary 68.35 (3) we can equip the mapping cone $\mathrm{Cone}(f\colon A \to X)$ with a natural CW-structure. In Exercise 81.2 we will show that the corresponding cellular chain complex is isomorphic to the algebraic mapping cone of the chain map $f_*\colon \mathrm{C}_*^{\mathrm{CW}}(A) \to \mathrm{C}_*^{\mathrm{CW}}(X)$.



Lemma 81.8. (Mapping Cone-LES Lemma) Let $\varphi\colon C_* \to D_*$ be a chain map between two chain complexes (C_n, c_n) and (D_n, d_n). We denote by $i\colon D_n \to \mathrm{M}(\varphi)_n = \mathrm{C}_{n-1} \oplus D_n$ the obvious natural inclusion map and we denote by $p\colon \mathrm{M}(\varphi)_n = \mathrm{C}_{n-1} \oplus D_n \to \mathrm{C}_{n-1}$ the obvious natural projection . Then the following sequence is exact

$$\ldots \;\to\; \mathrm{H}_n(C_*) \xrightarrow{\varphi_*} \mathrm{H}_n(D_*) \xrightarrow{i_*} \mathrm{H}_n(\mathrm{M}(\varphi)) \xrightarrow{p_*} \mathrm{H}_{n-1}(C_*) \xrightarrow{\varphi_*} \mathrm{H}_{n-1}(D_*) \;\to\; \ldots$$

Proof. For each $n \in \mathbb{N}_0$ we consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D_n & \xrightarrow{i_n} & \begin{matrix}\mathrm{C}_{n-1}\\ D_n\end{matrix} & \xrightarrow{p_n} & \mathrm{C}_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle \partial_n}\ \begin{pmatrix} -c_{n-1} & 0 \\ \varphi_{n-1} & d_n \end{pmatrix} & & \downarrow{\scriptstyle c_{n-1}} & & \\
0 & \longrightarrow & D_{n-1} & \xrightarrow[i_{n-1}]{} & \begin{matrix}\mathrm{C}_{n-2}\\ D_{n-1}\end{matrix} & \xrightarrow[p_{n-1}]{} & \mathrm{C}_{n-2} & \longrightarrow & 0.
\end{array}$$

We make the following observations:

(1) the horizontal sequences are exact,
(2) the left-hand square commutes, i.e. the maps i_* define a chain map,
(3) the right-hand square commutes up to a minus sign, i.e. we have

$$c_{n-1} \circ p_n \;=\; -p_{n-1} \circ \partial_n.$$

Fortunately this minus sign in the above equality does not change the fact that the maps p_n induce a map between homology groups.

Despite the minus sign in (3) we can use verbatim the same construction as on page 1620 to obtain connecting homomorphisms $\partial_n\colon \mathrm{H}_n(D_*) \to \mathrm{H}_n(C_*)$, $n \in \mathbb{N}_0$, and verbatim the

same proof as in the LES Proposition 74.9 shows that these connecting homomorphisms give rise to a long exact sequence of the form

$$\ldots \to \mathrm{H}_n(C_*) \xrightarrow{\partial_n} \mathrm{H}_n(D_*) \xrightarrow{i_*} \mathrm{H}_n(\mathrm{M}(\varphi)) \xrightarrow{p_*} \mathrm{H}_{n-1}(C_*) \xrightarrow{\partial_{n-1}} \mathrm{H}_{n-1}(D_*) \to \ldots$$

It follows easily from the definition of the connecting homomorphism on page 1620 that for each $n \in \mathbb{N}_0$ we have the equality

$$\partial_n = \varphi_* \colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*).$$

We have thus now shown that the sequence in the statement of the lemma is exact. ■

Now we are finally fully prepared to provide the proof of the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 81.4.

Proof of Proposition 81.4. Let $\varphi \colon C_* \to D_*$ be a chain map between chain complexes (C_*, c_*) and (D_*, d_*) of free abelian groups which induces an isomorphism on all homology groups. We need to show that φ is in fact a chain homotopy equivalence.

As above we denote by $(\mathrm{M}(\varphi)_*, \partial_*)$ the mapping cone of φ. We consider the long exact sequence from the Mapping Cone-LES Lemma 81.8:

$$\ldots \to \mathrm{H}_n(C_*) \xrightarrow{\varphi_*} \mathrm{H}_n(D_*) \xrightarrow{i_*} \mathrm{H}_n(\mathrm{M}(\varphi)) \xrightarrow{p_*} \mathrm{H}_{n-1}(C_*) \xrightarrow{\varphi_*} \mathrm{H}_{n-1}(D_*) \to \ldots$$

By our hypothesis the maps φ induce isomorphisms of homology groups, thus we see that $\mathrm{H}_n(\mathrm{M}(\varphi)) = 0$ for all $n \geq 0$. By Corollary 81.3 this means that we have a chain homotopy $\{P_n\}_{n\geq 0}$ from the identity map to the zero map. (Note that here we use that by our hypothesis on C_* and D_* each $\mathrm{M}(\varphi)_n = \mathrm{C}_{n-1} \oplus D_n$ is a free abelian group.)

Recall that this means that for each $n \in \mathbb{N}_0$ we have the equality

$$(*) \qquad P_{n-1} \circ \partial_n + \partial_{n+1} \circ P_n = \mathrm{id}_{\mathrm{C}_{n-1} \oplus D_n}.$$

Using the obvious matrix notation we write

$$P_n = \begin{pmatrix} P_n^{11} & P_n^{12} \\ P_n^{21} & P_n^{22} \end{pmatrix} \colon \underbrace{\begin{matrix} \mathrm{C}_{n-1} \\ D_n \end{matrix}}_{=\mathrm{M}(\varphi)_n} \to \underbrace{\begin{matrix} \mathrm{C}_n \\ D_{n+1}. \end{matrix}}_{=\mathrm{M}(\varphi)_{n+1}}$$

Now we can rewrite the above equality $(*)$ and we obtain that

$$\begin{pmatrix} P_{n-1}^{11} & P_{n-1}^{12} \\ P_{n-1}^{21} & P_{n-1}^{22} \end{pmatrix} \circ \begin{pmatrix} -c_{n-1} & 0 \\ \varphi_{n-1} & d_n \end{pmatrix} + \begin{pmatrix} -c_n & 0 \\ \varphi_n & d_{n+1} \end{pmatrix} \circ \begin{pmatrix} P_n^{11} & P_n^{12} \\ P_n^{21} & P_n^{22} \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{\mathrm{C}_{n-1}} & 0 \\ 0 & \mathrm{id}_{D_n} \end{pmatrix}.$$

Looking at the four entries separately we obtain the following four equalities:

$$\begin{aligned}
&\text{(a)} && -P_{n-1}^{11} \circ c_{n-1} + P_{n-1}^{12} \circ \varphi_{n-1} - c_n \circ P_n^{11} &&= \mathrm{id}_{\mathrm{C}_{n-1}}, \\
&\text{(b)} && P_{n-1}^{12} \circ d_n - c_n \circ P_n^{12} &&= 0, \\
&\text{(c)} && -P_{n-1}^{21} \circ c_{n-1} + P_{n-1}^{22} \circ \varphi_{n-1} + \varphi_n \circ P_n^{11} + d_{n+1} \circ P_n^{21} &&= 0, \\
&\text{(d)} && P_{n-1}^{22} \circ d_n + \varphi_n \circ P_n^{12} + d_{n+1} \circ P_n^{22} &&= \mathrm{id}_{D_n}.
\end{aligned}$$

We make the following observation:

(1) equation (b) implies that the homomorphisms $\{P_n^{12} \colon D_n \to \mathrm{C}_n\}_{n\in\mathbb{N}_0}$ define a chain map $D \to C$.

Now we claim that the chain map $\{P_n^{12}\}_{n\in\mathbb{N}_0}$ is a chain homotopy inverse to the chain map φ. We do so with the following two observations:

(2) equation (a) implies that $P^{12}\circ\varphi \simeq \mathrm{id}_{C_*}$ where the chain homotopy is given by the maps $P_n^{11}\colon \mathrm{C}_{n-1}\to C_n$,
(3) whereas equation (d) implies that $\varphi\circ P^{12}\simeq \mathrm{id}_{D_*}$, where the chain homotopy is given by the maps $P_n^{22}\colon D_n\to D_{n+1}$,

The last two statements together imply that φ is indeed a chain homotopy equivalence. ■

We conclude this section with the proof of Proposition 81.5.

Proof of Proposition 81.5. Let C_* and D_* be two chain complexes of free abelian groups and let $\iota\colon C_*\to D_*$ be a chain map with the following two properties:

(a) The chain map $\iota\colon C_*\to D_*$ induces an isomorphism of homology groups.
(b) For each $k\in\mathbb{N}_0$ the map $\iota\colon \mathrm{C}_k\to D_k$ is a monomorphism and the image is a subsummand of D_k.

We need to show that there exists a chain map $\pi\colon D_*\to \mathrm{C}_*$ with the following two properties:

(1) For each $k\in\mathbb{N}_0$ we have $\pi_k\circ\iota_k=\mathrm{id}_{\mathrm{C}_k}$.
(2) The maps ι and π are chain homotopy homotopy equivalences.

The key to proving the proposition is the following claim:

Claim. We set $\mathrm{C}_{-1}=D_{-1}=0$ and we define π_{-1} to be the zero homomorphism. Let $k\in\mathbb{Z}_{\geq -1}$. Suppose that for $j=-1,\ldots,k$ we are given maps $\pi_j\colon D_j\to\mathrm{C}_j$ such that for each $j\in\{0,\ldots,k\}$ we have the following equalities:

(1) $\pi_j\circ\iota_j=\mathrm{id}_{\mathrm{C}_j}$.
(2) $\partial\circ\pi_{j-1}=\pi_j\circ\partial$.

Then there exists a map $\pi_{k+1}\colon D_{k+1}\to\mathrm{C}_{k+1}$ such that the maps $\pi_0,\ldots,\pi_{k+1}$ have the above properties.

Proof. First note that by hypothesis we can pick for each $m\in\mathbb{N}$ a submodule $C_m^\dagger$ of D_m such that $D_m=\iota(\mathrm{C}_m)\oplus C_m^\dagger$. For orientation it is perhaps convenient to consider the following maps:

$$\begin{array}{ccc} D_{k+1}=\iota(\mathrm{C}_{k+1})\oplus C_{k+1}^\dagger & \xrightarrow{\partial_D} & \iota(\mathrm{C}_k)\oplus C_k^\dagger = D_k \\ {\scriptstyle\iota_{k+1}}\uparrow \quad \downarrow{\scriptstyle ?\pi_{k+1}?} & & {\scriptstyle\iota_k}\uparrow \quad \downarrow{\scriptstyle\pi_k} \\ \mathrm{C}_{k+1} & \xrightarrow[\partial_C]{} & \mathrm{C}_k \end{array}$$

Since $C_{k+1}^\dagger$ is a subgroup of the free abelian group D_{k+1} we know by the Subgroup-of-Free Abelian Group Lemma 51.2 that $C_{k+1}^\dagger$ is also a free abelian group. We pick a basis T for the free abelian group $C_{k+1}^\dagger$. Let $t\in T$. Since ι induces an isomorphism of homology groups we know that there exists a $c_{k+1}\in\mathrm{C}_{k+1}$ such that t and $\iota(c_{k+1})$ are homologous. We calculate that

$$\pi_k(\partial_D(t_{k+1})) \underset{\text{since } t_{k+1} \text{ and } \iota(c_{k+1}) \text{ are homologous we know that } \partial_D(t_{k+1})=\partial_D(\iota(c_{k+1}))}{=} \pi_k(\partial_D(\iota_{k+1}(c_{k+1}))) \underset{\text{since } \iota \text{ is a chain map}}{=} \pi_k(\iota_k(\partial_C(c_{k+1}))) \underset{\text{since } \pi_k\circ\iota_k=\mathrm{id}_{\mathrm{C}_k}}{=} \partial_C(c_{k+1}).$$

We set $\pi_{k+1}(t) := c_{k+1}$. Since T_{k+1} is a basis for the free abelian group $C_{k+1}^\dagger$ we obtain a homomorphism $\varphi\colon C_{k+1}^\dagger \to \mathrm{C}_{k+1}$. It is now straightforward to verify that the map

$$\begin{array}{rccc} \pi_{k+1}\colon D_{k+1} = \iota(\mathrm{C}_{k+1}) \oplus C_{k+1}^\dagger & \to & \mathrm{C}_{k+1} \\ \iota(c_{k+1}) + d_{k+1} & \mapsto & c_{k+1} + \varphi(d_{k+1}) \end{array}$$

has all the desired properties. ⊞

Since C_* and D_* are chain complexes of free abelian groups and since ι induces an isomorphism of homology groups we know by the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 81.4 that ι is a chain homotopy equivalence. By Property (2) of the claim the maps $\{\pi_n\colon D_n \to \mathrm{C}_n\}_{n\in\mathbb{N}_0}$ define a chain map. By Property (1) of the claim we have for each $k \in \mathbb{N}_0$ that $\pi_* \circ \iota_* = \mathrm{id}_{\mathrm{H}_k(C_*)}$. Thus we see that the chain map π_* also induces isomorphism of the homology groups. So once again we appeal to the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 81.4 to conclude that the maps π_* form a chain homotopy equivalence. ■

81.4. **Singular and cellular chain complexes.** In Chapter 80 we introduced the cellular chain complex and the cellular homology groups of a CW-complex. In the Singular-Cellular H_*-Isomorphism Proposition 80.4 we showed that cellular homology groups are naturally isomorphic to singular homology groups. At that point we did not say anything about the relationship between the cellular chain complex and the singular chain complex. We will rectify this in this section. The extra information that we gain in this section will stand us in good stead in Section 109.2 when we explore the relationship between cellular cohomology and singular cohomology.

First we recall some definitions and some notation, mostly from page 1770.

Definition. Let X be a CW-complex and let $n \in \mathbb{N}_0$.

(1) We denote by X^n the n-skeleton of X, i.e. the union of all cells of dimension $\leq n$.

(2) We write $\mathrm{C}_n^{\mathrm{CW}}(X) := \mathrm{H}_n(X^n, X^{n-1})$ and we denote by $d_n\colon \mathrm{C}_n^{\mathrm{CW}}(X) \to \mathrm{C}_{n-1}^{\mathrm{CW}}(X)$ the cellular boundary map from page 1769, i.e. d_n is the map that is given by the composition of the following two maps:

$$\begin{array}{ccccc} \mathrm{H}_n(X^n, X^{n-1}) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(X^{n-1}) & \to & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}). \\ & \uparrow & & & \\ & \text{connecting homomorphism of the long exact sequence of the pair } (X^n, X^{n-1}) & & & \end{array}$$

(3) We refer to

$$\ldots \xrightarrow{d_{n+2}} \mathrm{C}_{n+1}^{\mathrm{CW}}(X) \xrightarrow{d_{n+1}} \mathrm{C}_n^{\mathrm{CW}}(X) \xrightarrow{d_n} \mathrm{C}_{n-1}^{\mathrm{CW}}(X) \xrightarrow{d_{n-1}} \ldots$$

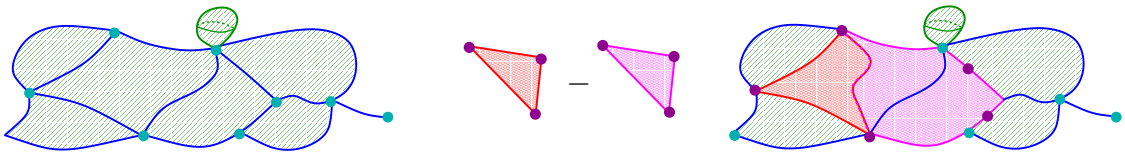
as the cellular chain complex. Its homology groups are denoted by $\mathrm{H}_n^{\mathrm{CW}}(X)$.

In the following we want to relate the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(X)$ to the singular chain complex $\mathrm{C}_*(X)$. There is no obvious "interesting" chain map between these two chain complexes. But as we will see in a minute, it turns out that there is a nice "intermediate" chain complex which maps into both of the above chain complexes:

Definition. Given a CW-complex X and given $n \in \mathbb{N}_0$ we define the intermediate cellular chain complex[986]

$$\mathrm{C}_n^{\mathrm{int}}(X) := \ker\left(\mathrm{C}_n(X^n) \xrightarrow{\partial} \mathrm{C}_{n-1}(X^n) \xrightarrow{\pi} \mathrm{C}_{n-1}(X^n, X^{n-1})\right)$$

where $\pi\colon \mathrm{C}_{n-1}(X^n) \to \mathrm{C}_{n-1}(X^n, X^{n-1})$ is the natural projection.



CW-complex X element of $\mathrm{C}_2^{\mathrm{int}}(X)$

Lemma 81.9.

(1) Given any CW-complex X the intermediate cellular chain complex $\mathrm{C}_*^{\mathrm{int}}(X)$ is a subcomplex of the singular chain complex $\mathrm{C}_*(X)$.
(2) If $f\colon X \to Y$ is a cellular map between two CW-complexes, then the induced map $f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ restricts to a map $\mathrm{C}_*^{\mathrm{int}}(X) \to \mathrm{C}_*^{\mathrm{int}}(Y)$.

Proof.

(1) First note that by definition we have for any $k \in \mathbb{N}_0$ the equality

$$(*) \qquad \mathrm{C}_k^{\mathrm{int}}(X) = \{\sigma \in \mathrm{C}_k(X^k) \,|\, \partial_k(\sigma) \in \mathrm{C}_{k-1}(X^{k-1})\}.$$

Now let $\sigma \in \mathrm{C}_n^{\mathrm{int}}(X)$. We need to show that $\partial_n(\sigma) \in \mathrm{C}_{n-1}^{\mathrm{int}}(X)$.
(a) By $(*)$ we know $\partial_n(\sigma) \in \mathrm{C}_{n-1}(X^{n-1})$.
(b) Next note that $\partial_{n-1}(\partial_n(\sigma)) = (\partial_{n-1} \circ \partial_n)(\sigma) = 0$. In particular we obtain that $\partial_{n-1}(\partial_n(\sigma)) \in \mathrm{C}_{n-2}(X^{n-2})$.
(c) The combination of (a), (b) and $(*)$ shows that $\partial_n(\sigma) \in \mathrm{C}_{n-1}^{\mathrm{int}}(X)$.

(2) Let $f\colon X \to Y$ be a cellular map between two CW-complexes. We obtain the following diagram:

$$\begin{array}{ccccc}
\mathrm{C}_n(X^n) & \xrightarrow{\partial} & \mathrm{C}_{n-1}(X^n) & \xrightarrow{\pi} & \mathrm{C}_{n-1}(X^n, X^{n-1}) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\mathrm{C}_n(Y^n) & \xrightarrow{\partial} & \mathrm{C}_{n-1}(Y^n) & \xrightarrow{\pi} & \mathrm{C}_{n-1}(Y^n, Y^{n-1}).
\end{array}$$

It follows from Lemma 72.9 that this diagram commutes. It follows immediately that $f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ restricts to a map $\mathrm{C}_*^{\mathrm{int}}(X) \to \mathrm{C}_*^{\mathrm{int}}(Y)$. ■

Lemma 81.10. Let X be a CW-complex. The maps

$$\begin{array}{rcl} p\colon \mathrm{C}_*^{\mathrm{int}}(X) & \to & \mathrm{C}_*^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1}) \\ \sigma & \mapsto & [\pi(\sigma)] \end{array}$$

are well-defined, in the sense that $\pi(\sigma)]$ is a cycle in $\mathrm{C}_*^{\mathrm{CW}}(X)$, and these maps form a chain map.

[986] Strictly speaking we only show in Lemma 81.9 that this is indeed a chain complex.

Proof. By definition of the boundary map of a relative chain complex we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{C}_n(X^n) & \xrightarrow{\quad\pi\quad} & \mathrm{C}_n(X^n, X^{n-1}) \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial} \\ \mathrm{C}_{n-1}(X^n) & \xrightarrow{\quad\pi\quad} & \mathrm{C}_{n-1}(X^n, X^{n-1}). \end{array}$$

The statements of the lemma follow easily from this observation. ■

The following proposition gives us in particular a new proof of the main statements of the Singular-Cellular H_*-Isomorphism Proposition 80.4.

Proposition 81.11. (Intermediate Chain Complex Proposition) Let X be a CW-complex.

- We denote by $i\colon \mathrm{C}_*^{\mathrm{int}}(X) \to \mathrm{C}_*(X)$ the natural inclusion map.
- We denote by $p\colon \mathrm{C}_*^{\mathrm{int}}(X) \to \mathrm{C}_*^{\mathrm{CW}}(X)$ the natural chain map from Lemma 81.10.

For each $n \in \mathbb{N}_0$ the natural maps

$$(1)\ \ i_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{int}}(X)) \to \underbrace{\mathrm{H}_n(\mathrm{C}_*(X))}_{=\mathrm{H}_n(X)} \quad \text{and} \quad (2)\ \ p_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{int}}(X)) \to \underbrace{\mathrm{H}_n(\mathrm{C}_*^{\mathrm{CW}}(X))}_{=\mathrm{H}_n^{\mathrm{CW}}(X)}$$

are isomorphisms.

Remark. The proof of the proposition is modelled on the proofs provided in [**SchubH1968**, p. 303] or alternatively [**Lü1998**, Lemma 4.2] and [**Lü1989**, p. 263], see also [**Wall1966a**, Lemma 1].

Proof of Proposition 81.11 (1). Let X be a CW-complex and let $n \in \mathbb{N}_0$. We need to show that the map

$$i_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{int}}(X)) \to \underbrace{\mathrm{H}_n(\mathrm{C}_*(X))}_{=\mathrm{H}_n(X)}$$

is an isomorphism. Before we start, let us recall that in the Skeleton-Homology Lemma 80.1 we proved the following two convenient facts:

(a) The inclusion $k\colon X^{n+1} \to X$ induces an isomorphism $k_*\colon \mathrm{H}_n(X^{n+1}) \xrightarrow{\cong} \mathrm{H}_n(X)$.
(b) The inclusion $k\colon X^n \to X$ induces an epimorphism $k_*\colon \mathrm{H}_n(X^n) \twoheadrightarrow \mathrm{H}_n(X)$.

Claim. Let $n \in \mathbb{N}_0$.
(i) The map $i_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{int}}(X)) \to \mathrm{H}_n(\mathrm{C}_*(X)) = \mathrm{H}_n(X)$ is a monomorphism.
(ii) The map $i_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{int}}(X)) \to \mathrm{H}_n(\mathrm{C}_*(X)) = \mathrm{H}_n(X)$ is an epimorphism.

Proof. We consider the following two commutative diagrams

$$\begin{array}{ccccc} \mathrm{C}_{n+1}^{\mathrm{int}}(X) & \overset{j}{\hookrightarrow} & \mathrm{C}_{n+1}(X^{n+1}) & \overset{k}{\hookrightarrow} & \mathrm{C}_{n+1}(X) \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} \\ \mathrm{C}_n^{\mathrm{int}}(X) & \overset{j}{\hookrightarrow} & \mathrm{C}_n(X^{n+1}) & \overset{k}{\hookrightarrow} & \mathrm{C}_n(X) \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} & & \overset{i}{\frown} & & \\ \mathrm{C}_n^{\mathrm{int}}(X) & \overset{j}{\hookrightarrow} & \mathrm{C}_n(X^n) & \overset{k}{\hookrightarrow} & \mathrm{C}_n(X) \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} \\ \mathrm{C}_{n-1}^{\mathrm{int}}(X) & \overset{j}{\hookrightarrow} & \mathrm{C}_{n-1}(X^n) & \overset{k}{\hookrightarrow} & \mathrm{C}_{n-1}(X). \end{array}$$

Here in each diagram the maps to the left are the obvious inclusion maps, for better or worse we refer to all of them as j. The maps to the right are induced by the inclusions $k\colon X^{n+1} \to X$ and $k\colon X^n \to X$. As we will now see, both statements of the claim follow easily from the diagrams together with (a) and (b) above:

(i) We consider the diagram to the left. Let $\sigma \in \mathrm{C}_n^{\mathrm{int}}(X)$ be a cycle with the property that $i(\sigma) \in \mathrm{C}_n(X)$ is null-homologous. It follows from (a) that $j(\sigma) \in \mathrm{C}_n(X^{n+1})$ is already null-homologous, i.e. there exists a $\tau \in \mathrm{C}_{n+1}(X^{n+1})$ with $\partial\tau = j(\sigma)$. By definition of $\mathrm{C}_n^{\mathrm{int}}(X)$ we know that $\sigma \in \mathrm{C}_n(X^n) \subset \mathrm{C}_n(X^{n+1})$. Thus we see that τ actually lies in

$$\mathrm{C}_{n+1}^{\mathrm{int}}(X) \;=\; \ker\big(\mathrm{C}_{n+1}(X^{n+1}) \xrightarrow{\partial} \mathrm{C}_n(X^{n+1}) \xrightarrow{\pi} \mathrm{C}_n(X^{n+1}, X^n)\big).$$

But this implies that σ is null-homologous in $\mathrm{C}_*^{\mathrm{int}}(X)$.

(ii) We consider the diagram to the right. Let $[\sigma] \in \mathrm{H}_n(X)$. By (b) we know that there exists a cycle $\tau \in \mathrm{C}_n(X^n)$ with $[k(\tau)] = [\sigma]$. Since $\partial\tau = 0$ we see that $\tau \in \mathrm{C}_n^{\mathrm{int}}(X)$. Since the bottom-left horizontal map is a monomorphism we see that $\tau \in \mathrm{C}_n^{\mathrm{int}}(X)$ is also a cycle. But this shows that $[\sigma] = [i(\tau)]$. ■

Proof of Proposition 81.11 (2). Let X be a CW-complex and let $n \in \mathbb{N}_0$. We need to show that the map

$$\begin{array}{rccc} p_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{int}}(X)) & \to & \mathrm{H}_n(\mathrm{C}_*^{\mathrm{CW}}(X)) \\ [\sigma] & \mapsto & [\pi(\sigma)] \end{array}$$

is an isomorphism. For orientation purposes we show the following simple commutative diagram:

$$\begin{array}{ccc} \mathrm{C}_n(X^n) & \xrightarrow{\quad\pi\quad} & \mathrm{C}_n(X^n, X^{n-1}) \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} \\ \mathrm{C}_{n-1}(X^n) & \xrightarrow{\quad\pi\quad} & \mathrm{C}_{n-1}(X^n, X^{n-1}). \end{array}$$

Furthermore it is convenient to recall the following definitions:

$$\begin{array}{rcl} \mathrm{C}_n^{\mathrm{int}}(X) & = & \ker(\pi \circ \partial\colon \mathrm{C}_n(X^n) \to \mathrm{C}_{n-1}(X^n, X^{n-1})) \\ \mathrm{C}_n^{\mathrm{CW}}(X) & = & \mathrm{H}_n(X^n, X^{n-1}) = \mathrm{H}_n(C_*(X^n, X^{n-1})). \end{array}$$

Claim 1. For each $n \in \mathbb{N}_0$ the map $p\colon \mathrm{C}_n^{\mathrm{int}}(X) \to \mathrm{C}_n^{\mathrm{CW}}(X)$ is an epimorphism.

Proof. Let $[\sigma] \in \mathrm{C}_n^{\mathrm{CW}}(X)$, i.e. let $\sigma \in \mathrm{C}_n^{\mathrm{CW}}(X)$ such that $\partial\sigma = 0$. Since the projection $\pi\colon \mathrm{C}_n(X^n) \to \mathrm{C}_n(X^n, X^{n-1})$ is an epimorphism there exists $\nu \in \mathrm{C}_n(X^n)$ with $\partial\nu = \sigma$. Using the above commutative diagram one sees easily that $\nu \in \mathrm{C}_n^{\mathrm{int}}(X)$. ⊞

Given $n \in \mathbb{N}_0$ we set

$$D_n \;:=\; \ker\big\{p\colon \mathrm{C}_n^{\mathrm{int}}(X) \xrightarrow{p} \mathrm{C}_n^{\mathrm{CW}}(X)\big\}.$$

Note that $\nu \in \mathrm{C}_n(X^n)$ lies in D_n if and only if the following two conditions are satisfied:

(a) $(\pi \circ \partial)(\nu) = 0$,

(b) $\pi(\nu) \in \mathrm{C}_n(X^n, X^{n-1})$ represents the zero element of $\mathrm{C}_n^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1})$.

Furthermore note that the groups D_* form a subcomplex of $\mathrm{C}_*^{\mathrm{int}}(X)$.

Claim 2. For each $n \in \mathbb{N}_0$ we have $\mathrm{C}_{n+1}(X^n) \subset D_{n+1}$.

Proof. For orientation purposes we consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{C}_{n+1}(X^n) & \hookrightarrow & \mathrm{C}_{n+1}(X^{n+1}) & \twoheadrightarrow^{\pi} & \mathrm{C}_{n+1}(X^{n+1}, X^n) \\
\downarrow\partial & & \downarrow\partial & & \downarrow\partial \\
\mathrm{C}_n(X^n) & \hookrightarrow & \mathrm{C}_n(X^{n+1}) & \twoheadrightarrow^{\pi} & \mathrm{C}_n(X^{n+1}, X^n).
\end{array}$$

Let $\Theta \in \mathrm{C}_n(X^{n+1})$. Since the diagram commutes and since the horizontal sequences are exact we see that Θ satisfies (a) and we see that $\pi(\Theta) = 0 \in \mathrm{C}_{n+1}(X^{n+1}, X^n)$, the latter statement implies that (b) is also satisfied. ⊞

Note that by Claim 1 we have a short exact sequence

$$0 \;\to\; D_* \;\to\; \mathrm{C}_n^{\mathrm{int}}(X) \xrightarrow{p} \mathrm{C}_n^{\mathrm{CW}}(X) \;\to\; 0$$

of chain complexes. Considering the corresponding long exact sequence of homology groups, see the LES Proposition 74.9, it suffices to prove the following claim.

Claim 3. The chain complex D_* is acyclic, i.e. all homology groups of D_* are zero.

Proof. For orientation purposes we now consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{C}_{n+1}(X^{n-1}) & \overset{i}{\hookrightarrow} & \mathrm{C}_{n+1}(X^n) & \overset{\pi}{\twoheadrightarrow} & \mathrm{C}_{n+1}(X^n, X^{n-1}) \\
\downarrow\partial & & \downarrow\partial & & \downarrow\partial \\
\mathrm{C}_n(X^{n-1}) & \overset{i}{\hookrightarrow} & \mathrm{C}_n(X^n) & \overset{\pi}{\twoheadrightarrow} & \mathrm{C}_n(X^n, X^{n-1}) \\
\downarrow\partial & & \downarrow\partial & & \downarrow\partial \\
\mathrm{C}_{n-1}(X^{n-1}) & \overset{i}{\hookrightarrow} & \mathrm{C}_{n-1}(X^n) & \overset{\pi}{\twoheadrightarrow} & \mathrm{C}_{n-1}(X^n, X^{n-1}).
\end{array}$$

So let $\mu \in \ker(\partial\colon D_n \to D_{n-1}) \subset D_n \subset \mathrm{C}_n^{\mathrm{int}}(X) \subset \mathrm{C}_n(X)$. We need to show that there exists a $\Theta \in D_{n+1}$ with $\partial\Theta = \mu$. By (b) we know that $\pi(\mu) \in \mathrm{C}_n(X^n, X^{n-1})$ represents the zero element of $\mathrm{C}_n^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1})$. It follows that there exists a $\gamma \in \mathrm{C}_{n+1}(X^n, X^{n-1})$ with $\partial\gamma = \pi(\mu)$. Since the top horizontal sequence is exact we can pick $\widetilde{\gamma} \in \mathrm{C}_{n+1}(X^n)$ with $\pi(\widetilde{\gamma}) = \gamma$. Note that

$$\pi(\partial\widetilde{\gamma} - \mu) \underset{\substack{\uparrow \\ \text{since } \partial\circ\pi = \pi\circ\partial}}{=} (\pi\circ\partial)(\widetilde{\gamma}) - \pi(\mu) = \underbrace{\partial(\pi(\widetilde{\gamma}))}_{=\gamma} - \pi(\mu) \underset{\substack{\uparrow \\ \text{since } \partial\gamma = \pi(\mu)}}{=} 0.$$

Since the middle horizontal sequence is exact there exists an $\alpha \in \mathrm{C}_n(X^{n-1})$ such that $i(\alpha) = \partial\widetilde{\gamma} - \mu$. Note that $i(\partial\alpha) = \partial(i(\alpha)) = i(\partial(\partial\widetilde{\gamma} - \mu)) = (\partial\partial\widetilde{\gamma} - \partial\mu) = 0$. Since i is a monomorphism we see that $\partial\alpha = 0$. As mentioned above, by the Skeleton-Homology Lemma 80.1 we know that $\mathrm{H}_n(X^{n-1}) = 0$. This implies that there exists an $\widetilde{\alpha} \in \mathrm{C}_{n+1}(X^{n-1})$ with $\partial\widetilde{\alpha} = \alpha$. Then

$$\partial(\widetilde{\gamma} - i(\widetilde{\alpha})) \underset{\substack{\uparrow \\ \text{since } \partial\circ i = i\circ\partial}}{=} \partial\widetilde{\gamma} - (i\circ\partial)(\widetilde{\alpha}) = \partial\widetilde{\gamma} - \underbrace{i(\alpha)}_{=\partial\widetilde{\gamma}-\mu} = \partial\widetilde{\gamma} - \partial\widetilde{\gamma} + \mu = \mu.$$

We set $\Theta := \widetilde{\gamma} - i(\widetilde{\alpha}) \in \mathrm{C}^{n+1}(X^n)$. We just showed that $\partial\Theta = \mu$. Since $\Theta \in \mathrm{C}^{n+1}(X^n)$ we obtain from Claim 2 that Θ actually lies in D_{n+1}. ■

The following proposition gives us the promised relationship between the singular and cellular chain complex of a given CW-complex.

Proposition 81.12. (Singular-Cellular Chain Complex Proposition) Let X be a CW-complex.

(1) The two natural chain maps
$$i\colon \mathrm{C}_*^{\mathrm{int}}(X) \hookrightarrow \mathrm{C}_*(X) \quad \text{and} \quad p\colon \mathrm{C}_*^{\mathrm{int}}(X) \twoheadrightarrow \mathrm{C}_*^{\mathrm{CW}}(X),$$
which are given by the obvious inclusion and the natural projection, are chain homotopy equivalences.

(2) There exists a chain homotopy equivalence $\mathrm{C}_*(X) \to \mathrm{C}_*^{\mathrm{CW}}(X)$.

Remark. In the proof of the Singular-Cellular Chain Complex Proposition 81.12 we will make use of the Subgroup-of-Free Abelian Group Lemma 51.2 (which we did not prove) which says that the subgroup of a free abelian group is again a free abelian group. This purely algebraic fact lets us conclude that the intermediate chain groups $\mathrm{C}_k^{\mathrm{int}}(X)$ are free abelian groups. In Section 170.2 we will use a somewhat different approach to relate the chain complexes $\mathrm{C}_*(X)$ and $\mathrm{C}_*^{\mathrm{CW}}(X)$ that does not make use of this algebraic fact.

Proof. Let X be a CW-complex.

(1) By definition each $\mathrm{C}_k^{\mathrm{int}}(X)$ is a subgroup of the free abelian group $\mathrm{C}_k(X)$. It follows from the Subgroup-of-Free Abelian Group Lemma 51.2 that each $\mathrm{C}_k^{\mathrm{int}}(X)$ itself is a free abelian group. In particular $\mathrm{C}_*^{\mathrm{int}}(X)$ is a chain complex of free abelian groups. By the Intermediate Chain Complex Proposition 81.11 we know that the chain maps $i\colon \mathrm{C}_*^{\mathrm{int}}(X) \hookrightarrow \mathrm{C}_*(X)$ and $p\colon \mathrm{C}_*^{\mathrm{int}}(X) \twoheadrightarrow \mathrm{C}_*^{\mathrm{CW}}(X)$ are quasi-isomorphisms. Since all chain complexes involved consist of free abelian groups we deduce from the Intermediate Chain Complex Proposition 81.11 that the chain maps are chain homotopy equivalences.

(2) This statement follows immediately from Statement (1) together with the fact, see Lemma 73.3, that being chain homotopy equivalent is indeed an equivalence relation. ■

81.5. **Abstractly isomorphic homology groups.** The following map shows that if we are given two chain complexes of free abelian groups, then any map on homology groups is induced by a chain map.

Proposition 81.13. (Lift-to-Chain Map-Proposition) Suppose $(\mathrm{C}_*, \partial_*)$ and (C'_*, ∂'_*) are two chain complexes of free abelian groups.

(1) If $\gamma_n\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(C'_*)$, $n \in \mathbb{N}_0$, is a family of homomorphisms, then there exists a chain map $f\colon C_* \to C'_*$ such that for every $n \in \mathbb{N}_0$ we have
$$f_* = \gamma_n\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(C'_*).$$

(2) If the maps γ_n in (1) are isomorphisms, then the chain maps f in (1) are chain homotopy equivalences.

Remark.

(1) We consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle n\mapsto(0,n)} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow[(m,n)\mapsto m+2n]{} & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

We view the top and bottom sequences as chain complexes C_* respectively D_* and we view the vertical maps as a chain map f. It is straightforward to see that f induces the zero map on homology but that it is not chain homotopic to the zero chain map. This shows that the chain map from the Lift-to-Chain Map-Proposition 81.13 is in general *not* unique up to chain homotopy.

(2) The combination of the Lift-to-Chain Map-Proposition 81.13 with the Singular-Cellular H_*-Isomorphism Proposition 80.4 gives us a new proof of the Singular-Cellular Chain Complex Proposition 81.12 (2).

Proof of the Lift-to-Chain Map-Proposition 81.13. Note that Statement (2) of the proposition is an immediate consequence of Statement (1), the fact that we are dealing with chain complexes of free abelian groups and the Intermediate Chain Complex Proposition 81.11. Thus it remains to prove Statement (1).

Let $(\mathrm{C}_*, \partial_*)$ and (C'_*, ∂'_*) be two chain complexes of free abelian groups. Furthermore let $\gamma_n\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(C'_*)$, $n \in \mathbb{N}_0$ be a family of homomorphisms. We have to show that there exists a chain map $f\colon C_* \to C'_*$ such that for every $n \in \mathbb{N}_0$ we have

$$f_* = \gamma_n\colon \mathrm{H}_n(C_*) \ \to \ \mathrm{H}_n(C'_*).$$

We start out by recalling some of the usual notation and basic observations regarding chain complexes. More precisely, for each $n \in \mathbb{N}_0$ we write

$$Z_n := \ker(\partial_n\colon \mathrm{C}_n \to \mathrm{C}_{n-1}), \quad B_n := \mathrm{im}(\partial_{n+1}\colon \mathrm{C}_{n+1} \to \mathrm{C}_n) \quad \text{and} \quad H_n = Z_n/B_n.$$

Recall that these groups form the following two types of short exact sequences

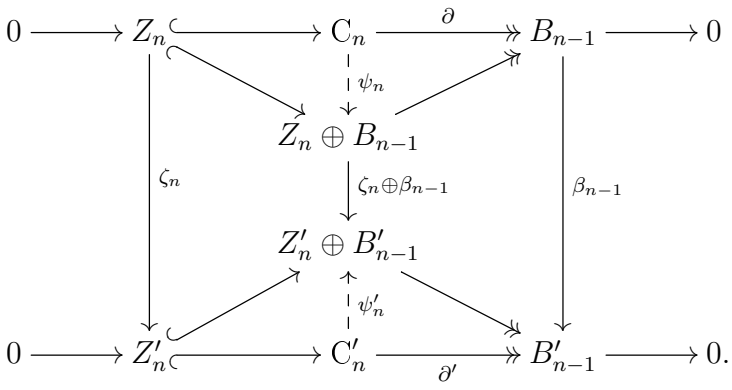
$$\begin{array}{lccccccccc} \text{(a)} & 0 & \to & Z_n & \hookrightarrow & \mathrm{C}_n & \xrightarrow{\partial_n} & B_{n-1} & \to & 0, \\ \text{(b)} & 0 & \to & B_n & \hookrightarrow & Z_n & \to & H_n & \to & 0. \end{array}$$

Similarly we define Z'_n, B'_n and H'_n for the chain complex C'_*. Evidently these groups also form short exact sequences that we denote by (a′) and (b′).

As we already mentioned on page 1791, since all the chain groups are free abelian it follows from the Subgroup-of-Free Abelian Group Lemma 51.2 that the subgroups Z_n, B_n, Z'_n and B'_n are also free abelian. In particular the sequences (b) and (b′) are free resolutions of H_n respectively H'_n. It follows from the Fundamental Theorem 81.1 of Homological Algebra that for each $n \in \mathbb{N}_0$ there exist homomorphisms β_n and ζ_n such that the following diagram commutes:

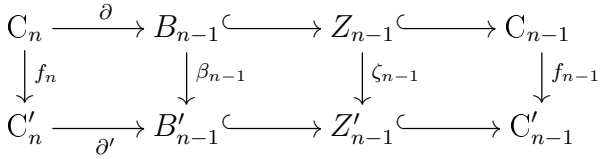
$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_n & \longrightarrow & Z_n & \longrightarrow & H_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \zeta_n} & & \downarrow{\scriptstyle \gamma_n} & & \\ 0 & \longrightarrow & B'_n & \longrightarrow & Z'_n & \longrightarrow & H'_n & \longrightarrow & 0. \end{array}$$

Our goal is to extend the maps $\zeta_n\colon Z_n \to Z'_n$ to a chain map $f\colon \mathrm{C}_n \to \mathrm{C}'_n$. To do so we consider the following diagram



Here the top sequence is precisely the short exact sequence (a) and the bottom sequence is the short exact sequence (a′). Here the unlabeled maps are the obvious inclusion and projection s. It follows immediately from Lemmas 78.1 and 78.2 that there exist isomorphisms $\psi_n\colon \mathrm{C}_n \xrightarrow{\cong} Z_n \oplus B_{n-1}$ and $\psi'_n\colon \mathrm{C}'_n \xrightarrow{\cong} Z'_n \oplus B'_{n-1}$ which make the diagram commute. Now we define $f_n\colon \mathrm{C}_n \to \mathrm{C}'_n$ to be vertical homomorphism given by the diagram, i.e. $f_n = (\psi'_n)^{-1} \circ (\zeta_n \oplus \beta_{n-1}) \circ \psi_n$.

Now we need to show that the maps f_n form a chain map. We consider the diagram



where all unlabeled maps are the obvious inclusion maps. The middle square commutes by the choice of β_{n-1} and ζ_{n-1}. We had arranged the maps f_n, f_{n-1} in such a way that the squares to the left and right commute. This shows that the rectangle commutes. But that was exactly what we needed to do to show that the maps f_n, $n \in \mathbb{N}_0$ form a chain map.

Finally note that it follows easily from the choice of ζ_n that the chain map f_n induces the given maps γ_n on homology. ■

We conclude this chapter with the following lemma which we will use on several occasions.

Lemma 81.14. (Finite-Rank Chain Complex Lemma) Let (C_*, c_*) be a free chain complex such that for each $k \in \mathbb{N}_0$ the homology group $\mathrm{H}_k(C)$ is finitely generated. Then there exists a chain complex (D_*, ∂_*) with the following two properties:

(1) The chain complexes $(\mathrm{C}_*, \partial_*)$ and (D_*, ∂_*) are chain homotopy equivalent.
(2) Each chain group D_k is a free abelian group of *finite rank*.

Proof. Let $k \in \mathbb{N}_0$. Recall that we assume that $\mathrm{H}_k(C_*)$ is finitely generated. It follows immediately from the Finitely Generated Abelian Group Classification Theorem 51.4 that there exists a chain complex D^k_* and an isomorphism $\varphi_k\colon \mathrm{H}_k(D^k_*) \to \mathrm{H}_k(C_*)$ such that only the chain groups D^k_k and D^k_{k+1} are non-zero and such that both chain groups are finitely generated free abelian groups.

We define $D_* = \bigoplus_{k\in\mathbb{N}_0} D_*^k$. It follows immediately from the Lift-to-Chain Map-Proposition 81.13 and 81.4 that the chain complex D_* is chain homotopy equivalent to (C_*, c_*). ■

Exercises for Chapter 81.

Exercise 81.1.

(a) We consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle n\mapsto(0,n)} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow[(m,n)\mapsto m+2n]{} & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

We view the top and bottom sequences as chain complexes C_* respectively D_* and we view the vertical maps as a chain map f.

(i) Convince yourself that f induces the zero map on homology.

(ii) Show that the chain map f is not chain homotopic to the zero chain map.

(b) Give examples of chain complexes C_* and D_* of free abelian groups and quasi-isomorphisms $f_*, g_*\colon C_* \to D_*$ that induce the same isomorphism on homology groups, but that are not chain homotopic.

Exercise 81.2. Let $f\colon A \to X$ be a map between topological spaces. We define the corresponding *mapping cone* as

$$\mathrm{Cone}(f\colon A\to X) \;:=\; ((A\times[0,1])\sqcup X)/\!\sim$$

where $(a,1)\sim f(a)$ for all $a\in A$ and $(a,0)\sim(b,0)$ for all $a,b\in A$. Now suppose that A and X are finite CW-complexes and that f is a cellular map.

We equip the mapping cone $\mathrm{Cone}(f\colon A\to X)$ with the natural CW-structure of the CW-Complex Cone Corollary 68.35. Show that the corresponding cellular chain complex is isomorphic to the algebraic mapping cone of the chain map $f_*\colon \mathrm{C}_*^{\mathrm{CW}}(A) \to \mathrm{C}_*^{\mathrm{CW}}(X)$.

Exercise 81.3. We consider the chain complex.

$$C_* \;=\; 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$$

and the zero chain complex D_*.

(a) Show that there exists a chain map $\varphi\colon C_* \to D_*$ which is a quasi-isomorphism.

(b) Show that the chain complexes C_* and D_* are not chain homotopy equivalent.

Exercise 81.4. Show that in the Lift-to-Chain Map-Proposition 81.13 we cannot drop the hypothesis that the chain complexes are free abelian groups. More precisely, give an example of chain complexes $(\mathrm{C}_*, \partial_*)$ and (C'_*, ∂'_*) and a sequence of homomorphisms $\gamma_n\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(C'_*)$, $n\in\mathbb{N}_0$, that is not realized by a chain map.

82. The Jordan-Brouwer Separation Theorem

We start out with the following proposition.

Theorem 82.1. (Sphere Complement-H_*-Theorem)

(1) For every injective map $h\colon \overline{B}^k \to S^n$ we have

$$\widetilde{\mathrm{H}}_i\big(S^n \setminus h(\overline{B}^k)\big) \;=\; 0 \quad \text{for all } i \in \mathbb{N}_0.$$

(2) For every injective map $h\colon S^k \to S^n$ we have

$$\widetilde{\mathrm{H}}_i\big(S^n \setminus h(S^k)\big) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = n-k-1, \\ 0, & \text{otherwise.} \end{cases}$$

Example.

(1) Let $K \subset S^3$ be a knot, i.e. let K be a 1-dimensional submanifold of $S^3 = \mathbb{R}^3 \cup \{\infty\}$ that is diffeomorphic to S^1. This means in particular that there exists a smooth embedding $h\colon S^1 \to S^3$ with $h(S^1) = K$. In Lemma 78.13 we had already determined that $\mathrm{H}_1(S^3 \setminus K) \cong \mathbb{Z}$. Now we can compute the higher homology groups as well. More precisely, the combination of the Sphere Complement-H_*-Theorem 82.1 (2) and the Reduced Homology Lemma 74.1 says that

$$\mathrm{H}_i(S^3 \setminus K) \;=\; \mathrm{H}_i(S^3 \setminus h(S^1)) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

(2) In the Sphere Complement-H_*-Theorem 82.1 we do not assume that h is in any way smooth. For example there are injective maps $h\colon S^1 \to S^2$ of infinite length, see e.g.

`https://en.wikipedia.org/wiki/Koch_snowflake`,

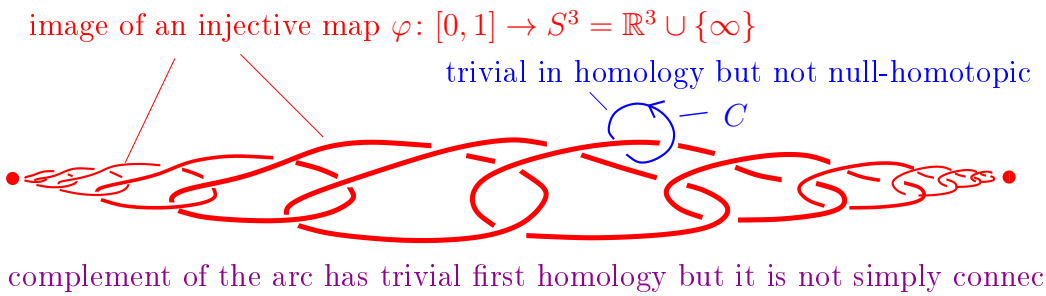
to which the proposition nonetheless applies. Reduced homology is only defined for non-empty topological spaces. It is implicit in the statement of the Sphere Complement-H_*-Theorem 82.1 that $S^n \setminus h(\overline{B}^k)$ is not the empty set. Even this statement is not entirely obvious since, as we have pointed out on several occasions, there are non-injective maps $h\colon \overline{B}^1 \to S^n$ that are in fact surjective.

(3) In the figure below we see the image of an injective map $\varphi\colon [0,1] \to S^3 = \mathbb{R}^3 \cup \{\infty\}$ and we see a closed curve $C \subset S^3 \setminus \varphi([0,1])$. It follows from the Sphere Complement-H_*-Theorem 82.1 (1) that $\mathrm{H}_1(S^3 \setminus \varphi([0,1])) = 0$, in particular C is null-homologous in $\mathrm{H}_1(S^3 \setminus \varphi([0,1])) = 0$.[987] But on [**Sti1993**, p. 152] it is shown that the curve C represents in fact a non-trivial element in $\pi_1(S^3 \setminus \varphi([0,1]))$.

To make the exposition more readable we prove the two parts of the Sphere Complement-H_*-Theorem 82.1 separately.

Proof of the Sphere Complement-H_*-Theorem 82.1 (1)**.** We write $I = [-1,1]$. By the Convex-to-Ball Proposition 2.20 (2) there exists a homeomorphism $I^k \xrightarrow{\cong} \overline{B}^k$. Therefore

[987] In Lemma 80.9 we saw that the boundary curve of an orientable surface with one boundary component is null-homologous. In Exercise 82.1 we will see that the closed curve C is in fact the boundary of an orientable surface that is embedded in $S^3 \setminus \varphi([0,1])$.



in the statement of the Sphere Complement-H$_*$-Theorem 82.1 (1) we can replace $\overline{B}^k$ by I^k. So we now want to show that for any injective map $h\colon I^k \to S^n$ we have

$$\widetilde{\mathrm{H}}_i(S^n \setminus h(I^k)) \;=\; 0 \quad \text{for all } i \in \mathbb{N}_0.$$

We will prove this statement by induction on k. We start out with $k = 0$. So let $h\colon I^0 = \{0\} \to S^n$ be an injective map. It follows from the stereographic projection defined in Lemma 2.18 that $S^n \setminus h(\{0\})$ is homeomorphic to $\mathbb{R}^n$. Hence we obtain from the Homotopy Equivalence-H$_*$-Corollary 73.9 (3) that $\widetilde{\mathrm{H}}_*(S^n \setminus h(\{0\})) = 0$.

Now we do the induction step. We assume we know the desired statement for all injective maps $I^{k-1} \to S^n$. Now let $h\colon I^k \to S^n$ be an injective map. We suppose that the desired conclusion does not hold, i.e. we suppose that $\widetilde{\mathrm{H}}_j(S^n \setminus h(I^k)) \neq 0$ for some j. We pick a non-trivial element $g_0 \in \widetilde{\mathrm{H}}_j(S^n \setminus h(I^k))$.

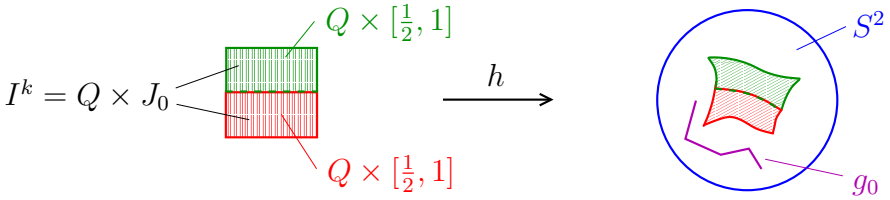
We write $Q = I^{k-1}$, $J_0 = I$ and we write

$$A_1 \;=\; S^n \setminus h(Q \times [0, \tfrac{1}{2}]) \quad \text{and} \quad A_2 \;=\; S^n \setminus h(Q \times [\tfrac{1}{2}, 1]).$$

Since h is injective we have

$$A_1 \cap A_2 \;=\; S^n \setminus h(I^k) \quad \text{and} \quad A_1 \cup A_2 \;=\; S^n \setminus h(Q \times \{\tfrac{1}{2}\}).$$

From the Compact Image Lemma 2.13 (3) we know that A_1 and A_2 are open subsets of



$A_1 \cup A_2$. Therefore we can apply the Mayer–Vietoris Theorem 78.5 and we obtain the long exact sequence

$$\ldots \to \underbrace{\widetilde{\mathrm{H}}_{j+1}(A_1 \cup A_2)}_{= 0 \text{ by induction}} \to \widetilde{\mathrm{H}}_j(A_1 \cap A_2) \;\to\; \widetilde{\mathrm{H}}_j(A_1) \oplus \widetilde{\mathrm{H}}_j(A_2) \to \underbrace{\widetilde{\mathrm{H}}_i(A_1 \cup A_2)}_{= 0 \text{ by induction}} \to \ldots$$

So we see that the middle map is an isomorphism. Thus there exists an $i \in \{1, 2\}$ such that the image of g_0 is non-trivial in $\widetilde{\mathrm{H}}_j(A_i)$. If $i = 1$, then we set $J_1 := [0, \frac{1}{2}]$, otherwise

we set $J_1 := [\frac{1}{2}, 1]$. We denote by

$$\varphi_1 \colon \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_0)) \;\to\; \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_1))$$

the inclusion induced map. By the above exact sequence φ_1 is an epimorphism. We denote by ρ_1 the inverse of the induced isomorphism

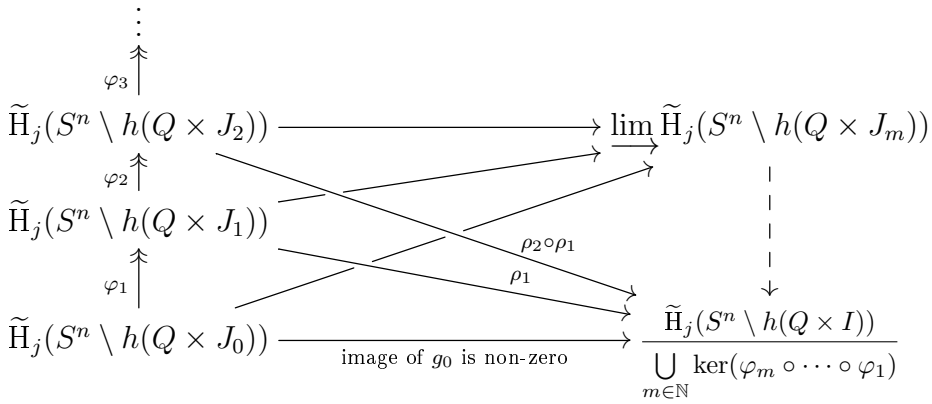
$$\widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_0))/\ker(\varphi_1) \;\to\; \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_1)).$$

Now we iterate this process by running the same argument with J_0 replaced by J_1 and so on. For each $m \in \mathbb{N}$ we can find an interval J_m of length $\frac{1}{2^m}$ that is contained in J_{m-1} such that for the inclusion induced homomorphism

$$\varphi_m \colon \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_{m-1})) \;\to\; \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_m))$$

we have that $g_m := \varphi_m(g_{m-1})$ is non-zero. We define ρ_m the same way from φ_m as we had obtained ρ_1 from φ_1.

By the definition of the direct limit we have the following commutative diagram:[988]



Since each J_m has half the length of J_{m-1} we conclude that the intersection $\bigcap_{m \in \mathbb{N}} J_m$ contains a unique point $t \in [0, 1]$.[989] We have

$$\varinjlim \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_m)) \;\underset{\uparrow}{\cong}\; \widetilde{\mathrm{H}}_j\Big(\bigcap_{m \in \mathbb{N}} S^n \setminus h(Q \times J_m)\Big) \;\underset{\uparrow}{=}\; \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times \{t\})) \;\underset{\uparrow}{=}\; 0.$$

by the Exhaustions-H$_*$-Proposition 79.4 since the subsets $S^n \setminus h(Q \times J_m)$ are open — since h is injective — by induction since $Q \times \{t\} \cong I^{k-1}$

But we have now obtained a contradiction, since the bottom horizontal map of the above commutative diagram has a non-trivial image but it factors through the trivial group. ■

Proof of the Sphere Complement-H$_*$-Theorem 82.1 (2). Recall that we have to show that for every injective map $h \colon S^k \to S^n$ we have isomorphisms

$$\widetilde{\mathrm{H}}_i\big(S^n \setminus h(S^k)\big) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = n - k - 1, \\ 0, & \text{otherwise.} \end{cases}$$

---

[988] Note that we have

$$\ker(\varphi_1) \subset \ker(\varphi_2 \circ \varphi_1) \subset \ker(\varphi_3 \circ \varphi_2 \circ \varphi_1) \subset \ldots$$

It follows that the union of these subgroups is again a subgroup.

[989] Why does this follow? What results from real analysis do we need to draw that conclusion?

We prove this statement by induction on k. For $k = 0$ we have

$$\widetilde{\mathrm{H}}_i(S^n \setminus h(S^0)) \underset{\text{since } S^n \setminus h(S^0) \text{ is homotopy equivalent to } S^{n-1}}{=} \widetilde{\mathrm{H}}_i(S^{n-1}) \underset{\text{by Proposition 74.4}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } i = n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Now suppose we know the statement for some $k \in \mathbb{N}$. Suppose we are given an injective map $h\colon S^{k+1} \to S^n$. We denote by $S^{k+1}_{\leq 0}$ the lower hemisphere of S^{k+1} and we denote by $S^{k+1}_{\geq 0}$ the upper hemisphere of S^{k+1}. Note that $S^{k+1}_{\leq 0} \cap S^{k+1}_{\geq 0} = S^k$. We apply the Mayer–Vietoris Theorem 78.5 to the decomposition of $S^n \setminus h(S^{k+1})$ into the open sets $S^n \setminus h(S^{k+1}_{\leq 0})$ and $S^n \setminus h(S^{k+1}_{\geq 0})$. We obtain a long exact sequence[991]

$$\cdots \to \widetilde{\mathrm{H}}_i(S^n \setminus h(S^k)) \to \underbrace{\widetilde{\mathrm{H}}_i((S^n \setminus h(S^{k+1}_{\leq 0}))}_{\substack{= 0 \text{ by part (1) and} \\ \text{since } S^{k+1}_{\leq 0} \cong \overline{B}^{k+1}}} \oplus \underbrace{\widetilde{\mathrm{H}}_i(S^n \setminus h(S^{k+1}_{\geq 0}))}_{\substack{= 0 \text{ by part (1) and} \\ \text{since } S^{k+1}_{\geq 0} \cong \overline{B}^{k+1}}} \to \widetilde{\mathrm{H}}_i(S^n \setminus h(S^{k+1})) \to \ldots.$$

Thus we see that

$$\widetilde{\mathrm{H}}_i(S^n \setminus h(S^{k+1})) \cong \widetilde{\mathrm{H}}_{i-1}(S^n \setminus h(S^k)) \underset{\text{by induction}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } i-1 = n-k-1, \\ 0, & \text{otherwise.} \end{cases}$$

■

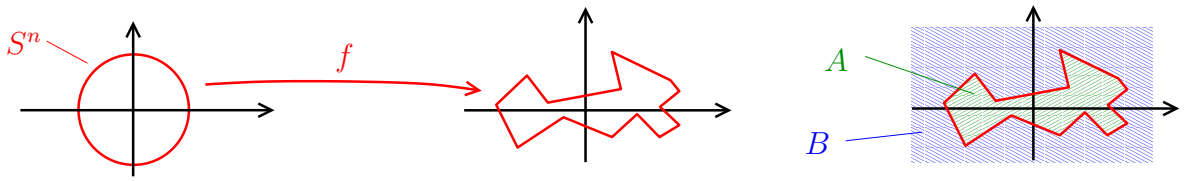
We have the following immediate corollary to the Sphere Complement-H_*-Theorem 82.1.

Corollary 82.2. (Jordan–Brouwer Separation Theorem) Let $n \in \mathbb{N}$. Furthermore let $f\colon S^n \to \mathbb{R}^{n+1}$ be an injective map.

(1) The complement $\mathbb{R}^{n+1} \setminus f(S^n)$ has precisely two path-components, one of which is bounded and one which is unbounded.

(2) If A and B are the two path-components of $\mathbb{R}^{n+1} \setminus f(S^n)$, then

$$\partial A = \partial B = A \cap B = f(S^n).$$



Proof.

(1) Let $n \in \mathbb{N}$. Furthermore let $f\colon S^n \to \mathbb{R}^{n+1}$ be an injective map. We use the inclusion $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \cup \{\infty\} = S^{n+1}$ to view $\mathbb{R}^{n+1}$ as a subspace of S^{n+1}. Now we see that

$$\widetilde{\mathrm{H}}_0(\mathbb{R}^{n+1} \setminus C) = \widetilde{\mathrm{H}}_0(\mathbb{R}^{n+1} \setminus f(S^n)) \underset{\text{see Exercise 78.12}}{\cong} \widetilde{\mathrm{H}}_0(S^{n+1} \setminus f(S^n)) \underset{\substack{\text{Homology-of-Sphere Complement} \\ \text{Theorem 82.1 (2)}}}{\cong} \mathbb{Z}$$

[990] Can you give a rigorous proof for the statement that the complement of two points in S^n is homotopy equivalent to S^{n-1}?

[991] Here we use, as in the proof of the Sphere Complement-H_*-Theorem 82.1 (1), implicitly the injectivity of h.

It follows from the Reduced Homology Lemma 74.1 (5) that $\mathbb{R}^{n+1} \setminus C$ has precisely two path-components. It remains to show that precisely one of them is unbounded.

Since S^n is compact there exists an $r \in \mathbb{R}$ such that $f(S^n) \subset B_r^{n+1}$. It is elementary to show that $\mathbb{R}^{n+1} \setminus B_r^{n+1}$ is path-connected. It follows that $\mathbb{R}^{n+1} \setminus B_r^{n+1}$ is contained in one of the two path-components. This shows that precisely one path-component is unbounded.

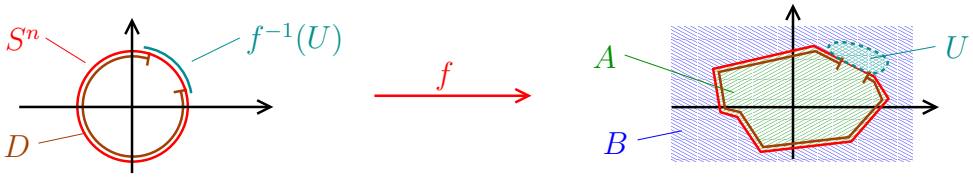
(2) First note that it follows from the Compact Image Lemma 2.13 and 1.21 that $\mathbb{R}^{n+1} \setminus f(S^n)$ is open. In particular, as we remarked on page 176, $\mathbb{R}^{n+} \setminus f(S^n)$ is locally path-connected. It follows easily that A and B are both open subsets of $\mathbb{R}^n$.

Claim. We have $\overline{A} = A \cup f(S^n)$.

Proof. Since B is open we know that $A \cup f(S^n) = \mathbb{R}^n \setminus B$ is closed. By definition of $\overline{A}$ this implies that $\overline{A} \subset A \cup f(S^n)$. Now let $P \in A \cup f(S^n)$. We need to show that $P \in \overline{A}$. Suppose that this is not the case. By the Interior-Closure Lemma 1.9 we know that $\overline{A}$ is closed. Thus our assumption that $P \notin \overline{A}$ implies that there exists an open subset $U \subset \mathbb{R}^{n+1}$ with $P \in U \subset \mathbb{R}^{n+1} \setminus \overline{A}$. By Exercise 82.5 there exists a subspace $D \subset S^n$, which is homeomorphic to $\overline{B}^n$ such that $S^n \setminus D \subset f^{-1}(U)$, i.e. such that $f(S^n \setminus D) \subset U$. We now obtain the following equalities:

$$\underbrace{\mathbb{R}^{n+1} \setminus f(D)}_{\substack{\text{path-connected by} \\ \text{the Homology-of-Sphere} \\ \text{Complement Theorem 82.1}}} = A \sqcup B \sqcup (f(S^n) \setminus f(D)) \overset{\substack{\text{since } f(S^n) \setminus f(D) = f(S^n \setminus D) \subset U \\ \downarrow}}{=} A \underset{\substack{\uparrow \\ \text{disjoint union since } U \cap A = \varnothing}}{\sqcup} (B \cup (U \cap \mathbb{R}^{n+1} \setminus f(D)).$$

We have thus written the path-connected space $\mathbb{R}^{n+1} \setminus f(D)$ as the disjoint union of the two open non-empty sets A and $B \cup (U \cap \mathbb{R}^{n+1} \setminus f(D))$. But by the Path-Connected Implies Connected Corollary 2.23 this is not possible. We have thus obtained a contradiction. ∎



For historic purposes we state the case $n = 1$ again with slightly different names. The following definition is a minute variation on the definition of a Jordan curve given on page 795.

Definition. A planar Jordan curve is the image of an injective map $S^1 \to \mathbb{R}^2$.

The following corollary is now just the case $n = 1$ of the Jordan–Brouwer Separation Theorem 82.2.

Corollary 82.3. (Jordan Curve Theorem) Let $C \subset \mathbb{R}^2$ be a planar Jordan curve.

(1) The complement $\mathbb{R}^2 \setminus C$ has precisely two path-components, one of which is bounded and one which is unbounded.

(2) If A and B are the two path-components of $\mathbb{R}^2 \setminus C$, then
$$\partial A \;=\; \partial B \;=\; A \cap B \;=\; C.$$

Remark.

(1) The Jordan Curve Theorem 82.3 might appear as "self-evident". To disabuse the reader of this idea recall that in Proposition 6.14 and Exercise 6.14 we saw that there exist planar Jordan curves with non-zero 2-dimensional Lebesgue measure.
(2) The Jordan Curve Theorem 82.3 takes its name from Camille Jordan [**Jor1887**][992] who formulated it in 1887. It seems like the first convincing proof was given by Oswald Veblen [**Veb1905**] in 1905.
(3) The general case of the Jordan–Brouwer Separation Theorem 82.2 was first proved by Luitzen Brouwer [**Brou1912b**] in 1912.

The Sphere Complement-H_*-Theorem 82.1, together with the Reduced Homology Lemma 74.1, says in particular, that if $h\colon S^{n-1} \to S^n$ is an injective map, then
$$\mathrm{H}_i(S^n \setminus h(S^{n-1})) \;\cong\; \begin{cases} \mathbb{Z}^2, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$
Put differently, homologically the topological space $S^n \setminus h(S^{n-1})$ looks like the disjoint union of two open n-dimensional balls. One might now suspect that $S^n \setminus h(S^{n-1})$ is in fact homeomorphic to the union of two open n-dimensional balls.
This is in fact true in dimension $n = 2$. More precisely Arthur Schönflies[993] proved in 1908 the following theorem. We refer to [**Tn1992**, Section 3] for a perhaps more readable proof of the theorem.[994]

Theorem 82.4. (Schönflies Theorem) Let $h\colon S^1 \to S^2$ be an injective map. Then h extends to a homeomorphism of S^2, i.e. there exists a homeomorphism $H\colon S^2 \to S^2$ such that $H|_{S^1} = h$. In particular both components of $S^2 \setminus h(S^1)$ are homeomorphic to open disks.

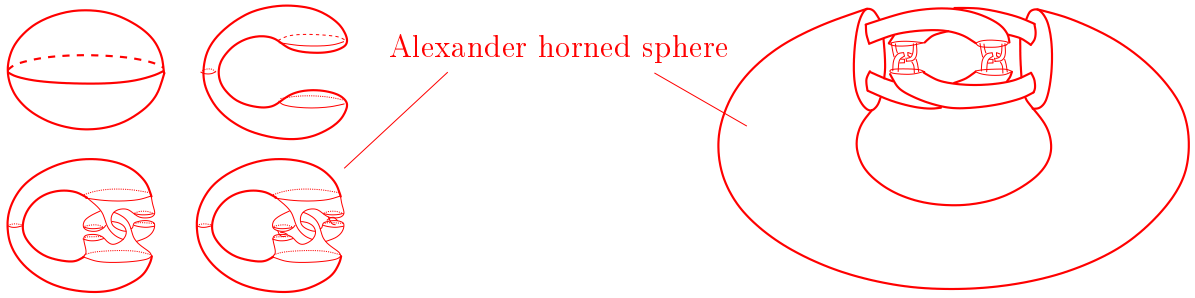
Remark.

(1) The Schönflies Theorem is often also called the Jordan-Schönflies Theorem.
(2) If $h\colon S^1 \to S^2$ is a smooth embedding, then the conclusion of the Schönflies Theorem 82.4 also follows from the much easier Proposition 55.18.

It is natural to ask whether the conclusion of the Schönflies Theorem also holds in higher dimensions. But this hope breaks down already one dimension higher. More precisely, James Alexander[995] showed in 1924 that there exists an injective map $S^2 \to \mathbb{R}^3$ such that one of the two components of the complement of $h(S^2)$ is not simply connected. The image $h(S^2)$ is called the *Alexander horned sphere*. It is illustrated in the figure below. A more detailed description of the Alexander horned sphere and the proof that the complement is

[992] Camille Jordan (1838-1922) was a French mathematician who most readers will already have encountered in linear algebra.
[993] Arthur Schönflies (1853–1928) was a German mathematician.
[994] More precisely, in [**Tn1992**, Section 3] a proof for the following statement is given:
Let $g\colon S^1 \to \mathbb{R}^2$ be an injective map. Then g extends to a homeomorphism of $\mathbb{R}^2$.
We leave it to the reader to deduce Theorem 82.4 from that statement.
[995] James Alexander (1888-1971) was an American mathematician. He also introduced the Alexander polynomial of a knot.

not simply connected is given in [**Al1924a**] and also [**Hat2002**, p. 170] and in [**DV2009**, p. 46]. In Exercise 82.2 we invite the reader to modify the Example shown in the figure below to produce an injective map $S^2 \to \mathbb{R}^3$ such that neither of the two components of the complement of $h(S^2)$ is simply connected.



We have now seen that injective maps $S^2 \to S^3$ can be pretty "wild". The map that defines the Alexander horned sphere is continuous, but it is certainly not smooth. So what happens if we restrict ourselves to smooth maps? In fact in the smooth setting we have the following theorem.

Theorem 82.5. (Generalized smooth Schönflies Theorem) Let $h\colon S^{n-1} \to S^n$ be a smooth embedding. If $n \neq 4$, then there exists an orientation-preserving diffeomorphism $H\colon S^n \to S^n$ with $H|_{S^{n-1}} = h$.

For $n = 2$ the statement follows fairly easily from the classification of smooth 2-dimensional smooth manifolds that we gave in the Surface Classification Theorem 55.4, see e.g. [**Hirs1976**, p. 207]. For $n = 3$ this statement was proved by James Alexander [**Al1924b**]. See also [**Hat3**, Chapter 1] and [**Schul2014**, Theorem 3.2.5] and for an exposition of the proof. For $n \geq 5$ this statement was independently proved by Barry Mazur [**Maz1959**] in 1959 and by Morton Brown [**BrownM1960**] in 1960.
The case $n = 4$ is missing in the Generalized Schönflies Theorem. In fact it is still unknown whether the statement holds for $n = 4$.

Question 82.6. Is it true that given any smooth embedding $h\colon S^3 \to S^4$ there exists a diffeomorphism $H\colon S^4 \to S^4$ with $H|_{S^3} = h$?

This question is very much open and there is not even a consensus whether one expects the answer to be yes or no. If you can answer this question you would almost certainly obtain a Fields medal.[996]

We conclude this section with two more consequences of the Sphere Complement-H_*-Theorem 82.1. The following theorem was first proved by Luitzen Brouwer [**Brou1911**] in 1911. It can be viewed as a significant strengthening of the Smooth Invariant of Domain Theorem 19.13 (1).

Theorem 82.7. (Topological Invariance of Domain Theorem) Let U be an open subset of $\mathbb{R}^n$.

(1) If $h\colon U \to \mathbb{R}^n$ is an injective map, then $h(U)$ is also an open subset of $\mathbb{R}^n$.[997]
(2) If $h\colon U \to \mathbb{R}^m$ be an injective map, then $m \geq n$.

[996] But hurry up, you can only get a Fields medal if you are under 40.
[997] How can one prove the theorem for $n = 1$ using only the methods from real analysis?

Remark. The Open Mapping Theorem from complex analysis, see [**Lang1999**, Theorem II.6.2], says that if $G \subset \mathbb{C}$ is an open subset and if $f\colon G \to \mathbb{C}$ is a non-constant *holomorphic* function, then $f(G)$ is also open. In fact, one can deduce the "Open Mapping Theorem" from the Topological Invariance of Domain Theorem 82.7 since by [**Jä2011**, Satz 10] every non-constant holomorphic function on an open subset of $\mathbb{C}$ is locally the power of an injective function.

In the proof of Theorem 82.7 we need the following elementary lemma.

Lemma 82.8. Let $U \subset \mathbb{R}^n$ be a subset. If U is open in $\mathbb{R}^n$, then every path-component of U is also an open subset of $\mathbb{R}^n$.

Proof. Let C be a path-component of the subset subset $U \subset \mathbb{R}^n$. Let $P \in C$. Since U is open there exists an $r > 0$ such that $B_r(P) \subset U$. But all points in $B_r(P)$ are path-equivalent to P, hence $B_r(P) \subset C$. ■

Proof of the Topological Invariance of Domain Theorem 82.7.

(1) Recall that by the homeomorphism from Lemma 2.18 we can view $\mathbb{R}^n$ as an open subset of $\mathbb{R}^n \cup \{\infty\} = S^n$. So we can view h as a map from U to S^n and we need to show that $h(U)$ is an open subset of S^n.

Let $x \in U$. We pick a closed n-ball B in U that contains x in the interior. It suffices to show that $h(B \setminus \partial B) \subset h(U)$ is open in S^n. Note that $S^n \setminus h(\partial B)$ is open by the Compact Image Lemma 2.13 (3). By Lemma 82.8 it therefore suffices to show that $h(B \setminus \partial B)$ is a path-component of $S^n \setminus h(\partial B)$. We will do so in the following claim.

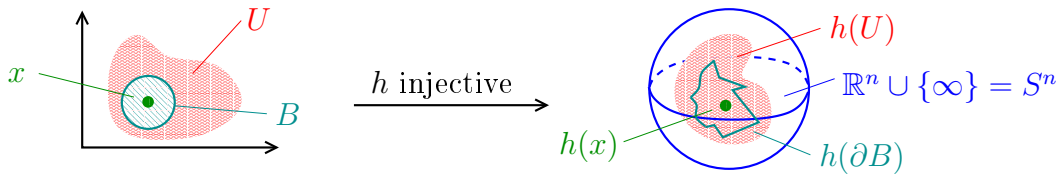
Claim. The topological space $S^n \setminus h(\partial B)$ has precisely two path-components namely the subsets $h(B \setminus \partial B)$ and $S^n \setminus h(B)$.

Proof. By the Sphere Complement-H$_*$-Theorem 82.1 (2) and the Reduced Homology Lemma 74.1 we know that

$$S^n \setminus h(\partial B) \;=\; h(B \setminus \partial B) \sqcup (S^n \setminus h(B))$$

has two path-components. It suffices to show that the two sets on the right-hand side are path-connected. The first set $h(B \setminus \partial B)$ is path-connected since it is the image of the path-connected topological space $B \setminus \partial B$, while the second set $S^n \setminus h(B)$ is path-connected by the Sphere Complement-H$_*$-Theorem 82.1 (1) and the Reduced Homology Lemma 74.1. ⊞

(2) Let $h\colon U \to \mathbb{R}^m$ be an injective map with $m \leq n$. We consider the inclusion given by composing h with the inclusion $i\colon \mathbb{R}^m \xrightarrow{x \mapsto (x,0)} \mathbb{R}^n$. By (1) we know that $(i \circ h)(U)$ is an open subset. But a subset of $\mathbb{R}^m$ is open under the inclusion $i\colon \mathbb{R}^m \to \mathbb{R}^n$ only if $m = n$. Thus we have shown that $m \geq n$. ■



Corollary 82.9. Let M be a non-empty closed n-dimensional topological manifold and let N be a connected n-dimensional topological manifold without boundary. Let $h\colon M \to N$ be a map.

(1) If $h\colon M \to N$ is locally injective, then h is also surjective.
(2) If h is injective, then h is already a homeomorphism.

Example. Corollary 82.9 implies in particular that there exists no injective map from the closed smooth manifold S^n into the smooth manifold $\mathbb{R}^n$.[998]

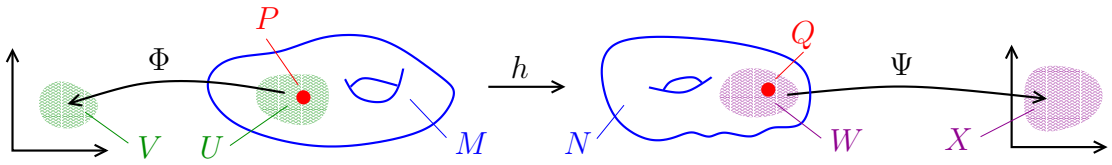
Proof. The second statement of the corollary is an immediate consequence of the first statement and the Compact-Hausdorff Proposition 2.17 (3). So it suffices to prove the first statement.

Let $h\colon M \to N$ be a locally injective map from a non-empty closed n-dimensional topological manifold M to a connected n-dimensional topological manifold N without boundary.

The image $h(M)$ is closed in N since M is compact and since N is Hausdorff. Since N is connected and since M is non-empty it suffices to show that $h(M)$ is also open in N.

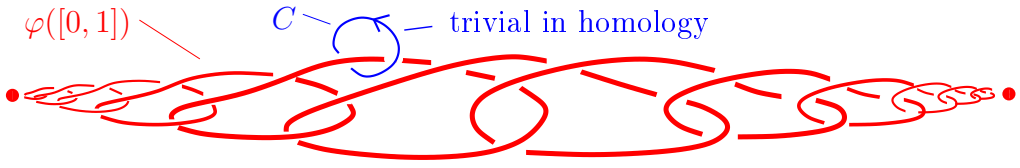
So let $Q \in h(M)$. We pick $P \in M$ with $h(P) = Q$. We pick a chart $\Phi\colon U \to V$ around P and we pick a chart $\Psi\colon W \to X$ around Q. After possibly replacing U by $U \cap h^{-1}(W)$ we can assume that $h(U) \subset W$. Since h is locally injective we can, after possibly shrinking U even more, assume that $h|_U$ is injective.

From Theorem 82.7 we know that $(\Psi \circ h \circ \Phi^{-1})(V)$ is an open subset of $\mathbb{R}^n$. Since Ψ is a homeomorphism we see that $h(\Phi^{-1}(V)) = h(U)$ is an open neighborhood of Q that is contained in $h(M)$. ■



Exercises for Chapter 82.

Exercise 82.1. In the figure below the exercise we see the image of an injective map $\varphi\colon [0,1] \to S^3 = \mathbb{R}^3 \cup \{\infty\}$ and we see a curve $C \subset S^3 \setminus \varphi([0,1])$. We know from the Sphere Complement-H_*-Theorem 82.1 (1) that x is null-homologous. Show that x is the boundary of an orientable surface Σ that is embedded in $S^3 \setminus \varphi([0,1])$. (In this context this can be seen as the converse to Exercise 80.6.)



[998] How can you show this for $n = 1$, just using methods from real analysis?

Exercise 82.2. Show that there exists an injective map $S^2 \to \mathbb{R}^3$ such that neither of the two components of the complement of $h(S^2)$ is simply connected.
Remark. You could try to modify the example shown in the figure on page 1815.

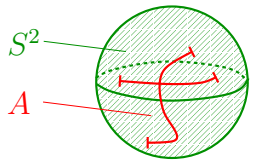
Exercise 82.3. Let M be a closed n-dimensional topological manifold, let X be a Hausdorff space, let $f\colon M \to X$ be a map and let $\Phi\colon B^n \to X$ be an embedding. Show that $f(M) \cap \Phi(B^n)$ is either empty or it equals $\Phi(B^n)$.
Remark. A typical example would be that X is a CW-complex and Φ is the characteristic map of an open n-cell.

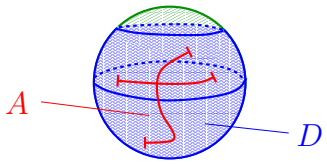
Exercise 82.4. Let $\varphi\colon S^1 \to \mathbb{R}^2$ be an injective map. Show that there exist distinct three points $x, y, z \in S^1$ such that $\varphi(x), \varphi(y), \varphi(z)$ from an equilateral triangle.
Hint. Use the Jordan–Brouwer Separation Theorem 82.2 for $n = 1$. *Remark.* This is a generalization of Exercise 82.4.

Exercise 82.5. Let $n \in \mathbb{N}_0$ and let $A \subset S^n$ be a closed subset with $A \neq S^n$. Show that there exists a subspace D of S^n with the following two properties:

(1) D is homeomorphic to $\overline{B}^n$.
(2) $A \subset D$.

Hint. First deal with the case that $P = (0, \dots, 0, 1) \in S^n \setminus A$.





83. Topological robotics ❀

In this chapter we will first study the matrix groups SO(3) and SU(2), viewed as topological spaces. In particular we will determine $\pi_1(\mathrm{SO}(3))$ and we will use this result to give an application to topological robotics.

83.1. The matrix groups SO(3) and SU(2).

We recall that the Matrix Group-Smooth Manifold Proposition 26.7 implies in particular that we can view SO(3) and SU(2) as closed 3-dimensional smooth manifolds. In this section we will see that the smooth manifolds SO(3) and SU(2) are diffeomorphic to smooth manifolds we are already very familiar with. We start out our discussion of SO(3) and SU(2) with some definitions.

Definition. We consider the set of matrices[999]

$$T := \left\{B \in \mathrm{SL}(2,\mathbb{C}) \,\middle|\, \mathrm{tr}(B) = 0 \text{ and } B = \overline{B}^T\right\}.$$

Note that T is a 3-dimensional real vector space. We equip it with the basis given by the so-called "Pauli matrices"[1000]

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad E_3 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},$$

and we equip T with the inner product given by

$$\begin{array}{rcl} T \times T & \mapsto & \mathbb{R} \\ (A,B) & \mapsto & g(A,B) := \frac{1}{2}\,\mathrm{tr}\left(A\overline{B}^T\right). \end{array}$$

We consider the isomorphism

$$\begin{array}{rcl} \Phi\colon \mathbb{R}^3 & \to & T \\ e_i & \mapsto & E_i. \end{array}$$

It follows from a straightforward calculation that Φ defines an isometry from $(\mathbb{R}^3, \langle\,,\,\rangle)$ to (T,g).[1001] It follows immediately from this observation that Φ induces an isomorphism $\mathrm{SO}(3) \to \mathrm{SIsom}(T,g)$, where $\mathrm{SIsom}(T,g)$ denotes the group of isometries of (T,g) of determinant $+1$. We use this isomorphism to identify these two groups.

Lemma 83.1. The map

$$\begin{array}{rcl} q\colon \mathrm{SU}(2) & \to & \mathrm{SIsom}(T,g) = \mathrm{SO}(3) \\ A & \mapsto & (B \mapsto ABA^{-1}) \end{array}$$

is a smooth map between smooth manifolds and it is a well-defined group homomorphism. Furthermore for $A, A' \in \mathrm{SU}(2)$ we have

$$q(A) = q(A') \iff A = \pm A'.$$

Example. Given $t \in \mathbb{R}$ we consider the matrix

$$A = \begin{pmatrix} \exp(-\mathrm{i}t) & 0 \\ 0 & \exp(\mathrm{i}t) \end{pmatrix} \in \mathrm{SU}(2).$$

---

[999] Recall that the trace $\mathrm{tr}(A)$ of a matrix is defined as the sum of all the diagonal entries.

[1000] Wolfgang Pauli (1900–1958) was Swiss-American theoretical physicist and one of the pioneers of quantum physics. These matrices were also independently introduced by Paul Dirac. Wikipedia writes: "In 1928, building on 2×2-spin matrices which he [Paul Dirac] discovered independently of Wolfgang Pauli's work on non-relativistic spin systems, [...] he proposed the Dirac equation as a relativistic equation of motion for the wave function of the electron. This work led Dirac to predict the existence of the positron, the electron's antiparticle [...]."

[1001] In the calculation we only have to show that $g(E_i,E_j) = \frac{1}{2}\,\mathrm{tr}(E_i\overline{E}_j^T) = \delta_{ij} = \langle e_i, e_j\rangle$ for all choices of $i,j \in \{1,2,3\}$. But this can be verified easily.

We want to determine $q(A) \in \mathrm{SO}(3)$. This means that we have to express the linear map $T \to T$ that is given by $B \mapsto ABA^{-1}$ with respect to the ordered basis $\{E_1, E_2, E_3\}$. A straightforward calculation shows that

$$\begin{aligned} AE_1A^{-1} &= E_1, \\ AE_2A^{-1} &= \begin{pmatrix} 0 & \exp(-2\mathrm{i}t) \\ \exp(2\mathrm{i}t) & 0 \end{pmatrix} = \cos(2t)\cdot E_2 + \sin(2t)\cdot E_3, \\ & \qquad \uparrow \text{ since } \exp(\mathrm{i}s) = \cos(s) + \mathrm{i}\sin(s) \\ AE_3A^{-1} &= \begin{pmatrix} 0 & -\mathrm{i}\cdot\exp(-2\mathrm{i}t) \\ \mathrm{i}\cdot\exp(2\mathrm{i}t) & 0 \end{pmatrix} \downarrow = -\sin(2t)\cdot E_2 + \cos(2t)\cdot E_3. \end{aligned}$$

But this implies that the homomorphism $T \to T$ given by conjugation with A, with respect to the ordered basis $\{E_1, E_2, E_3\}$ is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2t) & -\sin(2t) \\ 0 & \sin(2t) & \cos(2t) \end{pmatrix} \quad \begin{matrix} \text{put differently,} \\ \text{we have} \end{matrix} \quad q\begin{pmatrix} \exp(-\mathrm{i}t) & 0 \\ 0 & \exp(\mathrm{i}t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2t) & -\sin(2t) \\ 0 & \sin(2t) & \cos(2t) \end{pmatrix}.$$

Proof of Lemma 83.1. First we have to show that for $A \in \mathrm{SU}(2)$ and $B \in T$ we also have $ABA^{-1} \in T$. This is indeed the case since

$$\mathrm{tr}(ABA^{-1}) = \mathrm{tr}(B) = 0 \quad \text{and} \quad \big(\overline{ABA^{-1}}\big)^T = (\overline{A}^{-1})^T\overline{B}^T\overline{A}^T \underset{\substack{\uparrow \\ \text{since } B = \overline{B}^T \text{ and } A^{-1} = \overline{A}^T}}{=} A\overline{B}^TA^{-1}.$$

It follows easily from the definition that q is a group homomorphism. We also have to show that for $A \in \mathrm{SU}(2)$ we get indeed an isometry of (T, g). So let $B, C \in T$. Then we have indeed

$$\begin{aligned} g(q(A)(B), q(A)(C)) &= \tfrac{1}{2}\,\mathrm{tr}\left(ABA^{-1}\cdot\overline{ACA^{-1}}^T\right) = \tfrac{1}{2}\,\mathrm{tr}\left(AB\overbrace{A^{-1}\cdot(\overline{A}^{-1})^T}^{=\mathrm{id}}\overline{C}^T\overline{A}^T\right) \\ &= \tfrac{1}{2}\,\mathrm{tr}\left(AB\overline{C}^T\overline{A}^T\right) = \tfrac{1}{2}\,\mathrm{tr}\left(B\overline{C}^T\underbrace{\overline{A}^TA}_{=\mathrm{id}}\right) = \tfrac{1}{2}\,\mathrm{tr}\left(B\cdot\overline{C}^T\right). \end{aligned}$$

This shows that q defines group homomorphism $\mathrm{SU}(2) \to \mathrm{O}(3)$. By the Matrix Group Path-Component Proposition 2.37 we know that $\mathrm{SU}(2)$ is path-connected. The image $q(\mathrm{SU}(2))$ is a path-connected subset of $\mathrm{O}(3)$ that contains $q(\mathrm{id}) = \mathrm{id}$. It follows again the Matrix Group Path-Component Proposition 2.37 that $q(\mathrm{SU}(2))$ is contained in $\mathrm{SO}(3)$.

Finally let $A, A' \in \mathrm{SU}(2)$. If $A = -A'$ then we clearly have $q(A) = q(A')$. Now suppose that conversely we have $q(A) = q(A')$. We then have in particular $q(A)(E_i) = q(A')(E_i)$ for $i = 1, 2, 3$, i.e. we have $AE_iA^{-1} = A'E_i(A')^{-1}$ for $i = 1, 2, 3$. These equalities give us equations relating the four entries of the matrix A to the four entries of the matrix A'. An elementary calculation then shows that the only solutions are $A = \pm A'$. We leave the details to the reader.

This discussion shows that q is a well-defined map and that it is a group homomorphism. After one has unraveled the definitions it will also have become apparent that the map g is indeed a smooth map between smooth manifolds. ■

The following theorem is the main result of this section.

Theorem 83.2. (SU(2)-SO(3)-Covering Theorem) Let $q\colon \mathrm{SU}(2) \to \mathrm{SIsom}(T, g) = \mathrm{SO}(3)$ be the map from Lemma 83.1. Then the following two statements hold:

(1) The map q is a 2-fold covering map.
(2) There exist diffeomorphisms $f\colon S^3 \to \mathrm{SU}(2)$ and $g\colon \mathbb{R}\mathrm{P}^3 \to \mathrm{SO}(3)$ such that the following diagram commutes:

$$\begin{array}{ccc} S^3 & \xrightarrow[\cong]{f} & \mathrm{SU}(2) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{R}\mathrm{P}^3 & \xrightarrow[\cong]{g} & \mathrm{SO}(3), \end{array}$$

where we denote by $p\colon S^3 \to \mathbb{R}\mathrm{P}^3 = S^3/\pm 1$ the usual projection .

Remark. In Exercise 5.47 we gave a somewhat different homeomorphism between $\mathbb{R}\mathrm{P}^3$ and SO(3). In future discussions we will prefer the homeomorphism from the SU(2)-SO(3)-Covering Theorem 83.2.

Convention. We will use the diffeomorphism from the SU(2)-SO(3)-Covering Theorem 83.2 to make the identifications $S^3 = \mathrm{SU}(2)$ and $\mathbb{R}\mathrm{P}^3 = \mathrm{SO}(3)$.

In the proof of the SU(2)-SO(3)-Covering Theorem 83.2 we will need the following technical lemma.

Lemma 83.3. Let $q\colon M \to N$ be a map between two n-dimensional topological manifolds. Suppose the following conditions are satisfied:

(1) the map q is surjective,
(2) the map q is locally injective, i.e. for each $P \in M$ there exists a neighborhood U such that $q|_U$ is injective,
(3) for each point $Q \in N$ the preimage $q^{-1}(Q)$ is a finite set.

Then q is a covering map.

Proof. Since q is surjective we only have to show that each $Q \in N$ admits an open neighborhood that is uniformly covered. So let $Q \in N$. We denote its preimages by $P_1, \ldots, P_r$. Let $i \in \{1, \ldots, m\}$. It follows easily from (2) and the hypothesis that M is a topological manifold that we can pick a closed ball B_i which is a neighborhood around P_i and such that $q\colon B_i \to N$ is injective. Since the topological manifold N is in particular Hausdorff it follows from the Compact-Hausdorff Proposition 2.17 (3) that $q\colon B_i \to q(B_i)$ is a homeomorphism. It follows from the proof of Corollary 82.9 that $q(\overset{\circ}{B}_i)$ is an open neighborhood of Q. It is now straightforward to see that $U := \bigcap_{i=1}^{m} q(\overset{\circ}{B}_i)$ is a uniformly covered open neighborhood of Q. ■

Proof of the SU(2)-SO(3)-Covering Theorem 83.2. Somewhat confusingly the proof of the SU(2)-SO(3)-Covering Theorem 83.2 is given in three steps (a), (b) and (c).

(a) One can easily show that the map[1002]

$$\begin{array}{rcl} f\colon S^3 & \to & \mathrm{SU}(2) \\ (z,w) & \mapsto & \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \widetilde{f}\colon \mathrm{SU}(2) & \to & S^3 \\ \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} & \mapsto & (z,w) \end{array}$$

are inverses to one another and that both maps are smooth. This implies that both maps, in particular f, is a diffeomorphism.

(b) Now we want to show that $q\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a 2-fold covering map. By Lemma 83.1 we know that q is locally injective. But then it follows from Corollary 82.9 and the fact that SU(2) and SO(3) are closed 3-dimensional smooth manifolds and the fact that SO(3) is connected (which follows from the Matrix Group Path-Component Proposition 2.37 (3) together with Locally Path-Connected Lemma 2.40) that q is in fact also a surjection. Since the preimage of each $A \in \mathrm{SO}(3)$ consists of precisely two points in SU(2) we can apply Lemma 83.3 to deduce that q is indeed a 2-fold covering map.

(c) For any $(z,w) \in S^3$ we have $q(f(-z,-w)) = q(-f(z,w)) = q(f(z,w))$. Thus we see that the map

$$\begin{array}{rcl} g\colon \mathbb{R}\mathrm{P}^3 & \to & \mathrm{SO}(3) \\ [(z,w)] & \mapsto & q(f(z,w)) \end{array}$$

is well-defined. It follows immediately from the fact, obtained in (b), that q is surjective, that g is also surjective. Similarly it is straightforward to show, using (b), that the map g is injective. Since q, f and p are local diffeomorphisms and since locally g is given by a composition of q, f, p and their local inverses we see that g is a local diffeomorphism. But by the Smooth Pasting Proposition 19.23 (2) we know that a map between smooth manifolds that is a local diffeomorphism and a bijection is in fact a diffeomorphism.

Finally we note that it follows easily from the definitions that $g \circ p = q \circ f$, i.e. the diagram given in the theorem commutes. ■

From the SU(2)-SO(3)-Covering Theorem 83.2 and Corollary 48.18 we obtain immediately the following, perhaps initially somewhat surprising, corollary.

Corollary 83.4. We have

$$\pi_1(\mathrm{SU}(2)) \;=\; 0 \quad \text{and} \quad \pi_1(\mathrm{SO}(3)) \;\cong\; \mathbb{Z}_2.$$

In particular SU(2) is the universal covering of SO(3).[1003]

[1002] In Proposition 92.2 we will see that the map f arises naturally if we think of S^3 as the "unit quaternions".

[1003] The fact SU(2) is the universal covering of SO(3) plays an important role in theoretical physics and it is essential in understanding the notion of "spin" of an electron. At least that is a slogan that I remember from the days when I studied physics.

Remark. We can actually give an explicit description of the non-trivial element in the fundamental group $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$. More precisely, we consider the loop

$$\begin{array}{rcl} \gamma\colon [0,1] & \to & \mathrm{SO}(3) \\ t & \mapsto & \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi t) & \sin(2\pi t) \\ 0 & -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix} \end{array}$$

at the base point $\mathrm{id} \in \mathrm{SO}(3)$. The calculation on page 1819 shows that a lift of the path $\gamma\colon [0,1] \to \mathrm{SO}(3)$ to a path $[0,1] \to \mathrm{SU}(2)$ to the starting point $\mathrm{id} \in \mathrm{SU}(2)$ is given by

$$\begin{array}{rcl} \widetilde{\gamma}\colon [0,1] & \to & \mathrm{SO}(3) \\ t & \mapsto & \begin{pmatrix} \exp(-\mathrm{i}\pi t) & 0 \\ 0 & \exp(\mathrm{i}\pi t) \end{pmatrix}. \end{array}$$

Note that $\widetilde{\gamma}(1) = -\mathrm{id}$, i.e. $\widetilde{\gamma}$ is *not* a loop. Since $q\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a covering and since $\mathrm{SU}(2)$ is simply connected it follows from Lemma 48.15 that $[\gamma]$ is non-trivial in $\pi_1(\mathrm{SO}(3), \mathrm{id})$.

Corollary 83.4 also raises the following question.

Question 83.5. Let $n \geq 4$. What is $\pi_1(\mathrm{SO}(n))$?[1004][1005]

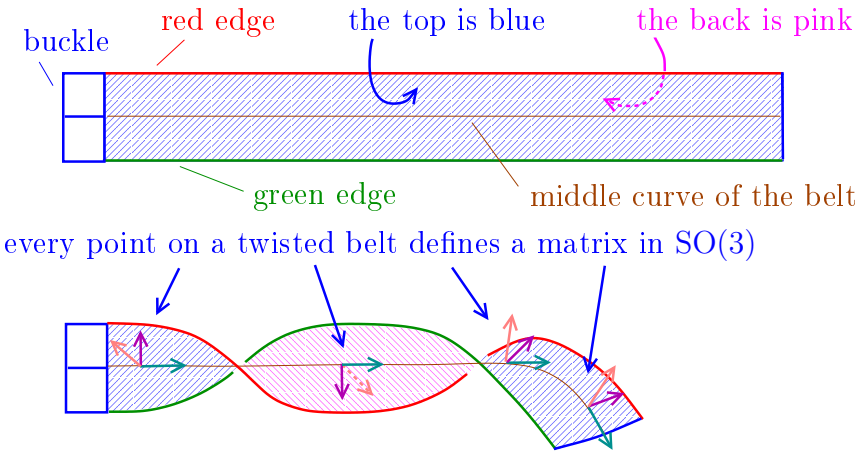
We will return to this question later on in Proposition 156.11.

83.2. **The belt trick ❀.** In this section we will see that one can visualize the non-trivial element in $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$ using a belt. We proceed as follows.

(1) First of all note that a matrix in $\mathrm{SO}(3)$ is precisely given by an ordered orthonormal basis, i.e. by a choice of an ordered set of three orthogonal vectors v_1, v_2, v_3 of length 1 in $\mathbb{R}^3$.
(2) We equip $\mathrm{SO}(3)$ with the base point given by the identity matrix, i.e. by the standard basis $\{e_1, e_2, e_3\}$.
(3) Now suppose we are given a belt with a buckle, a red edge and a green edge, with a blue top side and a pink lower side.
(4) Any point on a twisted belt defines a triple of orthogonal vectors:
   (a) the first vector lies in the plane defined by the belt and points away from the buckle,
   (b) the second vector also lies in the plane defined by the belt and points towards the red edge,
   (c) the third is orthogonal to the belt and it is attached to the top side of the belt.
(5) In the remainder of the discussion we fix the buckle in the xy-plane such that it points in the x-direction, so that to a point at the buckle we assign the identity matrix.
(6) We say that the end of the belt is in standard position if it lies in a plane parallel to the plane of the buckle and if it points into the x-direction. Put differently, the end of the belt is in standard position precisely if the matrix assigned to it is the identity matrix.
(7) When we walk on the middle curve of the belt from the buckle towards the other end of the belt, then we obtain a path $[0,1] \to \mathrm{SO}(3)$. In particular, if the end of

[1004] Recall that the Matrix Group Path-Component Proposition 2.37 (3) says that $\mathrm{SO}(n)$ is path-connected.
[1005] What is $\pi_1(\mathrm{SO}(2))$?

the belt is in standard position, then the path defined by the belt is in fact a loop in SO(3) at the base point id.

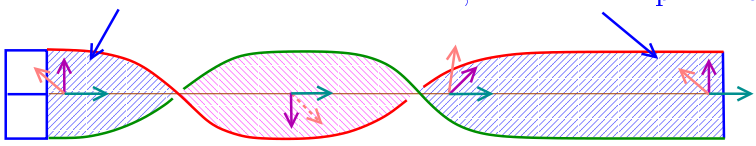


moving along the middle curve of the belt we get a path $[0,1] \to \mathrm{SO}(3)$

(8) When we hold the belt tightly without twisting it, then we assign the identity matrix to every point on the belt, i.e. the belt defines the constant loop $t \mapsto \mathrm{id} \in \mathrm{SO}(3)$.

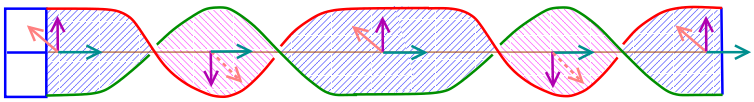
(9) Now we give the belt, that is still fixed at the buckle, a full twist. As in (7) this defines a loop $\gamma\colon [0,1] \to \mathrm{SO}(3)$, in fact it is straightforward to see that this loop is precisely the loop that we had considered on page 1823. In particular we have visualized the non-trivial element $[\gamma] \in \pi_1(\mathrm{SO}(3), \mathrm{id}) \in \mathbb{Z}_2$.

both points define the standard basis of $\mathbb{R}^3$, i.e. the base point id $\in \mathrm{SO}(3)$



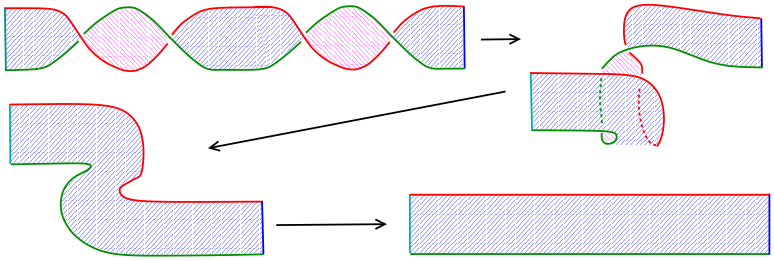
the path $\gamma\colon [0,1] \to \mathrm{SO}(3)$ defined by the matrices along the middle curve is a loop and thus defines an element $[\gamma] \in \pi_1(\mathrm{SO}(3), \mathrm{id})$

(10) The loop $\gamma * \gamma\colon [0,1] \to \mathrm{SO}(3)$ is given by performing another full twist along the belt. It follows from Corollary 83.4 that $[\gamma * \gamma] = [\gamma]^2$ is trivial in $\pi_1(\mathrm{SO}(3), \mathrm{id}) \cong \mathbb{Z}_2$, i.e. $\gamma * \gamma$ is null-homotopic.



the double full twist defines the loop $\gamma * \gamma$ which represents $[\gamma]^2 \in \pi_1(\mathrm{SO}(3), \mathrm{id})$

(10) One can see that the loop $\gamma * \gamma$ is null-homotopic as follows: one can untwist the belt while keeping the end in standard position, put differently, we can find a path-homotopy from the loop $\gamma * \gamma$ to the trivial loop. This homotopy is illustrated in the figure below.



The fact that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$ lies at the heart of some other tricks, in particular the "plate trick" and "Dirac's string trick". More information and some videos can be found here:

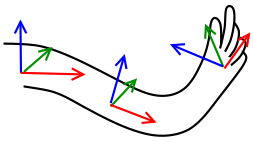
https://en.wikipedia.org/wiki/Plate_trick
https://www.youtube.com/watch?v=JaIR-cWk_-o
https://www.youtube.com/watch?v=JDJKfs3HqRg
https://www.youtube.com/watch?v=ICEIgznuHmg

83.3. **The plate trick ❀.** The plate trick is a variation on the belt trick. It describes the same phenomenon as the belt trick. The plate trick has the great advantage that the only tool it requires is the body of a volunteer. The disadvantage is that it is perhaps mathematically slightly less compelling.

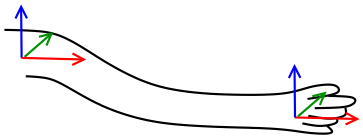
The basic idea is as in the belt trick. Namely we want to describe paths and loops in SO(3). This time we think of a path from the shoulder to ones left hand as a path in SO(3):

(a) the first vector lies on the arm and it points from the shoulder to the hand,
(b) the second vector also lies on the arm and it points from one side to the other,
(c) the third vector points away from the body.

If the fingers of the left hands are parallel to the shoulder, then we see a loop in SO(3).



the path from the shoulder
to the hand defines a path in SO(3)

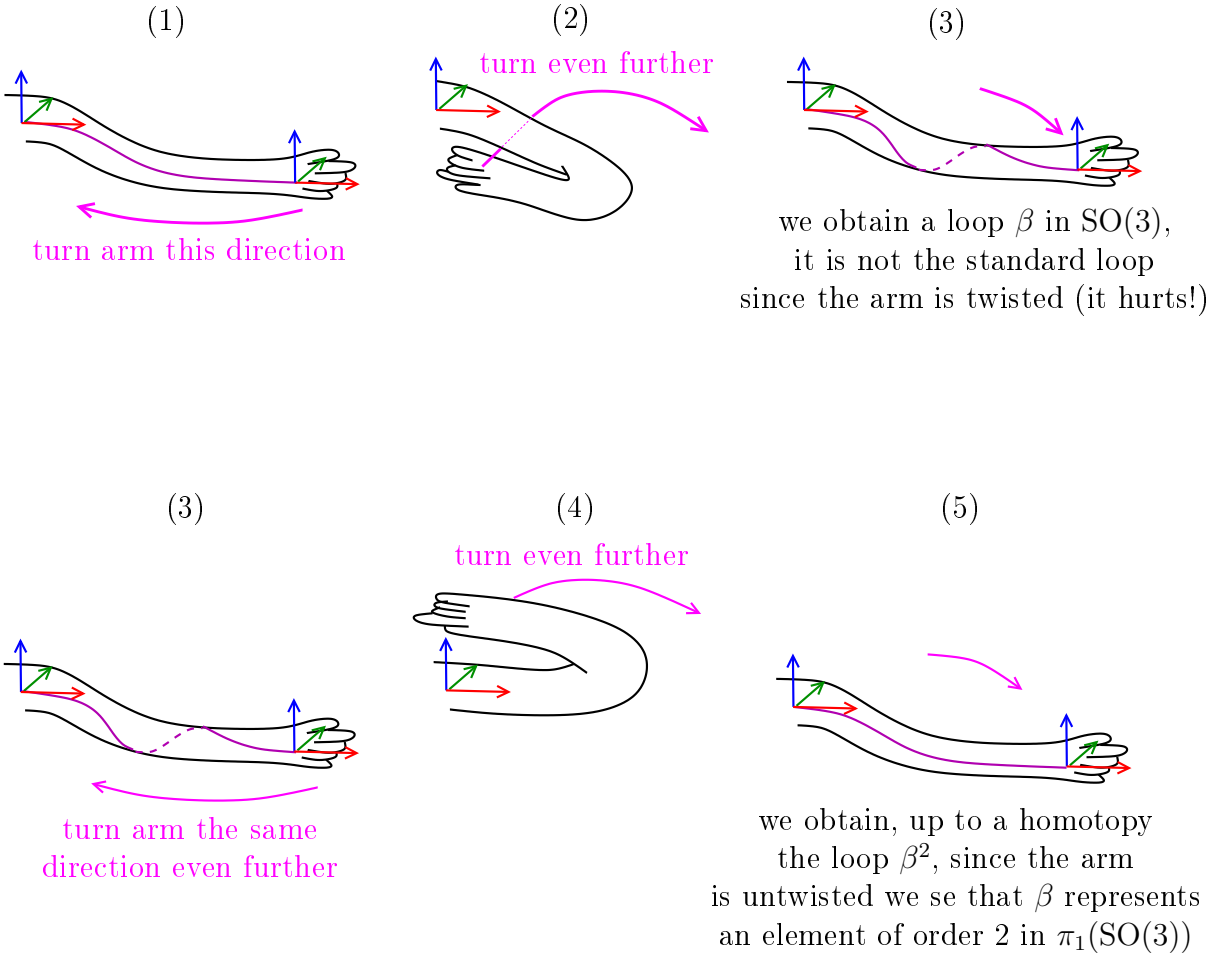


hand parallel to the shoulder
defines a loop in SO(3)

Now you, or your volunteer, does the following:

(1) You stretch your left arm and hands outwards such that the fingers are parallel to the shoulder.
(2) You rotate the left arm counterclockwise. The physiology of most arms will force the forearm to go under the arm.
(3) One continues till one does a 2π turn and the left arm is outstretched again. At this time we are describing a loop β in SO(3) which is non-constant since the arm is twisted. (If it doesn't hurt slightly, then the experiment is done incorrectly).

(4) Now one rotates the left arm still counterclockwise. The usual physiology will force the movement of the forearm to go over the arm.
(5) Again one continues till one has done another 2π turn. Up to a homotopy, keeping the end points fixed, the arm now describes $\beta * \beta$. But the arm turns out to be untwisted (which feels much better). This illustrates that β represents an element of order 2 in $\pi_1(\mathrm{SO}(3))$.



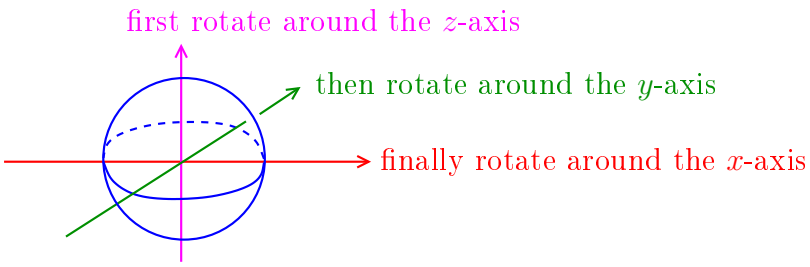
There are many more or less convincing illustrations and explanations of the plate trick. The most charming way of doing might be this one:

`http://ariwatch.com/VS/Algorithms/PlateTrick.htm`

83.4. **Topological robotics ❀.** In this section we want to discuss a "real-life" application of Corollary 83.4. Let us assume that we are given a solid ball and we are allowed to perform the following operations:

(1) first we can rotate the ball around the z-axis by an angle $\gamma \in S^1$,
(2) then we can rotate the ball around the y-axis by an angle $\beta \in S^1$, finally
(3) we can rotate the ball around the x-axis by an angle $\alpha \in S^1$.

Using these three operations, in that order, we can rotate the ball into any position. More precisely, we have the following lemma:



Lemma 83.6. The map

$$\begin{array}{rcl} \Phi\colon S^1\times S^1\times S^1 & \to & \mathrm{SO}(3) \\ (\alpha,\beta,\gamma) & \mapsto & \begin{array}{c}\text{rotation by } \alpha \\ \text{around the } x\text{-axis}\end{array} \circ \begin{array}{c}\text{rotation by } \beta \\ \text{around the } y\text{-axis}\end{array} \circ \begin{array}{c}\text{rotation by } \gamma \\ \text{around the } z\text{-axis}\end{array} \end{array}$$

is a surjection.[1006]

Remark. One can use Lemma 83.6 and a variation on the argument on page 1819 to give an alternative proof for the statement that $q\colon \mathrm{SU}(2)\to\mathrm{SO}(3)$ is surjective.

Sketch of proof. Let $A\in\mathrm{SO}(3)$ be matrix with columns v_1,v_2,v_3. After a rotation around the z-axis we can assume that v_1 lies in the xz-plane. After a rotation around the y-axis we can then arrange that v_1 lies in fact on the x-axis, i.e. we can arrange that $v_1=e_1$. Since $v_1=e_1,v_2,v_3$ is an orthonormal basis we see that v_2,v_3 lie in the yz-plane. After a rotation around the x-axis we can finally arrange that $v_2=e_2$ and $v_3=e_3$. ■

This lemma gives rise to the following practical problem: given a matrix $P\in\mathrm{SO}(3)$, can we continuously find parameters $\alpha(P),\beta(P),\gamma(P)\in S^1$ such that these parameters give back the matrix P? More precisely, we have the following question.

Question 83.7. Does there exist a continuous map $\Psi\colon \mathrm{SO}(3)\to S^1\times S^1\times S^1$ such that $\Phi\circ\Psi=\mathrm{id}_{\mathrm{SO}(3)}$?

It turns out that the answer is negative. More precisely, we have the following proposition.

Proposition 83.8. There is no continuous map $\Psi\colon \mathrm{SO}(3)\to S^1\times S^1\times S^1$ such that $\Phi\circ\Psi=\mathrm{id}_{\mathrm{SO}(3)}$.

This proposition falls into the area of "topological robotics", more information on this example and on topological robotics can be found in [**Gh2014**, p. 76]. Other aspects of topological robotics are discussed in the monograph [**Far2008**].

[1006] The map is also continuous. Why is that?

Proof. Let us suppose there exists a continuous map $\Psi\colon \mathrm{SO}(3) \to S^1 \times S^1 \times S^1$ such that $\Phi \circ \Psi = \mathrm{id}_{\mathrm{SO}(3)}$. Then we get the following commutative diagram

$$\begin{array}{ccc} & \overbrace{\pi_1(S^1 \times S^1 \times S^1)}^{\cong \mathbb{Z}^3} & \\ {\scriptstyle \Psi_*}\nearrow & & \searrow{\scriptstyle \Phi_*} \\ \underbrace{\pi_1(\mathrm{SO}(3))}_{\cong \mathbb{Z}_2} & \xrightarrow{\mathrm{id}_*} & \underbrace{\pi_1(\mathrm{SO}(3))}_{\cong \mathbb{Z}_2}. \end{array}$$

By Corollary 83.4 we have that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$ and from Corollary 48.18 we know that $\pi_1(S^1 \times S^1 \times S^1) \cong \mathbb{Z}^3$. Since $\mathbb{Z}_2$ is a finite group and since $\mathbb{Z}^3$ is torsion-free we see that the homomorphism $\Psi_*\colon \pi_1(\mathrm{SO}(3)) \to \pi_1(S^1 \times S^1 \times S^1)$ is the trivial map. But that is not possible since the horizontal map is an isomorphism of a non-trivial group. ■

83.5. **Linkages ❀.** We conclude this chapter with a somewhat unrelated but fun topic that also appears in topological robotics. We recall a few definitions.

Definition.

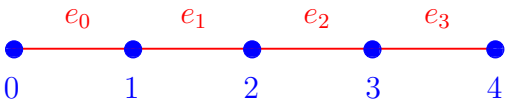
(1) An undirected abstract graph G is a triple (V, E, λ) where V is a non-empty set, E is a set and λ is a map

$$\lambda\colon E \ \to \ \{\text{subsets of } V \text{ with one or two elements}\}.$$

The elements of V are called vertices of G and the elements of E are called the edges of G. Furthermore, given $e \in E$ the points in $\lambda(e)$ are called the endpoints of e.

(2) Given $n \in \mathbb{N}$ we define the linear graph of length n to be the undirected abstract graph $G(n)$ which is given by the vertex set $V = \{0, \ldots, n\}$ and the edge set $\{e_0, \ldots, e_{n-1}\}$ with $\lambda(e_i) = \{i, i+1\}$ for $i = 0, \ldots, n-1$.

the linear graph $G(4)$



Definition.

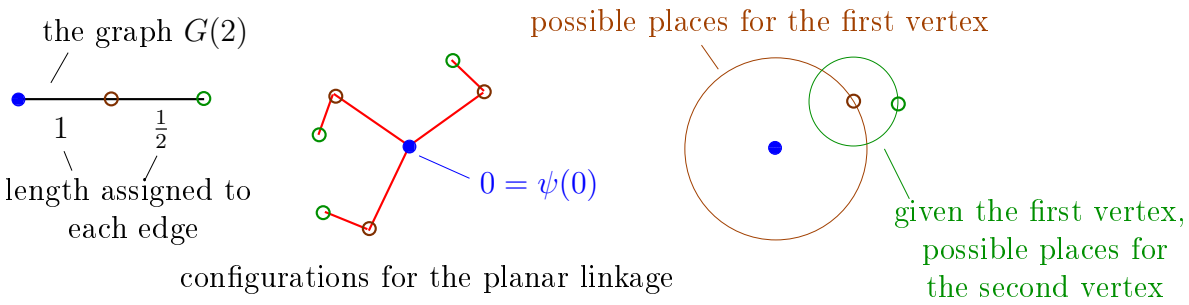
(1) A planar linkage is a triple $(G = (V, E), \lambda, \psi)$ consisting of a finite undirected abstract graph $G = (V, E)$, a function $\lambda\colon E \to \mathbb{R}_{>0}$ and a map $\psi\colon W \to \mathbb{C} = \mathbb{R}^2$ where W is a subset of V. Put differently, λ assigns a length to each edge and ψ represents a subset of the vertices by points in $\mathbb{C}$.

(2) A configuration of a planar linkage $(G = (V, E), \lambda, \psi)$ is a map $f\colon V \to \mathbb{R}^2 = \mathbb{C}$ such that $f|_W = \psi$ and such that for every edge e the distance between the endpoints of e is $\lambda(e)$.[1007]

(3) The set $C(G = (V, E), \lambda, \psi)$ of all configurations is called the configuration space of the linkage. We have an injective map

$$\begin{array}{rcl} C(G = (V, E), \lambda, \psi) & \to & \mathbb{C}^{|V|} := \mathrm{maps}(V \to \mathbb{C}) \\ f & \mapsto & (v \mapsto f(v)) \end{array}$$

thus we can view the configuration space $C(G = (V, E), \lambda, \psi)$ as a topological space.



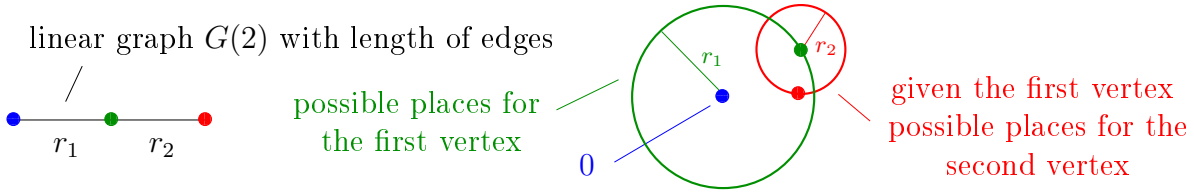
configurations for the planar linkage

In some cases the topological shape of the configuration space of a planar linkage is easy to describe. More precisely, we have the following lemma which is a straightforward consequence of the definitions and the Compact-Hausdorff Proposition 2.17 (3). We leave it to the reader to provide the proof.

Lemma 83.9. Let $G(m)$ be the linear graph with m edges. We consider a planar linkage of the form $(G(m), \lambda, \psi(0) = 0)$. For $i = 1, \ldots, m$ we write $r_i = \lambda(e_i)$. Then the map

$$\begin{array}{rcl} (S^1)^m = S^1 \times \cdots \times S^1 & \to & C(G(m), \lambda, \psi) \\ (z_1, \ldots, z_m) & \mapsto & \left(\begin{array}{rcl} \{0, \ldots, m\} & \to & \mathbb{C} \\ k & \mapsto & \sum_{i=1}^{k} r_i z_i \end{array} \right) \end{array}$$

is a homeomorphism.



For general linkages the topology of the configuration spaces are much harder to describe than for free planar linkages.

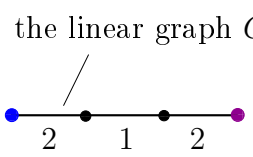
(1) Given $r_1, \ldots, r_m > 0$ we denote by $C(r_1, \ldots, r_m, P, Q)$ the configuration space corresponding to the planar linkage defined by $G(m)$, $\lambda(e_i) = r_i$ and by the conditions that $\psi(0) = P$ and $\psi(m) = Q$.
   (a) Using elementary arguments one can easily show that

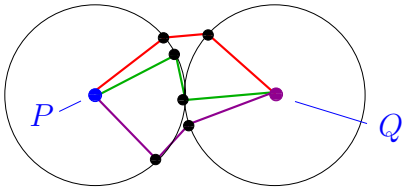
$$C(r_1, r_2, P, Q) \;=\; \begin{cases} \varnothing, & \text{if } r_1 + r_2 < d(P, Q), \\ \text{one point}, & \text{if } r_1 + r_2 = d(P, Q), \\ \text{two points}, & \text{if } r_1 + r_2 > d(P, Q). \end{cases}$$

   (b) Kevin Walker [**WalkK1985**, p. 35], see also [**Gh2014**, p. 12], showed in 1985 in his bachelor thesis that the surfaces of genus $0, 1, \ldots, 4$ are homeomorphic to configuration spaces of the type $(r_1, r_2, r_3, r_4, P, Q)$ for suitable values of $r_1, \ldots, r_4$ and points P and Q.

(2) Michael Kapovich and John Millson [**KaM2002**] showed that given any compact smooth manifold M there exists a planar linkage such that each component of the

[1007] In particular if e has only one endpoint, then we need $\lambda(e) = 0$.





configurations for the planar linkage

configuration space is homeomorphic to M. In particular this shows that any compact smooth manifold can be represented by a 2-dimensional physical object.

Exercises for Chapter 83.

Exercise 83.1. Let $q\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ be the homomorphism that we defined in Lemma 83.1. Show that

$$q\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},\quad q\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\quad q\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Remark. This calculation will actually turn out to be useful in Section 98.4.

84. The Hurewicz Theorem in dimension one

In this chapter we want to relate the first homology group to the fundamental group of a path-connected topological space X. Both are defined by mapping 1-dimensional smooth manifolds into X, hence it is reasonable to expect that there is a relationship. But in general these two groups associated to a topological space cannot be the same, since the first homology group is by definition abelian, whereas we saw on many occasions that fundamental groups can be non-abelian.
We start out with the following notation.

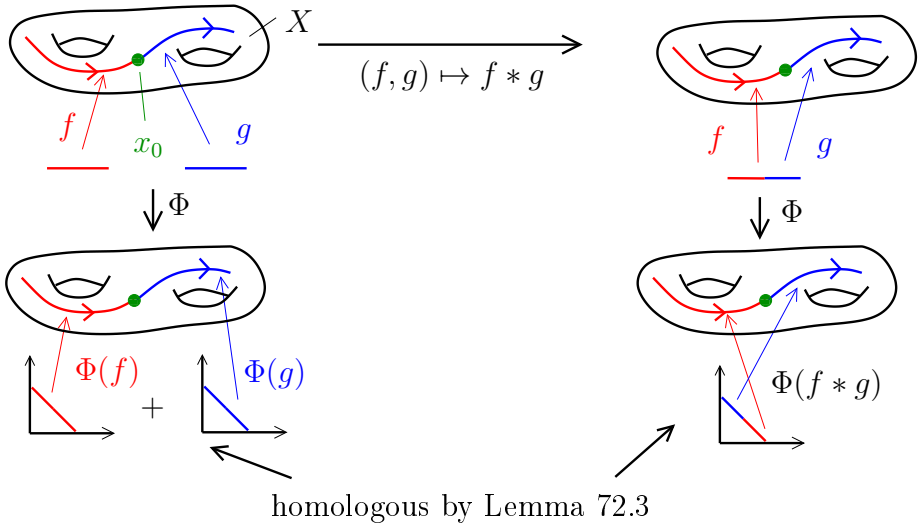
Notation. Let X be a topological space. Throughout this chapter we denote by Φ the following bijection:

$$\begin{array}{rcl} \Phi\colon \{\text{paths } [0,1] \to X\} & \to & \{\text{singular 1-simplices in } X\} \\ (\gamma\colon [0,1] \to X) & \mapsto & \begin{pmatrix} \Delta^1 & \to & X \\ (1-t,t) & \mapsto & \gamma(t) \end{pmatrix}. \end{array}$$

On many occasions we will use the trivial observation that for any path $\gamma\colon [0,1] \to X$ we have $\Phi(\overline{\gamma}) = \overline{\Phi(\gamma)}$.

Lemma 84.1. Let X be a topological space. If f and g are two paths in X such that the endpoint of f agrees with the starting point of g, then the singular 1-chains $\Phi(f) + \Phi(g)$ and $\Phi(f * g)$ are homologous.

Proof. The lemma follows quite easily from Lemma 72.3. We refer to the figure below for a sketch of the proof. We leave it to the reader to turn the sketch into a proper proof. ∎



Proposition 84.2. Let (X, x_0) be a pointed topological space.

(1) If γ is a loop in (X, x_0), then $\Phi(\gamma) \in \mathrm{C}_1(X)$ is a cycle.
(2) The map Φ induces a well-defined map

$$\begin{array}{rcl} \Phi_{(X,x_0)}\colon \pi_1(X,x_0) & \to & \mathrm{H}_1(X) \\ [\gamma\colon [0,1] \to X] & \mapsto & [\Phi(\gamma)]. \end{array}$$

(3) The map $\Phi_{(X,x_0)}\colon \pi_1(X,x_0) \to \mathrm{H}_1(X)$ is a group homomorphism.

(4) Let $x_1 \in X$ be some other point and let $\gamma\colon [0,1] \to X$ be a path from x_0 to x_1. We denote by

$$\begin{array}{rcl} \gamma_*\colon \pi_1(X,x_1) & \to & \pi_1(X,x_0) \\ [\varphi] & \mapsto & [\gamma * \varphi * \overline{\gamma}] \end{array}$$

the isomorphism that we introduced in the Change-of-Base Point Proposition 47.10. The following diagram commutes:

$$\begin{array}{ccc} \pi_1(X,x_1) & \xrightarrow{\Phi_{(X,x_1)}} & \\ \downarrow{\scriptstyle \gamma_*} & & \mathrm{H}_1(X). \\ \pi_1(X,x_0) & \xrightarrow{\Phi_{(X,x_0)}} & \end{array}$$

(5) Let *PTop* be the category of pointed topological spaces and let *Group* be the category of groups. The Hurewicz homomorphisms

$$\Phi_{(X,x_0)}\colon \pi_1(X,x_0) \to \mathrm{H}_1(X)$$

define a natural transformation from the functor

$$\begin{array}{rcl} \mathit{PTop} & \to & \mathit{Group} \\ (X,x_0) & \mapsto & \pi_1(X,x_0) \end{array} \quad \text{to the functor} \quad \begin{array}{rcl} \mathit{PTop} & \to & \mathit{Group} \\ (X,x_0) & \mapsto & \mathrm{H}_1(X). \end{array}$$

In plain English this means the following, given a map $f\colon X \to Y$ between topological spaces and given $x_0 \in X$ the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X,x_0) & \xrightarrow{f_*} & \pi_1(Y,f(x_0)) \\ {\scriptstyle \Phi_{(X,x_0)}}\downarrow & & \downarrow{\scriptstyle \Phi_{(Y,f(x_0))}} \\ \mathrm{H}_1(X) & \xrightarrow{f_*} & \mathrm{H}_1(Y). \end{array}$$

Proof.

(1) The first statements follows immediately from the definitions and the description of $\partial_1\colon \mathrm{C}_1(X) \to \mathrm{C}_0(X)$ given on page 1585.

(2) We denote by $\sim$ the equivalence relation on $[0,1]$ that is generated by $0 \sim 1$. We denote by $\alpha\colon \Delta^1 \to [0,1]/\!\sim$ the map that is given by $\alpha(1-t,t) = [t]$. Note that α defines an element in $\mathrm{H}_1([0,1]/\!\sim)$. Now suppose that $f,g\colon [0,1] \to X$ are two path-homotopic loops in x_0. We denote by $\widetilde{f},\widetilde{g}\colon [0,1]/\!\sim \to X$ the induced maps. Then

$$[\Phi(f)] \underset{\substack{\uparrow \\ \text{since } \Phi(f) \text{ and } \widetilde{f}\circ\alpha \text{ are same map}}}{=} \left[\widetilde{f}\circ\alpha\right] = \widetilde{f}_*([\alpha]) \underset{\substack{\uparrow \\ \text{by the Homotopic Maps-and-Homology Proposition 73.6, since } \widetilde{f} \text{ and } \widetilde{g} \text{ are homotopic maps } [0,1]/\!\sim \to X}}{=} \widetilde{g}_*([\alpha]) = \left[\widetilde{g}\circ\alpha\right] = [\Phi(g)].$$

(3) The third statement is an immediate consequence of Lemma 84.1.

(4) This statement follows quite easily from Lemma 72.3.

(5) The last statement follows easily from the definitions. ∎

Definition. Given a pointed topological space (X,x_0) we refer to the homomorphism

$$\begin{array}{rcl} \Phi_{(X,x_0)}\colon \pi_1(X,x_0) & \to & \mathrm{H}_1(X) \\ [\gamma\colon [0,1] \to X] & \mapsto & [\Phi(\gamma)] \end{array}$$

as the Hurewicz homomorphism.[1008]

It is clear that the Hurewicz homomorphism is in general neither injective nor surjective:

(1) The Hurewicz homomorphism is in general not injective since $\pi_1(X, x_0)$ is in many cases a non-abelian group whereas $\mathrm{H}_1(X)$ is an abelian group, and it is not possible that a non-abelian group is a subgroup of an abelian group.
(2) The Hurewicz homomorphism is in general not surjective, since $\pi_1(X, x_0)$ depends only on the path-component of x_0, while $\mathrm{H}_1(X)$ "sees" all path-components of X. But as we will see in the next lemma, this turns out to be the only obstacle to Φ being surjective.

Proposition 84.3. Let X be a topological space and let $x_0 \in X$. If X is path-connected, then the Hurewicz homomorphism

$$\Phi_{(X,x_0)} \colon \pi_1(X, x_0) \ \to \ \mathrm{H}_1(X)$$

is surjective.

In the proof of Proposition 84.3 we will need the following lemma.

Lemma 84.4. Let X be a set and let $x_1, \ldots, x_k$ and $y_1, \ldots, y_l$ be elements in X. If

$$\sum_{i=1}^{k} x_i \ - \ \sum_{i=1}^{l} y_i \ = \ 0 \ \in \ \mathbb{Z}^{(X)},$$

then $k = l$ and there exists a permutation $s \in S_k$ such that $y_i = x_{s(i)}$ for $i = 1, \ldots, k$.

Proof. Let X be a set and let $x_1, \ldots, x_k$ and $y_1, \ldots, y_l$ be elements in X such that

$$\underbrace{\sum_{i=1}^{k} x_i}_{=:a} \ = \ \underbrace{\sum_{i=1}^{l} y_i}_{=:b} \quad \in \ \mathbb{Z}^{(X)}.$$

It follows from Lemma 51.1 that there exists a unique homomorphism $\psi \colon \mathbb{Z}^{(X)} \to \mathbb{Z}$ with the property that $\psi(x) = 1$ for all $x \in X$. By definition we have $\psi(a) = k$ and $\psi(b) = l$. So from $a = b$ we deduce that $k = l$.

Note that it follows from applying Lemma 51.1 once again that there exists a unique homomorphism $\psi \colon \mathbb{Z}^{(X)} \to \mathbb{Z}$ with the property that $\psi(x_k) = 1$ and $\psi(x) = 0$ for all $x \in X \setminus \{x_k\}$. By definition we have $\psi(a) > 0$. From $\psi(b) = \psi(a) > 0$ it follows that there exists an $n \in \{1, \ldots, k\}$ such that $y_n = x_k$. We define $s(k) = n$. We obtain an equality

$$\sum_{i=1}^{k-1} x_i \ = \ \sum_{i \in \{1,\ldots,k\} \setminus \{s(k)\}} y_i \ \in \ \mathbb{Z}^{(X)}$$

and we iterate this procedure to define $s(k-1), \ldots, s(1)$. By construction the map $s \colon \{1, \ldots, k\} \to \{1, \ldots, k\}$ is a bijection ∎

Proof of Proposition 84.3. We denote by Ψ the inverse of the above bijection

$$\Phi \colon \{\text{paths } [0,1] \to X\} \ \to \ \{\text{singular 1-simplices in } X\}$$

We pick a homology class $z \in \mathrm{H}_1(X)$. Let $c = \sum_{i=1}^{k} a_i \cdot \sigma_i$ be a cycle in $\mathrm{C}_1(X)$ that represents z. It follows from Lemma 72.3 that, after possibly replacing a singular chain σ_i by $\overline{\sigma}_i$, we can assume that all the coefficients a_i are positive. Furthermore, since we do not require that

[1008] Witold Hurewicz (1904-1956) was a Polish mathematician.

the σ_i's are different, we can in fact suppose that $a_1 = \cdots = a_k = 1$. Next note that we have

$$0 \underset{\substack{\uparrow \\ \text{since } c \text{ is a cycle}}}{=} \partial_1(c) = \partial_1\Big(\sum_{i=1}^k \sigma_i\Big) \underset{\substack{\uparrow \\ \text{see discussion on page 1585}}}{=} \sum_{i=1}^k \sigma_i(0,1) - \sigma_i(1,0) = \sum_{i=1}^k \sigma_i(0,1) + \sum_{i=1}^k -\sigma_i(1,0).$$

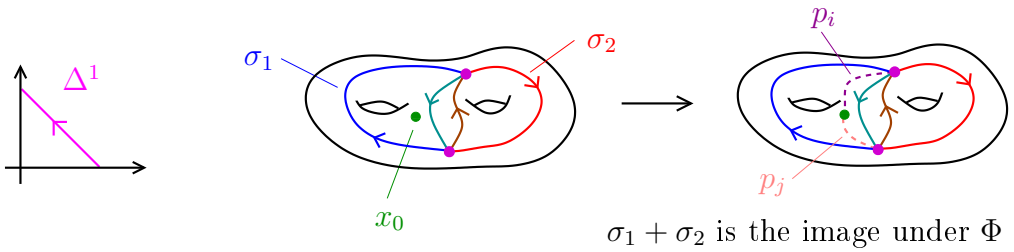
On the right hand side each "positive summand" needs to cancel with a "negative summand". This means, see also Lemma 84.4, that there exists a permutation $\theta \in S_k$ such that for each i the "endpoint" $\sigma_i(0,1)$ of σ_i is just the "starting point" $\sigma_{s(i)}(1,0)$ of $\sigma_{s(i)}$.

Since X is path-connected we can choose for each $i \in \{1,\dots,k\}$ a path p_i from the base point x_0 to the starting point $\sigma_i(1,0)$ of σ_i.
We obtain the following equality in $\mathrm{H}_1(X)$:

$$c = \sum_{i=1}^k \sigma_i \overset{\substack{\text{by Lemma 72.3 the singular chain} \\ \Phi(p_i)+\Phi(\overline{p_i}) = \Phi(p_i)+\overline{\Phi(p_i)} \\ \text{is a null-homologous cycle}}}{=} \sum_{i=1}^k \sigma_i + \sum_{i=1}^k \Phi(p_i) + \sum_{i=1}^k \Phi(\overline{p}_i) \overset{\substack{\text{since } \Phi\circ\Psi = \mathrm{id} \text{ and since permutation of the} \\ \text{summands does not change the sum}}}{=} \sum_{i=1}^k \big(\Phi(p_i) + \Phi(\Psi(\sigma_i)) + \Phi(\overline{p}_{s(i)})\big)$$

$$\underset{\substack{\uparrow \\ \text{by Lemma 84.1}}}{=} \sum_{i=1}^k \Phi\big(\underbrace{p_i * \Psi(\sigma_i) * \overline{p}_{s(i)}}_{\substack{\text{loop in } (X,x_0) \\ \text{by choice of } s}}\big) \underset{\substack{\uparrow \\ \text{by Lemma 84.1}}}{=} \Phi\Bigg(\underbrace{\prod_{i=1}^k p_i * \Psi(\sigma_i) * \overline{p}_{s(i)}}_{\text{loop in } (X,x_0)}\Bigg).$$

Thus we have shown that $z = [c] \in \Phi(\pi_1(X,x_0))$. This concludes the proof that the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_1(X,x_0) \to \mathrm{H}_1(X)$ is surjective. ■



$\sigma_1 + \sigma_2$ is the image under Φ of the product of the blue and the red loop

In the following we want to study to what degree the Hurewicz homomorphism fails to be injective. Before we can do so we need to recall some definitions and results from Section 53.5.

Definition. Let π be a group.

(1) For $x, y \in \pi$ we denote by $[x,y] := xyx^{-1}y^{-1}$ the commutator of x and y.
(2) We refer to

$$[\pi,\pi] := \langle\langle \{[x,y] \mid x,y \in \pi\} \rangle\rangle = \Big\{ \prod_{i=1}^n [x_i,y_i] \,\Big|\, x_1,y_1,\dots,x_n,y_n \in \pi \Big\}$$

as the commutator subgroup of π.
(3) We refer to $\pi_{\mathrm{ab}} := \pi/[\pi,\pi]$ as the abelianization of π.

In Section 53.5 we saw that the abelianization is "the largest abelian quotient" of π. More precisely, if $\alpha\colon \pi \to H$ is a homomorphism from π to an abelian group H, then the

Abelianization Proposition 53.20 says that there exists a unique homomorphism $\beta\colon \pi_{\mathrm{ab}} = \pi/[\pi,\pi] \to H$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi & \longrightarrow & \pi_{\mathrm{ab}} = \pi/[\pi,\pi] \\ & \searrow^{\alpha} & \downarrow{\scriptstyle \exists!\,\beta} \\ & & H. \end{array}$$

In particular, given a pointed topological space (X, x_0) the corresponding Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_1(X,x_0) \to \mathrm{H}_1(X)$ induces a homomorphism $\pi_1(X,x_0)_{\mathrm{ab}} \to \mathrm{H}_1(X)$ that we also denote by $\Phi_{(X,x_0)}$ and that we also refer to as the Hurewicz homomorphism. Now we can formulate the following proposition.

Theorem 84.5. (Hurewicz Theorem) Let (X, x_0) be a pointed topological space. If X is path-connected, then the Hurewicz homomorphism

$$\Phi_{(X,x_0)}\colon \pi_1(X,x_0)_{\mathrm{ab}} \;\to\; \mathrm{H}_1(X)$$

is a natural[1009] isomorphism.

Examples.

(1) If we compare our calculations of fundamental groups with our calculations of homology groups, then we see that we had encountered this isomorphism many times.
   (a) In Corollary 48.18 we showed that $\pi_1((S^1)^n) \cong \mathbb{Z}^n$, whereas on page 1744 we saw that $\mathrm{H}_1((S^1)^n) \cong \mathbb{Z}^n$.
   (b) For $n \geq 2$ we saw in Corollary 48.18 that $\pi_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}_2$ and in Lemma 78.8 we had calculated that $\mathrm{H}_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}_2$.
   (c) In the Fundamental Group-of-Surfaces Proposition 54.5 we showed that

$$\pi_1(\text{Klein bottle}) \;\cong\; \langle x, y \,|\, yxyx^{-1}\rangle.$$

   The abelianization of that group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$ which is isomorphic to the first homology of the Klein bottle that we had determined in Lemma 78.12.
   (d) In the Fundamental Group-of-Surfaces Proposition 54.5 we showed that for any $g \in \mathbb{N}$ we have

$$\pi_1(\text{surface of genus } g) \;\cong\; \langle x_1, y_1, \ldots, x_g, y_g \,|\, [x_1,y_1]\cdot\cdots\cdot[x_g,y_g]\rangle.$$

   The abelianization of this group is isomorphic to $\mathbb{Z}^{2g}$ which is precisely the first homology group that we had determined in Proposition 80.10.
   (e) Let X be a path-connected topological space and let $f\colon X \to X$ be a homeomorphism. By the Mapping Torus-Homology Lemma 78.22 we know that there exists a short exact sequence

$$0 \;\to\; \operatorname{coker}\big(\mathrm{H}_1(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_1(X)\big) \;\xrightarrow{i}\; \mathrm{H}_1(\mathrm{Tor}(X,f)) \;\to\; \mathbb{Z} \;\to\; 0.$$

   On the other hand it follows from the Mapping Torus-π_1-Proposition 50.6 and the remark on page 1129 that there exists an isomorphism $\psi\colon \pi_1(\mathrm{Tor}(X,f)) \cong$

[1009] Whenever we write "natural" one needs to work out what the categories and functors are. In this case we consider the category of path-connected topological spaces, the category of groups and the functors $(X,x_0) \to \pi_1(X,x_0)_{\mathrm{ab}}$ and $(X,x_0) \mapsto \mathrm{H}_1(X)$.

$\pi_1(X) \rtimes \mathbb{Z}$ where the action of $1 \in \mathbb{Z}$ on $\pi_1(X)$ is induced by f. Therefore we can put $\pi_1(\mathrm{Tor}(X,f))$ into a short exact sequence

$$0 \;\to\; \pi_1(X) \;\to\; \pi_1(\mathrm{Tor}(X,f)) \;\to\; \mathbb{Z} \;\to\; 0$$

Now we apply the abelianization functor to these groups and homomorphisms. We obtain the exact sequence[1010]

$$\pi_1(X)_{\mathrm{ab}} \;\to\; \pi_1(\mathrm{Tor}(X,f))_{\mathrm{ab}} \;\to\; \mathbb{Z} \;\to\; 0$$

which, using the Hurewicz Theorem 84.5, we can rewrite as

$$\mathrm{H}_1(X) \;\to\; \mathrm{H}_1(\mathrm{Tor}(X,f)) \;\to\; \mathbb{Z} \;\to\; 0.$$

This is, fortunately, consistent with the above short exact sequence.

(2) Note that we now have a new, and depending on one's taste, more satisfying proof of the Knot Group-Abelianization Corollary 59.25: The Hurewicz Theorem 84.5 together with Lemma 78.13 implies that given any knot $K \subset S^3$ the abelianization of $\pi_1(S^3 \setminus K)$ is isomorphic to $\mathbb{Z}$.

Proof of the Hurewicz Theorem 84.5. Let (X, x_0) be a pointed path-connected topological space. We write $\pi := \pi_1(X, x_0)$. By Proposition 84.2 we know that the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_{\mathrm{ab}} \to \mathrm{H}_1(X)$ is natural. Furthermore it follows from Proposition 84.3 that the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_{\mathrm{ab}} \to \mathrm{H}_1(X)$ is surjective. It remains to show that $\Phi_{(X,x_0)}$ is injective.

In the following we will make the identification $[0,1] = \Delta^1$ that is given by the homeomorphism $t \mapsto (1-t, t)$. In particular we will identify paths $[0,1] \to X$ with singular 1-simplices $\Delta^1 \to X$.

So let $f\colon [0,1] = \Delta^1 \to X$ be a loop in (X, x_0) such that $[f] = 0 \in \mathrm{H}_1(X)$. By definition this means that there exists a 2-chain $\sum_{j=1}^{s} n_j \sigma_j$ in X whose boundary is the singular 1-simplex f. Furthermore, since we do not demand that the σ_j are distinct, we can assume that all n_j are either -1 or 1. Given $j \in \{1, \dots, s\}$ and $k \in \{0,1,2\}$ we write[1011]

$$\mu_{(j,k)} := \sigma_j \circ i^2_k\colon [0,1] = \Delta^1 \to X \quad \text{ and } n_{(j,k)} := (-1)^k \cdot n_j.$$

We write $A = \{1, \dots, s\} \times \{0,1,2\}$. We have the following equality in $\mathrm{C}_1(X)$:

$$f \;=\; \partial_1\Big(\sum_{j=1}^{s} n_j \cdot \sigma_j\Big) \;\underset{\substack{\uparrow\\ \text{definition of } \partial_1}}{=}\; \sum_{j=1}^{s}\sum_{k=0}^{2} n_{(j,k)} \cdot \mu_{j,k} \;=\; \sum_{a \in A} n_a \cdot \mu_a.$$

By comparing the left and the right-hand side of this equality we see that there exists a $b \in A$ with $n_b \cdot \mu_b = 1 \cdot f$. We write $A_{\pm} = \{a \in A \setminus \{b\} \mid n_a = \pm 1\}$. Now we obtain from

[1010] In Exercise 53.6 we saw that if $\varphi\colon A \to B$ is an epimorphism, then the induced map $A_{\mathrm{ab}} \to B_{\mathrm{ab}}$ is also an epimorphism. We had also seen that there exist monomorphisms $\varphi\colon A \to B$ such that the induced map $A_{\mathrm{ab}} \to B_{\mathrm{ab}}$ is not a monomorphism.

[1011] Recall that for $j \in \{0, \dots, n\}$ we denote by i^n_j the map

$$\begin{array}{rcl} i^n_j\colon \Delta^{n-1} & \to & \Delta^n \\ (t_0, \dots, t_{n-1}) & \mapsto & (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{n-1}). \end{array}$$

the above equality that

$$\sum_{a\in A_+}\mu_a \;+\; \sum_{a\in A_-}(-1)\cdot\mu_a \;=\; 0.$$

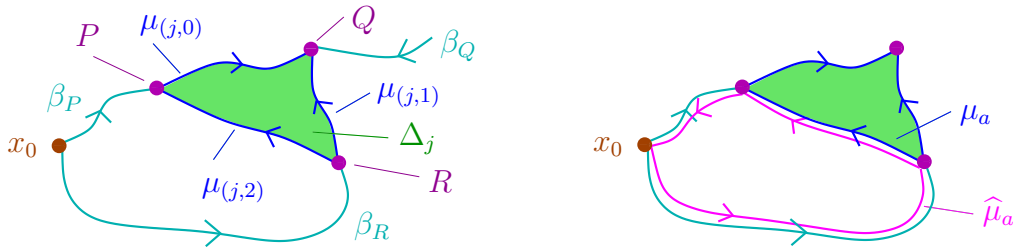
We make the following choices and introduce the following definitions:

(1) For each point P in X that is a starting or an endpoint of some μ_a we pick a path β_P from x_0 to P. For $P = x_0$ we just pick the constant path.

(2) Given $\mu_{(i,j)}\colon [0,1] = \Delta^1 \to X$ we write

$$\widehat{\mu}_{(i,j)} \;:=\; \beta_{\text{starting point of } \mu_{(i,j)}} * \mu_{(i,j)} * \overline{\beta}_{\text{endpoint of } \mu_{(i,j)}}.$$

Note that $\widehat{\mu}_{(i,j)}$ is a loop in (X, x_0), in particular it defines an element in the group $\pi = \pi_1(X, x_0)$ which, by a slight abuse of notation, we denote again by $\widehat{\mu}_{(i,j)}$.



We make the following elementary observation: for any $(i,j) \in \{1,\dots,s\}\times\{0,1,2\}$ we have the following equality in $\pi = \pi_1(X, x_0)$:

$$(*)\qquad \widehat{\mu}^{-1}_{(i,j)} \;:=\; \beta_{\text{starting point of } \overline{\mu}_{(i,j)}} * \overline{\mu}_{(i,j)} * \overline{\beta}_{\text{endpoint of } \overline{\mu}_{(i,j)}}.$$

In π_{ab} we have the following equalities:[1012]

$$e \;=\; \prod_{j=1}^{s} \beta_{\text{starting point of } \mu_{(j,0)}} * \Big(\overbrace{\mu_{(j,0)} * \overline{\mu}_{(j,1)} * \mu_{(j,2)}}^{\text{loop in the starting point of } \mu_{(j,0)}}\Big)^{n_j} * \overline{\beta}_{\text{starting point of } \mu_{(j,0)}}$$

$\uparrow$ since $\mu_{(j,0)} * \overline{\mu}_{(j,1)} * \mu_{(j,2)}$ is null-homotopic

$$= \prod_{j=1}^{s} \widehat{\mu}_{(j,0)}^{n_{(j,0)}} \cdot \widehat{\mu}_{(j,1)}^{n_{(j,1)}} \cdot \widehat{\mu}_{(j,2)}^{n_{(j,2)}} \qquad = \prod_{a\in\{1,\dots,k\}\times\{0,1,2\}} \widehat{\mu}_a^{n_a}$$

$\uparrow$ follows from inserting null-homotopic paths $\overline{\beta}_P * \beta_P$ together with the above equality $(*)$

$$= f \cdot \prod_{a\in A_-} \widehat{\mu}_a^{n_a} \cdot \widehat{\mu}_{\psi(a)}^{n_{\psi(a)}} \qquad = f \cdot \prod_{a\in A_-} \widehat{\mu}_a^{n_a} \cdot \widehat{\mu}_a^{-n_a} \;=\; f.$$

$\uparrow$ since $\{1,\dots,k\}\times\{0,1,2\} = (1,0) \sqcup A_- \sqcup \psi(A_-)$ since $\mu_{(1,0)} = f$ and since $\beta_{x_0} = c_{x_0}$

$\uparrow$ for $a \in A_-$ we have $\mu_{\psi(a)} = \mu_a$ and $n_a = -n_a$.

This completes the proof of the proposition.[1013] ■

For most topological spaces for which we had computed the fundamental group we had also determined the homology groups. But there is one rather interesting exception, namely the lens spaces. From Corollary 48.18 and the Hurewicz Theorem 84.5 we obtain immediately the following corollary.

[1012] In the notation we do not distinguish between loops and elements they represent in π_{ab}.

[1013] Where did we actually use that we work with π_{ab} instead of π?

Corollary 84.6. For any coprime $p, q \in \mathbb{N}$ we have

$$\mathrm{H}_1(\text{lens space } L(p,q)) \;\cong\; \mathbb{Z}_p.$$

Remark. The Hurewicz Theorem 84.5 is perhaps a mixed blessing. On the one hand it allows us to use the techniques that we had developed in earlier chapters to determine first homology groups. On the other hand it tells us that all the information that is contained in the first homology group is already contained in the fundamental group. So we do not get any new information. In particular, as we saw in Corollary 84.6, the first homology group cannot help us in our long-standing goal of distinguishing lens spaces with the same fundamental group.

Calculating (isomorphism types) of homology groups can get quite involved. But what is often even harder is to determine the maps on homology groups induced by maps between topological spaces. Using Proposition 84.2 (5) we can transfer calculations of induced maps on fundamental groups to obtain the corresponding induced maps on homology groups. For example we have the following lemma, which later on will be used on many occasions.

Lemma 84.7. (Action-on-Torus Homology Lemma) Let $n \in \mathbb{N}$. We denote by $T = \mathbb{R}^n/\mathbb{Z}^n = (S^1)^n$ the n-dimensional torus.

(1) We denote by $\sigma \in \mathrm{H}_1(S^1)$ the standard generator introduced on page 1677. Given $i \in \{1, \ldots, n\}$ we denote by

$$\begin{array}{rcl} p_i \colon S^1 & \to & (S^1)^n \\ z & \mapsto & (1, \ldots, 1, z, 1, \ldots, 1) \end{array}$$

the obvious i-th inclusion map. If we set $e_i := p_{i*}(\sigma)$, then $e_1, \ldots, e_n$ is a basis of $\mathrm{H}_1(T)$.

(2) Let $A \in \mathrm{GL}(n, \mathbb{Z})$ be a matrix. We denote by $T = \mathbb{R}^n/\mathbb{Z}^n$ the n-dimensional torus. By the Maps-on-Torus Lemma 19.37, the map

$$\begin{array}{rcl} f(A) \colon T = \mathbb{R}^n/\mathbb{Z}^n & \to & T = \mathbb{R}^n/\mathbb{Z}^n \\ [v] & \mapsto & [A \cdot v] \end{array}$$

is a homeomorphism. With respect to the above basis $e_1, \ldots, e_n$ of $\mathrm{H}_1(T)$ the induced map $f(A)_* \colon \mathrm{H}_1(T) \to \mathrm{H}_1(T)$ is given by multiplication by the matrix A.

Remark.

(1) In Exercise 117.6 we will study what self-maps of $H_2(S^2 \times S^2)$ can be realized by self-maps of $S^2 \times S^2$.

(2) Exercise 76.20 can be used to give an alternative proof of the Action-on-Torus Homology Lemma 84.7 (2).

Proof.

(1) This statement follows easily from the definitions, the Fundamental Group-via-Actions Theorem 48.16 and the Hurewicz Theorem 84.5.

(2) Let $\Theta\colon \pi_1(T,0) \to \mathbb{Z}^n$ be the isomorphism from the Fundamental Group-via-Actions Theorem 48.16. By Proposition 84.2 (5), the Fundamental Group-via-Actions Theorem 48.16 and Exercise 48.18 we obtain the following commutative diagram

$$\begin{array}{ccccc} \mathrm{H}_1(T) & \xleftarrow{\Phi_{(T,0)}} & \pi_1(T,0) & \xrightarrow[\cong]{\Theta} & \mathbb{Z}^n \\ {\scriptstyle f(A)}\downarrow & & \downarrow{\scriptstyle f(A)_*} & & \downarrow{\scriptstyle v\mapsto A\cdot v} \\ \mathrm{H}_1(T) & \xleftarrow{\Phi_{(T,0)}} & \pi_1(T,0) & \xrightarrow[\cong]{\Theta} & \mathbb{Z}^n. \end{array}$$

The desired statement follows from the observation that the horizontal isomorphism $\Theta \circ \Phi_{(T,0}\colon \mathrm{H}_1(T) \to \mathbb{Z}^n$ is precisely the one corresponding to the basis $e_1, \ldots, e_n$. ■

As another illustration of Proposition 84.2 (5) we also prove the following corollary.

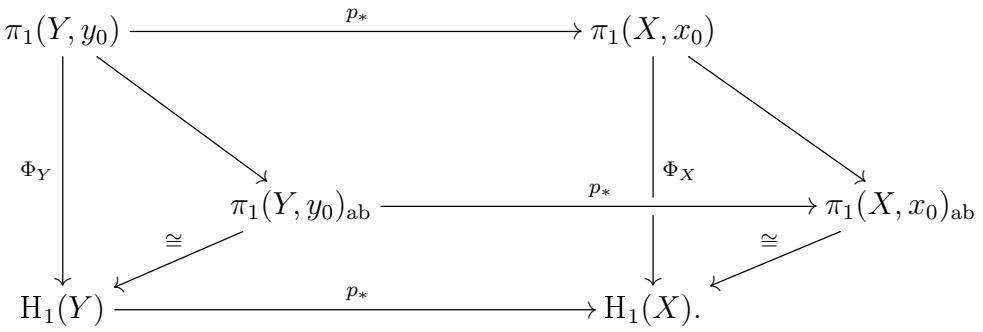
Corollary 84.8. Let $p\colon Y \to X$ be a finite-index covering of path-connected topological spaces. Then the cokernel of the map

$$p_*\colon \mathrm{H}_1(Y) \;\to\; \mathrm{H}_1(X)$$

is finite.

Remark. Let $p\colon Y \to X$ be a finite-index covering of path-connected topological spaces. Let $y_0 \in Y$. By Corollary 48.14 the induced map $p_*\colon \pi_1(Y,y_0) \to \pi_1(X,p(y_0))$ is a monomorphism. On the other hand it is in general not true that the induced homomorphism $p_*\colon \mathrm{H}_1(Y) \to \mathrm{H}_1(X)$ is a monomorphism. For example on page 1070 we saw that there exists a 2-fold covering map $p\colon T \to K$ from the torus T to the Klein bottle K. By Lemmas 78.11 and 78.12 we know that $\mathrm{H}_1(T) \cong \mathbb{Z}^2$ and that $\mathrm{H}_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. By Lemma 51.9 (2) there is no monomorphism from the group $\mathbb{Z}^2$ to $\mathbb{Z} \oplus \mathbb{Z}_2$.

Proof. We pick a base point y_0 of Y and we write $x_0 = p(y_0)$. We consider the following diagram of maps:



Here it follows from the Hurewicz Theorem 84.5 that the lower diagonal maps are isomorphisms. The "big rectangle in the back" commutes by Proposition 84.2 (5). The upper parallelogram commutes by the Abelianization Proposition 53.20 (7). Since the lower diagonal maps are isomorphisms it follows that the lower parallelogram also commutes.

By Lemma 48.15 and by our hypothesis that p is a finite covering we know that $p_*(\pi_1(Y,y_0))$ is a finite index subgroup of $\pi_1(X,x_0)$, i.e. $p_*\colon \pi_1(Y,y_0) \to \pi_1(X,x_0)$ has

a finite cokernel. It follows[1014] that $p_*\colon \pi_1(Y,y_0)_{\mathrm{ab}} \to \pi_1(X,x_0)_{\mathrm{ab}}$ also has a finite cokernel. From the commutativity of the lower parallelogram it follows immediately that the map $p_*\colon \mathrm{H}_1(Y) \to \mathrm{H}_1(X)$ has a finite cokernel as well. ■

Exercises for Chapter 84.

Exercise 84.1. Let $T = S^1 \times S^1$ be the torus. Let $\varphi, \psi\colon S^1 \to T$ be two smooth embeddings such that $\varphi_*([S^1]) = \psi_*([S^1]) \in \mathrm{H}_1(T)$. Show that φ and ψ are homotopic maps.

Exercise 84.2. Let $A \in \mathrm{SL}(2,\mathbb{Z})$. Prove the following statements:

(a) If $\mathrm{tr}(A) = 2$, then A is congruent over $\mathbb{Z}$ to a matrix of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

(b) If $\mathrm{tr}(A) = 2$, then A is congruent over $\mathbb{Z}$ to a matrix of the form $\begin{pmatrix} -1 & * \\ 0 & -1 \end{pmatrix}$.

(c) If $\mathrm{tr}(A) \in (-2,2)$, then A is a matrix of finite order in the group $\mathrm{SL}(2,\mathbb{Z})$.

(d) If $\mathrm{tr}(A) \notin [-2,2]$, then A has two real eigenvalues λ, λ^{-1} with $|\lambda| > 1$.

Exercise 84.3. Let $A \in \mathrm{SL}(2,\mathbb{Z})$. We consider the map

$$\begin{array}{rcl} f(A)\colon \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ [v] & \mapsto & [A \cdot v] \end{array}$$

and following the definition on page 467 we consider the corresponding mapping torus

$$\mathrm{Tor}(\mathbb{R}^2/S^2, f(A)) \;=\; ((\mathbb{R}^2/S^2) \times [0,1])/(x,0) \sim (f(A)(v),1).$$

(a) Show that $\mathrm{Tor}(\mathbb{R}^2/S^2, f(A))$ is a closed oriented 3-dimensional smooth manifold.

(b) Determine $\mathrm{rank}(\mathrm{H}_1(\mathrm{Tor}(\mathbb{R}^2/S^2, f(A))$ and $\mathrm{H}_2(\mathrm{Tor}(\mathbb{R}^2/S^2, f(A))$ in terms of the classification of Exercise 84.2.
Remark. You might want to use the Mapping Torus-Homology Proposition 78.20 and the Action-on-Torus Homology Lemma 84.7.

Exercise 84.4. Let $p\colon S^1 \times S^1 \to S^1$ and $q\colon S^1 \times S^1 \to S^1$ be the projection onto the first respectively second factor and let $\mu\colon S^1 \times S^1 \to S^1$ be the multiplication map. Show that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_1(S^1 \times S^1) & \xrightarrow{\quad\mu_*\quad} & \mathrm{H}_1(S^1). \\ & {\scriptstyle p_* \oplus q_*} \searrow \quad \nearrow {\scriptstyle (\varphi,\psi) \mapsto \varphi + \psi} & \\ & \mathrm{H}_1(S^1) \oplus \mathrm{H}_1(S^1) & \end{array}$$

Exercise 84.5. Let X and Y be path-connected topological spaces. Let $x \in X$ and let $y \in Y$. We denote by $i\colon X \to X \times Y$ the inclusion map that is given by $i(x) = (x,y)$ and similarly we denote by $j\colon Y \to X \times Y$ the inclusion map that is given by $j(y) = (x,y)$. Furthermore we denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the two natural projections Show that the map

$$i_* \oplus j_*\colon \mathrm{H}_1(X) \oplus \mathrm{H}_1(Y) \;\to\; \mathrm{H}_1(X \times Y)$$

[1014] This can be seen as follows: First we write $G = \pi_1(X,x_0)$ and $H = p_*(\pi_1(Y,y_0))$. Let $g_1, \ldots, g_k$ be coset representatives for G/H. One can easily verify that the images of $g_1, \ldots, g_k$ in G_{ab} have the property that $G_{\mathrm{ab}} = g_1 \mathrm{H}_{\mathrm{ab}} \cup \cdots \cup g_k \mathrm{H}_{\mathrm{ab}}$.

is an isomorphism and that the inverse is given by $p_* \oplus q_*$.
Hint. Use the Homotopy Groups-of-Product Proposition 71.8.

Exercise 84.6. Show that there exist maps $f, g\colon S^1 \times S^1 \to S^1 \times S^1$ such that $f \circ g$ and $g \circ f$ are not homotopic.
Remark. This exercise is the content of Exercise 38.5 (b). But with the methods of this chapter it is much easier to solve this exercise.

85. The Hurewicz Theorem in higher dimensions

The goal of this chapter is to formulate and prove the high-dimensional analogue of the Hurewicz Theorem 84.5 from Chapter 84. Furthermore in this chapter and in Chapter 86 we will discuss several applications of the higher-dimensional Hurewicz Theorem.

85.1. **The Hurewicz homomorphism in higher dimensions.** The Hurewicz Theorem 84.5 evidently raises the question whether there is a connection between higher homotopy groups and higher homology groups. As in the 1-dimensional case we first try to find a promising map from homotopy groups to homology groups.
Throughout the subsequent discussion the following notation is convenient.

Notation. We write $I = [0,1]$.

Next we introduce the following technical definition.

Definition. Let $n \in \mathbb{N}_0$. Let $\omega \in \mathrm{H}_n(\overline{B}^n, S^{n-1})$ be the standard generator as defined on page 1679. Let $\varphi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$ be the homeomorphism from page 158. We refer to $\varphi_*(\omega) \in \mathrm{H}_n(I^n, \partial I^n)$ as the standard generator of $\mathrm{H}_n(I^n, \partial I^n)$.

Now comes the actual definition we are interested in.

Definition. Let X be a topological space, let $x_0 \in X$ and let $n \geq 1$. We denote by $\omega \in \mathrm{H}_n(I^n, \partial I^n)$ the standard generator as defined above. We refer to the map

$$\begin{array}{rcccc} & & & & \text{isomorphism since } n \geq 1\text{, see page 1629} \\ \Phi_{(X,x_0)}\colon \pi_n(X,x_0) & \to & \mathrm{H}_n(X,x_0) & \xleftarrow{\cong} & \mathrm{H}_n(X) \\ [f\colon (I^n,\partial I^n) \to (X,x_0)] & \mapsto & f_*(\omega) & & \end{array}$$

as the Hurewicz homomorphism.[1015]

Remark. It follows quite easily from the explicit description of the standard generator of $\mathrm{H}_1(\overline{B}^1, S^0)$ on page 1679 that for $n = 1$ the above definition of the Hurewicz homomorphism agrees with the definition given on page 1832.

In the next lemma we show in particular that the Hurewicz homomorphism is in fact a homomorphism.

Lemma 85.1.

(1) Let X be a topological space. Given any $x_0 \in X$ and given any $n \geq 1$ the Hurewicz homomorphism
$$\Phi_{(X,x_0)}\colon \pi_n(X,x_0) \to \mathrm{H}_n(X)$$
is a homomorphism.

[1015] It follows from the Homotopic Maps-and-Homology Proposition 73.6 that the Hurewicz homomorphism is indeed well-defined.

(2) Let X be a topological space and let $n \geq 1$. Furthermore let $\gamma\colon I \to X$ be a path from a point x_0 to some point x_1. The following diagram commutes:

$$\begin{array}{ccc} \pi_n(X, x_1) & \xrightarrow{\Phi_{(X,x_1)}} & \mathrm{H}_n(X) \\ {\scriptstyle \gamma_*}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \pi_n(X, x_0) & \xrightarrow{\Phi_{(X,x_0)}} & \mathrm{H}_n(X). \end{array}$$

Here the vertical maps are defined in Propositions 47.10 and 71.5.

(3) As usual let *PTop* be the category of pointed topological spaces and let *Group* be the category of groups. The Hurewicz homomorphisms define a natural transformation from the functor

$$\begin{array}{ccc} \mathit{PTop} & \to & \mathit{Group} \\ (X, x_0) & \mapsto & \pi_n(X, x_0) \end{array} \quad \text{to the functor} \quad \begin{array}{ccc} \mathit{PTop} & \to & \mathit{Group} \\ (X, x_0) & \mapsto & \mathrm{H}_n(X). \end{array}$$

(4) Let X be a topological space, let $x_0 \in X$ and let $n \geq 1$. If we take the point of view from page 1565, i.e. if we view elements in $\pi_n(X, x_0)$ as homotopy classes of maps $(S^n, *) \to (X, x_0)$, then the Hurewicz homomorphism is given by the map

$$\begin{array}{ccc} \pi_n(X, x_0) & \to & \mathrm{H}_n(X) \\ [f\colon (S^n, *) \to (X, x_0)] & \mapsto & f_*(\omega) \end{array}$$

where $\omega \in \mathrm{H}_n(S^n)$ is the standard generator as defined on page 1677.

Proof.

(1) We denote by $\omega \in \mathrm{H}_n(I^n, \partial I^n)$ the standard generator. We need to show that for any two maps $f, g\colon (I^n, \partial I^n) \to (X, x_0)$ we have

$$[f_*(\omega)] + [g_*(\omega)] \;=\; [(f * g)_*(\omega)] \;\in\; \mathrm{H}_n(X, x_0).$$

But we showed precisely this equality in Lemma 79.12.

(2) Let $f\colon (S^n, *) \to (X, x_1)$ be a map. One easily sees that f and f^γ, viewed as maps $S^n \to X$ are homotopic. Thus it follows from the Homotopic Maps-and-Homology Proposition 73.6 that $f_*(\omega) = (f^\gamma)_*(\omega)$. But this means precisely that the given diagram commutes.

(3) This statement follows immediately from the functoriality of homology groups.

(4) This statement is a straightforward consequence of the Standard Generator-Relationship Proposition 76.5 (3). ∎

The following lemma gives in particular an affirmative answer to Question 71.17.

Lemma 85.2. For every $n \in \mathbb{N}$ the Hurewicz homomorphism

$$\Phi\colon \pi_n(S^n, *) \;\to\; \mathrm{H}_n(S^n) \cong \mathbb{Z}$$

sends $[\mathrm{id}_{S^n}\colon S^n \to S^n] \in \pi_n(S^n)$ to the standard generator of $\mathrm{H}_n(S^n)$, in particular the Hurewicz homomorphism is an epimorphism.

Proof. We denote by $\omega \in \mathrm{H}_n(S^n)$ the standard generator. By Lemma 85.1 (4) we have

$$\Phi(\underbrace{[\mathrm{id}_{S^n}\colon S^n \to S^n]}_{\in \pi_n(S^n, *)}) \;=\; (\mathrm{id}_{S^n})_*([\omega]) \;=\; [\omega].$$

Thus we see that the standard generator of $\mathrm{H}_n(S^n) \cong \mathbb{Z}$ lies in the image of the Hurewicz homomorphism. But this implies that the Hurewicz homomorphism $\Phi\colon \pi_n(S^n, *) \to \mathrm{H}_n(S^n)$ is an epimorphism. ■

The fairly simple-minded Lemma 85.2 actually already turns out to be quite useful. More precisely, we recall that for $\mathbb{A} = \mathbb{R}$ or $\mathbb{A} = \mathbb{C}$ and $k \leq l$ we can view $\mathbb{A}\mathrm{P}^k$ as a submanifold of $\mathbb{A}\mathrm{P}^l$. Now we can formulate the following application of Lemma 85.2.

Lemma 85.3. For $k < l$ the submanifold $\mathbb{R}\mathrm{P}^k$ is not a retract of $\mathbb{R}\mathrm{P}^l$.

Proof. We have

$$\pi_k(\mathbb{R}\mathrm{P}^k) \underset{\text{Coverings-}\pi_n\text{-Proposition 71.13}}{\cong} \pi_k(S^k) \overset{\text{Lemma 85.2}}{\neq} 0 \qquad \text{and} \qquad \pi_k(\mathbb{R}\mathrm{P}^l) \underset{\text{Coverings-}\pi_n\text{-Proposition 71.13}}{\cong} \pi_k(S^l) \overset{\text{Proposition 71.10}}{=} 0.$$

With this calculation, and the fact that π_n is functorial, the argument of Lemma 47.16 can be easily modified to show that $\mathbb{R}\mathrm{P}^k$ is not a retract of $\mathbb{R}\mathrm{P}^l$. ■

Lemma 85.3 raises the following question.

Question 85.4. Let $k < l$. Is the complex projective space $\mathbb{C}\mathrm{P}^k$ a retract of $\mathbb{C}\mathrm{P}^l$?

At this moment though, it is much more interesting to ask whether, given a path-connected topological space X and $n \geq 2$, the Hurewicz homomorphism

$$\Phi_{(X,x_0)}\colon \pi_n(X, x_0) \to \mathrm{H}_n(X)$$

is necessarily an isomorphism. But browsing through our examples shows that the Hurewicz homomorphism is in general neither a monomorphism nor an isomorphism:

(1) In Lemma 78.11 we showed that $\mathrm{H}_2(S^1 \times S^1) \cong \mathbb{Z}$ is non-zero, whereas on page 1573 we showed that $\pi_2(S^1 \times S^1) = 0$. This shows that the Hurewicz homomorphism $\pi_2(S^1 \times S^1) \to \mathrm{H}_2(S^1 \times S^1)$ is not an epimorphism.

(2) Here are two examples that show that in general the Hurewicz homomorphism is not a monomorphism:
   (a) We consider $X = \mathbb{R}\mathrm{P}^2$. By the Coverings-$\pi_n$-Proposition 71.13 and Lemma 85.2 we know that $\pi_2(\mathbb{R}\mathrm{P}^2) \cong \pi_2(S^2)$ is non-trivial. On the other hand we saw in Lemma 78.8 that $\mathrm{H}_2(\mathbb{R}\mathrm{P}^2) = 0$.
   (b) We consider $X = S^1 \vee S^2$. It follows immediately from Propositions 74.4 and 79.8 that $\mathrm{H}_2(S^1 \vee S^2) \cong \mathbb{Z}$. On the other hand we know by the Coverings-π_n-Proposition 71.13 and the discussion on page 1373 that $\pi_2(S^1 \vee S^2)$ is isomorphic to $\pi_2(\widetilde{X})$ where $\widetilde{X}$ is the real line "with infinitely many 2-spheres attached", see e.g. the figure on page 1573 for an illustration. Using the projections onto the various spheres and using Lemma 85.2 one can now show that $\pi_2(\widetilde{X})$ admits an epimorphism onto $\mathbb{Z}^\infty$.[1016] It follows from Lemma 51.9 (2) that there is no monomorphism from $\pi_2(X) = \pi_2(\widetilde{X})$ into $\mathrm{H}_2(X) \cong \mathbb{Z}$. In particular the Hurewicz homomorphism $\pi_2(X) \to \mathrm{H}_2(X)$, whatever it is precisely, cannot be a monomorphism.

The above examples show that we need to put some hypothesis on X for a higher Hurewicz homomorphism to be an isomorphism. This leads us to the following definition.

[1016] The proof of this statement is a slight variation on the proof of Exercise 71.3.

Definition. Let $k \in \mathbb{N}_0$. We say that a topological space X is k-connected, if $\pi_i(X) = 0$ for $i = 0, \dots, k$.[1017][1018]

The goal of this chapter is to prove the following theorem which can be viewed as the high-dimensional analogue of the original Hurewicz Theorem 84.5.

Theorem 85.5. (Hurewicz Theorem) Let (X, x_0) be a pointed topological space. Furthermore let $n \in \mathbb{N}_{\geq 2}$. If X is $(n-1)$-connected[1019], then the following two statements hold:

(1) we have $\widetilde{\mathrm{H}}_k(X; \mathbb{Z}) = 0$ for $k = 0, \dots, n-1$, and
(2) the Hurewicz homomorphism $\Phi_{(X,x_0)} \colon \pi_n(X, x_0) \to \mathrm{H}_n(X; \mathbb{Z})$ is an isomorphism.

Remark.

(1) On page 1844 we already saw that in general the hypothesis in the Hurewicz Theorem 85.5 that X is $(n-1)$-connected cannot be dropped.
(2) In Theorem 153.13 we will prove a relative version of the Hurewicz Theorem 85.5 that relates "relative homotopy groups" to relative homology groups.
(3) In Theorem 163.26 we will prove an addendum to the Hurewicz Theorem 85.5, which contains information on the map $\pi_{n+1}(X, x_0) \to \mathrm{H}_{n+1}(X; \mathbb{Z})$.

Using the Hurewicz Theorem 85.5 we can now determine some homotopy groups that had been eluding us for a long time.

Corollary 85.6.

(1) For any $n \in \mathbb{N}$ we have $\pi_n(S^n) \cong \mathbb{Z}$, in fact $\pi_n(S^n, *) = \mathbb{Z} \cdot [\mathrm{id}_{S^n}]$.
(2) For any $n \in \mathbb{N} \cup \{\infty\}$ we have $\pi_2(\mathbb{C}\mathrm{P}^n) \cong \mathbb{Z}$. In fact, if we make the identification $\mathbb{C}\mathrm{P}^1 = S^2$ from page 258 and if we use the convention from page 256 to consider $\mathbb{C}\mathrm{P}^1 = S^2$ as a subspace of $\mathbb{C}\mathrm{P}^n$, then $\pi_2(\mathbb{C}\mathrm{P}^n, *) = \mathbb{Z} \cdot [\mathrm{id}_{\mathbb{C}\mathrm{P}^1}]$.

Proof.

(1) Let $n \in \mathbb{N}$. By Proposition 71.10 we know that S^n is $(n-1)$-connected. Thus the statement follows immediately from the Hurewicz Theorems 84.5 and 85.5 and the fact that $\mathrm{H}_n(S^n; \mathbb{Z}) = \mathbb{Z} \cdot [S^n]$.
(2) Let $n \in \mathbb{N} \cup \{\infty\}$. We denote by $i \colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^n$ the obvious inclusion. From Lemma 85.1 (3) we obtain the following commutative diagram

$$\begin{array}{ccc} \pi_2(\mathbb{C}\mathrm{P}^1, *) & \xrightarrow{i_*} & \pi_2(\mathbb{C}\mathrm{P}^n, *) \\ {\scriptstyle \Phi_{(\mathbb{C}\mathrm{P}^1,*)}}\downarrow & & \downarrow{\scriptstyle \Phi_{(\mathbb{C}\mathrm{P}^n,*)}} \\ \mathrm{H}_2(\mathbb{C}\mathrm{P}^1; \mathbb{Z}) & \xrightarrow{i_*} & \mathrm{H}_2(\mathbb{C}\mathrm{P}^n; \mathbb{Z}). \end{array}$$

If $n \in \mathbb{N}$, then it follows immediately from Lemma 121.6 that the inclusion induced map $\mathrm{H}_2(\mathbb{C}\mathrm{P}^1; \mathbb{Z}) \to \mathrm{H}_2(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ is an isomorphism. For $n = \infty$ we need to appeal to the discussion on page 1772 to draw the same conclusion. Finally, since $[\mathbb{C}\mathrm{P}^1] = [S^2]$

[1017] For example, X is 0-connected if and only if X is path-connected and *non-empty*. Furthermore, X is 1-connected if and only if it is simply connected.
[1018] Since a k-connected space is in particularly 0-connected and thus path-connected we see that we can ignore base points.
[1019] Recall that by the definition on page 1845 this means that $\pi_i(X) = 0$ for $i = 0, \dots, n-1$.

is a generator of $\mathrm{H}_2(\mathbb{C}\mathrm{P}^1;\mathbb{Z}) = \mathrm{H}_2(S^2;\mathbb{Z})$ we obtain that $\pi_2(\mathbb{C}\mathrm{P}^n,*) = \mathbb{Z}\cdot[\mathrm{id}_{\mathbb{C}\mathrm{P}^1}]$ from the above commutative diagram. ■

On several occasions we will also need the following straightforward corollary to the Hurewicz Theorem 85.5.

Corollary 85.7. (Hurewicz Corollary) Let (X,x_0) be a pointed topological space and let $n\in\mathbb{N}$. We suppose that X is simply connected. If given every $k\in\{2,\dots,n\}$ we have $\pi_k(X,x_0)=0$ or $\mathrm{H}_k(X;\mathbb{Z})=0$, then the following two statements hold:

(1) for $k=1,\dots,n$ we have $\pi_k(X,x_0)=\mathrm{H}_k(X;\mathbb{Z})=0$,

(2) the Hurewicz homomorphism $\pi_{n+1}(X,x_0)\to\mathrm{H}_{n+1}(X;\mathbb{Z})$ is an isomorphism.

Proof. We prove the corollary by induction on n. First consider the case $n=1$. Since X is simply connected we obtain from the Hurewicz Theorem 84.5 that $\mathrm{H}_1(X;\mathbb{Z})=0$. Furthermore it follows from the Hurewicz Theorem 85.5 that $\Phi_{(X,x_0)}\colon\pi_2(X,x_0)\to\mathrm{H}_2(X;\mathbb{Z})$ is an isomorphism.

Now suppose the statement holds for $n-1$ with $n\geq 2$. Suppose that for every $k\in\{2,\dots,n\}$ we have $\pi_k(X)=0$ or $\mathrm{H}_k(X;\mathbb{Z})=0$. By induction we know that the homotopy and homology groups vanish up to degree $n-1$. Thus we can apply the Hurewicz Theorem 85.5 which says that $\pi_n(X)$ and $\mathrm{H}_n(X;\mathbb{Z})$ are isomorphic. Since one is zero, both are zero. Hence we can apply the Hurewicz Theorem 85.5 once more to see that the Hurewicz homomorphism $\pi_{n+1}(X,x_0)\to\mathrm{H}_{n+1}(X;\mathbb{Z})$ is an isomorphism. ■

As a nice application to the Hurewicz Corollary 85.7 we can prove the following lemma.

Lemma 85.8. Let X be a 1-dimensional CW-complex and let $x_0\in X$ be base point. Then $\pi_n(X,x_0)=0$ for every $n\geq 2$.

The content of this lemma is the same as for Lemma 71.15, the only difference is that this time the proof is more satisfying than before.

Proof. Let X be a 1-dimensional CW-complex, let $n\geq 2$ and let $x_0\in X$ be a base point. If necessary we can replace X by the path-component of x_0. In other words, without loss of generality we can assume that X is path-connected. We denote by $\widetilde{X}$ the universal covering of X. By the CW-complex Covering Proposition 68.37 we know that $\widetilde{X}$ is again a 1-dimensional CW-complex. It follows from the Homology-of-CW-Complex Proposition 80.6 (1) that $\mathrm{H}_n(\widetilde{X})=0$ for all $n\in\mathbb{N}_{\geq 2}$. Furthermore, by definition of a universal covering we know that $\widetilde{X}$ is simply connected. It follows from the Hurewicz Corollary 85.7 that $\pi_n(X,x_0)=0$ for every $n\geq 2$. ■

We conclude this section with the following proposition.

Proposition 85.9. Let $n\in\mathbb{N}$ and let $f\colon S^n\to S^n$ be a homeomorphism with $f(*)=*$. Then the following equality holds in $\pi_n(S^n,*)$:

$$[f] \;=\; \begin{cases} [\mathrm{id}_{S^n}], & \text{if } f \text{ is orientation-preserving,} \\ -[\mathrm{id}_{S^n}], & \text{if } f \text{ is orientation-reversing.} \end{cases}$$

Remark. Let $n\in\mathbb{N}$. Proposition 85.9 says that every orientation-preserving self-homeomorphism f of S^n with $f(*)=*$ is *homotopic*, rel the base point $*$, to the identity. It is not clear to me whether f is *isotopic*, rel the base point $*$, to the identity. Put differently, can one find a homotopy F from f to id such that each $F_t\colon S^n\to S^n$ is also a homeomorphism?

Proof. Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a homeomorphism with $f(*) = *$. By the naturality of the Hurewicz homomorphism, see Lemma 85.1 (3), we have the following commutative diagram

$$\begin{array}{ccc} \pi_n(S^n,*) & \xrightarrow{f_*} & \pi_n(S^n,*) \\ {\scriptstyle \Phi_{(S^n,*)}}\downarrow & & \downarrow{\scriptstyle \Phi_{(S^n,*)}} \\ \mathrm{H}_n(S^n;\mathbb{Z}) & \xrightarrow{f_*} & \mathrm{H}_n(S^n;\mathbb{Z}). \end{array}$$

By Proposition 71.10 we know that S^n is $(n-1)$-connected. Therefore the vertical maps are isomorphisms by the Hurewicz Theorems 84.5 and 85.5. The corollary now follows from Proposition 107.2 and the simple observation that $[f] = f_*([\mathrm{id}_{S^n}]) \in \pi_n(S^n,*)$. ■

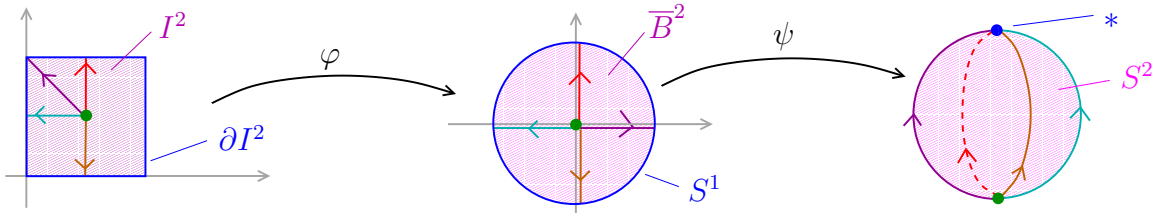
Now that we are fully convinced of the value of the Hurewicz Theorem 85.5 we turn to the proof thereof. We will discuss more applications in the Chapter 86.

85.2. **The group structure on $\pi_n(X,x_0)$.** Before we turn to the formulation of the high-dimensional Hurewicz Theorem we want to recall some conventions regarding higher homotopy groups and we want to introduce a useful approach to understanding the sum operation in the higher homotopy groups.
First we recall some conventions and notations that apply to any $n \in \mathbb{N}$.

(1) We write $* = (0,\dots,0,1) \in S^n$.
(2) We write $I = [0,1]$.
(3) We denote by $I^n \subset \mathbb{R}^n$ the cube in $\mathbb{R}^n$.
(4) We denote by ∂I^n the boundary of I^n viewed as a subset of $\mathbb{R}^n$.
(5) Let $\varphi\colon (I^n,\partial I^n) \to (\overline{B}^n, S^{n-1})$ be the homeomorphism from page 158.
(6) Let $\psi\colon (\overline{B}^n, S^{n-1}) \to (S^n,*)$ be the map from Lemma 5.20 that induces the usual identification $\overline{B}^n/S^{n-1} \to S^n$.
(7) As on page 1565 we use φ and ψ to make the identifications

$$\pi_n(X,x_0) = \begin{array}{c}\text{set (of homotopy classes)}\\ \text{of maps } (I^n,\partial I^n)\to(X,x_0)\\ \| \\ \text{set (of homotopy classes)}\\ \text{of maps } (\overline{B}^n, S^{n-1})\to(X,x_0).\end{array} = \begin{array}{c}\text{set (of homotopy classes)}\\ \text{of maps } (S^n,*)\to(X,x_0)\end{array}$$

The maps φ and ψ are illustrated in the figure below.



Now let (X,x_0) be a pointed topological space and let $n \in \mathbb{N}$. Recall that on page 1561 we defined the group structure on $\pi_n(X,x_0)$ by considering elements in the homotopy groups as homotopy classes of maps $(I^n,\partial I^n) \to (X,x_0)$. Under the above identifications it is at times difficult to see what the group structure actually looks like. We start out with considering inverses of homotopy classes.

Lemma 85.10. Let $n \in \mathbb{N}$ and let (X, x_0) be a pointed topological space.

(1) Let $f\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)$ be a map. If $\rho\colon \overline{B}^n \to \overline{B}^n$ is the reflection in some hyperplane of $\mathbb{R}^n$, then $-[f] = [f \circ \rho] \in \pi_n(X, x_0)$.

(2) Let $f\colon (S^n, *) \to (X, x_0)$ be some map. If $\rho\colon S^n \to S^n$ is the reflection in some hyperplane of $\mathbb{R}^{n+1}$ that contains the line[1020] $\mathbb{R} \cdot (0, \dots, 0, 1)$, then we have the equality $-[f] = [f \circ \rho] \in \pi_n(X, x_0)$.[1021]

Proof.

(1) Let $\varphi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$ be the homeomorphism from page 158 that we use to identify the two different points of view regarding the homotopy group $\pi_n(X, x_0)$. It follows from Proposition 71.1 (2) and from considering the definition of ρ that if ρ is the reflection in the $(x_2 = 0)$-hyperplane V, then $-[f] = [f \circ \rho]$.

Claim. If ψ is the reflection in some other hyperplane W, then ψ and ρ are homotopic as maps $(\overline{B}^n, S^{n-1}) \to (\overline{B}^n, S^{n-1})$.

Proof. The claim follows from Lemma 33.9. For the reader's convenience we recall the argument. First note that by the Orthogonal Action Lemma 5.26 (5) there exists an orthogonal matrix $A \in \mathrm{SO}(n)$ with $A \cdot V = W$. By the Matrix Group Path-Component Proposition 2.37 (3) we know that there exists a path $\gamma\colon [0,1] \to \mathrm{SO}(n)$ with $\gamma(0) = \mathrm{id}$ and $\gamma(1) = A$. The map

$$\begin{array}{rcl} \overline{B}^n \times [0,1] & \to & \overline{B}^n \\ (P, t) & \mapsto & \rho_{\gamma(t)\cdot V}(P) \end{array}$$

is the desired homotopy. ⊞

But this claim implies $[f \circ \psi] = [f \circ \rho]$ and thus we see that $[f \circ \psi] = -[f]$.

(2) This statement follows easily from Statement (1) and from considering the map $\psi\colon (\overline{B}^n, S^{n-1}) \to (S^n, *)$ from Lemma 5.20 that we use to identify the two different points of view regarding the homotopy group $\pi_n(X, x_0)$. ■

To consider the group structure on the higher homotopy groups we need the following new and fairly harmless definition.

Definition. Let $n \in \mathbb{N}$ and let M be an n-dimensional topological manifold.

(1) We say embeddings[1022] $\alpha_1, \dots, \alpha_m\colon \overline{B}^n \to M$ are (weakly) disjoint if the images (of the interiors) do not intersect.

(2) Let X be a topological space and let $x_0 \in X$. Given weakly disjoint embeddings $\alpha_1, \dots, \alpha_m\colon \overline{B}^n \to M$ and maps $f_1, \dots, f_m\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)$ we define[1023]

$$\begin{array}{rcl} \Phi(\alpha_1, \dots, \alpha_m, f_1, \dots, f_m)\colon M & \to & X \\ y & \mapsto & \begin{cases} f_i(z), & \text{if } y = \alpha_i(z) \text{ with } z \in \overline{B}^n, \\ x_0, & \text{otherwise.} \end{cases} \end{array}$$

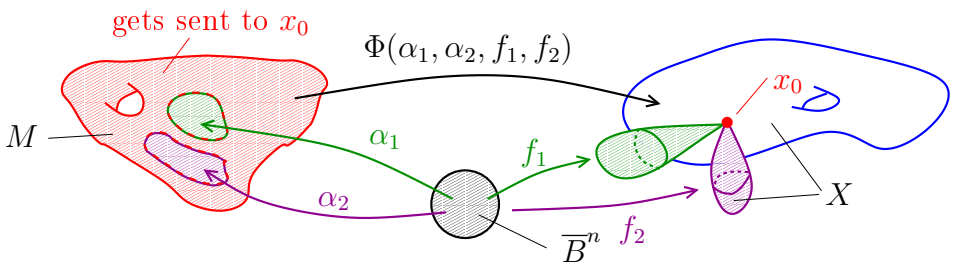
The following proposition gives a versatile tool for dealing with the group structure.

[1020] We can only consider such reflections since the point $* = (0, \dots, 0, 1)$ needs to be fixed.

[1021] Note that $\rho(*) = *$, hence $(f \circ \rho)(*) = x_0$.

[1022] We refer to page 718 for the definition of an embedding.

[1023] It follows from the Compact Image Lemma 2.13, the Closed-Compact Lemma 1.21, the Codimension-Zero Submanifold Proposition 19.38 and Pasting Proposition 2.6 (2) that this map is indeed continuous.



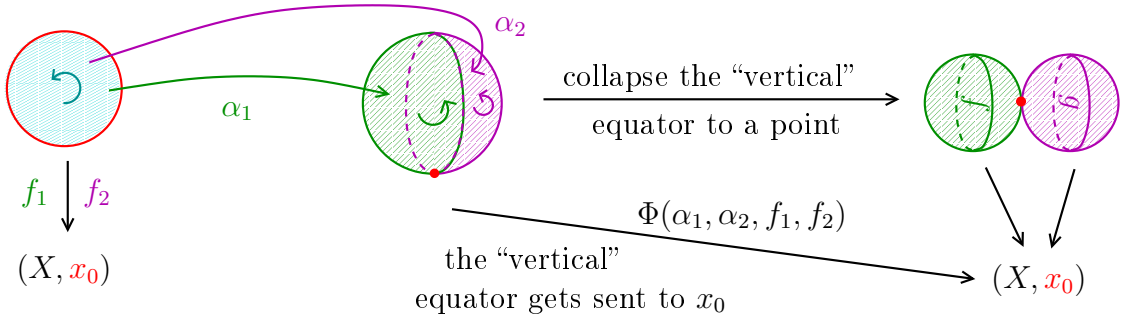
Proposition 85.11. (Addition-in-π_n-Proposition) Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_{\geq 2}$. Suppose we are given weakly disjoint smooth embeddings[1024] $\alpha_1, \ldots, \alpha_m \colon \overline{B}^n \to S^n$ such that the images of the B^n lie in $S^n \setminus \{*\}$. Also suppose we are given maps $f_1, \ldots, f_m \colon (\overline{B}^n, S^{n-1}) \to (X, x_0)$. Given $i \in \{1, \ldots, m\}$ we write $\epsilon_i = 1$ if α_i is orientation-preserving otherwise we write $\epsilon_i = -1$. Then we have the following equality in $\pi_n(X, x_0)$:

$$\big[\underbrace{\Phi(\alpha_1, \ldots, \alpha_m, f_1, \ldots, f_m) \colon (S^n, *) \to (X, x_0)}_{\in \pi_n(X, x_0)}\big] = \sum_{i=1}^{m} \epsilon_i \cdot \big[\underbrace{f_i \colon (\overline{B}^n, S^{n-1}) \to (X, x_0)}_{\in \pi_n(X, x_0)}\big].$$

Example. A slight modification of the map $\overline{B}^n \to S^n$ given in the Ball-Quotient Sphere Lemma 5.20 provides us with an orientation-preserving smooth embedding $\alpha_1 \colon \overline{B}^2 \to S^2$ that sends the disk to the "left hemisphere". Similarly we also obtain an orientation-preserving smooth embedding $\alpha_2 \colon \overline{B}^2 \to S^2$ that sends the disk to the "right hemisphere". It follows from the Addition-in-π_n-Proposition 85.11 that for any two maps $f_1, f_2 \colon (\overline{B}^2, S^1) \to (X, x_0)$ we have

$$[\Phi(\alpha_1, \alpha_2, f_1, f_2) \colon (S^2, *) \to (X, x_0)] = [f_1 \colon (\overline{B}^2, S^1) \to (X, x_0)] + [f_2 \colon (\overline{B}^2, S^1) \to (X, x_0)].$$

This situation is illustrated in the figure below. We obtain basically the same picture as we had already obtained on page 1566.



Remark. The statement of the Addition-in-π_n-Proposition 85.11 also holds for embeddings, i.e. we do not necessarily need that the embeddings are smooth. But in that case, in the proof below, one needs to replace the mainstream Smooth Ball Embedding Theorem 37.8 by the surprisingly difficult Theorem 132.10. In particular it is not clear to me

[1024] Note that here we work on purpose with "smooth embeddings" instead of "embeddings".

what goes into the proof of Theorem 132.10 (does it require the Hurewicz Theorem, are we in danger of using circular logic?). Therefore I prefer to stick to smooth embeddings.[1025]

In the proof of the Addition-in-π_n-Proposition 85.11 we need the following lemma which basically says the following: if we isotope the embeddings α_i, then the resulting Φ-maps are homotopic.

Lemma 85.12. Let $n \in \mathbb{N}$, let M be an n-dimensional smooth manifold and let Z be a subset of M. Furthermore suppose we are given a topological space X, a base point $x_0 \in X$ and maps $f_1, \ldots, f_m \colon (\overline{B}^n, S^{n-1}) \to (X, x_0)$. Finally suppose that for $i = 1, \ldots, m$ we are given homotopies

$$\Omega_i \colon \overline{B}^n \times [0,1] \ \to \ M \setminus Z$$

such that for each $t \in [0,1]$ the maps $\Omega_{1,t}, \ldots, \Omega_{m,t} \colon \overline{B}^n \to M$ are weakly disjoint embeddings, then the maps

$$\Phi(\underbrace{\Omega_{1,0}, \ldots, \Omega_{m,0}}_{\Omega \text{ at } t=0}, f_1, \ldots, f_m), \Phi(\underbrace{\Omega_{1,1}, \ldots, \Omega_{m,1}}_{\Omega \text{ at } t=1}, f_1, \ldots, f_m) \colon (M, Z) \ \to \ (X, x_0)$$

are homotopic as maps of pairs of topological spaces.

Proof. As before it follows from Lemmas 2.13 and 1.21, the Codimension-Zero Submanifold Proposition 19.38 and the Pasting Proposition 2.6 that the map

$$\begin{array}{rcl} H \colon M \times [0,1] & \to & X \\ (P,t) & \mapsto & \Phi(\Omega_{1,t}, \ldots, \Omega_{m,t}, f_1, \ldots, f_m)(P) \end{array}$$

is continuous. This map is then the desired homotopy. ∎

Sketch of a proof of the Addition-in-π_n-Proposition 85.11. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_{\geq 2}$. Furthermore let $f_1, \ldots, f_m \colon (\overline{B}^n, S^{n-1}) \to (X, x_0)$ be maps. Finally let $\alpha_1, \ldots, \alpha_m \colon \overline{B}^n \to S^n$ be weakly disjoint smooth embeddings such that the images of the B^n lie in $S^n \setminus \{*\}$. We start out with the following observations:

(1) Let $\rho \colon \overline{B}^n \to \overline{B}^n$ be the reflection in some hyperplane of $\mathbb{R}^n$. We replace every pair (α_i, f_i) for which α_i is orientation-reversing by the pair $(\alpha_i \circ \rho, f_i \circ \rho)$. It follows immediately from Lemma 85.10 that the desired equality holds for the new data if and only if it holds for the old data.

(2) If the α_i are not disjoint we make them disjoint by shrinking. More precisely, we consider the isotopies

$$\begin{array}{rcl} F_i \colon \overline{B}^n \times [0,1] & \to & M \\ (P,t) & \mapsto & \alpha_i(P \cdot (1 - \frac{t}{2})). \end{array}$$

Since the α_i were weakly disjoint we see that the resulting embeddings are now disjoint.

(3) In Lemma 5.20 we gave an explicit $(\overline{B}^n, S^{n-1}) \to (S^n, \{*\})$ and we showed that it restricts to a diffeomorphism $B^n \to S^n \setminus \{*\}$.

Keeping the above three observations in mind we obtain from Lemma 85.12 that it suffices to prove the following statement.

[1025] Recall that the absence of the word "topological" implies in our setup that smooth embeddings are smooth maps.

Statement. If $\alpha_1, \dots, \alpha_m \colon \overline{B}^n \to B^n$ are disjoint orientation-preserving smooth embeddings, then the following equality holds in $\pi_n(X, x_0)$:

$$\big[\underbrace{\Phi(\alpha_1, \dots, \alpha_m, f_1, \dots, f_m) \colon (\overline{B}^n, S^{n-1}) \to (X, x_0)}_{\in \pi_n(X, x_0)}\big] = \sum_{i=1}^{m} \big[\underbrace{f_i \colon (\overline{B}^n, S^{n-1}) \to (X, x_0)}_{\in \pi_n(X, x_0)}\big].$$

Perhaps the key observation in the proof is that the Smooth Ball Embedding Theorem 37.8 (where we use our hypothesis that $n \geq 2$ and here we use that work with smooth embeddings not just embeddings of balls) together with Lemma 85.12 implies that we only need to prove the statement for a single choice of disjoint orientation-preserving smooth embeddings $\alpha_1, \dots, \alpha_m$. Put differently, it remains to prove the following claim.

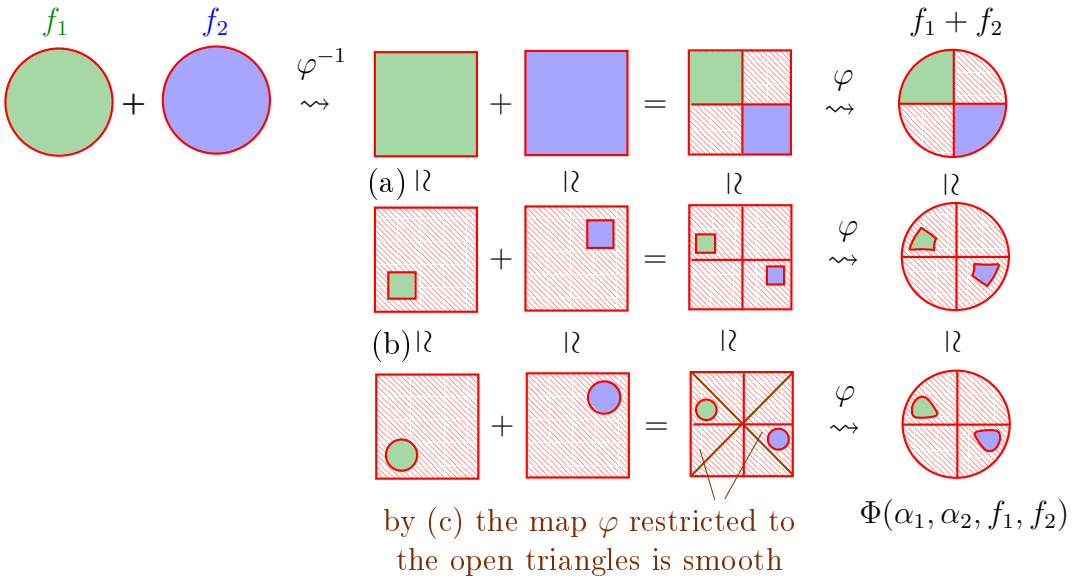
Claim. There are disjoint orientation-preserving smooth embeddings $\alpha_1, \dots, \alpha_m \colon \overline{B}^n \to B^n$ such that

$$\big[\underbrace{\Phi(\alpha_1, \dots, \alpha_m, f_1, \dots, f_m) \colon (\overline{B}^n, S^{n-1}) \to (X, x_0)}_{\in \pi_n(X, x_0)}\big] = \sum_{i=1}^{m} \big[\underbrace{f_i \colon (\overline{B}^n, S^{n-1}) \to (X, x_0)}_{\in \pi_n(X, x_0)}\big].$$

Proof. By induction it suffices prove the claim for $m = 2$. Even your famously verbose author does not feel like spelling out all the details. We leave it to the reader to piece together the argument using the figure below and the following observations:

(a) by the Addition-in-π_n-Proposition 85.11 (or alternatively Lemma 71.2) we can "move cubes" in I^n,
(b) by Lemma 14.2 the map $\varphi^{-1} \colon \overline{B}^n \to I^n$ is isotopic to the identity,
(c) the restriction of the map $\varphi \colon I^n \to \overline{B}^n$ to each open cone on one of the sides of the cube I^n is smooth.

Using Lemma 85.12 these moves translate into homotopies of the corresponding maps. ■

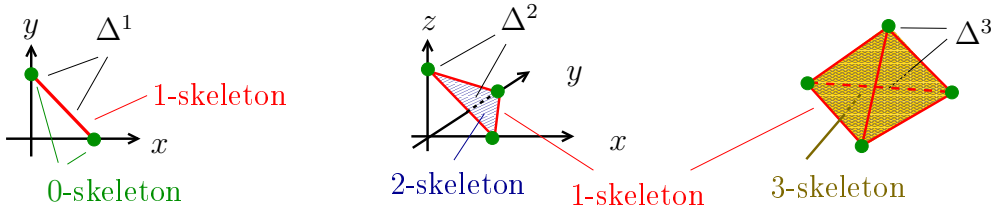


85.3. **The proof of the Hurewicz Theorem 85.5.** The proof of the Hurewicz Theorem 85.5 requires some preparations.

Convention. Let $k \in \mathbb{N}_0$. We view Δ^k as a CW-complex in the following way. Given $m \in \{0, \ldots, k\}$ the m-cells correspond precisely to all the maps

$$\begin{aligned} \overline{B}^m = \Delta^m &\to \Delta^k \\ (t_0, \ldots, t_m) &\mapsto \sum_{i=0}^{m} t_i e_{\varphi(i)} \end{aligned}$$

where φ is a strictly increasing map $\varphi\colon \{0, \ldots, m\} \to \{0, \ldots, k\}$. Note that the $(k-1)$-skeleton of Δ^k is given by $\partial\Delta^k$.

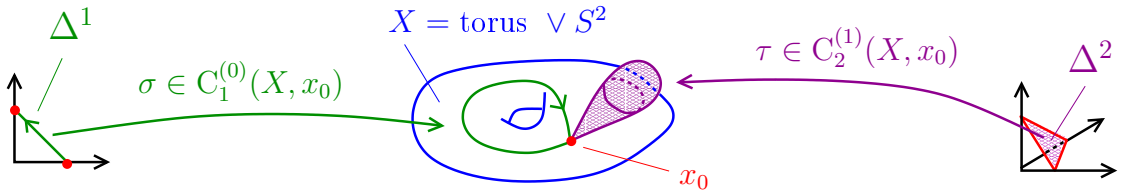


Definition. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_0$. Given $k \in \mathbb{N}_0$ we define[1026]

$$\mathrm{C}_k^{(n)}(X, x_0) := \begin{array}{l} \text{the subgroup of the relative chain group } \mathrm{C}_k(X, \{x_0\}) \text{ generated by} \\ \text{all singular simplices } \sigma\colon \Delta^k \to X \text{ with } \sigma(n\text{-skeleton of } \Delta^k) \subset \{x_0\}. \end{array}$$

These groups define a subcomplex of $\mathrm{C}_*(X, \{x_0\})$. In particular we obtain corresponding homology groups $\mathrm{H}_k^{(n)}(X, x_0)$ and the inclusion $\mathrm{C}_*^{(n)}(X, x_0) \to \mathrm{C}_*(X, \{x_0\})$ defines natural maps $\mathrm{H}_k^{(n)}(X, x_0) \to \mathrm{H}_k(X, \{x_0\})$.

Example. In the figure below we show a pointed topological space (X, x_0) and we consider an element $\sigma \in \mathrm{C}_1^{(0)}(X, x_0)$ and an element $\tau \in \mathrm{C}_2^{(1)}(X, x_0)$. From the pictures it becomes apparent that elements in $\mathrm{C}_n^{(n-1)}(X, x_0)$ are "close" to elements in $\pi_n(X, x_0)$. In particular it seems reasonable to expect that $\mathrm{H}_n^{(n-1)}(X, x_0)$ is a group that "sits between" $\mathrm{H}_n(X; \mathbb{Z})$ and $\pi_n(X, x_0)$.



In the following lemma we summarize a few basic facts about these new chain groups.

Lemma 85.13. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_0$. Furthermore let $k \in \{0, \ldots, n\}$. The following statements hold:

(1) We have $\mathrm{C}_k^{(n)}(X, x_0) = 0$,
(2) we have $\mathrm{H}_k^{(n)}(X, x_0) = 0$,
(3) any element in $\mathrm{C}_{k+1}^{(n)}(X, x_0)$ is a cycle.

[1026] Note that the notation $\mathrm{C}_k^{(n)}(X, x_0)$ instead of $\mathrm{C}_k^{(n)}(X, \{x_0\})$ is on purpose.

Proof. Let $k \in \{0, \dots, n\}$. Put differently, we have k with $n \geq k$. Note that this implies that Δ^k is the n-skeleton of Δ^k. In particular, by definition, $\mathrm{C}_k^{(n)}(X, x_0)$ is the subgroup of $\mathrm{C}_k(X, \{x_0\})$ generated by the constant maps $\Delta^k \to \{x_0\}$, but these are zero in the relative chain group $\mathrm{C}_k(X, \{x_0\})$. This concludes the proof of (1). The remaining two statements are immediate consequences of (1). ■

The following lemma is one of the key ingredients in the proof of the Hurewicz Theorem 85.5

Lemma 85.14. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}$. If X is n-connected, then the inclusion map

$$\mathrm{C}_*^{(n)}(X, x_0) \;\to\; \mathrm{C}_*(X, \{x_0\})$$

is a chain homotopy equivalence. In particular for any $k \in \mathbb{N}_0$ the induced map

$$\mathrm{H}_k^{(n)}(X, x_0) \;\to\; \mathrm{H}_k(X, \{x_0\})$$

is an isomorphism.

Proof. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}$. We suppose that X is n-connected. We recall the following notation from the Prism Boundary Lemma 73.7: given $t = 0$ and $t = 1$ and $k \in \mathbb{N}_0$ we denote by

$$\begin{aligned} \eta_t \colon \Delta^k &\to \Delta^k \times [0,1] \\ x &\mapsto (x, t) \end{aligned}$$

the obvious inclusion map. We start out with the following claim which basically says that any singular simplices $\Delta^k \to X$ can be homotoped consistently to a singular simplices that lie in $\mathrm{C}_k^{(n)}(X, x_0)$.

Claim. Given any $k \in \mathbb{N}_0$ and given any singular simplex $\sigma \colon \Delta^k \to X$ we can choose a map $P(\sigma) \colon \Delta^k \times [0,1] \to X$ such that the following four statements are satisfied:

(a) we have $P(\sigma) \circ \eta_0 = \sigma \colon \Delta^k \to X$,
(b) the singular k-simplex $P(\sigma) \circ \eta_1 \colon \Delta^k \to X$ lies in $\mathrm{C}_k^{(n)}(X, x_0)$,
(c) if $\sigma \in \mathrm{C}_k^{(n)}(X, x_0)$, then $P(\sigma) = \sigma \circ \pi$ where $\pi \colon \Delta^k \times [0,1] \to \Delta^k$ is the projection , and
(d) for any $j \in \{0, \dots, k\}$ we have

$$P(\sigma) \circ \underbrace{(i_j^k \times \mathrm{id}_{[0,1]})}_{\Delta^{k-1}\times[0,1] \to \Delta^k \times [0,1]} \;=\; P(\underbrace{\sigma \circ i_j^k}_{\Delta^{k-1} \to X}) \colon \Delta^{k-1} \times [0,1] \to X.$$

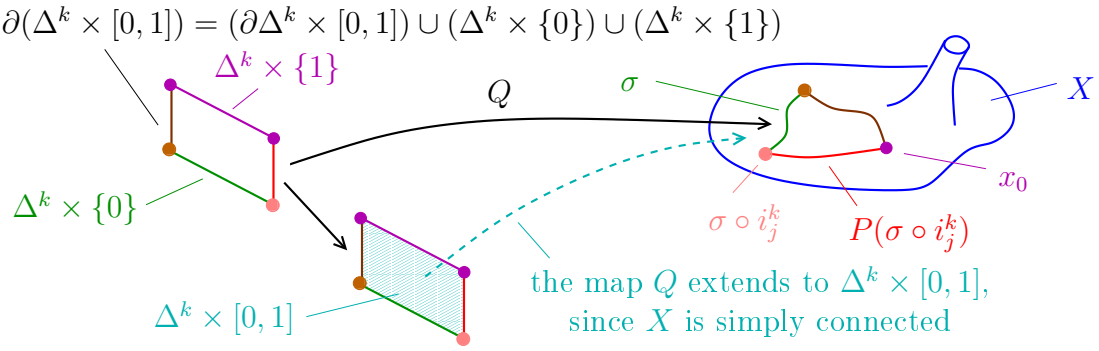
Proof. Property (d) already suggests that we should prove the claim by induction on k. It is convenient to view $k = -1$ as the (trivial) base case. Now let $k \in \mathbb{N}_0$ and suppose that we have proved the claim for all singular i-simplices with $i < k$. Now let $\sigma \colon \Delta^k \to X$ be a singular simplex. If $\sigma \in \mathrm{C}_k^{(n)}(X, x_0)$, then we set $P(\sigma) = \sigma \circ \pi$. In this case it is straightforward to verify that the conditions (a), (b) and (d) are satisfied.

Now consider the case that $\sigma \notin \mathrm{C}_k^{(n)}(X, x_0)$. We distinguish two cases.

(1) First we consider the case that $k \leq n$. By induction the maps $P(\sigma \circ i_j^k)$ have been defined for all j. We set $\partial(\Delta^k \times [0,1]) := (\partial\Delta^k \times [0,1]) \cup (\Delta^k \times \{0,1\})$ and we consider the map

$$Q \colon \partial(\Delta^k \times [0,1]) \;\to\; X$$

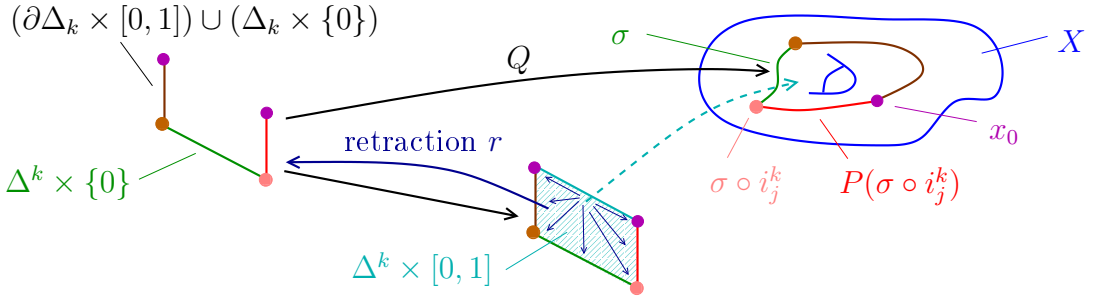
which is defined via σ on $\Delta^k \times \{0\}$, via the maps $P(\sigma \circ i^k_j)$ on $\partial\Delta^k \times [0,1]$[1027] and which sends all points in $\Delta^k \times \{1\}$ to x_0.[1028] (We refer to the figure below for an illustration of this map.) It follows almost immediately from the Convex-to-Ball Proposition 2.20 (2) that the pair $(\Delta^k \times [0,1], \partial(\Delta^k \times [0,1]))$ is homeomorphic to $(\overline{B}^{k+1}, S^k)$. Since $k \leq n$ and since X is n-connected we obtain from the Trivial-in-π_n-Lemma 71.4 that there exists an extension of Q to a map $\Delta^k \times [0,1] \to X$. We define $P(\sigma)$ to be any such extension. We obtain a map $P(\sigma)$ which is easily seen to have all the desired properties.



(2) Now we consider the case that $k > n$. Note that in this case condition (b) is satisfied "for free" by any map $\Delta^k \times [0,1] \to X$ that satisfies (d). Thus we only have to find a map $P(\sigma)$ that satisfies (a), (c) and (d). This is much easier than in (1). Indeed, similar to (1) we consider the map

$$G\colon (\Delta^k \times \{0\}) \cup (\partial\Delta^k \times [0,1]) \;\to\; X$$

that is defined via σ on $\Delta^k \times \{0\}$ and via the maps $P(\sigma \circ i^k_j)$ on $\partial\Delta^k \times [0,1]$. (We refer to the figure below for an illustration.) Note that by Lemma 70.6 (together with the homeomorphism $\overline{B}^k \cong \Delta^k$) there exists a retraction r from $\Delta^k \times [0,1]$ to the domain of G. Thus combining this retraction r with Q we obtain an extension of Q to map $\Delta^k \times [0,1] \to X$. We define $P(\sigma)$ to be this extension. It follows basically by construction that all the statements are satisfied. ⊞



[1027] Here we use that we assume that the maps $P(\mu)$ are defined for simplices of dimension $< k$.
[1028] It follows from Pasting Proposition 2.6 (2′) that this map is indeed continuous.

Now we turn to the actual proof of the statement that the inclusion map[1029]

$$\iota\colon \mathrm{C}_*^{(n)}(X,x_0) \;\to\; \mathrm{C}_*(X,\{x_0\})$$

is a chain homotopy equivalence. To do so we consider the map

$$\begin{aligned}\phi\colon \mathrm{C}_*(X,\{x_0\}) &\to \mathrm{C}_*^{(n)}(X,x_0)\\ (\sigma\colon \Delta^n\to X) &\mapsto \phi(\sigma) := \underbrace{P(\sigma)\circ\eta_1}_{\Delta^k\to\Delta^k\times[0,1]\to X}.\end{aligned}$$

that is determined by

It follows from (b) that ϕ does indeed take values in $\mathrm{C}_*^{(n)}(X,x_0)$ and it follows easily from (d) that ϕ is in fact a chain map. We want to show that ϕ is a chain homotopy inverse to ι. By (c) we have $\phi\circ\iota = \mathrm{id}$. Thus it remains to prove the following claim.

Claim. There exists a chain homotopy from $\iota\circ\phi$ to the identity of $\mathrm{C}_*(X,\{x_0\})$.

Proof. Let $k\in\mathbb{N}_0$. We recall that on page 1605 we introduced the singular $(k+1)$-chain

$$\Omega_k \;:=\; \sum_{j=0}^{k}(-1)^j\cdot\underbrace{[v_0,\dots,v_j,w_j,\dots,w_k]}_{\Delta^{k+1}\to\Delta^k\times[0,1]} \;\in\; \mathrm{C}_{k+1}(\Delta^k\times[0,1])$$

where $v_0,\dots,v_k,w_0,\dots,w_k$ denote the vertices of $\Delta^k\times[0,1]$, see page 1605 for details. The key property of these singular chains is that the Prism Boundary Lemma 73.7 says that for any $k\in\mathbb{N}_0$ we have

$$(*)\qquad \partial_{k+1}(\Omega_k)+\sum_{j=0}^{k}(-1)^j\cdot\underbrace{(i_j^k\times\mathrm{id}_{[0,1]})_*}_{\mathrm{C}_*(\Delta^{k-1}\times[0,1])\to\mathrm{C}_*(\Delta^k\times[0,1])}(\Omega_{k-1}) \;=\; \eta_1-\eta_0 \;\in\; \mathrm{C}_k(\Delta^k\times[0,1]).$$

For any $k\in\mathbb{N}_0$ we consider the map

$$\begin{aligned}D_k\colon \mathrm{C}_k(X,\{x_0\}) &\to \mathrm{C}_{k+1}(X,\{x_0\})\\ (\sigma\colon\Delta^k\to X) &\mapsto \underbrace{P(\sigma)_*}_{\mathrm{C}_*(\Delta^k\times[0,1])\to\mathrm{C}_*(X,\{x_0\})}(\underbrace{\Omega_k}_{\in\mathrm{C}_{k+1}(\Delta^k\times[0,1])}).\end{aligned}$$

that is determined by

We claim that these maps define a chain homotopy from $\iota\circ\phi$ to the identity of $\mathrm{C}_*(X,\{x_0\})$. Thus let $\sigma\colon\Delta^k\to X$ be a singular k-simplex. We need to show that $\partial D\sigma + D\partial\sigma = \phi(\sigma)-\sigma$. Indeed we have

$$\begin{aligned}\partial_{k+1}(D_k(\sigma))+D_{k-1}(\partial_k(\sigma)) &= \partial_{k+1}(D_k(\sigma))+D_{k-1}\Big(\sum_{j=0}^{k}(-1)^j\cdot\sigma\circ i_j^k\Big)\\ &= \partial_{k+1}(P(\sigma)_*(\Omega_k))+\sum_{j=0}^{k}(-1)^j\cdot P(\sigma\circ i_j^k)_*(\Omega_{k-1})\\ &\underset{\uparrow}{=} P(\sigma)_*\Big(\partial_{k+1}(\Omega_k)+\sum_{j=0}^{k}(-1)^j\cdot(i_j^k\times\mathrm{id}_{[0,1]})_*(\Omega_{k-1})\Big)\end{aligned}$$

in the first summand we used that $P(\sigma)_*$ is a chain map and
in the second summand we used that by (d) we have $P(i_j^k\circ\sigma) = P(\sigma)\circ(i_j^k\times\mathrm{id}_{[0,1]})$

$$\begin{aligned}&\underset{\uparrow}{=} P(\sigma)_*(\eta_1-\eta_0) \;=\; P(\sigma)\circ\eta_1 - P(\sigma)\circ\eta_0 \underset{\uparrow}{=} \phi(\sigma)-\sigma.\end{aligned}$$

by the above equality $(*)$ by (a) and definition of ϕ ∎

[1029]Here and throughout the proof we do not distinguish in the notation between a singular chain in $\mathrm{C}_k(X)$ and the element it represents in $\mathrm{C}_k(X,\{x_0\})$.

Before we can continue with the proof of the Hurewicz Theorem 85.5 we need to introduce another, slightly different point of view, regarding higher homotopy groups.

Convention. So let $n \in \mathbb{N}$. We consider the homeomorphism $\eta_n \colon (\Delta^n, \partial\Delta^n) \to (\overline{B}^n, S^{n-1})$ that is given by the composition of the following three maps:[1030]

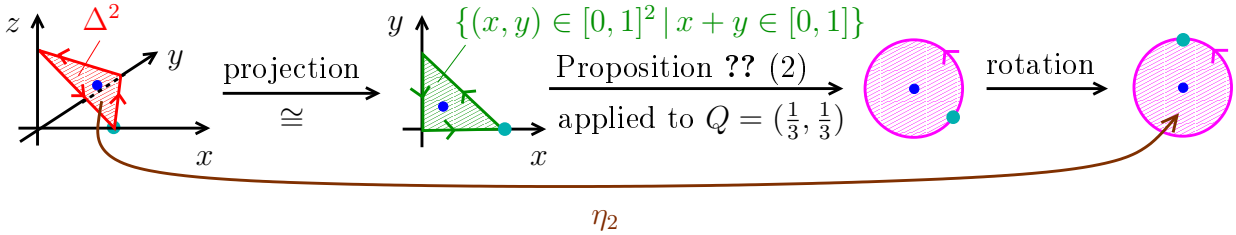
$$\eta_n \colon \Delta^n \xrightarrow{\cong} \left\{(x_1, \ldots, x_n) \in [0,1]^n \;\middle|\; \sum_{i=1}^{n} x_i \in [0,1]\right\} \xrightarrow[\text{with } Q = (\frac{1}{n+1}, \ldots, \frac{1}{n+1})]{\text{Proposition 2.20 (2)}} \overline{B}^n \xrightarrow[\text{by } A_n]{\text{multiplication}} \overline{B}^n.$$
$$(t_0, \ldots, t_n) \mapsto (t_0, \ldots, t_{n-1})$$

where $A_n \in \mathrm{O}(n)$ is a matrix which is chosen in such a way that we have the equality $\eta_n((1,0,\ldots,0)) = * := (0,\ldots,0,1)$.[10311032] We sketch the definition of η_n in the figure below.

We use the homeomorphism $\eta_n \colon (\Delta^n, \partial\Delta^n) \to (\overline{B}^n, S^{n-1})$ to identify these two pairs of topological spaces. In particular we use it to make the identification $\partial\Delta^n = S^{n-1}$. By a slight abuse of notation we write write $* := (1,0,\ldots,0) \in \Delta^n$. In particular for any $n \in \mathbb{N}_0$ we can extend the identifications from page 1847 to obtain the following identifications

$$\pi_n(X, x_0) \;=\; \begin{array}{c}\text{set (of homotopy classes)}\\ \text{of maps } (I^n, \partial I^n) \to (X, x_0)\\ \|\\ \text{set (of homotopy classes)}\\ \text{of maps } (\overline{B}^n, S^{n-1}) \to (X, x_0)\\ \|\\ \text{set (of homotopy classes)}\\ \text{of maps } (\Delta^n, \partial\Delta^n) \to (X, x_0).\end{array} \;=\; \begin{array}{c}\text{set (of homotopy classes)}\\ \text{of maps } (S^n, *) \to (X, x_0)\\ \|\\ \text{set (of homotopy classes)}\\ \text{of maps } (\partial\Delta^{n+1}, *) \to (X, x_0)\end{array}$$

As above we will use these identifications to pick at each given moment our most convenient point of view.



As an illustration, under the above identifications the Trivial-in-π_n-Lemma 71.4 translates into the following lemma.

Lemma 85.15. Let (X, x_0) be a pointed topological space and let $n \geq 1$. Furthermore let $f \colon (\partial\Delta^{n+1}, *) \to (X, x_0)$ be a map. Then f represents the trivial element in $\pi_n(X, x_0)$ if and only if there exists a map $F \colon \Delta^{n+1} \to X$ so that $F|_{\partial\Delta^{n+1}} = f$.

The following lemma is the other key technical ingredient in the proof of the Hurewicz Theorem 85.5.

[1030] Note that the composition of the first two homeomorphisms is precisely the homeomorphism from the Standard Simplex Homeomorphism Lemma 72.1.

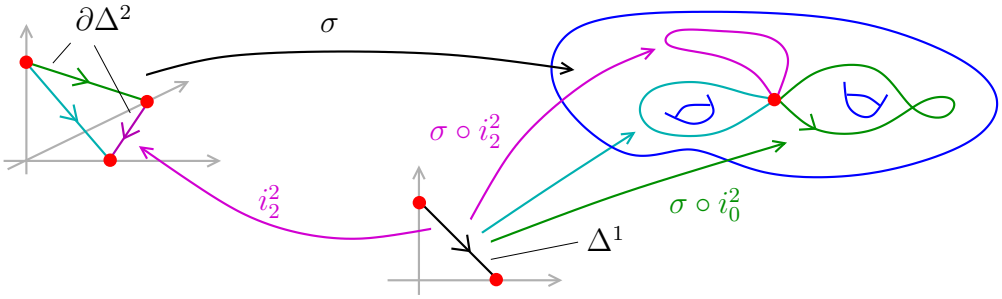
[1031] It follows from the Orthogonal Action Lemma 5.26 (3) that such a matrix exists.

[1032] The map η_n depends on the choice of the matrix A_n, this will not affect us at all.

Lemma 85.16. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_{\geq 2}$. Furthermore let $\sigma\colon \partial\Delta^{n+1} \to X$ be a map that sends the $(n-1)$-skeleton of $\partial\Delta^{n+1}$ to x_0. Then[10331034]

$$[\sigma\colon (\partial\Delta^{n+1}, *) \to (X, x_0)] \;=\; \sum_{j=0}^{n+1} (-1)^j \cdot [\sigma \circ i_j^{n+1}\colon (\Delta^n, \partial\Delta^n) \to (X, x_0)] \;\in\; \pi_n(X, x_0).$$

We refer to the figure below for an illustration for $n = 1$, even though this case is strictly speaking not allowed.



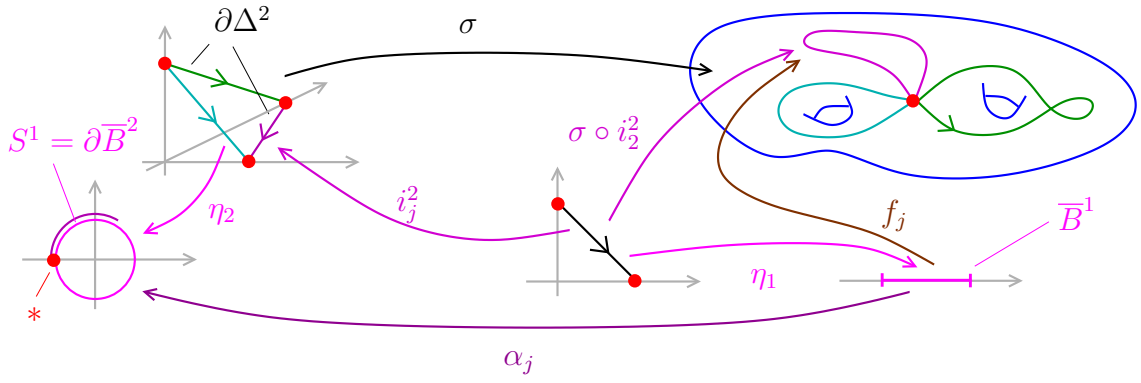
Proof. The reader might already have noticed that the statement of the lemma resembles the statement of the Addition-in-π_n-Proposition 85.11. Indeed, after some massaging we will be able to deduce the desired statement from the Addition-in-π_n-Proposition 85.11.

Given any $k \in \mathbb{N}_0$, let $\eta_k\colon (\Delta^k, \partial\Delta^k) \to (\overline{B}^k, S^{k-1})$ the homeomorphism from page 1856. Using our identifications the desired statement is now equivalent to the following statement.

Statement. Given $j \in \{0, \ldots, n+1\}$ we write $\alpha_i = \eta_{n+1} \circ i_j^{n+1} \circ \eta_n^{-1}\colon \overline{B}^n \to S^n = \partial\overline{B}^{n+1}$ and furthermore we write $f_j = \sigma \circ i_j^{n+1} \circ \eta_n^{-1}\colon \overline{B}^n \to X$. Then we have the following equality in $\pi_n(X, x_0)$:

$$\big[\underbrace{\Phi(\alpha_0, \ldots, \alpha_n, f_0, \ldots, f_n)\colon (S^n, *) \to (X, x_0)}_{\in \pi_n(X, x_0)}\big] \;=\; -\sum_{j=0}^{n+1} (-1)^j \cdot \big[\underbrace{f_j\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)}_{\in \pi_n(X, x_0)}\big].$$

We refer to the figure below for an illustration.



[1033]Note that our hypothesis that σ sends the $(n-1)$-skeleton of $\partial\Delta^{n+1}$ to x_0 implies in particular that for all $j \in \{0, \ldots, n+1\}$ we have $(\sigma \circ i_j^{n+1})(\partial\Delta^n) = \{x_0\}$.

[1034]Also note that by our base point $*$ is chosen in such a way that it lies on the $(n-1)$-skeleton of $\partial\Delta^{n+1}$.

First note that it follows from the discussion on page 2325 that the maps η_k is orientation-preserving if and only if k is even. Furthermore it is fairly straightforward to see that the maps i_j^{n+1} are orientation-preserving if and only if j is even. In summary, the embedding α_j is orientation-preserving if and only if j is odd.[1035]

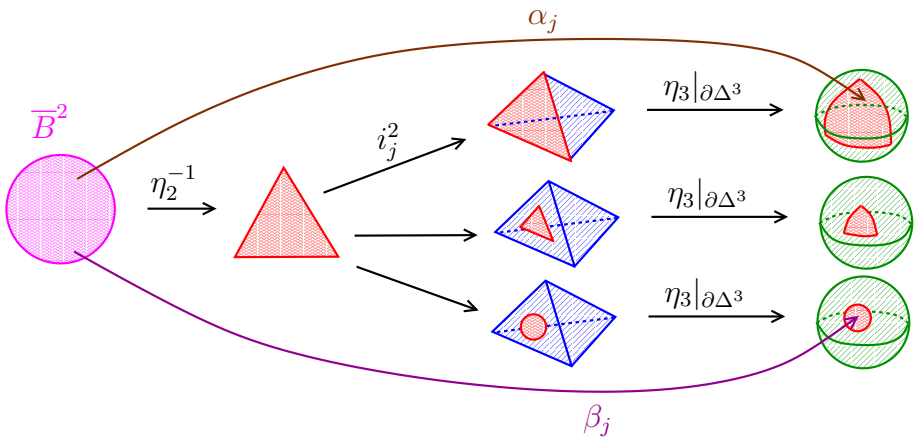
The only reason why we cannot apply the Addition-in-π_n-Proposition 85.11 immediately is that the maps α_j are embeddings but they are not smooth, so they are not smooth embeddings. Keeping Lemma in mind we see that it suffices to prove the following claim.

Claim. The maps $\alpha_0, \dots, \alpha_{n+1} \colon \overline{B}^n \to S^n$ are isotopic, through weakly disjoint embeddings, to smooth embeddings $\beta_0, \dots, \beta_{n+1}$.

Proof. For $j = 0, \dots, n+1$ we write $F_j = i_j^{n+1}(\mathring{\Delta}^n)$ and we write $\overline{F}_j = i_j^{n+1}(\Delta^n)$. Note that F_j is an n-dimensional submanifold of $\mathbb{R}^{n+1}$, we endow it with the usual orientation where the "normal vector sticks outward". Now this claim can be proved using the following observations:

(1) Similar to the discussion on page 1850 we can shrink the weakly disjoint embeddings $\alpha_j \colon \overline{B}^n \to \overline{F}_j$ to disjoint embeddings $\beta_j \colon \overline{B}^n \to F_j$.
(2) Using Lemma 14.2 we can isotope these embeddings $\beta_j \colon \overline{B}^n \to F_j$ to smooth embeddings[1036] $\gamma_i \colon \overline{B}^n \to F_j$.
(3) For each $j \in \{0, \dots, n+1\}$ the restriction of $\eta_{n+1} \colon \partial\Delta^{n+1} \to S^n$ to $\eta_{n+1} \colon F_j \to S^n$ is smooth and orientation-preserving.

Together with the figure below the reader should have no troubles assembling a full proof of the claim and the lemma using the above observations. ■



Now we can turn to the proof of the Hurewicz Theorem 85.5.

Proof of the Hurewicz Theorem 85.5. Let $n \in \mathbb{N}_{\geq 2}$ and let (X, x_0) be a pointed topological space. We suppose that X is $(n-1)$-connected.

Statement (1) of the theorem can be proved easily. Indeed, let $k \in \{0, \dots, n-1\}$. We have

[1035] Here we used the deep fact that precisely one out of the numbers n, $n+1$ is odd.
[1036] Recall that a smooth embedding is by definition smooth.

$$\widetilde{\mathrm{H}}_k(X;\mathbb{Z}) \underset{\text{Lemma 74.12}}{\cong} \mathrm{H}_k(X,\{x_0\};\mathbb{Z}) \underset{\substack{\text{by Lemma 85.14 since}\\ X \text{ is } (n-1)\text{-connected}}}{=} \mathrm{H}_k^{(n-1)}(X,x_0) \underset{\substack{\text{by Lemma 85.13 (2)}\\ \text{since } k \leq n-1}}{=} 0.$$

We turn to the much more interesting proof of Statement (2). As a reminder, we want to show that the map

$$\pi_n(X,x_0) \xrightarrow{\Phi_{(X,x_0)}} \mathrm{H}_n(X;\mathbb{Z})$$

is an isomorphism. The goal again is to bring the homology groups $\mathrm{H}_n^{(n-1)}(X,x_0)$ into the game. This leads us to the following claim.

Claim.

(1) The map

$$\begin{array}{rcl} \Psi_{(X,x_0)}\colon \pi_n(X,x_0) & \to & \mathrm{H}_n^{(n-1)}(X,x_0) \\ [f\colon (\Delta^n,\partial\Delta^n) \to (X,x_0)] & \mapsto & \big[\underbrace{f\colon \Delta^n \to X}_{\in \mathrm{C}_n^{(n-1)}(X,x_0)}\big] \end{array}$$

is well-defined and it is a homomorphism.

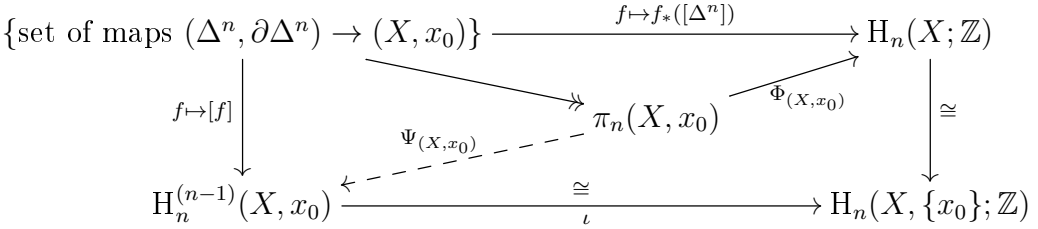
(2) The following diagram commutes up to the sign $(-1)^n$:

$$\begin{array}{ccc} \pi_n(X,x_0) & \xrightarrow[[f]\mapsto f_*([S^n])]{\Phi_{(X,x_0)}} & \mathrm{H}_n(X;\mathbb{Z}) \\ \Big\downarrow{\scriptstyle \Psi_{(X,x_0)}} & & \Big\downarrow{\scriptstyle \cong} \\ \mathrm{H}_n^{(n-1)}(X,x_0) & \xrightarrow[\iota]{\cong} & \mathrm{H}_n(X,\{x_0\};\mathbb{Z}) \end{array}$$

where ι is induced by the inclusion $\mathrm{C}_*^{(n-1)}(X,x_0) \to \mathrm{C}_n(X,\{x_0\})$.

(3) The right-vertical and the bottom-horizontal maps of the diagram in (2) are isomorphisms.

Proof. We prove the claim using a somewhat convoluted logic.[1037] We consider the following diagram



We make the following observations:

(a) It follows from $n \geq 2$ and from the long exact sequence of homology groups of the pair $(X,\{x_0\})$, see the Topological LES-Corollary 74.14, that the right-hand vertical map of both diagrams is an isomorphism.

(b) It follows from Lemma 85.14 and our hypothesis that X is $(n-1)$-connected that the bottom horizontal map ι of both diagrams is an isomorphism.

(c) Let $f\colon (\Delta^n,\partial\Delta^n) \to (X,x_0)$ be a map. The $(n-1)$-skeleton of Δ^n equals $\partial\Delta^n$, thus the map $f\colon \Delta^n \to X$ does indeed define an element in $\mathrm{C}_n^{(n-1)}(X,x_0)$. Furthermore by Lemma 85.13 (3) we know that every element in $\mathrm{C}_n^{(n-1)}(X,x_0)$ is a cycle, thus f

[1037] The convoluted logic is forced upon us since proving (1) as a stand-alone statement is actually slightly tricky.

defines an element in $\mathrm{H}_n^{(n-1)}(X, x_0)$. This shows that in the above diagram the left vertical map is well-defined.

(d) It is clear that the "outer rectangle" of the above diagram commutes.

(e) It follows from the Standard Generator-Relationship Proposition 76.5 (2) that the top small triangle in the above diagram commutes up to the sign $(-1)^n$.

(f) We want to show that $\Psi_{(X,x_0)}$ is well-defined and that it is a homomorphism. By (b) we know that ι is an isomorphism. Thus it suffices to show these statements for $\iota \circ \Psi_{(X,x_0)}$. But these now follow from (d) and (e) and the fact, see Lemma 85.1 (1), that the Hurewicz homomorphism $\Phi_{(X,x_0)}$ is well-defined and a homomorphism. ⊞

By the claim it suffices to show that the homomorphism

$$\begin{array}{rcl} \Psi_{(X,x_0)} \colon \pi_n(X, x_0) & \to & \mathrm{H}_n^{(n-1)}(X, x_0) \\ [f \colon (\Delta^n, \partial\Delta^n) \to (X, x_0)] & \mapsto & [f \colon \Delta^n \to X] \end{array}$$

is in fact an isomorphism. We will do so by defining an explicit inverse.

Claim.

(1) The map

$$\begin{array}{rcl} \Xi_{(X,x_0)} \colon \mathrm{C}_n^{(n-1)}(X, x_0) & \to & \pi_n(X, x_0) \\ (\sigma \colon (\Delta^n, \partial\Delta^n) \to (X, x_0)) & \mapsto & [\sigma \colon (\Delta^n, \partial\Delta^n) \to (X, x_0)] \end{array}$$

that is induced by

is well-defined.

(2) The map $\Xi_{(X,x_0)}$ from (1) descends to a well-defined map

$$\mathrm{H}_n^{(n-1)}(X, x_0) \to \pi_n(X, x_0).$$

(3) The map $\Xi_{(X,x_0)}$ is an inverse to $\Psi_{(X,x_0)}$.

Proof. Since the end is near we set out with renewed vigor to prove the claim.

(1) Recall that we know from the Homotopy Group-Abelian Proposition 71.3 and our hypothesis $n \geq 2$ that the group on the right is abelian. Therefore it follows from Lemma 51.1 that the map is well-defined.[1038]

(2) Let $\sigma \colon \Delta^{n+1} \to X$ be a singular $(n+1)$-simplex that defines an element in the group $\mathrm{C}_{n+1}^{(n-1)}(X, x_0)$. We need to show that $\Xi_{(X,X_0)}(\partial\sigma) = 0$. We calculate that we have the following equality in $\pi_n(X, x_0)$:

$$\Xi_{(X,X_0)}(\partial\sigma) = \Xi_{(X,X_0)}\Big(\sum_{j=0}^{n+1}(-1)^j \cdot \sigma \circ i_j^n\Big) = \sum_{j=0}^{n+1}(-1)^j \cdot [\overbrace{\sigma \circ i_j^{n+1}}^{(\Delta^n,\partial\Delta^n)\to(X,x_0)}] = -[\overbrace{\sigma|_{\partial\Delta^{n+1}}}^{(\partial\Delta^{n+1},*)\to(X,x_0)}] = 0.$$

by definition of $\Xi_{(X,x_0)}$ (↑) Lemma 85.16 (↑) Lemma 85.15 (↑)

(3) It follows immediately from the definitions that $\Xi_{(X,x_0)} \circ \Psi_{(X,x_0)}$ is the identity on $\pi_n(X, x_0)$. Furthermore it follows easily from Lemma 85.13 and the definitions that $\Psi_{(X,x_0)} \circ \Xi_{(X,x_0)}$ is the identity on a generating set for $\mathrm{H}_n^{(n-1)}(X, x_0)$. But by the claim and Lemma 85.1 (1) we know that $\Psi_{(X,x_0)}$ and $\Xi_{(X,x_0)}$ are both homomorphisms. Thus we see that $\Psi_{(X,x_0)} \circ \Xi_{(X,x_0)}$ is actually the identity on all of $\mathrm{H}_n^{(n-1)}(X, x_0)$.

[1038] Put differently, it would be a serious dereliction of duty to use the additive notation for a group structure, if the group was not known to be additive.

This concludes the proof of the claim and thus we are done with the proof of the Hurewicz Theorem 85.5. ■

Remark. The most awkward aspect of the proof of the Hurewicz Theorem 85.5 is that one needs to go back and forth between the different points of view regarding elements in higher homotopy groups, in particular the fact that we need the Addition-in-π_n-Proposition 85.11 and Lemma 85.16 is arguably aesthetically somewhat unpleasant.

In fact there is, in principle, an elegant way around this problem: One could prove the Hurewicz Theorem 85.5 using "cubical singular homology" instead of the usual singular homology. As the name suggests, in this theory the "building blocks" are singular cubes, i.e. maps from $[0,1]^n$ to a topological space X. (This theory is developed in [**Mass1991**, Chapter VII.2].) Since cubes are also used in the definition of the product structure on higher homotopy groups I would expect that many of the technical issues we had to deal with vanish magically.

The downside is that one still needs to show that "cubical singular homology" is naturally isomorphic to the usual singular homology. This can be shown using the Acyclic Model Theorem 113.16.

Exercises for Chapter 85. In the following four exercises we need the following definition.

Definition. Let X be a topological space. We say X is aspherical if X is path-connected and if all higher homotopy groups $\pi_n(X)$, $n \geq 2$, vanish.

Exercise 85.1. Let X be a connected 2-dimensional CW-complex with $\pi_2(X) = 0$. Show that X is aspherical.

Exercise 85.2. Let X be a topological space and let U and V be open subsets of X. Let $n \in \mathbb{N}_{\geq 2}$. Suppose that U and V are $(n-1)$-connected and that $U \cap V$ is n-connected. We pick a base point $x_0 \in U \cap V$ and we denote by $i\colon U \to X$ and $j\colon V \to X$ the inclusion maps. Show that the map

$$i_* \oplus j_*\colon \pi_n(U, x_0) \oplus \pi_n(V, x_0) \ \to \ \pi_1(X, x_0)$$

is an isomorphism.

Exercise 85.3. Let X be a CW-complex and let U and V be path-connected subcomplexes of X. Suppose that $U \cap V$ is path-connected and furthermore suppose that the inclusion induced maps $\pi_1(U \cap V) \to \pi_1(U)$ and $\pi_1(U \cap V) \to \pi_1(V)$ are monomorphisms. Finally suppose that U, V and $U \cap V$ are aspherical. Show that X is also aspherical.
Hint. You might want to use Propositions 53.24, 63.5 and the Excisive Triad-Proposition 109.12 (6).

Exercise 85.4. We start out with the following easy-to-digest definition. Let X be a topological space and let $k \in \mathbb{N}_0$. We say X is *k-connected*, if $\pi_i(X) = 0$ for $i = 0, \ldots, k$. Now let M and N be two closed oriented n-dimensional smooth manifolds. Suppose there exists a $k \in \mathbb{N}$ such that M and N are k-connected. Show that the connected sum $M \# N$ is also k-connected.

Exercise 85.5. Let X be an n-dimensional CW-complex and let Y be a subcomplex of X that is homotopy equivalent to S^n. Show that the inclusion induced map $\pi_n(Y) \to \pi_n(X)$ is a monomorphism.

Exercise 85.6. We denote by $*$ suitable base points of the spheres. Now let $f\colon S^n \to S^n$ be a map with $f(*) = *$. We suppose that f is a map of degree $k \in \mathbb{Z}$, i.e. f is a map such that the induced map $f_*\colon \mathrm{H}_n(S^n;\mathbb{Z}) \to \mathrm{H}_n(S^n;\mathbb{Z})$ is given by multiplication by k.

(a) Let $g\colon S^n \to S^m$ be a map. Show that $[g \circ f] = k \cdot [g] \in \pi_n(S^m)$.

(b) Show that if $g\colon S^m \to S^n$ is a map, then it is not necessarily the case that the equality $[f \circ g] = k \cdot [g] \in \pi_m(S^n)$ holds.
Hint. We view S^3 as a subset of $\mathbb{C}^2$ and we make the usual identification $\mathbb{C}\mathrm{P}^1 = S^1$. Now consider the following commutative diagram:

$$\begin{array}{ccc} S^3 & \xrightarrow{(z,w)\to[(z,w)]} & S^2 \\ {\scriptstyle (z,w)\mapsto(\overline{z},\overline{w})}\downarrow & & \downarrow{\scriptstyle [(z,w)]\mapsto[(\overline{z},\overline{w})]} \\ S^3 & \xrightarrow{(z,w)\to[(z,w)]} & S^2. \end{array}$$

(c) Let $g\colon S^{m-1} \to S^{n-1}$ be map. As usual we denote by $\Sigma(g)\colon S^m \to S^n$ the suspension of g. Show that $[f \circ \Sigma(g)] = k \cdot [\Sigma(g)] \in \pi_m(S^n)$.

Exercise 85.7. Given $n \in \mathbb{N}$ we denote by $L(n,1) = S^3/\mathbb{Z}_n$ the lens space, as defined on page 1075. For which $n \in \mathbb{N}$ is the Hurewicz homomorphism $\pi_3(L(n,1)) \to \mathrm{H}_3(L(n,1);\mathbb{Z})$ an isomorphism?

Exercise 85.8. Let X be a topological homology n-sphere, i.e. X is an n-dimensional topological manifold such that for any $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X) \cong \mathrm{H}_k(S^n)$. We denote by $\Sigma(X)$ the suspension of X as defined on page 451. Show that $\pi_i(\Sigma(X)) = 0$ for $i = 1, \ldots, n$.

Exercise 85.9. Let $n \in \mathbb{N}$. Let $\varphi\colon S^1 \to S^1$ be the map that is given by $z \mapsto z^n$. We consider the topological space $X := \overline{B}^2 \cup_\varphi S^1$. In other words, X is the CW-complex obtained by attaching a 2-cell to S^1 via the attaching map φ.

(a) Compute $\pi_1(X)$.

(b) Compute $\pi_2(X)$.

Exercise 85.10. Let $\varphi\colon S^1 \to S^5$ be a smooth embedding.

(a) Compute $\mathrm{H}_*(S^5 \setminus \varphi(S^1))$.

(b) Show that $\pi_3(S^5 \setminus \varphi(S^1)) \neq 0$.

Remark. For (b) you need to use that φ is a smooth embedding, the statement is in general not true, if we only assume that φ is an injective map. More precisely, in [**Haj2019**] it is shown that there exists an injective map $\varphi\colon S^1 \to S^5$ such that $\pi_3(S^5 \setminus \varphi(S^1)) = 0$.

Exercise 85.11. Show that there exists a closed 3-dimensional smooth manifold M such that $\pi_2(M)$ is infinitely generated.
Hint. You could try to imitate the topological space $S^1 \vee S^2$.

Exercise 85.12. Let X be a 0-connected countable CW-complex. Show that $\pi_2(X)$ is countable.
Remark. This answer Question 71.16 in the affirmative for $n = 2$.

86. Applications of the Hurewicz Theorem

In this chapter we want to spend some more time with the Hurewicz Theorems 84.5 and 85.5. In particular we want to discuss some applications thereof. This chapter is not entirely essential and can also be safely skipped at a first reading.

86.1. **The social choice problem ❀.** In this section we will consider the real-life problem that $n \in \mathbb{N}_{\geq 2}$ people have to agree in a democratic and reasonable way on one of the following:

(1) The temperature setting of a joint office.
(2) A vacation spot in Bavaria.
(3) A vacation spot on Earth.

More generally, the n people are supposed to agree on the following:

(4) A point in some topological space X.

One reasonable approach to solving this problem is the following. Each person gets to pick a point in X. We then need a map f which takes as input the n choices of points in X and which turns these choices into a common choice. In other words, we need a map

$$f\colon X^n = \underbrace{X \times \cdots \times X}_{n \text{ times}} \to X.$$

Our interpretation that the common choice should be chosen "democratically and reasonably" is that the map f should have the following three properties:

(a) A small change in choices of the people involved should correspond to a small change in the common choice.
(b) If everybody is of the same opinion, then evidently the common choice should be the point everybody picked.
(c) Everybody should be treated fairly, i.e. the social choice should not depend on the ordering of the n people.

In some cases it is easy to come up with a social choice. For example if X is a convex subset of some $\mathbb{R}^m$, e.g. if $X = \mathbb{R}^m$ itself, then given any $n \in \mathbb{N}$ the map[1039][1040]

$$\begin{array}{rcl} X^n & \to & X \\ (x_1, \ldots, x_n) & \mapsto & \frac{1}{n} \cdot \sum_{i=1}^{n} x_i \end{array}$$

evidently satisfies all of the above conditions. In particular this approach solves the problem for (1). If X is homeomorphic to a convex subset of some $\mathbb{R}^m$, then we can translate our problem into the previous situation and we can find a solution. In particular, Bavaria being to the best of my knowledge homeomorphic to a closed disk, we see that we have a solution to (2).[1041]

[1039] It follows easily from the convexity of X that the map does indeed take values in X.

[1040] For what choices of subsets X of $\mathbb{R}^m$ does the geometric mean $(x_1, \ldots, x_n) \mapsto (x_1 \cdot \cdots \cdot x_n)^{\frac{1}{n}}$ define a social choice?

[1041] The fact that Bavaria lost the Pfalz in 1946 greatly simplifies finding a solution to (2).

In the following we will now study the third problem in greater detail. At this point it is convenient to introduce the following definition from [**Eckm2004**], which is just a formalization of our wish list above.

Definition. Let X be a topological space and let $n \in \mathbb{N}$. A social choice of type n is a map

$$f\colon X^n \;=\; \underbrace{X \times \cdots \times X}_{n \text{ times}} \;\to\; X$$

such that the following three axioms are satisfied:

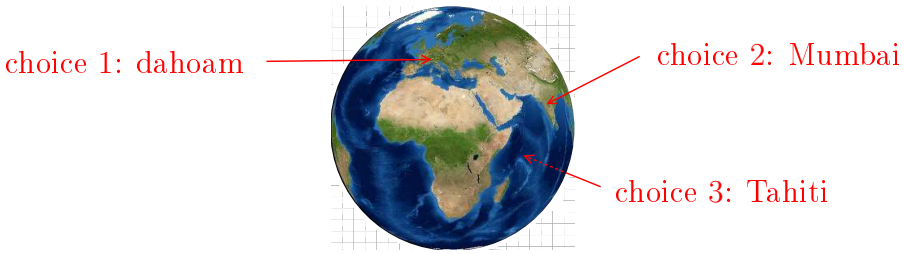
(a) The map f is continuous.
(b) For every $x \in X$ we have $f(x, \ldots, x) = x$.
(c) For every permutation $\sigma \in S_n$ and any $(x_1, \ldots, x_n) \in X^n$ we have

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \;=\; f(x_1, \ldots, x_n).$$

We say that a topological space is social, if for every $n \in \mathbb{N}$ it admits a social choice of type n.

Thus the question of whether there exists a "democratic and reasonable" way for n people to settle on a vacation spot on earth can be interpreted as the following question.

Question 86.1. Does there exist a social choice of type n on S^2?



As we will see, this question will be short-lived. The key to answer this question is the following proposition which is the main result of this section.

Proposition 86.2.

(1) Let X be a 0-connected space. If X admits a social choice of type n for some $n \in \mathbb{N}_{\geq 2}$, then given any $k \in \mathbb{N}$ the group[1042] $\pi_k(X)$ is abelian, and it is either zero or infinitely generated.
(2) Let X be a connected non-empty CW-complex that has the property that for every $k \in \mathbb{N}$ the number of k-cells is finite. If for every $n \in \mathbb{N}_0$ the topological space X admits a social choice of type $n \geq 2$, then for every $k \in \mathbb{N}$ we have $\pi_k(X) = 0$.

Remark.

(1) It follows from the Whitehead Theorem 162.1 that in Proposition 86.2 (2) we can upgrade the conclusion to the statement that X is actually contractible.
(2) In the context of sociology, Proposition 86.2 was first formulated in 1980 by the mathematical economist Graciela Chichilnisky [**Chic1980**]. It turns out that the proposition was already proved earlier by Beno Eckmann [**Eckm1954**] in 1954. The

[1042] In this section we will only deal with path-connected topological spaces and we will only be interested in the isomorphism type of the higher homotopy groups, thus we will use the Change-of-Base Point Proposition 71.5 as an excuse to ignore base points.

discussion of this section follows mostly [**Eckm2004**]. For more information and some generalizations we refer to [**Eckm2004**] and [**Weinb2004**].

We are now already in a position to answer Question 86.1 in the negative.

Corollary 86.3. Let $k \in \mathbb{N}$. The sphere S^k does not admit a social choice for any $n \in \mathbb{N}_{\geq 2}$.

Proof. By Corollary 85.6 we know that $\pi_k(S^k) \cong \mathbb{Z}$. Since $\mathbb{Z}$ is neither zero nor infinitely generated we obtain from Proposition 86.2 (1) that S^k does not admit a social choice for any $n \in \mathbb{N}_{\geq 2}$. ■

The idea behind the proof of Proposition 86.2 is quite simple. We need to see what effect the existence of a social choice on X has on the higher homotopy groups. Perhaps not surprisingly, the properties of a social choice give rise to very similar properties of the homotopy groups. This leads us to the following definition.

Definition. Let G be a group and let $n \in \mathbb{N}_{\geq 2}$. We say a map $\varphi\colon G^n \to G$ is a social homomorphism of type n if the following conditions are satisfied:

(a) The map φ is a homomorphism.
(b) For every $g \in G$ we have $\varphi(g, \ldots, g) = g$.
(c) For every permutation $\sigma \in S_n$ and any $(g_1, \ldots, g_n) \in G^n$ we have

$$\varphi\big(g_{\sigma(1)}, \ldots, g_{\sigma(n)}\big) \;=\; \varphi(g_1, \ldots, g_n).$$

Example. Evidently, if we consider $G = (\mathbb{Q}, +)$ or $G = (\mathbb{R}, +)$, then taking the arithmetic mean defines a social homomorphism.

The following lemma relates social choices on topological spaces to social homomorphisms.

Lemma 86.4. Let X be a path-connected topological space. If X admits a social choice of some type $n \geq 2$, then for every $k \in \mathbb{N}$ the homotopy group $\pi_k(X)$ admits a social homomorphism of type n.

Proof. To simplify the notation a little bit we only consider the case $n = 2$. Thus let $f\colon X \times X \to X$ be a social choice of type 2. Let $k \in \mathbb{N}$. We fix a base point $x_0 \in X$. We consider the map

$$\begin{array}{rcl} \Theta\colon \pi_k(X, x_0) \times \pi_k(X, x_0) & \to & \pi_k(X \times X, (x_0, x_0)) \\ ([\alpha\colon (S^k, *) \to (X, x_0)], [\beta\colon (S^k, *) \to (X, x_0)]) & \mapsto & \left[\begin{array}{rcl} (S^k, *) & \to & (X \times X, (x_0, x_0)) \\ z & \mapsto & (\alpha(z), \beta(z)) \end{array}\right]. \end{array}$$

Next we consider the map

$$\varphi\colon \pi_k(X, x_0) \times \pi_k(X, x_0) \xrightarrow{\Theta} \pi_k(X \times X, (x_0, x_0)) \xrightarrow{f_*} \pi_k(X, x_0).$$

It remains to show that $\varphi = f_* \circ \Theta$ is a social homomorphism of type 2. We do so in the following three steps.

(a) By the Homotopy Groups-of-Product Proposition 71.8 (2) we know that the map Θ is actually a homomorphism. It follows that $\varphi = f_* \circ \Theta$ is a homomorphism.
(b) It follows immediately from the fact that for every $x \in X$ we have $f(x, x) = x$ that for every $g \in \pi_k(X)$ we have $\varphi(g, g) = g$.
(c) Similarly it follows immediately from the fact that for every $x, y \in X$ we have $f(x, y) = f(y, x)$ that for every $g, h \in \pi_k(X, x_0)$ we have $\varphi(g, h) = \varphi(h, g)$. ■

In light of Lemma 86.4 we now need to understand, given $n \in \mathbb{N}_{\geq 2}$, which groups admit social homomorphisms of type n.

Definition. Let G be an abelian group.

(1) Let $n \in \mathbb{N}$. We say G is strongly divisible by n, if the map $g \mapsto n \cdot g$ is an isomorphism of the group G.
(2) We say G is strongly divisible, if G is strongly divisible by any $n \in \mathbb{N}$.

Remark. Usually an abelian group G is called *divisible* if for every $g \in G$ and $n \in \mathbb{N}$ there exists an $h \in G$ with $n \cdot h = g$. This will also be our definition later on page 2441. Clearly a group which is strongly divisible is also divisible. But the converse does not hold, for example the abelian group $(\mathbb{Q}/\mathbb{Z}, +)$ is divisible but it is not strongly divisible.

The following lemma gives a satisfying characterization of groups that admit a social homomorphism of type n.

Lemma 86.5. Let G be a group and let $n \in \mathbb{N}_{\geq 2}$. The following two statements are equivalent:

(1) The group G admits a social homomorphism of type n.
(2) The group G is abelian and it is strongly divisible by n.

Proof. In the following let G be a group and let $n \in \mathbb{N}_{\geq 2}$. Since we will shortly see that the groups involved are abelian we will use additive notation throughout the argument.

First we deal with the implication "(2) $\Rightarrow$ (1)". If G is abelian and if it is strongly divisible by n, then evidently the usual "arithmetic mean formula"

$$(g_1, \ldots, g_n) \;\mapsto\; \tfrac{1}{n}(g_1 + \cdots + g_n) := \begin{array}{c}\text{inverse of the isomorphism}\\ G \to G \text{ given by } g \mapsto n \cdot g,\\ \text{applied to } g_1 + \cdots + g_n \in G\end{array}$$

defines a social homomorphism of type n.

Next we prove the much more interesting implication "(1) $\Rightarrow$ (2)". Let $\varphi\colon G^n \to G$ be a social homomorphism of type $n \geq 2$. Throughout the proof we mean by (a), (b) and (c) the three defining properties of a social homomorphism. First note that it follows immediately from (c) that for any $g \in G$ we have the equalities

$$\varphi(g, 0, \ldots, 0) \;=\; \varphi(0, g, 0, \ldots, 0) \;=\; \varphi(0, \ldots, 0, g) \;\in\; G.$$

We denote this common value by $\psi(g)$. We make the following observations:

(i) It follows immediately from (a) that $\psi\colon G \to G$ is also a homomorphism.
(ii) For any $g, h \in G$ we have

$$\begin{array}{rcccl} & & \overset{\text{since } n \geq 2}{\downarrow} & & \overset{\text{by (a)}}{\downarrow} \\ \psi(g) + \psi(h) & = & \varphi(g, 0, \ldots, 0) + \varphi(0, h, 0, \ldots, 0) & = & \varphi(g, h, 0, \ldots, 0) \\ & = & \varphi(0, h, \ldots, 0) + \varphi(g, 0, 0, \ldots, 0) & = & \psi(h) + \psi(g). \\ & & \underset{\text{by (a)}}{\uparrow} & & \underset{\text{since } n \geq 2}{\uparrow} \end{array}$$

(iii) For any $g \in G$ we have

$$\underbrace{\psi(g) + \ldots \psi(g)}_{= n \cdot \psi(g)} \underset{\text{by definition of } \psi}{=} \varphi(g, 0, \ldots, 0) + \cdots + \varphi(0, \ldots, 0, g) \underset{\text{by (a)}}{=} \varphi(g, \ldots, g) \underset{\text{by (c)}}{=} g.$$

(iv) Let $g, h \in G$. We have

$$\begin{array}{lll} & \overset{\text{by (iii)}}{\downarrow} & \overset{\text{by (i)}}{\downarrow} \\ g+h \overset{}{=} & \psi(g+h)+\cdots+\psi(g+h) & = \psi(g)+\psi(h)+\cdots+\psi(g)+\psi(h) \\ = & \psi(h)+\psi(g)+\cdots+\psi(h)+\psi(g) & = \psi(h+g)+\cdots+\psi(h+g) \overset{}{=} h+g. \\ & \underset{\text{by (ii)}}{\uparrow} & \underset{\text{by (i)}}{\uparrow} \qquad\qquad\qquad\qquad \underset{\text{by (iii)}}{\uparrow} \end{array}$$

This shows that G is abelian.

(v) Since G is abelian we know that multiplication by n is a homomorphism of G.

(vi) It follows almost immediately from (i) and (iii) that ψ is an inverse to the map $G \to G$ that is given by $g \mapsto n \cdot g$. Together with (v) this shows that $g \mapsto n \cdot g$ does indeed define an isomorphism of G. ■

The following lemma basically tells us which finitely generated abelian groups are strongly divisible by some $n \in \mathbb{N}_{\geq 2}$.

Lemma 86.6. Let $n \in \mathbb{N}_{\geq 2}$.

(1) The group $\mathbb{Z}$ is not strongly divisible by n.

(2) Let $m \in \mathbb{N}$. The cyclic group $\mathbb{Z}_m$ is strongly divisible by n if and only if m and n are coprime.

(3) If a finitely generated group is strongly divisible by n, then it is isomorphic to a direct sum $\bigoplus_{i=1}^{k} \mathbb{Z}_{m_i}$ where each m_i is coprime to n.

Proof. The first two statements are basically trivial. Finally the last statement follows from the previous two statements together with the Finitely Generated Abelian Group Classification Theorem 51.4. ■

Now we have assembled everything we need for the proof of Proposition 86.2.

Proof of Proposition 86.2.

(1) Let X be a 0-connected space that admits a social choice of type n for some $n \in \mathbb{N}_{\geq 2}$. Let $k \in \mathbb{N}$. By Lemma 86.4 we know that the group $\pi_k(X)$ is abelian and that it is strongly divisible by n. It follows almost immediately from Lemma 86.6 that $\pi_k(X)$ is either zero or infinitely generated.

(2) Let X be a connected non-empty CW-complex that has the property that for every $k \in \mathbb{N}$ the number of k-cells is finite. We assume that X is social, i.e. we assume that given any $n \geq 2$ there exists a social choice of type n on X. By Lemma 86.4 we know that every homotopy group $\pi_k(X)$ is abelian and that it is strongly divisible by n.

Next note that by the CW-Skeleton-π_1-Proposition 69.13 and our hypothesis that X has finitely many 1-cells we know that $\pi_1(X)$ is finitely generated. It follows from the above together with Lemma 86.6 that $\pi_1(X) = 0$.

Now suppose that we already know that $\pi_i(X) = 0$ for $i = 1, \ldots, k$. By the Hurewicz Theorem 85.5 we know that $\pi_{k+1}(X) \cong \mathrm{H}_{k+1}(X)$. Since X has finitely many $(k+1)$-cells we obtain from the Homology-of-CW-Complex Proposition 80.6 that $\pi_{k+1}(X) \cong \mathrm{H}_{k+1}(X)$ is finitely generated. Basically the same argument as above shows that $\pi_{k+1}(X) = 0$. Iterating this procedure shows that all homotopy groups of X vanish. ■

We conclude this section with the following question.

Question 86.7. Does there exist a non-empty CW-complex that is social but that is not contractible?

We will return to this question later on in Section 163.6.

86.2. **The degree of a self-map of a sphere II.** Let $n \in \mathbb{N}_0$ and let $f\colon S^n \to S^n$ be map. On page 1684 we defined the degree of f as follows:

$$\deg(f) \;:=\; \deg\big(f_*\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n)\big).$$

In the Sphere Degree Lemma 76.11 (3) we showed that any two homotopic maps $S^n \to S^n$ have the same degree. Now we have the tools to prove the converse of this statement. In fact we have the following proposition.

Proposition 86.8. (Degree Isomorphism Proposition) For any $n \in \mathbb{N}$ the map

$$\begin{array}{rcl} \deg\colon \overbrace{[S^n, S^n]}^{\text{set of homotopy equivalence classes of maps } S^n \to S^n} & \to & \mathbb{Z} \\ [f\colon S^n \to S^n] & \to & \deg(f) \end{array}$$

is an isomorphism. (The set $[S^n, S^n]$ is a monoid through composition of maps.)

Proof. First note that we know by the Sphere Degree Lemma 76.11 (3) and (4) that the degree map is well-defined and that it is a monoid morphism. It remains to show that the degree map is a bijection. Now let $* \in S^n$ be the standard base point. We consider the following diagram

$$\begin{array}{ccccc} \langle (S^n,*),(S^n,*)\rangle & \xrightarrow{\cong} & [S^n,S^n] & \xrightarrow{\deg(f)} & \mathbb{Z} \\ \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle f\mapsto f_*([S^n])} & \nearrow{\scriptstyle \cong}_{k\cdot[S^n]\mapsto k} & \\ \pi_n(S^n,*) & \xrightarrow[[f]\mapsto f_*([S^n])]{\cong} & \mathrm{H}_n(S^n). & & \end{array}$$

We make the following observations and clarifications:

(1) By Corollary 70.10 we know that the map $\langle (S^n,*),(S^n,*)\rangle \to [S^n,S^n]$ is a bijection.
(2) By Theorem 85.5 we know that the bottom horizontal map is an isomorphism.
(3) The diagonal map is an isomorphism by Proposition 74.4.
(4) Finally note that it is clear that the diagram commutes. It follows from this observation and all of the above that the map $\deg\colon [S^n,S^n] \to \mathbb{Z}$ is indeed an isomorphism. ■

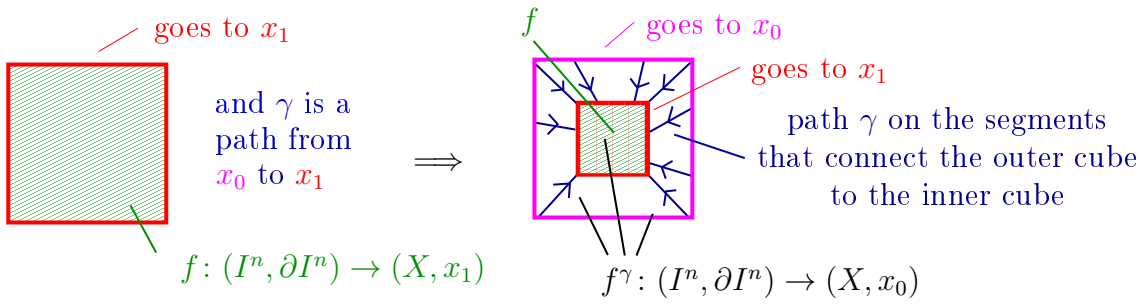
86.3. **Paths acting on homotopy groups.** In Propositions 47.10 and 71.5 we already saw that for path-connected topological spaces the *isomorphism types* of the homotopy groups do not depend on the choice of the base point.

In this section we want to discuss this statement in greater detail. We recall the following definition which appeared already in the proof of the Change-of-Base Point Proposition 71.5.

Definition. Let X be a topological space and let $n \in \mathbb{N}$. Let $\gamma\colon [0,1] \to X$ be a path from a point x_0 to a point x_1. Given a map $f\colon (I^n, \partial I^n) \to (X, x_1)$ we define the map

$$\begin{array}{rcl} f^\gamma\colon (I^n, \partial I^n) & \to & (X, x_0) \\ (t_1, \dots, t_n) & \mapsto & \begin{cases} f\big(2t_1 - \frac{1}{2}, \dots, 2t_n - \frac{1}{2}\big), & \text{if } (t_1, \dots, t_n) \in \big[\frac{1}{4}, \frac{3}{4}\big]^n, \\ \gamma(t), & \text{if } (t_1, \dots, t_n) \in \partial\big(\big[\frac{t}{4}, \frac{4-t}{4}\big]^n\big) \text{ for } t \in [0,1]. \end{cases} \end{array}$$

The definition of f^γ is illustrated in the figure below.



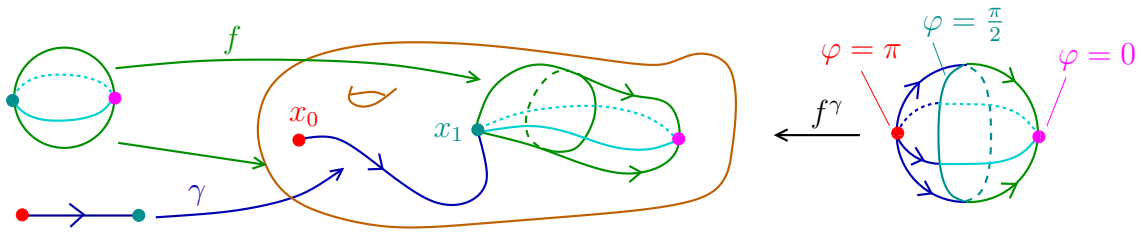
Example. We consider the case $n = 1$ in the above definition. In this situation the map $f\colon (I, \partial I) \to (X, x_1)$ is a loop. It follows almost immediately from the definitions that the new loop $f^\gamma\colon (I, \partial I) \to (X, x_0)$ equals, up to an orientation-preserving reparametrization, the loop $\gamma * f * \overline{\gamma}$ that is given by the product of the three paths γ, f and $\overline{\gamma}$. We refer to the figure below for an illustration.



Remark. We recall that on pages 1565 and 1847 we saw that we can interpret elements in a homotopy group $\pi_n(X, x_0)$ as homotopy classes of maps $(S^n, *) \to (X, x_0)$. Now let $\gamma\colon [0,1] \to X$ be a path from x_0 to x_1 and let $f\colon (S^n, *) \to (X, x_1)$ be a map. We define[1043]

$$\begin{array}{rcl} f^\gamma\colon S^n & \to & X \\ \underbrace{\cos(\varphi)\cdot(1,0,\dots,0) + \sin(\varphi)\cdot(0,v)}_{\varphi \in [0,\pi], v \in \mathbb{R}^{n-1}} & \mapsto & \begin{cases} f(\cos(2\varphi)\cdot(1,0,\dots,0) + \sin(2\varphi)\cdot(0,v)), & \text{if } \varphi \in [0, \frac{\pi}{2}] \\ \gamma(2 - \frac{2}{\pi}t), & \text{if } \varphi \in [\frac{\pi}{2}, \pi]. \end{cases} \end{array}$$

Of course one should only work with the formula if one is forced to at gun point. It is much more illuminating to look at the figure below where we try our best to visualize the map $f^\gamma\colon (S^n, *) \to (X, x_0)$. One can easily verify that, under the various identifications, the map f^γ corresponds to the construction on page 1869.



[1043] It follows easily from Pasting Proposition 2.6 (2′) that this map is continuous.

Before we continue we introduce the following basically self-explanatory definition.

Definition. Let G be a group and let A be a group (or module or vector space). An action of G on the group A (respectively module or vector space) is an action $G \times A \to A$ in the sense of the definition on page 248 such that for each $g \in G$ the map

$$\begin{array}{ccc} A & \to & A \\ a & \mapsto & g \cdot a \end{array}$$

is a group homomorphism (respectively module or vector space homomorphism).[1044]

The following proposition can be viewed as a more precise version of the Change-of-Base Point Proposition 71.5 and as a generalization of Proposition 47.20.

Proposition 86.9. Change-of-Base Point Proposition II) Let X be a topological space and let $n \in \mathbb{N}$.

(1) If $\gamma\colon [0,1] \to X$ is a path from x_0 to x_1, then the map

$$\begin{array}{rccc} \gamma_*\colon \pi_n(X,x_1) & \to & \pi_n(X,x_0) \\ [f] & \mapsto & [f^\gamma] \end{array}$$

is a well-defined isomorphism.

(2) If x_0 and x_1 are two points in X which lie in the same path-component of X, then $\pi_n(X,x_0)$ is isomorphic to $\pi_n(X,x_1)$.

(3) Let $\varphi\colon X \to Y$ be a map between topological spaces.

(a) If $\gamma\colon [0,1] \to X$ is a path from x_0 to x_1, then the following diagram commutes

$$\begin{array}{ccc} \pi_n(X,x_1) & \xrightarrow{\gamma_*} & \pi_n(X,x_0) \\ {\scriptstyle\varphi_*}\downarrow & & \downarrow{\scriptstyle\varphi_*} \\ \pi_n(Y,\varphi(x_1)) & \xrightarrow{(\varphi\circ\gamma)_*} & \pi_n(Y,\varphi(x_0)). \end{array}$$

(b) If $\varphi_*\colon \pi_n(X,x_0) \to \pi_n(Y,\varphi(x_0))$ is an isomorphism (epimorphism, monomorphism) for some base point $x_0 \in X$, then for any other base point x_1 in the same path component of X the map $\varphi_*\colon \pi_n(X,x_1) \to \pi_n(Y,\varphi(x_1))$ is also an isomorphism (epimorphism, monomorphism).[1045]

(4) Let γ and δ be two paths in X from x_0 to x_1. If γ and δ are path-homotopic in X, then the corresponding maps γ_* and δ_* from $\pi_n(X,x_1)$ to $\pi_n(X,x_0)$ agree.

(5) If γ is a path in X from x_0 to x_1 and if δ is a path in X from x_1 to x_2, then[1046]

$$(\underbrace{\gamma * \delta}_{\text{path from } x_0 \text{ to } x_2})_* = \gamma_* \circ \delta_*\colon \pi_n(X,x_2) \to \pi_n(X,x_0).$$

(6) The map

$$\begin{array}{rccc} \pi_1(X,x_0) \times \pi_n(X,x_0) & \to & \pi_n(X,x_0) \\ ([\gamma],[f]) & \mapsto & [\gamma]\cdot[f] := [f^\gamma] \end{array}$$

defines an action (in the sense of the definition on page 248) of the group $\pi_1(X,x_0)$ on the group $\pi_n(X,x_0)$.

[1044] In fact, since multiplication by g^{-1} is an inverse we see that these maps are actually isomorphisms.

(7) Let $f_0, f_1 \colon S^n \to X$ be two maps. We write $x_0 = f_0(*)$ and $x_1 = f_1(*)$. The following statement holds:

$$f_0 \text{ and } f_1 \text{ are homotopic} \iff \begin{array}{c}\text{there exists a path } \gamma \text{ from } x_0 \text{ to } x_1 \\ \text{such that } [f_1^\gamma] = [f_0] \in \pi_n(X, x_0).\end{array}$$

Remark. Let X be a topological space, let x_0, x_1 be two points in X and let γ and δ be two paths from x_0 to x_1. If X is simply connected, then Lemma 47.12 tells us that the two paths γ and δ are in fact homotopic. Therefore it follows from the Change-of-Base Point Proposition 86.9 (4) that $\gamma_* = \delta_* \colon \pi_n(X, x_0) \to \pi_n(X, x_1)$. In other words, if X is simply connected, then given any two base points x_0 and x_1 there exists in fact a natural[1047] isomorphism $\pi_n(X, x_0) \xrightarrow{\cong} \pi_n(X, x_1)$. In particular in this case it is almost always entirely safe to suppress the base point in the notation. By the Sphere-π_1-Proposition 47.13 this observation applies in particular to $X = S^k$ for $k \geq 2$.

Proof.

(0) Let $\gamma \colon [0,1] \to X$ be a path from x_0 to x_1. It is straightforward to see that the map

$$\begin{array}{rcl}\gamma_* \colon \pi_n(X, x_1) & \to & \pi_n(X, x_0) \\ [f] & \mapsto & [f^\gamma]\end{array}$$

is well-defined. It is also elementary, albeit potentially slightly painful, to see that γ_* is a group homomorphism. (See the corresponding discussion in the proof of the Change-of-Base Point Proposition 71.5.)

(4),(5) These two statements are basically elementary to verify. Nonetheless it is worth spending a second to convince oneself that in (5) we have the equality $(\gamma * \delta)_* = \gamma_* \circ \delta_*$ instead of $(\gamma * \delta)_* = \delta_* \circ \gamma_*$.

(1) Recall that in the Path-Homotopy Concatenation Proposition 47.5 (3) we saw that for any path δ the composition $\delta * \overline{\delta}$ is path-homotopic to the constant path. It follows easily from this observation together with (4) and (5) that γ_* is a bijection. Together with (0) we see that for any path $\gamma \colon [0,1] \to X$ from x_0 to x_1 the induced map $\gamma_* \colon \pi_n(X, x_1) \to \pi_n(X, x_0)$ is an isomorphism.

(2) This is an immediate consequence of (1)

(3) Statement (a) follows immediately from the definitions. Statement (b) is a consequence of the commutative diagram of (a) and the fact that the horizontal maps are isomorphisms by (1).

(6) It follows immediately from (4) and (5) that the map defines an action of $\pi_1(X, x_0)$ on the set $\pi_n(X, x_0)$. Furthermore (1) implies that this action is in fact an action on the group $\pi_n(X, x_0)$.

(7) The proof of this statement is a moderately complicated generalization of the proof of the Homotopies-of-Loops Proposition 50.16. We leave it to the reader to fill in the details. ∎

[1045] For $n = 1$ this is basically the content of Proposition 47.20.

[1046] Here it is worth remembering that by definition, see page 1052, the product path $\gamma * \delta$ is given by first traversing γ and then traversing δ.

[1047] As always, when the adjective "natural" gets used it is an amusing exercise to figure out what categories and functors are involved.

86.4. **Group rings.** Before we continue with our study of higher homotopy groups it is convenient to introduce the purely algebraic concept of a group ring.

Definition. Let G be a group and let R be a commutative ring. We denote the free R-module generated by the set G by $R[G]$. Put differently, with the notation from page 1147 we have

$$R[G] \;:=\; \Big\{ \sum_{i=1}^{m} r_i \cdot g_i \,\Big|\, r_1, \ldots, r_m \in R \text{ and } g_1, \ldots, g_m \in G \Big\}.$$

We equip $R[G]$ with the multiplication that is given by[1048]

$$\Big(\sum_{i=1}^{m} r_i \cdot g_i \Big) \cdot \Big(\sum_{j=1}^{n} s_j \cdot h_j \Big) \;:=\; \sum_{i=1}^{m} \sum_{j=1}^{n} \underbrace{r_i \cdot_R s_j}_{\in R} \cdot \underbrace{(g_i \cdot_G h_j)}_{\in G}.$$

We refer to $R[G]$ as the group ring of G with R-coefficients. (For $R = \mathbb{Z}$ we refer to $\mathbb{Z}[G]$ as the group ring of G.)

The following lemma summarizes a few basic facts about group rings.

Lemma 86.10. Let G be a group and let R be a commutative ring.

(1) The group ring $R[G]$ is an associative ring with a multiplicatively neutral element, namely $1 \cdot e$ where e denotes the trivial element in G.
(2) The group ring $R[G]$ is commutative if and only if G is an abelian group.
(3) If R is a field, then $R[G]$ is evidently an algebra over the field R.
(4) Given a unit $r \in R$ and an element $g \in G$ the product $r \cdot g$ is a unit in $R[G]$.
(5) If G has a torsion element, i.e. if there exists a non-trivial $g \in G$ such that $g^k = e$ for some $k \in \mathbb{N}$, then $R[G]$ has zero divisors.

Proof. All the statements are verified easily. Perhaps the most interesting statement is (5). We will not rob the reader of the pleasure of proving this in Exercise 86.3. ∎

Examples. Let R be a commutative ring.

(1) We consider the infinite cyclic group $\langle t \rangle$. In this case the group ring $R[\langle t \rangle]$ is isomorphic to the ring $R[t^{\pm 1}]$ of Laurent polynomials, more precisely the "obvious" map

$$\begin{array}{ccc} R[t^{\pm 1}] & \to & R[\langle t \rangle] \\ \sum_{i=r}^{i=s} a_i t^i & \mapsto & \sum_{i=r}^{i=s} a_i t^i \end{array}$$

is a ring isomorphism.
(2) More generally, if H is a free abelian group of rank m, then any choice of a basis $t_1, \ldots, t_m$ for H gives rise to a natural isomorphism from the multivariable Laurent polynomial ring $R[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ to the group ring $R[H]$.

The group rings of free groups are particularly interesting. Therefore we dedicate a lemma to them.

Lemma 86.11. Let $\langle T \rangle$ be the free group on a generating set T. Given any ring S and given any map $g\colon T \to S$ there exists a unique ring homomorphism $\varphi\colon \mathbb{Z}[\langle T \rangle] \to S$ with $\psi(t) = g(t)$ for all $t \in T$.

[1048] The multiplication is basically given by naive "multiplying out".

Remark. Given a commutative ring R the group ring $R[\langle T\rangle]$ is sometimes called the *free ring on the set S over the ring R*.

Proof. It follows immediately from Lemma 51.15 that there exists a unique group homomorphism $\varphi\colon \langle T\rangle \to S^* := \{\text{units of } S\}$ that extends the given map g. It is now straightforward to verify that the map

$$\begin{array}{rcl} \mathbb{Z}[\langle T\rangle] & \to & S \\ \sum_{i=1}^{m} r_i \cdot g_i & \mapsto & r_i \cdot \varphi(g_i) \end{array}$$

is a ring homomorphism that extends g and that it is the unique ring homomorphism with this desirable property. ■

As a side remark we mention the following beautiful conjectures. These say in particular that in some sense the statements of Lemma 86.10 (4) and (5) are optimal.

Conjecture 86.12. (Kaplansky Conjectures) Let G be a torsion-free group and let $\mathbb{F}$ be a field.

(1) The elements 0 and 1 are the only idempotents of $\mathbb{F}[G]$, i.e. they are the only elements in $\mathbb{F}[G]$ with $x^2 = x$.
(2) The group ring $\mathbb{F}[G]$ is a domain, i.e. it has no non-trivial zero divisors.
(3) If $\mathbb{F}$ is of characteristic zero, then the only units in $\mathbb{F}[G]$ are the trivial units, i.e. the elements of the form $f \cdot g$ with $f \in \mathbb{F} \setminus \{0\}$ and $g \in G$.
(4) The group ring $\mathbb{C}[G]$ can be embedded in a skew field.

Remark.

(1) The conjectures are named after the Canadian mathematician Irving Kaplansky [**Kapy1957, Kapy1970**]. The first conjecture is often also referred to as the *zero divisor conjecture*. They are formulated in [**Lü2002**, Conjecture 10.14], [**Lam1991**, p. 95] and [**FrG2005**, p. 729]. As is pointed out in [**Lam1991**, p. 95], an proof of Conjecture (3) for a field $\mathbb{F}$ implies that Conjecture (2) holds for the same field $\mathbb{F}$. It is elementary to see that Conjecture (4) implies Conjecture (2) and that Conjecture (2) implies Conjecture (1).
(2) In (3) we need to add the hypothesis that $\mathbb{F}$ is of characteristic zero since Giles Gardam [**Garda2021**, Theorem A] showed that there exists a torsion-free group G such that $\mathbb{F}_2[G]$ has non-trivial units.
(3) The Kaplansky Conjectures are very attractive since they are easy to state. But it is usually very very hard to prove non-trivial group-theoretic statements that are valid for all (torsion-free) groups. In particular all of the Kaplansky Conjectures are still wide open. We refer to [**Lü2009**, Chapter 4.5] for a summary of what is known regarding these conjectures.

We continue with the following elementary lemma.

Lemma 86.13. Let G be a group and let A be an abelian group (with additive notation for the group structure). Suppose we are given a G-action on the group A. Then the map[1049]

$$\begin{array}{rcl} \mathbb{Z}[G] \times A & \to & A \\ \left(\sum_{i=1}^{m} r_i \cdot g_i, a\right) & \mapsto & \sum_{i=1}^{m} r_i \cdot (\underbrace{g_i \cdot a}_{\in A}) \end{array}$$

defines a natural left $\mathbb{Z}[G]$-module structure on A.[1050]

Proof. The lemma is indeed elementary and trivial. We leave it to the reader to ponder what "natural" might mean in this context. ■

Before we continue we introduce or recall the following convention:

Convention. If a homotopy group is abelian, then sometimes we use additive notation for the group multiplication instead of the multiplicative notation.

This convention, Lemma 86.13, Propositions 71.3 and the Change-of-Base Point Proposition 86.9 lead us to the following lemma.

Lemma 86.14. Let (X, x_0) be a pointed topological space. If $n \geq 2$, then the map

$$\begin{array}{rcl} \mathbb{Z}[\pi_1(X, x_0)] \times \pi_n(X, x_0) & \to & \pi_n(X, x_0) \\ ([\gamma], [f]) & \mapsto & [f^\gamma] \end{array}$$

induced by

defines a natural $\mathbb{Z}[\pi_1(X, x_0)]$-left module structure on $\pi_n(X, x_0)$.

86.5. **Homotopy groups of $S^1 \vee S^2$.** In this section we want to consider the homotopy groups of $S^1 \vee S^2$ since they are quite instructive. On page 1844 we already saw that $\pi_2(S^1 \vee S^2)$ is very big, more precisely, we saw that it admits an epimorphism onto $\mathbb{Z}^\infty$. We will now see that if we take the view of Lemma 86.14, i.e. if we view $\pi_2(S^1 \vee S^2)$ as a module over the group ring $\mathbb{Z}[\pi_1(S^1 \vee S^2)]$, then the apparently messy group $\pi_2(S^1 \vee S^2)$ suddenly gets a simple description.

We explicitly construct $X = S^1 \vee S^2$ by identifying the point $i \in S^1$ with the point $(0, 0, -1) \in S^2$. In the following we evidently use the wedge point as the base point of $X = S^1 \vee S^2$. We refer to it by x_0. Finally we denote by $\gamma\colon [0, 1] \to S^1 \subset X$ the obvious counterclockwise loop that starts at i. Furthermore we denote by $\mathrm{id}_{S^2}\colon S^2 \to S^2 \subset X$ the obvious inclusion map.

Now we can formulate the following proposition which delivers the promised concise description of $\pi_2(S^1 \vee S^2)$.

Proposition 86.15. To simplify the formulation and the proof of the proposition we use the base point $* = (0, 0, -1)$ for S^2 and we use this base point to define the second homotopy groups. This convention has the advantage that $\mathrm{id}\colon (S^n, *) \to (S^1 \vee S^2, x_0)$ is a

[1049] Here we use a somewhat perilous notation, namely we use the symbol "$\cdot$" with two different meanings. First of all, for $g \in G$ and $a \in A$ we denote by $g \cdot a \in A$ the element defined by the G-action on A. Secondly, for $r \in \mathbb{Z}$ and $b \in A$ we write, as usual, $r \cdot b = \underbrace{b + \cdots + b}_{r\text{-times}}$ for $r \geq 0$ and $r \cdot b = \underbrace{-b - \cdots - b}_{(-r)\text{-times}}$ for $r < 0$.

[1050] If R is a non-commutative ring, then one needs to be a little careful when one talks about "R-modules". More precisely, there are now two different concepts one can consider: Given an abelian group M one can demand that for any $r, s \in R$ and $m \in M$ one has $(rs) \cdot m = r \cdot (s \cdot m)$ in which case one says that M is an "R-left module". Alternatively one could demand that for any $r, s \in R$ and $m \in M$ one has $(rs) \cdot m = s \cdot (r \cdot m)$ in which case one says that M is an "R-right module". This discussion is the analogue of the discussion on page 249.

map of based topological spaces. Using this convention the homomorphism

$$\begin{array}{rcl}\Psi\colon \mathbb{Z}[t^{\pm 1}] & \to & \pi_2(S^1 \vee S^2, x_0) \\ \sum_{i=1}^{n} a_i \cdot t^i & \mapsto & \sum_{i=1}^{n} a_i \cdot \left[(\mathrm{id}_{S^2})^{\gamma^i}\right]\end{array}$$

that is determined by

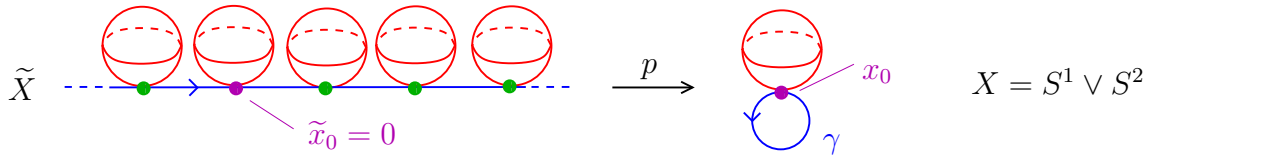
is an isomorphism of $\mathbb{Z}[t^{\pm 1}] = \mathbb{Z}[\langle t \rangle] = \mathbb{Z}[\pi_1(S^1 \vee S^2, x_0)]$-modules. In particular, as an abelian group, $\pi_2(S^1 \vee S^2)$ is isomorphic to $\mathbb{Z}^\infty$.

Proof. It follows immediately from the definitions that the map Ψ is indeed a homomorphism of $\mathbb{Z}[t^{\pm 1}] = \mathbb{Z}[\langle t \rangle] = \mathbb{Z}[\pi_1(S^1 \vee S^2, x_0)]$-modules. Thus it remains to show that Ψ is an isomorphism of abelian groups.

As on page 1373 we see that the universal covering of $X = S^1 \vee S^2$ is given by the map

$$\begin{array}{rcl} p\colon \overbrace{\big(\mathbb{R} \cup (\mathbb{Z} \times S^2)\big) \,/\, n \sim (n,(0,0,-1))}^{=:\widetilde{X}} & \to & S^1 \vee S^2 \\ P & \mapsto & \begin{cases} \exp(2\pi \mathrm{i}(P + \frac{1}{4})), & \text{if } P \in \mathbb{R} \\ Q, & \text{if } P = (n,Q) \text{ with } n \in \mathbb{Z}, Q \in S^2. \end{cases}\end{array}$$

We pick the base points $x_0 = i \in S^1 \subset X$ and $\widetilde{x}_0 = 0 \in \mathbb{R} \subset \widetilde{X}$. Even though we have seen this covering map on numerous occasions we show it again in the figure below. We



consider the following sequence of maps

$$(*) \qquad \mathbb{Z}[t^{\pm 1}] \xrightarrow{\Psi} \pi_2(X, x_0) \xrightarrow{\Xi} \pi_2(\widetilde{X}, \widetilde{x}_0) \xrightarrow{\Phi_{(\widetilde{X},\widetilde{x}_0)}} \mathrm{H}_2(\widetilde{X}; \mathbb{Z}) \xleftarrow{\Omega} \mathbb{Z}[t^{\pm 1}].$$

Here $\Phi_{(X,x_0)}$ denotes the Hurewicz homomorphism. Since $\widetilde{X}$ is simply connected we obtain from the Hurewicz Theorem 85.5 that it is an isomorphism. The reader will not have failed to notice that we did not define the two maps Ξ and Ω yet. We will do so now.

(1) First we recall that given a map $g\colon (S^2, *) \to (X, x_0)$ there exists by the Map Lifting Criterion 61.2 a unique lift $\widetilde{g}\colon (S^2, *) \to (\widetilde{X}, \widetilde{x}_0)$. Now we consider the map

$$\begin{array}{rcl}\Xi\colon \pi_2(X, x_0) & \to & \pi_2(\widetilde{X}, \widetilde{x}_0) \\ [g\colon (S^2, *) \to (X, x_0)] & \mapsto & \left[\widetilde{g}\colon (S^2, *) \to (\widetilde{X}, \widetilde{x}_0)\right].\end{array}$$

By the Coverings-π_n-Proposition 71.13 we know that Ξ is well-defined and that it is an isomorphism.

(2) Given $i \in \mathbb{Z}$ we denote by $f_i\colon S^2 \to X$ the "obvious" i-th inclusion map. In Lemma 79.7 we saw that the map

$$\begin{array}{rcl}\Omega\colon \mathbb{Z}[t^{\pm 1}] & \to & \mathrm{H}_2(\widetilde{X}; \mathbb{Z}) \\ \sum_{i=r}^{s} a_i \cdot t^i & \mapsto & \sum_{i=r}^{s} a_i \cdot f_i([S^2]).\end{array}$$

is an isomorphism.

We see in particular that in $(*)$ the three maps to the right are isomorphisms. Therefore it suffices to show that the maps $\Omega^{-1} \circ \Phi_{(X,x_0)} \circ \Xi \circ \Psi$ is the identity. In other words, it suffices to show the following claim.

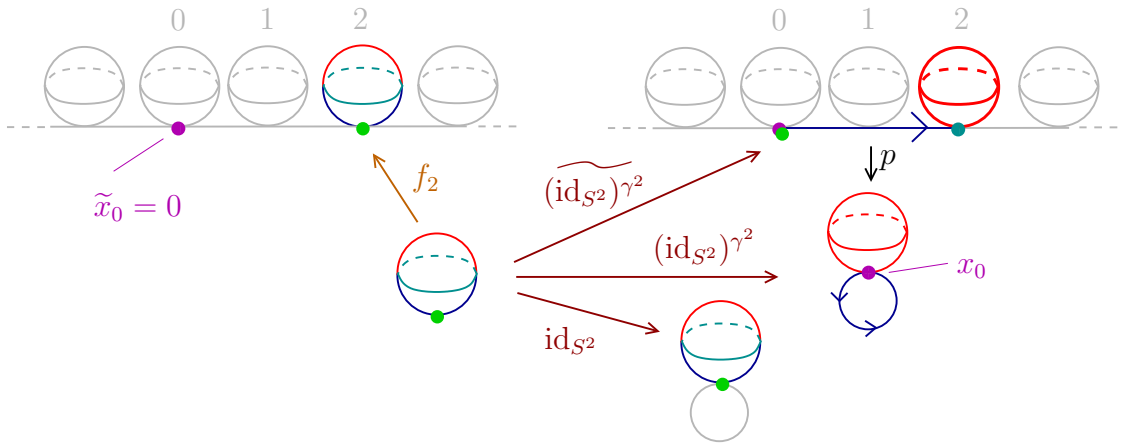
Claim. For any $n \in \mathbb{Z}$ we have

$$(\Phi_{(X,x_0)} \circ \Xi \circ \Psi)(t^n) \;=\; \Omega(t^n) \;\in\; \mathrm{H}_2(\widetilde{X};\mathbb{Z}).$$

Proof. So let $n \in \mathbb{Z}$. It follows immediately from the definitions of the various maps that the desired statement is equivalent to the statement that

$$\underbrace{\left(\widetilde{(\mathrm{id}_{S^2})^{\gamma^n}}\right)_*}_{\text{map } S^2 \to \widetilde{X}}[S^2] \;=\; \underbrace{f_n}_{\text{map } S^2 \to \widetilde{X}}([S^2]) \;\in\; \mathrm{H}_2(\widetilde{X};\mathbb{Z}).$$

Thus by the Homotopic Maps-and-Homology Proposition 73.6 it suffices to show that the two maps $(\mathrm{id}_{S^2})^{\gamma^n}$ and f_n are homotopic maps from S^2 to $\widetilde{X}$. For those readers who are still awake this should be basically obvious.[1051] As a digestion aid we illustrate both maps in the figure below. ■



In the Homology-of-CW-Complex Proposition 80.6 we saw that the homology groups of a finite CW-complex are finitely generated abelian groups. Since the discussion on page 1844 we had known that the corresponding statement is not true for homotopy groups. But we now also know that one should really view the homotopy groups of a pointed topological space (X, x_0) as modules over the group ring $\mathbb{Z}[\pi_1(X, x_0)]$. Thus the following question arises.

Question 86.16. Let X be a finite CW-complex. Are the homotopy groups $\pi_n(X, x_0)$, $n \geq 2$, finitely generated $\mathbb{Z}[\pi_1(X, x_0)]$-modules?

It turns out that the answer is yes if X is simply connected. More precisely, Jean-Pierre Serre [**Ser1953**][1052] proved the following theorem.

[1051] In fact this follows from the "⇐"-direction of the Change-of-Base Point Proposition 86.9 (7).

[1052] Jean-Pierre Serre (*1926) is a French mathematician. He was, and remains, the youngest person ever to be awarded the Fields Medal.

Theorem 86.17. (Serre) If X is a *simply-connected* topological space such that all homology groups are finitely generated (for example X could be a compact topological manifold or a finite CW-complex), then all homotopy groups are also finitely generated abelian groups.[1053]

Example. It follows in particular from Serre's Theorem 86.17 that the homotopy groups of spheres are finitely generated. Even this statement is highly non-trivial. In particular it is much better than our only result in this direction, namely Corollary 94.10. which says that homotopy groups of spheres are countable.

Nonetheless, it turns out that the answer to Question 86.16 is negative. In fact, one does not need to wander off particularly far to find such an example:

Proposition 86.18. The group $\pi_3(S^1 \vee S^2)$ is not finitely generated over the group ring $\mathbb{Z}[\pi_1(S^1 \vee S^2)] = \mathbb{Z}[t^{\pm 1}]$.

Proof. A proof for the proposition is for example sketched in [**Hat2002**, Chapter 4.2, Exercise 38]. ■

Remark. The Hilton-Milnor theorem, see [**Hilt1955**, Theorem A],[**Miln1972**] or [**WhdG1978**, Chapter XI], gives a calculation of the homotopy groups of a finite wedge of spheres, more precisely, these homotopy groups are computed in terms of the homotopy groups of the spheres involved. Unfortunately, at least at a first glance, it is not so clear how the module structure over the group ring can be read off from the Hilton-Milnor theorem.

Exercises for Chapter 86.

Exercise 86.1. Let X be a topological space that admits a social choice of some type $n \geq 2$. Is X necessarily connected?

Exercise 86.2. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_{\geq 2}$. We consider the Hurewicz homomorphism $\Phi\colon \pi_n(X, x_0) \to \mathrm{H}_n(X; \mathbb{Z})$. Show that for any $g \in \pi_1(X, x_0)$ and any $\sigma \in \pi_n(X, x_0)$ we have

$$\Phi(g \cdot \sigma) \;=\; \Phi(\sigma) \;\in \mathrm{H}_n(X; \mathbb{Z}).$$

Exercise 86.3. Let G be a group and let R be a commutative ring. Suppose there exists a non-trivial $g \in G$ such that $g^k = e$ for some $k \in \mathbb{N}$. Show that $R[G]$ has a zero divisor, i.e. show that there exist non-zero elements $p, q \in R[G]$ with $p \cdot q = 0$.
Hint. First consider the case that $G = \langle t \,|\, t^n \rangle$ is a finite cyclic group with $n \geq 2$ elements.

Exercise 86.4. Is the ring $\mathrm{M}(3 \times 3, \mathbb{R})$ of real (3×3)-matrices isomorphic, as a ring, to the real group ring $\mathbb{R}[G]$ of some group G?

Exercise 86.5. Let G be a finite group. We view $\mathbb{Q}[G]$ as a $\mathbb{Q}[G]$-left module via left multiplication. Show that $\mathbb{Q}[G]$ admits a submodule $V \neq 0$ with $V \neq \mathbb{Q}[G]$ that is isomorphic to the trivial $\mathbb{Q}[G]$-module $\mathbb{Q}$ where every $g \in G$ acts via the identity on $\mathbb{Q}$.

[1053] It follows immediately from the Hurewicz Theorem 85.5 that $\pi_2(X)$ is finitely generated. The fact that the higher homotopy groups are also finitely generated is much stronger.

Exercise 86.6. Let G be a finite group. Show that there exists a monomorphism from G to some unitary group $\mathrm{U}(n)$.

Exercise 86.7. We consider the ring $\mathrm{M}(2,\mathbb{R})$. Does there exist a finite set T and an epimorphism from the free ring $\mathbb{R}[\langle T\rangle]$ onto $\mathrm{M}(2,\mathbb{R})$?

Exercise 86.8. Let X be a simply connected space and let $f\colon X \to X$ be a self-homeomorphism. Recall that on page 467 we introduced the mapping torus

$$M \;:=\; (X \times [0,1])/(x,0) \sim (f(x),1).$$

Express the $\mathbb{Z}[\pi_1(M)]$-module $\pi_2(M)$ in terms of the homology groups of X and the action by f on the homology groups of X.

Exercise 86.9. Show that there exists a closed smooth manifold M such that $\pi_2(M)$ is infinitely generated as an abelian group.

87. The Euler characteristic

87.1. **The Euler characteristic and homology groups.** Let X be a finite CW-complex, i.e. let X be a CW-complex with finitely many cells. Recall that on page 1519 we defined the Euler characteristic $\chi(X)$ as follows:

$$\chi(X) \ = \ \sum_n (-1)^n \cdot \text{number of } n\text{-cells.}$$

A short glance on the examples in the figure below shows that the three different CW-structures for the torus provided in that figure have the same Euler characteristic. The following proposition explains why this is not a coincidence.

Proposition 87.1. (Euler Characteristic-H$_*$-Proposition) Let X be a finite CW-complex, then[1054]

$$\chi(X) \ = \ \sum_n (-1)^n \cdot \text{rank}(\text{H}_n(X)).$$

Examples.

(1) It follows either from the CW-structure of the n-dimensional sphere S^n with one 0-cell and one n-cell given on page 1472 or from the calculation of the homology groups in Proposition 74.4 that[1055]

$$\chi(S^n) \ = \ 1+(-1)^n \ = \ \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

(2) Let $n \in \mathbb{N}_0$. We want to determine the Euler characteristic of $\overline{B}^n$. One could of course write down a CW-structure for $\overline{B}^n$.[1056] But with our knowledge it is quicker to point out that follows immediately from the Euler Characteristic-H$_*$-Proposition 87.1 and the Homotopy Equivalence-H$_*$-Corollary 73.9 (3) that $\chi(\overline{B}^n) = 1$.

(3) Given any $k, n \in \mathbb{N}_0$ we deduce from the Homotopy Equivalence-H$_*$-Corollary 73.9, Proposition 74.4 and the Euler Characteristic-H$_*$-Proposition 87.1 that $\chi(S^k \times \overline{B}^n) = \chi(S^k) = 1+(-1)^k$.

(4) It follows either from the obvious generalizations of CW-structure provided in the figure below or alternatively from Proposition 80.10 that for the surface Σ_g of genus g and that for the non-orientable surface N_k of genus k we have

$$\chi(\Sigma_g) \ = \ 2-2g \qquad \text{and} \qquad \chi(N_k) \ = \ 2-k.$$

(5) It follows immediately from the CW-structures provided in Lemma 68.1 that for any $n \in \mathbb{N}_0$ we have

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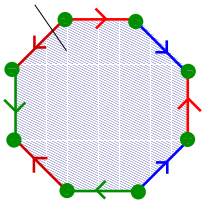
[1054] As a reminder, if G is a finitely generated abelian group, then it follows from the Finitely Generated Abelian Group Classification Theorem 51.4 that G is isomorphic to $\mathbb{Z}^r \oplus T$, where T is finite. We then call r the *rank* of G. Put differently, let $T \subset G$ be the torsion subgroup, then G/T is isomorphic to $\mathbb{Z}^r$ for an r, and this r is just the rank of G.

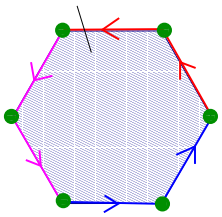
[1055] The statement that $\chi(S^2) = 2$ was proved, in a different language, by Leonhard Euler [**Eul1758**, p. 119]. This equality was already noted before by Francesco Maurolico [**Friedm2018**, p. 71] in 1537 and René Descartes [**Rabo2010**, p. 447] in 1628.

[1056] What is the "simplest" CW-structure you can find for $\overline{B}^n$?

surface Σ_g of genus $g=2$ | non-orientable surface N_k of genus $k=3$



1	number of 0-cells	1
$2 \cdot g$	number of 1-cells	k
1	number of 2-cells	1
$2-2\cdot g$	Euler characteristic	$2-k$



$$\chi(\mathbb{R}\mathrm{P}^n) \;=\; \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even} \end{cases} \qquad \text{and} \qquad \chi(\mathbb{C}\mathrm{P}^n) \;=\; n+1.$$

Before we turn to the proof of the Euler Characteristic-H_*-Proposition 87.1 we draw the following corollary.

Corollary 87.2. Let X and Y be finite CW-complexes. If X and Y are homotopy equivalent, then $\chi(X) = \chi(Y)$.

Proof. The corollary is an immediate consequence of the Euler Characteristic-H_*-Proposition 87.1 together with the Homotopy Equivalence-H_*-Corollary 73.9 (2). ■

Example. On page 1237 we introduced two examples of closed orientable 2-dimensional smooth manifolds for which we found it difficult to determine the genus. We show the same surfaces also in the figure below. Let us first consider the surface on the left: it is given identifying opposite sides of the hexagon E_6. The obvious CW-structures on the hexagon H_6 descends to a CW-structure on $E_6/\!\sim$. We make the following observations:

(1) the single 2-cell on E_6 defines a 2-cell for $E_6/\!\sim$,
(2) we identify every 1-cell of E_6 with the opposite 1-cell, thus $E_6/\!\sim$ has three 1-cells,
(3) we identify all 0-cells that differ by an even number of vertices, thus $E_6/\sim$ has precisely two 0-cells.

Summarizing we see that $E_6/\!\sim$ has Euler characteristic $2-3+1=0$. We had just seen on page 1879 that the surface of genus g has Euler characteristic $2-2g$. Thus we obtain from the Surface Classification Theorem 55.4 that $E_6/\!\sim$ is diffeomorphic to the surface of genus zero, i.e. the torus.

The same argument shows that $E_{10}/\!\sim$ has Euler characteristic $2-5+1=-2$ and that it is diffeomorphic to the surface of genus two. Evidently the same argument shows that if we take a regular $(4g+2)$-gon and if we identify opposite sides, then we obtain the surface of genus g.

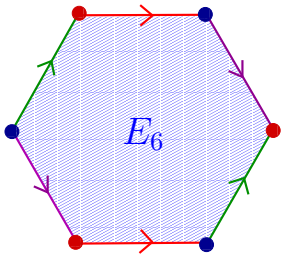
It is a great mental exercise to visualize that the surface on the left is a torus and that the surface on the right is a surface of genus two.

The key step in the proof of the Euler Characteristic-H_*-Proposition 87.1 is the following purely algebraic lemma.

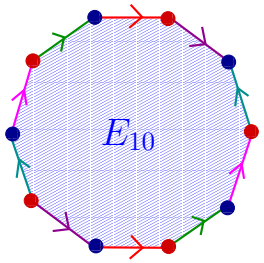
Lemma 87.3. Let

$$C_* \;:=\; 0 \;\to\; \mathrm{C}_k \xrightarrow{\partial_k} \mathrm{C}_{k-1} \xrightarrow{\partial_{k-1}} \;\dots\; \mathrm{C}_1 \xrightarrow{\partial_1} \mathrm{C}_0 \;\to\; 0$$

hexagon with opposite sides identified



dodecagon with opposite sides identified



in each case there are two equivalence classes of vertices: red and blue

be a chain complex of finitely generated abelian groups. Then the following equality holds:

$$\sum_{n=0}^{k}(-1)^n \cdot \operatorname{rank}(\mathrm{C}_n) \;=\; \sum_{n=0}^{k}(-1)^n \operatorname{rank}(\mathrm{H}_n(C_*)).$$

The same statement holds if the C_i are vector spaces over a field and if we replace ranks by dimensions.

Proof. For the above chain complex (C_n, ∂_n) we write for each $n \in \mathbb{N}_0$[1057]

$$Z_n := \ker(\partial_n), \quad B_n := \operatorname{im}(\partial_{n+1}) \quad \text{and} \quad H_n = Z_n/B_n.$$

These groups form the following two types of short exact sequences

(a) $$0 \hookrightarrow Z_n \to \mathrm{C}_n \xrightarrow{\partial_n} B_{n-1} \to 0,$$

and

(b) $$0 \hookrightarrow B_n \to Z_n \to H_n \to 0.$$

Now we see that

$$\begin{aligned}
\sum_{n=0}^{k}(-1)^n \cdot \operatorname{rank} \mathrm{C}_n
&\overset{\text{by (a) and the Rank Additivity Lemma 51.7 (2) we have } \operatorname{rank}\mathrm{C}_n = \operatorname{rank} Z_n + \operatorname{rank} B_{n-1}}{\downarrow} \\
&= \sum_{n=0}^{k}(-1)^n \cdot (\operatorname{rank} Z_n + \operatorname{rank} B_{n-1}) \\
&= \sum_{n=0}^{k}(-1)^n \cdot (\operatorname{rank} B_n + \operatorname{rank} H_n + \operatorname{rank} B_{n-1}) \\
&\overset{\text{by (b) and the Rank Additivity Lemma 51.7 (2) we have } \operatorname{rank} Z_n = \operatorname{rank} B_n + \operatorname{rank} H_n}{\uparrow} \\
&= \sum_{n=0}^{k}(-1)^n \cdot \operatorname{rank} H_n \;+ \sum_{n=0}^{k}(-1)^n \cdot \operatorname{rank} B_n + \sum_{n=0}^{k}(-1)^n \operatorname{rank} B_{n-1} \\
&\overset{\text{we do the substitution } m = n-1 \text{ and we use that } B_k = 0 \text{ and } B_{-1} = 0}{\uparrow} \\
&= \sum_{n=0}^{k}(-1)^n \cdot \operatorname{rank} H_n \;+ \sum_{n=0}^{k-1}(-1)^n \cdot \operatorname{rank} B_n - \sum_{m=0}^{k-1}(-1)^m \cdot \operatorname{rank} B_m \\
&= \sum_{n=0}^{k}(-1)^n \cdot \operatorname{rank} H_n.
\end{aligned}$$

Basically the same proof also applies if the C_i are vector spaces over a field and if we replace ranks by dimensions. ■

Now we can easily provide the proof of the Euler Characteristic-H_*-Proposition 87.1.

[1057] It is a common notation to call these groups Z_n and B_n, the names go back to the German words "Zykel" and "Bild".

Proof of the Euler Characteristic-H_*-Proposition 87.1. Let X be a finite k-dimensional CW-complex. We have

$$\begin{array}{rcccl} \chi(X) & = & \sum_{n=0}^{k}(-1)^n \cdot \text{number of } n\text{-cells} & \overset{\text{Skeleton-Homology Lemma 80.1}}{=} & \sum_{n=0}^{k}(-1)^n \cdot \operatorname{rank}(\mathrm{C}_n^{\mathrm{CW}}(X)) \\ & \underset{\text{Lemma 87.3}}{=} & \sum_{n=0}^{k}(-1)^n \cdot \operatorname{rank}(\mathrm{H}_n^{\mathrm{CW}}(X)) & \underset{\text{Singular-Cellular } \mathrm{H}_*\text{-Isomorphism Proposition 80.4}}{=} & \sum_{n=0}^{k}(-1)^n \cdot \operatorname{rank}(\mathrm{H}_n(X)). \end{array}$$

■

Using Lemma 87.3 we can also easily prove the following lemma which be will useful on several occasions.

Lemma 87.4. Let

$$0 \to A_k \to A_{k-1} \to \ldots \to A_1 \to A_0 \to 0$$

be an exact sequence of finitely generated abelian groups. Then

$$\sum_{n=0}^{k}(-1)^n \cdot \operatorname{rank} A_n = 0.$$

Proof. We can view this exact sequence of finitely generated abelian groups as a chain complex whose homology groups vanish. The lemma is therefore an immediate consequence of Lemma 87.3. ■

Definition.

(1) Given a topological space X and $n \in \mathbb{N}_0$ such that $\mathrm{H}_n(X)$ is a finitely generated abelian group we refer to

$$b_n(X) := \operatorname{rank}(\mathrm{H}_n(X))$$

as the n-th Betti number of X.[1058]

(2) Let X be a topological space. If all homology groups are finitely generated and if $\sum_n b_n(X)$ is finite, then we define the Euler characteristic of X to be

$$\chi(X) := \sum_n (-1)^n \cdot b_n(X).$$

Remark.

(1) It follows from the Euler Characteristic-H_*-Proposition 87.1 that this new definition of the Euler characteristic of a finite CW-complex agrees with the earlier definition.

(2) We will see in the Smooth Manifolds-Invariants Proposition 96.6 and alternatively on page 2997 that for every compact smooth manifold we can define the Euler characteristic. Furthermore we will see in the Topological Manifolds-Invariants Proposition 104.14 that we can also define the Euler characteristic for every compact topological manifold.

[1058] These numbers are named after the Italian mathematician Enrico Betti (1823-1892)

http://de.wikipedia.org/wiki/Enrico_Betti

who had studied an early version of homology groups.

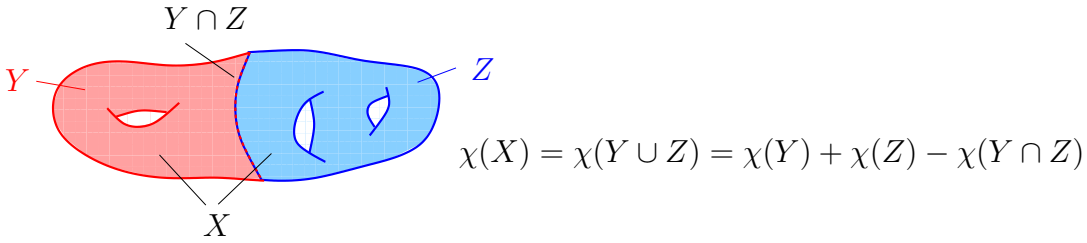
Example. Using the Mapping Torus-Homology Proposition 78.20 and using Lemma 87.4 several times we will prove the following statement in Exercise 87.3: Let X be a topological space that is homotopy equivalent to a finite CW-complex, then for any self-homeomorphism f of X we have

$$\chi(\mathrm{Tor}(X,f)) \;=\; 0.$$

87.2. **Properties of the Euler characteristic.** The Euler characteristic is an invariant which "sees less" than the homology groups. On the other hand we will now see that it behaves better than the homology groups under several natural operations.
The following lemma makes it possible to determine the Euler characteristic of a CW-complex from suitable decompositions.

Lemma 87.5. (Euler Characteristic-Addition Lemma) Let $X = Y \cup Z$ be a decomposition of a finite CW-complex X into two subcomplexes Y and Z. Then the following equality holds

$$\chi(X) \;=\; \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$



Proof. We give two proofs for the lemma:

(1) Given $n \in \mathbb{N}_0$ and given a subset W we denote by $n(W)$ the number of open n-cells that are contained in W. In our case we have

$$n(X) \;\underset{\uparrow}{=}\; n(Y) + n(Z) - n(Y \cap Z).$$

since every open n-cell of X lies in Y or Z, the n-cells that lie in $Y \cap Z$ get counted twice in $n(Y) + n(Z)$, so we have to subtract $n(Y \cap Z)$

By performing the alternating sum over all n we obtain immediately that

$$\chi(X) \;=\; \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$

We have thus proved the desired equality.

(2) The lemma also follows from applying Lemma 87.4 and the Euler Characteristic-H_*-Proposition 87.1 to the long exact sequence of homology groups that we obtain by applying the Mayer–Vietoris Theorem for CW-complexes 78.10 to the decomposition $X = Y \cup Z$. ∎

In the following we provide two examples to which we already know the answer by other means. Nonetheless, the approach using the Euler Characteristic-Addition Lemma 87.5 gives particularly short and pleasing arguments.

Example. Let $K \subset S^3$ be a knot. By the Tubular Neighborhood Theorem **??** there exists a tubular map $\Phi\colon \overline{B}^2 \times K$. We set $X_K := S^3 \setminus \Phi(B^2 \times K)$. By the Smooth Manifold-CW Structure Theorem 96.5 there exists a CW-structure for S^3 such that X_K and $\Phi(\overline{B}^2 \times S^1)$

are subcomplexes. We see that

$$0 \underset{\uparrow}{=} \chi(S^3) = \chi\big(X_K \cup \Phi(\overline{B}^2 \times S^1)\big) \underset{\uparrow}{=} \chi(X_K) + \chi\big(\overline{B}^2 \times S^1\big) - \chi\big(\partial\overline{B}^2 \times S^1\big) \underset{\uparrow}{=} \chi(X_K).$$

see page 1879 — Euler Characteristic-Addition Lemma 87.5 — see page 1879

Of course this calculation also follows from Lemma 100.16.

Example. As usual, given $g \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$ we denote by $\Sigma_{g,n}$ the surface Σ_g of genus g minus n open disks. Now we want to determine the Euler characteristic $\chi(\Sigma_{g,n})$. We could do so by using Propositions 87.1 and 80.10. But here we use an alternative approach. By the Smooth Manifold-CW Structure Theorem 96.5 there exists a CW-structure on Σ_g such that $\Sigma_{g,n}$ and the closed disks that we remove are subcomplexes. It now follows that

$$2-2g \underset{\uparrow}{=} \chi(\Sigma_g) = \chi\Big(\Sigma_{g,n} \cup \bigcup_{i=1}^{n} \overline{B}^2\Big) \underset{\uparrow}{=} \chi(\Sigma_{g,n}) + \sum_{i=1}^{n}\chi(\overline{B}^2) - \sum_{i=1}^{n}\chi(\partial\overline{B}^2) \underset{\uparrow}{=} \chi(\Sigma_{g,n}) + n.$$

see page 1879 — Euler Characteristic-Addition Lemma 87.5 — see page 1879

Thus we see that $\chi(\Sigma_{g,n}) = 2 - 2g - n$.

We record the above example in a lemma.

Lemma 87.6.

(1) Let $g \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$. If we denote by $\Sigma_{g,n}$ the surface Σ_g of genus g minus n open disks, then $\chi(\Sigma_{g,n}) = 2 - 2g - n$.
(2) Let Σ and Σ' be two compact orientable connected 2-dimensional smooth manifolds. If Σ and Σ' have the same number of boundary components and if $\chi(\Sigma) = \chi(\Sigma')$, then Σ and Σ' are diffeomorphic.

Proof.

(1) We proved this statement in the example preceding the lemma.
(2) This statement follows immediately from (1) and the classification of compact orientable connected 2-dimensional smooth manifolds, see the Surface Classification Theorem 55.4. ■

We continue with a calculation of the Euler characteristic of a connected sum.

Lemma 87.7. (Connected Sum-χ-Lemma) If M and N are two compact connected n-dimensional smooth manifolds. We have the following equality:[1059]

$$\chi(M\#N) = \chi(M) + \chi(N) + \begin{cases} -2, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We perform the following straightforward calculation:

$$\chi(M \setminus B^n) \underset{\uparrow}{=} \chi(M) - \chi(\overline{B}^n) + \chi(S^{n-1}) \underset{\uparrow}{=} \chi(M) + (-1)^{n-1}.$$

Euler Characteristic-Addition Lemma 87.5 — since $\chi(\overline{B}^n) = 1$ and $\chi(S^{n-1}) = 1 + (-1)^{n-1}$

[1059] Strictly speaking, if M and N are both orientable, then we need to assume that M and N are oriented to get a well-defined connected sum.

The same way we also see that $\chi(N \setminus B^n) = \chi(N) + (-1)^{n-1}$. It then follows, using the same type of arguments, that

$$\begin{aligned}\chi(M\#N) &= \chi(M \setminus B^n) + \chi(N \setminus B^n) - \chi(S^{n-1}) \\ &= \chi(M) + (-1)^{n-1} + \chi(N) + (-1)^{n-1} - (1 + (-1)^{n-1}) \\ &= \chi(M) + \chi(N) - 1 + (-1)^{n-1}.\end{aligned}$$

For completeness sake we point out that in the Connected Sum-H_*-Proposition 100.13 we will calculate the homology groups of the connected sum of two closed oriented smooth manifolds. Together the Euler Characteristic-H_*-Proposition 87.1 this gives an alternative proof of the above equality. ■

Examples.

(1) In the figure on page **??** we saw that the connected sum of two tori is the surface of genus two. By the Connected Sum-χ-Lemma 87.7 we obtain that[1060]

$$\chi(\text{surface of genus two}) = \underbrace{\chi(\text{torus})}_{=0} + \underbrace{\chi(\text{torus})}_{=0} - 2 = -2.$$

We had obtained the same result on page 1879 using a different method.

(2) Note that from the CW-structure for $\mathbb{R}P^2$ that we gave in the figure on page 1472 we read off that $\chi(\mathbb{R}P^2) = 1$. It follows from the Connected Sum-χ-Lemma 87.7 that for any $k \in \mathbb{N}$ we have

$$\chi(k \cdot \mathbb{R}P^2) = 1 - k.$$

Using the same argument as in Lemma 87.6 one can also show that

$$\chi(k \cdot \mathbb{R}P^2 \text{ minus } n \text{ open disks}) = 1 - k - n.$$

In the following we work our way through more constructions of CW-complexes and the effect on Euler characteristics. First we remind the reader of the following lemma.

Lemma 69.5. (Quotient-χ-Lemma) Let X be a finite CW-complex and let A be a subcomplex. We equip the quotient X/A with the CW-structure given by the CW-Complex Construction Lemma 68.32 (3). Then

$$\chi(X/A) = \chi(X) - \chi(A) + 1.$$

Next let X and Y be two finite CW-complexes. On page 1498 we introduced the product CW-structure which by Proposition 68.23 is indeed a CW-structure for the topological space $X \times Y$. The following lemma relates the Euler characteristics of X, Y and $X \times Y$ in an appealing way.[1061]

Lemma 87.8. (Product-χ-Lemma) Let X and Y be two finite CW-complexes, then

$$\chi(X \times Y) = \chi(X) \cdot \chi(Y).$$

Example. For any finite CW-complex X we have $\chi(S^1 \times X) = \chi(S^1) \cdot \chi(X) = 0$.

[1060] How many arguments can you find in the lecture notes that show that the Euler characteristic of the torus is zero?

[1061] Note that the lemma also gives an affirmative answer to the question posed in Exercise 69.2.

Proof. Let X and Y be two finite CW-complexes. Given $k \in \mathbb{N}_0$ we denote by c_k the number of k-cells of X and we denote by d_k the number of k-cells of Y.

We equip $X \times Y$ with the product CW-structure from page 1498. By construction we have

$$\#n\text{-cells of } X \times Y \;=\; \sum_{k+l=n} \#k\text{-cells of } X \cdot \#l\text{-cells of } Y \;=\; \sum_{k+l=n} c_k \cdot d_l.$$

It follows that

$$\chi(X) \cdot \chi(Y) \;=\; \Big(\sum_k (-1)^k \cdot c_k\Big) \cdot \Big(\sum_l (-1)^l \cdot d_l\Big) \;=\; \sum_n (-1)^n \sum_{k+l=n} c_k \cdot d_l \;=\; \chi(X \times Y). \;\blacksquare$$

Now let $p\colon \widetilde{X} \to X$ be a k-fold covering of path-connected topological spaces. We saw in Corollary 48.14 that the induced map $p_*\colon \pi_1(\widetilde{X}) \to \pi_1(X)$ is a monomorphism and we saw in Lemma 48.15 (3) that $p_*(\pi_1(\widetilde{X}))$ is a subgroup of index k of $\pi_1(X)$. There is no analogous statement for homology groups. For example the projection $p\colon S^2 \to S^2/\!\sim = \mathbb{R}\mathrm{P}^2$ is a 2-fold covering, but $\mathrm{H}_2(S^2) = \mathbb{Z}$ and $\mathrm{H}_2(\mathbb{R}\mathrm{P}^2) = 0$, which shows that the induced map on homology is in general not a monomorphism.

It is therefore perhaps surprising that we already saw in Proposition 69.4 that the Euler characteristic behaves well under finite coverings. For convenience we recall, a slightly simplified version, of the statement of Proposition 69.4.

Proposition 69.4. (Covering-χ-Proposition) Let $p\colon \widetilde{Y} \to Y$ be a finite covering of a finite connected CW-complex Y. Then $\widetilde{Y}$ is also a finite CW-complex with

$$\chi(\widetilde{Y}) \;=\; [\widetilde{Y} : Y] \cdot \chi(Y).$$

Example. We consider again the 2-fold covering $p\colon S^2 \to \mathbb{R}\mathrm{P}^2$. In this case we have by the calculation on page 1725 or alternatively in Proposition 80.11 that

$$\begin{array}{rcl} \mathrm{H}_0(S^2) &\cong& \mathbb{Z}, \\ \mathrm{H}_1(S^2) &\cong& 0, \\ \mathrm{H}_2(S^2) &\cong& \mathbb{Z}, \\ \mathrm{H}_i(S^2) &\cong& 0, \text{ for } i \geq 3, \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{H}_0(\mathbb{R}\mathrm{P}^2) &\cong& \mathbb{Z}, \\ \mathrm{H}_1(\mathbb{R}\mathrm{P}^2) &\cong& \mathbb{Z}_2, \\ \mathrm{H}_2(\mathbb{R}\mathrm{P}^2) &\cong& 0, \\ \mathrm{H}_i(\mathbb{R}\mathrm{P}^2) &\cong& 0, \text{ for } i \geq 3. \end{array}$$

Thus we see that the homology groups are quite different. Nonetheless we observe that $\chi(S^2) = 2$ is indeed twice $\chi(\mathbb{R}\mathrm{P}^2) = 1$, as we had determined on page 1879.

In Question 63.6 we asked how the genus of a surface changes if we take finite covers. We had answered this question already in the Subgroups-of-Surface Groups Proposition 64.6 (1). The proof of the following lemma is arguably more conceptual and easier to remember.

Lemma 87.9. Let Σ be the surface of genus g and let $p\colon \widetilde{\Sigma} \to \Sigma$ be a k-fold connected covering. Then[1062]

$$\mathrm{genus}(\widetilde{\Sigma}) \;=\; k \cdot (g-1) + 1.$$

Example. In the figure on page 1407 we illustrated a 3-fold covering of the surface of genus 4 over the surface of genus 2.

Remark. We had given an alternative proof of Lemma 87.9 in Exercise 64.6. The new proof using Euler characteristics is surely much more instructive than our previous proof.

[1062] Recall that by the Surface Classification Theorem 55.4 there exists precisely one $\widetilde{g}$ such that the closed, oriented 2-dimensional smooth manifold $\widetilde{\Sigma}$ is diffeomorphic to precisely the surface of genus $\widetilde{g}$.

Proof. We denote by $\widetilde{g}$ the genus of $\widetilde{\Sigma}$. The lemma follows immediately from the following two facts:

(1) By the remark on page 1879 we have $\chi(\Sigma) = 2 - 2g$ and $\chi(\widetilde{\Sigma}) = 2 - 2\widetilde{g}$.

(2) By Proposition 69.4 we have $\chi(\widetilde{\Sigma}) = k \cdot \chi(\Sigma)$. ∎

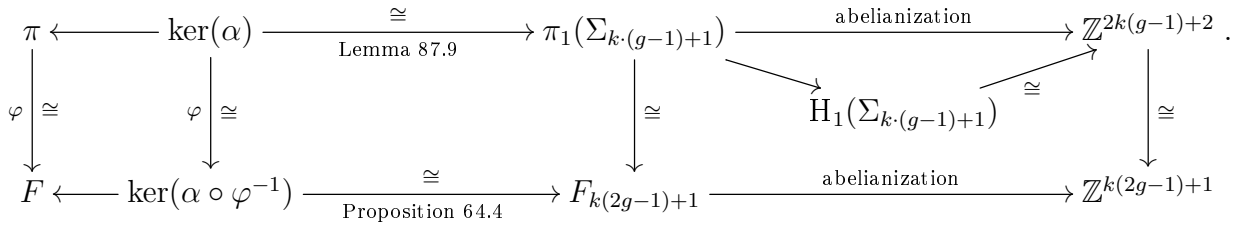
Now we can also give a new proof of the following proposition.

Proposition 64.17. If $g \geq 1$, then the group

$$\pi_1(\text{surface of genus } g) \;\cong\; \langle x_1, y_1, \ldots, x_g, y_g \,|\, [x_1, y_1] \cdot \cdots \cdot [x_g, y_g] \rangle$$

is not isomorphic to the free group F_{2g} on $2g$ generators.

Proof. Let Σ be a surface of genus $g \geq 1$. We write $\pi = \pi_1(\Sigma)$. Suppose that there exists an isomorphism $\varphi\colon \pi \to F$ where F is the free group on $2g$ generators. Let $\alpha\colon \pi \to \mathbb{Z}_k$ be an epimorphism for some $k \geq 2$.[1063] We denote by $p\colon \widetilde{\Sigma} \to \Sigma$ the covering which satisfies $p_*(\pi_1(\widetilde{\Sigma})) = \ker(\alpha)$ which is given by the Covering Existence Theorem 61.5. By Lemma 48.15 (3) this is a k-fold covering. We get the following commutative diagram:



It follows from $k \geq 2$ that $2k(g-1)+2 \neq k(2g-1)+1$, hence the two groups on the right cannot by isomorphic. We have thus obtained a contradiction. ∎

87.3. **Groups acting on spheres and surfaces.** We recall that on page 248 we said that a group G acts *freely* on a topological space X if for every $g \neq e$ and every $x \in X$ we have $g \cdot x \neq x$. For example for each $k \in \mathbb{N}$ the action

$$\begin{array}{rcl} \mathbb{Z}_k \times S^1 & \to & S^1 \\ (l, z) & \mapsto & \exp(2\pi \mathrm{i} l/k) \cdot z \end{array}$$

is free. We can easily generalize this example. More precisely, let S^{2n-1} be an odd-dimensional sphere. We view S^{2n-1} as subset of $\mathbb{R}^{2n} = \mathbb{C}^n$. Then it is straightforward to show that

$$\begin{array}{rcl} \mathbb{Z}_k \times S^{2n-1} & \to & S^{2n-1} \\ (l, (z_1, \ldots, z_n)) & \mapsto & (\exp(2\pi \mathrm{i} l/k) \cdot z_1, \ldots, \exp(2\pi \mathrm{i} l/k) \cdot z_n) \end{array}$$

is a free and continuous action of $\mathbb{Z}_k$ on S^{2n-1}.

It is less clear what finite groups can act freely and continuously on even-dimensional spheres. The group $\mathbb{Z}_2 \cong \{\pm 1\}$ acts of course freely and continuously on any sphere. The following proposition says that $\mathbb{Z}_2$ is in fact the only finite non-trivial group that can act freely and continuously on an even-dimensional sphere. This proposition and the above example give a complete answer to Question 48.8.

Proposition 87.10. If a finite group G acts freely and continuously on an even-dimensional sphere, then G is either trivial or $G \cong \mathbb{Z}_2$.

[1063] Why does such an epimorphism exist?

Proof. Let G be a finite group that acts freely and continuously on S^{2n}. By Lemma 48.5 the action is also discrete. By the Actions-Covering Proposition 48.9 the projection $S^{2n} \to S^{2n}/G$ is a covering of degree $|G|$. We obtain the equalities

$$2 \underset{\substack{\uparrow \\ \text{see page 1616}}}{=} \chi(S^{2n}) \underset{\substack{\uparrow \\ \text{Proposition 69.4}}}{=} |G| \cdot \chi(S^{2n}/G).$$

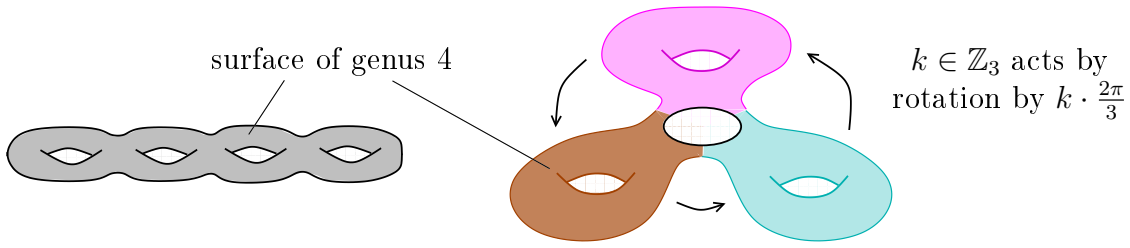
Itt follows that $|G|$ divides two, i.e. $|G| = 1$ or $|G| = 2$. In first case the group G is the trivial group and in the second case we have $G \cong \mathbb{Z}_2$.[1064] ■

We already gave a different proof of Proposition 87.10 in Exercise 76.21. But whereas the approach of Exercise 76.21 made very much use of the topology of spheres, the approach taken in the above proof generalizes to many other settings:

Proposition 87.11. Let $g \in \mathbb{N}_0$. If a finite group G acts freely and continuously on the surface Σ of genus g, then the order of G divides $\chi(G) = 2 - 2g$.

Proof. The proof of Proposition 87.11 is verbatim the same as the proof of Proposition 87.10. The only change is that we need to use the fact, shown on page 1879, that $\chi(\Sigma) = 2 - 2g$. ■

Example. Let Σ be the surface of genus four. By the remark on page 1879 we know that $\chi(\Sigma) = -6$. By Proposition 87.10 we know that the order of any finite group acting freely and continuously on Σ has to have an order that divides 6. In the figure below we sketch a free and continuous action of the group $\mathbb{Z}_3$ on Σ. In Exercise 87.10 we will see that given any n with $n|\chi(\Sigma)$ the cyclic group $\mathbb{Z}_n$ admits a free and continuous action on Σ.



87.4. **Topological graphs** (∗). Using the results of this chapter we can now easily prove the following lemma:

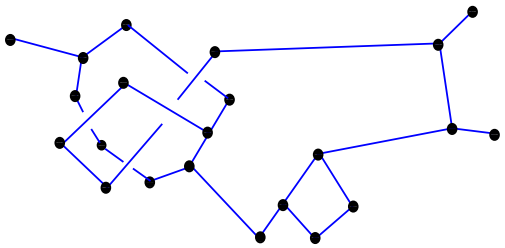
Lemma 87.12. Let G be a connected topological graph with v vertices and e edges. Then

$$\mathrm{H}_0(G) \cong \mathbb{Z}, \quad \mathrm{H}_1(G) \cong \mathbb{Z}^{e-v+1} \quad \text{and} \quad \mathrm{H}_i(G) = 0 \text{ for } i \geq 2.$$

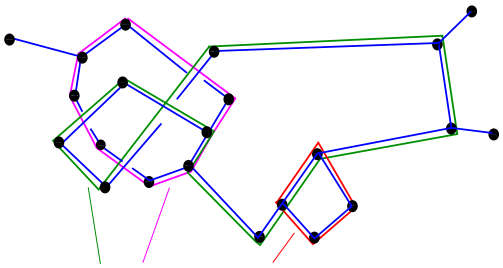
Example. In the figure below we see a connected topological graph G with 22 vertices and 24 edges. It follows from Lemma 87.12 that its first homology group is isomorphic to $\mathbb{Z}^3$. In the figure below we sketched three cycles which represent a generating set for $\mathrm{H}_1(G) \cong \mathbb{Z}^3$.

Proof. Let G be a connected topological graph with v vertices and e edges. Recall that by the discussion on page 1470 we can view G as a 1-dimensional CW-complex with v 0-cells and e 1-cells.

[1064] Why is every group with two elements isomorphic to $\mathbb{Z}_2$?



topological graph G with 22 vertices and 24 edges



generating set for $\mathrm{H}_1(G) \cong \mathbb{Z}^3$

Now that we have developed so many techniques for computing homology groups it is perhaps not surprising that we can provide three different proofs for the lemma.

(1) First of all, from the H_0-Proposition 72.5 together with the CW-Complex Properties Proposition 68.10 (7) it follows immediately that $\mathrm{H}_0(G) \cong \mathbb{Z}$. Furthermore, from the Homology-of-CW-Complex Proposition 80.6 it follows that $\mathrm{H}_i(G) = 0$ for $i \geq 2$ and that $\mathrm{H}_1(G)$ is torsion-free. From the Euler Characteristic-H_*-Proposition 87.1 we obtain that

$$\chi(X) \;=\; \sum_i (-1)^i \cdot \mathrm{rank}(\mathrm{H}_i(X)).$$

Putting everything together we see that

$$1 - \mathrm{rank}(\mathrm{H}_1(G)) \;=\; \chi(X) \;=\; \#0\text{-cells} - \#1\text{-cells} \;=\; v - e.$$

Since $\mathrm{H}_1(G)$ is torsion-free we obtain that $\mathrm{H}_1(G) \cong \mathbb{Z}^{e-k+1}$.

(2) We can also modify the argument of (1) as follows: instead of calculating $\mathrm{H}_1(G)$ using the Euler characteristic, we could also have determined that $\mathrm{H}_1(G) \cong \mathbb{Z}^{e-k+1}$ by combining the calculation of the fundamental group in the Fundamental Group-of-Graph Proposition 52.5 with the isomorphism of the Hurewicz Theorem 84.5.

(3) An alternative proof for Lemma 87.12 is given as follows: we have

$$\widetilde{\mathrm{H}}_j(G) \;\underset{\text{Quotient-by-Tree Proposition 69.8 (1) and the Homotopy Equivalence-}\mathrm{H}_*\text{-Corollary 73.9}}{\cong}\; \widetilde{\mathrm{H}}_j(\text{wedge of } 1-\chi(G) \text{ circles}) \;\underset{\text{Wedge-Homology Proposition 79.8}}{\cong}\; \bigoplus_{i=1}^{1-\chi(G)} \widetilde{\mathrm{H}}_j(S^1) \;\underset{\text{Proposition 74.4}}{\cong}\; \begin{cases} \mathbb{Z}^{1-\chi(G)}, & \text{if } j=1, \\ 0, & \text{else.} \end{cases}$$

It is a matter of taste which of the three proofs is the nicest. ■

Exercises for Chapter 87.

Exercise 87.1. Show that every CW-structure for the torus with one 0-cell has precisely one 2-cell.

Exercise 87.2.

(a) Let $k \in \mathbb{N}$. By Proposition 49.4 we know that the non-orientable surface N_k admits an essentially unique 2-fold covering $\Sigma \to N_k$ that is orientable. What is the genus of Σ?

(b) Let $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$. By Proposition 49.4 we know that the non-orientable surface $N_{k,m}$ admits an essentially unique 2-fold covering $\Sigma \to N_{k,m}$ that is orientable. What is the genus of Σ and what is the number of boundary components of Σ?
Remark. You could use Exercise 49.12.

Remark. These are precisely the same questions as in Exercise 55.11. But this time you can give a different, shorter proof using Euler characteristics.

Exercise 87.3. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. In the Mapping Torus-Homology Proposition 78.20 we showed that for any $n \in \mathbb{N}_0$ there exists a short exact sequence

$$0 \to \operatorname{coker}\big(\mathrm{H}_n(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_n(X)\big) \to \mathrm{H}_n(\mathrm{Tor}(X,f)) \to \ker\big(\mathrm{H}_{n-1}(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_{n-1}(X)\big) \to 0.$$

(a) Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be the torus. Given $A \in \mathrm{SL}(2,\mathbb{Z})$ we consider the homeomorphism

$$\begin{array}{rcl} f(A)\colon \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ [v] & \mapsto & [Av]. \end{array}$$

Determine the homology groups of $\mathrm{Tor}(T, f(-\mathrm{id}))$.
Remark. You can use the fact that under the usual identification $\mathrm{H}_1(T) = \mathbb{Z}^2$ the induced map $f(A)_*$ is given by multiplication by A.[1065]

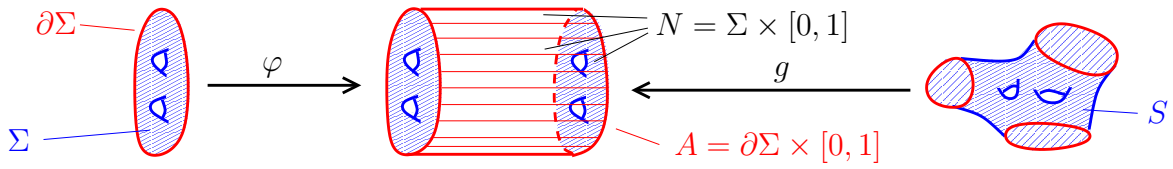
(b) Does there exist a matrix $A \in \mathrm{SL}(2,\mathbb{Z})$ such that $\mathrm{H}_1(\mathrm{Tor}(T, f(A))) \cong \mathbb{Z}$?

(c) Let X be a topological space that is homotopy equivalent to a finite CW-complex. Show that for any self-homeomorphism f of X we have

$$\chi(\mathrm{Tor}(X,f)) \;=\; 0.$$

Exercise 87.4. For a topological space X and $n \in \mathbb{N}_0$ we denote by $b_n(X) \in \mathbb{N}_0 \cup \{\infty\}$ the rank of $\mathrm{H}_n(X)$. Does there exist a topological space X such that $b_1(X) < \infty$ but such that for a *finite* cover $\widetilde{X}$ of X we have $b_1(\widetilde{X}) = \infty$?

Exercise 87.5. Given $g, n, \in \mathbb{N}_0$ we denote by $\Sigma_{g,n}$ the surface of genus g minus n open disks. Let $g, k, m, n \in \mathbb{N}_0$. Suppose there exists a smooth embedding $\Sigma_{g,m} \to \Sigma_{k,n}$. Show that $g \le k$.
Hint. Use the Codimension-Zero Submanifold Proposition 19.38.



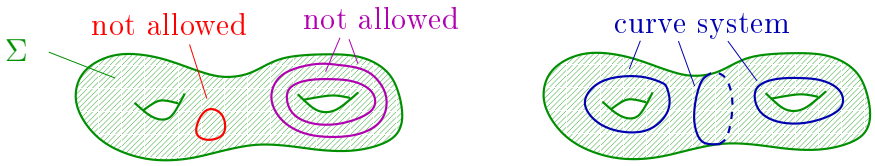
Exercise 87.6. Let Σ be a 2-dimensional smooth manifold. A *curve system* is a collection of disjoint curves (i.e. closed connected 1-dimensional smooth manifolds) $C_1, \dots, C_n$ on Σ with the following two properties:

(1) None of the C_i is the boundary of a disk on F.
(2) There is no annulus on F such that the boundary is the union of two curves.

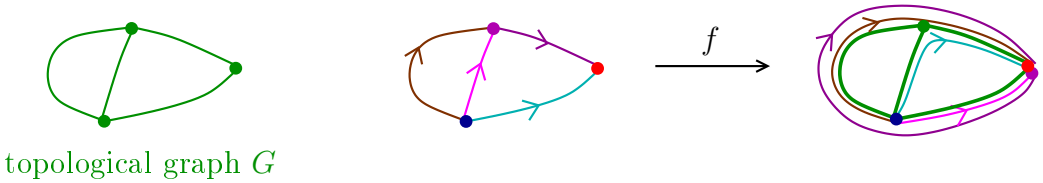
In the figure below on the right we see a curve system on the surface of genus two consisting of three curves.

[1065] Why does this hold?

(a) Is there a curve system consisting of four curves on the surface of genus two?
(b) Given $g \in \mathbb{N}_0$, what is the maximal number of curves in a curve system on the surface of genus g?



Exercise 87.7. We consider the topological graph G together with the map $f\colon G \to G$ shown in Exercise 87.7. Determine the determinant and the trace of the induced map $f_*\colon \mathrm{H}_1(G) \to \mathrm{H}_1(G)$.
Remark. In the proof of Lemma 87.12 we gave three different, more or less explicit, isomorphisms $\varphi\colon \mathrm{H}_1(G) \xrightarrow{\cong} \mathbb{Z}^2$. The challenge now is to find an isomorphism φ that is explicit enough so that one can calculate the map $\varphi \circ f_* \circ \varphi^{-1}\colon \mathbb{Z}^2 \to \mathbb{Z}^2$.



topological graph G

Exercise 87.8. Let M be a compact orientable connected 2-dimensional smooth manifold.
(a) Show that there exists a $g \in \mathbb{N}_0$ such that M admits g disjoint curves such that the complement is connected, but such that any $g+1$ disjoint curves disconnect M.
(b) Show that g equals the genus of M, as defined on page 1237.

Remark. This exercise gives an intrinsic definition of the genus of a compact orientable connected 2-dimensional smooth manifold.

Exercise 87.9. Let M be a closed non-orientable connected 2-dimensional smooth manifold. We define $k(M) \in \mathbb{N}_0$ to be the maximum $k \in \mathbb{N}_0 \cup \{\infty\}$ such that there exist k smooth embeddings of the Möbius band into M with the following two properties: the images are disjoint and the complement of images is connected.
(a) Let K be the Klein bottle. Show that $k(M) \geq 2$.
(b) Let M be a closed non-orientable connected 2-dimensional smooth manifold.
(i) Show that $k(M)$ is finite.
(ii) Show that $k(M)$ equals the genus of M, as defined on page 1237.
Remark. This exercise gives an intrinsic definition of the genus of a compact non-orientable connected 2-dimensional smooth manifold.

Exercise 87.10.
(a) Let $N_3 = \mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ be the non-orientable surface of genus three. Show that given any even $n \in \mathbb{N}$ there exists a connected orientable covering $p\colon \Sigma \to N_3$ of degree n.
(b) Let Σ be the surface of genus g. Show that given any $k \in \mathbb{N}$ that divides $\chi(\Sigma)$ there exists a continuous and free action by the cyclic group $\mathbb{Z}_n$ on Σ.

Remark. This exercise gives in particular a more systematic solution to Exercise 5.39.

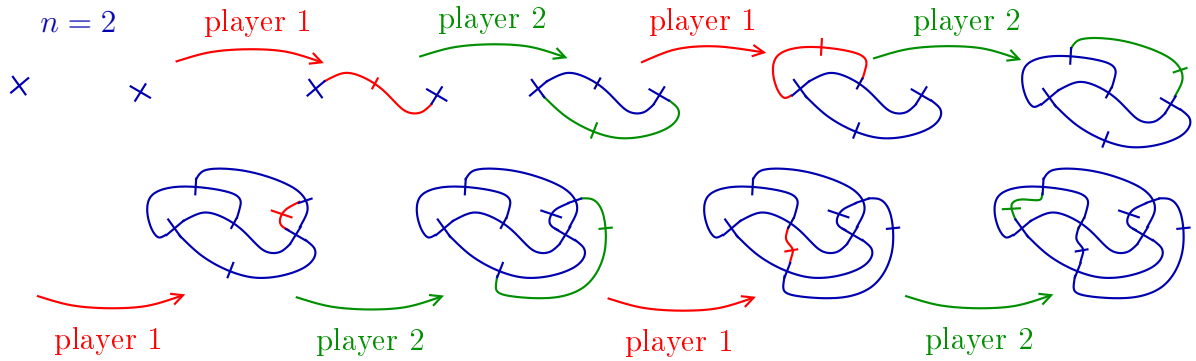
Exercise 87.11. Let Σ be the torus minus one open disk. Furthermore let $\alpha, \beta\colon S^1 \to \Sigma$ be two disjoint embeddings with $\alpha_*([S^1]) = \beta_*([S^1]) \in \mathrm{H}_1(\Sigma)$. We denote this homology class by x.

(a) Suppose that $x \neq 0$. Show that α is isotopic to β.
(b) Show that the conclusion of (a) does not necessarily hold, if we drop the hypothesis that $x \neq 0$.

Exercise 87.12. The two player game of "Brussels sprouts" works as follows:

(1) Initially one draws n disjoint crosses on a 2-dimensional smooth manifold.
(2) The two players take turns to do the following move: Join two free ends with a curve, without crossing any existing curve, and then put a short stroke across the new curve to create two new free ends.

Such a game with $n = 2$ is illustrated in the figure below.



(a) Suppose that the game takes place on the plane $\mathbb{R}^2$. Show that if n is odd, then the first player will always win and show that if n is even, then the second player invariably wins.
(b) Show that the conclusion of (a) also holds if the game takes place on a closed orientable 2-dimensional smooth manifold.

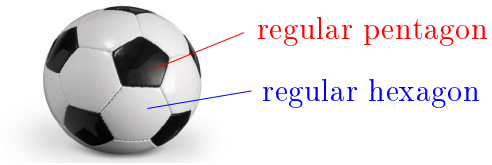
Remark. The game of "Brussels sprouts" is a variation on the more interesting game "sprouts" introduced by John Conway and Michal Paterson in the 1967. More information on "Sprouts" and "Brussels sprouts" can be found in [**FiG1991**, Chapter 6] and [**Gardn1989**, Chapter 1].

88. Applications of the Euler characteristic (∗)

In this chapter we show that the fairly simple concept of the Euler characteristic can be used to give answers to several questions about 3-dimensional objects. The reader is warned in that in this chapter at times we will sacrifice complete notational rigor for readability.

88.1. **Building a leather football ❀.** In this section we will discuss the shapes of footballs and the platonic solids. The results of this section will not be used later on and we will present some of the material in a slightly informal, somewhat less rigorous style.

Definition. An n-gon on the 2-dimensional sphere S^2 is called regular if the isometry group of the sphere acts transitively on the set of vertices and the set of faces of the n-gon. In particular all interior angles are the same. For example the classical leather football is made out of regular pentagons and hexagons.



Now we want to determine how many pentagons and hexagons are required to make a football. So suppose a football is made out of p spherical regular pentagons and h spherical regular hexagons where the vertices of the pentagons are precisely the vertices of hexagons. This endows S^2 with a CW-structure. We first note that each 1-cell bounds two 2-cells. Furthermore, elementary spherical geometry, see e.g. [**Bon2009**, p. 53] or [**Fen2001**, p. 260], says that the interior angle of a spherical pentagon and a spherical hexagon are larger than the corresponding Euclidean angles. Therefore we see that interior angles are larger than[1066] $\pi - \frac{2\pi}{5} = \frac{3}{5}\pi$. So at any vertex there are not more than three hexagons and pentagons that meet, which of course implies that there are precisely three hexagons and pentagons that meet. Therefore we see that

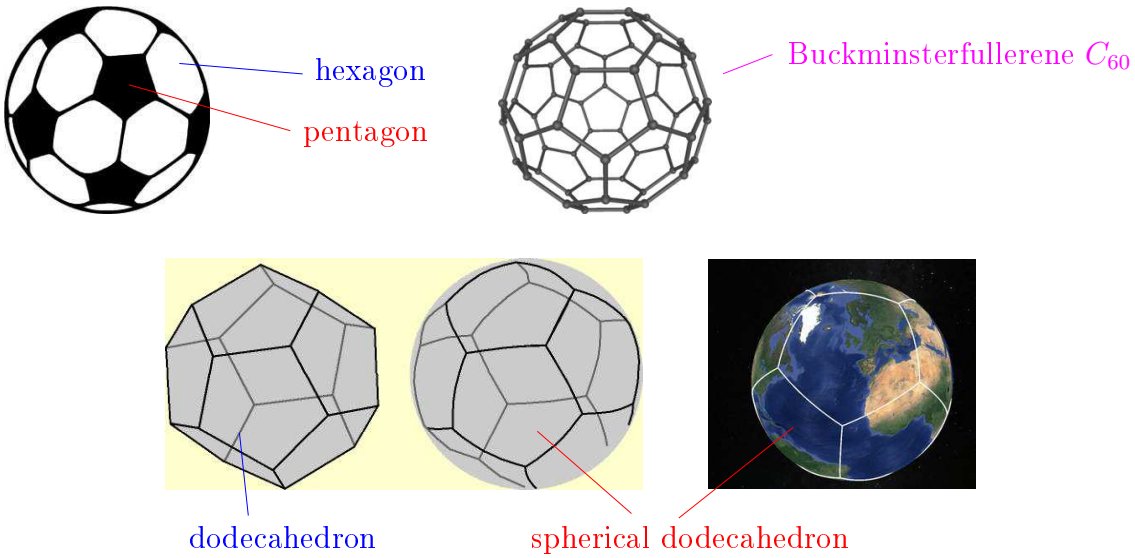
$$\begin{aligned} \#\text{2-cells} &= p + h \\ \#\text{1-cells} &= \tfrac{1}{2}(5p + 6h) \\ \#\text{0-cells} &= \tfrac{1}{3}(5p + 6h). \end{aligned}$$

By the Euler Characteristic-H_*-Proposition 87.1 and the above calculation of the Euler characteristic of S^2 we now have

$$2 \;=\; \chi(S^2) \;=\; \sum_{n=0}^{2}(-1)^n \cdot \#n\text{-cells} \;=\; \tfrac{1}{3}(5p+6h) - \tfrac{1}{2}(5p+6h) + (p+h) \;=\; \tfrac{1}{6}p.$$

Thus we see that a football has to have precisely 12 pentagons. On the other hand, the number of hexagons is not fixed. For example, the standard football has 20 hexagons, as pictured in the figure below on the left.
But in principle one could also build a football with zero hexagons, see the figure below one the right, and one obtains a spherical dodecahedron. One could also build with many hexagons, but they would be much less round since they would have bigger flat parts.

[1066] Where does that formula come from?



The 1-skeleton of this CW-structure were in fact already "invented by nature", more precisely, there exists a molecule, called the Buckminsterfullerene C_{60}, built out of 60 carbon atoms, which has exactly the same shape as a football. More information can be found at

`https://en.wikipedia.org/wiki/Buckminsterfullerene`

88.2. **Platonic solids ❀.** Recall that on page 157 we defined the *convex hull* of a subset $S \subset \mathbb{R}^n$ as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S. Since the intersection of convex sets is again convex we see that the convex hull of S is a convex subset of $\mathbb{R}^k$.

Definition.
(1) A convex polyhedron is the convex hull of finitely many points in $\mathbb{R}^k$.[1067] [1068][1069]
(2) The dimension of a convex polyhedron is defined as the minimal dimension of an affine subspace of $\mathbb{R}^k$ that contains the convex polyhedron.

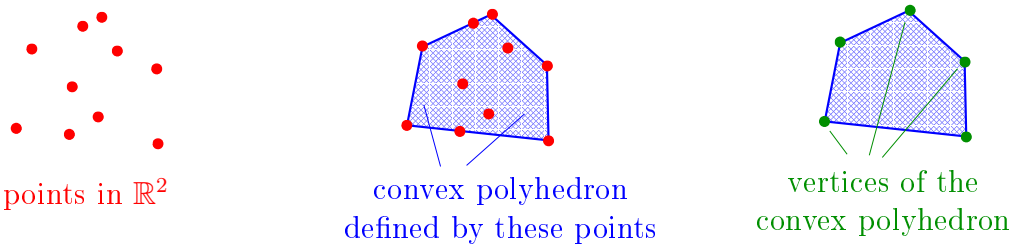
We leave the task of providing a proof of the following lemma to the meticulous reader.

Lemma 88.1. If P is a convex polyhedron in $\mathbb{R}^k$, then there exists a unique finite set V such that P is the convex hull of V, but such that for any proper subset W of V, P is not the convex hull of W.

Definition. Let P be a convex polyhedron in $\mathbb{R}^k$.
(1) We refer to the finite set of Lemma 88.1 as the vertices of the convex polyhedron P.

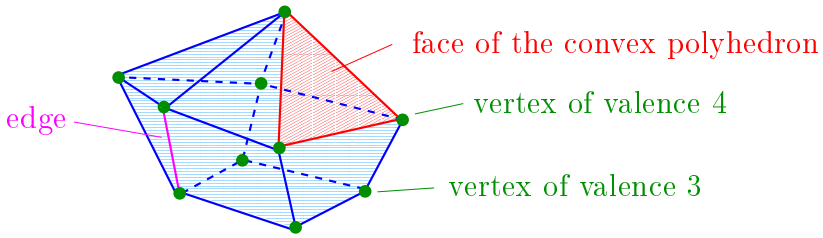
[1067] It is an amusing little exercise to show that this notion of a convex polyhedron agrees with the notion that we had originally defined on page 1552.
[1068] In the literature a convex polyhedron is often also called a polytope.
[1069] Recall that we saw in Exercise 2.38 (a) that the convex hull of points $P_1, \ldots, P_n \in \mathbb{R}^k$ is given by the set

$$\Big\{ \sum_{i=1}^{n} t_i \cdot P_i \,\Big|\, t_1, \ldots, t_i \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^{n} t_n = 1 \Big\}.$$



(2) If P is a k-dimensional convex polyhedron, then an edge of the convex polyhedron is the convex hull of two vertices that has the property that it lies on ∂P.[1070]
(3) If P is a k-dimensional convex polyhedron, then an m-dimensional face of the convex polyhedron is the convex hull of m vertices that has the property that it lies on ∂P. (In particular, by definition a 1-dimensional face is exactly the same as an edge.)
(4) An edge is a 1-dimensional face and sometimes we refer to a 2-dimensional face just as a face, without specifying the dimension.
(5) The valence of a vertex is defined as the number of edges that contain the vertex.

Example. The notion of a vertex, an edge, a face and the valence of a vertex are illustrated in the figure below.



Proposition 88.2. (Euler's Formula) Let P be a k-dimensional convex polyhedron in $\mathbb{R}^k$.
(1) The interior $\mathring{P}$ of P is non-empty and all faces of P are of dimension $\leq k-1$.
(2) The boundary ∂P admits a CW-structure such that for each $j \in \mathbb{N}_0$ the j-cells are precisely the j-dimensional faces of P.
(3)
$$\sum_{j=0}^{k-1}(-1)^j \cdot \#j\text{-dimensional faces} \;=\; 1+(-1)^{k-1}.$$

Remark. The 3-dimensional version of Proposition 88.2 gives us that for the boundary of a 3-dimensional convex polyhedron in $\mathbb{R}^3$ we have

$$\text{number of vertices} - \text{number of edges} + \text{number of faces} \;=\; 2.$$

This equality is known as Euler's Formula and was first formulated Leonhard Euler in 1752. Arguably Euler's Formula is historically the first result in topology. A famous book by Imre Lakatos [**Lak1976**] is dedicated to the challenge of finding the correct statement of Euler's formula and of providing a rigorous proof of Euler's Formula without employing the technical machinery, namely singular homology, that we used.

[1070] Here ∂P is the boundary of P viewed as a subset of $\mathbb{R}^k$.

Proof. We leave the pleasure of proving the first two statements to the reader as Exercise 88.1. Now we calculate that

$$\sum_{j=0}^{k-1}(-1)^j \cdot \#j\text{-dimensional faces} \underset{\text{by (1) and (2)}}{=} \chi(\partial P) \overset{\text{Convex-to-Ball Proposition 2.20 (2) together with (1)}}{=} \chi(S^{k-1}) \underset{\text{see page 1879}}{=} 1+(-1)^{k-1}.$$

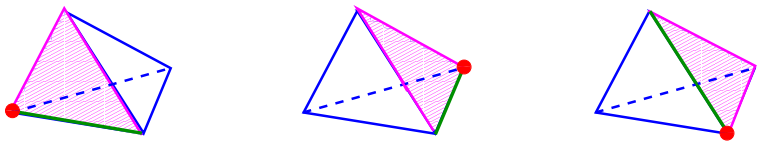
■

We move on to the next definitions.

Definition.

(1) Let P be a convex m-dimensional polyhedron in $\mathbb{R}^k$.
   (a) A rotational symmetry of P in $\mathbb{R}^k$ is a matrix $A \in \mathrm{SO}(k)$ such that $A \cdot P = P$. The rotational symmetry group $\mathrm{Sym}(P) \subset \mathrm{SO}(k)$ is defined as the group of all rotational symmetries of P.
   (b) A flag of P is a sequence $F_0 \subset F_1 \subset \cdots \subset F_{m-1}$ of faces of P such that each F_i is i-dimensional.
   (c) A convex polyhedron P is called regular if the rotational symmetry group $\mathrm{Sym}(P)$ acts transitively on the set of flags of P.

(2) Two convex polyhedra P and Q in $\mathbb{R}^k$ are called equivalent if there exists an $r > 0$, a point $v \in \mathbb{R}^k$, and an orthogonal matrix $A \in \mathrm{SO}(k)$ with $P = A \cdot (r \cdot Q) + v$.



three flags in the regular tetrahedron T

Examples.

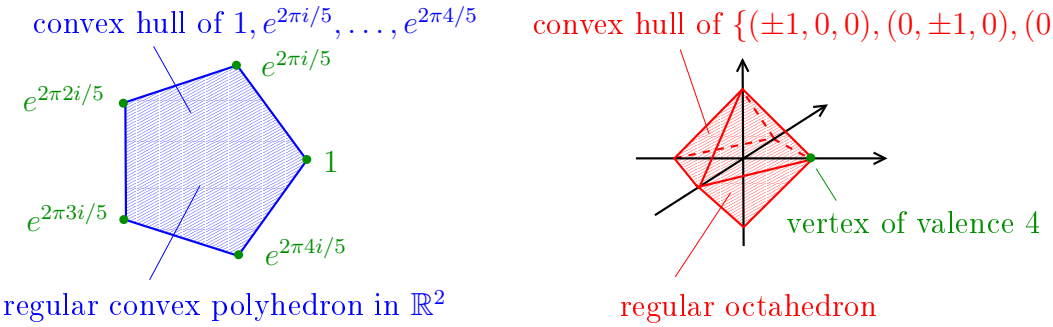
(1) Let T be the 3-dimensional polyhedron in $\mathbb{R}^3$ that has four 2-dimensional faces, each of which is an equilateral triangle. We show T together with three flags in the figure above. Some elementary geometry shows that T is, up to equivalence, uniquely defined and it is regular. We refer to T as the *regular tetrahedron*.

(2) The standard n-simplex Δ^n is an n-dimensional convex polyhedron in $\mathbb{R}^{n+1}$ with vertex set $\{(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\}$. Each vertex has valence n. It is straightforward to see that Δ^n is regular.

(3) Let $n \in \mathbb{N}_{\geq 3}$ and $r > 0$. The convex hull of the points $r, r \cdot e^{\pi \mathrm{i}/n}, \ldots, r \cdot e^{2\pi \mathrm{i}(n-1)/n}$ in $\mathbb{C} = \mathbb{R}^2$ is a regular convex polyhedron with vertices $r, r \cdot e^{2\pi \mathrm{i}/n}, \ldots, r \cdot e^{2\pi \mathrm{i}(n-1)/n}$, where each vertex has valence 2. We leave it to the reader to show that, up to equivalence, these are all regular 2-dimensional convex polyhedra in $\mathbb{R}^2$. We refer to the figure below for an illustration.

(4) The convex polyhedron spanned by $\{(\pm 1,0,0),(0,\pm 1,0),(0,0,\pm 1)\}$ is a regular 3-dimensional convex polyhedron where each vertex has valence 4. This convex polyhedron is usually called the *regular octahedron*. We refer to the figure below for an illustration.

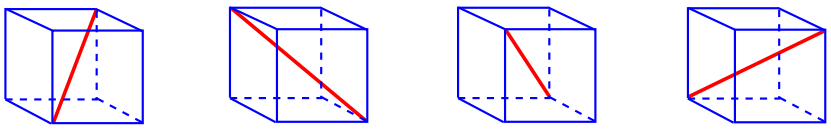
(5) We consider the cube $[-1,1]^3 \subset \mathbb{R}^3$. In [**Arm1988**, p. 38] it is shown that the group $\mathrm{Sym}([-1,1]^3)$ is isomorphic to S_4. More precisely, we denote by D the set of the



diagonals in the cube, i.e. the segments in $\mathbb{R}^3$ spanned by opposite vertices of the cube. Note that D has precisely four elements. We refer to the figure below for an illustration. Then the map

$$\begin{array}{rcl} \mathrm{Sym}([-1,1]^3) & \to & S_4 \cong \text{permutation group of } D \\ A & \mapsto & \begin{pmatrix} D & \to & D \\ d & \mapsto & A \cdot d \end{pmatrix} \end{array}$$

is shown in [**Arm1988**, p. 38] to be an isomorphism.[1071] It is clear that the cube is a regular convex polyhedron. Since it has six sides it is in our context also referred to as *regular hexahedron*.



(6) Let $z := 1 + \frac{\sqrt{5}}{2}$ be the golden ratio. We denote by D the polyhedron that is spanned by the 20 vertices

$$(\pm 1, \pm 1, \pm 1), \quad (0, \pm z, \pm \tfrac{1}{z}), \quad (\pm \tfrac{1}{z}, 0, \pm z), \quad \text{and} \quad (\pm z, \pm \tfrac{1}{z}, 0).$$

It is illustrated in the figure below. By [**Cox1948**, p. 53] this is a regular polyhedron. We refer to is as the *regular dodecahedron*. Note that it has 12 faces and 20 vertices. We can inscribe five cubes $C_1, \ldots, C_5$ into the regular dodecahedron. In [**Arm1988**, p. 40] it is shown that the map

$$\begin{array}{rcl} \mathrm{Sym}(D) & \to & A_5 \cong \text{positive permutations of } \{C_1, \ldots, C_5\} \\ A & \mapsto & (C_i \mapsto A \cdot C_i) \end{array}$$

is an isomorphism.[1072]

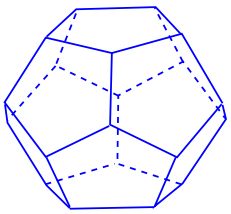
(7) Let $z := 1 + \frac{\sqrt{5}}{2}$ be the golden ratio. We denote by I the polyhedron that is spanned by the 12 vertices

$$(0, \pm 1, \pm z), \quad (\pm z, 0, \pm 1) \quad \text{and} \quad (\pm 1, \pm z, 0).$$

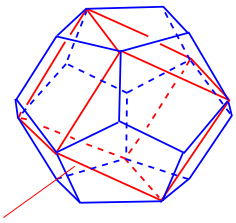
By [**Cox1948**, p. 52] this is a regular polyhedron. We refer to is as the *regular icosahedron*. Note that the regular dodecahedron has 20 vertices and 12 faces whereas

[1071] It is a good exercise to verify this statement.

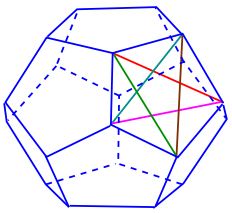
[1072] The regular dodecahedron, and its symmetry group, will play a prominent role later on. More precisely in Section 98.4 we will use these to construct the Poincaré homology sphere which is a closed 3-dimensional smooth manifold M such that $\pi_1(M)$ is a group of order 120, but with trivial abelianization.



regular dodecahedron

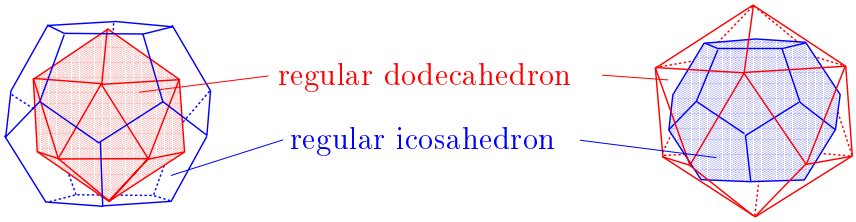


cube inscribed into the regular dodecahedron



the five diagonals of a face correspond precisely to one inscribed cube

the regular icosahedron has 12 vertices and 20 faces. In fact the regular icosahedron is dual to the regular dodecahedron in the following sense: up to a scaling factor and a rotation, the regular icosahedron is spanned by the centers of the faces of the regular dodecahedron and vice-versa. We refer to the figure below for an illustration of this fact. It follows that the symmetry group of the regular icosahedron I is in a suitable sense equal to the symmetry group of the regular dodecahedron D. In particular the group $\mathrm{Sym}(I)$ is isomorphic to A_5.



The following definition is one of the oldest in the history of mathematics.

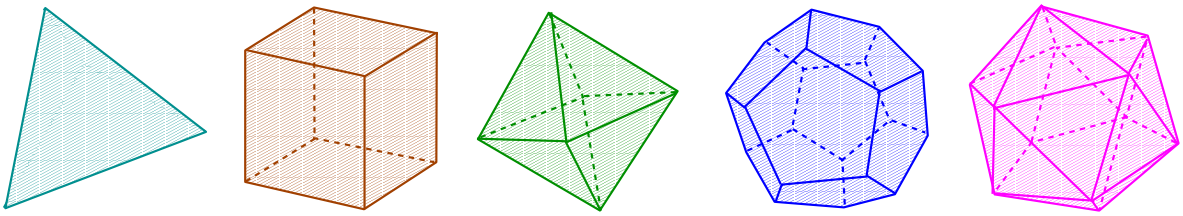
Definition. A Platonic solid is a regular 3-dimensional convex polyhedron in $\mathbb{R}^3$.

Theorem 88.3. Up to rotation and stretching the only regular convex 3-dimensional polyhedra in $\mathbb{R}^3$ are the platonic solids illustrated in the figure below.[1073] They have the following types of faces and valences and they have the following rotational symmetry groups:

	tetrahedron	hexahedron	octahedron	dodecahedron	icosahedron
number of faces	4	6	8	12	20
shape of face	3-gon	4-gon	3-gon	5-gon	3-gon
valence of vertex	4	3	4	3	5
rotational symmetry group	A_4	S_4	S_4	A_5	A_5

Sketch of proof. We will not provide a full proof for the theorem. We refer to [**Cox1948**, Section 6.7] for a full proof that the list of five platonic solids is indeed complete. Furthermore we refer to [**Arm1988**, p. 40] or alternatively to [**Ae2008**, Chapter 5.5] for a proof of the statement regarding the rotational symmetry groups.[1074]

[1074] We had sketched the proofs for the regular hexahedron, the regular dodecahedron and the regular icosahedron.



In the following we will show that any platonic solid fits into the table of the theorem. Put differently, we will show that given any platonic solid the number of faces, the shape of faces and the valence of a vertex corresponds to one of the five columns in the above table.

Thus let P be a regular convex 3-dimensional polyhedron in $\mathbb{R}^3$. We make the following observations:

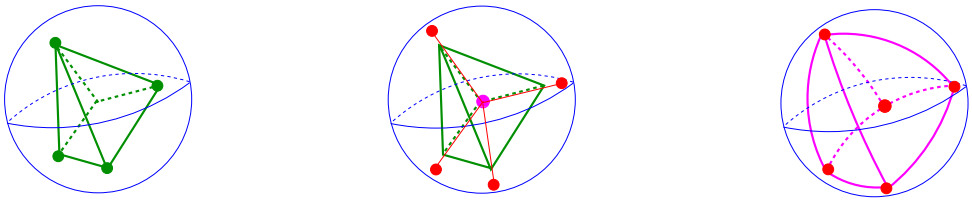
(1) It follows easily from the definition of a regular polyhedron that there exists an m such that each face of P is a regular m-gon.
(2) The number of faces that meet a vertex is the same for every vertex and it is at least three.

The following claim tells us the possible values for m and the number of faces that meet at a vertex.

Claim.

possible values for m	3	4	5
faces that meet at a vertex	3, 4, 5	4	3

Proof. First note that after a translation we can assume that the sum of all vertices is in the origin. Next we shrink the convex polyhedron so that it fits into the sphere S^2. Then we consider the projection of the vertices and edges of the convex polyhedron onto the sphere, see the figure below. Recall that the spherical interior angle of a regular m-gon is greater than the Euclidean angle of a regular m-gon. So it follows that in our case the interior angle is greater than the angle of a regular m-gon in $\mathbb{R}^3$, i.e. it is greater than $\pi - \frac{2\pi}{m}$. Since at any vertex at least three faces meet we obtain the restriction that $3(\pi - \frac{2\pi}{m}) < 2\pi$. It is now straightforward to see that we obtain the stated table. ⊞



Now we treat the cases $m = 3, 4, 5$ separately. We start out with the case $m = 3$, i.e. we assume that all the faces of the convex polyhedron are equilateral triangles. Let us first consider the case that three triangles meet at a vertex. Then we have

$$\#\text{faces} = k, \quad \#\text{edges} = \tfrac{1}{2} \cdot 3k \quad \text{and} \quad \#\text{vertices} = \tfrac{1}{3} \cdot 3k.$$

Now we see that

$$2 \underset{\text{Euler's Formula 88.2}}{=} \#\text{vertices} - \#\text{edges} + \#\text{faces} \underset{\text{by the above}}{=} k - \tfrac{3}{2}k + k = \tfrac{1}{2}k.$$

It follows that $k = 4$. Almost the same calculation shows that if four triangles meet at a vertex, then $k - \frac{3}{2}k + \frac{3}{4}k = 2$, i.e. $k = 8$. Finally by the above claim it remains to consider the case that five triangles meet at a vertex. In this case we see that $k - \frac{3}{2}k + \frac{3}{5}k = 2$, i.e. $k = 20$.

Next we consider the case $m = 4$, i.e. each face is a square. The same argument as above shows that the valence of a vertex can only be three and the same calculation shows that there have to be precisely six faces. Finally if $m = 5$, i.e. if each face is a regular pentagon, then as above we see that the valence of a vertex can only be three and that there have to be precisely twelve faces. ■

On page 1896 we had just seen that there exist infinitely many equivalence classes of regular convex polyhedra in $\mathbb{R}^2$. On the other hand we have just seen in Theorem 88.3 that in $\mathbb{R}^3$ there are only finitely many, namely precisely five equivalence classes of regular convex polyhedra in $\mathbb{R}^3$. This begs the question, what is the number in higher dimensions? It turns out that in $\mathbb{R}^4$ there are six equivalence classes of regular convex polyhedra, whereas in all higher dimensions there are only three equivalence classes of convex polyhedra. A nice exposition of these facts is given on the following website:

`http://math.ucr.edu/home/baez/platonic.html`

88.3. **Planar graphs ❀.** On pages 307 and 313 we defined the notions of an (ordered) abstract graph. By Lemma 7.5 and the discussion on page 313 we know that they have homeomorphic topological realizations. In this section we will mostly deal with undirected abstract graphs. Since this name is rather unwieldy we will refer to them just as graphs. For the convenience of the reader we recall the definition.

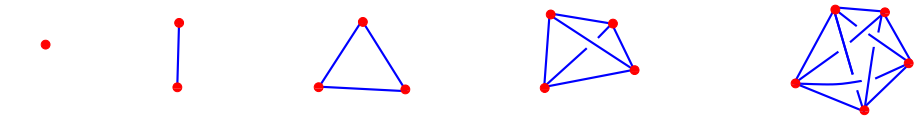
Definition. An (undirected abstract) graph G is a triple (V, E, φ) where V is a non-empty set, E is a set and φ is a map

$$\varphi\colon E \to \{\text{subsets of } V \text{ with one or two elements}\}.$$

The elements of V are called vertices of G and the elements of E are called the edges of G.

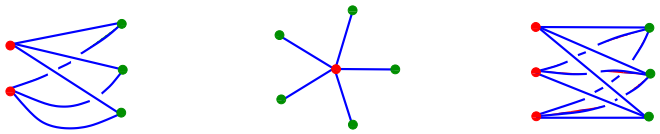
Examples.

(1) For $n \in \mathbb{N}$ we define the *complete graph* K_n as the abstract graph with vertex set $V = \{v_1, \dots, v_n\}$ and edge set $E = \{\{v_i, v_j\} \mid i, j \in \{1, \dots, n\} \text{ with } i \neq j\}$. Furthermore we define $\varphi\colon E \to V$ via $\varphi(e) = e$. Put differently, the complete graph K_n has n vertices and each vertex is connected to any other vertex by precisely one edge. In the figure below we show topological realizations of the complete graphs K_1, K_2, K_3, K_4 and K_5.



K_1 K_2 K_3 K_4 K_5

(2) For every choice of $m, n \in \mathbb{N}$ we define the *complete bipartite graph* $K_{m,n}$ as the abstract graph which is defined by the vertex set $V = \{x_1, \dots, x_m\} \cup \{y_1, \dots, y_n\}$, the edge set $E = \{\{x_i, y_j\} \mid i \in \{1, \dots, m\} \text{ and } j \in \{1, \dots, n\}\}$ and the map $\varphi\colon E \to V$ via $\varphi(e) = e$. Put differently, the complete bipartite graph $K_{m,n}$ consists of m "red-colored" vertices, n "green-colored" vertices and each "red-colored" vertex is connected to any "green-colored" vertex by precisely one edge. In the figure below we show topological realizations of the complete bipartite graphs $K_{2,3}$, $K_{1,5}$ and $K_{3,3}$.



$K_{2,3}$ $K_{1,5}$ $K_{3,3}$

Let $G = (V, E, \varphi)$ be a graph. On page 308 we introduced its topological realization $|G|$. It follows from Lemma 7.5 and the discussion on page 1470 that $|G|$ is naturally a 1-dimensional CW-complex where the 0-cells correspond to the vertices and the 1-cells correspond to the edges.
Let G be a finite graph. On page 312 we had already mentioned that there exists an embedding of its topological realization $|G|$ into $\mathbb{R}^3$. In Question 7.4 we asked, without really expecting a positive answer, whether one can always find an embedding of $|G|$ into $\mathbb{R}^2$. This leads us to the following dimension.

Definition. We say a finite graph G is planar if its topological realization $|G|$ admits an embedding $|G| \to \mathbb{R}^2$.

Examples.

(1) The above graphs K_1, K_2, K_3 and $K_{1,5}$ are planar. The above topological realizations of K_4 and $K_{2,3}$ are not planar. But as we see in the figure below both graphs admit planar realizations.



topological realizations of K_4 topological realizations of $K_{2,3}$

(2) One can easily verify that given an $m \in \mathbb{N}$ the bipartite graph $K_{2,m}$ is planar.

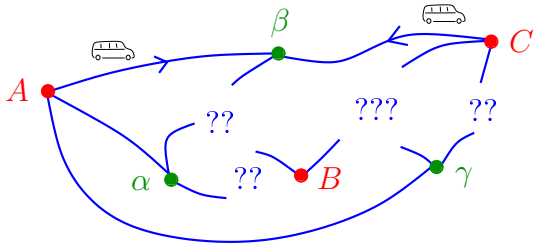
Now the following question arises.

Question 88.4. Are the graphs K_5 and $K_{3,3}$ also planar?

If this question sounds to abstract and bipartite graphs sound too otherworldly let me reformulate the question regarding $K_{3,3}$.

Question 88.5. Suppose we are in charge of finding efficient bus routes in a given city. Furthermore suppose the city has three main employers A, B and C. After work the employees, naturally, either want to go to the train station α, or the swimming pool β

or the shopping mall γ. Is it possible to find bus routes which connect each of the three employers with each of the three destinations such that the bus routes do not intersect?[1075]



To answer that question we need a few more definitions.

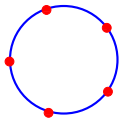
Definition. Let G be a graph.

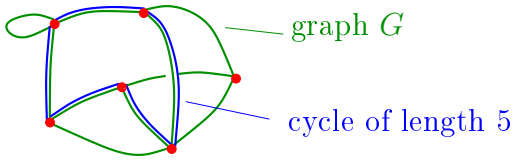
(1) Given $n \in \mathbb{N}$ we denote by X_n the "circular graph with n vertices and n edges", more precisely, X_n is the graph with

$$V(X_n) = \{1, \dots, n\} \quad \text{and} \quad E(X_n) = \{\{1,2\}, \{2,3\}, \dots, \{n-1, n\}, \{n, 1\}\}.$$

(2) A cycle of length n in G is an injective map $\varphi\colon X_n \to G$ of graphs in the sense of the definition given in Exercise 7.10.

(3) The girth of G is the shortest length of a cycle in G. If G has no cycles, then we define its girth as ∞.





Examples.

(1) The girth of the graph shown in the figure above to the right is one, since it has an edge where the two endpoints agree.

(2) For any $n \geq 3$ the girth of the complete graphs K_n equals three.

(3) For any $n, m \geq 2$ the girth of the complete bipartite graphs $K_{n,m}$ equals four.

The following gives an obstruction to a graph being planar.

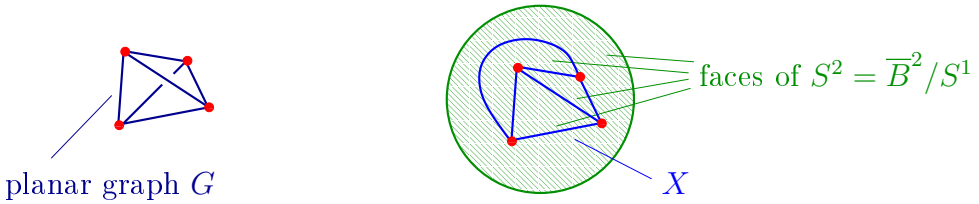
Proposition 88.6. Let G be a graph with v vertices and e edges. We denote by g the girth of G. If G is planar and if $g < \infty$, then

$$e \leq \frac{g}{g-2} \cdot (v-2).$$

Sketch of proof. Let G be a graph with v vertices and e edges. We suppose that the girth g of G is finite. Furthermore we suppose that G is planar, i.e. we suppose that there exists an embedding $\varphi\colon |G| \to \mathbb{R}^2$. We write $X = \varphi(|G|)$. Without loss of generality we can assume that X lies in $\overline{B}^2$.

Now we view X as a subset of the sphere $S^2 = \overline{B}^2/S^1$. We refer to the closures of the components of $S^2 \setminus X$ as *faces* and we denote by f the number of faces. We refer to the figure below for an illustration. We make the following observation:

[1075] It follows easily from Proposition 36.4 that the answer to this question does not depend on the precise location of A, B, C and α, β, γ, as long as they are distinct.



(1) The vertices, edges and faces define a CW-structure for $S^2 = \overline{B}^2/S^1$.[1076]
(2) We make the following two observations:
   (a) The boundary of each face is a cycle. By the definition of the girth each face is bounded by at least g edges.
   (b) Each edge lies in the boundary of at most two faces.

   Now we deduce that

$$\begin{aligned} g \cdot f &\leq \sum_{\text{faces } f} \text{number of edges in the boundary of } f \\ &= \text{number of pairs } (f,e) \text{ where } f \text{ is a face and } e \text{ an edge in the boundary of } f \\ &= \sum_{\text{edges } e} \text{number of faces that contain } e \text{ in the boundary } \leq 2e. \end{aligned}$$

   Here the first inequality follows from (a) and the last inequality follows from (b). Summarizing we see that $g \cdot f \leq 2e$.

Now we see that

$$2 \underset{\substack{\uparrow \\ \text{see page 1879}}}{=} \chi(S^2) = v - e + f \underset{\substack{\uparrow \\ \text{since } g \cdot f \leq 2e}}{\leq} v - e + \frac{2}{g}e = v - \frac{g-2}{g}e.$$

Solving for e we get the desired inequality $e \leq \frac{g}{g-2} \cdot (v-2)$. ■

The following proposition gives in particular a negative answer to Question 88.4.

Proposition 88.7. If a graph G contains the complete graph K_5 or the complete bipartite graph $K_{3,3}$ as a subgraph, then G itself cannot be planar.

Proof.

(1) Let $m \in \mathbb{N}_{\geq 3}$. It is straightforward to see that the complete graph K_m has m vertices and $\frac{1}{2}m(m-1)$ edges. Furthermore the girth is three. It is straightforward to verify that the inequality of Proposition 88.6 is only satisfied for $m = 3$ and $m = 4$.
(2) Let $m, n \in \mathbb{N}_{\geq 2}$. Again it easy to verify that the complete bipartite graph $K_{m,n}$ has $m + n$ vertices and $m \cdot n$ edges. Furthermore,as we pointed out above, the girth is four. It is straightforward to verify that the inequality of Proposition 88.6 is only satisfied if $m = 2$ or $n = 2$.
(3) The Proposition now follows from (1) and (2) and the observation that if a graph is planar, then any subgraph is also planar. ■

[1076] Here is the reason why it is the sketch of a proof: we do not prove that each face is in fact homeomorphic to a closed disk. This statement is "clear from the picture" and it can be proved using the Schönflies Theorem 82.4. But we will not attempt to give a rigorous proof.

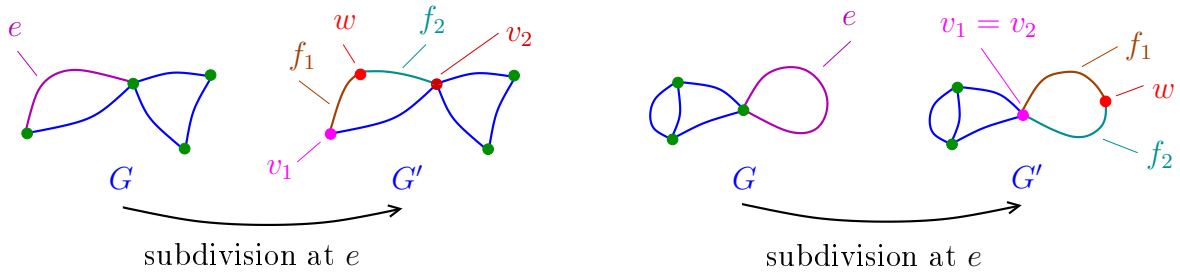
Remark. Let G be a graph that contains the complete graph K_7. By Proposition 88.7 the graph G cannot be planar. John Conway and Cameron Gordon proved in 1984 the following interesting variation: given any embedding $\varphi\colon |G| \to \mathbb{R}^3$ such that image is a polygon (i.e. it is the union of finitely many line segments) the image $\varphi(|G|)$ contains a non-trivial polygonal knot. More precisely, there exists a closed polygon C in $\varphi(|G|)$ such that $\pi_1(\mathbb{R}^3 \setminus C)$ is not isomorphic to $\mathbb{Z}$. We refer to **[CoG1983]** for the precise formulation of the theorem and the proof.

Now we will consider the question, to what degree the converse to Proposition 88.7 holds. First we need to introduce the notion of a subdivision of a graph. From the figure below it should become clear what is meant be a subdivision. As so often, the formal definition is a little cumbersome.

Definition. Let $G = (V, E, \varphi\colon E \to \mathcal{P}(V))$ be a graph.

(1) Let $e \in E$ be an edge. We say that a graph $G' = (V', E', \varphi'\colon E' \to \mathcal{P}(V'))$ is a subdivision of G at e if $V' = V \sqcup \{w\}$ and $E' = (E \setminus \{e\}) \sqcup \{f_1, f_2\}$ with $w \notin V$ and $f_1, f_2 \notin E$ such that $\varphi'(e') = e$ for $e' \in E \setminus \{e\}$, $\varphi'(f_1) = \{v_1, *\}$ and $\varphi'(f_2) = \{v_2, *\}$ where $\varphi(e) = \{v_1, v_2\}$.

(2) We say a graph G' is a subdivision of G if G' is obtained from G by a finite sequence of subdivisions along edges.



The following lemma says in particular that "planarity" is unaffected by subdivisions.

Lemma 88.8. Let G be a graph. If G' is a subdivision of G, then the topological realizations of G' and G are homeomorphic.

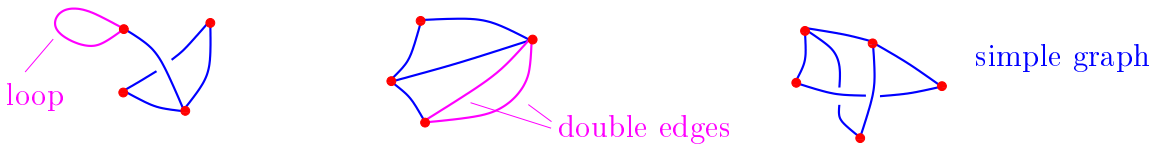
Proof. We leave it to the reader to provide the elementary proof of this lemma. ■

We move on to the final definition of this section.

Definition. We say a graph is simple if its girth is at least three.

Put differently, a graph G is simple if the following two conditions are satisfied:

(1) G has no loops, i.e. if there is no edge where the two endpoints agree,
(2) G does not have multiple edges, i.e. there are no two edges with the same endpoints.

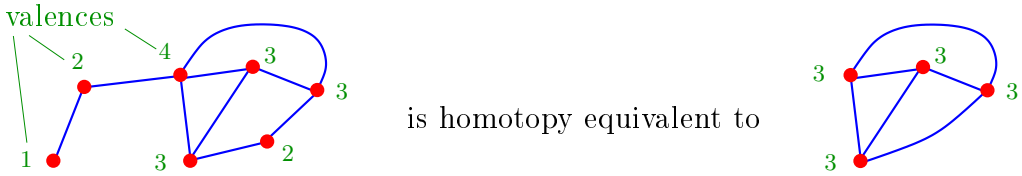


In the remainder of this section we restrict ourselves to graphs that are simple. We start out with the following observations.

(1) If a simple graph G admits a vertex v of valence[1077] one, then v is the endpoint of a single edge e. If we delete v and e we obtain a new graph G'. An argument as in the proof of Proposition 50.15 shows that the topological realization of G' is homotopy equivalent to the topological realization of G. It is straightforward to see that G is planar if and only if G' is planar.

(2) If a simple graph admits a vertex v of valence two, then v is the endpoint of two edges e_1 and e_2. We can combine e_1 and e_2 into one edge and eliminate v. This way we obtain a new graph with a homeomorphic topological realization.

This shows that for the most part we can restrict ourselves to simple graphs where every vertex has valence at least three.



If we restrict ourselves to simple graphs such that every vertex has valence at least three, then the graphs K_5 and $K_{3,3}$ turn out to be the only obstruction to being planar. More precisely the following theorem was proved by Kuratowski[1078] in 1930. We refer to [**BoM2008**, Chapter 10.5] for a proof.

Theorem 88.9. (Kuratowski's Reduction Theorem) Let G be a finite simple graph such that every vertex has valence at least three. Then the following statements are equivalent:

(1) The graph G is planar.
(2) The G contains neither a subdivision of K_5 nor a subdivision of $K_{3,3}$ as a subgraph.

Exercises for Chapter 88.

Exercise 88.1.

(a) Let P be a k-dimensional convex polyhedron in $\mathbb{R}^k$. Show that the interior $\overset{\circ}{P}$ of P is non-empty.

(b) Let Q be a k-dimensional convex polyhedron in some $\mathbb{R}^n$. Show that Q is homeomorphic to $\overline{B}^k$.
Hint. Use (a) and the Convex-to-Ball Proposition 2.20 (2).

(c) Let P be a k-dimensional convex polyhedron in some $\mathbb{R}^n$. Show that P admits a CW-structure such that the boundary ∂P is a subcomplex and such that for each $j \in \mathbb{N}_0$ the j-cells in ∂P are precisely the j-dimensional faces of P.

Exercise 88.2.

(a) Show that there exist injective maps $K_5 \to S^1 \times S^1$ and $K_{3,3} \to S^1 \times S^1$.

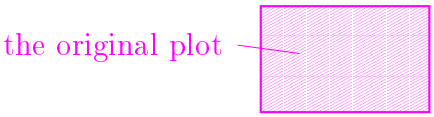
[1077] Recall that the *valence* of a vertex is defined as the number of edges that contain the vertex.
[1078] Kazimierz Kuratowski (1896-1980) was a Polish mathematician.

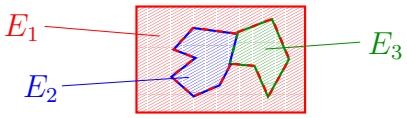
(b) Let $n \in \mathbb{N}_{\geq 6}$. What is the smallest g that you can find such that there exists an injective map from K_n to the surface of genus g?

Exercise 88.3. In his last will, a farmer, who owns a rectangular plot, wants to distribute his land among his sons as follows:

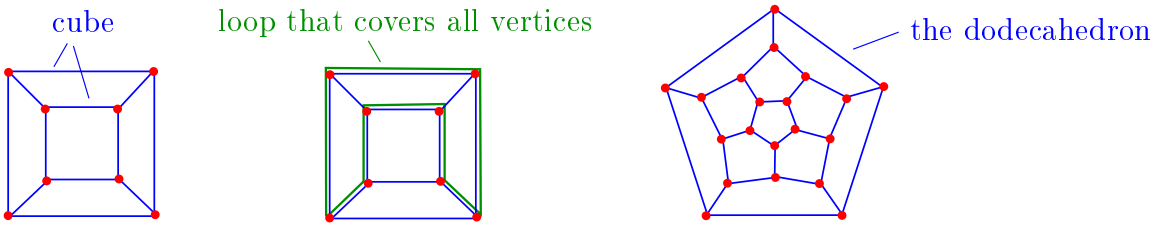
(1) every son gets a connected compact field with non-empty area,
(2) every son should be the neighbor of every other son.

What is maximal number of sons for which the farmer's last will can be fulfilled?
Remark. On the margin of the last will the farmer insists that the new distribution should give a CW-structure for the rectangular plot.





Exercise 88.4. Find an embedded loop on the edges of the dodecahedron that covers all vertices.
Remark. This is known as Hamilton's "icosian game". This puzzle was actually marketed in the 19th century as a board game.



89. Homology with coefficients

In this chapter we will introduce the homology groups of a topological space X with coefficients in a ring R. These homology groups will have many applications, for example they will be essential in relating the homology groups of a smooth manifold M to the de Rham cohomology groups of M. They will also be essential in proving theoretic statements, for example we will need to work with $\mathbb{F}_2$-coefficients to prove the higher-dimensional analogue of the Borsuk-Ulam Theorem 48.21. Before we can introduce the homology groups with coefficients we will need to introduce or recall some basic algebraic concepts.

89.1. The tensor product of abelian groups.

The key to defining homology groups with coefficients is the notion of the tensor product of two abelian groups.

Definition. Let A and B be abelian groups. We define the tensor product $A \otimes B$ to be the following abelian group:

$$A \otimes B := \left\{ \begin{array}{l} \text{free abelian group which is generated by all} \\ \text{symbols of the form } a \otimes b \text{ with } a \in A,\ b \in B \end{array} \right\} / N(A,B)$$

where $N(A,B)$ is the subgroup generated by the elements of the form

$$(a+a') \otimes b - (a \otimes b + a' \otimes b) \qquad \text{and} \qquad a \otimes (b+b') - (a \otimes b + a \otimes b')$$

with $a, a' \in A$ and $b, b' \in B$. For $a \in A$ and $b \in B$ we denote, by a slight abuse of notation, the image of $a \otimes b$ in $A \otimes B$ again by $a \otimes b$.

We summarize some elementary properties of the tensor product of two abelian groups in the following lemma.

Lemma 89.1. Let A and B be two abelian groups.

(1) For any $a, a' \in A$ and $b, b' \in B$ we have

$$\begin{array}{rcl} (a+a') \otimes b & = & a \otimes b + a' \otimes b \ \in \ A \otimes B \\ a \otimes (b+b') & = & a \otimes b + a \otimes b' \ \in \ A \otimes B. \end{array}$$

(2) For any $a \in A, b \in B$ and $m \in \mathbb{Z}$ we have

$$(m \cdot a) \otimes b \ = \ a \otimes (m \cdot b) \ = \ m \cdot (a \otimes b).$$

(3) Any element in $A \otimes B$ is of the form $\sum_{i=1}^{n} a_i \otimes b_i$.

(4) Let $n \in \mathbb{N}$. For any $c \in A \otimes \mathbb{Z}_n$ we have $n \cdot c = 0 \in A \otimes \mathbb{Z}_n$.

Proof. Let A and B be two abelian groups.

(1) This statement is an immediate consequence of the definition of the tensor product.

(2) Let $a \in A$ and $b \in B$. We first consider the case that $m > 0$. We have

$$(m \cdot a) \otimes b \ = \ \left(\sum_{i=1}^{m} a \right) \otimes b \ \underset{\text{by (1)}}{\uparrow} \ \sum_{i=1}^{m} a \otimes b \ = \ m \cdot (a \otimes b) \ \underset{\text{same argument backwards}}{\uparrow} \ \ldots \ = \ a \otimes (m \cdot b).$$

It follows easily from (1) that $0 \otimes b = a \otimes 0 = 0$ and that $(-a) \otimes b = a \otimes (-b) = -a \otimes b$. Using these observations we get Statement (2) for $m = 0$ and we can easily reduce the case $m < 0$ of (2) to the case $m > 0$ of (2).

(3) It follows immediately from the definition of the tensor product, together with the notational convention of "formal linear combinations" that we introduced on page 1147 that any element in $A \otimes B$ is of the form $\sum_{i=1}^{n} r_i \cdot (a_i \otimes b_i)$. But by (2) we know that $r_i \cdot (a_i \otimes b_i) = (r_i \cdot a_i) \otimes b_i$ thus we get the desired result.

(4) Let $n \in \mathbb{N}$ and let $c \in A \otimes \mathbb{Z}_n$. By (3) we can write $c = \sum_{i=1}^{m} a_i \otimes r_i$ with $a_1, \ldots, a_m \in A$ and $r_1, \ldots, r_m \in \mathbb{Z}_n$. Then

$$n \cdot c \;=\; n \cdot \Big(\sum_{i=1}^{m} a_i \otimes r_i \Big) \;\underset{\uparrow}{=}\; \sum_{i=1}^{m} a_i \otimes \underbrace{n \cdot r_i}_{=0} \;=\; 0.$$

by Statement (2)

■

Definition. Let A, B and C be abelian groups. We say that a map $f\colon A \times B \to C$ is bilinear, if for all $a, a' \in A$, $b, b' \in B$ the following equalities hold:

$$f(a + a', b) \;=\; f(a, b) + f(a', b) \quad \text{and} \quad f(a, b + b') \;=\; f(a, b) + f(a, b').$$

We leave it to the reader to provide the elementary proof of the following lemma.

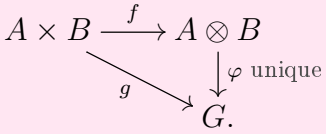
Lemma 89.2. Let A and B be abelian groups.

(1) The map

$$\begin{array}{rcl} f\colon A \times B & \to & A \otimes B \\ (a, b) & \mapsto & a \otimes b \end{array}$$

is bilinear.

(2) Given any abelian group G and any bilinear map $g\colon A \times B \to G$ there exists a unique homomorphism $\varphi\colon A \otimes B \to G$ such that the following diagram commutes:



(3) Given any abelian group G the map

$$\begin{array}{rcl} \mathrm{Hom}(A \otimes B, G) & \to & \text{set of bilinear maps } A \times B \to G \\ (\varphi\colon A \otimes B \to G) & \mapsto & (A \times B \xrightarrow{f} A \otimes B \xrightarrow{\varphi} G) \end{array}$$

is a bijection.

Remark. One can show quite easily that if a homomorphism $g\colon A \times B \to H$ is bilinear and if this homomorphism satisfies the universal property of Lemma 89.2 (2), then there exists an isomorphism $H \xrightarrow{\cong} A \otimes B$. We will not make use of this observation.

Before we state a few more properties of the tensor product it is convenient to introduce the following definition.

Definition. Let $f\colon A \to A'$ and $g\colon B \to B'$ be homomorphisms between abelian groups. We define $f \otimes g$ to be the homomorphism that is given by[1079]

$$\begin{array}{rcl} f \otimes g\colon A \otimes B & \to & A' \otimes B' \\ \sum_{i=1}^{n} a_i \otimes b_i & \mapsto & \sum_{i=1}^{n} f(a_i) \otimes g(b_i) \end{array}$$

[1079] It is straightforward to verify that the map $f \otimes g$ is indeed well-defined and that it is a homomorphism.

Now we can continue with the promised lemma on properties of the tensor product. On many occasions we will make use of it without explicitly referring to it.

Lemma 89.3. (Tensor Lemma) Let A, B, C and $A_i, i \in I$ be abelian groups.

(1) The maps
$$\begin{array}{rcl} A \otimes B & \to & B \otimes A \\ \sum_{i=1}^{n} a_i \otimes b_i & \mapsto & \sum_{i=1}^{n} b_i \otimes a_i \end{array} \quad \text{and} \quad \begin{array}{rcl} \left(\bigoplus_{i\in I} A_i\right) \otimes B & \to & \bigoplus_{i\in I} A_i \otimes B \\ \left(\sum_{i=1}^{n} a_i\right) \otimes b & \mapsto & \sum_{i=1}^{n} a_i \otimes b \end{array}$$
are well-defined and they are natural isomorphisms.

(2) The map
$$\begin{array}{rcl} (A \otimes B) \otimes C & \to & A \otimes (B \otimes C) \\ \sum_{j=1}^{n} (a_j \otimes b_j) \otimes c_j & \mapsto & \sum_{j=1}^{n} a_j \otimes (b_j \otimes c_j) \end{array}$$
is well-defined and it is a natural isomorphism.

(3) Let $k \in \mathbb{N}$. The maps[1080]
$$\begin{array}{rcl} \mathbb{Z}_k \otimes A & \to & A/kA \\ \sum_{i=1}^{m} n_i \otimes a_i & \mapsto & \sum_{i=1}^{m} n_i \cdot a_i \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{Z} \otimes A & \to & A \\ \sum_{i=1}^{m} n_i \otimes a_i & \mapsto & \sum_{i=1}^{m} n_i \cdot a_i. \end{array}$$
are well-defined and they are natural isomorphisms.

(4) Suppose $(I, \leq)$ is an ordered set and $(\{A_i\}_{i\in I}, \{\varphi_{ij}\}_{i\leq j})$ is a direct system of abelian groups. Then the map
$$\varinjlim (A_i \otimes B) \xrightarrow{\cong} \left(\varinjlim A_i\right) \otimes B$$
that is induced by the maps $A_i \to \varinjlim A_i$ is a natural isomorphism.

Proof. We refer to [**Lang1993**, Corollary XVI.2.2] for a proof for the distributivity in (1). The other statements in (1) to (3) follow fairly easily from the definitions and Lemma 89.1. We refer to [**Mun1984**, Chapter 50] for details. A proof of the last statement is given in [**Bou2007**, Chapter II.6.3] or alternatively in [**Mats1989**, Theorem A1]. ■

Convention. Given several abelian groups $A_1, \ldots, A_k$ we use the Tensor Lemma 89.3 as an excuse to write $A_1 \otimes A_2 \otimes \cdots \otimes A_k$, even though strictly speaking as a set this object does depend on the parenthesis.

Remark.

(1) As we mentioned in the statement of the Tensor Lemma 89.3, all the isomorphisms are natural, in the sense that they define natural transformations between appropriate functors. For example if we denote by $\mathcal{AbGroup}$ the category of abelian groups, then the isomorphisms of the Tensor Lemma 89.3 (3) define a natural transformation between the two functors
$$\begin{array}{rcl} \mathcal{AbGroup} & \to & \mathcal{AbGroup} \\ A & \mapsto & A \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathcal{AbGroup} & \to & \mathcal{AbGroup} \\ A & \mapsto & \mathbb{Z} \otimes A. \end{array}$$
As an example, for every $k \in \mathbb{N}$ and any abelian group A we get a commutative diagram

[1080] Note that for an abelian group A and $k \in \mathbb{Z}$ the set $kA := \{k \cdot a \,|\, a \in A\}$ is a subgroup of A and we can thus form the quotient A/kA.

$$\begin{array}{ccc} \mathbb{Z}\otimes A & \xrightarrow[n\otimes a\mapsto n\cdot a]{\cong} & A \\ {\scriptstyle \cdot k\otimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \cdot k} \\ \mathbb{Z}\otimes A & \xrightarrow[n\otimes a\mapsto n\cdot a]{\cong} & A. \end{array}$$

(2) We use these natural isomorphisms from the Tensor Lemma 89.3 to identify the various groups. Note that the Tensor Lemma 89.3 (2) allows us to write the tensor product $A\otimes B\otimes C$ unambiguously without having to worry about parentheses.

(3) In the Tensor Lemma 89.3 we saw that the tensor product commutes with forming direct sums. The analogous statement is not true if we replace direct sums by direct products. More precisely, in Exercise 89.11 we will see that

$$\Big(\prod_{n\in\mathbb{N}}\mathbb{Z}_n\Big)\otimes\mathbb{Q}\neq 0 \quad \text{but} \quad \prod_{n\in\mathbb{N}}\underbrace{(\mathbb{Z}_n\otimes\mathbb{Q})}_{=0,\text{ by the Tensor Lemma 89.3 (3)}} = 0.$$

Examples.

(1) Let $m,n\in\mathbb{N}$. We write $r=\gcd(m,n)$ and $s=m/r$. Then

$$\mathbb{Z}_m\otimes\mathbb{Z}_n \underset{\text{Tensor Lemma 89.3 (3)}}{\overset{\uparrow}{\cong}} \mathbb{Z}_n/m\mathbb{Z}_n = \mathbb{Z}_n/rs\mathbb{Z}_n = \mathbb{Z}_n/r\cdot(s\mathbb{Z}_n) \underset{\text{since } s \text{ is coprime to } n}{\overset{\uparrow}{=}} \mathbb{Z}_n/r\mathbb{Z}_n \xleftarrow[\cong]{\cdot n/r} Z_r = \mathbb{Z}_{\gcd(m,n)}.$$

(2) Let A be a finitely generated abelian group of rank r. By the Finitely Generated Abelian Group Classification Theorem 51.4 we have $A\cong\mathbb{Z}^r\oplus\bigoplus_{i=1}^n\mathbb{Z}_{a_i}$ where $a_1,\dots,a_n$ are non-zero natural numbers. For $S=\mathbb{Q}, S=\mathbb{R}$ or $S=\mathbb{C}$ we have

$$A\otimes S\cong\Big(\mathbb{Z}^r\oplus\bigoplus_{i=1}^n\mathbb{Z}_{a_i}\Big)\otimes S \underset{\text{Tensor Lemma 89.3 (1)}}{\overset{\uparrow}{=}} (\mathbb{Z}\otimes S)^r\oplus\Big(\bigoplus_{i=1}^n\mathbb{Z}_{a_i}\otimes S\Big) \underset{\text{Tensor Lemma 89.3 (3)}}{\overset{\uparrow}{\cong}} S^r\oplus\bigoplus_{i=1}^n S/a_iS \underset{\text{since } a_i\neq 0 \text{ in } S=\mathbb{Q},\mathbb{R},\mathbb{C}}{\overset{\uparrow}{=}} S^r.$$

(3) Let A be an abelian group, then

$$\ker\begin{pmatrix} A & \to & A\otimes\mathbb{Q} \\ a & \mapsto & a\otimes 1 \end{pmatrix} = \text{all torsion elements of } A.$$

If A is finitely generated, then this follows easily from the classification of finitely generated abelian groups together with the Tensor Lemma 89.3. If A is infinitely generated, then we can view it as the direct limit of finitely generated abelian groups and using the Tensor Lemma 89.3 the statement can be reduced to the finitely generated case. We leave the details to the reader.

(4) Recall that given a set W we denote by $\mathbb{Z}^{(W)}$ the free abelian group generated by W. Let A and B be two sets. We then have

$$\mathbb{Z}^{(A)}\otimes\mathbb{Z}^{(B)} = \Big(\bigoplus_{a\in A}\mathbb{Z}\cdot a\Big)\otimes\Big(\bigoplus_{b\in B}\mathbb{Z}\cdot b\Big) \underset{\text{Tensor Lemma 89.3 (1)}}{\overset{\downarrow}{=}} \bigoplus_{a\in A,b\in B}\mathbb{Z}\cdot a\otimes\mathbb{Z}\cdot b \underset{\text{Tensor Lemma 89.3 (3)}}{\overset{\downarrow}{=}} \bigoplus_{a\in A,b\in B}\mathbb{Z}\cdot(a,b) = \mathbb{Z}^{(A\times B)}.$$

(5) Let A and B be two finitely generated abelian groups. Using examples (1) and (2), the classification of finitely generated abelian groups and the Tensor Lemma 89.3 (1) and (4) one can now easily show that

$$\operatorname{rank}(A\otimes B) = \operatorname{rank}(A)\cdot\operatorname{rank}(B).$$

For future reference we record the following lemma:

Lemma 89.4. Let $f\colon A \to A'$ and $f'\colon A' \to A''$ be homomorphisms and let $g\colon B \to B'$ and $g'\colon B' \to B''$ be homomorphisms between abelian groups. Then we have

$$(f' \otimes g') \circ (f \otimes g) \;=\; (f' \circ f) \otimes (g' \circ g).$$

Proof. The statement follows immediately from the definitions. ■

The following elementary lemma will play a major role.

Lemma 89.5. (Tensor-with-Ring Lemma) Let R be a commutative ring.

(1) Given any abelian group A the map

$$\begin{aligned} (A \otimes R) \times R &\to A \otimes R \\ \left(\sum_{i=1}^{n} a_i \otimes b_i, r\right) &\mapsto \sum_{i=1}^{n} a_i \otimes b_i r \end{aligned}$$

defines an R-module structure on $A \otimes R$.

(2) If $f\colon A \to B$ is a homomorphism of abelian groups, then the map

$$\begin{aligned} f \otimes \mathrm{id}\colon A \otimes R &\to B \otimes R \\ \sum_i a_i \otimes r_i &\mapsto \sum_i f(a_i) \otimes r_i \end{aligned}$$

is a homomorphism of R-modules.

(3) The map

$$\begin{aligned} A &\mapsto A \otimes R \\ (f\colon A \to B) &\mapsto (f \otimes \mathrm{id}\colon A \otimes R \to B \otimes R) \end{aligned}$$

defines a functor from the category of abelian groups to the category of R-modules.

Proof. Once again the statements follow immediately from the definitions. ■

The Tensor-with-Ring Lemma 89.5 shows that tensoring with a commutative ring "turns" an abelian group into a module. In particular if we tensor with a field, we obtain a vector space. This is very convenient since sometimes it allows us to reduce problems about abelian groups into problems about vector spaces that we can handle with techniques from linear algebra.

For example we can now give a meaningful definition of the rank of any abelian group.

Definition. The rank of an abelian group A is defined as

$$\mathrm{rank}(A) \;:=\; \dim_{\mathbb{Q}}(A \otimes \mathbb{Q}).$$

Remark.

(1) Previously, on page 1152 we introduced the rank of finitely generated abelian groups. It follows from the discussion on page 1910 that for finitely generated abelian groups the two definitions of rank agree. Put differently, the above definition is an extension of the earlier notion of rank to all abelian groups.

(2) Let A be an abelian group. We can view A as a $\mathbb{Z}$-module and the above definition of the rank of the abelian group equals the rank of the corresponding $\mathbb{Z}$-module as introduced on page 99.

We conclude this section with the following lemma.

Lemma 89.6. (Tensor-Split-SES Lemma) Let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence and let G be an abelian group. If the short exact sequence splits, then

$$0 \to A \otimes G \xrightarrow{i\otimes \mathrm{id}} B \otimes G \xrightarrow{p\otimes \mathrm{id}} C \otimes G \to 0$$

is also a short exact sequence and it also splits.

Remark. In most applications we will consider a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

where C is a free abelian group. It then follows from Lemma 78.1 that the short exact sequence splits, which then implies by the Tensor-Split-SES Lemma 89.6 that for any abelian group G the sequence

$$0 \to A \otimes G \xrightarrow{i\otimes \mathrm{id}} B \otimes G \xrightarrow{p\otimes \mathrm{id}} C \otimes G \to 0$$

is also a short exact sequence which furthermore splits.

Proof. According to Splitting Lemma 78.2 we can without loss of generality suppose that the short exact sequence is of the form

$$0 \to A \xrightarrow{a\mapsto(a,0)} A \oplus C \xrightarrow{(a,c)\mapsto c} C \to 0.$$

If we tensor this short exact sequence with G, then using $(A \oplus C) \otimes G = A \otimes G \oplus C \otimes G$ we obtain the sequence

$$0 \to A \otimes G \xrightarrow{\sum_i a_i\otimes g_i \mapsto \left(\sum_i a_i\otimes g_i, 0\right)} A \otimes G \oplus C \otimes G \xrightarrow{\left(\sum_i a_i\otimes g_i, \sum_j c_j\otimes g_j\right)\mapsto \sum_j c_j\otimes g_j} C \otimes G \to 0,$$

which is evidently exact. It is also clear that the short exact sequence splits. ∎

89.2. The homology groups of a topological space with coefficients.

Definition. Let

$$\ldots \xrightarrow{\partial_{n+1}} \mathrm{C}_n \xrightarrow{\partial_n} \mathrm{C}_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} \mathrm{C}_1 \xrightarrow{\partial_1} \mathrm{C}_0 \to 0$$

be an algebraic chain complex.

(1) Let G be an abelian group. We tensor the chain complex with G, i.e. we consider the following sequence of homomorphisms:[1081]

$$\ldots \to \mathrm{C}_n \otimes G \xrightarrow{\partial_n\otimes \mathrm{id}} \mathrm{C}_{n-1} \otimes G \xrightarrow{\partial_{n-1}\otimes \mathrm{id}} \ldots \xrightarrow{\partial_2\otimes \mathrm{id}} \mathrm{C}_1 \otimes G \xrightarrow{\partial_1\otimes \mathrm{id}} \mathrm{C}_0 \otimes G \to 0.$$

For all i we have

$$(\partial_{i-1} \otimes \mathrm{id}) \circ (\partial_i \otimes \mathrm{id}) \underset{\text{Lemma 89.4}}{=} (\partial_{i-1} \circ \partial_i) \otimes \mathrm{id} = 0 \otimes \mathrm{id} = 0,$$

i.e. the above sequence of maps is again a chain complex. We denote the corresponding homology groups by $\mathrm{H}_n(C;G) := \mathrm{H}_n(C_* \otimes G, \partial_* \otimes \mathrm{id})$.

(2) If we tensor the chain complex C_* with a commutative ring R, then it follows from the discussion in the previous section that we obtain a chain complex of R-modules, i.e. all chain groups are R-modules and all chain maps are R-homomorphisms. In particular also all homology groups are R-modules.

Remark. Most of the results which we have proved for chain complexes of abelian groups hold, with the obvious modifications, also for chain complexes of R-modules. In particular one can easily convince oneself that Lemma 72.7, Lemma 73.2, Corollary 73.4 and Propositions 74.9 and the Connecting Homomorphism Lemma 74.8 also hold in this more general context. In the remainder of the course we will use these generalizations without further comment.

Now we are ready to define the homology groups of a topological space with coefficients in an abelian group G.

Definition. Let X be a topological space. For an abelian group G we define

$$\mathrm{C}_*(X;G) \quad := \quad \mathrm{C}_*(X) \otimes G.$$

As discussed above, we obtain a chain complex and we denote by

$$\mathrm{H}_k(X;G) \quad := \quad \mathrm{H}_k(\mathrm{C}_*(X;G), \partial \otimes \mathrm{id})$$

the corresponding homology groups. We call $\mathrm{H}_k(X;G)$ the k-th homology group of X with G-coefficients. We define the notions of "cycle", "homologous" and "null-homologous" in exactly the same way as on page 1587.

Example. We consider an electric circuit as a finite undirected abstract graph X where at the vertices we have resistors or capacitors. Kirchhoff's current rule, see e.g.

`https://en.wikipedia.org/wiki/Kirchhoff's_circuit_laws`

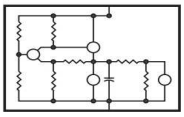
states that

> *at any node (vertex) in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node,*

or equivalently

> *the algebraic sum of currents in a network meeting at a point is zero.*

Put differently, a current defines a 1-cycle in $\mathrm{C}_1(X;\mathbb{R})$.[1082]

electric circuit 

We start out with the following three observations. Let X be a topological space.

(1) If the coefficients are given by a commutative ring R, e.g. if $R = \mathbb{Z}_n$ or $R = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, then, as we pointed out above, each $\mathrm{C}_n(X;R) = \mathrm{C}_n(X) \otimes R$ is an R-module, each boundary map is an R-module homomorphism, and each homology group $\mathrm{H}_n(X;R)$ is also an R-module.

(2) From the Tensor Lemma 89.3 (3) it follows immediately that we have natural isomorphisms $\mathrm{C}_*(X;\mathbb{Z}) \xrightarrow{\cong} \mathrm{C}_*(X)$ and $\mathrm{H}_*(X;\mathbb{Z}) \xrightarrow{\cong} \mathrm{H}_*(X)$. In the following we will identify these groups and we will go back and forth between these two notations, depending on which one is more convenient.

[1081] Note that if G has also the structure of a commutative ring, then this becomes a chain complex of G-modules.

[1082] This observation and the figure are due to Ghrist's excellent book on applied topology [**Gh2014**].

(3) Let G be an abelian group. Given any singular k-simplex σ, given any $a \in \mathbb{Z}$ and any $g \in G$ we have by definition of the tensor product that $a\sigma \otimes g = \sigma \otimes ag$. From this observation it follows that any element in $\mathrm{C}_k(X;G)$ can be written as

$$\sum_{i=1}^{n} \sigma_i \otimes g_i \quad \text{where } \sigma_1, \ldots, \sigma_n \text{ are singular } k\text{-simplices and } g_1, \ldots, g_n \in G.$$

Most results for homology groups of topological spaces carry over, with the obvious modifications, to the theory of homology groups with G-coefficients. For example, given an abelian group G we have the following statements:

(1) A map $f\colon X \to Y$ between two topological spaces X and Y induces a chain map

$$f_* \otimes \mathrm{id}\colon \mathrm{C}_*(X;G) \to \mathrm{C}_*(Y;G)$$

which in turn, for each $k \in \mathbb{N}_0$, induces a homomorphism

$$(f_* \otimes \mathrm{id})_*\colon \mathrm{H}_k(X;G) \to \mathrm{H}_k(Y;G).$$

To simplify the notation we usually denote this induced map on homology groups also by f_*.

(2) The k-th homology group with G-coefficients defines a functor from the category of topological spaces to the category of abelian groups.

(3) If R is a commutative ring, then a map $f\colon X \to Y$ between topological spaces induces a chain map of R-homomorphisms $\mathrm{C}_*(X;R) \to \mathrm{C}_*(Y;R)$ and R-homomorphisms $f_*\colon \mathrm{H}_k(X;R) \to \mathrm{H}_k(Y;R)$. Furthermore the k-th homology group with R-coefficients defines a functor from the category $\mathcal{Top}$ of topological spaces to the category $R-\mathcal{Mod}$ of R-modules.

(4) The augmentation map $\epsilon\colon \mathrm{H}_0(X;G) \to G$ is defined almost the same way as in Lemma 72.4. More precisely, it is given by the map

$$\begin{array}{rcl} \epsilon_X\colon \mathrm{H}_0(X;G) & \to & G \\ \left[\sum_{i=1}^{k} \sigma_i \otimes g_i\right] & \mapsto & \sum_{i=1}^{k} g_i. \end{array}$$

If X is path-connected, then the argument of the H_0-Proposition 72.5 implies immediately that the augmentation map $\epsilon\colon \mathrm{H}_0(X;G) \to G$ is an isomorphism. At times we will use this natural isomorphism to make the identification $\mathrm{H}_0(X;G) = G$.

(5) Precisely as on page 1611 we can use the augmentation map $\epsilon\colon \mathrm{H}_0(X;G) \to G$ to define reduced homology groups $\widetilde{\mathrm{H}}_k(X;G)$ with G-coefficients.

(6) Recall that we saw in the Homotopic Maps-and-Homology Proposition 73.6 that if $f, g\colon X \to Y$ are two homotopic maps between topological spaces, then the induced maps $f_*, g_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ are chain homotopic. By tensoring the chain homotopies we see that the corresponding induced maps $\mathrm{C}_*(X) \otimes G \to \mathrm{C}_*(Y) \otimes G$ are also chain homotopic, and we see that for any $n \in \mathbb{N}_0$ we have

$$f_* = g_*\colon \mathrm{H}_n(X;G) \ \to \ \mathrm{H}_n(Y;G).$$

In particular, if X and Y are homotopy equivalent topological spaces, then for any $n \in \mathbb{N}_0$ we have an isomorphism

$$\mathrm{H}_n(X;G) \ \xrightarrow{\cong} \ \mathrm{H}_n(Y;G).$$

(7) Given a pair (X, A) of topological spaces we can tensor the relative chain complex $\mathrm{C}_*(X, A)$ with G. We obtain the *relative chain complex* $\mathrm{C}_*(X, A; G) := \mathrm{C}_*(X, A) \otimes G$. The corresponding homology groups are the *relative homology groups* $\mathrm{H}_k(X, A; G)$.

(8) Let (X, B, A) be a triple of topological spaces. As pointed out on page 1625, the relative chain groups $\mathrm{C}_n(X, B; G)$ are free abelian groups. Thus it follows from the discussion on page 1912 that we obtain a short exact sequence of chain complexes $0 \to \mathrm{C}_*(B, A; G) \to \mathrm{C}_*(X, A; G) \to \mathrm{C}_*(X, B; G) \to 0$. By applying the obvious generalization of the LES Proposition 74.9 to this short exact sequence we obtain the following long exact sequence of homology groups with G-coefficients:

$$\ldots \to \mathrm{H}_n(B, A; G) \xrightarrow{i_*} \mathrm{H}_n(X, A; R) \xrightarrow{p_*} \mathrm{H}_n(X, B; G) \xrightarrow{\partial} \mathrm{H}_{n-1}(B, A; G) \to \ldots$$

(9) We have an obvious version of the Mayer–Vietoris Theorem 78.5 for homology groups with G-coefficients. The proof is basically identical to the proof of Theorem 78.5, we only have to note that when we consider the exact sequence

$$0 \to \mathrm{C}_*(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} \mathrm{C}_*(A) \oplus \mathrm{C}_*(B) \xrightarrow{i_A + i_B} \mathrm{C}_*^{\{A,B\}}(X) \to 0,$$

then it is a consequence of the discussion on page 1912 that the corresponding sequence with G-coefficients is also exact.

(10) We also have an obvious versions of the Excision Theorems 74.17 and 74.18, the proof for G-coefficients is almost verbatim the same as before.

(11) For a CW-complex X we can also consider the corresponding cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(X; G) := \mathrm{C}_*^{\mathrm{CW}}(X) \otimes G$ and the corresponding cellular homology groups $\mathrm{H}_*^{\mathrm{CW}}(X; G)$. Then, as in the Singular-Cellular H_*-Isomorphism Proposition 80.4, one can show that there exists a natural isomorphism $\mathrm{H}_*(X; G) \xrightarrow{\cong} \mathrm{H}_*^{\mathrm{CW}}(X; G)$. We can draw the following conclusions:
   (a) If X is an n-dimensional CW-complex, then $\mathrm{H}_i(X; G) = 0$ for $i \geq n + 1$.
   (b) If X has only finitely many n-cells, then given any field $\mathbb{F}$ the vector space $\mathrm{H}_n(X; \mathbb{F})$ is finite-dimensional.

(12) Let X be a topological space and let $X_1 \subset X_2 \subset X_3 \subset \ldots$ be a sequence of subsets such that $X = \bigcup_{i\in\mathbb{N}} X_i$ and such that one the following three conditions holds:
   (a) every compact subset of X is already contained in one of the X_i, or
   (b) each X_i is open in X, or
   (c) X is a CW-complex and each X_i is a subcomplex.

   Then a modest modification of the proof of the Exhaustions-H_*-Proposition 79.4 shows that for any $n \in \mathbb{N}_0$ the inclusion induced maps $\mathrm{H}_n(X_i; G) \to \mathrm{H}_n(X; G)$ induce an isomorphism

$$\varinjlim \mathrm{H}_n(X_i; G) \xrightarrow{\cong} \mathrm{H}_n(X; G).$$

(13) If X is a finite CW-complex and if $\mathbb{F}$ is a field, then the argument of the Euler Characteristic-H_*-Proposition 87.1 shows that

$$\chi(X) = \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(X; \mathbb{F})).$$

Examples.

(1) Let $n \in \mathbb{N}$. As on page 1472 we equip S^n with the CW-structure with one 0-cell and one n-cell. For any abelian group G it follows from (10), and similar to the discussion on page 1770, that

$$\mathrm{H}_k(S^n;G) \;\cong\; \mathrm{H}_k^{\mathrm{CW}}(S^n;G) \;\cong\; \begin{cases} G, & \text{if } k = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

The same result holds for reduced homology with G-coefficients except that in this case we have $\widetilde{\mathrm{H}}_0(S^n;G) = 0$.

(2) We want to determine homology groups of the real projective plane $\mathbb{R}\mathrm{P}^2$ with $\mathbb{F}_2$-coefficients. As usual we equip $\mathbb{R}\mathrm{P}^2$ with the CW-structure with one 0-cell, one 1-cell and one 2-cell that we introduced on page 1785. In the proof of Proposition 80.11 we saw that the cellular chain complex is given by

$$0 \;\to\; \mathbb{Z} \;\xrightarrow{\cdot 2}\; \mathbb{Z} \;\xrightarrow{0}\; \mathbb{Z} \;\to\; 0,$$

i.e. $\mathrm{H}_2(\mathbb{R}\mathrm{P}^2) = 0$, $\mathrm{H}_1(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}_2$ and $\mathrm{H}_0(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}$. On the other hand, if we tensor the above cellular chain complex with $\mathbb{F}_2$, then using the Tensor Lemma 89.3 (3) we obtain the chain complex

$$0 \;\to\; \mathbb{F}_2 \;\xrightarrow{\cdot 2 = 0}\; \mathbb{F}_2 \;\xrightarrow{0}\; \mathbb{F}_2 \;\to\; 0.$$

Using the isomorphism between cellular and singular homology with $\mathbb{F}_2$-coefficients we obtain that $\mathrm{H}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) = \mathbb{F}_2$, $\mathrm{H}_1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) = \mathbb{F}_2$ and $\mathrm{H}_0(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) = \mathbb{F}_2$.

Note that even though the homology groups of $\mathbb{R}\mathrm{P}^2$ with $\mathbb{Z}$-coefficients and $\mathbb{F}_2$-coefficients are quite different, in both cases the alternating sums of the dimensions and ranks give, as promised in the Euler Characteristic-H_*-Proposition 87.1 and in (10) above, the Euler characteristic. Indeed, we have

$$\mathrm{rank}(\mathrm{H}_0(X)) - \mathrm{rank}(\mathrm{H}_1(X)) + \mathrm{rank}(\mathrm{H}_2(X)) \;=\; 1 - 0 + 0 \;=\; 1,$$

and

$$\dim_{\mathbb{F}_2}(\mathrm{H}_0(X;\mathbb{F}_2)) - \dim_{\mathbb{F}_2}(\mathrm{H}_1(X;\mathbb{F}_2)) + \dim_{\mathbb{F}_2}(\mathrm{H}_2(X;\mathbb{F}_2)) \;=\; 1 - 1 + 1 \;=\; 1.$$

(3) Basically the same argument as in (3) shows that for any $n \in \mathbb{N}$ we have

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } k = 0, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, it is not difficult to see that for $m < n$ the inclusion induced maps $\mathrm{H}_k(\mathbb{R}\mathrm{P}^m;\mathbb{F}_2) \to \mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ are isomorphisms for $k = 0, \ldots, m$. It follows that for any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2) \;\underset{\substack{\uparrow \\ \text{see page 1915}}}{=}\; \varinjlim \mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \;\underset{\substack{\uparrow \\ \text{by the above}}}{=}\; \mathbb{F}_2.$$

Remark. Why should we study the homology groups of a topological space with G-coefficients? There are several answers one can give:

(1) At times it is easier to work over a field instead of working over the ring $\mathbb{Z}$. For example, if we only want to determine the Euler characteristic of a CW-complex we can now use a field, even a very simple one like $\mathbb{F}_2$, instead of working with the integers.

(2) If we work over the field $\mathbb{F}_2$ then many calculations become particularly simple, since we do not need to worry about signs. Even in the previous example we already saw that working with $\mathbb{F}_2$-coefficients can simplify calculations.
(3) Recall that given a smooth manifold M and $k \in \mathbb{N}_0$ we denote its de Rham cohomology groups $\mathrm{H}^k_{\mathrm{dR}}(M)$, which is in fact a real vector space. For each k we can now also study the homology groups $\mathrm{H}_k(M;\mathbb{R})$ which are also real vector spaces. The notations $\mathrm{H}^k_{\mathrm{dR}}(M)$ and $\mathrm{H}_k(M;\mathbb{R})$ look similar and the names "de Rham cohomology" and "homology" also sound similar. We will study the relationship between these two objects later on in Chapters 110 and 118.

We conclude this section with the following elementary lemma.

Lemma 89.7.

(1) Let X be a topological space and let R be a commutative ring with neutral element 1_R. The maps
$$\begin{array}{rcl} \varphi_*\colon \mathrm{C}_*(X) & \to & \mathrm{C}_*(X;R) \\ \left[\sum_{i=1}^k a_i \cdot \sigma_i\right] & \mapsto & \left[\sum_{i=1}^k a_i \cdot \sigma_i \otimes 1_R\right] \end{array}$$
form a chain map, in particular for each $n \in \mathbb{N}_0$ we obtain an induced map
$$\varphi_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(X;R).$$
(2) Let X be a topological space and let $\varphi\colon A \to B$ be a homomorphism of abelian groups. Then the maps
$$\begin{array}{rcl} \varphi_*\colon \mathrm{C}_*(X;A) & \to & \mathrm{C}_*(X;B) \\ \left[\sum_{i=1}^k \sigma_i \otimes a_i\right] & \mapsto & \left[\sum_{i=1}^k \sigma_i \otimes \varphi(a_i)\right] \end{array}$$
form a chain map, in particular for each $n \in \mathbb{N}_0$ we obtain an induced map
$$\varphi_*\colon \mathrm{H}_n(X;A) \to \mathrm{H}_n(X;B).$$
(3) Given a topological space X and $n \in \mathbb{N}_0$ the maps $A \mapsto \mathrm{H}_n(X;A)$ from (1) define a covariant functor from the category of abelian groups *AbGroup* to itself.
(4) The maps from (1) define a natural transformation from "homology with A-coefficients" to "homology with B-coefficients", i.e. given a map $f\colon X \to Y$ of topological spaces we get a commutative diagram
$$\begin{array}{ccc} \mathrm{H}_n(X;A) & \xrightarrow{f_*} & \mathrm{H}_n(Y;A) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \mathrm{H}_n(X;B) & \xrightarrow{f_*} & \mathrm{H}_n(Y;B). \end{array}$$

The above statements also generalize in an obvious way to pairs of topological spaces.

Proof. The lemma follows immediately from the definitions. ∎

89.3. **Exact functors.** The following question naturally arises from the discussion in the previous section:

Question 89.8. Given a topological space X and an abelian group G, what is the relationship between the usual homology groups of X and the homology groups with G-coefficients?

The example on page 1916 shows that for better or worse the naive guess that perhaps $\mathrm{H}_k(X;G) \cong \mathrm{H}_k(X) \otimes G$ is in general not correct. More precisely, we saw that

$$\mathbb{F}_2 = \mathrm{H}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \;\neq\; \mathrm{H}_2(\mathbb{R}\mathrm{P}^2) \otimes \mathbb{F}_2 = 0.$$

In the following section we will see that, despite this sobering example, one can always deduce the homology groups $\mathrm{H}_*(X;G)$ of a topological space X from the usual homology groups $\mathrm{H}_*(X)$. But the connection between these homology groups will turn out to be less naive than one might have thought initially.
Before we can hope to answer Question 89.8 we have to introduce many more definitions and algebraic tools. We start out with the following definition from category theory.

Definition. Let $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ be a covariant functor from the category $\mathcal{AbGroup}$ of abelian groups to itself.

(1) We say F is left-exact, if for every exact sequence
$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C$$
of abelian groups the sequence
$$0 \longrightarrow F(A) \xrightarrow{F(i)} F(B) \xrightarrow{F(j)} F(C)$$
is also exact.

(2) We say F is right-exact, if for every exact sequence
$$A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$
of abelian groups the sequence
$$F(A) \xrightarrow{F(i)} F(B) \xrightarrow{F(j)} F(C) \longrightarrow 0$$
is also exact.

(3) We say F is exact if it is left-exact and right-exact, put differently, the functor F is exact, if for every exact sequence
$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$
of abelian groups the sequence
$$0 \longrightarrow F(A) \xrightarrow{F(i)} F(B) \xrightarrow{F(j)} F(C) \longrightarrow 0$$
is also exact.

Remark. Let $\varphi\colon A \to B$ be a homomorphism between two abelian groups. Furthermore let $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ be a covariant functor from the category $\mathcal{AbGroup}$ of abelian groups to itself.

(1) If F is a left-exact functor, then "taking F" commutes with "taking kernels", more precisely, if $\varphi\colon A \to B$ is a homomorphism, then by applying F to the exact sequence $0 \to \ker(\varphi) \to A \to B$ we see that $F(\ker(\varphi)) = \ker(\varphi_*\colon F(A) \to F(B))$.

(2) Similarly, if F is right-exact, then "taking F" commutes with "taking cokernels", i.e. if $\varphi\colon A \to B$ is a homomorphism, then by applying F to the exact sequence $A \to B \to \operatorname{coker}(\varphi) \to 0$ we see that $F(\operatorname{coker}(\varphi)) = \operatorname{coker}(\varphi_*\colon F(A) \to F(B))$.

Lemma 89.9. Let G be an abelian group. If G is free abelian, then tensoring with G is exact.

Proof. Let
$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$
be a short exact sequence and let $G \cong \mathbb{Z}^{(S)}$ be a free abelian group. It follows from the Tensor Lemma 89.3 (1) and (3) that tensoring the above short exact sequence with $G \cong \mathbb{Z}^{(S)}$ results in an exact sequence that is isomorphic to the direct sum of S copies of the original sequence, hence it is again exact. ■

Example. In contrast to the previous lemma we will now see that tensoring with an abelian group G is in general not a *left-exact* functor. For example for any $k \in \mathbb{N}_{\geq 2}$ the sequence
$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot k} \mathbb{Z} \longrightarrow \mathbb{Z}_k$$
is exact, but after tensoring with $\mathbb{Z}_k$ we obtain, using the Tensor Lemma 89.3 (3), the sequence
$$0 \longrightarrow \mathbb{Z}_k \xrightarrow[=0]{\cdot k} \mathbb{Z}_k \longrightarrow \mathbb{Z}_k$$
that is not exact.

In contrast to the previous example we will now see that tensoring with an abelian group G is a *right-exact* functor.

Lemma 89.10. As usual we denote by $\mathcal{AbGroup}$ the category of abelian groups. For every abelian group G the functor
$$\begin{array}{ccc} \mathcal{AbGroup} & \to & \mathcal{AbGroup} \\ A & \mapsto & A \otimes G \end{array}$$
is right-exact.

Proof. Let
$$A \xrightarrow{i} B \xrightarrow{j} C \to 0$$
be an exact sequence of abelian groups and let G be an abelian group. We have to show that the sequence
$$A \otimes G \xrightarrow{i\otimes \mathrm{id}} B \otimes G \xrightarrow{j\otimes \mathrm{id}} C \otimes G \to 0$$
is also exact.

(a) Since $j\colon B \to C$ is an epimorphism it follows immediately from Lemma 89.1 (3) that the homomorphism
$$\begin{array}{rcl} j \otimes \mathrm{id}\colon B \otimes G & \to & C \otimes G \\ \sum_{i=1}^{n} b_i \otimes g_i & \mapsto & \sum_{i=1}^{n} j(b_i) \otimes g_i \end{array}$$
is also an epimorphism.

(b) By Lemma 89.4 we have $(j \otimes \mathrm{id}) \circ (i \otimes \mathrm{id}) = (j \circ i) \otimes \mathrm{id} = 0$, i.e. we have
$$\mathrm{im}(i \otimes \mathrm{id}\colon A \otimes G \to B \otimes G) \subset \ker(j \otimes \mathrm{id}\colon B \otimes G \to C \otimes G).$$

(c) By (b) it remains to show that
$$\ker(j \otimes \mathrm{id}\colon B \otimes G \to C \otimes G) \subset \mathrm{im}(i \otimes \mathrm{id}\colon A \otimes G \to B \otimes G).$$
Put differently it remains to show that the homomorphism
$$\Psi\colon B \otimes G/\mathrm{im}(i \otimes \mathrm{id}\colon A \otimes G \to B \otimes G) \to C \otimes G$$
that is induced by $j \otimes \mathrm{id}$, i.e. that is given by[1083]
$$[b \otimes g] \mapsto j(b) \otimes g,$$

is a monomorphism. We will do so by providing a left-inverse for Ψ, i.e. we will provide a map Φ with $\Phi \circ \Psi = \mathrm{id}$.

Claim. The map

$$\begin{array}{rcl} \varphi\colon C \times G & \to & B \otimes G / \operatorname{im}(i \otimes \mathrm{id}\colon A \otimes G \to B \otimes G) \\ (c, g) & \mapsto & b \otimes g \quad \text{where } b \in B \text{ is an element with } j(b) = c. \end{array}$$

is well-defined.

Proof. Let $(c, g) \in C \times G$. Since j is by hypothesis surjective we know that there exists a $b \in B$ with $j(b) = c$. We have to show that $\varphi(c, g)$ is independent of the choice of b. So let $b, b' \in B$ with $j(b) = j(b') = c$, i.e. with $j(b - b') = 0$. From the exactness of the original short exact sequence at B it follows that $b - b' = i(a)$ for an $a \in A$. Then we obtain that

$$b \otimes g - b' \otimes g \;=\; (b - b') \otimes g \;=\; i(a) \otimes g \in \operatorname{im}(i \otimes \mathrm{id}\colon A \otimes G \to B \otimes G).$$

Thus we have proved that φ is well-defined. ⊞

The map φ is easily seen to bilinear. It follows from Lemma 89.2 (2) that φ induces a well-defined map

$$\begin{array}{rcl} \Phi\colon C \otimes G & \to & B \otimes G / \operatorname{im}(i \otimes \mathrm{id}\colon A \otimes G \to B \otimes G) \\ \sum_{i=1}^{k} c_i \otimes g_i & \mapsto & \sum_{i=1}^{k} \varphi(c_i, g_i). \end{array}$$

By construction we have $(\Phi \circ \Psi)([b \otimes g]) = [b \otimes g]$ for all $b \in B$ and $g \in G$. But the elements $b \otimes g$ generate $B \otimes G$, hence $\Phi \circ \Psi = \mathrm{id}$, i.e. the map Φ is indeed a right-inverse to the map Ψ. ■

89.4. **The G-torsion of an abelian group.** Our final goal is to relate the homology groups $\mathrm{H}_*(C)$ of a chain complex to the homology groups $\mathrm{H}_*(C; G)$ of the same chain complex with G-coefficients. To state the theorem relating these groups we need to introduce the G-torsion of an abelian group.

We start our discussion with recalling the definition of a free abelian group and a (free) resolution, which we had introduced on pages 1147 and 1921.

Definition.

(1) Let S be a non-empty set. We refer to

$$\mathbb{Z}^{(S)} := \text{all maps from } S \text{ to } \mathbb{Z} \text{ which are non-zero for only finitely many } s \in S$$

as the free abelian group generated by S.

(2) A group G is called free abelian if G is isomorphic to $\mathbb{Z}^{(S)}$ for some set S.

(3) It follows easily from the definitions that a group G is free abelian if and only if there exists a subset $\mathcal{B} = \{b_i\}_{i \in I} \subset G$ with the following property: For any abelian group A and each choice of elements $a_i \in A, i \in I$ there exists a unique homomorphism $\varphi\colon G \to A$ such that $\varphi(b_i) = a_i$ for all $i \in I$. As on page 1149 we call such a subset $\mathcal{B}$ a basis of G.

[1083] Note that it is a consequence of (b) that Ψ is well-defined.

(4) A (free) resolution of an abelian group H is an exact sequence

$$\ldots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

where the F_i are (free) abelian groups. If there exists an $n \in \mathbb{N}_0$ with $F_n \neq 0$ but $F_m = 0$ for $m > n$, then we refer to n as the length of the resolution. If no such n exists, then we say that the length of the resolution is infinite.

Examples.

(1) Let $m \in \mathbb{N}$, then a free resolution of length 1 for $H = \mathbb{Z}_m$ is given by

$$0 \to \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \to \mathbb{Z}_m \to 0.$$

(2) Let G be a *finitely generated* abelian group. By the Finitely Generated Abelian Group Classification Theorem 51.4 there exists an $r \in \mathbb{N}_0$, non-zero natural numbers $a_1, \ldots, a_k$ and an isomorphism

$$\Phi\colon G \xrightarrow{\cong} \mathbb{Z}^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}_{a_i}.$$

Thus we obtain a free resolution[1084]

$$0 \to \mathbb{Z}^k \xrightarrow{0 \oplus \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_k \end{pmatrix}} \mathbb{Z}^r \oplus \mathbb{Z}^k \to \underbrace{\mathbb{Z}^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}_{a_i}}_{\cong G} \to 0.$$

In most cases, when we want to work with explicit examples, this type of free resolution will turn out to be the most useful free resolution.

The final example is so important that it becomes a definition.

Definition. We recall the following basic fact: if M is an abelian group, then by Lemma 51.1 there exists a unique homomorphism $\alpha(M)\colon \mathbb{Z}^{(M)} \to M$ such that for each $m \in M$ we have $\alpha(M)(m) = m$. Evidently this map is an epimorphism. Now let H be an abelian group. We consider the sequence

$$\mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \to 0.$$

The group $\mathbb{Z}^{(H)}$ is free abelian and $\alpha(H)$ is an epimorphism. In particular the sequence is exact. We set $H_1 := \ker\left(\mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H\right)$. Then

$$H_1 \hookrightarrow \mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \to 0$$

is exact. Now we consider

$$\mathbb{Z}^{(H_1)} \xrightarrow{\alpha(H_1)} \mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \to 0.$$

This sequence is exact and both the groups $\mathbb{Z}^{(H_1)}$ and $\mathbb{Z}^{(H)}$ are free abelian. Now we set $H_2 := \ker\left(\mathbb{Z}^{(H_1)} \xrightarrow{\alpha(H)} H\right)$ and we iterate this procedure. This way we obtain a free resolution of H.[1085] Note that this free resolution is unambiguously defined, we did not

[1084] Note that the homomorphism to the left is a monomorphism since $a_1, \ldots, a_k$ are non-zero.

make any choices along the way. Therefore we refer to this free resolution of H as the canonical free resolution of H.

The last example provides the proof for the following lemma.

Lemma 89.11. Every abelian group (not necessarily finitely generated) admits a canonical free resolution.

The following definition is one of the key definitions in homological algebra.

Definition. Let G and H be abelian groups. We denote by

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

the canonical free resolution from page 1922. We define the n-th G-torsion group of H as

$$\operatorname{Tor}_n(H,G) := \mathrm{H}_n(F_* \otimes G).$$

The following proposition is the key to calculating torsion-groups.

Proposition 89.12. (Tor-Free Resolution Proposition) Let G and H be abelian groups and let

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

be a free resolution of H. Then there exists a natural isomorphism

$$\operatorname{Tor}_n(H,G) \xrightarrow{\cong} \mathrm{H}_n(F_* \otimes G).$$

Example. In many situations we are only interested in determining the isomorphism type of $\operatorname{Tor}_n(H,G)$ where H is a given finitely generated abelian group. In such situations it is usually much more convenient to work with the free resolution given on page 1921. For example, let $m,n \in \mathbb{N}$. We want to determine the isomorphism type of $\operatorname{Tor}_i(\mathbb{Z}_m,\mathbb{Z}_n)$ for $i \in \mathbb{N}_0$. We will do so by doing the following three step process:

(1) We pick the free resolution $0 \to \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \to \mathbb{Z}_m \to 0$,
(2) we tensor this free resolution of $\mathbb{Z}_m$ with $\mathbb{Z}_n$ and from the Tensor Lemma 89.3 (3) we obtain the sequence $0 \to \mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_n \to \mathbb{Z}_m \otimes \mathbb{Z}_n \to 0$,
(3) we drop the term to the right and we compute the homology groups.

Now using an argument as in the calculation of the tensor product $\mathbb{Z}_m \otimes \mathbb{Z}_n$ on page 1910 we see that

$$\operatorname{Tor}_1(\mathbb{Z}_m,\mathbb{Z}_n) = \mathrm{H}_1\big(0 \to \mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_n \to 0\big) = \ker(\mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_n) \cong \mathbb{Z}_{\gcd(m,n)}$$

and

$$\operatorname{Tor}_0(\mathbb{Z}_m,\mathbb{Z}_n) = \mathrm{H}_0\big(0 \to \mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_n \to 0\big) = \operatorname{coker}\big(\mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_n\big) \cong \mathbb{Z}_{\gcd(m,n)}.$$

We will fill in the details in Exercise 89.10.

Proof. Let G and H be abelian groups and let

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

be a free resolution of H. We want to show that there exists a natural isomorphism

$$\operatorname{Tor}_n(H,G) \xrightarrow{\cong} \mathrm{H}_n(F_* \otimes G).$$

[1085] Note that this procedure does not stop, i.e. it produces an infinite free resolution.

We denote by (F'_*, f'_*) the canonical free resolution of H. Since F'_* is a free resolution we obtain from the Fundamental Theorem 81.1 of Homological Algebra that there exists an extension $\alpha_i, i \in \mathbb{N}$ of the identity map $\mathrm{id}\colon H \to H$ from the free resolutions F'_* to the free resolution F_*.

Since both resolutions are free we obtain from Corollary 81.2 that the maps α_* form a chain homotopy equivalence between the two chain complexes F'_* and F_*. One easily verifies that $\alpha_* \otimes \mathrm{id}\colon F_* \otimes G \to F'_* \otimes G$ is also a chain homotopy equivalence. It follows from this observation together with Corollary 73.4 that the map

$$\begin{array}{rcl} \alpha_n \otimes \mathrm{id}\colon \overbrace{\mathrm{H}_n(F'_* \otimes G)}^{=\mathrm{Tor}_n(H,G)} & \to & \mathrm{H}_n(F_* \otimes G) \\ \left[\sum_{i=1}^{k} c_i \otimes g_i\right] & \mapsto & \left[\sum_{i=1}^{k} \alpha_n(c_i) \otimes g_i\right] \end{array}$$

is an isomorphism. Using Statement (2) of the Fundamental Theorem 81.1 of Homological Algebra one can easily show that this map does not depend on the choice of the extension α'_*. ■

Remark. Using the Fundamental Theorem 81.1 of Homological Algebra we can now show that the torsion groups $\mathrm{Tor}_n(G, H)$ are "covariantly functorial in both entries". More precisely the following two statements hold:

(1) We fix a group G. Let $\beta\colon H \to H'$ be a homomorphism between two abelian groups. It follows from the Fundamental Theorem 81.1 of Homological Algebra applied to the canonical free resolutions of H and H', together with Lemma 73.2, that β induces for each n a canonical homomorphism

$$\beta_*\colon \mathrm{Tor}_n(H, G) \to \mathrm{Tor}_n(H', G).$$

One can now easily show that the maps given by

$$\begin{array}{rcl} H & \mapsto & \mathrm{Tor}_n(H, G) \\ (\beta\colon H \to H') & \mapsto & (\beta_*\colon \mathrm{Tor}_n(H, G) \to \mathrm{Tor}_n(H', G)) \end{array}$$

define a covariant functor from the category of abelian groups to the category of abelian groups.

(2) We fix a group H. Let $\alpha\colon G \to G'$ be a homomorphism between two abelian groups. Let F_* be the canonical free resolution of H. Then $\mathrm{id} \otimes \alpha$ defines a chain map from $F_* \otimes G$ to $F_* \otimes G'$ and we get an induced map

$$\alpha_*\colon \mathrm{Tor}_n(H, G) \to \mathrm{Tor}_n(H, G').$$

One can now easily show that the maps given by

$$\begin{array}{rcl} G & \mapsto & \mathrm{Tor}_n(H, G) \\ (\alpha\colon G \to G') & \mapsto & (\alpha_*\colon \mathrm{Tor}_n(H, G) \to \mathrm{Tor}_n(H, G')) \end{array}$$

define a covariant functor from the category of abelian groups to the category of abelian groups.

The following lemma shows that the 0-th torsion groups are objects that we are already familiar with.

Lemma 89.13. (Tor_0–Lemma) Let G and H be abelian groups, then there exists a natural isomorphism [1086]

$$\mathrm{Tor}_0(H,G) \xrightarrow{\cong} H \otimes G.$$

In the remainder of this course we will use the isomorphism from the Tor_0–Lemma 89.13 to identify the groups $\mathrm{Tor}_0(H,G)$ and $H \otimes G$.

Proof. Let G and H be abelian groups. Let

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

be the canonical free resolution of H. We know from Lemma 89.10 that the sequence

$$(*) \qquad F_1 \otimes G \xrightarrow{f_1 \otimes \mathrm{id}} F_0 \otimes G \xrightarrow{f_0 \otimes \mathrm{id}} H \otimes G \to 0$$

is exact. Now we obtain the following isomorphism

$$\begin{aligned} \mathrm{Tor}_0(H,G) &\overset{\text{by definition}}{=} \mathrm{H}_0\big(\cdots \to F_1 \otimes G \xrightarrow{f_1 \otimes \mathrm{id}} F_0 \otimes G \to 0\big) \\ &= \operatorname{coker}\big(F_1 \otimes G \xrightarrow{f_1 \otimes \mathrm{id}} F_0 \otimes G\big) \xrightarrow[\text{isomorphism since } (*) \text{ is exact}]{f_0 \otimes \mathrm{id}} H \otimes G. \end{aligned}$$

It is straightforward to verify that these isomorphisms define in fact a natural isomorphism from the functor $H \mapsto \mathrm{Tor}_0(H,G)$ to the functor $H \mapsto H \otimes G$ and also from the functor $G \mapsto \mathrm{Tor}_0(H,G)$ to the functor $G \mapsto H \otimes G$. ■

The following lemma shows that we do not need to worry about higher torsion groups for abelian groups.

Lemma 89.14.

(1) Every abelian group H admits a free resolution of length 1.
(2) For any two abelian groups H and G we have $\mathrm{Tor}_n(H,G) = 0$ for $n \geq 2$.

Proof.

(1) Let H be an abelian group. If H is finitely generated, then we already saw on page 1921 that H admits a free resolution of length 1. Now suppose that H is any abelian group. We have the exact sequence

$$0 \to \ker\big(\alpha(H)\colon \mathbb{Z}^{(H)} \to H\big) \to \mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \to 0$$

where $\alpha(H)$ is defined as in the proof of Lemma 89.11. It follows from the Subgroup-of-Free Abelian Group Lemma 51.2 that the group $\ker\big(\alpha(H)\colon \mathbb{Z}^{(H)} \to H\big)$, as the subgroup of the free abelian group $\mathbb{Z}^{(H)}$, is itself free abelian. So in this general case we also found a free resolution of length 1.

(2) Let G and H be abelian groups. By (1) there exists a free resolution of length 1, i.e. of the form

$$0 \to F_1 \to F_0 \to H \to 0.$$

Finally note that it now follows immediately from the Tor-Free Resolution Proposition 89.12 that $\mathrm{Tor}_n(H,G) \cong \mathrm{H}_n(F_* \otimes G) = 0$ for $n \geq 2$. ■

[1086] Note that "natural isomorphism" means in this context that given an abelian group G we have natural isomorphisms from the functor $H \mapsto \mathrm{Tor}_0(H,G)$ to the functor $H \mapsto H \otimes G$ and also from the functor $G \mapsto \mathrm{Tor}_0(H,G)$ to the functor $G \mapsto H \otimes G$.

Definition. Given two abelian groups H and G we now write $\operatorname{Tor}(H,G) := \operatorname{Tor}_1(H,G)$ and we refer to this group as as the G-torsion of H.

Remark. For many purposes the long discussion of G-torsion groups can be summarized in the following statement: it follows from the Tor-Free Resolution Proposition 89.12 that for a free resolution $0 \to F_1 \xrightarrow{i} F_0 \xrightarrow{j} H \to 0$ of an abelian group H we have

$$\operatorname{Tor}(H,G) \;\cong\; \ker\big(F_1 \otimes G \xrightarrow{i\otimes \mathrm{id}} F_0 \otimes G\big).$$

Now we summarize several properties of the torsion groups $\operatorname{Tor}(H,G)$ in the following lemma.

Lemma 89.15. (Tor-Properties Lemma) Let G, H, $\{\mathrm{H}_i\}_{i\in I}$ and $\{G_j\}_{j\in J}$ be abelian groups. Then the following holds:

(1) There exists a natural isomorphism $\operatorname{Tor}\big(\bigoplus_i \mathrm{H}_i, G\big) \cong \bigoplus_i \operatorname{Tor}(\mathrm{H}_i, G)$.
(2) There exists a natural isomorphism $\operatorname{Tor}\big(H, \bigoplus_j G_j\big) \cong \bigoplus_j \operatorname{Tor}(H, G_j)$.
(3) If H is a torsion-free abelian group, then $\operatorname{Tor}(H,G) = 0$.
(4) If G is a torsion-free abelian group, then $\operatorname{Tor}(H,G) = 0$.
(5) If T denotes the torsion subgroup[1087] of H, then $\operatorname{Tor}(H,G) = \operatorname{Tor}(T,G)$.
(6) For all $n \in \mathbb{N}$ we have a natural isomorphism $\operatorname{Tor}(\mathbb{Z}_n, G) \cong \ker\big(G \xrightarrow{\cdot n} G\big)$.
(7) For all $n, m \in \mathbb{N}$ we have $\operatorname{Tor}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_{\gcd(n,m)}$.
(8) Given any two finitely generated abelian groups G and H the corresponding torsion group $\operatorname{Tor}(H,G)$ is finite.[1088]

Remark. Using this lemma one can usually determine the G-torsion of an abelian group H without any problems. For example the groups $(\mathbb{Q},+)$, $(\mathbb{R},+)$ and $(\mathbb{C},+)$ are torsion-free abelian groups[1089] Thus it follows immediately from the Tor-Properties Lemma 89.15 (4) that for every abelian group H we have

$$\operatorname{Tor}(H,\mathbb{Q}) \;=\; \operatorname{Tor}(H,\mathbb{R}) \;=\; \operatorname{Tor}(H,\mathbb{C}) \;=\; 0.$$

Proof.

(1) For each $i \in I$ we choose a free resolution F^i_* of H_i. Then $\bigoplus_i F^i_*$ is a free resolution of $\bigoplus_i \mathrm{H}_i$. The statement follows easily from the natural isomorphisms

$$\Big(\bigoplus_i F^i_j\Big) \otimes G \;\cong\; \bigoplus_i (F^i_j \otimes G)$$

of the Tensor Lemma 89.3 together with Lemma 72.13 and the Tor-Free Resolution Proposition 89.12.

(2) We pick a free resolution F_* for H. As in (1) the statement follows easily from the natural isomorphisms

$$F_i \otimes \bigoplus_j G_j \;\cong\; \bigoplus_j F_i \otimes G_j$$

[1087] The torsion subgroup of an abelian group is defined as the subgroup of all elements of finite order.
[1088] This fact explains the name "torsion group".
[1089] Note though that in Lemma 51.10 we saw that the abelian group $(\mathbb{Q},+)$ is not free abelian.

of the Tensor Lemma 89.3 together with Lemma 72.13 and the Tor-Free Resolution Proposition 89.12.

(3) If H is a free abelian group, then $0 \to F_0 := H \xrightarrow{\mathrm{id}} H \to 0$ is already a free resolution, and it follows immediately from the Tor-Free Resolution Proposition 89.12 that $\mathrm{Tor}(H, G) = 0$. Later on we will only use the special case that H is a free abelian group. Therefore we do not provide the proof of the general case of a torsion-free abelian group which is not free. We refer to [**Hat2002**, p. 265] for the proof.

(4) By Lemma 89.14 (1) we can pick a free resolution $0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H$ of length 1 for H. First we consider the case that G is a free abelian group. In this case we have $G \cong \bigoplus_I \mathbb{Z}$ for an index set I. It follows that

$$\mathrm{Tor}(H, G) = \mathrm{Tor}\left(H, \bigoplus_I \mathbb{Z}\right) \underset{\text{by (1)}}{=} \bigoplus_I \mathrm{Tor}(H, \mathbb{Z}) \underset{\text{Tor-Free Resolution Proposition 89.12}}{\cong} \bigoplus_I \mathrm{H}_1(0 \to \underbrace{F_1 \otimes \mathbb{Z}}_{=F_1} \to \underbrace{F_0 \otimes \mathbb{Z}}_{=F_0} \to 0) = 0.$$

Now we no longer suppose that G is free abelian, we just suppose that G is an abelian group that is torsion-free. We denote by $\{G_i\}_{i \in I}$ the set of finitely generated subgroups of G. It follows from the Finitely Generated Abelian Group Classification Theorem 51.4 that each G_i is a free abelian group. We write $i \leq j$ if and only if $G_i \subset G_j$. Together with the inclusion maps the groups $\{G_i\}_{i \in I}$ form a direct system. We leave it to the reader to verify that

$$G = \varinjlim G_i.$$

We have

$$\begin{aligned} \mathrm{Tor}(H, G) = \mathrm{Tor}(H, \varinjlim G_i) &= \mathrm{H}_1\big(0 \to F_1 \otimes \varinjlim G_i \to F_0 \otimes \varinjlim G_i\big) \\ &\underset{\text{by the Tensor Lemma 89.3 (4)}}{=} \mathrm{H}_1\big(\varinjlim (0 \to F_1 \otimes G_i \to F_0 \otimes G_i \to 0)\big) \\ &\underset{\text{Direct Limit-Homology Lemma 79.6}}{=} \varinjlim \underbrace{\mathrm{H}_1(0 \to F_1 \otimes G_i \to F_0 \otimes G_i \to 0)}_{\substack{=0, \text{ by the above, since } G_i \text{ is a} \\ \text{free abelian group}}} = 0. \end{aligned}$$

(5) If H is a finitely generated abelian group, then it follows from the Finitely Generated Abelian Group Classification Theorem 51.4 that we can write H as a direct sum $F \oplus T$, where is T the torsion subgroup and where F is a free abelian group. It follows from (1) and (2) that

$$\mathrm{Tor}(H, G) \cong \mathrm{Tor}(F \oplus T, G) \underset{\text{by (1)}}{\cong} \mathrm{Tor}(F, G) \oplus \mathrm{Tor}(T, G) \underset{\text{by (3)}}{=} \mathrm{Tor}(T, G).$$

For the proof in the case that H is not finitely generated we once again refer to [**Hat2002**, p. 265] for a proof.

(6) We consider the free resolution

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}_n \to 0$$

for $H = \mathbb{Z}_n$. After tensoring with the abelian group G, and using the Tensor Lemma 89.3 we obtain the sequence

$$0 \to G \xrightarrow{\cdot n} G \to \mathbb{Z}_n \otimes G \to 0.$$

It follows from the Tor-Free Resolution Proposition 89.12 that $\mathrm{Tor}(\mathbb{Z}_n, G) = \ker\left(G \xrightarrow{\cdot n} G\right)$.[1090]

(7) We showed this isomorphism on page 1922. Alternatively one can deduce the statement using (6).

(8) Let G and H be two finitely generated abelian groups. It is a straightforward consequence of the Finitely Generated Abelian Group Classification Theorem 51.4, together with all of the above statements that the corresponding torsion group $\mathrm{Tor}(H, G)$ is finite. ■

89.5. **The Universal Coefficient Theorem.** Before we can state the Algebraic Universal Coefficient Theorem we need to introduce the following definition.

Definition. Let $(\mathrm{C}_*, \partial_*)$ be a chain complex and let G be an abelian group. Throughout this chapter, given $n \in \mathbb{N}_0$, we denote by μ the natural[1091] homomorphism[1092]

$$\begin{array}{rcl} \mu\colon \mathrm{H}_n(C_*) \otimes G & \to & \mathrm{H}_n(C_* \otimes G) \\ \sum_{i=1}^{m} [c_i] \otimes g_i & \mapsto & \left[\sum_{i=1}^{m} c_i \otimes g_i\right]. \end{array}$$

Now we can formulate and prove the following theorem which is the key result of the whole chapter.

Theorem 89.16. (Algebraic Universal Coefficient Theorem) Let (C_n, ∂_n) be a chain complex of free abelian groups and let G be an abelian group. Then for each $n \in \mathbb{N}_0$ there exists a natural homomorphism $p\colon \mathrm{H}_n(C; G) \to \mathrm{Tor}(\mathrm{H}_{n-1}(C), G)$ such that the following sequence is exact:

$$0 \to \mathrm{H}_n(C) \otimes G \xrightarrow{\mu} \mathrm{H}_n(C; G) \xrightarrow{p} \mathrm{Tor}(\mathrm{H}_{n-1}(C), G) \to 0.$$

Remark. The fact the homomorphisms μ and p are natural implies that given a chain map $f\colon C \to D$ and a homomorphism $\varphi\colon G \to G'$ between abelian groups the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}_n(C) \otimes G & \xrightarrow{\mu} & \mathrm{H}_n(C; G) & \xrightarrow{p} & \mathrm{Tor}(\mathrm{H}_{n-1}(C), G) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}_n(D) \otimes G' & \xrightarrow{\mu} & \mathrm{H}_n(D; G') & \xrightarrow{p} & \mathrm{Tor}(\mathrm{H}_{n-1}(D), G') & \longrightarrow & 0 \end{array}$$

[1090] The same argument, together with the Tor_0–Lemma 89.13 can be used to give a proof of the Tensor Lemma 89.3 (3), which back then was perhaps shoved under the carpet a little too quickly.

[1091] Here "natural" means that for a chain map $f_*\colon C_* \to D_*$ and a homomorphism $\varphi\colon G \to G'$ between abelian groups the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_n(C_*) \otimes G & \xrightarrow{\mu} & \mathrm{H}_n(C_* \otimes G) \\ \downarrow & & \downarrow \\ \mathrm{H}_n(D_*) \otimes G' & \xrightarrow{\mu} & \mathrm{H}_n(D_* \otimes G') \end{array}$$

where the vertical maps are induced by f and φ in the, hopefully, obvious way.

[1092] It is straightforward to show that the map is well-defined. We will discuss in Exercise 89.4 whether the "obvious converse" is also well-defined.

commutes. Here the vertical maps are all induced by the chain map $f\colon C \to D$ and the homomorphism $\varphi\colon G \to G'$.

Proof. Let (C_n, ∂_n) be a chain complex of free abelian groups. For each $n \in \mathbb{N}_0$ we write as always $Z_n := \ker(\partial_n)$ and $B_n := \operatorname{im}(\partial_{n+1})$. Note that it follows from the Subgroup-of-Free Abelian Group Lemma 51.2 that B_n and Z_n are free abelian groups.

We consider the following diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_n & \longrightarrow & \mathrm{C}_n & \xrightarrow{\partial_n} & B_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & Z_{n-1} & \longrightarrow & \mathrm{C}_{n-1} & \xrightarrow{\partial_{n-1}} & B_{n-2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

One sees easily that this diagram is commutative and that the horizontal sequences are exact. We can and will view the left and the right vertical columns as chain complexes. In other words, the above diagram is a short exact sequence of chain complexes. Recall that the groups B_n are free abelian groups, thus it follows from Lemmas 78.1 that each of the horizontal sequences splits.

Now let G be an abelian group. We tensor the above short exact sequence of chain complexes with G and we obtain the following diagram:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_n \otimes G & \longrightarrow & \mathrm{C}_n \otimes G & \xrightarrow{\partial_n \otimes \mathrm{id}} & B_{n-1} \otimes G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial_n \otimes \mathrm{id}} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & Z_{n-1} \otimes G & \longrightarrow & \mathrm{C}_{n-1} \otimes G & \xrightarrow{\partial_{n-1} \otimes \mathrm{id}} & B_{n-2} \otimes G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

This is again a sequence of chain complexes.

Since the original short exact sequences split it follows from Lemmas 89.6 that the horizontal sequences obtained from tensoring with G are still exact. Therefore we can apply the LES Proposition 74.9 to obtain a long exact sequence of homology groups. Since the vertical boundary maps on the left and on the right are the zero maps, we obtain the following long exact sequence

$$\ldots \to B_n \otimes G \xrightarrow{d_n} Z_n \otimes G \to \mathrm{H}_n(C; G) \xrightarrow{\partial_n} B_{n-1} \otimes G \xrightarrow{d_{n-1}} Z_{n-1} \otimes G \to \ldots$$

where the maps d_n are the connecting homomorphisms of the long exact sequence. It follows easily from the definition that the connecting homomorphism $d_n\colon B_n \otimes G \to Z_n \otimes G$ is the map that is given by $i_n \otimes \mathrm{id}\colon B_n \otimes G \to Z_n \otimes G$ where $i_n\colon B_n \to Z_n$ denotes the obvious inclusion map.[1093]

[1093] Why is this the case?

Using the Break-into-SES Lemma 78.4 the above discussion implies that we obtain the following short exact sequence

$$0 \to \operatorname{coker}\left(B_n \otimes G \xrightarrow{i_n \otimes \mathrm{id}} Z_n \otimes G\right) \to \mathrm{H}_n(C; G) \to \ker\left(B_{n-1} \otimes G \xrightarrow{i_{n-1} \otimes \mathrm{id}} Z_{n-1} \otimes G\right) \to 0.$$

We still have to show that the expressions left and right coincide with the desired expressions. We have natural isomorphisms

$$\begin{array}{ccccc} \operatorname{coker}\left(B_n \otimes G \xrightarrow{i_n \otimes \mathrm{id}} Z_n \otimes G\right) & \xleftarrow{\cong} & \operatorname{Tor}_0(\mathrm{H}_n(C), G) & \xrightarrow{\cong} & \mathrm{H}_n(C) \otimes G \\ \uparrow & & & & \uparrow \\ \text{by the Tor-Free Resolution Proposition 89.12 since } B_n \xrightarrow{i_n} Z_n \to \mathrm{H}_n(C) & & & & \operatorname{Tor}_0\text{–Lemma 89.13} \\ \text{is a free resolution of } \mathrm{H}_n(C) & & & & \end{array}$$

Similarly we have a natural isomorphism

$$\ker\left(B_{n-1} \otimes G \xrightarrow{i_{n-1} \otimes \mathrm{id}} Z_{n-1} \otimes G\right) \xrightarrow{\cong} \operatorname{Tor}(\mathrm{H}_{n-1}(C), G).$$

We have thus shown the existence of the desired short exact sequences. The statement that the homomorphism p is natural follows easily from the definitions and Statement (2) of the Fundamental Theorem 81.1 of Homological Algebra. We leave the verification of the details to the reader. ■

The following theorem is for us the main application of the Universal Coefficient Theorem 89.16.

Theorem 89.17. (Universal Coefficient Theorem) Let (X, A) be a pair of topological spaces and let G be an abelian group. Then for each $n \in \mathbb{N}_0$ there exists a natural homomorphism $\mathrm{H}_n(X, A; G) \to \operatorname{Tor}(\mathrm{H}_{n-1}(X, A), G)$ such that the following sequence is exact:

$$0 \to \mathrm{H}_n(X, A) \otimes G \xrightarrow{\mu} \mathrm{H}_n(X, A; G) \to \operatorname{Tor}(\mathrm{H}_{n-1}(X, A), G) \to 0.$$

Remark. Note that the naturality of the maps implies for every map $f\colon (X, A) \to (Y, B)$ between pairs of topological spaces and any homomorphism $\varphi\colon G \to G'$ between abelian groups we obtain the following *commutative* diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}_n(X, A) \otimes G & \xrightarrow{\mu} & \mathrm{H}_n(X, A; G) & \longrightarrow & \operatorname{Tor}(\mathrm{H}_{n-1}(X, A), G) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}_n(Y, B) \otimes G' & \xrightarrow{\mu} & \mathrm{H}_n(Y, B; G') & \longrightarrow & \operatorname{Tor}(\mathrm{H}_{n-1}(Y, B), G') & \longrightarrow & 0, \end{array}$$

where the vertical maps are the obvious (?) ones induced by f and φ.

Proof. Let (X, A) be a pair of topological spaces and let G be an abelian group. As we pointed out on page 1625, the chain groups $\mathrm{C}_n(X, A)$, $n \in \mathbb{N}_0$, are free abelian groups. Thus we can apply the Universal Coefficient Theorem 89.16 and we immediately obtain the desired short exact sequence. ■

Before we consider a few examples let us jot down the following almost immediate corollary.

Corollary 89.18. Let $f\colon (X,A)\to(Y,B)$ be a map between pairs of topological spaces and let G be an abelian group. Then

$$\begin{array}{c}\text{the induced map}\\ f_*\colon \mathrm{H}_n(X,A)\to\mathrm{H}_n(Y,B)\text{ is}\\ \text{an isomorphism for all } n\in\mathbb{N}_0\end{array}\quad\Longrightarrow\quad\begin{array}{c}\text{the induced map}\\ f_*\colon \mathrm{H}_n(X,A;G)\to\mathrm{H}_n(Y,B;G)\text{ is}\\ \text{an isomorphism for all } n\in\mathbb{N}_0.\end{array}$$

Proof. The corollary follows immediately from the naturality of the short exact sequence of the Universal Coefficient 89.17 together with the Five-Lemma 74.10. ■

Examples.

(1) We consider again the real projective space $\mathbb{R}\mathrm{P}^n$. In Proposition 80.11 we saw that

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n)\ \cong\ \begin{cases}\mathbb{Z}, & \text{if } k=0,\\ \mathbb{Z}_2, & \text{if } k \text{ is odd and } k<n,\\ 0, & \text{if } k \text{ is even and } 0<k\le n,\\ 0, & \text{if } k>n,\\ \mathbb{Z}, & \text{if } k=n \text{ and } n \text{ is odd.}\end{cases}$$

It follows from the Universal Coefficient Theorem 89.17, the discussion on page 1910 and the Tor-Properties Lemma 89.15 (7) that for any even number $2k\in\{0,\dots,n\}$ we have a short exact sequence

$$0\ \to\ \underbrace{\mathrm{H}_{2k}(\mathbb{R}\mathrm{P}^n)\otimes\mathbb{F}_2}_{\cong 0\otimes\mathbb{F}_2=0}\ \xrightarrow{\mu}\ \mathrm{H}_{2k}(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)\ \to\ \underbrace{\mathrm{Tor}(\mathrm{H}_{2k-1}(\mathbb{R}\mathrm{P}^n),\mathbb{F}_2)}_{\cong\mathrm{Tor}(\mathbb{F}_2,\mathbb{F}_2)\cong\mathbb{F}_2}\ \to\ 0$$

whereas for $2k+1\in\{0,\dots,n\}$ an odd number we have a short exact sequence

$$0\ \to\ \underbrace{\mathrm{H}_{2k+1}(\mathbb{R}\mathrm{P}^n)\otimes\mathbb{F}_2}_{\cong\mathbb{F}_2\otimes\mathbb{F}_2\cong\mathbb{F}_2\text{ or}\cong\mathbb{Z}\otimes\mathbb{F}_2\cong\mathbb{F}_2}\ \xrightarrow{\mu}\ \mathrm{H}_{2k+1}(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)\ \to\ \underbrace{\mathrm{Tor}(\mathrm{H}_{2k}(\mathbb{R}\mathrm{P}^n),\mathbb{F}_2)}_{\cong\mathrm{Tor}(0,\mathbb{F}_2)=0\ \text{ or }\cong\mathrm{Tor}(\mathbb{Z},\mathbb{F}_2)=0}\ \to\ 0.$$

Also it follows easily that $\mathrm{H}_k(\mathbb{R}\mathrm{P}^n)=0$ for $k>n$. Summarizing we obtain the following result:

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)\ \cong\ \begin{cases}\mathbb{F}_2, & \text{if } k=0,\dots,n,\\ 0, & \text{otherwise.}\end{cases}$$

Fortunately this calculation agrees with the calculation on page 1916.

(2) As in the Ball-Quotient Sphere Lemma 5.20 and as on page 258 we make the usual identifications $S^2=\overline{B}^2/S^1$ and $\mathbb{R}\mathrm{P}^2=\overline{B}^2/z\sim -z$ where $z\in S^1$. We denote by f the natural projection[1094]

$$f\colon \mathbb{R}\mathrm{P}^2=\overline{B}^2/z\sim -z\ \to\ S^2=\overline{B}^2/S^1.$$

As in the figure on page 1472 we equip $\mathbb{R}\mathrm{P}^2$ with the CW-structure with one 0-cell, one 1-cell and one 2-cell and as on page 1472 we equip S^2 with the CW-structure with one 0-cell and one 2-cell. With these CW-structures the above map f is given by collapsing the 1-cell to the unique 0-cell of S^2, in particular f is a cellular map. By the

[1094]Note that this is not a covering map and that this map has nothing to do with the more familiar projection $S^2\to\mathbb{R}\mathrm{P}^2$.

Singular-Cellular H_*-Isomorphism Proposition 80.4 and the discussion on page 1915 we obtain the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) & \xrightarrow{f_*} & \mathrm{H}_2(S^2;\mathbb{F}_2) \\ {\scriptstyle \Phi_{\mathbb{R}\mathrm{P}^2}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \Phi_{S^2}} \\ \mathrm{H}_2^{\mathrm{CW}}(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) & \xrightarrow{f_*} & \mathrm{H}_2^{\mathrm{CW}}(S^2;\mathbb{F}_2). \end{array}$$

As we saw on page 1916, the 2-cell is a generator of $\mathrm{H}_2^{\mathrm{CW}}(S^2;\mathbb{F}_2)$ and it gets sent to the 2-cell in S^2 which is, as we saw on page 1770, a generator of $\mathrm{H}_2^{\mathrm{CW}}(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$. Thus we see that the bottom map is an isomorphism, it follows that the top map is also an isomorphism. Summarizing the short exact sequences of the Universal Coefficient Theorem 89.17 give us the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overbrace{\mathrm{H}_2(\mathbb{R}\mathrm{P}^2)\otimes\mathbb{F}_2}^{=0} & \xrightarrow{\mu} & \overbrace{\mathrm{H}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)}^{\cong\mathbb{F}_2} & \longrightarrow & \overbrace{\mathrm{Tor}(\mathrm{H}_1(\mathbb{R}\mathrm{P}^2),\mathbb{F}_2)}^{\cong\mathbb{F}_2} & \longrightarrow & 0 \\ & & {\scriptstyle =0}\downarrow{\scriptstyle f_*} & & {\scriptstyle \cong}\downarrow{\scriptstyle f_*} & & {\scriptstyle =0}\downarrow{\scriptstyle f_*} & & \\ 0 & \longrightarrow & \underbrace{\mathrm{H}_2(S^2)\otimes\mathbb{F}_2}_{\cong\mathbb{F}_2} & \xrightarrow{\mu} & \underbrace{\mathrm{H}_2(S^2;\mathbb{F}_2)}_{\cong\mathbb{F}_2} & \longrightarrow & \underbrace{\mathrm{Tor}(\mathrm{H}_1(S^2),\mathbb{F}_2)}_{=0} & \longrightarrow & 0. \end{array}$$

Note that the two vertical maps to the left and right are the zero map, whereas the middle map is actually an isomorphism. This slightly weird fact will play a role in the No-Natural Splitting Proposition 89.24.

In the following corollary we give two instances where the Tor-terms in the Universal Coefficient Theorem 89.17 are zero, i.e. two instances where we can "pull out" the coefficients.

Corollary 89.19.

(1) Let (X, A) be a pair of topological spaces and let G be a subgroup of $(\mathbb{C}, +)$. Then for each $n \in \mathbb{N}_0$ the map

$$\mu\colon \mathrm{H}_n(X, A; \mathbb{Z}) \otimes G \;\to\; \mathrm{H}_n(X, A; G)$$

is a natural isomorphism.

(2) Let (X, A) be a pair of topological spaces and let $\mathbb{F}$ be a subfield of $\mathbb{C}$. Then for each $n \in \mathbb{N}_0$ we have $\mathrm{rank}(\mathrm{H}_n(X, A)) = \dim_{\mathbb{F}}(\mathrm{H}_n(X, A; \mathbb{F}))$.

Example. It follows immediately from Proposition 80.11 together with Corollary 89.19 and the Tensor Lemma 89.3 (3) that

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^{2n+1};\mathbb{R}) \;\cong\; \begin{cases} \mathbb{R}, & \text{if } k = 0, 2n+1, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad \mathrm{H}_k(\mathbb{R}\mathrm{P}^{2n};\mathbb{R}) \;\cong\; \begin{cases} \mathbb{R}, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof.

(1) This statement is an immediate consequence of the Universal Coefficient Theorem 89.17 and the fact, obtained in the Tor-Properties Lemma 89.15 (4), that the torsion-groups are zero for any subgroup of $(\mathbb{C}, +)$.

(2) To shorten the notation we consider only the case $A = \varnothing$. We have

$$\mathrm{rank}(\mathrm{H}_n(X)) \underset{\text{by the definition on page 1911}}{\overset{\uparrow}{=}} \dim_{\mathbb{Q}}(\mathrm{H}_n(X)\otimes\mathbb{Q}) \underset{\text{by (1)}}{\overset{\uparrow}{=}} \dim_{\mathbb{Q}}(\mathrm{H}_n(X))\otimes\mathbb{Q} \underset{\text{follows fairly easily from } \mathbb{Q}\subset\mathbb{F}}{\overset{\uparrow}{=}} \dim_{\mathbb{F}}(\mathrm{H}_n(X))\otimes\mathbb{F}.$$

■

Remark.

(1) Let (X, A) be a pair of topological spaces and let R be a subring of $\mathbb{C}$. Since the map $\mu\colon \mathrm{H}_n(X, A; \mathbb{Z}) \otimes R \to \mathrm{H}_n(X, A; R)$ is natural we can and we will in the following identify the groups $\mathrm{H}_n(X, A; \mathbb{Z}) \otimes R$ and $\mathrm{H}_n(X, A; R)$ using the isomorphism from Corollary 89.19.

(2) Given a pair (X, A) of topological spaces and $n \in \mathbb{N}_0$ we refer to[1095]

$$b_n(X, A) \;:=\; \mathrm{rank}(\mathrm{H}_n(X, A))$$

as the *n-th Betti number of* (X, A). For $A = \varnothing$ we write of course $b_n(X) := b_n(X, \varnothing)$. Note that it follows from Corollary 89.19 that for any subfield $\mathbb{F}$ of $\mathbb{C}$ we have $b_n(X, A) = \dim_{\mathbb{F}}(\mathrm{H}_n(X, A; \mathbb{F}))$.

(3) Let R be a subring of $\mathbb{C}$. the Tor-Properties Lemma 89.15 (4) and the Universal Coefficient Theorem 89.16 show that taking the tensor product with R is an exact functor in the sense of the definition on page 1918.

We conclude this section with the following lemma which we will need later on in the generalization of the Borsuk-Ulam Theorem 48.21 to higher dimensions.

Lemma 89.20. (Degree-and-Coefficients Lemma) Let $f\colon S^n \to S^n$ be a map and let $k \in \mathbb{N}$. Then the induced map

$$f_*\colon \mathrm{H}_n(S^n; \mathbb{Z}_k) \;\to\; \mathrm{H}_n(S^n; \mathbb{Z}_k)$$

is given by multiplication by $\deg(f)$.

Proof. Let $f\colon S^n \to S^n$ be a map and let $k \in \mathbb{N}$. Since the short exact sequence of the Universal Coefficient Theorem 89.17 is natural we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}_n(S^n) \otimes \mathbb{Z}_k & \xrightarrow{\mu} & \mathrm{H}_n(S^n; \mathbb{Z}_k) & \longrightarrow & \mathrm{Tor}(\mathrm{H}_{n-1}(S^n), \mathbb{Z}_k) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_* \otimes \mathrm{id}} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\
0 & \longrightarrow & \mathrm{H}_n(S^n) \otimes \mathbb{Z}_k & \xrightarrow{\mu} & \mathrm{H}_n(S^n; \mathbb{Z}_k) & \longrightarrow & \mathrm{Tor}(\mathrm{H}_{n-1}(S^n), \mathbb{Z}_k) & \longrightarrow & 0.
\end{array}$$

Note that $\mathrm{H}_{n-1}(S^n)$ is torsion-free, hence $\mathrm{Tor}(\mathrm{H}_{n-1}(S^n), \mathbb{Z}_k) = 0$ by the Tor-Properties Lemma 89.15. It follows from the exactness of the horizontal sequences that the horizontal maps μ on the left are isomorphisms. By definition the map $f_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$ is given by multiplication by $\deg(f)$, hence $f_* \otimes \mathrm{id}$ is also given by multiplication by $\deg(f)$. It now follows that the middle vertical map is also given by multiplication by $\deg(f)$. ■

89.6. **Splittings of the Universal Coefficient Theorem.** Our goal had been to determine the homology groups $\mathrm{H}_n(C; G)$ only in terms of the original groups $\mathrm{H}_k(C)$ and the abelian group G. The Algebraic Universal Coefficient Theorem 89.16 gave us a short exact sequence

$$0 \;\to\; \mathrm{H}_n(C) \otimes G \;\xrightarrow{\mu}\; \mathrm{H}_n(C; G) \;\xrightarrow{p}\; \mathrm{Tor}(\mathrm{H}_{n-1}(C), G) \;\to\; 0.$$

But in general, as we already saw on page 1715, given a short exact sequence, it is not possible to determine the middle group of a short exact sequence from the groups to the

[1095] Recall that on page 1911 we defined the rank of an arbitrary abelian group.

left and the right. But we saw in Splitting Lemma 78.2 that if a short exact sequence of abelian groups

$$0 \;\to\; A \;\xrightarrow{i}\; B \;\xrightarrow{p}\; C \;\to\; 0$$

splits, then the middle group B is the direct sum of the "outer" groups A and C. The question arises, whether the short exact sequence in the Universal Coefficient Theorem 89.16 splits. This is fortunately the case:

Proposition 89.21. (UCT-Splitting Proposition) Let (C_n, ∂_n) be a free chain complex[1096] and let G be an abelian group. Then for any $n \in \mathbb{N}_0$ the short exact sequence

$$0 \;\to\; \mathrm{H}_n(C) \otimes G \;\xrightarrow{\mu}\; \mathrm{H}_n(C;G) \;\to\; \mathrm{Tor}(\mathrm{H}_{n-1}(C), G) \;\to\; 0$$

of the Algebraic Universal Coefficient Theorem 89.16 splits. In particular there exists an isomorphism

$$\mathrm{H}_n(C;G) \;\cong\; \mathrm{H}_n(C) \otimes G \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(C), G).$$

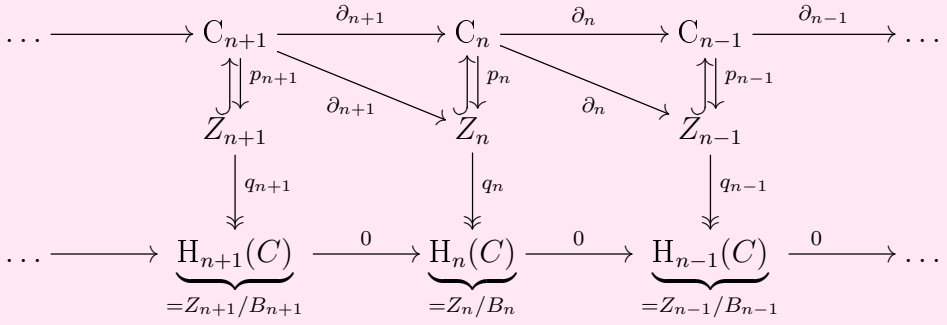
Proof. Let (C_n, ∂_n) be a chain complex of free abelian groups and furthermore let G be an abelian group. According to Splitting Lemma 78.2 it suffices to construct a homomorphism $\eta\colon \mathrm{H}_n(C;G) \to \mathrm{H}_n(C) \otimes G$ such that $\eta \circ \mu$ is the identity on $\mathrm{H}_n(C) \otimes G$.

For each $n \in \mathbb{N}_0$ we write as always $Z_n := \ker(\partial_n)$ and $B_n := \mathrm{im}(\partial_{n+1})$. We consider the short exact sequence

$$0 \;\to\; Z_n \;\hookrightarrow\; \mathrm{C}_n \;\xrightarrow{\partial_n}\; B_{n-1} \;\to\; 0.$$

The group $B_{n-1} \subset \mathrm{C}_{n-1}$ is a subgroup of a free abelian group, thus it is itself a free abelian group by the Subgroup-of-Free Abelian Group Lemma 51.2 . Therefore we can deduce from Lemma 78.1 that the above short exact sequence splits. In particular, according to Splitting Lemma 78.2, for each $n \in \mathbb{N}_0$ there exists a homomorphism $p_n\colon \mathrm{C}_n \to Z_n$ such that p_n is the identity on $Z_n \subset C_n$.

Claim. The following diagram commutes:



where the maps q_n are just the natural projections.

Proof. We start at C_{n+1}. If we go down and then right, then we end up with the zero map to $\mathrm{H}_n(C)$. On the other hand, if we first go right, then we end up in $B_n \subset Z_n$, but then the projection q_n to $\mathrm{H}_n(C) = Z_n/B_n$ is also the zero map. ⊞

The top and the bottom horizontal sequences of the claim are chain complexes. Since the diagram commutes we see that the vertical maps from top to bottom are chain maps.

[1096] A chain complex $(\mathrm{C}_*, \partial_*)$ is called *free*, if all the groups C_n are free abelian groups.

Now we tensor the top and the bottom horizontal sequences with G and we obtain the following diagram of chain complexes.

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & \mathrm{C}_{n+1} \otimes G & \xrightarrow{\partial_{n+1}\otimes \mathrm{id}} & \mathrm{C}_n \otimes G & \xrightarrow{\partial_n \otimes \mathrm{id}} & \mathrm{C}_{n-1} \otimes G & \xrightarrow{\partial_{n-1}\otimes \mathrm{id}} & \ldots \\
 & & \downarrow{\scriptstyle (q_{n+1}\circ p_{n+1})\otimes \mathrm{id}} & & \downarrow{\scriptstyle (q_n \circ p_n)\otimes \mathrm{id}} & & \downarrow{\scriptstyle (q_{n-1}\circ p_{n-1})\otimes \mathrm{id}} & & \\
\ldots & \longrightarrow & \mathrm{H}_{n+1}(C) \otimes G & \xrightarrow{0} & \mathrm{H}_n(C) \otimes G & \xrightarrow{0} & \mathrm{H}_{n-1}(C) \otimes G & \xrightarrow{0} & \ldots
\end{array}$$

The induced maps on the homology of these two chain complexes give us a homomorphism

$$\eta \colon \mathrm{H}_n(C; G) \;=\; \mathrm{H}_n(C_* \otimes G) \;\to\; \mathrm{H}_n(\text{lower sequence}) = \mathrm{H}_n(C) \otimes G.$$

It remains to prove the following claim.

Claim. The composition $\eta \circ \mu$ is the identity on $\mathrm{H}_n(C) \otimes G$.

Proof. Let $[\sigma] \otimes g \in \mathrm{H}_n(C) \otimes G$ where $\sigma \in Z_n$. Then

$$(\eta \circ \mu)([\sigma] \otimes g) \;=\; \eta([\sigma \otimes g]) \underset{\substack{\uparrow \\ \text{definition of } \eta}}{=} (q_n(p_n(\sigma))) \otimes g \underset{\substack{\uparrow \\ \text{since } p_n = \mathrm{id} \text{ on } Z_n}}{=} q_n(\sigma) \otimes g \;=\; [\sigma] \otimes g.$$

■

The following immediate corollary to the UCT-Splitting Proposition 89.21 shows in particular that the homology groups with G-coefficients of a topological space X are completely determined by the usual homology groups of X and the abelian group G.

Theorem 89.22. (UCT-Splitting Theorem) Let (X, A) be a pair of topological spaces and let G be an abelian group. Then for each $n \in \mathbb{N}_0$ the short exact sequence

$$0 \;\to\; \mathrm{H}_n(X, A) \otimes G \;\xrightarrow{\mu}\; \mathrm{H}_n(X, A; G) \;\to\; \mathrm{Tor}(\mathrm{H}_{n-1}(X, A), G) \;\to\; 0$$

from the Universal Coefficient Theorem 89.17 splits. In particular there exists an isomorphism

$$\mathrm{H}_n(X, A; G) \;\cong\; \mathrm{H}_n(X, A) \otimes G \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(X, A), G).$$

The following is an immediate corollary to the UCT-Splitting Theorem 89.22.

Corollary 89.23. Let (X, A) and (Y, B) be topological spaces and let G be an abelian group. Then

$$\begin{array}{c} \mathrm{H}_n(X, A) \text{ and } \mathrm{H}_n(Y, B) \text{ are} \\ \text{isomorphic for all } n \in \mathbb{N}_0 \end{array} \quad \Longrightarrow \quad \begin{array}{c} \mathrm{H}_n(Y, B; G) \text{ and } \mathrm{H}_n(X, A; G) \text{ and} \\ \text{isomorphic for all } n \in \mathbb{N}_0. \end{array}$$

Remark.

(1) Corollary 89.23 is in some sense a disappointment, it shows that homology groups with coefficients are not better at distinguishing topological spaces than ordinary homology groups.
(2) Corollary 89.23 is also an almost immediate consequence of Propositions 81.13 and 81.4.

The question now arises, whether given an abelian group G there exists in fact a *natural* isomorphism

$$\mathrm{H}_n(X; G) \;\cong\; \mathrm{H}_n(X) \otimes G \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(X), G).$$

The discussion on page 206 already shows that this sounds like a dubious idea. Indeed, the following proposition says that such a natural isomorphism does not exist for $G = \mathbb{F}_2$. More precisely, we have the following proposition.

Proposition 89.24. (No-Natural Splitting Proposition) It is not possible assign to each topology space X an isomorphism

$$\Phi_X \colon \mathrm{H}_n(X;\mathbb{F}_2) \ \to \ \mathrm{H}_n(X) \otimes \mathbb{F}_2 \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(X), \mathbb{F}_2)$$

such that for any map $f \colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_n(X;\mathbb{F}_2) & \xrightarrow[\cong]{\Phi_X} & \mathrm{H}_n(X) \otimes \mathbb{F}_2 \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(X), \mathbb{F}_2) \\ \downarrow f_* & & \downarrow f_* \quad \downarrow f_* \\ \mathrm{H}_n(Y;\mathbb{F}_2) & \xrightarrow[\cong]{\Phi_Y} & \mathrm{H}_n(Y) \otimes \mathbb{F}_2 \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(Y), \mathbb{F}_2). \end{array}$$

Proof. So suppose instead that such a map Φ_X exists for every topological space X. We consider the map

$$f \colon \mathbb{R}\mathrm{P}^2 = \overline{B}^2/z \sim -z \ \to \ S^2 = \overline{B}^2/S^1$$

from page 1930. We obtain the following diagram:

$$\begin{array}{ccc} \overbrace{\mathrm{H}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)}^{\cong \mathbb{F}_2} & \xrightarrow[\cong]{\Phi_{\mathbb{R}\mathrm{P}^2}} & \overbrace{\mathrm{H}_2(\mathbb{R}\mathrm{P}^2) \otimes \mathbb{F}_2}^{=0} \oplus \overbrace{\mathrm{Tor}(\mathrm{H}_1(\mathbb{R}\mathrm{P}^2), \mathbb{F}_2)}^{\cong \mathbb{F}_2} \\ {\scriptstyle f_*}\downarrow & & {\scriptstyle =0}\downarrow {\scriptstyle f_*} \quad {\scriptstyle =0}\downarrow {\scriptstyle f_*} \\ \underbrace{\mathrm{H}_2(S^2;\mathbb{F}_2)}_{\cong \mathbb{F}_2} & \xrightarrow[\cong]{\Phi_{\mathbb{R}\mathrm{P}^2}} & \underbrace{\mathrm{H}_2(S^2) \otimes \mathbb{F}_2}_{\cong \mathbb{F}_2} \oplus \underbrace{\mathrm{Tor}(\mathrm{H}_1(S^2), \mathbb{F}_2)}_{=0}. \end{array}$$

The horizontal maps are by hypothesis isomorphisms. We saw on page 1930 that the vertical map on the left is an isomorphism. On the other hand the two vertical maps on the right are zero since either the domain or the target is zero. Thus this diagram cannot possibly commute. We have thus found a contradiction to our assumption. ■

89.7. **The Eilenberg-Steenrod axioms for homology** $(*)$**.** In this section we give a very short introduction to the concept of (co-) homology theories which was introduced in the seminal book of Eilenberg–Steenrod [**ES1952**]. Before we can do so we introduce the following definition.

Definition. Let R be a commutative ring and let S be a set.

(1) An S-graded R-module is defined as an R-module M together with a direct sum decomposition $M = \bigoplus_{s \in S} M_s$ into R-modules M_s, $s \in S$. If $x \in M$ lies in some M_s, then we call x homogeneous and we refer to $\deg(x) := s$ as the degree of x.

(2) A morphism $f \colon M \to N$ between S-graded R-modules M and N is an R-module homomorphism such that for each $s \in S$ we have $f(M_s) \subset N_s$. Equivalently, it is a family of R-module homomorphisms $\{f_s \colon M_s \to N_s\}_{s \in S}$.

(3) The category $S\text{-}\mathcal{GradMod}_R$ of S-graded R-modules is the category where the objects are S-graded R-modules and the morphisms are the ones described in (2).

(4) Since abelian groups are for all intents and purposes the same as $\mathbb{Z}$-modules we define the category $S\text{-}\mathcal{GradAb}$ of S-graded abelian groups to be the category of $\mathbb{N}_0$-graded $\mathbb{Z}$-modules.

Example. Given a pair of topological spaces (X, Y) and an abelian group G we write

$$\mathrm{H}_*(X, Y; G) \;:=\; \bigoplus_{n\in\mathbb{N}_0} \mathrm{H}_n(X, Y; G)$$

and we view $\mathrm{H}_*(X, Y; G)$ as an $\mathbb{N}_0$-graded $\mathbb{Z}$-module.

The following definition goes back to [**ES1952**] and [**Miln1962b**].

Definition. A homology theory is a covariant functor

$$\mathcal{H}\colon \begin{array}{c}\text{category } \mathcal{PairTop} \text{ of}\\ \text{pairs of topological spaces}\end{array} \;\to\; \begin{array}{c}\text{category } \mathbb{N}_0\text{-}\mathcal{GradAb} \text{ of}\\ \mathbb{N}_0\text{-graded abelian groups}\end{array}$$

that satisfies the following axioms:[1097]

(1) (Homotopy invariant) If $f, g\colon (X, A) \to (Y, B)$ are homotopic maps between pairs of topological spaces, then $f_* = g_*$.

(2) (Connecting homomorphism) Given any pair (X, A) of topological spaces and any $n \in \mathbb{N}$ there exists a natural connecting homomorphism $\partial\colon \mathcal{H}_n(X, A) \to \mathcal{H}_{n-1}(A)$[1098] such that the sequence

$$\ldots \;\to\; \mathcal{H}_{n+1}(X, A) \;\xrightarrow{\partial}\; \mathcal{H}_n(A) \;\to\; \mathcal{H}_n(X) \;\to\; \mathcal{H}_n(X, A) \;\xrightarrow{\partial}\; \mathcal{H}_{n-1}(A) \;\to\; \ldots$$

is exact.

(3) (Excision) If U and A are subsets of a topological space X with $\overline{U} \subset \mathring{A}$, then for each $n \in \mathbb{N}_0$ the map $\mathcal{H}_n(X \setminus U, A \setminus U) \to \mathcal{H}_n(X, A)$ is an isomorphism.

(4) (Additivity) Let $\{(X_j, A_j)\}_{j\in J}$ be a family of pairs of topological spaces. We write $X = \bigsqcup_{j\in J} X_j$ and $A = \bigsqcup_{j\in J} A_j$. If given $j \in J$ we denote by $i_j\colon (X_j, A_j) \to (X, A)$ the obvious inclusion map, then for any $k \in \mathbb{N}_0$ the induced map

$$\bigoplus_{j\in J} i_{j*}\colon \bigoplus_{j\in J} \mathcal{H}_k(X_j, A_j) \;\to\; \mathcal{H}_k(X, A)$$

is an isomorphism.

(5) (Dimension axiom) We denote by $\star$ the topological space that consists of a single point. For $n \neq 0$ we have $\mathrm{H}_n(\star) = 0$.

Example. It follows from Proposition 74.15, the Topological LES-Corollary 74.14, the Excision Theorem 74.17, the obvious generalization of Lemma 72.14 and the Single Point-Homology Lemma 72.6 that singular homology is indeed a homology theory.

We conclude this chapter with a uniqueness theorem of homology theories. To formulate the theorem we need the following definition.

Definition. We call the category $\mathcal{FullPairCW}$

$$\begin{aligned}\mathrm{Ob}(\mathcal{FullPairCW}) &:= \text{all pairs of CW-complexes,}\\ \mathrm{Mor}((X, A), (Y, B)) &:= \text{all continuous maps } f\colon X \to Y \text{ with } f(A) \subset B,\end{aligned}$$

together with the usual composition of maps the full category of pairs of CW-complexes.[1099]

[1097] The axioms are usually called Eilenberg-Steenrod axioms.

[1098] Here we use the obvious notation that for a topological space Y we write $\mathcal{H}_n(Y) := \mathcal{H}_n(Y, \varnothing)$.

Now we can formulate the following uniqueness statement for homology theories.

Theorem 89.25. (Eilenberg-Steenrod Uniqueness Theorem) Let $\mathcal{H}$ and $\mathcal{H}'$ be two homology theories and let $\varphi\colon \mathcal{H}_0(\star) \to \mathcal{H}'_0(\star)$ be an isomorphism. Then there exists a natural isomorphism between $\mathcal{H}$ and $\mathcal{H}'$ viewed as functors

$$\begin{array}{c}\text{full category } \mathit{FullPairCW} \text{ of}\\ \text{pairs of CW-complexes}\end{array} \to \begin{array}{c}\text{category } \mathbb{N}_0\text{-}\mathit{GradAb} \text{ of}\\ \mathbb{N}_0\text{-graded abelian groups.}\end{array}$$

Sketch of a proof. Let G be an abelian group. We start our proof with the following observation: Given a pair (X, A) of CW-complexes we saw in the CW-Complex Cone Corollary 68.35 (3) that we can equip the mapping cone $\mathrm{Cone}(i\colon A \to X)$ of the inclusion map with a natural CW-structure. Given $n \in \mathbb{N}_0$ we define the n-th cellular homology $\mathrm{H}_n^{\mathrm{CW}}(X, A; G)$ as the reduced cellular homology with G-coefficients of the mapping cone.

the Singular-Cellular H_*-Isomorphism Proposition 80.4, together with the Mapping Cone-H_*-Lemma 78.15, the Singular-Cellular H_*-Isomorphism Proposition 80.4 and the discussion on page 1915 shows that for pairs of CW-complexes there exists a natural isomorphism between singular homology with G-coefficients and cellular homology with G-coefficients. Closer inspection of the proof shows that we can prove this statement only using the above Eilenberg-Steenrod axioms and the fact that $\mathrm{H}_0(\star; G) \cong G$.

Now let $\alpha\colon \mathcal{H}_0(\star) \to G$ be an isomorphism. By the above one can show that there exists a natural isomorphism from $\mathcal{H}$ to $\mathrm{H}_n^{\mathrm{CW}}(-; G)$. The same applies to $\mathcal{H}'$ and we obtain the theorem as long as φ is an isomorphism. This approach to the proof is developed in more detail in [**tD2008**, Chapter 12.2]. ■

Remark.

(1) Closely related results are also proved in [**ES1952**, p. 100] and [**Hu1966**, p. 52], see also [**GrH1981**, p. 138].
(2) The axioms of a cohomology theory are defined basically the same way with the obvious modifications and the analogue of the Eilenberg-Steenrod Uniqueness Theorem 89.25 also holds in this context, see [**DaK2001**, Theorem 1.31].
(3) If one drops the dimension axiom, then there are other (co-) homology theories, for example "topological K-theory". We refer to [**Karo2008**] for details.

Exercises for Chapter 89.

Exercise 89.1. Show that the abelianization functor from the category of groups to the category of abelian groups is right-exact and that it is not left-exact.

Exercise 89.2. Let $\mathcal{C}$ be the category of finitely generated abelian groups. Given an abelian group A we denote by $\mathrm{Tor}(A)$ as usual its torsion subgroup and furthermore we denote by $\mathrm{F}A := A/\mathrm{Tor}(A)$ the maximal torsion-free quotient. We consider the following two covariant functors:

$$\mathrm{F}\colon \begin{array}{ccc}\mathcal{C} & \mapsto & \mathcal{C}\\ A & \mapsto & \mathrm{F}A\end{array} \quad \text{and} \quad \mathrm{Tor}\colon \begin{array}{ccc}\mathcal{C} & \to & \mathcal{C}\\ A & \mapsto & \mathrm{Tor}(A).\end{array}$$

[1099] Note that we consider all continuous maps, we do not restrict ourselves to cellular maps. Thus *FullPairCW* is a full subcategory of the category *PairTop* of pairs of topological spaces.

(a) Show that the functor F is left-exact.
(b) Show that the functor F is not right-exact.
(c) Is the functor Tor left-exact?
(d) Is the functor Tor is right-exact?

Exercise 89.3. Let A be an $(m \times n)$-matrix over $\mathbb{Z}$ and let R be a commutative ring. Let $\varphi\colon \mathbb{Z} \to R$ be the ring homomorphism given by $\varphi(n) = n \cdot 1_R$. Finally let $\Theta_k\colon \mathbb{Z}^m \otimes R \to R^m$ and $\Theta_n\colon \mathbb{Z}^n \otimes R \to R^n$ be the natural isomorphisms provided by the Tensor Lemma 89.3. Show that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z}^n \otimes R & \xrightarrow{(v \mapsto Av) \otimes \mathrm{id}_R} & \mathbb{Z}^m \otimes R \\ {\scriptstyle \Theta_n}\downarrow & & \downarrow{\scriptstyle \Theta_m} \\ R^n & \xrightarrow{w \mapsto \varphi(A)w} & R^m. \end{array}$$

Exercise 89.4. Let $(\mathrm{C}_*, \partial_*)$ be a chain complex and let G be an abelian group. We consider the following map:

$$\begin{array}{rcl} \mathrm{H}_n(C_* \otimes G) & \to & \mathrm{H}_n(C_*) \otimes G \\ \left[\sum\limits_{i=1}^{n} c_i \otimes g_i\right] & \mapsto & \sum\limits_{i=1}^{n} [c_i] \otimes g_i \end{array}$$

Is this map well-defined?

Exercise 89.5. What are the isomorphism types of the homology groups of $\mathbb{R}\mathrm{P}^5$ with $\mathbb{F}_3$-coefficients?

Exercise 89.6. Let $A \to B \to C$ be a short exact sequence of abelian groups and let G be an abelian group. Show that there exists an exact sequence of the form

$$0 \to \mathrm{Tor}(A, G) \to \mathrm{Tor}(B, G) \to \mathrm{Tor}(C, G) \to A \otimes G \to B \otimes G \to C \otimes G \to 0.$$

Hint. Use the Snake Lemma that we proved in Exercise 74.9.
Remark. This exact sequence is sometimes called the *six-term sequence* in homological algebra.

Exercise 89.7. Let $n \in \mathbb{N}$. Let G be an abelian group (which is not necessarily finitely generated). Suppose G admits an element of order g, i.e. $n \cdot g = 0$ and $m \cdot g \neq 0$ for $m = 1, \ldots, n-1$. Show that $\mathrm{Tor}(G, \mathbb{Z}_n) \neq 0$.
Remark. You could make use of Exercise 89.6.

Exercise 89.8.

(a) Let G_* be a chain complex. (We do not assume that the chain groups G_k are finitely generated.) Suppose that for every $k \in \mathbb{N}_0$ and every prime $p \in \mathbb{N}$ we have $\mathrm{H}_k(G_*; \mathbb{Q}) = 0$ and $\mathrm{H}_k(G_*; \mathbb{Z}_p) = 0$. Show that for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(G_*) = 0$.
Remark. You might want to use Exercise 89.7.

(b) Let $f\colon C_* \to D_*$ be a chain map between two chain complexes. (We do not assume that the chain groups are finitely generated.) Suppose that for every $k \in \mathbb{N}_0$ and every prime $p \in \mathbb{N}$ the maps $\mathrm{H}_k(C_*; \mathbb{Q}) \to \mathrm{H}_k(D_*; \mathbb{Q})$ and $\mathrm{H}_k(C_*; \mathbb{Z}_p) \to \mathrm{H}_k(D_*; \mathbb{Z}_p)$ are isomorphisms. Show that for every $k \in \mathbb{N}_0$ the map $\mathrm{H}_k(C_*) \to \mathrm{H}_k(D_*)$ is an isomorphism.
Hint. Use the algebraic mapping cone that we introduced on page 1796.

Exercise 89.9. Determine $\operatorname{Tor}(\mathbb{Z}_n, \mathbb{Q}/\mathbb{Z})$.

Exercise 89.10. Let $m, n \in \mathbb{N}$. Show that

$$\operatorname{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{\gcd(m,n)}.$$

Exercise 89.11. Show that

$$\Big(\prod_{n\in\mathbb{N}} \mathbb{Z}_n\Big) \otimes \mathbb{Q} \neq 0.$$

Exercise 89.12. Let $k \in \mathbb{Z} \setminus \{0\}$ and let $\varphi_k \colon \Delta^1 \to S^1$ be the map given by $(1-t, t) \mapsto \exp(k \cdot 2\pi t \mathrm{i})$. Furthermore let $i \colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(S^1; \mathbb{R})$ be the homomorphism from Lemma 89.7 (1) that is induced by $\sigma \mapsto \sigma \otimes 1$. Show that

$$\big[\tfrac{1}{n} \otimes \varphi\big] \;=\; \tfrac{1}{n} \cdot i_*([S^1]) \;\in\; \mathrm{H}_1(S^1; \mathbb{R}).$$

We refer to the figure below for an illustration.



Exercise 89.13. Let $T = S^1 \times S^1$ and let K be the Klein bottle.

(a) Does there exist a map $f \colon K \to T$ such that $f_* \colon \mathrm{H}_2(K; \mathbb{F}_2) \to \mathrm{H}_2(T; \mathbb{F}_2)$ is an isomorphism?

(b) Does there exist a map $g \colon T \to K$ such that $g_* \colon \mathrm{H}_2(T; \mathbb{F}_2) \to \mathrm{H}_2(K; \mathbb{F}_2)$ is an isomorphism?

Exercise 89.14. Let $g \in \mathbb{N}$. We denote by N_g the non-orientable surface of genus g as defined on page 266. As a reminder, note that by Lemma **??** we know that N_g is diffeomorphic to the connected sum of g copies of $\mathbb{R}\mathrm{P}^2$. Show that

$$\mathrm{H}_i(N_g; \mathbb{F}_2) \;\cong\; \begin{cases} 0, & \text{if } i > 2, \\ \mathbb{F}_2, & \text{if } i = 0 \text{ or } i = 2, \\ \mathbb{F}_2^g, & \text{if } i = 1. \end{cases}$$

Remark. We calculated $\mathrm{H}_*(N_g)$ in Proposition 80.10.

90. The Künneth Theorem

In this chapter we want to study the following question:

Question 90.1. Let X and Y be two topological spaces. What is the connection between the homology groups of X and Y on the one hand and the homology groups of the product topological space $X \times Y$ on the other hand?

90.1. The tensor product of chain complexes.

First we want to find some connection between the chain complexes of X, Y and $X \times Y$. To do so we need the notion of the tensor product of chain complexes. Therefore we start out with a short excursion into homological algebra, i.e. into the study of chain complexes.

Lemma 90.2. Let $\mathcal{C} = (\mathrm{C}_*, \partial_*)$ and $\mathcal{C}' = (\mathrm{C}'_*, \partial'_*)$ be two chain complexes. Given $n \in \mathbb{N}_0$ we define

$$(\mathcal{C} \otimes \mathcal{C}')_n := \bigoplus_{p+q=n} \mathrm{C}_p \otimes \mathrm{C}'_q.$$

The consider the maps

$$d\colon (\mathcal{C} \otimes \mathcal{C}')_n \to (\mathcal{C} \otimes \mathcal{C}')_{n-1}$$

which are uniquely determined by

$$c_p \otimes c'_q \mapsto \partial c_p \otimes c'_q + (-1)^p \cdot c_p \otimes \partial' c'_q$$

for $c_p \in \mathrm{C}_p$ and $c'_q \in \mathrm{C}'_q$. The maps turn $\mathcal{C} \otimes \mathcal{C}'$ into a chain complex, called the tensor product $\mathcal{C} \otimes \mathcal{C}'$ of the chain complexes $\mathcal{C}$ and $\mathcal{C}'$.

We illustrate the definition of the tensor product of chain complexes in the following diagram:

$$\begin{array}{ccccc}
\mathrm{C}_0 \otimes \mathrm{C}'_2 & \xleftarrow{\partial_1 \otimes \mathrm{id}} & \mathrm{C}_1 \otimes \mathrm{C}'_2 & \xleftarrow{\partial_2 \otimes \mathrm{id}} & \mathrm{C}_2 \otimes \mathrm{C}'_2 \\
\downarrow{\scriptstyle \mathrm{id} \otimes \partial'_2} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial'_2} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial'_2} \\
\mathrm{C}_0 \otimes \mathrm{C}'_1 & \xleftarrow{\partial_1 \otimes \mathrm{id}} & \mathrm{C}_1 \otimes \mathrm{C}'_1 & \xleftarrow{\partial_2 \otimes \mathrm{id}} & \mathrm{C}_2 \otimes \mathrm{C}'_1 \\
\downarrow{\scriptstyle \mathrm{id} \otimes \partial'_1} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial'_1} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial'_1} \\
\mathrm{C}_0 \otimes \mathrm{C}'_0 & \xleftarrow{\partial_1 \otimes \mathrm{id}} & \mathrm{C}_1 \otimes \mathrm{C}'_0 & \xleftarrow{\partial_2 \otimes \mathrm{id}} & \mathrm{C}_2 \otimes \mathrm{C}'_0.
\end{array}$$

Proof. We still have to show that the maps d are indeed boundary maps of a chain complex, i.e. we have to verify that $d \circ d = 0$. This is the case since for any $c_p \in \mathrm{C}_p$ and $c'_q \in \mathrm{C}'_q$ we calculate that

$$\begin{aligned}
d(d(c_p \otimes c'_q)) &= d(\overbrace{\partial c_p}^{\in \mathrm{C}_{p-1}} \otimes c'_q) + d((-1)^p \cdot c_p \otimes \overbrace{\partial' c'_q}^{\in C'_{q-1}}) \\
&= \underbrace{\partial \partial c_p}_{=0} \otimes c'_q + (-1)^{p-1} \cdot \partial c_p \otimes \partial' c'_q + (-1)^p \cdot \partial c_p \otimes \partial' c'_q + (-1)^p \cdot c_p \otimes \underbrace{\partial' \partial' c'_q}_{=0} \\
&= (-1)^{p-1} \cdot \partial c_p \otimes \partial' c'_q - (-1)^{p-1} \cdot \partial c_p \otimes \partial' c'_q = 0.
\end{aligned}$$

Thus we have shown that $(\mathcal{C} \otimes \mathcal{C}', d)$ is indeed a chain complex.[1100] ∎

[1100] Note that in the calculation we made use of the extra-term $(-1)^p$ in the definition of the boundary map. If we defined the boundary map "naively" as $d(c_p \otimes c'_q) = \partial c_p \otimes c'_q + c_p \otimes \partial' c'_q$, then the result would not have been a chain complex.

Example. Let

$$\mathcal{C} = \cdots \to \mathrm{C}_n \xrightarrow{\partial_n} \mathrm{C}_{n-1} \xrightarrow{\partial_{n-1}} \mathrm{C}_{n-2} \to \dots \quad \text{and} \quad \mathcal{C}' = 0 \to \underset{\substack{\uparrow \\ \text{in degree } k}}{G} \to 0$$

be two chain complexes where the second consists of a single abelian group. In this case we have the equality

$$(\mathcal{C}_* \otimes \mathcal{C}'_*)_n = \mathcal{C}_{n-k} \otimes G,$$

i.e. the tensor product is just given by tensoring the chain groups of $\mathcal{C}$ by G and shifting the degrees by $-k$.

We conclude this section with the following lemma.

Lemma 90.3. Let $\mathcal{C}$ and $\mathcal{D}$ be two chain complexes.

(1) Given two chain maps $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{C}' \to \mathcal{D}'$ the induced map

$$\begin{aligned} f \otimes f'\colon \mathcal{C} \otimes \mathcal{C}' &\to \mathcal{D} \otimes \mathcal{D}' \\ \sum_{i=1}^{n} c_i \otimes c'_i &\mapsto \sum_{i=1}^{n} f(c_i) \otimes f'(c'_i) \end{aligned}$$

is also a chain map.

(2) If $f, g\colon \mathcal{C} \to \mathcal{D}$ and $f', g'\colon \mathcal{C}' \to \mathcal{D}'$ are chain homotopic maps, then the induced maps

$$f \otimes f', g \otimes g'\colon \mathcal{C} \otimes \mathcal{C}' \to \mathcal{D} \otimes \mathcal{D}'$$

are also chain homotopic.

(3) If $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{C}' \to \mathcal{D}'$ are homotopy equivalences, then the induced map

$$f \otimes f'\colon \mathcal{C} \otimes \mathcal{C}' \to \mathcal{D} \otimes \mathcal{D}'$$

is also a homotopy equivalence.

Proof. The proof of the lemma is elementary. For completeness' sake and for a good conscience we provide the proof of the first statement. So let $\mathcal{C}$ and $\mathcal{D}$ be two chain complexes and let $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{C}' \to \mathcal{D}'$ be chain maps. Let $c \in C_n$ and $c' \in \mathrm{C}'_l$, then

$$\begin{aligned} \partial(f(c)\otimes f'(c')) &\underset{\substack{\uparrow \\ \text{since } \deg(f(c)) = \deg(c) = k}}{=} \partial f(c)\otimes f'(c') + (-1)^k \cdot f(c) \otimes \partial f'(c') \\ &= (f\otimes f')\big(\partial c \otimes c' + (-1)^k \cdot c \otimes \partial c'\big) \end{aligned} \qquad \begin{aligned} &\underset{\substack{\uparrow \\ \text{since } f \text{ and } f' \text{ are chain maps}}}{=} f(\partial c)\otimes f'(c') + (-1)^k \cdot f(c) \otimes f'(\partial c') \\ &= (f\otimes f')(\partial(c\otimes c')). \end{aligned}$$

■

90.2. **The Eilenberg-Zilber Theorem.** The following theorem, which was first proved in 1953, relates the singular chain complex $\mathrm{C}_*(X \times Y)$ of the product of two topological spaces X and Y to the tensor product $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$ of the two individual chain complexes.

Theorem 90.4. (Eilenberg-Zilber Theorem) Let X and Y be topological spaces. There exist natural[1101] chain maps

$$\begin{aligned} \Upsilon\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) &\to \mathrm{C}_*(X \times Y) \\ \text{and} \qquad \Theta\colon \mathrm{C}_*(X \times Y) &\to \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y), \end{aligned}$$

with the following three properties:

(1) The maps are chain homotopy-inverses of one another. In particular, by Corollary 73.4, for each $n \in \mathbb{N}_0$ we have a natural isomorphism

$$\mathrm{H}_n(X \times Y) \xrightarrow[\cong]{\Theta_*} \mathrm{H}_n(\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)).$$

(2) For any $x \in X$ and any singular k-chain $\sigma\colon \Delta^k \to Y$ the singular chain $\Upsilon(\{x\} \otimes \sigma)$ equals the singular simplex $\Delta^k \to X \times Y$ that is given by $P \mapsto (x, \sigma(P))$.

(3) The same statement as in (2) also holds with the roles of X and Y swapped.

Proof. We postpone this technically intricate proof to Section 113.2. In that section the above statement follows from the EZ-Construction Theorem 113.2 together with the proof of the Eilenberg-Zilber Theorem provided on page 2496. ■

Since we just postponed the proof of the Eilenberg-Zilber 90.4 to significantly later let us prove a partial result that is much weaker, but that is good enough for many purposes. Namely we will prove an analogous statement for CW-complexes which will be sufficient for the calculations of homology groups that we want to do in this chapter. The proof of this special case is significantly easier to digest than the proof of the general case of the Eilenberg-Zilber Theorem.

Proposition 90.5. Let X and Y be two CW-complexes with finitely many cells in each dimension. Then there exists a natural[1102] isomorphism

$$C_*^{\mathrm{CW}}(X \times Y) \xrightarrow{\cong} C_*^{\mathrm{CW}}(X) \otimes C_*^{\mathrm{CW}}(Y)$$

of chain complexes.

Remark. Recall that, following the convention of page 1602, given two chain complexes C_* and D_* we write $C_* \simeq D_*$ if they are homotopy equivalent. Suppose that X and Y are CW-complexes which have only finitely many cells in each dimension. Then we have

$$\begin{array}{ccccccc} & & & & \text{Proposition 90.5} & & \\ & & & & \downarrow \cong & & \\ \mathrm{C}_*(X \times Y) & \underset{\uparrow}{\simeq} & \mathrm{C}_*^{\mathrm{CW}}(X \times Y) & \to & \mathrm{C}_*^{\mathrm{CW}}(X) \otimes \mathrm{C}_*^{\mathrm{CW}}(Y) & \underset{\uparrow}{\simeq} & \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y). \\ & & & & \text{Singular-Cellular Chain Complex Proposition 81.12 (2) and Lemma 90.3} & & \end{array}$$

For such X and Y this argument proves the existence of the chain homotopies in the statement of the Eilenberg-Zilber Theorem 90.4. But note that the chain homotopy equivalences of Lemma 90.3 are not natural, hence the above chain equivalences are not natural.

[1101] Here "natural" means that for maps $f\colon X \to X'$ and $g\colon Y \to Y'$ of topological spaces we obtain commutative diagrams

$$\begin{array}{ccc} \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) & \xrightarrow{\Upsilon} & \mathrm{C}_*(X \times Y) \\ \downarrow{\scriptstyle f_* \otimes f'_*} & & \downarrow{\scriptstyle (f \times g)_*} \\ \mathrm{C}_*(X') \otimes \mathrm{C}_*(Y') & \xrightarrow{\Upsilon} & \mathrm{C}_*(X' \times Y') \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{C}_*(X \times Y) & \xrightarrow{\Theta} & \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \\ \downarrow{\scriptstyle (f \times g)_*} & & \downarrow{\scriptstyle f_* \otimes f'_*} \\ \mathrm{C}_*(X' \times Y') & \xrightarrow{\Theta} & \mathrm{C}_*(X') \otimes \mathrm{C}_*(Y'). \end{array}$$

[1102] Here the notion "natural" means "natural" in the category of CW-complexes, i.e. the isomorphisms Φ_{XY} are only natural with respect to cellular maps $X \to X'$ and $Y \to Y'$, i.e. the commutative diagram from Footnote 1101 only exists for such cellular maps.

Proof. Let X and Y be two CW-complexes with finitely many cells in each dimension. On page 1497 we introduced a CW-structure on $X \times Y$ where the n-cells of $X \times Y$ are precisely of the form $e \times f$ where e is a p-cell of X and f is an $(n-p)$-cell of Y. We quickly recall the definition of the product CW-structure on $X \times Y$:

(1) Using the homeomorphism from the Convex-to-Ball Proposition 2.20 (2) we identify the n-dimensional closed ball $\overline{B}^n$ with the n-dimensional cube $I^n = [0,1]^n$.
(2) We put $(X \times Y)^0 = X^0 \times Y^0$.
(3) Now we suppose that the $(n-1)$-skeleton $(X \times Y)^{(n-1)}$ has already been defined. We consider a p-cell e in X which is given by a characteristic map $\varphi\colon I^p \to X^p$ and an $(n-p)$-cell f in Y which is given by a characteristic map $\psi\colon I^{n-p} \to Y^{n-p}$. Then we denote by $e \times f$ the n-cell in $X \times Y$ which is given by the attaching map

$$\begin{array}{rcl} \partial(I^n) = (\partial I^p \times I^{n-p}) \cup (I^p \times \partial I^{n-p}) & \to & (X \times Y)^{n-1} \\ (x,y) & \mapsto & (\varphi(x), \psi(y)). \end{array}$$

All these n-cells together define the n-skeleton of $X \times Y$.

Since X and Y have only finitely many cells in each dimension the above construction does indeed define a CW-structure on $X \times Y$.

Given $n \in \mathbb{N}_0$ we consider the map

$$C_n^{\mathrm{CW}}(X \times Y) \to C_p^{\mathrm{CW}}(X) \otimes C_{n-p}^{\mathrm{CW}}(Y)$$

which is induced by the property that for any p-cell e of X and any $(n-p)$-cell f of Y we have[1103]

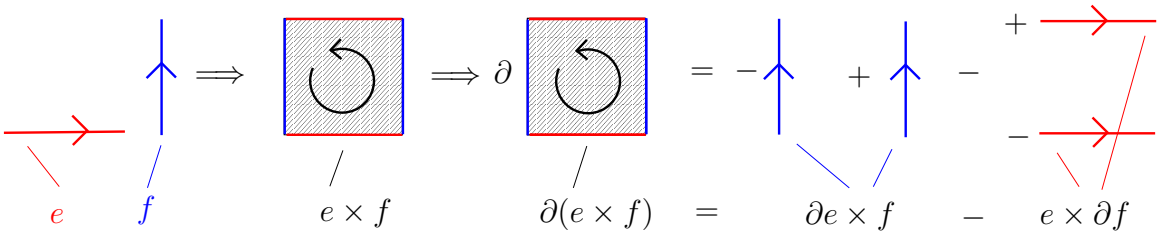
$$[e \times f] \mapsto [e] \otimes [f].$$

As on page 1910 one can show that this map is an isomorphism. It remains to prove that the above maps form a chain map. This statement in turn is equivalent to the following claim.

Claim. For any p-cell e of X and any q-cell f of Y we have

$$\underbrace{\partial(e \times f)}_{\in C_{p+q-1}^{\mathrm{CW}}(X \times Y)} = \underbrace{\partial e \otimes f}_{\in C_{p-1}^{\mathrm{CW}}(X) \otimes C_q^{\mathrm{CW}}(Y)} + (-1)^p \cdot \underbrace{e \otimes \partial f}_{\in C_p^{\mathrm{CW}}(X) \otimes C_{q-1}^{\mathrm{CW}}(Y)}.$$

Proof. It is possible to prove this formula using the description of the cellular boundary map in the Cellular Boundary Map-via-Degrees Proposition 80.8 and by doing a very careful sign-analysis. We illustrate this argument in the figure below.



Following [**Bre1993**, Theorem 12.2] we will give an alternative proof instead. Given a k-cell σ and a $(k-1)$-cell μ we denote by $[\mu : \sigma]$ the coefficient of μ in $\partial\sigma$. It follows quite easily from the Cellular Boundary Map-via-Degrees Proposition 80.8 that for $p, q \in \mathbb{N}_0$

[1103] We follow the convention of page 1777 where saw that cells naturally give rise to a basis of the cellular chain complex.

there exists a unique sign $\epsilon_{p,q} \in \{-1, 1\}$ such that for any CW-complexes K and L, any p-cell e of K, any q-cell f of L and any $(q-1)$-cell g of L we have

$$[e \times g : e \times f] \;=\; \epsilon_{p,q} \cdot [g : f].$$

It follows immediately from the definitions that $\epsilon_{0,q} = 1$ for all q.

Now let e be a p-cell of a CW-complex K and let f be a q-cell of a CW-complex L. We denote by τ the set of $(p-1)$-cells of K and we denote by ϕ the set of $(q-1)$-cells of L. We compute that

$$\begin{aligned}\partial(e \times f) \;&=\; \sum_{\tau}[\tau \times f : e \times f]\cdot(\tau \times f) \;+\; \sum_{\phi}[e \times \phi : e \times f]\cdot(e \times \phi)\\ &=\; \sum_{\tau}[\tau : e]\cdot(\tau \times f) \;+\; \sum_{\phi}\epsilon_{p,q}\cdot[\phi : f]\cdot(e \times \phi) \;=\; (\partial e) \times f \;+\; \epsilon_{p,q}\cdot e \times (\partial f).\end{aligned}$$

It remains to show that $\epsilon_{p,q} = (-1)^p$. We will prove this indirectly. More precisely, since we know that $\partial^2 = 0$ in the cellular chain complex we obtain that

$$\begin{aligned}0 \;=\; \partial\partial(e \times f) \;&=\; \overbrace{(\partial\partial e) \times f}^{=0} + \epsilon_{p-1,q}\cdot(\partial e) \times (\partial f) + \epsilon_{p,q}\cdot(\partial e) \times (\partial f) + \epsilon_{p,q-1}\epsilon_{p,q}\cdot e \times \overbrace{(\partial\partial f)}^{=0}\\ &= \epsilon_{p-1,q}\cdot(\partial e) \times (\partial f) + \epsilon_{p,q}\cdot(\partial e) \times (\partial f) \;=\; (\epsilon_{p-1,q} + \epsilon_{p,q})\cdot(\partial e) \times (\partial f).\end{aligned}$$

But this equality can hold for all CW-complexes K and L only if $\epsilon_{p,q} = -\epsilon_{p-1,q}$. Since $\epsilon_{0,q} = 1$ we obtain iteratively that $\epsilon_{p,q} = (-1)^p$. ∎

90.3. **The Künneth Theorem for chain complexes.** Our goal is to determine the homology groups of the product $X \times Y$ of two topological spaces in terms of the homology groups of X and Y. In light of the Eilenberg-Zilber Theorem 90.4, or alternatively, considering Proposition 90.5, we have to find a connection between the homology groups of two chain complexes $\mathcal{C}$, $\mathcal{C}'$ and the corresponding tensor product $\mathcal{C} \otimes \mathcal{C}'$.
We start out with the following elementary lemma.

Lemma 90.6. Let $\mathcal{C}$ and $\mathcal{C}'$ be two chain complexes. The map

$$\begin{aligned}\Omega\colon \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}') \;&\to\; \mathrm{H}_{p+q}(\mathcal{C} \otimes \mathcal{C}')\\ \sum_{i=1}^{r}[c_i] \otimes [c_i'] \;&\mapsto\; \sum_{i=1}^{r}[c_i \otimes c_i']\end{aligned}$$

is well-defined and it is natural with respect to the chain complexes $\mathcal{C}$ and $\mathcal{C}'$.

Proof. Let $\mathcal{C} = (C_n, \partial_n)$ and $\mathcal{C}' = (\mathrm{C}_n', \partial_n')$ be two chain complexes. First note that if $c_i \in \mathrm{C}_p$ and $c_i' \in \mathrm{C}_q$ are cycles, then

$$d(c_i \otimes c_i') \;=\; \underbrace{\partial c_i}_{=0} \otimes c_i' + (-1)^p c_i \otimes \underbrace{\partial c_i'}_{=0} \;=\; 0$$

i.e. $c_i \otimes c_i' \in \mathcal{C} \otimes \mathcal{C}'$ is indeed a cycle. Second, if $c_i + \partial e_i$ is an other representative of $[c_i]$ then

$$[(c_i + \partial e_i) \otimes c_i'] \underset{\substack{\uparrow\\ \text{since } \partial' c' = 0}}{=} [c_i \otimes c_i' + d(e_i \otimes c_i')] \;=\; [c_i \otimes c_i'].$$

The same way one can show that the value of Ω does not depend on the choice of representative of the class in $\mathrm{H}_q(\mathcal{C}')$. It is straightforward to verify that Ω is in fact a natural map. ∎

Example. In some instances the homomorphism Ω from Lemma 90.6 is an isomorphism. For example let $\mathcal{C} = (C_n, \partial_n)$ be a chain complex of free abelian groups such that all boundary maps $\mathrm{C}_n \to \mathrm{C}_{n-1}$ in $\mathcal{C}$ are the zero map. In this case we have for any chain complex $\mathcal{C}'$ that

$$\bigoplus_{n\in\mathbb{N}_0} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_{n-p}(\mathcal{C}'_*) \underset{\substack{\uparrow\\ \text{since all boundary maps}\\ \text{in } \mathcal{C} \text{ are zero}}}{=} \bigoplus_{n\in\mathbb{N}_0} \mathrm{C}_p \otimes \mathrm{H}_n(\mathcal{C}'_{*-p}) \underset{\substack{\uparrow\\ \text{by Theorem 89.16 and the}\\ \text{Tor-Properties Lemma 89.15 (3)}\\ \text{since all } \mathrm{C}_p \text{ are free abelian}}}{=} \bigoplus_{n\in\mathbb{N}_0} \mathrm{H}_n(\mathrm{C}_p \otimes \mathcal{C}'_{*-p}) \underset{\substack{\uparrow\\ \text{by the example on page 1941}\\ \text{since } \mathcal{C} \text{ is a direct sum}\\ \text{of trivial chain complexes}}}{=} \mathrm{H}_n(\mathcal{C}_* \otimes \mathcal{C}'_*).$$

Now we can formulate the following proposition which is the key algebraic result of this chapter.

Theorem 90.7. (Künneth Theorem for Chain Complexes)[1104] Let $\mathcal{C}$ and $\mathcal{C}'$ be two chain complexes. If $\mathcal{C}$ is free, i.e. if all chain groups of the chain complex $\mathcal{C}$ are free abelian groups, then there exists a natural short exact sequence

$$0 \to \bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}') \xrightarrow{\Omega} \mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') \to \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(\mathcal{C}), \mathrm{H}_q(\mathcal{C}')) \to 0.$$

If $\mathcal{C}'$ is also a free chain complex, then this short exact sequence splits.

Remark.

(1) The adjective "natural" means in this context that if $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{C}' \to \mathcal{D}'$ are chain maps, then we obtain a commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \displaystyle\bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}') & \xrightarrow{\Omega} & \mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') & \longrightarrow & \displaystyle\bigoplus_{p+q=n} \mathrm{Tor}(\mathrm{H}_p(\mathcal{C}), \mathrm{H}_q(\mathcal{C}')) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_*\otimes f'_*} & & \downarrow{\scriptstyle (f\otimes f')_*} & & \downarrow{\scriptstyle \mathrm{Tor}(f_*, f'_*)} & & \\
0 & \longrightarrow & \displaystyle\bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{D}) \otimes \mathrm{H}_q(\mathcal{D}') & \xrightarrow{\Omega} & \mathrm{H}_n(\mathcal{D} \otimes \mathcal{D}') & \longrightarrow & \displaystyle\bigoplus_{p+q=n} \mathrm{Tor}(\mathrm{H}_p(\mathcal{D}), \mathrm{H}_q(\mathcal{D}')) & \longrightarrow & 0.
\end{array}$$

Here the middle map is given by Lemma 90.3 (1) and the map on the right is given by the functoriality of the torsion-groups that we had discussed on page 1923.

(2) If $\mathcal{C}'$ is a chain complex which consists of a single abelian group G, i.e. if it is of the form $0 \to G \to 0$ with G in degree 0, then $\mathrm{H}_0(\mathcal{C}') = G$ and all other homology groups are zero. The Künneth Theorem together with the example on page 1941 thus gives us the short exact sequence

$$0 \to \mathrm{H}_n(\mathcal{C}) \otimes G \to \mathrm{H}_n(\underbrace{\mathcal{C} \otimes G}_{=\mathcal{C}\otimes\mathcal{C}'}) \to \mathrm{Tor}(\mathrm{H}_{n-1}(\mathcal{C}), G) \to 0,$$

i.e. we recover the Universal Coefficient Theorem 89.16 as a special case of the Künneth Theorem. In fact, as we will see, the proof of the Künneth Theorem is quite similar to the proof of the Universal Coefficient Theorem 89.16.

(3) The fact that the short exact sequence of the Künneth Theorem splits implies that the middle group is isomorphic to the direct sum of the two groups to the left and right. But, as we saw in the discussion of the Universal Coefficient Theorem 89.16, see the No-Natural Splitting Proposition 89.24, this isomorphism is not natural.

[1104] Hermann Künneth (1892-1975) was a German mathematician.

Proof. Let $\mathcal{C} = (C_n, \partial_n)$ and $\mathcal{C}' = (\mathrm{C}'_n, \partial'_n)$ be two chain complexes such that all C_n are free abelian groups. As we will see, the proof of the theorem is quite similar to the proof of the Universal Coefficient Theorem 89.16.

For $p \in \mathbb{N}_0$ we consider again

$$Z_p \;:=\; \ker(\partial_p \colon \mathrm{C}_p \to \mathrm{C}_{p-1}) \quad \text{and} \quad B_p \;:=\; \mathrm{im}(\partial_{p+1} \colon \mathrm{C}_{p+1} \to \mathrm{C}_p).$$

We denote by $i \colon Z_p \to \mathrm{C}_p$ the inclusion map. Furthermore we denote by $\mathcal{Z}$ the chain complex which is given by the groups Z_p where all the boundary maps are the zero map. The same way we obtain from the groups B_p a chain complex $\mathcal{B}$.

Claim. The sequence

$$0 \;\to\; \mathcal{Z} \otimes \mathcal{C}' \;\xrightarrow{i\otimes \mathrm{id}}\; \mathcal{C} \otimes \mathcal{C}' \;\xrightarrow{\partial \otimes \mathrm{id}}\; \mathcal{B} \otimes \mathcal{C}' \;\to\; 0$$

is a sequence of chain maps and it is exact at each degree.

Proof. It follows easily from the definitions of the boundary maps in the various chain complexes, see also the diagram on page 1928, that the homomorphisms $i \otimes \mathrm{id} \colon \mathcal{Z} \otimes \mathcal{C}' \to \mathcal{C} \otimes \mathcal{C}'$ and $\partial \otimes \mathrm{id} \colon \mathcal{C} \otimes \mathcal{C}' \to \mathcal{B} \otimes \mathcal{C}$ are chain maps. It remains to show that the sequence is exact at each degree. The argument is exactly the same as on page 1928. For the reader's convenience we recall the argument. First note that by definition the sequences

$$0 \;\to\; Z_p \;\xrightarrow{i}\; \mathrm{C}_p \;\xrightarrow{\partial}\; B_{p-1} \;\to\; 0$$

are exact. Also recall that by hypothesis each C_n is a free abelian group. It follows from the Subgroup-of-Free Abelian Group Lemma 51.2 that each B_n is also free abelian. By Lemma 78.1 this means that the above short exact sequence splits. But then it follows from the Tensor-Split-SES Lemma 89.6 that the sequence stays exact even after tensoring it with any C'_q. The claim is an immediate consequence of these observations.

From the LES Proposition 74.9 and the claim we thus obtain the following long exact sequence[1105]

$$\cdots \to \mathrm{H}_n(\mathcal{B}\otimes\mathcal{C}') \xrightarrow{d_n} \mathrm{H}_n(\mathcal{Z}\otimes\mathcal{C}') \to \mathrm{H}_n(\mathcal{C}\otimes\mathcal{C}') \to \mathrm{H}_{n-1}(\mathcal{B}\otimes\mathcal{C}') \xrightarrow{d_{n-1}} \mathrm{H}_{n-1}(\mathcal{Z}\otimes\mathcal{C}') \to \dots$$

The connecting homomorphism $d_n \colon \mathrm{H}_n(\mathcal{B}\otimes\mathcal{C}') \to \mathrm{H}_n(\mathcal{Z}\otimes\mathcal{C}')$ from the LES Proposition 74.9 is hereby, as in the proof of the Universal Coefficient Theorem 89.16, induced by the map $i \otimes \mathrm{id}$ where i denotes the inclusion $\mathcal{B} \to \mathcal{Z}$.

Since the boundary maps in the chain complexes $\mathcal{B}$ and $\mathcal{Z}$ are the zero maps it follows from the example on page 1945 that we obtain the following long exact sequence:

$$\dots \xrightarrow{i\otimes\mathrm{id}} \bigoplus_{p+q=n} Z_p \otimes \mathrm{H}_q(\mathcal{C}') \to \mathrm{H}_n(\mathcal{C}\otimes\mathcal{C}') \to \bigoplus_{p+q=n-1} B_p \otimes \mathrm{H}_q(\mathcal{C}') \xrightarrow{i\otimes\mathrm{id}} \bigoplus_{p+q=n-1} Z_p \otimes \mathrm{H}_q(\mathcal{C}') \to \dots$$

[1105] The index goes down by one in the map $\mathrm{H}_n(\mathcal{C}\otimes\mathcal{C}') \to \mathrm{H}_{n-1}(\mathcal{B}\otimes\mathcal{C}')$, since the map is induced by the short exact sequence $0 \to Z_p \to \mathrm{C}_p \to B_{p-1} \to 0$.

From the Break-into-SES Lemma 78.4 we obtain a short exact sequence

$$0 \to \bigoplus_{p+q=n} \overbrace{\operatorname{coker}\left(B_p \otimes \mathrm{H}_q(\mathcal{C}') \xrightarrow{i\otimes \mathrm{id}} Z_p \otimes \mathrm{H}_q(\mathcal{C}')\right)}^{\substack{= \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}'), \text{ since} \\ 0 \to B_p \to Z_p \to \mathrm{H}_p(\mathcal{C}) \to 0 \text{ is a free resolution}}} \to \mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') \to$$
$$\to \bigoplus_{p+q=n-1} \underbrace{\ker\left(B_p \otimes \mathrm{H}_q(\mathcal{C}') \xrightarrow{i\otimes \mathrm{id}} Z_p \otimes \mathrm{H}_q(\mathcal{C}')\right)}_{\substack{= \operatorname{Tor}(\mathrm{H}_p(\mathcal{C}), \mathrm{H}_q(\mathcal{C}')), \text{ since} \\ 0 \to B_p \to Z_p \to \mathrm{H}_p(\mathcal{C}) \to 0 \text{ is a free resolution}}} \to 0.$$

This proves the existence of the short exact sequence. Using Statement (2) of the Fundamental Theorem 81.1 of Homological Algebra one can show that this short exact sequence is natural. We leave it to the reader to verify the details.

Finally we suppose that $\mathcal{C}'_*$ is also a free chain complex. We leave it again to the reader to adapt the argument of the proof of the UCT-Splitting Proposition 89.21 to show that the above short exact sequence splits. Alternatively we refer to [**Mun1984**, Chapter 58] for details. ■

90.4. **The Künneth Theorem for topological spaces.** Now we return to our initial Question 90.1 of the chapter, namely, what is the relationship between the homology groups of the topological spaces X, Y and $X \times Y$? The following theorem gives a comprehensive answer.

Theorem 90.8. (Künneth Theorem for Topological Spaces) Let X and Y be topological spaces. Then there exists a short exact sequence

$$0 \to \bigoplus_{p+q=n} \mathrm{H}_p(X) \otimes \mathrm{H}_q(Y) \xrightarrow{\Omega} \mathrm{H}_n(X \times Y) \to \bigoplus_{p+q=n-1} \operatorname{Tor}(\mathrm{H}_p(X), \mathrm{H}_q(Y)) \to 0$$

with the following four properties:

(1) Both maps are natural with respect to the topological spaces X and Y.
(2) For $p = 0$ the map $\Omega\colon \mathrm{H}_0(X) \otimes \mathrm{H}_q(Y) \to \mathrm{H}_q(X \times Y)$ is the "obvious" map, namely for a point $x \in X$ and a singular cycle σ the image of $[x] \otimes [\sigma] \in \mathrm{H}_0(X) \otimes \mathrm{H}_q(Y)$ under Ω equals the image of $[\sigma]$ under the map $\mathrm{H}_q(Y) \to \mathrm{H}_q(\{x\} \times Y) \to \mathrm{H}_q(X \times Y)$, where the first map is induced by the natural homeomorphism and the second map is induced by the inclusion.
(3) The analogue of (2) also holds for $q = 0$.
(4) The above short exact sequence splits.[1106]

Remark. Both maps in the short exact sequence of the Künneth Theorem 90.8 for Topological Spaces are difficult to describe explicitly. At least for the former map we will partly get a nice description in the Topological Product Orientation Proposition 113.10.

[1106] But as in the case of the Universal Coefficient Theorem 89.16, see the No-Natural Splitting Proposition 89.24, the splitting is not natural.

Proof. We consider the following maps:

$$0 \to \bigoplus_{p+q=n} \mathrm{H}_p(X) \otimes \mathrm{H}_q(Y) \xrightarrow{\Omega} \mathrm{H}_n(\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X), \mathrm{H}_q(Y)) \to 0$$

$$\cong \Big\downarrow \Upsilon$$

$$\underbrace{\mathrm{H}_n(\mathrm{C}_*(X \times Y))}_{=H_n(X \times Y)}.$$

Here the sequence on the top is exact and natural by the Künneth Theorem 90.7 for Chain Complexes. The vertical map is a natural isomorphism by the Eilenberg-Zilber Theorem 90.4 together with Corollary 73.4. We obtain the desired short exact sequence from replacing the upper middle term by the lower middle term $\mathrm{H}_n(\mathrm{C}_*(X \times Y)) = \mathrm{H}_n(X \times Y)$. One can easily verify that this short exact sequence has all the desired properties. ■

Examples.

(1) Let X be a topological space. We want to determine the homology groups of $S^1 \times X$. The homology groups of S^1 are of course $\mathbb{Z}$ in the dimensions 0 and 1. For any abelian group A we have by the Tensor Lemma 89.3 a natural isomorphism $\mathbb{Z} \otimes A = A$ and by the Tor-Properties Lemma 89.15 we have $\mathrm{Tor}(\mathbb{Z}, A) = 0$. It follows from the Künneth Theorem 90.8 for topological spaces that there exists a natural isomorphism

$$\mathrm{H}_n(S^1 \times X) \xrightarrow{\cong} \mathrm{H}_{n-1}(X) \oplus \mathrm{H}_n(X).$$

This is of course precisely the same result as in Lemma 78.21.

It follows fairly easily from Statement (2) that the map $\mathrm{H}_n(X) \to \mathrm{H}_n(S^1 \times X)$ is the map induced by the projection $S^1 \times X \to X$. It is less clear what the map $\mathrm{H}_n(S^1 \times X) \to \mathrm{H}_{n-1}(X)$ should be. Some more effort shows that this map is precisely the map that we had described in detail in Exercise 75.4.

(2) Let $n \in \mathbb{N}$ and let $m \in \mathbb{N}$. We want to determine the homology groups of the generalized torus $(S^m)^n$. In fact using the Künneth Theorem 90.8 for topological spaces one can easily show that for any $k \in \mathbb{N}_0$ there exists an isomorphism

$$\mathrm{H}_i((S^m)^n) \cong \begin{cases} \mathbb{Z}^{\binom{n}{k}} = \mathbb{Z}^{\frac{n!}{k! \cdot (n-k)!}} & \text{if } i = km \text{ with } k \in \{0, \ldots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

In particular we have an isomorphism

$$\mathrm{H}_i((S^m)^2) \cong \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2m, \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{if } i = m, \\ 0, & \text{otherwise.} \end{cases}$$

We will fill in the details in Exercise 90.1.

(3) Let X and Y be two path-connected topological spaces. Then we have isomorphisms

$$\mathrm{H}_1(X \times Y) \underset{\uparrow}{\cong} \mathrm{H}_1(X) \otimes \mathrm{H}_0(Y) \oplus \mathrm{H}_0(X) \otimes \mathrm{H}_1(Y) \underset{\uparrow}{=} \mathrm{H}_1(X) \oplus \mathrm{H}_1(Y).$$

Künneth Theorem 90.8 for topological spaces — H_0-Proposition 72.5 and the Tensor Lemma 89.3

This is consistent with the product formula for fundamental groups proved in the Fundamental Group-of-Product Proposition 48.20 and with the isomorphism of the Hurewicz Theorem 84.5.

Now we discuss the product $S^2 \times S^4$ in slightly more detail. It follows easily from the Künneth Theorem 90.8 for topological spaces, Proposition 74.4, the H_0-Proposition 72.5, the Tensor Lemma 89.3 and the Tor-Properties Lemma 89.15 that

$$\mathrm{H}_k(S^2 \times S^4) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, 6, \\ 0, & \text{otherwise.} \end{cases}$$

But by the calculation on page 1772 these are precisely the homology groups of the complex projective space $\mathbb{C}\mathrm{P}^3$. Furthermore it follows from Propositions 47.13 and 48.20 that $S^2 \times S^4$ is simply connected and it follows from the discussion on page 1533 that $\mathbb{C}\mathrm{P}^3$ is also simply connected. So all the invariants we are aware of, and that we know how to compute[1107], agree for $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$. This raises the following question which we will answer in Corollary 121.9.

Question 90.9. Are the topological spaces $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ homeomorphic?

Exercises for Chapter 90.

Exercise 90.1. Let $n \in \mathbb{N}$ and let $m \in \mathbb{N}$. Show that for every $k \in \mathbb{N}_0$ there exists an isomorphism

$$\mathrm{H}_i((S^m)^n) \;\cong\; \begin{cases} \mathbb{Z}^{\binom{n}{k}} = \mathbb{Z}^{\frac{n!}{k!\cdot(n-k)!}} & \text{if } i = km \text{ with } k \in \{0, \ldots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 90.2. Let $n \in \mathbb{N}_0$. Show that

$$\sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} \;=\; 0.$$

Remark. There are many ways to prove this equality, one fun way of doing this is by computing the Euler characteristic of the n-dimensional torus in two different ways.

Exercise 90.3. We consider the oriented 2-dimensional smooth manifold Σ that is shown in the figure below. It has three oriented boundary components A, B, C. We pick orientation-preserving diffeomorphisms $\beta\colon S^1 \to B$ and $\gamma\colon S^1 \to C$. We consider

$$M \;:=\; (\Sigma \times S^1)/\sim \qquad \text{where } (\varphi(z), w) \sim (\psi(z), \overline{w}) \text{ for } (z, w) \in S^1 \times S^1.$$

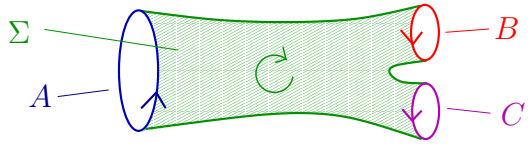
By the Gluing-Smooth Manifolds-Proposition **??** we know that M is a compact orientable 3-dimensional smooth manifold with boundary $\partial M = A \times S^1$. We denote by $i\colon \partial M \to M$

[1107] As of right now we do not know the higher homotopy groups of these two smooth manifolds, but these will not turn out to be very useful.

the inclusion. Show that there exist isomorphisms $\Phi\colon \mathbb{Z}\oplus\mathbb{Z}\to \mathrm{H}_1(\partial M)$ and $\Psi\colon \mathbb{Z}\oplus\mathbb{Z}_2\to \mathrm{H}_1(M)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{(a,b)\mapsto(2a,[b])} & \mathbb{Z}\oplus\mathbb{Z}_2 \\ {\scriptstyle\Phi}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle\Psi} \\ \mathrm{H}_1(\partial M) & \xrightarrow{i_*} & \mathrm{H}_1(M). \end{array}$$

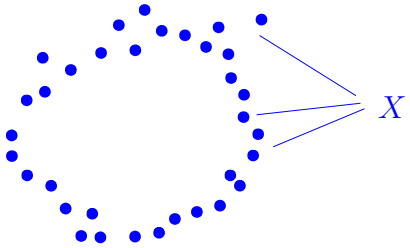
Remark. The oriented boundary components A, B, C define elements $a, b, c \in \mathrm{H}_1(\Sigma)$. On page 1784 we saw that $\mathrm{H}_1(\Sigma) \cong (\mathbb{Z}\cdot a \oplus \mathbb{Z}\cdot b \oplus \mathbb{Z}\cdot c)/\mathbb{Z}\cdot(a+b+c)$.



91. Applications of homology groups

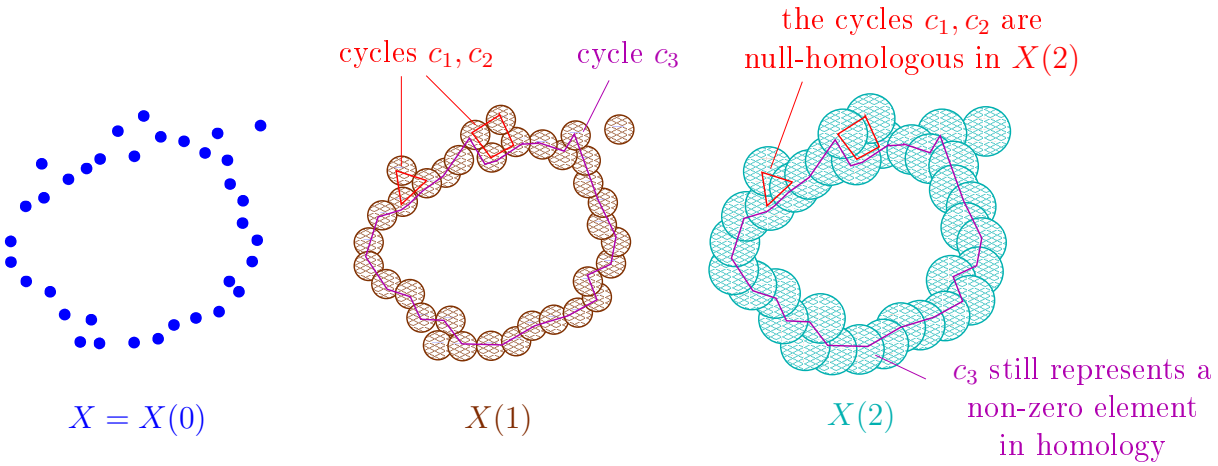
91.1. **Persistent homology ❀.** In this section we outline the idea behind "persistent homology". Persistent homology was introduced by Gunnar Carlsson [**Carl2009**] in 2009 More information and a much more careful treatment of persistent homology can for example be found in [**EdH2010**]. Some extra information can also be found in [**Gh2014**].

We consider the figure below. What do we see? If we forget about topology for a second, then we would say that we see a set X of dots that form a loop.



dots that appear to the eye to form a loop

If we view X as a topological space, then it is just a finite discrete subset of $\mathbb{R}^2$. In particular its homology groups are non-zero only in dimension 0. So how can we use the mathematical machine that we have developed over the last two semesters to "see the loop"?
Given $r \in [0, \infty)$ we denote by $X(r)$ the subset of $\mathbb{R}^2$ that is given by replacing each point $x \in X$ by the closed ball $\overline{B}_r(x)$ of radius r around x. The homeomorphism type of $X(r)$ changes with the radius r. For example in the figure below we see that at $r = 1$ all but one ball are joined up and we see that $\mathrm{H}_1(X(1)) \cong \mathbb{Z}^3$. We show three cycles c_1, c_2, c_3 in $X(1)$ which form a basis for $\mathrm{H}_1(X(1))$. Two of the generators, namely c_1, c_2 are formed "by accident", namely they arise from balls which are close by. This gets captured by the fact that they become null-homologous in $\mathrm{H}_1(X(2))$ whereas c_3 represents a non-trivial element for "much longer", i.e. for much larger values of r. Put differently, c_3 is a persistent element in homology, hence it captures global information.



Summarizing, given a subset X of $\mathbb{R}^2$ we can study the topological spaces $X(r)$ for any $r \in [0, \infty)$ and furthermore for any $r < s$ we can study the inclusion induced map $\mathrm{H}_n(X(r)) \to$

$\mathrm{H}_n(X(s))$. If there is a non-zero element in $\mathrm{H}_n(X(r))$ which stays non-zero for a "long time", then we say it is persistent and we hope that it captures something interesting.

The same approach can be applied to any finite subset of some $\mathbb{R}^n$. Pretty much any data one has in practice, e.g. blue pixels of a picture, measurements in an experiment, statistical data and so on, can be viewed as a finite subset of some $\mathbb{R}^n$, so in principle one can study its persistent homology. The real question is, whether one captures something significant or not. For example persistent homology was applied in **[SHP2016]**[1108] to study the "The topological shape of Brexit". It is perhaps debatable whether one can really obtain anything meaningful about the politics of Brexit using persistent homology.

91.2. **The transfer map.** Before we can give the next application of homology theory, namely the Borsuk-Ulam Theorem in higher dimensions, we need to introduce and study the transfer map in homology corresponding to a finite covering.

Let $p\colon \widetilde{X} \to X$ be a k-fold covering of a topological space X. Let $\sigma\colon \Delta^n \to X$ be a singular simplex. Since Δ^n is simply connected[1109] it follows from the Map Lifting Criterion 61.2 that there exist precisely k different lifts of σ to $\widetilde{X}$, i.e. there exist precisely k different singular simplices $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_k\colon \Delta^n \to \widetilde{X}$ with $p \circ \widetilde{\sigma}_i = \sigma$ for $i = 1, \ldots, k$. We write

$$p^*(\sigma) \ := \ \sum_{i=1}^{k} \widetilde{\sigma}_i.$$

Given any commutative ring R we refer to the map

$$\begin{array}{rcl} p^*\colon \mathrm{C}_*(X;R) & \to & \mathrm{C}_*(\widetilde{X};R) \\ \sum_{i=1}^{n} \sigma_i \otimes r_i & \mapsto & \sum_{i=1}^{n} p^*(\sigma_i) \otimes r_i \end{array}$$

as the *transfer map* of the covering. It is straightforward to verify that the transfer map is in fact a chain map. In particular it induces a map

$$p^*\colon \mathrm{H}_*(X;R) \ \to \ \mathrm{H}_*(\widetilde{X};R).$$

Finally suppose we are given a subset $A \subset X$. We write $\widetilde{A} := p^{-1}(A)$. One easily verifies that the transfer map $p^*\colon \mathrm{C}_n(X;R) \to \mathrm{C}_*(\widetilde{X};R)$ induces a chain map

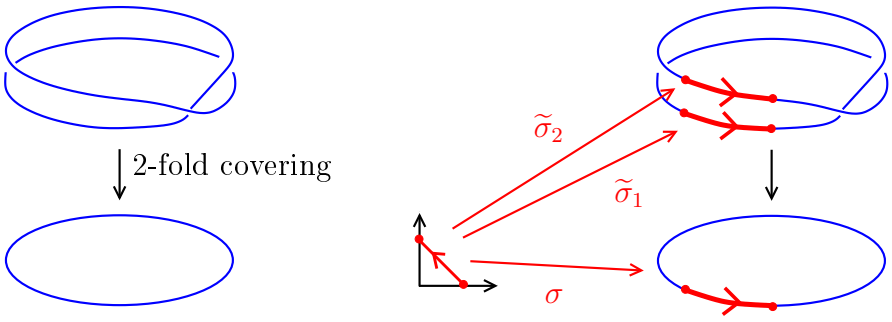
$$p^*\colon \mathrm{H}_*(X,A;R) \ \to \ \mathrm{H}_*(\widetilde{X},\widetilde{A};R).$$

Proposition 91.1. (Transfer Map Proposition) Let $p\colon \widetilde{X} \to X$ be a covering of degree $k \in \mathbb{N}$ of a topological space X and let $n \in \mathbb{N}_0$. Then the following hold:

(1) For any commutative ring R the transfer map $p^*\colon \mathrm{C}_n(X;R) \ \to \ \mathrm{C}_n(\widetilde{X};R)$ is a monomorphism.
(2) For any commutative ring R the map $p_* \circ p^*\colon \mathrm{H}_n(X;R) \to \mathrm{H}_n(X;R)$ is given by multiplication by k.
(3) The projection $p_*\colon \mathrm{H}_n(\widetilde{X};\mathbb{Q}) \to \mathrm{H}_n(X;\mathbb{Q})$ is an epimorphism.

[1108] Here is an easy way to tell that that paper was not written by mathematicians: In a mathematics paper the authors are invariably listed according to the alphabetic order of their last names. In most other sciences this is not the case.

[1109] To be precise, we also use that Δ^n is locally path-connected.



The analogous statements also hold in the relative setting.

Remark.

(1) If $p\colon \widetilde{X} \to X$ is a finite covering of path-connected topological spaces, then we saw in Corollary 84.8 that the map $p_*\colon \mathrm{H}_1(\widetilde{X}) \to \mathrm{H}_1(X)$ has finite cokernel. This implies in particular[1110] that $p_*\colon \mathrm{H}_1(\widetilde{X};\mathbb{Q}) \to \mathrm{H}_1(X;\mathbb{Q})$ is an epimorphism. Therefore we can view the Transfer Map Proposition 91.1 (3) as a generalization of Corollary 84.8.

(2) The conclusion of the Transfer Map Proposition 91.1 (3) does not hold if we drop the condition that the covering is finite. For example, we have an infinite covering $p\colon \mathbb{R} \to S^1$ but $0 = \mathrm{H}_1(\mathbb{R};\mathbb{Q}) \xrightarrow{p_*} \mathrm{H}_1(S^1;\mathbb{Q}) = \mathbb{Q}$ is not an epimorphism.

Proof. Let $p\colon \widetilde{X} \to X$ be a k-fold covering of a topological space X.

(1) Let $c \in \mathrm{C}_n(X;R)$ be non-trivial. We write $c = \sum_{i=1}^{n} \sigma_i \otimes r_i$ where $\sigma_1, \dots, \sigma_n$ are pairwise different singular simplices in X and where $r_1, \dots, r_n$ are all non-zero. Since each σ_i lifts to k different singular simplices and since for $i \neq j$ the lifts of σ_i and σ_j differ we see immediately from the definition that $p^*(\sigma)$ is also non-zero.

(2) Let R be a commutative ring. We denote by $p^*\colon \mathrm{C}_n(X;R) \to \mathrm{C}_n(\widetilde{X};R)$ the transfer map. Let $\sigma\colon \Delta^m \to X$ be a singular simplex. We denote the corresponding lifts by $\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_k$. Then

$$(p_* \circ p^*)(\sigma) \;=\; p_*(\widetilde{\sigma}_1 + \cdots + \widetilde{\sigma}_k) \;=\; \underbrace{p\circ\widetilde{\sigma}_1 + \cdots + p\circ\widetilde{\sigma}_k}_{k \text{ summands}} \;=\; \underbrace{\sigma + \cdots + \sigma}_{k \text{ summands}} \;=\; k\cdot\sigma.$$

We thus see that $p_* \circ p^*\colon \mathrm{C}_m(X) \to \mathrm{C}_m(X)$ is multiplication by k for any m. It follows easily that the map

$$\mathrm{H}_n(X;R) \xrightarrow{p^*} \mathrm{H}_n(\widetilde{X};R) \xrightarrow{p_*} \mathrm{H}_n(X;R)$$

is also given by multiplication by k.

(3) Since multiplication by $k \neq 0$ is an isomorphism of any rational vector space we obtain from (1) that $p_* \circ p^*\colon \mathrm{H}_n(X;\mathbb{Q}) \to \mathrm{H}_n(X;\mathbb{Q})$ is an isomorphism. In particular the second map, namely p_*, is an epimorphism.

The proofs in the relative setting are basically identical to the above. ∎

[1110] Why does this follow?

Proposition 91.2. (Double Covering-LES-Proposition) Let $p\colon \widetilde{X} \to X$ be a 2-fold covering. Then there exists a natural long exact sequence of the form

$$\ldots \longrightarrow \mathrm{H}_n(X;\mathbb{F}_2) \xrightarrow{p^*} \mathrm{H}_n(\widetilde{X};\mathbb{F}_2) \xrightarrow{p_*} \mathrm{H}_n(X;\mathbb{F}_2) \xrightarrow{\partial} \mathrm{H}_{n-1}(X;\mathbb{F}_2) \longrightarrow \ldots.$$

Here "natural" means that if we are given a commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y \end{array}$$

where $p\colon \widetilde{X} \to X$ and $q\colon \widetilde{Y} \to Y$ are 2-fold coverings, then the following diagram commutes:

$$\begin{array}{ccccccccc} \ldots \longrightarrow & \mathrm{H}_n(X;\mathbb{F}_2) & \xrightarrow{p^*} & \mathrm{H}_n(\widetilde{X};\mathbb{F}_2) & \xrightarrow{p_*} & \mathrm{H}_n(X;\mathbb{F}_2) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(X;\mathbb{F}_2) & \longrightarrow \ldots \\ & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \widetilde{f}} & & \downarrow{\scriptstyle \widetilde{f}} & & \downarrow{\scriptstyle f} & \\ \ldots \longrightarrow & \mathrm{H}_n(Y;\mathbb{F}_2) & \xrightarrow{q^*} & \mathrm{H}_n(\widetilde{Y};\mathbb{F}_2) & \xrightarrow{q_*} & \mathrm{H}_n(Y;\mathbb{F}_2) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(Y;\mathbb{F}_2) & \longrightarrow \ldots. \end{array}$$

Remark. Note that it follows from the Topological-Pullback Lemma 56.16 (3) a 2-fold covering $p\colon \widetilde{Y} \to Y$ together with a map $f\colon X \to Y$ of topological spaces naturally give rise to a commutative diagram as in the Double Covering-LES-Proposition 91.2.

Example. Now we will see that we can use the Double Covering-LES-Proposition 91.2 to calculate the homology groups $\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ just using the following basic facts:

(1) By the discussion on page 1916 we know that

$$\mathrm{H}_k(S^n;\mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } k=0 \text{ or } k=n, \\ 0, & \text{otherwise} \end{cases}$$

(2) By the obvious generalization of the H_0-Proposition 72.5 we know that $\mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) = \mathbb{F}_2$.

(3) Since $\mathbb{R}\mathrm{P}^n$ is an n-dimensional CW-complex we know from the discussion on page 1915 that $\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) = 0$ for $k > n$.

In the following discussion we write $\mathbb{F} = \mathbb{F}_2$ for space reasons. Applying the Double Covering-LES-Proposition 91.2 to the 2-fold covering $p\colon S^n \to \mathbb{R}\mathrm{P}^n$ we obtain the following long exact sequence:

$$\underbrace{\mathrm{H}_{n+1}(\mathbb{R}\mathrm{P}^n;\mathbb{F})}_{=0} \xrightarrow{\partial} \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_n(S^n;\mathbb{F})}_{=\mathbb{F}} \xrightarrow{p_*} \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{\partial} \mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_{n-1}(S^n;\mathbb{F})}_{=0} \to$$

which continues all the way to

$$\ldots \to \mathrm{H}_1(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_1(S^n;\mathbb{F})}_{=0} \xrightarrow{p_*} \mathrm{H}_1(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{\partial} \mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_0(S^n;\mathbb{F})}_{=\mathbb{F}} \xrightarrow{p_*} \underbrace{\mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{F})}_{=\mathbb{F}} \to 0.$$

Now we can make the following observations:

(1) If we work our way upward through the long exact sequence we see that the connecting homomorphisms give us isomorphisms

$$\mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^2;\mathbb{F}) \xrightarrow[\cong]{\partial} \mathrm{H}_{n-2}(\mathbb{R}\mathrm{P}^2;\mathbb{F}) \xrightarrow[\cong]{\partial} \dots \xrightarrow[\cong]{\partial} \mathrm{H}_1(\mathbb{R}\mathrm{P}^2;\mathbb{F}) \xrightarrow[\cong]{\partial} \mathrm{H}_0(\mathbb{R}\mathrm{P}^2;\mathbb{F}) \;\cong\; \mathbb{F}.$$

(2) Now that we know that $\mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \cong \mathbb{F}$ we obtain from the upper end of the long exact sequence that[1111]

$$\mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \mathrm{H}_n(S^n;\mathbb{F}) \quad \text{and} \quad \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{\partial} \mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F})$$

are also isomorphisms.

Summarizing we have shown that

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \;\cong\; \begin{cases} \mathbb{F}, & \text{if } k = 0,\dots,n, \\ 0, & \text{otherwise.} \end{cases}$$

This agrees of course with the result that we had obtained on page 1916.

Proof of the Double Covering-LES-Proposition 91.2. Let $p\colon \widetilde{X} \to X$ be a 2-fold covering. First we want to show that the promised long exact sequence in homology exists. By the LES Proposition 74.9 it suffices to show that the following sequence of chain maps

$$0 \;\to\; \mathrm{C}_*(X;\mathbb{F}) \xrightarrow{p^*} \mathrm{C}_*(\widetilde{X};\mathbb{F}) \xrightarrow{p_*} \mathrm{C}_*(X;\mathbb{F}) \;\to\; 0$$

is in fact exact. As in Lemma 48.4 we consider the continuous map

$$\begin{aligned} f\colon \widetilde{X} &\to \widetilde{X} \\ x &\mapsto \text{the unique other element of } p^{-1}(p(x)). \end{aligned}$$

Using the Map Lifting Criterion 61.2 we pick for each singular cell σ of X a lift $\widetilde{\sigma}$ to $\widetilde{X}$. We make the following observations which follow easily from the definitions and the Map Lifting Criterion 61.2:

(a) given any singular n-simplex σ we have $p_*(\widetilde{\sigma}) = \sigma$,
(b) given any singular n-simplex σ we have $p^*(\sigma) = \widetilde{\sigma} + f_*(\widetilde{\sigma})$,
(c) given any singular simplex μ of $\widetilde{X}$ we have $p_*(f_*(\mu)) = p_*(\mu)$,
(d) any singular singular n-simplex μ of $\widetilde{X}$ is of the form $\mu = \widetilde{\sigma}$ or $\mu = f_*(\widetilde{\sigma})$ for some singular n-simplex σ of X.

Now we turn to the proof of the exactness of the above sequence.

(1) We first show that the map $p_*\colon \mathrm{C}_n(\widetilde{X};\mathbb{F}) \to \mathrm{C}_n(X;\mathbb{F}) = \mathrm{C}_n(X) \otimes \mathbb{F}$ is surjective. So let $c = \sigma_1 \otimes b_1 + \dots + \sigma_k \otimes b_k \in \mathrm{C}_n(X;\mathbb{F})$. It follows from (a) that

$$p_*\big(\widetilde{\sigma}_1 \otimes b_1 + \dots + \widetilde{\sigma}_k \otimes b_k\big) \;=\; \sigma_1 \otimes b_1 + \dots + \sigma_k \otimes b_k \;=\; c.$$

(2) By the Transfer Map Proposition 91.1 we know that the transfer map p^* is a monomorphism.

[1111] Why does this follow, why are there no other possibilities?

(3) Next we show that $\mathrm{im}(p^*) \subset \ker(p_*)$. For any singular n-simplex σ of $\widetilde{X}$ we have by (c) above that[1112] $(p_* \circ p^*)(\sigma) = p_*(\widetilde{\sigma} + f_*(\widetilde{\sigma})) = 2\sigma$, but it follows from Lemma 89.1 (4) that any multiple of two is zero in $\mathrm{C}_*(X;\mathbb{F}) = \mathrm{C}_*(X) \otimes \mathbb{F}$.

(4) Finally we have to show that $\ker(p_*) \subset \mathrm{im}(p^*)$. So let $c \in \ker(p_*)$. By (d) we can write
$$c = (\widetilde{\sigma}_1 \otimes a_1 + f_*(\widetilde{\sigma}_1) \otimes b_1) + \cdots + (\widetilde{\sigma}_k \otimes a_k + f_*(\widetilde{\sigma}_k) \otimes b_k)$$
where $\sigma_1, \ldots, \sigma_k$ are pairwise different and where $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{F}$. By (c) we have
$$p(c) = \sigma_1 \otimes (a_1 + b_1) + \cdots + \sigma_k(a_k + b_k).$$
But this implies that $a_i = -b_i$ for $i = 1, \ldots, k$. But since we work over $\mathbb{F}$ we actually have $a_i = b_i$ for $i = 1, \ldots, k$. It follows from (b) that
$$c = p^*(\sigma_1 \otimes a_1 + \cdots + \sigma_k \otimes a_k),$$
as desired.

The fact the long exact sequence is indeed natural follows easily from the definitions and the naturality of the connecting homomorphism, see the Connecting Homomorphism Lemma 74.8 (2). ■

91.3. **The Borsuk-Ulam Theorem.** Our goal in this section is to give another proof of the following theorem:

Theorem 39.8. (Borsuk-Ulam Theorem) Let $n \in \mathbb{N}_0$. For every map $f\colon S^n \to \mathbb{R}^n$ there exists a pair of antipodal points x and $-x$ on S^n with $f(x) = f(-x)$.

Example. If our universe happens to be diffeomorphic to S^3, then there exists a pair of antipodal points where the temperature, distance to the Kneitinger and strength of the magnetic field agree.

Remark.

(1) We already gave a proof of the Borsuk-Ulam Theorem and in Sections 97.6 and 121.3 we will provide two other proofs of the Borsuk-Ulam Theorem 39.8. Also recall that in Theorem 48.21 we proved the Borsuk-Ulam Theorem in the special case that $n = 2$.

(2) l case $n = 2$.

(3) The Borsuk-Ulam Theorem 39.8 has many applications, in particular the Lusternik-Schnirelmann Theorem 40.1, the Ham-Sandwich Theorem 40.2, Pfister's Level Theorem 40.9 and the Necklace Theorem 40.11.

The key ingredient in the proof of the Borsuk-Ulam Theorem 39.8 is the following proposition.

Proposition 91.3. Let $f\colon S^n \to S^n$ be a map. If $f(-x) = -f(x)$ for all $x \in S^n$, then $\deg(f)$ is odd.

Remark. In the Smooth Odd Degree Proposition 39.9 we used a very different argument to provide a proof of the Odd Degree Proposition 39.9 under the stronger hypothesis that f is actually smooth.

Proof of the Odd Degree Proposition 91.3. Let $f\colon S^n \to S^n$ be a map such that $f(-x) = -f(x)$ for all $x \in S^n$. By the Degree-and-Coefficients Lemma 89.20 it suffices to show that $f_*\colon \mathrm{H}_n(S^n;\mathbb{Z}_2) \to \mathrm{H}_n(S^n;\mathbb{Z}_2)$ is an isomorphism.

[1112] That is of course basically the same argument as in the proof of the Transfer Map Proposition 91.1.

We start out with the observation that the map $f\colon S^n \to S^n$ evidently descends to a well-defined map

$$\begin{array}{rcl} \overline{f}\colon \mathbb{R}\mathrm{P}^n & \to & \mathbb{R}\mathrm{P}^n \\ [x] & \mapsto & [f(x)]. \end{array}$$

Claim. The diagram

$$\begin{array}{ccccccccccc} \longrightarrow \mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{p^*} & \mathrm{H}_k(S^n;\mathbb{Z}_2) & \xrightarrow{p_*} & \mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{\partial} & \mathrm{H}_{k-1}(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{p^*} & \mathrm{H}_{k-1}(S^n;\mathbb{Z}_2) & \longrightarrow \\ \downarrow{\scriptstyle \overline{f}_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \overline{f}_*} & & \downarrow{\scriptstyle \overline{f}_*} & & \downarrow{\scriptstyle f_*} & \\ \longrightarrow \mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{p^*} & \mathrm{H}_k(S^n;\mathbb{Z}_2) & \xrightarrow{p_*} & \mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{\partial} & \mathrm{H}_{k-1}(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{p^*} & \mathrm{H}_{k-1}(S^n;\mathbb{Z}_2) & \longrightarrow \end{array}$$

where the horizontal sequences are the exact sequences given by the Double Covering-LES-Proposition 91.2, commutes.

Proof. By the LES Proposition 74.9 it suffices to show that the following diagram commutes for every $k \in \mathbb{N}_0$:

$$\begin{array}{ccccccccc} 0 & \rightarrowtail & \mathrm{C}_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \overset{p^*}{\rightarrowtail} & \mathrm{C}_k(S^n;\mathbb{Z}_2) & \overset{p_*}{\twoheadrightarrow} & \mathrm{C}_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \twoheadrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{f}_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \overline{f}_*} & & \\ 0 & \rightarrowtail & \mathrm{C}_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \overset{p^*}{\rightarrowtail} & \mathrm{C}_k(S^n;\mathbb{Z}_2) & \overset{p_*}{\twoheadrightarrow} & \mathrm{C}_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \twoheadrightarrow & 0. \end{array}$$

Here the right-hand square commutes since by definition we have $p \circ f = \overline{f} \circ p$. Now we argue that the left-hand square commutes. So let $\sigma\colon \Delta^k \to \mathbb{R}\mathrm{P}^n$ be a singular k-simplex. We pick a point $P \in \Delta^k$. Furthermore we pick $Q \in S^n$ with $[Q] = \sigma(1,0,\ldots,0) \in \mathbb{R}\mathrm{P}^n$. We denote by $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ the two lifts of σ to S^n. Now we consider the singular simplex $\overline{f} \circ \sigma$. It remains to show that $p^*(\overline{f} \circ \sigma) = f \circ \widetilde{\sigma}_1 + f \circ \widetilde{\sigma}_2$. But this is indeed the case for the following reason:

(1) by construction we have $p \circ f = \overline{f} \circ p$ which implies that $f \circ \widetilde{\sigma}_1$ and $f \circ \widetilde{\sigma}_2$ are both lifts of $\overline{f} \circ \sigma$,
(2) it follows from $f(Q) = -f(Q)$ that $f \circ \widetilde{\sigma}_1$ and $f \circ \widetilde{\sigma}_2$ are two different lifts. ⊞

Now we consider the following diagram:

$$\begin{array}{ccccccccccc} \mathrm{H}_n(S^n;\mathbb{Z}_2) & \xleftarrow{p^*} & \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{\partial} & \ldots & \xrightarrow{\partial} & \mathrm{H}_1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{\partial} & \mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \overline{f}_*} & & \downarrow{\scriptstyle \overline{f}_*} & & & & \downarrow{\scriptstyle \overline{f}_*} & & \downarrow{\scriptstyle \overline{f}_*} \\ \mathrm{H}_n(S^n;\mathbb{Z}_2) & \xleftarrow{p^*} & \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{\partial} & \ldots & \xrightarrow{\partial} & \mathrm{H}_1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{\partial} & \mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2). \end{array}$$

We make the following observations:

(1) the diagram commutes by the previous claim,
(2) all the horizontal maps are isomorphisms by the discussion on page 1955,
(3) by the discussion on page 1594 the right vertical map is an isomorphism.

It follows that the left vertical map is an isomorphism. But that is exactly what we had wanted to prove. ■

Now we turn to the proof of the Borsuk-Ulam Theorem 39.8.

Proof of Theorem 39.8. Let $g\colon S^n \to \mathbb{R}^n$ be a map. Suppose there exists no pair of antipodal points x and $-x$ in S^n with $g(x) = g(-x)$. Put differently, suppose that for any

$x \in S^n$ we have $g(x) \neq g(-x)$. We consider the composition of the following two maps

$$\begin{array}{rcccl} \Phi\colon S^{n-1} \times [0,1] & \to & S^n_{\geq 0} & \to & S^{n-1}. \\ (y,t) & \mapsto & (y \cdot \sqrt{1-t^2}, t) & & \\ & & x & \mapsto & \underbrace{\frac{g(x)-g(-x)}{\|g(x)-g(-x)\|}} \\ & & & & \text{defined since } g(x) \neq g(-x) \end{array}$$

Now we see that

$$\text{odd number} \overset{\text{by definition we have } \Phi(-x,0) = -\Phi(x,0), \text{ thus it follows from the Odd Degree Proposition 91.3 that } \deg(\Phi_0) \text{ is odd}}{\downarrow =} \deg(\Phi_0) \underset{\text{since } \Phi_0 \text{ and } \Phi_1 \text{ are homotopic}}{\overset{\text{by the Sphere Degree Lemma 76.11 (3) since}}{\uparrow =}} \deg(\Phi_1) \underset{\text{by definition of } \Phi_1}{\overset{}{\uparrow =}} \deg(\text{constant map}) \overset{\text{Sphere Degree Lemma 76.11 (2)}}{\downarrow =} 0.$$

We have thus obtained a contradiction. ■

The Borsuk-Ulam Theorem 39.8 has many applications in quite different areas of mathematics. We present some of these applications in Chapter 40.

92. Division algebras and generalized projective spaces

92.1. **Division algebras.** We start out with the following purely algebraic definition.

Definition. An algebra over a field $\mathbb{F}$ is an $\mathbb{F}$-vector space A together with an $\mathbb{F}$-bilinear map

$$\begin{array}{ccc} A \times A & \to & A \\ (v, w) & \mapsto & v \cdot w. \end{array}$$

(1) We say that the algebra is commutative if for all $a, b \in A$ we have $a \cdot b = b \cdot a$.
(2) We say that the algebra is associative if for all $a, b, c \in A$ we have $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
(3) We say that the algebra has an identity if there exists an element $1 \in A$ with $1 \neq 0$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in A$.
(4) If A is an algebra with identity 1, then given $a \in A$ we say that $b \in A$ is an inverse of a if $a \cdot b = b \cdot a = 1$.
(5) A division algebra is an algebra such that for all $a \neq 0$ and $b \in A$ there exists an $x \in A$ with $a \cdot x = b$ and a $y \in A$ with $y \cdot a = b$.

Examples.

(1) The complex numbers $\mathbb{C} = \mathbb{R} \oplus \mathbb{R} \cdot \mathrm{i}$ with the usual multiplication are an algebra over $\mathbb{R}$ that is commutative and associative, it has an identity, and it is a division algebra.
(2) The set of $(n \times n)$-matrices $\mathrm{M}(n \times n, \mathbb{F})$ over a field $\mathbb{F}$, where the multiplication is given by the usual matrix multiplication, is an associative algebra over $\mathbb{F}$ with an identity. If $n \geq 2$ this algebra is clearly neither commutative nor is it a division algebra.
(3) The real vector space $\mathbb{R}^3$ together with the cross product

$$\begin{array}{ccc} \mathbb{R}^3 \times \mathbb{R}^3 & \to & \mathbb{R}^3 \\ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} & \mapsto & \begin{pmatrix} v_2 w_3 - w_2 v_3 \\ -(v_1 w_3 - w_1 v_3) \\ v_1 w_2 - w_1 v_2 \end{pmatrix} \end{array}$$

is an algebra over $\mathbb{R}$. We recall that the geometric interpretation of the cross product is given as follows: for $v, w \in \mathbb{R}^3$ the cross product $v \times w$ is the unique vector in $\mathbb{R}^3$ that satisfies the following three properties:
(a) the vector is orthogonal to v and w,
(b) the length equals the area of the parallelogram spanned by v and w,
(c) we have $\det(\underbrace{(v \;\; w \;\; v \times w)}_{3 \times 3\text{-matrix}}) \geq 0$.[1113]
We make the following observations:
(a) Clearly we have $v \times w = -w \times v$, hence the algebra is non-commutative.
(b) We have

$$(e_1 \times e_1) \times e_2 \;=\; 0 \times e_2 \;=\; 0 \;\neq\; -e_2 \;=\; e_1 \times e_3 \;=\; e_1 \times (e_1 \times e_2),$$

which shows that this algebra is non-associative.

[1113] If v and w are non-zero this just means that v, w and $v \times w$ form a positive basis for $\mathbb{R}^3$.

(c) It follows easily from the definition or the geometric interpretation that for $v \neq 0$ there is no vector w with $v \times w = v$, hence the algebra does not have an identity element.

(d) Finally it is straightforward to see that $(\mathbb{R}^3, \times)$ is not a division algebra. More precisely, one can easily show, using the geometric interpretation of the cross product, that given $a, b \in \mathbb{R}^3$ there exists a vector $x \in \mathbb{R}^3$ with $a \times x = b$ if and only if b is orthogonal to a.

We consider the next example in greater detail.

Definition. The quaternions $\mathbb{H}$ are defined as the 4-dimensional real vector space $\mathbb{R}^4$ spanned by $1, \mathrm{i}, \mathrm{j}, \mathrm{k}$ and the multiplication which is determined by

(1) $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$,
(2) $1 \cdot \mathrm{i} = \mathrm{i} \cdot 1 = \mathrm{i}, \quad 1 \cdot \mathrm{j} = \mathrm{j} \cdot 1 = \mathrm{j}, \quad 1 \cdot \mathrm{k} = \mathrm{k} \cdot 1 = \mathrm{k}$, and
(3) $\mathrm{i} \cdot \mathrm{j} = k$, $\mathrm{j} \cdot \mathrm{k} = \mathrm{i}$, $\mathrm{k} \cdot \mathrm{i} = \mathrm{j}$ and $\mathrm{j} \cdot \mathrm{i} = -\mathrm{k}$, $\mathrm{k} \cdot \mathrm{j} = -\mathrm{i}$, $\mathrm{i} \cdot \mathrm{k} = -\mathrm{j}$.

For a quaternion

$$z := a \cdot 1 + b \cdot \mathrm{i} + c \cdot \mathrm{j} + d \cdot \mathrm{k} \in \mathbb{H}$$

we define the conjugate

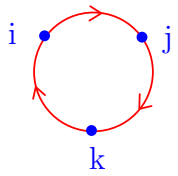
$$\overline{z} := a \cdot 1 - b \cdot \mathrm{i} - c \cdot \mathrm{j} - d \cdot \mathrm{k}$$

and the norm

$$|z| := \sqrt{a^2 + b^2 + c^2 + d^2}$$

the same way as we did for complex numbers.

Remark. The multiplication table for $\mathbb{H}$ can be summarized in the diagram shown in the figure below. Namely we show the three generators i, j and k. We have $\mathrm{i} \cdot \mathrm{j} = \mathrm{k}$ and the analogous statement holds for any *cyclic* permutation of $\{\mathrm{i}, \mathrm{j}, \mathrm{k}\}$. On the other hand for an anti-cyclic permutation we pick up a minus sign, e.g. we have $\mathrm{j} \cdot \mathrm{i} = -\mathrm{k}$.



The following lemma summarizes a few key properties of the quaternions. In particular we see that the conjugate and the norm of the quaternions behave in a quite similar fashion as the conjugate and norm of complex numbers.

Lemma 92.1. (Quaternion Lemma)

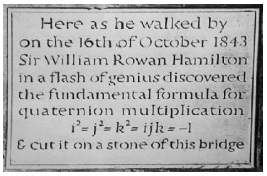
(1) The quaternions are a non-commutative algebra over $\mathbb{R}$ with identity.
(2) For any $z \in \mathbb{H}$ we have $\overline{z} + z \in \mathbb{R}$.
(3) For all $z, w \in \mathbb{H}$ we have

$$\overline{\overline{z}} = z, \quad \overline{z + w} = \overline{z} + \overline{w} \quad \text{and} \quad \overline{z \cdot w} = \overline{w} \cdot \overline{z}.$$

(4) If we make the obvious identification $\mathbb{H} = \mathbb{R}^4$, then the norm $|-|$ on $\mathbb{H}$ agrees with the Euclidean norm $\|-\|$ on $\mathbb{R}^4$.
(5) For every $z \in \mathbb{H}$ we have $z \cdot \overline{z} = |z|^2 \in \mathbb{R} = \mathbb{R} \cdot 1$.
(6) For every $z \neq 0$ a multiplicative inverse is given by $z^{-1} = \frac{\overline{z}}{|z|^2}$.

(7) For all $z, w \in \mathbb{H}$ we have $|z \cdot w| = |z| \cdot |w|$.
(8) The quaternions are associative.
(9) The quaternions are a division algebra.





Proof. Statements (1) to (3) follow from elementary calculations. We provide the details for the remaining statements:

(4) This statement follows immediately from the definition of $|z|$.
(5) This statement follows from a straightforward calculation.
(6) This statement follows immediately from (5).
(7) Let $z_1 = a_1 \cdot 1 + b_1 \cdot \mathrm{i} + c_1 \cdot \mathrm{j} + d_1 \cdot \mathrm{k}$ and $z_2 = a_2 \cdot 1 + b_2 \cdot \mathrm{i} + c_2 \cdot \mathrm{j} + d_2 \cdot \mathrm{k}$ be two quaternions. We see that

$$\begin{aligned} |z_1 \cdot z_2|^2 &= |(a_1 \cdot 1 + b_1 \cdot \mathrm{i} + c_1 \cdot \mathrm{j} + d_1 \cdot \mathrm{k}) \cdot (a_2 \cdot 1 + b_2 \cdot \mathrm{i} + c_2 \cdot \mathrm{j} + d_2 \cdot \mathrm{k})|^2 \\ &= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2)^2 + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)^2 + \\ &\quad + (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)^2 + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)^2 \\ &= (a_1^2 + b_1^2 + c_1^2 + d_1^2) \cdot (a_2^2 + b_2^2 + c_2^2 + d_2^2) \;=\; |z_1|^2 \cdot |z_2|^2. \end{aligned}$$

(8) There are several approaches to proving associativity:
  (a) Since the quaternions are an algebra we only have to check associativity for products of three of the generators $\{1, \mathrm{i}, \mathrm{j}, \mathrm{k}\}$ and that can be done easily by hand.
  (b) A tedious little calculation shows that the map

$$\begin{aligned} \mathbb{H} &\to \mathrm{M}(2 \times 2, \mathbb{C}) \\ \underbrace{z + \mathrm{j} \cdot w}_{\text{with } z, w \in \mathbb{C}} &\mapsto \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \end{aligned}$$

  is a ring homomorphism. Thus we see that $\mathbb{H}$ is isomorphic to a subring of the matrix ring $\mathrm{M}(2 \times 2, \mathbb{C})$, which is of course associative.
  (c) Finally one could use the more general result proved in Exercise 92.4.
(9) It remains to prove that $\mathbb{H}$ is in fact a division algebra. Thus let $a \neq 0$ and b be two elements in $\mathbb{H}$. We set $y := b \cdot a^{-1}$. Then

$$y \cdot a \;=\; (b \cdot a^{-1}) \cdot a \underset{\uparrow}{=} b \cdot (a^{-1} \cdot a) \;=\; b \cdot 1 \;=\; b.$$

since by (8) we know that $\mathbb{H}$ is associative

The same way we show that there exists an $x \in \mathbb{H}$ with $a \cdot x = b$. ∎

In the SU(2)-SO(3)-Covering Theorem 83.2 we saw that SU(2) is diffeomorphic to S^3. We give a refined version of this statement.

Proposition 92.2.
(1) The set $S^3 = \{z \in \mathbb{H} = \mathbb{R}^4 \,|\, |z| = 1\}$ forms a group under multiplication of quaternions.

(2) (a) The map $\mathbb{H} \times \mathbb{C} \to \mathbb{H}$ given by $(z, w) \mapsto z \cdot w$[1114] defines a complex vector space structure on $\mathbb{H}$.
(b) A basis for $\mathbb{H}$ viewed as a complex vector space is given by $\{1, \mathrm{j}\}$.
(c) Let $z + \mathrm{j} \cdot w \in \mathbb{H}$ with $z, w \in \mathbb{C}$. Left-multiplication by $z + \mathrm{j} \cdot w$ defines a complex linear map $\mathbb{H} \to \mathbb{H}$. With respect to the basis $\{1, \mathrm{j}\}$ this linear map is represented by the matrix $\begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}$.
(d) Left-multiplication by some $h \in \mathbb{H}$ defines a homomorphism of real vector spaces. If $h \neq 0$, then this homomorphism is represented, with respect to the standard basis of $\mathbb{H}$ as a real vector space, by a matrix in $\mathrm{GL}_+(4, \mathbb{R})$.

(3) The map

$$\begin{array}{rcl} f\colon S^3 & \to & \mathrm{SU}(2) \\ (z, w) = z + \mathrm{j} \cdot w & \mapsto & \underbrace{\begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}}_{\substack{\text{matrix that represents} \\ \text{left-multiplication by } z + \mathrm{j} \cdot w \text{ with} \\ \text{respect to the complex basis } \{1, \mathrm{j}\} \text{ of } \mathbb{H}}} \end{array}$$

is a diffeomorphism and it defines an isomorphism of groups.

Remark. Note that the map $f\colon S^3 \to \mathrm{SU}(2)$ from Proposition 92.2 satisfies in particular

$$f(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f(\mathrm{i}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad f(\mathrm{j}) = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad \text{and} \quad f(\mathrm{k}) = \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}.$$

Recall that in Lemma 83.1 we defined a homomorphism $q\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$. In Exercise 83.1 we calculated $q(f(\mathrm{i}))$, $q(f(\mathrm{j}))$ and $q(f(\mathrm{k}))$.

Proof.

(1) This statement follows immediately from the Quaternion Lemma 92.1 (5), (6) and (7).
(2) (a) This statement follows immediately from the observation that the obvious map $\mathbb{C} \to \mathbb{H}$ is a ring homomorphism.
(b) It follows from $\mathrm{k} = -\mathrm{j} \cdot \mathrm{i}$ that any quaternion $a \cdot 1 + b \cdot \mathrm{i} + c \cdot \mathrm{j} + d \cdot \mathrm{k}$ can be uniquely written as $(a \cdot 1 + b \cdot \mathrm{i}) + \mathrm{j} \cdot (c \cdot 1 - d \cdot \mathrm{i})$ where $a \cdot 1 + b \cdot \mathrm{i}$ and $c \cdot 1 - d \cdot \mathrm{i}$ are complex numbers. This shows that $\{1, j\}$ is a basis for $\mathbb{H}$, viewed as a complex vector space.
(c) First note that it follows from the associativity of quaternions that left-multiplication by a quaternion commutes with right-multiplication by a quaternion, in particular with right-multiplication by a complex number. This shows that left-multiplication by a quaternion is indeed a complex linear map.
Now let us turn to the statement regarding the representative matrix. This statement is elementary, but we think it is nonetheless worth doing. So we consider the basis $\{1, j\}$. Evidently we have $(z + \mathrm{j} \cdot w) \cdot 1 = 1 \cdot z + \mathrm{j} \cdot w$. Thus

[1114] Here we view $\mathbb{C}$ as a subset of $\mathbb{H}$ in the obvious way. Note that the order in the definition is important since $\mathbb{H}$ is not commutative.

the first column of the matrix is given by $\begin{pmatrix} z \\ w \end{pmatrix}$. More interestingly, we have

$$(z + j \cdot w) \cdot \mathrm{j} \underset{\uparrow}{=} z \cdot \mathrm{j} + \mathrm{j} \cdot w \cdot \mathrm{j} = \mathrm{j} \cdot \overline{z} + \overline{w} \cdot \mathrm{j} \cdot \mathrm{j} \underset{\uparrow}{=} \mathrm{j} \cdot \overline{z} + 1 \cdot (-\overline{w}).$$

both equalities follow from the fact that $\mathrm{i} \cdot \mathrm{j} = -\mathrm{j} \cdot \mathrm{i}$

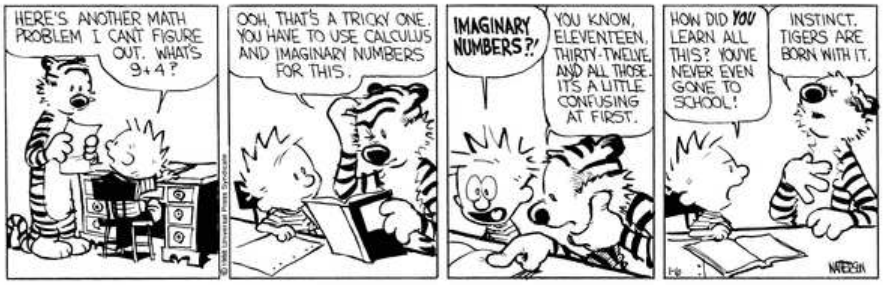
Thus the second column of the matrix is given by $\begin{pmatrix} \overline{z} \\ -\overline{w} \end{pmatrix}$.

(d) This statement follows easily from (c).

(3) We saw in the SU(2)-SO(3)-Covering Theorem 83.2 that the map $f\colon S^3 \to \mathrm{SU}(2)$ is a diffeomorphism. It follows from the associativity of quaternion multiplication and (2c), or alternatively from a tedious calculation, that the map

$$\begin{array}{rcl} f\colon \mathbb{H} & \to & \mathrm{M}(2 \times 2, \mathbb{C}) \\ (z, w) = z + \mathrm{j} \cdot w & \mapsto & \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \end{array}$$

is a ring homomorphism, in particular it preserves the ring structure. This implies that the given map $f\colon S^3 \to \mathrm{SU}(2)$ is a group homomorphism. ■



Before we move on to the octonions it is helpful to give an alternative description of the quaternions. Namely we can view the Hamiltonians as elements in $\mathbb{C} \oplus \mathbb{C}$ with the multiplication given by

$$(z_1, z_2) \cdot (w_1, w_2) := (z_1 w_1 - \overline{w}_2 z_2, z_2 \overline{w}_1 + w_2 z_1)$$

the conjugate given by

$$\overline{(w, z)} := (\overline{w}, -z)$$

and the norm given by

$$|(w, z)| := \sqrt{(w, z) \cdot \overline{(w, z)}}.$$

One easily verifies that this norm is multiplicative, i.e. given $a, b \in \mathbb{H} = \mathbb{C} \oplus \mathbb{C}$ one has $|a \cdot b| = |a| \cdot |b|$. An elementary calculation shows that the map $(w, z) \mapsto w + z \cdot \mathrm{j}$ defines an isomorphism between the two descriptions of $\mathbb{H}$, furthermore one can verify easily that this isomorphism preserves the conjugate and the norm.

Considering the above description of the quaternions one might hope to be lucky and just try to recycle the definitions:

Definition. The Cayley octonions $\mathbb{O}$ (or usually just octonions) are defined as $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ with the multiplication given by[1115]

$$(z_1, z_2) \cdot (w_1, w_2) := (z_1 w_1 - \overline{w}_2 z_2, z_2 \overline{w}_1 + w_2 z_1)$$

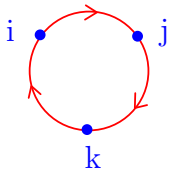
the conjugate given by $\overline{(w,z)} := (\overline{w}, -z)$

and the norm given by $|(w,z)| := \sqrt{(w,z)\cdot\overline{(w,z)}}.$

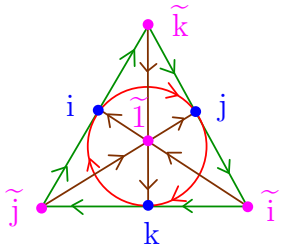
Remark. Octonionic multiplication can be summarized in the diagram shown in the figure below. More precisely, we write $\mathrm{i} = (\mathrm{i}, 0)$, $\mathrm{j} = (\mathrm{j}, 0)$, $\mathrm{k} = (\mathrm{k}, 0)$ and $\widetilde{1} = (0,1)$, $\widetilde{\mathrm{i}} = (0, \mathrm{i})$, $\widetilde{\mathrm{j}} = (0, \mathrm{j})$ and $\widetilde{\mathrm{k}} = (0, \mathrm{k})$. To the right of the figure below we show the *Fano plane*, which is a diagram with which has 7 points and 7 directed "lines". Similar to the discussion on page 1960 it encodes the multiplication where the product of two generators equals the third generator on the line with the positive sign if they are cyclic permutation of the given ordering and with a negative sign otherwise. For example we have

$$(\widetilde{\mathrm{j}} \cdot \mathrm{i}) \cdot \mathrm{j} = \widetilde{\mathrm{k}} \cdot \mathrm{j} = \widetilde{\mathrm{i}} \qquad \text{and} \qquad \widetilde{\mathrm{j}} \cdot (\mathrm{i} \cdot \mathrm{j}) = \widetilde{\mathrm{j}} \cdot \mathrm{k} = -\widetilde{\mathrm{i}}.$$

Note that this calculation also shows that the octonions are not associative. Finally we note that in [**Baez2001**, Chapter 2.1] the role of the Fano plane is explained in greater detail.



quaternionic multiplication



octonionic multiplication

The following lemma is the analogue of the Quaternion Lemma 92.1.

Lemma 92.3. (Octonion Lemma) The statements of the Quaternion Lemma 92.1 (1) to (7) also hold for octonions. For completeness' sake we spell out the precise statements:

(1) The octonions are a non-commutative algebra over $\mathbb{R}$ with identity.
(2) For any $z \in \mathbb{O}$ we have $\overline{z} + z \in \mathbb{R}$.
(3) For all $z, w \in \mathbb{O}$ we have

$$\overline{\overline{z}} = z, \quad \overline{z+w} = \overline{z} + \overline{w} \quad \text{and} \quad \overline{z \cdot w} = \overline{w} \cdot \overline{z}.$$

(4) If we make the obvious identification $\mathbb{O} = \mathbb{H} \oplus \mathbb{H} = \mathbb{R}^4 \oplus \mathbb{R}^4 = \mathbb{R}^8$, then the norm $|-|$ on $\mathbb{O}$ agrees with the Euclidean norm $\|-\|$ on $\mathbb{R}^8$.
(5) For every $z \in \mathbb{O}$ we have $z \cdot \overline{z} = |z|^2 \in \mathbb{R} = \mathbb{R} \cdot 1$.
(6) For every $z \neq 0$ a multiplicative inverse is given by $z^{-1} = \frac{\overline{z}}{|z|^2}$.
(7) For all $z, w \in \mathbb{O}$ we have $|z \cdot w| = |z| \cdot |w|$.

Furthermore, we have the following three statements:

(8) The octonions are *alternative*, this means that given any $x, y \in \mathbb{O}$ we have the equalities $(x \cdot y) \cdot y = x \cdot (y \cdot y)$ and $(x \cdot x) \cdot y = x \cdot (x \cdot y)$.
(9) Given any $x, y \in \mathbb{O}$ the algebra generated by 1, x and y is associative.

[1115] In fact these formulas also work for defining $\mathbb{C}$ as $\mathbb{R} \oplus \mathbb{R}$ where we equip $\mathbb{R}$ with the trivial involution.

(10) The octonions are a division algebra.

Remark.

(1) By the remark on page 98 we know that in an associative ring we can ignore parentheses when it comes to multiplication. Now that we are dealing with the octonions, which are not associative, we have to carefully keep track of parentheses.
(2) Our proof of the Octonion Lemma 92.3 (9) gets outsourced to a theorem of Artin. An alternative proof for the Octonion Lemma 92.3 (9), which does not rely on Artin's Theorem, is given in [**CSm2003**, p. 76].

Proof. As in the case of the Quaternion Lemma 92.1 the verification of (1) to (3) is totally elementary. The task of filling in the details is once again left to the reader. For later on it is helpful to consider the elementary prof of (4) in the notes:

(4) Given $(z, w) \in \mathbb{O}$ we calculate that

$$|(w,z)|^2 = (w,z)\cdot\overline{(w,z)} = (w,z)\cdot(\overline{w},-z) = (w\overline{w}+\overline{z}z, zw-zw) = \|w\|^2+\|z\|^2.$$

(5) This statement holds by definition of $|-|$.
(6) This statement follows immediately from (5).

Now we turn to the arguably more interesting proofs of the remaining four statements:

(7) It turns out that the proof of this statement is not totally straightforward. Thus let $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in $\mathbb{O}$. We need the following super-technical claim.

Claim. $$\underbrace{z_2\overline{w}_1\overline{z}_1\overline{w}_2 + w_2z_1w_1\overline{z}_2 - z_1w_1\overline{z}_2w_2 - \overline{w}_2z_2\overline{w}_1\overline{z}_1}_{=:S(w_2)} = 0.$$

Proof. Evidently, if w_2 is a real number, then we have $S(w_2) = 0$. Now suppose that w_2 is a purely imaginary quaternion, i.e. $w_2 = a\cdot \mathrm{i} + b\cdot \mathrm{j} + c\cdot \mathrm{j}$ for some $a, b, c \in \mathbb{R}$. In this case we calculate that

$$S(w_2) \overset{\substack{\text{since } w_2 \text{ is purely imaginary we have } \overline{w}_2 = -w_2\\ \downarrow}}{=} w_2(\underbrace{z_2\overline{w}_1\overline{z}_1 + z_1w_1\overline{z}_2}_{=:c}) - (\underbrace{z_2\overline{w}_1\overline{z}_1 + z_1w_1\overline{z}_2}_{=:c})w_2 = w_2\cdot c - c\cdot w_2 \underset{\substack{\uparrow\\ \text{we have } \overline{c}=c\text{, thus } c \text{ is a real number}}}{=} 0.$$

Finally note that for any $a, b \in \mathbb{H}$ we have $S(a+b) = S(a)+S(b)$. Since any quaternion is the sum of a real number and a purely imaginary we see that $S(w_2) = 0$ for any quaternion w_2. ⊞

Now we can provide the actual proof that the norm is multiplicative. Indeed, we calculate that

$$\begin{aligned}
|z \cdot w|^2 &= |(z_1, z_2) \cdot (w_1, w_2)|^2 = |(z_1 w_1 - \overline{w}_2 z_2, z_2 \overline{w}_1 + w_2 z_1)|^2 \\
&= (z_1 w_1 - \overline{w}_2 z_2) \cdot \overline{(z_1 w_1 - \overline{w}_2 z_2)} + (z_2 \overline{w}_1 + w_2 z_1) \cdot \overline{(z_2 \overline{w}_1 + w_2 z_1)} \\
&\quad \uparrow \text{see (4)} \\
&= (z_1 w_1 - \overline{w}_2 z_2) \cdot (\overline{w}_1 \overline{z}_1 - \overline{z}_2 w_2) + (z_2 \overline{w}_1 + w_2 z_1) \cdot (w_1 \overline{z}_2 + \overline{z}_1 \overline{w}_2) \\
&= (z_1 w_1) \cdot (\overline{w}_1 \overline{z}_1) - (z_1 w_1)(\overline{z}_2 w_2) - (\overline{w}_2 z_2) \cdot (\overline{w}_1 \overline{z}_1) + (\overline{w}_2 z_2) \cdot (\overline{z}_2 w_2) \\
&\quad + (z_2 \overline{w}_1)(w_1 \overline{z}_2) + (z_2 \overline{w}_1)(\overline{z}_1 \overline{w}_2) + (w_2 z_1) \cdot (w_1 \overline{z}_2) + (w_2 z_1) \cdot (\overline{z}_1 \overline{w}_2) \\
&= (z_1 w_1) \cdot (\overline{w}_1 \overline{z}_1) + (\overline{w}_2 z_2) \cdot (\overline{z}_2 w_2) + (z_2 \overline{w}_1)(w_1 \overline{z}_2) + (w_2 z_1) \cdot (\overline{z}_1 \overline{w}_2) \\
&\quad \uparrow \text{by the claim and the fact that the quaternions are associative} \\
&= (z_1 \overline{z}_1 + z_2 \overline{z}_2) \cdot (w_1 \overline{w}_1 + w_2 \overline{w}_2) = |(z_1, z_2)|^2 \cdot |(w_1, w_2)|^2 = |z|^2 \cdot |w|^2.
\end{aligned}$$

(8) Let $x, y \in \mathbb{O}$. We start out with the following claim.

Claim. We have the equalities $(x \cdot y) \cdot \overline{y} = x \cdot (y \cdot \overline{y})$ and $(x \cdot \overline{x}) \cdot y = x \cdot (\overline{x} \cdot y)$.

Proof. Let us first prove the first equality. We write $x = (x_1, x_2)$, $y = (y_1, y_2)$ with $x_1, x_2, y_1, y_2 \in \mathbb{H}$. Heroically we calculate that

$$\begin{aligned}
(xy) \cdot \overline{y} &= ((x_1, x_2)(y_1, y_2)) \cdot (\overline{y}_1, -y_2) = (x_1 y_1 - \overline{y}_2 x_2, y_2 x_1 + x_2 \overline{y}_1) \cdot (\overline{y}_1, -y_2) \\
&= ((x_1 y_1 - \overline{y}_2 x_2)\overline{y}_1 + \overline{y}_2 (y_2 x_1 + x_2 \overline{y}_1), (-y_2)(x_1 y_1 - \overline{y}_2 x_2) + (y_2 x_1 + x_2 \overline{y}_1) y_1) \\
&= ((|y_1|^2 + |y_2|^2) x_1, (|y_1|^2 + |y_2|^2) x_2) = (|y_1|^2 + |y_2|^2) \cdot (x_1, x_2) \\
&\quad \uparrow \text{since } \mathbb{H} \text{ is associative we have } (\overline{y}_2 x_2)\overline{y}_1 = \overline{y}_2 (y_2 x_1) \text{ and } y_2 (x_1 y_1) = (y_2 x_1) y_1 \\
&\quad \text{we also use that } \overline{y}_1 y_1 = |y_1|^2 \in \mathbb{R}, \text{ hence it commutes with any } z \in \mathbb{H} \\
&= |y|^2 \cdot x = x \cdot |y|^2 = x \cdot (y \overline{y}).
\end{aligned}$$

The proof of the second equality is basically the same. ⊞

Now, we use (2) to write $\overline{x} = -x + 2a$ and $\overline{y} = -y + 2b$ with $a, b \in \mathbb{R}$. We plug these equalities into the equalities of the claim and, since we are dealing with an algebra over the real numbers, we obtain the desired equalities $(x \cdot y) \cdot y = x \cdot (y \cdot y)$ and $(x \cdot x) \cdot y = x \cdot (x \cdot y)$.

(9) It is a non-trivial theorem of Emil Artin that any alternative algebra $\mathbb{A}$ has the property that given any two $x, y \in \mathbb{A}$ the algebra generated by 1, x and y is associative. A proof is for example provided in [**SchafR1966**, Theorem 3.1]. Alternatively see [**Baez2001**, p. 155].

(10) It remains to prove that $\mathbb{O}$ is in fact a division algebra. Since the octonions are not associative we cannot copy-paste the proof of the Quaternion Lemma 92.1 (9), since the latter proof used the associativity of the quaternions. Fortunately the spartan associativity result given in (8) is enough to give us the desired result.

Let $a, b \in \mathbb{O}$ with $a \neq 0$. We need to show that there exists an $x \in \mathbb{O}$ with $a \cdot x = b$ and a $y \in \mathbb{O}$ with $y \cdot a = b$. We set $x = a^{-1} \cdot b$. We calculate that

$$a \cdot x = a \cdot (a^{-1} \cdot b) = a \cdot \left(\tfrac{1}{|a|^2} \cdot \overline{a} \cdot b\right) = \tfrac{1}{|a|^2} \cdot a \cdot (\overline{a} \cdot b) \underset{\uparrow}{=} \tfrac{1}{|a|^2} \cdot (a \cdot \overline{a}) \cdot b = b.$$

by the claim in the proof of (8)

Basically the same calculation shows that we have $y \cdot a = b$. ■

Remark.

(1) The octonions were discovered[1116] by Arthur Cayley in 1845, shortly after Hamilton found the quaternions.
(2) The discussion in the previous pages shows that starting from $\mathbb{R}$ one can construct the complex numbers, which give rise to the quaternions and finally we obtain the octonions. When we move from the complex numbers to the quaternions we loose commutativity and when we go from the quaternions to the octonions we also loose associativity. Evidently the octonions are not commutative, but even worse, one can easily verify that the octonions are not even associative.[1117] The octonions are discussed in much greater detail in [**Baez2001**], [**CSm2003**, Chapter 6] and [**KaS1989**, Chapter 6].
(3) The definition of quaternions $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}$ out of the complex numbers and the definition of the octonions $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ out of the quaternions is a general procedure known as the *Cayley-Dickson construction* . This construction is discussed in greater detail in [**Baez2001**, Section 2.2]. In particular one can easily see that this process can be extended to define, basically by the same recipe, a 16-dimensional algebra $\mathbb{S} = \mathbb{O} \oplus \mathbb{O}$ with an identity, called the *sedenions*, this algebra is evidently neither commutative nor associative. On the other hand the properties stated in the Quaternion Lemma 92.1 (1)–(6) and the Octonion Lemma 92.3 (1)–(6) hold the same way, in particular $\mathbb{S}$ has multiplicative inverses, i.e. for each $a \in \mathbb{S} \setminus \{0\}$ there exists a $b \in \mathbb{S}$ with $a \cdot b = b \cdot a = 1$. But the proof of the Octonion Lemma 92.3 (7) and (8) do not carry over, since in the proof we made use of the fact that the quaternions are actually associative. In fact it turns out that the analogue of (7) does not hold for sedenions and that the sedenions are not a division algebra.[1118] [1119] We refer to [**More1997**] for more information on the failure of the sedenions to be a division algebra.

92.2. **Commutative division algebras.** After the discovery of quaternions and the octonions the question arose whether one can also find division algebras on $\mathbb{R}^n$ for some $n \neq 1, 2, 4, 8$. The following theorem, which was first proved by Heinz Hopf [**Hopf1940a, Hopf1940b, Hopf1964**] in 1940, is the best theorem that we can prove in this direction with our present knowledge.

Theorem 92.4. (Hopf Theorem on Commutative Division Algebras) Any finite-dimensional division algebra over $\mathbb{R}$ which is commutative and which has an identity is isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$.

Remark.

[1116]Or were they invented by Arthur Cayley? This question whether mathematics is "discovered" or "invented" is a source for endless philosophical discussions.
[1117]To quote from [**Frie2012**, p. 5] the immortal words: "here, associativity is obviously a concern".
[1118]In fact this construction can be iterated and one obtains algebras which are creatively called pathions (32-dimensional), chingons (64-dimensional), routons (128-dimensional), voudons (256-dimensional). We challenge the reader to come up with the explanations for these names. Alternatively the reader can look at [**dMar2002**, p. 7].
[1119]Perhaps more fruitfully, the Cayley-Dickson construction can be applied to the real numbers to give the complex numbers and it can also be applied to the complex numbers to give the quaternions.

(1) Note that in Theorem 92.4 we do not assume that the division algebra is associative. It is a nice exercise in algebra and real analysis to show, using the Fundamental Theorem of Algebra, the weaker statement that if V is a finite-dimensional commutative and *associative* division algebra over $\mathbb{R}$, then V is isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$.
(2) The statement of Theorem 92.4 does not hold if we drop the "finite dimension" condition. For example the field $\mathbb{R}(t)$ of rational functions over $\mathbb{R}$ is a commutative division algebra, but it is an infinite-dimensional real vector space.
(3) In Theorem 121.23 we will obtain further results on division algebras.

In the proof of Theorem 92.4 we will need the following basic lemma which might already be familiar from earlier algebra courses.

Lemma 92.5. Let A be a *finite-dimensional* algebra over a field $\mathbb{F}$. The following two statements are equivalent:

(1) A is a division algebra,
(2) A has no zero-divisors, i.e. for any $a, b \in A$ with $a \cdot b = 0$ we have $a = 0$ or $b = 0$.

Remark. Let A be a division algebra over a field $\mathbb{F}$. If A is associative, then it is clear that A has no zero-divisors. Indeed, if $a \cdot b = 0$ with $a \neq 0$, then we pick x with $x \cdot a = 1$ and we see that $0 = x \cdot 0 = x \cdot (a \cdot b) = (x \cdot a) \cdot b = 1 \cdot b = b$. The point about Lemma 92.5 is that it applies to non-associative algebras, as long as they are finite-dimensional.

Proof. Let A be a finite-dimensional algebra over a field $\mathbb{F}$. Let $a \in A$ be non-zero. Then the following holds:

$$\begin{aligned} \text{for any } y \in A \text{ there exists } x \in A \text{ with } ax = y &\Leftrightarrow \text{ the map } A \to A,\ x \mapsto ax \text{ is surjective} \\ &\Leftrightarrow \text{ the map } A \to A,\ x \mapsto ax \text{ is injective} \\ &\quad \uparrow \\ &\quad \text{since } x \mapsto ax \text{ is a homomorphism of finite-dimensional } \mathbb{F}\text{-vector spaces} \\ &\Leftrightarrow \text{ for any } b \neq 0 \text{ we have } ab \neq 0. \end{aligned}$$

The same statement holds for left-multiplication. The equivalence of (1) and (2) now follows easily from this observation and the definitions. ∎

Now we turn to the proof of Theorem 92.4.

Proof of Theorem 92.4. Let $(A, *)$ be a finite-dimensional division algebra over $\mathbb{R}$ which is commutative and which has an identity. Since A is a finite-dimensional vector space we can, after applying an isomorphism, assume that $A = \mathbb{R}^n$ for some $n \in \mathbb{N}$.

We first consider the case $n = 1$. We define $a := 1 * 1$. Since "$*$" is bilinear we have for all $x, y \in \mathbb{R}$ that $x * y = (x \cdot 1) * (y \cdot 1) = x \cdot (1 * 1) \cdot y = xay$. Since the algebra is a division algebra we obtain from Lemma 92.5 that $a \neq 0$. Then the map $\varphi(x) = ax$ defines an isomorphism from the algebra $(\mathbb{R}, *)$ to the algebra $(\mathbb{R}, \cdot)$.[1120]

Now suppose that $n \geq 2$. We first want to show that $n = 2$. To do this we consider the map

$$\begin{array}{rccl} f\colon S^{n-1} & \to & S^{n-1} & \\ x & \mapsto & f(x) & := \dfrac{x * x}{\|x * x\|}. \end{array}$$

[1120] Indeed, for $x, y \in \mathbb{R}$ we have $\varphi(x * y) = \varphi(x \cdot a \cdot y) = a \cdot x \cdot a \cdot y = (a \cdot x) \cdot (a \cdot y) = \varphi(x) \cdot \varphi(y)$.

This map is indeed well-defined, since for $x \neq 0$ it follows from our hypothesis on $(\mathbb{R}^n, *)$ and from Lemma 92.5 that $x * x \neq 0$. The map f is continuous since the multiplication map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is bilinear, hence continuous.[1121]

Since "$*$" is bilinear we have $f(-x) = f(x)$ for all $x \in \mathbb{R}^n$. It follows from the Topological-Quotient Proposition 5.15 (1b) that the map f descends to a continuous map

$$\begin{array}{rcl} \overline{f}\colon \mathbb{R}\mathrm{P}^{n-1} & \to & S^{n-1} \\ [x] & \mapsto & \overline{f}([x]) := f(x). \end{array}$$

Claim. The map $\overline{f}\colon \mathbb{R}\mathrm{P}^{n-1} \to S^{n-1}$ is injective.

Proof. So let $[x], [y] \in \mathbb{R}\mathrm{P}^{n-1} = S^{n-1}/\!\sim$. We have

$$\begin{array}{rcl} \underbrace{\overline{f}([x])}_{=f(x)} = \underbrace{\overline{f}([y])}_{=f(y)} \implies f(x) = f(y) & \implies & x * x = \alpha^2 \cdot (y * y) \text{ for } \alpha := \sqrt{\|x * x\| \cdot \|y * y\|^{-1}} \\ & \implies & 0 = x * x - \alpha^2 \cdot (y * y) \underset{\uparrow}{=} (x + \alpha y) * (x - \alpha y) \\ & & \qquad \text{since "}*\text{" is commutative and bilinear} \\ & \underset{\uparrow}{\implies} & x = \pm \alpha y \quad \underset{\uparrow}{\implies} \quad x = \pm y \quad \implies [x] = [y]. \\ & \text{Lemma 92.5} & \qquad \text{since } x, y \in S^{n-1} \end{array}$$

⊞

Since S^{n-1} is closed and connected (here we use that $n \geq 2$) and since $\mathbb{R}\mathrm{P}^{n-1}$ is a closed smooth manifold it follows from Corollary 82.9 that $\overline{f}\colon \mathbb{R}\mathrm{P}^{n-1} \to S^{n-1}$ is in fact a homeomorphism. In particular the fundamental groups of $\mathbb{R}\mathrm{P}^{n-1}$ and S^{n-1} are isomorphic. But it follows from the Sphere-π_1-Proposition 47.13 and Corollary 48.18 that this is only possible if $n = 2$.

It remains to show that the algebra $(A, *) = (\mathbb{R}^2, *)$ is isomorphic to the algebra $(\mathbb{C}, \cdot)$. We start out with the following claim.

Claim. There exists a $k \in A \setminus \mathbb{R} \cdot 1$ with $k * k = c \cdot 1$ for some $c \in \mathbb{R}_{<0}$.

Proof. We first pick a random element $j \in A \setminus \mathbb{R} \cdot 1$. Then 1 and j are a basis for $A = \mathbb{R}^2$ and therefore we can write $j * j = a \cdot 1 + b \cdot \mathrm{j}$ for some $a, b \in \mathbb{R}$. Then multiplying out and using this equality we see that

$$\underbrace{\left(j - \tfrac{b}{2} \cdot 1\right)}_{=:k} * \underbrace{\left(j - \tfrac{b}{2} \cdot 1\right)}_{=:k} = \underbrace{\left(a + \tfrac{1}{4} \cdot b^2\right)}_{=:c} \cdot 1.$$

We set $k := j - \frac{b}{2} \cdot 1$ and $c := a + \frac{1}{4} b^2$. We still need to show that $c < 0$. Suppose that $c \geq 0$. Then $k * k = c \cdot 1$ implies that $(k + \sqrt{c} \cdot 1) * (k - \sqrt{c} \cdot 1) = k * k - c \cdot 1 = 0$. From Lemma 92.5 we conclude that $k = \pm\sqrt{c} \cdot 1$. But this is not possible since $k = j - \frac{b}{2} \cdot 1$ is linearly independent from 1. ⊞

One can now easily verify that

$$\begin{array}{rcll} A & \to & \mathbb{C} & \\ x \cdot 1 + y \cdot \mathrm{k} & \mapsto & x + y \cdot \dfrac{\mathrm{i}}{\sqrt{-c}} & \text{with } x, y \in \mathbb{R} \end{array}$$

is an isomorphism from the algebra $(A, *)$ to the algebra $(\mathbb{C}, \cdot)$. ■

[1121] Indeed, the fact that the map is bilinear implies that there exists a real $n \times n$–matrix P such that for any $v, w \in \mathbb{R}^n$ we have $v \cdot w = v^T P w$, hence the map is continuous.

It is natural to ask what other division algebras exist. The following, purely algebraic, theorem classifies *associative* division algebras.

Theorem 92.6. (Frobenius Theorem) Any finite-dimensional division algebra over $\mathbb{R}$ which is associative and which has an identity is isomorphic to $\mathbb{R}, \mathbb{C}$ or to $\mathbb{H}$.

Remark. Note that in Exercise 92.6 we will see that there are many more *non-associative* division algebra structures on $\mathbb{R}^4$.

Proof. The statement was first proved by Frobenius in 1878. A short self-contained proof is for example given in [**Pal1968**]. ■

Finally let us record the following question.

Question 92.7. For which $n \in \mathbb{N}$, besides $n = 1, 2, 4, 8$ does $\mathbb{R}^n$ admit the structure of a division algebra?

We will answer this question in Theorem 121.25.

92.3. **Quaternionic and octonionic projective spaces.** In this section we introduce the quaternionic projective spaces $\mathbb{H}\mathrm{P}^n$ and we introduce the significantly more subtle octonionic projective space $\mathbb{O}\mathrm{P}^2$. These smooth manifolds get added to our ever expanding list of amusing topological spaces. At a later stage they will play an essential role.

On page 1472, given $n \in \mathbb{N}_0$, we introduced the real projective space $\mathbb{R}\mathrm{P}^n$ and the complex projective spaces $\mathbb{C}\mathrm{P}^n$. In the following we will introduce the quaternionic projective spaces $\mathbb{H}\mathrm{P}^n$. Since the quaternions $\mathbb{H}$ are non-commutative we do the constructions carefully.

Lemma 92.8. Given $v, w \in \mathbb{H}^{n+1} \setminus \{(0, \ldots, 0)\}$ we write $v \sim w$ if there exists a non-zero $h \in \mathbb{H}$ with $h \cdot v = w$. This defines an equivalence relation on $\mathbb{H}^{n+1} \setminus \{(0, \ldots, 0)\}$. The analogous statement also holds for $\mathbb{H}^\infty \setminus \{0\}$.

Proof.

(1) It is clear that $\sim$ is reflexive.
(2) Since $\mathbb{H}$ is a skew field we see that $\sim$ is symmetric.
(3) Finally suppose that $u \sim v$ and $v \sim w$. Thus there exist $g, h \in \mathbb{H} \setminus \{0\}$ with $gu = v$ and $hv = w$. Thus we see that

$$(hg)u \underset{\uparrow}{=} h(gu) = hv = w.$$

since the quaternions are associative!

■

Now we can define the quaternionic projective spaces.

Definition. Let $n \in \mathbb{N}_0$.

(1) For $(h_0, \ldots, h_n) \in \mathbb{H}^{n+1} \setminus \{0\}$ we denote by $[h_0 : \ldots : h_n]$ the corresponding equivalence class.
(2) Given $n \in \mathbb{N}$ we define the quaternionic projective space

$$\mathbb{H}\mathrm{P}^n := (\mathbb{H}^{n+1} \setminus \{(0, \ldots, 0)\})/\sim .$$

Similarly, we define the infinite-dimensional quaternionic projective space

$$\mathbb{H}\mathrm{P}^\infty := (\mathbb{H}^\infty \setminus \{0\})/\sim .$$

The following lemma summarizes some key properties of quaternionic projective spaces. $\mathbb{H}\mathrm{P}^n$. We will make use of some of them later on.

Lemma 92.9. (Quaternion-Projective Space Lemma) Let $n \in \mathbb{N}_0$.

(1) $\mathbb{H}\mathrm{P}^n$ is a closed orientable $4n$-dimensional smooth manifold.
(2) The map
$$\begin{array}{rcl} \mathbb{H}\mathrm{P}^1 & \to & \mathbb{H} \cup \{\infty\} = \mathbb{R}^4 \cup \{\infty\} \overset{\text{by Lemma 2.18}}{=} S^4 \\ [z_0 : z_1] & \mapsto & \begin{cases} z_0 \cdot z_1^{-1}, & \text{if } z_1 \neq 0, \\ \infty, & \text{if } [z_0 : z_1] = [1 : 0] \end{cases} \end{array}$$
is a homeomorphism.
(3) $\mathbb{H}\mathrm{P}^n$ admits a CW-structure that has exactly one cell in the dimensions $0, 4, 8, \ldots, 4n$.
(4) We have $\chi(\mathbb{H}\mathrm{P}^n) = n + 1$.
(5) $\mathbb{H}\mathrm{P}^n$ is 3-connected, i.e. we have $\pi_i(\mathbb{H}\mathrm{P}^3) = 0$ for $i = 1, 2, 3$.
(6) We have a homeomorphism $\varinjlim \mathbb{H}\mathrm{P}^n = \mathbb{H}\mathrm{P}^\infty$.
(7) We can equip $\mathbb{H}\mathrm{P}^\infty$ with a CW-structure that has exactly one cell in the dimensions $0, 4, 8, \ldots$.

Proof.

(1) This statement is proved in a very similar way as the Complex Projective Space-is-Complex Manifold Lemma 32.16.
(2) The proof of this statement is basically the same as the proof of Lemma 5.32 (2).
(3) The proof of this statement is an easy modification of the proof of Lemma 68.1.
(4) This statement follows immediately from (3).
(5) This follows from (3) and the CW-Skeleton-π_n-Proposition 71.9.
(6) This statement is proved the same way as Lemma 68.5 (1).
(7) This statement is proved basically the same way as Lemma 68.5 (3). ■

Since the octonions are not associative the definition on page 1472 cannot be applied to the octonions. But there is some hope in low dimensions. More precisely, as we will see shortly, one can use the octonions to construct smooth manifolds that have all the properties that one would naively expect $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$ to have.

Definition.

(1) We equip[1122]
$$T_2 := \mathbb{O}^2 \setminus \{(0,0)\}$$
with the equivalence relation[11231124]
$$(x_1, y_1) \sim (x_2, y_2) \iff \begin{array}{l} \text{there exists a } \lambda \in \mathbb{R}_{>0} \text{ such that} \\ y_1\overline{y_1} = \lambda \cdot y_2\overline{y_2} \text{ and } x_1\overline{y_1} = \lambda \cdot x_2\overline{y_2}. \end{array}$$
We set
$$\mathbb{O}\mathrm{P}^1 := T_2/\sim$$
and we refer to $\mathbb{O}\mathrm{P}^1$ as the octonionic projective line.
(2) We equip
$$T_3 := \big\{(x, y, z) \in \mathbb{O}^3 \setminus \{(0,0,0)\} \,\big|\, 1, x, y, z \text{ generate an associative algebra}\big\}$$

with the equivalence relation[1125]

$$(x_1,y_1,z_1)\sim(x_2,y_2,z_2) \iff \begin{array}{l}\text{there exists a } \lambda\in\mathbb{R}_{>0} \text{ such that}\\ y_1\overline{y_1}=\lambda\cdot y_2\overline{y_2},\quad x_1\overline{y_1}=\lambda\cdot x_2\overline{y_2},\quad x_1\overline{z_1}=\lambda\cdot x_2\overline{z_2},\\ y_1\overline{y_1}=\lambda\cdot y_2\overline{y_2},\quad y_1\overline{z_1}=\lambda\cdot y_2\overline{z_2},\quad z_1\overline{z_1}=\lambda\cdot z_2\overline{z_2}.\end{array}$$

We set

$$\mathbb{O}\mathrm{P}^2 := T_3/\sim$$

and we refer to $\mathbb{O}\mathrm{P}^2$ as the octonionic projective plane. In the literature the octonionic projective plane is often also called the Cayley projective plane.

(3) We denote the equivalence class of a point $(x,y)\in T_2$ by $[x:y]$ and we denote the equivalence class of a point $(x,y,z)\in T_3$ by $[x:y:z]$.

The following proposition hopefully convinces the reader that the octonion line $\mathbb{O}\mathrm{P}^1$ and the octonion plane $\mathbb{O}\mathrm{P}^2$ deserve their names.

Proposition 92.10. (Octonion Plane Proposition)

(1) The octonionic projective line $\mathbb{O}\mathrm{P}^1$ and the octonionic projective plane $\mathbb{O}\mathrm{P}^2$ are both compact.

(2) The map

$$\begin{array}{rcl}\mathbb{O}\mathrm{P}^1 & \to & \mathbb{O}\cup\{\infty\}=\mathbb{R}^8\cup\{\infty\}\overset{\text{by Lemma 2.18}}{=}S^8\\ [x:y] & \mapsto & \begin{cases}x\cdot y^{-1}, & \text{if } y\neq 0,\\ \infty, & \text{if } [x:y]=[1:0]\end{cases}\end{array}$$

is well-defined and it is a homeomorphism.

(3) The octonionic projective plane $\mathbb{O}\mathrm{P}^2$ is a closed orientable 16-dimensional smooth manifold.

(4) The map[1126]

$$\begin{array}{rcl}\mathbb{O}\mathrm{P}^1 & \to & \mathbb{O}\mathrm{P}^2\\ [x:y] & \mapsto & [x:y:0]\end{array}$$

is an embedding. We use it to view $\mathbb{O}\mathrm{P}^1$ as a subspace of $\mathbb{O}\mathrm{P}^2$.

(5) The octonionic projective plane $\mathbb{O}\mathrm{P}^2$ admits a CW-structure with precisely one cell in dimensions 0, 8 and 16, and which has no other cells. Furthermore this CW-structure can be chosen such that the 8-skeleton is precisely $\mathbb{O}\mathrm{P}^1$.

(6) We have $\chi(\mathbb{O}\mathrm{P}^2)=3$.

(7) $\mathbb{O}\mathrm{P}^2$ is 7-connected, i.e. we have $\pi_i(\mathbb{O}\mathrm{P}^2)=0$ for $i=1,2,\dots,7$.

Proof.

(1) It is straightforward to define equivalence relations on S^{15} and on S^{23} such that the octonionic projective line $\mathbb{O}\mathrm{P}^1$ is homeomorphic to $S^{15}/\sim$ and such that the

[1122] What happens if we use these definitions of $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$, but if we replace the octonions by our more familiar friends $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$?

[1123] Note that this is obviously an equivalence relation.

[1124] The definition of the equivalence relation is a priori asymmetric in the first and the second variable. Bu this asymmetry is an illusion, in fact if $y_1\overline{y_1}=\lambda\cdot y_2\overline{y_2}$ and $x_1\overline{y_1}=\lambda\cdot x_2\overline{y_2}$, then we also have

$$x_1\overline{x}_1 = x_1\overline{y}_1\overline{y}_1^{-1}y_1^{-1}y_1\overline{x}_1 = x_1\overline{y}_1(y_1\overline{y}_1)^{-1}\overline{x_1\overline{y}_1} = \lambda x_2\overline{y}_2(\lambda y_2\overline{y}_2)^{-1}\overline{\lambda x_2\overline{y}_2} = \lambda x_2\overline{y}_2\overline{y}_2^{-1}y_2^{-1}y_2\overline{x}_2 = \lambda x_2\overline{x}_2.$$

[1125] As for $\sim$ on T_2, this is basically by definition an equivalence relation.

[1126] It follows from the Octonion Lemma 92.3 (9) that this map is well-defined, i.e. given any point $(x_1,y_1)\in T_2$ we have indeed $(x_1,y_1,0)\in T_3$.

octonionic projective plane $\mathbb{O}\mathrm{P}^2$ is homeomorphic to $S^{23}/\sim$. It follows from the Compact Image Lemma 2.13 that $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$ are both compact.

(2) First we need to show that the given map $\Theta\colon \mathbb{O}\mathrm{P}^1 \to \mathbb{O}\cup\{\infty\}$ is well-defined. Thus suppose that we are given two points (x_1, y_1) and (x_2, y_2) in $\mathbb{O}^2 \setminus \{(0,0)\}$ that are equivalent. One can verify easily that $y_1 \neq 0$ if and only if $y_2 \neq 0$. By definition of Θ we only need to worry about the case that $y_1, y_2 \neq 0$. In this case we do indeed have

$$x_1 y_1^{-1} \underset{\uparrow}{=} x_1 \cdot \frac{1}{|y_1|^2} \cdot \overline{y}_1 \underset{\uparrow}{=} \frac{1}{y_1 \cdot \overline{y}_1} \cdot (x_1 \cdot \overline{y}_1) \underset{\uparrow}{=} \frac{1}{y_2 \cdot \overline{y}_2} \cdot (x_2 \cdot \overline{y}_2) \underset{\uparrow}{=} \ldots = x_2 y_2^{-1}.$$

Octonion Lemma 92.3 (7) and (6); by definition of "$\sim$" we have $y_1\overline{y_1} = \lambda \cdot y_2\overline{y_2}$ and $x_1\overline{y_1} = \lambda \cdot x_2\overline{y_2}$; same steps backwards

We have thus shown that Θ is well-defined. It is clear that the map Θ is a surjection. In Exercise 92.8 we will verify that Θ is also an injection. Next note that the argument in the proof of Lemma 5.32 (2) shows that Θ is continuous. We just saw in (1) that $\mathbb{O}\mathrm{P}^1$ is compact. Evidently $\mathbb{O}\cup\{\infty\} = S^8$ is Hausdorff. In summary we obtain from the Compact-Hausdorff Proposition 2.17 (3) that $\Theta\colon \mathbb{O}\mathrm{P}^1 \to \mathbb{O}\cup\{\infty\}$ is a homeomorphism.

(3) We start out with the following claim:

Claim. The map

$$\begin{array}{rcl} \Phi_z\colon \{[x:y:z] \in \mathbb{O}\mathrm{P}^2 \mid z \neq 0\} & \to & \mathbb{O}^2 \\ [x:y:z] & \mapsto & (xz^{-1}, yz^{-1}) \end{array}$$

is a homeomorphism between an open subset of $\mathbb{O}\mathrm{P}^2$ and $\mathbb{O}^2 = \mathbb{R}^{16}$.

Proof. We prove the claim in three mostly elementary steps.

(a) Basically the same argument as in (2) shows that the map Φ_z is well-defined.

(b) It follows easily from the definition of the quotient topology that the domain of the map Φ_z is indeed an open subset of $\mathbb{O}\mathrm{P}^2$.

(c) Finally we consider the map

$$\begin{array}{rcl} \Psi_z\colon \mathbb{O}^2 & \to & \{[x:y:z] \in \mathbb{O}\mathrm{P}^2 \mid z \neq 0\} \\ (x,y) & \mapsto & [x:y:1]. \end{array}$$

Note that here we use the non-trivial the Octonion Lemma 92.3 (9) to conclude that Ψ_z is actually well-defined, i.e. to conclude that $(x, y, 1)$ is actually an element in T_3. Furthermore note that it follows from the Topological-Quotient Proposition 5.15 (1b) that Ψ_z is continuous. It is clear that $\Phi_z \circ \Psi_z = \mathrm{id}$. Finally note that given $[x:y:z] \in \mathbb{O}\mathrm{P}^2$ with $z \neq 0$ we have

$$\Psi_z(\Phi_z([x:y:z])) = [xz^{-1}:yz^{-1}:1] \underset{\uparrow}{=} [x:y:z]$$

follows with $\lambda = |z|^2$ and from the observation that the hypothesis $(x, y, z) \in T_3$ together with the Octonion Lemma 92.3 (2) implies that the elements $x, y, z, \overline{x}, \overline{y}, \overline{z}$ generate an associative algebra

We have thus shown that Ψ_z is an inverse to Φ_z. This implies that Φ_z is indeed a homeomorphism. ⊞

Similarly we define charts Φ_x and Φ_y. It is straightforward to verify that the transition maps between the three charts are smooth and orientation-preserving. In particular these three charts define a smooth atlas for $\mathbb{O}\mathrm{P}^2$. We obtain from Lemma 18.2 that $\mathbb{O}\mathrm{P}^2$ is second-countable. In (1) we showed that $\mathbb{O}\mathrm{P}^2$ is compact. Finally in

Exercise 92.9 we will show that $\mathbb{O}\mathrm{P}^2$ is Hausdorff.[1127] The combination of all of the above shows that $\mathbb{O}\mathrm{P}^2$ is indeed a closed 16-dimensional smooth manifold. Finally, since the transition maps between the three charts are orientation-preserving we obtain from the Define Orientations-via Transition Maps Lemma 25.15 that $\mathbb{O}\mathrm{P}^2$ is actually orientable.

(4) It follows basically immediately from the discussion in Footnote 1124 that the given map $\mathbb{O}\mathrm{P}^1 \to \mathbb{O}\mathrm{P}^2$ is an injection. By (1) we know that $\mathbb{O}\mathrm{P}^1$ is compact and by (3) we know that $\mathbb{O}\mathrm{P}^2$ is Hausdorff. Thus it follows from the Compact-Hausdorff Proposition 2.17 (2) that the given map $\mathbb{O}\mathrm{P}^1 \to \mathbb{O}\mathrm{P}^2$ is an embedding.

(5) We view $\mathbb{O}\mathrm{P}^1 = S^8$ as a CW-complex with one 0-cell and one 8-cell in the obvious way. Next, similar to the proof of Lemma 68.1 we consider the attaching map

$$\begin{array}{rcl} \varphi\colon S^{15} = \{(x_1, x_2) \in \mathbb{O}^2 \,|\, |x_1|^2 + |x_2|^2 = 1\} & \to & \mathbb{O}\mathrm{P}^1 \\ (x_1, x_2) & \mapsto & [x_1 : x_2]. \end{array}$$

Furthermore we consider the map

$$\begin{array}{rcl} \Theta\colon \mathbb{O}\mathrm{P}^1 \cup_\varphi \overline{B}^{16} & \to & \mathbb{O}\mathrm{P}^2 \\ [P] & \mapsto & \begin{cases} [x_1 : x_2 : 0], & \text{if } P = [x_1 : x_2] \in \mathbb{O}\mathrm{P}^1, \\ [P : 1 - \|P\|], & \text{if } P \in \overline{B}^{16}. \end{cases} \end{array}$$

It follows from the Octonion Lemma 92.3 (8) that the map actually does take values in $\mathbb{O}\mathrm{P}^2$. We leave the slightly tedious task of verifying that the map is well-defined and a bijection to the reader. The left-hand side is compact and the right-hand side is Hausdorff by (2). Thus it follows from the Compact-Hausdorff Proposition 2.17 (3) that the map is a homeomorphism. But this implies that $\mathbb{O}\mathrm{P}^2$ admits a CW-structure with precisely three cells, one in dimensions 0, 8 and 16.

(6) Statement (6) is an immediate consequence of (5).

(7) Finally Statement (7) follows from (5) and the CW-Skeleton-π_n-Proposition 71.9. ■

It is natural to ask whether there exists a smooth manifold that deserves the name $\mathbb{O}\mathrm{P}^n$ for $n \geq 3$. In the proof of the Octonion Plane Proposition 92.10 we used that $n = 2$ has the property, provided by the Octonion Lemma 92.3 (9), that for any $x_1, \ldots, x_n \in \mathbb{O}$ the elements $1, x_1, \ldots, x_n$ generate an associative algebra. It is straightforward to see that this statement is not true for $n \geq 3$. Nonetheless, one can wonder whether there exists an alternative way to define a meaningful higher dimensional octonionic projective space. This slightly vague question can be formulated in many different ways. Here is one possible interpretation of this question.

Question 92.11. Let $n \geq 3$. Does there exist a closed orientable $8n$-dimensional smooth manifold that admits a CW-structure with precisely one cell in dimensions $0, 8, 16, \ldots, 8n$?

We will answer this question in Exercise 152.2.

the Octonion Plane Proposition 92.10 says in particular that the octonion plane $\mathbb{O}\mathrm{P}^2$ is a closed orientable 16-dimensional smooth manifold that is 7-connected and that has odd Euler characteristic. The following question is, to the best of my knowledge, open:

[1127] An arguably more elegant proof that $\mathbb{O}\mathrm{P}^2$ is Hausdorff is given in [**Lackm2012**].

Question 92.12. Does there exist an 8-connected closed orientable smooth manifold with odd Euler characteristic?

In [**Hoe2017**, Theorem 1.2 together with Proposition 4.2] it is shown that the dimension of an 8-connected closed orientable smooth manifold with odd Euler characteristic, if it exists, has to be divisible by 16.

Exercises for Chapter 92.

Exercise 92.1. Let $n \in \mathbb{N}_0$. Show that the following are equivalent statements:

(a) For every map $f\colon S^n \to \mathbb{R}^n$ there exists an $x \in S^n$ with $f(x) = f(-x)$.
(b) For every odd map $f\colon S^n \to \mathbb{R}^n$ there exists an $x \in S^n$ with $f(x) = 0$.
(c) There is no odd map $f\colon S^n \to S^{n-1}$.
(d) There is no map $f\colon \overline{B}^n \to S^{n-1}$ such that the restriction of f to $S^{n-1} \to S^{n-1}$ is odd.

Exercise 92.2. Let $n \in \mathbb{N}_0$. Show that the Borsuk-Ulam Theorem 39.8 implies the Brouwer Fixed Point Theorem 31.4.

Remark. From Exercise 39.8 you can pick your favorite formulation of the Borsuk-Ulam Theorem 39.8.

Exercise 92.3. Let $q_1, q_2 \in S^3 = \{h \in \mathbb{H} \,|\, |h| = 1\}$. We consider the map

$$\begin{array}{rcl} \mathbb{R}^4 = \mathbb{H} & \to & \mathbb{R}^4 = \mathbb{H} \\ z & \mapsto & q_1 z q_2^{-1}. \end{array}$$

Show that the map is linear and that it is represented by a matrix in SO(4).

Exercise 92.4. Let R be a ring. An involution on R is a map $R \to R$, written $z \mapsto \overline{z}$ with the following properties:

(1) $\overline{z+w} = \overline{z} + \overline{w}$ for all $z, w \in R$,
(2) $\overline{z \cdot w} = \overline{w} \cdot \overline{z}$ for all $z, w \in R$,
(3) $\overline{\overline{z}} = z$ for all $z \in R$.

Now let R be a ring with involution. We consider $S := R \oplus R$ with the obvious multiplication and the multiplication given by

$$(z_1, z_2) \cdot (w_1, w_2) \;:=\; (z_1 w_1 - \overline{w}_2 z_2, z_2 \overline{w}_1 + w_2 z_1)$$

and the involution given by

$$\overline{(w, z)} \;:=\; (\overline{w}, -z).$$

We say S is obtained from R via the Cayley-Dickson construction.

(a) Show that S is a ring with involution.
(b) Show that if R is commutative, then S is associative.

Exercise 92.5. Let $n \in \mathbb{Z}$. Show that the degree of the map

$$\begin{array}{rcl} S^3 = \{v \in \mathbb{H} \,|\, \|v\| = 1\} & \to & S^3 \\ z & \mapsto & z^n \end{array}$$

equals n.

Hint. You could use the Local-Global Degree Proposition 76.21.

Exercise 92.6. Let $\mathbb{A} = \mathbb{R} \oplus \mathbb{R} \cdot \mathrm{i} \oplus \mathbb{R} \cdot \mathrm{j} \oplus \mathbb{R} \cdot \mathrm{k}$. We define the multiplication on $\mathbb{A}$ the same way as we did for the quaternions, see page 1960, except that we set $\mathrm{i}^2 = -1 + \epsilon \cdot \mathrm{j}$.

(a) Show that $\mathbb{A}$ is a division algebra.
(b) Show that i has a left-inverse and a right-inverse.
(c) Show that the left-inverse and the right-inverse of i differ for sufficiently small $\epsilon > 0$.

Exercise 92.7. Show that the octonions $\mathbb{O}$ are not associative.

Exercise 92.8. Show that the map

$$\begin{array}{rcl} \mathbb{O}\mathrm{P}^1 & \to & \mathbb{O} \cup \{\infty\} \\ [x:y] & \mapsto & \begin{cases} x \cdot y^{-1}, & \text{if } y \neq 0, \\ \infty, & \text{if } [x:y] = [1:0] \end{cases} \end{array}$$

is an injection.

Exercise 92.9. Show that $\mathbb{O}\mathrm{P}^2$ is Hausdorff.
Hint. First show that $\mathbb{O}\mathrm{P}^2$ is in fact homeomorphic to a suitable quotient of the sphere $S^{23} = \{(x,y,z) \in \mathbb{O}^3 \,|\, |x|^2 + |y|^2 + |z|^2 = 1\}$ and then apply the Quotient Hausdorff Lemma 5.21.

Exercise 92.10. Let $n \in \mathbb{Z}$. We consider S^3 as a subspace of the quaternions $\mathbb{H}$. What is the degree of the map

$$\begin{array}{rcl} S^3 & \to & S^3 \\ z & \mapsto & z^n \,? \end{array}$$

Part IX

Simplicial Complexes

93. Simplicial complexes: Definition, examples and basic properties

In this section we introduce the notion of an abstract simplicial complex. Similar to the notion of an abstract graph these are purely combinatorial objects. The connection to topology comes from the fact that simplicial complexes admit topological realizations which can be used to combinatorially describe many interesting topological spaces. These topological realizations also lead to the notion of a simplicial complex which can be viewed as a special class of CW-complexes with particularly nice properties.

93.1. **Abstract simplicial complexes.** Before we do anything else we recall the following notation that we introduced on page 86.

Notation. Given a set X we denote by $\mathcal{P}(X)$ its power set, i.e. $\mathcal{P}(X)$ is the set of all subsets of X.

We continue with the following initially rather abstract definition.

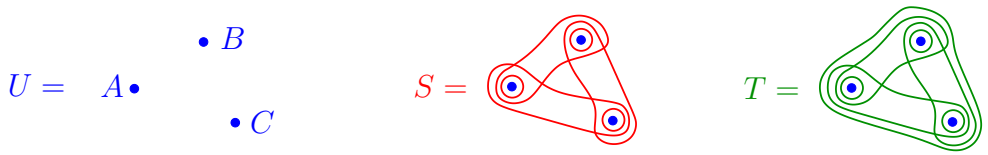
Definition. An abstract simplicial complex is a pair (V, S) where V is a set and S is a subset of the power set $\mathcal{P}(V)$ of V such that the following conditions are satisfied:

(1) each $s \in S$ is finite and non-empty,
(2) if $s \in S$, then any non-empty subset of s also lies in S,
(3) given any $v \in V$ the set $\{v\}$ lies in S.

We refer to the elements of V as the vertices of the abstract simplicial complex and we refer to the elements of S as the simplices of the abstract simplicial complex.

Examples.

(0) The pair $(\varnothing, \varnothing)$ is an abstract simplicial complex, it is called the *empty abstract simplicial complex*. Every other abstract simplicial complex is called *non-empty*.
(1) We consider the set $U = \{A, B, C\}$. It is straightforward to verify that the pair $X := (U, S)$ with $S = \{\{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}\}$ and that the pair $Y := (U, T)$ with $T = \mathcal{P}(U) \setminus \{\varnothing\}$ are abstract simplicial complexes. We illustrate these two abstract simplicial complexes in the figure below.[1128]



(2) Let $n \in \mathbb{N}$. The pairs
   (a) $D_n := (V, S)$ with $V = \{0, \ldots, n\}$ and $S = \mathcal{P}(V) \setminus \{\varnothing\}$, and
   (b) $S_{n-1} := (V, T)$ with $V = \{0, \ldots, n\}$ and $T = \mathcal{P}(V) \setminus \{\varnothing, V\}$,
   are abstract simplicial complexes.
(3) (a) Given $n \in \mathbb{N}_{\geq 3}$ the pair $R_n = (V, S)$ with

$$V := \mathbb{Z}_n \quad \text{and} \quad S := \{\{k\} \,|\, k \in \mathbb{Z}_n\} \cup \{\{k, k+1\} \,|\, k \in \mathbb{Z}_n\}$$

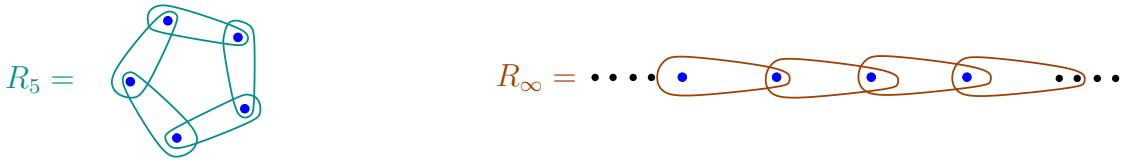
   is an abstract simplicial complex.
   (b) The pair $R_\infty := (V, S)$ with

$$V := \mathbb{Z} \quad \text{and} \quad S := \{\{n\} \,|\, n \in \mathbb{Z}\} \cup \{\{n, n+1\} \,|\, n \in \mathbb{Z}\}$$

[1128] To simplify the figures we will not draw the "trivial" simplices that are given by the vertices.

is an abstract simplicial complex.

We illustrate these two abstract simplicial complexes in the figure below.



Definition. Let $K = (V, S)$ be an abstract simplicial complex.

(1) We say $K = (V, S)$ is finite if V is finite.[1129] Similarly we say $K = (V, S)$ is countable if V is countable.
(2) If $s \in S$ has $k + 1$ elements, then s is called a k-dimensional simplex, or shorter, a k-simplex of (V, S).
(3) If $K = (V, S)$ is the empty abstract simplicial complex, then we define its dimension as -1. Now suppose that $K = (V, S)$ is non-empty. We define

$$\dim(K) \ := \ \max\{\#s - 1 \,|\, s \in S\} \ \in \ \mathbb{N}_0 \cup \{\infty\}.$$

In other words, the dimension of $K = (V, S)$ is the maximal dimension of a simplex of $K = (V, S)$.

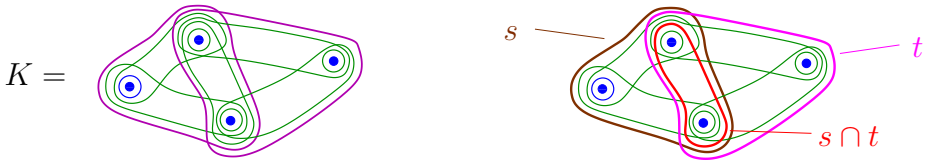
Examples.

(1) The above abstract simplicial complex (U, S) is 1-dimensional and the above abstract simplicial complex (U, T) is 2-dimensional.
(2) Let $n \in \mathbb{N}$. The abstract simplicial complex D_n is n-dimensional and the abstract simplicial complex S_{n-1} is $(n-1)$-dimensional
(3) The abstract simplicial complexes R_n and R_∞ are one-dimensional.

We continue with a few more harmless definitions and statements.

Definition. Let $K = (V, S)$ be an abstract simplicial complex. Let t be a simplex. A face of t is a simplex s with $s \subset t$.

Lemma 93.1. Let $K = (V, S)$ be an abstract simplicial complex. Furthermore let s and t be two simplices. Then either the intersection $s \cap t$ is empty, or the intersection $s \cap t$ is again a simplex and it is a face of s and t.



Proof. Let s and t be two simplices of an abstract simplicial complex $K = (V, S)$. If $s \cap t = \varnothing$ then there is nothing to show. Now suppose that $s \cap t \neq \varnothing$. Evidently $s \cap t$ is a non-empty subset of s and t. By definition of an abstract simplicial complex this means that $s \cap t$ is also a simplex. Furthermore, since $s \cap t \subset s$ and $s \cap t \subset t$ the intersection is a face of s and t. ■

[1129] Note that this implies that S is also finite. In fact since every $v \in V$ defines a simplex we see that an abstract simplicial complex is finite if and only if S is finite.

Definition. Let $L = (W, T)$ be an abstract simplicial complex. An abstract subcomplex of $L = (W, T)$ is a pair $K = (V, S)$ with $V \subset W$, with $S \subset T$ and such that $K = (V, S)$ is an abstract simplicial complex in its own right. On most occasions we just say "subcomplex" instead of "abstract subcomplex".

Examples.

(1) Basically by definition $X = (U, S)$ is a subcomplex of $Y = (U, T)$.
(2) Let $n \in \mathbb{N}$. Again, basically by definition S_{n-1} is a subcomplex of D_n.
(3) Let $K = (V, S)$ be an abstract simplicial complex and let $n \in \mathbb{N}_0$. We refer to V together with all simplices of dimension $\leq n$ as the *n-skeleton of the abstract simplicial complex*. It is basically clear from the definitions that the n-skeleton is a subcomplex. For convenience we also write $K^\infty := K$.
(4) Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$.
   (a) The pair $(s, \mathcal{P}(s) \setminus \{\varnothing\})$ is a subcomplex of K. Often we abbreviate this subcomplex to s.
   (b) The pair $(s, \mathcal{P}(s) \setminus \{\varnothing, s\})$ is a subcomplex of K. Usually we refer to this subcomplex as ∂s.

The following lemma gives us elementary ways to construct subcomplexes.

Lemma 93.2. Let $\{L_i = (W_i, T_i)\}_{i \in I}$ be a family of abstract simplicial complexes. The following statements hold:

(1) $$\bigcap_{i \in I} L_i := \left(\bigcap_{i \in I} W_i, \bigcap_{i \in I} V_i \right) \quad \text{and} \quad \bigcup_{i \in I} L_i := \left(\bigcup_{i \in I} W_i, \bigcup_{i \in I} V_i \right)$$
are abstract simplicial complexes.
(2) $\bigcap_{i \in I} L_i$ is a subcomplex of each L_i and each L_i in turn is a subcomplex of $\bigcup_{i \in I} L_i$.
(3) If each L_i is a subcomplex of some abstract simplicial complex K, then $\bigcap_{i \in I} L_i$ and $\bigcup_{i \in I} L_i$ are also subcomplexes of K.

Proof. The lemma is close to being a tautology. ∎

As the reader will know by now, not only do we want to study objects, we also want to study maps or morphisms between them. The following gives us the type of maps we are now interested in.

Definition. Let $K = (V, S)$ and $L = (W, T)$ be abstract simplicial complexes.

(1) A simplicial map $f \colon K \to L$ is defined as a map $f \colon V \to W$ such that for every $s \in S$ we have $f(s) \in T$. In particular a simplicial map induces a map $S \to T$.
(2) A simplicial isomorphism between $K = (V, S)$ and $L = (W, T)$ is a simplicial map that is a bijection on the set of vertices and on the set of simplices.

Examples.

(1) The identity map of $U = \{A, B, C\}$ defines a simplicial map from $X = (U, S)$ to $Y = (U, T)$.
(2) The map $f \colon \{0, 1, 2\} \to \{A, B, C\}$ given by $f(0) = A$, $f(1) = B$ and $f(2) = C$ defines a simplicial isomorphism between D_2 and (U, T) and it defines a simplicial

isomorphism between S_1 and (U, S). Similarly one sees that the abstract simplicial complex R_3 is simplicially isomorphic to S_1 and (U, S).

(3) The inclusion of a subcomplex into an abstract simplicial complex is, basically by definition, a simplicial map.

It is clear that the composition of simplicial maps is again a simplicial map. Thus it makes sense to introduce the following definition.

Definition. We call the category $\mathcal{AbsSimpCplx}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{AbsSimpCplx}) &:= \text{all abstract simplicial complexes,} \\ \mathrm{Mor}(K, L) &:= \text{all simplicial maps from } K \text{ to } L, \end{aligned}$$

together with the usual composition of maps the category of abstract simplicial complexes.

For future reference let us record the following lemma.

Lemma 93.3. Let K and L be two abstract simplicial complexes.

(1) If K and L are finite, then there exist only finitely many simplicial maps from K to L.

(2) If K is finite and if L is countable, then there exist only countably many simplicial maps from K to L.

Proof.

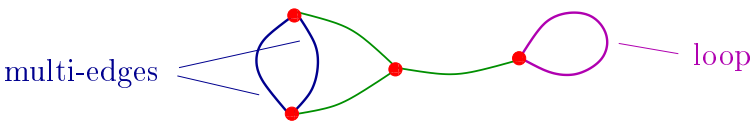
(1) This statement is obvious.

(2) Let $K = (V, S)$ and $L = (W, T)$ be two abstract simplicial complexes. We assume that K is finite and that L is countable. It follows from the Countability Lemma 0.8 (6) that there exist only countably many maps $V \to W$. Furthermore, since V is finite we see that given any map $V \to W$ there are only finitely many maps $S \to T$ such that the resulting map is a simplicial map. It follows from the Countability Lemma 0.8 (3) that there are indeed only countably many simplicial maps. ∎

Remark. The reader might have a certain sense of déjà vu: the definition of a 1-dimensional abstract simplicial complex sounds very similar to the definition of an undirected abstract graph which we gave on page 313. In fact some thought shows that the concepts are different since undirected graphs are more flexible. But once we rule out loops and multi-edges (see the figure below for a pictorial reminder of the definitions), then the concepts are basically the same. More precisely it follows easily from the definitions that the maps

$$\underbrace{(V, E, \varphi\colon E \to \mathcal{P}(V))}_{\substack{\text{undirected abstract graph} \\ \text{without loops \& multi-edges}}} \longmapsto \underbrace{(V, \varphi(E))}_{\substack{\text{1-dimensional abstract} \\ \text{simplicial complex}}} \quad \text{and} \quad \underbrace{(V, S \subset \mathcal{P}(V))}_{\substack{\text{1-dimensional abstract} \\ \text{simplicial complex}}} \longmapsto \underbrace{(V, S, \mathrm{id}\colon S \to \mathcal{P}(V))}_{\substack{\text{undirected abstract graph} \\ \text{without loops \& multi-edges}}}$$

define an isomorphism between the "category of undirected abstract graphs without loops and multi-edges" and the "category of 1-dimensional simplicial complexes".



93.2. **Ordered abstract simplicial complexes.** We recall the following definition from page 94.

Definition.

(1) A relation that is reflexive, transitive and antisymmetric is called a partial order.
(2) A partially ordered set is a set together with a partial order.
(3) A partial order that is connex is called a total order.
(4) A totally ordered set is a set together with a total order.
(5) In a totally ordered set $(P, \leq)$ we write $a < b$ if $a \leq b$ and $a \neq b$.

Examples.

(1) Let X be a set. We consider the power set $\mathcal{P}(X)$. The relation "$\subset$" is a partial order on $\mathcal{P}(X)$.
(2) The usual relation "$\leq$" on $\mathbb{Z}$ turns $\mathbb{Z}$ into a totally ordered set.
(3) Any subset of $\{A, B, \ldots, Z\}$ with the usual alphabetical ordering is a totally ordered set.

We continue with the following definition.

Definition.

(1) A total order on an abstract simplicial complex (V, S) is a total order on the vertex set S.
(2) An order on an abstract simplicial complex (V, S) is a partial order on V which has the property that the restriction to each simplex is actually a total order.
(3) An ordered abstract simplicial complex is an abstract simplicial complex together with an order.

Remark.

(1) Every totally ordered abstract simplicial complex is automatically also an ordered simplicial complex.
(2) By the Well-Ordering Theorem 0.7 we know that every set admits a total ordering. Thus every abstract simplicial complex can be turned into a totally ordered abstract simplicial complex.

The following definition is the obvious variation on the definition provided on page 1981.

Definition. We call the category $\mathit{OrdAbsSimpCplx}$ with

$$\begin{aligned} \mathrm{Ob}(\mathit{OrdAbsSimpCplx}) &:= \text{all ordered abstract simplicial complexes,} \\ \mathrm{Mor}(K, L) &:= \text{all order-preserving simplicial maps from } K \text{ to } L, \end{aligned}$$

together with the usual composition of maps the category of ordered abstract simplicial complexes.

93.3. **The topological realization of an abstract simplicial complex.** Since we are topologists we want to study abstract simplicial complexes with topological methods. The idea is to associate to an abstract simplicial complex a corresponding topological space. Before we can do so we need to introduce the following notation which is inspired by the definition of the free abelian group $\mathbb{Z}^{(S)}$ that we gave on page 1147.

Notation.

(1) Given any set V we write

$$\begin{array}{rcl} \mathbb{R}^V & := & \text{all maps from } V \text{ to } \mathbb{R} \\ \mathbb{R}^{(V)} & := & \text{all maps from } V \text{ to } \mathbb{R} \text{ which are non-zero for only finitely many } v \in V. \end{array}$$

We have the following conventions and observations:

(a) Given $v \in V$ we sometimes denote by v also the map $V \to \mathbb{R}$ which is given by $v \mapsto 1$ and $w \mapsto 0$ for $w \neq v$.

(b) One can easily verify that with this interpretation the set V forms a basis for the real vector space $\mathbb{R}^{(V)}$.

(c) If $V = \{v_1, \ldots, v_n\}$ is a finite set, then the map $v_i \mapsto e_i$ defines an isomorphism from $\mathbb{R}^V = \mathbb{R}^{(V)}$ to $\mathbb{R}^n$. In particular, if V is a finite set with a total order, then it follows from Exercise 0.4 that we have a natural isomorphism $\mathbb{R}^V \xrightarrow{\cong} \mathbb{R}^{|V|}$.

(2) If V is a subset of some set W, then we view $\mathbb{R}^{(V)}$ as a subset of $\mathbb{R}^{(W)}$ in the obvious way.

The following definition is the first step in our shift back to topology.

Definition. Let $K = (V, S)$ be some abstract simplicial complex. For a given n-simplex $s = \{v_0, \ldots, v_n\}$ we define

$$|s| := \left\{ \alpha \in \mathbb{R}^{(V)} \;\middle|\; \begin{array}{l} (1)\ \alpha(v) \geq 0 \text{ for all } v \in s \\ (2)\ \alpha(v) = 0 \text{ for all } v \notin s \\ (3)\ \text{we have } \sum_{v \in V} \alpha(v) = 1 \end{array} \right\} = \left\{ \underbrace{\sum_{i=0}^{n} t_i \cdot v_i}_{\in \mathbb{R}^{(V)}} \;\middle|\; \begin{array}{l} (1)\ \text{for each } i \text{ we have } t_i \geq 0 \\ (2)\ t_0 + \cdots + t_n = 1 \end{array} \right\}$$

Furthermore we set

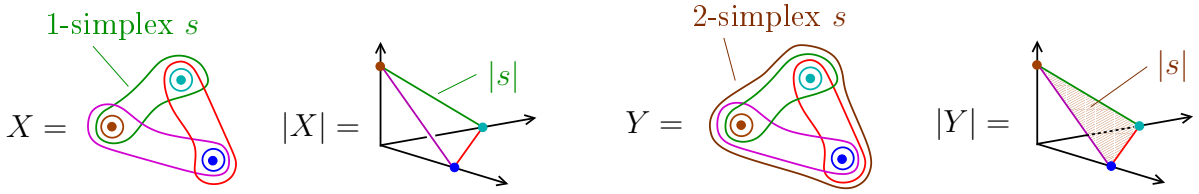
$$|K| := \bigcup_{s \in S} |s| \subset \mathbb{R}^{(V)}.$$

Sometimes, by a serious abuse of language, we refer to $|s|$ as a n-simplex in $|K|$.

Examples.

(1) We consider the abstract simplicial complexes $X = (U, S)$ and $Y = (U, T)$ as above. The corresponding sets $|X|$ and $|Y|$ are the subsets of $\mathbb{R}^U = \mathbb{R}^{\{A,B,C\}} = \mathbb{R}^3$ that are shown in the figure below.

(2) Let $n \in \mathbb{N}$. Given the abstract simplicial complex R_n the corresponding set $|R_n|$ is a subset of $\mathbb{R}^{\mathbb{Z}_n} = \mathbb{R}^{n-1}$. This means that for $n \geq 5$ we have serious problems with visualizing the set $|R_n|$. The problem is even bigger for the infinite abstract simplicial complex R_∞.



Lemma 93.4. Given any abstract simplicial complex $K = (V, S)$ we have

$$|K| = \left\{ \alpha \in \mathbb{R}^{(V)} \;\middle|\; \begin{array}{ll} (1) & \text{the set } \{v \in V \,|\, \alpha(v) \neq 0\} \text{ is a simplex of } K \\ (2) & \text{for every } v \in V \text{ we have } \alpha(v) \geq 0 \\ (3) & \text{we have } \sum_{v \in V} \alpha(v) = 1 \end{array} \right\}.$$

Proof. We leave the elementary task of proving the equality to the reader. Note though that one actually needs to use that $K = (V, S)$ is an abstract simplicial complex. ■

Note that as of right now $|K|$ is just a set. But rest assured, we will equip $|K|$ with a suitable topology shortly.[1130] But first let us introduce a convention.

Convention. Let $K = (V, S)$ be an abstract simplicial complex. As discussed above, we can view V as a subset of $\mathbb{R}^{(V)}$. In fact it follows easily from the definitions that $V \subset |K|$. Throughout these notes we use this observation to view V as a subset of $|K|$.

We continue with the following definition.

Definition. Let $K = (V, S)$ be an abstract simplicial complex. Let $k \in \mathbb{N}_0$ and let s be a k-simplex.

(1) If $v_0 \leq \cdots \leq v_k$ is a total order on the set of vertices of s, then we refer to the map

$$\begin{aligned} \Phi_s \colon \Delta^k = \left\{ (t_0, \ldots, t_k) \in \mathbb{R}^{k+1}_{\geq 0} \;\middle|\; \sum_{i=0}^{k} t_i = 1 \right\} &\to |K| \\ (t_0, \ldots, t_k) &\mapsto \sum_{i=0}^{k} t_i \cdot v_i \end{aligned}$$

as a characteristic map of the simplex s.

(2) If K is actually an ordered abstract simplicial complex, then we use the corresponding order on s and we write $\Phi_s^{\leq} \colon \Delta^k \to |K|$. In this case we can refer to $\Phi_s^{\leq}$ unambiguously as the characteristic map of the simplex s.

Remark.

(1) Note that if in the above definition of a characteristic map we use two different orderings of the vertices of s, then we get two maps $\Phi, \Psi \colon \Delta^k \to |K|$ with the same image and such that $\Psi^{-1} \circ \Phi \colon \Delta^k \to \Delta^k$ is a homeomorphism. In other words, any two characteristic maps differ by precomposing by a homeomorphism. In practice this means that any two characteristic maps exhibit the same properties and we do not need to distinguish between different characteristic maps.

(2) Given a k-simplex s we have, by definition, $\Phi_s(\Delta^k) = |s|$.

Definition. Let $K = (V, S)$ be an abstract simplicial complex.

(1) We say that a subset $U \subset |K|$ is open if for every simplex s of K the preimage $\Phi_s^{-1}(U)$ under a characteristic map[1131] Φ_s is open. An argument as in Exercise 2.37 shows that this defines a topology on $|K|$.

(2) We refer to the set $|K|$, together with this topology, as the topological realization of K.

[1130] A first idea might be to equip $\mathbb{R}^{(V)}$ with a suitable topology and then to endow $|K|$ with the subspace topology. But this is not really a good idea since this topology is artificial and does not reflect the structure of K. For example it would be rather unpleasant to have to show that a map out of K is continuous.

The following lemma summarizes a few basic properties of the topological realization of an abstract simplicial complex.

Lemma 93.5. (Simplex-Embedding Lemma) Let K be an abstract simplicial complex.

(1) A subset $A \subset |K|$ is closed if and only if for every simplex s of K the preimage $\Phi_s^{-1}(A)$ under a characteristic map Φ_s is closed.
(2) The topological realization $|K|$ is Hausdorff.
(3) For every k-simplex $s = \{v_0, \ldots, v_k\}$ of K any characteristic map $\Phi_s \colon \Delta^k \to |K|$ is a closed embedding. In particular the map $\Phi_s \colon \Delta^k \to |s|$ is a homeomorphism.

Proof. Let $K = (V, S)$ be an abstract simplicial complex.

(1) This statement follows almost immediately from the definition of the topology on $|K|$ together with the elementary the Image-Preimage Lemma 0.2 (8).
(2) Let P and Q be two disjoint points on $|K|$. Given $f, g \in \mathbb{R}^{(V)}$ we write

$$d(f,g) \;:=\; \sqrt{\sum_{v \in V} (f(v) - g(v))^2}.$$

One can easily verify that this is a metric on $\mathbb{R}^{(V)}$. By Proposition 1.13 we know that there exists an $\epsilon > 0$ such that the open ϵ-balls $B_\epsilon(P)$ and $B_\epsilon(Q)$ are disjoint in $\mathbb{R}^{(V)}$. It remains to show that $B_\epsilon(P) \cap |K|$ and $B_\epsilon(Q) \cap |K|$ are open in $|K|$. By definition of the topology on $|K|$ we need to prove the following claim.

Claim. Let $U := B_\epsilon(R)$ be an ϵ-ball in $\mathbb{R}^{(V)}$. For every simplex s the preimage $\Phi_s^{-1}(U \cap |K|)$ under a characteristic map Φ_s is open.

Proof. Let $s = \{v_0, \ldots, v_k\}$ be a k-simplex. We make the following observations:
(a) We have $\Phi_s^{-1}(U \cap |K|) = \Phi_s^{-1}(U)$.
(b) It follows from (a) and the observation that U is an open subset of the topological space $(\mathbb{R}^{(V)}, d)$ that it suffices to show that the map $\Phi_s \colon \Delta^k \to (\mathbb{R}^{(V)}, d)$ is continuous.
(c) We view $\Phi_s \colon \Delta^k \to \mathbb{R}^{(V)}$ as the composition of the maps

$$\Delta^k \;\xrightarrow{\Phi_s}\; \mathbb{R}^{k+1} = \mathbb{R}^{(s)} \;\hookrightarrow\; \mathbb{R}^{(V)}.$$

We consider the two maps separately.
(i) Note that $\mathbb{R}^{k+1} = \mathbb{R}^{(s)}$ is a finite-dimensional vector space and that the map $\Phi_s \colon \Delta^k \to \mathbb{R}^{(s)}$ is an affine linear map. It follows almost immediately from the Standard Simplex Homeomorphism Lemma 72.1 that this map is continuous.
(ii) It follows easily from the definition of topology on $\mathbb{R}^{k+1}$, via the usual Euclidean metric, and the definition of the metric d on $\mathbb{R}^{(V)}$, together with the simple-minded Exercise 2.13 (a), that the map $\mathbb{R}^{k+1} = \mathbb{R}^{(s)} \to \mathbb{R}^{(V)}$ is continuous.
Thus we see that the map $\Phi_s \colon \Delta^k \to (\mathbb{R}^{(S)}, d)$ itself is continuous. ⊞

[1131] It follows from the above remark that the statement that $\Phi_s^{-1}(U)$ is open does not depend on the choice of the characteristic map for the simplex s.

(3) Let s be a k-simplex of K. It follows almost immediately from the definitions that the map $\Phi_s\colon \Delta^k \to |K|$ defines a bijection $\Phi_s\colon \Delta^k \to |s|$. It follows immediately from the definition of the topology on $|K|$ that Φ_s is continuous. Evidently Δ^k is compact. By (1) we know that $|K|$ is Hausdorff. So once again the Compact-Hausdorff Proposition 2.17 (2) rides to our rescue and tells us that Φ_s is indeed an embedding and that the image is a closed subset. ■

The following is the simplicial analogue of the corresponding the Characteristic Maps Lemma 68.7 for CW-complexes.

Lemma 93.6. (Restrict-to-Simplex Lemma) Let $K = (V, S)$ be an abstract simplicial complex and let $f\colon |K| \to X$ be a map to some topological space X.

(1) If for each k-simplex $s \in S$ with characteristic map $\Phi_s\colon \Delta^k \to |K|$ the composition $f \circ \Phi_s\colon \Delta^k \to X$ is continuous, then the map f is continuous.

More generally we have the following continuity criterion:

(2) If for each simplex $s \in S$ there exists a k-simplex t with $s \subset t$ such that for some characteristic map $\Phi_t\colon \Delta^k \to |K|$ the composition $f \circ \Phi_t\colon \Delta^k \to X$ is continuous, then the map f is continuous.

Proof.

(1) This statement follows immediately from the definition of the topology on $|K|$.

(2) This statement follows immediately from (1) and the observation that characteristic maps are continuous. ■

The following proposition shows that not only do abstract simplicial complexes give rise to topological spaces, but simplicial maps also give rise to continuous maps between the topological realizations.

Proposition 93.7.

(1) If $f\colon K = (V, S) \to L = (W, T)$ is a simplicial map, then the map

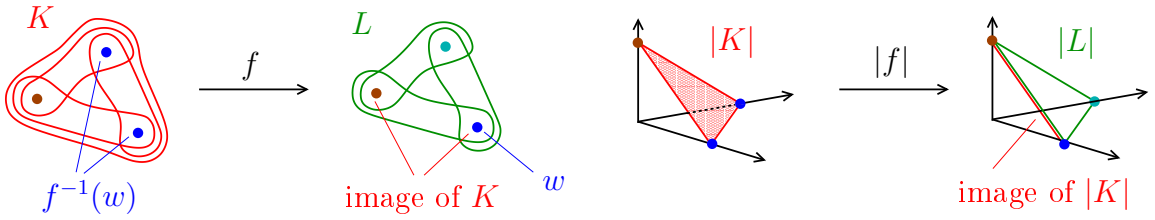
$$\begin{array}{rcl} |f|\colon |K| & \to & |L| \\ (\alpha\colon V \to \mathbb{R}) & \mapsto & \left(\begin{array}{ccc} W & \to & \mathbb{R} \\ w & \mapsto & \sum\limits_{v \in f^{-1}(w)} \alpha(v) \end{array} \right) \end{array}$$

is well-defined and continuous. We refer to the figure below for an illustration.

(2) Let $\mathcal{AbsSimpCplx}$ be the category of abstract simplicial complexes. The maps

$$\begin{array}{rcl} K & \mapsto & |K| \\ (f\colon K \to L) & \mapsto & (|f|\colon |K| \to |L|) \end{array}$$

define a covariant functor from the category $\mathcal{AbsSimpCplx}$ of abstract simplicial complexes to the category $\mathcal{Top}$ of topological spaces.



Remark.

(1) Let $f\colon K = (V, S) \to L = (W, T)$ be a simplicial map. It is elementary to see that, given a simplex $s = \{v_0, \ldots, v_k\}$ of K, the restriction of $|f|$ to $|s| \subset |K|$ is the map

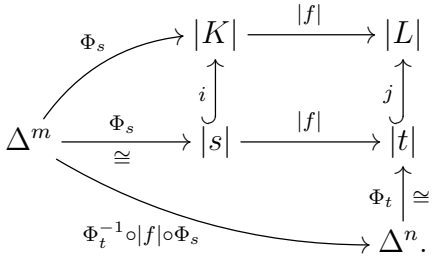
$$\begin{array}{rcl} |f|\colon |s| & \to & |f(s)| \\ \sum\limits_{i=0}^{k} t_i \cdot v_i & \mapsto & \sum\limits_{i=0}^{k} t_i \cdot f(v_i). \end{array}$$

(2) As we will see, the proof of Proposition 93.7 (1) is slightly delicate since the topology of the topological realization of a simplicial complex is designed, see the Restrict-to-Simplex Lemma 93.6, in such a way that it is straightforward to verify that a map *out* of the topological realization is continuous. It is much less clear how one can show that a map *to* a topological realization is continuous. The same health-warning applies also to CW-complexes.

Proof.

(1) Let $f\colon K = (V, S) \to L = (W, T)$ be a simplicial map. A priori we have written down a map $|f|\colon |K| \to \mathbb{R}^{(W)}$. But it is straightforward to verify, e.g. using Lemma 93.4, that the map $|f|$ does indeed take values in the subset $|L|$. It remains to show that $|f|\colon |K| \to |L|$ is continuous. By the Restrict-to-Simplex Lemma 93.6 (1) it suffices to show that for each m-simplex $s \in S$ with some characteristic map $\Phi_s\colon \Delta^m \to |K|$ the composition $f \circ \Phi_s\colon \Delta^m \to |L|$ is continuous.

Thus suppose we are given such s and Φ_s. We write $t := f(s)$. By definition of a simplicial map we know that t is some n-simplex of T. For clarity we consider the following commutative diagram:



We make the following clarifications and observations:

(a) We define $|s| \subset |K|$ and $|t| \subset |L|$ as in the Simplex-Embedding Lemma 93.5 (3). We equip both with the subspace topology coming from $|K|$ and $|L|$. By the super elementary Lemma 2.1 (2) we know that the inclusion maps $i\colon |s| \to |K|$ and $j\colon |t| \to |L|$ are continuous.

(b) It follows immediately from the definition of $|f|$ that the restriction of $|f|$ to $|s|$ takes values in $|t|$.

(c) We picked a characteristic map $\Phi_t\colon \Delta^n \to |t|$ for t.

(d) By the Simplex-Embedding Lemma 93.5 (3) we know that the characteristic map $\Phi_t\colon \Delta^n \to |t|$ is a homeomorphism. In particular the inverse $\Phi_t^{-1}\colon |t| \to \Delta^n$ exists and it is continuous.

(e) It follows almost immediately from the definitions of the maps Φ_s, Φ_t and $|f|$ that the map $\Phi_t^{-1} \circ |f| \circ \Phi_s\colon \Delta^m \to \Delta^n$ is an affine linear map. Hence it follows,

say from the Standard Simplex Homeomorphism Lemma 72.1, that the map is continuous.

(f) It follows immediately from the above that the map

$$|f| \circ \Phi_s \;=\; j \circ \Phi_t \circ (\Phi_t^{-1} \circ |f| \circ \Phi_s) \colon \Delta^m \;\to\; Y$$

is indeed continuous.

(2) Let $f \colon K \to L$ and $g \colon L \to M$ be simplicial maps. For any vertex v of M we have $f^{-1}(g^{-1}(v)) = (g \circ f)^{-1}(v)$. It follows immediately from this observation that $|g| \circ |f| = |g \circ f|$. It is now straightforward to see that the given maps define a covariant functor. ■

The following lemma can be viewed as an analogue of the Subcomplex Lemma 68.18, which we formulated in the context of CW-complexes.

Lemma 93.8. (Subcomplex Lemma) Let $K = (V, S)$ be a subcomplex of some given abstract simplicial complex $L = (W, T)$.

(1) The inclusion $\mathbb{R}^{(V)} \to \mathbb{R}^{(W)}$ restricts to an inclusion $|K| \to |L|$.
(2) The inclusion $|K| \to |L|$ is a closed embedding.
(3) Let $\{K_i\}_{i \in I}$ be a family of subcomplexes of an abstract simplicial complex K. Then

$$\Big|\bigcap_{i\in I} K_i\Big| \;:=\; \bigcap_{i\in I} |K_i| \quad \text{and} \quad \Big|\bigcup_{i\in I} K_i\Big| \;:=\; \bigcup_{i\in I} |K_i|.$$

are subcomplexes of K.

Remark. The statement and the proof of Lemma 93.8 is closely related to the content of the Subcomplex Lemma 68.18.

Convention. Let $K = (V, S)$ be a subcomplex of some given abstract simplicial complex $L = (W, T)$. We will use Lemma 93.8 to view $|K|$ as a subset of $|L|$. By Lemma 93.8 (2) we know that $|K|$ is in fact a closed subset of $|L|$.

Proof. Let $L = (W, T)$ be a simplicial complex and let $K = (V, S)$ be a subcomplex. We denote by $i \colon K \to L$ the inclusion map.

(1) It follows immediately from the definitions that the inclusion $\mathbb{R}^{(V)} \to \mathbb{R}^{(W)}$ restricts to an inclusion $|K| \to |L|$.
(3) This statement also follows immediately from the definitions. For readers who want to fill in the details, perhaps it is clearest to look at the description given in Lemma 93.4.
(2) Since $i \colon K \to L$ is a simplicial map we obtain from Proposition 93.7 (1) that the induced map $|i| \colon |K| \to |L|$ is continuous. Furthermore it follows easily from the definitions that the map $|i| \colon |K| \to |L|$ is in fact just the inclusion. In the remainder of the proof we will view $|K|$ as a subset of $|L|$.

By the Open -Injective Map Lemma 2.16 (1) it remains to show that the inclusion $|K| \to |L|$ is a closed map. Thus let $A \subset |K|$ be a closed subset. We need to show that A is also a closed subset of $|L|$. By the Simplex-Embedding Lemma 93.5 (1) and (3) it suffices to show that for each $t \in T$ the set $A \cap |t|$ is closed. Note that every simplex, in particular t itself, has only finitely many faces. Let $f_1, \ldots, f_m$ be

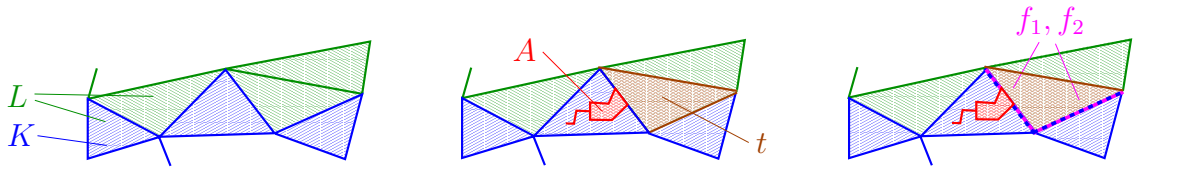
the faces of t that are contained in K. We consider the subsets $|f_1|, \ldots, |f_m|$ of $|K|$ and the subset $|t|$ of $|L|$ as defined on page 1983. We make the following observation:

(∗) It follows from the Simplex-Embedding Lemma 93.5 that the subspace topologies on $|f_i|$ coming from $|f_i| \subset |K|$ and $|f_i| \subset |t| \subset |L|$ agree and that $|f_i|$ is a closed subset of $|K|$ and a closed subset of $|t|$.

We have

$$\begin{aligned} A \cap |t| &= A \cap (|K| \cap |t|) && = A \cap \bigcup_{i=1}^{m} |f_i| = \bigcup_{i=1}^{m} (A \cap |f_i|) \\ &\uparrow \text{ since } A \subset |K| && \uparrow \text{ this follows from (3) since } K \text{ is a subcomplex} \\ &= \bigcup_{i=1}^{m} \text{closed subset of } |f_i| && = \bigcup_{i=1}^{m} \text{closed subset of } |t| = \text{closed subset of } |t|. \\ &\uparrow \text{ since } A \text{ is closed in } |K| \text{ and since} && \uparrow \text{ by the Open -Closed Inclusion Lemma 2.10 (2) since by the} \\ &\quad f_1, \ldots, f_m \in S \text{ are simplices of } K && \quad \text{Simplex-Embedding Lemma 93.5 (3) we know} \\ &&& \quad \text{that the } |f_i| \text{ are closed subsets of } |t| \end{aligned}$$

■



Example. Let $n \in \mathbb{N}$. We consider again the simplicial complex D_n and its subcomplex S_{n-1}. It follows almost immediately from the Simplex-Embedding Lemma 93.5 (3) that the map

$$\begin{aligned} \Psi \colon |D_n| &\to \Delta^n \\ \sum_{i=0}^{n} t_i \cdot \{i\} &\mapsto (t_0, \ldots, t_n) \end{aligned}$$

is a homeomorphism. By Proposition 93.7 (2) the map $|S_{n-1}| \to |D_n|$ is an embedding. Thus Ψ restricts to a homeomorphism $\Psi|_{|S_{n-1}|} \colon |S_{n-1}| \to \Psi(|S_{n-1}|)$ but basically by definition we have $\Psi(|S_{n-1}|) = \partial\Delta^n$. This shows that $|S_{n-1}|$ is homeomorphic to $\partial\Delta^n$.

Before we consider some examples we want to consider the topology for finite abstract simplicial complexes in greater detail. We recall that given any finite set X the vector space $\mathbb{R}^{(X)}$ is a finite-dimensional vector space. Furthermore recall that by the discussion on page 153 we know that any finite-dimensional real vector space has a natural topology.

Lemma 93.9. Let $K = (V, S)$ be an abstract simplicial complex. If K is finite, then the following two conclusions hold:

(1) The topological realization $|K|$ is compact.
(2) The topology of the topological realization $|K|$, defined on page 1984, agrees with the subspace topology which we obtain from viewing $|K|$ as a subset of the finite-dimensional real vector space $\mathbb{R}^V$.

Proof. Let $K = (V, S)$ be a finite abstract simplicial complex.

(1) By the Simplex-Embedding Lemma 93.5 we know in particular that for each k-simplex s the characteristic map $\Phi_s \colon \Delta^k \to |K|$ is continuous. Since Δ^k is compact we obtain from the Compact Image Lemma 2.13 that $|s| = \Phi_s(\Delta^k)$ is compact. By definition $|K|$ is the union of all the subsets $|s|$. Thus, since K is finite we see that $|K|$

is the union of finitely many compact subsets. By the Compact-Union-Intersection Lemma 1.20 (1) this implies that $|K|$ itself is compact.

(2) We write $V = \{v_1, \ldots, v_n\}$ and we identify the vector spaces $\mathbb{R}^V$ and $\mathbb{R}^n$ in the obvious way. We denote by $\mathcal{T}$ the topology on $|K|$ defined above and we denote by $\mathcal{R}$ the subspace topology which comes from $\mathbb{R}^V$. We need to show that the identity map defines a homeomorphism $(|K|, \mathcal{T}) \to (|K|, \mathcal{R})$. We make the following observations:
   (a) For any k-simplex $s \in S$ and any characteristic map $\Phi_s \colon \Delta^k \to |s|$ the corresponding map $\Delta^k \xrightarrow{\Phi_s} |s| \to |K| \to \mathbb{R}^V = \mathbb{R}^n$ is an affine linear map. It follows from the Standard Simplex Homeomorphism Lemma 72.1 that this map is continuous. It follows from this observation together with the Restrict-to-Simplex Lemma 93.6 (1) that the map $(|K|, \mathcal{T}) \to (|K|, \mathcal{R})$ is continuous.
   (b) In (1) we just argued that $(|K|, \mathcal{T})$ is compact.
   (c) The topological space $(|K|, \mathcal{R})$ is evidently Hausdorff.
   It follows from the Compact-Hausdorff Proposition 2.17 (3) that the identity map is indeed a homeomorphism. ■

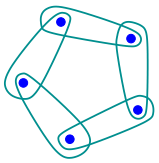
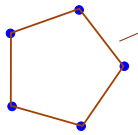
As we already noted above, the topological realizations of abstract simplicial complexes have a voracious appetite for dimensions. Fortunately, as well see shortly, in many cases we can reduce the dimensions drastically.

Lemma 93.10. Let $K = (V, S)$ be a finite abstract simplicial complex and let $\varphi \colon \mathbb{R}^V \to \mathbb{R}^n$ be a linear map such that the restriction of φ to $|K|$ is an injection. Then $\varphi \colon |K| \to \mathbb{R}^n$ is an embedding.

Proof. By Lemma 93.9 (1) we know that $|K|$ is compact. Furthermore, note that it follows from Lemma 93.9 (2) that the linear map $\varphi \colon \mathbb{R}^V \to \mathbb{R}^n$ restricts to a continuous map $\varphi \colon |K| \to \mathbb{R}^n$. Since $\mathbb{R}^n$ is Hausdorff it follows from the Compact-Hausdorff Proposition 2.17 (2) that $\varphi \colon |K| \to \mathbb{R}^n$ is indeed an embedding. ■

Example.

(1) Let $n \in \mathbb{N}_{\geq 3}$. As above we consider the abstract simplicial complex R_n. We consider the linear map $\varphi \colon \mathbb{R}^{\mathbb{Z}_n} \to \mathbb{R}^2$ that is given by $[k] \mapsto \exp(2\pi \mathrm{i} k/n)$. One can easily verify that the restriction of φ to $|R_n|$ is an injection. Thus we see that $|R_n|$ is homeomorphic to the regular n-gon that is shown in the figure below.

$R_5 =$  $|R_5|$ is homeomorphic to  regular 5-gon in $\mathbb{R}^2$

(2) Using the Closed Embedding-to-$\mathbb{R}^n$ Proposition 2.19 one can formulate a suitable generalization of Lemma 93.10 to infinite abstract simplicial complexes. We will not work this out in detail, but using this approach one can fairly easily show that the topological realization of the abstract simplicial complex R_∞ is homeomorphic to $\mathbb{R}$. We refer to the figure below for a very convincing illustration of this fact.

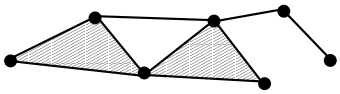
$R_\infty =$  $|R_\infty|$ is homeomorphic to

Given an n-simplex s of an abstract simplicial complex $K = (V, S)$ we defined on page 1984 the corresponding subset $|s| \subset |K|$, which, somewhat dangerously, we referred to as an n-simplex of $|K|$. The following definition gives us a convenient variation on that definition.

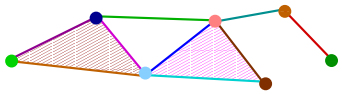
Definition. Let $K = (V, S)$ be an abstract simplicial complex. If we are given an n-simplex $s = \{v_0, \dots, v_n\} \in S$, then we refer to[1132]

$$\langle s \rangle := \left\{ \alpha \in \mathbb{R}^{(V)} \;\middle|\; \begin{array}{l} (1)\ \alpha(v) > 0 \text{ for all } v \in s \\ (2)\ \alpha(v) = 0 \text{ for all } v \notin s \\ (3)\ \text{we have } \sum_{v \in V} \alpha(v) = 1 \end{array} \right\} = \left\{ \underbrace{\sum_{i=0}^{n} t_i \cdot v_i}_{\in \mathbb{R}^{(V)}} \;\middle|\; \begin{array}{l} (1)\ \text{for each } i \text{ we have } t_i > 0 \\ (2)\ t_0 + \cdots + t_n = 1 \end{array} \right\}$$

as the corresponding open n-simplex of $|K|$.



simplicial complex



the corresponding open simplices

The following lemma summarizes a few properties of open simplices.

Lemma 93.11. (Open Simplex Lemma) Let $K = (V, S)$ be an abstract simplicial complex.

(1) Let $s \in S$ be a k-simplex.
   (a) The closure of the open simplex $\langle s \rangle$ is given by $|s|$.
   (b) Every characteristic map $\Phi_s \colon \Delta^k \to |s|$ restricts to a homeomorphism $\mathring{\Delta}^k \to \langle s \rangle$.

(2) The topological realization $|K|$ is the disjoint union of the open simplices, i.e.

$$|K| \;=\; \bigsqcup_{s \in S} \langle s \rangle.$$

   Put differently, given any $\alpha \in |K|$ there exists a unique simplex s with $\alpha \in \langle s \rangle$.

(3) For two distinct simplices $s \neq t$ we have $\langle s \rangle \cap \langle t \rangle = \varnothing$.

(4) The union of the open simplices is precisely $|K|$.

(5) If $J = (U, R)$ is a subcomplex of K, then

$$|J| \;=\; \bigsqcup_{r \in R} \langle r \rangle \;=\; \bigcup_{r \in R} |r| \quad \text{and for any } s \in S \setminus R \text{ we have } |J| \cap \langle s \rangle = \varnothing.$$

   In particular, if s is a simplex of K with $\langle s \rangle \cap |J| \neq \varnothing$, then s is a simplex of J which implies that $|s| \subset |J|$.

(6) Let s and t be simplices. If $|s| \subset |t|$, then $s \subset t$. In particular, either $\dim(s) < \dim(t)$ or $s = t$.

(7) Let $s \in S$ be a simplex. If $U \subset |K|$ is an open subset with $U \cap |s| \neq \varnothing$, then $U \cap \langle s \rangle \neq \varnothing$.

Proof.

[1132] Here we use the convention of page 1984 which allows us to view V as a subset of $\mathbb{R}^{(V)}$.

(1) Let $s \in S$ be a k-simplex and let $\Phi_s\colon \Delta^k \to |s|$ be a characteristic map. By the Simplex-Embedding Lemma 93.5 (3) we know that the map $\Phi_s\colon \Delta^k \to |s|$ is a homeomorphism and that $|s|$ is a closed subset of $|K|$. By definition we have $\langle s\rangle \subset |s|$.
   (b) It is clear that Φ_s restricts to a bijection $\mathring{\Delta}^k \to \langle s\rangle$. Thus it follows immediately that Φ_s restricts to a homeomorphism $\mathring{\Delta}^k \to \langle s\rangle$.
   (a) Since $|s|$ is a closed subset of $|K|$ we see that the closure of $\langle s\rangle$ in $|K|$ is contained in $|s|$. It is elementary to see that Δ^k is the closure of $\mathring{\Delta}^k$. Since $\Phi_s\colon \Delta^k \to |s|$ is a homeomorphism it follows almost immediately that $|s| = \Phi_s(\Delta^k)$ is indeed the closure of $\langle s\rangle = \Phi_s(\mathring{\Delta}^k)$.
(2) Let $\alpha \in |K|$. We set $s := \{v \in V \,|\, \alpha(v) \neq 0\}$. By definition of $|K|$ we know that s is a simplex. Evidently $\alpha \in \langle s\rangle$ and evidently s is unique.
(3) This statement follows immediately from (1).
(4) Conveniently enough this statement also follows immediately from (1).
(5) Let J be a subcomplex of K. If $r \in R$, then evidently we have $\langle r\rangle \subset |r| \subset |J|$. Keeping (1) in mind we see that it remains to show that if we have $s \in S$ with $|J| \cap \langle s\rangle \neq \varnothing$, then $s \in R$. Thus let s be a simplex of K with $|J| \cap \langle s\rangle \neq \varnothing$. It follows from (1), applied to the abstract simplicial complex $J = (U, R)$, that there exists a simplex r of J with $\langle s\rangle \cap \langle r\rangle \neq \varnothing$. But since r is also a simplex of K we obtain from (1), applied to K, that $s = r \in R$.
(6) Let s and t be simplices with $|s| \subset |t|$. We apply (4) to the subcomplex that is defined by s. We immediately obtain that $s \subset t$. If $s \subsetneq t$, then by definition we have $\dim(s) = \#S - 1 < \#T - 1 = \dim(t)$.
(7) Let $s \in S$ be a k-simplex and let $U \subset |K|$ be an open subset with $U \cap |s| \neq \varnothing$. Let $\Phi_s\colon \Delta^k \to |s|$ be a characteristic map. By definition of the topology on $|K|$ and by our hypothesis we know that $\Phi_s^{-1}(U)$ is an open non-empty subset of Δ^k. It is elementary to see that $\Phi_s^{-1}(U) \cap \mathring{\Delta}^k \neq \varnothing$. It now follows from (1b) that $U \cap \langle s\rangle \neq \varnothing$. ■

The following proposition reminds us of the CW-Complex-Finiteness Theorem 68.14 which gives an analogous statement for CW-complexes. The proposition can also be viewed as a converse to Lemma 93.9 (1).

Proposition 93.12. (Simplicial Complex-Finiteness Proposition) Let L be an abstract simplicial complex.

(1) If $X \subset |L|$ is a compact subset, then there exists a finite subcomplex K of L such that $X \subset |K|$.
(2) If $|L|$ is compact, then L is a finite abstract simplicial complex.

Remark. Let K be an abstract simplicial complex. Lemma 93.9 (1) together with the Simplicial Complex-Finiteness Proposition 93.12 shows that $|K|$ is compact if and only if K is finite. Later on in Proposition 94.6 we will prove the related result that $|K|$ is regionally compact if and only if K is "locally finite".

Proof. Let $L = (W, T)$ be an abstract simplicial complex. Clearly (2) is just a special case of (1). Thus it suffices to prove (1). Now let $X \subset |L|$ be a compact subset. For each simplex t of L with $\langle t\rangle \cap X \neq \varnothing$ we pick a point $y_t \in \langle t\rangle \cap X$. We denote by Y the set given by all the y_t. We equip Y with the subspace topology coming from $|L|$.

Claim.

(1) Every subset of Y, in particular Y itself, is a closed subset of $|L|$.
(2) Y is a discrete subset of $|L|$.

Proof.

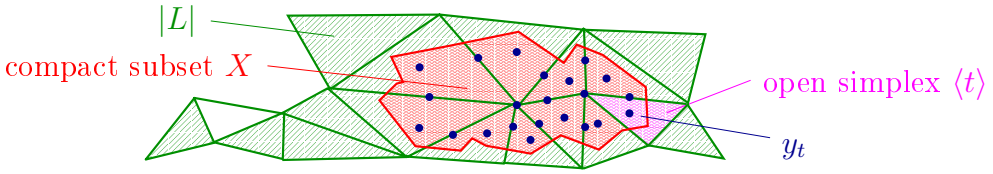
(1) Let $A \subset Y$ be a subset. We need to show that A is a closed subset of $|L|$. By the Simplex-Embedding Lemma 93.5 (1) and (3) it suffices to show that for every simplex t of $|L|$ the intersection $A \cap |t|$ is closed. By the Open Simplex Lemma 93.11 (5) we know that $|t|$ intersects only the interiors of the simplices s with $s \subset t$. Since there are only finitely many simplices[1133] contained in t we see that $A \cap |t|$ is finite, in particular it is compact. By the Simplex-Embedding Lemma 93.5 (2) we know that $|t|$ is Hausdorff. Hence we obtain from the Compact-Closed Lemma 1.21 (2) that $A \cap |t|$ is indeed closed.
(2) We need to show that the subspace topology of Y is the discrete topology. It follows from (1) and the Neighborhood Openness Criterion 1.5 that every subset of Y is in fact a closed subset of Y. But this implies that every subset of Y is in fact an open subset of Y. In other words, the subspace topology is indeed the discrete topology. ⊞

It follows from (1) and the definition of the subspace topology that Y is a closed subset of the compact set X. Furthermore by (2) we know that Y is a discrete subset of $|L|$. Since $Y \subset X \subset |L|$ we see that Y is also a discrete subset of X. Since X is compact we obtain from the Compact-Discrete Lemma 1.22 that Y is finite.

We define $S := \{t \in T \,|\, X \cap \langle t \rangle \neq \varnothing\}$. Note that $\#S = \#Y$. Thus, by the above discussion we know that S is finite. Now we set K to be the subcomplex that is given the union of all the simplices $t \in S$.[1134] Since t is finite we see that K is a finite subcomplex. Now we have

$$X \overset{\text{Open Simplex Lemma 93.11 (4)}}{\downarrow =} X \cap \bigcup_{t \in S} \langle t \rangle = \bigcup_{t \in S} X \cap \langle t \rangle \overset{\text{by definition of } t}{\downarrow \subset} \bigcup_{t \in S} \langle t \rangle \subset \bigcup_{t \in S} |s| \overset{\text{Lemma 93.8 (3)}}{\downarrow =} |K|.$$

■



We conclude this section with the following lemma which gives a criterion for a subset of a simplicial complex to be a subcomplex.

Lemma 93.13. Let $L = (W, T)$ be an abstract simplicial complex and let $X \subset |L|$ be a closed subset. If for every simplex $t \in T$ we have either $\langle t \rangle \cap X = \varnothing$ or $\langle t \rangle \subset X$, then there exists a subcomplex K of L with $|K| = X$.

[1133] This finiteness statement makes the proof of this proposition significantly simpler than the technically slightly tricky proof of the CW-Complex-Finiteness Theorem 68.14.
[1134] Strictly speaking we view the simplices as subcomplexes, and then take the union as defined in Lemma 93.2.

Proof. We consider

$$V \;:=\; W \cap X \quad \text{and} \quad S \;:=\; \{t \in T \,|\, \langle t\rangle \subset X\}.$$

Claim. We claim that (V, S) is a subcomplex of $L = (W, T)$.

Proof. Let $s \in S$ be a simplex and let r be a face of s. We need to show that $r \in S$, i.e. we need to show that $\langle r\rangle \subset X$. Basically by definition we have $\langle r\rangle \subset |s|$. Thus it suffices to show that $|s| \subset X$.

Now note note that by the Open Simplex Lemma 93.11 we know that $|s|$ is the closure of the subset $\langle s\rangle$. Since X is by hypothesis a closed subset of $|L|$ we obtain from elementary point-set topology, see e.g. Exercise 2.24 (a), that $|s| \subset X$. ⊞

Now we see that

$$\begin{array}{ccccccccc} |K| & = & \bigcup\limits_{t\in S} \langle t\rangle & = & \bigcup\limits_{\substack{t\in T \text{ with} \\ \langle t\rangle \subset X}} \langle t\rangle & = & \bigcup\limits_{t\in T} (\langle t\rangle \cap X) & = & X. \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ & & \text{Open Simplex} & & \text{definition} & & \text{by the hypothesis that} & & \text{Open Simplex} \\ & & & & \text{of } S & & \text{either } \langle t\rangle \cap X = \varnothing \text{ or } \langle t\rangle \subset X & & \\ & & \text{Lemma 93.11 (5)} & & & & & & \text{Lemma 93.11 (2)} \end{array}$$

■

93.4. **Simplicial complexes.** As on so many other occasions we start out this new section with a definition.

Definition.

(1) Let X be a topological space. A simplicial structure for X is a pair $(K, \Theta\colon |K| \to X)$ where the following holds:
   (a) K is an abstract simplicial complex,
   (b) Θ is a homeomorphism between the topological realization $|K|$ of K and the topological space X.

(2) A simplicial complex is a pair $(X, (K = (V, S), \Theta\colon |K| \to X))$ given by a topological space X together with a simplicial structure $(K = (V, S), \Theta\colon |K| \to X)$. We refer to X as the underlying topological space of the simplicial complex. To keep the notation at a manageable level we will often not distinguish in the notation between a simplicial complex and its underlying topological space.

(3) We say a simplicial complex is finite (respectively countable) if the corresponding abstract simplicial complex is *finite* (respectively *countable*).

(4) We say a simplicial complex is (totally) ordered if the corresponding abstract simplicial complex is (totally) ordered.

(5) Let $(X, (K = (V, S), \Theta\colon |K| \to X))$ be a simplicial complex. We introduce the following notions which are basically self-explanatory:
   (a) Given $v \in V$ we refer to $\Theta(v)$ as a vertex of the simplicial complex.
   (b) Let $s \in S$ be a k-simplex.
      (i) We refer to $\Theta(|s|) \subset X$ as a k-simplex of the simplicial complex. Furthermore we refer to $\Theta(\langle s\rangle) \subset X$ as an open k-simplex of the simplicial complex.
      (ii) If $\Phi_s\colon \Delta^k \to |s|$ is a characteristic map as defined on page 1984, then we refer to $\Theta \circ \Phi_s\colon \Delta^k \to X$ as a characteristic map of X.
      (iii) If t is a face of s, then we refer to $\Theta(|t|)$ as a face of the simplex $\Theta(|s|)$.

(c) We define the dimension of the simplicial complex to be the dimension of the corresponding abstract simplicial complex.
(d) Given $n \in \mathbb{N}_0$ we denote by $|K^n| \subset |K|$ the topological realization of the n-skeleton of K. We refer to $X^n := \Theta(|K^n|)$ as the n-skeleton of the simplicial complex X.

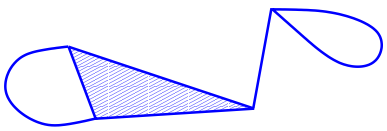
(6) Let $(Y, (L, \Theta\colon |L| \to Y))$ be a simplicial complex. A subcomplex is a simplicial complex $(X, (K, \Omega\colon |K| \to X))$ where X is a subspace of Y, K is a subcomplex of L and where the inclusion $X \to Y$ is precisely the map $\Theta \circ |i| \circ \Omega^{-1}$, where $i\colon K \to L$ is the inclusion map.

Remark.

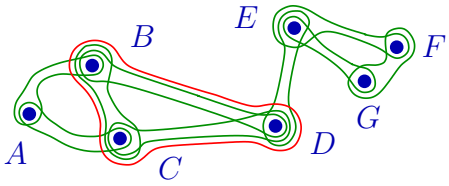
(1) In many cases the choice of a simplicial structure gets suppressed from the notation.
(2) In the literature a *simplicial structure* is often called a *triangulation*.

Examples.

(1) Given any abstract simplicial complex K its topological realization $|K|$ is a simplicial complex in an obvious way.
(2) On page 1989 we wrote down an explicit homeomorphism $|D_n| \xrightarrow{\cong} \Delta^n$ which restricts to a homeomorphism $|S_{n-1}| \xrightarrow{\cong} \partial\Delta^n$. These two homeomorphisms endow Δ^n and $\partial\Delta^n$ with a simplicial structure, called the *standard simplicial structure*.
(3) We consider the topological space X to the left of the figure below. To the right we show an abstract simplicial complex K with seven vertices such that X is homeomorphic to $|K|$.[1135] We have thus found a simplicial structure for X.



topological space X



abstract simplicial complex K

Before we continue with more examples of simplicial complexes we want to formulate the following lemma which summarizes a few basic properties of simplices in a simplicial complex.

Lemma 93.14. Let Y be a simplicial complex.

(1) Any k-simplex of Y is homeomorphic to Δ^k.
(2) The intersection of two simplices of Y is either empty or it is given by a single face of the two simplices.
(3) Let X be a subcomplex of Y and let s be a simplex of Y. If the corresponding open simplex intersects X, then $s \subset X$.
(4) If $X \subset Y$ is the union of simplices of X, then Y admits a natural structure of a subcomplex.

Proof.

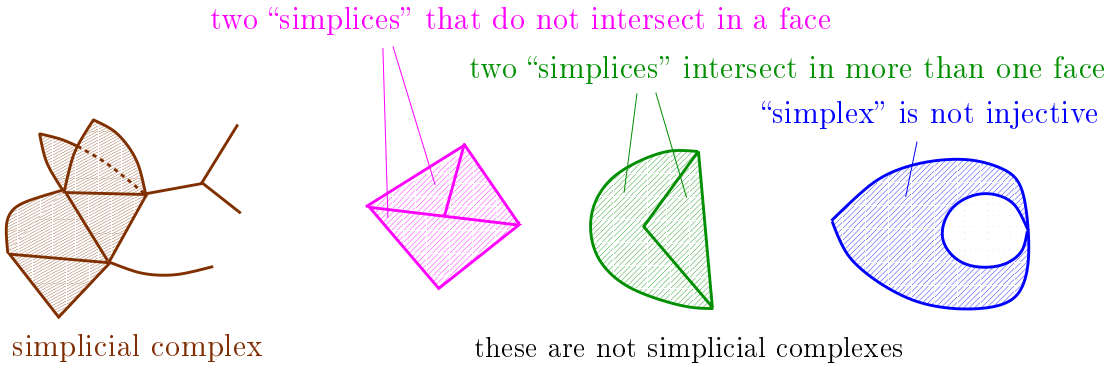
(1) This statement is a reformulation of the Simplex-Embedding Lemma 93.5.

[1135] What is the role of the vertices A, F and G? Are they really needed?

(2) This statement is an immediate consequence of Lemma 93.1.
(3) This follows from the Open Simplex Lemma 93.11 (5).
(4) This statement follows almost immediately from the definitions. ■

Remark.

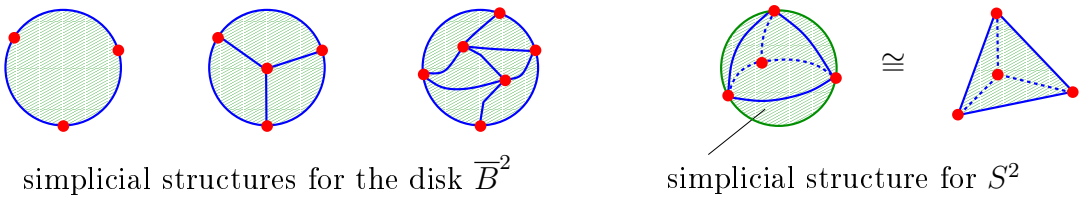
(1) In the following a simplicial structure for a topological space is often indicated by a suitable decomposition into simplices, i.e. into subsets which (look like they) are homeomorphic to some Δ^k. Some care though needs to be taken to make sure that one actually does define a simplicial structure. In particular one needs to ensure that the statements of Lemma 93.14 (1) and (2) are satisfied. We refer to the figure below for an illustration of examples and non-examples of simplicial complexes.
(2) In the following we often draw a simplicial complex and rely on the good will of the reader to read off the corresponding abstract simplicial complex.



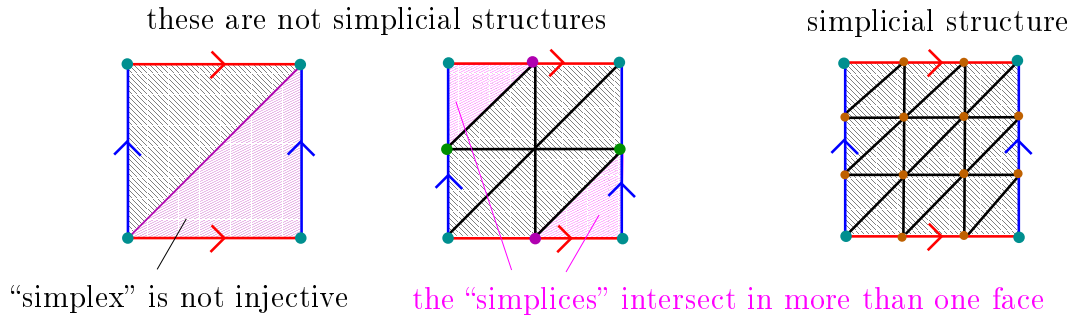
We continue with more examples of simplicial complexes.

Examples.

(1) In the figure below to the left we show several simplicial structures for the disk $\overline{B}^2$. Furthermore, in the figure below to the right we show a simplicial structure for the sphere S^2.



(2) In the figure below we show the torus $T = ([0,1] \times [0,1])/\sim$ together with three attempts at endowing it with a simplicial structure. Using Lemma 93.14 we see that the two attempts to the left do not define a simplicial structure. Only the fairly complicated attempt to the right actually provides us with a simplicial structure for the torus.
(3) Let $n \in \mathbb{N}$. We will now equip the closed ball $\overline{B}^n$ and the sphere S^n with a canonical simplicial structure. To do so we consider the simplicial complex D_n and its subcomplex S_{n-1} that we introduced on page 1978. On page 1989 we saw that there exists a canonical homeomorphism $f\colon |D_n| \to \Delta^n$ which restricts to a homeomorphism $|S_{n-1}| \to \partial\Delta^n$. Furthermore, in the Standard Simplex Homeomorphism Lemma 72.1



we gave an explicit homeomorphism $\Phi\colon \overline{B}^n \to \Delta^n$ that restricts to a homeomorphism $S^{n-1} \to \partial\Delta^n$. We see that $(D_n, \Phi^{-1} \circ f\colon |D_n| \to \overline{B}^n)$ is a simplicial structure for $\overline{B}^n$ and that $(S_{n-1}, \Phi^{-1} \circ f|_{|S_{n-1}|}\colon |S_{n-1}| \to S^{n-1})$ is a simplicial structure for S^{n-1}. We refer to these as the *canonical simplicial structures* of $\overline{B}^n$ and S^{n-1}.



It is by now abundantly clear that most simplicial complexes admit many different simplicial structures. Finally we consider maps between simplicial complexes. What comes next is basically self-explanatory and only written down out of a sense of duty.

Definition. Let $(X, (K, \Theta\colon |K| \to X))$ and $(Y, (L, \Omega\colon |L| \to Y))$ be two simplicial complexes.

(1) A map $f\colon X \to Y$ is called simplicial if there exists a simplicial map $g\colon K \to L$ such that $f = \Omega \circ |g| \circ \Theta^{-1}$.
(2) We say the two simplicial complexes are simplicially isomorphic if there exist simplicial maps between the two simplicial complexes that are inverses of one another.

Example. If Y is a simplicial complex and X is a subcomplex, then it follows immediately from the definitions that the inclusion $X \to Y$ is a simplicial map.

It follows immediately from the definitions that the composition of two simplicial maps is again simplicial. This leads us to the following definition.

Definition. We call the category $\mathcal{SimpCplx}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{SimpCplx}) &:= \text{all simplicial complexes,} \\ \mathrm{Mor}(K, L) &:= \text{all simplicial maps from } K \text{ to } L, \end{aligned}$$

together with the usual composition of maps the category of simplicial complexes. Similarly we define the category of (totally) ordered simplicial complexes.

We conclude this section with the following basic lemma.

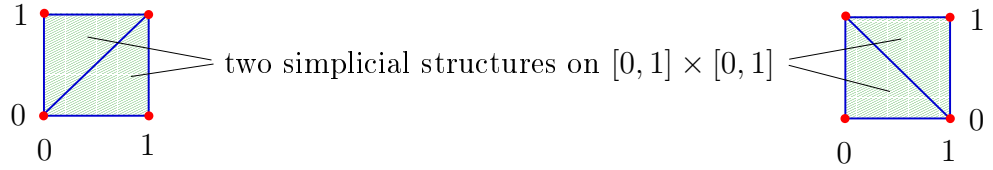
Lemma 93.15. Let $(X, (K = (V, S), \Theta\colon |K| \to X))$ and $(Y, (L = (W, T), \Omega\colon |L| \to Y))$ be two simplicial complexes. Given a simplicial map $\varphi\colon K \to L$ there exists a unique

simplicial map $\Phi\colon X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle\Theta}\downarrow & & \downarrow{\scriptstyle\Omega} \\ X & \xrightarrow{\Phi} & Y. \end{array}$$

Proof. The map Φ is given by $\Omega \circ |\varphi| \circ \Theta^{-1}$. Since Ω and Θ are homeomorphisms we see that Φ is also unique. ■

93.5. **The product of (abstract) simplicial complexes.** In this section we want to consider the product of (abstract) simplicial complexes. Let us first consider the simplicial complexes $X = Y = [0,1]$ with the "obvious" simplicial structure. As we see in the figure below, there are two very reasonable looking simplicial structures on the product $[0,1] \times [0,1]$.



It is not clear which of the two simplicial structures shown in the figure above is preferable or "more canonical". In fact to break the tie and to end up with a clear winner we need to work with ordered abstract simplicial complexes, as introduced on page 1978.
First we recall the following definition from page 92.

Definition. Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. Given (a, b) and (a', b') in $A \times B$ we define

$$(a,b) < (a',b') \quad :\Longleftrightarrow \quad \text{either } a < a' \text{ or } a = a' \text{ and } b < b'$$

and we define $(a,b) \leq (a',b')$ if either $(a,b) = (a',b')$ or $(a,b) < (a',b')$. We refer to "$\leq$" on $A \times B$ as the lexicographic order on $A \times B$.

Remark. Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. One can show easily that the lexicographic order is a partial order on $A \times B$ and similarly one can show easily that if $(A, \leq)$ and $(B, \leq)$ totally ordered sets, then the lexicographic order is also a total order.

Definition. Let $K = (V, S)$ and $L = (W, T)$ be two (totally) ordered abstract simplicial complexes. We define the product $K \times L$ of the abstract simplicial complexes to be the abstract simplicial complex[1136] that is given by the vertex set $V \times W$ and the simplex set[1137]

$$\left\{ \{(v_0,w_0),\dots,(v_k,w_k)\} \;\middle|\; \begin{array}{c} \{v_0,\dots,v_k\} \in S \text{ and } \{w_0,\dots,w_k\} \in T \text{ and} \\ \text{we have } v_0 \leq \cdots \leq v_k \text{ and } w_0 \leq \cdots \leq w_k \end{array} \right\}.$$

We equip $K \times L$ with the lexicographic order and we view $K \times L$ again as a (totally) ordered abstract simplicial complex.[1138]

Before we discuss examples we first state the following lemma which basically reassures us that we are doing something reasonable.

Lemma 93.16. (Simplicial Complex-Product Lemma) Suppose we are given by two ordered abstract simplicial complexes $K = (V, S)$ and $L = (W, T)$.

(1) The two natural projections $p\colon K \times L \to K$ and $q\colon K \times L \to L$ are simplicial maps.
(2) The ordered abstract simplicial complex $K \times L$ together with the natural projections from (1) is the product of K and L in the category $\mathcal{OrdAbsSimpCplx}$ of ordered abstract simplicial complexes, in the sense of the definition on page 211.
(3) The map
$$|p| \times |q|\colon |K \times L| \ \to \ |K| \times |L|$$
is a continuous bijection. If K and L are *finite*, then the map is in fact a homeomorphism.

Thus given two finite simplicial complexes K and L we can view the product also as a simplicial complex.

Remark.

(1) Lemma 68.24 teaches us that it is safer to stick to finite abstract simplicial complexes. In fact that is also all we ever need. For products of arbitrary abstract simplicial complexes we refer to [**FrPi1990a**, Proposition 4.3.5] and [**LW1969**, Corollary III.5.2]. Note though that one needs to tread carefully to make sure that the definitions and statements are indeed as desired.
(2) As mentioned earlier, the topology of the topological realization of an abstract simplicial complex is designed to make it easy to show that a map out of the topological realization is continuous. Furthermore, as we mentioned in Remark 5.2, the topology of a product of topological spaces makes it easy to find continuous maps to the product. This explains why in the Simplicial Complex-Product Lemma 93.16 we have a continuous map from left to right, but why we struggle to find a continuous map from right to left.

Proof.

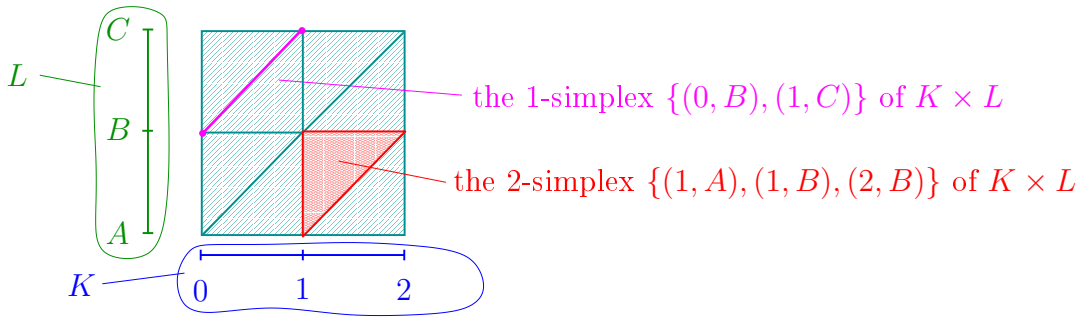
(1) It follows immediately from the definitions that the projections are indeed simplicial.
(2) We will not make use of this statement, thus we will outsource it to Exercise 93.10.
(3) It follows from (1) together with Proposition 93.7 and the the Topological-Product Proposition 5.1 (1a) that the map $|p| \times |q|\colon |K \times L| \to |K| \times |L|$ is continuous. We leave it to the reader to verify that the map is a bijection.

Finally suppose that K and L are finite. It follows from the Compact-Hausdorff Proposition 2.17 (3), together with Lemmas 93.5 (1) and 93.9 (1) and the Product Topology Properties Proposition 5.6 that the map is indeed a homeomorphism. ■

Examples.

[1136] One can indeed easily verify that this defines an abstract simplicial complex.
[1137] To rule out any potential confusion, the $v_0, \ldots, v_k$ and also the $w_0, \ldots, w_k$ do not need to be distinct vertices.
[1138] It follows almost immediately from the above remark that with this definition $K \times L$ is in fact a (totally) ordered abstract simplicial complex.

(1) We consider the two totally ordered abstract simplicial complexes K and L that are shown in the figure below. Here we use the obvious total orders given by $0 < 1 < 2$ and $A < B < C$. We also show the corresponding product $K \times L$.



(2) We equip S^1 with the canonical simplicial structure defined on page 1997. The product simplicial structure on the torus $S^1 \times S^1$ is simplicially isomorphic to the simplicial structure shown in the figure shown on page 1997 to the right.

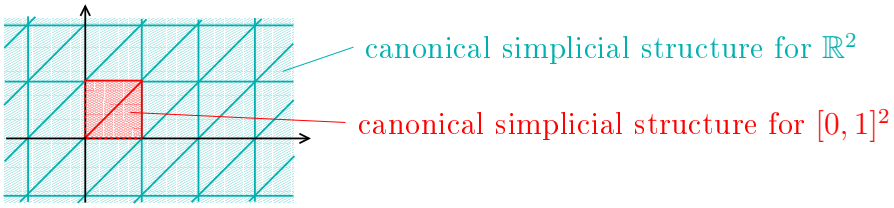
(3) Let I be the abstract simplicial complex given by $(V = \{(0, 1)\}, S = \{\{0\}, \{1\}, \{0, 1\}\})$. As in (1) we use the obvious total order given by $0 < 1$. We consider the product $I \times I$. Given any $n \in \mathbb{N}$ we can now view the cube $[0, 1]^n$ as a simplicial complex. For $n = 2$ we obtain the simplicial complex drawn in the figure shown on page 1998 on the left. We refer to this simplicial structure as the *canonical simplicial structure of* $[0, 1]^n$.

In plain English, the following lemma says that the $\mathbb{Z}^n$-translates of the canonical simplicial structure of $[0, 1]^n$ give a linear simplicial structure for $\mathbb{R}^n$.

Lemma 93.17. Let $n \in \mathbb{N}$. We denote by $(K = (V, S), \Theta\colon |K| \to [0, 1]^n)$ the canonical simplicial structure of $[0, 1]^n$. We define a new simplicial complex $\widetilde{K} = (\widetilde{V}, \widetilde{S})$ where

$$\widetilde{V} := \mathbb{Z}^n \quad \text{and} \quad \widetilde{S} := \{\{v_0 + w, \ldots, v_k + w\} \,|\, \{v_0, \ldots, v_k\} \in S \text{ and } w \in \mathbb{Z}^n\}.$$

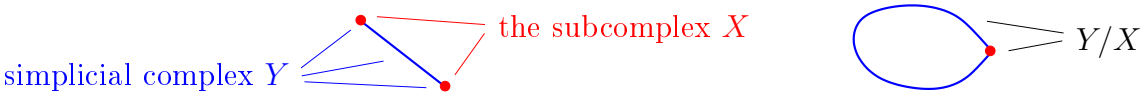
The (hopefully) obvious map $\widetilde{\Theta}\colon |\widetilde{K}| \to \mathbb{R}^n$ is a homeomorphism, in particular it equips $\mathbb{R}^n$ with the structure of a linear simplicial complex.



Proof. It is elementary to see that the map $\widetilde{\Theta}\colon |\widetilde{K}| \to \mathbb{R}^n$, once it is written down properly, is continuous and a bijection. It follows from the Closed Embedding-to-$\mathbb{R}^n$ Proposition 2.19 that the map is in fact a homeomorphism. ■

Arguably the most important ways to construct new topological spaces out of given ones are by taking products and by taking quotients. Within reasonable limits we saw in the CW-Complex Product Proposition 68.23 and the CW-Complex Construction Lemma 68.32 (3) that both operations can be done with CW-complexes. Furthermore, we just saw in the Simplicial Complex-Product Lemma 93.16 that we can take products of finite simplicial

complexes. Quotients on the other hand are a problem. In general, as is illustrated in the figure below, given a simplicial complex Y and a subcomplex X the simplicial structure of Y does not descend to a simplicial structure on the quotient Y/X.



Let us jot down the following question.

Question 93.18. Let Y be a simplicial complex and let X be a subcomplex. Does the quotient Y/X admit a simplicial structure?

In Corollary 93.26 we will give a partial answer to Question 93.18.

93.6. **The simplicial join of (abstract) simplicial complexes.** In this section we will continue with our task of finding useful constructions of (abstract) simplicial complexes that correspond to constructions that we already know from the world of topological spaces. More precisely, we want to consider the join construction from page 446 that so far we had not seriously used.

Definition. Let X and Y be two non-empty topological spaces. We define the join $X * Y$ to be the topological space obtained from $X \times [0,1] \times Y$, equipped with the product topology, by performing the following two types of identifications:

(1) for every $x \in X$ we identify all points in $\{x\} \times \{0\} \times Y$ to a single point and
(2) for every $y \in Y$ we identify all points in $X \times \{1\} \times \{y\}$ to a single point.

Furthermore, if one of X or Y is the empty topological space, then we define $X * Y$ to be the other topological space.

To get into the mood of thinking about joins let us first consider the following two examples.

Examples.

(1) Let X be a non-empty topological space and let $Y = \{1\}$ be the topological space given by a single point 1. We consider the maps

$$\begin{array}{rcl} X * \{1\} = (X \times [0,1] \times \{1\})/\sim & \to & \mathrm{Cone}(X) = (X \times [0,1])/(X \times \{1\}) \\ [(x,t,1)] & \mapsto & [(x,t)] \end{array}$$

One can easily verify that the map is a bijection. Furthermore, using the Topological-Quotient Proposition 5.15 (1b) and the Topological-Quotient Proposition 5.15 (1a) it is straightforward to see that this map and its inverse are continuous. Thus the map is a homeomorphism.

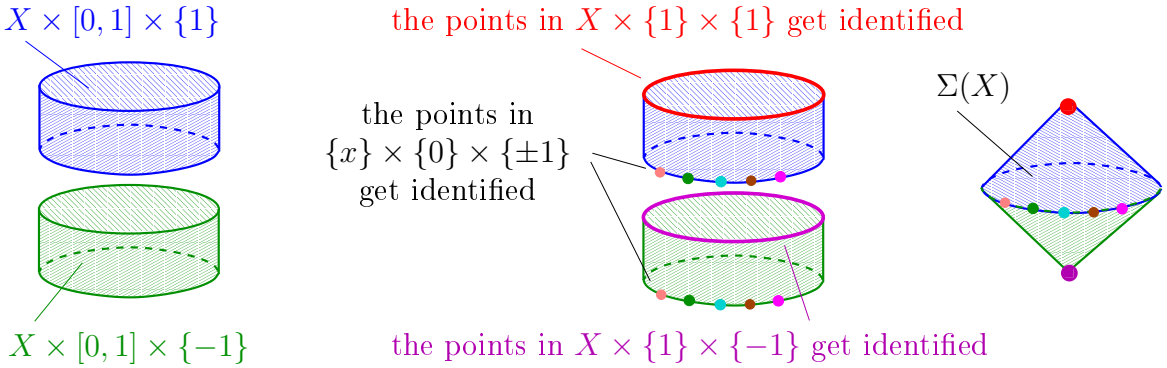
(2) Let X be a non-empty topological space and let $Y = \{-1, 1\}$ be the topological space given by the set with two elements ± 1 and the discrete topology. We also consider the suspension

$$\Sigma(X) \ := \ (X \times [-1,1])/\sim$$

which is defined by squashing all points in $X \times \{-1\}$ to a single point and by squashing all points in $X \times \{1\}$ to a single point. We consider the map

$$\begin{array}{rcl} X * \{-1,1\} = (X \times [0,1] \times \{-1,1\})/\sim & \to & \Sigma(X) = (X \times [-1,1])/\sim \\ [(x,t,\epsilon)] & \mapsto & [(x, \epsilon \cdot t)] \end{array}$$

With some minor effort one can show that this map is a bijection. Using the Topological-Quotient Proposition 5.15 (1b) and the Topological-Quotient Proposition 5.15 (1a) it is straightforward to see that this map is continuous. The fact that the inverse is also continuous needs slightly more effort, in fact this argument also requires Lemmas 5.23 and Lemma 5.36. We leave it to the reader to fill in the details. We refer to the figure below for $X = S^1$ for an illustration of the homeomorphism.



For completeness we note that for the empty topological space we have, basically by definition, that $\varnothing * \{1\} = \mathrm{Cone}(\varnothing)$ and $\varnothing * \{\pm 1\} = \Sigma(\varnothing)$

The following lemma gives us a few examples of joins which will play a role later on.

Lemma 93.19. (Join-Simplex Lemma)

(1) Given any $m, n \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} S^m * S^n & \to & S^{m+n+1} \\ [(x,t,y)] & \mapsto & \underbrace{\left(x \cdot \cos\left(\frac{\pi t}{2}\right), y \cdot \sin\left(\frac{\pi t}{2}\right)\right)}_{\in \mathbb{R}^{m+1} \times \mathbb{R}^{n+1} = \mathbb{R}^{m+n+2}} \end{array}$$

is a homeomorphism.[1139]

(2) Given any $m, n \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \Delta^m * \Delta^n & \to & \Delta^{m+n+1} \\ [(x_0, \ldots, x_m, t, y_0, \ldots, y_n)] & \mapsto & (t \cdot x_0, \ldots, t \cdot x_m, (1-t) \cdot y_0, \ldots, (1-t) \cdot y_n) \end{array}$$

is a homeomorphism.

(3) Given any $n \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \partial\Delta^n * \{\star\} & \to & \Delta^n \\ [(x,t,\star)] & \mapsto & x \cdot (1-t) + \left(\frac{1}{n+1}, \ldots, \frac{1}{n+1}\right) \cdot t \end{array}$$

Proof. The proof of (1), (2) and (3) is basically identical. In each case it is straightforward to verify that the map is well-defined and a bijection. It follows from the Topological-Quotient Proposition 5.15 (1b) that the given map is continuous. Finally it follows from Lemma 16.2 together with the Compact-Hausdorff Proposition 2.17 (3) that the given map is in fact a homeomorphism. ■

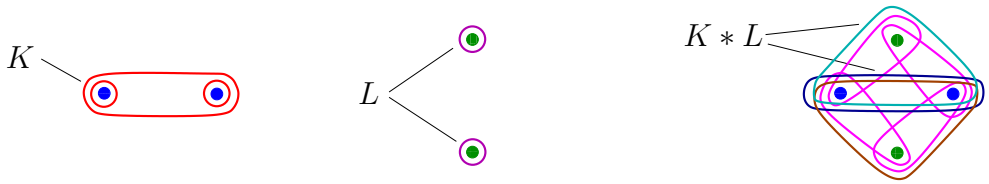
It turns out that in the simplicial setting we have an analogous construction.

[1139] In fact this statement is just the content of the Join of Spheres Lemma 16.4.

Definition. Let $K_i = (V_i, S_i)$, $i = 1, \ldots, m$ be abstract simplicial complexes. We define the simplicial join of $K_1, \ldots, K_m$ to be the abstract simplicial complex[1140][1141]

$$K_1 * \cdots * K_m := \left(V_1 \sqcup \cdots \sqcup V_m, \left\{ s_1 \sqcup \cdots \sqcup s_m \;\middle|\; \begin{array}{l} \text{for } i = 1, \ldots, m \text{ we have } s_i \in \overbrace{S_i \cup \{\varnothing\}}^{\subset \mathcal{P}(V_i)} \\ \text{and there exists at least one } s_i \neq \varnothing \end{array} \right\} \right).$$

This definition is illustrated, with limited success, in the figure below.



Remark. It follows immediately from the definitions that the join construction is associative, i.e. given abstract simplicial complexes K, L and M we have a natural isomorphism $(K * L) * M = K * (L * M)$ of abstract simplicial complexes.

The examples on page 2001 motivate the following definitions.

Definition. Let K be an abstract simplicial complex. Furthermore let $\{1\}$ be the abstract simplicial complex consisting of a single vertex 1 and finally let S_0 be the abstract simplicial complex $S_0 = (\{0, 1\}, \{\{0\}, \{1\}\})$ that we introduced on page 1978. We refer to

$$\mathrm{Cone}(K) := K * \{1\} \quad \text{respectively} \quad \Sigma(K) := K * S_0$$

as the simplicial cone on the abstract simplicial complex K respectively the simplicial suspension of the abstract simplicial complex K.

Example. Let $n \in \mathbb{N}_0$. We see that

$$\overbrace{\mathrm{Cone}\left(\{0, \ldots, n\}, \mathcal{P}(\{0, \ldots, n\}) \backslash \{\varnothing\}\right)}^{= \mathrm{Cone}(D_n), \text{ see definition of } D_n \text{ on page 1978}} \underset{\text{follows from the definition of the simplicial cone}}{=} \left(\{0, \ldots, n, *\}, \mathcal{P}(\{0, \ldots, n, *\} \backslash \{\varnothing\}\right) \underset{\text{induced by } * \mapsto n+1}{\overset{\cong}{\to}} D_{n+1}.$$

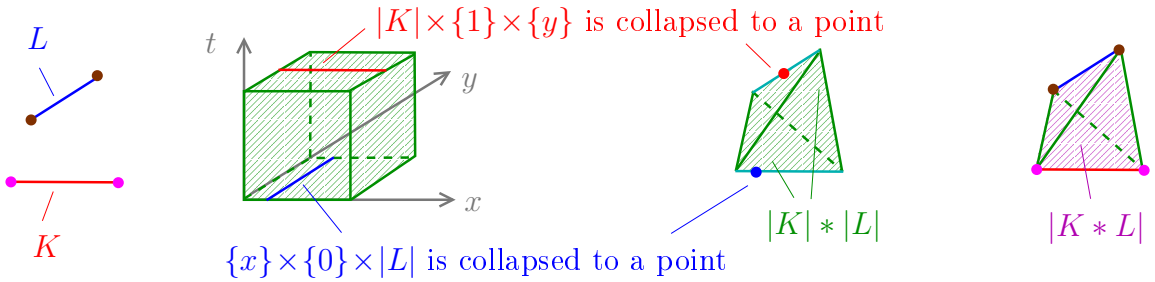
Lemma 93.20. (Join Lemma) Let K and L be two abstract simplicial complexes. If one of K or L is empty, then by definition we have a natural homeomorphism $|K| * |L| \cong |K * L|$. If K and L are non-empty, then the map

$$\begin{array}{rccl} \Theta\colon & \overbrace{(|K| \times [0,1] \times |L|)/\!\sim}^{= |K| * |L|} & \to & |K * L| \subset \mathbb{R}^{(V \sqcup W)} = \mathbb{R}^{(V)} \times \mathbb{R}^{(W)} \\ & [(x, r, y)] & \mapsto & (\underbrace{r \cdot x}_{\in \mathbb{R}^{(V)}}, \underbrace{(1-r) \cdot y}_{\in \mathbb{R}^{(W)}}) \end{array}$$

is a bijection and its inverse Θ^{-1} is continuous.[1142] Furthermore, if K and L are finite, then Θ is in fact a homeomorphism.

[1140] To avoid confusion it is perhaps worth recalling that the empty set is not a simplex.

[1141] One can easily verify that this is indeed an abstract simplicial complex.



Proof. Let $K=(V,S)$ and $L=(W,T)$ be two abstract simplicial complexes. If one of K or L is empty, then the statement is true basically by definition. Thus let us now assume that K and L are both non-empty. We consider the map

$$\begin{array}{rcl}\Theta\colon \overbrace{(|K|\times[0,1]\times|L|)/\!\sim}^{=|K|*|L|} & \to & |K*L|\subset\mathbb{R}^{(V\sqcup W)}=\mathbb{R}^{(V)}\times\mathbb{R}^{(W)}\\ [(x,r,y)] & \mapsto & (\underbrace{r\cdot x}_{\in\mathbb{R}^{(V)}},\underbrace{(1-r)\cdot y}_{\in\mathbb{R}^{(W)}}).\end{array}$$

We leave it to the reader to verify that the map does indeed take values in $|K*L|$ and that it is a bijection. Next we show that the inverse $\Theta^{-1}\colon |K*L|\to|K|*|L|$ is continuous.

Claim. The map $\Theta^{-1}\colon |K*L|\to|K|*|L|$ is continuous.

Proof. We use the continuity criterion from the Restrict-to-Simplex Lemma 93.6 (2) to show that Θ^{-1} is continuous. Since K and L are non-empty we see that every simplex of $K*L$ is contained in a simplex of the form $s\sqcup t$ where $s\in S$ and $t\in T$ are simplices. Thus let $s\in S$ be a k-simplex and let $t\in T$ be an l-simplex. Note that by the Join-Simplex Lemma 93.19 (2) we know that the map

$$\begin{array}{rcl}\Xi\colon \Delta^k*\Delta^l & \to & \Delta^{k+l+1}\\ [(x_0,\ldots,x_k,r,y_0,\ldots,y_l)] & \mapsto & (r\cdot x_0,\ldots,r\cdot x_k,(1-r)\cdot y_0,\ldots,(1-r)\cdot y_l)\end{array}$$

is a homeomorphism. Next we pick corresponding characteristic maps $\Phi_s\colon\Delta^k\to|K|$ and $\Phi_t\colon\Delta^l\to|L|$. It follows easily from the definitions that the map

$$\begin{array}{rcccl}\Phi_{s\sqcup t}\colon\Delta^{k+l+1} & \xrightarrow{\Xi^{-1}} & \overbrace{(\Delta^k\times[0,1]\times\Delta^l)/\!\sim}^{\Delta^k*\Delta^l} & \xrightarrow{\Omega} & |K*L|\subset\mathbb{R}^{(V\sqcup W)}=\mathbb{R}^{(V)}\times\mathbb{R}^{(W)}\\ & & [(x,r,y)] & \mapsto & (\underbrace{r\cdot\Phi_s(x)}_{\in\mathbb{R}^{(V)}},\underbrace{(1-r)\cdot\Phi_t(y)}_{\in\mathbb{R}^{(W)}})\end{array}$$

is a characteristic map for the simplex $s\sqcup t$. We obtain the following diagram

$$\begin{array}{ccccc}\Delta^{k+l+1} & \xrightarrow{\Phi_{s\sqcup t}} & |K*L| & \xrightarrow{\Theta^{-1}} & |K|*|L|\\ & \searrow^{\Xi^{-1}} & \uparrow{\scriptstyle\Omega} & \nearrow_{\Phi_s*\Phi_t} & \\ & & \Delta^k*\Delta^l & & \end{array}$$

[1142]This might be viewed as an odd formulation, but it is easier to write down the map Θ than to write down its inverse Θ^{-1}.

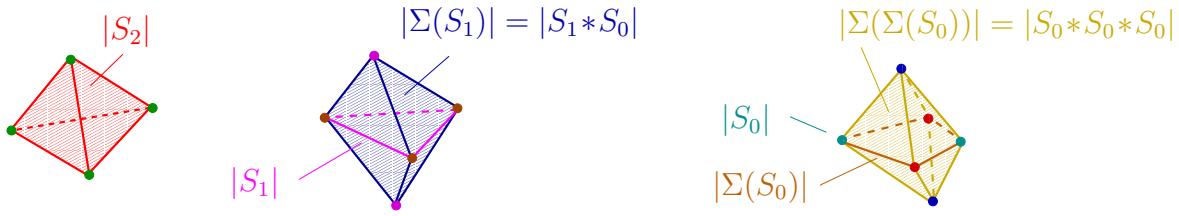
We make the following observations:

(1) It follows immediately from the definitions that the triangle to the left diagram commutes. Furthermore it is equally trivial to see that $\Theta \circ (\Phi_s * \Phi_t) = \Omega$, thus the triangle to the right also commutes.
(2) By the Join-Simplex Lemma 93.19 (2) we know that the left diagonal map is continuous.
(3) By Lemma 16.1 we know that the right diagonal map is continuous.
(4) It follows from the above that $\Theta^{-1} \circ \Phi_{s \sqcup t}$ is continuous.

It now follows from the Restrict-to-Simplex Lemma 93.6 (2) that $\Theta^{-1}\colon |K * L| \to |K| * |L|$ is continuous. ⊞

In the remainder of the proof we now assume that K and L are finite. We need to show that Θ is a homeomorphism. We already know that Θ is a bijection and that the inverse $\Theta^{-1}\colon |K * L| \to |K| * |L|$ is continuous. Furthermore, by the Simplex-Embedding Lemma 93.5 (2) together with Lemma 16.5 we know that $|K| * |L|$ is Hausdorff. Thus it follows, as so often, from the Compact-Hausdorff Proposition 2.17 (3) that Θ^{-1} is a homeomorphism. Thus Θ is also a homeomorphism. ■

Example. Let $n \in \mathbb{N}_0$. Recall that on page 1997 we gave an explicit simplicial structure $(S_n, \Theta\colon |S_n| \to S^n)$ for S^n. Now let $m, n \in \mathbb{N}$. By the Join-Simplex Lemma 93.19 (1) we know that S^{m+n+1} is homeomorphic to $S^m * S^n$. This observation, together with the Join Lemma 93.20, gives us lots of different simplicial structures on spheres. We sketch the resulting three simplicial structures for S^2 in the figure below.



Corollary 93.21. (Simplicial Cone-Suspension Corollary) If $X = (K, \Theta\colon |K| \to X)$ is a finite simplicial complex, then the suspension $\Sigma(X)$ and the cone $\mathrm{Cone}(X)$ admit natural simplicial structures where the corresponding abstract simplicial complexes are given by $\Sigma(K)$ and $\mathrm{Cone}(K)$.

Proof. The corollary follows immediately from the Join Lemma 93.20 together with the two examples on page 2002. ■

Remark. In Section 94.4 we will introduce the simplicial analogue of the mapping cylinder and mapping cone.

93.7. **Linear simplicial complexes.** In this short section we discuss an important special case of a simplicial complex.

Definition. A simplicial complex $(X, (K, \Theta\colon |K| \to X))$ is called linear if X is a subspace[1143] of some $\mathbb{R}^n$ and for each each simplex $s \in K$ the map $\Theta\colon |s| \to X$ is an affine linear map.

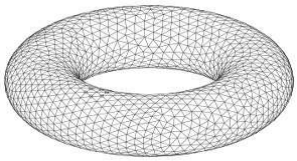
[1143] In particular we assume that X is endowed with the subspace topology coming from $\mathbb{R}^n$.

Example. Let $n \in \mathbb{N}$. The topological space $\mathbb{R}^n$ together with the canonical simplicial structure defined in Lemma 93.17 is clearly a linear simplicial complex.

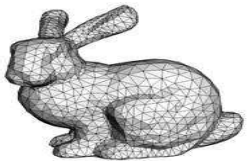
Remark.

(1) Different books have different definitions for "simplicial complexes". Our approach mostly follows the approach taken in [**Spa1995**, Chapter 3]. In many books what we call a "linear simplicial complex" is basically used as the definition of a simplicial complex, see e.g. [**Matv2006**, p. 6] or [**Bre1993**, Chapter 21].

(2) Linear simplicial complexes play an important role in computer visualizations. More precisely, in computer visualizations surfaces in $\mathbb{R}^3$ are usually approximated by linear simplicial complexes. These have the nice feature that they can be described by a fairly small amount of data. We refer to the figure below for an illustration.



linear simplicial structure for the torus



linear simplicial structure for a bunny

Lemma 93.22. Every finite simplicial complex is simplicially isomorphic to a finite linear simplicial complex.

Remark. An (abstract) simplicial complex is called *locally finite* if each simplex is the face of only finitely many simplices. With this definition one can formulate a much stronger version of Lemma 93.22: By [**FrPi1990a**, Theorem 3.3.15] a simplicial complex is homeomorphic to a linear simplicial complex if and only if it is finite-dimensional, countable and locally finite. We will discuss this statement in Exercise 93.8.

Proof. By definition a finite simplicial complex is homeomorphic to the topological realization $|K|$ of a finite abstract simplicial complex $K = (V, S)$. In Lemma 93.9 we showed that we can view $|K|$ as a linear simplicial complex in $\mathbb{R}^V = \mathbb{R}^m$. ■

We adopt the following notation.

Notation. Given[1144] $v_0, \ldots, v_k \in \mathbb{R}^n$ we write

$$|\{v_0, \ldots, v_k\}| := \left\{ \underbrace{\sum_{i=0}^{k} t_i \cdot v_i}_{\in \mathbb{R}^n} \;\middle|\; \begin{array}{l} (1) \text{ for each } i \text{ we have } t_i \geq 0 \\ (2)\ t_0 + \cdots + t_k = 1 \end{array} \right\}.$$

Remark. Let $v_0, \ldots, v_k \in \mathbb{R}^n$. Furthermore let $w_0, \ldots, w_m \in |\{v_0, \ldots, v_k\}|$. An elementary calculation shows that $|\{w_0, \ldots, w_m\}| \subset |\{v_0, \ldots, v_k\}|$. This inclusion can also be obtained from Exercise 2.38 (a).

Lemma 93.23.

[1144] We do not assume that $v_0, \ldots, v_k$ are distinct.

(1) Let K be a linear simplicial complex in $\mathbb{R}^n$.
   (a) Every k-simplex s of K is of the form $s = |\{v_0, \ldots, v_k\}|$ for some distinct vertices $v_0, \ldots, v_k$.
   (b) Every simplex of K is a convex subset of $\mathbb{R}^n$.
   (c) Let $v_0, \ldots, v_k$ be distinct vertices of K. The following three statements are equivalent:
      (i) $|\{v_0, \ldots, v_k\}|$ is a simplex of K,
      (ii) $\{v_0, \ldots, v_k\}$ is the vertex set of a simplex of K,
      (iii) $|\{v_0, \ldots, v_k\}| \subset K$.

(2) Let $f\colon K \to L$ be a simplicial map between two linear simplicial complexes. The restriction of f to any k-simplex $|\{v_0, \ldots, v_k\}|$ of K is given by

$$\begin{array}{rcl} |\{v_0, \ldots, v_k\}| & \to & |\{\varphi(v_0), \ldots, \varphi(v_k)\}| \\ \sum_{i=0}^{k} t_i \cdot v_i & \mapsto & \sum_{i=0}^{k} t_i \cdot \varphi(v_i). \end{array}$$

Proof.

(1) Almost all of these statements follow easily from the definitions. The only statement which needs a little bit of thought is the fact that (iii) implies (i). Since we will not make use of this statement we feel comfortable with the thought of leaving the details to the reader.

(2) This statement follows immediately from the definition of a simplicial map, see page 1980 and the definition of the topological realization of a simplicial map between abstract simplicial complexes, see Proposition 93.7. ■

93.8. **Simplicial structures and CW-structures I.** It is pretty clear that simplicial complexes and CW-complexes have structural similarities. In particular few readers will be surprised by the following lemma.

Lemma 93.24. (Simplicial-Implies-CW Lemma) Every (ordered) simplicial complex admits a (natural) CW-complex structure where given any $n \in \mathbb{N}$ the n-simplices of the simplicial structure are precisely the n-cells[1145] of the CW-structure. In particular the following statements hold:

(1) The simplicial subcomplexes are precisely the cellular subcomplexes,
(2) For each $n \in \mathbb{N}_0$ the n-skeleton of the simplicial complex equals the n-skeleton of the CW-complex.
(3) Any simplicial map $f\colon X \to Y$ between simplicial complexes is a cellular map between the corresponding CW-complexes.

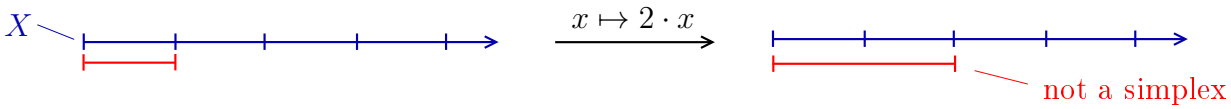
Remark. the Simplicial-Implies-CW Lemma 93.24 says in particular that there exists a functor

$$F\colon \begin{array}{c}\text{category } \mathcal{OrdAbsSimpCplx} \text{ of} \\ \text{ordered simplicial complexes}\end{array} \to \begin{array}{c}\text{category } \mathcal{CW} \text{ of} \\ \text{CW-complexes}\end{array}$$

which has the property that the functor keeps the underlying topological space fixed. The functor is faithful but not fully faithful. The latter is just a fancy way of saying that there exist cellular maps that are not simplicial. For example, consider the simplicial complex

[1145] Recall that cells and simplices are subsets, so it makes sense to talk of equality.

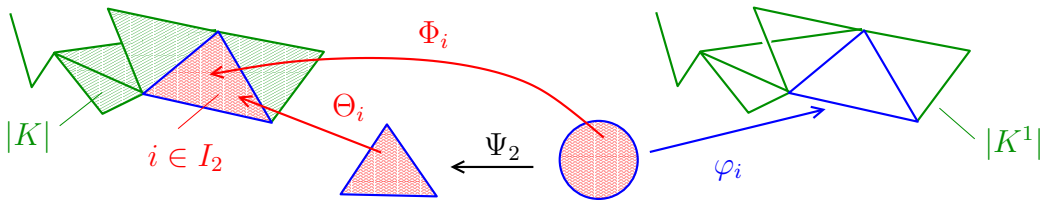
$X = [0, \infty)$ with the vertex set given by $\mathbb{N}_0$. We view X also as a CW-complex with the cellular structure that is induced from the simplicial structure. The map $f\colon X \to X$ given by $x \mapsto 2x$ is cellular but it is not simplicial. We refer to the figure below for an illustration.



Sketch of proof. Basically by definition of a simplicial complex it suffices to prove the statement for (ordered) abstract simplicial complexes and their topological realizations. The lemma now follows from fleshing out the following steps for a given (ordered) abstract simplicial complex $K = (V, S)$.

(a) Let $n \in \mathbb{N}_0$. Recall that in the Standard Simplex Homeomorphism Lemma 72.1 we had written down a specific homeomorphism $\Psi_n\colon (\overline{B}^n, S^{n-1}) \to (\Delta^n, \partial\Delta^n)$.
(b) Given $n \in \mathbb{N}_0$ we denote by K^n the n-skeleton of K. Recall that K^n is a subcomplex of K.
(c) Given $n \in \mathbb{N}_0$ let I_n be the set of n-simplices of K. For each $i \in I_n$ let $\Theta_i\colon \Delta^n \to |K|$ be a characteristic map as defined on page 1984. Recall that if K is ordered, then we saw on page 1984 that there is a natural choice of a characteristic map.
(d) Given $i \in I_n$ we write $\Phi_i := \Theta_i \circ \Psi_n\colon \overline{B}^n \to |K|$ and $\varphi_i := \Phi_i|_{S^{n-1}}\colon S^{n-1} \to |K^{n-1}|$. Note that φ_i takes values in $|K^{n-1}|$.
(e) Starting from $X^0 := V$ we can build up a CW-complex such that for each $n \in \mathbb{N}_0$ the attaching maps are precisely the maps $\varphi_i\colon S^{n-1} \to |K^{n-1}|$. The maps Ψ_i give rise to a map $\Xi_n\colon |X^n| \to |K^n|$ which is, basically by definition, a bijection.
(f) Eventually we get a CW-complex $X = \varinjlim X_n$ and the bijections Ξ_n can be combined to obtain a bijection $\Xi\colon |X| \to |K|$.
(g) It remains to show that Ξ is continuous. First note that it follows easily from the Restrict-to-Simplex Lemma 93.6 (1) together with the Characteristic Maps Lemma 68.7 (1) that Ξ is continuous. Furthermore, it follows from the Characteristic Maps Lemma 68.7 (4) together with the Simplex-Embedding Lemma 93.5 (3) that Ξ^{-1} is also continuous.
(h) In summary we have shown that $\Xi\colon |X| \to |K|$ is a homeomorphism.

It follows basically immediately from the construction of the CW-structure on $|K|$ that the properties (1), (2) and (3) are satisfied. ■



The Simplicial-Implies-CW Lemma 93.24 allows us to import many previous results about CW-complexes to the context of simplicial complexes. For example, the CW-Complex

Properties Proposition 68.10 (2) gives us a new proof that simplicial complexes are Hausdorff. More interesting of course are new results. The following corollary summarizes some of the nicest and most important ones.

Corollary 93.25.

(1) Simplicial complexes are normal.
(2) Simplicial complexes are locally contractible.[1146]
(3) A simplicial complex is connected if and only if it is path-connected.
(4) Let X be a simplicial complex and let A be a subcomplex. The following statements hold:
   (a) The inclusion $A \to X$ is a cofibration.
   (b) The map

$$\begin{array}{rcl} \varphi\colon \mathrm{Cone}(i\colon A \to X) & \to & X/A \\ P & \mapsto & \begin{cases} [P], & \text{if } P \in X, \\ [A], & \text{if } P = [(a,t)] \text{ with } a \in A \text{ and } t \in [0,1] \end{cases} \end{array}$$

   is a homotopy equivalence.
   (c) If A is contractible, then the projection $X \to X/A$ is a homotopy equivalence.
(5) If X is a finite simplicial complex, then

$$\chi(X) \;=\; \sum_{n \in \mathbb{N}_0} (-1)^n \cdot \text{number of } n\text{-simplices of } X.$$

Proof.

(1),(2) These two statements follow from the Simplicial-Implies-CW Lemma 93.24 together with the CW-Complex Properties Proposition 68.10 (2) and (6).
(3) This statement follows from (1) combined with Locally Path-Connected Lemma 2.40.
(4) (a) This statement follows from the Simplicial-Implies-CW Lemma 93.24 together with the Subcomplex Cofibration Theorem 17.9.
   (b) This follows from (a) together with the Cofibration-Quotient Lemma 17.7 (2).
   (c) This follows from (a) together with the Cofibration-Quotient Lemma 17.7 (3).
(5) This statement follows from the Simplicial-Implies-CW Lemma 93.24 together with the Euler Characteristic-H_*-Proposition 87.1. ■

Now we can give a partial answer to Question 93.18.

Corollary 93.26. Let Y be a simplicial complex and let X be a subcomplex. If Y is finite, then there exists a finite simplicial complex, namely $Y \cup \mathrm{Cone}(X)$, that is homotopy equivalent to Y/X.



[1146] In the rather unlikely event that the reader forgot what these properties are good for, recall that by Corollary 61.11 this means in particular that any path-connected simplicial complex admits a universal covering. In Exercise 93.9 we will see that any covering of a simplicial complex is again a simplicial complex.

Remark. We leave it as an entertainment for long winter nights to figure out whether the Intervall CW-Product Lemma 68.8 can be used to generalize Corollary 93.26 to infinite simplicial complexes.

Proof. Basically by definition we only need to deal with abstract simplicial complexes and their topological realizations. Thus let $L = (W, T)$ be a finite abstract simplicial complex and let $K = (V, S)$ be a subcomplex. We denote by $i\colon K \to L$ the inclusion map. Recall that i is, basically by definition, simplicial. Since K and L are finite we can consider the abstract simplicial complex $\operatorname{Cone}(i\colon K \to L)$ that we will introduce on page 2027. Now we see that

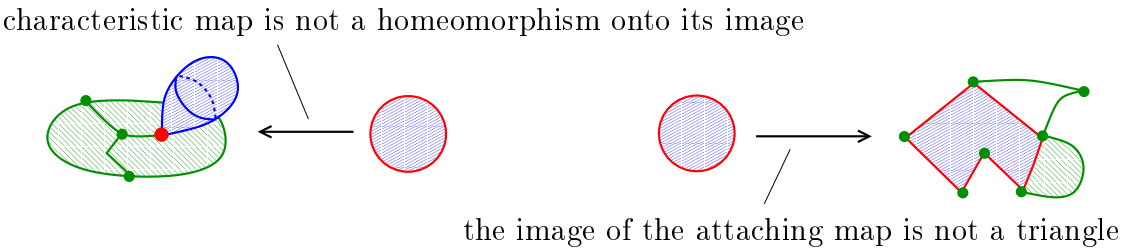
$$|L|/|K| \xleftarrow[\text{Corollary 93.25}]{\simeq} \operatorname{Cone}(|i|\colon |K| \to |L|) = |L| \cup_{|K|} \operatorname{Cone}(|K|) \xleftarrow[\text{Simplicial Cone-Suspension Corollary 93.21}]{\cong} \overbrace{|L| \cup_{|K|} |\operatorname{Cone}(K)|}^{=|\operatorname{Cone}(i\colon K \to L)|}$$

■

It is pretty clear that there is no naive converse to the Simplicial-Implies-CW Lemma 93.24. For example, the CW-complex constructed in the proof of the Simplicial-Implies-CW Lemma 93.24 has some special properties:

(1) It follows from the Simplex-Embedding Lemma 93.5 (2) that each characteristic map is an embedding.
(2) The CW-structure on the image of a characteristic map of an n-cell is equivalent to the one corresponding to the standard simplicial structure on $\partial\Delta^n$ that we defined on page 1995.

Both types of examples are illustrated in the figure below. Thus we see that many, arguably most, CW-structures do not come directly from simplicial structures.



It is perhaps more reasonable to ask, when does a CW-complex admit a simplicial structure? For example it is fairly obvious that the two cellular complexes shown in the figure above admit a simplicial structure. To formulate the relevant proposition we need the following definition.

Definition. A CW-complex is called regular if every characteristic map is an embedding.

Examples.

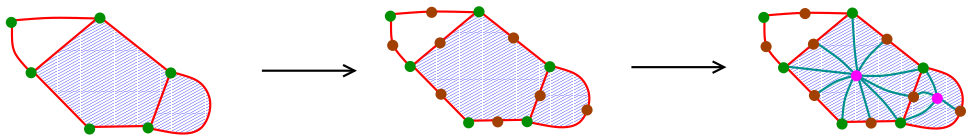
(1) The CW-complex shown in the figure above to the left is non-regular, whereas the CW-complex shown to the right is regular.
(2) Let X be a simplicial complex. We use the Simplicial-Implies-CW Lemma 93.24 to view X as CW-complex. It follows from the Simplex-Embedding Lemma 93.5 (3) that this CW-complex is regular.

Now we can formulate the promised proposition.

Proposition 93.27. Every (countable) regular CW-complex admits the structure of a (countable) simplicial complex.

Sketch of proof. We will not make use of this theorem, thus we refer to [**Geo2008**, Corollary 5.3.9] or alternatively to [**FrPi1990a**, Theorem 3.4.1] for a proof.

Here we only provide a sketch of a sketch of the proof. Thus let X be a regular CW-complex. We turn inductively the skeleta X^n into a simplicial complex. Suppose we already equipped X^{n-1} with a simplicial structure. For each n-cell of X^n we pick a point in the corresponding open cell and we use it to divide said n-cell into a suitable number of n-simplices. If the CW-complex X was countable, then it follows from the construction together with the Countability Lemma 0.8 that the simplicial complex is also countable. The reader should have no troubles with turning the above sketch of an argument into a formal proof. ■



regular CW-complex add point in each 1-cell add point in each 2-cell

The relatively strong hypothesis of Proposition 93.27 indicates that it is perhaps too much to hope that every CW-complex admits a simplicial structure. Indeed, this fear turns out to be fully justified:

Proposition 93.28. There exists a finite CW-complex that does not admit a simplicial structure.

Remark. Proposition 93.28 is rather disappointing. Later on in Proposition 94.12 and Theorem 104.22 we will see that every (finite) CW-complexes is homotopy equivalent to a (finite) simplicial complexes.

Proof. We consider the function

$$f\colon [0,1] \to \mathbb{R}, \quad t \mapsto \begin{cases} 0, & \text{if } t = 0, \\ t\cdot\cos(\frac{2\pi}{t}) & \text{if } t \in (0,1] \end{cases}$$

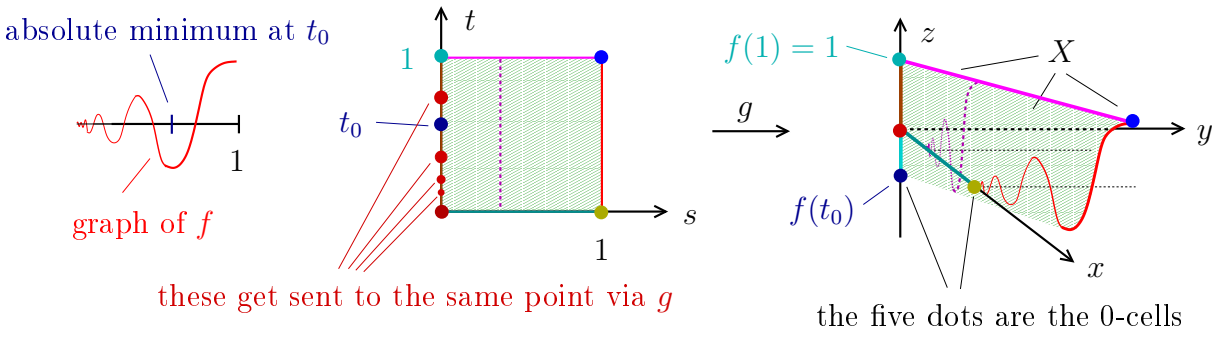
and we consider the map

$$g\colon [0,1]^2 \to \mathbb{R}^3, \quad (s,t) \mapsto (s, s\cdot t, f(t)).$$

Note that f assumes its maximum at $t = 1$ and that it assumes its minimum at a unique $t_0 \in (0,1)$. We set $X := g([0,1]^2)$. It is not particularly difficult to see that X admits a CW-structure with five 0-cells, five 1-cells and one 2-cell.[1147] We refer to the figure below for an illustration.

In [**FrPi1990a**, p. 128-130] a careful argument is given why X does not admit a simplicial structure. Other examples of finite CW-complexes that do not admit a simplicial structure are given in [**LW1969**, p. 81] (with a very sketchy proof) and [**Met1967**]. ■

[1147] The five 0-cells and the five 1-cells are clearly indicated in the figure below.



Exercises for Chapter 93.

Exercise 93.1. For each $g \geq 2$ give an explicit simplicial structure for the surface of genus g.

Exercise 93.2. Give an explicit simplicial structure for $\mathbb{R}\mathrm{P}^2$.
Remark. In Exercise 94.5 we will construct an explicit simplicial structure for all $\mathbb{R}\mathrm{P}^n$.

Exercise 93.3. In the figure on page 1997 we gave a simplicial structure for the torus $S^1 \times S^1$ with 18 2-simplices. Does there exist a simplicial structure with less than 18 2-simplices?

Exercise 93.4.

(a) Show that the set of isomorphism classes of finite abstract simplicial complexes is countable.

(b) Show that the set of isomorphism classes of countable abstract simplicial complexes is uncountable.

Exercise 93.5. We consider the abstract simplicial complex $K = (V, S)$ which is given by $V = \mathbb{Z} \cup \{*\}$ and $S = \{\{v\} \,|\, v \in V\} \cup \{\{*, n\} \,|\, n \in \mathbb{Z}\}$. Show that $|K|$ is not homeomorphic to a linear simplicial complex.
Remark. (V, S) is just the simplicial cone on the abstract simplicial complex with vertex set $\mathbb{Z}$ and which has no simplices of dimension ≥ 1.

Exercise 93.6. Let $m, n \in \mathbb{N}_0$. Show that the simplicial join $D_m * D_n$ of the abstract simplicial complexes D_m and D_n is simplicially isomorphic to D_{m+n+1}.

Exercise 93.7. Let K be a finite n-dimensional simplicial complex. Show that K is homeomorphic to a finite linear simplicial complex that is contained in $\mathbb{R}^{2n+1}$.
Remark. In Proposition 88.7 we already showed that there exist finite 1-dimensional simplicial complexes that cannot be embedded into $\mathbb{R}^{2n}$. In fact the van Kampen-Flores Theorem, which can be proved using the Borsuk-Ulam Theorem 39.8, says that given any $n \in \mathbb{N}$ there exists a finite n-dimensional simplicial complex K that does not admit a map $K \to \mathbb{R}^{2n}$ which is a homeomorphism onto its image. We refer to [**Mato2008**, Theorem 5.1.1] for details.

Exercise 93.8. Let $K = (V, S)$ be an abstract simplicial complex that is finite-dimensional and countable. Furthermore we assume that it is locally finite, which means that each simplex is the face of only finitely many simplices. Show that $|K|$ is homeomorphic to a

linear simplicial complex.
Hint. Use the Closed Embedding-to-$\mathbb{R}^n$ Proposition 2.19.

Exercise 93.9. Let K be a simplicial complex and let $p\colon \widetilde{K} \to K$ be a covering in the sense of the definition on page 1067. Show that $\widetilde{K}$ admits a structure of a simplicial complex such that p is simplicial.

Exercise 93.10.

(a) Let $K = (V, S)$ and $L = (W, T)$ be two ordered abstract simplicial complexes. Show that the ordered abstract simplicial complex $K \times L$ that we defined on page 1998, together with the two natural projections $p\colon K \times L \to K$ and $q\colon K \times L \to L$, is the product of K and L in the category $\mathcal{OrdAbsSimpCplx}$ of ordered abstract simplicial complexes.
(b) Is the category $\mathcal{OrdAbsSimpCplx}$ of ordered abstract simplicial complexes closed under colimits?

Exercise 93.11. Let $K = (V, S)$ be an abstract simplicial complex and let $J = (U, R)$ be a subcomplex such that for any choice of simplices $r \in R$ and $s \in S \setminus R$ we have $r \cap s = \varnothing$. Show that $|J|$ is an open subset of $|K|$.
Remark. Note that by Lemma 93.8 (2) we know that $|J|$ is always a closed subset of $|K|$.

Exercise 93.12. Let $K = (V, S)$ be an abstract simplicial complex. We say $K = (V, S)$ is *connected* if given any $v, v' \in V$ there exist 1-simplices $s_0, \ldots, s_k \in S$ such that $v \in s_0$, such that for every $i \in \{0, \ldots, k-1\}$ we have $s_i \cap s_{i+1} \neq \varnothing$ and such that $v' \in s_k$.[1148] Show that the following statements are equivalent:

(a) The abstract simplicial graph K is connected.
(b) Given any two simplices $t, t' \in S$ there exist simplices $s_0, \ldots, s_k \in S$ such that $t \subset s_0$, such that for every $i \in \{0, \ldots, k-1\}$ we have $s_i \cap s_{i+1} \neq \varnothing$ and such that $t' \in s_k$.
(c) The topological realization $|K|$ is connected.

For completeness we mention that Corollary 93.25 (3) says that (c) is equivalent to the following statement:

(d) The topological realization $|K|$ is path-connected.

Remark. The equivalence of (a) and (b) is straightforward.
Remark. For the equivalence of (b) and (c) use the Union Connected Lemma 2.28, Lemma 93.8 (2) and Exercise 93.11.

Exercise 93.13. Let $K = (V, S)$ and $L = (W, T)$ be abstract simplicial complexes. We denote by X and Y the corresponding CW-complexes as defined in the Simplicial-Implies-CW Lemma 93.24. Let $X \otimes Y$ be the topological space that is given by the product CW-structure, see page 1500. Show that there exists a homeomorphism $|K \times L| \to X \otimes Y$.
Remark. Note that here we certainly do not assume that K and L are finite.

Exercise 93.14. Let X be a topological space that admits the structure of a 1-dimensional CW-complex. Show that X also admits the structure of a 1-dimensional simplicial complex.

[1148] This definition is modelled on the definition of the connectedness of an abstract graph, see page 307.

94. The Simplicial Approximation Theorem

In Chapter 93 we covered in great detail the basics of (abstract) simplicial complexes. In this chapter we will formulate and prove the first major result on simplicial complexes, namely the Simplicial Approximation Theorem.

94.1. **The barycentric subdivision.** In this section we introduce the barycentric subdivision of an abstract simplicial complex. This construction allows us to replace an abstract simplicial complex by a "finer" abstract simplicial complex. This will be a major tool in the proof of several theorems.

Definition. Let $K = (V, S)$ be an abstract simplicial complex.

(1) For each k-simplex $s = \{v_0, \ldots, v_k\} \in S$ we define the barycenter of s to be the point
$$\underline{s} := \sum_{v \in s} \frac{1}{\dim(s)+1} \cdot v = \frac{1}{k+1} \cdot v_0 + \cdots + \frac{1}{k+1} \cdot v_k \in \langle s \rangle \subset \mathbb{R}^{(V)}.$$
Note that if $v \in V$ is a vertex, then the barycenter of the corresponding 0-simplex $\{v\}$ is v itself.

(2) Given $s, t \in S$ we write $\underline{s} \leq \underline{t}$ if $s \subset t$.

The following lemma summarizes a few properties of the barycenters.

Lemma 94.1. (Barycenter Lemma) Let $K = (V, S)$ be an abstract simplicial complex.

(1) The map $S \mapsto \mathbb{R}^{(V)}$ given by $s \mapsto \underline{s}$ is natural.
(2) For simplices $s, t \in S$ with $s \neq t$ we have $\underline{s} \neq \underline{t}$.
(3) For each 0-simplex s we have $s = \underline{s}$[1149] and for each simplex s of dimension ≥ 1 we have $\underline{s} \notin V$.

Proof.

(1) The most difficult part is to figure out what "natural" means in this context. We leave it to the reader to figure this out. Afterwards the statement is basically obvious.
(2) This statement follows from the Open Simplex Lemma 93.11 (2).
(3) This statement is trivial. (But it is nonetheless useful.) ■

The following lemma is much more interesting.

Lemma 94.2. Let $K = (V, S)$ be an abstract simplicial complex.

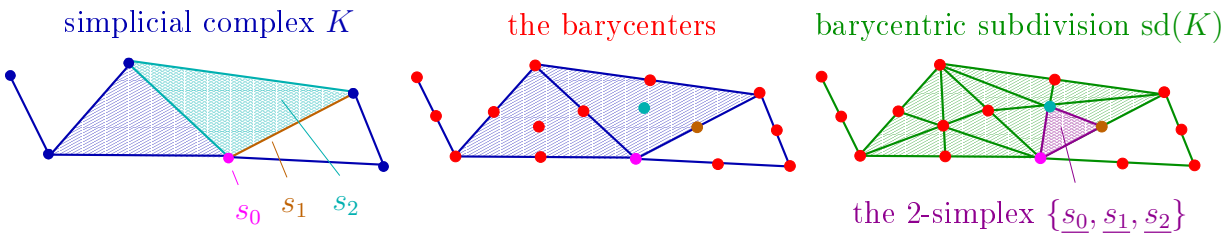
(1) The pair
$$\mathrm{sd}(K) = \left(\{\underline{s}\}_{s \in S}, \{\{\underline{s_0}, \ldots, \underline{s_k}\} \mid s_0 \subsetneq \cdots \subsetneq s_k\}\right)$$
is an abstract simplicial complex, called the barycentric subdivision of K.

(2) The relation "$\leq$" on the vertex set $\{\underline{s}\}_{s \in S}$ turns the barycentric subdivision $\mathrm{sd}(K)$ into an *ordered* abstract simplicial complex.

The definition of $\mathrm{sd}(K)$ is illustrated in the figure below.[1150]

[1149] Here we view, as usual, V as a subset of $\mathbb{R}^{(V)}$.

[1150] Here, as on many other occasions, we blur the difference between an abstract simplicial complex, its topological realization, and a linear simplicial complex that is simplicially isomorphic to the topological realization. In particular the illustrations shown in the figure below are dramatically dimensionally reduced.



Remark.

(1) Perhaps it is helpful to restate the definition of the barycentric subdivision $\mathrm{sd}(K)$ in other words: The vertices of $\mathrm{sd}(K)$ are precisely the barycenters of the original simplices and the simplices of the barycentric subdivision are given by the barycenters of an ascending sequence of simplices in K.

(2) The statement of Lemma 94.2 (2) is quite amazing: ex nihilo we created order where there was chaos.

(3) There is an alternative more formal, but basically equivalent definition of the barycentric subdivision. Namely, given an abstract simplicial complex $K = (V, S)$ one can consider the following abstract simplicial complex:

$$\widetilde{\mathrm{sd}}(K) \;=\; \{S, \{\{s_0, \ldots, s_k\} \,|\, s_0 \subsetneq \cdots \subsetneq\}\}.$$

The map $s \mapsto \underline{s}$ induces a natural isomorphism between the abstract simplicial complex $\widetilde{\mathrm{sd}}(K)$ and the abstract simplicial complex $\mathrm{sd}(K)$. Our first definition is arguably mnemonically superior, the second definition, which is for example used in [**dLon2013**, p. 179], is arguably purer.

Proof. The statements follow immediately from the definitions. ■

Notation. Let $K = (V, S)$ be an abstract simplicial complex. Given $i \in \mathbb{N}$ we refer to

$$\mathrm{sd}^i(K) \;:=\; \underbrace{\mathrm{sd}(\ldots \mathrm{sd}(\mathrm{sd}(K)) \ldots)}_{\text{sd applied } i \text{ times}}$$

as the i-th iterated barycentric subdivision of K.

The following lemma summarizes the most important properties of the barycentric subdivision of an abstract simplicial complex.

Lemma 94.3. (SD-Lemma) Let $K = (V, S)$ be an abstract simplicial complex.

(1) If K is finite, then $\mathrm{sd}(K)$ is also finite.

(2) The dimension of $\mathrm{sd}(K)$ equals the dimension of K.

(3) The map

$$\begin{array}{rcl} \mathfrak{V}\colon |\mathrm{sd}(K)| & \to & |K| \\ \alpha & \mapsto & \left(\begin{array}{ccc} V & \to & \mathbb{R}_{\geq 0} \\ v & \mapsto & \displaystyle\sum_{s \in S \text{ with } v \in s} \frac{1}{\dim(s)+1} \cdot \alpha(\underbrace{\underline{s}}_{\uparrow}) \end{array} \right) \end{array}$$

$\underline{s}$ is a vertex of $\mathrm{sd}(K)$, hence $\alpha(\underline{s})$ is defined

has the following properties:

(a) Let σ be a k-simplex of $\mathrm{sd}(K)$. By definition this means that $\sigma = \{\underline{s_0}, \ldots, \underline{s_k}\}$ where $s_0 \subsetneq \cdots \subsetneq s_k$ is an ascending sequence of simplices of K. The restriction of the map $\mathfrak{U}\colon |\mathrm{sd}(K)| \to |K|$ to $|\sigma|$ is given by[1151]

$$\begin{array}{ccc} |\sigma| & \to & |K| \\ \sum_{i=0}^{k} \lambda_i \cdot \underset{\uparrow}{\underline{s_i}} & \mapsto & \sum_{i=0}^{k} \lambda_i \cdot \underset{\uparrow}{\underline{s_i}} \\ \text{viewed as basis element of } \mathbb{R}^{(\mathrm{sd}(K))} & & \text{viewed as vector in } \mathbb{R}^{(V)} \end{array}$$

In particular we have $\mathfrak{U}(|\sigma|) \subset |s_k|$ and $\mathfrak{U}(\langle\sigma\rangle) \subset \langle s_k\rangle$.

(b) The map $\mathfrak{U}$ sends each vertex[1152] $\underline{s} \in \mathrm{sd}(K)$ to the corresponding point $\underline{s} \in |K|$.

(c) Given any k-simplex t of K the map $\mathfrak{U}$ restricts to a homeomorphism

$$\bigcup_{\substack{s = \{\underline{a_0}, \ldots, \underline{a_{k-1}}, \underline{t}\} \\ \text{a } k\text{-simplex of } \mathrm{sd}(K) \text{ where} \\ a_0 \subsetneq \cdots \subsetneq a_{k-1} \subsetneq t \\ \text{are simplices of } K}} |s| \xrightarrow{\cong} |t|.$$

(d) The map $\mathfrak{U}$ is a natural homeomorphism.

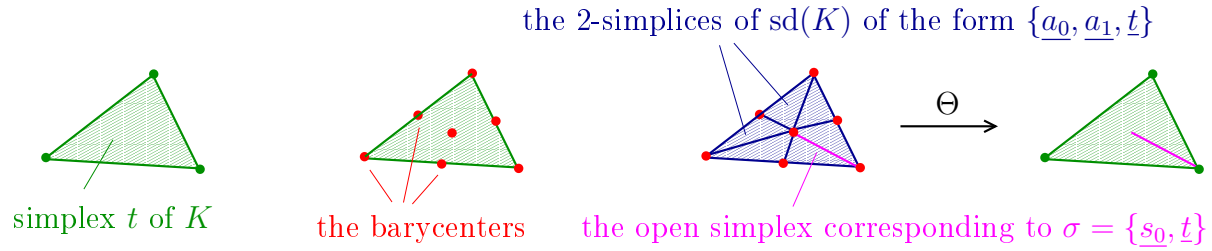
(4) If $f\colon K = (V, S) \to L = (W, T)$ is a simplicial map between two abstract simplicial complexes, then the map

$$\{\underline{s}\} \mapsto \{\underline{f(s)}\}$$

defines a simplicial map $f_*\colon \mathrm{sd}(K) \to \mathrm{sd}(L)$. This map has the property that the following diagram commutes:

$$\begin{array}{ccc} |\mathrm{sd}(K)| & \xrightarrow{\mathfrak{U}_K} & |K| \\ \downarrow{\scriptstyle |f_*|} & & \downarrow{\scriptstyle |f|} \\ |\mathrm{sd}(L)| & \xrightarrow{\mathfrak{U}_L} & |L|. \end{array}$$

(5) The maps from (4) turn $K \mapsto \mathrm{sd}(K)$ into a covariant functor from the category of abstract simplicial complexes to the category of (ordered) abstract simplicial complexes.



Given any abstract simplicial complex K and given any $i \in \mathbb{N}$ we will use the homeomorphism from the SD-Lemma 94.3 (3) to make the identification $|\mathrm{sd}^i(K)| = |K|$.

[1151] Note that this shows that $\mathfrak{U}$ does take values in $|K|$.

[1152] Here we use the convention from page 1984 that says that we view the vertex of an abstract simplicial complex as a subset of the topological realization.

Proof.

(1) This statement is basically clear.
(2) We will prove this statement in Exercise 94.3.
(3) Now we need to show that $\mho$ has all the promised properties:

(a) Let $\sigma = \{\underline{s_0}, \dots, \underline{s_k}\}$ be a k-simplex of $\mathrm{sd}(K)$. Given a point $P = \sum_{i=0}^{k} \lambda_i \cdot \underline{s_i}$ we calculate that

$$\mho(P) = \mho\Big(\sum_{i=0}^{k} \lambda_i \cdot \underline{s_i}\Big) = \sum_{i=0}^{k} \lambda_i \cdot \mho(\underline{s_i}) = \sum_{i=0}^{k} \lambda_i \cdot \mho\Big(\underbrace{\sum_{v \in s_i} \tfrac{1}{\dim(s_i)+1} \cdot v}_{=\underline{s_i}}\Big) = \sum_{i=0}^{k} \lambda_i \cdot \underline{s_i}.$$

Note that $\underline{s_0}, \dots, \underline{s_k}$ lie in $|s_k|$. Thus it follows from the calculation, together with the remark on page 2006, that $\mho(|\sigma|) \subset |s_k|$. Furthermore, if $P \in \langle\sigma\rangle$, then $\lambda_k > 0$, which in turn implies almost immediately that $\mho(P) \in \langle s_k\rangle$.[1153]

(b) This statement follows immediately from (a).

(c) Let t be a k-simplex t of K. We leave it to the reader to verify that $\mho$ restricts to a bijection

$$X := \bigcup_{s=\{\underline{a_0},\dots,\underline{a_{k-1}},\underline{t}\}} |s| \xrightarrow{\cong} |t|.$$

Note that the left-hand side X is actually a finite subcomplex of $|\mathrm{sd}(K)|$. Given a simplex $s = \{\underline{a_0}, \dots, \underline{a_{k-1}}, \underline{t}\}$ we pick a characteristic map $\Phi_s \colon \Delta^k \to |s|$. Furthermore we pick a characteristic map $\Phi_t \colon \Delta^k \to |t|$. It follows almost immediately from the definitions that the map $\Phi_t^{-1} \circ \mho \circ \Phi_s \colon \Delta^k \to \Delta^k$ is an affine linear map, in particular it is continuous. It follows from the Restrict-to-Simplex Lemma 93.6 (1) and the Simplex-Embedding Lemma 93.5 that the above map $\mho \colon X \to |t|$ is continuous. It now follows, as always, from the Compact-Hausdorff Proposition 2.17 (3) together with the Simplicial Complex-Finiteness Proposition 93.12 and the Simplex-Embedding Lemma 93.5 that the map $\mho \colon X \to |t|$ is a homeomorphism.

(d) Using (c) one can easily verify that $\mho$ is a bijection. Furthermore, it follows from the Restrict-to-Simplex Lemma 93.6 (1) together with (c) that $\mho$ is continuous. Finally, it follows from (c) that for each simplex t of K the map $\mho^{-1} \colon |t| \to |\mathrm{sd}(K)|$ is continuous. By the Restrict-to-Simplex Lemma 93.6 (1) this shows that $\mho^{-1} \colon |K| \to |\mathrm{sd}(K)|$ is continuous. We have thus shown that $\mho$ is indeed a homeomorphism. It follows easily from the definitions and the Barycenter Lemma 94.1 (1) that the homeomorphism is in fact natural.

(4) We leave the elementary task of verifying this statement to the reader.

(5) This statement follows almost immediately from the definitions. The only step which requires a short moment of reflection is to show that if $f \colon K \to L$ is a simplicial map, then the induced map $f_* \colon \mathrm{sd}(K) \to \mathrm{sd}(L)$ is actually order-preserving, i.e. it is a morphism in the category $\mathcal{OrdAbsSimpCplx}$ of ordered abstract simplicial complexes. ∎

The following definition is almost self-explanatory.

[1153] For the proof that $\mho(\langle\sigma\rangle) \subset \langle s_k\rangle$ it is arguably more efficient to work with the initial description of $\mho$.

Definition. Let $(X, (K, \Theta\colon |K| \to X))$ be a simplicial complex and let $i \in \mathbb{N}_0$. We define the i-th barycentric subdivision of (X, Θ) to be $(X, (\mathrm{sd}^i(K), \Theta\colon |\mathrm{sd}^i(K)| = |K| \to X))$

For completeness we state the following elementary lemma.

Lemma 94.4. For every $k \in \mathbb{N}_0$ the barycentric subdivision of a linear simplicial complex in $\mathbb{R}^n$ is again a linear simplicial complex in $\mathbb{R}^n$.

Proof. This statement follows almost immediately from the definition of the barycentric subdivision. ■

94.2. **Stars and links in simplicial complexes.** Given an (abstract) simplicial complex K and a simplex s we introduce and study the corresponding star $\mathrm{St}(K, s)$ and $\mathrm{Lk}(K, s)$. Both concepts are elementary, but both also play a major role in the study of (abstract) simplicial complexes.

Definition. Let $K = (V, S)$ be an abstract simplicial complex.

(1) Let $s \in S$ be a simplex. We introduce the following subsets of $|K|$:

$$\text{star } \mathrm{St}(K, s) := \bigcup_{t \in S \text{ with } s \subset t} |t| \quad \text{and the} \quad \text{open star } \mathring{\mathrm{St}}(K, s) := \bigcup_{t \in S \text{ with } s \subset t} \langle t \rangle.$$

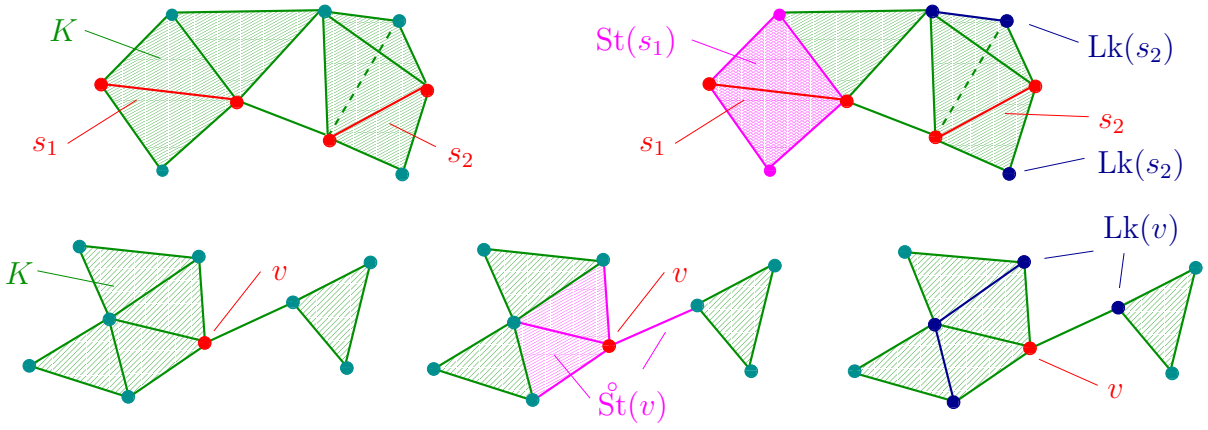
Furthermore we define the link of s to be

$$\mathrm{Lk}(K, s) := \bigcup_{\substack{t \in S \text{ with } s \cup t \in S \\ \text{but with } s \cap t = \varnothing}} |t|.$$

(2) Given a vertex v we use the obvious shorthand notations

$$\mathrm{St}(K, v) := \mathrm{St}(K, \{v\}), \quad \mathring{\mathrm{St}}(K, v) := \mathring{\mathrm{St}}(K, \{v\}) \text{ and } \mathrm{Lk}(K, v) := \mathrm{Lk}(K, \{v\}).$$

If K is understood, then we drop it from the notation. Furthermore we extend the definition in the obvious way to simplicial complexes.



Example. Let K be an abstract simplicial complex. We consider the simplicial cone $\mathrm{Cone}(K) = K * \{\star\}$ and the simplicial suspension $\Sigma(K) = K * \{\pm 1\}$ as defined on page 2003. Let $\epsilon \in \{-1, \}$. It follows easily from the definitions that $\mathrm{St}(\Sigma(K), \epsilon)$ is simplicially isomorphic to $\mathrm{Cone}(K)$ and that $\mathrm{Lk}(\Sigma(K), \epsilon) = K$.

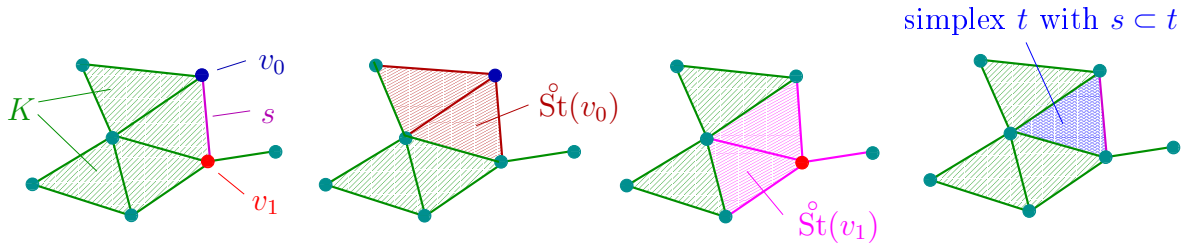
Lemma 94.5. (the Link Star Lemma) Let $(X, (K = (V,S), \Theta\colon |K| \to X))$ be a simplicial complex.

(1) For every simplex $s \in S$ the following statements hold:
  (a) We have the equality
$$\mathrm{Lk}(s) \;=\; \mathrm{St}(s) \setminus \bigcup_{r \in S \text{ with } r \subset s} \mathring{\mathrm{St}}(r).$$
  (b) The star $\mathrm{St}(s)$ and the link $\mathrm{Lk}(s)$ are subcomplexes of X. The corresponding abstract simplicial complexes are given by
$$\Big(\bigcup_{\substack{t \in S \text{ with} \\ s \subset t}} t, \{t \in S \,|\, s \cup t \in S\} \Big) \text{ and } \Big(\bigcup_{\substack{t \in S \text{ with} \\ s \subset t}} t \setminus s, \{t \in S \,|\, s \cup t \in S \text{ and } s \cap t = \varnothing\} \Big).$$
  Furthermore $X \setminus \mathring{\mathrm{St}}(s)$ is a subcomplex, the corresponding abstract simplicial complex is given by
$$(V \setminus s, \{t \in S \,|\, t \cap s = \varnothing\}.$$
  (c) The star $\mathrm{St}(s)$ is naturally simplicially isomorphic to the simplicial join of the link $\mathrm{Lk}(s)$ with the simplex s.[11541155]
  (d) (α) The open star $\mathring{\mathrm{St}}(s)$ is an open neighborhood of every point $P \in \Theta(\langle s \rangle)$, in particular $\mathring{\mathrm{St}}(s)$ is an open subset of X.
  (β) The star $\mathrm{St}(K,s)$ is a closed neighborhood of every point $P \in \Theta(\langle s \rangle)$, in particular $\mathrm{St}(K,s)$ is a closed subset of X.
  (e) (α) The star $\mathrm{St}(s)$ admits a deformation retraction to the simplex $\Theta(|s|)$.
  (β) The open star $\mathring{\mathrm{St}}(s)$ admits a deformation retraction to the open simplex $\Theta(\langle s \rangle)$.
  (f) Both $\mathrm{St}(s)$ and $\mathring{\mathrm{St}}(s)$ are contractible.
  (g) (α) The link $\mathrm{Lk}(s)$ is a deformation retract of $\mathrm{St}(s) \setminus \Theta(|s|)$.
  (β) Given any $P \in \Theta(\langle s \rangle)$ the subcomplex $\partial s * \mathrm{Lk}(s)$ is a deformation retract of $\mathrm{St}(s) \setminus \{P\}$.
  (h) If t is a simplex with $s \subset t$, then $\mathrm{St}(t) \subset \mathrm{St}(s)$.

(2) (a) Let $s \in S$ be a simplex. For $P \in \Theta(\langle s \rangle)$ and $v \in V$ we have $P \in \mathring{\mathrm{St}}(v)$ if and only if $v \in S$.
  (b) The open stars $\{\mathring{\mathrm{St}}(v)\}_{v \in V}$ defined by the vertices are an open cover of X.

(3) Given simplices $s_0, \ldots, s_n \in S$ the following equality holds:[1156]
$$\mathring{\mathrm{St}}(s_0) \cap \cdots \cap \mathring{\mathrm{St}}(s_n) \;=\; \begin{cases} \mathring{\mathrm{St}}(s_0 \cup \cdots \cup s_n), & \text{if } s_0 \cup \cdots \cup s_n \text{ is a simplex,} \\ \varnothing, & \text{otherwise.} \end{cases}$$
  (Note that in Exercise 94.1 we will see that the analogous statement for the stars does *not* hold.)

(4) Given vertices $v_0, \ldots, v_n \in V$ the following statement holds:
$$\mathring{\mathrm{St}}(v_0) \cap \cdots \cap \mathring{\mathrm{St}}(v_n) \neq \varnothing \iff \{v_0, \ldots, v_n\} \text{ is a simplex of } K = (V,S).$$



Proof. Basically by definition we can assume that we are dealing with the topological realization $X = |K|$ of an abstract simplicial complex $K = (V, S)$.

(1) Let $s \in S$ be a k-simplex.

(a) We have the following equalities:

$$\mathrm{St}(s) \setminus \bigcup_{r \subset s} \mathring{\mathrm{St}}(r) = \bigcup_{s \subset t} |t| \setminus \bigcup_{r \subset s} \bigcup_{r \subset t} \langle t \rangle \overset{\text{follows from Open Simplex Lemma 93.11 (5) applied to the subcomplex defined by } t}{=} \bigcup_{s \subset t} \bigsqcup_{u \subset t} \langle u \rangle \setminus \bigcup_{r \subset s} \bigsqcup_{r \subset u} \langle u \rangle$$

$$= \bigcup_{\substack{u \in S \text{ with } s \cup u \in S \\ \text{but with } s \cap u = \varnothing}} \langle u \rangle = \bigcup_{\substack{t \in S \text{ with } s \cup t \in S \\ \text{but with } s \cap t = \varnothing}} \bigsqcup_{u \subset t} \langle u \rangle = \underbrace{\bigcup_{\substack{t \in S \text{ with } s \cup t \in S \\ \text{but with } s \cap t = \varnothing}} |t|}_{= \mathrm{Lk}(s)}.$$

(b) It follows easily from the definitions, (a) and Lemma 93.14 (4) that the star $\mathrm{St}(s)$, that the link $\mathrm{Lk}(s)$ and that $|K| \setminus \mathring{\mathrm{St}}(s)$ are subcomplexes of X and that the corresponding abstract simplicial complexes are the ones we wrote down.

(c) This statement follows immediately from the description of the abstract simplicial complexes given in (b) and the definition of the simplicial join that we gave on page 2003.

(d) First we show that $\mathring{\mathrm{St}}(s)$ is an open subset of $|K|$. By the definition of the topology on $|K|$, see page 1984, we need to show that for every k-simplex u, with some characteristic map $\Phi_u \colon \Delta^k \to |u|$, the preimage $\Phi_u^{-1}(\mathring{\mathrm{St}}(s))$ is an open subset of Δ^k. In fact

$$\Phi_u^{-1}\big(\mathring{\mathrm{St}}(s)\big) \underset{\text{by definition}}{=} \Phi_u^{-1}\Big(\bigcup_{s \subset t} \langle t \rangle\Big) \underset{\substack{\text{Image-Preimage} \\ \text{Lemma 0.2 (6)}}}{=} \bigcup_{s \subset t} \big(\Phi_u^{-1}(\langle u \rangle)\big) \underset{\text{Open Simplex Lemma 93.11 (2)}}{=} \begin{cases} \Phi_u^{-1}(\langle t \rangle) = \Delta^k \setminus \partial \Delta^k, & \text{if } s \subset u \\ \varnothing, & \text{else.} \end{cases}$$

Since $\Delta^k \setminus \partial \Delta^k$ and $\varnothing$ are open subsets of Δ^k we are done. A very similar argument also shows that $\mathring{\mathrm{St}}(s)$ is a closed subset of $|K|$.

Now let $P \in \Theta(\langle s \rangle)$. By definition $P \in \mathring{\mathrm{St}}(s) \subset \mathrm{St}(s)$. Since $\mathring{\mathrm{St}}(s)$ is an open subset of $|K|$ we see that $\mathring{\mathrm{St}}(s)$ and $\mathrm{St}(s)$ are both neighborhoods of P.

[1154] Here we view $\mathrm{St}(s)$, $\mathrm{Lk}(s)$ and s as simplicial complexes.

[1155] In Exercise 94.2 we will prove the related statement that there exists a natural simplicial isomorphism $\mathrm{Lk}(K, s) * \partial s \cong \mathrm{Lk}(\mathrm{sd}(K), \underline{s})$.

[1156] To avoid confusion we point out that here, and in (4), we do not demand that the $s_0, \ldots, s_n$ are distinct.

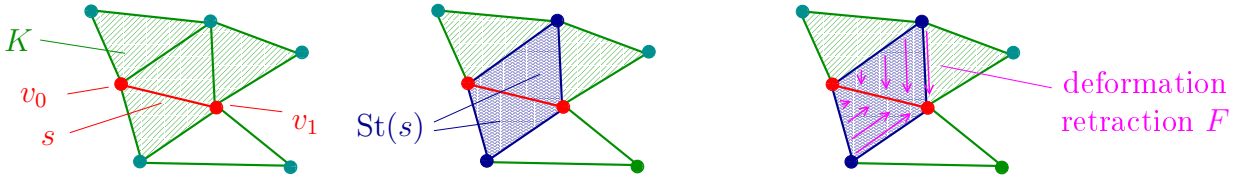
(e) (α) We write $s = \{v_0, \dots, v_k\}$. Note that by (1) and Lemma 93.8 we can view $\mathrm{St}(s)$ as a simplicial complex in its own right. Now let t be a simplex t with $s \subset t$. We write $t = \{v_0, \dots, v_k, v_{k+1}, \dots, v_l\}$. We consider the map

$$\begin{array}{rcl} F_t \colon |t| \times [0,1] & \to & |t| \\ \left(\sum\limits_{i=0}^{l} \lambda_i \cdot v_i, r\right) & \mapsto & \sum\limits_{i=0}^{k} \lambda_i \cdot v_i + (1-r) \cdot \sum\limits_{i=k+1}^{l} \lambda_i \cdot v_i + r \cdot \left(\sum\limits_{i=k+1}^{l} \lambda_i\right) \cdot v_k. \end{array}$$

It follows immediately from the definitions that these maps F_t combine to give a well-defined map $F \colon \mathrm{St}(s) \times [0,1] \to \mathrm{St}(s)$. Furthermore it is fairly straightforward to deduce from the fact that $\mathrm{St}(s)$ is a simplicial complex, the Simplicial-Implies-CW Lemma 93.24 and the Intervall CW-Product Lemma 68.8 (3) that F is actually continuous. Now that continuity is out of the way it is clear that F is a deformation retraction from $\mathrm{St}(s)$ to $|s|$.

(β) In (α) we just gave an explicit deformation retraction from $\mathrm{St}(s)$ to $|s|$. It follows basically from the definitions that this map restricts to a deformation retraction from $\mathring{\mathrm{St}}(s)$ to the open simplex $\langle s \rangle$.

(f) This statement follows immediately from (e), together with the Simplex-Embedding Lemma 93.5 (3) and the observation that Δ^k and $\mathring{\Delta}^k$ are contractible.



(g) Let $P \in \Theta(\langle s \rangle)$. The proofs that the link $\mathrm{Lk}(s)$ is a deformation retract of $\mathrm{St}(s) \backslash |s|$ and that the subcomplex $\partial s * \mathrm{Lk}(s)$ is a deformation retract of $\mathrm{St}(s) \backslash \{P\}$ is quite similar to the proof of (e). We will fill in the details in Exercise 94.4.

(h) If t is a simplex with $s \subset t$, then it follows immediately from the definitions that $\mathrm{St}(t) \subset \mathrm{St}(s)$.

(2) (a) Let $s \in S$ be a simplex, let $P \in \langle s \rangle$ and let $v \in V$. We have

$$P \in \mathring{\mathrm{St}}(v) = \bigcup_{t \in S \text{ with } v \in t} \langle t \rangle \iff \langle s \rangle \subset \bigcup_{t \in S \text{ with } v \in t} \langle t \rangle \iff v \in s.$$

both statements follow from the Open Simplex Lemma 93.11 (2)

(b) Recall that the Open Simplex Lemma 93.11 (2) says that given any point $P \in K$ there exists an $s \in S$ with $P \in \langle s \rangle$. By (a) we know that for any vertex v of s we have $P \in \langle s \rangle \subset \mathring{\mathrm{St}}(v)$. This shows that the open stars corresponding to the vertices cover all of K. By (1d) we know that this cover is in fact an open cover.

(3) Let $s_0, \dots, s_n \in S$. We have the following equality:

$$\bigcap_{i=0}^{n} \mathring{\mathrm{St}}(s_i) \overset{\text{by definition}}{=} \bigcap_{i=0}^{n} \bigcup_{t \in S \text{ with } s_i \subset t} \langle t \rangle \overset{\text{Open Simplex Lemma 93.11 (5)}}{=} \bigcup_{t \in S \text{ with } s_0, \dots, s_n \subset t} \langle t \rangle \overset{\text{by definition of an open star}}{=} \begin{cases} \mathring{\mathrm{St}}(s_0 \cup \dots \cup s_n), & \text{if } s_0 \cup \dots \cup s_n \in S, \\ \varnothing, & \text{otherwise.} \end{cases}$$

(4) Given $v_0, \dots, v_n \in V$ the following holds:

$$\mathring{\mathrm{St}}(v_0) \cap \dots \cap \mathring{\mathrm{St}}(v_n) \neq \varnothing \quad \overset{\text{by (3)}}{\uparrow} \quad \Leftrightarrow \quad \begin{array}{c}\text{there exists a } t \in S \\ \text{with } v_0, \dots, v_n \in t\end{array} \quad \overset{\text{by definition of an abstract simplicial complex}}{\uparrow} \quad \Leftrightarrow \quad \{v_0, \dots, v_n\} \text{ is a simplex.}$$

■

Definition. An abstract simplicial complex $K = (V, S)$ is called locally finite if given any vertex $v \in V$ there exist only finitely many simplices that contain v.

The following proposition can be viewed as an analogue of Proposition 68.11.

Proposition 94.6. Let $K = (V, S)$ be an abstract simplicial complex. The following three statements are equivalent:

(1) The abstract simplicial complex K is locally finite.
(2) For every vertex $v \in V$ the star $\mathrm{St}(K, \{v\})$ is a finite simplicial complex.
(3) For every vertex $v \in V$ the link $\mathrm{Lk}(K, \{v\})$ is a finite simplicial complex.
(4) The topological realization $|K|$ is regionally compact.

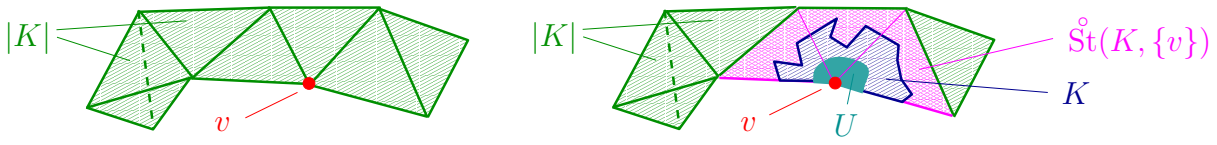
Proof. Let $K = (V, S)$ be an abstract simplicial complex.

We start out with (1) ⇔ (2) ⇔ (3). This equivalence of statements follows almost immediately from the definitions and the observation that an abstract simplicial complex is finite if and only if it has finitely many vertices.

We continue with (3) ⇒ (4). We need to show that $|K|$ is regionally compact. By Lemma 2.41 (2) together with the Simplex-Embedding Lemma 93.5 (2) it suffices to show that every point $P \in |K|$ admits a compact neighborhood. By the Open Simplex Lemma 93.11 there exists a simplex $s \in S$ such that P is contained in the open simplex $\langle s \rangle$. By the Link Star Lemma 94.5 (1d) we know that $\mathrm{St}(K, s)$ is a neighborhood of P. Since $\mathrm{St}(K, s)$ is a finite simplicial complex we obtain from Lemma 93.9 that $\mathrm{St}(K, s)$ is compact.

Finally we prove (4) ⇒ (3). Now we assume that $|K|$ is regionally compact. Let $v \in V$ be a vertex. By the Link Star Lemma 94.5 (1d) we know that the open star $\mathring{\mathrm{St}}(K, \{v\})$ is an open neighborhood of v. Since $|K|$ is regionally compact there exists a compact neighborhood X of v with $X \subset \mathring{\mathrm{St}}(K, \{v\})$. Since X is compact we know by the Simplicial Complex-Finiteness Proposition 93.12 that there exists a finite subcomplex L of K such that X is contained in $|L|$.

Since X is a neighborhood of v we know that there exists an open neighborhood U of v with $v \in U \subset X$. It follows from Open Simplex Lemma 93.11 (7) that for any simplex s with $v \in s$ we have $U \cap \langle s \rangle \neq \varnothing$. But by the Open Simplex Lemma 93.11 (5) that implies $|s| \subset |L|$. In other words, we see that $\mathrm{St}(K, \{v\}) \subset |L|$. Since L is finite we see that $\mathrm{St}(K, \{v\})$ is a finite simplicial complex. ■



We recall the following definition from page 285.

Definition. The diameter of a non-empty subset A of some $\mathbb{R}^n$ is defined as

$$\operatorname{diam}(A) \;:=\; \sup\{\|a-b\| \,|\, a,b \in A\} \;\in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

It is pretty clear that the barycentric subdivision of a linear simplicial complex results in more but smaller simplices. The following lemma is one way of formulating precisely that the simplices shrink.

Lemma 94.7. If X is a finite linear non-empty simplicial complex in $\mathbb{R}^n$, then

$$\lim_{k\to\infty} \max\{\operatorname{diam}(\underbrace{\operatorname{St}(\operatorname{sd}^k(X), v)}_{\uparrow}) \,|\, v \text{ is a vertex of } \operatorname{sd}^k(X)\} \;=\; 0.$$

underlying topological space of X and $\operatorname{sd}^k(X)$ is the same



the maximal diameter of a star shrinks

Proof. Let Y be a finite linear non-empty d-dimensional simplicial complex in $\mathbb{R}^n$. We define the *mesh of* Y as

$$\operatorname{mesh}(Y) \;:=\; \max\{\operatorname{diam}(s) \,|\, s \text{ is a simplex of } Y\}.$$

The lemma follows from the SD-Lemma 94.3 (1) and (2), Lemma 94.4 and the following claim.

Claim.

(1) For every vertex v of Y we have $\operatorname{diam}(\operatorname{St}(v)) \leq 2 \cdot \operatorname{mesh}(Y)$.
(2) For every $k \in \mathbb{N}_0$ we have

$$\operatorname{mesh}(\operatorname{sd}^k(Y)) \;\leq\; \left(\tfrac{d}{d+1}\right)^k \cdot \operatorname{mesh}(Y).$$

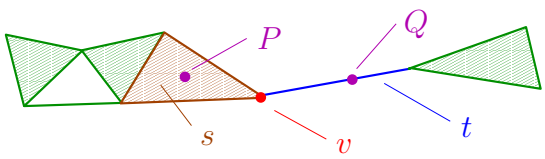
Proof.

(1) Let v be a vertex of Y. We need to show that $\operatorname{diam}(\operatorname{St}(v)) \leq 2 \cdot \operatorname{mesh}(Y)$. Thus let $P, Q \in \operatorname{St}(v)$. By definition of the star $\operatorname{St}(v)$ there exist simplices s and t with $v, P \in s$ and with $v, Q \in t$. We see that

$$\|P-Q\| \underset{\uparrow}{\leq} \|P-v\| + \|Q-v\| \underset{\uparrow}{\leq} \operatorname{diam}(s) + \operatorname{diam}(t) \underset{\uparrow}{\leq} 2\operatorname{mesh}(Y).$$

triangle inequality; since $v, P \in s$ and $v, Q \in t$; definition of mesh

(2) Once the reader has untangled the definitions it will have become clear that this statement is nothing but a reformulation of Lemma 74.28. ∎



94.3. **The Simplicial Approximation Theorem.** In this section we will state and prove the Simplicial Approximation Theorem which is one of the main tools in the study of simplicial complexes. Before we can state the theorem we need to introduce one last basically self-explanatory notion.

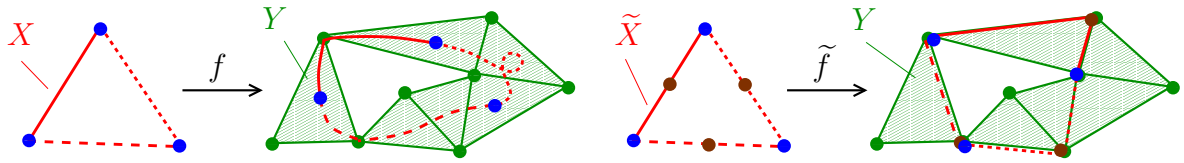
Definition. Let X be an (abstract) simplicial complex and let A be a subcomplex. We refer to (X, A) as a pair of (abstract) simplicial complexes.

Now we are ready to formulate the main theorem of this section.

Theorem 94.8. (Simplicial Approximation Theorem) Let (X, A) and (Y, B) be pairs of simplicial complexes. Let $f\colon (X, A) \to (Y, B)$ be a map of pairs of topological spaces. If X is finite, then there exists an iterated barycentric subdivision $(\widetilde{X}, \widetilde{A})$ of (X, A) and a simplicial map $\widetilde{f}\colon (\widetilde{X}, \widetilde{A}) \to (Y, B)$ with the following properties:

(1) The map $\widetilde{f}$ is homotopic to f as a map $(X, A) \to (Y, B)$ of pairs of topological spaces.
(2) For every vertex v of $\widetilde{X}$ we have $f(\mathrm{St}(\widetilde{X}, v)) \subset \mathrm{St}(Y, \widetilde{f}(v))$.

Example. We consider the simplicial complexes X and Y and the map $f\colon X \to Y$ that are illustrated in the figure below. This map is actually *not* homotopic to a simplicial map.[1157] But after a single barycentric subdivision we see that f is homotopic to a simplicial map $\widetilde{f}\colon \mathrm{sd}(X) \to Y$.



Remark.

(1) The first statement of the Simplicial Approximation Theorem 94.8 is of course somewhat similar to the statement of the Cellular Approximation Theorem 70.15. But since simplicial complexes and simplicial maps are more rigid than their cellular siblings there are some important differences:
   (a) On the negative side the Simplicial Approximation Theorem 94.8 only applies to finite domains and we need to allow for several barycentric subdivisions.
   (b) On the plus side the output is much stronger since simplicial maps are much more restrictive than cellular maps. We will make use of this shortly.
(2) The second statement of the Simplicial Approximation Theorem 94.8 essentially says that the new map $\widetilde{f}$ does not stray too far away from the original map f.

Before we give a proof of the Simplicial Approximation Theorem 94.8, let us point out an important consequence. On page 1844 we saw that homotopy groups of fairly simple topological spaces, e.g. $S^1 \vee S^2$, can be uncomfortably large, for example we saw that $\pi_2(S^1 \vee S^2)$ is not necessarily finitely generated. In Question 71.16 we asked whether the homotopy groups of a countable CW-complex are actually countable. The following proposition gives us some evidence for a positive answer.

[1157] Why not?

Proposition 94.9. If X is a simplicial complex with countably many simplices, then for every $n \in \mathbb{N}$ and every $x_0 \in X$ the group $\pi_n(X, x_0)$ is countable.

Proof. Let X be a simplicial complex and let $n \in \mathbb{N}$. By the discussion on page 1997 we can pick a finite simplicial structure for S^n. Let $*$ be a vertex of S^n. By the Change-of-Base Point Proposition 71.5 it suffices to show that for a single vertex x_0 of X the group $\pi_n(X, x_0)$ is countable.

Next note that by the Simplicial Approximation Theorem 94.8, applied to the maps $(S^n, \{*\}) \to (X, x_0)$ we have a surjection

$$\bigcup_{i \in \mathbb{N}_0} \underbrace{\{\text{set of simplicial maps } (\mathrm{sd}^i(S^n), \{*\}) \to (X, x_0)\}}_{\text{countable set by Lemma 93.3 (2)}} \twoheadrightarrow \underbrace{[(S^n, *), (X, x_0)]}_{\substack{\text{set of homotopy classes} \\ \text{of maps } (S^n, *) \to (X, x_0)}} = \pi_n(X, x_0).$$

By Lemma 93.3 (2) together with the Countability Lemma 0.8 (3) the set on the left-hand side is countable. It follows from the Countability Lemma 0.8 (3) that the set on the right-hand side, i.e. $\pi_n(X, x_0)$, is also countable. ■

It is not immediately clear how one could possibly generalize the proof of Proposition 94.9 from simplicial complexes to CW-complexes. In particular at the moment we still lack the tools to tackle Question 71.16 completely. We will return to Question 71.16 with Proposition 104.21.

Before we turn to the proof of the Simplicial Approximation Theorem 94.8 let us record the following immediate corollary to Proposition 94.9.

Corollary 94.10. For every $n, k \in \mathbb{N}$ the group $\pi_n(S^k)$ is countable.

Now we turn to the proof of the Simplicial Approximation Theorem 94.8.

Proof of the Simplicial Approximation Theorem 94.8. In an attempt to keep the notation at a manageable level we will assume for simplicity that $A = B = \varnothing$. We leave it to the reader to modify the subsequent proof to provide a proof for the general case.

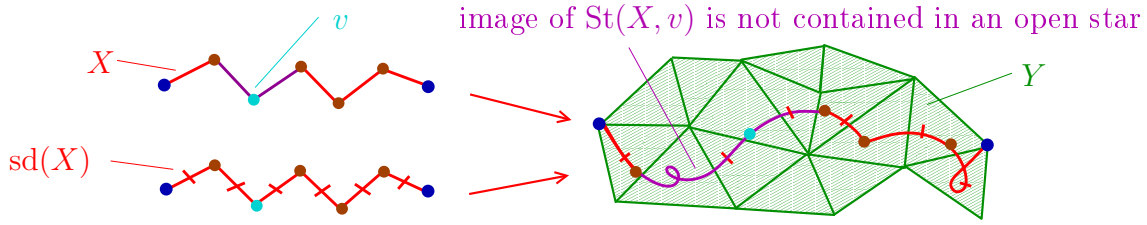
Thus let $(X, (K = (V, S), \Theta\colon |K| \to X))$ and $(Y, (L = (W, T), \Omega\colon |L| \to Y))$ be two simplicial complexes. For convenience we identify V with its image $\Theta(V) \subset X$ and we identify W with its image $\Omega(W) \subset Y$. Furthermore let $f\colon X \to Y$ be a continuous map. We make the following preparations.

(1) Recall that we assume that X is finite. Evidently we can assume that X is non-empty.
(2) It follows from the hypothesis that X is finite and the combination of Lemma 93.9, the Compact Image Lemma 2.13 and the Simplicial Complex-Finiteness Proposition 93.12 that $f(X)$ is contained in a finite subcomplex of Y. Thus we can assume that Y is also a finite simplicial complex.
(3) Since X and Y are finite it follows from Lemma 93.22 that we can and will assume that X and Y are linear simplicial complexes contained in some $\mathbb{R}^m$ respectively some $\mathbb{R}^n$. Recall that this means, by definition, that X and Y are equipped with the subspace topology coming from $\mathbb{R}^m$ and $\mathbb{R}^n$. In particular we can use the usual Euclidean metric on $\mathbb{R}^m$ to view X as a metric space.

We start out with the following claim.

Claim 1. There exists a $k \in \mathbb{N}_0$ such that for every vertex v of $\mathrm{sd}^k(X)$ there exists a $w \in W$ with $f(\mathrm{St}(\mathrm{sd}^k(X), v)) \subset \mathring{\mathrm{St}}(Y, w)$.

Proof. Recall that by the Link Star Lemma 94.5 (2) we know that the open stars $\{\mathring{\mathrm{St}}(w)\}_{w \in W}$, form an open cover of Y. It follows that the sets $U_w := f^{-1}(\mathring{\mathrm{St}}(w)) \subset X$ form an open cover of X. Since X is compact we obtain from the Lebesgue Lemma 6.5 that there exists an $\epsilon > 0$ such that for every subset A of X with $\mathrm{diam}(A) < \epsilon$ there exists a $w \in W$ with $A \subset U_w$. By Lemma 94.7 there exists a $k \in \mathbb{N}_0$ such that for every vertex v of $\mathrm{sd}^k(X)$ we have $\mathrm{diam}(\mathrm{St}(\mathrm{sd}^k(X), v)) < \epsilon$. In other words, this k has the desired property. ⊞



By the SD-Lemma 94.3 (1) and Lemma 94.4 we know that $\mathrm{sd}^k(X)$ is again a finite linear simplicial complex in $\mathbb{R}^m$. Thus we can safely replace X by $\mathrm{sd}^k(X)$. We are now in the following setting:

(∗) For every $v \in V$ there exists a $w \in W$ with $f(\mathrm{St}(X, v)) \subset \mathring{\mathrm{St}}(Y, w)$. In other words, there exists a map $\varphi\colon V \to W$ with the property that for every vertex $v \in V$ we have $f(\mathrm{St}(X, v)) \subset \mathring{\mathrm{St}}(Y, \varphi(v))$.

We continue the proof with the following claim.

Claim 2. For every simplex $s \in S$ the image $\varphi(s)$ is a simplex in T.

Proof. Let $s = \{v_0, \ldots, v_k\}$ be a simplex of $K = (V, S)$. Note that

$$
\begin{array}{ll}
 & \text{Link Star Lemma 94.5 (4)} \\
 & \downarrow \\
v_0, \ldots, v_k \text{ form a simplex in } X & \Rightarrow \mathring{\mathrm{St}}(v_0) \cap \ldots \cap \mathring{\mathrm{St}}(v_k) \neq \varnothing \quad \Rightarrow f\big(\mathring{\mathrm{St}}(v_0) \cap \ldots \cap \mathring{\mathrm{St}}(v_k)\big) \neq \varnothing \\
 & \Rightarrow f(\mathring{\mathrm{St}}(v_0)) \cap \ldots \cap f(\mathring{\mathrm{St}}(v_k)) \neq \varnothing \quad \Rightarrow \mathring{\mathrm{St}}(\varphi(v_0)) \cap \ldots \cap \mathring{\mathrm{St}}(\varphi(v_k)) \neq \varnothing \\
 & \uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\
 & \text{Image-Preimage Lemma 0.2 (4)} \qquad\qquad\qquad\qquad\qquad\qquad \text{by choice of } \varphi \text{ we have } f(\mathrm{St}(v_i)) \subset \mathring{\mathrm{St}}(\varphi(v_i)) \\
 & \Rightarrow \{\varphi(v_0), \ldots, \varphi(v_k)\} \text{ is a simplex in } T. \\
 & \uparrow \\
 & \text{Link Star Lemma 94.5 (4)}
\end{array}
$$

⊞

Claim 2 says that $\varphi\colon V \to W$ defines a simplicial map $K = (V, S) \to L = (W, T)$. It follows from Lemma 93.15 that there exists a simplicial map $g\colon X \to Y$ with $g|_V = \varphi$. We claim that g is the desired map.

First note that it follows immediately from the construction of φ and g that for any vertex v of $\mathrm{sd}(X)$ we have $f(\mathrm{St}(X, v)) \subset \mathrm{St}(Y, g(v))$. Thus it remains to show that g is in fact homotopic to our original map f. To do so we consider the usual interpolation between the maps f and g:[1158]

$$
\begin{array}{rcl}
F\colon X \times [0,1] & \to & \mathbb{R}^n \\
(x, t) & \mapsto & f(x) \cdot (1 - t) + g(x) \cdot t.
\end{array}
$$

[1158] Note that we showed in the Convex Homotopy Lemma 14.1 that this map is actually continuous.

It remains to show the map F actually takes values in the subset $Y \subset \mathbb{R}^n$. For each $x \in X$ we know that $f(x) \in Y$ and $g(x) \in Y$. We need to show that for each $t \in [0,1]$ the point $f(x) \cdot (1-t) + g(x) \cdot t$ also lies in Y. In our setting this follows from the observation that simplices are convex, see Lemma 93.23 (1b), together with the following claim.

Claim 3. For every $x \in X$ there exists a $t \in T$ such that both $f(x)$ and $g(x)$ are contained in $\Omega(|t|)$.

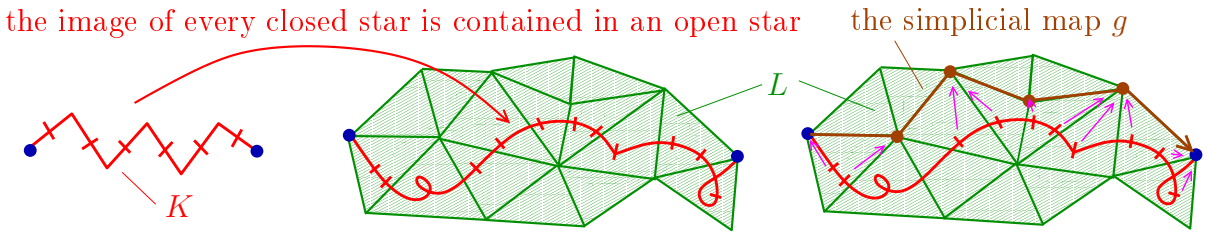
Proof. Now let $x \in X$. By the Open Simplex Lemma 93.11 (2) there exists a unique simplex $s = \{v_0, \ldots, v_k\}$ such that $x \in \Theta(\langle s \rangle)$. We see that

$$
\begin{array}{l}
\qquad\qquad\qquad\qquad \text{by the Link Star Lemma 94.5 (3), since } s = \{v_0, \ldots, v_k\} \\
x \in \Theta(\langle s \rangle) \;\overset{\downarrow}{\Rightarrow}\; x \in \mathring{\mathrm{St}}(v_0) \cap \ldots \cap \mathring{\mathrm{St}}(v_k) \;\Rightarrow\; f(x) \in f\big(\mathring{\mathrm{St}}(v_0) \cap \ldots \cap \mathring{\mathrm{St}}(v_k)\big) \\
\quad \underset{\uparrow}{\Rightarrow}\; f(x) \in f\big(\mathring{\mathrm{St}}(v_0)\big) \cap \ldots \cap f\big(\mathring{\mathrm{St}}(v_k)\big) \;\underset{\uparrow}{\Rightarrow}\; f(x) \in \mathring{\mathrm{St}}(\varphi(v_0)) \cap \ldots \cap \mathring{\mathrm{St}}(\varphi(v_k)) \\
\quad \text{Image-Preimage Lemma 0.2 (4)} \qquad\qquad \text{by choice of } \varphi \text{ we have } f(\mathrm{St}(v_i)) \subset \mathring{\mathrm{St}}(\varphi(v_i)) \\
\quad \underset{\uparrow}{\Rightarrow} \text{ there exists a } t \in T \text{ which contains } \varphi(s) = \{\varphi(v_0), \ldots, \varphi(v_k)\} \text{ and with } f(x) \in \Omega(\langle t \rangle). \\
\quad \text{Link Star Lemma 94.5 (3)}
\end{array}
$$

It remains to show that $g(x) \in \Omega(|t|)$. In fact we see that

$$
\begin{array}{ccccccc}
x \in \Theta(\langle s \rangle) & \underset{\uparrow}{\Rightarrow} & x \in \Theta(|s|) & \underset{\uparrow}{\Rightarrow} & g(x) \in \Omega(|\varphi(s)|) & \underset{\uparrow}{\Rightarrow} & g(x) \in \Omega(|t|). \\
\text{by definition of } \langle s \rangle \text{ and } |s| & & & \text{follows from the discussion on page 1987} & & \text{since } \varphi(s) \subset t &
\end{array}
$$

We have thus found the desired simplex t. ∎



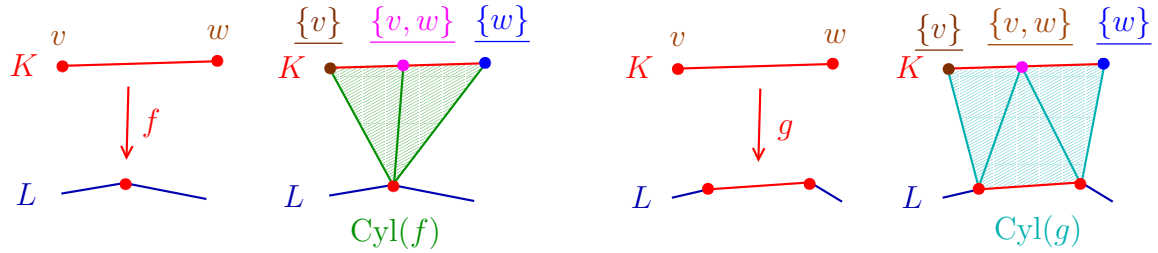
94.4. **Mapping cylinders and mapping cones of simplicial maps.** In this section we will see that there exist simplicial analogues of the mapping cylinder and the mapping cone that we introduced on page 454 and 457. The definitions of these simplicial analogues that we give are taken from [**Kas2014**].

Definition. Let $f\colon K \to L$ be a simplicial map between two abstract simplicial complexes. The simplicial mapping cylinder $\mathrm{Cyl}(f\colon K \to L)$ is the abstract simplicial complex that is given by the following data:

(1) The vertex set is the disjoint union of the vertex set of the barycentric subdivision $\mathrm{sd}(K)$ of K and the vertex set of L.

(2) The set of simplices is given by

$$\left\{ \sigma \sqcup \tau \;\middle|\; \begin{array}{c} \sigma = \varnothing \text{ or } \sigma \text{ a simplex of } \mathrm{sd}(K) \;\&\; \tau = \varnothing \text{ or } \tau \text{ a simplex of } L \text{ or such that} \\ \text{for every vertex } v = \underset{\uparrow}{\underline{s}} \in \sigma \text{ and every vertex } w \in \tau \text{ we have } w \in f(s) \\ \text{where } s \text{ is a simplex of } K \end{array} \right\}.$$



Let $f\colon K \to L$ be a simplicial map between two finite abstract simplicial complexes. One might expect that the next lemma states, say under the hypothesis that K is finite, that the topological realization $|\operatorname{Cyl}(f\colon K \to L)|$ of the simplicial mapping cylinder is homeomorphic to the mapping cylinder $\operatorname{Cyl}(|f|\colon |K| \to |L|)$ of the map between the topological realizations. We will prove a weaker statement, but which nonetheless says that the simplicial mapping cylinder has all the essential properties we really need.

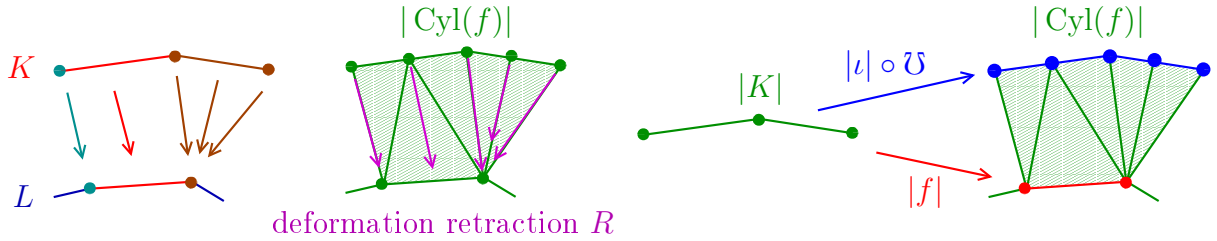
Lemma 94.11. (Simplicial Mapping Cylinder Lemma) Let $f\colon K \to L$ be a simplicial map between two abstract simplicial complexes.

(1) The simplicial mapping cylinder $\operatorname{Cyl}(f\colon K \to L)$ contains $\mathrm{sd}(K)$ and L as subcomplexes.

We now assume that K is actually a finite abstract simplicial complex. We denote by $\iota\colon \mathrm{sd}(K) \to \operatorname{Cyl}(f\colon K \to L)$ the natural inclusion and we denote by $\mho\colon |K| \to |\mathrm{sd}(K)|$ the natural homeomorphism from the SD-Lemma 94.3.

(2) There exists a deformation retraction $R\colon |\operatorname{Cyl}(f)| \times [0,1] \to |\operatorname{Cyl}(f)|$ from $|\operatorname{Cyl}(f)|$ to the subset $|L|$ such that $R_1 \circ \iota \circ \mho = f\colon |K| \to |\operatorname{Cyl}(f)|$.

(3) The map $|f|\colon |K| \to |\operatorname{Cyl}(f)|$ is homotopic to the map $|\iota| \circ \mho\colon |K| \to |\operatorname{Cyl}(f)|$.



Remark. It is not clear to me whether the statement of the Simplicial Mapping Cylinder Lemma 94.11 (2) also holds if K is infinite. For example one might be able to get stronger results using Lemma 70.13. But for our purposes it is good enough to know that the statement holds if K is finite.

Sketch of proof. Let $f\colon K \to L$ be a simplicial map between two abstract simplicial complexes.

(1) By taking $\tau = \varnothing$ respectively $\sigma = \varnothing$ one sees immediately that $\mathrm{sd}(K)$ and L are subcomplexes of $\mathrm{Cyl}(f\colon K \to L)$.

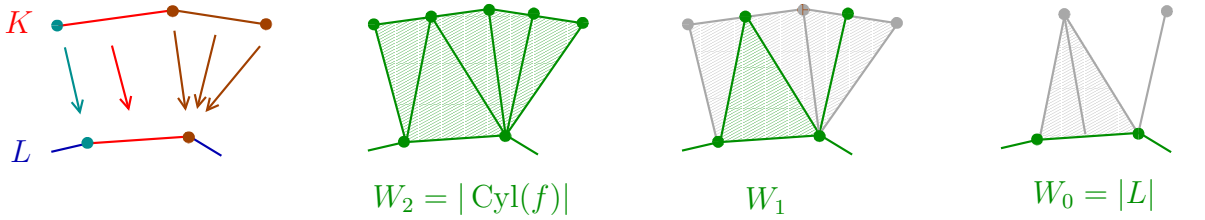
We now assume that K is actually a finite abstract simplicial complex.

(2) Given $r \in \mathbb{N}_0$ we denote by W_r the topological realization of the maximal subcomplex of $\mathrm{Cyl}(f\colon K \to L)$ that contains all vertices of L and that contains all vertices $\mathrm{sd}(K)$ of the form $\underline{s}$ where s is a $\geq r$-simplex of K. We leave it to the reader to verify that for each $r \in \mathbb{N}_0$ there exists a unique homotopy

$$F_r\colon W_r \times [0,1] \;\to\; W_r$$

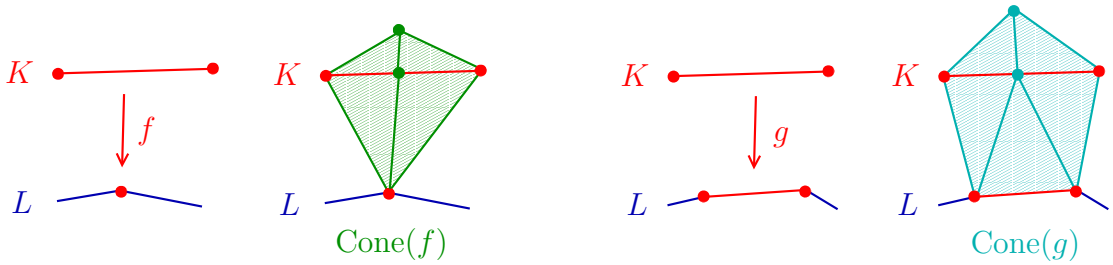
with the following properties:
(a) F_r is a homotopy rel W_{r-1}.
(b) Given any vertex $\underline{s}$ that lies in W_r but that does not lie in W_{r-1} the homotopy is given by $F_r(\underline{s},t) = \underline{s}\cdot(1-t) + \underline{f(s)}\cdot t$.
(c) The restriction of F_r to each simplex takes values in a simplex and it is linear.
Note that F_r is in particular a deformation retraction from W_r to W_{r-1}. The composition of the homotopies $F_{\dim(K)},\dots,F_1$ and F_0 has the desired properties.

(3) This statement follows immediately from (2). ∎



We conclude this section with the following definition.

Definition. Let $f\colon K \to L$ be a simplicial map between two abstract simplicial complexes. The simplicial mapping cone $\mathrm{Cone}(f\colon K \to L)$ is defined as the union of the simplicial mapping cylinder $\mathrm{Cyl}(f\colon K \to L)$ and the simplicial cone $\mathrm{Cone}(\mathrm{sd}(K))$ that we defined on page 2003.



In Section 94.5 we will see that the concept of a simplicial mapping cone is indeed useful.

94.5. **Simplicial structures and CW-structures II.** Recall that in the Simplicial-Implies-CW Lemma 93.24 we saw that every (ordered) simplicial complex admits a (natural) CW-complex structure where given any $n \in \mathbb{N}$ the n-simplices of the simplicial structure are precisely the n-cells of the CW-structure. The converse of this statement does not hold,

in fact in Proposition 93.28 we showed that there exists a CW-complex that does not admit a simplicial structure.
Nonetheless, in this section we will prove a partial converse to the Simplicial-Implies-CW Lemma 93.24, namely we will show that every CW-complex is *homotopy equivalent* to a simplicial complex. More precisely, we will prove the following proposition.

Proposition 94.12.

(1) Every finite CW-complex is homotopy equivalent to a finite simplicial complex of the same dimension.
(2) Every countable CW-complex is homotopy equivalent to a countable simplicial complex of the same dimension.
(3) Every CW-complex is homotopy equivalent to a simplicial complex of the same dimension.

Remark. Alternative proofs for Proposition 94.12 are given in [**Hat2002**, Theorem 2C.5], [**LW1969**, Theorem 6.1], [**WhdJ1949b**, Theorem 13] and [**Gra1975**, Corollary 16.44].

Most of the work in the proof of Proposition 94.12 is done in the following lemma.

Lemma 94.13. Let X be a CW-complex that is homotopy equivalent to the topological realization of an abstract simplicial complex K. Furthermore let $\varphi\colon S^{k-1} \to X$ be a map. There exists an $i \in \mathbb{N}_0$ a simplicial map $f\colon \mathrm{sd}^i(S_{k-1}) \to K$ such that $X \cup_\varphi \overline{B}^k$ is homotopy equivalent to the topological realization of the simplicial mapping cone $\mathrm{Cone}(f\colon \mathrm{sd}^i(S_{k-1}) \to K)|$.

The key ingredient in the proof of Lemma 94.13 is the Homotopy Pushout Theorem which we recall for the reader's convenience.

Theorem 17.19. (Homotopy Pushout Theorem) Suppose we are given the following commutative diagram of maps between topological spaces:

$$\begin{array}{ccccc} Y & \xleftarrow{f} & A & \xhookrightarrow{i} & X \\ {\scriptstyle\varphi_Y}\downarrow & & \downarrow{\scriptstyle\varphi_A} & & \downarrow{\scriptstyle\varphi_X} \\ Y' & \xleftarrow{f'} & A' & \xhookrightarrow{i'} & X'. \end{array}$$

If the vertical maps are homotopy equivalences and if the maps i and i' to the right are closed cofibrations, then the induced map

$$Y \cup_A X \;\to\; Y' \cup_{A'} X'$$

between the pushouts is a homotopy equivalence.

Proof of Lemma 94.13. We start out with the following preparations:

(1) By the discussion on page 1989 there exists a homeomorphism $g\colon |D_k| \to \overline{B}^k$ which restricts to a homeomorphism $g\colon |S_{k-1}| \to S^{k-1}$.
(2) Given any abstract simplicial complex L and given any $i \in \mathbb{N}_0$ we always denote by $\mho\colon |L| \to |\mathrm{sd}^i(L)|$ the homeomorphism from the SD-Lemma 94.3 (3).
(3) By the Simplicial Approximation Theorem 94.8 there exists an $i \in \mathbb{N}_0$, a simplicial map $f\colon \mathrm{sd}^i(S_{k-1}) \to K$ and a homotopy $F\colon |\mathrm{sd}^i(S_{k-1})| \times [0,1] \to |K|$ such that $F_0 = \theta \circ \varphi \circ g \circ \mho^{-1}$ and such that $F_1 = |f|$.

(4) We denote by $j\colon |K| \to |\operatorname{Cyl}(f\colon \operatorname{sd}^i(S_{k-1}) \to K)|$ the inclusion map. Note that by the Simplicial Mapping Cylinder Lemma 94.11 (2) we know that j is a homotopy equivalence.

(5) Finally note that it follows immediately from the Simplicial Mapping Cylinder Lemma 94.11 (3) that there exists a homotopy $G\colon |\operatorname{sd}^i(S_{k-1})| \times [0,1] \to |\operatorname{Cyl}(f\colon \operatorname{sd}^i(S_{k-1}) \to K))|$ with $G_0 = j \circ |f|$ and such that $G_1 = |\iota| \circ \mho$ where $\iota\colon \operatorname{sd}^{i+1}(S_{k-1}) \to \operatorname{Cyl}(f\colon \operatorname{sd}^i(S_{k-1}) \to K)$ is the natural inclusion.

We consider the following commutative diagram:

$$
\begin{array}{ccccc}
X & \xleftarrow{\varphi} & S^{k-1} & \hookrightarrow & \overline{B}^k \\
\downarrow{\scriptstyle \theta} & & \cong\downarrow{\scriptstyle g^{-1}} & & \cong\downarrow{\scriptstyle g^{-1}} \\
|K| & \xleftarrow{\theta\circ\varphi\circ g} & |S_{k-1}| & \hookrightarrow & |\operatorname{Cone}(S_{k-1})| \\
\downarrow{\scriptstyle \mathrm{id}} & & \cong\downarrow{\scriptstyle \mho} & & \cong\downarrow{\scriptstyle \mho} \\
|K| & \xleftarrow{\theta\circ\varphi\circ g\circ\mho^{-1}} & |\operatorname{sd}^i(S_{k-1})| & \hookrightarrow & |\operatorname{Cone}(\operatorname{sd}^i(S_{k-1}))| \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle x\mapsto(x,0)} & & \downarrow{\scriptstyle x\mapsto(x,0)} \\
|K| & \xleftarrow{F} & |\operatorname{sd}^i(S_{k-1})| \times [0,1] & \hookrightarrow & |\operatorname{Cone}(\operatorname{sd}^i(S_{k-1}))| \times [0,1] \\
\uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle x\mapsto(x,1)} & & \uparrow{\scriptstyle x\mapsto(x,1)} \\
|K| & \xleftarrow{|f|} & |\operatorname{sd}^i(S_{k-1})| & \hookrightarrow & |\operatorname{Cone}(\operatorname{sd}^i(S_{k-1}))| \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
|\operatorname{Cyl}(f)| & \xleftarrow{j\circ|f|} & |\operatorname{sd}^i(S_{k-1})| & \hookrightarrow & |\operatorname{Cone}(\operatorname{sd}^i(S_{k-1}))| \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle x\mapsto(x,0)} & & \downarrow{\scriptstyle x\mapsto(x,0)} \\
|\operatorname{Cyl}(f)| & \xleftarrow{G} & |\operatorname{sd}^i(S_{k-1})| & \hookrightarrow & |\operatorname{Cone}(\operatorname{sd}^i(S_{k-1}))| \times [0,1] \\
\uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle x\mapsto(x,1)} & & \uparrow{\scriptstyle x\mapsto(x,1)} \\
|\operatorname{Cyl}(f)| & \xleftarrow{|\iota|\circ\mho} & |\operatorname{sd}^i(S_{k-1})| & \hookrightarrow & |\operatorname{Cone}(\operatorname{sd}^i(S_{k-1}))| \\
\mathrm{id}\downarrow & & \cong\downarrow{\scriptstyle \mho} & & \cong\downarrow{\scriptstyle \mho} \\
|\operatorname{Cyl}(f)| & \xleftarrow{|\iota|} & |\operatorname{sd}^{i+1}(S_{k-1})| & \hookrightarrow & |\operatorname{Cone}(\operatorname{sd}^{i+1}(S_{k-1}))|.
\end{array}
$$

It follows from (4) and elementary observations that all of the vertical maps are homotopy equivalences. Furthermore note that it follows immediately from the Manifold Boundary-Cofibration Proposition 17.8 and the Product-Cofibration Proposition 17.11 that all of the right horizontal maps are closed cofibrations. Thus it follows from the Homotopy Pushout Theorem 17.19, applied altogether eight times, that the induced maps between the various pushouts have homotopy inverses. Combining these eight homotopy equivalences, respectively their homotopy inverses, we obtain the promised homotopy equivalence

$$X \cup_{S^{k-1}} \overline{B}^k \;\to\; \underbrace{|\operatorname{Cyl}(f)| \cup_{|\operatorname{sd}^i(S_{k-1})|} |\operatorname{Cone}(\operatorname{sd}^{i+1}(S_{k-1}))|}_{=\operatorname{Cone}(f\colon \operatorname{sd}^i(S_{k-1}) \to K)}|. \qquad \blacksquare$$

Proof of Proposition 94.12 (1). Let X be a *finite* CW-complex. An induction argument over the number of cells, using Lemma 94.13, shows that X is homotopy equivalent to a finite simplicial complex of the same dimension. $\blacksquare$

Sketch of a proof of Proposition 94.12 (2) and (3). Now let X be any (countable) CW-complex. Without loss of generality we can and will assume that X is connected and non-empty.

Claim. There exists an ascending sequence $K_{-1} \subset K_0 \subset K_1 \subset K_2 \subset \dots$ of abstract simplicial complexes and a sequence of homotopy equivalences $\theta_n \colon X^n \to |K_n|$, $n \in \mathbb{Z}_{\geq -1}$ such that for each $n \in \mathbb{N}_0$ there exists a homotopy $F_n \colon X^n \times [0,1] \to |K_{n+1}|$ between the maps $X^n \xrightarrow{\theta_n} |K_n| \to |K_{n+1}|$ and $X^n \to X^{n+1} \xrightarrow{\theta_{n+1}} |K_{n+1}|$.

Proof. We take K_{-1} to be the empty simplicial complex. Assume that we have already constructed $K_{-1}, \dots, K_n$ and $\theta_{-1}, \dots, \theta_n$. We denote by $\{\varphi_i \colon S^n \to X^n\}_{i \in I}$ the attaching maps of the $(n+1)$-cells of X. By basically the same argument as in the proof of Lemma 94.13 we obtain $d_i \in \mathbb{N}_0$, simplicial maps $f_i \colon \operatorname{sd}^{d_i}(S_n) \to K_n$ and a homotopy equivalence

$$\theta_{n+1} \colon \underbrace{X^n \cup_{\bigsqcup \varphi_i} \bigsqcup_{i \in I} \overline{B}^{n+1}}_{=X^{n+1}} \to \underbrace{\left| \operatorname{Cyl}\left(\bigsqcup_{i \in I} f \colon \bigsqcup_{i \in I} \operatorname{sd}^{d_i}(S_n) \to K \right) \cup \bigsqcup_{i \in I} \operatorname{Cone}\left(\operatorname{sd}^{d_i+1}(S_n) \right) \right|}_{=:K_{n+1}}$$

such that the maps $X^n \xrightarrow{\theta_n} |K_n| \to |K_{n+1}|$ and $X^n \to X^{n+1} \xrightarrow{\theta_{n+1}} |K_{n+1}|$ are homotopic. ⊞

We set $K := \bigcup_{n \in \mathbb{N}_0} K_n$. If X is countable, then K is also countable. Thus it remains to show that X is homotopy equivalent to $|K|$.

(1) As on page1546 we equip $[0, \infty)$ with the CW-structure where the 0-skeleton is given by $\mathbb{N}_0$ and we refer to the CW-complex

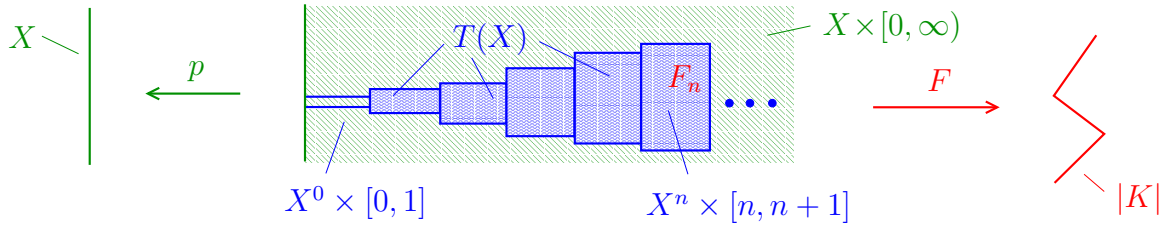
$$T(X) \;:=\; \bigcup_{n \in \mathbb{N}_0} X^n \times [n, n+1] \quad \subset \; X \times [0, \infty)$$

as the *telescope of* X.

(2) The homotopies F_n define a map $F \colon T(X) \to |K|$ which is continuous by the Characteristic Maps Lemma 68.7.

(3) Furthermore the projections $X^n \times [0,1] \to X^n$ define a map $p \colon T(X) \to X$ which is also continuous by the Characteristic Maps Lemma 68.7.

We pick $P \in X^0$. It follows fairly easily from the Direct Limit-π_n-Proposition 71.11 and the claim that for every $n \in \mathbb{N}$ the maps $F_* \colon \pi_n(T(X), P) \to \pi_n(|K|, P)$ and $p_* \colon \pi_n(T(X), P) \to \pi_n(X, P)$ are isomorphisms. It follows from the Whitehead Theorem 162.1 that $F \colon T(X) \to |K|$ and $p \colon T(X) \to X$ are homotopy equivalences. We have thus shown that X is homotopy equivalent to $|K|$.[1159] ■



[1159] We leave it to the industrious reader to figure out whether one can avoid the cheap exit via the Whitehead Theorem 162.1 by using say the Telescope Lemma 70.14 in a clever fashion.

Exercises for Chapter 94.

Exercise 94.1. Let $K = (V, S)$ be an abstract simplicial complex. By the Link Star Lemma 94.5 (3) we know that for any choice of simplices $s_0, \ldots, s_n \in S$ the following equality regarding *open* stars holds:

$$\mathring{\mathrm{St}}(s_0) \cap \cdots \cap \mathring{\mathrm{St}}(s_n) \;=\; \begin{cases} \mathring{\mathrm{St}}(s_0 \cup \cdots \cup s_n), & \text{if } s_0 \cup \cdots \cup s_n \text{ is a simplex,} \\ \varnothing, & \text{otherwise.} \end{cases}$$

Show that in general the analogous statement for stars does *not* hold.

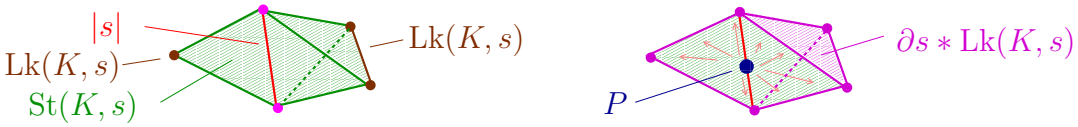
Exercise 94.2. Let K be an abstract simplicial complex and let s be a k-simplex. Show that there exists a natural simplicial isomorphism $\mathrm{Lk}(K, s) * \partial s \cong \mathrm{Lk}(\mathrm{sd}(K), \underline{s})$.

Exercise 94.3. Let $K = (V, S)$ be an abstract simplicial complex. Show that the dimension of the barycentric subdivision $\mathrm{sd}(K)$ equals the dimension of K.

Exercise 94.4. Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$.

(a) Show that the link $\mathrm{Lk}(s)$ is a deformation retract of $\mathrm{St}(s) \setminus |s|$.
Hint. Consider the proof of the Link Star Lemma 94.5 (e).

(b) Let $P \in \langle s \rangle$. Show that $\partial s * \mathrm{Lk}(s)$ is a deformation retract of $\mathrm{St}(s) \setminus \{P\}$.
Hint. Turn the idea of "pushing away from P" into a rigorous argument.



Exercise 94.5. Let $n \in \mathbb{N}$. As on page 1978 we consider the abstract simplicial complex $S_n := (V, T)$ with $V = \{0, \ldots, n+1\}$ and $T = \mathcal{P}(V) \setminus \{\varnothing, V\}$. Note that given a k-simplex s of S_n the set $V \setminus s$ is an $(n-k)$-simplex. Next we consider the barycentric subdivision $\mathrm{sd}(S_n)$ together with the map

$$\begin{array}{rccc} f\colon & \mathrm{sd}(S_n) & \to & \mathrm{sd}(S_n) \\ & \underline{s} & \mapsto & \underline{V \setminus s}. \end{array}$$

(a) Show that f is a simplicial map.

(b) Show that $S_n/s \sim f(s)$, interpreted in the obvious way, is an abstract simplicial complex with $2^{n+1} - 1$ vertices.

(c) Sketch the abstract simplicial complex $S_n/s \sim f(s)$ for $n = 1$ and $n = 2$.

(d) Show that the topological realization of $S_n/s \sim f(s)$ is homeomorphic to the real projective space $\mathbb{R}\mathrm{P}^n = S^n/x \sim -x$.

Remark. It is not known what is the "smallest" simplicial structure for the real projective spaces $\mathbb{R}\mathrm{P}^n$. It is shown in **[ArM1991]** that any simplicial structure for $\mathbb{R}\mathrm{P}^n$ needs to have at least $\frac{1}{2}(n+1)(n+2)$ vertices. But all known simplicial structures seem to have about 2^n vertices.

Exercise 94.6. Let $n \in \mathbb{N}$ and let $k \in \{1, \ldots, n-1\}$. Use the Simplicial Approximation Theorem 94.8 to show that $\pi_k(S^n) = 0$.
Remark. This statement is of course the content of Proposition 71.10 which we proved using the Cellular Approximation Theorem 70.15.

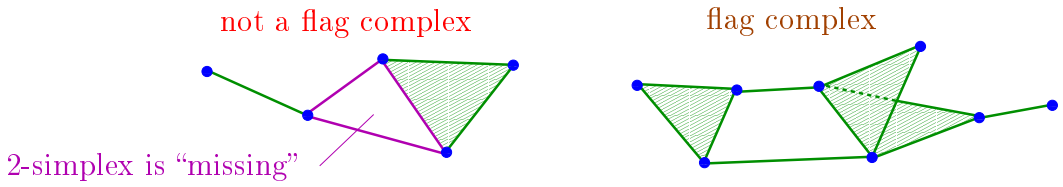
Exercise 94.7. We say an abstract simplicial complex $K = (V, S)$ is a *flag complex* if the following condition is satisfied:

(∗) If $v_0, \ldots, v_n \in V$ have the property that $\{v_i, v_j\} \in S$ for all $i, j \in \{0, \ldots, n\}$, then $\{v_0, \ldots, v_n\} \in S$.

More loosely speaking, K is a flag complex, if a simplex exists whenever its 1-skeleton exists. We refer to the figure below for an illustration of the definition.

(a) Show that every flag complex is determined by its 1-skeleton. More precisely, let $K = (V, S)$ and $L = (W, T)$ be two abstract simplicial complexes. Suppose that $\varphi\colon K^1 \xrightarrow{\cong} L^1$ is a simplicial isomorphism between the 1-skeleta. We suppose that K and L are flag complexes. Show that φ can be extended to a simplicial isomorphism $K \xrightarrow{\cong} L$.

(b) Let K be an abstract simplicial complex. Show that the barycentric subdivision $\mathrm{sd}(K)$ is a flag complex.



Exercise 94.8. We recall the following definition from page 369. We say a family $\{U_i\}_{i \in I}$ of subsets of a set X has *order* $\leq n$ if for any $J \subset I$ with $\#J > n$ we have $\bigcap_{j \in J} U_j = \varnothing$.

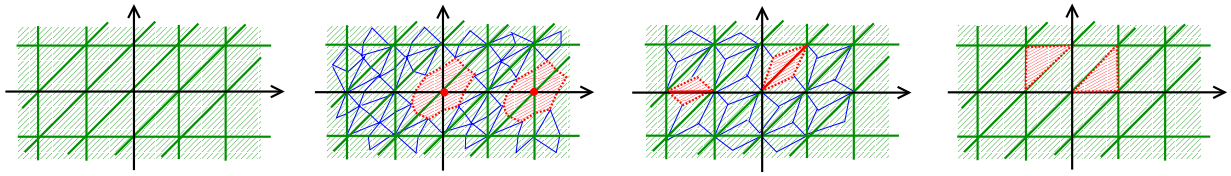
(a) Let $K = (V, S)$ be an n-dimensional abstract simplicial complex. Given $i \in \{0, \ldots, n\}$ we denote by K_i the abstract simplicial complex one obtains from barycentric subdivision of all simplices of degree $> i$.[1160] As usual we use (a slight generalize of) the SD-Lemma 94.3 to identify the various topological realizations, i.e. we write $|K| = |K_0| = \cdots = |K_n|$. We denote by $\mathcal{V}_i$ the collection of open stars of the i-simplices of K in $|K_i|$. Show that the sets in $\mathcal{V}_0 \cup \cdots \cup \mathcal{V}_n$ form an open cover of $|K| = |K_0| = \cdots = |K_n|$ of order $n + 1$.

Next we recall another definition from page 369. Let X be a non-empty topological space. We say $\mathrm{cdim}(X) \leq n$ if every open cover of X admits an open refinement of order $n + 1$.

(b) Show that every finite simplicial complex X of dimension n has $\mathrm{cdim}(X) \leq n$.

(c) Show that every compact subspace X of $\mathbb{R}^n$ has $\mathrm{cdim}(X) \leq n$.

Remark. This approach gives a new proof for Theorem 11.3.



[1160] We did not define this construction, but the reader should have no troubles making sense of it.

95. Simplicial homology

In this chapter we introduce the simplicial homology groups of an (ordered) abstract simplicial complex. As we will see, the definition is refreshingly straightforward. Later in the chapter we discuss the relationship between simplicial and singular homology.

Before we get into details it is convenient to recall the following definition from basic group theory which we will need on numerous occasions throughout this chapter.

Definition. Let $n \in \mathbb{N}$. Given a permutation $\sigma \in S_n$ we refer to[1161]

$$\operatorname{sign}(\sigma) \;:=\; \begin{array}{c}\text{determinant of the } (n \times n)\text{-matrix for which the} \\ (i,\sigma(i))\text{-entries are one and all the other entries are zero}\end{array}$$

as the sign of the permutation.

The following lemma summarizes the two key properties of the sign of a permutation.

Lemma 95.1. Let $n \in \mathbb{N}$.

(1) The map $\operatorname{sign}\colon S_n \to \{-1, 1\}$ is a homomorphism.
(2) If σ is a transposition, then $\operatorname{sign}(\sigma) = -1$.

Proof. The proof follows immediately from standard properties of the determinant. ■

Definition. Let X be a finite set and let $\sigma \in \operatorname{Bij}(X)$. We set $n := \#X$. We pick a bijection $\varphi\colon \{1, \dots, n\} \to X$. We define $\operatorname{sign}(\sigma) = \operatorname{sign}(\varphi^{-1} \circ \sigma \circ \varphi)$. It follows easily from Lemma 95.1 that this definition is independent of the choice of the bijection φ.

Throughout this chapter we adopt the following convention.

Convention. Given $n \in \mathbb{N}$ we identify the permutation group $S_n = \operatorname{Bij}\{1, \dots, n\}$ with the group $\operatorname{Bij}\{0, \dots, n-1\}$ in the obvious way.

95.1. Simplicial homology of an ordered abstract simplicial complex.

In this section we will define the simplicial chain complex and the simplicial homology groups of an *ordered* abstract simplicial complex. This is going to be surprisingly easy. In the subsequent section we will then deal with abstract simplicial complex that are not equipped with an order.

We start out with the following elementary observation which we will use frequently throughout this chapter.

Observation. Let $(T, \leq)$ be a totally ordered set and let $v_0, \dots, v_n \in T$. By Exercise 0.4 there exists a unique permutation $\sigma \in S_{n+1}$ such that $v_{\sigma(0)} < \dots < v_{\sigma(n)}$.

After this preliminary remark we can now turn to an interesting definition.

Definition. Let $K = (V, S)$ be an ordered abstract simplicial complex.[1162] Given $n \in \mathbb{N}_0$ we define

$$\mathrm{C}_n^{\mathrm{simp},\leq}(K) \;:=\; \text{free abelian group generated by the set of } n\text{-simplices of } K.$$

[1161] Surely the reader will have seen the definition of the sign of a permutation. We leave it to the reader to show that the previous definition agrees with our definition.

Furthermore we consider the boundary map

$$\begin{array}{rcl} \partial_n \colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_{n-1}^{\mathrm{simp},\leq}(K) \\ \underbrace{\{v_0 < \cdots < v_n\}}_{\substack{\text{any } n\text{-simplex } s \text{ is uniquely} \\ \text{of the form } s = \{v_0, \ldots, v_n\} \\ \text{with } v_0 < \cdots < v_n}} & \mapsto & \sum_{i=0}^{n} (-1)^i \cdot \{v_0, \ldots, v_{i-1}, \widehat{v_i}, v_{i+1}, \ldots, v_n\}. \end{array}$$

Lemma 95.2. Let $K = (V, S)$ be an ordered abstract simplicial complex. Given any $n \in \mathbb{N}_0$ we have $\partial_{n-1} \circ \partial_n = 0$. In other words, $(C_*^{\mathrm{simp},\leq}(K), \partial_*)$ is a chain complex.

Proof. The proof of the lemma is almost verbatim the same as the proof of Proposition 72.2. We leave it to the reader to make the necessary minute modifications. ■

The reader is by now well aware of the fact that such a lemma leads straight to the following definition.

Definition. Let K be an ordered abstract simplicial complex. We refer to the chain complex $(\mathrm{C}_*^{\mathrm{simp},\leq}(K), \partial_*)$ as the simplicial chain complex. Given $n \in \mathbb{N}_0$ we define the n-th simplicial homology group of K as the n-th homology group of the simplicial chain complex, i.e. we define

$$\mathrm{H}_n^{\mathrm{simp},\leq}(K) \;:=\; \frac{\ker\left(\partial_n \colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp},\leq}(K)\right)}{\mathrm{im}\left(\partial_{n+1} \colon \mathrm{C}_{n+1}^{\mathrm{simp},\leq}(K) \to \mathrm{C}_n^{\mathrm{simp},\leq}(K)\right)}.$$

Example. We consider the 1-dimensional abstract simplicial complex $K = (V, S)$ shown in the figure below. It has five vertices $\{A, B, C, D, E\}$, it has the corresponding five 0-simplices and it has five 1-simplices. We view $K = (V, S)$ as an ordered abstract simplicial complex in the obvious way, i.e. we have $A < B < C < D < E$. The corresponding simplicial chain complex is given as follows:

$$0 \;\to\; \begin{array}{l} \mathbb{Z} \cdot \{A, B\} \\ \mathbb{Z} \cdot \{B, C\} \\ \mathbb{Z} \cdot \{C, D\} \\ \mathbb{Z} \cdot \{D, E\} \\ \mathbb{Z} \cdot \{C, E\} \end{array} \xrightarrow{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}} \begin{array}{l} \mathbb{Z} \cdot \{A\} \\ \mathbb{Z} \cdot \{B\} \\ \mathbb{Z} \cdot \{C\} \\ \mathbb{Z} \cdot \{D\} \\ \mathbb{Z} \cdot \{E\} \end{array} \;\to\; 0.$$

We leave it as an elementary exercise in linear algebra to show that

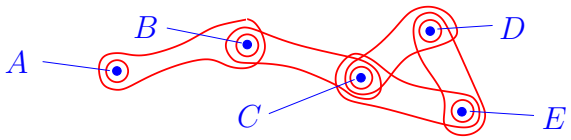
$$\mathrm{H}_n^{\mathrm{simp},\leq}(K) \;\cong\; \begin{cases} 0, & \text{if } n > 1, \\ \mathbb{Z}, & \text{if } n = 0, 1. \end{cases}$$

Furthermore one can easily show that

$$\mathrm{H}_1^{\mathrm{simp},\leq}(K) \;=\; \ker\left(\partial_1 \colon \mathrm{C}_1^{\mathrm{simp},\leq}(K) \to \mathrm{C}_0^{\mathrm{simp},\leq}(K)\right) \;=\; \mathbb{Z} \cdot [\{C, D\} + \{D, E\} - \{C, E\}].$$

As always, not only do we want to associate groups to our given objects, but we really want to construct a functor. This is done in the following lemma.

[1162] Recall that according to the definition on page 1982 "ordered" means that the vertex set V is equipped with a partial order such that the restriction to each simplex is a total order.



Lemma 95.3.

(1) Given an order-preserving simplicial map $f\colon K = (V,S) \to L = (W,T)$ between ordered abstract simplicial complexes we define

$$\begin{array}{rcl} f_*\colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n^{\mathrm{simp},\leq}(L) \\ \{v_0,\dots,v_n\} & \mapsto & \begin{cases} \{f(v_0),\dots,f(v_n)\}, & \text{if } f(v_0),\dots,f(v_n) \text{ are pairwise different,} \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

These maps form a chain map, thus they induce maps $f_*\colon \mathrm{H}_n^{\mathrm{simp},\leq}(K) \to \mathrm{H}_n^{\mathrm{simp},\leq}(L)$.

(2) The maps

$$\begin{array}{rcl} (K,\leq) & \mapsto & \mathrm{C}_*^{\mathrm{simp},\leq}(K) \\ (f\colon (K,\leq) \to (L,\leq)) & \mapsto & (f_*\colon \mathrm{C}_*^{\mathrm{simp},\leq}(K) \to \mathrm{C}_*^{\mathrm{simp},\leq}(L)) \end{array}$$

define a covariant functor from the category *OrdAbsSimpCplx* of ordered abstract simplicial complexes to the category *ChCplx* of chain complexes. In particular, for each $n \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} (K,\leq) & \mapsto & \mathrm{H}_n^{\mathrm{simp},\leq}(K) \\ (f\colon (K,\leq) \to (L,\leq)) & \mapsto & (f_*\colon \mathrm{H}_n^{\mathrm{simp},\leq}(K) \to \mathrm{H}_n^{\mathrm{simp},\leq}(L)) \end{array}$$

define a covariant functor from the category *OrdAbsSimpCplx* of ordered abstract simplicial complexes to the category *AbGroup* of abelian groups.

Proof. The statements follow easily from the definitions. We leave it to the reader to fill in the details. ■

Naturally the question arises, to what degree do the simplicial homology groups of an ordered abstract simplicial complex depend on the choice of the order. As the following lemma shows, the answer is, not very much.

Lemma 95.4. Let $K = (V,S)$ be an abstract simplicial complex and let "$\leq$" and "$\widetilde{\leq}$" be two orders on K. The maps

$$\begin{array}{rcl} \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n^{\mathrm{simp},\widetilde{\leq}}(K) \\ \{v_0 < \dots < v_n\} & \mapsto & \underbrace{\operatorname{sign}(\sigma) \cdot \{v_{\sigma(0)} \widetilde{<} \dots \widetilde{<} v_{\sigma(n)}\}}_{\substack{\sigma \in S_n \text{ is the unique permutation} \\ \text{such that } v_{\sigma(0)} \widetilde{\leq} \dots \widetilde{\leq} v_{\sigma(n)}}} \end{array}$$

define an isomorphism of chain complexes.

Proof. The proof of the lemma is an appealing little calculation that will be done in Exercise 95.1. ■

95.2. **Simplicial homology.** In the last section we introduced simplicial homology for *ordered* abstract simplicial complexes. In this section we will introduce simplicial homology for "unordered" abstract simplicial complexes. Evidently the definition is going to be very similar, but as we will see, the definition is ever so more subtle.

Definition. Let $K = (V, S)$ be an abstract simplicial complex and let $n \in \mathbb{N}_0$.

(1) An ordered n-simplex of K is an $(n+1)$-tuple $(v_0, \ldots, v_n) \in V^{n+1}$ such that $\{v_0, \ldots, v_n\}$ is an n-simplex of $K = (V, S)$.

(2) Given an ordered n-simplex $(v_0, \ldots, v_n)$ we denote by $\lfloor (v_0, \ldots, v_n) \rfloor := \{v_0, \ldots, v_n\}$ the corresponding n-simplex of K.

(3) Let $n \in \mathbb{N}_0$. We define

$$\widehat{\mathrm{C}}_n^{\mathrm{simp}}(K) := \text{free abelian group generated by the set of ordered } n\text{-simplices of } K.$$

Furthermore we define the n-th simplicial chain group to be

$$\mathrm{C}_n^{\mathrm{simp}}(K) := \widehat{\mathrm{C}}_n^{\mathrm{simp}}(K)/I_n$$

where I_n is the subgroup of $\widehat{\mathrm{C}}_n^{\mathrm{simp}}(K)$ generated by all elements of the form

$$(v_0, \ldots, v_n) - \mathrm{sign}(\sigma) \cdot (v_{\sigma(0)}, \ldots, v_{\sigma(n)}) \qquad \text{where } \sigma \in S_{n+1} = \mathrm{Bij}\{0, \ldots, n\}$$

Given an ordered simplex $(v_0, \ldots, v_n)$ we denote its image in $\mathrm{C}_n^{\mathrm{simp}}(K)$ by $[v_0, \ldots, v_n]$.

Remark. Let $K = (V, S)$ be an abstract simplicial complex. Since the set of vertices is in a natural bijection with the set of 0-simplices and since the permutation group S_1 is trivial we see that we have a natural identification $\mathrm{C}_0^{\mathrm{simp}}(K) = \mathbb{Z}^{(V)}$.

Lemma 95.5. Let $K = (V, S)$ be an abstract simplicial complex. If "$\leq$" is an order on $K = (V, S)$, then for each $n \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n^{\mathrm{simp}}(K) \\ \{v_0 < \cdots < v_n\} & \mapsto & [v_0, \ldots, v_n] \end{array}$$

is a natural isomorphism.

Proof. Let $n \in \mathbb{N}_0$. First note that it follows immediately from the definitions that the map is natural.

It remains to show that the given map is an isomorphism. We denote by P_n the set of ordered n-simplices, as defined above. Note that by definition we have $\widehat{\mathrm{C}}_n^{\mathrm{simp}}(K) = \mathbb{Z}^{(P_n)}$. In the following we denote by Q_n the set of all ordered n-simplices $(v_0, \ldots, v_n)$ such that $v_0 < \cdots < v_n$. By Lemma 51.1 there exist unique homomorphisms as follows:

$$\begin{array}{rcl} \Phi\colon \mathbb{Z}^{(P_n)} & \to & \mathbb{Z}^{(Q_n)} \\ (v_0, \ldots, v_n) & \mapsto & \mathrm{sign}(\sigma) \cdot (v_{\sigma(0)}, \ldots, v_{\sigma(n)}) \end{array}$$

here $\sigma \in S_{n+1}$ is the unique permutation with $v_{\sigma(0)} < \cdots < v_{\sigma(n)}$

and

$$\begin{array}{rcl} \Psi\colon \mathbb{Z}^{(Q_n)} & \to & \mathbb{Z}^{(P_n)} \\ (v_0, \ldots, v_n) & \mapsto & (v_0, \ldots, v_n). \end{array}$$

Furthermore we denote by $\pi\colon \mathbb{Z}^{(P_n)} \to \mathbb{Z}^{(P_n)}/I_n = \mathrm{C}_n^{\mathrm{simp}}(K)$ the obvious projection. It follows easily from Lemma 95.1 that $\Phi\colon \mathbb{Z}^{(P_n)} \to \mathbb{Z}^{(Q_n)}$ descends to a homomorphism $\overline{\Phi}\colon \mathbb{Z}^{(P_n)}/I_n \to \mathbb{Z}^{(Q_n)}$. Note that it is an immediate consequence of the definitions that $\overline{\Phi} \circ \pi irc \Psi$ is the identity on the basis Q_n of $\mathbb{Z}^{(Q_n)}$ and that $\pi \circ \Psi \circ \overline{\Phi}$ is the identity on the generating set $\pi(P_n)$ of $\mathbb{Z}^{(P_n)}/I_n$. This shows that $\pi \circ \Psi$ and $\overline{\Phi}$ are isomorphisms. The promised statements follow readily from this observation. ∎

Lemma 95.6. Let K be an abstract simplicial complex.

(1) Given any $n \in \mathbb{N}_0$ the group $\mathrm{C}_n^{\mathrm{simp}}(K)$ is a free abelian group where the rank is given by the cardinality of the set S_n of n-simplices of K. In fact, if we choose for each n-simplex $s \in S_n$ an ordered n-simplex $\widetilde{s}$ with $\lfloor \widetilde{s} \rfloor = s$, then $\{[\widetilde{s}]\}_{s \in S_n}$ is a basis for $\mathrm{C}_n^{\mathrm{simp}}(K)$.

(2) Let $n \in \mathbb{N}$. There exists a unique homomorphism

$$\partial_n \colon \mathrm{C}_n^{\mathrm{simp}}(K) \quad \to \quad \mathrm{C}_{n-1}^{\mathrm{simp}}(K)$$

with the property that given any ordered n-simplex $(v_0, \ldots, v_n)$ of K we have

$$\partial_n([v_0, \ldots, v_n]) \quad = \quad \sum_{i=0}^{n} (-1)^i \cdot [v_0, \ldots, v_{i-1}, \widehat{v_i}, v_{i+1}, \ldots, v_n].$$

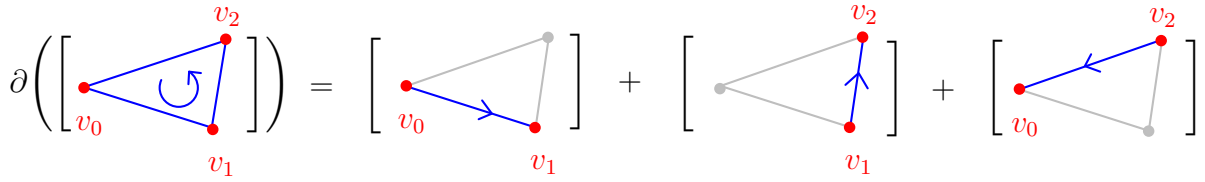
(3) For every $n \in \mathbb{N}_0$ we have $\partial_{n-1} \circ \partial_n = 0$.

Remark. Let $K = (V, S)$ be an abstract simplicial complex and let $n \in \mathbb{N}_0$. Lemma 95.5 and Lemma 95.6 (1) say in particular that $\mathrm{C}_n^{\mathrm{simp}}(K)$ is a free abelian group. If $K = (V, S)$ is an ordered abstract simplicial complex, then this free abelian group has a natural basis. In the absence of an order there is only a natural basis where each element of the basis is well-defined "up to sign".

Example. For an ordered simplex (v_0, v_1, v_2) in an abstract simplicial complex K we see that

$$\partial([v_0, v_1, v_2]) \quad = \quad [v_1, v_2] - [v_0, v_2] + [v_0, v_1] \quad = \quad [v_0, v_1] + [v_1, v_2] + [v_2, v_0].$$

In other words, if we consider the figure below, then we see that the boundary is exactly what we would naively had thought.



Proof.

(1) It follows from the Well-Ordering Theorem 0.7 that we can equip $K = (V, S)$ with an order. Thus the desired statement is now an almost immediate consequence of Lemma 95.5

(2) Let $n \in \mathbb{N}$. By Lemma 51.1 there exists a unique homomorphism

$$\partial_n \colon \widehat{\mathrm{C}}_n^{\mathrm{simp}}(K) \quad \to \quad \mathrm{C}_{n-1}^{\mathrm{simp}}(K)$$

with the property that given any ordered n-simplex $(v_0, \ldots, v_n)$ of K we have

$$\partial_n((v_0, \ldots, v_n)) \quad = \quad \sum_{i=0}^{n} (-1)^i \cdot [v_0, \ldots, v_{i-1}, \widehat{v_i}, v_{i+1}, \ldots, v_n].$$

It remains to show that ∂_n factors through $\mathrm{C}_n^{\mathrm{simp}}(K) = \widehat{\mathrm{C}}_n^{\mathrm{simp}}(K)/I_n$. In other words, we need to show that ∂_n vanishes on I_n. Since the permutation group S_n is generated by transpositions and since transpositions have the sign -1 it remains to prove the following claim.

Claim. For any $0 \leq k < l \leq n$ and any ordered n-simplex $(v_0, \ldots, v_n)$ of K we have
$$\partial_n(v_0, \ldots, v_l, \ldots, v_k, \ldots, v_n) \;=\; -\partial_n(v_0, \ldots, v_k, \ldots, v_l, \ldots, v_n).$$

Proof. Let $(v_0, \ldots, v_n)$ be an ordered n-simplex and let $0 \leq k < l \leq n$. We calculate that

$$\begin{aligned}
&\partial_n(.., v_l, .., v_k, ..) \;= \\
&= \sum_{i \neq k,l} (-1)^i \cdot \underbrace{[.., v_l, .., \widehat{v_i}, .., v_k, ..]}_{=-[..,v_k,..,\widehat{v_i},..,v_l,..]} + (-1)^k \cdot \underbrace{[.., \widehat{v_l}, .., v_k, ..]}_{\substack{= (-1)^{l-k-1} \cdot [., v_k, \ldots, \widehat{v_l}, .] \\ \text{since } l\text{-}k\text{-}1 \text{ transpositions} \\ \text{are used}}} + (-1)^l \cdot \underbrace{[.., v_l, .., \widehat{v_k}, ..]}_{\substack{= (-1)^{l-k-1} \cdot [., \widehat{v_k}, \ldots, v_l, .] \\ \text{since } l\text{-}k\text{-}1 \text{ transpositions} \\ \text{are used}}} \\
&= -\sum_{i \neq k,l} (-1)^i \cdot [.., v_k, .., \widehat{v_i}, .., v_l, ..] - (-1)^l \cdot [.., v_k, .., \widehat{v_l}, ..] - (-1)^k \cdot [.., \widehat{v_k}, .., v_l, ..] \\
&= -\partial_n(.., v_k, .., v_l, ..).
\end{aligned}$$

(3) The proof of this lemma is almost verbatim the same as the proof of Proposition 72.2. We leave the pleasant task of copy-pasting the proof to the reader. ■

Lemma 95.6 leads us right away to the following definition.

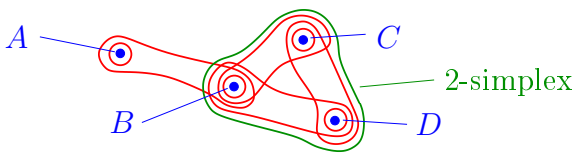
Definition. Let K be an abstract simplicial complex. We refer to the chain complex $(\mathrm{C}_*^{\mathrm{simp}}(K), \partial_*)$ as the simplicial chain complex. Given $n \in \mathbb{N}_0$ we define the n-th simplicial homology group of K as the n-th homology group of the simplicial chain complex, i.e. we define
$$\mathrm{H}_n^{\mathrm{simp}}(K) \;:=\; \frac{\ker\left(\partial_n \colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\right)}{\mathrm{im}\left(\partial_{n+1} \colon \mathrm{C}_{n+1}^{\mathrm{simp}}(K) \to \mathrm{C}_n^{\mathrm{simp}}(K)\right)}.$$

Examples.

(1) We consider the 2-dimensional abstract simplicial complex $K = (V, S)$ shown in the figure below. It has four vertices $\{A, B, C, D\}$, it has the corresponding four 0-simplices, it has four 1-simplices and it has one 2-simplex. The corresponding simplicial chain complex is given as follows:

$$0 \;\to\; \mathbb{Z} \cdot [B, C, D] \xrightarrow{\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}} \begin{matrix} \mathbb{Z} \cdot [A, B] \\ \mathbb{Z} \cdot [B, C] \\ \mathbb{Z} \cdot [B, D] \\ \mathbb{Z} \cdot [C, D] \end{matrix} \xrightarrow{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}} \begin{matrix} \mathbb{Z} \cdot [A] \\ \mathbb{Z} \cdot [B] \\ \mathbb{Z} \cdot [C] \\ \mathbb{Z} \cdot [D] \end{matrix} \;\to\; 0.$$

Elementary linear algebra shows that
$$\mathrm{H}_n^{\mathrm{simp}}(K) \;\cong\; \begin{cases} 0, & \text{if } n > 0, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$



(2) If $K = (V, S)$ is an abstract simplicial complex with no simplices of dimension ≥ 1, then it follows immediately from the definitions that for $n \in \mathbb{N}_0$ we have

$$\mathrm{H}_n^{\mathrm{simp}}(K) \;\cong\; \begin{cases} \mathrm{C}_0^{\mathrm{simp}}(K) = \mathbb{Z}^{(V)}, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The following is the obvious analogue of Lemma 95.3.

Lemma 95.7.

(1) Given a simplicial map $f\colon K = (V, S) \to L = (W, T)$ we define

$$\begin{array}{rcl} f_*\colon \mathrm{C}_n^{\mathrm{simp}}(K) & \to & \mathrm{C}_n^{\mathrm{simp}}(L) \\ [v_0, \ldots, v_n)] & \mapsto & \begin{cases} [f(v_0), \ldots, f(v_n)], & \text{if } f(v_0), \ldots, f(v_n) \text{ are pairwise different,} \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

These maps form a chain map, thus they induce maps $f_*\colon \mathrm{H}_n^{\mathrm{simp}}(K) \to \mathrm{H}_n^{\mathrm{simp}}(L)$.

(2) The maps

$$\begin{array}{rcl} K & \mapsto & \mathrm{C}_*^{\mathrm{simp}}(K) \\ (f\colon K \to L) & \mapsto & (f_*\colon \mathrm{C}_*^{\mathrm{simp}}(K) \to \mathrm{C}_*^{\mathrm{simp}}(L)) \end{array}$$

define a covariant functor from the category $\mathcal{AbsSimpCplx}$ of abstract simplicial complexes to the category $\mathcal{ChCplx}$ of chain complexes. For each $n \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} K & \mapsto & \mathrm{H}_n^{\mathrm{simp}}(K) \\ (f\colon K \to L) & \mapsto & (f_*\colon \mathrm{H}_n^{\mathrm{simp}}(K) \to \mathrm{H}_n^{\mathrm{simp}}(L)) \end{array}$$

define a covariant functor from the category $\mathcal{AbsSimpCplx}$ of abstract simplicial complexes to the category $\mathcal{AbGroup}$ of abelian groups.

Proof. As in the case of the proof of Lemma 95.3 we leave it to the reader to provide the proof. ■

For completeness we state the following elementary lemma.

Lemma 95.8. Let $K = (V, S)$ be an ordered abstract simplicial complex. The maps

$$\begin{array}{rcl} \Omega_n\colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n^{\mathrm{simp}}(K) \\ \{v_0 < \cdots < v_n\} & \mapsto & [v_0, \ldots, v_n] \end{array}$$

define a natural isomorphism of chain complexes. In particular they induce natural isomorphisms

$$\Omega_*\colon \mathrm{H}_n^{\mathrm{simp},\leq}(K) \;\to\; \mathrm{H}_n^{\mathrm{simp}}(K).$$

Remark. In the following we will discuss the simplicial homology groups $\mathrm{H}_n^{\mathrm{simp}}(K)$ in greater detail. By Lemma 95.8 the subsequent statements have obvious analogues for the simplicial homology groups $\mathrm{H}_n^{\mathrm{simp},\leq}(K)$.

Proof. A very short moment of reflection shows that the given map is a natural chain map. We deduce from Lemma 95.6 (1) that the chain map is in fact an isomorphism of chain complexes. ■

The next lemma can be viewed as an analogue of the Skeleton-Homology Lemma 80.1 (2) and (3).

Lemma 95.9. Let K be an abstract simplicial complex and let $m \in \mathbb{N}_0 \cup \{\infty\}$. As usual we denote by K^m the m-skeleton of K. Furthermore let $n \in \mathbb{N}_0$.

(1) If K has only finitely many n-simplices, then $\mathrm{H}_n^{\mathrm{simp}}(K)$ is finitely generated.
(2) If $n > m$, then $\mathrm{H}_n^{\mathrm{simp}}(K^m) = 0$.
(3) If $n < m$, then the inclusion induced map $\mathrm{H}_n^{\mathrm{simp}}(K^m) \to \mathrm{H}_n^{\mathrm{simp}}(K)$ is an isomorphism.

Analogous statements also hold for ordered abstract simplicial complexes and the homology groups $\mathrm{H}_n^{\mathrm{simp},\leq}(K^m)$ and $\mathrm{H}_n^{\mathrm{simp},\leq}(K)$.

Proof.

(1) This statement follows easily from the definitions together with Lemma 51.9 (2b).
(2) This statement follows from the observation that $\mathrm{C}_n^{\mathrm{simp}}(K^m) = 0$ for $n > m$.
(3) This statement follows almost immediately from the observation that by definition we have $\mathrm{C}_i^{\mathrm{simp}}(K^m) = \mathrm{C}_i^{\mathrm{simp}}(K)$ for $i = 0, \ldots, m$. ■

It is clear that given any finite abstract simplicial complex one can compute the corresponding homology groups by a brute force calculation. But for large examples this can be very inefficient. Furthermore, it is rather unsatisfactory to outsource calculations to a computer. It is much more entertaining to try to do calculations cleverly on one's own. In the following we will get to know a few approaches to calculating simplicial homology groups. We will develop more techniques in Exercises 95.3 and 95.7.

As for singular homology, some calculations, statements and proofs become easier if we consider *reduced* simplicial homology.

Definition. Let K be an abstract simplicial complex.

(1) We refer to the map[1163]
$$\begin{array}{rcl} \epsilon\colon \mathrm{C}_0^{\mathrm{simp}}(K) & \to & \mathbb{Z} \\ \left[\sum_{i=1}^m a_i \cdot [v_i]\right] & \mapsto & \sum_{i=1}^m a_i \end{array}$$
as the augmentation.
(2) We consider the following generalized[1164] chain complex
$$\ldots \xrightarrow{\partial_3} \mathrm{C}_2^{\mathrm{simp}}(K) \xrightarrow{\partial_2} \mathrm{C}_1^{\mathrm{simp}}(K) \xrightarrow{\partial_1} \underset{\text{degree } 0}{\mathrm{C}_0^{\mathrm{simp}}(K)} \xrightarrow{\epsilon} \underset{\text{degree } -1}{\mathbb{Z}} \to 0.$$
We introduce the following two definitions.
(a) We refer to this chain complex as the augmented simplicial chain complex $\widetilde{\mathrm{C}}_*^{\mathrm{simp}}(K)$ of K.
(b) Let $k \in \mathbb{N}_0$. We define the k-th reduced simplicial homology group $\widetilde{\mathrm{H}}_k^{\mathrm{simp}}(K)$ of K to be the k-th homology group of the augmented chain complex.

Remark. The above definition also makes sense for the empty abstract simplicial complex $\varnothing$. Furthermore, similar to the Reduced Homology Lemma 74.1 (0) we see that $\mathrm{H}_{-1}^{\mathrm{simp}}(\varnothing) = \mathbb{Z}$ and $\mathrm{H}_i^{\mathrm{simp}}(\varnothing) = 0$ for $i \neq -1$.

[1163] Morally the group $\mathbb{Z}$ is the free abelian group on the "empty" simplex.
[1164] It requires about one second to verify that this is indeed a generalized chain complex.

The following lemma is the analogue of the Reduced Homology Lemma 74.1 (4). It basically says that in the following it suffices to calculate reduced simplicial homology groups.

Lemma 95.10. (Reduced Homology Lemma) Let $K = (V, S)$ be an abstract simplicial complex.

(1) Given any vertex $v \in V$ the map

$$\begin{array}{rcl} \widetilde{\mathrm{H}}_0^{\mathrm{simp}}(K) \oplus \mathbb{Z} & \to & \mathrm{H}_0^{\mathrm{simp}}(K) \\ [\sigma] \oplus n & \mapsto & [\sigma] + [n \cdot v] \end{array}$$

is an isomorphism. In particular, if K is non-empty, then we have an isomorphism $\widetilde{\mathrm{H}}_0^{\mathrm{simp}}(K) \oplus \mathbb{Z} \cong \mathrm{H}_0^{\mathrm{simp}}(K)$.

(2) For every $n \geq 1$ we have the equality $\mathrm{H}_n^{\mathrm{simp}}(K) = \widetilde{\mathrm{H}}_n^{\mathrm{simp}}(K)$.

Proof. The proof is basically identical to the already rather straightforward proof of the Reduced Homology Lemma 74.1 (4). ∎

By the Cone Contractibility Lemma 16.7 we know that the cone on a topological space is contractible, in particular we know by the Homotopy Equivalence-H_*-Corollary 73.9 (3) that its reduced homology groups vanish. The following lemma can be viewed as a simplicial analogue of this observation.

Lemma 95.11. (Cone Homology Lemma) Let K be an abstract simplicial complex. Let $\mathrm{Cone}(K)$ be the simplicial cone as defined on page 2003. Given any $k \in \mathbb{N}_0$ we have

$$\widetilde{\mathrm{H}}_k^{\mathrm{simp}}(\mathrm{Cone}(K)) \;=\; 0.$$

Proof. Let $K = (V, S)$ be an abstract simplicial complex. We set $L := \mathrm{Cone}(K)$. Note that it follows from Lemma 73.5 that it remains to prove that the identity map $\mathrm{id}\colon \mathrm{C}_*^{\mathrm{simp}}(L) \to \mathrm{C}_*^{\mathrm{simp}}(L)$ is homotopic to the zero map. In other words, it suffices to prove the following claim.

Claim. We can find "diagonal" maps P_n, $n \in \mathbb{Z}_{\geq -1}$, to obtain a diagram

$$\begin{array}{ccccccccccccccc}
\ldots & \longrightarrow & C_3^{\mathrm{simp}}(L) & \xrightarrow{\partial_3} & C_2^{\mathrm{simp}}(L) & \xrightarrow{\partial_2} & C_1^{\mathrm{simp}}(L) & \xrightarrow{\partial_1} & C_0^{\mathrm{simp}}(L) & \xrightarrow{\partial_0 := \epsilon} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle P_3} & \downarrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle P_2} & \downarrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle P_1} & \downarrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle P_0} & \downarrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle P_{-1}} & \downarrow{\scriptstyle \mathrm{id}} & \\
\ldots & \longrightarrow & C_3^{\mathrm{simp}}(L) & \xrightarrow[\partial_3]{} & C_2^{\mathrm{simp}}(L) & \xrightarrow[\partial_2]{} & C_1^{\mathrm{simp}}(L) & \xrightarrow[\partial_1]{} & C_0^{\mathrm{simp}}(L) & \xrightarrow[\partial_0 := \epsilon]{} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

such that for every $n \in \mathbb{Z}_{\geq -1}$ we have the equality

$$(*) \qquad \partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; \mathrm{id}\,.$$

Proof. In the following we denote the cone point of $\mathrm{Cone}(L)$ by "$\star$". To prove the claim we consider the map

$$\begin{array}{rcl} P_{-1}\colon \mathbb{Z} & \to & \mathrm{C}_0^{\mathrm{simp}}(L) \\ n & \mapsto & n \cdot [\star] \end{array}$$

and for $n \geq 0$ we consider the map[1165]

$$\begin{array}{rcl} P_n\colon \mathrm{C}_n^{\mathrm{simp}}(L) & \to & \mathrm{C}_{n+1}^{\mathrm{simp}}(L) \\ ][v_0, \ldots, v_n] & \mapsto & \begin{cases} [\star, v_0, \ldots, v_n], & \text{if all } v_i \neq \star, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

We claim that $(*)$ holds for all $n \in \mathbb{Z}_{\geq -1}$. The case $n = -1$ can be verified easily. Thus it remains to show that $(*)$ holds if $n \in \mathbb{N}_0$. In fact, given $[v_0, \ldots, v_n] \in \mathrm{C}_n^{\mathrm{simp}}(L)$ with $v_0, \ldots, v_n \in V$ we immediately see that

$$\underbrace{[v_0, \ldots, v_n] + \sum_{i=1}^{n+1} (-1)^i \cdot [\star, v_0, \ldots, \widehat{v_{i-1}}, \ldots, v_n]}_{=(\partial_{n+1} \circ P_n)([v_0, \ldots, v_n])} + \underbrace{\sum_{i=0}^{n} (-1)^i \cdot [\star, v_0, \ldots, \widehat{v_i}, \ldots, v_n]}_{=(P_{n-1} \circ \partial_n)([v_0, \ldots, v_n])} = [v_0, \ldots, v_n].$$

Furthermore, given $[\star, v_0, \ldots, v_{n-1}] \in \mathrm{C}_n^{\mathrm{simp}}(L)$ with $v_0, \ldots, v_{n-1} \in V$ it is equally straightforward to verify that the equality holds. ■

Next we want to calculate the simplicial homology groups of the abstract simplicial complexes D_n and S_n that we introduced on page 1978. By the Reduced Homology Lemma 95.10 it suffices to compute the reduced simplicial homology groups.

Lemma 95.12. Given any $n \in \mathbb{N}$ and given any $k \in \mathbb{N}_0$ we have

$$\widetilde{\mathrm{H}}_k^{\mathrm{simp}}(D_n) \;=\; 0 \quad \text{and} \quad \widetilde{\mathrm{H}}_k^{\mathrm{simp}}(S_n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof.

(1) Given any $n \in \mathbb{N}$ and given any $k \in \mathbb{N}_0$ we calculate that

$$\widetilde{\mathrm{H}}_k^{\mathrm{simp}}(D_n) \;\underset{\text{see page 2003}}{\cong}\; \widetilde{\mathrm{H}}_k^{\mathrm{simp}}(\mathrm{Cone}(D_{n-1})) \;\underset{\text{by the Cone Homology Lemma 95.11}}{=}\; 0.$$

(2) Let $n \in \mathbb{N}_0$. We consider the following diagram:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{C}_n^{\mathrm{simp}}(S_n) & \xrightarrow{\partial} & \mathrm{C}_{n-1}^{\mathrm{simp}}(S_n) & \xrightarrow{\partial} & \ldots & \xrightarrow{\partial} & \mathrm{C}_0^{\mathrm{simp}}(S_n) & \xrightarrow{\epsilon} & \mathbb{Z} \longrightarrow 0 \\
 & & \downarrow & & \downarrow{=} & & \downarrow{=} & & & & \downarrow{=} & & \downarrow{=} \\
0 & \longrightarrow & \mathrm{C}_{n+1}^{\mathrm{simp}}(D_{n+1}) & \longrightarrow & \mathrm{C}_n^{\mathrm{simp}}(D_{n+1}) & \xrightarrow{\partial} & \mathrm{C}_{n-1}^{\mathrm{simp}}(D_{n+1}) & \xrightarrow{\partial} & \ldots & \xrightarrow{\partial} & \mathrm{C}_0^{\mathrm{simp}}(D_{n+1}) & \xrightarrow{\epsilon} & \mathbb{Z} \longrightarrow 0 \\
 & & \downarrow{\cong} & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & \ldots & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & 0 \longrightarrow 0.
\end{array}$$

We make the following clarifications and comments:

(a) The top two horizontal sequences are the reduced chain complexes of the abstract simplicial complexes S_n and D_{n+1}.

(b) Recall that by definition the simplices of S_n and D_{n+1} are the same except that D_{n+1} has a single $(n+1)$-simplex whereas S_n has no $(n+1)$-simplices. This observation explains the vertical maps.

(c) The diagram clearly commutes and it is clear that the vertical sequences are exact.

(d) It follows from the above that we can apply the LES Proposition 74.9 and we obtain a long exact sequence of homology groups. Together with (1) we obtain the reduced simplicial homology groups of S_n. ■

[1165] One can easily verify that the map P_n is actually well-defined.

In the Suspension-H_*-Proposition 76.7 we studied the singular homology groups of a suspension. Statement (2) of the following lemma is the simplicial analogue of the Suspension-H_*-Proposition 76.7.

Lemma 95.13. (Join-Homology Lemma) Let K be a non-empty abstract simplicial complex.

(1) Let $n \in \mathbb{N}_0$ and let S_n be the abstract simplicial complex that we introduced on page 1978. For every $k \in \mathbb{N}_0$ there exists a natural isomorphism
$$\widetilde{\mathrm{H}}_k^{\mathrm{simp}}(K * S_n) \;\cong\; \widetilde{\mathrm{H}}_{k-(n+1)}^{\mathrm{simp}}(K).$$

(2) We denote by $\Sigma(K)$ the simplicial suspension of K as defined on page 2003. For every $k \in \mathbb{N}_0$ there exists an isomorphism
$$\widetilde{\mathrm{H}}_k^{\mathrm{simp}}(\Sigma(K)) \;\cong\; \widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(K).$$

Example. Let $n \in \mathbb{N}$. Given any $k \in \mathbb{N}_0$ we see that
$$\widetilde{\mathrm{H}}_k\big(\overbrace{\Sigma(\Sigma(\ldots\Sigma(S_0)))}^{n\text{-fold simplicial suspension}}\big) \;\underset{\uparrow}{\cong}\; \widetilde{\mathrm{H}}_{k-n}(S_0) \;\underset{\uparrow}{\cong}\; \begin{cases} \mathbb{Z}, & \text{if } k-n=0, \\ 0, & \text{otherwise.} \end{cases}$$
Join-Homology Lemma 95.13 applied n times see the discussion on page 2041

Proof. Since the author has no intention of monopolizing all the fun we will leave the proof of Statement (1) as Exercise 95.2 to the reader. Note that Statement (2) is an immediate consequence of Statement (1) since $\Sigma(K)$ is naturally isomorphic to $K * S_0$. ∎

Next we extend the definition of simplicial homology groups in the obvious way to simplicial complexes:

Definition. Let $(X, (K = (V, S), \Theta\colon |K| \to X))$ be a simplicial complex. Given $n \in \mathbb{N}_0$ we define
$$\mathrm{H}_n^{\mathrm{simp}}(X) \;:=\; \mathrm{H}_n^{\mathrm{simp}}(K).$$
Similarly, given a simplicial map $f\colon X = (X, \Theta) \to Y = (L, \Omega)$ between two simplicial complexes we define the induced map $f_*\colon \mathrm{H}_n^{\mathrm{simp}}(X) \to \mathrm{H}_n^{\mathrm{simp}}(Y)$ as the induced map $f_*\colon \mathrm{H}_n^{\mathrm{simp}}(K) \to \mathrm{H}_n^{\mathrm{simp}}(L)$.

Finally we introduce simplicial homology with coefficients in the obvious way. More precisely, we have the following definition which is the obvious variation on the definition given in Section 89.2.

Definition. Let G be an abelian group and let $n \in \mathbb{N}_0$. Given an abstract simplicial complex K we set
$$\mathrm{H}_n^{\mathrm{simp}}(K; G) \;:=\; \mathrm{H}_n(\mathrm{C}_*^{\mathrm{simp}}(K) \otimes G).$$
If K is actually ordered we set $\mathrm{H}_n^{\mathrm{simp},\leq}(K; G) := \mathrm{H}_n(\mathrm{C}_*^{\mathrm{simp},\leq}(K) \otimes G)$. Finally given a simplicial complex $(X, (K = (V, S), \Theta\colon |K| \to X))$ we define, following the above convention, $\mathrm{H}_n^{\mathrm{simp}}(X; G) := \mathrm{H}_n^{\mathrm{simp}}(K; G)$.

Surely there is no need to state all the self-evident properties of simplicial homology with coefficients. Let us just mention that it follows from the Tensor Lemma 89.3 (3) that given any abstract simplicial complex K and given any $n \in \mathbb{N}_0$ the map
$$\begin{array}{ccc} \mathrm{H}_n^{\mathrm{simp}}(K; \mathbb{Z}) & \to & \mathrm{H}_n^{\mathrm{simp}}(K) \\ \Big[\sum\limits_{i=0}^{k} s_i \otimes m_i\Big] & \mapsto & \Big[\sum\limits_{i=0}^{k} s_i \cdot m_i\Big] \end{array}$$

is a natural isomorphism. Throughout these notes we will use this isomorphism to make the identification $\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{Z}) = \mathrm{H}_n^{\mathrm{simp}}(K)$.

95.3. **The Simplicial Künneth Theorem.** Let $(K,\leq)$ and $(L,\leq)$ be two ordered abstract simplicial complexes. Recall that on page 1998 we introduced the corresponding product $K \times L$. The following theorem, which can be viewed as the simplicial analogue of the Künneth Theorem 90.8 for Topological Spaces, computes the simplicial homology groups of $K \times L$ in terms of the simplicial homology groups of K and L.

Theorem 95.14. (Simplicial Künneth Theorem) Let $(K,\leq)$ and $(L,\leq)$ be two ordered abstract simplicial complexes. There exists a short exact sequence

$$0 \to \bigoplus_{p+q=n} \mathrm{H}_p^{\mathrm{simp},\leq}(K) \otimes \mathrm{H}_q^{\mathrm{simp},\leq}(L) \xrightarrow{\Omega} \mathrm{H}_n^{\mathrm{simp},\leq}(K \times L) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p^{\mathrm{simp},\leq}(K), \mathrm{H}_q^{\mathrm{simp},\leq}(L)) \to 0$$

with the following four properties:

(1) Both maps are natural with respect to the ordered abstract simplicial complexes K and L.
(2) For $p = 0$ the map $\Omega\colon \mathrm{H}_0^{\mathrm{simp},\leq}(K) \otimes \mathrm{H}_q^{\mathrm{simp},\leq}(L) \to \mathrm{H}_q^{\mathrm{simp},\leq}(K \times L)$ is the "obvious" map, namely for any vertex v of K and any simplicial cycle $\sigma \in \mathrm{C}_q^{\mathrm{simp},\leq}(L)$ the image of the class $[v] \otimes [\sigma] \in \mathrm{H}_0^{\mathrm{simp},\leq}(K) \otimes \mathrm{H}_q^{\mathrm{simp},\leq}(L)$ under Ω equals the image of $[\sigma]$ under the map $\mathrm{H}_q^{\mathrm{simp},\leq}(L) \to \mathrm{H}_q^{\mathrm{simp},\leq}(\{v\} \times L) \to \mathrm{H}_q^{\mathrm{simp},\leq}(K \times L)$, where the first map is induced by the natural isomorphism and the second map is induced by the inclusion.
(3) The analogue of (2) also holds for $q = 0$.
(4) The above short exact sequence splits.

We will not really make use of the Simplicial Künneth Theorem 95.14, thus we will only provide a sketch of the proof. A full proof is given in [**FePi2011**, Theorem III.6.4].

Sketch of proof. Using the Acyclic Model Theorem 113.16, together with Exercise 95.5, one can prove a simplicial analogue of the Eilenberg-Zilber Theorem 90.4, i.e. one can prove that there exists a natural chain map

$$\Upsilon\colon C_*^{\mathrm{simp},\leq}(K) \otimes C_*^{\mathrm{simp},\leq}(L) \ \to \ C_*^{\mathrm{simp},\leq}(K \times L)$$

with the following three properties:

(1) The chain map is a chain homotopy equivalence.
(2) Given any vertex v of K and given any ordered simplex $(w_0,\ldots,w_k)$ of L we have $\Upsilon([v] \otimes [w_0,\ldots,w_k]) = [(v,w_0),\ldots,(v,w_k)]$.
(3) The analogue of (2) with the roles of K and L swapped also holds.

The precise construction of such a map is given in [**FePi2011**, Chapter III.6]. Next we consider the following maps:

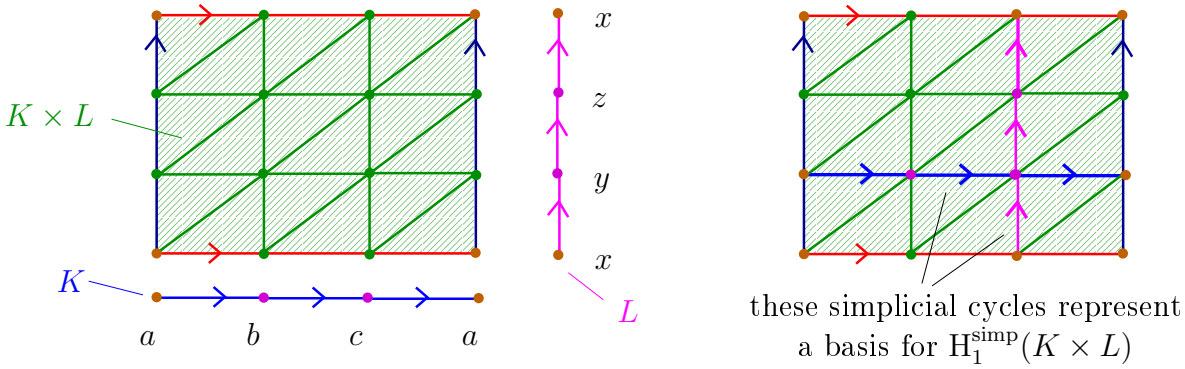
$$\begin{array}{ccccccccc} & & & & \mathrm{H}_n^{\mathrm{simp},\leq}(\mathrm{C}_*^{\mathrm{simp},\leq}(K)\otimes \mathrm{C}_*^{\mathrm{simp},\leq}(L)) & & & & \\ & & & \overset{\Omega}{\nearrow} & \downarrow \cong \Upsilon & \searrow & & & \\ 0 & \to & \bigoplus_{p+q=n} \mathrm{H}_p^{\mathrm{simp},\leq}(K)\otimes \mathrm{H}_q^{\mathrm{simp},\leq}(L) & & & & \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p^{\mathrm{simp},\leq}(K), \mathrm{H}_q^{\mathrm{simp},\leq}(L)) & \to & 0. \\ & & & \searrow & \underbrace{\mathrm{H}_n^{\mathrm{simp},\leq}(\mathrm{C}_*^{\mathrm{simp},\leq}(K\times L))}_{=\mathrm{H}_n(K\times L)} & \nearrow & & & \end{array}$$

Here the sequence on the top is the natural short exact sequence provided by the Künneth Theorem 90.7 for Chain Complexes. It follows from the above together with Corollary 73.4 that the vertical map is a natural isomorphism. It is now fairly straightforward to see that we obtain the desired short exact sequence from replacing the upper middle term by the lower middle term $\mathrm{H}_n^{\mathrm{simp},\leq}(\mathrm{C}_*^{\mathrm{simp},\leq}(K\times L)) = \mathrm{H}_n^{\mathrm{simp},\leq}(K\times L)$. ■

Example. We consider the ordered abstract simplicial spaces K and L that are shown in the figure below. Both are simplicially isomorphic to the simplicial circle S_1. Furthermore we consider the "simplicial torus" $K \times L$. It follows from the Simplicial Künneth Theorem 95.14 that

$$\mathrm{H}_i^{\mathrm{simp}}(K\times L) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2, \\ \mathbb{Z}^2, & \text{if } i = 1, \\ 0, & \text{if } i > 2. \end{cases}$$

Furthermore it follows easily from Statement (2) of the Simplicial Künneth Theorem 95.14 that the two singular cycles shown in the figure below form a basis for $H_1^{\mathrm{simp},\leq}(K\times L)$.



95.4. **The topological invariance of simplicial homology I.** The following question arises naturally.

Question 95.15. Let $(X, (K, \Theta\colon |K| \to X))$ be a simplicial complex and let $n \in \mathbb{N}_0$. Does the isomorphism type of the simplicial homology group $\mathrm{H}_n^{\mathrm{simp}}(X)$ only depend on the homeomorphism type of X?

For example, we know by the discussion on page 1997 and on page 2005 that the topological realizations of S_n and of the n-fold simplicial suspension of S_0 are both homeomorphic to

S^n and conveniently enough we know by Lemma 95.12 and the last example of the previous section that the homology groups are indeed isomorphic.

The following "research program" summarizes the traditional approach to tackling Question 95.15.

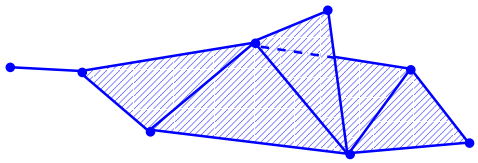
Program 95.16.

(1) Show that any two (let us say finite) abstract simplicial complexes with homeomorphic topological realizations are related by a finite sequence of "moves".
(2) Show that the isomorphism type of simplicial homology groups is unchanged under these "moves".

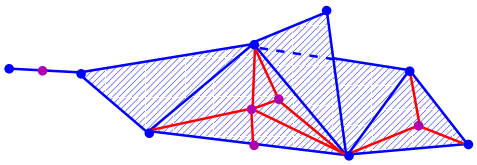
This idea leads straight to the definition of a subdivision of a simplicial complex.

Definition. We say that a simplicial complex $(X, (K = (V, S), \Omega\colon |K| \to X))$ is a subdivision of some simplicial complex $(X, (L = (W, T), \Theta\colon |L| \to X)$ if the following two conditions are satisfied:

(1) Every simplex of $|K|$ is contained in some simplex of $|L|$. More precisely, given any simplex $s \in S$ of K there exists a simplex $t \in T$ of L such that $\Omega(|s|) \subset \Theta(|t|)$.
(2) For every k-simplex s of K and every l-simplex t of L with $\Omega(|s|) \subset \Theta(|t|)$ the map $\Phi_t^{-1} \circ \Theta \circ \Omega \circ \Phi_s\colon \Delta^k \to |s| \subset |t| \to \Delta^l$ is affine linear.



simplicial complex



subdivision of the simplicial complex

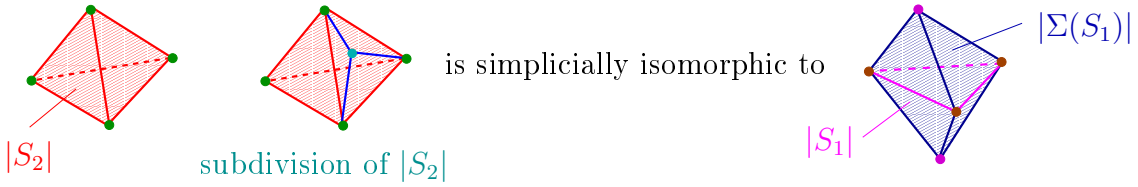
Remark.

(1) Note that for linear simplicial complexes condition (3) is automatically satisfied.
(2) The definition of a subdivision we give agrees with the definition in most sources, see e.g. [**Hud1969**, p. 5], [**Pon1952**, p. 36], [**RS1972**, p. 15], [**Gla1970**, p. 7], [**Spa1995**, p. 121] and [**Zee1963b**, p. I.4].
(3) In [**Mun1984**, p. 83] the definition of a subcomplex has the extra condition that every simplex of X equals the union of finitely many simplices of Y. It follows from [**Hud1969**, Lemma 1.2] together with the Simplicial Complex-Finiteness Proposition 93.12 (2) that the above definition agrees with the definition provided in [**Mun1984**, p. 83].
(4) The definition of a subdivision is somewhat unsatisfactory. In the spirit that abstract simplicial complexes should be purely combinatorial objects there should be a "topology free" definition of the subdivision of an abstract simplicial complexes, which then mirrors the topological definition given above. We will return to the question of how to work completely combinatorially in Section 96.6.

Examples.

(1) It follows from the SD-Lemma 94.3 (3a) and (3c) that the barycentric subdivision $\mathrm{sd}(X)$ of a simplicial complex X is indeed a subdivision in the sense of the above definition.

(2) In the figure below we see that a subdivision of the simplicial complex $|S_2|$ is simplicially isomorphic to $\Sigma(S_1)$.



(3) Let $n \in \mathbb{N}_0$. By the Join-Simplex Lemma 93.19 (3) we know that the map

$$\begin{array}{rcl} f\colon \partial\Delta^n * \{\star\} & \to & \Delta^n \\ [(x,t,\star)] & \mapsto & x\cdot(1-t) + \left(\frac{1}{n+1},\dots,\frac{1}{n+1}\right)\cdot t \end{array}$$

is a homeomorphism. It follows easily from this observation that the simplicial complex $(S_{n-1} * \{\star\}, |S_{n-1} * \{\star\}| = \partial\Delta^n * \{\star\} \xrightarrow{f} \Delta^n)$ is a subdivision of the simplicial complex $(D_n, \Omega\colon |D_n| \to \Delta^n)$. We refer to the figure below for an illustration.



For completeness' sake we mention the following result.

Proposition 95.17.

(1) Let X and Y be two finite linear simplicial complexes. If $|X| = |Y|$, then there exist subdivisions X' of X and Y' of Y which are simplicially isomorphic.
(2) If we are given two subdivisions X' and X'' of some finite linear simplicial complex X, then there exists a subdivision Y of X which is a subdivision of X' and X''.

Proof.

(1) This statement is proved in [**Gla1970**, Corollary I.4] and [**Zee1963b**, Corollary I.1].
(2) This statement follows immediately from (1) together with Lemma 93.22. ■

The following theorem gives us part (2) of Program 95.16. We will not provide a proof, instead we refer to [**Mun1966a**, Theorem 17.2] for a detailed proof.

Theorem 95.18. Let X be a simplicial complex and let X' be a subdivision of X. If $\varphi\colon X' \to X$ is a simplicial approximation of the identity map $X' \to X$, then the induced map $\varphi_*\colon \mathrm{H}_n^{\mathrm{simp}}(X') \to \mathrm{H}_n^{\mathrm{simp}}(X)$ is an isomorphism for every $n \in \mathbb{N}_0$.

So an obvious candidate for the "moves" of Program 95.16 is a combination of subdivisions and simplicial isomorphisms. Now that we have dealt with part (2) of Program 95.16, let us try to tackle part (1). In fact this part is the following conjecture, which came to be known as the "Hauptvermutung". It was originally formulated by Heinrich Tietze [**Tie1908**, p. 13f] in 1908. The conjecture was also raised as a question by Ernst Steinitz [**Steini1908**, p. 32] in 1908.

Conjecture 95.19. (Hauptvermutung) Let X and Y be two finite simplicial complexes. If the underlying topological spaces are homeomorphic, then there exist subdivisions of X and Y which are simplicially isomorphic.

For better or worse it turned out that the Hauptvermutung was too optimistic. More precisely, the Hauptvermutung was eventually disproved by John Milnor [**Miln1961**, Theorems 1 and 2] in 1961.

Theorem 95.20. There exist two finite linear simplicial complex X and Y for which the underlying topological spaces are homeomorphic, but such that there are no subdivisions of X and Y which are simplicially isomorphic.

Remark. In fact a few years later Rob Kirby and Larry Siebenmann [**KSi1977**] (see also [**Ran1996**, Chapter 4]) showed that given any $n \geq 5$ there exist such finite abstract simplicial complexes such that the underlying topological space is a compact n-dimensional topological manifold.

Proof. The theorem is proved in [**Miln1961**, Theorems 1 and 2]. Alternative proofs are given in [**CoheM1973**, p. 84], [**Stal1965b**, p. 252] and [**Ran1996**, Chapter 2]. ■

Even though Part (1) of Program 95.16 did not work out, it turns out that Question 95.15 nonetheless has a positive answer. More precisely we have the following theorem.

Theorem 95.21. Let X and Y be two simplicial complexes. If the underlying topological spaces X and Y are homeomorphic, then for every $n \in \mathbb{N}_0$ the simplicial homology groups $\mathrm{H}_n^{\mathrm{simp}}(X)$ and $\mathrm{H}_n^{\mathrm{simp}}(Y)$ are isomorphic.

Using the Simplicial Approximation Theorem 94.8 and several other tools from the theory of simplicial complexes one can give a direct proof of Theorem 95.21. This approach is for example pursued in [**Mun1966a**, Corollary 18.2], [**Pra2007**, Chapter 2.2] and [**Gib2010**, p. 136].

We will instead take an indirect approach to proving Theorem 95.21, namely we will show that given a simplicial complex the *simplicial* and *singular* homology groups are isomorphic. By the functoriality of *singular* homology the isomorphism type of *singular* homology groups only depends on the homeomorphism type of the topological space. With this observation in mind we see that Theorem 95.21 is a consequence of the following theorem.

Theorem 95.22. Let X be a simplicial complex. For every $n \in \mathbb{N}_0$ there exists a natural[1166] isomorphism $\Phi_X \colon \mathrm{H}_n^{\mathrm{simp}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$.

We postpone the proof of Theorem 95.22 to the next section. In fact, in the next section we will formulate and prove a more precise version of Theorem 95.22. But before we delve into the detailed arguments let us first discuss the respective advantages and disadvantages of simplicial homology.

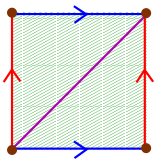
[1166] Here "natural" means that if $f \colon X \to Y$ is a simplicial map, then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_n^{\mathrm{simp}}(X) & \xrightarrow{f_*} & \mathrm{H}_n^{\mathrm{simp}}(Y) \\ \downarrow{\scriptstyle \Phi_X} & & \downarrow{\scriptstyle \Phi_L} \\ \mathrm{H}_n(X) & \xrightarrow{f_*} & \mathrm{H}_n(Y). \end{array}$$

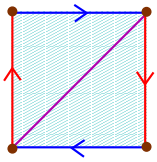
In some textbooks, see e.g. [**Matv2006, Mun1984, SZ1994, HY1988**], homology groups are initially introduced via simplicial homology. This approach has several advantages:

(1) The definition of simplicial homology is much more elementary and intuitive than the definition of singular homology.
(2) One can immediately compute the simplicial homology groups of interesting simplicial complexes, e.g. as we will see in Exercise 95.10, with enough patience and a little bit of cleverness one can compute the simplicial homology groups of abstract simplicial complexes that correspond to the surface of genus g.

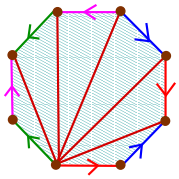
Nonetheless there are serious disadvantages to this approach. A first problem, which is already apparent in the case of the torus, is that even for relatively simple examples the simplicial structures tend to have a large number of simplices, thus the corresponding chain complexes tend to be fairly large and brute force computations by hand become unpleasant. It turns out that there are some ways around this, for example one can extend the theory of simplicial homology from simplicial complexes to "Δ-complexes", which tend to have a significantly smaller number of simplices. Loosely speaking a Δ-complex is a topological space that is decomposed into simplices but now we only demand that the interiors of the simplices are embedded. To give a flavor of the theory we show three Δ-complexes in the figure below. The theory of Δ-complexes is developed in detail in [**Hat2002**, Chapter 2.1].



Δ-complex structure for the torus



Δ-complex structure for $\mathbb{R}\mathrm{P}^2$



Δ-complex structure for the surface of genus 2

Here are more serious reasons why we, and most modern authors, decided against starting out with simplicial homology and why we do not develop the theory of simplicial homology in greater detail:

(1) For slightly more complicated topological spaces it is rather laborious to compute simplicial homology groups. For example we challenge the reader to compute the simplicial homology groups of the projective spaces $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ without reverting to cellular homology.
(2) Most topological spaces one wants to study (e.g. the complement of a knot in S^3), do not come with an obvious simplicial structure, it can be quite cumbersome to show that a simplicial structure exists, let alone to give an explicit simplicial structure.
(3) For a given topological space X which admits a simplicial structure we know by Theorem 95.21 that we have a well-defined *isomorphism type* of simplicial homology groups, but there is no single group that we can associate to X.
(4) Simplicial homology is functorial with respect to simplicial maps. Most maps are not simplicial. We do have the Simplicial Approximation Theorem 94.8 as a tool. But to use it in a sensible fashion one needs to show that the simplicial approximations of a given continuous map induce the same map on simplicial homology.

It should have become clear by now that simplicial homology comes with a Pandora's box of issues and problems. Overall, the initial extra effort in developing the theory of singular homology groups eventually pays of handily. The theory of singular homology groups is much more natural and much more flexible than the rather pedestrian theory of simplicial homology groups.
We conclude this section with the following piece of wisdom which was handed down to the author from his advisor:

> *For actual calculations it is best to work with very small objects, for proving general statements it is best to work with very large objects.*

95.5. **The topological invariance of simplicial homology II.** In this section we will first show that given an abstract simplicial complex the simplicial homology groups are naturally isomorphic to singular homology groups. Afterwards we will try to find a natural chain homotopy equivalence.
We start out with the first key result of this section.

Theorem 95.23. (Simplicial-Singular Homology Isomorphism Theorem) Let $(K, \Theta\colon |K| \to X)$ be a simplicial complex. For each $n_0 \in \mathbb{N}$ there exists a natural isomorphism $\mathrm{H}_n^{\mathrm{simp}}(K) \to \mathrm{H}_n(X)$.

The proof of the Simplicial-Singular Homology Isomorphism Theorem 95.23 builds on a proposition which relates the simplicial chain complex to the cellular chain complex. Before we state the proposition we recall that given a CW-complex we consider $\mathrm{C}_n^{\mathrm{CW}}(X) := \mathrm{H}_n(X^n, X^{n-1})$ together with the boundary map d_n which is given by the composition

$$\mathrm{H}_n(X^n, X^{n-1}) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(X^{n-1}) \xrightarrow{j_{n-1}} \mathrm{H}_{n-1}(X^{n-1}, X^{n-2})$$
$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{=:d_n}$$

where ∂_n is the connecting map in the long exact sequence of the pair (X^n, X^{n-1}) and where j_{n-1} is the natural map from absolute to relative homology.
Now we can formulate the promised proposition.

Proposition 95.24. Let K be an abstract simplicial complex. Given an ordered n-simplex $s = (v_0, \ldots, v_n)$ of K we denote by

$$\begin{aligned} \Phi_s^{\leq}\colon \Delta^n = \left\{(t_0, \ldots, t_n) \in \mathbb{R}_{\geq 0}^{n+1} \,\middle|\, \sum_{i=0}^n t_i = 1\right\} &\to |s| \subset |K| = X \\ (t_0, \ldots, t_n) &\mapsto \sum_{i=0}^n t_i \cdot v_i \end{aligned}$$

the corresponding characteristic map. The maps[1167]

$$\begin{aligned} \Upsilon_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) &\to \mathrm{C}_n^{\mathrm{CW}}(|K|) \\ [s] &\mapsto (\Phi_s^{\leq}\colon \Delta^n \to |K|) \end{aligned}$$

form a natural isomorphism of chain complexes.

Proof of Proposition 95.24. Let $K = (V, S)$ be an abstract simplicial complex. We write $X := |K|$. Given $n \in \mathbb{N}_0$ we denote by S_n the set of ordered n-simplices of K. Next

[1167]Recall that in the Simplicial-Implies-CW Lemma 93.24 we showed that the simplicial complex $|K|$ admits a natural CW-structure.

we consider the following diagram

$$\begin{array}{ccc}
\mathrm{C}_n^{\mathrm{simp}}(K) := \Big(\bigoplus_{s\in S_n} \mathbb{Z}\cdot s\Big)/I_n & \xrightarrow{[s]\mapsto[\Phi_s^{\leq}]} & \mathrm{H}_n(X^n, X^{n-1}) = \mathrm{C}_n^{\mathrm{CW}}(X) \\
\Big\downarrow{\scriptstyle \partial_n} & \swarrow{\scriptstyle \partial} & \Big\downarrow{\scriptstyle d_n} \\
\mathrm{C}_{n-1}^{\mathrm{simp}}(K) := \Big(\bigoplus_{s\in S_{n-1}} \mathbb{Z}\cdot s\Big)/I_{n-1} & \xrightarrow{[s]\mapsto[\Phi_s^{\leq}]} & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}) = \mathrm{C}_{n-1}^{\mathrm{CW}}(X).
\end{array}$$

$$\mathrm{H}_{n-1}(X^{n-1}) \xrightarrow{j_{n-1}} \mathrm{H}_{n-1}(X^{n-1}, X^{n-2})$$

We make the following clarifications and observations:

(1) If s is an ordered k-simplex, then it follows immediately from the definition of the CW-structure on X that $\Phi_s^{\leq}(\Delta^k) \subset X^k$ and that $\Phi_s^{\leq}(\partial\Delta^k) \subset X^{k-1}$. In particular this shows that $\Phi_s^{\leq}$ represents an element in $\mathrm{H}_k(X^k, X^{k-1})$.
(2) It follows fairly easily from (1) together with Lemma 51.1 and Proposition 76.20 that the horizontal maps are well-defined.
(3) Basically the same argument as in the Skeleton-Homology Lemma 80.1 (1) shows that the horizontal maps are isomorphisms.
(4) As explained above, the top diagonal map $\partial\colon \mathrm{H}_n(X^n, X^{n-1}) \to \mathrm{H}_{n-1}(X^{n-1})$ is the connecting homomorphism of the pair (X^n, X^{n-1}) and the bottom diagonal map $j_{n-1}\colon \mathrm{H}_{n-1}(X^{n-1}) \to \mathrm{H}_{n-1}(X^{n-1}, X^{n-2})$ is the natural map from absolute homology to relative homology. The map d_n is by definition the composition of these two maps. In other words, the triangle commutes by definition.
(5) It follows fairly immediately from the definition of the connecting homomorphism, see the Topological LES-Proposition 74.13, and the definition of the maps $\Phi_s^{\leq}$ that the pentagon to the left commutes.
(6) It follows from (4) and (5) that the outer square of the diagram also commutes.
(7) The horizontal maps are basically by definition natural.

The above discussion shows that the horizontal maps define a natural isomorphism of chain complexes. ∎

For the reader's convenience we recall the following proposition.

Proposition 80.4. (Singular-Cellular H_*-Isomorphism Proposition) Given any CW-complex X and given any $n \in \mathbb{N}_0$ there exists a uniquely determined isomorphism

$$\Phi_X\colon \mathrm{H}_n(X) \xrightarrow{\cong} \mathrm{H}_n^{\mathrm{CW}}(X)$$

with the property that the following diagram commutes:[1168]

$$\begin{array}{ccccc}
 & & \mathrm{H}_n(X) & & \\
 & \nearrow & & \searrow{\scriptstyle \Phi_X} & \\
\mathrm{H}_n(X^n) & & & & \mathrm{H}_n^{\mathrm{CW}}(X). \\
 & \searrow{\scriptstyle j_n} & & \nearrow & \\
 & & \ker(d_n) & &
\end{array}$$

Furthermore this map Φ_X is natural.

Now we can put everything together to obtain a proof of the Simplicial-Singular Homology Isomorphism Theorem 95.23.

Proof of Theorem 95.23. It follows immediately from the definitions that we only need to consider the case of the topological realization of an abstract simplicial complex K. We equip $|K|$ with the CW-structure provided by the Simplicial-Implies-CW Lemma 93.24. Now given any $n \in \mathbb{N}_0$ we have the following two natural isomorphisms:

$$\underset{\text{natural isomorphism } \Upsilon_K \text{ given by Proposition 95.24}}{\mathrm{H}_n^{\mathrm{simp}}(K)} \xrightarrow[\uparrow]{\cong} \mathrm{H}_n^{\mathrm{CW}}(|K|) \xleftarrow[\uparrow]{\cong} \underset{\text{natural isomorphism } \Phi_{|K|} \text{ given by the Singular-Cellular } \mathrm{H}_*\text{-Isomorphism Proposition 80.4}}{\mathrm{H}_n(|K|).}$$

The natural isomorphism $\Phi_X^{-1} \circ \Upsilon_K$ is the natural isomorphism of our desires. ■

One problem with the Simplicial-Singular Homology Isomorphism Theorem 95.23 is that it does not give us an explicit isomorphism between the homology groups. This issue gets mostly resolved by the following more sophisticated theorem.

Theorem 95.25. (Simplicial-Singular Chain Complex Theorem) Let K be an *ordered* abstract simplicial complex. The maps

$$\begin{array}{rcl} \Theta_n \colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n(|K|) \\ s & \mapsto & (\Phi_s^{\leq} \colon \Delta^n \to |K|) \end{array}$$

are a natural[1169] chain homotopy equivalence. In particular for each $n \in \mathbb{N}_0$ the induced map

$$\Theta_* \colon \mathrm{H}_n^{\mathrm{simp},\leq}(K) \to \mathrm{H}_n(|K|)$$

is a natural isomorphism.

Remark. The Simplicial-Singular Chain Complex Theorem 95.25 can actually be quite useful in practice. Let us elaborate. In practice, given a topological space X one would like to perform the following three types of tasks:

(1) Determine the isomorphism type of the homology groups $\mathrm{H}_k(X)$.
(2) Determine explicit cycles in $\mathrm{C}_k(X)$ that represent a basis or a generating set for the homology groups $\mathrm{H}_k(X)$.
(3) Calculate induced maps $\mathrm{H}_k(X) \to \mathrm{H}_k(Y)$ with respect to, say, given bases or generating sets for the homology groups $\mathrm{H}_k(X)$ and $\mathrm{H}_k(Y)$.

In fact in many situations, e.g. in applications of the Mayer–Vietoris Theorem 78.5, these problems feed on one another. Even though by now we have a long list of tools, the fact that the singular chain groups $\mathrm{C}_*(X)$ are basically always of infinite, in fact mostly of uncountable, rank makes it hard to attack the above problems directly.

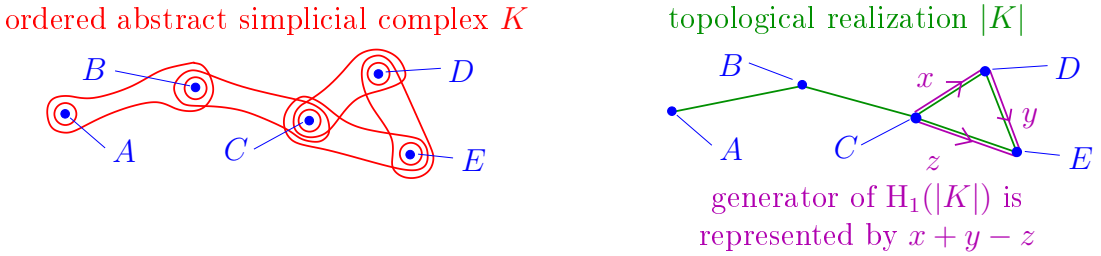
Fortunately, in many situations the Simplicial-Singular Chain Complex Theorem 95.25 allows us to translate the above problems into tasks in a simplicial chain complex. For reasonable abstract simplicial complexes, these problems in turn boil down to elementary problems in linear algebra that usually can be solved easily.

[1168] Here we again denote by $j_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ the obvious map. In the long exact sequence of the pair (X^n, X^{n-1}) we see that the image of $\mathrm{H}_n(X^n)$ in $\mathrm{H}_n(X^n, X^{n-1})$ lies in the kernel of the connecting homomorphism ∂_n. This implies that the map $j_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ takes values in $\ker(d_n)$.
[1169] Here "natural" means with respect to the category of *ordered* abstract simplicial complexes.

Example. We consider the 1-dimensional ordered abstract simplicial complex K shown in the figure below which we had already considered on page 2036. By the discussion on page 2036 we know that

$$\mathrm{H}_1^{\mathrm{simp},\leq}(K) = \ker\left(\partial_1 \colon \mathrm{C}_1^{\mathrm{simp},\leq}(K) \to \mathrm{C}_0^{\mathrm{simp},\leq}(K)\right) = \mathbb{Z} \cdot [\{C,D\} + \{D,E\} - \{C,E\}].$$

We write $x = \Theta_1(\{C,D\})$, $y = \Theta_1(\{D,E\})$ and $z = \Theta_1(\{C,E\})$. It follows from the Simplicial-Singular Chain Complex Theorem 95.25 and the above discussion that $x+y-z$ represents a generator of $\mathrm{H}_1(|K|)$.[1170]



Proof of the Simplicial-Singular Chain Complex Theorem 95.25. Let K be an ordered abstract simplicial complex. It follows almost immediately from the definitions that the maps

$$\begin{array}{rcl} \Theta_n \colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n(|K|) \\ s & \mapsto & (\Phi_s^{\leq} \colon \Delta^n \to |K|) \end{array}$$

define a natural chain map. We leave it to the skeptical reader to verify that this statement is indeed true and basically trivial. It remains to show that the maps Θ_n define a chain homotopy equivalence. By the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 81.4 it now suffices to prove the following claim.

Claim. Let K be an ordered abstract simplicial complex. For every $n \in \mathbb{N}_0$ the induced map

$$\Theta_* \colon \mathrm{H}_n^{\mathrm{simp},\leq}(K) \to \mathrm{H}_n(|K|)$$

is an isomorphism.

Proof. We start out with two observations. Let K be an ordered abstract simplicial complex, let $m \in \mathbb{N}_0 \cup \{\infty\}$ and let $n \in \mathbb{N}_0$. We write $X = |K|$. It follows from Lemma 95.9 and the Skeleton-Homology Lemma 80.1 that the following statements hold:

(i) The two inclusion induced maps

$$\begin{array}{rcl} \mathrm{H}_n^{\mathrm{simp},\leq}(K^m) & \to & \mathrm{H}_n^{\mathrm{simp},\leq}(K) \\ \mathrm{H}_n(|K^m|) & \to & \mathrm{H}_n(|K|) \end{array}$$ are both isomorphisms whenever $n < m$.

(ii) If $n > m$, then $\begin{array}{rcl} \mathrm{H}_n^{\mathrm{simp},\leq}(K^m) & = & 0 \\ \mathrm{H}_n(|K^m|) & = & 0. \end{array}$

It follows immediately from (i) that it suffices to prove the claim for finite-dimensional ordered abstract simplicial complexes. As usual, for finite-dimensional ordered abstract simplicial complexes we proceed by induction on the dimension. The case of an empty and of a 0-dimensional ordered abstract simplicial complex is trivial. So suppose we have

[1170] What other methods do you know for proving this statement?

shown the statement for all ordered abstract simplicial complexes of dimension $\leq m-1$. Now let K be an m-dimensional ordered abstract simplicial complex. We write $X := |K|$ and we denote by $i\colon K^{m-1} \to K$ and $i\colon X^{m-1} \to X$ the inclusion maps. It follows from (i) and (ii), the naturality of Θ and our induction hypothesis that we only need to worry about the two maps

$$\Theta_*\colon \mathrm{H}_m^{\mathrm{simp},\leq}(K) \ \to \ \mathrm{H}_m(X) \quad \text{and} \quad \Theta_*\colon \mathrm{H}_{m-1}^{\mathrm{simp},\leq}(K) \ \to \ \mathrm{H}_{m-1}(X).$$

To take care of this map we consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}_m^{\mathrm{simp},\leq}(K) & \hookrightarrow & \mathrm{C}_m^{\mathrm{simp},\leq}(K) & \xrightarrow{\partial} & \mathrm{H}_{m-1}^{\mathrm{simp},\leq}(K^{m-1}) & \xrightarrow{i_*} & \mathrm{H}_{m-1}^{\mathrm{simp},\leq}(K) \longrightarrow 0 \\
 & & \downarrow{\scriptstyle \Theta_*} & & \cong\downarrow{\scriptstyle \Psi_*} & & \downarrow{\scriptstyle \cong\ \Theta_*} & & \downarrow{\scriptstyle \Theta_*} \\
 & & & & \mathrm{C}_m^{\mathrm{CW}}(X) & & & & \\
 & & & & \downarrow{\scriptstyle \mathrm{id}} & & & & \\
0 & \longrightarrow & \mathrm{H}_m(X) & \longrightarrow & \mathrm{H}_m(X, X^{m-1}) & \xrightarrow{\partial} & \mathrm{H}_{m-1}(X^{m-1}) & \xrightarrow{i_*} & \mathrm{H}_{m-1}(X) \longrightarrow 0.
\end{array}$$

We make the following observations and clarifications:

(1) First we show that the horizontal maps on the top actually make sense. Since $K = K^m$ we see that $\mathrm{H}_m^{\mathrm{simp},\leq}(K)$ is a subgroup of $\mathrm{C}_m^{\mathrm{simp},\leq}(K)$. In fact by definition it equals $\ker(\partial\colon \mathrm{C}_m^{\mathrm{simp},\leq}(K) \to \mathrm{C}_{m-1}^{\mathrm{simp},\leq}(K))$. Furthermore note, that basically by definition, given $s \in \mathrm{C}_m^{\mathrm{simp},\leq}(K)$ we have $\partial s \in \mathrm{C}_{m-1}^{\mathrm{simp},\leq}(K^{m-1})$. Since $\partial \circ \partial = 0$ we see that ∂s defines an element in $\mathrm{H}_{m-1}^{\mathrm{simp},\leq}(K^{m-1})$.

(2) Using (1) and using the explicit definitions of simplicial homology groups it is not hard to see that the top horizontal sequence is exact.

(3) It follows from $X = X^m$ and the Skeleton-Homology Lemma 80.1 that $\mathrm{H}_i(X, X^{m-1}) = 0$ for $i \neq m$. Thus we see that the bottom horizontal sequence is a segment of the long exact sequence of the pair (X, X^{m-1}).

(4) It follows easily from the definition of the maps Θ and Ψ and the explicit description of the connecting homomorphism $\partial\colon \mathrm{H}_m(X, X^{m-1}) \to \mathrm{H}_{m-1}(X^{m-1})$ that we gave in the Topological LES-Proposition 74.13 that the two squares to the left commute. It follows from the naturality of Θ_* that the square to the right commutes.

(5) It follows from induction that the vertical map $\Theta_*\colon \mathrm{H}_{m-1}^{\mathrm{simp},\leq}(K^{m-1}) \to \mathrm{H}_{m-1}(X^{m-1})$ to the right is an isomorphism.

(6) By Proposition 95.24 we know that Ψ_* is an isomorphism.

(7) It follows from all of the above together with the Five-Lemma 74.10 that the vertical maps $\Theta_*\colon \mathrm{H}_m^{\mathrm{simp},\leq}(K) \to \mathrm{H}_m(X)$ and $\Theta_*\colon \mathrm{H}_{m-1}^{\mathrm{simp},\leq}(K) \to \mathrm{H}_{m-1}(X)$ to the left and right are isomorphisms. ■

Given an abstract simplicial complex K together with an order "$\leq$" on K we denote by $\Omega_{\leq,k}\colon \mathrm{C}_k^{\mathrm{simp},\leq}(K) \to \mathrm{C}_k^{\mathrm{simp}}(K)$ the natural chain isomorphism given by Lemma 95.8. Together with the Simplicial-Singular Chain Complex Theorem 95.25 we obtain an isomorphism

$$\mathrm{H}_k^{\mathrm{simp}}(K) \ \xleftarrow{\Omega_{\leq,k}^{-1}} \ \mathrm{H}_k^{\mathrm{simp},\leq}(K) \ \xrightarrow{\Theta_{\leq,k}} \ \mathrm{H}_k(|K|).$$

A priori this isomorphism depends on the choice of the order "$\leq$". The next lemma shows that fortunately this is not the case.

Lemma 95.26. Let K be an abstract simplicial complex and let "$\leq$" and "$\subseteq$" be two orders on K. The two chain maps

$$\begin{aligned} & \mathrm{C}_k^{\mathrm{simp}}(K) \xrightarrow{\Omega_{\leq,k}^{-1}} \mathrm{C}_k^{\mathrm{simp},\leq}(K) \xrightarrow{\Theta_{\leq,k}} \mathrm{C}_k(|K|) \\ \text{and} \quad & \mathrm{C}_k^{\mathrm{simp}}(K) \xrightarrow{\Omega_{\subseteq,k}^{-1}} \mathrm{C}_k^{\mathrm{simp},\subseteq}(K) \xrightarrow{\Theta_{\subseteq,k}} \mathrm{C}_k(|K|) \end{aligned}$$

are chain homotopic. In particular they induce the same maps on homology.

Proof. Let $K = (V, S)$ be an abstract simplicial complex. We start with the following preparations:

(1) For each $i \in \mathbb{N}_0$ we pick once and for all for each i-simplex $s \in S$ an ordered i-simplex $\widetilde{s}$ with $\lfloor \widetilde{s} \rfloor = s$. We denote by $\widetilde{S}_i$ the set given by all these ordered i-simplices.

(2) We introduce the following convention. If for some $i \in \mathbb{N}_0$ we are given for every $\widetilde{s} \in \widetilde{S}_i$ a singular chain $P_i(\widetilde{s}) \in \mathrm{C}_{i+1}(|\widetilde{s}|)$ then we denote by P_i the map

$$\begin{aligned} \mathrm{C}_i^{\mathrm{simp}}(K) &\to \mathrm{C}_{i+1}(|K|) \\ \widetilde{s} &\mapsto P_i(\widetilde{s}) \quad \text{viewed as an element in } \mathrm{C}_{i+1}(|K|) \text{ via the inclusion } |\widetilde{s}| \subset |K| \end{aligned}$$

Now let "$\leq$" and "$\subseteq$" be two orders on $\Delta = (V, S)$.

Claim. Let $l \in \mathbb{N}$. Suppose that for every $i \in \{0, \ldots, l-1\}$ and for every $\widetilde{s} \in \widetilde{S}_i$ we are given a singular chain $P_i(\widetilde{s}) \in \mathrm{C}_{i+1}(|\widetilde{s}|)$ such that for every $i \in \{0, \ldots, l-1\}$ we have the equality

$$(*) \qquad \partial_{i+1} \circ P_i + P_{i-1} \circ \partial_i \;=\; \Theta_{\leq,i} \circ \Omega_{\leq,i}^{-1} - \Theta_{\subseteq,i} \circ \Omega_{\subseteq,i}^{-1} \colon \mathrm{C}_i^{\mathrm{simp}}(K) \to \mathrm{C}_i(|K|).$$

Then for every $\widetilde{s} \in \widetilde{S}_l$ we can also find $P_l(s) \in \mathrm{C}_{l+1}(|\widetilde{s}|)$ such that the above equality holds for $i = 0, \ldots, l$.

Proof. Let $\widetilde{s} \in \widetilde{S}_l$. Note that we have

$$\begin{aligned} \partial_l((-P_{l-1} \circ \partial_l + \Theta_{\leq,l} \circ \Omega_{\leq,l}^{-1} - \Theta_{\subseteq,l} \circ \Omega_{\subseteq,l}^{-1})(\widetilde{s})) &\overset{\text{since } \Theta_{\leq,i} \circ \Omega_{\leq,i}^{-1} \text{ and } \Theta_{\subseteq,i} \circ \Omega_{\subseteq,i}^{-1} \text{ are chain maps}}{=} (-\partial_l \circ P_{l-1} + \Theta_{\leq,l-1} \circ \Omega_{\leq,l-1}^{-1} - \Theta_{\subseteq,l-1} \circ \Omega_{\subseteq,l-1}^{-1})(\partial_l(\widetilde{s})) \\ &= (P_{l-2} \circ \partial_{l-1})(\partial_l(\widetilde{s})) = P_{l-2}(\partial_{l-1} \circ \partial_l)(\widetilde{s}) = 0. \end{aligned}$$

(the last equality: by equality $(*)$ for $i = l-1$)

Note that by construction all terms are contained in $\mathrm{C}_l(|\widetilde{s}|) \subset \mathrm{C}_l(|K|)$. For orientation it is helpful to consider the following diagram

$$\begin{array}{ccccccc} & & \mathrm{C}_l^{\mathrm{simp}}(\widetilde{s}) & \xrightarrow{\partial_l} & \mathrm{C}_{l-1}^{\mathrm{simp}}(\widetilde{s}) & \xrightarrow{\partial_{l-1}} & \mathrm{C}_{l-1}^{\mathrm{simp}}(\widetilde{s}) \\ & \overset{P_l}{\swarrow} & \downarrow & \overset{P_{l-1}}{\swarrow} & \downarrow & \overset{P_{l-1}}{\swarrow} & \downarrow \\ \mathrm{C}_{l+1}(|\widetilde{s}|) & \xrightarrow[\partial_{l+1}]{} & \mathrm{C}_l(|\widetilde{s}|) & \xrightarrow[\partial_l]{} & \mathrm{C}_{l-1}(|\widetilde{s}|) & \xrightarrow[\partial_{l-1}]{} & \mathrm{C}_{l-2}(|\widetilde{s}|) \end{array}$$

Note that we are dealing with $l \in \mathbb{N}_0$. This implies that $\mathrm{H}_l(|\widetilde{s}|) = 0$. But this means that every cycle is in fact a boundary. In particular the above cycle is the boundary of some $P_{l+1}(\widetilde{s}) \in \mathrm{C}_{l+1}(|\widetilde{s}|)$, i.e. we have

$$(\partial_{l+1}(P_{l+1}(\widetilde{s})) \;=\; (-P_{l-1} \circ \partial_l + \Theta_{\leq,l} \circ \Omega_{\leq,l}^{-1} - \Theta_{\subseteq,l} \circ \Omega_{\subseteq,l}^{-1})(\widetilde{s}).$$

By construction all these $P_{l+1}(\widetilde{s})$ have the desired property. ⊞

Now we can easily find the desired chain homotopy. Indeed, for every 0-simplex s we set $P_0(s) = 0 \in \mathrm{C}_1(|s|)$. Since $\Omega_{\subseteq,0} = \Omega_{\subseteq,0} \colon \mathrm{C}_0^{\mathrm{simp}}(K) \to \mathrm{C}_0(|K|)$ we see that can start the machine provided by the claim and iteratively we construct maps P_i which are precisely the promised chain homotopy. ■

The following is now an almost immediate corollary to Lemma 95.8, the Simplicial-Singular Chain Complex Theorem 95.25 and Lemma 95.26.

Corollary 95.27. Let K be an abstract simplicial complex. We pick an order "$\leq$" on K. The maps

$$\Xi_k := \Theta_{\leq,k} \circ \Omega_{\leq,k}^{-1} \colon \mathrm{H}_k^{\mathrm{simp}}(K) \ \to \ \mathrm{H}_k(|K|),$$

which by Lemma 95.26 do not depend on the choice of the order "$\leq$", define a natural isomorphism between functors from the category *AbsSimpCplx* of abstract simplicial complexes (the key here is that the word "ordered" is missing) to the category *AbGroup* of abelian groups.

It is worth repeating the following:

(1) Given any abstract simplicial complex K and given any $k \in \mathbb{N}_0$ there exists a natural isomorphism $\mathrm{H}_k^{\mathrm{simp}}(K) \to \mathrm{H}_k(|K|)$ of homology groups.
(2) If K is an ordered abstract simplicial, then there is in fact a natural chain homotopy equivalence $\mathrm{C}_*^{\mathrm{simp},\leq}(K) \to \mathrm{C}_*(|K|)$.

The alert reader will have noticed that given an abstract simplicial complex we gave two natural isomorphisms $\mathrm{H}_n^{\mathrm{simp}}(K) \to \mathrm{H}_n(|K|)$. The following proposition shows that the two isomorphisms agree.

Proposition 95.28. Let K be an abstract simplicial complex and let $n_0 \in \mathbb{N}$. We write $X = |K|$. We denote by

$$\Upsilon_* \colon \mathrm{C}_*^{\mathrm{simp}}(K) \ \to \ \mathrm{C}_*^{\mathrm{CW}}(X) \quad \text{and} \quad \Phi_X \colon \mathrm{H}_n(X) \ \to \ \mathrm{H}_n^{\mathrm{CW}}(X)$$

the isomorphisms given by Propositions 95.24 and 80.4. Next we pick an order "$\leq$" on K. We consider the chain homotopy equivalences

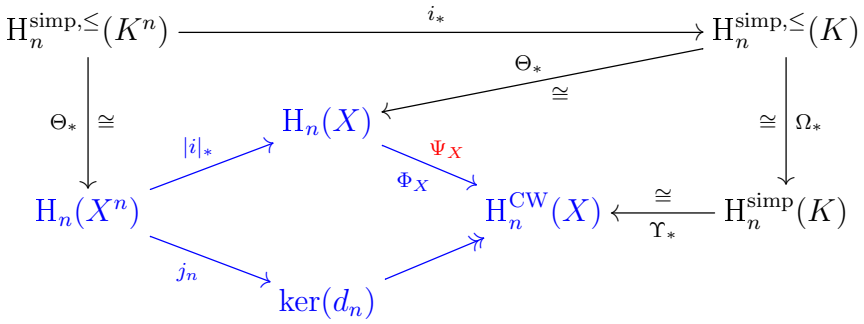
$$\Omega \colon \mathrm{C}_*^{\mathrm{simp},\leq}(K) \ \to \ \mathrm{C}_*^{\mathrm{simp}}(K) \quad \text{and} \quad \Theta \colon \mathrm{C}_*^{\mathrm{simp},\leq}(K) \ \to \ \mathrm{C}_*(|K|)$$

from Lemma 95.8 and the Simplicial-Singular Chain Complex Theorem 95.25. With this notation the following equality holds:

$$\Phi_X^{-1} \circ \Upsilon_* \ = \ \Theta_* \circ \Omega_*^{-1} \colon \mathrm{H}_n^{\mathrm{simp}}(K) \ \to \ \mathrm{H}_n(X).$$

Notation. Let K be an abstract simplicial complex and let $n_0 \in \mathbb{N}$. By a slight abuse of notation we denote by $\Theta_K \colon \mathrm{H}_n^{\mathrm{simp}}(K) \to \mathrm{H}_n(|K|)$ the natural isomorphism given by the Simplicial-Singular Homology Isomorphism Theorem 95.23 and Proposition 95.28.

Proof. We write $\Psi_X := \Upsilon_* \circ \Omega_* \circ \Theta_*^{-1}$. We need to show that $\Psi_X = \Phi_X$. To do so we consider the following diagram:



We make the following clarifications and observations:

(1) The map $i\colon K^n \to K$ is the inclusion map.
(2) The triangle shaped quadrangle to the upper left commutes by the naturality of Θ_*.
(3) By the definition of Ψ_X the quadrangle to the right commutes.
(4) It follows easily from the definitions of the various maps that the "outer diagram" commutes, i.e. the two maps from $\mathrm{H}_n^{\mathrm{simp},\leq}(K^n)$ to $\mathrm{H}_n^{\mathrm{CW}}(X)$ agree.
(5) Since Υ_*, Ω_* and Θ_* are isomorphisms we see that Ψ_X is an isomorphism and we see, in combination with (2), (3) and (4), that Ψ_X makes the parallelogram commute. But by the characterization of Φ_X in the Singular-Cellular H_*-Isomorphism Proposition 80.4 this means that $\Psi_X = \Phi_X$. ■

95.6. **Relative simplicial homology.** In this section, given an (ordered) abstract simplicial complex K together with a subcomplex L we introduce the relative simplicial homology groups $\mathrm{H}_k^{\mathrm{simp}}(K,L)$ and $\mathrm{H}_k^{\mathrm{simp},\leq}(K,L)$. This section contains no surprises and can be skipped easily. We will make little use of relative simplicial homology groups, except that they will come in handy in the proof of the Simplicial Poincaré Duality Theorem 103.4.

Definition. Let K be an abstract simplicial complex and let L be a (possibly empty) subcomplex of K.

(1) Given $k \in \mathbb{N}_0$ we define

$$\mathrm{C}_k^{\mathrm{simp}}(K,L) \;:=\; \mathrm{C}_k^{\mathrm{simp}}(K)/\mathrm{C}_k^{\mathrm{simp}}(L).$$

The usual boundary map $\partial_k\colon \mathrm{C}_k^{\mathrm{simp}}(K) \to \mathrm{C}_{k-1}^{\mathrm{simp}}(K)$ descends to a boundary map $\partial_k\colon \mathrm{C}_k^{\mathrm{simp}}(K,L) \to \mathrm{C}_{k-1}^{\mathrm{simp}}(K,L)$ and we can thus define the corresponding relative simplicial homology groups $\mathrm{H}_k^{\mathrm{simp}}(K,L)$.
(2) If "$\leq$" is an order on K, then we endow L with the corresponding order and we introduce $\mathrm{C}_k^{\mathrm{simp},\leq}(K,L)$ and $\mathrm{H}_k^{\mathrm{simp},\leq}(K,L)$ in the obvious way.

Lemma 95.29. Let K be an abstract simplicial complex and let L be a subcomplex of K. The sequence

$$\begin{array}{ccccccccccc} \ldots & \to & \mathrm{H}_k^{\mathrm{simp}}(L) & \to & \mathrm{H}_k^{\mathrm{simp}}(K) & \to & \mathrm{H}_k^{\mathrm{simp}}(K,L) & \xrightarrow{\partial_k} & \mathrm{H}_{k-1}^{\mathrm{simp}}(L) & \to & \ldots \\ & & & & & & [\sigma] & \mapsto & [\partial_k(\sigma)] & & \end{array}$$

is exact and the connecting homomorphism is natural. The analogous statement also holds if K is equipped with an order.

Proof. The lemma is an immediate consequence of the LES Proposition 74.9. ■

We conclude this short section with the following lemma.

Lemma 95.30. (Simplicial-Singular Homology Isomorphism Lemma) Let K be an abstract simplicial complex and let L be a subcomplex of K.

(1) Let "$\leq$" be an order on K. The natural maps

$$\begin{array}{rcl} \mathrm{C}_k^{\mathrm{simp},\leq}(K,L) & \to & \mathrm{C}_k^{\mathrm{simp}}(K,L) \\ [\sigma] & \mapsto & [\Omega(\sigma)] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{C}_k^{\mathrm{simp},\leq}(K,L) & \to & \mathrm{C}_k(|K|,|L|) \\ [\sigma] & \mapsto & [\Theta(\sigma)] \end{array}$$

are chain homotopy equivalences. Furthermore the natural maps

$$\begin{array}{rcl} \mathrm{H}_k^{\mathrm{simp},\leq}(K,L) & \to & \mathrm{H}_k^{\mathrm{simp}}(K,L) \\ [\sigma] & \mapsto & \Omega(\sigma)] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{H}_k^{\mathrm{simp},\leq}(K,L) & \to & \mathrm{H}_k(|K|,|L|) \\ [\sigma] & \mapsto & [\Theta(\sigma)] \end{array}$$

are isomorphisms.

(2) The map $\Xi := \Omega \circ \Theta^{-1} \colon \mathrm{H}_k^{\mathrm{simp}}(K,L) \to \mathrm{H}_k(|K|,|L|)$ does not depend on the choice of the order "$\leq$" on K, and it defines a natural isomorphism.

(3) The following diagram commutes:

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & \mathrm{H}_k^{\mathrm{simp},\leq}(K) & \longrightarrow & \mathrm{H}_k^{\mathrm{simp},\leq}(K,L) & \xrightarrow{\partial_k} & \mathrm{H}_{k-1}^{\mathrm{simp},\leq}(L) & \longrightarrow & \mathrm{H}_{k-1}^{\mathrm{simp},\leq}(K) \longrightarrow \ldots \\ & & \Xi\downarrow & & \Xi\downarrow & & \Xi\downarrow & & \Xi\downarrow \\ \ldots & \longrightarrow & \mathrm{H}_k(|K|) & \longrightarrow & \mathrm{H}_k(|K|,|L|) & \xrightarrow{\partial_k} & \mathrm{H}_{k-1}(|L|) & \longrightarrow & \mathrm{H}_{k-1}(|K|) \longrightarrow \ldots \end{array}$$

Proof. We consider the following diagram:

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & \mathrm{H}_k^{\mathrm{simp}}(L) & \longrightarrow & \mathrm{H}_k^{\mathrm{simp}}(K) & \longrightarrow & \mathrm{H}_k^{\mathrm{simp}}(K,L) & \xrightarrow{\partial_k} & \mathrm{H}_{k-1}^{\mathrm{simp}}(L) \longrightarrow \mathrm{H}_{k-1}^{\mathrm{simp}}(K) \longrightarrow \ldots \\ & & \Omega\uparrow & & \Omega\uparrow & & \Omega\uparrow & & \Omega\uparrow \\ \ldots & \longrightarrow & \mathrm{H}_k^{\mathrm{simp},\leq}(L) & \longrightarrow & \mathrm{H}_k^{\mathrm{simp},\leq}(K) & \longrightarrow & \mathrm{H}_k^{\mathrm{simp},\leq}(K,L) & \xrightarrow{\partial_k} & \mathrm{H}_{k-1}^{\mathrm{simp},\leq}(L) \longrightarrow \mathrm{H}_{k-1}^{\mathrm{simp},\leq}(K) \longrightarrow \ldots \\ & & \Theta\downarrow & & \Theta\downarrow & & \Theta\downarrow & & \Theta\downarrow \\ \ldots & \longrightarrow & \mathrm{H}_k(|L|) & \longrightarrow & \mathrm{H}_k(|K|) & \longrightarrow & \mathrm{H}_k(|K|,|L|) & \xrightarrow{\partial_k} & \mathrm{H}_{k-1}(|L|) \longrightarrow \mathrm{H}_{k-1}(|K|) \longrightarrow \ldots \end{array}$$

It follows easily from the definitions and the naturality of the connecting homomorphisms, see the LES Proposition 74.9, that the diagram commutes. By Lemma 95.29 and by the Topological LES-Corollary 74.14 we know that the horizontal sequences are exact. Furthermore by Lemma 95.8 and the Simplicial-Singular Chain Complex Theorem 95.25 we know that the outer two vertical maps are isomorphisms. Now it follows from the Five-Lemma 74.10 that the middle vertical maps are also isomorphisms. It follows from the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 81.4 that the corresponding chain maps are chain homotopy equivalences. This concludes the proof Statement (1).

The same argument as in the proof of Corollary 95.27 shows that the composition of the middle vertical maps does not depend on the choice of the order "$\leq$".[1171]

Finally Statement (3) follows from the fact that the above diagram commutes. ■

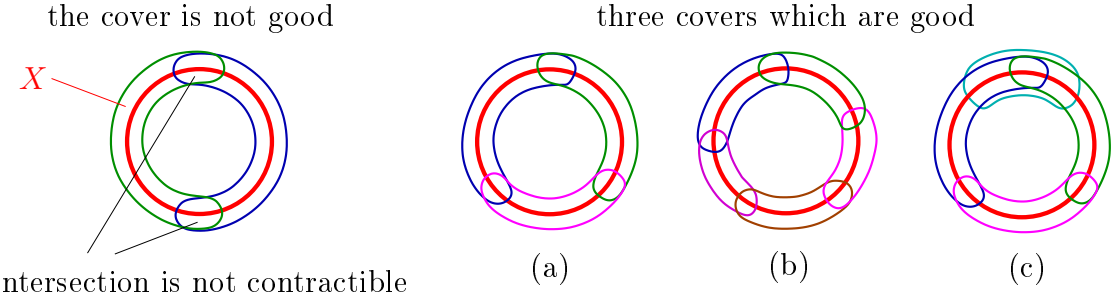
95.7. **The Nerve Theorem.** In this section we will state the "Nerve Theorem". In many situations it gives an effective way for showing that a given topological space is homotopy equivalent to a simplicial complex.

Definition. Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a cover of X.

(1) We say the cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X is good if for every choice of $i_1, \ldots, i_m \in I$ the intersection $U_{i_1} \cap \cdots \cap U_{i_m}$ is either empty or contractible.

(2) We say the cover $\mathcal{U}$ of X is point-finite if each point $x \in X$ is contained in only finitely many U_i.[1172]

Example. In the figure below we show the topological space $X = S^1$ with one cover which is not good and three covers which are actually good.



As we will see, the following lemma is a straightforward consequence of the Link Star Lemma 94.5.

Lemma 95.31. (Open Stars-Good Cover Lemma) Let $K = (V, S)$ be an abstract simplicial complex. The open stars $\{\mathring{\mathrm{St}}(v)\}_{v\in V}$ are an open good point-finite cover.



Proof. Let $K = (V, S)$ be an abstract simplicial complex. Let $P \in |K|$. By the Open Simplex Lemma 93.11 (2) we know that there exists a unique $s \in S$ with $P \in \langle s \rangle$. Furthermore, by the Link Star Lemma 94.5 (2) we know that $P \in \mathring{\mathrm{St}}(v)$ if and only if v is a vertex of s. This discussion, together with the Link Star Lemma 94.5 (2), shows that the open stars $\mathring{\mathrm{St}}(v)$ form an open point-finite cover of $|K|$. It remains to show that the cover

[1171]Note that Exercise 74.3 shows that one cannot reduce this statement "Five Lemma"-style to the fact that we already know that the composition of the outer two vertical maps is independent of the choice of the order "$\leq$".

[1172]Note that this notion is weaker than the notion of a locally finite cover that we introduced on page 114.

is good. Thus let $v_0, \dots, v_n \in V$. We have

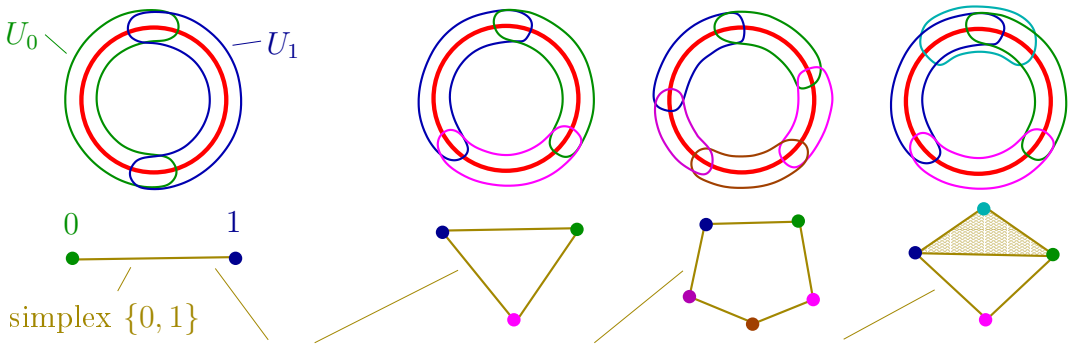
$$\mathring{\mathrm{St}}(v_0) \cap \dots \cap \mathring{\mathrm{St}}(v_n) \underset{\text{Link Star Lemma 94.5 (3)}}{=} \begin{cases} \overbrace{\mathring{\mathrm{St}}(\{v_0, \dots, v_n\})}^{\text{contractible by the Link Star Lemma 94.5 (1f)}}, & \text{if } \{v_0, \dots, v_n\} \text{ is a simplex,} \\ \varnothing, & \text{otherwise.} \end{cases}$$

We have thus shown that the open stars $\{\mathring{\mathrm{St}}(v)\}_{v \in V}$ are a good point-finite cover. ■

As we will see shortly, the Open Stars-Good Cover Lemma 95.31 has a partial converse, namely the mysteriously named Nerve Theorem. To state this converse we need the following definition.

Definition. Let X be a set and let $\mathcal{U} = \{U_i\}_{i \in I}$ be a family of subsets of X. We refer to the abstract simplicial complex that is given by the vertex set I and the simplex set[1173] $\{\{i_1, \dots, i_r\} \,|\, i_1, \dots, i_r \in I \text{ with } U_{i_1} \cap \dots \cap U_{i_m} \neq \varnothing\}$ as the nerve complex $N(\mathcal{U})$ of the covering.

Example. In the figure below we revisit the four covers of S^1 that we showed in the figure on page 2061. This time we also show the topological realizations of the corresponding nerve complexes. As is clear from the figure, for the cover to the left, which is not good, the topological realization of the nerve complex does not look at all like S^1. On the other hand, for the three covers to the right, which are good, the topological realization of the nerve complex is homeomorphic or at least homotopy equivalent to S^1. As we will see in a minute, that is not a coincidence.



To formulate the Nerve Theorem of this section it is convenient to introduce the following definition.

Definition. We say that a map $f\colon X \to Y$ between path-connected topological spaces is a weak homotopy equivalence if there exists $x_0 \in X$ such that for every $k \in \mathbb{N}_0$ the induced map $f_*\colon \pi_k(X, x_0) \to \pi_k(Y, f(x_0))$ is an isomorphism.

By the Homotopy-π_n-Proposition 71.7 (2) a homotopy equivalence between path-connected topological spaces is a weak homotopy equivalence. The following theorem says, rather amazingly, that for CW-complexes the converse holds.

[1173] It follows easily from the definitions that this is indeed an abstract simplicial complex.

Theorem 162.1. (Whitehead Theorem) Let $f\colon X \to Y$ be a map between two connected CW-complexes.[1174] If $f\colon X \to Y$ is a weak homotopy equivalence, then f is a homotopy equivalence.

As is clear from the numbering, we will prove the Whitehead Theorem 162.1 much later in these notes. But, as we will see, this theorem greatly facilitates the discussion of the Nerve Theorem.

Recall that in the Open Stars-Good Cover Lemma 95.31 we showed that every simplicial complex admits a good cover. We can now formulate and prove the Nerve Theorem which can in particular be viewed as a partial converse to the Open Stars-Good Cover Lemma 95.31.

Theorem 95.32. (Nerve Theorem) Let X be a path-connected topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open good point-finite cover of X. The following two statements hold:

(1) There exists a weak homotopy equivalence $|N(\mathcal{U})| \to X$ from the topological realization of the nerve complex $N(\mathcal{U})$ to X. In fact there exists a weak homotopy equivalence $f\colon |N(\mathcal{U})| \to X$ such that for any simplex $\{i_0, \ldots, i_k\}$ of $N(\mathcal{U})$ we have $f(|\{i_0, \ldots, i_k\}|) \subset U_{i_0} \cap \cdots \cap U_{i_k}$.
(2) If X is furthermore a CW-complex, then any map $f\colon |N(\mathcal{U})| \to X$ as in (1) is in fact a homotopy equivalence.

Remark.

(1) The Nerve Theorem has its origins in the work of Karol Borsuk [**Bor1948**] from 1948 and Jean Leray [**Ler1950**] from 1950. As stated the theorem was basically proved by André Weil [**Weil1952**, p. 141] in 1952.
(2) Let X be a CW-complex. It follows from the CW-Complex Properties Proposition 68.10 (8) that the conclusion of the Nerve Theorem 95.32 also holds if $\mathcal{U} = \{U_i\}_{i\in I}$ is a family of *subcomplexes* which form a good point-finite cover of X.
(3) There are many variations on the statement and on the proof of the Nerve Theorem 95.32.
   (a) In [**Bj1995**, Theorem 10.6] and [**Bj2003003**, Theorem 6] a Nerve Theorem is formulated for the case that X is a regular CW-complex[1175] and the U_i are subcomplexes. In this case the cover does not need to be point-finite.
   (b) Other versions are formulated in [**Nago2007**, Theorem 3.3] and [**BoS1973**, Theorem 8.2.1].
   (c) An elegant proof of the Nerve Theorem 95.32 for the case of a finite cover is given in [**Koz2008**, Theorem 15.21].

Proof. Statement (2) of the Nerve Theorem is an immediate consequence of Statement (1) together with the Whitehead Theorem 162.1. Thus it remains to prove Statement (1). In the discussion below we will only sketch an argument. A full proof of Statement (1) is given in [**McCor1967**, Theorem 1].

[1174] Note that we do not assume that the map f is cellular.

[1175] We refer to page 2010 for the definition of a regular CW-complex, note that simplicial complexes are regular CW-complexes.

Thus let X be a topological space and let $\mathcal{U} = \{U_v\}_{v\in V}$ be an open good point-finite cover of X. We adopt the following notation:

(1) We set
$$S \;:=\; \{\{v_0,\dots,v_n\}\,|\,v_0,\dots,v_n\in V \text{ with } U_{v_0}\cap\dots\cap U_{v_n}\neq\varnothing\}.$$
Recall that by definition the nerve complex is given by $N(\mathcal{U}) := (V,S)$. For convenience we write $N = N(\mathcal{U})$.

(2) Given $s=\{v_0,\dots,v_n\}\in S$ we set $U_s := U_{v_0}\cap\dots\cap U_{v_n}$.

(3) We consider the barycentric subdivision
$$\mathrm{sd}(N) \;=\; \big(\{\underline{s}\}_{s\in S}, \{\{\underline{s_0},\dots,\underline{s_k}\}\,|\,s_0\subsetneq\dots\subsetneq s_k\}\big).$$
By the SD-Lemma 94.3 we have a natural homeomorphism $|\mathrm{sd}(N)|\to|N|$ which we use to identify these two topological spaces.

Next we define inductively a map $F\colon|\mathrm{sd}(N)|\to X$ which satisfies the following condition:

($\star$) For every simplex $\{\underline{s_0},\underline{s_1},\dots,\underline{s_k}\}$ of $\mathrm{sd}(N)$ we have
$$F\big(\big|\{\underline{s_0},\underline{s_1},\dots,\underline{s_k}\}\big|\big) \;\subset\; U_{s_0}.$$

We construct F inductively as follows:

(1) For each $\underline{s}$ we pick a point $F(\underline{s})\in U_s$.

(2) Suppose that F is defined on the $(n-1)$-skeleton of $|\mathrm{sd}(N)|$. We want to extend F to the n-skeleton of $|\mathrm{sd}(N)|$. Let $t=\{\underline{s_0},\dots,\underline{s_n}\}$ be an n-simplex of $\mathrm{sd}(N)$ and let $\Phi_t^{\leq}\colon\Delta^n\to|\mathrm{sd}(N)|$ be the corresponding characteristic map. Note that
$$(F\circ\Phi_t^{\leq})(\partial\Delta^n) \;=\; \bigcup_{i=0}^{n} \underbrace{F\big(\big|\{\underline{s_0},\dots,\widehat{\underline{s_i}},\dots,\underline{s_n}\}\big|\big)}_{\subset U_{s_0}\text{ or }\subset U_{s_1}\text{ by }(\star)} \;\subset\; U_{s_0}.$$
by definition we have for $s\subset t$ that $U_t\subset U_s$, in particular we have $U_{s_1}\subset U_{s_0}$

By hypothesis we know that U_{s_0} is contractible. It follows from the Homotopy-π_n-Proposition 71.7 and the Trivial-in-π_n-Lemma 71.4 together with the fact, shown in the Standard Simplex Homeomorphism Lemma 72.1 (3), that the pair $(\Delta^n,\partial\Delta^n)$ is homeomorphic to $(\overline{B}^n,S^{n-1})$, that $F\circ\Phi_t^{\leq}\colon\partial\Delta^n\to U_{s_0}$ extends to a map $G\colon\Delta^n\to U_{s_0}$. Now we define F on $|t|=|\{\underline{s_0},\dots,\underline{s_n}\}|$ to be the map $G\circ\Phi_t^{-1}$. Note that this map still satisfies ($\star$).

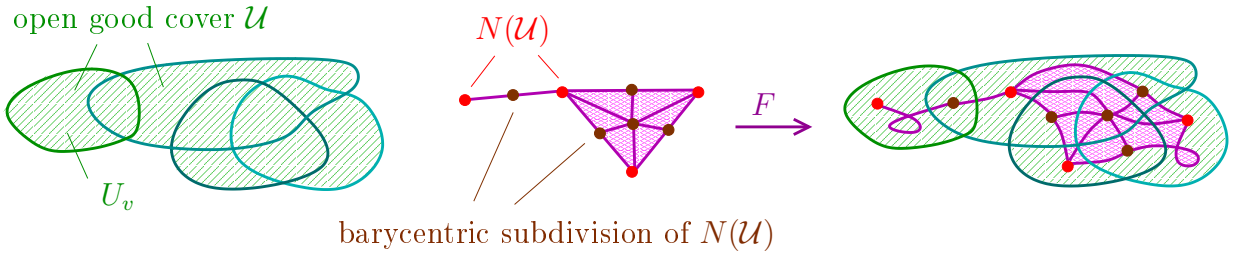
It follows almost immediately from the SD-Lemma 94.3 and ($\star$) that $F\colon|N|=|\mathrm{sd}(N)|\to X$ has the desired property that for every simplex $\{v_0,\dots,v_k\}$ of $N=N(\mathcal{U})$ we have the inclusion $F(|\{v_0,\dots,v_k\}|)\subset U_{v_0}\cap\dots\cap U_{v_k}$.

It remains to show that F is a weak homotopy equivalence. Unfortunately this is where our discussion of the proof peters out. We suggest to use the Weak Homotopy-vs-Homology Isomorphisms Proposition 161.6 below to show that F is a weak homotopy equivalence. One should be able to employ this proposition using the following two approaches:

(1) One uses the Mayer–Vietoris 54.1 and the HNN-Seifert–van Kampen Theorem 57.3, and suitable analogues for simplicial complexes, to show that F induces an isomorphism of homotopy groups.

(2) One uses cleverly the naturality of the long exact sequence coming from the Mayer–Vietoris Theorem 78.5 (or more elegantly, the naturality of the Mayer–Vietoris spectral sequence which we alluded to on page 1720) to show that the lift $\widetilde{F}$ of F to the universal covers of $|\operatorname{sd}(N)|$ and X induces an isomorphism of all homology groups.

We leave it as a rather challenging exercise to the reader to make the above approach work. Even the case of a finite cover is already rather delicate. ■



For completeness we also state the following variation on the Nerve Theorem 95.33.

Theorem 95.33. (Nerve Theorem) Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an good open cover of X. If X is paracompact, then there exists a homotopy equivalence $|N(\mathcal{U})| \to X$.

Proof. Since X is paracompact it follows from the Paracompact-Partition of Unity Theorem 10.9 that there exists a partition of unity $\{\varphi_i\}_{i\in I}$ such that for each $i \in I$ we have $\operatorname{supp}(\varphi_i) \subset U_i$. We define

$$\begin{array}{rcl} G\colon X & \to & |N(\mathcal{U})| \\ x & \mapsto & \begin{pmatrix} I & \to & \mathbb{R} \\ i & \mapsto & \varphi_i(x) \end{pmatrix}. \end{array}$$

In [**Hat2002**, Proposition 4G.2] it is shown that the map G is a homotopy inverse to the map $F\colon |N(\mathcal{U})| \to X$ that we constructed in the proof of the Nerve Theorem 95.32. ■

One of the many fun applications of the Nerve Theorem 95.32 is the following theorem which was first proved by Eduard Helly [**Hel1923**] in 1923 using much more elementary techniques. Our proof is arguably technological overkill, but it has the advantage that from our point of view it is very conceptual.

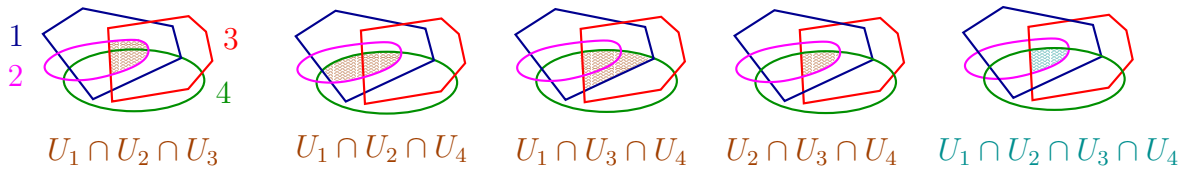
Theorem 95.34. (Helly's Theorem) Let $n \in \mathbb{N}_0$ and let $U_1, \ldots, U_m$ be convex open subsets of $\mathbb{R}^n$. If any choice of $n+1$ of those sets has non-empty intersection, then the intersection of all those sets is non-empty, i.e. we also have

$$U_1 \cap \cdots \cap U_m \neq \varnothing.$$

Examples.

(1) For $n = 1$ and $m = 3$ Helly's Theorem 95.34 says that if we are given three open intervals such that any two of them have a non-empty intersection, then the intersection of the three open intervals is non-empty.

(2) For $n = 2$ and $m = 4$ Helly's Theorem 95.34 says if we are given four convex open subsets of $\mathbb{R}^2$, such that any three of them have non-empty intersection, then the intersection of all four subsets is also non-empty. This situation is illustrated in the figure below.



Remark.

(1) In our formulation of Helly's Theorem 95.34 we require that the sets U_i are open. This is in fact just an artifact of our proof. In fact the statement holds without this assumption. We refer to [**Mato2002**, Theorem 1.3.2] for an elementary proof of this more general statement.
(2) As we mentioned above, our topological proof of Helly's Theorem 95.34 is strictly speaking more difficult than necessary. But the advantage of the topological approach taken in these notes is that it is easier to generalize this approach to non-convex subsets.
(3) We refer to [**Eckh1993**] for a long list of references for proofs of Helly's Theorem and for a discussion of the relevance of Helly's Theorem.

In our proof of Helly's Theorem we will need the following proposition which is interesting in its own right.

Proposition 95.35. If M is an open subset of $\mathbb{R}^n$, then $\mathrm{H}_k(M) = 0$ for $k \geq n$.

Proof. The proposition follows from a fun argument using the Mayer–Vietoris Theorem 78.5. We invite the reader to fill in the details in Exercise 95.15. ■

Proof of Helly's Theorem 95.34. Let $n \in \mathbb{N}_0$. We prove the theorem by induction on m. For $m \leq n+1$ there is nothing to show. Now suppose that the theorem holds for some $m \geq n+1$. Let $\mathcal{U} := \{U_1, \ldots, U_{m+1}\}$ be a family of convex open subsets of $\mathbb{R}^n$ such that the intersection of any $n+1$ of those subsets has non-empty intersection. By induction hypothesis we know that the following holds:

(i) The intersection of any m of the sets in $\mathcal{U}$ is non-empty.

We need to show that $U_1 \cap \cdots \cap U_{m+1} \neq \varnothing$. We do a proof by contradiction. So we assume that the following holds:

(ii) $U_1 \cap \cdots \cap U_{m+1} = \varnothing$.

Next we consider the corresponding nerve complex $N(\mathcal{U})$.

(1) It follows immediately from (i) and (ii) that the nerve complex $N(\mathcal{U})$ is simplicially isomorphic to the simplicial complex S_{m-1} that we introduced on page 1978.
(2) As we mentioned before, one of the key properties of convex subsets is that any intersection of convex subsets is again convex, in particular any intersection is contractible. Thus $\mathcal{U}$ is an open good finite cover of $U_1 \cup \cdots \cup U_{m+1}$.

Now we see that

$$\begin{array}{ccccccccc} \mathbb{Z} & \cong & \mathrm{H}_{m-1}^{\mathrm{simp}}(S_{m-1}) & \cong & \mathrm{H}_{m-1}^{\mathrm{simp}}(N(\mathcal{U})) & \cong & \mathrm{H}_{m-1}(|N(\mathcal{U})|) & \overset{\downarrow}{\cong} & \mathrm{H}_{m-1}\big(\underbrace{U_1 \cup \cdots \cup U_{m+1}}_{\text{open subset of } \mathbb{R}^n}\big) & = & 0. \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ \text{Lemma 95.12} & & \text{by (1)} & & \text{Simplicial-Singular Homology Isomorphism Theorem 95.23} & & & & \text{by Proposition 95.35, since } m-1 \geq n & & \end{array}$$

by the Nerve Theorem 95.32 which we can apply by (2), together with the Homotopy Equivalence-H_*-Corollary 73.9

We have thus obtained a contradiction to (ii). ■

The Nerve Theorem 95.32 gives a practical tool for showing that a given topological space is homotopy equivalent to a simplicial complex. It is natural to go through familiar classes of topological spaces and to see whether it applies. Fortunately the next theorem shows that it applies to everybody's favorite topological spaces, namely smooth manifolds.

Theorem 95.36. (Smooth Manifold-Good Cover Theorem)

(1) Every compact smooth manifold admits an open good finite cover.
(2) Every smooth manifold admits an open good point-finite cover.

Remark. To the best of my knowledge it is not known whether every topological manifold admits an open cover that is good, let alone good and point-finite.

Sketch of proof. Let M be an n-dimensional smooth manifold. There are many approaches to proving the desired statements.[1176]

(1) If M is a smooth manifold without boundary, then in the Smoothly Good Cover Theorem **??** we proved a stronger statement, namely we showed that one can find a finite (respectively locally finite) smoothly good cover. Here, according to the definition on page **??**, "smoothly good" means that in the definition of a good cover we replace "contractible" by "diffeomorphic to $\mathbb{R}^n$".
(2) In the Smooth Manifold-Simplicial Structure Theorem 96.2 we will see that M admits a smooth simplicial structure, which by the Open Stars-Good Cover Lemma 95.31 gives us the desired result.
(3) A detailed proof of the theorem (at least for closed smooth manifolds) is given in [**Weil1952**, p. 120], [**vGl1985**] and [**GuH2019**, Theorem 5.3.2].
(4) Proofs with less detail are given in [**BoT1982**, Theorem 5.1], [**KaL1987**, Chapter VI.3] and [**Lee2009**, Theorem 10.17].

In all approaches, except for the second one, the cases of smooth manifolds with non-empty boundary are not really treated. It is left as a challenge to the reader to extend the results from smooth manifolds without boundary to smooth manifolds with boundary. ■

We obtain the following corollary from the combination of the Smooth Manifold-Good Cover Theorem 95.36, the Nerve Theorem 95.32 and the Simplicial-Implies-CW Lemma 93.24.

Corollary 95.37. Every (compact) smooth manifold is weakly homotopy equivalent to a (finite) simplicial complex, in particular it is weakly homotopy equivalent to a (finite) CW-complex.

[1176] It is not clear to me whether (2) and (3) give smoothly good covers, that is why we did not mention them in the discussion of the Smoothly Good Cover Theorem **??**.

Later, in the Smooth Manifold-CW Structure Theorem 96.5 , we will see that a much stronger version of Corollary 95.37 holds.

95.8. **The simplicial subdivision map.** Let $(K, \leq)$ be an ordered abstract simplicial complex. By the Simplicial-Singular Homology Isomorphism Theorem 95.23 and the SD-Lemma 94.3 we know that the simplicial homology groups of K and of its barycentric subdivision $\mathrm{sd}(K)$ are isomorphic.
In this section we will strengthen this result. More precisely, we will give a natural map $\sigma\colon \mathrm{sd}(K) \to K$ and we will give a natural chain map

$$u_*\colon \mathrm{C}_*^{\mathrm{simp},\leq}(K) \ \to \ \mathrm{C}_*^{\mathrm{simp},\leq}(\mathrm{sd}(K))$$

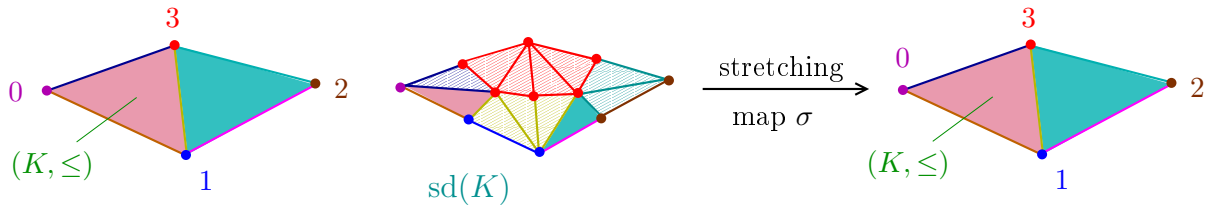
such that the chain maps σ_* and u_* are chain homotopy inverses of one another. These two constructions will prove their worth later on when we study continuous self-maps $f\colon |K| \to |K|$. More precisely, by the Simplicial Approximation Theorem 94.8 we know that a given continuous map $f\colon |K| \to |K|$ is homotopic to a simplicial map $g\colon |\mathrm{sd}^i(K)| \to |K|$, thus we get an induced chain map $g_*\colon \mathrm{C}_*^{\mathrm{simp},\leq}(\mathrm{sd}^i(K)) \to \mathrm{C}_*^{\mathrm{simp},\leq}(K)$. Unfortunately this is a chain map between *different* chain complexes. But if we combine g_* with an i-fold iteration of u_*, then we obtain an *endomorphism* $\mathrm{C}_*^{\mathrm{simp},\leq}(\mathrm{sd}^i(K)) \to \mathrm{C}_*^{\mathrm{simp},\leq}(\mathrm{sd}^i(K))$ of a chain complex.
We start out with the following lemma.

Lemma 95.38. (Stretching Map Lemma) Let $(K, \leq)$ be an ordered abstract simplicial complex.

(1) There exists a unique simplicial map $\sigma\colon \mathrm{sd}(K) \to K$ with the following property: given an n-simplex $s = \{v_0 < \cdots < v_n\}$ of K the map σ sends the corresponding vertex $\underline{s}$ of $\mathrm{sd}(K)$ to the vertex v_n of K.
(2) The stretching map $\sigma\colon \mathrm{sd}(K) \to K$ is order-preserving. (Here we equip $\mathrm{sd}(K)$ with the natural order introduced in Lemma 94.2 (2).)
(3) The stretch map σ is natural. Here "natural" means that given any order-preserving simplicial map $f\colon (K, \leq) \to (L, \leq)$ between two ordered abstract simplicial complexes and given any $n \in \mathbb{N}_0$ the following diagram commutes:[1177]

$$\begin{array}{ccc} \mathrm{sd}(K) & \xrightarrow{f_*} & \mathrm{sd}(L) \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ K & \xrightarrow{f_*} & L. \end{array}$$

We refer to this map σ as the stretching map.



[1177] Recall that by the SD-Lemma 94.3 (5) we know that the maps $K \mapsto \mathrm{sd}(K)$ define a covariant functor from the category $\mathcal{OrdAbsSimpCplx}$ of ordered abstract simplicial complexes to itself.

Proof. It follows easily from the definitions that the given map on the set of vertices induces a simplicial map, that this map is order-preserving and that this map is natural. We leave it to the reader to fill in the details. ■

In basically all cases the stretching map $\sigma\colon \mathrm{sd}(K) \to K$ is not a simplicial isomorphism, in particular the induced map on the simplicial chain complexes is not an isomorphism. Our goal now is to show the "best possible result", namely that the induced chain map $\sigma_*\colon \mathrm{C}_*^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \to \mathrm{C}_*^{\mathrm{simp},\leq}(K)$ is a chain homotopy equivalence. We will do so by giving an explicit chain map in the other direction.

Definition. Let $(K,\leq)$ be an ordered abstract simplicial complex.

(1) Given an n-simplex $s = \{v_0 < \cdots < v_n\}$ of K, given[1178] $\sigma \in \mathrm{Bij}\{0,\ldots,n\}$ and given $m \in \{0,\ldots,n\}$ we write

$$s_{\sigma,m} \;:=\; \{v_{\sigma(0)},\ldots,v_{\sigma(m)}\}.$$

Note that $s_{\sigma,n} = s$. Furthermore, we set

$$s_\sigma \;:=\; \{\overbrace{\underline{s_{\sigma,n}}}^{=\underline{s}}, \underline{s_{\sigma,n-1}},\ldots,\underline{s_{\sigma,0}}\}.$$

Note that s_σ is an n-simplex of $\mathrm{sd}(K)$.

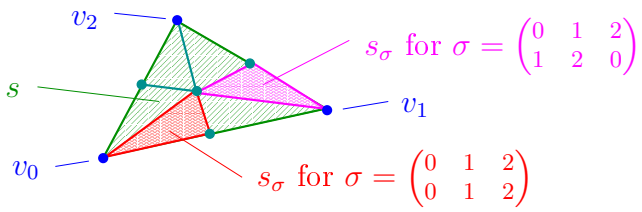
(2) Given an n-simplex s of K we define

$$u_n(s) \;:=\; \sum_{\sigma\in\mathrm{Bij}\{0,\ldots,n\}} \mathrm{sign}(\sigma)\cdot s_\sigma \;\in\; \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)).$$

We refer to the resulting map[1179]

$$\begin{array}{rccc} u_n\colon & \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \\ & s & \mapsto & u_n(s) \end{array}$$

as the simplicial subdivision map.



The following lemma can be viewed as an analogue of the Chain Subdivision Lemma 74.24 and Lemma 74.25.

Lemma 95.39. (Chain Subdivision Lemma) Let $(K,\leq)$ be an ordered abstract simplicial complex. The simplicial subdivision maps

$$u_n\colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) \;\to\; \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)), \quad n \in \mathbb{N}_0$$

[1178] Given $n \in \mathbb{N}_0$ we denote by $\mathrm{Bij}\{0,\ldots,n\}$ the group of bijections of the set $\{0,\ldots,n\}$. Note that we have an obvious isomorphism $\mathrm{Bij}\{0,\ldots,n\} \cong S_{n+1}$. Given $\sigma \in \mathrm{Bij}\{0,\ldots,n\}$ we define $\mathrm{sign}(\sigma)$ basically the same way as we did on page 2035.

[1179] Recall that $\mathrm{C}_n^{\mathrm{simp},\leq}(K)$ is by definition the free abelian group generated by the set of simplices. Thus to specify a homomorphism $\varphi\colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) \to A$ to some abelian group we only need to specify $\varphi(s)$ for each n-simplex s of K.

are a natural chain map. Here "natural" means that given any order-preserving simplicial map $f\colon (K,\leq) \to (L,\leq)$ between two ordered abstract simplicial complexes and given any $n \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \xrightarrow{\quad f_* \quad} & \mathrm{C}_n^{\mathrm{simp},\leq}(L) \\ {\scriptstyle u_n}\downarrow & & \downarrow{\scriptstyle u_n} \\ \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)) & \xrightarrow{\quad f_* \quad} & \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(L)). \end{array}$$

Proof. Let $(K,\leq)$ be an ordered abstract simplicial complex. We need to show that the maps

$$u_n\colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) \;\to\; \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)), \quad n \in \mathbb{N}_0$$

are a natural chain map. In fact it follows immediately from the definitions and the Barycenter Lemma 94.1 (1) that the maps u_n are natural. Thus it remains to prove that the maps u_n are a chain map. We will do so in the following claim.

Claim. For any $n \in \mathbb{N}_0$ and any n-simplex s of K we have the following equality:

$$\partial u_n(s) \;=\; u_{n-1}(\partial s).$$

Proof. We introduce the following notation. Given an n-simplex $t = \{w_0 < \cdots < w_n\}$ and given $i \in \{0,\ldots,n\}$ we write $t(i) := \{w_0 < \cdots < \widehat{w}_i < \cdots < w_n\}$, i.e. $t(i)$ is the $(n-1)$-simplex given by striking out the vertex w_i. Now let $s = \{v_0 < \cdots < v_n\}$ be an n-simplex of K. We perform the following slightly messy calculation:

$$\partial u_n(s) \overset{\text{by definition of } u_n}{=} \partial\Big(\sum_{\sigma\in\mathrm{Bij}\{0,\ldots,n\}} \mathrm{sign}(\sigma)\cdot s_\sigma\Big) \overset{\text{definition of } \partial}{=} \sum_{i=0}^{n}(-1)^i \cdot \sum_{\sigma\in\mathrm{Bij}\{0,\ldots,n\}} \mathrm{sign}(\sigma)\cdot(s_\sigma)(i) \overset{\text{split the sum into } i=0 \text{ and } i=1,\ldots,n}{=}$$

$$= \sum_{\sigma\in\mathrm{Bij}\{0,\ldots,n\}} \mathrm{sign}(\sigma)\cdot(s_\sigma)(0) + \sum_{i=1}^{n}(-1)^i \cdot \sum_{\sigma\in\mathrm{Bij}\{0,\ldots,n\}} \mathrm{sign}(\sigma)\cdot(s_\sigma)(i)$$

second summand vanishes since for a fixed $i \in \{1,\ldots,n\}$ and for $\mu,\nu \in \mathrm{Bij}\{0,\ldots,n\}$ we have $(s_\mu)(i) = (s_\nu)(i)$ if $\mu(i-1) = \nu(i)$, if $\mu(i) = \nu(i-1)$ and if all other values of μ and ν are the same, thus given the fixed $i \in \{1,\ldots,n\}$ we can pair up the elements in $\mathrm{Bij}\{0,\ldots,n\}$ to give the same contribution $(s_\sigma)(i)$ to the sum, but with opposite sign

$$= \sum_{\sigma\in\mathrm{Bij}\{0,\ldots,n\}} \mathrm{sign}(\sigma)\cdot(s_\sigma)(0) \overset{\text{follows from considering the bijection}}{=} \sum_{\tau\in\mathrm{Bij}\{0,\ldots,n-1\}} \sum_{i=0}^{n}(-1)^i\cdot\mathrm{sign}(\tau)\cdot(s(i))_\tau \overset{\text{by definition of } \partial \text{ and } u_{n-1}}{=} u_{n-1}(\partial s).$$

$$\{0,\ldots,n\}\times\mathrm{Bij}\{0,\ldots,n-1\} \to \mathrm{Bij}\{0,\ldots,n\}$$

$$(i,\tau) \mapsto \left(\begin{array}{ccc} \{0,\ldots,n\} & \to & \{0,\ldots,n\} \\ j & \mapsto & \begin{cases} i, & \text{if } j=0, \\ j-1, & \text{if } j\in\{1,\ldots,i\}, \\ j, & \text{if } j>i. \end{cases} \end{array}\right)$$

■

The following lemma finally relates the stretching maps to the simplicial subdivision maps.

Lemma 95.40. (Stretching-Subdivision Lemma) Let $(K, \leq)$ be an ordered abstract simplicial complex. As before we denote by $\sigma\colon \operatorname{sd}(K) \to K$ the stretching map introduced in the Stretching Map Lemma 95.38.

(1) The induced chain map

$$\sigma_*\colon \mathrm{C}_*^{\mathrm{simp},\leq}(\operatorname{sd}(K)) \to \mathrm{C}_*^{\mathrm{simp},\leq}(K)$$

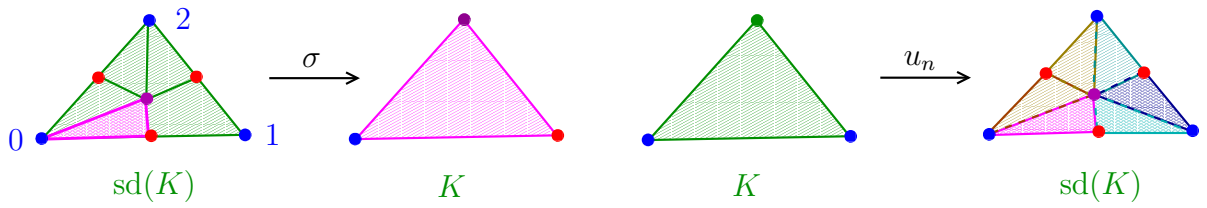
and the simplicial subdivision maps

$$u_n\colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) \to \mathrm{C}_n^{\mathrm{simp},\leq}(\operatorname{sd}(K)), \quad n \in \mathbb{N}_0$$

are chain homotopy inverses of one another.

(2) The maps σ_* and the maps u_* are chain homotopy equivalences.

(3) The maps $\sigma_*\colon \mathrm{H}_*^{\mathrm{simp},\leq}(\operatorname{sd}(K)) \to \mathrm{H}_*^{\mathrm{simp},\leq}(K)$ and $u_*\colon \mathrm{H}_*^{\mathrm{simp},\leq}(K) \to \mathrm{H}_*^{\mathrm{simp},\leq}(\operatorname{sd}(K))$ are natural isomorphisms and they are inverses of one another.



Remark. The Stretching-Subdivision Lemma 95.40 (2) can be viewed as an analogue of Lemma 74.26 where we proved a similar statement for the singular subdivision map.

Proof. Statement (2) of the lemma is an immediate consequence of Statement (1). Furthermore Statement (3) follows from Statements (1) and (2) together with Corollary 73.4 and the naturality statements of the Stretching Map Lemma 95.38 and the Chain Subdivision Lemma 95.39. Thus it remains to prove Statement (1).

Let $(K, \leq)$ be an ordered abstract simplicial complex. It follows easily from the definitions that for each $n \in \mathbb{N}_0$ the map $\sigma_* \circ u_n\colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) \to \mathrm{C}_n^{\mathrm{simp},\leq}(K)$ is the identity. Thus it remains to prove that $u_* \circ \sigma_*$ is chain homotopic to the identity.

The definition of the chain homotopy requires some preparations and extra definitions. Let $n \in \mathbb{N}_0$ and let $t = \{\underline{s_0}, \ldots, \underline{s_n}\}$ be an n-simplex of $\operatorname{sd}(K)$ with $\#s_n = n+2$. For convenience we set $s_{-1} := \varnothing$. Since $\#s_n = n+2$ and since for each $i \in \{0, \ldots, n\}$ we have $s_{i-1} \subsetneq s_i$ we see that there exists a unique $i \in \{0, \ldots, n\}$ with $\#s_i = \#s_{i-1} + 2$. Thus there exist two vertices $v < w$ of K such that $s_i = s_{i-1} \sqcup \{v, w\}$. We say that t can be *expanded upward* if w is larger than every element in s_{i-1}. In this case we set

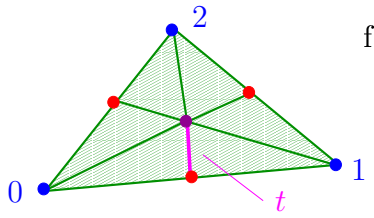
$$\Phi(t) := \overbrace{\{\underline{s_0}, \ldots, \underline{s_{i-1}}, \underline{s_{i-1} \cup \{w\}}, \underline{s_i}, \ldots \underline{s_n}\}}^{(n+1)\text{-simplex of } \operatorname{sd}(K)}$$

and

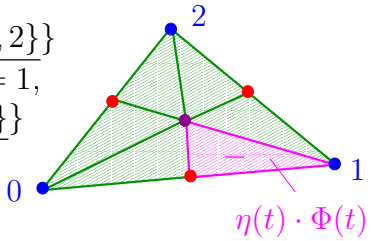
$$\eta(t) := \operatorname{sign}(\sigma) \text{ where } \sigma \in \operatorname{Bij}\{0, \ldots, n\} \text{ with } \Phi(t) = (s_n)_\sigma.$$

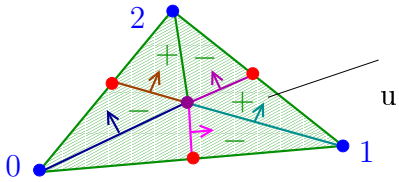
notation from page 2069

These definitions are illustrated in the figures below.

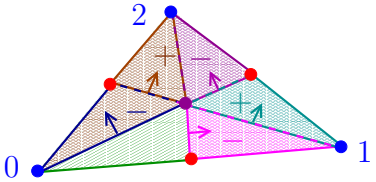


for the 1-simplex $t = \{\underline{\{0,1\}}, \underline{\{0,1,2\}}\}$
we have $i = 0$ and $v = 0$ and $w = 1$,
thus $\Phi(t) = \{\underline{\{1\}}, \underline{\{0,1\}}, \underline{\{0,1,2\}}\}$
and we see that $\eta(t) = -1$





arrow indicates
upward expansion



For each $n \in \mathbb{N}_0$ we consider the following map

$$\begin{array}{rcl} P_n \colon \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)) & \to & \mathrm{C}_{n+1}^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \\ t = \{\underline{s_0}, \ldots, \underline{s_n}\} & \mapsto & \begin{cases} \eta(t) \cdot \Phi(t), & \text{if } \#s_n = n+2 \text{ and if } t \text{ can be expanded upward,} \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

We leave it to the reader's unrelenting zest for verifying technical details to show that these maps P_n form a chain homotopy between the identity and the chain map $\sigma_* \circ u_*$. ■

We defined the simplicial subdivision map for *ordered* abstract simplicial complexes. As we will see shortly, the choice of an ordering is irrelevant.

Definition. Let $(K, \leq)$ be an abstract simplicial complex.

(1) Let $s = (v_0, \ldots, v_n)$ be an ordered n-simplex of K. Given $m \in \{0, \ldots, n\}$ we set

$$F_m(s) \ := \ \{v_0, \ldots, v_m\}.$$

Furthermore we set

$$F(s) \ := \ \big(\underline{F_0(s)}, \underline{F_1(s)}, \ldots, \underline{F_n(s)}\big).$$

Note that $F(s)$ is an ordered n-simplex of $\mathrm{sd}(K)$.

(2) Note that any two ordered n-simplices $s = (v_0, \ldots, v_n)$ and $t = (w_0, \ldots, w_n)$ of K with $\{v_0, \ldots, v_n\} = \{w_0, \ldots, w_n\}$ there exist a unique $\sigma \in \mathrm{Bij}\{0, \ldots, n\}$ such that $v_i = w_{\sigma(i)}$ for all $i = 0, \ldots, n$. We set $\mathrm{sign}(s,t) := \mathrm{sign}(\sigma)$.

(3) Given an ordered n-simplex $s = (v_0, \ldots, v_n)$ we set

$$u_n(s) \ := \ \sum_{\substack{t = (w_0, \ldots, w_n) \text{ an ordered} \\ n\text{-simplex of } K \text{ with} \\ \{v_0, \ldots, v_n\} = \{w_0, \ldots, w_n\}}} (-1)^{\mathrm{sign}(s,t)} \cdot F(t)$$

More precisely, we have the following lemma.

Lemma 95.41. (Chain Subdivision Lemma) Let K be an abstract simplicial complex.

(1) For every $n \in \mathbb{N}_0$ the map
$$\begin{array}{ccc} \mathrm{C}_n^{\mathrm{simp}}(K) & \to & \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(K)) \\ [s] & \mapsto & [u_n(s)] \end{array}$$
is well-defined. We refer to this map again as the *simplicial subdivision map*.

(2) The maps u_n are natural in the sense that given a simplicial map $f\colon K \to L$ the following diagram commutes
$$\begin{array}{ccc} \mathrm{C}_n^{\mathrm{simp}}(K) & \xrightarrow{f_*} & \mathrm{C}_n^{\mathrm{simp}}(L) \\ \downarrow{\scriptstyle u_n} & & \downarrow{\scriptstyle u_n} \\ \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(K)) & \xrightarrow{f_*} & \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(L)). \end{array}$$

(3) If K is equipped with an order "$\leq$", then for every $n \in \mathbb{N}_0$ the following diagram commutes:
$$\begin{array}{ccc} \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \xrightarrow{u_n} & \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{C}_n^{\mathrm{simp}}(K) & \xrightarrow{u_n} & \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(K)). \end{array}$$
Here the vertical maps are the natural isomorphisms given by Lemma 95.5.

(4) The maps $u_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(K))$ form a chain homotopy equivalence. In fact, if we pick an order on K and we consider the corresponding stretching map $\sigma\colon \mathrm{sd}(K) \to K$, then the maps u_n and the maps $\sigma_*\colon \mathrm{C}_*^{\mathrm{simp}}(\mathrm{sd}(K)) \to \mathrm{C}_*^{\mathrm{simp}}(K)$ are chain homotopy inverses.

(5) The induced maps $u_*\colon \mathrm{H}_n^{\mathrm{simp}}(K) \to \mathrm{H}_n(\mathrm{sd}^{\mathrm{simp}}(K))$ are natural isomorphisms.

Remark. Note in contrast the stretching maps certainly do depend on the choice of an order.

Proof. The first three statements follow easily from the definitions. The only statement which perhaps needs a little bit of thought is the first statement. We leave it to the reader to fill in the details. Let us turn to the proof of Statement (4). As mentioned before, by the Well-Ordering Theorem 0.7 we can pick an order "$\leq$" on K. Let $n \in \mathbb{N}_0$. By (2) we have the following commutative diagram:
$$\begin{array}{ccc} \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \xrightarrow{u_n} & \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{C}_n^{\mathrm{simp}}(K) & \xrightarrow{u_n} & \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(K)). \end{array}$$
By the Stretching-Subdivision Lemma 95.40 we know that the top horizontal maps form a chain homotopy equivalence and that a chain homotopy inverse is given by the stretching maps. It follows that the analogous statements hold for the bottom horizontal maps. Finally Statement (5) follows from (2) and (4) together with Corollary 73.4. ■

In the remainder of this section we want to discuss the relationship of the stretching map and the simplicial division map with various maps on the topological realizations. First we recall that given an abstract simplicial complex K we gave in the SD-Lemma 94.3 (3)

an explicit natural homeomorphism $\mho\colon |\operatorname{sd}(K)| \to |K|$. This allows us to formulate the following lemma.

Lemma 95.42. Let $(K, \leq)$ be an abstract simplicial complex. We denote by $\sigma\colon \operatorname{sd}(K) \to K$ the stretching map from the Stretching Map Lemma 95.38. The map $|\sigma| \circ \mho^{-1}\colon |K| \to |K|$ is homotopic to the identity.

Proof. Since we will not make use of Lemma 95.42, therefore we outsource the proof thereof to Exercise 95.17. ■

We recall some more maps and statements from earlier on.

(1) Let K be an ordered abstract simplicial complex. By the Simplicial-Singular Chain Complex Theorem 95.25 we know that the maps

$$\begin{aligned} \Theta_K\colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) &\to \mathrm{C}_n(|K|) \\ s &\mapsto (\Phi_s^{\leq}\colon \Delta^n \to |K|) \end{aligned}$$

are a natural chain homotopy equivalence.

(2) Given a topological space X and $n \in \mathbb{N}_0$ we denote by $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$ the singular subdivision map as defined in page 1640.

Now we can formulate the following proposition.

Proposition 95.43. For every ordered abstract simplicial complex $(K, \leq)$ and every $n \in \mathbb{N}_0$ the following diagram commutes:

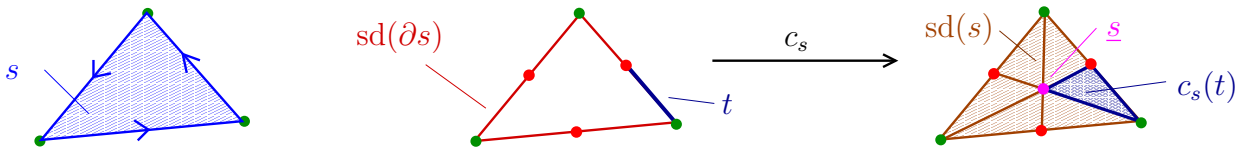
$$\begin{array}{ccc} \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \xrightarrow{u_n} & \mathrm{C}_n^{\mathrm{simp},\leq}(\operatorname{sd}(K)) \\ \Big\downarrow{\scriptstyle \Theta_K} & & \Big\downarrow{\scriptstyle \Theta_{\operatorname{sd}(K)}} \\ & & \mathrm{C}_n(|\operatorname{sd}(K)|) \\ & & \Big\downarrow{\scriptstyle \mho_*} \\ \mathrm{C}_n(|K|) & \xrightarrow{u_n} & \mathrm{C}_n(|K|). \end{array}$$

Furthermore all maps form natural chain homotopy equivalences.

The key step in proving Proposition 95.43 is to give an alternative description of the simplicial subdivision map that resembles the definition of the singular subdivision map which we gave on page 1640. We start out with the following definition.

Definition. Let $(K, \leq)$ be an ordered abstract simplicial complex. Given a k-simplex s of K we define the "coning map"

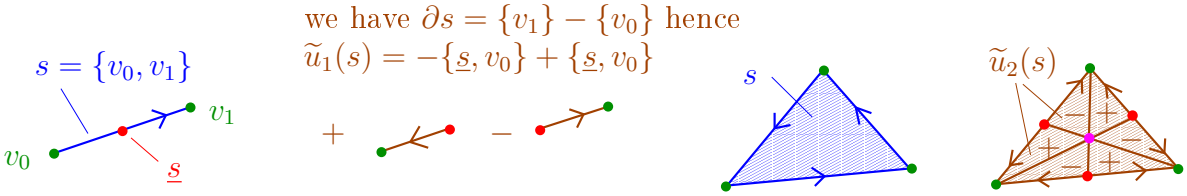
$$\begin{aligned} c_s\colon \mathrm{C}_i^{\mathrm{simp},\leq}(\operatorname{sd}(\partial s)) &\to \mathrm{C}_{i+1}^{\mathrm{simp},\leq}(\operatorname{sd}(s)) \\ &\mapsto \{\underline{s}\} \cup t. \end{aligned}$$



Definition. Let $(K, \leq)$ be an ordered abstract simplicial complex. We define the map [1180] $u_0\colon \mathrm{C}_0^{\mathrm{simp},\leq}(K) \to \mathrm{C}_0^{\mathrm{simp},\leq}(\operatorname{sd}(K))$ to be the obvious inclusion. Furthermore, for $n \geq 1$ we

define inductively the following maps:

$$\begin{array}{rccl} \widetilde{u}_n\colon & \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \\ & s & \mapsto & (-1)^n \cdot c_s\big(\underbrace{u_{n-1}(\partial s)}_{\in \mathrm{C}_{n-1}^{\mathrm{simp},\leq}(\mathrm{sd}(\partial s))}\big). \end{array}$$



The following lemma gives the promised alternative description of the simplicial subdivision maps.

Lemma 95.44. Let $(K, \leq)$ be an ordered abstract simplicial complex. Given any n-simplex s of K we have $\widetilde{u}_n(s) = u_n(s)$.

Proof of Lemma 95.44. Let $(K, \leq)$ be an ordered abstract simplicial complex. We prove the lemma by induction on n. For $n = 0$ the statement is basically obvious. Now suppose that the statement holds for some $n \in \mathbb{N}_0$. Let $s = \{v_0 < \cdots < v_{n+1}\}$ be some $(n+1)$-simplex. Given $i \in \{0, \ldots, n+1\}$ we write $s(i) := \{v_0, \ldots, \widehat{v_i}, \ldots, v_{n+1}\}$. We calculate that

$$\begin{array}{rcl} u_{n+1}(s) & \overset{\text{definition of } \widetilde{u}_{n+1}}{=} & (-1)^{n+1} \cdot c_s(\widetilde{u}_n(\partial s)) \\ & \overset{\text{by definition of } \partial s}{=} & (-1)^{n+1} \cdot c_s\Big(u_n\Big(\sum_{i=0}^{n+1}(-1)^i \cdot s(i)\Big)\Big) \\ & \overset{\text{by induction and definition of } c_s}{=} & (-1)^{n+1} \cdot \sum_{i=0}^{n+1}(-1)^i \cdot \sum_{\sigma \in \mathrm{Bij}\{0,\ldots,n\}} \mathrm{sign}(\sigma) \cdot \{\underline{s}, s(i)_\sigma\} \\ & = & \sum_{\tau \in \mathrm{Bij}\{0,\ldots,n+1\}} \mathrm{sign}(\tau) \cdot s_\tau \;=\; u_n(s). \end{array}$$

see definition on page 2069

here we use the following bijection:

$$\Psi\colon \{0, \ldots, n+1\} \times \mathrm{Bij}\{0, \ldots, n\} \to \mathrm{Bij}\{0, \ldots, n+1\}$$

$$(i, \sigma) \mapsto \left(\begin{array}{rcl} \{0, \ldots, n+1\} & \to & \{0, \ldots, n+1\} \\ j & \mapsto & \begin{cases} j, & \text{if } j \in \{0, \ldots, i-1\}, \\ i, & \text{if } j = n+1, \\ j+1, & \text{if } j \in \{i, \ldots, n\} \end{cases} \end{array} \right)$$

and we use that $\mathrm{sign}(\Psi(i, \sigma)) = (-1)^{i+1} \cdot (-1)^{n+2} \cdot \mathrm{sign}(\sigma)$

We have thus proved the induction step. ∎

Now we can easily prove Proposition 95.43.

Proof of Proposition 95.43. It follows immediately from Lemma 95.44 together with the definitions of the various maps that the diagram commutes.

By Lemmas 95.40 and 74.26 we know that the horizontal maps form chain homotopy equivalences. Furthermore by the Simplicial-Singular Chain Complex Theorem 95.25 the top vertical maps also form chain homotopy equivalences. Finally the map $\mho$ is a homeomorphism. Thus the bottom vertical maps $\mho_*$ also form a chain homotopy equivalence. ∎

We conclude with the following "unordered" version of Proposition 95.43.

[1180] Recall that in Lemma 94.2 (2) we saw that $\mathrm{sd}(K)$ is naturally an ordered abstract simplicial complex.

Proposition 95.45. For every abstract simplicial complex K and every $n \in \mathbb{N}_0$ the following diagram commutes:[1181]

$$\begin{array}{ccc} \mathrm{H}_n^{\mathrm{simp}}(K) & \xrightarrow{u_n} & \mathrm{H}_n^{\mathrm{simp}}(\mathrm{sd}(K)) \\ & & \downarrow \Theta_{\mathrm{sd}(K)} \\ \Theta_K \downarrow & & \mathrm{H}_n(|\,\mathrm{sd}(K)|) \\ & & \downarrow \mathfrak{o}_* \\ \mathrm{H}_n(|K|) & \xrightarrow{u_n} & \mathrm{H}_n(|K|). \end{array}$$

Furthermore all maps are natural.

Proof. The proposition follows easily from the fact that every abstract simplicial complex admits an order together with Lemma 95.8 and Proposition 95.45 and together with Proposition 95.28. ■

Remark. For the record we point out that some of the definitions and results in this section are at least implicit in [**Deo2018**, Theorem 4.9.1] and [**Mun1966a**, p. 99].

Exercises for Chapter 95.

Exercise 95.1. Let $K = (V, S)$ be an abstract simplicial complex and let "$\leq$" and "$\widetilde{\leq}$" be two orders on K. Show that the maps

$$\begin{array}{rcl} \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n^{\mathrm{simp},\widetilde{\leq}}(K) \\ \{v_0 < \cdots < v_n\} & \mapsto & \underbrace{\mathrm{sign}(\sigma) \cdot \{v_{\sigma(0)} \widetilde{<} \ldots \widetilde{<} v_{\sigma(n)}\}}_{\substack{\sigma \in S_n \text{ is the unique permutation} \\ \text{such that } v_{\sigma(0)} \widetilde{<} \ldots \widetilde{<} v_{\sigma(n)}}} \end{array}$$

define an isomorphism of chain complexes.

Exercise 95.2. Let K be a non-empty abstract simplicial complex, let $n \in \mathbb{N}_0$ and let S_n be the abstract simplicial complex that we introduced on page 1978. Show that for every $k \in \mathbb{N}_0$ there exists a natural isomorphism

$$\widetilde{\mathrm{H}}_k^{\mathrm{simp}}(K * S_n) \;\cong\; \widetilde{\mathrm{H}}_{k-(n+1)}^{\mathrm{simp}}(K).$$

Exercise 95.3. Let $K = (V, S)$ be an abstract simplicial complex. Recall that on page 2013 we said that $K = (V, S)$ is *connected* if given any $v, v' \in V$ there exist 1-simplices $s_0, \ldots, s_k \in S$ such that $v \in s_0$, such that for every $i \in \{0, \ldots, k-1\}$ we have $s_i \cap s_{i+1} \neq \varnothing$ and such that $v' \in s_k$. Show that if K is a connected non-empty abstract simplicial complex, then given any abelian group G the map

$$\begin{array}{rcl} \mathrm{H}_0^{\mathrm{simp}}(K; G) & \to & G \\ \left[\sum_{i=0}^{k} \{v_i\} \otimes g_i\right] & \mapsto & \sum_{i=0}^{k} g_i \end{array}$$

is a natural isomorphism

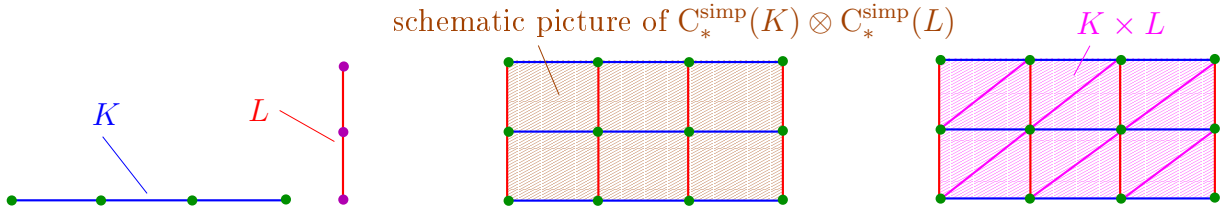
Remark. This is the simplicial analogue of the H_0-Proposition 72.5.

[1181] Here the vertical maps Θ are the natural isomorphisms introduced on page 2058.

Exercise 95.4. In this exercise we give a down-to-earth proof of the Simplicial Künneth Theorem 95.14 in the special case that we are dealing with 1-dimensional abstract simplicial complexes.

(a) Let K and L be two ordered 1-dimensional abstract simplicial complexes. Show that the tensor product of the chain complexes $\mathrm{C}_*^{\mathrm{simp}}(K)$ and $\mathrm{C}_*^{\mathrm{simp}}(L)$, as defined on page 1940, is chain homotopy equivalent to the chain complex $\mathrm{C}_*^{\mathrm{simp}}(K \times L)$.
(b) Prove (a) for arbitrary finite-dimensional abstract simplicial complexes.
(c) Prove (a) for arbitrary abstract simplicial complexes.

Remark. Together with the Künneth Theorem 90.7 for chain complexes this exercise makes it possible to calculate the simplicial homology groups of $K \times L$ directly in terms of the simplicial homology groups of K and L.



Exercise 95.5. Let $p, q \in \mathbb{N}_0$. We consider the product $D_p \times D_q$ of the ordered abstract simplicial complexes defined on page 1978. Show that

$$\mathrm{H}_i^{\mathrm{simp}}(D_p \times D_q) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Remark. This exercise goes into the proof of the Simplicial Künneth Theorem 95.14. In particular you are not allowed to use this theorem.

Exercise 95.6. Show that for any abstract simplicial complex K and any simplex s of K we have for every $i \in \mathbb{N}$ that $\mathrm{H}_i^{\mathrm{simp}}(\mathrm{St}(K, s)) = 0$.

Exercise 95.7.

(a) Formulate and prove a Mayer–Vietoris style theorem for simplicial homology, along the lines of the Mayer–Vietoris Theorem 78.5.
(b) Use the exact sequence from (a) to provide a new solution to Exercise 95.2.

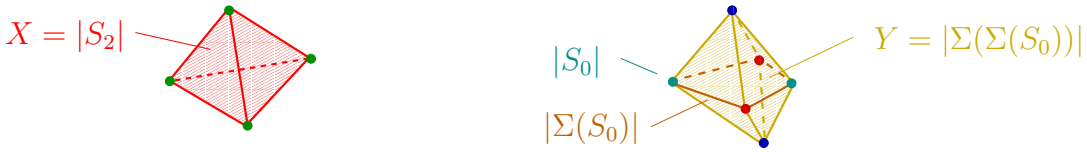
Exercise 95.8. Let K be a simplicial complex and let A and B be two subcomplexes with $A \cup B = K$. Show that for each $k \in \mathbb{N}_0$ the natural map $\mathrm{H}_k^{\mathrm{simp}}(A, A \cap B) \to \mathrm{H}_k^{\mathrm{simp}}(X, B)$ is an isomorphism.
Remark. This statement can be viewed as the Simplicial Excision Theorem. The biggest difference to the singular setting is that in the simplicial setting the proof is almost embarrassingly easy.

Exercise 95.9. Let K be an abstract simplicial complex and let L be a subcomplex of K. Show that for each $k \in \mathbb{N}_0$ there exists a natural isomorphism

$$\mathrm{H}_k^{\mathrm{simp}}(K, L) \;\cong\; \widetilde{\mathrm{H}}_k^{\mathrm{simp}}\big(\underbrace{K \cup_L \mathrm{Cone}(L)}_{\text{mapping cone of } L \to K}\big).$$

Exercise 95.10. Compute the simplicial homology groups of an abstract simplicial complex K such that $|K|$ is homeomorphic to the surface Σ of genus two.
Hint. As we saw in the figure on page **??**, we can view Σ as the connected sum of two tori and you could use Exercise 95.7.

Exercise 95.11. Show that the two simplicial complexes $X = |S_2|$ and $Y = |\Sigma(\Sigma(S_0))|$ which are shown in the figure below admit subdivisions X' and Y' which are simplicially isomorphic.



Exercise 95.12. Let $K = (V, S)$ be an abstract simplicial complex. Let $\mathcal{U} = \{\mathring{\mathrm{St}}(v)\}_{v \in V}$ be the open cover of the topological realization $|K|$ that is given by the open stars. What is the relationship between the original abstract simplicial complex K and the nerve complex $N(\mathcal{U})$?

Exercise 95.13. Formulate and prove a simplicial analogue of the Mayer–Vietoris Theorem 78.5.
Remark. The proof is significantly easier than the original proof of the Mayer–Vietoris Theorem 78.5.

Exercise 95.14. Let K and L be two ordered abstract simplicial complexes. Show that for each $k \in \mathbb{N}_0$ there exists a short exact sequence of the following form:

$$0 \;\to\; \widetilde{\mathrm{H}}_k^{\mathrm{simp}}(K * L) \;\to\; \widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(K) \oplus \widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(L) \;\to\; \widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(K \times L) \;\to\; 0.$$

Remark. This statement is the simplicial analogue of Exercise 78.20. In fact at least if K and L are finite, then we can reduce the simplicial statement to the singular statement. The challenge is to give a purely combinatorial argument.

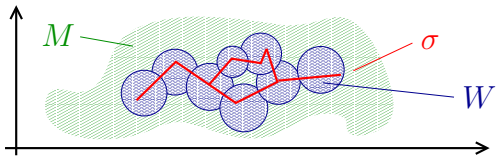
Exercise 95.15. Let $n \in \mathbb{N}_0$.

(a) Let W be an open subset of $\mathbb{R}^n$ that is the union of finitely many convex open subsets. Show that $\mathrm{H}_k(W) = 0$ for $k \geq n$.
Hint. Use the Mayer–Vietoris Theorem 78.5 and do an induction argument on the number of convex open subsets. As on many other occasions we point out the convenient fact that the intersection of convex subsets of $\mathbb{R}^n$ is again convex.

(b) Let M be an open subset of $\mathbb{R}^n$ and let $\sigma \in \mathrm{C}_n(M)$ be a cycle. Show that there exists an open subset W of $\mathbb{R}^n$ that is the union of finitely many convex open subsets such that $\sigma \in \mathrm{C}_n(W)$ and such that $W \subset M$.

(c) Let M be an open subset of $\mathbb{R}^n$. Show that $\mathrm{H}_k(M) = 0$ for $k \geq n$.

Exercise 95.16. Let $P_1, \ldots, P_s$ be points in $\mathbb{R}^2$ with $s \geq 3$. We assume that any three of those points are contained in an open disk of radius r. Show that all the given points are contained in an open disk of radius r.



Exercise 95.17. Let $(K, \leq)$ be an abstract simplicial complex. Let $\sigma\colon \mathrm{sd}(K) \to K$ be the stretching map from the Stretching Map Lemma 95.38. Show that the map $|\sigma| \circ \mathfrak{V}^{-1}\colon |K| \to |K|$ is homotopic to the identity.
Remark. Use the Interval CW-Product Corollary 68.9 to overcome any continuity anxieties that might arise.

Exercise 95.18. Let K be an abstract simplicial complex. Show that the maps

$$\begin{array}{rcl} \mathrm{C}_k^{\mathrm{simp}}(K) \otimes \mathbb{Q} & \to & \mathrm{C}_k(|K|) \otimes \mathbb{Q} \\ [v_0, \ldots, v_k] \otimes r & \mapsto & \frac{1}{(k+1)!} \cdot \sum_{\sigma \in S_{k+1}} \cdot \left(\begin{array}{rcl} \Delta^k & \mapsto & |K| \\ (t_0, \ldots, t_k) & \mapsto & \sum_{i=0}^{k} t_i \cdot v_{\sigma(i)} \end{array} \right) \otimes r \end{array}$$

are a natural chain homotopy equivalence.

Exercise 95.19. Let $n \in \mathbb{N}_0$. Let $K = ((V, S), \Theta\colon |K| \to \Delta^n)$ be a subdivision of the standard simplex Δ^n. We denote by $e_0, \ldots, e_n \in V$ the vertices that get sent to the standard vertices of Δ^n. Let Z be a finite set and let $\varphi\colon V \to Z$ be a map. We refer to φ as a *coloring of* K.

(1) We say φ satisfies *Sperner's condition* if $\varphi\colon V \to Z$ is injective on $\{e_0, \ldots, e_n\} \subset V$ and if for each $v \in V$ with $\Theta(v) \in |\Theta(\{e_{r_1}, \ldots, e_{r_s}\})|$ we have $\varphi(v) \in \{\varphi(e_{r_1}), \ldots, \varphi(e_{r_s})\}$. In other words, the vertices $e_0, \ldots, e_n$ have distinct colors, and if a vertex lies on a face spanned by some of the e_i, then the vertex has the color of one of the e_i.
(2) We say an n-simplex s of K is a *rainbow simplex* if $\varphi\colon s \to Z$ is injective, put differently, if the colors of vertices of s are all different.

We want to prove the following statement:
Sperner's Lemma. *If φ satisfies Sperner's condition, then the number of rainbow simplices is odd.*
To prove this statement we introduce some notation:

(3) We set $\Delta^Z := \left\{ \sum_{z \in Z} t_z \cdot z \mid \sum_{z \in Z} t_z = 1 \text{ and each } t_z \geq 0 \right\} \subset \mathbb{R}^Z$.
(4) We denote by $f^\varphi\colon |K| \to \Delta^Z$ the unique map which agrees with φ on V and which is affine linear on each $|s| \subset |K|$.
(5) We define $\partial|K| := \Theta^{-1}(\partial\Delta^n)$ and we define $\partial\Delta^Z$ the same way as we defined $\partial\Delta^n$.
(6) We write

$$\deg(\varphi) \;:=\; \deg\Big(\underbrace{\mathrm{H}_n(|K|, \partial|K|; \mathbb{F}_2)}_{\cong \mathbb{F}_2} \to \underbrace{\mathrm{H}_n(\Delta^Z, \partial\Delta^Z; \mathbb{F}_2)}_{\cong \mathbb{F}_2}\Big) \;\in\; \mathbb{F}_2.$$

Sperner's Lemma follows from solving the following two exercises:

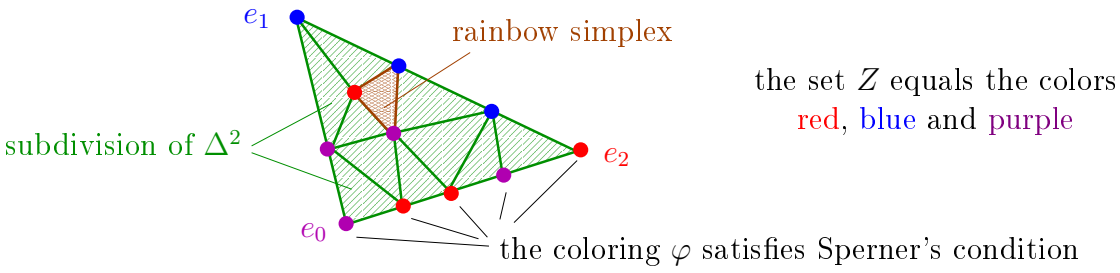
(a) Show that if φ satisfies Sperner's condition, then $\deg(\varphi) \neq 0$.
Hint. Use Exercise 76.22 to prove this statement by induction on n.

(b) Show that

$$\deg(\varphi) \;=\; \text{number of rainbox simplices } \in \mathbb{F}_2.$$

Hint. The n-simplices of K give rise to a generator of $\mathrm{H}_n(|K|, \partial|K|; \mathbb{F}_2)$.

Remark. The conclusion of this slightly challenging exercise is known as Sperner's Lemma. It was first formulated and proved by Emanuel Sperner [**Spe1928**, p. 267] in 1928. Sperner's Lemma can be proved in a purely combinatorial fashion, see [**AZ2014**, Chapter 27.6], [**Pra2006**, Theorem 2.26] or [**Harz1978**, Satz 12.2]. Sperner's Lemma can be used to give a purely combinatorial proof of the Brouwer Fixed Point Theorem 31.4, we refer to [**Harz1978**, Chapter 12] or [**Pra2006**, p. 81] for details. Sperner's Lemma also has fun applications, for example it can be applied to achieve "rental harmony" [**Su1999**]. Finally we refer to [**DGMM2019**] for a survey on Sperner's Lemma, its cousin Tucker's Lemma, and its many applications and ramifications.



96. Simplicial and PL-structures on manifolds

In this chapter we will see that smooth manifolds admit a simplicial structure. In fact we will see that they admit particularly nice simplicial structures. There will be many benefits. For example we can almost immediately draw conclusions on the homology groups and homotopy groups of smooth manifolds.

96.1. **Simplicial structures on smooth manifolds.** We recall the following definition from page 1994.

Definition. A simplicial structure for a given topological space X is defined as a pair $(K = (V, S), \Theta\colon |K| \to X)$ where the following holds:

(1) K is an abstract simplicial complex,
(2) Θ is a homeomorphism between the topological realization $|K|$ of K and the topological space X.

Given a simplex $s \in S$ we refer to $\Theta(|s|)$ as a simplex of X and we refer to $\Theta(\langle s\rangle)$ as an open simplex of X.

In this section we want to consider simplicial structures on smooth manifolds. Since the smooth structure on smooth manifolds is important for many arguments and constructions it is reasonable to consider simplicial structures that reflect the smooth structure. In the following the idea is to consider simplicial structures (K, Θ) such that for each k-simplex the characteristic map $\Theta \circ \Phi_s\colon \Delta^k \to M$ is a smooth embedding. Since Δ^k is not a smooth manifold it takes a little bit of an effort to explain what a "smooth embedding" is supposed to be in this case.

We start out with the following definition. The first part just recalls the definitions introduced on pages 549 and 532.

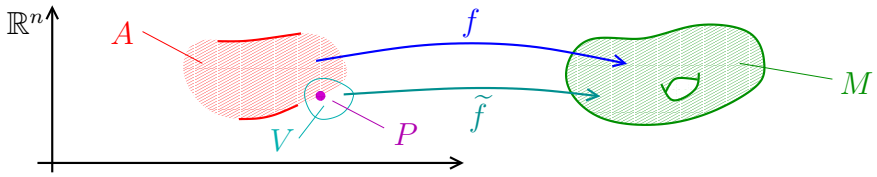
Definition. Let $n \in \mathbb{N}_0$ and let A be a subset of $\mathbb{R}^n$ for which there exists an open subset U of $\mathbb{R}^n$ with $U \subset A \subset \overline{U}$. Furthermore let M be a smooth manifold.

(1) We say a map $f\colon A \to M$ is smooth at a point $P \in A$, if there exists an open neighborhood V of P in $\mathbb{R}^n$ and a smooth map $\widetilde{f}\colon V \to \mathbb{R}^m$ which coincides with f on $A \cap V$. We define $\mathrm{D}f_P := D\widetilde{f}_P\colon \mathbb{R}^n \to \mathrm{T}_{f(P)}M$.[1182] We refer to the figure below for an illustration.
(2) We say a map $f\colon A \to M$ is an immersion if f is smooth at every $P \in A$ and if given any $P \in A$ the differential $\mathrm{D}f_P$ has rank n.[1183]
(3) We say a map $f\colon A \to M$ is a smooth embedding if it is an immersion and an embedding.

Example. Given $k \in \mathbb{N}_0$ we consider the map

$$\begin{aligned}\Xi_k\colon \triangle_k := \Big\{(t_1,\dots,t_k) \in [0,1]^k \;\Big|\; \sum_{j=1}^{k} t_j \in [0,1]\Big\} &\to \mathbb{R}^{k+1}\\ (t_1,\dots,t_k) &\mapsto \Big(t_1,\dots,t_k,1-\sum_{j=1}^{k} t_j\Big).\end{aligned}$$

[1182] We leave it to the reader to verify, using our restrictive choice of domain A and using Exercise 2.33 (1), that the definition of $\mathrm{D}f_P$ does not depend on the choice of the smooth extension $\widetilde{f}$.
[1183] If A is in fact an open subset of $\mathbb{R}^n$, then this definition of an immersion agrees of course with the definition given on page 651.



This map has two interesting properties:

(1) By picking the obvious extension of Ξ_k to a map $\mathbb{R}^k \to \mathbb{R}^{k+1}$ we see that Ξ_k is an immersion.

(2) It follows easily from the Compact-Hausdorff Proposition 2.17 (3) that the map Ξ_k defines a homeomorphism $\Xi_k \colon \triangle_k \to \Delta^k$.

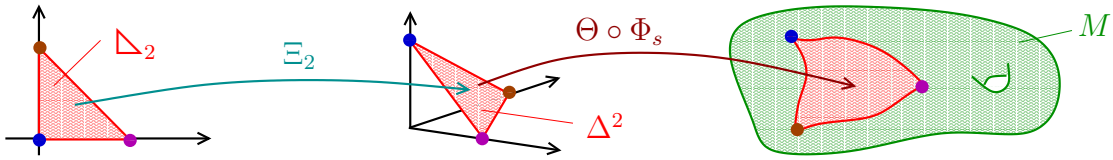
The combination of (1) and (2) shows that Ξ_k is an embedding.

This leads us to the following definition.

Definition. Let M be a smooth manifold.

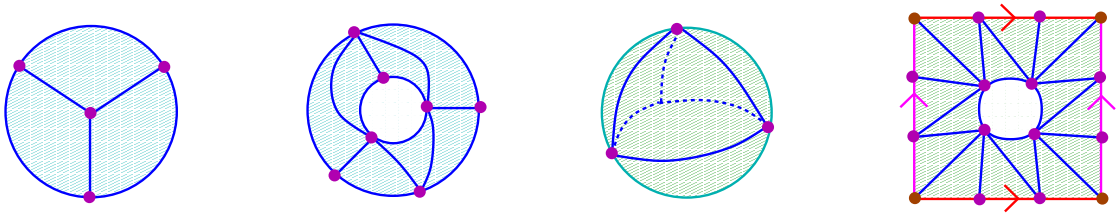
(1) Let $k \in \mathbb{N}_0$. We say that a map $\Phi \colon \Delta^k \to M$ is a smooth embedding if the composition $\Phi \circ \Xi_k \colon \triangle_k \to M$ is a smooth embedding.

(2) We say that a given simplicial structure $(K, \Theta \colon |K| \to M)$ is smooth if for each k-simplex s the corresponding characteristic map $\Theta \circ \Phi_s \colon \Delta^k \to M$ is a smooth embedding.



Remark. At first glance there are different definitions of "smooth simplicial structure" in the literature. For example our definition of a "smooth simplicial structure" is at first glance different from the one given in [**Mun1966a**, p. 81]. But [**Mun1966a**, Theorem 8.4] shows that our definition is in fact equivalent to the definition used in Munkres' book. Also, it is not difficult to see that our definition is equivalent to the one used in Whitney's book [**Why1957**].

Example. In the figure below we show smooth simplicial structures for the closed disk, the annulus, the 2-dimensional sphere and for the torus minus an open disk.



The following lemma contains two basic statements about (smooth) simplicial structures of smooth manifolds.

Lemma 96.1. Let M be a smooth manifold which is equipped with a smooth simplicial structure. If M is n-dimensional, then the dimension of each simplex is at most n.

Remark. In the Manifold-Simplex Proposition 98.1 we will prove several other statements regarding simplicial structures on smooth manifolds. In particular we will give an alternative proof of Lemma 96.1. Furthermore we will show that any (smooth) simplicial structure on a manifold M restricts to a (smooth) simplicial structure on the smooth manifold ∂M.

Proof. Let s be a k-simplex of the given smooth simplicial structure. Combining the characteristic map of the simplex with a chart we obtain a map from an open non-empty subset of $\mathbb{R}^k$ to $\mathbb{R}^n$. If the simplicial structure is smooth this map is in fact an embedding. It follows almost immediately from the Smooth Invariance of Domain Theorem 19.13 that $k \leq n$. ■

Now we can formulate the following theorem.

Theorem 96.2. (Smooth Manifold-Simplicial Structure Theorem)

(1) Every smooth manifold admits a countable smooth simplicial structure.
(2) If M is a smooth manifold with boundary ∂M, then any smooth simplicial structure for ∂M can be *extended* to a smooth simplicial structure for M.
(3) If M is a smooth manifold and if N is a proper submanifold of M, then there exists a smooth simplicial structure for M that *restricts* to a smooth simplicial structure on N.
(4) If $M = A \cup B$ is a decomposition into two n-dimensional submanifolds such that $A \cap B$ is a union of boundary components of A and a union of boundary components of B, then we can find a smooth simplicial structure on M such that A and B are subcomplexes.

Remark.

(1) Note that the conclusion of the Smooth Manifold-Simplicial Structure Theorem 96.2 (3) is somewhat weaker than the conclusion of the Smooth Manifold-Simplicial Structure Theorem 96.2 (2), it does not say that we can extend a given smooth simplicial structure on N to a smooth simplicial structure on M.
(2) By [**Bin1983**, Theorem XVIII.3.C] any PL-structure on the boundary of a 3-dimensional topological manifold can be extended to a PL-structure on the 3-dimensional topological manifold.
(3) Many more subtle results on finding smooth simplicial structures on smooth manifolds can be found in [**Vero1984**, Chapter 7].

Proof. We start out with giving precise references for the three statements.

(1) This statement is contained in [**Mun1966a**, Theorem 10.6]. For smooth manifolds without boundary the statement is also shown in [**Why1957**, Chapter IV.12]. For closed smooth manifolds, with a slightly weaker conclusion, the statement was first proved by Whitehead [**WhdJ1940**, Theorem 7] in 1940, building on earlier work of Cairns [**Cai1934**]. Other proofs for (some types of) smooth manifolds are given in [**Cai1961, Freu1939**] and [**Muk2015**, Theorem 9.5.1]. A sketch of an argument is also given in [**Thu1997**, Theorem 3.10.2].

(2) This statement is also contained in [**Mun1966a**, Theorem 10.6], alternatively see the discussion on [**WhdJ1940**, p. 822f].
(3) If N is a closed submanifold of M, then this statement can be proved by solving [**Mun1966a**, Problem 10.14]. The general case can be deduced from [**Vero1984**, Theorem 7.8], once one has unraveled the statement [**Vero1984**, Theorem 7.8].
(4) First we apply (1) to equip the smooth manifold $A \cap B$ with a simplicial structure. Next we use (2) to extend the smooth simplicial structure to A and B. It is fairly straightforward to verify that putting these two smooth simplicial structures together we obtain a smooth simplicial structure for $M = A \cup B$.

To give the reader a sense of how one can possibly construct smooth simplicial structures on smooth manifolds we provide a sketch for the proof of Statement (1). The sketch is partially inspired by the argument given by Whitney [**Why1957**, Chapter IV.12].

For simplicity let M be a closed n-dimensional smooth manifold. Recall that by the Euclidean Embedding Theorem 27.1 and the Smooth Embedding Theorem 24.10 we can view M as a submanifold of $\mathbb{R}^m$ for some $m \geq n$. By the General Tubular Neighborhood Theorem **??** we can pick a tubular neighborhood $p\colon W \to M$. (For the purpose of this sketchy proof we might as well assume that the tubular neighborhood is given by $W = \overline{B}^{m-n} \times M$ and that $p\colon \overline{B}^{m-n} \times M \to M$ is the obvious projection.) We equip $\mathbb{R}^m$ with the canonical simplicial structure defined in Lemma 93.17.

We make the following, ever so slightly vague observation.

Observation. Using a suitable generalization of the Transversality Theorem 42.4 and using the definition of a smooth map and a smooth submanifold ("locally everything is close to being linear") we can shrink the cubes to a suitably small size and we can find an isotopy of M such that the resulting simplicial structure (V, S) for $\mathbb{R}^m$ satisfies the following two conditions:[1184]

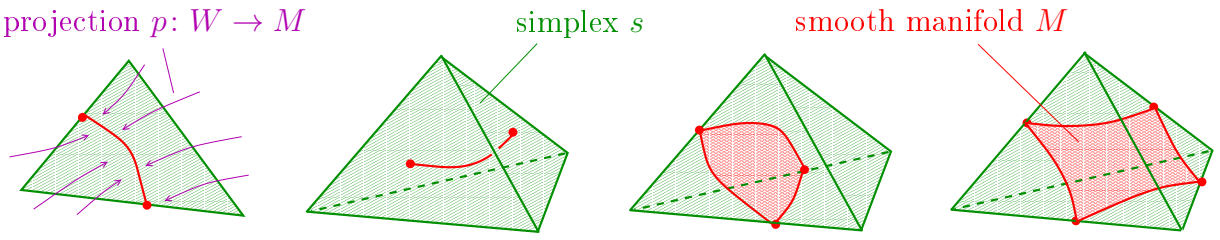
(a) For every simplex $s \in S$ the intersection of s with M is "close" to the transverse[1185] intersection of an n-dimensional affine space with s.
(b) Every simplex s with $s \cap M \neq \varnothing$ is contained in the tubular neighborhood W and the restriction of the projection $p\colon W \to M$ to s is "close" to being an affine linear map.

We refer to the figure below for an illustration.

We set $k := m - n$. In the following we spell out two consequences of (a).

(1) No simplex of dimension less than k intersects M.
(2) If a k-simplex s intersects M, then it does so in a unique point $P = P_s$, which we know by (1) lies necessarily in the interior of s.

[1184] In our notation we blur the difference between the abstract simplices of the simplicial structure and the corresponding simplices in $\mathbb{R}^n$.
[1185] Here by "transverse" we mean the following: if s is an l-dimensional simplex, then it is contained in an l-dimensional affine subspace. The transverse intersection of an n-dimensional affine space with an l-dimensional affine space in $\mathbb{R}^m$ is of dimension $m + l - n$, where negative dimension is understood to be the empty set.



$n = 1$ and $m = 2$ $n = 1$ and $m = 3$ $n = 2$ and $m = 3$ $n = 2$ and $m = 3$

Next we make the following constructions.

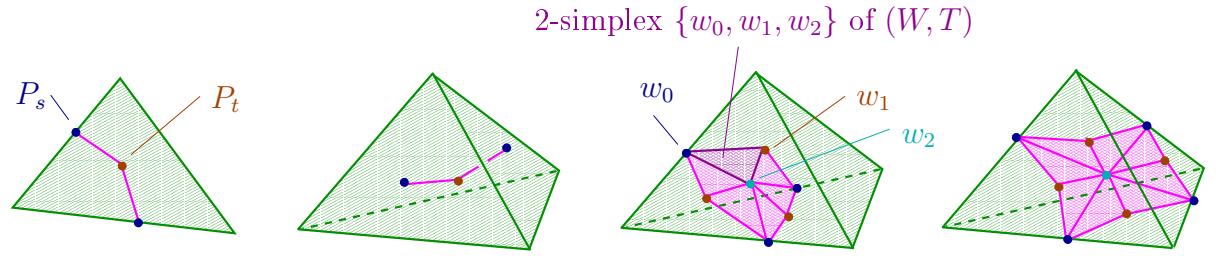
(3) For each l-simplex t with $l > k$ which has the property that the corresponding open simplex $\langle t \rangle$ intersects M we pick a point P_t that lies in the intersection of M with the interior of t.

(4) We consider the abstract simplicial complex

$$L := (W, T) \qquad \text{where } W := \{s \in S \,|\, \langle s \rangle \cap M \neq \varnothing\} \text{ and}$$
$$T := \{(w_0, \ldots, w_l) \,|\, w_0, \ldots, w_l \in S \text{ and } w_0 \subsetneq \ldots \subsetneq w_l\}.$$

(5) Let $\Theta\colon |L| \to \mathbb{R}^m$ be the unique map with $\Theta(w) = P_w$ for $w \in W$ and which is affine linear on each simplex $|t|$.

Some of these definitions are illustrated in the figure below. We continue with the following



$n = 1$ and $m = 2$ $n = 1$ and $m = 3$ $n = 2$ and $m = 3$ $n = 2$ and $m = 3$

observations:

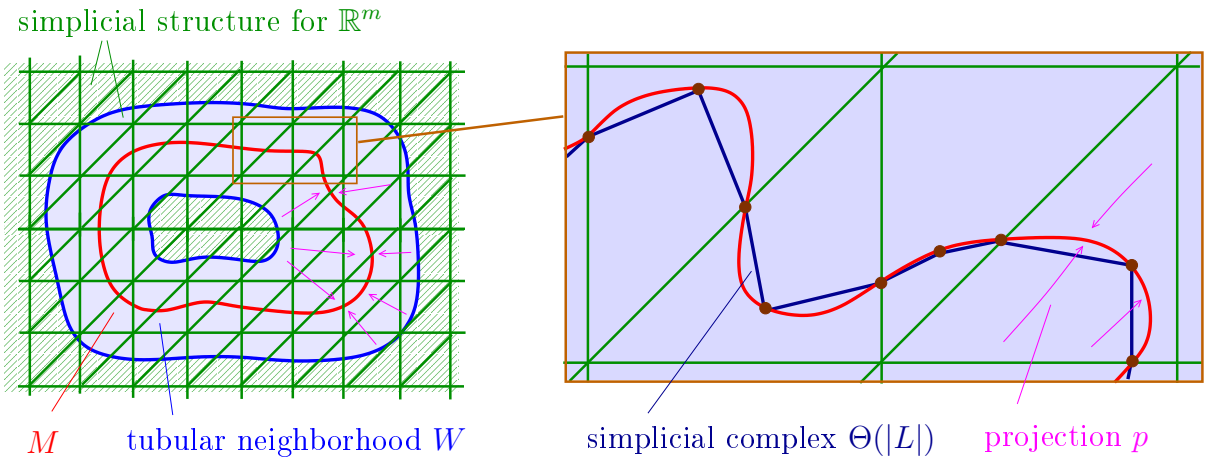
(6) It follows from (a) and (b) that the map $p\colon W \to M$ restricts to a homeomorphism $\Theta(|L|) \to M$. Furthermore, for each simplex $t \in T$ the map $p\colon \Theta(|t|) \to M$ is a submersion, i.e. at each point the differential has full rank.

(7) By (6) we now know that $(L = (W, T), p \circ \Theta\colon |L| \to M)$ is a smooth simplicial structure for M.

We refer to the figure below for a more "global" illustration of the above argument. Evidently this outline has several serious issues. For example we never made it clear what we mean by "close". Nonetheless we feel that with enough effort one can turn the above sketch of an argument into a proper proof.

We refer to [**Why1957**, Chapter IV.12] for a very detailed discussion how a variation on the above approach can be made to work. ■

Just for fun we mention that in our sketch of a proof of the Smooth Manifold-Simplicial Structure Theorem 96.2 we basically proved the following proposition which says that any



submanifold of $\mathbb{R}^m$ can be "arbitrarily" approximated by a linear simplicial subcomplex of $\mathbb{R}^m$. We will not make use of this proposition, hence we have no qualms about the rather sketchy proof.

Proposition 96.3. Let M be a closed smooth submanifold of $\mathbb{R}^m$. Given any $\epsilon > 0$ there exists a linear simplicial complex $(L = (W, T), \Theta\colon |L| \to X)$ with $X \subset \mathbb{R}^m$ and an isotopy $F\colon X \times [0,1] \to \mathbb{R}^m$ with the following properties:

(1) We have $F_0 = \mathrm{id}$.
(2) The map F_1 is a homeomorphism $F_1\colon X \to M$ and it has the property that $F_1 \circ \Theta$ defines a smooth simplicial structure for M.
(3) For every $x \in X$ the map $[0,1] \to \mathbb{R}^m$ given by $t \mapsto F(x,t)$ is affine linear.
(4) For every $x \in X$ we have $\|x - F_1(x)\| < \epsilon$.

Example. Proposition 96.3 says in particular that we can approximate any knot M in $\mathbb{R}^3$ by a "simplicial knot" X. We refer to the figure below for an illustration.



Sketch of proof. We continue with the notation and results in the proof of the Smooth Manifold-Simplicial Structure Theorem 96.2. The $\epsilon > 0$ corresponds to a suitable interpretation of "close" in the formulation of (a) and (b). We set $X := \Theta(|L|)$. The isotopy F is the map which interpolates the identity id_X and the projection $p\colon X \to M$. ■

In Theorem 95.20 we saw that there exist compact spaces which admit two "inequivalent" simplicial structures, in other words, in Theorem 95.20 we saw that the Hauptvermutung 95.19 is false in general. Interestingly, the "smooth" analogue of the Hauptvermutung 95.19 is actually correct. More precisely, the following theorem holds.

Theorem 96.4. Let M and N be smooth manifolds and suppose we are given two smooth simplicial structures $(K, \Theta\colon |K| \to M)$ and $(L, \Omega\colon |L| \to N)$. If M and N are diffeomorphic, then there are subdivisions of K and L which are simplicially isomorphic.

Proof. This statement is shown in [**WhdJ1940**, Theorem 8] and alternatively in [**Mun1966a**, Theorem 10.5] (see also [**Mun1975**, Theorem 10.13]). It is perhaps worth pointing out that we do not need to assume that M is compact. ■

Theorem 96.4 will come in handy at a later stage. But for the foreseeable future it is of no real interest to us.

We conclude this section with picking up some low-hanging fruit that we obtain almost for free from the Smooth Manifold-Simplicial Structure Theorem 96.2. First we have the following theorem.

Theorem 96.5. (Smooth Manifold-CW Structure Theorem) Every (compact) n-dimensional smooth manifold M admits a (finite) CW-structure with the following properties:

(1) every cell has dimension at most n,
(2) the boundary ∂M is a subcomplex.

Furthermore, if $M = A \cup B$ is a decomposition into two n-dimensional submanifolds such that $A \cap B$ is a union of boundary components of A and a union of boundary components of B, then we can find a CW-structure on M which has all the above properties and such that A and B are subcomplexes.

Remark.

(1) Theorems 96.2 and 96.5 can be viewed as a significant strengthening of Corollary 95.37. Later on, in the Topological Manifold-Simplicial Complex Retract Theorem 104.13 together with the CW-Complex Domination Proposition 162.8, and in the Morse Functions-Cells Proposition 138.6 and in the Smooth Manifold-CW-Homotopy Type Proposition 139.8 we will get to know two more methods for equipping a smooth manifold with CW-structure, at least up to a homotopy equivalence.
(2) Note that the Smooth Manifold-CW Structure Theorem 96.5 is stated for smooth manifolds and *not* for topological manifolds. The situation for topological manifolds is in fact much more complicated and we will deal with it later on in Section 104.6.

Proof. The theorem follows immediately from the Smooth Manifold-Simplicial Structure Theorem 96.2 together with Lemma 96.1 and the Simplicial-Implies-CW Lemma 93.24. ■

Almost for free we now obtain the following proposition which gives us some control over the "size" of invariants of smooth manifolds.

Proposition 96.6. (Smooth Manifolds-Invariants Proposition) Let M be an n-dimensional 0-connected smooth manifold.

(1) For every $k > n$ and for every abelian group G we have $\mathrm{H}_k(M; G) = 0$.
(2) The higher homotopy groups of M are countable.

If M is compact then the following further statements hold:

(3) The fundamental group of M is finitely presented.
(4) All homology groups of M are finitely generated abelian groups. Furthermore, given any field $\mathbb{F}$ the homology groups $\mathrm{H}_k(M; \mathbb{F})$ are finite-dimensional.

Remark.

(1) Note that in Exercise 86.9 we saw that there exists a closed smooth manifold M such that $\pi_2(M)$ is infinitely generated as an abelian group. In other words, the Smooth Manifolds-Invariants Proposition 96.6 (2) is in a sense optimal.
(2) On page 2997 we will give an alternative proof for some of the Statements (1), (3) and (4) of the Topological Manifolds-Invariants Proposition 104.14 using "handle decompositions". An alternative approach to proving these three statements is also given by using the Smoothly Good Cover Theorem **??**. We will carry out this idea in Exercise 96.8.
(3) In Chapter 98 we will see that the existence of smooth simplicial structures on smooth manifolds can be used to extract even more information on homology groups of compact smooth manifolds.
(4) In the Topological Manifolds-Invariants Proposition 104.14 we will extend the Smooth Manifolds-Invariants Proposition 96.6 to compact *topological* manifolds.

Proof.

(1) This statement follows from the Smooth Manifold-CW Structure Theorem 96.5 together with the Homology-of-CW-Complex Proposition 80.6 (1) and the discussion on page 1915.
(2) This statement follows from the Smooth Manifold-Simplicial Structure Theorem 96.2 together with Proposition 94.9.
(3) This statement follows from the Smooth Manifold-CW Structure Theorem 96.5 together with the CW-Skeleton-π_1-Proposition 69.13 (3).
(4) It follows from the Smooth Manifold-CW Structure Theorem 96.5 together with the Homology-of-CW-Complex Proposition 80.6 (2) that all homology groups are finitely generated. Furthermore it follows from the discussion on page 1915 that each $\mathrm{H}_k(M;\mathbb{F})$ is finite-dimensional. ■

We can even use our results to say something meaningful about fundamental groups and homology groups for non-compact smooth manifolds.

Proposition 96.7. Let M be a 0-connected smooth manifold.
(3′) The fundamental group of M is countable.
(4′) All homology groups $\mathrm{H}_k(M)$ are countable abelian groups. Furthermore, given any field $\mathbb{F}$ the dimension of each homology groups $\mathrm{H}_k(M;\mathbb{F})$ is countable.[1186]

Remark.

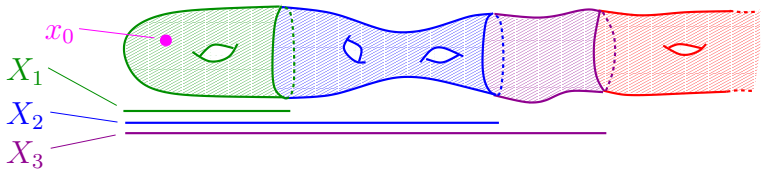
(1) In the Countable Fundamental Group Proposition 49.1 we already showed that fundamental groups of (possibly non-compact) topological manifolds are countable. But it seems difficult to adapt the approach of the Countable Fundamental Group Proposition 49.1 to deal with homology groups.
(2) In Proposition 104.15 we will extend Proposition 96.7 to *topological* manifolds.

Proof. Let M be a 0-connected smooth manifold. By the Push-Away-from-Boundary Corollary 28.5 we know that M is homotopy equivalent to $M \setminus \partial M$. It follows from this

[1186]Note that in contrast we will see on page 2476 that the "cohomology groups" of a non-compact smooth manifold can be uncountable.

fact, together with the Homotopy-π_1-Proposition 50.3 and the Homotopy Equivalence-H_*-Corollary 73.9, that we might as well assume that M has no boundary. Since M has no boundary we know by the Compact Submanifold Exhaustion Proposition 31.2 that there exists a sequence $X_1, X_2, \ldots$ of n-dimensional smooth submanifolds of M with the following three properties:

(1) For each $i \in \mathbb{N}$ we have $X_i \subset \overset{\circ}{X}_{i+1}$.
(2) Each M_i is compact and 0-connected.
(3) We have $\bigcup_{i\in\mathbb{N}} \overset{\circ}{X}_i = M$.



We pick a base point $x_0 \in X_1$. Given $i \leq j$ we denote by $\varphi_{ij}\colon X_i \to X_j$ the inclusion map. We then see that

$$\pi_1(M,x_0) \underset{\uparrow}{\cong} \varinjlim \pi_1(X_i) \underset{\uparrow}{\cong} \Big(\underset{i\in I}{*}\pi_1(X_i,x_0)\Big)/\langle\langle\{\varphi_{ij*}(g)\cdot g^{-1} \mid i\leq j \text{ and } g\in\pi_1(X_i,x_0)\}\rangle\rangle.$$

Direct Limit Proposition 56.7 — proof of the Direct Limit Existence Proposition 56.1 (2)

By the Smooth Manifolds-Invariants Proposition 96.6 we know that each $\pi_1(X_i, x_0)$ is finitely presented, in particular countable. It follows from the above discussion that $\pi_1(M, x_0)$ is countable.

Similarly we see that given any $k \in \mathbb{N}_0$ we have the following isomorphisms:

$$\mathrm{H}_k(M) \underset{\uparrow}{\cong} \varinjlim \mathrm{H}_k(X_i) \underset{\uparrow}{\cong} \Big(\bigoplus_{i\in I}\mathrm{H}_k(X_i)\Big)/\{\varphi_{ij*}(g) - g \mid \ i\leq j \text{ and } g\in \mathrm{H}_k(X_i)\}.$$

Homology-via-Exhaustions Proposition 79.4 — proof of the Direct Limit Existence Proposition 56.1 (3)

It is now clear that $\mathrm{H}_k(M)$ is countable. A very similar argument also works if we work with coefficients in a field $\mathbb{F}$. ■

On several occasions we will also study relative homology groups of a compact smooth manifold M, for example we will be interested in the relative homology groups $\mathrm{H}_k(M, \partial M)$. The following lemma, together with the Smooth Manifolds-Invariants Proposition 96.6 (4), gives us some control over these invariants.

Lemma 96.8. Let X be a topological space, let $A \subset X$ be a subset and let $k \in \mathbb{N}_0$.

(1) If the homology groups $\mathrm{H}_k(X)$ and $\mathrm{H}_{k-1}(A)$ are finitely generated, then so is the relative homology group $\mathrm{H}_k(X, A)$.
(2) Let $\mathbb{F}$ be a field. If the homology groups $\mathrm{H}_k(X;\mathbb{F})$ and $\mathrm{H}_{k-1}(A;\mathbb{F})$ are finite-dimensional, then so is the relative homology group $\mathrm{H}_k(X, A;\mathbb{F})$.

Proof. The first statement follows from the long exact sequence of the pair (X, A) together with the Rank Additivity Lemma 51.7. The second statement is proved in a similar fashion. ■

96.2. **PL-manifolds.** Throughout these notes we have so far only encountered two flavors of manifolds, namely topological manifolds and smooth manifolds. In this section we will introduce a new variety of manifolds, namely we will introduce the concept of PL-manifolds. In a nutshell a PL-manifold is a topological manifold together with a well-behaved simplicial structure.

For the reader's convenience we recall the following definition from page 2048.

Definition. We say that a simplicial complex Y is a subdivision of some simplicial complex X if the following three conditions are satisfied:

(1) the underlying topological spaces are the same,
(2) every simplex of Y is contained in some simplex of X,
(3) for every l-simplex t of Y and every k-simplex s of X which contains t the map $\Phi_s^{-1} \circ \Phi_t \colon \Delta^l \to \Delta^k$ is affine linear.

Remark. Let X be a simplicial complex and let Y be a subdivision. In Exercise 96.3 we will see that every k-simplex of X is the union of finitely many simplices of Y of dimension $\leq k$. This shows that our definition of a subdivision agrees with the definition given in [**Mun1966a**, p. 83]. It also shows that if we use the Simplicial-Implies-CW Lemma 93.24 to view X and Y as CW-complexes, then the identity map $X \to Y$ is cellular.

We continue with the following definition.

Definition. Let X and Y be simplicial complexes. We say a map $f \colon X \to Y$ is piecewise linear, if there exist subdivisions X' and Y' such that $f \colon X' \to Y'$ is simplicial.[1187]

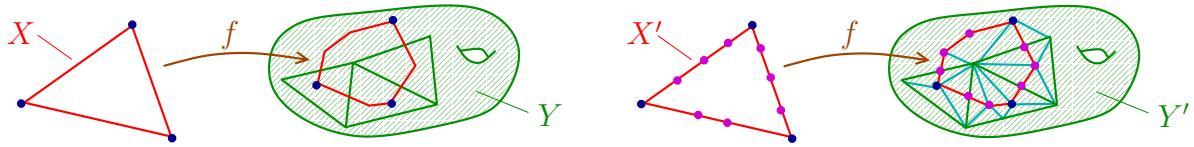
Remark. The definition of a piecewise linear map is the one given in [**Zee1963b**, p. 6] and [**Gla1970**, p. 13]. It follows from [**RS1972**, Theorem 2.14] and [**RS1972**, Example 1.5(4)] that the definition of a piecewise linear map on [**RS1972**, p. 5] is equivalent to the above definition. Furthermore, by [**Zee1963b**, Lemma I.7] the definition of a piecewise linear map on [**Hud1969**, p. 2] is also equivalent to the above definition.

Warning. Many of our favorite reference, e.g. [**Zee1963b**], mostly deals with finite simplicial complexes. Thus on several occasions we will also restrict to the case of finite simplicial complex. We leave it to the undaunted reader to figure out to what degree the results of this section also generalize to infinite simplicial complexes.

We start out with some examples of piecewise linear maps.

Examples.

(1) In the figure below we show a map $f \colon X \to Y$ between two finite linear simplicial complexes which is not simplicial, but which is piecewise linear.



(2) Let $f \colon X \to Y$ be a map between linear simplicial complexes. If X is a finite linear simplicial complex, then it follows from the Simplicial Approximation Theorem 94.8 that f is homotopic to a piecewise linear map.

[1187] Colloquially one usually says "PL" instead of "piecewise linear".

From the definitions it is not entirely clear that the composition of two piecewise linear maps is again piecewise linear. Fortunately, with some effort one can show that this is indeed the case.

Proposition 96.9. The composition of two piecewise linear maps between *finite* simplicial complexes is again piecewise linear.

Proof. The proposition is proved in [**Zee1963b**, Lemma I.2] and [**Gla1970**, Theorem I.6]. ■

Remark. As we hinted at above, both references for the proof of Proposition 96.9 do indeed assume that the simplicial complexes involved are finite. It is not clear to me whether the statement holds for maps between arbitrary simplicial complexes.

We move on to the following basically self-explanatory definition.

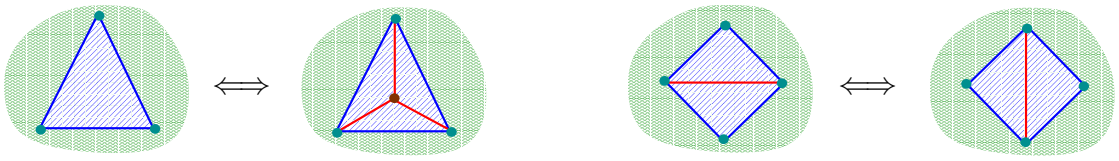
Definition. Let X and Y be two simplicial complexes.

(1) A map $f\colon X \to Y$ is called a PL-homeomorphism if f is piecewise linear, if f is a bijection and if f^{-1} is also piecewise linear.[1188]
(2) We say X and Y are PL-homeomorphic if there exists a PL-homeomorphism $X \to Y$.

We continue with two examples of PL-homeomorphisms.

Examples.

(1) If X' is a subdivision of a simplicial complex X, then it follows immediately from the definition that the map $X' \to X$ is a PL-homeomorphism.
(2) If $f\colon X \to Y$ is a map such that there exists subdivision X' of X and Y' of Y such that the map $f\colon X' \to Y'$ is a simplicial isomorphism, then f is evidently a PL-homeomorphism.
(3) (a) In the figure below we show to the left 2-dimensional linear simplicial complexes X_1 and Y_1 which are related by adding a vertex to a 2-dimensional simplex. It follows from (1) that X_1 and Y_1 are PL-homeomorphic.
    (b) In the figure below we show to the right 2-dimensional simplicial complexes X_2 and Y_2. If we subdivide X_2 and Y_2 by adding the intersection point of the two diagonals, then the resulting two linear simplicial complexes are simplicially isomorphic. By (2) this shows that X_2 and Y_2 are PL-homeomorphic.



$|X_1|$ $|Y_1|$ $|X_2|$ $|Y_2|$

(4) Let $n \in \mathbb{N}$. We consider the two linear simplicial complexes $\mathrm{Cone}(\partial\Delta^n)$ and Δ^n. It follows from the discussion on page 2049 that Δ^n admits a subdivision, given by adding a single vertex, which is simplicially isomorphic to $\mathrm{Cone}(\partial\Delta^n)$. By (1) this shows that $\mathrm{Cone}(\partial\Delta^n)$ and Δ^n are PL-homeomorphic.

[1188] For the record we state that it follows from Proposition 93.7 that every piecewise linear map is continuous.

(5) By Proposition 95.17 (1) we know that any two finite linear simplicial complexes X and Y with $|X| = |Y|$ are PL-homeomorphic.

Later on we will need the following elementary lemma.

Lemma 96.10. If $f_1 \colon X_1 \to Y_1$ and $f_2 \colon X_2 \to Y_2$ are two PL-homeomorphisms between two finite simplicial complexes, then the map

$$f_1 * f_2 \colon X_1 * X_2 \;\to\; Y_1 * Y_2$$

is also a PL-homeomorphism.

Proof. We leave it to the reader to provide the proof. ■

Before we give the definition of a PL-manifold let us introduce the following convention.

Convention. Let $n \in \mathbb{N}_0$. On page 1978 we introduced the abstract simplicial complexes

(1) $D_n := (V, S)$ with $V = \{0, \ldots, n\}$ and $S = \mathcal{P}(V) \setminus \{\varnothing\}$, and
(2) $S_{n-1} := (V, T)$ with $V = \{0, \ldots, n\}$ and $T = \mathcal{P}(V) \setminus \{\varnothing, V\}$.

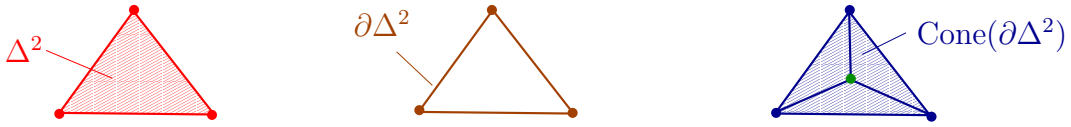
On page 1989 we saw that there exists a canonical homeomorphism $f \colon |D_n| \to \Delta^n$ which restricts to a homeomorphism $|S_{n-1}| \to \partial\Delta^n$. Unless we say explicitly something else we will always use these homeomorphisms to view Δ^n and $\partial\Delta^n$ as simplicial complexes.

The above convention leads us to our next definition.

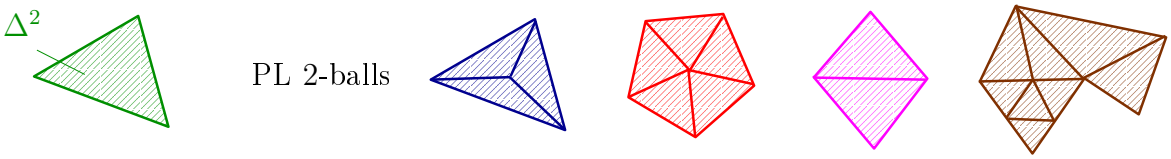
Definition. Let $n \in \mathbb{N}$.

(1) A PL n-ball is defined as a simplicial complex that is PL-homeomorphic to the simplicial complex Δ^n.
(2) A PL n-sphere is defined as a simplicial complex that is PL-homeomorphic to the simplicial complex $\partial\Delta^{n+1}$.

Example. Let $n \in \mathbb{N}_0$. In Exercise 96.2 we will show that $\mathrm{Cone}(\partial\Delta^n)$ is a PL n-ball.



Remark. In practice it can be a little tricky to show that a given simplicial complex is in fact a PL n-ball or a PL n-sphere. But in most cases one can verify, with enough patience and dexterity, that if a simplicial complex "looks" like an n-ball or an n-sphere, then it is in fact a PL n-ball respectively a PL n-sphere. For example it is entertaining to show that all the simplicial complexes shown in the figure below are in fact PL 2-balls. Shortly we will get to know a more systematic way for finding PL-homeomorphisms.



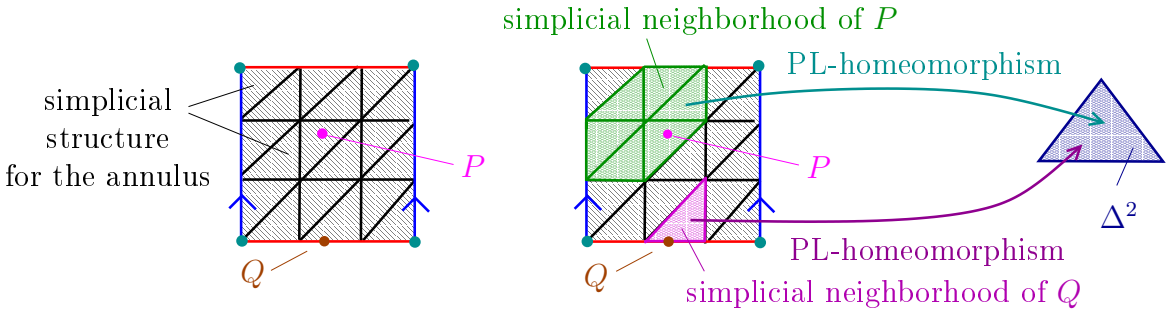
Now we can provide the definition of a PL-manifold. The definition is modelled on the definition of a topological manifold that we gave on page 511. The key difference is that we replace the words "neighborhood", "ball" and "homeomorphic" by "simplicial neighborhood", "PL-ball" and "PL-homeomorphic".

Definition.

(1) Let X be a simplicial complex and $P \in X$. A simplicial neighborhood is a subcomplex of X that is a neighborhood of P in the usual sense as defined on page 113.
(2) An n-dimensional PL-structure on a topological space X is defined as a simplicial structure $(K = (V, S), \Theta\colon |K| \to X)$ such that every point $P \in X$ admits a simplicial neighborhood that is a PL n-ball.
(3) An n-dimensional PL-manifold is a pair (X, Θ) where X is a topological space and Θ is an n-dimensional PL-structure with countably many simplices.

Examples.

(1) In the figure below we see to the left a simplicial structure for the annulus $S^1 \times [0, 1]$. The illustrations to the right of the figure should convince the reader that the simplicial structure for the annulus is a PL-structure.



(2) Let $n \in \mathbb{N}_0$. The canonical simplicial structure on Δ^n is basically by definition a PL-structure. Recall that in the Standard Simplex Homeomorphism Lemma 72.1 we gave an explicit homeomorphism $\Phi\colon \overline{B}^n \to \Delta^n$. Thus $(D_n, \Phi^{-1}\colon \Delta_n = |D_n| \to \overline{B}^n)$ is a PL-structure for $\overline{B}^n$. Similar to the discussion on page 1997 we will refer to it as the *canonical* PL-*structure* of $\overline{B}^n$.
(3) Let $n \in \mathbb{N}_0$. We consider $\partial\Delta^{n+1}$ with the usual simplicial structure defined on page 2092. Given any vertex v the star $\mathrm{St}(\partial\Delta^n, v)$ is, basically by definition, simplicially isomorphic to $\mathrm{Cone}(\partial\Delta^n)$. It follows from the example on page 2092 that $\mathrm{St}(\partial\Delta^n, v)$ is a PL n-ball. This shows that $\partial\Delta^n$ is an n-dimensional PL-manifold. Recall that in the Standard Simplex Homeomorphism Lemma 72.1 we gave an explicit homeomorphism $\Phi\colon S^n \to \partial\Delta^{n+1}$. This shows that $(S_n, \Phi^{-1}\colon \partial\Delta_{n+1} = |S_n| \to S^n)$ defines a PL-structure on S^n. We refer to it as the *canonical* PL-*structure* on S^n.

Proposition 96.11. Let X and Y be two simplicial complexes. If X is an n-dimensional PL-manifold and if X and Y are PL-homeomorphic, then Y is also an n-dimensional PL-manifold.

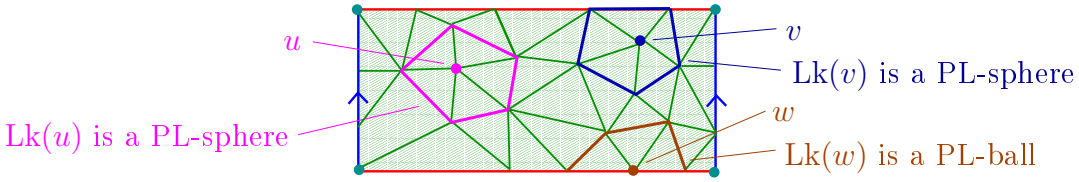
Example. It follows from Proposition 96.11 that any PL n-ball and any PL n-sphere is indeed an n-dimensional PL-manifold.

Proof. This proposition is less trivial than it might appear initially. Fortunately a proof is given in [**Gla1970**, Theorem II.2]. ■

The following proposition gives an alternative characterization of PL-manifolds.

Proposition 96.12. (PL-Manifold Characterization Proposition) We consider a simplicial complex $(K = (V, S), \Theta\colon |K| \to X)$. The following statements are equivalent:

(1) (X, Θ) is an n-dimensional PL-manifold.
(2) For every vertex $v \in V$ the link $\mathrm{Lk}(K, v)$ is a PL $(n-1)$-sphere or a PL $(n-1)$-ball.
(3) For every vertex $v \in V$ the star $\mathrm{St}(K, v)$ is a PL n-ball.
(4) For every k-simplex $s \in S$ the link $\mathrm{Lk}(K, s)$ is a PL $(n-k-1)$-sphere or a PL $(n-k-1)$-ball.
(5) For every k-simplex $s \in S$ the star $\mathrm{St}(K, s)$ is a PL $(n-k)$-ball.



Proof.

(1)⇒(2) This implication is shown in [**Hud1969**, p. 26] (alternatively see also [**Zee1963b**, p. III.2]).

(2)⇒(3) Let $v \in V$. First we consider the case that the link $\mathrm{Lk}(v)$ is a PL $(n-1)$-sphere. In this case we have the following PL-homeomorphisms:

$$\mathrm{St}(v) \underset{\uparrow}{\cong} \{v\} * \mathrm{Lk}(v) \underset{\uparrow}{\cong} \{v\} * \text{PL } (n-1)\text{-sphere} \underset{\uparrow}{\cong} \{v\} * \partial\Delta^n \underset{\uparrow}{\cong} \Delta^n.$$

Link Star Lemma 94.5 (1c) — since (2) holds — Lemma 96.10 — see page 2091

The case that the link $\mathrm{Lk}(v)$ is a PL $(n-1)$-ball is dealt with the same way, except that this time we use the basically trivial fact, first pointed out on page 2003 that $\{v\} * \Delta^{n-1}$ is in fact simplicially isomorphic to Δ^n. We have thus shown that (2) implies (3).

(3)⇒(1) This implication follows immediately from the Link Star Lemma 94.5 (1).

(2)⇒(4) We prove the statement by induction on k. Note that the case $k = 0$ is precisely Statement (2). Now suppose that the statement holds for some $k-1$. Let $K = (V, S)$ be an abstract simplicial complex and let $t = \{v_0, \ldots, v_k\}$ be a k-simplex. We consider the $(k-1)$-simplex $s = \{v_0, \ldots, v_{k-1}\}$. By induction hypothesis we know that $\mathrm{Lk}(K, s)$ is a PL $(n-k)$-sphere or a PL $(n-k)$-ball. As we pointed out on page 2093, it follows from Proposition 96.11 that $\mathrm{Lk}(K, s)$ is in fact an $(n-k)$-dimensional PL-manifold. Now we see that

$$\mathrm{Lk}(K, t) \underset{\uparrow}{=} \mathrm{Lk}(\mathrm{Lk}(K, s), v_k) \underset{\uparrow}{=} \mathrm{Lk}\left(\begin{matrix}(n-k)\text{-dimensl.}\\ \text{PL-manifold}\end{matrix}, v_k\right) \underset{\uparrow}{=} \begin{matrix}\text{PL } (n-k-1)\text{-sphere}\\ \text{or a PL } (n-k-1)\text{-ball.}\end{matrix}$$

follows easily from definitions — by induction hypothesis — by (1)⇒(2)

(4)⇒(5) The proof of this statement is basically identical to the proof of (2)⇒(3).

(5)⇒(3) This implication holds for the trivial reason that every vertex is in particular a 0-simplex. ■

Remark.

(1) The notion of a PL-manifold gets defined differently by different authors. There is the usual question whether or not one wants a manifold to be second-countable or not which in our case translates into the question whether or not one wants the simplicial complex to be countable. But if we ignore this rather technical issue, then we see the following:
   (a) The above definition above is basically the one given in [**Hud1969**, p. 20] and [**Zee1963b**, p. II.10].
   (b) It follows from the PL-Manifold Characterization Proposition 96.12 (1)⇔(2) that our definition is equivalent to the definition provided in [**Bry2002**, p. 223] and [**Stal1967**, Definition 4.4.9].
   (c) It follows from the PL-Manifold Characterization Proposition 96.12 (1)⇔(5) that our definition is equivalent to the definition given in [**Geo2008**, p. 134].
   (d) It is a consequence of [**RS1972**, Theorem 2.2] that our definition is equivalent to the one given in [**RS1972**, p. 7].
   (e) In the literature an n-dimensional PL-manifold is sometimes defined as a pair $(X, \{\Phi_i \colon U_i \to V_i\})$ where X is an n-dimensional topological manifold and where $\{\Phi_i \colon U_i \to V_i\}$ is an atlas for X such that given any $i, j \in I$ the corresponding transition map $\Phi_j \circ \Phi_i^{-1} \colon V_i \cap \Phi_i(U_j) \to V_j \cap \Phi_j(U_j)$ is "piecewise linear". Here, in this context we say that a map $f \colon U \to V$ between open subsets of $\mathbb{R}^n$ is *piecewise linear* if any point $P \in U$ there exists a linear simplicial complex $X \subset U$ which is a neighborhood of P such that the restriction of f to each simplex of X is linear. This definition is for example used in [**GoS1999**, p. 7], [**Rudy2016**, p. 16] and [**HiM1974**, p. 8]. It seems to me that the discussions on [**Hud1969**, p. 82] and [**Ded1962**, p. 370] show that these two concepts of PL-manifolds basically agree. It would be nice though to have a more direct reference for this statement.
(2) A PL-structure for a topological space is sometimes called a *piecewise linear triangulation*, or shorter, a PL-*triangulation*.

the PL-Manifold Characterization Proposition 96.12, and drawing enough pictures, shows that the following definition captures the right intuition.

Definition. We say a PL-manifold is closed if X is compact and for every vertex the link $\mathrm{Lk}(v)$ is a PL-sphere.

We have no intention of going deep into the theory of PL-manifolds. Instead we refer the interested reader to [**Hud1969, Zee1963b, Gla1970, RS1972**] for a much more detailed introduction to PL-manifolds. For example these references contain the definition of the boundary a PL-manifold and a discussion of PL-submanifolds and the existence of "regular neighborhoods" (see also the next section) of subcomplexes.
As an aside we mention only one result, namely suppose that M is a PL-manifold. Similar to the statements of the Smooth Manifold-Simplicial Structure Theorem 96.2 (2) and (3), one can extend a given PL-structure on the boundary to a PL-structure on M and one can extend a given PL-structure on a proper submanifold to a PL-structure of M. We refer to [**Arm1967**] for the precise statements and proofs.

96.3. **Regular Neighborhoods.** In this short section we introduce the notion of a regular neighborhood of a subcomplex of a simplicial complex. This is an object that gets used

commonly in topology and thus we want to give a definition and we want state to some of the main properties of regular neighborhoods. Otherwise we will barely make use of this concept.
First we give a slight generalization of the concept of the barycentric subdivision of an abstract simplicial complex.

Definition. Let $K = (V, S)$ be an abstract simplicial complex and let $J = (U, R)$ be a finite subcomplex. We define the barycentric subdivision $\mathrm{sd}_J(K)$ of K away from J by barycentric subdividing every simplex that is not contained in J. More formally, and much less readably, $\mathrm{sd}_J(K)$ is defined as follows:

$$\begin{aligned} \text{vertices of } \mathrm{sd}_J(K) &:= U \cup \{\underline{s}\}_{s \in S \setminus R} \\ \text{simplices of } \mathrm{sd}_J(K) &:= \left\{ r \cup \{\underline{s_0}, \ldots, \underline{s_k}\} \;\middle|\; \begin{array}{c} \text{where } r \in R \cup \{\varnothing\} \text{ and } s_0, \ldots, s_k \in S \\ \text{such that } r \subsetneq s_0 \subsetneq \cdots \subsetneq s_k \end{array} \right\}. \end{aligned}$$

It follows immediately from the definitions that $\mathrm{sd}_J(K)$ is an abstract simplicial complex and that for $J = \varnothing$ we obtain the usual barycentric subdivision $\mathrm{sd}(K)$.



The obvious generalization of the SD-Lemma 94.3 holds in the above context. But there is certainly no need for formulating the generalizations. Let us move on to the definition of regular neighborhoods.

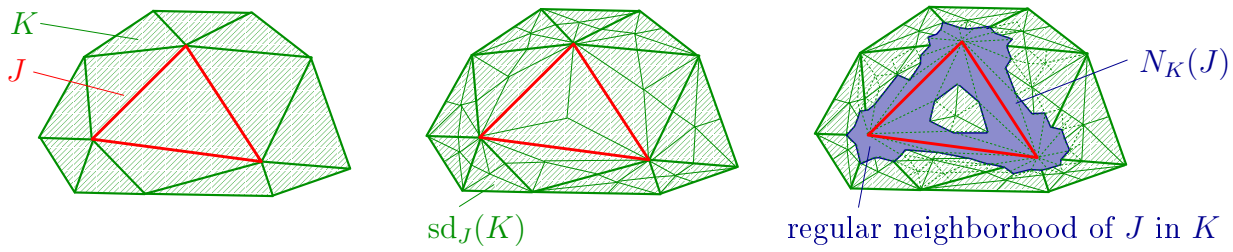
Definition. Let $(K = (V, S), \Theta\colon |K| \to X)$ be a simplicial complex and let $J = (U, R)$ be a subcomplex. We set

$$S_K(J) := \{s \in \mathrm{sd}_J(\mathrm{sd}_J(K)) \,|\, \text{there exists an } r \in R \text{ with } r \subset s\}.$$

Now we define

$$N_K(J) := \bigcup_{s \in S_K(J)} \Theta(|s|) \quad \text{and} \quad \mathring{N}_K(J) := \bigcup_{s \in S_K(J)} \Theta(\langle s \rangle).$$

We refer to $N_K(J)$ as the regular neighborhood of J in K.



Remark. The notion of a regular neighborhood gets defined differently by different authors. Our definition of *the* regular neighborhood of J in K is an example of *a* regular neighborhood as introduced in [**RS1972**, p. 33] or alternatively in [**Gla1970**, Theorem II.15n].

The following proposition states a few key properties of regular neighborhoods of subcomplexes of PL-manifolds.

Proposition 96.13. (Simplicial Regular Neighborhood Proposition) Let $(K = (V,S), \Theta\colon |K| \to M)$ be a compact PL-manifold and let $J = (U,R)$ be a subcomplex with $\Theta(|J|) \subset M \setminus \partial M$.

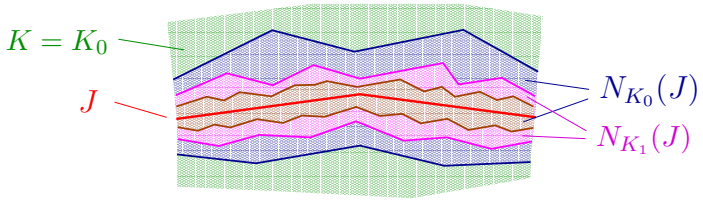
(1) $N_K(J)$ is a compact PL-manifold and it is a submanifold of the topological manifold M.
(2) (a) $\mathring{N}_K(J)$ is an open subset of M.
    (b) The boundary of $N_K(J)$ as a subset of M equals $N_K(J) \setminus \mathring{N}_K(J)$.
    (c) $N_K(J)$ and $\mathring{N}_K(J)$ are both neighborhoods of $\Theta(|J|)$.
(3) $\Theta(|J|)$ is a deformation retract of $N_K(J)$.
(4) $\partial N_K(J)$ is a deformation retract of $N_K(J) \setminus \Theta(|J|)$.

Sketch of proof. Evidently we can just assume that $M = |K|$ and that $\Theta = \mathrm{id}$. As mentioned above, our definition of the regular neighborhood is an example of a regular neighborhood as defined in [**RS1972**, p. 33]. Thus we can use the results from [**RS1972**].

(1) This statement is explicitly and implicitly proved in [**RS1972**, Propostion 3.10].
(2) We leave the fairly routine verification of these statements to the reader.
(3) This statement is proved in [**RS1972**, Corollary 3.30 (ii)], see also [**Gla1970**, p. 77].
(4) We only provide an outline of a proof. First we define a sequence of abstract simplicial complexes via setting $K_0 := K$ and iteratively via setting $K_{n+1} := \mathrm{sd}_J(K_n)$ for $n \in \mathbb{N}$. Similarly to the SD-Lemma 94.3 (3) we see that these abstract simplicial complexes define again naturally simplicial structures for M. Using a suitable variation on Lemma 94.7 one can show that $\bigcap_{n\in\mathbb{N}_0} N_{K_n}(J) = |J|$. The key input at this point is [**RS1972**, Corollary 3.18] which implies that for every $n \in \mathbb{N}_0$ there exists a homeomorphism $F_n\colon N_{K_n}(J) \times [0,1] \to N_{K_n}(J) \setminus \mathring{N}_{K_{n+1}}(J)$ such that for every $P \in N_{K_n}(J)$ we have $F_n(P,0) = P$ and $F_n(P,1) \in \partial N_{K_{n+1}}(J)$. Next we want to combine all these maps F_n to obtain a homeomorphism $\partial N_K(J) \times [0,1] \to N_K(J) \setminus |J|$. To do so we set $s_0 = 0$, $s_1 = \frac{1}{2}$, $s_2 = \frac{3}{4}$, $s_3 = \frac{7}{8}$ and so on. Furthermore, given $n \in \mathbb{N}_0$ we define $\Xi_n\colon \partial N_K(J) \to \partial N_{K_n}(J)$ via $\Xi_n(P) = F_{n-1}(\ldots(F_1(F_0(P,1),1),\ldots,1)$. Finally we consider the map

$$\begin{array}{rcl} \partial N_K(J)\times[0,1) & \to & N_K(J)\setminus|J| \\ (P,t) & \mapsto & F_n(\Xi_n(P), 2^{n+1}\cdot(t-s_n)), \quad \text{if } t\in[s_n,s_{n+1}) \text{ for some } n\in\mathbb{N}_0. \end{array}$$

Some thought shows that this map is actually a homeomorphism. The desired statement follows from the observation that $\partial N_K(J) = \partial N_K(J) \times \{0\}$ is a deformation retract of $N_K(J) \times [0,1)$. The more carefully one looks at this outline, the more little exercises one detects which need to be carried out to turn this argument into a proper proof. ■



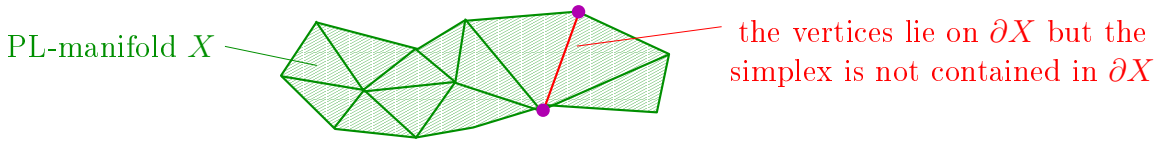
This concludes our discussion of regular neighborhoods. This concept is discussed in much greater detail in [**WhdJ1939**], [**RS1972**, Chapter 3], [**Gla1970**, Chapter III.B], [**Hud1969**, Chapter 2] and [**Zee1963b**, Chapter 3]. Some properties are also discussed in [**Hem1976**, p. 7f]. In particular we refer to [**RS1972**, Theorems 3.8 and 3.24] and [**Gla1970**, Theorem II.16n] for some uniqueness theorems which explain to what degree the regular neighborhood depends on the choice of simplicial structures on a given PL-manifold. Unfortunately it is not always immediately apparent that the various definitions in the above literature are mutually compatible.

96.4. **The interplay between topological, smooth and PL-manifolds.** Now that the PL-manifolds joined the playground, let us see how they interact with our old friends, the topological and smooth manifolds. Since morally speaking topological manifolds have the least amount of structure it is not surprising that PL-manifolds are indeed topological manifolds:

Proposition 96.14. Let X be an *n-dimensional* PL-*manifold*. The following statements hold:

(1) X is an n-dimensional topological manifold.
(2) A vertex v of the PL-structure lies on the boundary of the topological manifold X if and only if the link $\mathrm{Lk}(v)$ is a PL-ball.

Remark. Let X be a PL-manifold and let s be a simplex such that all vertices lie on the boundary ∂X of X, viewed as a topological manifold. As we see in the figure below, this piece of information does not imply that s is actually contained in ∂X.



Proof. Let $(K = (V, S), \Theta\colon |K| \to X)$ be an *n-dimensional* PL-*manifold*. Without loss of generality we can assume that $\Theta = \mathrm{id}$, in particular we can assume that $|K| = X$.

(1) In the following we will show that X is an n-dimensional topological manifold.
  (a) By the Simplex-Embedding Lemma 93.5 we know that X is Hausdorff.
  (b) By definition of a PL-manifold we know that S is countable, in particular V is countable. Furthermore note that the PL-Manifold Characterization Proposition 96.12 says that for each $v \in V$ the open star $\mathring{\mathrm{St}}(v)$ is homeomorphic to a subset of $\mathbb{R}^n$. From the Second Countability Lemma 9.1 (1) and (2) we obtain that each open star is second-countable. Thus it follows from the Link Star Lemma 94.5 (2) that X is the countable union of countably many second-countable open subsets. Hence we obtain from the Second Countability Lemma 9.1 (4) that X itself is second-countable.
  (c) By definition of a PL-manifold we know that for each $P \in X$ there exists in particular a neighborhood U of P and a homeomorphism $\Phi\colon U \to \overline{B}^n$. On page 513 we showed that $\overline{B}^n$ itself is an n-dimensional topological manifold. It it now elementary to see that there exists a chart, in the sense of the definition on page 511, for X at P.

We have thus verified that X is indeed an n-dimensional topological manifold.

(2) Let $v \in V$ be a vertex. We calculate that

$$\begin{array}{ccccc} \mathrm{H}_n(X, X\setminus\{v\}) & \underset{\uparrow}{\cong} & \mathrm{H}_n(\mathrm{St}(v), \mathrm{St}(v)\setminus\{v\}) & \underset{\uparrow}{\cong} & \mathrm{H}_n(\mathrm{St}(v), \mathrm{Lk}(v)) \\ & \text{by the Excision Theorem 74.18 which we can apply by the Link Star Lemma 94.5 (1d)} & & \text{Link Star Lemma 94.5 (1f) and Corollary 74.16 (2a)} & \end{array}$$

$$\begin{array}{ccc} \underset{\uparrow}{\cong}\ \widetilde{\mathrm{H}}_{n-1}(\mathrm{Lk}(v)) & \cong & \begin{cases} \mathbb{Z}, & \text{if } \mathrm{Lk}(v) \text{ is a PL } (n-1)\text{-sphere,} \\ 0, & \text{if } \mathrm{Lk}(v) \text{ is a PL } (n-1)\text{-ball.} \end{cases} \\ \text{by the long exact sequence in reduced homology of the pair } (\mathrm{St}(v), \mathrm{Lk}(v)) & & \\ \text{together with the Link Star Lemma 94.5 (1g)} & & \end{array}$$

It follows from the Manifold-Local Homology Lemma 75.1 that $v \in \partial X$ if and only if $\mathrm{Lk}(v)$ is a PL $(n-1)$-ball. ■

Next we turn to the relationship between smooth manifolds and PL-manifolds. The following theorem says that any smooth manifold admits an essentially natural PL-structure.

Theorem 96.15.

(1) Every smooth simplicial structure on an n-dimensional smooth manifold is an n-dimensional PL-structure.
(2) If M is a closed smooth manifold, then for every smooth simplicial structure the corresponding PL-manifold is also closed.
(3) Every smooth manifold admits the structure of a PL-manifold.
(4) Any two smooth simplicial structures on a smooth manifold are PL-homeomorphic.

Proof.

(1) It follows from [**WhdJ1940**, Theorem 5] (see also the argument of [**Mun1966a**, p. 82]) together with the PL-Manifold Characterization Proposition 96.12 that every smooth simplicial structure on a smooth manifold is a PL-structure in the above sense.
(2) This statement follows immediately from Proposition 96.14, together with fact, proved in the Topological Manifold Boundary Proposition 75.2 (2), that a smooth manifold is closed as a smooth manifold if and only if it is closed as a topological manifold.
(3) This statement follows from (1) together with the Smooth Manifold-Simplicial Structure Theorem 96.2 where we showed that every smooth manifold admits a countable *smooth* simplicial structure.
(4) This statement follows immediately from Theorem 96.4. ■

Theorem 96.15 says in particular that given any $n \in \mathbb{N}_0$ the map

$$\begin{array}{rccc} \mathfrak{w}_n\colon & \left\{\begin{array}{c}\text{diffeomorphism classes of closed}\\ n\text{-dimensional smooth manifolds}\end{array}\right\} & \to & \left\{\begin{array}{c}\text{PL-homeomorphism classes of closed}\\ n\text{-dimensional PL-manifolds.}\end{array}\right\} \\ & [M] & \mapsto & [(M, \text{smooth simplicial structure})] \end{array}$$

is well-defined. It is natural to ask whether $\mathfrak{w}_n$ is a bijection. It turns out that the answer is complicated. More precisely, we have the following theorem.

Theorem 96.16.

(1) For $n = 1, \ldots, 6$ the map $\mathfrak{w}_n$ is a bijection.

(2) For $n = 7$ (and many other dimensions) the map $\mathfrak{w}_n$ is not a monomorphism.
(3) For $n = 10$ (and many other dimensions) the map $\mathfrak{w}_n$ is not an epimorphism. In fact there exists a closed 10-dimensional PL-manifold that does not admit a single smooth structure.

Proof. The theorem is a combination of several major results in topology. Later in Section **??** we will discuss this theorem in much greater detail and we will give all the necessary references. In particular we will restate (1) as the PL-Smoothing Theorem **??**. ■

We turn our focus back to the relationship between topological and PL-manifolds. In light of Proposition 96.14 it is perhaps tempting to think that a PL-manifold is the same as a topological manifold together with some simplicial structure. But it turns out that the situation is more delicate.

Theorem 96.17. There exists a simplicial structure on S^5 that is not a PL-structure.

The key to proving Theorem 96.17 is the following very subtle and very difficult theorem which was proved independently by Jim Cannon [**Cann1979**] and Richard Edwards [**Edw1980, Edw2006**] in the late 1970s. We also refer to [**Dav1986**, Corollary VII.2.C] and [**Lat1979**] for alternative accounts of the proof. It is also worth reading Jim Cannon's reminiscences [**Cann1994**].

Theorem 96.18. (Double Suspension Theorem) Let Y be an n-dimensional PL-manifold. If Y is a homology sphere, i.e. if $\mathrm{H}_k(Y;\mathbb{Z}) \cong \mathrm{H}_k(S^n;\mathbb{Z})$ for all k, then the *double* suspension $\Sigma(\Sigma(Y))$ is homeomorphic to S^{n+2}.

The proof of Theorem 96.17 also requires the following lemma.

Lemma 96.19. Let $n \in \mathbb{N}_{\geq 2}$ and let M be a closed 0-connected n-dimensional topological manifold. If $\pi_1(M)$ is non-trivial, then the suspension $\Sigma(M)$ is not an $(n+1)$-dimensional topological manifold.

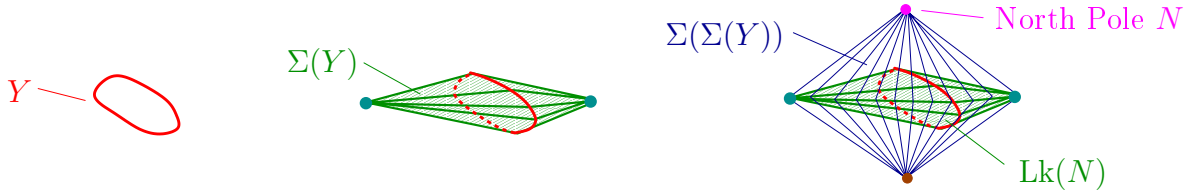
Proof. We outsource the proof to the very amusing Exercise 76.9. ■

With these preparations it is now quite easy to provide a proof of Theorem 96.17.

Proof of Theorem 96.17.

(1) By Proposition 98.11 we know that there exists a closed orientable 3-dimensional smooth manifold Y that is a homology 3-sphere, but such that $\pi_1(Y)$ is a non-trivial group.
(2) By Theorem 96.15 we can equip Y with a PL-structure.
(3) It follows from (2) and the Double Suspension Theorem 96.18 that $\Sigma(\Sigma(Y))$ is homeomorphic to S^5.
(4) By (2) we know that Y is equipped with a simplicial structure. We equip the double suspension $\Sigma(\Sigma(Y))$ with the simplicial structure given by "the double suspension" of the simplicial structure. More precisely, we equip $\Sigma(\Sigma(Y))$ with the simplicial structure given by applying the Simplicial Cone-Suspension Corollary 93.21 twice.
(5) Let N be the "North Pole" of the second suspension. It follows from the discussion on page 2018 that $\mathrm{Lk}(\Sigma(\Sigma(Y)), N) = \Sigma(Y)$.
(6) Since $\pi_1(Y)$ is non-trivial we obtain from Lemma 96.19 that $\Sigma(Y)$ is not even a 4-dimensional topological manifold. Therefore we obtain from Proposition 96.14 that $\mathrm{Lk}(\Sigma(\Sigma(Y)), N) = \Sigma(Y)$ is not a 4-dimensional PL-manifold, let alone a PL 4-sphere.

(7) We obtain from the above, together with the PL-Manifold Characterization Proposition 96.12, that $\Sigma(\Sigma(Y))$ admits a simplicial structure that is not a PL-structure.
(8) In (3) we saw that $\Sigma(\Sigma(Y))$ is homeomorphic to S^5. It follows from (7) that S^5 admits a simplicial structure that is not a PL-structure. ■



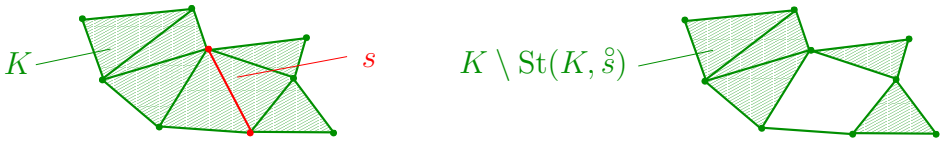
Towards the end of these notes, in Section **??**, we will study the relationship between topological manifolds, smooth manifolds and PL-manifolds in greater detail.

96.5. **Stellar subdivisions and the Alexander-Newman Theorem.** The notion of a PL-homeomorphism, and thus also the notion of a PL-manifold, relies on the notion of a subdivision which, as discussed on page 2048, is not combinatorial. In the final two sections of this chapter we will discuss how one can turn the theory of PL-homeomorphisms and PL-manifolds into a purely combinatorial theory.
First, given an abstract simplicial complex we introduce, inspired by the definition on page 2018 and by the Link Star Lemma 94.5 (1b), the combinatorial notion of the star and the link of a simplex.

Definition. Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$ be a simplex. We introduce the following three subcomplexes[1189] of K:

$$\begin{aligned}
\mathrm{St}(K,s) &:= \Big(\bigcup_{t\in S \text{ with } s\subset t} t,\ \{t \in S \,|\, s\cup t \in S\}\Big)\\
\mathrm{Lk}(K,s) &:= \Big(\bigcup_{t\in S \text{ with } s\subset t} t \setminus s, \{t \in S \,|\, s\cup t \in S \text{ and } s\cap t = \varnothing\}\Big)\\
K\setminus \mathrm{St}(K,\mathring{s}) &:= \begin{cases} (V\setminus\{v\}, \{t\in S \,|\, s\not\subset t\}), & \text{if } s=\{v\} \text{ is a 0-simplex,}\\ (V, \{t\in S \,|\, s\not\subset t\}), & \text{otherwise.}\end{cases}
\end{aligned}$$

We refer to the figure below for an illustration of the definition of $K \setminus \mathrm{St}(K, \mathring{s})$.



Remark. Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$.

(1) The vigilant reader will have noticed that we defined $\mathrm{St}(K, s)$ and $\mathrm{Lk}(K, s)$ twice, namely on page 2018 as suitable subsets of $|K|$ and now as abstract subcomplexes. The vigilant reader surely will have no troubles in figuring out which of the two definitions we are working with at any given moment.
(2) The notation "$K \setminus \mathrm{St}(K, \mathring{s})$" is meant to be suggestive, but note that on its own $\mathring{s}$ makes no sense.

Next we recall the following definitions from page 1980 and page 2014.

[1189] One can easily verify that all three are indeed subcomplexes.

Definition. Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$ be a simplex.

(1) (a) If s is a 0-simplex, then we define $\partial s = (\varnothing, \varnothing)$.
(b) If s is not a 0-simplex, then we define ∂s to be the abstract simplicial complex $(s, \mathcal{P}(s) \setminus \{\varnothing, s\})$.

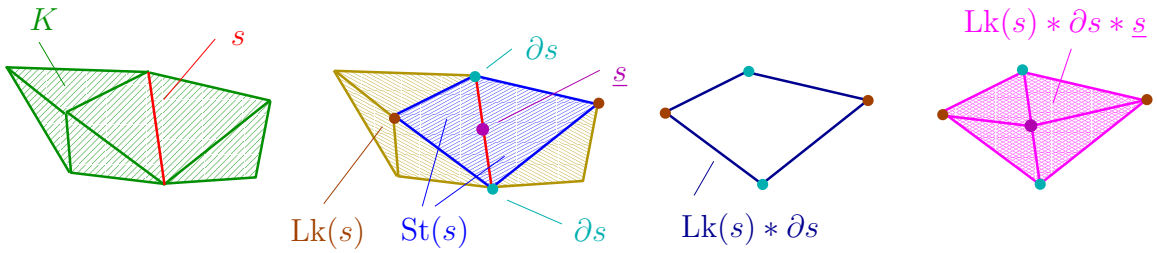
(2) We denote by $\underline{s} := \sum_{v \in s} \frac{1}{\dim(s)+1} \cdot v \in |K|$ the barycenter.[1190]

The following lemma summarizes several basic relations between the objects we just introduced.

Lemma 96.20. Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$. The following statements hold:

(1) $\mathrm{St}(K, s) = \mathrm{Lk}(K, s) * s$.
(2) The union of the subcomplexes $K \setminus \mathrm{St}(K, \mathring{s})$ and $\mathrm{St}(K, s) = \mathrm{Lk}(K, s) * s$ equals K and the intersection equals $\mathrm{Lk}(K, s) * \partial s$.
(3) The intersection of the abstract simplicial complexes $K \setminus \mathrm{St}(K, \mathring{s})$ and $\mathrm{Lk}(K, s) * \partial s * \underline{s}$ also equals $\mathrm{Lk}(K, s) * \partial s$.

Proof. We leave it to the reader to verify that all the statement follow indeed immediately from the definitions. ∎



Now we define the notion of a "stellar subdivision at a simplex s". The basic idea is to add a vertex in the interior of s and subdivide all the simplices that contain s accordingly. Slightly more precisely, the "stellar subdivision at a simplex s" is defined as replacing the subcomplex $\mathrm{St}(K, s) = \mathrm{Lk}(K, s) * s$ by the abstract simplicial complex $\mathrm{Lk}(K, s) * \partial s * \underline{s}$. Even more precisely, it is defined as follows.

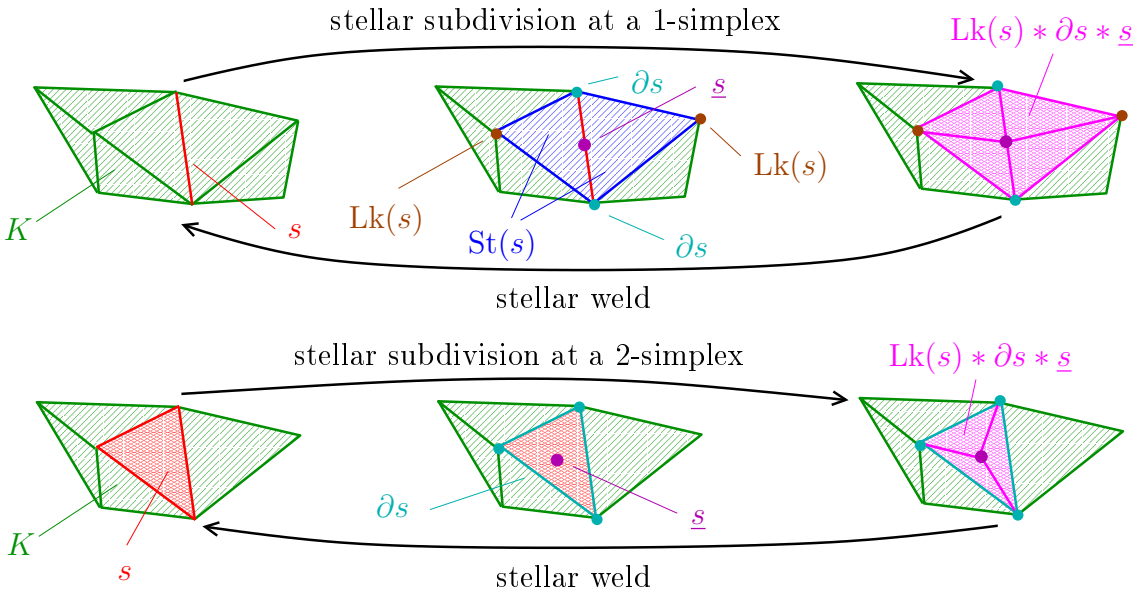
Definition.

(1) Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$. We define the stellar subdivision of K at s to be the abstract simplicial complex[1191]

$$\sigma_s(K) \ := \ (K \setminus \mathrm{St}(K, \mathring{s})) \cup \mathrm{Lk}(K, s) * \partial s * \underline{s}.$$

(2) Let K be an abstract simplicial complex. If K' is a stellar subdivision of K, then we say that K is a stellar weld of K'.
(3) We say two abstract simplicial complexes K and L are stellar equivalent if there exists a finite sequence $K = K_0, K_1, \ldots, K_m = L$ of abstract simplicial complexes such that each K_{i+1} is simplicially isomorphic to either a stellar subdivision or a stellar weld of K_i.

[1190] The meticulous reader will notice that in this section, on several occasions, we will use the Barycenter Lemma 94.1.



Remark.

(1) We follow the naming conventions used in **[Lic1999]** which differ slightly from the ones used in say **[Hud1969, Zee1963b, RS1972]**. More precisely, what we call a *stellar subdivision* is called an *elementary stellar subdivision* in **[Hud1969, Zee1963b, RS1972]**.

(2) It follows easily from the definitions and the Barycenter Lemma 94.1 (3) that the stellar subdivision at a 0-simplex is the identity.

Lemma 96.21. Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$. If $\mathrm{Lk}(K, s)$ is finite, then there exists a natural PL-homeomorphism $|K| \to |\sigma_s(K)|$.

Remark. The proof of Lemma 96.21 shows that $|\sigma_s(K)|$ is essentially a subdivision of $|K|$. This shows that the name "stellar *subdivision*" is justified.

Proof. Let $K = (V, S)$ be a finite abstract simplicial complex and let $s \in S$. Recall that on page 2091 we saw that there exists a natural PL-homeomorphism $\varphi\colon |s| \to |\partial s * \underline{s}|$ which is the identity on the common subset $|\partial s|$. It follows from Lemmas 93.20 and 96.10, together with our hypothesis that $\mathrm{Lk}(K, s)$ is finite, that the map $\mathrm{id}_{\mathrm{Lk}(K,s)}$ and φ give rise to a natural PL-homeomorphism $f\colon |\mathrm{Lk}(K, s) * s| \to |\mathrm{Lk}(K, s) * \partial s * \underline{s}|$ which has the following property:

(∗) The map f is the identity on the common subset $|\mathrm{Lk}(K, s) * \partial s|$.

Next we consider the two maps

$$\Phi\colon |K| \to |\sigma_s(K)| \qquad P \mapsto \begin{cases} P, & \text{if } P \in |K \setminus \mathrm{St}(K, \overset{\circ}{s})|, \\ f(P), & \text{if } P \in |\mathrm{St}(K, s)| \end{cases}$$

$$\Psi\colon |\sigma_s(K)| \to |K| \qquad P \mapsto \begin{cases} P, & \text{if } P \in |K \setminus \mathrm{St}(K, \overset{\circ}{s})|, \\ f^{-1}(P), & \text{if } P \in |\mathrm{Lk}(K, s) * \partial s * \underline{s}|. \end{cases}$$

[1191] It follows from Lemma 93.2 (1) that $\sigma_s(K)$ is indeed an abstract simplicial complex.

We make the following observations:

(1) It follows from Lemma 96.20 (2) and (3) together with (∗) that both maps are well-defined.
(2) By Lemma 93.8 together with Pasting Proposition 2.6 we know that both maps are continuous.
(3) Basically by construction the maps are inverses of one another.

In summary we have shown that both maps are homeomorphisms. It remains to show that $\Phi\colon |K| \to |\sigma_s(K)|$ is in fact a PL-homeomorphism. To do so, note that it follows easily from the definition of the map Φ that the simplicial complex $(\sigma_s(K), \Phi^{-1}\colon |\sigma_s(K)| \to |K|)$ is a subdivision of $|K|$. With respect to this subdivision the map Φ is evidently a simplicial isomorphism. Thus we see that Φ is in fact a PL-homeomorphism. ■

The following rather amazing theorem is one of the key bridges between topology and combinatorics.

Theorem 96.22. (Alexander-Newman Theorem) Let K and L be two finite abstract simplicial complexes. The following two statements are equivalent:

(1) The abstract simplicial complexes K and L are stellar equivalent.
(2) The simplicial complexes $|K|$ and $|L|$ are PL-homeomorphic.

Proof. The "(1)⇒(2)"-direction is just the combination of Lemma 96.21 and Proposition 96.9. The much harder "(2)⇒(1)"-direction was first proved by James Alexander [**Al1930**, Theorem 15:1] in 1930, building on work of Max Newman [**New1926**]. More modern expositions of the proof are given in [**Gla1970**, Theorem II.17] and in [**Lic1999**, Theorem 4.5]. ■

According to [**Lic1999**, p. 311] and [**Mel2018**, p. 1] the following question, posed in particular in [**Hud1969**, p. 14] and [**Pac1991**, Problem 3.3], is still open. An affirmative answer for 2-dimensional abstract simplicial complexes was given by Jürgen Ewald [**Ewa1986**] in 1986.

Question 96.23. Let K and L be two finite abstract simplicial complexes such that the simplicial complexes $|K|$ and $|L|$ are PL-homeomorphic. Do there exist iterated stellar subdivisions of K and L which are simplicially isomorphic?

96.6. **Combinatorial manifolds and Pachner's Theorem.** The Alexander-Newman Theorem 96.22 and Propositions 96.12 and 96.14 (2) motivate the following definitions.

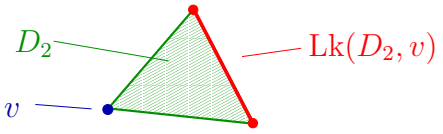
Definition.

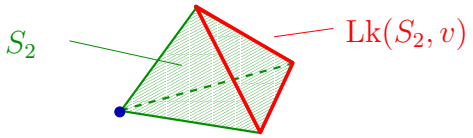
(1) We say that two finite abstract simplicial complexes K and L are combinatorially homeomorphic if K and L are stellar equivalent.
(2) Let $n \in \mathbb{N}_0$. We say that a finite abstract simplicial complex is a combinatorial n-ball if it is combinatorially homeomorphic to D_n and we say it is a combinatorial n-sphere if it is combinatorially homeomorphic to S_n.[1192]
(3) Let K be a finite abstract simplicial complex. We say K is an n-dimensional combinatorial manifold if for every vertex v the link $\mathrm{Lk}(K, v)$ is a combinatorial $(n-1)$-sphere or a combinatorial $(n-1)$-ball.
(4) We say an n-dimensional combinatorial manifold K is closed if for every vertex v the link $\mathrm{Lk}(K, v)$ is a combinatorial $(n-1)$-sphere.

.

Examples. Let $n \in \mathbb{N}$. We consider the abstract simplicial complexes D_n and S_n.

(1) For every vertex v of D_n we have, basically by definition, that $\mathrm{Lk}(D_n, v)$ is simplicially isomorphic to D_{n-1}. Thus we see that D_n is an n-dimensional combinatorial manifold.
(2) Similarly, for every vertex v of S_n we have, again basically by definition, that $\mathrm{Lk}(S_n, v)$ is simplicially isomorphic to S_{n-1}. Thus we obtain that S_n is a closed n-dimensional combinatorial manifold.





Remark.

(1) From what I can see, there is no combinatorial analogue of the notion that a map between two finite abstract simplicial complexes is piecewise linear.
(2) In the literature, see e.g. [**Hud1969**, p. 26] and [**Gla1970**, p. 18], a *combinatorial manifold* is often defined as a PL-manifold. Even though the concept of a "wrong definition" does not exist, it seems to me that it is more appropriate to define a combinatorial manifold as above, namely to define it immediately as a purely combinatorial object.

The statement of the following proposition, which can be viewed as an analogue of Proposition 96.11, does not come as a surprise.

Proposition 96.24. Let K and L be two finite abstract simplicial complexes. If K is a (closed) n-dimensional combinatorial manifold and if L is combinatorially homeomorphic to K, then L is a (closed) n-dimensional combinatorial manifold.

Example. It follows from Proposition 96.24 and the previous examples that every combinatorial n-ball is indeed an n-dimensional combinatorial manifold and that every combinatorial n-sphere is indeed a closed n-dimensional combinatorial manifold.

Proof. A short moment's thought shows that this statement does indeed need a proof, it does not follow immediately from the definitions. In fact the proposition is a consequence of [**Gla1970**, Lemma on p. 19]. ■

By definition the link of any 0-simplex in a closed combinatorial manifold is a combinatorial sphere. The following lemma, which is an analogue of the PL-Manifold Characterization Proposition 96.12 (2)⇒(4), generalizes this statement to higher dimensional simplices.

Lemma 96.25. Let K be a closed n-dimensional combinatorial manifold. If s is a k-simplex, then the link $\mathrm{Lk}(K, s)$ is a combinatorial $(n-k-1)$-sphere.

Proof. The proof of this lemma is almost the same as the proof of the PL-Manifold Characterization Proposition 96.12 (2)⇒(4). The only slight difference is that now we need to use the fact, mentioned above, that every combinatorial k-sphere is a closed k-dimensional combinatorial manifold. ■

[1192]We refer to page 1978 for the definition of the abstract simplicial complexes S_n and D_n

By definition two combinatorially homeomorphic combinatorial manifolds are related by a sequence of stellar subdivisions and stellar welds. In the following we will see that they are also related by the more elegant "bistellar moves".

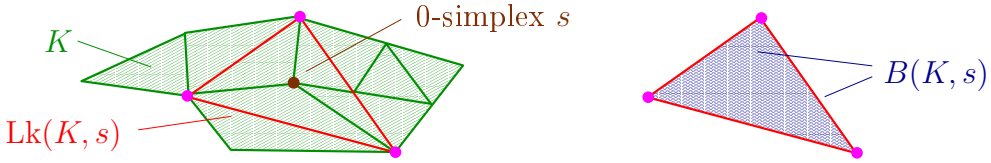
To introduce the notion of a bistellar move we need the following basic definition.

Definition. Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$ be a simplex.

(1) Let $m \in \mathbb{N}_0$. We say that $\mathrm{Lk}(K, s) =: (U, R)$ is a non-trivial m-sphere in K if the following two conditions are satisfied:
   (a) we have $\#U = m + 2$ and $R = \mathcal{P}(U) \setminus \{\varnothing, U\}$,[1193]
   (b) the vertex set U is not a simplex of K.
   In this case we denote by $B(K, s)$ the abstract simplicial complex $(U, \mathcal{P}(U) \setminus \{\varnothing\})$. We illustrate the definition in the figure below.

(2) We say that $\mathrm{Lk}(K, s)$ is a non-trivial (-1)-sphere in K if it is the empty subcomplex. In this case we define $B(K, s)$ to be abstract simplicial complex consisting of the single vertex $\underline{s}$.

As usual, if K is understood then we write $B(s)$ instead of $B(K, s)$.



We move on to the next lemma which can be viewed as the fourth part of Lemma 96.20.

Lemma 96.26. Let $K = (V, S)$ be a closed n-dimensional combinatorial manifold and let $s \in S$ be a k-simplex. We suppose that the link $\mathrm{Lk}(K, s)$ is a non-trivial $(n - k - 1)$-sphere in K. Then the intersection of the abstract simplicial complexes $K \setminus \mathrm{St}(K, \mathring{s})$ and $\partial s * B(K, s)$ equals $\mathrm{Lk}(K, s) * \partial s$.

Proof. Again we leave the task of verifying the lemma to the reader. ■

Suppose we are in the setting of Lemma 96.26. Loosely speaking the *bistellar move along* s is defined as the result of replacing the subcomplex

$$\mathrm{St}(K, s) \;=\; \underset{\cong D_k}{s} \;*\; \underset{\cong S_{n-k-1}}{\mathrm{Lk}(K, s)} \quad \text{by the abstract simplicial complex} \quad \underset{\cong S_{k-1}}{\partial s} \;*\; \underset{\cong D_{n-k}}{B(K, s)}.$$

More precisely, we have the following definition.

Definition. Let $K = (V, S)$ be a closed n-dimensional combinatorial manifold and let $s \in S$ be a k-simplex such that $\mathrm{Lk}(K, s)$ is a non-trivial $(n-k-1)$-sphere in K. We define

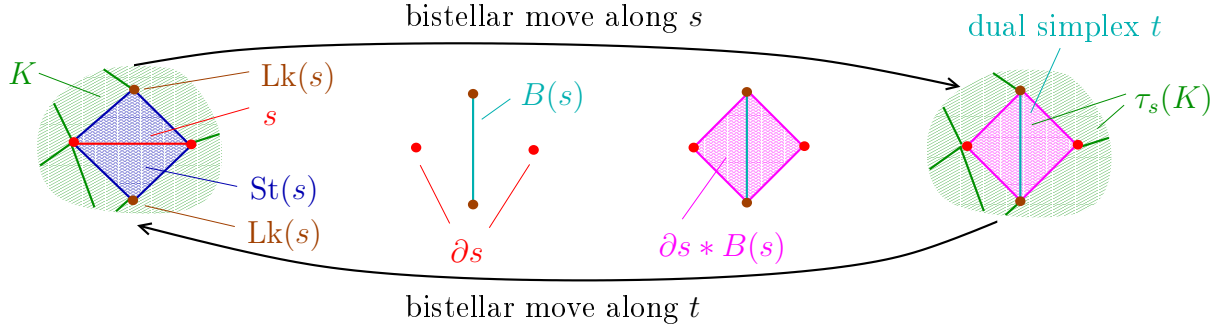
$$\tau_s(K) \;:=\; (K \setminus \mathrm{St}(K, \mathring{s})) \cup \partial s * B(K, s).$$

We refer to the unique $(n - k)$-simplex in $B(K, s)$ as the dual simplex. We say $\tau_s(K)$ is obtained from K by a bistellar move along s.

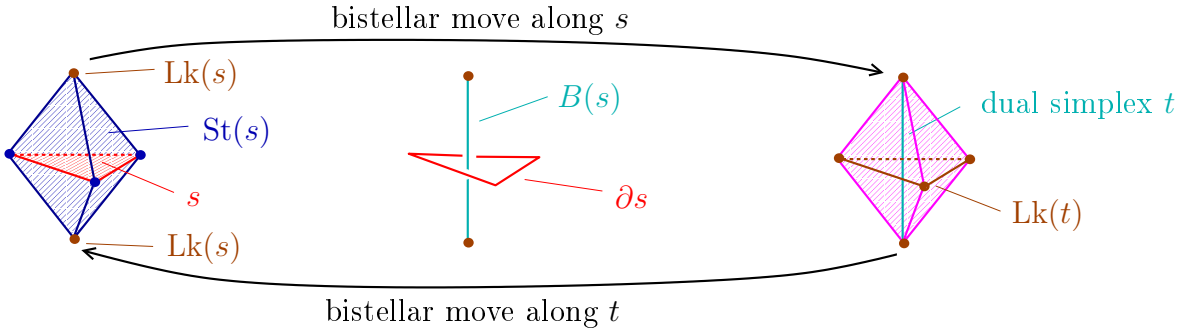
[1193] This just means that $\mathrm{Lk}(K, s)$ is simplicially isomorphic to the abstract simplicial complex S_m. It is worth recalling that Lemma 96.25 says that the link of a k-simplex in a closed n-dimensional combinatorial manifold is combinatorially homeomorphic to S_{n-k-1}, which is a much weaker condition than demanding that the link is simplicially isomorphic to S_{n-k-1}.

Examples.

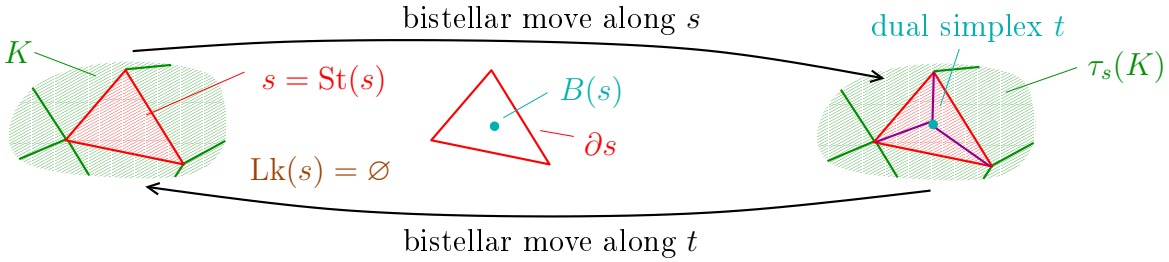
(1) In the figure below we show a bistellar move along a 1-simplex s in a 2-dimensional combinatorial manifold.



(2) In the figure below we show a bistellar move along a 2-simplex s in a 3-dimensional combinatorial manifold.



(3) In the figure below we show a bistellar move along a 2-simplex s in a 2-dimensional combinatorial manifold. In this case $\mathrm{Lk}(s) = \varnothing$, which by the above convention, is a non-trivial (-1)-sphere.



Note that in each example we see that the bistellar move along the dual simplex actually gets us back to the original abstract simplicial complex. In the next lemma we will see that this is not a coincidence.

The following lemma summarizes a few key properties of bistellar moves.

Lemma 96.27. (Bistellar Lemma) Let $K = (V, S)$ be a closed n-dimensional combinatorial manifold and let $s \in S$ be a k-simplex such that $\mathrm{Lk}(K, s)$ is a non-trivial $(n-k-1)$-sphere in K. We denote by t the dual simplex. The following statements hold:

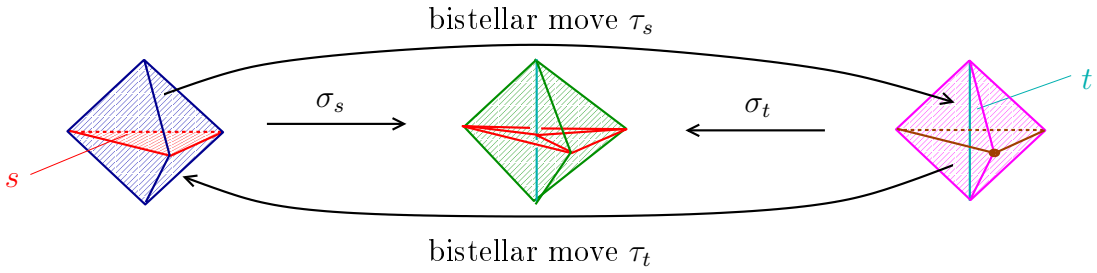
(1) The dual simplex t is an $(n-k)$-simplex in $\tau_s(K)$ with the following properties:
  (a) $\partial t = \mathrm{Lk}(K, s)$,
  (b) $\mathrm{Lk}(\tau_s(K), t) = \partial s$,
  (c) (i) if $s = \{v\}$ is a 0-simplex, then $B(\tau_s(K), t) = (\{\underline{t}\}, \{\underline{t}\})$,
    (ii) if s is not a 0-simplex, then $B(\tau_s(K), t) = s$,[1194]
  (d) $\tau_s(K) \setminus \mathrm{St}(\tau_s(K), \mathring{t}) = K \setminus \mathrm{St}(K, \mathring{s})$.

  Furthermore ∂s is a non-trivial $(k-1)$-sphere in $\tau_s(K)$.

(2) $\sigma_s(K)$ is naturally simplicially isomorphic to $\sigma_t(\tau_s(K))$.

(3) $\tau_s(K)$ is a closed n-dimensional combinatorial manifold which is naturally combinatorially homeomorphic to K.

(4) $\tau_t(\tau_s(K))$[1195] is naturally simplicially isomorphic to K.



Remark. The Bistellar Lemma 96.27 shows that in particular that if two closed n-dimensional combinatorial manifold are bistellar equivalent, then they are also stellar equivalent. Furthermore the Bistellar Lemma 96.27 gives us also the justification for the name "bistellar move".

Proof.

(1) We leave the elementary verification of these statements to the reader. Note that the slightly awkward fact that we need to deal with 0-simplices separately is due to the fact that already in the definitions on page 2106 and page 2101 we were forced to treat 0-simplices slightly differently.

(2) This statement follows easily from (1). Indeed, we see that

$$\sigma_t(\tau_s(K)) \underset{\text{by definition}}{=} \overbrace{(\tau_s(K) \setminus \mathrm{St}(\tau_s(K), \mathring{t}))}^{= K \setminus \mathrm{St}(K, \mathring{s}) \text{ by (1)}} \cup \overbrace{\mathrm{Lk}(\tau_s(K), t) * \partial t * \underline{t}}^{= \partial s * \mathrm{Lk}(K,s) * \underline{t} \text{ by (1)}} \underset{\text{by Lemma 96.20 the map } \underline{s} \mapsto \underline{t} \text{ defines a simplicial isomorphism}}{\overset{\cong}{\to}} \sigma_s(K).$$

(3) It follows from (2) that $\tau_s(K)$ is naturally stellar equivalent to K. In other words, $\tau_s(K)$ is naturally combinatorially homeomorphic to K. It follows from Proposition 96.24 that $\tau_s(K)$ is also a closed n-dimensional combinatorial manifold.

[1194] Here we view the simplex s as an abstract simplicial complex in its own right.

[1195] It follows from (1) that we can form $\tau_t(\tau_s(K))$.

(4) First assume that s is not a 0-simplex. In this case we see that

$$\tau_t(\tau_s(K)) \underset{\text{by definition}}{=} \overbrace{(\tau_s(K) \setminus \mathrm{St}(\tau_s(K), \overset{\circ}{t}))}^{= K \setminus \mathrm{St}(K, \overset{\circ}{s}) \text{ by (1)}} \cup \overbrace{\partial t * B(\tau_s(K), t)}^{=\mathrm{Lk}(K,s)*s \text{ by (1)}} \underset{\text{Lemma 96.20}}{=} K.$$

Finally we deal with the case that $s = \{v\}$ is a 0-simplex. Note that in this case the vertex set of $\tau_t(\tau_s(K))$ equals $(V \setminus \{v\}) \cup \{\underline{t}\}$. The obvious bijection between the vertex sets of $\tau_t(\tau_s(K))$ and of K, i.e. the map given by $\underline{t} \mapsto v$, induces, by the same argument as above, a simplicial isomorphism $\tau_t(\tau_s(K)) \to K$. ■

Note that the Bistellar Lemma 96.27 (4) implies in particular that "being related by a sequence of bistellar moves" is an equivalence relation. This observation motivates the following definition.

Definition. We say that two closed n-dimensional combinatorial manifold are bistellar equivalent if one can be obtained from the other via a finite sequence of bistellar moves and simplicial isomorphisms.

The following theorem can be viewed as a refinement of the Alexander-Newman Theorem 96.22 for closed combinatorial manifolds. The theorem was proved in 1991 by Udo Pachner [**Pac1991**].

Theorem 96.28. (Pachner's Theorem) Let K and L be two closed combinatorial manifolds. The following two statements are equivalent:

(1) The abstract simplicial complexes K and L are bistellar equivalent.
(2) The abstract simplicial complexes K and L are combinatorially homeomorphic (i.e. they are stellar equivalent).

Remark.

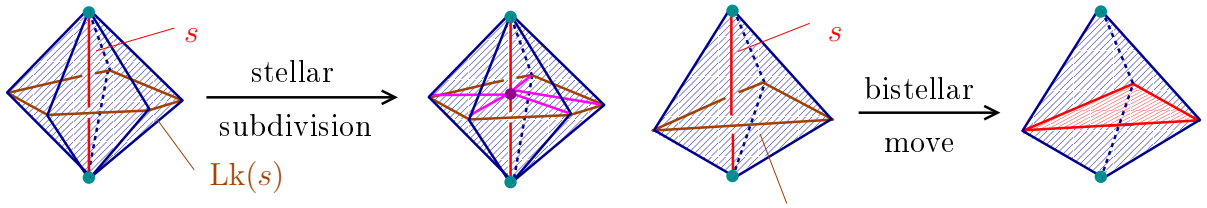
(1) Pachner's Theorem 96.28 is the reason why bistellar moves are often called *Pachner moves*.
(2) An analogue of Pachner's Theorem 96.28 for combinatorial manifolds that are not closed is given in [**Pac1991**, Theorem 6.3] and [**Lic1997b**, Theorem 5.10].
(3) The Alexander-Newman Theorem 96.22 and Pachner's Theorem 96.28 can be viewed as an analogue of the Reidemeister Moves Theorem 59.21 which says that any two knot diagrams for a given knot are related by a finite sequence of Reidemeister moves.

Proof. The "(1)⇒(2)"-direction is an immediate consequence of the Bistellar Lemma 96.27. The much more interesting "(2)⇒(1)"-direction is proved in [**Pac1991**, Theorem 5.5]. An exposition of the proof is also given in [**Lic1999**, Theorem 5.9]. For combinatorial manifolds of dimension ≤ 3 a more elementary proof is given in [**BW1996**, Theorem 4.11] and implicitly in [**TV1992**]. Furthermore for combinatorial manifolds of dimension ≤ 4 a proof is also given in [**Pac1978**]. ■

Let us quickly compare the statements and roles of the Alexander-Newman Theorem 96.22 and Pachner's Theorem 96.28.

(1) The Alexander-Newman Theorem 96.22 is more general in so far as it applies to any two finite simplicial complexes whereas Pachner's Theorem 96.28 only deals with closed combinatorial manifolds.

(2) Now we restrict ourselves to the class of closed n-dimensional combinatorial manifolds.
   (a) A stellar subdivision consists in replacing $\mathrm{Lk}(K,s) * s$ by $\mathrm{Lk}(K,s) * \partial s * \underline{s}$. As should become clear from considering the figure below, for $n \geq 3$ there are infinitely many simplicial isomorphism types of links of simplices that can appear. Thus we see that even for closed n-dimensional combinatorial manifolds there are infinitely many[1196] "combinatorially distinct" stellar subdivisions.
   (b) On the other hand a Pachner move can only be applied if the link $\mathrm{Lk}(K,s)$ is in fact simplicially isomorphic to some S_k, $k = 0, \ldots, n$. Recall that if a Pachner move is possible, then it replaces $\mathrm{Lk}(K,s) * s$ by $\partial s * B(K,s)$. Thus we see that there are only $n+1$ "combinatorially distinct" moves.



stellar subdivision is always possible

$\mathrm{Lk}(s)$ needs to be some S_k

Now, at the end the end of this chapter we have obtained the following sequence of maps:

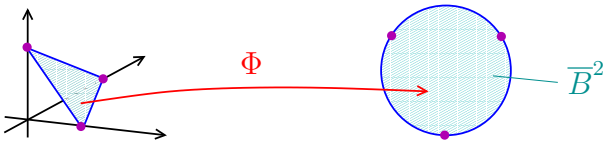
$$\begin{array}{c}
\{\text{diffeomorphism classes of closed } n\text{-dimensional smooth manifolds}\} \\
\Big\downarrow \boldsymbol{w}_n \quad \text{by Theorem 96.16 a bijection for } n \leq 6 \\
\{\text{PL-homeomorphism classes of closed } n\text{-dimensional PL-manifolds}\} \\
\Big\uparrow |K| \\
\Big\updownarrow K \quad \text{a bijection by the Alexander-Newman Theorem 96.22} \\
\left\{ \begin{array}{c} \text{combinatorial homeomorphism classes of} \\ \text{closed } n\text{-dimensional combinatorial manifolds} \end{array} \right\} \\
\Big\uparrow \quad \text{identity by Pachner's Theorem 96.28} \\
\{\text{bistellar equivalence classes of closed } n\text{-dimensional combinatorial manifolds}\}
\end{array}$$

This shows in particular that for dimensions ≤ 6 the classification of closed smooth manifolds boils down to (very difficult!) combinatorics.
In particular the sequence of maps gives us an approach to defining invariants of closed n-dimensional PL-manifolds (or equivalently smooth manifolds for $n \leq 6$): we "just" need to define an invariant for closed n-dimensional combinatorial manifolds that is invariant under the $n+1$ bistellar moves. This approach to defining invariants has been used by Vladimir Turaev and Oleg Viro [**TV1992**] and John Barrett and Bruce Westbury [**BW1996**] in the 3-dimensional setting and by Christopher Douglas and David Reutter [**DR2018**] in the 4-dimensional setting.
Finally we mention that the significance of Pachner's Theorem 96.28 in mathematics and theoretical physics ("loop quantum gravity theory") is nicely explained in [**Lic1997b, Lic1999**] and [**Kle2018**].

[1196] We leave it to the reader to fill the expression "combinatorially distinct" with meaning.

Exercises for Chapter 96.

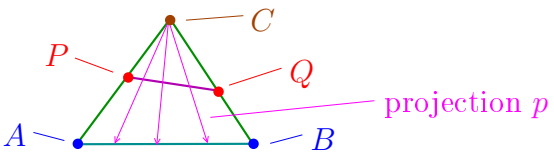
Exercise 96.1. We equip the smooth manifold $\overline{B}^2$ with the canonical simplicial structure $\Phi\colon \Delta^2 \to \overline{B}^2$ introduced on page 1997. Is this a smooth simplicial structure for the smooth manifold $\overline{B}^2$?



Exercise 96.2. Let $n \in \mathbb{N}_0$. Show that $\mathrm{Cone}(\partial\Delta^n)$ is PL-homeomorphic to Δ^n. We refer to the figure on page 2092 for an illustration.

Exercise 96.3. Let X be a simplicial complex and let Y be a subdivision. Show that every k-simplex of X is the union of finitely many simplices of Y of dimension $\leq k$.

Exercise 96.4. We consider the triangle Δ_{ABC} shown in the figure below. Let $P \in \overline{AC}$ and $Q \in \overline{BC}$ be two points such that the segment $\overline{PQ}$ is not parallel to the segment $\overline{AB}$. Let p be the radial projection from C of the segment $\overline{PQ}$ to the base $\overline{AB}$ of the triangle. We equip $\overline{PQ}$ and $\overline{AB}$ with the obvious simplicial structures. Show that the map p is not piecewise linear.
Remark. The fact that the radial projection p is not piecewise linear is called the "standard mistake" in [**Zee1963b**, p. I.6] and [**RS1972**, p. 6].



Exercise 96.5. We consider the two abstract simplicial complexes K and L shown in the figure below.

(a) Show that L is not a stellar subdivision of K.
(b) Show "by hand" that K and L are stellar equivalent.



Exercise 96.6. Use the Alexander-Newman Theorem 96.22 to prove Pachner's Theorem 96.28 in the 2-dimensional case.

Exercise 96.7. Let K be an abstract simplicial complex. Show that if K is a closed n-dimensional combinatorial manifold, then the maximal dimension of a simplex of K is precisely n.
Hint. You could do an induction on n.

Exercise 96.8. Let M be an n-dimensional smooth manifold. In the Smoothly Good Cover Theorem **??** we showed that M admits a smoothly good finite open cover. In plain English this means that there exists an open cover $U_1, \ldots, U_k$ of M such that the intersection of any selection of these sets is diffeomorphic to $\mathbb{R}^n$. Use this result to give an alternative proof of Statements (1), (2) and (4) of the Smooth Manifolds-Invariants Proposition 96.6 for M. More precisely, prove the following statements:

(1) For every $k > n$ and for every abelian group G we have $\mathrm{H}_k(M; G) = 0$.
(3) The fundamental group of M is finitely presented.
(4) All homology groups of M are finitely generated abelian groups. Furthermore, given any field $\mathbb{F}$ the homology groups $\mathrm{H}_k(M; \mathbb{F})$ are finite-dimensional.

Remark. You should make use of the calculations of the invariants for $\mathbb{R}^n$, the Mayer–Vietoris Theorem 78.5, the Seifert-van Kampen Theorem 54.1 and the HNN-Seifert-van Kampen Theorem 57.3.

97. The Lefschetz Fixed Point Theorem

In this chapter we will state and prove the Lefschetz Fixed Point Theorem which gives a criterion for a self-map to have a fixed point. This theorem can be viewed as a far reaching generalization of the Brouwer Fixed Point Theorem 31.4.

97.1. **The trace of an endomorphism.** In the formulation of the Lefschetz Fixed Point Theorem we need the notion of the trace of suitable endomorphisms. Even though this notion is surely well-known to the reader we recall the definition and a few properties of the trace of an endomorphism. This short section will facilitate the discussion below. Nonetheless, the author will not hold it against the reader if the reader decides to skip this section.

Definition. Let R be a ring. Given an $(n \times n)$-matrix $A = (a_{ij})$ over R we define its trace to be

$$\operatorname{tr}(A) \;:=\; \sum_{i=1}^{n} a_{ii}.$$

Lemma 97.1. Let R be a ring and let A and B be two $(n \times n)$-matrices over R. If R is commutative, then we have

$$\operatorname{tr}(A \cdot B) \;=\; \operatorname{tr}(B \cdot A).$$

Proof. We perform the following simple calculation:

$$\operatorname{tr}(A\cdot B) = \operatorname{tr}\Big(\text{matrix with } (i,j)\text{-entry } \sum_{k=1}^{n} a_{ik}\cdot b_{kj}\Big) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}\cdot b_{ki} \underset{\text{since } R \text{ is commutative}}{\overset{\uparrow}{=}} \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ki}\cdot a_{ik} \underset{\text{same calculation backwards}}{\overset{\downarrow}{=}} \operatorname{tr}(B\cdot A).$$

∎

Lemma 97.2. Let R be a commutative domain, let V be a free R-module of rank n and let $\varphi\colon V \to V$ be an endomorphism. We pick a basis for V. Then

$$\operatorname{tr}(\varphi) \;:=\; \text{trace of the matrix representing } \varphi \text{ with the respect to the given basis}$$

does not depend on the choice of the basis. We refer to $\operatorname{tr}(\varphi)$ as the trace of the endomorphism.

Notation. Given a commutative domain R, a free R-module V of finite rank and an endomorphism $\varphi\colon V \to V$, depending on the amount of space we have, we use one of the following three notations:

$$\operatorname{tr}(\varphi) \;=\; \operatorname{tr}(\varphi\colon V \to V) \;=\; \operatorname{tr}(\varphi \circlearrowleft V).$$

Example. Let R be a commutative domain and let V be a free R-module of finite rank. It follows immediately from the definitions that $\operatorname{tr}(\mathrm{id}_V) = \operatorname{rank}(V)$.

Proof. Let R be a commutative domain. Let $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ be two bases for V. Since R is a commutative domain the rank of V is well-defined, see page 99. Thus we see that $m = n$. Now let $P = (p_{ij})$ be the transition matrix, i.e. $P = (p_{ij})$ is the matrix that is uniquely determined by the condition that given any $i \in \{1, \ldots, n\}$ we have $v_i = \sum_{j=1}^{n} p_{ji} \cdot w_j$. The "opposite" transition matrix is evidently an inverse to P, in particular

we see that P is invertible. We denote by A and B the matrices representing φ with respect to the two bases. We see that

$$\operatorname{tr}(B) \underset{\substack{\uparrow\\ \text{an elementary calculation}\\ \text{shows that } B\cdot P = P\cdot A}}{=} \operatorname{tr}(P\cdot A\cdot P^{-1}) \underset{\substack{\uparrow\\ \text{by Lemma 97.1 since } R \text{ is commutative and since all}\\ \text{matrices are square matrices of the same size}}}{=} \operatorname{tr}(P^{-1}\cdot P\cdot A) = \operatorname{tr}(A).$$

■

In many of the applications below we will consider the case that R is a field. But on several occasions the case $R = \mathbb{Z}$ is also of interest.

Lemma 97.3. Let $\varphi\colon V \to V$ be an endomorphism of a finitely generated free abelian group and let R be a commutative domain. Let $\gamma\colon \mathbb{Z} \to R$ be the natural ring homomorphism with $\gamma(1_{\mathbb{Z}}) = 1_R$ from page 205. Then

$$\gamma\big(\operatorname{tr}\big(\underbrace{\varphi\colon V\to V}_{\substack{\text{homomorphism of}\\ \text{free } \mathbb{Z}\text{-modules}}}\big)\big) = \operatorname{tr}\big(\underbrace{\varphi\otimes \mathrm{id}_R\colon V\otimes R\to V\otimes R}_{\substack{\text{homomorphism of free } R\text{-modules}\\ \text{by the Tensor-with-Ring Lemma 89.5}}}\big) \in R.$$

Proof. Let $\{v_1,\dots,v_n\}$ be a basis for the free abelian group V and let A be the $(n\times n)$-matrix that represents φ with respect to this basis. It is straightforward to verify that $v_1\otimes 1,\dots,v_n\otimes 1$ is a basis for the R-module $V\otimes R$ and that $\gamma(A)$ is the matrix that represents the endomorphism $\varphi\otimes\mathrm{id}$. Since the trace commutes with ring homomorphisms we obtain the desired equality $\gamma(\operatorname{tr}(A)) = \operatorname{tr}(\gamma(A))$. ■

Sometimes it is also convenient to consider the trace of an endomorphism of a finitely generated abelian group. This leads us to the following definition.

Definition.

(1) Given a finitely generated abelian group H we refer to

$$\mathrm{F}H := H/\operatorname{Tor}(H) \quad \text{where } \operatorname{Tor}(H) \text{ denotes the torsion subgroup of } H$$

as the maximal torsion-free quotient of H.

(2) Given a homomorphism $\alpha\colon A\to B$ between two finitely generated abelian groups we denote the obvious induced map $\mathrm{F}A\to \mathrm{F}B$ by α as well.

Lemma 97.4. Given a finitely generated abelian group H the maximal torsion-free quotient $\mathrm{F}H = H/\operatorname{Tor}(H)$ of H is a free abelian group.

Proof. A very elementary exercise shows that $\mathrm{F}H = H/\operatorname{Tor}(H)$ is a torsion-free abelian group. Since H is finitely generated we obtain from the Finitely Generated Abelian Group Classification Theorem 51.4 that $\mathrm{F}H$ is in fact a free abelian group. ■

Remark.

(1) It follows easily from the definition that the map $H\to \mathrm{F}H$ defines a functor from the category of finitely generated abelian groups to the category of free abelian groups.

(2) It might sound silly to mention the Finitely Generated Abelian Group Classification Theorem 51.4 explicitly in the proof of Lemma 97.4, but it is worth noting that for commutative rings R which are not PIDs there is usually no meaningful functor from the category of finitely generated R-modules to the category of free R-modules.

(3) The hypothesis that H is finitely generated is of course necessary, for example the maximal torsion-free quotient of the abelian group $(\mathbb{Q},+)$ is again $(\mathbb{Q},+)$, which is certainly not a free abelian group.

Definition. Given an endomorphism $\varphi\colon H\to H$ of a finitely generated abelian group we define

$$\operatorname{tr}(\varphi) \;=\; \operatorname{tr}\begin{pmatrix} \mathrm{F}H & \to & \mathrm{F}H \\ [h] & \mapsto & [\varphi(h)] \end{pmatrix}.$$

We conclude this elementary section with the following basic lemma which is a variation on Lemma 97.3.

Lemma 97.5. Let H be a finitely generated abelian group and let $\varphi\colon H\to H$ be an endomorphism. Then the following equality holds:

$$\operatorname{tr}(\varphi\colon H\to H) \;=\; \operatorname{tr}(\varphi\otimes\mathrm{id}\colon H\otimes\mathbb{Q}\to H\otimes\mathbb{Q}) \;\in\; \mathbb{Q}.$$

Proof. By the Finitely Generated Abelian Group Classification Theorem 51.4 we can write $H=F\oplus T$ where F is a free abelian group and T is a torsion group. We denote by ψ the map $F\xrightarrow{f\mapsto f\oplus 0}F\oplus T\xrightarrow{\varphi}F\oplus T\xrightarrow{f\oplus t\mapsto f}F$. Then we see that

$$\operatorname{tr}(\varphi\colon H\to H)\underset{\text{follows easily from the definitions}}{\uparrow=}\operatorname{tr}(\psi\colon F\to F)\underset{\text{by Lemma 97.3}}{\uparrow=}\operatorname{tr}(\psi\otimes\mathrm{id}\colon F\otimes\mathbb{Q}\to F\otimes\mathbb{Q})\underset{\text{by the Tensor Lemma 89.3}}{\uparrow=}\operatorname{tr}(\varphi\otimes\mathrm{id}\colon H\otimes\mathbb{Q}\to H\otimes\mathbb{Q}).$$

∎

97.2. The Lefschetz number.

In this section we return to topology.

Definition. Let $\mathbb{F}$ be a field.

(1) Let X be a topological space. We say the homology of X is $\mathbb{F}$-finite if all homology groups $\mathrm{H}_i(X;\mathbb{F})$ with $\mathbb{F}$-coefficients are finite-dimensional vector spaces and if all but finitely many homology groups $\mathrm{H}_i(X;\mathbb{F})$ are zero.

(2) Let X be a topological space such that the homology is $\mathbb{F}$-finite. Let $\varphi\colon X\to X$ be a map. We define the $\mathbb{F}$-Lefschetz number of φ to be[1197]

$$\Lambda(\varphi,\mathbb{F}) \;:=\; \sum_{n\in\mathbb{N}_0}(-1)^n\cdot\operatorname{tr}(\varphi_*\colon \mathrm{H}_n(X;\mathbb{F})\to\mathrm{H}_n(X;\mathbb{F})).$$

Remark. In Exercise 97.1 we will see that the $\mathbb{F}$-Lefschetz number depends only rather mildly on the choice of the field $\mathbb{F}$.

In general it can be tricky to determine (the trace of) an induced map on homology. The following lemma gives us some basic (partial) calculations of Lefschetz numbers.

Lemma 97.6.

(1) If X is a finite CW-complex, then for any field $\mathbb{F}$ we have $\Lambda(\mathrm{id}_X,\mathbb{F})=\chi(X)$.

(2) Let X be a 0-connected space and let $\varphi\colon X\to X$ be a map. For every field $\mathbb{F}$ we have $\operatorname{tr}(\varphi_*\colon \mathrm{H}_0(X;\mathbb{F})\to\mathrm{H}_0(X;\mathbb{F}))=1$.

(3) Let $\mathbb{F}$ be a field and let X be a 0-connected space with $\mathbb{F}$-finite homology. If $\varphi\colon X\to X$ is a constant map, then $\Lambda(\varphi,\mathbb{F})=1$.

[1197] Recall that on page 1914 we pointed out that the induced map $\varphi_*\colon \mathrm{H}_n(X;\mathbb{F})\to\mathrm{H}_n(X;\mathbb{F})$ is an endomorphism of an $\mathbb{F}$-vector space. Since the vector space $\mathrm{H}_n(X;\mathbb{F})$ is by hypothesis finite-dimensional we see that the trace is in fact defined.

Proof.

(1) We calculate that

$$\Lambda(\mathrm{id}_X, \mathbb{F}) \underset{\text{by definition}}{\uparrow} \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \mathrm{tr}(\mathrm{id}_* \circlearrowleft \mathrm{H}_n(X;\mathbb{F})) \underset{\text{see example on page 2113}}{\uparrow} \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \dim_{\mathbb{F}}(\mathrm{H}_n(X;\mathbb{F})) \underset{\text{see page 1915}}{\uparrow} \chi(X).$$

(2) This statement follows immediately from the obvious generalizations of the H_0-Proposition 72.5 and the discussion on page 1594 to homology with $\mathbb{F}$-coefficients.

(3) This statement follows immediately from (2). ■

The following lemma is an immediate and quite useful consequence of the Homotopic Maps-and-Homology Proposition 73.6 and the discussion on page 1914.

Lemma 97.7. Let $\mathbb{F}$ be a field and let X be a topological space such that the homology is $\mathbb{F}$-finite. Let $\varphi, \psi\colon X \to X$ be two maps. If φ and ψ are homotopic, then the Lefschetz numbers agree, i.e. $\Lambda(\varphi, \mathbb{F}) = \Lambda(\psi, \mathbb{F})$.

At times it is more efficient to work with the "usual" homology groups, i.e. the homology groups with $\mathbb{Z}$-coefficients. This leads to the following unexciting definition.

Definition.

(1) Let X be a topological space. We say the homology of X is $\mathbb{Z}$-finite if all homology groups $\mathrm{H}_i(X)$ are finitely generated abelian groups and if all but finitely many homology groups $\mathrm{H}_i(X)$ are zero.

(2) Let X be a topological space such that the homology is $\mathbb{Z}$-finite. Let $\varphi\colon X \to X$ be a map. We define the Lefschetz number of φ to be

$$\Lambda(\varphi) \;:=\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \mathrm{tr}(\varphi_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(X)).$$

The following lemma relates the above Lefschetz number $\Lambda(\varphi)$ to the Lefschetz number $\Lambda(\varphi, \mathbb{Q})$.

Lemma 97.8. Let X be a topological space and let $\varphi\colon X \to X$ be a map.

(1) Let $k \in \mathbb{N}_0$. We have $\mathrm{rank}(\mathrm{H}_k(X)) = \dim(\mathrm{H}_k(X;\mathbb{Q}))$. Furthermore, if $\mathrm{H}_k(X)$ is finitely generated, then we have

$$\mathrm{tr}(\varphi_*\colon \mathrm{H}_k(X) \to \mathrm{H}_k(X)) \;=\; \mathrm{tr}(\varphi_*\colon \mathrm{H}_k(X;\mathbb{Q}) \to \mathrm{H}_k(X;\mathbb{Q})).$$

(2) If the homology of X is $\mathbb{Z}$-finite, then the homology of X is also $\mathbb{Q}$-finite and we have the equality $\Lambda(\varphi) = \Lambda(\varphi, \mathbb{Q})$.

Proof. Statement (2) is an immediate consequence of Statement (1). Thus let us prove the first statement. Let $k \in \mathbb{N}_0$. The equality $\mathrm{rank}(\mathrm{H}_k(X)) = \dim(\mathrm{H}_k(X;\mathbb{Q}))$ was shown in Corollary 89.19 (2). Now assume that $\mathrm{H}_k(X)$ is finitely generated. We have the following equalities:

$$\mathrm{tr}\left(\mathrm{H}_k(X) \xrightarrow{\varphi_*} \mathrm{H}_k(X)\right) \underset{\text{by Lemma 97.5}}{=} \mathrm{tr}\left(\mathrm{H}_k(X)\otimes\mathbb{Q} \xrightarrow{\varphi_*\otimes\mathrm{id}} \mathrm{H}_k(X)\otimes\mathbb{Q}\right) \underset{\substack{\text{by the naturality of the short exact sequence of the Universal Coefficient Theorem 89.17 and the observation that by Lemma 89.15 (4) the Tor-groups with } \mathbb{Q}\text{-coefficients vanish}}}{=} \mathrm{tr}\left(\mathrm{H}_k(X;\mathbb{Q}) \xrightarrow{\varphi_*} \mathrm{H}_k(X;\mathbb{Q})\right).$$

■

97.3. **The formulation of the Lefschetz Fixed Point Theorems.** Now we can finally formulate the long promised Lefschetz Fixed Point Theorem. This theorem, with somewhat different language, was first proved in 1926 by Solomon Lefschetz [**Lef1926**, p. 48].[1198]

Theorem 97.9. (Lefschetz Fixed Point Theorem I) Let X be a finite simplicial complex and let $\varphi\colon X \to X$ be a map. If there exists a field $\mathbb{F}$ such that the $\mathbb{F}$-Lefschetz number $\Lambda(\varphi, \mathbb{F})$ is non-zero, then φ has a fixed point.

We will prove the Lefschetz Fixed Point Theorem 97.9 in the next section. First we will see that one of our earlier great successes, namely the Sphere-Degree Fixed Point Theorem Theorem 38.11, is just a special case of the Lefschetz Fixed Point Theorem 97.9.

Theorem 38.11. (Sphere-Degree Fixed Point Theorem) Let $n \in \mathbb{N}$ and let $\varphi\colon S^n \to S^n$ be a map. If $\deg(\varphi) \neq (-1)^{n+1}$, then φ has a fixed point.

Proof. We consider the topological space S^n. By the discussion on page 1997 we know that S^n admits the structure of a finite simplicial complex. Let $\varphi\colon S^n \to S^n$ be a map. We calculate that

$$\Lambda(\varphi, \mathbb{Q}) \overset{\text{by Lemmas 97.8 we have } \Lambda(\varphi,\mathbb{Q}) = \Lambda(\varphi)}{\downarrow} = \underbrace{\operatorname{tr}\left(\mathrm{H}_0(S^n) \xrightarrow{\varphi_*} \mathrm{H}_0(S^n)\right)}_{=1 \text{ by Lemma 97.6 (2)}} + (-1)^n \cdot \underbrace{\operatorname{tr}\left(\mathrm{H}_n(S^n) \xrightarrow{\varphi_*} \mathrm{H}_n(S^n)\right)}_{=\deg(\varphi) \text{ by definition of the degree}} = 1 + (-1)^n \cdot \deg(\varphi).$$

The Lefschetz Fixed Point Theorem 97.9 thus says that if $1 + (-1)^n \cdot \deg(\varphi) \neq 0$, then φ has a fixed point. But that is exactly what we wanted to show. ■

For many applications the following variation on the Lefschetz Fixed Point Theorem 97.9 is particularly useful.

Theorem 97.10. (Lefschetz Fixed Point Theorem II) Let M be a compact smooth manifold and let $\varphi\colon M \to M$ be a map. If there exists a field $\mathbb{F}$ such that the $\mathbb{F}$-Lefschetz number $\Lambda(\varphi, \mathbb{F})$ is non-zero, then φ has a fixed point.

Remark.

(1) Later on in Theorem 104.12 we will prove the Lefschetz Fixed Point Theorem also for finite CW-complexes and compact topological manifolds. In fact not only is the statement of Theorem 104.12 more general than the statement of Theorem 97.10, the proof of Theorem 104.12 will also be self-contained, since we will not make use of the Smooth Manifold-Simplicial Structure Theorem 96.2 for which we provided only a sketch of a proof.
(2) Reading (1) one might be tempted to think that the conclusion of the Lefschetz Fixed Point Theorem holds for every compact space. But in Exercise 97.5 we will see that this is certainly *not* the case.
(3) We will prove a generalization of the Lefschetz Fixed Point Theorem 97.10 in the Lefschetz-Hopf Theorem 128.5.

Proof. Let M be a compact smooth manifold. By the Smooth Manifold-Simplicial Structure Theorem 96.2 we know that M admits a finite simplicial structure. Thus we can apply the Lefschetz Fixed Point Theorem 97.9 and we are done. ■

[1198] Solomon Lefschetz (1884-1972) was an American mathematician.

Corollary 97.11. Let M be a compact connected smooth manifold such that $\mathrm{H}_n(M)$ is torsion for all $n \geq 1$. Then every map $\varphi\colon M \to M$ has a fixed point.

Proof. Let M be a compact connected smooth manifold such that $\mathrm{H}_n(M)$ is torsion for all $n \geq 1$. We calculate that

$$\Lambda(\varphi, \mathbb{Q}) \overset{\text{by Lemma 97.8}}{=} \underbrace{\mathrm{tr}(\varphi_*\colon \mathrm{H}_0(M) \to \mathrm{H}_0(M))}_{=1 \text{ by Lemma 97.6 (2)}} + \sum_{n\in\mathbb{N}} (-1)^n \cdot \underbrace{\mathrm{tr}(\varphi_*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M))}_{=0, \text{ since } \mathrm{H}_n(M) \text{ is torsion}} = 1.$$

It follows from the Lefschetz Fixed Point Theorem 97.10 that φ has a fixed point. ■

Examples.

(1) If we apply Corollary 97.11 to the compact connected smooth manifold $M = \overline{B}^n$, then we recover the Brouwer Fixed Point Theorem 31.4, which we had already proved on page 784 and on page 1618.

(2) In Proposition 80.11 we computed the homology groups of the real projective spaces $\mathbb{R}\mathrm{P}^n$. These calculations show in particular that the hypotheses of Corollary 97.11 are satisfied for *even*-dimensional real projective spaces $\mathbb{R}\mathrm{P}^{2n}$. Thus we see that every self-map $f\colon \mathbb{R}\mathrm{P}^{2n} \to \mathbb{R}\mathrm{P}^{2n}$ has a fixed point. (This gives an alternative solution of Exercise 76.12.) We will consider the case of *odd*-dimensional real projective spaces in Exercise 97.3.

In the next section we will provide a proof of the Lefschetz Fixed Point Theorem 97.9 and afterwards we will consider two more elaborate applications of the Lefschetz Fixed Point Theorem 97.9. There are many other pretty consequences of the Lefschetz Fixed Point Theorem 97.9 which we cannot mention in these modest notes. In fact the book by Brown [**BrownR1971**] is completely dedicated to the proof of the Lefschetz Fixed Point Theorem 97.9 and its numerous applications and generalizations.

97.4. **Proof of the Lefschetz Fixed Point Theorem 97.9.** One of the keys to proving the Lefschetz Fixed Point Theorem 97.9 is the following result in linear algebra which is a not-so-distant relative to Lemma 87.3.

Theorem 97.12. (Hopf Trace Formula)

(1) Let $\mathbb{F}$ be a field. If

$$C_* := 0 \to \mathrm{C}_k \xrightarrow{\partial_k} \mathrm{C}_{k-1} \xrightarrow{\partial_{k-1}} \dots \xrightarrow{\partial_2} \mathrm{C}_1 \xrightarrow{\partial_1} \mathrm{C}_0 \to 0$$

is a chain complex of finite-dimensional $\mathbb{F}$-vector spaces, then given any chain map $\varphi_*\colon C_* \to \mathrm{C}_*$ we have

$$\sum_{i=0}^{k} (-1)^i \cdot \mathrm{tr}(\varphi_i\colon \mathrm{C}_i \to \mathrm{C}_i) = \sum_{i=0}^{k} (-1)^i \cdot \mathrm{tr}(\varphi_*\colon \mathrm{H}_i(C_*) \to \mathrm{H}_i(C_*)).$$

(2) The analogue of (1) also holds if we start out with a chain complex of finitely generated free abelian groups.

The key to proving the Hopf Trace Formula 97.12 is the following lemma.

Lemma 97.13. Let $\mathbb{F}$ be a field. Suppose we are given a commutative diagram of homomorphisms of finite-dimensional $\mathbb{F}$-vector spaces of the following type:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \longrightarrow & 0. \end{array}$$

If the horizontal sequences are exact, then

$$\operatorname{tr}(\beta\colon B \to B) \;=\; \operatorname{tr}(\alpha\colon A \to A) + \operatorname{tr}(\gamma\colon C \to C).$$

Proof. Let $a_1, \ldots, a_m$ be a basis for A and let $c_1, \ldots, c_n$ be a basis for C. For $i = 1, \ldots, n$ we pick a $\overline{c}_i \in B$ with $p(\overline{c}_i) = c_i$. It follows easily from our hypothesis that the horizontal sequences are exact that $i(a_1), \ldots, i(a_m), \overline{c}_1, \ldots, \overline{c}_n$ form a basis for B. Furthermore it is elementary to show, using the hypothesis that the diagram commutes, that the matrix of β with respect to this basis is of the following form:

$$\begin{pmatrix} \text{matrix that} & * \;\; * \\ \text{represents } \alpha & * \;\; * \\ 0 \;\; 0 & \text{matrix that} \\ 0 \;\; 0 & \text{represents } \gamma \end{pmatrix}$$

where each "$*$" indicates some element in $\mathbb{F}$.

It is now clear that $\operatorname{tr}(\beta) = \operatorname{tr}(\alpha) + \operatorname{tr}(\gamma)$. ∎

Now that Lemma 97.13 is in our toolbox the proof of the Hopf Trace Formula 97.12 is almost embarrassingly similar to the proof of Lemma 87.3.

Proof of Hopf Trace Formula 97.12. First we note that Statement (2) follows almost immediately from Statement (1) applied to $\mathbb{F} = \mathbb{Q}$, together with Lemma 97.5 and the Universal Coefficient Theorem 89.16. We leave it to the reader to fill in the details.

Thus it remains to prove Statement (1). For the given chain complex $(\mathrm{C}_*, \partial_*)$ over some field $\mathbb{F}$ we write for each $n \in \mathbb{N}_0$

$$Z_n := \ker(\partial_n), \quad B_n := \operatorname{im}(\partial_{n+1}) \quad \text{and} \quad H_n = Z_n/B_n.$$

We write $\varphi_{B_n} := \varphi_n|_{B_n}$ and $\varphi_{Z_n} := \varphi_n|_{Z_n}$. For symmetry reasons we also write $\varphi_{\mathrm{C}_n} := \varphi_n$ and $\varphi_{H_n} := \varphi_*\colon H_n \to H_n$. These $\mathbb{F}$-vector spaces and homomorphisms form the following two types of commutative diagrams of short exact sequences

(a)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_n & \longrightarrow & \mathrm{C}_n & \xrightarrow{\partial_n} & B_{n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi_{Z_n}} & & \downarrow{\scriptstyle\varphi_{\mathrm{C}_n}} & & \downarrow{\scriptstyle\varphi_{B_{n-1}}} & & \\ 0 & \longrightarrow & Z_n & \longrightarrow & \mathrm{C}_n & \xrightarrow{\partial_n} & B_{n-1} & \longrightarrow & 0 \end{array}$$

and

(b)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_n & \longrightarrow & Z_n & \longrightarrow & H_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi_{B_n}} & & \downarrow{\scriptstyle\varphi_{Z_n}} & & \downarrow{\scriptstyle\varphi_{H_n}} & & \\ 0 & \longrightarrow & B_n & \longrightarrow & Z_n & \xrightarrow{\partial_n} & H_n & \longrightarrow & 0. \end{array}$$

Now we see that

$$\begin{aligned}
\sum_{n=0}^{k}(-1)^n\cdot\operatorname{tr}(\varphi_{\mathrm{C}_n}) &\overset{\text{by (a) and Lemma 97.13 we know that } \operatorname{tr}(\varphi_{\mathrm{C}_n})=\operatorname{tr}(\varphi_{Z_n})+\operatorname{tr}(\varphi_{B_{n-1}})}{\downarrow} \\
&= \sum_{n=0}^{k}(-1)^n\cdot(\operatorname{tr}(\varphi_{Z_n})+\operatorname{tr}(\varphi_{B_{n-1}})) \\
&= \sum_{n=0}^{k}(-1)^n\cdot(\operatorname{tr}(\varphi_{B_n})+\operatorname{tr}(\varphi_{H_n})+\operatorname{tr}(\varphi_{B_{n-1}})) \\
&\overset{\text{by (b) and Lemma 97.13 we know that } \operatorname{tr}(\varphi_{Z_n})=\operatorname{tr}(\varphi_{B_n})+\operatorname{tr}(\varphi_{H_n})}{\uparrow} \\
&= \sum_{n=0}^{k}(-1)^n\cdot\operatorname{tr}(\varphi_{H_n}) + \sum_{n=0}^{k}(-1)^n\cdot\operatorname{tr}(\varphi_{B_n}) + \sum_{n=0}^{k}(-1)^n\operatorname{tr}(\varphi_{B_{n-1}}) \\
&= \sum_{n=0}^{k}(-1)^n\cdot\operatorname{tr}(\varphi_{H_n}) + \sum_{n=0}^{k-1}(-1)^n\cdot\operatorname{tr}(\varphi_{B_n}) - \sum_{m=0}^{k-1}(-1)^m\cdot\operatorname{tr}(\varphi_{B_m}) \\
&\overset{\text{we do the substitution } m=n-1 \text{ and we use that } B_k=0 \text{ and } B_{-1}=0}{\uparrow} \\
&= \sum_{n=0}^{k}(-1)^n\cdot\operatorname{tr}(\varphi_{H_n}).
\end{aligned}$$

■

Now we are fully prepared for tackling the proof of the Lefschetz Fixed Point Theorem 97.

Proof of the Lefschetz Fixed Point Theorem 97.9. Let X be a finite simplicial complex and let $\varphi\colon X\to X$ be a map *without* a fixed point. Furthermore let $\mathbb{F}$ be a field. We need to show that

$$\Lambda(\varphi,\mathbb{F}) \;:=\; \sum_{n\in\mathbb{N}_0}(-1)^n\cdot\operatorname{tr}(\varphi_*\colon \mathrm{H}_n(X;\mathbb{F})\to\mathrm{H}_n(X;\mathbb{F})) \;=\; 0.$$

The basic idea for the proof is quite simple:

> We need to upgrade the fact that $\varphi(x)\neq x$ for every $x\in X$ to the statement that for a suitable chain complex C_* that calculates $\mathrm{H}_*(X)$ we also have $\varphi(\sigma)\neq\sigma$ for every element σ of a suitable basis. This implies that the trace on the chain group level is zero, hence we get the desired statement from the Hopf Trace Formula 97.12 (1).

For better or worse turning the idea into a proper proof requires a substantial effort. First of all we note that since X is a finite simplicial complex we can use Lemma 93.9 to view X as a subspace of some $\mathbb{R}^n$. In particular we can and will equip X with the usual Euclidean metric $d(x,y):=\|x-y\|_{\mathbb{R}^n}$.

Claim 1. There exists an iterated barycentric subdivision Y of X such that for every vertex w of Y we have $\varphi(\mathrm{St}(Y,w))\cap\mathrm{St}(Y,w)=\varnothing$.

Proof. The claim is proved in several easy steps:

(1) Since X is a finite simplicial complex we know by Lemma 93.9 that X is compact.
(2) From the compactness of X together with our hypothesis that for every $x\in X$ we have $\varphi(x)\neq x$ we obtain through some elementary arguments, as provided in Exercise 6.3, that there exists an $\epsilon>0$ such that for every $x\in X$ we have $d(\varphi(x),x)\geq\epsilon$
(3) By Lemma 94.7 there exists a $j\in\mathbb{N}_0$ such that for every vertex v of the iterated barycentric subdivision $\mathrm{sd}^j(X)$ we have $\operatorname{diam}(\mathrm{St}(\mathrm{sd}^j(X),v))<\frac{\epsilon}{2}$.
(4) It follows from (2) and (3) that $\varphi(\mathrm{St}(\mathrm{sd}^j(X),w))\cap\mathrm{St}(\mathrm{sd}^j(X),w)=\varnothing$. Indeed, let $x,y\in\mathrm{St}(\mathrm{sd}^j(X),w)$. We calculate that

$$d(\varphi(x),y) \underset{\text{triangle inequality}}{\overset{\uparrow}{\geq}} d(\varphi(x),x)-d(x,y) \underset{\text{by (2) and (3)}}{\overset{\uparrow}{\geq}} \epsilon-\tfrac{\epsilon}{2} \geq \tfrac{\epsilon}{2} > 0.$$

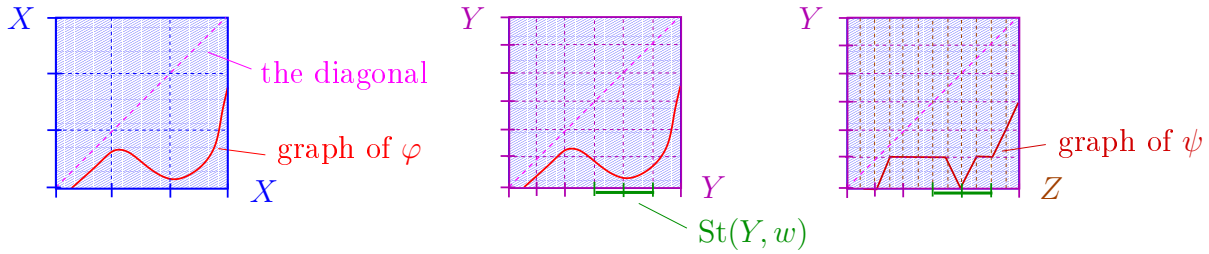
This implies that $\varphi(\mathrm{St}(\mathrm{sd}^j(X), w)) \cap \mathrm{St}(\mathrm{sd}^j(X), w) = \varnothing$.

Thus the iterated barycentric subdivision $Y := \mathrm{sd}^j(K)$ has the desired property. ⊞

Next we apply the Simplicial Approximation Theorem 94.8 to the map $\varphi\colon Y \to Y$. We obtain an $i \in \mathbb{N}_{\geq 0}$ and a simplicial map $\psi\colon Z := \mathrm{sd}^i(Y) \to Y$ with the following properties:

(A) The map ψ is homotopic to φ as a map $Y \to Y$.
(B) For every vertex w of $Z = \mathrm{sd}^i(Y)$ we have $\varphi(\mathrm{St}(\underbrace{\mathrm{sd}^i(Y)}_{=Z}, w)) \subset \mathrm{St}(Y, \psi(w))$.

It follows from (A) and Lemma 97.7 that the Lefschetz numbers of φ and ψ agree.



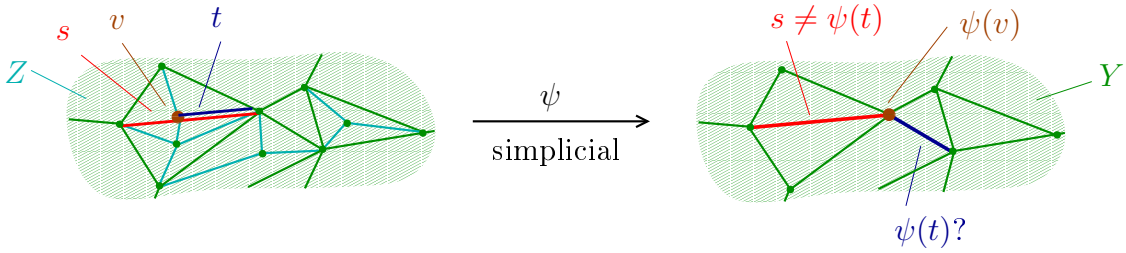
The map $\psi\colon Z \to Y$ is a simplicial map between two different simplicial complexes. The following claim gets as close as possible to saying that ψ does not fix a simplex.

Claim 2. For every simplex s of Y and every simplex t of Z with $t \subset s$ we have $\psi(t) \neq s$.

Proof. Let s be a simplex of Y and let t be a simplex of Z with $t \subset s$. Let v be a vertex of t. (It might be helpful to consider the figure below.) We make the following deductions:

v a vertex of the simplex t of Z
$\implies t \subset \mathrm{St}(Z, v) \implies \varphi(t) \subset \varphi(\mathrm{St}(Z, v)) \implies \varphi(t) \subset \mathrm{St}(Y, \psi(v))$
by the definition of $\mathrm{St}(Z, v)$, see page 2018 — by (B)
$\implies \varphi(t) \cap \varphi(\mathrm{St}(Y, \psi(v))) = \varnothing \implies t \cap \mathrm{St}(Y, \psi(v)) = \varnothing$
by Claim 1 — Image-Preimage Lemma 0.2 (4)
$\implies t \cap \mathrm{St}(Y, \psi(t)) = \varnothing \implies \psi(t) \neq s.$
since $\psi(v)$ is a vertex of $\psi(t)$ we have $\mathrm{St}(Y, \psi(t)) \subset \mathrm{St}(Y, \psi(v))$ — we have $t \subset \mathrm{St}(Y, s)$, indeed this follows from $t \subset s$ and from $s \subset \mathrm{St}(Y, s)$, which in turn holds by definition of $\mathrm{St}(Y, s)$

⊞



Next, given $n \in \mathbb{N}_0$ let

$$u_n\colon \mathrm{C}_n^{\mathrm{simp}}(Y) \to \mathrm{C}_n^{\mathrm{simp}}(\underbrace{\mathrm{sd}^i(Y)}_{=Z})$$

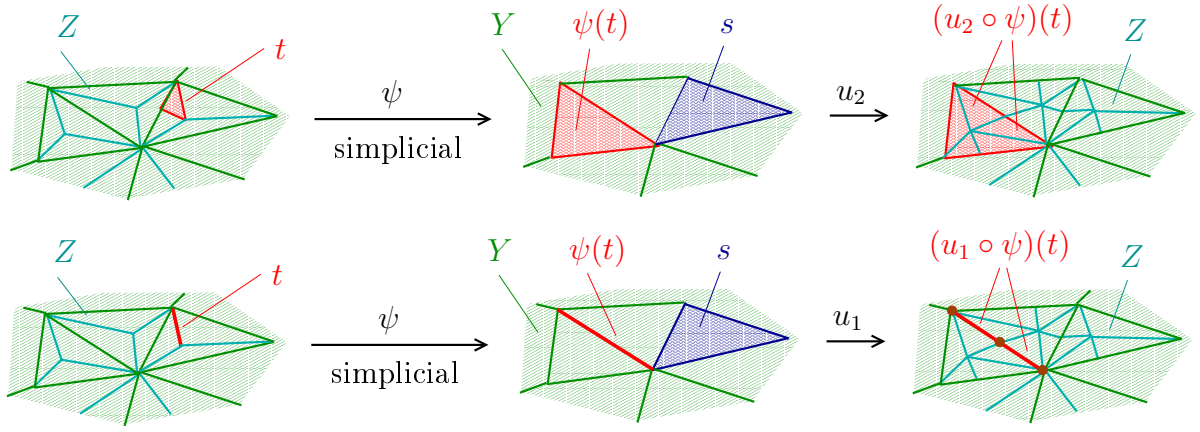
be the i-fold iterated simplicial subdivision map that we introduced in the Chain Subdivision Lemma 95.41.

Claim 3. For each $n \in \mathbb{N}_0$ the map $u_n \circ \psi_* \colon \mathrm{C}_n^{\mathrm{simp}}(Z) \to \mathrm{C}_n^{\mathrm{simp}}(Z)$ has trace zero.

Proof. Let T_n be the set of n-simplices of Z. It follows from Lemma 95.5 that T_n, with some choice of ordering for each $t \in T_n$, is a basis of $\mathrm{C}_n^{\mathrm{simp}}(Z)$. Now let $t \in T_n$. If $\psi(t)$ is not an n-simplex of Y, then by definition we have $\psi_*(t) = 0$. Now suppose that $\psi(t)$ is an n-simplex of Y. Then we see that

$$\begin{aligned}
(u_n \circ \psi_*)(t) &\overset{\text{by definition of the simplicial subdivision map}}{=} \text{linear combination of } n\text{-simplices of } Z \text{ contained in } \psi(t) \\
&= \text{linear combination of } n\text{-simplices of } Z \text{ not equal to } t. \\
&\quad \text{here we need to distinguish two cases:}[1199] \\
&\quad (1)\ \text{If } t \text{ is contained in some } n\text{-simplex } s \text{ of } Y, \text{ then the conclusion follows from Claim 2.} \\
&\quad (2)\ \text{If } t \text{ is not contained in any } n\text{-simplex of } Y, \text{ then it is in particular} \\
&\qquad \text{not contained in the } n\text{-simplex } \psi(t) \text{ of } Y \\
&= \text{linear combination of elements in } T_n \setminus \{t\}.
\end{aligned}$$

This calculation shows that, with respect to the basis T_n, each diagonal entry of the map $u_n \circ \psi_*$ is zero. In particular the trace of the map $u_n \circ \psi_* \colon \mathrm{C}_n^{\mathrm{simp}}(Z) \to \mathrm{C}_n^{\mathrm{simp}}(Z)$ is zero. ⊞



In the following we will use simplicial homology with $\mathbb{F}$-coefficients. The definition should be clear to the reader and it should be clear that some of the statements which we cite below, which were formulated for homology without coefficients, also have obvious generalizations to homology with $\mathbb{F}$-coefficients.

Claim 4. The following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{H}_n^{\mathrm{simp}}(Z;\mathbb{F}) & \xrightarrow{\psi_*} & \mathrm{H}_n^{\mathrm{simp}}(Y;\mathbb{F}) & \xrightarrow{u_n} & \mathrm{H}_n^{\mathrm{simp}}(Z;\mathbb{F}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{H}_n(Z;\mathbb{F}) & \xrightarrow{\psi_*} & \mathrm{H}_n(Z;\mathbb{F}) & \xrightarrow{u_n} & \mathrm{H}_n(Z;\mathbb{F})
\end{array}$$

here the vertical maps are the natural isomorphism defined on page 2058.

Proof. It turns out that there is not much to say. The rectangle to the left commutes by the naturality of the vertical maps and the rectangle to the right commutes by Proposition 95.45. ⊞

[1199] The two cases are illustrated in the figure below.

Now we can quickly conclude the proof of the Lefschetz Fixed Point Theorem 97.9. Indeed, we see that

$$\begin{array}{lll}
\Lambda(\varphi,\mathbb{F}) \overset{\text{Lemma 97.7}}{=} \Lambda(\psi,\mathbb{F}) = \sum_{i\in\mathbb{N}_0}(-1)^i\cdot\operatorname{tr}(\psi_*\circlearrowleft \mathrm{H}_i(Z;\mathbb{F})) & \overset{\text{by Lemma 74.27 the singular subdivision maps } u_* \text{ are chain homotopic to the identity, thus the induced map on homology is the identity}}{=} & \sum_{i\in\mathbb{N}_0}(-1)^i\cdot\operatorname{tr}(u_*\circ\psi_*\circlearrowleft \mathrm{H}_i(Z;\mathbb{F})) \\
= \sum_{i\in\mathbb{N}_0}(-1)^i\cdot\operatorname{tr}(u_*\circ\psi_*\circlearrowleft \mathrm{H}_i^{\mathrm{simp}}(Z;\mathbb{F})) & & = \sum_{i\in\mathbb{N}_0}(-1)^i\cdot\operatorname{tr}(u_*\circ\psi_*\circlearrowleft \mathrm{C}_i^{\mathrm{simp}}(Z;\mathbb{F})) \\
\uparrow \text{ by Claim 4} & & \uparrow \text{ by the Hopf Trace Formula 97.12 (1)} \\
= \sum_{i\in\mathbb{N}_0}(-1)^i\cdot\operatorname{tr}(u_*\circ\psi_*\circlearrowleft \mathrm{C}_i^{\mathrm{simp}}(Z)) & & = 0. \\
\uparrow \text{ by Lemma 97.3 and since by definition } \mathrm{C}_i^{\mathrm{simp}}(Z;\mathbb{F}) = \mathrm{C}_i^{\mathrm{simp}}(Z)\otimes\mathbb{F} & & \uparrow \text{ by Claim 3}
\end{array}$$

We have thus proved that if φ has no fixed points, then $\Lambda(\varphi,\mathbb{F}) = 0$. ∎

97.5. **Vector fields on smooth manifolds.** In this section we will apply the Lefschetz Fixed Point Theorem 97.9 to the study of the (non-)existence of certain vector fields on smooth manifolds.

For the reader's convenience we recall the following definition from page 639.

Definition. Let M be an n-dimensional smooth manifold.

(1) A vector field on M is a map v which assigns to each $P \in M$ a vector $v(P) \in \mathrm{T}_P M$ such that the following condition is satisfied: For every chart[1200] $\Phi\colon U \to V$ from an open set U of M to an open set V of $\mathbb{R}^n$ the map

$$\begin{array}{ccccc}
V & \to & \mathbb{R}^n & & \\
P & \mapsto & \mathrm{D}\Phi_{\Phi^{-1}(P)}\big(\underbrace{v(\Phi^{-1}(P))}_{\in \mathrm{T}_{\Phi^{-1}(P)}M}\big) & \subset & \mathrm{T}_P V \underset{\uparrow}{=} \mathbb{R}^n \\
 & & & & \text{identification from Proposition 23.9}
\end{array}$$

is continuous. We say that the vector field v is smooth if for every chart the above map is smooth.

(2) Given a vector field v on M we say that $x \in M$ is a zero of v if $v(x) = 0$.

One of the goals of this section is to address the following time-honored question.

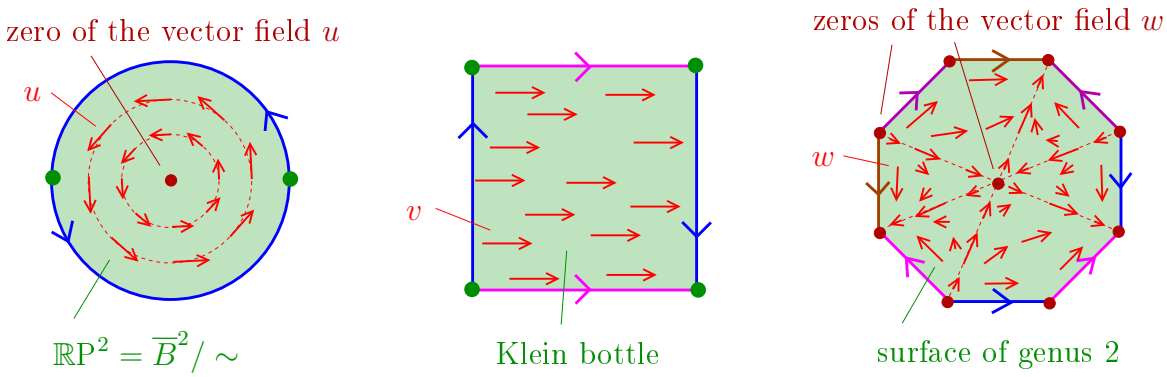
Question 23.19 Which closed connected 2-dimensional smooth manifolds admit a nowhere-vanishing vector field?

Example. In the figure below we show vector fields on the projective plane $\mathbb{R}\mathrm{P}^2$, on the Klein bottle and the surface of genus 2. The vector field on the Klein bottle is nowhere-vanishing, whereas the vector field on $\mathbb{R}\mathrm{P}^2$ and the surface of genus 2 both have zeros.

The attentive reader will not have failed to notice that it is the right moment to recall one of the highlights from one of the earlier chapters:

Theorem 38.12. (Hairy Ball Theorem) Every vector field on an even-dimensional sphere vanishes on at least one point.

[1200] Recall that for a smooth manifold we mean by a "chart" a chart from the given smooth structure.



Sketch of proof. Let v be a nowhere-vanishing vector field on a sphere S^n. By "traveling along the great circle defined by v" we can find a homotopy $F\colon S^n \times [0,1] \to S^n$ from the identity to the antipodal map $\rho\colon S^n \to S^n$ given by $\rho(x) = -x$. In particular we see that $(-1)^{n+1} = \deg(\rho) = \deg(\mathrm{id}) = 1$. Thus we see that n is odd. ■

Remark. Regarding Question 23.19 the Hairy Ball Theorem 38.12 says in particular that S^2 does not admit a nowhere-vanishing vector field.

To address Question 23.19 it is also helpful to add a few ways to construct new vector fields to our toolbox.

Lemma 97.14.

(1) Let $f\colon M \to N$ be a map between smooth manifolds which is a local diffeomorphism. Let v be a vector field on N. The map

$$P \mapsto (\mathrm{D}f_P)^{-1}\big(\underbrace{v(f(P))}_{\in \mathrm{T}_{f(P)}(N)}\big)$$

is a vector field on M that we denote by f^*v. A point $P \in M$ is a zero of f^*v if and only if $f(P)$ is a zero of v.

(2) Let M be a smooth manifold and let v be a vector field on M. Suppose $p\colon \widetilde{M} \to M$ is a covering. Then p^*v is a vector field on $\widetilde{M}$.

(3) Let M be a smooth manifold and let v be a vector field on M. Suppose we are given a group G that acts freely, properly and smoothly on M such that for every $g \in G$ we have $(g\cdot\colon M \to M)^*(v) = v$. Then there exists a unique vector field u on M/G such that $v = p^*u$ where $p\colon M \to M/G$ is the projection .[1201]

Proof.

(1) The mature reader will have no troubles with providing a proof for this statement.
(2) This statement follows from (1) and the Manifold Covering Proposition 49.2 (2a).
(3) The proof of this statement is very similar to the proof of (1) and it is also left to the reader. ■

Lemma 97.14 allows us to answer Question 23.19 in some special cases:

[1201] Recall that by the Manifold Quotient-by-Group Action Proposition 19.32 we know that M/G is a smooth manifold and that the projection $p\colon M \to M/G$ is a local diffeomorphism.

Examples.

(1) If v is a vector field on $\mathbb{R}\mathrm{P}^2$ with n zeros, then it follows from Lemma 97.14, applied to the 2-fold covering $p\colon S^2 \to \mathbb{R}\mathrm{P}^2 = S^2/x \sim -x$ that S^2 admits a vector field with $2n$ zeros. By the Hairy Ball Theorem 38.12 we know that $2n > 0$. In other words, we have just shown that every vector field on $\mathbb{R}\mathrm{P}^2$ has at least one zero.

(2) On the smooth manifold $\mathbb{R}^2$ we now consider the constant "horizontal" vector field $v(x,y) = (1,0) \in T_{(x,y)}\mathbb{R}^2 = \mathbb{R}^2$. As discussed in Lemma 5.25 and on page 1075 we can view the torus and the Klein bottle as quotients of $\mathbb{R}^2$ by a group G that acts freely, properly and smoothly. Furthermore, in both cases one can easily verify that the given G-action preserves the vector field v. This shows that the nowhere-vanishing vector field v descends to a nowhere-vanishing vector field on the torus and the Klein bottle. For the Klein bottle we obtain precisely the vector field shown in the figure on page 2124.

The following theorem is the main result of this section.

Theorem 97.15. (Euler Characteristic-Vector Field Theorem) Let M be a closed smooth manifold. If M admits a nowhere-vanishing vector field, then $\chi(M) = 0$.

Remark.

(1) On page 1879 we saw that for even n the Euler characteristic $\chi(S^n)$ equals 2. Thus we see that the Euler Characteristic-Vector Field Theorem 97.15 is a generalization of the Hairy Ball Theorem 38.12.

(2) In Smooth Euler Characteristic-Vector Field Theorem 45.22 we proved the same statement for closed orientable smooth manifolds.

(3) For 2-dimensional smooth manifolds a low-tech proof of the Euler Characteristic-Vector Field Theorem 97.15 involving "negative, neutral and positive charges" is given in [**Thu1997**, Proposition 1.3.3].

(4) Let M be a closed connected 2-dimensional smooth manifold that is not diffeomorphic to the torus or the Klein bottle. It follows from the calculations on pages 1879 and 1885 that $\chi(M) \neq 0$. Thus we obtain from the Euler Characteristic-Vector Field Theorem 97.15 that M does not admit a nowhere-vanishing vector field. Together with the above discussion we now have a complete answer to Question 23.19.

(5) In the Nowhere Vanishing Vector Field Existence Theorem 129.6 we will see that the converse to the Euler Characteristic-Vector Field Theorem 97.15 holds. Namely we will see that if M is a closed connected smooth manifold with $\chi(M)$, then M admits a nowhere-vanishing vector field.

(6) In Corollary 129.8 we will prove a suitable extension of the Euler Characteristic-Vector Field Theorem 97.15 to the setting of smooth manifolds with non-empty boundary.

The key idea behind the proof of the Euler Characteristic-Vector Field Theorem 97.15 is similar to the proof of the Hairy Ball Theorem 38.12, namely we want to use the nowhere-vanishing vector field v on M to construct an interesting self-map of M. This is done in the following proposition.

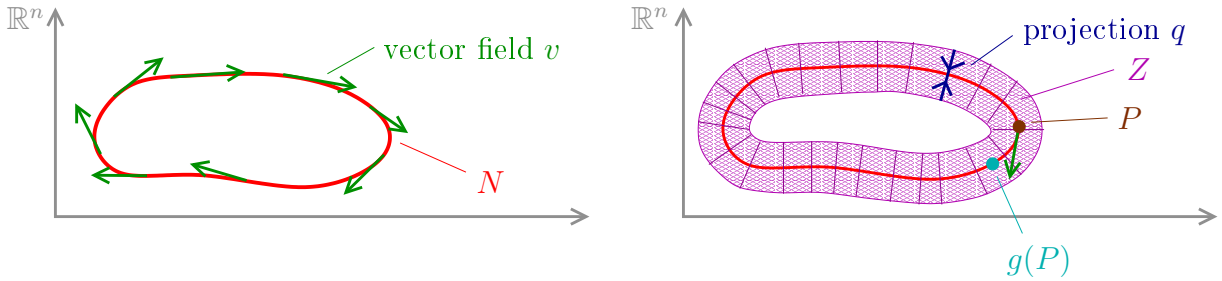
Proposition 97.16. Let M be a closed smooth manifold. If there exists a nowhere-vanishing vector field, then there exists a map $g\colon M \to M$ that is homotopic to the identity and which has no fixed points.

Proof. The proposition follows immediately from the combination of the Replace-by-Smooth Vector Field Proposition 29.6, the Flow Theorem 35.1 and the Flow Fixed Point Theorem 35.4.

Recall that our proof of the Flow Theorem 35.1 was built on the Fundamental Theorem 35.7 for Autonomous ODEs which we did not prove. Therefore, for the reader's peace of mind we will sketch in the following a self-contained proof of the proposition. By the Euclidean Embedding Theorem 27.1 (1) together with the Smooth Embedding Theorem 24.10 we can and will view M as a submanifold of some $\mathbb{R}^n$. It follows from Theorem **??** that there exists an $\epsilon > 0$ such that the following two conditions are satisfied:

(a) The subset $Z := \{P + w \,|\, P \in M \text{ and } w \in (\widetilde{\mathrm{T}}_P M)^\perp \text{ with } \|w\| \leq \epsilon\}$ of $\mathbb{R}^n$ is a neighborhood of M.

(b) The map
$$\begin{array}{rcl} q\colon Z & \to & M \\ P + w & \mapsto & P \end{array} \quad \text{where } P \in M \text{ and } w \in (\widetilde{\mathrm{T}}_P M)^\perp \text{ with } \|w\| \leq \epsilon$$
is well-defined and continuous.



Next we consider the map
$$\begin{array}{rcl} F\colon M \times \mathbb{R}_{\geq 0} & \to & \mathbb{R}^n \\ (P, t) & \mapsto & P + t \cdot v(P). \end{array}$$

We make the following observations:

(1) As in the Convex Homotopy Lemma 14.1 we see that F is, unsurprisingly, continuous.

(2) It follows from (a), together with the fact that M is in particular compact and together with some elementary point-set theoretic arguments along the lines of Exercise 5.9, that there exists a $\mu > 0$ such that $F(M \times [0, \mu]) \subset Z$.

(3) We set $g := q \circ F_\mu\colon M \to M$.

(4) By (1) we know that $q \circ F\colon M \times [0, \mu] \to M$ defines a homotopy between the identity and g.

(5) It remains to show that g has no fixed points. Thus let $P \in M$. By (b) we can write $F_\mu(P) = P + \mu \cdot v(P) =: Q + w$ with $Q \in M$ and $w \in (\widetilde{\mathrm{T}}_Q M)^\perp$ with $\|w\| \leq \epsilon$. Since $\widetilde{\mathrm{T}}_P M \cap \widetilde{\mathrm{T}}_P M^\perp = \{0\}$ and since $v(P)$ is a non-zero vector in $\widetilde{\mathrm{T}}_P M$ we see that $P \neq Q$. By definition we have $g(P) = Q$. Thus we have shown that $g(P) \neq P$.

This concludes the second proof of the proposition. ∎

Now it is straightforward to provide the proof of the Euler Characteristic-Vector Field Theorem 97.15.

Proof of the Euler Characteristic-Vector Field Theorem 97.15. Let M be a closed smooth manifold. Suppose there exists a nowhere-vanishing vector field v on M. By Proposition 97.16 there exists a map $g\colon M \to M$ that is homotopic to the identity and which has no fixed points. Now we calculate that

$$\chi(M) \overset{\text{Lemma 97.6 (1)}}{\overset{\downarrow}{=}} \Lambda(\mathrm{id}\colon M \to M) \overset{\text{by Lemma 97.7 since } f \text{ and id are homotopic}}{\overset{\uparrow}{=}} \Lambda(f\colon M \to M) \overset{\text{Lemma 97.8 (2)}}{\overset{\uparrow}{=}} \Lambda(f\colon M \to M, \mathbb{Q}) \overset{\text{by the claim and the Lefschetz Fixed Point Theorem II 97.10}}{\overset{\downarrow}{=}} 0.$$

■

As a teaser we state the following theorem which we will prove later on. This proposition is basically the converse to the Euler Characteristic-Vector Field Theorem 97.15.

Theorem 129.6.(Nowhere Vanishing Vector Field Existence Theorem) Let M be a closed connected smooth manifold. If $\chi(M) = 0$, then M admits a nowhere-vanishing smooth vector field.

In Chapter 129 we will not only provide a proof of the Nowhere Vanishing Vector Field Existence Theorem 129.6 but we will also formulate and prove a significant generalization of the Euler Characteristic-Vector Field Theorem 97.15, namely the Poincaré-Hopf Theorem 129.7 which, given a vector field, relates the type of zeros which occur to the Euler characteristic of the smooth manifold.

97.6. **The Borsuk-Ulam Theorem II ❀.** In this section we will see how to use the Lefschetz Fixed Point Theorem 97.9 to give an alternative proof of the Borsuk-Ulam Theorem 39.8. First we remind the reader of the statement of the Borsuk-Ulam Theorem 39.8.

Theorem 39.8. (Borsuk-Ulam Theorem) Let $n \in \mathbb{N}_0$. For every map $g\colon S^n \to \mathbb{R}^n$ there exists a pair of antipodal points x and $-x$ on S^n with $g(x) = g(-x)$.

Remark. Let us remind the reader that the Borsuk-Ulam Theorem 39.8 has many beautiful applications, in particular the Lusternik-Schnirelmann Theorem 40.1, the Ham-Sandwich Theorem 40.2, Pfister's Level Theorem 40.9 and the Necklace Theorem 40.11.

The key to our second proof of the Borsuk-Ulam Theorem 39.8 is the following proposition.

Proposition 97.17. Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a map. If $f(-x) = -f(x)$ for all $x \in S^n$, then the Lefschetz number $\Lambda(f)$ is even.

Examples.

(1) Let $A \in \mathrm{O}(n+1)$. We consider the map $f\colon S^n \to S^n$ that is given by $f(x) := A \cdot x$. Evidently we have $f(-x) = -f(x)$ for all $x \in S^n$. Furthermore it follows from the Sphere Degree Lemma 76.11 (6) together with Lemma 97.6 (2) that $\Lambda(f) = 1 + (-1)^n \cdot \det(A)$.

(2) In Exercise 97.13 we will see that given any $n \in \mathbb{N}$ and given any *even* $m \in \mathbb{Z}$ there exists a map $f\colon S^n \to S^n$ with $\Lambda(f) = m$ and such that $f(-x) = -f(x)$ for all $x \in S^n$.

Remark. The proof of Proposition 97.17 is an expanded write-up of the arguments in [**Deo2018**, Theorem 4.9.17] and [**Arm1983**, Chapter 9.3].

The following gives us a criterion for showing that a trace is even.

Lemma 97.18. Let V be a finitely generated free $\mathbb{Z}$-module and let $\rho\colon V \to V$ be an isomorphism with the following two properties:

(i) There exists a subset T of V such that $T \cap \rho(T) = \varnothing$ and such that $T \cup \rho(T)$ is a basis for V.

(ii) We have $\rho \circ \rho = \mathrm{id}_V$.

Let $\varphi\colon V \to V$ be a homomorphism. If φ commutes with ρ, then $\mathrm{tr}(\varphi\colon V \to V)$ is even.

Proof. We equip V with the basis $T \sqcup \rho(T)$. It follows immediately from (ii) that

$$\text{matrix representing } \rho \text{ with respect to the basis } T \sqcup \rho(T) \;=\; \begin{matrix} \\ T \\ \rho(T) \end{matrix} \begin{matrix} \begin{matrix} T & \rho(T) \end{matrix} \\ \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \end{matrix}.$$

Now let $\varphi\colon V \to V$ be a homomorphism that commutes with ρ. We write

$$\text{matrix representing } \varphi \text{ with respect to the basis } T \sqcup \rho(T) \;=:\; \begin{matrix} \\ T \\ \rho(T) \end{matrix} \begin{matrix} \begin{matrix} T & \rho(T) \end{matrix} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{matrix}.$$

Since φ commutes with g we see that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Multiplying out both sides we see in particular that $A = D$. But this implies evidently that the trace of the above matrix representing φ is even. Thus we have shown that $\mathrm{tr}(\varphi)$ is even. ∎

Proof of Proposition 97.17. Let $f\colon S^n \to S^n$ be a map which has the property that $f(-x) = -f(x)$ for all $x \in S^n$. We denote by $\rho\colon S^n \to S^n$ the map that is given by $\rho(x) = -x$. Note that our hypothesis on f is equivalent to the statement that f commutes with ρ. We need to show that $\Lambda(f)$ is even. Similar to the proof of the Lefschetz Fixed Point Theorem 97.9 on page 2120 we intend to proceed as follows:

> We want to upgrade the hypothesis that $f \circ \rho = \rho \circ f$ to a statement about an induced map on a suitable simplicial chain complex C_* such that we can conclude from Lemma 97.18 and the Hopf Trace Formula 97.12 (2) that $\Lambda(f)$ is even.

To carry out this program we do the following: We equip S^n with the simplicial structure given by the homeomorphisms

$$\begin{array}{ccccc} & & & & \text{here } S_0 \text{ is the abstract simplicial complex with vertex set } \{0,1\} \\ & & & & \downarrow \\ S^n & \xleftarrow[\uparrow]{\cong} & S^0 * \cdots * S^0 & = & |S_0| * \cdots * |S_0| \quad \xrightarrow[\uparrow]{\cong} \quad |S_0 * \cdots * S_0|. \\ & & \text{homeomorphism given by the Join-Simplex Lemma 93.19 (1)} & & \text{homeomorphism given by the Join Lemma 93.20} \end{array}$$

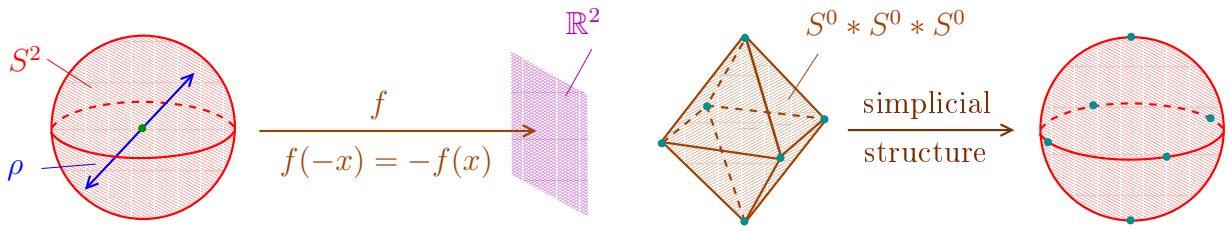
We introduce the following notation:

(1) We write $K := S_0 * \cdots * S_0$ and we denote by $\Psi\colon |K| \to S^n$ the above homeomorphism.

(2) Given any $k \in \mathbb{N}_0$ we denote by $\mathfrak{V}\colon |\mathrm{sd}^k(K)| \to |K|$ the natural homeomorphism given by the SD-Lemma 94.3 (3).

(3) By a slight abuse of notation we denote by $\rho\colon K \to K$ the map that is induced by the reflection on the simplices of all the copies of S_0.

(4) Note that by the SD-Lemma 94.3 (5) the simplicial map $\rho\colon K \to K$ induces a simplicial map $\mathrm{sd}^k(K) \to \mathrm{sd}^k(K)$ which we also denote by ρ.



Claim 1. There exists a $k \in \mathbb{N}_0$ and a *simplicial* map $\varphi\colon \mathrm{sd}^k(K) \to K$ with the following properties:

(1) the map $\Psi \circ |\varphi| \circ \mathfrak{V}^{-1} \circ \Psi^{-1}\colon S^n \to S^n$ is homotopic to f,
(2) the map φ commutes with the maps ρ, more precisely, the following diagram commutes

$$\begin{array}{ccc} \mathrm{sd}^k(K) & \xrightarrow{\varphi} & K \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ \mathrm{sd}^k(K) & \xrightarrow{\varphi} & K. \end{array}$$

Proof. As we will see, the proof of Claim 1 follows from a careful reading of the proof of the Simplicial Approximation Theorem 94.8.

(a) First note that Claim 1 of the proof of the Simplicial Approximation Theorem 94.8 shows that there exists a $k \in \mathbb{N}_0$ such that for every vertex v of $\mathrm{sd}^k(K)$ there exists a vertex w of K with $f((\Psi \circ \mathfrak{V})(\mathrm{St}(|\mathrm{sd}^k(K)|, v))) \subset \Psi(\mathring{\mathrm{St}}(|K|, w))$. We denote by V the vertex set of $\mathrm{sd}^k(K)$ and we denote by W the vertex set of K. By the above there exists a map $\varphi\colon V \to W$ with the following property:
($*$) For every vertex $v \in V$ we have $f((\Psi \circ \mathfrak{V})(\mathrm{St}(|\mathrm{sd}^k(K)|, v))) \subset \Psi(\mathring{\mathrm{St}}(|K|, \varphi(v)))$.
(b) In the proof of the Simplicial Approximation Theorem 94.8 we have shown that every map $\varphi\colon V \to W$ that satisfies ($*$) is a simplicial map and that it has the property that $\Psi \circ |\varphi| \circ \mathfrak{V}^{-1} \circ \Psi^{-1}$ is homotopic to f.
(c) It follows immediately from the definition of Ψ and the naturality of $\mathfrak{V}$ that the following two diagrams commute:

$$\begin{array}{ccc} |K| & \xrightarrow{\rho} & |K| \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\Psi} \\ S^n & \xrightarrow{\rho} & S^n \end{array} \qquad \text{and} \qquad \begin{array}{ccc} |\mathrm{sd}^k(K)| & \xrightarrow{\rho} & |\mathrm{sd}^k(K)| \\ {\scriptstyle\mathfrak{v}}\downarrow & & \downarrow{\scriptstyle\mathfrak{v}} \\ |K| & \xrightarrow{\rho} & |K|. \end{array}$$

(d) Since all maps called ρ satisfy the condition that $\rho \circ \rho = \mathrm{id}$ we obtain easily from (a) and (c) that we can pick a map $\varphi\colon V \to W$ which satisfies ($*$) and such that $\varphi \circ \rho = \rho \circ \varphi$.
(e) By (b) we know that φ satisfies (1). Furthermore, by (d) we know that the simplicial map φ also satisfies (2). ⊞

Before we move on to the next claim we introduce and recall some notation:

(1) Now we write $g := \Psi \circ |\varphi| \circ \mathfrak{V}^{-1} \circ \Psi^{-1}$.

(2) We denote by $u_*\colon \mathrm{C}_*^{\mathrm{simp}}(K) \to \mathrm{C}_*^{\mathrm{simp}}(\mathrm{sd}^k(K))$ the k-fold iterated simplicial subdivision map that we introduced in the Chain Subdivision Lemma 95.41.

Claim 2. For every $j \in \mathbb{N}_0$ there exists an isomorphism $\mu\colon \mathrm{H}_j^{\mathrm{simp}}(\mathrm{sd}^k(K)) \to \mathrm{H}_j(S^n)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_j^{\mathrm{simp}}(\mathrm{sd}^k(K)) & \xrightarrow{u_j \circ \varphi_*} & \mathrm{H}_j^{\mathrm{simp}}(\mathrm{sd}^k(K)) \\ \mu \downarrow \cong & & \cong \downarrow \mu \\ \mathrm{H}_j(S^n) & \xrightarrow[(\Psi \circ |\varphi| \circ \mho^{-1} \circ \Psi^{-1})_*]{g_*} & \mathrm{H}_j(S^n). \end{array}$$

Proof. We consider the following much more elaborate diagram:

$$\begin{array}{ccccc} \mathrm{H}_j^{\mathrm{simp}}(\mathrm{sd}^k(K)) & \xrightarrow{\varphi_*} & \mathrm{H}_j^{\mathrm{simp}}(K) & \xrightarrow{u_j} & \mathrm{H}_j^{\mathrm{simp}}(\mathrm{sd}^k(K)) \\ \downarrow{\scriptstyle \Theta_{\mathrm{sd}^k(K)}} & & \downarrow{\scriptstyle \Theta_K} & & \downarrow{\scriptstyle \Theta_{\mathrm{sd}^k(K)}} \\ \mathrm{H}_j(|\mathrm{sd}^k(K)|) & \xrightarrow{|\varphi|_*} & \mathrm{H}_j(|K|) & \xrightarrow{u_j} & \mathrm{H}_j(|K|) \\ \downarrow{\scriptstyle (\Psi \circ \mho)_*} & & & \searrow^{\Psi_*} & \downarrow{\scriptstyle \Psi_*} \\ \mathrm{H}_j(S^n) & & \xrightarrow[(\Psi \circ |\varphi| \circ \mho^{-1} \circ \Psi^{-1})_*]{g_*} & & \mathrm{H}_j(S^n). \end{array}$$

We make the following observations and clarifications:

(1) The maps Θ are the natural isomorphisms introduced on page 2058.
(2) We denote by $u_*\colon \mathrm{C}_*(|K|) \to \mathrm{C}_*(|K|)$ the k-fold iterated singular subdivision map that we introduced on page 1640.
(3) It follows from Proposition 95.45, applied iteratively k times, that the rectangle to the top right commutes.
(4) The quadrilateral at the bottom commutes by definition.
(5) The triangle at the bottom right commutes by Lemma 74.27.
(6) The square to the top left commutes by the naturality of Θ.

It follows from the above that the isomorphism $\mu := \Psi_* \circ \mho_* \circ \Theta_{\mathrm{sd}^k(K)}$ has the desired property. ⊞

To shorten the notation a little bit we now write $L := \mathrm{sd}^k(K)$. We calculate that

$$\begin{aligned} \Lambda(f) \overset{\text{by Lemma 97.7 and Claim 1 (1)}}{\downarrow =} \Lambda(g) &= \sum_{j \in \mathbb{N}_0} (-1)^j \cdot \mathrm{tr}\left(g_* \circlearrowleft \mathrm{H}_i(S^n)\right) \overset{\text{by Claim 2}}{\downarrow =} \sum_{j \in \mathbb{N}_0} (-1)^j \cdot \mathrm{tr}\left((u_j \circ \varphi_*) \circlearrowleft \mathrm{H}_i^{\mathrm{simp}}(L)\right) \\ &= \sum_{j \in \mathbb{N}_0} (-1)^j \cdot \mathrm{tr}\left((u_j \circ \varphi_*) \circlearrowleft \mathrm{C}_i^{\mathrm{simp}}(L)\right). \\ &\quad \uparrow \text{by the Hopf Trace Formula 97.12 (2)} \end{aligned}$$

At this point we see that it suffices to prove the following claim.

Claim 3. For each $j \in \mathbb{N}_0$ the trace $\mathrm{tr}\left((u_j \circ \varphi_*) \circlearrowleft \mathrm{C}_j^{\mathrm{simp}}(L)\right)$ is even.

Proof. Evidently the plan is to apply Lemma 97.18. We make a few more preparations:

(1) We consider the following diagram:

$$\begin{array}{ccccc} \mathrm{C}_j^{\mathrm{simp}}(L) & \xrightarrow{g_*} & \mathrm{C}_j^{\mathrm{simp}}(K) & \xrightarrow{u_j} & \mathrm{C}_j^{\mathrm{simp}}(L) \\ \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \rho_*} \\ \mathrm{C}_j^{\mathrm{simp}}(L) & \xrightarrow{g_*} & \mathrm{C}_j^{\mathrm{simp}}(K) & \xrightarrow{u_j} & \mathrm{C}_j^{\mathrm{simp}}(L). \end{array}$$

It follows from Claim 1 (2) that the square to the left commutes. It follows from the naturality of the simplicial subdivision maps that the square to the right commutes.

(2) Note that it follows immediately from the definition of the map $\rho\colon K \to K$ that $\rho\colon K \to K$ does not fix a single simplex of K and that $\rho \circ \rho = \mathrm{id}_K$. Since the map $K \mapsto L = \mathrm{sd}^k(K) = L$ is functorial we see that $\rho\colon L \to L$ has the same properties, i.e. $\rho\colon L \to L$ does not fix a single simplex of L and it satisfies $\rho \circ \rho = \mathrm{id}_L$.

(3) We denote by S_j the set of j-simplices of L. It follows immediately from (2) that there exists a subset T_j of S_j with the following properties:

$$\text{(i)} \qquad T_j \cap \rho(T_j) \;=\; \varnothing \qquad\qquad \text{and} \qquad\qquad \text{(ii)} \qquad T_j \cup \rho(T_j) \;=\; S_j.$$

We equip each $t \in T_j$ with some order and we use these orders to view T_j as a subset of $\mathrm{C}_j^{\mathrm{simp}}(L)$. It follows from (ii) and Lemma 95.5 that $T_j \cup \rho(T_j)$ is a basis of $\mathrm{C}_j^{\mathrm{simp}}(L)$.

Now we apply Lemma 97.18 to $V = \mathrm{C}_j^{\mathrm{simp}}(L)$, $T = T_j$, $\rho = \rho_*$ and $\varphi = \varphi$. It follows from (1), (2) and (3) that all the hypotheses of Lemma 97.18 are satisfied. Thus we obtain that the trace of $u_j \circ \varphi_*$ is indeed even. ∎

The argument for deducing the Borsuk-Ulam Theorem 39.8 from Proposition 97.17 is almost identical to the argument on page 1957 for deducing the Borsuk-Ulam Theorem 39.8 from the Odd Degree Proposition 91.3.

Proof of the Borsuk-Ulam Theorem 39.8. Let $g\colon S^n \to \mathbb{R}^n$ be a map. Suppose there exists no pair of antipodal points x and $-x$ in S^n with $g(x) = g(-x)$. Put differently, suppose that for any $x \in S^n$ we have $g(x) \neq g(-x)$. We consider the composition of the following two maps

$$\begin{array}{rccccl} \Phi\colon S^{n-1} \times [0,1] & \to & & S^n_{\geq 0} & \to & S^{n-1}. \\ (y,t) & \mapsto & (y \cdot \sqrt{1-t^2}, t) & & & \\ & & & x & \mapsto & \underbrace{\frac{g(x)-g(-x)}{\|g(x)-g(-x)\|}}_{\text{defined since } g(x) \neq g(-x)} \end{array}$$

Now we see that

$$\begin{array}{ccccccc} \text{even number} & \overset{\downarrow}{=} & \Lambda(\Phi_0) & = & \Lambda(\Phi_1) & = & \Lambda(\text{constant map}) \;=\; 1. \\ & & \text{by Lemma 97.7 since } \Phi_0 \text{ and } \Phi_1 \text{ are homotopic} & & \text{by definition of } \Phi_1 & & \text{Lemma 97.6 (3)} \end{array}$$

by definition we have $\Phi(-x,0) = -\Phi(x,0)$, thus it follows from Proposition 97.17 that $\Lambda(\Phi_0)$ is even

We have thus obtained a contradiction. ∎

Exercises for Chapter 97.

Exercise 97.1. Let X be a finite simplicial complex, let $\varphi\colon X \to X$ be a map and let $\mathbb{F}$ be a field. We denote by $\gamma\colon \mathbb{Z} \to \mathbb{F}$ the ring homomorphism that is given by $\gamma(n) = n \cdot 1_{\mathbb{F}}$. Show that

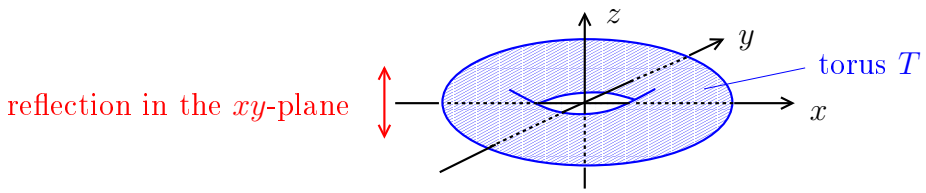
$$\Lambda(X, \mathbb{F}) \;=\; \gamma(\Lambda(X)).$$

Hint. Use Exercise 89.3.

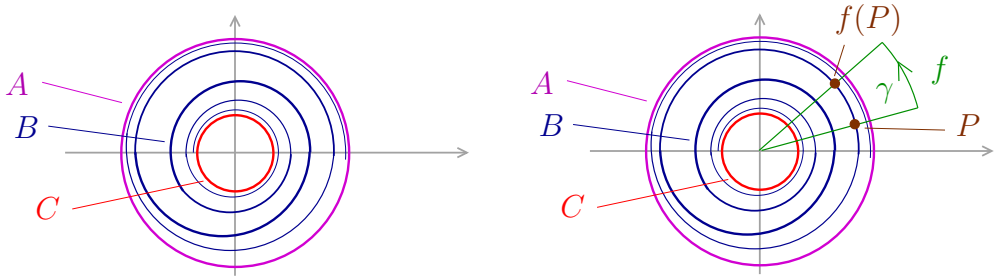
Exercise 97.2. Let $n \in \mathbb{N}$. Show that there exists a compact smooth manifold M and a map $\varphi\colon M \to M$ with Lefschetz number $\Lambda(\varphi) = n$ such that φ has precisely one fixed point.

Exercise 97.3. Let $n \in \mathbb{N}$. Does the real projective space $\mathbb{R}\mathrm{P}^{2n+1}$ admit a self-map without fixed points?

Exercise 97.4. We consider the torus $T := S^1 \times S^1$ embedded in $\mathbb{R}^3$ as shown in the figure below. Let $f\colon T \to T$ be the map that is given by reflection in the xy-plane. Is f homotopic to a map without fixed points?



Exercise 97.5. We consider the topological space $X = A \cup B \cup C$ that is shown in the figure below. Here A is a circle of radius 3, C is a circle of radius 1 around the origin and B is the image of an embedding $\mathbb{R} \to \mathbb{R}^2$ which spirals towards A and C.



(a) Give a rigorous description of X.
(b) Show that X is compact.
(c) Compute the homology groups of X.

Fix some "angle" $\gamma \in (0, \pi)$. Let $f\colon X \to X$ be the map that is given by rotation by the angle γ on A and C and which is given by sending a point $s \cdot \exp(\mathrm{i}\varphi)$ on B to the "next" point on B of the form $t \cdot \exp(\mathrm{i}(\varphi + \gamma))$ with $s > t$.

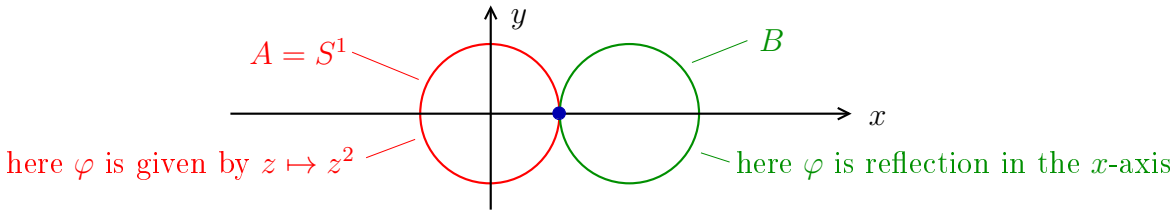
(d) Show that f is continuous.
(e) Show that f has no fixed points.
(f) Show that the Lefschetz number $\Lambda(f) \neq 0$.

This long but fun exercise shows that in the formulation of the Lefschetz Fixed Point Theorems it does not suffice to work with compact spaces that are say $\mathbb{Q}$-homology finite. We need the topological space to be "sufficiently nice".

Exercise 97.6. We consider the topological space $X = A \cup B$ that is shown in the figure below. Let $\varphi\colon X \to X$ be the map that is given by $z \mapsto z^2$ on A and that is given by reflection along the x-axis on B.

(a) Show that the Lefschetz number $\Lambda(\varphi, \mathbb{F}) = 0$ for every field $\mathbb{F}$.
(b) Show that any map ψ that is homotopic to φ actually has a fixed point.

Remark. It is easy to see that X admits a simplicial structure. Thus the exercise shows that the converse to the Lefschetz Fixed Point Theorem 97.9 does not hold.



Exercise 97.7.

(1) Let $A \in \mathrm{GL}(n, \mathbb{C})$ be a matrix. We denote by $\varphi_A\colon \mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^{n-1}$ the induced map given by $\varphi_A([P]) := [A \cdot P]$.
   (a) Show that the Lefschetz number $\Lambda(\varphi_A) = n$.
   Hint. Use the Matrix Group Path-Component Proposition 2.37 (4). The super conscientious reader might also want to consult the discussion on page 426 to show that a certain map is indeed continuous.
   (b) Use (a) to show that A has an eigenvalue.
(2) What does the approach in (1) say about the existence of real eigenvalues of real matrices?

Exercise 97.8. Let $n \in \mathbb{N}_0$ be *even*.

(a) Show that the complex projective space $\mathbb{C}\mathrm{P}^n$ admits a self-map without fixed points.
Remark. In Exercise 121.10 we will prove the much more interesting converse, namely we will show that for every *odd* n every self-map of $\mathbb{C}\mathrm{P}^n$ has a fixed point.
(b) Does there exist a nowhere-vanishing vector field on $\mathbb{C}\mathrm{P}^n$?

Exercise 97.9. For which $n \in \mathbb{N}$ does the real projective space $\mathbb{R}\mathrm{P}^n$ admit a nowhere-vanishing vector field?

Exercise 97.10. Does the Klein bottle K admit two vector fields v and w such that at every point $P \in K$ the two vectors $v(P), w(P) \in \mathrm{T}_P K$ are linearly independent?

Exercise 97.11. In the figure on page 2124 we saw that the surface Σ of genus 2 supports a vector field with two zeros. What does such a vector field look like if we consider Σ as a submanifold of $\mathbb{R}^3$ in the usual way?

Exercise 97.12. Does the surface of genus 2 admit a vector field with a single zero?

Exercise 97.13. Let $m \in \mathbb{Z}$ be even.

(a) Show that there exists a map $f\colon S^1 \to S^1$ with Lefschetz number $\Lambda(f) = m$ and such that for every $x \in S^1$ we have $f(-x) = -f(x)$.
(b) Now prove (a) for higher-dimensional spheres.

Exercise 97.14. Let X be a finite CW-complex and let $\varphi\colon X \to X$ be a self-map. Furthermore let $p\colon \widetilde{X} \to X$ be a finite covering. By the Lift-to-Universal Coverings Proposition 61.13 we know that there exists a map $\widetilde{\varphi}\colon \widetilde{X} \to \widetilde{X}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & X. \end{array}$$

Show that

$$\Lambda(\widetilde{\varphi}\colon \widetilde{X} \to \widetilde{X}) \;=\; [\widetilde{X} : X] \cdot \Lambda(\varphi\colon X \to X).$$

Hint. Use cellular homology and use the Hopf Trace Formula, i.e. use Theorem 97.12.

Exercise 97.15. Let M be a compact topological manifold. Suppose we are given two compact codimension-zero submanifolds A and B such that $A \cap B = \partial A = \partial B$. Furthermore let $f\colon M \to M$ be a map that restricts to self-maps of A and B. Show that

$$\Lambda(f\colon M \to M) \;=\; \Lambda(f\colon A \to A) + \Lambda(f\colon B \to B) - \Lambda(f\colon A \cap B \to A \cap B).$$

Hint. Use the Mayer–Vietoris Theorem 78.9 for Manifolds and use the Hopf Trace Formula, i.e. use Theorem 97.12.
Remark. In the Topological Manifolds-Invariants Proposition 104.14 we will see that the homology groups of A, B and M are finitely generated and that there exists an n such that they vanish in dimension $> n$.

Exercise 97.16. Let M be a smooth manifold and let v be a vector field on M. Show that the zero set $\{x \in M \mid v(x) = 0\}$ is a closed subset of M.
Hint. Use Lemma 1.8 (2b).

Exercise 97.17. A *Lie group* is a smooth manifold X with empty boundary together with a group structure such that the two maps

$$\begin{array}{ccc} X \times X & \to & X \\ (x, y) & \mapsto & x \cdot y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X & \to & X \\ x & \mapsto & x^{-1} \end{array}$$

are smooth. For example it is not difficult to show that the matrix groups $\mathrm{SO}(n)$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ with the usual multiplication are Lie groups. Show that the Euler characteristic of any closed Lie group of dimension ≥ 1 is zero.

Exercise 97.18. For which compact connected 2-dimensional smooth manifolds M can you find a nowhere-vanishing vector field v with the additional property that for each $P \in \partial M$ we have $v(P) \in \mathrm{T}_P M$?
Remark. We had posed this question already as Exercise 23.11, but now you actually have the tools to give a complete answer.

98. Homology groups of pseudomanifolds and smooth manifolds

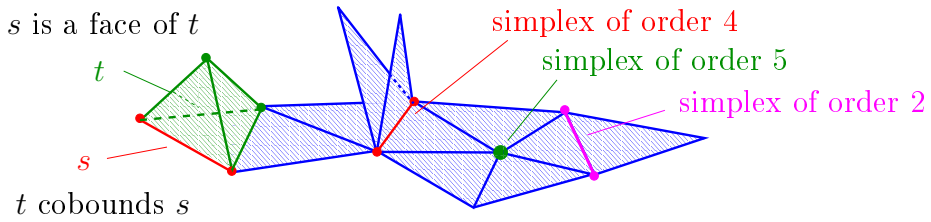
Let M be an n-dimensional smooth manifold. By the Smooth Manifold-Simplicial Structure Theorem 96.2 we know that M admits a smooth simplicial structure. In the Smooth Manifolds-Invariants Proposition 96.6 we used this fact to show that for every $k > n$ we have $\mathrm{H}_k(M) = 0$. In this chapter we will squeeze more information out of the existence of a simplicial structure. In particular this will allow us to calculate $\mathrm{H}_n(M;\mathbb{Z})$ and it will allow us to make some statements regarding $\mathrm{H}_{n-1}(M;\mathbb{Z})$.

98.1. **The top-dimensional homology group of a pseudomanifold.** In this chapter we will see that any simplicial structure of an n-dimensional topological manifold has several special properties. This will lead us to the notion of an *n-dimensional pseudomanifold* which can be viewed as a simplicial analogue of a manifold. We will then prove several statements regarding the simplicial homology groups of an n-dimensional pseudomanifold. In the subsequent section we will then use the calculations from this chapter to prove the promised results on homology groups of smooth manifolds.
First we recall and introduce some definitions.

Definition. Let $K = (V, S)$ be an abstract simplicial complex.

(1) Let t be a simplex of K. As on page 1979 we define a face of t to be a simplex s with $s \subset t$. In this setting we also say that t cobounds s.
(2) Let $s \in S$ be a k-simplex of K. We say s has order r, if there exist precisely r simplices of dimension $k+1$ which contain s as a face.

As before we sometimes use the same terms when we talk about topological realizations and simplicial complexes.

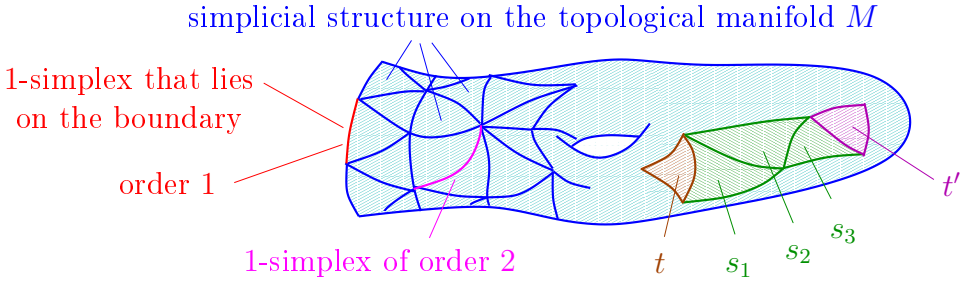


Proposition 98.1. (Manifold-Simplex Proposition) Let M be an n-dimensional topological manifold that is equipped with a simplicial structure.

(1) There is no simplex of dimension $> n$ and every simplex is the face of an n-simplex.
(2) The simplicial structure has finitely many simplices if and only if it has finitely many n-simplices.
(3) The simplicial structure on M restricts to a simplicial structure on the $(n-1)$-dimensional topological manifold ∂M. If M is a smooth manifold and if we are given a smooth simplicial structure, then the restriction of the simplicial structure to the smooth manifold ∂M is also a smooth simplicial structure.
(4) Let s be an $(n-1)$-simplex of M. Then

$$\text{order of } s \;=\; \begin{cases} 1, & \text{if } s \text{ lies on the boundary,} \\ 2, & \text{otherwise.} \end{cases}$$

(5) If M is connected, then given any two n-simplices t and t' there exists a sequence $t = s_0, s_1, \ldots, s_k = t'$ of n-simplices such that any two consecutive simplices have a common $(n-1)$-dimensional face.



Remark.

(1) In Lemma 96.1 we gave a different proof why the maximal dimension of a simplex of a smooth simplicial structure of an n-dimensional smooth manifold equals n.

(2) A rather different proof of the Manifold-Simplex Proposition 98.1 (5) is given in [**Mun1984**, Corollary 65.2].

The key idea behind the proof of the Manifold-Simplex Proposition 98.1 is to squeeze the maximal amount of information out of the local homology groups. To do so we first recall the following definition from page 1655.

Definition. Given a topological space X and given a point $P \in X$ the relative homology group $\mathrm{H}_j(X, X \setminus \{P\})$ is called the n-th local homology group of X at the point P.

We recall the calculation of the local homology groups of a topological manifold.

Lemma 75.1. (Manifold-Local Homology Lemma) Let M be an n-dimensional topological manifold and let $P \in M$. For every $j \in \mathbb{N}_0$ we have

$$\mathrm{H}_j(M, M \setminus \{P\}) \cong \begin{cases} \mathbb{Z}, & \text{if } P \in M \setminus \partial M \text{ and } j = n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since in a minute we will calculate a different local homology group it is perhaps helpful to remind us of the proof of the lemma. For simplicity we only consider the case that $P \in M \setminus \partial M$. In this case we can pick a chart $\Phi\colon U \to B^n$ for P with $\Phi(P) = 0$. We then have

$$\mathrm{H}_j(M, M\setminus\{P\}) \underset{\text{Excision Theorem 74.18}}{\leftarrow} \mathrm{H}_j(U, U\setminus\{P\}) \underset{\substack{\text{isomorphism by the long exact sequence in reduced homology of the pair } (B^n, B^n\setminus\{0\}) \text{ and the fact that } \widetilde{\mathrm{H}}_*(B^n) = 0 \text{ by page 1612}}}{\overset{\Phi_*}{\underset{\cong}{\longrightarrow}}} \mathrm{H}_j(B^n, B^n\setminus\{0\}) \underset{\uparrow}{\overset{\partial}{\longrightarrow}} \widetilde{\mathrm{H}}_{j-1}(B^n\setminus\{0\}) \underset{\substack{\text{isomorphism since } S^{n-1} \text{ is homotopy equivalent to } B^n\setminus\{0\}}}{\leftarrow} \widetilde{\mathrm{H}}_{j-1}(S^{n-1}).$$

The statement now follows from the calculation of the reduced homology groups of spheres given in Proposition 74.4. ∎

The next proposition calculates the local homology groups of a simplicial complex. This calculation is also interesting in its own right.

Proposition 98.2. Let $K = (V, S)$ be an abstract simplicial complex and let $P \in |K|$. By the Open Simplex Lemma 93.11 there exists a unique simplex $s \in S$ such that P is a point in the open simplex $\langle s \rangle$. We set $m := \dim(s)$. If $\mathrm{Lk}(K, s)$ is a finite simplicial complex, then for every $j \in \mathbb{N}_0$ we have[1202][1203]

$$\mathrm{H}_j(|K|, |K| \setminus \{P\}) \;\cong\; \widetilde{\mathrm{H}}_{j-1-m}(\mathrm{Lk}(K, s)).$$

Proof. We perform the following calculation:

$$\begin{array}{ccccc}
\mathrm{H}_j(|K|, |K| \setminus \{P\}) & \overset{\cong}{\leftarrow} & \mathrm{H}_j(\mathrm{St}(s), \mathrm{St}(s) \setminus \{P\}) & \overset{\cong}{\rightarrow} & \widetilde{\mathrm{H}}_{j-1}(\mathrm{St}(s) \setminus \{P\}) \\
& & \uparrow & & \uparrow \\
& & \widetilde{\mathrm{H}}_{j-1}(S^{m-1} * \mathrm{Lk}(s)) & \cong & \widetilde{\mathrm{H}}_{j-1-m}(\mathrm{Lk}(s)) \\
& & \uparrow & & \uparrow \\
& & \widetilde{\mathrm{H}}_{j-1}(\partial s * \mathrm{Lk}(s)) & \cong & \widetilde{\mathrm{H}}_{j-1}(\partial s * \mathrm{Lk}(s))
\end{array}$$

by the long exact sequence in reduced homology of the pair $(\mathrm{St}(s), \mathrm{St}(s) \setminus \{P\})$ and the observation, see the Link Star Lemma 94.5 (1f), that $\mathrm{St}(s)$ is contractible, which implies that its reduced homology vanishes

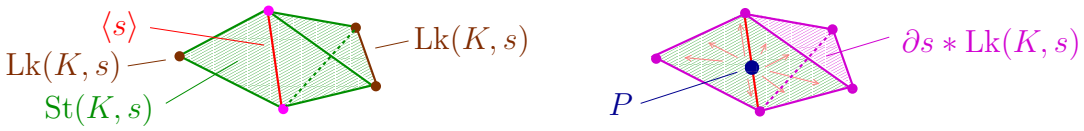
by the Link Star Lemma 94.5 (1d) we know that $\mathrm{St}(s)$ is a neighborhood, thus we can apply the Excision Theorem 74.18 (2)

by the Link Star Lemma 94.5 (1g) together with Homotopy Equivalence-H_*-Corollary 73.9

since s is an m-simplex we know by Simplex-Embedding Lemma 93.5 and the standard Simplex Homeomorphism Lemma 72.1 (3) that ∂s is homeomorphic to S^{m-1}

by Corollary 76.8 we have $\widetilde{\mathrm{H}}_{l+m}(S^{m-1} * X) \cong \widetilde{\mathrm{H}}_l(X)$ whenever X is compact and Hausdorff, we can apply this result since $\mathrm{Lk}(s)$ is by hypothesis finite

■



Now we can finally provide the proof of the Manifold-Simplex Proposition 98.1.

Proof of the Manifold-Simplex Proposition 98.1. Basically by definition of a simplicial structure it suffices to consider the case that the n-dimensional topological manifold M is the topological realization $M = |K|$ of some abstract simplicial complex $K = (V, S)$. Note that by the Topological Manifolds-Local Properties Lemma 18.8 we know that the topological manifold $M = |K|$ is regionally compact. Thus it follows from Proposition 94.6 that the abstract simplicial complex K is locally finite and that the link of any vertex is a finite simplicial complex. This implies in particular that we can apply Proposition 98.2 without worries.

(1) It suffices to prove the following claim.

Claim. Given any simplex s, the maximal dimension of a simplex that contains s is precisely given by n.

Proof. Let s be a simplex of K. As we mentioned in the beginning of the proof, we know that K is locally finite. This implies that there exists a simplex t of maximal dimension that contains s. We set $m := \dim(t)$. Now let $P \in \langle t \rangle$ be a point. We

[1202] It can happen that $\mathrm{Lk}(K, s) = \varnothing$. Recall that by definition we have $\widetilde{\mathrm{H}}_{-1}(\varnothing) \cong \mathbb{Z}$.

[1203] I do not know whether the statement holds in general without the condition that $\mathrm{Lk}(K, s)$ is finite.

calculate that

$$\begin{array}{ccccc} \mathbb{Z} & \cong & \widetilde{\mathrm{H}}_{-1}(\mathrm{Lk}(K,t)) & \cong & \mathrm{H}_m(|K|,|K|\backslash\{P\}) \\ & & \uparrow & & \downarrow \\ & & \text{since there are no simplices of dimension} > m \text{ which contain } t \text{ we see that } \mathrm{Lk}(K,t)=\varnothing & & \text{Manifold-Local Homology Lemma 75.1 applied to } |K|=M \end{array} \quad \cong \begin{cases} \mathbb{Z}, & \text{if } P \in M\backslash\partial M \text{ and } m=n, \\ 0, & \text{otherwise.} \end{cases}$$

Comparing the left-hand side with the right-hand side shows that $m = n$. ⊞

(2) This statement is an almost immediate consequence of (1). Indeed, we only have to show that if K has finitely many n-simplices, then it has only finitely many simplices. But every n-simplex has only finitely many faces and by (1) we see that every simplex of K is the face of an n-simplex. Thus we are done.

(3) First note that if we are given a smooth simplicial structure on a smooth manifold that restricts to a simplicial structure on ∂M, then it follows easily from the definitions that this simplicial structure on ∂M is actually a smooth simplicial structure. Thus, to prove (3), it remains to show that the simplicial structure on $|K| = M$ restricts to a simplicial structure on the topological manifold ∂M.

If M is a smooth manifold, then we know by the Smooth Manifold Boundary Proposition 19.26 (3a) that ∂M is a closed subset of M. In the general case that M is a topological manifold the same statement follows from the Topological Manifold Boundary Proposition 75.2.

Now that we know that ∂M is a closed subset of M we obtain the desired statement from Lemma 93.13 together with the following claim.

Claim. Let s be an m-simplex of K. Either $\langle s\rangle \cap \partial M = \varnothing$ or $\langle s\rangle \subset \partial M$.

Proof. Let s be a simplex. For any point $P \in \langle s\rangle$ we perform almost the same calculation as in (1):

$$\begin{array}{ccccc} \widetilde{\mathrm{H}}_{n-1-m}(\mathrm{Lk}(K,s)) & \cong & \mathrm{H}_n(|K|,|K|\backslash\{P\}) \\ & & \downarrow \\ & & \text{Manifold-Local Homology Lemma 75.1 applied to } |K|=M \end{array} \quad \cong \begin{cases} \mathbb{Z}, & \text{if } P \in M\backslash\partial M, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from this calculation that $\langle s\rangle \cap \partial M = \varnothing$ if $\widetilde{\mathrm{H}}_{n-1-m}(\mathrm{Lk}(K,s)) \cong \mathbb{Z}$ and that $\langle s\rangle \subset \partial M$ if $\widetilde{\mathrm{H}}_{n-1-m}(\mathrm{Lk}(K,s)) = 0$. ⊞

(4) Let s be an $(n-1)$-simplex of M. We denote by r the order of s. We pick a point $P \in \langle s\rangle$. Similar to (1) we calculate that

$$\begin{array}{ccccc} \mathbb{Z}^{r-1} & \cong & \widetilde{\mathrm{H}}_0(\mathrm{Lk}(K,s)) & \cong & \mathrm{H}_n(|K|,|K|\setminus\{P\}) \\ & & \uparrow & & \downarrow \\ & & \text{since } s \text{ is an } (n-1)\text{-simplex and since there are no simplices of dimension} > n \text{ we see that } \mathrm{Lk}(K,s) & & \text{Lemma 75.1 applied to } |K|=M \\ & & \text{consists of } r \text{ points, the isomorphism now follows from the Reduced Homology Lemma 74.1 (5)} & & \end{array} \quad \cong \begin{cases} \mathbb{Z}, & \text{if } P \in M\setminus\partial M, \\ 0, & \text{otherwise.} \end{cases}$$

Comparing the left-hand side with the right-hand side gives us the desired statement.

(5) Now we assume that M is in fact connected. We say that two n-simplices t and t' of K are *equivalent* if there exists a sequence $t = s_0, s_1, \ldots, s_k = t'$ of n-simplices such

that any two consecutive simplices have a common $(n-1)$-dimensional face. Recall that our task is to show that any two n-simplices are equivalent in the above sense.

The key to doing so is the following claim:

Claim. Let t and t' be two n-simplices. If $t \cap t' \neq \varnothing$, then t and t' are actually equivalent in the above sense.

Proof. Given $m \in \{0, \dots, n-1\}$ we say that two n-simplices x and x' are *m-equivalent* if there exists a sequence $x = s_0, s_1, \dots, s_k = x'$ of n-simplices such that any two consecutive simplices have a common face of dimension $\geq m$. It suffices to show that if we are given $m \in \{0, \dots, n-2\}$ such that any two n-simplices in K are m-equivalent, then any two n-simplices in K are also $(m+1)$-equivalent. In fact evidently it suffices to show that if we are given two n-simplices x and x' which have an m-simplex s in common, then x and x' are $(m+1)$-equivalent. To do so we first note that since $\dim(x) > \dim(s)$ we can pick a vertex w of $x \cap \mathrm{Lk}(K, s)$ and similarly we can pick a vertex w' of $x' \cap \mathrm{Lk}(K, s)$.

Now we want to show that $\mathrm{Lk}(K, s)$ is connected. To do so we pick a point $P \in \langle s \rangle$ and we perform basically the same calculation as in the previous arguments:

$$\widetilde{\mathrm{H}}_0(\mathrm{Lk}(K,s)) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_{m+1}(|K|, |K| \setminus \{P\}) = \widetilde{\mathrm{H}}_{m+1}(M, M \setminus \{P\}) \underset{\uparrow}{=} 0.$$

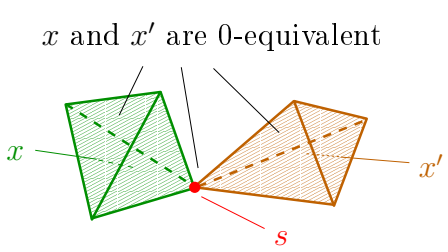
Proposition 98.2 applied to m-simplex s and $j = m+1$

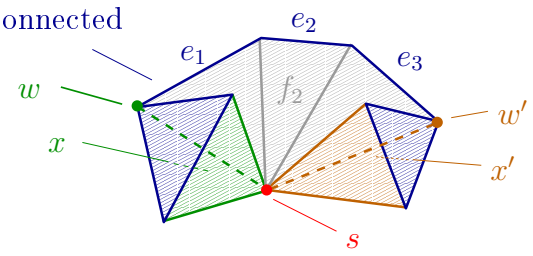
by the Manifold-Local Homology Lemma 75.1 since $m \leq n-2$

It follows from this calculation together with the Reduced Homology Lemma 74.1 and Exercise 93.12 that the abstract simplicial complex $\mathrm{Lk}(K, s)$ is in fact connected in the sense of the definition on page 2013. This means in our setting that there exist 1-simplices $e_1, \dots, e_k \in S$ of $\mathrm{Lk}(K, s)$ such that the following holds:

($*$) We have $w \in e_1$, for every $i \in \{1, \dots, k-1\}$ we have $e_i \cap e_{i+1} \neq \varnothing$ and we have $w' \in e_k$.

Let $i \in \{1, \dots, k\}$. Since e_i is a simplex in $\mathrm{Lk}(K, s)$ there exists a simplex f_i of K with $e_i \cup s \subset f_i$. By (1) we can in fact find an n-simplex $\widetilde{f}_i$ of K with $e_i \cup s \subset f_i \subset \widetilde{f}_i$. We set $\widetilde{f}_0 := x$ and $\widetilde{f}_{k+1} := x'$. Note that it follows from ($*$) that any two consecutive $\widetilde{f}_i$ and $\widetilde{f}_{i+1}$ have s and a vertex of $\mathrm{Lk}(K, s)$ in common, i.e. they have an $(m+1)$-simplex in common. This shows that x and x' are in fact $(m+1)$-connected. ⊞





We pick an equivalence class of n-simplices of K and we denote by J the subcomplex of K given by the union of the n-simplices in said equivalence class. Since J is a subcomplex we know by Lemma 93.8 that $|J|$ is a closed subset of $|K| = M$. Furthermore it follows from the claim together with Exercise 93.11 that $|J|$ is also

an open subset of $|K| = M$. By hypothesis $|K| = M$ is connected. Thus we see that $K = J$, which implies that any two n-simplices in K are in fact equivalent. ■

Our main goal is to say something meaningful about homology groups of smooth manifolds. The idea is to use the Manifold-Simplex Proposition 98.1 and the fact that we can calculate singular homology groups via simplicial homology groups. Since we will only use the information given by the Manifold-Simplex Proposition 98.1 it is convenient to introduce the following definition.

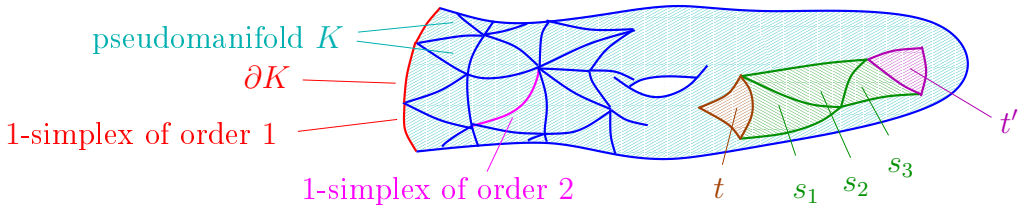
Definition. Let $n \in \mathbb{N}_0$. An n-dimensional pseudomanifold is a non-empty abstract simplicial complex K such the following conditions are satisfied:

(1) Every simplex of K is a face of an n-simplex of K.[1204]
(2) Every $(n-1)$-simplex has order one or two.
(3) Given any two n-simplices t and t' there exists a sequence $t = s_0, s_1, \ldots, s_k = t'$ of n-simplices such that any two consecutive simplices have a common $(n-1)$-dimensional face.

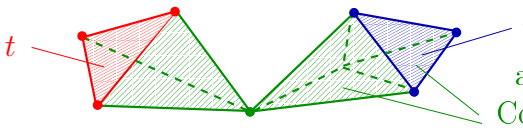
Given a pseudomanifold K we define ∂K to be the subcomplex that is given by the union of all $(n-1)$-simplices of order 1. We say a pseudomanifold K is closed if $\partial K = \varnothing$ and if K is finite.

Examples.

(1) In the figure below we illustrate the definition of a 2-dimensional pseudomanifold.



(2) In the figure below we show an abstract simplicial complex that satisfies (1) and (2) but that does not satisfy (3). This example shows that Condition (3) is stronger than requiring ordinary connectedness.



abstract simplicial complex that satisfies Conditions (1) and (2) of a pseudomanifold but which does not satisfy Condition (3)

Remark.

(1) The definition of a pseudomanifold we give seems to be the one which is most frequently used, it goes back to work of Luitzen Brouwer [**Brou1976**, p. 477] in 1912.
(2) the Manifold-Simplex Proposition 98.1 says that if we are given a simplicial structure $(K, \Theta\colon |K| \to M)$ of a connected non-empty n-dimensional topological manifold M, then K is an n-dimensional pseudomanifold.
(3) In Exercise 98.4 we will show that if K is an n-dimensional pseudomanifold with $n \in \{1, 2\}$, then $|K|$ is a closed n-dimensional smooth manifold.

[1204] Note that this condition implies that K is necessarily n-dimensional.

(4) Let K be a finite n-dimensional pseudomanifold. As usual we denote by $\Sigma(K)$ the suspension as defined on page 2003. As we will see in Exercise 98.1, it follows very easily from the definitions that $\Sigma(K)$ is an $(n+1)$-dimensional pseudomanifold with $\partial\Sigma(K) = \Sigma(\partial K)$.

Now assume that the fundamental group $\pi_1(|K|)$ of the topological realization $|K|$ is non-trivial. It follows from the Simplicial Cone-Suspension Corollary 93.21 together with Lemma 96.19 that the topological realization $|\Sigma(K)|$ of the suspension $\Sigma(K)$ is not a topological manifold. This shows that for $n \geq 3$ not all n-dimensional pseudomanifolds arise from topological manifolds.

(5) In the literature there exist several, not always entirely equivalent, notions of a pseudomanifold. In general, as we just saw in (3), the topological realizations of pseudomanifolds are not topological manifolds. But many tools and results on topological manifolds can be generalized to the context of (topological realization of) pseudomanifolds, see e.g. [**KiW2006**, Chapter 4]. Also note that in the literature an n-dimensional pseudomanifold is sometimes called an *n-circuit*.

Next we recall and expand a definition from page 2038.

Definition. Let $K = (V, S)$ be an abstract simplicial complex.

(1) Let $n \in \mathbb{N}_0$. An ordered n-simplex of K is an $(n+1)$-tuple $(v_0, \ldots, v_n) \in V^{n+1}$ such that $\{v_0, \ldots, v_n\} \in S$. We write $\lfloor (v_0, \ldots, v_n) \rfloor := \{v_0, \ldots, v_n\}$.

(2) An oriented n-simplex is a pair (ϵ, s) where $\epsilon \in \{\pm 1\}$ and s is an ordered n-simplex of K. As in (1) we write $\lfloor (\epsilon, s) \rfloor := \lfloor s \rfloor$ and we refer to $\lfloor (\epsilon, s) \rfloor$ as the underlying simplex. Sometimes we also say that (ϵ, s) is an orientation of the n-simplex s.

(3) We say two oriented n-simplices $(\mu, s = (v_0, \ldots, v_n))$ and $(\nu, t = (w_0, \ldots, w_n))$ are equivalent if

$$\mu \;=\; \nu \cdot \underset{\uparrow}{\operatorname{sign}}(\sigma) \qquad \text{where } \sigma \in \operatorname{Bij}\{0, \ldots, n\} \text{ with } v_i = w_{\sigma(i)} \text{ for } i = 0, \ldots, n.$$

sign of σ as defined on page 2035

It follows immediately from Lemma 95.1 that this is indeed an equivalence relation on the set of oriented n-simplices.

(4) An orientation of an n-simplex s is an equivalence class of oriented n-simplices with underlying simplex s.

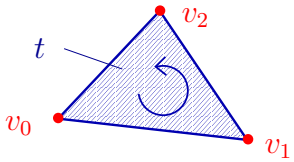
Definition. Let $K = (V, S)$ be an abstract simplicial complex and let $(\epsilon, t = (v_0, \ldots, v_n))$ be an oriented n-simplex. Let s be a codimension-one face of t. If v_i is the one vertex of t that is missing in s, then we define the induced orientation of s to be

$$((-1)^i \cdot \epsilon, (v_0, \ldots, \widehat{v_i}, \ldots, v_n)).$$

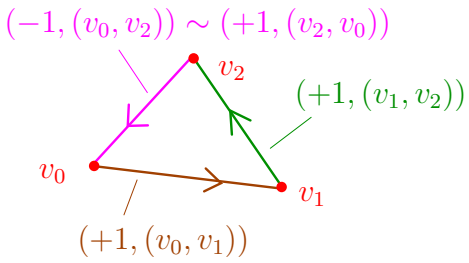
Example. In the figure below we show the oriented 2-simplex $(+1, (v_0, v_1, v_2))$ together with the orientations induced on all of the three codimension-one faces.

Lemma 98.3. Let $K = (V, S)$ be an abstract simplicial complex, let t be a simplex of K and let s be a codimension-one face of t. Any two equivalent orientations of t induce equivalent orientations of s.

Proof. This statement follows easily from the properties of the sign of a bijection, see Lemma 95.1. We leave it to the reader to fill in the details. ∎



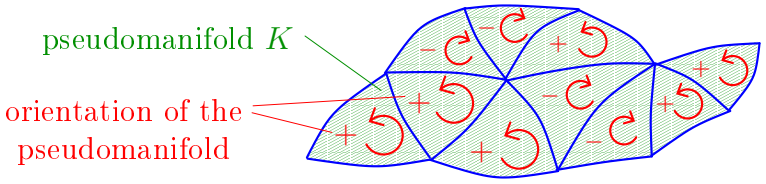
the oriented 2-simplex $(+1,(v_0,v_1,v_2))$ induces the following orientations of its codimension-one faces



Definition. Let K be an n-dimensional pseudomanifold.

(1) An orientation is a choice of an orientation for each n-simplex of K such that for any $(n-1)$-simplex s of order 2 the two cobounding n-simplices induce opposite orientations on s.

(2) We say K is orientable if K admits an orientation.



In the following we will study the simplicial homology groups of pseudomanifolds. Before we do so, let us recall the definition of simplicial homology that we gave on page 2036 and in Lemma 95.6.

Definition. Let $K=(V,S)$ be an abstract simplicial complex and let $n \in \mathbb{N}_0$. We set

$$\mathrm{C}_n^{\mathrm{simp}}(K) := \left\{ \begin{array}{c} \text{free abelian group generated} \\ \text{by all ordered } n\text{-simplices of } K \end{array} \right\} \Big/ (v_0,\ldots,v_n) \sim \mathrm{sign}(\sigma)\cdot(v_{\sigma(0)},\ldots,v_{\sigma(n)}).$$

In the discussion below, we will often not distinguish in the notation between an ordered simplex and the element it represents in $\mathrm{C}_k^{\mathrm{simp}}(K)$. The boundary map is given by

$$\begin{array}{rcl} \partial_n \colon \mathrm{C}_n^{\mathrm{simp}}(K) & \to & \mathrm{C}_{n-1}^{\mathrm{simp}}(K) \\ (v_0,\ldots,v_n) & \mapsto & \sum\limits_{i=0}^{n} (-1)^i \cdot (v_0,\ldots,v_{i-1},\widehat{v_i},v_{i+1},\ldots,v_n). \end{array}$$

Now we can provide the promised theorem.

Theorem 98.4. (Pseudomanifold Homology Theorem) Let K be an n-dimensional pseudomanifold. The following statements hold:

(1) We have an isomorphism as follows:

$$\mathrm{H}_n^{\mathrm{simp}}(K) \cong \left\{ \begin{array}{ll} \mathbb{Z}, & \text{if } K \text{ is closed and orientable,} \\ 0, & \text{otherwise.} \end{array} \right.$$

Furthermore, if K is closed and orientable, and if $\{(\epsilon_w, Y_w)\}_{w\in W}$ is an orientation of K (here W denotes the set of n-simplices of K), then a generator of $\mathrm{H}_n^{\mathrm{simp}}(K) \cong \mathbb{Z}$ is represented by the simplicial cycle $\sum\limits_{w\in W} \epsilon_w \cdot Y_w \in \mathrm{C}_n^{\mathrm{simp}}(K)$.

(2) If K is finite, then we also have an isomorphism as follows:

$$\mathrm{H}_{n-1}^{\mathrm{simp}}(K) \;\cong\; \begin{cases} \text{free abelian group}, & \text{if } K \text{ is orientable or if } \partial K \neq \varnothing, \\ \text{free abelian group} \oplus \mathbb{Z}_2, & \text{if } K \text{ is non-orientable and } \partial K = \varnothing. \end{cases}$$

(3) If K is finite, then

$$\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, \text{ if } K \text{ is closed, regardless whether or not } K \text{ is orientable,} \\ 0, \quad \text{otherwise.} \end{cases}$$

Furthermore, if K is in fact closed and if $Y_1, \ldots, Y_m$ is a set of ordered n-simplices of K such that for each n-simplex s there exists a unique $i \in \{1, \ldots, m\}$ with $s = \lfloor Y_i \rfloor$, then the unique non-zero element of $\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2) \cong \mathbb{F}_2$ is represented by the simplicial cycle $Y_1 \otimes 1 + \cdots + Y_m \otimes 1 \in \mathrm{C}_n^{\mathrm{simp}}(K;\mathbb{F}_2) = \mathrm{C}_n^{\mathrm{simp}}(K) \otimes \mathbb{F}_2$.

Theorem 98.4 leads us to the following definition.

Definition. Let K be a closed n-dimensional pseudomanifold.

(1) If K is equipped with an orientation $\{(\epsilon_w, Y_w)\}_{w \in W}$, then we refer to the simplicial homology class

$$[K] \;:=\; \Big[\sum_{w \in W} \epsilon_w \cdot Y_w \Big] \;\in\; \mathrm{H}_n^{\mathrm{simp}}(K)$$

as the fundamental class of K.

(2) We refer to the unique non-zero element in $\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2) \cong \mathbb{F}_2$ as the $\mathbb{F}_2$-fundamental class $[K]_{\mathbb{F}_2}$.

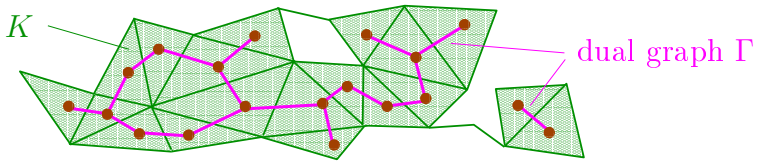
For clarity we break the proof of Theorem 98.4 into three parts. For the proof of the first part it is convenient to introduce the following definitions which we will use on a few other occasions.

Definition. Let $K = (V, S)$ be an n-dimensional abstract simplicial complex such that every $(n-1)$-simplex has order ≤ 2.

(1) We denote by W the set of n-simplices of K.

(2) We denote by E_2 the set of $(n-1)$-simplices of order 2.

(3) Let $\Gamma = (W, E_2, \varphi)$ be the undirected abstract graph with vertex set W, with edge set E_2 and where $\varphi\colon E_2 \to \mathcal{P}(W)$ is the map that associates to each edge $e \in E_2$ the set $\{w \in W \,|\, e \text{ is a face of } w\}$. Note that for $e \in E_2$ the set $\varphi(e)$ has precisely two elements. This means that Γ is indeed an undirected abstract graph in the sense of the definition on page 313. We call Γ the dual graph of the abstract simplicial complex K. We refer to the figure below for an illustration.



Proof of Theorem 98.4 (1). Let K be an n-dimensional pseudomanifold. Our goal is to calculate $\mathrm{H}_n^{\mathrm{simp}}(K)$. We start out with the following easy but useful observation which

we will use throughout this proof without referring to it again explicitly:

$$\mathrm{H}_n^{\mathrm{simp}}(K) \underset{\uparrow}{=} \ker\big(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\big).$$

K has no simplices of dimension $n+1$, hence $\mathrm{C}_{n+1}^{\mathrm{simp}}(K) = 0$

To determine the right hand side of the above we need to introduce some notation.

(1) We denote by E the set of $(n-1)$-simplices of K. Furthermore we denote by E_1 the set of $(n-1)$-simplices of order 1 and similarly we denote by E_2 the set of $(n-1)$-simplices of order 2. Note that it follows from the hypothesis that K is an n-dimensional pseudomanifold that $E = E_1 \sqcup E_2$.

(2) We denote by W the set of n-simplices of K.

(3) Let $\Gamma = (W, E_2, \varphi)$ be the dual graph of K. Note that part (3) of the definition of a pseudomanifold implies that the undirected abstract graph Γ is connected in the sense of the definition on page 315.

(4) For each $(n-1)$-simplex $e \in E$ we pick, once and for all, an ordered $(n-1)$-simplex X_e with $\lfloor X_e \rfloor = e$. We use Lemma 95.5 to view $\{X_e\}_{e\in E}$ as a basis for $\mathrm{C}_{n-1}^{\mathrm{simp}}(K)$.

(5) We consider the two obvious projections

$$p_1\colon \overbrace{\bigoplus_{e\in E} \mathbb{Z}\cdot X_e}^{=\mathrm{C}_{n-1}^{\mathrm{simp}}(K)} \to \bigoplus_{e\in E_1} \mathbb{Z}\cdot X_e \quad \text{and} \quad p_2\colon \overbrace{\bigoplus_{e\in E} \mathbb{Z}\cdot X_e}^{=\mathrm{C}_{n-1}^{\mathrm{simp}}(K)} \to \bigoplus_{e\in E_2} \mathbb{Z}\cdot X_e$$

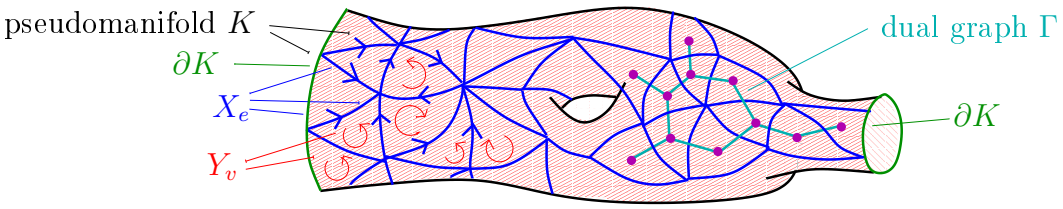
Since $E = E_1 \sqcup E_2$ we see for $s \in \mathrm{C}_{n-1}^{\mathrm{simp}}(K)$ that $s = 0$ if and only if $p_1(s) = 0$ and $p_2(s) = 0$.

Now suppose that for each $w \in W$ we are given an oriented n-simplex Y_w with $\lfloor Y_w \rfloor = w$. We introduce the following conventions.

(6) As above we use Lemma 95.5 to view $\{Y_w\}_{w\in W}$ as a basis for $\mathrm{C}_n^{\mathrm{simp}}(K)$.

(7) Given $e \in E$ and $w \in W$ we set[1205]

$$\mu_{ew} = \begin{cases} 0, & \text{if } e \text{ is not a face of } w, \\ +1, & \text{if } e \text{ is a face of } w \text{ and } [(+1, Y_w)] \text{ induces } [(+1, X_e)], \\ -1, & \text{if } e \text{ is a face of } w \text{ and } [(+1, Y_w)] \text{ induces } [(-1, X_e)]. \end{cases}$$



Claim 1. Given $\sum_{w\in W} a_w \cdot Y_w \in \mathrm{C}_n^{\mathrm{simp}}(K)$ we have

(a) $p_1\Big(\partial_n\Big(\sum_{w\in W} a_w \cdot Y_w\Big)\Big) = 0 \;\Leftrightarrow\;$ for all $w \in W$ that cobound an $e \in E_1$ we have $a_w = 0$

(b) $p_2\Big(\partial_n\Big(\sum_{w\in W} a_w \cdot Y_w\Big)\Big) = 0 \;\Leftrightarrow\;$ for all $w \neq w' \in W$ that cobound some $e \in E_2$ we have $\mu_{ew} \cdot a_w = -\mu_{ew'} \cdot a_{w'}$.

[1205] Note that if e is a face of w, then precisely one of the two cases does indeed occur.

Proof. We start out our proof of Claim 1 with the following calculation:

$$\partial_n\Big(\sum_{w\in W} a_w\cdot Y_w\Big) = \sum_{w\in W} a_w\cdot\partial_n(Y_w) = \sum_{w\in W} a_w\cdot \sum_{\substack{e\in E \text{ such that}\\ e \text{ is a face of } w}} \mu_{ew}\cdot X_e = \sum_{e\in E} X_e\cdot \sum_{\substack{w\in W \text{ such that}\\ e \text{ is a face of } w}} \mu_{ew}\cdot a_w.$$

↑ follows immediately from the definition of ∂_n, see page 2142, definition of induced orientations, see page 2141, and the above definition of μ_{ew}

Now we consider the two projections.

(a) Note that an element $s \in \mathrm{C}_{n-1}^{\mathrm{simp}}(K)$ lies in the kernel of p_1 if and only if for all $e \in E_1$ the X_e-coefficient of s vanishes. Thus let $e \in E_1$. Let $w \in W$ be the unique n-simplex that cobounds e. Using the above calculation we see that

$$X_e\text{–coefficient of } \partial_n\Big(\sum_{w\in W} a_w\cdot Y_w\Big) \;=\; \mu_{ew}\cdot a_w.$$

The promised result follows from this discussion.

(b) Next let $e \in E_2$. Let $w \neq w'$ be the two n-simplices that cobound e. Similar to the above we see that

$$X_e\text{–coefficient of } \partial_n\Big(\sum_{w\in W} a_w\cdot Y_w\Big) \;=\; \mu_{ew}\cdot a_w + \mu_{ew'}\cdot a_{w'}.$$

As in (a) we see that this calculation gives us the desired result.

Now note that Claim 1 follows almost immediately from the above discussion.

Claim 2. Every element $\sum_{w\in W} a_w\cdot Y_w$ that lies in the kernel of the map

$$\mathrm{C}_n^{\mathrm{simp}}(K) \xrightarrow{\partial_n} \mathrm{C}_{n-1}^{\mathrm{simp}}(K) = \bigoplus_{e\in E}\mathbb{Z}\cdot X_e \xrightarrow{p_2} \bigoplus_{e\in E_2}\mathbb{Z}\cdot X_e$$

has the property that for all $w, w' \in W$ we have $|a_w| = |a_{w'}|$.

Proof. We consider the map

$$\begin{array}{rcl} f\colon W & \to & \mathbb{N}_0 \\ w & \mapsto & |a_w|. \end{array}$$

It follows from Claim 1 that for every $e \in E_2$ and $w, w' \in \varphi(e)$ we have $|a_w| = |a_{w'}|$. Since the abstract graph (W, E_2, φ) is connected we obtain from Lemma 7.6 that $f\colon W \to \mathbb{N}_0$ is constant. ⊞

Claim 3. Given any $w_0 \in W$ the map

$$\begin{array}{rcl} \Theta_{w_0}\colon \ker\big(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\big) & \to & \mathbb{Z} \\ \sum_{w\in W} a_w\cdot Y_w & \mapsto & a_{w_0} \end{array}$$

is a monomorphism.

Proof. It is clear that the given map is a homomorphism. It follows immediately from Claim 2 that the map Θ_{w_0} is a monomorphism. ⊞

Now we can prove Statement (1) of the proposition.

(i) We need to show that if K is infinite, then $\mathrm{H}_n^{\mathrm{simp}}(K) = 0$. So suppose that K is infinite. It follows from the fact that K is finite-dimensional and the fact that every simplex is contained in an n-simplex that W is also infinite. Now let $\sum_{w\in W} a_w\cdot Y_w$

be an element in $\ker\big(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\big)$. Since W is infinite and since by definition every element in $\mathrm{C}_n^{\mathrm{simp}}(K)$ is a finite linear combination of ordered simplices we see that there exists a $w_0 \in W$ with $a_{w_0} = 0$. But then it follows from Claim 2 that $a_w = 0$ for all $w \in W$.

(ii) We need to show that if $\partial K \neq \varnothing$, then $\mathrm{H}_n^{\mathrm{simp}}(K) = 0$. Thus suppose that $\partial K \neq \varnothing$. By definition of ∂K this implies that $E_1 \neq \varnothing$. We pick some $e_0 \in E_1$. Let w_0 be the unique n-simplex that cobounds e_0. Now let $\sum_{w\in W} a_w \cdot Y_w$ be an element in $\ker\big(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\big)$. By Claim 1 we have $a_{w_0} = 0$. But then it follows from Claim 2 that $a_w = 0$ for all $w \in W$.

(iii) Now assume that $\mathrm{H}_n^{\mathrm{simp}}(K) \neq 0$. We need to show that K is actually orientable. To do so it suffices to prove the following claim.

Claim. If $\sum_{w\in W} a_w \cdot Y_w$ is a non-zero element in $\ker\big(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\big)$, then the orientations[1206] $\{(\mathrm{sign}(a_w), Y_w)\}_{w\in W}$ define an orientation of K.

Proof. Let $\sum_{w\in W} a_w \cdot Y_w$ be a non-zero element in $\ker\big(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\big)$. Let e be an $(n-1)$-simplex of order 2, i.e. let $e \in E_2$. Let $w \neq w'$ be the two cobounding n-simplices of e. By Claim 1 we know that $\mu_{ew} \cdot a_w = -\mu_{ew'} \cdot a_{w'}$. It follows from Claim 2 that we also have the equality $\mu_{ew} \cdot \mathrm{sign}(a_w) = -\mu_{ew'} \cdot \mathrm{sign}(a_{w'})$. But basically by definition that means that $(\mathrm{sign}(a_w), Y_w)$ and $(\mathrm{sign}(a_{w'}), Y_{w'})$ induce opposite orientations on the $(n-1)$-simplex e. ⊞

(iv) Finally we suppose that K is closed and orientable. Let $\{(\mu_w, Y_w)\}_{w\in W}$ be an orientation of K. We need to show that $\mathrm{H}_n^{\mathrm{simp}}(K) \cong \mathbb{Z}$ and that $\sum_{w\in W} \mu_w \cdot Y_w$ is a generator of $\mathrm{H}_n^{\mathrm{simp}}(K)$.

Since K is by definition non-empty we can pick some $w_0 \in W$. We consider the maps

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\Psi} & \ker(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)) & \xrightarrow{\Theta_{w_0}} & \mathbb{Z} \\ r & \mapsto & r \cdot \sum_{w\in W} \mu_w \cdot Y_w & & \end{array}$$

It follows almost immediately from Claim 1 that the map Ψ to the left is well-defined, in the sense that it takes values in $\ker(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K))$. Furthermore it is clear that the composition $\Theta_{w_0} \circ \Psi$ is given by multiplication by μ_w, in particular the composition $\Theta_{w_0} \circ \Psi$ is an isomorphism. By Claim 3 we know that Θ_{w_0} is a monomorphism. It follows from this discussion that Ψ is an isomorphism. But that is exactly what we needed to show. ■

The proof of Statement (2) of Theorem 98.4 will require the following elementary lemma.

Lemma 98.5. Let A, B, C and D be abelian groups. We denote by $p_1\colon B \oplus C \to B$ the projection. Now suppose we are given an exact sequence

$$A \xrightarrow{\alpha} B \oplus C \xrightarrow{\varphi} D$$

[1206] Here recall that given a non-zero integer $a \in \mathbb{Z} \setminus \{0\}$ we denote by $\mathrm{sign}(a) := \frac{a}{|a|} \in \{\pm 1\}$ its sign. Also note that we know by hypothesis that at least one $a_w \neq 0$, but then it follows from Claim 2 that all $a_w \neq 0$, in particular it makes sense to consider the signs of the a_w.

of homomorphisms such that the following conditions are satisfied:

(1) the group $B/p_1(\alpha(A))$ is torsion,
(2) the group D is torsion-free,
(3) the map $p_1 \circ \alpha \colon A \to B$ is a monomorphism.

Then the map

$$\begin{array}{rcl} \Theta \colon B/p_1(\alpha(A)) \oplus \ker(C \xrightarrow{c \mapsto \varphi((0,c))} D) & \to & \dfrac{\ker(\varphi \colon B \oplus C \to D)}{\operatorname{im}(\alpha \colon A \to B \oplus C)} \\ ([b], c) & \mapsto & [(b,c)] \end{array}$$

is an isomorphism.

Proof. proof of Lemma 98.5] From our hypotheses (1) and (2) and the exactness of the sequence we easily deduce the following statement:

(∗) The map $B \xrightarrow{b \mapsto \varphi((b,0))} D$ is the zero map.

Note that (∗) implies that if $\varphi((b,c)) = 0$, then we also have $\varphi((0,c)) = 0$. From this observation it follows immediately that Θ is an epimorphism. Now suppose we are given $b \in B$ and $c \in \ker(C \to D)$ with $\Theta([b],c) = 0$. This means that there exists an $a \in A$ with $\alpha(a) = (b,c)$. By the exactness of the sequence and by (∗) we know that there exists an $\widetilde{a} \in A$ with $\alpha(\widetilde{a}) = (b,0)$. From (3) we obtain that $a = \widetilde{a}$. Thus we see that $(b,c) = \alpha(a) = \alpha(\widetilde{a}) = (b,0)$. This shows that $[b] = 0 \in B/p_1(\alpha(A))$ and $c = 0$. In other words, we have shown that Θ is a monomorphism. ■

We move on to the proof of Statement (2) of Theorem 98.4.

Proof of Theorem 98.4 (2). Let K be a finite n-dimensional pseudomanifold. Recall that we need to prove the following statements:

(a_1) If K is orientable and closed, then $\mathrm{H}^{\mathrm{simp}}_{n-1}(K)$ is a free abelian group.
(a_2) If $\partial K \neq \varnothing$, then $\mathrm{H}^{\mathrm{simp}}_{n-1}(K)$ is a free abelian group.
(b) If K is non-orientable and if $\partial K = \varnothing$, then $\mathrm{H}^{\mathrm{simp}}_{n-1}(K) \cong F \oplus \mathbb{Z}_2$ where F is a free abelian group.

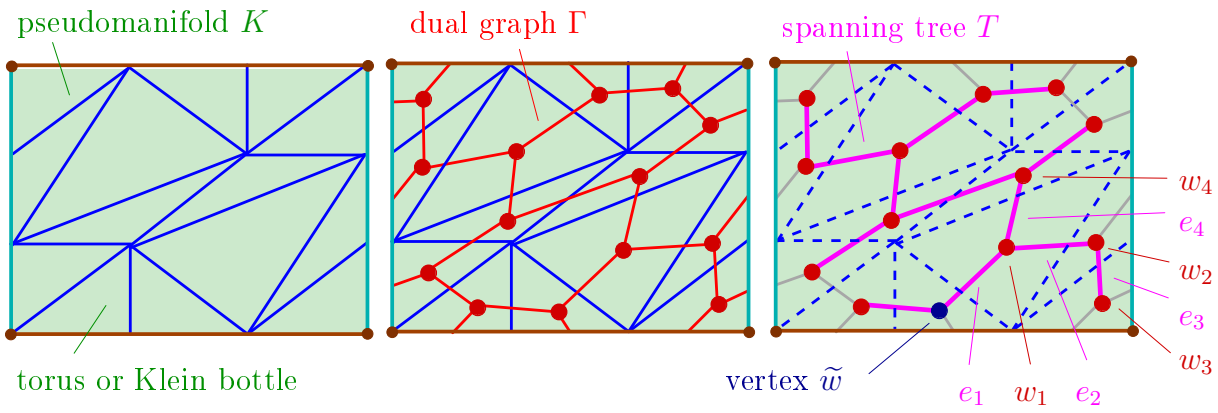
In the following we will use freely the notation that we introduced in the proof of Statement (1). In particular we consider again the abstract graph $\Gamma = (W, E_2, \varphi)$. Since K is finite we know that Γ is finite, hence by the Spanning Tree Existence Proposition 7.10 (1) we know that there exists a spanning tree T of Γ. Recall that this means that T is connected, that $\chi(T) = 1$ and, by the Spanning Tree Existence Proposition 7.10 (2) it means that T contains every vertex. We pick $\widetilde{w} \in W$. By Lemma 7.8 we know that we can write $W = \{\widetilde{w}, w_1, \ldots, w_k\}$ and $E_2 = \{e_1, \ldots, e_k\}$ such that the following statements hold:

(1) For each $i \in \{1, \ldots, k\}$ the vertex w_i is an endpoint of the edge e_i.
(2) If some vertex w_i is an endpoint of some e_j, then $j \leq i$.

We introduce the following notation. We denote by

$$p_1 \colon \mathrm{C}^{\mathrm{simp}}_{n-1}(K) \to \bigoplus_{i=1}^{k} \mathbb{Z} \cdot e_i \quad \text{and} \quad p_2 \colon \mathrm{C}^{\mathrm{simp}}_{n-1}(K) \to \bigoplus_{e \in E \setminus \{e_1, \ldots, e_k\}}^{k} \mathbb{Z} \cdot e_i$$

the natural projections.
The key to the proof of Theorem 98.4 (2) is the following observation, which is an immediate consequence of Claim 1 together with the clever choice of $w_1, \ldots, w_k$ and $e_1, \ldots, e_k$.



Observation. The map $\bigoplus_{i=1}^{k}\mathbb{Z}\cdot w_i \xrightarrow{\partial_n} \mathrm{C}_{n-1}^{\mathrm{simp}}(K) \xrightarrow{p_1} \bigoplus_{i=1}^{k}\mathbb{Z}\cdot e_i$ is represented by a matrix of the form

$$D := \begin{pmatrix} \pm 1 & * & \dots & * \\ 0 & \pm 1 & * & \vdots \\ 0 & 0 & \pm 1 & * \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}$$

(a_1) First we assume that K is orientable and closed. Let $\{(\epsilon_w, Y_w)\}_{w\in W}$ be an orientation of K. We set $\mu := \sum_{w\in W}\epsilon_w\cdot Y_w$. Since $\epsilon_{\widetilde{w}} \in \{\pm 1\}$ we see that $Y_{w_1},\dots,Y_{w_k},\mu$ is a basis for $\mathrm{C}_n^{\mathrm{simp}}(K)$. This time we consider the following maps

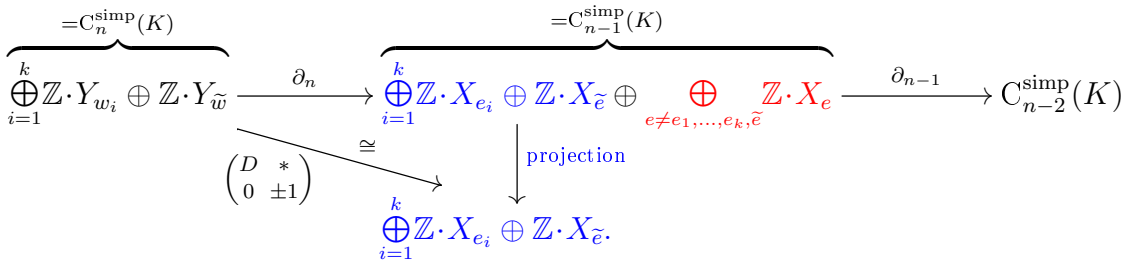
$$\begin{array}{ccccccc} \overbrace{\bigoplus_{i=1}^{k}\mathbb{Z}\cdot Y_{w_i}\oplus\mathbb{Z}\cdot\mu}^{=\mathrm{C}_n^{\mathrm{simp}}(K)} & \xrightarrow[\mu\mapsto 0]{\partial_n} & \overbrace{\bigoplus_{i=1}^{k}\mathbb{Z}\cdot X_{e_i}\oplus\bigoplus_{e\neq e_1,\dots,e_k}\mathbb{Z}\cdot X_e}^{=\mathrm{C}_{n-1}^{\mathrm{simp}}(K)} & \xrightarrow{\partial_{n-1}} & \mathrm{C}_{n-2}^{\mathrm{simp}}(K) \\ \downarrow \text{projection} & \nearrow \partial_n & \downarrow \text{projection } p_2 & & \\ \bigoplus_{i=1}^{k}\mathbb{Z}\cdot Y_{w_i} & \xrightarrow[\cong]{\text{matrix } D} & \bigoplus_{i=1}^{k}\mathbb{Z}\cdot X_{e_i}. & & \end{array}$$

As we already discussed in the proof of Statement (1), since K is closed we obtain from Claim 1 that $\partial_n(\mu) = 0$. Thus the image of ∂_n equals the image of the diagonal map. Furthermore, the bottom horizontal map is represented by the matrix D from the observation, in particular we see that the bottom horizontal map is an isomorphism. From Lemma 98.5 we now obtain the following isomorphism:

$$\underbrace{\ker\left(\bigoplus_{e\neq e_1,\dots,e_k}\mathbb{Z}\cdot X_e \xrightarrow{\partial_{n-1}} \mathrm{C}_{n-2}^{\mathrm{simp}}(K)\right)}_{\substack{\text{subgroup of a free abelian group thus by} \\ \text{Lemma 51.9 (3) it is a free abelian group}}} \xrightarrow{\cong} \underbrace{\frac{\ker\left(\mathrm{C}_{n-1}^{\mathrm{simp}}(K) \xrightarrow{\partial_{n-1}} \mathrm{C}_{n-2}^{\mathrm{simp}}(K)\right)}{\mathrm{im}\left(\mathrm{C}_n^{\mathrm{simp}}(K) \xrightarrow{\partial_n} \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\right)}}_{=\mathrm{H}_{n-1}^{\mathrm{simp}}(K)}.$$

(a_2) Now we assume that $\partial K \neq \varnothing$, i.e. we assume that $E_1 \neq \varnothing$. We pick an $\widetilde{e} \in E_1$. Let $\widetilde{w} \in W$ be the unique n-simplex that cobounds $\widetilde{e}$. We pick $w_1,\dots,w_k$ and $e_1,\dots,e_k$

accordingly. We consider the following maps

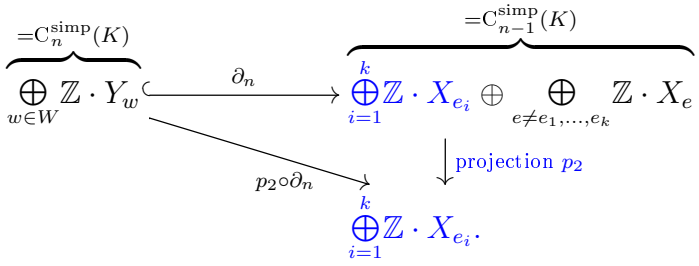


It follows easily from the Observation and Claim 1 that the diagonal map is an isomorphism. It now follows from Lemma 98.5 that

$$\underbrace{\ker\left(\bigoplus_{e\neq e_1,\dots,e_k,\widetilde{e}}\mathbb{Z}\cdot X_e\xrightarrow{\partial_{n-1}}\mathrm{C}_{n-2}^{\mathrm{simp}}(K)\right)}_{\substack{\text{subgroup of a free abelian group thus by}\\ \text{Lemma 51.9 (3) it is a free abelian group}}}\xrightarrow{\cong}\underbrace{\frac{\ker\left(\mathrm{C}_{n-1}^{\mathrm{simp}}(K)\xrightarrow{\partial_{n-1}}\mathrm{C}_{n-2}^{\mathrm{simp}}(K)\right)}{\mathrm{im}\left(\mathrm{C}_n^{\mathrm{simp}}(K)\xrightarrow{\partial_n}\mathrm{C}_{n-1}^{\mathrm{simp}}(K)\right)}}_{=\mathrm{H}_{n-1}^{\mathrm{simp}}(K)}.$$

But the group on the right hand side is the subgroup of a finitely generated free abelian group, so it follows from Lemma 51.9 (3) that it is itself a finitely generated free abelian group.

(b) Finally we assume that K is closed and non-orientable. First we consider the following maps:



By our hypothesis that K is closed and non-orientable, together with Statement (1), we know that $\mathrm{H}_n^{\mathrm{simp}}(K)=0$. This means that the horizontal map is a monomorphism. On the other hand, for rank reasons the diagonal map $p_2\circ\partial_n$ has a kernel. In other words, there exists some non-zero $\mu=\sum_{w\in W}a_w\cdot Y_w\in\mathrm{C}_n^{\mathrm{simp}}(K)$ with $p_2(\partial_n(\mu))=0$. By Claim 2 we know that the value $|a_w|$ is constant. Thus after dividing by this constant (which is non-zero since μ is non-zero) we can arrange that each $|a_w|=1$.

Since K is by hypothesis closed, i.e. since each edge has order 2, we see, say by the calculation in the proof of Claim 1 or by direct inspection, that for each $e\in E$ the X_e-coefficient of $\partial_n(\mu)$ lies in $\{0,\pm 2\}$. Furthermore since ∂_n is a monomorphism we know that $\partial_n(\mu)\neq 0$. Thus there exists some $f\in E$ such that the f-coefficient of $\partial_n(\mu)$ equals ± 2. From $p_2(\partial_n(\mu))=0$ we obtain that $f\neq e_1,\dots,e_k$. Next we

consider the following maps:

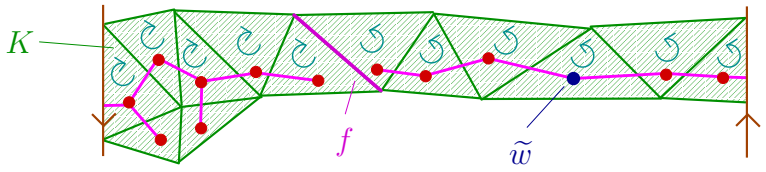
$$\overbrace{\bigoplus_{i=1}^{k}\mathbb{Z}\cdot Y_{w_i}\oplus\mathbb{Z}\cdot\mu}^{=\mathrm{C}_n^{\mathrm{simp}}(K)}\xrightarrow{\partial_n}\overbrace{\bigoplus_{i=1}^{k}\mathbb{Z}\cdot X_{e_i}\oplus\mathbb{Z}\cdot X_f\oplus\bigoplus_{e\neq e_1,\dots,e_k,f}\mathbb{Z}\cdot X_e}^{=\mathrm{C}_{n-1}^{\mathrm{simp}}(K)}\xrightarrow{\partial_{n-1}}\mathrm{C}_{n-2}^{\mathrm{simp}}(K)$$

$$\begin{pmatrix} D & * \\ 0 & \pm 2 \end{pmatrix}\cong\bigoplus_{i=1}^{k}\mathbb{Z}\cdot X_{e_i}\oplus\mathbb{Z}\cdot X_f \quad\text{(projection)}$$

Finally we note that it follows from the above together with Lemma 98.5 that the following map is an isomorphism:

$$\underbrace{\ker\left(\bigoplus_{e\neq e_1,\dots,e_k,f}\mathbb{Z}\cdot X_e\to\mathrm{C}_{n-2}^{\mathrm{simp}}(K)\right)}_{\text{subgroup of a free abelian group thus by Lemma 51.9 (3) it is a free abelian group}}\oplus\underbrace{\mathbb{Z}^{k+1}\Big/\begin{pmatrix} D & * \\ 0 & \pm 2 \end{pmatrix}\mathbb{Z}^{k+1}}_{\cong\mathbb{Z}_2\text{ by the observation}}\to\underbrace{\frac{\ker\left(\mathrm{C}_{n-1}^{\mathrm{simp}}(K)\to\mathrm{C}_{n-2}^{\mathrm{simp}}(K)\right)}{\mathrm{im}\left(\mathrm{C}_{n}^{\mathrm{simp}}(K)\to\mathrm{C}_{n-1}^{\mathrm{simp}}(K)\right)}}_{=\mathrm{H}_{n-1}^{\mathrm{simp}}(K)}.$$

We have thus finished the proof of Theorem 98.4 (2). ■



Finally we get to the proof of the last statement of Theorem 98.4 (3).

Proof of Theorem 98.4 (3). Let K be a finite n-dimensional pseudomanifold. First we point out that it would be a fun exercise to modify the proof of (1) to obtain the promised result. But in these notes we will not pursue this approach. Instead we will do a little trick. First we consider the case that K is closed. In this case we perform the following calculation:

$$\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2)\ \cong\ \mathrm{H}_n^{\mathrm{simp}}(K)\otimes\mathbb{F}_2\oplus\mathrm{Tor}(\mathrm{H}_{n-1}^{\mathrm{simp}}(K),\mathbb{F}_2)\ \cong\ \begin{cases}\overbrace{\mathbb{Z}\otimes\mathbb{F}_2\oplus\mathrm{Tor}(\mathbb{Z}^l,\mathbb{F}_2)}^{\text{if } K \text{ is orientable}}\\ \underbrace{0\otimes\mathbb{F}_2\oplus\mathrm{Tor}(\mathbb{Z}^l\oplus\mathbb{Z}_2,\mathbb{F}_2)}_{\text{if } K \text{ is non-orientable}}\end{cases}\ \cong\ \mathbb{F}_2.$$

by the Algebraic Universal Coefficient Theorem 89.16 applied to the chain complex $\mathrm{C}_*^{\mathrm{simp}}(K)$ — by (1) and (3) — Tensor Lemma 89.3 and 89.15

Now suppose that $Y_1,\dots,Y_m$ is a set of ordered n-simplices of K such that for each n-simplex s of K there exists a unique $i\in\{1,\dots,m\}$ with $s=\lfloor Y_i\rfloor$. One easily verifies that the simplicial chain $Y_1\otimes 1+\cdots+Y_m\otimes 1\in\mathrm{C}_n^{\mathrm{simp}}(K;\mathbb{F}_2)=\mathrm{C}_n^{\mathrm{simp}}(K)\otimes\mathbb{F}_2$ is a simplicial cycle. Since it is non-zero we see that it equals necessarily the unique non-zero element of $\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2)=\ker(\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2)\to\mathrm{H}_{n-1}^{\mathrm{simp}}(K;\mathbb{F}_2))\cong\mathbb{F}_2$.

Finally if the pseudomanifold K is not closed, then the above approach shows almost immediately that $\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2)=0$. With a slight sigh of relief we note that we have now finally managed to prove all three statements of Theorem 98.4. ■

98.2. **The top-dimensional homology group of a smooth manifold.** As the reader surely expected, the main application of Theorem 98.4 is the following theorem which gives us lots of interesting extra information on the homology groups of smooth manifolds.

Theorem 98.6. (Smooth Manifold-Top Homology Theorem) Let M be a connected non-empty n-dimensional smooth manifold. The following statements hold:

(1) We have an isomorphism as follows:

$$\mathrm{H}_n(M;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } M \text{ is closed and orientable,} \\ 0, & \text{otherwise.} \end{cases}$$

(2) If M is compact, then we also have an isomorphism as follows:

$$\mathrm{H}_{n-1}(M;\mathbb{Z}) \cong \begin{cases} \text{free abelian group}, & \text{if } M \text{ is orientable or if } \partial M \neq \varnothing, \\ \text{free abelian group} \oplus \mathbb{Z}_2, & \text{if } M \text{ is non-orientable and closed.} \end{cases}$$

In both cases the free abelian group is finitely generated.

(3) If M is compact, then

$$\mathrm{H}_n(M;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } M \text{ is closed, regardless whether or not } M \text{ is orientable,} \\ 0, & \text{otherwise.} \end{cases}$$

Examples.

(1) Let $k \in \mathbb{N}$. We consider $k \cdot \mathbb{R}\mathrm{P}^2$, i.e. we consider the connected sum of k copies of $\mathbb{R}\mathrm{P}^2$. By Exercise **??** we know that $k \cdot \mathbb{R}\mathrm{P}^2$ is non-orientable. In Proposition 80.10 we showed that

$$\mathrm{H}_1(\underbrace{k \cdot \mathbb{R}\mathrm{P}^2}_{=N_k};\mathbb{Z}) \cong \mathbb{Z}^{k-1} \oplus \mathbb{Z}_2 \quad \text{and} \quad \mathrm{H}_2(k \cdot \mathbb{R}\mathrm{P}^2;\mathbb{Z}) = 0.$$

This is reassuringly consistent with the Smooth Manifold-Top Homology Theorem 98.6.

(2) Recall that the Möbius band M is a compact non-orientable 2-dimensional smooth manifold with non-empty boundary. Since M is homotopy equivalent to S^1 we see that $\mathrm{H}_1(M;\mathbb{Z}) \cong \mathrm{H}_1(S^1;\mathbb{Z}) \cong \mathbb{Z}$. This example illustrates that in the Smooth Manifold-Top Homology Theorem 98.6 the "$\mathbb{Z}_2$"-term for a non-orientable smooth manifold only appears if M is closed.

(3) In Chapter 105 we will make sense of the notion of an orientable topological manifold. We will prove the generalization of the Smooth Manifold-Top Homology Theorem 98.6 to the setting of topological manifolds in Theorems 106.1, 106.2 and 106.3 and Proposition 106.22. These new results completely subsume the Smooth Manifold-Top Homology Theorem 98.6. Nonetheless it is worth covering the above proof since it is fairly geometric, whereas the latter proofs for topological manifolds are rather unintuitive.

Proof of the Smooth Manifold-Top Homology Theorem 98.6. Let M be a connected non-empty n-dimensional smooth manifold. Note that by the Smooth Manifold-Simplicial Structure Theorem 96.2 we know that M admits a smooth simplicial structure $(K = (V,S), \Theta\colon |K| \to M)$. By the Manifold-Simplex Proposition 98.1 we know that K is an n-dimensional pseudomanifold. Evidently now we would like to apply Theorem 98.4 and call it a day.

But to do so we need to show that the hypotheses on the smooth manifolds do indeed match the corresponding hypotheses for pseudomanifolds. Let us start with the good news:

(1) By Lemma 93.9 (1) and the Simplicial Complex-Finiteness Proposition 93.12 we know that M is compact if and only if K is finite.
(2) From the Manifold-Simplex Proposition 98.1 (1), (3) and (4) we obtain that $\partial M = \varnothing$ if and only if $\partial K = \varnothing$.

Here is the bad news: We still have to relate the (non-) orientability of the smooth manifold M to the (non-) orientability of the abstract simplicial complex K. This will be done in the subsequent Proposition 98.7. ■

The following proposition gives us the last piece of the proof of the Smooth Manifold-Top Homology Theorem 98.6.

Proposition 98.7. (Pseudomanifold-Orientation Proposition) Let $n \in \mathbb{N}$ and let M be a connected non-empty n-dimensional smooth manifold.[1207] Furthermore let $(K = (V,S), \Theta\colon |K| \to M)$ be a smooth simplicial structure. The map

$$\left\{\begin{array}{c}\text{orientations of the smooth}\\ \text{manifold } M \text{ in the sense of the}\\ \text{definition on page 675}\end{array}\right\} \xrightarrow{\cong} \left\{\begin{array}{c}\text{orientations of the}\\ \text{pseudomanifold } K \text{ in the sense}\\ \text{of the definition on page 2142}\end{array}\right\}.$$

$$\begin{array}{rl} & \text{for each } n\text{-simplex } s \text{ we pick a corresponding}\\ & \text{ordered simplex } Y_s \text{ and we consider}\\ \text{orientation of } M \mapsto & \text{the oriented } n\text{-simplex } [(\epsilon_s, Y_s)] \text{ where}\\ & \epsilon_s = +1 \text{ if } \Theta \circ \Phi^{\leq}_{Y_s} : \mathring{\Delta}^n \to M \text{ is orientation-pres.}\\ & \epsilon_s = -1 \text{ if } \Theta \circ \Phi^{\leq}_{Y_s} : \mathring{\Delta}^n \to M \text{ is orientation-rev.}\end{array}$$

is a natural bijection.[1208] In particular the smooth manifold M is orientable if and only if the pseudomanifold K is orientable.

Proof of Proposition 98.7. In the following we will provide a proof of Proposition 98.7. To preserve the author's sanity we will skip a few technical calculations. We leave it to the reader to fill in the details.

Throughout the proof let M be a non-empty n-dimensional smooth manifold and let $(K = (V,S), \Theta\colon |K| \to M)$ be a smooth simplicial structure. First note that the case $n = 0$ is trivial, since every 0-dimensional smooth manifold and every 0-dimensional pseudomanifold is orientable. Thus in the following we assume that $n > 0$. Before we do anything else we recall and introduce a few definitions and we make a few basic comments regarding these definitions.

(a) Let $k \in \mathbb{N}_0$. On page 1581 we introduced

$$\mathring{\Delta}^k := \{(t_0, \ldots, t_k) \in \Delta^k \,|\, \text{all coordinates are non-zero}\}.$$

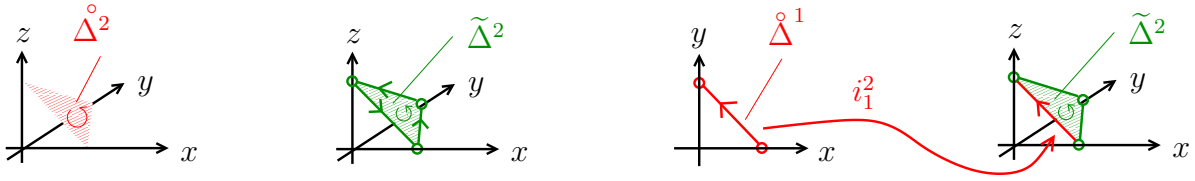
Now we also define

$$\widetilde{\Delta}^k := \{(t_0, \ldots, t_k) \in \Delta^k \,|\, \text{at least two coordinates are non-zero}\}.$$

[1207] We do not impose any other conditions, in particular neither do we assume that M is compact nor do we assume that $\partial M = \varnothing$.
[1208] We leave it to the reader to figure out what "natural" means in this context.

(b) Similar to the discussion on page 1581 we view $\mathring{\Delta}^k$ and $\widetilde{\Delta}^k$ in the obvious way[1209] as smooth manifolds. Note that $\mathring{\Delta}^k$ has no boundary and that the boundary of $\widetilde{\Delta}^k$ equals $\widetilde{\Delta}^k \cap \partial\Delta^k$. We equip the smooth manifolds $\mathring{\Delta}^k$ and $\widetilde{\Delta}^k$ with the orientation where for each $P \in \mathring{\Delta}^k$ a basis $v_1, \ldots, v_k$ of $\widetilde{\mathrm{T}}_P\mathring{\Delta}^k$ is positive if the vectors $\{(1, \ldots, 1), v_1, \ldots, v_k\}$ are a positive basis for $\mathbb{R}^{k+1}$. As always we endow the boundary $\partial\widetilde{\Delta}^k$ of the oriented smooth manifold $\widetilde{\Delta}^k$ with the orientation given by the Boundary Orientation Lemma 25.17.

(c) Note that for any $j \in \{0, \ldots, k\}$ the usual j-th face map $i^k_j \colon \Delta^{k-1} \to \Delta^k$ restricts to an embedding $i^k_j \colon \mathring{\Delta}^{k-1} \to \partial\widetilde{\Delta}^k \subset \widetilde{\Delta}^k$. Using the Basics-of-Orientations Lemma 25.13 (2) it is straightforward to verify that the map $i^k_j \colon \mathring{\Delta}^{k-1} \to \partial\widetilde{\Delta}^k$ is orientation-preserving if and only if j is even.



(d) Let $k \in \mathbb{N}_0$ and let $Y = (v_0, \ldots, v_k)$ be an ordered k-simplex of $K = (V, S)$. On page 1984 we introduced the corresponding characteristic map

$$\begin{aligned} \Phi^{\leq}_Y \colon \Delta^k = \left\{ (t_0, \ldots, t_k) \in \mathbb{R}^{k+1}_{\geq 0} \;\middle|\; \sum_{i=0}^k t_i = 1 \right\} &\to |K| \\ (t_0, \ldots, t_k) &\mapsto \sum_{i=0}^k t_i \cdot v_i. \end{aligned}$$

(e) It follows almost immediately from the definition of a "smooth" simplicial structure, see page 2082, that for every ordered k-simplex $Y = (v_0, \ldots, v_k)$ the map $\Theta \circ \Phi^{\leq}_Y \colon \widetilde{\Delta}^n \to M$ is an immersion.

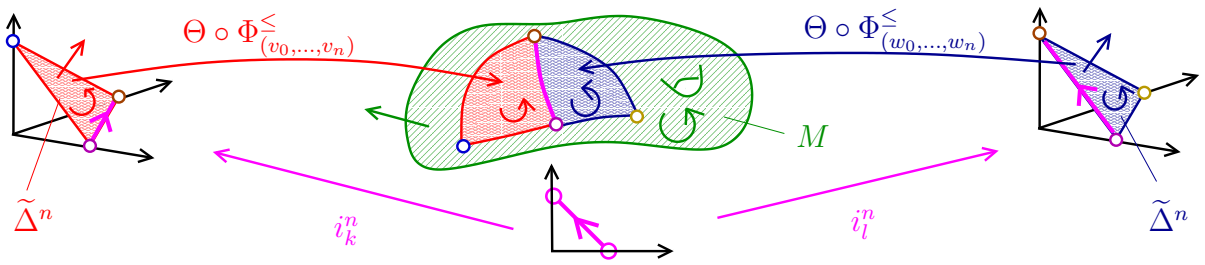
Now we turn to the actual proof of the proposition.

First we assume that we are given an orientation of the smooth manifold M. We need to equip the pseudomanifold $K = (V, S)$ with a natural orientation. We say that an ordered n-simplex $Y = (v_0, \ldots, v_n)$ is *positive* if the map $\Theta \circ \Phi^{\leq}_Y \colon \mathring{\Delta}^n \to M$ is orientation-preserving in the sense of the definition on page 682. Since $n \geq 1$ we see easily that, by possibly swapping the order of two vertices, we can find for each n-simplex $s \in S$ a positive ordered n-simplex Y_s with $\lfloor Y_s \rfloor = s$. It remains to prove the following claim.

Claim 1. The oriented n-simplices $[(+1, Y_s)]$ form an orientation of the pseudomanifold K.

Proof. Let e be an $(n-1)$-simplex of K of order 2 and let s, t be the two n-simplices that cobound e. We need to show that the orientations $[(+1, Y_s)]$ and $[(+1, Y_t)]$ induce opposite orientations on e. We write $Y_s = (v_0, \ldots, v_n)$ and we write $Y_t = (w_0, \ldots, w_n)$. Let v_k be

[1209] Slightly more precisely, a smooth atlas for $\mathring{\Delta}^k$ is given by projection to the first k coordinates, and for $\widetilde{\Delta}^k$ we obtain a smooth atlas by the same projection and an affine linear map that sends points on a component of $\widetilde{\Delta}^k \cap \partial\Delta^k$ into the $(k-1)$-hyperplane.



the vertex missing in e and let w_l be the vertex missing in e. We consider the maps

$$\mathring{\Delta}^{n-1} \xrightarrow{i^n_k} \widetilde{\Delta}^n \xrightarrow{\Theta\circ\Phi^{\leq}_{(v_0,\ldots,v_n)}} M \xleftarrow{\Theta\circ\Phi^{\leq}_{(w_0,\ldots,w_n)}} \widetilde{\Delta}^n \xleftarrow{i^n_l} \mathring{\Delta}^{n-1}.$$

The two maps $\mathring{\Delta}^{n-1} \to M$ are immersions and the images coincide, thus we obtain a self-diffeomorphism of $\mathring{\Delta}^{n-1}$. A little bit of thought, using (c) and using the Basics-of-Orientations Lemma 25.13 (2), shows that this map is orientation-preserving if and only if $k+l+1$ is even.[1210] Next we need to associate to each orientation of the pseudomanifold K an orientation of the smooth manifold M. Thus suppose we are given an orientation $\{(\mu_s, Y_s)\}_{s\in S}$ of the pseudomanifold K. Since $n > 0$ we can and will assume that for each n-simplex s we have $\mu_s = 1$.

Claim 2. The smooth manifold $M \setminus \Theta(K^{n-2})$ has a natural orientation.[1211]

Proof. Let $P \in M \setminus \Theta(K^{n-2})$. Since every $(n-1)$-simplex is the face of an n-simplex we see that there exists an n-simplex s with $P \in \Theta(|s|)$. Since $P \notin \Theta(K^{n-2})$ we see that there exists a $Q \in \widetilde{\Delta}^n$ such that $P = \Phi^{\leq}_{Y_s}(Q)$. Since $\Phi^{\leq}_{Y_s} \colon \widetilde{\Delta}^n \to M$ is an immersion we know that the induced map $\mathrm{T}_Q\widetilde{\Delta}^k \to \mathrm{T}_P M$ is an isomorphism. Thus it makes sense to equip $\mathrm{T}_P M$ with the image of the positive orientation of $\widetilde{\Delta}^n$ under the above isomorphism. We need to show that the definition of the orientation does not depend on the choice of s. If $P \in M \setminus \Theta(K^{n-1})$, then we know from Open Simplex Lemma 93.11 that s is unique, hence we are done. Otherwise there exists an $(n-1)$-simplex e such that $P \in \Theta(\langle e \rangle)$. If the order of e equals one, then s is again unique and again we are done. Finally if the order of e equals two, then it follows from the hypothesis that $\{(\mu_s, Y_s)\}_{s\in S}$ is an orientation of K. A reasonably elementary calculation shows that the orientation on $\mathrm{T}_P M$ does not depend on the choice of the cobounding n-simplex. Finally we note that it is fairly straightforward to show that these orientations of the tangent spaces define an orientation, in the sense of the definition on page 675, of the smooth submanifold $M \setminus \Theta(K^{n-2})$. ⊞

We move on to the next claim.

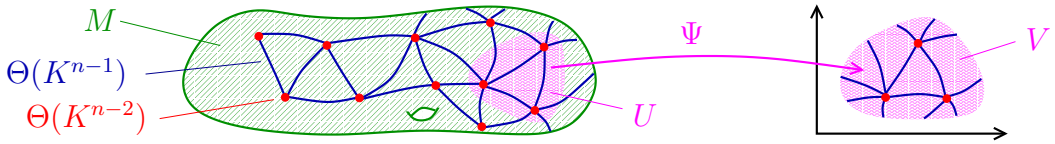
Claim 3. For every open connected subset U of M the following holds:

(a) The set $U \cap (M \setminus \Theta(K^{n-2}))$ is connected.
(b) The closure of $U \cap (M \setminus \Theta(K^{n-2}))$ equals U.

[1210] Also note that the fact that $\mathring{\Delta}^n$ is dense in $\widetilde{\Delta}^n$ implies that the map $\Theta \circ \Phi^{\leq}_{\widetilde{Y}} \colon \widetilde{\Delta}^n \to M$ is also orientation-preserving.

[1211] Note that by Lemma 93.8 we know that $M \setminus \Theta(K^{n-2})$ is an open subset of M, in particular it is a smooth submanifold of M.

Proof. For $i = 0, \ldots, n-2$ we set $N_i = U \cap (M \setminus \Theta(K^{i-1}))$ and we set $Y_i = \Theta(K^i) \cap N_i$. Note that for each $i \in \{1, \ldots, n-2\}$ we have $N_i = N_{i-1} \setminus Y_{i-1}$. It follows easily from the definition of a smooth simplicial structure together with Lemma 93.8 that each N_i is an n-dimensional smooth manifold and that each Y_i is a proper i-dimensional smooth submanifold of N_i. Since $i \leq n-2$ we obtain iteratively from the future Corollary 42.6 and the hypothesis that $N_0 = U$ is path-connected that each N_i is path-connected. Finally note that we know by Exercise 19.15 and the fact that $\dim(Y_i) = i < n = \dim(N_i)$ that the closure of $N_i \setminus Y_i$ is N_i. Note that the statements of Claim 3 follow almost immediately from these considerations.



Finally let $P \in M$. It follows quite easily from Claim (3a) that we can find a chart $\Psi\colon U \to V$ around P such that V is a connected open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ and such that Ψ is orientation-preserving for some $Q \in M \setminus \Theta(K^{n-2})$. Furthermore it follows from Claim (3a) and the Basics-of-Orientations Lemma 25.13 (2) that the restriction of Ψ to $U \setminus \Theta(K^{n-2})$ is orientation-preserving at every point. Now we equip all $Q \in U$ with the orientation induced by Ψ and the standard orientation of $\mathbb{R}^n$ respectively $\mathbb{H}^n$. It follows easily from Claim (3b) that these orientations do not depend on the choice of Ψ and that these orientations of the tangent spaces define an orientation of the smooth manifold M.

In the above discussion we constructed natural maps

$$\{\text{orientations of the smooth manifold } M\} \leftrightarrow \{\text{orientations of the pseudomanifold } K\}$$

in both directions. It follows reasonably easily from the definitions that the two maps are inverses of one another, thus both maps are natural bijections. ■

For manifolds with non-empty boundary the following theorem contains lots of extra information.

Theorem 98.8. Let M be a connected non-empty n-dimensional smooth manifold. The following statements hold:

(1) We have an isomorphism as follows:

$$\mathrm{H}_n(M, \partial M; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } M \text{ is orientable and compact,} \\ 0, & \text{otherwise.} \end{cases}$$

(2) If M is compact, then we also have an isomorphism as follows:

$$\mathrm{H}_{n-1}(M, \partial M; \mathbb{Z}) \cong \begin{cases} \text{free abelian group}, & \text{if } M \text{ is orientable,} \\ \text{free abelian group} \oplus \mathbb{Z}_2, & \text{if } M \text{ is non-orientable.} \end{cases}$$

In both cases the free abelian group is finitely generated.

(3) If M is compact, then we have $\mathrm{H}_n(M, \partial M; \mathbb{F}_2) \cong \mathbb{F}_2$.

Remark.

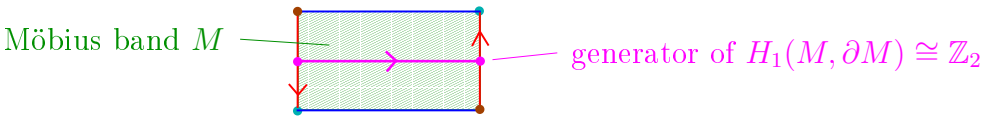
(1) Note that if M is a connected non-empty smooth manifold with $\partial M = \varnothing$, then the statements of Theorem 98.8 are contained in the Smooth Manifold-Top Homology Theorem 98.6.

(2) We will prove the generalization of the Smooth Manifold-Top Homology Theorem 98.6 to the setting of topological manifolds in Theorems 106.1 and 106.2 and Proposition 106.22.

Example. Let M be the Möbius band. Recall that M is a compact non-orientable 2-dimensional smooth manifold. In Exercise 74.7 we already saw that

$$\mathrm{H}_k(M, \partial M; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}_2, & \text{if } k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

This calculation is of course consistent with Theorem 98.8.



Proof. Let M be a connected non-empty n-dimensional smooth manifold. We dealt with the case $\partial M = \varnothing$ in the Smooth Manifold-Top Homology Theorem 98.6 so we might as well assume that $\partial M \neq \varnothing$.

In the following we first sketch what we think should be the "correct" proof of Theorem 98.8 (1). Afterwards we will give the full details for an alternative proof.

(a) As before one uses the Smooth Manifold-Simplicial Structure Theorem 96.2 to equip M with some smooth simplicial structure $(K, \Theta\colon |K| \to M)$. It follows from the Manifold-Simplex Proposition 98.1 that K is an n-dimensional pseudomanifold such that ∂M corresponds precisely to ∂K.

(b) By the Simplicial-Singular Homology Isomorphism Lemma 95.30 we have a natural isomorphism $\mathrm{H}_n^{\mathrm{simp}}(K, \partial K) \xrightarrow{\cong} \mathrm{H}_n(|K|, |\partial K|; \mathbb{Z})$.

(c) It is fairly straightforward to modify the argument of Theorem 98.4 to show that one has an isomorphism

$$\mathrm{H}_n^{\mathrm{simp}}(K, \partial K) \cong \begin{cases} \mathbb{Z}, & \text{if } K \text{ is orientable and finite,} \\ 0, & \text{otherwise.} \end{cases}$$

(d) By Proposition 98.7 we know that M is orientable if and only if K is orientable. The desired Statement (1) now follows from the combination of all of the above.

The above approach is surely convincing and it is relatively straightforward to work out the details. Furthermore the same approach can also be employed to settle Statements (2) and (3).

Now we turn to our second approach to proving the proposition. This second approach has the advantage that this time we can provide all the details. Given M we consider its double $\mathrm{D}M = M \cup_{\partial M = \partial M'} M'$ where M' is a second copy of M. We refer to page 1667 for details and we refer to the figure below for an illustration. Recall that by the Doubling-Manifold Lemma 75.12 we know the following:

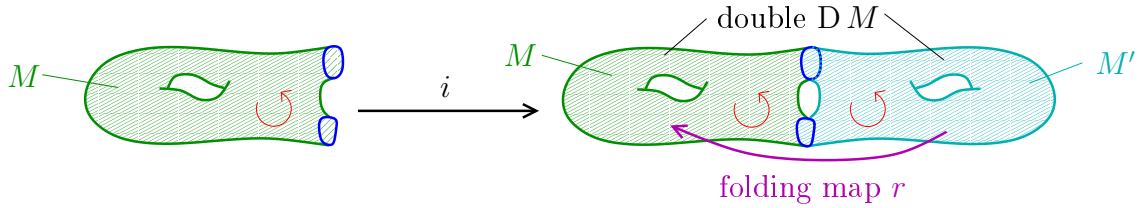
(a) The double $\mathrm{D}M$ is also an n-dimensional smooth manifold such that $M \subset \mathrm{D}M$ is a smooth submanifold.

(b) The boundary of the double $\mathrm{D}M$ is empty.

(c) M is orientable if and only if $\mathrm{D}M$ is orientable.

(d) M is connected if and only if $\mathrm{D}M$ is connected.

(e) M is compact if and only if $\mathrm{D}M$ is compact.



In the following we denote by $i\colon M \to \mathrm{D}M$ the natural inclusion and we denote by $r\colon \mathrm{D}M \to M$ the folding map that we introduced on page 1668. Given $k \in \mathbb{N}_0$ we consider the following maps:

$$\begin{array}{ccccccccccc} & & & & & & \mathrm{H}_k(M,\partial M) & & & & \\ & & & & & & \downarrow\cong & & & & \\ \ldots & \longrightarrow & \mathrm{H}_k(M) & \underset{r_*}{\overset{i_*}{\hookrightarrow}} & \mathrm{H}_k(\mathrm{D}M) & \longrightarrow & \mathrm{H}_k(\mathrm{D}M,M) & \xrightarrow{\partial} & \mathrm{H}_{k-1}(M) & \underset{r_*}{\overset{i_*}{\hookrightarrow}} & \mathrm{H}_{k-1}(\mathrm{D}M) \longrightarrow \ldots \end{array}$$

We make the following observations:

(f) The horizontal sequence is the long exact sequence in homology corresponding to the pair $(\mathrm{D}M, M)$.

(g) Since $r \circ i = \mathrm{id}_M$ we have $r_* \circ i_* = \mathrm{id}_{\mathrm{H}_k(M)}$, which implies that $i_*\colon \mathrm{H}_k(M) \to \mathrm{H}_k(\mathrm{D}M)$ is a monomorphism.

(h) By the Doubling-Manifold Lemma 75.12 we know that the vertical map is an isomorphism.

Next we recall some of our earlier calculations:

(i) Since $\partial M \neq \varnothing$ we obtain from the Smooth Manifold-Top Homology Theorem 98.6 that $\mathrm{H}_n(M) = 0$ and that $\mathrm{H}_{n-1}(M)$ is a free abelian group.

(j) By the Smooth Manifold-Top Homology Theorem 98.6 we have

$$\mathrm{H}_n(\mathrm{D}M) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } \mathrm{D}M \text{ is closed and orientable,} \\ 0, & \text{otherwise.} \end{cases}$$

(k) By the Smooth Manifold-Top Homology Theorem 98.6 and since $\partial(\mathrm{D}M) = \varnothing$ we have

$$\mathrm{H}_{n-1}(\mathrm{D}M) \;\cong\; \begin{cases} \text{free abelian group}, & \text{if } \mathrm{D}M \text{ is orientable,} \\ \text{free abelian group} \oplus \mathbb{Z}_2, & \text{if } \mathrm{D}M \text{ is non-orientable.} \end{cases}$$

Now we turn to the actual proofs of the three statements of the proposition.

(1) By the above we have an exact sequence

$$0 \longrightarrow \underbrace{\mathrm{H}_n(M)}_{=0} \underset{r_*}{\overset{i_*}{\hookrightarrow}} \mathrm{H}_n(\mathrm{D}M) \longrightarrow \underbrace{\mathrm{H}_n(\mathrm{D}M, M)}_{\cong \mathrm{H}_n(M,\partial M)} \xrightarrow{\partial} 0.$$

The desired statement now follows from (a)–(h) together with (i) and (j).

(2) By the above we have an exact sequence:

$$0 \longrightarrow \underbrace{\mathrm{H}_{n-1}(M)}_{\text{free abelian}} \underset{r_*}{\overset{i_*}{\rightleftarrows}} \mathrm{H}_{n-1}(\mathrm{D}M) \longrightarrow \underbrace{\mathrm{H}_{n-1}(\mathrm{D}M, M)}_{\cong \mathrm{H}_{n-1}(M,\partial M)} \xrightarrow{\partial} 0.$$

Since $r_* \circ i_* = \mathrm{id}_{\mathrm{H}_{n-1}(M)}$ we obtain from Splitting Lemma 78.2 and the above discussion that $\mathrm{H}_{n-1}(\mathrm{D}M) \cong \mathrm{H}_{n-1}(M, \partial M) \oplus$ free abelian group. The desired statement follows from (a)–(h) together with (k) and Lemma 51.9.

(3) We can deduce this statement from the Smooth Manifold-Top Homology Theorem 98.6 (3) using a slight variation on the above arguments. Alternatively we can deduce (3) from (1) and (2) using the Universal Coefficient Theorem 89.17, the same way that we deduced the Smooth Manifold-Top Homology Theorem 98.6 (3) from the Smooth Manifold-Top Homology Theorem 98.6 (1) and (2). ■

98.3. **Homology groups of lens spaces.** In this section we will apply the results from the previous sections to complete the long overdue calculation of the homology groups of lens spaces.

Lemma 98.9. (Lens Space Homology Lemma) Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. For the corresponding lens space $L(p, q)$ we have

$$\mathrm{H}_n(L(p,q); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 3, \\ 0, & \text{if } n = 2 \text{ or } n > 3, \\ \mathbb{Z}_p, & \text{if } n = 1. \end{cases}$$

Remark. An alternative approach to computing the homology groups of lens spaces will be given in Exercise 146.2.

The calculation of this lemma is rather disappointing, it shows that homology groups cannot distinguish between two lens spaces of the form $L(p, q)$ and $L(p, r)$. In particular we have made no progress at all on Question 48.7.

Proof. We start out with the following observations:

(1) It follows from the Actions-Covering Proposition 48.9 and the definition of a lens space that $L(p, q)$ is finitely covered by S^3. It follows from $\chi(S^3) = 0$ together with Proposition 69.4 that $\chi(L(p, q)) = 0$.
(2) Note that on page 1075 we pointed out that every lens space $L(p, q)$ is a closed orientable connected 3-dimensional smooth manifold.
(3) We already know from the H_0-Proposition 72.5 that $\mathrm{H}_0(L(p, q)) \cong \mathbb{Z}$.
(4) By Corollary 84.6 we know that $\mathrm{H}_1(L(p, q)) \cong \mathbb{Z}_p$.
(5) By (2) and the Smooth Manifolds-Invariants Proposition 96.6 (or alternatively the Top-Homology Theorem 106.3) we know that $\mathrm{H}_n(L(p, q)) = 0$ for $n \geq 4$.
(6) By the Smooth Manifold-Top Homology Theorem 98.6 (1) and (2) (or alternatively by the Orientation-Top Homology Theorem 106.1) we know that $\mathrm{H}_3(L(p, q)) \cong \mathbb{Z}$.

Thus at the moment we are still missing $\mathrm{H}_2(L(p,q))$. Now note that

$$\operatorname{rank}\mathrm{H}_2(L(p,q)) \overset{\text{by the Euler Characteristic-H}_*\text{-Proposition 87.1}}{\downarrow} = \underbrace{\chi(L(p,q))}_{=0} - \underbrace{\operatorname{rank}\mathrm{H}_0(L(p,q))}_{=1} + \underbrace{\operatorname{rank}\mathrm{H}_1(L(p,q))}_{=0} + \underbrace{\operatorname{rank}\mathrm{H}_3(L(p,q))}_{=1} = 0.$$

Furthermore we know by the Smooth Manifold-Top Homology Theorem 98.6 (2) that $\mathrm{H}_2(L(p,q))$ is a finitely generated free abelian group. Therefore we see that $\mathrm{H}_2(L(p,q)) = 0$. ■

98.4. **The Poincaré Homology Sphere.** We start out with the following almost self-explanatory definition of a homology sphere.

Definition.

(1) We say that an n-dimensional topological manifold X is a topological homology n-sphere if for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X;\mathbb{Z}) \cong \mathrm{H}_k(S^n;\mathbb{Z})$. Sometimes we also just say homology n-sphere.
(2) Sometimes, for disambiguation we also refer to such a topological manifold as an integral homology n-sphere.
(3) If X is actually a smooth manifold, then we refer to X as a smooth homology n-sphere.

Lemma 98.10. Let M be a smooth manifold. If M is a homology n-sphere, then M is closed, orientable and connected.

Proof. The lemma is an immediate consequence of the H_0-Proposition 72.5 and the Smooth Manifold-Top Homology Theorem 98.6. ■

It follows from the Topological 1-Dimensional Manifold Classification Theorem 22.1 and the Surface Classification Theorem 55.4 and the calculation of homology groups in Proposition 80.10 that for $n = 1, 2$ every homology n-sphere is homeomorphic to the standard sphere S^n. In 1900 Henri Poincaré [**Poi1900**],[**PGL1953**, p. 370] claimed that the same statement also holds for $n = 3$, i.e. he claimed that every homology 3-sphere is in fact homeomorphic to S^3.[1212] Shortly afterwards, in 1904, Henri Poincaré [**Poi1904, PGL1953**] himself found a counterexample to this claim. More precisely, he proved the following proposition.

Proposition 98.11. There exists a 3-dimensional smooth manifold M that is a homology 3-sphere, but such that $\pi_1(M)$ is a non-trivial group.

Remark. In Exercise 98.13 and in Proposition 141.5 we will prove the higher-dimensional analogue of Proposition 98.11.

Now we turn to the proof of Proposition 98.11. By now there are many proofs. In the following we will give Poincaré's example of such a smooth manifold. Later on in Exercise 100.4 we will give a different proof of Proposition 98.11.

[1212] In fact in [**PGL1953**, p. 498] he wrote "Pour ne pas trop allonger ce travail, je me bornerai à énoncer le théorème suivant dont la démonstration demanderait quelques développements: Tout polyèdre qui a tous ses nombres de Betti égaux Ã si et tous ses tableaux T_q bilatères est simplement connexe, c'est-à-dire homéomorphe à l'hypersphère.

To construct Poincaré's example we need to recall a few definitions and results from earlier on.

Definition.

(1) Let $z := 1 + \frac{\sqrt{5}}{2}$ be the golden ratio. As on page 1897 let D be the regular dodecahedron that is the convex hull of the 20 vertices

$$(\pm 1, \pm 1, \pm 1), \quad (0, \pm z, \pm \tfrac{1}{z}), \quad (\pm \tfrac{1}{z}, 0, \pm z), \quad \text{and} \quad (\pm z, \pm \tfrac{1}{z}, 0).$$

Furthermore let $\mathrm{Sym}(D)$ be the subgroup of $\mathrm{SO}(3)$ that preserves the regular dodecahedron, i.e. that preserves the set of these 20 vertices.

(2) Let $T := \{A \in \mathrm{M}(2 \times 2, \mathbb{C}) \,|\, A = \overline{A}^T \text{ and } \mathrm{tr}(A) = 0\}$. This is a 3-dimensional real vector space which we equip with the basis that is given by the matrices

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad E_3 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

Furthermore we equip T with the form $(A, B) \mapsto g(A, B) := \frac{1}{2} \mathrm{tr}\left(A\overline{B}^T\right)$. Next recall that on page 1819 we showed that the map $T \to \mathbb{R}^3$ given by $E_i \to e_i$ defines an isometry $(T, g) \to (\mathbb{R}^3, \langle\ ,\ \rangle)$ which we now use to make the identification $\mathrm{SIsom}(T, g) = \mathrm{SO}(3)$. Furthermore, in Lemma 83.1 we showed that the map

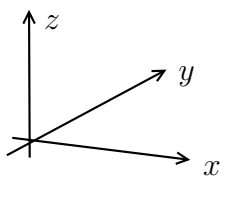
$$\begin{array}{rccl} q\colon & \mathrm{SU}(2) & \to & \mathrm{SIsom}(T, g) = \mathrm{SO}(3) \\ & A & \mapsto & (B \mapsto ABA^{-1}) \end{array}$$

is a well-defined group homomorphism, that it is an epimorphism and that the kernel is given by $\pm \mathrm{id}$.

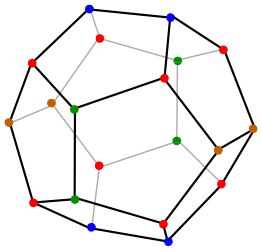
(3) We make the identification $S^3 = \{z \in \mathbb{H} = \mathbb{R}^4 \,|\, |z| = 1\}$, i.e. we view S^3 as the set of quaternions of length 1. By Proposition 92.2 (3) we know that the following map is a diffeomorphism and a group isomorphism:

$$\begin{array}{rccl} & f\colon S^3 & \to & \mathrm{SU}(2) \\ (z, w) = z + \mathrm{j} \cdot w & & \mapsto & \underbrace{\begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}}_{\substack{\text{matrix that represents} \\ \text{left-multiplication by } z + \mathrm{j} \cdot w \text{ with} \\ \text{respect to the complex basis } \{1, \mathrm{j}\} \text{ of } \mathbb{H}}} \end{array}$$

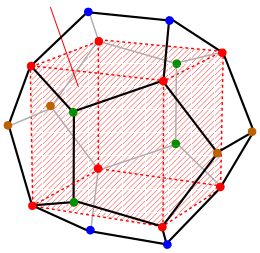
(4) We refer to $\Gamma := (q \circ f)^{-1}(\mathrm{Sym}(D)) \subset S^3$ as the binary icosahedral group.



dodecahedron



inscribed cube



Remark. It might be a little confusing that we refer to Γ as the binary *icosahedral* group. As we discussed on page 1898, the symmetry group of a regular icosahedron I equals the

symmetry group of a regular dodecahedron D. Therefore in the above definition of the binary icosahedral group we could replace $\mathrm{Sym}(D)$ by $\mathrm{Sym}(I)$, which is more logical and more common. But since it is a little easier to describe $\mathrm{Sym}(D)$ geometrically we prefer to work with the regular dodecahedron.

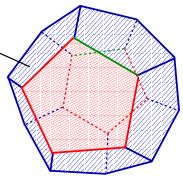
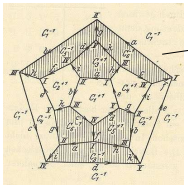
Proposition 98.12.

(1) The binary icosahedral group Γ is a group with 120 elements.
(2) The binary icosahedral group Γ has trivial abelianization.
(3) The quotient $M := S^3/\Gamma$ is a smooth homology 3-sphere with non-trivial fundamental group.

Definition. The smooth manifold S^3/Γ is called the Poincaré Homology Sphere.

Remark. The Poincaré Homology Sphere is ubiquitous in topology and can be described in many different ways. The different avatars and also the history of the Poincaré Homology Sphere are discussed in detail in [**SeW1933**], [**Rolf1990**, Chapter 10.E], [**Vol2013**] and [**KSc1979**]. In fact [**KSc1979**] provides altogether *eight* different ways to construct the Poincaré Homology Sphere. In the following we outline a few of these alternative descriptions of the Poincaré Homology Sphere.

(1) In 1904, when Henri Poincaré discovered the eponymous sphere, he gave a different description in terms of gluing two "handlebodies". It is shown in [**SeW1933**] that this topological space is homeomorphic to the topological space we describe next in (2).
(2) Let D be the "solid" regular dodecahedron. Now identify each pentagon on the boundary with the opposite pentagon via a $\pi/5$ twist in the "clockwise" direction. As is explained in [**KSc1979**], the resulting topological space is in fact homeomorphic to the Poincaré Homology Sphere as we defined it above.

dodecahedron   description of the Poincaré homology sphere in [**?**]

(3) On page 3140 we will give two descriptions of the Poincaré Homology Sphere as the result of a "surgery" on a knot respectively a link.

Proof.

(1) Let $z := 1 + \frac{\sqrt{5}}{2}$ be the golden ratio. Recall that D is the polyhedron that is spanned by the 20 vertices

$$(\pm 1, \pm 1, \pm 1), \quad (0, \pm z, \pm \tfrac{1}{z}), \quad (\pm \tfrac{1}{z}, 0, \pm z), \quad \text{and} \quad (\pm z, \pm \tfrac{1}{z}, 0).$$

In the figures on page 1898 and on 2160 we show that one can inscribe five cubes $C_1, \ldots, C_5$ into the regular dodecahedron. As we already mentioned on page 1897, it is shown in [**Arm1988**, p. 40] that the map

$$\begin{array}{rcl} \mathrm{Sym}(D) & \to & A_5 \cong \text{positive permutations of } \{C_1, \ldots, C_5\} \\ A & \mapsto & (C_i \mapsto A \cdot C_i) \end{array}$$

is an isomorphism. We consider the maps : $\mathrm{SU}(2) \to \mathrm{SIsom}(T,g) = \mathrm{SO}(3)$ and $f\colon S^3 \to \mathrm{SU}(2)$ that we defined above and we recall that the binary icosahedral group is defined as $\Gamma := (q\circ f)^{-1}(\mathrm{Sym}(D)) \subset S^3$. As we mentioned above, in Lemma 83.1 we showed that $\ker(q\circ f) = \pm 1 \in S^3 = \{z \in \mathbb{H} = \mathbb{R}^4 \,|\, |z| = 1\}$. Furthermore it follows from the SU(2)-SO(3)-Covering Theorem 83.2 (1) that $q\circ f$ is an epimorphism. Since $\mathrm{Sym}(D) \cong A_5$ has $\frac{1}{2}5! = 60$ elements and since $q\circ f$ has a kernel with two elements we see that Γ is a group with 120 elements.

(2) We need to show that the abelianization of Γ is trivial.[1212][1213] By (1) we know that the group $\Gamma \subset S^3 = \{z \in \mathbb{H} = \mathbb{R}^4 \,|\, |z| = 1\}$ fits into a short exact sequence

$$1 \;\to\; \{\pm 1\} \;\xrightarrow{\rho}\; \Gamma \;\xrightarrow{q\circ f}\; \mathrm{Sym}(D) \;\to\; 1.$$

By the Abelianization Proposition 53.20 we get an induced exact sequence

$$\{\pm 1\} \;\xrightarrow{\rho}\; \Gamma_{\mathrm{ab}} \;\xrightarrow{q\circ f}\; (\mathrm{Sym}(D))_{\mathrm{ab}} \;\to\; 1.$$

Almost every algebra book worth its money shows that $A_5 \cong \mathrm{Sym}(D)$ is a simple group, see e.g. [**Bog2008**, Theorem 11.3]. This fact implies that the abelianization of $\mathrm{Sym}(D)$ is trivial. Thus it suffices to prove the following claim.

Claim. The quaternion $-1 \in S^3 = \{z \in \mathbb{H} = \mathbb{R}^4 \,|\, |z| = 1\}$ represents the trivial element in Γ_{ab}.

Proof. First recall that a straightforward explicit calculation, which you and I carried out in Exercise 83.1 and on page 1960, shows that

$$(q\circ f)(\mathrm{i}) = q\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i}\end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1\end{pmatrix}}_{=:A} \text{ and } (q\circ f)(\mathrm{j}) = q\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1\end{pmatrix}}_{=:B}.$$

It follows immediately from the explicit description of the vertices of the regular dodecahedron D in (1) that the two matrices A and B to the right actually lie in $\mathrm{Sym}(D)$. Next note that it follows from the above calculation, that $\mathrm{i} = (q\circ f)^{-1}(A)$ and $\mathrm{j} = (q\circ f)^{-1}(B)$. In particular we see that $\mathrm{i},\mathrm{j} \in (q\circ f)^{-1}(\mathrm{Sym}(D)) = \Gamma$. By definition of the quaternion multiplication on page 1960 we have

$$[\mathrm{i},\mathrm{j}] \;=\; \underbrace{\mathrm{i}\cdot\mathrm{j}}_{=-\mathrm{j}\cdot\mathrm{i}} \cdot \mathrm{i}^{-1}\cdot\mathrm{j}^{-1} \;=\; -\mathrm{j}\cdot\mathrm{i}\cdot\mathrm{i}^{-1}\cdot\mathrm{j}^{-1} \;=\; -1.$$

We have thus shown that -1 is a commutator of elements in Γ. By definition of Γ_{ab} this implies that -1 represents indeed the trivial element in Γ_{ab}. ⊞

(3) Recall that it follows from Proposition 92.2 (3) that $\Gamma := (q\circ f)^{-1}(\mathrm{Sym}(D))$ is a finite subgroup of the *group* $S^3 = \{z \in \mathbb{H} = \mathbb{R}^4 \,|\, |z| = 1\}$. Since subgroups act freely on groups we see that Γ acts freely on S^3. Since Γ is a finite subgroup it now follows from the Manifold Quotient-by-Group Action Proposition 19.32 that $M := S^3/\Gamma$ is a closed 3-dimensional smooth manifold. It remains to show that $M = S^3/\Gamma$ is a

[1213] An argument similar to our proof of (2) is also provided in [**Bre1993**, p. 354] and in [**Toen2017**, p. 133].

homology sphere. First we perform the following calculation:

$$\mathrm{H}_1(S^3/\Gamma) \overset{\text{by the Hurewicz Theorem 84.5}}{\cong} \pi_1(S^3/\Gamma)_{\mathrm{ab}} \overset{\text{by the Sphere-}\pi_1\text{-Proposition 47.13 and the Fundamental Group-via-Actions Theorem 48.16}}{\cong} \Gamma_{\mathrm{ab}} \overset{\text{by (2)}}{=} 0.$$

Next note that by the Smooth Manifolds-Invariants Proposition 96.6 we have $\mathrm{H}_i(M) = 0$ for $i \geq 4$. Furthermore note that by construction Γ is contained in $\mathrm{SU}(2)$. Thus we see that the action of Γ on S^3 is orientation-preserving.[1214] Therefore it follows from the Orientation-Action Proposition 25.19 that $M = S^3/\Gamma$ is actually orientable. This implies in turn by the Smooth Manifold-Top Homology Theorem 98.6 that $\mathrm{H}_3(M) \cong \mathbb{Z}$. Thus it remains to show that $\mathrm{H}_2(M) = 0$. By the Smooth Manifold-Top Homology Theorem 98.6 (2) we know that $\mathrm{H}_2(M)$ is a free abelian group. So it suffices to show that $b_2(M) = 0$. In fact we have

$$b_2(M) \overset{\text{since } b_0(M) = b_3(M) = 1 \text{ and } b_1(M) = 0}{=} \chi(M) \overset{\text{Proposition 69.4 and the Smooth Manifold-CW Structure Theorem 96.5}}{=} \frac{1}{k} \cdot \chi(S^3) \overset{\text{see page 1879}}{=} 0.$$

■

Remark. Let M be a homology 3-sphere with *finite* fundamental group. As is pointed out in [**Kerv1969**, p. 67], it follows from the work of Michio Suzuki [**Suz1955**] or alternatively John Milnor [**Miln1957b**] that $\pi_1(M)$ is isomorphic to the binary icosahedral group. In fact, it follows from the much more fancy Elliptisation Theorem **??** that such M is actually diffeomorphic to the Poincaré Homology Sphere.

We conclude this section with the statement of a conjecture that was formulated by Henri Poincaré [**PGL1953**, p. 498] [**Poi1904**].

Conjecture 98.13. (Poincaré Conjecture) If M is a homology 3-sphere that is simply connected, then M is homeomorphic to S^3.

We will discuss the Poincaré Conjecture and various variations thereof in Chapter **??**.

98.5. **The topology of smooth manifolds.** In this section we provide two results on the topology of smooth manifolds. Let us formulate our first result.

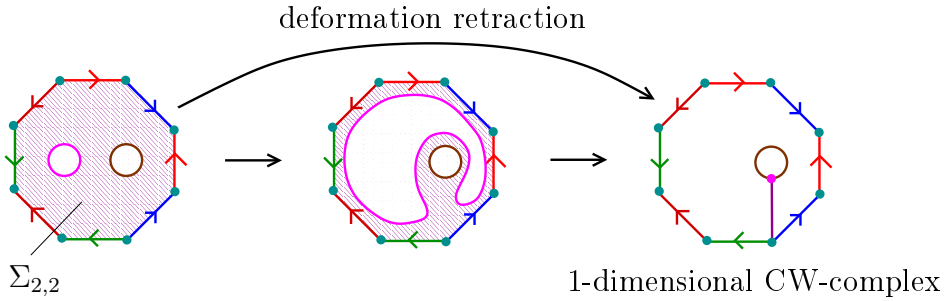
Proposition 98.14. (No Top-Cell Proposition) Let M be a connected non-empty n-dimensional smooth manifold. If M is non-compact or if M has non-empty boundary, then M is homotopy equivalent to an $(n-1)$-dimensional simplicial complex, in particular M is homotopy equivalent to an $(n-1)$-dimensional CW-complex.

Examples.

(1) Given $g \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$ we denote as usual by $\Sigma_{g,k}$ the surface of genus g with k open disks removed. In the figure below we sketch a deformation retraction from $\Sigma_{2,2}$ to a CW-complex of Euler characteristic -4.
More generally, almost the same argument shows that $\Sigma_{g,k}$ with $k \geq 1$ admits a deformation retraction to a CW-complex with k 0-cells and $2g + 2(k-1)$ 1-cells, i.e. a 1-dimensional CW-complex of Euler characteristic $k - (2g + 2(k-1)) = -2g - k + 2$.

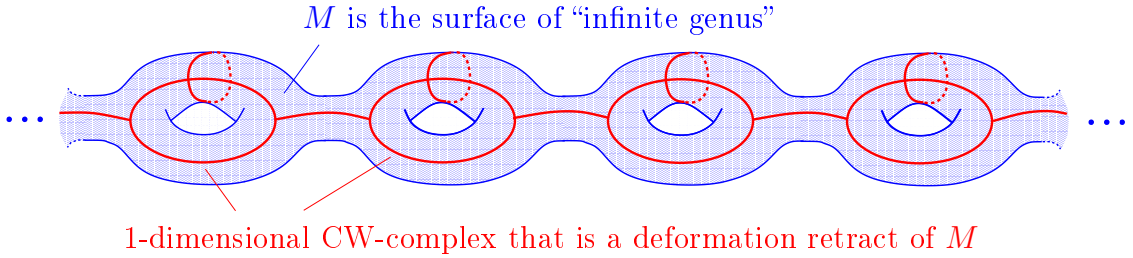
[1214] Since the action of Γ is free it follows also from Exercise 5.34 that the action is orientation-preserving.

Using the Quotient-by-Tree Proposition 69.8 (2) we also see that $\Sigma_{g,k}$ is homotopy equivalent to the wedge of $2g+k-1$ circles.



(2) The non-compact smooth manifold that is given by removing a point from the torus is easily seen to be homotopy equivalent to the wedge of two circles.

(3) In the figure below we show the surface M of infinite genus, which is non-compact, and a 1-dimensional CW-complex that is deformation retract of M. It is admittedly not overly obvious why the 1-dimensional CW-complex is indeed a deformation retract of M.



Remark. Let M be a compact connected n-dimensional smooth manifold with non-empty boundary. By the No Top-Cell Proposition 98.14 we now know that M is homotopy equivalent to a finite $(n-1)$-dimensional CW-complex which implies by the Homology-of-CW-Complex Proposition 80.6 that $\mathrm{H}_n(M)=0$ and that $\mathrm{H}_{n-1}(M)$ is a finitely generated free abelian group. This is of course consistent with the Smooth Manifold-Top Homology Theorem 98.6.

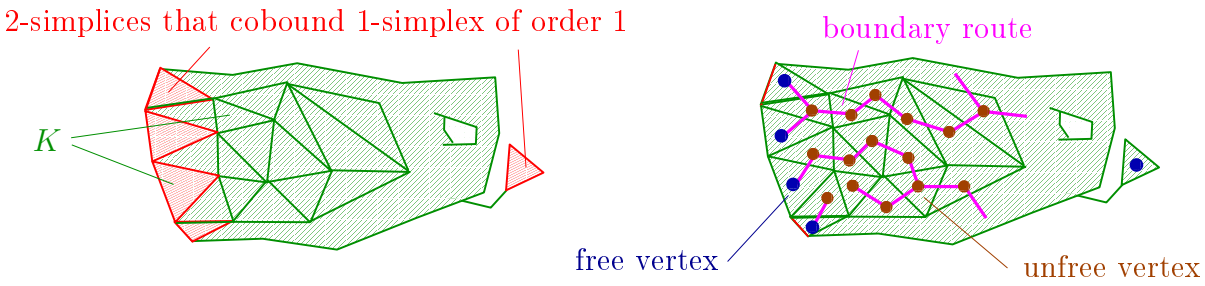
The proof of the No Top-Cell Proposition 98.14 requires some preparations.

Definition. Let $K=(V,S)$ be an n-dimensional abstract simplicial complex such that every $(n-1)$-simplex has order ≤ 2. We denote by W the set of n-simplices and we denote by E the set of $(n-1)$-simplices.

(1) We say an n-simplex is free if it admits a codimension one face of order 1. Otherwise we say that the n-simplex is unfree.

(2) A boundary route is a subgraph $\widetilde{\Gamma}=(W,\widetilde{F},\varphi)$ of the dual graph[1215] $\Gamma=(W,E_2,\varphi)$ with the following property: every n-simplex of K, i.e. every vertex of the dual graph, is connected in $\widetilde{\Gamma}$ to a free n-simplex.

[1215] We refer to page 2143 for the definition of the dual graph.



Lemma 98.15. Let $K = (V, S)$ be an n-dimensional abstract simplicial complex that admits a boundary route. There exists a subcomplex $K' = (V', S')$ of $K = (V, S)$ with the following three properties:

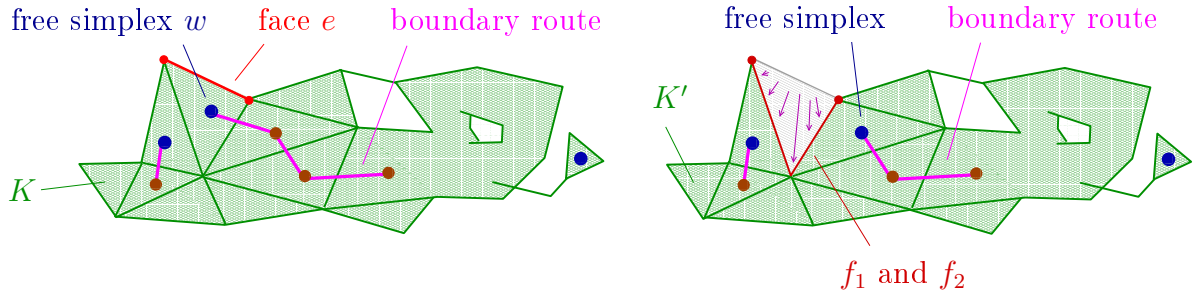
(1) $K' = (V', S')$ admits a boundary route.
(2) $|K'|$ is a deformation retract of $|K|$.
(3) The number of n-simplices of K' is less than the number of n-simplices of K.

Proof of Lemma 98.15. Let $K = (V, S)$ be an n-dimensional abstract simplicial complex that admits a boundary route $\widetilde{\Gamma} = (W, \widetilde{F}, \varphi)$. Since K is n-dimensional we know that it contains at least one n-simplex and since K admits a boundary route we see that there exists an n-simplex w that is free. By definition this means that w admits a codimension-one face e of order 1. We denote by $f_1, \ldots, f_n$ the remaining codimension-one faces of w. We set

$$K' \ := \ (V, S \setminus \{w, e\}).$$

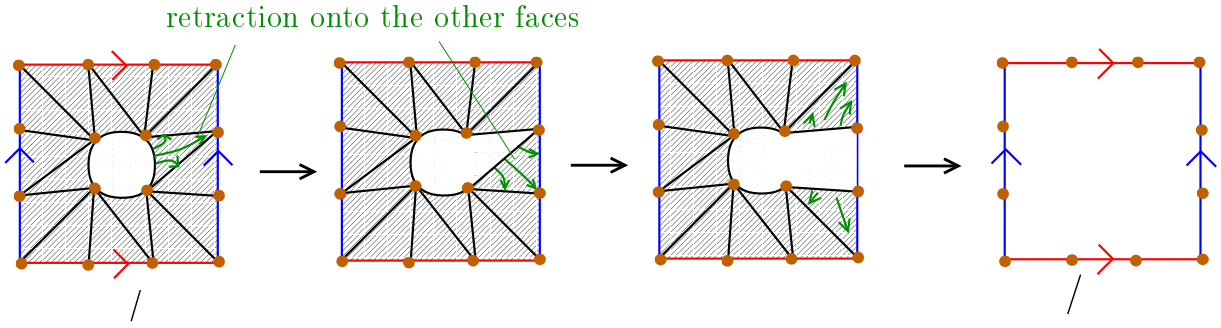
Note that K' is indeed a subcomplex of K. A slight generalization of Exercise 15.13 shows that there exists a deformation retraction from $|w|$ to $\bigcup_{i=1}^{n} |f_i|$. It follows from Lemma 93.8 together with the Homotopy Stacking Lemma 14.3 that this deformation retraction together with the constant homotopy on $|K'|$ defines a deformation retraction from $|K|$ to $|K'|$.

Since every $(n-1)$-simplex of K has order ≤ 2 we see see that every n-simplex of K' that contains one of the f_i as a face is a free n-simplex of K'. Using this observation it is now straightforward to see a boundary route for K' is given by $(W \setminus \{w\}, \{\widetilde{f} \in \widetilde{F} \mid \widetilde{f} \not\subset w\}, \varphi)$. ■



Proof of the No Top-Cell Proposition 98.14 in the compact case. Let M be a compact connected non-empty n-dimensional smooth manifold with non-empty boundary. Recall that by the Smooth Manifold-Simplicial Structure Theorem 96.2 we know that M admits a simplicial structure $(K = (V, S), \Theta\colon |K| \to M)$ such that ∂M is a subcomplex. We need to show that $|K|$ is homotopy equivalent to a finite $(n-1)$-dimensional simplicial complex.

Let $\Gamma = (W, E_2, \varphi)$ be the dual graph of K. By the Manifold-Simplex Proposition 98.1 (5) we know that Γ is connected. Thus it follows from the Spanning Tree Existence Proposition 7.10 (1) that Γ admits a spanning tree T. By definition T is connected and by the Spanning Tree Existence Proposition 7.10 we know that a spanning tree contains all vertices. Since $\partial M \neq \varnothing$ we know by the Manifold-Simplex Proposition 98.1 that $\partial K \neq \varnothing$. Thus we see that T is in fact a boundary route for K. This means that we can start up the machine given by Lemma 98.15. Iteratively applying Lemma 98.15 we get the desired deformation retraction. ■



simplicial structure for the surface of genus one with one boundary component

1-dimensional simplicial complex

Proof of the No Top-Cell Proposition 98.14 in the non-compact case. Let M be a non-compact connected n-dimensional smooth manifold. It follows from [**NR2004**, Theorem 2.2], or alternatively [**WhdJ1961b**, Lemma 2.1], together with Proposition 94.12 that M is homotopy equivalent to an $(n-1)$-dimensional simplicial complex.

We will not attempt to give a full proof of our own, but we will sketch an outline of a proof. Evidently we again use the fact that by the Smooth Manifold-Simplicial Structure Theorem 96.2 we know that we can equip M with a simplicial structure $(K = (V, S), \Theta\colon |K| \to M)$. It is fairly straightforward to show that there exists a sequence $Y_1, Y_2, \ldots$ of non-empty subcomplexes of K with the following properties:

(1) The sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $Y_i \subset Y_{i+1}$.
(2) Each X_i is a finite n-dimensional pseudomanifold.
(3) We have $\bigcup\limits_{i\in\mathbb{N}} X_i = K$.

Now, with a considerable effort one can modify the above arguments used in the compact case to show that there exists a sequence of $(n-1)$-dimensional subcomplexes $X_i \subset Y_i$ with the following properties:

(4) For each $i \in \mathbb{N}$ we have $X_i \subset X_{i+1}$.
(5) Each $|X_i|$ is a deformation retract of $|Y_i|$.

Finally we set $X := \bigcup\limits_{i\in\mathbb{N}} X_i$. Note that X is an $(n-1)$-dimensional subcomplex of Y. It remains to show that X is a deformation retract of Y.

We pick a base point $* \in$. Using the above one can show, reasonably easily, that the inclusion induced map $\pi_n(X, *) \to \pi_n(Y, *)$ is an isomorphism for every $n \in \mathbb{N}$. It follows from the Whitehead Deformation Retract Proposition 162.2[1216], which is a variation on the

[1216]My apologies for referring to a future result.

Whitehead Theorem 162.1, that X is indeed a deformation retract of Y. We leave it as a charming challenge to the reader to turn this idea into a rigorous proof. ■

We give a new proof of the following proposition.

Proposition 64.5. Let M be a connected 2-dimensional smooth manifold. If M is non-compact, then $\pi_1(M)$ is a free group.

Proof. Let M be a connected 2-dimensional smooth manifold that is non-compact. By the No Top-Cell Proposition 98.14 we know that M is homotopy equivalent to a 1-dimensional CW-complex. It follows from the Homotopy-π_1-Proposition 50.3 (2) and Proposition 69.9 (1) that $\pi_1(M)$ is a free group. ■

We conclude this chapter with the following proposition.

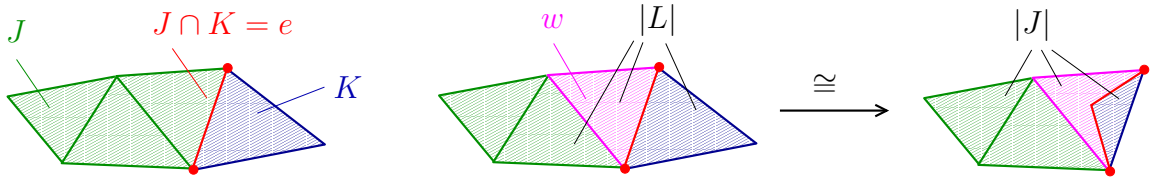
Proposition 98.16. (Single Top-Cell Proposition) Let M be a compact connected n-dimensional smooth manifold. There exists a CW-structure for M with a single n-cell.

Remark. For fans of small CW-complexes we also recall Proposition 70.4 which says that any 0-connected CW-complex X is homotopy equivalent to a CW-complex Y with a single 0-cell which has the property that for any $n \geq 2$ the number of n-cells of X equals the number of n-cells of Y.

In the proof of the Single Top-Cell Proposition 98.16 we will make use of the following lemma.

Lemma 98.17. Let L be an n-dimensional pseudomanifold that contains two subcomplexes J and K. If K consists of a single n-simplex, if $J \cap K$ consists of a single $(n-1)$-simplex e and if e has order 1 in J, then there exists a homeomorphism $(|L|, |\partial L|) \to (|J|, |\partial J|)$ of pairs of topological spaces.

Sketch of a proof of Lemma 98.17. Let w be the one n-simplex of J that cobounds e. It is elementary, albeit ever-so-slightly painful, to write down an explicit homeomorphism $|w| \cup |K| \to |w|$ that is the identity on $|\partial w| \setminus (|\partial w| \cap |K|)$. Together with the identity on $|J| \setminus |w|$ this homeomorphism gives us the desired homeomorphism from $|L|$ to $|J|$. It follows almost immediately from the definitions that this homeomorphism restricts to a homeomorphism $|\partial L| \to |\partial J|$. We leave it to the reader to fill in the details. ■



Sketch of a proof of the Single Top-Cell Proposition 98.16. We sketch a proof of the proposition. The statement is also proved with fewer details but more authority in [**Mc2002**, Theorem 5.1].

Let M be a closed connected non-empty n-dimensional smooth manifold. By the Smooth Manifold-Simplicial Structure Theorem 96.2 we know that M admits a simplicial structure $(K = (V, S), \Theta\colon |K| \to M)$. We need to show that $|K|$ admits a CW-structure with a single n-cell.

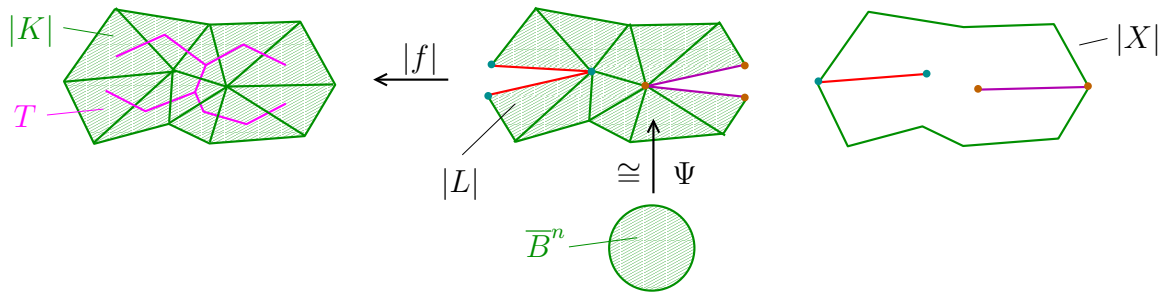
Let $\Gamma = (W, E_2, \varphi)$ be the dual graph of K. By the Manifold-Simplex Proposition 98.1 (5) we know that Γ is connected. Thus it follows from the Spanning Tree Existence Proposition 7.10 (1) that Γ admits a spanning tree T. Note that by the Spanning Tree Existence Proposition 7.10 (2) we know that T contains every vertex of Γ.

Let X be the subcomplex of K that is given by removing all n-simplices and by removing all $(n-1)$ that do not correspond to an edge of the spanning tree T. Note that X is an $(n-1)$-dimensional simplicial complex. Furthermore let L be the abstract simplicial complex that is given by the "disjoint union of the n-simplices of K and gluing them along the $(n-1)$-simplices corresponding to the edges of T". It is not difficult, but also not particularly instructive, to write down a completely rigorous definition of L. Note that L comes with a natural simplicial map $f\colon L \to K$.

Next note that using an induction argument based on Lemma 98.17, and which also makes use of the Standard Simplex Homeomorphism Lemma 72.1, it is rather straightforward to show that there exists a homeomorphism $\Psi\colon (\overline{B}^n, S^{n-1}) \to (\Delta^n, \partial\Delta^n) \to (|L|, |\partial L|)$. Finally, using the Compact-Hausdorff Proposition 2.17 (3) one can verify that the map

$$\begin{array}{rcl} & & \text{where } x \sim (\Psi \circ |f|)(x) \text{ for every } x \in S^{n-1} \\ & & \downarrow \\ (\overline{B}^n \sqcup |X|)/\!\sim & \to & |K| \\ [P] & \mapsto & \begin{cases} (\Psi \circ |f|)(P), & \text{if } P \in \overline{B}^n, \\ P, & \text{if } P \in |X| \subset |K| \end{cases} \end{array}$$

is a homeomorphism. The left-hand side admits an obvious CW-structure with a single n-cell. Thus we have shown that $|K|$ admits a CW-structure with a single n-cell. ∎



Exercises for Chapter 98.

Exercise 98.1. Let K be a finite n-dimensional pseudomanifold.

(a) Show that the suspension $\Sigma(K)$, as defined on page 2003, is an $(n+1)$-dimensional pseudomanifold.
(b) Show that if K is closed, then $\Sigma(K)$ is closed.
(c) Show that if K is orientable in the sense of the definition on page 2142, then $\Sigma(K)$ is orientable.

Exercise 98.2. Let K be an n-dimensional pseudomanifold such that ∂K is an $(n-1)$-dimensional pseudomanifold. Let $i\colon \partial K \to K$ be the inclusion map. We consider the cone $K \cup_{\partial K} \mathrm{Cone}(\partial K)$. Show that $\mathrm{Cone}(i\colon \partial K \to K)$ is a closed n-dimensional pseudomanifold.

Exercise 98.3. Let K be an m-dimensional pseudomanifold and let L be an n-dimensional pseudomanifold. We equip K and L with a total order and we define the product $K \times L$ as on page 1998.

(a) Show that $K \times L$ is an $(m+n)$-dimensional pseudomanifold.
(b) Show that $K \times L$ is closed if and only if both K and L are closed.
(c) Show that $K \times L$ is orientable if and only if both K and L are orientable.

Exercise 98.4.

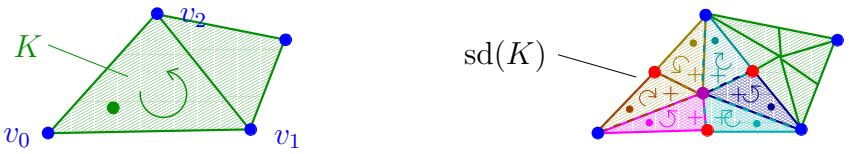
(a) Let K be a 1-dimensional pseudomanifold. Show that $|K|$ is a closed 1-dimensional smooth manifold.
(b) Let K be a 2-dimensional pseudomanifold. Show that $|K|$ is a closed 2-dimensional smooth manifold.

Remark. Part (a) is fairly straightforward. The proof of part (b) is more delicate. One way to prove part (b) is to imitate the proof, given in Propositions 18.7 and 19.18, that the surface Σ_g of genus g is closed 2-dimensional smooth manifold.

Exercise 98.5. Let $K = (V, S)$ be an n-dimensional pseudomanifold.

(a) Show that the barycentric subdivision $\mathrm{sd}(K)$ is an n-dimensional pseudomanifold.
(b) Suppose that K is equipped with an orientation. Let $t = \{\underline{s_0}, \ldots, \underline{s_n}\}$ be an n-simplex of $\mathrm{sd}(K)$. For $i = 0, \ldots, n$ we denote by v_i the vertex of s_i that is not contained in s_{i-1}. Let $\epsilon \in \{-1, 1\}$ be the sign such that $(\epsilon, (v_0, \ldots, v_n))$ matches the given orientation of K. We equip the n-simplex t of $\mathrm{sd}(K)$ with the orientation given by $(\epsilon, (\underline{s_0}, \ldots, \underline{s_n}))$. Show that the above orientations of the n-simplices of $\mathrm{sd}(K)$ define an orientation for $\mathrm{sd}(K)$.

We refer to the figure below for an illustration.

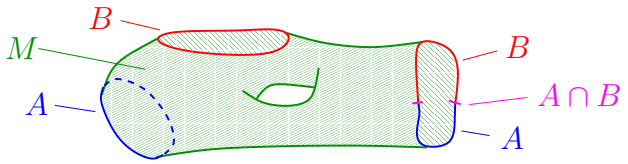


Exercise 98.6. Let M be a compact connected non-empty smooth manifold. Furthermore let $(K = (V, S), \Theta\colon |K| \to M)$ be a simplicial structure for M. Let Γ be the dual graph as defined on page 2143. Is there a relationship between the homology groups $\mathrm{H}_1(\Gamma; \mathbb{Z})$ and the groups $\mathrm{H}_1(M; \mathbb{Z})$ or $\mathrm{H}_1(M, \partial M; \mathbb{Z})$? For example is there a relationship between the ranks or the minimal number of generators?

Exercise 98.7. Let M be a compact orientable n-dimensional smooth manifold. Suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. Show that $\mathrm{H}_{n-1}(M, A)$ is a free abelian group.

Exercise 98.8. Let M be a compact connected n-dimensional smooth manifold and let $p \neq 2$ be a prime. Determine $\mathrm{H}_n(M, \partial M; \mathbb{F}_p)$ and $\mathrm{H}_n(M; \mathbb{F}_p)$.

Exercise 98.9. Let $k, l \in \mathbb{N}$ and let $N_{k,l}$ be $N_k = k \cdot \mathbb{R}\mathrm{P}^2$ minus l open disks. Determine $\mathrm{H}_1(N_{k,l}, \partial N_{k,l}; \mathbb{Z})$.



Exercise 98.10. Let K be an infinite n-dimensional pseudomanifold such that $\partial K = \varnothing$. Show that $\mathrm{H}_{n-1}^{\mathrm{simp}}(K)$ is a free abelian group.
Remark. This exercise will require a serious effort.

Exercise 98.11. Let $p, q \in \mathbb{N}$ be coprime. Show that every orientation-reversing diffeomorphism of the lens space $L(p,q)$ has at least one fixed point.

Exercise 98.12. Let X be a connected 2-dimensional smooth manifold with infinite fundamental group (We do not assume that X is compact.) Show that X is aspherical, i.e. show that $\pi_i(X) = 0$ for $i \geq 2$.
Hint. First use the Hurewicz Theorem 85.5 and the results from this section to show that $\pi_2(X) = 0$.
Remark. This exercise gives in particular a new proof of the fact that given a surface Σ_g of genus $g \geq 1$ we have $\pi_i(\Sigma_g) = 0$ for $i \geq 2$. We had initially proved this statement on page 1575 using the rather difficult results from Chapter 66.

Exercise 98.13. Let M be a 3-dimensional smooth homology 3-sphere with non-trivial fundamental group.[1217] Furthermore let $n \geq 4$. We pick a smooth embedding $\varphi\colon \overline{B}^3 \to M$ and we consider $W := (M \setminus \varphi(B^3) \times \overline{B}^{n-3}$. By Proposition **??** we can equip W with the structure of an orientable n-dimensional smooth manifold. Show that the double $\mathrm{D}W$ is an n-dimensional smooth homology n-sphere with non-trivial fundamental group.

Exercise 98.14. Let Σ be a compact oriented connected 2-dimensional smooth manifold. We consider the product $N := \Sigma \times [0,1]$ and we consider the subset $A := \partial\Sigma \times [0,1] \subset N$. Let $\varphi\colon \Sigma \to \Sigma \times \{0\} \subset N$ be the obvious inclusion map. Now let $g\colon S \to N$ be a map from a compact oriented connected 2-dimensional smooth manifold with $g(\partial S) \subset A$ such that $\varphi_*([\Sigma]) = g_*([S])$. Show that $\chi(S) \leq \chi(\Sigma)$.
Remark. In a slightly simplified version the exercise says that the surface $\Sigma \times \{0\}$ is the "simplest" surface in N representing the homology class $[\Sigma \times \{0\}]$.

[1217] By Proposition 98.11 we know that such M exists.

99. Simplicial cohomology

In this chapter we will introduce the simplicial cohomology groups of an (ordered) abstract simplicial complex. This chapter can be seen as a continuation of our discussions of abstract simplicial complexes in Chapters 93, 94 and 95. The objects in this chapter are for the most part purely combinatorial and for the most part this chapter can be read independently of the chapters on singular cohomology. We will only make use of the purely algebraic statements in Sections 108.1, 108.2 and 108.3.
As for simplicial homology, the study of simplicial cohomology is great fun in its own right. But as we will see, it is also a very convenient tool for calculating singular cohomology groups and for determining the often intractable cup and cap products.
In Chapter 103 we will prove the Simplicial Poincaré Duality Theorem 103.4 for pseudo-manifolds, which gives us in particular a Poincaré Duality theorem for smooth manifolds.

99.1. **Simplicial chain complexes.** In the following section we will introduce simplicial cohomology groups. Before we do so it is convenient to recall several definitions and results on simplicial chain complexes.
We start out with the following definition from page 2036.

Definition. Let $K = (V, S)$ be an ordered abstract simplicial complex.[1218] Given $k \in \mathbb{N}_0$ we define

$$\mathrm{C}_k^{\mathrm{simp},\leq}(K) := \text{free abelian group generated by the set of } k\text{-simplices of } K.$$

Furthermore we consider the boundary map

$$\begin{aligned} \partial_k \colon \mathrm{C}_k^{\mathrm{simp},\leq}(K) &\to \mathrm{C}_{k-1}^{\mathrm{simp},\leq}(K) \\ \underbrace{\{v_0 < \cdots < v_k\}}_{\substack{\text{any } k\text{-simplex } s \text{ is uniquely} \\ \text{of the form } s = \{v_0,\dots,v_k\} \\ \text{with } v_0 < \cdots < v_k}} &\mapsto \sum_{i=0}^{k} (-1)^i \cdot \{v_0, \dots, v_{i-1}, \widehat{v_i}, v_{i+1}, \dots, v_k\}. \end{aligned}$$

By Lemma 95.2 we know that $(\mathrm{C}_*^{\mathrm{simp},\leq}(K), \partial_*)$ is a chain complex, thus we can define the corresponding simplicial homology groups $\mathrm{H}_*^{\mathrm{simp},\leq}(K)$.

The above definition is straightforward, but it does have the major disadvantage that it requires the choice of an order on K. In most situations there is no natural choice of an order. In the absence of an order we are led to the following more elaborate definition which initially we gave on page 2038 and in Lemma 95.6.

Definition. Let $K = (V, S)$ be an abstract simplicial complex.

(1) Let $k \in \mathbb{N}_0$. An ordered k-simplex of K is a $(k+1)$-tuple $(v_0, \dots, v_k) \in V^{k+1}$ such that $\{v_0, \dots, v_k\}$ is a k-simplex of $K = (V, S)$. In the following we write $\lfloor (v_0, \dots, v_k) \rfloor := \{v_0, \dots, v_k\}$.

[1218] Recall that, according to the definition on page 1982, an *order* on an abstract simplicial complex (V, S) is a partial order on V that has the property that the restriction to each simplex is actually a total order.

(2) Let $k \in \mathbb{N}_0$. We define the k-th simplicial chain group

$$\mathrm{C}_k^{\mathrm{simp}}(K) := \begin{array}{c}\text{free abelian group on the set}\\ \text{of ordered } k\text{-simplices of } K\end{array} \Big/ \begin{array}{c}(v_0,\dots,v_k) - \mathrm{sign}(\sigma)\cdot(v_{\sigma(0)},\dots,v_{\sigma(k)})\\ \text{where } \sigma \in S_{k+1} = \mathrm{Bij}\{0,\dots,k\}\end{array}$$

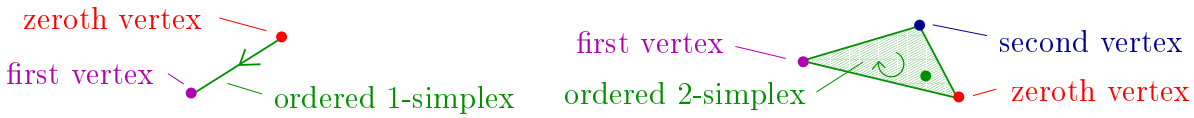
(3) Given an ordered simplex $(v_0,\dots,v_k)$ we denote its image in $\mathrm{C}_k^{\mathrm{simp}}(K)$ by $[v_0,\dots,v_k]$.

(4) We consider the map

$$\begin{aligned}\partial_k \colon \mathrm{C}_k^{\mathrm{simp}}(K) &\to \mathrm{C}_{k-1}^{\mathrm{simp}}(K)\\ [v_0,\dots,v_k] &\mapsto \sum_{i=0}^{k} (-1)^i \cdot [v_0,\dots,v_{i-1},\widehat{v_i},v_{i+1},\dots,v_k].\end{aligned}$$

(5) By Lemma 95.6 we know that $(\mathrm{C}_*^{\mathrm{simp}}(K),\partial_*)$ is a chain complex, thus we can define the corresponding simplicial homology groups $\mathrm{H}_*^{\mathrm{simp}}(K)$.

Notation. In pictures we denote the ordering of the simplices of a 1-simplex in the obvious way by an arrow pointing from the first to the second vertex. Furthermore, given an ordered 2-simplex we indicate the first vertex by a dot and the ordering of the remaining vertices by an oriented 3/4-circle. We refer to the figure below for an illustration.



We recall the following lemma.

Lemma 95.6. Let K be an abstract simplicial complex. If we choose for each k-simplex $s \in S$ an ordered k-simplex $\widetilde{s}$ with $\lfloor \widetilde{s} \rfloor = s$, then the $[\widetilde{s}]$ form a basis for $\mathrm{C}_k^{\mathrm{simp}}(K)$.

We conclude this short section with the following proposition that combines several earlier statements on the relationship between the above two simplicial chain complexes and also the singular chain complex of the topological realization.

Proposition 99.1.

(1) Let $(K = (V,S), \leq)$ be an ordered abstract simplicial complex.

(a) We consider the following maps:

$$\begin{array}{ccccc}\mathrm{C}_k^{\mathrm{simp}}(K) & \xleftarrow{\Omega_{\leq,k}} & \mathrm{C}_k^{\mathrm{simp},\leq}(K) & \xrightarrow{\Theta_{\leq,k}} & \mathrm{C}_k(|K|)\\ [v_0,\dots,v_k] & \longleftarrow & s = \{v_0 < \cdots < v_k\} & \longmapsto & (\Phi_s^{\leq} \colon \Delta^k \to |K|).\end{array}$$

map given by sending e_i to v_i

The maps $\Omega_{\leq,k}$ to the left define a natural isomorphism of chain complexes whereas the maps $\Theta_{\leq,k}$ to the right define a natural chain homotopy equivalence.

(b) There exist maps

$$\pi_k \colon \mathrm{C}_k(|K|) \to \mathrm{C}_k^{\mathrm{simp},\leq}(K)$$

which have the following two properties:

(i) The maps π_k form a chain map which form a chain homotopy equivalence.

(ii) For each $k \in \mathbb{N}_0$ we have $\pi_k \circ \Theta_k = \mathrm{id}_{\mathrm{C}_k^{\mathrm{simp},\leq}(K)}$.

(2) Let K be an abstract simplicial complex and let "$\leq$" and "$\subseteq$" be two orders on K. The two chain maps

$$\begin{array}{ccccccc} & & \mathrm{C}_k^{\mathrm{simp}}(K) & \xrightarrow{\Omega^{-1}_{\leq,k}} & \mathrm{C}_k^{\mathrm{simp},\leq}(K) & \xrightarrow{\Theta_{\leq,k}} & \mathrm{C}_k(|K|) \\ \text{and} & & \mathrm{C}_k^{\mathrm{simp}}(K) & \xrightarrow{\Omega^{-1}_{\subseteq,k}} & \mathrm{C}_k^{\mathrm{simp},\subseteq}(K) & \xrightarrow{\Theta_{\subseteq,k}} & \mathrm{C}_k(|K|) \end{array}$$

are chain homotopic.

(3) Let G be an abelian group. The chain homotopy equivalence class of (2) induces a natural isomorphism

$$\Xi_{k*}\colon \mathrm{H}_k^{\mathrm{simp}}(K;G) \;\to\; \mathrm{H}_k(|K|;G).$$

Proof.

(1) (a) This statement is the combination of Lemma 95.8 and the Simplicial-Singular Chain Complex Theorem 95.25.

(b) It follows almost immediately from the definitions that for each $k \in \mathbb{N}_0$ the map $\mathrm{C}_k^{\mathrm{simp},\leq}(K) \to \mathrm{C}_k(|K|)$ is a monomorphism and that the image is a summand. The desired statement is now an immediate consequence of (a) and Proposition 81.5.

(2) This statement is the content of Lemma 95.26.

(3) It follows immediately from (2) and a slight generalization of Lemma 73.2 that any choice of order on K induces the same map on homology with G-coefficients, i.e. the map Ξ_{k*} is well-defined. It follows easily from (1) and Corollary 73.4 that the map Ξ_{k*} is a natural isomorphism. ■

99.2. **The definition of the simplicial cohomology groups.** In this section we will introduce the simplicial cohomology groups of an (ordered) abstract simplicial complex. The discussion in the previous section also shows that it will be necessary to consider two slightly different types of simplicial cohomology groups.
We start out with a decidedly dull definition.

Definition. Let K be an (ordered) abstract simplicial complex and let G be an abelian group. For each $k \in \mathbb{N}_0$ we write

$$\mathrm{C}^k_{\mathrm{simp}}(K;G) := \mathrm{Hom}(\mathrm{C}_k^{\mathrm{simp}}(K), G) \quad \text{and} \quad \mathrm{C}^k_{\mathrm{simp},\leq}(K;G) := \mathrm{Hom}(\mathrm{C}_k^{\mathrm{simp},\leq}(K), G).$$

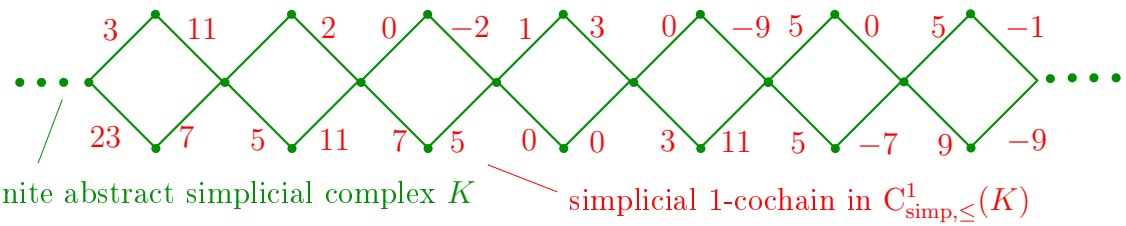
Elements of $\mathrm{C}^k_{\mathrm{simp}}(K;G)$ and $\mathrm{C}^k_{\mathrm{simp},\leq}(K;G)$ are called simplicial k-cochains.

Remark. Let G be an abelian group, let K be an ordered abstract simplicial complex and let $k \in \mathbb{N}_0$.

(1) It follows from Lemma 108.1 that a simplicial k-cochain in $\mathrm{C}^k_{\mathrm{simp},\leq}(K;G)$ is the same data as assigning an element of G to *every* k-simplex. Note that in contrast to the setting of singular cochains one can actually hope to "see" simplicial cochains. For example see the figure below for a simple-minded illustration.

(2) In contrast, it follows easily from the Tensor Lemma 89.3 that a chain in $\mathrm{C}_k^{\mathrm{simp},\leq}(K) \otimes G$ is the same data as assigning an element of G to *finitely many* k-simplices.

Now we introduce the following notation.



Notation.

(1) Let K be an order abstract simplicial complex and let s be an ordered k-simplex. We denote by $s^*\colon \mathrm{C}_k^{\mathrm{simp},\leq}(K) \to \mathbb{Z}$ the map that is given by $s^*(s) = 1$ and $s^*(t) = 0$ for any k-simplex $t \neq s$.

(2) Let K be an abstract simplicial complex and let $s = (v_0, \ldots, v_k)$ be an ordered k-simplex. It follows immediately from the definitions that

$$\begin{array}{rcl} s^*\colon \mathrm{C}_k^{\mathrm{simp}}(K) & \to & \mathbb{Z} \\ [w_0, \ldots, w_k] & \mapsto & \begin{cases} 0, & \text{if } \{v_0, \ldots, v_k\} \neq \{w_0, \ldots, w_k\}, \\ \mathrm{sign}(\sigma), & \text{if there exists a } \sigma \in \mathrm{Bij}\{0, \ldots, k\} \\ & \qquad \text{with } w_i = v_{\sigma(i)} \text{ for } = 0, \ldots, k \end{cases} \end{array}$$

is well-defined, in other words, s^* is a well-defined cochain in $\mathrm{C}^k_{\mathrm{simp}}(K;\mathbb{Z})$.

In the following we will mostly study the simplicial cohomology of finite (ordered) abstract simplicial complexes. In this context the following lemma is very helpful.

Lemma 99.2. Let $k \in \mathbb{N}_0$.

(1) Let $K = (V, S)$ be an ordered abstract simplicial complex. If K is *finite*, then the set $\{s^* \mid s \text{ a } k\text{-simplex}\}$ is a basis for $\mathrm{C}^k_{\mathrm{simp},\leq}(K;\mathbb{Z})$.

(2) Let $K = (V, S)$ be an abstract simplicial complex. We choose for each k-simplex s an ordered k-simplex $\widetilde{s}$ with $\lfloor \widetilde{s} \rfloor = s$. If K is finite, then the $\widetilde{s}^*$ form a basis for $\mathrm{C}^k_{\mathrm{simp}}(K;\mathbb{Z})$.

Proof. This follows immediately from Lemma 108.1 together with the definition of $\mathrm{C}_k^{\mathrm{simp},\leq}(K)$ and together with Lemma 95.6. ■

Now that we have gained a little bit of familiarity with cochains let us move on to the definition of the simplicial cochain complex and the simplicial cohomology groups.

Definition. Let K be an abstract simplicial complex and let G be an abelian group. For each $k \in \mathbb{N}_0$ we refer to

$$\delta_k := \partial_{k+1}^*\colon \underbrace{\mathrm{Hom}(\mathrm{C}_k^{\mathrm{simp}}(K), G)}_{=\mathrm{C}^k_{\mathrm{simp}}(K;G)} \to \underbrace{\mathrm{Hom}(\mathrm{C}_{k+1}^{\mathrm{simp}}(K), G)}_{=\mathrm{C}^{k+1}_{\mathrm{simp}}(K;G)}$$

as the coboundary map. Any simplicial k-cochain in $\ker(\delta_k\colon \mathrm{C}^k_{\mathrm{simp}}(K;G) \to \mathrm{C}^{k+1}_{\mathrm{simp}}(K;G))$ is called a simplicial k-cocycle. We refer to the cochain complex

$$\ldots \leftarrow \mathrm{C}^{k+1}_{\mathrm{simp}}(K;G) \xleftarrow{\delta_k} \mathrm{C}^k_{\mathrm{simp}}(K;G) \xleftarrow{\delta_{k-1}} \mathrm{C}^{k-1}_{\mathrm{simp}}(K;G) \leftarrow \ldots \xleftarrow{\delta_0} \mathrm{C}^0_{\mathrm{simp}}(K;G) \leftarrow 0$$

as the simplicial cochain complex of K with G-coefficients. We define the k-th simplicial cohomology group of K with G-coelfficients as follows:

$$\mathrm{H}^k_{\mathrm{simp}}(K;G) := \mathrm{H}^k(\mathrm{C}^{\mathrm{simp}}_*(K);G) = \frac{\ker(\delta_k\colon \mathrm{C}^k_{\mathrm{simp}}(K;G) \to \mathrm{C}^{k+1}_{\mathrm{simp}}(K;G))}{\mathrm{im}(\delta_{k-1}\colon \mathrm{C}^{k-1}_{\mathrm{simp}}(K;G) \to \mathrm{C}^k_{\mathrm{simp}}(K;G))}.$$

If K is actually an ordered abstract simplicial complex, then we define the cochain complex $\mathrm{C}^*_{\mathrm{simp},\leq}(K;G)$ and the simplicial cohomology groups $\mathrm{H}^k_{\mathrm{simp},\leq}(K;G)$ in the obvious way.
Given a finite abstract simplicial complex it is just an elementary, albeit at times surprisingly intricate, problem in linear algebra to determine the simplicial cohomology groups and to give explicit examples of simplicial cocycles.

Example. We consider the "simplicial circle" C with vertex set $V = \{a, b, c\}$ shown in the figure below. We consider the ordered 1-simplices $s := (a, b)$, $t := (b, c)$ and $u := (c, a)$. We see that the simplicial chain complex $\mathrm{C}^{\mathrm{simp}}_*(C;\mathbb{Z})$ is of the following form:[1219]

$$\begin{array}{ccc} 0 \to \mathrm{C}^{\mathrm{simp}}_1(C) \xrightarrow{\partial_1} \mathrm{C}^{\mathrm{simp}}_0(C) \to 0 & \text{thus the dual} & 0 \leftarrow \mathrm{C}^1_{\mathrm{simp}}(C;\mathbb{Z}) \xleftarrow{\delta_0} \mathrm{C}^0_{\mathrm{simp}}(C;\mathbb{Z}) \leftarrow 0 \\ s \mapsto b - a & \text{chain complex} & u^* - s^* \leftarrow\!\shortmid\ a^* \\ t \mapsto c - b & \mathrm{C}^*_{\mathrm{simp}}(C;\mathbb{Z}) & s^* - t^* \leftarrow\!\shortmid\ b^* \\ u \mapsto a - c. & \text{is given by} & t^* - u^* \leftarrow\!\shortmid\ c^*. \end{array}$$

An elementary calculation shows that

$$\mathrm{H}^0_{\mathrm{simp}}(C;\mathbb{Z}) = \ker(\delta_0) = \mathbb{Z} \cdot (a^* + b^* + c^*).$$

We turn to the slightly more tricky calculation of $\mathrm{H}^1_{\mathrm{simp}}(C;\mathbb{Z})$. First note that it is elementary to show that

$$\mathrm{im}\left(\delta_0\colon \mathrm{C}^0_{\mathrm{simp}}(C;\mathbb{Z}) \to \mathrm{C}^1_{\mathrm{simp}}(C;\mathbb{Z})\right) = \{k \cdot s^* + l \cdot t^* + m \cdot u^* \mid k + l + m = 0\}.$$

Now let us consider the cochain $\theta = t^*\colon \mathrm{C}^{\mathrm{simp}}_1(C) \to \mathbb{Z}$. (In other words, θ is the cochain that is given by $\theta(s) = 0$, $\theta(t) = 1$ and $\theta(u) = 0$.) An elementary argument, see e.g. Exercise 72.13, shows that the map

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\cong} & \mathrm{H}^1_{\mathrm{simp}}(C;\mathbb{Z}) = \mathrm{coker}(\delta_0) \\ n & \mapsto & n \cdot [\theta] \end{array}$$

is an isomorphism.[1220] Note that on several occasions throughout this chapter we will make use of the above simplicial cocycle $\theta = t^* \in \mathrm{C}^1_{\mathrm{simp}}(C;\mathbb{Z})$.

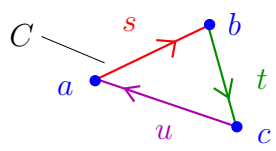
We state the next example as a lemma.

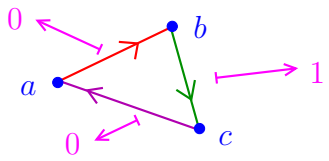
[1219] Usually it takes a while to get used to coboundaries. Thus let us show in detail that we have $\delta_0(a^*) = u^* - s^* \in \mathrm{C}^1_{\mathrm{simp}}(C;\mathbb{Z}) = \mathrm{Hom}(\mathrm{C}^{\mathrm{simp}}_1(C), \mathbb{Z})$. We need to show that $\delta_0(a^*) = u^* - s^*$ are the same homomorphism. By Lemma 95.6 we know that $\{s, t, u\}$ is a basis for $\mathrm{C}^{\mathrm{simp}}_1(C)$. So we need to show that the two homomorphisms take the same values on s, t and u. We calculate that

$$\delta_0(a^*)(s) \underset{\text{definition of } \delta_0}{=} a^*(\partial_1 s) \underset{\text{definition of } \partial_1}{=} a^*(b - a) = a^*(b) - a^*(a) = -1 = u^*(s) - s^*(s) = (u^* - s^*)(s).$$

The same way we proceed with t and u.

[1220] Of course there are many other choices of a cochain θ that would work. We picked this particular one since later on it works well in the pictures.





cocycle $\theta\colon \mathrm{C}_1^{\mathrm{simp}}(K) \to \mathbb{Z}$ represents a generator of $\mathrm{H}^1_{\mathrm{simp}}(K;\mathbb{Z})$

Lemma 99.3. Let G be an abelian group and let $K = (V, S)$ be an abstract simplicial complex.

(1) Given any $g \in G$ the constant cochain $\varphi_g\colon \mathrm{C}_0^{\mathrm{simp}}(K) \to G$ given by $\varphi_g(v) = g$ for every $v \in V$ is a cocycle.
(2) If K is non-empty and connected in the sense of the definition on page 2013, then $\mathrm{H}^0_{\mathrm{simp}}(K;G) = \{\varphi_g \,|\, g \in G\}$. In particular $\mathrm{H}^0_{\mathrm{simp}}(K;G) \cong G$.

Proof.

(1) Let $g \in G$. We need to show that $\delta_0(\varphi_g) = 0 \in \mathrm{C}^1_{\mathrm{simp}}(K) = \mathrm{Hom}(\mathrm{C}_1^{\mathrm{simp}}(K), G)$. In other words, by definition of $\mathrm{C}_1^{\mathrm{simp}}(K)$ we need to show that for every ordered 1-simplex (v_0, v_1) of K we have $(\delta_0\varphi_g)([v_0, v_1]) = 0$. Indeed, we calculate that

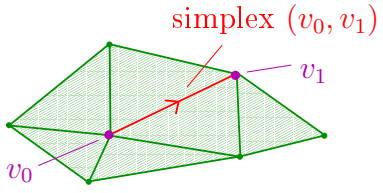
$$\delta_0(\varphi_g)([v_0,v_1]) \underset{\text{definition of } \delta_0}{=} \varphi_g(\partial_1([v_0,v_1])) \underset{\text{definition of } \partial_1}{=} \varphi_g(v_1 - v_0) = \varphi_g(v_1) - \varphi_g(v_0) = g - g = 0.$$

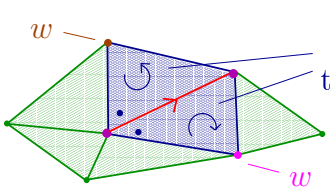
(2) We will prove the second statement in Exercise 99.2. ∎

We conclude this section with the following explicit calculation of the coboundary map. This result will come in handy in the following chapter.

Lemma 99.4. Let $K = (V, S)$ be an abstract simplicial complex and let $(v_0, \ldots, v_k)$ be an ordered k-simplex of K. We have

$$\delta_k([v_0,\ldots,v_k]^*) = (-1)^{k+1} \cdot \sum_{\substack{w \in V \text{ such that} \\ \{v_0,\ldots,v_k,w\} \text{ is} \\ \text{a } (k+1)\text{-simplex}}} [v_0,\ldots,v_k,w]^* \in \mathrm{C}^{k+1}_{\mathrm{simp}}(K;\mathbb{Z}).$$





the coboundary $\delta_1([v_0, v_1]^*)$ sends these two ordered 2-simplices to $+1$ and all the other ones to zero

Proof. We need to show that both sides define the same map $\mathrm{C}^{\mathrm{simp}}_{k+1}(K) \to \mathbb{Z}$. Basically by definition of $\mathrm{C}^{\mathrm{simp}}_{k+1}(K)$ this means that we need to show both sides provide the same value for any given $[y_0, \ldots, y_{k+1}]$. So suppose we are given an ordered $(k+1)$-simplex $(y_0, \ldots, y_{k+1})$ of K. After possibly reordering the y_i we can arrange that there exists an $m \in \{0, \ldots, k\}$ such that $y_i = v_i$ for $i = 0, \ldots, m-1$ and such that $\{y_m, \ldots, y_{k+1}\} \cap \{v_m, \ldots, v_k\} = \varnothing$. We calculate that

$$\begin{array}{rcl}
(\delta_k([v_0,\dots,v_k]^*)([y_0,\dots,y_{k+1}]) & \overset{\text{definition of } \delta_k}{\overset{\downarrow}{=}} & [v_0,\dots,v_k]^*(\partial_{k+1}([y_0,\dots,y_{k+1}])) \overset{\text{definition of } \partial_{k+1}}{\overset{\downarrow}{=}} \\
& = & \sum_{i=0}^{k+1}(-1)^i\cdot[v_0,\dots,v_k]^*([y_0,\dots,\widehat{y_i},\dots,y_{k+1}]) \\
& = & \sum_{i=0}^{k+1}\begin{cases} 0, & \text{if } i<k+1 \text{ or if } m\neq k \\ (-1)^{k+1}, & \text{if } i=k+1 \text{ and if } m=k. \end{cases} \\
& \overset{\text{follows from the choice of } m}{\uparrow} &
\end{array}$$

It follows easily from the definitions that the cochain on the right hand side of the lemma leads to the same result. ■

99.3. **Properties of simplicial cohomology groups.** In the following lemma we will see that simplicial cohomology groups define contravariant functors. Before we can formulate that lemma let us recall a definition from Lemmas 95.3 and 95.7. Namely, given a simplicial map $f\colon K=(V,S)\to L=(W,T)$ between abstract simplicial complexes we consider the following chain map:

$$\begin{array}{rcl}
f_*\colon \mathrm{C}_k^{\mathrm{simp}}(K) & \to & \mathrm{C}_k^{\mathrm{simp}}(L) \\
[v_0,\dots,v_k] & \mapsto & \begin{cases} [f(v_0),\dots,f(v_k)], & \text{if } f(v_0),\dots,f(v_k) \text{ are pairwise different,} \\ 0, & \text{otherwise.} \end{cases}
\end{array}$$

When we are dealing with an order-preserving simplicial map between ordered abstract simplicial complexes, then we define the induced map $\mathrm{C}_k^{\mathrm{simp},\leq}(K)\to\mathrm{C}_k^{\mathrm{simp},\leq}(L)$ in the analogous way.

Lemma 99.5. Let $k\in\mathbb{N}_0$ and let G be an abelian group.

(1) The maps

$$\begin{array}{rcl}
(K,\leq) & \mapsto & \mathrm{H}^k_{\mathrm{simp},\leq}(K;G) \\
(f\colon(K,\leq)\to(L,\leq)) & \mapsto & \left(\begin{array}{rcl} f_*\colon \mathrm{H}^k_{\mathrm{simp},\leq}(K;G) & \to & \mathrm{H}^k_{\mathrm{simp},\leq}(L;G) \\ [\varphi\colon \mathrm{C}_k^{\mathrm{simp}}(K)\to G] & \mapsto & [\varphi\circ f_*] \end{array}\right)
\end{array}$$

define a contravariant functor from the category *OrdAbsSimpCplx* of ordered abstract simplicial complexes to the category *AbGroup* of abelian groups.

(2) The maps

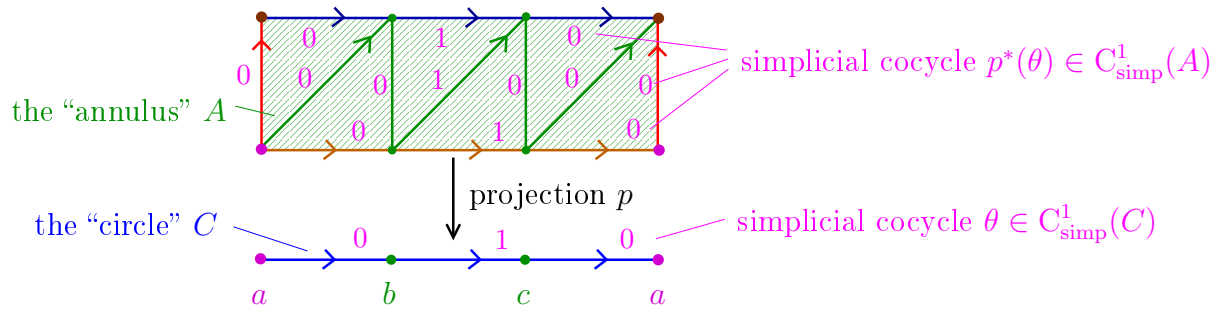
$$\begin{array}{rcl}
K & \mapsto & \mathrm{H}^k_{\mathrm{simp}}(K;G) \\
(f\colon K\to L) & \mapsto & \left(\begin{array}{rcl} f_*\colon \mathrm{H}^k_{\mathrm{simp}}(K;G) & \to & \mathrm{H}^k_{\mathrm{simp}}(L;G) \\ [\varphi\colon \mathrm{C}_k^{\mathrm{simp}}(K)\to G] & \mapsto & [\varphi\circ f_*] \end{array}\right)
\end{array}$$

define a contravariant functor from the category *AbsSimpCplx* of abstract simplicial complexes to the category *AbGroup* of abelian groups.

Proof. The lemma is an immediate consequence of Lemmas 95.3, 95.7 and 108.8. ■

Example. In the figure below we show the "simplicial annulus" A together with the natural projection p to the simplicial circle C. Furthermore we also show the simplicial cocycle $\theta\in\mathrm{C}^1_{\mathrm{simp}}(C;\mathbb{Z})$ that we introduced on page 2175 and we show the corresponding cocycle $p^*(\theta)\in\mathrm{C}^1_{\mathrm{simp}}(A)$.

The following proposition relates the two flavors of simplicial cohomology groups to one another and it relates them to the singular cohomology groups of the topological realization.



Proposition 99.6. (Simplicial-Singular Cohomology Isomorphism Proposition) Let G be an abelian group.

(1) Let $(K, \leq)$ be an ordered abstract simplicial complex.

(a) The natural maps

$$\begin{aligned} \Theta^*_{\leq,k} \colon \mathrm{C}^k(|K|;G) &\to \mathrm{C}^k_{\mathrm{simp},\leq}(K;G) \\ (\varphi \colon \mathrm{C}_k(|K|) \to G) &\mapsto \left(\mathrm{C}^{\mathrm{simp}}_{k,\leq}(K) \xrightarrow{\Theta_{\leq,k}} \mathrm{C}_k(|K|) \xrightarrow{\varphi} G\right) \end{aligned}$$

are a cochain homotopy equivalence. In particular the maps induce natural isomorphisms

$$\Theta^*_{\leq,k} \colon \mathrm{H}^k(|K|;G) \to \mathrm{H}^k_{\mathrm{simp},\leq}(K;G)$$

(b) Every simplicial cocycle $\varphi \in \mathrm{C}^k_{\mathrm{simp},\leq}(K;G)$ "extends" to a singular cocycle of $\mathrm{C}^k(|K|;G)$. More precisely, there exists a cocycle $\psi \in \mathrm{C}^k(|K|;G)$ such that $\Theta^*_{\leq,k}(\psi) = \varphi$, i.e. such that $\psi \circ \Theta_{\leq,k} = \varphi \colon \mathrm{C}^{\mathrm{simp}}_k(K) \to G$.

(2) Let K be an abstract simplicial complex. We pick an order "$\leq$" for K. The composition of the two maps

$$\mathrm{H}^k(|K|;G) \xrightarrow{\Theta^*_{\leq}} \mathrm{H}^k_{\mathrm{simp},\leq}(K;G) \xrightarrow{\Omega^{-1}_{\leq,*}} \mathrm{H}^k_{\mathrm{simp}}(K;G)$$

does not depend on the choice of "$\leq$" and it defines a natural isomorphism

$$\Xi^*_k \colon \mathrm{H}^k(|K|;G) \to \mathrm{H}^k_{\mathrm{simp}}(K;G).$$

Proof.

(1) (a) This statement follows immediately from Proposition 99.1 (1) and the Dual Cochain Map Lemma 108.8.

(b) Let $\varphi \in \mathrm{C}^k_{\mathrm{simp},\leq}(K;G)$ be a simplicial cocycle. By Proposition 99.1 (2) there exist maps $\pi_k \colon \mathrm{C}_k(|K|) \to \mathrm{C}^{\mathrm{simp},\leq}_k(K)$ which have the following two properties:

(i) The maps π_k form a chain map.

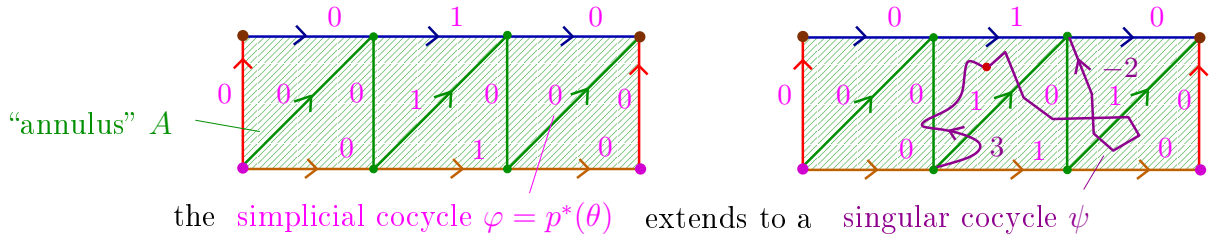
(ii) For each $k \in \mathbb{N}_0$ we have $\pi_k \circ \Theta_k = \mathrm{id}_{\mathrm{C}^{\mathrm{simp},\leq}_k(K)}$.

We set $\psi := \pi^*_k(\varphi)$. By (i) this is a cocycle and by (ii) we have $\psi \circ \Theta_{\leq,k} = \varphi$.

(2) This statement follows immediately from Proposition 99.1 (1) and the Dual Cochain Map Lemma 108.8. ■

Example. Let us rephrase the statement of the Simplicial-Singular Cohomology Isomorphism Proposition 99.6 (1b). It says that given any simplicial cocycle $\varphi \in \mathrm{C}^k_{\mathrm{simp},\leq}(K;G)$

there exists a singular cocycle $\psi \in \mathrm{C}^k(|K|;G)$ with $[\Theta^*_{\leq,k}(\psi)] = [\varphi] \in \mathrm{H}^k_{\mathrm{simp},\leq}(K;G)$ and which, restricted to the singular simplices provided by the simplices of K, coincides with φ. We try to illustrate this statement in the figure below.



We continue with the following lemma which is just a special case of Lemma 109.6.

Lemma 99.7. Let G be an abelian group and let $k \in \mathbb{N}_0$.

(1) Given any abstract simplicial complex K the map

$$\begin{aligned} \langle\ ,\ \rangle\colon \mathrm{H}^k_{\mathrm{simp}}(K;G) \times \mathrm{H}^{\mathrm{simp}}_k(K) &\to G \\ ([\varphi],[\sigma]) &\mapsto \langle [\varphi],[c]\rangle_K := \varphi(\sigma), \end{aligned}$$

called the *Kronecker pairing*, is well-defined and bilinear.

(2) Given any simplicial map $f\colon K \to L$ between two abstract simplicial complexes, given any $\sigma \in \mathrm{H}^{\mathrm{simp}}_k(K)$ and given any $\varphi \in \mathrm{H}^k_{\mathrm{simp}}(L;G)$ we have

$$\langle f^*(\varphi),\sigma\rangle_K = \langle \varphi, f_*(\sigma)\rangle_L.$$

With the above lemma we can now formulate the following theorem which is an immediate consequence of the purely algebraic Universal Coefficient Theorem 110.11 for Cohomology Groups.

Theorem 99.8. (Universal Coefficient Theorem for Simplicial Cohomology) Let K be an abstract simplicial complex and let G be an abelian group. For each $k \in \mathbb{N}_0$ there exists a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(\mathrm{H}^{\mathrm{simp}}_{k-1}(K),G) & \longrightarrow & \mathrm{H}^k_{\mathrm{simp}}(K;G) & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}(\mathrm{H}^{\mathrm{simp}}_k(K),G) & \longrightarrow & 0 \\ & & & & \varphi & \mapsto & \begin{pmatrix} \mathrm{H}^{\mathrm{simp}}_k(K) \to G \\ \sigma \mapsto \langle \varphi,\sigma\rangle_K \end{pmatrix} & & \end{array}$$

which is natural in the abstract simplicial complex K and natural in the abelian group G. This short exact sequence splits and there exists therefore an isomorphism

$$\mathrm{H}^k_{\mathrm{simp}}(K;G) \cong \mathrm{Ext}(\mathrm{H}^{\mathrm{simp}}_{k-1}(K),G) \oplus \mathrm{Hom}(\mathrm{H}^{\mathrm{simp}}_k(K),G).$$

The analogous statement for ordered abstract simplicial complexes and the corresponding simplicial homology and cohomology groups also holds.

Remark.

(1) The Universal Coefficient Theorem 99.8 for Simplicial Cohomology is a double-edged sword. It allows us to compute the isomorphism types of simplicial cohomology groups in terms of the much more friendly simplicial homology groups. But it also says, somewhat disappointingly, that the isomorphism types of simplicial cohomology groups do not contain any information that is not already contained in the isomorphism types of the simplicial homology groups.

(2) The Universal Coefficient Theorem 99.8 for Simplicial Cohomology is also somewhat less powerful than it might appear. The theorem can be very convenient for determining the isomorphism types of simplicial cohomology groups. But if the Ext-terms are non-zero, then the theorem does not provide an approach to finding explicit cocycles representing cohomology classes.

99.4. **The simplicial cup product.** We had seen that one the remarkable features of singular cohomology is that it comes with a cup product and that it interacts with singular homology via the cap product. Before we attempt to introduce simplicial analogues let us remind us of the definitions from pages 2503, 2513 and 2551.

Definition.

(1) For $i = 0, \dots, n$ we write $v_i := (0, \dots, 1, 0, \dots, 0) \in \mathbb{R}^{n+1}$.

(2) For $a_0, \dots, a_k \in \Delta^n$ we consider the map

$$\begin{array}{rcl} [a_0, \dots, a_k] \colon \Delta^k & \to & \Delta^n \\ (\lambda_0, \dots, \lambda_k) & \mapsto & \sum_{j=0}^{k} \lambda_j \cdot a_j. \end{array}$$

(3) Let X be a topological space and let R be a commutative ring.

(a) Let $\varphi \in \mathrm{C}^p(X; R)$ and $\psi \in \mathrm{C}^q(X; R)$ be cochains. We define the cup product of φ and ψ to be the singular cochain

$$\begin{array}{rcl} \varphi \cup \psi \colon \mathrm{C}_{p+q}(X) & \to & R \\ (\sigma \colon \Delta^{p+q} \to X) & \mapsto & \underbrace{\varphi(\sigma \circ [v_0, \dots, v_p])}_{\in R} \cdot \underbrace{\psi(\sigma \circ [v_p, \dots, v_{p+q}])}_{\in R}. \end{array}$$

(b) Given a singular cochain $\varphi \in \mathrm{C}^k(X; R)$ and a singular simplex $\sigma \colon \Delta^n \to X$ with $k \leq n$ we define the cap product of φ and σ to be the singular chain

$$\varphi \cap \sigma := \underbrace{\varphi(\sigma \circ [v_0, \dots, v_k])}_{\in R} \cdot \underbrace{\sigma \circ [v_k, \dots, v_n]}_{\text{map } \Delta^{n-k} \to X}.$$

For $k > n$ we define $\varphi \cap \sigma = 0$.

In this and in the coming section we will introduce simplicial analogues of the above definitions. Evidently the idea is to imitate the above definitions and to define the cup and cap products on the level of simplicial (co-) chains. A short moment's thought shows that on the (co-) chain level one needs to work with *ordered* abstract simplicial complexes.
First we have the following definition which is clearly inspired by our second definition of the cup product in singular cohomology that we gave on page 2513.

Definition. Let $(K, \leq)$ be an ordered abstract simplicial complex and let R be a commutative ring. Given $\varphi \in \mathrm{C}^p_{\mathrm{simp},\leq}(K; R)$ and $\psi \in \mathrm{C}^q_{\mathrm{simp},\leq}(K; R)$ we consider the cup product

$$\begin{array}{rcl} \varphi \cup \psi \colon \mathrm{C}^{\mathrm{simp},\leq}_{p+q}(K) & \to & R \\ \{v_0 < \cdots < v_p < \cdots < v_{p+q}\} & \mapsto & \underbrace{\varphi(\{v_0 < \ldots < v_p\})}_{\in R} \cdot \underbrace{\psi(\{v_p < \ldots < v_{p+q}\})}_{\in R}. \end{array}$$

The following lemma is the analogue of Coboundary-Cup Product Lemma 114.2.

Lemma 99.9. (Coboundary-Cup Product Lemma) Let $(K, \leq)$ be an ordered abstract simplicial complex and let R be a commutative ring. Furthermore let $\varphi \in \mathrm{C}^p_{\mathrm{simp},\leq}(K;R)$ and $\psi \in \mathrm{C}^q_{\mathrm{simp},\leq}(K;R)$. Then

$$\delta(\varphi \cup \psi) \;=\; \delta\varphi \cup \psi \,+ (-1)^p \cdot \varphi \cup \delta\psi \;\in\; \mathrm{C}^{p+q+1}_{\mathrm{simp},\leq}(K;R).$$

In the following we will state several lemmas that are the simplicial counterparts of earlier results on the singular cup product. In most cases the proofs are, with an appropriate change of notation, basically identical. For sake of illustration we provide the proof of the Coboundary-Cup Product Lemma 99.9 and we invite the reader to compare the proof to the proof of the original Coboundary-Cup Product Lemma 114.2.

Proof. Let $\varphi \in \mathrm{C}^p_{\mathrm{simp},\leq}(K;R)$ and $\psi \in \mathrm{C}^q_{\mathrm{simp},\leq}(K;R)$ be simplicial cochains. We have to prove the following equality:

$$\delta(\varphi \cup \psi) \;=\; \delta\varphi \cup \psi \,+ (-1)^p \cdot \varphi \cup \delta\psi \;\in\; \mathrm{C}^{p+q+1}_{\mathrm{simp},\leq}(K;R) = \mathrm{Hom}(\mathrm{C}^{\mathrm{simp},\leq}_{p+q+1}(K), R).$$

So let $s = \{v_0 < \cdots < v_{p+q+1}\}$ be a simplicial $(p+q+1)$-simplex. We have the following three equalities in R:[1221]

$$\begin{aligned}
\text{(a) } (\delta(\varphi\cup\psi))(s) &= (\varphi\cup\psi)\Big(\sum_{j=0}^{p+q+1}(-1)^j\cdot\{v_0<\ldots<\widehat{v_j}<\ldots<v_{p+q+1}\}\Big)\\
&= \sum_{j=0}^{p}(-1)^j\cdot(\varphi\cup\psi)(\{v_0<\ldots<\widehat{v_j}<\ldots<v_{p+q+1}\})\\
&\quad + (-1)^p\cdot\sum_{k=1}^{q+1}(-1)^k\cdot(\varphi\cup\psi)(\{v_0<\ldots<\widehat{v_{k+p}}<\ldots<v_{p+q+1}\})\\
&= \sum_{j=0}^{p}(-1)^j\cdot\varphi(\{v_0<\ldots<\widehat{v_j}<\ldots<v_{p+1}\})\cdot\psi(\{v_p<\ldots<v_{p+q+1}\})\\
&\quad + (-1)^p\cdot\sum_{k=1}^{q+1}(-1)^k\cdot\varphi(\{v_0<\ldots<v_p\})\cdot\psi(\{v_p<\ldots<\widehat{v_k}<\ldots<v_{p+q+1}\}).\\
\text{(b) } (\delta\varphi\cup\psi)(s) &= (\delta\varphi)(\{v_0<\ldots<v_{p+1}\})\cdot\psi(\{v_p<\ldots<v_{p+q+1}\})\\
&= \varphi\Big(\sum_{j=0}^{p+1}(-1)^j\cdot\{v_0<\ldots<\widehat{v_j}<\ldots<v_{p+1}\}\Big)\cdot\psi(\{v_p<\ldots<v_{p+q+1}\})\\
&= \sum_{j=0}^{p+1}(-1)^j\cdot\varphi(\{v_0<\ldots<\widehat{v_j}<\ldots<v_{p+1}\})\cdot\psi(\{v_p<\ldots<v_{p+q+1}\})
\end{aligned}$$

and

$$\begin{aligned}
\text{(c) } (-1)^p\cdot(\varphi\cup\delta\psi)(s) &= (-1)^p\cdot\varphi(\{v_0<\ldots<v_p\})\cdot(\delta\psi)(\{v_p<\ldots<v_{p+q+1}\})\\
&= (-1)^p\cdot\varphi(\{v_0<\ldots<v_p\})\cdot\psi\Big(\sum_{k=0}^{q+1}(-1)^k\cdot\{v_p<\ldots<\widehat{v_k}<\ldots<v_{p+q+1}\}\Big)\\
&= (-1)^p\cdot\sum_{k=0}^{q+1}(-1)^k\cdot\varphi(\{v_0<\ldots<v_p\})\cdot\psi(\{v_p<\ldots<\widehat{v_k}<\ldots<v_{p+q+1}\}).
\end{aligned}$$

Now we observe that if we take the sum of (b) and (c), then the $(j = p+1)$-summand in (b) cancels with the $(k = 0)$-summand in (c). But after this observation it is obvious that the desired equality (b)+(c)=(a) holds. ■

[1221] Here in the argument we use on several occasions that for a simplicial n-cochain ϕ and an $(n+1)$-simplex $t = \{w_0 < \ldots < w_{n+1}\}$ we have by definition

$$(\delta_{n+1}\phi)(t) \;=\; \phi(\partial_n t) \;=\; \phi\Big(\sum_{l=0}^{n+1}(-1)^l\cdot\{w_0<\ldots<\widehat{w_l}<\ldots<w_{n+1}\}\Big).$$

In light of the Coboundary-Cup Product Lemma 99.9 it is now straightforward to see that the cup product descends to a map on simplicial cohomology.

Lemma 99.10. Let R be a commutative ring. For every ordered abstract simplicial complex $(K, \leq)$ the map

$$\begin{array}{rcl} \cup\colon \mathrm{H}^p_{\mathrm{simp},\leq}(K;R) \times \mathrm{H}^q_{\mathrm{simp},\leq}(K;R) & \to & \mathrm{H}^{p+q}_{\mathrm{simp},\leq}(K;R) \\ ([\varphi],[\psi])] & \mapsto & [\varphi] \cup [\psi] := [\varphi \cup \psi] \end{array}$$

is well-defined. We refer to this map again as the cup product.

Proof. This lemma follows almost immediately from the Coboundary-Cup Product Lemma 99.9. For more details we refer to the proof of Lemma 114.3. In that proof we just need to replace Coboundary-Cup Product Lemma 114.2 by the Coboundary-Cup Product Lemma 99.9. ■

The simplicial cup product is an interesting invariant in its own right and as we will see shortly, for finite abstract simplicial complexes it can be computed effectively. As for simplicial homology the question arises whether the isomorphism type of the simplicial cup product is an invariant of the topological realization. As we discussed in Sections 95.4 and 95.5, arguably the cleanest way of showing this is by proving that the simplicial cup product corresponds to the singular cup product. This is precisely what we will do in the first part of the following proposition, which can be viewed as an addendum to the Simplicial-Singular Cohomology Isomorphism Proposition 99.6.

Proposition 99.11. Let K be an abstract simplicial complex and let R be a commutative ring.

(1) Let "$\leq$" be an order on K. Let $\Theta_{\leq,k}\colon \mathrm{C}_k^{\mathrm{simp},\leq}(K) \to \mathrm{C}_k(|K|)$ be the natural chain map that we introduced in Proposition 99.1. The following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p(|K|;R) \times \mathrm{H}^q(|K|;R) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^{p+q}(|K|;R) \\ {\scriptstyle\Theta^*_\leq}\downarrow{\scriptstyle\cong} \quad {\scriptstyle\cong}\downarrow{\scriptstyle\Theta^*_\leq} & & {\scriptstyle\cong}\downarrow{\scriptstyle\Theta^*_\leq} \\ \mathrm{H}^p_{\mathrm{simp},\leq}(K;R) \times \mathrm{H}^q_{\mathrm{simp},\leq}(K;R) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^{p+q}_{\mathrm{simp},\leq}(K;R). \end{array}$$

(2) There exists a unique map

$$\mathrm{H}^p_{\mathrm{simp}}(K;R) \times \mathrm{H}^q_{\mathrm{simp}}(K;R) \xrightarrow{\quad\cup\quad} \mathrm{H}^{p+q}_{\mathrm{simp}}(K;R)$$

such that for every choice of order "$\leq$" for K the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p_{\mathrm{simp}}(K;R) \times \mathrm{H}^q_{\mathrm{simp}}(K;R) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^{p+q}_{\mathrm{simp}}(K;R) \\ {\scriptstyle\Omega^*_\leq}\downarrow{\scriptstyle\cong} \quad {\scriptstyle\cong}\downarrow{\scriptstyle\Omega^*_\leq} & & {\scriptstyle\cong}\downarrow{\scriptstyle\Omega^*_\leq} \\ \mathrm{H}^p_{\mathrm{simp},\leq}(K;R) \times \mathrm{H}^q_{\mathrm{simp},\leq}(K;R) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^{p+q}_{\mathrm{simp},\leq}(K;R). \end{array}$$

In other words, we can define the cup product on $\mathrm{H}^*_{\mathrm{simp}}(K;R)$ in a way that is independent of the choice of an order "$\leq$".

(3) The following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p(|K|;R) \times \mathrm{H}^q(|K|;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(|K|;R) \\ {\scriptstyle \Xi_p^*}\downarrow{\scriptstyle \cong} \qquad {\scriptstyle \cong}\downarrow{\scriptstyle \Xi_q^*} & & {\scriptstyle \cong}\downarrow{\scriptstyle \Xi_{p+q}^*} \\ \mathrm{H}^p_{\mathrm{simp}}(K;R) \times \mathrm{H}^q_{\mathrm{simp}}(K;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}_{\mathrm{simp}}(K;R). \end{array}$$

Remark. Let K be an abstract simplicial complex. In the presence of an order on K we have, by definition, a natural simplicial cup product on the cochain groups $\mathrm{C}^*_{\mathrm{simp},\leq}(K;R)$. In the absence of an order we do not have a natural simplicial cup product on the cochain groups $\mathrm{C}^*_{\mathrm{simp}}(K;R)$. But Proposition 99.11 says that we do have a natural simplicial cup product on the cohomology groups $\mathrm{H}^*_{\mathrm{simp}}(K;R)$.

Proof.

(1) It follows follows immediately from the definitions of the simplicial cup product and the definition of the singular cup product that we gave on page 2513 and that we recalled on page 2180 that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{C}^p(|K|;R) \times \mathrm{C}^q(|K|;R) & \xrightarrow{\cup} & \mathrm{C}^{p+q}(|K|;R) \\ {\scriptstyle \Theta_\leq^*}\downarrow \qquad \downarrow{\scriptstyle \Theta_\leq^*} & & \downarrow{\scriptstyle \Theta_\leq^*} \\ \mathrm{C}^p_{\mathrm{simp},\leq}(K;R) \times \mathrm{C}^q_{\mathrm{simp},\leq}(K;R) & \xrightarrow{\cup} & \mathrm{C}^{p+q}_{\mathrm{simp},\leq}(K;R). \end{array}$$

The desired statement follows immediately from this observation.

(2) Surely there is a heroic proof for this statement that does not involve singular cohomology. But to lessen the pain let us give a proof that makes the most use of our previous results. Thus we consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^p_{\mathrm{simp}}(K;R) \times \mathrm{H}^q_{\mathrm{simp}}(K;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}_{\mathrm{simp}}(K;R) \\ {\scriptstyle (\Omega_\leq^{-1})_*}\uparrow{\scriptstyle \cong} \qquad {\scriptstyle \cong}\uparrow{\scriptstyle (\Omega_\leq^{-1})_*} & & {\scriptstyle \cong}\uparrow{\scriptstyle (\Omega_\leq^{-1})_*} \\ \mathrm{H}^p_{\mathrm{simp},\leq}(K;R) \times \mathrm{H}^q_{\mathrm{simp},\leq}(K;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}_{\mathrm{simp},\leq}(K;R) \\ {\scriptstyle \Theta_\leq^*}\uparrow{\scriptstyle \cong} \qquad {\scriptstyle \cong}\uparrow{\scriptstyle \Theta_\leq^*} & & {\scriptstyle \cong}\uparrow{\scriptstyle \Theta_\leq^*} \\ \mathrm{H}^p(|K|;R) \times \mathrm{H}^q(|K|;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(|K|;R). \end{array}$$

By Proposition 99.1 together with the Cochain Map Lemma 108.5 we see that the composition of the two vertical maps does not depend on the choice of "$\leq$". By (1) we know that the lower square commutes. Now we define the cup product on $\mathrm{H}^*_{\mathrm{simp}}(K;R)$ via the singular cup product and we see that it makes the upper square commute for any choice of "$\leq$".

(3) This statement follows immediately from the definition of the natural isomorphisms Ξ^*, see the Simplicial-Singular Cohomology Isomorphism Proposition 99.6 (3). ∎

Remark. Proposition 99.11 shows in particular that if we are given a simplicial complex $(X, \Theta\colon |K| \to X)$, then we can compute the initially intractable singular cup product on $\mathrm{H}^*(X;R)$ via the simplicial cup product on $\mathrm{H}^*_{\mathrm{simp}}(K;\mathbb{R})$. If K is finite and explicitly given, then at least in principle one can compute the simplicial cup product on $\mathrm{H}^*_{\mathrm{simp}}(K;R)$ by brute force. This approach though is not a cure-all. For example given any $n \in \mathbb{N}$ we will want to determine the cup product on $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ and $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$. By the Smooth Manifold-Simplicial Structure Theorem 96.2 we know that the complex projective spaces

$\mathbb{C}\mathrm{P}^n$ and the real projective spaces $\mathbb{R}\mathrm{P}^n$ admit simplicial structures. But we run into the following issues:

(1) It seems to be difficult to find explicit simplicial structures for the complex projective spaces $\mathbb{C}\mathrm{P}^n$. So we cannot even get started.
(2) Explicit simplicial structures for smooth manifolds tend to have many simplices. For example in Exercise 94.5 we gave an explicit simplicial structure for the real projective space $\mathbb{R}\mathrm{P}^n$ with $2^{n+1}-1$ vertices and it seems difficult to find simplicial structures with a significantly smaller number of vertices.

Later on, in Propositions 121.7 and 121.7 we will compute these cup products using the Simplicial Poincaré Duality Theorem 103.4.

To do calculations of cup products we need to deal with specific cohomology classes and in fact also with explicit cocycles. In the next lemma we will see that closed oriented pseudomanifolds are a rich source of interesting explicit cocycles. Before we can state this lemma it is convenient to recall some definitions from page 2142 and page 2143.

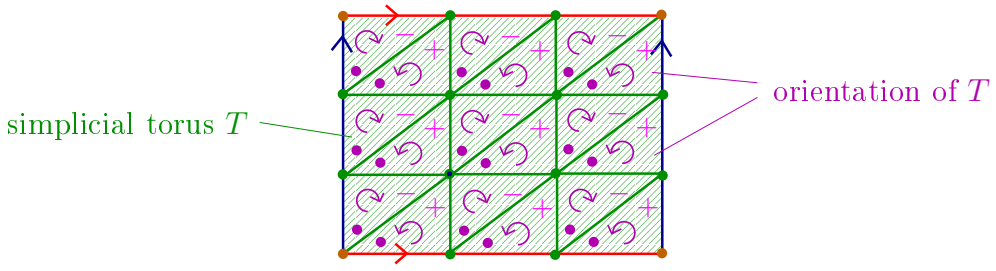
Definition. Let $K = (V, S)$ be an n-dimensional pseudomanifold.

(1) We denote by S_n the set of n-simplices of K.
(2) An orientation of an n-simplex s of K is an equivalence class of a pair $(\epsilon, \widetilde{s})$ where $\epsilon \in \{-1, 1\}$ and $\widetilde{s}$ is an ordered n-simplex with $\lfloor \widetilde{s} \rfloor = s$.
(3) An orientation of K is a choice of an orientation for each n-simplex of K such that for any $(n-1)$-simplex s of order 2 the two cobounding n-simplices induce opposite orientations on s.
(4) If K is a closed pseudomanifold that is equipped with an orientation $\{(\epsilon_w, Y_w)\}_{w \in S_n}$, then we refer to the simplicial homology class

$$[K] \;:=\; \Big[\sum_{w \in S_n} \epsilon_w \cdot Y_w \Big] \;\in\; \mathrm{H}_n^{\mathrm{simp}}(K)$$

as the fundamental class of K.

We illustrate the definitions for the simplicial torus in the figure below.



Now we can formulate the following lemma.

Lemma 99.12. Let $K = (V, S)$ be a closed connected n-dimensional pseudomanifold that is equipped with an orientation $\{(\epsilon_w, Y_w)\}_{w \in S_n}$ in the sense of the definition on page 2142.

(1) (a) For any $s, t \in S_n$ we have $[\epsilon_s \cdot Y_s^*] = [\epsilon_t \cdot Y_t^*] \in \mathrm{H}^n_{\mathrm{simp}}(K; \mathbb{Z})$. We refer to this cohomology class as the dual fundamental class $[K]^* \in \mathrm{H}^n_{\mathrm{simp}}(K; \mathbb{Z})$.
    (b) We have $\mathrm{H}^n_{\mathrm{simp}}(K; \mathbb{Z}) = \mathbb{Z} \cdot [K]^*$.
(2) We have $\langle [K]^*, [K] \rangle_K = 1$.

Proof.

(1) It follows from Theorem 98.4, and the discussion on page 2148 that the simplicial chain complex of K is of the form

$$0 \to \underbrace{\mathbb{Z}\cdot\Big[\sum_{w\in S_n}\epsilon_w\cdot Y_w\Big]\oplus\widetilde{C}_n}_{=\mathrm{C}_n^{\mathrm{simp}}(K)} \xrightarrow[\partial_n]{\begin{pmatrix}0 & D\\ 0 & 0\end{pmatrix}} \underbrace{\widetilde{C}_{n-1}\oplus\widehat{\mathrm{C}}_{n-1}}_{=\mathrm{C}_{n-1}^{\mathrm{simp}}(K)} \to \dots$$

such that $D\colon \widetilde{C}_n \to \widetilde{C}_{n-1}$ is an isomorphism. It follows easily that the map

$$\begin{array}{rcccc} \mathrm{H}^n_{\mathrm{simp}}(K;\mathbb{Z}) = \operatorname{coker}(\delta_{n-1}) & \to & \operatorname{Hom}\Big(\mathbb{Z}\cdot\Big[\sum\limits_{w\in S_n}\epsilon_w\cdot Y_w\Big],\mathbb{Z}\Big) & \to & \mathbb{Z}\\ \varphi & \mapsto & \text{restriction of } \varphi & & \\ & & & \varphi & \mapsto \quad \varphi\Big(\sum\limits_{w\in S_n}\epsilon_w\cdot Y_w\Big) \end{array}$$

is an isomorphism. The desired statements follow almost immediately from this observation.[1222]

(2) This statement follows immediately from the definitions. ∎

Now let us try to determine the simplicial cup product of the simplicial torus.

Example. We consider the simplicial circle C with vertex set $V = \{a, b, c\}$ and the simplicial circle $D = \{x, y, z\}$ with the obvious total order on the vertex sets. Furthermore we consider the simplicial torus $T = C \times D$, where we define the product as on page 1998. We equip T with the orientation shown in the figure on page 2184. We denote by $p\colon C\times D \to C$ and $q\colon C \times D \to D$ the natural projections. We equip the vertex set of $T = C \times D$ with the lexicographic ordering. In the figure below we show $T = C \times D$ together with the projections.

Let us first consider the simplicial cohomology of T. Let $\theta \in \mathrm{C}^1_{\mathrm{simp}}(C)$ be the simplicial cocycle that we introduced on page 2175. We have the following equalities:

$$\mathrm{H}^i_{\mathrm{simp}}(T;\mathbb{Z}) \;=\; \begin{cases} \mathbb{Z}\cdot[\text{constant cochain } v\mapsto 1], & \text{if } i=0,\\ \mathbb{Z}\cdot[p^*(\theta)]\oplus\mathbb{Z}\cdot[q^*(\theta)], & \text{if } i=1,\\ \mathbb{Z}\cdot[T]^*, & \text{if } i=2,\\ 0, & \text{if } i>2. \end{cases}$$

This can be obtained either through a slightly heroic linear algebra exercise, or, more systematically, through the combination of Lemma 99.12, the Simplicial Künneth Theorem 95.14 and the Universal Coefficient Theorem 99.8 for Simplicial Cohomology. Now we can calculate our first interesting simplicial cup product:

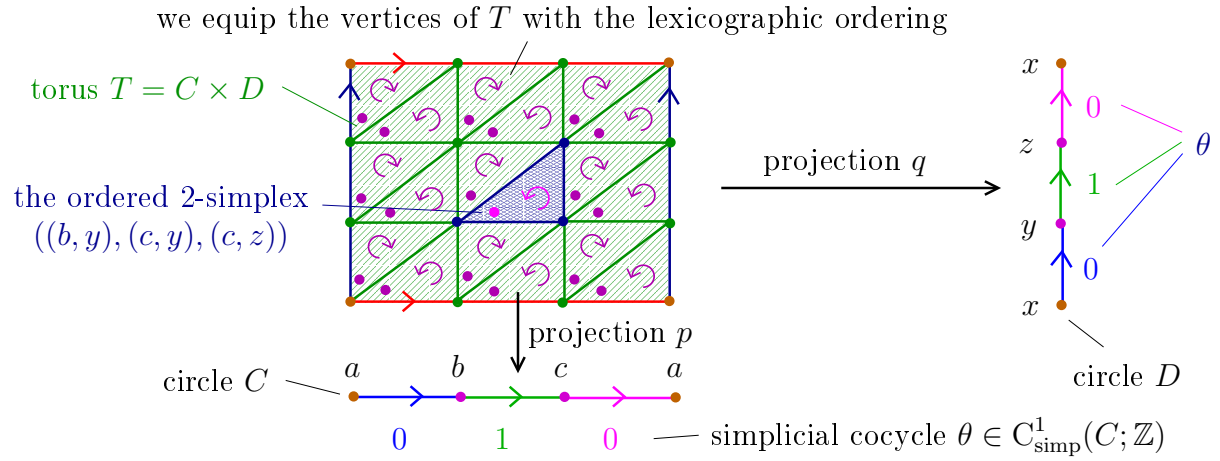
$$[p^*(\theta)]\cup[q^*(\theta)] \;=\; [p^*(\theta)\cup q^*(\theta)] \underset{\text{by inspection}}{=} [\overbrace{((b,y),(c,y),(c,z))}^{\text{ordered 2-simplex}}{}^*] \underset{\text{by Lemma 99.12}}{=} [T]^* \;\in\; \mathrm{H}^2_{\mathrm{simp}}(T;\mathbb{Z}).$$

[1222] The above argument is related to the content of Exercise 108.1.

Similar calculations gives us the following table for the cup product:[1223]

$$\begin{pmatrix} [p^*(\theta)] \cup [p^*(\theta)] & [p^*(\theta)] \cup [q^*(\theta)] \\ [q^*(\theta)] \cup [p^*(\theta)] & [q^*(\theta)] \cup [q^*(\theta)] \end{pmatrix} = \begin{pmatrix} 0 & [T]^* \\ -[T]^* & 0 \end{pmatrix}.$$

This is the simplicial analogue of the calculation performed in Lemma 114.14.



In the following we state several properties of the simplicial cup product that are the analogues of the corresponding properties of the singular cup product.

Proposition 99.13. Let $K = (V,S)$ be an abstract simplicial complex and let R be a commutative ring. The following statements hold:

(1) The cup product on $\mathrm{H}^*_{\mathrm{simp}}(K;R)$ is R-bilinear and associative.

(2) The abelian group $\mathrm{H}^*_{\mathrm{simp}}(K;R) = \bigoplus_{n\in\mathbb{N}_0} \mathrm{H}^n_{\mathrm{simp}}(K;R)$ together with the map

$$\cup\colon \mathrm{H}^*_{\mathrm{simp}}(K;R) \times \mathrm{H}^*_{\mathrm{simp}}(K;R) \to \mathrm{H}^*_{\mathrm{simp}}(K;R)$$

is a ring where the multiplicatively neutral element is given by

$$1_K \;:=\; \big[\text{map } V \to R \text{ given by } v \mapsto 1\big] \;\in\; \mathrm{H}^0_{\mathrm{simp}}(K;R).$$

Proof. We pick an order "$\leq$" for K. As in the proof of the Cup Product-Ring Proposition 114.7 we see that the cup product on the cochain level is associative. It now follows easily that all the given statements hold for $(\mathrm{H}^*_{\mathrm{simp},\leq}(K;R),\cup)$. By Proposition 99.11 (2) the statements now also hold for $(\mathrm{H}^*_{\mathrm{simp}}(K;R),\cup)$. ■

Proposition 99.14. Let K be an abstract simplicial complex and let R be a commutative ring. For any $\varphi \in \mathrm{H}^p_{\mathrm{simp}}(K;R)$ and any $\psi \in \mathrm{H}^q_{\mathrm{simp}}(K;R)$ we have

$$\varphi \cup \psi \;=\; (-1)^{pq} \cdot \psi \cup \varphi \;\in\; \mathrm{H}^{p+q}_{\mathrm{simp}}(K;R).$$

Proof. We pick an order "$\leq$" for K. By Proposition 99.11 (2) it suffices to prove the following claim.

[1223] In fact the remaining three entries of the matrix can also be obtained easily from Proposition 99.14 and Lemma 99.15.

Claim. For any $\varphi \in \mathrm{H}^p_{\mathrm{simp},\leq}(K;R)$ and $\psi \in \mathrm{H}^q_{\mathrm{simp},\leq}(K;R)$ we have

$$\varphi \cup \psi \;=\; (-1)^{pq} \cdot \psi \cup \varphi \;\in\; \mathrm{H}^{p+q}_{\mathrm{simp},\leq}(K;R).$$

Proof. The proof of the claim happens at the "cochain level". The argument is basically identical to the proof of the Cup Product-Commutativity Proposition 114.8. The only real difference is that one needs to replace the product topological space $\Delta^n \times [0,1]$ by the product of the ordered abstract simplicial complexes $D_n = (\{0,\ldots,n\}, \mathcal{P}(\{0,\ldots,n\}\backslash\{\varnothing\}))$ and $D_1 = (\{0,1\},\{\{0\},\{1\}\})$, as defined in the Simplicial Complex-Product Lemma 93.16. We leave it to the reader to make the necessary modifications to the argument. ■

Lemma 99.15. Let R be a commutative ring.

(1) Let $f\colon K \to L$ be a simplicial map between two abstract simplicial complexes. For any $\varphi \in \mathrm{H}^p_{\mathrm{simp}}(L;R)$ and $\psi \in \mathrm{H}^q_{\mathrm{simp}}(L;R)$ we have

$$f^*(\varphi) \cup f^*(\psi) \;=\; f^*(\varphi \cup \psi).$$

In particular the map

$$f^*\colon \mathrm{H}^*_{\mathrm{simp}}(L;R) \;\to\; \mathrm{H}^*_{\mathrm{simp}}(K;R)$$

is a homomorphism of graded rings which satisfies $f^*(1_L) = 1_K$.

(2) The maps

$$\begin{aligned} K &\mapsto (\mathrm{H}^*_{\mathrm{simp}}(K;R), \cup) \\ (f\colon K \to L) &\mapsto (f^*\colon \mathrm{H}^*_{\mathrm{simp}}(L;R) \to \mathrm{H}^*_{\mathrm{simp}}(K;R)) \end{aligned}$$

define a contravariant functor from the category $\mathcal{AbsSimpCplx}$ of abstract simplicial complexes to the category $\mathcal{GrRing}$ of graded rings.

Proof.

(1) Let $f\colon K = (V,S) \to L = (W,T)$ be a simplicial map between two abstract simplicial complexes. By Exercise 0.5 we can pick total orders "$\leq_K$" on K and "$\leq_L$" on L such that the map $f\colon V \to W$ is order-preserving. It follows basically immediately from the definitions that for $\varphi \in \mathrm{C}^p_{\mathrm{simp},\leq_L}(L;R)$ and $\psi \in \mathrm{C}^q_{\mathrm{simp},\leq_L}(L;R)$ we have the equality $f^*(\varphi) \cup f^*(\psi) = f^*(\varphi \cup \psi) \in \mathrm{C}^{p+q}_{\mathrm{simp},\leq_K}(K)$. The statements now follow from Proposition 99.11 (2).

(2) This statement follows from (1) together with Lemma 99.5 (2). ■

99.5. **The simplicial cap product.** In this section we will introduce the simplicial cap product. In contrast to the simplicial cup product it is perhaps less immediately clear what the simplicial cap product is good for. But it will have its moment of glory in Chapter 100 when we state and prove the Simplicial Poincaré Duality Theorem 103.4.

Definition. Let $(K,\leq)$ be an ordered abstract simplicial complex and let R be a commutative ring. Given an n-simplex $s = \{v_0 < \cdots < v_n\}$, given $k \leq n$ and given a simplicial cochain $\varphi \in \mathrm{C}^k_{\mathrm{simp},\leq}(K;R)$ we define

$$\varphi \cap s \;:=\; \underbrace{\varphi(\{v_0 < \cdots < v_k\})}_{\in R} \cdot \underbrace{\{v_k < \cdots < v_n\}}_{\in \mathrm{C}^{\mathrm{simp},\leq}_{n-k}(K)} \;\in\; \mathrm{C}^{\mathrm{simp},\leq}_{n-k}(K;R).$$

We extend this definition to simplicial R-chains and we obtain the cap product

$$\begin{array}{rcl} \cap\colon \mathrm{C}^k_{\mathrm{simp},\leq}(X;R) \times \mathrm{C}_n^{\mathrm{simp},\leq}(X;R) & \to & \mathrm{C}_{n-k}^{\mathrm{simp},\leq}(X;R) \\ \varphi \cap r \cdot \sigma & \mapsto & r \cdot \underbrace{(\varphi \cap \sigma)}_{\in R}. \end{array}$$

For $k > n$ we define the cap product to be the zero map.

The next lemma says that the cap product defines a map on simplicial (co-) homology.

Lemma 99.16. Let $(K, \leq)$ be an ordered abstract simplicial complex. Given any commutative ring R and given any $k, n \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{simp},\leq}(K;R) \times \mathrm{H}_n^{\mathrm{simp},\leq}(K;R) & \to & \mathrm{H}_{n-k}^{\mathrm{simp},\leq}(K;R) \\ ([\varphi],[\sigma]) & \mapsto & [\varphi \cap \sigma] \end{array}$$

is well-defined.

Proof. The same argument as in the proof of Lemma 116.1 shows that given an n-simplex $s = \{v_0 < \cdots < v_n\}$, given $k \leq n$ and given any simplicial cochain $\varphi \in \mathrm{C}^k_{\mathrm{simp},\leq}(K;R)$ we have the following equality:

$$\partial(\varphi \cap \sigma) \;=\; (-1)^k \cdot (-\delta\varphi \cap \sigma \,+\, \varphi \cap \partial\sigma) \;\in\; \mathrm{C}_{n-k-1}(X;R).$$

As in the proof of Lemma 116.2 we see that the lemma follows easily from this equality. ∎

The next proposition is an analogue of Proposition 99.11.

Proposition 99.17. Let K be an abstract simplicial complex and let R be a commutative ring.

(1) Let "$\leq$" be an order on K. Let $\Theta_{\leq,k}\colon \mathrm{C}_k^{\mathrm{simp},\leq}(K) \to \mathrm{C}_k(|K|)$ be the natural chain map that we introduced in Proposition 99.1. The following diagram commutes:[1224]

$$\begin{array}{ccc} \mathrm{H}^p(|K|;R) \times \mathrm{H}_n(|K|;R) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{n-p}(|K|;R) \\ {\scriptstyle \Theta^*_\leq}\downarrow{\scriptstyle \cong} \quad {\scriptstyle \cong}\uparrow{\scriptstyle \Theta_*^\leq} & & {\scriptstyle \cong}\uparrow{\scriptstyle \Theta_*^\leq} \\ \mathrm{H}^p_{\mathrm{simp},\leq}(K;R) \times \mathrm{H}_n^{\mathrm{simp},\leq}(K;R) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{n-p}^{\mathrm{simp},\leq}(K;R). \end{array}$$

(2) There exists unique map

$$\mathrm{H}^p_{\mathrm{simp}}(K;R) \times \mathrm{H}_n^{\mathrm{simp}}(K;R) \xrightarrow{\quad\cap\quad} \mathrm{H}_{n-p}^{\mathrm{simp}}(K;R)$$

such that for any choice of order "$\leq$" for K the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p_{\mathrm{simp}}(K;R) \times \mathrm{H}_n^{\mathrm{simp}}(K;R) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{n-p}^{\mathrm{simp}}(K;R) \\ {\scriptstyle \Omega^*_\leq}\downarrow{\scriptstyle \cong} \quad {\scriptstyle \cong}\uparrow{\scriptstyle \Omega_*^\leq} & & {\scriptstyle \cong}\uparrow{\scriptstyle \Omega_*^\leq} \\ \mathrm{H}^p_{\mathrm{simp},\leq}(K;R) \times \mathrm{H}_n^{\mathrm{simp},\leq}(K;R) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{n-p}^{\mathrm{simp},\leq}(K;R). \end{array}$$

In other words, we can define the cap product on $\mathrm{H}^*_{\mathrm{simp}}(K;R)$ and $\mathrm{H}_*^{\mathrm{simp}}(K;R)$ in a way that is independent of the choice of an order "$\leq$".

(3) The following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p(|K|;R) \times \mathrm{H}_n(|K|;R) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{n-p}(|K|;R) \\ {\scriptstyle \Xi^*}\downarrow{\scriptstyle \cong} \qquad {\scriptstyle \cong}\uparrow{\scriptstyle \Xi_*} & & {\scriptstyle \cong}\uparrow{\scriptstyle \Xi_*} \\ \mathrm{H}^p_{\mathrm{simp}}(K;R) \times \mathrm{H}^{\mathrm{simp}}_n(K;R) & \xrightarrow{\quad\cap\quad} & \mathrm{H}^{\mathrm{simp}}_{n-p}(K;R). \end{array}$$

Sketch of proof.

(1) It follows almost immediately from the definition that the corresponding diagram on the level of simplicial (co-) chains commutes. This implies that the given diagram commutes.
(2) As in the proof of Proposition 99.11 (2) this statement follows from (1) together with Proposition 99.1, the Homotopy Equivalence-H_*-Corollary 73.9 and the Cochain Map Lemma 108.5.
(3) This statement follows immediately from the definition of the natural isomorphisms Ξ_* and Ξ^*, see Propositions 99.1 (3) and 99.6 (3). ■

Example. As on page 2185 we consider again the simplicial torus $T = C \times D$. We equip $T = (V, S)$ with the orientation $\{(\epsilon_w, Y_w)\}_{w \in S_2}$ shown in the figures on page 2184 and 2190 and we denote by $p\colon C \times D \to C$ the natural projection. Furthermore, we again make use of the simplicial cocycle $\theta \in \mathrm{C}^1_{\mathrm{simp}}(C;\mathbb{Z})$ that we first encountered on page 2175. We introduce the following extra notation:

(1) We denote by $j\colon D \to C \times D$ the simplicial map given by $v \mapsto (c, v)$.
(2) We denote by $\sigma := \{x,y\} + \{y,z\} + \{z,x\} \in \mathrm{C}^{\mathrm{simp},\leq}_1(D)$ the "obvious" simplicial cycle. Note that it represents the simplicial fundamental class $[D] \in \mathrm{H}^{\mathrm{simp},\leq}_1(D)$.

It follows immediately from the definitions that

$$p^*(\theta) \cap \sum_{w \in S_2} \epsilon_w \cdot Y_w \;=\; j_*(\sigma) \;\in\; \mathrm{C}^{\mathrm{simp},\leq}_1(C \times D)$$

thus in terms of (co-) homology we have

$$p^*([\theta]) \cap [T] \;=\; j_*([D]) \;\in\; \mathrm{H}^{\mathrm{simp},\leq}_1(C \times D).$$

We refer to the figure below for an illustration.

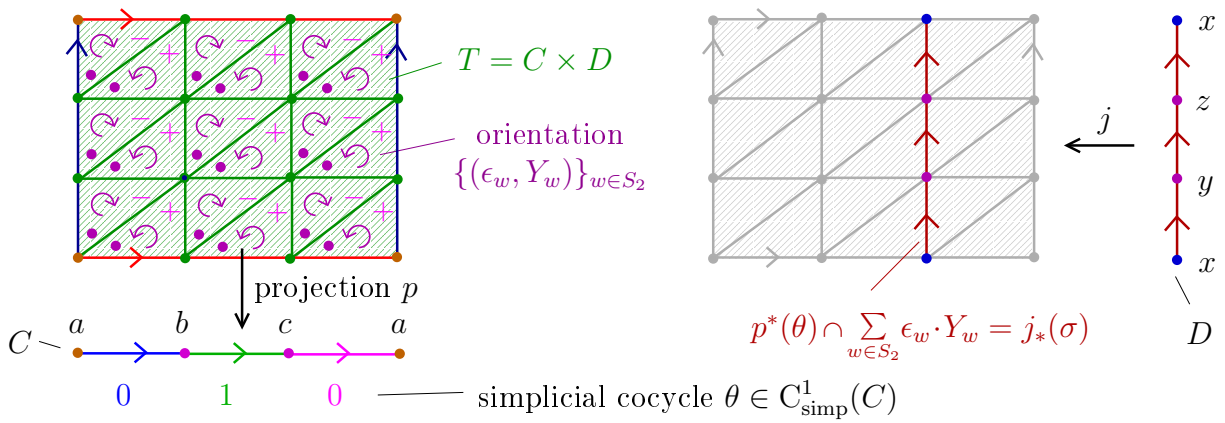
In the following we state three lemmas that summarize the key properties of the simplicial cap product. All of these statements are analogues of singular statements. First, the following lemma is an analogue of the Cap Product-Kronecker Lemma 116.4.

Lemma 99.18. Let K be a connected non-empty abstract simplicial complex. Given any $\varphi \in \mathrm{H}^p_{\mathrm{simp}}(K;R)$ and $\sigma \in \mathrm{H}^{\mathrm{simp}}_p(K;R)$ the following equality holds:

$$\varphi \cap \sigma \;=\; \langle \varphi, \sigma \rangle_K \;\in\; H^{\mathrm{simp}}_0(K;R) \underset{\uparrow}{=} R.$$

as in Exercise 95.3 there is a natural identification

[1224] Here the expression "the diagram commutes" is understood in the sense of the convention from page 2538.



Proof. First we pick an order "$\leq$" for the abstract simplicial complex K. Next we observe that the statement for $\varphi \in \mathrm{H}^p_{\mathrm{simp},\leq}(K;R)$ and $\sigma \in \mathrm{H}^{\mathrm{simp},\leq}_p(K;R)$ follows immediately from the definitions. Now the statement follows from Proposition 99.17 (2). ■

The following lemma relates the simplicial cup product to the simplicial cap product. It can be viewed as an analogue of the Cup-Cap Lemma 116.7.

Lemma 99.19. Let K be an abstract simplicial complex and let R be a commutative ring. For any $\varphi \in \mathrm{H}^p_{\mathrm{simp}}(K;R)$, any $\psi \in \mathrm{H}^q_{\mathrm{simp}}(K;R)$ and any $\sigma \in \mathrm{H}^{\mathrm{simp}}_n(K;R)$ we have

$$\varphi \cap (\psi \cap \sigma) \;=\; (\psi \cup \varphi) \cap \sigma \;\in\; \mathrm{H}^{\mathrm{simp}}_{n-p-q}(K;R).$$

Sketch of proof. We pick an order "$\leq$" for K. With basically the same, quite elementary, argument as in the proof of the Cup-Cap Lemma 116.7 we see that for any $\varphi \in \mathrm{H}^p_{\mathrm{simp},\leq}(K;R)$, any $\psi \in \mathrm{H}^q_{\mathrm{simp},\leq}(K;R)$ and any $\sigma \in \mathrm{H}^{\mathrm{simp},\leq}_n(K;R)$ we have

$$\varphi \cap (\psi \cap \sigma) \;=\; (\psi \cup \varphi) \cap \sigma \;\in\; \mathrm{H}^{\mathrm{simp},\leq}_{n-p-q}(K;R).$$

The desired statement now follows from Proposition 99.17 (2). ■

Finally we have the following analogue of the Cap Product-Naturality Lemma 116.8.

Lemma 99.20. Let R be a commutative ring and let $f\colon K \to L$ be a simplicial map between two abstract simplicial complexes. Given any $p, n \in \mathbb{N}_0$ the following diagram commutes

$$\begin{array}{ccccc} \mathrm{H}^p_{\mathrm{simp}}(K;R) & \times & \mathrm{H}^{\mathrm{simp}}_n(K;R) & \xrightarrow{\cap} & \mathrm{H}^{\mathrm{simp}}_{n-p}(K;R) \\ {\scriptstyle f^*}\uparrow & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathrm{H}^p_{\mathrm{simp}}(L;R) & \times & \mathrm{H}^{\mathrm{simp}}_n(L;R) & \xrightarrow{\cap} & \mathrm{H}^{\mathrm{simp}}_{n-p}(L;R), \end{array}$$

i.e. for any $\varphi \in \mathrm{H}^p_{\mathrm{simp}}(L;R)$ and any $\sigma \in \mathrm{H}^{\mathrm{simp}}_n(K;R)$ we have

$$\varphi \cap f_*(\sigma) \;=\; f_*(\underbrace{f^*(\varphi) \cap \sigma}_{\in \mathrm{H}^{\mathrm{simp}}_{n-p}(K;R)}) \;\in\; \mathrm{H}^{\mathrm{simp}}_{n-p}(L;R).$$

Proof. Let $f\colon K = (V,S) \to L = (W,T)$ be a simplicial map between two abstract simplicial complexes. By Exercise 0.5 we can pick total orders "$\leq_K$" on K and "$\leq_L$" on L such that the map $f\colon V \to W$ is order-preserving. It follows basically immediately from

the definitions that for any $\varphi \in \mathrm{C}^p_{\mathrm{simp},\leq L}(L;R)$ and $\sigma \in \mathrm{C}^{\mathrm{simp},\leq K}_n(K;R)$ we have the equality $f_*(\sigma) = f_*(f^*(\varphi)\cap\sigma)$. The promised statement now follows from Proposition 99.17 (2). ■

We conclude this chapter with a short calculation which in particular shows that at times simplicial cup products can be computed using simplicial cap products.

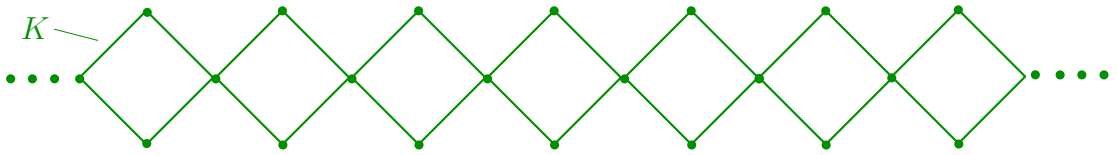
Example. Let us look one last time at the simplicial torus $T = C \times D$ with the natural projections $p\colon C \times D \to C$ and $q\colon C \times D \to D$. We will now see how we can obtain the simplicial cup product $p^*(\theta) \cup q^*(\theta)$, that we initially computed on page 2186, from the calculation of the simplicial cap product on page 2189. Indeed, under the natural identification $H_0^{\mathrm{simp}}(T;R) = R = H_0^{\mathrm{simp}}(D;R)$ from Exercise 95.3 we have the following equalities:

$$\begin{array}{lllll}
\langle p^*([\theta])\cup q^*([\theta]),[T]\rangle_T & \overset{\text{by Lemma 99.18}}{=} (p^*([\theta])\cup q^*([\theta]))\cap[T] & \overset{\text{Lemma 99.19}}{=} q^*([\theta])\cap(p^*([\theta])\cap[T]) & \overset{\text{see calculation on page 2189}}{=} q^*([\theta])\cap j_*([D]) & \\
= \langle q^*([\theta]), j_*([D])\rangle_T & = \langle [\theta], (q_*\circ j_*)([D])\rangle_D & = \langle [\theta],[D]\rangle_D & = 1. & \\
\uparrow & \uparrow & \uparrow & \uparrow & \\
\text{by Lemma 99.18} & \text{by Lemma 99.7 (2)} & \text{since } q\circ j = \mathrm{id}_D & \text{by definition of } \theta \text{ and } [D] &
\end{array}$$

In Chapter 100 we will state the Simplicial Poincaré Duality Theorem 103.4. Given a closed oriented PL-manifold K it allows for the calculation of simplicial cap products of the form $\varphi \cap [K]$. With tricks as above one can use this calculation of a simplicial cap product to determine several interesting simplicial cup products. In particular in Chapter 121 we will use this approach to calculate the (simplicial) cup products of the real and complex projective spaces.

Exercises for Chapter 99.

Exercise 99.1. We consider the abstract simplicial complex K shown in the figure below. Determine the isomorphism type of $\mathrm{H}^1_{\mathrm{simp}}(K;\mathbb{Z})$.



Exercise 99.2. Let $K = (V,S)$ be a non-empty abstract simplicial complex. We assume that K is connected. Recall that by the definition on page 2013 this means that given any $v, v' \in V$ there exist 1-simplices $s_0, \ldots, s_k \in S$ such that $v \in s_0$, such that for every $i \in \{0,\ldots,k-1\}$ we have $s_i \cap s_{i+1} \neq \varnothing$ and such that $v' \in s_k$. Now let G be an abelian group. Given any $g \in G$ we consider the simplicial cochain $\varphi_g\colon \mathrm{C}_0^{\mathrm{simp}}(K) \to G$ that is given by $\varphi_g(v) = g$ for every vertex $v \in V$. Show that the map

$$\begin{array}{ccc} G & \to & \mathrm{H}^0_{\mathrm{simp}}(K;G) \\ g & \mapsto & \varphi_g \end{array}$$

is a natural isomorphism.

Remark. This statement can be viewed is the cohomological analogue of Exercise 95.3.

Exercise 99.3. Let K be an abstract simplicial complex and let L be a subcomplex of K. Given an abelian group G and given $k \in \mathbb{N}_0$ we define

$$\mathrm{H}^k_{\mathrm{simp}}(K, L; G) \;:=\; \mathrm{H}^k(\mathrm{C}^{\mathrm{simp}}_*(K, L), G).$$

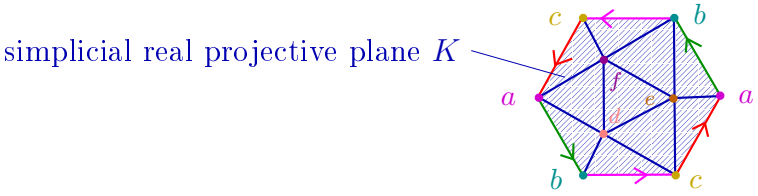
Show that there exists a natural isomorphism

$$\mathrm{H}^k_{\mathrm{simp}}(K, L; G) \;\cong\; \widetilde{\mathrm{H}}^k_{\mathrm{simp}}(\underbrace{K \cup_L \mathrm{Cone}(L)}_{\text{mapping cone of } L \to K}; G).$$

Exercise 99.4. In the figure below we show an abstract simplicial complex K whose topological realization $|K|$ is homeomorphic to the real projective plane $\mathbb{R}\mathrm{P}^2$.

(a) Show that for $i = 0, 1, 2$ we have $\mathrm{H}^i_{\mathrm{simp}}(K; \mathbb{F}_2) \cong \mathbb{F}_2$ and identify in each dimension a cocycle that represents the unique non-zero element.

(b) Compute the simplicial cup product $\mathrm{H}^1_{\mathrm{simp}}(K; \mathbb{F}_2) \times \mathrm{H}^1_{\mathrm{simp}}(K; \mathbb{F}_2) \xrightarrow{\cup} \mathrm{H}^2_{\mathrm{simp}}(K; \mathbb{F}_2)$. In other words, determine whether it is trivial or non-trivial.



Exercise 99.5. Let $g \in \mathbb{N}$. First pick a suitable abstract simplicial complex K that has the property that its topological realization is homeomorphic to the surface of genus g. Afterwards compute the cup product

$$\mathrm{H}^1_{\mathrm{simp}}(K; \mathbb{Z}) \times \mathrm{H}^1_{\mathrm{simp}}(K; \mathbb{Z}) \;\to\; \mathrm{H}^2_{\mathrm{simp}}(K; \mathbb{Z}).$$

Exercise 99.6. We consider the topological space

$$X = \overline{B}^2/\!\sim \qquad \text{where } z \sim z \cdot \exp(2\pi\,\mathrm{i}/3) \text{ for } z \in S^1.$$

(a) Determine the isomorphism types of $\mathrm{H}_i(X; \mathbb{F}_3)$ and $\mathrm{H}^i(X; \mathbb{F}_3)$.

(b) Give an explicit simplicial structure K for X.

(c) Compute the simplicial cup product $\mathrm{H}^1_{\mathrm{simp}}(K; \mathbb{F}_3) \times \mathrm{H}^1_{\mathrm{simp}}(K; \mathbb{F}_3) \xrightarrow{\cup} \mathrm{H}^2_{\mathrm{simp}}(K; \mathbb{F}_3)$.

100. The fundamental class of smooth manifolds

Let M be a compact *orientable* connected non-empty n-dimensional smooth manifold. By Theorem 98.8 we now know that we have $\mathrm{H}_n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$. In this chapter we will see that a *choice of an orientation* naturally determines a generator $[M]$ of $\mathrm{H}_n(M, \partial M; \mathbb{Z})$, called the fundamental class of M. As the name already suggests, this homology class is of fundamental importance in the study of smooth manifolds.

100.1. **The definition of the fundamental class.** For our definition of fundamental classes we need to recall the following notation.

Notation. Let $n \in \mathbb{N}_0$. As on page 1581 we consider

$$\begin{aligned} \partial\Delta^n &:= \{(t_0, \dots, t_n) \in \Delta^n \,|\, \text{at least one coordinate is zero}\} \\ \mathring{\Delta}^n &:= \{(t_0, \dots, t_n) \in \Delta^n \,|\, \text{all coordinates are non-zero}\}. \end{aligned}$$

We view $\mathring{\Delta}^n$ as a smooth manifold and we equip it with the orientation defined on page 1581. Loosely speaking it is the orientation given by the "outward pointing vector $(1, \dots, 1)$".

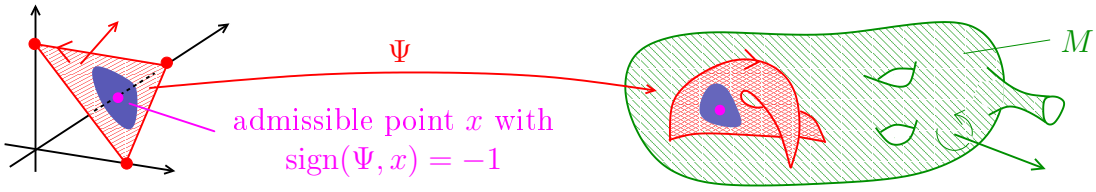
We move on to the following somewhat technical definition.

Definition. Let M be an oriented n-dimensional smooth manifold and let $\Psi\colon \Delta^n \to M$ be a map.

(1) We say that a point $x \in \mathring{\Delta}^n$ is admissible for Ψ if $\Psi^{-1}(\Psi(x)) = \{x\}$ and if there exists an open connected neighborhood U of $x \in \mathring{\Delta}^n$ such that the map $\Psi|_U\colon U \to M$ is a smooth embedding.

(2) Given an admissible point x for Ψ we define

$$\mathrm{sign}(\Psi, x) = \begin{cases} +1, & \text{if } \Psi \text{ is orientation-preserving at } x, \\ -1, & \text{if } \Psi \text{ is orientation-reversing at } x. \end{cases}$$



Example. In many cases we will consider embeddings $\Psi\colon \Delta^n \to M$ such that the restriction of Ψ to $\mathring{\Delta}^n$ is a smooth embedding. In this case every point $x \in \mathring{\Delta}^n$ is admissible for Ψ, and we have

$$\mathrm{sign}(\Psi, x) = \begin{cases} +1, & \text{if } \Psi\colon \mathring{\Delta}^n \to M \text{ is orientation-preserving}, \\ -1, & \text{if } \Psi\colon \mathring{\Delta}^n \to M \text{ is orientation-reversing}. \end{cases}$$

Also note that it is clear that any non-empty smooth manifold admits such a smooth embedding.

The following theorem is the main technical result of this chapter.

Theorem 100.1. Let $n \in \mathbb{N}$ and let M be a compact oriented connected n-dimensional smooth manifold.

(1) For every $x \in M \setminus \partial M$ the inclusion induced map

$$\omega_x \colon \underbrace{\mathrm{H}_n(M, \partial M)}_{\cong \mathbb{Z} \text{ by Theorem 98.8}} \to \underbrace{\mathrm{H}_n(M, M \setminus \{x\})}_{\cong \mathbb{Z} \text{ by the Manifold-Local Homology Lemma 75.1}}$$

is an isomorphism.

(2) There exists a unique $[M] \in \mathrm{H}_n(M, \partial M)$ with the following property: For every map $\Psi \colon \Delta^n \to M$ and every admissible $x \in \mathring{\Delta}^n$ for Ψ we have the following equality in $\mathrm{H}_n(M, M \setminus \{\Psi(x)\})$:[1225][1226]

$$\underbrace{\omega_{\Psi(x)}}_{\text{map } \mathrm{H}_n(M,\partial M) \to \mathrm{H}_n(M, M\setminus\{\Psi(x)\}) \text{ as in (1)}}([M]) \quad = \quad \mathrm{sign}(\Psi, x) \cdot [\, \underbrace{\Psi \colon \Delta^n \to M}_{\in \mathrm{C}_n(M, M\setminus\{\Psi(x)\})} \,].$$

(3) The uniquely determined element $[M] \in \mathrm{H}_n(M, \partial M)$ from (2) is a generator, in particular we have $\mathrm{H}_n(M, \partial M) = \mathbb{Z} \cdot [M]$.

Proof of Theorem 100.1 (1) for closed manifolds. Let $n \in \mathbb{N}$ and let M be a closed oriented connected n-dimensional smooth manifold. By the Smooth Manifold-Simplicial Structure Theorem 96.2 we can pick a smooth simplicial structure $(K = (V, S), \lambda \colon |K| \to M)$ for the smooth manifold M. Since V is finite we can easily pick a total order "$\leq$" on the vertex set V.

In the following we will use the results from Section 98.1 to construct an explicit singular cycle in $\mathrm{C}_n(M)$ which represents a generator of $\mathrm{H}_n(M)$. We will use this cycle to prove (1). Afterwards we will show that the homology class $[M]$ that it represents actually has all the properties stated in (2).

In the following we will make heavy use of the objects we introduced in Section 98.1. Even if the reader might no longer be completely familiar with all the technical definitions, the overall flow of the argument should still be easy to follow. Without further ado, let us construct the promised singular cycle.

(a) Given an m-simplex s of K we consider the embedding

$$\begin{aligned} \Phi_s^{\leq} \colon \quad \Delta^m &\to |K| \\ (t_0, \ldots, t_m) &\mapsto \sum_{i=0}^{m} t_i \cdot \underbrace{i}_{\uparrow}\text{-th vertex of } s. \end{aligned}$$

here we use the total order on V to order the vertices of s

(b) We denote by $s_1, \ldots, s_k$ the n-simplices of K.

[1225] Note that by definition of a smooth embedding, see page 651, we know that $\Psi(x) \subset M \setminus \partial M$. This implies that we have an induced map $\mathrm{H}_n(M, \partial M) \to \mathrm{H}_n(M, M \setminus \{\Psi(x)\}$.

[1226] Since x is an admissible point for Ψ we know that $\Psi^{-1}(\Psi(x)) = \{x\}$. Note that this implies in particular that $\Psi(\partial \Delta^n) \subset M \setminus \{\Psi(x)\}$ which in turn implies that Ψ is a cycle in $\mathrm{C}_*(M, M \setminus \{\Psi(x)\})$.

(c) Let $i \in \{1, \dots, k\}$. It follows from the definition of a smooth simplicial structure, see page 2082, that the map $\lambda \circ \Phi^{\leq}_{s_i} \colon \mathring{\Delta}^n \to M$ is a smooth embedding. Now we set

$$\epsilon_i := \begin{cases} +1, & \text{if } \lambda \circ \Phi^{\leq}_{s_i} \colon \mathring{\Delta}^n \to M \text{ is orientation-preserving,} \\ -1, & \text{if } \lambda \circ \Phi^{\leq}_{s_i} \colon \mathring{\Delta}^n \to M \text{ is orientation-reversing.} \end{cases}$$

(d) By Proposition 98.7 we know that the oriented n-simplices $[(\epsilon_1, s_1)], \dots, [(\epsilon_k, s_k)]$ form an orientation, in the sense of the definition on page 2142, for the pseudo-manifold K.

(e) Next we consider the maps

$$\begin{array}{ccccccc} & & \text{isomorphism by Lemma 95.8} & & \text{chain homotopy equivalence by the Simplicial-Singular Chain Complex Theorem 95.25} & & \\ & & \downarrow & & \downarrow & & \\ \mathrm{C}_m^{\mathrm{simp}}(K) & \leftarrow & \mathrm{C}_m^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_m(|K|) & \xrightarrow{\lambda_*} & \mathrm{C}_m(M) \\ \underbrace{[v_0, \dots, v_m]}_{=[s]} & \mapsfrom & \underbrace{\{v_0 < \dots < v_m\}}_{=s} & \mapsto & \Phi^{\leq}_s & & \end{array}$$

(f) It follows from (d) together with Theorem 98.4 (1) that $\epsilon_1 \cdot [s_1] + \dots + \epsilon_k \cdot [s_k]$ is a cycle in the simplicial chain complex $\mathrm{C}^{\mathrm{simp}}_*(K)$ and we know that it represents a generator of $\mathrm{H}^{\mathrm{simp}}_n(K)$.

(g) It now follows from (e) and (f) that $c := \epsilon_1 \cdot \lambda \circ \Phi^{\leq}_{s_1} + \dots + \epsilon_k \cdot \lambda \circ \Phi^{\leq}_{s_k} \in \mathrm{C}_n(M)$ is a singular cycle in $\mathrm{C}_*(M)$ and that it represents a generator of $\mathrm{H}_n(M)$.

Now we turn to the actual proof of (1). As a reminder, recall that we need to prove the following claim.

Claim. For every $x \in M$ the map

$$\omega_x \colon \mathrm{H}_n(M) \to \mathrm{H}_n(M, M \setminus \{x\})$$

is an isomorphism.

Proof. First we take care of the special case of a point $y \in M$ which can be written as $y = (\lambda \circ \Phi^{\leq}_{s_i})(P)$ for some $i \in \{1, \dots, k\}$ and some $P \in \mathring{\Delta}^n$. We consider the following two maps:

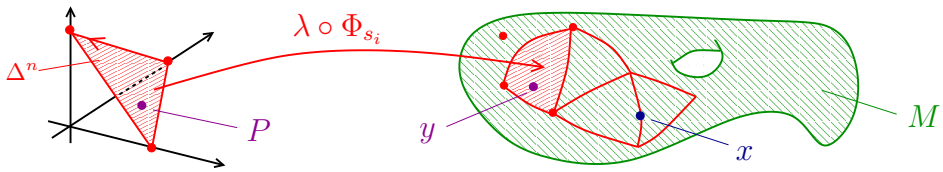
$$\begin{array}{ccccccc} & & & & \text{isomorphism by the Manifold-Local Homology Lemma 75.1} & & \\ & & & & \downarrow & & =\mathbb{Z}\cdot[\mathrm{id}_{\Delta^n}], \text{by Exercise 76.1} \\ \mathrm{H}_n(M) & \xrightarrow{\omega_y} & \mathrm{H}_n(M, M \setminus \{y\}) & \xleftarrow[\cong]{(\lambda \circ \Phi^{\leq}_{s_i})_*} & \overbrace{\mathrm{H}_n(\Delta^n, \Delta^n \setminus \{P\})} & & . \\ [c] & \mapsto & [c] = [\lambda \circ \Phi^{\leq}_{s_i}] & \longleftarrow & [\mathrm{id}_{\Delta^n}] & & \\ & & \uparrow & & & & \\ & & \text{since all the other } \lambda \circ \Phi^{\leq}_{s_j} \text{ are zero in } \mathrm{C}_n(M, M \setminus \{y\}) & & & & \end{array}$$

The above data shows that the image of $[c]$ in $\mathrm{H}_n(M, M \setminus \{y\})$ equals the image of a generator of $\mathrm{H}_n(\Delta^n, \Delta^n \setminus \{P\})$. Since the map to the right is an isomorphism we see that the map $\omega_y \colon \mathrm{H}_n(M) \to \mathrm{H}_n(M, M \setminus \{y\})$ is an epimorphism. Since M is closed orientable connected and non-empty we know by Theorem 98.8 that the group to the left is also isomorphic to $\mathbb{Z}$. Thus we see that both groups are isomorphic to $\mathbb{Z}$ we see that the map ω_y is actually an isomorphism.

Now let $x \in M$ be any point. Since M is connected we obtain from Proposition 36.4 that there exists a diffeomorphism $f \colon M \to M$ with $f(x) = y$. Now we consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_n(M) & \xrightarrow{\omega_x} & \mathrm{H}_n(M, M\setminus\{x\}) \\ {\scriptstyle f_*}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle f_*} \\ \mathrm{H}_n(M) & \xrightarrow[\cong]{\omega_y} & \mathrm{H}_n(M, M\setminus\{y\}). \end{array}$$

By the above we know that the bottom map is an isomorphism. Since f is a diffeomorphism the vertical maps are isomorphisms. Since the diagram commutes we see that the top horizontal map is also an isomorphism. ■



The proof of Theorem 100.1 (2) is a technically more intricate variation on the proof of Theorem 100.1 (1).

Proof of Theorem 100.1 (2) and (3) for closed manifolds. We continue with the notation of the proof of Theorem 100.1 (1). Note that the uniqueness part of Statement (2) is basically an immediate consequence of Statement (1). Thus to prove Statement (2) it remains to prove existence. In the following we will now show that

$$[M] := [c] = [\epsilon_1 \cdot \lambda \circ \Phi^{\leq}_{s_1} + \cdots + \epsilon_k \cdot \lambda \circ \Phi^{\leq}_{s_k}] \in \mathrm{H}_n(M)$$

has the desired properties. Before we do so, note though that in (1) we already saw that $[c]$ is a generator of $\mathrm{H}_n(M)$. Thus once we have shown that $[c]$ has the desired properties we have also shown Statement (3).

First let us now prove the following claim.

Claim. For every map $\Psi\colon \Delta^n \to M$ and every admissible $x \in \mathring{\Delta}^n$ we have

$$\omega_{\Psi(x)}([M]) = \mathrm{sign}(\Psi, x) \cdot [\underbrace{\Psi\colon \Delta^n \to M}_{\in \mathrm{C}_n(M, M\setminus\{\Psi(x)\})}].$$

Proof. Similar to the proof of the claim in Statement (1) we first deal with a special case. We pick an n-simplex s_i and we consider the corresponding map $\Upsilon := \lambda \circ \Phi^{\leq}_{s_i}\colon \Delta^n \to M$. Note that since we are given a smooth simplicial structure we know that the restriction of $\Upsilon = \lambda \circ \Phi^{\leq}_{s_i}\colon \Delta^n \to M$ to $\mathring{\Delta}^n$ is a smooth embedding. In particular any $x \in \mathring{\Delta}^n$ is admissible. Next note that we have the following equalities in $\mathrm{H}_n(M, M\setminus\{\Upsilon(x)\})$:

$$\omega_{\Upsilon(x)}([M]) \underset{\text{definition of } [M]}{=} [\epsilon_1 \cdot \lambda \circ \Phi^{\leq}_{s_1} + \cdots + \epsilon_k \cdot \lambda \circ \Phi^{\leq}_{s_k}] \underset{\text{since all other } \Phi^{\leq}_{s_j} \text{ are zero in } \mathrm{C}_n(M, M\setminus\{\Upsilon(x)\})}{=} [\epsilon_i \cdot \underbrace{\lambda \circ \Phi^{\leq}_{s_i}}_{=\Upsilon}] \underset{\text{since } \epsilon_i = \mathrm{sign}(\Upsilon, x)}{=} \mathrm{sign}(\Upsilon, x) \cdot [\Upsilon].$$

This concludes the proof of the claim for the special case $\Upsilon = \lambda \circ \Phi^{\leq}_{s_i}$ and any $x \in \mathring{\Delta}^n$.

Now we turn to the general case. Let $\Psi\colon \Delta^n \to M$ be a map and let $x \in \mathring{\Delta}^n$ be an admissible point. By definition we have $\Psi^{-1}(\Psi(x)) = \{x\}$ and by definition there exists an open connected neighborhood U of $x \in \mathring{\Delta}^n$ such that the map $\Psi|_U\colon U \to M$ is a smooth embedding. We pick an orientation-preserving embedding $\theta\colon \overline{B}^n \to U$ with $\theta(0) = x$.

Next we consider the embedding $\widetilde{\Upsilon} := \Upsilon \circ \theta \colon \overline{B}^n \to M$, with $\Upsilon = \lambda \circ \Phi_{s_i}^{\leqslant}$ as above, and we consider the embedding $\widetilde{\Psi} = \Psi \circ \theta \colon \overline{B}^n \to M$. First we make the following assumption:

(∗) The maps $\widetilde{\Upsilon}$ and $\widetilde{\Psi}$ are both orientation-preserving embeddings or they are both orientation-reversing embeddings.

In this case it follows from the Smooth Ball Embedding Theorem 37.8 that there exists a diffeomorphism f of M with the following two properties:

(i) the map f is homotopic to the identity,
(ii) $f \circ \widetilde{\Upsilon} = \widetilde{\Psi}$.

We consider the following commutative diagram[1227]

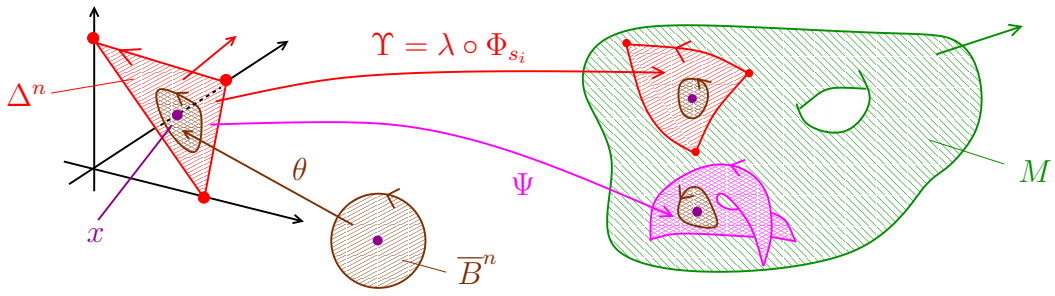
$$\begin{array}{ccccc}
 & & \mathrm{H}_n(M, M\setminus\{\Upsilon(x)\}) & \xleftarrow[\cong]{\omega_{\Upsilon(x)}} & \mathrm{H}_n(M) \\
 & \overset{\Upsilon_*}{\nearrow} & \uparrow{\scriptstyle \widetilde{\Upsilon}_*} & & \\
\mathrm{H}_n(\Delta^n, \Delta^n\setminus\{x\}) & \xleftarrow[\theta_*]{\cong} & \mathrm{H}_n(\overline{B}^n, \overline{B}^n\setminus\{0\}) & & \\
 & \underset{\Psi_*}{\searrow} & \downarrow{\scriptstyle \widetilde{\Psi}_*} & & \\
 & & \mathrm{H}_n(M, M\setminus\{\Psi(x)\}) & \xleftarrow[\omega_{\Psi(x)}]{\cong} & \mathrm{H}_n(M).
\end{array}
\qquad
\begin{array}{ccc}
 & & \\
 & & \\
 & & \\
\cong & f_* & f_*{=}\mathrm{id} \\
 & & \\
 & & \\
 & &
\end{array}$$

We make the following observations and clarifications:

(a) It follows from (i) and from the Homotopic Maps-and-Homology Proposition 73.6 that the map $f_* \colon \mathrm{H}_n(M) \to \mathrm{H}_n(M)$ is the identity.
(b) Note that by Theorem 100.1 we know that the maps $\omega_{\Upsilon(x)}$ and $\omega_{\Psi(x)}$ are isomorphisms.
(c) Note that it follows immediately from the Excision Theorem 74.18 that the map $\theta_* \colon \mathrm{H}_n(\overline{B}^n, \overline{B}^n \setminus \{0\}) \to \mathrm{H}_n(\Delta^n, \Delta^n \setminus \{x\})$ is an isomorphism.
(d) By (c) we can now set $\sigma := \theta_*^{-1}([\mathrm{id}_{\Delta^n}]) \in \mathrm{H}_n(\overline{B}^n, \overline{B}^n \setminus \{0\})$.

We make the following observations:

$$\begin{array}{lrcll}
 & \omega_{\Upsilon(x)}([M]) & = & \mathrm{sign}(\Upsilon, x) \cdot [\Upsilon \colon \Delta^n \to M] & \text{by the above special case} \\
\Longrightarrow & \omega_{\Upsilon(x)}([M]) & = & \mathrm{sign}(\Upsilon, x) \cdot \widetilde{\Upsilon}_*(\sigma) & \text{by (d), since } \widetilde{\Upsilon}_* = \Upsilon_* \circ \theta_* \\
\Longrightarrow & \omega_{\Psi(x)}([M]) & = & \mathrm{sign}(\Upsilon, x) \cdot \widetilde{\Psi}_*(\sigma) & \text{by (a) and (b)} \\
\Longrightarrow & \omega_{\Psi(x)}([M]) & = & \mathrm{sign}(\Psi, x) \cdot \widetilde{\Psi}_*(\sigma) & \text{by (∗)} \\
\Longrightarrow & \omega_{\Psi(x)}([M]) & = & \mathrm{sign}(\Psi, x) \cdot [\Psi \colon \Delta^n \to M] & \text{by (d), since } \widetilde{\Psi}_* = \Psi_* \circ \theta_*.
\end{array}$$



[1227] Since Ψ is admissible we know that $\Psi^{-1}(\Psi(x)) = \{x\}$. This implies that Ψ does indeed induce a map $\Psi_* \colon \mathrm{H}_n(\Delta^n, \Delta^n \setminus \{x\}) \to \mathrm{H}_n(M, M \setminus \{\Psi(x)\})$.

Finally we suppose that $(*)$ does not hold. Let

$$\begin{aligned} \rho\colon \Delta^n &\to \Delta^n \\ (t_0, t_1, t_2, \dots, t_n) &\mapsto (t_1, t_0, t_2, \dots, t_n) \end{aligned}$$

be the map that swaps the first two coordinates. Since ρ is orientation-preserving we now see that $\widetilde{\Upsilon}$ and $\widetilde{\Psi} \circ \rho$ satisfy $(*)$. Now we see that

$$\begin{aligned} & & \omega_{(\Psi\circ\rho)(\rho(x))}([M]) &= \operatorname{sign}(\Psi\circ\rho, \rho(x)) \cdot \overbrace{[\Psi\circ\rho\colon \Delta^n \to M]}^{=\Psi_*(\rho_*[\mathrm{id}_{\Delta^n}])} && \text{since } \widetilde{\Upsilon} \text{ and } \widetilde{\Psi}\circ\rho \text{ satisfy } (*) \\ & & & & & \text{we can apply the above} \\ \implies & & \omega_{\Psi(x)}([M]) &= \operatorname{sign}(\Psi\circ\rho, \rho(x)) \cdot \Psi_*(-[\mathrm{id}_{\Delta^n}]) && \text{by Exercise 76.2} \\ \implies & & \omega_{\Psi(x)}([M]) &= (-\operatorname{sign}(\Psi, x)) \cdot \Psi_*(-[\mathrm{id}_{\Delta^n}]) && \text{since } \rho \text{ is orientation-reversing} \\ \implies & & \omega_{\Psi(x)}([M]) &= \operatorname{sign}(\Psi, x) \cdot \underbrace{[\Psi\colon \Delta^n \to M]}_{=\Psi_*([\mathrm{id}_{\Delta^n}])}. \end{aligned}$$

■

We have now proved Theorem 100.1 for all closed smooth manifolds. It remains to prove Theorem 100.1 for smooth manifold with non-empty boundary.

Proof of Theorem 100.1 for manifolds with non-empty boundary. Let M be a compact oriented connected n-dimensional smooth manifold with non-empty boundary. Let M' be another copy of M. We consider the double $\mathrm{D}M = M \cup_{\partial M = \partial M'} M'$ of M, as defined on page 1667. Since $\partial M \neq \varnothing$ we obtain from the Doubling-Manifold Lemma 75.12 that $\mathrm{D}M$ is a closed oriented n-dimensional smooth manifold such that the map $M \to \mathrm{D}M$ is an orientation-preserving embedding.

(1) Let $x \in M \setminus \partial M$. We consider the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{H}_n(M, \partial M) & \xrightarrow{\cong} & \mathrm{H}_n(\mathrm{D}M, M') & \xleftarrow{\cong} & \mathrm{H}_n(\mathrm{D}M) \\ \downarrow & & & \searrow & \downarrow\cong \\ \mathrm{H}_n(M, M\setminus\{x\}) & & \xrightarrow{\qquad\cong\qquad} & & \mathrm{H}_n(\mathrm{D}M, \mathrm{D}M\setminus\{x\}). \end{array}$$

As we saw in the Doubling-Manifold Lemma 75.12 (6), it follows from the Excision Theorem 75.10 that the top left horizontal map is an isomorphism. Basically the same argument shows that the bottom horizontal map is an isomorphism. Furthermore note that in the proof of Theorem 98.8 we saw that the top right horizontal map is an isomorphism. By the above discussion of closed smooth manifolds we know that the right vertical map is an isomorphism. It now follows, as desired, that the left vertical map is also an isomorphism.

(2) By the discussion of the closed case we can consider the class $[\mathrm{D}M] \in \mathrm{H}_n(\mathrm{D}M)$. We define $[M] \in \mathrm{H}_n(M, \partial M)$ to be the preimage of $[\mathrm{D}M]$ under the composition of the above two horizontal isomorphisms. A slight variation on the argument provided in (1) now shows that $[M]$ has all the desired properties. We leave it to the reader to fill in the details. ■

Theorem 100.1 leads us to the following definition.

Definition. Let $n \in \mathbb{N}$ and let M be a compact oriented n-dimensional smooth manifold.

(1) We denote by $N_1, \ldots, N_k$ the components of M. Furthermore for $i = 1, \ldots, k$ we now denote by $[N_i] \in \mathrm{H}_n(N_i, \partial N_i)$ the homology class from Theorem 100.1. We refer to

$$[M] \;:=\; \sum_{i=1}^{k} [N_i] \;\in\; \mathrm{H}_n(M, \partial M) \;\underset{\uparrow}{=}\; \bigoplus_{i=1}^{k} \mathrm{H}_n(N_i, \partial N_i).$$

isomorphism given by Lemma 72.14

as the fundamental class of M.

(2) We refer to any cycle in $\mathrm{C}_n(M, \partial M)$ that represents the fundamental class $[M]$ as a fundamental cycle.

Remark. In Chapter 105 we will introduce the notion of an orientation on a topological manifold and on page 2351 we will introduce fundamental classes for oriented topological manifolds. After a few reinterpretations one sees that the Smooth-Topological Orientation Proposition 105.11 together with the Orientation-Top Homology Theorem 106.1 provide in particular an alternative proof of Theorem 100.1.

Conveniently enough, the proof of Theorem 100.1 actually gives an algorithm for determining explicit fundamental cycles.

Algorithm 100.2. Let $n \in \mathbb{N}$ and let M be a compact oriented non-empty n-dimensional smooth manifold.

(1) By the Smooth Manifold-Simplicial Structure Theorem 96.2 we can pick a smooth simplicial structure $(K = (V, S), |K| \xrightarrow{\lambda} M)$. Since V is finite we can pick a total order "$\leq$" on V.

(2) We denote by $s_1, \ldots, s_k$ the n-simplices of K.

(3) For $i \in \{1, \ldots, k\}$ we consider the characteristic map

$$\begin{aligned} \Phi_{s_i}^{\leq} \colon \quad \Delta^n &\to |K| \\ (t_0, \ldots, t_n) &\mapsto \sum_{j=0}^{n} t_j \cdot \underset{\uparrow}{j}\text{-th vertex of } s_i \end{aligned}$$

we use the total order on V to order the vertices of s_i

and we set

$$\epsilon_i \;:=\; \begin{cases} +1, & \text{if } \lambda \circ \Phi_{s_i}^{\leq} \colon \mathring{\Delta}^n \to M \text{ is orientation-preserving,} \\ -1, & \text{if } \lambda \circ \Phi_{s_i}^{\leq} \colon \mathring{\Delta}^n \to M \text{ is orientation-reversing.} \end{cases}$$

(4) The cycle $\epsilon_1 \cdot \lambda \circ \Phi_{s_1}^{\leq} + \cdots + \epsilon_k \cdot \lambda \circ \Phi_{s_k}^{\leq} \in \mathrm{C}_n(M)$ is a fundamental cycle of M.

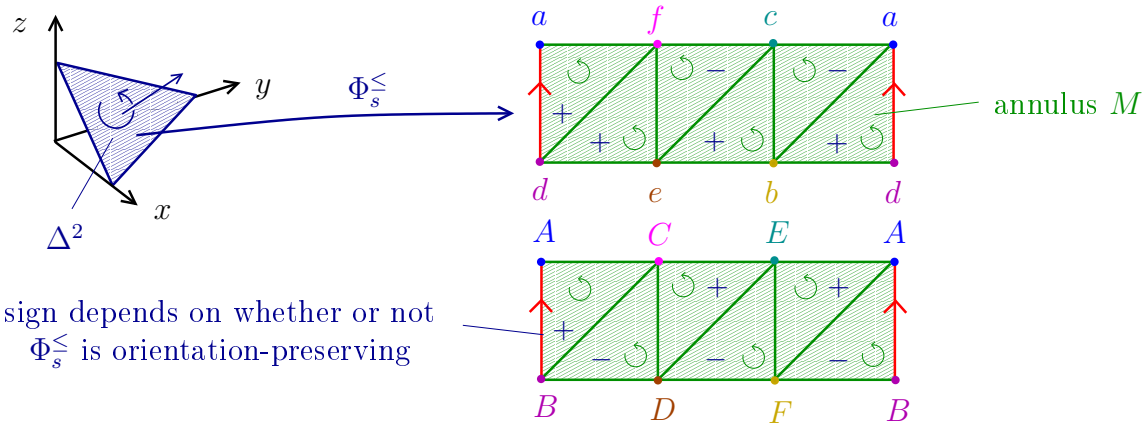
Proof. It is basically clear that it suffices to prove the validity of the algorithm for the *connected* case. Next note that if M is closed, then it follows immediately from the proof of Theorem 100.1 (1) that $\epsilon_1 \cdot \lambda \circ \Phi_{s_1}^{\leq} + \cdots + \epsilon_k \cdot \lambda \circ \Phi_{s_k}^{\leq} \in \mathrm{C}_n(M)$ is a cycle. Furthermore, in the proof of Theorem 100.1 (2) we saw that this cycle represents the fundamental class $[M] \in \mathrm{H}_n(M)$ of M. If M has non-empty boundary, then we equip the double $\mathrm{D}M$ with the simplicial structure given by "doubling" the simplicial structure of M. It is now straightforward to verify that the above algorithm also works in that case. We leave it to the reader to fill in the details. ∎

Example. In the figure below we consider the annulus $M = ([0, 3] \times [0, 1])/(0, y) \sim (3, y)$ equipped with an orientation and equipped with a smooth simplicial structure. In the

figure below we use two different ways to name the vertices, namely once the vertex set is $V = \{a, \dots, f\}$ and once it is $V = \{A, \dots, F\}$. In both cases we equip these sets with the obvious total order. We now see that the singular chains

$$\begin{aligned}
\mu = \; & +\Phi^{\leq}_{\{a,d,f\}} - \Phi^{\leq}_{\{d,e,f\}} + \Phi^{\leq}_{\{c,e,f\}} - \Phi^{\leq}_{\{b,c,e\}} + \Phi^{\leq}_{\{a,b,c\}} - \Phi^{\leq}_{\{a,b,d\}} \in \mathrm{C}_n(M, \partial M) \\
\nu = \; & +\Phi^{\leq}_{\{A,B,C\}} - \Phi^{\leq}_{\{B,C,D\}} + \Phi^{\leq}_{\{C,D,E\}} - \Phi^{\leq}_{\{D,E,F\}} + \Phi^{\leq}_{\{A,E,F\}} - \Phi^{\leq}_{\{A,B,F\}} \in \mathrm{C}_n(M, \partial M)
\end{aligned}$$

are both fundamental cycles of M.



Even though Algorithm 100.2 is very explicit, it also has some drawbacks. For example simplicial structures tend to have many simplices, which in the above algorithm leads to cycles with many summands. In a minute we will state a result that addresses this issue. But first let us prove a lemma which in particular says that, in order to show that we are given the fundamental class of a compact oriented *connected* smooth manifold, we only have to verify the defining property for a single choice of a map $\Psi\colon \Delta^n \to M$ and an admissible point $x \in \mathring{\Delta}^n$.

Lemma 100.3. (Fundamental Class-Characterization Lemma) Let M be a compact oriented non-empty n-dimensional smooth manifold and let $\varphi \in \mathrm{H}_n(M, \partial M)$. We denote by $N_1, \dots, N_k$ the components of M. The following statements are equivalent:

(1) φ is the fundamental class of M.

(2) For each $i \in \{1, \dots, k\}$ there exists a map $\Psi\colon \Delta^n \to N_i$ and an admissible point $x \in \mathring{\Delta}^n$ such that the following equality holds in $\mathrm{H}_n(M, M \setminus \{\Psi(x)\})$:

$$\underset{\text{map } \mathrm{H}_n(M,\partial M) \to \mathrm{H}_n(M, M\setminus\{\Psi(x)\})}{\omega_{\Psi(x)}(\varphi)} \;=\; \mathrm{sign}(\Psi, x) \cdot \underbrace{[\Psi\colon \Delta^n \to N_i \to M]}_{\in \mathrm{C}_n(M, M\setminus\{\Psi(x)\})}.$$

(3) For each $i \in \{1, \dots, k\}$ there exists a point $y \in N_i \setminus \partial_i N$ such that

$$\omega_y(\varphi) \;=\; \omega_y([M]).$$

Proof. A short moment of reflection shows that it suffices to prove the lemma in the case that M is connected. Next we note that the (1)⇒(2)-direction and (1)⇒(3)-direction are of course tautologies. We turn to the proof of the (2)⇒(1)-direction. We have the following

two equalities in $\mathrm{H}_n(M, M \setminus \{x\})$:

$$\omega_{\Psi(x)}(\varphi) \underset{\text{by hypothesis}}{\underset{\uparrow}{=}} \operatorname{sign}(\Psi, x) \cdot [\Psi\colon \Delta^n \to M] \underset{\text{by definition of } [M]}{\underset{\uparrow}{=}} \omega_{\Psi(x)}([M]).$$

Since we assume that M is connected we know by Theorem 100.1 that $\omega_{\Psi(x)}$ is an isomorphism. Thus it follows from the above that $\varphi = [M]$.

Finally we consider the (3)⇒(1)-direction. This statement follows again from the fact that ω_y is an isomorphism. ■

The following proposition now gives us a practical criterion for showing that a given cycle represents the fundamental class.

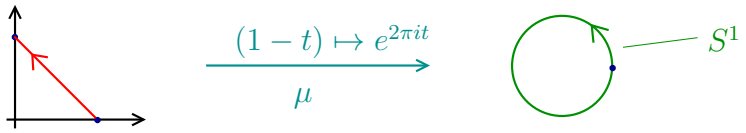
Proposition 100.4. (Fundamental Class-Criterion Proposition) Let M be a compact oriented connected n-dimensional smooth manifold. Let $r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m \in \mathrm{C}_n(M, \partial M)$ be a cycle. Suppose there exists a $j \in \{1, \ldots, m\}$ such that the following three conditions are satisfied:

(1) The map $\sigma_j\colon \mathring{\Delta}^n \to M$ is an orientation-preserving smooth embedding.[1228]
(2) There exists an $x \in \sigma_j(\Delta^n) \setminus \sigma_j(\partial\Delta^n)$ which is not contained in the image of any other $\sigma_i\colon \Delta^n \to M$.
(3) $r_j = 1$.

Then the cycle $r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m$ represents the fundamental class of the oriented smooth manifold, i.e. we have

$$[r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m] \;=\; [M] \;\in\; \mathrm{H}_n(M, \partial M).$$

Example. It follows easily from the Fundamental Class-Criterion Proposition 100.4 and our orientation convention, see page 681, that the singular 1-simplex $\mu\colon \Delta^1 \to S^1$ given by $\mu(1-t, t) = \exp(2\pi i t)$ represents the fundamental class of the oriented smooth manifold S^1.



Proof of the Fundamental Class-Criterion Proposition 100.4. Let M be a compact oriented connected n-dimensional smooth manifold, let $\sum_{i=1}^{m} r_i \cdot \sigma_i \in \mathrm{C}_n(M, \partial M)$ be a cycle and let $j \in \{1, \ldots, m\}$ such that conditions (1)–(3) above are satisfied. We pick $x \in \sigma_j(\mathring{\Delta}^n)$ as in (2). Now we see that we have the following equalities in $\mathrm{H}_n(M, M \setminus \{x\})$:

$$\omega_x([r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m]) \;=\; [\overbrace{r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m}^{\in \mathrm{C}_n(M, M\setminus\{x\})}] \underset{\text{since (2) implies that for } i \neq j \text{ we have } \sigma_i(\Delta^n) \subset M \setminus \{x\}}{\underset{\uparrow}{=}} [r_j \cdot \sigma_j] \;=\; [\sigma_j\colon \Delta^n \to M].$$

By hypothesis $\sigma_j\colon \Delta^n \to M$ restricts to an orientation-preserving embedding $\sigma_j\colon \mathring{\Delta}^n \to M$. Since $x \in \sigma_j(\Delta^n) \setminus \sigma_j(\partial\Delta^n)$ we see that x is an admissible point for σ_j. Thus all the

[1228] Note that we do *not* place any conditions on the restriction of σ_j to $\partial\Delta^n$.

hypotheses of the Fundamental Class-Characterization Lemma 100.3 hold, and therefore we conclude from the Fundamental Class-Characterization Lemma 100.3 that the cycle $r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m \in \mathrm{C}_n(M, \partial M)$ is indeed a fundamental cycle of M. ■

100.2. **Examples of fundamental classes.** In this section we will give explicit cycles representing the fundamental classes of $\overline{B}^n$, of S^n and of surfaces of genus ≥ 1. At this stage it is convenient to recall the following notation.

Notation.

(1) We consider the projection

$$\begin{array}{rcl} p\colon \Delta^n & \to & \triangle_n := \left\{(t_1, \ldots, t_n) \in [0,1]^n \;\middle|\; \sum_{j=1}^{n} t_j \in [0,1]\right\} \\ (t_0, \ldots, t_n) & \mapsto & (t_0, \ldots, t_{n-1}). \end{array}$$

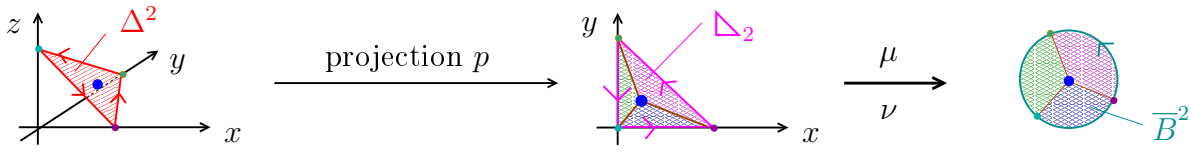
One can easily verify that the restriction $p\colon \overset{\circ}{\triangle}_n \to \overset{\circ}{\Delta}^n$ is orientation-preserving if and only if n is even.

(2) (a) Let $\mu\colon \triangle_n \to \overline{B}^n$ be the explicit homeomorphism with $\mu(\frac{1}{n+1}, \ldots, \frac{1}{n+1}) = 0$ that is given by the Convex-to-Ball Proposition 2.20 (2). Note that the restriction of μ to a map $\overset{\circ}{\triangle}_n \to B^n$ is not smooth. But it is easily seen to be an orientation-preserving smooth embedding on an open dense subset of $\overset{\circ}{\triangle}$.

(b) We denote by $\Phi\colon \Delta^n \to \triangle_n$ the natural projection. Given $P \in \overset{\circ}{\triangle}_n$ we consider the maps

$$\begin{array}{rcl} f_P\colon \overset{\circ}{\triangle}_n & \to & \mathbb{R}^n \\ (x_1, \ldots, x_n) & \mapsto & \frac{((x_1,\ldots,x_n)-P)}{x_1 \cdot \cdots \cdot x_n \cdot (1-x_1-\cdots-x_n)} \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \mathbb{R}^n & \to & B^n \\ (x_1, \ldots, x_n) & \mapsto & \frac{x_1^2+\cdots+x_n^2}{1+x_1^2+\cdots+x_n^2} \cdot (x_1, \ldots, x_n) \end{array}$$

For $P = (\frac{1}{n+1}, \ldots, \frac{1}{n+1})$ we denote the homeomorphism by $\nu\colon \triangle_n \to \overline{B}^n$. In Exercise 72.2 we saw that this map has the very neat feature that the restriction $\overset{\circ}{\triangle}_n \to B^n$ is actually an orientation-preserving diffeomorphism.



The following lemma is an immediate consequence of the above and the Fundamental Class-Characterization Lemma 100.3.

Lemma 100.5. Let $n \in \mathbb{N}$. The cycles $(-1)^n \cdot (\mu \circ p\colon \Delta^n \to \overline{B}^n) \in \mathrm{C}_n(\overline{B}^n, S^{n-1})$ and $(-1)^n \cdot (\nu \circ p\colon \Delta^n \to \overline{B}^n) \in \mathrm{C}_n(\overline{B}^n, S^{n-1})$ are both representatives of the fundamental class $[\overline{B}^n] \in \mathrm{H}_n(\overline{B}^n, S^{n-1})$.

Let $n \in \mathbb{N}$. The eagle-eyed reader will have noticed that by now we have introduced the symbol $[S^n] \in \mathrm{H}_n(S^n)$ twice:

(1) On page 1677 we introduced the "standard generator" $[S^n] = [\alpha - \beta] \in \mathrm{H}_n(S^n)$. Here α is given by

$$\begin{array}{rcccl} \Delta^n & \xrightarrow{\mu \circ p} & \overline{B}^n & \to & S^n \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \\ & & x & \mapsto & \left(x, \sqrt{1 - \|x\|^2}\right) \end{array}$$

and $\beta\colon \Delta^n \to S^n$ is the composition of α with reflection in the hyperplane defined by $x_{n+1} = 0$.

(2) We endow S^n with the orientation introduced on page 681, namely given $P \in S^n$ we say a basis $v_1, \ldots, v_n$ of $\mathrm{T}_P M = \widetilde{\mathrm{T}}_P M$ is positive if $P, v_1, \ldots, v_n$ is a positive basis for $\mathbb{R}^{n+1}$. With this orientation of S^n we now also have the corresponding fundamental class $[S^n] \in \mathrm{H}_n(S^n)$.

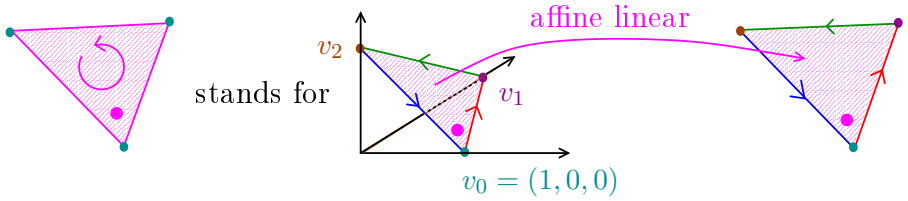
Fortunately the next lemma says that our various sign conventions work out in such a way that the two definitions of $[S^n]$ agree.

Lemma 100.6. Let $n \in \mathbb{N}$. The standard generator $[S^n] \in \mathrm{H}_n(S^n)$ that we introduced on page 1677 agrees with the fundamental class of the oriented smooth manifold S^n.

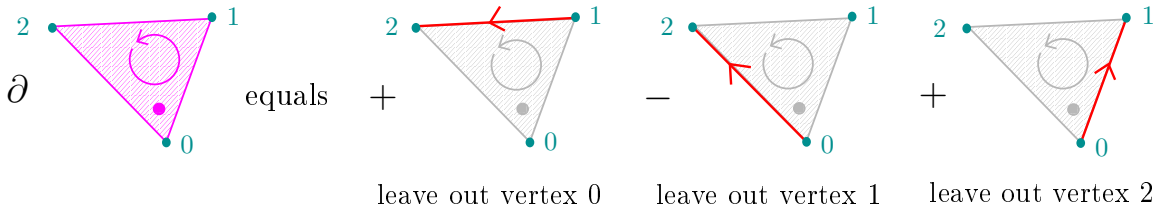
Proof. It follows from the above discussion, and some elementary arguments that the map $\alpha\colon \Delta^n \to S^n$ is an embedding and that it is an orientation-preserving embedding on an open subset of $\mathring{\Delta}^n$. The desired equality now follows almost immediately from the Fundamental Class-Criterion Proposition 100.4 applied to the cycle $\alpha - \beta$. ■

We move on to giving explicit representatives of fundamental classes for surfaces. This will require the introduction of more notation.

Notation. We will give many examples of fundamental classes of 2-dimensional smooth manifolds. Since we want to be very careful what singular simplex we mean by a given picture we adopt the following convention: A triangle T with a dot in a corner and a directional arrow in the center means the unique affine linear map $\Delta^2 \to T$ which sends $(1, 0, 0)$ to the vertex decorated with the dot and which respects the orientation of the vertices. We illustrate this notation in the figure below.

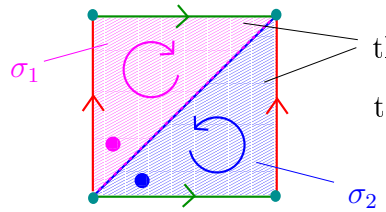


Convention. We also take some artistic liberties and allow triangles where the boundary is not given by segments and where the boundary consists of curves. In this case we expect that the boundary curves are parametrized with constant speed. Since the precise definition of the boundary of a singular 2-simplex is often a source of confusion we recall the definition in the figure below.



Example. In the figure below we show the torus $T = ([0, 1] \times [0, 1])/\!\sim$. We equip T with the orientation coming from the standard orientation on $(0, 1) \times (0, 1) \subset T$, which we can also view as an open subset of $\mathbb{R}^2$. In the figure we also show two singular 2-simplices σ_1

and σ_2. It is straightforward to verify that $-\sigma_1+\sigma_2$ is a cycle. (In the verification it is very helpful to keep the figure below in mind.) The summand σ_2 satisfies the conditions imposed in the Fundamental Class-Criterion Proposition 100.4, thus we see that $-\sigma_1+\sigma_2$ represents a fundamental class of T, in particular it represents a generator of $\mathrm{H}_2(T)$. This is of course the same cycle that we had already initially encountered on page 1589. We already saw in Lemma 78.11 (2) in a somewhat ad hoc argument that it represents a generator of $\mathrm{H}_2(T)$.



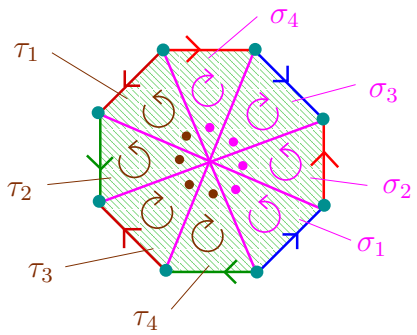
the torus T

the 2-cycle $-\sigma_1+\sigma_2 \in \mathrm{C}_2(T)$ represents the fundamental class of the torus

Examples. In the figure below we show the surface of genus 2 with eight singular 2-simplices $\sigma_1,\sigma_2,\sigma_3,\sigma_4$ and $\tau_1,\tau_2,\tau_3,\tau_4$. The interior of the octagon is an open subset of $\mathbb{R}^2$ and we equip the surface of genus 2 with the orientation coming from the standard orientation of $\mathbb{R}^2$. It is a fun exercise to verify that the singular 2-chain

$$\sigma_1+\sigma_2-\sigma_3-\sigma_4+\tau_1+\tau_2-\tau_3-\tau_4$$

is indeed a cycle. It follows easily from the Fundamental Class-Criterion Proposition 100.4 that the singular 2-chain is a fundamental cycle, i.e. it represents a fundamental class of the surface of genus 2.



$\sigma_1+\sigma_2-\sigma_3-\sigma_4$
$+\quad \tau_1+\tau_2-\tau_3-\tau_4$
is a cycle and represents the fundamental class

100.3. **Properties of the fundamental class.** In this section we will studies properties of the fundamental class. Before we can do so we will have plug a few holes in our definitions, namely we have not yet dealt with non-empty 0-dimensional manifolds and with the empty manifold.

First let us recall the following remark from page 675.

Definition. An orientation for a 0-dimensional smooth manifold M

Now we define the corresponding fundamental class.

Definition. Let $(M=\{P_1,\ldots,P_k\},\epsilon\colon M\to\{\pm1\})$ be a compact oriented non-empty 0-dimensional smooth manifold. The corresponding fundamental class $[M]$ is defined as the homology class

$$[M] := \sum_{i=1}^{k}\epsilon(P_i)\cdot[P_i] \in \mathrm{H}_0(M).$$

Finally let us consider the empty manifold. As we mentioned on page 514, basically by definition the empty set is a topological and smooth manifold of any dimension. Note that for the empty manifold $\varnothing$ we have the *equality* $\mathrm{H}_0(\varnothing) = \mathrm{H}_1(\varnothing) = \mathrm{H}_2(\varnothing) = \cdots = \{0\}$. This leads us to the following definition.

Definition. We define the fundamental class $[\varnothing]$ of the empty manifold to be $0 \in \{0\}$. Note that by the above the fundamental class $[\varnothing]$ is an element in every homology group of $\varnothing$.

Remark. The 0-dimensional oriented smooth manifolds and the empty manifold are a nuisance for a conscientious writer. In the following we will not always treat these cases explicitly and rely on the good will of the reader to deal with these on their own.

Now let us continue with serious mathematics. The following lemma tells us the behavior of the fundamental class under diffeomorphisms.

Lemma 100.7. Let $n \in \mathbb{N}_0$, let M and N be compact oriented connected n-dimensional smooth manifolds and let $f\colon M \to N$ be a diffeomorphism. We have[1229]

$$f_*([M]) \;=\; \begin{cases} [N], & \text{if } f \text{ is orientation-preserving,} \\ -[N], & \text{if } f \text{ is orientation-reversing.} \end{cases}$$

Proof. If $n = 0$, then the statement follows immediately from the definitions. As we will see, for $n \in \mathbb{N}$ the statement follows basically immediately from the definition property of the fundamental class. Indeed, we set $\epsilon := +1$ if f is orientation-preserving, otherwise we set $\epsilon := -1$. Let $\Psi\colon \Delta^n \to M$ be an embedding such that the map $\Psi\colon \mathring{\Delta}^n \to M$ is orientation-preserving. We pick some $x \in \mathring{\Delta}^n$. We consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}_n(M, \partial M) & \xrightarrow{\omega_{\Psi(x)}} & \mathrm{H}_n(M, M \setminus \{\Psi(x)\}) \\ \downarrow f_* & & \downarrow f_* \\ \mathrm{H}_n(N, \partial N) & \xrightarrow{\omega_{(f\circ\Psi)(x)}} & \mathrm{H}_n(N, N \setminus \{(f\circ\Psi)(x)\}). \end{array}$$

Now we see that

$$\begin{array}{rrcll} & \omega_{\Psi(x)}([M]) & = & \operatorname{sign}(\Psi, x) \cdot [\Psi], & \text{by definition of } [M] \\ \Longrightarrow & \omega_{(f\circ\Psi)(x)}(f_*([M])) & = & \operatorname{sign}(\Psi, x) \cdot [f \circ \Psi], & \text{since the diagram commutes} \\ \Longrightarrow & \omega_{(f\circ\Psi)(x)}(f_*([M])) & = & \epsilon \cdot \operatorname{sign}(f \circ \Psi, x) \cdot [f \circ \Psi], & \text{by definition of } \epsilon \\ \Longrightarrow & f_*([M] & = & \epsilon \cdot [N], & \text{by the Characterization Lemma 100.3.} \end{array}$$

■

Given an oriented smooth manifold M we denote, as usual, by $-M$ the same smooth manifold but with the opposite orientation. Now we can formulate the following lemma.

Lemma 100.8. For every compact oriented n-dimensional smooth manifold M we have

$$[-M] \;=\; -[M] \;\in\; \mathrm{H}_n(M, \partial M).$$

Proof. This statement follows basically immediately from Lemma 100.7 since the identity map $M \to -M$ is orientation-reversing. ■

[1229] Note that this statement now also makes sense for the empty manifold.

On page 689 we saw that the boundary of an oriented smooth manifold inherits a natural orientation. For compact smooth manifolds the following proposition gives the relationship between the fundamental class of the smooth manifold and of its boundary.

Proposition 100.9. (Fundamental Class-of-Boundary Proposition) Let M be a compact oriented n-dimensional smooth manifold with boundary. The connecting homomorphism

$$\partial\colon \mathrm{H}_n(M,\partial M) \;\to\; \mathrm{H}_{n-1}(\partial M)$$

of the long exact sequence in homology of the pair $(M,\partial M)$ has the property that

$$[M] \;\mapsto\; [\partial M].$$



Remark. In the Fundamental Class-of-Boundary Proposition 106.27 II we will prove the analogue of the Fundamental Class-of-Boundary Proposition 100.9 for topological manifolds.

Proof. Let M be a compact oriented n-dimensional smooth manifold with boundary components $N_1,\ldots,N_k$. By the Smooth Manifold-Simplicial Structure Theorem 96.2 the smooth manifold M admits a smooth simplicial structure $(K=(V,S),\lambda\colon |K|\to M)$. By the Manifold-Simplex Proposition 98.1 (3) we know that there exists a subcomplex $J=(U,R)$ of $=(V,S)$ such that $(J=(U,R),\lambda\colon |J|\to\partial M)$ is a smooth simplicial structure for ∂M. We pick some total order "$\leq$" on V. Next we introduce the following notation for K:

(1) We denote by S_n the set of n-simplices of K.
(2) For each $s\in S_n$ we denote by $\Phi_s^{\leq}\colon \Delta^n\to|K|$ the corresponding characteristic map.
(3) For each $s\in S_n$ we set

$$\epsilon_s \;:=\; \begin{cases} +1, & \text{if } \lambda\circ\Phi_s^{\leq}\colon \mathring{\Delta}^n\to M \text{ is orientation-preserving,}\\ -1, & \text{if } \lambda\circ\Phi_s^{\leq}\colon \mathring{\Delta}^n\to M \text{ is orientation-reversing.}\end{cases}$$

Next we introduce the analogous notions for the subcomplex J:

(4) We denote by R_{n-1} the set of $(n-1)$-simplices of R.
(5) For each $r\in R_{n-1}$ we denote by $\Psi_r^{\leq}\colon \Delta^{n-1}\to|J|$ the corresponding characteristic map.
(6) For each $r\in R_{n-1}$ we set

$$\mu_r \;:=\; \begin{cases} +1, & \text{if } \lambda\circ\Psi_r^{\leq}\colon \mathring{\Delta}^{n-1}\to\partial M \text{ is orientation-preserving,}\\ -1, & \text{if } \lambda\circ\Psi_r^{\leq}\colon \mathring{\Delta}^{n-1}\to\partial M \text{ is orientation-reversing.}\end{cases}$$

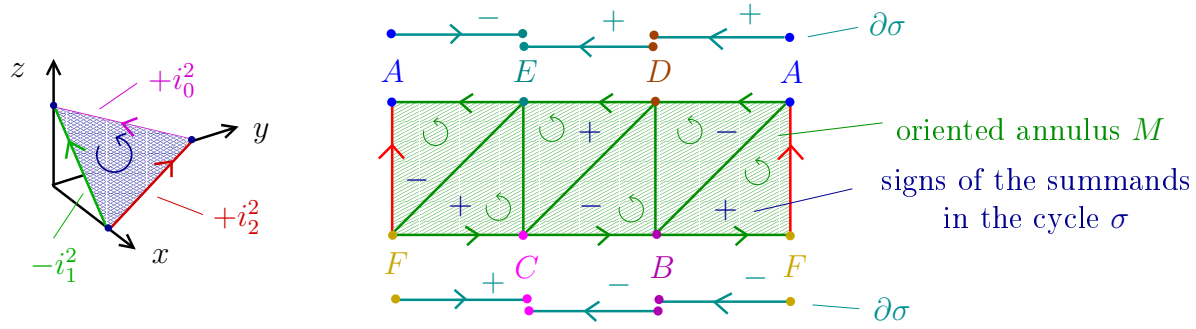
Now we see that

$$\partial[M] \overset{\text{Algorithm 100.2}}{=} \partial\Big[\sum_{s\in S_n} \epsilon_s \cdot \lambda \circ \Phi_s^{\leq}\Big] \overset{\text{Topological LES-Proposition 74.13}}{=} \Big[\sum_{s\in S_n} \epsilon_s \cdot \lambda \circ \partial_n(\Phi_s^{\leq})\Big] \overset{\text{definition of } \partial_n \colon \mathrm{C}_n(M) \to \mathrm{C}_{n-1}(M)}{=} \Big[\sum_{s\in S_n} \epsilon_s \cdot \lambda \circ \sum_{j=0}^{n} (-1)^j \cdot \Phi_s^{\leq} \circ i_j^n\Big]$$

since we are given a cycle in $\mathrm{C}_n(M, \partial M)$ the contributions of singular simplices for which the image does not lie in ∂M cancel

$$= \Big[\sum_{\substack{s\in S_n \text{ and } j\in\{0,\dots,n\} \\ \text{such that} \\ (\Phi_s^{\leq} \circ i_j^n)(\Delta^{n-1}) \subset \partial M}} (-1)^j \cdot \epsilon_s \cdot \lambda \circ \Phi_s^{\leq} \circ i_j^n\Big] = \Big[\sum_{r\in R} \mu_r \cdot \lambda \circ \Psi_r^{\leq}\Big] = [\partial M].$$

by the Manifold-Simplex Proposition 98.1 (4) we have a bijection of singular simplices that appear in the sums, furthermore the corresponding signs match by the discussion on page 2153 — Algorithm 100.2 ■



The following straightforward corollary says that we can "extend" fundamental cycles from the boundary to the total manifold.

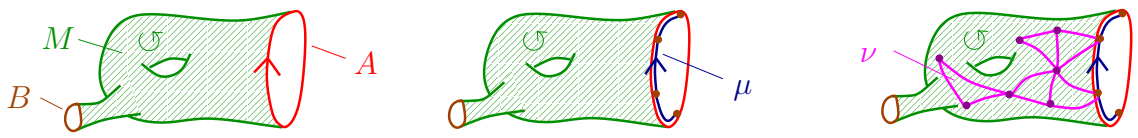
Corollary 100.10. Let M be a compact oriented n-dimensional smooth manifold. Let A be a union of components of ∂M. We write $B := \partial M \setminus A$. We consider the map

$$\mathrm{C}_n(M, \partial M) \xrightarrow{\partial_n} \mathrm{C}_{n-1}(\partial M) \xrightarrow{\mathrm{id}} \mathrm{C}_{n-1}(A \sqcup B) \xrightarrow{\cong} \mathrm{C}_{n-1}(A) \oplus \mathrm{C}_{n-1}(B) \twoheadrightarrow \mathrm{C}_{n-1}(A).$$

$$=: \Theta_A$$

Given any fundamental cycle $\mu \in \mathrm{C}_{n-1}(A)$ there exists a chain $\nu \in \mathrm{C}_n(M)$ such that ν represents a fundamental cycle in $\mathrm{C}_n(M, \partial M)$ and such that $\Theta_A(\nu) = \mu$.

Proof. First note that it is basically clear that it suffices to prove the statement for the case $A = \partial M$. Next, pick a singular chain $\widetilde{\nu} \in \mathrm{C}_n(M)$ that represents a fundamental cycle in $\mathrm{C}_n(M, \partial M)$. By Propositions 100.9 and 74.13 we know that the cycles $\partial_n(\widetilde{\nu})$ and μ are homologous in $\mathrm{C}_{n-1}(\partial M)$. This means that there exists a $\sigma \in \mathrm{C}_n(\partial M)$ with $\partial\sigma = \partial_n(\widetilde{\nu}) - \mu$. Now we set $\nu := \widetilde{\nu} - \sigma \in \mathrm{C}_n(M)$. Since ν and $\widetilde{\nu}$ represent the same element in $\mathrm{C}_n(M, \partial M)$ we see that ν is also a fundamental cycle. Finally we note that by construction we have $\partial_n(\nu) = \partial_n(\widetilde{\nu}) - \partial_n(\sigma) = \mu$. ■



Next we record the following useful corollary to the Fundamental Class-of-Boundary Proposition 100.9.

Corollary 100.11. (Fundamental Class-of-Boundary Corollary) Let M be a compact oriented connected n-dimensional smooth manifold. We denote by $i\colon \partial M \to M$ the inclusion map and we denote by $N_1, \ldots, N_k$ the components of ∂M.

(1) We have
$$\ker\big(i_*\colon \underbrace{\mathrm{H}_{n-1}(\partial M)}_{=\mathbb{Z}\cdot[N_1]\oplus\cdots\oplus\mathbb{Z}\cdot[N_k]} \to \mathrm{H}_{n-1}(M)\big) \;=\; \mathbb{Z}\cdot\underbrace{[\partial M]}_{\substack{\uparrow\\ \text{we have } [\partial M]=[N_1]+\cdots+[N_k]}} \subset \mathrm{H}_{n-1}(\partial M).$$

(2) Let $j \in \{1, \ldots, k\}$. We consider the following maps:
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}_{n-1}(\partial M)/\mathbb{Z}\cdot[\partial M] & \xrightarrow{i_*} & \mathrm{H}_{n-1}(M) & \longrightarrow & \mathrm{coker}(\mathrm{H}_{n-1}(\partial M) \xrightarrow{i_*} \mathrm{H}_{n-1}(M)) & \longrightarrow & 0. \\ & & \cong\uparrow & & & & & & \\ & & \mathbb{Z}\cdot[N_1]\oplus\ldots\oplus\widehat{\mathbb{Z}\cdot[N_j]}\oplus\ldots\oplus\mathbb{Z}\cdot[N_k] & & & & & & \end{array}$$
The following two statements hold:
(a) The horizontal sequence is exact and it splits.
(b) The vertical map is an isomorphism.

(3) If M has precisely one boundary component, then $i_*\colon \mathrm{H}_{n-1}(\partial M) \to \mathrm{H}_{n-1}(M)$ is the zero map.

(4) The boundary ∂M is not a retract of M.

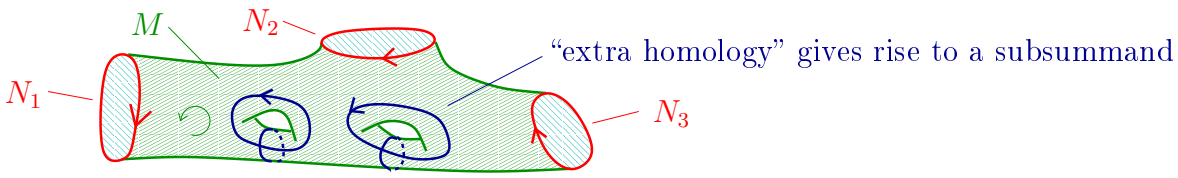
Remark.

(1) The special case that M is a 2-dimensional smooth manifold is foreshadowed by Exercise 80.14.

(2) In Proposition 74.7 we showed that the sphere $S^{n-1} = \partial\overline{B}^{n-1}$ is not a retract of the closed ball $\overline{B}^n$. the Fundamental Class-of-Boundary Corollary 100.11 (3) can thus be viewed as a generalization of Proposition 74.7 from $\overline{B}^n$ to the case of compact oriented connected smooth manifolds with non-empty boundary.

(3) In Corollary 106.28 we will extend the statement of the Fundamental Class-of-Boundary Corollary 100.11 to topological manifolds. Also note that in the Boundary Non-Retraction Proposition 31.3 we gave a very different proof of the Fundamental Class-of-Boundary Corollary 100.11.

Example. In the figure below we show a compact oriented n-dimensional smooth manifold with three boundary components N_1, N_2 and N_3. the Fundamental Class-of-Boundary Corollary 100.11 says that *any two* boundary components generate a *subsummand* of $\mathrm{H}_{n-1}(M)$. Furthermore in $\mathrm{H}_{n-1}(M)$ we have the equality $[N_1] + [N_2] + [N_3] = 0$.



Proof. Let M be a compact oriented n-dimensional smooth manifold. We denote by $i\colon \partial M \to M$ the inclusion map.

(1) Using the Fundamental Class-of-Boundary Proposition 100.9 and the long exact sequence in homology of the pair $(M, \partial M)$ we obtain that the following sequence is exact:

$$\underbrace{\mathrm{H}_n(M, \partial M)}_{=\mathbb{Z}\cdot[M] \text{ by Theorem 100.1}} \xrightarrow{[M]\mapsto[\partial M]} \mathrm{H}_{n-1}(\partial M) \xrightarrow{i_*} \mathrm{H}_{n-1}(M).$$

Statement (1) is now an immediate consequence.

(2) (a) It follows from (1) that the horizontal sequence is exact. By the Smooth Manifold-Top Homology Theorem 98.6 (2) the group $\mathrm{H}_{n-1}(M)$ is free abelian. It follows from Lemma 51.9 that the group $\operatorname{coker}(i_*\colon \mathrm{H}_{n-1}(\partial M) \to \mathrm{H}_{n-1}(M)) \cong \ker(\mathrm{H}_{n-1}(M) \to \mathrm{H}_{n-1}(M, \partial M))$ is also free abelian. It now follows from Lemma 78.1 that the short exact sequence splits.

(b) It follows immediately from $[\partial M] = [N_1] + \cdots + [N_k]$ that the vertical map is an isomorphism.

(3) Now we assume that ∂M has precisely one component. By Theorem 100.1 we now have $\mathrm{H}_{n-1}(\partial M) = \mathbb{Z} \cdot [\partial M]$. Thus this statement is now an immediate consequence of (1).

(4) Suppose there exists a retraction $r\colon M \to \partial M$, i.e. a map with $r \circ i = \mathrm{id}_{\partial M}$. We consider the following commutative diagram

$$\begin{array}{ccccc} & & \mathrm{H}_{n-1}(M) & & \\ & \overset{i_*}{\nearrow} & & \overset{r_*}{\searrow} & \\ \mathrm{H}_{n-1}(\partial M) & & \xrightarrow{(r\circ i)_*=\mathrm{id}} & & \mathrm{H}_{n-1}(\partial M). \end{array}$$

We just saw in (1) that i_* is not injective. But this leads immediately to a contradiction. ∎

In the next lemma we study the fundamental classes of codimension-zero submanifolds.

Lemma 100.12. (Codimension Zero-Fundamental Class Lemma) Let M be a compact oriented n-dimensional smooth manifold. Furthermore let $W \subset M$ be a compact n-dimensional submanifold. We write $\mathring{W} = W \setminus \partial W$ and we equip W with the orientation given by the Basics-of-Orientations Lemma 25.13 (6). The following two statements hold:

(1) The inclusion induced map $\mathrm{H}_n(W, \partial W) \to \mathrm{H}_n(M, M \setminus \mathring{W})$ is an isomorphism.

(2) The images of the fundamental classes $[M] \in \mathrm{H}_n(M, \partial M)$ and $[W] \in \mathrm{H}_n(W, \partial W)$ under the maps

$$\mathrm{H}_n(M, \partial M) \longrightarrow \mathrm{H}_n(M, M \setminus \mathring{W}) \xleftarrow{\cong} \mathrm{H}_n(W, \partial W)$$

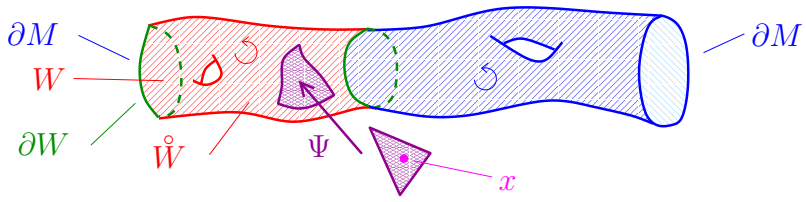
$\uparrow$ isomorphism by the Excision Theorem 75.10

agree.

Proof.

(1) The first statement is an immediate consequence of the Excision Theorem 75.10.

(2) We denote by $\Phi\colon \mathrm{H}_n(M, \partial M) \to \mathrm{H}_n(W, \partial W)$ the map given in Statement (2). We have to show that $\Phi([M])$ is the fundamental class for W. So let $\Psi\colon \Delta^n \to W$ be a



map and let $x \in \mathring{\Delta}^n$ be an admissible point. We consider the following commutative diagram

$$\begin{array}{ccccc}
 & & \overset{\Phi}{\curvearrowright} & & \\
\mathrm{H}_n(M, \partial M) & \longrightarrow & \mathrm{H}_n(M, M \setminus \mathring{W}) & \xleftarrow{\cong} & \mathrm{H}_n(W, \partial W) \\
\downarrow{\scriptstyle \omega_{\Psi(x)}} & & & & \downarrow{\scriptstyle \omega_{\Psi(x)}} \\
\mathrm{H}_n(M, M \setminus \{\Psi(x)\}) & & \xleftarrow[\text{isomorphism by the Excision Theorem 74.18}]{\cong} & & \mathrm{H}_n(W, W \setminus \{\Psi(x)\}) \\
 & {\scriptstyle \Psi_*} \nwarrow & & \nearrow {\scriptstyle \Psi_*} & \\
 & & \mathrm{H}_n(\Delta^n, \Delta^n \setminus \{x\}). & &
\end{array}$$

The desired statement now follows easily from the defining properties of fundamental classes and the above discussions. We leave the few remaining details to the reader. ■

One of the most popular approaches to computing the homology groups of some topological space X is to write X as the union $X = A \cup B$ of "simpler topological spaces A and B" in a suitable way and to use the Mayer–Vietoris Theorem 78.5. The catch of the Mayer–Vietoris Theorem 78.5 is that not only does one need to understand the homology groups of A, B and $A \cap B$, one also needs to understand the inclusion induced maps on homology groups, which can be quite difficult. Fortunately the Fundamental Class-of-Boundary Corollary 100.11 (2) now gives an interesting calculation of an inclusion induced map of homology groups. In the following two sections we will use the Fundamental Class-of-Boundary Corollary 100.11 (2) to study the homology groups of the connected sum of two smooth manifolds and to study the homology groups of knot complements.

100.4. **The homology groups of the connected sum of two smooth manifolds.** Let M and N be two compact oriented connected non-empty n-dimensional smooth manifolds. We recall the definition of the connected sum $M \# N$ that we gave on page **??**. First we pick an orientation-preserving smooth embedding $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and we pick an orientation-reversing smooth embedding $\psi\colon \overline{B}^n \to N \setminus \partial N$. Then we defined the connected sum of M and N as follows:

$$M \# N \;:=\; \big(M \setminus \varphi\big(B^n\big)\big) \sqcup \big(N \setminus \psi\big(B^n\big)\big) \,/\, \big\{\varphi(P) = \psi(P) \,\big|\, P \in S^{n-1}\big\}.$$

We refer to the Connected Sum-Proposition **??** for some key properties of $M \# N$. Recall that in the Connected Sum-π_1-Proposition 52.12 we showed that $\pi_1(M \# N)$ is isomorphic to the free product of $\pi_1(M)$ and $\pi_1(N)$. The following proposition says that a very similar result holds for the homology groups of $M \# N$.

Proposition 100.13. (Connected Sum-H_*-Proposition) Let M and N be two closed oriented connected non-empty n-dimensional smooth manifolds. For any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(M\#N) \;\cong\; \begin{cases} \mathrm{H}_k(M) \oplus \mathrm{H}_k(N), & \text{if } k \neq 0, n, \\ \mathbb{Z}, & \text{if } k = 0, n. \end{cases}$$

Moreover, for $k \neq 0, n$ the inclusion maps induce isomorphisms

$$\mathrm{H}_k(M) \oplus \mathrm{H}_k(N) \;\xleftarrow{\cong}\; \mathrm{H}_k(M \setminus B^n) \oplus \mathrm{H}_k(N \setminus B^n) \;\xrightarrow{\cong}\; \mathrm{H}_k(M\#N)$$

where $B^n \subset M$ and $B^n \subset N$ denote the open balls used in the definition of the connected sum of M and N.

Remark. In Exercise 100.8 we will consider the homology groups of $M\#N$ if we drop the condition that the smooth manifolds are closed.

The key to the proof of the Connected Sum-H_*-Proposition 100.13 is the following lemma.

Lemma 100.14. Let X and Y be two compact orientable connected n-dimensional smooth manifolds such that $\partial X \cong S^{n-1}$ and $\partial Y \cong S^{n-1}$. We pick a diffeomorphism $\varphi\colon \partial X \to \partial Y$.

(1) There exists an isomorphism

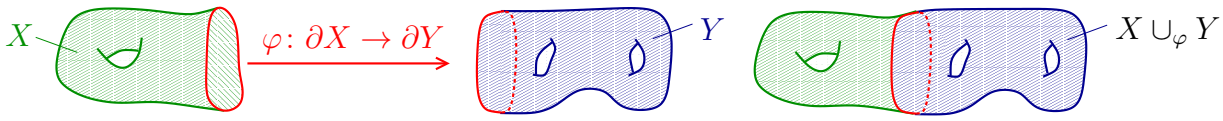
$$\mathrm{H}_n(X \cup_\varphi Y) \;\cong \mathbb{Z}.$$

Furthermore for every $k < n$ the inclusion maps $X \to X \cup_\varphi Y$ and $Y \to X \cup_\varphi Y$ induce an isomorphism

$$\widetilde{\mathrm{H}}_k(X) \oplus \widetilde{\mathrm{H}}_k(Y) \;\xrightarrow{\cong}\; \widetilde{\mathrm{H}}_k(X \cup_\varphi Y).$$

(2) In particular for the case that $Y = \overline{B}^n$ we obtain that for any $k < n$ the inclusion map $X \to X \cup_\varphi \overline{B}^n$ induces an isomorphism

$$\widetilde{\mathrm{H}}_k(X) \;\xrightarrow{\cong}\; \widetilde{\mathrm{H}}_k(X \cup_\varphi \overline{B}^n).$$



Proof of Lemma 100.14. The second statement is an immediate consequence of the first statement and the fact that for every $k < n$ we have $\widetilde{\mathrm{H}}_k(\overline{B}^n) = 0$. Thus it suffices to prove the first statement. To simplify the notation we write $S^{n-1} = \partial X = \partial Y$. We also write $Z = X \cup_\varphi Y$. It follows from the Mayer-Vietoris Theorem 78.9 for Manifolds[1230] that there exists a long exact sequence

$$\to \widetilde{\mathrm{H}}_k(S^{n-1}) \xrightarrow{i_*\oplus -i_*} \widetilde{\mathrm{H}}_k(X) \oplus \widetilde{\mathrm{H}}_k(Y) \xrightarrow{i_*+i_*} \widetilde{\mathrm{H}}_k(Z) \xrightarrow{\partial_k} \widetilde{\mathrm{H}}_{k-1}(S^{n-1}) \to \dots$$

where i always stands for the corresponding inclusion map. By Proposition 74.4 we have $\widetilde{\mathrm{H}}_n(S^n) \cong \mathbb{Z}$ and $\widetilde{\mathrm{H}}_k(S^n) = 0$ for $k \neq n$. We see immediately that for any $k < n$ the inclusion maps induce an isomorphism $\widetilde{\mathrm{H}}_k(X) \oplus \widetilde{\mathrm{H}}_k(Y) \to \widetilde{\mathrm{H}}_k(Z)$.

Now we turn to the n-th homology of Z. First note that by Theorem 98.8 we have $\widetilde{\mathrm{H}}_n(X) = \widetilde{\mathrm{H}}_n(Y) = 0$. Therefore the above long exact sequence gives rise to an exact

[1230] Note that here secretly we also used the Gluing-Smooth Manifolds-Proposition **??**

sequence

since S^{n-1} is the unique boundary component of X we deduce from Fundamental Class-of-Boundary Corollary 100.11 that the map $i_*\colon \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \to \widetilde{\mathrm{H}}_{n-1}(X)$ is the zero map, the same way we also see that the map $i_*\colon \mathrm{H}_{n-1}(S^{n-1}) \to \mathrm{H}_{n-1}(Y)$ is the zero map

$$\underbrace{\widetilde{\mathrm{H}}_n(X)\oplus\widetilde{\mathrm{H}}_n(Y)}_{=0 \text{ by Theorem 98.8}} \to \widetilde{\mathrm{H}}_n(Z) \xrightarrow{\partial_n} \underbrace{\widetilde{\mathrm{H}}_{n-1}(S^{n-1})}_{=\mathbb{Z}\cdot[S^{n-1}]} \xrightarrow{i_*\oplus -i_*} \widetilde{\mathrm{H}}_{n-1}(X)\oplus\widetilde{\mathrm{H}}_{n-1}(Y) \to \widetilde{\mathrm{H}}_{n-1}(Z) \to 0.$$

∂_n: isomorphism by the above

All the desired statements follow from the above data. ∎

Now we can easily provide the proof of the Connected Sum-H_*-Proposition 100.13.

Proof of the Connected Sum-H_*-Proposition 100.13. Let M and N be two closed oriented connected non-empty n-dimensional smooth manifolds. As required in the definition of the connected sum of M and N we pick an orientation-preserving smooth embedding $\varphi\colon \overline{B}^n \to M$ and we pick an orientation-reversing smooth embedding $\psi\colon \overline{B}^n \to N$. Recall that

$$M\#N \;=\; \big(M \setminus \varphi\big(B^n\big)\big) \sqcup \big(N \setminus \psi\big(B^n\big)\big) \,/\, \big\{\varphi(P) = \psi(P) \,\big|\, P \in S^{n-1}\big\}.$$

We write $X = M \setminus \varphi(B^n)$ and $Y = N \setminus \psi(B^n)$. Note that it follows immediately from Lemma 100.14 (1) that $\mathrm{H}_n(M\#N) = \mathrm{H}_n(X \cup_{S^{n-1}} Y) \cong \mathbb{Z}$. Now let $k < n$. Note that in this case we have

$$\mathrm{H}_k(M) \oplus \mathrm{H}_k(N) \;\xleftarrow{\cong}\; \mathrm{H}_k(\underbrace{M \setminus \varphi(B^n)}_{=X}) \oplus \mathrm{H}_k(\underbrace{N \setminus \psi(B^n)}_{=Y}) \;\xrightarrow{\cong}\; \mathrm{H}_k(M\#N)$$

Lemma 100.14 (1) applied to $M = X \cup_{S^{n-1}} \overline{B}^n$ and $N = Y \cup_{S^{n-1}} \overline{B}^n$

Lemma 100.14 (1) applied to $M\#N = X \cup_{S^{n-1}} Y$

∎

Examples.

(1) Let $T = S^1 \times S^1$ be the torus. In the figure on page **??** we saw that the connected sum $T\#T$ is diffeomorphic to the surface Σ_2 of genus 2. More generally, the same argument shows that the connected sum of g copies of the torus T is diffeomorphic to the surface Σ_g of genus g. Together with the calculation of the homology groups of the torus in Lemma 78.11 we now see that

$$\mathrm{H}_k(\Sigma_g) \;\cong\; \mathrm{H}_k(\underbrace{T\#\ldots\#T}_{g \text{ copies}}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, \\ \mathbb{Z}^{2g}, & \text{if } k = 1, \\ 0, & \text{if } k > 2. \end{cases}$$

This agrees of course with the result obtained in Proposition 80.10.

(2) We consider the complex manifold $\mathbb{C}\mathrm{P}^2$. On page 1772 we showed that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^2) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \\ 0, & \text{otherwise.} \end{cases}$$

As always we equip $\mathbb{C}\mathrm{P}^2$ with the natural orientation coming from the Complex Manifold-Orientation Proposition 32.23. We can now form the connected sum $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$. It follows from the Connected Sum-H_*-Proposition 100.13 that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 4, \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases}$$

For the complex projective space $\mathbb{C}\mathrm{P}^2$ we use the common notation that we write $\overline{\mathbb{C}\mathrm{P}}^2$ instead of $-\mathbb{C}\mathrm{P}^2$. Since the homology groups do not depend on the orientation we have the same homology groups for $\overline{\mathbb{C}\mathrm{P}}^2$ and the calculation shows that the homology groups of $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$ agree.

Recall that on page 1948 we showed, using the Künneth Theorem 90.8 that

$$\mathrm{H}_k(S^2 \times S^2) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 4, \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases}$$

This fact, and our above calculations shows, that the homology groups of the connected sums $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$, $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$ and of $S^2 \times S^2$ are all isomorphic. Furthermore, note that by Propositions 47.13, 48.20, 52.12 and the discussion on page 1533 we know that all of these topological spaces are simply connected. So this raises the following intriguing question.

Question 100.15. Are the closed 4-dimensional smooth manifolds $S^2 \times S^2$, $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$ homeomorphic?

We will answer this question in Corollary 133.12 and Proposition 133.13.

100.5. **Homology groups of knot complements.** We return to the ever-popular study of knots in S^3.

Definition. Let $K \subset S^3$ be a knot. By the Tubular Neighborhood Theorem **??** there exists a smooth embedding $\Phi\colon \overline{B}^2 \times K \to S^3$ such that $\Phi(0, P) = P$ for all $P \in K$. We refer to

$$X_K := S^3 \setminus \Phi(B^2 \times K)$$

as the exterior of K. If K is oriented, then we pick Φ to be orientation-preserving and, following the definition on page 1305, we refer to any oriented submanifold $\Phi(S^1 \times \{P\})$ as a meridian μ_K of K.

The following lemma gives us in particular the long-awaited calculation of the homology groups of knot complements.

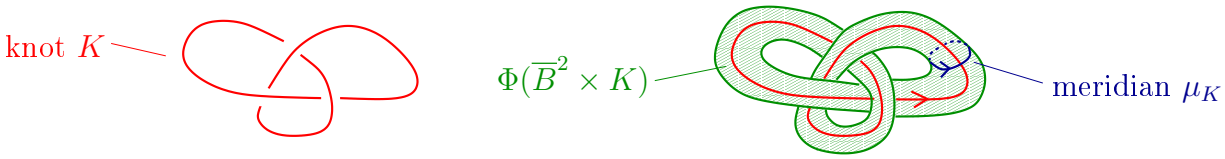
Lemma 100.16. For any knot $K \subset S^3$ the following statements hold:

(1) The exterior X_K of the knot K is a compact orientable connected 3-dimensional smooth manifold and the boundary ∂X_K is diffeomorphic to the torus.
(2) The diffeomorphism type of X_K as an oriented smooth manifold does not depend on the choice of Φ.

(3) The exterior X_K is a deformation retract of the knot complement $S^3 \setminus K$.
(4) We pick an orientation for K. For every $n \in \mathbb{N}_0$ we have[1231]

$$\underset{\text{induced by the inclusion } X_K \to S^3 \setminus K}{\mathrm{H}_n(S^3 \setminus K) \xleftarrow[\uparrow]{\cong} \mathrm{H}_n(X_K)} = \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z} \cdot [\mu_K], & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

where μ_K denotes a meridian of K.



Remark. In Lemma 78.13 we already showed that $\mathrm{H}_0(X_K) = \mathbb{Z}$ and that $\mathrm{H}_1(X_K) = \mathbb{Z} \cdot [\mu_K]$. In fact in the proof of Lemma 100.16 (4) below we will follow the same approach as in Lemma 78.13, except that now can overcome the hurdle which prevented us back then from computing all homology groups.

Proof. Let $K \subset S^3$ be an oriented knot and let $P \in K$. We pick a tubular neighborhood $\overline{B}^2 \times K$ for K and we pick $Q \in S^1$. We write $\mu_K = S^1 \times \{P\}$ and $\gamma = \{Q\} \times K$.

(1) This statement is basically a consequence of the Codimension-Zero Smooth Embedding Proposition 24.16.
(2) This statement is an immediate consequence of Proposition **??**.
(3) A deformation retraction from $S^3 \setminus K$ to X_K is given by the map

$$\begin{array}{rcl} (S^3 \setminus K) \times [0,1] & \to & X_K = S^3 \setminus (B^2 \times K) \\ (x,t) & \mapsto & \begin{cases} x, & \text{if } x \in Y = S^3 \setminus (\overline{B}^2 \times K), \\ (((1-t)r + t)\exp(\mathrm{i}\alpha), z), & \text{if } x = (r \cdot \exp(\mathrm{i}\alpha), z) \in \overline{B}^2 \times K. \end{cases} \end{array}$$

It follows easily from the Homotopy Stacking Lemma 14.3 and the Codimension-Zero Smooth Embedding Proposition 24.16 that this map is indeed continuous.
(4) First note that by the Homotopy Equivalence-H_*-Corollary 73.9, the Deformation Retract-Homotopy Equivalence Lemma 15.5 and by (3) we know that the inclusion $X_K \to S^3 \setminus K$ induces an isomorphism on all homology groups.

Now we proceed with almost the same argument as in the proof of Lemma 78.13. Namely, we consider the decomposition

$$S^3 = \underbrace{(S^3 \setminus (B^2 \times K))}_{=X_K} \cup_{S^1 \times K} (\overline{B}^2 \times K)$$

and we apply the Mayer-Vietoris Theorem 78.9 with reduced homology groups. We obtain the following long exact sequence:

$$\begin{array}{rcccccccc} \to & \mathrm{H}_3(\partial X_K) & \xrightarrow{i_*-i_*} & \mathrm{H}_3(X_K) & \oplus & \mathrm{H}_3(\overline{B}^2 \times K) & \xrightarrow{i_*+i_*} & \mathrm{H}_3(S^3) & \xrightarrow{\partial_3} \\ \to & \mathrm{H}_2(\partial X_K) & \xrightarrow{i_*-i_*} & \mathrm{H}_2(X_K) & \oplus & \mathrm{H}_2(\overline{B}^2 \times K) & \xrightarrow{i_*+i_*} & \mathrm{H}_2(S^3) & \xrightarrow{\partial_2} \\ \to & \mathrm{H}_1(\partial X_K) & \xrightarrow{i_*-i_*} & \mathrm{H}_1(X_K) & \oplus & \mathrm{H}_1(\overline{B}^2 \times K) & \xrightarrow{i_*+i_*} & \mathrm{H}_1(S^3) & \xrightarrow{\partial_1} 0. \end{array}$$

[1231] We will give an alternative approach to calculating these homology groups in Exercise 100.5.

Now we throw in everything we know about the above setting:

$$\begin{array}{ccccccccc}
\to & \underbrace{\mathrm{H}_3(\partial X_K)}_{=0} & \xrightarrow{i_*-i_*} & \mathrm{H}_3(X_K) & \oplus & \underbrace{\mathrm{H}_3(\overline{B}^2\times K)}_{=0} & \xrightarrow{i_*+i_*} & \underbrace{\mathrm{H}_3(S^3)}_{=\mathbb{Z}\cdot[S^3]} & \xrightarrow{\partial_3} \\
\to & \underbrace{\mathrm{H}_2(\partial X_K)}_{=\mathbb{Z}\cdot[\partial X_K]} & \xrightarrow{i_*-i_*} & \mathrm{H}_2(X_K) & \oplus & \underbrace{\mathrm{H}_2(\overline{B}^2\times K)}_{=0} & \xrightarrow{i_*+i_*} & \underbrace{\mathrm{H}_2(S^3)}_{=0} & \xrightarrow{\partial_2} \\
 & & \uparrow & & & & & & \\
 & & \text{by the Fundamental Class-of-Boundary Corollary 100.11 (2) this is the zero map} & & & & & & \\
\to & \underbrace{\mathrm{H}_1(\partial X_K)}_{=\mathbb{Z}\cdot[\mu_K]\oplus\mathbb{Z}\cdot[\gamma]} & \xrightarrow{i_*-i_*} & \mathrm{H}_1(X_K) & \oplus & \underbrace{\mathrm{H}_1(\overline{B}^2\times K)}_{=\mathbb{Z}\cdot[\gamma]} & \xrightarrow{i_*+i_*} & \underbrace{\mathrm{H}_1(S^3)}_{=0} & \xrightarrow{\partial_1} 0. \\
 & & \uparrow & & & & & & \\
 & & \text{we have } [\mu_K]\mapsto([\mu_K],0) \text{ and } [\gamma]\mapsto(?,[\gamma]) & & & & & &
\end{array}$$

Note that we obtain from the above discussion that the connecting homomorphism $\partial_3\colon \mathbb{Z}\cdot[S^3] = \mathrm{H}_3(S^3) \to \mathrm{H}_2(\partial X_K) = \mathbb{Z}\cdot[\partial X_K]$ is an epimorphism. Since every epimorphism $\mathbb{Z}\to\mathbb{Z}$ is necessarily an isomorphism we see that the connecting homomorphism is in fact an isomorphism. Thus we obtain that $\mathrm{H}_3(X_K) = \mathrm{H}_2(X_K) = 0$. At the lower end we obtain easily, as we explained in the proof of Lemma 78.13, that $\mathrm{H}_1(X_K) = \mathbb{Z}\cdot[\mu_K]$. ■

100.6. **The $\mathbb{F}_2$-fundamental class.** In this final section we introduce the $\mathbb{F}_2$-fundamental class for any compact (not necessarily orientable) smooth manifold. Truth be told, this notion will only play a fairly minor role in the subsequent chapters.

Now let M be a compact connected n-dimensional smooth manifold. Recall that by Theorem 98.8 (3) we know that, regardless of whether or not M is orientable, we have $\mathrm{H}_n(M,\partial M;\mathbb{F}_2)\cong\mathbb{F}_2$. This leads us to the following definition.

Definition. Let M be a compact n-dimensional smooth manifold. We denote by $N_1,\dots,N_k$ the components of M. We refer to

$$[M]_{\mathbb{F}_2} := \sum_{i=1}^{k} \text{the unique non-zero element of } \mathrm{H}_n(N_i,\partial N_i;\mathbb{F}_2) \in \underbrace{\mathrm{H}_n(M,\partial M;\mathbb{F}_2)}_{=\bigoplus_{i=1}^{k}\mathrm{H}_n(N_i,\partial N_i;\mathbb{F}_2)}$$

as the $\mathbb{F}_2$-fundamental class of M.

Remark. In the $\mathbb{F}_2$-Fundamental Class Corollary 106.5 we will generalize this notion to compact *topological* manifolds.

The following proposition is the "$\mathbb{F}_2$"-analogue of some of the results from Section 100.1.

Proposition 100.17. Let M be a compact n-dimensional smooth manifold.

(1) Let $(K=(V,S),\lambda\colon |K|\to M)$ be a smooth simplicial structure for M. We pick a total order "$\leq$" on V. We denote by $s_1,\dots,s_k$ the n-simplices of K. Furthermore for $i\in\{1,\dots,k\}$ we consider the characteristic map $\Phi^{\leq}_{s_i}\colon\Delta^n\to|K|$. With this notation the singular chain

$$\lambda\circ\Phi^{\leq}_{s_1}\otimes 1+\cdots+\lambda\circ\Phi^{\leq}_{s_k}\otimes 1 \in \mathrm{C}_n(M)\otimes\mathbb{F}_2 = \mathrm{C}_n(M;\mathbb{F}_2)$$

is a cycle which represents the $\mathbb{F}_2$-fundamental class of M.

(2) Let $x \in M \setminus \partial M$.
   (a) The image of the $\mathbb{F}_2$-fundamental class $[M]_{\mathbb{F}_2} \in \mathrm{H}_n(M, \partial M; \mathbb{F}_2)$ under the natural map $\mathrm{H}_n(M, \partial M; \mathbb{F}_2) \to \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2)$ is the unique non-trivial element.
   (b) If M is connected, then the natural map $\mathrm{H}_n(M, \partial M; \mathbb{F}_2) \to \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2)$ is an isomorphism.

Sketch of proof. It is straightforward to see that it suffices to consider the case that M is connected. Furthermore, using a doubling argument one can fairly easily reduce the proof of the statement to the case of a closed n-dimensional smooth manifold M. Next note that a slight variation on the argument of Theorem 98.4 shows that

$$\sigma \;:=\; \lambda \circ \Phi^{\leq}_{s_1} \otimes 1 + \cdots + \lambda \circ \Phi^{\leq}_{s_k} \otimes 1 \;\in\; \mathrm{C}_n(M; \mathbb{F}_2)$$

is indeed a cycle in the chain complex $\mathrm{C}_*(M; \mathbb{F}_2) = \mathrm{C}_*(M) \otimes \mathbb{F}_2$. Next note that the argument of the proof of Theorem 100.1 (1) shows that for every $x \in M$ the natural map $\mathrm{H}_n(M; \mathbb{F}_2) \to \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2)$ sends $[\sigma]$ to the unique non-zero element. Furthermore note that by Theorem 98.8 (3) we know that $\mathrm{H}_n(M; \mathbb{F}_2) \cong \mathbb{F}_2$. Finally note that this implies that the natural map $\mathrm{H}_n(M; \mathbb{F}_2) \to \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2)$ is indeed an isomorphism and that $[\sigma]$ is indeed a generator of $\mathrm{H}_n(M; \mathbb{F}_2)$. ■

The following proposition is the $\mathbb{F}_2$-analogue of the Fundamental Class-Criterion Proposition 100.4.

Proposition 100.18. Let M be a compact connected n-dimensional smooth manifold. Let $(\sigma_1 + \cdots + \sigma_m) \otimes 1 \in \mathrm{C}_n(M, \partial M; \mathbb{F}_2) = \mathrm{C}_n(M, \partial M) \otimes \mathbb{F}_2$ be a cycle. Suppose there exists a $j \in \{1, \ldots, m\}$ such that the following two conditions are satisfied:

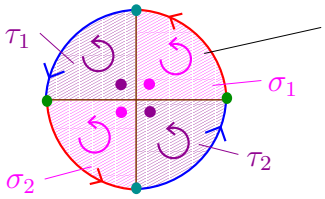
(1) The map $\sigma_j \colon \overset{\circ}{\Delta}{}^n \to M$ is a smooth embedding.
(2) There exists an $x \in \sigma_j(\Delta^n) \setminus \sigma_j(\partial \Delta^n)$ which is not contained in the image of any other $\sigma_i \colon \Delta^n \to M$.

Then the cycle $(\sigma_1 + \cdots + \sigma_m) \otimes 1$ represents the $\mathbb{F}_2$-fundamental class of M.

Proof. The proof of this proposition is almost the same as the proof of the Fundamental Class-Criterion Proposition 100.4. We leave it to the reader to make the necessary modifications. ■

Examples.

(1) Now we will give a representative for the $\mathbb{F}_2$-fundamental class for the real projective plane $\mathbb{R}\mathrm{P}^2$. More precisely, we consider the four singular simplices $\sigma_1, \sigma_2, \tau_1$ and τ_2 of $\mathbb{R}\mathrm{P}^2$ shown in the figure below, where we use a hopefully obvious variation on the notation introduced in the figure on page 2203. It is straightforward to see that the singular 2-chain $(\sigma_1 + \sigma_2 + \tau_1 + \tau_2) \otimes 1 \in \mathrm{C}_2(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$ is a cycle. Using Proposition 100.18 we see that it represents the $\mathbb{F}_2$-fundamental class of the real projective plane.



projective plane $\mathbb{R}\mathrm{P}^2 = \overline{B}^2 / \sim$

the 2-cycle $(\sigma_1 + \sigma_2 + \tau_1 + \tau_2) \otimes 1 \in \mathrm{C}_2(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$ represents the $\mathbb{F}_2$-fundamental class of the real projective plane

(2) In Exercise 100.13 we will write down a representative for the $\mathbb{F}_2$-fundamental class of the Klein bottle.

The following proposition is the analogue of the Fundamental Class-of-Boundary Proposition 100.9.

Proposition 100.19. Let M be a compact n-dimensional smooth manifold. The connecting homomorphism
$$\partial\colon \mathrm{H}_n(M, \partial M; \mathbb{F}_2) \ \to \ \mathrm{H}_{n-1}(\partial M; \mathbb{F}_2)$$
of the long exact sequence in homology with $\mathbb{F}_2$-coefficients of the pair $(M, \partial M)$ has the property that
$$[M]_{\mathbb{F}_2} \ \mapsto \ [\partial M]_{\mathbb{F}_2}.$$

Proof. The proof is the obvious modification of the previous proof. ∎

Finally we have the following analogue of the Fundamental Class-of-Boundary Corollary 100.11.

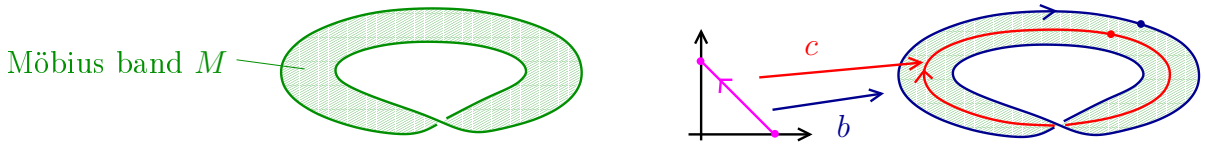
Corollary 100.20. Let M be a compact n-dimensional smooth manifold. We denote by $i\colon \partial M \to M$ the inclusion map.

(1) We have
$$\ker\big(i_*\colon \mathrm{H}_{n-1}(\partial M) \ \to \ \mathrm{H}_{n-1}(M)\big) \ = \ \mathbb{Z} \cdot [\partial M]_{\mathbb{F}_2} \ \in \ \mathrm{H}_{n-1}(\partial M; \mathbb{F}_2).$$
(2) If M has precisely one boundary component, then $i_*\colon \mathrm{H}_{n-1}(\partial M; \mathbb{F}_2) \to \mathrm{H}_{n-1}(M; \mathbb{F}_2)$ is the zero map.
(3) The boundary ∂M is not a retract of M.

Proof. Once again the proof is basically the same as the proof of the analogous statement for ordinary homology groups. ∎

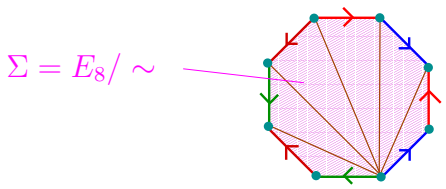
Remark. It is perhaps instructive to consider the good old Möbius band M. We denote by $i\colon \partial M \to M$ the inclusion map. Furthermore let $[b] \in \mathrm{H}_1(M)$ and $[c] \in \mathrm{H}_1(\partial M)$ be the generators shown in the figure below. We end up with the following situation:
$$\begin{array}{ccc} \mathrm{H}_1(\partial M) & \xrightarrow{[b]\mapsto 2\cdot[c]} & \mathrm{H}_1(M) \\ \cong\downarrow{\scriptstyle [\sigma]\mapsto[\sigma\otimes 1]} & & \cong\downarrow{\scriptstyle [\sigma]\mapsto[\sigma\otimes 1]} \\ \mathrm{H}_1(\partial M; \mathbb{F}_2) & \xrightarrow{[b\otimes 1]\mapsto[2\cdot c\otimes 1]=[c\otimes 2]=0} & \mathrm{H}_1(M; \mathbb{F}_2). \end{array}$$

Note that the vertical maps are isomorphisms by the Universal Coefficient Theorem 89.17. We see that the top horizontal map is an injective map, in fact it is basically given by multiplication by 2. On the other hand we see that the bottom horizontal map is the zero map.



Exercises for Chapter 100.

Exercise 100.1. We consider the surface $\Sigma = E_8/\!\sim$ of genus two as shown in the figure below. Show that there exists a cycle of the form $\epsilon_1 \cdot \sigma_1 + \cdots + \epsilon_6 \cdot \sigma_6 \in \mathrm{C}_2(E_8/\!\sim)$ such that the images of the six singular simplices $\sigma_1, \ldots, \sigma_6$ are precisely the six triangles shown in the figure.



Exercise 100.2. Let M be a compact oriented connected n-dimensional smooth manifold with *at least two* boundary components. Let C be one of the boundary components. By the Fundamental Class-of-Boundary Corollary 100.11 we know that the inclusion induced map $i_*\colon \mathrm{H}_{n-1}(C) \to \mathrm{H}_{n-1}(M)$ admits a left-inverse, i.e. there exists a homomorphism $\varphi\colon \mathrm{H}_{n-1}(M) \to \mathrm{H}_{n-1}(C)$ such that $\varphi \circ i_* = \mathrm{id}_{\mathrm{H}_{n-1}(C)}$. Also note that in Lemma 55.13 we showed that in the case $n = 2$ a stronger statement holds, namely C is a retract of M. Is it true in general that C is necessarily a retract of M?

Exercise 100.3. Let M be a closed oriented connected n-dimensional smooth manifold and let $C \subset M$ be a closed oriented connected $(n-1)$-dimensional submanifold of M. We assume that C is non-separating. Recall that according to the definition on page 1246 this means that $M \setminus C$ is connected. By the Tubular Neighborhood Theorem **??** we can pick a tubular neighborhood $[-1,1] \times C$. We write $W := M \setminus ((-1,1) \times C)$.

(a) Show that there exists an exact sequence of the form

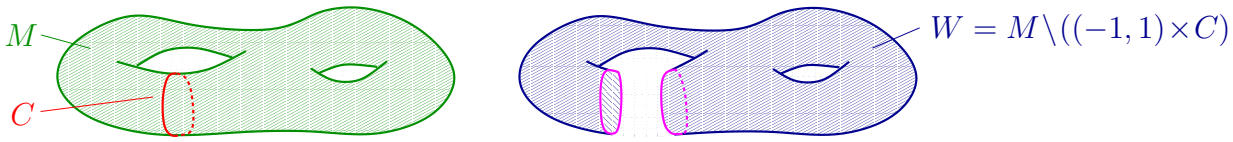
$$0 \;\to\; \mathrm{H}_{n-1}(\underbrace{M \setminus ((-1,1) \times C)}_{=W}) \;\to\; \mathrm{H}_{n-1}(M) \;\to\; \mathrm{H}_{n-2}(C) \;\to\; \ldots$$

Hint. Consider a Meyer–Vietoris sequence.

(b) Use (a) and Exercise 100.2 to show that the map $i_*\colon \mathrm{H}_{n-1}(C) \to \mathrm{H}_{n-1}(M)$ admits a left-inverse.

Remark. This implies in particular that $i_*([C])$ is a non-trivial element of $\mathrm{H}_{n-1}(M)$.

(c) Does the conclusion of (b) also hold if we replace the hypothesis that M is closed by the hypothesis that M is compact?



Exercise 100.4. Let $K = \gamma(S^1) \subset S^3$ be a knot and let $\Phi\colon \overline{B}^2 \times K \to S^3$ be a tubular map for K. As on page 2213 we refer to $X_K = S^3 \setminus \Phi(B^2 \times K)$ as the exterior of the knot K and we refer to $\mu_K = \Phi(S^1 \times \{P\})$ as a meridian of K. In Lemma 78.13, or alternatively

in Lemma 100.16, we saw that $[\mu_K]$, viewed as an element in $\mathrm{H}_1(X_K)$, is a generator of $\mathrm{H}_1(X_K)$.

(a) Show that there exists an $n \in \mathbb{Z}$ such that the curve

$$\begin{array}{rcl} S^1 & \to & \partial X_K = \Phi(S^1 \times K) \\ z & \mapsto & \Phi(z^n, \gamma(z)) \end{array}$$

represents a generator of $\ker(\mathrm{H}_1(\partial X_K) \to \mathrm{H}_1(X_K))$. We refer to the image of this curve as a *longitude* λ_K of K.

(b) Sketch longitudes for the three knots that are shown in the figure below.

(c) Now suppose that we are given two knots K and J. We define X_K, X_J, μ_K, μ_J and λ_K and λ_J as above. We pick a diffeomorphism $\varphi\colon \partial X_K \to \partial X_J$ with $\varphi_*(\mu_K) = \lambda_J$ and $\varphi_*(\lambda_K) = \mu_K$. We denote by $Y = X_K \cup_\varphi X_J$ the result of gluing X_K to X_J along the boundary. Show that Y is a homology sphere, i.e. show that $\mathrm{H}_1(Y) = \mathrm{H}_2(Y) = 0$ and $\mathrm{H}_3(Y) \cong \mathbb{Z}$.

Remark. If K and J are two non-trivial knots, then it follows from Proposition **??** (which in turn is a consequence of the famous Loop Theorem **??**) together with the Amalgamated Product-Monomorphism Proposition 53.24, that the fundamental group of Y is always non-trivial. This shows that Y is a homology sphere with non-trivial fundamental group. In other words we have now found an alternative proof of Proposition 98.11.



trivial knot trefoil figure-8 knot

Exercise 100.5. In this exercise we will consider an alternative approach to computing the homology groups of a knot complement. Let $K \subset S^3$ be a knot. We pick a tubular neighborhood $\overline{B}^2 \times K$ and we consider the exterior $X_K = S^3 \setminus (B^2 \times K)$. Now do the following:

(a) Show that the inclusion induced maps

$$\mathrm{H}_*(S^3, K) \xrightarrow{\cong} \mathrm{H}_*(S^3, \overline{B}^2 \times K) \xleftarrow{\cong} \mathrm{H}_*(X_K, \partial X_K)$$

are both isomorphisms.

(b) Compute the homology groups of the pair $(X_K, \partial X_K)$.

(c) Determine, using (b), the homology groups of X_K.

Exercise 100.6. We define an *m-component link* to be a 1-dimensional submanifold of S^3 that is diffeomorphic to the disjoint union of m copies of S^1. We pick a tubular neighborhood $\overline{B}^2 \times L$ and we define the exterior of L as $X_L := S^3 \setminus (B^2 \times L)$. We denote the components of L by $L_1, \ldots, L_m$. For each $i \in \{1, \ldots, m\}$ we pick a point $P_i \in L_i$ and we define $\mu_i := S^1 \times \{P_i\}$.

(a) Show that $\mathrm{H}_1(X_L) = \mathbb{Z} \cdot [\mu_1] \oplus \cdots \oplus \mathbb{Z} \cdot [\mu_m]$.

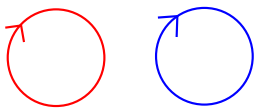
(b) Determine $\mathrm{H}_2(X_L)$.

Remark. There are basically two ways to go about doing this exercise, you can modify the proof of Lemma 100.16 or you could generalize the approach taken in Exercise 100.5.

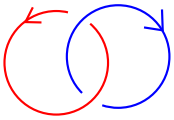
Exercise 100.7.

(a) In the figure below we see two 2-component links, namely the unlink and the Hopf link. Are these two links isotopic?

(b) Let $L \subset S^3$ be an m-component link. Show that the weight of the group $\pi_1(S^3 \setminus L)$, as defined in Exercise 59.13, equals m.



2-component trivial link



Hopf link

Exercise 100.8. Let M and N be two oriented, connected n-dimensional smooth manifolds. We suppose that M is closed and that N has *non-empty* boundary. Express the homology groups of $M\#N$ in terms of the homology groups of M and N.

Exercise 100.9. Let $f\colon M \to N$ be a map between two compact orientable connected n-dimensional smooth manifolds with non-empty boundary. We assume that $f\colon \partial M \to \partial N$ is a homeomorphism. Show that f is a surjection.

Exercise 100.10. Let M be a compact oriented n-dimensional smooth manifold and let $p\colon \widetilde{M} \to M$ be a finite covering. By the Manifold Covering Proposition 49.2 we can equip $\widetilde{M}$ with the structure of a compact oriented smooth manifold such that $p\colon \widetilde{M} \to M$ is an orientation-preserving map. Let $p^*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(\widetilde{M})$ be the transfer map that we introduced on page 1952. Show that $p^*([M]) = \left[\widetilde{M}\right] \in \mathrm{H}_n(\widetilde{M}, \partial\widetilde{M})$.

Exercise 100.11. Let Σ be a closed orientable connected 2-dimensional smooth manifold and let $f\colon \Sigma \to \Sigma$ be a diffeomorphism. We consider the corresponding mapping torus $M := \mathrm{Tor}(\Sigma, f)$. Note that we know by the Manifold-Mapping Torus Lemma 16.25 that M is a closed connected 3-dimensional smooth manifold.

(a) Suppose that if f is orientation-preserving, then $\mathrm{rank}(\mathrm{H}_2(M)) = \mathrm{rank}(\mathrm{H}_1(M))$.

(b) Suppose that if f is orientation-reversing, then $\mathrm{rank}(\mathrm{H}_2(M)) = \mathrm{rank}(\mathrm{H}_1(M)) - 1$.

Remark. You might want to use the Mapping Torus-Homology Proposition 78.20.

Exercise 100.12. Give examples of closed oriented connected non-empty smooth manifolds M and N of the same dimension such that $\pi_2(M\#N)$ is not isomorphic to $\pi_2(M) \oplus \pi_2(N)$.

Exercise 100.13. Give an example of a representative for the $\mathbb{F}_2$-fundamental class for the Klein bottle.

Exercise 100.14. Can you find a representative for the $\mathbb{F}_2$-fundamental class of $\mathbb{R}\mathrm{P}^2$ with two or three summands?

Exercise 100.15. Let $n \in \mathbb{N}$ and let M be a compact oriented n-dimensional smooth manifold. The image of the fundamental class $[M] \in \mathrm{H}_n(M, \partial M)$ under the natural map

$\mathrm{H}_n(M,\partial M) \to \mathrm{H}_n(M,\partial M;\mathbb{R})$ is called the *real fundamental class*, denoted by $[M]_{\mathbb{R}}$. Let $(K=(V,S),\Theta\colon |K| \to M)$ be a smooth simplicial structure. Given an ordered n-simplex $s=(v_0,\ldots,v_n)$ of K we denote by $\Phi_s\colon \Delta^n \to |K|$ the corresponding characteristic map given by $(t_0,\ldots,t_n) \mapsto t_0\cdot v_0 + \cdots + t_n \cdot v_n$.

(a) Show that

$$\sigma \;:=\; \sum_{\substack{s=(v_0,\ldots,v_n)\ \text{ordered } n\text{-simplex}\\ \text{such that } \Theta\circ\Phi_s\colon \mathring{\Delta}^n \to M\\ \text{is orientation-preserving}}} (\Theta\circ\Phi_s\colon \Delta^n \to M) \;\in\; \mathrm{C}_n(M)$$

represents a cycle in $\mathrm{C}_n(M,\partial M)$.

(b) Show that $[\sigma]\otimes\frac{2}{n!} \in \mathrm{C}_n(M,\partial M;\mathbb{R}) = \mathrm{C}_n(M,\partial M)\otimes\mathbb{R}$ represents $[M]_{\mathbb{R}}$.

Exercise 100.16. Let $n\in\mathbb{N}$ and let M and N be two compact oriented connected non-empty n-dimensional smooth manifolds. We pick an orientation-preserving smooth embedding $\varphi\colon \overline{B}^n_2 \to M\setminus\partial M$ and furthermore we pick an orientation-reversing smooth embedding $\psi\colon \overline{B}^n_2 \to N\setminus\partial N$. We use φ and ψ to define the connected sum

$$M\#N \;:=\; \big(M\setminus\varphi\big(B^n\big)\big) \sqcup \big(N\setminus\psi\big(B^n\big)\big) \,/\, \big\{\varphi(P)=\psi(P)\,\big|\,P\in S^{n-1}\big\}.$$

We refer to the map

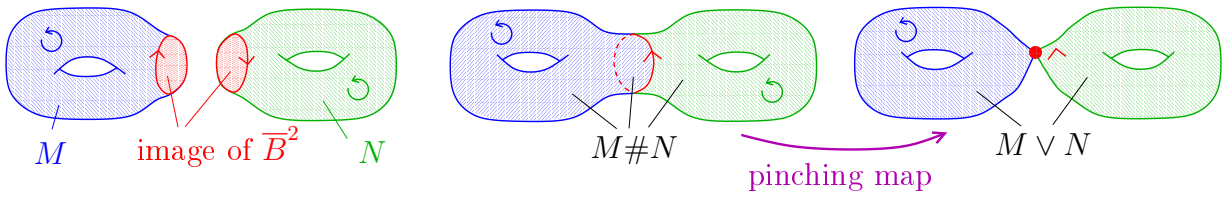
$$\begin{array}{rcl} p\colon M\#N & \to & M\vee N \\ P & \mapsto & \begin{cases} \varphi(x\cdot 2(t-1)), & \text{if } P=\varphi(x\cdot t) \text{ with } x\in S^{n-1} \text{ and } t\in[1,2] \\ \psi(x\cdot 2(t-1)), & \text{if } P=\psi(x\cdot t) \text{ with } x\in S^{n-1} \text{ and } t\in[1,2] \\ P, & \text{otherwise.} \end{cases} \end{array}$$

as the *pinching map*. Show that for $i=0,\ldots,n-1$ the induced map

$$p_*\colon \mathrm{H}_i(M\#N) \;\to\; \mathrm{H}_i(M\vee N)$$

is an isomorphism and that $p_*([M\#N]) = [M]+[N] \in \mathrm{H}_n(M\vee N) = \mathrm{H}_n(M)\oplus\mathrm{H}_n(N)$.

Remark. We leave it to the reader to show that for $M=N=S^n$ this pinching map is homotopic to the pinching map that we introduced on page 1565.



101. The degree of a map between oriented smooth manifolds II

In Chapters 38 and 39 we used Sard's Theorem 31.1 and several other tricks from differential topology to introduce the degree of a map between closed oriented connected non-empty smooth manifolds of the same dimension. In this chapter we will use a completely different approach to define the degree. The new approach is perhaps less intuitive, but since it builds on the machinery we have developed over the last chapters we can now quickly develop the theory of degrees.

101.1. **The definition of the degree of a map between compact smooth manifolds.** Let $n \in \mathbb{N}$ and let M be a compact oriented connected non-empty n-dimensional smooth manifold. Recall that in Theorem 100.1 and the Fundamental Class-Characterization Lemma 100.3 we proved that there exists a unique class $[M] \in \mathrm{H}_n(M, \partial M)$ such that the following holds:

(1) We have $\mathrm{H}_n(M, \partial M) = \mathbb{Z} \cdot [M]$.

(2) For every smooth embedding[1232] $\Psi\colon \Delta^n \to M$ and for every point $x \in \mathring{\Delta}^n$ with $\Psi^{-1}(\Psi(x)) = \{x\}$ we have the following equality in $\mathrm{H}_n(M, M \setminus \{\Psi(x)\})$:

$$\underset{\text{this is the natural map } \mathrm{H}_n(M,\partial M)\to\mathrm{H}_n(M,M\setminus\{\Psi(x)\}),\text{ which is in fact an isomorphism by Theorem 100.1 (1)}}{\omega_{\Psi(x)}([M])} \quad = \quad \operatorname{sign}(\Psi, x) \cdot [\, \underbrace{\Psi\colon \Delta^n \to M}_{\in \mathrm{C}_n(M, M\setminus\{\Psi(x)\})} \,].$$

In this short chapter we will explore the following definition.

Definition. Let $f\colon M \to N$ be a map between two compact oriented connected non-empty n-dimensional smooth manifolds. We assume that $f(\partial M) \subset \partial N$. We denote by $\deg(f) \in \mathbb{Z}$ the unique integer with

$$\underset{\text{since } f(\partial M) \subset \partial N \text{ we have an induced map } f_*\colon \mathrm{H}_n(M,\partial M) \to \mathrm{H}_n(N,\partial N)}{f_*([M])} \quad = \quad \deg(f) \cdot [N] \quad \in \quad \mathrm{H}_n(N, \partial N) = \mathbb{Z} \cdot [N].$$

We refer to $\deg(f)$ as the degree of f.

Remark.

(1) For a map $f\colon S^n \to S^n$ the above degree is, by definition, exactly the same as the degree that we defined on page 1684. Furthermore, if f is smooth and M and N are closed, then it follows from the Local-Global Degree Proposition 101.6 below that the above definition of the degree agrees with the definitions given on 899 on page 926.

(2) On page 2379 we will generalize the notion of degree from smooth manifolds to topological manifolds.

The following lemma generalizes several statements of the Sphere Degree Lemma 76.11.

Lemma 101.1. (Manifold Degree Lemma) Let $n \in \mathbb{N}$ and let L, M and N be compact oriented connected non-empty n-dimensional smooth manifolds.

(1) $\deg(\mathrm{id}_M) = 1$.

[1232] Recall that on page 2082 we defined what it means for a map $\Delta^n \to M$ to be a smooth embedding.

(2) For every map $f\colon M \to N$ such that $f(\partial M) \subset \partial N$ we have the following equality:[1233]
$$f_*([\partial M]) \;=\; \deg(f\colon M \to N) \cdot [\partial N] \;\in\; \mathrm{H}_{n-1}(\partial N).$$
In particular, if ∂M and ∂N are connected and non-empty, then we obtain the equality $\deg(f\colon \partial M \to \partial N) = \deg(f\colon M \to N)$.
(3) Let $f\colon M \to N$ be a map with $f(\partial M) \subset \partial N$. If f is not surjective, then $\deg(f) = 0$.
(4) Let $f, g\colon M \to N$ be two maps with $f(\partial M) \subset \partial N$ and $g(\partial M) \subset \partial N$. If f is homotopic to g, then $\deg(f) = \deg(g)$.
(5) For any two maps $f\colon L \to M$ and $g\colon M \to N$ with $f(\partial L) \subset \partial M$ and $g(\partial M) \subset \partial N$ we have $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.

Proof.

(1) This statement is trivial.
(2) We consider the following diagram:
$$\begin{array}{ccc} \mathrm{H}_n(M, \partial M) & \xrightarrow[[M]\mapsto[\partial M], \text{ by the Fundamental Class-of-Boundary Proposition 100.9}]{\partial_n} & \mathrm{H}_{n-1}(\partial M) \\ \downarrow f_* & & f_* \downarrow \\ \mathrm{H}_n(N, \partial N) & \xrightarrow[[N]\mapsto[\partial N], \text{ by the Fundamental Class-of-Boundary Proposition 100.9}]{\partial_n} & \mathrm{H}_{n-1}(\partial N) \end{array}$$
where the horizontal maps are given by the connecting homomorphisms of the pairs $(M, \partial M)$ and $(N, \partial N)$. By the Topological LES-Proposition 74.13 we know that the connecting homomorphism are natural, in other words, the diagram commutes. The statement now follows from the Fundamental Class-of-Boundary Proposition 100.9.
(3) So let $f\colon M \to N$ be a non-surjective map with $f(\partial M) \subset \partial N$. We pick a point P in N that is not in the image of f. Note that $N \setminus \{P\}$ is an open subset of N. Thus, by the discussion on page 553, we can view it as an n-dimensional smooth manifold with $\partial(N \setminus \{P\}) = \partial N \cap (N \setminus \{P\}) = \partial N \setminus \{P\}$. Also note that it follows easily from $n \geq 1$ that $N \setminus \{P\}$ is non-compact.

Now we consider the following very simple commutative diagram:
$$\begin{array}{ccc} \mathrm{H}_n(M, \partial M) & \xrightarrow{\quad f_* \quad} & \mathrm{H}_n(N, \partial N) \\ & {\scriptstyle f_*} \searrow \quad \nearrow & \\ & \underbrace{\mathrm{H}_n(N \setminus \{P\}, \partial(N \setminus \{P\}))} & \end{array}$$
$= 0$ by the Smooth Manifold-Top Homology Theorem 98.6, since $N \setminus \{P\}$ is non-compact and connected

It follows immediately that $f_*\colon \mathrm{H}_n(M, \partial M) \to \mathrm{H}_n(N, \partial N)$ is the zero map.
(4) This statement follows immediately from the Homotopic Maps-and-Homology Proposition 73.6.
(5) Exactly as in the proof of the Sphere Degree Lemma 76.11 (4) we have
$$(g \circ f)_*([L]) \;=\; g_*(f_*([L])) \;=\; g_*(\deg(f) \cdot [M]) \;=\; \deg(g) \cdot \deg(f) \cdot [N].$$
This shows that $\deg(g \circ f) = \deg(g) \cdot \deg(f)$. ■

[1233] This statement can be viewed as a generalization of the Boundary Degree Proposition 39.6.

We move on to the following definition.

Definition. Let $f\colon M \to N$ be a smooth map between two smooth manifolds of the same dimension. Let $P \in M \setminus \partial M$. We say f is a local diffeomorphism at P if there exists an open neighborhood U of P such that $f|_U$ is an embedding.[1234]

The following proposition is much more interesting than the previous lemma. It gives in particular a practical criterion for a map to have degree ± 1.

Proposition 101.2. (Local-Global Degree Proposition 101.2 Let $n \in \mathbb{N}$ and let $f\colon M \to N$ be a map between two compact oriented connected non-empty n-dimensional smooth manifolds such that $f^{-1}(\partial N) = \partial M$. We suppose that there exists an $x \in M \setminus \partial M$ such that $f^{-1}(f(x)) = \{x\}$ and we suppose that f is a local diffeomorphism at x. Then

$$\deg(f) \;=\; \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } x \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$

Proof. Let U be an open connected neighborhood of x such that $f\colon U \to N$ is an embedding. We set

$$\epsilon \;:=\; \begin{cases} +1, & \text{if } f\colon U \to N \text{ is orientation-preserving} \\ -1, & \text{if } f\colon U \to N \text{ is orientation-reversing.} \end{cases}$$

Note that we need to show that $\deg(f) = \epsilon$. Since $f^{-1}(\partial N) = \partial M$ we can pick a smooth embedding $\Psi\colon \Delta^n \to U$ such that $x \in \Psi(\mathring{\Delta}^n)$. Note that $\Psi^{-1}(x)$ is an admissible point for Ψ and note that it follows from our hypothesis $f^{-1}(f(x)) = \{x\}$ that $\Psi^{-1}(x)$ is also an admissible point for $f \circ \Psi$. We consider the following commutative diagram[1235]

$$\begin{array}{ccc} \mathrm{H}_n(M, \partial M) & \xrightarrow[\cong]{\omega_x} & \mathrm{H}_n(M, M \setminus \{x\}) \\ \downarrow{\scriptstyle f_*} & & {\scriptstyle f_*}\downarrow \\ \mathrm{H}_n(N, \partial N) & \xrightarrow[\cong]{\omega_{f(x)}} & \mathrm{H}_n(N, N \setminus \{f(x)\}). \end{array}$$

After these preparations we can now turn to the actual proof:

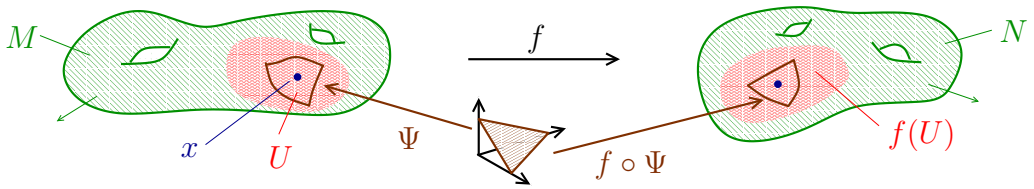
since $x \in \Psi(\mathring{\Delta}^n)$ and Ψ is an embedding we see that $\Psi\colon \Delta^n \to M$ defines an element of $\mathrm{H}_n(M, M\setminus\{x\})$

$$\begin{array}{rrcll} & \omega_x([M]) & = & \mathrm{sign}(\Psi, x) \cdot [\overset{\downarrow}{\Psi}\colon \Delta^n \to M] & \text{by definition of } [M] \\ \Rightarrow & f_*(\omega_x([M])) & = & \mathrm{sign}(\Psi, x) \cdot f_*([\Psi]) & \text{applying } f_* \\ \Rightarrow & \omega_{f(x)}(f_*([M])) & = & \mathrm{sign}(\Psi, x) \cdot [f \circ \Psi] & \text{since the diagram commutes} \\ \Rightarrow & \omega_{f(x)}(f_*([M])) & = & \epsilon \cdot \mathrm{sign}(f \circ \Psi, x) \cdot [f \circ \Psi] & \text{since } \mathrm{sign}(f \circ \Psi, x) = \epsilon \cdot \mathrm{sign}(\Psi, x) \\ \Rightarrow & \omega_{f(x)}(\deg(f) \cdot [N]) & = & \epsilon \cdot \omega_{f(x)}([N]) & \text{by definition of } \deg(f) \text{ and } [N] \\ \Rightarrow & \deg(f) \cdot [N] & = & \epsilon \cdot [N] & \text{by Theorem 100.1 (1)} \\ \Rightarrow & \deg(f) & = & \epsilon & \text{since } \mathrm{H}_n(N) = \mathbb{Z} \cdot [N]. \end{array}$$

■

[1234] Note that it follows from the Smooth Invariance of Domain Theorem 19.13 that this definition of a "local diffeomorphism at a point" is consistent with the definition given on page 547.

[1235] Note that our hypothesis that $f^{-1}(f(x)) = \{x\}$ implies that $f(M \setminus \{x\}) \subset N \setminus \{f(x)\}$. Also note that $x \in M \setminus \partial M$ and $f(x) \in N \setminus \partial N$. In particular this implies that the various inclusion induced maps actually make sense.



101.2. **Properties of the degree.** We can now restate and reprove the Degree Realization Proposition 38.13 from earlier on.

Proposition 38.13 (Degree Realization Proposition) Let $n \in \mathbb{N}$ and let M be a closed oriented connected non-empty n-dimensional smooth manifold. Given any $k \in \mathbb{Z}$ there exists a smooth map $f\colon M \to S^n$ with $\deg(f) = k$.

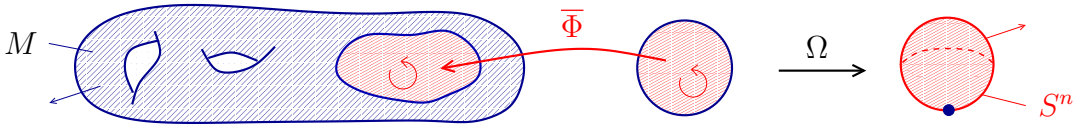
Proof. [1236]The proof is almost identical to the proof we had given on page 911. Let M be a closed oriented connected non-empty n-dimensional smooth manifold and let $k \in \mathbb{Z}$. As we mentioned on page 1687, by Lemma 76.10 and the Suspension-Degree Lemma 76.12 there exists a map $g\colon S^n \to S^n$ of degree k. It follows from this fact and the Manifold Degree Lemma 101.1 (4) that it suffices to prove the following claim.

Claim. There exists a map $f\colon M \to S^n$ of degree $+1$.

Proof. First we pick an orientation-preserving smooth embedding $\Phi\colon \overline{B}^n \to M$. It follows fairly easily from the Twice Quotient Lemma 5.23 (2), and the Compact-Hausdorff Proposition 2.17 (3) together with the Compact Image Lemma 2.13 and the Quotient Hausdorff Lemma 5.21, that the induced map $\overline{\Phi}\colon \overline{B}^n/S^{n-1} \to M/(M \setminus \Phi(B^n))$ is a homeomorphism. We denote by $\Omega\colon \overline{B}^n/S^{n-1} \to S^n$ the explicit homeomorphism from the Ball-Quotient Sphere Lemma 5.20. By the Ball-Quotient Sphere Lemma 5.20 (3) this homeomorphism has the property that the restriction of Ω to B^n is in fact an orientation-preserving smooth embedding. We consider the map $f\colon M \to S^n$ that is given by

$$M \longrightarrow M/(M \setminus \Phi(B^n)) \xrightarrow{\overline{\Phi}^{-1}} \overline{B}/S^{n-1} \xrightarrow{\Omega} S^n.$$
$$=:f$$

It follows almost immediately from the Local-Global Degree Proposition 101.2 that f is a map of degree $+1$. ■



To put the previous proposition into context we recall the following theorem. It says that for maps to S^n the converse to the Manifold Degree Lemma 101.1 (3) holds.

Theorem 38.16. (Hopf Degree Theorem) Let M be a closed oriented connected non-empty n-dimensional smooth manifold and let $f, g\colon M \to S^n$ be two maps. If[1237] $\deg(f) = \deg(g)$, then the maps f and g are actually homotopic.

[1237]When we formulated the Hopf Degree Theorem 38.16 we used the definition of the degree that we gave on 899. On page 2222 we pointed out that it follows from the Local-Global Degree Proposition 101.6 that

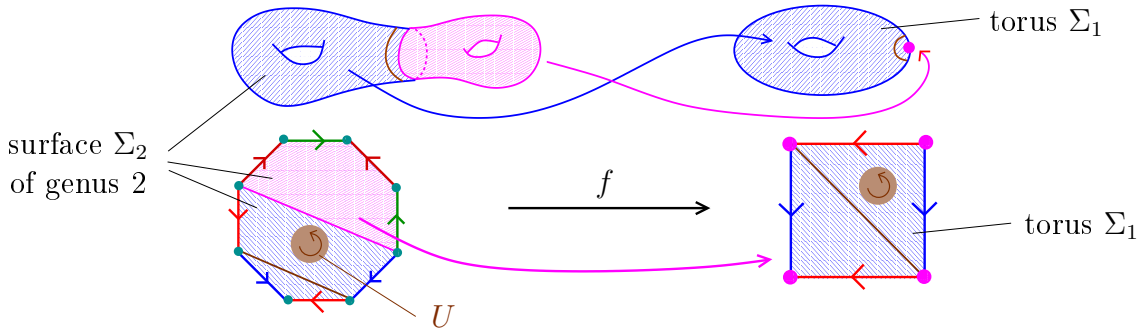
For the record we also state the following open question, see [**Kir1997**, Problem 5.26], which was initially posed by Heinz Hopf.

Question 101.3. (Hopf) Let M be a closed orientable smooth manifold M. Is every map $f\colon M \to M$ of degree ± 1 a homotopy equivalence?

In the following discussion, given $g \in \mathbb{N}_0$, we denote as usual by Σ_g the surface of genus g. We move on to our next list of interesting examples of maps of no-zero degree.

Lemma 101.4. Let $k \in \mathbb{N}_0$. For every $g \geq k$ there exists a degree-one map from Σ_g to Σ_k.

Proof. We prove the lemma for $g = 2$ and $k = 1$. It should be clear how to modify the discussion for other values of $g \geq k$. Let us consider the map from the surface Σ_2 of genus two to the torus Σ_1 that is shown in the figure below. In fact in the figure below we show the map twice, on top we show a slightly informal description of the map, at the bottom we give a more formal description. Note that the hypotheses of the Local-Global Degree Proposition 101.2 are satisfied. Thus we see that $\deg(f) = 1$. ■



The following question, which extends Question 38.15, arises naturally.

Question 101.5. Suppose that $g < k$. Does there exist a degree-one map from Σ_g to Σ_k?

We will be able to give an answer to this question later on in Corollary 101.9.

But first let us move on to the next proposition, which is a generalization of the Local-Global Degree Proposition 101.2.

Proposition 101.6. (Local-Global Degree Proposition) Let $f\colon M \to N$ be a map between two compact oriented connected non-empty n-dimensional smooth manifolds such that $f^{-1}(\partial N) = \partial M$.

(1) Let $y \in N \setminus \partial N$ such that the following condition is satisfied:
   (∗) The preimage $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$ and f is a local diffeomorphism around each x_i.

   Then
$$\deg(f) \;=\; \sum_{i=1}^{m} \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } x_i \\ -1, & \text{if } f \text{ is orientation-reversing at } x_i. \end{cases}$$

(2) If f is smooth, then every regular value in $N \setminus \partial N$ satisfies (∗).
(3) If f is smooth, then the set of regular values in $N \setminus \partial N$ has full measure in N.[1238]

[1238] the earlier definition of the degree agrees with the definition of the degree in terms of fundamental classes that we are using right now.

Remark. In Proposition 106.32 we state a result that generalizes the Local-Global Degree Proposition 101.6 (1) in the sense that we no longer demand that f is a local diffeomorphism around each x_i.

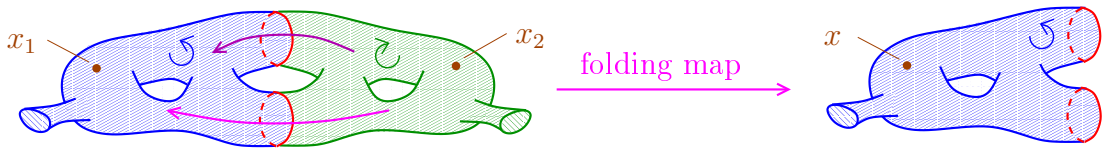
Example. Let M be a compact oriented n-dimensional smooth manifold and let W be the union of some components of ∂M. As on page 1667 we consider the double

$$\mathrm{D}_W M \;=\; (M \times \{1\}) \cup_{W\times\{1\}=W\times\{2\}} (M \times \{2\})$$

of M along W. By the Doubling-Manifold Lemma 75.12 we know that $\mathrm{D}_W M$ is naturally a compact oriented n-dimensional smooth manifold. We consider the folding map

$$\begin{array}{rcl} f\colon \mathrm{D}_W M & \to & M \\ [(P,i)] & \mapsto & P. \end{array}$$

If we take any $x \in M \setminus \partial M$, then we see that $f^{-1}(P)$ consists of precisely two points, namely $x_1 = [(x,1)]$ and $x_2 = [(x,2)]$. The folding map is an orientation-preserving local diffeomorphism around x_1 and it is an orientation-reversing local diffeomorphism around x_2. Thus we obtain from the Local-Global Degree Proposition 101.6 that the degree of the folding map is zero.



Since the two parts of the Local-Global Degree Proposition 101.6 are of very different nature we break the proof of the Local-Global Degree Proposition 101.6 into two parts.

Proof of the Local-Global Degree Proposition 101.6 (1). Let $f\colon M \to N$ be a map between two compact oriented connected non-empty n-dimensional smooth manifolds such that $f^{-1}(\partial N) = \partial M$. Let $y \in N \setminus \partial N$ such that $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$ and such that f is a local diffeomorphism around each x_i.

For $i = 1, \ldots, m$ we pick an open connected neighborhood U_i of x_i such that $f\colon U_i \to N$ is an embedding. Note that we can arrange that for any $i \neq j$ we have $x_j \notin U_i$. For $i = 1, \ldots, m$ we set

$$\epsilon_i \;:=\; \begin{cases} +1, & \text{if } f\colon U_i \to N \text{ is orientation-preserving} \\ -1, & \text{if } f\colon U_i \to N \text{ is orientation-reversing.} \end{cases}$$

Note that we need to show that $\deg(f) = \epsilon_1 + \cdots + \epsilon_m$.

Let $i \in \{1, \ldots, m\}$. Since $y \in N \setminus \partial N$ and since by hypothesis we have $f^{-1}(\partial N) = \partial M$ we know that $x_i \in M \setminus \partial M$. Thus we can pick a smooth embedding $\Psi_i\colon \Delta^n \to U_i$ such

[1238] We refer to page 762 for the definition of a subset of full measure in N and, more usefully, we refer to Propositions 30.2 and 30.3 for the main properties of subsets of full measure.

that $x_i \in \Psi_i(\mathring{\Delta}^n)$. We consider the following commutative diagram

$$\begin{array}{ccc}
\mathrm{H}_n(M,\partial M) & \xrightarrow{\quad =:\mu \quad} & \mathrm{H}_n(M, M\setminus\{x_1,\dots,x_m\}) \\
\Big\downarrow{\scriptstyle f_*} & \searrow^{\oplus\omega_{x_i}} \quad \swarrow_{\oplus\nu_i}^{\cong} & \Big\downarrow{\scriptstyle f_*} \\
\mathrm{H}_n(N,\partial N) & \xrightarrow[\cong]{\omega_{f(x)}} & \mathrm{H}_n(N, N\setminus\{y\})
\end{array}
\qquad \bigoplus_{i=1}^m \mathrm{H}_n(M, M\setminus\{x_i\})$$

We make the following clarifications and observations:

(1) The map $\mu\colon \mathrm{H}_n(M,\partial M) \to \mathrm{H}_n(M, M\setminus\{x_1,\dots,x_m\})$ and furthermore all the maps $\nu_i\colon \mathrm{H}_n(M, M\setminus\{x_1,\dots,x_m\}) \to \mathrm{H}_n(M, M\setminus\{x_i\})$ are induced by the natural maps on the pairs of topological spaces. Note that here we use that $x_1,\dots,x_m \in M\setminus\partial M$, that $y \in N\setminus\partial N$ and that $f(\partial M)\subset \partial N$.

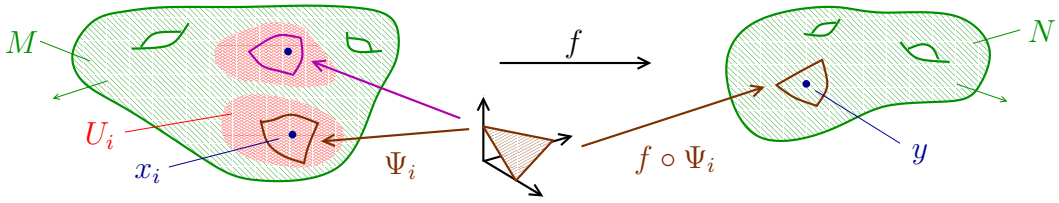
(2) It follows easily from the Excision Theorem 74.18, see also Exercise 75.1, that the diagonal map $\bigoplus_{i=1}^m \nu_i\colon \mathrm{H}_n(M, M\setminus\{x_1,\dots,x_m\}) \to \bigoplus_{i=1}^m \mathrm{H}_n(M, M\setminus\{x_i\})$ is an isomorphism.

(3) Let $i\in\{1,\dots,m\}$. Note that we arranged that $\Psi_i(\partial\Delta^n)\subset M\setminus\{x_1,\dots,x_m\}$. In particular $\Psi_i\colon \Delta^n \to M$ defines an element in $\mathrm{H}_n(M, M\setminus\{x_1,\dots,x_m\})$.

Now we see that

$$\begin{aligned}
& \bigoplus_{i=1}^m \omega_{x_i}([M]) = \bigoplus_{i=1}^m \operatorname{sign}(\Psi_i,x_i)\cdot[\Psi_i] && \text{by definition of } [M] \\
\Rightarrow\quad & \Big(\bigoplus_{i=1}^m \omega_{x_i}\Big)([M]) = \Big(\bigoplus_{i=1}^m \nu_i\Big)\Big(\sum_{j=1}^m \operatorname{sign}(\Psi_j,x_j)\cdot[\Psi_j]\Big) && \text{since } \nu_i([\Psi_j]) = \delta_{ij}\cdot[\Psi_i] \\
\Rightarrow\quad & \mu([M]) = \sum_{j=1}^m \operatorname{sign}(\Psi_j,x_j)\cdot[\Psi_j] && \text{since } \oplus\nu_i \text{ is an isomorphism and since the triangle commutes} \\
\Rightarrow\quad & f_*(\mu([M])) = \sum_{j=1}^m \operatorname{sign}(\Psi_j,x_j)\cdot f_*([\Psi_j]) && \text{applying } f_* \\
\Rightarrow\quad & \omega_{f(x)}(f_*([M])) = \sum_{j=1}^m \operatorname{sign}(\Psi_j,x_j)\cdot[f\circ\Psi_j] && \text{since the diagram commutes} \\
\Rightarrow\quad & \omega_{f(x)}(f_*([M])) = \sum_{j=1}^m \epsilon_j\cdot\operatorname{sign}(f\circ\Psi_j,x_j)\cdot[f\circ\Psi_j] && \text{since } \operatorname{sign}(f\circ\Psi_j,x_j)=\epsilon_j\cdot\operatorname{sign}(\Psi_j,x_j) \\
\Rightarrow\quad & \omega_{f(x)}(\deg(f)\cdot[N]) = \sum_{j=1}^m \epsilon_j\cdot\omega_{f(x)}([N]) && \text{by definition of } \deg(f) \text{ and } [N] \\
\Rightarrow\quad & \deg(f)\cdot[N] = \sum_{j=1}^m \epsilon_j\cdot[N] && \text{by Theorem 100.1 (1)} \\
\Rightarrow\quad & \deg(f) = \sum_{j=1}^m \epsilon_j && \text{since } \mathrm{H}_n(N,\partial N)=\mathbb{Z}\cdot[N].
\end{aligned}$$

■



Proof of the Local-Global Degree Proposition 101.6 (2) and (3). Let $f\colon M \to N$ be a smooth map between two compact oriented connected non-empty n-dimensional smooth manifolds M and N such that $f^{-1}(\partial N) = \partial M$.

(2) Let $y \in N \setminus \partial N$ be a regular value. It follows from Lemma 26.2 (2) that $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$. Let $i \in \{1, \ldots, m\}$. Note that it follows from $f^{-1}(\partial N) = \partial M$ that $x_i \in M \setminus \partial M$. Since x_i is a regular point we obtain from Lemma 26.2 (1) that f is a local diffeomorphism around x_i.

(3) By Sard's Theorem 31.1 together with the Measure-Zero Properties Proposition 30.2 we know that the set of regular values of f that are contained in $N \setminus \partial N$ has full measure in N. ■

The following proposition gives us many examples of degrees of maps between smooth manifolds.

Proposition 101.7. (Covering Map-Degree Proposition) Let $p\colon \widetilde{M} \to M$ be a finite covering of compact oriented connected non-empty smooth manifolds. If p is orientation-preserving, then

$$\deg(p) \;=\; [\widetilde{M} : M].$$

Proof. It turns out that there are two quick proofs for the proposition. In fact the desired statement follows immediately from the Local-Global Degree Proposition 101.6. Alternatively, if we denote by $p^*\colon \mathrm{H}_n(M, \partial M) \to \mathrm{H}_n(\widetilde{M}, \partial\widetilde{M})$ the transfer map that we introduced on page 1952, then we also see that

$$p_*([\widetilde{M}]) \underset{\text{Exercise 100.10}}{\uparrow} p_*(p^*([M])) \underset{\text{Transfer Map Proposition 91.1}}{\uparrow} [\widetilde{M} : M] \cdot [M].$$

■

The following proposition is the key to answering our ancient Question 38.15 and our much more recent Question 101.5.

Proposition 101.8. (Degree One-Epimorphism Proposition) Let $f\colon M \to N$ be a map between compact oriented connected non-empty n-dimensional smooth manifolds such that $f(\partial M) \subset \partial N$. If $\deg(f) = \pm 1$, then $f_*\colon \pi_1(M) \to \pi_1(N)$ is an epimorphism.

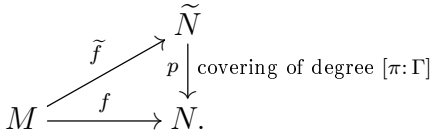
The statement of the Degree One-Epimorphism Proposition 101.8 is somewhat surprising, in so far as it takes as input fairly simple information, namely the induced map on n-th homology, which is an abelian group, and gives as output information about fundamental groups, which are usually much harder to deal with. Also it is a priori not clear why the homology in the maximal dimension should have any impact on fundamental groups.

Proof. Let $f\colon M \to N$ be a map between compact oriented connected non-empty n-dimensional smooth manifolds such that $f(\partial M) \subset \partial N$. We fix a base point $P \in M$ and we write $Q = f(P)$. Furthermore we write $\pi = \pi_1(N, Q)$ and $\Gamma = f_*(\pi_1(M, P))$.

By the Covering Existence Theorem 61.5[1239] there exists a path-connected covering $p\colon (\widetilde{N}, \widetilde{Q}) \to (N, Q)$ such that $p_*(\pi_1(\widetilde{N}, \widetilde{Q})) = \Gamma$ and such that $[\widetilde{N} : N] = [\pi : \Gamma]$. By the Manifold Covering Proposition 49.2 we can equip the covering space $\widetilde{N}$ with the structure of

[1239] Here we use that according to the discussion on pages 1352 any smooth manifold is locally simply connected.

a compact oriented smooth manifold such that the projection $p\colon \widetilde{N} \to N$ is an orientation-preserving map and such that we have $\partial\widetilde{N} = p^{-1}(\partial N)$. Since by design we know that $p_*(\pi_1(\widetilde{N}, \widetilde{Q})) = \Gamma = f_*(\pi_1(M, P))$ we obtain from the Map Lifting Criterion 61.2 that the map $f\colon (M, P) \to (N, Q)$ lifts to a map $\widetilde{f}\colon (M, P) \to (\widetilde{N}, \widetilde{Q})$. We can summarize the situation in the following simple commutative diagram



First suppose that Γ is a subgroup of π of infinite index. It follows immediately from $[\widetilde{N} : N] = [\pi : \Gamma]$ and Lemma 48.3 (5) that $\widetilde{N}$ is non-compact. Therefore we obtain from the Smooth Manifold-Top Homology Theorem 98.6 that $\mathrm{H}_n(\widetilde{N}) = 0$. But this shows that $f_* = p_* \circ \widetilde{f}_*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(N)$ factors through the trivial group. Hence $\deg(f) = 0$. But this is a contradiction to our hypothesis that $\deg(f) = \pm 1$.

So now we know that Γ is necessarily a finite-index subgroup of π. It follows from $[\widetilde{N} : N] = [\pi : \Gamma]$ and Lemma 48.3 (4) that $\widetilde{N}$ is compact. Therefore, by the above we now know that $\widetilde{N}$ is a compact oriented connected n-dimensional smooth manifold. We deduce that

$$\pm 1 \;=\; \deg(f) \underset{\uparrow}{=} \deg(\widetilde{f}) \cdot \deg(p) \underset{\uparrow}{=} \deg(\widetilde{f}) \cdot [\widetilde{N} : N] \;=\; \deg(\widetilde{f}) \cdot [\Gamma\colon \pi].$$

Manifold Degree Lemma 101.1 — Covering Map-Degree Proposition 101.7

This shows that $[\Gamma\colon \pi] = 1$, i.e. $\Gamma = \pi$. ■

Now we can give a negative answer to Questions 38.15 and 101.5.

Corollary 101.9. Let $g, k \in \mathbb{N}_0$. If there exists a degree-one map from Σ_g to Σ_k, then $g \geq k$.

Proof. Suppose that there exists a degree-one map from Σ_g to Σ_k. By the Degree One-Epimorphism Proposition 101.8 we obtain an epimorphism $\pi_1(\Sigma_g) \to \pi_1(\Sigma_k)$. In particular we obtain an epimorphism from the abelianization $\pi_1(\Sigma_g)$ onto the abelianization of $\pi_1(\Sigma_k)$. By the Fundamental Group-of-Surfaces Proposition 54.5 the former is isomorphic to $\mathbb{Z}^{2g}$ and the latter is isomorphic to $\mathbb{Z}^{2k}$. It follows from the Rank Additivity Lemma 51.7 that $g \geq k$. ■

We conclude this chapter with several questions. Recall that it follows from Lemma 101.4 that there exists a degree-one map $S^2 \times S^2 \to S^4$. It is natural to ask whether the converse holds. This leads us to the following question which we will answer in Lemma 117.5.

Question 101.10. Does there exist a degree-one map from the 4-dimensional sphere S^4 to $S^2 \times S^2$?

Later, in Corollary 119.14, we will also address the following question.

Question 101.11. Does there exist a degree-one map from $S^1 \times S^2$ to $\mathbb{R}\mathrm{P}^3$?

It is also interesting to study degrees of self-maps of smooth manifolds. For example note that it follows from Lemmas 76.10 and the Suspension-Degree Lemma 76.12 that given any $k \in \mathbb{Z}$ and given any $n \in \mathbb{N}$ there exists a map $f\colon S^n \to S^n$ of degree k. Also note that it follows easily from the Covering Map-Degree Proposition 101.7 that the torus $T = S^1 \times S^1$

admits a self-map of any degree. This naturally leads us to the question whether similar phenomena hold for surfaces of higher genus.

Question 101.12.

(1) Let $g \geq 2$. Does there exist a self-map $f\colon \Sigma_g \to \Sigma_g$ of degree -1?
(2) Let $g \geq 2$ and let $k \geq 2$. Does there exist a self-map $f\colon \Sigma_g \to \Sigma_g$ of degree k?
(3) Does there exist a self-map $f\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ of degree -1?

We will address the first question in Exercise 101.3 and we will address the last two questions later on in Proposition 114.17 and Corollary 121.10.
Finally we conclude this chapter with two conjectures that were formulated by Heinz Hopf in 1931. We refer to [**Hau1987, Neum1953**] for some more information and partial results.

Conjecture 101.13. (Hopf Conjecture) Let M be a closed oriented connected smooth manifold and let $f\colon M \to M$ be a map.

(1) If $\deg(f) = \pm 1$, then $f_*\colon \pi_1(M) \to \pi_1(M)$ is an isomorphism.
(2) If $\deg(f) = \pm 1$, then f is a homotopy equivalence.

Remark. Let M be a closed oriented connected smooth manifold and let $f\colon M \to M$ be a map with $\deg(f) = \pm 1$.

(1) By the Degree One-Epimorphism Proposition 101.8 we know that f_* is an epimorphism. Thus the Hopf Conjecture 101.13 (2) leads to the group-theoretic question, under what circumstances is a surjective automorphism of a group also injective. In particular this leads to the definition of a Hopfian group that we gave on page 1416.
(2) In Proposition **??** we will see that in general f is not homotopic to a homeomorphism. Thus in general "homotopy equivalence" is the best we can hope for. Note though that if M is in fact aspherical, see page 1861 for the definition, then the Borel Conjecture **??** predicts that f is actually homotopic to a homeomorphism.

Exercises for Chapter 101.

Exercise 101.1. Let M be a compact oriented n-dimensional smooth manifold with non-empty boundary. Show that there exists a map $f\colon M \to \overline{B}^n$ with $f(\partial M) \subset S^{n-1}$ of degree $+1$.
Remark. For $n \geq 2$ one can combine this result with Exercise 76.5 to get maps of arbitrary degree.

Exercise 101.2. Given an example of a map $f\colon M \to N$ between closed oriented connected non-empty n-dimensional smooth manifolds such that $f_*\colon \pi_1(M) \to \pi_1(N)$ is an epimorphism, but such that the degree of f is zero.

Exercise 101.3. Given $g \in \mathbb{N}$ we denote by Σ_g the surface of genus g.

(a) Let $g \geq 2$. Does there exist a self-map $f\colon \Sigma_g \to \Sigma_g$ of degree -1?
(b) Let $g \geq 1$ and let $k \geq 2$. Does there exist a self-map $f\colon \Sigma_g \to \Sigma_g$ of degree k?

Exercise 101.4. Given $g \in \mathbb{N}$ we denote by Σ_g the surface of genus g. Show that if there exists a map $f\colon \Sigma_g \to \Sigma_k$ of non-zero degree, then $g \geq k$.

Exercise 101.5. Let $f\colon M \to N$ be a map between two closed oriented non-empty n-dimensional smooth manifolds. Suppose that there exists a compact smooth manifold W with $\partial W = M$ and a smooth map $g\colon W \to N$ with $g|_M = f$. Show that $\deg(f) = 0$.
Remark. This exercise is almost identical to Exercise 38.11, but now you should be able to give a much shorter proof.

Exercise 101.6. Let $f\colon M \to N$ be a map between two closed connected non-empty n-dimensional smooth manifolds. Furthermore let $p\colon \widetilde{N} \to N$ be a finite covering. As in the Topological-Pullback Lemma 56.16 we consider the pullback

$$\widetilde{M} \;:=\; f^*\widetilde{N} \;=\; \{(x,y) \in M \times \widetilde{N} \,|\, f(x) = p(y)\}$$

together with the following commutative diagram:

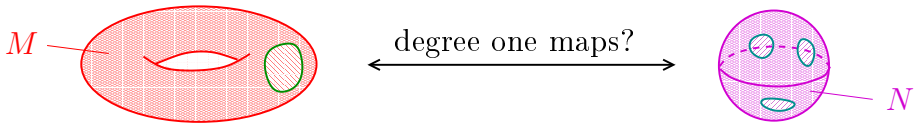
$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{(x,y)\mapsto y} & \widetilde{N} \\ {\scriptstyle =:q}\downarrow{\scriptstyle (x,y)\mapsto x} & \overset{=:F}{\nearrow} & \downarrow{\scriptstyle p} \\ M & \xrightarrow{\quad f \quad} & N. \end{array}$$

Note that by the Topological-Pullback Lemma 56.16 (1) we know that q is again a covering with $\deg(q) = \deg(p)$.

(a) We suppose that M and N are oriented. We equip the smooth manifolds $\widetilde{M}$ and $\widetilde{N}$ with the corresponding orientations given by the Manifold Covering Proposition 49.2. Show that $\deg(F\colon \widetilde{M} \to \widetilde{N}) = \deg(f\colon M \to N)$.
(b) Suppose that the induced map $f_*\colon \mathrm{H}_n(M;\mathbb{F}_2) \to \mathrm{H}_n(N;\mathbb{F}_2)$ is an isomorphism. Show that $F_*\colon \mathrm{H}_n(\widetilde{M};\mathbb{F}_2) \to \mathrm{H}_n(\widetilde{N};\mathbb{F}_2)$ is also an isomorphism.

Exercise 101.7. Let M be the torus minus one open disk and let N be S^2 minus three open disks.

(a) Show that there is no degree one map $f\colon M \to N$.
(b) Show that there is no degree one map $g\colon N \to M$.



Exercise 101.8. Given $m \in \mathbb{N}$ we denote by N_m the non-orientable surface of genus m, as defined on page 266. Let $k \in \mathbb{N}$.

(a) Let $g \geq k$. Show that there exists a map $f\colon N_g \to N_k$ such that the induced map $\mathrm{H}_2(N_g;\mathbb{F}_2) \to \mathrm{H}_2(N_k;\mathbb{F}_2)$ is an isomorphism.
(b) Let $g < k$. Show that there does not exist a map $f\colon N_g \to N_k$ such that the induced map $\mathrm{H}_2(N_g;\mathbb{F}_2) \to \mathrm{H}_2(N_k;\mathbb{F}_2)$ is an isomorphism.
Remark. Recall that in Exercise 89.14 we saw that $\mathrm{H}_1(N_m;\mathbb{F}_2) \cong \mathbb{F}_2^m$.

Exercise 101.9. Let X be a topological space. Given $\varphi \in \mathrm{H}_k(X;\mathbb{R})$ we define

$$\|\varphi\|_{\mathbb{R}} \;:=\; \inf\Big\{ \sum_{i=1}^n |r_i| \,\Big|\, \sigma = \underbrace{\sum_{i=1}^n \sigma_i \otimes r_i}_{\text{with } \sigma_i \neq \sigma_j \text{ for } i \neq j} \text{ is a cycle with } [\sigma] = \varphi \in \mathrm{H}_k(X;\mathbb{R}) \Big\}.$$

In the following let M be a closed oriented connected non-empty n-dimensional smooth manifold M. Let $[M] \in \mathrm{H}_n(M)$ be the fundamental class and let $i\colon \mathrm{H}_1(M) \to \mathrm{H}_1(M;\mathbb{R})$ be the homomorphism from Lemma 89.7 (1). We define the *simplicial volume of* M as

$$\|M\|_{\mathbb{R}} \;:=\; \|i_*([M])\| \;\in \mathbb{R}_{\geq 0}.$$

(a) Show that $\|S^1\|_{\mathbb{R}} = 0$.

(b) Let $n \in \mathbb{N}_{\geq 2}$. What is $\|S^n\|_{\mathbb{R}}$?

(c) Let $f\colon M \to N$ be a map between closed oriented connected non-empty n-dimensional smooth manifolds. Show that

$$\|N\|_{\mathbb{R}} \;\leq\; \tfrac{1}{\deg(f)} \cdot \|M\|_{\mathbb{R}}$$

where $\deg(f)$ denotes the degree of f as defined on page 2222.

(d) Let M and N be two closed oriented connected non-empty n-dimensional smooth manifolds. We suppose that M and N are in fact homotopy equivalent. Show that $\|M\|_{\mathbb{R}} = \|N\|_{\mathbb{R}}$.

(e) Let N be a closed oriented non-empty n-dimensional smooth manifold. Furthermore let $p\colon M \to N$ be a d-fold covering. Show that $\|M\|_{\mathbb{R}} = d \cdot \|N\|_{\mathbb{R}}$.

(f) Let $g \in \mathbb{N}_0$. As usual we denote by Σ_g the surface of genus $g \geq 1$. Show that $\|\Sigma_g\|_{\mathbb{R}} \leq 2g - 2$.

Hint. Make use of (f).

Remark. In [**Frig2017**, Corollary 7.5] or alternatively [**Thu1978**, Theorem 6.2] it is shown that we also have the reverse inequality, i.e. we have $\|\Sigma_g\|_{\mathbb{R}} = 2g - 2$.

Remark. The invariant $\|\varphi\|_{\mathbb{R}}$ is the "real analogue" of the invariant $\|\varphi\|_{\mathbb{Z}}$ introduced in Exercise 76.14.

102. Homology classes and smooth (sub-)manifolds

In Chapter 100 we saw that every compact oriented n-dimensional smooth manifold N comes naturally with a fundamental class $[N] \in \mathrm{H}_n(N, \partial N)$. In Chapters 100 and 101 we already saw several interesting applications of the fundamental class and we now feel like we have a good understanding of fundamental classes.

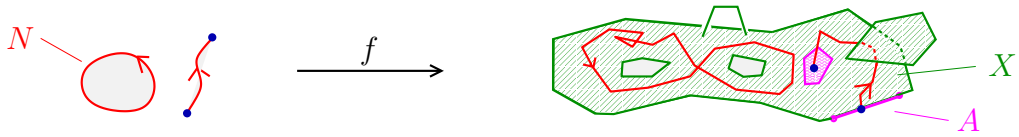
In this chapter we want to study to which degree homology classes can be understood in terms of fundamental classes. More precisely, we want to study the following two questions:

(1) Given a topological space X and a homology class $\varphi \in \mathrm{H}_n(X)$, can we always find a closed oriented n-dimensional smooth manifold N and a map $f\colon N \to X$ such that $f_*([N]) = \varphi$?
(2) If X is in fact a smooth manifold, can we find an $f\colon N \to X$ as in (1) that is actually a smooth embedding?

102.1. Realizing homology classes by smooth manifolds.

Let us start out with the following friendly definition.

Definition. Let X be a topological space, let $A \subset X$ be a subset and let $\varphi \in \mathrm{H}_n(X, A)$ be a homology class. We say φ can be realized by a smooth manifold if there exists a compact oriented n-dimensional smooth manifold N and a map $f\colon (N, \partial N) \to (X, A)$ such that $f_*([N]) = \varphi$.



Examples. Let (X, x_0) be a path-connected pointed topological space and let $n \in \mathbb{N}$. Recall that the Hurewicz homomorphism is defined as the map

$$\begin{array}{rcl} \Phi_{(X,x_0)}\colon \pi_n(X, x_0) & \to & \mathrm{H}_n(X) \\ [f\colon S^n \to X] & \mapsto & f_*([S^n]). \end{array}$$

Thus we see that every homology class that lies in the image of the Hurewicz homomorphism is realized by a smooth manifold. But as we remarked on page 1844, in general the Hurewicz homomorphism is certainly not an epimorphism.

In this and the subsequent section we want to study the following question.

Question 102.1. Let X be a topological space and let $A \subset X$ be a subset. For which $n \in \mathbb{N}_0$ can homology classes in $\mathrm{H}_n(X, A)$ be realized by smooth manifolds?

The case $n = 0$ is evidently trivial. So let us continue with the case $n = 1$.

Proposition 102.2. Let X be a topological space. Every class in $\mathrm{H}_1(X)$ is realized by a smooth manifold.

Proof. The proposition is basically an immediate consequence of Proposition 84.3. Nonetheless we prefer to do prove the proposition here "by hand", since this proof will also give us some ideas for how to approach the case of higher-dimensional homology classes.

Thus suppose we are given a homology class $\varphi \in \mathrm{H}_1(X)$. We pick a singular chain $c = a_1 \cdot \sigma_1 + \cdots + a_k \cdot \sigma_k \in \mathrm{C}_k(X)$ that represents φ. Note that we do not necessarily

demand that the σ_j are different. In fact note that by "taking multiple copies of the singular simplices σ_j" we can and will arrange that $a_j \in \{-1, 1\}$, $j = 1, \dots, k$. Next we will see that we can arrange that all $a_j = 1$. To do so, given a singular 1-simplex $\sigma\colon \Delta^1 \to X$ we denote by $\overline{\sigma}\colon \Delta^1 \to X$ the singular 1-simplex that is given by $\overline{\sigma}(t, 1-t) := \sigma(1-t, t)$. In Lemma 72.3 we saw that $-\sigma$ and $\overline{\sigma}$ are homologous. Thus, after replacing each summand $-\sigma_j$ by $\overline{\sigma}_j$ we can assume that all $a_j = 1$.

Next note that we have

$$0 \underset{\uparrow}{=} \partial_1(c) = \partial_1\Big(\sum_{j=1}^{k} \sigma_j\Big) \underset{\uparrow}{=} \sum_{j=1}^{k} \sigma_j(0,1) - \sigma_j(1,0) = \sum_{j=1}^{k} \sigma_j(0,1) + \sum_{j=1}^{k} -\sigma_j(1,0).$$

since c is a cycle — by definition of ∂_1, see also page 1585

On the right hand side each "positive summand" needs to cancel with a "negative summand". As in the proof of Proposition 84.3 this means, see also Lemma 84.4, that there exists a permutation $\theta \in S_k$ such that for each $j \in \{1, \dots, k\}$ the "endpoint" $\sigma_j(0,1)$ of σ_j is just the "starting point" $\sigma_{\theta(j)}(1,0)$ of $\sigma_{\theta(j)}$. We introduce the following notation:

(1) We consider the topological space

$$N := (\{1, \dots, k\} \times \Delta^1)/\sim \quad \text{where for } j \in \{1, \dots, k\} \text{ we have } \sigma_j(0,1) \sim \sigma_{\theta(j)}(1,0).$$

(2) For $j = 1, \dots, k$ we denote by $\Psi_j\colon \Delta^1 \xrightarrow{x \mapsto (j,x)} \{1, \dots, k\} \times \Delta^1$ the j-th inclusion map.
(3) We denote by $p\colon \{1, \dots, k\} \times \Delta^1 \to N$ the projection .
(4) It follows immediately from the definitions and the Topological-Quotient Proposition 5.15 (1b) that the map

$$\begin{array}{rcl} f\colon N = (\{1, \dots, k\} \times \Delta^1)/\sim & \to & X \\ [(j,z)] & \mapsto & \sigma_j(z) \end{array}$$

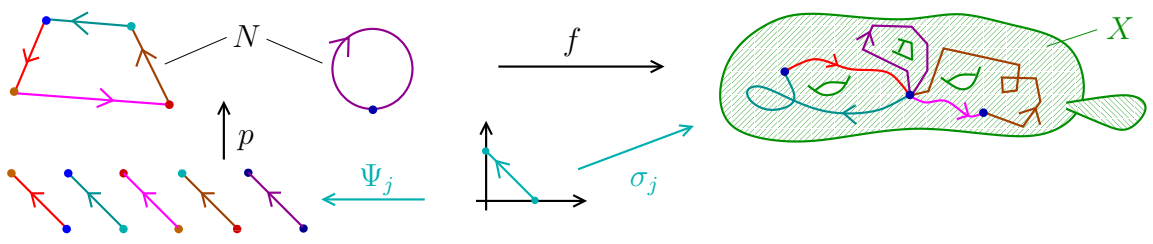
is well-defined and continuous.

It is now straightforward to verify that N can be equipped with the structure of a closed 1-dimensional smooth manifold and that we can equip N with an orientation such that for each $j \in \{1, \dots, k\}$ the map $p \circ \Psi_j\colon \mathring{\Delta}^1 \to N$ is an orientation-preserving embedding. Finally we see that

$$f_*([N]) \underset{\uparrow}{=} f_*([p \circ \Psi_1 + \dots + p \circ \Psi_k]) \underset{\uparrow}{=} [\sigma_1 + \dots + \sigma_k] = c.$$

follows from the Fundamental Class-Criterion Proposition 100.4 applied to any of the maps $p \circ \Psi_j$ — by definition of f and $p \circ \Psi_j$

■



Evidently the idea now is to work our way up to higher dimensions. In fact we have the following proposition.

Proposition 102.3. (Realization-by-Surface Proposition) Let X be a topological space. Every class in $\mathrm{H}_2(X)$ is realized by a smooth manifold.

Proof. The basic idea of the proof of the Realization-by-Surface Proposition 102.3 is similar to the proof of Proposition 102.2. Nonetheless, ironing out the details is much more messy, hence we postpone the proof of the case $n = 2$ to Section 102.2. Alternatively see also [**Cal2009**, p. 3] for a quick sketch of the proof. ■

Remark. In Lemma 102.9 we will see that any naive attempt of generalizing the ideas behind the proofs of Propositions 102.2 and 102.3 to higher dimensions runs into serious problems.

We move on to the following proposition that summarizes a few generalities on realizing homology classes by smooth manifolds.

Proposition 102.4. Let $k \in \mathbb{N}$.

(1) Let X be a topological space and let $A \subset X$ be a subset. We suppose that X is path-connected. If $A = \varnothing$ or if $k \geq 2$, then every homology class in $\mathrm{H}_k(X, A)$ that can be realized by a smooth manifold can also be realized by a *connected* smooth manifold.

(2) Suppose that for every topological space X every homology class in $\mathrm{H}_k(X)$ can be realized by a smooth manifold. Then also for every pair (X, A) of topological spaces every relative homology class in $\mathrm{H}_k(X, A)$ can be realized by a smooth manifold.

Proof. In order not to disrupt the flow of the conversation too much we postpone the somewhat technical proof of Proposition 102.4 to Section 102.3. ■

Note that Propositions 102.2 and 102.3 together with Proposition 102.4 give us a complete, and affirmative, answer to Question 102.1 for the dimensions $k = 1$ and $k = 2$.

We move on to the following much more fancy theorem which was first proved by René Thom [**Tho1954b, Tho2007**] in 1954.[1240] The proof of the theorem uses methods that go well beyond our present abilities. We will not make use of the theorem.

Theorem 102.5. (Thom's Theorem I)

(1) If X is a topological space, then every homology class of dimension ≤ 6 can be realized by a smooth manifold.

(2) Given any $n \geq 7$ there exists a topological space X and a homology class in $\mathrm{H}_n(X)$ that is not realized by a smooth manifold.

(3) Given any $n \in \mathbb{N}_0$ there exists a $k \in \mathbb{N}$ such that for any topological space X and any homology class $\sigma \in \mathrm{H}_n(X)$ the class $k \cdot \sigma$ can be realized by a smooth manifold.

Remark. The statement of Thom's Theorem 102.5 solved what was known as the *Steenrod problem*, see [**Eil1949**, p. 257].

Proof. As mentioned before, the theorem is proved in [**Tho1954b**, Theorem III.3, III.4 and III.9]. A more modern account of the proof is given in Theorems IV.7.35, IV.7.36, IV.7.37 and IV.7.38 together with Remark IV.7.40 of [**Rudy1998**]. In fact the topological space

[1240] René Thom (1923-2002) was a French mathematician who received the Fields Medal in 1958. A short outline of the key ideas and contributions of René Thom can be found in [**Sul2004**]

X in (2) is given by the $(n-7)$-fold suspension of the Eilenberg-Maclane space $K(\mathbb{Z}_3, 2)$, which we will introduce later in Chapter 163. ■

For completeness we also mention $\mathbb{F}_2$-coefficients.

Theorem 102.6. (Thom's Theorem II) Let X be a path-connected topological space and let $A \subset X$ be a subset. Given any $\varphi \in \mathrm{H}_n(X, A; \mathbb{F}_2)$ there exists a compact connected n-dimensional smooth manifold N and a map $f\colon N \to X$ with $f(\partial N) \subset A$ such that $f_*([N]_{\mathbb{F}_2}) = \varphi$.

Proof. First let us consider the case $A = \varnothing$ and for the time being let us allow possibly disconnected smooth manifolds. In this setting, for $n = 1$ and $n = 2$ the statement follows from straightforward modifications of the proofs of Propositions 102.2 and 102.3. In fact, much more interestingly, the statement now holds in all dimensions by the above work of René Thom [**Tho1954b**, Theorem III.2], see also [**Rudy1998**, Theorem IV.7.33]. The statements about relative homology and connectedness follow from the obvious variations on Proposition 102.4. ■

We conclude this section with the following remark.

Remark.

(1) In Chapter 105 we will introduce the notion of an orientation of a topological manifold. Furthermore in Chapter 106 we will introduce the notion of the fundamental class $[N] \in \mathrm{H}_n(N, \partial N)$ of a compact oriented n-dimensional topological manifold N. It now makes sense to ask whether a homology class can be realized by a topological manifold. By [**Rudy1998**, Theorem IV.7.35 and Remark IV.3.740] there exist homology classes in any dimension ≥ 7 that cannot by realized by topological manifolds. But by [**Rudy1998**, Theorem IV.7.38] there do exist homology classes that can be realized by topological manifolds but which cannot be realized by smooth manifolds.

(2) Lowell Jones [**JonL1973**, Proposition 7.3] (see also Norman Levitt [**Levit1972**, Theorem 7.1]) showed that every homology class can be represented by a "Poincaré Duality space".

102.2. Proof of the Realization-by-Surface Proposition 102.3.

In this section we prove the following proposition from the last section.

Proposition 102.3. (Realization-by-Surface Proposition) Let X be a topological space. Every class in $\mathrm{H}_2(X)$ is realized by a smooth manifold.

The proof of the Realization-by-Surface Proposition 102.3 is modelled on the proof of Proposition 102.2. But, as we will see, the proof is technically more difficult.

We start out our proof with the more general setting of some n-dimensional class. Later on we will specialize to the case that $n = 2$. It is even worthwhile keeping the case $n = 1$ in mind, since even for $n = 1$ the argument below differs slightly from the argument provided in the proof of Proposition 102.2.

Thus let X be a topological space, let $n \in \mathbb{N}$ and let $c = a_1 \cdot \sigma_1 + \cdots + a_k \cdot \sigma_k$ be an n-dimensional singular chain in X. We do not necessarily demand that the σ_j are different. In fact note that by "taking multiple copies of the singular simplices σ_j" we can and will

arrange that $a_j \in \{-1,1\}$, $j = 1,\dots,k$.[1241] the Realization-by-Surface Proposition 102.3 is more difficult than the proof of Proposition 102.2. We set $T := \{1,\dots,k\} \times \{0,\dots,n\}$.

Now assume that c is a cycle. We see that we have the following equality in the free abelian group $\mathrm{C}_{n-1}(X)$:

$$0 \underset{\text{since } c \text{ is a cycle}}{=} \partial_1(c) = \partial_1\Big(\sum_{j=1}^{k} a_j \cdot \sigma_j\Big) = \sum_{j=1}^{k}\sum_{r=0}^{n} (-1)^r \cdot a_j \cdot \sigma_j \circ \underset{\text{here } i_r^n\colon \Delta^{n-1} \to \Delta^n \text{ is the } r\text{-th face map}}{i_r^n} = \sum_{(j,r)\in T} (-1)^r \cdot a_j \cdot \sigma_j \circ i_r^n.$$

It follows from Lemma 84.4 that there exists a decomposition $T = T_1 \sqcup T_2$ into two disjoint subsets together with a bijection

$$\begin{array}{rcl} \Theta = (\theta_1,\theta_2)\colon T_1 & \to & T_2 \\ (j,r) & \mapsto & \Theta(j,r) = (\theta_1(j,r),\theta_2(j,r)) \end{array}$$

such that the following condition is satisfied: For every $(j,r) \in T_1$ we have

$$\begin{array}{rcl} \sigma_j \circ i_r^n & = & \sigma_{\theta_1(j,r)} \circ i^n_{\theta_2(j,r)} \\ \text{and} \quad (-1)^r \cdot a_j & = & -(-1)^{\theta_2(j,r)} \cdot a_{\theta_1(j,r)}. \end{array}$$

Next we introduce some notation:

(1) We consider the topological space

$$Y(\Theta) := (\{1,\dots,k\} \times \Delta^n)/\sim \quad \begin{array}{l} \text{where for each } (j,r) \in T_1 \text{ and each } P \in \Delta^{n-1} \\ \text{we have } (j, i_r^n(P)) \sim (\theta_1(j,r), i^n_{\theta_2(j,r)}(P)). \end{array}$$

Informally speaking $Y(\Theta)$ is given by taking one n-simplex for each singular simplex σ_j and gluing these simplices along $(n-1)$-dimensional faces.[1242]

(2) For $j = 1,\dots,k$ we denote by $\Psi_j\colon \Delta^n \xrightarrow{x \mapsto (j,x)} \{1,\dots,k\} \times \Delta^n$ the j-th inclusion map.

(3) We denote by $p\colon \{1,\dots,k\} \times \Delta^n \to Y(\Theta)$ the projection .

(4) It follows immediately from the definitions and the Topological-Quotient Proposition 5.15 (1b) that the map

$$\begin{array}{rcl} f\colon Y(\Theta) = (\{1,\dots,k\} \times \Delta^n)/\sim & \to & X \\ [(j,P)] & \mapsto & \sigma_j(P) \end{array}$$

is well-defined and continuous.

(5) We set

$$\mu := \sum_{j=1}^{k} a_j \cdot p \circ \Psi_j \in \mathrm{C}_n(Y(\Theta)).$$

We can now formulate the following lemma.

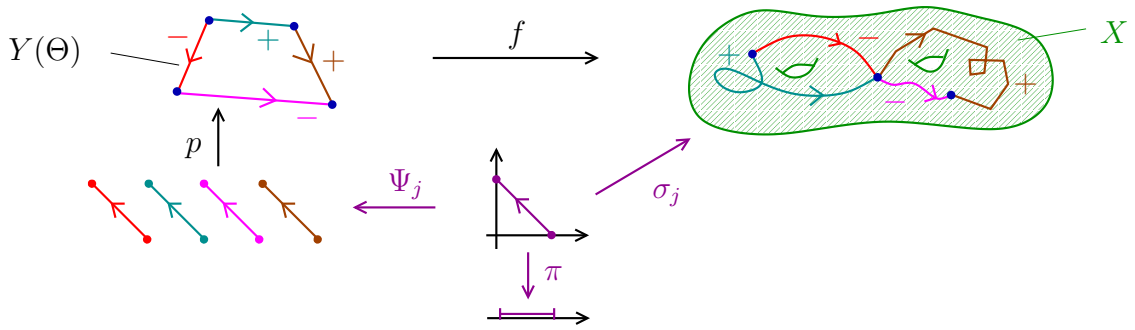
Lemma 102.7.

(1) $\mu \in \mathrm{C}_n(Y(\Theta))$ is a cycle.

(2) $f_*([\mu]) = [c] \in \mathrm{H}_n(X)$.

[1241] Note that if n is even, then it follows from Exercise 72.12 that we cannot arrange that all a_j are of the same sign. This is one, but not the main, reason why the proof of

[1242] It might seem that $Y(\Theta)$ has an "obvious simplicial structure", but that is not necessarily the case, for example after gluing there might be two n-dimensional simplices that intersect in more than just one $(n-1)$-dimensional face.



Proof. First we show that μ is a cycle. This follows from the following straightforward calculation:

$$\begin{aligned}\partial\mu &= \partial\Big(\sum_{j=1}^{k} a_j \cdot p\circ\Psi_j\Big) = \sum_{j=1}^{k}\sum_{r=0}^{n}(-1)^r\cdot a_j\cdot p\circ\Psi_j\circ i_r^n = \sum_{(j,r)\in T}(-1)^r\cdot a_j\cdot p\circ\Psi_j\circ i_r^n\\
&= \sum_{(j,r)\in T_1}(-1)^r\cdot a_j\cdot p\circ\Psi_j\circ i_r^n + \sum_{(j,r)\in T_1}(-1)^{\theta_2(j,r)}\cdot a_{\theta_1(j,r)}\cdot p\circ\Psi_{\theta_1(j,r)}\circ i^n_{\theta_2(j,r)}\end{aligned}$$

since $T = T_1 \sqcup T_2$ and since $\Theta = (\theta_1,\theta_2)\colon T_1 \to T_2$ is a bijection

$$= \sum_{(j,r)\in T_1}\underbrace{(-1)^r\cdot a_j\cdot p\circ\Psi_j\circ i_r^n + (-1)^{\theta_2(j,r)}\cdot a_{\theta_1(j,r)}\cdot p\circ\Psi_{\theta_1(j,r)}\circ i^n_{\theta_2(j,r)}}_{=0 \text{ since } (-1)^r\cdot a_j = -(-1)^{\theta_2(j,r)}\cdot a_{\theta_1(j,r)} \text{ and } p\circ\Psi_j\circ i_r^n = p\circ\Psi_{\theta_1(j,r)}\circ i^n_{\theta_2(j,r)}\colon \Delta^{n-1}\to Y(\Theta)} = 0.$$

It follows immediately from the definitions that $f_*(\mu) = c \in \mathrm{C}_n(X)$. In particular we have $f_*([\mu]) = [c] \in \mathrm{H}_n(X)$. ∎

We move on to the next lemma, which together with Lemma 102.7 provides the long-desired proof of the Realization-by-Surface Proposition 102.3. Note that, as a bonus, we also obtain a new proof of Proposition 102.2.

Lemma 102.8. We continue with the above notation. If $n = 1$ or if $n = 2$, then the topological space $Y(\Theta)$ admits the structure of a closed oriented 2-dimensional smooth manifold such that $[\mu] \in \mathrm{H}_2(Y(\Theta))$ is the corresponding fundamental class.

Remark. In Lemma 102.9 below we will see that for $n = 3$ the topological space $Y(\Theta)$ is in general not a topological manifold.

Sketch of a proof of Lemma 102.8. The proof of Lemma 102.8 for the case $n = 1$ is very elementary and thus left to the reader.

In the following we sketch a proof of Lemma 102.8 for the case $n = 2$. As we will see, the proof is quite similar to the proof, given in Propositions 18.7 and 19.18, that the surface Σ_g of genus g is indeed a closed orientable 2-dimensional smooth manifold.

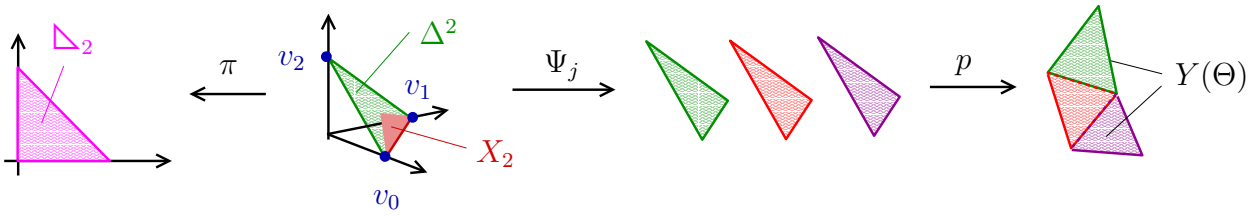
First note that it follows easily from the Quotient-Compact Connected Lemma 5.16 (1) that $Y(\Theta)$ is compact. We leave it to the reader to verify, say using the Quotient Hausdorff Lemma 5.21, that $Y(\Theta)$ is Hausdorff.

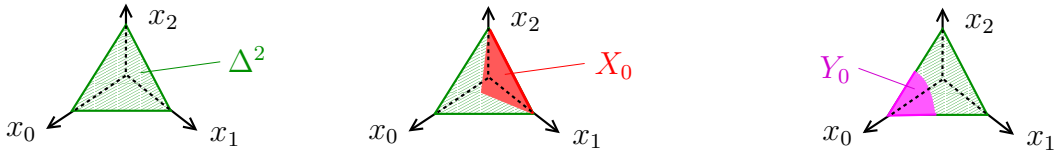
Next we construct a finite atlas for $Y(\Theta)$. To do so we introduce the following notation:

- Let $\pi\colon \Delta^2 \to \triangle_2$ be the projection . Note that in the Standard Simplex Homeomorphism Lemma 72.1 we saw that π is a homeomorphism.
- We denote by $v_0 = (1,0,0)$, $v_1 = (0,1,0)$ and $v_2 = (0,0,1)$ the three vertices of Δ^2.

- Let $j \in \{0,1,2\}$. In the following will define two open subsets X_j and Y_j of Δ^2. In the subsequent definition we denote by w, w' the two vertices that are not equal to v_j. Now we introduce the following notation:
  - ▷ We denote by X_j the union of the open interval from w to w' and the open triangle in Δ^2 spanned by w, w' and the barycenter $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ of Δ^2.
  - ▷ We denote by Y_j the intersection of $\Delta^2 \subset \mathbb{R}^3$ with the open ball in $\mathbb{R}^3$ of radius $\frac{1}{2}$ around the j-th vertex v_j.

  We refer to the figures below for an illustration of the above definitions.





Next we sketch the construction of three types of charts for the topological space $Y(\Theta)$. We leave it to the reader to fill in the details in the discussions below.

(1) For each $j \in \{1, \dots, k\}$ we let $\alpha_j := \pi \circ \Psi_j^{-1} \circ p^{-1} \colon U_j := p(\mathring{\Delta}_j) \to V_j := \mathring{\Delta}_2 \subset \mathbb{R}^2$ be the "obvious" chart.

(2) Let $(j,r) \in T_1$. We write $\Theta(j,r) =: (j',r')$. Similar to the proof of the Surface-is-Topological Manifold Proposition 18.7 one can easily write down an explicit chart

$$\beta_{j,r} \colon U_{j,r} := \underbrace{p(\Psi_i(X_r)) \cup p(\Psi_{j'}(X_{r'}))}_{\subset Y(\Theta)} \quad \to \quad V_{j,r} \subset \mathbb{R}^2$$

such that the maps $\beta_{j,r} \circ p \circ \Psi_j^{-1} \circ \pi^{-1} \colon \Delta_2 \to \mathbb{R}^2$ and $\beta_{j,r} \circ p \circ \Psi_{j'}^{-1} \circ \pi^{-1} \colon \Delta_2 \to \mathbb{R}^2$ are both affine linear and such that they are both orientation-preserving if and only if $(-1)^r \cdot a_j = 1$ and such that they are both orientation-reversing if and only if $(-1)^r \cdot a_j = -1$.

(3) Let $j \in \{1, \dots, k\}$ and let $r \in \{0,1,2\}$. An open neighborhood of $p(\Psi_j(v_r))$ is given by

$$\widetilde{U}_{j,r} \;:=\; \bigcup_{\substack{(j',r') \in T \text{ such that} \\ p(\Psi_{j'}(v_{r'})) = p(\Psi_j(v_r))}} p(\Psi_{j'}(Y_{r'})).$$

Again similar to the proof of Propositions 18.7 one can now write down an explicit chart $\gamma_{j,r} \colon \widetilde{U}_{j,r} \to B^2_{\frac{1}{2}}$ such that on each $p(\Psi_r(Y,s))$ the map is a concatenation of π, $(p \circ \Psi_r)^{-1}$, translation, rotation and a map of the form $\exp(\mathrm{i}t) \mapsto \exp(\mathrm{i}wt)$ for some $w \in \mathbb{R}_{\geq 0}$. In fact we can take $w = \frac{2\pi}{n}$ where $n \in \mathbb{N}$ is the number of terms (j',r') that appear in the above definition of $\widetilde{U}_{j,r}$.

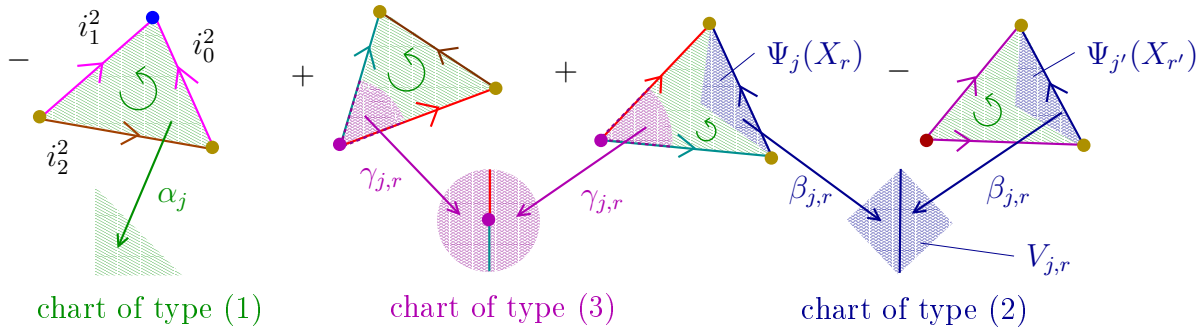
We leave it to the reader to verify that all of these maps are homeomorphisms and that they form an atlas for $Y(\Theta)$. Note that by Lemma 18.2 (2) we now also know that $Y(\Theta)$

is second-countable. We have thus verified that $Y(\Theta)$ is a closed 2-dimensional topological manifold.

Next note that, as in the proof of Proposition 19.18, one can now verify easily that the transition maps between the given charts are indeed smooth and that they are orientation-preserving. Not surprisingly, once again we leave the details to the reader. Thus in summary we obtain from the above discussion, together with Lemmas 18.2 and 25.15, that $Y(\Theta)$ is a smooth manifold that is endowed with an orientation such that all charts are orientation-preserving.

Recall that in Lemma 102.7 we showed that $\mu = a_1 \cdot p \circ \Psi_1 + \cdots + a_k \cdot p \circ \Psi_k \in \mathrm{C}_2(Y(\Theta))$ is a cycle. It follows almost immediately from the Fundamental Class-Criterion Proposition 100.4 that μ represents the fundamental class of the closed oriented smooth manifold $Y(\Theta)$. ■

four simplices $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ with gluings indicated by colors



We conclude this section with the following lemma which shows that the above approach will not work in higher dimensions.

Lemma 102.9. Given any $n \in \mathbb{N}_{\geq 3}$ there exists a topological space X together with a cycle $c = a_1 \cdot \sigma_1 + \cdots + a_k \cdot \sigma_k \in \mathrm{C}_n(X)$ such that for any choice of Θ the topological space $Y(\Theta)$ is not a closed topological manifold, let alone a closed smooth manifold.

Sketch of proof. Let $n \in \mathbb{N}_{\geq 3}$ and let M be the $(n-1)$-dimensional torus. Furthermore let $(K, \Xi\colon |K| \to M)$ be a smooth simplicial structure for M, which we know exists by the Smooth Manifold-Simplicial Structure Theorem 96.2, or by some more down-to-earth argument.

By Propositions 98.1 and 98.7 we know that K is a closed orientable $(n-1)$-dimensional pseudomanifold. Next we consider the suspension $\Sigma(K)$, as defined on page 2003. Note that by the Simplicial Cone-Suspension Corollary 93.21 we know that $X := |\Sigma(K)|$ is homeomorphic to $\Sigma(M)$.

Next recall that by the fairly straightforward Exercise 98.1 we know that $\Sigma(K)$ is a closed n-dimensional pseudomanifold that admits an orientation $\{(\epsilon_1, Y_1), \ldots, (\epsilon_k, Y_k)\}$. Now let $\varphi = \epsilon_1 \cdot Y_1 + \cdots + \epsilon_k \cdot Y_k \in \mathrm{C}_n^{\mathrm{simp}}(\Sigma(K))$ be the cycle from Theorem 98.4. Next note that after picking some total order on the vertex set of $\Sigma(K)$ we obtain from Lemmas 95.8 and the Simplicial-Singular Chain Complex Theorem 95.25 a corresponding singular cycle $c = a_1 \cdot \sigma_1 + \cdots + a_k \cdot \sigma_k \in \mathrm{C}_n(X) = \mathrm{C}_n(|\Sigma(K)|)$. We set $T := \{1, \ldots, k\} \times \{0, \ldots, n\}$. We leave it to the dedicated reader to verify that there is essentially only one choice of a decomposition $T = T_1 \sqcup T_2$ and a bijection $\Theta = (\theta_1, \theta_2)\colon T_1 \to T_2$ such that the condition

from page 2238 holds, i.e. such that for every $(j,r) \in T_1$ we have

$$\begin{aligned} \sigma_j \circ i_r^n &= \sigma_{\theta_1(j,r)} \circ i_{\theta_2(j,r)}^n \\ \text{and} \quad (-1)^r \cdot a_j &= -(-1)^{\theta_2(j,r)} \cdot a_{\theta_1(j,r)}. \end{aligned}$$

Furthermore we leave it to the reader to verify that $Y(\Theta)$ is actually homeomorphic to $|\Sigma(K)| \cong \Sigma(M)$.

Finally we point out that it follows from Exercise 76.8, together with the fact that $\mathrm{H}_1(M) = \mathrm{H}_1((S^1)^n) \neq 0$ and together with our hypothesis $n \geq 3$ that $X = |\Sigma(K)| \cong \Sigma(M)$ is not a closed topological manifold. ■

102.3. **Proof of Proposition 102.4.** The sole goal of this section is to prove Proposition 102.4. For convenience we prove the two parts of the proposition separately.

Proposition 102.4.

(1) Let $k \in \mathbb{N}$, let X be a topological space and let $A \subset X$ be a subset. We suppose that X is path-connected. If $A = \varnothing$ or if $k \geq 2$, then every homology class in $\mathrm{H}_k(X, A)$ that can be realized by a smooth manifold can also be realized by a *connected* smooth manifold.

Proof of Proposition 102.4 (1). To simplify the discussion we only consider the case that $A = \varnothing$. The case $A \neq \varnothing$ is proved in a completely analogous fashion, the only difference is that the notation becomes even more messy. We will point out though explicitly where we need that $k \geq 2$ for $A \neq \varnothing$.

Let X be a path-connected topological space and let $\theta \in \mathrm{H}_k(X)$ be a homology class. Suppose there exists a closed oriented k-dimensional smooth manifold W and a map $f\colon W \to X$ with $f_*([W]) = \theta$. If W is connected, then there is nothing to show. In the following we will deal with the case that W has precisely two components M and N. The general case is dealt with the same way.

We make the following preparations.

(1) As usual we denote by $\overline{B}_3^k \subset \mathbb{R}^k$ the closed ball of radius 3 around the origin. We pick an orientation-preserving smooth embedding $\varphi\colon \overline{B}_3^k \to M$ and we pick an orientation-reversing smooth embedding $\psi\colon \overline{B}_3^k \to N$. We write $P := \varphi(0)$ and $Q := \psi(0)$.

(2) We use the restrictions of φ and ψ to $\overline{B}^k$ to form, following the definition on page **??**, the connected sum $M\#N$, i.e. we consider

$$M\#N \;=\; ((M \setminus \varphi(B^k)) \sqcup (N \setminus \psi(B^k)))/\!\sim \qquad \text{where } \varphi(z) \sim \psi(z) \text{ for } z \in S^{k-1}.$$

(3) We recall a few statements from the Connected Sum-Proposition **??**:
   (a) The connected sum $M\#N$ is a closed k-dimensional smooth manifold which can be oriented in such a way that the two natural inclusions $M \setminus \varphi(B^k) \to M\#N$ and $N \setminus \psi(B^k) \to M\#N$ are both orientation-preserving smooth embeddings.
   (b) If $k \geq 2$ or if at least one of M or N is closed, then $M\#N$ is connected.
   Note though that for $A \neq \varnothing$ we have to deal with the connected sum of compact manifolds M and N that are not necessarily closed. So here the argument breaks down for $A \neq \varnothing$ and $k = 1$, since the connected sum of two copies of $\overline{B}^1$ is actually not connected.

(4) We consider the topological space $Z := (M \sqcup [-1,1] \sqcup N)/\sim$ where we make the identifications $P \sim -1$ and $1 \sim Q$.
(5) We denote by $i\colon M \to Z$ and $j\colon N \to Z$ the obvious maps. It is easy to see that they are in fact embeddings. Next we denote by $p\colon Z \to M$ the map that is given by the identity on M and which sends all other points to P. Similarly we define $q\colon Z \to N$.
(6) We consider the following maps:

$$\begin{array}{ccc} & & \mathrm{H}_k(M) \oplus \mathrm{H}_k(N) \\ & \overset{\text{isomorphism since inclusions are homotopy equivalences}}{\swarrow} & \qquad\qquad \searrow^{i_* \oplus j_*} \\ \mathrm{H}_k((M \sqcup [-1,1))/\sim) \oplus \mathrm{H}_k((N \sqcup (-1,1])/\sim) & \xrightarrow[\text{Mayer–Vietoris Theorem 78.5}]{\text{isomorphism by the}} & \mathrm{H}_k(Z). \end{array}$$

It follows that the right diagonal map $i_* \oplus j_*\colon \mathrm{H}_k(M) \oplus \mathrm{H}_k(N) \to \mathrm{H}_k(Z)$ is an isomorphism. It is now straightforward to see that the inverse of $i_* \oplus j_*$ is given by $p_* \oplus q_*\colon \mathrm{H}_k(Z) \to \mathrm{H}_k(M) \oplus \mathrm{H}_k(N)$.
(7) We consider the map

$$\begin{array}{rcl} \Theta\colon M\#N & \to & Z \\ x & \mapsto & \begin{cases} [x], & \text{if } x \in M \setminus \varphi(B_3^k), \\ [\varphi(3(r-2)\cdot y)], & \text{if } x = \varphi(r \cdot y) \text{ with } r \in [2,3] \text{ and } y \in S^{k-1}, \\ [-r+1], & \text{if } x = \varphi(r \cdot y) \text{ with } r \in [1,2] \text{ and } y \in S^{k-1}, \\ [r-1], & \text{if } x = \psi(r \cdot y) \text{ with } r \in [1,2] \text{ and } y \in S^{k-1}, \\ [\psi(3(r-2)\cdot y)], & \text{if } x = \psi(r \cdot y) \text{ with } r \in [2,3] \text{ and } y \in S^{k-1}, \\ [x], & \text{if } x \in N \setminus \psi(B_3^k). \end{cases} \end{array}$$

It follows easily from the Pasting Proposition 2.6 (2) that $\Theta\colon M\#N \to Z$ is actually continuous.
(8) It follows almost immediately from the Local-Global Degree Proposition 101.2 together with (3a) that the maps $p \circ \Theta\colon M\#N \to M$ and $q \circ \Theta\colon M\#N \to N$ both have degree one. In other words, we have the equality

$$(p_* \oplus q_*)(\Theta_*([M\#N])) = [M] + [N] \in \mathrm{H}_k(M) \oplus \mathrm{H}_k(N) = \mathrm{H}_k(M \sqcup N) = \mathrm{H}_k(W).$$

(9) By hypothesis the topological space X is path-connected. This implies that there exists a path $\gamma\colon [-1,1] \to X$ with $\gamma(-1) = P$ and $\gamma(1) = Q$. Using γ we can extend the map $f\colon W = M \sqcup N \to X$ to a map $g\colon Z \to X$ in the obvious way.

By (3) the connected sum $M\#N$ is a closed oriented *connected* smooth manifold. Thus it remains to prove the following claim.

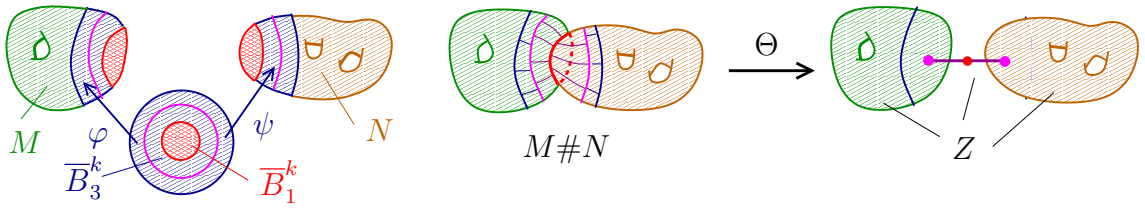
Claim. We have $(g \circ \Theta)_*([M\#N]) = \theta \in \mathrm{H}_k(X)$.

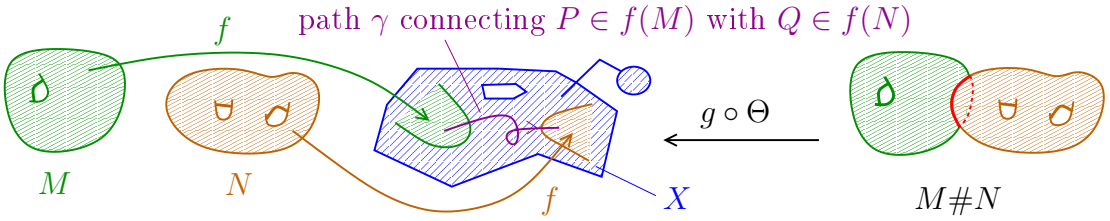
Proof. It follows from the above that we have the following diagram:

$$\begin{array}{ccccc} & & \mathrm{H}_k(M) \oplus \mathrm{H}_k(N) & \xrightarrow{=} & \mathrm{H}_k(M \sqcup N) = \mathrm{H}_k(W) \\ & & {\scriptstyle p_* \oplus q_*}\uparrow \cong \quad \cong \downarrow {\scriptstyle i_* \circ j_*} & \overset{f_*}{\searrow} & \downarrow {\scriptstyle f_*} \\ \mathrm{H}_k(M\#N) & \xrightarrow{\Theta_*} & \mathrm{H}_k(Z) & \xrightarrow{g_*} & \mathrm{H}_k(X). \end{array}$$

It follows from the definition of g together with (6) that the diagram commutes. Now we see that we have the following equalities in $\mathrm{H}_k(X)$:

$$(g\circ\Theta)_*([M\#N]) \underset{\text{by (8) and (6)}}{=} g_*(i_*([M])+j_*([N])) \underset{\text{since the diagram commutes}}{=} f_*([M])+f_*([N]) = f_*(\underbrace{[M\sqcup N]}_{=[W]}) = \theta. \qquad \blacksquare$$





We move on to the second statement of Proposition 102.4.

Proposition 102.4.

(2) Let $k\in\mathbb{N}$. Suppose that for every topological space X every homology class in $\mathrm{H}_k(X)$ can be realized by a smooth manifold. Then also for every pair (X,A) of topological spaces every relative homology class in $\mathrm{H}_k(X,A)$ can be realized by a smooth manifold.

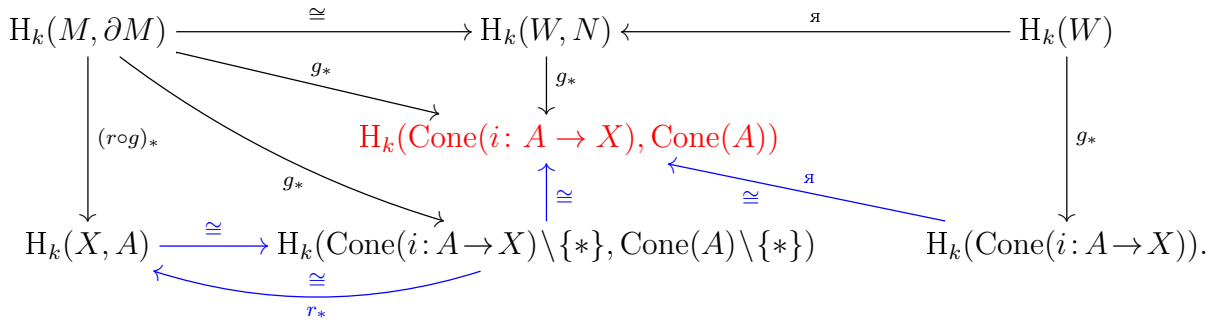
Proof. Let $k\in\mathbb{N}$. We make the following hypothesis:

(∗) For every topological space Y every homology class in $\mathrm{H}_k(Y)$ can be realized by a smooth manifold.

Now let (X,A) be a pair of topological spaces and let $\varphi\in\mathrm{H}_k(X,A)$ be a relative homology class. We denote by $i\colon A\to X$ the inclusion map. We consider the mapping cone

$$\mathrm{Cone}(i\colon A\to X) = (\underbrace{\mathrm{Cone}(A)}_{=(A\times[0,1])/A\times\{0\}}\sqcup X)/\!\sim \text{ where } [(a,1)]\sim i(a)=a \text{ for } a\in A.$$

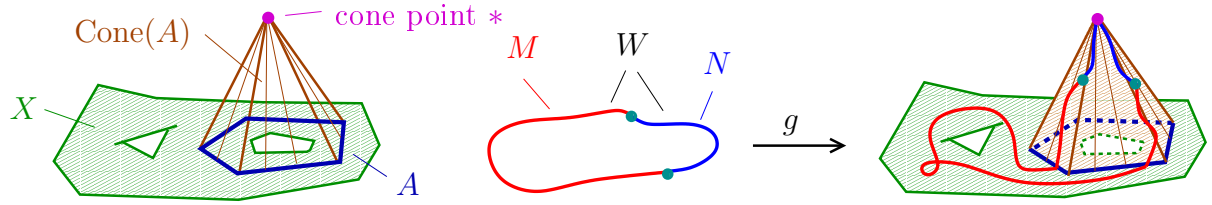
We denote by $*[A\times\{0\}]\in\mathrm{Cone}(i\colon A\to X)$ the cone point. We consider the following diagram:



We make the following explanations, clarifications and observations:

(1) By the Mapping Cone Basics Lemma 16.13 (2) and (3) we can and will view $\mathrm{Cone}(A)$ and X as subsets of $\mathrm{Cone}(i\colon A \to X)$.
(2) All maps that are not specified explicitly are inclusion induced maps.
(3) The maps я are the natural maps from absolute to relative homology.
(4) Note that by the Mapping Cone Basics Lemma 16.13 (6) there exists a natural deformation retraction from $\mathrm{Cone}(i\colon A \to X) \setminus \{*\}$ to X. We denote by $r\colon \mathrm{Cone}(i\colon A \to X) \setminus \{*\} \to X$ the corresponding retraction. Note that it restricts to a retraction from $\mathrm{Cone}(A) \setminus \{*\}$ to A.
(5) By the proof of the Mapping Cone-H_*-Lemma 78.15 we know that the four maps at the bottom, shown in blue, are isomorphisms.
(6) By (5) together with our hypothesis $(*)$ we know that there exists a closed oriented k-dimensional smooth manifold W and a map $g\colon W \to \mathrm{Cone}(i\colon A \to X)$ such that $я(g_*([W]))$ equals the image of our given class φ in the central red group $\mathrm{H}_k(\mathrm{Cone}(i\colon A \to X), \mathrm{Cone}(A))$.
(7) Since $\mathrm{Cone}(i\colon A \to X) \setminus \{*\})$ and $\mathrm{Cone}(A) \setminus A$ are open subsets that cover $\mathrm{Cone}(i)$ we obtain from Exercise 31.3 that there exist two compact codimension-zero submanifolds M and N of W such that $M \cup N = W$, $M \cap N = \partial M = \partial N$, such that $M \subset g^{-1}(\mathrm{Cone}(i\colon A \to X) \setminus \{*\})$ and such that $N \subset g^{-1}(\mathrm{Cone}(A) \setminus A)$.
(8) By (7) we have $g(N) \subset \mathrm{Cone}(A)$ and $g(\partial M) \subset \mathrm{Cone}(A)$. This implies that the two maps in relative homology denoted by g_*, namely the diagonal map to the left and the vertical map in the middle, actually do exist.
(9) By the Codimension Zero-Fundamental Class Lemma 100.12 (2) we know that the images of $[M]$ and $[W]$ in $\mathrm{H}_k(W, N)$ agree.

It is clear that the diagram commutes. It is now straightforward to verify that the compact oriented n-dimensional smooth manifold M together with the map $r \circ g\colon (M, \partial M) \to (X, A)$ has all the desired properties. ■



102.4. **Representing homology classes by submanifolds I.** In the previous sections we studied the following definition.

Definition. Let X be a topological space, let $A \subset X$ be a subset and let $\varphi \in \mathrm{H}_k(X, A)$ be a homology class. We say φ can be realized by a smooth manifold if there exists a compact oriented k-dimensional smooth manifold N and a map $f\colon (N, \partial N) \to (X, A)$ such that $f_*([N]) = \varphi$.

If the topological space X we are interested in is in fact a smooth manifold it is natural to ask whether one can actually find maps f that are smooth embeddings. This leads us to the following definition.

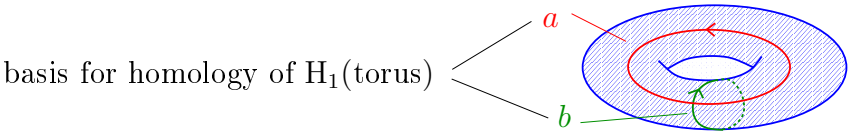
Definition. Let M be a smooth manifold and let A be a union of components of ∂M.

(1) Let N be a compact oriented proper k-dimensional submanifold of M with $\partial N \subset A$. By a slight abuse of notation, we denote by $[N]$ the image of the fundamental class $[N]$ under the inclusion induced map $\mathrm{H}_k(N, \partial N) \to \mathrm{H}_k(M, A)$.

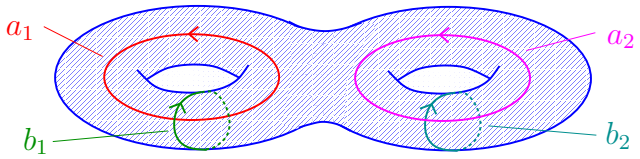
(2) We say that a homology class $\varphi \in \mathrm{H}_k(M, A)$ is represented by a submanifold if there exists a compact oriented proper k-dimensional submanifold N with $\partial N \subset A$ such that $[N] = \varphi$.

Examples.

(1) We first consider the 2-dimensional torus $T = S^1 \times S^1$. Under the usual isomorphism $\mathrm{H}_1(S^1 \times S^1) \cong \mathbb{Z}^2$ the basis element $(1, 0)$ is represented by the oriented submanifold $S^1 \times \{1\}$ and the basis element $(0, 1)$ is represented by the oriented submanifold $\{1\} \times S^1$. This situation is illustrated in the figure below on the left.

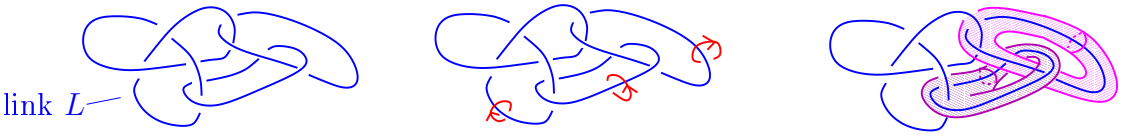


(2) We consider the surface Σ_2 of genus 2. As in the figure on page **??** we view Σ_2 as the connected sum $T \# T$ of two tori. If we use the calculation on page 2212 to determine an isomorphism $\mathrm{H}_1(\Sigma_2) \cong \mathbb{Z}^4$, then it follows from (1) that the basis elements are represented by the submanifolds a_1, b_1, a_2, b_2 that are illustrated in the figure below.[1243] [1244]



basis for homology of H_1(surface of genus 2)

(3) Let $L = L_1 \cup \cdots \cup L_m$ be an m-component link in S^3. Those readers who solved Exercise 100.6 will see that the m meridians represent a basis for $\mathrm{H}_1(S^3 \setminus L) \cong \mathbb{Z}^m$ and that any choice of $m - 1$ tori $S^1 \times L_i$ represent a basis for $\mathrm{H}_2(S^3 \setminus L) \cong \mathbb{Z}^{m-1}$.



meridians represent basis for $\mathrm{H}_1(S^3 \setminus L)$

two out of three tori represent basis for $\mathrm{H}_2(S^3 \setminus L)$

(4) Any zero class $0 \in \mathrm{H}_k(M)$ is represented by the empty submanifold.

(5) We consider the complex projective space $\mathbb{C}\mathrm{P}^n$. We saw on page 1473 that for $k \leq n$ the map

[1243] Another point of view is that the submanifolds a_1, b_1, a_2, b_2 are given by deforming the curves x_1, y_1, x_2, y_2 that are shown in the figure on page 1225.

[1244] What do these submanifolds look like in the formal picture of the surface of genus 2 as the quotient $E_8/\sim$ of the Euclidean octagon?

$$\begin{array}{rcl} \mathbb{C}\mathrm{P}^k & \to & \mathbb{C}\mathrm{P}^n \\ [v_0 : \ldots : v_k] & \mapsto & [v_0 : \ldots : v_k : 0 : \ldots : 0] \end{array}$$

is a smooth embedding. We use this map to view $\mathbb{C}\mathrm{P}^k$ as a submanifold of $\mathbb{C}\mathrm{P}^n$. It follows from the discussion on page 1772 and the natural isomorphism of the Singular-Cellular H_*-Isomorphism Proposition 80.4 that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z} \cdot [\mathbb{C}\mathrm{P}^i], & \text{if } k = 2i \text{ for some } i \in \{0, \ldots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Put differently, the generators of the non-trivial homology groups of the complex projective space $\mathbb{C}\mathrm{P}^n$ are represented by the complex projective spaces $\mathbb{C}\mathrm{P}^i$ with $i = 0, \ldots, n$.

(6) Let M be a compact oriented connected k-dimensional smooth manifold. By Theorem 98.8 we have $\mathrm{H}_n(M, \partial M) = \mathbb{Z} \cdot [M]$. It follows from Lemma 100.8 that the classes $\pm[M]$ are represented by M, viewed as its own submanifold, equipped with the two possible orientations. Since M is connected we know by Exercise 19.19 that M admits no other non-empty n-dimensional proper submanifold. Thus we see that the classes $k \cdot [M]$ with $k \notin \{\pm 1\}$ are not represented by submanifolds.

For completeness' sake we state the following lemma.

Lemma 102.10. Let M be a smooth manifold, let A be a union of components of ∂M and let $\varphi \in \mathrm{H}_k(M, A)$. If N is a proper submanifold that represents φ, then ∂N represents the class $\partial_k(\varphi) \in \mathrm{H}_{k-1}(A)$ where $\partial_k \colon \mathrm{H}_k(M, A) \to \mathrm{H}_{k-1}(A)$ is the connecting homomorphism of the long exact sequence of the pair (M, A).

Proof. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_k(M, A) & \xrightarrow[\varphi \mapsto \partial_k(\varphi)]{\partial_k} & \mathrm{H}_{k-1}(A) \\ {\scriptstyle i_*}\uparrow & & \uparrow{\scriptstyle i_*} \\ \mathrm{H}_k(N, \partial N) & \xrightarrow[[N] \mapsto [\partial N]]{\partial_k} & \mathrm{H}_{k-1}(\partial N) \end{array}$$

where the horizontal maps are given by the connecting homomorphisms and where the vertical maps are induced by the inclusion maps. By the Topological LES-Proposition 74.13 we know that the connecting homomorphism are natural, in other words, the diagram commutes. The statement now follows from the Fundamental Class-of-Boundary Proposition 100.9. ∎

We are naturally lead to the following question which can be viewed as a refinement of Question 102.1 in the context of smooth manifolds.

Question 102.11. Let M be a smooth manifold and let A be a union of components of ∂M. For which $k \in \mathbb{N}_0$ can homology classes in $\mathrm{H}_k(M, A)$ be represented by submanifolds?

The following proposition, which can be viewed as analogue of Proposition 102.4 (2) says that the relative problem is equivalent to the absolute problem.

Proposition 102.12. Let $k \in \mathbb{N}$. Suppose that for every (orientable) n-dimensional smooth manifold every homology class in $\mathrm{H}_k(M)$ can be represented by a submanifold.

Then also for every (orientable) n-dimensional smooth manifold M and every union A of components of ∂M every relative homology class in $\mathrm{H}_k(M, A)$ can be represented by a submanifold.

Proof. Let A be the union of some components of ∂M and let $\sigma \in \mathrm{H}_k(M, A)$. Let M' be a second copy of M. As on page 1667 we consider the double of M along A, i.e. we consider $\mathrm{D}_A M = M \cup_{A=A'} M'$. Note that by the Doubling-Manifold Lemma 75.12 we know that $\mathrm{D}_A M$ is an (orientable) n-dimensional smooth manifold which contains A as a submanifold. Next let $D\colon \mathrm{H}_k(M, A) \to \mathrm{H}_k(\mathrm{D}_A M)$ be the doubling homomorphism given by the Doubling-Homology Lemma 75.16.

By our hypothesis there exists a closed oriented submanifold W of $\mathrm{D}_A M$ such that $[W] = D(\sigma)$. It follows from the Transversality Theorem 42.4 and the Homotopic Maps-and-Homology Proposition 73.6 that we can arrange that W is transverse to A. Now we set $N := W \cap M$. It follows from Exercise 42.8, which is a variation on the Submanifold Transversal Intersection Theorem 42.1, that N is a proper submanifold of W. It remains to prove the following claim.

Claim. We have $[N] = \sigma \in \mathrm{H}_k(M, A)$.

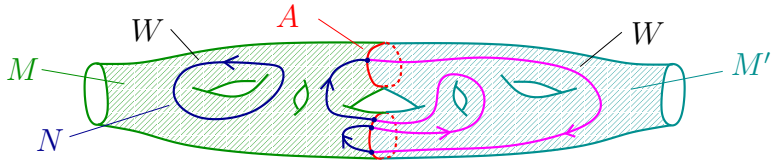
Proof. We denote by $i\colon W \to \mathrm{D}_A M$ the inclusion map. Now we consider the following diagram

$$\begin{array}{ccccc}
\mathrm{H}_k(N, \partial N) & \xrightarrow[\cong]{\alpha} & \mathrm{H}_n(W, W \setminus \mathring{N}) & \xleftarrow{\beta} & \mathrm{H}_k(W) \\
\downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\
\mathrm{H}_k(M, A) & \xrightarrow[\cong]{\gamma} & \mathrm{H}_k(\mathrm{D}_A M, M') & \xleftarrow{\delta} & \mathrm{H}_k(\mathrm{D}_A M).
\end{array}$$

$$\mathrm{H}_k(M, A) \xrightarrow{\quad D \quad} \mathrm{H}_k(\mathrm{D}_A M)$$

Here α, β, γ and δ are the obvious maps. Now we make the following observations:

$$\begin{array}{rrcll}
 & i_*([W]) & = & D(\sigma) & \text{by choice of } W \\
\Longrightarrow & i_*(\beta([W])) & = & \delta(D(\sigma)) & \text{since the diagram commutes} \\
\Longrightarrow & i_*(\alpha([N]) & = & \gamma(\sigma) & \text{by Lemmas 100.12 and 75.16 (3)} \\
\Longrightarrow & \gamma(i_*([N])) & = & \gamma(\sigma) & \text{since the diagram commutes} \\
\Longrightarrow & i_*([N]) & = & \sigma & \text{since } \gamma \text{ is an isomorphism by the Excision Theorem 75.10}
\end{array}$$ ■



One obvious approach to Question 102.11 is as follows: in Section 102.1 we already saw that in many cases, given a class $\varphi \in \mathrm{H}_k(M, A)$, there exists a compact oriented k-dimensional smooth manifold N together with a map $f\colon (N, \partial N) \to (X, A)$ such that $f_*([N]) = \varphi$. The question now is the following: can we replace f by some proper embedding? The following proposition, which we will prove in Section 102.6, addresses this issue:

Proposition 102.13. Let $n \in \mathbb{N}$, let M be an n-dimensional smooth manifold and let A be a union of components of ∂M. Furthermore let N be a compact oriented k-dimensional smooth manifold and let $\varphi\colon (N, \partial N) \to (M, A)$ be a map.

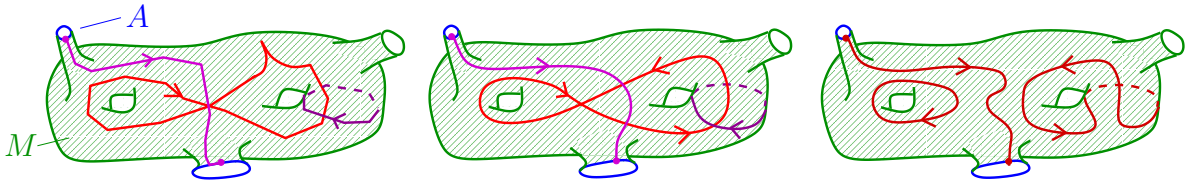
(1) If $n > 2k$, then $\varphi\colon (N, \partial N) \to (M, A)$ is homotopic, as a map of pairs of topological spaces, to a proper embedding $\psi\colon (N, \partial N) \to (M, A)$. In particular by the Smooth Embedding Theorem 24.10 (2) $\psi(N)$ is a proper submanifold of M and by Proposition 74.15 we have $\varphi_*([N]) = [\psi(N)] \in \mathrm{H}_k(M, A)$.

(2) If $n = 2k$, then there exists a compact oriented proper k-dimensional submanifold W of N with $\partial W \subset A$ such that $\varphi_*([N]) = [W] \in \mathrm{H}_k(M, A)$.[1245]

It is worth stating the following immediate corollary to Propositions 102.2 and 102.13.

Corollary 102.14. Let M be an m-dimensional smooth manifold with $m \geq 2$ and let A be the union of some components of ∂M. Given any homology class in $\sigma \in \mathrm{H}_1(M, A)$ there exists a compact oriented proper 1-dimensional submanifold P with $\partial P \subset A$ such that $[P] = \sigma$.

Example. Let M be a 2-dimensional smooth manifold and let $\sigma \in \mathrm{H}_1(M)$ be a homology class. For fun we illustrate in the figure below the three main steps in the proof of Proposition 102.13 in this special case.



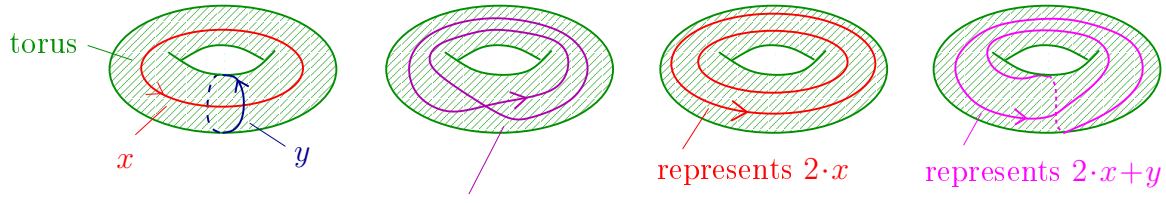
by Proposition 102.2 we can represent the homology class by some map $\varphi\colon N \to M$

isotope to get smooth proper embedding without triple points

resolve intersections

Next we want to consider the question whether we can represent a homology class by a *connected* submanifold. For tori we have the following ambiguous answer.

Lemma 102.15. Let $T = S^1 \times S^1$ be the torus. We denote by $x = [S^1 \times \{1\}]$ and $y = [\{1\} \times S^1]$ the obvious basis for $\mathrm{H}_1(T)$. A non-zero class $r \cdot x + s \cdot y$ can be represented by a connected submanifold if and only if $\gcd(r, s) = 1$.



$2 \cdot x$ can be realized by a map from a connected manifold

Proof. We, i.e. you, will provide the proof in Exercise 102.4. For fun we will deal with the analogous problem for surfaces of higher genus in Exercise 102.5. ■

[1245] Note that we do not claim that W is diffeomorphic to N.

The following proposition, which can be viewed as an analogue of Proposition 102.4 (1), shows that in most settings we will not have troubles finding connected submanifolds. As for the previous proposition we postpone the proof to Section 102.7.

Proposition 102.16. Let $n \in \mathbb{N}$, let M be a connected n-dimensional smooth manifold and let A be a union of components of ∂M. Furthermore let N be a compact oriented k-dimensional submanifold with $\partial N \subset N$. We suppose that $n - k \geq 2$. If $n \geq 2$ or if $A = \varnothing$, then there exists a compact oriented *connected* submanifold W of M with $\partial W \subset A$ such that $[W] = [N] \in \mathrm{H}_k(M, A)$.

Remark. As we just saw in Lemma 102.15, in general the conclusion of Proposition 102.16 fails to hold if $k = n-1$. Note thought that in this setting some control over connectedness can be obtained using [**Tur2002**, Lemma 1.2].

The combination of Propositions 102.3, 102.13 and 102.16 gives us the following pretty corollary.

Corollary 102.17. Let M be an n-dimensional smooth manifold with $n \geq 4$ and let $A \subset \partial M$ be a union of boundary components. Every class in $\mathrm{H}_2(M, A)$ is represented by a connected submanifold.

Remark. Let M be a compact 4-dimensional smooth manifold and let $\sigma \in \mathrm{H}_2(M)$. By Corollary 102.17 it makes sense to the define the *genus of* σ:

$$\mathrm{genus}(\sigma) \ := \ \begin{array}{c}\text{minimal genus of a closed oriented connected} \\ \text{smooth submanifold } \Sigma \text{ of } M \text{ with } [\Sigma] = \sigma.\end{array}$$

As an example let us consider the 4-dimensional smooth manifold $\mathbb{C}\mathrm{P}^2$. By the discussion on page 2247 we know that $\mathrm{H}_2(\mathbb{C}\mathrm{P}^2) = \mathbb{Z} \cdot [\mathbb{C}\mathrm{P}^1]$. Let $n \in \mathbb{Z}$. We consider

$$W_n \ := \ \{[x : y : z] \in \mathbb{C}\mathrm{P}^2 \,|\, x^n + y^n + z^n = 0\}.$$

In Exercise 32.12 we showed that W_n is a closed 1-dimensional complex submanifold of the 2-dimensional complex manifold $\mathbb{C}\mathrm{P}^2$. In particular this shows that W_n is a closed 2-dimensional submanifold of $\mathbb{C}\mathrm{P}^2$. In Exercise 126.6 we will show that $[W_n] = n \cdot [\mathbb{C}\mathrm{P}^1]$. As is pointed out in [**GoS1999**, p. 38], it is a consequence of the "adjunction formula", that W_n is diffeomorphic to the surface of genus $\frac{1}{2}(n-1)(n-2)$. In particular we have the inequality $\mathrm{genus}(n \cdot [\mathbb{C}\mathrm{P}^1]) \leq \frac{1}{2}(n-1)(n-2)$. The Thom Conjecture predicted that equality holds. This conjecture was proved in 1994 by Peter Kronheimer and Tom Mrowka [**KrM1994**] using "Seiberg-Witten Theory". In general though it is still a very difficult problem to determine the genus of the homology class of a given smooth manifold.

Corollary 102.17 suffers from the rather annoying defect that it does not apply to the case of 3-dimensional smooth manifolds. Perhaps rather surprisingly this gets taken care of by coming "from the other direction", namely this case gets taken care of by the following proposition.

Proposition 130.4. Let M be a compact orientable n-dimensional smooth manifold and let $A \subset \partial M$ be a union of boundary components. Every class in $\mathrm{H}_{n-1}(M, A)$ can be represented by a submanifold.

Sketch of proof. As the number already suggests, we will not prove this statement in this chapter. In the following we will just provide the flavor of the argument which is meant

to serve as an amuse-bouche. For simplicity we assume that M is closed. Now suppose we are given a homology class $\sigma \in \mathrm{H}_{n-1}(M) = \mathrm{H}_{n-1}(M;\mathbb{Z})$. By the Poincaré Duality Theorem 119.1 we have an isomorphism $\mathrm{PD}_M \colon \mathrm{H}_{n-1}(M;\mathbb{Z}) \to \mathrm{H}^1(M;\mathbb{Z})$. Thus we obtain the Poincaré dual $\mathrm{PD}_M(\sigma) \in \mathrm{H}^1(M;\mathbb{Z})$. We will see that cohomology classes in $\mathrm{H}^1(M;\mathbb{Z})$ give rise to smooth maps $f \colon M \to K(\mathbb{Z},1) = S^1$. (Here $K(G,n)$ stands for an Eilenberg-Maclane space of type $K(G,n)$, the precise definition is irrelevant right now, except that in the discussion below it indicates some similarities between initially quite different looking arguments.) The desired submanifold is then given by the preimage of a regular value of f. ■

The analogue for codimension-two homology classes also holds:

Proposition ??. Let M be a compact orientable n-dimensional smooth manifold and let $A \subset \partial M$ be a union of boundary components. Every class in $\mathrm{H}_{n-2}(M,A)$ can be represented by a submanifold.

Sketch of proof. The logic of the proof is somewhat similar to the idea behind the proof of the Codimension-One Submanifold Existence Proposition 130.4. This time the Poincaré Duality Theorem 119.1 gives us an isomorphism $\mathrm{PD}_M \colon \mathrm{H}_{n-2}(M;\mathbb{Z}) \to \mathrm{H}^2(M;\mathbb{Z})$. Thus the given homology class in $\mathrm{H}_{n-2}(M;\mathbb{Z})$ corresponds to a cohomology class in $\mathrm{H}^2(M;\mathbb{Z})$. We will see that such cohomology classes give rise to maps $f \colon M \to K(\mathbb{Z},2) = \mathbb{C}\mathrm{P}^\infty$. Since M is compact we obtain a map $f \colon M \to \mathbb{C}\mathrm{P}^m$ for sufficiently large m. In this occasion the desired submanifold of M is given by the preimage of $\mathbb{C}\mathrm{P}^{m-1} \subset \mathbb{C}\mathrm{P}^m$. ■

Optimistically one might hope that the pattern of the previous two propositions extends to codimension-three homology classes. But Statement (2) of the following theorem shows that this is not the case.

Theorem 102.18. (Thom's Theorem III)

(1) Let M be a compact orientable n-dimensional smooth manifold and let A be a union of boundary components. Given $k \leq 6$ with $k < n$ every class in $\mathrm{H}_k(M,A)$ can be represented by a submanifold.
(2) There exists a 10-dimensional closed smooth manifold[1246] and a class in $\mathrm{H}_7(M)$ that cannot be represented by a submanifold.

Proof.

(1) First let us deal with the case that $A = \varnothing$. In this case, for $k \leq 5$ the statement can be found in [**Tho1954a**, Théorème 6] and also in [**Tho1954b**, Théorème II.27]. The statement for $k = 6$ can be found in the footnote on page 173 of [**Tho2007**]. The relative case now follows from the above together with Proposition 102.12.
(2) This statement was proved by Christian Bohr, Bernhard Hanke and Dieter Kotschick, see [**BHK2002**, Theorem 1]. Their work builds on earlier work of René Thom [**Tho1954b**, p. 62f]. ■

For completeness' sake we now also discuss homology with $\mathbb{F}_2$-coefficients. Given a compact (possibly non-orientable) proper k-dimensional submanifold N of a manifold M we

[1246] It follows from Statement (1) together with Propositions 130.4 and **??** that for a compact orientable n-dimensional smooth manifold of dimension $n \leq 9$ any homology class of dimension $< n$ can be represented by a submanifold. Thus the example in Statement (2) is "dimensionally minimal".

denote by $[N]_{\mathbb{F}_2}$ the image of the $\mathbb{F}_2$-fundamental class N under the inclusion induced map $\mathrm{H}_k(N, \partial N; \mathbb{F}_2) \to \mathrm{H}_k(M, \partial M; \mathbb{F}_2)$. We say N *represents the homology class* $[N]_{\mathbb{F}_2}$.
Surely it does not come as a surprise that we have the following analogue of Corollaries 102.14 and 102.17 for $\mathbb{F}_2$-coefficients. We will skip the proof since it is essentially the same.

Proposition 102.19. Let M be an n-dimensional smooth manifold and let $A \subset \partial M$ be a union of boundary components.

(1) If $n \geq 2$, then every class in $\mathrm{H}_1(M, A; \mathbb{F}_2)$ can be represented by a submanifold.
(2) If $n \geq 4$, then every class in $\mathrm{H}_2(M, A; \mathbb{F}_2)$ can be represented by a connected submanifold.

The following proposition is an analogue of the Codimension-One Submanifold Existence Proposition 130.4.

Proposition ??. Let M be a compact n-dimensional smooth manifold and let A be a union of boundary components. Every class in $\mathrm{H}_{n-1}(M, A; \mathbb{F}_2)$ can be represented by a submanifold.

Remark. This result will have its moment of glory when we will employ it in our proof, provided in Section **??**, that $\pi_{n+2}(S^n) \cong \mathbb{Z}_2$ for $n \geq 3$.

Sketch of proof. The proof of this proposition is quite similar to the proof of Proposition **??**. The Co-Homology Symmetry Theorem Theorem 119.6 with $\mathbb{F}_2$-coefficients gives us an isomorphism $\mathrm{PD}_M \colon \mathrm{H}_{n-1}(M; \mathbb{F}_2) \to \mathrm{H}^1(M; \mathbb{F}_2)$. Thus given the homology class in $\mathrm{H}_{n-1}(M; \mathbb{F}_2)$ corresponds to a cohomology class in $\mathrm{H}^1(M; \mathbb{F}_2)$. We will see that such cohomology classes give rise to maps $f \colon M \to K(\mathbb{F}_2, 1) = \mathbb{R}\mathrm{P}^\infty$. Since M is compact we obtain a map $M \to \mathbb{R}\mathrm{P}^m$ for sufficiently large m. Now the desired submanifold is given by the preimage of $\mathbb{R}\mathrm{P}^{m-1} \subset \mathbb{R}\mathrm{P}^m$. ∎

Remark. Note that this author is not aware of an analogue of Proposition **??**.

Finally let us see once again what René Thom has to say:

Theorem 102.20. (Thom's Theorem IV) Let M be a compact n-dimensional smooth manifold and let A be a union of boundary components.

(1) If $n \leq 5$, then any class in $\mathrm{H}_{n-2}(X, A; \mathbb{F}_2)$ can be represented by a submanifold.
(2) If $n \leq 7$, then any class in $\mathrm{H}_{n-3}(X, A; \mathbb{F}_2)$ can be represented by a submanifold.
(3) For $k \leq \lfloor \frac{n}{2} \rfloor$ any class in $\mathrm{H}_k(X, A; \mathbb{F}_2)$ can be represented by a submanifold.

Proof. In the case $A = \varnothing$ the three statements are precisely the content of [**Tho2007**, Théorème II.26]. The relative case follows from the $\mathbb{F}_2$-analogue of Proposition 102.12. Finally note that Statement (3) also follows from Theorem 102.5 together with the $\mathbb{F}_2$-analogue of Proposition 102.13 (2). ∎

We conclude this section with the following remark.

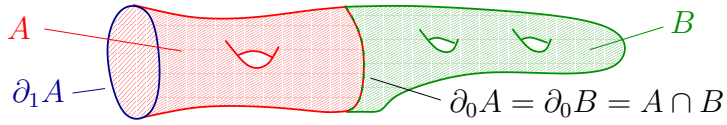
Remark. As we already mentioned on page 2237, later in Chapter 105 we will introduce the notion of an orientation of a topological manifold and in Chapter 106 we will introduce the notion of the fundamental class $[N] \in \mathrm{H}_n(N, \partial N)$ of a compact oriented n-dimensional topological manifold N. Thus it makes sense to ask which homology classes of a given topological manifold can be represented by submanifolds. This question is discussed in [**FNOP2019**].

In the remainder of this chapter we will provide the proofs of Proposition 102.13 and 102.16.

102.5. **Rearranging maps from smooth manifolds.** Let (X, A) be a pair of topological spaces. In this section we will state and prove a proposition which allows us to replace certain compact oriented smooth manifolds M and maps $f\colon (M, \partial M) \to (X, A)$ by a different compact oriented smooth manifold N together with a map $g\colon (N, \partial N) \to (X, A)$ such that $f_*([M]) = g_*([N])$. This procedure will come in handy in the proofs of Proposition 102.13 and 102.16.
The following definition is a special case of a more general concept introduced on page 570.

Definition. Let M be a compact n-dimensional smooth manifold. We say M is decomposed into submanifolds A and B if the following conditions hold:

(1) $M = A \cup B$.
(2) A and B are both compact n-dimensional submanifolds of M.
(3) We have $A \cap B = \partial_0 A = \partial_0 B$.



Now we can state the one result of this section.

Proposition 102.21. Let M and $\widetilde{M}$ be two compact oriented n-dimensional smooth manifolds. Suppose that M is decomposed into two submanifolds A and B and that $\widetilde{M}$ is decomposed into two submanifolds $\widetilde{A}$ and $\widetilde{B}$. We equip $A \cap B$ and $\widetilde{A} \cap \widetilde{B}$ with the orientation coming from A respectively $\widetilde{A}$. Suppose that we are given an orientation-preserving diffeomorphism $\Theta\colon A \cap B \to \widetilde{A} \cap \widetilde{B}$. Next we set

$$N := (A \sqcup \widetilde{B})/x \sim \Theta(x) \text{ for } x \in A \cap B \quad \text{and } \widetilde{N} := (\widetilde{A} \sqcup B)/\Theta(y) \sim y \text{ for } y \in A \cap B.$$

Note that it follows from the Gluing-Smooth Manifolds-Proposition **??** that N and $\widetilde{N}$ are both compact oriented n-dimensional smooth manifolds such that the obvious maps $A \to N$ and $\widetilde{A} \to \widetilde{N}$ are orientation-preserving embeddings. Next let (X, S) be a pair of topological space and furthermore let $f\colon (M, \partial M) \to (X, S)$ and $\widetilde{f}\colon (\widetilde{M}, \partial\widetilde{M}) \to (X, S)$ be two maps such that we have $\widetilde{f} \circ \Theta = f\colon A \cap B \to X$. We define[1247]

$$g\colon N = (A \sqcup \widetilde{B})/\sim \to X \qquad [P] \mapsto \begin{cases} f(P), & \text{if } P \in A, \\ \widetilde{f}(P), & \text{if } P \in \widetilde{B} \end{cases}$$

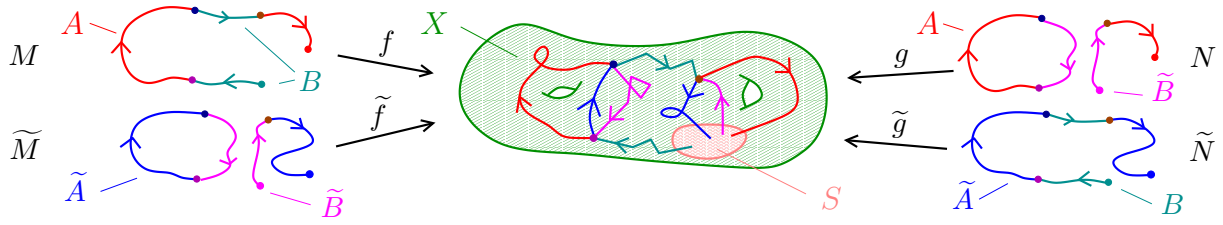
and

$$\widetilde{g}\colon \widetilde{N} = (\widetilde{A} \sqcup B)/\sim \to X \qquad [P] \mapsto \begin{cases} \widetilde{f}(P), & \text{if } P \in \widetilde{A}, \\ f(P), & \text{if } P \in B. \end{cases}$$

With this notation we have

$$f_*([M]) + \widetilde{f}_*([\widetilde{M}]) \;=\; g_*([N]) + \widetilde{g}_*([\widetilde{N}]) \;\in\; \mathrm{H}_n(X, S).$$

[1247] It follows from $\widetilde{f} \circ \Theta = f$ that both maps are continuous.



Proof. We pick a fundamental cycle $\mu \in \mathrm{C}_{n-1}(A \cap B)$ for the closed oriented $(n-1)$-dimensional smooth manifold $A \cap B$. We consider the map

$$\mathrm{C}_n(A, \partial A) \xrightarrow{\partial_n} \mathrm{C}_{n-1}(\overbrace{\partial_0 A \sqcup \partial_1 A}^{=\partial A}) \xrightarrow{\cong} \mathrm{C}_{n-1}(\overbrace{A \cap B}^{=\partial_0 A}) \oplus \mathrm{C}_{n-1}(\partial_1 A) \twoheadrightarrow \mathrm{C}_{n-1}(A \cap B).$$
$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{=:\omega_A}$$

By Corollary 100.10 we know that there exists a singular chain $\nu_A \in \mathrm{C}_n(A)$ such that ν_A represents a fundamental cycle in $\mathrm{C}_n(A, \partial A)$ and such that $\omega_A(\nu_A) = \mu$. With the same notation we also obtain some $\nu_B \in \mathrm{C}_n(B)$ with $\omega_B(\nu_B) = \mu$.

Next we consider $\widetilde{\mu} := \Theta_*(\mu) \in \mathrm{C}_{n-1}(\widetilde{A} \cap \widetilde{B})$. As above we pick corresponding singular chains $\widetilde{\nu}_{\widetilde{A}} \in \mathrm{C}_n(\widetilde{A})$ and $\widetilde{\nu}_{\widetilde{B}} \in \mathrm{C}_n(\widetilde{B})$. Now we see that we have the following equalities in $\mathrm{H}_n(X, S)$:

it follows from $\omega_A(\nu_A) = \mu = \omega_B(\nu_B)$ that $\nu_A - \nu_B$ is a cycle in $\mathrm{C}_n(M, \partial M)$ using the Fundamental Class-Characterization Lemma 100.3 (3)⇒(1) one now can easily verify that $\nu_A - \nu_B$ is a fundamental cycle of M, the same argument shows that $\widetilde{\nu}_{\widetilde{A}} - \widetilde{\nu}_{\widetilde{B}}$ is a fundamental cycle of $\widetilde{M}$

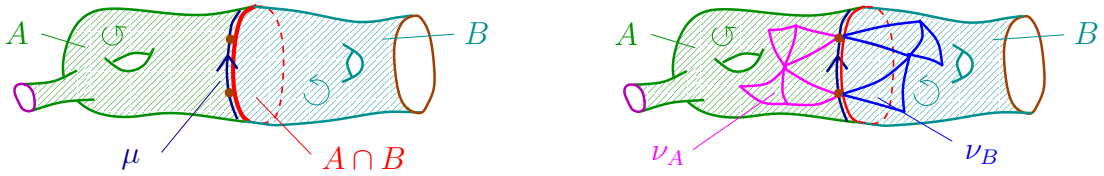
$$\begin{aligned}
f_*([M]) + \widetilde{f}_*([\widetilde{M}]) &\overset{\downarrow}{=} f_*([\nu_A - \nu_B]) + \widetilde{f}_*([\widetilde{\nu}_{\widetilde{A}} - \widetilde{\nu}_{\widetilde{B}}]) \qquad = [f_*(\nu_A) - f_*(\nu_B) + \widetilde{f}_*(\widetilde{\nu}_{\widetilde{A}}) - \widetilde{f}_*(\widetilde{\nu}_{\widetilde{B}})] \\
&= [f_*(\nu_A) - \widetilde{f}_*(\widetilde{\nu}_{\widetilde{B}}) + \widetilde{f}_*(\widetilde{\nu}_{\widetilde{A}}) - f_*(\nu_B)] \overset{\uparrow}{=} [f_*(\nu_A) - \widetilde{f}_*(\widetilde{\nu}_{\widetilde{B}})] + [\widetilde{f}_*(\widetilde{\nu}_{\widetilde{A}}) - f_*(\nu_B)]
\end{aligned}$$

it follows fairly easily from $\widetilde{f} \circ \Theta = f$ and $\widetilde{\mu} = \Theta_*(\mu)$ that $f_*(\nu_A) - \widetilde{f}_*(\widetilde{\nu}_{\widetilde{B}})$ and $\widetilde{f}_*(\widetilde{\nu}_{\widetilde{A}}) - f_*(\nu_B)$ are both cycles in $\mathrm{C}_n(X, S)$

$$= [g_*(\nu_A) - g_*(\widetilde{\nu}_{\widetilde{B}})] + [\widetilde{g}_*(\widetilde{\nu}_{\widetilde{A}}) - \widetilde{g}_*(\nu_B)] \;=\; g_*(\underbrace{[\nu_A - \widetilde{\nu}_B]}_{=[N]}) + \widetilde{g}_*(\underbrace{[\widetilde{\nu}_{\widetilde{A}} - \widetilde{\nu}_B]}_{=[\widetilde{N}]}).$$

$\uparrow$ by definition of g and $\widetilde{g}$

■

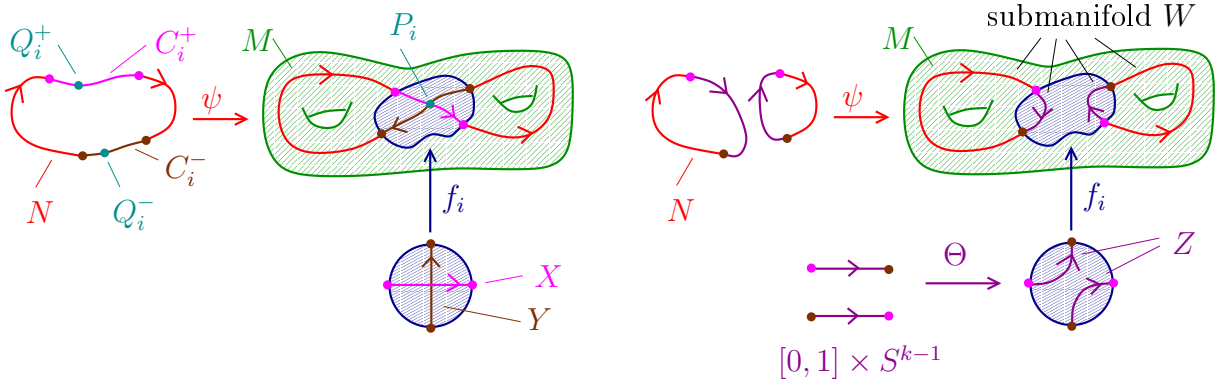


102.6. **Proofs of Proposition 102.13.** For the reader's convenience we recall the statement of Proposition 102.13.

Proposition 102.13. Let $n \in \mathbb{N}$, let M be an n-dimensional smooth manifold and let A be a union of components of ∂M. Furthermore let N be a compact oriented k-dimensional smooth manifold and let $\varphi\colon (N, \partial N) \to (M, A)$ be a map.

(1) If $n > 2k$, then $\varphi\colon (N, \partial N) \to (M, A)$ is homotopic, as a map of pairs of topological spaces, to a proper embedding $\psi\colon (N, \partial N) \to (M, A)$. In particular $\psi(N)$ is a proper submanifold of M and we have $\varphi_*([N]) = [\psi(N)] \in \mathrm{H}_k(M, A)$.

(2) If $n = 2k$, then there exists a compact oriented proper k-dimensional submanifold W of N with $\partial W \subset A$ such that $\varphi_*([N]) = [W] \in \mathrm{H}_k(M, A)$.



Proof. Let $n \in \mathbb{N}$, let M be an n-dimensional smooth manifold and let A be a union of components of ∂M. Furthermore let N be a compact oriented k-dimensional smooth manifold and let $\varphi\colon (N, \partial N) \to (M, A)$ be a map.

First let us consider the case $n > 2k$. In this case it follows from the Whitney Approximation Theorem 29.1 (3) together with Theorem **??** (2) that $\varphi\colon (N, \partial N) \to (M, A)$ is homotopic, as a map of pairs of topological spaces, to a proper embedding $\psi\colon (N, \partial N) \to (M, A)$. It follows from the Smooth Embedding Theorem 24.10 (2) that $\psi(N)$ is a proper submanifold of M and that $\psi\colon N \to \psi(N)$ is a diffeomorphism. Thus we see that

$$\varphi_*([N]) \underset{\substack{\uparrow \\ \text{Proposition 74.15}}}{=} \psi_*([N]) \underset{\substack{\uparrow \\ \text{since } \psi \text{ is a diffeomorphism}}}{=} [\psi(N)] \in \mathrm{H}_k(M, A).$$

Now let us deal with the much more subtle case $n = 2k$. From the Whitney Approximation Theorem 29.1 (3) together with Theorem **??** (1) we now obtain that the map φ is homotopic, as a map of pairs of topological spaces, to a proper immersion $\psi\colon N \to M$ which has furthermore the property that there exist points $P_1, \ldots, P_m$ in $M \setminus \partial M$ with the following properties:

(1) For each $i \in \{1, \ldots, m\}$ the preimage $\psi^{-1}(P_i)$ consists of precisely two points $Q_i^\pm$ which furthermore satisfy that $\psi_*(\mathrm{T}_{Q_i^+} N) + \psi_*(\mathrm{T}_{Q_i^-} N) = \mathrm{T}_{P_i}(M)$.

(2) The restriction of ψ to $N \setminus \{Q_1^\pm, \ldots, Q_m^\pm\}$ is injective.

In the following we write

$$X := \left\{ (x, 0) \in \overline{B}^{2k} \,\middle|\, x \in \overline{B}^k \right\} \quad \text{and} \quad Y := \left\{ (0, y) \in \overline{B}^{2k} \,\middle|\, y \in \overline{B}^k \right\}.$$

We view X and Y as submanifolds of $\overline{B}^{2k}$. We equip X and Y with the obvious orientation. It follows easily from the Smooth Embedding Theorem 24.10 (2) together with the Submanifold Transversal Intersection Theorem 42.1 (1) that we can find the following:

(1) Disjoint k-dimensional submanifolds $\mathrm{C}_1^{\pm}, \dots, \mathrm{C}_m^{\pm}$ of N such that for each $i \in \{1, \dots, m\}$ we have $Q_i^{\pm} \in \mathring{C}_i^{\pm} = \mathrm{C}_i^{\pm}$. Note that the restriction of ψ to any $\mathring{C}_i^{\pm}$ is in fact a smooth embedding.

(2) Disjoint embeddings $f_1, \dots, f_m \colon \overline{B}^{2k} \to M \setminus \partial M$ such that for each $i \in \{1, \dots, m\}$ we have

$$\psi(\mathrm{C}_i^+) \;=\; f_i(X), \quad \psi(\mathrm{C}_i^-) \;=\; f_i(Y), \quad \text{and } f_i(\overline{B}^{2k}) \cap \psi(N) \;=\; f_i(X) \cup f_i(Y).$$

Furthermore, after possibly precomposing the maps f_i with reflections in hyperplanes we can assume that the maps $X \to f_i(X) \overset{\psi}{\leftarrow} \mathrm{C}_i^+$ and $Y \to f_i(Y) \overset{\psi}{\leftarrow} \mathrm{C}_i^-$ are orientation-preserving.

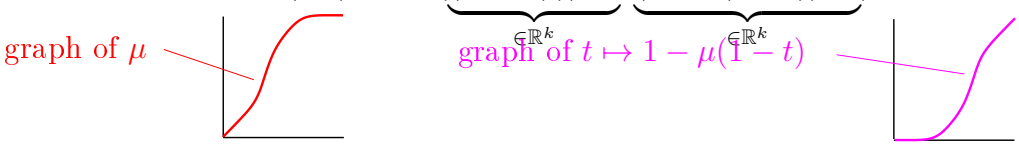
Claim 1. There exists a proper smooth embedding $\Theta \colon [0,1] \times S^{k-1} \to \overline{B}^{2k}$ with the following three properties:

(1) For $t \in [0, \frac{1}{4}]$ and $z \in S^{k-1}$ we have $\Theta(t,z) = ((1-t)z, 0) \in \mathbb{R}^k \times \mathbb{R}^k$. Furthermore the map $\Theta_0 \colon \{0\} \times S^{k-1} \to S^{k-1} \times \{0\}$ is orientation-preserving.

(2) For $t \in [\frac{3}{4}, 1]$ and $z \in S^{k-1}$ we have $\Theta(t,z) = (0, t \cdot z) \in \mathbb{R}^k \times \mathbb{R}^k$. Furthermore the map $\Theta_0 \colon \{1\} \times S^{k-1} \to \{0\} \times S^{k-1}$ is orientation-reversing.

Proof. To prove the claim first note that, using the Smooth Transition Function Lemma 19.4 one can easily find a smooth function $\mu \colon [0,1] \to [0,1]$ such that $\mu(t) = t$ for $t \in [0, \frac{1}{4}]$, and such that $\mu(t) = 1$ for $t \in [\frac{3}{4}, 1]$ and such that $\mu'(t) > 0$ for any $t \in (0, \frac{3}{4})$. We leave it to the reader to verify that the following map has all the desired properties:

$$\begin{aligned} \Theta \colon [0,1] \times S^{k-1} &\to \overline{B}^{2k} \subset \mathbb{R}^k \times \mathbb{R}^k \\ (t,z) &\mapsto (\underbrace{(1-\mu(t)) \cdot z}_{\in \mathbb{R}^k}, \underbrace{(1-\mu(1-t)) \cdot z}_{\in \mathbb{R}^k}). \end{aligned}$$



⊞

Next we set $Z := \Theta([0,1] \times S^{k-1})$ and we set

$$W \;:=\; \psi\Big(\underbrace{N \setminus \bigcup_{i=1}^{m} (\mathring{C}_i^+ \cup \mathring{C}_i^-)}_{=:\widehat{N}}\Big) \cup \bigcup_{i=1}^{m} f_i(Z).$$

We leave it to the reader to verify that W is in fact a compact proper k-dimensional submanifold of M with $\partial W \subset A$ which can be oriented in such a way that the embedding $\psi \colon \widehat{N} \to W$ is orientation-preserving. It remains to prove the following claim.

Claim 2. We have $\varphi_*([N]) = [W] \in \mathrm{H}_k(X, A)$.

Proof. As usual we denote by $-W$ the same smooth manifold but with the opposite orientation. By Lemma 100.8 we have $[-W] = -[W]$. Thus it suffices to show that

$\varphi_*([N]) + [-W] = 0$. Now we perform the following calculation:

$$\varphi_*([N])+[-W] \overset{\text{by Proposition 74.15 since } \psi \text{ is homotopic to } \varphi}{\downarrow} = \psi_*([N])+[-W] = \psi_*\Big(\underbrace{\Big[\widehat{N} \cup \coprod_{i=1}^{m} (C_i^+ \cup C_i^-)\Big]}_{N}\Big) + \underbrace{\Big[-\psi(\widehat{N}) \cup - \coprod_{i=1}^{m} f_i(Z)\Big]}_{=W}$$

$$\overset{\text{Proposition 102.21}}{\downarrow} = \underbrace{(\psi \cup \psi)_*(\overbrace{[\widehat{N} \cup_{\partial_0 \widehat{N} = \partial_0 \widehat{N}} -\widehat{N}]}^{\text{the double } D_{\partial_0 \widehat{N}} \widehat{N}})}_{\substack{\text{the map } \psi \cup \psi \text{ factors through the} \\ \text{folding map } D_{\partial_0 \widehat{N}} \widehat{N} \to \widehat{N} \\ \text{which has degree zero by} \\ \text{the discussion on page 2227}}} + \sum_{i=1}^{m} \underbrace{(\psi \cup f_i)_*((Q_i^+ \cup Q_i^-) \cup Z)}_{\substack{= 0, \text{since } \psi \cup f_i \text{ takes values in } f_i(\overline{B}^{2k}) \\ \text{and since } \mathrm{H}_k(f_i(\overline{B}^{2k})) = 0}} = 0.$$

■

102.7. **The internal connected sum and the proof of Proposition 102.16.** We start with the slightly technical construction of an internal connected sum of two oriented submanifolds of a given smooth manifold. This construction is interesting in its own right, and it will be the key ingredient for the proof of Proposition 102.16.

Construction 102.22. Let M be a connected n-dimensional smooth manifold and let N_0 and N_1 be two compact oriented proper k-dimensional submanifolds of M. We suppose that $n - k \geq 2$. We pick points $P_0 \in N_0 \setminus \partial N_0$ and $P_1 \in N_1 \setminus \partial N_1$. We proceed with the following steps:

(1) Suppose we are given a smooth embedding $\gamma\colon [0,1] \to M \setminus \partial M$ with the following properties:
   (a) $\gamma(0) = P_0$ and $\gamma(1) = P_1$.
   (b) If we set $C := \gamma([0,1])$, then $C \cap N_0 = \{P_0\}$ and $C \cap N_1 = \{P_1\}$.
   (c) C hits N_0 and N_1 transversally, in the sense that $\mathrm{T}_{P_0} C \cap \mathrm{T}_{P_0} N_0 = \{0\}$ and $\mathrm{T}_{P_1} C \cap \mathrm{T}_{P_1} N_1 = \{0\}$.
   Note that it follows from Corollary 42.6 together with Theorem **??** that it follows from our hypothesis $n - k \geq 2$ that such γ always exists.

(2) Next we write $D := \{(x,0) \in \mathbb{R}^k \times \mathbb{R}^{n-1-k} \,|\, x \in \overline{B}^k\} \subset \overline{B}^{n-1}$. We challenge the courageous reader to show, say using a suitable variation on the Tubular Neighborhood Theorem **??**, that there exists an embedding $\Phi\colon \overline{B}^{n-1} \times C \to M$ such that $\Phi(\overline{B}^{n-1} \times C) \cap N_0 = \Phi(D \times \{P_0\})$ and such that $\Phi(\overline{B}^{n-1} \times C) \cap N_1 = \Phi(D \times \{P_1\})$.

(3) Using Exercise 19.12 one can easily find a smooth embedding $\Theta\colon S^{k-1} \times [0,1] \to \overline{B}^{k-1} \times C$ with the following two properties:
   (a) For $z \in S^{k-1}$ and $t \in [0, \frac{1}{4}]$ we have $\Theta(z,t) = ((1-t) \cdot z, P_0)$.
   (b) For $z \in S^{k-1}$ and $t \in [\frac{3}{4}, 1]$ we have $\Theta(z,t) = (t \cdot z, P_1)$.
   We set $Z := \Theta(S^{k-1} \times [0,1])$.

There are now two cases to consider.

Case (1) Among the two maps $\Phi\colon D \times \{P_0\} \to N_0$ and $\Phi\colon D \times \{P_1\} \to N_1$ one is orientation-preserving and one is orientation-reversing. In this case we set

$$N_0 \#_\gamma N_1 \;:=\; (N_0 \setminus \Phi(D \times \{P_0\})) \cup (N_1 \setminus \Phi(D \times \{P_1\}))) \cup \Phi(D \times C).$$

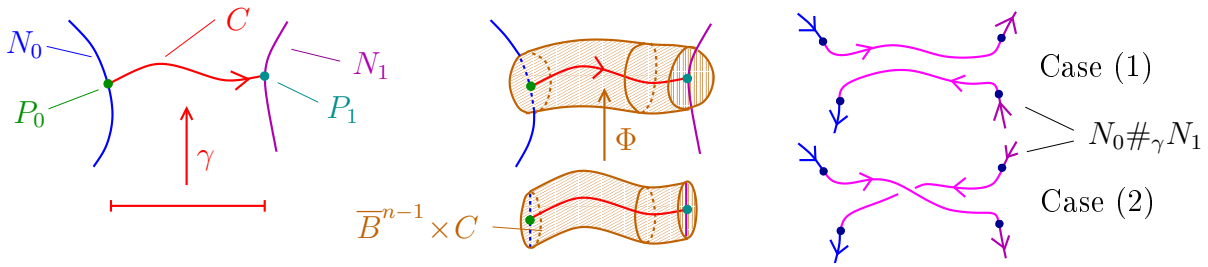
Case (2) If both maps $\Phi\colon D\times\{P_0\}\to N_0$ and $\Phi\colon D\times\{P_1\}\to N_1$ are orientation-preserving or if both are orientation-reversing then we add a "half-twist" to get us back into the game. More precisely, we consider the map [1248]

$$\begin{array}{rcl}\Omega\colon D\times C & \to & \overline{B}^{n-1}\times C\\ ((\underbrace{x_1,x_2,\ldots,x_k}_{\in\mathbb{R}^k},\underbrace{(0,0)}_{\in\mathbb{R}\times\mathbb{R}^{n-k-2}}),\gamma(t)) & \mapsto & ((\underbrace{\cos(t)\cdot x_1,x_2,\ldots,x_k}_{\in\mathbb{R}^k},\underbrace{(\sin(t)\cdot x_1,0)}_{\in\mathbb{R}\times\mathbb{R}^{n-k-2}}),\gamma(t)).\end{array}$$

Note that Ω is the identity on $D\times\{P_0\}$ and that Ω restricts to an orientation-reversing self-diffeomorphism of $D\times\{P_1\}$. Now we set

$$N_0\#_\gamma N_1 \;:=\; (N_0\setminus\Phi(D\times\{P_0\}))\cup(N_1\setminus\Phi(D\times\{P_1\}))\cup(\Phi\circ\Omega)(D\times C).$$

It is straightforward to verify that in both cases $N_0\#_\gamma N_1$ is a k-dimensional submanifold of M. Furthermore one sees easily that in both cases we can equip the smooth manifold $N_0\#_\gamma N_1$ with an orientation in such a way that the inclusions $N_0\setminus\Phi(D\times\{P_0\})\to N_0\#_\gamma N_1$ and $N_1\setminus\Phi(D\times\{P_1\})\to N_0\#_\gamma N_1$ are both orientation-preserving. We refer to $N_0\#_\gamma N_1$ as an *internal connected sum of* N_0 *and* N_1 *along* γ or as a *tubing of* N_0 *and* N_1 *along* γ.[1249]



The following lemma justifies the name "internal connected sum".

Lemma 102.23. We continue with the above notation. The internal connected sum $N_01\#_\gamma N_1$, viewed as a smooth manifold in its own right, is diffeomorphic to the connected sum $N_0\#N_1$.

Proof. The lemma can be proved fairly easily using the Smooth Collar Neighborhood Theorem 28.3 applied to $N_0\setminus\Phi(D\times\{P_0\})$ and $N_1\setminus\Phi(D\times\{P_1\})$. We leave it to the reader to fill in the details. ■

Example. In the figure below we show two disjoint oriented knots J and J. By definition both are submanifolds of S^3. We also show a path γ that connects K to J and we show a corresponding internal connected sum $K\#_\gamma J$. On page 1319 we introduced the connected sum $K\#J$. The resulting image, see e.g. the figure on page 1319 looks quite similar to what we see in the figure below. With some effort one can show that given any two oriented knots the corresponding connected sum $K\#J$ is smoothly isotopic to any internal connected sum $K\#_\gamma J$. In fact sometimes this approach is used as the definition of the connected sum of two oriented knots.

We recall the statement of proposition that we are actually supposed to prove.

[1248] Note that here we use again, in a most subtle way, our hypothesis $n-k\geq 2$.

[1249] The construction also depends on the choice of Φ and Θ. We do not claim that the isotopy type of $N_0\#_\gamma N_1$ is well-defined.



Proposition 102.16. Let $n \in \mathbb{N}$, let M be a connected n-dimensional smooth manifold and let A be a union of components of ∂M. Furthermore let N be a compact oriented k-dimensional submanifold with $\partial N \subset N$. We suppose that $n - k \geq 2$. If $n \geq 2$ or if $A = \varnothing$, then there exists a compact oriented *connected* submanifold W of M with $\partial W \subset A$ such that $[W] = [N] \in \mathrm{H}_k(M, A)$.

Sketch of proof. To simplify the notation we assume that N has precisely two components N_0 and N_1. As explained in Construction 102.22, since $n - k \geq 2$ we can perform an internal connected sum $N_0 \#_\gamma N_1$. As in the proof of Proposition 102.13 one can deduce, using Proposition 102.21, that $[N_0 \#_\gamma N_1] = [N_0] + [N_1] \in \mathrm{H}_k(M, A)$.

Next, as we pointed out in Lemma 102.23, the internal connected sum $N_0 \#_\gamma N_1$ is diffeomorphic to the connected sum $N_0 \# N_1$. If $n \geq 2$, then it follows immediately from the Connected Sum-Proposition **??** that the connected sum $N_0 \# N_1$ is connected. Finally, if $A = \varnothing$, then we know that both N_0 and N_1 are closed. Thus it follows again from the Connected Sum-Proposition **??** that $N_0 \# N_1$ is connected. ■

Exercises for Chapter 102.

Exercise 102.1. Let M be a closed 3-dimensional smooth manifold and let $\sigma \in \mathrm{H}_2(M)$ be a homology class. By the Realization-by-Surface Proposition 102.3 there exists a closed oriented 2-dimensional smooth manifold F and a map $\varphi\colon F \to M$ such that $\varphi_*([F]) = \varphi$. Can you modify the proof of Proposition 102.13 to show that there exists a closed oriented 2-dimensional smooth manifold G and an *embedding* $\psi\colon G \to M$ such that $\psi_*([G]) = \sigma$? Identify the issues you run into and try to resolve them.

Exercise 102.2. Let $n \in \mathbb{N}$. As on page 255 we consider the complex projective space $\mathbb{C}\mathrm{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$. By the discussion on page 1772 we know that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k \in \{0, 2, \ldots, 2n\}, \\ 0, & \text{otherwise.} \end{cases}$$

We consider the map $f\colon \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ given by $f([z_0 : \cdots : z_n]) = [\overline{z}_0 : \cdots : \overline{z}_n]$. For each $k \in \{0, 2, \ldots, 2n\}$ determine the degree of the induced map $f_*\colon \mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^n)$.

Exercise 102.3. Let $g \in \mathbb{N}_0$. As usual we denote by Σ_g the surface of genus g. Let M be a smooth manifold and let $P \in M$. Now let $k \in \mathbb{N}_0$. Show that if there exists a map $f\colon \Sigma_k \to M \times \Sigma_g$ with $f_*([\Sigma_k]) = [\{P\} \times \Sigma_g] \in \mathrm{H}_2(M \times \Sigma_g)$, then $k \geq g$.

Exercise 102.4. Let $T = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$ be the torus. We denote by $x = [S^1 \times \{1\}]$ and $y = [\{1\} \times S^1]$ the obvious basis for $\mathrm{H}_1(T)$. Let $(r, s) \neq (0, 0) \in \mathbb{Z}^2$. Show that the following two statements are equivalent:

(a) The homology class $r \cdot x + s \cdot y$ can be represented by a connected submanifold of T.

(b) $\gcd(r, s) = 1$.

Hint. For the "(a)⇒(b)"-direction you might want to use the classification of 2-dimensional smooth manifolds.

Exercise 102.5. Let $g \in \mathbb{N}$. We consider the surface Σ of genus g. Let $\varphi \neq 0 \in \mathrm{H}_1(\Sigma)$. Show that the following two statements are equivalent:

(a) φ can be represented by a connected submanifold of Σ.
(b) φ is a primitive class in $\mathrm{H}_1(\Sigma)$.

Remark. The case $g = 1$ is evidently just the content of Exercise 102.4.

103. The Simplicial Poincaré Duality Theorem

In this chapter we will prove one of the most beautiful and powerful theorems in the theory of abstract simplicial complexes, namely the Simplicial Poincaré Duality Theorem 103.4. This theorem has many applications, arguably the most interesting ones concern the singular (co-) homology groups of smooth manifolds.

103.1. **Simplicial Homology Manifolds.** To state the Simplicial Poincaré Duality Theorem we will need the following notion.

Definition.

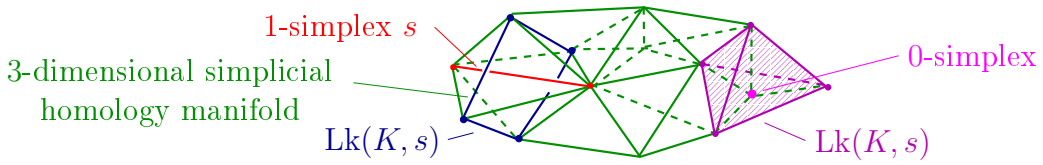
(1) We say that an abstract simplicial complex L is an simplicial homology m-sphere if L is m-dimensional, if L is finite and if for every $i \in \mathbb{N}_0$ we have

$$\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(L) \cong \begin{cases} \mathbb{Z}, & \text{if } i = m, \\ 0, & \text{else.} \end{cases}$$

Furthermore we say that L is a simplicial homology ball if $\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(L) = 0$ for all $i \in \mathbb{N}_0$.

(2) We say that an abstract simplicial complex K is an n-dimensional simplicial homology manifold if K is n-dimensional, if K is finite and if for every $k \in \mathbb{N}_0$ and every k-simplex s of K the link $\mathrm{Lk}(K, s)$ is a simplicial homology $(n-1-k)$-sphere or if it is a simplicial homology ball.

(3) We say a simplicial homology manifold K is closed, if K is finite and if for every k-simplex s of K the link $\mathrm{Lk}(K, s)$ is a simplicial homology $(n-1-k)$-sphere.



Remark. Note that by the discussion on page 2042 the empty abstract simplicial complex is a simplicial homology (-1)-sphere. Furthermore it is a straightforward consequence of the Reduced Homology Lemma 95.10 (1) that the empty abstract simplicial complex is in fact the only simplicial homology (-1)-sphere. It follows in particular that every simplex in a closed n-dimensional simplicial homology manifold has dimension at most n.

The following lemma gives a reformulation of the definition of a simplicial homology manifold.

Lemma 103.1. Let K be an abstract simplicial complex. As always we denote by $\mathrm{sd}(K)$ its barycentric subdivision. The following two statements are equivalent:

(1) K is a closed n-dimensional simplicial homology manifold.

(2) For every vertex v[1250] of the barycentric subdivision $\mathrm{sd}(K)$ the link $\mathrm{Lk}(\mathrm{sd}(K), v)$ is a simplicial homology $(n-1)$-sphere.

Proof. The lemma follows immediately from the second part of the following claim.

[1250] It is worth remembering that by definition the vertices of $\mathrm{sd}(K)$ are precisely the barycenters of the simplices of K.

Claim. Let K be an abstract simplicial complex and let s be a k-simplex of K.

(a) There is a simplicial isomorphism $\mathrm{Lk}(K,s) * \partial s \cong \mathrm{Lk}(\mathrm{sd}(K), \underline{s})$.

(b) For each $i \in \mathbb{N}_0$ we have $\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(\mathrm{Lk}(K,s)) \cong \widetilde{\mathrm{H}}_{i+k}^{\mathrm{simp}}(\mathrm{Lk}(\mathrm{sd}(K), \underline{s}))$.

Proof.

(a) This statement is actually the content of Exercise 94.2. For the reader's convenience we sketch the proof. So let us consider the map

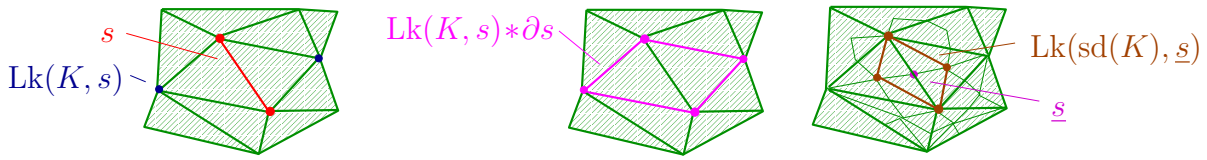
$$\begin{array}{rcl} \mathrm{Lk}(K,s) * \partial s & \to & \mathrm{Lk}(\mathrm{sd}(K), \underline{s}) \\ v & \mapsto & \begin{cases} s \setminus \{v\}, & \text{if } v \in \partial s, \\ \underline{s \sqcup \{v\}}, & \text{if } v \in \mathrm{Lk}(K,s). \end{cases} \end{array}$$

It is elementary to verify that this map on the level of vertices induces an isomorphism of abstract simplicial complexes.

(b) For any $i \in \mathbb{N}_0$ we have

$$\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(\mathrm{Lk}(K,s)) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_{i+k}^{\mathrm{simp}}(\mathrm{Lk}(K,s) * \partial s) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_{i+k}^{\mathrm{simp}}(\mathrm{Lk}(\mathrm{sd}(K), \underline{s})).$$

Join-Homology Lemma 95.13 (1) by (a) ■



Slightly annoyingly the notion of a "simplicial homology manifold" is now the third definition of a "manifold-like" structure on an abstract simplicial complex. Namely on page 2140 we introduced the notion of a pseudomanifold and on page 2093 we introduced the notion of a PL-manifold. For the reader's and also the author's convenience we now recall these two earlier notions. Thus let K be a finite non-empty abstract simplicial complex.

(1) K is a closed n-dimensional pseudomanifold if the following conditions are satisfied:
   (a) Every simplex of K is a face of an n-simplex of K.
   (b) Every $(n-1)$-simplex has order two, i.e. it is the face of precisely two n-dimensional faces.
   (c) Given any two n-simplices t and t' there exists a sequence $t = s_0, s_1, \ldots, s_k = t'$ of n-simplices such that any two consecutive simplices have a common $(n-1)$-dimensional face.

(2) By the PL-Manifold Characterization Proposition 96.12 we know that K is a closed n-dimensional PL-manifold if and only if for every k-simplex $s \in S$ the link $\mathrm{Lk}(K,s)$ is a PL $(n-k-1)$-sphere, i.e. $|\mathrm{Lk}(K,s)|$ is PL-homeomorphic to the simplicial complex $\partial\Delta^{n-k}$.[1251]

The following proposition summarizes everything anybody ever wanted to know about the relationship between this new concept and our two earlier concepts.

[1251] By the Alexander-Newman Theorem 96.22 the notion of a PL-manifold is basically equivalent to the notion of a combinatorial manifold that we introduced on page 2104. Since the notion of a PL-manifold is more commonly used we stick in the discussion to PL-manifold even though the notion of a combinatorial manifold would be slightly more appropriate.

Proposition 103.2.

(1) A closed connected non-empty n-dimensional simplicial homology manifold K is a closed n-dimensional pseudomanifold with $\partial K = \varnothing$.
(2) There exists a closed pseudomanifold that is not a simplicial homology manifold.
(3) Every closed non-empty n-dimensional PL-manifold is a closed n-dimensional simplicial homology manifold.
(4) There exists a closed simplicial homology manifold that is not a PL-manifold.

Remark. Slightly more informally Proposition 103.2 can be summarized as follows:

$$\text{PL-manifold} \quad \overset{\Longrightarrow}{\not\Longleftarrow} \quad \text{simplicial homology manifold} \quad \overset{\Longrightarrow}{\not\Longleftarrow} \quad \text{pseudomanifold.}$$

Proof.

(1) Let K be a closed connected non-empty n-dimensional simplicial homology manifold. We want to show that K is a closed n-dimensional pseudomanifold. The well-versed reader will easily spot that arguments below are at times almost identical to the arguments provided in the proof of the Manifold-Simplex Proposition 98.1.

Let us prove that the three conditions of a pseudomanifold, which we recalled above, are satisfied.

(a) Let s be a k-simplex of K. By hypothesis $\widetilde{\mathrm{H}}^{\mathrm{simp}}_{n-k-1}(\mathrm{Lk}(K,s)) \neq 0$. This implies immediately that s is the face of an n-simplex.

(b) Let s be an $(n-1)$-simplex of K. By hypothesis $\widetilde{\mathrm{H}}^{\mathrm{simp}}_{0}(\mathrm{Lk}(K,s)) \cong \mathbb{Z}$, which implies by the Reduced Homology Lemma 95.10 that $\mathrm{H}^{\mathrm{simp}}_{0}(\mathrm{Lk}(K,s)) \cong \mathbb{Z}^2$. Recall that on page 2261 we pointed out that the maximal dimension of a simplex of K is n. Since s is $(n-1)$-dimensional we see that $\mathrm{Lk}(K,s)$ is a 0-dimensional simplicial complex. Now we see that it consists of precisely two vertices, which implies that there are precisely two n-simplices which contain s as a face.

(c) We say that two n-simplices t and t' of K are *equivalent* if there exists a sequence $t = s_0, s_1, \ldots, s_k = t'$ of n-simplices such that any two consecutive simplices have a common $(n-1)$-dimensional face. Recall that our task is to show that any two n-simplices are equivalent in the above sense.
The key to doing so is the following claim:

Claim. Let t and t' be two n-simplices. If $t \cap t' \neq \varnothing$, then t and t' are actually equivalent in the above sense.

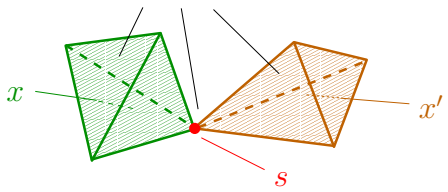
Proof. Given $m \in \{0, \ldots, n-1\}$ we say that two n-simplices x and x' are *m-equivalent* if there exists a sequence $x = s_0, s_1, \ldots, s_k = x'$ of n-simplices such that any two consecutive simplices have a common face of dimension $\geq m$. It suffices to show that if we are given $m \in \{0, \ldots, n-2\}$ such that any two n-simplices in K are m-equivalent, then any two n-simplices in K are also $(m+1)$-equivalent. In fact evidently it suffices to show that if we are given two n-simplices x and x' which have an m-simplex s in common, then x and x' are $(m+1)$-equivalent. To do so we first note that since $\dim(x) > \dim(s)$ we can pick a vertex w of $x \cap \mathrm{Lk}(K,s)$ and similarly we can pick a vertex w' of $x' \cap \mathrm{Lk}(K,s)$.

By hypothesis we know that $\widetilde{\mathrm{H}}_0^{\mathrm{simp}}(\mathrm{Lk}(K,s)) = 0$. It follows from this fact together with the Reduced Homology Lemma 74.1 and Exercise 93.12 that the abstract simplicial complex $\mathrm{Lk}(K,s)$ is in fact connected. This means in our setting that there exist 1-simplices $e_1, \dots, e_k \in S$ of $\mathrm{Lk}(K,s)$ such that the following holds:

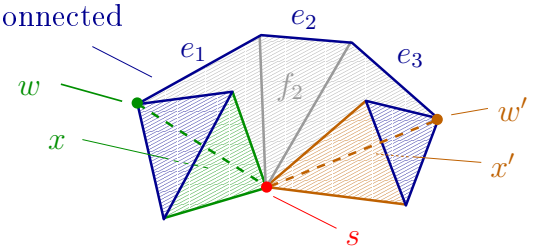
(∗) We have $w \in e_1$, for every $i \in \{1, \dots, k-1\}$ we have $e_i \cap e_{i+1} \neq \varnothing$ and we have $w' \in e_k$.

Let $i \in \{1, \dots, k\}$. Since e_i is a simplex in $\mathrm{Lk}(K,s)$ there exists a simplex f_i of K with $e_i \cup s \subset f_i$. By (a) we can in fact find an n-simplex $\widetilde{f}_i$ of K with $e_i \cup s \subset f_i \subset \widetilde{f}_i$. We set $\widetilde{f}_0 := x$ and $\widetilde{f}_{k+1} := x'$. Note that it follows from (∗) that any two consecutive $\widetilde{f}_i$ and $\widetilde{f}_{i+1}$ have s and a vertex of $\mathrm{Lk}(K,s)$ in common, i.e. they have an $(m+1)$-simplex in common. This shows that x and x' are in fact $(m+1)$-connected. ⊞

x and x' are 0-equivalent



$\mathrm{Lk}(K,s)$ is connected



Now let t and t' be any two n-simplices of K. By hypothesis K is connected. Together with (a) this implies easily that there exists a sequence of n-simplices $t = u_0, u_1, \dots, u_k = t'$ such that any two consecutive n-simplices share at least one vertex. By the claim any two consecutive n-simplices are equivalent. Hence t and t' are equivalent.

(2) Let K be a closed n-dimensional pseudomanifold. It follows from the discussion on page 2141 that the suspension $\Sigma(K) = K * \{\pm 1\}$ is a closed $(n+1)$-dimensional pseudomanifold. By the discussion on page 2018 we know that $\mathrm{Lk}(\Sigma(K), \{-1\}) = K$. Thus if $\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(K) \neq 0$ for two different dimensions, e.g. if $K = S_1 \times S_1$ is the simplicial torus, then $\Sigma(K)$ is not a simplicial homology manifold.

(3) Let K be a closed n-dimensional PL-manifold and let s be a k-simplex. We see that

$$\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(\mathrm{Lk}(K,s)) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_i(|\mathrm{Lk}(K,s)|) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_i(S^{n-k-1}) \cong \begin{cases} \mathbb{Z}, & \text{if } i = n-k-1, \\ 0, & \text{else.} \end{cases}$$

by the Simplicial-Singular Homology Isomorphism Theorem 95.23

since $\mathrm{Lk}(K,s)$ is a PL $(n-k-1)$-sphere we know that $|\mathrm{Lk}(K,s)|$ is homeomorphic to $\partial\Delta^{n-k}$ in particular $|\mathrm{Lk}(K,s)|$ is homeomorphic to S^{n-k-1}

(4) By Proposition 98.11 we know that there exists a closed orientable 3-dimensional smooth manifold Y that is a homology 3-sphere, but such that $\pi_1(Y)$ is a non-trivial group. By the Smooth Manifold-Simplicial Structure Theorem 96.2 and Theorem 96.15 we can equip Y with a PL-structure $(K, \Theta\colon |K| \to Y)$. We consider the suspension $\Sigma(K) = K * \{\pm 1\}$. We want to show that $\Sigma(K)$ is a closed 4-dimensional

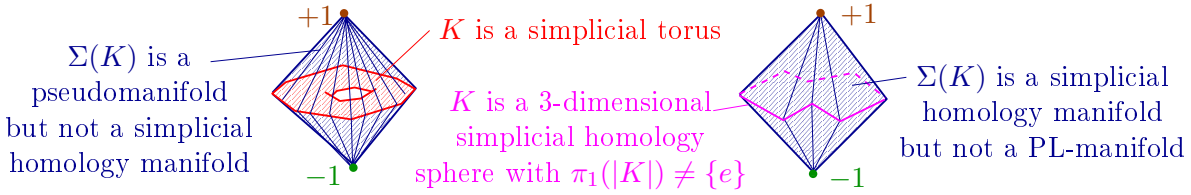
simplicial homology manifold but that it is not a PL-manifold. It turns out that the simplices of $\Sigma(K) = K * \{\pm 1\}$ come in the following three flavors:

(a) Any k-simplex s of K is also a k-simplex of $\Sigma(K)$. We have, almost by definition, that $\mathrm{Lk}(\Sigma(K), s) = \Sigma(\mathrm{Lk}(K, s))$. It follows from this observation, together with (1) and the Join-Homology Lemma 95.13 (2) that $\mathrm{Lk}(\Sigma(K), s)$ is a simplicial homology $(3-k)$-sphere.

(b) Given a k-simplex s of K we obtain the $(k+1)$-simplices $s \cup \{-1\}$ and $s \cup \{1\}$ of $\Sigma(K)$. Note that it follows almost immediately from the various definitions that $\mathrm{Lk}(\Sigma(K), s \cup \{-1\}) = \mathrm{Lk}(K, s) = \mathrm{Lk}(\Sigma(K), s \cup \{1\})$. Thus we obtain from (1) that $\mathrm{Lk}(\Sigma(K), s \cup \{-1\})$ and $\mathrm{Lk}(\Sigma(K), s \cup \{1\})$ are both simplicial homology $(3-(k+1))$-spheres.

(c) Finally we need to consider the 0-simplices ± 1 of $\Sigma(K) = K * \{\pm 1\}$. As we discussed on page 2018, we have $\mathrm{Lk}(\Sigma(K), \{\pm 1\}) = K$. Since K is a homology 3-sphere we see that $\mathrm{Lk}(\Sigma(K), \{\pm 1\}) = K$ are simplicial homology 3-spheres. But since $\pi_1(|K|) = \pi_1(Y)$ is by construction a non-trivial group we see that $|\mathrm{Lk}(\Sigma(K), \{\pm 1\})| = |K|$ are not homeomorphic to the 3-sphere, let alone are $\mathrm{Lk}(\Sigma(K), \{\pm 1\})$ PL 3-spheres or PL-balls.

Thus we see that $\Sigma(K)$ is a 4-dimensional simplicial homology manifold, but $\Sigma(K)$ is not a PL-manifold. ■



We conclude this section with the following lemma.

Lemma 103.3. Let K be an abstract simplicial complex. If $|K|$ is an n-dimensional topological manifold with $\partial(|K|) = \varnothing$, then K is an n-dimensional simplicial homology manifold.

Proof. Let s be a k-simplex of K. We need to show that $\mathrm{Lk}(K, s)$ is a simplicial homology $(n-k-1)$-sphere. We pick a point P in the corresponding open simplex $\langle s \rangle$. We perform the following calculation:

$$\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(\mathrm{Lk}(K,s)) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_{i+k+1}(|K|, |K| \setminus \{P\}) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } i+k+1 = n, \\ 0, & \text{else.} \end{cases}$$

by Proposition 98.2

by the Manifold-Local Homology Lemma 75.1, since $|K|$ is an n-dimensional topological manifold with $\partial(|K|) = \varnothing$

As a parenthetical remark we point out that alternatively we could have proved the lemma using Lemma 103.1 instead of using Proposition 98.2. ■

103.2. **The statement of the Simplicial Poincaré Duality Theorem.** Let K be a closed connected n-dimensional simplicial homology manifold. By Proposition 103.2 (1)

we know that K is in particular a closed n-dimensional pseudomanifold. In particular it makes sense to talk of orientations of K. We recall the notion of a fundamental class.

(1) If K is oriented, then by the definition on page 2143 we have a fundamental class $[K] \in \mathrm{H}_n^{\mathrm{simp}}(K)$. In the following, given a commutative ring R we denote by $[K]$ also the image of $[K]$ under the natural map $\mathrm{H}_n^{\mathrm{simp}}(K) \xrightarrow{[\sigma] \mapsto [\sigma \otimes 1_R]} \mathrm{H}_n^{\mathrm{simp}}(K;R)$.

(2) Regardless of whether or not K is orientable we defined on page 2143 the $\mathbb{F}_2$-fundamental class $[K]_{\mathbb{F}_2} \in \mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2)$.

The following theorem is the main theorem on simplicial (co-) homology groups of simplicial homology manifolds.

Theorem 103.4. (Simplicial Poincaré Duality Theorem) Let K be a closed n-dimensional simplicial homology manifold and let $k \in \mathbb{N}_0$. If K is orientable, then for every choice of orientation and every commutative ring R the map

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{simp}}(K;R) & \to & \mathrm{H}^{\mathrm{simp}}_{n-k}(K;R) \\ \varphi & \mapsto & \varphi \cap [K] \end{array}$$

is an isomorphism. Regardless of whether or not K is orientable the map

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{simp}}(K;\mathbb{F}_2) & \to & \mathrm{H}^{\mathrm{simp}}_{n-k}(K;\mathbb{F}_2) \\ \varphi & \mapsto & \varphi \cap [K]_{\mathbb{F}_2} \end{array}$$

is an isomorphism.

Remark. One might think or hope that the Simplicial Poincaré Duality Theorem 103.4 also holds for closed (orientable) pseudomanifolds, but as we will see in Exercise 103.1 this is not the case. Nonetheless, it turns out that on pseudomanifolds one can introduce more refined (co-) homology groups, namely intersection homology, that do satisfy an analogue of the Poincaré Duality Theorem. We refer to [**Ban2007**] for details.

The proof of the Simplicial Poincaré Duality Theorem 103.4 is evidently non-trivial and somewhat lengthy. Thus we will first draw several interesting conclusions before we head to the proof.

Example. Let K be a closed orientable n-dimensional simplicial homology manifold. Let us see what the Simplicial Poincaré Duality Theorem 103.4 tells us for $k = 1$. We have the following isomorphisms:

$$\begin{array}{rcl} \mathrm{H}^{\mathrm{simp}}_{n-1}(K) & \underset{\substack{\uparrow \\ \text{Poincaré Duality Theorem 103.4}}}{\cong} & \mathrm{H}^1_{\mathrm{simp}}(K;\mathbb{Z}) \underset{\substack{\uparrow \\ \text{Universal Coefficient Theorem 99.8}}}{\cong} \mathrm{Hom}(\mathrm{H}^{\mathrm{simp}}_1(K),\mathbb{Z}) \oplus \mathrm{Ext}(\overbrace{\mathrm{H}^{\mathrm{simp}}_0(K)}^{\text{free abelian group by Exercise 95.3}},\mathbb{Z}) \\ & \underset{\substack{\uparrow \\ \text{Ext-Calculation Lemma 110.5 (3)}}}{\cong} & \mathrm{Hom}(\mathrm{H}^{\mathrm{simp}}_1(K),\mathbb{Z}) \underset{\substack{\uparrow \\ \text{by Lemma 95.9 (1) we know that the group the group } \mathrm{H}^{\mathrm{simp}}_1(K) \text{ is finitely} \\ \text{generated, hence it is isomorphic to } \mathbb{Z}^k \oplus T \text{ for some } k \in \mathbb{N} \text{ and some finite group } T}}{\cong} \mathrm{Hom}(\mathbb{Z}^k \oplus T,\mathbb{Z}) \cong \mathbb{Z}^k. \end{array}$$

We have thus shown that $\mathrm{H}^{\mathrm{simp}}_{n-1}(K)$ is a free abelian group. In the more general setting of closed orientable pseudomanifolds we had obtained the same result in Theorem 98.4, but using a rather different argument.

Our richest source of simplicial homology manifolds are smooth manifolds. We obtain the following corollary to the Simplicial Poincaré Duality Theorem 103.4.

Theorem 103.5. (Poincaré Duality Theorem) Let M be a closed n-dimensional smooth manifold and let $k \in \mathbb{N}_0$. If M is orientable, then for every choice of orientation and every commutative ring R the map

$$\begin{array}{rcl} \mathrm{H}^k(M;R) & \to & \mathrm{H}_{n-k}(M;R) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

is an isomorphism. Regardless of whether or not M is orientable the map

$$\begin{array}{rcl} \mathrm{H}^k(M;\mathbb{F}_2) & \to & \mathrm{H}_{n-k}(M;\mathbb{F}_2) \\ \varphi & \mapsto & \varphi \cap [M]_{\mathbb{F}_2} \end{array}$$

is an isomorphism.

Proof. One can show easily (see e.g. the elementary future the Cup-Cap Disjoint Union Lemma 116.9) that we can assume that M is actually connected. Recall that by the Smooth Manifold-Simplicial Structure Theorem 96.2 we know that M admits a smooth simplicial structure $(K = (V,S), f\colon |K| \to M)$. Without loss of generality we can assume that $|K| = M$. We denote by $\Xi_*\colon \mathrm{H}_k^{\mathrm{simp}}(K;R) \to \mathrm{H}_k(M;R)$ and $\Xi^*\colon \mathrm{H}^k(M;R) \to \mathrm{H}_k^{\mathrm{simp}}(K;R)$ the natural isomorphisms provided by Proposition 99.1 (1) and 99.6 (3). By Lemma 103.3 we know that K is a closed n-dimensional simplicial homology manifold.

Let us first deal with the case that M is equipped with a natural orientation. By Proposition 98.7 we know that the simplicial homology manifold K inherits an orientation. Next we consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{simp}}(K;R) & \xrightarrow[\cong]{\cap[K]} & \mathrm{H}^{\mathrm{simp}}_{n-k}(K;R) \\ {\scriptstyle \Xi^*}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \Xi_*} \\ \mathrm{H}^k(M;R) & \xrightarrow{\cap[M]} & \mathrm{H}^{\mathrm{simp}}_{n-k}(M;R). \end{array}$$

We make the following observations:

(1) By Algorithm 100.2 we know that $\Xi_*([K]) = [M]$.
(2) It follows from (1) and Proposition 99.17 (3) that the diagram commutes.
(3) The top horizontal map is an isomorphism by the Simplicial Poincaré Duality Theorem 103.4.

Since the vertical maps are isomorphisms we obtain from the above that the bottom horizontal map is, as promised, an isomorphism.

The proof for the statement regarding $\mathbb{F}_2$-coefficients is almost identical. We leave it to the reader to make the necessary straightforward modifications to the above argument. ■

There are many applications of the Poincaré Duality Theorem 103.5. In the following we will state two results. Since we have a predilection for smooth manifolds we state them in the context of smooth manifolds, but the statements also hold in the admittedly more general setting of simplicial homology manifolds.

Theorem 119.6. (Co-Homology Symmetry Theorem) Let M be a closed n-dimensional smooth manifold and let $k \in \mathbb{N}_0$.

(1) We have $\dim_{\mathbb{F}_2}(\mathrm{H}_k(M;\mathbb{F}_2)) = \dim_{\mathbb{F}_2}(\mathrm{H}_{n-k}(M;\mathbb{F}_2))$.
(2) If M is orientable, then for any field $\mathbb{F}$ we have $\dim_{\mathbb{F}}(\mathrm{H}_k(M;\mathbb{F})) = \dim_{\mathbb{F}}(\mathrm{H}_{n-k}(M;\mathbb{F}))$.

Proposition 119.7. The Euler characteristic of any closed odd-dimensional smooth manifold is zero.

As the reader will have noticed, the numbers of these results correspond to Chapter 119. In fact, in Chapter 119 we will prove a more general Poincaré Duality Theorem 119.1 that applies also to topological manifolds, possibly with boundary. In Section 119.2 we discuss many consequences of the Poincaré Duality Theorem 119.1. For closed smooth manifolds all statements in Section 119.2 can also be deduced from the above Poincaré Duality Theorem 103.5. In particular this way we obtain the above results. To avoid repetition, and for the sake of brevity, we do not spell out the proofs twice.

There are many other applications of the Poincaré Duality Theorem 103.5. Arguably one of the most interesting applications is that the Poincaré Duality Theorem 103.5 allows us to compute the cup product of the real and complex projective spaces. This will be done in Chapter 121.

Remark. The Poincaré Duality Theorem 103.5 has two significant limitations and one aesthetic issue.

(1) The theorem only deals with closed smooth manifolds. In fact the approach taken in the proof of Poincaré Duality Theorem 103.4 can be generalized, with some effort, to deal with compact (oriented) smooth manifolds with boundary. We refer to [**SZ1994**, Satz 14.8.8] for details.
(2) The Poincaré Duality Theorem 103.5 applies only to smooth manifolds. There is though some room for improvement. More precisely, the Simplicial Poincaré Duality Theorem 103.4 together with Lemma 103.3 shows that we have Poincaré Duality for any closed oriented topological manifold that admits a simplicial structure. But by Manolescu's Theorem 104.35 we know that for every $n \geq 5$ there exists a closed orientable n-dimensional topological manifold that does not admit a simplicial structure. Thus the Poincaré Duality Theorem 103.4 does not imply that Poincaré Duality holds for all closed topological manifolds.
(3) Even for smooth manifolds the proof of the Poincaré Duality Theorem 103.5 is aesthetically problematic since it makes use of the somewhat artificial choice of a simplicial structure.

All three issues will be addressed by the Poincaré Duality Theorem 119.1 that we will state and prove in Chapter 119. Nonetheless, the (proof) of the Poincaré Duality Theorems 103.4 and 103.5 also have two advantages over the Poincaré Duality Theorem 119.1:

(1) The proofs in this chapter are much more visual, whereas the proof of the Poincaré Duality Theorem 119.1 is rather formal and it is difficult to get an intuition for "why" and "how" it works.
(2) More importantly, much later in Chapter **??** we will introduce a new invariant, namely Reidemeister torsion. The intermediate results in our proof of Poincaré Duality Theorem 103.4 will show that Reidemeister torsion of closed smooth manifolds is symmetric.

103.3. The barycentric subdivision.

The proof of the Simplicial Poincaré Duality Theorem 103.4 relies on a clever use of the barycentric subdivision of the simplicial homology

manifold. In this section we will recall the main definitions and statements on barycentric subdivisions which we will need later on in this chapter.
We recall the following definitions from pages 2014 and 2072.

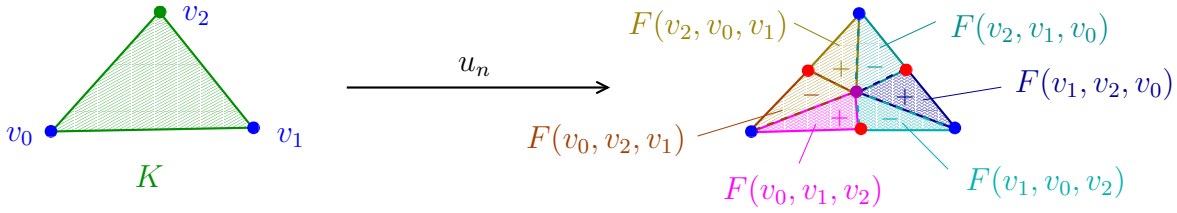
Definition. Let $K = (V, S)$ be an abstract simplicial complex.

(1) For each k-simplex $s = \{v_0, \ldots, v_k\} \in S$ we define the barycenter of s to be the point
$$\underline{s} \;:=\; \sum_{v \in s} \frac{1}{\dim(s)+1} \cdot v \;=\; \frac{1}{k+1} \cdot v_0 + \cdots + \frac{1}{k+1} \cdot v_k \;\in\; \mathbb{R}^{(V)}.$$

(2) The barycentric subdivision of K is defined to be the following abstract simplicial complex:
$$\mathrm{sd}(K) \;=\; \big(\{\underline{s}\}_{s \in S}, \{\{\underline{s_0}, \ldots, \underline{s_k}\} \,|\, s_0 \subsetneq \cdots \subsetneq s_k\}\big).$$

(3) Given an ordered k-simplex $(v_0, \ldots, v_k)$ of K we consider the following ordered k-simplex of $\mathrm{sd}(K)$:
$$F(v_0, \ldots, v_k) \;:=\; \big(\underline{\{v_0\}}, \underline{\{v_0, v_1\}}, \ldots, \underline{\{v_0, \ldots, v_k\}}\big).$$

(4) Given an ordered k-simplex $s = (v_0, \ldots, v_k)$ of K we set
$$u_k(s) \;:=\; \sum_{\sigma \in \mathrm{Bij}\{0,\ldots,k\}} (-1)^{\mathrm{sign}(\sigma)} \cdot [F(v_{\sigma(0)}, \ldots, v_{\sigma(k)})] \;\in\; \mathrm{C}_k^{\mathrm{simp}}(\mathrm{sd}(K)).$$
We refer to the resulting map
$$\begin{array}{rcl} u_k \colon \mathrm{C}_k^{\mathrm{simp}}(K) & \to & \mathrm{C}_k^{\mathrm{simp}}(\mathrm{sd}(K)) \\ [s] & \mapsto & u_k(s) \end{array}$$
as the simplicial subdivision map. By the Chain Subdivision Lemma 95.41 (1) we know that these subdivision maps are actually well-defined and that they form a chain map.



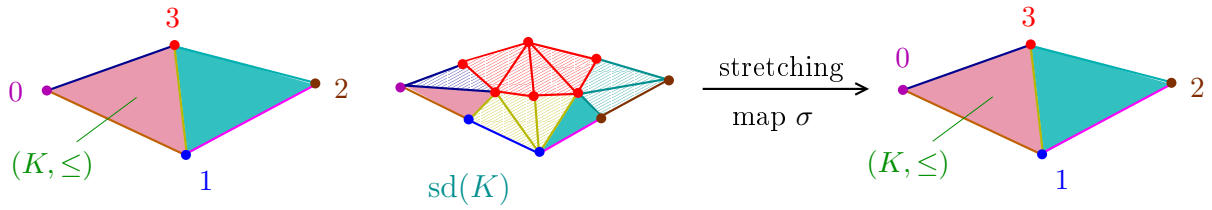
Remark. Let $K = (V, S)$ be an abstract simplicial complex. Note that it follows easily from the definitions that given any n-simplex t of $\mathrm{sd}(K)$ there exists a unique ordered n-simplex $(v_0, \ldots, v_n)$ of K with $\lfloor F(v_0, \ldots, v_n) \rfloor = t$.

The following lemma shows that subdivision maps induce isomorphisms on simplicial (co-) homology groups.

Lemma 95.38. Let $(K, \leq)$ be an ordered abstract simplicial complex.

(1) There exists a unique simplicial map $\sigma \colon \mathrm{sd}(K) \to K$, called the *stretching map*, with the following property: given an n-simplex $s = \{v_0 < \cdots < v_n\}$ of K the map σ sends the corresponding vertex $\underline{s}$ of $\mathrm{sd}(K)$ to the vertex v_n of K.

The following lemma is an excerpt of the Chain Subdivision Lemma 95.41 together with a little dose of the Dual Cochain Map Lemma 108.8 (4).



Lemma 95.41. Let K be an abstract simplicial complex. We pick an order on K and we consider the corresponding stretching map $\sigma\colon \mathrm{sd}(K) \to K$.

(5) For every choice of some $k \in \mathbb{N}_0$ and of some abelian group G the two induced maps $u_*\colon \mathrm{H}_k^{\mathrm{simp}}(K; G) \to \mathrm{H}_k^{\mathrm{simp}}(\mathrm{sd}(K); G)$ and $u^*\colon \mathrm{H}^k_{\mathrm{simp}}(\mathrm{sd}(K); G) \to \mathrm{H}^k_{\mathrm{simp}}(K; G)$ are isomorphisms and the inverses are given by σ_* and σ^*.

Next we turn to barycentric subdivisions of pseudomanifolds. At this point it is convenient to introduce the following definition.

Definition. Let $K = (V, S)$ be a closed oriented n-dimensional pseudomanifold. Given an ordered n-simplex t of K we write

$$\mathrm{sign}(t) := \begin{cases} +1, & \text{if the oriented } n\text{-simplex } [(+1, t)] \text{ corresponds to the orientation,} \\ -1, & \text{otherwise.} \end{cases}$$

The following lemma contains everything anybody ever wanted to know about barycentric subdivisions of closed pseudomanifolds.

Lemma 103.6. Let $K = (V, S)$ be a closed n-dimensional pseudomanifold.

(1) The barycentric subdivision $\mathrm{sd}(K)$ is a closed n-dimensional pseudomanifold.

(2) Suppose that K is equipped with an orientation. Let t be an n-simplex of $\mathrm{sd}(K)$. As we pointed out on page 2269, there exists a unique ordered n-simplex $(v_0, \ldots, v_n)$ of K with $\lfloor F(v_0, \ldots, v_n) \rfloor = t$. We equip the n-simplex t of $\mathrm{sd}(K)$ with the orientation given by $(\mathrm{sign}(t), F(v_0, \ldots, v_n))$.
   (a) The above orientations of the n-simplices define an orientation for $\mathrm{sd}(K)$.
   (b) We have $u_*([K]) = [\mathrm{sd}(K)] \in \mathrm{H}_n^{\mathrm{simp}}(\mathrm{sd}(K))$.

(3) If K is oriented, then given any $k \in \mathbb{N}_0$ and given any commutative ring R the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{simp}}(K; R) & \xrightarrow{\cap [K]} & \mathrm{H}^{\mathrm{simp}}_{n-k}(K; R) \\ {\scriptstyle u^*}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle u_*} \\ \mathrm{H}^k_{\mathrm{simp}}(\mathrm{sd}(K); R) & \xrightarrow{\cap [\mathrm{sd}(K)]} & \mathrm{H}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K); R). \end{array}$$

The analogues of (2) and (3) hold for arbitrary closed n-dimensional pseudomanifolds if we work with $\mathbb{F}_2$-coefficients and the simplicial $\mathbb{F}_2$-fundamental classes $[K]_{\mathbb{F}_2}$ and $[\mathrm{sd}(K)]_{\mathbb{F}_2}$.

Sketch of proof. Statements (1) and (2a) follow almost immediately from the definitions, the proof is outsourced to Exercise 98.5. Furthermore Statement (2b) can be deduced easily from the definition of the fundamental class on page 2143 and the definition of the subdivision map $u_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(K))$. We leave it to the reader to fill in the details.

Next we turn to the proof of (3). We pick an order "$\leq$" on K and we denote by $\sigma\colon \mathrm{sd}(K) \to K$ the corresponding stretching map. Note that by the above the Chain Subdivision Lemma 95.41 (5) we know that $u_* = \sigma_*^{-1}$ and $u^* = (\sigma^*)^{-1}$. It follows that it suffices to show that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{simp}}(K;R) & \xrightarrow{\quad\cap[K]\quad} & \mathrm{H}^{\mathrm{simp}}_{n-k}(K;R) \\ {\scriptstyle\sigma^*}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\uparrow{\scriptstyle\sigma_*} \\ \mathrm{H}^k_{\mathrm{simp}}(\mathrm{sd}(K);R) & \xrightarrow{\cap[\mathrm{sd}(K)]} & \mathrm{H}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K);R). \end{array}$$

Now let $\varphi \in \mathrm{H}^k_{\mathrm{simp}}(K;R)$. We see that we have the following equalities:

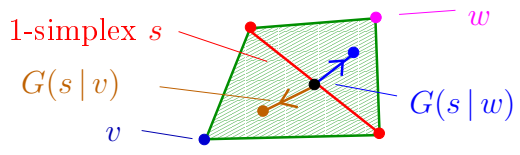
$$\sigma_*(\sigma^*(\varphi) \cap [\mathrm{sd}(K)]) \underset{\substack{\uparrow\\ \text{by Lemma 99.20, here we use}\\ \text{that } \sigma\colon \mathrm{sd}(K)\to K \text{ is a simplicial map}}}{=} \varphi \cap \sigma_*([\mathrm{sd}(K)]) \underset{\substack{\uparrow\\ \text{by (2b)}}}{=} \varphi \cap [K].$$

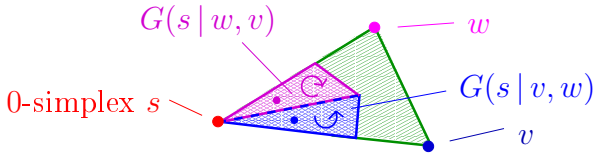
Finally we leave it to the reader to verify that the $\mathbb{F}_2$-analogues also hold. ∎

103.4. **The dual chain complex I.** In this section, given a pseudomanifold K, we will introduce the dual chain complex $\mathrm{D}^{\mathrm{simp}}_*(K)$. This is the main new concept needed for the proof of the Simplicial Poincaré Duality Theorem 103.4.

Definition. Let $K = (V, S)$, let $s \in S$ be a k-simplex and let $v_1, \ldots, v_l \in V$ such that $s \cup \{v_1, \ldots, v_l\}$ is a $(k+l)$-simplex of K. We set

$$G(s \,|\, v_1, \ldots, v_l) = \overbrace{\big(\underline{s}, \underline{s \cup \{v_1\}}, \ldots, \underline{s \cup \{v_1, \ldots, v_l\}}\big)}^{\text{ordered } l\text{-simplex of } \mathrm{sd}(K)}.$$





Remark. Note that for an ordered k-simplex $s = (v_0, \ldots, v_k)$ of K we have by definition $G(\{v_0\} \,|\, v_1, \ldots, v_k) = F(v_0, \ldots, v_k)$.

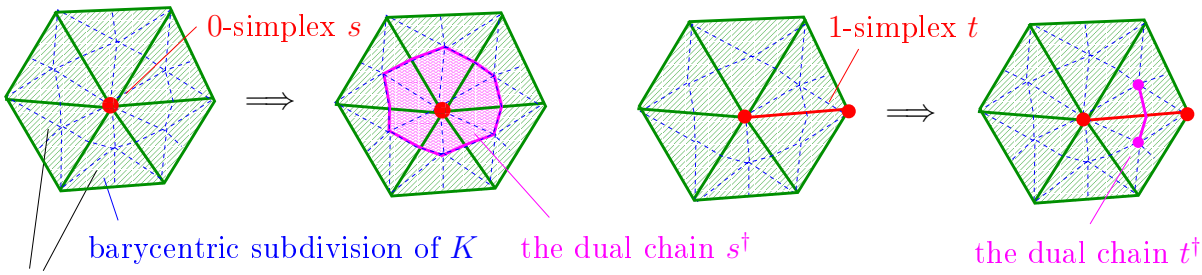
Definition. Let $K = (V, S)$ be a closed oriented n-dimensional pseudomanifold.

(1) Let $s = (v_0, \ldots, v_k)$ be an ordered k-simplex of K. We define the dual chain of s to be the following simplicial chain in $\mathrm{C}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K))$:

$$s^\dagger := \sum_{\substack{(v_{k+1}, \ldots, v_n) \in V^{n-k} \text{ such that} \\ \{v_0, \ldots, v_k, v_{k+1}, \ldots, v_n\} \in S_n}} \mathrm{sign}((v_0, \ldots, v_k, v_{k+1}, \ldots, v_n)) \cdot [\, \underbrace{G(s \,|\, v_{k+1}, \ldots, v_n)}_{\text{ordered } (n-k)\text{-simplex of } \mathrm{sd}(K)} \,].$$

(2) We define

$$\mathrm{D}^{\mathrm{simp}}_{n-k}(K) = \begin{array}{c}\text{subgroup of } \mathrm{C}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K)) \text{ generated by the } s^\dagger,\\ \text{where } s \text{ runs over all } k\text{-simplices of } K.\end{array}$$



part of the pseudomanifold K

Lemma 103.7. Let $K = (V, S)$ be a closed oriented n-dimensional pseudomanifold and let $k \in \mathbb{N}_0$.

(1) Given two ordered k-simplices s and $\widetilde{s}$ with $[s] = [\widetilde{s}] \in \mathrm{C}_k^{\mathrm{simp}}(K)$ we have $s^\dagger = \widetilde{s}^\dagger$.

(2) Let "$\leq$" be an order on $K = (V, S)$. The dual chains

$$\{s^\dagger \,|\, s = (v_0, \ldots, v_k) \text{ an ordered } k\text{-simplex of } K \text{ with } v_0 < \cdots < v_k\}$$

form a basis for $\mathrm{D}_{n-k}^{\mathrm{simp}}(K)$.

Proof. The first statement of the lemma follows easily from the definitions. Thus let us turn to the proof of the second statement. We pick an order "$\leq$" on $K = (V, S)$. We set

$$Y \;:=\; \{(v_0, \ldots, v_k) \in V^{k+1} \,|\, \{v_0, \ldots, v_k\} \in S_k \text{ and } v_0 < \cdots < v_k\}.$$

Recall that by the Manifold-Simplex Proposition 98.1 we know that in a closed n-dimensional pseudomanifold any simplex is the face of an n-simplex. It follows that given any $y = (v_0, \ldots, v_k) \in Y$ we can pick an ordered $(n-k)$-simplex $\widetilde{y} = (w_0, \ldots, w_{n-k})$ of K such that $v_k = w_0$, such that $\{v_0, \ldots, v_k = w_0, \ldots, w_{n-k}\} \in S_n$ and such that $w_1 < \cdots < w_{n-k}$. We make the following observation:

$(*)$ It follows easily from the definition of an order that for $y_1 \neq y_2 \in Y$ we have $\widetilde{y}_1 \neq \widetilde{y}_2$.

Now we consider the following maps:

$$\begin{array}{ccccc}
\bigoplus_{y \in Y} \mathbb{Z} \cdot y & \xrightarrow{y \mapsto y^\dagger} & \mathrm{D}_{n-k}^{\mathrm{simp}}(K) & \xhookrightarrow{\iota} & \mathrm{C}_{n-k}^{\mathrm{simp}}(\mathrm{sd}(K)) \\
\Big\downarrow{\scriptstyle y \,\updownarrow\, \widetilde{y}} & & & & \Big\uparrow{\scriptstyle \Phi} \;\; \Big\downarrow{\scriptstyle \Phi^{-1}} \\
\bigoplus_{\{\widetilde{y} \,|\, y \in Y\}} \mathbb{Z} \cdot \widetilde{y} & \xleftarrow[p]{\text{projection}} & \bigoplus_{\substack{(w_0, \ldots, w_{n-k}) \\ \text{ordered} \\ (n\text{-}k)\text{-simplex of } K}} \mathbb{Z} \cdot (w_0, \ldots, w_{n-k}) & & F(w_0, \ldots, w_{n-k}) \,\updownarrow\, (w_0, \ldots, w_{n-k})
\end{array}$$

We make the following observations:

(a) The right vertical map is an isomorphism. This follows from the observation, which we basically already mentioned on page 2269, that given any ordered $(n-k)$-simplex $(\underline{s}_0, \ldots, \underline{s}_{n-k})$ of $\mathrm{sd}(K)$ with $s_0 \subsetneq s_1 \subsetneq \cdots \subsetneq s_{n-k}$ there exists a unique ordered n-simplex $(w_0, \ldots, w_{n-k})$ of K with $F(w_0, \ldots, w_{n-k}) = (\underline{s}_0, \ldots, \underline{s}_{n-k})$.

(b) It follows immediately from the definitions of the various maps that the diagram commutes.

Now we turn to the actual proof of the second statement of the lemma. First note that it follows from (1) that $\{y^\dagger\}_{y\in Y}$ form a generating set for $\mathrm{D}^{\mathrm{simp}}_{n-k}(K)$. Secondly note that it follows from (a) and (b) that the map $p\circ\Phi^{-1}\circ\iota\colon \mathrm{D}^{\mathrm{simp}}_{n-k}(K)\to \bigoplus_{\{\widetilde{y}\,|\,y\in Y\}}\mathbb{Z}\cdot\widetilde{y}$ sends $y^\dagger$ to $\widetilde{y}$. It follows from $(*)$ that the $\{y^\dagger\}_{y\in Y}$ are linearly independent. Thus we have shown that the given dual chains $\{y^\dagger\}_{y\in Y}$ are a basis for $\mathrm{D}^{\mathrm{simp}}_{n-k}(K)$. ■

To fully appreciate our next lemma it is worth recalling the following lemma.

Lemma 99.4. Let $K=(V,S)$ be an abstract simplicial complex. Given any ordered k-simplex $s=(v_0,\ldots,v_k)$ we have the following equality:

$$\delta_k([v_0,\ldots,v_k]^*) \;=\; (-1)^{k+1}\cdot \sum_{\substack{w\in V \text{ such that}\\ \{v_0,\ldots,v_k,w\} \text{ is}\\ \text{a } (k+1)\text{-simplex}}} [v_0,\ldots,v_k,w]^* \;\in\; \mathrm{C}^{k+1}_{\mathrm{simp}}(K;\mathbb{Z}).$$

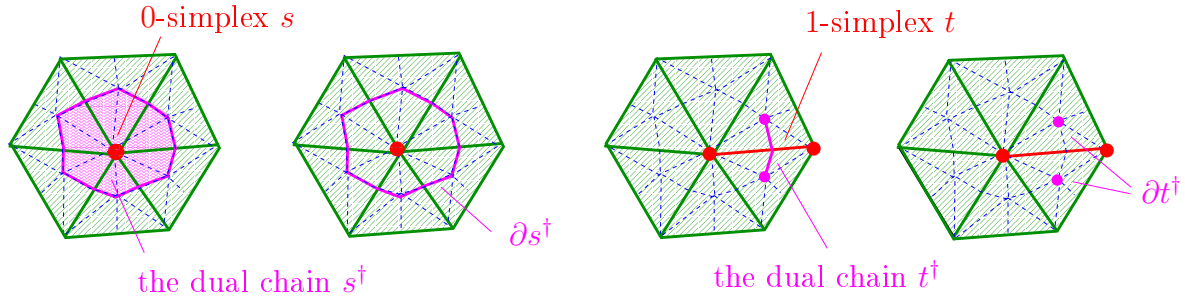
Now we can state the promised lemma. Its statement perfectly matches the statement of Lemma 99.4.

Lemma 103.8. Let $K=(V,S)$ be an abstract simplicial complex that is an orientable n-dimensional pseudomanifold.

(1) Given any ordered k-simplex $s=(v_0,\ldots,v_k)$ we have the following equality:

$$\partial_{n-k}(\underbrace{[v_0,\ldots,v_k]^\dagger}_{\in \mathrm{C}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K))}) \;=\; \sum_{\substack{w\in V \text{ such that}\\ \{v_0,\ldots,v_k,w\}\in S_{k+1}}} [v_0,\ldots,v_k,w]^\dagger \;\in\; \mathrm{C}^{\mathrm{simp}}_{n-k-1}(\mathrm{sd}(K)).$$

(2) The groups $\mathrm{D}^{\mathrm{simp}}_*(K)$ form a subcomplex of $\mathrm{C}^{\mathrm{simp}}_*(\mathrm{sd}(K))$.



Proof. The second statement is an immediate consequence of the first statement. Thus we just need to prove the first statement. We start out with the following elementary observation.

Observation. Let $s=(v_0,\ldots,v_k)$ be some ordered k-simplex of K and furthermore let $(v_{k+1},\ldots,v_n)\in V^{n-k}$ such that $s\cup\{v_{k+1},\ldots,v_n\}=\{v_0,\ldots,v_k,v_{k+1},\ldots,v_n\}\in S_n$. Given any $i\in\{1,\ldots,n-k-1\}$ we have by definition

$$G(s\,|\,v_{k+1},..,v_n) \text{ with } i\text{-th vertex removed} = (\underline{s},..,\underline{s\cup\{v_{k+1},..,v_{k+i-1}\}},\underline{s\cup\{v_{k+1},..,v_{k+i+1}\}},..,\underline{s\cup\{v_{k+1},..,v_n\}}).$$

Furthermore it follows immediately from the definitions that we have

$$\begin{aligned} G(s\,|\,v_{k+1},..,v_n) \text{ with 0-th vertex removed} &= G(s\cup\{v_{k+1}\}\,|\,v_{k+2},\ldots,v_n)\\ G(s\,|\,v_{k+1},..,v_n) \text{ with } (n-k)\text{-th vertex removed} &= G(s\,|\,v_{k+1},v_{k+2},\ldots,v_{n-1}).\end{aligned}$$

Now we turn to the actual proof of the first statement. Thus let $s = (v_0, \dots, v_k)$ be an ordered k-simplex of K. We see that

definition of $s^\dagger$ and definition of the simplicial boundary map

$$\partial s^\dagger \overset{\downarrow}{=} \sum_{\substack{(v_{k+1},\dots,v_n)\in V^{n-k} \text{ such that} \\ \{v_0,\dots,v_k,v_{k+1},\dots,v_n\}\in S_n}} \sum_{i=0}^{n-k} (-1)^i \cdot \operatorname{sign}(v_0,\dots,v_n) \cdot \begin{bmatrix} G(s \,|\, v_{k+1},\dots,v_n) \text{ with} \\ i\text{-th vertex removed} \end{bmatrix}$$

$$= \sum_{\substack{(v_{k+1},\dots,v_n)\in V^{n-k} \text{ such that} \\ \{v_0,\dots,v_k,v_{k+1},\dots,v_n\}\in S_n}} \sum_{i=0 \text{ or } i=n-k} (-1)^i \cdot \operatorname{sign}(v_0,\dots,v_n) \cdot \begin{bmatrix} G(s \,|\, v_{k+1},\dots,v_n) \text{ with} \\ i\text{-th vertex removed} \end{bmatrix}$$

given a fixed $(v_{k+1},\dots,v_n) \in V^{n-k}$ with $\{v_0,\dots,v_k,v_{k+1},\dots,v_n\} \in S_n$ and given $i \in \{1,\dots,n-k-1\}$ it follows almost immediately from the above observation that the contributions by $((v_{k+1},\dots,v_{k+i},v_{k+i+1},\dots,v_n),i)$ and $((v_{k+1},\dots,v_{k+i+1},v_{k+i},\dots,v_n),i+1)$ cancel, in other words, for $i = 1,\dots,n-k-1$ all terms cancel

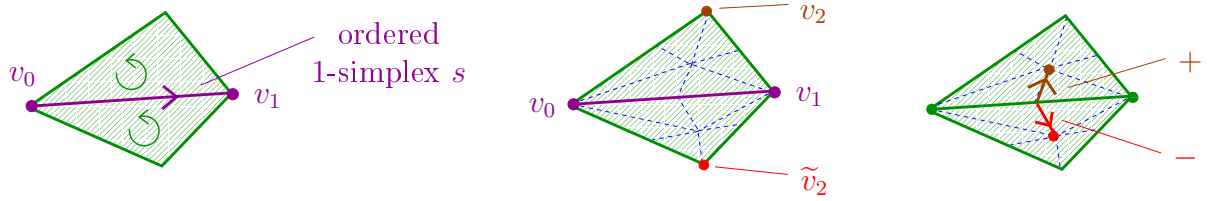
$$= \sum_{\substack{(v_{k+1},\dots,v_n)\in V^{n-k} \text{ such that} \\ \{v_0,\dots,v_k,v_{k+1},\dots,v_n\}\in S_n}} \operatorname{sign}(v_0,\dots,v_n) \cdot \begin{bmatrix} G(s \,|\, v_{k+1},\dots,v_n) \text{ with} \\ 0\text{-th vertex removed} \end{bmatrix}$$

by definition of a closed pseudomanifold every $(n-1)$-simplex has order 2, since the pseudomanifold is oriented we obtain from the above observation that all contributions for $i = n-k$ cancel

$$= \sum_{\substack{v_{k+1}\in V \text{ such that} \\ \{v_0,..,v_k,v_{k+1}\}\in S_{k+1}}} \sum_{\substack{(v_{k+2},..,v_n)\in V^{n-k-1} \text{ such that} \\ \{v_0,..,v_k,v_{k+1},v_{k+2},..,v_n\}\in S_n}} \operatorname{sign}(v_0,..,v_n) \cdot [G(s \cup \{v_{k+1}\} \,|\, v_{k+2},..,v_n)]$$

follows from fact that any subset of a simplex is again a simplex, we also use the above observation to rewrite the summand

$$= \sum_{\substack{w\in V \text{ such that} \\ \{v_0,\dots,v_k,w\}\in S_{k+1}}} [v_0,\dots,v_k,w]^\dagger. \qquad \blacksquare$$



Using 103.8 we can now easily prove the following proposition which is one of the key steps in the proof of the Simplicial Poincaré Duality Theorem 103.4.

Proposition 103.9. Let $K = (V, S)$ be a closed oriented n-dimensional pseudomanifold.

(1) For each $k \in \mathbb{N}_0$ the map

$$\begin{array}{rccc} \Lambda\colon & \operatorname{Hom}(\mathrm{C}_k^{\mathrm{simp}}(K), \mathbb{Z}) & \to & \mathrm{D}_{n-k}^{\mathrm{simp}}(K) \\ & \sum_{i=1}^m a_i \cdot s_i^* & \mapsto & \sum_{i=1}^m a_i \cdot s_i^\dagger \end{array}$$

here s_i is a k-simplex of K and we define $s_i^* \colon \mathrm{C}^k_{\mathrm{simp}}(K) \to \mathbb{Z}$ as on page 2174

is a natural isomorphism of abelian groups.

(2) For each $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}(\mathrm{C}_k^{\mathrm{simp}}(K), \mathbb{Z}) & \xrightarrow[\Lambda]{s^* \mapsto s^\dagger} & \mathrm{D}_{n-k}^{\mathrm{simp}}(K) \\ \downarrow{\scriptstyle (-1)^k \cdot \delta_k = (-1)^k \cdot \partial_{k+1}^*} & & \downarrow{\scriptstyle \partial_{n-k}} \\ \mathrm{Hom}(\mathrm{C}_{\mathrm{simp}}^{k+1}(K), \mathbb{Z}) & \xrightarrow[\Lambda]{s^* \mapsto s^\dagger} & \mathrm{D}_{n-k-1}^{\mathrm{simp}}(K). \end{array}$$

(3) The maps

$$\begin{array}{rcl} \Lambda_* \colon \mathrm{H}_{\mathrm{simp}}^k(K; \mathbb{Z}) & \to & \mathrm{H}_{n-k}(\mathrm{D}_*^{\mathrm{simp}}(K)) \\ \left[\sum_{i=1}^m a_i \cdot s_i^*\right] & \mapsto & \left[\sum_{i=1}^m a_i \cdot s_i^\dagger\right] \end{array}$$

are well-defined and they are natural isomorphisms.

Proof.

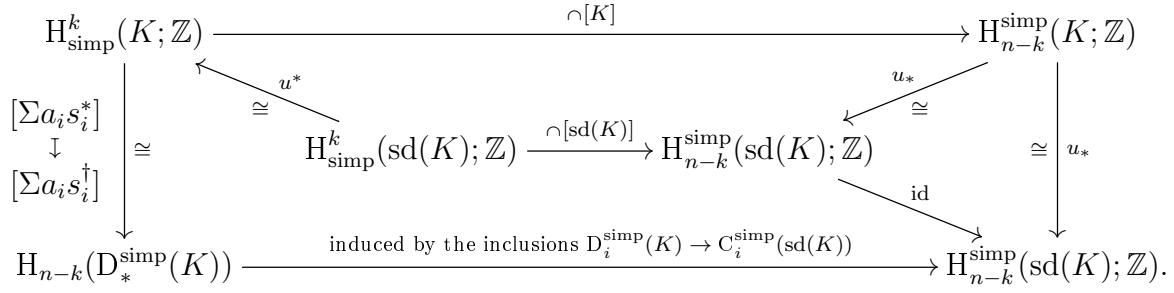
(1) First note that K is by definition a finite abstract simplicial complex. Thus it follows immediately from Lemma 99.2 and Lemma 103.7 (2) that the maps Λ are isomorphisms. Basically by definition the maps Λ are natural.

(2) Let $k \in \mathbb{N}_0$ and let $s = (v_0, \ldots, v_k)$ be an ordered k-simplex of K. We calculate that

$$\partial_{n-k}(\Lambda(s)) = \partial_{n-k}([v_0, .., v_k]^\dagger) \overset{\text{Lemma 103.8}}{=} \sum_{\substack{w \in V \text{ such that} \\ \{v_0, \ldots, v_k, w\} \in S_{k+1}}} [v_0, \ldots, v_k, w]^\dagger \overset{\text{Lemma 99.4}}{=} (-1)^k \cdot \Lambda(\delta_k([v_0, .., v_k]^*).$$

(3) In (2) we showed that the maps Λ are cochain maps "up to sign", i.e. they are generalized cochain maps in the sense of the discussion on page 2398. As we pointed out on page 2398, this sign issue is completely irrelevant and we still get a well-defined map on cohomology groups. Furthermore, by (1) we know that the maps on each cochain level are isomorphisms, thus we get in fact an isomorphism of cohomology groups. ■

It is a good moment to keep stock of where we are in the proof of the Simplicial Poincaré Duality Theorem 103.4. Let K be a closed oriented n-dimensional simplicial homology manifold and let $k \in \mathbb{N}_0$. Let us consider the following diagram:



Our main goal is to show that the top horizontal map is an isomorphism. By now we already know the following:

(1) By Proposition 103.9 we now know that the left vertical map is an isomorphism.

(2) By the Chain Subdivision Lemma 95.41 we know that the subdivision maps u_* and u^* are isomorphisms.

(3) By Lemma 103.6 (3) we know that the upper quadrilateral commutes.

Thus to prove the core of the Simplicial Poincaré Duality Theorem 103.4 it remains to carry out the following steps:

(4) We need to show that the lower left part of the diagram commutes.

(5) We need to show that the bottom horizontal map is an isomorphism.

As we will see in a second, Step (4) follows from a fairly straightforward calculation. Step (5) is much more difficult and we postpone Step (5) to the next section. Note though that Step (5) is particularly interesting since so far we have only used that K is a closed n-dimensional pseudomanifold. We did not yet use the hypothesis that we are dealing with a simplicial homology manifold.

As promised, here is the lemma that takes care of Step (4).

Lemma 103.10. Given any closed oriented n-dimensional pseudomanifold K the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{simp}}(K;\mathbb{Z}) & \xleftarrow[\cong]{u^*} & \mathrm{H}^k_{\mathrm{simp}}(\mathrm{sd}(K);\mathbb{Z}) \\ {\scriptstyle [\Sigma a_i s_i^*]} \updownarrow {\scriptstyle [\Sigma a_i s_i^\dagger]} \Big\downarrow \cong & & \Big\downarrow {\scriptstyle \cap[\mathrm{sd}(K)]} \\ \mathrm{H}_{n-k}(\mathrm{D}^{\mathrm{simp}}_*(K)) & \longrightarrow & \mathrm{H}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K);\mathbb{Z}). \end{array}$$

Proof. Let $K = (V, S)$ be a closed oriented n-dimensional pseudomanifold. Given two k-simplices s and t of K we write $\underline{s} \leq \underline{t}$ if $s \subset t$. As we pointed out in Lemma 94.2, this order on the vertex set of $\mathrm{sd}(K)$ turns $\mathrm{sd}(K)$ naturally into an ordered abstract simplicial complex. In the following we use the natural isomorphisms $\mathrm{C}^{\mathrm{simp},\leq}_k(\mathrm{sd}(K)) \to \mathrm{C}^{\mathrm{simp}}_k(\mathrm{sd}(K))$ and $\mathrm{C}^k_{\mathrm{simp}}(\mathrm{sd}(K);\mathbb{Z}) \to \mathrm{C}^k_{\mathrm{simp},\leq}(\mathrm{sd}(K);\mathbb{Z})$ to identify these two (co-) chain complexes and throughout the proof we will work with $\mathrm{C}^{\mathrm{simp}}_k(\mathrm{sd}(K))$ and $\mathrm{C}^k_{\mathrm{simp}}(\mathrm{sd}(K))$.

We set

$$\mu := \sum_{\substack{(u_0,\dots,u_n) \text{ ordered} \\ n\text{-simplex of } K}} \mathrm{sign}(u_0,\dots,u_n) \cdot [\underline{\{u_0\}}, \underline{\{u_0,u_1\}}, \dots, \underline{\{u_0,\dots,u_n\}}] \in \mathrm{C}^{\mathrm{simp}}_n(\mathrm{sd}(K)).$$

It follows immediately from Lemma 103.6 and Theorem 98.4 that μ is a simplicial cycle that represents the simplicial fundamental class $[\mathrm{sd}(K)] \in \mathrm{H}^{\mathrm{simp}}_n(\mathrm{sd}(K))$.

Next we pick an order "$\leq$" on K and we denote by $\sigma\colon \mathrm{sd}(K) \to K$ the corresponding stretching map, as defined on page 2269. Recall that by the Chain Subdivision Lemma 95.41 we know that $\sigma^*\colon \mathrm{H}^k_{\mathrm{simp}}(K;\mathbb{Z}) \to \mathrm{H}^k_{\mathrm{simp}}(\mathrm{sd}(K);\mathbb{Z})$ and $u^*\colon \mathrm{H}^k_{\mathrm{simp}}(\mathrm{sd}(K);\mathbb{Z}) \to \mathrm{H}^k_{\mathrm{simp}}(K;\mathbb{Z})$ are inverses of one another. Thus it remains to prove the following claim.[1252]

[1252] Here, as on many other occasions we will use the remark on page 2046 to make the identification $\mathrm{C}^{\mathrm{simp}}_*(\mathrm{sd}(K)) = \mathrm{C}^{\mathrm{simp}}_*(\mathrm{sd}(K);\mathbb{Z})$.

Claim. The following diagram commutes:[1253]

$$\begin{array}{ccc} \mathrm{C}^k_{\mathrm{simp}}(K;\mathbb{Z}) & \xrightarrow{\quad\sigma^*\quad} & \mathrm{C}^k_{\mathrm{simp}}(\mathrm{sd}(K);\mathbb{Z}) \\ \Sigma a_i s_i^* \downarrow \Sigma a_i s_i^\dagger \;\cong & & \downarrow \cap\mu \\ \mathrm{D}^{\mathrm{simp}}_{n-k}(K) & \hookrightarrow & \mathrm{C}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K)). \end{array}$$

Proof. The verification of the claim consists of a careful, but at the end of the day elementary, calculation. Note that by Lemma 99.2 (2) it suffices to show that for every ordered k-simplex $s = (v_0, \dots, v_k)$ of K with $v_0 < \cdots < v_k$ we have the following equality:

$$\sigma^*(s^*) \cap \mu \;=\; s^\dagger \;\in\; \mathrm{C}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K)).$$

To verify this equality it is helpful to first do one specific calculation separately. Namely let $(z_0, \dots, z_k)$ be some other ordered k-simplex of S. We perform the following little calculation:

$$\sigma^*(s^*)([\underline{\{z_0\}}, \dots, \underline{\{z_0, \dots, z_k\}}]) \overset{\text{definition of } \sigma^*}{=} s^*(\sigma_*([\underline{\{z_0\}}, \dots, \underline{\{z_0, \dots, z_k\}}]))$$

$$= \begin{cases} (v_0, \dots, v_k)^*(z_0, \dots, z_k), & \text{if } z_0 < \dots < z_k, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & \text{if } (v_0, \dots, v_k) = (z_0, \dots, z_k), \\ 0, & \text{otherwise.} \end{cases}$$

by the definition of the stretching map we see that $\{\sigma(\underline{\{z_0\}}), \dots, \sigma(\underline{\{z_0, \dots, z_k\}})\}$ is a k-simplex only if $z_0 < \cdots < z_k$, furthermore if $z_0 < \cdots < z_k$, then $\sigma(\underline{\{z_0\}}, \dots, \underline{\{z_0, \dots, z_k\}})) = (z_0, \dots, z_k)$

Now we can verify that $\sigma^*(s^*) \cap \mu = s^\dagger$. Indeed:

$$\sigma^*(s^*) \cap \mu \;=\; \sigma^*(s^*) \cap \Big(\sum_{\substack{(z_0,..,z_n) \text{ ordered} \\ n\text{-simplex of } K}} \mathrm{sign}(z_0,..,z_n) \cdot [\underline{\{z_0\}}, \underline{\{z_0, z_1\}}, .., \underline{\{z_0, .., z_n\}}] \Big)$$

(definition of μ)

by definition of the cap product, here we use that $\underline{\{z_0\}} < \underline{\{z_0, z_1\}} < \dots < \underline{\{z_0, .., z_n\}}$

$$= \sum_{\substack{(z_0,..,z_n) \text{ ordered} \\ n\text{-simplex of } K}} \mathrm{sign}(z_0,..,z_n) \cdot \sigma^*(s^*)([\underline{\{z_0\}}, .., \underline{\{z_0, .., z_k\}}]) \cdot \underbrace{[\underline{\{z_0, .., z_{k+1}\}}, .., \underline{\{z_0, .., z_n\}}]}_{= G(\{z_0,..,z_k\} \,|\, z_{k+1},..,z_n)}$$

$$= \sum_{\substack{(z_0,..,z_n) \text{ ordered} \\ n\text{-simplex of } K \text{ with} \\ (z_0,..,z_k) = (v_0,..,v_k)}} \mathrm{sign}(z_0,..,z_n) \cdot G(\underbrace{\{v_0,..,v_k\}}_{=s} \,|\, z_{k+1},..,z_n) \;=\; s^\dagger.$$

by the above calculation

by definition of $s^\dagger$, see page 2271

■

103.5. **The dual chain complex II.** As mentioned in the last section, the following proposition is the last major puzzle-piece in the proof of the Simplicial Poincaré Duality Theorem 103.4.

Proposition 103.11. Let K be a closed n-dimensional simplicial homology manifold.

(1) The inclusion map $\mathrm{D}^{\mathrm{simp}}_*(K) \to \mathrm{C}^{\mathrm{simp}}_*(\mathrm{sd}(K))$ is a chain homotopy equivalence.

[1253] It follows from Proposition 103.9 and the formula in the proof of Lemma 99.16 that the two vertical maps are (co-) chain maps "up to the sign $(-1)^k$".

(2) For every abelian group G the induced maps $\mathrm{H}_k(\mathrm{D}^{\mathrm{simp}}_*(K)\otimes G)\to \mathrm{H}^{\mathrm{simp}}_k(\mathrm{sd}(K);G)$ are isomorphisms.

Remark.

(1) Note that in Proposition 103.11 we finally work with simplicial homology manifolds and not just with pseudomanifolds.
(2) Note that in Proposition 103.11 we do not demand that K is orientable.
(3) In Section 103.7 we will see that the dual chain complex $\mathrm{D}^{\mathrm{simp}}_*(K)$ can be viewed as the cellular chain complex of a "dual CW-structure" on $|K|$. The proof of Proposition 103.11 is in fact similar in spirit to the proof, given in the Singular-Cellular H_*-Isomorphism Proposition 80.4 and the Singular-Cellular Chain Complex Proposition 81.12 (2), that the cellular chain complex of a CW-complex X is chain homotopy equivalent to the singular chain complex of X.

Definition. Let $K=(V,S)$ be an n-dimensional abstract simplicial complex.

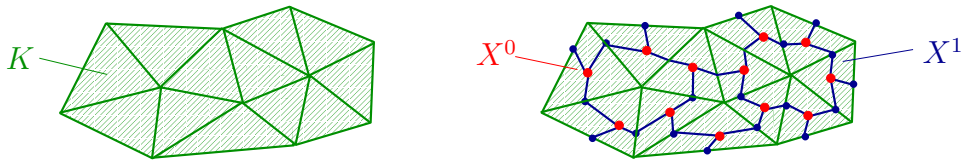
(1) Given $k\in\mathbb{N}_0$ we define

$$X^k := \begin{array}{c}\text{all simplices of } \mathrm{sd}(K) \text{ of the form } G(s\,|\,v_1,\ldots,v_l)\\ \text{where } s \text{ is a simplex of } S \text{ of dimension at least } n-k\end{array}$$

It follows immediately from the definitions that X^k can be viewed as a k-dimensional subcomplex of $\mathrm{sd}(K)$ and that $X^n=\mathrm{sd}(K)$.

(2) Let $k\in\mathbb{N}_0$. Similar to the discussion on page 1769 we consider the map

$$\begin{array}{rcl} d_k\colon \mathrm{H}^{\mathrm{simp}}_k(X^k,X^{k-1}) & \to & \mathrm{H}^{\mathrm{simp}}_k(X^{k-1},X^{k-2})\\ [\sigma] & \mapsto & [\partial_k(\sigma)].\\ \uparrow & & \\ \text{where } \sigma\in \mathrm{C}^{\mathrm{simp}}_k(X^k) & & \end{array}$$



The following lemma can be viewed as a partial analogue of the Skeleton-Homology Lemma 80.1.

Lemma 103.12. Let $K=(V,S)$ be a closed n-dimensional simplicial homology manifold and let $k\in\mathbb{N}_0$.

(1) For $i\neq k$ we have $\mathrm{H}^{\mathrm{simp}}_i(X^k,X^{k-1})=0$.
(2) The map

$$\begin{array}{rcl}\Theta\colon \mathrm{D}^{\mathrm{simp}}_k(K) & \to & \mathrm{H}^{\mathrm{simp}}_k(X^k,X^{k-1})\\ \sigma & \mapsto & [\sigma]\end{array}$$

is well-defined, i.e. every chain $\sigma\in\mathrm{D}^{\mathrm{simp}}_k(K)$ does indeed represent a cycle in $\mathrm{C}^{\mathrm{simp}}_k(X^k,X^{k-1})$. Furthermore the map Θ is an isomorphism.

(3) The maps Θ introduced in (2) define an isomorphism of chain complexes

$$(\mathrm{D}^{\mathrm{simp}}_k(K),\partial_k) \to (\mathrm{H}^{\mathrm{simp}}_k(X^k,X^{k-1}),d_k).$$

Proof. Let $K=(V,S)$ be a closed n-dimensional simplicial homology manifold. Given $l\in\mathbb{N}_0$ we denote by S_l the set of l-simplices of K. Given an $(n-k)$-simplex $s\in S_{n-k}$ we

define the following:[1254]

$$\begin{aligned} Z(s) &= \{ \text{ all simplices of } \mathrm{sd}(K) \text{ of the form } G(t \,|\, v_1, \ldots, v_l) \text{ with } s \subset t \} \subset X^k \\ \partial Z(s) &= \{ \text{ all simplices of } \mathrm{sd}(K) \text{ of the form } G(t \,|\, v_1, \ldots, v_l) \text{ with } s \subsetneq t \} \subset X^{k-1}. \end{aligned}$$



Claim 1. Let s be an $(n-k)$-simplex of K. Given any $l \in \mathbb{N}$ we have

$$\widetilde{\mathrm{H}}_l^{\mathrm{simp}}(Z(s)) = 0 \qquad \text{and} \qquad \widetilde{\mathrm{H}}_l^{\mathrm{simp}}(\partial Z(s)) \cong \begin{cases} \mathbb{Z}, & \text{if } l = k-1, \\ 0, & \text{otherwise.} \end{cases}$$

In particular $\partial Z(s)$ is a simplicial homology $(k-1)$-sphere.

Proof. Let s be an $(n-k)$-simplex of K. We start out with the following little observation:

(∗) Note that $Z(s)$ and $\mathrm{sd}(\partial s)$ are subcomplexes of $\mathrm{St}(\mathrm{sd}(K), \underline{s})$ and that $\partial Z(s)$ and $\mathrm{sd}(\partial s)$ are subcomplexes of $\mathrm{Lk}(\mathrm{sd}(K), \underline{s})$. It is now elementary to verify that the natural maps on the vertex sets induce simplicial isomorphisms

$$Z(s) * \mathrm{sd}(\partial s) \xrightarrow{\cong} \mathrm{St}(\mathrm{sd}(K), \underline{s}) \quad \text{and} \quad \partial Z(s) * \mathrm{sd}(\partial s) \xrightarrow{\cong} \mathrm{Lk}(\mathrm{sd}(K), \underline{s}).$$

We leave it to the reader to go over the details. It might be helpful to have a look at the figure on page 2279 to do a reality check.

Now let us turn to the actual calculation of the simplicial homology groups of $Z(s)$ and $\partial Z(s)$. Thus let $l \in \mathbb{N}_0$. We perform the following two calculations:

$$\widetilde{\mathrm{H}}_l^{\mathrm{simp}}(Z(s)) \underset{\uparrow}{=} \widetilde{\mathrm{H}}_{l+n-k}^{\mathrm{simp}}(Z(s) * \mathrm{sd}(\partial s)) \underset{\uparrow}{=} \widetilde{\mathrm{H}}_{l+n-k}^{\mathrm{simp}}(\overbrace{\mathrm{St}(\mathrm{sd}(K), \underline{s})}^{=\mathrm{Cone}(\mathrm{Lk}(\mathrm{sd}(K), \underline{s}))}) \underset{\uparrow}{=} 0.$$

since s is an $(n-k)$-simplex we obtain from the Chain Subdivision Lemma 95.41 that $\widetilde{\mathrm{H}}_{n-k-1}^{\mathrm{simp}}(\mathrm{sd}(\partial s)) \cong \mathbb{Z}$ and that $\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(\mathrm{sd}(\partial s)) = 0$ for $i \neq n-k-1$, the statement now follows from Exercise 95.14

by (∗)

by the Link Star Lemma 94.5 together with the Cone Homology Lemma 95.11

$$\widetilde{\mathrm{H}}_l^{\mathrm{simp}}(\partial Z(s)) \underset{\downarrow}{=} \widetilde{\mathrm{H}}_{l+n-k}^{\mathrm{simp}}(\partial Z(s) * \mathrm{sd}(\partial s)) \underset{\downarrow}{=} \widetilde{\mathrm{H}}_{l+n-k}^{\mathrm{simp}}(\mathrm{Lk}(\mathrm{sd}(K), \underline{s})) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, \text{ if } l+n-k=n-1, \\ 0, \text{ otherwise.} \end{cases}$$

by our hypothesis that K is an n-dimensional simplicial homology manifold together with Lemma 103.1 we know that $\mathrm{Lk}(\mathrm{sd}(K), \underline{s})$ is an $(n-1)$-dimensional simplicial homology sphere ⊞

Claim 2. Let $k \in \mathbb{N}_0$. The map

$$\bigoplus_{s \in S_{n-k}} j_s \colon \bigoplus_{s \in S_{n-k}} \mathrm{C}_*^{\mathrm{simp}}(Z(s), \partial Z(s)) \to \mathrm{C}_*^{\mathrm{simp}}(X^k, X^{k-1}),$$

[1254] Note that it follows immediately from the definitions that $Z(s)$ are precisely the simplices that appear as summands in $s^\dagger \in \mathrm{C}_k^{\mathrm{simp}}(K)$ and that $\partial Z(s)$ are precisely the simplices that appear as summands in $\partial s^\dagger \in \mathrm{C}_{k-1}^{\mathrm{simp}}(K)$.

induced by the inclusions, is an isomorphism of chain complexes.

Proof. Since each of the inclusion induced maps $\mathrm{C}_*^{\mathrm{simp}}(Z(s), \partial Z(s)) \to \mathrm{C}_*^{\mathrm{simp}}(X^k, X^{k-1})$ is a chain map we see that the given map is indeed a chain map. Thus it suffices to show that the map is an isomorphism in each degree. Now let $l \in \mathbb{N}_0$. We consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{s \in S_{n-k}} \mathrm{C}_l^{\mathrm{simp}}(\partial Z(s)) & \xrightarrow{\alpha} & \bigoplus_{s \in S_{n-k}} \mathrm{C}_l^{\mathrm{simp}}(Z(s)) & \longrightarrow & \bigoplus_{s \in S_{n-k}} \mathrm{C}_l^{\mathrm{simp}}(Z(s), \partial Z(s)) & \longrightarrow & 0 \\
 & & \big\downarrow =:\varphi & & \big\downarrow =:\psi & & \big\downarrow & & \\
0 & \longrightarrow & \mathrm{C}_l^{\mathrm{simp}}(X^{k-1}) & \longrightarrow & \mathrm{C}_l^{\mathrm{simp}}(X^k) & \longrightarrow & \mathrm{C}_l^{\mathrm{simp}}(X^k, X^{k-1}) & \longrightarrow & 0. \\
 & & & & \big\downarrow p & & & & \\
 & & & & \bigoplus_{s \in S_{n-k}} \bigoplus_{(v_0, \ldots, v_l)} \mathbb{Z} \cdot G(s | v_0, \ldots, v_l) & & & &
\end{array}$$

We make the following clarifications and observations:

(a) The horizontal sequences are evidently exact.
(b) The vertical maps between the first and second row are induced by the inclusions $\partial Z(s) \to X^{k-1}$ and $Z(s) \to X^k$.
(c) It is basically clear that the middle vertical map ψ is an epimorphism.
(d) Let $G(t \,|\, v_0, \ldots, v_l)$ be an l-simplex in X^{k-1}. By definition of X^{k-1} this means that t is a simplex of K of dimension $\geq n - (k-1)$. For any $(n-k)$-simplex s with $s \subset t$ we have by definition that $G(t \,|\, v_0, \ldots, v_l)$ is a simplex of $\partial Z(s)$. This shows that the left vertical map φ is also an epimorphism.
(e) The vertical map p is the natural projection given by the fact that the bottom group is generated by a subset of the set of l-simplices of X^k.
(f) It follows easily from the definitions that $\ker(p \circ \psi) \subset \mathrm{im}(\alpha)$.
(g) It follows from (f) that $\ker(\psi) \subset \mathrm{im}(\alpha)$.

It now follows from (a), (b), (c) and (g) together with an elementary diagram chase, see Exercise 74.12, that the right vertical map is indeed an isomorphism. ⊞

Claim 3. For every ordered $(n-k)$-simplex s of K we have $\mathbb{Z} \cdot [s^\dagger] = \mathrm{H}_k^{\mathrm{simp}}(Z(s), \partial Z(s))$.

Proof. Let $s = (v_0, \ldots, v_{n-k})$ be an ordered $(n-k)$-simplex of K. By Proposition 103.2 we know that K is a closed n-dimensional pseudomanifold. By definition this implies that s is the face of an n-simplex. We pick vertices $w_1, \ldots, w_k$ of K such that $(v_0, \ldots, v_{n-k}, w_1, \ldots, w_k)$ is an n-simplex of K. Next we consider the following diagram:

$$\begin{array}{ccccccc}
\overbrace{\widetilde{\mathrm{H}}_k^{\mathrm{simp}}(Z(s))}^{=0 \text{ by Claim 1}} & \longrightarrow & \mathrm{H}_k^{\mathrm{simp}}(Z(s), \partial Z(s)) & \xrightarrow[\cong]{\partial_k} & \overbrace{\widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(\partial Z(s))}^{\cong \mathbb{Z} \text{ by Claim 1}} & \longrightarrow & \overbrace{\widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(Z(s))}^{=0 \text{ by Claim 1}} \\
 & & \big\uparrow {\scriptstyle n \mapsto n \cdot [s^\dagger]} \quad =:\varphi & & \big\downarrow \iota & & \\
 & & \mathbb{Z} & \xleftarrow[\text{projection } p]{} & \widetilde{\mathrm{C}}_{k-1}^{\mathrm{simp}}(\partial Z(s)). & &
\end{array}$$

We make the following clarifications and observations:

(a) It follows from Claim 1 that the connecting homomorphism ∂_k on the top is an isomorphism.
(b) Note that $\partial Z(s)$ is a $(k-1)$-dimensional simplicial complex. Therefore we have the *inclusion* $\iota\colon \widetilde{\mathrm{H}}^{\mathrm{simp}}_{k-1}(\partial Z(s)) \to \widetilde{\mathrm{C}}^{\mathrm{simp}}_{k-1}(\partial Z(s))$.
(c) It follows immediately from the definitions that $\partial s^\dagger$ lies in $\mathrm{C}^{\mathrm{simp}}_{k-1}(\partial Z(s))$. In particular $s^\dagger$ defines an element $\mathrm{H}^{\mathrm{simp}}_k(Z(s), \partial Z(s))$. In other words, the vertical map φ is defined.
(d) We denote by $p\colon \mathrm{C}^{\mathrm{simp}}_{k-1}(\partial Z(s)) \to \mathbb{Z}$ the projection from the free abelian group $\mathrm{C}^{\mathrm{simp}}_{k-1}(\partial Z(s))$ onto the $G(s \cup \{w_{k+1}\} \mid w_{k+2}, \dots, w_n)$-coefficient.
(e) It follows easily from Lemma 103.8 (1) together with the various definitions that the map $p \circ \iota \circ \partial_k \circ \varphi\colon \mathbb{Z} \to \mathbb{Z}$ is multiplication by ± 1.
(f) By Claim 1 we have $\widetilde{\mathrm{H}}^{\mathrm{simp}}_{k-1}(\partial Z(s)) \cong \mathbb{Z}$.
(g) It follows easily from (d) and (e) that $\partial_k \circ \varphi$ is an isomorphism. Together with (1) we see that φ is an isomorphism. ⊞

As the reader will have noticed, the above claims basically take care of the three statements of the proposition:

(1) It follows from Claim 1 and 2 that for $i \neq k$ we have $\mathrm{H}^{\mathrm{simp}}_i(X^k, X^{k-1}) = 0$.
(2),(3) We consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{D}^{\mathrm{simp}}_k(K) & \xrightarrow[\cong]{s^\dagger \mapsto [s^\dagger]} & \bigoplus_{s \in S_{n-k}} \mathrm{H}^{\mathrm{simp}}_k(Z(s), \partial Z(s)) & \xrightarrow[\cong]{} & \mathrm{H}^{\mathrm{simp}}_k(X^k, X^{k-1}) \\
\downarrow{\scriptstyle \partial_k} & & & & \downarrow{\scriptstyle d_k} \\
\mathrm{D}^{\mathrm{simp}}_{k-1}(K) & \xrightarrow[\cong]{s^\dagger \mapsto [s^\dagger]} & \bigoplus_{s \in S_{n-k+1}} \mathrm{H}^{\mathrm{simp}}_{k-1}(Z(s), \partial Z(s)) & \xrightarrow[\cong]{} & \mathrm{H}^{\mathrm{simp}}_{k-1}(X^{k-1}, X^{k-2}).
\end{array}$$

We make the following clarifications and observations:
(a) By Lemma 103.7 we know that the $\{s^\dagger\}_{s \in S_{n-l}}$ form a basis for each $\mathrm{D}^{\mathrm{simp}}_l(K)$.
(b) The composition of the horizontal maps equals the map Θ that is induced by the maps $\mathrm{D}^{\mathrm{simp}}_k(K) \to \mathrm{C}^{\mathrm{simp}}_k(X^k) \to \mathrm{C}^{\mathrm{simp}}_k(X^k, X^{k-1})$.
(c) It follows immediately from the definitions that the diagram commutes.
(d) By Claim 3 we know that the left horizontal maps are isomorphisms and by Claim 2 we know the right horizontal maps are isomorphisms.

This discussion shows that the maps

$$\begin{array}{rccc}
\Theta\colon & \mathrm{D}^{\mathrm{simp}}_k(K) & \to & \mathrm{H}^{\mathrm{simp}}_k(X^k, X^{k-1}) \\
& \sigma & \mapsto & [\sigma]
\end{array}$$

define an isomorphism $(\mathrm{D}^{\mathrm{simp}}_*(K), \partial_*) \to (\mathrm{H}^{\mathrm{simp}}_k(X^k, X^{k-1}), d_k)_{k \in \mathbb{N}_0}$ of chain complexes. ■

The following lemma is also a partial analogue of the Skeleton-Homology Lemma 80.1.

Lemma 103.13. Let $K = (V, S)$ be a closed n-dimensional simplicial homology manifold and let $k \in \mathbb{N}_0$.

(1) For every $i > k$ we have $\mathrm{H}^{\mathrm{simp}}_i(X^k) = 0$.

(2) For every $i < k$ the inclusion induced map $\mathrm{H}_i^{\mathrm{simp}}(X^k) \to \mathrm{H}_i^{\mathrm{simp}}(\mathrm{sd}(K))$ is an isomorphism.

Proof.

(1) Note that X^k is an abstract simplicial complex of dimension k. It follows immediately from the definition of simplicial homology that $\mathrm{H}_i^{\mathrm{simp}}(K) = 0$ for $i > k$.
(2) We fix $i \in \mathbb{N}_0$. Let $m > i$. We consider the following excerpt from the long exact sequence in simplicial homology of the pair (X^{m+1}, X^m):

$$\dots \to \underbrace{\mathrm{H}_{i+1}^{\mathrm{simp}}(X^{m+1}, X^m)}_{=0 \text{ by Lemma 103.12 (1) since } i<m} \to \mathrm{H}_i^{\mathrm{simp}}(X^m) \to \mathrm{H}_i^{\mathrm{simp}}(X^{m+1}) \to \underbrace{\mathrm{H}_i^{\mathrm{simp}}(X^{m+1}, X^m)}_{=0 \text{ by Lemma 103.12 (1) since } i<m} \to \dots$$

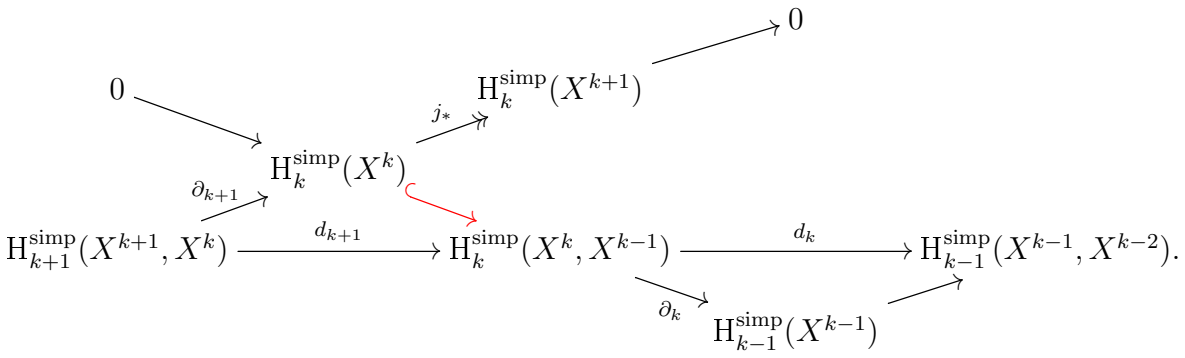
Now let $k > i$. The above discussion immediately implies that the following inclusion induced maps are isomorphisms: $\mathrm{H}_i^{\mathrm{simp}}(X^k) \xrightarrow{\cong} \mathrm{H}_i^{\mathrm{simp}}(X^{k+1}) \xrightarrow{\cong} \dots \xrightarrow{\cong} \mathrm{H}_i^{\mathrm{simp}}(X^n)$. The desired statement follows from the fact that $X^n = \mathrm{sd}(K)$. ■

Now we are finally in a position to provide the long overdue proof of Proposition 103.11.

Proof of Proposition 103.11. Let K be a closed n-dimensional simplicial homology manifold and let G be an abelian group. Recall that we need to prove the following two statements:

(1) The inclusion map $\mathrm{D}_*^{\mathrm{simp}}(K) \to \mathrm{C}_*^{\mathrm{simp}}(\mathrm{sd}(K))$ is a chain homotopy equivalence.
(2) The induced maps $\mathrm{H}_k(\mathrm{D}_*^{\mathrm{simp}}(K) \otimes G) \to \mathrm{H}_k^{\mathrm{simp}}(\mathrm{sd}(K); G)$ are isomorphisms.

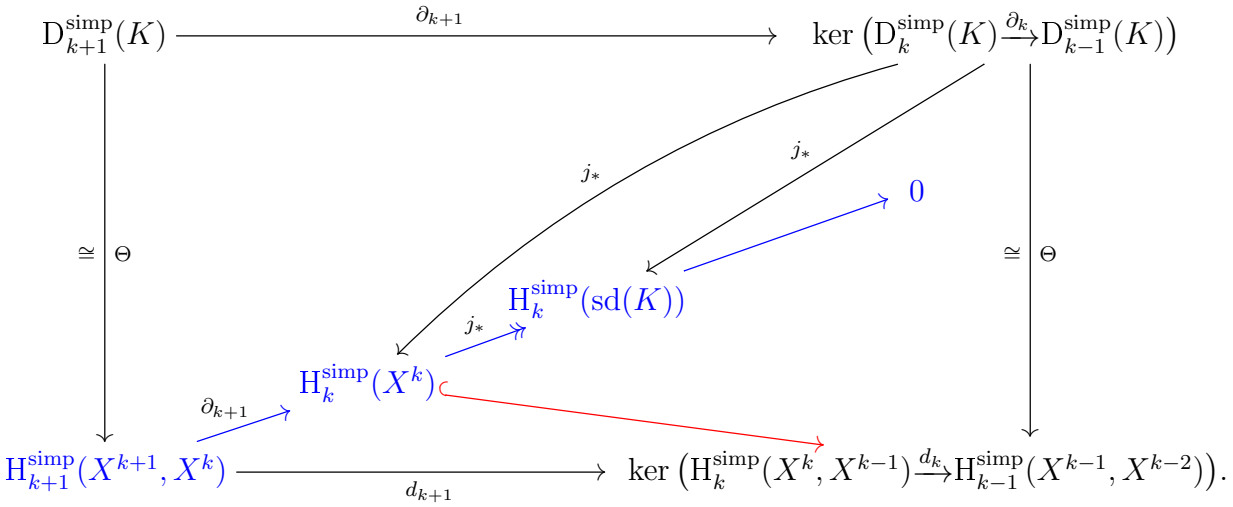
Similarly to the proof of the Singular-Cellular H_*-Isomorphism Proposition 80.4 we consider the following diagram:



We make the following observations:

(1) It follows from Lemma 95.29, i.e. from the long exact sequence in simplicial homology, together with Lemma 103.12 (1) and Lemma 103.13 (1) that all diagonal sequences are exact.
(2) It follows from the definition of d_k that the map $\mathrm{H}_k^{\mathrm{simp}}(X^k) \to \mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1})$ takes values in $\ker(d_k \colon \mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1}) \to \mathrm{H}_{k-1}^{\mathrm{simp}}(X^{k-1}, X^{k-2}))$.

We move on to the main diagram of this proof:



We make the following clarification and observations:

(3) All the maps called j_* are induced by inclusions on the level of chain groups. In fact note that every map in the diagram is either induced by taking boundaries or it is induced by an inclusion. It follows almost immediately from this observation that the diagram commutes.

(4) The vertical maps are isomorphisms by Lemma 103.12.

(5) By (1) the map $\mathrm{H}^{\mathrm{simp}}_k(X^k) \to \ker(d_k \colon \mathrm{H}^{\mathrm{simp}}_k(X^k, X^{k-1}) \to \mathrm{H}^{\mathrm{simp}}_{k-1}(X^{k-1}, X^{k-2}))$ is a monomorphism.

(6) It follows from (1), together with Lemma 103.13, that the diagonal blue sequence

$$\mathrm{H}^{\mathrm{simp}}_{k+1}(X^{k+1}, X^k) \xrightarrow{\partial_{k+1}} \mathrm{H}^{\mathrm{simp}}_k(X^k) \xrightarrow{j_*} \mathrm{H}^{\mathrm{simp}}_k(\mathrm{sd}(K)) \to 0$$

is exact.

An amusing little diagram chase now shows that the sequence

$$\mathrm{D}^{\mathrm{simp}}_{k+1}(K) \xrightarrow{\partial_{k+1}} \ker\left(\partial_k \colon \mathrm{D}^{\mathrm{simp}}_k(K) \to \mathrm{D}^{\mathrm{simp}}_{k-1}(K)\right) \xrightarrow{j_*} \mathrm{H}^{\mathrm{simp}}_k(\mathrm{sd}(K)) \to 0$$

is also exact. In other words we have proved the following statement:

(0) The inclusion map $\mathrm{D}^{\mathrm{simp}}_*(K) \to \mathrm{C}^{\mathrm{simp}}_*(\mathrm{sd}(K))$ induces for each $k \in \mathbb{N}_0$ an isomorphism $\mathrm{H}_k(\mathrm{D}^{\mathrm{simp}}_*(K)) \to \mathrm{H}^{\mathrm{simp}}_k(\mathrm{sd}(K))$.

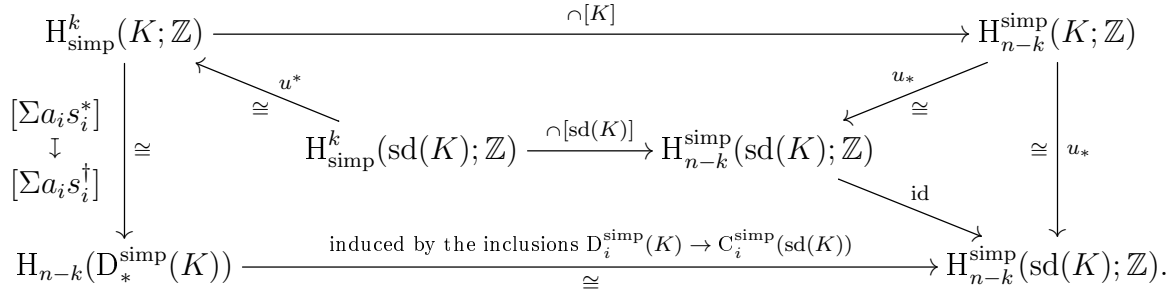
Now let us turn to the two statements we actually need to prove.

(1) It follows from (0) and the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 81.4 that the chain map $\mathrm{D}^{\mathrm{simp}}_*(K) \to \mathrm{C}^{\mathrm{simp}}_*(\mathrm{sd}(K))$ is in fact a chain homotopy equivalence.

(2) Finally note that it follows either from (1), or from (0) together with the Algebraic Universal Coefficient Theorem 89.16 and the Five Lemma 74.10, that the maps $\mathrm{H}_k(\mathrm{D}^{\mathrm{simp}}_*(K); G) \to \mathrm{H}^{\mathrm{simp}}_*(\mathrm{sd}(K); G)$ are also isomorphisms. Alternatively we could have carried through the above arguments with G-coefficients without any hiccup. ■

103.6. **The proof of the Simplicial Poincaré Duality Theorem 103.4.** In this section we combine the results from the previous sections to finally provide the proof of the Simplicial Poincaré Duality Theorem 103.4. For the most part we are now just saying that the program, set out on page 2275 has been completed successfully.

Let K be a closed n-dimensional pseudomanifold. First we assume that K is orientable and that we are given an orientation for K. In the following we will only work with the commutative ring $R = \mathbb{Z}$. The proof for the general case is basically the same, except that one has to carry slightly more notation around. As on page 2275 we consider the following diagram:



In the last sections we have shown the following:

(1) By Proposition 103.9 we now know that the left vertical map is an isomorphism.
(2) By the Chain Subdivision Lemma 95.41 we know that the subdivision maps u_* and u^* are isomorphisms.
(3) By Lemma 103.6 (3) we know that the upper quadrilateral commutes.
(4) By Lemma 103.10 we know that the lower left part of the diagram commutes.
(5) Finally we now know by Proposition 103.11 that the bottom horizontal map is an isomorphism.

The combination of the above five statements implies that the map

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{simp}}(K;\mathbb{Z}) & \to & \mathrm{H}^{\mathrm{simp}}_{n-k}(K;\mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [K] \end{array}$$

is, as promised and desired, an isomorphism.

Finally suppose that we are dealing with any closed n-dimensional pseudomanifold, in other words we drop the hypothesis that K is orientable. We need to show that the map

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{simp}}(K;\mathbb{F}_2) & \to & \mathrm{H}^{\mathrm{simp}}_{n-k}(K;\mathbb{F}_2) \\ \varphi & \mapsto & \varphi \cap [K]_{\mathbb{F}_2} \end{array}$$

is an isomorphism. The argument is almost identical to the previous argument. In fact the only place where we ever really used the orientation was in the definition of $s^\dagger$ on page 2271 and the resulting map $\mathrm{C}^k_{\mathrm{simp}}(K;\mathbb{Z}) \to \mathrm{D}^{\mathrm{simp}}_k(K)$ that is given by $s^* \mapsto s^\dagger$. A little bit of thought shows that the same recipe defines a map $\mathrm{C}^k_{\mathrm{simp}}(K;\mathbb{F}_2) \to \mathrm{D}^{\mathrm{simp}}_k(K) \otimes \mathbb{F}_2$ and the argument of Proposition 103.9 shows that this map is actually an isomorphism of cochain complexes. In the remainder of the argument we did not really use the orientation, so these parts stay virtually untouched. For example Proposition 103.11 says that the maps $\mathrm{H}_{n-k}(\mathrm{D}^{\mathrm{simp}}_*(K) \otimes \mathbb{F}_2) \to \mathrm{H}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K);\mathbb{F}_2)$ are isomorphisms. We leave it to the reader to turn this sketch of a proof into a proper argument.

103.7. **The dual CW-structure of a smooth manifold.** Let M be a closed smooth manifold that is equipped with a smooth simplicial structure $(K = (V, S), \Theta\colon |K| \to M)$. Some of the figures above, e.g. the figures on pages 2272 and 2279, suggest that the dual chains correspond to an alternative decomposition of M into cells. As we will see shortly, that is indeed the case.
First we recall some of the notation that we introduced earlier.

Notation. Let $K = (V, S)$ be an n-dimensional pseudomanifold.

(1) Given $m \in \mathbb{N}_0$ we denote by S_m the set of m-simplices of K.
(2) Let $s \in S_m$ be an m-simplex and let $v_1, \ldots, v_l \in V$ such that $s \cup \{v_1, \ldots, v_l\} \in S_{m+l}$. As on page 2271 we consider the following ordered l-simplex of $\mathrm{sd}(K)$:
$$G(s \,|\, v_1, \ldots, v_l) \;:=\; \big(\underline{s}, \underline{s \cup \{v_1\}}, \ldots, \underline{s \cup \{v_1, \ldots, v_l\}}\big).$$
(3) Let $k \in \mathbb{N}_0$ and let $s \in S_{n-k}$ be an $(n-k)$-simplex. As on page 2279 we consider
$$Z(s) \;:=\; \{ \text{ all simplices of } \mathrm{sd}(K) \text{ of the form } G(t \,|\, v_1, \ldots, v_l) \text{ with } s \subset t\}.$$
Furthermore, as on page 2278 we consider the following:
$$X^k \;:=\; \text{the subcomplex of } \mathrm{sd}(K) \text{ given by all } Z(s) \text{ with } s \in S_{n-k}.$$

Now we can formulate the main result of this section.

Proposition 103.14. Let M be a closed n-dimensional smooth manifold. Furthermore let $(K = (V, S), \Theta\colon |K| \to M)$ be a smooth simplicial structure.

(1) For each $(n-k)$-simplex s we have a homeomorphism $|Z(s)| \cong \overline{B}^k$.
(2) There exists a CW-structure on M such that for each $k \in \mathbb{N}_0$ the following holds:
  (a) The k-cells are given by the $|Z(s)|$ with $s \in S_{n-k}$.
  (b) The k-skeleton is given by $|X^k|$.
(3) If we equip M with the CW-structure from (2), then there exists a natural isomorphism $\mathrm{D}_k^{\mathrm{simp}}(K) \to \mathrm{C}_k^{\mathrm{CW}}(M)$ of chain complexes.

We refer to the CW-structure described in Proposition 103.14 as the *dual* CW-*structure* of the smooth manifold. We refer to the figure below for an illustration.



Sketch of proof. Let M be a closed n-dimensional smooth manifold. Furthermore let $(K = (V, S), \Theta\colon |K| \to M)$ be a smooth simplicial structure. Note that we know by Theorem 96.15 that $(K = (V, S), \Theta\colon |K| \to M)$ is a closed n-dimensional PL-manifold. This implies that we might as well prove the statement under the hypothesis that we are dealing with a closed n-dimensional PL-manifold. Furthermore note that it follows immediately from the definition of a PL-manifold, see page 2093, together with the SD-Lemma 94.3 that the barycentric subdivision $\mathrm{sd}(K)$ is also a closed n-dimensional PL-manifold.

Now let us turn to the proof of the three statement.

(1) One could prove this statement in one go, but it seems didactically opportune to first consider the two simplest cases.

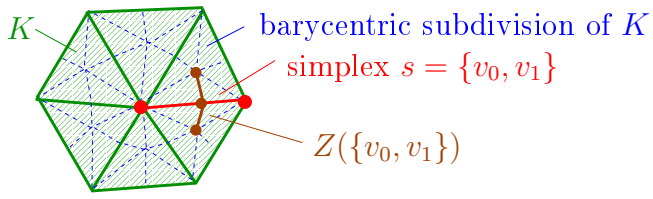
(a) Let $s = \{v_0\}$ be a 0-simplex of K. We have the following homeomorphisms:

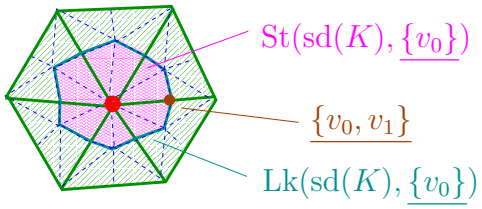
$$\begin{array}{ccccc} |Z(\{v_0\})| & = & |\operatorname{St}(\operatorname{sd}(K), \underline{\{v_0\}})| & = & |\text{ PL } n\text{-ball }| & \cong & \overline{B}^n. \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{follows immediately} & & \text{by the PL-Manifold Characterization} & & \text{since PL-homeomorphic} & & \\ \text{from the definitions} & & \text{Proposition 96.12 since } \operatorname{sd}(K) \text{ is} & & \text{implies homeomorphic} & & \\ & & \text{an } n\text{-dimensional PL-manifold} & & & & \end{array}$$

(b) Next let $s = \{v_0, v_1\}$ be a 1-simplex of K. Note that

$$\begin{array}{ccccc} & & \text{by the PL-Manifold Characterization Proposition 96.12} & & \\ & & \text{since } \operatorname{sd}(K) \text{ is a closed } n\text{-dimensional PL-manifold} & & \\ |Z(\{v_0, v_1\})| & \overset{\downarrow}{=} & |\operatorname{St}(\operatorname{Lk}(\operatorname{sd}(K), \underline{\{v_0\}}), \underline{\{v_0, v_1\}})| & \overset{\downarrow}{=} & |\operatorname{St}(\text{PL } (n-1)\text{-sphere}, \underline{\{v_0, v_1\}})| \\ & & = |\operatorname{St}(\text{closed } (n-1)\text{-dimensional PL-manifold}, \underline{\{v_0, v_1\}})| & & \\ & & \uparrow & & \\ & & \text{as we pointed out on page 2093, it follows from Proposition 96.11 that a} & & \\ & & \text{PL } (n-1)\text{-sphere is a closed } (n-1)\text{-dimensional PL-manifold} & & \\ & \cong & |\text{PL } (n-1)\text{-ball}| & \cong & \overline{B}^{n-1}. \\ & \uparrow & & \uparrow & \\ & \text{PL-Manifold Characterization} & & \text{since PL-homeomorphic implies homeomorphic} & \\ & \text{Proposition 96.12} & & & \end{array}$$

(c) Finally let $s = \{v_0, \dots, v_k\}$ be any k-simplex of K. Similar to (b) we see that $Z(s)$ is the star of an iterated sequence of k links. Applying the PL-Manifold Characterization Proposition 96.12 multiple times we see that $Z(s)$ is homeomorphic to $\overline{B}^{n-k}$. We leave it to the reader to write down this notationally messy argument.





(2) This statement follows fairly immediately from (1). We leave it to the reader to fill in the details.

(3) Given each $k \in \mathbb{N}_0$ we consider the map

$$\begin{array}{ccccc} \mathrm{D}_k^{\mathrm{simp}}(K) & \to & \mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1}) & \xrightarrow[\Xi]{\downarrow} & \mathrm{H}_k(|X^k|, |X^{k-1}|) = \mathrm{C}_k^{\mathrm{CW}}(M). \\ \sigma & \mapsto & [\sigma] & & \\ & & \text{natural isomorphism given by Lemma 95.30} & & \end{array}$$

Note that by Lemma 103.12 we know that the left maps are well-defined and that they form an isomorphism of chain complexes. We obtain immediately from the Simplicial-Singular Homology Isomorphism Lemma 95.30 (3) that the right maps also form an isomorphism of chain complexes. The composition of the two chain isomorphisms thus gives us the promised natural isomorphism of chain complexes. ■

Exercises for Chapter 103.

Exercise 103.1. Let T be the simplicial 2-dimensional torus. By Exercise 98.1 we know that the suspension $\Sigma(T)$ is a closed orientable 3-dimensional pseudomanifold. Show that the conclusion of the Simplicial Poincaré Duality Theorem 103.4 does not hold for $\Sigma(T)$.

Exercise 103.2. Let K be a simplicial complex. Show that K is an n-dimensional simplicial homology manifold if and only if the topological realization $|K|$ is an n-dimensional homology manifold. Recall by the definition on page 1672 a topological space X is called an n-dimensional homology manifold if for each $x \in X$ we have

$$\mathrm{H}_k(X, X \setminus \{x\}; \mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 103.3.

(a) Let M be a connected closed 3-dimensional smooth manifold. Show that if M is non-orientable, then $\mathrm{H}_1(M; \mathbb{Z})$ is infinite.
(b) Is the hypothesis in (a) that M is closed necessary?

Exercise 103.4. Let M and N be two closed orientable connected non-empty 3-dimensional smooth manifolds such that $\pi_1(M)$ is isomorphic to $\pi_1(N)$. Show that for all $i \in \mathbb{N}_0$ there exists an isomorphism $\mathrm{H}_i(M; \mathbb{Z}) \cong \mathrm{H}_i(N; \mathbb{Z})$.

Exercise 103.5. Let W be a closed orientable connected 4-dimensional smooth manifold with Euler characteristic $\chi(W) = -3$.

(a) Show that W admits a finite connected covering $\widetilde{W}$ with $\chi(\widetilde{W}) < -27$.
(b) Is the hypothesis in (a) that W is orientable necessary?

Exercise 103.6. Let M be a closed orientable connected odd-dimensional smooth manifold and let $f\colon M \to M$ be a map such that the induced map $f_*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M)$ is the identity. Show that the Lefschetz number

$$\Lambda(\varphi) \;=\; \sum_{n \in \mathbb{N}_0} (-1)^n \cdot \mathrm{tr}(\varphi_*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M)),$$

equals zero.

Exercise 103.7. Let $n \in \mathbb{N}$ and let M be a closed oriented connected non-empty n-dimensional smooth manifold. Suppose that there exists a map $f\colon S^n \to M$ of degree ± 1, i.e. such that $f_*([S^n]) = [M] \in \mathrm{H}_n(M)$. Show that $\mathrm{H}_i(M) = 0$ for $i = 1, \ldots, n-1$.
Remark. In the Degree One-Epimorphism Proposition 101.8 we showed under the same hypotheses that we also have $\pi_1(M) = 0$.

Exercise 103.8.

(1) Give a suitable definition of an n-dimensional simplicial homology manifold *with boundary*.

(2) Let K be a connected orientable n-dimensional simplicial homology manifold with non-empty boundary ∂K. We consider $M := K \cup_{\partial K} \mathrm{Cone}(\partial K)$, i.e. we consider the simplicial mapping cone of the inclusion $L \to K$.

(a) Show that $M = K \cup_{\partial K} \mathrm{Cone}(\partial K)$ is a *closed* orientable n-dimensional pseudo-manifold.

(b) Show that

$$\mathrm{H}_i(\mathrm{D}_k(M)) \;\cong\; \begin{cases} 0, & \text{if } k = n, \\ \mathrm{H}_i^{\mathrm{simp}}(\mathrm{sd}(K)), & \text{otherwise.} \end{cases}$$

Remark. This is a non-trivial variation on Proposition 103.11.

(c) Use the above together with Proposition 103.9 and Exercise 99.3 to show that for every $i \in \mathbb{N}_0$ there exists a natural isomorphism $\mathrm{H}^i_{\mathrm{simp}}(K, \partial K; \mathbb{Z}) \to \mathrm{H}^{\mathrm{simp}}_{n-i}(K; \mathbb{Z})$.

Part X

Homology groups of topological manifolds

104. Borsuk's Theorem, ENRs and ANRs

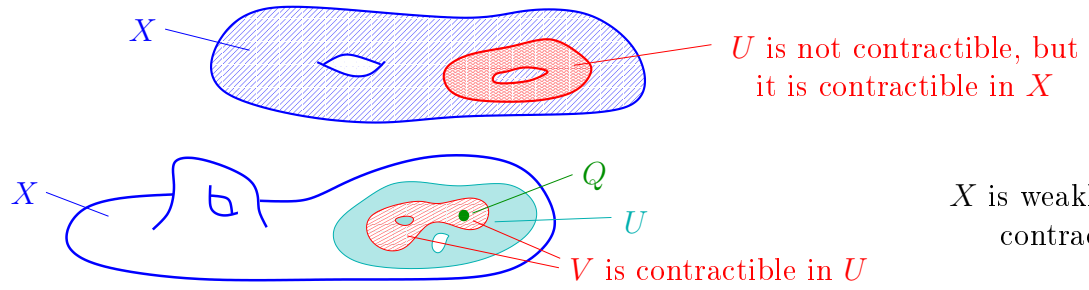
In this chapter we will state and prove Borsuk's Theorem 104.3 which will give us many goodies:

(1) We can prove the Lefschetz Fixed Point Theorem also for CW-complexes and topological manifolds.[1255]
(2) We can get some control over the invariants of compact topological manifolds, for example we will be able to show that the homology groups of a compact topological manifold are finitely generated and that the fundamental groups are finitely presented.

104.1. **Neighborhood retracts.** We recall two definitions from pages 431 and 175 and we introduce two related definitions.

Definition. Let X be a topological space.

(1) We say X is contractible if X is homotopy equivalent to a point.
(2) We say X is locally contractible if given any $Q \in X$ and given any open neighborhood U of Q there exists an open neighborhood V of Q that is contained in U and that is contractible.
(3) We say that a subset U of X is contractible in X if the inclusion map $U \to X$ is homotopic to a constant map.[1256]
(4) We say X is weakly locally contractible if given any $Q \in X$ and given any open neighborhood U of Q there exists an open neighborhood V of Q that is contained in U and that is contractible in U.[1257]



Examples.

(1) It follows almost immediately from the definitions that a topological space that is locally contractible is also weakly locally contractible.
(2) By the Topological Manifolds-Local Properties Lemma 18.8 (1) every topological manifold is locally homeomorphic to some non-empty convex subset of $\mathbb{R}^n$. Since non-empty convex subsets are contractible we see that every topological manifold is locally contractible.

[1255] So far, in Chapter 97 we only proved the Lefschetz Fixed Point Theorem for simplicial complexes and smooth manifolds.

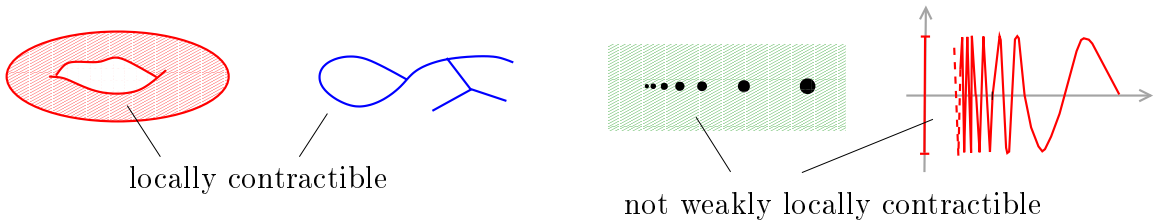
[1256] Note that the Contractibility Criterion Lemma 15.2 says that a topological space X is contractible if and only if the identity map is homotopic to a constant map.

[1257] What we call a "weakly locally contractible" topological space is referred to in many books, see e.g. [**Bor1967, Saka2013**], as a "locally contractible" topological space. Since on page 175 we defined once and for all what it means that a topological space is "locally P" we stick with our more cumbersome name.

(3) In the CW-Complex Properties Proposition 68.10 (6) we showed that CW-complexes are locally contractible. In particular topological graphs are locally contractible. (Note that we had also shown in Exercise 61.1 that topological graphs are locally contractible.)
(4) It is easy to show that $\mathbb{R}^2 \setminus \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$ is not weakly locally contractible.
(5) It is an amusing little exercise to show that the Topologist's Sine Curve

$$\big\{(0,y) \,\big|\, y \in [-1,1]\big\} \cup \big\{(x, \sin(\tfrac{1}{x})) \,\big|\, x \in (0, \tfrac{1}{\pi}]\big\} \subset \mathbb{R}^2,$$

which we had initially introduced on page 164, is not weakly locally contractible.
(6) It is an equally amusing (but not entirely trivial) exercise to come up with a topological space that is weakly locally contractible but not locally contractible.



Next we recall the following definition from pages 433 and 1061.

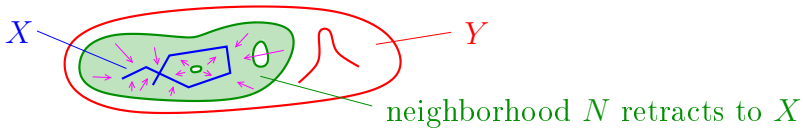
Definition. Let X be a topological space. We say a subset $A \subset X$ is a retract of X if there exists a map $r\colon X \to A$ with $r(a) = a$ for all $a \in A$.

In our discussions of retracts the following two basically trivial observations will at times be helpful.

Observation 104.1. Let Z be a topological space and let K be a subset.
(1) Let $K \subset U \subset V \subset Z$ be subsets. If K is a retract of V, then K is also a retract of U.
(2) If K is a retract of a neighborhood, then it is also a retract of an open neighborhood.

Proof. The first statement follows from the observation that the restriction of any retraction $r\colon V \to K$ to U is again a retraction. The second statement follows immediately from the first statement. ■

Definition. Let Y be a topological space and let X be a subset. We say X is a neighborhood retract in Y if there exists an open neighborhood N of X in Y such that X is a retract of N.[1258]



Examples.

[1258]By Observation 104.1 (2) this definition is equivalent to asking that there exists any neighborhood (not necessarily closed) that admits a retract to X. This formulation is often the definition of a neighborhood retract that is used in the literature. For our discussion it is convenient to already assume that the neighborhood is open.

(1) Let $K \subset S^3$ be a knot. By the Tubular Neighborhood Theorem **??** we know that there exists a tubular neighborhood $\overline{B}^2 \times K$ of K in S^3. Since $K = \{0\} \times K$ is evidently a retract of $\overline{B}^2 \times K$ we see that K is a neighborhood retract in S^3.
(2) Later on we will generalize (1) as follows. It is a consequence of the Regular Neighborhood Theorem 123.3, which is a fairly immediate consequence of the General Tubular Neighborhood Theorem **??**, that every compact proper submanifold X of a given smooth manifold M is a neighborhood retract in M.
(3) We will see in Exercise 104.2 that $X = \{0\} \cup \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$, viewed as a subset of $Y = \mathbb{R}$, is not a neighborhood retract.

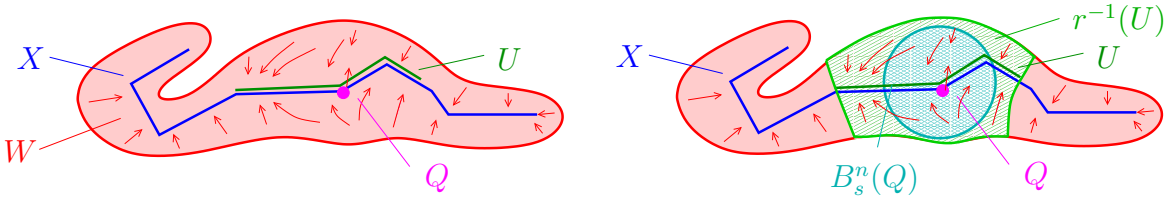
We continue with the following fairly elementary lemma.

Lemma 104.2. Let $n \in \mathbb{N}_0$ and let X be a subset of $\mathbb{R}^n$. If X is a neighborhood retract in $\mathbb{R}^n$, then the topological space X is weakly locally contractible.

Proof. Let W be a neighborhood of X that admits a retraction $r\colon W \to X$. Let $Q \in X$ and let U be a neighborhood of Q in X. Since W is a neighborhood of X there exists an $s > 0$ such that $B_s^n(Q) \subset r^{-1}(U)$. We denote by $F\colon B_s^n(Q) \times [0,1] \to B_s^n(Q)$ the obvious homotopy from the identity to the constant map. Finally we consider the map

$$\begin{array}{rcl} (B_s^n(Q) \cap X) \times [0,1] & \to & U \\ (x,t) & \mapsto & r\big(\underbrace{F(x,t)}_{\in B_s^n(Q) \subset r^{-1}(U)}\big). \end{array}$$

Evidently this map is a homotopy from the identity to a constant map. In particular $V := B_s^n(Q) \cap X$ is contractible in U. ■



It is rather stunning that for closed subsets of $\mathbb{R}^n$ the converse to Lemma 104.2 holds. More precisely, the following theorem was proved by Karol Borsuk [**Bor1932**] in 1932.

Theorem 104.3. (Borsuk's Theorem) Let $n \in \mathbb{N}_0$ and let X be a closed subset of $\mathbb{R}^n$. If the topological space X is weakly locally contractible, then X is a neighborhood retract in $\mathbb{R}^n$.[1259]

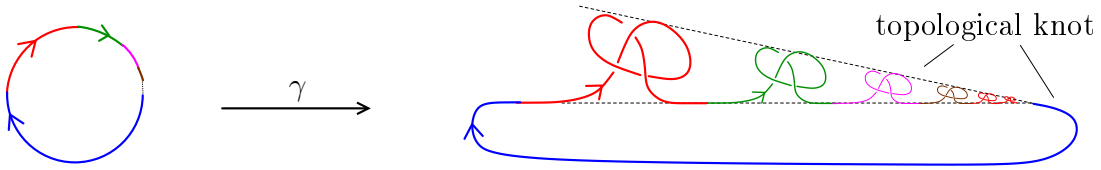
Remark.

(1) Borsuk's Theorem 104.3 can be viewed as a topological analogue of the Neighborhood Projection Theorem 28.1.
(2) Proofs of Borsuk's Theorem 104.3 are also given in [**Bre1993**, Theorem E.3] and [**Hat2002**, Theorem A.7].

Example. We define a *topological knot* to be the image of an embedding $\gamma\colon S^1 \to \mathbb{R}^3$. If the embedding is smooth, then we obtain the usual notion of a knot in $\mathbb{R}^3$. But once we drop

[1259] Is is perhaps worth stressing that the proposition does *not* claim that X is a *deformation retract* of a neighborhood of X.

smoothness one can get weird examples. For example, as we discussed on page 659, using the Compact-Hausdorff Proposition 2.17 (3) it is not difficult to show that the example shown in the figure below is indeed a topological knot.[1260]

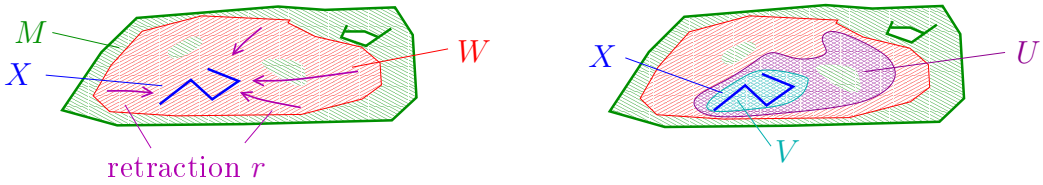


Evidently S^1 is locally contractible and by the Compact Image Lemma 2.13 together with the Compact-Closed Lemma 1.21 we know that $\gamma(S^1)$ is a closed subset of $\mathbb{R}^3$. Thus it follows from Borsuk's Theorem 104.3 that every topological knot is a neighborhood retract in $\mathbb{R}^3$. It is a mindboggling exercise to try to visualize a neighborhood of the above topological knot together with a retraction to $\gamma(S^1)$.

We conclude this discussion of Borsuk's Theorem 104.3 with a proposition that generalizes the statement of Borsuk's Theorem 104.3 from $\mathbb{R}^n$ to topological manifolds and that furthermore gives another rather technical conclusion. This second conclusion will play a significant role later on in our proof of the Generalized Alexander Duality Theorem 124.2.

Proposition 104.4. Let M be a topological manifold and let $X \subset M$ be a closed subset that is weakly locally contractible.

(1) There exists an open neighborhood $W \subset M$ of X for which there exists a retraction $r\colon W \to X$.
(2) We can find W and r such that furthermore the following condition is satisfied: given any neighborhood U of X with $X \subset U \subset W$ there exists an open neighborhood V with $X \subset V \subset U$ such that the inclusion $i\colon V \to U$ and the retraction $r\colon V \to X \subset U$ are homotopic maps from V to U.



Proof. Let M be a topological manifold and let $X \subset M$ be a closed subset that is weakly locally contractible. By the Euclidean Embedding Theorem 27.1 there exists an $n \in \mathbb{N}$ and a proper embedding $f\colon M \to \mathbb{R}^n$.[1261] By the definition of a proper map, see page 650, this implies that $f(M)$ is a closed subset of $\mathbb{R}^n$. It follows from the elementary the Open -Closed Inclusion Lemma 2.10 that $f(X)$ is a closed subset of $\mathbb{R}^n$.

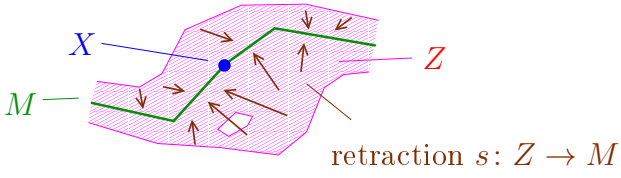
The above discussion shows that we only need to consider the case that M is a closed subset of $\mathbb{R}^n$ and that $X \subset M$ is a closed subset of $\mathbb{R}^n$. We now apply Borsuk's Theorem 104.3 not once, but twice:

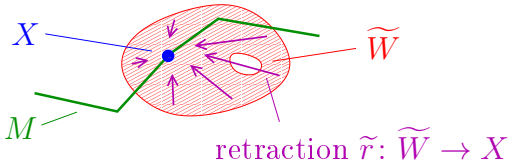
[1260] It is also not difficult, but rather pointless, to write down such an embedding explicitly.
[1261] Note that one could also use the Euclidean Embedding Theorem 11.13 for Topological Spaces together with the Covering Dimension-of-Manifolds Proposition 27.7.

(a) Since M is a topological manifold we know by the Topological Manifolds-Local Properties Lemma 18.8 that M is weakly locally contractible. Since M is a closed subset of $\mathbb{R}^n$ we know by Borsuk's Theorem 104.3 that there exists an open neighborhood $Z \subset \mathbb{R}^n$ of M for which there exists a retraction $s\colon Z \to X$.

(b) Since X is weakly locally contractible we know again by Borsuk's Theorem 104.3 that there exists an open neighborhood $\widetilde{W} \subset \mathbb{R}^n$ of X for which there exists a retraction $\widetilde{r}\colon \widetilde{W} \to X$. After possibly replacing $\widetilde{W}$ by $\widetilde{W} \cap Z$ we can assume that $\widetilde{W} \subset Z$.

We set $W := \widetilde{W} \cap M$ and $r := \widetilde{r}|_W\colon W \to X$. By the definition of the subspace topology we know that W is indeed an open neighborhood of $X \subset M$. Thus we have proved (1).





Now let us show that the above W and r actually also satisfy Statement (2). Thus let U be a neighborhood of $X \subset M$ with $X \subset U \subset W$. By definition of the subspace topology there exists an open subset $\widetilde{U}$ of $\mathbb{R}^n$ with $U = \widetilde{U} \cap M$. After possibly replacing $\widetilde{U}$ by $\widetilde{U} \cap \widetilde{W} \cap s^{-1}(U)$ we might as well assume that $X \subset \widetilde{U} \subset \widetilde{W}$ and that $s(\widetilde{U}) \subset U$. We consider the homotopy

$$\begin{array}{rcl} \widetilde{F}\colon \widetilde{U} \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & (1-t)\cdot x + t \cdot \widetilde{r}(x) \end{array}$$

from the identity $\mathrm{id}_{\widetilde{U}}$ to the retraction $\widetilde{r}\colon \widetilde{U} \to X$. Note that this is a homotopy rel X.

Claim. There exists an open neighborhood $\widetilde{V}$ of $X \subset \mathbb{R}^n$, that is contained in $\widetilde{U}$, such that $\widetilde{F}(\widetilde{V} \times [0,1]) \subset \widetilde{U}$.

Proof. We consider the open subset $Y := \widetilde{F}^{-1}(\widetilde{U}) \subset \widetilde{U} \times [0,1]$. Since $X \subset \widetilde{U}$ and since $\widetilde{F}$ is a homotopy rel X we see that Y contains $X \times [0,1]$. Since $[0,1]$ is compact and since Y is open one sees, as in the proof of the Product Topology Properties Proposition 5.6, that for any $x \in X$ there exists an open neighborhood R_x of $x \in \widetilde{U}$ such that $R_x \times [0,1] \subset Y$. We set $\widetilde{V} := \bigcup_{x \in X} R_x$. This is an open neighborhood of $X \subset \mathbb{R}^n$ that is contained in $\widetilde{U}$ such that $\widetilde{F}(\widetilde{V} \times [0,1]) \subset \widetilde{U}$. ⊞

We set $V := \widetilde{V} \cap M$. By definition of the subspace topology this is an open neighborhood of X. Evidently $X \subset V \subset U$. Finally we consider the map

$$\begin{array}{rcl} F\colon U \times [0,1] & \to & M \\ (x,t) & \mapsto & \underbrace{s(\underbrace{\widetilde{F}(x,t)}_{\in \widetilde{U} \subset Z})}_{\in U}. \end{array}$$

Note that this map takes values in $s(\widetilde{U}) = U$. Furthermore note that this is a homotopy from $F_0 = i\colon V \to U$ to $F_1 = r\colon V \to U$. We have thus shown that W and r satisfy the desired condition. ■

104.2. **Proof of the Borsuk's Theorem 104.3.** Before we provide the proof of Borsuk's Theorem 104.3 it is convenient to introduce one definition and to prove one lemma.

Definition. Let $k \in \mathbb{N}_0$ and let $n \in \mathbb{N}$. A $\frac{1}{2^k}$-cube is a subset C of $\mathbb{R}^n$ of the form

$$C \;=\; \left[\tfrac{a_1}{2^k}, \tfrac{a_1+1}{2^k}\right] \times \cdots \times \left[\tfrac{a_n}{2^k}, \tfrac{a_n+1}{2^k}\right] \quad \text{with } a_1, \ldots, a_n \in \mathbb{Z}.$$

Put differently, a $\frac{1}{2^k}$-cube is a cube in $\mathbb{R}^n$ with side length $\frac{1}{2^k}$ and where the coordinates of all vertices lie in $\frac{1}{2^k} \cdot \mathbb{Z}$.

Lemma 104.5. Let $x \in \mathbb{R}^n$ and let $r > 0$. If $k \in \mathbb{N}_0$ is chosen such that $\frac{1}{2^k} < \frac{1}{3\sqrt{n}} \cdot r$, then there exist finitely many $\frac{1}{2^k}$-cubes $W_1, \ldots, W_m$ such that

$$x \;\in\; \text{interior of } (W_1 \cup \cdots \cup W_m) \;\subset\; W_1 \cup \cdots \cup W_m \;\subset\; B_r^n(x).$$

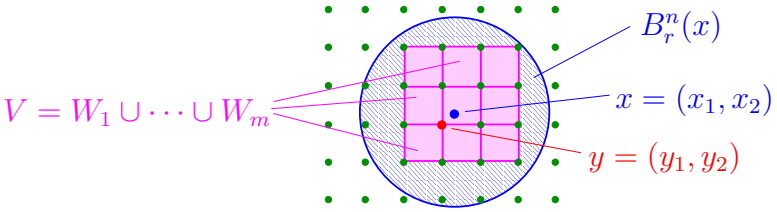
Proof. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and let $r > 0$. Furthermore let $k \in \mathbb{N}_0$ such that $\frac{1}{2^k} < \frac{1}{3\sqrt{n}} \cdot r$. We write $\epsilon = \frac{1}{2^k}$. For $i = 1, \ldots, n$ we denote by y_i the result of rounding x_i down to the closest number in $\mathbb{Z} \cdot \epsilon$. Clearly we have

$$(x_1, \ldots, x_n) \in \big[y_1, y_1{+}\epsilon\big] \times \cdots \times \big[y_n, y_n{+}\epsilon\big] \subset \text{interior of } \underbrace{\big[y_1{-}\epsilon, y_1{+}2\epsilon\big] \times \ldots \times \big[y_n{-}\epsilon, y_n{+}2\epsilon\big]}_{=:V}.$$

Since V is the union of finitely many ϵ-cubes it suffices to show that V is contained in $B_r^n(x)$. Thus let $y \in V$. We have

$$d(y,x) \underset{\substack{\uparrow \\ \text{since } x \in V}}{\leq} \text{diameter of } V \underset{\substack{\uparrow \\ \text{see page 285}}}{=} \sqrt{n} \cdot 3\epsilon \;=\; \sqrt{n} \cdot \frac{3}{2^k} \underset{\substack{\uparrow \\ \text{choice of } k}}{<} r$$

which means by definition that $y \in B_r^n(x)$. Thus we have shown that $V \subset B_r^n(x)$. ∎

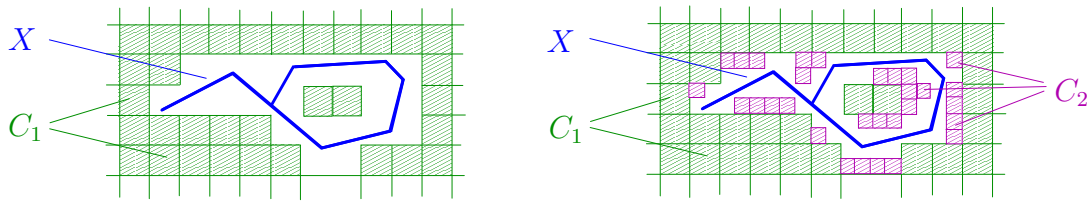


Now we turn to the actual proof of Borsuk's Theorem 104.3

Proof of Borsuk's Theorem 104.3. Let X be a closed, weakly locally contractible subset of $\mathbb{R}^n$. By Observation 104.1 (2) it suffices to show that there exists a neighborhood W of X in $\mathbb{R}^n$ that admits a retraction $W \to X$. Evidently we only have to consider the case that X is non-empty. We introduce the following notation.

(1) We define C_1 to be the union of all 1-cubes that do not intersect X.
(2) We define C_2 to be the union of all $\frac{1}{2}$-cubes that do not intersect $\mathring{C}_1$ and that do not intersect X.
(3) We iterate this process, i.e. for any $k \geq 3$ we define C_n to be the union of all $\frac{1}{2^k}$-cubes that do not intersect the interior of $C_1 \cup \cdots \cup C_{k-1}$ and that do not intersect X.

Claim 1.



(1) We have $\bigcup_{k\in\mathbb{N}} C_k = \mathbb{R}^n \setminus X$.
(2) Given any $x_0 \in \mathbb{R} \setminus X$ there exists an open neighborhood U that intersects only finitely many cubes of $C_1, C_2, \ldots$.

Proof. Let $x_0 \in \mathbb{R}^n \setminus X$. Since X is closed we know that $\mathbb{R}^n \setminus X$ is open. This means that there exists an $r > 0$ such that $B_r^n(x_0) \subset \mathbb{R}^n \setminus X$. We pick $k \in \mathbb{N}_0$ such that $\frac{\sqrt{n}}{3\cdot 2^k} < r$. By Lemma 104.5 there exist finitely many $\frac{1}{2^k}$-cubes $W_1, \ldots, W_m$ such that

$$x_0 \in \underbrace{\text{interior of } W_1 \cup \cdots \cup W_m}_{:=U} \subset B_r^n(x).$$

This implies in particular that x_0 is contained in C_n or it is contained already in one of $C_1, \ldots, C_{k-1}$. Furthermore it is fairly easy to see that if U intersects a $\frac{1}{2^m}$-cube with $m \geq k$ it already intersects a $\frac{1}{2^k}$-cube. It follows from this observation and the iterative definition of the C_i that U intersects only finitely many cubes of $C_1, C_2, \ldots$. ⊞

We view each C_i as a CW-complex in the obvious way: the n-cells correspond precisely to the cubes and the lower dimensional cells correspond precisely to the faces of the cubes. For each k the union $C_1 \cup \cdots \cup C_n$ is also a CW-complex in an obvious way and finally we can also view the union $C := \bigcup_{k\in\mathbb{N}} C_k = \mathbb{R}^n \setminus X$ as a CW-complex. We leave it to the reader to verify, using Statement (2) of the claim and using Lemma 1.8 (2), that the topology on this CW-complex, given by the weak topology defined on page 1479, agrees with the usual subspace topology on $\mathbb{R}^n \setminus X$.

We continue with the following definition: given a cell σ of C and a map $f\colon \sigma \to X$ we define[12621263]

$$\rho(f) := \max\{d(z, f(z)) \,|\, z \in \sigma\}.$$

We start with the iterative definition of a subset of $\mathbb{R}^n$ together with a map to X.

(a) We denote by A_0 the 0-skeleton of C. Next we define a map $r_0\colon A_0 \to X$. Let $a \in A_0$. Since X is a closed subset of $\mathbb{R}^n$ it follows from the Distance-Minimizer Lemma 6.4 that we can find a point $r_0(a) \in X$ with $d(a, r_0(a)) = d(a, X)$.
(b) Now suppose inductively that we have defined a map $r_i\colon A_i \to X$ on a union A_i of some of the i-cells of C. We define the next set A_{i+1} and the corresponding map $r_{i+1}\colon A_{i+1} \to X$ as follows:
(b_1) Let σ be an $(i+1)$-cell of C for which r_i is defined on $\partial\sigma$. Suppose that r_i can be extended to a map $g\colon \sigma \to X$ that agrees with r_i on $\partial\sigma$. Then we pick a map

[1262]Given $a, b \in \mathbb{R}^n$ we denote by $d(a, b) = \|a - b\|$ the usual Euclidean distance.
[1263]Note that σ is compact, hence the maximum exists by the Compact Image Lemma 2.13 (2).

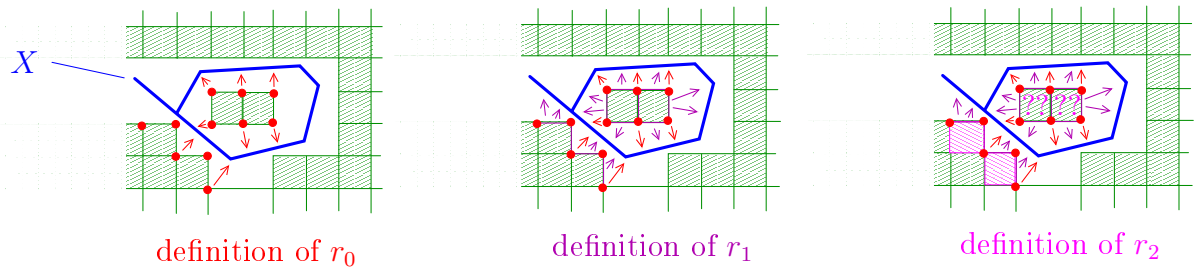
$r_{i+1}\colon \sigma \to X$ with $r_{i+1}|_{\partial\sigma} = r_i$ such that

$$\rho(r_{i+1}) \;\leq\; 2 \cdot \underbrace{\inf\{\rho(g) \,|\, g\colon \sigma \to X \text{ is a map with } g|_{\partial\sigma} = r_i|_{\partial\sigma}\}}_{\geq d(\sigma, X) \text{ which in turn is } > 0 \text{ by Lemma 6.3, here we use again that } X \subset \mathbb{R}^n \text{ is closed}}.$$

(b_2) We denote by A_{i+1} the union of all $(i+1)$-cells as in (b_1) and we define r_{i+1} on A_{i+1} as in (b_1).

(c) We set $A := \bigcup_{i\in\mathbb{N}} A_i \cup X$ and we define $r\colon A \to X$ to be the unique map that equals r_i on each A_i and that is the identity on X.

We refer to the figure below for an illustration. It remains to show that the set A is a



neighborhood of X and that the map $r\colon A \to X$ is continuous. Both statements follow from the following claim.

Claim 2. Given any $x \in X$ and any $\epsilon_0 > 0$ there exists an open neighborhood U of x that is contained in A and such that $\|\underbrace{r(x)}_{=x} - r(y)\| < \epsilon_0$ for all $y \in U$, i.e. such that $r(U) \subset B^n_{\epsilon_0}(x)$.

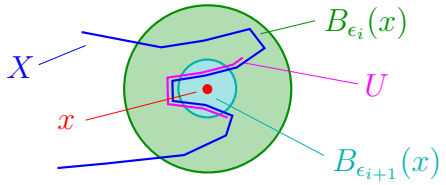
Proof. So let $x \in X$ and let $\epsilon_0 > 0$. We adopt the following notation: given $\mu > 0$ we write $W(\mu) = X \cap B^n_\mu(x)$.

Subclaim. We can find $\epsilon_1, \ldots, \epsilon_{2n}$ with the following two properties:

(1) for each $i \in \{0, \ldots, 2n-1\}$ we have $\epsilon_{i+1} < \frac{1}{3} \cdot \epsilon_i$,

(2) for every $i = 1, \ldots, 2n$ the subset $W(\epsilon_i)$ is contractible in $W(\epsilon_{i-1})$.

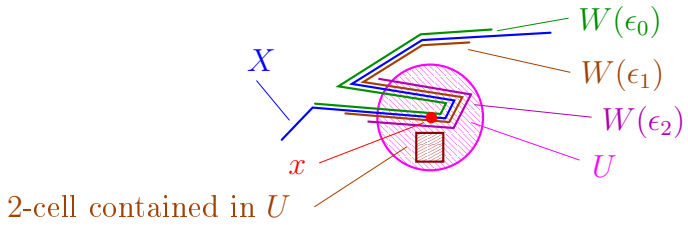
So suppose that we have already found $\epsilon_0, \ldots, \epsilon_i$ for some $i \in \{0, \ldots, 2n-1\}$ with the desired properties. Since X is weakly locally contractible we can find an open neighborhood U of x in X that is contained in $W(\epsilon_i)$ and that is contractible in $W(\epsilon_i)$. Since U is open there exists an $\epsilon_{i+1} < \frac{1}{3}\epsilon_i$ such that $W(\epsilon_{i+1}) \subset U$. Since U is contractible in $W(\epsilon_i)$ the subset $W(\epsilon_{i+1})$ is also contractible in $W(\epsilon_i)$. We refer to the figure below for an illustration. This concludes the proof of the subclaim.



We set $U := B^n_{\frac{1}{4}\epsilon_{2n}}(x)$. It suffices to prove the following two subclaims.

Subclaim 1. For each $i = 0, \dots, n$ the map r is defined for each i-cell σ of the CW-complex C that lies in U and for each such σ we have $r(\sigma) \subset W(\epsilon_{2n-2i})$.

Subclaim 2. The union of X with all cells of C that lie in U is a neighborhood of x.



First we prove Subclaim 1. We prove Subclaim 1 iteratively for $i = 0, 1, \dots, n$. We start out with $i = 0$. Thus we have to show that $r(U \cap A_0) \subset W(\epsilon_{2n})$. Therefore let $y \in U \cap A_0$. Then we have

$$d(x, r(y)) \underset{\uparrow}{\leq} d(x,y) + d(y, r_0(y)) \underset{\uparrow}{\leq} d(x,y) + d(y,x) = 2d(x,y) \underset{\uparrow}{<} 2 \cdot \tfrac{1}{4} \cdot \epsilon_{2n} = \tfrac{1}{2} \cdot \epsilon_{2n}.$$

triangle inequality and $r(y) = r_0(y)$ — by (a) since $x \in X$ — since $y \in U = B^n_{\frac{1}{4}\epsilon_{2n}}(x)$

But since $r(y) \in X$ this means precisely that $r(y) \in W(\epsilon_{2n}) = X \cap B^n_{\epsilon_{2n}}(x)$. This shows that Subclaim 1 holds for $i = 0$.

Now suppose that Subclaim 1 holds for some $i \in \{0, \dots, n-1\}$. Let σ be an $(i+1)$-cell of the CW-complex C inside U. Since $r(\partial\sigma) \subset W(\epsilon_{2n-2i})$ and since $W(\epsilon_{2n-2i})$ is contractible in $W_{2n-2i-1}$ it follows from the Convex-to-Ball Proposition 2.20 and the Homotopy-π_n-Proposition 71.7 (3) that we can extend $r|_{\partial\sigma} = r_i|_{\partial\sigma}$ to a map $s\colon \sigma \to W(\epsilon_{2n-2i-1})$. By definition of A_{i+1} this implies that $\sigma \subset A_{i+1}$. In particular r is defined on σ. It remains to show that $r(\sigma) \subset W(\epsilon_{2n-2i-2})$. Thus let $y \in \sigma$.

We have

$$d(x, r(y)) \underset{\uparrow}{\leq} d(x,y) + d(y, r_{i+1}(y)) \underset{\uparrow}{\leq} \tfrac{1}{4}\epsilon_{2n} + \rho(r_{i+1}) \underset{\uparrow}{\leq} \tfrac{1}{4}\epsilon_{2n} + 2 \cdot \inf\left\{ \rho(g) \,\middle|\, \begin{array}{c} g\colon \sigma \to X \text{ with} \\ g|_{\partial\sigma} = r_i|_{\partial\sigma} \end{array} \right\}.$$

by the triangle inequality and since $r(y) = r_{i+1}(y)$ — since $y \in \sigma \subset U = B^n_{\frac{1}{4}\epsilon_{2n}}(x)$ and by definition of $\rho(r_{i+1})$ — by (b_1)

$$\underset{\uparrow}{\leq} \tfrac{1}{4}\epsilon_{2n} + 2 \cdot \underbrace{\max\{d(z, s(z)) \mid z \in \sigma\}}_{=\rho(s)} \underset{\uparrow}{\leq} \tfrac{1}{4}\epsilon_{2n} + 2 \cdot \max\{d(z,x) + d(x, s(z)) \mid z \in \sigma\}$$

since s is such a map — triangle inequality

$$\underset{\uparrow}{\leq} \tfrac{1}{4}\epsilon_{2n} + 2\big(\tfrac{1}{4}\epsilon_{2n} + \epsilon_{2n-2i-1}\big) = \tfrac{3}{4}\epsilon_{2n} + 2\epsilon_{2n-2i-1} \underset{\uparrow}{\leq} \tfrac{1}{12}\epsilon_{2n-2i-2} + \tfrac{2}{3}\epsilon_{2n-2i-2} \leq \epsilon_{2n-2i-2}.$$

since $\sigma \subset U = B^n_{\frac{1}{4}\epsilon_{2n}}(x)$ and since for each $z \in \sigma$ we have $s(z) \in W(\epsilon_{2n-2i-1}) \subset B^n_{\epsilon_{2n-2i-1}}(x)$ — since for each j we have $\epsilon_{j+1} \leq \frac{1}{3}\epsilon_j$

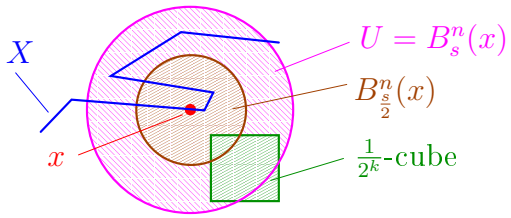
This concludes the proof that $r(\sigma) \subset W(\epsilon_{2n-2i-2})$. In particular this concludes the proof of Subclaim 1.

Finally it remains to prove Subclaim 2. We set $s = \frac{1}{4}\epsilon_{2n}$. As a reminder, we have to show that the union of X with all cells of C that lie in $B^n_s(x)$ is a neighborhood of x. Let W be a $\frac{1}{2^k}$-cube with $W \cap B^n_{\frac{s}{2}}(x) \neq \varnothing$ and $W \cap (\mathbb{R}^n \setminus B^n_s(x)) \neq \varnothing$. It follows that the diameter of W is at least $\frac{s}{2}$ which in turn implies that $\frac{\sqrt{n}}{2^k} \geq \frac{s}{2}$. We set $k_0 = \lfloor \ln_2(\frac{2\sqrt{n}}{s}) \rfloor$.

We see that

$$\begin{aligned}
&\text{union of } X \text{ with all cells of } C \text{ that lie in } B^n_s(x)\\
&\quad\supset \text{union of } X \text{ with all cubes of } C \text{ that lie in } B^n_s(x)\\
&\quad\supset B^n_s(x) \setminus \text{union of all cubes of } C \text{ that are not completely contained in } B^n_s(x)\\
&\qquad\uparrow \text{ since } \mathbb{R}^n \setminus X = C\\
&\quad= B^n_s(x) \setminus \text{union of all cubes of } C \text{ that are not completely contained in } B^n_s(x)\\
&\qquad\qquad \text{which do not contain the point } x \text{ but which intersect } B^n_{\frac{s}{s}}(x)\\
&\quad\supset B^n_s(x) \setminus \underbrace{\text{union of all } \tfrac{1}{2^k}\text{-cubes with } k \le k_0 \text{ of } C \text{ that are not contained in } B^n_s(x)}\\
&\qquad\uparrow \qquad\qquad \text{which do not contain the original point } x \text{ but which do intersect } B^n_{\frac{s}{s}}(x).}_{\text{finite union of cubes, hence a closed subset}}\\
&\quad \text{definition of } k_0
\end{aligned}$$

The last set is clearly an (open) neighborhood of x. But then the first set is also a neighborhood of x. This concludes the proof of Subclaim 2. ■



104.3. **Euclidean Neighborhood Retracts.** Borsuk's Theorem 104.3 has many applications. For example, much later, it will be a key ingredient in the proof of the Generalized Alexander Duality Theorem 124.2.
In this section we will discuss several fairly direct consequences of Borsuk's Theorem 104.3. For the following discussion it is convenient to introduce the following definition.

Definition. A topological space X is called a euclidean neighborhood retract (or short ENR) if there exists an embedding $f\colon X \to \mathbb{R}^n$ and an open neighborhood U of $f(X)$ such that $f(X)$ is a retract of U.

Remark. Let $i\colon X \to \mathbb{R}^n$ be an embedding of a topological space. The property of X being a retract of an open neighborhood, rather surprisingly, does not depend on the choice of the embedding. More precisely, [**tD2008**, Proposition 18.4.1] says the following. If X is an ENR, then given any embedding $f\colon X \to Y$ into a metric space Y there exists an open neighborhood U of $f(X)$ such that X is a retract of U. We will not make use of this pretty statement.

In the following we will mostly work with the following slight reformulation of Borsuk's Theorem.

Theorem 104.6. (Borsuk's Theorem) Let X be a topological space which is weakly locally contractible and which admits an embedding $f\colon X \to \mathbb{R}^n$.

(1) If $f(X)$ is closed, then X is an ENR.
(2) If X is regionally compact (e.g. if X is compact), then X is an ENR.

Proof.

(1) This statement follows immediately from Borsuk's Theorem 104.6.

(2) Now assume that X is regionally compact. If X is compact, then we know of course by the Compact-Closed Lemma 1.21 (2) that $f(X)$ is compact and we can apply (1). Slightly more interestingly, if X is regionally compact, then we know by Exercise 104.6 that there exists an embedding $g\colon X \to \mathbb{R}^{n+1}$ such that $g(X)$ is a *closed* subset of $\mathbb{R}^{n+1}$. Thus we obtain again from (1) that X is an ENR. ■

Definition. Let X be a topological space and let $\mathcal{C}$ be a class of topological spaces (e.g. $\mathcal{C}$ could be the class of CW-complexes). We say X is a retract of an element in $\mathcal{C}$ if there exists a topological space $Y \in \mathcal{C}$, a map $i\colon X \to Y$ and a map $r\colon Y \to X$ with $r \circ i = \mathrm{id}_X$.[1264]

In this chapter our key interest in ENR and thus in Borsuk's Theorem 104.3 stems from the following fairly simple proposition.

Proposition 104.7. (ENR-Simplicial Complex Retract Proposition)

(1) If $K \subset \mathbb{R}^n$ is a compact neighborhood retract, then K is the retract of a finite n-dimensional simplicial complex.

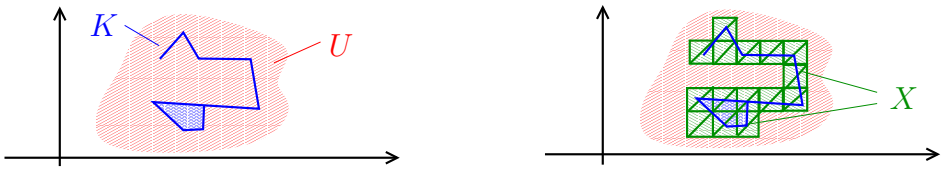
(2) Every compact ENR is a retract of a finite simplicial complex.

Proof.

(1) Let K be a compact subset of $\mathbb{R}^n$ which is a retract of some open neighborhood V. The idea is to define a simplicial complex X to be the union of finitely many sufficiently small cubes. To do so we set $r := d(K, \mathbb{R}^n \setminus V)$. Since K is compact and since $\mathbb{R}^n \setminus V$ is closed it follows from Lemma 6.3 that $r > 0$. We pick $k \in \mathbb{N}_0$ such that $\frac{1}{2^k} < \frac{1}{\sqrt{n}} \cdot r$. We denote by X the union of all $\frac{1}{2^k}$-cubes that intersect K. (See page 2295 for the definition of a $\frac{1}{2^k}$-cube.) Since K is compact we see that X is the union of finitely[1265] many $\frac{1}{2^k}$ cubes. The same argument as in the proof of Lemma 104.5 shows, by our choice of k, that each $\frac{1}{2^k}$-cube is contained in V.

We equip each cube with the canonical simplicial structure that we introduced on page 2000. As in Lemma 93.17 we see that the union of the cubes X also admits a simplicial structure. It follows from Observation 104.1 (1) that K is a retract of the finite simplicial complex X.

(2) This statement follows immediately from (1). ■



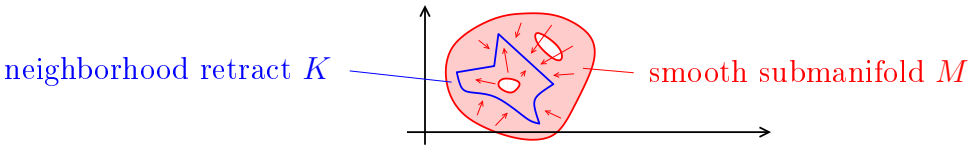
The statement of the following proposition is quite similar to the statement of the ENR-Simplicial Complex Retract Proposition 104.7, it just replaces simplicial complexes by smooth manifolds. We will not make much use of Proposition 104.8, except that it will come in handy at one occasion later on.

[1264] Note that Exercise 2.43 implies that such a map $i\colon X \to Y$ is necessarily an embedding.

[1265] Why is that? Why do we need only finitely many such cubes?

Proposition 104.8.

(1) If $K \subset \mathbb{R}^n$ is a compact neighborhood retract, then K is the retract of a compact n-dimensional smooth submanifold of $\mathbb{R}^n$.
(2) Every compact ENR is a retract of a compact smooth manifold.



Proof.

(1) Let K be a compact subset of $\mathbb{R}^n$ that is a retract of some open neighborhood V. Since V is an open subset of $\mathbb{R}^n$ it is in particular an n-dimensional smooth manifold. Therefore we can apply the Compact Submanifold Exhaustion Proposition 31.2 and we obtain a sequence $M_1, M_2, \dots$ of subsets of V with the following three properties:
   (a) the sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $M_i \subset M_{i+1}$,
   (b) each M_i is a compact n-dimensional smooth submanifold of V,
   (c) we have $\bigcup_{i\in\mathbb{N}} \mathring{M}_i = V$ where $\mathring{M}_i := M_i \setminus \partial M_i$.

   We make the following observations:
   (i) By Lemma 120.1 (3) each $\mathring{M}_i$ is an open subset of V and thus also of $\mathbb{R}^n$.
   (ii) It follows from (a), (c) and (i), the hypothesis that K is compact and Lemma 2.12 that there exists an $i \in \mathbb{N}$ such that $f(K) \subset M_i$.
   (iii) Since V is an open subset of $\mathbb{R}^n$ every smooth submanifold of V is in fact also a smooth submanifold of $\mathbb{R}^n$.

   It follows from this discussion and property (2) and Observation 104.1 (1) that the smooth manifold $M := M_i$ has all the desired properties.
(2) This statement follows immediately from (1). ∎

The following proposition shows that many of our favorite topological spaces are ENRs.

Proposition 104.9. (Manifolds-ENR Proposition)

(1) Every topological manifold is an ENR.
(2) Every countable CW-complex that is regionally compact and finite-dimensional is an ENR.

Remark.

(1) By the CW-Complex Compactness Corollary 68.15 we know that a finite CW-complex is compact, in particular regionally compact. Thus the Manifolds-ENR Proposition 104.9 (2) implies that every finite CW-complex is an ENR.
(2) By Exercise 68.8 we know that a CW-complex is regionally compact if and only if it is locally finite.

The proof of the Manifolds-ENR Proposition 104.9 (2) relies very much on the following lemma which is perhaps interesting in its own right.

Lemma 104.10. Let X be a CW-complex. If X is countable, regionally compact and finite-dimensional, then X admits an embedding into some $\mathbb{R}^n$.

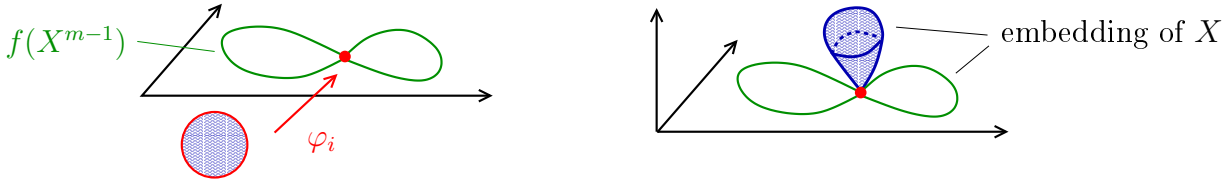
Remark. The special case of Lemma 104.10 provided by finite CW-complexes is precisely the content of Exercise 68.11.

Proof of Lemma 104.10. The theorem in its full generality is proved in [**FrPi1990b**, Theorem A]. In the discussion below we will only deal with the much simpler case that X is actually a finite CW-complex. We prove this case by induction on the dimension of the CW-complex X. The case $\dim(X) = -1$ is of course trivial. So suppose we already know the statement for all finite CW-complexes of dimension $\leq m-1$. Now let X be a finite m-dimensional CW-complex. As usual we denote by X^{m-1} the m-skeleton of X. We denote by $\varphi_1, \ldots, \varphi_k \colon S^{m-1} \to X^{m-1}$ the attaching maps of the m-cells of X. By induction there exists an $n \in \mathbb{N}_0$ and an embedding $f \colon X^{m-1} \to \mathbb{R}^n$. We consider the map

$$\overbrace{\left(X^{m-1} \sqcup \bigsqcup_{i=1}^{k} \overline{B}_i^m\right)/\sim}^{=X} \to \mathbb{R}^n \times \overbrace{\mathbb{R}^{m+1} \times \cdots \times \mathbb{R}^{m+1}}^{k \text{ copies}}$$

$$[x] \mapsto \begin{cases} (f(x), 0, \ldots, 0), & \text{if } x \in X^{m-1}, \\ (r \cdot f(\varphi_i(v)), 0, \ldots, \underbrace{rv \cdot (1-r), 1-r}_{\text{in the } i\text{-th } \mathbb{R}^{m+1}\text{-factor}}, \ldots, 0), & \text{if } x = r \cdot v \in \overline{B}_i^m \text{ where } \\ & r \in [0,1] \text{ and } v \in S^{m-1}. \end{cases}$$

It follows from Lemma 5.19 (2) that the somewhat risqué notation for points in $\overline{B}_i^m$ actually leads to a meaningful and continuous map on each $\overline{B}_i^m$. Note that from the Characteristic Maps Lemma 68.7 (4) we obtain that the overall map is continuous. Our hypothesis that X is a finite CW-complex implies by the CW-Complex Compactness Corollary 68.15 that X is compact. Finally it is elementary to verify that the map is actually injective. Thus, in summary, we can appeal to everybody's best friend, i.e. the Compact-Hausdorff Proposition 2.17 (2), to conclude the proof that this map is in fact an embedding. ■



Proof of the Manifolds-ENR Proposition 104.9.

(1) Just for fun we first give a separate, more down to earth argument, for the case of a closed smooth manifold M. By the Euclidean Embedding Theorem 27.1 and the Smooth Embedding Theorem 24.10 we can view M as a closed submanifold of some $\mathbb{R}^n$. Now we apply Theorem 28.1 to conclude that $M \subset \mathbb{R}^n$ admits an open neighborhood that retracts to M.

Now we turn to the general case. Thus let M be a topological manifold. By the Euclidean Embedding Theorem 27.1 we know that there exists an embedding $f \colon M \to \mathbb{R}^n$ such that $f(M)$ is a closed subset of $\mathbb{R}^n$. As we discussed on page 2290, every topological manifold is weakly locally contractible. Thus we can apply Borsuk's Theorem 104.6 to deduce that M is indeed an ENR.

(2) Let X be a CW-complex that is countable, regionally compact and finite-dimensional. By Lemma 104.10 we know that there exists an embedding $f\colon X \to \mathbb{R}^n$. Furthermore, as we discussed on page 2291, it follows almost immediately from the CW-Complex Properties Proposition 68.10 (6) that the CW-complex X is weakly locally contractible. Thus we can once again apply Borsuk's Theorem 104.6 (2) to conclude that X is an ENR. ■

Recall that in Theorem 97.9 and 97.10 we formulated and proved the Lefschetz Fixed Point Theorem in the realm of finite simplicial complexes and compact smooth manifolds. Our aim now is to extend the Lefschetz Fixed Point Theorem to compact ENRs. To do so we first need to convince ourselves that in this setting the Lefschetz number is actually defined. This follows from the following lemma.

Lemma 104.11. If X is a compact ENR, then for every field $\mathbb{F}$ the homology of X is $\mathbb{F}$-finite.

Proof. Let X be a compact ENR and let $\mathbb{F}$ be a field. It follows from Propositions 104.7 that X is a retract of a finite simplicial complex Y. Recall that this means that there exists a map $i\colon X \to Y$ and a map $r\colon Y \to X$ with $r \circ i = \mathrm{id}_X$.

(1) It follows from the Simplicial-Implies-CW Lemma 93.24 together with the Homology-of-CW-Complex Proposition 80.6 and the discussion on page 1915 that the homology of Y is $\mathbb{F}$-finite.
(2) It follows from the functoriality of homology groups and the fact that $r \circ i = \mathrm{id}_X$ that for every $k \in \mathbb{N}$ the map $r_*\colon \mathrm{H}_k(Y;\mathbb{F}) \to \mathrm{H}_k(X;\mathbb{F})$ is an epimorphism.

It follows immediately from (1) and (2) that the homology of X is $\mathbb{F}$-finite. ■

Now that we know that the Lefschetz number of every self-map of a compact ENR is defined we can now formulate the following theorem.

Theorem 104.12. (Lefschetz Fixed Point Theorem III) Let X be a compact ENR (e.g. X could be a compact topological manifold or a finite CW-complex) and let $\varphi\colon X \to X$ be a map. If the $\mathbb{F}$-Lefschetz number $\Lambda(\varphi, \mathbb{F})$ is non-zero for some field $\mathbb{F}$, then φ has a fixed point.

Remark.

(1) The above Lefschetz Fixed Point Theorem 104.12 together with the Manifolds-ENR Proposition 104.9 also gives a different, now totally self-contained, proof of the Lefschetz Fixed Point Theorem 97.10 for compact smooth manifolds.
(2) For compact orientable topological manifolds we will provide a generalization of the above Lefschetz Fixed Point Theorem 104.12 in the Lefschetz-Hopf Theorem 128.9.

Proof. Let X be a compact ENR. By the ENR-Simplicial Complex Retract Proposition 104.7 we know that X is a retract of a finite simplicial complex Y. Recall that this means that there exists a map $i\colon X \to Y$ and a map $r\colon Y \to X$ with $r \circ i = \mathrm{id}_X$. By the very harmless Exercise 2.43 we know that i is an embedding. Thus in the following we will view X as a subspace of Y. Now let $\varphi\colon X \to X$ be a map. We consider the map $i \circ \varphi \circ r\colon Y \to Y$.

Claim. We have $\Lambda(i \circ \varphi \circ r, \mathbb{F}) = \Lambda(\varphi, \mathbb{F})$.

Proof. Let $k \in \mathbb{N}_0$. We consider the following sequence of maps

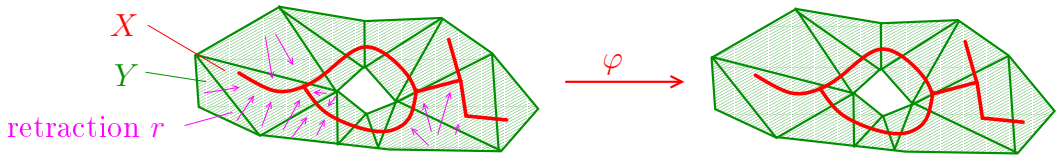
$$0 \longrightarrow \ker(r_*) \hookrightarrow \mathrm{H}_k(Y;\mathbb{F}) \underset{i_*}{\overset{r_*}{\rightleftarrows}} \mathrm{H}_k(X;\mathbb{F}) \longrightarrow 0.$$

Note that we have the equality $r_* \circ i_* = (r \circ i)_* = \mathrm{id}_* = \mathrm{id}_{\mathrm{H}_k(X;\mathbb{F})}$. This observation implies that the horizontal sequence is exact and that it splits. As in Splitting Lemma 78.2 we see that this implies that we can write $\mathrm{H}_k(Y;\mathbb{F}) = i_*(\mathrm{H}_k(X;\mathbb{F})) \oplus \ker(r_*)$. With slightly suggestive notation we see that

$$\mathrm{tr}\left(\mathrm{H}_k(Y;\mathbb{F}) \xrightarrow{i_* \circ \varphi_* \circ r_*} \mathrm{H}_k(Y;\mathbb{F})\right) = \mathrm{tr}\Bigg(\underbrace{\begin{pmatrix}\mathrm{id}\\0\end{pmatrix}(\varphi_*)\begin{pmatrix}\mathrm{id} & 0\end{pmatrix}}_{=\begin{pmatrix}\varphi_* & 0\\0 & 0\end{pmatrix}}\Bigg) = \mathrm{tr}\left(\mathrm{H}_k(X;\mathbb{F}) \xrightarrow{\varphi_*} \mathrm{H}_k(X;\mathbb{F})\right).$$

The desired formula for the Lefschetz numbers follows immediately from this claim. ⊞

From our hypothesis, the claim and the Lefschetz Fixed Point Theorem 97.9 for simplicial complexes we obtain that $i \circ \varphi \circ r \colon Y \to Y$ admits a fixed point $y \in Y$. This fixed point y lies necessarily in the image of the last map, i.e. it lies in $i(X) = X$. It is clear that the point y is also a fixed point for φ. ■



Remark. In the possibly unlikely event that the reader wants to know more about ENRs we refer to [**Dol1980**, Chapter IV.8] for more information.

104.4. **Invariants of compact topological manifolds.** There are several ways how one can get a grip on the topology of a (compact) *smooth* manifold M:

(a) In Corollary 95.37 we used the fact that (compact) smooth manifolds admit good (finite) covers together with the Nerve Theorem 95.32 to show that M is weakly homotopy equivalent to a finite simplicial complex.

(b) In the Smooth Manifold-CW Structure Theorem 96.5 we used the fact that M is diffeomorphic to a smooth submanifold of some $\mathbb{R}^n$ to show that M actually admits a (finite) smooth simplicial structure.

(c) Later on, in the Morse Functions-Cells Proposition 138.6 and in the Smooth Manifold-CW-Homotopy Type 139.8, we will use "Morse theory" to show that M is homotopy equivalent to a (finite) CW-complex X with $\dim(X) \leq \dim(M)$.

These results, together with basic facts about invariants of finite CW-complexes, see the CW-Skeleton-π_1-Proposition 69.13 (3) and the Homology-of-CW-Complex Proposition 80.6 (2), imply the following excerpt of Proposition 96.6.

Proposition 96.6. (Smooth Manifolds-Invariants Proposition) Let M be a 0-connected smooth manifold.

(1) For any $k > n$ we have $\mathrm{H}_k(M) = 0$ and for every abelian group G we also have $\mathrm{H}_k(M; G) = 0$.
(2) The higher homotopy groups of M are countable.

Furthermore, if M is compact, then the following statements hold:

(3) The fundamental group of M is finitely presented.
(4) All homology groups of M are finitely generated abelian groups. Furthermore, given any field $\mathbb{F}$ the homology groups $\mathrm{H}_k(M; \mathbb{F})$ are finite-dimensional.

To the best of the author's understanding none of the above approaches (a), (b) and (c) can be carried over to topological manifolds.
It seems like the only tool we have for handling "the size" of invariants of compact topological manifolds is the following theorem, which is an immediate consequence of the Manifolds-ENR Proposition 104.9 together with the ENR-Simplicial Complex Retract Proposition 104.7.

Theorem 104.13. (Topological Manifold-Simplicial Complex Retract Theorem) Every compact topological manifold is a retract of a finite simplicial complex.

Fortunately this is enough to extend most of the results of Proposition 96.6:

Proposition 104.14. (Topological Manifolds-Invariants Proposition) Let M be a compact 0-connected topological manifold. The following statement holds:

(1) There exists an $l \in \mathbb{N}_0$ such that for every $k > l$ we have $\mathrm{H}_k(M) = 0$ and such that for every abelian group G we have $\mathrm{H}_k(M; G) = 0$.
(2) The higher homotopy groups of M are countable.
(3) The fundamental group of M is finitely presented.
(4) All homology groups $\mathrm{H}_k(M)$ are finitely generated abelian groups. Furthermore, given any field $\mathbb{F}$ the homology groups $\mathrm{H}_k(M; \mathbb{F})$ are finite-dimensional.

Remark.

(1) Let M be an n-dimensional topological manifold.
   (a) In the Top-Homology Theorem 106.3 we will see that in (1) we can take $l = \dim(M)$.
   (b) If M is in fact a smooth manifold, then the Smooth Manifold-Top Homology Theorem 98.6 allows us to compute the homology group $\mathrm{H}_n(M)$. In the Top-Homology Theorem 106.3 we will prove the analogous result without the hypothesis that M is smooth.
(2) the Topological Manifolds-Invariants Proposition 104.14 together with Lemma 96.8 often allows us to control the "size" of relative homology groups of a compact topological manifold. For example for a compact topological manifold M all relative homology groups $\mathrm{H}_k(M, \partial M)$ are finitely generated.
(3) We will deal with non-compact topological manifolds in Theorem 104.19 below.

Proof of the Topological Manifolds-Invariants Proposition 104.14. [1266] Let M be a compact 0-connected topological manifold, let $\mathbb{F}$ be a field, let G be an abelian group

[1266] The argument below is quite similar to the proof of Lemma 104.11.

and let $n \in \mathbb{N}$. We pick a base point $x_0 \in M$. It follows from Propositions 104.7 and 104.9 (1) that M is a retract of a finite simplicial complex Y. Recall that this means that there exists a map $i\colon M \to Y$ and a map $r\colon Y \to M$ with $r \circ i = \mathrm{id}_M$. We write $y_0 = i(x_0)$. Furthermore we denote the dimension of the simplicial complex by l. We make the following observations:

(a) By Proposition 94.9 we know that $\pi_n(Y, y_0)$ is countable.
(b) It follows from the Simplicial-Implies-CW Lemma 93.24 together with the CW-Skeleton-π_1-Proposition 69.13 that $\pi_1(Y, y_0)$ is finitely presented.
(c) It follows from the Simplicial-Implies-CW Lemma 93.24 together with the Homology-of-CW-Complex Proposition 80.6 and the discussion on page 1915 that the homology groups $\mathrm{H}_k(Y)$ are finitely generated, that the homology groups $\mathrm{H}_k(Y; \mathbb{F})$ are finite-dimensional and that $\mathrm{H}_k(Y) = 0$ and $\mathrm{H}_k(Y; G) = 0$ for $k > l$.

Now we turn to the invariants of M.

(1) Let $k \in \mathbb{N}_0$. It follows from the functoriality of homology groups and the fact that $r \circ i = \mathrm{id}_M$ that the maps $r_*\colon \mathrm{H}_k(Y) \to \mathrm{H}_k(M)$ and $\mathrm{H}_k(Y; G) \to \mathrm{H}_k(M; G)$ are epimorphisms. In particular if the groups to the left vanish, then so do the groups to the right. This observation together with (c) implies (1).
(4) The argument in (1) also shows that if $\mathrm{H}_k(Y)$ is finitely generated, then so is $\mathrm{H}_k(M)$. Similarly, if $\mathrm{H}_k(Y; \mathbb{F})$ is finite-dimensional, then so is $\mathrm{H}_k(M; \mathbb{F})$. Thus we see that (c) implies (4).
(2) As in (1) we see $r_*\colon \pi_n(Y, y_0) \to \pi_n(M, x_0)$ is an epimorphism. It follows from (a) and the Countability Lemma 0.8 that $\pi_n(M, x_0)$ is also countable.
(3) The logic of (1), (2) and (4) shows immediately that $\pi_1(M, x_0)$ is *finitely generated*. The statement that $\pi_1(M, x_0)$ is *finitely presented* requires a little more thought. More precisely, by the functoriality of fundamental groups we know that $\pi_1(M, x_0)$ is a *retract* of $\pi_1(Y, y_0)$. By (b) we know that the group $\pi_1(Y, y_0)$ is finitely presented. Thus we obtain from the purely group-theoretic the Finite Presentation-Retract Lemma 53.16 that $\pi_1(M, x_0)$ is also finitely presented. ■

Finally we can also say something non-trivial about possibly non-compact topological manifolds, namely we have the following analogue of Proposition 96.7.

Proposition 104.15. Let M be a 0-connected topological manifold.
(3′) The fundamental group of M is countable.
(4′) All homology groups $\mathrm{H}_k(M)$ are countable abelian groups. Furthermore, given any field $\mathbb{F}$ the dimension of each homology groups $\mathrm{H}_k(M; \mathbb{F})$ is countable.

Remark. In the Countable Fundamental Group Proposition 49.1 we already showed that fundamental groups of (possibly non-compact) topological manifolds are countable. But it seems difficult to adapt the approach of the Countable Fundamental Group Proposition 49.1 to deal with homology groups.

Proof. Let M be a 0-connected topological manifold. We pick $x_0 \in M$. By the Manifolds-ENR Proposition 104.9 we know that M is an ENR. Recall that this means that there exists an embedding $f\colon X \to \mathbb{R}^n$ and an open neighborhood U of $f(X)$ such that $f(X)$ is a retract of U.

Note that U is an open subset of $\mathbb{R}^n$. In particular U is a smooth manifold. It follows from Proposition 96.7 that $\pi_1(U, x_0)$ and all homology groups $\mathrm{H}_k(U)$ are countable.

As in the proof of the Topological Manifolds-Invariants Proposition 104.14 we see that $\pi_1(M, x_0)$ is isomorphic to a quotient of $\pi_1(U, x_0)$ and we see that each $\mathrm{H}_k(M)$ is isomorphic to a quotient of $\mathrm{H}_k(U)$. It follows from the above that $\pi_1(U, x_0)$ and all homology groups $\mathrm{H}_k(M)$ are countable. The same logic also applies to the homology groups $\mathrm{H}_k(M; \mathbb{F})$. ■

104.5. **ANRs and the homotopy type of topological manifolds.** In this section we will see in particular that all topological manifolds are homotopy equivalent to locally finite simplicial complexes, in particular, they are homotopy equivalent to CW-complexes. In contrast to the previous sections we will not attempt to prove the statements of this section, instead we try our best to give a good overview of the literature.
We start out with the following definition.

Definition. We say that a topological space X is dominated by a topological space Y if there exist maps $i\colon X \to Y$ and $r\colon Y \to X$ such that $r \circ i$ is homotopic to the identity id_X.

Example.

(1) If a topological space X is a retract of some topological space Y, then evidently X is dominated by Y.
(2) It follows from the ENR-Simplicial Complex Retract Proposition 104.7 together with the Manifolds-ENR Proposition 104.9 that every compact topological manifold and every finite CW-complex is dominated by a finite simplicial complex.

The following proposition is the most powerful result on dominations that we actually prove in these notes.

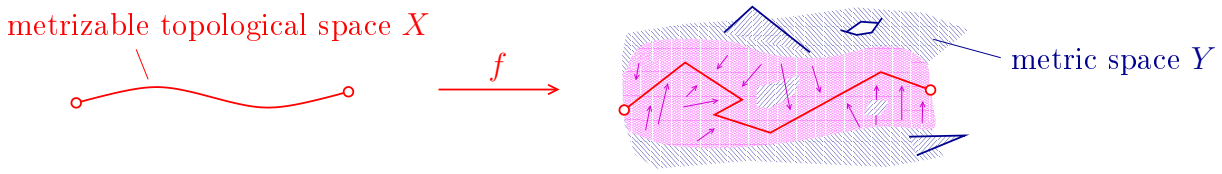
Proposition 162.8 (CW-Complex Domination Proposition). Let X be a topological space. If X is dominated by a CW-complex, then X is homotopy equivalent to a CW-complex.

Remark. Suppose a topological space X is a retract of some CW-complex Y. By the CW-Complex Domination Proposition 162.8 we now know that X is homotopy equivalent to a CW-complex. One might hope that X should actually admit the structure of a CW-complex. But in Exercise 104.5 we will see that in general this is *not* the case.

Proof. We will prove the CW-Complex Domination Proposition 162.8 in Section 162.3, using the Whitehead Theorem 162.1. Alternatively see the proofs of Theorems 104.19, 104.22 and 104.23 for alternative proofs. ■

It turns out that there exist several powerful generalizations of the CW-Complex Domination Proposition 162.8. To formulate these generalizations it is helpful to introduce the following definition.

Definition. A topological space X is called an absolute neighborhood retract (or short ANR) if X is metrizable and if the following condition is satisfied: whenever $f\colon X \to Y$ is an embedding into a metric space Y such that $f(X)$ is a closed subset of Y, then $f(X)$ is a neighborhood retract in Y.



Remark.

(1) Like in many other settings in the literature there exist various equivalent approaches to defining an ANR, see e.g. [**MaS1982**, Chaper I.3.1(ii)] and [**FrPi1990a**, Proposition A.6.2]. The definition above is the one given in [**FrPi1990a**, p. 281].

(2) Let X be an ANR. If X admits a closed embedding into some $\mathbb{R}^n$, then it follows immediately from the definitions that X is an ENR. Nonetheless, not every ANR is an ENR. For example we will see shortly that the disjoint union $X = \bigsqcup_{k\in\mathbb{N}} \overline{B}^k$ is an ANR, but it follows easily from the Topological Invariance of Domain Theorem 82.7 that X does not admit an embedding into any $\mathbb{R}^n$.

(3) Some references use a definition of an ANR which differs slightly from ours. More precisely, in [**Han1951**, p. 390] and [**Miln1959a**, p. 272] an ANR is supposed to satisfy the extra condition that it is separable, i.e. that it contains a countable subset that is dense. Note that in Exercise 9.8 we saw that a metrizable topological space is separable if and only if it is second-countable. Thus, since in these notes we are more familiar with the concept of second-countable topological spaces we will use that notion in the remainder of our discussion of ANRs.

(4) The theory of ANRs is developed in great detail in [**Hu1965, Bor1967, Chig1996**].

We will soon see what ANRs are good for. But first let us quote two theorems which give us many examples of ANRs.

Theorem 104.16. (Dugundji Extension Theorem) Let V be a real vector space and $\|-\|\colon V \to \mathbb{R}_{\geq 0}$ a norm on V. We equip V with the corresponding metric and we use this metric to view V as a topological space. With respect to this topology every convex subset of V is an ANR.[1267]

Example. The only consequence of the Dugundji Extension Theorem 104.16 that we will use is that every convex subset of some $\mathbb{R}^n$ is an ANR.

Proof. This theorem was proved by James Dugundji [**Dug1951**, Corollary 4.2] in 1952. A proof is also given in [**MaS1982**, Theorem I.3.1], [**Maye1989**, Theorem I.5.7] and [**Saka2013**, Theorem 6.1.1]. ■

Theorem 104.17. (Hanner's Theorem) Let X be a topological space which is second-countable and metrizable. If X is the union of a family $\{U_i\}_{i\in I}$ of open subsets such that each U_i is an ANR, then X itself is an ANR.

Proof. This theorem was proved by Olof Hanner [**Han1951**, Theorem 3.3] in 1950. An exposition of the proof is also given in [**Fera2004**, Theorem 2.77] and in [**Hu1965**, Theorem III.8.1]. ■

[1267] Note that there are no other hidden conditions on the subset except for convexity, in particular we do not demand that the subset is open, closed or compact.

Corollary 104.18. Every topological manifold is an ANR.

Proof. Let X be an n-dimensional topological manifold. By definition X is second-countable. By the Manifold Metrization Proposition 18.10 we know that X is metrizable. In the Topological Manifolds-Local Properties Lemma 18.8 (1) we showed that X is locally homeomorphic to some non-empty convex subset of $\mathbb{R}^n$. By the Dugundji Extension Theorem 104.16 every non-empty convex subset of $\mathbb{R}^n$ is an ANR. In other words, we have shown that X admits an open cover by ANRs. Thus we obtain from Hanner's Theorem 104.17 that X is indeed an ANR. ■

The following amazing theorem now tells us what ANRs are good for.

Theorem 104.19. Let X be a topological space. The following statements are equivalent:

(1) X is dominated by a countable CW-complex.
(2) X is homotopy equivalent to a countable CW-complex.
(3) X is homotopy equivalent to a countable simplicial complex.
(3′) X is homotopy equivalent to a countable simplicial complex that is locally finite.
(4) X is homotopy equivalent to an ANR which is second-countable.

Furthermore the following refinement of the equivalence of (2) and (3) holds: If X is homotopy equivalent to an n-dimensional countable CW-complex, then it is also homotopy equivalent to an n-dimensional countable simplicial complex, and vice versa.

Proof. The fact that all the statements are equivalent goes well beyond the scope of these notes. Thus we just give the necessary references. The implications (3′)⇒(3)⇒(2)⇒(1) are of course trivial. Next note that the equivalence of (2) and (3) is a consequence of the Simplicial-Implies-CW Lemma 93.24 and Proposition 94.12. As is explained in [**Miln1959a**, Theorem 1], the implication (1)⇒(3′) follows from work of James Whitehead [**WhdJ1950**, Theorem 24] and the implication (3′)⇒(4)⇒(1) follows from the work of Olof Hanner [**Han1951**, Corollary 3.5 and Theorem 6.1]. Alternatively we refer to [**FrPi1990a**, Theorem 5.2.1] or [**LW1969**, p. 137] for a proof that the five statements are equivalent. ■

The above results now give us several corollaries on topological manifolds.

Corollary 104.20.

(1) Every topological manifold is homotopy equivalent to a countable locally finite simplicial complex.
(2) The homology groups, the fundamental group and the higher homotopy groups of a 0-connected topological manifold are countable.

Remark. Let M be a 0-connected topological manifold. By Corollary 104.20 we now know that $\pi_1(M)$ is countable. A significantly more down-to-earth proof for this statement is given in [**Lee2000**, Proposition 7.21].

Proof.

(1) This statement follows from Corollary 104.18 together with Theorem 104.19 (4)⇒(3′) and the fact that topological manifolds are by definition second-countable.

(2) Let X be a 0-connected topological manifold. By (1) we know that X is homotopy equivalent to a 0-connected countable simplicial complex Y. It follows from Proposition 94.9 and the Simplicial-Implies-CW Lemma 93.24 together with the Homology-of-CW-Complex Proposition 80.6 that all homotopy groups[1268] and all homology groups of Y are countable. Finally note that it is a consequence of the Homotopy-π_n-Proposition 71.7 and the Homotopy Equivalence-H_*-Corollary 73.9 that the same statements hold for X. ■

By Proposition 86.15 we know that $\pi_2(S^1 \vee S^2)$ is isomorphic to $\mathbb{Z}^\infty$, in particular it is infinitely generated. This result, foreshadowed by our discussion on page 1573, shows that homotopy groups of finite CW-complexes can be infinitely generated. In Question 71.16 we asked, somewhat apprehensively, whether the homotopy groups of a countable CW-complex are necessarily countable. With a slight sigh of relief we can now give an affirmative answer to this question.

Proposition 104.21. If X is a CW-complex with countably many cells, then for any $n \in \mathbb{N}$ and any $x_0 \in X$ the group $\pi_n(X, x_0)$ is countable.

Proof. This follows from the combination of Theorem 104.19 (1)⇒(4) together with the Homotopy-π_n-Proposition 71.7 and Proposition 94.9. ■

Theorem 104.19 deals with (second-) countable objects. The following theorem is the analogue of Theorem 104.19 without any cardinality restrictions and, on the flip side, without any cardinality information.

Theorem 104.22. If X is a topological space, then the following statements are equivalent:

(1) X is dominated by a CW-complex.
(2) X is homotopy equivalent to a CW-complex.
(3) X is homotopy equivalent to a simplicial complex.
(4) X is homotopy equivalent to an ANR.

Proof. The implication "(1)⇒(2)" is precisely the statement of the CW-Complex Domination Proposition 162.8. The fact that (2) and (3) are equivalent is a consequence of the Simplicial-Implies-CW Lemma 93.24 and Proposition 94.12. The statement that in fact all four statements are equivalent is proved in [**FrPi1990a**, Theorem 5.2.1]. ■

Theorem 104.22, which gives no control of the "size" of the objects is not all that helpful. It is much more interesting to ask whether the analogue of Theorem 104.19 holds if we replace "countable" by "finite". In fact the following theorem holds.

Theorem 104.23. Let X be a topological space. The following statements are equivalent:

(2) X is homotopy equivalent to a finite CW-complex.
(3) X is homotopy equivalent to a finite simplicial complex.
(4) X is homotopy equivalent to a compact ANR.

Furthermore the following refinement of the equivalence of (2) and (3) holds: If X is homotopy equivalent to a finite n-dimensional CW-complex, then it is also homotopy equivalent to a finite n-dimensional simplicial complex, and vice versa.

[1268] For the fundamental group we could alternatively use the argument given on page 1537.

Proof. The fact that (2) and (3) are equivalent is a consequence of the Simplicial-Implies-CW Lemma 93.24 and Proposition 94.12. Next note that one can show, with some effort, that it is a consequence of the Dugundji Extension Theorem 104.16 that every finite simplicial complex is an ANR, we refer to [**LW1969**, p. 210] and [**HNV2004**, p. 472] for details. This observation gives us the implication (3)⇒(4). Finally the implication (4)⇒(2) was shown by James West [**Wes1977**] in 1977. ■

The attentive reader will have noticed that the dog that did not bark is the (1)⇒(2) implication. It turns out that there is a good reason why it did not bark. The following theorem, proved by C.T.C. Wall shows that in general the statement one would hope for is actually incorrect.

Theorem 104.24. There exists a topological space that is dominated by a finite CW-complex but that is not homotopy equivalent to a finite CW-complex.

Proof. The statement is proved in [**Wall1965a**, p. 66]. Alternatively see [**Var1989**, p. 163] and [**FeR2001**, Theorem 3.1]. More precisely, in [**Wall1965a, Var1989**] it is shown that to a topological space X that is dominated by a finite CW-complex one can associate an invariant in the "K-group $\widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$" which vanishes if and only if X is homotopy equivalent to a finite CW-complex. It is shown in the above references that there exist examples where the invariant is indeed non-zero. Interestingly it is conjectured, see e.g. [**KrL2005**, p. 179], that the group $\widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ is zero, whenever $\pi_1(X)$ is a torsion-free group. ■

From this discussion we obtain the following corollary.

Corollary 104.25. Every compact topological manifold is homotopy equivalent to a finite simplicial complex.

Proof. This statement follows immediately from Corollary 104.18 together with Theorem 104.23 (4)⇒(3). A very different proof for the corollary was given by Rob Kirby and Larry Siebenmann in [**KSi1969**, Chapter III.2] and [**KSi1977**, p. 744]. ■

Let M be a (compact) n-dimensional topological manifold. By the above we now know that M is homotopy equivalent to a (finite) simplicial complex X. But the above results do not say anything about the dimension of the simplicial complex X. But it turns out, that at least for compact topological manifolds this can mostly be fixed.

Theorem 104.26.

(1) Every closed n-dimensional topological manifold is homotopy equivalent to a finite n-dimensional simplicial complex.
(2) Every compact connected n-dimensional topological manifold with non-empty boundary is homotopy equivalent to a finite $(n-1)$-dimensional simplicial complex.

Remark. It seems like a nice project to figure out whether Theorem 104.26 (2) can be generalized to non-compact topological manifolds.

Sketch of proof. First note that the 1-dimensional case follows immediately from the classification result proved in the Topological 1-Dimensional Manifold Classification Theorem 22.1. Furthermore the 2-dimensional case and the 3-dimensional case follow from

Radó's Theorem 104.29 and Moise's Theorem 104.30 below. Thus we can now assume that $n \geq 4$.

First note that by Theorem 104.22 (2)⇔(3) it suffices to prove the statement with simplicial complexes replaced by CW-complexes.

Next we cite the following theorem of C.T.C. Wall [**Wall1966a**, Corollary 5.1] (see also [**Wall1965a**, Theorem E] and [**Var1989**, Chapter 5.2]):

> **Wall's Finiteness Theorem.** *Let Y be a connected finite* CW*-complex such that there exists an $r \in \mathbb{N}_0$ such that all twisted cohomology groups $\mathrm{H}^s(Y;V)$ vanish for $s > r$. (The precise definition of twisted cohomology is irrelevant to our discussion, so stay with us.) Now* [**Wall1966a**, Corollary 5.1] *says that if these conditions are satisfied and if $r \geq 3$, then Y is homotopy equivalent to a finite r-*dimensional CW*-complex.*[1269]

Theorem 104.26 is now a consequence of Wall's Theorem and some reasonably standard facts in algebraic topology. In the discussion below we will outline the argument, the detailed proof for reducing Theorem 104.26 to Wall's Theorem is given in [**FNOP2019**, Theorem 4.1].

Now let X be a compact orientable n-dimensional topological manifold. It follows from Theorem 104.23 that X is homotopy equivalent to a finite CW-complex Y. It follows from "Poincaré Duality with twisted coefficients", see [**FNOP2019**, Theorem A.15], that for any $r \in \mathbb{N}_0$ and any $\mathbb{Z}[\pi_1(X, x_0)]$-left module V we have $\mathrm{H}^r(Y;V) \cong \mathrm{H}^r(X;V) \cong \mathrm{H}_{n-r}(X, \partial X; V)$. If $r > n$, i.e. if $n-r < 0$, then we have $\mathrm{H}_{n-r}(X, \partial X; V) = 0$. Furthermore, if X is connected and if $\partial X \neq 0$ and if $r = n$, i.e. if $n - r = 0$ we have, by an argument similar to the one given on page 1625, that $\mathrm{H}_{n-r}(X, \partial X; V) = 0$. The desired statement now follows from these calculations together with Wall's Theorem.

Finally we consider the case of a compact non-orientable n-dimensional topological manifold. The argument is basically the same, except that one needs Poincaré Duality for non-orientable manifolds. For untwisted coefficients this can be found in [**DaK2001**, Theorem 5.7]. For twisted coefficients this follows from an argument as in [**FNOP2019**, Theorem A.15], but it is conceivable that nobody ever bothered to work out the details. ■

We conclude this section with the following proposition which can be viewed as an analogue of the Single Top-Cell Proposition 98.16.

Proposition 104.27. Every closed connected non-empty n-dimensional topological manifold is homotopy equivalent to a finite n-dimensional CW-complex which has a single 0-cell and which has a single n-cell.

Proof. Let X be a closed connected non-empty n-dimensional topological manifold. We pick an embedding $\varphi\colon \overline{B}^n \to X$. It follows from Proposition 75.3 that $Y := X \setminus \varphi(B^n)$ is a compact connected non-empty n-dimensional topological manifold with $\partial Y = \varphi(S^{n-1})$. By Theorem 104.26 we know that there exists a finite $(n-1)$-dimensional CW-complex Z

[1269] To the best of my knowledge it seems to be unknown whether the statement holds for $r = 2$.

and a homotopy equivalence $h\colon Y \to Z$. We consider the following diagram:

$$\begin{array}{ccccc} \overline{B}^n & \xleftarrow{\quad i \quad} & S^{n-1} & \xrightarrow{\quad f \quad} & Y \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle h} \\ \overline{B}^n & \xleftarrow{\quad i \quad} & S^{n-1} & \xrightarrow{\quad \varphi\circ f \quad} & Z. \end{array}$$

Note that the diagram commutes, note that the vertical maps are homotopy equivalences and note that it follows from the Manifold Boundary-Cofibration Proposition 17.8 that the inclusion map $i\colon S^{n-1} \to \overline{B}^n$ is a closed cofibration. Thus it follows from the Homotopy Pushout Theorem 17.19 that the pushouts $\overline{B}^n \cup_{S^{n-1}} Y$ and $\overline{B}^n \cup_{S^{n-1}} Z$ are homotopy equivalent. By Lemma 5.36 the former pushout is homeomorphic to X. Furthermore by the CW-Complex Construction Lemma 68.32 the latter pushout admits a CW-structure with a single n-cell and no cells of dimension $> n$. Finally note that it follows from Proposition 70.4 that we can in fact find a CW-complex that is homotopy equivalent to X that contains not only a single n-cell but that also contains only a single 0-cell. ■

104.6. **Topological manifolds and CW-structures.** Recall that by the Smooth Manifold-CW Structure Theorem 96.5 we know that every smooth manifold admits a simplicial structure. By Corollaries 104.20 and 104.25 we now know that every (compact) topological manifold is homotopy equivalent to a (finite) CW-complex. The following question naturally arises.

Question 104.28.

(1) Does every topological manifold admit a simplicial structure?
(2) Does every topological manifold admit a CW-structure?

In the following we will survey what is known regarding Question 104.28. Let us start out with some positive statements.

Theorem 104.29. (Radó's Theorem) Every 2-dimensional topological manifold admits a PL-structure, in particular it admits a simplicial and a CW-structure.

Proof. In 1926 it was shown by Tibor Radó [**Rad1926**] that every 2-dimensional topological manifold admits a simplicial structure. It is elementary to see that the simplicial structure of any 1-dimensional topological manifold is in fact a PL-structure. It follows from the PL-Manifold Characterization Proposition 96.12 that the simplicial structure of any 2-dimensional topological manifold is in fact a PL-structure.

Note that we already mentioned the above result of Tibor Radó in the proof of the Surface-Smooth Structure Existence Theorem 55.5. We refer to that proof to alternative references. ■

Theorem 104.30. (Moise's Theorem) Every 3-dimensional topological manifold admits a PL-structure, in particular it admits a simplicial and a CW-structure.

Proof. This theorem was first proved in 1952 by Edwin Moise [**Moi1952**]. Alternative proofs are given in [**Moi1977**, p. 252 and 253], [**Moi1952**, Theorem 6], [**Bin1983**, Theorem XVIII.3.B], [**GNR2021**], [**Ham1976**, Theorem 2], [**Shal1984**]. ■

Now we turn to the higher dimensional setting. In this setting the stories between CW-structures and simplicial structures diverge. In the following we first consider CW-structures

on high-dimensional topological manifolds. It turns out that the 4-dimensional case is a no-show, in the sense that the following question is still open.

Question 104.31. Does every closed 4-dimensional topological manifold admit a CW-structure?

Once we move beyond dimension 4, the situation becomes somewhat clearer again.

Theorem 104.32. (Kirby-Quinn-Siebenmann Theorem) Every *closed* topological manifold of dimension ≥ 5 has a CW-structure.

Proof. For dimensions ≥ 6 this theorem was proved in the 1970s by Rob Kirby and Larry Siebenmann [**KSi1977**, Essay III.2]. In 1982 this result was extended to the 5-dimensional case by Frank Quinn [**Qu1982**, Theorem 2.3.1]. ■

Note that the Kirby-Quinn-Siebenmann Theorem 104.32 only deals with *closed* topological manifolds. The following question is still open.

Question 104.33. Let $n \in \mathbb{N}_{\geq 5}$. Does every *compact* n-dimensional topological manifold admit a CW-structure?

Now what about simplicial structures?

Theorem 104.34. There exists a closed 4-dimensional topological manifold that does not admit a simplicial structure.

Proof. This theorem was proved by Andrew Casson, see [**AkM1990**, page xvi], in the 1980s building on the work of Mike Freedman [**Fre1982**]. An account of the proof is also given in [**Sav2012**, Theorem 18.3]. ■

The question whether topological manifolds of dimension ≥ 5 admit a simplicial structure had been open for a very long time and it was finally resolved in 2013 by Ciprian Manolescu [**Man2016a**].

Theorem 104.35. (Manolescu's Theorem) For every $n \in \mathbb{N}_{\geq 5}$ there exists a closed n-dimensional topological manifold that does not admit a simplicial structure.



Proof. The theorem is proved in [**Man2016a**]. A more relaxed exposition of the key ideas of the proof is also given in [**Man2016b**]. Note that, as is explained in [**Man2016a, Man2016b**], in dimension 5 the examples are necessarily non-orientable, whereas for every $n \geq 5$ there exists a closed orientable n-dimensional topological manifold that does not admit a simplicial structure.[1270] ■

In the unlikely event that the reader got slightly confused by the various theorems and counterexamples we now summarize some of the key statements about the topology of

[1270] We will explain later on in Chapter 105 what it means for a topological manifold to be orientable.

compact topological manifolds in the following table:

	$n=1,2,3$	$n=4$	$n\geq 5$
each closed top. n-mfd. is homotopy equiv. to a finite CW-cplx.	✓	✓	✓
each closed top. n-mfd. admits a CW-structure	✓	?	✓
each compact top. n-mfd. is homotopy equ. to a finite CW-cplx.	✓	✓	✓
each compact top. n-manifold admits a CW-structure	✓	?	?
each compact top. n-manifold admits a simplicial structure	✓	✗	✗

Finally we note that we will discuss the existence and uniqueness of PL-structures and smooth structures on topological manifolds in greater detail in Chapter **??**.

104.7. **The Euler characteristic of compact topological manifolds.** The Euler characteristic of a finite CW-complex is one of the most convenient invariants and often it gives a convenient approach to calculating invariants of CW-complexes. As we just saw, it is not clear whether (compact) topological manifolds admit CW-structures. But fortunately we assembled just enough results on compact topological manifolds to be able to develop the theory of Euler characteristics of compact topological manifolds.
The first part of the following definition was already introduced on page 1932.

Definition.

(1) Given a pair (X,A) of topological spaces and $n\in\mathbb{N}_0$ we refer to

$$b_n(X,A) \;:=\; \operatorname{rank}(\mathrm{H}_n(X,A))$$

as the n-th Betti number of (X,A). For $A=\varnothing$ we write of course $b_n(X):=b_n(X,\varnothing)$.

(2) Let (X,A) be a pair of topological spaces. If all $b_n(X,A)$ are finite and if the sum $\sum_{n\in\mathbb{N}_0} b_n(X,A)$ is finite, then we define the Euler characteristic of (X,A) to be

$$\chi(X,A) \;:=\; \sum_{n\in\mathbb{N}_0}(-1)^n\cdot b_n(X,A).$$

For $A=\varnothing$ we write $\chi(X):=\chi(X,\varnothing)$.

Remark.

(1) It follows from the Euler Characteristic-H_*-Proposition 87.1 that for a finite CW-complex the above definition agrees with the original definition of the Euler characteristic of a finite CW-complex.

(2) Let X be a compact topological manifold and let A be a compact submanifold or let A be a union of components of ∂X. It follows from Propositions 75.2 (3) and (4) and the Topological Manifolds-Invariants Proposition 104.14 (0) and (4), together with Lemma 96.8, that the Euler characteristic $\chi(X,A)$ is defined.

It turns out that the formulas we proved before for the Euler characteristic of CW-complexes also hold for topological manifolds. More precisely, we have the following proposition, of which the first three statements are the analogue of Lemmas 87.8 and 87.5 and of Proposition 69.4 in our new context.

Proposition 104.36. (Manifold-χ-Proposition)

(1) Let X and Y be homotopy equivalent topological spaces. If the Euler characteristic is defined for one of the two topological spaces, then it is defined for the other topological space and we have
$$\chi(X) \;=\; \chi(Y).$$
(2) Let X be and Y be two compact topological manifolds, then
$$\chi(X \times Y) \;=\; \chi(X) \cdot \chi(Y).$$
(3) Let $X = Y \cup Z$ be a decomposition of a compact topological manifold into two compact submanifolds Y and Z such that $Y \cap Z$ is also a compact submanifold of X. Then the following equality holds
$$\chi(X) \;=\; \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$
(4) Let X be a compact topological manifold and let $A \subset X$ be a subspace that is a compact topological manifold in its own right.[1271] Then
$$\chi(X, A) \;=\; \chi(X) \,-\, \chi(A).$$
Furthermore, for any field $\mathbb{F}$ we have
$$\chi(X, A) \;=\; \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(X, A; \mathbb{F})).$$
(5) Let X be a compact topological manifold and let $p\colon \widetilde{X} \to X$ be a finite covering. Then
$$\chi(\widetilde{X}) \;=\; \big[\widetilde{X} : X\big] \cdot \chi(X).$$

Sketch of proof.

(1) This statement follows immediately from the Homotopy Equivalence-H_*-Corollary 73.9.
(2) This equality can be proved using the Künneth Theorem 90.7 for Chain Complexes. We will fill in the details in Exercise 104.9.
(3) We will not make use of this statement. Therefore we hide, with a good conscience, the proof in Exercise 104.10.
(4) Let X be a compact topological manifold and let $A \subset X$ be a subspace that is a compact topological manifold in its own right.
   (a) It follows immediately from considering the long exact sequence in homology of the pair (X, A) together with Lemma 87.4 that $\chi(X, A) = \chi(X) - \chi(A)$.
   (b) Let $\mathbb{F}$ be a field. Let Y be a compact topological manifold. From Corollary 104.25 we know that Y is homotopy equivalent to a finite CW-complex Z. We observe that

$$\chi(Y) \;\overset{\text{Homotopy Equivalence-H}_*\text{-Corollary 73.9}}{\underset{\text{see page 1915}}{=}}\; \chi(Z) \;\underset{\uparrow}{=}\; \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(Z; \mathbb{F})) \;\underset{\uparrow}{=}\; \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(Y; \mathbb{F})).$$
by the generalization of the Homotopy Equivalence-H_*-Corollary 73.9 to $\mathbb{F}$-coefficients, see page 1914

[1271] For example A could be compact submanifold or A could be a union of boundary components.

Using this observation we can now prove that

$$\chi(X,A) \underset{\text{by (a)}}{\underset{\uparrow}{=}} \chi(X) - \chi(A) \underset{\text{by the above observation}}{\underset{\uparrow}{=}} \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(X;\mathbb{F})) - \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(A;\mathbb{F}))$$
$$\underset{\text{by the long exact sequence of the pair } (X,A) \text{ with } \mathbb{F}\text{-coefficients and Lemma 87.3}}{\underset{\uparrow}{=}} \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(X,A;\mathbb{F})).$$

(5) Let X be a compact topological manifold and let $p\colon \widetilde{X} \to X$ be a finite covering. Without loss of generality we can assume that X and $\widetilde{X}$ are connected. By Corollary 104.25 there exists a homotopy equivalence $f\colon X \to Y$, where Y is a finite CW-complex. By the Covering Existence Theorem 61.5 there exists a connected covering $q\colon \widetilde{Y} \to Y$ such that $q_*(\pi_1(\widetilde{Y})) = f_*(p_*(\pi_1(\widetilde{X})))$. Using the Map Lifting Criterion 61.2 one can now easily show that $f\colon X \to Y$ lifts to map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ and that this map is again a homotopy equivalence[1272]. Now we obtain that

$$\chi(\widetilde{X}) \underset{\text{by (1) since } \widetilde{X} \text{ and } \widetilde{Y} \text{ are homotopy equivalent}}{\underset{\uparrow}{=}} \chi(\widetilde{Y}) \underset{\text{Proposition 69.4 and Lemma 48.15}}{\underset{\uparrow}{=}} \big[\pi_1(Y) : q_*(\pi_1(\widetilde{Y}))\big] \cdot \chi(Y) \underset{\text{by (1) we have } \chi(Y) = \chi(X) \text{ since } X \text{ and } Y \text{ are homotopy equivalent}}{\underset{\uparrow}{\overset{\downarrow}{=}}} \big[\pi_1(X) : p_*(\pi_1(\widetilde{X}))\big] \cdot \chi(X)$$
$$\underset{\text{Lemma 48.15}}{\underset{\uparrow}{=}} \big[\widetilde{X} : X\big] \cdot \chi(X).$$

$\blacksquare$

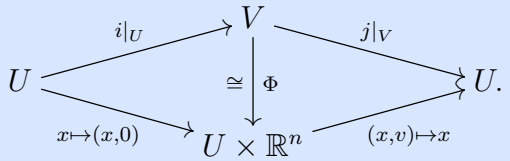
104.8. **Microbundles.** In this section we will give a short introduction to the concept of a normal microbundle of a submanifold of a topological manifold. A normal microbundle can be viewed as an analogue of the normal bundle of a smooth submanifold of a smooth manifold. We will not make use of microbundles. So this section we can safely be skipped.

We start out with the definition of a microbundle, which was introduced by John Milnor [**Miln1964a**, Chapter 2].

Definition. A microbundle consists of two topological spaces B and E and two maps $i\colon B \to E$ and $j\colon E \to B$ such that the following two conditions are satisfied:

(1) we have $j \circ i = \mathrm{id}_B$,

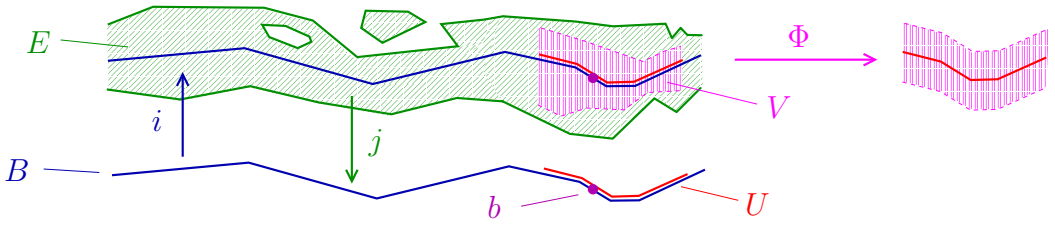
(2) for every $b \in B$ there exists an open neighborhood U of $b \in B$ and an open neighborhood V of $i(b) \in E$ with $i(U) \subset V$ and $j(V) \subset U$ and there exists a homeomorphism $\Phi\colon V \to U \times \mathbb{R}^n$ such that the following diagram commutes:



Such a map Φ is called a local trivialization.

[1272]Here the point is that not only the homotopy equivalences but also the homotopies lift to the covering spaces. We leave the details to the reader.

We refer to B as the base space of the microbundle, we refer to E as the total spaces of the microbundle and refer to n as the fiber dimension of the microbundle.[1273]



Examples.

(1) Every $\mathbb{R}^n$-bundle (in particular every vector bundle) is evidently a microbundle. But the notion of a microbundle is much less restrictive. For example we only have control over a neighborhood of $i(B) \subset E$ and the local trivializations do not have to be compatible in any way.

(2) Let M be a topological manifold. One can easily show, see also [**Miln1964a**, Lemma 2.1], that the maps

$$\begin{array}{rcl} i\colon M & \to & M \times M \\ x & \mapsto & (x,x) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} j\colon M \times M & \to & M \\ (x,y) & \mapsto & x \end{array}$$

form a microbundle. This microbundle is called the *tangent microbundle of* M.[1274] This definition and this name is inspired by the Total Tangent Space-Diagonal Embedding Proposition 45.16 which says given a closed smooth submanifold of $\mathbb{R}^n$ there exists a smooth embedding of the tangent bundle $\widetilde{\mathrm{T}}M$ into the product $M \times M$ with with $(x,0) \mapsto (x,x)$ for each $x \in M$.

As we just discussed, $\mathbb{R}^n$-bundles are microbundles. Somewhat surprisingly, in the settings we are interested in, every microbundle is essentially an $\mathbb{R}^n$-bundle.

Theorem 104.37. (Kister-Mazur Theorem) Let $i\colon B \to E$ and $j\colon E \to B$ be a microbundle. If B is a locally finite, finite dimensional simplicial complex or if B is a topological manifold, then there exists an open neighborhood U of $i(B) \subset E$ such that the restriction of j to U is an $\mathbb{R}^n$-bundle.

Proof. This theorem was proved by James Kister [**Kis1964**, Corollary 1 and Theorem 2] (see also [**Kis1963**]) and independently by Barry Mazur. The proof is also written up as [**Kupe2017b**, Theorem 3.7]. ■

Definition. Let X be a topological manifold and let Y be a submanifold of X. A normal microbundle of Y in X consists of a neighborhood U of $Y \subset X$ and a retraction $j\colon U \to M$ such that the inclusion $i\colon Y \to U$ and the retraction $j\colon U \to Y$ form a microbundle.

Remark.

(1) Note that if Y admits a normal microbundle, then Y is in particular a neighborhood retract in X in the sense of the definition on page 2291.

(2) A normal microbundle can be viewed as an analogue of the normal bundle of a smooth submanifold of a smooth manifold.

[1273] Using the local homology groups $\mathrm{H}_k(j^{-1}(b), j^{-1}(b), i(b))$ one can show that n is well-defined.

Colin Rourke and Brian Sanderson [**RS1968**, Theorem 3.12] gave an example of a topological submanifold of $S^9 \times S^{19}$ that is homeomorphic to S^{19} and that does not admit a normal microbundle. But fortunately the following theorem says that normal microbundles exist if the codimension of the submanifold is sufficiently large.

Theorem 104.38. Let X be an $(n+q)$-dimensional topological manifold and let Y be an n-dimensional proper submanifold of X. If there exists a $j \in \{0,1,2\}$ with $n \leq q+1+j$ and $q \geq 5+j$, then Y admits a normal microbundle restricting to a normal microbundle of $\partial N \subset \partial M$. Furthermore, if $n \leq q+j$, then this normal microbundle is unique up to isotopy.

Proof. This theorem is [**Ster1975**, Theorem 4.5]. ■

We will not develop the theory of microbundles any further. We refer to [**Scor2005**] and [**KSi1977**] for more information and the role they play in the study of topological manifolds.

Exercises for Chapter 104.

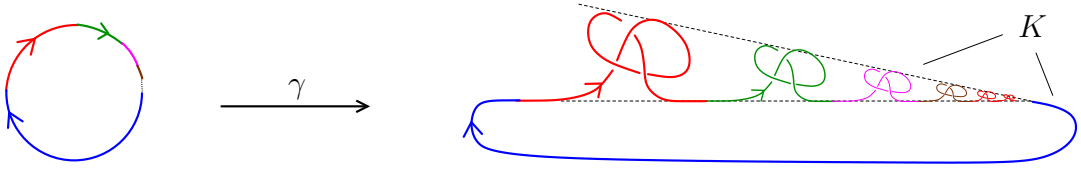
Exercise 104.1.

(1) Show that there are only countably many homotopy types of finite CW-complexes.
Hint. Use Theorem 104.23.
(2) Show that there are uncountably many homotopy types of countable CW-complexes.
(3) Let $k, n \in \mathbb{N}$.
   (a) Are there only finitely many homotopy types of k-dimensional CW-complexes with at most n cells?
   (b) Are there only finitely many homotopy types of k-dimensional simplicial complexes with at most n simplices?

Exercise 104.2. We consider $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ as a subset of $Y = \mathbb{R}$. Show that there is no open neighborhood of X in Y that admits a retraction onto X.

Exercise 104.3. We consider the embedding $\gamma\colon S^1 \to \mathbb{R}^3$ that is shown in the figure below. We set $K := \gamma(S^1)$.

(a) Show that $\pi_1(\mathbb{R}^3 \setminus K)$ is an infinitely generated group.
Remark. The rigor of the solution is bounded from above by the precision of the statement of the problem.
(b) What do you think are the homology groups of $\mathbb{R}^3 \setminus K$?
(c) Show that there is no neighborhood U of K that admits a deformation retraction to K.
Remark. Write to the author if you have a solution.



Exercise 104.4. Show that there exist embeddings $f\colon S^1 \to S^3$ which do not admit a tubular neighborhood, i.e. for which there is no map $\Phi\colon \overline{B}^2 \times S^1 \to S^3$ with $f(P) = \Phi(P, 0)$ for all $P \in S^1$.

Exercise 104.5. Let C be the Cantor set as defined on page 130. Note that basically by construction the complement of C in $[0,1]$ is the disjoint union of countably many open intervals $\{(a_n - \epsilon_n, a_n + \epsilon_n)\}_{n\in\mathbb{N}}$. We set

$$X \;:=\; \underbrace{\{(c,0) \in \mathbb{R}^2 \,|\, c \in C\}}_{=C\times\{0\}\subset\mathbb{R}^2} \cup \bigcup_{n\in\mathbb{N}} \overline{B}^2_{\epsilon_n}((a_n,0)) \;\subset\; \mathbb{R}^2.$$

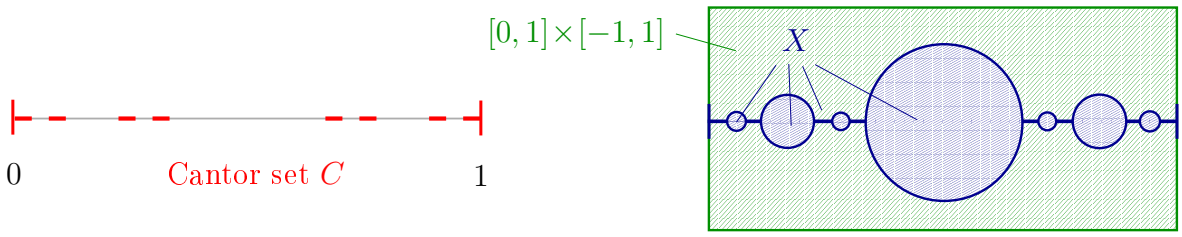
(We refer to the figure below for an illustration.) In the following we will show that X is a retract of a finite CW-complex but that X itself is not a CW-complex.

(a) Show that X is a deformation retract of the CW-complex $[0,1] \times [-1,1] \subset \mathbb{R}^2$.
(b) Show that X is compact.

Given a path-connected topological space Y we define

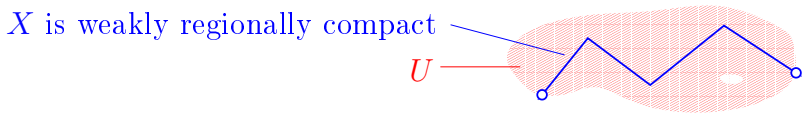
$$S(Y) := \{y \in Y \,|\, Y \setminus \{y\} \text{ is not path-connected}\}.$$

(c) Show that $S(X) = \{(c,0) \,|\, c \in C \setminus \{0,1\}\}$.
(d) Let Y be a finite CW-complex. Show that $S(Y)$ has only finitely many path-components.
(e) Show that X does not admit the structure of a CW-complex.



Exercise 104.6. We say that a topological space X is *weakly regionally compact* if every point $x \in X$ admits a compact neighborhood. Now let $X \subset \mathbb{R}^n$ be a subset that is weakly regionally compact. In the following we will show that there exists an embedding $g\colon X \to \mathbb{R}^{n+1}$ such that $g(X)$ is a *closed* subset of $\mathbb{R}^{n+1}$.

(a) Show that there exists an open subset $U \subset \mathbb{R}^n$ with $U \cap \overline{X} = X$.
(b) We set $C := \overline{X} \setminus X = \overline{X} \setminus U$. Note that it follows from the Interior-Closure Lemma 1.9 that C is a closed subset of $\mathbb{R}^n$ and note that it follows from Lemma 6.3 that $d(x, C) > 0$ for every $x \in X$. Now we consider the map $f\colon X \to \mathbb{R}$ given by $x \mapsto d(x, C)$. Show that the map $g\colon X \to \mathbb{R}^{n+1}$ given by $x \mapsto (x, \frac{1}{f(x)})$ is an embedding such that $g(X)$ is a closed subset of $\mathbb{R}^{n+1}$.



Exercise 104.7. Let X be a compact space that is a retract of a finite simplicial complex X. Show that X is an ENR.
Remark. This is the, very easy, converse to the ENR-Simplicial Complex Retract Proposition 104.7.

Exercise 104.8. Let X be a compact space. Is $\mathrm{H}_1(X)$ necessarily finitely generated?

Exercise 104.9. Let X be and Y be two compact topological manifolds. Show that

$$\chi(X \times Y) \;=\; \chi(X) \cdot \chi(Y).$$

Exercise 104.10. Let $X = Y \cup Z$ be a decomposition of a compact topological manifold into two compact submanifolds Y and Z such that $\partial Y = \partial Z$ is a submanifold of X. Show that

$$\chi(X) \;=\; \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$

105. Orientations of topological manifolds

In Section 25.2 we introduced the notion of an orientation of a smooth manifold. The basic idea is that an orientation of a smooth manifold is a "continuous" choice of orientations for each tangent space. Since topological manifolds do not come with tangent spaces there is no obvious extension of this definition of an orientation to the context of topological manifolds. In this chapter we will use "orientations of local homology groups" to introduce the concept of an orientation on a topological manifold.

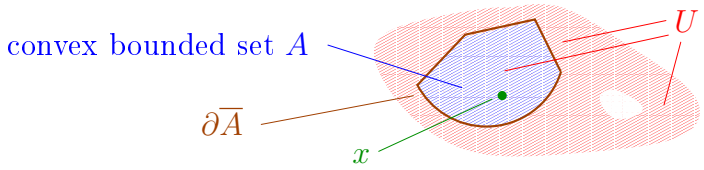
105.1. **Relative homology and convex subsets of $\mathbb{R}^n$.** The next lemma is at first glance a little dull, but i will come in handy on several occasions later on in this chapter. In all likelihood it is best to have a quick glance at the statement and to move on to the next section.

Lemma 105.1. (Complement-of-Convex Homology Lemma) Let $n \in \mathbb{N}$, let U be an open subset of $\mathbb{R}^n$, let A be a convex bounded subset of U with $\overline{A} \subset U$ and let $x \in A$. Then for every abelian group G and every $i \in \mathbb{N}_0$ the following inclusion induced maps are isomorphisms:

$$\begin{array}{lrcl}
(1) & \mathrm{H}_i(\mathbb{R}^n \setminus A; G) & \to & \mathrm{H}_i(\mathbb{R}^n \setminus \{x\}; G) \\
(2) & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus A; G) & \to & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}; G) \\
(3) & \mathrm{H}_i(U, U \setminus A; G) & \to & \mathrm{H}_i(U, U \setminus \{x\}; G).
\end{array}$$

If $\mathring{A}$ is non-empty and if $x \in \mathring{A}$ is a point, then also the following inclusion induced maps are isomorphisms

$$\begin{array}{lrcl}
(4) & \mathrm{H}_i(\partial\overline{A}; G) & \to & \mathrm{H}_i(\mathbb{R}^n \setminus \{x\}; G) \\
(5) & \mathrm{H}_i(\mathbb{R}^n, \partial\overline{A}; G) & \to & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}; G)
\end{array}$$



Proof. Let U be an open subset of $\mathbb{R}^n$, let A be a convex bounded subset of U with $\overline{A} \subset U$ and let $x \in A$. To simplify the notation we drop the coefficients G from the notation and without loss of generality we assume that $x = 0$. Now let $i \in \mathbb{N}_0$.

(1) We have to show that the inclusion induced map $\mathrm{H}_i(\mathbb{R}^n \setminus A) \to \mathrm{H}_i(\mathbb{R}^n \setminus \{0\})$ is an isomorphism. We had already stated this fact in Exercise 73.3. Now that we actually need the statement we give the full details.

Since A is bounded we can pick an $r > 0$ such that $A \subset B_r^n(0)$. We consider the following map:

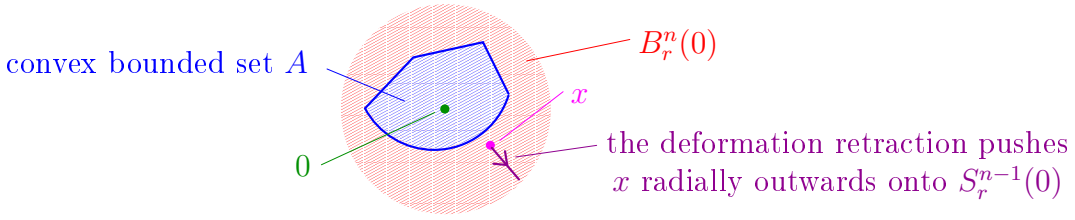
$$\begin{array}{rcl}
F\colon (\mathbb{R}^n \setminus A) \times [0,1] & \to & \mathbb{R}^n \\
(x,t) & \mapsto & \begin{cases} x, & \text{if } x \in \mathbb{R}^n \setminus B_r^n(0), \\ x \cdot \underbrace{\big((1-t) + t \cdot \frac{r}{\|x\|}\big)}_{\geq 1 \text{ since } r > \|x\|}, & \text{if } x \in \overline{B}_r^n(0) \setminus A. \end{cases}
\end{array}$$

Note that if $y \notin A$, then it follows from the convexity of A and the fact that $0 \in A$, that for any $s \geq 1$ the point sy cannot lie in A either. This shows that the above map actually takes values in $\mathbb{R}^n \setminus A$.

Now it is obvious that the map F defines a deformation retraction from $\mathbb{R}^n \setminus A$ to $\mathbb{R}^n \setminus B_r^n(0)$.[1275] [1276] Clearly $\mathbb{R}^n \setminus B_r^n(0)$ is also a deformation retract of $\mathbb{R}^n \setminus \{0\}$. Now we consider the following commutative diagram of inclusion induced maps:

$$\begin{array}{ccc} \mathrm{H}_i(\mathbb{R}^n \setminus A) & \longrightarrow & \mathrm{H}_i(\mathbb{R}^n \setminus \{0\}) \\ & \nwarrow \quad \nearrow & \\ & \mathrm{H}_i(\mathbb{R}^n \setminus B_r^n(0)). & \end{array}$$

It follows from the Homotopy Equivalence-H_*-Corollary 73.9 and the above discussion that the two diagonal maps are isomorphisms. It follows that the horizontal map is also an isomorphism.



(2) Now we consider the following commutative diagram of long exact sequences

$$\begin{array}{ccccccccccc} \ldots \longrightarrow \mathrm{H}_i(\mathbb{R}^n \setminus A) & \longrightarrow & \mathrm{H}_i(\mathbb{R}^n) & \longrightarrow & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus A) & \longrightarrow & \mathrm{H}_{i-1}(\mathbb{R}^n \setminus A) & \longrightarrow & \mathrm{H}_{i-1}(\mathbb{R}^n) \to \ldots \\ \downarrow \cong & & \downarrow = & & \downarrow \cong & & \downarrow \cong & & \downarrow = \\ \ldots \to \mathrm{H}_i(\mathbb{R}^n \setminus \{x\}) & \to & \mathrm{H}_i(\mathbb{R}^n) & \to & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) & \to & \mathrm{H}_{i-1}(\mathbb{R}^n \setminus \{x\}) & \to & \mathrm{H}_{i-1}(\mathbb{R}^n) \to \ldots \end{array}$$

By (1) the first and the fourth vertical maps are isomorphisms. The second and the fifth vertical map are the identity. It follows from the Five Lemma 74.10 that the middle vertical map is also an isomorphism.

(3) We consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}_i(U, U \setminus A) & \longrightarrow & \mathrm{H}_i(U, U \setminus \{x\}) \\ \downarrow & & \downarrow \\ \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus A) & \longrightarrow & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \end{array}$$

of inclusion induced homomorphisms. Since $\overline{A} \subset U$ we deduce from the Excision Theorem 74.18 that the two vertical maps are isomorphisms. By (2) we know that the bottom horizontal map is an isomorphism. Hence the top horizontal map is also an isomorphism.

[1275] Note that the fact that $0 \in A$ and the convexity of A imply that for any $x \notin A$ and any $s \geq 1$ we have $s \cdot x \notin A$, in particular the map F does indeed take values in $\mathbb{R}^n \setminus A$.

[1276] As always, checking that a map is a deformation retraction also requires showing that the map is continuous. In this case the map is continuous by the Pasting Proposition 2.6 and the fact that $B_r^n(0) \setminus A$ is a closed subset of $\mathbb{R}^n \setminus A$.

Now we suppose that $\mathring{A}$ is non-empty and we suppose that x is a point in $\mathring{A}$. It follows from Exercise 1.23 that without loss of generality we can assume that $x = 0$ and that $A = \overline{A}$.

(4) Since A is compact, convex with non-empty boundary we can apply Lemma 14.2. Thus we now know that there exists an isotopy

$$F\colon A \times [0,1] \;\to\; \mathbb{R}^n$$

from the inclusion map $i\colon A \to \mathbb{R}^n$ to a homeomorphism $\varphi\colon A \to \overline{B}^n$ that has the following two properties:
(a) $F_1 = \varphi$ restricts to a homeomorphism $\partial A \to S^{n-1}$, and
(b) F has the property that $F(0,t) = 0$ for all $t \in [0,1]$.
Now we have the following diagram

$$\begin{array}{ccccc} & & \mathrm{H}_i(\partial A) & & \\ & \overset{\cong}{\swarrow}{\scriptstyle \varphi_*} & \downarrow{\scriptstyle \varphi_*} & \searrow{\scriptstyle (i|_{\partial A})_*} & \\ \mathrm{H}_i(S^{n-1}) & \xrightarrow[\cong]{} & \mathrm{H}_i(\mathbb{R}^n \setminus \{0\}) & \xrightarrow[=]{} & \mathrm{H}_i(\mathbb{R}^n \setminus \{0\}). \end{array}$$

Here all the unmarked maps are induced by inclusions. The right-hand triangle commutes by the Homotopy Equivalence-H_*-Corollary 73.9 since, by the above property (b), the isotopy F restricts to an isotopy $\partial A \times [0,1] \to \mathbb{R}^n \setminus \{0\}$. The left horizontal map is an isomorphism since S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$. It is now clear that the right-hand diagonal map is also an isomorphism.

(5) With the same argument as in (2) we can reduce the proof of this statement to (4). ■

105.2. **The local homology of topological manifolds.** We recall the following definition from page 1655.

Definition. Given a topological space X and a point $x_0 \in X$ the homology groups $\mathrm{H}_n(X, X \setminus \{x_0\}; R)$ is called the n-local homology group of X at the point x_0.

In this section we will study the local homology of topological manifolds, in particular we want to gain a good understanding of generators of these groups.
Some of the following definitions and conventions we had already introduced earlier.

Definition. Let $n \in \mathbb{N}$.

(1) We equip $\mathbb{R}^n$ with the orientation given by the standard basis $e_1, \ldots, e_n$. With this orientation we can view any open subset of $\mathbb{R}^n$ as an oriented smooth manifold.

(2) We write

$$\partial\Delta^n \;=\; \bigcup_{i=0}^{n} \{(t_0, \ldots, t_n) \in \Delta^n \,|\, t_i = 0\}.$$

and we refer to $\mathring{\Delta}^n := \Delta^n \setminus \partial\Delta^n$ as the interior of Δ^n.

(3) As on page 1581 we say that a basis $\{v_1, \ldots, v_n\}$ for $\mathrm{T}_P\mathring{\Delta}^n$ is positive, if the ordered set of vectors $\{(1, \ldots, 1), v_1, \ldots, v_n\}$ is a positive basis for $\mathbb{R}^{n+1}$. This equips $\mathring{\Delta}^n$ with the structure of an oriented smooth manifold.

(4) Given a subset V of $\mathbb{R}^n$ we say that a map $f\colon V \to \mathbb{R}^k$ is affine linear if there exists a $P \in \mathbb{R}^k$ and a $k \times n$–matrix A such that $f(v) = P + Av$ for all $v \in V$.



orientations of the interiors of the standard simplices

Example. The projection

$$\begin{array}{rcl} p\colon \Delta^n & \to & \mathbb{R}^n \\ (x_1, \ldots, x_{n+1}) & \mapsto & (x_1, \ldots, x_n) \end{array}$$

is an injective affine linear map. One can easily verify that the restriction of this map to the interior $\mathring{\Delta}^n$ is orientation-preserving if and only if n is even.



projection is orientation-reversing projection is orientation-preserving

We continue with the following lengthy lemma.

Lemma 105.2. Let R be a commutative ring.

(1) For any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(\Delta^n, \partial\Delta^n; R) \;=\; \begin{cases} 0, & \text{if } k \neq n, \\ R \cdot [\mathrm{id}_{\Delta^n} \otimes 1], & \text{if } k = n. \end{cases}$$

(2) Let $\Phi\colon \Delta^n \to \mathbb{R}^n$ be an injective affine linear map such that 0 lies in the image of $\mathring{\Delta}^n$ and such that the restriction of Φ to $\mathring{\Delta}^n$ is orientation-preserving. Then for any $k \in \mathbb{N}_0$ the induced map

$$\Phi_*\colon \mathrm{H}_k(\Delta^n, \partial\Delta^n; R) \;\to\; \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R)$$

is an isomorphism and this map does not depend on the choice of Φ.

(3) Let $P \in \mathbb{R}^n$. We pick $r > 0$ such that $B_r^n(0)$ contains P. Then for any $k \in \mathbb{N}_0$ the inclusion induced maps

$$\mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R) \;\leftarrow\; \mathrm{H}_k\big(\mathbb{R}^n, \mathbb{R}^n \setminus B_r^n(0); R\big) \;\leftarrow\; \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R)$$

are isomorphisms and the resulting isomorphism

$$\mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R) \;\xrightarrow{\cong}\; \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R)$$

does not depend on the choice of r.

(4) Let $P \in \mathbb{R}^n$ be a point. If $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ is the reflection in an affine hyperplane through P, then the induced map

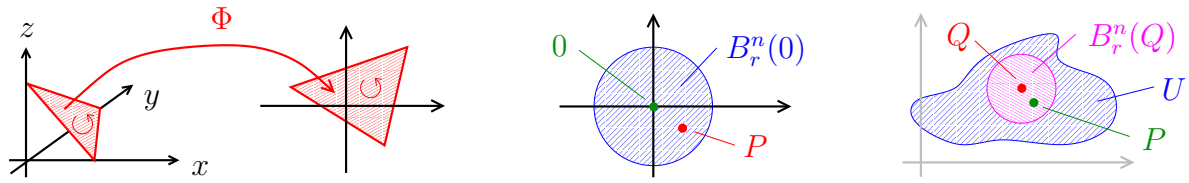
$$\rho_*\colon \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R) \to \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R)$$

is given by multiplication by minus -1. The same conclusion also holds if we replace $\mathbb{R}^n$ by any open ball centered at P.

(5) Let $U \subset \mathbb{R}^n$ be an open set and let $\overline{B}_r^n(Q) \subset U$ be a closed ball. Then for any $P \in B_r^n(Q)$ the inclusion induced maps

$$\begin{array}{ccc} \mathrm{H}_k(U, U \setminus B_r^n(Q); R) & \longrightarrow & \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus B_r^n(Q); R) \\ \downarrow & & \downarrow \\ \mathrm{H}_k(U, U \setminus \{P\}; R) & \longrightarrow & \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R) \end{array}$$

are isomorphisms and they form a commutative diagram.



Proof.

(1) For $k \neq n$ the statement is proved the same way as Lemma 76.1. Next recall that in the Identity-is-Generator Lemma 76.2 (2) we saw that the identity map $\mathrm{id}\colon \Delta^n \to \Delta^n$ represents a generator of $\mathrm{H}_n(\Delta^n, \partial\Delta^n; \mathbb{Z}) \cong \mathbb{Z}$. Basically the same proof shows that for any commutative ring R the element $\mathrm{id}_{\Delta^n} \otimes 1 \in \mathrm{C}_n(\Delta^n, \partial\Delta^n; R)$ represents a generator of the homology group $\mathrm{H}_n(\Delta^n, \partial\Delta^n; R) \cong R$.

(2) Let $\Phi\colon \Delta^n \to \mathbb{R}^n$ be an injective affine linear map such that 0 lies in the image of $\mathring{\Delta}^n$ and such that the restriction of Φ to $\mathring{\Delta}^n$ is orientation-preserving. Note that it follows from the Compact-Hausdorff Proposition 2.17 (2) that Φ is an embedding.

Then we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}_k(\Delta^n, \partial\Delta^n; R) & \xrightarrow{\Phi_*} & \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R). \\ {\scriptstyle \Phi_*}\downarrow & & \uparrow \\ \mathrm{H}_k(\Phi(\Delta^n), \Phi(\partial\Delta^n); R) & \longrightarrow & \mathrm{H}_k(\mathbb{R}^n, \Phi(\partial\Delta^n); R) \end{array}$$

where the bottom and the right vertical map are induced by inclusions. Here the left vertical map is an isomorphism since Φ is an embedding.

Since Φ is an injective affine linear map we see that $\Phi(\Delta^n)$ is a convex subset of $\mathbb{R}^n$ with non-empty interior. By our hypothesis the point 0 lies in the interior of $\Phi(\Delta^n)$. It follows from the Complement-of-Convex Homology Lemma 105.1 (5) that the right vertical map is an isomorphism. Clearly $\Phi(\Delta^n) \to \mathbb{R}^n$ is a homotopy equivalence, hence it follows from Corollary 74.16 that the bottom horizontal map is an isomorphism. Since the diagram commutes we have thus now shown, as desired, that the top horizontal map is also an isomorphism.

If $\Psi\colon \Delta^n \to \mathbb{R}^n$ is another such injective affine linear orientation-preserving map, then it follows from elementary linear algebra that there exists some $P \in \mathbb{R}^n$ and a matrix A with *positive* determinant such that

$$\Phi \;=\; (v \mapsto P + A \cdot v) \circ \Psi\colon \Delta^n \to \mathbb{R}^n.$$

We write $\Omega = A \circ \Psi$. It suffices to show that Ψ and Ω and also Ω and Φ induce the same maps $\mathrm{H}_k(\Delta^n, \partial\Delta^n; R) \to \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R)$.

(a) By the Matrix Group Path-Component Proposition 2.37 (1) we know that $\{A \in \mathrm{GL}(n,\mathbb{R}) \mid \det(A) > 0\}$ is path-connected. This implies in particular that there exists a path $\gamma\colon [0,1] \to \mathrm{GL}(n,\mathbb{R})$ with $\gamma(0) = \mathrm{id}$ and $\gamma(1) = A$. We consider the map

$$\begin{aligned} F\colon \Delta^n \times [0,1] &\to \mathbb{R}^n \\ (Q,t) &\mapsto \gamma(t) \cdot \Psi(Q). \end{aligned}$$

Since by hypothesis we have $0 \notin \Phi(\partial\Delta^n)$ and since $\gamma(t)(0) = 0$ we see that $\gamma(t)(\partial\Delta^n) \subset \mathbb{R}^n \setminus \{0\}$ for all t. In particular we deduce that F defines a homotopy between Ψ and Ω as maps $(\Delta^n, \partial\Delta^n) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. But by Proposition 74.15 this means that $\Psi_* = \Omega_*$ as maps on relative homology.

(b) Now we compare Ω and $\Phi = (w \mapsto P + w) \circ \Omega$. We consider the map

$$\begin{aligned} G\colon \Delta^n \times [0,1] &\to \mathbb{R}^n \\ (Q,t) &\mapsto (P \cdot t + \Omega(Q)). \end{aligned}$$

Note that $0 \in \Omega(\mathring{\Delta}^n) = A(\Psi(\mathring{\Delta}^n))$ and that $0 \in \Phi(\mathring{\Delta}^n)$ which implies by $\Phi = P + \Omega$ that $-P \in \Omega(\mathring{\Delta}^n)$. Since $\Omega(\mathring{\Delta}^n)$ is convex we see that G defines a homotopy between Ω and Φ as maps $(\Delta^n, \partial\Delta^n) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. But again by Proposition 74.15 this means that $\Omega_* = \Phi_*$ as maps on relative homology.

(3) The fact that both inclusion induced maps are isomorphisms is an immediate consequence of the Complement-of-Convex Homology Lemma 105.1 (2). Now suppose we are given another such $s > 0$. Without loss of generality we can assume that $r \geq s$. We obtain the following commutative diagram:

$$\begin{array}{ccccc} & & \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus B_r^n(0); R) & & \\ & \overset{\cong}{\swarrow} & \downarrow & \overset{\cong}{\searrow} & \\ \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R) & & & & \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R). \\ & \underset{\cong}{\nwarrow} & & \underset{\cong}{\nearrow} & \\ & & \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus B_s^n(0); R) & & \end{array}$$

All maps are induced by inclusions, so the diagram commutes. By the Complement-of-Convex Homology Lemma 105.1 the diagonal maps are isomorphisms. It is now obvious that the composition of the top maps and the composition of the bottom maps agree.

(4) Let $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ be the reflection in an affine hyperplane V through a point P. We denote by $f\colon \mathbb{R}^n \to \mathbb{R}^n$ the map that is given by addition by P and furthermore we write $W = f^{-1}(V)$. Note that W goes through the origin. Finally we denote by $\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ the reflection in the hyperplane W. We consider the following diagram

$$\begin{array}{ccccccc} \mathrm{H}_{n-1}(S^{n-1}) \otimes R & \overset{\mu}{\underset{\cong}{\to}} & \mathrm{H}_{n-1}(S^{n-1}; R) & \xrightarrow{f_*} & \mathrm{H}_{n-1}(\mathbb{R}^n \setminus \{P\}; R) & \overset{\partial}{\underset{\cong}{\leftarrow}} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R) \\ \downarrow{\scriptstyle \sigma_* \otimes \mathrm{id}} & & \downarrow{\scriptstyle \sigma_*} & & \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \rho_*} \\ \mathrm{H}_{n-1}(S^{n-1}) \otimes R & \overset{\mu}{\underset{\cong}{\to}} & \mathrm{H}_{n-1}(S^{n-1}; R) & \xrightarrow{f_*} & \mathrm{H}_{n-1}(\mathbb{R}^n \setminus \{P\}; R) & \overset{\partial}{\underset{\cong}{\leftarrow}} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R). \end{array}$$

where μ is the natural map from page 1927 and where the right-hand horizontal maps are the connecting homomorphisms from the Topological LES-Corollary 74.14.

It follows from the Sphere Degree Lemma 76.11 that the vertical map on the left is given by multiplication by -1. It is straightforward to verify that the diagram commutes and that all horizontal maps are isomorphisms. Furthermore the horizontal maps on the top and bottom are identical. It thus follows that the vertical map on the right is also given by multiplication by -1.

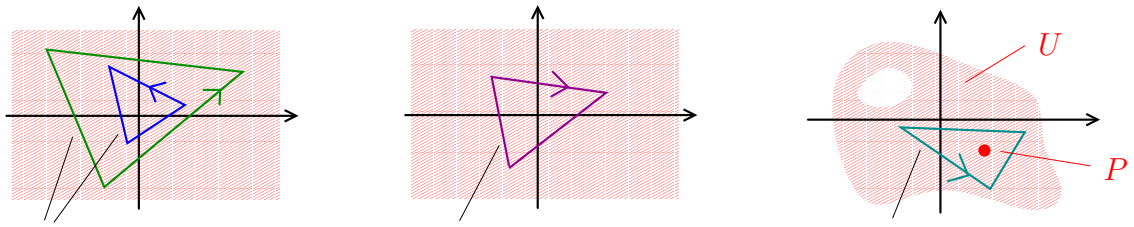
(5) It is clear that the diagram commutes since all maps are induced by inclusions. The horizontal maps are isomorphisms by the Excision Theorem 74.18 and the vertical maps are isomorphisms by the Complement-of-Convex Homology Lemma 105.1. ■

Convention. We use the isomorphisms from Lemma 105.2 to identify the various relative homology modules. In particular, for any open set U of $\mathbb{R}^n$, any open ball $B_r(P)$ with $\overline{B}_r(P) \subset U$, any point $Q \in U$ and any commutative ring R we make the identifications

$$R = \mathrm{H}_n(\Delta^n, \partial\Delta^n; R) = \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R) = \mathrm{H}_n(U, U \setminus B_r(P); R) = \mathrm{H}_n(U, U \setminus \{Q\}; R).$$

Furthermore, we refer to the image of the generator $[\mathrm{id}_{\Delta^n} \otimes 1] \in \mathrm{H}_n(\Delta^n, \partial\Delta^n; R)$ as the standard generator of the various relative homology modules.

Example. We refer to the figure below for an illustration of representatives of the standard generators.[1277] Here, for simplicity, given a singular 2-simplex $\Delta^2 \to \mathbb{R}^2$ we only sketch the image of $\partial\Delta^2$ with the given orientation. Finally note that it follows from Lemma 105.2 (4) that the singular simplex illustrated in the middle of the figure below represents *minus* the standard generator.



standard generator for $\mathrm{H}_2(\mathbb{R}^2, \mathbb{R}^2 \setminus \{0\}; \mathbb{Z})$ | minus the standard generator | standard generator for $\mathrm{H}_2(U, U \setminus \{P\}; \mathbb{Z})$

Lemma 105.3. Let R be a commutative ring, let U be an open subset of $\mathbb{R}^n$, let $P \in U$ and let $\sigma\colon \Delta^n \to U$ be a singular n-simplex with the following properties:

(1) the point P lies in the image of $\mathring{\Delta}^n$,

(2) the restriction of σ to a map $\mathring{\Delta}^n \to U$ is a diffeomorphism onto its image.

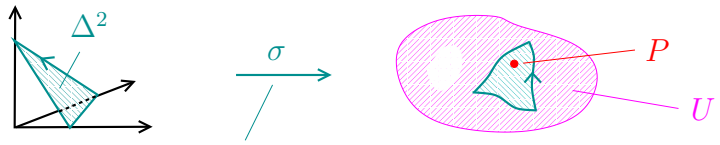
Then $\sigma \otimes 1 \in \mathrm{C}_n(U) \otimes R$ represents a generator of $\mathrm{H}_n(U, U \setminus \{P\}; R)$.[1278] Furthermore, if

(3) the restriction of σ to a map $\mathring{\Delta}^n \to U$ is orientation-preserving,

then $\sigma \otimes 1$ represents the standard generator of $\mathrm{H}_n(U, U \setminus \{P\}; R)$.

[1277] At this occasion it is good to remember that if $\sigma\colon \Delta^k \to X$ is a map with $\sigma(\partial\Delta^n) \subset A$, then σ is a cycle in $\mathrm{C}_k(X, A)$, i.e. it represents a class in $\mathrm{H}_k(X, A)$.

[1278] Note that we do not make any assumptions on the behavior of σ restricted to $\partial\Delta^n$. If one looks very carefully, then one realizes that one has to convince oneself that $\sigma(\partial\Delta^n)$ is a subset of $U \setminus \{P\}$, i.e. one has to show that it follows from (1) and (2), that $P \notin \sigma(\partial\Delta^n)$. How can one prove this statement?



σ represents the standard generator of $\mathrm{H}_2(U, U \setminus \{P\}; \mathbb{Z})$

Proof. We denote by $Q \in \mathring{\Delta}^n$ the point with $\sigma(Q) = P$. To simplify the notation we work throughout the proof with the coefficients $R = \mathbb{Z}$. Using the fact that the inclusion induced map

$$\mathrm{H}_n(U, U \setminus \{P\}) \xrightarrow{\cong} \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\})$$

is an isomorphism by the Excision Theorem 74.18 we can without loss of generality assume that $U = \mathbb{R}^n$. As always we denote by $\mathrm{D}\sigma_Q \colon \mathrm{T}_Q \mathring{\Delta}^n \to \mathrm{T}_P U = \mathbb{R}^n$ the differential of σ at the point Q. We denote by

$$\begin{array}{rcl} f \colon \Delta^n & \to & \mathbb{R}^n \\ x & \mapsto & (\mathrm{D}\sigma_Q)\underbrace{(x - Q)}_{\in \mathrm{T}_Q \Delta^n} + P \end{array}$$

the linear approximation of σ at the point Q. We make the following two observations:

(1) Since $\sigma \colon \mathring{\Delta}^n \to U$ is a diffeomorphism onto its image we obtain that $\mathrm{D}\sigma_Q$ is an isomorphism of vector spaces, in particular f is an injective affine linear map.

(2) Basically the same argument as in the proof of Lemma 76.15 shows that there exists an open neighborhood V of Q in $\mathring{\Delta}^n$ such that the restrictions of σ and f to maps

$$\sigma, f \colon (V, V \setminus \{Q\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{P\})$$

are homotopic.

Now consider the following diagram

$$\begin{array}{ccccccccc} \mathrm{H}_n(\Delta^n, \partial\Delta^n) & \longrightarrow & \mathrm{H}_n(\Delta^n, \Delta^n \setminus \{Q\}) & \longleftarrow & \mathrm{H}_n(V, V \setminus \{Q\}) & \xrightarrow{=} & \mathrm{H}_n(V, V \setminus \{Q\}) & \longrightarrow & \mathrm{H}_n(\Delta^n, \Delta^n \setminus \{Q\}) \\ \downarrow{\scriptstyle \sigma_*} & & \downarrow{\scriptstyle \sigma_*} & & \downarrow{\scriptstyle \sigma_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}) & \xrightarrow{=} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}) & \xrightarrow{=} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}) & \xrightarrow{=} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}) & \xrightarrow{=} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}). \end{array}$$

It is clear that the first two squares and the last square commute. Furthermore the third square commutes by the above observation (2) together with Proposition 74.15. As in the proof of Lemma 105.2 (2) we see that the last vertical map is an isomorphism. It now follows easily that $\sigma \in \mathrm{C}_n(\mathbb{R}^n)$ represents a generator of $\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\})$.

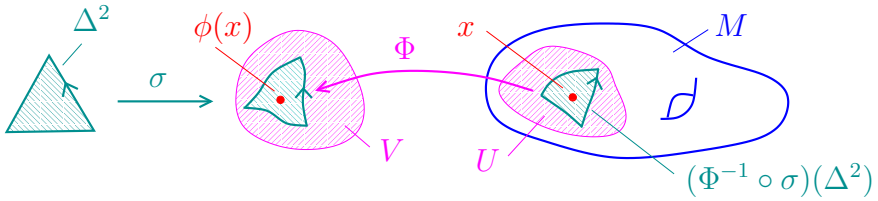
Finally suppose that the restriction of σ to a map $\mathring{\Delta}^n \to U$ is orientation-preserving. In this case, by definition of orientation-preserving, the map f is also orientation-preserving, therefore by the above commutative diagram the singular n-simplex σ does indeed represent the standard generator of $\mathrm{H}_n(U, U \setminus \{P\})$. ∎

We continue with the following lemma which is almost identical to the Manifold-Local Homology Lemma 75.1.

Lemma 105.4. Let M be an n-dimensional topological manifold and let R be a commutative ring. Then for every $x \in M \setminus \partial M$ and any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(M, M \setminus \{x\}; R) \;\cong\; \begin{cases} R, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if $\Phi\colon U \to V$ is a chart around x and if $\sigma\colon \Delta^n \to V$ represents a generator of $\mathrm{H}_n(V, V \setminus \{\Phi(x)\}; R)$, then $\Phi^{-1} \circ \sigma$ represents a generator of $\mathrm{H}_n(M, M \setminus \{x\}; R)$.



Remark. Let $n \in \mathbb{N}_0$. A topological space X that has the property that for each $x \in X$ we have

$$\mathrm{H}_k(X, X \setminus \{x\}; \mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise} \end{cases}$$

is often called a *k-dimensional homology manifold*. By Lemma 105.4 such topological spaces can be viewed as generalizations of closed topological manifolds.

Let Y be an homology n-sphere.[1279] By the Suspension-H_*-Proposition 76.7 we know that the suspension $\Sigma(Y)$ of Y is an $(n+1)$-dimensional homology manifold. On the other hand we saw in Exercise 76.9 that the suspension $\Sigma(Y)$ is *not* a topological manifold.[1280] Recall that the Double Suspension Theorem 96.18 says that the *double* suspension $\Sigma(\Sigma(Y))$ of any homology n-sphere is actually homeomorphic to S^{n+2}.

The proof of Lemma 105.4 is basically identical to the proof of the Manifold-Local Homology Lemma 75.1. We recall the argument just in case some readers might have forgotten.

Proof. Let M be an n-dimensional topological manifold, let R be a commutative ring and let $x \in M \setminus \partial M$. Since $x \notin \partial M$ we can pick a chart $\Phi\colon U \to V$ such that V is an open subset of $\mathbb{R}^n$. We write $y = \Phi(x)$. For any $k \in \mathbb{N}_0$ we then have

$$\mathrm{H}_k(M, M \setminus \{x\}; R) \;\underset{\uparrow}{\overset{\cong}{\longleftarrow}}\; \mathrm{H}_k(U, U \setminus \{x\}; R) \;\underset{\uparrow}{\overset{\Phi_*}{\underset{\cong}{\longrightarrow}}}\; \mathrm{H}_k(V, V \setminus \{y\}; R) \;\underset{\uparrow}{\overset{\cong}{\longrightarrow}}\; \begin{cases} R, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Excision Theorem 74.18 — since Φ is a homeomorphism — see Lemma 105.2 (5)

The last statement regarding generators now follows immediately from the above isomorphisms. ∎

[1279] Recall that by the definition on page 2159 an homology n-sphere is an n-dimensional topological manifold X such that for any $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X; \mathbb{Z}) \cong \mathrm{H}_k(S^n; \mathbb{Z})$. On page 2161 we saw that there exists a smooth homology 3-sphere with non-trivial fundamental group, namely the Poincaré Homology Sphere.

[1280] To be super-precise we showed that it is not an $(n+1)$-dimensional topological manifold, but it follows easily from the Topological Manifold-Dimension Proposition 75.4 that if $\Sigma(Y)$ was a topological manifold it would need to be $(n+1)$-dimensional.

105.3. **R-Sections on topological manifolds.** In Section 25.2 we defined the notion of an oriented smooth manifold via orientations of the tangent spaces. The notion of a tangent space requires the smooth structure on a smooth manifold, put differently, topological manifolds do not come with tangent spaces. In the next section we will see that we can still define the notion of an orientation of a topological manifold. Furthermore we will see that for smooth manifolds this new notion of an orientation will be equivalent to the previous notion of an orientation.

The key to defining the new notion of orientation is the following, initially rather obscure definition.

Definition. Let M be an n-dimensional topological manifold and let R be a commutative ring.

(1) An R-section for M at $x \in M \setminus \partial M$ is an element $\mu_x \in \mathrm{H}_n(M, M \setminus \{x\}; R) \cong R$.

(2) Let $W \subset M$ and let $x \in W$. Given any $k \in \mathbb{N}_0$ we denote by ρ_x the natural map

$$\rho_x \colon \mathrm{H}_k(M, M \setminus W; R) \;\to\; \mathrm{H}_k(M, M \setminus \{x\}; R).$$

Given $\varphi \in \mathrm{H}_k(M, M \setminus W; R)$ we often write $\varphi_x := \rho_x(\varphi)$, i.e. the above map is also written as

$$\varphi \;\mapsto\; \varphi_x.$$

(3) Let A be a subset of $M \setminus \partial M$. (In the following we will mostly be interested in the case $A = M \setminus \partial M$). An R-section for M along A is a choice for each $x \in A$ of an R-section $\mu_x \in \mathrm{H}_n(M, M \setminus \{x\}; R)$ at x such that the following "continuity condition" is satisfied: for each $x \in A$ there exists an open neighborhood U of x in A and a homology class $\mu_U \in \mathrm{H}_n(M, M \setminus U; R)$ such that for each $y \in U$ the image of μ_U under the natural map

$$\rho_y \colon \mathrm{H}_n(M, M \setminus U; R) \;\to\; \mathrm{H}_n(M, M \setminus \{y\}; R)$$

equals μ_y.

(4) We denote by $\Gamma_R^A(M)$ the set of all R-sections of M along A.

For $A = M \setminus \partial M$ we say "section" instead of "section along A" and we write $\Gamma_R(M)$ instead of $\Gamma_R^{M \setminus \partial M}(M)$.

Example. In the figure below we attempt to illustrate the definition of a $\mathbb{Z}$-section on a surface M. Here, for simplicity, given a singular 2-simplex $\Delta^2 \to M$ we only sketch the image of $\partial\Delta^2$ with the given orientation.

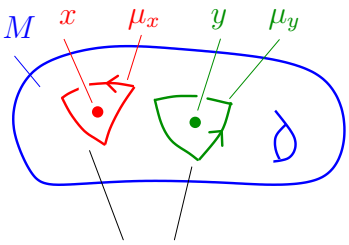
(1) In the figure below we show to the left two points x and y together with singular 2-simplices representing $\mu_x \in \mathrm{H}_2(M, M \setminus \{x\}; \mathbb{Z})$ and $\mu_y \in \mathrm{H}_2(M, M \setminus \{y\}; \mathbb{Z})$. Furthermore we sketch an open set U that contains x and y and we sketch a homology class $\mu_U \in \mathrm{H}_2(M, M \setminus U; \mathbb{Z})$ that gets sent to μ_x and μ_y via the natural maps[1281]

$$\mathrm{H}_n(M, M \setminus U; \mathbb{Z}) \xrightarrow{\rho_x} \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{Z}) \quad \text{and} \quad \mathrm{H}_n(M, M \setminus U; \mathbb{Z}) \xrightarrow{\rho_y} \mathrm{H}_n(M, M \setminus \{y\}; \mathbb{Z}).$$

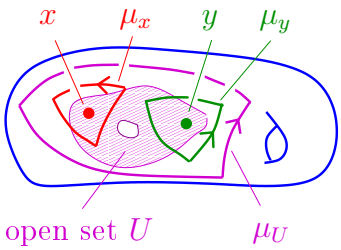
In fact these $\mathbb{Z}$-sections at x and y form part of a $\mathbb{Z}$-section of M.

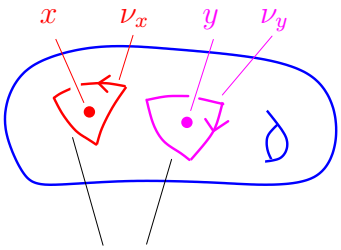
(2) In the figure below we show to the right two $\mathbb{Z}$-sections ν_x and ν_y that cannot be part of a $\mathbb{Z}$-section of M.

[1281] Note that in this example the group $\mathrm{H}_2(M, M \setminus U; \mathbb{Z})$ is not isomorphic to $\mathbb{Z}$, i.e. the two maps are not isomorphisms.



here the $\mathbb{Z}$-sections at the points correspond to a $\mathbb{Z}$-section on M





here the $\mathbb{Z}$-sections at the points do *not* come from a $\mathbb{Z}$-section on M

Lemma 105.5. Let M be an n-dimensional topological manifold and let R be a commutative ring.

(1) Given any subset $A \subset M \setminus \partial M$ the set $\Gamma_R^A(M)$ admits a unique R-module structure such that for each $y \in A$ the map $\Gamma_R^A(M) \to \mathrm{H}_n(M, M \setminus \{y\}; R)$ that is given by $\{\alpha_x\}_{x\in A} \mapsto \alpha_y$ is an R-module homomorphism.

(2) For every subset $A \subset M \setminus \partial M$ (often we will take $A = M \setminus \partial M$) the map

$$\begin{array}{rcl} \mathrm{H}_n(M, M \setminus A; R) & \to & \Gamma_R^A(M) \\ \alpha & \mapsto & \{\alpha_x\}_{x\in A} \end{array}$$

is a well-defined homomorphism of R-modules.

Remark. In the Homology-Section Theorem 106.10 we will see that in many settings the converse to Lemma 105.5 holds.

Proof.

(1) We will prove this statement in Exercise 105.2.

(2) Let $A \subset M \setminus \partial M$ be a subset and let $\alpha \in \mathrm{H}_n(M, M \setminus A; R)$. Let $x \in A$. We need to show that there exists an open neighborhood U of x in A and a homology class $\varphi \in \mathrm{H}_n(M, M \setminus U; R)$ such that for each $y \in U$ we have $\rho_x(\varphi) = \alpha_x$. But this is trivial, we just take $U = A$ and $\varphi = \alpha$. ■

The following proposition contains much more interesting results on $\Gamma_R^A(M)$.

Proposition 105.6. (Sections Proposition) Let M be a topological manifold and let A be a *connected* subset of $M \setminus \partial M$.[1282]

(1) Let R be a commutative ring and let $\{\alpha_x\}_{x\in A}$ and $\{\beta_x\}_{x\in A}$ be two R-sections for M along A. If there exists an $x \in A$ with $\alpha_x = \beta_x$, then the two R-sections agree.

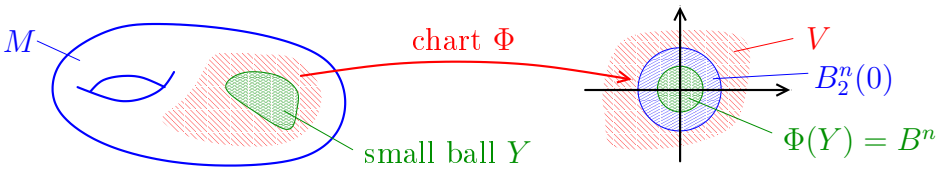
(2) Let $\{\alpha_x\}_{x\in A}$ be a $\mathbb{Z}$-section for M along M. Suppose there exists a $y \in A$, an $m \in \mathbb{Z}$ and a $\beta_y \in \mathrm{H}_n(M, M \setminus \{y\}; \mathbb{Z})$ such that $\alpha_y = m \cdot \beta_y$, then there exists a $\mathbb{Z}$-section $\{\beta_x\}_{x\in A}$ along A such that $\alpha_x = m \cdot \beta_x$ for every $x \in A$.

(3) Let $\{\alpha_x\}_{x\in A}$ be a $\mathbb{Z}$-section along A for M. If there exists a $y \in A$ such that α_y is a generator of $\mathrm{H}_n(M, M\setminus\{y\}; \mathbb{Z}) \cong \mathbb{Z}$, then α_x is a generator of $\mathrm{H}_n(M, M\setminus\{x\}; \mathbb{Z}) \cong \mathbb{Z}$ for every $x \in A$.

[1282] In some applications we will use that by Exercise 18.19 we know that if M is a connected topological manifold, then $M \setminus \partial M$ is also connected.

Before we provide the proof of the Sections Proposition 105.6 we need to prove one little lemma which in turn requires the following definition.

Definition. Let M be an n-dimensional topological manifold. We say that $Y \subset M$ is a small ball in M if there exists a chart $\Phi\colon U \to V$ such that $B_2^n(0) \subset V$ and such that $\Phi(Y) = B^n$.



Lemma 105.7. (Small Ball-Lemma) Let M be an n-dimensional topological manifold.

(1) Given any point $x \in M \setminus \partial M$ and any neighborhood U of x there exists a small ball contained in U that contains x.

Let R be a commutative ring.

(2) For any point y in a small ball Y and any $k \in \mathbb{N}_0$ the map

$$\rho_y\colon \mathrm{H}_k(M, M \setminus Y; R) \;\to\; \mathrm{H}_k(M, M \setminus \{y\}; R)$$

is an isomorphism.

(3) Let A be a subset of $M \setminus \partial M$, let $\{\alpha_x\}_{x\in A}$ be an R-section along A and let $x \in A$. There exists a small ball Y in $M \setminus \partial M$ that contains x and a $\varphi \in \mathrm{H}_k(M, M \setminus Y; R)$ such that for every $y \in Y \cap A$ we have $\rho_y(\varphi) = \alpha_y \in \mathrm{H}_n(M, M \setminus \{y\}; R)$.

Proof. Let M be an n-dimensional topological manifold.

(1) The proof of the first statement is elementary.

(2) Now let y be a point that is contained in a small ball Y. To simplify the notation we drop the commutative ring R from the notation. We consider the long exact sequence of the triple $(M, M \setminus \{y\}, M \setminus Y)$ given by the Topological LES-Corollary 74.14:

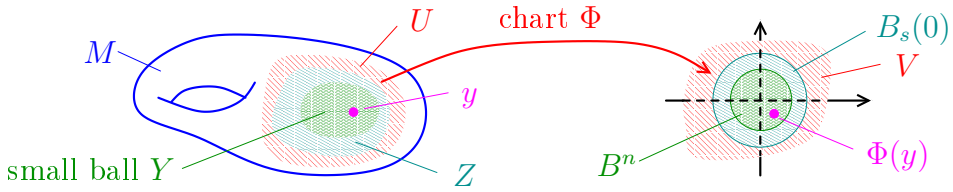
$$\to \mathrm{H}_k(M\setminus\{y\}, M\setminus Y) \to \mathrm{H}_k(M, M\setminus Y) \to \mathrm{H}_k(M, M\setminus\{y\}) \to \mathrm{H}_{k-1}(M\setminus\{y\}, M\setminus Y) \to$$

We see immediately that it suffices to show that all the relative homology groups $\mathrm{H}_k(M \setminus \{y\}, M \setminus Y)$ vanish. Since Y is a small ball we can pick a chart $\Phi\colon U \to V$ such that $B_2^n(0) \subset V$ and such that $\Phi(Y) = B^n$. We pick $s \in (1, 2)$ and we write $Z = \Phi^{-1}(B_s^n(0))$. Then

$$\mathrm{H}_k(M\setminus\{y\}, M\setminus Y) \xleftarrow[\uparrow]{\cong} \mathrm{H}_k(Z\setminus\{y\}, Z\setminus Y) \xrightarrow[\cong]{\Phi_*} \mathrm{H}_k(B_s^n(0)\setminus\{\Phi(y)\}, B_s^n(0)\setminus B^n) \underset{\uparrow}{=} 0.$$

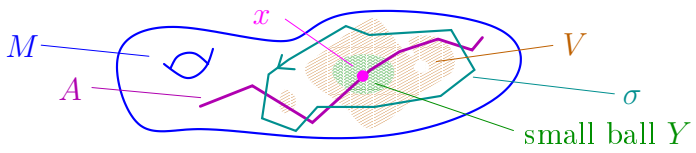
Excision Theorem 74.18

by Corollary 74.16, since $B_s^n(0) \setminus B^n$ is a deformation retract of $B_s^n(0) \setminus \{\Phi(y)\}$



(3) Let A be a subset of M, let $\{\alpha_x\}_{x\in A}$ be an R-section along A and let $x \in A$. By definition of an R-section along A and by definition of the subspace topology there exists an open neighborhood V of $x \in M \setminus \partial M$ and a singular chain $\sigma \in \mathrm{C}_n(M;R)$ that represents a cycle in $\mathrm{C}_n(M, M \setminus (V \cap A);R)$ such that for each $y \in V \cap A$ we have $\alpha_y = \rho_y([\sigma]) \in \mathrm{H}_n(M, M \setminus \{y\};R)$.

Since σ represents a cycle in the relative chain complex $\mathrm{C}_*(M, M \setminus (V \cap A);R)$ we know that $\partial\sigma \in \mathrm{C}_{n-1}(M \setminus (V \cap A);R)$. It follows from the fact that Δ^{n-1} is compact together with the Compact Image Lemma 2.13 and the Compact-Closed Lemma 1.21 that there exists an open neighborhood $U \subset M$ of x with $U \subset V$ such that $\partial\sigma \in \mathrm{C}_{n-1}(M \setminus U;R)$. In other words, σ represents a cycle in $\mathrm{C}_n(M, M \setminus U)$. By (1) there exists a small ball Y with $x \in Y \subset U$. It follows immediately from the functoriality of homology groups that the homology class $[\sigma] \in \mathrm{H}_n(M, M \setminus Y;R)$ has all the desired properties. ■



Proof of the Sections Proposition 105.6. Let M be an n-dimensional topological manifold, let A be a connected subset of $M \setminus \partial M$ and let R be a commutative ring.

(1) Let $\{\alpha_x\}_{x\in A}$ and $\{\beta_x\}_{x\in A}$ be two R-sections for M along A. We assume that there exists a $z \in A$ with $\alpha_z = \beta_z$. We consider the R-section along A that is given by $\gamma_x = \alpha_x - \beta_x$ for $x \in A$. Next we consider the following two subsets of A:

$$V := \{y \in A \,|\, \gamma_y = 0\} \quad \text{and} \quad W := \{y \in A \,|\, \gamma_y \neq 0\}.$$

We need to show that $V = A$. Since V is by hypothesis non-empty and since A is connected it suffices to show that V and W are both open. In other words, it suffices to prove the following claim.

Claim. Let $x \in A$. There exists an open neighborhood U of x in A such that either γ_y is zero for every $z \in U$ or γ_y is non-zero for every $z \in U$.

Proof. Let $x \in A$. It follows immediately from the Small Ball-Lemma 105.7 (3) that there exists a small ball Y in $M \setminus \partial M$ that contains x and a $\varphi \in \mathrm{H}_k(M, M \setminus Y;R)$ such that for every $y \in Y \cap A$ we have $\rho_y(\varphi) = \alpha_y \in \mathrm{H}_n(M, M \setminus \{y\};R)$. Furthermore, note that by the Small Ball-Lemma 105.7 (2) we know that for every $y \in Y$ the map ρ_y is an isomorphism. But this implies that $\gamma_x = 0$ if and only if $\gamma_y = 0$ for all $y \in Y$. ⊞

(2) Let $n \in \mathbb{N}$. We start out with the following definition: Given an element g of a group G with $G \cong \mathbb{Z}$ we say that g is divisible by n if there exists an $h \in G$ with $n \cdot h = g$. In this case we write $g/n := h$. Note that h is well-defined since $G \cong \mathbb{Z}$.

Let $\{\alpha_x\}_{x\in A}$ be a $\mathbb{Z}$-section for M along A. Given $n \in \mathbb{N}$ we consider the two sets

$$\begin{aligned} V &:= \{y \in A \,|\, \alpha_y \text{ is divisible by } n\}, \qquad \text{and} \\ W &:= \{y \in A \,|\, \alpha_y \text{ is not divisible by } n\}. \end{aligned}$$

The same argument as in (1) one shows that V and W are open.

If V is non-empty, then $V = M \setminus \partial M$ and it is straightforward to verify that $\beta_y := \alpha_y/n$, $y \in A$, is in fact a $\mathbb{Z}$-section along A. We leave it to the reader to fill in the details.

(3) Let $\{\alpha_x\}_{x\in A}$ be a $\mathbb{Z}$-section for M along A. Let $y \in A$ be a point such that α_y is a generator of $\mathrm{H}_n(M, M\setminus\{y\}; \mathbb{Z}) \cong \mathbb{Z}$. Now let $z \in A$. If α_z is not a generator, then there exists a $\beta_z \in \mathrm{H}_n(M, M\setminus\{z\}; \mathbb{Z}) \cong \mathbb{Z}$ and some $n \geq 2$ such that $\alpha_z = n \cdot \beta_z$. By (2) we can extend β_z to a $\mathbb{Z}$-section $\{\beta_x\}_{x\in A}$ such that $\alpha_x = n \cdot \beta_x$ for every $x \in A$. But this implies that α_y is not a generator. So we have obtained a contradiction. ■

105.4. **Orientations of topological manifolds.** After the technical preparations from the last section we can now finally introduce the notion of an orientation of a topological manifold.

Definition. Let M be an n-dimensional topological manifold and let R be a commutative ring.

(1) An R-section for M at a point $x \in M \setminus \partial M$ that is in fact a generator of the R-module $\mathrm{H}_n(M, M \setminus \{x\}; R) \cong R$ is called an *R-orientation for M at $x \in M$*.

(2) An *R-section for M* that is an R-orientation at each point $x \in M \setminus \partial M$ is called an *R-orientation of M*.

(3) The topological manifold M is called *R-orientable* if it admits an R-orientation.

(4) Let M and N be two n-dimensional topological manifolds that are equipped with R-orientations $\{\alpha_x\}_{x\in M\setminus\partial M}$ and $\{\beta_x\}_{x\in N\setminus\partial N}$. Let $f\colon M \to N$ be a map that is a local homeomorphism. We say the map f is *orientation-preserving* if the following holds: for any open set $U \subset M$, such that $f\colon U \to f(U)$ is a homeomorphism and such that $f(U)$ is open, and any $x \in U \setminus \partial M$ the image of α_x under the map[1283] [1284] [1285]

these two maps are isomorphisms by the Excision Theorem 74.18

$$\mathrm{H}_n(M, M\setminus\{x\}) \xleftarrow[\cong]{\downarrow} \mathrm{H}_n(U, U\setminus\{x\}) \xrightarrow[\cong]{f_*} \mathrm{H}_n(f(U), f(U)\setminus\{f(x)\}) \xrightarrow[\cong]{\downarrow} \mathrm{H}_n(N, N\setminus\{f(x)\}).$$

equals $\beta_{f(x)}$. Otherwise we call f *orientation-reversing*.

Example. Let M be a 0-dimensional topological manifold. Note that for each $x \in M$ we have a natural isomorphism

$$\mathbb{Z} \xrightarrow[\uparrow]{n\mapsto n\cdot[x]} \mathrm{H}_n(\{x\}) \xrightarrow[\uparrow]{\cong} \mathrm{H}_0(M, M\setminus\{x\}).$$

H_0-Proposition 72.5 — inclusion induced

Thus we see that an orientation of a 0-dimensional topological manifold is basically the same as a map $M \to \{\pm 1\}$. In other words, we see that we have the same concept of an orientation as we remarked on page 675 for 0-dimensional smooth manifolds.

We leave the proof of the following lemma as an amusing exercise to the reader.

[1283] Since $f\colon U \to f(U)$ is a homeomorphism we have $f(U \setminus \{x\}) = f(U) \setminus \{f(x)\}$.

[1284] It follows from the fact that $f\colon U \to f(U)$ is a homeomorphism, the fact that $x \in U \setminus \partial U$ and the Topological Manifold Boundary Proposition 75.2 that $f(x) \notin \partial N$.

[1285] We suppress the R-coefficients from the notation for space reasons.

Lemma 105.8. Let M and N be two n-dimensional topological manifolds and furthermore let $f, g\colon M \to N$ be two homeomorphisms. If f and g are isotopic, then f is orientation-preserving if and only if g is orientation-preserving.

Before we start with the discussions of orientations in detail we get the following technical lemma out of the way.

Lemma 105.9. Let M be an-dimensional topological manifold.

(1) Let R be a commutative ring and let $A \subset M \setminus \partial M$ be a *connected* subset. If M is R-orientable, then for every $x \in M \setminus \partial M$ the map

$$\begin{array}{rcl} \Phi\colon \Gamma_R^A(M) & \to & \overbrace{\mathrm{H}_n(M, M \setminus \{x\}; R)}^{\cong R} \\ \{\alpha_y\}_{y\in A} & \mapsto & \alpha_x \end{array}$$

is an isomorphism of R-modules.

(2) If M is non-orientable and connected, then $\Gamma_{\mathbb{Z}}(M) = 0$.

Proof. Let M be a connected n-dimensional topological manifold.

(1) Let R be a commutative ring such that M is R-orientable. We pick an R-orientation $\{\mu_y\}_{y\in M\setminus\partial M}$ for M. Next let $A \subset M\setminus\partial M$ be a connected subset and let $x \in M\setminus\partial M$. By definition μ_x is a generator of the R-module $\mathrm{H}_n(M, M \setminus \{x\}; R) \cong R$, i.e. every element in $\mathrm{H}_n(M, M\setminus\{x\}; R)$ is of the form $r\cdot\mu_x$ for some unique $r \in R$. We consider the map

$$\begin{array}{rcl} \Psi\colon \mathrm{H}_n(M, M \setminus \{x\}; R) & \to & \Gamma_R^A(M) \\ r \cdot \mu_x & \mapsto & \{r \cdot \mu_y\}_{y\in A}. \end{array}$$

Evidently we have $\Phi \circ \Psi = \mathrm{id}$. It follows from our hypothesis that A is connected and from the Sections Proposition 105.6 (1) that $\Psi \circ \Phi$ is also the identity.

(2) Now we suppose that M admits a non-zero $\mathbb{Z}$-section $\{\alpha_x\}_{x\in M\setminus\partial M}$. We need to show that M is orientable. We pick $y \in M \setminus \partial M$ such that $\alpha_y \neq 0$. We pick a generator $\beta_y \in \mathrm{H}_n(M, M \setminus \{y\}; \mathbb{Z}) \cong \mathbb{Z}$. Then $\alpha_y = n \cdot \beta_y$ for some $n \in \mathbb{Z}$. It follows again from our hypothesis that M is connected and from the Sections Proposition 105.6 (2) that there exists a $\mathbb{Z}$-section $\{\beta_x\}_{x\in M\setminus\partial M}$ such that $\alpha_x = n \cdot \beta_x$ for all $x \in M \setminus \partial M$. It follows immediately from the Sections Proposition 105.6 (3) that $\{\beta_x\}_{x\in M\setminus\partial M}$ is an orientation for M. ■

In the following lemma we collect some basic facts about $\mathbb{Z}$-orientations.

Lemma 105.10. Let M be a topological manifold.

(1) For any point $x \in M \setminus \partial M$ there exist precisely two $\mathbb{Z}$-orientations for M at x.

(2) If M is connected and orientable, then M admits precisely two $\mathbb{Z}$-orientations.

Proof. The first statement is an immediate consequence of the fact, proved in Lemma 105.4, that for any $x \in M \setminus \partial M$ we have $\mathrm{H}_n(M, M \setminus \{x\}; \mathbb{Z}) \cong \mathbb{Z}$. The second statement follows easily from the Sections Proposition 105.6 and the definitions. ■

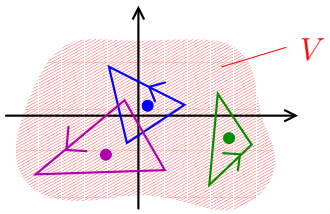
Examples.

(1) Let V be a connected open subset of $\mathbb{R}^n$. For any $x \in V$ we define

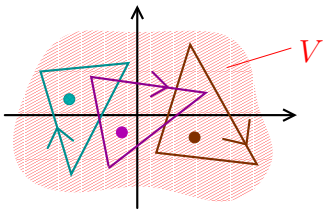
$$\mu_x \;:=\; \text{the standard generator of } \mathrm{H}_n(V, V \setminus \{x\}; \mathbb{Z}) \text{ as defined on page 2328.}$$

We claim that the μ_x define a $\mathbb{Z}$-orientation for V. Indeed, let $x \in V$. We pick an $r > 0$ such that $\overline{B}_r(x) \subset V$. We set $U := B_r(x)$. We denote by μ_U the standard generator of $\mathrm{H}_n(V, V \setminus U; \mathbb{Z})$ as defined on page 2328. It follows from Lemma 105.2 (5) that μ_U has the desired property. We refer to this $\mathbb{Z}$-orientation as the *standard $\mathbb{Z}$-orientation* of V.

By Lemma 105.10 (2) the topological manifold V admits precisely two different $\mathbb{Z}$-orientations. We have already given one orientation above. By Lemma 105.2 the other one is given by a reflection of the standard generators in hyperplanes. In the figure below we sketch the two different orientations of V.

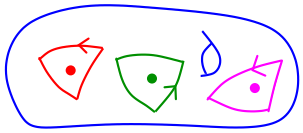


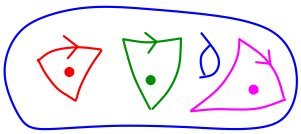
standard generators
for $\mathrm{H}_2(V, V \setminus \{x\}; \mathbb{Z})$



the negative of the
standard generators

(2) In the figure below we sketch the two different orientations of the torus.





(3) Let R be a commutative ring. Recall that for $n \geq 1$ we denote by $[S^n] \in \mathrm{H}_n(S^n; \mathbb{Z})$ the standard generator that we introduced on page 1677. It follows from the Universal Coefficient Theorem 89.17 together with the Tor-Properties Lemma 89.15 (3) that the natural map

$$\mathrm{H}_n(S^n) \otimes R \ \to \ \mathrm{H}_n(S^n; R)$$

is an isomorphism. By a slight abuse of notation we denote by $[S^n]$ also the image of $[S^n] \otimes 1 \in \mathrm{H}_n(S^n; R)$ under the above map. Note that $[S^n] \in \mathrm{H}_n(S^n; R) \cong R$ is a generator. We refer to it as the *standard generator of* $\mathrm{H}_n(S^n; R)$. For any $x \in S^n$ it follows easily from the long exact sequence of homology groups with R-coefficients of the pair $(S^n, S^n \setminus \{x\})$ that the natural map

$$\mathrm{H}_n(S^n; R) \ \to \ \mathrm{H}_n(S^n, S^n \setminus \{x\}; R)$$

is an isomorphism. We denote by $\mu_x \in \mathrm{H}_n(S^n, S^n \setminus \{x\}; R)$ the image of $[S^n]$. By the above μ_x is an R-orientation at x. In fact all these R-orientations $\{\mu_x\}_{x \in S^n}$ define an R-orientation of S^n. Indeed, in the definition on page 2335 we just need to set $U = S^n$ and $\mu_U = [S^n] \in \mathrm{H}_n(S^n, \underbrace{S^n \setminus S^n}_{=\varnothing}; \mathbb{Z})$. We refer to this orientation of S^n as the *standard R-orientation of S^n*.

The following proposition shows in particular that for smooth manifolds the notion of orientability from page 675 agrees with the above notion of orientability.

Proposition 105.11. (Smooth-Topological Orientation Proposition)

(1) Give any smooth manifold M there exists a natural bijection

$$\left\{ \begin{array}{l} \text{orientations on } M \text{ in} \\ \text{the sense of page 675} \end{array} \right\} \xrightarrow{\cong} \{\, \mathbb{Z}\text{-orientations on } M \,\}$$

such that the following two statements hold:

(a) For any open subsets of some $\mathbb{R}^n$ the above bijections sends the standard orientation from page 675 to the above standard $\mathbb{Z}$-orientation.

(b) If $f\colon M \to N$ is a local diffeomorphism between oriented connected smooth manifolds, then the following two statements are equivalent:

(i) f preserves the orientations on M and N in the sense of page 675,

(ii) f preserves the $\mathbb{Z}$-orientations on M and N.

(2) Given an n-dimensional smooth manifold M we have

$$M \text{ is orientable in the sense of page 675} \iff M \text{ is } \mathbb{Z}\text{-orientable.}$$

In the proof of the Smooth-Topological Orientation Proposition 105.11 we will use the following lemma.

Lemma 105.12. Let $\Psi\colon V \to V'$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. Then for any $x \in V$ we have

$$\begin{array}{c} \Psi \text{ preserves the standard} \\ \mathbb{Z}\text{-orientation of } V \text{ at the point } x \end{array} \iff \begin{array}{c} \Psi \text{ preserves the orientation of } V \text{ in the} \\ \text{sense of page 675 at the point } x. \end{array}$$

Proof. Let $\Psi\colon V \to V'$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$ and let $x \in V$. We write $x' = \Psi(x)$. Given $w \in \mathbb{R}^n$ we denote by t_w the translation $\mathbb{R}^n \to \mathbb{R}^n$ that is given by $v \mapsto v + w$. We set $\epsilon := 1$ if Ψ preserves the orientation of V in the sense of page 675 at the point x. Otherwise we set $\epsilon := -1$. We consider the following diagram

$$\begin{array}{ccccc} \mathrm{H}_n(V, V\backslash\{x\}; \mathbb{Z}) & \xleftarrow{t_{x*}} & \mathrm{H}_n(t_{-x}(V), t_{-x}(V)\backslash\{0\}) & \longrightarrow & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\}) \\ \downarrow{\scriptstyle \Psi_*} & & \downarrow{\scriptstyle (t_x \circ \Psi \circ t_{-x'})_*} & & \downarrow{\scriptstyle \cdot\epsilon} \\ \mathrm{H}_n(V', V'\backslash\{x'\}; \mathbb{Z}) & \xleftarrow{t_{x'*}} & \mathrm{H}_n(t_{-x}(V), t_{-x}(V)\backslash\{0\}) & \longrightarrow & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\}). \end{array}$$

We make the following observations:

(1) It follows easily from the definitions that each translation t_v is orientation-preserving in both senses of an orientation.

(2) It follows from the Local Diffeomorphism-Degree Proposition 76.14 together with (1) that the square at the right commutes.

(3) It is clear that the square at the left commutes.

(4) The statement of the lemma follows from (2) and (3). ∎

Sketch of the proof of Proposition 105.11. First, as we saw on page 2335, the case of 0-dimensional smooth manifolds is basically a tautology, at least as long as all statements are interpreted in a reasonable way. Thus in the following we assume that we are dealing with smooth manifolds of dimension ≥ 1.

Next note that Statement (2) is just a reader-friendly condensation of Statement (1). Thus we need to prove Statement (1). We define the promised natural bijections. Thus let M be an n-dimensional smooth manifold where $n \geq 1$. We first suppose that M is oriented in the sense of page 675. Let $x \in M \setminus \partial M$. We pick a chart $\Phi\colon U \to V$ around x

that is orientation-preserving in the sense of page 675. We define

$$\mu_x \;:=\; \Phi_*^{-1}\big(\text{standard generator of } \mathrm{H}_n(V, V\setminus\{\Phi(x)\};\mathbb{Z}) \text{ as defined on page 2328}\big).$$

It follows from Lemma 105.12 that this definition does not depend on the choice of the orientation-preserving chart Φ. We claim that the classes $\{\mu_x\}_{x\in M\setminus\partial M}$ form a $\mathbb{Z}$-orientation for M. We only have to show that these classes satisfy the "continuity condition". We claim that for any $x \in M \setminus \partial$ a sufficiently small ball U containing x has the desired property. The proof of this fact is very similar to the arguments used in the proof of the Sections Proposition 105.6. We leave it to the reader to fill in the details.

Now we define the inverse map. So suppose we are given a $\mathbb{Z}$-orientation $\{\mu_x\}_{x\in M\setminus\partial M}$ for M. Note that by the Basics-of-Orientations Lemma 25.13 we only need to find an orientation for $M \setminus \partial M$. Thus let $x \in M \setminus \partial M$. We pick a chart $\Phi\colon U \to V$ around x such that V is connected. It follows from Lemma 105.2 that, after possibly reflecting V in a hyperplane, we can assume that Φ preserves the $\mathbb{Z}$-orientation at x, i.e. it sends the given $\mathbb{Z}$-orientation μ_x to the standard orientation of the open subset V of $\mathbb{R}^n$ that we introduced on page 2337. We define

$$\text{orientation of } \mathrm{T}_xM \;:=\; \Phi^*(\text{standard orientation of } \mathrm{T}_{\Phi(x)}V = \mathbb{R}^n).$$

It follows from Lemma 105.12 that this definition does not depend on the choice of Φ. As before, using small balls, one can show that this defines an orientation of M in the sense of page 675. We leave the verification of the details once again to the surely highly motivated reader.

It is basically clear, that by construction, the bijections have the desired properties (a) and (b). This concludes the proof of Statement (1). ■

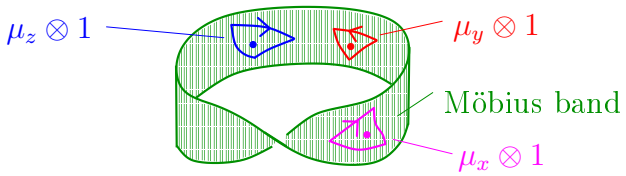
Convention. the Smooth-Topological Orientation Proposition 105.11 says in particular that for smooth manifolds the notion of orientability that we introduced on page 675 agrees with the notion of $\mathbb{Z}$-orientability that we had just introduced. For $R = \mathbb{Z}$ we will therefore often drop the "R" from the definitions on page 2335, i.e. often we just say orientation at a point, orientation and orientable.

Remark. It follows basically from the definitions that under the equivalence of the Smooth-Topological Orientation Proposition 105.11 the orientations of S^n introduced on page 681 and page 2337 agree.

The following proposition shows that the notion of R-orientability is more flexible than the previous notion of orientability.

Proposition 105.13. ($\mathbb{F}_2$-Orientability Proposition) Every topological manifold is $\mathbb{F}_2$-orientable.

Example. The $\mathbb{F}_2$-Orientability Proposition 105.13 says in particular that the real projective plane, the Klein bottle and the Möbius band are $\mathbb{F}_2$-orientable.



Proof. Let M be an n-dimensional topological manifold. Let $x \in M \backslash \partial M$. By Lemma 105.4 we know that $\mathrm{H}_n(M, M \backslash \{x\}; \mathbb{F}_2) \cong \mathbb{F}_2$. We define μ_x to be the unique non-trivial element in $\mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2) \cong \mathbb{F}_2$. Using the Small Ball-Lemma 105.7 it is straightforward to show that the continuity condition is satisfied, in particular we have defined an $\mathbb{F}_2$-orientation on M. ■

Lemma 105.14. Let M be a topological manifold. If M is orientable, then it is R-orientable for any commutative ring R.

Proof. Let M be an n-dimensional orientable topological manifold and let R be a commutative ring. Let $x \in M \setminus \partial M$. It follows from the Universal Coefficient Theorem 89.17, together with the fact that $\mathrm{H}_{n-1}(M, M \setminus \{x\}; \mathbb{Z}) = 0$ obtained in Lemma 105.4, that the natural map

$$\begin{array}{rcl} \mu\colon \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{Z}) \otimes R & \to & \mathrm{H}_n(M, M \setminus \{x\}; R) \\ \sum\limits_{i=1}^{k} [\sigma_i] \otimes r_i & \mapsto & \Big[\sum\limits_{i=1}^{k} \sigma_i \otimes r_i\Big] \end{array}$$

is an isomorphism.

Now we pick an orientation $\{\alpha_x\}_{x \in M \setminus \partial M}$ for M. For each $x \in M \setminus \partial M$ we denote by $\widetilde{\alpha}_x \in \mathrm{H}_n(M, M \setminus \{x\}; R)$ the image of $\alpha_x \otimes 1$ under the above isomorphism. It is straightforward to verify that these generators $\{\widetilde{\alpha}_x\}_{x \in M \setminus \partial M}$ define an R-orientation for the topological manifold M. ■

We continue with the following proposition which is a variation on Propositions 49.4 and 49.5, which were formulated in the context of smooth manifolds.

Proposition 105.15. (Orientation Covering-Proposition)

(1) To each topological manifold M we can naturally[1286] associate an oriented topological manifold $\widetilde{M}$ together with a 2-fold covering $p\colon \widetilde{M} \to M$ and to each local homeomorphism $f\colon M \to N$ we can associate an orientation-preserving local homeomorphism $\widetilde{f}\colon \widetilde{M} \to \widetilde{N}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{f}} & \widetilde{N} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ M & \xrightarrow{f} & N. \end{array}$$

(2) Let M be a connected topological manifold and let $p\colon \widetilde{M} \to M$ be the 2-fold covering constructed in (1).

(a) If M is orientable, then there exists a unique orientation-preserving homeomorphism to $\Theta\colon \widetilde{M} \to M \sqcup -M$ such that the following diagram commutes:

$$\begin{array}{ccccc} \widetilde{M} & & \xrightarrow[\cong]{\Theta} & & M \sqcup -M \\ & {\scriptstyle p}\searrow & & \swarrow{\scriptstyle q} & \\ & & M & & \end{array}$$

where q is the obvious map given by the identity on each component of $M \sqcup -M$.

(b) If M is non-orientable, then $\widetilde{M}$ is connected.

(3) Let M be a topological manifold. The map

$$\begin{array}{rccl} \Xi_M\colon & \widetilde{M} & \to & \widetilde{M} \\ & x & \mapsto & \text{the unique other points of } p^{-1}(p(x)) \end{array}$$

is the unique orientation-reversing self-homeomorphism $\Xi_M\colon \widetilde{M} \to \widetilde{M}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow[\cong]{\Xi_M} & \widetilde{M} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & M. & \end{array}$$

Furthermore, given a local homeomorphism $f\colon M \to N$ we have $\widetilde{f} \circ \Xi_M = \Xi_N \circ \widetilde{f}$.

(4) Let M be a connected non-orientable topological manifold. Suppose that we are given two connected 2-fold coverings $p\colon \widetilde{M} \to M$ and $q\colon \widehat{M} \to M$ such that $\widetilde{M}$ and $\widehat{M}$ are orientable. There exists a homeomorphism $\Phi\colon \widetilde{M} \to \widehat{M}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\Phi} & \widehat{M} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & M. & \end{array}$$

If $\widetilde{M}$ and $\widehat{M}$ are oriented, then there exists a unique orientation-preserving Φ as above.

Definition. Let M be a connected non-orientable topological manifold without boundary. We refer to the connected 2-fold covering $p\colon \widetilde{M} \to M$ which we construct in the proof the Orientation Covering-Proposition 105.15 as the orientation covering of M.

Given a non-orientable connected smooth manifold M with orientation cover $p\colon \widetilde{M} \to M$ we discussed in Proposition 49.7 at length the subgroup of $\pi_1(M)$ defined by $\pi_1(\widetilde{M})$. These results generalize without problems to the present setting of topological manifold. In particular we now have the following generalization of the definition from page 1114.

Definition. Let M be a connected topological manifold and let $x_0 \in M$.

(1) If M is non-orientable, then we refer to the epimorphism

$$w_1\colon \pi_1(M, x_0) \ \to \ \pi_1(M, x_0)/p_*(\pi_1(\widetilde{M}, \widetilde{x}_0)) = \mathbb{Z}_2,$$

which is given by the obvious analogue of Proposition 49.7, as the orientation character of M.

(2) If M is orientable, then we define the orientation character of M to be the trivial homomorphism $w_1\colon \pi_1(M, x_0) \to \mathbb{Z}_2$.

Sketch of proof. The proof is very similar to the proof of Propositions 49.4 and 49.5.

(1) Let M be a topological manifold.
   (a) For $x \in M \setminus \partial M$ we define an orientation at x to be a $\mathbb{Z}$-orientation for M at the point x.

[1286]Since the definition of $\widetilde{M}$ is a little more messy than in the case of smooth manifolds we prefer to hide it in the proof.

(b) Let $x \in \partial M$. An orientation-neighborhood for x is a triple (U, μ, x) where U is an open neighborhood of $x \in M$ and μ is a $\mathbb{Z}$-orientation on U. We say that two orientation-neighborhoods (U, μ, x) and (V, ν, x) are equivalent if the orientations μ and ν agree on $U \cap V$. For the purpose of this proof we define an orientation for x to be an equivalence class of orientation-neighborhoods for x.

We consider the set

$$\widetilde{M} := \{\mu_x \mid x \in M \text{ and } \mu_x \text{ is an orientation for } M \text{ at the point } x\}$$

and the projection $p\colon \widetilde{M} \to M$ that is given by $\mu_x \mapsto x$ for $x \in M \setminus \partial M$ and $[(U, \mu, x)] \mapsto x$ for $x \in \partial M$. It follows from Lemma 105.10 (1) that for every $x \in M \setminus \partial M$ the preimage $p^{-1}(x)$ has precisely two elements. We leave it to the reader to verify that for every $x \in \partial M$ the preimage $p^{-1}(x)$ also has precisely two elements.

We equip $\widetilde{M}$ with a topology similar to the proof of Proposition 49.4. It is then straightforward to see that $p\colon \widetilde{M} \to M$ is a 2-fold covering. Similar to the proof of Proposition 49.4 one can show that $\widetilde{M}$ admits a natural orientation.

Now let $f\colon M \to N$ be a local homeomorphism. We leave it to the reader to define the map $\widetilde{f}\colon \widetilde{M} \to \widetilde{N}$ that has the promised properties.[1287]

(2) The proof of this statement is almost identical to the proof of the Orientation Covering-Proposition 49.5 (2).

(3) The proof of this statement is identical to the very terse proof of the Orientation Covering-Proposition 49.5 (3).

(4) Not surprisingly the proof of this statement is also identical to the proof of the Orientation Covering-Proposition 49.5 (4).

We leave it to the reader to flesh out all the details. ■

Finally we conclude this section with the following corollary which is closely related to the Non-Orientable $\mathbb{Z}_2$-Epimorphism Corollary 49.8.

Corollary 105.16. (Non-Orientable $\mathbb{Z}_2$-Epimorphism Corollary) Let M be a connected topological manifold without boundary. If M is non-orientable, then there exists an epimorphism $\pi_1(M) \to \mathbb{Z}_2$.

Proof. The proof of the corollary is basically identical to the proof of the Non-Orientable $\mathbb{Z}_2$-Epimorphism Corollary 49.8, we just need to replace the Orientation Covering-Proposition 49.5 by the Orientation Covering-Proposition 105.15. ■

Remark. Many other results on orientations of smooth manifolds can now be extended to topological manifolds. For example the proof of the Orientation-Flipping Proposition 37.4 that we gave in Section 49.5 also applies verbatim to the setting of topological manifolds.. We leave it to the reader to work out the precise statements and proofs.

105.5. **Induced orientations.** In this final section we will show that various constructions of new topological manifolds out of a given oriented topological manifold are naturally equipped with orientations. For the most part the statements sound reasonable and familiar from the setting of smooth manifolds. Nonetheless, the constructions of the induced representations for topological manifolds are at times quite different from the constructions

[1287] It is obvious how to define the map if $\partial M = \partial N = \varnothing$. Dealing with the boundaries is slightly awkward.

of induced orientations for smooth manifolds. Truth be told, the reader will not miss out on much by just moving on to the next chapter.

Lemma 105.17. (Topological Manifold-Induced Orientation Lemma) Let R be a commutative ring and let M be a topological manifold that is equipped with an R-orientation.

(1) Let N be a topological manifold and let $f\colon N \to M$ be a map. If $f\colon N \setminus \partial N \to M$ is a local homeomorphism, then there exists a unique R-orientation on N such that f is orientation-preserving.
(2) If N is a submanifold of codimension zero (for example N could be an open subset of M), then N admits a unique R-orientation such that the inclusion map $N \to M$ is orientation-preserving.
(3) If $p\colon \widetilde{M} \to M$ is a covering, then there exists a unique orientation on $\widetilde{M}$ that turns p into an orientation-preserving map.

Proof. To simplify the notation we suppress the ring R from the notation. Thus let M be an n-dimensional topological manifold that is equipped with an orientation $\{\alpha_x\}_{x\in M\setminus\partial M}$.

(1) Let N be a topological manifold and let $f\colon N \to M$ be a local homeomorphism. Let $x \in N \setminus \partial N$. We pick an open neighborhood U of x such that $f\colon U \to f(U)$ is a homeomorphism and such that the image $f(U)$ is an open subset of M. We define $\beta_x \in \mathrm{H}_n(N, N\setminus\{x\}; R)$ to be the image of α_x under the map[1288]

$$\begin{array}{ccccccc} & & \text{these two maps are isomorphisms by the Excision Theorem 74.18} & & & & \\ & & \downarrow & & & & \downarrow \\ \mathrm{H}_n(M, M\setminus\{f(x)\}) & \xleftarrow{\cong} & \mathrm{H}_n(f(U), f(U)\setminus\{f(x)\}) & \xrightarrow[\cong]{f_*^{-1}} & \mathrm{H}_n(U, U\setminus\{x\}) & \xrightarrow{\cong} & \mathrm{H}_n(N, N\setminus\{x\}). \end{array}$$

We leave it to the reader to verify that these classes β_x define an orientation on N. By definition the map f is orientation-preserving with respect to the orientations on M and N. It is elementary to verify that the orientation on N is uniquely determined.
(2) This statement follows from (1) since the inclusion map of a codimension-zero submanifold is, basically by definition, a local homeomorphism.
(3) This statement also follows from (1) since a covering map is a local homeomorphism. ■

The following lemma, which can be viewed as an analogue of the Doubling-Manifold Lemma 75.12 (5), says in particular that the double of an R-orientable topological manifold is again R-orientable.

Lemma 105.18. (Double-Orientability Lemma) Let M be a connected topological manifold with non-empty boundary and let R be a commutative ring. Then

$$M \text{ is } R\text{-orientable} \iff \text{the double } \mathrm{D}M \text{ is } R\text{-orientable.}$$

Furthermore, if M is equipped with an R-orientation, then $\mathrm{D}M$ admits a unique R-orientation such that the inclusion map $M \to \mathrm{D}M$ is orientation-preserving.

Sketch of proof. Let M be a connected topological manifold. In the following we write $\mathrm{D}M = (M \sqcup M')/\!\sim$ where M' is another copy of M. We apply the Topological Collar

[1288] Since $f\colon U \to f(U)$ is a homeomorphism we have $f(U \setminus \{x\}) = f(U) \setminus \{f(x)\}$. Furthermore it follows from the fact that $x \in U \setminus \partial U$ and the Topological Manifold Boundary Proposition 75.2 that $f(x) \notin \partial N$.

Neighborhood Theorem 75.5 to obtain a collar neighborhood $[-2,0]\times\partial M$ of ∂M in M. To simplify the notation we henceforth drop the coefficient ring R from the notation.

If $\mathrm{D}M$ is orientable, then it follows immediately from the Topological Manifold-Induced Orientation Lemma 105.17 that the submanifold M is also orientable.

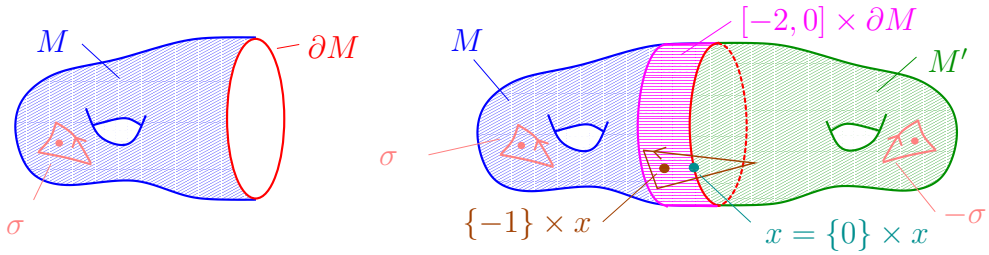
Now suppose that M is equipped with an orientation $\{\mu_x\}_{x\in M\setminus\partial M}$. We define orientations for the points of $\mathrm{D}M$ in $M\setminus\partial M$, $M'\setminus\partial M'$ and in $\partial M=\partial M'$ separately.

(1) Given $x\in M\setminus\partial M$ we denote by α_x the image of μ_x under the inclusion induced isomorphism $\mathrm{H}_n(M,M\setminus\{x\})\to\mathrm{H}_n(\mathrm{D}M,\mathrm{D}M\setminus\{x\})$ given by the Excision Theorem 75.10.
(2) We denote by $r\colon \mathrm{D}M\to\mathrm{D}M$ the obvious homeomorphism given by interchanging the two copies of M in $\mathrm{D}M$. Given $x\in M'\setminus\partial M'$ we define $\alpha_x:=-r_*^{-1}(\alpha_{r(x)})$.
(3) Finally let $x\in\partial M$. We define $\alpha_x\in\mathrm{H}_n(\mathrm{D}M,\mathrm{D}M\setminus\{x\})$ to be the image of the class $\alpha_{\{-1\}\times x}\in\mathrm{H}_n(\mathrm{D}M,\mathrm{D}M\setminus(\{-1\}\times x))$, which we defined in (1), under the isomorphism

$$\mathrm{H}_n(\mathrm{D}M,\mathrm{D}M\setminus\{(-1,x)\}\xleftarrow[\uparrow]{\cong}\mathrm{H}_n(\mathrm{D}M,\mathrm{D}M\setminus([-1,0]\times\{x\}))\xrightarrow[\uparrow]{\cong}\mathrm{H}_n(\mathrm{D}M,\mathrm{D}M\setminus\overbrace{\{(0,x)\}}^{=x}).$$

these maps are isomorphisms by an excision argument together with Complement-of-Convex Homology Lemma 105.1

We leave it as a slightly annoying exercise to the indefatigable reader to show that $\{\alpha_x\}_{x\in\mathrm{D}M}$ does indeed define an orientation on $\mathrm{D}M$. By construction of the orientation on $\mathrm{D}M$ the inclusion $M\to\mathrm{D}M$ is orientation-preserving. The uniqueness of the orientation on $\mathrm{D}M$ is a consequence of the Sections Proposition 105.6 (1) and of our hypothesis that M is connected. ∎



Finally let M be an n-dimensional topological manifold equipped with an orientation $\{\mu_x\}_{x\in M\setminus\partial M}$. Now we will see that the given orientation induces an orientation on ∂M.

Proposition 105.19. (Topological Manifold-Boundary Orientation Proposition) Let R be a commutative ring. Given any topological manifold M there exists a natural map

$$\{R\text{-orientations of } M\}\ \to\ \{R\text{-orientations of } \partial M\}.$$

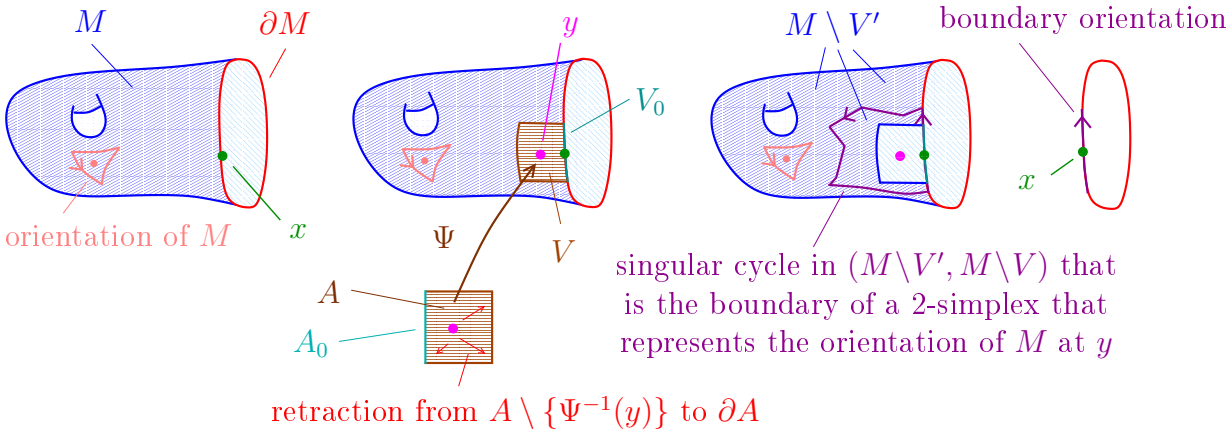
Here natural means that if $f\colon M\to N$ is a homeomorphism, then the following diagram commutes:

$$\begin{array}{ccc}\{R\text{-orientations of } M\} & \longrightarrow & \{R\text{-orientations of } \partial M\}\\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*}\\ \{R\text{-orientations of } N\} & \longrightarrow & \{R\text{-orientations of } \partial N\}.\end{array}$$

Convention. If M is an R-oriented topological manifold, then invariably we will equip ∂M with the R-orientation provided by the construction in the proof of the Topological Manifold-Boundary Orientation Proposition 105.19.

Proof. To simplify the notation we will only deal with case $R = \mathbb{Z}$, which is anyway the only case we care about. We suppose we are given an orientation $\{\mu_y\}_{y\in M\setminus\partial M}$ for M. In the following we will define a natural orientation for ∂M.

We set $n := \dim(M)$ and we write $A = [0,1]\times[-1,1]^n$, we write $A_0 := \{0\}\times[-1,1]^{n-1}$ and we write $A' = (0,1)\times(-1,1)^{n-1}$. Now let $x \in \partial M$. Since M is a topological manifold and since $x \in \partial M$ we can pick a chart $\Phi\colon U \to [0,2)\times(-2,2)^{n-1}$ with $\Phi(x) = (0,0)$. We write $\Psi := \Phi^{-1}$. Furthermore we write $V = \Psi(A)$, $V_0 = \Psi(A_0)$ and we write $V' = \Psi(A')$. Finally we pick a point $y \in V'$.



We consider the following sequence of maps

$$\begin{array}{ccccc} \mathrm{H}_n(M, M\setminus\{y\}) & \xleftarrow{(1)} & \mathrm{H}_n\big(M, M\setminus V'\big) & \xrightarrow[(2)]{\partial} & \mathrm{H}_{n-1}(M\setminus V', (M\setminus V')\setminus\{x\}) \\ & & & & \uparrow{\scriptstyle (3)} \\ & & & & \mathrm{H}_{n-1}(V_0, V_0\setminus\{x\}) \xrightarrow[(4)]{} \mathrm{H}_{n-1}(\partial M, \partial M\setminus\{x\}). \end{array}$$

Here all the maps are induced by the obvious inclusions of pairs of topological spaces, except for the second map which is the connecting homomorphism of the long exact sequence of the triple $(M, M\setminus V', (M\setminus V')\setminus\{x\})$ that is provided by the Topological LES-Proposition 74.13.

Claim 1. All the maps in the above diagram are isomorphisms.

Proof. In the following we treat the four maps separately.

(1) It is elementary to see that $A\setminus\Psi^{-1}(y)$ is a deformation retract of ∂A. It follows that $M\setminus V'$ is a deformation retract of $M\setminus\{y\}$.[1289] We obtain from Corollary 74.16 that (1) an isomorphism.

[1289] Looking at the figure on page 2345 this statement sounds reasonable. But the statement nonetheless requires a proof. So let r be a deformation retraction from $A\setminus\Psi^{-1}(y)$ to ∂A. We consider the map

$$\begin{array}{rcl} s\colon (M\setminus\{y\})\times[0,1] & \to & M\setminus\{y\} \\ (P,t) & \mapsto & \begin{cases} P, & \text{if } P \notin V\setminus\{y\}, \\ \Psi(r(\Psi^{-1}(P),t)), & \text{if } P \in V\setminus\{y\}. \end{cases} \end{array}$$

(2) We consider the long exact sequence of the triple $(M, M \setminus V', (M \setminus V') \setminus \{x\})$ that is provided by the Topological LES-Proposition 74.13:

$$\begin{aligned} \mathrm{H}_n(M, (M \setminus V') \setminus \{x\}) \quad &\to \mathrm{H}_n(M, M \setminus V') \xrightarrow{\partial} \mathrm{H}_{n-1}(M \setminus V', M \setminus V) \to \\ \to \mathrm{H}_{n-1}(M, (M \setminus V') \setminus \{x\}). \end{aligned}$$

Similarly to (1) one sees that $(M \setminus V') \setminus \{x\}$ is a deformation retract of M. Therefore it follows from Corollary 74.16 that the groups at the two extremes are zero. In other words, the connecting homomorphism is an isomorphism.

(3) It follows quite easily from the Excision Theorem 74.18 that (3) is an isomorphism.

(4) Replace (3) by (4) in the previous sentence.

(5) The triangle is given by inclusion induced maps, hence it commutes. It now follows from (3) and (4) that (5) is also an isomorphism. ⊞

Given $x \in \partial M$ we now define $\nu_x \in \mathrm{H}_{n-1}(\partial M, \partial M \setminus \{x\})$ to be the image of the generator $\mu_y \in \mathrm{H}_n(M, M \setminus \{y\})$ under the above sequence of isomorphisms. We continue with our next claim.

Claim 2.

(1) The definition of ν_x is independent of the choice of y.

(2) The definition of ν_x is independent of the choice of Φ.

Proof.

(1) We need to show that for any $z \in \Psi(V')$ we have

$$\begin{array}{c} \text{preimage of } \mu_y \text{ under the map} \\ \mathrm{H}_n(M, M \setminus V') \xrightarrow{\cong} \mathrm{H}_n(M, M \setminus \{y\}) \end{array} = \begin{array}{c} \text{preimage of } \mu_z \text{ under the map} \\ \mathrm{H}_n(M, M \setminus V') \xrightarrow{\cong} \mathrm{H}_n(M, M \setminus \{z\}). \end{array}$$

But this statement can be shown in a similar fashion as the Sections Proposition 105.6 (1). We leave it to the reader to fill in the details.

(2) Suppose we are given a different chart than Φ. We define W, W_0, W' in the obvious way. By possibly taking a third chart whose image lies in the intersection of V and W we can assume, without loss of generality, that $W \subset V$. (We refer to the figure below for an illustration.) Furthermore by (1) we can work with a point $y \in W$ for both charts. We consider the following diagram

$$\begin{array}{ccc} \mathrm{H}_n(M, M \setminus \{y\}) & \xleftarrow[(1)]{\cong} & \mathrm{H}_n\big(M, M \setminus V'\big) \\ \cong \uparrow (1) & \swarrow & \downarrow (2)\ \partial \\ \mathrm{H}_n\big(M, M \setminus W'\big) & \leftarrow & \mathrm{H}_{n-1}(M \setminus V', (M \setminus V') \setminus \{x\}) \\ (2)\ \partial \downarrow & \swarrow & \uparrow (3)\ \cong \\ \mathrm{H}_{n-1}(M \setminus W', (M \setminus W') \setminus \{x\}) & & \mathrm{H}_{n-1}(V_0, V_0 \setminus \{x\}) \\ (3)\ \uparrow \cong & \searrow & \downarrow (4) \\ \mathrm{H}_{n-1}(W_0, W_0 \setminus \{x\}) & \xrightarrow[(4)]{} & \mathrm{H}_{n-1}(\partial M, \partial M \setminus \{x\}). \end{array}$$

Note that $V \setminus \{y\}$ and $M \setminus \Psi(A')$ are closed subsets of $M \setminus \{y\}$. (Hmm, why is that?). It follows from the Homotopy Stacking Lemma 14.3 that the map s is continuous.

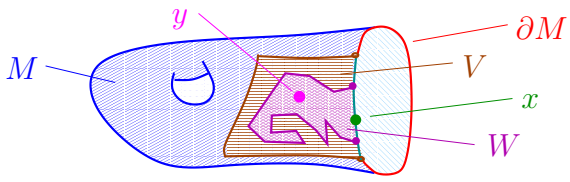
Here the two "outer paths" from the top left to the lower right correspond to the two "competing" definitions of ν_x. The dashed arrows are given by inclusions. We make the following observations:

(a) The triangle on top commutes since all maps are induced by inclusions.
(b) The upper parallelogram commutes since the connecting homomorphism is natural, see the Topological LES-Proposition 74.13 (3) for details.
(c) The lower parallelogram commutes since all the maps are induced by inclusions. (Note that the reversal of the direction of the arrow is not a typo.)
(d) The triangle on the bottom commutes since all maps are induced by inclusions.

We saw above that all maps (1), (2), (3) and (4) are isomorphisms. By going from the top to the bottom we see that all the dashed arrows are isomorphisms.

Since all pieces of the diagram commute and since all maps are isomorphisms we see that the total diagram commutes. This shows that the two "competing" definitions of ν_x agree. ⊞

Using Claim 1 it is now straightforward to show that the $\{\nu_x\}_{x\in\partial M}$ form an orientation for ∂M and that this orientation is natural. ■



Remark. Let M be an n-dimensional smooth manifold that is equipped with an orientation in the sense of Section 25.2. On page 689 and page 688 we had equipped the $(n-1)$-dimensional smooth manifold ∂M with an orientation in the sense of Section 25.2. If one takes the strenuous journey through all conventions and definitions one sees that the boundary orientations from page 689 and the proof of the Topological Manifold-Boundary Orientation Proposition 105.19 are compatible. More precisely, for any oriented n-dimensional M we have the following commutative "diagram"

$$\begin{array}{ccc} \text{oriented smooth manifold } M & \xmapsto{\text{take boundary}} & \text{oriented smooth manifold } \partial M \\ \Big\downarrow \text{Smooth-Topological Orientation Proposition 105.11} & & \Big\downarrow \text{Smooth-Topological Orientation Proposition 105.11} \\ \mathbb{Z}\text{-oriented smooth manifold } M & \xmapsto{\text{take boundary}} & \mathbb{Z}\text{-oriented smooth manifold } \partial M. \end{array}$$

Remark. Given topological manifolds M and N we can form the product $M \times N$. If M and N are oriented, then one would expect that $M \times N$ comes with a natural orientation. This is indeed the case, but the definition of this natural orientation requires more tools than we have at the moment. Thus we postpone the definition of the product orientation to Section 113.5.

Exercises for Chapter 105.

Exercise 105.1. We view $\mathbb{R}$ as a topological manifold and we equip it with the standard $\mathbb{Z}$-orientation that we had introduced on page 2337. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a map that is a local homeomorphism and that is orientation-preserving. Show that f is strictly monotonously increasing.

Exercise 105.2. Let M be an n-dimensional topological manifold, let R be a commutative ring and let $A \subset M \setminus \partial M$ be a subset.

(a) Show that for R-sections $\{\alpha_x\}_{x\in A}$ and $\{\beta_x\}_{x\in A}$ along A the collection $\{\alpha_x + \beta_x\}_{x\in A}$ is also an R-section along A.

(b) Show that the set $\Gamma_R^A(M)$ admits a unique R-module structure such that for each $y \in A$ the map $\Gamma_R^A(M) \to \mathrm{H}_n(M, M \setminus \{y\}; R)$ given by $\{\alpha_x\}_{x\in A} \mapsto \alpha_y$ is an R-module homomorphism.

Exercise 105.3. Let M be an oriented n-dimensional topological manifold. Furthermore let $f, g\colon M \to M$ be two homeomorphisms. Show that if f is orientation-preserving and if g is homeotopic to f, then g is also orientation-preserving.
Remark. In the Smooth Isotopy-Orientation Proposition 34.2 we proved the analogue in the setting of smooth manifolds.

Exercise 105.4. Let M be an oriented n-dimensional topological manifold, let A and B be disjoint unions of boundary components of M and let $f\colon A \to B$ be a homeomorphism. We equip A and B with the boundary orientations, as defined in the Topological Manifold-Boundary Orientation Proposition 105.19.

(a) If f is orientation-reversing, then $M/a \sim f(a)$ admits an orientation such that the inclusion $M \setminus (A \cup B) \to M/a \sim f(a)$ is orientation-preserving.

(b) If M is connected and if f is not orientation-reversing, then $M/a \sim f(a)$ is non-orientable.

Remark. This is the long overdue proof of the Gluing-Topological Manifolds-Proposition 75.8 (6).

Exercise 105.5. Let M be a compact oriented 1-dimensional topological manifold. We equip ∂M with the orientation coming from the Topological Manifold-Boundary Orientation Proposition 105.19. Recall that on page 2335 we saw that an orientation on the 0-dimensional topological manifold ∂M is naturally the same as a map $\epsilon\colon \partial M \to \{\pm 1\}$. Show that

$$\sum_{P\in\partial M} \epsilon(P) \;=\; 0.$$

Remark. This exercise can be viewed as the analogue of the Boundary-of-Compact Smooth 1-Manifold Proposition 25.18 for topological manifolds.

106. The fundamental class of topological manifolds

In several previous chapters, e.g. in Chapter 96 and 98, we have assembled lots of information on homology groups of smooth manifolds. Let us recall some of the results. Thus let M be a connected n-dimensional smooth manifold.

(1) In the Smooth Manifolds-Invariants Proposition 96.6 (2) we saw, using the Smooth Manifold-CW Structure Theorem 96.5 , that $\mathrm{H}_i(M;\mathbb{Z}) = 0$ for $i > n$.
(2) Furthermore, in the Smooth Manifold-Top Homology Theorem 98.6 and Theorem 98.8 we showed that

$$\mathrm{H}_n(M;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } M \text{ closed and orientable,} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mathrm{H}_n(M,\partial M;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } M \text{ compact and orientable,} \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if M is compact and orientable, then we used (2) to define the corresponding fundamental class $[M] \in \mathrm{H}_n(M,\partial M;\mathbb{Z})$ which turned out to be very useful tool.
The key tool in proving the above statements was the fact, established in the Smooth Manifold-Simplicial Structure Theorem 96.2, that smooth manifolds admit simplicial structures. As we discussed in Section 104.6, in general topological manifolds do not admit simplicial structures, so if we want to replicate the above results for topological manifolds we will have to use a very different approach.
In this chapter we will indeed manage to reprove all of the above statements for topological manifolds. The advantage of the new approach is that it is more general, it covers topological manifolds, and that it does not require any extra topological input, i.e. it does not require the existence of a simplicial or cellular structure. The flip side of this chapters approach is that the proofs are arguably less intuitive.

106.1. **The fundamental class of an oriented topological manifold.** We recall the following notation that we introduced on page 2349.

Notation. Let M be a topological manifold, let R be a commutative ring and finally let $x \in M \setminus \partial M$. Given any $k \in \mathbb{N}_0$ we denote by ρ_x the obvious map

$$\rho_x \colon \mathrm{H}_k(M,\partial M;R) \to \mathrm{H}_k(M, M\setminus\{x\};R).$$

Given $\alpha \in \mathrm{H}_k(M,\partial M;R)$ we often write $\alpha_x := \rho_x(\alpha)$, i.e. the above map is also written as

$$\alpha \mapsto \alpha_x.$$

The following three theorems contain most of the main results of this chapter.

Theorem 106.1. (Orientation-Top Homology Theorem) Let M be a compact connected non-empty n-dimensional topological manifold. Then the following statements are equivalent:

(1) the topological manifold M is orientable,
(2) $\mathrm{H}_n(M,\partial M;\mathbb{Z}) \neq 0$,
(3) $\mathrm{H}_n(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}$,

(4) for every $x \in M \setminus \partial M$ the map

$$\begin{array}{rcl} \rho_x \colon \mathrm{H}_n(M, \partial M; \mathbb{Z}) & \to & \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{Z}) \\ \alpha & \mapsto & \alpha_x \end{array}$$

is an isomorphism.

Theorem 106.2. (Top-$\mathbb{F}_2$-Homology Theorem) Let M be a compact connected non-empty n-dimensional topological manifold M. Then the following two statements hold:

(1) $\mathrm{H}_n(M, \partial M; \mathbb{F}_2) \cong \mathbb{F}_2$, and

(2) for every $x \in M \setminus \partial M$ the map

$$\begin{array}{rcl} \rho_x \colon \mathrm{H}_n(M, \partial M; \mathbb{F}_2) & \to & \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2) \\ \alpha & \mapsto & \alpha_x \end{array}$$

is an isomorphism.

Theorem 106.3. (Top-Homology Theorem) Let M be a connected non-empty n-dimensional topological manifold and let R be a commutative ring. The following three statements hold:

(1) For every $i > n$ we have $\mathrm{H}_i(M; R) = 0$ and $\mathrm{H}_i(M, \partial M; R) = 0$.

(2)
$$\mathrm{H}_n(M; R) \cong \begin{cases} R, & \text{if } M \text{ is closed and } R\text{-orientable,} \\ 0, & \text{otherwise.} \end{cases}$$

(3)
$$\mathrm{H}_n(M, \partial M; R) \cong \begin{cases} R, & \text{if } M \text{ is compact and } R\text{-orientable,} \\ 0, & \text{otherwise.} \end{cases}$$

Remark.

(1) As mentioned before, the above theorems generalize most of the results of the Smooth Manifolds-Invariants Proposition 96.6 (2), the Smooth Manifold-Top Homology Theorem 98.6 and Theorem 98.8 from smooth manifolds to topological manifolds.

(2) One might be tempted to consider the Top-Homology Theorem 106.3 (3) as "intuitively clear". After all, how could an n-dimensional object possibly have homology groups in higher dimensions? To shatter this belief, fix some $n \geq 2$. We consider the following subset of $\mathbb{R}^n$:[1290]

$$\begin{array}{rcl} X & := & \bigcup\limits_{n \in \mathbb{N}} n\text{-dimensional sphere of radius } \frac{1}{n} \text{ around } (\frac{1}{n}, 0, \ldots, 0) \\ & = & \bigcup\limits_{n \in \mathbb{N}} \{x \in \mathbb{R}^n \mid \|x - (\frac{1}{n}, 0, \ldots, 0)\| = \frac{1}{n^2}\}. \end{array}$$

In [**BaM1962**] it is shown, rather shockingly, that for every q with $q \equiv 1 \bmod (r-1)$ the group $\mathrm{H}_q(X; \mathbb{Q})$ is not only non-zero, it is even uncountable.

We will not immediately prove these three theorems, but we will first draw many conclusions, before we head towards the somewhat intricate proofs of the above three theorems.

[1290] As the reader will surely have noticed, this example can be viewed as a high-dimensional analogue of the Hawaiian Earings that we introduced in Exercise 56.1.

Definition. Let M be a compact n-dimensional topological manifold and let R be a commutative ring. An *R-fundamental class of M* is an element $\beta \in \mathrm{H}_n(M, \partial M; R)$ with the property that for any $x \in M \setminus \partial M$ the element $\beta_x \in \mathrm{H}_n(M, M \setminus \{x\}; R)$ is a generator. For $R = \mathbb{Z}$ we just say *fundamental class*.

Remark. Let M be a compact n-dimensional topological manifold and let R be a commutative ring.

(1) Let $[M]$ be a $\mathbb{Z}$-fundamental class for M. The image of $[M] \otimes 1$ under the natural map $\mu\colon \mathrm{H}_n(M; \mathbb{Z}) \otimes R \to \mathrm{H}_n(M; R)$ is an R-fundamental class. Indeed, let $x \in M \setminus \partial M$. From Lemma 89.7 (4) we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_n(M; \mathbb{Z}) \otimes R & \xrightarrow{\mu} & \mathrm{H}_n(M; R) \\ \downarrow & & \downarrow \\ \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{Z}) \otimes R & \xrightarrow{\mu} & \mathrm{H}_n(M, M \setminus \{x\}; R) \end{array}$$

By Lemma 105.4 and the Universal Coefficient Theorem 89.17 the bottom horizontal map is an isomorphism. Since the image of the $\mathbb{Z}$-fundamental class $[M]$ under the obvious map $\mathrm{H}_n(M; \mathbb{Z}) \to \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{Z})$ is a generator it follows from the commutative diagram that the image of $\mu([M] \otimes 1)$ under the obvious map $\mathrm{H}_n(M; R) \to \mathrm{H}_n(M, M \setminus \{x\}; R)$ is indeed a generator.

(2) If M has connected components $M_1, \ldots, M_k$ and if for $i = 1, \ldots, k$ we are given R-fundamental classes $[M_i]_R \in \mathrm{H}_n(M_i, \partial M_i; R)$, then

$$\sum_{i=1}^{k} [M_i]_R \ \in \ \mathrm{H}_n(M, \partial M; R) = \bigoplus_{i=1}^{k} \mathrm{H}_n(M_i, \partial M_i; R)$$

is easily seen to be an R-fundamental class of M. The fundamental classes for the M_i are unique if and only if the fundamental class for M is unique.

Example. Let M be the empty topological space. As we mentioned on page 514, basically by definition the empty set is a topological manifold of any dimension. Note that for the empty manifold $\varnothing$ we have the *equality* $\mathrm{H}_0(\varnothing) = \mathrm{H}_1(\varnothing) = \mathrm{H}_2(\varnothing) = \cdots = \{0\}$. By definition this element 0 is a fundamental class of the empty manifold and it is evidently unique. We write $[\varnothing] := 0$. This is consistent with the discussion on page 2205.

The following corollary to the Orientation-Top Homology Theorem 106.1 says in particular that an orientation determines and is in turn uniquely determined by a fundamental class.

Corollary 106.4. (Fundamental Class-Corollary) Let M be a compact n-dimensional topological manifold.

(1) If $[M]$ is a fundamental class, then $\{[M]_x\}_{x \in M \setminus \partial M}$ defines an orientation for M.
(2) If $\{\alpha_x\}_{x \in M \setminus \partial M}$ is an orientation for M, then there exists a unique fundamental class $[M] \in \mathrm{H}_n(M, \partial M; \mathbb{Z})$ with $[M]_x = \alpha_x$ for all $x \in M \setminus \partial M$.

Furthermore, if M is connected, non-empty and orientable, then the following two statements hold:

(3) Every fundamental class $[M] \in \mathrm{H}_n(M, \partial M; \mathbb{Z})$ is a generator of $\mathrm{H}_n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$.
(4) Every generator of $\mathrm{H}_n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}^{1291}$ is a fundamental class.

Proof. Let M be a compact non-empty n-dimensional topological manifold. By considering the components of M separately we see that without loss of generality we can assume that M is connected. It is also clear that we only have to study the case that M is non-empty.

(1) Let $[M]$ be a fundamental class for the topological manifold M. It follows from Lemma 105.5 (2) that the homology classes $\{\rho_x([M])\}_{x\in M\setminus\partial M}$ form a section on M. By definition of a fundamental class each $\rho_x([M])$ is a generator. Therefore the section $\{\rho_x([M])\}_{x\in M\setminus\partial M}$ is indeed an orientation.

(2) Suppose we are given an orientation $\{\alpha_x\}_{x\in M\setminus\partial M}$ for M. By the Orientation-Top Homology Theorem 106.1 (1) $\Rightarrow$ (4) we know that for every $x \in M \setminus \partial M$ the map

$$\rho_x \colon \mathrm{H}_n(M, \partial M; \mathbb{Z}) \ \to \ \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{Z})$$

is an isomorphism. Now pick a $y \in M$. We write $[M] := \rho_y^{-1}(\alpha_y)$. Since ρ_y is an isomorphism the class $[M]$ is uniquely determined. So it remains to show that $[M]_x = \alpha_x$ for all $x \in M \setminus \partial M$. By Lemma 105.5 (2) the classes $\{[M]_x\}_{x\in M\setminus\partial M}$ form a $\mathbb{Z}$-section. It follows from the Sections Proposition 105.6 that $[M]_x = \alpha_x$ for all $x \in M \setminus \partial M$.

(3) This statement follows immediately from the definition of a fundamental class and the Orientation-Top Homology Theorem 106.1 (1) $\Rightarrow$ (4).

(4) The last statement is an immediate consequence of Lemma 105.5 (2) and the Orientation-Top Homology Theorem 106.1 (1) $\Rightarrow$ (4). ■

We also have the following corollary to the Top-$\mathbb{F}_2$-Homology Theorem 106.2 which deals with $\mathbb{F}_2$-coefficients.

Corollary 106.5. ($\mathbb{F}_2$-Fundamental Class Corollary) Let M be a compact, non-empty n-dimensional topological manifold.

(1) If M is connected, then[1292]

$$[M]_{\mathbb{F}_2} \ := \ \text{the unique non-zero element of } \mathrm{H}_n(M, \partial M; \mathbb{F}_2) \cong \mathbb{F}_2$$

is the unique $\mathbb{F}_2$-fundamental class of M.

(2) If M has connected components $M_1, \ldots, M_k$, then

$$[M]_{\mathbb{F}_2} \ := \ \sum_{i=1}^{k} [M_i]_{\mathbb{F}_2} \ \in \ \mathrm{H}_n(M, \partial M; \mathbb{F}_2) = \bigoplus_{i=1}^{k} \mathrm{H}_n(M_i, \partial M_i; \mathbb{F}_2)$$

is the unique $\mathbb{F}_2$-fundamental class of M.

Proof. It follows immediately from the Top-$\mathbb{F}_2$-Homology Theorem 106.2 (2) that the class $[M]_{\mathbb{F}_2}$ that we had just defined in (1) is indeed an $\mathbb{F}_2$-fundamental class for M and that it is unique. As we pointed out on page 2351, the second statement is an immediate consequence of the first statement. ■

Remark. Let R be a commutative ring. We consider the empty topological manifold $\varnothing$. Note that we have the equality of sets $\mathrm{H}_0(\varnothing; R) = \mathrm{H}_1(\varnothing; R) = \mathrm{H}_2(\varnothing; R) = \cdots = \{0\}$. As

[1291] Here we use that in the Orientation-Top Homology Theorem 106.1 we saw that for a compact orientable, non-empty n-dimensional topological manifold we have $\mathrm{H}_n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$.

[1292] Here in the formulation we already use the Top-$\mathbb{F}_2$-Homology Theorem 106.2 (1) to guarantee that $\mathrm{H}_n(M, \partial M; \mathbb{F}_2) \cong \mathbb{F}_2$.

on page 2205 we now define the fundamental class $[\varnothing]$ of the empty topological manifold $\varnothing$ to be the unique element in the group $\mathrm{H}_0(\varnothing;R) = \mathrm{H}_1(\varnothing;R) = \mathrm{H}_2(\varnothing;R) = \cdots = \{0\}$.

Let M be a compact non-empty n-dimensional *smooth* manifold. On page 2215 we introduced the corresponding $\mathbb{F}_2$-fundamental class $[M]_{\mathbb{F}_2} \in \mathrm{H}_n(M,\partial M;\mathbb{F}_2)$. Furthermore, if M is in fact oriented, then on page 2199 we introduced the corresponding fundamental class $[M] \in \mathrm{H}_n(M,\partial M;\mathbb{Z})$. We will now see that the previous definitions are consistent with the definitions from this chapter.

First let us consider the case of $\mathbb{F}_2$-fundamental classes. This case is easy: it follows basically immediately from the $\mathbb{F}_2$-Fundamental Class Corollary 106.5 (1) that the two definitions actually agree.

Now let us assume that the smooth manifold M is oriented in the sense of Section 25.2. In Theorem 100.1 we defined the fundamental class $[M]$ in $\mathrm{H}_n(M,\partial M;\mathbb{Z})$. On the other hand, in the Smooth-Topological Orientation Proposition 105.11 we saw that the orientation of M also determines a $\mathbb{Z}$-orientation and thus we obtain a fundamental class $[M] \in \mathrm{H}_n(M,\partial M;\mathbb{Z})$ by the Fundamental Class-Corollary 106.4. Not surprisingly we will now see that these two definitions agree.

Lemma 106.6. The two definitions of the fundamental class of a compact oriented non-empty smooth manifold agree.

Remark. Let M be a compact oriented non-empty smooth manifold. In Algorithm 100.2 and the Fundamental Class-Criterion Proposition 100.4 we developed some techniques for finding explicit cycles representing the fundamental class of the oriented smooth manifold M. By Lemma 106.6 these approaches also work for finding explicit cycles representing the fundamental class of a smooth manifold that is equipped with a $\mathbb{Z}$-orientation.

Proof. Let M be a compact n-dimensional smooth manifold that is oriented in the sense of Section 25.2. The case $n = 0$ is basically trivial. Hence we now only consider the case $n \geq 1$. As we will see, the lemma follows almost immediately from the definitions.

Now let us first recall the construction, in the proof of the Smooth-Topological Orientation Proposition 105.11, of the corresponding $\mathbb{Z}$-orientation of M. Let $x \in M \setminus \partial M$. We pick a chart $\Phi\colon U \to V$ around x that is orientation-preserving in the sense of page 675. We define

$$\mu_x \;:=\; \Phi_*^{-1}\big(\text{standard generator of } \mathrm{H}_n(V, V\setminus\{\Phi(x)\};\mathbb{Z}), \text{ as defined on page 2328}\big).$$

In the proof of the Smooth-Topological Orientation Proposition 105.11 we saw this definition of μ_x does not depend on the choice of the orientation-preserving chart. Furthermore we saw these classes $\{\mu_x\}_{x\in M\setminus\partial M}$ define a $\mathbb{Z}$-orientation for M.

We denote by $\alpha \in \mathrm{H}_n(M;\mathbb{Z})$ the fundamental class of Theorem 100.1. By definition of the fundamental class from the Fundamental Class-Corollary 106.4 it now remains to prove the following claim.

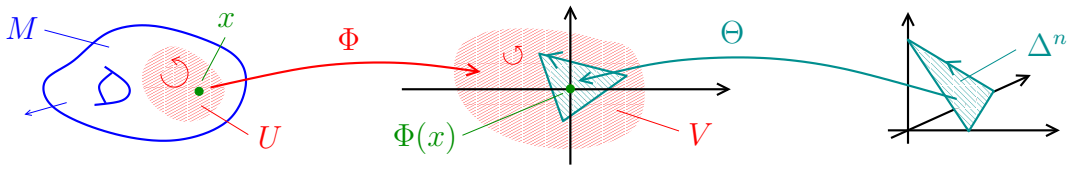
Claim. Given any $x \in M \setminus \partial M$ we have $\alpha_x = \mu_x \in \mathrm{H}_n(M, M\setminus\{x\})$.

Proof. Let $x \in M \setminus \partial M$. We pick a chart $\Phi\colon U \to V$ around x with $\Phi(x) = 0$ that is orientation-preserving. Furthermore let $\Theta\colon \Delta^n \to V$ be an affine linear map such that 0

lies in the image of $\Theta(\mathring{\Delta}^n)$ and which has the property that $\Theta\colon \mathring{\Delta}^n \to V$ is orientation-preserving. We see that

$$\alpha_x \underset{\text{follows from Theorem 100.1 and the fact that } \Phi^{-1}\circ\Theta \text{ is orientation-preserving}}{=} [\Phi^{-1}\circ\Theta\colon \Delta^n \to M] \overset{\text{[}\Theta\colon \Delta^n \to V\text{] is by definition a standard generator of } \mathrm{H}_n(V, V\setminus\{0\};\mathbb{Z})}{=} \Phi_*^{-1}([\text{standard generator of } \mathrm{H}_n(V, V\setminus\{0\};\mathbb{Z})]) = \mu_x.$$

■



We continue with the following two propositions which can be viewed as a generalizations of the Fundamental Class-Criterion Proposition 100.4 and 100.18.

Proposition 106.7. Let M be a compact oriented connected n-dimensional topological manifold. Let $r_1\cdot\sigma_1 + \cdots + r_m\cdot\sigma_m \in \mathrm{C}_n(M,\partial M)$ be a cycle. Suppose there exists a $j\in\{1,\ldots,m\}$ such that the following three conditions are satisfied:

(1) The map $\sigma_j\colon \mathring{\Delta}^n \to M$ is an orientation-preserving embedding.[1293]
(2) There exists an $x\in\sigma_j(\Delta^n)\setminus\sigma_j(\partial\Delta^n)$ which is not contained in the image of any other $\sigma_i\colon \Delta^n\to M$.
(3) $r_j = 1$.

Then the cycle $r_1\cdot\sigma_1 + \cdots + r_m\cdot\sigma_m$ represents the fundamental class of the oriented topological manifold.

Proposition 106.8. Let M be a compact connected n-dimensional topological manifold. Let $(\sigma_1+\cdots+\sigma_m)\otimes 1 \in \mathrm{C}_n(M,\partial M;\mathbb{F}_2) = \mathrm{C}_n(M,\partial M)\otimes\mathbb{F}_2$ be a cycle. Suppose there exists a $j\in\{1,\ldots,m\}$ such that the following two conditions are satisfied:

(1) The map $\sigma_j\colon \mathring{\Delta}^n \to M$ is an embedding.
(2) There exists an $x\in\sigma_j(\Delta^n)\setminus\sigma_j(\partial\Delta^n)$ which is not contained in the image of any other $\sigma_i\colon \Delta^n\to M$.

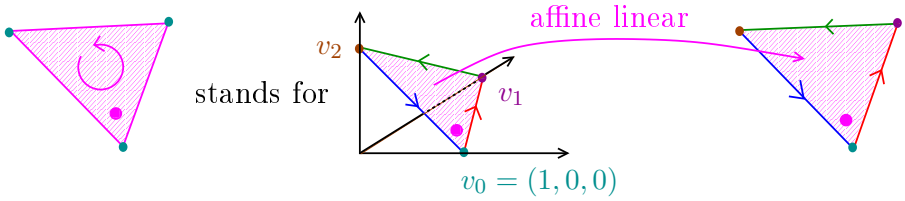
Then the cycle $(\sigma_1+\cdots+\sigma_m)\otimes 1$ represents the $\mathbb{F}_2$-fundamental class of M.

Proof. We leave it to the reader to modify the proofs of Propositions 100.4 and 100.18 to obtain the above result. ■

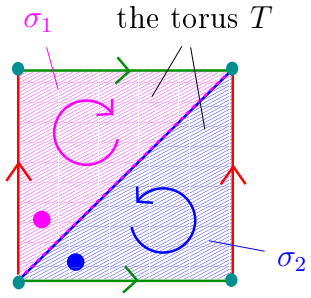
For the readers convenience we recall the discussion of cycles representing fundamental classes of the torus and the surface of genus 2 which we already gave in the figures on 2204 and 2204. There is always the outside chance that a reader decided to give Chapter 100 a pass.

Example. We will give some examples of fundamental classes of 2-dimensional topological manifolds. We use the notation from the figure below to indicate a singular simplex. In the figure below we now show cycles representing the fundamental classes of the torus

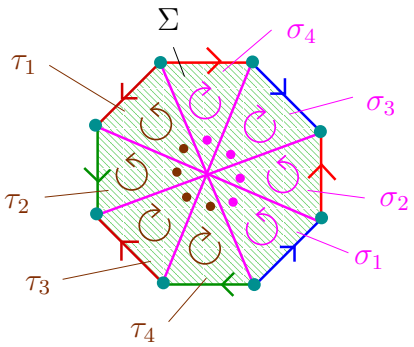
[1293] We equip Δ^n, and thus also its submanifold $\mathring{\Delta}^n$, with the $\mathbb{Z}$-orientation which is given by considering the orientation given by $[\mathrm{id}_{\Delta^n}]\in\mathrm{H}_n(\Delta^n,\Delta^n\setminus\{x\})$ for each $x\in\mathring{\Delta}^n$.

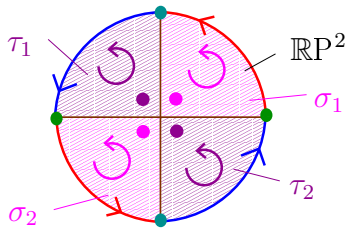


and the surface of genus 2 and we show a cycle representing the $\mathbb{F}_2$-fundamental class of $\mathbb{R}\mathrm{P}^2 = \overline{B}^2/z \sim -z$.



$[T] = [-\sigma_1 + \sigma_2]$





$[\mathbb{R}\mathrm{P}^2]_{\mathbb{F}_2} = [(\sigma_1+\tau_1+\sigma_2+\tau_2)\otimes 1]$

$$[\Sigma] = [\sigma_1+\sigma_2-\sigma_3-\sigma_4+\tau_1+\tau_2-\tau_3-\tau_4]$$

106.2. **Lifts of sections on topological manifolds.** In this section we will prove the fundamental theorem of topological manifolds which lies at the heart of the proofs of Theorems 106.1, 106.2 and 106.3. Before we can state the lemma we introduce the following helpful notation.

Notation. Let M be a topological manifold without boundary, let R be a commutative ring and let $A \subset M$ be a subset.

(1) We write $\mathrm{H}_n(M|A;R) := \mathrm{H}_n(M, M\setminus A; R)$. Given a point $x \in M$ we use the obvious abbreviation $\mathrm{H}_n(M|x;R) := \mathrm{H}_n(M|\{x\};R)$.
(2) Given a homology class $\alpha \in \mathrm{H}_n(M|A;R)$ and a point $x \in A$ we denote by α_x the image of α under the obvious map $\mathrm{H}_n(M|A;R) \to \mathrm{H}_n(M|x;R)$.
(3) Let $\{\alpha_x\}_{x\in M}$ be an R-section on M and let $A \subset M$ be a subset. We call a class $\beta \in \mathrm{H}_n(M|A;R)$ a lift to A if for all $x \in A$ we have $\beta_x = \alpha_x$.
(4) As on page 2331 we denote by $\Gamma_R^A(M)$ the set of all R-sections of M along A.

The following lemma follows immediately from the definitions. But since we will refer to it on several occasions it is good to write it down.

Lemma 106.9. Let M be a topological manifold without boundary, let R be a commutative ring and let $A \subset B \subset M$ be subsets:

(1) Since $M\setminus B \subset M\setminus A$ we have an inclusion induced map $\mathrm{H}_n(M|B;R) \to \mathrm{H}_n(M|A;R)$.
(2) Let $\{\alpha_x\}_{x\in M}$ be an R-section on M. If $\mu_B \in \mathrm{H}_n(M|B;R)$ is a lift to B, then the image of μ_B under the above map $\mathrm{H}_n(M|B;R) \to \mathrm{H}_n(M|A;R)$ is a lift of the R-section to the subset A.

The following theorem is the main technical result of this chapter.

Theorem 106.10. (Homology-Section Theorem) Let M be an n-dimensional topological manifold with empty boundary and let $A \subset M$ be a compact subset. Let R be a commutative ring. Then the following hold:

(i) The map
$$\begin{array}{rcl} \mathrm{H}_n(M|A;R) & \to & \Gamma^A_M(R) \\ \varphi & \mapsto & \{\varphi_x\}_{x\in A} \end{array}$$
is an isomorphism. In other words, if $\{\alpha_x\}_{x\in A}$ is an R-section for M along A, then there exists a unique lift $\alpha^M_A = \alpha_A$ to A, i.e. there exists a unique homology class $\alpha^M_A = \alpha_A \in \mathrm{H}_n(M|A;R)$ such that for each $x \in A$ we have $\alpha_{A,x} = \alpha_x$.

(ii) For $i > n$ we have $\mathrm{H}_i(M|A;R) = 0$.

Remark. To avoid any nervousness later on we remark that the statements (i) and (ii) hold trivially for $A = \varnothing$ since $\mathrm{H}_n(M|\varnothing;R) = \mathrm{H}_n(M, M;R) = 0$.

Corollary 106.11. Let M be an n-dimensional topological manifold with empty boundary, let R be a commutative ring and let $A \subset M$ be a compact connected subset. If M is R-orientable, then for every $x \in A$ the map
$$\mathrm{H}_n(M, M\setminus A;R) \;\to\; \mathrm{H}_n(M, M\setminus\{x\};R) \;\underset{\uparrow}{\cong}\; R$$
isomorphic by Lemma 105.4
is an isomorphism.

Proof. We consider the following diagram:
$$\begin{array}{ccc} \mathrm{H}_n(M, M\setminus A;R) & \xrightarrow{\varphi\mapsto\varphi_x} & \mathrm{H}_n(M, M\setminus\{x\};R) \\ & {\scriptstyle \varphi\mapsto\{\varphi_x\}_{x\in A}} \searrow \quad \nearrow {\scriptstyle \{\varphi_x\}_{x\in A}\mapsto\varphi_x} & \\ & \Gamma^A_M(R) & \end{array}$$
The diagram commutes by definition. Since A is compact we know by the Homology-Section Theorem 106.10 (i) that the left diagonal map is an isomorphism. Furthermore, since A is connected we obtain from Lemma 105.9 that the right diagonal map is an isomorphism. Thus we see that the horizontal map is an isomorphism, as promised. ■

For later on we also record the following corollary to the Homology-Section Theorem 106.10 .

Corollary 106.12. Let M be an oriented n-dimensional topological manifold with empty boundary and let R be a commutative ring. We suppose that M is equipped with an R-orientation. We use the notation from the Homology-Section Theorem 106.10 .

(1) Let $K \subset L \subset M$ be compact subsets. We denote by $f_{KL}\colon (M, M\setminus L) \to (M, M\setminus K)$ the inclusion map of pairs. The map $f_{KL*}\colon \mathrm{H}_n(M, M\setminus L) \to \mathrm{H}_n(M, M\setminus K)$ sends α^M_L to α^M_K.

(2) Let U be an open subset of M and let $A \subset U$ be a compact subset. We use the Topological Manifold-Induced Orientation Lemma 105.17 to view U as an oriented topological manifold. If $i\colon U \to M$ denotes the inclusion map, then $i_*(\alpha^U_A) = \alpha^M_A$.

Proof. The lemma follows easily from the functoriality of homology groups and the uniqueness statement of the Homology-Section Theorem 106.10 . ■

Now we turn to the proof of the Homology-Section Theorem 106.10 . The very rough idea behind the proof of the fundamental theorem of topological manifolds can be summarized as follows:

(1) Using Mayer–Vietoris sequences we want to show that if the compact subsets A, B and $A \cap B$ have the desired properties, then so does $A \cup B$.
(2) Using charts we want to reduce the proof of the theorem to the case that $M = \mathbb{R}^n$.
(3) Finally for $M = \mathbb{R}^n$ we want to reduce the proof of the theorem to compact convex subsets A of $\mathbb{R}^n$.

The following lemmas give us the tools to put this idea into practice.

Lemma 106.13. Let M be an n-dimensional topological manifold, let R be a commutative ring and let A and B be two compact subsets of M. If the statements (i) and (ii) from the Homology-Section Theorem 106.10 hold for A, B and $A \cap B$, then they also hold for $A \cup B$.

Proof. Let M be an n-dimensional topological manifold, let R be a commutative ring and let A and B be two compact subsets of M such that A, B and $A \cap B$ satisfy statements (i) and (ii). To simplify the notation we drop R from the notation. We consider the inverted Mayer–Vietoris Sequence from Theorem 78.7

$$\cdots \to \overbrace{\mathrm{H}_{n+1}(M|A \cap B)}^{=0, \text{ by hypothesis on } A \cap B} \to \mathrm{H}_n(M|A \cup B) \xrightarrow[\alpha \mapsto (\alpha, -\alpha)]{\Phi} \mathrm{H}_n(M|A) \oplus \mathrm{H}_n(M|B) \xrightarrow[(\alpha, \beta) \mapsto \alpha + \beta]{\Psi} \mathrm{H}_n(M|A \cap B) \to \dots$$

Now we can prove that $A \cup B$ also satisfies statements (i) and (ii).

(ii) For $i > n$ the group $\mathrm{H}_i(M|A \cup B)$ is stuck, by our hypothesis on A, B and $A \cap B$, between groups that are zero, thus we see that $\mathrm{H}_i(M|A \cup B) = 0$ for $i > n$.

(i) Now suppose we are given a section $\{\alpha_x\}_{x \in M}$. We denote by α_A, α_B and $\alpha_{A \cap B}$ the unique lifts to A, B and $A \cap B$ given by our hypothesis. By Lemma 106.9 the images of α_A and α_B in $\mathrm{H}_n(M|A \cap B)$ are lifts to $A \cap B$. By the uniqueness of the lift $\alpha_{A \cap B}$ these images thus equal $\alpha_{A \cap B}$. By the exactness of the above sequence we see that $(\alpha_A, -\alpha_B) = \Phi(\alpha_{A \cup B})$ for some $\alpha_{A \cup B} \in \mathrm{H}_n(M|A \cup B)$. We claim that $\alpha_{A \cup B}$ is indeed a lift to $A \cup B$. So let $x \in A \cup B$. First we consider the case that $x \in A$. We have the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}_n(M|A \cup B) & \xrightarrow{[\alpha_{A \cup B} \mapsto \alpha_A]} & \mathrm{H}_n(M|A) \\ & \searrow & \downarrow \\ & & \mathrm{H}_n(M|x). \end{array}$$

We see that the image of $\alpha_{A \cup B}$ in $\mathrm{H}_n(M|x)$ is the image of α_A in $\mathrm{H}_n(M|x)$, hence it equals α_x. The same argument applies to the case that $x \in B$.

It remains to prove the uniqueness of the lift $\alpha_{A \cup B}$. If $\beta \in \mathrm{H}_n(M|A \cup B)$ is another lift, then it follows immediately from Lemma 106.9 and the uniqueness of α_A and α_B that $\Phi(\beta) = (\alpha_A, -\alpha_B)$. By the exactness of the sequence we see that we already have the equality $\alpha_{A \cup B} = \beta$ in $\mathrm{H}_n(M|A \cup B)$. ∎

Later on we will need the following variation on Lemma 106.13.

Lemma 106.14. Let M be an n-dimensional topological manifold, let R be a commutative ring. Let $\mathcal{A}$ be a family of compact subsets of M with the following two properties:

(1) Statements (i) and (ii) from the Homology-Section Theorem 106.10 hold for each $A \in \mathcal{A}$.
(2) The family $\mathcal{A}$ is closed under intersections, i.e. whenever $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.

Then the statements hold for all finite unions of elements in $\mathcal{A}$.

Proof. Given $k \in \mathbb{N}$ we write

$$\mathcal{U}_k := \{\text{all subsets of } M \text{ that are the union of at most } k \text{ subsets in } \mathcal{A}\}.$$

We claim that for every $k \in \mathbb{N}$ statements (i) and (ii) hold for all elements in $\mathcal{U}_k$. We prove this claim by induction on k.

(1) If $k = 1$, then the claim holds by hypothesis (1).
(2) Suppose that the claim holds for some k. Let $A_1, \dots, A_k, A_{k+1} \in \mathcal{A}$. We write $B = A_2 \cup \dots \cup A_{k+1}$. Note that

$$A_1 \cap B \;=\; A_1 \cap (A_2 \cup \dots \cup A_{k+1}) \;=\; \bigcup_{i=2}^{k+1} \underbrace{(A_1 \cap A_i)}_{\in \mathcal{A} \text{ by (2)}} \;\in \mathcal{U}_k.$$

Thus by induction hypothesis we see that the statements hold for A_1, B and also for $A_1 \cap B$. So the statements also hold for $A_1 \cup B = A_1 \cup \dots \cup A_{k+1}$ by Lemma 106.13. ■

Our next goal is to prove the Homology-Section Theorem 106.10 for $M = \mathbb{R}^n$. We will proceed in several steps.

Lemma 106.15. Let R be a commutative ring and let $\{\mu_x\}_{x \in \mathbb{R}^n}$ be an R-section on $\mathbb{R}^n$, let $A \subset \mathbb{R}^n$ be a convex bounded subset and let $\beta \in \mathrm{H}_n(\mathbb{R}^n | A; R)$. If $\mu_y = \beta_y$ for some $y \in A$, then $\mu_x = \beta_x$ for all $x \in A$, i.e. β is a lift to A.

Proof. The proof is very similar to the proof of the Sections Proposition 105.6. The following are the two key observations:

(1) a convex set is path-connected and thus connected,
(2) since A is convex, for any $x \in A$ and any $\epsilon > 0$ the intersection $A \cap B_\epsilon(x)$ is the intersection of two convex bounded sets, hence convex and bounded.

In the proof of the Sections Proposition 105.6 we now need to replace the Small Ball-Lemma 105.7 by the Complement-of-Convex Homology Lemma 105.1. We leave it to the reader to modify the proof of the Sections Proposition 105.6 to obtain a proof of Lemma 106.15. ■

Lemma 106.16. Statements (i) and (ii) from the Homology-Section Theorem 106.10 hold for $M = \mathbb{R}^n$, any commutative ring R and any convex compact subset A of $\mathbb{R}^n$.

Proof. Let A be a convex compact subset of $\mathbb{R}^n$ and let R be a commutative ring. As pointed out above, if $A = \varnothing$, then there is nothing to prove. Thus we can assume that $A \neq \varnothing$.

Now we can prove that A satisfies the statements (i) and (ii).

(i) Let $\{\alpha_x\}_{x\in\mathbb{R}^n}$ be a section for $\mathbb{R}^n$. Since $A \neq \varnothing$ we can pick a $y \in A$. We denote by $\alpha_A \in \mathrm{H}_n(\mathbb{R}^n|A)$ the preimage of α_y under the isomorphism

$$\mathrm{H}_n(\mathbb{R}^n|A;R) \;\to\; \mathrm{H}_n(\mathbb{R}^n|\{y\};R)$$

given by the Complement-of-Convex Homology Lemma 105.1. It follows from Lemma 106.15 that $\alpha_{A,x} = \alpha_x$ for all $x \in A$, i.e. we have shown that α_A is a lift to A. The uniqueness of the lift α_A follows again from the above isomorphism.

(ii) It follows the Complement-of-Convex Homology Lemma 105.1 and 75.1 that $\mathrm{H}_i(\mathbb{R}^n|A;R) = \mathrm{H}_i(\mathbb{R}^n|\{y\};R) = 0$ for every $i > n$. ■

Lemma 106.17. Every subset of $\mathbb{R}^n$ that is the union of finitely many convex compact sets satisfies the statements (i) and (ii) from the Homology-Section Theorem 106.10 .

Proof. We consider

$$\mathcal{A} := \{\text{all convex compact subsets of } \mathbb{R}^n\}.$$

This family of subsets of $\mathbb{R}^n$ is closed under intersections.[1294] By Lemma 106.16 the two statements hold for each $A \in \mathcal{A}$. The lemma is now an immediate consequence of Lemma 106.14. ■

The following lemma is arguably the hardest step towards proving the fundamental theorem of topological manifolds.

Lemma 106.18. Statements (i) and (ii) from the Homology-Section Theorem 106.10 hold for $M = \mathbb{R}^n$, any commutative ring R and any compact subset A of $\mathbb{R}^n$.

Proof. To simplify the notation we again drop the R from the notation. Let $\{\alpha_x\}_{x\in\mathbb{R}^n}$ be a section for $\mathbb{R}^n$ and let $A \subset \mathbb{R}^n$ be an arbitrary compact set.

We first prove statement (i). We pick a closed ball B that contains A. We denote by $\alpha_B \in \mathrm{H}_n(\mathbb{R}^n|B)$ the lift constructed in Lemma 106.16. By Lemma 106.9 the image α_A of the class α_B under the map $\mathrm{H}_n(\mathbb{R}^n|B) \to \mathrm{H}_n(\mathbb{R}^n|A)$ is a lift to A. To complete the proof of statement (i) we still need to prove the uniqueness of α_A.

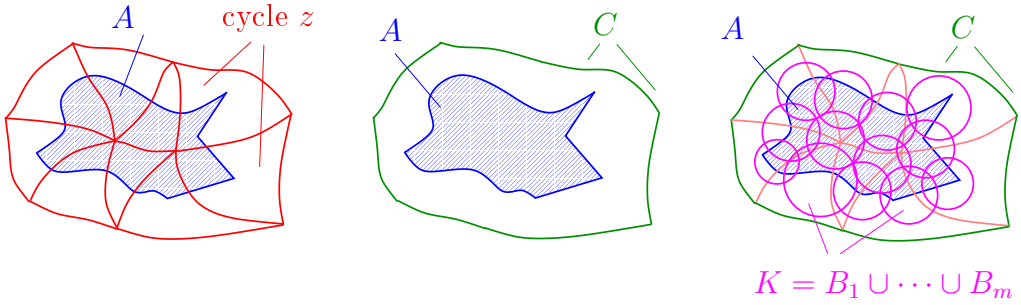
This uniqueness statement is equivalent to showing that if $\beta \in \mathrm{H}_n(\mathbb{R}^n|A)$ is a class such that $\beta_x = 0$ for all $x \in A$, then β is already zero. So let $\beta \in \mathrm{H}_n(\mathbb{R}^n|A)$. We pick a singular n-chain $z \in \mathrm{C}_n(\mathbb{R}^n)$ that represents β. We denote by $C \subset \mathbb{R}^n \setminus A$ the union of the images of the singular simplices appearing in $\partial z \subset \mathrm{C}_{n-1}(\mathbb{R}^n \setminus A)$. Note that C is the union of finitely many compact sets, hence C itself is compact.

Claim. There exist finitely many closed balls $B_1, \ldots, B_m$ with the following three properties:

(1) each B_i is contained in $\mathbb{R}^n \setminus C$,
(2) each B_i contains a point $x_i \in A$,
(3) we have $A \subset B_1 \cup \cdots \cup B_m$.

[1294] Here we use one of the most convenient properties of convex sets: the intersection of two convex sets is again convex.

Proof. Since C is compact and since $A \cap C = \varnothing$ we can find for each $x \in A$ an $r_x > 0$ with $B_{2r_x}(x) \subset \mathbb{R}^n \setminus C$. Since A is compact there exist finitely many points $x_1, \ldots, x_m$ in A such that A is contained in $B_{r_{x_1}}(x_1) \cup \cdots \cup B_{r_{x_m}}(x_m)$. It is now clear that $B_i := \overline{B}_{r_{x_i}}(x_i)$, $i = 1, \ldots, m$ have the desired property. We refer to the figure below for an illustration. ⊞



We write $K := B_1 \cup \cdots \cup B_m$. By (1) the singular n-chain $z \in \mathrm{C}_n(\mathbb{R}^n)$ also represents an element $\gamma \in \mathrm{H}_n(\mathbb{R}^n|K)$. Note that for each $x \in A$ we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_n(\mathbb{R}^n|K) & \xrightarrow{\gamma=[z]\mapsto\beta=[z]} & \mathrm{H}_n(\mathbb{R}^n|A) \\ & {\scriptstyle \gamma\mapsto\gamma_x} \searrow \quad \swarrow {\scriptstyle \beta\mapsto\beta_x} & \\ & \mathrm{H}_n(\mathbb{R}^n|x). & \end{array}$$

It follows in particular from the commutative diagram that it suffices to show that we have $\gamma = 0 \in \mathrm{H}_n(\mathbb{R}^n|K)$. We do so as follows:

(1) By our hypothesis on β and by the above commutative diagram we know that for each $i \in \{1, \ldots, m\}$ we have $\gamma_{x_i} = \beta_{x_i} = 0$.
(2) Let $i \in \{1, \ldots, m\}$. Since $\gamma_{x_i} = 0$ and since B_i is convex and bounded it follows from Lemma 106.15 (applied to the zero section and to the image of $\gamma \in \mathrm{H}_n(\mathbb{R}^n|B_i)$) that $\gamma_x = 0$ for all $x \in B_i$.
(3) By (2) we know that $\gamma_x = 0$ for all $x \in K = B_1 \cup \cdots \cup B_m$.
(4) But now it follows from Lemma 106.17 that $\gamma = 0$.

This concludes the proof of statement (i).

Finally we need to show statement (ii) for A, i.e. we need to show that $\mathrm{H}_i(\mathbb{R}^n|A) = 0$ for $i > n$. So let $\beta \in \mathrm{H}_i(\mathbb{R}^n|A)$. As above we pick a singular n-chain $z \in \mathrm{C}_n(\mathbb{R}^n)$ that represents β. We define C, K and γ as above. By Lemma 106.17 we have $\gamma = 0 \in \mathrm{H}_i(\mathbb{R}^n|K)$. But as before this implies that also $\beta = 0 \in \mathrm{H}_i(\mathbb{R}^n|A)$. ■

Lemma 106.19. Let M be an n-dimensional topological manifold with empty boundary and let $A \subset M$ be a compact subset. If there exists a chart $\Phi \colon U \to \mathbb{R}^n$ such that A is contained in U, then statements (i) and (ii) hold for A.

Proof. The lemma is a fairly straightforward consequence of Lemma 106.18 and the fact that we have isomorphisms

$$\begin{array}{ccccc} \mathrm{H}_n(M, M \setminus A; R) & \xleftarrow{\cong} & \mathrm{H}_n(U, U \setminus A; R) & \xrightarrow{\Phi} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \Phi(A); R) \\ \uparrow & & \uparrow & & \uparrow \\ \text{Excision Theorem 74.18} & & & & \text{isomorphism since } \Phi \text{ is a homeomorphism} \end{array}$$

We leave it to the reader to fill in the details. ■

Now we can finally complete the proof of the Homology-Section Theorem 106.10 .

Proof of the Homology-Section Theorem 106.10 . Let M be an n-dimensional topological manifold with empty boundary. We consider

$$\mathcal{A} := \text{all compact subsets } A \text{ of } M \text{ for which there exists a chart } \Phi\colon U \to \mathbb{R}^n \text{ with } A \subset U.$$

By Lemma 106.19 statements (i) and (ii) hold for all subsets in $\mathcal{A}$. It is clear that $\mathcal{A}$ is closed under intersections. Thus it follows from Lemma 106.14 that the statements (i) and (ii) hold for all finite unions of sets in $\mathcal{A}$. Therefore it suffices to prove the following claim.

Claim. Every compact subset A of M is the union of finitely many sets in $\mathcal{A}$.

Proof. Let A be a compact subset of M. It is clear that given any point $x \in A$ there exists a chart $\Phi_x\colon U_x \to V_x$ around x such that the image is some open ball in $\mathbb{R}^n$. But since each open ball in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ we can find around each $x \in A$ a chart around x such that the image V_x equals $\mathbb{R}^n$. Now we write $A_x := A \cap \Phi_x^{-1}(\overline{B}_1^n(0))$ and $U'_x := \Phi_x^{-1}(B_1^n(0))$. Since A is compact we can find finitely many $x_1, \dots, x_m$ in A such that A is contained in $U'_{x_1} \cup \cdots \cup U'_{x_m}$. By construction the corresponding charts $\Phi_i := \Phi_{x_i}\colon U_{x_i} \to \mathbb{R}^n$ and the sets $A_i := A_{x_i}$ have the desired properties. ⊞



It follows from Lemma 106.19 that statements (i) and (ii) hold for all the A_i and for all arbitrary intersections of the A_i. But then Lemma 106.13 implies that they also hold for $A = A_1 \cup \cdots \cup A_m$. ■

106.3. **Proof of the Top-Homology Theorem 106.3.** Now we are in a position that we can provide the long-delayed proofs of Theorems 106.1, 106.2 and 106.3. We start out with the proof of the Top-Homology Theorem 106.3 since we will use that theorem in the proof of the Orientation-Top Homology Theorem 106.1.

The key extra ingredient in the proof of the Top-Homology Theorem 106.3 is the following proposition.

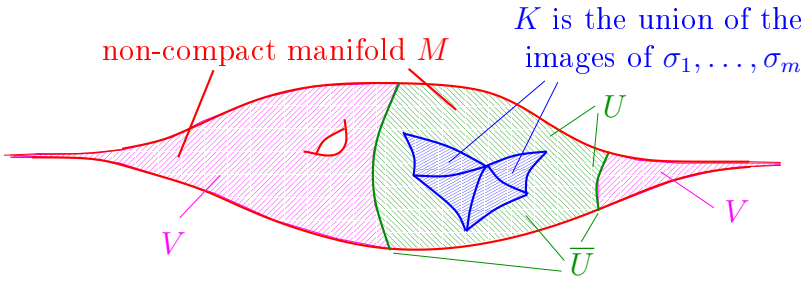
Proposition 106.20. Let M be a connected n-dimensional topological manifold and let R be a commutative ring. If M is non-compact and if $\partial M = \varnothing$, then $\mathrm{H}_k(M; R) = 0$ for every $k \geq n$.

Proof. Let M be a connected non-compact n-dimensional topological manifold such that $\partial M = \varnothing$, let R be a commutative ring and let $k \geq n$. Let $\alpha \in \mathrm{H}_k(M; R)$. We need to show that $\alpha = 0$. We represent α by a cycle $z = \sum_{i=1}^{m} \sigma_i \otimes r_i \in \mathrm{C}_k(M; R) = \mathrm{C}_k(M) \otimes R$. The union of the images of $\sigma_1, \dots, \sigma_m$ is a compact subset K of M. Since K is compact it follows from Exercise 106.1 that there exists an open subset $U \subset M$ that contains K and such that the closure $\overline{U}$ is compact. We consider the open subset $V = M \setminus \overline{U}$ and we consider the long exact sequence of the triple $(M, U \cup V, V)$, see the Topological LES-Corollary 74.14. It fits

into the following commutative diagram

$$\begin{array}{ccccccc} \ldots \longrightarrow \mathrm{H}_{k+1}(M, U\cup V; R) & \xrightarrow{\partial} & \mathrm{H}_k(U\cup V, V; R) & \longrightarrow & \mathrm{H}_k(M, V; R) & \longrightarrow & \ldots \\ & & \cong\uparrow & & \uparrow & & \\ & & \mathrm{H}_k(U; R) & \xrightarrow{[z]\mapsto[z]=\alpha} & \mathrm{H}_k(M; R). & & \end{array}$$

Here the left vertical map is an isomorphism since $U \cup V$ is the disjoint union of the open sets U and V.[1295]



Now we consider the modules to the top-left and top-right of the diagram. We have

$$\mathrm{H}_{k+1}(M, U\cup V; R) \;=\; \mathrm{H}_{k+1}(M|\overbrace{M\setminus(U\cup V)}^{=\overline{U}\setminus U}; R) \;\underset{\uparrow}{=}\; 0 \quad \text{if } k > n-1$$

by the Homology-Section Theorem 106.10 (ii) since M has no boundary and since $\overline{U}$ and $\overline{U}\setminus U$ are compact

$$\mathrm{H}_k(M, V; R) \;=\; \mathrm{H}_k(M|\underbrace{M\setminus V}_{=\overline{U}}; R) \qquad \underset{\downarrow}{=}\; 0 \quad \text{if } k > n.$$

After these preparations we come to the actual proof that $\alpha = [z] = 0 \in \mathrm{H}_k(M; R)$. We first consider the case $k > n$. We deduce from the above exact sequence and the above calculations that $\mathrm{H}_k(U; R) = 0$. But z represents an element in $\mathrm{H}_k(U; R)$. It follows that $[z] = 0$ in $\mathrm{H}_k(M; R)$.

Now we consider the case $k = n$. It follows from the above discussion that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_n(U\cup V, V; R) & \hookrightarrow & \mathrm{H}_n(M, V; R) \\ \cong\uparrow & & \uparrow \\ \mathrm{H}_n(U; R) & \xrightarrow{[z]\mapsto[z]=\alpha} & \mathrm{H}_n(M; R). \end{array}$$

Thus it suffices to show that the image of α in $\mathrm{H}_n(M, V; R)$ is zero. We showed in Lemma 105.5 (2) that $\{\alpha_x\}_{x\in M}$ is an R-section of M.[1296] Since M is non-compact we can find a point x outside of the compact set K. The class $\alpha_x \in \mathrm{H}_n(M, M\setminus\{x\}; R)$ is again represented by z which lies in $K \subset M\setminus\{x\}$. It is thus clear that $\alpha_x = 0 \in \mathrm{H}_n(M, M\setminus\{x\}; R)$. But since M is connected it follows from the Sections Proposition 105.6 (1) that $\{\alpha_x\}_{x\in M}$ is the zero section. Both the image of α in $\mathrm{H}_n(M, V; R)$ and the zero-class are lifts of the

[1295] One could at this point refer to the Excision Theorem 74.17 but this isomorphism follows already from the observation that $\mathrm{C}_*(U\cup V) = \mathrm{C}_*(U)\oplus\mathrm{C}_*(V)$ as chain complexes, since U and V are both open subsets of $U \cup V$.

[1296] Here we used a second time that M has no boundary.

zero section to $\mathrm{H}_n(M, V; R)$. By the Homology-Section Theorem 106.10 (i) both lifts agree, i.e. the image of α in $\mathrm{H}_n(M, V; R)$ is zero. ■

Now we can easily provide the proof of the Top-Homology Theorem 106.3.

Proof of the Top-Homology Theorem 106.3. Let M be a connected n-dimensional topological manifold and let R be a commutative ring. Recall that we need to prove the following three statements:

(1) (a) For every $i > n$ we have $\mathrm{H}_i(M; R) = 0$.
(b) For every $i > n$ we have $\mathrm{H}_i(M, \partial M; R) = 0$.

(2)
$$\mathrm{H}_n(M; R) \cong \begin{cases} R, & \text{if } M \text{ is closed and } R\text{-orientable,} \\ 0, & \text{otherwise.} \end{cases}$$

(3)
$$\mathrm{H}_n(M, \partial M; R) \cong \begin{cases} R, & \text{if } M \text{ is compact and } R\text{-orientable,} \\ 0, & \text{otherwise.} \end{cases}$$

Let us start out with the proof of (1) and (2). Note that (1b) is an immediate consequence of (1a), applied to M and ∂M, and the long exact sequence in homology of the pair $(M, \partial M)$. Thus it remains to prove (1a) and (2). In the following we prove (1a) and (2) together in one go. First suppose that M is not closed. The hypothesis that M is not closed means that M is not compact or that it has a non-empty boundary. If M is not compact without boundary, then we had just proved in Proposition 106.20 that $\mathrm{H}_i(M; R) = 0$ for $i \geq n$. If M has non-empty boundary, then we saw in Corollary 75.7 that M is homotopy equivalent to a non-compact n-dimensional topological manifold N without boundary. It follows from the Homotopy Equivalence-H_*-Corollary 73.9 and Proposition 106.20 that for any $i \geq n$ we have $\mathrm{H}_i(M; R) \cong \mathrm{H}_i(N; R) = 0$.

Now suppose that M is closed. Let $i \geq n$. We have

$$\mathrm{H}_i(M; R) = \mathrm{H}_i(M, \overbrace{M \setminus M}^{=\varnothing}; R) = \mathrm{H}_i(M|M; R) \underset{\uparrow}{\cong} \begin{cases} 0, & \text{if } i > n, \\ R, & \text{if } i = n. \end{cases}$$

by the Homology-Section Theorem 106.10 (ii) and Corollary 106.11 since M is compact and since M has no boundary

Now we turn to the proof of (3). If $\partial M = \varnothing$, then this is just the statement of (2). So let us now assume that $\partial M \neq \varnothing$. Similar to the proof of Theorem 98.8 we will prove (3) by reducing it to (2) via the doubling construction. We denote by $\mathrm{D}M$ the double of M as introduced on page 1667. We consider the following diagram:

$$\underbrace{\mathrm{H}_n(M)}_{=0} \xrightarrow{\iota_*} \mathrm{H}_n(\mathrm{D}M) \longrightarrow \mathrm{H}_n(\mathrm{D}M, M) \longrightarrow \mathrm{H}_{n-1}(M; R) \xhookrightarrow{\iota_*} \mathrm{H}_{n-1}(\mathrm{D}M)$$
$$\mathrm{H}_n(\mathrm{D}M) \underset{\cong}{\uparrow} \mathrm{H}_i(M, \partial M)$$

We make the following observations and clarifications:

- We denote by $\iota\colon M \to \mathrm{D}M$ the inclusion map.
- The top sequence is just the long exact sequence of the pair $(\mathrm{D}M, M)$.
- Since $\partial M \neq \varnothing$ we know by (2) that $\mathrm{H}_n(M) = 0$.

- The vertical map is induced by the inclusion and it is an isomorphism by the Excision Theorem 75.10 for Topological Manifolds, see also the Doubling-Manifold Lemma 75.12 (6).
- By Lemma 75.11 we have a retraction $r\colon \mathrm{D}M \to M$ such that $r \circ \iota = \mathrm{id}_M$. It follows from the functoriality of homology groups that $\iota_*\colon \mathrm{H}_{n-1}(M) \to \mathrm{H}_{n-1}(\mathrm{D}M)$ is a monomorphism.

In summary, we have shown that $\mathrm{H}_n(\mathrm{D}M) \cong \mathrm{H}_n(M, \partial M)$. The desired statement now follows from this observation, the calculation in (2) and the fact, see Lemmas 75.12 and 105.18, that M is compact and R-orientable if and only if $\mathrm{D}M$ is closed and R-orientable. ■

106.4. **Proof of Theorems 106.1 and 106.2.** The key to proving Theorems 106.1 and 106.2 is the following proposition which later on we will apply to $R = \mathbb{Z}$ and $R = \mathbb{F}_2$.

Proposition 106.21. Let M be a connected, compact n-dimensional topological manifold and let R be a commutative ring. Then the following hold:

(1) If M is R-orientable, then for any $x \in M \setminus \partial M$ the map

$$\mathrm{H}_n(M, \partial M; R) \;\to\; \mathrm{H}_n(M, M \setminus \{x\}; R) \;\underset{\text{isomorphic by Lemma 105.4}}{\overset{\cong}{\uparrow}}\; R$$

is an isomorphism.

(2) If M is non-orientable, then $\mathrm{H}_n(M, \partial M; \mathbb{Z}) = 0$.

Proof. Let M be a connected, compact n-dimensional topological manifold and let R be a commutative ring.

We first consider the case that M is actually closed. Recall that we denote by $\Gamma_R(M)$ the set of all R-sections of M. By Lemma 105.5 (2) we have a well-defined R-module homomorphism

$$\begin{aligned} \mathrm{H}_n(M; R) &\mapsto \Gamma_R(M) \\ \alpha &\mapsto \{\alpha_x\}_{x \in M}. \end{aligned}$$

Since M is closed we can apply the Homology-Section Theorem 106.10 (i) to the special case of the compact subset $A = M$ and we obtain that the above map is an isomorphism. The desired statements (1) and (2) now both follow from Lemma 105.9.

Now we consider the case that M is not closed, i.e. we suppose that $\partial M \neq \varnothing$. As usual we denote by $\mathrm{D}M$ the double of M. The idea now is to prove the desired statements for M by reducing them to the equivalent statements for the closed topological manifold $\mathrm{D}M$ that we had just proved.

As before we think of the double as $\mathrm{D}M = (M \sqcup M')/\sim$ where M' is another copy of M. We consider the maps

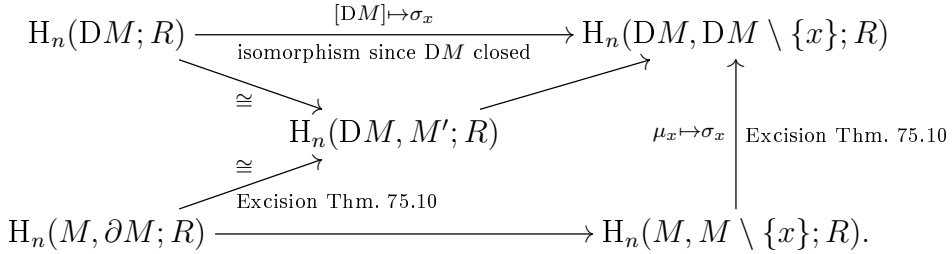
$$\underbrace{\mathrm{H}_n(M'; R)}_{=0 \text{ by Thm. } 106.3} \longrightarrow \mathrm{H}_n(\mathrm{D}M; R) \longrightarrow \mathrm{H}_n(\mathrm{D}M, M'; R) \longrightarrow \mathrm{H}_{n-1}(M'; R) \xrightarrow[\text{by } 75.13]{\text{injective}} \mathrm{H}_{n-1}(\mathrm{D}M; R)$$

$$\mathrm{H}_n(\mathrm{D}M, M'; R) \;\overset{\cong}{\uparrow}\; \mathrm{H}_n(M, \partial M; R), \qquad \text{with } \Phi \text{ from } \mathrm{H}_n(\mathrm{D}M; R) \text{ to } \mathrm{H}_n(M, \partial M; R),$$

where on the top we have written down the long exact sequence of the pair $(\mathrm{D}M, M')$. Note that the vertical map is an isomorphism by the Excision Theorem 75.10. The map Φ is defined to be the unique homomorphism that makes the triangle commute.

It follows immediately from the information provided in the above diagram that the map $\Phi\colon \mathrm{H}_n(\mathrm{D}M;R) \to \mathrm{H}_n(M,\partial M;R)$ is an isomorphism. Now we turn to the proof of the two statements of the proposition.

(1) Now suppose that M is R-orientable. Let $x \in M \setminus \partial M$. We consider the following diagram



Here all maps are the obvious inclusion induced maps. This implies that the diagram commutes. Since $\mathrm{D}M$ is closed and since it is R-orientable by the Double-Orientability Lemma 105.18 we know from the first part of the proof that the top horizontal map is an isomorphism. Furthermore by the above discussion we know that the top-left diagonal map is an isomorphism. Finally two of the maps are isomorphisms by the Excision Theorem 75.10. It now follows immediately that the bottom horizontal map is also an isomorphism. But that is exactly what we were supposed to show.

(2) If M is non-orientable, then it follows from the Double-Orientability Lemma 105.18 that the double $\mathrm{D}M$ is non-orientable either. We obtain that

$$\mathrm{H}_n(M,\partial M;\mathbb{Z}) \underset{\text{see above}}{\cong} \mathrm{H}_n(\mathrm{D}M;\mathbb{Z}) \underset{\substack{\text{since } \mathrm{D}M \text{ is closed and non-orientable and}\\ \text{since we have proved (1) in the closed case}}}{=} 0.$$

■

Remark. Note that the second statement of Proposition 106.21 only deals with integer coefficients. A more general statement is given by [**Hat2002**, Theorem 3.26]. This theorem says that if M is a closed connected n-dimensional topological manifold that is *not* R-orientable, then for any $x \in M$ the map $\rho_x\colon \mathrm{H}_n(M;R) \to \mathrm{H}_n(M, M\setminus\{x\};R) \cong R$ is injective with image $\{r \in R \mid 2r = 0\}$. For example for $R = \mathbb{Z}$ we have $\{r \in \mathbb{Z} \mid 2r = 0\} = \{0\}$, i.e. the map ρ_x is the zero map and by injectivity we have $\mathrm{H}_n(M;\mathbb{Z}) = 0$. In the case of closed topological manifolds we can thus view [**Hat2002**, Theorem 3.26] as a generalization of Proposition 106.21 (2). We will not make use of [**Hat2002**, Theorem 3.26].

Now we can finally provide the proofs of Theorems 106.1 and 106.2.

Proof of the Orientation-Top Homology Theorem 106.1. Let M be a compact connected non-empty n-dimensional topological manifold. We have to show that the following statements are equivalent:

(1) the topological manifold M is orientable,
(2) $\mathrm{H}_n(M,\partial M;\mathbb{Z}) \neq 0$,
(3) $\mathrm{H}_n(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}$,
(4) for every $x \in M \setminus \partial M$ the map

$$\rho_x\colon \mathrm{H}_n(M,\partial M;\mathbb{Z}) \to \mathrm{H}_n(M, M\setminus\{x\};\mathbb{Z})$$

is an isomorphism.

We have (1) $\Rightarrow$ (4) by Proposition 106.21 (1) applied with $R = \mathbb{Z}$. Since M is non-empty we obtain from the Topological Manifold Boundary Proposition 75.2 (5) that there exists a point $x \in M \setminus \partial M$. Therefore we have (4) $\Rightarrow$ (3) by Lemma 105.4. Evidently we have (3) $\Rightarrow$ (2). Furthermore we have (2) $\Rightarrow$ (1) by Proposition 106.21 (2). ■

Proof of the Top-$\mathbb{F}_2$-Homology Theorem 106.2. Let M be a compact connected non-empty n-dimensional topological manifold. We have to show the following two statements:

(1) $\mathrm{H}_n(M, \partial M; \mathbb{F}_2) \cong \mathbb{F}_2$, and
(2) for every $x \in M \setminus \partial M$ the map

$$\rho_x \colon \mathrm{H}_n(M, \partial M; \mathbb{F}_2) \quad \to \quad \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2)$$

is an isomorphism.

By the $\mathbb{F}_2$-Orientability Proposition 105.13 we know that M admits an $\mathbb{F}_2$-orientation. Statement (2) follows immediately from this fact and Proposition 106.21 (1) applied to $R = \mathbb{F}_2$. As in the proof of the Orientation-Top Homology Theorem 106.1, the first statement is an immediate consequence of the second statement, together with the fact that there exists a point in $M \setminus \partial M$ and Lemma 105.4. ■

106.5. **Further properties of topological manifolds.** Let M be a compact n-dimensional topological manifold. With the Orientation-Top Homology Theorem 106.1 we now completely understand the n-dimensional homology of M. In this section we will obtain some results on the $(n-1)$-dimensional homology of M which generalize the results of the Smooth Manifold-Top Homology Theorem 98.6 (2) and Theorem 98.8 (2) from smooth manifolds to topological manifolds.

Proposition 106.22. Let M be a connected compact n-dimensional topological manifold.

$$\text{(1)} \qquad \mathrm{H}_{n-1}(M; \mathbb{Z}) \cong \begin{cases} \text{free abelian group}, & \text{if } M \text{ is orientable or if } \partial M \neq \varnothing, \\ \text{free abelian group} \oplus \mathbb{Z}_2, & \text{if } M \text{ is non-orientable and closed.} \end{cases}$$

$$\text{(2)} \quad \mathrm{H}_{n-1}(M, \partial M; \mathbb{Z}) \cong \begin{cases} \text{free abelian group}, & \text{if } M \text{ is orientable,} \\ \text{free abelian group} \oplus \mathbb{Z}_2, & \text{if } M \text{ is non-orientable.} \end{cases}$$

In all cases the free abelian group is finitely generated.

Remark.

(1) As a reality check it is good to recall the calculation of the homology groups of closed 2-dimensional topological manifolds that we gave in Proposition 80.10. There we showed for any $g \in \mathbb{N}_0$ and any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_n(\Sigma_g) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}^{2g}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0 \end{cases} \qquad \text{and} \qquad \mathrm{H}_n(k \cdot \mathbb{R}\mathrm{P}^2) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}_2 \oplus \mathbb{Z}^{k-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0, \end{cases}$$

where, as always, we denote by Σ_g the surface of genus g. Fortunately these calculations are coherent with Proposition 106.22.

(2) It follows immediately from the Universal Coefficient Theorem 89.17, the Orientation-Top Homology Theorem 106.1, Proposition 106.22 and the Tor-Properties Lemma 89.15 that for any closed orientable n-dimensional topological manifold and any abelian group G we have $\mathrm{H}_n(M;G) \cong G$.

In the proof of Proposition 106.22 we will use the following lemma which is an immediate consequence of the Finitely Generated Abelian Group Classification Theorem 51.4, and of the Tor-Properties Lemma 89.15.

Lemma 106.23. Let A be a finitely generated abelian group. Then the following holds:

$$A \text{ is a free abelian group} \quad \Longleftrightarrow \quad \mathrm{Tor}(A, \mathbb{Z}_p) = 0 \text{ for all primes } p.$$

Now we can provide the proof of Proposition 106.22. First we consider Statement (1). For that statement we consider the two cases that the topological manifold is closed and that it has non-empty boundary separately.

Proof of Proposition 106.22 (1) I. Let M be a closed connected n-dimensional topological manifold. By the Topological Manifolds-Invariants Proposition 104.14 (4) we know that $\mathrm{H}_{n-1}(M;\mathbb{Z})$ is a finitely generated abelian group.

We first consider the case that M is orientable. We want to show that $\mathrm{H}_{n-1}(M;\mathbb{Z})$ is a free abelian group. By Lemma 106.23 it suffices to show that $\mathrm{Tor}(\mathrm{H}_{n-1}(M;\mathbb{Z}), \mathbb{F}_p) = 0$ for all primes p.

So let p be a prime. We have

$$\mathbb{F}_p \overset{\substack{\text{by Proposition 106.21 (1), since by Lemma 105.14 we know that } M \text{ is also } \mathbb{F}_p\text{-orientable} \\ \downarrow}}{\cong} \mathrm{H}_n(M;\mathbb{F}_p) \underset{\substack{\uparrow \\ \text{UCT-Splitting Theorem 89.22}}}{\cong} \mathrm{H}_n(M;\mathbb{Z}) \otimes \mathbb{F}_p \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(M;\mathbb{Z}), \mathbb{F}_p) \underset{\substack{\uparrow \\ \text{by the Orientation-Top Homology Theorem 106.1 (3) we have } \mathrm{H}_n(M;\mathbb{Z}) \cong \mathbb{Z}}}{\cong} \mathbb{F}_p \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(M;\mathbb{Z}), \mathbb{F}_p).$$

It follows from Lemma 51.9 that $\mathrm{Tor}(\mathrm{H}_{n-1}(M;\mathbb{Z}), \mathbb{F}_p) = 0$.

Now we turn to the case that M is non-orientable. By the Orientation Covering-Proposition 105.15 there exists a connected 2-fold covering $q\colon \widetilde{M} \to M$ such that $\widetilde{M}$ is orientable. We consider the maps

$$\begin{tikzcd} \mathrm{H}_{n-1}(M;\mathbb{Z}) \arrow[r, "q^*"] \arrow[rr, bend right, "\cdot 2"'] & \mathrm{H}_{n-1}(\widetilde{M};\mathbb{Z}) \arrow[r, "q_*"] & \mathrm{H}_{n-1}(M;\mathbb{Z}). \end{tikzcd}$$

Here q^* denotes the transfer map that we introduced on page 1952 and q_* denotes of course the map induced by the projection q. By the Transfer Map Proposition 91.1 the composition of q^* and q_* is given by multiplication by 2. Put differently, the above diagram commutes.

Since $\widetilde{M}$ is orientable we obtain from the case that we had just proved that $\mathrm{H}_{n-1}(\widetilde{M};\mathbb{Z})$ is torsion-free. Now we see that

$$\text{torsion subgroup of } \mathrm{H}_{n-1}(M;\mathbb{Z}) \underset{\substack{\uparrow \\ \text{since } \mathrm{H}_{n-1}(\widetilde{M};\mathbb{Z}) \text{ is torsion-free}}}{\subset} \ker(q^*) \subset \ker(q_* \circ q^*) \underset{\substack{\uparrow \\ \text{by the above commutative diagram}}}{\subset} \{a \in \mathrm{H}_{n-1}(M;\mathbb{Z}) \mid 2a = 0\}.$$

From these inclusions and the classification of finitely generated abelian groups we obtain easily that $\mathrm{H}_{n-1}(M;\mathbb{Z}) \cong (\mathbb{Z}_2)^r \oplus \mathbb{Z}^k$ for some $r \in \mathbb{N}_0$ and some $k \in \mathbb{N}_0$.

It remains to show that $r = 1$. In light of the Tor-Properties Lemma 89.15 it suffices to show that we have $\mathrm{Tor}(\mathrm{H}_{n-1}(M;\mathbb{Z}), \mathbb{F}_2) \cong \mathbb{F}_2$. The argument is very similar to the

argument in the orientable case. More precisely, we have

$$\begin{array}{ccccccc} & & & \text{UCT-Splitting Theorem 89.22} & & & \\ & & & \downarrow & & & \\ \mathbb{F}_2 & \cong & \mathrm{H}_n(M;\mathbb{F}_2) & \cong & \mathrm{H}_n(M;\mathbb{Z}) \otimes \mathbb{F}_2 \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(M;\mathbb{Z}),\mathbb{F}_2) & \cong & 0 \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(M;\mathbb{Z}),\mathbb{F}_2). \\ & \uparrow & & & & \uparrow & \\ & \mathbb{F}_2\text{-Orientability Proposition 105.13} & & & & & \\ & \text{and the Top-}\mathbb{F}_2\text{-Homology Theorem 106.2} & & & \text{we have } \mathrm{H}_n(M;\mathbb{Z}) = 0 & & \end{array}$$

■

Now we deal with topological manifolds with non-empty boundary.

Proof of Proposition 106.22 (1) II. Let M be a connected, compact n-dimensional topological manifold with non-empty boundary. We need to show that $\mathrm{H}_{n-1}(M;\mathbb{Z})$ is a free abelian group. There are two approaches to the proof of this statement.

(1) We can imitate the proof of the closed case, or
(2) we reduce the proof of the proposition to the case of closed topological manifolds that we had just dealt with.

We will pursue the second approach. First recall that by the Double-Orientability Lemma 105.18 the double $\mathrm{D}M$ is orientable if and only if M is orientable.

We first consider the case that M is orientable. By Proposition 106.22 we know that $\mathrm{H}_{n-1}(\mathrm{D}M;\mathbb{Z})$ is a finitely generated free abelian group. It follows from Lemma 75.13 that the group $\mathrm{H}_{n-1}(M;\mathbb{Z})$ is isomorphic to a subgroup of $\mathrm{H}_{n-1}(\mathrm{D}M;\mathbb{Z})$, hence by Lemma 51.9 (3) it is also a finitely generated free abelian group.

Now we consider the case that M is non-orientable. We denote by M' a copy of M and as usual we denote by $\mathrm{D}M = M \cup_{\partial M = \partial M'} M'$ the double. From the Mayer–Vietoris Theorem 78.9 for Manifolds we obtain the following long exact sequence:

$$\cdots \to \mathrm{H}_n(\mathrm{D}M;\mathbb{Z}) \to \mathrm{H}_{n-1}(\partial M;\mathbb{Z}) \to \mathrm{H}_{n-1}(M;\mathbb{Z}) \oplus \mathrm{H}_{n-1}(M';\mathbb{Z}) \to \mathrm{H}_{n-1}(\mathrm{D}M;\mathbb{Z}) \to \dots$$

We make the following observations:

(1) By the Topological Manifold Boundary Proposition 75.2 (3) and (4) we know that ∂M is a closed $(n-1)$-dimensional topological manifold. Thus we obtain from the Orientation-Top Homology Theorem 106.1 that $\mathrm{H}_{n-1}(\partial M;\mathbb{Z})$ is isomorphic to a free abelian group $\mathbb{Z}^r$ for some $r \in \mathbb{N}_0$. (We do not have any control over the orientability of ∂M and the number of components of ∂M but this does not matter.)
(2) Since $\mathrm{D}M$ is non-orientable we know from the Orientation-Top Homology Theorem 106.1 that $\mathrm{H}_n(\mathrm{D}M;\mathbb{Z}) = 0$ and we know from Proposition 106.22 that $\mathrm{H}_{n-1}(\mathrm{D}M;\mathbb{Z}) \cong \mathbb{Z}^s \oplus \mathbb{Z}_2$ for some $s \in \mathbb{N}_0$.
(3) Since $M = M'$ we have $\mathrm{H}_{n-1}(M;\mathbb{Z}) = \mathrm{H}_{n-1}(M';\mathbb{Z})$. By the classification of finitely generated abelian groups we can pick an isomorphism $\mathrm{H}_{n-1}(M;\mathbb{Z}) \cong F \oplus T$ where F is a free abelian group and T is a torsion group.

We can thus rewrite the above sequence as follows:

$$0 \;\to\; \mathbb{Z}^r \;\xrightarrow{\varphi}\; F \oplus F \oplus T \oplus T \;\to\; \mathbb{Z}^s \oplus \mathbb{Z}_2 \;\to\; \dots$$

The map $\varphi\colon \mathbb{Z}^r \to F \oplus F \oplus T \oplus T$ is a monomorphism, but since $\mathbb{Z}^r$ is torsion-free and since T is finite we see that $\mathrm{im}(\varphi) \cap (T \oplus T)$ is trivial. But this implies that $T \oplus T$ is isomorphic to a subgroup of $\mathbb{Z}^s \oplus \mathbb{Z}_2$. But this is only possible if $T = 0$. This concludes the proof that $\mathrm{H}_{n-1}(M;\mathbb{Z})$ is a free abelian group. ■

Proof of Proposition 106.22 (2). First we note that by Propositions 75.2 (3) and (4) and 104.14 (4) together with Lemma 96.8 (1) we know that $\mathrm{H}_{n-1}(M,\partial M;\mathbb{Z})$ is finitely generated. Now there are two approaches to proving Statement (2):

(1) We can use basically exactly the argument as in the proof of Theorem 98.8 (2) to reduce, via a doubling construction, the study of relative homology groups to the study of absolute homology groups. The only difference is that we need to replace the Smooth Manifold-Top Homology Theorem 98.6 by Statement (1) of the present proposition.

(2) Alternatively we can adopt the argument of Statement (1). First we assume that M is orientable. By Lemma 106.23 it suffices to show that $\mathrm{Tor}(\mathrm{H}_{n-1}(M,\partial M;\mathbb{Z}),\mathbb{F}_p)=0$ for all primes p. So let p be a prime. We have

$$\begin{array}{ccccc} & & \overset{\text{UCT-Splitting Theorem 89.22}}{\downarrow} & & \\ \mathbb{F}_p & \cong \mathrm{H}_n(M,\partial M;\mathbb{F}_p) & \cong & \mathrm{H}_n(M,\partial M;\mathbb{Z})\otimes\mathbb{F}_p \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(M,\partial M;\mathbb{Z}),\mathbb{F}_p) & \\ & & & \cong \mathbb{F}_p \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(M,\partial M;\mathbb{Z}),\mathbb{F}_p). & \\ & \uparrow & & \uparrow & \\ & \text{by Proposition 106.21 (1), since by Lemma 105.14 } M \text{ is also } \mathbb{F}_p\text{-orientable} & & \text{by the Orientation-Top Homology Theorem 106.1 (3) we have } \mathrm{H}_n(M,\partial M;\mathbb{Z})\cong\mathbb{Z} & \end{array}$$

It follows from Lemma 51.9 that $\mathrm{Tor}(\mathrm{H}_{n-1}(M,\partial M;\mathbb{Z}),\mathbb{F}_p)=0$. The case that M is non-orientable is very similar, we leave it to the reader to make the necessary modifications. ■

106.6. **The fundamental class and induced orientations.** In Section 105.5 we saw that an orientation on a topological manifold M induces orientations on codimension-zero submanifolds of M, on the boundary ∂M and on covering spaces of M. If M is compact, then M admits a fundamental class and we will see in this section how it is related to the fundamental classes of the various corresponding oriented topological manifolds.

First of all, in the Topological Manifold-Induced Orientation Lemma 105.17 we saw that a codimension-zero submanifold W of an oriented topological manifold M inherits an orientation. For compact topological manifolds the following lemma, which is the analogue of the Codimension Zero-Fundamental Class Lemma 100.12, shows the relationship between the corresponding fundamental classes.

Lemma 106.24. (Codimension Zero-Fundamental Class Lemma) Let M be a compact oriented n-dimensional topological manifold. Furthermore let $W\subset M$ be a compact non-empty codimension-zero submanifold. We write $\mathring{W}=W\setminus\partial W$. The following two statements hold:

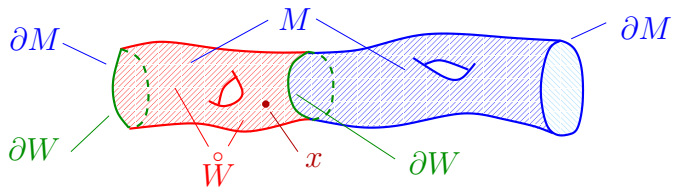
(1) The inclusion induced map $\mathrm{H}_n(W,\partial W;\mathbb{Z})\to\mathrm{H}_n(M,M\setminus\mathring{W};\mathbb{Z})$ is an isomorphism.

(2) The image of the fundamental classes $[M]\in\mathrm{H}_n(M,\partial M;\mathbb{Z})$ and $[W]\in\mathrm{H}_n(W,\partial W;\mathbb{Z})$ under the maps

$$\mathrm{H}_n(M,\partial M;\mathbb{Z}) \longrightarrow \mathrm{H}_n(M,M\setminus\mathring{W};\mathbb{Z}) \longleftarrow \mathrm{H}_n(W,\partial W;\mathbb{Z})$$

agree.

Remark. In Lemma 120.4 we will generalize the Codimension Zero-Fundamental Class Lemma 106.24 to the case that W is a codimension-zero submanifold *with corner*.



Proof.

(1) The first statement is an immediate consequence of the Excision Theorem 75.10.

(2) We denote by $\Phi\colon \mathrm{H}_n(M, \partial M; \mathbb{Z}) \to \mathrm{H}_n(W, \partial W; \mathbb{Z})$ the map given in Statement (2). We have to show that $\Phi([M])$ is the fundamental class for W. But this follows immediately from the Fundamental Class-Corollary 106.4 and from considering for each $x \in W \setminus \partial W$ the following commutative diagram

$$\begin{array}{ccccc}
 & & \overset{\Phi}{\frown} & & \\
\mathrm{H}_n(M, \partial M; \mathbb{Z}) & \longrightarrow & \mathrm{H}_n(M, M \setminus \mathring{W}; \mathbb{Z}) & \overset{\cong}{\longleftarrow} & \mathrm{H}_n(W, \partial W; \mathbb{Z}) \\
\downarrow & \swarrow & & & \downarrow \\
\mathrm{H}_n(M, M \setminus \{x\}; \mathbb{Z}) & & \underset{\text{isomorphism by the Excision Theorem 74.18}}{\overset{\cong}{\longleftarrow}} & & \mathrm{H}_n(W, W \setminus \{x\}; \mathbb{Z}).
\end{array}$$

We leave the few remaining details to the reader. ■

In the Topological Manifold-Induced Orientation Lemma 105.17 (3) we showed that a covering space $\widetilde{M}$ of an oriented topological manifold M inherits an orientation from M. The following proposition can now be viewed as a generalization of Exercise 100.10 and the Covering Map-Degree Proposition 101.7.

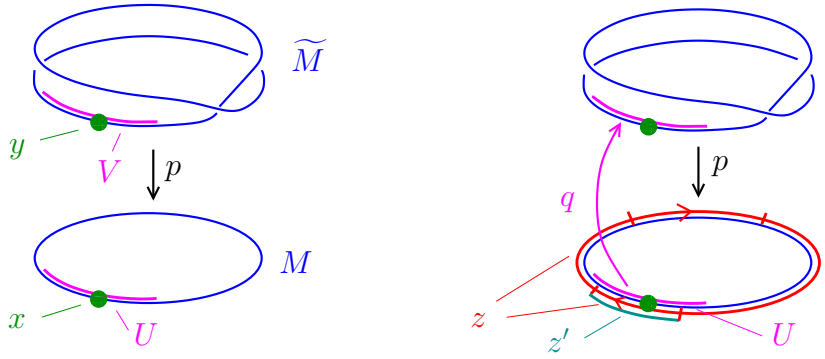
Proposition 106.25. (Fundamental Class-Covering Proposition) Let M be a compact oriented non-empty n-dimensional topological manifold and let $p\colon \widetilde{M} \to M$ be a finite covering. We denote by $[M]$ the fundamental class of M and we denote by $p^*\colon \mathrm{H}_n(M; \mathbb{Z}) \to \mathrm{H}_n(\widetilde{M}; \mathbb{Z})$ the transfer map that we introduced on page 1952. $p^*([M])$ equals the fundamental class $\big[\widetilde{M}\big]$ of the oriented topological manifold $\widetilde{M}$ and we have $p_*\big(\big[\widetilde{M}\big]\big) = \big[\widetilde{M} : M\big] \cdot [M]$.

Proof. Let M be a compact n-dimensional topological manifold. To simplify the notation we assume that M is closed. Furthermore let $p\colon \widetilde{M} \to M$ be a finite covering and let $\{\mu_x\}_{x \in M}$ be an orientation for M. We denote by $\{\nu_x\}_{x \in \widetilde{M}}$ the orientation on $\widetilde{M}$ that we defined in the Topological Manifold-Induced Orientation Lemma 105.17 (3).

We want to show that $p^*([M])$ equals the fundamental class $\big[\widetilde{M}\big]$. By the Fundamental Class-Corollary 106.4 it suffices to show that for all $y \in \widetilde{M}$ we have $p^*([M])_y = \nu_y \in \mathrm{H}_n(\widetilde{M}, \widetilde{M} \setminus \{y\}; \mathbb{Z})$. So let $y \in \widetilde{M}$. We pick an open neighborhood V around y such that $U := p(V)$ is uniformly covered. We write $x = p(y)$ and we denote by $q\colon U \to V$ the inverse of the homeomorphism $p\colon V \to U$.

We pick a representative $z = a_1 \cdot \sigma_1 + \cdots + a_m \cdot \sigma_m \in \mathrm{C}_n(M)$ for $[M]$. By applying the Comfortable Covering Proposition 74.23 to the *open* cover of M given by $\{U, M \setminus \{x\}\}$ we

can arrange that the image of each σ_i lies in U or in $M \setminus \{x\}$. We denote by z' the sum of all summands in z such that the image of σ_i lies in U.



A moment's thought shows that in $\mathrm{H}_n(M, M \setminus \{x\}; \mathbb{Z})$ we have $[z] = [z']$ and that in $\mathrm{H}_n(\widetilde{M}, \widetilde{M} \setminus \{y\})$ we have $[p^*(z)] = [q_*(z')]$. From these observations we deduce that the following diagram commutes:

$$\begin{array}{ccccccc}
\mathrm{H}_n(M;\mathbb{Z}) & & \xrightarrow{\qquad\qquad p^* \qquad\qquad} & & & & \mathrm{H}_n(\widetilde{M};\mathbb{Z}) \\
\downarrow{\scriptstyle [M]=[z]\mapsto[z]=[z']=[\mu_x]} & & & & & & \downarrow{\scriptstyle [p^*z]\mapsto[p^*z]=[q_*(z')]} \\
\mathrm{H}_n(M, M\setminus\{x\};\mathbb{Z}) & \xleftarrow{\cong} & \mathrm{H}_n(U, U\setminus\{x\};\mathbb{Z}) & \xrightarrow[q_*]{\cong} & \mathrm{H}_n(V, V\setminus\{y\};\mathbb{Z}) & \xrightarrow{\cong} & \mathrm{H}_n(\widetilde{M}, \widetilde{M}\setminus\{y\};\mathbb{Z}).
\end{array}$$

$$\mu_x \mapsto \nu_y$$

Note that the bottom horizontal maps to the left and right are isomorphisms by the Excision Theorem 74.18 and the middle horizontal map is an isomorphism since q is a homeomorphism. It follows from this diagram that we indeed have $p^*([M])_y = \nu_y$.

Finally we have

$$p_*\big([\widetilde{M}]\big) \underset{\text{by the above}}{=} p_*(p^*([M])) \underset{\text{Transfer Map Proposition 91.1}}{=} [\widetilde{M} : M] \cdot [M].$$

■

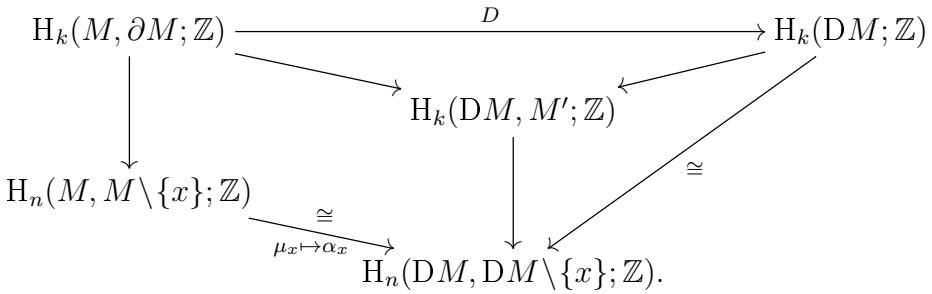
In the Double-Orientability Lemma 105.18 we saw that the double of an oriented topological manifold with non-empty boundary comes with a natural orientation. The following lemma following relates the various fundamental classes in this construction.

Lemma 106.26. Let M be a compact oriented n-dimensional topological manifold with non-empty boundary. If we equip $\mathrm{D}M$ with the orientation coming from the Double-Orientability Lemma 105.18, then

$$[\mathrm{D}M] \;=\; D([M]) \;\in\; \mathrm{H}_n(\mathrm{D}M;\mathbb{Z}),$$

where $D\colon \mathrm{H}_n(M, \partial M;\mathbb{Z}) \to \mathrm{H}_n(\mathrm{D}M;\mathbb{Z})$ denotes the doubling homomorphism that we introduced in the Doubling-Homology Lemma 75.16.

Proof. Let $\{\mu_x\}_{x \in M\setminus\partial M}$ be an orientation for M. We denote by $\{\alpha_x\}_{x\in \mathrm{D}M}$ the corresponding orientation for $\mathrm{D}M$ that we had constructed in the Double-Orientability Lemma 105.18. To simplify the notation we write $\mathrm{D}M = (M \sqcup M')/\!\sim$ where M' is a copy of M and we view M as a submanifold of the double. By the Topological Manifold Boundary Proposition 75.2 (5) there exists an $x \in M \setminus \partial M$. We consider the following diagram



Here all maps are the obvious maps for pairs of topological spaces. The lemma follows from the following observations:

(1) The top triangle commutes by the Doubling-Homology Lemma 75.16 (3). The parallelogram on the left and the triangle on the right commute by the functoriality of homology.
(2) The homomorphism to the bottom left is an isomorphism by the Excision Theorem 75.10 for topological Manifolds.
(3) The right diagonal map is an isomorphism by the Orientation-Top Homology Theorem 106.1.
(4) By definition $[M]$ gets sent to $\mu_x \in \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{Z})$ and by definition μ_x gets sent to α_x. It follows from the commutativity of the diagram that $D([M])$ gets sent to α_x. But since the right diagonal map is an isomorphism it follows, from the definition of $[\mathrm{D}M]$, that $[\mathrm{D}M] = D([M])$. ■

In the Topological Manifold-Boundary Orientation Proposition 105.19 we saw that the boundary of an oriented topological manifold inherits a natural orientation. The following proposition can now be seen as a generalization of the Fundamental Class-of-Boundary Proposition 100.9 from smooth manifolds to topological manifolds.

Proposition 106.27. (Fundamental Class-of-Boundary Proposition II) Let M be a compact n-dimensional topological manifold with non-empty boundary.

(1) We denote by $\partial_n \colon \mathrm{H}_n(M, \partial M; \mathbb{Z}) \to \mathrm{H}_{n-1}(\partial M; \mathbb{Z})$ the connecting homomorphism of the long exact sequence of the pair $(M, \partial M)$. If M is oriented, then[1297]
$$\partial_n([M]) \;=\; [\partial M] \;\in\; \mathrm{H}_{n-1}(\partial M; \mathbb{Z}).$$
(2) Similarly, for the connecting homomorphism $\partial_n \colon \mathrm{H}_n(M, \partial M; \mathbb{F}_2) \to \mathrm{H}_{n-1}(\partial M; \mathbb{F}_2)$ we have
$$\partial_n([M]_{\mathbb{F}_2}) \;=\; [\partial M]_{\mathbb{F}_2} \;\in\; \mathrm{H}_{n-1}(\partial M; \mathbb{F}_2).$$

Proof. As a health warning we point out that in this proof we make heavy use of the notation introduced in the Topological Manifold-Boundary Orientation Proposition 105.19. We only provide the proof of Statement (1); the proof of Statement (2) is almost identical.

Let $\{\mu_y\}_{y \in M \setminus \partial M}$ be the orientation of for M. Next let $x \in \partial M$. We pick a chart $\Phi \colon U \to [0, 2) \times (-2, 2)^{n-1}$ with $\Phi(x) = (0, 0)$. We write $V' = \Phi^{-1}((0, 1) \times (-1, 1)^{n-1})$.

[1297] Note that ∂M is a closed $(n-1)$-dimensional topological manifold by the Topological Manifold Boundary Proposition 75.2 (3) and (4).

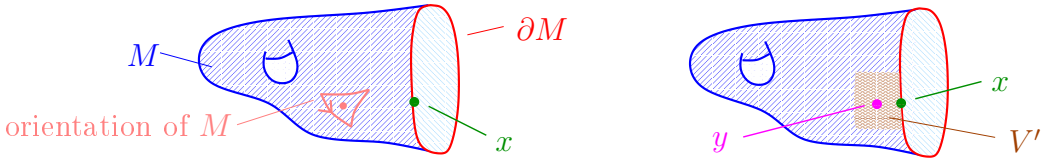
Finally we pick a point $y \in V'$. We consider the following diagram:

$$\begin{array}{ccccccc}
 & & & \overset{\mu_y \mapsto \nu_x}{\curvearrowright} & & & \\
\mathrm{H}_n(M, M\backslash\{y\}) & \xleftarrow[\cong]{} & \mathrm{H}_n\big(M, M\backslash V'\big) & \xrightarrow[\cong]{\partial_n} & \mathrm{H}_{n-1}(M\backslash V', (M\backslash V')\backslash\{x\}) & \xleftarrow[\cong]{} & \mathrm{H}_{n-1}(\partial M, \partial M\backslash\{x\}) \\
\mu_y \updownarrow [M] & \nearrow & & & & \nwarrow & \nu_x \updownarrow [\partial M] \\
\mathrm{H}_n(M, \partial M) & & & \xrightarrow{\qquad\qquad\partial_n\qquad\qquad} & & & \mathrm{H}_{n-1}(\partial M).
\end{array}$$

We make the following observations and clarifications:

(1) All the unmarked maps are induced by the obvious inclusions of pairs of topological space.
(2) In the proof of the Topological Manifold-Boundary Orientation Proposition 105.19 we showed that the three horizontal blue maps on the top, connecting $\mathrm{H}_n(M, M\backslash\{y\})$ to $\mathrm{H}_{n-1}(\partial M, \partial M \setminus \{x\})$, are isomorphisms.
(3) By definition of the orientation $\{\nu_x\}_{x\in\partial M}$ of ∂M the composition of the top horizontal maps sends μ_x to ν_y.
(4) The maps ∂_n are the connecting homomorphism of the long exact sequence of the triple $(M, M \setminus V', (M\backslash V')\backslash\{x\})$ respectively of the pair $(M, \partial M)$.
(5) The triangles are given by inclusion induced maps, hence they both commute.
(6) Finally note that it follows from the naturality of the connecting homomorphisms, see the Topological LES-Proposition 74.13 (3), that the quadrilateral commutes.

It follows from the above and the defining property of $[M]$ that for any $x \in \partial M$ the image of $\partial_n([M]) \in \mathrm{H}_{n-1}(M)$ under the map $\mathrm{H}_{n-1}(\partial M) \to \mathrm{H}_{n-1}(\partial M, \partial M \setminus \{x\})$ equals ν_x. But that means that $\partial_n([M]) = [\partial M]$. ■



Remark. Given a compact oriented n-dimensional topological manifold the conclusion of the Fundamental Class-of-Boundary Proposition 106.27 can also be summarized in the following commutative "diagram"

$$\begin{array}{ccc}
\mathbb{Z}\text{-oriented topological manifold } M & \xmapsto{\text{take boundary}} & \mathbb{Z}\text{-oriented topological manifold } \partial M \\
\Big\downarrow \text{take fundamental class} & & \Big\downarrow \text{take fundamental class} \\
[M] & \xmapsto[\mathrm{H}_n(M,\partial M;\mathbb{Z})\to\mathrm{H}_{n-1}(\partial M;\mathbb{Z})]{\text{apply connecting homomorphism}} & \partial[M] = [\partial M].
\end{array}$$

Example. We will illustrate the Fundamental Class-of-Boundary Proposition 106.27 with the Möbius band M. By the $\mathbb{F}_2$-Orientability Proposition 105.13 and the Top-$\mathbb{F}_2$-Homology Theorem 106.2 it has an $\mathbb{F}_2$-fundamental class $[M] \in \mathrm{H}_2(M, \partial M; \mathbb{F}_2)$. In the figure below we consider the Möbius band with two singular 1-simplices α_1, α_2 and two singular 2-simplices

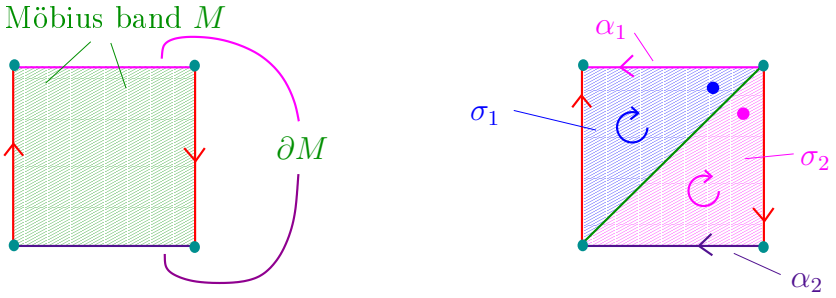
σ_1, σ_2. In this case we have

$$\partial\big(\underbrace{\sigma_1 + \sigma_2}_{=:\mu}\big) \;=\; \underbrace{\alpha_1 + \alpha_2}_{\in \mathrm{C}_1(\partial M)}.$$

In particular, $\mu \otimes 1$ is a cycle in $\mathrm{C}_2(M, \partial M; \mathbb{F}_2)$. It easily follows from Proposition 100.18 that $\mu \otimes 1$ represents the $\mathbb{F}_2$-fundamental class in $\mathrm{H}_2(M, \partial M; \mathbb{F}_2)$. Furthermore, if we denote by $\partial\colon \mathrm{H}_2(M, \partial M; \mathbb{F}_2) \to \mathrm{H}_1(\partial M; \mathbb{F}_2)$ the connecting homomorphism of the long exact sequence in $\mathbb{F}_2$-homology of the pair $(M, \partial M)$, then

$$\partial[M] \;=\; \partial[\mu \otimes 1] \;\underset{\text{Topological LES-Proposition 74.13}}{\uparrow}\; [\partial\mu \otimes 1] \;\underset{\text{by the above}}{\uparrow}\; \big[(\alpha_1 + \alpha_2) \otimes 1\big],$$

but the last class is clearly[1298] an $\mathbb{F}_2$-fundamental class for the circle ∂M.



The following corollary is the generalization of the Fundamental Class-of-Boundary Corollary 100.11 to the case of topological manifolds.

Corollary 106.28. Let M be a compact n-dimensional topological manifold. Then the following two statements hold:

(1) We have

$$\ker\big(\mathrm{H}_{n-1}(\partial M; \mathbb{F}_2) \to \mathrm{H}_{n-1}(M; \mathbb{F}_2)\big) \;=\; \mathbb{F}_2 \cdot [\partial M]_{\mathbb{F}_2} \;\in\; \mathrm{H}_{n-1}(\partial M; \mathbb{F}_2).$$

Furthermore, if M is oriented, then

$$\ker\big(\mathrm{H}_{n-1}(\partial M; \mathbb{Z}) \to \mathrm{H}_{n-1}(M; \mathbb{Z})\big) \;=\; \mathbb{Z} \cdot [\partial M] \;\in\; \mathrm{H}_{n-1}(\partial M; \mathbb{Z}).$$

(2) Let M be a compact n-dimensional topological manifold with non-trivial boundary. Then the boundary ∂M is not a retract of M.

Remark. In the Boundary Non-Retraction Proposition 31.3 we gave a very different proof of Corollary 106.28 (2) for smooth manifolds.

Proof. The proof is verbatim the same as the proof of the Fundamental Class-of-Boundary Corollary 100.11, the only difference in the proof is that we need to replace the Fundamental Class-of-Boundary Proposition 100.9 by the Fundamental Class-of-Boundary Proposition 106.27.

As a reminder, the first statement is proved using the long exact sequence in homology of the pair $(M, \partial M)$ together with the Fundamental Class-of-Boundary Proposition 106.27. The second statement follows from the first statement and the usual functoriality argument,

[1298] Why is this clear?

this time we employ the functoriality of homology with $\mathbb{F}_2$-coefficients and the fact that $[\partial M]_{\mathbb{F}_2}$ is non-zero. ■

Example. It follows from Corollary 106.28 that the boundary of the Möbius band M is not a retract of the Möbius band. A more down-to-earth proof is given as follows: we pick a base point $x_0 \in \partial M$ and we denote by α and β the loops shown in the figure below. Note that α represents a generator of $\pi_1(\partial M, x_0) \cong \mathbb{Z}$ and that β represents a generator of $\pi_1(M, x_0) \cong \mathbb{Z}$. We denote by $i\colon \partial M \to M$ the inclusion map. Note that $i_*([\alpha]) = 2 \cdot [\beta]$. If there was a retraction $r\colon M \to \partial M$, then we would obtain a commutative diagram

$$\begin{array}{ccc} & \pi_1(M,x_0) & \\ {\scriptstyle i_*}\nearrow & & \searrow{\scriptstyle r_*} \\ \pi_1(\partial M, x_0) & \xrightarrow[(r\circ i)_*]{} & \pi_1(\partial M, x_0) \end{array}$$

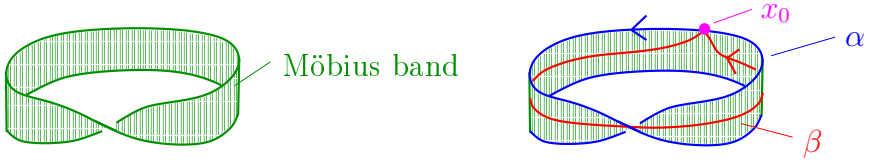
which by the above gives rise to the commutative diagram

$$\begin{array}{ccc} & \mathbb{Z} & \\ {\scriptstyle \cdot 2}\nearrow & & \searrow{\scriptstyle ??} \\ \mathbb{Z} & \xrightarrow[\mathrm{id}]{} & \mathbb{Z}, \end{array}$$

but such a commutative diagram cannot exist. Looking more carefully one sees that the two proofs that ∂M is not a retract of the Möbius band M are in some sense the same proof. More precisely, we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{Z} = \pi_1(\partial M, x_0) & \xrightarrow{i_*} & \pi_1(M, x_0) = \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z} = \mathrm{H}_1(\partial M; \mathbb{Z}) & \xrightarrow{i_*} & \mathrm{H}_1(M; \mathbb{Z}) = \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{F}_2 = \mathrm{H}_1(\partial M; \mathbb{F}_2) & \xrightarrow{i_*} & \mathrm{H}_1(M; \mathbb{F}_2) = \mathbb{F}_2. \end{array}$$

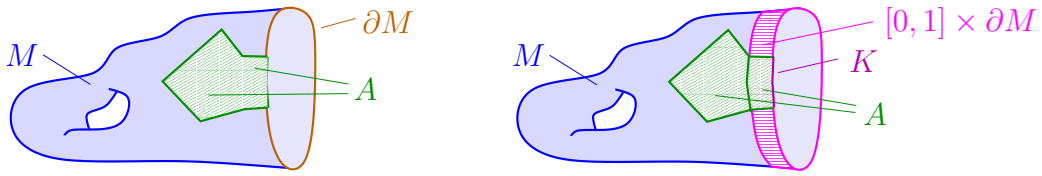
Here the top vertical maps are given by the Hurewicz homomorphism from page 1832 and the bottom vertical maps are given by the map $[\alpha] \mapsto [\alpha \otimes 1]$ from page 1927. Since both maps are natural, see Proposition 84.2 (5) and page 1927, we see that the diagram commutes. The above observation that the top horizontal map is given by multiplication by 2 translates into the fact that the bottom horizontal map is the zero map.



Remark. In the Topological Product Orientation Proposition 113.10 we will deal with the difficult question of the fundamental class of a product of topological manifolds.

106.7. **Lifts of homology orientations for topological manifolds with boundary (∗).** In this very technical section we generalize the Homology-Section Theorem 106.10 (i) to the case of topological manifolds with boundary. Unless the reader is blessed with a surplus of energy it is best to skip this section.

Definition. Let M be a topological manifold. We say a compact subset $A \subset M$ is a product close to the boundary if there exists a collar neighborhood $[0,1] \times \partial M$ for the boundary ∂M and a compact subset $K \subset \partial M$ such that $A \cap ([0,1] \times \partial M) = [0,1] \times K$.



Remark. If M is a topological manifold with *empty boundary*, then every compact subset is in fact a product close to the boundary.

We leave the task of providing a proof of the following elementary lemma to the reader.

Lemma 106.29. Let M be a topological manifold and let $[0,1]\times\partial M$ be a collar neighborhood (which always exists by the topological Smooth Collar Neighborhood Theorem 75.5). Given any compact subset K of M there exists a compact subset A that contains K and that is a product close to the boundary.

Throughout this section we will need the following notation.

Notation. Given a subset A of a topological manifold M we write $A_\partial := A \cap (M \setminus \partial M)$. In other words, we have $A_\partial = A \setminus \partial M$. Note that $(M \setminus A) \cup \partial M = M \setminus A_\partial$.

Now we can formulate the main result of this section.

Theorem 106.30. Let M be an n-dimensional topological manifold and let $A \subset M$ be a compact subset that is a product close to the boundary. Let R be a commutative ring. If $\{\alpha_x\}_{x\in M\setminus\partial M}$ is an R-section for M, then there exists a unique lift $\alpha_A^M = \alpha_A$ to A_∂, i.e. there exists a unique class $\alpha_A^M = \alpha_A \in \mathrm{H}_n(M, M \setminus A_\partial; R)$ such that for each $x \in A_\partial$ we have $\alpha_{A,x} = \alpha_x$.

Proof. Let M be an n-dimensional topological manifold and let R be a commutative ring. Furthermore let $A \subset M$ be a compact subset that is a product close to the boundary. We introduce the following notation and conventions:

(1) As usual we denote by $\mathrm{D}M$ the double of M. As before we think of the double as $\mathrm{D}M = (M \sqcup M')/\sim$ where M' is another copy of M. We recall that by the Doubling-Manifold Lemma 75.12 we know that $\mathrm{D}M$ is an n-dimensional topological manifold with empty boundary such that $M \subset \mathrm{D}M$ is a submanifold.

(2) By definition there exists a collar neighborhood $[0,1] \times \partial M$ for the boundary ∂M and a compact subset $K \subset \partial M$ such that $A \cap ([0,1] \times \partial M) = [0,1] \times K$.

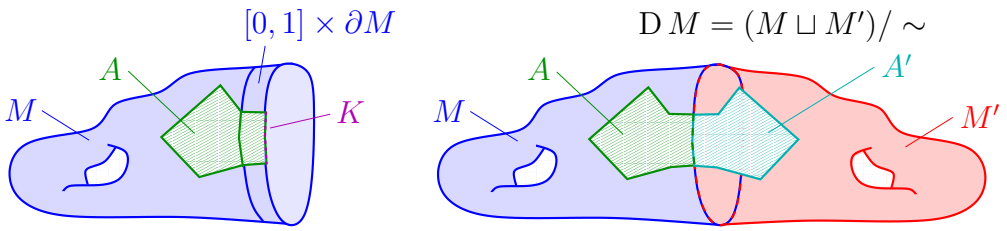
(3) We denote by $DA = A \cup A' \subset \mathrm{D}M$ the double of A. As DA is the union of two compact sets we see that DA itself is compact.

(4) Given $t \in (0,1]$ we write $A_t := A \setminus ([0,t) \times K) = A \cap (M \setminus ([0,t) \times \partial M))$. It follows easily from the Compact-Closed Lemma 1.21 that A_t is again compact.

(5) It follows from the Excision Theorem 75.10 for topological manifolds that for any $P \in M \setminus \partial M$ the inclusion induced map

$$\mathrm{H}_n(M, M \setminus \{P\}; R) \;\to\; \mathrm{H}_n(\mathrm{D}M, \mathrm{D}M \setminus \{P\}; R)$$

is an isomorphism. We will use this isomorphism to identify these two homology groups.



The idea now is the same as in the proof of Proposition 106.21. Namely we want to prove the desired statement for M by reducing it to a corresponding statement for the topological manifold $\mathrm{D}M$ that has no boundary.

Claim. The inclusion induced map

$$\beta\colon \mathrm{H}_n(M, M \setminus A_\partial; R) \ \to \ \mathrm{H}_n(\mathrm{D}M, (M \setminus A) \cup M'; R)$$

is an isomorphism.

Proof. Recall that A is a product close to the boundary. Using this fact and using a gentle generalization of the Excision Theorem 75.10 for topological manifolds one can easily show that the inclusion induced map is an isomorphism. We leave it to the reader to fill in the details. ⊞

We denote by Φ the composition of the following two maps

$$\mathrm{H}_n(\mathrm{D}M, \mathrm{D}M \setminus DA; R) \ \to \ \mathrm{H}_n(\mathrm{D}M, (M \setminus A) \cup M'; R) \xrightarrow{\beta^{-1}} \mathrm{H}_n(M, M \setminus A_\partial; R).$$

After all these preparations we now turn to the actual proof of the theorem. Thus let $\{\alpha_x\}_{x \in M \setminus \partial M}$ be an R-section for M. As in the proof of the Double-Orientability Lemma 105.18 we see that there exists an R-section $\{\alpha_x\}_{x \in \mathrm{D}M}$ for $\mathrm{D}M$ that restricts to the given R-section on M. By the Homology-Section Theorem 106.10 (i) there exists a lift $\mu_{DA} \in \mathrm{H}_n(\mathrm{D}M, \mathrm{D}M \setminus DA; R)$ of the R-section on $\mathrm{D}M$. We claim that $\Phi(\mu_{DA})$ is a lift for M. So let $P \in A_\partial$. We have the following commutative diagram

$$\begin{array}{ccccc}
 & & \overset{\Phi}{\frown} & & \\
\mathrm{H}_n(\mathrm{D}M, \mathrm{D}M \setminus DA; R) & \longrightarrow & \mathrm{H}_n(\mathrm{D}M, (M \setminus A) \cup M'; R) & \xleftarrow[\cong]{\beta} & \mathrm{H}_n(M, M \setminus A_\partial; R) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{H}_n(\mathrm{D}M, \mathrm{D}M \setminus \{P\}; R) & \xrightarrow{\mathrm{id}} & \mathrm{H}_n(\mathrm{D}M, \mathrm{D}M \setminus \{P\}; R) & \xleftarrow{\cong} & \mathrm{H}_n(M, M \setminus \{P\}; R).
\end{array}$$

Since all maps are induced by inclusions of pairs of topological spaces we see that the diagram commutes. By definition of μ_{DA} the left-hand vertical map sends μ_{DA} to α_P. But this implies that the right-hand vertical map sends $\Phi(\mu_{DA})$ to $\alpha_P \in \mathrm{H}_n(M, M \setminus \{P\}; R)$.

Now we turn to the proof of the uniqueness of a lift. The following claim is the key to proving uniqueness.

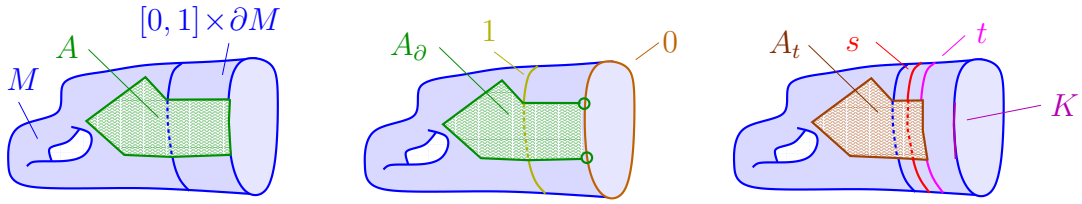
Claim. Let $t \in (0,1)$. Note that $A_t \subset A_\partial$. The two inclusion induced maps

$$\begin{aligned}
\gamma\colon \mathrm{H}_n(M, M \setminus A_\partial; R) \ &\to \ \mathrm{H}_n(M, M \setminus A_t; R) \\
\delta\colon \mathrm{H}_n(M, M \setminus A_t; R) \ &\to \ \mathrm{H}_n(\mathrm{D}M, \mathrm{D}M \setminus A_t; R)
\end{aligned}$$

are isomorphisms.

Proof. It follows immediately from the Excision Theorem 75.10 for topological manifolds that the map δ is an isomorphism. Regarding the map γ, note that it follows from the long exact sequence of the triple $(M, M\setminus A_t, M\setminus A_\partial)$, see the Topological LES-Proposition 74.13, that it suffices to show that $\mathrm{H}_i(M\setminus A_t, M\setminus A_\partial; R) = 0$ for all i. We pick $s \in (t,1)$. We see that indeed we have

$$\mathrm{H}_i(M\setminus A_t, M\setminus A_\partial; R) \overset{\cong}{\underset{\text{by the Excision Theorem 74.17, here we excise everything in the complement of } ([0,s)\times\partial M)\cap A}{\longleftarrow}} \mathrm{H}_i(([0,s)\times\partial M)\setminus(K\times[t,s)), ([0,s)\times\partial M)\setminus((0,s)\times K); R) \underset{\text{follows from Corollary 74.16 since } ([0,s)\times\partial M)\setminus((0,s)\times K) \text{ is a deformation retract of } ([0,s)\times\partial M)\setminus([t,s)\times K}{=} 0. \quad \boxplus$$



Now let $\mu, \nu \in \mathrm{H}_n(M, M\setminus A_\partial; R)$ be two lifts of the R-section $\{\alpha_x\}_{x\in M\setminus\partial M}$. With the same arguments as before one can convince oneself easily of the fact that $\delta^{-1}(\gamma(\mu))$ and $\delta^{-1}(\gamma(\nu))$ are lifts of the R-section $\{\alpha_x\}_{x\in \mathrm{D}M}$. By the Homology-Section Theorem 106.10 these two lifts are identical. But then it follows from the claim that also $\mu = \nu$. ■

We conclude this long and tiring chapter with the following lemma that is a generalization of Corollary 106.12.

Lemma 106.31. Let M be an oriented n-dimensional topological manifold and let R be a commutative ring. We suppose that M is equipped with an R-orientation. We use the notation from Theorem 106.30.

(1) Let $K \subset L \subset M$ be compact subsets that are products close to the boundary. We denote by $f_{KL}\colon (M, M\setminus L) \to (M, M\setminus K)$ the inclusion map of pairs. The map $f_{KL*}\colon \mathrm{H}_n(M, M\setminus L) \to \mathrm{H}_n(M, M\setminus K)$ sends α_L^M to α_K^M.

(2) Let U be an open subset of M and let $A \subset U$ be a compact subset that is a product close to the boundary. We use the Topological Manifold-Induced Orientation Lemma 105.17 to view U as an oriented topological manifold. If $i\colon U \to M$ denotes the inclusion map, then $i_*(\alpha_A^U) = \alpha_A^M$.

Proof. The lemma follows easily from the functoriality of homology groups and the uniqueness statement of Theorem 106.30. ■

106.8. **The degree of a map between oriented topological manifolds.** Recall that on page 2222 we used the fundamental class to define the degree of a map between smooth manifolds. Now that we also managed to introduce the fundamental class for topological manifolds we can generalize the earlier definition to the setting of topological manifolds. More precisely, we have the following definition.

Definition. Let $f\colon M \to N$ be a map between two compact oriented connected non-empty n-dimensional topological manifolds. We assume that $f(\partial M) \subset \partial N$. We denote by $\deg(f) \in \mathbb{Z}$ the unique integer with

by the Fundamental Class-Corollary 106.4 (3)

$$f_*([M]) \;=\; \deg(f)\cdot[N] \;\in\; \mathrm{H}_n(N,\partial N) \;\overset{\downarrow}{=}\; \mathbb{Z}\cdot[N].$$

since $f(\partial M)\subset \partial N$ we have an induced map $f_*\colon \mathrm{H}_n(M,\partial M)\to \mathrm{H}_n(N,\partial N)$

We refer to $\deg(f)$ as the degree of f.

Remark.

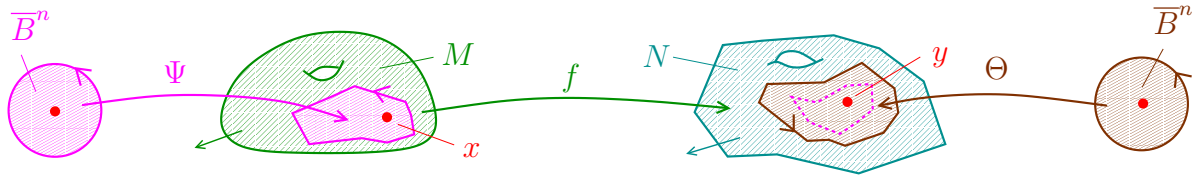
(1) For a map $f\colon S^n\to S^n$ the above degree is, by definition, exactly the same as the degree that we defined on page 1684.
(2) Given a map between *smooth* manifolds it follows from Lemma 106.6 that the above definition of the degree agrees with the definition given on pages 2222.
(3) Given a *smooth* map between closed *smooth manifolds* it follows from Lemma 106.6 and the discussion below that the above definition of the degree agrees with the definition given on page 899.

In Propositions 76.21 and 101.6 we already saw that in some settings we can calculate the "global" degree of a map by "local" degrees. To formulate the analogue in our present setting we need the following straightforward definition.

Definition. Let $f\colon M\to N$ be a map between two oriented n-dimensional topological manifolds and let $x\in M\setminus\partial M$ such that $f(x)\in N\setminus\partial N$. We set $y:=f(x)$. We suppose that there exists an open neighborhood U of x such that $f(z)\neq f(x)$ for every $z\in U\setminus\{x\}$. We pick an embedding $\Psi\colon \overline{B}^n\to U\subset M$ with $\Phi(0)=x$ and we pick an embedding $\Theta\colon \overline{B}^n\to M$ with $\Psi(0)=y$ and with $f(\Psi(\overline{B}^n))\subset\Theta(\overline{B}^n)$. We assume that Ψ and Θ are either both orientation-preserving or they are both orientation-reversing. We define the local degree $\deg(f,x)$ as follows:

$$\deg(f,x) \;:=\; \deg\big((\Theta^{-1}\circ f\circ\Psi)_*\colon \mathrm{H}_n(\overline{B}^n,\overline{B}^n\setminus\{0\})\to \mathrm{H}_n(\overline{B}^n,\overline{B}^n\setminus\{0\})\big).$$

We leave it to the reader to verify that $\deg(f,x)$ is well-defined.



Remark.

(1) It is fairly straightforward to show that for $M=N=S^n$ the above local degree agrees with the definition given on page 1694.
(2) Let $f\colon M\to N$ be a map between two oriented n-dimensional topological manifolds. Let $x\in M\setminus\partial M$ be a point such that f is a local homeomorphism around x. It follows easily from the definitions that

$$\deg(f,x) \;=\; \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } x\\ -1, & \text{if } f \text{ is orientation-reversing at } x.\end{cases}$$

This observation, together with the Smooth-Topological Orientation Proposition 105.11 implies that for a smooth map $f\colon M\to N$ between smooth manifolds and a point

$x \in M \setminus \partial M$ at which f is a local diffeomorphism the above definition of the local degree agrees with the definition given on page 898.

It follows from the above remark that the following proposition is a generalization of the Local-Global Degree Propositions 76.21 and 101.6.

Proposition 106.32. (Local-Global Degree Proposition) Let $f\colon M \to N$ be a map between two closed oriented connected n-dimensional topological manifolds M and N and let $x \in M$. Let $y \in N$ such that $f^{-1}(y)$ consists of finitely many points $x_1, \dots, x_m$. Then

$$\deg(f\colon M \to N) \;=\; \sum_{i=1}^{m} \deg(f, x_i).$$

Proof of Proposition 106.32. The proof of the proposition is similar to the proofs of Propositions 76.21 and 101.6. Thus, in a vein attempt to keep these notes concise, we will not provide a proof. Instead we hope that the reader will take the baton and carry out the proof. ■

106.9. **Applications.** The results from this chapter makes it possible to read several sections which are placed in the part on simplicial complexes. For example, with the results from this chapter one can now read the following sections:

(1) Section 98.3, in which we calculate the homology groups of lens spaces.
(2) Section 98.4, in which we construct the Poincaré Homology Sphere.
(3) Section 100.4, in which we calculate the homology groups of the connected sum of two closed oriented smooth manifolds.
(4) Section 100.5, in which we calculate the homology groups of a knot complement.

Furthermore, using Lemma 106.6 and a little bit of mental agility one can also read Chapter 101, where we introduce and study the degree of a map between closed oriented connected non-empty smooth manifolds. Finally, Chapter 102, in which we discuss the question, which homology classes can be represented by (sub-) manifolds, is now also accessible to the reader.

Exercises for Chapter 106.

Exercise 106.1. Let M be a connected n-dimensional topological manifold without boundary and let K be a compact subset. Show that there exists an open subset $U \subset M$ that contains K and such that the closure $\overline{U}$ is compact.

Exercise 106.2. Let M be a closed orientable connected non-empty topological manifold of dimension $n \geq 3$. Now we know that $\mathrm{H}_n(M;\mathbb{Z}) \cong \mathbb{Z}$ and that $\mathrm{H}_{n-1}(M;\mathbb{Z})$ is torsion-free. Furthermore we know that $\mathrm{H}_0(M;\mathbb{Z}) \cong \mathbb{Z}$. Can we also say something about the other homology groups? More precisely, are there any restrictions on $\mathrm{H}_{n-2}(M;\mathbb{Z})$ or can any finitely generated abelian group be isomorphic to $\mathrm{H}_{n-2}(M;\mathbb{Z})$ for some such smooth manifold?

Exercise 106.3.

(a) Let M be a compact, connected n-dimensional topological manifold with non-empty boundary. Show that $\mathrm{H}_{n-1}(M;\mathbb{Z}) \cong \mathbb{Z}^k$ for some $k \in \mathbb{N}_0$.

Hint. You can use that we already obtained some results for closed smooth manifolds, in particular we have a good understanding of $\mathrm{H}_{n-1}(\mathrm{D}M;\mathbb{Z})$. You could start with the case that M is orientable and afterwards you could consider the case that M is non-orientable.

(b) Let M be a connected n-dimensional topological manifold. Suppose that M is non-compact. Does it follow that $\mathrm{H}_{n-1}(M;\mathbb{Z})$ is a free abelian group? It suffices to give a short justification for your answer.

Exercise 106.4. Let $X = S^1 \cup \{*\}$ be the topological space that is given by the circle "with two 1's", where the topology is defined in the same way as we defined the topology on the "line with two 0's" on page 122. It is straightforward to see that X is locally homeomorphic to open subsets of $\mathbb{R}$, in fact one can easily find a finite atlas for X. Furthermore we know by the Second Countability Lemma 9.1 (4) that X is second-countable. But X is not a topological manifold since it is not Hausdorff. The argument of Exercise 78.5 shows that $\mathrm{H}_1(X;\mathbb{F}_2) \cong \mathbb{F}_2^2$. This shows that in the proof of the Top-$\mathbb{F}_2$-Homology Theorem 106.2 at some point we must have used that topological manifolds are Hausdorff. Where did we make use of this fact?

Exercise 106.5. Let X be a topological space, let $n \in \mathbb{N}_0$ and let G be an abelian group.

(1) A *locally finite singular n-chain* is a (possibly infinite) sum $\sum_{i\in I} a_i \cdot (\sigma_i \colon \Delta^n \to X)$ with $a_i \in \mathbb{Z}$, such that for every $x \in X$ there exists an open neighborhood U of x such that $\{i \in I \mid \sigma_i(\Delta^n) \cap U \neq \varnothing\}$ is finite. We define $\mathrm{C}_n^{\mathrm{lf}}(X)$ as the abelian group given by all locally finite singular n-chains. Note that the usual boundary map defines in an obvious way a boundary map $\mathrm{C}_n^{\mathrm{lf}}(X) \otimes G \to \mathrm{C}_{n-1}^{\mathrm{lf}}(X) \otimes G$. We define the *$n$-th locally finite homology group* $\mathrm{H}_n^{\mathrm{lf}}(X;G)$ to be the n-th homology of the corresponding chain complex.

(2) We consider the set $\mathcal{K}(X)$ of all compact subsets of X. The set $\mathcal{K}(X)$ together with the relation "$\subset$" defined by inclusion is a directed set since the union of two compact subsets is again compact. Given any two compact subsets $K \subset L$ we get an induced map $\mathrm{H}_n(X, X \setminus L; G) \to \mathrm{H}_n(X, X \setminus K; G)$. These homology groups form an *inverse* system over the directed set $(\mathcal{K}(X), \subset)$ and we can form the inverse limit[1299]

$$\overleftarrow{\mathrm{H}}_n(X;G) := \varprojlim_{K\in\mathcal{K}(X)} \mathrm{H}_n(X, X \setminus K; G).$$

In the following we will see that these two types of homology groups are often naturally isomorphic.

(a) Let *TopPropMap* be the category of topological spaces with proper maps, i.e. the objects are all topological spaces and the morphisms are continuous maps that are proper in the sense of the definition on page 180, i.e. maps such that preimage of compact sets are compact. Show that both homology groups define a covariant functor from *TopPropMap* to the category *AbGroup* of abelian groups.

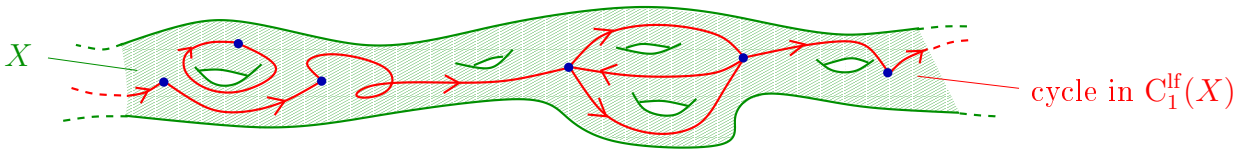
(b) Give an explicit natural map $\mathrm{H}_n^{\mathrm{lf}}(X;G) \to \overleftarrow{\mathrm{H}}_n(X;G)$ which is the identity for all compact spaces.

[1299] To the best of my knowledge there is no established name for this natural object.

(c) We assume that there exists a nested sequence $K_1 \subset K_2 \subset \dots$ of compact subsets of X such that $X = \bigcup_{i\in\mathbb{N}} \mathring{K}_n$. Show that the natural map $\mathrm{H}_n^{\mathrm{lf}}(X;G) \to \overleftarrow{\mathrm{H}}_n(X;G)$ from (b) is an isomorphism.

(d) Let $k \in \mathbb{N}$. Compute $\mathrm{H}_n^{\mathrm{lf}}(\mathbb{R}^k;\mathbb{Z})$.

Remark. Locally finite homology is discussed in detail in[**HR1996**, p. 30-34] and [**Spa1993**]. Note that statement (c) is [**Spa1993**, Theorem 10.1], also see [**HR1996**, p 34] for a related result. For completeness sake we note that it follows with some effort from [**Bre1997**, Corollary V.12.21 and pages 288-292] that for suitable topological spaces these two homology groups are naturally isomorphic to Borel-Moore homology.



Exercise 106.6. Let M be a connected n-dimensional topological manifold with $\partial M = \varnothing$, let $K \subset M$ be a compact subset and let $x_0 \in K$ be a point.

(a) Show that there exists a *connected* compact subset $L \subset M$ with $K \subset L$.

(b) Show that the map

$$\begin{array}{rcl} \mathrm{Im}(\mathrm{H}_n(M, M\setminus L;\mathbb{Z}) \to \mathrm{H}_n(M, M\setminus K;\mathbb{Z})) & \to & \mathrm{H}_n(M, M\setminus\{x_0\};\mathbb{Z}) \\ [\sigma] & \mapsto & [\sigma] \end{array}$$

is an isomorphism.

Hint. Use Corollary 106.11.

Exercise 106.7. Let M be a connected orientable non-empty n-dimensional topological manifold such that $\partial M = \varnothing$. As in the above Exercise 106.5 we now consider the group $\overleftarrow{\mathrm{H}}_n(M;\mathbb{Z}) := \varprojlim_{K\in\mathcal{K}(M)} \mathrm{H}_n(M, M\setminus K;\mathbb{Z})$. Show that in our situation this group is naturally isomorphic to $\mathbb{Z}$. We refer to the corresponding natural generator as the fundamental class $[M] \in \overleftarrow{\mathrm{H}}_n(M;\mathbb{Z})$.

Hint. Use Exercise 106.6.

Exercise 106.8. Let X be a topological space. Given $\varphi \in \mathrm{H}_k(X;\mathbb{R})$ we define

$$\|\varphi\|_{\mathbb{R}} := \inf\Big\{ \sum_{i=1}^n |r_i| \;\Big|\; \sigma = \underbrace{\sum_{i=1}^n \sigma_i \otimes r_i}_{\text{with } \sigma_i \neq \sigma_j \text{ for } i\neq j} \text{ is a cycle with } [\sigma] = \varphi \in \mathrm{H}_k(X;\mathbb{R}) \Big\}.$$

In the following let M be a closed oriented connected non-empty n-dimensional topological manifold M. Let $[M] \in \mathrm{H}_n(M)$ be the fundamental class and let $i\colon \mathrm{H}_1(M) \to \mathrm{H}_1(M;\mathbb{R})$ be the homomorphism from Lemma 89.7 (1). We define the *simplicial volume of* M as

$$\|M\|_{\mathbb{R}} := \|i_*([M])\| \in \mathbb{R}_{\geq 0}.$$

(a) Show that $\|S^1\|_{\mathbb{R}} = 0$.

(b) Show that $\|(S^1)^2\|_{\mathbb{R}} = 0$.

(c) Let $f\colon M \to N$ be a map between closed oriented connected non-empty n-dimensional topological manifolds. Show that

$$\|N\|_{\mathbb{R}} \leq \frac{1}{\deg(f)} \cdot \|M\|_{\mathbb{R}}$$

where $\deg(f)$ denotes the degree of f as defined on page 2379.

(d) Let M and N be two closed oriented connected non-empty n-dimensional topological manifolds. We suppose that M and N are in fact homotopy equivalent. Show that $\|M\|_{\mathbb{R}} = \|N\|_{\mathbb{R}}$.

(e) Let N be a closed oriented non-empty n-dimensional topological manifold. Furthermore let $p\colon M \to N$ be a d-fold covering. Show that $\|M\|_{\mathbb{R}} = d \cdot \|N\|_{\mathbb{R}}$.

(f) Let M be a compact oriented n-dimensional topological manifold with non-empty connected boundary. Show that $\|M\| \geq \frac{1}{n+1} \cdot \|\partial M\|$.

Remark. For the most part this exercise is the "topological analogue" of Exercise 101.9.

Part XI

Singular Cohomology

107. Chirality

In this section we will take a short break from overly abstract concepts and we study chirality of objects in $\mathbb{R}^3$, knots and topological manifolds. We will run into several questions which will motivate us to develop new tools in the study of topological spaces.

107.1. **Chiral objects.** In physics or chemistry an object X in $\mathbb{R}^3$ is called *amphichiral* (or often short *achiral*) if it agrees with its mirror image, otherwise it is called *chiral*. To make this definition more precise we have to explain what we mean by "mirror image" and what we mean by "agree".
First of all, given a hyperplane $H \subset \mathbb{R}^3$ through the origin we denote by $\rho_H \colon \mathbb{R}^3 \to \mathbb{R}^3$ the reflection in the hyperplane H.[1300] Sometimes in this chapter we will extend ρ_H to a map from $S^3 = \mathbb{R}^3 \cup \{\infty\}$ to itself by setting $\rho_H(\infty) = \infty$.
The following lemma says that any two reflections in a hyperplane through the origin differ by multiplication by a matrix in SO(3).

Lemma 107.1. If H and G are two hyperplanes in $\mathbb{R}^3$ through the origin, then there exists a matrix $A \in \mathrm{SO}(3)$ such that for all $v \in \mathbb{R}^3$ we have

$$\rho_G(v) \;=\; A \cdot \rho_H(v).$$

Sketch of proof. With respect to the standard basis of $\mathbb{R}^3$ the maps ρ_G and ρ_H are represented by orthogonal matrices P and Q with determinant -1. The matrix $A = PQ^{-1}$ then lies in SO(3) and for any $v \in \mathbb{R}^3$ we have $\rho_G(v) = Pv = A \cdot Qv = A \cdot \rho_H(v)$. ■

The above discussion leads us to the following definitions.

Definition. Let X be a subset of $\mathbb{R}^3$.

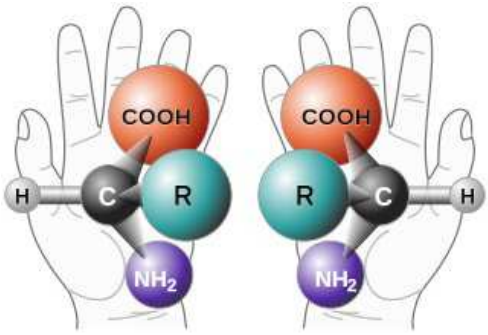
(1) We say another subset Y of $\mathbb{R}^3$ agrees with X if there exists a matrix $A \in \mathrm{SO}(3)$ and a point $P \in \mathbb{R}^3$ such that $X = A \cdot Y + P = \{A \cdot y + P \,|\, y \in Y\}$.
(2) For every hyperplane $H \subset \mathbb{R}^3$ we refer to $\rho_H(X)$ as the mirror image of X.[1301]
(3) We say X is amphichiral if its mirror image agrees with X, otherwise we say X is chiral.

Examples.

(1) Human hands are chiral, indeed, the left-hand does not agree with the right-hand. In fact the term chirality is derived from the Greek word for hand, $\chi\epsilon\iota\rho$.[1302]
(2) Many molecules are chiral, see e.g. the figure below for an example of a chiral molecule.[1303]

107.2. **Chiral knots.** Now we turn our focus to knots. We recall some of the relevant definitions from pages 1301 and 1315.

[1300] How is the reflection in a hyperplane defined?
[1301] To be precise, the definition of the mirror image depends on the choice of the hyperplane. By Lemma 107.1 reflections in two different hyperplanes differ by multiplication by a matrix in SO(3), i.e. the mirror images with respect to different hyperplanes agree in the above sense.
[1302] The same word is also the etymological root for the German word "Chirurg".
[1303] In fact, how can you show that the molecule is chiral?



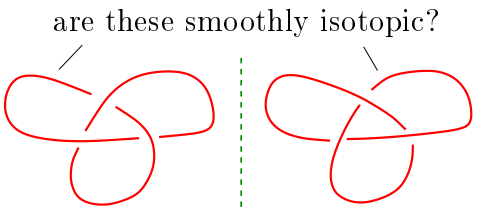
source: https://en.wikipedia.org/wiki/Chirality

Definition.

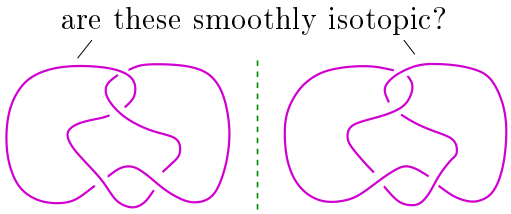
(1) A knot is a 1-dimensional submanifold of $S^3 = \mathbb{R}^3 \cup \{\infty\}$ that is diffeomorphic to S^1.

(2) Given a knot $K \subset S^3$ the reflection of K in any hyperplane of $\mathbb{R}^4$ is called mirror of K and denoted by K^{mir}. Note that in Exercise 59.1 we showed that, up to a smooth isotopy, the definition of K^{mir} does not depend on the choice of the hyperplane.

Example. In the figure below we show the trefoil and its mirror and we also show the figure-8 knot and its mirror. In both cases the question arises whether the original knot and its mirror are smoothly isotopic.



the trefoil K and its mirror K^{m}



the figure-8 knot J and its mirror J^{m}

Next we recall the following definition from page 1316.

Definition. We say that a knot K in $\mathbb{R}^3$ is amphichiral if it is smoothly isotopic to its mirror image, otherwise we call the knot K chiral.

Example. In the figure on page 1316 we showed that the figure-8 knot J is smoothly isotopic to its mirror image J^*, i.e. the figure-8 knot is amphichiral.

We recall the following question on which so far we have made zero progress.

Question 59.9. Is the trefoil chiral?

The question that naturally arises is, what does this discussion have to do with homology groups? How can we address this question using the tools of algebraic topology? In the following section we will turn Question 59.9 into a question about homology groups.

107.3. Chiral topological manifolds.

Definition. Let M be a compact oriented topological manifold. We say that M is amphichiral if M admits an orientation-reversing self-homeomorphism, otherwise we say that M is chiral.

In the next proposition, which is a variation on Lemma 100.7, we will give several criteria for a self-homeomorphism to be orientation-preserving.

Proposition 107.2. Let M be a compact oriented connected n-dimensional topological manifold and let $f\colon M \to M$ be a homeomorphism. Then the following three statements are equivalent:

(1) The map f is orientation-preserving in the sense of the definition on page 2335.

(2) There exists a point $x \in M \setminus \partial M$ and an open neighborhood U of x, e.g. $U = M$, such that[1304]
$$f_*(\underbrace{\text{orientation at } x}_{\in \mathrm{H}_n(U, U\setminus\{x\};\mathbb{Z})}) = \underbrace{\text{orientation at } f(x)}_{\in \mathrm{H}_n(f(U), f(U)\setminus\{f(x)\};\mathbb{Z})}.$$

(3) We have $f_*([M]) = [M]$.

Suppose M is in fact a smooth manifold and that f is a diffeomorphism. We equip M with the orientation in the sense of Section 25.2 coming from the Smooth-Topological Orientation Proposition 105.11. Then the above are equivalent to the following statement:

(4) The map f is orientation-preserving in the sense of Section 25.2.

Analogously the above statements also hold if in (1) and (4) we replace "orientation-preserving" by "orientation-reversing" and if in (2) and (3) we add a minus sign.

For the most part the proof of Proposition 107.2 boils down to the following lemma which is also useful in its own right.

Lemma 107.3. Let $f\colon M \to N$ be a map between compact oriented connected n-dimensional topological manifolds such that $f(\partial M) \subset \partial N$. Let $x \in M \setminus \partial M$ and let U be an open neighborhood of x with the property that f restricts to a homeomorphism $f\colon U \to f(U)$ and such that $f^{-1}(f(x)) = \{x\}$. Then
$$\begin{aligned}\deg(f\colon M \to N) &= \deg(f_*\colon \mathrm{H}_n(M, M\setminus\{x\};\mathbb{Z}) \to \mathrm{H}_n(N, N\setminus\{f(x)\};\mathbb{Z})\\ &= \deg(f_*\colon \mathrm{H}_n(U, U\setminus\{x\};\mathbb{Z}) \to \mathrm{H}_n(f(U), f(U)\setminus\{f(x)\};\mathbb{Z}).\end{aligned}$$

Proof. Let M and N be compact connected n-dimensional topological manifolds that are equipped with orientations $\{\mu_x\}_{x\in M\setminus\partial M}$ and $\{\nu_x\}_{x\in N\setminus\partial N}$. Furthermore let f, $x \in M \setminus \partial M$ and U as above. We consider the following commutative diagram
$$\begin{array}{ccc}
\mathrm{H}_n(M, \partial M;\mathbb{Z}) & \xrightarrow{\quad f_*\quad} & \mathrm{H}_n(N, \partial N;\mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathrm{H}_n(M, M\setminus\{x\};\mathbb{Z}) & \xrightarrow{\quad f_*\quad} & \mathrm{H}_n(N, N\setminus\{f(x)\};\mathbb{Z}) \\
\uparrow & & \uparrow \\
\mathrm{H}_n(U, U\setminus\{x\};\mathbb{Z}) & \xrightarrow{\quad f_*\quad} & \mathrm{H}_n(f(U), f(U)\setminus\{f(x)\};\mathbb{Z}).
\end{array}$$

[1304] Recall that by the construction of the Topological Manifold-Induced Orientation Lemma 105.17 the orientation on M induces an orientation on any open subset.

We make the following observations:

(1) the top vertical maps are isomorphisms by the Orientation-Top Homology Theorem 106.1,
(2) the bottom vertical maps are isomorphisms by the Excision Theorem 74.18,
(3) it follows from the Fundamental Class-Corollary 106.4 that the vertical maps send the fundamental classes $[M]$ and $[N]$ to the orientations μ_x respectively $\nu_{f(x)}$.

The desired equalities follow easily from these observations.[1305] ■

Now we can also provide a proof for Proposition 107.2.

Sketch of a proof. The equivalence of (1), (2) and (3) follows quite easily from Lemma 107.3. Now suppose M that is in fact a smooth manifold and that f is a diffeomorphism. The equivalence of (2) and (4) is a straightforward consequence of Lemma 105.12 and the Excision Theorem 74.18. Once again we leave it to the reader to fill in the details. ■

Examples. Let $n \in \mathbb{N}_0$.

(1) It follows from the Sphere Degree Lemma 76.11 (6) that the map

$$\begin{array}{rcl} \rho\colon S^n & \to & S^n \\ (x_0, \dots, x_{n-1}, x_n) & \mapsto & (x_0, \dots, x_{n-1}, -x_n), \end{array}$$

i.e. the map that is given by reflection in the $x_n = 0$ hyperplane, is orientation-reversing. This shows that the sphere S^n is amphichiral.

(2) Let M be a closed, oriented topological manifold. It follows from (1) and the naturality of the Künneth Theorem 90.8 for topological spaces that the map

$$\begin{array}{rcl} S^n \times M & \to & S^n \times M \\ (x, P) & \mapsto & (\rho(x), P), \end{array}$$

is orientation-reversing. This shows that the product of the sphere S^n with any closed, oriented topological manifold is amphichiral.

Now we want to relate the notion of amphichiral knots to the notion of amphichiral smooth manifolds. Before we can do so we recall the following definition.

Definition. Let $K \subset \mathbb{R}^3$ be a knot. We pick a tubular neighborhood $\overline{B}^2 \times K$ as provided by the Tubular Neighborhood Theorem **??**. We refer to $X_K := S^3 \setminus (B^2 \times K)$ as the exterior of K.[1306]



The following proposition together with Proposition 107.2 now turns Question 59.9 into a question about topological spaces.

[1305] Where on earth did we use the hypothesis that $f^{-1}(f(x)) = \{x\}$?

[1306] As we pointed out in Lemma 100.16, the exterior X_K is a compact orientable smooth manifold and the diffeomorphism type of X_K as an oriented smooth manifold is well-defined, i.e. it is independent of the choice of the tubular neighborhood.

Proposition 107.4. Let $K \subset \mathbb{R}^3$ be a knot. If K is amphichiral, then the knot exterior X_K is also amphichiral.

Remark. We had just seen in Proposition 107.4 that the exterior of an amphichiral knot is amphichiral. The converse to that statement is also true, i.e. if the exterior of a knot is amphichiral, then the knot is also amphichiral. But this direction is much harder to prove. To the best of my knowledge it follows from a very difficult theorem by Cameron Gordon and John Luecke [**GL1989**, p. 371] from 1989.

Proof. Let $K \subset \mathbb{R}^3$ be an amphichiral knot. By the Tubular Neighborhood Theorem **??** we can pick a tubular neighborhood $\nu K = \overline{B}^2 \times K$. As on page 2213 we consider the knot exterior $X_K := S^3 \setminus (B^2 \times K)$. By Proposition 107.2 it suffices to show that there exists a homeomorphism $f\colon X_K \to X_K$ with $f_*([X_K]) = -[X_K]$.

We denote by ρ the reflection in a hyperplane of $\mathbb{R}^3$. As usual we extend ρ to a self-homeomorphism of $S^3 = \mathbb{R}^3 \cup \{\infty\}$ by setting $\rho(\infty) = \infty$. We write $K^{\mathrm{mir}} = \rho(K)$ and $\nu K^{\mathrm{mir}} = \rho(\nu K)$. By hypothesis there exists a smooth isotopy from K^{mir} to K. By the Isotopy Extension Theorem 36.1 this smooth isotopy extends to a diffeotopy of S^3. This means that we have a smooth map

$$F\colon S^3 \times [0,1] \;\to\; S^3$$

such that $F_0 = \mathrm{id}$, such that each F_t is a diffeomorphism and such that $F_1(K^{\mathrm{mir}}) = K$. We write $\Phi := F_1\colon S^3 \to S^3$. Then $\Phi(\nu K^{\mathrm{mir}})$ is also a tubular neighborhood for K and by Proposition **??** there exists an orientation-preserving diffeomorphism $\Psi\colon S^3 \to S^3$ with $\Psi(\Phi(\nu K^{\mathrm{mir}})) = \nu K$. We write $f := \Psi \circ \Phi \circ \rho$.

Now we consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}_3(S^3) \arrow[r, "\rho_*"] \arrow[d, "\cong"] & \mathrm{H}_3(S^3) \arrow[r, "\Phi_*"] \arrow[d, "\cong"] & \mathrm{H}_3(S^3) \arrow[r, "\Psi_*"] \arrow[d, "\cong"] & \mathrm{H}_3(S^3) \arrow[d, "\cong"] \\
\mathrm{H}_3(S^3, \nu K) \arrow[r, "\rho_*"] & \mathrm{H}_3(S^3, \nu K^{\mathrm{mir}}) \arrow[r, "\Phi_*"] & \mathrm{H}_3(S^3, \Phi(\nu K^{\mathrm{mir}})) \arrow[r, "\Psi_*"] & \mathrm{H}_3(S^3, \nu K) \\
\mathrm{H}_3(X_K, \partial X_K) \arrow[u, "\cong"] \arrow[r, "\rho_*"] \arrow[rrr, bend right, "f_* = (\Psi\circ\Phi\circ\rho)_*"'] & \mathrm{H}_3(X_{K^{\mathrm{mir}}}, \partial X_{K^{\mathrm{mir}}}) \arrow[u, "\cong"] \arrow[r, "\Phi_*"] & \mathrm{H}_3(\Phi(X_{K^{\mathrm{mir}}}), \Phi(\partial X_{K^{\mathrm{mir}}})) \arrow[u, "\cong"] \arrow[r, "\Psi_*"] & \mathrm{H}_3(X_K, \partial X_K). \arrow[u, "\cong"]
\end{tikzcd}$$

Here the upper left vertical map is an isomorphism by the long exact sequence of the pair $(S^3, \nu K)$ and the fact that $\mathrm{H}_i(\nu K) = 0$ for $i = 2, 3$ since $\nu K = K \times B^2$ is homotopy equivalent to S^1. Exactly the same argument shows that in fact all upper vertical maps are isomorphisms. The lower vertical maps are induced by the inclusion of pairs of topological spaces. All of them are isomorphisms by the Excision Theorem for Smooth Manifolds 75.10.

By Proposition 107.2 we are done once we have shown that the composition of three bottom horizontal maps is given by multiplication by -1. By the commutative diagram it suffices to prove the analogous statement for the composition of the top three horizontal maps. It follows from the Sphere Degree Lemma 76.11 (6) that the top-left horizontal map is given by multiplication by -1. It is a consequence of the Homotopic Maps-and-Homology Proposition 73.6 that the top-middle horizontal map is the identity, since Φ is homotopic

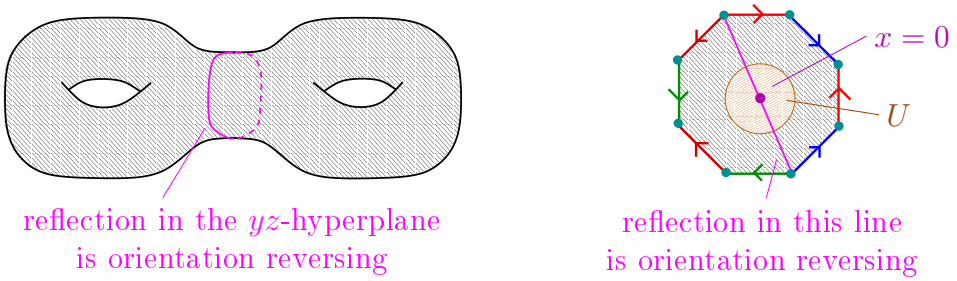
to the identity. Finally using Proposition 107.2 we see that the top-right horizontal map is the identity since Ψ is orientation-preserving. ■

Now we continue with our discussion of examples of amphichiral smooth manifolds.

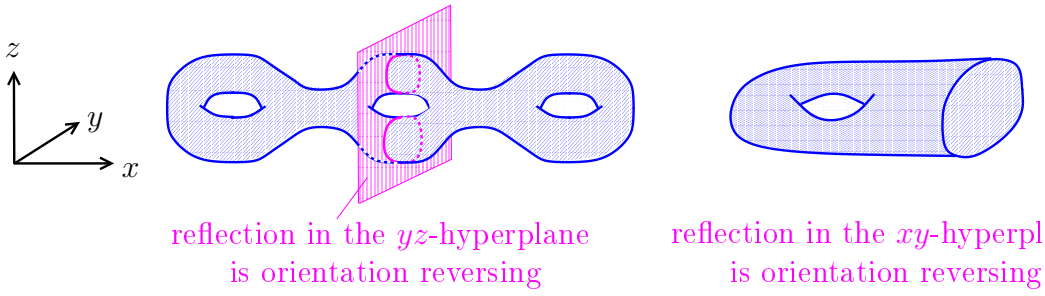
Examples.

(1) We had just seen on page 2389 that topological manifolds of the type $S^n \times M$ are amphichiral. It follows that the torus $S^1 \times S^1$ is amphichiral.

(2) The surface of genus 2 is amphichiral. Pictorially an orientation-reversing self-homeomorphism is given by "reflection in the yz-plane", see the figure below. In the "octagon-picture" an orientation-reversing self-homeomorphism is given by the reflection f in the line illustrated in the figure below. It follows easily from Proposition 107.2 applied to the point $x = 0$ and U an open disk around x together with Lemma 105.2 that the map f is indeed orientation-reversing.



(3) All other compact oriented 2-dimensional smooth manifolds are also amphichiral. We show some of the orientation-reversing maps in the figure below. In Exercise 107.2 we will discuss how these maps can be described rigorously.



(4) Example (2) can also be generalized as follows: if M is a compact oriented topological manifold, then its double $\mathrm{D}M = (M \sqcup M')/\sim$ is amphichiral. Indeed, the map $f\colon \mathrm{D}M \to \mathrm{D}M$ that swaps the two copies of M is orientation-reversing. This follows immediately from the definition of the orientation of $\mathrm{D}M$ that we gave in the Double-Orientability Lemma 105.18.

The long list of examples of amphichiral smooth manifolds raises the following question.

Question 107.5. Are there compact oriented topological manifolds that are chiral?

We end this chapter with a short discussion of lens spaces. We recall the following definition.

Definition. Let $p \in \mathbb{N}$ and let $q \in \mathbb{N}$ be coprime to p. On page 1075 we defined the lens space

$$L(p,q) := S^3/\mathbb{Z}_p = \big\{(z_1,z_2) \in \mathbb{C}^2 \,\big|\, |z_1|^2 + |z_2|^2 = 1\big\}/(z_1,z_2) \sim (e^{2\pi \mathrm{i}/p} \cdot z_1, e^{2\pi \mathrm{i} q/p} \cdot z_2).$$

Furthermore, on page 1075 we had equipped $L(p,q)$ with the orientation which turns the natural projection $S^3 \to L(p,q)$ into an orientation-preserving map.

Now let $q, r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$. In the Lens Spaces-Diffeomorphisms Lemma 48.6 we showed that if $q \equiv \pm r^{\pm 1} \bmod p$, then $L(p,q)$ and $L(p,r)$ are diffeomorphic.[1307] The following lemma is now a slight refinement of the Lens Spaces-Diffeomorphisms Lemma 48.6.

Lemma 107.6. Let $p \in \mathbb{N}$ and let $q, r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$.

(1) In the case that $q \equiv r^{\pm 1} \bmod p$ there exists an orientation-preserving diffeomorphism $L(p,q) \to L(p,r)$.
(2) On the other hand, if $q \equiv -r^{\pm 1} \bmod p$, then there exists an orientation-reversing diffeomorphism $L(p,q) \to L(p,r)$.

Proof. Let $p \in \mathbb{N}$ and let $q, r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$. We suppose that $q \equiv \pm r^{\pm 1} \bmod p$. As in the proof of the Lens Spaces-Diffeomorphisms Lemma 48.6 we define a diffeomorphism

as follows:

$$\begin{array}{lll} & \Phi\colon S^3 & \to \quad S^3 \\ \text{Case 1: } q \equiv r \bmod p. & (z_1,z_2) & \mapsto \quad (z_1,z_2) \\ \text{Case 2: } q \equiv -r \bmod p. & (z_1,z_2) & \mapsto \quad (z_1,\overline{z_2}) \\ \text{Case 3: } q \equiv r^{-1} \bmod p. & (z_1,z_2) & \mapsto \quad (z_2,z_1) \\ \text{Case 4: } q \equiv -r^{-1} \bmod p. & (z_1,z_2) & \mapsto \quad (z_2,\overline{z_1}) \end{array}$$

In all four cases it is straightforward to verify that the map descends to a diffeomorphism $L(p,q) \to L(p,r)$. It follows fairly easily from the Sphere Degree Lemma 76.11 and Proposition 107.2 that the diffeomorphism Φ is orientation-preserving in cases 1 and 3 and that it is orientation-reversing in cases 2 and 4.[1308] The lemma now follows from the fact that the projections $S^3 \to L(p,q)$ and $S^3 \to L(p,r)$ are orientation-preserving. ∎

Example. Note that $-2^{-1} \equiv -3 \equiv 2 \bmod 5$. Thus we obtain from Lemma 107.6 that there exists an orientation-reversing diffeomorphism $L(5,2) \to L(5,2)$, i.e. the lens space $L(5,2)$ is amphichiral.

The argument of the previous example does not work for the lens space $L(3,1)$. So the following question arises:

Question 107.7. Is the lens space $L(3,1)$ chiral?

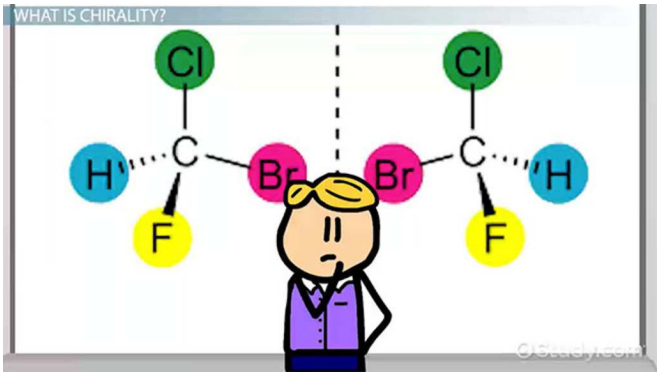
With our present knowledge we are not able to answer Questions 59.9, 107.5 and 107.7. The key to answering these three questions will be the cohomology groups that we introduce in Chapter 108.

[1307] We still don't know whether the converse holds.

[1308] For example in case 3 the map is given by $(x_1,x_2,x_3,x_4) \mapsto (x_3,x_4,x_1,x_2)$ which is the composition of two reflections in hyperplanes, namely in the $x_1 = x_3$ and the $x_2 = x_4$ hyperplane.

Exercises for Chapter 107.

Exercise 107.1. Why is the molecule shown in the figure below chiral? How can you show rigorously that the two molecules left and right cannot be turned into one another using a translation and a matrix in SO(3)?



Exercise 107.2.

(a) Give a rigorous proof for the statement that the surface of genus 3 is amphichiral. More precisely, give a precise description of the surface of genus 3 (e.g. as a dodecagon with appropriate identifications) and indicate carefully an orientation-reversing self-homeomorphism.

(b) Do the same exercise for the genus 2 surface with two open disks removed.

Exercise 107.3. Which connected 1-dimensional topological manifolds are chiral?
Remark. In Exercise 43.16 we studied the 2-dimensional case.

108. The cohomology groups of a topological space

In this chapter we will introduce the cohomology groups $\mathrm{H}^k(X;G)$ of a topological space together with a choice of an abelian group G. These study of the cohomology will keep us busy for a very long time. For many mathematicians it takes a long time till they feel fully comfortable with cohomology group since elements of cohomology groups are much less visual than elements of homology groups. Perhaps, before we embark on our new adventure it is good quote John von Neumann who said

"In mathematics you don't understand things. You just get used to them."

108.1. The Hom-functor.

We start out with the following obvious notation.

Notation. Given a group A and an abelian group G we write

$$\mathrm{Hom}(A,G) \;=\; \{\text{all group homomorphisms } A \to G\}.$$

The set of homomorphisms $\mathrm{Hom}(A,G)$ has the structure of an abelian group with the sum of two homomorphisms $\varphi,\psi\colon A\to G$ given by the group homomorphism that is defined by $(\varphi+\psi)(a) := \varphi(a)+\psi(a)$.

Example. We leave it to the reader to provide a proof for the statement that for any $n\in\mathbb{N}$ and any abelian group G we have a natural isomorphism $\mathrm{Hom}(\mathbb{Z}_n,G)\cong\ker(G\xrightarrow{\cdot n}G)$. Note that this implies in particular that for any $m\in\mathbb{N}$ we have $\mathrm{Hom}(\mathbb{Z}_n,\mathbb{Z}_m)\cong\mathbb{Z}_{\gcd(n,m)}$.

On several occasions we will implicitly and explicitly use the following lemma which is an immediate consequence of Lemma 51.1.

Lemma 108.1. Let G be an abelian group. If S is a non-empty set, then the homomorphism[1309] [1310] [1311]

$$\begin{array}{rcl} \mathrm{Hom}(\mathbb{Z}^{(S)},G) & \to & \{\text{all maps from the set } S \text{ to } G\} \\ (\varphi\colon \mathbb{Z}^{(S)}\to G) & \mapsto & \begin{pmatrix} S & \to & G \\ s & \mapsto & \varphi(s) \end{pmatrix} \end{array}$$

is an isomorphism. In particular the map

$$\begin{array}{rcl} \mathrm{Hom}(\mathbb{Z},G) & \to & G \\ \varphi & \mapsto & \varphi(1) \end{array}$$

is an isomorphism.

[1309] Recall that on page 1147, given a non-empty set S we defined

$$\mathbb{Z}^S \;:=\; \text{all maps from } S \text{ to } \mathbb{Z}.$$

and we defined the free abelian group generated by S to be the subgroup

$$\mathbb{Z}^{(S)} \;:=\; \text{all maps from } S \text{ to } \mathbb{Z} \text{ which are non-zero for only finitely many } s\in S.$$

Following the notation introduced on page 1147 we view each $s\in S$ as an element in $\mathbb{Z}^{(S)}$.

[1310] Why do we write $\mathbb{Z}^{(S)}$ in the lemma? Should it not be $\mathbb{Z}^S$?

[1311] Note that the group structure on G turns both sides of the lemma into a group, so the given map clearly respects this group structure, i.e. the given map is a homomorphism.

Definition. Let G be an abelian group and let $f\colon A \to B$ be a group homomorphism. Then f induces a homomorphism

$$\begin{array}{rcl} f^*\colon \operatorname{Hom}(B,G) & \to & \operatorname{Hom}(A,G) \\ (\varphi\colon B \to G) & \mapsto & \begin{pmatrix} A & \to & G \\ a & \mapsto & \varphi(f(a)) \end{pmatrix}. \end{array}$$

Example. Let G be an abelian group.

(1) If A is a subgroup of a group B and if $i\colon A \to B$ is the inclusion map, then for each homomorphism $\varphi\colon B \to G$ the homomorphism $i^*\varphi\colon A \to G$ is just the restriction of φ to $A \subset B$.
(2) If S is a subset of a set T, then we have a natural inclusion $i\colon \mathbb{Z}^{(S)} \to \mathbb{Z}^{(T)}$ and it follows from Lemma 108.1 that the map $i^*\colon \operatorname{Hom}(\mathbb{Z}^{(T)}, G) \to \operatorname{Hom}(\mathbb{Z}^{(S)}, G)$ is an epimorphism. Indeed, we just extend the map $S \to G$ to a map $T \to G$ by assigning the trivial element in G to all elements in $T \setminus S$.

It follows immediately from the definitions that

$$(\mathrm{id}_A)_* = \mathrm{id}_{\operatorname{Hom}(A,G)}, \quad \text{for all groups } A$$

and

$$(g \circ f)^* = f^* \circ g^* \quad \text{for all homomorphisms } f\colon A \to B \text{ and } g\colon B \to C.$$

Put differently, given an abelian group G the maps

$$\begin{array}{rcl} A & \mapsto & \operatorname{Hom}(A,G) \\ (f\colon A \to B) & \mapsto & (f^*\colon \operatorname{Hom}(B,G) \to \operatorname{Hom}(A,G)) \end{array}$$

define a *contravariant* functor from the category of groups to the category of abelian groups.

Remark. If R is a commutative ring, for example if $R = \mathbb{Z}_n, R = \mathbb{Z}, R = \mathbb{Q}$ or $R = \mathbb{R}$, then for any group C the set of homomorphisms $\operatorname{Hom}(C,R)$ forms an R-module in an obvious way. More precisely, given a homomorphism $\varphi\colon C \to R$ and $r \in R$ we define $r\varphi$ to be the homomorphism given by $c \mapsto r \cdot \varphi(c)$. Now we see easily that the above maps $A \mapsto \operatorname{Hom}(A,R)$ and $f \mapsto f^*$ define a contravariant functor from the category of groups to the category of R-modules.

The following lemma might be familiar from linear algebra. We refer to [**Rot2009**, Theorem 2.32(i)] for more details.

Lemma 108.2. Let $\{A_i\}_{i\in I}$ be a family of groups and let G be an abelian group. Then the map

$$\begin{array}{rcl} \operatorname{Hom}\left(\bigoplus_{i\in I} A_i, G\right) & \to & \prod_{i\in I} \operatorname{Hom}(A_i,G) \\ f & \mapsto & \prod_{i\in I} f|_{A_i} \end{array}$$

is an isomorphism.

The following rather technical lemma will be used later on.

Lemma 108.3. Let $\{\{A_i\}_{i\in\mathbb{N}}, \{\varphi_{ij}\colon A_i \to A_j\}_{i\le j}\}$ be a direct system of groups and let G be an abelian group. Then

$$\{\{\operatorname{Hom}(A_i,G)\}_{i\in\mathbb{N}}, \{\varphi_{ij}^*\colon \operatorname{Hom}(A_j,G) \to \operatorname{Hom}(A_i,G)\}_{i\le j}\}$$

is an inverse system of abelian groups (or modules if G is a commutative ring), and we have a natural isomorphism

$$\operatorname{Hom}\left(\varinjlim A_i, G\right) \xrightarrow{\cong} \varprojlim \operatorname{Hom}(A_i, G)$$

of abelian groups (or modules if G is a commutative ring).

Proof. As so often for arguments regarding limits, the proof follows mostly from unraveling the definitions. We refer to [**Rot2009**, Proposition 5.26] for more details. ■

We continue with the following lemma which we will use on many occasions.

Lemma 108.4. Let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of abelian groups and let G be an abelian group. If C is free abelian, then the dual sequence

$$0 \to \operatorname{Hom}(C, G) \xrightarrow{p^*} \operatorname{Hom}(B, G) \xrightarrow{i^*} \operatorname{Hom}(A, G) \to 0$$

is also exact.

Proof. It follows from the hypothesis that C is a free abelian and from Lemma 78.1 that the initial short exact sequence splits. But this means that, according to Splitting Lemma 78.2, that there exists an isomorphism $\Phi\colon B \to A \oplus C$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\quad i \quad} & B & \xrightarrow{\quad p \quad} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \cong}{\scriptstyle \Phi} & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & A & \xrightarrow{a \mapsto (a,0)} & A \oplus C & \xrightarrow{(a,c) \mapsto c} & C & \longrightarrow & 0.
\end{array}$$

By applying the contravariant $\operatorname{Hom}(-, G)$ functor we obtain the upper half of the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \operatorname{Hom}(A,G) & \xleftarrow{\quad i^* \quad} & \operatorname{Hom}(B,G) & \xleftarrow{\quad p^* \quad} & \operatorname{Hom}(C,G) & \longleftarrow & 0 \\
 & & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle \cong}{\scriptstyle \Phi^*} & & \uparrow{\scriptstyle =} & & \\
0 & \longleftarrow & \operatorname{Hom}(A,G) & \xleftarrow{(a \mapsto (a,0))^*} & \operatorname{Hom}(A \oplus C, G) & \xleftarrow{((a,c) \mapsto c)^*} & \operatorname{Hom}(C,G) & \longleftarrow & 0 \\
 & & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle (f \oplus) \mapsto (a \oplus c \mapsto f(a) + g(c))} & & \uparrow{\scriptstyle =} & & \\
0 & \longleftarrow & \operatorname{Hom}(A,G) & \xleftarrow{f \oplus g \mapsto f} & \operatorname{Hom}(A,G) \oplus \operatorname{Hom}(C,G) & \xleftarrow{g \mapsto 0 \oplus g} & \operatorname{Hom}(C,G) & \longleftarrow & 0.
\end{array}$$

One can easily verify that the lower half of the diagram commutes. Furthermore it is evident that the bottom sequence is exact. Finally it is straightforward to see that the vertical maps are isomorphisms we obtain that the upper sequence is also exact. But that is exactly what we wanted to prove. ■

108.2. Cochain complexes.

Now we introduce the notion of a cochain complex and of its cohomology groups. A cochain complex is exactly the same as a chain complex except that the boundary map now increases the degree by one instead of lowering it by one.

Definition.

(1) A cochain complex is a sequence

$$\leftarrow D^n \xleftarrow{\delta_{n-1}} D^{n-1} \xleftarrow{\delta_{n-2}} \dots \xleftarrow{\delta_1} D^1 \xleftarrow{\delta_0} D^0 \leftarrow 0$$

of maps between abelian groups such that for all $i \in \mathbb{N}_0$ we have $\delta_{i+1} \circ \delta_i = 0$. We refer to the maps δ_* as coboundary maps.

(2) We define the n-th cohomology group of a cochain complex (D^*, δ_*) to be

$$\mathrm{H}^n(D) \ := \ \frac{\ker(\delta_n \colon D^n \to D^{n+1})}{\mathrm{im}(\delta_{n-1} \colon D^{n-1} \to D^n)}.$$

Example. In case the reader has seen differential forms already we can give an example. Given a smooth manifold M and given $n \in \mathbb{N}_0$ we write

$$\Omega^n(M) \ := \ \{\text{all smooth } n\text{-forms on } M\}.$$

On page **??** we introduced the differentials $d_n \colon \Omega^n(M) \to \Omega^{n+1}(M)$. By the Differentials-on-Smooth Manifolds Proposition **??** we now obtain a cochain complex. The corresponding cohomology groups are called the de Rham cohomology groups of M. We studied this example in great detail in Chapter **??**.

Now we will see that most of the definitions and algebraic results for chain complexes can be generalized almost verbatim to cochain complexes.

Definition.

(1) A cochain map $f \colon C^* \to D^*$ between two cochain complexes consists of a family $\{f_n \colon \mathrm{C}^n \to D^n\}_{n \in \mathbb{N}_0}$ of maps that commute with the coboundary maps, i.e. such that for all $n \in \mathbb{N}_0$ we have

$$\underset{\delta_n \colon D^n \to D^{n+1}}{\delta_n} \circ f_n \ = \ f_{n+1} \circ \underset{\delta_n \colon \mathrm{C}^n \to C^{n+1}}{\delta_n} \colon \mathrm{C}^n \ \to \ D^{n+1}.$$

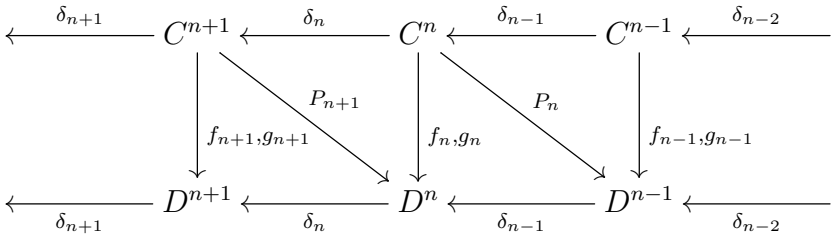
(2) Let $f, g \colon C^* \to D^*$ be two cochain maps between two cochain complexes C^* and D^*. A cochain homotopy $P = \{P_n\}_{n \in \mathbb{N}_0}$ between the cochain maps f and g consists of a family $\{P_n \colon \mathrm{C}^n \to D^{n-1}\}_{n \in \mathbb{N}_0}$ of maps such that for each $n \in \mathbb{N}_0$ we have

$$\delta_{n-1} \circ P_n + P_{n+1} \circ \delta_n \ = \ f_n - g_n.$$

If there exists a cochain homotopy between cochain maps f and g, then we say that f and g are cochain homotopic.

(3) A cochain map $f \colon C^* \to D^*$ between two cochain complexes C^* and D^* is called a cochain homotopy equivalence if there exists a cochain map $g \colon D^* \to C^*$ such that $f \circ g$ is cochain homotopic to id_{C^*} and $g \circ f$ is cochain homotopy to id_{D^*}.

Remark. For orientation we summarize in the following diagram all objects arising in the definition of a cochain homotopy:



The statements of the next lemma surely do not come as much of a surprise.

Lemma 108.5. (Cochain Map Lemma)

(1) If $f\colon C^* \to D^*$ is a cochain map between cochain complexes C^* and D^*, then the map
$$\begin{array}{rcl} f_*\colon \mathrm{H}^n(C) & \to & \mathrm{H}^n(D) \\ [c] & \mapsto & [f_n(c)] \end{array}$$
is well-defined.

(2) Two cochain maps that are cochain homotopic induce the same map on cohomology groups.

(3) Every cochain homotopy equivalence induces an isomorphism of cohomology groups.

(4) Cochain complexes together with the cochain maps form a category, called the category of cochain complexes.

Proof. The first statement is proved the same way as Lemma 72.7 and the second statement is proved the same way as Lemma 73.2. The proof of the third statement is basically the same as the proof of Corollary 73.4. Finally the last statement follows immediately from the definitions. ■

Remark. Let C^* and D^* be cochain complexes. On a few rare occasions we will need to deal with the following slight generalization of a cochain map. A *generalized cochain maps* is a family $\{f_n\colon \mathrm{C}^n \to D^n\}_{n\in\mathbb{N}_0}$ of maps that commute with the coboundary maps up to sign, i.e. such that for every $n \in \mathbb{N}_0$ there exists an $\epsilon \in \{-1, 1\}$ such that $\delta_n \circ f_n = f_{n+1} \circ \delta_n$. One can easily verify that the conclusion of the Cochain Map Lemma 108.5 (1) also holds for generalized cochain maps.

One of the most important algebraic constructions in homology theory was that we could associate to a short exact sequence of chain complexes a long exact sequence of homology groups. We have an almost identical construction for short exact sequences of cochain complexes and their cohomology groups. We will discuss this construction in detail:

Definition. Let
$$0 \to C^* \xrightarrow{f} B^* \xrightarrow{g} A^* \to 0$$
be a short exact sequence of cochain complexes. In particular for each $n \in \mathbb{N}_0$ we have a commutative diagram
$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^{n+1} & \xrightarrow{f_{n+1}} & B^{n+1} & \xrightarrow{g_{n+1}} & A^{n+1} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \delta_n} & & \uparrow{\scriptstyle \delta_n} & & \uparrow{\scriptstyle \delta_n} & & \\ 0 & \longrightarrow & C^n & \xrightarrow{f_n} & B^n & \xrightarrow{g_n} & A^n & \longrightarrow & 0 \end{array}$$

where the horizontal sequences are exact. Now we will define a connecting homomorphism

$$\delta\colon \mathrm{H}^n(A) \to \mathrm{H}^{n+1}(C).$$

The definition is almost verbatim the same as the definition on page 1620 of the connecting homomorphism in homology. More precisely, we perform the following steps:

(1) Let $z \in \mathrm{H}^n(A)$.
(2) We pick $\alpha \in A^n$ with $z = [\alpha]$.
(3) By the surjectivity of g_n we have $\alpha = g_n(\beta)$ for some $\beta \in B^n$.
(4) From the commutativity of the diagram and the exactness of the horizontal sequences we see that $\delta_n(\beta) = f_{n+1}(\gamma)$ for some uniquely determined $\gamma \in C^{n+1}$.
(5) We put $\delta_n(z) := [\gamma] \in \mathrm{H}^{n+1}(C)$.

The following proposition is now the obvious analogue of the Connecting Homomorphism Lemma 74.8 and the LES Proposition 74.9. We leave it to the reader to make the necessary changes to the proof.

Proposition 108.6. (Connecting Homomorphism Proposition)

(1) The connecting homomorphisms are well-defined and they are indeed homomorphisms.[1312]
(2) The connecting homomorphisms are natural in the sense that commuting diagrams of short exact sequences of cochain complexes give rise to commuting diagrams of cohomology groups involving the connecting homomorphisms.
(3) If

$$0 \to C^* \xrightarrow{f} B^* \xrightarrow{g} A^* \to 0$$

is a short exact sequence of cochain complexes, then

$$\cdots \to \mathrm{H}^n(C) \xrightarrow{f_*} \mathrm{H}^n(B) \xrightarrow{g_*} \mathrm{H}^n(A) \xrightarrow{\delta} \mathrm{H}^{n+1}(C) \xrightarrow{f_*} \mathrm{H}^{n+1}(B) \xrightarrow{g_*} \dots$$

forms an exact sequence.

Finally the following lemma says that cohomology commutes with direct sums and direct products. The proof of the lemma is identical to the proof of Lemma 72.13 that we did not give.

Lemma 108.7. If $\{C_a\}_{a\in A}$ is a family of cochain complexes, then the family of inclusion maps $C_a \to \bigoplus_{a\in A} C_a$, $a \in A$, and $C_a \to \prod_{a\in A} C_a$, $a \in A$, induce isomorphisms

$$\bigoplus_{a\in A} \mathrm{H}^n(C_a) \xrightarrow{\cong} \mathrm{H}^n\Big(\bigoplus_{a\in A} C_a\Big) \quad \text{and} \quad \prod_{a\in A} \mathrm{H}^n(C_a) \xrightarrow{\cong} \mathrm{H}^n\Big(\prod_{a\in A} C_a\Big).$$

108.3. **Dual cochain complexes.** Now we will present the arguably most important source of cochain complex.

Definition. Let

$$(C_*, \partial_*)\colon \quad \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

[1312] Furthermore, if the cochain complexes are cochain complexes over a commutative ring R, then the connecting homomorphism is in fact an R-module homomorphism.

be a chain complex and let G be an abelian group. For each $n \in \mathbb{N}_0$ we write

$$\delta_n \;:=\; \partial_{n+1}^*\colon \operatorname{Hom}(C_n, G) \;\to\; \operatorname{Hom}(C_{n+1}, G).$$

We obtain the following cochain complex[1313]

$$\leftarrow \operatorname{Hom}(C_{n+1}, G) \xleftarrow{\delta_n} \operatorname{Hom}(C_n, G) \xleftarrow{\delta_{n-1}} \ldots \xleftarrow{\delta_1} \operatorname{Hom}(C_1, G) \xleftarrow{\delta_0} \operatorname{Hom}(C_0, G) \leftarrow 0.$$

We refer to it as cochain complex dual to (C_*, ∂_*) and we denote it by $\operatorname{Hom}(C_*, G)$. We define the n-th cohomology group of the chain complex (C_*, ∂_*) with G-coefficients as the cohomology of the cochain complex $(\operatorname{Hom}(C_*, G), \delta_*)$, i.e. we define

$$\mathrm{H}^n(C; G) \;:=\; \frac{\ker(\delta_n\colon \operatorname{Hom}(C_n, G) \to \operatorname{Hom}(C_{n+1}, G))}{\operatorname{im}(\delta_{n-1}\colon \operatorname{Hom}(C_{n-1}, G) \to \operatorname{Hom}(C_n, G))}.$$

Remark.

(1) We had just defined $\delta_n := \partial_{n+1}^*$, i.e. the coboundary map in the cochain complex is the dual of the boundary map in the chain complex. This is the same definition that is used in most books on algebraic topology, see e.g. [**Mun1984**, p. 251] or [**Hat2002**, p. 189]. In the book by Bredon [**Bre1993**, p. 321] the coboundary is defined as $\delta_n := (-1)^{n+1} \cdot \partial_{n+1}^*$.

The different conventions do not affect the definition of the cohomology groups, but they will affect the sign conventions in some formulas later on. In particular if one cites results from other books one needs to be careful about the sign conventions used.

(2) If R is a commutative ring and if (C_*, ∂_*) is a chain complex, then it follows immediately from the remark on page 2395 that the corresponding cohomology groups $\mathrm{H}^n(C; R)$ come naturally with the structure of R-modules.

Many of the statements about chain complexes and their homology groups hold in a slightly modified version also for cohomology groups. For example we have the following purely algebraic lemma.

Lemma 108.8. (Dual Cochain Map Lemma) Let G be an abelian group.

(1) Let $f = \{f_n\colon C_n \to D_n\}_{n\in\mathbb{N}_0}$ be a chain map between chain complexes C_* and D_*, then for each $n \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} f^*\colon \operatorname{Hom}(D_n, G) & \to & \operatorname{Hom}(C_n, G) \\ (\varphi\colon D_n \to G) & \mapsto & \big(\underbrace{C_n \xrightarrow{f_n} D_n \xrightarrow{\varphi} G}_{=\varphi\circ f_n = f_n^*\varphi}\big) \end{array}$$

is a cochain map, in particular it induces a well-defined map

$$f^*\colon \mathrm{H}^n(D; G) \;\to\; \mathrm{H}^n(C; G).$$

(2) For each $n \in \mathbb{N}_0$ the maps

$$C_* \;\mapsto\; \mathrm{H}^n(C; G) \quad \text{and} \quad f \;\mapsto\; f^*$$

[1313] It follows the functoriality property of $\operatorname{Hom}(-, G)$ that $\delta_n \circ \delta_{n-1} = \partial_{n+1}^* \circ \partial_n^* = (\partial_n \circ \partial_{n+1})^* = 0$, i.e. this is indeed a cochain complex.

define a *contravariant* functor from the category $\mathcal{ChCplx}$ of chain complexes to the category $\mathcal{AbGroup}$ of abelian groups. (If G is a commutative ring, then this is in fact a functor to the category of G-modules.)

(3) Let $f, g\colon C_* \to D_*$ be two chain maps. If they are chain homotopic, then the induced cochain maps $f^*, g^*\colon \mathrm{Hom}(D_*, G) \to \mathrm{Hom}(C_*, G)$ are cochain homotopic and therefore the induced maps $f^*, g^*\colon \mathrm{H}^n(D; G) \to \mathrm{H}^n(C, G)$ are identical.

(4) If $f_*\colon C_* \to D_*$ is a chain homotopy equivalence, then for every $n \in \mathbb{N}_0$ the induced maps $f^*\colon \mathrm{H}^n(D; G) \to \mathrm{H}^n(C; G)$ are isomorphisms.

Proof.

(1) Let $f = \{f_n\colon C_n \to D_n\}_{n\in\mathbb{N}_0}$ be a chain map. This means that for each $n \in \mathbb{N}_0$ we have

$$f_{n-1} \circ \partial_n \;=\; \partial_n \circ f_n$$

which implies by the fact that $A \mapsto \mathrm{Hom}(A, G)$ is a *contravariant* functor that

$$\underbrace{\partial_n^*}_{=\delta_{n-1}} \circ f_{n-1}^* \;=\; f_n^* \circ \underbrace{\partial_n^*}_{=\delta_{n-1}}.$$

We have thus shown that f^* is a cochain map. The second statement follows immediately from the Cochain Map Lemma 108.5.

(2) This statement follows easily from the definitions.

(3) Let $\{P_n\colon C_n \to D_{n+1}\}_{n\in\mathbb{N}_0}$ be a chain homotopy $P = \{P_n\}_{n\in\mathbb{N}_0}$ between f and g. Recall that this means that for each $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; f_n - g_n.$$

We set $Q_n := P_{n-1}^*\colon \mathrm{Hom}(D_n, G) \to \mathrm{Hom}(C_{n-1}, G)$. It follows again from the fact that $A \mapsto \mathrm{Hom}(A, G)$ is a *contravariant* functor that

$$\underbrace{P_n^*}_{=Q_{n+1}} \circ \underbrace{\partial_{n+1}^*}_{=\delta_n} + \underbrace{\partial_n^*}_{=\delta_{n-1}} \circ \underbrace{P_{n-1}^*}_{=Q_n} \;=\; f_n^* - g_n^*.$$

This shows that the maps $Q_n, n \in \mathbb{N}_0$ form a cochain homotopy. It follows from the Cochain Map Lemma 108.5 that the induced maps on cohomology groups agree.

(4) The last statement follows immediately from (2) and (3). ∎

In Lemma 89.7 (2) we basically saw that for any chain complex C_* and any homomorphism $f\colon A \to B$ of abelian groups the maps

$$\begin{array}{rcl} f_*\colon C_* \otimes A & \to & C_* \otimes B \\ \left[\sum_{i=1}^k \sigma_i \otimes a_i\right] & \mapsto & \left[\sum_{i=1}^k \sigma_i \otimes f(a_i)\right] \end{array}$$

form a chain map, in particular for each $n \in \mathbb{N}_0$ we obtain an induced map

$$f_*\colon \mathrm{H}_n(C_* \otimes A) \;\to\; \mathrm{H}_n(C_* \otimes B).$$

Similarly, it is straightforward to check that the maps

$$\begin{array}{rcl} f_*\colon \mathrm{Hom}(C_n, A) & \to & \mathrm{Hom}(C_n, B) \\ \varphi & \mapsto & f \circ \varphi \end{array}$$

define a cochain map, in particular we get for each $n \in \mathbb{N}_0$ an induced map

$$f_*\colon \mathrm{H}^n(C_*; A) \;\to\; \mathrm{H}^n(C_*; B)$$

The following lemma basically says that these maps have all the properties one could possibly hope for. The lemma can be viewed as a cohomological analogue of Lemma 89.7.

Lemma 108.9.

(1) Given a chain complex C_* and $n \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} A & \mapsto & \mathrm{H}^n(C_*; A) \\ (f\colon A \to B) & \mapsto & (f_*\colon \mathrm{H}^n(C_*; A) \to \mathrm{H}^n(C_*; B)) \end{array}$$

form a covariant functor from the category of abelian groups to the category of abelian groups. In other words, "cohomology is covariant in the coefficients".

(2) Given a group homomorphism $f\colon A \to B$ the induced maps f_* define a natural transformation from the contravariant functor $C_* \mapsto \mathrm{H}^n(C_*; A)$ to the contravariant functor $C_* \mapsto \mathrm{H}^n(C_*; B)$.

Proof. The two statements follow basically immediately from the definitions. ■

108.4. **The singular cohomology groups of topological spaces.** Now we apply the somewhat dry algebra from the last section to our favorite chain complex, namely the singular chain complex of a pair of topological spaces.

Definition. Let (X, A) be a pair of topological spaces and let G be an abelian group.

(1) For each $n \in \mathbb{N}_0$ we write

$$\mathrm{C}^n(X, A; G) := \mathrm{Hom}(\mathrm{C}_n(X, A), G) \text{ and } \delta_n := \partial_{n+1}^*\colon \mathrm{C}^n(X, A; G) \to \mathrm{C}^{n+1}(X, A; G).$$

We use the following notation:

(a) We refer to the elements of $\mathrm{C}^n(X, A; G)$ as singular n-cochains,

(b) we refer to the elements in the kernel of $\delta_n\colon \mathrm{C}^n(X, A; G) \to \mathrm{C}^{n+1}(X, A; G)$ as singular n-cocycles and

(c) we refer to the elements in the image of $\delta_{n-1}\colon \mathrm{C}^{n-1}(X, A; G) \to \mathrm{C}^n(X, A; G)$ as singular n-coboundaries.

When $n \in \mathbb{N}_0$ is understood then sometimes we just write δ instead of δ_n.

(2) We refer to the cochain complex

$$\ldots \leftarrow \mathrm{C}^{n+1}(X, A; G) \xleftarrow{\delta_n} \mathrm{C}^n(X, A; G) \xleftarrow{\delta_{n-1}} \mathrm{C}^{n-1}(X, A; G) \leftarrow \ldots \xleftarrow{\delta_0} C^0(X, A; G) \leftarrow 0$$

as the singular cochain complex of (X, A) with G-coefficients. We define the n-th singular cohomology group of (X, A) with G-coefficients[1314] as follows

$$\mathrm{H}^n(X, A; G) := \mathrm{H}^n(\mathrm{C}_*(X, A); G) = \frac{\ker(\delta_n\colon \mathrm{C}^n(X, A; G) \to \mathrm{C}^{n+1}(X, A; G))}{\mathrm{im}(\delta_{n-1}\colon \mathrm{C}^{n-1}(X, A; G) \to \mathrm{C}^n(X, A; G))}.$$

(3) If $A = \varnothing$ then we just write $\mathrm{C}^n(X; G) = \mathrm{C}^n(X, \varnothing; G)$ and similar for cohomology groups.

(4) If $G = \mathbb{Z}$ then often we drop the coefficients from the notation, e.g. often we write $\mathrm{H}^n(X, A) = \mathrm{H}^n(X, A; \mathbb{Z})$.

Remark. As we had already pointed out on page 2400, if R is a commutative ring, e.g. if $R = \mathbb{Z}_n, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $R = \mathbb{C}$, then the cohomology groups $\mathrm{H}^n(X, A; R)$ can in fact be viewed as R-modules and the coboundary maps are R-module homomorphisms.

[1314] In most cases we drop the adjective "singular" and we just talk of the n-th cohomology group of (X, A) with coefficients in G. We only refer to it as singular cohomology when we want to distinguish it from de Rham cohomology.

In the following we first discuss the definition and examples of the absolute cohomology groups $\mathrm{H}^*(X;G)$. A little later we will discuss the arguably even more confusing relative cohomology groups $\mathrm{H}^*(X,A;G)$. We start this discussion of absolute cohomology groups with the following remark.

Remark. Let G be an abelian group.

(1) Let X be a topological space and let $n \in \mathbb{N}_0$. We denote by $\mathrm{S}_n(X)$ the set of singular n-simplices. We then have natural identifications

$$\mathrm{C}^n(X;G) \;=\; \mathrm{Hom}(\mathrm{C}_n(X),G) \underset{\substack{\uparrow\\ \text{by definition } \mathrm{C}_n(X)=\mathbb{Z}^{(\mathrm{S}_n(X))}}}{=} \mathrm{Hom}\left(\mathbb{Z}^{(\mathrm{S}_n(X))},G\right) \underset{\substack{\uparrow\\ \text{identification given by Lemma 108.1}}}{=} \text{maps from } \mathrm{S}_n(X) \text{ to } G.$$

Put differently, a singular n-cochain is precisely the same as assigning to each singular n-simplex an element in G. We will use this bijection throughout the course without explicitly referring to it.

(2) Let X be a topological space. We recall that, following the discussion on page 1583, we can identify the singular 0-simplices in X with the points in X. According to the discussion in (1) we can now make the identification

$$\mathrm{C}^0(X;G) \;=\; \{\text{all maps from } X \text{ to } G\}.$$

Before we develop the theory of cohomology groups in detail we first want to consider some explicit examples. As a starter we consider cochains and cocycles on the topological spaces $\mathbb{R}$ and S^1.

Examples.

(1) We start out with the topological space $X = \mathbb{R}$. We consider the two functions

$$\begin{array}{rcl} \alpha_{\mathbb{R}}\colon \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x \end{array} \quad \text{and} \quad \begin{array}{rcl} \alpha_{\mathbb{Z}}\colon \mathbb{R} & \to & \mathbb{Z} \\ x & \mapsto & \lfloor x \rfloor := \max\{n \in \mathbb{Z} \,|\, n \le x\}. \end{array}$$

As pointed out above, these two functions define singular 0-cochains. We denote these 0-cochains again by $\alpha_{\mathbb{R}}$ and $\alpha_{\mathbb{Z}}$. For any singular 1-simplex $\mu\colon \Delta^1 \to \mathbb{R}$ we have

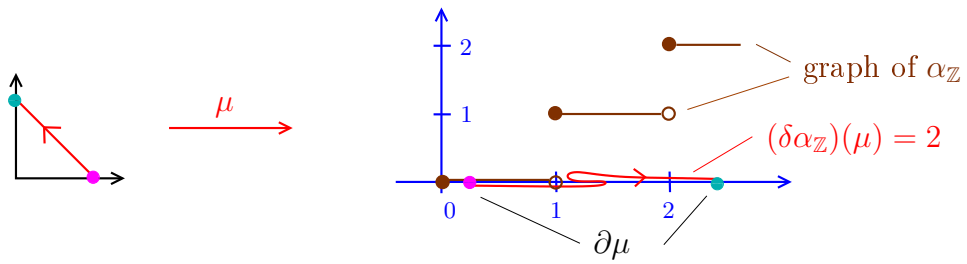
$$(\delta\alpha_{\mathbb{R}})(\mu) \underset{\substack{\uparrow\\ \text{definition of } \delta}}{=} \alpha_{\mathbb{R}}(\partial\mu) \underset{\substack{\uparrow\\ \text{see page 1585}}}{=} \alpha_{\mathbb{R}}(\mu(0,1)-\mu(1,0)) \underset{\substack{\uparrow\\ \text{definition of } \alpha_{\mathbb{R}}}}{=} \mu(0,1)-\mu(1,0)$$

and, slightly more interesting, it follows immediately from the above discussion and the definition of $\alpha_{\mathbb{Z}}$ that

$$(\delta\alpha_{\mathbb{Z}})(\mu) \;=\; \alpha_{\mathbb{Z}}(\partial\mu) \;= \begin{cases} \#\{n \in \mathbb{Z} \,|\, \mu(1,0) < n \le \mu(0,1)\}, & \text{if } \mu(0,1) \ge \mu(1,0) \\ -\#\{n \in \mathbb{Z} \,|\, \mu(0,1) < n \le \mu(1,0)\}, & \text{if } \mu(1,0) \ge \mu(0,1). \end{cases}$$

In a somewhat informal way we can say that the cochain $\delta\alpha_{\mathbb{Z}}\colon \mathrm{C}_1(\mathbb{R}) \to \mathbb{Z}$ applied to a singular 1-simplex μ measures how many integers the simplex μ "crosses from left to right". We refer to the figure below for an illustration of $\delta\alpha_{\mathbb{Z}}$.

(2) Now we consider the topological space $X = S^1$. As usual we denote by $p\colon \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi\,\mathrm{i}t)$ the universal covering of S^1. Let $\sigma\colon \Delta^1 \to S^1$ be a singular 1-simplex. Since Δ^1 is simply connected it follows from the Covering Existence Theorem 61.5 that σ lifts to a map $\widetilde{\sigma}\colon \Delta^1 \to \mathbb{R}$. The lift is not unique, but it is straightforward



to see that any two lifts differ by addition by an integer. For a singular 1-simplex $\sigma\colon \Delta^1 \to S^1$ we pick a lift $\widetilde{\sigma}\colon \Delta^1 \to \mathbb{R}$ of σ and we define

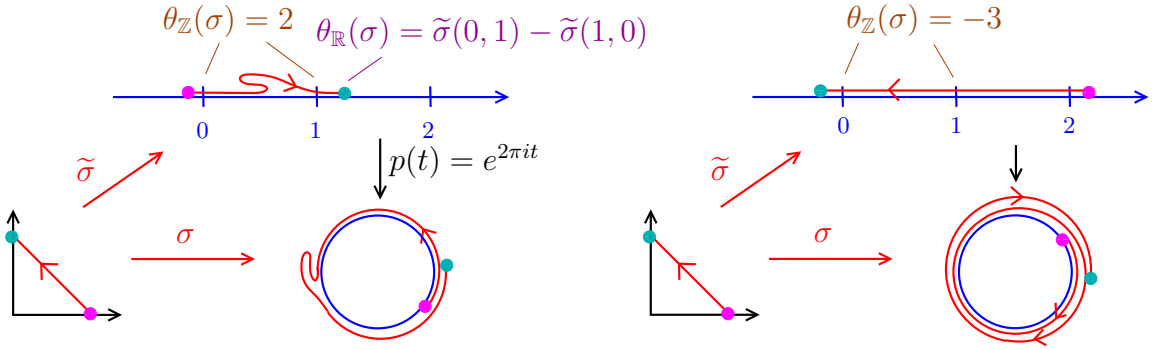
$$\theta_{\mathbb{R}}(\sigma) \;:=\; (\delta\alpha_{\mathbb{R}})(\underset{\in C_1(\mathbb{R})}{\widetilde{\sigma}}) \;=\; \widetilde{\sigma}(0,1) - \widetilde{\sigma}(1,0)$$

and we also define

$$\theta_{\mathbb{Z}}(\sigma) \;:=\; (\delta\alpha_{\mathbb{Z}})(\widetilde{\sigma}) \;=\; \begin{cases} \#\{n \in \mathbb{Z} \,|\, \widetilde{\sigma}(1,0) < n \leq \widetilde{\sigma}(0,1)\}, & \text{if } \widetilde{\sigma}(0,1) \geq \widetilde{\sigma}(1,0) \\ -\#\{n \in \mathbb{Z} \,|\, \widetilde{\sigma}(0,1) < n \leq \widetilde{\sigma}(1,0)\}, & \text{if } \widetilde{\sigma}(1,0) \geq \widetilde{\sigma}(0,1). \end{cases}$$

Since any two lifts of σ differ by addition by an integer we see that $\theta_{\mathbb{R}}(\sigma)$ and $\theta_{\mathbb{Z}}(\sigma)$ are both well-defined, i.e. independent of the choice of the lift $\widetilde{\sigma}$. The definitions of $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are illustrated in the figure below.[1315]

As we had explained on page 2403, these two maps $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ actually define singular 1-cochains in $C^1, \mathbb{R})$ and $C^1; \mathbb{Z})$.



As an example we consider the singular 1-simplex

$$\mu\colon \Delta^1 \;\to\; S^1 \quad \text{that is defined by} \quad \mu(1-t,t) \;=\; \exp(2\pi i t).$$

A lift of μ to the above universal covering $\mathbb{R} \to S^1$ is given by the singular 1-simplex

$$\widetilde{\mu}\colon \Delta^1 \;\to\; \mathbb{R} \quad \text{that is defined by} \quad \widetilde{\mu}(1-t,t) \;=\; t.$$

It follows immediately from the definitions that $\theta_{\mathbb{R}}(\mu) = \theta_{\mathbb{Z}}(\mu) = 1$.

[1315] Here is an alternative description of the singular 1-cochain θ: We denote by $\Phi\colon [0,1] \to \Delta^1$ the homeomorphism that is given by $\Phi(t) = (1-t,t)$. Then for any singular 1-simplex $\sigma\colon \Delta^1 \to S^1$ we have

$$\theta_{\mathbb{R}}(\sigma) \;=\; \frac{1}{2\pi\,\mathrm{i}} \int\limits_{\sigma\circ\Phi} \frac{1}{z}\, dz,$$

where the right-hand side denotes the path integral of the holomorphic function $\frac{1}{z}$ over the continuous path $\sigma \circ \Phi\colon [0,1] \to S^1$. We leave it as an amusing exercise to verify this equality using complex analysis.

The following lemma shows that $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are actually quite interesting singular 1-cochains.

Lemma 108.10. We continue with the notation from the previous example.

(1) The singular 1-cochains $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are cocycles, in particular $\theta_{\mathbb{R}}$ defines an element in $\mathrm{H}^1(S^1;\mathbb{R})$ and $\theta_{\mathbb{Z}}$ defines an element in $\mathrm{H}^1(S^1;\mathbb{Z})$.

(2) The singular 1-cochain cochains $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are not the coboundary of a singular 0-cochain, i.e. the cohomology classes $[\theta_{\mathbb{R}}] \in \mathrm{H}^1(S^1;\mathbb{R})$ and $[\theta_{\mathbb{Z}}] \in \mathrm{H}^1(S^1;\mathbb{Z})$ are non-trivial.

Proof.

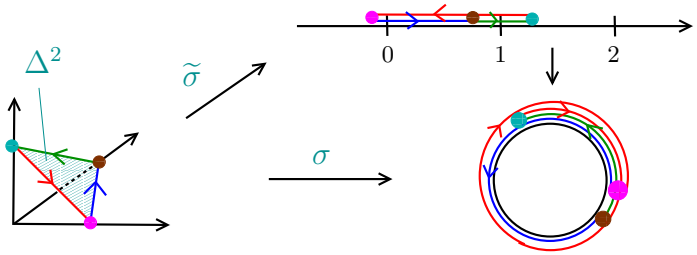
(1) We will first prove that the singular 1-cochain $\theta_{\mathbb{R}}$ is a cocycle. Thus we have to show that $\delta_1\theta_{\mathbb{R}} = 0 \in \mathrm{C}^2(S^1,\mathbb{R}) = \mathrm{Hom}(\mathrm{C}_2(S^1),\mathbb{R})$. So let $\sigma\colon \Delta^2 \to S^1$ be a singular 2-simplex. We have to show that $(\delta_1\theta_{\mathbb{R}})(\sigma) = \theta_{\mathbb{R}}(\partial\sigma) = 0$.

As before we denote by $p\colon \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi i t)$ the universal covering of S^1. Since Δ^2 is simply connected we can apply the Map Lifting Criterion 61.2 to obtain a lift of $\sigma\colon \Delta^2 \to S^1$ to a map $\widetilde{\sigma}\colon \Delta^2 \to \mathbb{R}$. We refer to the figure below for an illustration. Then

$$\theta_{\mathbb{R}}(\partial\sigma) \underset{\uparrow}{=} \theta_{\mathbb{R}}\Big(\sum_{j=0}^{2}(-1)^j \cdot \sigma \circ i_j^2\Big) \underset{\uparrow}{=} (\delta\alpha_{\mathbb{R}})\Big(\overbrace{\sum_{j=0}^{2}(-1)^j \cdot \widetilde{\sigma} \circ i_j^2}^{=\partial\widetilde{\sigma}}\Big) \underset{\uparrow}{=} \alpha_{\mathbb{R}}(\partial(\partial\widetilde{\sigma})) \underset{\uparrow}{=} 0.$$

definition of ∂ — by definition of $\theta_{\mathbb{R}}$ and since $\widetilde{\sigma} \circ i_j^2$ is a lift of $\sigma \circ i_j^2$ — by definition of $\delta\alpha_{\mathbb{R}}$ and $\partial\widetilde{\sigma}$ — since $\partial \circ \partial = 0$

Exactly the same argument applies to $\theta_{\mathbb{Z}}$.



(2) We still need to show that the singular 1-cochain cochains $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are not the coboundary of a singular 0-cochain. So let β be a singular 0-cochain. As usual we view β as a function $S^1 \to \mathbb{R}$. We consider again the singular 1-simplex $\mu\colon \Delta^1 \to S^1$ that is defined by $\mu(1-t,t) = \exp(2\pi i t)$. Then

$$(\delta_0\beta)(\mu) = \beta(\partial(\mu)) = \beta\big(\underbrace{\mu(0,1)}_{=1\in S^1} - \underbrace{\mu(1,0)}_{=1\in S^1}\big) = 0.$$

But in the previous example we saw that $\theta_{\mathbb{R}}(\mu) = \theta_{\mathbb{Z}}(\mu) = 1$. ■

Remark. Note that in Lemma 108.10 we have just given a mysteriously simple proof that $\mathrm{H}^1(S^1;\mathbb{R}) \neq 0$ and also that $\mathrm{H}^1(S^1;\mathbb{Z}) \neq 0$.

We continue with another example of cochains.

Example. We return to the topological space $X = \mathbb{R}$. We fix a point $y \in \mathbb{R}$. We consider the cochain

$$\begin{array}{rcl} \varphi_y\colon \mathrm{C}_0(\mathbb{R}) & \to & \mathbb{Z} \end{array}$$

that is determined by the function

$$\begin{array}{rcl} \mathbb{R} & \to & \mathbb{Z} \\ x & \mapsto & \begin{cases} 0, & \text{if } x \leq y, \\ 1, & \text{else.} \end{cases} \end{array}$$

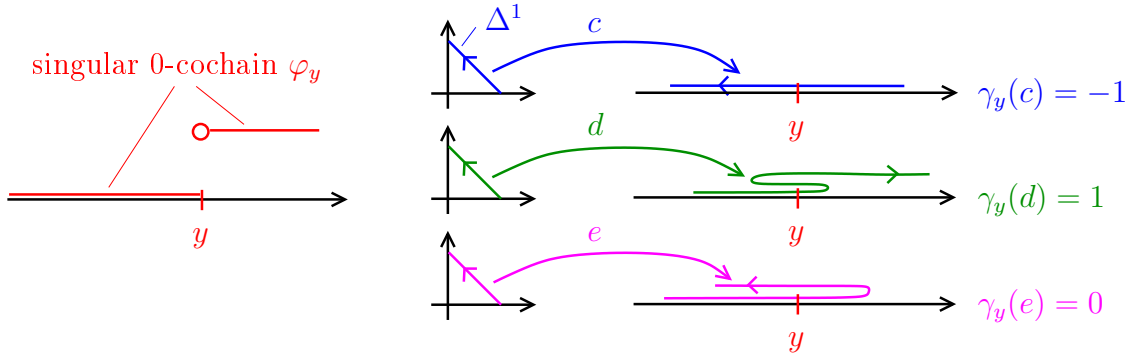
For any singular 1-simplex $d\colon \Delta^1 \to X = \mathbb{R}$ we have

$$(\delta_0\varphi_y)(d) \;=\; \varphi_y(\partial d) \;\underset{\substack{\uparrow \\ \text{since } \partial d = 1\cdot d(0,1) + (-1)\cdot d(1,0) \\ \text{and since } \varphi_y\colon \mathrm{C}_0(\mathbb{R}) \to \mathbb{Z} \text{ is linear}}}{=}\; \varphi_y(d(0,1)) - \varphi_y(d(1,0)) \;=\; \begin{cases} 0, & \text{if } d(0,1), d(1,0) \leq y, \\ 0, & \text{if } d(0,1), d(1,0) \geq y, \\ 1, & \text{if } d(1,0) \leq y < d(0,1) \\ -1, & \text{if } d(0,1) \leq y < d(1,0). \end{cases}$$

In a slightly informal way we can say that the homomorphism $\delta_0\varphi_y\colon \mathrm{C}_1(\mathbb{R}) \to \mathbb{Z}$ applied to a singular 1-simplex d equals $+1$ if the simplex "crosses y from right to left" and it equals -1 if the simplex "crosses y from left to right". We will make use of this singular 1-cochain on several occasions, hence we give it its own notation, namely we write

$$\gamma_y \;:=\; \delta_0\varphi_y \;\in\; \mathrm{C}^1(\mathbb{R};\mathbb{Z}).$$

We illustrate the singular 1-cochain $\gamma_y = \delta_0\varphi_y \in \mathrm{C}^1(\mathbb{R};\mathbb{Z})$ in the figure below.



After this long discussion of explicit cochains and cohomology classes we want to turn to more general results. The only two instances where we could compute homology groups without any problems was the 0-th homology group of a path-connected space, see the H_0-Proposition 72.5, and the homology groups of a point, see the Single Point-Homology Lemma 72.6. Not surprisingly we can also deal with the same situations with cohomology groups.

First of all we have the following analogue of the H_0-Proposition 72.5.

Proposition 108.11. (H^0-Proposition) Let X be a non-empty path-connected topological space and let G be an abelian group. Then the map

$$\begin{array}{rcl} G & \to & \mathrm{H}^0(X;G) \\ g & \mapsto & [\text{0-cochain that assigns to each } x \in X \text{ the value } g] \end{array}$$

is a well-defined map that is in fact an isomorphism. For any point $P \in X$ an inverse is given by the map

$$\begin{array}{rcl} \mathrm{H}^0(X;G) & \to & G \\ [f] & \mapsto & f(P) \end{array}$$

Remark. At times we will use the isomorphism from the H^0-Proposition 108.11 to identify the 0-th cohomology $\mathrm{H}^0(X;G)$ of a path-connected topological space X with G.

Proof. We consider the "right-hand end" of the chain complex of $\mathrm{C}_*(X)$ and of the cochain complex $\mathrm{Hom}(\mathrm{C}_*(X),G)$:

$$\mathrm{C}_1(X) \xrightarrow{\partial_1} \mathrm{C}_0(X) \to 0 \quad \text{and} \quad \mathrm{Hom}(\mathrm{C}_1(X),G) \xleftarrow{\delta_0=\partial_1^*} \mathrm{Hom}(\mathrm{C}_0(X),G) \leftarrow 0.$$

Now we calculate that

$$\begin{aligned}
\mathrm{H}^0(X;G) &= \ker \left\{ \begin{array}{ccc} \mathrm{Hom}(\mathrm{C}_0(X),G) & \to & \mathrm{Hom}(\mathrm{C}_1(X),G) \\ (f\colon \mathrm{C}_0(X) \to G) & \mapsto & (f \circ \partial_1\colon \mathrm{C}_1(X) \to G) \end{array} \right\} \\
&\overset{\text{since the singular 1-simplices form a basis of } \mathrm{C}_1(X)}{\downarrow =} \\
&\{\text{maps } f\colon X \to G \mid \underbrace{f(\partial_1(\sigma))}_{=f(\sigma(0,1))-f(\sigma(1,0))} = 0 \text{ for all singular 1-simplices } \sigma\colon \Delta^1 \to X\} \\
&\overset{\text{by Lemma 108.1 we have the natural identification } \mathrm{Hom}(\mathrm{C}_0(X),G) = \text{maps from } X \text{ to } G}{\uparrow =} \\
&\text{all constant maps } X \to G \\
&\overset{\text{since } X \text{ is path-connected and since } \Delta^1 \cong [0,1]}{\uparrow =} \\
&G. \\
&\overset{\text{since } X \text{ is non-empty}}{\uparrow =}
\end{aligned}$$

The remaining statements of the proposition can be verified easily. ∎

The following lemma is an analogue of the Single Point-Homology Lemma 72.6.

Lemma 108.12. (Single Point-Cohomology Lemma) Let $X = \{x\}$ be a topological space that consists of a single point x. For any abelian group G we have

$$\mathrm{H}^n(X;G) = \left\{ \begin{array}{ll} 0, & \text{if } n \geq 1, \\ G, & \text{if } n = 0. \end{array} \right.$$

Proof. We outsource the proof to Exercise 108.3. ∎

We want to conclude this section with a short discussion of relative cohomology classes. Before we do so we make the following observations:

Observation.

(1) Given an abelian group C, a subgroup D and an abelian group G we have a natural isomorphism

$$\begin{array}{rcl} \mathrm{Hom}(C/D,G) & \xrightarrow{\cong} & \text{all homomorphisms } C \to G \text{ that vanish on } D \\ (\varphi\colon C/D \to G) & \mapsto & (C \to C/D \xrightarrow{\varphi} G). \end{array}$$

We will use this isomorphism to identify the groups left and right.

(2) If S is a set and T is a subset, then for any abelian group G the identification

$$\mathrm{Hom}(\mathbb{Z}^{(S)},G) = \text{all maps from } S \text{ to } G$$

from Lemma 108.1 restricts to an isomorphism

$$\left\{ f \in \mathrm{Hom}(\mathbb{Z}^{(S)},G) \,\middle|\, f \text{ vanishes on } \mathbb{Z}^{(T)} \right\} = \text{all maps } S \to G \text{ that are trivial on } T.$$

(3) It follows immediately from Lemma 108.1 and from (1) and (2) that given a pair of topological spaces (X,A) and an abelian group G we have for every $n \in \mathbb{N}_0$ a natural

isomorphism

$$\underbrace{\mathrm{Hom}(\mathrm{C}_n(X,A),G)}_{=\mathrm{C}^n(X,A;G)=\mathrm{Hom}(\mathrm{C}_n(X)/\mathrm{C}_n(A),G)} \;=\; \text{all maps } \mathrm{S}_n(X)\to G \text{ that are trivial on } \mathrm{S}_n(A).$$

We will often use this natural isomorphism to identify the left-hand side with the right-hand side.

Examples.

(1) In Exercise 108.4 we will modify the argument of the H^0-Proposition 108.11 to show that for any path-connected topological space with non-empty subset A we have

$$\mathrm{H}^0(X,A;G) \;=\; 0 \quad \text{for any abelian group } G.$$

(2) We write $I=[0,1]$. In this example we want to construct a non-trivial element in $\mathrm{H}^1(I,\partial I)$. We consider the function

$$\begin{array}{rcl} f\colon I & \to & \mathbb{Z} \\ x & \mapsto & \begin{cases} 0, & \text{if } x\in[0,\frac{1}{2}) \\ 1, & \text{otherwise} \end{cases} \end{array}$$

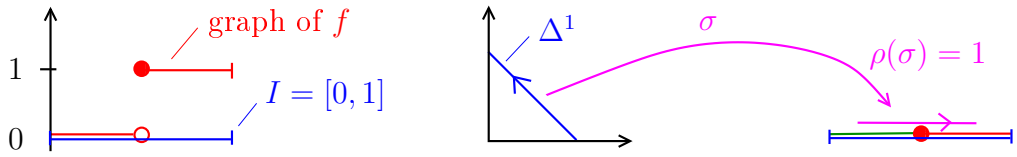
which we use to define the singular 1-cochain

$$\begin{array}{rcl} \rho\colon \mathrm{C}_1(I) & \to & \mathbb{Z} \\ (\sigma\colon \Delta^1\to I) & \mapsto & f(\sigma(0,1))-f(\sigma(1,0)). \end{array}$$

Loosely speaking, similar to the singular 1-cochain φ_y from page 2406 the singular 1-cochain $\rho(\sigma)$ measures whether σ crosses $\frac{1}{2}$, and if yes, in what direction. We refer to the figure below for an illustration. Since Δ^1 is connected and since $\{0,1\}$ is discrete it follows from the Connected-to-Discrete Lemma 2.27 that any singular 1-simplex $\sigma\colon\Delta^1\to\partial I=\{0,1\}$ is constant. In particular in this case we have $\rho(\sigma)=0$. Thus we see, using the above observation, that ρ defines a map

$$\rho\colon \mathrm{C}_1(I,\partial I) \;\to\; \mathbb{Z}.$$

In Exercise 108.8 we will see that ρ is a singular 1-cocycle and that it defines a non-trivial element in $\mathrm{H}^1(I,\partial I;\mathbb{Z})$.



108.5. **Basic properties of cohomology groups.** Now we will see that maps between pairs of topological spaces induce maps on cohomology groups.

Definition. Let G be an abelian group and let $f\colon(X,A)\to(Y,B)$ be a map between pairs of topological spaces. Recall that f induces a chain map $f_*\colon\mathrm{C}_*(X,A)\to\mathrm{C}_*(Y,B)$. For each $n\in\mathbb{N}_0$ we denote by

$$f^*\colon \mathrm{H}^n(Y,B;G) \;\to\; \mathrm{H}^n(X,A;G)$$

the induced map of cohomology groups that we obtain from the Dual Cochain Map Lemma 108.8 (1).

Example. We will see in Exercise 108.6 that any map $f\colon X \to Y$ between two path-connected non-empty topological spaces induces an isomorphism $f^*\colon \mathrm{H}^0(Y;G) \to \mathrm{H}^0(X;G)$. A very similar argument shows that if Y is path-connected and X is non-empty, then $f^*\colon \mathrm{H}^0(Y;G) \to \mathrm{H}^0(X;G)$ is a monomorphism.

Notation. Let X be a topological space and let A be a subset. Furthermore let G be an abelian group. We denote by $i\colon A \to X$ the inclusion map. Sometimes we refer to $i^*\colon \mathrm{H}^n(X;G) \to \mathrm{H}^n(A;G)$ as the restriction map.

We summarize some basic properties of the induced maps in the following lemma.

Lemma 108.13. (Homotopic Maps-H*-Lemma) Let G be an abelian group and let $n \in \mathbb{N}_0$.

(1) The maps
$$\begin{array}{rcl}(X,A) & \mapsto & \mathrm{H}^n(X,A;G)\\ (f\colon (X,A)\to(Y,B)) & \mapsto & (f^*\colon \mathrm{H}^n(Y,B;G)\to \mathrm{H}^n(X,A;G))\end{array}$$
define a contravariant functor from the category of pairs of topological spaces to the category of abelian groups. (If G is a commutative ring, then this is in fact a functor to the category of G-modules.)

(2) Let $f,g\colon (X,A)\to(Y,B)$ be maps between pairs of topological spaces. If f and g are homotopic, then they induce the same maps
$$f^* = g^*\colon \mathrm{H}^n(Y,B;G) \to \mathrm{H}^n(X,A;G).$$

(3) If $f\colon X \to Y$ is a homotopy equivalence between two topological spaces, then the induced map
$$f^*\colon \mathrm{H}^n(Y;G) \xrightarrow{\cong} \mathrm{H}^n(X;G)$$
is an isomorphism. In particular the cohomology groups of homotopy equivalent topological spaces are isomorphic.

(4) Let X be a topological space, let $A \subset B$ be subsets of X and let $n \in \mathbb{N}_0$. If A is a deformation retract of B, then the inclusion induced map
$$\mathrm{H}^n(X,B;G) \to \mathrm{H}^n(X,A;G)$$
is an isomorphism. Similarly, if B is a deformation retract of X, then the inclusion induced map
$$\mathrm{H}^n(X,A;G) \to \mathrm{H}^n(B,A;G)$$
is an isomorphism.

Example. The Homotopic Maps-H*-Lemma 108.13 (4), together with the Single Point-Cohomology Lemma 108.12 and the discussion on page 432 implies that for any $n \geq 1$, any abelian group G and any $k \in \mathbb{N}_0$ we have
$$\mathrm{H}^n(\mathbb{R}^k;G) = \mathrm{H}^n\big(\overline{B}^k;G\big) = \mathrm{H}^n\big(B^k;G\big) = \mathrm{H}^n\big([0,1]^k;G\big) = \mathrm{H}^n\big(\Delta^k;G\big) = 0.$$

Proof.

(1) This statement follows immediately from the Dual Cochain Map Lemma 108.8 (2) together with the fact that the map $(X,A) \mapsto \mathrm{C}_*(X,A)$ defines a covariant functor

from the category *PairTop* of pairs of topological spaces to the category *ChCplx* of chain complexes.

(2) This statement is an immediate consequence of Proposition 74.15 together with Lemma 108.8 (3).

(3) This statement follows from (1) and (2).

(4) This statement is proved almost verbatim the same way as Corollary 74.16 (2a) and (2b). ■

Many of the statements about homology groups can be modified to give corresponding statements about cohomology groups. For example we have the following lemma.

Lemma 108.14. Let X be a topological space and let G be an abelian group.

(1) We denote by $\{X_j\}_{j\in J}$ the path-components of X. The inclusion maps $\iota_j\colon X_j \to X$, $j \in J$, induce an isomorphism

$$\prod_{j\in J} \iota_j^*\colon \mathrm{H}^n(X;G) \xrightarrow{\cong} \prod_{j\in J} \mathrm{H}^n(X_j;G).$$

(2) Suppose that X has finitely many path-components $X_1, \ldots, X_k$. If for $j = 1, \ldots, k$ we pick a point $P_j \in X_j$, then the map

$$\begin{array}{rcl} \mathrm{H}^0(X;G) & \to & G^k = \overbrace{G \times \cdots \times G}^{k \text{ times}} \\ [f] & \mapsto & (f(P_1), \ldots, f(P_k)) \end{array}$$

is an isomorphism.

Proof. Let X be a topological space with path-components $\{X_j\}_{j\in J}$ and let G be an abelian group. We have the following isomorphisms of cochain complexes:

$$\begin{array}{ccccc} \mathrm{Hom}(\mathrm{C}_*(X), G) & \xrightarrow{\left(\bigoplus_{j\in J} \iota_{j*}\right)^*} & \mathrm{Hom}\left(\bigoplus_{j\in J} \mathrm{C}_*(X_j), G\right) & \xrightarrow{\cong} & \prod_{j\in J} \mathrm{Hom}(\mathrm{C}_*(X_j), G). \\ \uparrow & & & & \uparrow \\ \text{Lemma 72.14} & & & & \text{Lemma 108.2} \end{array}$$

The first statement now follows easily from this observation together with the fact that cohomology commutes with direct products, see Lemma 108.7.

The second statement follows from the first statement together with the calculation of 0-th cohomology groups of path-connected topological spaces in the H^0-Proposition 108.11. ■

The following lemma is the analogue of the Topological LES-Proposition 74.13.

Lemma 108.15. (Cohomological LES-Lemma) Given a triple (X, B, A) of topological spaces and an abelian group G we obtain a long exact sequence

$$\ldots \xrightarrow{\delta_{n-1}} \mathrm{H}^n(X, B; G) \xrightarrow{p^*} \mathrm{H}^n(X, A; G) \xrightarrow{i^*} \mathrm{H}^n(B, A; G) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(X, B; G) \xrightarrow{p^*} \ldots$$

hereby the maps are given as follows:

(1) p^* is the map induced by the map $p\colon (X, A) \to (X, B)$,

(2) i^* is the map induced by the map $i\colon (B, A) \to (X, A)$,

(3) the connecting homomorphism $\delta_n\colon \mathrm{H}^n(B,A;G) \to \mathrm{H}^{n+1}(X,B;G)$ is natural and it is given by the map

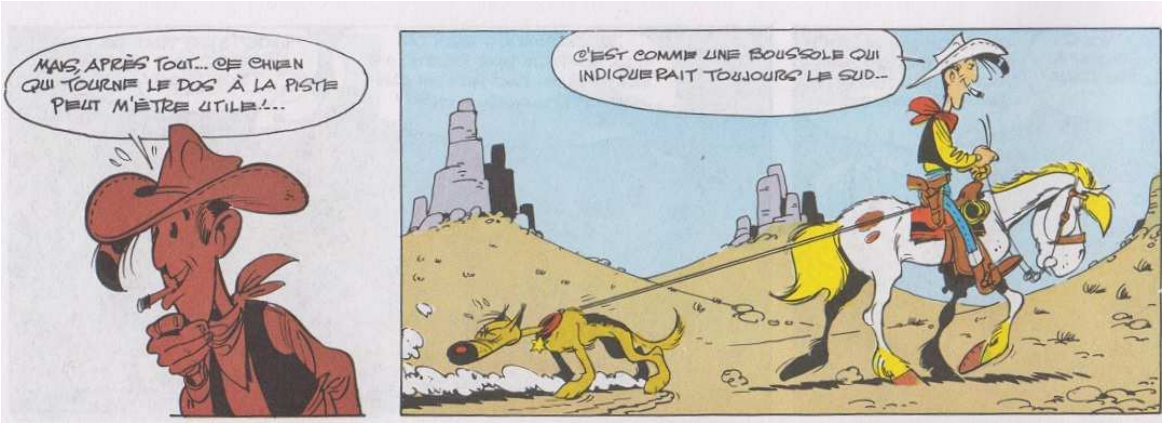
$$\begin{array}{rcl} \mathrm{H}^n(B,A;G) & \to & \mathrm{H}^{n+1}(X,B;G) \\ [f\colon \mathrm{C}_n(B,A)\to G] & \mapsto & \left[\begin{array}{rcl} \mathrm{C}_{n+1}(X,B) & \to & G \\ \sigma & \mapsto & f(q(\partial_n(\sigma))) \end{array}\right] \\ & & \uparrow \\ & & \text{with } \sigma \in \mathrm{C}_{n+1}(X) \end{array}$$

here $q\colon \mathrm{C}_n(X) \to \mathrm{C}_n(B)$ is the map given by sending all singular simplices that do not lie in B to zero.

Remark.

(1) An equivalent approach to expressing the connecting homomorphism is given as follows:
   (a) take a cocycle $f \in \mathrm{C}^n(B,A;G) = \mathrm{Hom}(\mathrm{C}_n(B,A),G)$,
   (b) denote by $\pi\colon \mathrm{C}_n(B) \to \mathrm{C}_n(B,A)$ the projection ,
   (c) denote by $\widetilde{f}$ the result of extending $f\circ\pi\colon \mathrm{C}_n(B) \to G$ to a map $\mathrm{C}_n(X) \to G$ by sending all singular simplices that do not lie in B to zero (sometimes we call $\widetilde{f}$ the *trivial extension* of $f\circ\pi$),
   (d) then consider the coboundary $\delta\widetilde{f} \in \mathrm{C}^{n+1}(X;G)$, since f is a cocycle it follows that $\delta\widetilde{f}$ actually defines an element in $\mathrm{C}^{n+1}(X,B;G)$ that we also denote by $\delta\widetilde{f}$, and which turns out to be a cocycle,
   (e) then $\delta[f] = [\delta\widetilde{f}] \in \mathrm{H}^{n+1}(X,B;G)$.

(2) One of the difficulties with the long exact sequence of cohomology groups is that all maps seem to point the wrong way. You should use the Lucky Luke approach to cohomology: if Rantanplan (i.e. your intuition) points one way, you should take the opposite direction.



(3) The most frequent application of the Cohomological LES-Lemma 108.15 is that given a pair (X,B) of topological spaces, i.e. given the triple $(X,B,\varnothing)$ and given an abelian group G we obtain a natural long exact sequence

$$\ldots \xrightarrow{\delta_{n-1}} \mathrm{H}^n(X,B;G) \xrightarrow{p^*} \mathrm{H}^n(X;G) \xrightarrow{i^*} \mathrm{H}^n(B;G) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(X,B;G) \xrightarrow{p^*} \ldots$$

Proof. Let (X, B, A) be a triple of topological spaces. As in the proof of the Topological LES-Proposition 74.13 we note that

$$0 \;\to\; \mathrm{C}_*(B, A) \;\xrightarrow{i_*}\; \mathrm{C}_*(X, A) \;\xrightarrow{p_*}\; \mathrm{C}_*(X, B) \;\to\; 0$$

is a short exact sequence of chain complexes. Now let G be an abelian group. Recall that we pointed out on page 1625 that the chain groups $\mathrm{C}_*(X, B)$ are free abelian groups. Thus we deduce from Lemma 108.4 that the following induced sequence of cochain complexes is also exact:

$$0 \;\to\; \mathrm{Hom}(\mathrm{C}_*(X, B), G) \;\xrightarrow{p^*}\; \mathrm{Hom}(\mathrm{C}_*(X, A), G) \;\xrightarrow{i^*}\; \mathrm{Hom}(\mathrm{C}_*(B, A), G) \;\to\; 0.$$

The lemma now follows easily from the Connecting Homomorphism Proposition 108.6 and the construction of the connecting homomorphism. We leave the details to the reader. ■

Examples.

(1) Let X be a path-connected topological space and let A be any *non-empty* subset A of X. We denote by $i\colon A \to X$ the inclusion map. By the above remark we have a long exact sequence

$$0 \;\to\; \mathrm{H}^0(X, A; G) \;\to\; \mathrm{H}^0(X; G) \;\underset{\uparrow}{\xrightarrow{i^*}}\; \mathrm{H}^0(A; G) \;\to\; \dots$$

monomorphism, see page 2409

Thus it follows that $\mathrm{H}^0(X, A; G) = 0$. In Exercise 108.4 we will provide an alternative proof for this fact using only the definitions of cohomology groups.

(2) Let $b \in \mathbb{R}_{>0}$. We consider the long exact sequence in cohomology with $\mathbb{Z}$-coefficients of the pair $(\mathbb{R}, \mathbb{R} \setminus [-b, b])$. Together with (1) and Lemma 108.14 we obtain the following commutative diagram where the top horizontal sequence is exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^0(\mathbb{R}; \mathbb{Z}) & \xrightarrow{i^*} & \mathrm{H}^0(\mathbb{R} \setminus [-b, b]; \mathbb{Z}) & \xrightarrow{\delta} & \mathrm{H}^1(\mathbb{R}, \mathbb{R} \setminus [-b, b]; \mathbb{Z}) & \longrightarrow & \mathrm{H}^1(\mathbb{R}; \mathbb{Z}) \\
 & & \downarrow{\scriptstyle [f] \mapsto f(-b-1)} & & \downarrow{\scriptstyle [f] \mapsto (f(-b-1), f(b+1))} & & & & \downarrow{\scriptstyle \mathrm{id}} \\
 & & \mathbb{Z} & \xrightarrow[m \mapsto (m,m)]{} & \mathbb{Z}^2 & & & & 0.
\end{array}$$

The cokernel of the bottom horizontal map is easily seen to be isomorphic to $\mathbb{Z}$ and it is generated by the element $(0, 1) \in \mathbb{Z}^2$. Put differently, the cokernel of the map $i^*\colon \mathrm{H}^0(\mathbb{R}; \mathbb{Z}) \to \mathrm{H}^0(\mathbb{R} \setminus [-b, b]; \mathbb{Z})$ is represented by the singular 0-cochain given by

$$\begin{array}{rcl}
\psi\colon \mathbb{R} \setminus [-b, b] & \to & \mathbb{Z} \\
x & \mapsto & \begin{cases} 0, & \text{if } x \le -b, \\ 1, & \text{if } x \ge b. \end{cases}
\end{array}$$

Now we want to determine $\delta\psi \in \mathrm{H}^1(\mathbb{R}, \mathbb{R} \setminus [-b, b]; \mathbb{Z})$. It is perhaps helpful to consider the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Hom}(\mathrm{C}_1(\mathbb{R}, \mathbb{R} \setminus [-b, b]), \mathbb{Z}) & \xrightarrow{p^*} & \mathrm{Hom}(\mathrm{C}_1(\mathbb{R}), \mathbb{Z}) & \xrightarrow{i^*} & \mathrm{Hom}(\mathrm{C}_1(\mathbb{R} \setminus [-b, b]), \mathbb{Z}) & \to & 0 \\
 & & \uparrow{\scriptstyle \delta_0} & & \uparrow{\scriptstyle \delta_0} & & \uparrow{\scriptstyle \boldsymbol{\delta_0}} & & \\
0 & \longrightarrow & \mathrm{Hom}(\mathrm{C}_0(\mathbb{R}, \mathbb{R} \setminus [-b, b], \mathbb{Z}) & \xrightarrow{p^*} & \mathrm{Hom}(\mathrm{C}_0(\mathbb{R}), \mathbb{Z}) & \xrightarrow{i^*} & \underbrace{\mathrm{Hom}(\mathrm{C}_0(\mathbb{R} \setminus [-b, b], \mathbb{Z})}_{\text{contains } \psi} & \longrightarrow & 0.
\end{array}$$

We start with the map $\psi\colon \mathbb{R} \setminus [-b,b] \to \mathbb{Z}$. We need to pick an extension to a map $\mathbb{R} \to \mathbb{Z}$. We might as well take the map $\varphi_0\colon \mathbb{R} \to \mathbb{Z}$ from page 2406. Then it follows from the discussion of the connecting homomorphism that the coboundary $\gamma_0 := \delta\varphi_0$ represents an element in $\mathrm{H}^1(\mathbb{R}, \mathbb{R} \setminus [-b,b]; \mathbb{Z})$, and that $\delta\psi = [\gamma_0]$ is a generator of $\mathrm{H}^1(\mathbb{R}, \mathbb{R} \setminus [-b,b]; \mathbb{Z}) \cong \mathbb{Z}$.

108.6. **Reduced cohomology groups.** We saw on several occasions that the reduced homology groups $\widetilde{\mathrm{H}}_k(X)$ of a non-empty topological space can be convenient for expressing calculations and results. In this last section we quickly introduce the reduced cohomology groups.

Definition. Let X be a topological space. By Lemma 72.4 (1) we know that the following is a chain complex

$$\begin{array}{ccccccccccc} \ldots & \xrightarrow{\partial_3} & \mathrm{C}_2(X) & \xrightarrow{\partial_2} & \mathrm{C}_1(X) & \xrightarrow{\partial_1} & \mathrm{C}_0(X) & \xrightarrow{\epsilon} & \mathbb{Z} & \to & 0. \\ & & & & & & \sum_{i=1}^k n_i\sigma_i & \mapsto & \sum_{i=1}^k n_i. & & \end{array}$$

Let G be an abelian group. In a fashion similar to reduced homology we now define the k-th reduced cohomology group $\widetilde{\mathrm{H}}^k(X;G)$ of X to be the k-th cohomology group of the above chain complex.

The following lemma is the analogue of Lemmas 74.1, 74.2 and 74.12. For brevity's sake we leave the proof to the reader.

Lemma 108.16. (Reduced Cohomology Lemma) Let G be an abelian group.

(1) Let $f\colon X \to Y$ be a map between two topological spaces. For every $n \in \mathbb{N}_0$ the map f induces a map $f^*\colon \widetilde{\mathrm{H}}^n(Y;G) \to \widetilde{\mathrm{H}}^n(X;G)$ in the obvious way. For every $n \in \mathbb{N}_0$ the maps

$$\begin{array}{ccc} X & \mapsto & \widetilde{\mathrm{H}}^n(X;G) \\ (f\colon X \to Y) & \mapsto & (f^*\colon \widetilde{\mathrm{H}}^n(Y;G) \to \widetilde{\mathrm{H}}^n(X;G)) \end{array}$$

define a contravariant functor from the category of topological spaces to the category of abelian groups.

(2) For each $n \in \mathbb{N}_0$ the maps

$$\begin{array}{ccc} \mathrm{H}^n(X;G) & \to & \widetilde{\mathrm{H}}^n(X;G) \\ [\varphi] & \mapsto & [\varphi] \end{array}$$

define a natural transformation from the contravariant functor $X \mapsto \mathrm{H}^n(X;G)$ to the contravariant functor that is given by $X \mapsto \widetilde{\mathrm{H}}^n(X;G)$.

(3) For each topological space the map

$$\begin{array}{ccc} G & \to & \mathrm{H}^0(X;G) \\ g & \mapsto & [\text{cochain that assigns to each singular 0-simplex the element } g] \end{array}$$

is well-defined. If X is non-empty, this map is a monomorphism that splits.

(4) If X is a non-empty topological space then the maps from (2) and (3) define a natural short exact sequence

$$0 \to G \to \mathrm{H}^n(X;G) \to \widetilde{\mathrm{H}}^n(X;G) \to 0$$

that splits.

(5) For every non-empty topological space X we have

$$\mathrm{H}^n(X;G) \;\cong\; \begin{cases} \widetilde{\mathrm{H}}^0(X;G) \oplus G, & \text{if } n = 0, \\ \widetilde{\mathrm{H}}^n(X;G), & \text{if } n \neq 0. \end{cases}$$

The isomorphism is natural for $n \geq 1$.

(6) If X is a topological space with $n \in \mathbb{N}$ path-components, then $\widetilde{\mathrm{H}}^0(X;G) \cong G^{n-1}$.

(7) Given a topological space X, a point $x_0 \in X$ and $n \in \mathbb{N}_0$ we consider the map

$$\mathrm{H}^n(X, \{x_0\}; G) \longrightarrow \mathrm{H}^n(X, \varnothing; G) \xrightarrow{=} \mathrm{H}^n(X;G) \longrightarrow \widetilde{\mathrm{H}}^n(X;G).$$

For any $n \in \mathbb{N}_0$ this map forms a natural isomorphism from the contravariant functor $(X, x_0) \mapsto \widetilde{\mathrm{H}}_n(X;G)$ to the contravariant functor $(X, x_0) \mapsto \mathrm{H}^n(X, x_0; G)$.

(8) Any homotopy equivalence between two topological spaces induces isomorphisms of reduced cohomology.

Finally the following lemma plays the role of the Topological LES-Corollary 74.14.

Lemma 108.17. Let X be a topological space and let $B \subset X$ be a non-empty subset. Then given any abelian group G there exists a natural long exact sequence

$$\ldots \xrightarrow{\delta_{n-1}} \mathrm{H}^n(X, B; G) \xrightarrow{p^*} \widetilde{\mathrm{H}}^n(X;G) \xrightarrow{i^*} \widetilde{\mathrm{H}}^n(B;G) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(X, B; G) \xrightarrow{p^*} \ldots$$

Proof. The lemma follows immediately from the Cohomological LES-Lemma 108.15 applied to the triple $(X, B, \{P\})$ where P is a point in B, together with the Reduced Cohomology Lemma 108.16 (7). ■

108.7. **Isomorphic homology groups implies isomorphic cohomology groups.** In this short section we will see that the isomorphism type of cohomology groups cannot be used to extract more information from chain complexes than we already extracted from homology groups.

Proposition 108.18. Suppose $(\mathrm{C}_*, \partial_*)$ and (C'_*, ∂'_*) are two chain complexes of free abelian groups. Suppose that for each $n \in \mathbb{N}_0$ the homology groups $\mathrm{H}_n(C_*)$ and $\mathrm{H}_n(C'_*)$ are isomorphic. Then for each $n \in \mathbb{N}_0$ and any abelian group G the cohomology groups $\mathrm{H}^n(C;G)$ and $\mathrm{H}^n(C';G)$ are also isomorphic.

Proof. Suppose $(\mathrm{C}_*, \partial_*)$ and (C'_*, ∂'_*) are two chain complexes of free abelian groups. By our hypothesis there exist isomorphisms $\gamma_n \colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(C'_*)$, $n \in \mathbb{N}_0$. By the Lift-to-Chain Map-Proposition 81.13 these isomorphisms can be realized by a chain map $f \colon C_* \to C'_*$. But by the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 81.4 this chain map is a chain homotopy equivalence. The statement of the proposition follows immediately from the Dual Cochain Map Lemma 108.8. ■

This proposition gives us a rather sobering corollary. It says that if our only goal is to distinguish (pairs of) topological spaces, then cohomology groups do not contain more information than homology groups.

Corollary 108.19. Let (X, A) and (Y, B) be topological spaces and let G be an abelian group. Then

$$\begin{array}{c}\mathrm{H}_n(X,A) \text{ and } \mathrm{H}_n(Y,B) \text{ are} \\ \text{isomorphic for all } n \in \mathbb{N}_0\end{array} \implies \begin{array}{c}\mathrm{H}^n(Y,B;G) \text{ and } \mathrm{H}^n(X,A;G) \text{ are} \\ \text{isomorphic for all } n \in \mathbb{N}_0.\end{array}$$

Proof. This corollary follows immediately from Proposition 108.18 and the observation from page 1625 that relative singular chain groups are free abelian groups. ■

The previous corollary does not specify how the isomorphism of the cohomology groups is given. If we already start out with a map $f\colon (X, A) \to (Y, B)$ that induces an isomorphism of homology groups, then we can be more precise, namely we have the following corollary which is similar in spirit to Corollary 89.18.

Corollary 108.20. Let $f\colon (X, A) \to (Y, B)$ be a map between pairs of topological spaces and let G be an abelian group. Then

$$\begin{array}{c}\text{the induced map} \\ f_*\colon \mathrm{H}_n(X,A) \to \mathrm{H}_n(Y,B) \text{ is} \\ \text{an isomorphism for all } n \in \mathbb{N}_0\end{array} \implies \begin{array}{c}\text{the induced map} \\ f^*\colon \mathrm{H}^n(Y,B;G) \to \mathrm{H}^n(X,A;G) \text{ is} \\ \text{an isomorphism for all } n \in \mathbb{N}_0.\end{array}$$

Remark. In Proposition 110.13 we will see that the converse to Corollary 108.20 also holds.

Proof. Let $f\colon (X, A) \to (Y, B)$ be a map between pairs of topological spaces and let G be an abelian group. If $f_*\colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(Y, B)$ is an isomorphism for all $n \in \mathbb{N}_0$, then it follows from the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 81.4 that $f_*\colon \mathrm{C}_*(X, A) \to \mathrm{C}_*(Y, B)$ is already a chain homotopy equivalence. (Here, as in the previous proof we use that relative singular chain groups are free abelian groups.) Now we get the desired conclusion from the Dual Cochain Map Lemma 108.8. ■

The upshot of this short section is that we will need to work to justify spending so many pages on cohomology.

Exercises for Chapter 108.

Exercise 108.1. Let $f\colon B \to C$ be a homomorphism between abelian groups such that $\mathrm{im}(f)$ is a summand of C. Let $\{k_1, \ldots, k_m\}$ be a generating set for $\ker(f)$. Finally suppose that we are given homomorphisms $\varphi_1, \ldots, \varphi_m\colon B \to \mathbb{Z}$ such that for all $i, j \in \{1, \ldots, m\}$ we have $\varphi_i(k_j) = \delta_{ij}$. Show that the map

$$\begin{array}{rcl}\Phi\colon \mathbb{Z}^m & \to & \mathrm{coker}(f\colon \mathrm{Hom}(C,\mathbb{Z}) \to \mathrm{Hom}(B,\mathbb{Z})) \\ e_i & \mapsto & [\varphi_i]\end{array}$$

is an isomorphism.

Exercise 108.2. Let $\varphi\colon C_1 \to C_0$ be a homomorphism between two finitely generated free abelian groups.

(a) Suppose that A is an $(n \times m)$-matrix that represents φ. Show that the transpose A^T represents the map $\varphi^*\colon \mathrm{Hom}(C_0, \mathbb{Z}) \to \mathrm{Hom}(C_1, \mathbb{Z})$.

(b) We consider the chain complex $C_* := 0 \to C_1 \xrightarrow{\varphi} C_0 \to 0$. Show that

$$\mathrm{Tor}(\mathrm{H}^1(C_*;\mathbb{Z})) \cong \mathrm{Tor}(\mathrm{H}_1(C)) \quad \text{and} \quad \mathrm{FH}^1(C_*;\mathbb{Z}) \cong \mathrm{FH}_1(C).$$

Remark. Statement (b) can be seen as a special case of the Universal Coefficient Theorem 110.11 for Cohomology Groups that we will prove later on.

Exercise 108.3. Let $X = \{x\}$ be a topological space that consists of a single point x. Show that for any abelian group G we have

$$\mathrm{H}^n(X;G) \;\cong\; \begin{cases} 0, & \text{if } n \geq 1, \\ G, & \text{if } n = 0. \end{cases}$$

Exercise 108.4. Let X be a path-connected topological space and let A be a non-empty subset of X. Show that

$$\mathrm{H}^0(X, A; G) \;=\; 0,$$

using only the definitions of cohomology groups.

Exercise 108.5. Let X be a path-connected topological space and let A and B be two disjoint non-empty subsets of X. Give an explicit example of a map $f\colon X \to \mathbb{Z}$ such that the map

$$\begin{array}{rcl} \mathrm{C}_1(X, A \cup B) & \to & \mathbb{Z} \\ [\sigma\colon \Delta^1 \to X] & \mapsto & f(\sigma((0,1)) - f(\sigma((1,0)) \end{array}$$

is cocycle that represents a non-zero element in $\mathrm{H}^1(X, A; \mathbb{Z})$.

Exercise 108.6. Show that any map $f\colon X \to Y$ between two path-connected non-empty topological spaces induces an isomorphism $f^*\colon \mathrm{H}^0(Y;G) \to \mathrm{H}^0(X;G)$.

Exercise 108.7. Let X be a topological space. Is the 0-th homology group $\mathrm{H}_0(X;\mathbb{Z})$ necessarily isomorphic to the 0-th cohomology group $\mathrm{H}^0(X;\mathbb{Z})$?

Exercise 108.8. We write $I = [0,1]$. As on page 2408 we consider the function

$$\begin{array}{rcl} f\colon I & \to & \mathbb{Z} \\ x & \mapsto & \begin{cases} 0, & \text{if } x \in [0, \frac{1}{2}) \\ 1, & \text{otherwise.} \end{cases} \end{array}$$

This map gives rise to the following singular 1-cochain:

$$\begin{array}{rcl} \rho\colon \mathrm{C}_1(I) & \to & \mathbb{Z} \\ (\sigma\colon \Delta^1 \to I) & \mapsto & f(\sigma(0,1)) - f(\sigma(1,0)). \end{array}$$

As we pointed out on page 2408, the map ρ actually defines a cochain $\rho\colon \mathrm{C}_1(I, \partial I) \to \mathbb{Z}$.

(a) Show that $\rho \in \mathrm{C}_1(I, \partial I)$ is a singular 1-cocycle.
(b) Show that ρ defines a non-trivial element in $\mathrm{H}^1(I, \partial I; \mathbb{Z})$.

Exercise 108.9. Let $X = \mathbb{R}^2$. We consider the singular 2-cochain $\varphi\colon \mathrm{C}_2(\mathbb{R}^2; \mathbb{F}_2)$ that is given by

$$\varphi(\sigma) \;:=\; \begin{cases} 1, & \text{if } (0,0) \text{ lies in the interior of the convex hull} \\ & \text{of the three points } \sigma(1,0,0), \sigma(0,1,0), \sigma(0,0,1), \\ 0, & \text{otherwise.} \end{cases}$$

Is φ a cocycle?

Exercise 108.10. Let $f\colon S^2 \to \mathbb{F}_2$ be a function such that $f(-x) = 1 - f(x)$ for all $x \in S^2$. Given a singular simplex $\tau\colon \Delta^1 \to S^2$ we write $\mu(\tau) = 0$ if f is constant on $\tau(\Delta^1)$ and otherwise we write $\mu(\tau) = 1$. We denote by $p\colon S^2 \to \mathbb{R}\mathrm{P}^2 = S^2/x \sim -x$ the projection . We consider the following cochain

$$\begin{array}{rcl} \varphi\colon \mathrm{C}_2(\mathbb{R}\mathrm{P}^2) & \to & \mathbb{F}_2 \\ (\sigma\colon \Delta^2 \to \mathbb{R}\mathrm{P}^2) & \mapsto & \begin{array}{c}\text{pick a lift } \widetilde{\sigma}\colon \Delta^2 \to S^2 \text{ of } \sigma \text{ and calculate} \\ \mu(\widetilde{\sigma} \circ i_0^2) + \mu(\widetilde{\sigma} \circ i_1^2) + \mu(\widetilde{\sigma} \circ i_2^2).\end{array} \end{array}$$

(a) Show that φ is well-defined, i.e. show that $\varphi(\sigma)$ does not depend on the choice of the lift $\widetilde{\sigma}$.
Remark. Note that if $\widetilde{\sigma}\colon \Delta^2 \to S^2$ is a lift of σ, then the only other one is given by $\nu \circ \widetilde{\sigma}$ where $\nu\colon S^2 \to S^2$ is the map defined by $\nu(x) = -x$ for all $x \in S^2$.
(b) Show that φ is a cocycle in $\mathrm{C}^2(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$.
(c) Does φ represent a non-zero cohomology class?

Exercise 108.11. Suppose $(\mathrm{C}_*, \partial_*)$ and (C'_*, ∂'_*) are two chain complexes. We do *not* assume that the chain groups are necessarily free abelian groups. Suppose that for each $n \in \mathbb{N}_0$ the homology groups $\mathrm{H}_n(C_*)$ and $\mathrm{H}_n(\mathrm{C}'_*)$ are isomorphic. Does it follow that for each $n \in \mathbb{N}_0$ and any abelian group G the cohomology groups $\mathrm{H}^n(\mathrm{C}_*; G)$ and $\mathrm{H}^n(\mathrm{C}'_*; G)$ are also isomorphic?

109. Properties of cohomology groups

After the disappointment of Corollary 108.19 one might be tempted to drop cohomology groups and move on to a different subject. But later on we will see that cohomology groups can be very interesting after all. The key, as we will see, is that cohomology is a contravariant functor whereas homology is a covariant functor.
Therefore we now persevere with the study of cohomology groups. In this chapter we want to extend the following deep theorems and theories from the homological setup to the cohomological setting:

(1) The Excision Theorem 74.17,
(2) the isomorphism between singular homology and cellular homology from the Singular-Cellular H_*-Isomorphism Proposition 80.4,
(3) the Mayer-Vietoris Theorem 78.5.

These theorems will allow us in particular to give many more calculations of cohomology groups.

109.1. Excision theorems for cohomology groups.

We start out with the analogue of the Excision Theorem 74.17.

Theorem 109.1. (Excision Theorem for Cohomology Groups) Let X be a topological space and let G be an abelian group.

(1) Let $Z \subset A \subset X$ be subsets such that the closure of Z is contained in the interior of A. Then the inclusion
$$i\colon (X \setminus Z, A \setminus Z) \quad \to \quad (X, A)$$
induces for each $n \in \mathbb{N}_0$ an isomorphism
$$i^*\colon \mathrm{H}^n(X, A; G) \quad \xrightarrow{\cong} \quad \mathrm{H}^n(X \setminus Z, A \setminus Z; G).$$
(2) Let K be a subset of X and let U be a neighborhood of the closure of K (e.g. let X be Hausdorff, K compact and U open), then the inclusion $(U, U \setminus K) \to (X, X \setminus K)$ induces for each $n \in \mathbb{N}_0$ an isomorphism
$$\mathrm{H}^n(X, X \setminus K; G) \quad \xrightarrow{\cong} \quad \mathrm{H}^n(U, U \setminus K; G).$$

Proof.

(1) In the original Excision Theorem 74.17 we showed that the inclusion induced map
$$i_*\colon \mathrm{H}_n(X \setminus Z, A \setminus Z) \quad \to \quad \mathrm{H}_n(X, A)$$
is an isomorphism for every $n \in \mathbb{N}_0$. But then it follows immediately from Corollary 108.20 that the inclusion induced maps
$$i^*\colon \mathrm{H}^n(X, A; G) \quad \xrightarrow{\cong} \quad \mathrm{H}^n(X \setminus Z, A \setminus Z; G).$$
are also isomorphisms.
(2) By the same argument as in the proof of Theorem 74.18 one can easily deduce Statement (2) from Statement (1). ∎

Remark. Using Corollary 108.20 we also obtain the "cohomological" versions of the various other excision theorems, e.g. Theorems 74.18, 74.31 and 75.10. For example, let X be an m-dimensional topological manifold and let B be a compact m-dimensional submanifold of

X. We write $\mathring{B} := B \setminus \partial_0 B$.[1316] For every $n \in \mathbb{N}_0$ and any abelian group G the inclusion induced map

$$\mathrm{H}^n(X, B; G) \xrightarrow{\cong} \mathrm{H}^n\big(X \setminus \mathring{B}, \partial_0 B; G\big)$$

is an isomorphism.

Psychologically it is often helpful to replace relative homology groups by the homology groups of a quotient space. In the Homology-of-Quotients Proposition 74.20 we had seen that we can perform this switch for reasonable subsets. The following proposition does the same for cohomology groups.

Proposition 109.2. (Cohomology-of-Quotients Proposition) Let G be an abelian group. Let X be a topological space and let $A \subset X$ be a non-empty subset. We denote by $p\colon (X, A) \to (X/A, A/A)$ the natural projection. We consider the following map:

$$\widetilde{\mathrm{H}}^n(X/A; G) \xleftarrow[\cong]{\text{Reduced Cohomology Lemma 108.16 (7)}} \mathrm{H}^n(X/A, A/A; G) \xrightarrow{p^*} \mathrm{H}^n(X, A; G)$$

The following two statements hold:

(1) The composition of the above maps from left to right define a natural transformation from the contravariant functor $(X, A) \mapsto \widetilde{\mathrm{H}}^n(X, A)$ to the contravariant functor $(X, A) \mapsto \mathrm{H}^n(X, A; G)$.

(2) If A is a non-empty good subset[1317], then the above map $\widetilde{\mathrm{H}}^n(X/A; G) \to \mathrm{H}^n(X, A; G)$ is an isomorphism.

Proof. At this stage we have all the tools to rewrite the proof of the Homology-of-Quotients Proposition 74.20. In particular we can use the above Excision Theorem 109.1 for Cohomology Groups to replace the Excision Theorem 74.17. We leave it to the reader to verify that nothing goes wrong. ■

Example. Let G be an abelian group. From the calculation of the cohomology groups of finitely many points, see Lemmas 108.12 and 108.14, we obtain that

$$\mathrm{H}^k(S^0; G) \cong \begin{cases} G^2, & \text{if } k = 0, \text{ and} \\ 0, & \text{if } k \neq 0. \end{cases}$$

Similar to the proof of Proposition 74.4 we can use the Cohomology-of-Quotients Proposition 109.2 to calculate, via an induction argument, that for $n \geq 1$ we have

$$\mathrm{H}^k(S^n; G) \cong \begin{cases} G, & \text{if } k = 0 \text{ or } k = n, \text{ and} \\ 0, & \text{if } k \neq 0, n. \end{cases}$$

We will give an alternative calculation of the cohomology groups of spheres in the next section using cellular cohomology.

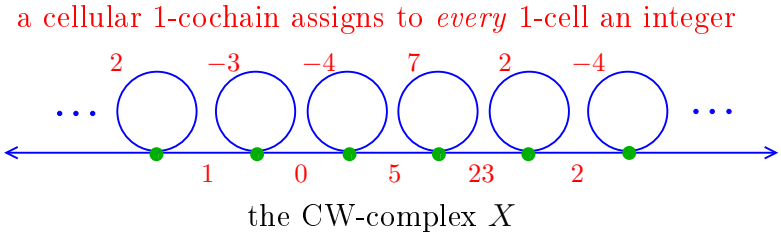
[1316] We refer to page 1658 for the definition of $\partial_0 B$.

[1317] Recall according to the definition on page 1613 this means that A is closed and that there exists an open neighborhood U of A in X such that A is a deformation retract of U.

109.2. **Cellular cohomology.** In Chapter 80 we introduced the cellular homology of a CW-complex. We saw that it is naturally isomorphic to singular homology and we saw that in many examples it is straightforward to compute cellular homology. Now we will repeat the trick and introduce cellular cohomology.
First we recall some definitions and some notation. Let X be a CW-complex.

(1) Recall that given $n \in \mathbb{N}_0$ we denote by X^n the n-skeleton of X.
(2) We write $\mathrm{C}_n^{\mathrm{CW}}(X) := \mathrm{H}_n(X^n, X^{n-1})$ and we denote by $d_n \colon \mathrm{C}_n^{\mathrm{CW}}(X) \to \mathrm{C}_{n-1}^{\mathrm{CW}}(X)$ the cellular boundary map from page 1769.
(3) We refer to
$$\ldots \xrightarrow{d_{n+1}} \mathrm{C}_{n+1}^{\mathrm{CW}}(X) \xrightarrow{d_{n+1}} \mathrm{C}_n^{\mathrm{CW}}(X) \xrightarrow{d_n} \mathrm{C}_{n-1}^{\mathrm{CW}}(X) \xrightarrow{d_{n-1}} \ldots$$
as the *cellular chain complex*.
(4) Let G be an abelian group. We remark that for a finite CW-complex we have
$$\mathrm{Hom}\left(\mathrm{C}_n^{\mathrm{CW}}(X), G\right) \underset{\uparrow}{\cong} \text{maps from the set of } n\text{-cells to } G \;=\; G^{\#\, n\text{-cells}}.$$
Lemmas 80.1 and 108.1
We refer to elements in $\mathrm{Hom}\left(\mathrm{C}_n^{\mathrm{CW}}(X), G\right)$ as *cellular n-cochains*.

Example. Given a CW-complex X an integral cellular 1-cochain in $\mathrm{Hom}\left(\mathrm{C}_1^{\mathrm{CW}}(X), \mathbb{Z}\right)$ assigns to *every* 1-cell an integer. We refer to the figure below for an illustration. In contrast an element in $\mathrm{C}_1^{\mathrm{CW}}(X)$ can be viewed as a formal linear combination of 1-cells which is the same as saying that an element in $\mathrm{C}_1^{\mathrm{CW}}(X)$ assigns a non-zero integer to only *finitely many* 1-cells.



the CW-complex X

Definition. Let G be an abelian group and let $n \in \mathbb{N}_0$.

(1) Let X be a CW-complex. We write $f_n := d_{n+1}^*$. We refer to
$$\cdots \leftarrow \mathrm{Hom}\left(\mathrm{C}_{n+1}^{\mathrm{CW}}(X), G\right) \xleftarrow{f_n} \mathrm{Hom}\left(\mathrm{C}_n^{\mathrm{CW}}(X), G\right) \leftarrow \mathrm{Hom}\left(\mathrm{C}_{n-1}^{\mathrm{CW}}(X), G\right) \xleftarrow{f_{n-1}} \ldots$$
as the cellular cochain complex. We define the n-th cellular cohomology group of X with G-coefficients as
$$\mathrm{H}_{\mathrm{CW}}^n(X; G) := \mathrm{H}^n\left(\mathrm{C}_*^{\mathrm{CW}}(X), G\right) = \frac{\ker\left(f_n \colon \mathrm{Hom}(\mathrm{C}_n^{\mathrm{CW}}(X), G) \to \mathrm{Hom}(\mathrm{C}_{n+1}^{\mathrm{CW}}(X), G)\right)}{\mathrm{im}\left(f_{n-1} \colon \mathrm{Hom}(\mathrm{C}_{n-1}^{\mathrm{CW}}(X), G) \to \mathrm{Hom}(\mathrm{C}_n^{\mathrm{CW}}(X), G)\right)}.$$
(2) If $f \colon X \to Y$ is a cellular map, then by Lemma 80.3 we get an induced chain map $f_* \colon \mathrm{C}_*^{\mathrm{CW}}(X) \to \mathrm{C}_*^{\mathrm{CW}}(Y)$ and thus we obtain from the Dual Cochain Map Lemma 108.8 (1) an induced map $f^* \colon \mathrm{H}_{\mathrm{CW}}^n(Y; G) \to \mathrm{H}_{\mathrm{CW}}^n(X; G)$.

For completeness' sake we jot down the following fairly self-evident lemma.

Lemma 109.3. Let G be an abelian group and let $n \in \mathbb{N}_0$. The maps

$$X \mapsto \mathrm{H}^n_{\mathrm{CW}}(X;G) \quad \text{and} \quad (f\colon X \to Y) \mapsto (f^*\colon \mathrm{H}^n_{\mathrm{CW}}(Y;G) \to \mathrm{H}^n_{\mathrm{CW}}(X;G))$$

define a contravariant functor from the category of CW-complexes to the category of abelian groups.

Proof. This lemma follows immediately from Lemmas 80.3 and 108.8 (2). ■

Perhaps not surprisingly we have the following theorem which plays a similar role as the Singular-Cellular H_*-Isomorphism Proposition 80.4.

Proposition 109.4. (Singular-Cellular H^*-Isomorphism Proposition) For every CW-complex X, any abelian group G and any $n \in \mathbb{N}_0$ there exists a natural isomorphism[1318]

$$\mathrm{H}^n(X;G) \xrightarrow{\cong} \mathrm{H}^n_{\mathrm{CW}}(X;G).$$

Remark. An alternative proof of this theorem is given in [**Hat2002**, p. 203].

First proof of Proposition 109.4. Let X be a CW-complex. Recall that on page 1801 we introduced the intermediate cellular chain complex that is given by

$$\mathrm{C}^{\mathrm{int}}_n(X) := \ker\big(\mathrm{C}_n(X^n) \xrightarrow{\partial} \mathrm{C}_{n-1}(X^n) \to \mathrm{C}_{n-1}(X^n, X^{n-1})\big) \quad \text{for } n \in \mathbb{N}_0.$$

We consider the two *natural* chain maps

$$i\colon \mathrm{C}^{\mathrm{int}}_*(X) \hookrightarrow \mathrm{C}_*(X) \quad \text{and} \quad p\colon \mathrm{C}^{\mathrm{int}}_*(X) \twoheadrightarrow \mathrm{C}^{\mathrm{CW}}_*(X),$$

which are given by the obvious inclusion and the obvious projection. In the Singular-Cellular Chain Complex Proposition 81.12 we had seen that these two chain maps are chain homotopy equivalences. Thus it follows from the Dual Cochain Map Lemma 108.8 (4) that we get isomorphisms

$$\mathrm{H}^n(X;G) = \mathrm{H}^n(\mathrm{C}_*(X);G) \xrightarrow[\cong]{i^*} \mathrm{H}^n(\mathrm{C}^{\mathrm{int}}_*(X);G) \xleftarrow[\cong]{p^*} \mathrm{H}^n(\mathrm{C}^{\mathrm{CW}}_*(X);G) = \mathrm{H}^n_{\mathrm{CW}}(X;G).$$

As pointed out above, the two chain maps i and p are natural. Therefore we see that the above isomorphism $\mathrm{H}^n(X;G) \to \mathrm{H}^n_{\mathrm{CW}}(X;G)$ is in fact natural. ■

Alternative proof of Proposition 109.4 without "natural". Let X be a CW-complex. We denote by $\mathrm{C}_* = \mathrm{C}_*(X)$ the singular chain complex and we denote by $D_* = \mathrm{C}^{\mathrm{CW}}_*(X)$ the cellular chain complex. In the Singular-Cellular H_*-Isomorphism Proposition 80.4 we showed that all the homology groups of the chain complexes C_* and D_* are isomorphic. It follows from Proposition 108.18 that for any abelian group G and any $n \in \mathbb{N}_0$ the cohomology groups $\mathrm{H}^n(X;G) = \mathrm{H}^n(\mathrm{C}_*;G)$ and $\mathrm{H}^n_{\mathrm{CW}}(X;G) = \mathrm{H}^n(D_*;G)$ are isomorphic. This approach to proving the Singular-Cellular H^*-Isomorphism Proposition 109.4 circumvents the use of the Singular-Cellular Chain Complex Proposition 81.12. But it is not clear how this approach can be used to show that there exists a *natural* isomorphism. Of course for many applications this is not an issue. ■

[1318]Here "natural" means that there exists a natural transformation between the contravariant functors $X \mapsto \mathrm{H}^n(X;G)$ and $X \mapsto \mathrm{H}^n_{\mathrm{CW}}(X;G)$ from the category of CW-complexes to the category of abelian groups.

Examples. Let G be an abelian group.

(1) We use cellular cohomology to once again compute the cohomology groups of spheres. As we saw on page 1472 we can view S^n, $n \geq 1$, as a CW-complex with one 0-cell and one other cell of dimension n. Furthermore the cellular boundary maps are the zero maps.[1319] Thus we see that the cellular chain complex is of the form

$$0 \to \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \to 0.$$

The dual cochain complex is then given by

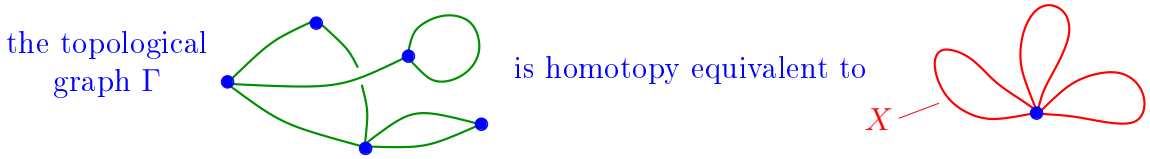
$$0 \leftarrow \underbrace{\mathrm{Hom}(\mathbb{Z}, G)}_{=G} \xleftarrow{0} \cdots \xleftarrow{0} \underbrace{\mathrm{Hom}(\mathbb{Z}, G)}_{=G} \leftarrow 0.$$

Summarizing, precisely as on page 2419 we obtain that

$$\mathrm{H}^k(S^n; G) \cong \begin{cases} G, & \text{if } k = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Let Γ be a connected finite topological graph. It follows from Proposition 50.15 (2) that Γ is homotopy equivalent to a CW-complex X with one 0-cell and $\chi(\Gamma) + 1$ 1-cells, see the figure below. The unique cellular boundary map in the cellular chain complex of X is zero by Lemma 80.7. Summarizing we see that for any $n \in \mathbb{N}_0$ we have

$$\mathrm{H}^n(\Gamma; G) \underset{\text{Homotopic Maps-H}^*\text{-Lemma 108.13}}{\overset{\uparrow}{\cong}} \mathrm{H}^n(X; G) \underset{\text{Singular-Cellular H}^*\text{-Isomorphism Proposition 109.4}}{\overset{\downarrow}{\cong}} \mathrm{H}^n_{\mathrm{CW}}(X; G) \underset{\text{since the coboundary map is zero}}{\overset{\uparrow}{\cong}} \mathrm{Hom}(\mathrm{C}^{\mathrm{CW}}_n(X); G) \cong \begin{cases} G, & \text{if } n=0, \\ G^{\chi(\Gamma)+1}, & \text{if } n=1, \\ 0, & \text{if } n \geq 2. \end{cases}$$



(3) Given $g \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$ we denote by $\Sigma_{g,m}$ the surface of genus g minus m open disks. On page 2164 we had argued that for $m \geq 1$ the topological space $\Sigma_{g,m}$ is homotopy equivalent to the wedge of $2g + m - 1$ circles. The previous example, together with the Homotopic Maps-H*-Lemma 108.13, thus gives us the calculation of the cohomology groups of $\Sigma_{g,m}$ for $m \geq 1$.

(4) Let Σ_g be the surface of genus g. In the figure on page 1472 we showed that we can view Σ_g as a CW-complex with one 0-cell, $2g$ 1-cells and one 2-cell. On page 1780 we saw that the cellular boundary maps are all zero. Hence for any abelian group G and any $n \in \mathbb{N}_0$ we have

$$\mathrm{H}^n(\Sigma_g; G) \underset{\text{Singular-Cellular H}^*\text{-Isomorphism Proposition 109.4}}{\overset{\uparrow}{\cong}} \mathrm{H}^n_{\mathrm{CW}}(\Sigma_g; G) = \mathrm{Hom}\left(\mathrm{C}^{\mathrm{CW}}_n(\Sigma_g); G\right) \cong \begin{cases} G, & \text{if } n = 0 \text{ or } n = 2, \\ G^{2g}, & \text{if } n = 1, \text{ and} \\ 0, & \text{if } n \neq 0, 1, 2. \end{cases}$$

[1319] For $n \geq 2$ this follows by definition, for $n = 1$ this is a consequence of Lemma 80.7.

In all the examples so far the cohomology groups are isomorphic to the corresponding homology groups that we had calculated earlier. Now we will see an example where the homology and cohomology groups differ.

Examples.

(1) We consider the n-dimensional projective space $\mathbb{R}\mathrm{P}^n$. In the proof of Proposition 80.11 we saw that there exists a CW-structure on $\mathbb{R}\mathrm{P}^n$ such that the corresponding cellular chain complex is of the following form:

$$0 \to \underset{\substack{\uparrow \\ \text{degree } n}}{\mathbb{Z}} \xrightarrow{\cdot(1+(-1)^n)} \mathbb{Z} \to \ldots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

which allowed us to compute the homology groups of $\mathbb{R}\mathrm{P}^n$. Now let G be an abelian group. We dualize this chain complex, i.e. we apply the functor $\mathrm{Hom}(-, G)$. Using the isomorphism $\mathrm{Hom}(\mathbb{Z}, G) \to G$ from Lemma 108.1 we see that the cochain complex is isomorphic to the following cochain complex:

$$0 \leftarrow G \xleftarrow{\cdot(1+(-1)^n)} G \leftarrow \ldots \xleftarrow{0} G \xleftarrow{\cdot 2} G \xleftarrow{0} G \leftarrow 0.$$

For $G = \mathbb{Z}$ we thus obtain from the Singular-Cellular H^*-Isomorphism Proposition 109.4 that for any $k \in \{0, \ldots, n\}$ we have

$$\mathrm{H}^k(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is even and } k \neq 0, \\ 0, & \text{if } k \text{ is odd and } k \neq n, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd} \end{cases} \quad \text{but } \mathrm{H}_k(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd and } k \neq n, \\ 0, & \text{if } k \text{ is even and } k \neq 0, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd} \end{cases}$$

where on the right-hand side we recall the calculation of the homology groups using the above cellular chain complex, see also Proposition 80.11. Therefore we see that for $n \geq 2$ the isomorphism types of the homology groups and the cohomology groups of $\mathbb{R}\mathrm{P}^n$ with $\mathbb{Z}$-coefficients do *not* coincide.

(2) To add to the confusion we also compute the cohomology of $\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients. The corresponding dual cellular cochain complex is then given by

$$0 \to \mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2 \leftarrow \ldots \leftarrow \mathbb{F}_2 \xleftarrow{\cdot 2=0} \mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2 \to 0$$

and we see that

$$\mathrm{H}^k(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } k = 0, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

If we compare this result with the calculation on page 1916 of the homology of $\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients we see that we get the same result. More precisely, the homology groups and the cohomology groups of $\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients are isomorphic.

One can easily modify the above argument and apply it to $\mathbb{R}\mathrm{P}^\infty$, which we can view as a CW-complex with precisely one cell in each dimension, see 1480. We then obtain easily that for each $k \in \mathbb{N}_0$ we have

$$\mathrm{H}^k(\mathbb{R}\mathrm{P}^\infty; \mathbb{F}_2) \cong \mathbb{F}_2$$

(3) We recall that in Lemma 68.1 we showed that we can view $\mathbb{C}\mathrm{P}^n$ as a CW-complex which admits exactly one cell in dimensions $0, 2, \ldots, 2n$ and which admits no odd-dimensional cells. It follows easily from the Singular-Cellular H^*-Isomorphism Proposition 109.4 that

$$\mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \ldots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 68.6 (2) we can view the infinite-dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty$ as a CW-complex that has precisely one cell in each even dimension and no cells in odd dimensions. Using the Singular-Cellular H^*-Isomorphism Proposition 109.4 we obtain that

$$\mathrm{H}^k(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k \text{ even}, \\ 0, & \text{if } k \text{ odd}. \end{cases}$$

Given $k < l \in \mathbb{N} \cup \{\infty\}$ we have an inclusion $\mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^l$. Since the isomorphism of the Singular-Cellular H^*-Isomorphism Proposition 109.4 is natural and since the cellular chain complexes of $\mathbb{C}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^\infty$ agree up to the dimension $2n+1$ we obtain almost immediately that the inclusion induced maps $\mathrm{H}^k(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \to \mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ are actually isomorphisms up to dimension $2n$.

We conclude this section with the following fairly reasonably immediate corollary to the Singular-Cellular H^*-Isomorphism Proposition 109.4. This corollary can be viewed as an analogue of the Homology-of-CW-Complex Proposition 80.6 and Proposition 96.6.

Corollary 109.5. Let X be a topological space. Suppose we are in one of the following two situations:

(1) X is homotopy equivalent to a retract of a finite CW-complex,
(2) X is a compact topological manifold.

Then for every $n \in \mathbb{N}_0$ the cohomology group $\mathrm{H}^n(X;\mathbb{Z})$ is a finitely generated group.

Proof. Statement (2) is an immediate consequence of Statement (1) and the Topological Manifold-Simplicial Complex Retract Theorem 104.13. We will prove Statement (1) in Exercise 109.1. ■

109.3. **Explicit generators of cohomology groups.** In the previous section we had used cellular cohomology to compute the isomorphism types of the cohomology groups of many different topological spaces. But this calculation of the isomorphism types does not give us any explicit generators of the cohomology groups. For example on page 2404 we introduced a singular 1-cochain $\theta_{\mathbb{Z}} \colon \mathrm{C}_1(S^1) \to \mathbb{Z}$ and we showed in Lemma 108.10 that $\theta_{\mathbb{Z}}$ defines a non-zero cohomology class $[\theta_{\mathbb{Z}}] \in \mathrm{H}^1(S^1;\mathbb{Z})$. By the discussion on page 2422 we now also know that $\mathrm{H}^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$, but we do not know whether $[\theta_{\mathbb{Z}}] \in \mathrm{H}^1(S^1;\mathbb{Z})$ is a generator.

On the other hand in earlier chapters we had obtained a pretty good understanding of generators of homology groups. The idea now is to translate information on homology groups to information on cohomology groups. The key tool relating homology and cohomology is given by the Kronecker pairing that we now introduce.

Lemma 109.6. Let C_* be a chain complex.

(1) Let G be an abelian group. The map

$$\begin{array}{rcl} \langle\ ,\ \rangle\colon \mathrm{H}^n(C;G) \times \mathrm{H}_n(C) & \to & G \\ ([\varphi],[c]) & \mapsto & \langle [\varphi],[c]\rangle := \underset{\uparrow}{\varphi(c)} \end{array}$$

here $\varphi \in \mathrm{Hom}(C_n, G)$ and $c \in C_n$, so $\varphi(c) \in G$ is defined

is well-defined and bilinear.[1320] In particular, if (X, A) is a pair of topological spaces, then the map

$$\begin{array}{rcl} \langle\ ,\ \rangle_X\colon \mathrm{H}^n(X,A;G) \times \mathrm{H}_n(X,A) & \to & G \\ ([\varphi],[c]) & \mapsto & \langle [\varphi],[c]\rangle_X := \varphi(c) \end{array}$$

is well-defined and bilinear. We refer to both pairings as the Kronecker pairing.[13211322]

(2) If $\varphi\colon G_1 \to G_2$ is a group homomorphism then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^n(C;G_1) \times \mathrm{H}_n(C) & \xrightarrow{\langle -,-\rangle} & G_1 \\ \downarrow{\scriptstyle \varphi_* \times \mathrm{id}} & & \downarrow{\scriptstyle \varphi} \\ \mathrm{H}^n(C;G_2) \times \mathrm{H}_n(C) & \xrightarrow{\langle -,-\rangle} & G_2. \end{array}$$

(3) Let $f\colon C_* \to D_*$ be a chain map between two chain complexes and let G be an abelian group. Then for any $c \in \mathrm{H}_n(C)$ and $\varphi \in \mathrm{H}^n(D;G)$ we have

$$\langle f^*(\varphi), c\rangle \ = \ \langle \varphi, f_*(c)\rangle.$$

In particular, suppose that $f\colon (X,A) \to (Y,B)$ is a map between two pairs of topological spaces. Let $c \in \mathrm{H}_n(X,A)$ and let $\varphi \in \mathrm{H}^n(Y,B;G)$. Then

$$\langle f^*(\varphi), c\rangle_X \ = \ \langle \varphi, f_*(c)\rangle_Y.$$

Proof. Let C_* be a chain complex.

(1) Let G be an abelian group. Let $\varphi \in \mathrm{Hom}(C_n, G)$ be a cocycle and let $c \in C_n$ be a cycle. Furthermore let $\alpha \in \mathrm{Hom}(C_{n-1}, G)$ and let $d \in \mathrm{C}_{n+1}$. Then the following equalities hold in G:

$$(\varphi + \delta\alpha)(c) \underset{\substack{\uparrow \\ \text{definition of } \delta}}{=} \varphi(c) + \alpha(\partial c) \underset{\substack{\uparrow \\ \text{since } c \text{ is a cycle}}}{=} \varphi(c) \quad \text{and} \quad \varphi(c + \partial d) \underset{\substack{\uparrow \\ \text{definition of } \delta}}{=} \varphi(c) + (\delta\varphi)(d) \underset{\substack{\uparrow \\ \text{since } \varphi \text{ is a cocycle}}}{=} \varphi(c).$$

This calculation shows that the definition of the Kronecker pairing does not depend on the choices of the representatives of the cohomology and the homology classes. Finally note that it is clear that the Kronecker pairing is bilinear.

[1320] In general, given a commutative ring R a map $\Theta\colon U \times V \to W$ between R-modules is called *bilinear*, if for all $u, u' \in U$, $v, v' \in V$ and $r \in R$ we have

$$\begin{array}{rclcrcll} \Theta(u+u',v) & = & \Theta(u,v) + \Theta(u',v) & \text{and} & \Theta(ru,v) & = & r\Theta(u,v), & \text{and} \\ \Theta(u,v+v') & = & \Theta(u,v) + \Theta(u,v') & \text{and} & \Theta(u,rv) & = & r\Theta(u,v). & \end{array}$$

In our special case we consider the case of $\mathbb{Z}$-modules.

[1321] Often we drop the subscript X in the notation.

[1322] Leopold Kronecker (1823–1891) was a German mathematician who worked on number theory. I don't know how his name got attached to this pairing.

(2) This statement follows immediately from the definitions.
(3) The statement also follows immediately from the definitions. Indeed, if $[c] \in \mathrm{H}_n(C)$ and $[\varphi] \in \mathrm{H}^n(D;G)$, then

$$\langle f^*([\varphi]),[c]\rangle \;=\; \langle [f^*\varphi],[c]\rangle \;=\; (f^*\varphi)(c) \;=\; \varphi(f_*(c)) \;=\; \langle [\varphi],[f_*(c)]\rangle \;=\; \langle [\varphi], f_*([c])\rangle.$$

The statement regarding maps between pairs of topological spaces is an immediate consequence of the purely algebraic statement. ■

Now we state the lemma which will allow us in many situations to show that given cocycles form a basis for cohomology groups.

Lemma 109.7. (Kronecker-Basis Lemma) Let X be a topological space and let $n \in \mathbb{N}_0$. Suppose that $\mathrm{H}^n(X;\mathbb{Z})$ and $\mathrm{H}_n(X;\mathbb{Z})$ are free abelian groups of rank k and suppose we are given $\varphi_1,\ldots,\varphi_k \in \mathrm{H}^n(X;\mathbb{Z})$ and $c_1,\ldots,c_k \in \mathrm{H}_n(X;\mathbb{Z})$. If

$$\det\Big(\underbrace{\langle \varphi_i, c_j\rangle_{i,j=1,\ldots,k}}_{=\,k\times k\text{–matrix}}\Big) \;=\; \pm 1,$$

then $\varphi_1,\ldots,\varphi_k \in \mathrm{H}^n(X;\mathbb{Z})$ and $c_1,\ldots,c_k \in \mathrm{H}_n(X;\mathbb{Z})$ are both bases.

Example. Let $X = S^1$. We denote by $\theta_{\mathbb{Z}}\colon \mathrm{C}_1(S^1) \to \mathbb{Z}$ the singular 1-cocycle that we introduced on page 2404. We consider the singular cycle $\mu\colon \Delta^1 \to S^1$ that is given by $(1-t,t) \mapsto \exp(2\pi\,\mathrm{i}t)$. As on page 2404 we see that

$$\langle [\theta_{\mathbb{Z}}],[\mu]\rangle \;=\; \theta_{\mathbb{Z}}(\mu) \;=\; \alpha_{\mathbb{Z}}\left(\text{boundary of lift } \begin{array}{ccc} \widetilde{\mu}\colon \Delta^1 & \to & \mathbb{R} \\ (1-t,t) & \mapsto & t \end{array} \text{ of } \mu\right) \;=\; 1.$$

From Proposition 74.4 we know that $\mathrm{H}_1(S^1;\mathbb{Z}) \cong \mathbb{Z}$ and by the discussion on page 2419 we know that $\mathrm{H}^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$. Therefore it follows from the Kronecker-Basis Lemma 109.7 that $[\theta_{\mathbb{Z}}] \in \mathrm{H}^1(S^1;\mathbb{Z})$ is in fact a generator.[1323] In particular we have an equality $\mathrm{H}^1(S^1;\mathbb{Z}) = \mathbb{Z}\cdot[\theta_{\mathbb{Z}}]$.

Remark. the Kronecker-Basis Lemma 109.7 is really quite generous, it gives us as output that both sets are bases. In particular to apply the lemma we do not even have to verify that the c_i form a basis.

The Kronecker-Basis Lemma 109.7 is an immediate consequence of the following purely algebraic lemma.

Lemma 109.8. Let A and B be two free abelian groups of rank k, let $a_1,\ldots,a_k \in A$ and let $b_1,\ldots,b_k \in B$. Furthermore let $\langle\ ,\ \rangle\colon A\times B \to \mathbb{Z}$ be a bilinear pairing. If

$$\det(\langle a_i,b_j\rangle_{i,j=1,\ldots,k}) \;=\; \pm 1,$$

then both $a_1,\ldots,a_k$ and $b_1,\ldots,b_k$ are bases.

Proof. We denote by S the free abelian group generated by the set $\{a_1,\ldots,a_k\}$ and we denote by T the free abelian group generated by the set $\{b_1,\ldots,b_k\}$. We denote by $i\colon S \to A$ and by $j\colon T \to B$ the obvious maps given by $a_i \mapsto a_i$ and $b_i \mapsto b_i$. We denote by

[1323] It also follows from the Kronecker-Basis Lemma 109.7 that the singular 1-simplex $\mu\colon \Delta^1 \to S^1$ represents a generator for $\mathrm{H}_1(S^1) \cong \mathbb{Z}$. We had obtained that result in Lemma 76.4.

$\Phi\colon A \to \operatorname{Hom}(B,\mathbb{Z})$ the homomorphism given by $a \mapsto (b \mapsto \langle a,b\rangle)$.[1324] The claim follows from the following three observations:

(1) Let $\gamma\colon G \to H$ be a homomorphism between two finitely generated free abelian groups of the same rank. Lemma 51.9 (5) says that if γ is an epimorphism, then γ it is already an isomorphism.

(2) We consider the following sequence of homomorphisms

$$S \xrightarrow{i} A \xrightarrow{\Phi} \operatorname{Hom}(B,\mathbb{Z}) \xrightarrow{j^*} \operatorname{Hom}(T,\mathbb{Z}).$$

With respect to the basis $a_1,\dots,a_k$ of S and the dual basis $b_1^*,\dots,b_k^*$ of $\operatorname{Hom}(T,\mathbb{Z})$ this linear map is represented by the matrix $\langle a_i,b_j\rangle_{i,j=1,\dots,k}$. Thus by our hypotheses this map is an isomorphism. Furthermore by our hypotheses each of the individual groups is a free abelian group of rank k.

(3) Since the map in (2) is an isomorphism it is in particular an epimorphism. Starting from the right and using (1) three times in a row we see that all maps are in fact isomorphisms. In particular we have $S = A$. The same argument turned around shows that $T = B$. ■

Now we will use the Kronecker-Basis Lemma 109.7 to determine explicit bases for the first cohomology groups of surfaces. We start out with the torus.

Lemma 109.9. We consider the two projection s

$$\begin{array}{rcl} p\colon S^1\times S^1 & \to & S^1 \\ (v,w) & \mapsto & v \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon S^1\times S^1 & \to & S^1 \\ (v,w) & \mapsto & w. \end{array}$$

Then $p^*([\theta_{\mathbb{Z}}])$ and $q^*([\theta_{\mathbb{Z}}])$ form a basis for $\mathrm{H}^1(S^1\times S^1;\mathbb{Z}) \cong \mathbb{Z}^2$.

Proof. We denote as usual by $\mu\colon \Delta^1 \to S^1$ our favorite singular 1-cycle of S^1 that is given by $\mu(1-t,t) = \exp(2\pi i t)$. For simplicity we write $\theta = \theta_{\mathbb{Z}}$. We denote by

$$\begin{array}{rcl} i\colon S^1 & \to & S^1\times S^1 \\ z & \mapsto & (z,1) \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon S^1 & \to & S^1\times S^1 \\ z & \mapsto & (1,z) \end{array}$$

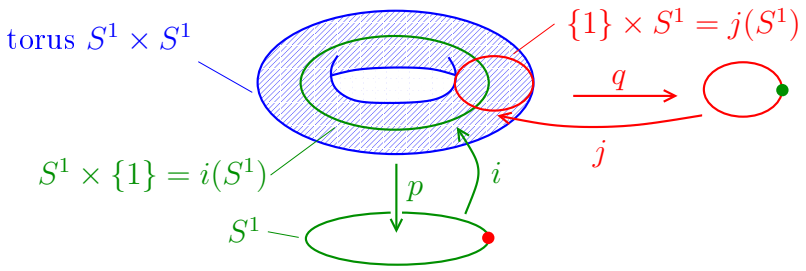
the two obvious inclusion maps. We calculate that

$$\begin{pmatrix} \langle [p^*\theta],[i_*(\mu)]\rangle & \langle [p^*\theta],[j_*(\mu)]\rangle \\ \langle [q^*\theta],[i_*(\mu)]\rangle & \langle [q^*\theta],[j_*(\mu)]\rangle \end{pmatrix} \overset{\text{since } \langle [p^*\theta],[i_*(\mu)]\rangle = (p^*\theta)(i_*(\mu)) = \theta(p_*(i_*(\mu)))\text{ and similarly for the other terms}}{\downarrow} = \begin{pmatrix} \theta(p_*(i_*(\mu))) & \theta(p_*(j_*(\mu))) \\ \theta(q_*(i_*(\mu))) & \theta(q_*(j_*(\mu))) \end{pmatrix} \underset{\text{since } p\circ i = \mathrm{id} \text{ and } q\circ j = \mathrm{id} \text{ and since } p\circ j \equiv 1 \in S^1 \text{ and } q\circ i \equiv 1 \in S^1 \text{ here we denote by 1 also the constant singular 1-simplex given by } (1-t,t)\mapsto 1}{\uparrow} = \begin{pmatrix} \theta(\mu) & \theta(1) \\ \theta(1) & \theta(\mu) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We know from Lemma 78.11 that $\mathrm{H}_1(S^1\times S^1;\mathbb{Z}) \cong \mathbb{Z}^2$. Furthermore we saw on page 2422 that $\mathrm{H}^1(S^1\times S^1;\mathbb{Z}) \cong \mathbb{Z}^2$. The above calculation, together with the Kronecker-Basis Lemma 109.7 shows that $p^*([\theta_{\mathbb{Z}}])$ and $q^*([\theta_{\mathbb{Z}}])$ form a basis for $\mathrm{H}^1(S^1\times S^1;\mathbb{Z})$. ■

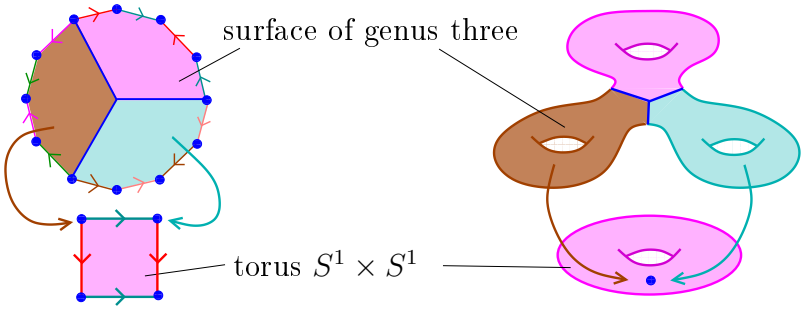
Together with the H^0-Proposition 108.11 we have now found explicit generators for the cohomology groups of the torus in dimensions 0 and 1. The following question arises:

[1324] The fact that $\langle\ ,\ \rangle$ is bilinear implies that $b \mapsto \langle a,b\rangle$ is a homomorphism and that $\Phi\colon A \to \operatorname{Hom}(B,\mathbb{Z})$ is a homomorphism.



Question 109.10. Can we find an explicit generator of $\mathrm{H}^2(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z}$?

Now we turn to surfaces of higher genus. As usual, given $g \in \mathbb{N}$ we denote by Σ_g the surface of genus g. In the figure below we show projections $p_i \colon \Sigma_g \to S^1 \times S^1$, $i = 1, \dots, g$.[1325] The following lemma is a generalization of Lemma 109.9.

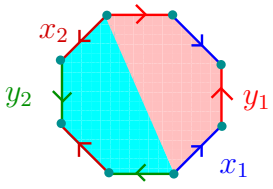


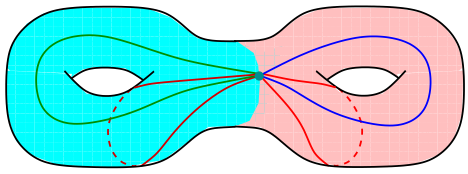
Lemma 109.11. We denote by $\alpha, \beta \in \mathrm{H}^1(S^1 \times S^1; \mathbb{Z})$ the basis given by Lemma 109.9. A basis for $\mathrm{H}^1(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}^{2g}$ is given by $p_i^*(\alpha), p_i^*(\beta)$, $i = 1, \dots, g$.

Proof. As we will see, the proof of the lemma is very similar to the proof of Lemma 109.9. For $i = 1, \dots, g$ we write $\varphi_i = p_i^*(\alpha)$ and $\psi_i := p_i^*(\beta)$. We consider the singular 1-cycles $x_1, \dots, x_g$ and $y_1, \dots, y_g$ that are indicated in the figure below. As in the proof of Lemma 109.9 it follows easily from the definitions that (for $g = 2$) we have

$$\begin{pmatrix} \langle \varphi_1, x_1 \rangle & \langle \varphi_1, y_1 \rangle & \langle \varphi_1, x_2 \rangle & \langle \varphi_1, y_2 \rangle \\ \langle \psi_1, x_1 \rangle & \langle \psi_1, y_1 \rangle & \langle \psi_1, x_2 \rangle & \langle \psi_1, y_2 \rangle \\ \langle \varphi_2, x_1 \rangle & \langle \varphi_2, y_1 \rangle & \langle \varphi_2, x_2 \rangle & \langle \varphi_2, y_2 \rangle \\ \langle \psi_2, x_1 \rangle & \langle \psi_2, y_1 \rangle & \langle \psi_2, x_2 \rangle & \langle \psi_2, y_2 \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The lemma now follows immediately from the Kronecker-Basis Lemma 109.7. ∎





109.4. Excisive triads and the Mayer–Vietoris Theorem for cohomology groups.

Before we state and prove the Mayer–Vietoris Theorem for cohomology groups it is helpful to introduce the following definition.

[1325] In fact for $g = 2$ we already saw this map in the figure on page 1064.

Definition.

(1) A triad of topological spaces is a triple (X, A_1, A_2) where X is a topological space and where both A_1 and A_2 are subsets of X.[1326]

(2) A map $f\colon (X, A_1, A_2) \to (Y, B_1, B_2)$ of triads is a map $f\colon X \to Y$ with $f(A_1) \subset B_1$ and $f(A_2) \subset B_2$.

(3) Given a triad (X, A_1, A_2) of topological spaces we consider the following subgroup of $\mathrm{C}_n(A_1 \cup A_2)$:[1327]

$$\mathrm{C}_n^{\{A_1,A_2\}}(A_1 \cup A_2) := \left\{ \sum_{j=1}^{k} a_j \sigma_j \in \mathrm{C}_n(A_1 \cup A_2) \;\middle|\; \begin{array}{l} \text{for each } j \text{ there exists an } i \in \{1,2\} \\ \text{such that the image of } \sigma_j \text{ lies in } A_i \end{array} \right\}.$$

The usual boundary map on the groups $\mathrm{C}_*(A_1 \cup A_2)$ restricts to a boundary map on $\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2)$, so we can view $\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2)$ as a chain complex. We denote the corresponding homology groups by $\mathrm{H}_*^{\{A_1,A_2\}}(A_1 \cup A_2)$.

(4) We say a triad (X, A_1, A_2) is excisive if the inclusion map

$$\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2) \;\to\; \mathrm{C}_*(A_1 \cup A_2)$$

is a chain homotopy equivalence of chain complexes.

Example. In Exercise 109.2 we will see that the triad $(X, A_1, A_2) := (\mathbb{R}, (-\infty, 0], (0, \infty))$ is *not* excisive.

After this negative example we want to collect some positive examples of excisive triads.

Proposition 109.12. (Excisive Triad-Proposition) Let (X, A_1, A_2) be a triad of topological spaces. The triad is excisive if one of the following conditions is satisfied:

(1) $A_1 \subset A_2$ or $A_2 \subset A_1$,

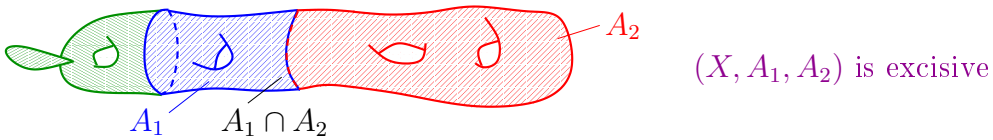
(2) $A_1 = \varnothing$ or $A_2 = \varnothing$ or $A_1 = A_2$,

(3) $A_1 \cup A_2 = \mathring{A}_1 \cup \mathring{A}_2$ where we denote by $\mathring{A}_1$ and $\mathring{A}_2$ the interiors of A_1 and A_2 in $A_1 \cup A_2$,

(4) A_1 and A_2 are both open subsets of $A_1 \cup A_2$,

(5) $A_1 \cup A_2$ is a topological manifold and A_1 and A_2 are codimension-zero submanifolds which are both closed subsets and such that each component of $A_1 \cap A_2$ is a boundary component of A_1 and a boundary component of A_2,[1328]

(6) X is a CW-complex and A_1 and A_2 are both subcomplexes.



In the proof of the Excisive Triad-Proposition 109.12 we will need the following lemma which gives us several criteria for a triad to be excisive.

[1326] This definition should not be confused with the definition of a *triple* of topological spaces, which was defined as a triple (X, B, A) where X is a topological space and $A \subset B \subset X$ are subsets. In a triad we do not demand that the third set is contained in the second set.

[1327] The definition is basically the same as on page 1638 except that this time we do not demand that the interiors of A_1 and A_2 cover all of X.

[1328] In Exercise 109.5 we will state a generalization of this statement.

Lemma 109.13. (Excisive Characterization Lemma) Let (X, A_1, A_2) be a triad of topological spaces. The following statements are equivalent:

(1) the triad is excisive,
(2) for every $n \in \mathbb{N}_0$ the inclusion induced map
$$\mathrm{H}_n^{\{A_1,A_2\}}(A_1 \cup A_2) \;\to\; \mathrm{H}_n(A_1 \cup A_2)$$
is an isomorphism,
(3) for every $n \in \mathbb{N}_0$ the inclusion induced map
$$\mathrm{H}_n(A_1, A_1 \cap A_2) \;\to\; \mathrm{H}_n(A_1 \cup A_2, A_2)$$
is an isomorphism.

Now suppose that the triad is excisive.

(a) Given any abelian group G the statements (2) and (3) also hold for G-coefficients.
(b) Given any abelian group G the analogue of (2) also holds for cohomology groups, i.e. given any $n \in \mathbb{N}_0$ and given any abelian group G the inclusion induced map $\mathrm{H}^n(A_1 \cup A_2, A_2; G) \to \mathrm{H}^n(A_1, A_1 \cap A_2; G)$ is an isomorphism.

Proof. Let (X, A_1, A_2) be a triad of topological spaces. The equivalence of (1) and (2) is an immediate consequence of Corollary 73.4 and the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 81.4.

Now we turn to the proof that (2) and (3) are equivalent. We write $Y = A_1 \cup A_2$. We consider the following sequences of chain complexes:

$$\begin{array}{ll}
\text{(I)} & 0 \;\to\; \mathrm{C}_*^{\{A_1,A_2\}}(Y) \;\to\; \mathrm{C}_*(Y) \;\to\; \mathrm{C}_*(Y)/\mathrm{C}_*^{\{A_1,A_2\}}(Y) \;\to\; 0 \\
\text{(II)} & 0 \;\to\; \underbrace{\mathrm{C}_*(A_1)/\mathrm{C}_*(A_1 \cap A_2)}_{=\mathrm{C}_*(A_1, A_1 \cap A_2)} \;\to\; \underbrace{\mathrm{C}_*(Y)/\mathrm{C}_*(A_2)}_{=\mathrm{C}_*(Y, A_2)} \;\to\; \mathrm{C}_*(Y)/\mathrm{C}_*^{\{A_1,A_2\}}(Y) \;\to\; 0.
\end{array}$$

The top sequence is exact by definition. It is easy to see that the bottom sequence is also exact. We can thus apply the LES Proposition 74.9 to get the corresponding long exact sequences of homology groups:

$$\begin{aligned}
&\text{(I)} \;\to \mathrm{H}_n\big(\mathrm{C}_*^{\{A_1,A_2\}}(Y)\big) \to \mathrm{H}_n(Y) \to \mathrm{H}_n\big(\mathrm{C}_*(Y)/\mathrm{C}_*^{\{A_1,A_2\}}(Y)\big) \xrightarrow{\partial} \mathrm{H}_{n-1}\big(\mathrm{C}_*^{\{A_1,A_2\}}(Y)\big) \to \\
&\text{(II)} \to \mathrm{H}_n(A_1, A_1 \cap A_2) \to \mathrm{H}_n(Y, A_2) \to \mathrm{H}_n\big(\mathrm{C}_*(Y)/\mathrm{C}_*^{\{A_1,A_2\}}(Y)\big) \xrightarrow{\partial} \mathrm{H}_{n-1}(A_1, A_1 \cap A_2) \to
\end{aligned}$$

But using these long exact sequences we see that

$$\text{(2) holds} \underset{\text{long exact sequence (I)}}{\iff} \mathrm{H}_n\big(\mathrm{C}_*(Y)/\mathrm{C}_*^{\{A_1,A_2\}}(Y)\big) = 0 \text{ for all } n \in \mathbb{N}_0 \underset{\text{long exact sequence (II)}}{\iff} \text{(3) holds.}$$

Finally we turn to the proofs of the addenda (a) and (b). Thus suppose that (X, A_1, A_2) is excisive and that G is an abelian group.

(a) The fact that the map in (2) is also an isomorphism with G-coefficients follows from basically the same argument as above. Furthermore, if (2) holds for G-coefficients, then again the same argument as above shows that also (3) holds with G-coefficients.
(b) This statement follows from the above "(1) $\Rightarrow$ (3)" implication together with Corollary 108.20. ∎

Now we turn to the proof of the Excisive Triad-Proposition 109.12.

Proof of the Excisive Triad-Proposition 109.12. Let (X, A_1, A_2) be a triad of topological spaces.

(1) If $A_1 \subset A_2$, then $\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2) = \mathrm{C}_*(A_2) = \mathrm{C}_*(A_1 \cup A_2)$, so the triad is excisive. The same argument applies if $A_2 \subset A_1$.
(2) This case is just a special case of (1).
(3) If $A_1 \cup A_2 = \mathring{A}_1 \cup \mathring{A}_2$, then the desired statement follows immediately from Proposition 81.7.
(4) The case that A_1 and A_2 are both open subsets of $A_1 \cup A_2$ is a special case of (3).
(5) Suppose that $A_1 \cup A_2$ is a compact topological manifold and that A_1 and A_2 are submanifolds such that each component of $A_1 \cap A_2$ is a boundary component of A_1 and a boundary component of A_2. The Excision Theorem 75.10 for topological manifolds implies that for every $n \in \mathbb{N}_0$ the inclusion induced map

$$\mathrm{H}_n(A_1, A_1 \cap A_2) \;\to\; \mathrm{H}_n(A_1 \cup A_2, A_2)$$

is an isomorphism. By the Excisive Characterization Lemma 109.13 (3) $\Rightarrow$ (1) this implies that the triad (X, A_1, A_2) is excisive.
(6) Suppose that X is a CW-complex and that A_1 and A_2 are both subcomplexes. The proof of the Excision Theorem 75.10 for Topological Manifolds can easily be modified to give the corresponding statement for CW-complexes. More precisely, in the proof of the Excision Theorem 75.10 for Topological Manifolds we need to replace the Smooth Collar Neighborhood Theorem 28.3 by the CW-Complex Properties Proposition 68.10 (8). Furthermore we need to replace the Smooth Manifold Boundary Proposition 19.26 (3) by the Subcomplex Lemma 68.18 (2). We then proceed as in (5). We leave it to the reader to fill in the details. ■

Now we can formulate the following theorem, which in view of the Excisive Triad-Proposition 109.12 is a generalization of the various Mayer–Vietoris Theorems 78.5, 78.9, and 78.10 that we had formulated earlier.

Theorem 109.14. (Mayer–Vietoris Theorem) Given any excisive triad (X, A, B) with $X = A \cup B$ there exists a natural connecting homomorphism

$$\partial_n \colon \mathrm{H}_n(X) \;\to\; \mathrm{H}_{n-1}(A \cap B)$$

such that the following sequence[1329]

$$\cdots \to \mathrm{H}_n(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \xrightarrow{i_A + i_B} \mathrm{H}_n(X) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A \cap B) \to \dots$$

is exact, where the various maps denoted by i are the obvious inclusion induced maps. The same connecting homomorphism and long exact sequence also exists for homology with coefficients in any abelian group G.

[1329] The fact that the connecting homomorphism is natural implies given a map $f \colon (X, A, B) \to (Y, C, D)$ of excisive triads with $X = A \cup B$ and $Y = C \cup D$ the following diagram commutes:

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & \mathrm{H}_n(A \cap B) & \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} & \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) & \xrightarrow{i_A + i_B} & \mathrm{H}_n(X) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A \cap B) & \longrightarrow & \dots \\
 & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \\
\dots & \longrightarrow & \mathrm{H}_n(C \cap D) & \xrightarrow{i_{C\cap D} \oplus -i_{C\cap D}} & \mathrm{H}_n(C) \oplus \mathrm{H}_n(D) & \xrightarrow{i_C + i_D} & \mathrm{H}_n(Y) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(C \cap D) & \longrightarrow & \dots
\end{array}$$

Proof. Let (X, A, B) be a triad with $X = A \cup B$. In the proof of the original Mayer–Vietoris Theorem 78.5 we showed that the sequence

$$0 \to \mathrm{C}_*(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} \mathrm{C}_*(A) \oplus \mathrm{C}_*(B) \xrightarrow{i_A + i_B} \mathrm{C}_*^{\{A,B\}}(X) \to 0$$

of chain maps is exact. We deduce from the LES Proposition 74.9 that we get a long exact sequence of the form

$$\cdots \to \mathrm{H}_n(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \xrightarrow{i_A + i_B} \mathrm{H}_n^{\{A,B\}}(X) \xrightarrow{p_n} \mathrm{H}_{n-1}(A \cap B) \to \ldots$$

Recall that the connecting homomorphism is natural by the LES Proposition 74.9. Now we suppose that (X, A, B) is in fact excisive. By definition, and since $X = A \cup B$, this means that the inclusion map

$$\mathrm{C}_*^{\{A,B\}}(X) \quad \to \quad \mathrm{C}_*(X)$$

is a chain homotopy equivalence, hence for every $n \in \mathbb{N}_0$ it induces a natural isomorphism

$$\Phi_n \colon \mathrm{H}_n^{\{A,B\}}(X) \quad \xrightarrow{\cong} \quad \mathrm{H}_n(X).$$

For each $n \in \mathbb{N}_0$ we now replace $\mathrm{H}_n^{\{A,B\}}(X)$ by $\mathrm{H}_n(X)$ and we write $\partial_n := p_n \circ \Phi_n^{-1}$ and we finally get the desired natural long exact sequence.

The argument for homology with G-coefficients is very similar, see also page 1915. We leave the details to the dedicated reader. ■

We also have the following analogue for cohomology groups.

Theorem 109.15. (Mayer–Vietoris Theorem for Cohomology Groups) Let (X, A, B) be an excisive triad with $X = A \cup B$ and let G be an abelian group. Then there exists a natural[1330] long exact sequence

$$\cdots \to \mathrm{H}^n(X; G) \xrightarrow{i^* \oplus i^*} \mathrm{H}^n(A; G) \oplus \mathrm{H}^n(B; G) \xrightarrow{i^* \oplus -i^*} \mathrm{H}^n(A \cap B; G) \to \mathrm{H}^{n+1}(X; G) \to \ldots$$

where the maps i^* are the restrictions to the various subspaces.

Proof. Let (X, A, B) be an excisive triad with $X = A \cup B$ and let G be an abelian group. We start out with the following claim.

Claim. The inclusion map

$$\mathrm{C}_*^{\{A,B\}}(X) \quad \to \quad \mathrm{C}_*(X)$$

induces for each $n \in \mathbb{N}_0$ a natural isomorphism

$$\mathrm{H}^n(X; G) \quad \to \quad \mathrm{H}^n\big(\mathrm{C}_*^{\{A,B\}}(X); G\big).$$

Proof. The claim follows immediately from the definition of an excisive triad together with the Dual Cochain Map Lemma 108.8 (3). ⊞

Now we turn to the actual proof of the Mayer–Vietoris Theorem for Cohomology Groups. Exactly as in the proof of the Mayer–Vietoris Theorem 109.14 we start out with the fact that the following sequence is exact:

$$0 \to \mathrm{C}_*(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} \mathrm{C}_*(A) \oplus \mathrm{C}_*(B) \xrightarrow{i_A + i_B} \mathrm{C}_*^{\{A,B\}}(X) \to 0.$$

The groups $\mathrm{C}_*^{\{A,B\}}(X)$ are clearly free abelian groups. Thus we deduce from Lemma 108.4 and the Connecting Homomorphism Proposition 108.6 that there exists a natural long exact

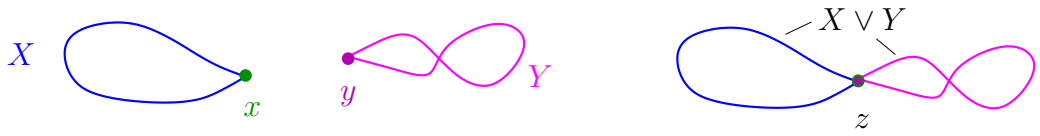
[1330] The meaning of "natural" is evidently totally analogous to the interpretation given in Footnote 1329.

sequence

$$\to \underbrace{\mathrm{H}^n\big(\mathrm{C}_*^{\{A,B\}}(X);G\big)}_{=\mathrm{H}^n(X;G)\ \text{by the claim}} \to \mathrm{H}^n(A;G)\oplus\mathrm{H}^n(B;G) \to \mathrm{H}^n(A\cap B;G) \to \underbrace{\mathrm{H}^{n+1}\big(\mathrm{C}_*^{\{A,B\}}(X);G\big)}_{=\mathrm{H}^{n+1}(X;G)\ \text{by the claim}} \to \quad \blacksquare$$

As an application of the Mayer–Vietoris Theorem 109.15 for Cohomology Groups we will determine the cohomology groups of the wedge of two topological spaces. We will not make use of this example.

Lemma 109.16. (Wedge Excisive Lemma) Let X and Y be topological spaces and let $x \in X$ and $y \in Y$ be good points.[1331] We use these points to define the wedge $X \vee Y$. Then $(X \vee Y, X, Y)$ is an excisive triad.



Proof. We denote by z the point in $X \vee Y$ that is given by identifying x with y. For each $k \in \mathbb{N}_0$ we consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}_k(Y,\{y\};\mathbb{Z}) & \longrightarrow & \mathrm{H}_k(X,\{x\};\mathbb{Z}) \oplus \mathrm{H}_k(Y,\{y\};\mathbb{Z}) & \longrightarrow & \mathrm{H}_k(X,\{x\};\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}_k(Y,\{y\};\mathbb{Z}) & \longrightarrow & \mathrm{H}_k(X\vee Y,\{z\};\mathbb{Z}) & \longrightarrow & \mathrm{H}_k(X\vee Y, Y;\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

where all the maps are the obvious maps. The top horizontal sequence is evidently exact. The bottom horizontal sequence is an excerpt from the long exact sequence of the triple $(\{y\} = \{z\}, Y, X \vee Y)$, see the Topological LES-Proposition 74.13. The excerpt shown is also exact since $(Y,\{y\})$ is a retract of $(X \vee Y,\{z\})$, which implies that for any $l \in \mathbb{N}_0$ the inclusion induced map $\mathrm{H}_l(Y,\{y\};\mathbb{Z}) \to \mathrm{H}_l(X \vee Y,\{z\};\mathbb{Z})$ is a monomorphism.

The middle vertical map in the above diagram is an isomorphism by the Wedge-H_*-Proposition 79.8 and the left vertical map is the identity. It follows from the Five Lemma 74.10 that the right vertical hand map is an isomorphism. We obtain from the Excisive Characterization Lemma 109.13 (3) $\Rightarrow$ (1) that $(X \vee Y, X, Y)$ is an excisive triad. $\blacksquare$

We prove the following proposition that is the analogue of the Wedge-H_*-Proposition 79.9.

Proposition 109.17. (Wedge-H^*-Proposition) Let $\{A_k\}_{k\in K}$ be a family of topological spaces. For each $k \in K$ suppose that we are given a good point $a_k \in A_k$. Let G be an abelian group. Given $j \in K$ we denote by

$$i_j\colon A_j \to \bigvee_{k\in K} A_k \quad \text{respectively} \quad p_j\colon \bigvee_{k\in K} A_k \to A_j$$

[1331] Recall that according to the definition on page 1171 we say that a point x in a topological space X is *good*, if $\{x\}$ is a closed subset and if there exists an open neighborhood U of x such that $\{x\}$ is a deformation retract of U.

the natural inclusion map respectively the natural projection that we introduced on page 1134. Then for every $n \in \mathbb{N}_0$ the map

$$\prod_{k\in K} i_k^* \colon \widetilde{\mathrm{H}}^n\Big(\bigvee_{k\in K} A_k; G\Big) \;\to\; \prod_{k\in K} \widetilde{\mathrm{H}}^n(A_k; G)$$

is an isomorphism where the inverse is given by

$$\prod_{k\in K} p_k^* \colon \prod_{k\in K} \widetilde{\mathrm{H}}^n(A_k; G) \;\to\; \widetilde{\mathrm{H}}^n\Big(\bigvee_{k\in K} A_k; G\Big)$$

Proof.

(1) First we consider the case we are given just two topological spaces. By the Wedge Excisive Lemma 109.16 we can use the Mayer–Vietoris Theorem 109.15 for Cohomology Groups to prove, completely analogously to the Wedge-H_*-Proposition 79.8, the desired statement.

(2) The case that we are dealing with finitely many topological spaces follows from (1) by induction.

(3) The case that K is infinite is more tricky. There are two possible approaches:

(a) In the proof of the Wedge-H_*-Proposition 79.9, that deals with the corresponding statement for homology groups, we gave two different proofs. The first proof generalizes to the present setting. It is not clear to me how the second proof can be generalized. But fortunately providing one proof is enough.[1332]

(b) Alternatively we can consider the map

$$\bigoplus_{k\in K} i_{k*} \colon \bigoplus_{k\in K} \widetilde{C}_n(A_k) \;\to\; \widetilde{C}_n\Big(\bigvee_{k\in K} A_k\Big).$$

By the Wedge-H_*-Proposition 79.9 this chain map induces an isomorphism of chain complexes. It follows from the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 81.4 that this map is in fact a chain homotopy equivalence. It follows from the Dual Cochain Map Lemma 108.8 that the above map induces an isomorphism of dual cochain complexes. The desired statement now follows from throwing Lemmas 108.2 and 108.7 into the bargain. The same logic applies to the projections. ■

Recall that in the Suspension-H_*-Proposition 76.7 we showed that for any $k \in \mathbb{Z}$ and any topological space X we have a natural isomorphism

$$\Sigma_X \colon \widetilde{\mathrm{H}}_k(X) \;\xrightarrow{\cong}\; \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)).$$

Not surprisingly we have a totally analogous statement in cohomology:

Lemma 109.18. (Suspension-H^*-Lemma) Let $k \in \mathbb{Z}$ and let G be an abelian group. Given any topological space X we have a natural isomorphism

$$\Sigma_X \colon \widetilde{\mathrm{H}}^k(\Sigma(X); G) \;\xrightarrow{\cong}\; \widetilde{\mathrm{H}}^{k+1}(X; G)$$

which has the property that for any $\varphi \in \mathrm{H}^k(X; G)$ and any $\sigma \in \mathrm{H}_k(X)$ we have

$$\langle \Sigma_X(\varphi), \Sigma_X(\sigma)\rangle_{\Sigma(X)} \;=\; \langle \varphi, \sigma\rangle_X \;\in\; G.$$

[1332] In fact the proof of the Wedge-H_*-Proposition 79.9 that we sketched in Exercise 79.2 can also be generalized to the present context.

Proof. The construction of the isomorphism Σ_X is basically the same as the construction of the corresponding isomorphism in cohomology. In the Suspension-H_*-Proposition 76.7 we just need to replace the Mayer–Vietoris Theorem 78.5 by the Mayer–Vietoris Theorem 109.15 for Cohomology Groups. We leave it to the reader to fill in the details. ■

Example. Let $n \in \mathbb{N}$. Let $\Sigma(S^n) \to S^{n+1}$ be the homeomorphism from page 452. On page 1683 we showed that

$$\Sigma_{S^n}([S^n]) \;=\; (-1)^{n+1} \cdot [S^{n+1}] \;\in\; \mathrm{H}_{n+1}(S^{n+1};\mathbb{Z}).$$

From the equality in the Suspension-H^*-Lemma 109.18 we obtain the corresponding equality

$$\Sigma_{S^n}([S^n]^*) \;=\; (-1)^{n+1} \cdot [S^{n+1}]^* \;\in\; \mathrm{H}^{n+1}(S^{n+1};\mathbb{Z}).$$

Exercises for Chapter 109.

Exercise 109.1. Let X be a topological space which is homotopy equivalent to a retract of a finite CW-complex. Show that for every $n \in \mathbb{N}_0$ the cohomology group $\mathrm{H}^n(X;\mathbb{Z})$ is a finitely generated group.

Exercise 109.2. Show that the triad $(X, A_1, A_2) := (\mathbb{R}, (-\infty, 0], (0, \infty))$ is *not* excisive.

Exercise 109.3. Use cellular cohomology to determine the isomorphism types of the cohomology groups of the Klein bottle with $\mathbb{F}_2$-coefficients, with $\mathbb{F}_3$-coefficients and with $\mathbb{Z}$-coefficients.

Exercise 109.4. Let $\{A_k\}_{k\in K}$ be a family of CW-complexes. Suppose that for each $k \in K$ suppose that we are given a point in the 0-skeleton of A_k. Provide a proof for the Wedge-H^*-Proposition 109.17 using cellular cohomology. More precisely, provide a proof along the following lines:

(a) Equip $X := \bigvee_{k\in K} A_k$ with the CW-structure provided by the CW-Complex Construction Lemma 68.32.
(b) Determine the relationship between the reduced cellular chain complexes of X and of the A_i.
(c) Make use of the Singular-Cellular Chain Complex Proposition 81.12.
(d) Make use of Lemmas 108.2 and 108.7.

Exercise 109.5. Let (X, A_1, A_2) be a triad of topological spaces. We suppose that $A_1 \cup A_2$ is a topological manifold and that A_1 and A_2 are codimension-zero submanifolds which are both closed subsets. Furthermore we suppose that we have $A_1 \cup A_2 = (A_1 \setminus \partial A_1) \cup (A_2 \setminus \partial A_2) \cup C$ where each component of C is a boundary component of A_1 and a boundary component of A_2. Show that (X, A_1, A_2) is an excisive triad.
Remark. This exercise is a generalization of the Excisive Triad-Proposition 109.12 (5).

110. The Universal Coefficient Theorems for Cohomology Groups

110.1. **The Ext group.** One of the questions that arises from the last chapter is, how can we explain the erratic relationship between homology groups and cohomology groups that we saw on page 2423. Furthermore it would be interesting to know whether given a chain complex one can determine the cohomology groups with G-coefficients from the usual homology group.

These questions are purely in the realm of algebra. More precisely, let $(\mathrm{C}_*, \partial_*)$ be a chain complex of free abelian groups and let G an abelian group. We want to find a connection between the cohomology groups $\mathrm{H}^*(C;G)$ and the homology groups $\mathrm{H}_*(C)$. This problem, and later on its solution, is very similar to the problem we faced in Chapter 89 where we showed that we can determine homology groups of a chain complex with G-coefficients from the usual homology group of the chain complex.

As usual we denote by $\mathcal{AbGroup}$ the category of abelian groups. On page 1918 we defined what it means for a *covariant* functor $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ to be exact, left-exact and right-exact. Basically the same way we can introduce these notions for *contravariant* functors $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$.

Definition. A contravariant functor $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ is called left-exact[1333] if for all exact sequences

$$A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

of abelian groups the sequence

$$0 \longrightarrow F(C) \xrightarrow{F(j)} F(B) \xrightarrow{F(i)} F(A)$$

is also exact. Furthermore a contravariant functor $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ is called right-exact, if for all exact sequences

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C$$

the sequence

$$F(C) \xrightarrow{F(j)} F(B) \xrightarrow{F(i)} F(A) \longrightarrow 0$$

is also exact.

In Lemma 89.10 we saw that the covariant functor $-\otimes G$ is right-exact but that in general it is not left-exact. With the contravariant functor $\mathrm{Hom}(-,G)$ the situation is precisely the converse. More precisely, in the next lemma we will show that this functor is left-exact and immediately afterwards we will give an example that shows that the contravariant functor $\mathrm{Hom}(-,G)$ is in general not right-exact.

Lemma 110.1. (Hom-Left Exact Lemma) For any abelian group G the contravariant functor $\mathrm{Hom}(-,G)$ is left-exact.

Proof. Let G be an abelian group and let

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be an exact sequence of abelian groups. We need to show that the sequence

$$0 \longrightarrow \mathrm{Hom}(C,G) \xrightarrow{\beta^*} \mathrm{Hom}(B,G) \xrightarrow{\alpha^*} \mathrm{Hom}(A,G)$$

[1333] Admittedly one could also refer to this definition as "right"-exact, since the direction changes. But the definition we give is the one which is commonly used in the literature.

is also exact. It is clear that β^* is injective and that $\alpha^* \circ \beta^* = (\beta \circ \alpha)^* = 0$, i.e. we have $\operatorname{im}(\beta^*) \subset \ker(\alpha^*)$. It remains to show that $\ker(\alpha^*) \subset \operatorname{im}(\beta^*)$. So let $f\colon B \to G$ be a homomorphism such that $\alpha^*(f) = 0$. We consider the following diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0. \\ & & \downarrow{\scriptstyle f} & \searrow^{\pi} & \uparrow{\scriptstyle \overline{\beta}}\ {\scriptstyle [b]\mapsto\beta(b)} & & \\ & & G & \xleftarrow[f(b)\hookleftarrow[b]]{\overline{f}} & B/\alpha(A) & & \end{array}$$

We make the following observations:

(1) Since $\alpha^*(f)$ we know that f vanishes on $\alpha(A)$. We denote by $\overline{f}\colon B/\alpha(A) \to G$ the resulting homomorphism given by $[b] \mapsto f(b)$.
(2) We denote $\pi\colon B \to B/\alpha(A)$ the projection.
(3) By the exactness of the original sequence the homomorphism β induces an isomorphism $\overline{\beta}\colon B/\alpha(A) \to C$.
(4) By construction the above diagram commutes.
(5) Now we see that $f = \overline{f} \circ \overline{\beta}^{-1} \circ \beta$, i.e. we have $f = \beta^*(\overline{f} \circ \overline{\beta}^{-1})$. ■

Now we give the example, promised above, that shows that the contravariant functor $\operatorname{Hom}(-, G)$ is in general not right-exact.

Example. We consider the chain complex

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{p} \mathbb{Z}_2$$

where $p\colon \mathbb{Z} \to \mathbb{Z}_2$ denotes the natural projection. We apply the contravariant functor $\operatorname{Hom}(-, \mathbb{F}_2)$. The resulting cochain complex is of the form[1334]

$$\mathbb{F}_2 \xrightarrow[\cong]{p^*} \mathbb{F}_2 \xrightarrow[=0]{(\cdot 2)^*} \mathbb{F}_2 \to 0.$$

Since this cochain complex is not exact it follows that the functor $\operatorname{Hom}(-, \mathbb{F}_2)$ is not right-exact.

In Chapter 89.4 we introduced the torsion groups $\operatorname{Tor}(H, G)$ which "measure" to what degree the functor $- \otimes G$ fails to be a left-exact functor. The same way we now introduce the so called Ext-groups $\operatorname{Ext}(H, G)$ which "measure" to what degree the functor $\operatorname{Hom}(-, G)$ fails to be right-exact.

Definition. Let G and H be abelian groups. We denote by

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

the canonical free resolution from page 1922. We define the n-th G-Ext-group of H as

$$\operatorname{Ext}^n(H, G) := \mathrm{H}^n(F_*; G).$$

[1334]Here we use the identification $\operatorname{Hom}(\mathbb{Z}, G) = G$ from Lemma 108.1 and we use that multiplication by 2 is the zero endomorphism of $\mathbb{F}_2$.

Remark. Let $n \in \mathbb{N}_0$. On page 1923 we saw that $\mathrm{Tor}_n(H, G)$ is covariantly functorial in both G and H. Now we will see that the groups $\mathrm{Ext}_n(H, G)$ are contravariantly functorial in H and covariantly functorial in G:

(1) Let $\alpha\colon H \to H'$ be a homomorphism between abelian groups and let G be an abelian group. Then it follows from the Fundamental Theorem 81.1 of Homological Algebra together with the definition of the G-Ext groups that α induces a homomorphism

$$\begin{aligned} \alpha^*\colon \mathrm{Ext}^n(H', G) &\to \mathrm{Ext}^n(H, G). \\ \text{It is now straightforward to show that} & \\ H &\mapsto \mathrm{Ext}^n(H, G) \\ (\alpha\colon H \to H') &\mapsto (\alpha^*\colon \mathrm{Ext}^n(H', G) \to \mathrm{Ext}^n(H, G)) \end{aligned}$$

is a *contravariant* functor from the category of abelian groups to the category of abelian groups.

(2) On the other hand, if $\beta\colon G \to G'$ is a homomorphism between abelian groups then one sees easily that β induces a homomorphism

$$\begin{aligned} \beta_*\colon \mathrm{Ext}^n(H, G) &\to \mathrm{Ext}^n(H, G') \\ \text{Furthermore one can easily prove that} & \\ G &\mapsto \mathrm{Ext}^n(H, G) \\ (\beta\colon G \to G') &\mapsto (\beta_*\colon \mathrm{Ext}^n(H, G) \to \mathrm{Ext}^n(H, G')) \end{aligned}$$

is a *covariant* functor from the category of abelian groups to the category of abelian groups.

The following proposition is the analogue of the Tor-Free Resolution Proposition 89.12.

Proposition 110.2. (Ext-Free Resolution Proposition) Let G and H be abelian groups and let

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

be a free resolution of H. Then for each $n \in \mathbb{N}_0$ there exists a canonical isomorphism

$$\mathrm{Ext}^n(H, G) \xrightarrow{\cong} \mathrm{H}^n(F_*, G).$$

In particular if

$$0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

is a free resolution of H of length one, then there exist canonical isomorphisms

$$\begin{aligned} \mathrm{Ext}^1(H, G) &\xrightarrow{\cong} \mathrm{coker}\left(\mathrm{Hom}(F_0, G) \xrightarrow{f_1^*} \mathrm{Hom}(F_1, G)\right) \\ \text{and} \quad \mathrm{Ext}^0(H, G) &\xrightarrow{\cong} \ker\left(\mathrm{Hom}(F_0, G) \xrightarrow{f_1^*} \mathrm{Hom}(F_1, G)\right) \end{aligned}$$

Proof. The proof is almost the same as the proof of the Tor-Free Resolution Proposition 89.12. We leave it to the reader to make the necessary changes. ■

In Lemma 89.14 we saw that the higher torsion groups of abelian groups vanish. The same statement, with basically the same proof, also holds for Ext groups. More precisely, we have the following corollary.

Corollary 110.3. For every choice of abelian groups G and H we have $\mathrm{Ext}^n(H, G) = 0$ for $n \geq 2$.

Proof. As we pointed out in the proof of Lemma 89.14 any abelian group H admits a free resolution of length 2:

$$0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0.$$

The corollary now follows immediately from the Ext-Free Resolution Proposition 110.2. ■

In the Tor_0–Lemma 89.13 we saw that the 0-th Torsion functor is an old acquaintance. More precisely we saw that for all abelian groups G and H there exists a natural isomorphism $\mathrm{Tor}_0(H,G) \xrightarrow{\cong} H \otimes G$. The following lemma gives us the analogous result for the 0-th Ext functor.

Lemma 110.4. Let G and H be abelian groups .There exists a natural[1335] isomorphism

$$\mathrm{Ext}^0(H,G) \xrightarrow{\cong} \mathrm{Hom}(H,G).$$

Proof. The proof of Lemma 110.4 is, not surprisingly, rather similar to the proof of the Tor_0–Lemma 89.13. Let G and H be abelian groups. Let

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

be the canonical free resolution of H. We know from the Hom-Left Exact Lemma 110.1 that the sequence

$$(*) \qquad 0 \to \mathrm{Hom}(H,G) \xrightarrow{f_0^*} \mathrm{Hom}(F_0,G) \xrightarrow{f_1^*} \mathrm{Hom}(F_1,G)$$

is exact. Now we obtain the following isomorphism

$$\begin{aligned} \mathrm{Ext}_0(H,G) &\overset{\text{by definition}}{=} \mathrm{H}^0\big(\cdots \leftarrow \mathrm{Hom}(F_1,G) \xleftarrow{f_1^*} \mathrm{Hom}(F_0,G) \leftarrow 0\big) \\ &= \ker\big(\mathrm{Hom}(F_1,G) \xleftarrow{f_1^*} \mathrm{Hom}(F_0,G)\big) \xrightarrow[\text{isomorphism since } (*) \text{ is exact}]{f_0^*} \mathrm{Hom}(H,G). \end{aligned}$$

It is straightforward to verify that these isomorphisms are in fact natural. ■

Definition. Given abelian groups G and H we refer to

$$\mathrm{Ext}(H,G) := \mathrm{Ext}^1(H,G)$$

as *the* G-Ext-group of H.

Remark. Let G and H be two abelian groups. An *extension of H by G* is a short exact sequence of the form

$$0 \longrightarrow G \xrightarrow{\alpha} K \xrightarrow{\beta} H \longrightarrow 0.$$

[1335] Here by "natural" we mean that these isomorphisms define a natural isomorphism from the contravariant functor $H \mapsto \mathrm{Ext}^0(H,G)$ to the contravariant functor $H \mapsto \mathrm{Hom}(H,G)$ and that they also define a natural transformation from the covariant functor $G \mapsto \mathrm{Ext}^0(H,G)$ to the covariant functor $G \mapsto \mathrm{Hom}(H,G)$.

We say that two such extensions given by K and K' are *equivalent* if there exists a homomorphism $\varphi\colon K \to K'$ that makes the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \xrightarrow{\alpha} & K & \xrightarrow{\beta} & H & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{\varphi} & & \downarrow{=} & & \\ 0 & \longrightarrow & G & \xrightarrow{\alpha'} & K' & \xrightarrow{\beta'} & H & \longrightarrow & 0. \end{array}$$

In [**HS1997**, Theorem II.3.4] it is shown that there is a natural bijection from the set of equivalence classes of extensions of H by G with the set $\mathrm{Ext}(H,G)$. This fact is the origin of the name "Ext-group".

In light of Corollary 110.3 and Lemma 110.4 the remaining task is to compute the Ext-group $\mathrm{Ext}(H,G) = \mathrm{Ext}_1(H,G)$. The following lemma allows us in particular to determine the Ext-group for all finitely generated abelian groups.

Lemma 110.5. (Ext-Calculation Lemma) Let G, H and $G_1, \ldots, G_k, \mathrm{H}_1, \ldots, \mathrm{H}_k$ be abelian groups. The following statements hold:

(1) There exists a natural isomorphism

$$\mathrm{Ext}\left(\bigoplus_{i=1}^{k} \mathrm{H}_i, G\right) \;\cong\; \bigoplus_{i=1}^{k} \mathrm{Ext}(\mathrm{H}_i, G).$$

(2) There exists a natural isomorphism

$$\mathrm{Ext}\left(H, \bigoplus_{i=1}^{k} G_i\right) \;\cong\; \bigoplus_{i=1}^{k} \mathrm{Ext}(H, G_i).$$

(3) If H is a free abelian group, then $\mathrm{Ext}(H,G) = 0$.

(4) For all $n \in \mathbb{N}$ we have a natural isomorphism $\mathrm{Ext}(\mathbb{Z}_n, G) \cong G/nG$.

(5) If H is a finitely generated abelian group, then we have a natural isomorphism

$$\mathrm{Ext}(H,\mathbb{Z}) \cong \text{torsion subgroup of } H := \{h \in H \mid \text{there exists an } n \in \mathbb{N} \text{ with } nh = 0\}.$$

In particular $\mathrm{Ext}(H,\mathbb{Z})$ is a finite group.

(6) For $n, m \in \mathbb{N}$ we have $\mathrm{Ext}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_{\gcd(n,m)}$.

Remark.

(a) The first two statements of the lemma also generalize to the case of infinitely many groups, but some care needs to be taken. Indeed, some of the finite direct sums are secretly finite direct products and hence they generalize to infinite direct products. More precisely, the following holds:

(1′) If $\{\mathrm{H}_i\}_{i\in I}$ is a family of abelian groups and if G is an abelian group, then

$$\mathrm{Ext}\left(\bigoplus_{i\in I} \mathrm{H}_i, G\right) \;\cong\; \prod_{i\in I} \mathrm{Ext}(\mathrm{H}_i, G).$$

(2′) If H is an abelian group and if $\{G_i\}_{i\in I}$ is a family of abelian groups, then

$$\mathrm{Ext}\left(H, \prod_{i\in I} G_i\right) \;\cong\; \prod_{i\in I} \mathrm{Ext}(H, G_i).$$

We refer to [**Rot2009**, Propositions 7.21 and 7.22] for a proof. We will not make use of these more general statements.

(b) The statements of the Ext-Calculation Lemma 110.5 look very similar to the statements of the Tor-Properties Lemma 89.15. But there are some differences:
   (i) In the Tor-Properties Lemma 89.15 we saw that for every abelian group we have $\mathrm{Tor}(H, \mathbb{Z}) = 0$ whereas we have now shown in the Ext-Calculation Lemma 110.5 (5) that for every finite abelian group we have $\mathrm{Ext}(H, \mathbb{Z}) \cong H$.
   (ii) In the Tor-Properties Lemma 89.15 we saw that $\mathrm{Tor}(H, G) = 0$ not only whenever H is a free abelian group but also for all torsion-free abelian groups H. Here the situation is different, the statement of the Ext-Calculation Lemma 110.5 (3) does not generalize to torsion-free abelian groups. For example in [**Rot2009**, p. 428] it is shown that there exists an abelian group G such that $\mathrm{Ext}(\mathbb{Q}, G) \neq 0$. But this issue is of no concern to us.

Proof. The first three statements are proved almost the same way as we proved the Tor-Properties Lemma 89.15. We leave it to the reader to fill in the details. We turn to the final three statements.

(4) Let G be an abelian group and let $n \in \mathbb{N}$. We have

$$\mathrm{Ext}(\mathbb{Z}_n, G) \overset{\text{Ext-Free Resolution Proposition 110.2 applied to the free resolution } 0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}_n \to 0}{=} \mathrm{coker}\left(\mathrm{Hom}(\mathbb{Z}, G) \xrightarrow{(\cdot n)^*} \mathrm{Hom}(\mathbb{Z}, G)\right) \underset{\text{by the isomorphism } \mathrm{Hom}(\mathbb{Z}, G) = G \text{ given by } f \mapsto f(1)}{=} \mathrm{coker}\left(G \xrightarrow{\cdot n} G\right) = G/nG.$$

(5) This statement follows immediately from (1), (3) and (4) together with the Finitely Generated Abelian Group Classification Theorem 51.4.
(6) This statement follows immediately from (4) and the calculation on page 1922. ■

We have one more general result for calculating Ext-groups. Before we can state the result we need one more definition.

Definition. An abelian group G is called divisible if for every $g \in G$ and $n \in \mathbb{N}$ there exists an $h \in G$ with $n \cdot h = g$.

Examples.

(1) The groups $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are divisible.
(2) Every quotient of a divisible group is easily seen to be divisible again. For example the quotient group $\mathbb{Q}/\mathbb{Z}$ is also divisible.[1336] This example also shows, using the notation of the definition, that h is not necessarily unique. For example, for $g = [0]$ and $n = 3$ we could take $h = [0]$ or $h = [\frac{1}{3}]$.
(3) The group $\mathbb{Z}$ is not divisible, and the only finite group that is divisible is the trivial group.

Proposition 110.6. Let G and H be abelian groups. If G is divisible, then $\mathrm{Ext}(H, G) = 0$.

In the proof of Proposition 110.6 it is useful to introduce the following notion.

Definition. An abelian group G is called injective if for all abelian groups A and B, every injective homomorphism $\varphi\colon B \to A$ and every homomorphism $\beta\colon B \to G$ there exists a

[1336] Odd as it may sound, this example will actually be important later in Section 148.4.

homomorphism $\alpha\colon A \to G$ such that the following diagram commutes

$$\begin{array}{ccccc} 0 & \longrightarrow & B & \overset{\varphi}{\hookrightarrow} & A \\ & & {\scriptstyle\beta}\downarrow & \swarrow_{\alpha} & \\ & & G. & & \end{array}$$

The following lemma shows why we are interested in injective groups.

Lemma 110.7. Let G and H be abelian groups. If G is injective, then $\mathrm{Ext}(H, G) = 0$.

Proof. Let H be an abelian group. As we pointed out in the proof of Lemma 89.14 the abelian group H admits a free resolution of length 2:

$$0 \to F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} H \to 0.$$

By the Ext-Free Resolution Proposition 110.2 the group $\mathrm{Ext}(H, G)$ is isomorphic to the cokernel of the map

$$\mathrm{Hom}\begin{pmatrix} F_1 \\ \downarrow \\ G \end{pmatrix} \xleftarrow{\varphi_1^*} \mathrm{Hom}\begin{pmatrix} F_0 \\ \downarrow \\ G \end{pmatrix}.$$

But by definition of G being injective this map φ_1^* is in fact an epimorphism. Thus we have shown that $\mathrm{Ext}(H, G) = 0$. ■

To provide a proof for Proposition 110.6 it remains to prove the following proposition.

Proposition 110.8. Every abelian group that is divisible is also injective.

Remark. It is straightforward to prove the converse to Proposition 110.8, namely every injective abelian group is also divisible.

In the proof of Proposition 110.8 we will need Zorn's Lemma which we now recall for the reader's convenience.

Lemma 0.6. (Zorn's Lemma) Suppose a partially ordered set $(P, \leq)$ has the property that every chain has an upper bound in P. Then the set P contains at least one maximal element.

We also recall the definitions in the statement of Zorn's Lemma:

(1) A *partially ordered set* $(P, \leq)$ is a set together with a preorder $\leq$ (see page 1256) that has the property that $a \leq b$ and $b \leq a$ implies that $a = b$.
(2) A *chain* in P is a subset Q such that for any two elements $q, q' \in Q$ we have $q \leq q'$ or $q' \leq q$.
(3) An *upper bound* of a subset Q is an element $q \in Q$ such that $p \leq q$ for all $q \in Q$.
(4) A *maximal element* of a subset Q is an element $q \in Q$ such that there is no $r \in Q$ with $q < r$.[1337]

We refer to [**Cie1997**, Theorem 4.3.4] for a proof of Zorn's Lemma. Perhaps it is more fun to look at Zorn's Lemma in "The Simpsons", see

`http://www.simonsingh.net/Simpsons_Mathematics/zorns-lemma/`

Now we can give the proof of Proposition 110.8.

[1337] Aren't "upper bound" and "maximal element" the same?

Proof of Proposition 110.8. Let G be an abelian group that is divisible. Suppose we are given a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & B & \overset{\varphi}{\hookrightarrow} & A \\ & & \downarrow{\scriptstyle \beta} & \swarrow{\scriptstyle ?\,\alpha\,?} & \\ & & G. & & \end{array}$$

Since φ is a monomorphism we can assume that B is in fact a subgroup of A and that φ is the inclusion map.

We denote by P the set of all pairs (C_i, γ_i) where C_i is a subgroup of A which contains B and where $\gamma_i \colon \mathrm{C}_i \to G$ is a homomorphism such that $\gamma_i|_B = \beta$. This family of pairs is non-empty since (B, β) satisfies the required conditions. We define $(\mathrm{C}_i, \gamma_i) < (\mathrm{C}_j, \gamma_j)$ if $\mathrm{C}_i \subset \mathrm{C}_j$ and if $\gamma_j|_{\mathrm{C}_i} = \gamma_i$. Given a chain $(\mathrm{C}_j, \gamma_j)_{j \in J}$ in P we get an upper bound by setting

$$\begin{array}{rcl} C := \bigcup\limits_{j \in J} \mathrm{C}_j \text{ and } \gamma \colon C & \to & G \\ c & \mapsto & \gamma_j(c) \text{ if } j \in J \text{ with } c \in \mathrm{C}_j. \end{array}$$

Since $(\mathrm{C}_j, \gamma_j)_{j \in J}$ is a chain one can easily verify that C is in fact a subgroup of A which contains B and that γ is a homomorphism with $\gamma|_B = \beta$.

Therefore we can apply Zorn's Lemma to this family and we obtain a maximal pair (C_m, γ_m). It remains to prove the following claim.

Claim. We have $\mathrm{C}_m = A$.

Proof. Suppose that this is not the case. Then there exists an $a \in A \setminus \mathrm{C}_m$. We denote by $\langle\langle \mathrm{C}_m, a \rangle\rangle$ the subgroup of A generated by C_m and a, i.e. $\langle\langle \mathrm{C}_m, a \rangle\rangle$ is the smallest subgroup of A that contains C_m and a. To obtain a contradiction it suffices to show that there exists a homomorphism $\delta \colon \langle\langle \mathrm{C}_m, a \rangle\rangle \to G$ that extends γ_m. We distinguish two cases:

(1) If there exists no $n \in \mathbb{N}$ such that $n \cdot a \in \mathrm{C}_m$, then one can show easily that we have $\langle\langle \mathrm{C}_m, a \rangle\rangle = \mathrm{C}_m \oplus \langle a \rangle$. We extend γ_m to $\langle\langle \mathrm{C}_m, a \rangle\rangle = \mathrm{C}_m \oplus \langle a \rangle$ by setting $\delta(a) = 0$.

(2) Otherwise there exists a minimal $n \in \mathbb{N}$ such that $n \cdot a \in \mathrm{C}_m$. We denote by $r \in \mathbb{N}_0$ the order of $n \cdot a \in \mathrm{C}_m$. Here we say that the order is 0 if $n \cdot a \in \mathrm{C}_m$ has infinite order. We consider the maps

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}_r & \overset{\Phi}{\to} & \mathrm{C}_m \oplus \mathbb{Z}_{nr} & \overset{\Psi}{\to} & \langle\langle \mathrm{C}_m, a \rangle\rangle & \to & 0 \\ & & k & \mapsto & (k \cdot n \cdot a, -k \cdot n) & & & & \\ & & & & (c, l) & \mapsto & c + l \cdot a & & \end{array}$$

It is clear that Φ is injective and that $\Psi \circ \Phi = 0$. Note that the image of Ψ lies a priori in A, but it is straightforward to see that the image of Ψ is precisely $\langle\langle \mathrm{C}_m, a \rangle\rangle$. By definition of n we have that $\ker(\Psi) \subset \mathrm{im}(\Phi)$. Put differently, the sequence is exact. Since G is divisible there exists an $h \in G$ such that $n \cdot h = \gamma(n \cdot a)$. We denote by $\alpha \colon \mathrm{C}_m \oplus \mathbb{Z}_{nr} \to G$ the homomorphism that is given by setting $\alpha(c) = \gamma(c)$ for $c \in \mathrm{C}_m$ and $\alpha(k) = k \cdot h$ for $k \in \mathbb{Z}$. One can easily verify that $\mathrm{im}(\Phi)$ is contained in $\ker(\alpha)$. It is now straightforward to show that the homomorphism

$$\begin{array}{rcccl} \delta \colon \langle\langle \mathrm{C}_m, a \rangle\rangle & \to & (\mathrm{C}_m \oplus \mathbb{Z}_{nr}) / \mathrm{im}(\Phi) & \to & G \\ x & \mapsto & [\Psi^{-1}(x)] & & \\ & & [y] & \mapsto & \alpha(y) \end{array}$$

has the desired properties.

This concludes the proof of the claim and thus of the lemma. ■

The following proposition gives arguably slightly weird example.

Proposition 110.9. The group $\mathrm{Ext}(\mathbb{Q},\mathbb{Z})$ is isomorphic to the additive group $(\mathbb{R},+)$.

Remark. Note that this proposition is not totally pointless, for example it follows from the existence of Moore spaces, see the Moore Space Existence Proposition 79.11, that there exist topological spaces X with $\mathrm{H}_n(X) \cong (\mathbb{Q},+)$. Thus even unexpected abelian groups can appear in topology.

Proof. This proposition is proved in [**Wie1969**]. Alternatively the proposition is almost proved in [**Bre1997**, Lemma 14.8]. We also refer to [**Boar2010**] for the calculation of $\mathrm{Ext}(\mathbb{Q},\mathbb{Z})$. ■

This example also shows that Ext-groups can be quite large. In fact the following proposition holds.

Proposition 110.10. Let G be an abelian group. If G is not a finitely generated group, then $\mathrm{Hom}(G,\mathbb{Z})$ is uncountable or $\mathrm{Ext}(G,\mathbb{Z})$ is uncountable.

Proof. We will barely make use of this proposition, thus we refer to [**Hat2002**, Proposition 3.F.12] instead for a proof. ■

110.2. **The Universal Coefficient Theorem for Cohomology Groups.** Finally we want to explore the connection between the cohomology groups of a chain complex C and the usual homology groups of the chain complex. The key to relating homology groups to cohomology groups is given by the following evaluation homomorphism.

Definition. Let $(\mathrm{C}_*,\partial_*)$ be a chain complex and let G be an abelian group. We refer to the map[1338]

$$\begin{array}{rcl} \mathrm{ev}\colon \mathrm{H}^n(C;G) & \to & \mathrm{Hom}(\mathrm{H}_n(C),G) \\ [\varphi\colon \mathrm{C}_n \to G] & \mapsto & \left(\begin{array}{rcl} \mathrm{H}_n(C) & \to & G \\ [c] & \mapsto & \langle [\varphi],[c]\rangle = \varphi(c) \end{array} \right) \end{array}$$

as the evaluation homomorphism.

Remark. It follows immediately from Lemma 109.6 (3) that the evaluation homomorphisms in natural in the chain complex C_* in the sense that if $f\colon C_* \to D_*$ is a chain map, then for any $n \in \mathbb{N}_0$ and any abelian group G the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}^n(C;G) & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}(\mathrm{H}_n(C),G) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (f_*)^*} \\ \mathrm{H}^n(D;G) & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}(\mathrm{H}_n(D),G). \end{array}$$

Furthermore, it follows easily from the definitions that the evaluation map is natural in G, in the sense that if $\varphi\colon G_1 \to G_2$ is homomorphism of abelian groups, then for any $n \in \mathbb{N}_0$

[1338] This map is well-defined by Lemma 109.6.

and any chain complex C_* the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}^n(C;G_1) & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}(\mathrm{H}_n(C),G_1) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \mathrm{H}^n(C;G_2) & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}(\mathrm{H}_n(C),G_2). \end{array}$$

Here the map on the left is the one we had introduced on page 2401.

Recall that in Lemma 78.8 we saw that $\mathrm{H}_2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) = 0$ and that on page 2423 we showed that $\mathrm{H}^2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) \cong \mathbb{Z}_2$. These calculations imply that the evaluation homomorphism $\mathrm{ev}\colon \mathrm{H}^n(X;G) \to \mathrm{Hom}(\mathrm{H}_n(X);G)$ is in general not a monomorphism. The following theorem relates homology groups and cohomology groups via the evaluation homomorphism and the Ext-groups.

Theorem 110.11. (Universal Coefficient Theorem for Cohomology Groups) Let $(\mathrm{C}_*,\partial_*)$ be a chain complex of free abelian groups and let G be an abelian group. For each $n \in \mathbb{N}_0$ there exists a map

$$\epsilon\colon\ \mathrm{Ext}(\mathrm{H}_{n-1}(C),G)\ \to\ \mathrm{H}^n(C;G)$$

which is natural in the chain complex C_* and natural in the abelian group G and which has the property that the following sequence is exact:

$$0\ \to\ \mathrm{Ext}(\mathrm{H}_{n-1}(C),G)\ \xrightarrow{\epsilon}\ \mathrm{H}^n(C;G)\ \xrightarrow{\mathrm{ev}}\ \mathrm{Hom}(\mathrm{H}_n(C),G) \to 0.$$

This short exact sequence splits and there exists therefore an isomorphism[1339]

$$\mathrm{H}^n(C;G)\ \cong\ \mathrm{Ext}(\mathrm{H}_{n-1}(C),G) \oplus \mathrm{Hom}(\mathrm{H}_n(C),G).$$

Before we provide the proof of the above Universal Coefficient Theorem for Cohomology Groups 110.11 let us formulate the most important application, namely let us apply it to the chain complex of a pair of topological spaces.

Theorem 110.12. (Universal Coefficient Theorem for Cohomology Groups) Let (X,A) be a pair of topological spaces and let G be an abelian group. For each $n \in \mathbb{N}_0$ there exists a map

$$\epsilon\colon\ \mathrm{Ext}(\mathrm{H}_{n-1}(X,A),G)\ \to\ \mathrm{H}^n(X,A;G)$$

which is natural in (X,A), which is natural in the abelian group G and which has the property that the following sequence is exact:

$$0\ \to\ \mathrm{Ext}(\mathrm{H}_{n-1}(X,A),G)\ \xrightarrow{\epsilon}\ \mathrm{H}^n(X,A;G)\ \xrightarrow{\mathrm{ev}}\ \mathrm{Hom}(\mathrm{H}_n(X,A),G) \to 0.$$

This short exact sequence splits and there exists therefore an isomorphism

$$\mathrm{H}^n(X,A;G)\ \cong\ \mathrm{Ext}(\mathrm{H}_{n-1}(X,A),G) \oplus \mathrm{Hom}(\mathrm{H}_n(X,A),G).$$

Remark. The Universal Coefficient Theorem 110.12 for Cohomology Groups is the analogue of the Algebraic Universal Coefficient Theorem 89.16 together with the UCT-Splitting Theorem 89.22.

[1339] Once again the two maps in the short exact sequence is natural, whereas the isomorphism is not natural. More precisely, the discussion on page 1935 can easily be modified to show that there exists no *natural* isomorphism

$$\mathrm{H}^n(C;G)\ \cong\ \mathrm{Ext}(\mathrm{H}_{n-1}(C),G) \oplus \mathrm{Hom}(\mathrm{H}_n(C),G).$$

We will discuss this issue in Exercise 110.1.

Proof of Theorem 110.12 assuming Theorem 110.11. As the reader will have noticed there is really not much to say, except that we recall that we pointed out on page 1625 that the chain groups $\mathrm{C}_n(X, A)$, $n \in \mathbb{N}_0$ are free abelian groups. ■

Remark.

(1) Here is one straightforward but interesting consequence to the Universal Coefficient Theorem 110.12 for Cohomology Groups. Given any pair (X, A) of topological spaces we see that

$$\mathrm{H}^1(X;\mathbb{Z}) \underset{\text{Universal Coefficient Theorem 110.12}}{\cong} \mathrm{Ext}(\mathrm{H}_0(X,A),\mathbb{Z}) \oplus \mathrm{Hom}(\mathrm{H}_1(X,A),\mathbb{Z}) \underset{\text{by the Ext-Calculation Lemma 110.5 and Corollary 72.15}}{=} \mathrm{Hom}(\mathrm{H}_1(X,A),\mathbb{Z}) \underset{\text{since } \mathrm{Hom}(G,\mathbb{Z}) \text{ is always torsion free}}{=} \text{torsion-free.}$$

(2) The Universal Coefficient Theorem 110.12 for Cohomology Groups implies in particular that topological spaces with isomorphic homology groups also have isomorphic cohomology groups. In other words, if our only goal is to distinguish homeomorphism types of topological spaces, then cohomology groups will not be of much help.

(3) The fact that cohomology groups are determined by homology groups is also immediate consequence of the combination of Propositions 81.13 and 81.4 together with the Cochain Map Lemma 108.5.

Now we turn to the proof of the purely algebraic Universal Coefficient Theorem 110.11 for Cohomology Groups. The proof is, perhaps not surprisingly, rather similar to the proofs of the Universal Coefficient Theorem 89.17 and the UCT-Splitting Proposition 89.21.

Proof. Let $(\mathrm{C}_*, \partial_*)$ be a chain complex of free abelian groups. For each $n \in \mathbb{N}_0$ we write as always $Z_n := \ker(\partial_n)$ and $B_n := \mathrm{im}(\partial_{n+1})$. Note that it follows from the Subgroup-of-Free Abelian Group Lemma 51.2 that B_n and Z_n are free abelian groups.

As in the proof of the Universal Coefficient Theorem 89.17 we consider the following short exact sequence of vertical chain complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_n & \longrightarrow & \mathrm{C}_n & \xrightarrow{\partial_n} & B_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & Z_{n-1} & \longrightarrow & \mathrm{C}_{n-1} & \xrightarrow{\partial_{n-1}} & B_{n-2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Now let G be an abelian group. We apply the functor $\mathrm{Hom}(-, G)$ to the above short exact sequence of chain complexes and we obtain the following diagram:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}(B_{n-1}, G) & \xrightarrow{\partial_n^*} & \mathrm{Hom}(C_n, G) & \longrightarrow & \mathrm{Hom}(Z_n, G) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle \partial_n^*} & & \uparrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathrm{Hom}(B_{n-2}, G) & \xrightarrow{\partial_{n-1}^*} & \mathrm{Hom}(C_{n-1}, G) & \longrightarrow & \mathrm{Hom}(Z_{n-1}, G) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

This is again a sequence of cochain complexes. Since the Z_n are free abelian we obtain from Lemma 108.4 that the above horizontal sequences of cochain complexes are still exact. Therefore we can apply the Connecting Homomorphism Proposition 108.6 to obtain the following long exact sequence

$$\to \operatorname{Hom}(Z_{n-1}, G) \xrightarrow{d_n} \operatorname{Hom}(B_{n-1}, G) \to \mathrm{H}^n(C; G) \xrightarrow{\partial_n} \operatorname{Hom}(Z_n, G) \xrightarrow{d_{n-1}} \operatorname{Hom}(B_n, G) \to \dots$$

where the maps d_n are the connecting homomorphisms of the long exact sequence. One can show easily that the connecting homomorphism $d_n \colon \operatorname{Hom}(Z_n, G) \to \operatorname{Hom}(B_n, G)$ is the map $d_n = i_n^*$ where $i_n \colon B_n \to Z_n$ is the inclusion map. By the Break-into-SES Lemma 78.4 we obtain the following short exact sequence

$$0 \to \operatorname{coker}(\operatorname{Hom}(Z_{n-1},G) \to \operatorname{Hom}(B_{n-1},G)) \to \mathrm{H}^n(C;G) \to \ker(\operatorname{Hom}(Z_n,G) \to \operatorname{Hom}(B_n,G)) \to 0.$$

We have to show that the expressions left and right coincide with the desired expressions. Both equalities follow immediately from the Ext-Free Resolution Proposition 110.2 (together with Lemma 110.4 for the right-hand term) and the observation that for each $l \in \mathbb{N}_0$ a free resolution of $\mathrm{H}_l(C)$ is given by $0 \to B_l \xrightarrow{i_l} Z_l \to \mathrm{H}_l(C)$. We have thus shown the existence of the desired short exact sequence

$$0 \;\to\; \operatorname{Ext}(\mathrm{H}_{n-1}(C), G) \;\xrightarrow{\epsilon}\; \mathrm{H}^n(C; G) \;\xrightarrow{\mathrm{ev}}\; \operatorname{Hom}(\mathrm{H}_n(C), G) \to 0.$$

The statement that the homomorphism ϵ to the left is natural follows easily from the definitions and the Fundamental Theorem 81.1 (2) of Homological Algebra. We leave the verification of the details to the reader.

It remains to show that the above short exact sequence splits. By definitions this means that we need to construct a homomorphism $\eta \colon \operatorname{Hom}(\mathrm{H}_n(C), G) \to \mathrm{H}^n(C; G)$ such that $\mathrm{ev} \circ \eta$ is the identity on $\operatorname{Hom}(\mathrm{H}_n(C), G)$.

We consider again the short exact sequence

$$0 \;\to\; Z_n \;\hookrightarrow\; \mathrm{C}_n \;\xrightarrow{\partial_n}\; B_{n-1} \;\to\; 0.$$

Since B_{n-1} is a free abelian group we obtain from Lemmas 78.1 and 78.2 that there exists a homomorphism $p_n \colon \mathrm{C}_n \to Z_n$ such that p_n is the identity on $Z_n \subset C_n$.

In the proof of the UCT-Splitting Proposition 89.21 we showed that the following diagram commutes:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & \mathrm{C}_{n+1} & \xrightarrow{\partial_{n+1}} & \mathrm{C}_n & \xrightarrow{\partial_n} & \mathrm{C}_{n-1} & \xrightarrow{\partial_{n-1}} & \dots \\
 & & \downarrow\uparrow p_{n+1} & \searrow^{\partial_{n+1}} & \downarrow\uparrow p_n & \searrow^{\partial_n} & \downarrow\uparrow p_{n-1} & & \\
 & & Z_{n+1} & & Z_n & & Z_{n-1} & & \\
 & & \downarrow q_{n+1} & & \downarrow q_n & & \downarrow q_{n-1} & & \\
\dots & \longrightarrow & \underbrace{\mathrm{H}_{n+1}(C)}_{=Z_{n+1}/B_{n+1}} & \xrightarrow{0} & \underbrace{\mathrm{H}_n(C)}_{=Z_n/B_n} & \xrightarrow{0} & \underbrace{\mathrm{H}_{n-1}(C)}_{=Z_{n-1}/B_{n-1}} & \xrightarrow{0} & \dots
\end{array}$$

where the maps q_n are just the natural projections.

The top and the bottom horizontal sequences of the above diagram are chain complexes. Since the diagram commutes we see that the vertical maps from top to bottom are chain maps.

Now we apply the functor $\mathrm{Hom}(-,G)$ to the top and the bottom horizontal sequences and we obtain the following diagram of cochain complexes.

$$\begin{array}{ccccccccc}
\ldots & \longleftarrow & \mathrm{Hom}(C_{n+1},G) & \xleftarrow{\partial_{n+1}^*} & \mathrm{Hom}(C_n,G) & \xleftarrow{\partial_n^*} & \mathrm{Hom}(C_{n-1},G) & \longleftarrow & \ldots \\
 & & \uparrow{\scriptstyle (q_{n+1}\circ p_{n+1})^*} & & \uparrow{\scriptstyle (q_n\circ p_n)^*} & & \uparrow{\scriptstyle (q_{n-1}\circ p_{n-1})^*} & & \\
\ldots & \longleftarrow & \mathrm{Hom}(\mathrm{H}_{n+1}(C),G) & \xleftarrow{0} & \mathrm{Hom}(\mathrm{H}_n(C),G) & \xleftarrow{0} & \mathrm{Hom}(\mathrm{H}_{n-1}(C),G) & \longleftarrow & \ldots
\end{array}$$

The induced maps on the cohomology groups of these two cochain complexes give us a homomorphism

$$\eta\colon\ \mathrm{Hom}(\mathrm{H}_n(C),G) = \mathrm{H}^n(\text{lower sequence}) \ \to\ \mathrm{H}^n(\text{upper sequence}) = \mathrm{H}^n(C;G).$$

It remains to prove the following claim.

Claim. The composition $\mathrm{ev}\circ\eta$ is the identity on $\mathrm{Hom}(\mathrm{H}_n(C),G)$.

Proof. Let $\varphi\colon \mathrm{H}_n(C)\to G$ be homomorphism. We need to show that $(\mathrm{ev}\circ\eta)(\varphi)=\varphi$ in $\mathrm{Hom}(\mathrm{H}_n(C),G)$. So let $[\sigma]\in \mathrm{H}_n(C)$ with $\sigma\in Z_n$. Then

$$\big((\mathrm{ev}\circ\eta)(\varphi)\big)([\sigma]) \underset{\text{definition of ev}}{=} \langle \overbrace{\eta(\varphi)}^{\in \mathrm{H}^n(C;G)}, [\sigma]\rangle \underset{\text{definition of } \eta \text{ and } \langle\,,\,\rangle}{=} (\overbrace{\varphi\circ q_n\circ p_n}^{\in \mathrm{Hom}(C_n,G)})(\sigma) \underset{\text{since } p_n=\mathrm{id} \text{ on } Z_n}{=} (\varphi\circ q_n)(\sigma) \underset{\text{since } q_n(\sigma)=[\sigma]}{=} \varphi([\sigma]).$$

■

Example. We consider again the projective space $P := \mathbb{R}\mathrm{P}^2$. As a reminder, we showed on page 1725 that $\mathrm{H}_2(P)=0$, $\mathrm{H}_1(P)\cong\mathbb{Z}_2$ and $\mathrm{H}_0(P)\cong\mathbb{Z}$. It follows from the Universal Coefficient Theorem 110.12 and from the properties of the Ext-group that we had listed in the Ext-Calculation Lemma 110.5 that

$$\begin{array}{lllll}
\mathrm{H}^2(P;\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_2(P),\mathbb{Z}) \oplus \mathrm{Ext}(\mathrm{H}_1(P),\mathbb{Z}) & \cong \mathrm{Hom}(0,\mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z}_2,\mathbb{Z}) & = 0\oplus\mathbb{Z}_2 = \mathbb{Z}_2 \\
\mathrm{H}^1(P;\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_1(P),\mathbb{Z}) \oplus \mathrm{Ext}(\mathrm{H}_0(P),\mathbb{Z}) & \cong \mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z},\mathbb{Z}) & = 0\oplus 0 = 0 \\
\mathrm{H}^0(P;\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_0(P),\mathbb{Z}) & \cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) & \cong \mathbb{Z}
\end{array}$$

and

$$\begin{array}{lllll}
\mathrm{H}^2(P;\mathbb{Z}_2) \cong \mathrm{Hom}(\mathrm{H}_2(P),\mathbb{Z}_2)\oplus\mathrm{Ext}(\mathrm{H}_1(P),\mathbb{Z}_2) & \cong \mathrm{Hom}(0,\mathbb{Z}_2)\oplus\mathrm{Ext}(\mathbb{Z}_2,\mathbb{Z}_2) & = 0\oplus\mathbb{Z}_2 = \mathbb{Z}_2 \\
\mathrm{H}^1(P;\mathbb{Z}_2) \cong \mathrm{Hom}(\mathrm{H}_1(P),\mathbb{Z}_2)\oplus\mathrm{Ext}(\mathrm{H}_0(P),\mathbb{Z}_2) & \cong \mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z}_2)\oplus\mathrm{Ext}(\mathbb{Z},\mathbb{Z}_2) & = \mathbb{Z}_2\oplus 0 = \mathbb{Z}_2 \\
\mathrm{H}^0(P;\mathbb{Z}_2) \cong \mathrm{Hom}(\mathrm{H}_0(P),\mathbb{Z}_2) & \cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_2) & = \mathbb{Z}_2.
\end{array}$$

Fortunately this coincides with the calculations from page 2423.

Example. We return to the study of lens spaces. In the Lens Space Homology Lemma 98.9 we saw that

$$\mathrm{H}_n(L(p,q);\mathbb{Z}) \ \cong\ \begin{cases} \mathbb{Z}, & \text{if } n=0,3, \\ 0, & \text{if } n=2 \text{ or } n>3, \\ \mathbb{Z}_p, & \text{if } n=1. \end{cases}$$

As in the previous example we can now use the Universal Coefficient Theorem 110.12 and the Ext-Calculation Lemma 110.5 to compute the cohomology groups of $L(p,q)$. We leave

it to the reader to verify that[1340]

$$\mathrm{H}^n(L(p,q);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0,3 \\ 0, & \text{if } n = 1 \text{ or } n > 3, \\ \mathbb{Z}_p, & \text{if } n = 2. \end{cases}$$

This shows, as was already foreshadowed by Corollary 108.19, that cohomology groups also cannot distinguish between two lens spaces of the form $L(p,q)$ and $L(p,r)$. In particular we have not made any progress on Question 48.7 since the introduction of the fundamental group. All the extra invariants, namely higher homotopy groups, homology groups and cohomology groups (with any coefficients) have proved useless.

We conclude this section with the following proposition that gives in particular a converse to Corollary 108.20.

Proposition 110.13.

(1) Let C_* be a chain complex. The following two statements are equivalent:
   (a) $\mathrm{H}_k(C_*) = 0$ for all $k \in \mathbb{N}_0$.
   (b) $\mathrm{H}^k(C_*;\mathbb{Z}) = 0$ for all $k \in \mathbb{N}_0$.

(2) Let $f\colon D_* \to E_*$ be a chain map. The following two statements are equivalent:
   (a) All the induced map $f_*\colon \mathrm{H}_k(D_*) \to \mathrm{H}_k(E_*)$ are isomorphisms.
   (b) All the induced map $f^*\colon \mathrm{H}^k(E_*;\mathbb{Z}) \to \mathrm{H}^k(D_*;\mathbb{Z})$ are isomorphisms.

Proof.

(1) We start out with the following claim.

Claim. Let A be an abelian group. If A is non-trivial, then $\mathrm{Hom}(A,\mathbb{Z})$ or $\mathrm{Ext}(A,\mathbb{Z})$ is non-trivial.

Proof. Let A be a non-trivial abelian group. If A is finitely generated, then it follows from the Finitely Generated Abelian Group Classification Theorem 51.4 of Finitely Generated Abelian Groups together with the Ext-Calculation Lemma 110.5 that $\mathrm{Hom}(A,\mathbb{Z})$ or $\mathrm{Ext}(A,\mathbb{Z})$ is non-trivial. Furthermore, if A is infinitely generated, then the desired conclusion follows from Proposition 110.10. ⊞

The equivalence of (a) and (b) follows immediately from the claim together with the Universal Coefficient Theorem 110.11 for Cohomology Groups.

(2) Let $f\colon D_* \to E_*$ be a chain map. We denote by $(\mathrm{M}(f),\partial_*)$ the corresponding algebraic mapping cone as defined on page 1796. By the Mapping Cone-LES Lemma 81.8 there exists a long exact sequence of the form

$$\ldots \to \mathrm{H}_k(D_*) \xrightarrow{f_*} \mathrm{H}_k(E_*) \to \mathrm{H}_k(\mathrm{M}(f)) \to \mathrm{H}_{k-1}(D_*) \xrightarrow{f_*} \ldots$$

Basically the same argument also shows that there exists a long exact sequence of the form

$$\ldots \leftarrow \mathrm{H}^k(D_*;\mathbb{Z}) \xleftarrow{f^*} \mathrm{H}_k(E_*;\mathbb{Z}) \leftarrow \mathrm{H}_n(\mathrm{M}(f);\mathbb{Z}) \leftarrow \mathrm{H}_{n-1}(D_*;\mathbb{Z}) \xrightarrow{f^*} \ldots$$

Thus we see that the equivalence of (2a) and (2b) is an immediate consequence of the equivalence of (1a) and (1b) applied to $C_* = \mathrm{M}(f)$. ■

[1340]Note that the $\mathbb{Z}_p$ shifted from $n = 1$ in homology to $n = 2$ in cohomology.

110.3. Cohomology groups of topological manifolds.

Proposition 110.14. (Top-Cohomology Proposition) Let M be a connected n-dimensional topological manifold. Let G be an abelian group.

(1) Let $x \in M \setminus \partial M$. For $i \neq n$ we have $\mathrm{H}_i(M, M \setminus \{x\}; G) = 0$. Furthermore the map
$$\mathrm{ev}\colon \mathrm{H}^n(M, M \setminus \{x\}; G) \ \to \ \mathrm{Hom}(\mathrm{H}_n(M, M \setminus \{x\}; \mathbb{Z}), G)$$
is an isomorphism.
(2) If M is compact but not closed, then $\mathrm{H}^n(M; G) = 0$.
(3) If M is closed orientable and non-empty, then $\mathrm{H}^n(M; G) \cong G$.
(4) If M is closed non-orientable and non-empty, then $\mathrm{H}^n(M; G) \cong G/2G$.
(5) For any $k > n$ we have $\mathrm{H}^k(M; G) = 0$.

Proof. Let M be a compact connected n-dimensional topological manifold and let G be an abelian group.

(1) This statement follows immediately from Lemma 105.4 and the Universal Coefficient Theorem 110.12.
(2) We suppose that M has non-empty boundary. Then
$$\mathrm{H}^n(M; G) \ \underset{\text{Universal Coefficient Theorem 110.12}}{\overset{\uparrow}{\cong}} \ \mathrm{Hom}(\mathrm{H}_n(M; \mathbb{Z}), G) \oplus \mathrm{Ext}(\mathrm{H}_{n-1}(M; \mathbb{Z}), G) \ \underset{\substack{\text{by the Top-Homology Theorem 106.3 we have } \mathrm{H}_n(M; \mathbb{Z}) = 0 \text{ and} \\ \text{by Proposition 106.22 (1) we have } \mathrm{H}_{n-1}(M; \mathbb{Z}) \cong \mathbb{Z}^k \text{ for some } k \in \mathbb{N}_0}}{\overset{\uparrow}{=}} \ \mathrm{Ext}(\mathbb{Z}^k, G) \ \overset{\text{Ext-Calculation Lemma 110.5}}{\overset{\downarrow}{=}} \ 0.$$
(3) Suppose that M is closed orientable and non-empty. The argument is very similar to the proof of (2). We just need to use that we know from the Orientation-Top Homology Theorem 106.1 that $\mathrm{H}_n(M; \mathbb{Z}) \cong \mathbb{Z}$ and that from Proposition 106.22 (1) we know that $\mathrm{H}_{n-1}(M; \mathbb{Z}) \cong \mathbb{Z}^k$ for some k.
(4) Next suppose that M is closed non-orientable and non-empty. This time we use that we know from the Orientation-Top Homology Theorem 106.1 and Proposition 106.22 (2) that $\mathrm{H}_n(M; \mathbb{Z}) \cong 0$ and that $\mathrm{H}_{n-1}(M; \mathbb{Z}) \cong \mathbb{Z}^k \oplus \mathbb{Z}_2$ for some k. Here we also use the Ext-Calculation Lemma 110.5 (4) which says that $\mathrm{Ext}(\mathbb{Z}_2, G) \cong G/2G$.
(5) So let $k > n$. This time we use that by the Top-Homology Theorem 106.3 we know that $\mathrm{H}_k(M; \mathbb{Z}) = 0$ and that we know from Theorems 106.1 and 106.3 that $\mathrm{H}_{k-1}(M; \mathbb{Z})$ is either zero or isomorphic to $\mathbb{Z}$. The desired statement follows again from the Universal Coefficient Theorem 110.12 and the fact that by the Ext-Calculation Lemma 110.5 we have $\mathrm{Ext}(\mathbb{Z}, G) = 0$. ■

Lemma 110.15. (Dual Fundamental Class Lemma) Let M be a compact oriented connected non-empty n-dimensional topological manifold. Then we have $\mathrm{H}^n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$. Furthermore there exists a unique generator $[M]^* \in \mathrm{H}^n(M, \partial M; \mathbb{Z})$ which satisfies $\langle [M]^*, [M] \rangle_M = 1$.[1341]

Proof. Let M be a compact oriented connected n-dimensional non-empty topological manifold.

[1341] Here $[M] \in \mathrm{H}_n(M, \partial M; \mathbb{Z})$ denotes the fundamental class provided by the Fundamental Class-Corollary 106.4.

It follows from the Universal Coefficient Theorem 110.12 for Cohomology Groups together with Proposition 106.22 that the map

$$\mathrm{H}^n(M,\partial M;\mathbb{Z}) \xrightarrow{\mathrm{ev}} \mathrm{Hom}(\underbrace{\mathrm{H}_n(M,\partial M;\mathbb{Z})}_{=\mathbb{Z}\cdot[M]},\mathbb{Z})$$

is an isomorphism. By the Fundamental Class-Corollary 106.4 (3) we have the equality $\mathrm{H}_n(M,\partial M;\mathbb{Z}) = \mathbb{Z}\cdot[M]$. Hence there exists a unique element $[M]^* \in \mathrm{H}^n(M,\partial M;\mathbb{Z})$ with $\mathrm{ev}([M]^*)([M]) = 1$, i.e. with $\langle [M]^*, [M]\rangle_M = 1$. ■

Definition.

(1) Given a compact oriented connected non-empty n-dimensional topological manifold M we refer to the uniquely determined element $[M]^* \in \mathrm{H}^n(M,\partial M;\mathbb{Z})$ from the Dual Fundamental Class Lemma 110.15 as the dual fundamental class of M.

(2) Given a compact oriented n-dimensional non-empty topological manifold M with finitely many components $M_1,\ldots,M_k$ we define its dual fundamental class to be

$$[M]^* := [M_1]^* + \cdots + [M_k]^* \in \underbrace{\mathrm{H}^n(M_1,\partial M_1;\mathbb{Z}) \oplus \cdots \oplus \mathrm{H}^n(M_k,\partial M_k;\mathbb{Z})}_{=\mathrm{H}^n(M,\partial M;\mathbb{Z})}.$$

Example. We consider the smooth manifold S^1 with the usual orientation. We saw on page 2201 that the singular simplex $\mu\colon \Delta^1 \to S^1$ that is given by $(1-t,t) \mapsto \exp(2\pi\,\mathrm{i}t)$ represents the fundamental class of S^1. On page 2404 we introduced a cocycle $\theta_{\mathbb{Z}} \in \mathrm{C}^1;\mathbb{Z})$ and we had explicitly calculated that $\langle [\theta_{\mathbb{Z}}], [S^1]\rangle = \theta_{\mathbb{Z}}(\mu) = 1$. This shows immediately that $[\theta_{\mathbb{Z}}] \in \mathrm{H}^1(S^1;\mathbb{Z})$ is the dual fundamental class of S^1.

Later on we will need the following lemma.

Lemma 110.16. Let $f\colon M \to N$ be a map between closed oriented connected non-empty n-dimensional topological manifolds. Then

$$f^*([N]^*) \;=\; \deg(f)\cdot[M]^*.$$

Proof. Let $f\colon M \to N$ be a map between closed, oriented connected n-dimensional non-empty topological manifolds. It follows from the Dual Fundamental Class Lemma 110.15 that any class in $\varphi = \mathrm{H}^n(M;\mathbb{Z})$ is of the form $\varphi = k\cdot[M]^*$ for some $k \in \mathbb{Z}$ and it follows from $\langle [M]^*, [M]\rangle = 1$ and the bilinearity of the Kronecker pairing that $k = \langle \varphi, [M]\rangle_M$. In our case we have

$$\langle f^*([N]^*), [M]\rangle \underset{\uparrow}{=} \langle [N]^*, f_*([M])\rangle \underset{\uparrow}{=} \langle [N]^*, \deg(f)\cdot[N]\rangle \underset{\uparrow}{=} \deg(f)\cdot\underbrace{\langle [N]^*, [N]\rangle}_{=1} = \deg(f).$$

Lemma 109.6 (3) — by definition of $\deg(f)$ — since $\langle\,,\,\rangle$ is bilinear

It follows from the above discussion that $f^*([N]^*) = \deg(f)\cdot[M]^*$. ■

Next we record the following useful consequence of the Universal Coefficient Theorem 110.12 for Cohomology Groups:

Proposition 110.17. (First Cohomology Proposition) Let X be a path-connected topological space, let $x_0 \in X$ and let G be an abelian group. Then the Hurewicz homomorphism $\pi_1(X,x_0) \to \mathrm{H}_1(X)$ from page 1832 and the evaluation homomorphism

$\mathrm{ev}\colon \mathrm{H}^1(X;G) \to \mathrm{Hom}(\mathrm{H}_1(X);G)$ induce natural isomorphisms

$$\mathrm{H}^1(X;G) \overset{\cong}{\to} \mathrm{Hom}(\mathrm{H}_1(X);G) \overset{\cong}{\to} \mathrm{Hom}(\pi_1(X,x_0),G).$$

One often uses the natural isomorphisms from the First Cohomology Proposition 110.17 to identify these groups.

Proof. As we pointed out on page 2444 and Proposition 84.2 (5), both maps are natural. So it remains to show that both maps are isomorphisms.

We insert one extra map into the above sequence of maps to see that the maps are indeed isomorphisms:

$$\mathrm{H}^1(X;G) \underset{\uparrow}{\overset{\mathrm{ev}}{\longrightarrow}} \mathrm{Hom}(\mathrm{H}_1(X);G) \overset{\text{Hurewicz Theorem 84.5}}{\underset{\cong}{\downarrow}} \mathrm{Hom}(\pi_1(X,x_0)_{\mathrm{ab}},G) \underset{\uparrow}{\overset{\cong}{\longleftarrow}} \mathrm{Hom}(\pi_1(X,x_0),G).$$

isomorphism by Theorem 110.12 since $\mathrm{H}_0(X) \cong \mathbb{Z}$ by the H_0-Proposition 72.5

by the Abelianization Proposition 53.20 (2), since G is abelian

■

110.4. **Cohomology with field coefficients.** Before we continue we introduce a slight variation on the Kronecker pairing defined in Lemma 109.6:

Definition. Let C_* be a chain complex and let R be a commutative ring. We consider the map

$$\begin{array}{rcl} \langle\ ,\ \rangle\colon \mathrm{H}^n(C;R) \times \mathrm{H}_n(C;R) & \to & R \\ \left([\varphi], \left[\sum_{i=1}^k c_i \otimes a_i\right]\right) & \mapsto & \sum_{i=1}^k \varphi(c_i)\cdot a_i. \end{array}$$

We leave it to the reader to verify that this map is well-defined and bilinear. We refer to this pairing also as the Kronecker pairing.

The next proposition relates homology and cohomology with coefficients in a field $\mathbb{F}$.

Proposition 110.18. (Cohomology-Field Coefficients Proposition) Let (X,A) be a pair of topological spaces and let $\mathbb{F}$ be a field. The map

$$\begin{array}{rcl} \mathrm{ev}\colon \mathrm{H}^n(X,A;\mathbb{F}) & \to & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_n(X,A;\mathbb{F}),\mathbb{F}) \\ \varphi & \mapsto & \left(\begin{array}{rcl} \mathrm{H}_n(X,A;\mathbb{F}) & \to & \mathbb{F} \\ \sigma & \mapsto & \langle\varphi,\sigma\rangle \end{array}\right) \end{array}$$

is a natural isomorphism of $\mathbb{F}$-vector spaces.[1342]

We postpone the proof of the Cohomology-Field Coefficients Proposition 110.18 to the next section. In most applications we are only interested in the following corollary to the Cohomology-Field Coefficients Proposition 110.18.

[1342] Note that "natural" makes sense since $(X,A) \mapsto \mathrm{H}^k(X,A;\mathbb{F})$ and $(X,A) \mapsto \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_k(X,A;\mathbb{F}),\mathbb{F})$ both define *contravariant* functors from the category of pairs of topological spaces to the category of $\mathbb{F}$-vector spaces.

Corollary 110.19. Let (X, A) be a pair of topological space and let $k \in \mathbb{N}_0$ such that $\mathrm{H}_k(X, A; \mathbb{Z})$ and $\mathrm{H}_{k-1}(X, A; \mathbb{Z})$ are finitely generated. (By the Topological Manifolds-Invariants Proposition 104.14 (4) we know if X and A are compact topological manifolds, then this condition is satisfied.)

(1) Given any field $\mathbb{F}$ the homology and cohomology groups $\mathrm{H}_k(X, A; \mathbb{F})$ and $\mathrm{H}^k(X, A; \mathbb{F})$ are finite-dimensional $\mathbb{F}$-vector spaces with

$$\dim_{\mathbb{F}}(\mathrm{H}_k(X, A; \mathbb{F})) \;=\; \dim_{\mathbb{F}}(\mathrm{H}^k(X, A; \mathbb{F})).$$

(2) For any subfield $\mathbb{F}$ of $\mathbb{C}$ we have

$$\dim_{\mathbb{F}}(\mathrm{H}_k(X, A; \mathbb{F})) \;=\; \dim_{\mathbb{F}}(\mathrm{H}^k(X, A; \mathbb{F})) \;=\; b_n(X, A).$$

Remark. Note that in Corollary 110.19 we only claim that the homology and cohomology groups are abstractly isomorphic. We do not claim that there is a natural isomorphism. In fact it does not make sense to ask for a natural isomorphism since the homology groups of a topological space X with $\mathbb{F}$-coefficients are covariant in X whereas the cohomology groups of X with $\mathbb{F}$-coefficients are contravariant in X.

Proof. First note that Statement (2) is an immediate consequence of Statement (1) and the discussion on page 1932. So let turn to the proof of Statement (1). To simplify the notation we only consider the case $A = \varnothing$. Let X be a topological space and let $k \in \mathbb{N}_0$ such that $\mathrm{H}_k(X; \mathbb{Z})$ and $\mathrm{H}_{k-1}(X; \mathbb{Z})$ are finitely generated. Let p be a prime number. By the Finitely Generated Abelian Group Classification Theorem 51.4 we can make the identifications

$$\mathrm{H}_{k-1}(X; \mathbb{Z}) \;=\; \mathbb{Z}^r \oplus \bigoplus_{i=1}^{\alpha} \mathbb{Z}_{a_i} \oplus \bigoplus_{i=1}^{\beta} \mathbb{Z}_{b_i} \quad \text{and} \quad \mathrm{H}_k(X; \mathbb{Z}) \;=\; \mathbb{Z}^s \oplus \bigoplus_{i=1}^{\gamma} \mathbb{Z}_{c_i} \oplus \bigoplus_{i=1}^{\delta} \mathbb{Z}_{d_i}$$

where all the a_i, b_i, c_i and d_i are non-zero, and where each a_i and each c_i is divisible by p and none of the b_i and d_i are divisible by p. We start out with the following claim:

Claim. We have

$$\dim_{\mathbb{F}}(\mathrm{H}_k(X, \mathbb{F})) \;=\; \begin{cases} s, & \text{if } \mathbb{F} \text{ is a field of characteristic zero,} \\ s + \alpha + \gamma, & \text{if } \mathbb{F} \text{ is a field of characteristic } p. \end{cases}$$

Proof. First assume that $\mathbb{F}$ is a field of characteristic zero. In this case we have

$$\mathrm{H}_k(X; \mathbb{F}) \;\underset{\uparrow}{\cong}\; \mathrm{H}_k(X; \mathbb{Z}) \otimes \mathbb{F} \;=\; \left(\mathbb{Z}^s \oplus \bigoplus_{i=1}^{\gamma} \mathbb{Z}_{c_i} \oplus \bigoplus_{i=1}^{\delta} \mathbb{Z}_{d_i}\right) \otimes \mathbb{F} \;\underset{\uparrow}{\cong}\; \mathbb{F}^s.$$

by Theorem 89.17 and the Tor-Properties Lemma 89.15 (4) since $\mathbb{F}$ is torsion-free — see page 1910

Now suppose that $\mathbb{F}$ is a field of characteristic p. We make the following observation. For any $n \in \mathbb{N}$ we have

$$\mathbb{Z}_n \otimes \mathbb{F} \underset{\uparrow}{=} \operatorname{coker}\left(\mathbb{F} \xrightarrow{\cdot n} \mathbb{F}\right) = \begin{cases} \mathbb{F}, & \text{if } p|n, \\ 0, & \text{else} \end{cases} \quad \text{and} \quad \operatorname{Tor}(\mathbb{Z}_n, \mathbb{F}) \underset{\uparrow}{=} \ker\left(\mathbb{F} \xrightarrow{\cdot n} \mathbb{F}\right) = \begin{cases} \mathbb{F}, & \text{if } p|n, \\ 0, & \text{else.} \end{cases}$$

Tensor Lemma 89.3 (3) — Tor-Properties Lemma 89.15 (6)

Then

$$\mathrm{H}_k(X; \mathbb{F}) \;\underset{\uparrow}{\cong}\; \mathrm{H}_k(X; \mathbb{Z}) \otimes \mathbb{F} \oplus \operatorname{Tor}(\mathrm{H}_{k-1}(X; \mathbb{Z}), \mathbb{F}) \;\underset{\uparrow}{\cong}\; \mathbb{F}^s \oplus \mathbb{F}^{\gamma} \oplus \mathbb{F}^{\alpha}.$$

by Theorem 89.17 — by the Tor-Properties Lemma 89.15, the discussion on page 1910 and the above observations

⊞

Now we have

$$\dim_{\mathbb{F}}(\mathrm{H}^k(X;\mathbb{F})) \underset{\uparrow}{=} \dim_{\mathbb{F}}(\mathrm{Hom}(\mathrm{H}_k(X;\mathbb{F}),\mathbb{F})) \underset{\uparrow}{=} \dim_{\mathbb{F}}(\mathrm{H}_k(X;\mathbb{F}))$$

Cohomology-Field Coefficients Proposition 110.18 since by the claim $\mathrm{H}_k(X;\mathbb{F})$ is finite-dimensional

We could also prove the equality $\dim_{\mathbb{F}}(\mathrm{H}^k(X;\mathbb{F})) = \dim_{\mathbb{F}}(\mathrm{H}_k(X;\mathbb{F}))$ without appealing to the Cohomology-Field Coefficients Proposition 110.18. Indeed, using Theorem 110.12, the fact that for any abelian group G we have $\mathrm{Hom}(\mathbb{Z}_n, G) \cong \ker(G \xrightarrow{\cdot n} G)$ and Proposition 110.6 one can easily compute $\dim_{\mathbb{F}}(\mathrm{H}^k(X;\mathbb{F}))$ and one ends up with the same calculation as in the above claim. We leave it to the reader to fill in the details. ■

110.5. **Proof of the Cohomology-Field Coefficients Proposition 110.18 (∗).** In this section we give the proof of the Cohomology-Field Coefficients Proposition 110.18. Before we can provide the proof we need to prepare the ground.

Let $\mathbb{F}$ be a field. We define a *chain complex over* $\mathbb{F}$ to be a chain complex (V_*, v_*) where all chain groups are $\mathbb{F}$-vector spaces and all boundary maps are $\mathbb{F}$-homomorphisms. For example, if $(\mathrm{C}_*, \partial_*)$ is a chain complex of abelian groups, then $(C_* \otimes \mathbb{F}, \partial_* \otimes \mathrm{id})$ is easily seen to be a chain complex over $\mathbb{F}$.

Given a chain complex (V_*, v_*) over the field $\mathbb{F}$ and given $n \in \mathbb{N}_0$ its n-homology

$$\mathrm{H}_n(V) \;:=\; \frac{\ker(v_n\colon V_n \to V_{n-1})}{\mathrm{im}(v_{n+1}\colon V_{n+1} \to V_n)}.$$

is an $\mathbb{F}$-vector space in a natural way.

We also define cochain complexes and cochain maps over $\mathbb{F}$ in the obvious way. For example, if (C_*, c_*) is a chain complex, then the dual cochain complex $(\mathrm{Hom}(\mathrm{C}_*(X,\mathbb{F})), c^*)$ is a cochain complex over $\mathbb{F}$.

Given a chain complex (V_*, v_*) over $\mathbb{F}$ we obtain a cochain complex by considering $(\mathrm{Hom}_{\mathbb{F}}(V_*;\mathbb{F}), v^*)$ and we define its n-th cohomology group to be the $\mathbb{F}$-vector space

$$\mathrm{H}^n(V;\mathbb{F}) \;:=\; \frac{\ker(v_{n+1}^*\colon \mathrm{Hom}_{\mathbb{F}}(V_n,\mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(V_{n+1},\mathbb{F}))}{\mathrm{im}(v_n^*\colon \mathrm{Hom}_{\mathbb{F}}(V_{n-1},\mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(V_n,\mathbb{F}))}.$$

Theorem 110.20. Let $\mathbb{F}$ be a field and let (V_*, v_*) be a chain complex over $\mathbb{F}$. Then the map

$$\begin{array}{rcl} \mathrm{ev}_{\mathbb{F}}\colon \mathrm{H}^k(V;\mathbb{F}) & \mapsto & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_k(V),\mathbb{F}). \\ (\varphi\colon V_k \to \mathbb{F}) & \to & \begin{pmatrix} \mathrm{H}_k(V) & \to & \mathbb{F} \\ [c] & \mapsto & \varphi(c) \end{pmatrix} \end{array}$$

is well-defined and it is an isomorphism of $\mathbb{F}$-vector spaces.

Proof. This theorem is in fact identical to Theorem **??**. If the reader did not make it to Theorem **??**, then note that the proof of Theorem 110.20 is a variation, and mostly a simplification, of the proof of the Universal Coefficient Theorem 110.11 for Cohomology Groups. ■

We will also need the following lemma.

Lemma 110.21. Let C be an abelian group and let $\mathbb{F}$ be a field. Then the map[1343]

$$\begin{array}{rcl} \Phi_C\colon \operatorname{Hom}_{\mathbb{Z}}(C,\mathbb{F}) & \to & \operatorname{Hom}_{\mathbb{F}}(C\otimes\mathbb{F},\mathbb{F}) \\ (\varphi\colon C\to\mathbb{F}) & \mapsto & \left(\begin{array}{rcl} C\otimes\mathbb{F} & \to & \mathbb{F} \\ \sum\limits_{i=1}^{k} c_i\otimes a_i & \mapsto & \sum\limits_{i=1}^{k}\varphi(c_i)\cdot a_i \end{array}\right) \end{array}$$

is an isomorphism.

Proof. It is straightforward to verify that an inverse map to Φ_C is given by the following homomorphism of $\mathbb{F}$-vector spaces:

$$\begin{array}{rcl} \Psi_C\colon \operatorname{Hom}_{\mathbb{F}}(C\otimes\mathbb{F},\mathbb{F}) & \to & \operatorname{Hom}_{\mathbb{Z}}(C,\mathbb{F}) \\ (\varphi\colon C\otimes\mathbb{F}\to\mathbb{F}) & \mapsto & \big(C \xrightarrow{c\mapsto c\otimes 1} C\otimes\mathbb{F} \xrightarrow{\varphi} \mathbb{F}\big). \end{array}$$

■

Corollary 110.22. Let C_* be a chain complex and let $\mathbb{F}$ be a field. The maps

$$\Phi_{\mathrm{C}_k}\colon \operatorname{Hom}_{\mathbb{Z}}(C_k,\mathbb{F}) \to \operatorname{Hom}_{\mathbb{F}}(\mathrm{C}_k\otimes\mathbb{F},\mathbb{F}) \quad \text{with } k\in\mathbb{N}_0$$

from the previous lemma define an isomorphism of cochain complexes of $\mathbb{F}$-vector spaces, in particular for each $k\in\mathbb{N}_0$ they induce an isomorphism

$$\Phi\colon \mathrm{H}^k(C;\mathbb{F}) \to \mathrm{H}^k(C\otimes\mathbb{F};\mathbb{F})$$

of $\mathbb{F}$-vector spaces.

Proof. By Lemma 110.21 the maps Φ_{C_k}, $k\in\mathbb{N}_0$, are isomorphisms of the cochain vector spaces. It is straightforward to see that these maps are also cochain maps. ■

Now we can finally provide the proof of the Cohomology-Field Coefficients Proposition 110.18.

Proof of the Cohomology-Field Coefficients Proposition 110.18. Let (X,A) be a pair of topological spaces and let $\mathbb{F}$ be a field. We write $\mathrm{C}_* = \mathrm{C}_*(X,A)$. Given $k\in\mathbb{N}_0$ we consider the diagram

$$\begin{array}{ccc} & \mathrm{H}^k(C\otimes\mathbb{F};\mathbb{F}) & \\ {\scriptstyle\Phi}\nearrow & & \searrow{\scriptstyle \mathrm{ev}_{\mathbb{F}}} \\ \mathrm{H}^k(C;\mathbb{F}) & \xrightarrow[\mathrm{ev}]{} & \operatorname{Hom}_{\mathbb{F}}(\mathrm{H}_n(C\otimes\mathbb{F}),\mathbb{F}). \end{array}$$

It follows easily from the definitions that for any representative in $\mathrm{H}^k(C;\mathbb{F})$ the image under the composition of the two diagonal maps agrees with the image under the map ev. This implies that ev is well-defined and that the diagram commutes.

Also note that the left diagonal map is an isomorphism by Corollary 110.22 whereas the right diagonal map is an isomorphism by Theorem 110.20. In particular the bottom map is an isomorphism. But that is exactly what we had wanted to show. ■

[1343]Note that to the left we consider homomorphisms of $\mathbb{Z}$-modules, i.e. homomorphisms of abelian groups whereas to the right we consider homomorphisms of $\mathbb{F}$-vector spaces.

110.6. **Change of coefficients in cohomology.** Let $(\mathrm{C}_*, \partial_*)$ be a chain complex of free abelian groups and let G be an abelian group. By the Algebraic Universal Coefficient Theorem 89.16 we can relate the homology groups of $C_* \otimes G$ to the homology groups of C_*. Furthermore, by the Universal Coefficient Theorem for Cohomology Groups 110.11 we can relate the cohomology groups $\mathrm{H}^*(\mathrm{C}_*; G)$ to the homology groups $\mathrm{H}_*(C)$. The following question arises.

Question 110.23. Can we relate the cohomology groups $\mathrm{H}^*(C; G)$ to the cohomology groups $\mathrm{H}^*(C; \mathbb{Z})$?

It turns out that the question is slightly subtle. To formulate the answer we need the following lemma.

Lemma 110.24. Let G be an abelian group.

(1) For any abelian group H the map

$$\begin{array}{rcl} \mu\colon \mathrm{Hom}(H,\mathbb{Z}) \otimes G & \to & \mathrm{Hom}(H,G) \\ \sum_{i=1}^m \varphi_i \otimes g_i & \mapsto & \left(\begin{array}{ccc} H & \to & G \\ h & \mapsto & \sum_{i=1}^m \varphi_i(h) \cdot g_i \end{array} \right) \end{array}$$

is a well-defined *natural* homomorphism.

(2) Given a chain complex $(\mathrm{C}_*, \partial_*)$ the map

$$\begin{array}{rcl} \mathrm{H}^n(C;\mathbb{Z}) \otimes G & \stackrel{\mu}{\to} & \mathrm{H}^n(C;G) \\ \sum_{i=1}^m [\varphi_i] \otimes g_i & \mapsto & \sum_{i=1}^m [\mu(\varphi_i \otimes g_i)] \end{array}$$

is well-defined.

Proof. The proof is elementary and is left to the reader. ∎

The following theorem gives an affirmative answer to Question 110.23, albeit only under an extra hypothesis.

Theorem 110.25. (Universal Coefficient Theorem) Let $(\mathrm{C}_*, \partial_*)$ be a chain complex of free abelian groups and let G be an abelian group. We suppose that all cohomology groups $\mathrm{H}^n(\mathrm{C}_*; \mathbb{Z})$ are finitely generated. Then for each $n \in \mathbb{N}_0$ there exists a short exact sequence of the form

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{H}^n(C;\mathbb{Z}) \otimes G & \stackrel{\mu}{\to} & \mathrm{H}^n(C;G) & \to & \mathrm{Tor}(\mathrm{H}^{n+1}(C;\mathbb{Z}), G) & \to & 0. \\ & & [\varphi] \otimes g & \mapsto & [\mu(\varphi \otimes g)] & & & & \end{array}$$

Furthermore the short exact sequence splits.[1344]

The key to proving the Universal Coefficient Theorem 110.25 is the following elementary but subtle lemma.

Lemma 110.26. Let G be an abelian group. If H is a free abelian group of *finite rank*, then the map

$$\mu\colon \mathrm{Hom}(H,\mathbb{Z}) \otimes G \to \mathrm{Hom}(H,G)$$

[1344] As always, the splitting is not natural. In fact in this theorem we do *not* even claim that the short exact sequence is natural. The reason is that the construction of the short exact sequence involves a choice of the chain complex D_* and we do not intend to discuss the question how that choice affects the map on the right hand side.

is an isomorphism.

Proof. Let $v_1, \dots, v_n$ be a basis for the finitely generated free abelian group H. We denote by $\varphi_1, \dots, \varphi_n$ the dual basis of $\mathrm{Hom}(H, \mathbb{Z})$. In other words, these are the homomorphisms that are uniquely determined by $\varphi_i(v_j) = \delta_{ij}$. We leave it to the reader to verify that the map

$$\begin{aligned} \nu\colon \mathrm{Hom}(H, G) &\to \mathrm{Hom}(H, \mathbb{Z}) \otimes G \\ \psi &\mapsto \sum_{i=1}^{n} \varphi_i \otimes \psi(v_i) \end{aligned}$$

is an inverse to the homomorphism μ. ■

We will also need the following lemma.

Lemma 110.27. Let $(\mathrm{C}_*, \partial_*)$ be a chain complex of free abelian groups. If all cohomology groups $\mathrm{H}^n(\mathrm{C}_*; \mathbb{Z})$ are finitely generated, then all homology groups $\mathrm{H}_n(C_*)$ are also finitely generated.

Proof of Lemma 110.27. Let $(\mathrm{C}_*, \partial_*)$ be a chain complex of free abelian groups such that all cohomology groups $\mathrm{H}^n(\mathrm{C}_*; \mathbb{Z})$ are finitely generated. By the Universal Coefficient Theorem 110.11 for Cohomology Groups there exists an isomorphism

$$\mathrm{H}^n(C; \mathbb{Z}) \cong \mathrm{Ext}(\mathrm{H}_{n-1}(C), \mathbb{Z}) \oplus \mathrm{Hom}(\mathrm{H}_n(C), \mathbb{Z}) \quad \text{for every } n \in \mathbb{N}_0.$$

We see that all the groups $\mathrm{Ext}(\mathrm{H}_{n-1}(C), \mathbb{Z})$ and $\mathrm{Hom}(\mathrm{H}_n(C), \mathbb{Z})$ are finitely generated, in particular they are countable. But by Proposition 110.10 this implies that all the homology groups $\mathrm{H}_n(C_*)$ are finitely generated. ■

Proof of the Universal Coefficient Theorem 110.25. Let G be an abelian group. Furthermore let $(\mathrm{C}_*, \partial_*)$ be a chain complex of free abelian groups such that all cohomology groups $\mathrm{H}^n(\mathrm{C}_*; \mathbb{Z})$ are finitely generated. It follows from Lemma 110.27 that all homology groups $\mathrm{H}_n(C_*)$ are also finitely generated. Thus we obtain from the Finite-Rank Chain Complex Lemma 81.14 that there exists a chain homotopy equivalence $h\colon C_* \to D_*$ from $(\mathrm{C}_*, \partial_*)$ to a chain complex (D_*, ∂_*) such that each chain group D_n is a finitely generated free abelian group. Now we consider the following diagram:

$$\begin{array}{ccccccccc}
0 \to & \overbrace{\mathrm{H}_n(\mathrm{Hom}(D_*, \mathbb{Z}))}^{=\mathrm{H}^n(D;\mathbb{Z}) \text{ by def.}} \otimes G & \xrightarrow{[\varphi]\otimes g \mapsto [\varphi \otimes g]} & \mathrm{H}_n(\mathrm{Hom}(D_*, \mathbb{Z}); G) & \to & \mathrm{Tor}(\overbrace{\mathrm{H}_{n+1}(\mathrm{Hom}(D_*, \mathbb{Z}))}^{=\mathrm{H}^{n+1}(D;\mathbb{Z}) \text{ by def.}}, G) & \to 0. \\
& \uparrow{=} & & \downarrow{\cong}\,\mu & & \uparrow{=} & & & \\
& \mathrm{H}^n(D; \mathbb{Z}) \otimes G & \xrightarrow{\mu} & \mathrm{H}_n(\mathrm{Hom}(D_*, G)) & & \mathrm{Tor}(\mathrm{H}^{n+1}(D; \mathbb{Z}), G) & & & \\
& \uparrow{=} & & \uparrow{=} & & \uparrow{=} & & & \\
& \mathrm{H}^n(D; \mathbb{Z}) \otimes G & \xrightarrow{\mu} & \mathrm{H}^n(D_*; G) & & \mathrm{Tor}(\mathrm{H}^{n+1}(D; \mathbb{Z}), G) & & & \\
& \downarrow{\cong}\,h^* & & \downarrow{\cong}\,h^* & & \downarrow{\cong}\,h^* & & & \\
& \mathrm{H}^n(C; \mathbb{Z}) \otimes G & \xrightarrow{\mu} & \mathrm{H}^n(C; G) & & \mathrm{Tor}(\mathrm{H}^{n+1}(C; \mathbb{Z}), G). & & &
\end{array}$$

We make the following clarifications:

(1) The horizontal sequence on the top is the short exact sequence obtained from the Algebraic Universal Coefficient Theorem 89.16 which we applied to the chain complex $E_* = \mathrm{Hom}(D_{-*}, \mathbb{Z})$. We need to invert the sign of the degrees twice, thus we end up with the degree $n + 1 = -((-n) - 1)$ on the right hand side. We know by the UCT-Splitting Proposition 89.21 that the short exact sequence splits.

(2) It follows from Lemma 110.26 that the middle vertical map between the first two rows is an isomorphism. Here we use that each chain group of D_* is a free abelian group of finite rank.
(3) One can easily verify that the diagram commutes.
(4) By the Dual Cochain Map Lemma 108.8 (4) the bottom vertical maps are isomorphisms.

It is now clear that we obtain the desired short exact sequence. ■

The following theorem gives a practical "topological" version of the Universal Coefficient Theorem 110.25.

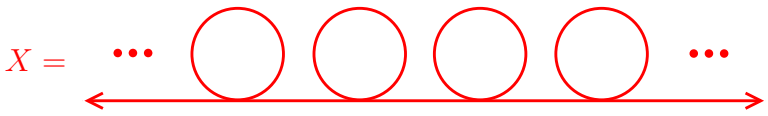
Theorem 110.28. (Universal Coefficient Theorem) Let X be a topological space and let G be an abelian group. If all homology groups $\mathrm{H}_n(X)$ of X are finitely generated (e.g. X could be a compact topological manifold or X could be a CW-complex that has only finitely many cells in each dimension[1345]), then for each $n \in \mathbb{N}_0$ there exists a short exact sequence of the form

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{H}^n(X;\mathbb{Z}) \otimes G & \xrightarrow{\mu} & \mathrm{H}^n(X;G) & \to & \mathrm{Tor}(\mathrm{H}^{n+1}(X;\mathbb{Z}),G) & \to & 0. \\ & & [\varphi] \otimes g & \mapsto & [\mu(\varphi \otimes g)] & & & & \end{array}$$

Furthermore the short exact sequence splits.

Proof. This theorem follows immediately from applying the Universal Coefficient Theorem 110.25 to the singular chain complex of X. ■

Remark. In general the hypothesis on X that we put into the Universal Coefficient Theorem 110.28 cannot be dropped. For example consider the infinite string of circles X as shown in the figure below and we take $G = \mathbb{Q}$. By the Tor-Properties Lemma 89.15 (4) we know that all Tor-groups $\mathrm{Tor}(H, \mathbb{Q})$ vanish. In Exercise 110.10 we will see that the groups $\mathrm{H}^1(X;\mathbb{Z}) \otimes \mathbb{Q}$ and $\mathrm{H}^1(X;\mathbb{Q})$ are not isomorphic. This shows that in this setting the sequence stated in the Universal Coefficient Theorem 110.28 is not exact.



the groups
$H^1(X;\mathbb{Z}) \otimes \mathbb{Q}$ and $H^1(X;\mathbb{Q})$
are not isomorphic

The following lemma gives a convenient interpretation of the map μ appearing in the Universal Coefficient Theorem 110.28.

Lemma 110.29. Let X be a topological space, let G be an abelian group and let $f\colon \mathbb{Z} \to G$ be a homomorphism. The following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^n(X;\mathbb{Z}) & & \\ {\scriptstyle a \mapsto a \otimes f(1)}\downarrow & \searrow^{f_*} & \\ \mathrm{H}^n(X;\mathbb{Z}) \otimes G & \xrightarrow[\mu]{} & \mathrm{H}^n(X;G). \end{array}$$

[1345] By the Topological Manifolds-Invariants Proposition 104.14 (4) and the Homology-of-CW-Complex Proposition 80.6 (1) such topological spaces qualify.

Proof. The statement follows immediately from explicitly writing down the homomorphisms. ■

Example. Let $n \in \mathbb{N} \cup \{\infty\}$. We denote by $f\colon \mathbb{Z} \to \mathbb{Z}_2$ the obvious group homomorphism. It follows from the Universal Coefficient Theorem 110.28, from Lemma 110.29, from the calculations of the cohomology groups $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ on page 2424, and from the properties of Tor-groups, see the Tor-Properties Lemma 89.15, that for any $k \in \mathbb{N}_0$ the map $f_*\colon \mathrm{H}_*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \to \mathrm{H}_*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}_2)$ is an epimorphism.

We conclude this chapter with the following lemma which unwittingly gets used frequently.

Lemma 110.30. We denote by $\varphi\colon \mathbb{Z} \to \mathbb{Q}$ the obvious inclusion. Let $n \in \mathbb{N}_0$.

(1) Let X be a topological space such that all homology groups are finitely generated. We suppose that $\mathrm{H}^n(X;\mathbb{Z})$ is a finitely generated free abelian group. If $v_1, \ldots, v_m$ is a basis of $\mathrm{H}^n(X;\mathbb{Z})$, then $\varphi_*(v_1), \ldots, \varphi_*(v_m)$ is a basis of $\mathrm{H}^n(X;\mathbb{Q})$.

(2) Let $f\colon X \to Y$ be a map between topological spaces. We suppose that all homology groups of X and Y are finitely generated and we suppose that $\mathrm{H}^n(X;\mathbb{Z})$ and $\mathrm{H}^n(Y;\mathbb{Z})$ are free abelian groups. We consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^n(Y;\mathbb{Z}) & \xrightarrow{f^*} & \mathrm{H}^n(X;\mathbb{Z}) \\ {\scriptstyle\varphi_*}\downarrow & & \downarrow{\scriptstyle\varphi_*} \\ \mathrm{H}^n(Y;\mathbb{Q}) & \xrightarrow{f^*} & \mathrm{H}^n(X;\mathbb{Q}). \end{array}$$

Suppose $v_1, \ldots, v_r$ is a basis for $\mathrm{H}^n(X;\mathbb{Z})$ and that $w_1, \ldots, w_s$ is a basis for $\mathrm{H}^n(Y;\mathbb{Z})$. The matrix that represents $f^*\colon \mathrm{H}^n(Y;\mathbb{Z}) \to \mathrm{H}^n(X;\mathbb{Z})$ with respect to the given bases also represents $f^*\colon \mathrm{H}^n(Y;\mathbb{Q}) \to \mathrm{H}^n(X;\mathbb{Q})$ with respect to the bases $\varphi_*(v_1), \ldots, \varphi_*(v_r)$ and $\varphi_*(w_1), \ldots, \varphi_*(w_s)$.

Proof.

(1) This statement follows easily from the Universal Coefficient Theorem 110.28 together with the Tor-Properties Lemma 89.15 (4) and Lemma 110.29.

(2) The fact that the diagram commutes is an immediate consequence of Lemma 108.9. The remaining statement is just a psychologically helpful reinterpretation of the fact that the diagram commutes. ■

Exercises for Chapter 110.

Exercise 110.1. Let C_* be a chain complex of free abelian groups and let $n \in \mathbb{N}$. Show that there does not exist a *natural* isomorphism

$$\mathrm{H}^n(C;G) \;\cong\; \mathrm{Ext}(\mathrm{H}_{n-1}(C), G) \oplus \mathrm{Hom}(\mathrm{H}_n(C), G).$$

Exercise 110.2. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of finitely generated abelian groups. We assume that A_0 and A_1 are free abelian groups. Show that there exists a connected CW-complex X such that for any $n \in \mathbb{N}$ we have $\mathrm{H}^n(X;\mathbb{Z}) \cong A_n$.
Hint. Use Proposition 79.10.
Remark. The statement is not true if we drop the hypothesis that the groups are finitely generated, in fact as is shown in [**Luc1978**] (see also [**KaW1961**]) there is no topological space X with $\mathrm{H}^n(X;\mathbb{Z}) \cong \mathbb{Q}$ for some $n \in \mathbb{N}_0$.

Exercise 110.3. Let $(P, \leq)$ be a partially ordered set and let Q be a subset. Is an upper bound necessarily a maximal element? And conversely, is a maximal element necessarily an upper bound?

Exercise 110.4. Let X and Y are topological spaces. Suppose that $\mathrm{H}_n(X;\mathbb{Z}) \cong \mathrm{H}_n(Y;\mathbb{Z})$ for some $n \in \mathbb{N}_0$. Does it follow that $\mathrm{H}^n(X;\mathbb{Z})$ and $\mathrm{H}^n(Y;\mathbb{Z})$ are also isomorphic?

Exercise 110.5. Redo Exercise 109.3. More precisely, use the Universal Coefficient Theorem 110.12 for Cohomology Groups to determine the isomorphism types of the cohomology groups of the Klein bottle with $\mathbb{F}_2$-coefficients, with $\mathbb{F}_3$-coefficients and with $\mathbb{Z}$-coefficients.

Exercise 110.6. Give an explicit example of a cocycle $\varphi\colon \mathrm{C}_1(\mathbb{R}\mathrm{P}^2) \to \mathbb{F}_2$ that represents a generator of $\mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \cong \mathbb{F}_2$.

Exercise 110.7. Let M be a compact oriented connected n-dimensional topological manifold with connected non-empty boundary. Show that $\delta([\partial M]^*) = [M]^* \in \mathrm{H}^n(M, \partial M;\mathbb{Z})$ where δ denotes the connecting homomorphism in the long exact sequence of cohomology groups of the pair $(M, \partial M)$.
Hint. Use the Fundamental Class-of-Boundary Proposition 106.27.

Exercise 110.8.

(a) Let G and H be $\mathbb{Q}$-vector spaces. Let $\varphi\colon G \to H$ be a homomorphism of abelian groups. Show that φ is also a homomorphism of $\mathbb{Q}$-vector spaces.
(b) Show that the abelian groups $\mathbb{Q}$ and $\mathbb{Q}^2$ are not isomorphic.
(c) Show that the abelian groups $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic.
(d) Use (a) and (b), together with the existence of Moore spaces, see the Moore Space Existence Proposition 79.11, and together with the Universal Coefficient Theorem 110.12 for Cohomology Groups and Proposition 110.9 to conclude that there exist topological spaces X and Y such that $\mathrm{H}^n(X;\mathbb{Z}) \cong \mathrm{H}^n(Y;\mathbb{Z})$ for all $n \in \mathbb{N}_0$ but such that $\mathrm{H}_1(X;\mathbb{Z})$ is not isomorphic to $\mathrm{H}_1(Y;\mathbb{Z})$.

Remark. This exercise shows that in Corollary 108.19 the roles of homology and cohomology are not symmetric.

Exercise 110.9. Let H be a free abelian group and let G be an abelian group. As in Lemma 110.26 we consider the map

$$\begin{array}{ccc} \mathrm{Hom}(H,\mathbb{Z}) \otimes G & \to & \mathrm{Hom}(H,G) \\ \sum\limits_{i=1}^{m} \varphi_i \otimes g_i & \mapsto & \left(\begin{array}{ccc} H & \to & G \\ h & \mapsto & \sum\limits_{i=1}^{m} \varphi_i(h) \cdot g_i \end{array} \right). \end{array}$$

Show that this map is in general not an isomorphism.

Exercise 110.10. We consider the topological space X shown in the figure on page 2458. Show that the map

$$\mu\colon \mathrm{H}^1(X;\mathbb{Z}) \otimes \mathbb{Q} \to \mathrm{H}^1(X;\mathbb{Q})$$

from Lemma 110.24 is not an isomorphism.

Exercise 110.11. Show that there exists a topological space X such that the two groups $\mathrm{H}^1(X;\mathbb{Z}) \otimes \mathbb{Q}$ and $\mathrm{H}^1(X;\mathbb{Q})$ are not isomorphic.
Hint. You could make use of Proposition 79.10 to find unusual topological spaces.

Exercise 110.12. Let X be a topological space which is $\mathbb{Z}$-finite in the sense of the definition on page 2116, i.e. all homology groups $\mathrm{H}_i(X)$ are finitely generated abelian groups and all but finitely many homology groups $\mathrm{H}_i(X)$ are zero. Let $\varphi\colon X \to X$ be a map. Show that

$$\underbrace{\sum_{k\in\mathbb{N}_0} (-1)^k \cdot \mathrm{tr}(\varphi_*\colon \mathrm{H}_k(X;\mathbb{Z}) \to \mathrm{H}_k(X;\mathbb{Z}))}_{\text{Lefschetz number } \Lambda(\varphi) \text{ as defined on page 2116}} = \sum_{k\in\mathbb{N}_0} (-1)^k \cdot \mathrm{tr}(\varphi^*\colon \mathrm{H}^k(X;\mathbb{Z}) \to \mathrm{H}^k(X;\mathbb{Z})).$$

111. Cohomology groups and limits

111.1. **Direct limits of direct systems.** In the intermediate future direct limits of direct systems and inverse limits of inverse systems will play a large role. To fortify us for these discussions we recall the definitions and we prove some fairly elementary but useful lemmas regarding limits.
First we recall the following definition from page 1256.

Definition. A directed set is a set I together with a relation $\leq$ on I which satisfies the following three conditions:

(1) the relation is reflexive, i.e. for any $i \in I$ we have $i \leq i$.
(2) the relation is transitive, i.e. for any $i, j, k \in I$ we have
$$i \leq j \text{ and } j \leq k \implies i \leq k.$$
(3) for any $i, j \in I$ there exists a $k \in I$ with $i \leq k$ and $j \leq k$.

Examples.

(a) Evidently the natural numbers $\mathbb{N}$ together with the usual "$\leq$" relation form a directed set $(\mathbb{N}, \leq)$.
(b) Let M be a set. We denote by $\mathcal{P}(M)$ the power set of M, i.e. the set of all subsets of M. Then the relation given by "$\subseteq$" turns $\mathcal{P}(M)$ into a directed set. To prove this claim it only remains to verify (3). But this also holds, since for any $X, Y \in \mathcal{P}(M)$, i.e. for any subsets X, Y of M we can take $Z = X \cup Y$ and we then have $X \subseteq Z$ and $Y \subseteq Z$.

Next we recall the following definition from page 1257.

Definition. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. A direct system over the directed set[1352] $(I, \leq)$ is a family of objects $\{X_i\}_{i \in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ij} \colon X_i \to X_j\}$ for all $i, j \in I$ with $i \leq j$ such that the following two conditions are satisfied:

(1) $f_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$,
(2) $f_{ik} = f_{jk} \circ f_{ij}$ for all $i, j, k \in I$ with $i \leq j \leq k$.

Example. Let $\mathcal{C}$ be a category and let
$$X_1 \xrightarrow{g_1} X_2 \xrightarrow{g_2} X_3 \xrightarrow{g_3} X_4 \xrightarrow{g_4} \dots$$
be a sequence of morphisms in the category $\mathcal{C}$. For $i \in \mathbb{N}_0$ we set $f_{ii} = \mathrm{id}$ and for $i \leq j$ we set $f_{ij} = g_{j-1} \circ \dots \circ g_{i+1} \circ g_i \colon X_i \to X_j$. It is clear that $(\{X_i\}_{i \in \mathbb{N}}, \{f_{ij}\}_{i \leq j})$ is a direct system over the directed set $(\mathbb{N}, \leq)$. By a slight abuse of notation we refer to such a sequence $(\{X_i\}_{i \in \mathbb{N}}, \{g_i \colon X_i \to X_{i+1}\}_{i \in \mathbb{N}})$ also as a direct system.

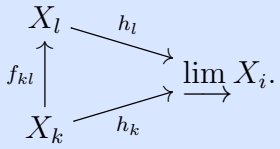
Finally we also recall the definition of a direct limit from page 1258.

Definition. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. Suppose we are given a direct system $(\{X_i\}_{i \in I}, \{f_{ij}\}_{i \leq j})$ in the category $\mathcal{C}$. A direct limit of the direct system is

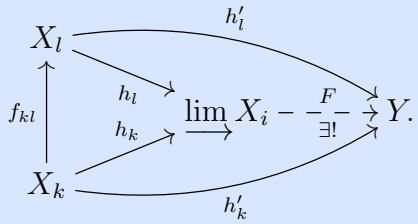
[1352] On page 1257 we had also considered direct systems that are defined over preordered sets, but for the remainder of this course we are only interested in direct systems over directed sets.

an object $\varinjlim X_i$ in $\mathcal{C}$ together with morphisms $h_i\colon X_i \to \varinjlim X_i, i \in I$ in the category $\mathcal{C}$ such that the following two conditions are satisfied:

(1) For all $k \leq l$ we have $h_l \circ f_{kl} = h_k\colon X_k \to \varinjlim X_i$, i.e. for all $k \leq l$ the following diagram commutes



(2) If we are given another object Y and morphisms $h'_i\colon X_i \to Y, i \in I$ that satisfy (1), then there exists a unique morphism $F\colon \varinjlim X_i \to Y$ such that for all $k \leq l$ the following diagram commutes



As we pointed out on page 1258, the direct limit of direct system, if it exists, is in an appropriate sense unique up to isomorphism.

The following proposition is a variation on the Direct Limit Existence Proposition 56.1.

Proposition 111.1. (Direct Limit Existence Proposition) Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ij}\}_{i\leq j})$ be a direct system over one of the following categories:

(1) the category of abelian groups,
(2) the category of R-modules, where R is a commutative ring,
(3) the category of chain complexes,
(4) the category of cochain complexes.

Then[1353]

$$\varinjlim X_i \;:=\; \Big(\bigsqcup_{i\in I} X_i\Big)/\sim \quad \text{where } x \sim f_{ij}(x) \text{ for all } i \leq j,\ x \in X_i$$

with the obvious maps $X_i \to \varinjlim X_i$ is a direct limit. Here we view $\varinjlim X_i$ as a group as follows: if we are given $a \in X_i$ and $b \in X_j$, then by definition of a directed set there exists a $k \in I$ with $i \leq k$ and $j \leq k$, and we define[1354]

$$[a] + [b] \;:=\; \underbrace{\big[\underbrace{f_{ik}(a)}_{\in X_k} + \underbrace{f_{jk}(b)}_{\in X_k}\big]}_{\text{addition in } X_k} \;\in\; \varinjlim X_i.$$

Depending on the category we are dealing with we can also equip $\varinjlim X_i$ with an R-module structure or we can equip it with (co) boundary maps.

[1353] The construction of the direct limit that we gave in the Direct Limit Existence Proposition 56.1 was different, and arguably more complicated, since there we had considered direct systems over arbitrary preordered sets.

[1354] It is straightforward to verify that this definition does not depend on the choice of k.

Proof. This explicit construction of the direct limit was already given on page 1262. The proof of the proposition is a not overly exciting exercise in going through all the definitions. We leave this task to the reader. ■

Now we give three lemmas which will be helpful in getting direct limits "under control".

Lemma 111.2. Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ij}\colon X_i \to X_j\}_{i\leq j})$ be a direct system over I in one of the four categories that we had considered in the Direct Limit Existence Proposition 111.1. Then the following four statements hold:

(1) Every element in $\varinjlim X_i$ lies in the image of one of the maps $X_j \to \varinjlim X_i$.

(2) Let $a_j \in X_j$ and let $a_k \in X_k$. The images of a_j and a_k in $\varinjlim X_i$ agree if and only if there exists an $l \in I$ with $l \geq j$ and $l \geq k$ such that $f_{jl}(a_j) = f_{kl}(a_k) \in X_l$.

(3) Let $a_j \in X_j$. The image of a_j in $\varinjlim X_i$ is zero if and only if there exists a $k \geq j$ such that $f_{jk}(a_j) = 0 \in X_k$.

(4) If all maps $f_{ij}\colon X_i \to X_j$ are isomorphisms, then for any $j \in I$ the obvious map $X_j \to \varinjlim X_i$ is an isomorphism.

Proof.

(1) The first statement follows immediately from the explicit description of the direct limit given in the Direct Limit Existence Proposition 111.1.

(2) The second statement will be proved in Exercise 111.1. For the exercise it is useful to note that the symbol "$\sim$" in the definition of $\varinjlim X_i$ in the Direct Limit Existence Proposition 111.1 is only a relation and we really have to work with the equivalence relation generated by this relation, see page 258 for more information.

(3) The third statement is a special case of the second statement.

(4) The last statement will be proved in Exercise 111.1. ■

The second lemma is useful for simplifying calculations.

Lemma 111.3. (Limit-over-Subset Lemma) Let $(I, \leq)$ be a directed set. Furthermore let $(\{X_i\}_{i\in I}, \{f_{ij}\colon X_i \to X_j\}_{i\leq j})$ be a direct system over I in one of the four categories that we had considered in the Direct Limit Existence Proposition 111.1. Let J be a subset of I such that $(J, \leq)$ is again a directed set.

(1) There exists a unique map

$$\varinjlim_{j\in J} X_j \longrightarrow \varinjlim_{i\in I} X_i$$

such that for any $j \in J$ the following diagram commutes:

$$\begin{array}{ccc} & X_j & \\ \swarrow & & \searrow \\ \varinjlim_{j\in J} X_j & \longrightarrow & \varinjlim_{i\in I} X_i. \end{array}$$

(2) If J has the property that given any $i \in I$ there exists a $j \in J$ with $i \leq j$, then the above map

$$\varinjlim_{j\in J} X_j \longrightarrow \varinjlim_{i\in I} X_i$$

is an isomorphism.

The first part of the lemma is just an immediate consequence of the definition of the direct limit over the directed set J. As we will see, the second statement is a fairly straightforward consequence of the next lemma. But it is also a good exercise to prove the above second statement "by hand".

The following definition is a generalization of the definition on page 1753.

Definition. Let $(I, \leq)$ and $(J, \leq)$ be directed sets.

(1) A morphism between the directed sets $(I, \leq)$ and $(J, \leq)$ is a map $\varphi\colon I \to J$ such that for any $i_1 \leq i_2$ in I we have $\varphi(i_1) \leq \varphi(i_2)$.

(2) A morphism $\varphi\colon I \to J$ is called cofinal if given any $j \in J$ there exists an $i \in I$ with $\varphi(i) \geq j$.

(3) Let $(\{K_i\}_{i\in I}, \{\chi_{i_1 i_2}\}_{i_1\leq i_2})$ and $(\{L_j\}_{j\in J}, \{\lambda_{j_1 j_2}\}_{j_1\leq j_2})$ be direct systems in one of the four categories that we had considered in the Direct Limit Existence Proposition 111.1. A homomorphism between the direct systems is a family $\{f_i\colon K_i \to L_{\varphi(i)}\}_{i\in I}$ of homomorphisms that has the property that for any $i_1 \leq i_2$ the following diagram commutes:

$$\begin{array}{ccc} K_{i_2} & \xrightarrow{f_{i_2}} & L_{\varphi(i_2)} \\ {\scriptstyle \chi_{i_1 i_2}}\uparrow & & \uparrow{\scriptstyle \lambda_{\varphi(i_1)\varphi(i_2)}} \\ K_{i_1} & \xrightarrow{f_{i_1}} & L_{\varphi(i_1)} \end{array}$$

(4) The maps $K_i \xrightarrow{f_i} L_{\varphi(i)} \to \varinjlim L_j$ induce, by the universal property of a direct limit, a map

$$\varinjlim_{i\in I} K_i \to \varinjlim_{j\in J} L_j$$

which is uniquely determined by the property that for any $i \in I$ the following diagram commutes

$$\begin{array}{ccc} K_i & \xrightarrow{f_i} & L_{\varphi(i)} \\ \downarrow & & \downarrow \\ \varinjlim_{i\in I} K_i & \longrightarrow & \varinjlim_{j\in J} L_j. \end{array}$$

Lemma 111.4. (Cofinal-Isomorphism Lemma) Let $\{f_i\colon K_i \to L_{\varphi(i)}\}_{i\in I}$ be a homomorphism between two direct systems $(\{K_i\}_{i\in I}, \{\chi_{i_1 i_2}\}_{i_1\leq i_2})$ and $(\{L_j\}_{j\in J}, \{\lambda_{j_1 j_2}\}_{j_1\leq j_2})$ in one of the four categories that we had considered in the Direct Limit Existence Proposition 111.1. Suppose the following two conditions are satisfied:

(1) the morphism $\varphi\colon I \to J$ is cofinal, and

(2) for each $i \in I$ the map $f_i\colon K_i \to L_{\varphi(i)}$ is an isomorphism.

Then the induced map

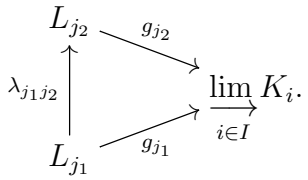
$$\varinjlim_{i\in I} K_i \to \varinjlim_{j\in J} L_j$$

is an isomorphism.

Sketch of proof. Let $j \in J$. Since $\varphi\colon I \to J$ is cofinal we can pick $i \in I$ with $f(i) \geq j$. We define

$$\begin{aligned} g_j\colon L_j &\to \varinjlim_{i\in I} K_i \\ x &\mapsto [f_i^{-1}(\lambda_{j,f(i)}(x))]. \end{aligned}$$

It is straightforward to verify that this map does *not* depend on the choice of i. Using this observation it is elementary to verify that for any $j_1 \leq j_2$ the following diagram commutes:



But by the universal property of the direct limit this implies that we get an induced

$$\varinjlim_{j\in J} L_j \to \varinjlim_{i\in I} K_i.$$

We leave it to the reader to verify that this map is an inverse to the map under consideration. ■

111.2. **Inverse limits of inverse systems.** We recall the following definition from page 1273.

Definition. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. An inverse system in the category $\mathcal{C}$ over I is a family of objects $\{X_i\}_{i\in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ji}\colon X_j \to X_i\}$ for all $i, j \in I$ with $i \leq j$[1355] such that the following two conditions are satisfied:

(1)	f_{ii}	$=$	id_{X_i}	for all $i \in I$,
(2)	f_{ki}	$=$	$f_{ji} \circ f_{kj}$	for all $i, j, k \in I$ with $i \leq j \leq k$.

Example. Let $\mathcal{C}$ be a category and let

$$\ldots \xrightarrow{g_4} X_4 \xrightarrow{g_3} X_3 \xrightarrow{g_2} X_2 \xrightarrow{g_1} X_1$$

be a sequence of morphisms in the category $\mathcal{C}$. For $i \in \mathbb{N}$ we set $f_{ii} = \mathrm{id}$ and for $i \leq j$ we set $f_{ji} = g_i \circ \cdots \circ g_{j-1}\colon X_j \to X_i$. It is clear that $(\{X_i\}_{i\in I}, \{f_{ji}\}_{i\leq j})$ is an inverse system over the directed set $(\mathbb{N}, \leq)$. It is also clear that any inverse system over $(\mathbb{N}, \leq)$ arises that way. By a slight abuse of notation we refer to such a sequence $(\{X_i\}_{i\in\mathbb{N}}, \{g_i\colon X_{i+1} \to X_i\}_{i\in\mathbb{N}})$ also as an inverse system.

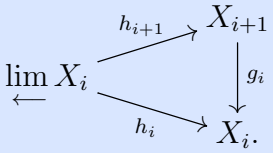
Using the convention from the previous example we can now reformulate in our context the original definition of an inverse limit from page 1273.

Definition. Let $(\{X_i\}_{i\in\mathbb{N}}, \{g_i\colon X_{i+1} \to X_i\}_{i\in\mathbb{N}})$ be an inverse system in a category $\mathcal{C}$. An inverse limit of the inverse system is an object $\varprojlim X_i$ in $\mathcal{C}$ together with morphisms

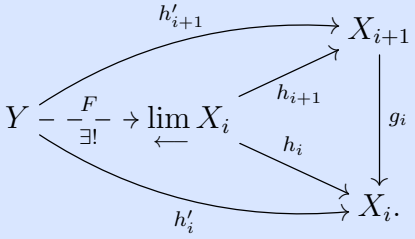
[1355] So in an inverse system, given $i \leq j$ we have a morphism $X_j \to X_i$, whereas in a direct system we had a morphism $X_i \to X_j$.

$h_i\colon \varprojlim X_i \to X_i$, $i \in I$, in the category $\mathcal{C}$ such that the following two conditions are satisfied:

(1) For all $i \in \mathbb{N}$ we have $g_i \circ h_{i+1} = h_i\colon \varprojlim X_i \to X_i$, i.e. for all $i \in \mathbb{N}$ the following diagram commutes



(2) If we are given another object Y and morphisms $h'_i\colon Y \to X_i$, $i \in \mathbb{N}$, that satisfy (1), then there exists a unique morphism $F\colon Y \to \varprojlim X_i$ such that for all $i \in \mathbb{N}$ the following diagram commutes



Throughout the remainder of this course we will only be interested in inverse systems over the directed set $(\mathbb{N}, \leq)$. In the following proposition we give an explicit description of inverse limits for this special case. The proposition can be viewed as a slight variation on the Inverse Limit Existence Proposition 56.15.

Proposition 111.5. (Inverse Limit Existence Proposition) Let $(\{X_i\}_{i\in\mathbb{N}}, \{g_i\colon X_{i+1} \to X_i\}_{i\in\mathbb{N}})$ be an inverse system over one of the following categories:

(1) the category of abelian groups,
(2) the category of R-modules, where R is a commutative ring,
(3) the category of chain complexes,
(4) the category of cochain complexes.

Then
$$\varprojlim X_i \;:=\; \Big\{(x_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} X_i \;\Big|\; g_i(x_{i+1}) = x_i \text{ for all } i \in \mathbb{N}\Big\}$$
with the natural projections $\varprojlim X_i \to X_i$ is a direct limit.

Proof. This proposition is implicitly proved on page 1279. ■

Remark. the Inverse Limit Existence Proposition 111.5 can be summarized as saying that, under the hypothesis of the proposition, we have

$$\varprojlim X_i \;=\; \text{all infinite sequences } (x_1, x_2, x_3, \dots) \text{ of the form } \begin{array}{ccc} x_3 & \in & X_3 \\ \downarrow & & \downarrow g_2 \\ x_2 & \in & X_2 \\ \downarrow & & \downarrow g_1 \\ x_1 & \in & X_1. \end{array}$$

Example. For each $i \in \mathbb{N}$ we define $X_i := \mathbb{Z}^i$ and we denote by

$$\begin{array}{rcl} g_i\colon \mathbb{Z}^{i+1} & \to & \mathbb{Z}^i \\ (x_1, \dots, x_{i+1}) & \mapsto & (x_1, \dots, x_i) \end{array}$$

the projection . Then $(\{\mathbb{Z}^i\}_{i\in\mathbb{N}}, \{g_i\}_{i\in\mathbb{N}})$ forms an inverse system in the category of abelian groups. As we had already pointed out on page 1279, using the description of the inverse limit provided in the Inverse Limit Existence Proposition 111.5 one can show fairly easily that

$$\varprojlim_{i\in\mathbb{N}} \mathbb{Z}^i \;=\; \{(x_1, x_2, x_3, \dots) \mid x_j \in \mathbb{Z}\} \;\cong\; \mathbb{Z}^{\mathbb{N}}.$$

We conclude this section with the following analogue of the Direct Limit Stabilization Lemma 56.2.

Lemma 111.6. (Indirect Limit Stabilization Lemma) Let $f_n\colon X_n \to X_{n+1}$ be a sequence of morphisms in any category. Suppose there exists an $N \in \mathbb{N}$ such that all f_n for $n \geq N$ are isomorphisms. Then $\varprojlim X_n$ exists and the natural map $\varprojlim X_n \to X_N$ is an isomorphism.

Proof. We leave it to the reader to prove this elementary lemma. ■

111.3. **Cohomology groups and inverse limits.** In Propositions 56.7 and 79.4 we saw that in favorable situations the fundamental group and the homology groups of a union of topological spaces are the direct limit of the corresponding fundamental groups and homology groups. In this section we will try to extend these results to cohomology groups. The results will not only be useful for computing cohomology groups, but they will also play an essential role in many proofs later on. For example the proof that de Rham cohomology is isomorphic to singular cohomology with real coefficients will build on the somewhat technical results of this section.
We recall that the Exhaustions-H_*-Proposition 79.4 was a consequence of the following two lemmas. The first lemma is of topological nature.

Lemma 79.5. (Exhaustion-Chain Complex Lemma) Let X be a topological space and let $X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subsets such that every compact subset of X is already contained in one of the X_i. Then the chain complexes $\mathrm{C}_*(X_i), i \in \mathbb{N}$ together with the inclusion induced chain maps $\mathrm{C}_*(X_i) \to \mathrm{C}_*(X_j)$ for $i \leq j$ form a direct system of chain complexes. Furthermore the inclusion induced maps $\mathrm{C}_*(X_i) \to \mathrm{C}_*(X)$ induce an isomorphism

$$\varinjlim \mathrm{C}_*(X_i) \;\xrightarrow{\cong}\; \mathrm{C}_*(X)$$

of chain complexes.

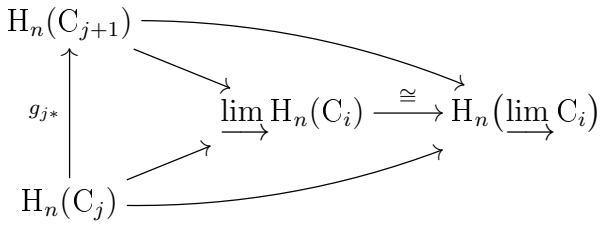
The second lemma is purely algebraic.

Lemma 79.6. (Direct Limit-Homology Lemma) Let $(\{\mathrm{C}_i\}_{i\in\mathbb{N}}, \{g_i\colon \mathrm{C}_i \to \mathrm{C}_{i+1}\}_{i\in\mathbb{N}})$ be a direct system of chain complexes. Then for each $n \in \mathbb{N}_0$ we obtain an induced direct system $(\{\mathrm{H}_n(\mathrm{C}_i)\}_{i\in\mathbb{N}}, \{g_{i*}\}_{i\in\mathbb{N}})$ of abelian groups. Furthermore, given any $n \in \mathbb{N}_0$ the maps $\mathrm{H}_n(\mathrm{C}_i) \to \mathrm{H}_n\big(\varinjlim \mathrm{C}_i\big)$ induce an isomorphism

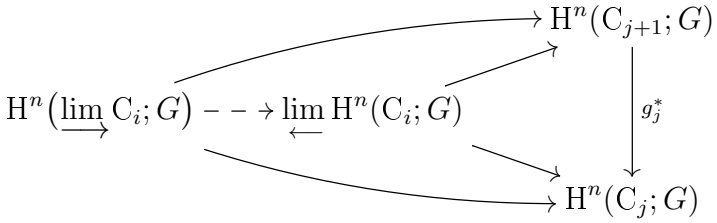
$$\varinjlim \mathrm{H}_n(\mathrm{C}_i) \;\xrightarrow{\cong}\; \mathrm{H}_n\big(\varinjlim \mathrm{C}_i\big).$$

Let $(\{\mathrm{C}_i\}_{i\in\mathbb{N}}, \{g_i\colon \mathrm{C}_i \to \mathrm{C}_{i+1}\}_{i\in\mathbb{N}})$ be a direct system of chain complexes. The last lemma can be summarized as follows: for every $j \in \mathbb{N}$ and any $n \in \mathbb{N}_0$ we have the following

diagram



Now we turn to cohomology groups. Let G be an abelian group. Whereas homology is a covariant functor, cohomology is a contravariant functor. This implies that from the above direct system of chain complexes we obtain for each $n \in \mathbb{N}_0$ an *inverse system* $(\{\mathrm{H}^n(\mathrm{C}_i; G)\}_{i\in\mathbb{N}}, \{g_i^*\colon \mathrm{H}^n(\mathrm{C}_{i+1}; G) \to \mathrm{H}^n(\mathrm{C}_i; G)\}_{i\in\mathbb{N}})$ of abelian groups. Furthermore the maps $\mathrm{C}_i \to \varinjlim \mathrm{C}_i$ induce maps $\mathrm{H}^n(\varinjlim \mathrm{C}_i; G) \to \mathrm{H}^n(\mathrm{C}_i; G)$. Put differently, for any $j \in \mathbb{N}$ we are in the following situation



where the dotted arrow exists by the definition of the inverse limit of the groups $\mathrm{H}^n(\mathrm{C}_i; G)$. The question is whether the dotted arrow is in fact an isomorphism, put differently, the following question arises.

Question 111.7. Let $(\{\mathrm{C}_i\}_{i\in\mathbb{N}}, \{g_i\colon \mathrm{C}_i \to \mathrm{C}_{i+1}\}_{i\in\mathbb{N}})$ be a direct system of chain complexes. Is the above map

$$\mathrm{H}^n\big(\varinjlim \mathrm{C}_i; \mathbb{Z}\big) \;\to\; \varprojlim \mathrm{H}^n(\mathrm{C}_i; \mathbb{Z})$$

always an isomorphism?

It turns out that this is not necessarily the case. To formulate the relationship between the two sides of the above question it is useful to rewrite the inverse limit from the Inverse Limit Existence Proposition 111.5. More precisely, given an inverse system $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ of abelian groups we consider the map

$$\begin{array}{rcl} \mu\colon \prod\limits_{i\in\mathbb{N}} D_i & \to & \prod\limits_{i\in\mathbb{N}} D_i \\ (x_i)_{i\in\mathbb{N}} & \mapsto & (x_i - g_i(x_{i+1}))_{i\in\mathbb{N}}. \end{array}$$

the Inverse Limit Existence Proposition 111.5 can be reformulated as saying that

$$\varprojlim D_i \;=\; \ker(\mu).$$

This leads us to the following definition.

Definition. Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ be an inverse system of abelian groups. Using the above notation we define[1356]

$$\varprojlim{}^1 D_i \;:=\; \operatorname{coker}(\mu).$$

Example. We consider the inverse system

$$D_*\colon \ \ldots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \ \to 0.$$

In Exercise 111.2 we will see that $\varprojlim D_* = 0$ and $\varprojlim^1 D_* = 0$. Furthermore, we also consider the inverse system

$$E_*\colon \ \ldots \xrightarrow{\cdot 5} \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \ \to 0.$$

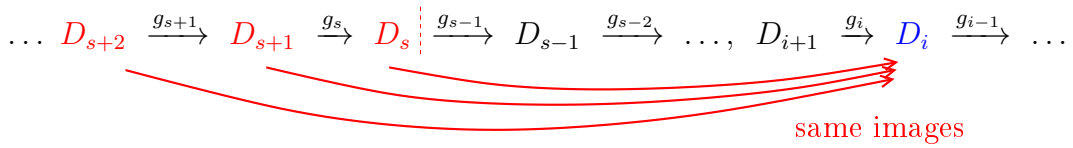
In Exercise 111.2 we will see that $\varprojlim^1 E_* \neq 0$. Furthermore in Exercise 111.3 we will see that this inverse system arises "in nature", i.e. we will see that there exists a nested sequence of topological spaces X_i such that for each $i \in \mathbb{N}$ we have $\mathrm{H}^1(X_i;\mathbb{Z}) \cong \mathbb{Z}$ and such that for each $i \in \mathbb{N}$ the map $\mathrm{H}^1(X_{i+1};\mathbb{Z}) \to \mathrm{H}^1(X_i;\mathbb{Z})$ is given by multiplication by $i+1$.

We cannot proceed without introducing the following somewhat technical definition.

Definition. We say an inverse system of abelian groups $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ satisfies the Mittag-Leffler condition if given any $i \in \mathbb{N}$ there exists an $n \geq i$ such that for all $s \geq n$ we have

$$\mathrm{im}(g_i \circ \cdots \circ g_{s-1}\colon D_s \to D_i) \ = \ \mathrm{im}(g_i \circ \cdots \circ g_{n-1}\colon D_n \to D_i).$$

.



Example. It is straightforward to verify that the above inverse system D_* satisfies the Mittag-Leffler condition whereas the above inverse system E_* does not satisfy the Mittag-Leffler condition.

In many situations one can ignore the precise definition of the Mittag-Leffler condition and instead one can appeal to the following lemma which in fact follows immediately from the definitions.

Lemma 111.8. Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ be an inverse system of abelian groups. If there exists an $m \in \mathbb{N}$ such that for all $i \in \mathbb{N}$ the map $g_i\colon D_{i+1} \to D_i$ is an epimorphism, then the inverse system satisfies the Mittag-Leffler condition.

Our main interest in the Mittag-Leffler condition arises from the fact that if a direct system satisfies this condition, then the pesky $\varprojlim^1$-limit vanishes.

Lemma 111.9. Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ be an inverse system of abelian groups. If this inverse system satisfies the Mittag-Leffler condition, then

$$\varprojlim^1 D_i \ = \ 0.$$

[1356] If R is a commutative ring and if $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ is an inverse system of R-modules, then $\varprojlim^1 D_i$ is also an R-module in an obvious way.

Proof. Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ be an inverse system of abelian groups that satisfies the Mittag-Leffler condition. This means that given given any $i \in \mathbb{N}$ there exists an $n(i) \geq i$ such that for all $s \geq n(i)$ we have

$$\operatorname{im}(g_i \circ \cdots \circ g_{s-1}\colon D_s \to D_i) \;=\; \operatorname{im}(g_i \circ \cdots \circ g_{n(i)-1}\colon D_{n(i)} \to D_i).$$

Without loss of generality we can assume that $n(i)$ is monotonously increasing in i. We have to show that the map

$$\begin{array}{rcl} \mu\colon \prod\limits_{i\in\mathbb{N}} D_i & \to & \prod\limits_{i\in\mathbb{N}} D_i \\ (x_i)_{i\in\mathbb{N}} & \mapsto & (x_i - g_i(x_{i+1}))_{i\in\mathbb{N}} \end{array}$$

is an epimorphism.

Given $j \geq i$ we now write $f_{ji} := g_i \circ \cdots \circ g_{j-1}\colon D_j \to D_i$. First we consider $(a_i)_{i\in\mathbb{N}} \in \prod\limits_{i\in\mathbb{N}} D_i$ with the property that for all $i \in \mathbb{N}$ we have $a_i \in \operatorname{im}(f_{n(i),i}\colon D_{n(i)} \to D_i)$. To show that $\{a_i\}_{i\in\mathbb{N}}$ lies in the image of μ it suffices to prove the following claim.

Claim 1. There exist $b_i \in \operatorname{im}(f_{n(i),i}\colon D_{n(i)} \to D_i)$ with $i \in \mathbb{N}$ such that for each i we have $a_i = b_i - g_i(b_{i+1})$.

Proof. We define the b_i inductively. We set $b_0 = a_0$. Suppose $b_0, \ldots, b_k$ are already defined with the desired properties. By our hypothesis we have $a_k \in \operatorname{im}(f_{n(k),k})$ and by our choice of b_k we have $b_k \in \operatorname{im}(f_{n(k),k})$. By the Mittag-Leffler condition and since we arranged that $n(k+1) \geq n(k)$ we have $\operatorname{im}(f_{n(k+1),k}) = \operatorname{im}(f_{n(k),k})$. Thus can we pick $c \in D_{n(k+1)}$ with $f_{n(k+1),k}(c) = a_k - b_k$. Now we put

$$b_{k+1} \;=\; -f_{n(k+1),k+1}(c).$$

Then

$$b_k - g_{k+1}(b_{k+1}) = b_k + g_{k+1}(f_{n(k+1),k+1}(c)) = b_k + \underbrace{(g_{k+1} \circ f_{n(k+1),k+1})}_{=f_{(n(k+1),k)}}(c) = b_k + (a_k - b_k) = a_k.$$

We have thus shown that $b_0, \ldots, b_{k+1}$ have the desired properties. ⊞

Now let $(b_i)_{i\in\mathbb{N}} \in \prod\limits_{i\in\mathbb{N}} D_i$ be an arbitrary element. For each $i \in \mathbb{N}$ we set

$$c_i \;=\; b_i + \sum_{j=i+1}^{n(i)} f_{j,i}(b_j).$$

Claim 2. For each $i \in \mathbb{N}$ we have $b_i - (c_i - g_i(c_{i+1})) \in \operatorname{im}(f_{n(i),i})$.

Proof. Let $i \in \mathbb{N}$. We have

$$\begin{aligned} b_i - (c_i - g_i(c_{i+1})) &= b_i - b_i - \sum_{j=i+1}^{n(i)} f_{j,i}(b_j) + g_i\Big(b_{i+1} + \sum_{j=i+2}^{n(i+1)} f_{j,i+1}(b_j)\Big) \\ &= -\sum_{j=i+1}^{n(i)} f_{j,i}(b_j) + f_{i+1,i}(b_{i+1}) + \sum_{j=i+2}^{n(i+1)} f_{j,i}(b_j) \;=\; \sum_{j=n(i)+1}^{n(i+1)} f_{j,i}(b_j). \end{aligned}$$

But by definition of $n(i)$ we know that for each $j \in \{n(i)+1, \ldots, n(i+1)\}$ we have $f_{j,i}(b_j) \in \operatorname{im}(f_{n(i),i})$. ⊞

It follows from the claim and the above discussion that $(b_i - (c_i - g_i(c_{i+1})))_{i\in\mathbb{N}}$ lies in the image of μ. Evidently $(c_i - g_i(c_{i+1}))_{i\in\mathbb{N}}$ lies in the image of μ. Since μ is a homomorphism we see that $(b_i)_{i\in\mathbb{N}}$ also lies in the image of μ. ■

We continue with the following definition.

Definition. Let R be a commutative ring and let $(X_i, f_i)_{i\in\mathbb{N}}$ and $(Y_i, g_i)_{i\in\mathbb{N}}$ be inverse systems of R-modules. A map of inverse systems from (X_i, f_i) to (Y_i, g_i) consists of a sequence of module homomorphisms $u_i\colon X_i \to Y_i$, $i \in \mathbb{N}$ such that for each i we have $g_i \circ u_i = u_{i-1} \circ f_i$.

Remark. Note that a map $u\colon (X_i, f_i) \to (Y_i, g_i)$ between two inverse systems induces a map $u_*\colon \varprojlim X_i \to \varprojlim Y_i$ and also a map $u_*\colon \varprojlim{}^1 X_i \to \varprojlim{}^1 Y_i$ in the obvious way.[1357]

Now we can formulate the following proposition.

Proposition 111.10. (Direct Limit-Cohomology Proposition) Let $(\{\mathrm{C}_i^*\}_{i\in\mathbb{N}}, \{f_i\colon \mathrm{C}_{i+1}^* \to \mathrm{C}_i^*\}_{i\in\mathbb{N}})$ be an inverse system of cochain complexes over $\mathbb{N}$. If for each $k \in \mathbb{N}_0$ the inverse system of cochain groups $\{\mathrm{C}_i^k\}_{i\in\mathbb{N}}$ satisfies the Mittag-Leffler condition, then there exists a natural[1358] short exact sequence of the form

$$0 \;\to\; \varprojlim_{i\in\mathbb{N}}{}^1 \mathrm{H}^{n-1}(\mathrm{C}_i^*) \;\to\; \mathrm{H}^n\big(\varprojlim_{i\in\mathbb{N}} \mathrm{C}_i^*\big) \;\to\; \varprojlim_{i\in\mathbb{N}} \mathrm{H}^n(\mathrm{C}_i^*) \;\to\; 0$$

where the right-hand map is the obvious map.[1359] If R is a commutative ring and if the C_i^* are cochain complexes of R-modules, then the maps in the short exact sequence are natural R-module homomorphisms.

Lemma 111.11. Given any short exact sequence

$$0 \;\to\; (X_i, f_i) \;\xrightarrow{u_i}\; (Y_i, g_i) \;\xrightarrow{v_i}\; (Z_i, h_i) \;\to\; 0$$

of inverse systems of abelian groups there exists a natural exact sequence

$$0 \to \varprojlim X_i \xrightarrow{u_*} \varprojlim Y_i \xrightarrow{v_*} \varprojlim Z_i \xrightarrow{\delta} \varprojlim{}^1 X_i \xrightarrow{u_*} \varprojlim{}^1 Y_i \xrightarrow{v_*} \varprojlim{}^1 Z_i \to 0.$$

If R is a commutative ring and if all cochain complexes and all maps are in the category of R-modules, then the maps in short exact sequence are natural R-module homomorphisms.

Proof. We consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod_{i\in\mathbb{N}} X_i & \xrightarrow{u} & \prod_{i\in\mathbb{N}} Y_i & \xrightarrow{v} & \prod_{i\in\mathbb{N}} Z_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} & & \\
0 & \longrightarrow & \prod_{i\in\mathbb{N}} X_i & \xrightarrow{u} & \prod_{i\in\mathbb{N}} Y_i & \xrightarrow{v} & \prod_{i\in\mathbb{N}} Z_i & \longrightarrow & 0.
\end{array}$$

[1357] Whatever obvious means.

[1358] As so often it requires a few moments of pondering to figure out what "natural" is actually supposed to mean.

[1359] Here it takes a little effort to figure out what the "obvious" map actually is supposed to be. By definition an inverse limit of cochain complexes comes with cochain maps $\varprojlim \mathrm{C}_i^* \to \mathrm{C}_i^*$. Since cohomology of cochain complexes is covariant in cochain maps we get induced maps $\mathrm{H}^n(\varprojlim \mathrm{C}_i) \to \mathrm{H}^n(\mathrm{C}_i^*)$ which then, by definition of the inverse limit, give rise to a map $\mathrm{H}^n(\varprojlim \mathrm{C}_i) \to \varprojlim \mathrm{H}^n(\mathrm{C}_i^*)$.

The horizontal sequences are evidently exact. We can view the rows as chain complexes. The desired natural exact sequence now follows from theSnake Lemma that we proved in Exercise 74.9. ■

Proof of Proposition 111.10. Let $(\{\mathrm{C}_i^*\}_{i\in\mathbb{N}}, \{f_i\}_{i\in\mathbb{N}})$ be an inverse system of cochain complexes such that for each $k \in \mathbb{N}_0$ the inverse system of cochain groups $\{\mathrm{C}_i^k\}_{i\in\mathbb{N}}$ satisfies the Mittag-Leffler condition. In the following, given $m \in \mathbb{N}_0$ and $i \in \mathbb{N}$, we denote by $B_i^m = \mathrm{im}(\mathrm{C}_i^{m-1} \to \mathrm{C}_i^m)$ the image of the $(m-1)$-st coboundary map and we denote by $Z_i^m = \ker(\mathrm{C}_i^m \to \mathrm{C}_i^{m+1})$ the kernel of the m-th coboundary map. These give rise to the following two short exact sequences of inverse systems

$$\begin{array}{lccccccccc} \text{(i)} & & 0 & \to & B_i^m & \to & Z_i^m & \to & \mathrm{H}^m(\mathrm{C}_i) & \to & 0 \\ \text{(ii)} & & 0 & \to & Z_i^m & \to & C_i^m & \to & B_i^{m+1} & \to & 0. \end{array}$$

Let $m \in \mathbb{N}_0$. By our hypothesis the inverse systems $\{\mathrm{C}_i^m\}_{i\in\mathbb{N}}$ satisfy the Mittag-Leffler condition. It is straightforward to see that then the inverse system $\{B_i^m\}_{i\in\mathbb{N}}$ also satisfies the Mittag-Leffler condition. It follows from Lemma 111.9 that

$$\varprojlim{}^1 B_i^{m+1} \;=\; 0 \quad \text{and} \quad \varprojlim{}^1 C_i^m \;=\; 0.$$

(Here and throughout the proof all limits are taken over $i \in \mathbb{N}$.) Using this data we obtain from Lemma 111.11, applied to the two short exact sequences (i) and (ii) of inverse systems above, the following two exact sequences:

$$0 \to \varprojlim B_i^m \to \varprojlim Z_i^m \to \varprojlim \mathrm{H}^m(\mathrm{C}_i) \xrightarrow{\delta} \underbrace{\varprojlim{}^1 B_i^m}_{=0} \to \varprojlim{}^1 Z_i^m \to \varprojlim{}^1 \mathrm{H}^m(\mathrm{C}_i) \to 0$$

$$0 \to \varprojlim Z_i^m \to \varprojlim \mathrm{C}_i^m \to \varprojlim B_i^{m+1} \xrightarrow{\delta} \varprojlim{}^1 Z_i^m \to \underbrace{\varprojlim{}^1 \mathrm{C}_i^m}_{=0} \to \varprojlim{}^1 B_i^{m+1} \to 0.$$

From these sequence we deduce that the maps

$$\begin{array}{lcl} \text{(i)} & \varprojlim{}^1 Z_i^m \;\to\; \varprojlim{}^1 \mathrm{H}^m(\mathrm{C}_i) \\ \text{(ii)} & \varprojlim Z_i^m \;\to\; Z^m\big(\varprojlim \mathrm{C}_i\big) := \ker\big(\varprojlim \mathrm{C}_i^m \to \varprojlim B_i^{m+1}\big) \end{array}$$

are isomorphisms and that the following sequences are exact

$$\begin{array}{lccccccccc} \text{(i')} & & 0 & \to & \varprojlim B_i^m & \to & \varprojlim Z_i^m & \to & \varprojlim \mathrm{H}^m(\mathrm{C}_i) & \to & 0 \\ \text{(ii')} & & 0 & \to & \underbrace{B^m\big(\varprojlim \mathrm{C}_i\big)}_{=:\mathrm{im}(\varprojlim \mathrm{C}_i^{m-1} \to \varprojlim B_i^m)} & \to & \varprojlim B_i^m & \to & \varprojlim{}^1 Z_i^{m-1} & \to & 0. \end{array}$$

From (ii') and the isomorphism (ii) we obtain the inclusions

$$B^m\big(\varprojlim \mathrm{C}_i\big) \;\subset\; \varprojlim B_i^m \;\subset\; \varprojlim Z_i^m \;=\; Z^m\big(\varprojlim \mathrm{C}_i\big)$$

which gives rise to the short exact sequence[1360]

$$0 \to \underbrace{\frac{\varinjlim B_i^m}{B^m(\varinjlim \mathrm{C}_i)}}_{=\varinjlim {}^1\mathrm{H}^{m-1}(\mathrm{C}_i) \text{ by (ii') and (i)}} \to \underbrace{\frac{Z^m(\varinjlim \mathrm{C}_i)}{B^m(\varinjlim \mathrm{C}_i)}}_{=\mathrm{H}^m(\varinjlim \mathrm{C}_i) \text{ by def.}} \to \underbrace{\frac{\varinjlim Z_i^m}{\varinjlim B_i^m}}_{=\varinjlim \mathrm{H}^m(\mathrm{C}_i) \text{ by (i')}} \to 0.$$

■

111.4. The cohomology group of an ascending sequence of topological spaces.

The following proposition can be viewed as the analogue of the Exhaustions-H_*-Proposition 79.4 for cohomology groups.

Proposition 111.12. (Cohomology-via-Exhaustions Proposition) Let X be a topological space and let $X_1 \subset X_2 \subset X_3 \subset \ldots$ be a sequence of subsets such that $X = \bigcup_{i\in\mathbb{N}} X_i$ and such that one the following three conditions holds:

(1) every compact subset of X is contained in one of the X_i, or
(2) each X_i is open in X, or
(3) X is a CW-complex and each X_i is a subcomplex.

Then for any $n \in \mathbb{N}_0$ and any abelian group R there exists a natural short exact sequence

$$0 \to \varprojlim_{i\in\mathbb{N}} {}^1\mathrm{H}^{n-1}(X_i; R) \xrightarrow{\tau} \mathrm{H}^n(X; R) \to \varprojlim_{i\in\mathbb{N}} \mathrm{H}^n(X_i; R) \to 0$$

where the right-hand map is induced by the restriction maps $\mathrm{H}^n(X; R) \to \mathrm{H}^n(X_i; R)$. If R is a commutative ring, then all maps are R-module homomorphisms.

Remark. In many situations the term $\varprojlim^1 \mathrm{H}^{n-1}(X_i; R)$ will be zero. But it can be non-zero and one should view the proposition as a warning that in general it is not true that "limits commute with everything".

Proof. It follows easily from the Finiteness the CW-Complex-Finiteness Theorem 68.14 and Lemma 56.8 we only have to prove the proposition under the hypothesis (1). Therefore suppose that we are given a topological space X and subspaces $X_1 \subset X_2 \subset X_3 \subset \ldots$ with $X = \bigcup_{i\in\mathbb{N}} X_i$ and such that the following condition holds:

(1) every compact subset of X is contained in one of the X_i.

Furthermore let R be a commutative ring.

For $i \in \mathbb{N}$ we now consider the cochain complex $D_i^* := \mathrm{Hom}(\mathrm{C}_*(X_i), R)$ and for $i \in \mathbb{N}$ we denote by $f_i \colon D_{i+1}^* \to D_i^*$ the map that is induced by the inclusion $X_i \to X_{i+1}$.

Claim. For every choice of $k \in \mathbb{N}$ the inverse system of cochain groups that is given by $(\{D_i^k\}_{i\in\mathbb{N}}, \{f_i \colon D_{i+1}^k \to D_i^k\}_{i\in\mathbb{N}})$ satisfies the Mittag-Leffler condition.

Proof. Let $k \in \mathbb{N}_0$. By Lemma 111.8 it suffices to show that for every $i \in \mathbb{N}$ the map

$$D_{i+1}^k = \mathrm{Hom}(\mathrm{C}_n(X_{i+1}); R) \xrightarrow{f_i} \mathrm{Hom}(\mathrm{C}_n(X_i); R) = D_i^k$$

[1360] Here we use that a filtration $P \subset Q \subset R$ of three abelian groups induces a short exact sequence $0 \to Q/P \to R/P \to R/Q \to 0$.

is an epimorphism. Since $X_i \subset X_{i+1}$ we know that the set of singular chains in X_i is a subset of the set of singular chains in X_{i+1}. It follows from the discussion on page 2395 that the map $\mathrm{Hom}(\mathrm{C}_n(X_{i+1}); R) \to \mathrm{Hom}(\mathrm{C}_n(X_i); R)$ is indeed an epimorphism. ⊞

Let $n \in \mathbb{N}_0$. From the Direct Limit-Cohomology Proposition 111.10 we obtain the short exact sequence

$$0 \ \to \ \varprojlim{}^1 \mathrm{H}^{n-1}(D_i^*) \ \to \ \mathrm{H}^n\big(\varprojlim D_i^*\big) \ \to \ \varprojlim \mathrm{H}^n(D_i^*) \ \to \ 0$$

of R-modules. Let $i \in \mathbb{N}$. We make the following two observations:

(1) By definition we have $\mathrm{H}^k(D_i^*) = \mathrm{H}^k(X_i; R)$ for every $k \in \mathbb{N}_0$.

(2) We have natural isomorphisms

$$\varprojlim D_i^* \ \underset{\text{by definition of the } D_i}{=} \ \varprojlim \mathrm{Hom}(\mathrm{C}_*(X_i); R) \ \underset{\text{Lemma 108.3}}{=} \ \mathrm{Hom}\big(\varinjlim \mathrm{C}_*(X_i), R\big) \ \underset{\text{Lemma 79.5 and hypothesis (1)}}{=} \ \mathrm{Hom}(\mathrm{C}_*(X), R).$$

Combining all of the above we obtain our desired short exact sequence

$$0 \ \to \ \varprojlim{}^1 \mathrm{H}^{n-1}(X_i; R) \ \to \ \mathrm{H}^n(X; R) \ \to \ \varprojlim \mathrm{H}^n(X_i; R) \ \to \ 0.$$ ■

We provide a few examples for the above proposition.

Examples.

(1) As in Lemma 79.7 we consider the real line with infinitely many 2-dimensional spheres attached, i.e. we consider

$$X \ := \ (\mathbb{R} \cup S^2 \times \mathbb{Z})/\!\sim$$

where given $i \in \mathbb{Z}$ we identify $i \in \mathbb{R}$ with $((0,0,-1), i) \in S^2 \times \mathbb{Z}$. (See the figure below on the left.) Furthermore, given $k \in \mathbb{N}$ we consider the open subset

$$X_k \ := \ \big(\big(-k - \tfrac{1}{2}, k + \tfrac{1}{2}\big) \cup S^2 \times \{-k, \ldots, k\}\big)/\!\sim .$$

We leave it to the reader to verify that we can make the identifications[1361]

$$\mathrm{H}^n(X_k) \ = \ \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z}^{2k+1}, & \text{if } n = 2, \\ 0, & \text{else.} \end{cases}$$

such that for each $k \in \mathbb{N}$ the inclusion maps $X_k \to X_{k+1}$ induce the identity map on $\mathbb{Z} = \mathrm{H}^0(X_{k+1}) \to \mathrm{H}^0(X_k) = \mathbb{Z}$ and such that the inclusions induce the map

$$\begin{array}{rcl} \mathbb{Z}^{2k+3} = \mathrm{H}^2(X_{k+1}) & \to & \mathrm{H}^2(X_k) = \mathbb{Z}^{2k+1} \\ (a_1, \ldots, a_{2k+3}) & \mapsto & (a_2, \ldots, a_{2k+2}) \end{array}$$

It follows from Lemma 111.8 and the above calculations that for each $n \in \mathbb{N}_0$ the direct system of cohomology groups $\{\mathrm{H}^n(X_k)\}_{k \in \mathbb{N}}$ satisfies the Mittag-Leffler condition. We obtain that

$$\mathrm{H}^n(X) \ \underset{\substack{\text{Cohomology-via-Exhaustions} \\ \text{Proposition 111.12 and} \\ \text{Lemma 111.9}}}{=} \ \varprojlim \mathrm{H}^n(X_k) \ = \ \varprojlim \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z}^{2k+1}, & \text{if } n = 2, \\ 0, & \text{else} \end{cases} \ \underset{\text{see page 2468}}{=} \ \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z}^{\mathbb{N}}, & \text{if } n = 2, \\ 0, & \text{else.} \end{cases}$$

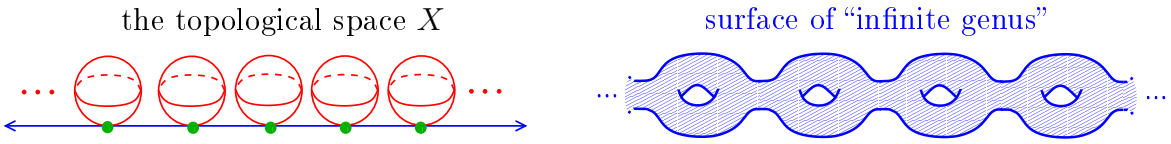
[1361] One possible approach to showing these statements is to use the Mayer–Vietoris Theorem 109.15 for Cohomology Groups.

Here we recall that $\mathbb{Z}^{\mathbb{N}} = \{(x_1, x_2, \dots) \mid x_i \in \mathbb{Z}\}$. Note that in Lemma 79.7 we saw that $\mathrm{H}_2(X) \cong \mathbb{Z}^{(\mathbb{N})}$. In this case we see that $\mathrm{H}_2(X) \ncong \mathrm{H}^2(X)$ since the former group is countable whereas the latter group is uncountable.[1362]

(2) We consider again the surface Σ_∞ of infinite genus that we had already encountered on page 1269 and in Lemma 80.4. (See also the figure below on the right.) Using a variation on example (1) and the proof of Lemma 56.11 one can show that

$$\mathrm{H}^n(\Sigma_\infty) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z}^{\mathbb{N}}, & \text{if } n = 1, \\ 0, & \text{if } n \geq 2. \end{cases}$$

We leave the task of providing full details to the valiant reader. Note that this shows that the first cohomology group is uncountable. In particular we see that the cohomology groups with $\mathbb{Z}$-coefficients of a non-compact smooth manifold do not need to be countable. This is in contrast to Proposition 96.7 where we showed that the homology groups with $\mathbb{Z}$-coefficients of a non-compact smooth manifold are always countable.



We conclude this chapter with the following variation on the Cohomology-via-Exhaustions Proposition 111.12. It can be viewed as the cohomological analogue of the Skeleton-Homology Lemma 80.1 (3).

Proposition 111.13. Let X be a CW-complex and let $n \in \mathbb{N}$. For every $k > n$ and every abelian group G the inclusion $i\colon X^k \to X$ induces an isomorphism

$$i^*\colon \mathrm{H}^n(X; G) \;\to\; \mathrm{H}^n(X^k; G).$$

Remark. We will give a very different proof of this proposition in Proposition **??**.

Proof. By the Cohomology-via-Exhaustions Proposition 111.12 there exists a natural short exact sequence

$$0 \;\to\; \varprojlim_{j\in\mathbb{N}}{}^1\mathrm{H}^{n-1}(X_j; G) \;\xrightarrow{\tau}\; \mathrm{H}^n(X; G) \;\to\; \varprojlim_{j\in\mathbb{N}} \mathrm{H}^n(X_j; G) \;\to\; 0$$

where the right-hand map is induced by the restriction maps $\mathrm{H}^n(X; G) \to \mathrm{H}^n(X_i; G)$. It follows immediately from the Indirect Limit Stabilization Lemma 111.6 and Lemma 111.8 that is suffices to prove the following claim.

Claim. For any $j \geq k > n$ the inclusion induced map $\mathrm{H}^n(X^j; G) \to \mathrm{H}^n(X^k; G)$ is an isomorphism.

Proof. It follows from the Skeleton-Homology Lemma 80.1, the Universal Coefficient Theorem 110.12 for Cohomology Groups that for every $m \geq n$ we have $\mathrm{H}^n(X^{m+1}, X^m; G) = 0$. Using the long exact sequence in cohomology of the pairs $(X^{m+1}, X^m), (X^{m+2}, X^m), \dots$

[1362] Of course we could have obtained the same observation from the Universal Coefficient Theorem 110.12 for Cohomology Groups.

one can now easily show that for any $j \geq k > n$ the inclusion induced map $\mathrm{H}^n(X^j; G) \to \mathrm{H}^n(X^k; G)$ is an isomorphism. ■

Exercises for Chapter 111.

Exercise 111.1. Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i \in I}, \{f_{ij} \colon X_i \to X_j\}_{i \leq j})$ be a direct system of abelian groups over I.

(a) Let $a_j \in X_j$ and let $a_k \in X_k$. Prove the following statement: if the images of a_j and a_k in $\varinjlim X_i$ agree, then there exists an $l \in I$ with $l \geq j$ and $l \geq k$ such that $f_{jl}(a_j) = f_{kl}(a_k) \in X_l$.

(b) Show that if all maps $f_{ij} \colon X_i \to X_j$ are isomorphisms, then for any $j \in I$ the obvious map $X_j \to \varinjlim X_i$ is an isomorphism.

Exercise 111.2. We consider the inverse systems

$$D_* \colon \ \ldots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \ \to 0$$

and

$$E_* \colon \ \ldots \xrightarrow{\cdot 5} \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \ \to 0.$$

(a) Show that $\varprojlim D_* = 0$,

(b) show that $\varprojlim{}^1 D_* = 0$,

(c) show that $\varprojlim{}^1 E_* \neq 0$.

Exercise 111.3. Show that there exists a nested sequence of topological spaces X_i such that for each $i \in \mathbb{N}$ we have $\mathrm{H}^1(X_i; \mathbb{Z}) \cong \mathbb{Z}$ and such that for each $i \in \mathbb{N}$ the map $\mathrm{H}^1(X_{i+1}; \mathbb{Z}) \to \mathrm{H}^1(X_i; \mathbb{Z})$ is given by multiplication by $i + 1$.
Put differently, we want topological spaces and maps such that the inverse system

$$\mathrm{H}^1(X_4) \ \to \ \mathrm{H}^1(X_3) \ \to \ \mathrm{H}^1(X_2) \ \to \ \mathrm{H}^1(X_1)$$

is isomorphic to the inverse system E_* from Exercise 111.2.

112. Cohomology with compact support and bounded cohomology

112.1. The definition of cohomology with compact support and basic properties. In this section we introduce "cohomology with compact support", which is an interesting variation on the usual cohomology groups. Later on it will be a key tool in proving the Poincaré Duality Theorem, which is the most important result regarding homology and cohomology groups of topological manifolds.
We start out with the following definition.

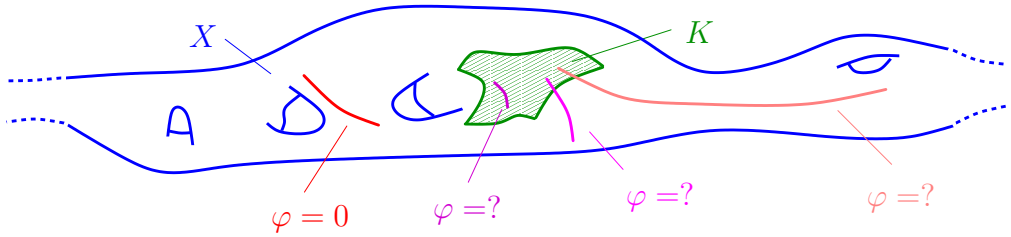
Definition. Let X be a topological space and let G be an abelian group.

(1) Given a subset $A \subset X$ we say that a cochain $\varphi \in \mathrm{C}^i(X;G) = \mathrm{Hom}(\mathrm{C}_i(X), G)$ vanishes on A if $\varphi(c) = 0$ for all singular i-simplices $c\colon \Delta^i \to X$ in A, i.e. if φ vanishes on the image of $\mathrm{C}_i(A) \to \mathrm{C}_i(X)$.

(2) Given a subset $B \subset X$ we say that a cochain vanishes outside B if the cochain vanishes on $X \setminus B$.

(3) We say that a cochain $\varphi \in \mathrm{C}^i(X;G) = \mathrm{Hom}(\mathrm{C}_i(X), G)$ has compact support if there exists a compact set $K \subset X$ such that φ vanishes outside of K. We denote the set of all i-cochains with compact support by $\mathrm{C}^i_{\mathrm{c}}(X;G)$.

φ is a cohomology class that vanishes outside of a compact subset K



Remark. Let X be a topological space and let G be an abelian group.

(1) Let A be a subset of X and let $n \in \mathbb{N}_0$. The inclusion $(X, \varnothing) \to (X, X \setminus A)$ induces an epimorphism $\mathrm{C}_n(X) \to \mathrm{C}_n(X, X \setminus A)$ and thus it also induces a monomorphism $\mathrm{C}^n(X, X \setminus A; G) \to \mathrm{C}^n(X;G)$. We have

$$\begin{aligned}\mathrm{im}(\mathrm{C}^n(X, X\setminus A; G) \to \mathrm{C}^n(X;G)) &= \text{all maps of the form } \mathrm{C}_n(X) \to \overbrace{\mathrm{C}_n(X, X\setminus A)}^{=\mathrm{C}_n(X)/\mathrm{C}_n(X\setminus A)} \to G \\ &= \{\varphi \in \mathrm{Hom}(\mathrm{C}_n(X), G) \,|\, \varphi \text{ vanishes on } X \setminus A\} \\ &= \{\varphi \in \mathrm{Hom}(\mathrm{C}_n(X), G) \,|\, \varphi \text{ vanishes outside } A\}.\end{aligned}$$

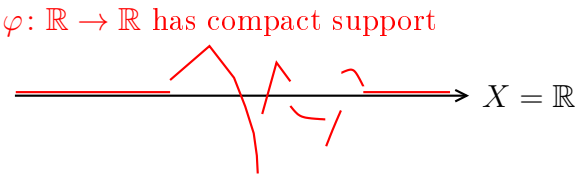
(2) On page 2403 we had made the identification

$$\mathrm{C}^0(X;G) \;=\; \{\text{maps from } X \text{ to } G\}.$$

With this identification we can now write

$$\mathrm{C}^0_{\mathrm{c}}(X;G) \;=\; \left\{ \text{maps } f\colon X \to G \,\middle|\, \begin{array}{c}\text{there exists a compact subset } K \\ \text{with } f|_{X\setminus K} \equiv 0\end{array} \right\}.$$

The first question that arises is whether cochains with compact support give again rise to cohomology groups. The following lemma shows in particular that this is indeed the case.



Lemma 112.1. Let X be a topological space and let G be an abelian group.

(1) A cochain $\varphi \in \mathrm{C}^i(X;G)$ has compact support if and only if there exists a compact subset K such that φ lies in the image of the map $\mathrm{C}^i(X, X \setminus K; G) \to \mathrm{C}^i(X;G)$.
(2) The set of cochains that have compact support form a subcomplex $(\mathrm{C}^*_{\mathrm{c}}(X;G);\delta)$ of the cochain complex $(\mathrm{C}^*(X;G);\delta)$.
(3) For any compact subset K of X the map $i_K\colon \mathrm{C}^*(X, X\setminus K;G) \to \mathrm{C}^*_{\mathrm{c}}(X;G)$ from (1) is a cochain map.

Proof. Let X be a topological space and let G be an abelian group. The first statement of the lemma follows immediately from the above remark.

We turn to the proof of the second statement.

Claim. Let $A, B \subset X$ be two subsets.

(a) If $\varphi \in \mathrm{C}^i(X;G)$ vanishes outside A and if $\psi \in \mathrm{C}^i(X;G)$ vanishes outside B, then for any $r, s \in \mathbb{Z}$ the linear combination $r\varphi + s\psi \in \mathrm{C}^i(X;G)$ vanishes outside the union $A \cup B$.
(b) If $\varphi \in \mathrm{C}^i(X;G)$ vanishes outside A, then $\delta\varphi \in \mathrm{C}^{i+1}(X;G)$ also vanishes outside A.

Proof. The first statement is obvious. Now we turn to the proof of the second statement. Suppose we are given a cochain $\varphi \in \mathrm{C}^i(X;G)$ that vanishes outside A. We need to show that $\delta\varphi \in \mathrm{C}^{i+1}(X;G)$ also vanishes outside A. So let c be a singular $(i+1)$-simplex in $X \setminus A$. Then

$$(\delta\varphi)(c) \;=\; \varphi(\partial c) \;=\; \sum_{j=0}^{n} (-1)^j \cdot \varphi(c \circ i_j^n) \;\underset{\uparrow}{=}\; 0.$$

image of each $c \circ i_j^n$ also lies in $X \setminus A$ and φ vanishes outside A

The second statement is now an immediate consequence of the above claim and the observation that given two compact subsets $K, L \subset X$ the union $K \cup L$ is again a compact subset of X.

Finally the last statement is an immediate consequence of the definitions. ∎

Definition. Let X be a topological space and let G be an abelian group. We define the n-th cohomology group with compact support $\mathrm{H}^n_{\mathrm{c}}(X;G)$ to be the cohomology of the cochain complex given by cochains with compact support, i.e. we define

$$\mathrm{H}^n_{\mathrm{c}}(X;G) \;:=\; \frac{\ker(\delta_n\colon \mathrm{C}^n_{\mathrm{c}}(X;G) \to \mathrm{C}^{n+1}_{\mathrm{c}}(X;G))}{\mathrm{im}(\delta_{n-1}\colon \mathrm{C}^{n-1}_{\mathrm{c}}(X;G) \to \mathrm{C}^n_{\mathrm{c}}(X;G))}.$$

Remark.

(1) If X is a compact space, then every cochain has compact support, in particular the cohomology groups with compact support are just the usual cohomology groups.

(2) Let X be a path-connected topological space. Then

$$\mathrm{H}^0_{\mathrm{c}}(X;G) \underset{\uparrow}{=} \left\{ X \xrightarrow{f} G \;\middle|\; \begin{array}{l} f \text{ is constant and there exists a} \\ \text{compact subset } K \text{ with } \varphi|_{X\setminus K} \equiv 0 \end{array} \right\} = \begin{cases} 0, & \text{if } X \text{ non-compact,} \\ G, & \text{if } X \text{ compact.} \end{cases}$$

see proof of the H^0-Proposition 108.11 and description of $\mathrm{C}^0_{\mathrm{c}}(X;G)$ on page 2478.

Example. We consider the topological space $X = \mathbb{R}$. As on page 2406 we consider the singular 0-cochain

$$\varphi_0 \colon \mathrm{C}_0(X) \;\to\; \mathbb{Z}$$

that is uniquely determined by the property that for any singular 0-simplex $x \in X$ we have

$$x \;\mapsto\; \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

It is clear that this 0-cochain does *not have* compact support. As on page 2406 we also consider the coboundary $\gamma_0 := \delta\varphi_0 \in \mathrm{C}^1(\mathbb{R};\mathbb{Z})$. Since γ_0 is a coboundary, it is in particular a cocycle. By definition $\gamma_0 = \delta\varphi_0$ is uniquely determined by the property that for any singular 1-simplex $d\colon \Delta^1 \to \mathbb{R}$ we have

$$(\delta\varphi_0)(d) \;=\; \varphi_0(\partial d) \underset{\uparrow}{=} \varphi_0(d(0,1)) - \varphi_0(d(1,0)).$$

see page 1585

We claim that $\gamma_0 = \delta\varphi_0$ has compact support. In fact, we will now see that we can take K to be the compact set given by $K := \{0\}$. Indeed, if $d\colon \Delta^1 \to \mathbb{R}\setminus\{0\}$ is a singular 1-simplex, then it follows from the fact that Δ^1 is homeomorphic to the interval $[0,1]$ and the intermediate value theorem that either $d(0,1)$ and $d(1,0)$ both lie to the left of 0 or both lie to the right of 0. In both cases we obtain from the above formula that $\gamma_0(d) = (\delta\varphi_0)(d) = 0$. Thus we see that $\gamma_0 = \delta\varphi_0$ defines an element in $\mathrm{H}^1_{\mathrm{c}}(\mathbb{R})$.

Now we claim that $[\gamma_0] \in \mathrm{H}^1_{\mathrm{c}}(\mathbb{R};\mathbb{Z})$ is non-zero. This means that we have to show that there is no $\psi \in \mathrm{C}^0_{\mathrm{c}}(\mathbb{R};\mathbb{Z})$ with $\delta\psi = \gamma_0$. So let $\psi \in \mathrm{C}^0_{\mathrm{c}}(\mathbb{R};\mathbb{Z})$. Following the discussion on page 2478 we view ψ as a function $\mathbb{R} \to \mathbb{Z}$ which is zero outside of a compact subset of $\mathbb{R}$. This implies that there exists a $C \geq 0$ such that $\psi(x) = 0$ for all $x \notin [-C, C]$. Now we consider the singular 1-chain $d\colon \Delta^1 \to \mathbb{R}$ that is given by $d(1-t,t) = -2C + 4C \cdot t$ with $t \in [0,1]$. (See the figure below for an illustration.) Then we have

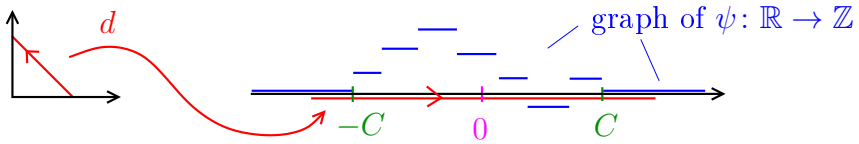
$$\gamma_0(d) \;=\; (\delta\varphi_0)(d) \;=\; \varphi_0(d(0,1)) - \varphi_0(d(1,0)) \;=\; \varphi_0(-2C) - \varphi_0(2C) \;=\; 0 - 1 \;=\; -1$$

but on the other hand we have

$$(\delta\psi)(d) \;=\; \psi(d(0,1)) - \psi(d(1,0)) \;=\; \psi(-2C) - \psi(2C) \underset{\uparrow}{=} 0 - 0 \;=\; 0.$$

since ψ is zero outside $[-C, C]$

This concludes the proof that $\gamma_0 = \delta\varphi_0$ is non-trivial in $\mathrm{H}^1_{\mathrm{c}}(\mathbb{R};\mathbb{Z})$.



The following lemma is an almost immediate consequence of Lemma 112.1 (1).

Lemma 112.2. Let X be a topological space and let G be an abelian group. Given any $\varphi \in \mathrm{H}^i_{\mathrm{c}}(X;G)$ there exists a compact subset K such that φ lies in the image of the map $\mathrm{H}^i(X, X \setminus K; G) \to \mathrm{H}^i_{\mathrm{c}}(X;G)$.

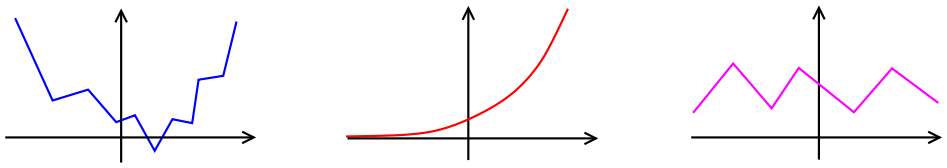
Proof. Let $[\varphi] \in \mathrm{H}^i_{\mathrm{c}}(X;G)$. By Lemma 112.1 (1) there exists a compact subset K such that φ lies in the image of the map $\mathrm{C}^i(X, X\setminus K; G) \to \mathrm{C}^i_{\mathrm{c}}(X;G)$. Since $\mathrm{C}^*(X, X\setminus K;G)$ can be viewed as a subcomplex of $\mathrm{C}^*_{\mathrm{c}}(X;G)$ we see that φ is also a cocycle in $\mathrm{C}^*(X, X \setminus K;G)$, in particular φ lies in the image of the map $\mathrm{H}^i(X, X \setminus K;G) \to \mathrm{H}^i_{\mathrm{c}}(X;G)$. ■

We conclude this section with the discussion of induced maps on cohomology groups with compact support.

Let $f\colon X \to Y$ be a map between topological spaces and let G be an abelian group. We saw that f induces a map $f^*\colon \mathrm{C}^*(Y;G) \to \mathrm{C}^*(X;G)$. But in general this map does *not* restrict to a map $f^*\colon \mathrm{C}^*_{\mathrm{c}}(Y;G) \to \mathrm{C}^*_{\mathrm{c}}(X;G)$. For example, if $f\colon \mathbb{R} \to \{P\}$ is the map that sends every point in $\mathbb{R}$ to the point P and if $\varphi\colon \mathrm{C}_0(\{P\}) \to \mathbb{Z}$ is the map given by $P \mapsto 1$, then $f^*\varphi\colon \mathrm{C}_0(\mathbb{R}) \to \mathbb{Z}$ corresponds to the map that takes the value 1 on all of $\mathbb{R}$, in particular it does not have compact support.

A moment of reflection shows that if we want to get induced maps on cohomology groups with compact support, we have to restrict ourselves to proper maps. For the reader's convenience we recall the definition from page 180.

Definition. A map $f\colon X \to Y$ between topological spaces is called proper if the preimage of every compact subset of Y is a compact subset of X.



Lemma 112.3. Let G be an abelian group.

(1) Let $f\colon X \to Y$ be a map between topological spaces. If f is a proper map, then the cochain map $f^*\colon \mathrm{C}^*(Y;G) \to \mathrm{C}^*(X;G)$ restricts to a cochain map
$$f^*\colon \mathrm{C}^*_{\mathrm{c}}(Y;G) \to \mathrm{C}^*_{\mathrm{c}}(X;G).$$
In particular f^* induces a well-defined homomorphism
$$\mathrm{H}^*_{\mathrm{c}}(Y;G) \to \mathrm{H}^*_{\mathrm{c}}(X;G).$$

(2) The maps $X \mapsto \mathrm{H}^n_{\mathrm{c}}(X;G)$ and $f \mapsto f^*$ define a contravariant functor
$$\text{category } \mathit{TopPropMap} \text{ where objects are topological spaces and morphisms are proper maps} \to \text{category } \mathit{AbGroup} \text{ of abelian groups.}$$

(3) Homeomorphic topological spaces have isomorphic cohomology groups with compact support.

Proof. Let $f\colon X \to Y$ be a proper map and let G be an abelian group. Let $\varphi\colon \mathrm{C}_n(Y) \to G$ be a cocycle with compact support. We pick a compact subset K such that φ vanishes on $Y \setminus K$. Then $f^*\varphi$ vanishes on $f^{-1}(Y \setminus K) = X \setminus f^{-1}(K)$, i.e. $f^*\varphi$ vanishes outside of the

subset $f^{-1}(K)$. But $f^{-1}(K)$ is compact since f is proper. This concludes the proof of the first statement. The second statement and the third statement are obvious. ■

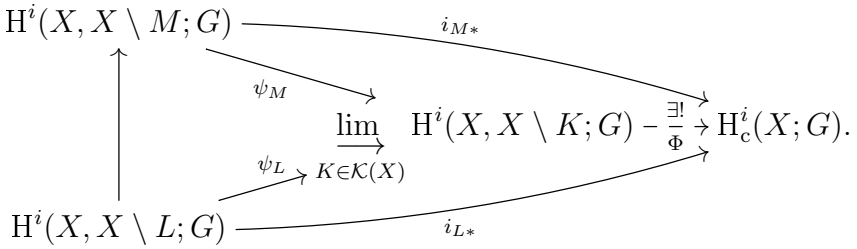
112.2. **Cohomology with compact support and direct limits.** Given a topological space X we consider the set $\mathcal{K}(X)$ of all compact subsets of X. The set $\mathcal{K}(X)$ together with the relation "$\subset$" defined by inclusion is a directed set since the union of two compact subsets is again compact. Furthermore, given any two compact subsets $K \subset L$ and any $i \in \mathbb{N}_0$ we get induced maps[1363]

$$\mathrm{H}^i(X, X \setminus K; G) \;\to\; \mathrm{H}^i(X, X \setminus L; G).$$

Since cohomology groups are contravariantly functorial we see that these cohomology groups form a *direct* system over the directed set $(\mathcal{K}(X), \subset)$ and we can form the direct limit

$$\varinjlim_{K \in \mathcal{K}(X)} \mathrm{H}^i(X, X \setminus K; G).$$

Recall that given any compact subset $K \subset X$ we have by Lemma 112.1 (3) a natural map $i_{K*} \colon \mathrm{H}^i(X, X \setminus K; G) \to \mathrm{H}^i_{\mathrm{c}}(X; G)$. Given compact subsets $L \subset M$ these maps give rise to the following commutative diagram:



Since this diagram commutes for all compact subsets $L \subset M$ we get the induced dotted arrow by the universal property of the direct limit.
The following proposition says that the above dotted arrow is in fact an isomorphism. This gives in particular an often convenient reformulation of the concept of cohomology with compact support.

Proposition 112.4. ($\mathrm{H}^n_{\mathrm{c}}$-via-Direct Limits Proposition) Let X be a topological space and let G be an abelian group. Then the above map

$$\Phi \colon \varinjlim_{K \in \mathcal{K}(X)} \mathrm{H}^i(X, X \setminus K; G) \to \mathrm{H}^i_{\mathrm{c}}(X; G)$$

defines a natural isomorphism.[1364]

Proof. We need to show that the map Φ in the proposition is a bijection.

We first show that Φ is surjective. So let $\varphi \in \mathrm{H}^i_{\mathrm{c}}(X; G)$. By Lemma 112.2 there exists a compact subset L such that φ lies in the image of $\mathrm{H}^i(X, X \setminus L; G) \to \mathrm{H}^i(X; G)$. It

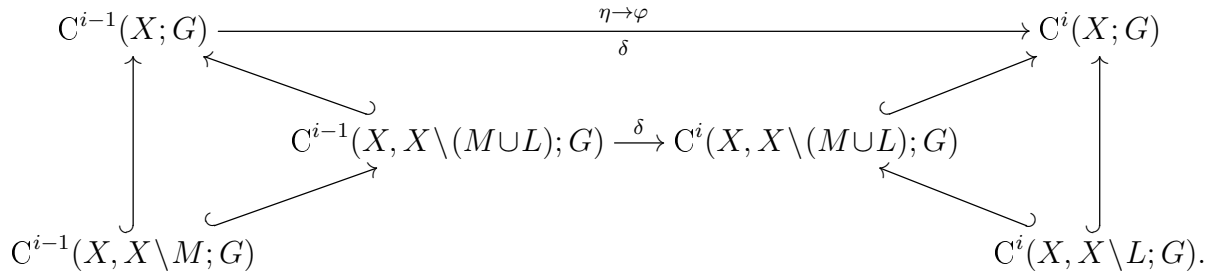
[1363]The most confusing part about cohomology is to figure out which way the maps go. In this case $K \subset L$ implies $X \setminus L \subset X \setminus K$ which means that we have a map of pairs $(X, X \setminus L) \to (X, X \setminus K)$ given by inclusion, which by the discussion on page 2408 implies that we do indeed get an induced map $\mathrm{H}^i(X, X \setminus K; G) \to \mathrm{H}^i(X, X \setminus L; G)$.

[1364]Here the term "natural" refers to the fact that both sides define a contravariant functor from the category where objects are topological spaces and morphisms are proper maps to the category of abelian groups.

follows immediately from the above commutative diagram that φ also lies in the image of $\Phi\colon \varinjlim \mathrm{H}^i(X, X \setminus K; G) \to \mathrm{H}^i_{\mathrm{c}}(X; G)$.

Now we show that the map Φ is injective. In the subsequent discussion, given a subset A of X, we identify the image of the monomorphism $\mathrm{C}^j(X, A; G) \to \mathrm{C}^j(X; G)$ with $\mathrm{C}^j(X, A; G)$, i.e. we view each $\mathrm{C}^j(X, A)$ as a subgroup of $\mathrm{C}^j(X)$.

Now suppose we are given an element c in the kernel of Φ. By Lemma 111.2 (1) we know that the element c lies in the image of some $\mathrm{H}^i(X, X \setminus L; G)$. We pick a representative $\varphi \in \mathrm{C}^i(X, X \setminus L; G)$. Since φ represents the zero class in $\mathrm{H}^i_{\mathrm{c}}(X; G)$ there exists a compactly supported $(i-1)$-cocycle η with $\delta\eta = \varphi$. We pick a compact subset M of X such that η vanishes on $X \setminus M$, i.e. according to the above notation and the discussion on page 2478 we have $\eta \in \mathrm{C}^{i-1}(X, X \setminus M)$. We consider the following commutative diagram:



It follows immediately from the diagram that in $\mathrm{C}^i(X, X \setminus (M \cup L); G)$ we have $\delta\eta = \varphi$, which shows that $[\varphi]$ represents the zero class in $\mathrm{H}^i(X, X \setminus (M \cup L); G)$. It follows from Lemma 111.2 (2) or alternatively from the commutative diagram from page 2482 that $\psi_L([\varphi]) = c$ is zero. ∎

Example. Now we turn to a key example, namely for each $k \in \mathbb{N}_0$ we consider the cohomology with compact support $\mathrm{H}^k_{\mathrm{c}}(\mathbb{R}^n; \mathbb{Z})$ of $\mathbb{R}^n$. We have

$$\begin{array}{ccccc}
 & & & & \vdots \\
 & & & & \uparrow \\
 & \text{Proposition 112.4} & & \text{Limit-over-Subset Lemma 111.3} & \mathrm{H}^k(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}_2(0); \mathbb{Z}) \\
 & \downarrow & & \downarrow & \uparrow \\
\mathrm{H}^k_{\mathrm{c}}(\mathbb{R}^n; \mathbb{Z}) & = & \varinjlim_{K \in \mathcal{K}(\mathbb{R}^n)} \mathrm{H}^k(\mathbb{R}^n, \mathbb{R}^n \setminus K; \mathbb{Z}) & = & \varinjlim_{s \in \mathbb{N}} \mathrm{H}^k(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}_1(0); \mathbb{Z}) \\
 & & & & \uparrow \\
 & & & & \mathrm{H}^k(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}_0(0); \mathbb{Z}) \\
 & \cong & \mathrm{H}^k(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}_0(0); \mathbb{Z}) & \cong & \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases} \\
 & \uparrow & & \uparrow & \\
 & \text{by Corollary 74.16 (2) all vertical maps in the direct system are isomorphisms, hence we can apply Lemma 111.2} & & \text{same argument as in the proof of Lemma 105.4} &
\end{array}$$

We draw three interesting conclusions from this example:

(1) This example shows that cohomology with compact support can be used to show that the topological spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic for $n \neq m$.

(2) The example also shows that cohomology with compact support, in contrast to basically all other invariants we have studied so far, is not the same for homotopy equivalent topological spaces.

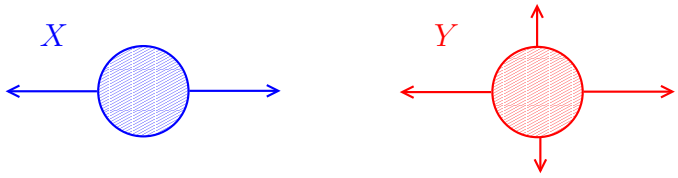
(3) Finally we consider the case $n = 1$ in more detail. We consider again the singular 1-cochain $\gamma_0 := \delta\varphi_0$ from page 2406. On pages 2413 and 2480 we saw that γ_0 defines

an element in each $\mathrm{H}^1(\mathbb{R}, \mathbb{R} \setminus [-s,s]; \mathbb{Z})$ and an element in $\mathrm{H}^1_{\mathrm{c}}(\mathbb{R}; \mathbb{Z})$. Furthermore on page 2413 we saw that for each $s \in N$ the class $[\gamma_0] \in \mathrm{H}^1(\mathbb{R}, \mathbb{R} \setminus [-s,s]; \mathbb{Z})$ is a generator. The above argument now implies that $[\gamma_0] \in \mathrm{H}^1_{\mathrm{c}}(\mathbb{R}; \mathbb{Z})$ is also a generator.

Example. In Exercise 112.2 we will use cohomology with compact support to show that the topological spaces

$$X \;:=\; \overline{B}_1(0) \cup x\text{-axis} \quad \text{and} \quad Y \;:=\; \overline{B}_1(0) \cup x\text{-axis} \cup y\text{-axis}$$

that are illustrated in the figure below are not homeomorphic.



In the previous examples cohomology with compact support turns out to be "bigger" than ordinary cohomology. But there are also examples "in the other direction":

Example. We consider again the topological space

$$X \;:=\; (\mathbb{R} \cup S^2 \times \mathbb{Z})/\!\sim$$

where given $i \in \mathbb{Z}$ we identify $i \in \mathbb{Z} \subset \mathbb{R}$ with $((0,0,-1), i) \in S^2 \times \mathbb{Z}$. In Exercise 112.6 we saw that

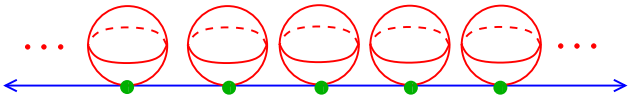
$$\mathrm{H}^2_{\mathrm{c}}(X) \;\cong\; \mathbb{Z}^{(\mathbb{N})},$$

in particular $\mathrm{H}^2_{\mathrm{c}}(X)$ is countable, whereas on page 2475 we saw that

$$\mathrm{H}^2(X) \;\cong\; \mathbb{Z}^{\mathbb{N}}$$

is uncountable.

the topological space X



$\mathrm{H}^2(X) \cong \mathbb{Z}^{\mathbb{N}}$ but $\mathrm{H}^2_{\mathrm{c}}(X) \cong \mathbb{Z}^{(\mathbb{N})}$

In the following we will see the rather surprising fact that under fairly mild hypotheses an inclusion $U \to V$ of topological spaces induces for every $n \in \mathbb{N}_0$ a map $\mathrm{H}^n_{\mathrm{c}}(U) \to \mathrm{H}^n_{\mathrm{c}}(V)$ which "goes the same way".

Definition. Recall that given a topological space X we denote by $\mathcal{K}(X)$ the set of all compact subsets of X. Suppose U is an open subset of a topological space V and suppose that V is Hausdorff. We denote by $i_{UV*} = i_* \colon \mathrm{H}^n_{\mathrm{c}}(U) \to \mathrm{H}^n_{\mathrm{c}}(V)$ the composition of the following maps

$$\mathrm{H}^n_{\mathrm{c}}(U) \;\overset{\substack{\mathrm{H}^n_{\mathrm{c}}\text{-via-Direct Limits Proposition 112.4}\\ \downarrow}}{=}\; \varinjlim_{K \in \mathcal{K}(U)} \mathrm{H}^n(U, U \setminus K) \;\overset{\cong}{\leftarrow}\; \varinjlim_{K \in \mathcal{K}(U)} \mathrm{H}^n(V, V \setminus K) \;\to\; \varinjlim_{K \in \mathcal{K}(V)} \mathrm{H}^n(V, V \setminus K) \;\overset{\substack{\mathrm{H}^n_{\mathrm{c}}\text{-via-Direct Limits Proposition 112.4}\\ \downarrow}}{=}\; \mathrm{H}^n_{\mathrm{c}}(V).$$

by the Excision Theorem 74.18 (2) the map $\mathrm{H}^n(V, V \setminus K) \to \mathrm{H}^n(U, U \setminus K)$ is an isomorphism for every compact subset K of U

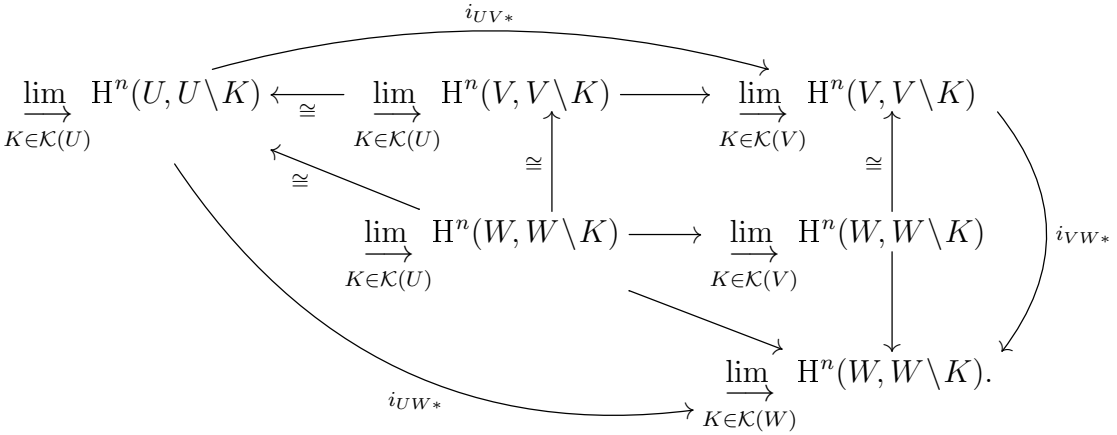
given by the Limit-over-Subset Lemma 111.3 (1) and fact that by the Compact Image Lemma 2.13 we have $\mathcal{K}(U) \subset \mathcal{K}(V)$

Note that, confusingly, the induced map on cohomology with compact support "goes the same direction" as the inclusion map $i\colon U \to V$.

Lemma 112.5. Given a topological space X that is Hausdorff the above construction defines for each $n \in \mathbb{N}_0$ a *covariant* functor

$$\begin{array}{c}\text{category where the objects are open subsets}\\ \text{of } X \text{ and where the morphisms are inclusions}\end{array} \to \begin{array}{c}\text{category of}\\ \text{abelian groups.}\end{array}$$

Proof. Let X be a topological space that is Hausdorff and let $U \subset V \subset W$ be open subsets. Let $n \in \mathbb{N}_0$. We have to show that $i_{UW*} = i_{VW*} \circ i_{UV*}\colon \mathrm{H}^n_{\mathrm{c}}(U) \to \mathrm{H}^n_{\mathrm{c}}(W)$. We consider the following diagram:



It follows from the explicit description of the direct limits in the Direct Limit Existence Proposition 111.1 and the uniqueness statement in the universal property of the direct limits that the two smaller triangles and the rectangle commute. Hence the diagram commutes and we get the desired statement. ■

We conclude this section with the following proposition.

Proposition 112.6. (Exhaustion-$\mathrm{H}^*_{\mathrm{c}}$-Proposition) Let X be a topological space that is Hausdorff. Let $X_i, i \in \mathbb{N}$ be a sequence of subsets such that the following holds:

(1) the sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $X_i \subset X_{i+1}$,
(2) each X_i is open in X,
(3) and $X = \bigcup_{i\in\mathbb{N}} X_i$.

Then for any $n \in \mathbb{N}_0$ the above inclusion induced maps $\mathrm{H}^n_{\mathrm{c}}(X_i) \to \mathrm{H}^n_{\mathrm{c}}(X)$ induce an isomorphism

$$\varinjlim \underbrace{\mathrm{H}^n_{\mathrm{c}}(X_i)}_{\uparrow} \xrightarrow{\cong} \mathrm{H}^n_{\mathrm{c}}(X).$$

by Lemma 112.5 this is a direct system with the above inclusion induced maps

Proof. Using the $\mathrm{H}^n_{\mathrm{c}}$-via-Direct Limits Proposition 112.4 we see that we have to show that the map

$$\Phi\colon \varinjlim_{i\in\mathbb{N}} \varinjlim_{K\in\mathcal{K}(X_i)} \mathrm{H}^n(X_i, X_i \setminus K) \xrightarrow{\cong} \varinjlim_{K\in\mathcal{K}(X)} \mathrm{H}^n(X, X \setminus K)$$

is an isomorphism. We need to show that Φ is injective and surjective. The key to the proof of these two statements are the following observations:

(a) it follows from (1) and (2), together with Lemma 2.12, that every compact subset L of X is already contained in some X_i,
(b) it follows from the Excision Theorem 74.18 (2) that for each compact subset $L \subset X_i$ we get an inclusion induced isomorphism $\mathrm{H}^n(X, X \setminus L) \to \mathrm{H}^n(X_i, X_i \setminus L)$.

Now we prove that Φ is surjective, the proof that Φ is injective is fairly similar and we leave that part of the proof to the reader.

So let c be an element of the limit on the right-hand side. By Lemma 111.2 there exists a compact subset L of X and some $\varphi \in \mathrm{H}^n(X, X \setminus L)$ such that c is the image of φ under the map $\mathrm{H}^n(X, X \setminus L) \to \varinjlim \mathrm{H}^n(X, X \setminus K)$. By (a) there exists an $i \in \mathbb{N}$ with $L \subset X_i$. We consider the following commutative diagram

$$\begin{array}{ccc} \varinjlim_{i \in \mathbb{N}} \varinjlim_{K \in \mathcal{K}(X_i)} \mathrm{H}^n(X_i, X_i \setminus K) & \xrightarrow{\Phi} & \varinjlim_{K \in \mathcal{K}(X)} \mathrm{H}^n(X, X \setminus K) \\ \uparrow \beta & & \uparrow \\ \mathrm{H}^n(X_i, X_i \setminus L) & \xleftarrow[\cong]{\alpha} & \mathrm{H}^n(X, X \setminus L). \end{array}$$

The bottom horizontal map is an isomorphism by (b). Since the diagram is commutative we now see that $\Phi(\beta(\alpha(\varphi))) = c$. ∎

112.3. **Bounded cohomology** (∗). We had just seen that with a modest modification of the original definition of cohomology we obtain a new invariant which for non-compact spaces behaves quite differently from the usual cohomology groups.
In this section we will introduce another variation on cohomology, namely bounded cohomology. We will not make use of it later on, but it does play an important role in topology. A much more detail discussion of bounded cohomology is for example provided in [**Löh2010**].

Definition. Let X be a topological space.

(1) Let $C \in \mathbb{R}$. We say that a cochain $\rho \in \mathrm{C}_n(X;\mathbb{R}) = \mathrm{Hom}(\mathrm{C}_n(X), \mathbb{R})$ is C-bounded if for all singular n-simplices $\sigma\colon \Delta^n \to X$ we have $|\rho(\sigma)| \leq C$.
(2) We say that a cochain $\rho \in \mathrm{C}_n(X;\mathbb{R}) = \mathrm{Hom}(\mathrm{C}_n(X), \mathbb{R})$ is bounded if it is C-bounded for some $C \in \mathbb{R}$.
(3) Given $i \in \mathbb{N}_0$ we write

$$\mathrm{C}^n_{\mathrm{b}}(X;\mathbb{R}) \;:=\; \{\text{all bounded cochains in } \mathrm{C}^n(X;\mathbb{R})\}.$$

Examples.

(1) Let X be a topological space. The 0-cochain that assigns to each point $x \in X$ the same value $c \in \mathbb{R}$ is bounded.
(2) The singular 1-cochain $\theta_{\mathbb{R}} \in \mathrm{C}^1;\mathbb{R})$ defined on page 2404 is *not* bounded. Indeed, given $n \in \mathbb{N}$ we denote by $\mu_n\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $(1-t, t) \mapsto \exp(2\pi \mathrm{i} n t)$. It follows easily from the definition of $\theta_{\mathbb{R}}$ that $\theta_{\mathbb{R}}(\mu) = n$.

Lemma 112.7. Let X be a topological space. The bounded cochains $\mathrm{C}_\mathrm{b}^*(X)$ form a subcomplex of the cochain complex $(\mathrm{C}^*(X), \delta)$.

Proof. Suppose that $\rho \in \mathrm{C}^n(X;\mathbb{R})$ is a cochain that is C-bounded. For any $\varphi\colon \Delta^{n+1} \to X$ we obtain that

$$|(\delta\rho)(\varphi)| \;=\; |\rho(\partial\varphi)| \;\underset{\uparrow}{=}\; \left|\sum_{j=0}^{n+1}(-1)^j \cdot \rho(\varphi \circ i_j^{n+1})\right| \;\underset{\uparrow}{\leq}\; \sum_{j=0}^{n+1}|\rho(\varphi \circ i_j^{n+1})| \;\underset{\uparrow}{\leq}\; (n+2)\cdot C.$$

definition of ∂ and linearity of ρ — triangle inequality — since ρ is C-bounded

This shows that $\delta\rho$ is $C \cdot (n+2)$-bounded, in particular it is bounded. This shows that bounded cochains $\mathrm{C}_\mathrm{b}^*(X)$ form a subcomplex of the cochain complex $(\mathrm{C}^*(X), \delta)$. ■

This lemma leads us straight to the following definition.

Definition. Let X be a topological space and let $n \in \mathbb{N}_0$. The *n-th bounded cohomology group of X* is defined as

$$\mathrm{H}_\mathrm{b}^n(X;\mathbb{R}) \;:=\; \frac{\ker(\delta_n\colon \mathrm{C}_\mathrm{b}^n(X;\mathbb{R}) \to \mathrm{C}_\mathrm{b}^{n+1}(X;\mathbb{R}))}{\mathrm{im}(\delta_{n-1}\colon \mathrm{C}_\mathrm{b}^{n-1}(X;\mathbb{R}) \to \mathrm{C}_\mathrm{b}^n(X;\mathbb{R}))}.$$

Examples.

(1) The proof of the H^0-Proposition 108.11 can be used, without any changes, to show that 0-th bounded cohomology of a path-connected topological space agrees with 0-th cohomology with real coefficients.[1365]

(2) Every cochain on a topological space with finitely many points is bounded. Thus it follows from the Single Point-Cohomology Lemma 108.12 that for a topological space $\{x_0\}$ consisting of a single point x_0 we have

$$\mathrm{H}_\mathrm{b}^n(\{x_0\};\mathbb{R}) \;=\; \mathrm{H}^n(\{x_0\};\mathbb{R}) \;=\; \begin{cases} 0, & \text{if } n \geq 1, \\ \mathbb{R}, & \text{if } n = 0. \end{cases}$$

Proposition 112.8.

(1) Let $f\colon X$ *to* Y be a map between two topological spaces. The induced map $f^*\colon \mathrm{C}^*(Y;\mathbb{R}) \to \mathrm{C}^*(X;\mathbb{R})$ on singular cochains restricts to a cochain map $f^*\colon \mathrm{C}_\mathrm{b}^*(Y;\mathbb{R}) \to \mathrm{C}_\mathrm{b}^*(X;\mathbb{R})$. In particular the map f^* induces a map

$$f^*\colon \mathrm{H}_\mathrm{b}^n(Y;\mathbb{R}) \;\to\; \mathrm{H}_\mathrm{b}^n(X;\mathbb{R}).$$

(2) For each $n \in \mathbb{N}_0$ the maps $X \mapsto \mathrm{H}_\mathrm{b}^n(X;\mathbb{R})$ and $f \mapsto f^*$ define a contravariant functor from the category of topological spaces to the category of real vector spaces.

(3) Homotopic maps induce the same maps on bounded cohomology.

(4) If $f\colon X \to Y$ is a homotopy equivalence, then the induced map

$$f^*\colon \mathrm{H}_\mathrm{b}^n(Y;\mathbb{R}) \;\xrightarrow{\cong}\; \mathrm{H}_\mathrm{b}^n(X;\mathbb{R})$$

is in fact an isomorphism.

[1365] Is this statement also true for topological spaces that are not path-connected?

Proof.

(1) It is obvious that the pullback of a bounded cohomology class is again bounded. The first statement is an immediate consequence of this observation.

(2) This statement is obvious.

(3) Let $f, g\colon X \to Y$ be two maps between topological spaces X and Y. Furthermore let $F\colon X \times [0,1] \to Y$ be a homotopy between the maps f and g. As on page 1608 we define

$$P_n\colon \mathrm{C}_n(X) \;\to\; \mathrm{C}_{n+1}(Y)$$

to be the map that is given by

$$(\sigma\colon \Delta^n \to X) \;\mapsto\; \sum_{j=0}^{n} (-1)^j \cdot F \circ (\sigma \times \mathrm{id}_{[0,1]}) \circ \underbrace{[v_0, \dots, v_j, w_j, \dots, w_n]}_{\Delta^{n+1} \to \Delta^n \times [0,1]}.$$

In the proof of the Homotopic Maps-and-Homology Proposition 73.6 we saw that the maps P_n define a chain homotopy between f_* and g_*. It follows from this explicit description of P_n that for any cochain $\varphi \in \mathrm{C}^{n+1}(Y;\mathbb{R})$ that is C-bounded the chain $P^*\varphi \in \mathrm{C}^n(X;\mathbb{R})$ is $(n+1)\cdot C$-bounded, the map $P^*\colon \mathrm{C}^{n+1}(Y;\mathbb{R}) \to \mathrm{C}^n(X;\mathbb{R})$ restricts to a map $\mathrm{C}_\mathrm{b}^{n+1}(Y;\mathbb{R}) \to \mathrm{C}_\mathrm{b}^n(X;\mathbb{R})$. As in the proof of the Dual Cochain Map Lemma 108.8 (3) one sees easily that the maps P^* form a cochain homotopy between $f^*, g^*\colon \mathrm{C}_\mathrm{b}^*(Y;\mathbb{R}) \to \mathrm{C}_\mathrm{b}^*(X;\mathbb{R})$. The third statement now follows from the Cochain Map Lemma 108.5.

(4) This statement follows immediately from (2) and (3). ∎

So far it seems like bounded cohomology might be quite similar to usual cohomology with real coefficients. It turns out that this guess is dramatically misguided.
For example the following theorem was proved by Mikhail Gromov [**Grom1983**, p. 40], see also [**Iv1987**, Theorem 4.3].[1366]

Theorem 112.9. Let $f\colon X \to Y$ be a map between two connected CW-complexes. If the induced map $f_*\colon \pi_1(X) \to \pi_1(Y)$ is an epimorphism of fundamental groups such that the kernel $\ker(f_*\colon \pi_1(X) \to \pi_1(Y))$ is an amenable group, then for any $n \in \mathbb{N}_0$ the induced map

$$f^*\colon \mathrm{H}_\mathrm{b}^n(Y;\mathbb{R}) \;\xrightarrow{\cong}\; \mathrm{H}_\mathrm{b}^n(X;\mathbb{R})$$

is an isomorphism.

Remark. In the formulation of Theorem 112.9 we used the notion of an "amenable group". We will not give the definition of an "amenable group", instead we refer to [**Pie1984**] or [**CSC2010**, Section 4] for the definition. What matters to us are the following two facts:

(1) Groups that admit a finite index subgroup that is solvable are amenable, see [**Pie1984**, Propositions 13.3 and 13.4]. In particular finite groups and solvable groups are amenable.

(2) Groups that contain a free group on two generators are not amenable, see [**Pie1984**, Proposition 14.1].

Now we can formulate the following rather surprising corollary.

[1366] Mikhail Gromov (*1943) is a Russian-French mathematician who is easily one of the most influential geometers alive. He won the Abel prize in 2009.

Corollary 112.10. Let X be a connected CW-complex such that $\pi_1(X)$ is amenable, e.g. such that $\pi_1(X)$ is solvable. Then $\mathrm{H}_{\mathrm{b}}^n(X;\mathbb{R}) = 0$ for any $n \geq 1$.

Example. It follows from Corollary 112.10 that the higher bounded cohomology groups of the n-tori $(S^1)^n$, the spheres S^n and lens spaces all vanish. In particular this new invariant is also not useful in our attempt to answer Question 48.7.

Proof. We consider the map $f\colon X \to \{x_0\}$ that is given by sending every point in X to the point x_0. Then for any $n \geq 1$ we have

$$\begin{array}{ccccc} \mathrm{H}_{\mathrm{b}}^n(X;\mathbb{R}) & \xrightarrow[\cong]{f^*} & \mathrm{H}_{\mathrm{b}}^n(\{x_0\};\mathbb{R}) & = & 0. \\ \uparrow & & \uparrow & & \\ \text{by Theorem 112.9 since} & & \text{see above example} & & \\ \ker(f_*) = \pi_1(X) \text{ is amenable} & & & & \end{array}$$

We have thus proved the desired result. ∎

The following vanishing result is in fact fairly easy to show, see e.g. [**Frig2017**, Chapter 2].

Lemma 112.11. For any CW-complex X we have $\mathrm{H}_{\mathrm{b}}^1(X;\mathbb{R}) = 0$.

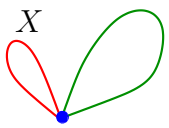
On the other hand, if the fundamental group is not amenable, then we also see some totally unexpected behavior as is shown in the following proposition.

Proposition 112.12. Let X be the wedge of two circles. Then $\mathrm{H}_{\mathrm{b}}^2(X;\mathbb{R})$ and $\mathrm{H}_{\mathrm{b}}^3(X;\mathbb{R})$ are vector spaces of uncountable dimension.

Proof. The statement regarding $\mathrm{H}_{\mathrm{b}}^2(X;\mathbb{R})$ is proved in [**EF1997**]. An alternative fairly straightforward proof, using quasi-morphisms as introduced on page 1166, is given in [**Roll2009**, Corollary 2.3]. Furthermore the statement regarding $\mathrm{H}_{\mathrm{b}}^3(X;\mathbb{R})$ is proved in [**So1997**, Theorem 3]. ∎

Somewhat shockingly the following question is still open:

Question 112.13. Let X be the wedge of two circles. What are the bounded cohomology groups $\mathrm{H}_{\mathrm{b}}^k(X;\mathbb{R}) \neq 0$ for $k \geq 4$? Is any of the groups non-zero? Are all of them non-zero? Are all of them of uncountable dimension?



$$\begin{aligned} \dim(H_b^1(X;\mathbb{R})) &= 0 \\ \dim(H_b^2(X;\mathbb{R})) &= \infty \\ \dim(H_b^3(X;\mathbb{R})) &= \infty \\ \dim(H_b^4(X;\mathbb{R})) &= ??? \end{aligned}$$

Exercises for Chapter 112.

Exercise 112.1. We consider $X = \mathbb{R}$ and the singular 0-cochain

$$\varphi_0\colon \mathrm{C}_0(X) \to \mathbb{Z}$$

that is uniquely determined by the property that for any singular 0-simplex $x \in X = \mathbb{R}$ we have

$$x \mapsto \begin{cases} 1, & \text{if } x \leq 0, \\ 0, & \text{if } x > 0. \end{cases}$$

On page 2480 we saw that $\gamma_0 = \delta\varphi_0 \colon \mathrm{C}_1(\mathbb{R}) \to \mathbb{Z}$ defines a class in $\mathrm{H}^1_{\mathrm{c}}(\mathbb{R};\mathbb{Z})$. Prove, only using the definitions, that this class is non-zero.

Exercise 112.2. What is the cohomology with compact support of the topological spaces

$$X \;:=\; \overline{B}_1(0) \cup x\text{-axis} \quad \text{and} \quad Y \;:=\; \overline{B}_1(0) \cup x\text{-axis} \cup y\text{-axis}$$

that are illustrated in the figure on page 2484.

Exercise 112.3.

(a) Give an example of a path-connected topological space X such that $\mathrm{H}^1(X) = 0$ but such that $\mathrm{H}^1_{\mathrm{c}}(X) \cong \mathbb{Z}^{(\mathbb{N})}$. You do not have to justify why your example has the correct properties.

(b) Give an example of a path-connected topological space X such that $\mathrm{H}^2(X) = 0$ but such that $\mathrm{H}^2_{\mathrm{c}}(X) \cong \mathbb{Z}^3$. You do not have to justify why your example has the correct properties.

Exercise 112.4. Let X be a topological space. We consider the set $\mathcal{K}(X)$ of all compact subsets of X. The set $\mathcal{K}(X)$ together with the relation "$\subset$" defined by inclusion is a directed set since the union of two compact subsets is again compact. Given any two compact subsets $K \subset L$ we get an induced map $\mathrm{H}_n(X, X \setminus L; G) \to \mathrm{H}_n(X, X \setminus K; G)$. As in Exercise 106.5 we define

$$\overleftarrow{\mathrm{H}}_n(X;\mathbb{Z}) \;:=\; \varprojlim_{K \in \mathcal{K}(X)} \mathrm{H}_n(X, X \setminus K; \mathbb{Z}).$$

(a) Recall that on page 2479 we introduced cohomology group with compact support $\mathrm{H}^n_{\mathrm{c}}(X;\mathbb{Z}) = \mathrm{H}^n_{\mathrm{c}}(X;\mathbb{Z})$. Define a useful pairing $\mathrm{H}^n_{\mathrm{c}}(X) \times \overleftarrow{\mathrm{H}}_n(X;\mathbb{Z}) \to \mathbb{Z}$.
Remark. You could make use of Exercise 56.21.

(b) Use (a) to show that $\mathrm{H}^{\mathrm{lf}}_1(\mathbb{R}) \neq 0$.

Exercise 112.5. Let M be a *compact* topological manifold. We write $\mathring{M} = M \setminus \partial M$. The cohomology group $\mathrm{H}^n_{\mathrm{c}}(\mathring{M})$ is homeomorphic to some other cohomology group that might look more familiar. What is it? You do not have to justify your answer.

Exercise 112.6. Let us once again consider the topological space

$$X \;:=\; (\mathbb{R} \cup (S^2 \times \mathbb{Z}))/\!\sim$$

where given $i \in \mathbb{Z}$ we identify $i \in \mathbb{Z} \subset \mathbb{R}$ with $((0,0,-1), i) \in S^2 \times \mathbb{Z}$. On page 2475 we saw that $\mathrm{H}^2(X) \cong \mathbb{Z}^{\mathbb{N}}$. Show that $\mathrm{H}^2_{\mathrm{c}}(X) \cong \mathbb{Z}^{(\mathbb{N})}$.

Exercise 112.7. Show that for every path-connected topological space X the boundary cohomology $\mathrm{H}^1_b(X;\mathbb{R}) = 0$.

113. The Eilenberg-Zilber theorem and the cross product

In this chapter we will to try to find if one can define a "product" in singular homology and/or in singular cohomology. The key to doing so will be the Eilenberg-Zilber Theorem 90.4 that we had stated before without providing a proof. In this chapter we will give a slightly more precise statement and we provide the proof. In this chapter and in Chapter 114 we will see how it can be used to find product structures on homology and cohomology groups.

113.1. **The Eilenberg-Zilber theorem.** We start out by recalling and introducing some definitions.

Definition.

(1) Let $\mathcal{C} = (\mathrm{C}_*, \partial_*)$ and $\mathcal{C} = (\mathrm{C}'_*, \partial'_*)$ be two chain complexes. On page 1940 we defined the tensor product $\mathcal{C} \otimes \mathcal{C}'$ of the chain complexes where the n-th chain group is defined as

$$(\mathcal{C} \otimes \mathcal{C}')_n := \bigoplus_{p+q=n} \mathrm{C}_p \otimes \mathrm{C}'_q$$

and where the boundary homomorphisms are given by

$$\begin{array}{rcl} \partial^{\otimes} \colon (\mathcal{C} \otimes \mathcal{C}')_n & \to & (\mathcal{C} \otimes \mathcal{C}')_{n-1} \\ c_p \otimes c'_q & \mapsto & \partial c_p \otimes c'_q + (-1)^p \cdot c_p \otimes \partial' c'_q. \end{array}$$

(2) Given a point x in a topological space X we denote by x also the singular 0-simplex given by sending the unique point of Δ^0 to x.

(3) Given two topological spaces X and Y we refer to the maps

$$\begin{array}{rcl} \mathrm{C}_0(X) \otimes \mathrm{C}_0(Y) & \to & \mathrm{C}_0(X \times Y) \\ x \otimes y & \mapsto & (x, y) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{C}_0(X \times Y) & \to & \mathrm{C}_0(X) \otimes \mathrm{C}_0(Y) \\ (x, y) & \mapsto & x \otimes y \end{array}$$

as the canonical maps. We refer to the identity maps on $\mathrm{C}_0(X) \otimes \mathrm{C}_0(Y)$ and on $\mathrm{C}_0(X \times Y)$ also as the canonical maps.

The following theorem is a slightly more precise formulation of the original Eilenberg-Zilber Theorem 90.4.

Theorem 113.1. (Eilenberg-Zilber Theorem)

(1) Given any topological spaces X and Y there exist natural chain maps

$$\begin{array}{rcl} \Upsilon \colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) & \to & \mathrm{C}_*(X \times Y) \\ \text{and} \quad \Theta \colon \mathrm{C}_*(X \times Y) & \to & \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y), \end{array}$$

which on the 0-level are the above canonical maps.

(2) The maps Υ and Θ are unique up to natural chain homotopies.[1367]

(3) The maps Υ and Θ are naturally[1368] chain homotopy inverses of one another.

We refer to the maps Υ and Θ as *Eilenberg-Zilber* maps. In the notation and our discussions we will usually ignore the fact that these maps are well-defined only up to a natural

[1367]Here "natural chain homotopy" means that given two such maps Υ and Υ' there exists a choice of chain homotopies $(\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_k \to \mathrm{C}_{k+1}(X \times Y)$, which is natural in X and Y, between Υ and Υ'. An analogous statement holds for maps Θ and Θ'.

[1368]Here "naturally" means that $\Upsilon \circ \Theta$ and $\Theta \circ \Upsilon$ are naturally homotopic to the identity maps, i.e. there exist natural chain homotopies from $\Upsilon \circ \Theta$ to the identity map and from $\Theta \circ \Upsilon$ to the identity map.

homotopy. We will only work with the properties of Υ and Θ stated in the theorem, so a different choice of maps Υ and Θ does not affect anything we say afterwards.

113.2. **Proof of the Eilenberg-Zilber Theorem 113.1.** The proof of the Eilenberg-Zilber Theorem 113.1 breaks up into three parts: we first construct Υ, then we construct Θ and finally we show that these maps are unique in the above sense and that they have the desired properties.
We start out with the following definition.

Definition. A topological space X is called acyclic if $\mathrm{H}_0(X;\mathbb{Z}) \cong \mathbb{Z}$ and if all higher homology groups are zero.

Example. In the Homotopy Equivalence-H_*-Corollary 73.9 we showed that every contractible topological space is acyclic. In particular every standard simplex Δ^n is contractible and thus acyclic. Furthermore, all products $\Delta^p \times \Delta^q$ of standard simplices are contractible and thus acyclic.[1369]

We also fix some notation.

Notation. Let X be a topological space and let $x \in X$. If $\sigma\colon \Delta^n \to Y$ is a singular n-simplex in a topological space Y, then we denote by $x \times \sigma\colon \Delta^n \to X \times Y$ the singular n-simplex given by $v \mapsto (x, \sigma(v))$. We use the analogous notation for the roles of X and Y swapped.

Now we can formulate the following theorem.

Theorem 113.2. (EZ-Construction Theorem) Given any two topological spaces X and Y and any $p, q \in \mathbb{N}_0$ there exists a homomorphism

$$\Upsilon_{p,q}\colon \mathrm{C}_p(X) \otimes \mathrm{C}_q(Y) \ \to \ \mathrm{C}_{p+q}(X \times Y)$$

such that the following conditions are satisfied:

(1) for $x \in X$ and $\sigma\colon \Delta^q \to Y$ we have $\Upsilon_{0,q}(x \otimes \sigma) = x \times \sigma$, the same statement also holds with the roles of X and Y swapped,

(2) the map $\Upsilon_{p,q}$ is natural, i.e. if $f\colon X' \to X$ and $g\colon Y' \to Y$ are maps between topological spaces, then for any $a \in \mathrm{C}_p(X')$ and any $b \in \mathrm{C}_q(Y')$ we have

$$(f \times g)_*(\underbrace{\Upsilon_{p,q}(a \otimes b)}_{\in \mathrm{C}_{p+q}(X' \times Y')}) \ = \ \underbrace{\Upsilon_{p,q}(f_*(a) \otimes g_*(b))}_{\in \mathrm{C}_{p+q}(X \times Y)},$$

(3) the maps $\Upsilon_{p,q}$ form a chain map, i.e. for any $a \in \mathrm{C}_p(X)$ and any $b \in \mathrm{C}_q(Y)$ we have

$$\partial(\Upsilon_{p,q}(a \otimes b)) \ = \ \underbrace{\Upsilon_{p-1,q}(\partial a \otimes b) + (-1)^p \cdot \Upsilon_{p,q-1}(a \otimes \partial b)}_{= \Upsilon(\partial^{\otimes}(a \otimes b))}.$$

Proof. We prove the existence of the maps $\Upsilon_{p,q}$ by induction on $p \cdot q$. For clarity we henceforth suppress the subscript from $\Upsilon_{p,q}$.

[1369] This can be seen in many ways, for example pick points $x \in \Delta^p$ and $y \in \Delta^q$, it is straightforward to show that there exists a deformation retraction r from Δ^p to $\{x\}$ and that there exists a deformation retraction s from Δ^q to $\{y\}$. But these two deformation retractions also define a deformation retraction from $\Delta^p \times \Delta^q$ to (x, y).

Clearly for $p = 0$ or $q = 0$ the maps from (1) have the properties (2) and (3). So now suppose that we are given $p, q \in \mathbb{N}_0$ with $p \cdot q > 0$ such that the maps Υ have been defined for all p', q' with $p' \cdot q' < p \cdot q$.

Given any $n \in \mathbb{N}_0$ we denote by $\mathrm{id}_n \colon \Delta^n \to \Delta^n$ the identity map. We first want to come up with a reasonable definition of $\Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q)$. If we want (3) to be satisfied we have to define $\Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q) = z$ where $z \in \mathrm{C}_{p+q}(\Delta^p \times \Delta^q)$ satisfies

$$(*) \qquad \partial z \;=\; \Upsilon(\partial\,\mathrm{id}_p \otimes \mathrm{id}_q) + (-1)^p \cdot \Upsilon(\mathrm{id}_p \otimes \partial\,\mathrm{id}_q) \;\in\; \mathrm{C}_{p+q-1}(\Delta^p \times \Delta^q).$$

As we pointed out on page 2492, the product $\Delta^p \times \Delta^q$ is acyclic, in particular we have $\mathrm{H}_{p+q-1}(\Delta^p \times \Delta^q) = 0$, which means that every singular $(p+q-1)$-chain *that is a cycle* is in fact the boundary of some $(p+q)$-chain z. So we have to show that the right-hand side of $(*)$ is indeed a cycle. We compute that

$$\begin{aligned}
&\overbrace{\partial\big(\Upsilon(\partial\,\mathrm{id}_p \otimes \mathrm{id}_q) + (-1)^p \cdot \Upsilon(\mathrm{id}_p \otimes \partial\,\mathrm{id}_q)\big)}^{\text{boundary of our hypothetical } \Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q)} \overset{\text{linearity of } \partial}{\downarrow}{=}\; \partial(\Upsilon(\partial\,\mathrm{id}_p \otimes \mathrm{id}_q)) + (-1)^p \cdot \partial(\Upsilon(\mathrm{id}_p \otimes \partial\,\mathrm{id}_q)) \\
&= \Upsilon(\underbrace{\partial\partial\,\mathrm{id}_p}_{=0} \otimes \mathrm{id}_q) + (-1)^{p-1} \cdot \Upsilon(\partial\,\mathrm{id}_p \otimes \partial\,\mathrm{id}_q) \\
&\quad + (-1)^p \cdot \Upsilon(\partial\,\mathrm{id}_p \otimes \partial\,\mathrm{id}_q) + (-1)^p \cdot (-1)^p \cdot \Upsilon(\mathrm{id}_p \otimes \underbrace{\partial\partial\,\mathrm{id}_q}_{=0}) = 0.
\end{aligned}$$

$\uparrow$ by our induction hypothesis we can apply the boundary formula (3) to $\Upsilon(\partial\,\mathrm{id}_p \otimes \mathrm{id}_q)$ and $\Upsilon(\mathrm{id}_p \otimes \partial\,\mathrm{id}_q)$

This discussion gives us the definition of $\Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q) := z$.

Now let X and Y be topological spaces and let $\varphi \colon \Delta^p \to X$ and $\psi \colon \Delta^q \to Y$ be singular simplices. If we want (2) to be satisfied then we have to define

$$\Upsilon(\underbrace{\varphi \otimes \psi}_{\in \mathrm{C}_p(X) \otimes \mathrm{C}_q(Y)}) \;:=\; (\varphi \times \psi)_*(\underbrace{\Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q)}_{\in \mathrm{C}_{p+q}(\Delta^p \times \Delta^q)}) \;\in\; \mathrm{C}_{p+q}(X \times Y).$$

We extend this map linearly to all of $\mathrm{C}_p(X) \otimes \mathrm{C}_q(Y)$. It follows almost immediately from the definition of Υ that the map Υ is natural in X and Y.

It remains to prove (3). This property holds basically by definition. But for completeness' sake we carry out the proof. So let $\varphi \colon \Delta^p \to X$ and $\psi \colon \Delta^q \to Y$ be singular simplices. We then have the following equality in $\mathrm{C}_{p+q-1}(X \times Y)$:

$$\begin{aligned}
\partial(\Upsilon(\varphi \otimes \psi)) \;&=\; \partial\big((\varphi \times \psi)_*(\Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q))\big) \overset{\text{since } (\varphi \times \psi)_* \text{ is a chain map}}{\downarrow}{=}\; (\varphi \times \psi)_*(\partial(\overbrace{\Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q)}^{=z})) \\
&=\; (\varphi \times \psi)_*(\Upsilon(\partial\,\mathrm{id}_p \otimes \mathrm{id}_q)) + (-1)^p \cdot (\varphi \times \psi)_*(\Upsilon(\mathrm{id}_p \otimes \partial\,\mathrm{id}_q)) \\
&\quad \uparrow \text{ by definition of } z \text{ and the linearity of } (\varphi \times \psi)_* \\
&=\; (\Upsilon(\varphi_*(\partial\,\mathrm{id}_p) \otimes \psi_*(\mathrm{id}_q))) + (-1)^p \cdot \Upsilon(\varphi_*(\mathrm{id}_p) \otimes \psi_*(\partial\,\mathrm{id}_q)) \\
&\quad \uparrow \text{ by induction we can apply the naturality of } \Upsilon \\
&=\; \Upsilon(\partial(\varphi) \otimes \psi) + (-1)^p \cdot \Upsilon(\varphi \otimes \partial\psi)) \\
&\quad \uparrow \text{ since } \varphi_* \text{ and } \psi_* \text{ are chain maps and since } \varphi_*(\mathrm{id}_p) = \varphi \text{ and } \psi_*(\mathrm{id}_q) = \psi
\end{aligned}$$

■

Now we turn to the construction of the second map in the Eilenberg-Zilber Theorem 113.1.

Theorem 113.3. (EZ-Inverse Theorem) Given any two topological spaces X and Y and any $k \in \mathbb{N}_0$ there exists a homomorphism

$$\Theta_k \colon \mathrm{C}_k(X \times Y) \ \to \ (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_k \ = \ \bigoplus_{p+q=k} \mathrm{C}_p(X) \otimes \mathrm{C}_q(Y)$$

such that the following conditions are satisfied:

(1) the map Θ_0 is the canonical map on the 0-level, i.e. for every $x \in X$ and $y \in Y$ we have $\Theta_0((x,y)) = x \otimes y$,

(2) the map Θ_k is natural in X and Y, i.e. for any two maps $f \colon X' \to X$ and $g \colon Y' \to Y$ of topological spaces and any $c \in \mathrm{C}_k(X' \times Y')$ we have

$$\Theta_k((f \times g)_*(c)) \ = \ (f_* \otimes g_*)(\Theta_k(c)),$$

(3) the maps Θ_* form a chain map, i.e. for any $k \in \mathbb{N}_0$ we have

$$\partial_k^{\otimes} \circ \Theta_k \ = \ \Theta_{k-1} \circ \partial_k.$$

In the proof of the EZ-Inverse Theorem 113.3 we will need the following lemma.

Lemma 113.4. Let X and Y be two acyclic topological spaces. For every $k \geq 1$ we have

$$\mathrm{H}_k(\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)) \ = \ 0.$$

Furthermore we have an isomorphism

$$\begin{array}{rcl} \epsilon \colon \mathrm{H}_0(\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)) & \xrightarrow{\cong} & \mathbb{Z} \\ \Big[\sum_{i=1}^{k} \underbrace{a_i}_{\in \mathbb{Z}} \cdot \underbrace{x_i}_{\in X} \otimes \underbrace{y_i}_{\in Y} \Big] & \mapsto & \sum_{i=1}^{k} a_i. \end{array}$$

Proof. The lemma is an immediate consequence of our hypotheses, the H_0-Proposition 72.5 and the Künneth Theorem 90.7 for Chain Complexes. (Note that this Künneth Theorem is purely algebraic.) ■

Proof of the EZ-Inverse Theorem 113.3. As we will see the idea behind the proof of the EZ-Inverse Theorem 113.3 is very similar to the idea of the proof of Theorem 113.2. We start out with the outline of proof:

(a) We use (1) to define Θ_0,

(b) we suppose the map $\Theta_0, \ldots, \Theta_{k-1}$ are defined, we then use (3) to make an educated guess for $\Theta_k(d_k)$ where we consider
  (i) the topological spaces $X = Y = \Delta^k$ and
  (ii) the singular k-simplex $d_k \colon \Delta^k \to \Delta^k \times \Delta^k$ that is given by the diagonal map $d_k(x) = (x,x)$.

(c) we then use the naturality statement of (2) to define Θ_k for all topological spaces X and Y and all singular simplices in $\mathrm{C}_*(X \times Y)$.

Now we carry out the above program. First of all, given a 0-simplex (x,y) in $X \times Y$ we define $\Theta_0((x,y)) = x \otimes y$. Then we extend this definition linearly to get the desired map $\Theta_0 \colon \mathrm{C}_0(X \times Y) \to \mathrm{C}_0(X) \otimes \mathrm{C}_0(Y)$.

Now suppose we have already constructed the maps $\Theta_0, \ldots, \Theta_{k-1}$. We want to define Θ_k and we start out by defining $\Theta_k(d_k \colon \Delta^k \to \Delta^k \times \Delta^k)$. Since we need (3) to be satisfied we need to define $\Theta_k(d_k) := c$ where $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_k$ satisfies $\partial_k^{\otimes}(c) = \Theta_{k-1}(\partial d_k)$.

In the chain complex $\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)$ we calculate that

$$\partial^{\otimes}(\overbrace{\Theta_{k-1}(\partial_k(d_k))}^{\text{defined by induction}}) \;\underset{\text{by induction we can apply (3)}}{=}\; \Theta_{k-2}(\underbrace{\partial_{k-1}\partial_k}_{=0} d_k) \;=\; 0,$$

i.e. the chain $\Theta_{k-1}(\partial_k d_k) \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_{k-1}$ is in fact a cycle. Since Δ^k is acyclic it follows immediately from Lemma 113.4 that this cycle is null-homologous. (For $k = 1$ one easily verifies, using the explicit description of Θ_0, that the cycle lies in the kernel of the isomorphism $\epsilon\colon \mathrm{H}_0(\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)) \to \mathbb{Z}$.) This means that there exists a singular chain $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_k$ with $\partial^{\otimes}(c) = \Theta_{k-1}(\partial_k d_k)$. We define $\Theta_k(d_k) := c$.

Now we define Θ_k for arbitrary topological spaces X and Y and an arbitrary singular k-simplex $\sigma\colon \Delta^k \to X \times Y$. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the natural projections. Note that we have a commutative diagram

$$\begin{array}{ccc} & & \Delta^k \times \Delta^k \\ & \overset{d_k}{\nearrow} & \big\downarrow {\scriptstyle (p\circ\sigma)\times(q\circ\sigma)} \\ \Delta^k & & \\ & \underset{\sigma}{\searrow} & X \times Y, \end{array}$$

i.e. we have $\sigma = ((p\circ\sigma)\times(q\circ\sigma))\circ d_k = ((p\circ\sigma)\times(q\circ\sigma))_*(d_k)$. For the naturality condition (2) to be satisfied we need to define

$$\Theta_k(\sigma) \;:=\; ((p\circ\sigma)_* \otimes (q\circ\sigma)_*)(\Theta_k(d_k)) \;\in\; (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_k.$$

With this definition one readily verifies that the map Θ_k is natural in X and Y.

Finally we have to show that Θ_k satisfies the boundary formula (3). So as before let $\sigma\colon \Delta^k \to X \times Y$ be a singular k-simplex. Then

$$\begin{array}{lll} & \text{since } (p\circ\sigma)_* \otimes (q\circ\sigma)_*\colon \mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) & \\ \text{by definition of } \Theta_k(\sigma) & \text{is a chain map by Lemma 90.3} & \\ \partial^{\otimes}\Theta_k(\sigma) \overset{\downarrow}{=} \partial^{\otimes}\big(((p\circ\sigma)_* \otimes (q\circ\sigma)_*)(\Theta_k(d_k))\big) & \overset{\downarrow}{=} ((p\circ\sigma)_* \otimes (q\circ\sigma)_*)(\partial^{\otimes}\Theta_k(d_k)) & \\ = ((p\circ\sigma)_* \otimes (q\circ\sigma)_*)(\Theta_{k-1}(\partial d_k)) & = \Theta_{k-1}\big(((p\circ\sigma)\times(q\circ\sigma))_*(\partial d_k)\big) & \\ \uparrow \text{ by definition of } \Theta_k(d_k) & \uparrow \text{ by naturality of } \Theta_{k-1} & \\ = \Theta_{k-1}\partial\big(((p\circ\sigma)\times(q\circ\sigma))_*(d_k)\big) & = \Theta_{k-1}\partial\sigma. & \\ \uparrow \text{ since } ((p\circ\sigma)\times(q\circ\sigma))_* \text{ is a chain map} & \uparrow \text{ since } \sigma = ((p\circ\sigma)\times(q\circ\sigma))\circ d_k & \end{array}$$

We have now verified that Θ_k has all the desired properties. ∎

As we saw in the proofs of Theorem 113.2 and the EZ-Inverse Theorem 113.3, the natural chain maps $\Upsilon\colon \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y) \to \mathrm{C}_*(X\times Y)$ and $\Theta\colon \mathrm{C}_*(X\times Y) \to \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)$ depend on many choices. The next theorem says that different choices in the proofs lead to natural chain maps that are naturally homotopy equivalent to one another.

Theorem 113.5. Let X and Y be topological spaces. Any two natural chain maps

$$\mathrm{C}_*(X \times Y) \;\to\; \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$$

that are the canonical maps on the 0-level are in fact naturally chain homotopic. The same conclusion also holds for any two natural chain maps

$$\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)\to\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y),\ \ \mathrm{C}_*(X\times Y)\to\mathrm{C}_*(X\times Y),\ \ \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)\to\mathrm{C}_*(X\times Y)$$

that are the canonical maps on the 0-level.

Proof. The proof of this theorem is in many ways quite similar to the proofs of the EZ-Construction Theorem 113.2 and the EZ-Inverse 113.3.

In the following we show the first statement. The proofs of the other three statements are very similar. So let

$$F,G\colon \mathrm{C}_*(X\times Y)\ \to\ \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)$$

be two natural chain maps that are the canonical maps on the 0-chains, i.e. which satisfy $F((x,y))=x\otimes y=G((x,y))$ for all $x\in X, y\in Y$. Given any $k\in\mathbb{Z}_{\geq -1}$ we have to define a map

$$P_k\colon \mathrm{C}_k(X\times Y)\ \to\ (\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y))_{k+1}$$

that satisfies the following three properties:

(1) the map P_{-1} is the zero map,
(2) the map P_k is natural in X and Y,
(3) we have $\partial^{\otimes}\circ P_k+P_{k-1}\circ\partial=F-G$.

So suppose we have already defined the maps $P_{-1},\dots,P_{k-1}$ with $k\geq 0$. As before we denote by $d_k\colon\Delta^k\to\Delta^k\times\Delta^k$ the singular k-simplex that is given by the diagonal map $d_k(x):=(x,x)$. We first want to define $P_k(d_k)\in(\mathrm{C}_*(\Delta^k)\otimes\mathrm{C}_*(\Delta^k)_{k+1}$. In the chain complex $\mathrm{C}_*(\Delta^k)\otimes\mathrm{C}_*(\Delta^k)$ we calculate that

$$\begin{aligned}\partial^{\otimes}((F-G-P_{k-1}\partial)(d_k)) &= \partial^{\otimes}F(d_k)-\partial^{\otimes}G(d_k)-\partial^{\otimes}P_{k-1}(\partial d_k)\\ &\overset{\uparrow}{=} F(\partial d_k)-G(\partial d_k)-(F-G-P_{k-2}\partial)(\partial d_k)=P_{k-2}\partial\partial d_k=0.\end{aligned}$$

since F and G are chain maps and since inductively we know that $\partial^{\otimes}\circ P_{k-1}+P_{k-1}\circ\partial=F-G$

It follows from Lemma 113.4 that there exists a $(k+1)$-chain $c\in(\mathrm{C}_*(\Delta^k)\otimes\mathrm{C}_*(\Delta^k))_{k+1}$ with $\partial^{\otimes}(c)=(F-G-P_{k-1}\partial)(d_k)$. (For $k=0$ we need to show that $\epsilon((F-G)(d_0))=0$, to conclude this we need to use that F and G agree on 0-simplices.) We define $P_k(d_k):=c$.

Now let X and Y be two topological spaces and let $\sigma\in\mathrm{C}_k(X\times Y)$. As in the proof of the EZ-Construction Theorem 113.2 we denote by $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ the natural projections and we set

$$P_k(\sigma)\ :=\ ((p\circ\sigma)_*\otimes(q\circ\sigma)_*)(P_k(d_k))\ \in\ (\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y))_{k+1}.$$

Basically by definition this map is natural in X and Y. As in the proof of the EZ-Construction Theorem 113.2 one easily verifies that condition (3) is satisfied. ■

Now we can easily provide the proof of the Eilenberg-Zilber Theorem 113.1.

Proof of the Eilenberg-Zilber Theorem 113.1. We pick natural chain maps

$$\Upsilon\colon \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)\ \to\ \mathrm{C}_*(X\times Y)$$

and

$$\Theta\colon \mathrm{C}_*(X\times Y)\ \to\ \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)$$

as provided by the EZ-Construction Theorem 113.2 and the EZ-Inverse Theorem 113.3. These maps are the canonical maps on the 0-level. By Theorem 113.5 the maps Υ and Θ are thus unique up to a natural chain homotopy.

It remains to show that the maps Υ and Θ are naturally chain homotopy inverses of one another. The map $\Theta \circ \Upsilon \colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$ is a natural chain map that is the canonical map on the 0-level. But this is also the case for the identity map. It follows from Theorem 113.5 that $\Theta \circ \Upsilon$ is naturally chain homotopic to the identity. The same argument shows that $\Upsilon \circ \Theta$ is naturally chain homotopic to the identity. ■

113.3. **The relative Eilenberg–Zilber map Υ.** In this short section we introduce a relative version of the Eilenberg–Zilber map Υ. We start out with the following elementary but essential remark.

Remark. Let X and Y be topological spaces and let $A \subset X$ and $B \subset Y$ be subsets. We denote by $i \colon A \to X$ and $j \colon B \to Y$ the inclusion maps. Then the diagram

$$\begin{array}{ccc} \mathrm{C}_*(A) \otimes \mathrm{C}_*(B) & \xhookrightarrow{i_* \otimes j_*} & \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \\ \downarrow \Upsilon & & \downarrow \Upsilon \\ \mathrm{C}_*(A \times B) & \xhookrightarrow{(i \times j)_*} & \mathrm{C}_*(X \times Y) \end{array}$$

commutes by the naturality of Υ. In the following we do not explicitly mention the names of the various inclusion induced maps.

Proposition 113.6. Let X and Y be topological spaces and let $A \subset X$ and $B \subset Y$ be subsets.

(1) The Eilenberg-Zilber map Υ induces a natural chain homotopy equivalence[1370]

$$\begin{array}{rcl} \mathrm{C}_*(X, A) \otimes \mathrm{C}_*(Y, B) & \to & \mathrm{C}_*(X \times Y)/(\mathrm{C}_*(A \times Y) + \mathrm{C}_*(X \times B)) \\ \sum_{i=1}^{n} [c_i] \otimes [d_i] & \mapsto & \sum_{i=1}^{n} [\Upsilon(c_i \otimes d_i)]. \end{array}$$

(2) The Eilenberg-Zilber map Υ descends to a well-defined chain map

$$\begin{array}{rcl} \mathrm{C}_*(X, A) \otimes \mathrm{C}_*(Y, B) & \to & \mathrm{C}_*(X \times Y, A \times Y \cup X \times B) \\ \sum_{i=1}^{n} [c_i] \otimes [d_i] & \mapsto & \sum_{i=1}^{n} \big[\Upsilon(c_i \otimes d_i)\big]. \end{array}$$

Proof. Let X and Y be topological spaces and let $A \subset X$ and $B \subset Y$ be subsets.

(1) We consider the following diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{C}_*(A) \otimes \mathrm{C}_*(Y) & \to & \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) & \longrightarrow & \mathrm{C}_*(X, A) \otimes \mathrm{C}_*(Y) & \longrightarrow & 0 \\ & & \downarrow \Upsilon & & \downarrow \Upsilon & & \downarrow \Upsilon & & \\ 0 & \longrightarrow & \mathrm{C}_*(A \times Y) & \longrightarrow & \mathrm{C}_*(X \times Y) & \longrightarrow & \mathrm{C}_*(X \times Y)/\mathrm{C}_*(A \times Y) & \to & 0. \end{array}$$

The top sequence is given by tensoring a short exact sequence by the free abelian groups $\mathrm{C}_*(Y)$, Lemma 89.9 implies that the resulting sequence is again exact. The bottom sequence is exact by definition. The two left vertical maps are homotopy equivalences by Theorem 113.1 and the diagram commutes by the naturality of Υ, as explained in the remark preceding the proposition. Finally note that it follows from

[1370] Note that $\mathrm{C}_*(A \times Y) + \mathrm{C}_*(X \times B)$ is in fact a subcomplex of the chain complex $\mathrm{C}_*(X \times Y)$, so the right-hand side is indeed a chain complex.

the Two-Out-Of-Three-Homotopy Equivalence Corollary 81.6 that the right vertical map is also a chain homotopy equivalence.

We consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{C}_*(X,A)\otimes\mathrm{C}_*(B) & \longrightarrow & \mathrm{C}_*(X,A)\otimes\mathrm{C}_*(Y) & \longrightarrow & \mathrm{C}_*(X,A)\otimes\mathrm{C}_*(Y,B) & \longrightarrow & 0 \\ & & \downarrow\Upsilon & & \downarrow\Upsilon & & \downarrow\Upsilon & & \\ 0 & \longrightarrow & \mathrm{C}_*(X\times B)/\mathrm{C}_*(A\times B) & \longrightarrow & \mathrm{C}_*(X\times Y)/\mathrm{C}_*(A\times Y) & \longrightarrow & \dfrac{\mathrm{C}_*(X\times Y)}{\mathrm{C}_*(A\times Y)+\mathrm{C}_*(X\times B)} & \longrightarrow & 0. \end{array}$$

As above, it follows from the fact that each $\mathrm{C}_k(X,A)$ is a free abelian group and Lemma 89.9 that the top sequence is exact. It follows easily from[1371]

$$\mathrm{C}_*(X\times B)\cap\mathrm{C}_*(A\times Y) \;=\; \mathrm{C}_*((X\times B)\cap(A\times Y)) \;=\; \mathrm{C}_*(A\times B)$$

that the bottom sequence is also exact. The middle vertical map is a chain homotopy equivalence by the above discussion. Exactly the same argument also shows that the left vertical map is a chain homotopy equivalence. The diagram commutes by the naturality of Υ. It follows from the Two-Out-Of-Three-Homotopy Equivalence Corollary 81.6 that the right vertical map is also a homotopy equivalence.

(2) The second statement follows immediately from (1) together with the observation that $\mathrm{C}_*(A\times Y)+\mathrm{C}_*(X\times B)$ is a subcomplex of $\mathrm{C}_*(A\times Y\cup X\times B)$. ■

113.4. **The cross product on homology.** Recall that one of our goals is to find some type of product structure in homology or cohomology. We start our search for a product structure in homology, since we are more comfortable with homology groups.
The following definition already gives us some type of product structure on homology.

Definition. Let (X,A) and (Y,B) be two pairs of topological spaces. We refer to the map[1372]

$$\begin{array}{rcl} \times\colon \mathrm{H}_p(X,A)\otimes\mathrm{H}_q(Y,B) & \xrightarrow{\times} & \mathrm{H}_{p+q}(X\times Y, A\times Y\cup X\times B). \\ [c]\otimes[d] & \mapsto & [c]\times[d] := [\Upsilon(c\otimes d)] \end{array}$$

as the cross product.

In the following lemma we summarize some elementary properties of the cross product.

Lemma 113.7. (Cross Product Lemma)

(1) Let X and Y be two topological spaces and let $x\in X$ be a point. We denote by $f\colon Y\to X\times Y$ the map that is given by $y\mapsto(x,y)$ and we denote by $[x]\in\mathrm{H}_0(X)$ the homology class defined by x. Then for any $q\in\mathbb{N}_0$ the maps

$$\mathrm{H}_q(Y) \underset{f_*}{\overset{\sigma\mapsto[x]\times\sigma}{\rightrightarrows}} \mathrm{H}_q(X\times Y)$$

agree. The statement holds with the roles of X and Y reversed.

(2) The cross product is natural, i.e. if $f\colon(X,A)\to(X',A')$ and $g\colon(Y,B)\to(Y',B')$ are maps between pairs of topological spaces, then for any $p,q\in\mathbb{N}_0$ the following

[1371] Why does this equality of chain complexes hold?

[1372] Note that the map is well-defined by Lemmas 90.6 and 113.6 (2) and it is independent of the choice of Υ by Theorem 113.1 (2).

diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_p(X,A)\otimes\mathrm{H}_q(Y,B) & \xrightarrow{\times} & \mathrm{H}_{p+q}(X\times Y, A\times Y\cup X\times B) \\ \downarrow{\scriptstyle f_*} \quad \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle (f\times g)_*} \\ \mathrm{H}_p(X',A')\otimes\mathrm{H}_q(Y',B') & \xrightarrow{\times} & \mathrm{H}_{p+q}(X'\times Y', A'\times Y'\cup X'\times B'). \end{array}$$

Proof. The lemma follows immediately from the definitions and the properties of Υ given in the EZ-Construction Theorem 113.2 (1) and (2). ■

We have the following variation on the Künneth Theorem 90.8 for topological spaces.

Theorem 113.8. (The Künneth Theorem) Let (X,A) and (Y,B) be pairs of topological spaces. We assume that one of the following two statements holds:

(1) A and B are open subsets (note that A and B can possibly be empty), or
(2) (X,A) and (Y,B) are pairs of CW-complexes.

Then for each $n\in\mathbb{N}$ there exists a natural short exact sequence

$$\begin{aligned} 0 \;\to\; \bigoplus_{p+q=n} \mathrm{H}_p(X,A)\otimes\mathrm{H}_q(Y,B) \;&\xrightarrow{\times}\; \mathrm{H}_n(X\times Y, A\times Y\cup X\times B) \\ &\to \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X,A),\mathrm{H}_q(Y,B)) \;\to\; 0. \end{aligned}$$

Furthermore the above short exact sequence splits.

Examples.

(1) Let M be an m-dimensional topological manifold and let N be an n-dimensional topological manifold. Let $x\in M\setminus\partial M$ and let $y\in N\setminus\partial N$. Then it follows from the relative Künneth Theorem 113.8 (2), Lemma 105.4 and the Tor-Properties Lemma 89.15 that the cross product defines an isomorphism

$$\mathrm{H}_m(M,M\setminus\{x\})\otimes\mathrm{H}_n(N,N\setminus\{y\}) \xrightarrow{\times} \mathrm{H}_{m+n}(M\times N, \underbrace{(M\setminus\{x\})\times N\cup M\times(N\setminus\{y\})}_{=(M\times N)\setminus\{(x,y)\}}).$$

(2) Let X and Y be path-connected topological spaces. Let $x\in X$ and let $y\in Y$. We denote by $i\colon X\to X\times Y$ the inclusion map that is given by $i(x)=(x,y)$ and similarly we denote by $j\colon Y\to X\times Y$ the inclusion map that is given by $j(y)=(x,y)$. It follows from the H_0-Proposition 72.5, the Cross Product Lemma 113.7 (1) and the Künneth Theorem 113.8 that the map

$$i_*\oplus j_*\colon \mathrm{H}_1(X;\mathbb{Z})\oplus\mathrm{H}_1(Y;\mathbb{Z}) \;\to\; \mathrm{H}_1(X\times Y;\mathbb{Z})$$

is an isomorphism.[1373] Let G be any abelian group. Furthermore note that it follows easily from the above, the naturality of the Universal Coefficient Theorem 89.17 and the Tor-Properties Lemma 89.15 (3) that the map

$$i_*\oplus j_*\colon \mathrm{H}_1(X;G)\oplus\mathrm{H}_1(Y;G) \;\to\; \mathrm{H}_1(X\times Y;G)$$

is also an isomorphism.

[1373] We had given an alternative proof of this fact in Exercise 84.5.

Proof. We consider the following maps

$$0 \to \bigoplus_{p+q=n} \mathrm{H}_p(X,A)\otimes \mathrm{H}_q(Y,B) \to \mathrm{H}_n(\mathrm{C}_*(X,A)\otimes \mathrm{C}_*(Y,B)) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X,A),\mathrm{H}_q(Y,B)) \to 0.$$
$$\downarrow \Upsilon$$
$$\mathrm{H}_n(\mathrm{C}_*(X\times Y)/(\mathrm{C}_*(A\times Y)+\mathrm{C}_*(X\times B)))$$
$$\downarrow$$
$$\mathrm{H}_n(\mathrm{C}_*(X\times Y)/\mathrm{C}_*(A\times Y\cup X\times B))$$
$$\downarrow =$$
$$\mathrm{H}_n(X\times Y, A\times Y\cup X\times B)$$

We first note that the top sequence is exact by the Künneth Theorem for Chain Complexes 90.7. (Here we use again that relative chain groups are free abelian groups.) Here the top vertical map is an isomorphism by Proposition 113.6. Recall that we assume that one of the following two statements holds:

(1) A and B are open subsets, or
(2) (X,A) and (Y,B) are pairs of CW-complexes.

We then know from the Excisive Triad-Proposition 109.12 (4) respectively (6) that the inclusion induced map

$$\underbrace{\mathrm{C}_*^{\{A\times Y, X\times B\}}(A\times Y\cup X\times B)}_{=\mathrm{C}_*(A\times Y)+\mathrm{C}_*(X\times B)} \to \mathrm{C}_*(A\times Y\cup X\times B)$$

is a chain homotopy equivalence. The same argument as in the proof of Corollary 74.16 shows that this piece of information implies that the second vertical map is indeed an isomorphism. It follows immediately from the definition of the cross product that the induced "diagonal map"

$$\bigoplus_{p+q=n} \mathrm{H}_p(X,A)\otimes \mathrm{H}_q(Y,B) \to \mathrm{H}_n(X\times Y, A\times Y\cup X\times B)$$

is precisely the cross product. ∎

113.5. **Product orientations of topological manifolds.** In this second to last section we want to consider products of topological manifolds. First let us recall the following fact.

Proposition 113.9. Let M be an m-dimensional topological manifold and let N be an n-dimensional topological manifold. Then the following hold:

(1) The product $M\times N$ is an $(m+n)$-dimensional topological manifold.
(2) We have $\partial(M\times N) = (\partial M\times N)\cup(M\times \partial N)$.

Proof. This proposition follows from the combination of the Topological Manifold Product Proposition 18.4 and the Topological Manifold Boundary Proposition 75.2. ∎

The cross product, that we just introduced, allows us to formulate the following proposition.

Proposition 113.10. (Topological Product Orientation Proposition) Let M be an m-dimensional topological manifold and let N be an n-dimensional topological manifold.

Suppose M is equipped with an orientation $\{\mu_x\}_{x\in M\setminus\partial M}$ and that N is equipped with an orientation $\{\nu_y\}_{y\in N\setminus\partial N}$.

(1) Given $(x,y)\in(M\times N)\setminus\partial(M\times N)$ we denote by $\rho_{(x,y)}$ the image of $\mu_x\otimes\nu_y$ under the isomorphism

$$\mathrm{H}_m(M,M\setminus\{x\})\otimes\mathrm{H}_n(N,N\setminus\{y\}) \xrightarrow{\times} \mathrm{H}_{m+n}(M\times N,(M\times N)\setminus\{(x,y)\})$$

from page 2499. Then these classes $\rho_{(x,y)}$ define an orientation on $M\times N$.

(2) Suppose both M and N are compact and connected, then the map

$$\mathrm{H}_m(M,\partial M)\otimes\mathrm{H}_n(N,\partial N) \xrightarrow{\times} \mathrm{H}_{m+n}(M\times N,\partial(M\times N))$$

given by the cross product is an isomorphism and it has the property that

$$[M]\times[N] \;=\; [M\times N] \;\in\; \mathrm{H}_{m+n}(M\times M,\partial(M\times N))$$

where $[M\times N]$ denotes the fundamental class of the compact topological manifold $M\times N$ equipped with the orientation defined in (1).

Proof. Let M be an m-dimensional topological manifold and let N be an n-dimensional topological manifold.

(1) The first statement of the proposition is a straightforward consequence of the definitions and the naturality of the cross product, see the Cross Product Lemma 113.7. For completeness' sake we sketch the proof. To simplify the notation we assume that $\partial M=\partial N=\varnothing$. Now let $(x,y)\in M\times N$. We pick an open neighborhood U of x and a class $\mu_U\in\mathrm{H}_m(M,M\setminus U)$ and an open neighborhood V of y and a class $\nu_V\in\mathrm{H}_n(N,N\setminus V)$ as given by the definition on page 2331. By the naturality of the cross product we obtain the following commutative diagram

$$\begin{array}{ccc}
\mathrm{H}_m(M,M\setminus U)\times\mathrm{H}_n(N,N\setminus V) & \xrightarrow{\times} & \mathrm{H}_{m+n}(M\times N,M\times N\setminus(U\times V)) \\
{\scriptstyle \mu_U\mapsto\mu_x,\ \nu_V\mapsto\nu_y}\downarrow & & \downarrow \\
\mathrm{H}_m(M,M\setminus\{x\})\times\mathrm{H}_n(N,N\setminus\{y\}) & \xrightarrow[(\mu_x,\nu_y)\mapsto\rho_{(x,y)}]{\times} & \mathrm{H}_{m+n}(M\times N,M\times N\setminus\{(x,y)\}).
\end{array}$$

It is now straightforward to verify that $\phi_{U\times V}:=\mu_U\times\nu_V$ has the properties desired by the definition on page 2331.

(2) Now suppose that both M and N are compact. By the Product Topology Properties Proposition 5.6, the product $M\times N$ is then also compact. Now let $x\in M\setminus\partial M$ and let $y\in N\setminus\partial N$. By the Smooth Manifold Product Proposition 20.6 we have $(x,y)\in(M\times N)\setminus\partial(M\times N)$. Furthermore, by the Cross Product Lemma 113.7 we have the following commutative diagram

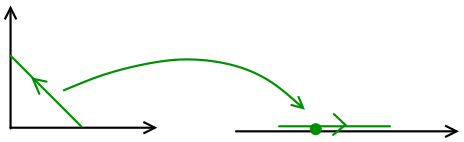
$$\begin{array}{ccc}
\mathrm{H}_m(M,\partial M)\otimes\mathrm{H}_n(N,\partial N) & \xrightarrow{\times} & \mathrm{H}_{m+n}(M\times N,\partial(M\times N)) \\
\downarrow & & \downarrow \\
\mathrm{H}_m(M,M\setminus\{x\})\otimes\mathrm{H}_n(N,N\setminus\{y\}) & \xrightarrow{\times} & \mathrm{H}_{m+n}(M\times N,M\times N\setminus\{(x,y)\}).
\end{array}$$

The vertical maps are isomorphisms by the Orientation-Top Homology Theorem 106.1. On page 2499 we saw that the bottom horizontal map is an isomorphism. It follows

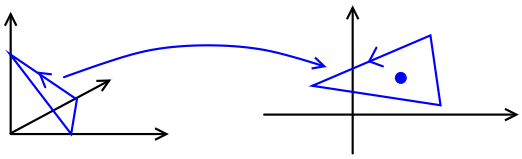
that, as claimed, the top horizontal map is also an isomorphism. The statement regarding the fundamental classes is an immediate consequence of the commutativity of the diagram and the definitions. ■

Definition. Let M and N be oriented topological manifolds. We refer to the orientation on $M \times N$ defined in the Topological Product Orientation Proposition 113.10 (1) as the product orientation.

Remark. Recall that given any open subset $U \subset \mathbb{R}^k$, $k \in \mathbb{N}_0$, we introduced on page 2337 the standard orientation for U viewed as a topological manifold. (See also the figure below for a reminder.) The following lemma says that the above definition of the product orientation gives us the "expected" answer when we deal with open subsets of $\mathbb{R}^k$'s.



standard generator for $\mathbb{R}$



standard generator for $\mathbb{R}^2$

Proposition 113.11. Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open non-empty subsets. We equip U and V with the standard orientation. Then the product orientation of $U \times V$, defined above, agrees with the standard orientation of $U \times V \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$.

We postpone the proof of Proposition 113.11 to the next section.

Examples.

(1) Let M and N be two oriented smooth manifolds. We have the following diagram

$$\begin{array}{ccc} \text{oriented smooth manifolds } M, N & \overset{\substack{\text{Smooth-Topological}\\ \text{Orientation}\\ \text{Proposition 105.11}}}{\rightsquigarrow} & \text{oriented topological manifolds } M, N \\ \Big\downarrow \text{page 695} & & \Big\downarrow \text{see above} \\ \text{oriented smooth manifold } M \times N & \overset{\textit{Proposition } 105.11}{\rightsquigarrow} & \text{oriented topological manifold } M \times N. \end{array}$$

It follows fairly easily from Proposition 113.11 that this diagram commutes.

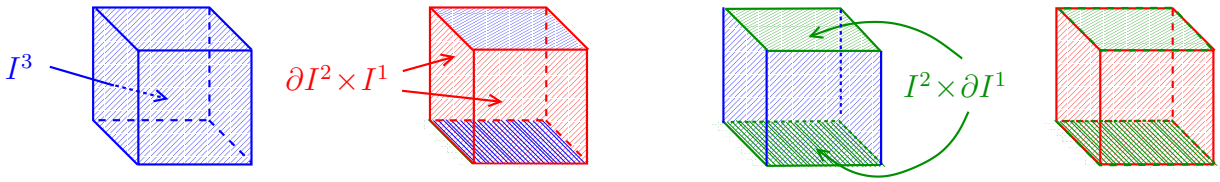
(2) We write $I = [-1, 1]$. We view I as a 1-dimensional topological manifold and we equip I with the orientation coming from $\mathbb{R}$. Given $k \in \mathbb{N}_0$ we write

$$\partial I^k \ := \ \{(x_1, \ldots, x_k) \,|\, \text{there exists an } i \text{ with } x_i = -1 \text{ or } x_i = 1\}.$$

Given any $m, n \in \mathbb{N}$ one sees easily that

$$\partial(I^m \times I^n) \ = \ (\partial I^m \times I^n) \cup (I^m \times \partial I^n) \ = \ \partial(I^{m+n}).$$

It follows from the Topological Manifold Product Proposition 18.4 and the above



discussion that we can view each $I^k = I \times \cdots \times I$ as a k-dimensional topological manifold with boundary given by the above subset ∂I^k. We equip I^k with the product orientation. It follows immediately from Proposition 113.11 that the orientation on the open subset $I^k \setminus \partial I^k \subset \mathbb{R}^k$ is the standard orientation.[1374] As usual we denote by $[I^k] \in \mathrm{H}_k(I^k, \partial I^k; \mathbb{Z})$ the fundamental class.

Now let $m, n \in \mathbb{N}_0$. We get a cross product

$$\overbrace{\mathrm{H}_m(I^m, \partial I^m)}^{=\mathbb{Z}\cdot[I^m]} \otimes \overbrace{\mathrm{H}_n(I^n, \partial I^n)}^{=\mathbb{Z}\cdot[I^n]} \quad \to \quad \overbrace{\mathrm{H}_{m+n}(I^{m+n}, \partial I^{m+n})}^{=\mathbb{Z}\cdot[I^{n+n}]}$$

which by the Topological Product Orientation Proposition 113.10 (2) has the property that

$$[I^m] \times [I^n] \quad = \quad [I^{m+n}].$$

113.6. **The shuffle product and the proof of Proposition 113.11** $(*)$**.** In this section we will provide the proof of Proposition 113.11. The difficulty in the proof is that the definition of the cross product relies on the map Υ from the EZ-Construction Theorem 113.2. We gave a non-constructive proof of the existence of the map Υ. In this section we give an *explicit* description of a map ∇ which has all the properties that we had asked for in the EZ-Construction Theorem 113.2. We will then use this map ∇ to prove Proposition 113.11. First we recall some notation from page 1584.

Notation. For $i = 0, \ldots, n$ we write

$$v_i \ := \ (0, \ldots, 1, 0, \ldots, 0) \ \in \ \mathbb{R}^{n+1}.$$

Furthermore for $a_0, \ldots, a_k \in \Delta^n$ we consider the map

$$\begin{array}{rcl} [a_0, \ldots, a_k] \colon \Delta^k & \to & \Delta^n \\ \sum_{j=0}^{k} \lambda_j \cdot v_j & \mapsto & \sum_{j=0}^{k} \lambda_j \cdot a_j. \end{array}$$

The map $[a_0, \ldots, a_k]$ thus sends the standard k-simplex Δ^k to the simplex "spanned by $a_0, \ldots, a_k$" in Δ^n.

Definition. Let $m, n \in \mathbb{N}_0$.

(1) An (m,n)-shuffle is a pair (μ, ν) of disjoint sets of integers

$$1 \leq \mu_1 < \mu_2 < \cdots < \mu_m \leq m+n \quad \text{and} \quad 1 \leq \nu_1 < \nu_2 < \cdots < \nu_n \leq m+n.$$

We denote by $\mathrm{sign}(\mu, \nu)$ the sign of the permutation $(\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n)$. We denote by $\eta^\mu \colon \Delta^{m+n} \to \Delta^m$ the unique affine linear map which satisfies $\eta^\mu(v_i) = v_j$ for $\mu_j \leq i < \mu_{j+1}$.[1375] The same way we define $\eta^\nu \colon \Delta^{m+n} \to \Delta^n$.

[1374] Note that the homeomorphism $f \colon (I^k, \partial I^k) \to (\overline{B}^k, S^{k-1})$ from the Convex-to-Ball Proposition 2.20 (2) has the property that the restriction to a dense open subset of $I^k \setminus \partial I^k \to B^k$ is an orientation-preserving diffeomorphism. In particular this homeomorphism is thus an orientation-preserving homeomorphism from the oriented topological manifold I^k to the oriented topological manifold $\overline{B}^k$.

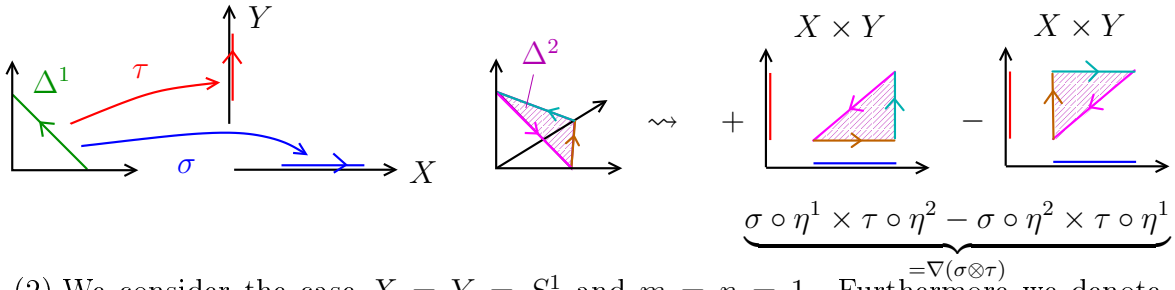
(2) Given two topological spaces X and Y we define

$$\begin{aligned} \nabla_{mn}\colon \mathrm{C}_m(X)\otimes \mathrm{C}_n(Y) &\to \mathrm{C}_{m+n}(X\times Y) \\ (\sigma\otimes\tau) &\mapsto \sum_{(\mu,\nu)} \operatorname{sign}(\mu,\nu)\cdot \underbrace{(\overbrace{\sigma\circ\eta^{\mu}}^{\Delta^{m+n}\to X}\times \overbrace{\tau\circ\eta^{\nu}}^{\Delta^{m+n}\to Y})}_{\text{map }\Delta^{m+n}\to X\times Y} \end{aligned}$$

where we take the sum over all (m,n)-shuffles.

Examples.

(1) We consider the case $X=Y=\mathbb{R}$ and $m=n=1$. There exist two $(1,1)$-shuffles, namely $\{\mu_1=1,\nu_1=2\}$ and $\{\mu_1=2,\nu_1=1\}$. Note that $\eta^1\colon\Delta^2\to\Delta^1$ equals the map $[v_0,v_1,v_1]$ and that $\eta^2\colon\Delta^2\to\Delta^1$ equals the map $[v_0,v_0,v_1]$. In the figure below we illustrate $\nabla_{11}(\sigma\otimes\tau)$ for two affine linear singular 1-simplices $\sigma\colon\Delta^1\to X=\mathbb{R}$ and $\tau\colon\Delta^1\to Y=\mathbb{R}$.



(2) We consider the case $X=Y=S^1$ and $m=n=1$. Furthermore we denote by $\mu\colon\Delta^1\to S^1$ the singular 1-simplex that is given by $\mu(1-t,t)=\exp(2\pi\,\mathrm{i}t)$. On page 2201 we saw that μ represents the fundamental class of S^1 with the standard orientation. Comparing the figures on page 2204 and on page 2504 we see that the shuffle $\nabla(\mu\otimes\mu)$ is precisely the usual representative for the fundamental class of the 2-dimensional torus $S^1\times S^1=([0,1]\times[0,1])/\!\sim$.

Proposition 113.12. The maps ∇ that we had just defined have all the properties that we had mentioned in the EZ-Construction Theorem 113.2, in other words, the maps $\nabla\colon \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)\to\mathrm{C}_*(X\times Y)$ are natural chain maps that on the 0-level equal the canonical maps that we introduced on page 2491.

Proof. It follows immediately from the definition that the maps ∇_{mn} are natural and that on the 0-level they agree with the canonical maps. It is elementary (albeit rather painful) to verify that these maps are chain maps. We refer to [**Dol1956**, p. 184] and [**EMc1953**] for details and proofs. ■

Now we can finally provide the proof of Proposition 113.11.

Proof of Proposition 113.11. It suffices to prove the statement for $U=\mathbb{R}^m$ and for $V=\mathbb{R}^n$. Given any $k\in\mathbb{N}_0$ we consider the singular chain $\sigma_k=(-1)^k\cdot[v_0,\ldots,v_k,0]$.

[1375] Here we implicitly use that $\mu_0=0$ and $\mu_{m+1}=m+n+1$.

(In plain English the singular simplex $[v_0, \ldots, v_k, 0]\colon \Delta^k \to \mathbb{R}^k$ is just given by the obvious projection.) We make the following observation:

(∗) Let $z \in \mathbb{R}^k$ be any point that lies in the interior of the image of σ_k. It follows from the discussion on page 2325 that the chain σ_k represents the standard generator of $\mathrm{H}_k(\mathbb{R}^k, \mathbb{R}^k \setminus \{z\})$.

Next we consider the (m,n)-shuffle that is given by the sets $\mu_0 = \{1, \ldots, m\}$ and by $\nu_0 = \{m+1, \ldots, m+n\}$. It follows easily from the definitions that

$$\eta^{\mu_0} = [v_0, \ldots, v_m, \ldots, v_m]\colon \Delta^{m+n} \to \Delta^m \quad \text{and} \quad \eta^{\nu_0} = [v_m, \ldots, v_m, \ldots, v_{m+n}]\colon \Delta^{m+n} \to \Delta^n.$$

Also note that it is a straightforward consequence of the definitions that we have the following equality of singular $(m+n)$-chains in $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$:

$$(\sigma_m \circ \eta^{\mu_0}) \times (\sigma_n \circ \eta^{\nu_0}) \;=\; (-1)^m \cdot (-1)^n \cdot [v_0, \ldots, v_{m-1}, v_m, \ldots, v_{m+n-1}, 0] \;=\; \sigma_{m+n}.$$

We pick a point $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$ that lies in the interior of the image of σ_{m+n}. Note that x lies in the interior of the image of σ_m and that y lies in the interior of the image of σ_n.

It follows from the observation (∗), the definition of the product orientation on page 2502 and the Sections Proposition 105.6 (1) that it suffices to prove the following claim.

Claim. The following equality holds in $\mathrm{H}_{m+n}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} \setminus \{(x,y)\})$:

$$[\sigma_m] \times [\sigma_n] \;=\; [\sigma_{m+n}].$$

Proof. We have

$$\begin{aligned}
[\sigma_m] \times [\sigma_n] \;&\underset{\text{Proposition 113.12 and Theorem 113.5}}{\uparrow=}\; [\nabla_{mn}(\sigma_m \otimes \sigma_n)] \;\underset{\text{by definition}}{\uparrow=}\; \Big[\sum_{(\mu,\nu)} \operatorname{sign}(\mu,\nu) \cdot (\sigma_m \circ \eta^\mu \times \sigma_n \circ \eta^\nu)\Big] \\
&\underset{\text{for all other shuffles } (\mu,\nu) \text{ the point } (x,y) \text{ does not lie in the image of } \sigma_m \circ \eta^\mu \times \sigma_n \circ \eta^\nu}{\uparrow=}\; \Big[\underbrace{\operatorname{sign}(\mu_0,\nu_0)}_{=1} \cdot \underbrace{(\sigma_m \circ \eta^{\mu_0} \times \sigma_n \circ \eta^{\nu_0})}_{=\sigma_{m+n}}\Big] \;=\; [\sigma_{m+n}].
\end{aligned}$$

This concludes the proof of the claim and thus of the proposition. ∎

113.7. **The Acyclic Model Theorems.** In this section we will prove two "Acyclic Model Theorems" which on many occasions will allow us to construct maps and homotopies. We start out with the following fairly friendly Acyclic Model Theorem. We will need this particular Acyclic Model Theorem later on in Chapter 151 when we construct Steenrod operations.

Theorem 113.13. (Acyclic Model Theorem) Let $\mathcal{T}op$ be the category of topological spaces and let $\mathcal{C}$ be the category of generalized[1376] chain complexes. Let $G\colon \mathcal{T}op \to \mathcal{C}$ be a covariant functor. Furthermore let S be a natural transformation from the functor $X \mapsto \mathrm{C}_*(X)$ to the functor $X \mapsto G(X)$. If $\mathrm{H}_i(G(\Delta^k)) = 0$ for any $k \in \mathbb{N}_0$ and any $i \in \mathbb{N}_0$, then there exists a natural chain homotopy P from S to the zero map. In other words, for any topological space X and any $k \in \mathbb{N}$ there exists a natural map $P_X\colon \mathrm{C}_k(X) \to G_{k+1}(X)$, such that for any $k \in \mathbb{N}_0$ we have

$$\partial \circ P_X + P_X \circ \partial \;=\; S_X\colon \mathrm{C}_k(X) \;\to\; G_k(X).$$

[1376] Here "generalized" just means, see page 1592, that we also allow chain groups in negative degrees.

Sketch of proof. The proof of this theorem is very similar to the "naturally chain homotopic statement" of Theorem 113.5. First recall that the fact that S is a natural transformation means that for any topological space X we get a commutative diagram as follows:

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & \mathrm{C}_2(X) & \xrightarrow{\partial_2} & \mathrm{C}_1(X) & \xrightarrow{\partial_1} & \mathrm{C}_0(X) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S} & & \\
\ldots & \longrightarrow & G_2(X) & \xrightarrow{\partial_2} & G_1(X) & \xrightarrow{\partial_1} & G_0(X) & \xrightarrow{\partial_0} & G_{-1}(X) & \xrightarrow{\partial_{-1}} & G_{-2}(X) & \xrightarrow{\partial_{-2}} & \ldots
\end{array}$$

Now we consider $\mathrm{id}_{\Delta^0} \in \mathrm{C}_0(\Delta^0)$. It is a cycle in the chain complex $\mathrm{C}_*(\Delta^0)$. Since the above diagram commutes we see that $S(\mathrm{id}_{\Delta^0})$ is also cycle in $G_0(\Delta^0)$. Since we assume that $\mathrm{H}_0(G(\Delta^0)) = 0$ we see that there exists some $\mu_1 \in G_1(\Delta^0)$ with $\partial_1(\mu_1) = S(\mathrm{id}_{\Delta^0})$. Now, given any topological space X we define

$$\begin{array}{rcl}
P_0 \colon \mathrm{C}_0(X) & \to & G_1(X) \\
\sum_{i=1}^m a_i \cdot \underset{\uparrow}{f_i} & \mapsto & \sum_{i=1}^m a_i \cdot f_{i*}(\mu_1). \\
\text{map } \Delta^0 \to X & &
\end{array}$$

We iterate this procedure. Namely suppose that for $i = 0, \ldots, k-1$ we have already constructed natural maps $P_i \colon \mathrm{C}_i(X) \to G_{i+1}(X)$ such that $\partial \circ P_i + P_{i-1} \circ \partial = S_i$ for $i = 0, \ldots, k-1$. For reference we consider the following diagram:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & \mathrm{C}_{k+1}(X) & \xrightarrow{\partial_{k+1}} & \mathrm{C}_k(X) & \xrightarrow{\partial_k} & \mathrm{C}_{k-1}(X) & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle S_{k+1}} & \swarrow{\scriptstyle P_k} & \downarrow{\scriptstyle S_k} & \swarrow{\scriptstyle P_k - 1} & \downarrow{\scriptstyle S_{k-1}} & & \\
\ldots & \longrightarrow & G_{k+1}(X) & \xrightarrow[\partial_{k+1}]{} & G_k(X) & \xrightarrow[\partial_k]{} & G_{k-1}(X) & \xrightarrow{\partial_{k-1}} & \ldots
\end{array}$$

Next we need to construct a suitable map P_k. We calculate that

$$\begin{array}{rcl}
\partial_k((S_k - P_{k-1} \circ \partial)(\mathrm{id}_{\Delta^k})) & = & \partial S_k(\mathrm{id}_{\Delta^k}) - \partial P_{k-1}(\partial\, \mathrm{id}_{\Delta^k}) \\
 & \underset{\uparrow}{=} & S_{k-1}(\partial\, \mathrm{id}_{\Delta^k}) - (S_{k-1} - P_{k-2}\partial)(\partial\, \mathrm{id}_{\Delta^k}) \;=\; (P_{k-2} \circ \overbrace{\partial \circ \partial}^{=0})(\mathrm{id}_{\Delta^k}) \;= 0\,. \\
 & \text{since } S \text{ is a chain map and since } \partial \circ P_{k-1} + P_{k-2} \circ \partial = S_{k-1} &
\end{array}$$

Since we assume that $\mathrm{H}_k(G(\Delta^k)) = 0$ we see that there exists some $\mu_{k+1} \in G_{k+1}(\Delta^k)$ with $\partial_{k+1}(\mu_{k+1}) = (S_k - P_{k-1} \circ \partial)(\mathrm{id}_{\Delta^k})$. Now, given any topological space X we define

$$\begin{array}{rcl}
P_k \colon \mathrm{C}_k(X) & \to & G_{k+1}(X) \\
\sum_{i=1}^m a_i \cdot \underset{\uparrow}{f_i} & \mapsto & \sum_{i=1}^m a_i \cdot f_{i*}(\mu_{k+1}). \\
\text{map } \Delta^k \to X & &
\end{array}$$

Using the fact that S is natural one can easily verify that P_k has the desired properties. ∎

The above version of the Acyclic Model Theorem will not work for the situations that we need to deal with in the near future. There are two problems with the above Acyclic Model Theorem:

(1) On several occasions we will be interested in natural transformations of the form $\mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$, but for $X = \Delta^k$ the chain complexes to the right does

not satisfy the condition that the homology groups for $X = \Delta^k$ vanish in dimensions ≥ 0.

(2) We will also need to deal with situations where to the left we have more complicated chain complexes, e.g. we might want to study natural transformations of the type $\mathrm{C}_*(X) \times \mathrm{C}_*(X) \to \mathrm{C}_*(X \times X)$.

To formulate a version of the Acyclic Model Theorem that deals with the above two issues we need a few more definitions.

Definition.

(1) An augmented chain complex is a chain complex

$$(\mathrm{C}_*, \partial_*) = \ldots \xrightarrow{\partial_3} \mathrm{C}_2 \xrightarrow{\partial_2} \mathrm{C}_1 \xrightarrow{\partial_1} \mathrm{C}_0 \to 0$$

together with an epimorphism $\epsilon \colon \mathrm{C}_0 \to \mathbb{Z}$ such that $\epsilon \circ \partial_1 \colon \mathrm{C}_1 \to \mathbb{Z}$ is the zero map.

(2) We say that an augmented chain complex $((\mathrm{C}_*, \partial_*), \epsilon \colon \mathrm{C}_0 \to \mathbb{Z})$ is acyclic if the generalized chain complex

$$\ldots \xrightarrow{\partial_3} \mathrm{C}_2 \xrightarrow{\partial_2} \mathrm{C}_1 \xrightarrow{\partial_1} \mathrm{C}_0 \xrightarrow{\epsilon} \underset{\substack{\uparrow \\ \text{degree } -1}}{\mathbb{Z}} \to 0$$

is acyclic, i.e. if all homology groups are zero.

(3) Let $((\mathrm{C}_*, \partial_*), \epsilon)$ and $((\mathrm{C}'_*, \partial'_*), \epsilon')$ be two augmented chain complexes. A chain map of augmented chain complexes is a chain map $f_* \colon C_* \to \mathrm{C}'_*$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{C}_0 & \xrightarrow{\epsilon} & \mathbb{Z} \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle =} \\ \mathrm{C}'_0 & \xrightarrow{\epsilon'} & \mathbb{Z}. \end{array}$$

(4) We refer to the category $\mathcal{A}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{A}) &= \text{augmented chain complexes} \\ \mathrm{Mor}((C, \epsilon), (C', \epsilon')) &= \text{set of chain maps of augmented chain complexes from } (C, \epsilon) \text{ to } (C', \epsilon') \end{aligned}$$

as the category of augmented chain complexes.

Examples.

(1) Let X be a non-empty topological space. We denote by $\epsilon_X \colon C_0(X) \to \mathbb{Z}$ the augmentation map that we introduced on page 1590, i.e. ϵ_X is the map that sends each singular 0-simplex to 1. It follows from Lemma 72.4 (1) that $(C_*(X), \epsilon_X)$ is an augmented chain complex. If X is a contractible topological space, e.g. if $X = \Delta^k$, then it follows from the Reduced Homology Lemma 74.1 (7) that the corresponding augmented chain complex is acyclic.

(2) If (C_*, μ) and (D_*, ν) are two augmented chain complexes then the tensor product $C_* \otimes D_*$ of chain complexes, as defined in Lemma 90.2, together with the map

$$\begin{aligned} \mu \otimes \nu \colon (C_* \otimes D_*)_0 = C_0 \otimes D_0 &\to \mathbb{Z} \\ \sum_{i=1}^k c_i \otimes d_i &\mapsto \sum_{i=1}^k \mu(c_i) \cdot \nu(d_i) \end{aligned}$$

is easily seen to be an augmented chain complex.[1377]

Definition. We say a chain complex C_* is almost acyclic if $\mathrm{H}_i(C_*) = 0$ for $i \geq 1$ and if $\mathrm{H}_0(C_*) \cong \mathbb{Z}$.

Example. If X is a contractible topological space, then it follows from the Homotopy Equivalence-H_*-Corollary 73.9 that $\mathrm{C}_*(X)$ is almost acyclic.

Proposition 113.14. Let C_* and D_* be two chain complexes. If C_* and D_* are almost acyclic and if C_* is free, i.e. if each chain group C_n is a free abelian group, then $C_* \otimes D_*$ is also almost acyclic.

Proof. Let $n \in \mathbb{N}_0$. Since C_* is free we can apply the Künneth Theorem for Chain Complexes 90.7 which gives us the following short exact sequence:

$$0 \to \bigoplus_{p+q=n} \mathrm{H}_p(C_*) \otimes \mathrm{H}_q(D_*) \xrightarrow{\Theta} \mathrm{H}_n(C_* \otimes D_*) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(C_*), \mathrm{H}_q(D_*)) \to 0.$$

The proposition is an immediate consequence of this short exact sequence and of some very elementary statements regarding tensor products and torsion groups, see the Tensor Lemma 89.3 and 89.15. ■

The following lemma gives a convenient criterion for showing that augmented chain complexes are acyclic.

Lemma 113.15. Let $((\mathrm{C}_*, \partial_*), \epsilon)$ be an augmented chain complex. If the chain complex $(\mathrm{C}_*, \partial_*)$ is almost acyclic, then the augmented chain complex $((\mathrm{C}_*, \partial_*), \epsilon)$ is acyclic.

Proof. We denote by $\widetilde{C}_*$ the generalized chain complex

$$\ldots \xrightarrow{\partial_3} \mathrm{C}_2 \xrightarrow{\partial_2} \mathrm{C}_1 \xrightarrow{\partial_1} \mathrm{C}_0 \xrightarrow{\epsilon} \!\!\!\rightarrow \mathbb{Z} \longrightarrow 0.$$

We need to show that this chain complex is acyclic. It is clear that for $i \geq 1$ we have $\mathrm{H}_i(\widetilde{C}_*) = \mathrm{H}_i(C_*) = 0$. It remains to show that $\mathrm{H}_0(\widetilde{C}_*) = 0$. Since $\mathbb{Z}$ is a free abelian group and since $\epsilon\colon \mathrm{C}_0 \to \mathbb{Z}$ is an epimorphism we can pick a map $s\colon \mathbb{Z} \to \mathrm{C}_0$ with $\epsilon \circ s = \mathrm{id}_{\mathbb{Z}}$. As in the Reduced Homology Lemma 74.1 (4) one can easily show that the map

$$\begin{array}{rcl} \mathrm{H}_0(\widetilde{C}_*) \oplus \mathbb{Z} & \to & \mathrm{H}_0(C_*) \\ [\sigma] \oplus n & \mapsto & [\sigma] + s(n) \end{array}$$

is an isomorphism. Since $\mathrm{H}_0(C_*) \cong \mathbb{Z}$ it follows from Exercise 51.3 (b) (see also more generally Proposition 51.11) that $\mathrm{H}_0(\widetilde{C}_*) = 0$. ■

Definition. Let $\mathcal{K}$ be a category (in all applications $\mathcal{K}$ will be the category of non-empty topological spaces or the category of tuples of non-empty topological spaces) and let

[1377] Indeed, by definition $(C_* \otimes D_*)_1$ is generated by elements $c_1 \otimes d_0$ with $c_1 \in C_1$ and $d_0 \in D_0$ and by elements $c_0 \otimes d_1$ with $c_0 \in C_0$ and $d_1 \in D_1$. We calculate that

$$((\mu \otimes \nu) \circ \partial_1)(c_1 \otimes d_0) = (\mu \otimes \nu)(\partial c_1 \otimes d_0 + (-1)^1 \cdot c_1 \otimes \underbrace{\partial d_0}_{=0}) = (\mu \otimes \nu)(\partial c_1 \otimes d_0) = \mu(\partial c_1) \cdot \nu(d_0) = 0 \cdot \nu(d_0) = 0.$$

The same argument shows that $((\mu \otimes \nu) \circ \partial_1)(c_0 \otimes d_1) = 0$.

$F\colon \mathcal{K} \to \mathcal{A}$ be a covariant functor from $\mathcal{K}$ to the category $\mathcal{A}$ of augmented chain complexes. Let $\mathcal{M}$ be a collection of objects of $\mathcal{K}$.

(1) We say the functor F is acyclic relative to $\mathcal{M}$ if given any $X \in \mathcal{M}$ the augmented chain complex $F(X)$ is acyclic.
(2) We say the functor F is free relative to $\mathcal{M}$ if for each $k \in \mathbb{N}_0$ there exists $\mathcal{M}_k \subset \mathcal{M}$ and if for each $M \in \mathcal{M}_k$ there exists an element $\sigma_M \in F_k(M)$[1378] such that given any $X \in \mathcal{K}$ the following two statements hold:
   (a) The set
$$\{\underbrace{F(f)}_{\uparrow}(\sigma_M) \in F_k(X) \mid \ M \in \mathcal{M}_k \text{ and } f \in \operatorname{Mor}(M,X)\}$$
morphism $f\colon M \to X$ induces a homomorphism $F(f)\colon F_k(M) \to F_k(X)$
   is a basis for the abelian group $F_k(X)$.[1379]
   (b) The description of the set in (a) has no double elements, much more precisely, for $(M,f) \neq (M',f')$ we have $F(f)(\sigma_M) \neq F(f')(\sigma_{M'})$.

Remark. The objects in a collection $\mathcal{M}$ as in the previous definition are often called *models*.

The definition of "free relative to $\mathcal{M}$" is arguably somewhat intimidating. The following example should convince the reader that in practice it is not that bad.

Example. Let $\mathcal{K}$ be the category of non-empty topological spaces and let F be the covariant functor that assigns to each $X \in \mathcal{K}$ the augmented singular chain complex $(C_*(X), \epsilon_X)$. We consider
$$\mathcal{M} \ := \ \{\Delta^k \mid k \in \mathbb{N}_0\} \ = \ \text{the set of all standard simplices.}$$
We claim that F is acyclic and free relative to $\mathcal{M}$:

(1) By the Reduced Homology Lemma 74.1 (7) we know that for every $k \in \mathbb{N}_0$ we have $\widetilde{\mathrm{H}}_*(\Delta^k) = 0$. But by Lemma 113.15 this just means that $\mathcal{M}$ is acyclic relative to F.
(2) Let $k \in \mathbb{N}_0$. We set $\mathcal{M}_k = \{\Delta^k\}$ and we consider $\sigma_k = \mathrm{id}_{\Delta^k} \in \mathrm{C}_k(\Delta^k)$. Now recall that given any topological space X and given any $k \in \mathbb{N}_0$ we have
$$\begin{aligned} \mathrm{C}_k(X) &\overset{\text{by definition of } \mathrm{C}_k(X), \text{ see page 1583}}{=} \text{free abelian group generated by the set of singular } k\text{-simplices in } X \\ &= \text{free abelian group generated by the set of all maps } \underbrace{f\colon \Delta^k \to X}_{=f_*(\mathrm{id}_{\Delta^k})}. \end{aligned}$$
by definition of a singular k-simplex, see page 1583
   But this means that the set $\{f_*(\mathrm{id}_{\Delta^k})\}_{f\colon \Delta^k \to X}$ is a basis for $\mathrm{C}_k(X)$.

Now we can formulate the main theorem of this section.

Theorem 113.16. (Acyclic Model Theorem) Let $\mathcal{K}$ be a category with a collection of objects $\mathcal{M}$. Furthermore let $F, G\colon \mathcal{K} \to \mathcal{A}$ be two covariant functors from $\mathcal{K}$ to the

[1378] Here $F_k(M)$ is the k-chain group of the chain complex $F(M)$.
[1379] In particular this implies that each $F_k(X)$ is a free abelian group.

category of augmented chain complexes. If F is free relative to $\mathcal{M}$ and if G is acyclic relative to $\mathcal{M}$, then the following statements hold:

(1) There exists a natural transformation T from F to G. In other words, given any object X of $\mathcal{K}$ we have a chain map $T_X\colon F(X) \to G(X)$ of augmented chain complexes such that for any morphism $f\colon X \to Y$ in $\mathcal{K}$ the following diagram commutes:[1380]

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle T_X}\downarrow & & \downarrow{\scriptstyle T_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y). \end{array}$$

(2) Given two natural transformations S_X and T_X from F to G there is a natural chain homotopy P_X between them.

Proof. The proof is, ultimately very much the same as the proof of Theorem 113.5. We decline to provide a proof and we follow the lead given by Munkres [**Mun1984**, p. 185], namely like Munkres we assign the proof as an exercise to the reader. Alternatively the reader can find the proof in [**Spa1995**, p. 169], [**Vic1994**, Theorem 5.3] or [**Dol1980**, p. 175]. ■

As an example for the Acyclic Model Theorem 113.16 we will prove a statement that we will anyway need shortly. To formulate the application we need the following definition.

Definition. A diagonal approximation is a natural chain map

$$\varphi\colon \mathrm{C}_*(X) \ \to \ \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$$

such that for each $x \in X$, viewed as a singular 0-simplex, we have

$$\varphi(x) \ = \ x \otimes x \ \in \ \underbrace{\mathrm{C}_0(X) \otimes \mathrm{C}_0(X)}_{=(\mathrm{C}_*(X)\otimes \mathrm{C}_*(X))_0}.$$

Proposition 113.17. (Diagonal Approximation Uniqueness Proposition) Given any two diagonal approximations Φ and Ψ there exists a natural chain homotopy equivalence between Φ and Ψ.

Proof. We let $\mathcal{K}$ be the category of non-empty topological spaces. As on page 2509 we consider the collection

$$\mathcal{M} \ := \ \{\Delta^k \,|\, k \in \mathbb{N}_0\} \ = \ \text{the set of all standard simplices.}$$

We consider the covariant functors[1381]

$$\begin{array}{rcl} F\colon \mathcal{K} & \to & \mathcal{A} \\ X & \mapsto & (\mathrm{C}_*(X), \epsilon_X) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G\colon \mathcal{K} & \to & \mathcal{A} \\ X & \mapsto & (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X), \epsilon_X \otimes \epsilon_X). \end{array}$$

Now let $\Phi, \Psi\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ be two diagonal approximations. By definition these are natural transformations from F to G. The hypothesis that Φ and Ψ are diagonal approximations implies that Φ and Ψ define chain maps of augmented chain complexes.

By the Acyclic Model Theorem 113.16 (2) it remains to prove the following claim.

[1380] Note that the zero map is *not* a chain map of augmented chain complexes, since it does not respect the augmentation.

[1381] It follows from the discussion on page 2507 that both F and G are indeed functors to the category of augmented chain complexes.

Claim.

(1) The functor F is free relative to $\mathcal{M}$.
(2) The functor G is acyclic relative to $\mathcal{M}$.

Proof.

(1) Conveniently enough we showed on page 2509 that F is free relative to $\mathcal{M}$.
(2) We need to show that given any $k \in \mathbb{N}_0$ the corresponding augmented chain complex $(\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k), \epsilon_X \otimes \epsilon_X)$ is acyclic. By Lemma 113.15 it suffices to show that the chain complex $\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)$ is almost acyclic. But this is a consequence of the fact that $\mathrm{C}_*(\Delta^k)$ is almost acyclic, see page 2508, and Proposition 113.14. ■

Exercises for Chapter 113.

Exercise 113.1. Let M and N be topological manifolds and let $\varphi \in \mathrm{H}_k(M)$ and $\psi \in \mathrm{H}_l(N)$. If φ is realized by a closed oriented k-dimensional submanifold A and if ψ is realized by a closed oriented l-dimensional submanifold B, then the cross product $\varphi \times \psi \in \mathrm{H}_{k+l}(M \times N)$ is realized by the closed $(k+l)$-dimensional submanifold $A \times B$.

Exercise 113.2. Let X and Y be topological spaces. Show that for any $\sigma \in \mathrm{H}_0(X)$ and $\tau \in \mathrm{H}_0(X)$ we have

$$\epsilon_X(\sigma) \cdot \epsilon_Y(\tau) \;=\; \epsilon_{X \times Y}(\sigma \times \tau).$$

Here the maps ϵ are as always the various augmentation maps.

Exercise 113.3. Let M be the Möbius band. We consider the 3-dimensional topological manifold $M \times [0,1]$.

(a) What is the homeomorphism type of $\partial(M \times [0,1])$?
(b) Is the topological manifold $M \times [0,1]$ orientable?

Exercise 113.4. Let $f\colon A \to M$ and $g\colon B \to N$ be maps between closed oriented connected non-empty topological manifolds with $\dim(A) = \dim(M)$ and $\dim(B) = \dim(N)$. We consider the degree of maps between closed oriented connected non-empty topological manifolds of the same dimension as defined on page 2379. Show that

$$\deg(f \times g\colon A \times B) \;=\; \deg(f\colon A \to M) \cdot \deg(g\colon B \to N).$$

Remark. For smooth manifolds we gave a very very different proof in Exercise 38.17.

114. The cup product

114.1. **A first definition of the cup product.** Let X be a topological space. The cross product from page 2498 applied to the same topological spaces $X = Y$ gives us in particular the cross product[1382]

$$\begin{array}{rcl} \times\colon \overbrace{\mathrm{H}_p(\mathrm{C}_*(X)) \otimes \mathrm{H}_q(\mathrm{C}_*(X))}^{=\mathrm{H}_p(X)\otimes\mathrm{H}_q(X)} & \to & \mathrm{H}_{p+q}(\mathrm{C}_*(X)\otimes\mathrm{C}_*(X)) \xrightarrow{\Upsilon_*} \overbrace{\mathrm{H}_{p+q}(\mathrm{C}_*(X\times X))}^{=\mathrm{H}_{p+q}(X\times X)}. \\ [c]\otimes[d] & \mapsto & [c\otimes d]. \end{array}$$

This is very nice, but this construction does not give us a product on the homology groups of X itself, since we end up in the homology of a different topological space.
Now we try our luck with cohomology groups. Before we can study maps on cohomology groups we need a little bit of algebraic preparation.

Definition. Let $\mathcal{C}_*$ and $\mathcal{D}_*$ be chain complexes.

(1) Given $p, q \in \mathbb{N}_0$ we denote by

$$\Pi_{p,q}\colon (\mathcal{C}_* \otimes \mathcal{D}_*)_{p+q} = \bigoplus_{i+j=p+q} \mathcal{C}_i \otimes \mathcal{D}_j \to \mathcal{C}_p \otimes \mathcal{D}_q$$

the natural projection.

(2) Let R be a commutative ring. We denote by Ξ the map

$$\begin{array}{l} \mathrm{Hom}(\mathcal{C}_p, R)\times\mathrm{Hom}(\mathcal{D}_q, R) \to \mathrm{Hom}((\mathcal{C}\otimes\mathcal{D})_{p+q}, R) \\ (\varphi,\psi) \mapsto \left((\mathcal{C}_*\otimes\mathcal{D}_*)_{p+q} \xrightarrow{\Pi_{p,q}} \mathcal{C}_p\otimes\mathcal{D}_q \xrightarrow{\varphi\otimes\psi} R\otimes R \to R,\ a\otimes b \mapsto a\cdot b \right). \end{array}$$

Lemma 114.1. Let $\mathcal{C}_*$ and $\mathcal{D}_*$ be chain complexes. For any commutative ring R the map

$$\begin{array}{rcl} \mathrm{H}^p(\mathcal{C}_*; R) \times \mathrm{H}^q(\mathcal{D}; R) & \to & \mathrm{H}^{p+q}(\mathcal{C}_*\otimes\mathcal{D}_*; R) \\ ([\varphi],[\psi]) & \mapsto & [\Xi(\varphi,\psi)] \end{array}$$

is well-defined.

Proof. We leave the verification of the lemma to the reader. ∎

We return to topology. So let X be a topological space. We consider the map[1383]

$$\begin{array}{rcl} \overbrace{\mathrm{H}^p(X;\mathbb{Z})\otimes\mathrm{H}^q(X;\mathbb{Z})}^{=\mathrm{H}^p(X;\mathbb{Z})\otimes\mathrm{H}^q(X;\mathbb{Z})} & \xrightarrow{\Xi} & \mathrm{H}^{p+q}(\mathrm{C}_*(X)\otimes\mathrm{C}_*(X);\mathbb{Z}) \xrightarrow{\Theta^*} \overbrace{\mathrm{H}^{p+q}(\mathrm{C}_*(X\times X);\mathbb{Z})}^{=\mathrm{H}^{p+q}(X\times X;\mathbb{Z})}. \\ [\varphi]\otimes[\psi] & \mapsto & [\varphi\otimes\psi\circ\Pi_{p,q}]. \end{array}$$

Summarizing we get a map

$$\mathrm{H}^p(X;\mathbb{Z}) \otimes \mathrm{H}^q(X;\mathbb{Z}) \to \mathrm{H}^{p+q}(X\times X;\mathbb{Z})$$

which is sometimes called the *cross product on cohomology*. Again, this is nice, but exactly as in the case of the cross product on homology, we end up in the cohomology of $X \times X$, but we really would like to end up in the cohomology of X.

[1382] Here $\Upsilon\colon \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y) \to \mathrm{C}_*(X\times Y)$ denotes the Eilenberg-Zilber map from Theorem 113.1.

[1383] Here $\Theta\colon \mathrm{C}_*(X\times Y) \to \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)$ denotes the Eilenberg-Zilber map from Theorem 113.1 (1). It follows from Theorem 113.1 (2) and the Dual Cochain Map Lemma 108.8 (3) that the definition below does not depend on the choice of Θ.

Now we have to find a way how to go back from the topological space $X \times X$ to the original space X. In general there does not exist an "interesting"[1384] map $X \times X \to X$. But, as we saw before, there does exist an interesting map in the opposite direction, namely the diagonal map

$$\begin{array}{rcl} d\colon X & \to & X \times X \\ x & \mapsto & (x,x). \end{array}$$

This diagonal map d induces a map

$$d_*\colon \mathrm{H}_*(X;\mathbb{Z}) \to \mathrm{H}_*(X \times X;\mathbb{Z})$$

on homology, and, much more interesting in our context, it also induces a map

$$d^*\colon \mathrm{H}^*(X \times X;\mathbb{Z}) \to \mathrm{H}^*(X;\mathbb{Z}).$$

on cohomology groups that goes in the "opposite direction". This discussion leads us to the following definition.

Definition. Let X be a topological space. We define the cup product[1385] on the cohomology groups of X as the composition of the following maps:

$$\mathrm{H}^p(X;\mathbb{Z}) \times \mathrm{H}^q(X;\mathbb{Z}) \to \mathrm{H}^p(X;\mathbb{Z}) \otimes \mathrm{H}^q(X;\mathbb{Z}) \xrightarrow{\Theta^* \circ \Xi} \mathrm{H}^{p+q}(X \times X;\mathbb{Z}) \xrightarrow{d^*} \mathrm{H}^{p+q}(X;\mathbb{Z}).$$

For the resulting pairing we use the notation

$$\begin{array}{rcl} \mathrm{H}^p(X;\mathbb{Z}) \times \mathrm{H}^q(X;\mathbb{Z}) & \to & \mathrm{H}^{p+q}(X;\mathbb{Z}) \\ (a,b) & \mapsto & a \mathbin{\widetilde{\cup}} b \end{array}$$

and we refer to $a \mathbin{\widetilde{\cup}} b$ as the cup product of a and b.

114.2. **An alternative definition of the cup product.** The definition of the cup product in the previous section was very natural and we could now develop the theory of the cup product using that definition. If one takes that route then one has to deal with the awkward fact that the Eilenberg-Zilber map Θ is not very explicit.
Now we give an alternative definition of the cup product which is more ad hoc, but which works surprisingly well in practice.

Definition. Let X be a topological space and let R be a commutative ring. Furthermore let $\varphi \in \mathrm{C}^p(X;R)$ and $\psi \in \mathrm{C}^q(X;R)$ be cochains. We define the cup product of φ and ψ to be the singular cochain[1386]

$$\varphi \cup \psi\colon \mathrm{C}_{p+q}(X) \to R$$

that is given by

$$(\sigma\colon \Delta^{p+q} \to X) \mapsto \underbrace{\varphi(\sigma \circ [v_0,\ldots,v_p])}_{\in R} \cdot \underbrace{\psi(\sigma \circ [v_p,\ldots,v_{p+q}])}_{\in R}.$$

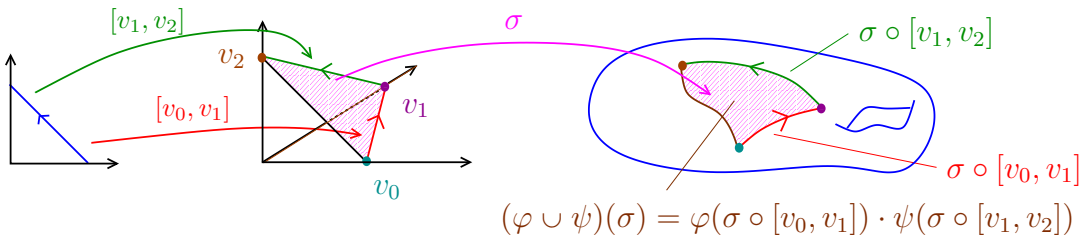
The[1387] cup product thus defines a map

$$\cup\colon \mathrm{C}^p(X;R) \times \mathrm{C}^q(X;R) \to \mathrm{C}^{p+q}(X;R).$$

[1384]The projection from $X \times X$ onto one of the two factors does not count as "interesting", since it does not "mix" the two factors. In fact in Exercise 114.1 we will see that the map

$$\mathrm{H}_*(X) \otimes \mathrm{H}_*(X) \xrightarrow{\times} \mathrm{H}_*(X \times X) \xrightarrow{p_*} \mathrm{H}_*(X),$$

where $p\colon X \times X \to X$ denotes the projection on the first factor, is the zero map.
[1385]The name "cup product" comes from the notation "$a \mathbin{\widetilde{\cup}} b$" for the composition since $\cup$ looks like a cup.



Remark. The sign conventions for the cup product for cohomology classes $\varphi \in \mathrm{H}^p(X; R)$ and $\psi \in \mathrm{H}^q(X; R)$ differ in the literature. More precisely, given a topological space X and cochains $\varphi \in \mathrm{C}^p(X; R)$ and $\psi \in \mathrm{C}^q(X; R)$ the cup product $\varphi \cup \psi$, applied to a singular $(p+q)$-simplex $\sigma\colon \Delta^{p+q} \to X$ is defined as follows:

		$(\varphi \cup \sigma)(\sigma)$
(a)	Hatcher [**Hat2002**, p. 206]	$\varphi(\sigma \circ [v_0, \ldots, v_p]) \cdot \psi(\sigma \circ [v_p, \ldots, v_{p+q}])$
(b)	Bredon [**Bre1993**, p. 328]	$(-1)^{pq} \cdot \varphi(\sigma \circ [v_0, \ldots, v_p]) \cdot \psi(\sigma \circ [v_p, \ldots, v_{p+q}])$
(c)	Dold [**Dol1956**, p. 222]	$(-1)^{pq} \cdot \varphi(\sigma \circ [v_0, \ldots, v_p]) \cdot \psi(\sigma \circ [v_p, \ldots, v_{p+q}])$
(d)	Spanier [**Spa1995**, p. 251]	$\varphi(\sigma \circ [v_0, \ldots, v_p]) \cdot \psi(\sigma \circ [v_p, \ldots, v_{p+q}])$
(e)	Munkres [**Mun1984**, p. 288]	$\varphi(\sigma \circ [v_0, \ldots, v_p]) \cdot \psi(\sigma \circ [v_p, \ldots, v_{p+q}])$
(f)	Greenberg-Harper [**GrH1981**, p. 195]	$\varphi(\sigma \circ [v_0, \ldots, v_p]) \cdot \psi(\sigma \circ [v_p, \ldots, v_{p+q}])$

So we see that our sign convention agrees with the conventions of Hatcher, Spanier, Munkres and Greenberg-Harper, but they differ by the sign $(-1)^{pq}$ from the conventions used in Bredon and Dold.

Example. Since the second definition of the cup product is initially strange and not very intuitive we consider an example in slightly more detail. In the following we consider the torus $T = S^1 \times S^1 = ([0,1] \times [0,1])/\sim$ with the two singular 2-simplices σ_1 and σ_2 illustrated in the figure on page 2515 which we already saw in the figure on page 2204. We recall the following notation:

(1) we denote by p respectively q the projection $S^1 \times S^1 \to S^1$ onto the first respectively second factor,
(2) we denote by $\mu\colon \Delta^1 \to S^1$ the singular 1-simplex given by $\mu(1-t, t) = \exp(2\pi\,\mathrm{i}t)$,
(3) furthermore we denote by $\nu\colon \Delta^1 \to S^1$ the constant singular 1-simplex that is given by $\nu(1-t, t) = 1$.
(4) finally we recall that on page 2404 we introduced a cochain $\theta := \theta_{\mathbb{Z}} \in \mathrm{C}^1; \mathbb{Z})$ and we showed that $\theta(\mu) = 1$, also note that $\theta(\nu) = 0$.

[1386]Here, when we write $[v_0, \ldots, v_p]\colon \Delta^p \to \Delta^{p+q}$ and $[v_p, \ldots, v_{p+q}]\colon \Delta^q \to \Delta^{p+q}$, we use the notation from page 2503.
[1387]In the literature, given a singular n-simplex $\sigma\colon \Delta^n \to X$ one often writes ${}_p\lfloor\sigma = \sigma \circ [v_0, \ldots, v_p]$ and $\sigma\rfloor_p = \sigma \circ [v_{n-p}, \ldots, v_n]$.

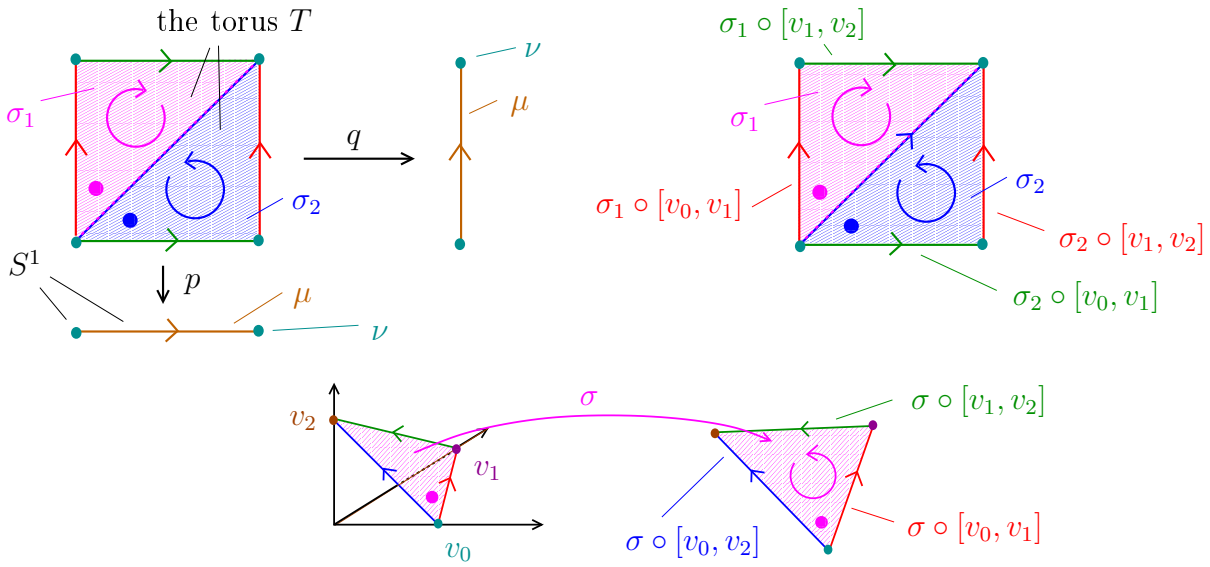
We want to determine the values of $p^*(\theta) \cup q^*(\theta) \in \mathrm{C}^2(T;\mathbb{Z}) = \mathrm{Hom}(\mathrm{C}_2(T),\mathbb{Z})$ on the singular 2-simplices σ_1 and σ_2. We have

$$\begin{aligned}(p^*(\theta) \cup q^*(\theta))(\sigma_1) &= p^*(\theta)(\sigma_1 \circ [v_0,v_1]) \cdot q^*(\theta)(\sigma_1 \circ [v_1,v_2]) \\ &= \theta(\underbrace{p \circ \sigma_1 \circ [v_0,v_1]}_{=\nu}) \cdot \theta(\underbrace{q \circ \sigma_1 \circ [v_1,v_2]}_{=\nu}) \quad = \quad 0 \cdot 0 \quad = \quad 0.\end{aligned}$$

Much more interestingly we have

$$\begin{aligned}(p^*(\theta) \cup q^*(\theta))(\sigma_2) &= p^*(\theta)(\sigma_2 \circ [v_0,v_1]) \cdot q^*(\theta)(\sigma_2 \circ [v_1,v_2]) \\ &= \theta(\underbrace{p \circ \sigma_2 \circ [v_0,v_1]}_{=\mu}) \cdot \theta(\underbrace{q \circ \sigma_2 \circ [v_1,v_2]}_{=\mu}) \quad = \quad 1 \cdot 1 \quad = \quad 1.\end{aligned}$$

To aid the reader we also recall in the figure below the notation that we had initially introduced on page 2203 and that we use in the figure on page 2515.



The definition of the cup product on cochains might look strange and arbitrary. But the following lemma shows that it behaves well with respect to coboundary maps.

Lemma 114.2. (Coboundary-Cup Product Lemma) Let X be a topological space and let R be a commutative ring. Furthermore let $\varphi \in \mathrm{C}^p(X;R)$ and $\psi \in \mathrm{C}^q(X;R)$. Then

$$\delta(\varphi \cup \psi) \quad = \quad \delta\varphi \cup \psi \; + (-1)^p \cdot \varphi \cup \delta\psi \quad \in \quad \mathrm{C}^{p+q+1}(X;R).$$

Remark. Note that a similar looking statement holds for differential forms. More precisely, given a smooth manifold M and two differential forms $\omega \in \Omega^p(M)$ and $\eta \in \Omega^q(M)$ it follows from the Differentials-on-Smooth Manifolds Proposition **??** (1) that

$$d(\omega \wedge \eta) \quad = \quad d\omega \wedge \eta \; + \; (-1)^p \cdot \omega \wedge d\eta \quad \in \quad \Omega^{p+q+1}(M).$$

The (co-) boundary formula of Coboundary-Cup Product Lemma 114.2 is of course also very similar to the definition of the boundary map in the tensor product of two chain complexes.

Proof. The lemma follows from an elementary, albeit slightly confusing, calculation. Indeed, let $\varphi \in \mathrm{C}^p(X;R)$ and $\psi \in \mathrm{C}^q(X;R)$ be cochains. We have to prove the following

equality:

$$\delta(\varphi\cup\psi) \;=\; \delta\varphi\cup\psi \,+(-1)^p\cdot\varphi\cup\delta\psi \;\in\; \mathrm{C}^{p+q+1}(X;R)=\mathrm{Hom}(\mathrm{C}_{p+q+1}(X),R).$$

So let $\sigma\colon \Delta^{p+q+1}\to X$ be a singular $(p+q+1)$-simplex. We have the following three equalities in R:[1388]

$$\begin{aligned}
\text{(a)}\quad (\delta(\varphi\cup\psi))(\sigma) &= (\varphi\cup\psi)\Big(\sum_{j=0}^{p+q+1}(-1)^j\cdot\sigma\circ i_j^{p+q+1}\Big)\\
&= (\varphi\cup\psi)\Big(\sum_{j=0}^{p}(-1)^j\cdot\sigma\circ i_j^{p+q+1}\Big)+(\varphi\cup\psi)\Big(\sum_{j=p+1}^{p+q+1}(-1)^j\cdot\sigma\circ i_j^{p+q+1}\Big)\\
&= \sum_{j=0}^{p}(-1)^j\cdot(\varphi\cup\psi)(\sigma\circ i_j^{p+q+1})+(-1)^p\cdot\sum_{k=1}^{q+1}(-1)^k\cdot(\varphi\cup\psi)(\sigma\circ i_{k+p}^{p+q+1})\\
&= \sum_{j=0}^{p}(-1)^j\cdot\varphi(\underbrace{\sigma\circ[v_0,\dots,v_{p+1}]\circ i_j^{p+1}}_{\text{map }\Delta^p\to X})\cdot\psi(\underbrace{\sigma\circ[v_{p+1},\dots,v_{p+q+1}]}_{\text{map }\Delta^q\to X})\\
&\quad+(-1)^p\cdot\sum_{k=1}^{q+1}(-1)^k\cdot\varphi(\underbrace{\sigma\circ[v_0,\dots,v_p]}_{\text{map }\Delta^p\to X})\cdot\psi(\underbrace{\sigma\circ[v_p,\dots,v_{p+q+1}]\circ i_k^{q+1}}_{\text{map }\Delta^q\to X}).
\end{aligned}$$

$$\begin{aligned}
\text{(b)}\quad (\delta\varphi\cup\psi)(\sigma) &= (\delta\varphi)(\sigma\circ[v_0,\dots,v_{p+1}])\cdot\psi(\sigma\circ[v_{p+1},\dots,v_{p+q+1}])\\
&= \varphi\Big(\sum_{j=0}^{p+1}(-1)^j\cdot\underbrace{\sigma\circ[v_0,\dots,v_{p+1}]\circ i_j^{p+1}}_{\text{map }\Delta^p\to X}\Big)\cdot\psi(\underbrace{\sigma\circ[v_{p+1},\dots,v_{p+q+1}]}_{\text{map }\Delta^q\to X})\\
&= \sum_{j=0}^{p+1}(-1)^j\cdot\varphi(\sigma\circ[v_0,\dots,v_{p+1}]\circ i_j^{p+1})\cdot\psi(\sigma\circ[v_{p+1},\dots,v_{p+q+1}])
\end{aligned}$$

and

$$\begin{aligned}
\text{(c)}\quad (-1)^p\cdot(\varphi\cup\delta\psi)(\sigma) &= (-1)^p\cdot\varphi(\sigma\circ[v_0,\dots,v_p])\cdot(\delta\psi)(\sigma\circ[v_p,\dots,v_{p+q+1}])\\
&= (-1)^p\cdot\varphi(\underbrace{\sigma\circ[v_0,\dots,v_p]}_{\text{map }\Delta^p\to X})\cdot\psi\Big(\sum_{k=0}^{q+1}(-1)^k\cdot\underbrace{\sigma\circ[v_p,\dots,v_{p+q+1}]\circ i_k^{q+1}}_{\text{map }\Delta^q\to X}\Big)\\
&= (-1)^p\cdot\sum_{k=0}^{q+1}(-1)^k\cdot\varphi(\sigma\circ[v_0,\dots,v_p])\cdot\psi(\sigma\circ[v_p,\dots,v_{p+q+1}]\circ i_k^{q+1}).
\end{aligned}$$

Now we observe that if we take the sum of (b) and (c), then the $(j=p+1)$-summand in (b) cancels with the $(k=0)$-summand in (c). But after this observation it is obvious that the desired equality (b)+(c)=(a) holds. ■

Lemma 114.3. Let X be a topological space and let R be a commutative ring. Then the map

$$\begin{aligned}\cup\colon \mathrm{H}^p(X;R)\times\mathrm{H}^q(X;R) &\to \mathrm{H}^{p+q}(X;R)\\ ([\varphi],[\psi])] &\mapsto [\varphi]\cup[\psi]:=[\varphi\cup\psi]\end{aligned}$$

is well-defined.

Definition. We refer to the pairing of Lemma 114.3 also as the cup product.

As we will see, the proof of Lemma 114.3 is basically identical to the proof of Proposition **??**.

[1388]Here in the argument we use on several occasions that for a singular n-cochain ϕ and a singular $(n+1)$-simplex μ we have by definition

$$(\delta\phi)(\mu) \;=\; \phi(\partial\mu) \;=\; \phi\Big(\sum_{l=0}^{n+1}(-1)^l\cdot\mu\circ i_l^{n+1}\Big).$$

Proof. Let $\varphi \in \mathrm{C}^p(X;R)$ and $\psi \in \mathrm{C}^q(X;R)$ be cocycles.

(1) It follows immediately from Coboundary-Cup Product Lemma 114.2 that $\varphi \cup \psi$ is again a cocycle.

(2) If $\varphi' = \varphi + \delta\tau$ is another representative of $[\varphi]$, then

$$\varphi' \cup \psi - \varphi \cup \psi \underset{\uparrow}{=} (\varphi' - \varphi) \cup \psi = \delta\tau \cup \psi \underset{\uparrow}{=} \delta(\tau \cup \psi).$$

since "$\cup$" is clearly bilinear; by Coboundary-Cup Product Lemma 114.2 and since $\delta\psi = 0$

This shows that $\varphi \cup \psi$ and $\varphi' \cup \psi$ represent the same cohomology class, i.e. the cup product does not depend on the choice of the representative of the cohomology class of the first entry.

(3) The proof that the cup product does not depend on the choice of the representative of the second cohomology class is basically the same as the proof of (2). ■

The following proposition now says that the original definition of the cup product and the new definition actually agree.

Proposition 114.4. (Definitions-Cup Product Proposition) Let X be a topological space. Then for any $\varphi \in \mathrm{H}^p(X;\mathbb{Z})$ and any $\psi \in \mathrm{H}^q(X;\mathbb{Z})$ we have

$$\varphi \mathbin{\widetilde{\cup}} \psi = \varphi \cup \psi \in \mathrm{H}^{p+q}(X;\mathbb{Z}).$$

In the proof of the Definitions-Cup Product Proposition 114.4 we need the following definition and the subsequent lemma.

Definition. A diagonal approximation consists of a natural chain map

$$\Phi\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$$

for each topological space X, such that for each $x \in X$, viewed as a singular 0-simplex, the following equality holds in $(\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_0 = \mathrm{C}_0(X) \otimes \mathrm{C}_0(X)$:

$$\Phi(x) = x \otimes x.$$

Example. Given any topological space X we denote by $d\colon X \to X \times X$ the diagonal map given by $d(x) = (x,x)$. Then the induced chain maps

$$\Theta \circ d_*\colon \mathrm{C}_*(X) \xrightarrow{d_*} \mathrm{C}_*(X \times X) \underset{\uparrow}{\xrightarrow{\Theta}} \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$$

Eilenberg-Zilber map from Theorem 113.1

form a diagonal approximation in the above sense.

The following lemma gives another diagonal approximation which will be very useful for doing actual calculations.

Lemma 114.5. (Alexander-Whitney Lemma) Given any topological space X we consider the map

$$\Delta\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$$

which for each singular n-simplex $\sigma\colon \Delta^n \to X$ is given by

$$\sigma \mapsto \sum_{p=0}^{n} \sigma \circ [v_0, \ldots, v_p] \otimes \sigma \circ [v_p, \ldots, v_n].$$

These maps form a diagonal approximation.

The diagonal approximation from the Alexander-Whitney Lemma 114.5 is called the *Alexander-Whitney diagonal approximation.* [13891390]

Proof. It follows almost immediately from the definition that the maps Δ are natural. Furthermore it is obvious that on the 0-chains the maps are of the desired form. It remains to show that for a given topological space X the maps actually form a chain map.

This follows from an elementary and uneventful calculation. For completeness' sake we provide the argument. Let $\mu\colon \Delta^n \to X$ be a singular n-simplex. Given $i_1, \ldots, i_r$ we write $[i_1, \ldots, i_r] := \mu \circ [v_{i_1}, \ldots, v_{i_r}]$. We also use the convention that the empty singular simplex $[\,]$ is 0. We calculate that

$$\begin{aligned}\Delta(\partial[0,\ldots,n]) &= \Delta\Big(\sum_{i=0}^{n}(-1)^i\cdot[0,\ldots,\widehat{i},\ldots n]\Big)\\ &= \underset{\uparrow}{\sum_{0\le p<i\le n}}(-1)^i\cdot[0,..,p]\otimes[p,..,\widehat{i},..,n]+\sum_{0\le i<q\le n}(-1)^i\cdot[0,..,\widehat{i},..,q]\otimes[q,..,n]\end{aligned}$$

in the second sum we secretly used the substitution $q = p+1$

and

$$\begin{aligned}\partial(\Delta[0,\ldots,n]) &= \partial\Big(\sum_{p=0}^{n}[0,\ldots,p]\otimes[p,\ldots,n]\Big)\\ &= \underset{\uparrow}{\sum_{0\le i\le p\le n}}(-1)^i\cdot[0,..,\widehat{i},..,p]\otimes[p,..,n]+\sum_{0\le p\le i\le n}(-1)^i\cdot[0,..,p]\otimes[p,..,\widehat{i},..,n].\end{aligned}$$

here we use the sign convention for the boundary in a tensor product

We see that the difference between the two terms equals

$$\begin{aligned}&\sum_{i=p}(-1)^i\cdot[0,\ldots,\widehat{i},\ldots,p]\otimes[p,\ldots,n]+\sum_{i=p}(-1)^i\cdot[0,\ldots,p]\otimes[p,\ldots,\widehat{i},\ldots,n]\\ =\ &\underset{\uparrow}{\sum_{i=1}^{n}}(-1)^i\cdot[0,\ldots,i-1]\otimes[i,\ldots,n]+\sum_{i=0}^{n-1}(-1)^i\cdot[0,\ldots,i]\otimes[i+1,\ldots,n]\end{aligned}$$

here we use that $[\,]=0$

$$= \sum_{i=1}^{n}(-1)^i\cdot[0,\ldots,i-1]\otimes[i,\ldots,n]+\sum_{j=1}^{n}(-1)^{j-1}\cdot[0,\ldots,j-1]\otimes[j,\ldots,n]\ =\ 0.$$

We have thus shown that Δ is indeed a chain map. ■

Proposition 114.6.

(1) There exist diagonal approximations.
(2) Given any two diagonal approximations Φ and Ψ there exists a natural chain homotopy equivalence between Φ and Ψ.

Proof.

(1) We had just given two examples of diagonal approximations.
(2) The uniqueness statement was proved in Proposition 114.6. ■

Now we are ready to provide the proof of the Definitions-Cup Product Proposition 114.4.

Proof of Proposition 114.4. We start out with the following claim.

[1389] James Alexander (1888-1971) was an American mathematician whom we had already met on page 1814.
[1390] Hassler Whitney (1907–1989) was an American mathematician. We will encounter him again later on.

Claim. For any two cochains $\varphi \in \mathrm{C}^p(X;\mathbb{Z})$ and $\psi \in \mathrm{C}^q(X;\mathbb{Z})$ we have[1391]

$$\varphi \cup \psi \;=\; (\varphi \otimes \psi) \circ \Pi_{p,q} \circ \Delta \;\in\; \mathrm{C}^{p+q}(X;\mathbb{Z}) = \mathrm{Hom}(\mathrm{C}_{p+q}(X),\mathbb{Z}).$$

Proof. We have to show that the two sides are the same maps $\mathrm{C}_{p+q}(X) \to \mathbb{Z}$. Therefore let $\sigma\colon \Delta^{p+q} \to X$ be a singular $(p+q)$-simplex. We calculate that

$$\begin{aligned}
(\varphi \otimes \psi \circ \Pi_{p,q} \circ \Delta)(\sigma) &\overset{\text{definition of the Alexander-Whitney diagonalization}}{\downarrow} \\
&= (\varphi \otimes \psi)\Big(\Pi_{p,q}\Big(\sum_{i+j=p+q} \overbrace{\sigma \circ [v_0,\dots,v_i] \otimes \sigma \circ [v_i,\dots,v_{p+q}]}^{\in \mathrm{C}_i(X)\otimes \mathrm{C}_j(X)}\Big)\Big) \\
&= (\varphi \otimes \psi)(\sigma \circ [v_0,\dots,v_p] \otimes \sigma \circ [v_q,\dots,v_{p+q}]) \\
&= \varphi(\sigma \circ [v_0,\dots,v_p]) \cdot \psi(\sigma \circ [v_p,\dots,v_n]) \\
&= (\varphi \cup \psi)(\sigma).
\end{aligned}$$

⊞

Now suppose that we are given cocycles $\varphi \in \mathrm{C}^p(X;\mathbb{Z})$ and $\psi \in \mathrm{C}^q(X;\mathbb{Z})$. We obtain the following equality in $\mathrm{H}^{p+q}(X;\mathbb{Z})$:[1392]

$$\begin{aligned}
[\varphi \smile \psi] \overset{\text{by definition}}{\downarrow} [(\varphi \otimes \psi)\circ\Pi_{p,q}\circ(\Theta\circ d_*)] &= (\Theta\circ d_*)^*\big(\overbrace{[(\varphi \otimes \psi)\circ\Pi_{p,q}]}^{\in \mathrm{H}^{p+q}(\mathrm{C}_*(X)\otimes \mathrm{C}_*(X))}\big) \\
&= \Delta^*\big([(\varphi \otimes \psi)\circ\Pi_{p,q}]\big) = [(\varphi \otimes \psi)\circ\Pi_{p,q}\circ\Delta] \overset{\text{by the claim}}{\uparrow} = [\varphi \cup \psi].
\end{aligned}$$

$\uparrow$ by Proposition 114.6 the maps $\Theta\circ d_*, \Delta\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$ are chain homotopic so by the Dual Cochain Map Lemma 108.8 (3) they induce the same map on cohomology

We have thus obtained the desired equality of cohomology classes. ■

Henceforth we will mostly work with the second definition of the cup product. In the following proposition we summarize some of the key properties of the cup product.

Proposition 114.7. (Cup Product-Ring Proposition) Let X be a topological space and let R be a commutative ring.

(1) The cup product on $\mathrm{H}^*(X;R)$ is R-bilinear and associative.

(2) The abelian group $\mathrm{H}^*(X;R) = \bigoplus_{n\in\mathbb{N}_0} \mathrm{H}^n(X;R)$ together with the map

$$\cup\colon \mathrm{H}^*(X;R) \times \mathrm{H}^*(X;R) \to \mathrm{H}^*(X;R)$$

is a ring[1393] where the multiplicatively neutral element is given by

$$1_X \;:=\; \big[\text{constant map } X \to R \text{ given by } x \mapsto 1\big] \;\in\; \mathrm{H}^0(X;R).$$

Definition. Given a topological space X and given a commutative ring R we refer to $(\mathrm{H}^*(X;R), \cup)$ as the *R-cohomology ring of X*.

Proof. Let X be a topological space.

(1) It is clear from the second definition of the cup product that it is R-bilinear. It is also almost obvious, using the second definition, that the cup product is associative, but for completeness' sake we carry out the argument. Thus let $\alpha \in \mathrm{C}^p(X;R)$,

[1391] Here $\Delta\colon \mathrm{C}_{p+q}(X) \to (\mathrm{C}_*(X)\otimes\mathrm{C}_*(X))_{p+q}$ denotes the above Alexander-Whitney diagonalization.

[1392] Note that it follows from Lemma 114.1 that for two cocycles φ and ψ the composition $(\varphi\otimes\psi)\circ\Pi_{p,q}$ is indeed a cocycle, i.e. $(\varphi\otimes\psi)\circ\Pi_{p,q}$ defines an element in $\mathrm{H}^{p+q}(\mathrm{C}_*(X)\otimes\mathrm{C}_*(X);\mathbb{Z})$.

[1393] Recall that on page 98 we stated that in the lecture notes a ring is understood to be associative and it is understood to have a multiplicatively neutral element. We do not demand that a ring is commutative.

$\beta \in \mathrm{C}^q(X;R)$ and $\gamma \in \mathrm{C}^r(X;R)$ be cochains and let $\sigma\colon \Delta^{p+q+r} \to X$ be a singular simplex. Then

$$\begin{aligned}((\alpha \cup \beta) \cup \gamma)(\sigma) &= (\alpha \cup \beta)(\sigma \circ [v_0, \dots, v_{p+q}]) \cdot \gamma(\sigma \circ [v_{p+q}, \dots, v_{p+q+r}]) \\ &= \alpha(\sigma \circ [v_0, \dots, v_p]) \cdot \beta(\sigma \circ [v_p, \dots, v_{p+q}]) \cdot \gamma(\sigma \circ [v_{p+q}, \dots, v_{p+q+r}]) \\ &= (\alpha \cup (\beta \cup \gamma))(\sigma). \\ &\quad \uparrow \\ &\quad \text{same argument backwards}\end{aligned}$$

Thus we see that the cup product is already associative for cochains, so it is in particular associative for cohomology classes.

(2) Let $\eta\colon X \to R$ be the map that assigns to each point in X the value $1 \in R$. As usual we view η as a singular 0-cochain. It follows from the H^0-Proposition 108.11 that η is a 0-cocycle and that we obtain an element $1_X := [\eta] \in \mathrm{H}^0(X;R)$. It is clear from the definition that it is the neutral element with respect to the cup product. Together with (1) we have now shown that the cup product defines a ring structure on $\mathrm{H}^*(X;R)$. ■

Now that we have shown that $(\mathrm{H}^*(X;R), \cup)$ is a ring the question arises whether it is commutative or whether it is not commutative. It turns out that the answer lies somewhere in between. More precisely, we have the following proposition.

Proposition 114.8. (Cup Product-Commutativity Proposition) Let X be a topological space and let R be a commutative ring. For any $a \in \mathrm{H}^k(X;R)$ and $b \in \mathrm{H}^l(X;R)$ we have

$$a \cup b \;=\; (-1)^{kl} \cdot b \cup a.$$

Remark.

(1) The type of (non-) commutativity we encounter in the Cup Product-Commutativity Proposition 114.8 is the same as the one we encountered when we studied the wedge product of alternating forms and the wedge product on de Rham cohomology, see Lemma **??** and Proposition **??**.

(2) If $R = \mathbb{F}_2$, then we are in the pretty situation that $-1 = 1$, i.e. the cup product is commutative. Put differently, the $\mathbb{F}_2$-cohomology ring $\mathrm{H}^*(X;\mathbb{F}_2)$ of a topological space X is a commutative ring.

(3) The proof for the Cup Product-Commutativity Proposition 114.8 that we provide below is based on the second definition of the cup product. It is elementary but unfortunately unpleasant to read. Later, in Exercise 117.1, we will provide a different, more conceptual proof based on the first definition of the cup product.

Proof. We give a proof of the proposition following [**Hat2002**, p. 216-217]. In Hatcher's book more motivations for the steps in the proof are given.

So let $\varphi \in \mathrm{C}^k(X;R)$ and $\psi \in \mathrm{C}^l(X;R)$ be two cocycles. Given a singular n-simplex $\sigma\colon \Delta^n \to X$ we denote by $\overline{\sigma}$ the singular n-simplex obtained from σ by precomposing σ with the affine linear map $[v_n, \dots, v_0]\colon \Delta^n \to \Delta^n$ that reverses the order of the vertices. Furthermore we write $\epsilon_n = (-1)^{n(n+1)/2}$. We note that for any $m, n \in \mathbb{N}_0$ an elementary calculation shows that $\epsilon_{m+n} = (-1)^{mn} \cdot \epsilon_m \epsilon_n$ and that $\epsilon_1 = -1$.

Claim. The maps

$$\begin{array}{rcl}\rho\colon \mathrm{C}_n(X) & \to & \mathrm{C}_n(X)\\ \sigma & \mapsto & \epsilon_n\cdot\overline{\sigma}\end{array}$$

form a chain map that is chain homotopic to the identity.

We first show that the claim actually implies the statement of the proposition. Thus let $[\varphi]\in \mathrm{H}^k(X;R)$ and $[\psi]\in \mathrm{H}^l(X;R)$. Since by the claim ρ is chain homotopic to the identity it follows from the Dual Cochain Map Lemma 108.8 that the induced map $\rho^*\colon \mathrm{C}^*(X;R)\to \mathrm{C}^*(X;R)$ induces the identity on cohomology groups. So to deduce the statement of the proposition it suffices to show that

$$\rho^*(\psi\cup\varphi) \;=\; (-1)^{kl}\cdot\rho^*\varphi\cup\rho^*\psi \;\in\; \mathrm{C}^{k+l}(X;R).$$

Now let σ be a $(k+l)$-chain. Then

$$\begin{array}{rcl}(\rho^*(\psi\cup\varphi))(\sigma) & = & (\psi\cup\varphi)(\rho_*(\sigma)) \;=\; (\psi\cup\varphi)(\epsilon_{k+l}\cdot\sigma\circ[v_{k+l},\dots,v_0])\\ & = & \epsilon_{k+l}\cdot\psi(\sigma\circ[v_{k+l},\dots,v_k])\cdot\varphi(\sigma\circ[v_k,\dots,v_0])\\ & = & (-1)^{kl}\cdot\varphi(\epsilon_k\cdot\sigma\circ[v_k,\dots,v_0])\cdot\psi(\epsilon_l\cdot\sigma\circ[v_{k+l},\dots,v_k])\\ & & \uparrow\\ & & \text{since } \epsilon_{k+l}=(-1)^{kl}\cdot\epsilon_k\cdot\epsilon_l \text{ and since } R \text{ is a } \textit{commutative} \text{ ring}\\ & = & (-1)^{kl}\cdot(\rho^*\varphi\cup\rho^*\psi)(\sigma).\end{array}$$

We have thus shown the statement of the proposition modulo the claim.

Proof. It is straightforward to verify that ρ is a chain map, indeed, for any singular n-simplex σ we have

$$\begin{array}{rcl}\partial\rho(\sigma) & = & \partial(\epsilon_n\cdot\sigma\circ[v_n,\dots,v_0]) \;=\; \epsilon_n\cdot\sum_{i=0}^{n}(-1)^i\cdot\sigma\circ[v_n,\dots,\widehat{v_{n-i}},\dots,v_0],\\ \rho\partial(\sigma) & = & \rho\Big(\sum_{i=0}^{n}(-1)^i\cdot\sigma\circ[v_0,\dots,\widehat{v_i},\dots,v_n]\Big) \;\underset{\uparrow}{=}\; \epsilon_{n-1}\cdot\sum_{j=0}^{n}(-1)^{n-j}\cdot\sigma\circ[v_n,\dots,\widehat{v_{n-j}},\dots,v_0].\\ & & \qquad\qquad \text{definition of } \rho \text{ and substitution } j=n-i\end{array}$$

From $\epsilon_n=(-1)^n\cdot\epsilon_{n-1}$ we now obtain that $\partial\rho=\rho\partial$, i.e. we have shown that ρ is a chain map.

We still have to show that ρ is chain homotopic to the identity. We proceed with ideas similar to the ideas used in the proof of the Homotopic Maps-and-Homology Proposition 73.6. As in that proof, given $i\in\{0,\dots,n\}$ we write

$$\begin{array}{rcl}v_i & := & (0,\dots,1,0,\dots,0)\times\{0\} \;\in\; \mathbb{R}^{n+1}\times\{0\},\\ \text{and}\qquad w_i & := & (0,\dots,1,0,\dots,0)\times\{1\} \;\in\; \mathbb{R}^{n+1}\times\{1\}.\end{array}$$

We justify the reassignment of v_i to a different point by saying that we identify Δ^n with $\Delta^n\times\{0\}\in\mathbb{R}^{n+1}\times\mathbb{R}$. We denote by $\pi\colon\Delta^n\times[0,1]\to\Delta^n$ the projection . We consider the map

$$\begin{array}{rcl}P\colon \mathrm{C}_n(X) & \to & \mathrm{C}_{n+1}(X)\\ \sigma & \mapsto & \sum_{i=0}^{n}(-1)^i\cdot\epsilon_{n-i}\cdot(\sigma\circ\pi)\circ[v_0,\dots,v_i,w_n,\dots,w_i].\end{array}$$

Now we want to show that $\partial P+P\partial=\rho-\mathrm{id}$. We calculate

$$\begin{array}{rcl}(P\partial)(\sigma) & = & \sum_{i<j}(-1)^i(-1)^j\cdot\epsilon_{n-i-1}\cdot(\sigma\circ\pi)\circ[v_0,\dots,v_i,w_n,\dots,\widehat{w_j},\dots,w_i]\\ & & +\sum_{i>j}(-1)^{i-1}(-1)^j\cdot\epsilon_{n-i}\cdot(\sigma\circ\pi)\circ[v_0,\dots,\widehat{v_j},\dots,v_i,w_n,\dots,w_i],\end{array}$$

On the other hand we have

$$\begin{aligned}(\partial P)(\sigma) &= \sum_{j\le i}(-1)^i(-1)^j\cdot\epsilon_{n-i}\cdot(\sigma\circ\pi)\circ[v_0,\dots,\widehat{v_j},\dots,v_i,w_n,\dots,w_i]\\ &\quad+\sum_{j\ge i}(-1)^i(-1)^{i+1+n-j}\cdot\epsilon_{n-i}\cdot(\sigma\circ\pi)\circ[v_0,\dots,v_i,w_n,\dots,\widehat{w_j},\dots,w_i].\end{aligned}$$

Now using that $\epsilon_{n-i}=(-1)^{n-i}\epsilon_{n-i-1}$ we see that the sum over the $j\neq i$ terms agrees precisely with $-P\partial$ calculated above (with the roles of i and j reversed). It remains to show that the sum over the $i=j$ terms equals $(\rho-\mathrm{id})(\sigma)$. This is indeed the case since

$$\begin{aligned}&\text{sum of the } i=j \text{ summands in the above formula for } (\partial P)(\sigma)\\ &= \epsilon_n\cdot(\sigma\circ\pi)\circ[w_n,\dots,w_0]+\sum_{i=1}^{n}\epsilon_{n-i}\cdot(\sigma\circ\pi)\circ[v_0,\dots,v_{i-1},w_n,\dots,w_i]\\ &\quad+\sum_{i=0}^{n-1}(-1)^{n+i+1}\epsilon_{n-i}\cdot(\sigma\circ\pi)\circ[v_0,\dots,v_i,w_n,\dots,w_{i+1}]-(\sigma\circ\pi)\circ[v_0,\dots,v_n]\\ &\underset{\uparrow}{=}\ \epsilon_n\cdot(\sigma\circ\pi)\circ[w_n,\dots,w_0]-(\sigma\circ\pi)\circ[v_0,\dots,v_n] \;=\; (\rho-\mathrm{id})(\sigma).\end{aligned}$$

in the second sum we substitute $j=i+1$, we obtain a new sign $(-1)^{n+j}\epsilon_{n-j+1}=-\epsilon_{n-j}$, thus we see that the two sums over i cancel

We have now completed the proof that $\partial P+P\partial=\rho-\mathrm{id}$, in particular we have shown the claim and thus also the proposition. ■

Let $k\in\mathbb{N}$ be odd. It follows from the Cup Product-Commutativity Proposition 114.8 that given any topological space X and any $\varphi\in\mathrm{H}^k(X;\mathbb{Z})$ we have $\varphi\cup\varphi=-(1)^{k\cdot k}\cdot\varphi\cup\varphi$, put differently, we have $2\cdot(\varphi\cup\varphi)=0$. Thus we see that $\varphi\cup\varphi$ is 2-torsion. The next lemma makes the amusing observation that for $k=1$ this "square" is in fact always zero.

Lemma 114.9. Let X be a topological space. For any $\varphi\in\mathrm{H}^1(X;\mathbb{Z})$ we have $\varphi\cup\varphi=0$.

Remark. Note that there is no higher-dimensional analogue of Lemma 114.9. For example on [**Hat2002**, p. 403] it is shown that given any odd k with $k\ge 3$ there exists a cohomology class $\gamma\in\mathrm{H}^k(\mathbb{R}\mathrm{P}^\infty\times\mathbb{R}\mathrm{P}^\infty;\mathbb{Z})$ with $\gamma^m\neq 0$ for all $m\in\mathbb{N}$.

Proof. Let X be a topological space and let $\varphi\in\mathrm{C}^1(X;\mathbb{Z})$ be a cocycle. We need to show that $\varphi\cup\varphi\in\mathrm{C}^2(X;\mathbb{Z})$ is a coboundary. We consider the 1-dimensional cochain

$$\begin{aligned}\mu\colon \mathrm{C}_1(X) &\to \mathbb{Z}\\ (\sigma\colon\Delta^1\to X) &\mapsto \tfrac{1}{2}(\underbrace{\varphi(\sigma)-\varphi(\sigma)^2}_{\in 2\mathbb{Z}}).\end{aligned}$$

that is determined by

It remains to prove the following claim.

Claim. We have $\varphi\cup\varphi=\delta\mu\colon \mathrm{C}_2(X)\to\mathbb{Z}$.

Proof. We only need to prove this equality for singular 2-simplices. So suppose we are given $\tau\colon\Delta^2\to X$. Given $j\in\{0,1,2\}$ we write $i_j=i_j^2\colon\Delta^1\to\Delta^2$. Note that

$$(*)\qquad \varphi(\tau\circ i_0)-\varphi(\tau\circ i_1)+\varphi(\tau\circ i_2) \;=\; \varphi(\partial\tau) \underset{\substack{\uparrow\\ \text{definition of }\delta\varphi}}{=} (\delta\varphi)(\tau) \underset{\substack{\uparrow\\ \text{since }\varphi\text{ is a cocycle}}}{=} 0.$$

Now we compute that

$$\begin{aligned}(\delta\mu)(\tau) &= \mu(\partial\tau) = \mu\big((\tau\circ i_0)-(\tau\circ i_1)+(\tau\circ i_0)\big)\\ &= \tfrac{1}{2}\big(\varphi(\tau\circ i_0)-\varphi(\tau\circ i_1)+\varphi(\tau\circ i_2)+\varphi(\tau\circ i_0)^2-\varphi(\tau\circ i_1)^2+\varphi(\tau\circ i_2)^2\big)\\ &= \tfrac{1}{2}\big(\varphi(\tau\circ i_0)^2-(\varphi(\tau\circ i_0)+\varphi(\tau\circ i_2))^2+\varphi(\tau\circ i_2)^2\big)\\ &\quad\uparrow\\ &\quad\text{by }(*)\text{ we have }\varphi(\tau\circ i_1)=\varphi(\tau\circ i_0)+\varphi(\tau\circ i_2)\\ &= \varphi(\tau\circ i_2)\cdot\varphi(\tau\circ i_0) = (\varphi\cup\varphi)(\tau).\\ &\quad\uparrow\qquad\qquad\uparrow\\ &\quad\text{elementary algebra}\qquad\text{definition of the cup-product}\end{aligned}$$

This concludes the proof of the claim and thus of the lemma. ■

We continue with the following lemma that shows that the cup product behaves well under maps between topological spaces.

Lemma 114.10. (Cup Product-Naturality Lemma) Let R be a commutative ring.

(1) Let $f\colon X\to Y$ be a map between topological spaces. For any $c\in \mathrm{H}^p(Y;R)$ and any $d\in \mathrm{H}^q(Y;R)$ we have
$$f^*(c)\cup f^*(d) = f^*(c\cup d).$$
In particular the map
$$f^*\colon \mathrm{H}^*(Y;R) \to \mathrm{H}^*(X;R)$$
is a homomorphism of graded rings[1394] which satisfies $f^*(1_Y)=1_X$.

(2) The maps
$$\begin{aligned} X &\mapsto (\mathrm{H}^*(X;R),\cup)\\ (f\colon X\to Y) &\mapsto (f^*\colon \mathrm{H}^*(Y;R)\to \mathrm{H}^*(X;R))\end{aligned}$$
define a contravariant functor from the category *Top* of topological spaces to the category *GrRing* of graded rings.

Proof. The lemma follows basically immediately from the definitions. But for completeness' sake we write down the proof. So let $c=[\varphi]\in \mathrm{H}^p(Y;R)$ and let $b=[\psi]\in \mathrm{H}^q(Y;R)$. Furthermore let $\sigma\colon \Delta^{p+q}\to X$ be a singular $(p+q)$-chain. Then

$$\begin{aligned}(f^*\varphi\cup f^*\psi)(\sigma) &= (f^*\varphi)(\sigma\circ[v_0,\dots,v_p])\cdot(f^*\psi)(\sigma\circ[v_p,\dots,v_{p+q}])\\ &= \varphi(f\circ(\sigma\circ[v_0,\dots,v_p]))\cdot\psi(f\circ(\sigma\circ[v_p,\dots,v_{p+q}]))\\ &= \varphi((f\circ\sigma)\circ[v_0,\dots,v_p))\cdot\psi((f\circ\sigma)\circ[v_p,\dots,v_{p+q}])\\ &= (\varphi\cup\psi)(f\circ\sigma) = (f^*(\varphi\cup\psi))(\sigma).\end{aligned}$$

Finally note that it follows immediately from the definitions of 1_X, 1_Y and f^* that we have the equality $f^*(1_Y)=1_X$. Statement (2) follows immediately from Statement (1) and the fact that cohomology groups with R-coefficients form a contravariant functor of abelian groups. ■

We can use the Cup Product-Naturality Lemma 114.10 to calculate the cup product of the wedge of finitely many spheres.

Lemma 114.11. For any wedge of finitely many spheres all cup products in degrees ≥ 1 are zero.

[1394] Recall that a *graded ring* is a ring R together with a decomposition $R=\bigoplus_{n\in\mathbb{N}_0} R_n$ such that for any $m,n\in\mathbb{N}_0$ the multiplication map restricts to a map $R_m\times R_n\to R_{m+n}$. A homomorphism $f\colon R\to S$ between graded rings is a ring homomorphism with $f(R_n)\subset S_n$ for all $n\in\mathbb{N}_0$

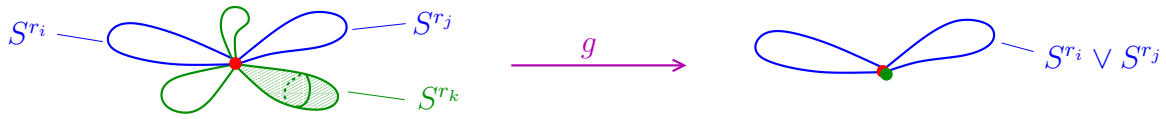
Proof. Let $S^{r_1} \vee \cdots \vee S^{r_k}$ be the wedge of finitely many spheres and let R be a commutative ring. For each $i \in \{1, \ldots, k\}$ we denote by $p_i \colon S^{r_1} \vee \cdots \vee S^{r_k} \to S^{r_i}$ the natural projection. By the Wedge-H*-Proposition 109.17 we know that for every $n \in \mathbb{N}$ the map

$$p_1^* \oplus \cdots \oplus p_k^* \colon \mathrm{H}^n(S^{r_1}; R) \oplus \cdots \oplus \mathrm{H}^n(S^{r_k}; R) \xrightarrow{\cong} \mathrm{H}^n(S^{r_1} \vee \cdots \vee S^{r_k}; R)$$

is an isomorphism. It remains to show that for any choice of $i, j \in \{1 \ldots, k\}$ with $r_i > 0$ and $r_j > 0$ we have $p_i^*([S^{r_i}]^*) \cup p_j^*([S^{r_j}]^*) = 0$. We prove the statement for $i \neq j$. The proof of the statement for $i = j$ is basically the same. So suppose that $i \neq j$. We denote by $g \colon S^{r_1} \vee \cdots \vee S^{r_k} \to S^{r_i} \vee S^{r_j}$ the natural projection and we denote by $q_i \colon S^{r_i} \vee S^{r_j} \to S^{r_i}$ and $q_j \colon S^{r_j} \vee S^{r_j} \to S^{r_j}$ the natural projections. Then

$$p_i^*([S^{r_i}]^*) \cup p_j^*([S^{r_j}]^*) \underset{\text{since } q_i \circ g = p_i \text{ and } q_j \circ g = p_j}{\overset{\uparrow}{=}} g^*(q_i^*([S^{r_i}]^*)) \cup g^*(q_j^*([S^{r_j}]^*)) \underset{\text{Cup Product-Naturality Lemma 114.10}}{\overset{\uparrow}{=}} g^*(\overbrace{q_i^*([S^{r_i}]^*) \cup q_j^*([S^{r_j}]^*)}^{\in \mathrm{H}^{r_i + r_j}(S^{r_i} \vee S^{r_j}) = 0}) = 0.$$

We have thus proved that all cup products in degrees ≥ 1 are zero. ∎



Later on, after we will have found methods for computing the cup product, we will often use the following corollary to show that certain topological spaces are not homotopy equivalent.

Corollary 114.12. Let X and Y be topological spaces. If X and Y are homotopy equivalent, then for any commutative ring R there exists an isomorphism

$$(\mathrm{H}^*(Y; R), \cup) \xrightarrow{\cong} (\mathrm{H}^*(X; R), \cup)$$

of graded rings.

Proof. Let $f \colon X \to Y$ be a homotopy equivalence. In the Homotopic Maps-H*-Lemma 108.13 we saw that the induced maps $f^* \colon \mathrm{H}^n(Y; R) \to \mathrm{H}^n(X; R)$ on cohomology *groups* are isomorphisms. But it follows from the Cup Product-Naturality Lemma 114.10 that these isomorphisms preserve the ring structure, i.e. the map f^* is in fact an isomorphism of graded rings. ∎

114.3. **The cup product of the torus.** Evidently our goal now is to determine the cup product of some of our favorite different topological spaces. We start out with two elementary observations regarding the cup product.

Observation 114.13. Let R be a commutative ring.

(1) Let X be a path-connected topological space. It follows from the H^0-Proposition 108.11 that $\mathrm{H}^0(X; R) = R \cdot 1_X$. For any $r \cdot 1_X \in \mathrm{H}^0(X; R)$ and $\varphi \in \mathrm{H}^k(X; R)$ we have

$$(r \cdot 1_X) \cup \varphi \underset{\text{since } \cup \text{ is } R\text{-bilinear}}{\overset{\uparrow}{=}} r \cdot (1_X \cup \varphi) \underset{\text{Cup Product-Ring Proposition 114.7}}{\overset{\uparrow}{=}} r \cdot \varphi \underset{\text{same argument backwards}}{\overset{\uparrow}{=}} \varphi \cup (r \cdot 1_X).$$

(2) If M is an m-dimensional topological manifold, then for any p, q with $p+q > m$ the cup product $\mathrm{H}^p(M;R) \times \mathrm{H}^q(M;R)$ is zero, since it takes values in the group $\mathrm{H}^{p+q}(M;R)$, which is zero by the Top-Homology Theorem 106.3.
(3) By the Cup Product-Ring Proposition 114.7 the cup product is R-bilinear, so it suffices to determine the cup product on a generating system.

In this section we want to determine the cup product on the torus $T = S^1 \times S^1$. By Observation 114.13 it remains to determine the cup product

$$\cup\colon \mathrm{H}^1(T;\mathbb{Z}) \times \mathrm{H}^1(T;\mathbb{Z}) \ \to\ \mathrm{H}^2(T;\mathbb{Z}).$$

We first need to understand the cohomology groups $\mathrm{H}^1(T;\mathbb{Z})$ and $\mathrm{H}^2(T;\mathbb{Z})$ explicitly. We recall some notation and results from before:

(1) We denote by p respectively q the projection $S^1 \times S^1 \to S^1$ onto the first respectively second factor. Furthermore, we recall that on page 2404 we introduced a cochain $\theta := \theta_{\mathbb{Z}} \in \mathrm{C}^1;\mathbb{Z})$, we showed in Lemma 108.10 that θ is a cocycle and we showed on page 2426 that θ represents a generator of $\mathrm{H}^1(S^1;\mathbb{Z})$.
(2) In Lemma 109.9 we showed that $\mathrm{H}^1(T;\mathbb{Z}) \cong \mathbb{Z}^2$ and that the two cohomology classes $\alpha := p^*([\theta])$ and $\beta := q^*([\theta])$ form a basis for $\mathrm{H}^1(S^1 \times S^1;\mathbb{Z})$.
(3) On page 2204 we showed that the cycle $-\sigma_1 + \sigma_2$, with the notation from the figure on page 2204, represents the fundamental class of T.
(4) In Lemma 109.9 and 110.15 we showed that

$$\begin{array}{rcl} \mathrm{H}^2(T;\mathbb{Z}) & \to & \mathbb{Z} \\ [\varphi] & \mapsto & \langle \varphi, [T] \rangle = \varphi(-\sigma_1 + \sigma_2) \end{array}$$

is in fact an isomorphism. We will use this isomorphism to make the identification $\mathrm{H}^2(T;\mathbb{Z}) = \mathbb{Z}$.

By Oberservation 114.13 (3) and the above discussion it suffices to determine all entries in the following matrix

$$\begin{pmatrix} \langle \alpha \cup \alpha, [T] \rangle & \langle \alpha \cup \beta, [T] \rangle \\ \langle \beta \cup \alpha, [T] \rangle & \langle \beta \cup \beta, [T] \rangle \end{pmatrix}.$$

Now we determine these four entries separately.

(1) We first consider the entry $\langle \alpha \cup \beta, [T] \rangle$. Fortunately we already did all the work on page 2515, more precisely, we have

$$\langle \alpha \cup \beta, [T] \rangle = (p^*\theta \cup q^*\theta)(-\sigma_1 + \sigma_2) = (p^*\theta \cup q^*\theta)(-\sigma_1) + (p^*\theta \cup q^*\theta)(\sigma_2) \underset{\uparrow}{=} 0 + 1 = 1.$$

see calculations on page 2515

(2) We could do the same argument as in (1) with α and β reversed. But it is easier to apply the Cup Product-Commutativity Proposition 114.8 which gives us that $\beta \cup \alpha = (-1)^{1 \cdot 1} \cdot \alpha \cup \beta = -\alpha \cup \beta = -1$.
(3) There are many different ways for showing that $\alpha \cup \alpha = 0$:
  (a) We could do the calculation as in (1), but that is the least efficient approach.
  (b) We could use the symmetry from the Cup Product-Commutativity Proposition 114.8 to argue that $\alpha \cup \alpha = -\alpha \cup \alpha$, and hence $\alpha \cup \alpha = 0$ since $\mathrm{H}^2(T;\mathbb{Z})$ is torsion-free.
  (c) We could just use Lemma 114.9 and remove all the fun.

(d) We can also argue that

$$\langle \alpha \cup \alpha, [T] \rangle \;=\; \langle [p^*\theta] \cup [p^*\theta], [T] \rangle \;\underset{\uparrow}{=}\; \langle p^*([\theta] \cup [\theta]), [T] \rangle \;\underset{\uparrow}{=}\; 0.$$

Cup Product-Naturality Lemma 114.10 since $[\theta] \cup [\theta] \in \mathrm{H}^2(S^1;\mathbb{Z}) = 0$

(4) The same way as in (3) we see that $\beta \cup \beta = 0$.

Summarizing we have proved the following lemma.

Lemma 114.14. Let T be the 2-dimensional torus. We use the above notation. The cup product $\mathrm{H}^1(T;\mathbb{Z}) \times \mathrm{H}^1(T;\mathbb{Z}) \to \mathrm{H}^2(T;\mathbb{Z})$ with respect to the basis α, β of $\mathrm{H}^1(T;\mathbb{Z})$ and the identification $\mathrm{H}^2(T;\mathbb{Z}) = \mathbb{Z} \cdot [T]^* = \mathbb{Z}$[1395]is given by the matrix

$$\begin{pmatrix} \alpha \cup \alpha & \alpha \cup \beta \\ \beta \cup \alpha & \beta \cup \beta \end{pmatrix} \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus we have shown that the cup product on cohomology is in general non-zero and that it is in general not commutative.

Remark.

(1) We have just shown that the 2-cocycle $p^*(\theta) \cup q^*(\theta) \in \mathrm{C}^2(T;\mathbb{Z})$ represents a generator of $\mathrm{H}^2(T;\mathbb{Z})$. We have thus answered Question 109.10.

(2) We have now performed enough calculations to see that the cohomology rings contain more information than the homology or the cohomology groups. More precisely, it follows from Lemma 78.11 and Propositions 74.4 and 79.8 that for the 2-dimensional torus $S^1 \times S^1$ and for $S^1 \vee S^1 \vee S^2$ we have

$$\mathrm{H}_i(S^1 \times S^1;\mathbb{Z}) \;\cong\; \mathrm{H}_i(S^1 \vee S^1 \vee S^2;\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2, \\ \mathbb{Z}^2, & \text{if } i = 1, \\ 0, & \text{if } i \geq 3. \end{cases}$$

We deduce from the Universal Coefficient Theorem 110.12 for Cohomology Groups that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ also have isomorphic cohomology groups. In Lemma 114.14 we saw that the cup product on $\mathrm{H}^1(S^1 \times S^1;\mathbb{Z})$ is non-zero, whereas we showed in Lemma 114.11 that the cup product on $\mathrm{H}^1(S^1 \vee S^1 \vee S^2;\mathbb{Z})$ is zero.[1396]

We conclude this section with the following slightly subtle lemma.

Lemma 114.15. Let $f\colon T \to T$ be a homeomorphism of the torus $T = S^1 \times S^1$. If f is orientation-preserving, then

$$\det\big(f_*\colon \mathrm{H}_1(T;\mathbb{Z}) \to \mathrm{H}_1(T;\mathbb{Z})\big) \;=\; \det\big(f^*\colon \mathrm{H}^1(T;\mathbb{Z}) \to \mathrm{H}^1(T;\mathbb{Z})\big) \;=\; +1.$$

Conversely, if f is orientation-reversing, then both determinants equal -1.

Proof. In Exercise 114.2 we will use the above calculation of the cup product of the torus to prove the lemma. ■

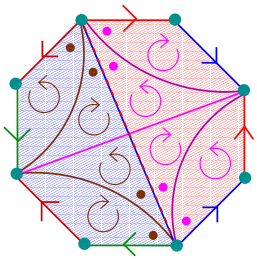
[1395]Here $[T]^* \in \mathrm{H}^2(T;\mathbb{Z})$ denotes the dual fundamental class that we introduced on page 2451, it is uniquely determined by $\langle [T]^*, [T] \rangle = 1$.

[1396]Of course we do not need to use cup products to distinguish the torus $S^1 \times S^1$ and the wedge $S^1 \vee S^1 \vee S^2$, this can already be done using fundamental groups. Nonetheless it is encouraging to see that we found a new way to prove this statement.

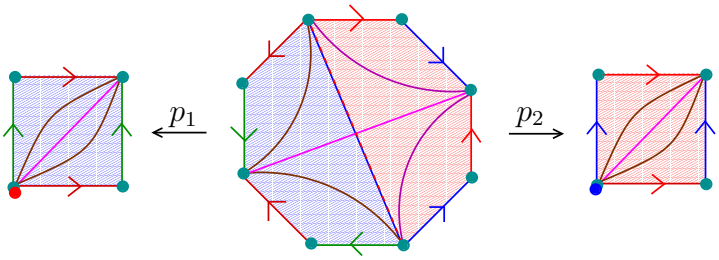
114.4. **The cup product of the surface of genus 2.** In this section we calculate the cup product for the surface Σ of genus 2. By Observation 114.13 it remains to determine the cup product

$$\mathrm{H}^1(\Sigma;\mathbb{Z}) \times \mathrm{H}^1(\Sigma;\mathbb{Z}) \ \to \ \mathrm{H}^2(\Sigma;\mathbb{Z})$$

on a basis of $\mathrm{H}^1(\Sigma;\mathbb{Z})$. We recall that on page 2428 we introduced for $i = 1, 2$ a map $p_i \colon \Sigma \to T = S^1 \times S^1$. We refer to the figure below on the right for a reminder of the definition of the maps. We denote by $\alpha, \beta \in \mathrm{H}^1(T;\mathbb{Z})$ the basis from Lemma 109.9 that



fundamental class of Σ



projections from Σ to the torus

we had also used in the previous section.
In Lemma 109.11 we saw that the map

$$p_1^* \oplus p_2^* \colon \mathrm{H}^1(T;\mathbb{Z}) \oplus \mathrm{H}^1(T;\mathbb{Z}) \ \to \ \mathrm{H}^1(\Sigma;\mathbb{Z})$$

is an isomorphism. We deduce that a basis for $\mathrm{H}^1(\Sigma;\mathbb{Z}) \cong \mathbb{Z}^4$ is given by $\alpha_i := p_i^*(\alpha)$, $\beta_i := p_i^*(\beta)$, $i = 1, 2$. Now we can use the definition of the cup product and the explicit description of a representative of the fundamental class $[\Sigma] \in \mathrm{H}_2(\Sigma;\mathbb{Z})$ in the figure above on the left (see also the figure on page 2204 for a more detailed discussion) to compute the cup product $\mathrm{H}^1(\Sigma;\mathbb{Z}) \times \mathrm{H}^1(\Sigma;\mathbb{Z}) \to \mathrm{H}^2(\Sigma;\mathbb{Z}) = \mathbb{Z}$.
A not particularly exciting calculation leads us eventually to the following lemma.[1397]

Lemma 114.16. Let Σ be the surface of genus 2. We use the above notation. The cup product $\mathrm{H}^1(\Sigma;\mathbb{Z}) \times \mathrm{H}^1(\Sigma;\mathbb{Z}) \to \mathrm{H}^2(\Sigma;\mathbb{Z})$ with respect to the basis $\alpha_1, \beta_1, \alpha_2, \beta_2$ of $\mathrm{H}^1(\Sigma;\mathbb{Z})$ and the identification $\mathrm{H}^2(\Sigma;\mathbb{Z}) = \mathbb{Z} \cdot [\Sigma]^* = \mathbb{Z}$ is given by the matrix

$$\begin{pmatrix} \alpha_1 \cup \alpha_1 & \alpha_1 \cup \beta_1 & \alpha_1 \cup \alpha_2 & \alpha_1 \cup \beta_2 \\ \beta_1 \cup \alpha_1 & \beta_1 \cup \beta_1 & \beta_1 \cup \alpha_2 & \beta_1 \cup \beta_2 \\ \alpha_2 \cup \alpha_1 & \alpha_2 \cup \beta_1 & \alpha_2 \cup \alpha_2 & \alpha_2 \cup \beta_2 \\ \beta_2 \cup \alpha_1 & \beta_2 \cup \beta_1 & \beta_2 \cup \alpha_2 & \beta_2 \cup \beta_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

An analogous statement holds for any surface of genus $g \geq 1$, i.e. the cup product on the first cohomology of the surface of genus $g \geq 1$ can be represented by a block diagonal matrix consisting of g copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Remark. In the next section we will obtain a more conceptual proof of Lemma 114.16.

Before we continue with more calculations of cup products we return to a question that we had asked earlier on.

[1397] We have to evaluate the cup products of the explicit 1-cochains on the eight singular 2-simplices illustrated in the figure above. It is straightforward to see that all 16 cup products evaluate to zero on the four central singular 2-simplices. For the remaining cup products the calculation is almost the same as for the torus on page 2515.

Question 101.12.
(2) Let $g \geq 2$ and let $k \geq 2$. Does there exist a self-map $f\colon \Sigma_g \to \Sigma_g$ of degree k?[1398]

Now we have the tools to answer this question. In this context it is useful to introduce the following definition.

Definition. A closed oriented connected non-empty topological manifold M is called flexible if given any $k \in \mathbb{Z}$ there exists a map $f\colon M \to M$ of degree k.

Example. In Lemma 76.10 we saw that S^1 is flexible. The combination of this fact with the Degree-Suspension Lemma 76.12 shows that in fact any sphere S^n, $n \geq 1$ is flexible.

Now we can determine which closed oriented connected 2-dimensional smooth manifolds are flexible. More precisely, we have the following proposition which gives a negative answer to Question 101.12 (2).

Proposition 114.17. Let $g \geq 1$. We denote by Σ_g the surface of genus g.
(1) The torus $\Sigma_1 = S^1 \times S^1$ is flexible.
(2) For $g \geq 2$ the surface Σ_g is not flexible. In fact if $k \neq -1, 0, 1$, then there is no map $\Sigma_g \to \Sigma_g$ of degree k.

Proof.

(1) We first show that the torus $S^1 \times S^1$ is flexible. So let $k \in \mathbb{Z}$. We consider the map $f\colon S^1 \to S^1$ that is given by $z \mapsto z^k$. We claim that $f \times \mathrm{id}\colon S^1 \times S^1 \to S^1 \times S^1$ has degree k. In fact by the naturality of the short exact sequence of the Künneth-Theorem 90.8 we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_1(S^1) \otimes \mathrm{H}_1(S^1) & \xrightarrow[\cong]{\Theta} & \mathrm{H}_2(S^1 \times S^1) \\ \downarrow{\scriptstyle f_* \otimes \mathrm{id}} & & \downarrow{\scriptstyle (f \times \mathrm{id})_*} \\ \mathrm{H}_1(S^1) \otimes \mathrm{H}_1(S^1) & \xrightarrow[\cong]{\Theta} & \mathrm{H}_2(S^1 \times S^1) \end{array}$$

where the horizontal maps are isomorphisms. By Lemma 76.10 the map on the left is given by multiplication by k. Hence the map on the right is also given by multiplication by k.

(2) Now let $g \geq 2$ and let $f\colon \Sigma_g \to \Sigma_g$ be a map of degree $k \neq 0$. We have to show that $k \in \{\pm 1\}$. We write $\pi := \pi_1(\Sigma_g)$. We start out with the following claim.

Claim. The map $f_*\colon \pi \to \pi$ is an epimorphism.

Proof. First note that the argument in the proof of the Degree One-Epimorphism Proposition 101.8 shows that $f_*(\pi)$ is a finite-index subgroup of π. (Here we used that $k \neq 0$.) We denote by $n \in \mathbb{N}$ the index of the subgroup $f_*(\pi) \subset \pi$. In the proof of Proposition 64.17 we used the multiplicativity of the Euler characteristic under finite covers to show that the abelianization of $f_*(\pi)$ is isomorphic to $\mathbb{Z}^{2n\cdot(g-1)+2}$. On the other hand the epimorphism $\pi \to f_*(\pi)$ induces evidently an epimorphism of

[1398] Recall that if $f\colon M \to N$ is a map between two closed, oriented connected, non-empty n-dimensional topological manifolds, then the degree $\deg(f) \in \mathbb{Z}$ is defined as the unique integer $\deg(f)$ that satisfies $f_*([M]) = \deg(f) \cdot [N]$.

the abelianization of π onto the abelianization of $f_*(\pi)$, i.e. we have an epimorphism from $\mathbb{Z}^{2g}$ onto $\mathbb{Z}^{2n\cdot(g-1)+2}$. Since $g \geq 2$ this is only possible if $n = 1$. ⊞

Claim. The map $f^*\colon \mathrm{H}^1(\Sigma_g;\mathbb{Z}) \to \mathrm{H}^1(\Sigma_g;\mathbb{Z})$ is an isomorphism.

Proof. By the first claim we know that the map $f_*\colon \pi \to \pi$ is an epimorphism. It follows from the naturality of the Hurewicz Isomorphism, see Propositions 84.3 and 84.2 (5), that the map $f_*\colon \mathrm{H}_1(\Sigma_g) \to \mathrm{H}_1(\Sigma_g)$ is also an epimorphism. But any epimorphism from a finitely generated abelian group to itself is in fact an isomorphism[1399], so we see that $f_*\colon \mathrm{H}_1(\Sigma_g;\mathbb{Z}) \to \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ is in fact an isomorphism. By the First Cohomology Proposition 110.17 the evaluation homomorphism $\mathrm{ev}\colon \mathrm{H}^1(\Sigma_g;\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_1(\Sigma_g;\mathbb{Z}),\mathbb{Z})$ is a natural isomorphism, i.e. we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1(\Sigma_g;\mathbb{Z}) & \xrightarrow[\cong]{\mathrm{ev}} & \mathrm{Hom}(\mathrm{H}_1(\Sigma_g;\mathbb{Z}),\mathbb{Z}) \\ \big\downarrow{\scriptstyle f^*} & & \cong\big\downarrow{\scriptstyle (f_*)^*} \\ \mathrm{H}^1(\Sigma_g;\mathbb{Z}) & \xrightarrow[\cong]{\mathrm{ev}} & \mathrm{Hom}(\mathrm{H}_1(\Sigma_g;\mathbb{Z}),\mathbb{Z}). \end{array}$$

The right-hand map is an isomorphism, it follows from the commutative diagram that the left-hand map $f^*\colon \mathrm{H}^1(\Sigma_g;\mathbb{Z}) \to \mathrm{H}^1(\Sigma_g;\mathbb{Z})$ is also an isomorphism. ⊞

By Lemma 114.16 we know that there exist $a, b \in \mathrm{H}^1(\Sigma_g;\mathbb{Z})$ with $a \cup b = [\Sigma_g]^*$. Since $f^*\colon \mathrm{H}^1(\Sigma_g;\mathbb{Z}) \to \mathrm{H}^1(\Sigma_g;\mathbb{Z})$ is an isomorphism we can in particular find cohomology classes $c, d \in \mathrm{H}^1(\Sigma_g;\mathbb{Z})$ with $f^*(c) = a$ and $f^*(d) = b$. Now we have

$$[\Sigma_g]^* = a \cup b = f^*(c) \cup f^*(d) \underset{\uparrow}{=} f^*(c \cup d) \overset{\downarrow}{=} f^*(m \cdot [\Sigma_g]^*) \underset{\uparrow}{=} \deg(f) \cdot m \cdot [\Sigma_g]^*.$$

since $c \cup d \in \mathrm{H}^2(\Sigma_g;\mathbb{Z})$ and since $[\Sigma_g]^* \in \mathrm{H}^2(\Sigma_g;\mathbb{Z})$ is a generator such an m exists

Cup Product-Naturality Lemma 114.10 — by Lemma 110.16 we have $f^*([\Sigma_g]^*) = \deg(f) \cdot [\Sigma_g]^*$

Evidently this is only possible if $\deg(f) \in \{\pm 1\}$. ■

114.5. **The cup product of the real projective plane (∗).** Now we want to understand the cup product in the $\mathbb{F}_2$-cohomology of the real projective plane $\mathbb{R}\mathrm{P}^2$. Recall that on page 2423 we saw that $\mathrm{H}^i(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) = \mathbb{F}_2$[1400] for $i = 0, 1, 2$ and that the $\mathbb{F}_2$-cohomology vanishes in all other dimensions. By Observation 114.13 it remains to determine the cup product on the first cohomology.

To compute the cup product we need to get a good understanding of the cohomology groups. Recall that on page 2216 we had seen that $(\sigma_1+\sigma_2+\tau_1+\tau_2)\otimes 1 \in \mathrm{C}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ is a representative of the $\mathbb{F}_2$-fundamental class $[\mathbb{R}\mathrm{P}^2]$ of $\mathbb{R}\mathrm{P}^2 = \overline{B}^2/\!\sim$.[1401] (See the figure below for the definition of $\sigma_1, \sigma_2, \tau_1$ and τ_2.) It follows from the Cohomology-Field Coefficients Proposition 110.18 and the above that the map

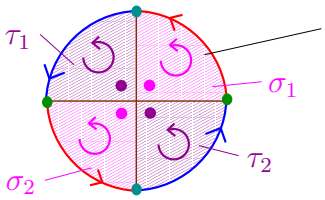
$$\begin{array}{rcl} \eta\colon \mathrm{H}^2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) & \to & \mathbb{F}_2 \\ [\varphi \in \mathrm{C}^2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)] & \mapsto & [\varphi(\sigma_1)+\varphi(\sigma_2)+\varphi(\tau_1)+\varphi(\tau_2)] \end{array}$$

[1399] Why is that?

[1400] Why is it safe to write $\mathrm{H}^i(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) = \mathbb{F}_2$? Would it not be much more appropriate to write $\mathrm{H}^i(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \cong \mathbb{F}_2$?

[1401] Here we use the identification $\mathbb{R}\mathrm{P}^2 = \overline{B}^2/\!\sim$ from page 258.

is an isomorphism.



projective plane $\mathbb{R}\mathrm{P}^2 = \overline{B}^2/\sim$

the 2-cycle $(\sigma_1 + \sigma_2 + \tau_1 + \tau_2) \otimes 1 \in \mathrm{C}_2(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$ represents the $\mathbb{F}_2$-fundamental class of the real projective plane

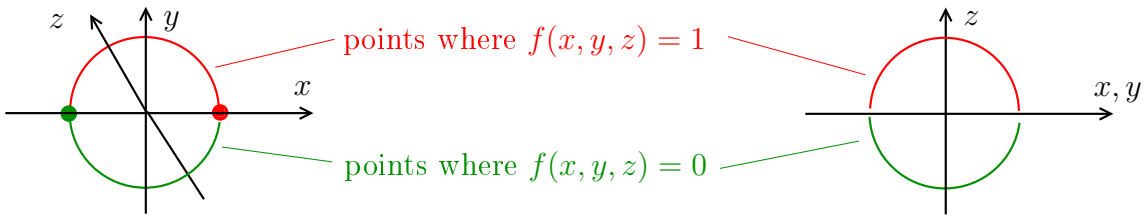
Now we also need to find an explicit representative of $x \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$. We will find such a representative in a similar fashion to the construction of the singular 1-cocycle $\theta_{\mathbb{Z}} \in \mathrm{H}^1(S^1; \mathbb{Z})$ on page 2404. More precisely, we consider the map

$$\begin{array}{rcl} f\colon S^2 & \to & \mathbb{F}_2 \\ (x,y,z) & \mapsto & \begin{cases} 1, & \text{if } (z>0) \text{ or } (z=0 \text{ and } y>0) \text{ or } (z=y=0 \text{ and } x=1), \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

Note that for any point $P \in S^2$ we have $f(-P) = 1 - f(P)$.[1402] For a singular 1-simplex $\sigma\colon \Delta^1 \to \mathbb{R}\mathrm{P}^2$ we pick a lift $\widetilde{\sigma}\colon \Delta^1 \to S^2$ of σ and we define

$$\varphi(\sigma) \;:=\; f\big(\widetilde{\sigma}(0,1)\big) - f\big(\widetilde{\sigma}(1,0)\big) \;\in\; \mathbb{F}_2.$$

We refer to the figure below for an illustration.



Note that φ is well-defined since the only other lift of σ is given by multiplying $\widetilde{\sigma}$ by -1,[1403] but then it follows from $f(-P) = 1 - f(P)$ that the difference of the evaluations at the endpoints does not change. (Here we use that we work with $\mathbb{F}_2$-coefficients instead of $\mathbb{Z}$-coefficients.) The argument of Lemma 108.10 can easily be adapted to show that the singular 1-cochain $\varphi\colon \mathrm{C}_1(\mathbb{R}\mathrm{P}^2) \to \mathbb{F}_2$ is a cocycle.

Example. We consider the map $\mu\colon \Delta^1 \to \mathbb{R}\mathrm{P}^2 = S^2/\sim$ given by $\mu(1-t,t) = [\exp(\pi \mathrm{i} t)]$. Then $\mu \in \mathrm{C}_1(\mathbb{R}\mathrm{P}^2)$. Thus we can apply the Kronecker pairing $\mathrm{H}^1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2) \times \mathrm{H}_1(\mathbb{R}\mathrm{P}^2) \to \mathbb{F}_2$ from Lemma 109.6 to $[\varphi] \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$ and $[\mu] \in \mathrm{H}_1(\mathbb{R}\mathrm{P}^2)$ and we calculate that

$$\langle [\varphi], [\mu] \rangle \;=\; \varphi(\mu) \;\underset{\uparrow}{=}\; f(\exp(\pi \mathrm{i})) - f(\exp(0)) \;=\; 1 \in \mathbb{F}_2.$$

since a lift of $\mu(1-t,t) = [\exp(\pi \mathrm{i} t)]$ to S^2 is given by $\widetilde{\mu}(1-t,t) = \exp(\pi \mathrm{i} t)$

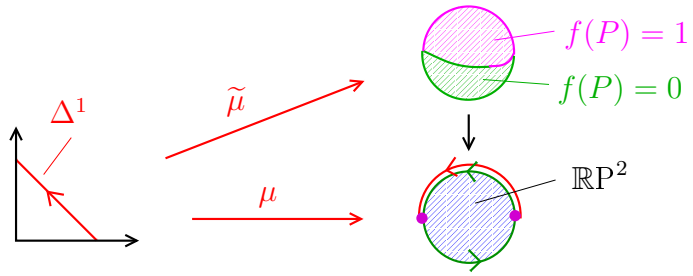
Since this is non-zero we have now actually shown two statements at once: the cohomology class $[\varphi] \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$ is non-zero and we have also certified that the homology class $[\mu] \in \mathrm{H}_1(\mathbb{R}\mathrm{P}^2)$ is non-zero.[1404]

Now we have the tools to prove the following lemma.

[1402] In fact for the subsequent discussion any map $f\colon S^2 \to \mathbb{F}_2$ with this property would work. We just picked this example for concreteness.

[1403] Why is that true?

[1404] Do you know any other methods for showing that $[\mu] \neq 0 \in \mathrm{H}_1(\mathbb{R}\mathrm{P}^2)$?



Lemma 114.18. Under the identifications $\mathrm{H}^i(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) = \mathbb{F}_2$ the cup product

$$\begin{array}{rcl} \cup\colon \mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \times \mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) & \to & \mathrm{H}^2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \\ \mathbb{F}_2 \times \mathbb{F}_2 & \to & \mathbb{F}_2 \\ (a,b) & \mapsto & a \cdot b. \end{array}$$

is given by

Remark. Note that the calculation in Lemma 114.18 that the statement of Lemma 114.9 does not hold for $\mathbb{F}_2$-coefficients.

Proof. By the discussion above it suffices to show that $\eta([\varphi] \cup [\varphi]) \neq 0$. Put differently, by definition of η we need to show that

$$(\varphi \cup \varphi)(\sigma_1) + (\varphi \cup \varphi)(\sigma_2) + (\varphi \cup \varphi)(\tau_1) + (\varphi \cup \varphi)(\tau_2) \;=\; 1 \;\in \mathbb{F}_2.$$

In Exercise 114.4 we will determine all four summands and we will see that the sum is indeed 1. ■

The above calculation of the cup product of the real projective plane $\mathbb{R}\mathrm{P}^2$ is very explicit and hands on, but not very conceptual. Later on, in the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 121.16, we will give a very different calculation of the cup product on $\mathbb{F}_2$-cohomology of all real projective spaces $\mathbb{R}\mathrm{P}^n$.

Exercises for Chapter 114.

Exercise 114.1. Let X be a topological space. We consider the map

$$\Phi\colon \mathrm{H}_*(X) \otimes \mathrm{H}_*(X) \xrightarrow{\times} \mathrm{H}_*(X \times X) \xrightarrow{p_*} \mathrm{H}_*(X)$$

where $p\colon X \times X \to X$ denotes the projection on the first factor. Show that Φ is the trivial map.

Exercise 114.2. Let $f\colon T \to T$ be a homeomorphism of the torus $T = S^1 \times S^1$. Show that if f orientation-preserving, then the following two statements hold:

(a) $f_*\colon \mathrm{H}_1(T;\mathbb{Z}) \to \mathrm{H}_1(T;\mathbb{Z})$ has determinant one,
(b) $f^*\colon \mathrm{H}^1(T;\mathbb{Z}) \to \mathrm{H}^1(T;\mathbb{Z})$ has determinant one.

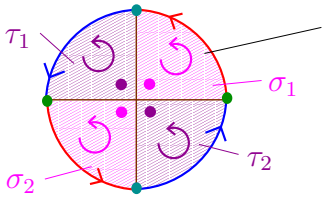
Exercise 114.3.

(a) Use cup products to show that there is no map $f\colon S^1 \times S^1 \to S^2$ of degree one, i.e. there is no map that induces an isomorphism on second homology.
(b) Given $g \in \mathbb{N}_0$ we denote by Σ_g the surface of genus g. Let $g < k$. Use cup product to show that there is no degree-one map from Σ_g to Σ_k.

Remark. This exercise shows that cup products can also be used to answer Question 101.5. Note that in Corollary 101.9 we used the fundamental class to prove the above statements.

Exercise 114.4. We consider $\mathbb{R}\mathrm{P}^2 = S^2/\!\sim$ which we identify with $\overline{B}^2/\!\sim$ via the homeomorphism that is induced by the projection $S^2 \to \overline{B}^2$. Let $\varphi \in \mathrm{C}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ be the cocycle that we defined on page 2530. Determine *each of the four summands* in the sum

$$(\varphi \cup \varphi)(\sigma_1) + (\varphi \cup \varphi)(\sigma_2) + (\varphi \cup \varphi)(\tau_1) + (\varphi \cup \varphi)(\tau_2).$$



projective plane $\mathbb{R}\mathrm{P}^2 = \overline{B}^2/\!\sim$

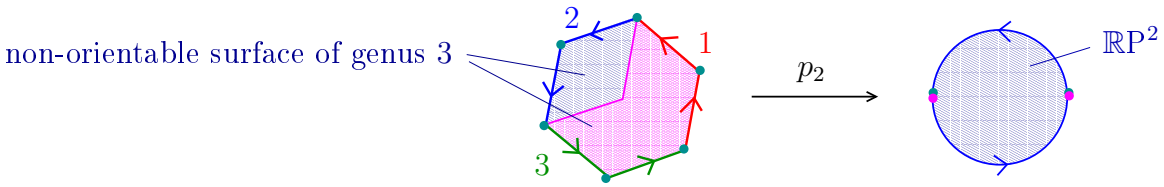
the 2-cycle $(\sigma_1 + \sigma_2 + \tau_1 + \tau_2) \otimes 1 \in \mathrm{C}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ represents the $\mathbb{F}_2$-fundamental class of the real projective plane

Exercise 114.5. Let K be the Klein bottle.

(a) Is the cup product $\mathrm{H}^1(K;\mathbb{Z}) \times \mathrm{H}^1(K;\mathbb{Z}) \to \mathrm{H}^2(K;\mathbb{Z})$ is trivial?
(b) Show that the cup product $\mathrm{H}^1(K;\mathbb{F}_2) \times \mathrm{H}^1(K;\mathbb{F}_2) \to \mathrm{H}^2(K;\mathbb{F}_2)$ is non-trivial.

Exercise 114.6. Let $k \in \mathbb{N}$. We denote by N_k the non-orientable surface of genus k as defined on page 266. For $i = 1, \dots, k$ we denote by $p_i \colon N_k \to \mathbb{R}\mathrm{P}^2$ the map that is illustrated in the figure below.

(a) Show that the map $p_1^* \oplus \cdots \oplus p_k^* \colon \mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \oplus \cdots \oplus \mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \to \mathrm{H}^1(N_k;\mathbb{F}_2)$ is an isomorphism.
Hint. You could use cellular cohomology.
(b) Let $x \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ be the unique non-trivial element. For $i = 1, \dots, k$ we write $y_i = p_i^*(x)$. Furthermore we denote by $[N_k]^* \in \mathrm{H}^2(N_k;\mathbb{F}_2)$ the unique non-trivial element. Show that for any $i, j \in \{1, \dots, k\}$ we have $y_i \cup y_j = \delta_{ij} \cdot [N_k]^*$.
(c) Let $g < k$. Use cup product to show that there is no map $f \colon N_g \to N_k$ that induces an isomorphism $f_* \colon \mathrm{H}_2(N_g;\mathbb{F}_2) \to \mathrm{H}_2(N_k;\mathbb{F}_2)$.
Remark. This statement is the non-orientable analogue of Exercise 114.3. Note that we had proved this statement in Exercise 101.8 using an alternative approach. Both are fun!



Exercise 114.7. Given a matrix $A \in \mathrm{M}(n \times n, \mathbb{F}_2)$ we define the bilinear form

$$\begin{array}{rccc} \lambda(A) \colon & \mathbb{F}_2^n \times \mathbb{F}_2^n & \to & \mathbb{F}_2 \\ & (v, w) & \mapsto & v^T A w. \end{array}$$

We consider the two matrices $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ over $\mathbb{F}_2$. Next we consider the torus $T = S^1 \times S^1$ and the Klein bottle $K = N_2$. By a slight variation on Lemma 114.14 and by Exercise 114.6 we know that the cup products $\mathrm{H}^1(T;\mathbb{F}_2) \times \mathrm{H}^1(T;\mathbb{F}_2) \to \mathrm{H}^2(T;\mathbb{F}_2)$ and $\mathrm{H}^1(K;\mathbb{F}_2) \times \mathrm{H}^1(K;\mathbb{F}_2) \to \mathrm{H}^2(K;\mathbb{F}_2)$ are isometric to $\lambda(P)$ respectively $\lambda(Q)$.

(a) Show that the forms $\lambda(P)$ and $\lambda(Q)$ are not isometric.

(b) Show that there is no map $f\colon T \to K$ such that the map $f_*\colon \mathrm{H}_2(T;\mathbb{F}_2) \to \mathrm{H}_2(K;\mathbb{F}_2)$ is an isomorphism.

Remark. This approach gives an alternative, and arguably more systematic solution, to Exercise 114.7.

115. The relative cup product

Our goal in this chapter is to introduce a relative version of the cup product. More precisely, given an excisive triad (X, A, B) and a commutative ring R we will introduce a cup product

$$\mathrm{H}^p(X, A; R) \times \mathrm{H}^q(X, B; R) \xrightarrow{\quad\cup\quad} \mathrm{H}^{p+q}(X, A \cup B; R)$$

that is functorial in an appropriate sense and that equals the previous definition of the cup product for the special case that $A = B = \varnothing$. This relative cup product will help us in determining the cup products of wedges of topological spaces and of the suspension of a topological space. Also it will help us to determine the mysterious sounding Lusternik-Schnirelmann category.

115.1. **Definition of the relative cup product.** In this section we will work a lot with relative cochains. Before we do so we recall the following convention from page 2407.

Convention.

(1) Given an abelian group C, a subgroup D and an abelian group G we have a natural isomorphism

$$\begin{array}{rcl} \mathrm{Hom}(C/D, G) & \xrightarrow{\cong} & \{f\colon C \to G \,|\, f \text{ vanishes on } D\} \\ (\varphi\colon C/D \to G) & \mapsto & (C \to C/D \xrightarrow{\varphi} G). \end{array}$$

We will use this isomorphism to identify the groups left and right.

(2) In particular, given a pair of topological spaces (X, A), a commutative ring R and $n \in \mathbb{N}_0$ we make the identification

$$\mathrm{C}^n(X, A; R) \xrightarrow{\cong} \{f\colon \mathrm{C}_n(X) \to R \,|\, f(\sigma) = 0 \text{ for all } \sigma\colon \Delta^n \to A\}.$$

The definition of the relative cup product requires some preparation. We start out with the following definition.

Definition. Let (X, A, B) be a triad of topological spaces. As in the definition on page 2429 we write

$$\mathrm{C}_n^{\{A,B\}}(A \cup B) := \left\{ \sum_{j=1}^{k} n_j \sigma_j \in \mathrm{C}_n(A \cup B) \;\middle|\; \begin{array}{c} \text{for each } j \text{ the image of } \sigma_j \\ \text{lies in } A \text{ or it lies in } B \end{array} \right\}$$

The usual boundary map on the groups $\mathrm{C}_*(A \cup B)$ restricts to a boundary map on the groups $\mathrm{C}_*^{\{A,B\}}(A \cup B)$. Given a commutative ring R we consider

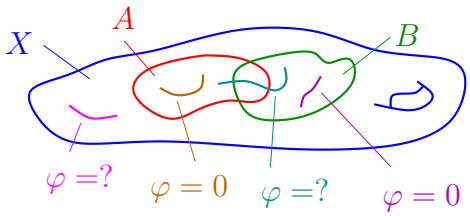
$$\mathrm{C}^n(X, \{A, B\}; R) := \ker\left(\mathrm{C}^n(X; R) \to \mathrm{Hom}\Big(\underbrace{\mathrm{C}_n^{\{A,B\}}(A \cup B)}_{\text{subgroup of } \mathrm{C}_n(X)}, R\Big)\right).$$

This is again a cochain complex and we denote the corresponding cohomology groups by $\mathrm{H}^*(X, \{A, B\}; R)$.

Remark. With the above notation we could write that

$$\mathrm{C}^n(X, \{A, B\}; R) = \{\varphi\colon \mathrm{C}_n(X) \to R \,|\, \varphi(\sigma) = 0 \text{ for any } \sigma\colon \Delta^n \to A \text{ and any } \sigma\colon \Delta^n \to B\}.$$

The definition of $\mathrm{C}^n(X, \{A, B\}; R)$ is illustrated in the figure below.



$$\varphi \in \mathrm{H}^1(X, \{A, B\}; R)$$

Lemma 115.1. Let (X, A, B) be an excisive triad of topological spaces. Given any commutative ring R the natural[1405] map

$$\mathrm{C}^*(X, A \cup B; R) \quad \to \quad \mathrm{C}^*(X, \{A, B\}; R)$$

induces for every $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}^n(X, A \cup B; R) \quad \xrightarrow{\cong} \quad \mathrm{H}^n(X, \{A, B\}; R).$$

Proof. We have the following commutative diagram of short exact sequences of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{C}_*^{\{A,B\}}(A \cup B) & \longrightarrow & \mathrm{C}_*(X) & \longrightarrow & \dfrac{\mathrm{C}_*(X)}{\mathrm{C}_*^{\{A,B\}}(A \cup B)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle =\mathrm{id}} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{C}_*(A \cup B) & \longrightarrow & \mathrm{C}_*(X) & \longrightarrow & \mathrm{C}_*(X, A \cup B) & \longrightarrow & 0. \end{array}$$

It is clear that the groups in the first and second column are free abelian. On page 1625 we already remarked that relative chain groups are free abelian, and the same same argument shows that the top right group is free abelian, in fact a basis is given by all singular simplices of X that are not contained in A or B. By our hypothesis the triad (X, A, B) is excisive. By definition this means that the left vertical map is a chain homotopy equivalence. The middle vertical map is the identity. It follows from this discussion together with the Two-Out-Of-Three-Homotopy Equivalence Corollary 81.6 that the right vertical map is also a chain homotopy equivalence.

Recall that by the convention on page 2534 we have an identification

$$\mathrm{Hom}\left(\frac{\mathrm{C}_n(X)}{\mathrm{C}_n^{\{A,B\}}(A \cup B)}, R\right) \;=\; \mathrm{C}^n(X, \{A, B\}; R).$$

The lemma follows from the above vertical chain homotopy equivalence, the above observation and the Dual Cochain Map Lemma 108.8. ∎

The following lemma is the motivation for introducing the rather technical cohomology modules $\mathrm{H}^*(X; \{A, B\}; R)$.

Lemma 115.2. Let (X, A, B) be a triad of topological spaces and let R be a commutative ring. Then the map

$$\begin{array}{rcl} \mathrm{C}^p(X, A; R) \times \mathrm{C}^q(X, B; R) & \to & \mathrm{C}^{p+q}(X, \{A, B\}; R) \\ (\varphi, \psi) & \mapsto & \left[\begin{array}{rcl} \mathrm{C}_{p+q}(X) & \to & R \\ \sigma & \mapsto & \varphi(\sigma \circ [v_0, \ldots, v_p]) \cdot \psi(\sigma \circ [v_p, \ldots, v_{p+q}]) \end{array}\right]. \end{array}$$

[1405] What is the "natural" map?

is well-defined and it induces an R-bilinear map

$$\cup\colon \mathrm{H}^p(X,A;R)\times \mathrm{H}^q(X,B;R) \;\to\; \mathrm{H}^{p+q}(X,\{A,B\};R).$$

Proof. The lemma follows easily from the following claim.

Claim. Given $\varphi\in \mathrm{C}^p(X,A;R)$ and $\psi\in \mathrm{C}^q(X,B;R)$ the homomorphism

$$\begin{array}{rcl} \mathrm{C}_{p+q}(X) & \to & R \\ (\sigma\colon \Delta^{p+q}\to X) & \mapsto & \varphi(\sigma\circ[v_0,\dots,v_p])\cdot\psi(\sigma\circ[v_p,\dots,v_{p+q}]) \end{array}$$

lies actually in $\mathrm{C}^{p+q}(X,\{A,B\})$.

Proof. By the remark on page 2534 we have to show that the above cochain vanishes on all singular $(p+q)$-simplices that lie entirely in A or that lie entirely in B. But this is clear:

(1) given $\sigma\colon \Delta^{p+q}\to A$ the first factor $\varphi(\sigma\circ[v_0,\dots,v_p])$ is necessarily zero since we have $\varphi\in \mathrm{C}^p(X,A;R)$, and similarly,
(2) given $\sigma\colon \Delta^{p+q}\to B$ the second factor $\psi(\sigma\circ[v_p,\dots,v_{p+q}])$ is necessarily zero since we have $\psi\in \mathrm{C}^q(X,B;R)$.

This concludes the proof of the claim and thus also of the lemma. ∎

Definition. Let (X,A,B) be an excisive triad of topological spaces and let R be a commutative ring. We refer to the composition

$$\mathrm{H}^p(X,A;R)\times \mathrm{H}^q(X,B;R) \xrightarrow{(c,d)\mapsto c\cup d} \mathrm{H}^{p+q}(X,\{A,B\};R) \xleftarrow[\uparrow]{\cong} \mathrm{H}^{p+q}(X,A\cup B;R)$$

isomorphism by Lemma 115.1

again as a cup product (or sometimes as a relative cup product) and we denote it as usual by the "$\cup$"-symbol.

Remark. Let X be a topological space, let $A,B\subset X$ be subsets and let R be a commutative ring. Suppose that $\mathrm{C}_*^{\{A,B\}}(A\cup B)=\mathrm{C}_*(A\cup B)$. (This is for example the case if $A=B=\varnothing$ or more generally, if A is a subset of B or if B is a subset of A.) Then the above definition of the cup product is just the "naive" definition given by

$$\begin{array}{rcl} \mathrm{C}^p(X,A;R)\times \mathrm{C}^q(X,B;R) & \to & \mathrm{C}^{p+q}(X,A\cup B;R) \\ (\varphi,\psi) & \mapsto & \left(\begin{array}{rcl} \mathrm{C}_{p+q}(X,A\cup B) & \to & R \\ \sigma & \mapsto & \underbrace{\varphi(\sigma\circ[v_0,\dots,v_p])}_{\in R}\cdot\underbrace{\psi(\sigma\circ[v_p,\dots,v_{p+q}])}_{\in R} \end{array}\right). \end{array}$$

In the following we summarize a few properties of the relative cup product. We start out with the obvious generalization of the Cup Product-Commutativity Proposition 114.8.

Proposition 115.3. (Cup Product-Commutativity Proposition) Let (X,A,B) be an excisive triad of topological spaces and let R be a commutative ring. For every $\varphi\in \mathrm{H}^k(X,A;R)$ and $\psi\in \mathrm{H}^l(X,B;R)$ we have

$$\varphi\cup\psi \;=\; (-1)^{kl}\cdot\psi\cup\varphi \;\in\; \mathrm{H}^{k+l}(X,A\cup B;R).$$

Proof. In Exercise 115.1 we will see that a modification of the proof of the Cup Product-Commutativity Proposition 114.8 provides us with a proof of the Cup Product-Commutativity Proposition 115.3. ∎

Note that the first part of following proposition is a generalization of the Cup Product-Naturality Lemma 114.10.

Proposition 115.4. (Cup Product-Naturality Proposition)

(1) Let (X, A, B) and (Y, C, D) be two excisive triads and let R be a commutative ring. Let $f\colon (X, A, B) \to (Y, C, D)$ be a map. Then for all $p, q \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p(Y, C; R) \times \mathrm{H}^q(Y, D; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(Y, C \cup D; R) \\ \downarrow{\scriptstyle f^* \times f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{H}^p(X, A; R) \times \mathrm{H}^q(X, B; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(X, A \cup B; R). \end{array}$$

(2) Let (X, A, B) be an excisive triad. If $\varphi\colon R \to S$ is a homomorphism between commutative rings, then for all $p, q \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p(X, A; R) \times \mathrm{H}^q(X, B; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(X, A \cup B; R) \\ \downarrow{\scriptstyle \varphi_* \times \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \mathrm{H}^p(X, A; S) \times \mathrm{H}^q(X, B; S) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(X, A \cup B; S). \end{array}$$

Here the vertical maps are the maps defined on page 2401.

Proof.

(1) It follows easily from the definitions that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}^p(Y, C; R) \times \mathrm{H}^q(Y, D; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(Y, \{C, D\}; R) & \xleftarrow{\cong} & \mathrm{H}^{p+q}(Y, C \cup D; R) \\ \downarrow{\scriptstyle f^* \times f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{H}^p(X, A; R) \times \mathrm{H}^q(X, B; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(X, \{A, B\}; R) & \xleftarrow{\cong} & \mathrm{H}^{p+q}(X, A \cup B; R). \end{array}$$

The desired statement follows immediately from this observation.

(2) This statement follows, similar to (1), basically immediately from the definitions. ■

We record a particularly useful special case of the Cup Product-Naturality Proposition 115.4 (1) as a corollary.

Corollary 115.5. (Cup Product-Naturality Corollary) Let (X, A, B) be an excisive triad and let R be a commutative ring. Then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p(X, A; R) \times \mathrm{H}^q(X, B; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(X, A \cup B; R) \\ \downarrow & & \downarrow \\ \mathrm{H}^p(X; R) \times \mathrm{H}^q(X; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(X; R). \end{array}$$

In particular, if $\alpha \in \mathrm{H}^p(X; R)$ lies in the image of $\mathrm{H}^p(X, A; R)$ and if $\beta \in \mathrm{H}^q(X; R)$ lies in the image of $\mathrm{H}^q(X, B; R)$ and if $A \cup B = X$, then $\alpha \cup \beta = 0$.

Proof. The first part of the corollary follows immediately from the Cup Product-Naturality Proposition 115.4 (1) applied to the obvious map $f\colon (X, \varnothing, \varnothing) \to (X, A, B)$ of excisive triads.

We turn to the proof of the second part. We denote by $\sigma\colon \mathrm{H}^p(X, A; R) \to \mathrm{H}^p(X; R)$, $\tau\colon \mathrm{H}^p(X, B; R) \to \mathrm{H}^p(X; R)$ and by $\upsilon\colon \mathrm{H}^{p+q}(X, A \cup B; R) \to \mathrm{H}^{p+q}(X; R)$ the obvious maps.

Let $\widehat{\alpha} \in \mathrm{H}^p(X, A; R)$ and $\widehat{\beta} \in \mathrm{H}^q(X, B; R)$. We write $\alpha = \sigma(\widehat{\alpha})$ and $\beta = \tau(\widehat{\beta})$. We suppose that $A \cup B = X$. Then

$$\alpha \cup \beta \;=\; \sigma(\widehat{\alpha}) \cup \tau(\widehat{\beta}) \;\underset{\text{by the commutative diagram}}{=}\; \upsilon\big(\widehat{\alpha} \cup \widehat{\beta}\big) \;\underset{\substack{\text{since } \widehat{\alpha} \cup \widehat{\beta} \text{ lies in the group} \\ \mathrm{H}^{p+q}(X, A \cup B; R) = \mathrm{H}^{p+q}(X, X; R) = 0}}{=}\; \upsilon(0) \;=\; 0.$$

■

Before we state the next lemma we introduce one more convention.

Convention. Let A, B, C, D, X and Y be sets, let $f\colon C \to A$, $g\colon B \to D$ and $\mu\colon X \to Y$ be maps and let $\varphi\colon A \times B \to X$ and $\psi\colon C \times D \to Y$ be maps. We say that the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{\varphi} & X \\ {\scriptstyle f}\uparrow \quad \downarrow{\scriptstyle g} & & \quad {\scriptstyle \mu}\downarrow \\ C \times D & \xrightarrow{\psi} & Y \end{array}$$

commutes, if for all $c \in C$ and $b \in B$ we have

$$\psi(c, g(b)) \;=\; \mu(\varphi(f(c), b)).$$

A similar convention holds with the directions of f and g reversed. Furthermore, if we are dealing with group homomorphisms, then there is an obvious notion of "commutes up to a given sign".

Using this convention we can now formulate our next lemma.

Lemma 115.6. (Cup Product-Connecting Homomorphism Lemma) Let (X, A) be a pair of topological spaces and let R be a commutative ring. We denote by $i\colon A \to X$ the inclusion map and we denote by δ the connecting homomorphisms in the long exact sequence in cohomology corresponding to the pair (X, A). For any $p, q \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^q(A; R) \times \mathrm{H}^p(A; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(A; R) \\ {\scriptstyle \delta}\downarrow \quad \uparrow{\scriptstyle i^*} & & \quad \downarrow{\scriptstyle \delta} \\ \mathrm{H}^{q+1}(X, A; R) \times \mathrm{H}^p(X; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q+1}(X, A; R) \end{array}$$

and the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}^p(A; R) \times \mathrm{H}^q(A; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(A; R) \\ {\scriptstyle i^*}\uparrow \quad \downarrow{\scriptstyle \delta} & & \quad \downarrow{\scriptstyle \delta} \\ \mathrm{H}^p(X; R) \times \mathrm{H}^{q+1}(X, A; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q+1}(X, A; R) \end{array}$$

up to the sign $(-1)^p$.

Remark. In Lemma 132.18 we will formulate a generalization of the Cup Product-Connecting Homomorphism Lemma 115.6.

Proof. For no particular reason we prove the second statement. The first statement can either be proved the same way or it can be deduced from the second statement using the Cup Product-Commutativity Proposition 115.3. We pick cocycles $\varphi \in \mathrm{C}^p(X; R)$ and $\psi \in \mathrm{C}^q(A; R)$. In the following we denote by $\widetilde{\psi}\colon \mathrm{C}^q(X) \to R$ the extension of ψ that is defined to be zero on all singular k-simplices that do not lie in the subset A. We have the

following equalities:

$$\begin{array}{lllll}
\delta([i^*\varphi]\cup[\psi]) & = \delta([i^*\varphi\cup\psi]) & \overset{\text{by the discussion of the connecting homomorphism in the remark on page 2411 and since } \varphi\cup\psi \text{ is the trivial extension of } i^*\varphi\cup\psi\colon \mathrm{C}_{p+q}(A)\to R \text{ to a homomorphism } \mathrm{C}_{p+q}(X)\to R}{=} [\delta(\varphi\cup\widetilde{\psi})] & \overset{\text{Coboundary-Cup Product Lemma 114.2}}{=} [\delta\varphi\cup\widetilde{\psi}+(-1)^p\cdot(\varphi\cup\delta\widetilde{\psi})] \\
= (-1)^p\cdot[\varphi\cup\delta\widetilde{\psi}] & = (-1)^p\cdot([\varphi]\cup[\delta\widetilde{\psi}]) & = (-1)^p\cdot([\varphi]\cup\delta[\psi]). & \\
\text{since } \varphi \text{ is a cocycle} & & \text{again by the remark on page 2411} &
\end{array}$$

This equality implies that the diagram commutes up to the sign $(-1)^p$. ■

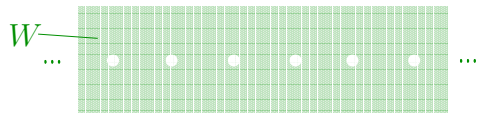
Finally, recall that on page 2479 we introduced cohomology with compact support $\mathrm{H}^*_{\mathrm{c}}(X;\mathbb{Z})$. For the insatiable reader we also provide the definition of the cup product on cohomology with compact support.

Definition. Let X be a topological space, let $\varphi\in\mathrm{H}^k_{\mathrm{c}}(X;\mathbb{Z})$ and let $\psi\in\mathrm{H}^l_{\mathrm{c}}(X;\mathbb{Z})$. Recall that by Lemma 112.2 there exists a compact subset $K\subset X$ and a cohomology class $\widetilde{\varphi}\in\mathrm{H}^k(X,X\setminus K;\mathbb{Z})$ such that the image of $\widetilde{\varphi}$ in $\mathrm{H}^k_{\mathrm{c}}(X;\mathbb{Z})$ equals φ. Analogously we find a compact set L and $\widetilde{\psi}\in\mathrm{H}^l(X,X\setminus L;\mathbb{Z})$. We define

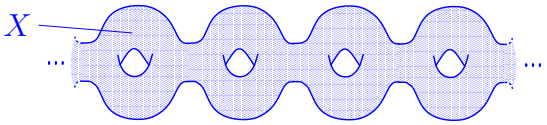
$$\varphi\cup\psi \;:=\; \text{the image of } \varphi\cup\psi\in\mathrm{H}^{k+l}(X,X\setminus(K\cup L)) \text{ in } \mathrm{H}^{k+l}_{\mathrm{c}}(X;\mathbb{Z}).$$

It follows easily from the Cup Product-Naturality Corollary 115.5 that this definition does not depend on any of the choices. We refer to $\varphi\cup\psi$ as the cup product in cohomology with compact support.

The cup product on cohomology with compact support can be very useful for tackling non-compact spaces. For example in Exercise 115.5 we will make of use of the cup product on cohomology with compact support to show that the topological manifold $W=\mathbb{R}^2\setminus\{(n,0)\,|\,n\in\mathbb{Z}\}$ is not homeomorphic to the surface X of infinite genus that is shown in the figure below. On a more theoretical side we will make of the above definition in Theorem 127.9 when we will learn how to compute cup products on topological manifolds by "intersecting cycles".



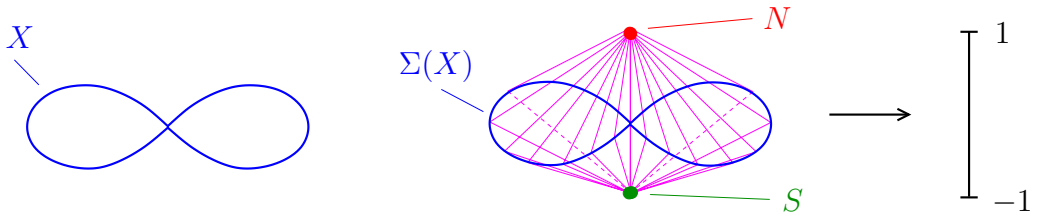
cup product on $\mathrm{H}^*_{\mathrm{c}}(W;\mathbb{Z})$ is trivial



cup product on $\mathrm{H}^*_{\mathrm{c}}(X;\mathbb{Z})$ is non-trivial

115.2. **Calculations using the relative cup product.** In this section we will use the Cup Product-Naturality Proposition 115.4 (1) to calculate cup products of several topological spaces. We start out with the suspension $\Sigma(X)=(X\times[-1,1])/\sim$ of a topological space X. We refer to page 451 for the precise definition and we refer to the figure below for a reminder.

The following lemma shows that the cup product of the suspension of a topological space is quite dull.



Lemma 115.7. (Suspension-Cup Product Lemma) Let X be a path-connected topological space and let R be a commutative ring.

(1) We have

$$\mathrm{H}^k(\Sigma(X);R) \;\cong\; \begin{cases} R, & \text{if } k=0,\\ 0, & \text{if } k=1,\\ \mathrm{H}^{k-1}(X;R), & \text{if } k\geq 2. \end{cases}$$

(2) For any $k,l\geq 1$ the cup product

$$\mathrm{H}^k(\Sigma(X);R)\times \mathrm{H}^l(\Sigma(X);R) \;\to\; \mathrm{H}^{k+l}(\Sigma(X);R)$$

is the zero map.

Proof. We denote by $p\colon X\times[-1,1]\to\Sigma(X)$ the natural projection. We denote by $N:=[X\times\{1\}]$ the "North Pole" and we denote by $S:=[X\times\{-1\}]$ the South Pole. We refer to the figure above for an illustration. We write $U=\Sigma(X)\setminus\{S\}$ and $V=\Sigma(X)\setminus\{N\}$. By the same argument as in the Cone Contractibility Lemma 16.7 that U admits deformation retraction from U to the North Pole N. This shows that U is contractible. The same argument shows that V is contractible.

(1) Note that U and V are open subsets of $\Sigma(X)$.[1406] It follows from the Excisive Triad-Proposition 109.12 (4) that $(\Sigma(X),U,V)$ is excisive. Therefore we can apply the Mayer–Vietoris Theorem 109.15 for Cohomology Groups and we obtain the following long exact sequence

$$\ldots\to\mathrm{H}^i(\Sigma(X);R)\to\mathrm{H}^i(U;R)\oplus\mathrm{H}^i(V;R)\to\mathrm{H}^i(U\cap V;R)\to\mathrm{H}^{i+1}(\Sigma(X);R)\to\ldots$$

The desired isomorphism now follows easily from the above long exact sequence and the following:
(a) above we had observed that U and V are contractible,
(b) the intersection $U\cap V$ is homeomorphic to $X\times(-1,1)$, in particular it is homotopy equivalent to X,
(c) Lemmas 108.14 and 108.12 and the discussion of induced maps on 0-th cohomology on page 2409.

(2) Now let $k,l\geq 1$. It follows immediately from the long exact sequence of cohomology groups corresponding to the pair $(\Sigma(X),U)$, the fact that U is homotopy equivalent to a point (together with Lemmas 108.12 and 108.13) that the inclusion induced map $\mathrm{H}^k(\Sigma(X),U;R)\to\mathrm{H}^k(\Sigma(X);R)$ is an isomorphism. Evidently the same also holds if we replace U by V.

By the Cup Product-Naturality Corollary 115.5, and the fact that $(\Sigma(X),U,V)$ is excisive, we have the following commutative diagram

[1406] Why is that?

$$\begin{array}{ccc} \mathrm{H}^k(\Sigma(X), U; R) \times \mathrm{H}^l(\Sigma(X), V; R) & \xrightarrow{\cup} & \mathrm{H}^{k+l}(\Sigma(X), \overbrace{U \cup V}^{=\Sigma(X)}; R) \\ \downarrow\cong & & \downarrow \\ \mathrm{H}^k(\Sigma(X); R) \times \mathrm{H}^l(\Sigma(X); R) & \xrightarrow{\cup} & \mathrm{H}^{k+l}(\Sigma(X); R). \end{array}$$

Since $U \cup V = \Sigma(X)$ we see that the cohomology group on the top-right is zero. Furthermore, since the vertical map on the left is an isomorphism we see that the cup product on the bottom is zero. ■

Let X and Y be two topological spaces and let $x \in X$ and $y \in Y$ be good points. (See page 1171 for the definition of a good point.) We use these points to define the wedge $X \vee Y$ of X and Y. Furthermore let R be a commutative ring. We denote by $p\colon X \vee Y \to X$ and $q\colon X \vee Y \to Y$ the natural projections. By the Wedge-H*-Proposition 109.17 we know that for every $n \in \mathbb{N}$ the map

$$p^* \oplus q^*\colon \mathrm{H}^n(X; R) \oplus \mathrm{H}^n(Y; R) \xrightarrow{\cong} \mathrm{H}^n(X \vee Y; R)$$

is an isomorphism. The following lemma determines the cup product of $X \vee Y$ in terms of the cup products of X and Y. It can be viewed as a generalization of Lemma 114.11.

Lemma 115.8. (Wedge-Cup Product Lemma) We continue with the above notation. Then for any $\alpha, \beta \in \mathrm{H}^*(X; R)$ and $\varphi, \psi \in \mathrm{H}^*(Y; R)$ in degrees ≥ 1 the following statements hold in $\mathrm{H}^*(X \vee Y; R)$:

$\cup$	$p^*(\beta)$	$q^*(\psi)$
$p^*(\alpha)$	$p^*(\alpha \cup \beta)$	0
$q^*(\varphi)$	0	$q^*(\varphi \cup \psi)$

Proof. The statement regarding the "diagonal" terms of the table follows immediately from the Cup Product-Naturality Lemma 114.10. It remains to prove that the off-diagonal terms are zero. We show that the top-right entry is zero. The proof that the bottom-left entry is zero is basically identical.

Thus let $\alpha \in \mathrm{H}^k(X; R)$ and $\psi \in \mathrm{H}^l(Y; R)$ with $k, l \geq 1$. We view X and Y as subsets of $X \vee Y$. We denote by $f\colon (X \vee Y, \varnothing) \to (X \vee Y, X)$ and $g\colon (X \vee Y, \varnothing) \to (X \vee Y, Y)$ the two inclusion maps of pairs of topological spaces.

Claim. There exist $\widetilde{\alpha} \in \mathrm{H}^k(X \vee Y, Y; R)$ with $g^*(\widetilde{\alpha}) = p^*(\alpha)$ and $\widetilde{\psi} \in \mathrm{H}^l(X \vee Y, X; R)$ with $f^*(\widetilde{\psi}) = q^*(\psi)$.

Proof. To prove the claim we consider the following two diagrams[1407]

$$\begin{array}{ccc} \mathrm{H}^k(X \vee Y; R) & \xleftarrow{g^*} & \mathrm{H}^k(X \vee Y, Y; R) \\ {\scriptstyle p^*}\uparrow & & \downarrow \\ \mathrm{H}^k(X; R) & \longleftarrow & \mathrm{H}^k(X, \{x\}; R) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{H}^l(X \vee Y; R) & \xleftarrow{f^*} & \mathrm{H}^l(X \vee Y, X; R) \\ {\scriptstyle q^*}\uparrow & & \downarrow \\ \mathrm{H}^l(Y; R) & \longleftarrow & \mathrm{H}^l(Y, \{y\}; R). \end{array}$$

Here the maps that are unlabeled are induced by the obvious inclusions. It is easy to verify that the diagrams commute. It follows easily from the long exact sequence in cohomology

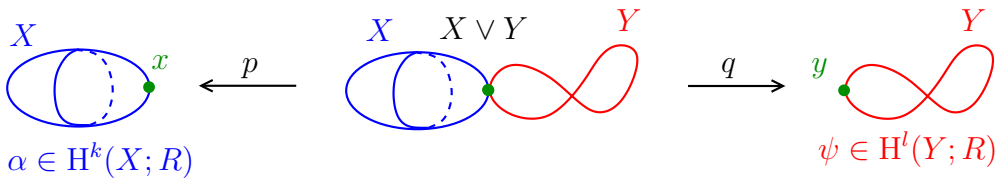
[1407]Recall that in $X \vee Y$ the points x and y get identified, so we can view x as an element in Y and we can view y as an element in X.

of a pair of topological spaces that the two bottom horizontal maps are isomorphisms. Recall that in the Wedge Excisive Lemma 109.16 we showed that the triad $(X \vee Y, X, Y)$ is excisive. Therefore it follows from the Excisive Characterization Lemma 109.13 that the vertical maps on the right are isomorphisms. The claim is an immediate consequence of these observations. ⊞

Since, as we had just seen, the triad $(X \vee Y, X, Y)$ is excisive we can apply the Cup Product-Naturality Corollary 115.5 and we obtain that

$$p^*(\alpha) \cup q^*(\psi) \;=\; g^*(\widetilde{\alpha}) \cup f^*(\widetilde{\psi}) \underset{\uparrow}{=} 0.$$

by the Cup Product-Naturality Corollary 115.5 since $X \cup Y = X \vee Y$ ■



Example. Now we can also give a new proof of Lemma 114.16, more precisely, we can give a new and somewhat more conceptual proof of the calculation of the cup product of the surface Σ of genus 2. First we recall that on page 2428 we introduced two projections $p_i \colon \Sigma \to T_i = S^1 \times S^1$ for $i = 1, 2$. It follows easily from Lemma 107.3 that these projections are degree-one maps, i.e. they send the fundamental class $[\Sigma]$ to the fundamental class $[T_i]$. In Lemma 109.11 we saw that the map

$$p_1^* \oplus p_2^* \colon \mathrm{H}^1(T_1; \mathbb{Z}) \oplus \mathrm{H}^1(T_2; \mathbb{Z}) \;\to\; \mathrm{H}^1(\Sigma; \mathbb{Z})$$

is an isomorphism. We make the following observations:

(1) Let $\varphi, \psi \in \mathrm{H}^1(T_i; \mathbb{Z})$, $i = 1, 2$. Then

$$\langle \varphi \cup \psi, [T_i] \rangle_{T_i} \underset{\uparrow}{=} \langle \varphi \cup \psi, p_{i*}([\Sigma]) \rangle_{T_i} \underset{\uparrow}{=} \langle p_i^*(\varphi \cup \psi), [\Sigma] \rangle_{\Sigma} \underset{\uparrow}{=} \langle p_i^*(\varphi) \cup p_i^*(\psi), [\Sigma] \rangle_{\Sigma}.$$

since p_i is of degree one — Lemma 109.6 (3) — Cup Product-Naturality Lemma 114.10

(2) We denote by $q_i \colon T_1 \vee T_2 \to T_i$, $i = 1, 2$ the obvious projections. The two projections p_1, p_2 give rise to a map $p_1 \vee p_2 \colon \Sigma \to T_1 \vee T_2$ which has the property that $q_i \circ (p_1 \vee p_2) = p_i$, $i = 1, 2$. (See the figure below for an illustration.) Now let $\varphi_1 \in \mathrm{H}^1(T_1; \mathbb{Z})$ and $\varphi_2 \in \mathrm{H}^1(T_2; \mathbb{Z})$. Then

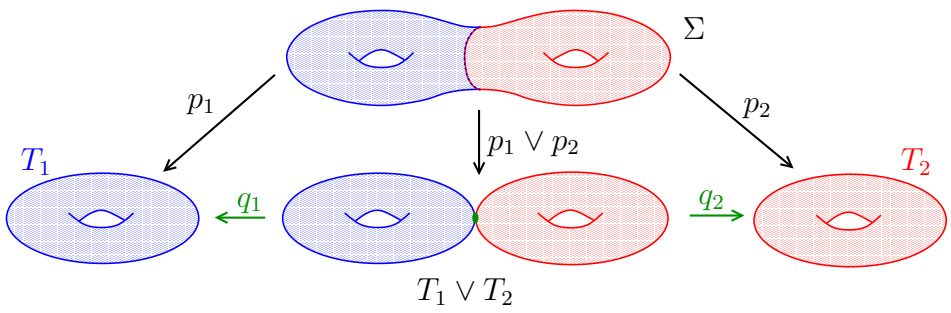
$$p_1^*(\varphi_1) \cup p_2^*(\varphi_2) \underset{\uparrow}{=} (p_1 \vee p_2)^*(q_1^*(\varphi_1) \cup q_2^*(\varphi_2)) \underset{\uparrow}{=} (p_1 \vee p_2)^*(0) \;=\; 0.$$

by the Cup Product-Naturality Lemma 114.10, since $q_i \circ (p_1 \vee p_2) = p_i$ — Wedge-Cup Product Lemma 115.8

Precisely the same proof shows also that $p_2^*(\varphi_2) \cup p_1^*(\varphi_1) = 0$.

These two claims, together with the previous calculation of the cup product on the torus $T = S^1 \times S^1$, see Lemma 114.14, give a new proof of Lemma 114.16.

115.3. **The Lusternik-Schnirelmann category.** We start out with the following definition.



Definition. Let M be an m-dimensional smooth manifold.

(1) A parametrization for M is a smooth map $\Psi\colon V \to M$ with the following properties:
   (a) the domain V is either an open ball in $\mathbb{R}^m$, or an open rectangle, or all of $\mathbb{R}^m$,[1408]
   (b) the map Ψ is a diffeomorphism onto its image.

(2) We say a family $\{\Psi_i\colon V_i \to M\}_{i\in I}$ of parametrizations covers M if $\bigcup_{i\in I} \Psi_i(V_i) = M$.

The following slightly vague question arises:

Question 115.9. For a given smooth manifold M, how many parametrizations do we need to cover all of M?

Examples.

(1) Let $N = (0, \dots, 0, 1)$ be the North Pole and let $S = (0, \dots, 0, -1)$ be the South Pole of the n-dimensional sphere S^n. We can cover S^n by the two open subsets $S^n \setminus \{N\}$ and $S^n \setminus \{S\}$. Stereographic projection, see the figure on page 155, gives rise to two diffeomorphisms $\Phi_1\colon S^n \setminus \{N\} \to \mathbb{R}^n$ and $\Phi_2\colon S^n \setminus \{S\} \to \mathbb{R}^n$. The inverses of these two maps define parametrizations that cover S^n.

Since S^n is not diffeomorphic to any of the admissible domains of a parametrization we see that the above collection of parametrizations is optimal in the sense that we cannot parametrize S^n with fewer parametrizations.

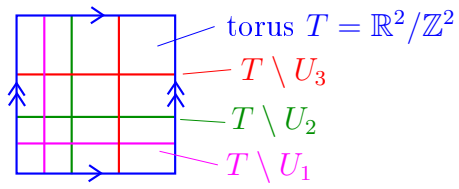
(2) We consider the n-torus $(S^1)^n = \mathbb{R}^n/\mathbb{Z}^n$. We denote by $p\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ the projection . We pick real numbers $0 < s_1 < s_2 < \cdots < s_{n+1} < 1$. For $i = 1, \dots, n+1$ we consider the map

$$\begin{array}{rccc} \Psi_i\colon & \overbrace{(s_i, s_i+1)\times\ldots\times(s_i, s_i+1)}^{=:V_i} & \to & \mathbb{R}^n/\mathbb{Z}^n \\ & (t_1, \dots, t_n) & \mapsto & p(t_1, \dots, t_n). \end{array}$$

For $i = 1, \dots, n+1$ we write $U_i = \Psi_i(V_i)$. It is straightforward to see that the maps $\Psi_1, \dots, \Psi_{n+1}$ are parametrizations that cover $\mathbb{R}^n/\mathbb{Z}^n$.[1409] We illustrate the images of the parametrizations for the 2-torus in the figure below.

[1408] The condition on V sounds slightly arbitrary, and it is, but it is reasonable that for practical purposes one prefers domains that are "easy to describe".

[1409] Hmm, why is it true that $U_1 \cup \cdots \cup U_{n+1} = \mathbb{R}^n/\mathbb{Z}^n$?



(3) For $i = 1, \ldots, n+1$ we consider the map

$$\begin{array}{rcl} \Psi_i \colon \mathbb{R}^n & \to & \mathbb{R}\mathrm{P}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim \\ (x_1, \ldots, x_n) & \mapsto & \underbrace{[x_1 : \ldots : x_{i-1} : 1 : x_i : \ldots : x_n]}_{\text{point in } \mathbb{R}\mathrm{P}^n \text{ represented by } (x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_n)}. \end{array}$$

It is straightforward to see that the maps $\Psi_1, \ldots, \Psi_{n+1}$ are parametrizations that cover the real projective space $\mathbb{R}\mathrm{P}^n$.

(4) Almost the same construction as in (3) shows that we can cover the complex projective space $\mathbb{C}\mathrm{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$ by $n+1$ parametrizations such that the domain of each parametrization is diffeomorphic to $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

The following question is now a more precise version of Question 115.9.

Question 115.10.

(1) Can we cover $(S^1)^n$ by fewer than $n+1$ parametrizations?
(2) Can we cover $\mathbb{R}\mathrm{P}^n$ by fewer than $n+1$ parametrizations?
(3) Can we cover $\mathbb{C}\mathrm{P}^n$ by fewer than $n+1$ parametrizations?

In the remainder of this chapter we will study a generalization of these questions and we will see how cup products can be used to approach this question and its generalizations. We will eventually answer Question 115.10 in Corollary 117.23 and Proposition 121.18.
Now we will generalize this discussion to arbitrary topological spaces by introducing the following definition.

Definition.

(1) We say that a map $f \colon Y \to X$ between topological spaces is null-homotopic if it is homotopic to a constant map.
(2) The Lusternik-Schnirelmann category $\mathrm{cat}(X)$[1410] of a topological space X is defined as the smallest $n \in \mathbb{N}_0$ such that there exists an open cover $U_1, \ldots, U_{n+1}$ of X such that each inclusion map $U_i \to X$ is null-homotopic.[14111412] If no such n exists, then we define $\mathrm{cat}(X) = \infty$.

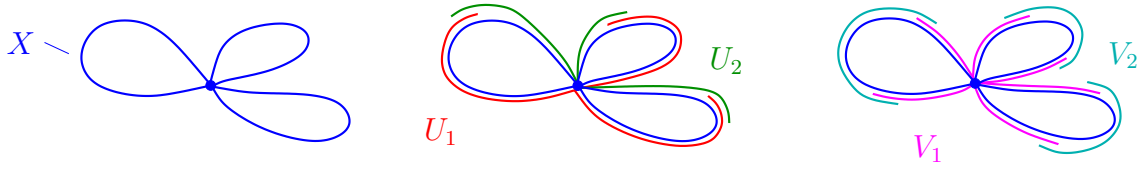
Remark.

[1410]The word "category" with its modern meaning was first used by Eilenberg-Maclane [**EMc1945**] in 1945. On the other hand the term "Lusternik-Schnirelmann category" was already established by 1941, see [**Fox1941**].
[1411]Recall that according to the definition on page 418 a map $f \colon U \to X$ is called null-homotopic if it is homotopic to a constant map.
[1412]Note the fairly well-hidden "+1" in the subscript of U_{n+1}. Sometimes, especially in the older literature the Lusternik-Schnirelmann category is defined without the adjustment given by the "+1".

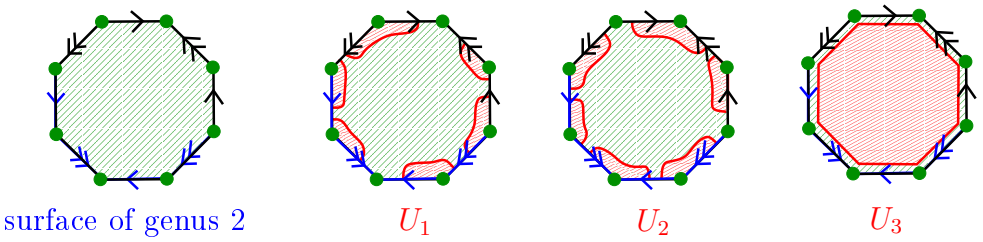
(1) Suppose that a given smooth manifold M can be covered by $n+1$ parametrizations. Since the domain of a parametrization is contractible and since a parametrization is a diffeomorphism onto its image we see that $\mathrm{cat}(M) \leq n$.[1413]
(2) The sets U_i in the definition of the Lusternik-Schnirelmann category do not have to be contractible, for example if X itself is contractible, then for any subset $U \subset X$ the inclusion map $U \to X$ is null-homotopic.
(3) The Lusternik-Schnirelmann category plays an important role in topological robotics, see e.g. [**Far2008**] for more details.

Examples.

(1) For a non-empty topological space X we have $\mathrm{cat}(X) = 0$ if and only if $\mathrm{id}\colon X \to X$ is null-homotopic, i.e. if X is contractible.
(2) The above discussion of parametrizations of S^n together with Statement (1) shows that $\mathrm{cat}(S^n) = 1$.
(3) (a) In the figure below we consider the wedge X of $n \geq 1$ circles together with two open covers $X = U_1 \cup U_2$ and also $X = V_1 \cup V_2$. Note that all the inclusion maps $U_i \to X$ and also $V_i \to X$ are null-homotopic. So this gives us two separate proofs that $\mathrm{cat}(X) \leq 1$. Together with Statement (1) this shows that we have $\mathrm{cat}(X) = 1$.
(b) Using the obvious generalization of the sets V_1 and V_2 shown in the figure below one can easily show that the Lusternik-Schnirelmann category of any non-empty wedge of spheres has Lusternik-Schnirelmann category equal to 1.



(4) The discussion preceding Question 115.10 gives us the inequalities $\mathrm{cat}((S^1)^n) \leq n$, $\mathrm{cat}(\mathbb{R}\mathrm{P}^n) \leq n$ and also $\mathrm{cat}(\mathbb{C}\mathrm{P}^n) \leq n$.
(5) For any $g \geq 2$ the surface Σ of genus g satisfies $\mathrm{cat}(\Sigma) \leq 2$, see the figure below for an illustration in the case $g = 2$.



We move on to more general statements.

Proposition 115.11. Let X and Y be topological spaces. If X and Y are homotopy equivalent, then $\mathrm{cat}(X) = \mathrm{cat}(Y)$.

[1413] Hmm, why is the image of a parametrization an open subset of the smooth manifold?

Proof. We outsource the proof of the proposition to Exercise 115.9. Alternatively the diffident reader is referred to [**CLOT2003**, Theorem 1.30] for the proof. ■

The following proposition gives interesting upper bounds for many examples. In particular it gives a unified approach to the above Examples (3)-(5).

Proposition 115.12.
(1) If X is a connected n-dimensional simplicial complex X, then $\operatorname{cat}(X) \leq n$.
(2) If X is a connected n-dimensional CW-complex, then $\operatorname{cat}(X) \leq n$.
(3) If X is a connected n-dimensional smooth manifold, then $\operatorname{cat}(X) \leq n$.
(4) If X is a connected n-dimensional topological manifold, then $\operatorname{cat}(X) \leq n$.

In the proof of Proposition 115.12 we make use of the following lemma.

Lemma 115.13. Let X be a topological space and let $U \subset X$ be a subset with components $\{U_i\}_{i \in I}$. If each inclusion $U_i \to X$ is null-homotopic and if X is path-connected, then the inclusion $U \to X$ is also null-homotopic.

Proof. By hypothesis there exists for each $i \in I$ a $P_i \in X$ such that the inclusion map $U_i \to X$ is homotopic to the constant map which sends all points in U_i to P_i. Now pick $Q \in X$. Since X is path-connected we can find paths that connect the P_i to Q. Using these paths we can construct a homotopy from the inclusion map $U \to X$ to the constant map that sends all points to Q. ■

Proof.

(1) We only have to deal with the case that $X = |K|$ is the topological realization of a connected n-dimensional abstract simplicial complex. We consider the barycentric subdivision $\operatorname{sd}(K)$ as defined in Lemma 94.2. By the SD-Lemma 94.3 (3d) we know that the topological realization $|\operatorname{sd}(K)|$ is homeomorphic to $|K|$. Thus it suffices to show that $\operatorname{cat}(|\operatorname{sd}(K)|) \leq n$. For $i = 0, \ldots, n$ we set

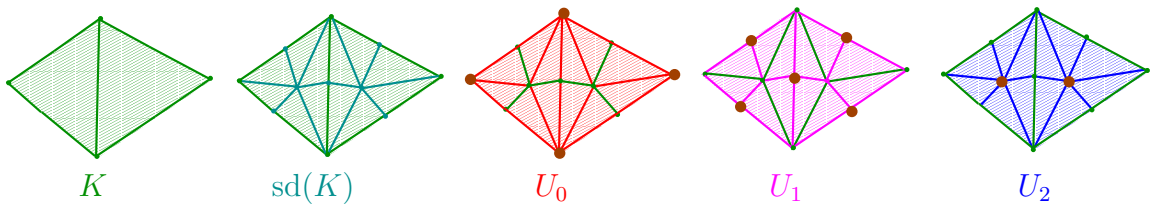
$$S_i \;:=\; \text{set of } i\text{-simplices of } K \qquad \text{and} \qquad U_i \;=\; \bigcup_{s \in S_i} \mathring{\operatorname{St}}(\operatorname{sd}(K), \underline{s}).$$

In other words, U_i is the union of the open stars in the barycentric subdivision corresponding to the vertices of the barycentric subdivision that are given by the barycenters of the i-simplices of K. We fix $i \in \{0, \ldots, n\}$. We make the following observations:
(a) By the Link Star Lemma 94.5 (1d) we know that each $\mathring{\operatorname{St}}(\operatorname{sd}(K), \underline{s})$ is contractible.
(b) It follows easily from the Link Star Lemma 94.5 (3) that for any two $s \neq t \in S_i$ we have $\mathring{\operatorname{St}}(\operatorname{sd}(K), s) \cap \mathring{\operatorname{St}}(\operatorname{sd}(K), t) = \varnothing$.

It follows almost immediately from (a) and (b), together with Lemma 115.13 that the inclusion $U_i \to |\operatorname{sd}(K)|$ is null-homotopic.

Furthermore it follows from the Link Star Lemma 94.5 (2b) that the U_i, $i = 0, \ldots, n$ form an open cover of $|\operatorname{sd}(K)|$. Thus we shown that $\operatorname{cat}(|\operatorname{sd}(K)|) \leq n$.

(2) By Proposition 94.12 we know that every connected n-dimensional CW-complex is homotopy equivalent to a connected n-dimensional simplicial complex. Thus the statement follows from (1) together with Proposition 115.11.



(3) Let X be a connected n-dimensional smooth manifold X. By the Smooth Manifold-Simplicial Structure Theorem 96.2 we know that X has the structure of an n-dimensional simplicial complex. Thus the statement follows from (1).
(4) Let X be a connected n-dimensional smooth manifold. There are two different approach to showing that $\mathrm{cat}(X) \leq n$:
   (a) By the Smooth Manifold-Simplicial Structure Theorem 96.2 we know that X has the structure of an n-dimensional simplicial complex. Thus the statement follows from (1).
   (b) In the proof of the Small Atlas Proposition 27.8 we implicitly showed that X admits an atlas $\{\Phi_i \colon U_i \to V_i\}_{i=1,\ldots,n+1}$ with $n+1$ charts with the property that for each component C of any U_i the map $\Phi_i|_C \colon C \to V_i$ factors through a contractible set. Since X is connected we see that each Φ_i is homotopic to a constant map.
(5) Let X be a connected n-dimensional topological manifold. There are again two different approaches to showing that $\mathrm{cat}(X) \leq n$:
   (a) At least if X is compact, then by Theorem 104.26 we know that X is homotopy equivalent to a connected n-dimensional simplicial complex. So the statement follows again from (1) together with Proposition 115.11.
   (b) The approach of (4a) also works in the topological context. ■

Remark. If X and Y are two connected CW-complexes with finite Lusternik-Schnirelmann categories, then $\mathrm{cat}(X \times Y) \leq \mathrm{cat}(X) + \mathrm{cat}(Y)$, see e.g. [**Fox1941**, Theorem 9] or [**CLOT2003**, Theorem 1.37]. The Ganea Conjecture from 1971 predicted that for any CW-complex X one has $\mathrm{cat}(X \times S^n) = \mathrm{cat}(X) + \mathrm{cat}(S^n) = \mathrm{cat}(X) + 1$. But this conjecture was eventually disproved by Norio Iwase [**Iw1998, Iw2002**] in 1998.

As we saw in the above examples and Proposition 115.12, one can obtain an upper bound on the Lusternik-Schnirelmann category of a given topological space by writing down explicit open subsets. To obtain lower bounds on the Lusternik-Schnirelmann category we introduce the following definition.

Definition. Given a topological space X we define its cup length $\mathrm{cl}(X) \in \mathbb{N}_0 \cup \{\infty\}$ as follows:

$$\mathrm{cl}(X) := \max\left\{ n \in \mathbb{N}_0 \,\middle|\, \begin{array}{c}\text{there exists a commutative ring } R \text{ and } R\text{-cohomology} \\ \text{classes } x_1, \ldots, x_n \text{ in degrees} \geq 1 \text{ with } x_1 \cup \cdots \cup x_n \neq 0\end{array} \right\}.$$

Examples.

(1) If X is an n-dimensional CW-complex, then it follows immediately from the Singular-Cellular H^*-Isomorphism Proposition 109.4 that $\mathrm{cl}(X) \leq n$.

(2) In Lemma 114.14 and Lemma 114.16 we saw that for any surface Σ of genus $g \geq 1$ there exist $x, y \in \mathrm{H}^1(\Sigma; \mathbb{Z})$ with $x \cup y \neq 0$, which together with (1) shows that $\mathrm{cl}(\Sigma) = 2$.
(3) In Lemma 114.18 we saw that there exists an $x \in \mathrm{H}_1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$ with $x \cup x \neq 0$, which shows that $\mathrm{cl}(\mathbb{R}\mathrm{P}^2) = 2$.

The following proposition shows that the cup length is a lower bound on the Lusternik-Schnirelmann category.

Proposition 115.14. For every topological space X we have

$$\mathrm{cl}(X) \ \leq \ \mathrm{cat}(X).$$

Example. It follows immediately from Proposition 115.14 and the above examples that

$$\mathrm{cat}(S^1 \times S^1) \ = \ \mathrm{cat}(\text{surface of genus } g \geq 2) \ = \ \mathrm{cat}(\mathbb{R}\mathrm{P}^2) \ = \ 2.$$

Proof. Let X be a topological space, let R be a commutative ring and let $x_j \in \mathrm{H}^{s_j}(X; R)$, $j = 1, \ldots, n$ be cohomology classes in dimensions $s_j \geq 1$ such that $x_1 \cup x_2 \cup \cdots \cup x_n \neq 0$. We have to show that $\mathrm{cat}(X) \geq n$.

Thus let $k \leq n$ and let $U_1, \ldots, U_k$ be open subsets in X with $U_1 \cup \cdots \cup U_k = X$ such that each inclusion map $U_i \to X$ is null-homotopic. We have to show that $U_1 \cup \cdots \cup U_k$ is a *proper* subset of X.

Claim. For any $j \in \{1, \ldots, k\}$ the obvious map $\rho_j \colon \mathrm{H}^{s_j}(X, U_j; R) \to \mathrm{H}^{s_j}(X; R)$ is in fact an isomorphism.

Proof. By our hypothesis the inclusion map $i_j \colon U_j \to X$ is null-homotopic, it follows that the restriction map $i_j^* \colon \mathrm{H}^k(X; R) \to \mathrm{H}^k(U_j; R)$ is the zero map for $k \geq 1$ and an epimorphism for $k = 0$.[1414] We deduce from this observation, the long exact sequence of cohomology groups corresponding to the pair (X, U_j) and the fact that $s_j \geq 1$ that the obvious map $\rho_j \colon \mathrm{H}^{s_j}(X, U_j; R) \to \mathrm{H}^{s_j}(X; R)$ is in fact an isomorphism. ⊞

Given $j \in \{1 \ldots, k\}$ we write $y_j := \rho_j^{-1}(x_j) \in \mathrm{H}^{s_j}(X, U_j; R)$ and we set $s = s_1 + \cdots + s_k$. By the Cup Product-Naturality Corollary 115.5[1415] we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^{s_1}(X, U_1; R) \times \cdots \times \mathrm{H}^{s_k}(X, U_k; R) & \xrightarrow{\cup} & \mathrm{H}^{s}(X, U_1 \cup \cdots \cup U_k; R) \\ \Big\downarrow {\scriptstyle (y_1, \ldots, y_k)} \atop {\scriptstyle \downarrow} \atop {\scriptstyle (x_1, \ldots, x_k)} & & \Big\downarrow \\ \mathrm{H}^{s_1}(X; R) \times \cdots \times \mathrm{H}^{s_k}(X; R) & \xrightarrow{\cup} & \mathrm{H}^{s}(X; R). \end{array}$$

By our hypotheses the cup product $x_1 \cup \cdots \cup x_k$ is non-zero, which then implies, by the commutativity of the diagram, that the cup product $y_1 \cup \cdots \cup y_k$ is also non-zero. But this implies that the group $\mathrm{H}^s(X, U_1 \cup \cdots \cup U_k; R)$ is non-zero, which in turn implies that $U_1 \cup \cdots \cup U_k$ is a proper subset of X. ■

[1414] This can be seen as follows: the map $i_j \colon U_j \to X$ is homotopic to a constant map that takes the value $P_j \in X$. We denote by $c_j \colon U_j \to \{P_j\}$ the constant map and we denote by $k_j \colon \{P_j\} \to X$ the inclusion map. We statement now follows from $i_j^* = (k_j \circ c_j)^* = c_j^* \circ k_j^*$ and Lemmas 108.13 and 108.12.

[1415] Strictly speaking Corollary 115.5 was formulated for the cup product of two terms, but one can easily reduce the general case of k terms to the special case of 2 terms.

Proposition 115.14 gives us a tool to attack Question 115.10. In fact we saw right after Proposition 115.14 that we can now answer the question for the 2-dimensional torus $(S^1)^2$ and the real projective plane $\mathbb{R}\mathrm{P}^2$.
The following questions naturally arise:

Question 115.15.
(1) What is the $\mathbb{Z}$-cohomology ring of $(S^1)^n$?
(2) What is the $\mathbb{F}_2$-cohomology ring of $\mathbb{R}\mathrm{P}^n$?
(3) What is the $\mathbb{Z}$-cohomology ring of $\mathbb{C}\mathrm{P}^n$?

We will answer these questions in Propositions 117.22, 121.16 and 121.7.
There are many other questions that we cannot answer in these notes, in fact which are still open. As an example we conclude this introduction to the Lusternik-Schnirelmann category of a topological space with the following conjecture which is formulated as a question in [**Rudy2017**, Question 1.1].

Conjecture 115.16. (Rudyak Conjecture) Let M and N be two closed oriented connected n-dimensional topological manifolds. If there exists a map $f\colon M \to N$ of degree one, then $\mathrm{cat}(M) \geq \mathrm{cat}(N)$.

Exercises for Chapter 115.

Exercise 115.1. Let (X, A, B) be an excisive triad of topological spaces and let R be a commutative ring. Show that for every $\varphi \in \mathrm{H}^k(X, A; R)$ and $\psi \in \mathrm{H}^l(X, B; R)$ the following equality holds:

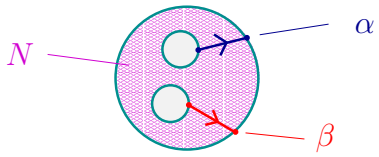
$$\varphi \cup \psi \;=\; (-1)^{kl} \cdot \psi \cup \varphi \;\in\; \mathrm{H}^{k+l}(X, A \cup B; R).$$

Hint. Evidently you should study the proof of the Cup Product-Commutativity Proposition 114.8 carefully.

Exercise 115.2. Let M be the torus minus one open disk. Show that the cup product $\mathrm{H}^1(M, \partial M; \mathbb{Z}) \times \mathrm{H}^1(M, \partial M; \mathbb{Z}) \to \mathrm{H}^2(M, \partial M; \mathbb{Z})$ is non-trivial.
Hint. Reduce this statement to the calculation of the cup product of the torus.

Exercise 115.3. Let N be the disk $\overline{B}^2$ minus two open disks.

(a) Show that the two homology classes $\alpha, \beta \in \mathrm{H}_1(N, \partial N; \mathbb{Z})$ shown in the figure below are a basis for $\mathrm{H}_1(N, \partial N; \mathbb{Z}) \cong \mathbb{Z}^2$.
(b) We denote by $\varphi, \psi \in \mathrm{H}^1(N, \partial N; \mathbb{Z}) = \mathrm{Hom}(\mathrm{H}_1(N, \partial N; \mathbb{Z}), \mathbb{Z})$ the basis dual to α, β. Show that $\alpha \cup \beta = 0 \in \mathrm{H}^2(N, \partial N; \mathbb{Z})$.
Hint. Show that there exist open subsets A and B with the following properties:
(i) $\partial N \subset A$ and $\partial N \subset B$.
(ii) $A \cup B = N$.
(iii) φ lies in the image of $\mathrm{H}^1(N, A; \mathbb{Z}) \to \mathrm{H}^1(N, \partial N; \mathbb{Z})$ and ψ lies in the image of $\mathrm{H}^1(N, B; \mathbb{Z}) \to \mathrm{H}^1(N, \partial N; \mathbb{Z})$.
(c) Show that the cup product $\mathrm{H}^1(N, \partial N; \mathbb{Z}) \times \mathrm{H}^1(N, \partial N; \mathbb{Z}) \to \mathrm{H}^2(N, \partial N; \mathbb{Z})$ is the zero map.

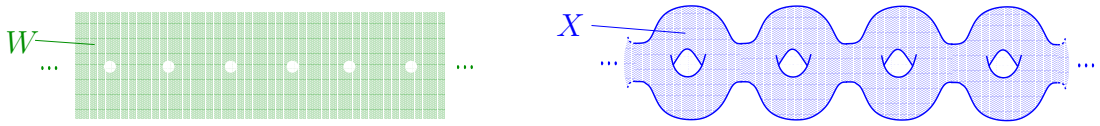


Exercise 115.4. Let M be the torus minus one open disk and let N be the disk $\overline{B}^2$ minus two open disks. Use the calculations from Exercise 115.2 and 115.3 to show that there is no degree one map $g\colon N \to M$.
Remark. This gives an alternative solution to Exercise 101.7 (b).

Exercise 115.5. As in Exercise 115.5 we consider $W = \mathbb{R}^2 \setminus \{(n,0)\,|\,n \in \mathbb{Z}\}$ and we consider the surface X of infinite genus that is shown in the figure below.

(a) Show that the cup product $\mathrm{H}^1_{\mathrm{c}}(W;\mathbb{Z}) \times \mathrm{H}^1_{\mathrm{c}}(W;\mathbb{Z}) \to \mathrm{H}^2_{\mathrm{c}}(W;\mathbb{Z})$, that we defined on page 2539, is the zero map.
(b) Show that the cup product $\mathrm{H}^1_{\mathrm{c}}(X;\mathbb{Z}) \times \mathrm{H}^1_{\mathrm{c}}(X;\mathbb{Z}) \to \mathrm{H}^2_{\mathrm{c}}(X;\mathbb{Z})$ is non-zero.
Hint. You could use the pairing $\mathrm{H}^2_{\mathrm{c}}(X;\mathbb{Z}) \times \mathrm{H}^{\mathrm{lf}}_2(X) \to \mathbb{Z}$ from Exercise 106.5 to detect that an element in $\mathrm{H}^2_{\mathrm{c}}(X;\mathbb{Z})$ is non-zero.
(c) Show that X and W are not homeomorphic.
(d) Does this argument imply that X and W are not homotopy equivalent?

Remark. In Exercise 55.14 we showed the weaker statement that W and X, viewed as 2-dimensional smooth manifolds, are not diffeomorphic.



Exercise 115.6. Let X and Y be non-empty topological spaces. Show that we have the following lower bound on the cup length of $X \times Y$:

$$\mathrm{cl}(X \times Y) \;\geq\; \max\{\mathrm{cl}(X), \mathrm{cl}(Y)\}.$$

Exercise 115.7. Let X and Y be topological spaces. Show that if X and Y are homotopy equivalent, then the cup lengths of X and Y agree, i.e. we have $\mathrm{cl}(X) = \mathrm{cl}(Y)$.

Exercise 115.8. Determine the Lusternik-Schnirelmann category for all compact (not necessarily connected) 2-dimensional smooth manifolds.

Exercise 115.9. Let X and Y be topological spaces. Show that if X and Y are homotopy equivalent, then $\mathrm{cat}(X) = \mathrm{cat}(Y)$.

116. The cap product

In this chapter we introduce the cap product that relates the homology and the cohomology groups. It can be viewed as a generalization of the Kronecker pairing. Initially it might not be clear that the cap product is good for, but later on we will see that it plays an essential role in many calculations and in the formulation of the Poincaré Duality Theorem.

116.1. **The definition of the cap product and calculations.** We start out by introducing a convention that simplifies our notation later on.

Convention. Let R be a commutative ring and let S be a non-empty set. We consider

$$R^{(S)} \ := \ \text{all maps from } S \text{ to } R \text{ which are non-zero for only finitely many } s \in S$$

with the obvious R-module structure. We refer to $R^{(S)}$ as the free R-module generated by S. For example, the free $\mathbb{Z}$-module generated by S is precisely the same as the free abelian group generated by S. We use the same convention as on page 1147 to identify elements in $R^{(S)}$ with finite formal linear combinations $r_1s_1+\cdots+r_ks_k$ where $r_1,\ldots,r_k \in R$ and $s_1,\ldots,s_k \in S$. By the Tensor Lemma 89.3 the map

$$\begin{array}{rcl} \mathbb{Z}^{(S)} \otimes R & \to & R^{(S)} \\ \sum_{i=1}^n n_i \cdot s_i \otimes r_i & \mapsto & \sum_{i=1}^n \underbrace{n_i r_i}_{\in R} \cdot s_i \end{array}$$

is an isomorphism. We use this isomorphism to identify the left-hand side with the right-hand side. In particular, given a topological space X and given $k \in N_0$ we write

$$\mathrm{C}_k(X;R) \ := \ \mathbb{Z}^{S_k(X)} \otimes R \ = \ R^{S_k(X)} \ = \ \left\{ \sum_{i=1}^n r_i \cdot \sigma_i \,\middle|\, r_1,\ldots,r_n \in R \text{ and } \sigma_i \colon \Delta^k \to X \right\},$$

where $S_k(X)$ denotes the set of singular k simplices in X.

The following definition is inspired by the second definition of the cup product that we gave on page 2513.

Definition. Let X be a topological space and let R be a commutative ring. Given a singular cochain $\varphi \in \mathrm{C}^k(X;R)$ and a singular simplex $\sigma \colon \Delta^l \to X$ with $k \leq l$ we define the cap product as[1416]

$$\varphi \cap \sigma \ := \ \underbrace{\varphi(\sigma \circ [v_0,\ldots,v_k])}_{\in R} \cdot \underbrace{\sigma \circ [v_k,\ldots,v_l]}_{\text{map } \Delta^{l-k} \to X}.$$

We extend this definition to singular R-chains and we obtain the cap product

$$\begin{array}{rcl} \cap \colon \mathrm{C}^k(X;R) \times \mathrm{C}_l(X;R) & \to & \mathrm{C}_{l-k}(X;R) \\ \varphi \cap r \cdot \sigma & \mapsto & r \cdot \underbrace{(\varphi \cap \sigma)}_{\in R}. \end{array}$$

For $k > l$ we define the cap product to be the zero map.

As for the cup product, the first thing we want to verify is that this map on the cochain and chain level defines a map on the level of cohomology groups and homology groups. The key to doing so is the following elementary lemma.

[1416] Here we use, as usual, the notation from page 2503.

Lemma 116.1. Let X be a topological space and let R be a commutative ring. Furthermore let $\varphi \in \mathrm{C}^k(X;R)$ and let $\sigma \in \mathrm{C}_l(X;R)$. Then

$$\partial(\varphi \cap \sigma) \;=\; (-1)^k \cdot (-\delta\varphi \cap \sigma \;+\; \varphi \cap \partial\sigma) \;\in\; \mathrm{C}_{l-k-1}(X;R).$$

In particular this implies that $\varphi \cap \partial\sigma$ and $\delta\varphi \cap \sigma$ are homologous.

Proof. For $k > l$ the statement is trivial. So assume that $k \leq l$. Clearly it suffices to prove the equality for the case that $\sigma\colon \Delta^l \to X$ is a singular l-simplex. This case follows from an elementary calculation. Indeed, we have

$$\begin{aligned}
\text{(a)} \qquad \varphi \cap \partial\sigma \;&=\; \sum_{i=0}^{k} (-1)^i \cdot \varphi(\sigma \circ [v_0, \ldots, \widehat{v_i}, \ldots, v_{k+1}]) \cdot \sigma \circ [v_{k+1}, \ldots, v_l] \\
&\qquad + \sum_{i=k+1}^{l} (-1)^i \cdot \varphi(\sigma \circ [v_0, \ldots, v_k]) \cdot \sigma \circ [v_k, \ldots, \widehat{v_i}, \ldots, v_l] \\
\text{(b)} \qquad \delta\varphi \cap \sigma \;&=\; \sum_{i=0}^{k+1} (-1)^i \cdot \varphi(\sigma \circ [v_0, \ldots, \widehat{v_i}, \ldots, v_{k+1}]) \cdot \sigma \circ [v_{k+1}, \ldots, v_l] \\
\text{(c)} \qquad (-1)^k \cdot \partial(\varphi \cap \sigma) \;&=\; \sum_{i=k}^{l} (-1)^i \cdot \varphi(\sigma \circ [v_0, \ldots, v_k]) \cdot \sigma \circ [v_k, \ldots, \widehat{v_i}, \ldots, v_l].
\end{aligned}$$

Note that the summand of (b) corresponding to $i = k+1$ equals *minus* the summand of (c) corresponding to $i = k$. Adding (b) and (c) and comparing it to (a) we see that

$$\varphi \cap \partial\sigma \;=\; \delta\varphi \cap \sigma + (-1)^k \cdot \partial(\varphi \cap \sigma)$$

which gives the desired equality. ∎

The following lemma says that "$\cap$" descends to a map on homology and cohomology.

Lemma 116.2. Let X be a topological space and let R be a commutative ring. Then for any $k, l \in \mathbb{N}_0$ the map

$$\begin{array}{ccc} \mathrm{H}^k(X;R) \times \mathrm{H}_l(X;R) & \to & \mathrm{H}_{l-k}(X;R) \\ ([\varphi],[\sigma]) & \mapsto & [\varphi \cap \sigma] \end{array}$$

is well-defined and R-bilinear.

We refer to the pairing from Lemma 116.2 as the *cap product*.[1417]

Proof. For $k > l$ the statement is trivial. So assume that $k \leq l$. The statement now follows easily from Lemma 116.1. Indeed, let $\varphi \in \mathrm{C}^k(X;R)$ be a cocycle and let $\sigma \in \mathrm{C}_l(X;R)$ be a cycle.

(1) It follows from Lemma 116.1 that $\varphi \cap \sigma$ is a cycle, i.e. it represents a homology class.
(2) If $\sigma' = \sigma + \partial\tau$ is another representative of $[\sigma]$, then

$$\varphi \cap \sigma - \varphi \cap \sigma' \;=\; \varphi \cap (\sigma - \sigma') \;=\; \varphi \cap \partial\tau \;\underset{\uparrow}{=}\; \pm\partial(\varphi \cap \tau).$$

by Lemma 116.1 and since $\delta\varphi = 0$

This shows that $\varphi \cap \sigma$ and $\varphi \cap \sigma'$ represent the same homology class, i.e. the cap product does not depend on the choice of the representative of the homology class.

[1417] The name "cap product" comes from the notation "$a \cap b$" for the composition since $\cap$ looks like, just a little bit, like the "a" in cap. The other reason is that the cup product was first introduced, and mathematicians needed a different name for a somewhat related object. Also mathematicians are well-known for their great sense of humor.

(3) The proof that the cap product does not depend on the choice of representative of the cohomology class is basically the same as the proof of (2).

This shows that the map is well-defined. It is clear that the map is indeed R-bilinear. ■

Remark. We already saw on page 2514 that the definition of the cup product in the literature is not uniform. Rather depressingly, the situation is even worse for the cap product, there is an endless variety of definitions of the cap product in the literature. More precisely, given a cochain $\varphi \in \mathrm{C}^k(X;R)$ and a singular simplex $\sigma\colon \Delta^l \to X$ the following definitions of $\varphi \cap \sigma$ can be found in the literature:[1418]

		$\varphi \cap \sigma$
(a)	Hatcher [**Hat2002**, p. 239]	$\varphi(\sigma \circ [v_0, \dots, v_k]) \cdot \sigma \circ [v_k, \dots, v_l]$
(b)	Bredon [**Bre1993**, p. 335]	$(-1)^{k(l-k)} \cdot \varphi(\sigma \circ [v_0, \dots, v_k]) \cdot \sigma \circ [v_k, \dots, v_l]$
(c)	Dold [**Dol1956**, p. 335]	$(-1)^{k(l-k)} \cdot \sigma \circ [v_0, \dots, v_{l-k}] \cdot \varphi(\sigma \circ [v_{l-k}, \dots, v_l])$
(d)	Spanier [**Spa1995**, p. 254]	$\sigma \circ [v_0, \dots, v_{l-k}] \cdot \varphi(\sigma \circ [v_{l-k}, \dots, v_l])$
(e)	Munkres [**Mun1984**, p. 389]	$\sigma \circ [v_0, \dots, v_{l-k}] \cdot \varphi(\sigma \circ [v_{l-k}, \dots, v_l])$

To the best of my understanding, see [**Spa1995**, p. 250], the singular chains

$$\varphi(\sigma \circ [v_0, \dots, v_k]) \cdot \sigma \circ [v_k, \dots, v_l] \quad \text{and} \quad (-1)^{k(l-k)} \cdot \sigma \circ [v_0, \dots, v_{l-k}] \cdot \varphi(\sigma \circ [v_{l-k}, \dots, v_l])$$

are homologous. This shows that on the level of homology and cohomology groups the definition (a), which is also our definition, agrees with (c), but that the sign differs for (b), (d) and (e). For the most part this is not an issue, but one needs to be careful when one cites formulas from the literature. For example, as we had already mentioned on page 2400, Bredon [**Bre1993**] defined the coboundary map as

$$\delta_k^{\mathrm{Br}} \;:=\; (-1)^{k+1} \cdot \partial_{k+1}^* \;=\; (-1)^{k+1} \cdot \delta_k$$

and as we saw above, the cap product is also defined differently. Thus our Lemma 116.1, which says that for $\varphi \in \mathrm{C}^k(X;R)$ and $\sigma\colon \Delta^l \to X$ we have

$$\partial(\varphi \cap \sigma) \;=\; (-1)^k \cdot (\varphi \cap \partial\sigma \;-\; \delta\varphi \cap \sigma)$$

becomes the following equality in [**Bre1993**, Proposition VI.5.1]:[1419]

$$\partial(\varphi \cap \sigma) \;=\; (-1)^k \cdot \varphi \cap \partial\sigma \;+\; \delta^{\mathrm{Br}}(\varphi) \cap \sigma.$$

The following lemma follows immediately from the definitions.

Lemma 116.3. Let X be a topological space and let R be a commutative ring. Then for any $\sigma \in \mathrm{H}_k(X;R)$ we have

$$1_X \cap \sigma \;=\; \sigma.$$

For future applications we state the special case $k = l$ in the cap product in the following lemma.

[1418]The definition of the cap product given by Greenberg-Harper in [**GrH1981**, p. 205] agrees with our definition, *except*, that they use a different notation, namely they cap a homology class with a cohomology class instead of capping a cohomology class with a homology class. In other words, in our setting they write $\sigma \cap \varphi$ instead of $\varphi \cap \sigma$.

[1419]The lesson you should draw is that if you ever have to write algebraic topology lectures notes, stick to a single book and do not try to mix books.

Lemma 116.4. (Cap Product-Kronecker Lemma) Let X be a topological space and let R be a commutative ring.

(1) The composition

$$\mathrm{H}^k(X;R) \times \mathrm{H}_k(X;R) \xrightarrow{\cap} \mathrm{H}_0(X;R) \xrightarrow[\text{from page 1914}]{\text{augmentation } \epsilon} R$$

is precisely the Kronecker pairing from pages 2425 and 2452.

(2) If X is path-connected, and if we make the usual identification $\mathrm{H}_0(X;R) = R$ given by the augmentation map, see the H_0-Proposition 72.5, then for any $\varphi \in \mathrm{H}^k(X;R)$ and $\sigma \in \mathrm{H}_k(X;R)$ we have

$$\varphi \cap \sigma \;=\; \langle \varphi, \sigma \rangle \;\in\; R.$$

Proof. Both statements follow immediately from recalling all the relevant definitions. ■

Example. We consider some examples of the cap product in detail to get some feeling for what it does.

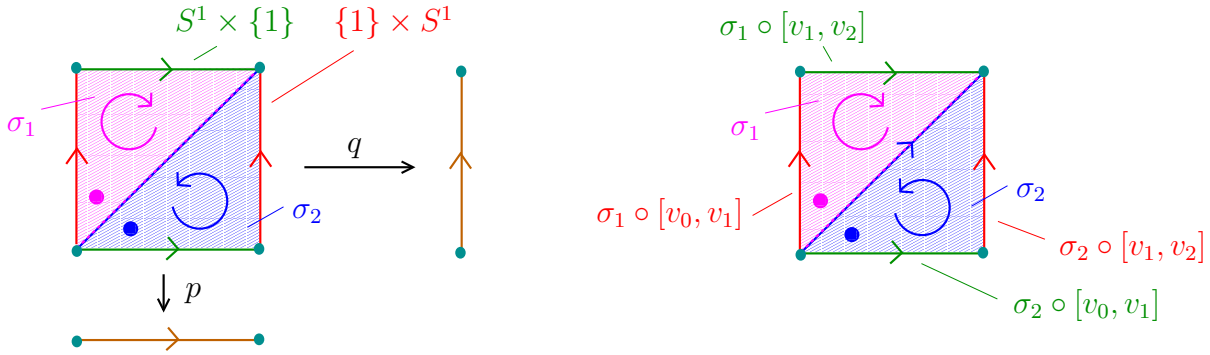
(1) Let $T = S^1 \times S^1 = ([0,1] \times [0,1])/\!\sim$ be the torus. We recall the following notations and results that we have by now used on many occasions. (We refer to the figure below for some of the notation.)

(a) We denote by σ_1, σ_2 the singular 2-simplices from the figure below. As we pointed out on page 2204, the singular 2-chain $-\sigma_1 + \sigma_2$ is a representative of the fundamental class $[T]$.

(b) We denote by p respectively q the projection $S^1 \times S^1 \to S^1$ onto the first respectively second factor.

(c) We denote by $\theta := \theta_{\mathbb{Z}} \in \mathrm{C}^1;\mathbb{Z})$ the singular 1-cocycle that we introduced on page 2404.

(d) As before we denote by $\mu\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $\mu(1-t,t) = \exp(2\pi\,\mathrm{i}t)$. By the discussion on page 2201 we know that μ represents the fundamental class $[S^1]$ of S^1 equipped with the standard orientation. By a slight abuse of notation we also denote by $[1 \times S^1]$ the image of the fundamental class under the monomorphism $\mathrm{H}_1(1 \times S^1;\mathbb{Z}) \to \mathrm{H}_1(S^1 \times S^1;\mathbb{Z})$.



We calculate that

$$p^*(\theta) \cap \sigma_1 \;=\; \underbrace{p^*(\theta)(\sigma_1 \circ [v_0, v_1])}_{=\theta(p\circ\sigma_1\circ[v_0,v_1])=0} \cdot \sigma_1 \circ [v_1, v_2] \;=\; 0$$

and similarly we calculate that

$$p^*(\theta) \cap \sigma_2 \;=\; \underbrace{p^*(\theta)(\sigma_2 \circ [v_0, v_1])}_{=\theta(p\circ\sigma_2\circ[v_0,v_1])=1} \cdot \sigma_2 \circ [v_1, v_2] \;= \sigma_2 \circ [v_1, v_2] = 1 \times \mu.$$

Putting these calculations together we obtain that

$$[p^*(\theta)] \cap [T] \;=\; [p^*(\theta) \cap (-\sigma_1 + \sigma_2)] \;=\; [1 \times S^1].$$

Basically the same calculation shows that

$$[q^*(\theta)] \cap [T] \;=\; [q^*(\theta) \cap (-\sigma_1 + \sigma_2)] \;=\; -[S^1 \times 1].$$

So we see that the map

$$\cap[T] \colon \mathrm{H}^1(T; \mathbb{Z}) \;\to\; \mathrm{H}_1(T; \mathbb{Z})$$

is non-trivial, in fact we have just shown that this map satisfies

$$\begin{array}{rcl} [p^*(\theta)] & \mapsto & [1 \times S^1] \\ [q^*(\theta)] & \mapsto & -[S^1 \times 1] \end{array}$$

which shows that "capping with the fundamental class" is in this case an isomorphism.

(2) In Exercise 116.2 we will deal with the surface Σ of genus 2 and there we will also see that the map

$$\cap[\Sigma] \colon \mathrm{H}^1(\Sigma; \mathbb{Z}) \;\to\; \mathrm{H}_1(\Sigma; \mathbb{Z})$$

is an isomorphism.

(3) Finally we consider again the real projective plane $\mathbb{R}\mathrm{P}^2 = \overline{B}^2/\!\sim$. We introduce the following notation:

(a) We denote by $\varphi \in \mathrm{C}^1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$ the cocycle that we introduced on page 2530.

(b) We denote by $\gamma \colon \Delta^1 \to \mathbb{R}\mathrm{P}^2 = \overline{B}^2/\!\sim$ the singular 1-simplex that is given by $\gamma(t, 1-t) = [\exp(\pi i t)]$. Note that γ is a cycle in $\mathrm{C}_1(\mathbb{R}\mathrm{P}^2)$ and note that $\varphi(\gamma) = 1$.

(c) In the figure on page 2216 we already saw a representative for the $\mathbb{F}_2$-fundamental class, but we will work with a different one that is "smaller" and that is shown in the figure below. In the figure below we see the singular 2-simplex σ that sends the face $[v_0, v_2]$ to a point P and the singular 2-simplex τ that sends all points on Δ^2 to P. It is straightforward to see that $(\sigma + \tau) \otimes 1 \in \mathrm{C}_2(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$ is a cycle. Using Proposition 100.18 we see that it represents the $\mathbb{F}_2$-fundamental class of $\mathbb{R}\mathrm{P}^2$.

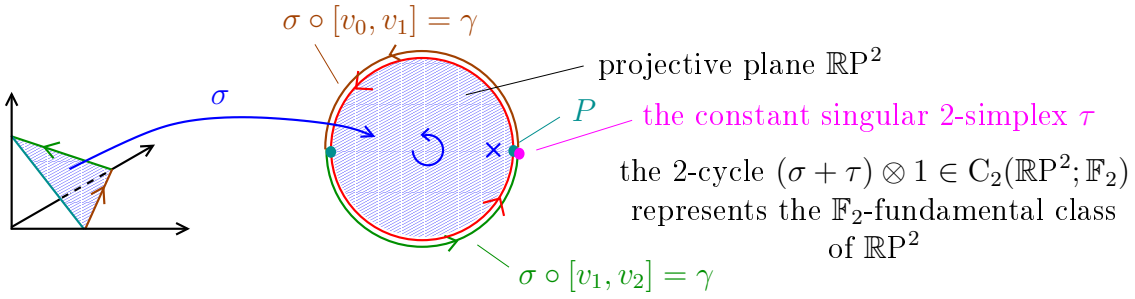
It is straightforward to see that

$$[\varphi] \cap [\mathbb{R}\mathrm{P}^2] \;=\; [\varphi] \cap [(\sigma + \tau) \otimes 1] \;=\; [\underbrace{\varphi \cap \sigma}_{=\gamma} \otimes 1 + \underbrace{\varphi \cap \tau}_{=0} \otimes 1] \;=\; [\gamma \otimes 1]$$

We saw on page 2530 that $[\gamma \otimes 1]$ is the unique non-trivial element of $\mathrm{H}_1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$. Thus in this case we see that the map

$$\cap[\mathbb{R}\mathrm{P}^2] \colon \mathrm{H}^1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2) \;\to\; \mathrm{H}_1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$$

is also an isomorphism.



Remark. In the last three examples we saw for the closed 2-dimensional smooth manifolds $F = S^1 \times S^1$, $F = \Sigma$, $F = \mathbb{R}\mathrm{P}^2$ and for $R = \mathbb{Z}$ respectively $R = \mathbb{F}_2$ that the map

$$- \cap [F]_R \colon \mathrm{H}^1(F; R) \ \to \ \mathrm{H}_1(F; R)$$

is an isomorphism. In Chapter 119 we will see that it is not a coincidence. More precisely we will see that this observation can be generalized to all closed topological manifolds.

116.2. **The relative cap product and more properties of the cap product.** After considering the various examples of the cap product we continue with developing the theory. Similar to the case of the cup product, we intend to extend the notion of cap product to relative homology and cohomology groups. As in the case of the cup product it requires some delicate footwork to get to the right notion of a relative cap product. We start out with the following rather technical definition.

Definition. Let (X, A, B) be a triad of topological spaces and let R be a commutative ring. We write

$$\mathrm{C}_n(X, \{A, B\}; R) \ := \ \frac{\mathrm{C}_n(X) \otimes R}{\mathrm{C}_n^{\{A,B\}}(A \cup B) \otimes R}.$$

The boundary map on $\mathrm{C}_*(X; R)$ descends to a boundary map on $\mathrm{C}_*(X, \{A, B\}; R)$ which we can thus view as a chain complex. We denote the corresponding homology groups by $\mathrm{H}_*(X, \{A, B\}; R)$.

This technical definition allows us to define a first version of a relative cap product.

Lemma 116.5. Let (X, A, B) be a triad of topological spaces, let $k \leq l$ and let R be a commutative ring. Then the map

$$\begin{aligned} \cap \colon \mathrm{C}^k(X, A; R) \times \mathrm{C}_l(X, \{A, B\}; R) \ &\to \ \mathrm{C}_{l-k}(X, B; R) \\ (\varphi, [\sigma \colon \Delta^l \to X]) \ &\mapsto \ \big[\underbrace{\varphi(\sigma \circ [v_0, \dots, v_k])}_{\in R} \cdot \underbrace{\sigma \circ [v_k, \dots, v_l]}_{\text{map } \Delta^{l-k} \to X}\big] \end{aligned}$$

is well-defined and it descends to a map

$$\cap \colon \mathrm{H}^k(X, A; R) \times \mathrm{H}_l(X, \{A, B\}; R) \ \to \ \mathrm{H}_{l-k}(X, B; R)$$

that is R-bilinear.

Proof. We first show that the cap product on (co-) chains is well-defined. By definition of

$$\mathrm{C}_n(X, \{A, B\}; R) \ := \ \frac{\mathrm{C}_n(X) \otimes R}{\mathrm{C}_n^{\{A,B\}}(A \cup B) \otimes R} \ = \ \frac{\mathrm{C}_n(X) \otimes R}{\mathrm{C}_n(A) \otimes R + \mathrm{C}_n(B) \otimes R}$$

we have to show that if we are given a singular l-simplex $\sigma\colon \Delta^l \to X$ such that

(1) the image of σ lies in A, or
(2) the image of σ lies in B,

then the product $\varphi(\sigma \circ [v_0, \ldots, v_k]) \cdot \sigma \circ [v_0, \ldots, v_k]$ is zero in $\mathrm{C}_{l-k}(X, B; R)$. But this is indeed the case:

(1) first, if the image of σ lies in A, then it follows immediately from the hypothesis that φ lies in $\mathrm{C}^k(X, A; R)$ that $\varphi(\sigma \circ [v_0, \ldots, v_k]) = 0$, and
(2) if the image of σ lies in B, then the image of $\sigma \circ [v_0, \ldots, v_k]$ also lies in B. Hence the term is zero in $\mathrm{C}_{l-k}(X, B; R)$.

This concludes the proof of the first part of the lemma.

The argument that the map on (co-) chain complexes descends to a map on the level of homology and cohomology groups is the same as in the proof of Lemma 116.2. It is obvious that the map $\cap$ is R-bilinear. ■

Evidently we are not very interested in the group $\mathrm{H}_n(X, \{A, B\}; R)$. The following lemma allows us to replace this group by the much more agreeable group $\mathrm{H}_n(X, A \cup B; R)$, at least for reasonable triads.

Lemma 116.6. If (X, A, B) is an excisive triad of topological spaces, then given any commutative ring R the obvious map

$$\mathrm{C}_*(X, \{A, B\}; R) \quad \to \quad \mathrm{C}_*(X, A \cup B; R)$$

induces for every $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X, \{A, B\}; R) \quad \xrightarrow{\cong} \quad \mathrm{H}_n(X, A \cup B; R).$$

Proof. In the proof of Lemma 115.1 we already showed that the map

$$\mathrm{C}_*(X, \{A, B\}) \ \to \ \mathrm{C}_*(X, A \cup B)$$

on chain complexes is a chain homotopy equivalence. By tensoring all maps and chain homotopies with R we see that the map

$$\mathrm{C}_*(X, \{A, B\}; R) \ \to \ \mathrm{C}_*(X, A \cup B; R)$$

is also a chain homotopy equivalence. The second statement is then an immediate consequence of Corollary 73.4. ■

Lemma 116.6 leads to the following definition.

Definition. Let (X, A, B) be an excisive triad of topological spaces, let $k \leq l$ and let R again be a commutative ring. We refer to the map

$$\mathrm{H}^k(X, A; R) \times \mathrm{H}_l(X, A \cup B; R) \quad \xleftarrow[\uparrow]{\cong} \quad \mathrm{H}^k(X, A; R) \times \mathrm{H}_l(X, \{A, B\}; R) \quad \xrightarrow[\uparrow]{\cap} \quad \mathrm{H}_{l-k}(X, B; R)$$

isomorphism given by Lemma 116.6 — map from Lemma 116.5

as the cap product (sometimes we refer to it as the relative cap product) that we also denote by "$\cap$". For $k > l$ we define the cap product to be the zero map.

Remark. In many examples, e.g. if $A = \varnothing$, or $B = \varnothing$ or $A = B$, then the slightly mysterious first map in the definition of the cap product is just the identity. In particular,

for any subset A of X we obtain cap products

$$\mathrm{H}^k(X;R) \ \times \ \mathrm{H}_l(X,A;R) \ \stackrel{\cap}{\longrightarrow} \ \mathrm{H}_{l-k}(X,A;R)$$

and

$$\mathrm{H}^k(X,A;R) \ \times \ \mathrm{H}_l(X,A;R) \ \stackrel{\cap}{\longrightarrow} \ \mathrm{H}_{l-k}(X;R)$$

and both of them are induced, on the (co-) chain level, by the formula

$$(\varphi, \sigma\colon \Delta^l \to X) \quad \mapsto \quad \varphi(\sigma\circ[v_0,\dots,v_k])\cdot\sigma\circ[v_k,\dots,v_l].$$

This observation also shows that we can easily generalize the Cap Product-Kronecker Lemma 116.4 to the relative case in the obvious way. For example, if X is path-connected, and if we make the usual identification $\mathrm{H}_0(X;\mathbb{Z}) = \mathbb{Z}$ given by the augmentation map, see the H_0-Proposition 72.5, then for any $\varphi \in \mathrm{H}^k(X,A;\mathbb{Z})$ and $\sigma \in \mathrm{H}_k(X,A;\mathbb{Z})$ we have

$$\varphi\cap\sigma \ = \ \langle\varphi,\sigma\rangle \ \in \ \mathbb{Z}.$$

For the record we point out that it follows from this observation and the Dual Fundamental Class Lemma 110.15 that if M is a compact oriented path-connected n-dimensional topological manifold, then for any point $P \in M$ we have

$$[M]^* \cap [M] \ = \ [P]$$

and $[M]^*$ is the only element in $\mathrm{H}^n(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}$ that has that property.

Example. Let $X = S^1 \times [0,1]$ be the annulus. In the figure below we show two singular 2-simplices σ_1, σ_2. It follows easily from the Fundamental Class-Criterion Proposition 100.4 that $-\sigma_1+\sigma_2$ represents the fundamental class $[X] \in \mathrm{H}_2(X,\partial X;\mathbb{Z})$ of the annulus X with the usual orientation. We denote by $p\colon X \to [0,1]$ the projection . As on page 2408 we consider the function

$$\begin{array}{rcl} f\colon [0,1] & \to & \mathbb{Z} \\ x & \mapsto & \left\{ \begin{array}{ll} 0, & \text{if } x \in [0,\frac{1}{2}) \\ 1, & \text{otherwise} \end{array} \right. \end{array}$$

which we use to define the singular 1-cochain

$$\begin{array}{rcl} \rho\colon \mathrm{C}_1([0,1]) & \to & \mathbb{Z} \\ (\sigma\colon \Delta^1 \to [0,1]) & \mapsto & f(\sigma(0,1)) - f(\sigma(1,0)). \end{array}$$

It follows from the discussion on page 2408 that $p^*(\rho) \in \mathrm{C}^1(X,\partial X;\mathbb{Z})$ is a cochain, thus it represents an element in $\mathrm{H}^1(X,\partial X;\mathbb{Z})$. We calculate that

$$\begin{array}{rcl} [p^*(\rho)]\cap\sigma_1 & = & \underbrace{p^*(\sigma_1\circ[v_0,v_1])}_{=1}\cdot\underbrace{\sigma_1\circ[v_1,v_2]}_{=[S^1\times 1]} \ = \ [S^1\times 1] \\ [p^*(\rho)]\cap\sigma_2 & = & \underbrace{p^*(\sigma_2\circ[v_0,v_1])}_{=0}\cdot\sigma_2\circ[v_1,v_2] \ = \ 0. \end{array}$$

Adding up we see that

$$[p^*(\rho)]\cap[X] \ = \ [p^*(\rho)\cap(-\sigma_1+\sigma_2)] \ = \ -[S^1\times 1].$$

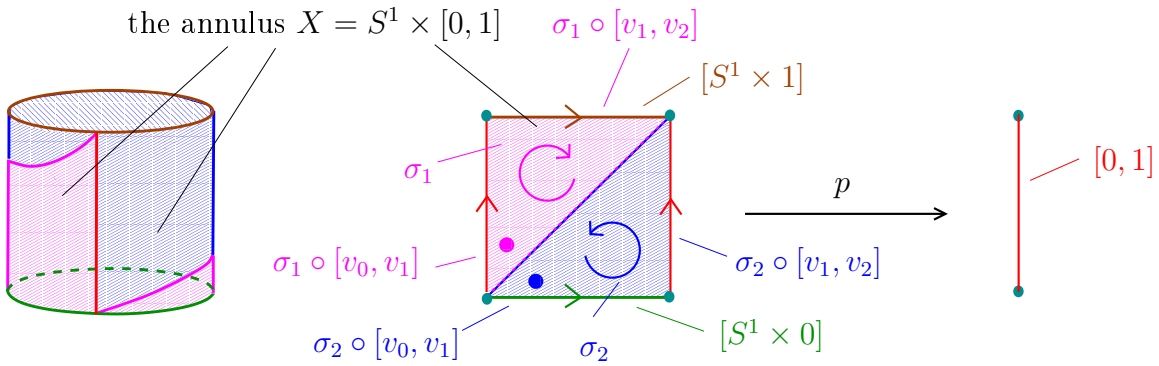
Since the inclusion $S^1\times 1 \to X$ is a homotopy equivalence we see that $\mathrm{H}_1(X;\mathbb{Z}) \cong \mathbb{Z}$ and that $[S^1\times 1]$ is a generator. A straightforward argument, e.g. using the long exact sequence of the pair $(X,\partial X)$, shows that $\mathrm{H}^1(X,\partial X;\mathbb{Z}) \cong \mathbb{Z}$. The above calculation thus shows that the map

$$\begin{array}{rcl} \mathrm{H}^1(X,\partial X;\mathbb{Z}) & \to & \mathrm{H}_1(X;\mathbb{Z}) \\ \varphi & \mapsto & \varphi\cap[X] \end{array}$$

is an epimorphism. Since both groups are isomorphic to $\mathbb{Z}$ we see that the map is in fact an isomorphism and as a bonus we obtain that $[\rho] \in \mathrm{H}^1(X, \partial X; \mathbb{Z})$ is a generator. In Exercise 116.3 we will show that the map

$$\begin{array}{ccc} \mathrm{H}^1(X;\mathbb{Z}) & \to & \mathrm{H}_1(X, \partial X; \mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [X] \end{array}$$

is also an isomorphism.



The following lemma relates the cup product to the cap product.

Lemma 116.7. Cup-Cap Lemma Let X be a topological space and let R be a commutative ring.

(1) Let $\varphi \in \mathrm{C}^k(X;R)$, $\psi \in \mathrm{C}^l(X;R)$ and let $\sigma \in \mathrm{C}_n(X;R)$. Then[1420]
$$\varphi \cap (\psi \cap \sigma) \ = \ (\psi \cup \varphi) \cap \sigma \ \in \ \mathrm{C}_{n-k-l}(X;R).$$

(2) If $A \subset X$ is a subset and $C, D \in \{\varnothing, A\}$ we have for any $\varphi \in \mathrm{H}^k(X, C; R)$, any $\psi \in \mathrm{H}^l(X, D; R)$ and any $\sigma \in \mathrm{H}_n(X, C \cup D; R)$ that
$$\varphi \cap (\psi \cap \sigma) \ = \ (\psi \cup \varphi) \cap \sigma \ \in \ \mathrm{H}_{n-k-l}(X;R).$$

(3) If $A \subset X$ is a subset and $C, D \in \{\varnothing, A\}$ we have for any $\varphi \in \mathrm{H}^k(X, C; R)$, any $\psi \in \mathrm{H}^l(X, D; R)$ and any $\sigma \in \mathrm{H}_{k+l}(X, C \cup D; R)$ that
$$\langle \varphi, \psi \cap \sigma \rangle \ = \ \langle \psi \cup \varphi, \sigma \rangle \ \in \ R.$$

[1420]Note that on the right-hand side it says $(\psi \cup \varphi) \cap \sigma$ instead of $(\varphi \cup \psi) \cap \sigma$. In fact if one picks a different sign convention for the cap product this sign goes away. For example with the sign conventions used by Bredon, see [**Bre1993**, Chapter 5], the equation becomes
$$\varphi \cap (\psi \cap \sigma) \ = \ (\varphi \cup \psi) \cap \sigma$$
which looks much more appealing. Once we consider (co-) homology classes we obtain from the Cup Product-Commutativity Proposition 114.8 that the difference between these two expressions $(\psi \cup \varphi) \cap \sigma$ and $(\varphi \cup \psi) \cap \sigma$ is just a factor of $(-1)^{kl}$.

Proof. The case $k+l>n$ is trivial. So suppose that $k+l\leq n$.

(1) So let $\varphi\in \mathrm{C}^k(X;R)$, $\psi\in \mathrm{C}^l(X;R)$. It suffices to consider the case that $\sigma\colon \Delta^n\to X$ is a singular n-simplex. We have

$$\begin{aligned}\varphi\cap(\psi\cap\sigma) &= \varphi\cap\big(\psi(\sigma\circ[v_0,\ldots,v_l])\cdot\sigma\circ[v_l,\ldots,v_n]\big)\\ &= \psi(\sigma\circ[v_0,\ldots,v_l])\cdot\varphi\cap(\sigma\circ[v_l,\ldots,v_n])\\ &= \psi(\sigma\circ[v_0,\ldots,v_l])\cdot\varphi(\sigma\circ[v_l,\ldots,v_{l+k}])\cdot\sigma\circ[v_{l+k},\ldots,v_n]\\ &= (\psi\cup\varphi)(\sigma\circ[v_0,\ldots,v_{l+k}])\cdot\sigma\circ[v_{l+k},\ldots,v_n]\\ &= (\psi\cup\varphi)\cap\sigma.\end{aligned}$$

(2) The statement regarding the (co-) homology classes follows almost immediately from the above result on (co-) chains obtained in (1).

(3) This statement follows immediately from (2) together with the Cap Product-Kronecker Lemma 116.4. ■

Now suppose we are given a map of excisive triads of topological spaces. We get induced maps on homology and cohomology and the following lemma tells us the relationship of these induced maps and the cap product.

Lemma 116.8. (Cap Product-Naturality Lemma) Let R be a commutative ring.

(1) If $f\colon (X,A,B)\to(\widetilde{X},\widetilde{A},\widetilde{B})$ is a map between two excisive triads, then for any $k,l\in\mathbb{N}_0$ the following diagram commutes[1421]

$$\begin{array}{ccccc}\mathrm{H}^k(X,A;R) & \times & \mathrm{H}_l(X,A\cup B;R) & \xrightarrow{\cap} & \mathrm{H}_{l-k}(X,B;R)\\ {\scriptstyle f^*}\uparrow & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*}\\ \mathrm{H}^k(\widetilde{X},\widetilde{A};R) & \times & \mathrm{H}_l(\widetilde{X},\widetilde{A}\cup\widetilde{B};R) & \xrightarrow{\cap} & \mathrm{H}_{l-k}(\widetilde{X},\widetilde{B};R),\end{array}$$

i.e. for any $\varphi\in\mathrm{H}^k(\widetilde{X},\widetilde{A};R)$ and any $\sigma\in\mathrm{H}_l(X,A\cup B;R)$ we have

$$\varphi\cap f_*(\sigma) \;=\; f_*(\underbrace{f^*(\varphi)\cap\sigma}_{\in\mathrm{H}_{l-k}(X,B;R)}) \;\in\; \mathrm{H}_{l-k}(\widetilde{X},\widetilde{B};R).$$

(2) Let X and $\widetilde{X}$ be path-connected topological spaces and let $B\subset X$ and $\widetilde{B}\subset\widetilde{X}$ be subsets. If $f\colon(X,B)\to(\widetilde{X},\widetilde{B})$ is a map, then for any $\varphi\in\mathrm{H}^k(\widetilde{X},\widetilde{B};R)$ and any $\sigma\in\mathrm{H}_k(X,B;R)$ we have

$$\varphi\cap f_*(\sigma) \;=\; f^*(\varphi)\cap\sigma \;\in\; \underbrace{\mathrm{H}_0(X;R)=R=\mathrm{H}_0(\widetilde{X};R)}_{\substack{\text{identification from}\\ \text{the H}_0\text{-Proposition 72.5}}}.$$

Remark. Note that by the Cap Product-Kronecker Lemma 116.4 (1) we can view the above as a generalization of Lemma 109.6 (3).

Proof. The first statement follows immediately from the definitions. We leave the details to the skeptical reader. The second statement is an immediate consequence of the first statement. ■

[1421] Here "the diagram commutes" is understood in the sense of the convention from page 2538.

The next lemma says that cup and cap products behave in the obvious way with respect to the path-components.

Lemma 116.9. (Cup-Cap Disjoint Union Lemma) Let X be a topological space and let R be a commutative ring. We denote by $\{X_j\}_{j\in J}$ the path-components of X. By Lemmas 72.14 and 108.14 the inclusion maps $\iota_j\colon X_j \to X$, $j \in J$, induce for any $n \in \mathbb{N}_0$ isomorphisms

$$\Phi := \bigoplus_{j\in J}(\iota_j)_*\colon \bigoplus_{j\in J}\mathrm{H}_n(X_j;G) \xrightarrow{\cong} \mathrm{H}_n(X;G)$$

and

$$\Psi := \prod_{j\in J}\iota_j^*\colon \mathrm{H}^n(X;G) \xrightarrow{\cong} \prod_{j\in J}\mathrm{H}^n(X_j;G).$$

Then the following two statements hold:

(1) For any $i \neq j$ and any $\alpha \in \mathrm{H}^m(X_i;R)$ and $\beta \in \mathrm{H}^n(X_j;R)$ we have
$$\Psi^{-1}(\alpha) \cup \Psi^{-1}(\beta) = 0.$$
(2) For any $i \neq j$ and any $\alpha \in \mathrm{H}^m(X_i;R)$ and $\beta \in \mathrm{H}_n(X_j;R)$ we have
$$\Psi^{-1}(\alpha) \cap \Phi(\beta) = 0.$$

The same type of statements hold for cup and cap products on relative (co-) homology groups.

Proof. The statements in the lemma follow easily from the definitions. Therefore we leave the proof of the lemma to the reader. ■

In the next lemma of the chapter we relate the relative cap product to (co-) boundary maps on (co-) homology. More precisely, we have the following lemma which is similar in spirit to the Cup Product-Connecting Homomorphism Lemma 115.6.

Lemma 116.10. (Cap Product-Connecting Homomorphism Lemma) Let R be a commutative ring.

(1) Let (X, B, A) be a triple of topological spaces. For any $n, k \in \mathbb{N}_0$ the following diagram commutes:
$$\begin{array}{ccc} \mathrm{H}^k(B,A;R) \times \mathrm{H}_{n-1}(B,A;R) & \xrightarrow{\cap} & \mathrm{H}_{n-k-1}(B;R) \\ \downarrow{\scriptstyle \delta} & \uparrow{\scriptstyle \partial} & \downarrow{\scriptstyle i_*} \\ \mathrm{H}^{k+1}(X,B;R) \times \mathrm{H}_n(X,B;R) & \xrightarrow{\cap} & \mathrm{H}_{n-k-1}(X;R). \end{array}$$
Here $i\colon B \to X$ denotes the inclusion map and δ and ∂ denote the connecting homomorphisms in the long exact sequences in (co-) homology corresponding to the triple (X, B, A).
(2) Let (X, A, B) be an excisive triad and let $n \in \mathbb{N}_0$. Note that by the Excisive Characterization Lemma 109.13 the map $j_*\colon \mathrm{H}_{n-1}(B, A\cap B;R) \to \mathrm{H}_{n-1}(A\cup B, A;R)$ is an isomorphism. For any $k \in \mathbb{N}_0$ the following diagram commutes up to the sign

$(-1)^{k+1}$:

$$\begin{array}{ccc}
\mathrm{H}^k(X,A;R)\times\mathrm{H}_n(X,A\cup B;R) & \xrightarrow{\ \cap\ } & \mathrm{H}_{n-k}(X,B;R) \\
{\scriptstyle i^*}\downarrow \quad {\scriptstyle \partial}\downarrow & & \\
\mathrm{H}^k(A\cup B,A;R)\times\mathrm{H}_{n-1}(A\cup B,A;R) & & \Big\downarrow{\scriptstyle \partial} \\
{\scriptstyle j^*}\downarrow\cong \quad {\scriptstyle (j_*)^{-1}}\downarrow\cong & & \\
\mathrm{H}^k(B,A\cap B;R)\times\mathrm{H}_{n-1}(B,A\cap B;R) & \xrightarrow{\ \cap\ } & \mathrm{H}_{n-k-1}(B;R).
\end{array}$$

Here $i\colon (A\cup B,A)\to(X,A)$ and $j\colon (B,A\cap B)\to(A\cup B,A)$ denote the inclusion maps and ∂ denotes the connecting homomorphisms in the long exact sequences in homology corresponding to the triple $(X,A\cup B,A)$ respectively the pair (X,B).

Proof.

(1) We pick $\psi\in\mathrm{H}^k(B,A;R)$ and $\tau\in\mathrm{H}_n(B,A;R)$. We make the following observations:
   (a) We can find $\sigma\in\mathrm{C}_n(B;R)$ such that the image of σ in $\mathrm{C}_n(B,A;R)$ is a cycle that represents τ.
   (b) We can pick a singular cochain $\varphi\in\mathrm{C}^k(B;R)=\mathrm{Hom}(\mathrm{C}_k(B),R)$ that vanishes on $\mathrm{C}_k(A)$ and such that the corresponding element in $\mathrm{C}^k(B,A;R)$ represents ψ. We denote by $\widetilde{\varphi}\colon \mathrm{C}_k(X)\to R$ the extension of φ that is defined to be zero on all singular k-simplices that do not lie in B.

   We have the following equalities:

$$\delta\psi\cap\tau \underset{\uparrow}{=} \left[\delta\widetilde{\varphi}\right]\cap[\sigma] \overset{\downarrow}{=} \left[\delta\widetilde{\varphi}\cap\sigma\right] \underset{\uparrow}{=} [\widetilde{\varphi}\cap\partial\sigma] = [\varphi\cap\partial\sigma] \overset{\downarrow}{=} [\varphi]\cap[\partial\sigma] \underset{\uparrow}{=} \psi\cap\partial\tau.$$

by the discussion of relative cap products on page 2558

Cohomological LES-Lemma 108.15 and the discussion on page 2411

by Lemma 116.1 the classes $\delta\varphi\cap\sigma$ and $\varphi\cap\partial\sigma$ are homologous

Topological LES Proposition 74.13

   This equality implies that the diagram commutes.[1422]

(2) We pick $\psi\in\mathrm{H}^k(X,A;R)$ and $\tau\in\mathrm{H}_n(X,A\cup B;R)$. We make the following observations:
   (a) It follows immediately from our hypothesis that the triad (X,A,B) is excisive together with Lemma 116.6 that we can find $\sigma\in\mathrm{C}_n(X;R)$ such that $\partial\sigma=\alpha+\beta$ with $\alpha\in\mathrm{C}_{n-1}(A;R)$ and $\beta\in\mathrm{C}_{n-1}(B;R)$ and such that the image of σ in $\mathrm{C}_n(X,A\cup B;R)$ is a cycle that represents τ.
   (b) As on page 2534 we make the identification

$$\mathrm{C}^k(X,A;R) \xrightarrow{=} \text{ all homomorphisms } \mathrm{C}_k(X)\to R \text{ that vanish on } \mathrm{C}_k(A).$$

   We can pick a singular cochain $\varphi\in\mathrm{C}^k(X;R)=\mathrm{Hom}(\mathrm{C}_k(X),R)$ that vanishes on $\mathrm{C}_k(A)$ and such that the corresponding element in $\mathrm{C}^k(X,A;R)$ is a cocycle that represents ψ. Note that $\delta\varphi=0$.

[1422] With the various quotient groups, lifts, extensions, inclusions and so on it is admittedly quite tricky to figure out in between where any of the equalities hold.

We have the following equalities:

$$\begin{array}{lcll}
\partial(\psi\cap\tau) & \overset{\text{choice of } \sigma \text{ and definition of relative cap product}}{=} & \partial([\varphi\cap\sigma]) \overset{\text{Topological LES-Proposition 74.13}}{=} [\partial(\varphi\cap\sigma)] & \overset{\text{Lemma 116.1}}{=} (-1)^k\cdot[-\delta\varphi\cap\sigma+\varphi\cap\partial\sigma] \\
& \overset{\uparrow}{=} & (-1)^k\cdot[\varphi\cap\alpha+\varphi\cap\beta]) & \overset{\uparrow}{=} (-1)^k\cdot[\varphi\cap\beta]) \\
& & \text{since } \delta\varphi=0 \text{ and } \partial\sigma=\alpha+\beta & \text{since } \varphi \text{ vanishes on } \mathrm{C}_k(A) \text{ and since } \alpha\circ[v_0,\dots,v_k]\in\mathrm{C}_k(A) \\
& \overset{\uparrow}{=} & (-1)^k\cdot([j^*i^*\varphi]\cap j_*^{-1}(\partial[\sigma])). & \\
& & \text{by the Topological LES-Proposition 74.13 we have } j_*([\beta])=\partial[\sigma] &
\end{array}$$

This equality implies that the diagram commutes. ■

As we had mentioned before, a frequently used trick in the study of topological manifolds is to replace a topological manifold with non-empty boundary by its double. When doing this trick the following lemma can at times be useful.

Lemma 116.11. Let M be a topological manifold. We denote by $r\colon \mathrm{D}M\to M$ the folding map from page 1668 and given $l\in\mathbb{N}_0$ we denote by $D\colon \mathrm{H}_l(M,\partial M;\mathbb{Z})\to\mathrm{H}_l(\mathrm{D}M;\mathbb{Z})$ the doubling homomorphism that we introduced in the Doubling-Homology Lemma 75.16. Then for any $\phi\in\mathrm{H}^k(M;\mathbb{Z})$ and any $\sigma\in\mathrm{H}_l(M,\partial M;\mathbb{Z})$ we have

$$r^*\phi\cap D(\sigma) \;=\; D(\phi\cap\sigma) \;\in\; \mathrm{H}_{l-k}(\mathrm{D}M;\mathbb{Z}).$$

Proof. We will show that the desired equality actually holds already on the (co-) chain level. So let $\phi\in\mathrm{C}^k(M;\mathbb{Z})$ and let $\sigma\colon\Delta^l\to M$ be a singular l-simplex. If $k>l$, then both sides of the lemma are trivially zero. Hence we can assume that $k\le l$. For $i=1,2$ we denote by $j_i\colon M\to M\times\{i\}\subset\mathrm{D}M$ the obvious map. Then

$$\begin{array}{lcll}
r^*\phi\cap D(\sigma) & \overset{\text{by definition of } D(\sigma)}{\overset{\downarrow}{=}} & r^*\phi\cap(j_1\circ\sigma-j_2\circ\sigma) & \overset{\text{since } \cap \text{ is bilinear}}{\overset{\downarrow}{=}} r^*\phi\cap j_1\circ\sigma - r^*\phi\cap j_2\circ\sigma \\
& = & \phi(\underbrace{r\circ j_1\circ\sigma}_{=\sigma}\circ[v_0,\dots,v_k])\cdot j_1\circ\sigma\circ[v_k,\dots,v_l] - \phi(\underbrace{r\circ j_2\circ\sigma}_{=\sigma}\circ[v_0,\dots,v_k])\cdot j_2\circ\sigma\circ[v_k,\dots,v_l] & \\
& \overset{\uparrow}{=} & \text{definition of cap product and definition of } r^*\phi & \\
& = & \phi(\sigma\circ[v_0,\dots,v_k])\cdot j_1\circ(\sigma\circ[v_k,\dots,v_l]) - \phi(\sigma\circ[v_0,\dots,v_k])\cdot j_2\circ(\sigma\circ[v_k,\dots,v_l]) & \\
& = & j_{1*}(\phi\cap\sigma) - j_{2*}(\phi\cap\sigma) \;=\; D(\phi\cap\sigma). & \\
& \overset{\uparrow}{=} & \text{definition of } j_{1*},\, j_{2*} \text{ and the cap product} &
\end{array}$$

■

Exercises for Chapter 116.

Exercise 116.1. Let $X=S^1\vee S^2$. We denote by $p\colon X\to S^1$ and $q\colon X\to S^2$ the natural projections and we denote by $i\colon S^1\to X$ and $j\colon S^2\to X$ the obvious inclusion maps. Determine $p^*([\theta_{\mathbb{Z}}])\cap j_*([S^2])$.

Exercise 116.2. We denote by Σ the surface of genus two. Furthermore we denote by $p_1,p_2\colon\Sigma\to T=S^1\times S^1$ the two projections and we denote by $p,q\colon S^1\times S^1\to S^1$ the projection onto the first respectively the second factor. As usual we write $\alpha=p^*(\theta)$ and $\beta=q^*(\theta)$. We saw that $[p_1^*(\alpha)],[p_1^*(\beta)],[p_2^*(\alpha)],[p_2^*(\beta)]$ form a basis for $\mathrm{H}^1(\Sigma;\mathbb{Z})$.

Furthermore we denote by $x_1, y_1, x_2, y_2 \in \mathrm{H}_1(\Sigma;\mathbb{Z})$ the standard basis. For each of the four basis elements of $\mathrm{H}^1(\Sigma;\mathbb{Z})$ determine the cap product with $[\Sigma]$ in terms of the basis for $\mathrm{H}_1(\Sigma;\mathbb{Z})$.
Remark. This exercise is not as bad as it might sound initially.

Exercise 116.3. We consider the annulus $X = S^1 \times [0,1]$. Determine the following map:

$$\begin{array}{rcl} \mathrm{H}^1(X;\mathbb{Z}) & \to & \mathrm{H}_1(X, \partial X;\mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [X] \end{array}$$

More precisely, give a basis for $\mathrm{H}^1(X;\mathbb{Z})$ and $\mathrm{H}_1(X,\partial X;\mathbb{Z})$ and express the map $- \cap [X]$ in terms of that basis.

Exercise 116.4. Let X be a topological space and let $k, l \in \mathbb{N}_0$. We consider the cohomology with compact support group $\mathrm{H}^k_{\mathrm{c}}(X;\mathbb{Z})$ and as in Exercise 106.5 we consider the group

$$\overleftarrow{\mathrm{H}}_l(X;\mathbb{Z}) \;:=\; \varprojlim_{K \in \mathcal{K}(X)} \mathrm{H}_n(X, X \setminus K; G).$$

Show that there exists a natural cap product

$$\cap \colon \mathrm{H}^k_{\mathrm{c}}(X;\mathbb{Z}) \times \overleftarrow{\mathrm{H}}_l(X;\mathbb{Z}) \;\to\; \mathrm{H}_{l-k}(X;\mathbb{Z})$$

that equals the usual cap product if X is compact.
Remark. Use the description of cohomology with compact support that is provided by the $\mathrm{H}^n_{\mathrm{c}}$-via-Direct Limits Proposition 112.4.

117. The Product Theorem and the Künneth Theorem for cohomology

In this chapter we will deal with the cup and cap product of product topological spaces. In particular at the end of the chapter we will have determined the cup product of the product $S^m \times S^n$ of two spheres and of the n-dimensional torus $(S^1)^n$. Furthermore we will have determined the Lusternik-Schnirelmann category of the n-dimensional torus.

117.1. **The Product Theorem.** The following theorem relates the cross product to the cap product and the cup product.

Theorem 117.1. (Product Theorem) Let (X, A) and (Y, B) be two pairs of topological spaces such that the triad $(X \times Y, A \times Y, X \times B)$ is excisive. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the natural projections. Furthermore let[1423] $A_1, A_2 \in \{\varnothing, A\}$ and let $B_1, B_2 \in \{\varnothing, B\}$. Then for any $\alpha \in \mathrm{H}^k(X, A_1; \mathbb{Z})$, $\mu \in \mathrm{H}_m(X, A_1 \cup A_2; \mathbb{Z})$, $\beta \in \mathrm{H}^l(Y, B_1; \mathbb{Z})$ and $\nu \in \mathrm{H}_n(Y, B_1 \cup B_2; \mathbb{Z})$ we have the equality[1424]

$$\big(p^*(\alpha) \cup q^*(\beta)\big) \cap (\mu \times \nu) = (-1)^{\deg(\alpha)\cdot\deg(\beta)+\deg(\beta)\cdot\deg(\mu)} \cdot (\alpha \cap \mu) \times (\beta \cap \nu)$$

in $\mathrm{H}_{m+n-k-l}(X \times Y, X \times B_2 \cup A_2 \times Y; \mathbb{Z})$.



Remark. Even though the equality in the Product Theorem 117.1 is easy to write down, it is much harder to figure out in which (co-) homology groups the various objects actually live. In fact the equality is understood as follows:

$$\Big(\underbrace{\underbrace{p^*(\alpha)}_{\in \mathrm{H}^k(X\times Y, A_1\times Y)} \cup \underbrace{q^*(\beta)}_{\in \mathrm{H}^l(X\times Y, X\times B_1)}}_{\in \mathrm{H}^{k+l}(X\times Y, A_1\times Y \cup X\times B_2)}\Big) \cap \overbrace{(\mu\times\nu)}^{\text{lies in } \mathrm{H}_{m+n}(X\times Y, (A_1\cup A_2)\times Y \cup X\times(B_1\cup B_2))} = (-1)^{\deg(\alpha)\cdot\deg(\beta)+\deg(\beta)\cdot\deg(\mu)} \cdot \underbrace{\underbrace{(\alpha \cap \mu)}_{\in \mathrm{H}_{m-k}(X, A_2)} \times \underbrace{(\beta \cap \nu)}_{\in \mathrm{H}_{n-l}(Y, B_2)}}_{\in \mathrm{H}_{m+n-k-l}(X\times Y, X\times B_2 \cup A_2\times Y)}.$$

In this formula we have to ensure that all products are in fact defined. This follows easily from the Excisive Triad-Proposition 109.12, our hypothesis that $A_1, A_2 \in \{\varnothing, A\}$ and $B_1, B_2 \in \{\varnothing, B\}$ and our hypothesis that the triad $(X \times Y, A \times Y, X \times B)$ is excisive.

Remark.

(1) The condition in the Product Theorem 117.1 that the triad $(X \times Y, A \times Y, X \times B)$ needs to be excisive is arguably a little awkward. By the Excisive Triad-Proposition 109.12 this condition is satisfied if we are in one of the following situations:
   (a) $A = \varnothing$ or $B = \varnothing$,
   (b) (X, A) and (Y, B) are pairs of CW-complexes,

[1423] By $A_1, A_2 \in \{\varnothing, A\}$ we mean that $A_1 = \varnothing$ or $A_1 = A$ and that $A_2 = \varnothing$ or $A_2 = A$.

[1424] Here, and throughout this section, we denote by p also the map of pairs $(X \times Y, A \times Y) \to (X, A)$ given by projection onto the first factor and we denote by q the map of pairs $(X \times Y, X \times B) \to (Y, B)$ given by projection onto the second factor.

(c) $(X, A) = (X, \partial X)$ and $(Y, B) = (Y, \partial Y)$ where X and Y are topological manifolds,[1425]

(d) A and B are open subsets of X respectively Y.

(2) In fact a more general statement of the Product Theorem 117.1 holds if we deal with excisive triads (X, A_1, A_2) and (Y, B_1, B_2) and (co-) homology classes $\alpha \in \mathrm{H}^k(X, A_1; \mathbb{Z})$, $\mu \in \mathrm{H}_m(X, A_2; \mathbb{Z})$, $\beta \in \mathrm{H}^l(Y, B_1; \mathbb{Z})$ and $\nu \in \mathrm{H}_n(Y, B_2; \mathbb{Z})$. But some care needs to be taken to see which other triads need to be excisive to ensure that all the cap and cup products are defined. We will not attempt to formulate this more general statement and we will not make use of it. (See also [**Dol1956**, p. 240 and p. 244] for a few more details.)

(3) The cross product is defined the same way in all of the usual text books in algebraic topology. But we saw on pages 2514 and 2553 that in the literature different sign conventions are used for the cup product and the cap product. These also lead to different formulations of the Product Theorem 117.1. More precisely, given topological spaces X and Y and $\alpha \in \mathrm{H}^k(X; \mathbb{Z})$, $\mu \in \mathrm{H}_m(X; \mathbb{Z})$, $\beta \in \mathrm{H}^l(Y; \mathbb{Z})$ and $\nu \in \mathrm{H}_n(Y; \mathbb{Z})$ the following equations can be found in the literature:

[**Bre1993**, p. 337] $(p^*(\alpha) \cup q^*(\beta)) \cap (\mu \times \nu) = (-1)^{\deg(\beta)\cdot\deg(\mu)} \cdot (\alpha \cap \mu) \times (\beta \cap \nu)$

[**Dol1956**, p. 240] $(p^*(\alpha) \cup q^*(\beta)) \cap (\mu \times \nu) = (-1)^{\deg(\beta)\cdot\deg(\mu)} \cdot (\alpha \cap \mu) \times (\beta \cap \nu)$

[**Spa1995**, p. 255] $(p^*(\alpha) \cup q^*(\beta)) \cap (\mu \times \nu) = (-1)^{\deg(\alpha)\cdot(\deg(\nu)-\deg(\beta))} \cdot (\alpha \cap \mu) \times (\beta \cap \nu).$

With the aforementioned sign conventions the formulas of Dold and Spanier are consistent with our formula, whereas the formula of Bredon differs from the formula of Dold and Spanier by the term $(-1)^{\deg(\alpha)\cdot\deg(\nu)+\deg(\beta)\cdot\deg(\mu)}$.[1426][1427]

(4) The conclusion of the Product Theorem 117.1 also holds, with verbatim the same proof, if throughout we replace $\mathbb{Z}$-coefficients by $\mathbb{F}_2$-coefficients.

The proof of the Product Theorem 117.1 is somewhat lengthy and technical. Therefore we first discuss some consequences of Theorem 117.1 which will then hopefully convince us that it is worth the effort to prove Theorem 117.1.

The following is one of the key calculations of the cup product and the cap product. As we will see later, many other calculations can be reduced to this calculation.

[1425] Let X be an m-dimensional topological manifold and let Y be an n-dimensional smooth manifold. It follows from Propositions 18.4 and 75.2 that $W := \partial(X \times Y) = \partial X \times Y \cup X \times \partial Y$ is an $(m + n - 1)$-dimensional topological manifold. It is straightforward to verify that $\partial X \times Y$ and $X \times \partial Y$ are submanifolds of W such that

$$(\partial X \times Y) \cap (X \times \partial Y) = \partial X \times \partial Y = \partial(\partial X \times Y) = \partial(X \times \partial Y).$$

Therefore it follows from the Excisive Triad-Proposition 109.12 (5) that the triad $(X \times Y, \partial X \times Y, X \times \partial Y)$ is excisive.

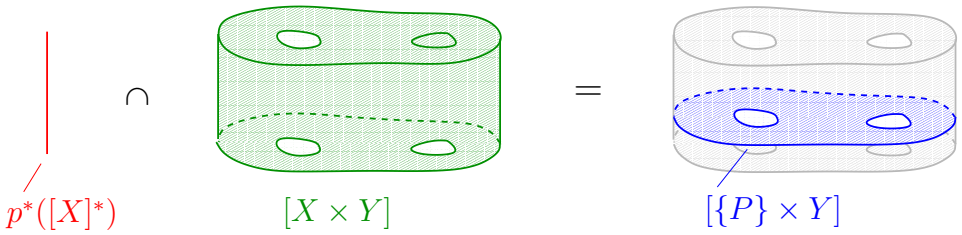
[1426] Mathematicians with a long experience of teaching cohomology groups sidestep all issues with signs by working with $\mathbb{F}_2$-coefficients, see [**Hau2014**].

[1427] We leave it to the never-flagging enthusiasm of the reader to figure out whether the formulas agree with the one given by Greenberg-Harper [**GrH1981**, p. 262].

Proposition 117.2. (Fundamental Class-Product Proposition) Let X be a compact oriented connected m-dimensional topological manifold and let Y be a compact oriented connected n-dimensional topological manifold. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the natural projections. Then the following two statements hold:

(1)
$$\underbrace{p^*([X]^*)}_{\in \mathrm{H}^m(X\times Y,\partial X\times Y)} \cup \underbrace{q^*([Y]^*)}_{\in \mathrm{H}^n(X\times Y,X\times\partial Y)} = \underbrace{[X\times Y]^*}_{\in \mathrm{H}^{m+n}(X\times Y,\partial(X\times Y))}.$$

(2) If $P \in X$ and $Q \in Y$ are points, then[1428]
$$\overbrace{p^*([X]^*)}^{\in \mathrm{H}^m(X\times Y,\partial X\times Y)} \cap \overbrace{[X\times Y]}^{\in \mathrm{H}_{m+n}(X\times Y,\partial(X\times Y))} = \overbrace{[\{P\}\times Y]}^{\in \mathrm{H}_n(X\times Y,X\times\partial Y)}$$
and similarly
$$\underbrace{q^*([Y]^*)}_{\in \mathrm{H}^n(X\times Y,X\times\partial Y)} \cap \underbrace{[X\times Y]}_{\in \mathrm{H}_{m+n}(X\times Y,\partial(X\times Y))} = (-1)^{mn} \cdot \underbrace{[X\times\{Q\}]}_{\in \mathrm{H}_m(X\times Y,\partial X\times Y)}.$$



Remark. As we pointed out above, the conclusion of the Product Theorem 117.1 also holds if we work with $\mathbb{F}_2$-coefficients. It follows easily that there is also an $\mathbb{F}_2$-coefficient analogue of the Fundamental Class-Product Proposition 117.2.

Example. If we apply the Fundamental Class-Product Proposition 117.2 (2) to $X = S^1$ and $Y = S^1$ we see that
$$p^*([S^1]^*) \cap [S^1 \times S^1] = [1 \times S^1] \quad \text{and} \quad q^*([S^1]^*) \cap [S^1 \times S^1] = -[S^1 \times 1].$$
This matches our down-to-earth calculation from page 2555.

Proof. Let X be a compact oriented connected m-dimensional topological manifold and let Y be a compact oriented connected n-dimensional topological manifold. Furthermore let P be a point in X and let Q be a point in Y. We start out with the following two observations:

(a) For any compact oriented connected k-dimensional topological manifold M and any point $R \in M$ we obtain from the discussion on page 2558 that $[M]^* \cap [M] = [R]$ and that $[M]^*$ is the unique element in $\mathrm{H}^k(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$ with $[M]^* \cap [M] = [R]$ in $\mathrm{H}_0(M; \mathbb{Z}) = \mathbb{Z} \cdot [R]$.

(b) By the Topological Product Orientation Proposition 113.10 we have $[X \times Y] = [X] \times [Y]$.

[1428]Here we use the following suggestive notation: We can view $\{P\} \times Y$ as a submanifold of $X \times Y$. It is canonically homeomorphic to Y, thus can view $\{P\} \times Y$ as a compact oriented topological manifold. We denote by $[\{P\} \times Y]$ its fundamental class, and by a slight abuse of notation we also denote by $[\{P\} \times Y]$ the image of the fundamental class under the inclusion $\{P\} \times Y \to X \times Y$.

Now we turn to the actual proof of the two statements.

(1) We perform the following calculation in $\mathrm{H}_0(X \times Y; \mathbb{Z})$:

$$\begin{array}{ccccc} & & & \overset{\text{observation (b)}}{\downarrow} & \\ & (p^*([X]^*) \cup q^*([Y]^*)) \cap ([X \times Y]) & \overset{}{=} & (p^*([X]^*) \cup q^*([Y]^*)) \cap ([X] \times [Y]) & \\ \overset{}{=} & (-1)^{mn+mn} \cdot ([X]^* \cap [X]) \times ([Y]^* \cap [Y]) & \overset{}{=} & [P] \times [Q] & \overset{}{=} [(P,Q)]. \\ \uparrow & & & \uparrow & \uparrow \\ \text{by Theorem 117.1 applied} & & & \text{observation (a)} & \text{see the Cross Product Lemma 113.7 (2)} \\ \text{to } (X, A_1, A_2) = (X, \partial X, \partial X) & & & & \\ \text{and } (Y, B_1, B_2) = (Y, \partial Y, \partial Y) & & & & \end{array}$$

This implies, by observation (a), that $p^*([X]^*) \cup q^*([Y]^*) = [X \times Y]^*$.

(2) Let $P \in X$. Then

$$\begin{array}{ccccc} p^*([X]^*) \cap ([X \times Y]) & \overset{}{=} & (p^*([X]^*) \cup q^*(1_Y)) \cap ([X] \times [Y]) & & \\ \uparrow & & & & \\ \text{by the Cup Product-Naturality Proposition 115.4 (1) we have } q^*(1_Y) = 1_{X \times Y} \text{ and} & & & & \\ \text{by observation (b) we have } [X \times Y] = [X] \times [Y] & & & & \\ \overset{}{=} & ([X]^* \cap [X]) \times (1_Y \cap [Y]) & \overset{}{=} & [P] \times [Y] & \overset{}{=} [\{P\} \times Y]. \\ \uparrow & & \uparrow & & \uparrow \\ \text{by Theorem 117.1 applied to } (Y, B_1, B_2) = (Y, \varnothing, \partial Y) & & \text{by Lemma 116.3} & & \text{Cross Product} \\ \text{note that here we use that } 1_Y \text{ lies in } \mathrm{H}^0(Y; \mathbb{Z}) & & \text{and observation (a)} & & \text{Lemma 113.7 (2)} \end{array}$$

This proves the first equality of (2). Now let $Q \in Y$. Then, very similar to the above, we have

$$\begin{aligned} q^*([Y]^*) \cap ([X \times Y]) &= (p^*(1_X) \cup q^*([Y]^*)) \cap ([X] \times [Y]) \\ &= (-1)^{mn} \cdot (1_X \cap [X]) \times ([Y]^* \cap [Y]) \\ &= (-1)^{mn} \cdot [X] \times [Q] = (-1)^{mn} \cdot [X \times \{Q\}]. \end{aligned}$$

■

Using the Fundamental Class-Product Proposition 117.2 (1) we can now easily compute the cup product of the product $S^m \times S^n$ of two spheres.

Lemma 117.3. Let $m, n \in \mathbb{N}$. We denote by $p\colon S^m \times S^n \to S^m$ and by $q\colon S^m \times S^n \to S^n$ the two natural projections. Let $1 \in \mathrm{H}^0(S^m \times S^n; \mathbb{Z})$ be the neutral element from the Cup Product-Ring Proposition 114.7.

(1) The cohomology of $S^m \times S^n$ is given as follows:

$$\mathrm{H}^*(S^m \times S^n; \mathbb{Z}) = \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot p^*([S^m]^*) \oplus \mathbb{Z} \cdot q^*([S^n]^*) \oplus \mathbb{Z} \cdot [S^m \times S^n]^*.$$

(2) The multiplication table for the cup product is given by

$\cup$	1	$p^*([S^m]^*)$	$q^*([S^n]^*)$	$[S^m \times S^n]^*$
1	1	$p^*([S^m]^*)$	$q^*([S^n]^*)$	$[S^m \times S^n]^*$
$p^*([S^m]^*)$	$p^*([S^m]^*)$	0	$[S^m \times S^n]^*$	0
$q^*([S^n]^*)$	$q^*([S^n]^*)$	$(-1)^{mn} \cdot [S^m \times S^n]^*$	0	0
$[S^m \times S^n]^*$	$[S^m \times S^n]^*$	0	0	0.

Here the table shows the cup product of the term in the first column with the term in the first row (and not the other way round).

Remark.

(1) For $m = n = 1$, i.e. for the torus $S^1 \times S^1$ we had calculated the cup product in Lemma 114.14.

(2) If we use the Fundamental Class-Product Proposition 117.2 (1) with $\mathbb{F}_2$-coefficients then we can also determine the cup product on $\mathrm{H}^*(S^m \times S^n; \mathbb{F}_2)$ and we obtain basically the same result.

Proof.

(1) Using Proposition 74.4, the Künneth Theorem 90.8 for topological spaces and the Universal Coefficient Theorem 110.12 for Cohomology Groups one can easily determine the isomorphism type of $\mathrm{H}^*(S^m \times S^n; \mathbb{Z})$. Furthermore, using the Kronecker-Basis Lemma 109.7 and the homology classes given by the fundamental classes of S^m, S^n and $S^m \times S^n$ it is straightforward to verify that the given cohomology classes do indeed form a basis for $\mathrm{H}^*(S^m \times S^n; \mathbb{Z})$.

(2) By Observation 114.13 it remains to show that the red-colored and the blue-colored entries are correct.

(a) First we consider the red-colored entries. We have

$$p^*([S^m]^*) \cup p^*([S^m]^*), \underset{\uparrow}{=} p^*([S^m]^* \cup [S^m]^*) \underset{\uparrow}{=} p^*(0) = 0.$$

Cup Product-Naturality Lemma 114.10 — since $[S^m]^* \cup [S^m]^* \in \mathrm{H}^{2m}(S^m; \mathbb{Z}) = 0$

The same argument also shows that $q^*([S^n]^*) \cup q^*([S^n]^*) = 0$.

(b) We turn to the blue-colored entries. We calculate that

$$(-1)^{mn} \cdot q^*([S^n]^*) \cup p^*([S^m]^*) \underset{\uparrow}{=} p^*([S^m]^*) \cup q^*([S^n]^*) \underset{\uparrow}{=} [S^m \times S^n]^*.$$

Cup Product-Commutativity Proposition 114.8 — Fundamental Class Product Proposition 117.2 (1) ■

In the following we consider the wedge $X := S^2 \vee S^2 \vee S^4$ and we consider the product $Y := S^2 \times S^2$. It follows from Lemma 78.11 and Propositions 74.4 and 79.8 and the Universal Coefficient Theorem 110.12 for Cohomology Groups that $X = S^2 \vee S^2 \vee S^4$ and $Y = S^2 \times S^2$ have isomorphic homology and cohomology groups. Furthermore one obtains easily from Propositions 47.13, 48.20 and 52.3 that both X and Y are simply connected. It is quite easy to show that $X = S^2 \vee S^2 \vee S^4$ and $Y = S^2 \times S^2$ are not homeomorphic. Indeed, suppose there exists a homeomorphism $\phi\colon S^2 \vee S^2 \vee S^4 \to S^2 \times S^2$. We denote by P we wedge point of $X = S^2 \vee S^2 \vee S^4$ and we write $Q = f(P)$. Then $X \setminus \{P\}$ and $Y \setminus \{Q\}$ would also be homeomorphic. But the former topological space has three path-components whereas the latter has just one path-component.[1429] We refer to the figure below for a dimensionally reduced illustration.



The following lemma is much deeper than the above discussions.

Lemma 117.4. The topological spaces $S^2 \vee S^2 \vee S^4$ and $S^2 \times S^2$ are not even homotopy equivalent.

[1429] How can you prove rigorously that given any point $Q \in S^2 \times S^2$ the complement $S^2 \times S^2 \setminus \{Q\}$ is still path-connected?

Proof. We had just seen in Lemma 117.3 that there exist cohomology classes α and β in $\mathrm{H}^2(S^2 \times S^2; \mathbb{Z})$ with $\alpha \cup \beta \neq 0$. On the other hand in Lemma 114.11 we saw that the cup product

$$\cup\colon \mathrm{H}^2(S^2 \vee S^2 \vee S^4; \mathbb{Z}) \times \mathrm{H}^2(S^2 \vee S^2 \vee S^4; \mathbb{Z}) \ \to\ \mathrm{H}^4(S^2 \vee S^2 \vee S^4; \mathbb{Z})$$

is the zero-map. It follows from Corollary 114.12 that the topological spaces $S^2 \vee S^2 \vee S^4$ and $S^2 \times S^2$ are not homotopy equivalent. ■

In the Degree Realization Proposition 38.13 we saw that there exists a degree-one map from $S^2 \times S^2$ onto the 4-dimensional sphere S^4. In Question 101.10 we had asked whether there exists a degree-one map from the 4-dimensional sphere S^4 to $S^2 \times S^2$. Now we can answer this question in the negative.

Lemma 117.5. There is no map $f\colon S^4 \to S^2 \times S^2$ of non-zero degree.

Proof. Let $f\colon S^4 \to S^2 \times S^2$ be a map. We have to show that $\deg(f) = 0$. By Lemma 110.16 it suffices to show that $f^*([S^2 \times S^2]^*) = 0$. In Lemma 117.3 we showed that there exist cohomology classes $a, b \in \mathrm{H}^2(S^2 \times S^2; \mathbb{Z})$ such that $a \cup b = [S^2 \times S^2]^* \in \mathrm{H}^4(S^2 \times S^2; \mathbb{Z})$. It follows that

$$\begin{array}{ccccccc} f^*([S^2 \times S^2]^*) & = & f^*(a \cup b) & = & f^*(a) \cup f^*(b) & = & 0 \cup 0 \ = 0. \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \text{since } [S^2 \times S^2]^* = a \cup b & & \text{Cup Product-Naturality Lemma} & & \text{since } f^*(a), f^*(b) \in \mathrm{H}^2(S^4; \mathbb{Z}) = 0 \end{array}$$

■

117.2. **Proof of the Product Theorem 117.1.** Now that we are fully convinced of the value of Theorem 117.1 it is time to actually provide a proof thereof.
Initially we introduced only the tensor product $C_* \otimes D_*$ of two chain complexes C_* and D_*. The definition of the boundary map in the tensor product $C_* \otimes D_*$ is *not* symmetric in the two terms. But it is straightforward to see that the tensor product of chain complexes is associative. More precisely, given chain complexes $\mathrm{C}^1_*, \ldots, \mathrm{C}^k_*$ the boundary map on the tensor product $\mathrm{C}^1_* \otimes \cdots \otimes \mathrm{C}^k_*$ is always determined by

$$\partial(c_1 \otimes c_2 \otimes \cdots \otimes c_k) \ = \ \sum_{i=1}^{k} (-1)^{\deg(c_1)+\cdots+\deg(c_{i-1})} \cdot c_1 \otimes \cdots \otimes \partial c_i \otimes \cdots \otimes c_k,$$

regardless of the way how one breaks the tensor product into the tensor product of pairs of two chain complexes.
The following lemma is basically the same as Lemma 90.3, but we state it here for the convenience of the reader.

Lemma 117.6. Let $f_i\colon \mathrm{C}^i_* \to D^i_*$, $i = 1, \ldots, k$ be chain maps between chain complexes. The map

$$f_1 \otimes \cdots \otimes f_k\colon \mathrm{C}^1_* \otimes \cdots \otimes \mathrm{C}^k_* \ \to\ D^1_* \otimes \cdots \otimes D^k_*$$

that is determined by

$$c_1 \otimes \cdots \otimes c_k \ \mapsto\ f(c_1) \otimes \cdots \otimes f(c_k)$$

is also a chain map.

Remark. Even the definition of $f_1 \otimes \cdots \otimes f_k$ is, oddly enough, not the same everywhere. For example [**Bre1993**, p. 315] adds a sign to the map.

Proof. Since in the proof of the Product Theorem 117.1 there are lots of steps later on which can make us nervous, let us convince ourselves that the statement is indeed correct. We have

$$\begin{aligned}\partial\big((f_1\otimes\ldots\otimes f_k)(c_1\otimes\ldots\otimes c_k)\big) &= \partial(f_1(c_1)\otimes\ldots\otimes f_k(c_k)) \\ &= \sum_{i=1}^{k}(-1)^{\deg(f_1(c_1))+\cdots+\deg(f_{i-1}(c_{i-1}))}\cdot f_1(c_1)\otimes\ldots\otimes\overbrace{f_i(\partial c_i)}^{=\partial f_i(c_i)}\otimes\ldots\otimes f_k(c_k) \\ &= (f_1\otimes\ldots\otimes f_k)\Big(\sum_{i=1}^{k}(-1)^{\deg(c_1)+\cdots+\deg(c_{i-1})}\cdot c_1\otimes\ldots\partial c_i\otimes\ldots\otimes c_k\Big) \\ &\quad\uparrow \\ &\quad\text{since } \deg(f_i(c_i))=\deg(c_i) \\ &= (f_1\otimes\ldots\otimes f_k)(\partial(c_1\otimes\ldots\otimes c_k)).\end{aligned}$$

■

Note though that the naive swapping of two factors in the tensor product of two chain complexes is not necessarily a chain map. But we have the following lemma.

Lemma 117.7. Let A_* and B_* be chain complexes. The unique map

$$\tau\colon A_*\otimes B_* \;\to\; B_*\otimes A_*$$

that for $a\in A_k$ and $b\in B_l$ is given by

$$a\otimes b \;\mapsto\; (-1)^{kl}\cdot b\otimes a$$

is a chain map.

Remark. In Exercise 117.1 we will see that we can use Lemma 117.7 to give a new proof of the Cup Product-Commutativity Proposition 114.8, i.e. we can give a new proof that the cup product is (anti-) commutative.

Proof. The difficult bit about the lemma is coming up with the statement. The verification is now almost trivial. So let $a\in A_k$ and $b\in B_l$. Then we have

$$\begin{aligned}\tau(\partial(a\otimes b)) &= \tau\big(\partial a\otimes b+(-1)^k\cdot a\otimes\partial b\big) \\ &= (-1)^{(k-1)l}\cdot b\otimes\partial a+(-1)^{k+k(l-1)}\cdot\partial b\otimes a \\ &= \partial((-1)^{kl}\cdot b\otimes a) \;=\; \partial(\tau(a\otimes b)).\end{aligned}\qquad\begin{aligned}&= \tau(\partial a\otimes b)+(-1)^k\cdot\tau(a\otimes\partial b) \\ &= (-1)^{kl}\cdot\partial b\otimes a+(-1)^{l+kl}\cdot b\otimes\partial a\end{aligned}$$

■

Now we have to introduce a plethora of new definitions and notations.

Notation.

(1) We denote by $\mathbb{Z}$ also the chain complex given by the group $\mathbb{Z}$ in degree zero and where all other chain groups are zero.

(2) Given a chain complex C_* and $k\in\mathbb{N}_0$ we denote by $\mathrm{C}_*[k]$ the chain complex which is the result of "shifting C_* by k to the left", i.e. it is the chain complex for which the m-th chain groups $(\mathrm{C}_*[k])_m$ is given by C_{m-k}.[1430] For example, $\mathbb{Z}[3]$ is the following chain complex:

$$\ldots \;\to\; 0 \;\to\; \underset{\substack{\uparrow\\ \text{degree 3}}}{\mathbb{Z}} \;\to\; 0 \;\to\; 0 \;\to\; 0.$$

(3) If $\alpha\colon C_*\to D_*$ is a chain map, then for any $k\in\mathbb{N}_0$ we denote by $\alpha[k]\colon \mathrm{C}_*[k]\to D_*[k]$ the obvious shift of the chain map. Evidently this is again a chain map.

(4) Let C_* be a chain complex and let $k\in\mathbb{N}_0$.

(a) Given a cocycle $\varphi\colon \mathrm{C}_k\to\mathbb{Z}$ we denote by $\Pi_k(\varphi)\colon C_*\to\mathbb{Z}[k]$ the map that is given by φ on C_n and zero on all other chain groups.

(b) Conversely, given a chain map $\psi\colon C_* \to \mathbb{Z}[k]$ we denote by $\psi_k\colon \mathrm{C}_k \to \mathbb{Z}$ the map that is the k-th map of the chain map.

Lemma 117.8. Let C_* be a chain complex and let $k \in \mathbb{N}_0$. The following statements hold:

(1) Let $\varphi\colon \mathrm{C}_k \to \mathbb{Z}$ be a cocycle. The map $\Pi_k(\varphi)\colon C_* \to \mathbb{Z}[k]$ is a chain map.
(2) Let $\psi\colon C_* \to \mathbb{Z}[k]$ be a chain map. The map $\psi_k\colon \mathrm{C}_k \to \mathbb{Z}$ is a cocycle.
(3) Let $\alpha, \beta\colon C_* \to \mathbb{Z}[k]$ be chain maps. If α and β are chain homotopic, then α_k and β_k are cohomologous.

Proof.

(1) Let $\varphi\colon \mathrm{C}_k \to \mathbb{Z}$ be a cocycle. We consider the following diagram:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \mathrm{C}_{k+1} & \xrightarrow{\partial} & \mathrm{C}_k & \xrightarrow{\partial} & \mathrm{C}_{k-1} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \dots . \end{array}$$

The hypothesis that φ is a cocycle means that $\delta\varphi = \varphi \circ \partial = 0$, which implies immediately that $\Pi_k(\varphi)$ is a chain map.

(2) The proof of Statement (2) is almost the same as the proof of Statement (1).

(3) Let $\alpha, \beta\colon C_* \to \mathbb{Z}[k]$ be chain maps and let $\{P_m\colon \mathrm{C}_m \to \mathbb{Z}[k]_{m+1}\}_{m\in\mathbb{N}_0}$ be a chain homotopy between α and β. We summarize all relevant maps in the following diagram:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \mathrm{C}_{k+1} & \xrightarrow{\partial} & \mathrm{C}_k & \xrightarrow{\partial} & \mathrm{C}_{k-1} & \longrightarrow & \dots \\ & & \downarrow & \swarrow_{P_k} & \downarrow{\scriptstyle \alpha_k-\beta_k} & \swarrow_{P_{k-1}} & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \dots . \end{array}$$

By definition of a chain homotopy we obtain that $\alpha_k - \beta_k = P_{k-1} \circ \partial$. But this means that $\alpha_k - \beta_k = \delta P_{k-1}$, i.e. α_k and β_k are cohomologous. ■

Lemma 117.9. Let C_* be a chain complex and let $k \in \mathbb{Z}$. The maps

$$\begin{array}{rcl} \eta_m\colon (\mathbb{Z}[k] \otimes \mathrm{C}_*)_m = \mathbb{Z} \otimes \mathrm{C}_{m-k} & \to & (\mathrm{C}_*[k])_m = \mathrm{C}_{m-k} \\ a \otimes c_{m-k} & \mapsto & (-1)^{k(m-k)} \cdot a \cdot c_{m-k} \end{array}$$

form an isomorphism

$$\eta\colon \mathbb{Z}[k] \otimes \mathrm{C}_* \xrightarrow{\cong} \mathrm{C}_*[k]$$

of chain complexes.

Example. Let $k, l \in \mathbb{N}_0$. The chain map

$$\eta\colon \mathbb{Z}[k] \otimes \mathbb{Z}[l] \to \mathbb{Z}[k+l]$$

is determined by the fact that for $a \in (\mathbb{Z}[k])_k = \mathbb{Z}$ and $b \in (\mathbb{Z}[l])_l = \mathbb{Z}$ we have

$$\eta(a \otimes b) = (-1)^{kl} \cdot a \cdot b.$$

[1430] Once again, conventions in the literature differ. It is fairly common to define the boundary maps on $\mathrm{C}_*[k]$ to be the boundary maps of C_* multiplied by $(-1)^k$, see e.g. [**Weib1994**, p. 10].

Proof. It follows from the Tensor Lemma 89.3 (3) that each η_m is an isomorphism. But we still need to verify that the maps η_m form a chain map $\eta\colon \mathbb{Z}[k] \otimes C_* \to \mathrm{C}_*[k]$. Thus let $k \in \mathbb{N}_0$, let $a \in (\mathbb{Z}[k])_k = \mathbb{Z}$ and let $c_{m-k} \in \mathrm{C}_{m-k}$. Then

$$\begin{array}{rcccl} & & \text{since } a \text{ is of degree } k \text{ and since } \partial a = 0 & & \text{since } \partial c_{m-k} \text{ is of degree } m-k-1 \\ \eta_{m-1}(\partial(a \otimes c_{m-k})) & \overset{\downarrow}{=} & \eta_{m-1}((-1)^k \cdot a \otimes \partial c_{m-k}) & \overset{\downarrow}{=} & (-1)^k \cdot (-1)^{k(m-1-k)} \cdot a \cdot \partial c_{m-k} \\ & = & \partial((-1)^{k(m-k)} \cdot a \cdot c_{m-k}) & = & \partial(\eta_m(a \otimes c_{m-k})). \end{array}$$

It follows immediately that the maps η_m form a chain map. ■

Notation.

(1) Given topological spaces X and Y we denote by $\Upsilon\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X \times Y)$ and $\Theta\colon \mathrm{C}_*(X \times Y) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$ the chain maps from the Eilenberg-Zilber Theorem 113.1.

(2) Given a topological space W we denote by $\Delta_W\colon \mathrm{C}_*(W) \to \mathrm{C}_*(W) \otimes \mathrm{C}_*(W)$ the Alexander-Whitney diagonalization from page 2518. Recall that Δ_W is also a chain map.

In the following two lemmas we will reinterpret the cup product and the cap product as chain maps.

Lemma 117.10. Let W be a topological space and let $\alpha \in \mathrm{C}^k(W;\mathbb{Z})$ and $\beta \in \mathrm{C}^l(W;\mathbb{Z})$ be cocycles. Then we have the following equality of chain maps:

$$(-1)^{kl} \cdot \Pi_{k+l}\underbrace{(\alpha \cup \beta)}_{\text{cocycle in } \mathrm{C}^{k+l}(W;\mathbb{Z})} = \eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Delta_W\colon \mathrm{C}_*(W) \to \mathbb{Z}[k+l].$$

Proof. We introduce the following notation:

(1) We denote by $\mu\colon \mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$ the obvious isomorphism of abelian groups given by $\mu(a \otimes b) = a \cdot b$.

(2) We denote by $\Pi_{k,l}\colon (\mathrm{C}_*(W) \otimes \mathrm{C}_*(W))_{k+l} \to \mathrm{C}_k(W) \otimes \mathrm{C}_l(W)$ the natural projection.

In the proof of the Definitions-Cup Product Proposition 114.4 on page 2519 we showed that

$$\alpha \cup \beta = \mu \circ (\alpha \otimes \beta) \circ \Pi_{k,l} \circ \Delta_W\colon \mathrm{C}_{k+l}(W) \to \mathbb{Z}.$$

The lemma follows immediately from this equality and the following claim.

Claim. [1431]

$$(-1)^{kl} \cdot \mu \circ (\alpha \otimes \beta) \circ \Pi_{k,l} = \underbrace{(\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)))_{k+l}}_{\substack{\text{the restriction to the } (k+l)\text{-th chain group of the} \\ \text{chain map } \mathrm{C}_*(W) \otimes \mathrm{C}_*(W) \to \mathbb{Z}[k] \otimes \mathbb{Z}[l] \to \mathbb{Z}[k+l] \\ a \otimes b \mapsto (-1)^{kl} \cdot ab}}\colon (\mathrm{C}_*(W) \otimes \mathrm{C}_*(W))_{k+l} \to \mathbb{Z}.$$

Proof. Let $\sigma_i \in \mathrm{C}_i(W)$ and $\sigma_j \in \mathrm{C}_j(W)$ with $i + j = k + l$. If $(i,j) \neq (k,l)$, then both maps are zero. So now suppose that $(i,j) = (k,l)$. Then it is clear that both maps send $\sigma_k \otimes \sigma_l$ to $(-1)^{kl} \cdot \alpha(\sigma_k) \cdot \beta(\sigma_l)$. (Note that the sign $(-1)^{kl}$ for the map on the right-hand side stems from the definition of the chain map η.) ■

[1431] Here, given two chain complexes $\mathcal{C}_*$ and $\mathcal{D}_*$ the map $\Pi_{k,l}\colon \mathcal{C}_* \otimes \mathcal{D}_* \to \mathcal{C}_k \otimes \mathcal{B}_l$ is the natural projection that we introduced on page 2512.

Given a topological space W and given $\varphi \in \mathrm{C}^k(W;\mathbb{Z})$ we obtain for each $m \in \mathbb{N}_0$ a map

$$\begin{array}{rcl} \varphi \cap -\colon \mathrm{C}_m(W) & \to & \mathrm{C}_{m-k}(W) = (\mathrm{C}_*(W)[k])_m \\ \sigma & \mapsto & \varphi \cap \sigma. \end{array}$$

Unfortunately in general these maps do *not* form a chain map from the chain complex $\mathrm{C}_*(W)$ to the chain complex $\mathrm{C}_*(W)[k]$. This motivates introducing the following slight variation on the cap product.

Definition. Let W be a topological space. Given $\varphi \in \mathrm{C}^m(W;\mathbb{Z})$ and given $\sigma \in \mathrm{C}_n(W)$ we define[1432]

$$\varphi \mathbin{\widetilde{\cap}} \sigma \ := \ (-1)^{m(n-m)} \cdot \varphi \cap \sigma.$$

In particular for $m = n$ we have $\varphi \mathbin{\widetilde{\cap}} \sigma = \varphi \cap \sigma$.

Lemma 117.11. Let W be a topological space.

(1) If $\varphi \in \mathrm{C}^m(W;\mathbb{Z})$ is a cocycle, then the map

$$\begin{array}{rcl} \varphi \mathbin{\widetilde{\cap}} -\colon \mathrm{C}_*(W) & \to & \mathrm{C}_*(W)[m] \\ \sigma & \mapsto & \varphi \mathbin{\widetilde{\cap}} \sigma \end{array}$$

is a chain map.

(2) If $\varphi, \psi \in \mathrm{C}^m(W;\mathbb{Z})$ are cohomologous cocycles, then the chain maps $\varphi \mathbin{\widetilde{\cap}} -$ and $\psi \mathbin{\widetilde{\cap}} -$ are chain homotopic.

Proof. Let W be a topological space. We start out with the following claim.

Claim. For any cochain $\tau \in \mathrm{C}^l(W;\mathbb{Z})$ and any chain $\mu \in \mathrm{C}_n(W;\mathbb{Z})$ we have

$$\partial(\tau \mathbin{\widetilde{\cap}} \mu) \ = \ (-1)^{l+n} \cdot \delta\tau \mathbin{\widetilde{\cap}} \mu + \tau \mathbin{\widetilde{\cap}} \partial\mu.$$

Proof. The claim is an immediate consequence of Lemma 116.1. Since we are worried about signs we write down the detailed calculation. We have

$$\begin{aligned} \partial(\tau \mathbin{\widetilde{\cap}} \mu) &\underset{\text{definition of } \widetilde{\cap}}{\uparrow=} (-1)^{l(n-l)} \cdot \partial(\tau \cap \mu) \underset{\text{Lemma 116.1}}{\uparrow=} (-1)^{l(n-l)} \cdot (-1)^l \cdot (-\delta\tau \cap \mu + \tau \cap \partial\mu) \\ &= (-1)^{l(n-l)+l} \cdot \big(- (-1)^{(l+1)\cdot(n-l-1)} \cdot \delta\tau \mathbin{\widetilde{\cap}} \mu + (-1)^{l\cdot(n-l-1)} \cdot \tau \mathbin{\widetilde{\cap}} \partial\mu \big) \\ &= (-1)^{l+n} \cdot \delta\tau \mathbin{\widetilde{\cap}} \mu + \tau \mathbin{\widetilde{\cap}} \partial\mu. \end{aligned}$$

⊞

Now we turn to the actual proof of the two statements.

(1) Let $\varphi \in \mathrm{C}^m(W;\mathbb{Z})$ be a cocycle. Furthermore let $\sigma \in \mathrm{C}_n(W)$. By the definition of the cap product and by definition of the shifting of chain complexes we have $\varphi \cap \sigma \in \mathrm{C}_{n-m}(W) = (\mathrm{C}_*(W)[m])_n$. This shows that $\varphi \mathbin{\widetilde{\cap}} -$ preserves degrees. It remains to show that the map $\varphi \mathbin{\widetilde{\cap}} -$ commutes with the boundary maps. But this is an immediate consequence of the above claim and the fact that φ is a cocycle.

(2) So let $\varphi, \psi \in \mathrm{C}^m(W;\mathbb{Z})$ be cocycles. We assume they are cohomologous which means that there exists a $\tau \in \mathrm{C}^{m-1}(W;\mathbb{Z})$ such that $\varphi = \psi + \delta\tau$. For any $k \in \mathbb{N}_0$ we consider the map

$$\begin{array}{rcl} P_k\colon \mathrm{C}_k(W) & \to & (\mathrm{C}_*(W)[m])_{k+1} = \mathrm{C}_{k-m+1}(W) \\ \sigma & \mapsto & (-1)^{m+k} \cdot \tau \mathbin{\widetilde{\cap}} \sigma. \end{array}$$

[1432] If we compare this definition with the discussion on page 2553 we see that the definition of $\widetilde{\cap}$ is precisely the definition of the cap product in Bredon [**Bre1993**] and Dold [**Dol1956**].

We calculate that for any $\sigma \in \mathrm{C}_k(W)$ we have

$$\begin{aligned} \varphi \mathbin{\widetilde{\cap}} \sigma - \psi \mathbin{\widetilde{\cap}} \sigma \;&=\; \delta\tau \mathbin{\widetilde{\cap}} \sigma \;\overset{\text{by the claim}}{\overset{\downarrow}{=}}\; (-1)^{m+k} \cdot \partial(\tau \mathbin{\widetilde{\cap}} \sigma) + (-1)^{m+k-1} \cdot \tau \mathbin{\widetilde{\cap}} \partial\sigma \\ &=\; (\partial \circ P_k)(\sigma) + (P_{k-1} \circ \partial)(\sigma). \end{aligned}$$

This shows that the maps $\{P_k\}_{k\in\mathbb{N}_0}$ form a chain homotopy between the chain maps $\varphi \mathbin{\widetilde{\cap}} -$ and $\psi \mathbin{\widetilde{\cap}} -$. ■

In the proof of Theorem 117.1 we will also need the following reformulation of the cap product.

Proposition 117.12. Let W be a topological space and let $\varphi \in \mathrm{C}^m(W;\mathbb{Z})$ be a cocycle. Then we have the following equality of chain maps

$$\varphi \mathbin{\widetilde{\cap}} - \;=\; \eta \circ (\Pi_m(\varphi) \otimes \mathrm{id}) \circ \Delta_W \colon \mathrm{C}_*(W) \;\to\; \mathrm{C}_*(W)[m].$$

Proof. Let W be a topological space, let $\varphi \in \mathrm{C}^m(W)$ be a cocycle and let $\sigma\colon \Delta^n \to W$ be a singular n-simplex. We have the following equalities in $\mathrm{C}_*(W)[m]_n = \mathrm{C}_{n-m}(W)$:

$$\begin{aligned} \big(\eta \circ (\Pi_m(\varphi) \otimes \mathrm{id}) \circ \Delta_W\big)(\sigma) \;&\overset{\text{definition of the Alexander-Whitney diagonalization}}{\overset{\downarrow}{=}}\; \eta\Big((\Pi_m(\varphi) \otimes \mathrm{id})\Big(\sum_{j=0}^{n} \sigma \circ [v_0,\dots,v_j] \otimes \sigma \circ [v_j,\dots,v_n]\Big)\Big) \\ &=\; \eta\big(\underbrace{\varphi(\sigma \circ [v_0,\dots,v_m])}_{\in \mathbb{Z}[m]_m} \otimes \underbrace{\sigma \circ [v_m,\dots,v_n]}_{\in \mathrm{C}_{n-m}}\big) \\ &\overset{\text{by definition of the chain map } \Pi_m(\varphi)\otimes\mathrm{id}\colon \mathrm{C}_*(W)\otimes\mathrm{C}_*(W)\to\mathbb{Z}[m]\otimes\mathrm{C}_*(W)}{\uparrow} \\ &=\; (-1)^{m(n-m)} \cdot \underbrace{\varphi(\sigma\circ[v_0,\dots,v_m])\cdot\sigma\circ[v_m,\dots,v_n]}_{=\varphi\cap\sigma} \;=\; \varphi \mathbin{\widetilde{\cap}} \sigma. \\ &\qquad\uparrow \text{ definition of } \eta \text{ in Lemma 117.9} \qquad\qquad \uparrow \text{ definition of } \mathbin{\widetilde{\cap}} \text{ on page 2574} \end{aligned}$$

We have thus verified that the desired equality holds. ■

The following statement is just the definition of the cross product.

Tautology 117.13. Let X and Y two topological spaces. For any $[\mu] \in \mathrm{H}_m(X;\mathbb{Z})$ and $[\nu] \in \mathrm{H}_n(Y;\mathbb{Z})$ we have

$$[\mu] \times [\nu] \;=\; [\Upsilon(\mu \otimes \nu)].$$

We need one more slightly delicate chain map before we can finally turn to the proof of the Product Theorem 117.1.

Lemma 117.14. Let C_* and D_* be chain complexes and let $k, l \in \mathbb{N}_0$. The map

$$\Psi_{k,l}\colon \mathrm{C}_*[k] \otimes D_*[l] \;\to\; (C_* \otimes D_*)[k+l]$$

which is defined by the property that for $c_m \in \mathrm{C}_m$ and $d_n \in D_n$ we have

$$c_m \otimes d_n \;\mapsto\; (-1)^{k(l+n)} \cdot c_m \otimes d_n.$$

is an isomorphism of chain complexes.

Proof. First note that it is clear that $\Psi_{k,l}$ preserves degrees and that on each degree it is an isomorphism. It remains to show that the map $\Psi_{k,l}$ is a chain map. Let $c_m \in \mathrm{C}_m$ and

$d_n \in D_n$. We calculate that

$$\begin{aligned}
\Psi_{k,l}(\partial(c_m \otimes d_n)) &\overset{\text{since } c_m \text{ has degree } m+k \text{ in } \mathrm{C}_*[k]}{\downarrow} \\
&= \Psi_{k,l}(\partial c_m \otimes d_n + (-1)^{m+k} \cdot c_m \otimes \partial d_n) \\
&= (-1)^{k(l+n)} \cdot \partial c_m \otimes d_n + (-1)^{m+k} \cdot (-1)^{k(l+n-1)} \cdot c_m \otimes \partial d_n \\
&= (-1)^{k(l+n)} \cdot (\partial c_m \otimes d_n + (-1)^m \cdot c_m \otimes \partial d_n) \underset{\text{since } c_m \text{ has degree } m \text{ in } C_*}{\overset{\uparrow}{=}} (-1)^{k(l+n)} \cdot \partial(c_m \otimes d_n) \\
&= \partial((-1)^{k(l+n)} \cdot c_m \otimes d_n)) = \partial(\Psi_{k,l}(c_m \otimes d_n)).
\end{aligned}$$

This calculation implies that $\Psi_{k,l}$ is a chain map. ■

The following theorem is the main technical result of this section. As we will see shortly, the Product Theorem 117.1 is a straightforward consequence of Theorem 117.15.

Theorem 117.15. Let X and Y be topological spaces. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the projection s. Let $\alpha \in \mathrm{C}^k(X;\mathbb{Z})$ and $\beta \in \mathrm{C}^l(Y;\mathbb{Z})$ be cocycles. Then the chain maps

$$(-1)^{kl} \cdot ((p^*\alpha \cup q^*\beta) \mathbin{\widetilde{\cap}} -) \circ \Upsilon \quad \text{and} \quad \Upsilon[k+l] \circ \Psi_{k,l} \circ (\alpha \mathbin{\widetilde{\cap}} - \otimes \beta \mathbin{\widetilde{\cap}} -)$$

from $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$ to $\mathrm{C}_*(X \times Y)[k+l]$ are chain homotopic.

Proof. Let X and Y be topological spaces and let $\alpha \in \mathrm{C}^k(X;\mathbb{Z})$ and $\beta \in \mathrm{C}^l(Y;\mathbb{Z})$ be cocycles. We consider the following diagram of chain maps

$$\begin{array}{ccccc}
\mathrm{C}_*X \otimes \mathrm{C}_*Y & \xrightarrow{\Upsilon} & \mathrm{C}_*(X\times Y) & \xrightarrow{\Delta_{X\times Y}} & \mathrm{C}_*(X\times Y)\otimes\mathrm{C}_*(X\times Y) \\
\downarrow{\scriptstyle \Delta_X\otimes\Delta_Y} & & & & \downarrow{\scriptstyle \Theta\otimes\mathrm{id}} \\
\mathrm{C}_*X\otimes\mathrm{C}_*X\otimes\mathrm{C}_*Y\otimes\mathrm{C}_*Y & \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} & \mathrm{C}_*X\otimes\mathrm{C}_*Y\otimes\mathrm{C}_*X\otimes\mathrm{C}_*Y & \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\Upsilon} & \mathrm{C}_*X\otimes\mathrm{C}_*Y\otimes\mathrm{C}_*(X\times Y) \\
\downarrow{\scriptstyle \Pi_k(\alpha)\otimes\mathrm{id}\otimes\Pi_l(\beta)\otimes\mathrm{id}} & & \downarrow{\scriptstyle \Pi_k(\alpha)\otimes\Pi_l(\beta)\otimes\mathrm{id}\otimes\mathrm{id}} & & \downarrow{\scriptstyle \Pi_k(\alpha)\otimes\Pi_l(\beta)\otimes\mathrm{id}} \\
\mathbb{Z}[k]\otimes\mathrm{C}_*X\otimes\mathbb{Z}[l]\otimes\mathrm{C}_*Y & \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} & \mathbb{Z}[k]\otimes\mathbb{Z}[l]\otimes\mathrm{C}_*X\otimes\mathrm{C}_*Y & \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\Upsilon} & \mathbb{Z}[k]\otimes\mathbb{Z}[l]\otimes\mathrm{C}_*(X\times Y) \\
\downarrow{\scriptstyle \eta\otimes\eta} & & \downarrow{\scriptstyle \eta\otimes\mathrm{id}\otimes\mathrm{id}} & & \downarrow{\scriptstyle \eta\otimes\mathrm{id}} \\
 & & \mathbb{Z}[k+l]\otimes(\mathrm{C}_*X\otimes\mathrm{C}_*Y) & \xrightarrow{\mathrm{id}\otimes\Upsilon} & \mathbb{Z}[k+l]\otimes\mathrm{C}_*(X\times Y) \\
 & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\
\mathrm{C}_*X[k]\otimes\mathrm{C}_*Y[l] & \xrightarrow{\Psi_{k,l}} & (\mathrm{C}_*X\otimes\mathrm{C}_*Y)[k+l] & \xrightarrow{\Upsilon[k+l]} & \mathrm{C}_*(X\times Y)[k+l].
\end{array}$$

Claim 1. The above diagram commutes up to chain homotopies.

Proof. We consider the various smaller squares and rectangles.

(1) The two paths along the top rectangle from the top-left corner to the bottom-right corner are both natural chain maps. Furthermore, both have the property that for $x \in X$ and $y \in Y$ the element $x \otimes y \in \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$ gets sent to $x \otimes y \otimes (x, y)$ in $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \otimes \mathrm{C}_*(X \times Y)$. In particular both paths agree on the 0-level.

It follows from Acyclic Model Theorem 113.16 (2) that the rectangle commutes up to a chain homotopy. More precisely, we apply the Acyclic Model Theorem 113.16 as follows:

(a) We take $\mathcal{C}$ to be the category of tuples (X, Y) of topological spaces.

(b) We consider the two functors $F\colon (X,Y) \mapsto (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y), \epsilon_X \otimes \epsilon_Y)$ and $G\colon (X,Y) \mapsto (\mathrm{C}_*(X \times Y), \epsilon_{X\times Y})$.
(c) We take $\mathcal{M} = \{\Delta^k \times \Delta^l\}_{k,l\in\mathbb{N}_0}$.
(d) We consider the two natural transformations that are given by the two extremal paths in the diagram (shifted downward in degrees).

(2) It is clear that the two squares between the second and the third row commute.
(3) It is straightforward to verify that the two squares at the bottom right commute.
(4) It remains to consider the big square to the bottom left of the diagram. Starting with $a \in (\mathbb{Z}[k])_k = \mathbb{Z}$, $b \in \mathrm{C}_m(X)$, $c \in (\mathbb{Z}[l])_l = \mathbb{Z}$ and $d \in \mathrm{C}_n(Y)$ and keeping in mind the sign conventions of Lemmas 117.7, 117.9 and 117.14 we get the following results for $a \otimes b \otimes c \otimes d$:
(a) going down-right gives $(-1)^{km} \cdot (-1)^{ln} \cdot (-1)^{k(l+n)} \cdot ac \cdot (b \otimes d)$,
(b) going right-down-down gives $(-1)^{ml} \cdot (-1)^{kl} \cdot (-1)^{(k+l)(m+n)} \cdot ac \cdot (b \otimes d)$.
One easily verifies that the signs agree. So we see that the bottom left square also commutes. ⊞

In light of the previous claim it suffices to prove the following claim.

Claim 2.
(1) the "left-bottom route" equals the chain map $\Upsilon[k+l] \circ \Psi_{k,l} \circ ((\alpha \mathbin{\widetilde{\cap}} -) \otimes (\beta \mathbin{\widetilde{\cap}} -))$,
(2) the "top-right route" is chain homotopic to $(-1)^{kl} \cdot ((p^*\alpha \cup q^*\beta) \mathbin{\widetilde{\cap}} -) \circ \Upsilon$.

Proof. We start out with the proof of Statement (1). First we note that

$$(\alpha \mathbin{\widetilde{\cap}} -) \otimes (\beta \mathbin{\widetilde{\cap}} -) \underset{\text{by Proposition 117.12}}{\overset{\uparrow}{=}} (\eta \circ (\Pi_k(\alpha) \otimes \mathrm{id}) \circ \Delta_X) \otimes (\eta \circ (\Pi_l(\beta) \otimes \mathrm{id}) \circ \Delta_Y) \underset{\text{follows from the observation that for chain maps } a,b,c \text{ and } d \text{ we have } (a\circ b)\otimes(c\circ d) = (a\otimes b)\circ(c\otimes d)}{\overset{\uparrow}{=}} (\eta \otimes \eta) \circ ((\Pi_k(\alpha) \otimes \mathrm{id} \otimes \Pi_l(\beta) \otimes \mathrm{id})) \circ (\Delta_X \otimes \Delta_Y).$$

If we compose this equality with $\Upsilon[k+l] \circ \Psi_{k,l}$ then we obtain the "bottom left route" in the above diagram. This concludes the proof of Statement (1).

We continue with the proof of Statement (2). In that proof we will make use of the following subclaim.

Subclaim. The two cocycles[1433]
$$(\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta)_{k+l} \quad \text{and} \quad (-1)^{kl} \cdot p^*(\alpha) \cup q^*(\beta)$$
in $\mathrm{C}^{k+l}(X \times Y; \mathbb{Z})$ are cohomologous.

We consider the following diagram

$$\begin{array}{ccccc} \mathrm{C}_{k+l}(X\times Y) & \hookrightarrow & \mathrm{C}_*(X\times Y) & \xrightarrow{\Theta} & \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y) \\ & & \downarrow{\scriptstyle \Delta_{X\times Y}} & \nearrow{\scriptstyle p_*\otimes q_*} & \downarrow{\scriptstyle \Pi_k(\alpha)\otimes\Pi_l(\beta)} \\ & & \mathrm{C}_*(X\times Y)\otimes\mathrm{C}_*(X\times Y) & \xrightarrow[\Pi_k(p^*(\alpha))\otimes\Pi_l(q^*(\beta))]{} & \mathbb{Z}[k]\otimes\mathbb{Z}[l] \xrightarrow[\eta]{} \mathbb{Z}[k+l]. \end{array}$$

[1433] Recall that $\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta$ is a chain map $\mathrm{C}_*(X \times Y) \to \mathbb{Z}[k+l]$. Furthermore recall that the map $(\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta)_{k+l}\colon \mathrm{C}_{k+l}(X \times Y) \to (\mathbb{Z}[k+l])_{k+l} = \mathbb{Z}$ is the $(k+l)$-th map of the chain map.

Note that by the above discussion all maps except for the top left map are chain maps. It follows immediately from the definitions that the bottom triangle of chain maps commutes "on the nose". The top triangle of chain maps commutes, up to a chain homotopy, by Theorem 113.5.[1434] It follows from Lemma 117.8 (3) that the restrictions of these two chain maps to maps $\mathrm{C}_{k+l}(X \times Y) \to \mathbb{Z} = \mathbb{Z}[k+l]$ are cohomologous cochains. But the restriction of the "top route" equals evidently the left-hand side of the subclaim. On the other hand, by Lemma 117.10 the restriction of the "bottom route" equals the right-hand side of the subclaim. This concludes the proof of the subclaim.

Now we see that

$$
\begin{aligned}
(-1)^{kl} \cdot ((p^*\alpha \cup q^*\beta) \mathbin{\widetilde{\cap}} -) \circ \Upsilon
&\overset{\text{above subclaim and Lemma 117.11 (2)}}{\simeq} \big((\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta)_{k+l} \mathbin{\widetilde{\cap}} -\big) \circ \Upsilon \\
&= \eta \circ \big(\Pi_{k+l}((\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta)_{k+l}) \otimes \mathrm{id}\big) \circ \Delta_{X\times Y} \circ \Upsilon \\
&\quad \uparrow \text{Proposition 117.12 applied to the } (k+l)\text{-cocycle } (\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta)_{k+l} \\
&= \eta \circ \big(\Pi_{k+l}((\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)))_{k+l}) \circ \Theta \otimes \mathrm{id}\big) \circ \Delta_{X\times Y} \circ \Upsilon \\
&\quad \uparrow \text{since for every chain map } \Omega\colon \mathrm{C}_*(X) \times \mathrm{C}_*(Y) \to \mathbb{Z}[k+l] \text{ we} \\
&\quad \text{have } \Pi_{k+l}((\Omega \circ \Theta)_{k+l}) = \Pi_{k+l}(\Omega_{k+l}) \circ \Theta \text{ as maps } \mathrm{C}_*(X \times Y) \to \mathbb{Z}[k+l], \\
&\quad \text{applies this observation to } \Omega = \eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \\
&= \eta \circ ((\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta) \otimes \mathrm{id}) \circ \Delta_{X\times Y} \circ \Upsilon \\
&\quad \uparrow \text{here we use the equality of chain maps} \\
&\quad \Pi_{k+l}((\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)))_{k+l}) = \eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta))\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathbb{Z}[k+l] \\
&\quad \text{which is obvious once one unravels the definitions} \\
&= \text{the "top right route" in the above diagram.} \\
&\quad \uparrow \text{follows from } (f \otimes \mathrm{id}) \circ (g \otimes \mathrm{id}) = (g \circ f) \otimes \mathrm{id}
\end{aligned}
$$

This concludes the proof of the claim and thus also of the theorem. ∎

Now we can finally provide the proof of Theorem 117.1.

Proof of Theorem 117.1. To simplify the notation we only consider the special case that $A = B = \varnothing$. For the general case one needs to take more care, we refer to [**Dol1956**, p. 240 and p. 244] for details. So let X and Y be two topological spaces. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the projection s. Let $[\alpha] \in \mathrm{H}^k(X;\mathbb{Z})$, $[\beta] \in \mathrm{H}^l(Y;\mathbb{Z})$ and let $[\mu] \in \mathrm{H}_m(X;\mathbb{Z})$ and $[\nu] \in \mathrm{H}_n(Y;\mathbb{Z})$. We have to show that

$$
\Big(\underbrace{[p^*(\alpha)]}_{\in \mathrm{H}^k(X\times Y)} \cup \underbrace{[q^*(\beta)]}_{\in \mathrm{H}^l(X\times Y)}\Big) \cap \underbrace{\big([\mu] \times [\nu]\big)}_{\in \mathrm{H}_{m+n}(X\times Y)} = (-1)^{kl+lm} \cdot \underbrace{([\alpha] \cap [\mu])}_{\in \mathrm{H}_{m-k}(X)} \times \underbrace{([\beta] \cap [\nu])}_{\in \mathrm{H}_{n-l}(Y)}
$$

[1434] More precisely, given $x \in X$ and $y \in Y$ both routes along the triangle send (x, y) to $x \otimes y$. Furthermore both routes are natural chain maps. Thus we can apply Theorem 113.5.

We have the following equality in $\mathrm{H}_{m+n-k-l}(X\times Y)$:

$$\begin{aligned}
(p^*([\alpha])\cup q^*([\beta]))\cap([\mu]\times[\nu]) &\overset{\text{Tautology 117.13 and the definition of } \widetilde{\cap}\text{, see page 2574}}{=} (-1)^{(k+l)(m+n-k-l)}\cdot([p^*\alpha\cup q^*\beta])\ \widetilde{\cap}\ [\Upsilon(\mu\otimes\nu)]\\
&= (-1)^{(k+l)(m+n-k-l)}\cdot(((p^*\alpha\cup q^*\beta)\ \widetilde{\cap}\ -)\circ\Upsilon)_*([\mu\otimes\nu])\\
&= (-1)^{(k+l)(m+n-k-l)}\cdot((-1)^{kl}\cdot\Upsilon[k+l]\circ\Psi_{k,l}\circ(\alpha\ \widetilde{\cap}\ -\otimes\beta\ \widetilde{\cap}\ -))_*([\mu\otimes\nu])\\
&\overset{\text{by Theorem 117.15 and the fact that chain homotopic maps induce the same map on homology groups}}{=}\\
&= (-1)^{(k+l)(m+n-k-l)}\cdot(-1)^{kl}\cdot\Big[\Upsilon[k+l]\circ\Psi_{k,l}\circ\Big(\underbrace{\alpha\ \widetilde{\cap}\ \mu}_{\substack{=(-1)^{k(m-k)}\cdot\alpha\cap\mu\\ \in \mathrm{C}_{m-k}(X)[k]}}\otimes\underbrace{\beta\ \widetilde{\cap}\ \nu}_{\substack{=(-1)^{l(n-l)}\cdot\beta\cap\nu\\ \in \mathrm{C}_{n-l}(Y)[l]}}\Big)\Big]\\
&= (-1)^{(k+l)(m+n-k-l)}\cdot(-1)^{kl}\cdot(-1)^{k(l+n-l)}\cdot(-1)^{k(m-k)}\cdot(-1)^{l(n-l)}\cdot\big[\Upsilon(\alpha\cap\mu\otimes\beta\cap\nu)\big]\\
&\overset{\text{definition of } \Psi_{k,l}\text{, see page 2575, and of the shifting from } \Upsilon \text{ to } \Upsilon[k+l]}{=}\\
&= (-1)^{kl+lm}\cdot[\alpha\cap\mu\times\beta\cap\nu] = (-1)^{kl+lm}\cdot([\alpha]\cap[\mu]\times[\beta]\cap[\nu]).\\
&\overset{\text{cleaning up signs and Tautology 117.13}}{}
\end{aligned}$$

■

117.3. **The Künneth Theorem for cohomology groups.** Given two topological spaces X and Y the Künneth Theorem 113.8 for topological spaces determines the homology groups of $X\times Y$ in terms of the homology groups of X and Y. The following proposition gives us, under some mild hypothesis, a similar answer for cohomology groups.

Theorem 117.16. (The Künneth Theorem for cohomology groups) Let X and Y be two topological spaces. We denote by $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ the natural projections. If all the homology groups of X are finitely generated[1435], then given any $n\in\mathbb{N}_0$ there exists a short exact sequence

$$0\to\bigoplus_{k+l=n}\mathrm{H}^k(X;\mathbb{Z})\otimes\mathrm{H}^l(Y;\mathbb{Z})\to\mathrm{H}^n(X\times Y;\mathbb{Z})\to\bigoplus_{k+l=n+1}\mathrm{Tor}(\mathrm{H}^k(X;\mathbb{Z}),\mathrm{H}^l(Y;\mathbb{Z}))\to 0$$
$$\varphi\otimes\psi\mapsto p^*\varphi\cup q^*\psi$$

which is natural with respect to the topological spaces X and Y. If all the homology groups of Y are also finitely generated, then the short exact sequence splits (albeit not naturally).

Remark. Let X and Y be two topological spaces. In the literature the map

$$\begin{aligned}\mathrm{H}^k(X;\mathbb{Z})\times\mathrm{H}^l(Y;\mathbb{Z}) &\to \mathrm{H}^{k+l}(X\times Y;\mathbb{Z})\\ (\varphi,\psi) &\mapsto p^*\varphi\cup q^*\psi\end{aligned}$$

is often referred to as the *cohomology cross product* and it is written as $\varphi\times\psi := p^*\varphi\cup q^*\psi$. We will not make use of this notation.

The proof of the theorem rests on a purely algebraic theorem that, given two chain complexes $\mathcal{C}$ and $\mathcal{C}'$, relates the cohomology groups of $\mathcal{C}$ and $\mathcal{C}'$ to the cohomology groups of $\mathcal{C}\otimes\mathcal{C}'$. Before we can state the theorem we need to introduce one more definition, which is closely related to the definitions introduced on page 2512.

[1435] By this we mean that for every $k\in\mathbb{N}_0$ the abelian group $\mathrm{H}_k(X;\mathbb{Z})$ is finitely generated.

Definition. Let $\mathcal{C}$ and $\mathcal{C}'$ be two chain complexes and let $p, q \in \mathbb{N}_0$.

(1) We denote by
$$\Pi_{p,q}\colon (\mathcal{C}_* \otimes \mathcal{C}'_*)_{p+q} = \bigoplus_{i+j=p+q} \mathrm{C}_i \otimes \mathrm{C}'_j \;\to\; \mathrm{C}_p \otimes \mathrm{C}'_q$$
the natural projection.

(2) We denote by $\mu\colon \mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$ the obvious isomorphism given by $\mu(a \otimes b) = ab$.

(3) We consider the map:
$$\begin{aligned}\Xi_{p,q}\colon \mathrm{Hom}(\mathrm{C}_p, \mathbb{Z}) \otimes \mathrm{Hom}(\mathrm{C}'_q, \mathbb{Z}) \;&\to\; \mathrm{Hom}\left((\mathcal{C} \otimes \mathcal{C}')_{p+q}, \mathbb{Z}\right)\\ \varphi_p \otimes \varphi'_q \;&\mapsto \left(\bigoplus_{i+j=p+q} \mathrm{C}_i \otimes \mathrm{C}'_j \xrightarrow{\Pi_{p,q}} \mathrm{C}_p \otimes \mathrm{C}'_q \xrightarrow{\varphi_p \otimes \varphi'_q} \mathbb{Z} \otimes \mathbb{Z} \xrightarrow{\mu} \right).\end{aligned}$$

(4) It is straightforward to verify that the map $\Xi_{p,q}$ induces a well-defined map on cohomology groups, i.e. that the map
$$\begin{aligned}\Xi_{p,q}\colon \mathrm{H}^p(\mathcal{C}; \mathbb{Z}) \otimes \mathrm{H}^q(\mathcal{C}'; \mathbb{Z}) \;&\to\; \mathrm{H}^{p+q}(\mathcal{C} \otimes \mathcal{C}'; \mathbb{Z})\\ ([\varphi_p] \otimes [\varphi'_q]) \;&\mapsto\; [\Xi_{p,q}(\varphi_p \otimes \varphi'_q)]\end{aligned}$$
is well-defined. Furthermore it follows immediately from the definition that $\Xi_{p,q}$ is natural[1436] with respect to $\mathcal{C}$ and $\mathcal{C}'$.

Now we can state the promised purely algebraic theorem.

Theorem 117.17. (The Künneth Theorem for cochain complexes) Let $\mathcal{C}$ and $\mathcal{C}'$ be chain complexes. If $\mathcal{C}$ is a free chain complex and if all homology groups of $\mathcal{C}$ are finitely generated, then for every $n \in \mathbb{N}_0$ there exists a short exact sequence
$$0 \to \bigoplus_{p+q=n} \mathrm{H}^p(\mathcal{C}; \mathbb{Z}) \otimes \mathrm{H}^q(\mathcal{C}'; \mathbb{Z}) \xrightarrow{\Xi_{p,q}} \mathrm{H}^n(\mathcal{C} \otimes \mathcal{C}'; \mathbb{Z}) \to \bigoplus_{p+q=n+1} \mathrm{Tor}(\mathrm{H}^p(\mathcal{C}; \mathbb{Z}), \mathrm{H}^q(\mathcal{C}; \mathbb{Z})) \to 0$$
which is natural with respect to $\mathcal{C}$ and $\mathcal{C}'$. If all the chain groups of $\mathcal{C}'$ are also free abelian and all the homology groups of $\mathcal{C}'$ are also finitely generated, then the short exact sequence splits (albeit not naturally).

In the proof of the Künneth Theorem 117.17 for cochain complexes we need one technical lemma. To state the lemma we recall that on page 1940 we introduced the tensor product of two chain complexes. Basically the same definition also allows us to define the tensor product of two cochain complexes. With this preamble we can formulate the next lemma.

Lemma 117.18. Let $\mathcal{C}_*$ and $\mathcal{C}'_*$ be two chain complexes. We assume that all chain groups of $\mathcal{C}$ are finitely generated free abelian groups. Then the map
$$\begin{aligned}\Omega\colon \mathrm{Hom}(\mathcal{C}_*, \mathbb{Z}) \otimes \mathrm{Hom}(\mathcal{C}'_*, \mathbb{Z}) \;&\to\; \mathrm{Hom}(\mathcal{C}_* \otimes \mathcal{C}'_*, \mathbb{Z})\\ (\varphi_p\colon \mathrm{C}_p \to \mathbb{Z}) \otimes (\varphi'_q\colon C'_q \to \mathbb{Z}) \;&\mapsto\; (\mu \circ (\varphi_p \otimes \varphi'_q) \circ \Pi_{p,q}\colon (\mathcal{C}_* \otimes \mathcal{C}'_*)_{p+q} \to \mathbb{Z})\end{aligned}$$
is an isomorphism of cochain complexes.

[1436] In this case this means that for chain maps $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{C}' \to \mathcal{D}'$ the following diagram commutes:
$$\begin{array}{ccc}\mathrm{H}^p(\mathcal{D}; \mathbb{Z}) \otimes \mathrm{H}^q(\mathcal{D}'; \mathbb{Z}) & \xrightarrow{\Xi_{p,q}} & \mathrm{H}^{p+q}(\mathcal{D} \otimes \mathcal{D}'; \mathbb{Z})\\ \downarrow{\scriptstyle f^* \otimes (f')^*} & & \downarrow{\scriptstyle (f \otimes f')^*}\\ \mathrm{H}^p(\mathcal{C}; \mathbb{Z}) \otimes \mathrm{H}^q(\mathcal{C}'; \mathbb{Z}) & \xrightarrow{\Xi_{p,q}} & \mathrm{H}^{p+q}(\mathcal{C} \otimes \mathcal{C}'; \mathbb{Z}).\end{array}$$

Proof. We start out the proof of the lemma with the following elementary well-known observation:

(∗) For finitely many abelian groups $A_1, \dots, A_k$ we have a canonical isomorphism

$$\operatorname{Hom}(A_1 \oplus \cdots \oplus A_k, \mathbb{Z}) \;=\; \operatorname{Hom}(A_1, \mathbb{Z}) \oplus \cdots \oplus \operatorname{Hom}(A_k, \mathbb{Z}).$$

Now we turn to the actual proof of the lemma. It is elementary to verify that the map Ω is in fact a cochain map. So it remains to show that for every $n \in \mathbb{N}_0$ the map

$$\begin{aligned} \bigoplus_{p+q=n} \operatorname{Hom}(\mathrm{C}_p, \mathbb{Z}) \otimes \operatorname{Hom}(C'_q, \mathbb{Z}) &\to \operatorname{Hom}\Big(\bigoplus_{p+q=n} \mathrm{C}_p \otimes \mathrm{C}'_q, \mathbb{Z}\Big) \\ \varphi_p \otimes \varphi'_q &\mapsto \mu \circ (\varphi_p \otimes \varphi'_q) \circ \Pi_{p,q} \end{aligned}$$

is an isomorphism. Applying (∗) to the right-hand side we see that it suffices to prove the following claim.

Claim. For every choice of $p, q \in \mathbb{N}_0$ the map

$$\begin{aligned} \operatorname{Hom}(\mathrm{C}_p, \mathbb{Z}) \otimes \operatorname{Hom}(C'_q, \mathbb{Z}) &\to \operatorname{Hom}(\mathrm{C}_p \otimes C'_q, \mathbb{Z}) \\ \varphi_p \otimes \varphi'_q &\mapsto \mu \circ (\varphi_p \otimes \varphi'_q) \end{aligned}$$

is an isomorphism.

Proof. Let $p, q \in \mathbb{N}_0$. By our hypothesis on $\mathcal{C}_*$ the chain group C_p is a finitely generated free abelian group. Therefore we can find a basis $a_1, \dots, a_k$ for C_p. We denote by $\widehat{a}_1, \dots, \widehat{a}_k$ the dual basis of $\operatorname{Hom}(\mathrm{C}_p, \mathbb{Z})$. We can thus write

$$\operatorname{Hom}(\mathrm{C}_p, \mathbb{Z}) = \bigoplus_{i=1}^{k} \mathbb{Z} \cdot \widehat{a}_i \quad \text{and} \quad \mathrm{C}_p \otimes \mathrm{C}'_q = \bigoplus_{i=1}^{k} \mathbb{Z} \cdot a_i \otimes \mathrm{C}'_q.$$

Applying (∗) to the above decompositions and both sides of the claim one sees easily that the given homomorphism is an isomorphism. ■

Now we can finally prove the Künneth Theorem 117.17 for cochain complexes.

Proof of the Künneth Theorem 117.17 for cochain complexes. We only provide an outline of the proof. Let $\mathcal{C} = (\mathrm{C}_*, c_*)$ and $\mathcal{C}' = (\mathrm{C}'_*, c'_*)$ be chain complexes such that all the chain groups C_n are free abelian groups and such that all the homology groups $\mathrm{H}_k(\mathcal{C}; \mathbb{Z})$ are finitely generated.

The idea is that one can turn any cochain complex into a chain complex by reversing the signs in the degrees. Thus one can try to reduce the theorem to the Künneth Theorem 90.7. Towards the end of the proof we need to insert Lemma 117.18 to make everything work.

The proof proceeds in three steps:

(1) Since $\mathcal{C}$ is a free chain complex and since all homology groups of $\mathcal{C}$ are finitely generated we can use the Finite-Rank Chain Complex Lemma 81.14 to replace $\mathcal{C}$ by a chain complex such that all chain groups are finitely generated free abelian groups.

(2) Given $n \in \mathbb{Z}_{\leq 0}$ we write $E_n := \operatorname{Hom}(\mathrm{C}_{-n}, \mathbb{Z})$, $e_n := \partial^*_{-n+1} \colon E_n \to E_{n-1}$ and similarly we write $E'_n := \operatorname{Hom}(\mathcal{C}'_{-n}, \mathbb{Z})$ and $e'_n := \partial^*_{-n+1} \colon E'_n \to E'_{n-1}$. It follows immediately from the definitions that $\mathcal{E}_* := (E_*, e_*)$ and $\mathcal{E}'_* := (E'_*, e'_*)^*)$ form chain complexes in negative degrees. (We defined chain complexes to be given by chain groups indexed by $n \in \mathbb{N}_0$ but evidently the same theory also works for chain groups indexed by $\mathbb{Z}$ and $\mathbb{Z}_{\leq 0}$.) Since the groups E_n are free abelian we can apply the Künneth Theorem 90.7

(which also applies to chain complexes in non-positive degrees) to obtain for each $n \in \mathbb{Z}_{\leq 0}$ a natural short exact sequence

$$0 \to \bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{E}) \otimes \mathrm{H}_q(\mathcal{E}') \to \mathrm{H}_n(\mathcal{E} \otimes \mathcal{E}') \to \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(\mathcal{E}), \mathrm{H}_q(\mathcal{E}')) \to 0.$$
$$[\varphi] \otimes [\psi] \quad :$$

Put differently, for each $n \in \mathbb{N}_0$ we have a natural short exact sequence

$$0 \to \bigoplus_{-p-q=n} \underbrace{\mathrm{H}_{-p}(\mathcal{E})}_{=\mathrm{H}^p(\mathcal{C};\mathbb{Z})} \otimes \underbrace{\mathrm{H}_{-q}(\mathcal{E}')}_{=\mathrm{H}^q(\mathcal{E}';\mathbb{Z})} \to \mathrm{H}_{-n}(\mathcal{E} \otimes \mathcal{E}') \to \bigoplus_{-p-q=n+1} \mathrm{Tor}(\underbrace{\mathrm{H}_{-p}(\mathcal{E})}_{=\mathrm{H}^p(\mathcal{C};\mathbb{Z})}, \underbrace{\mathrm{H}_{-q}(\mathcal{E}')}_{=\mathrm{H}^p(\mathcal{C};\mathbb{Z})}) \to 0.$$

(3) For each $n \in \mathbb{N}_0$ we have isomorphisms

$$\mathrm{H}_{-n}(\mathcal{E}_* \otimes \mathcal{E}'_*) = \mathrm{H}^n(\mathcal{E}_{-*} \otimes \mathcal{E}'_{-*}) = \mathrm{H}^n\big(\overbrace{\mathrm{Hom}(\mathcal{C}_*, \mathbb{Z})}^{=\mathcal{E}_{-*}} \otimes \overbrace{\mathrm{Hom}(\mathcal{C}'_*, \mathbb{Z})}^{=\mathcal{E}'_{-*}}\big) \xrightarrow{\Omega} \mathrm{H}^n(\mathcal{C}_* \otimes \mathcal{C}'_*; \mathbb{Z}).$$
$$\uparrow \text{ by definition} \qquad \uparrow \text{ by definition} \qquad \uparrow \text{ by Lemma 117.18 and our hypothesis on } \mathcal{C}_*$$

We use this isomorphism to replace the middle term of the second short exact sequence in (2) and we obtain the desired short exact sequence.

Finally suppose that all the chain groups of $\mathcal{C}'$ are also free abelian and that all the homology groups of $\mathcal{C}'$ are also finitely generated. As in (1) we can use the Finite-Rank Chain Complex Lemma 81.14 to replace $\mathcal{C}$ by a chain complex such that all chain groups are finitely generated free abelian groups. It follows that $\mathcal{E}_*$ is a free chain complex which implies, by the last statement of the Künneth Theorem 90.7, that the short exact sequence splits. ■

Now we are in a position to prove the Künneth Theorem 117.16 for cohomology groups.

Proof of the Künneth Theorem 117.16 for cohomology groups. Let X and Y be two topological spaces such that all the homology groups of X are finitely generated. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the natural projections. Throughout the proof we use all the definitions that we introduced in Section 117.2.

It follows from the Eilenberg-Zilber Theorem 113.1, together with comparing the conclusion of the Künneth Theorem 117.17 for cochain complexes with the desired statement of the Künneth Theorem 117.16 for cohomology groups, that it remains to prove the following claim.

Claim. For any $k, l \in \mathbb{N}_0$ the map

$$\begin{aligned} \mathrm{H}^k(X;\mathbb{Z}) \otimes \mathrm{H}^l(Y;\mathbb{Z}) &\to \mathrm{H}^{k+l}(X \times Y;\mathbb{Z}) \\ [\alpha] \otimes [\beta] &\mapsto [p^*\alpha \cup q^*\beta] \end{aligned}$$

agrees with the map that is given by

$$[\alpha] \otimes [\beta] \mapsto [(\alpha \otimes \beta) \circ \Pi_{k,l} \circ \Theta].$$

Proof. Let $\alpha \in \mathrm{C}^k(X;\mathbb{Z})$ and $\beta \in \mathrm{C}^l(Y;\mathbb{Z})$ be cocycles. We have to show that the two cocycles

$$\underbrace{p^*\alpha \cup q^*\beta}_{=:\aleph} \quad \text{and} \quad \underbrace{(\alpha \otimes \beta) \circ \Pi_{k,l}}_{=:\beth} \circ \Theta$$

are cohomologous.

As usual, given a topological space W we denote by $\epsilon_W\colon \mathrm{C}_0(W) \to \mathbb{Z}$ the augmentation map. Clearly ϵ_W is a cocycle so by Lemma 117.8 (2) we can extend ϵ_W to a chain map

$\mathrm{C}_*(W) \to \mathbb{Z}$ which, by a slight abuse of notation, we also denote by ϵ_W. It follows from the Eilenberg-Zilber Theorem 113.1, Theorem 117.15 and Lemma 117.8 (2) that the maps

$$\underbrace{(-1)^{kl} \cdot \epsilon_{X\times Y}[k+l] \circ ((p^*\alpha \cup q^*\beta) \mathbin{\widetilde{\cap}} -)}_{=:\aleph'}\colon \mathrm{C}_{k+l}(X \times Y) \to \mathbb{Z}[k+l] = \mathbb{Z}$$

and

$$\underbrace{\epsilon_{X\times Y}[k+l] \circ \Upsilon[k+l] \circ \Psi_{k,l} \circ (\alpha \mathbin{\widetilde{\cap}} - \otimes \beta \mathbin{\widetilde{\cap}} -)}_{=:\beth'} \circ \Theta\colon \mathrm{C}_{k+l}(X \times Y) \to \mathbb{Z}[k+l] = \mathbb{Z}$$

are cocycles that are cohomologous. It remains to prove the following two subclaims.

Subclaim. We have

$$\aleph' = (-1)^{kl} \cdot \aleph\colon \mathrm{C}_{k+l}(X \times Y) \to \mathbb{Z}.$$

Let $\sigma \in \mathrm{C}_{k+l}(X \times Y)$. Then

$$\aleph'(\sigma) \underset{\uparrow}{=} (-1)^{kl} \cdot \epsilon_{X\times Y}((p^*\alpha \cup q^*\beta) \mathbin{\widetilde{\cap}} \sigma) \underset{\uparrow}{=} (-1)^{kl} \cdot (p^*\alpha \cup q^*\beta)(\sigma) = (-1)^{kl} \cdot \aleph(\sigma).$$

under the identification $\mathbb{Z}[k+l] = \mathbb{Z}$ we have $\epsilon_{X\times Y}[k+l] = \epsilon_{X\times Y}$

by definition of $\widetilde{\cap}$, see page 2574, together with the Cap Product-Kronecker Lemma 116.4 (1)

This concludes the proof of the subclaim.

Subclaim. We have

$$\beth' = (-1)^{kl} \cdot \beth\colon (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_{k+l} \to \mathbb{Z}.$$

We have to show that both sides take the same value on any tensor product of the form $\sigma_i \otimes \sigma_j$ with $i+j = k+l$ and $\sigma_i \in \mathrm{C}_i(X)$ and let $\sigma_j \in \mathrm{C}_j(Y)$. It follows easily from the definition that both sides are zero for $\sigma_i \otimes \sigma_j$ with $(i,j) \neq (k,l)$. So let $\sigma_k \in \mathrm{C}_k(X)$ and let $\sigma_l \in \mathrm{C}_l(Y)$. Then

$$\begin{aligned}
\beth'(\sigma_k \otimes \sigma_l) &\underset{\uparrow}{=} (\epsilon_{X\times Y} \circ \Upsilon)[k+l] \circ \Psi_{k,l} \circ (\alpha \mathbin{\widetilde{\cap}} \sigma_k \otimes \beta \mathbin{\widetilde{\cap}} \sigma_l) \\
&\quad \text{since } \epsilon_{X\times Y}[k+l] \circ \Upsilon[k+l] = (\epsilon_{X\times Y} \circ \Upsilon)[k+l] \\
&\underset{\uparrow}{=} (-1)^{kl} \cdot (\epsilon_X \otimes \epsilon_Y) \circ (\alpha \cap \sigma_k \otimes \beta \cap \sigma_l) && \underset{\uparrow}{=} (-1)^{kl} \cdot \alpha(\sigma_k) \cdot \beta(\sigma_l) \\
&\quad \text{we use } \epsilon_{X\times Y} \circ \Upsilon = \epsilon_X \otimes \epsilon_Y\colon (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_0 \to \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}, && \text{by the Cap Product-Kronecker Lemma 116.4} \\
&\quad \text{definition of } \Psi_{k,l} \text{ in Lemma 117.14 and the definition of } \widetilde{\cap} \text{ on page 2574} \\
&= (-1)^{kl} \cdot ((\alpha \otimes \beta) \circ \Pi_{k,l})(\sigma_k \otimes \sigma_l) && = (-1)^{kl} \cdot \beth(\sigma_k \otimes \sigma_l).
\end{aligned}$$

This concludes the proof of the second subclaim, thus also of the claim and therefore also of the theorem. ∎

117.4. **The cohomology ring of product topological spaces.** In this section we will determine the cohomology ring of the m-dimensional torus $(S^1)^m$ and more generally, under some mild technical hypothesis, we will determine the cohomology ring of a product $X \times Y$ of topological spaces in terms of the cohomology rings of X and Y.
We recall the following definition.

Definition.

(1) A graded ring is a ring R together with a decomposition $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ such that for any $m, n \in \mathbb{N}_0$ the multiplication map restricts to a map $R_m \times R_n \to R_{m+n}$. We say that an element r of R that lies in some R_n is homogeneous and we refer to $\deg(r) := n$ as the degree of r.

(2) A homomorphism $\varphi\colon A \to B$ between graded rings is a ring homomorphism that preserves the gradings, i.e. that satisfies $\varphi(A_n) \subset B_n$ for all $n \in \mathbb{N}_0$.
(3) A graded ring $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ such that for any two homogeneous elements a and b the equality $a \cdot b = (-1)^{\deg(a)\cdot\deg(b)} \cdot b \cdot a$ holds is called a superalgebra.

Examples.

(1) the Cup Product-Commutativity Proposition 114.8 says in particular that for any topological space X the cohomology ring $(\mathrm{H}^*(X;\mathbb{Z}), \cup)$ is a superalgebra. Furthermore the Cup Product-Naturality Lemma 114.10 implies that for any map $f\colon X \to Y$ the induced map $f^*\colon \mathrm{H}^*(Y;\mathbb{Z}) \to \mathrm{H}^*(X;\mathbb{Z})$ is a homomorphism of superalgebras.
(2) The polynomial ring $\mathbb{F}_2[a] = \bigoplus_{i\in\mathbb{N}_0} \mathbb{F}_2 \cdot a^i$ is a superalgebra if we equip the ring with the grading that is determined by setting $\deg(a^i) := i$.
(3) The polynomial ring $\mathbb{Z}[x] = \bigoplus_{i\in\mathbb{N}_0} \mathbb{Z} \cdot x^i$ is a superalgebra if we equip the ring with the grading that is determined by setting $\deg(x^i) := 2i$.
(4) More generally, the polynomial ring $\mathbb{Z}[x_1, \ldots, x_k]$ in k variables $x_1, \ldots, x_k$ can be viewed as a superalgebra by setting $\deg(x_1^{r_1} \cdot \cdots \cdot x_k^{r_k}) = 2(r_1 + \cdots + r_k)$.
(5) The ring $\mathbb{Z}[x]/(x^2) = \mathbb{Z} \oplus \mathbb{Z} \cdot x$ is a superalgebra by viewing elements in $\mathbb{Z}$ as elements of degree 0 and by viewing elements in $\mathbb{Z} \cdot x$ as elements of degree 1. Note that the map
$$\begin{array}{rcl} \mathbb{Z}[x]/(x^2) & \to & \mathrm{H}^*(S^1;\mathbb{Z}) \\ a \cdot 1 + b \cdot x & \mapsto & a \cdot 1_{S^1} + b \cdot [S^1]^* \end{array}$$
is an isomorphism of superalgebras.
(6) We consider the ring
$$\Lambda[x_1, \ldots, x_k] := \mathbb{Z}[x_1, \ldots, x_k]/\text{ideal generated by } x_1^2, \ldots, x_k^2, \{x_i x_j + x_j x_i\}_{i,j=1,\ldots,k}.$$
In this ring we use the symbol "$\wedge$" for multiplication instead of "$\cdot$". For example for any $i, j \in \{1, \ldots, k\}$ we have $x_i \wedge x_i = 0$ and $x_i \wedge x_j = -x_j \wedge x_i$. This ring can be viewed in a unique way as a superalgebra such that $\deg(x_j) = 1$ for $j = 1, \ldots, k$. The ring $\Lambda[x_1, \ldots, x_k]$ is often referred to as the *exterior algebra on generators* $x_1, \ldots, x_k$, see e.g. [**Hat2002**, p. 217] and [**Mac1975**, p. 183].[1437] We leave it as an amusing exercise to verify that $\Lambda[x_1, \ldots, x_k]$ is a free abelian group of rank 2^k where a basis is given by
$$\{x_{n_1} \wedge \cdots \wedge x_{n_l} \,|\, n_1 < \cdots < n_l\}.$$

In Exercise 110.2 we saw that any sequence $\{A_n\}_{n\geq 1}$ of finitely generated abelian groups, with A_1 torsion-free, can appear as the sequence of cohomology groups $\mathrm{H}^*(X;\mathbb{Z})$ of some CW-complex. It is natural to ask whether any superalgebra structure can be realized as

[1437] If we replace the $\mathbb{Z}$-coefficients by $\mathbb{R}$-coefficients we obtain a ring that we denote by $\Lambda_{\mathbb{R}}[x_1, \ldots, x_k]$. This ring is isomorphic to the ring $\wedge^*(\mathbb{R}^k)^* = \bigoplus_{j=0}^{k} \wedge^j (\mathbb{R}^k)^*$ of alternating forms on $\mathbb{R}^k$, where the ring structure on $\wedge^*(\mathbb{R}^k)^*$ is given by the wedge product of alternating forms. More precisely a ring isomorphism is given by
$$\begin{array}{rcl} \Lambda_{\mathbb{R}}[x_1, \ldots, x_k] & \to & \wedge^*(\mathbb{R}^k)^* \\ x_{i_1} \wedge \cdots \wedge x_{i_r} & \to & dy_{i_1} \wedge \cdots \wedge dy_{i_r} \end{array}$$
where $dy_i\colon \mathbb{R}^k \to \mathbb{R}$ is the projection onto the i-th coordinate.

the cohomology ring of a topological space. More precisely, we have the following question, to which we will return later with Proposition 152.6.

Question 117.19. Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ be superalgebra such that ${}_0 \cong \mathbb{Z}$, R_1 is torsion-free and such that all R_n are finitely generated. Does there exist a topological space X such that $(\mathrm{H}^*(X;\mathbb{Z}), \cup)$ is isomorphic to R?

Since there is not much to say right now about this question we continue with the next definition.

Definition. The tensor product of two superalgebras $A = \bigoplus_{n\in\mathbb{N}_0} A_n$ and $B = \bigoplus_{n\in\mathbb{N}_0} B_n$ is defined as the graded abelian group

$$A \otimes B := \bigoplus_{n\in\mathbb{N}_0} \underbrace{\bigoplus_{p+q=n} A_p \otimes B_q}_{=:(A\otimes B)_n}$$

where for $a_k \in A_k, a_l \in A_l, b_m \in B_m$ and $b_n \in B_n$ the multiplication is given by

$$(a_k \otimes b_l) \cdot (a_m \otimes b_n) := (-1)^{lm} \cdot a_k \cdot a_m \otimes b_l \cdot b_n.$$

Lemma 117.20.

(1) The tensor product of two superalgebras is again a superalgebra.

(2) The tensor product of superalgebras is associative, i.e. for any three superalgebras A, B and C the map

$$\begin{array}{rcl} (A \otimes B) \otimes C & \to & A \otimes (B \otimes C) \\ (a \otimes b) \otimes c & \mapsto & a \otimes (b \otimes c) \end{array}$$

that is induced by

is an isomorphism of superalgebras.

It follows immediately from Lemma 117.20 (2) that we can ignore parentheses when we consider the tensor product of several superalgebras.

Proof. Let A, B and C be superalgebras and let $a_k \in A_k, a_l \in A_l, b_m \in B_m$ and $b_n \in B_n$.

(1) We have the following equalities:

$$\begin{aligned} (a_k \otimes b_l)\cdot(a_m \otimes b_n) &= (-1)^{lm} \cdot a_k \cdot a_m \otimes b_l \cdot b_n = (-1)^{lm+km+ln} \cdot a_m \cdot a_k \otimes b_n \cdot b_l \\ &= (-1)^{lm+km+ln+kn} \cdot (a_m \otimes b_n) \cdot (a_k \otimes b_l) \\ &= (-1)^{(m+n)(k+l)} \cdot (a_m \otimes b_n) \cdot (a_k \otimes b_l) \\ &= (-1)^{\deg(a_m\otimes b_n)\cdot\deg(a_k\otimes b_l)} \cdot (a_m \otimes b_n) \cdot (a_k \otimes b_l). \end{aligned}$$

It follows immediately that $A \otimes B$ is again a superalgebra.

(2) It suffices to show that the indicated map is a homomorphism of superalgebras. Thus let $c_r \in \mathrm{C}_r$ and $c_s \in \mathrm{C}_s$. Then

$$\begin{aligned} ((a_k \otimes b_l) \otimes c_r)\cdot((a_m \otimes b_n) \otimes c_s) &= (-1)^{r(m+n)} \cdot ((a_k \otimes b_l)\cdot(a_m \otimes b_n)) \otimes (c_r \cdot c_s) \\ &= (-1)^{r(m+n)+lm} \cdot (a_k \cdot a_m \otimes b_l \cdot b_n \otimes c_r \cdot c_s) \\ &= (-1)^{m(l+r)} \cdot (a_k \cdot a_m \otimes (b_l \otimes c_r) \cdot (b_n \otimes c_s)) \\ &= (a_k \otimes (b_l \otimes c_r)) \cdot (a_m \otimes (b_n \otimes c_s)). \end{aligned}$$

■

Examples.

(1) As above we view the polynomial rings $\mathbb{F}_2[a]$ and $\mathbb{F}_2[b]$ as superalgebras by setting $\deg(a^i) = \deg(b^i) = i$. We consider the map

$$\begin{array}{rcl} \mathbb{F}_2[a] \otimes \mathbb{F}_2[b] & \to & \mathbb{F}_2[a,b] \\ a^r \otimes b^s & \mapsto & a^r \cdot b^s. \end{array}$$

Using the Tensor Lemma 89.3 (1) it is straightforward to show that this map is an isomorphism of abelian groups. One can easily verify that the map is a homomorphism of superalgebras. Summarizing we have convinced ourselves that this map is in fact an isomorphism of superalgebras.

(2) As before we view the polynomial rings $\mathbb{Z}[x_j]$, $j = 1, \ldots, k$ as superalgebras with $\deg(x_j^i) = 2i$. We consider the map

$$\begin{array}{rcl} \mathbb{Z}[x_1] \otimes \cdots \otimes \mathbb{Z}[x_k] & \to & \mathbb{Z}[x_1, \ldots, x_k] \\ x_1^{r_1} \otimes \cdots \otimes x_k^{r_k} & \mapsto & x_1^{r_1} \cdot \cdots \cdot x_k^{r_k}. \end{array}$$

The same argument as in (1) shows that this map is an isomorphism of superalgebras.

(3) As above we view the rings $\mathbb{Z}[x_j]/(x_j^2)$ as superalgebras with $\deg(x_j) = 1$. We leave it as an amusing exercise to verify that the map

$$\begin{array}{rcl} \mathbb{Z}[x_1]/(x_1^2) \otimes \cdots \otimes \mathbb{Z}[x_k]/(x_k^2) & \to & \Lambda[x_1, \ldots, x_k] \\ x_1^{r_1} \otimes \cdots \otimes x_k^{r_k} & \mapsto & x_1^{r_1} \wedge \cdots \wedge x_k^{r_k} \end{array}$$

is an isomorphism of superalgebras.

The following proposition explains our sudden interest in tensor products of superalgebras.

Proposition 117.21. (Cohomology Ring-of-Product Proposition) Let X and Y be two topological spaces. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the natural projections.

(1) The map

$$\begin{array}{rcl} \Phi\colon \mathrm{H}^*(X;\mathbb{Z}) \otimes \mathrm{H}^*(Y;\mathbb{Z}) & \to & \mathrm{H}^*(X \times Y;\mathbb{Z}) \\ \varphi \otimes \psi & \mapsto & p^*\varphi \cup q^*\psi \end{array}$$

is a homomorphism of superalgebras.

(2) If all the homology groups of X are finitely generated and torsion-free, then the homomorphism from (1) is in fact an isomorphism.

Proof. Let X and Y be two topological spaces.

(1) Let $\varphi_k \in \mathrm{H}^k(X;\mathbb{Z})$, $\psi_l \in \mathrm{H}^l(Y;\mathbb{Z})$, $\varphi_m \in \mathrm{H}^m(X;\mathbb{Z})$ and $\psi_n \in \mathrm{H}^n(Y;\mathbb{Z})$. Then

$$\begin{array}{ll} \overbrace{\Phi\big((-1)^{lm} \cdot (\varphi_k \cup \varphi_m) \otimes (\psi_l \cup \psi_n)\big)}^{=\Phi((\varphi_k \otimes \psi_l) \cdot (\varphi_m \otimes \psi_n))} & = (-1)^{lm} \cdot p^*(\varphi_k \cup \varphi_m) \cup q^*(\psi_l \cup \psi_n) \\ = (-1)^{lm} \cdot (p^*(\varphi_k) \cup p^*(\varphi_m) \cup q^*(\psi_l) \cup q^*(\psi_n)) & = \underbrace{p^*(\varphi_k) \cup q^*(\psi_l) \cup p^*(\varphi_m) \cup q^*(\psi_n)}_{=\Phi(\varphi_k \otimes \psi_l) \cup \Phi(\varphi_m \otimes \psi_n))}. \\ \uparrow & \uparrow \\ \text{Cup Product-Commutativity Proposition 114.8} & \text{Cup Product-Naturality Lemma 114.10} \end{array}$$

It follows immediately that Φ is a homomorphism of superalgebras.

(2) Now suppose that all the homology groups of X are finitely generated and torsion-free. This hypothesis allows us to apply the Künneth Theorem 117.16 for cohomology

groups and we obtain the following short exact sequence:

$$0 \to \bigoplus_{k+l=n} \mathrm{H}^k(X;\mathbb{Z}) \otimes \mathrm{H}^l(Y;\mathbb{Z}) \to \mathrm{H}^n(X \times Y;\mathbb{Z}) \to \bigoplus_{k+l=n+1} \mathrm{Tor}(\mathrm{H}^k(X;\mathbb{Z}), \mathrm{H}^l(Y;\mathbb{Z})) \to 0.$$
$$\varphi \otimes \psi \mapsto p^*\varphi \cup q^*\psi$$

Since the homology groups of X are torsion-free we obtain from the Universal Coefficient Theorem 110.12 for Cohomology Groups that the cohomology groups $\mathrm{H}^k(X;\mathbb{Z})$ are also torsion-free. But by the Tor-Properties Lemma 89.15 this implies that all the Tor-terms on the right-hand side of the short exact sequence vanish. ■

The following proposition in particular provides an answer to Question 115.15 (1).

Proposition 117.22. (Cohomology Ring-of-Torus Proposition) Let $n \in \mathbb{N}_0$. We denote by $T = (S^1)^n$ the n-dimensional torus. For $i = 1, \ldots, n$ we denote by $p_i \colon T \to S^1$ the projection onto the i-th factor. Then there exists a unique isomorphism

$$\psi \colon \Lambda[x_1, \ldots, x_n] \to \mathrm{H}^*(T;\mathbb{Z})$$

of superalgebras which has the property that for each $i \in \{1, \ldots, n\}$ we have

$$\psi(x_i) = p_i^*([S^1]^*).$$

Proof. We consider the following isomorphism of superalgebras

$$\Lambda[x_1, \ldots, x_n] \xleftarrow[\uparrow]{\cong} \mathbb{Z}[x_1]/(x_1^2) \otimes \ldots \otimes \mathbb{Z}[x_k]/(x_k^2) \xrightarrow[\uparrow]{\cong} \mathrm{H}^*(S^1;\mathbb{Z}) \otimes \ldots \otimes \mathrm{H}^*(S^1;\mathbb{Z}) \xrightarrow[\uparrow]{\cong} \mathrm{H}^*((S^1)^n;\mathbb{Z}).$$

isomorphism from page 2586 — isomorphism from page 2584 — Cohomology Ring-of-Product Proposition 117.21

It is clear from the definitions of the various maps that the map from left to right sends each x_i to $p_i^*([S^1]^*)$. The composition of the isomorphisms thus gives us the desired isomorphism. The uniqueness follows immediately from the fact that the x_i generate $\Lambda[x_1, \ldots, x_n]$ as a ring. ■

The following corollary gives in particular a negative answer to Question 115.10.

Corollary 117.23. For the n-dimensional torus $T = (S^1)^n$ the Lusternik-Schnirelmann category $\mathrm{cat}(T)$ equals n.

Proof. Given any $n \in \mathbb{N}_0$ we have

$$n \overset{\text{see page 2545}}{\underset{\downarrow}{\geq}} \text{Lusternik-Schnirelmann category } \mathrm{cat}((S^1)^n) \overset{\text{by Proposition 115.14}}{\underset{\downarrow}{\geq}} \text{cup length } \mathrm{cl}((S^1)^n) \underset{\uparrow}{\geq} n.$$

by the Cohomology Ring-of-Torus Proposition 117.22 there exists
somorphism $\psi \colon \Lambda[x_1, \ldots, x_n] \to \mathrm{H}^*((S^1)^n;\mathbb{Z})$ of graded rings,
furthermore by the discussion on page 2584 we know that $x_1 \wedge \cdots \wedge x_n \neq 0$,
since ψ is an isomorphism we deduce that $\psi(x_1) \cup \cdots \cup \psi(x_n) \neq 0$, i.e. $\mathrm{cl}((S^1)^n) \geq n$ ■

We conclude this chapter with the following variation on the Cohomology Ring-of-Product Proposition 117.21.

Proposition 117.24. Let X and Y be two topological spaces. We denote by $p \colon X \times Y \to X$ and $q \colon X \times Y \to Y$ the natural projections. If all the homology groups of X are finitely generated, then the map

$$\Phi \colon \mathrm{H}^*(X;\mathbb{F}_2) \otimes \mathrm{H}^*(Y;\mathbb{F}_2) \to \mathrm{H}^*(X \times Y;\mathbb{F}_2)$$
$$\varphi \otimes \psi \mapsto p^*\varphi \cup q^*\psi$$

is an isomorphism of superalgebras.

In the proof of Proposition 117.24 we will need the following theorem which is a variation on the Künneth Theorem 117.17 for cochain complexes.

Theorem 117.25. Let $\mathcal{C}$ and $\mathcal{C}'$ be chain complexes. If $\mathcal{C}$ is a free chain complex and if all homology groups of $\mathcal{C}$ are finitely generated, then for every $n \in \mathbb{N}_0$ there exists a natural isomorphism

$$\bigoplus_{k+l=n} \mathrm{H}^k(\mathcal{C};\mathbb{F}_2) \otimes \mathrm{H}^l(\mathcal{C}';\mathbb{F}_2) \xrightarrow{\Xi} \mathrm{H}^n(\mathcal{C} \otimes \mathcal{C}';\mathbb{F}_2).$$

The proof of Theorem 117.25 is evidently very similar to the proof of the Künneth Theorem 117.17 for cochain complexes. We refer to [**Mun1984**, Theorem 60.6] for a proof.

Proof of Proposition 117.24. The proof is almost the same as the proof of the Cohomology Ring-of-Product Proposition 117.21. More precisely, we first note that if in the proof of the Künneth Theorem 117.16 for cohomology groups we replace the Künneth Theorem 117.17 for cochain complexes by Theorem 117.25 we obtain that the map

$$\begin{aligned} \bigoplus_{k+l=n} \mathrm{H}^k(X;\mathbb{F}_2) \otimes \mathrm{H}^l(Y;\mathbb{F}_2) &\to \mathrm{H}^n(X \times Y;\mathbb{F}_2) \\ \varphi \otimes \psi &\mapsto p^*\varphi \cup q^*\psi \end{aligned}$$

is an isomorphism. But the same argument as in the proof of the Cohomology Ring-of-Product Proposition 117.21 shows that this map is in fact a homomorphism of superalgebras. ■

Exercises for Chapter 117.

Exercise 117.1. Let X be a topological space. Recall that the Cup Product-Commutativity Proposition 114.8 says that for any $a \in \mathrm{H}^k(X;\mathbb{Z})$ and $b \in \mathrm{H}^l(X;\mathbb{Z})$ we have

$$a \cup b \;=\; (-1)^{kl} \cdot b \cup a.$$

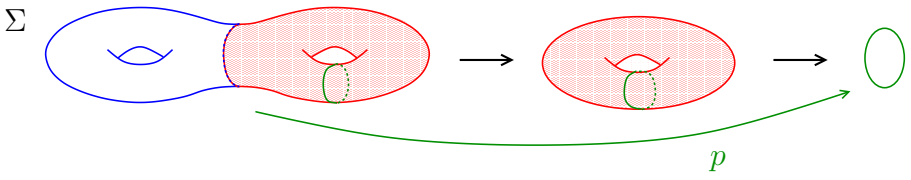
Provide a proof of this statement using the original definition of the cup product from page 2513 and using Lemma 117.7.

Exercise 117.2. Let $g \in \mathbb{N}$ and denote by Σ_g the surface of genus g. Show that the Lusternik-Schnirelmann category of $S^1 \times \Sigma_g$ is greater or equal than three.

Exercise 117.3. Let Σ be the surface of genus 2, let $q\colon \Sigma \times S^1 \to \Sigma$ be the projection and let $p\colon \Sigma \to S^1$ be the map shown in the figure below. Determine the cap product

$$q(p^*([S^1]^*)) \cap [\Sigma \times S^1] \;\in\; \mathrm{H}_2(\Sigma \times S^1;\mathbb{Z})$$

up to sign.



Exercise 117.4. Let $n \in \mathbb{N}_0$. We write $T := (S^1)^n$. Let $\varphi\colon T \to T$ be map. We pick a basis for $\mathrm{H}_1(T) \cong \mathbb{Z}^n$ and we denote by A the $(n \times n)$-matrix that represents the map $\varphi_*\colon \mathrm{H}_1(T) \to \mathrm{H}_1(T)$ with respect to the given basis. Show that

$$\underbrace{\sum_{k=0}^{n} (-1)^k \cdot \mathrm{tr}(\varphi_*\colon \mathrm{H}_k(T) \to \mathrm{H}_k(T))}_{\text{Lefschetz number } \Lambda(\varphi)} \;=\; \det(\mathrm{id} - A).$$

Hint. Use the cohomological reformulation of the Lefschetz number given in Exercise 110.12 and use the Cohomology Ring-of-Torus Proposition 117.22.
Remark. It follows from this calculation together with the Lefschetz Fixed Point Theorem 97.9 that unless 1 is an eigenvalue of A the map φ has a fixed point.

Exercise 117.5. Let $n \in \mathbb{N}$ and let W be a compact orientable $(n+1)$-dimensional topological manifold such that ∂W is homeomorphic to the n-torus $(S^1)^n$. Show that the inclusion induced map $\mathrm{H}_1(\partial W; \mathbb{Z}) \to \mathrm{H}_1(W; \mathbb{Z})$ is not an injection.
Hint. Consider the cup product on $(S^1)^n$ and consider the interplay of the fundamental classes of ∂W and W.

Exercise 117.6. Given $n \in \mathbb{N}$ we denote by $p\colon S^n \times S^n \to S^n$ and $q\colon S^n \times S^n \to S^n$ the two obvious projections. Let $\varphi \in \mathrm{H}^n(S^n; \mathbb{Z})$ be a generator. We know, for example by the Cohomology Ring-of-Product Proposition 117.21, that $p^*(\varphi)$ and $q^*(\varphi)$ form a basis for $\mathrm{H}^n(S^n \times S^n; \mathbb{Z})$. Given a map $f\colon S^n \times S^n \to S^n \times S^n$ we denote by $A(f)$ the matrix representing the automorphism f^* of $\mathrm{H}^n(S^n \times S^n; \mathbb{Z})$ with respect to the basis $p^*(\varphi)$ and $q^*(\varphi)$.

(a) Let $B \in \mathrm{SL}(2, \mathbb{Z})$. Show that there exists a homeomorphism $f\colon S^1 \times S^1 \to S^1 \times S^1$ with $A(f) = B$.
Hint. Use the Action-on-Torus Homology Lemma 84.7.

(b) Now let $f\colon S^n \times S^n \to S^n \times S^n$ be a homeomorphism. Show that if n is even, then $A(f)$ is given by one of the following eight matrices

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}.$$

118. The relationship between de Rham cohomology and singular cohomology

In this chapter we want to find the connection between the de Rham cohomology groups from Chapter **??** and the singular cohomology groups that we defined in Chapter 108. To avoid a repetitive use of suffixes we will use the following convention.

Convention. In this chapter by a "manifold" we always mean a smooth manifold *without boundary*, unless we say explicitly otherwise.

118.1. **Smooth cohomology theories.** To a smooth manifold M we can associate the singular cohomology groups $\mathrm{H}^i(M;\mathbb{R})$ and also the de Rham cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(M)$. Rereading the chapters on these cohomology groups one notices many common features, for example they both satisfy a Mayer–Vietoris type theorem. Now we want to put both cohomologies in a common framework.

Definition. A smooth cohomology theory is a contravariant functor[14381439]

$$\begin{array}{rcl} \mathcal{H}^*\colon \text{category of smooth manifolds} & \to & \text{category of } \mathbb{N}_0\text{-graded real vector spaces} \\ M & \mapsto & \mathcal{H}^*(M) = \bigoplus_{n\in\mathbb{N}_0} \mathcal{H}^n(M) \\ (f\colon M\to N) & \mapsto & \{f^* = \mathcal{H}^n(f)\colon \mathcal{H}^n(N)\to\mathcal{H}^n(M)\}_{n\in\mathbb{N}_0} \end{array}$$

that satisfies the following three axioms:

(1) (Smooth homotopy invariance) Let $f,g\colon M\to N$ be smooth maps between smooth manifolds. If f and g are smoothly homotopic, then for all $n\in\mathbb{N}_0$ we have

$$\mathcal{H}^n(f) = \mathcal{H}^n(g)\colon \mathcal{H}^n(N) \;\to\; \mathcal{H}^n(M).$$

(2) (Mayer–Vietoris sequence) If M is a smooth manifold and U and V are open subsets such that $M = U\cup V$, then there exists a natural long exact sequence of the form

$$\ldots \to \mathcal{H}^k(M) \xrightarrow{j_U^*\oplus -j_V^*} \mathcal{H}^k(U)\oplus\mathcal{H}^k(V) \xrightarrow{i_U^*+i_V^*} \mathcal{H}^k(U\cap V) \xrightarrow{\delta} \mathcal{H}^{k+1}(M) \to \ldots$$

where $j_U\colon U\to M$, $j_V\colon V\to M$, $i_U\colon U\cap V\to U$ and $i_V\colon U\cap V\to V$ denote the inclusion maps.

(3) (Limit property) Let M be a smooth manifold and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets that satisfy the following three conditions:
   (a) for each $i\in\mathbb{N}$ the closure $\overline{U}_i$ of U_i is compact,
   (b) for each $i\in\mathbb{N}$ we have $U_i\subset U_{i+1}$, and
   (c) $M = \bigcup_{i\in\mathbb{N}} U_i$.
   Then for every $n\in\mathbb{N}_0$ there exists a natural short exact sequence

$$0 \;\to\; \varprojlim{}^1\, \mathcal{H}^{n-1}(U_i) \;\xrightarrow{\tau}\; \mathcal{H}^n(M) \;\to\; \varprojlim \mathcal{H}^n(U_i) \;\to\; 0$$

   where the right-hand map is induced by the maps $\mathcal{H}^n(M)\to\mathcal{H}^n(U_i)$ which in turn are induced by the inclusions $U_i\to M$.

[1438] Given a smooth map $f\colon M\to N$ the contravariant functor provides for each $n\in\mathbb{N}_0$ a morphism $\mathcal{H}^n(f)\colon\mathcal{H}^n(N)\to\mathcal{H}^n(M)$. As usual, when there is no danger of confusion we just write f^* instead of $\mathcal{H}^n(f)$.

[1439] We refer to page 1935 for the definition of the category of $\mathbb{N}_0$-graded real vector spaces.

This time it is easy to give examples for our new definition.

Proposition 118.1.
(1) Singular cohomology with real coefficients is a smooth cohomology theory.
(2) De Rham cohomology is a smooth cohomology theory.

Proof.

(1) It follows from the Homotopic Maps-H*-Lemma 108.13, the Mayer–Vietoris Theorem 109.15, together with the Excisive Triad-Proposition 109.12 and the Cohomology-via-Exhaustions Proposition 111.12 (1), that singular cohomology with real coefficients is a smooth cohomology theory.
(2) It follows from the Homotopic Maps-De Rham Cohomology Proposition **??**, Theorem **??** and the De Rham Cohomology-via-Exhaustions Proposition 118.2, which we will prove in the next section, that de Rham cohomology is a smooth cohomology theory. ■

To show that de Rham cohomology is indeed a smooth cohomology theory we will need to prove the following proposition, which is very similar in nature to the Cohomology-via-Exhaustions Proposition 111.12 for singular cohomology.

Proposition 118.2. (De Rham Cohomology-via-Exhaustions Proposition) Let M be a smooth manifold and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets of M that satisfy the following three conditions:
(1) for each $i \in \mathbb{N}$ the closure $\overline{U}_i$ of U_i is compact,
(2) for each $i \in \mathbb{N}$ we have $U_i \subset U_{i+1}$, and
(3) $M = \bigcup_{i\in\mathbb{N}} U_i$.
Then for every $n \in \mathbb{N}_0$ there exists a natural short exact sequence

$$0 \;\to\; \varprojlim{}^1 \mathrm{H}^{n-1}_{\mathrm{dR}}(U_i) \;\xrightarrow{\tau}\; \mathrm{H}^n_{\mathrm{dR}}(M) \;\to\; \varprojlim \mathrm{H}^n_{\mathrm{dR}}(U_i) \;\to\; 0$$

where the right-hand map is induced by the obvious maps $\mathrm{H}^n_{\mathrm{dR}}(M) \to \mathrm{H}^n_{\mathrm{dR}}(U_i)$.

Proof. For each $k \in \mathbb{N}_0$ we consider the inverse system

$$\Omega^k(U_1) \;\leftarrow\; \Omega^k(U_2) \;\leftarrow\; \Omega^k(U_3) \;\leftarrow\; \dots$$

where all the maps are induced by the inclusion, put differently, they are all restriction maps.

Claim. The inverse system satisfies the Mittag-Leffler condition from page 2470.

Proof. Given any $j \geq i$ we denote by $\rho_{ji}\colon \Omega^k(U_j) \to \Omega^k(U_i)$ the restriction map. Now let $i \in \mathbb{N}$. We have to show that there exists an $n \geq i$ such that for all $r \geq n$ we have

$$\operatorname{im}\big(\rho_{ri}\colon \Omega^k(U_r) \to \Omega^k(U_i)\big) \;=\; \operatorname{im}\big(\rho_{ni}\colon \Omega^k(U_n) \to \Omega^k(U_i)\big).$$

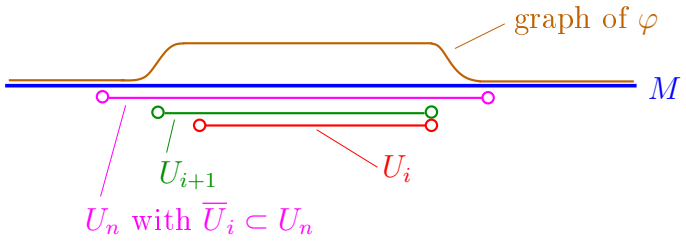
Since $\overline{U}_i$ is compact, since the U_j are nested and since $\overline{U}_i$ is contained in the union of all U_j there exists an n such that $\overline{U}_i \subset U_n$. We refer to the figure below for an illustration. We claim that n has the desired property. So let $r \geq n$. We have to show that

$$\operatorname{im}(\rho_{ri}\colon \Omega^k(U_r) \to \Omega^k(U_i)) \;=\; \operatorname{im}(\rho_{ni}\colon \Omega^k(U_n) \to \Omega^k(U_i)).$$

Clearly we have the inclusion "$\subset$". It remains to prove the reverse inclusion. So suppose $\eta \in \Omega^k(U_i)$ is the restriction of some $\mu \in \Omega^k(U_n)$. Note that U_n and $M \setminus \overline{U}_i$ form an open cover for M. It follows immediately from the Manifold-Partitions of Unity Existence Theorem 21.1 that there exists a smooth function $\varphi\colon M \to \mathbb{R}$ with $\varphi|_{\overline{U}_i} \equiv 1$ and such that $\operatorname{supp}(\varphi) \subset U_n$. Let $r \geq n$. We see that

$$\eta = \mu|_{U_i} \underset{\substack{\uparrow \\ \text{since } \varphi \equiv 1 \text{ on } U_i}}{=} (\varphi \cdot \mu)|_{U_i} = \overbrace{\epsilon(\varphi \cdot \mu, U_r)}^{\text{smooth by Lemma ??}}|_{U_i}.$$

We have thus shown that η is also the restriction of some smooth form on U_r. ⊞



By the claim we can apply the Direct Limit-Cohomology Proposition 111.10 to the inverse system $(\{\Omega^*(U_i)\}_{i\in\mathbb{N}}, \{\rho_{ij}\}_{i\geq j})$ of cochain complexes over $\mathbb{N}$ and we obtain the short exact sequence

$$0 \to \underbrace{\varprojlim_{i\in\mathbb{N}}{}^{1} \mathrm{H}^{n-1}(\Omega^*(U_i))}_{=\mathrm{H}^{n-1}_{\mathrm{dR}}(U_i)} \to \underbrace{\mathrm{H}^n\Big(\overbrace{\varprojlim_{i\in\mathbb{N}} \Omega^*(U_i)}^{=\Omega^*(M), \text{ see page 1280}} \Big)}_{=\mathrm{H}^n_{\mathrm{dR}}(M)} \to \underbrace{\varprojlim_{i\in\mathbb{N}} \mathrm{H}^n(\Omega^*(U_i))}_{=\mathrm{H}^n_{\mathrm{dR}}(U_i)} \to 0.$$

We have thus obtained the desired short exact sequence. ■

118.2. **Isomorphisms between smooth cohomology theories.** One might now be optimistic and hope that smooth cohomology theories are necessarily isomorphic. As it stands this thought though is too optimistic. For example other smooth cohomology theories are given by $\mathcal{H}^n(M) = 0$ for all $n \in \mathbb{N}_0$ or by $\mathcal{H}^n(M) = \mathrm{H}^{n+5}(M;\mathbb{R})$ or by $\mathcal{H}^n(M) = \mathrm{H}^n(M;\mathbb{R}) \oplus \mathrm{H}^n(M;\mathbb{R})$. This shows that if we want to prove that smooth cohomology theories are isomorphic we have to add some extra hypotheses, for example we could demand that the 0-th cohomology groups of a point are isomorphic.

Notation. In the following $\star$ denotes the topological space consisting of a single point. Evidently $\star$ is a 0-dimensional smooth manifold.

On page 203 we introduced the notion of a natural transformation between two contravariant functors. For the reader's convenience recall the definition since it will play an essential role in the following discussion.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two contravariant functors. A natural transformation between the contravariant functors F and G assigns to each object $X \in \mathrm{Ob}(\mathcal{C})$ a morphism $\Phi_X\colon F(X) \to G(X)$ in $\mathcal{D}$ such that for each morphism

$f\colon X \to Y$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{array}{ccc} F(Y) & \xrightarrow{F^*(f)} & F(X) \\ {\scriptstyle \Phi_Y}\downarrow & & \downarrow{\scriptstyle \Phi_X} \\ G(Y) & \xrightarrow{G^*(f)} & G(X). \end{array}$$

If all the maps Φ_X are isomorphisms we also say it is a natural isomorphism.

Example. Given $n \in \mathbb{N}_0$ we consider the two contravariant functors $X \mapsto \mathrm{H}^n(X;\mathbb{Z})$ and $X \mapsto \mathrm{Hom}(\mathrm{H}_n(X),\mathbb{Z})$ from the category of topological spaces to the category of abelian groups. It follows from the discussion on page 2444 that the evaluation homomorphisms $\mathrm{ev}\colon \mathrm{H}^n(X;\mathbb{Z}) \to \mathrm{Hom}(\mathrm{H}_n(X),\mathbb{Z})$ define a natural transformation between the two contravariant functors, i.e. given any map $f\colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^n(Y;\mathbb{Z}) & \xrightarrow{f^*} & \mathrm{H}^n(X;\mathbb{Z}) \\ {\scriptstyle \mathrm{ev}}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}} \\ \mathrm{Hom}(\mathrm{H}_n(Y),\mathbb{Z}) & \xrightarrow{f^*} & \mathrm{Hom}(\mathrm{H}_n(X),\mathbb{Z}). \end{array}$$

With this notation and this definition we can now formulate the following theorem.

Theorem 118.3. (Smooth Cohomology Theory Isomorphism Theorem) Let $\mathcal{H}^*$ and $\mathcal{K}^*$ be smooth cohomology theories and let $t\colon \mathcal{H}^* \to \mathcal{K}^*$ be a natural transformation. Suppose t has the following three properties:

(i) the natural transformation t is compatible with the connecting homomorphism in the Mayer-Vietoris sequence, i.e. with the obvious notation we always have a commutative diagram

$$\begin{array}{ccccccccc} \ldots \longrightarrow \mathcal{H}^n(M) & \xrightarrow{j_U^*\oplus -j_V^*} & \mathcal{H}^n(U)\oplus\mathcal{H}^n(V) & \xrightarrow{i_U^*+i_V^*} & \mathcal{H}^n(U\cap V) & \xrightarrow{\delta} & \mathcal{H}^{n+1}(M) \longrightarrow \ldots \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t\oplus t} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} \\ \ldots \longrightarrow \mathcal{K}^n(M) & \xrightarrow{j_U^*\oplus -j_V^*} & \mathcal{K}^n(U)\oplus\mathcal{K}^n(V) & \xrightarrow{i_U^*+i_V^*} & \mathcal{K}^n(U\cap V) & \xrightarrow{\delta} & \mathcal{K}^{n+1}(M) \longrightarrow \ldots \end{array}$$

(ii) the natural transformation t is compatible with the map τ appearing in the limit property, i.e. with the obvious notation we always have a commutative diagram[1440]

$$\begin{array}{ccccccccc} 0 \longrightarrow & \varprojlim{}^1\,\mathcal{H}^{n-1}(U_i) & \xrightarrow{\tau} & \mathcal{H}^n(M) & \longrightarrow & \varprojlim \mathcal{H}^n(U_i) & \longrightarrow 0 \\ & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & \\ 0 \longrightarrow & \varprojlim{}^1\,\mathcal{K}^{n-1}(U_i) & \xrightarrow{\tau} & \mathcal{K}^n(M) & \longrightarrow & \varprojlim \mathcal{K}^n(U_i) & \longrightarrow 0, \end{array}$$

(iii) for all $n \in \mathbb{N}_0$ the map $t^n(\star)\colon \mathcal{H}^n(\star) \to \mathcal{K}^n(\star)$ is an isomorphism.

Then for every smooth manifold M and every $n \in \mathbb{N}_0$ the map

$$t\colon \mathcal{H}^n(M) \to \mathcal{K}^n(M)$$

is an isomorphism.

[1440] Note that the left and right vertical maps are given by the remark on page 2472.

The proof of the Smooth Cohomology Theory Isomorphism Theorem 118.3 rests on the following useful little lemma, which incidentally also solves Exercise 1.30.

Lemma 118.4. (Rational Open Balls Lemma)

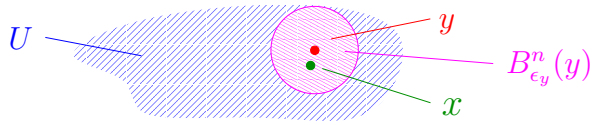
(1) Every open subset of $\mathbb{R}^n$ is the union of open balls of the form $B^n_\epsilon(x)$ where ϵ is rational and $x \in \mathbb{Q}^n$.
(2) Every open subset of $\mathbb{R}^n$ is the union of countably many open balls.

Proof of Lemma 118.4.

(1) Let U be an open subset of $\mathbb{R}^n$. For $x \in U$ we set $\mu_x := \sup\{\mu \in [0,1] \mid B^n_\mu(x) \subset U\}$. The hypothesis that U is open implies that $\mu_x > 0$. We pick a rational number $\epsilon_x \in (\frac{1}{2}\mu_x, \mu_x)$. Note that $B^n_{\epsilon_x}(x) \subset U$. It remains to prove the following claim.

Claim. The subset U is the union of all $B^n_{\epsilon_x}(x)$ with $x \in U \cap \mathbb{Q}^n$.

Proof. Let $y \in U$. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ we can pick $x \in U \cap \mathbb{Q}^n$ with $\|x-y\| < \frac{1}{4}\epsilon_y$. Note that this is equivalent to saying that $y \in B^n_{\frac{1}{4}\epsilon_y}(x)$. Thus it remains to prove that $B^n_{\frac{1}{4}\epsilon_y}(x) \subset B^n_{\epsilon_x}(x)$. Recall that we pointed out above that $B^n_{\epsilon_y}(y) \subset U$. Since $\|x-y\| < \frac{1}{4}\epsilon_y$ we have by the triangle inequality that $B^n_{\frac{3}{4}\epsilon_y}(x) \subset B^n_{\epsilon_y}(y)$. Thus $B^n_{\frac{3}{4}\epsilon_y}(x) \subset U$. By definition of μ_x this implies that $\frac{3}{4}\epsilon_y \leq \mu_x$. Hence we have $\frac{1}{4}\epsilon_y \leq \frac{3}{8}\epsilon_y \leq \frac{1}{2}\mu_x \leq \epsilon_x$. But then we also have $B^n_{\frac{1}{4}\epsilon_y}(x) \subset B^n_{\epsilon_x}(x)$, as desired. ⊞



(2) This statement follows immediately from (1). ■

Proof of Theorem 118.3. Let $\mathcal{H}^*$ and $\mathcal{K}^*$ be smooth cohomology theories, which means they both satisfy

(α) smooth homotopy invariance,
(β) the Mayer-Vietoris sequence,
(γ) the limit property.

Furthermore let $t\colon \mathcal{H}^* \to \mathcal{K}^*$ be a natural transformation that has the above properties (i), (ii) and (iii). We need to prove the following statement:

Statement. Let $m \in \mathbb{N}_0$. For every m-dimensional smooth manifold M and every $n \in \mathbb{N}_0$ the map

$$t^n(M)\colon \mathcal{H}^n(M) \to \mathcal{K}^n(M)$$

is an isomorphism.

We start out with the following two observations:

(A) Suppose M is an m-dimensional smooth manifold and U and V are open subsets with $M = U \cup V$. If all t^n are isomorphisms for U, V and $U \cap V$, then it follows from (β), (i) and the Five Lemma 74.10 that all the t^n are therefore isomorphisms for M.
(B) Let M be an m-dimensional smooth manifold and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets that satisfy the following three conditions:
 (a) for each $i \in \mathbb{N}$ the closure $\overline{U}_i$ of U_i is compact,

(b) for each $i \in \mathbb{N}$ we have $U_i \subset U_{i+1}$, and
(c) $M = \bigcup_{i\in\mathbb{N}} U_i$.

If all the t^n are isomorphisms for all U_i, then it follows from (γ) and the Five Lemma 74.10 that all the t^n are therefore isomorphisms for M.

After these preparations we will prove the desired statement in four easy steps:

(1) Let M be an open convex subset of $\mathbb{R}^m$. We pick an inclusion $f\colon \star \to M$. As in the proof of the Poincaré Lemma **??** we see that f is a smooth homotopy equivalence. Given $n \in \mathbb{N}_0$ we consider the diagram

$$\begin{array}{ccc} \mathcal{H}^n(M) & \xrightarrow{f^*} & \mathcal{H}^n(\star) \\ {\scriptstyle t}\downarrow & & \cong\downarrow{\scriptstyle t} \\ \mathcal{K}^n(M) & \xrightarrow{f^*} & \mathcal{K}^n(\star). \end{array}$$

The diagram commutes since t is a natural transformation. It follows easily from (α) that the horizontal maps are isomorphisms. By (iii) the right vertical map is also an isomorphism. It follows that the left vertical map is, as desired, an isomorphism

(2) We claim that t is an isomorphism for every subset M of $\mathbb{R}^m$ that is the union of k open convex subsets of $\mathbb{R}^m$. We prove this by induction on the number k of open convex subsets:
   (a) If $k = 1$, then M itself is an open convex subset of $\mathbb{R}^m$, hence the statement follows from (1).
   (b) Suppose the statement holds whenever M is the union of $k-1$ open convex subsets. Now let M be the union of k open convex subsets $U_1, \ldots, U_k$. We write $V = U_1 \cup \cdots \cup U_{k-1}$. By induction the statement holds for V and also for U_k. Furthermore since the intersection of two convex open subsets is again convex and open we see that $V \cap U_k = (U_1 \cap U_k) \cup \cdots \cup (U_{k-1} \cap U_k)$ is the union of $k-1$ open convex subsets, i.e. by induction the statement holds for $V \cap U_k$. But then it follows from (A) that the statement also holds for U.

(3) Let M be an open subset of $\mathbb{R}^m$. By the Rational Open Balls Lemma 118.4 we can write M as the union of countably many open balls $B_i, i \in \mathbb{N}$. For $i \in \mathbb{N}$ we define $U_i := \bigcup_{j\le i} B_j$. By (2) the statement holds for each U_i, but then it follows from (B) that the statement holds for M.

(4) Finally let M be any m-dimensional smooth manifold. We say a subset W of M is *small* if it is open, if $\overline{W}$ is compact and if W is contained in the domain of some chart for M. We make the following observations:
   (a) Since $\mathcal{H}$ and $\mathcal{K}$ are diffeomorphism invariants we obtain from (2) that the statement holds for every small subset of W (viewed as a smooth manifold in its own right).
   (b) The same induction argument as in (2) shows that the statement holds for finite unions of small subsets (here the key observation is that the intersection of two small subsets is again small).[1441]

[1441]Here we implicitly use the following statement: if A and B are subsets of M such that $\overline{A}$ and $\overline{B}$ are compact, then $\overline{A \cap B}$ and $\overline{A \cup B}$ are also compact. We will prove this statement in Exercise 118.2.

(c) Finally it follows easily from the fact that the smooth manifold M is second-countable, the existence of charts around every point and Lemma 1.30 that there exist small sets $W_i, i \in \mathbb{N}$ which cover M. As in (3) we consider $U_i := \bigcup_{j \leq i} W_j$. We had just seen in (b) that the statement holds for all the U_i. But then it follows from (B) that the statement holds for M. ■

118.3. **Smooth singular cohomology.** One might think that it follows immediately from Proposition 118.1 and the Smooth Cohomology Theory Isomorphism Theorem 118.3 (together with the calculations of the cohomology groups of a point in the Single Point-Cohomology Lemma 108.12 and on page **??**) that the singular real valued cohomology and the de Rham cohomology groups of smooth manifolds are always isomorphic.
But recall that to apply the Smooth Cohomology Theory Isomorphism Theorem 118.3 we need to find a natural transformation

$$t\colon \text{de Rham cohomology} \quad \to \quad \text{real valued singular cohomology}$$

that has the property that

$$\mathrm{H}^n_{\mathrm{dR}}(\star) \quad \xrightarrow{\cong} \quad \mathrm{H}^n(\star;\mathbb{R}) \qquad \text{for all } n \in \mathbb{N}_0.$$

Here is an idea: in Section **??** we had seen that given a compact oriented n-dimensional smooth manifold W and a smooth differential n-form ω on W there exists an integral

$$\int_W \omega \ \in \ \mathbb{R}$$

that has many good properties. So given some smooth manifold M (with no restrictions on the dimension) we could try to define a map

$$\begin{array}{rcl} t\colon \Omega^n(M) & \to & \mathrm{Hom}(\mathrm{C}_n(M);\mathbb{R}) \\ \omega & \mapsto & \left(\begin{array}{rcl} \mathrm{C}_n(M) & \to & \mathbb{R} \\ \sum_{i=1}^k a_i \cdot (f_i\colon \Delta^n \to M) & \mapsto & \sum_{i=1}^k a_i \cdot \int_{\Delta^n} f_i^*\omega \end{array} \right). \end{array}$$

Unfortunately, there are several problems with this "definition":

(1) The first question that arises is the following: what does it mean to integrate a form over Δ^n? After all, it is not clear in what sense Δ^n is a smooth manifold. With some luck that should not be too much of an issue, since Δ^n is clearly pretty close to being a smooth manifold.
(2) Let us suppose that we have resolved (1). Now let $f\colon \Delta^n \to M$ be a singular n-chain. If the map f is not smooth, then $f^*\omega$ just does not make any sense.[1442]
(3) After fixing the previous issues, why should t be a cochain map?

Arguably the most serious issue is (2). The only way around it is to only work with smooth singular simplices. Let us recall the following definitions.

[1442]Recall that if $f\colon N \to M$ is a smooth map between smooth manifolds and ω is a smooth differential n-form on M, then $f^*\omega$ is defined to be the differential n-form that for $v_1, \ldots, v_n \in \mathrm{T}_P N$ has the value

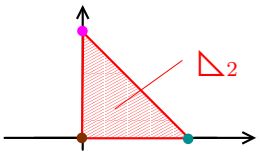
$$(f^*\omega)_P(v_1, \ldots, v_n) \ = \ \omega_{f(P)}(\mathrm{D}f_P(v_1), \ldots, \mathrm{D}f_P(v_n)).$$

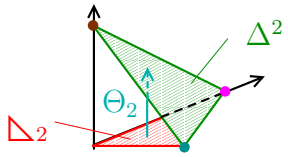
But if f is not differentiable, then there is no differential $\mathrm{D}f_P\colon \mathrm{T}_P N \to \mathrm{T}_{f(P)} M$.

Definition. Let $n \in \mathbb{N}_0$.

(1) As in the Standard Simplex Homeomorphism Lemma 72.1 we consider the planar simplex
$$\triangle_n := \left\{(t_1, \ldots, t_n) \in \mathbb{R}^n_{\geq 0} \;\middle|\; \sum_{i=1}^n t_i \in [0,1]\right\}.$$

(2) In the Standard Simplex Homeomorphism Lemma 72.1 we saw that the map[1443]
$$\begin{aligned} \Theta_n \colon \triangle_n &\to \Delta^n \\ (t_1, \ldots, t_n) &\mapsto \left(t_1, \ldots, t_n, 1 - \sum_{i=1}^n t_i\right) \end{aligned}$$
is a homeomorphism.





Now we turn to smooth manifolds. Continuing with the above definition we can now introduce the notion of a smooth singular simplex.

Definition. Let M be a smooth manifold.

(3) Recall that by the definition on page 549 a map $f \colon \triangle_n \to M$ is smooth if given any $a \in \triangle_n$ there exists an open neighborhood U in $\mathbb{R}^n$ of a and a smooth map $\widetilde{f} \colon U \to M$ which coincides with f on A. We define the differential of f at $a \in \triangle_n$ to be the differential of $\widetilde{f}$.[1444]

(4) We say a singular n-simplex $\sigma \colon \Delta^n \to M$ is smooth if the map $\sigma \circ \Theta_n \colon \triangle_n \to M$ is smooth. We illustrate the definition in the figure below.

(5) We write
$$\mathrm{C}_n^{\text{smooth}}(M) := \begin{array}{c}\text{the free abelian group generated by the} \\ \text{set of smooth singular } n\text{-simplices in } M.\end{array}$$
This is a subgroup of $\mathrm{C}_n(M)$ and it is straightforward to verify that the boundary map $\partial_n \colon \mathrm{C}_n(M) \to \mathrm{C}_{n-1}(M)$ restricts to a map $\partial_n \colon \mathrm{C}_n^{\text{smooth}}(M) \to \mathrm{C}_{n-1}^{\text{smooth}}(M)$.

(6) Given an abelian group G we write
$$\mathrm{C}^n_{\text{smooth}}(M; G) := \mathrm{Hom}\left(\mathrm{C}_n^{\text{smooth}}(M), G\right)$$
and we define $\delta_n \colon \mathrm{C}^n_{\text{smooth}}(M; G) \to \mathrm{C}^{n+1}_{\text{smooth}}(M; G)$ as the map $\delta_n := \partial_{n+1}^*$.

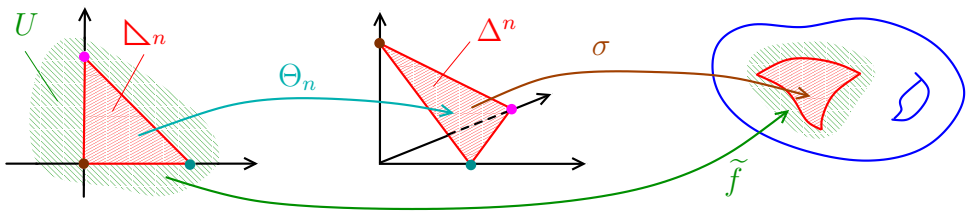
(7) We define the n-th smooth singular cohomology group of a smooth manifold M with G-coefficients as
$$\mathrm{H}^n_{\text{smooth}}(M; G) := \frac{\ker\left(\delta_n \colon \mathrm{C}^n_{\text{smooth}}(M; G) \to \mathrm{C}^{n+1}_{\text{smooth}}(M; G)\right)}{\mathrm{im}\left(\delta_{n-1} \colon \mathrm{C}^{n-1}_{\text{smooth}}(M; G) \to \mathrm{C}^n_{\text{smooth}}(M; G)\right)}.$$

Remark.

(1) Let $f \colon M \to N$ be a *smooth* map between two smooth manifolds. Evidently the induced map $f_* \colon \mathrm{C}^n(M) \to \mathrm{C}^n(N)$ restricts to a map $f_* \colon \mathrm{C}_n^{\text{smooth}}(M) \to \mathrm{C}_n^{\text{smooth}}(N)$. In particular we get an induced map $f^* \colon \mathrm{C}^n_{\text{smooth}}(N, G) \to \mathrm{C}^n_{\text{smooth}}(M, G)$. Thus we

[1443] Put differently, Θ_n is the inverse of the obvious projecton map $\Delta^n \to \triangle_n$.

[1444] It is straightforward to verify that this definition does not depend on the choice of U and $\widetilde{f}$.



obtain for each $n \in \mathbb{N}_0$ an induced map

$$f^* \colon \mathrm{H}^n_{\mathrm{smooth}}(N; G) \ \to \ \mathrm{H}^n_{\mathrm{smooth}}(M; G).$$

(2) Every map $\Delta^n \to \star$ is smooth, so there is no difference between ordinary singular cohomology and smooth singular cohomology for smooth manifold $\star$.

Proposition 118.5. Smooth singular cohomology with real coefficients is a smooth cohomology theory.

Sketch of proof. We need to verify that smooth singular cohomology with real coefficients satisfies the properties of a smooth cohomology theory.

(0) It is clear that smooth singular cohomology with real coefficients is a contravariant functor from the category of smooth manifolds to the category of $\mathbb{N}_0$-graded real vector spaces.

(1) Suppose $f, g \colon M \to N$ are two smooth maps between two smooth manifolds and suppose we are given a *smooth* homotopy F between f and g. Some thought shows that the chain homotopy

$$P_n \colon \mathrm{C}_n(M) \ \to \ \mathrm{C}_{n+1}(N), \quad n \in \mathbb{N}_0$$

from the proof of the Homotopic Maps-and-Homology Proposition 73.6 restricts to a chain homotopy

$$\mathrm{C}^{\mathrm{smooth}}_n(M) \ \to \ \mathrm{C}^{\mathrm{smooth}}_{n+1}(N), \quad n \in \mathbb{N}_0.$$

Now we can once again appeal to the Dual Cochain Map Lemma 108.8 to conclude that the induced maps on smooth singular cohomology are the same.

(2) Let M be a smooth manifold and let U and V be open subsets with $M = U \cup V$. The key to proving the existence of the Mayer–Vietoris sequence for ordinary singular cohomology was to show that the inclusion map

$$\mathrm{C}^{\{U,V\}}_*(M) \ \to \ \mathrm{C}_*(M)$$

is a chain homotopy equivalence. This in turn we showed using the Comfortable Covering Proposition 74.23. In the proof of the Comfortable Covering Proposition 74.23 we introduced the subdivision maps

$$u_n \colon \mathrm{C}_n(M) \ \to \ \mathrm{C}_n(M), \quad n \in \mathbb{N}_0.$$

It follows immediately from the definitions that these maps restrict to maps

$$u_n \colon \mathrm{C}^{\mathrm{smooth}}_n(M) \ \to \ \mathrm{C}^{\mathrm{smooth}}_n(M), \quad n \in \mathbb{N}_0.$$

Using this observation the proof of the existence of a Mayer–Vietoris sequence for smooth singular cohomology is otherwise identical to the proof for ordinary singular cohomology.

(3) The limit property is proved the same way as the Cohomology-via-Exhaustions Proposition 111.12 (1). We only have to replace the fact, proved in the Exhaustion-Chain Complex Lemma 79.5 that $\mathrm{C}_*(M) = \varinjlim \mathrm{C}_*(U_i)$ by the observation that the same argument as in the Exhaustion-Chain Complex Lemma 79.5 shows that $\mathrm{C}_*^{\mathrm{smooth}}(M) = \varinjlim \mathrm{C}_*^{\mathrm{smooth}}(U_i)$. ■

Now we can easily show that smooth singular cohomology with real coefficients and ordinary singular cohomology with real coefficients are naturally isomorphic. More precisely, we have the following proposition.

Proposition 118.6. There exists a natural isomorphism from smooth singular cohomology with real coefficients to ordinary singular cohomology with real coefficients. In particular for any smooth manifold M and any $n \in \mathbb{N}_0$ we have a natural isomorphism

$$\mathrm{H}^n(M;\mathbb{R}) \xrightarrow{\cong} \mathrm{H}^n_{\mathrm{smooth}}(M;\mathbb{R}).$$

Proof. For any smooth manifold M we have the natural inclusion

$$\mathrm{C}_*^{\mathrm{smooth}}(M) \to \mathrm{C}_*(M)$$

which evidently respects the boundary maps. These maps thus give rise to natural maps

$$\mathrm{C}^*(M;\mathbb{R}) \to \mathrm{C}^*_{\mathrm{smooth}}(M;\mathbb{R})$$

which respect the coboundary maps. Thus we obtain a natural map

$$t(M)\colon \mathrm{H}^*(M;\mathbb{R}) \to \mathrm{H}^*_{\mathrm{smooth}}(M;\mathbb{R}).$$

As we pointed out above, for $\star$ there is no difference between smooth and ordinary singular cohomology, i.e.

$$t(\star)\colon \mathrm{H}^*(\star;\mathbb{R}) \to \mathrm{H}^*_{\mathrm{smooth}}(\star;\mathbb{R})$$

is an isomorphism. Since the natural transformation is induced by the inclusion map it follows easily that the natural transformation is compatible with the connecting homomorphism in the Mayer-Vietoris sequence and that it is compatible with the map τ appearing in the limit property. The proposition now follows immediately from the Smooth Cohomology Theory Isomorphism Theorem 118.3. ■

Now we also want to relate smooth singular cohomology to de Rham cohomology. The key for doing so is the following definition and the subsequent proposition.

Definition.

(1) Given $x \in \triangle_n$ we define $\widetilde{\mathrm{T}}_x\triangle_n := \mathbb{R}^n$. Using this definition it makes sense to talk of smooth n-forms on $\triangle_n$.

(2) Given a smooth n-form α on $\triangle_n$ we define[1445]

$$\int_{\triangle_n} \alpha := \int_{x\in\triangle_n} \alpha_x(e_1,\ldots,e_n)\,d\lambda.$$

(3) Let M be a smooth manifold and let $f\colon \triangle_n \to M$ be a smooth map. Since $\triangle$ is regular closed, i.e. since it equals the closure of its interior, we obtain from the argument of Lemma 19.2 that it makes sense to consider the map $\mathrm{D}f_x\colon \mathbb{R}^n \to \mathrm{T}_{f(x)}M$ for all $x \in \triangle_n$.

(4) Given a smooth manifold M, given a smooth differential n-form ω on M and a smooth singular n-simplex $\sigma\colon \Delta^n \to M$ we use (1), (2) and (3) to define[1446]

$$\int_\sigma \omega \;:=\; (-1)^n \cdot \int_{\triangle_n} (\sigma \circ \Theta_n)^*(\omega).$$

We extend this definition linearly to define the integral of the smooth differential n-form ω over a smooth singular n-chain $\sum_{i=1}^{k} a_i \circ \sigma_i$.

Proposition 118.7. (Stokes-for-Simplices Proposition) Let M be a smooth manifold, let ω be a smooth differential $(n-1)$-form on M and let $\sigma\colon \Delta^n \to M$ be a smooth singular n-simplex. Then[1447]

$$\int_\sigma d\omega \;=\; \int_{\partial\sigma} \omega.$$

All the hard work for the proof of the Stokes-for-Simplices Proposition 118.7 is done in the following lemma.

Lemma 118.8. For $i = 1, \ldots, n$ we consider the map

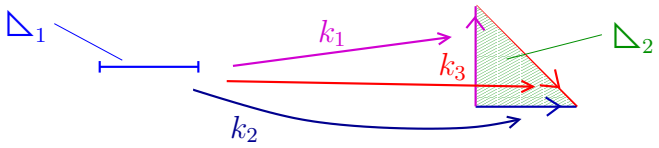
$$\begin{array}{rcl} k_i\colon \triangle_{n-1} & \to & \triangle_n \\ (x_1, \ldots, x_{n-1}) & \to & (x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1}). \end{array}$$

Furthermore we consider the map

$$\begin{array}{rcl} k_{n+1}\colon \triangle_{n-1} & \to & \triangle_n \\ (x_1, \ldots, x_{n-1}) & \to & (x_1, \ldots, x_{n-1}, 1 - x_1 - \cdots - x_{n-1}) \end{array}$$

Given any smooth $(n-1)$-form α on $\triangle_n$ we have

$$\int_{\triangle_n} d\alpha \;=\; \sum_{i=1}^{n} (-1)^i \cdot \int_{\triangle_{n-1}} k_i^*\alpha + (-1)^{n+1} \cdot \int_{\triangle_{n-1}} k_{n+1}^*\alpha.$$



Proof. Let α be a smooth $(n-1)$-form on $\triangle_n$ By the Alternating Forms-Basis Proposition **??** we can write

$$\alpha \;=\; \sum_{j=1}^{n} f_j \cdot dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

where $f_1, \ldots, f_n\colon \triangle_n \to \mathbb{R}$ are smooth functions. First let us calculate the differential of the n-form $d\alpha$.

[1445] The Lebesgue integral is defined since $\triangle_n$ is a compact subset of $\mathbb{R}^n$ and since the integrand is continuous.

[1446] Recall that on page 1581 we had equipped $\mathring{\Delta}^n$ with an orientation. The factor $(-1)^n$ is motivated by the fact that the restriction of the map $\Theta_n\colon \triangle_n \to \Delta^n$ to the interiors is orientation-preserving if and only if n is even.

[1447] Note that on the left we integrate a smooth differential n-form over a smooth singular n-simplex and that on the right we integrate a smooth $(n-1)$-form over a smooth singular $(n-1)$-chain.

Claim 1. For each $x \in \Delta_n$ we have

$$(d\alpha)_x(e_1, \ldots, e_n) \;=\; \sum_{i=1}^{n} (-1)^{i-1} \cdot \frac{\partial f_i}{\partial x_i}(x).$$

Proof. We performed the calculation already in the proof of Stokes' Theorem **??**. But there is no harm done in doing so again. So let us do so:

$$d\alpha \;\overset{\text{see definition on page ??}}{\downarrow}\; \sum_{j=1}^{n} df_j \cdot dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \;\overset{\text{1-Form-via-}dx^i\text{-Lemma ??}}{\downarrow}\; \sum_{j=1}^{n} \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j} \cdot dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

$$= \sum_{j=1}^{n} (-1)^i \cdot \frac{\partial f_j}{\partial x_j} \cdot dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^n$$

$$\uparrow \text{ follows from } dx^i \wedge dx^i = 0 \text{ and } dx^i \wedge dx^j = -dx^j \wedge dx^i$$

Thus for each $x \in \Delta_n$ we have

$$(d\alpha)_x(e_1, \ldots, e_n) \;=\; \sum_{i=1}^{n} (-1)^{i-1} \cdot \frac{\partial f_i}{\partial x_i}(x) \cdot \underbrace{(dx^1 \wedge \cdots \wedge dx^n)(e_1, \ldots, e_n)}_{=1} \;=\; \sum_{i=1}^{n} (-1)^{i-1} \cdot \frac{\partial f_i}{\partial x_i}(x).$$

⊞

We adopt the following notation: given $i \in \{1, \ldots, n\}$ and given $x \in \mathbb{R}^n$ we denote by $\varphi_i(x) \in \mathbb{R}^{i-1}$ the first $i-1$ coordinates and we denote by $\psi_i(x) \in \mathbb{R}^{n-i}$ the last $n-i$ coordinates. Furthermore we denote by $|x|$ the sum of the coordinates of x. Now we see that

$$\int_{\Delta_n} d\alpha \;=\; \int_{x \in \Delta_n} (d\alpha)_x(e_1, \ldots, e_n) \;=\; \sum_{i=1}^{n} (-1)^{i-1} \cdot \int_{x \in \Delta_n} \frac{\partial f_i}{\partial x_i}(x)\, d\lambda \;\overset{\text{Fubini Theorem ??}}{\downarrow}\; =$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \cdot \int_{s \in \Delta_{n-1}} \underbrace{\int_{x_i=0}^{x_i=1-|s|} \frac{\partial f_i}{\partial x_i}(\varphi_i(s), x_i, \psi_i(s))\, dx_i}_{\substack{\text{can be calculated using the} \\ \text{Fundamental Theorem of Calculus}}} d\lambda$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \cdot \int_{s \in \Delta_{n-1}} (f_i(\varphi_i(s), 1-|s|, \psi_i(s)) - f_i(\varphi_i(s), 0, \psi_i(s))\, d\lambda$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \cdot \int_{s \in \Delta_{n-1}} \underbrace{f_i(\varphi_i(s), 0, \psi_i(s))}_{=k_i(s)}\, d\lambda + \sum_{i=1}^{n} (-1)^{i-1} \cdot \int_{s \in \Delta_{n-1}} (f_i(\varphi_i(s), 1-|s|, \psi_i(s))\, d\lambda.$$

It remains to recognize the terms. We will do so in the following two claims.

Claim 2. Given any $i \in \{1, \ldots, n\}$ we have

$$\int_{s \in \Delta_{n-1}} f_i(k_i(s))\, d\lambda \;=\; \int_{\Delta_{n-1}} k_i^* \alpha.$$

Proof. We calculate that

$$\begin{aligned}\int_{\Delta_{n-1}} k_i^*\alpha &= \int_{s\in\Delta_{n-1}} (k_i^*\alpha)_s(e_1,\dots,e_{n-1})\,d\lambda = \int_{s\in\Delta_{n-1}} \alpha_{k_i(s)}(k_{i*}(e_1),\dots,k_{i*}(e_{n-1}))\,d\lambda\\ &= \int_{s\in\Delta_{n-1}} \sum_{j=1}^{n} f_j(k_i(s))\cdot \underbrace{(dx^1\wedge\cdots\wedge\widehat{dx^j}\wedge\cdots\wedge dx^n)(e_1,\dots,\widehat{e_i},\dots,e_{n-1})}_{=\delta_{ij}\text{, where }\delta_{ij}\text{ denotes the Kronecker delta}}\,d\lambda\\ &= \int_{s\in\Delta_{n-1}} f_i(k_i(s))\,d\lambda.\end{aligned}$$

⊞

Claim 3. $\displaystyle\sum_{i=1}^{n}\int_{s\in\Delta_{n-1}} (-1)^{i-1}\cdot f_i(\varphi_i(s),1-|s|,\psi_i(s))\,d\lambda = (-1)^{n+1}\cdot\int_{\Delta_{n-1}} k_{n+1}^*\alpha.$

Proof. We calculate that

$$\begin{aligned}\int_{\Delta_{n-1}} k_{n+1}^*\alpha &= \int_{s\in\Delta_{n-1}} (k_{n+1}^*\alpha)_s(e_1,\dots,e_{n-1})\,d\lambda = \int_{s\in\Delta_{n-1}} \alpha_{k_{n+1}(s)}(k_{n+1,*}(e_1),\dots,k_{n+1,*}(e_{n-1}))\,d\lambda\\ &= \int_{s\in\Delta_{n-1}} \sum_{j=1}^{n} f_j(k_{n+1}(s))\cdot\underbrace{(dx^1\wedge\cdots\wedge\widehat{dx^j}\wedge\cdots\wedge dx^n)(e_1-e_n,\dots,e_{n-1}-e_n)}_{=(-1)^{n-j}}\,d\lambda\\ &= \sum_{j=1}^{n}\int_{s\in\Delta_{n-1}} (-1)^{n-j}\cdot f_j(k_{n+1}(s))\,d\lambda = \sum_{j=1}^{n}\int_{s\in\Delta_{n-1}} (-1)^{n-j}\cdot f_j\underbrace{(\varphi_n(s),1-|s|,\psi_n(s))}_{=k_{n+1}(s)}\,d\lambda\\ &= \sum_{j=1}^{n}\int_{s\in\Delta_{n-1}} (-1)^{n-j}\cdot f_j(\varphi_j(s),1-|s|,\psi_j(s))\,d\lambda.\end{aligned}$$

↑ follows from the Lebesgue Transformation Theorem ?? applied to the diffeomorphism that swaps the j-th and the n-th coordinate

This calculation of course implies the claim. ■

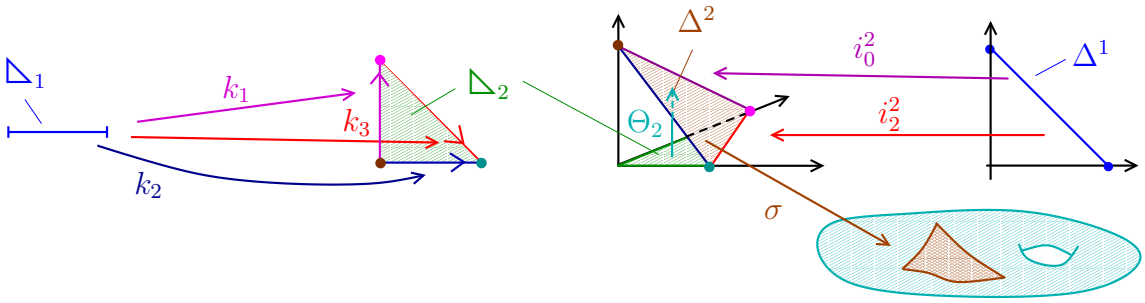
Proof of the Stokes-for-Simplices Proposition 118.7. Let M be a smooth manifold, let ω be a smooth differential $(n-1)$-form on M and let $\sigma\colon\Delta^n\to M$ be a smooth singular n-simplex. For $j\in\{0,\dots,n\}$ we consider as usual the map

$$\begin{aligned} i_j^n\colon\Delta^{n-1} &\to \Delta^n\\ (t_0,\dots,t_{n-1}) &\mapsto (t_0,\dots,t_{j-1},0,t_j,\dots,t_{n-1}).\end{aligned}$$

The proposition follows from the following calculation:

$$\begin{aligned}\int_\sigma d\omega &\overset{\text{by the definition on page 2600}}{=} (-1)^n\cdot\int_{\Delta_n}(\sigma\circ\Theta_n)^*(d\omega) \overset{\text{by the Differentials-on-Smooth Manifolds Proposition ??}}{=} (-1)^n\cdot\int_{\Delta_n} d((\sigma\circ\Theta_n)^*(\omega)) \overset{\text{Lemma 118.8 applied to }\alpha=(\sigma\circ\Theta_n)^*(d\omega)}{=}\\ &= \sum_{j=1}^{n}(-1)^{n+i}\cdot\int_{\Delta_{n-1}}\underbrace{k_j^*((\sigma\circ\Theta_n)^*(\omega))}_{(\sigma\circ\Theta_n\circ k_j)^*(\omega)} + (-1)^{2n+1}\cdot\int_{\Delta_{n-1}}\underbrace{k_{n+1}^*((\sigma\circ\Theta_n)^*(\omega))}_{(\sigma\circ\Theta_n\circ k_j)^*(\omega)}\\ &= (-1)^{n-1}\cdot\sum_{l=0}^{n}(-1)^l\cdot\int_{\Delta_{n-1}}(\sigma\circ i_l^n\circ\Theta_{n-1})^*(\omega) = (-1)^{n-1}\cdot\sum_{l=0}^{n}(-1)^l\cdot\int_{\sigma\circ i_l^n}\omega = \int_{\partial\sigma}\omega.\end{aligned}$$

↑ since $\Theta_n\circ k_j = i_{j+1}^{n-1}\circ\Theta_{n-1}\colon\Delta_{n-1}\to\Delta^n$; ↑ by the definition on page 2600; ↑ by definition of $\partial\sigma$ ■



118.4. **The isomorphism between singular and de Rham cohomology.** After the preparations in the previous sections we can now easily prove the theorem that we had been aiming for.

Theorem 118.9. (De Rham Theorem) There exists a natural isomorphism $u = u_M$ from de Rham cohomology to singular cohomology with real coefficients. In particular for any smooth manifold M and any $k \in \mathbb{N}_0$ we have a natural isomorphism

$$u_M \colon \mathrm{H}^k_{\mathrm{dR}}(M) \xrightarrow{\cong} \mathrm{H}^k(M;\mathbb{R})$$

which has the property that for any closed differential k-form ω and any smooth singular k-chain σ that is a cycle we have

$$\big\langle \underbrace{u_M([\omega])}_{\in \mathrm{H}^k(M;\mathbb{R})}, \underbrace{[\sigma]}_{\in \mathrm{H}_k(M;\mathbb{Z})} \big\rangle_M = \int_\sigma \omega.$$

(Here we stress, that according to the convention from page 2590 it is understood that the smooth manifolds have empty boundary.)

Remark. A rather different proof for Theorem 118.9 is given in [**War1983**, p. 205] and [**Ram2005**, Chapte 4.6]. More precisely, in [**War1983, Ram2005**] the equivalence of de Rham cohomology and real valued singular cohomology is proved by relating both to another type of cohomology theory, namely sheaf cohomology. We also refer to [**BoT1982**, Theorems 8.9 and 15.8], [**Dup1978**, Theorem 1.15], [**Lau2011**, Section 6.7.6] and [**Ful1995**, Section 15C] for other proofs.

Proof. We denote by s the natural isomorphism from real-valued singular cohomology to real-valued smooth singular cohomology that we had constructed in the proof of Proposition 118.6.

Now we construct a natural isomorphism t from de Rham cohomology to real-valued smooth singular cohomology. For any smooth manifold M and any $k \in \mathbb{N}_0$ we have, by the above discussion, a map

$$\begin{array}{rcl} \Omega^k(M) & \to & \mathrm{C}^k_{\mathrm{smooth}}(M;\mathbb{R}) \\ \omega & \mapsto & \begin{pmatrix} \mathrm{C}^{\mathrm{smooth}}_k(M) & \to & \mathbb{R} \\ \sigma & \mapsto & \int_\sigma \omega \end{pmatrix} \end{array}$$

Note that it follows basically immediately from the definitions that this map is natural in M. Next note that it is an immediate consequence of the Stokes-for-Simplices Proposition 118.7 that for given M all these maps together form a cochain map $\Omega^*(M) \to \mathrm{C}^*_{\mathrm{smooth}}(M;\mathbb{R})$.

Summarizing we see that we have constructed a natural map

$$t(M)\colon \mathrm{H}^*_{\mathrm{dR}}(M) \;\to\; \mathrm{H}^*_{\mathrm{smooth}}(M;\mathbb{R}).$$

Furthermore, in the special case of $M = \star$ the map

$$t(\star)\colon \mathrm{H}^*_{\mathrm{dR}}(\star) \;\to\; \mathrm{H}^*_{\mathrm{smooth}}(\star;\mathbb{R})$$

is easily seen to be an isomorphism. Once one has unraveled all the definitions it becomes apparent that this natural transformation is compatible with the connecting homomorphism in the Mayer-Vietoris sequence and that it is compatible with the map τ appearing in the limit property. Now we obtain from the Smooth Cohomology Theory Isomorphism Theorem 118.3 that t is a natural isomorphism from de Rham cohomology to real-valued smooth singular cohomology.

Finally we set $u := s^{-1} \circ t$. By the above u is a natural isomorphism from de Rham cohomology to singular cohomology with real coefficients. It follows easily from the definitions of s and t that for any closed differential k-form ω and any smooth singular k-chain σ that is a cycle we have

$$\langle \underbrace{u([\omega])}_{\in \mathrm{H}^k(M;\mathbb{R})}, \underbrace{[\sigma]}_{\in \mathrm{H}_k(M;\mathbb{Z})} \rangle_M \;=\; \int_\sigma \omega. \qquad \blacksquare$$

The following proposition summarizes a few useful properties of the natural isomorphism $u_M\colon \mathrm{H}^k_{\mathrm{dR}}(M) \to \mathrm{H}^k(M;\mathbb{R})$ from the above de Rham Theorem 118.9.

Proposition 118.10.

(1) Let M be a closed oriented connected n-dimensional smooth manifold. Furthermore let $\omega \in \Omega^n(M)$ be a smooth differential n-form.
   (a) We have $\langle u_M([\omega]), [M]\rangle_M \;=\; \int_M \omega$.
   (b) If $\int_M \omega = 1$, then $u_M([\omega]) = [M]^* \in \mathrm{H}^n(M;\mathbb{R})$ where $[M]^*$ denotes the dual fundamental class that we introduced on page 2451.

(2) Let $f\colon M \to N$ be a smooth map between closed oriented connected n-dimensional smooth manifolds and let ω be a smooth differential n-form on N with $\int_N \omega = 1$. Then

$$\int_M f^*\omega \;=\; \underbrace{\deg(f\colon M \to N)}_{\text{as defined on page 2222 or page 2379}}.$$

Proof.

(1) (a) Let M be a closed oriented connected n-dimensional smooth manifold and let $\omega \in \Omega^n(M)$ be a smooth differential n-form. As we explained in Algorithm 100.2, there exist smooth singular simplices $\varphi_1, \ldots, \varphi_k\colon \Delta^n \to M$ with the following properties:

- We have $\bigcup_{i=1}^{k} \varphi_i(\Delta^n) = M$.
- For any $i, j \in \{1, \ldots, k\}$ we have $\varphi_i(\Delta^n) \cap \varphi_j(\Delta^n) = \varphi_i(\partial\Delta^n) \cap \varphi_j(\partial\Delta^n)$.
- If for each $\in \{1, \ldots, k\}$ we set $\epsilon_i = +1$ if φ_i is orientation-preserving and $\epsilon_i = -1$ otherwise, then the singular chain $\sigma := \epsilon_1 \cdot \varphi_1 + \cdots + \epsilon_k \cdot \varphi_k$ is a singular cycle that represents the fundamental class $[M]$.

As above we denote by $\Theta_n\colon \triangle_n \to \Delta^n$ the inverse of the obvious projecton map. We calculate that

$$\int_M \omega \overset{\text{follows fairly easily from (i)– (ii) and our orientation conventions}}{=} \sum_{i=1}^{k}(-1)^n \cdot \epsilon_i \cdot \int_{\triangle_n} (\varphi_i \circ \Theta_n)^*(\omega) \overset{\text{see definition on page 2600}}{=} \int_\sigma \omega \overset{\text{by Theorem 118.9}}{=} \langle u_M([\omega]), [M]\rangle_M.$$

(b) This statement follows immediately from (a) and the definition of the dual fundamental class.

(2) Let $f\colon M \to N$ be a smooth map between closed oriented connected n-dimensional smooth manifolds and let ω be a smooth differential n-form on N with $\int_N \omega = 1$.

$$\begin{array}{lllll} \int_M f^*\omega & \overset{\text{by (1a)}}{=} \langle u_M(f^*([\omega])), [M]\rangle_M & \overset{\text{since } u \text{ are } \textit{natural} \text{ isomorphisms}}{=} \langle f^*(u_N([\omega])), [M]\rangle_M & \overset{\text{Lemma 109.6 (3)}}{=} \langle u_N([\omega]), f_*([M])\rangle_N & \\ & \overset{\text{by definition of } \deg(f)}{=} \deg(f) \cdot \langle u_N([\omega]), [N]\rangle_N & \overset{\text{by (1a)}}{=} \deg(f) \cdot \int_N \omega & \overset{\text{by choice of } \omega}{=} \deg(f). & \end{array}$$

■

Theorem 118.9 allows us to transport information from de Rham cohomology to real-valued singular cohomology and vice-versa. For example the many computational techniques that we had developed, e.g. cellular cohomology, for singular cohomology can now be used to calculate isomorphism types of de Rham cohomology groups. In particular we can easily prove the following corollary.

Corollary 118.11. If M is a closed, oriented connected, non-empty n-dimensional smooth manifold, then the map

$$\begin{array}{rcl} \mathrm{H}^n_{\mathrm{dR}}(M) & \to & \mathbb{R} \\ [\omega] & \mapsto & \int_M \omega \end{array}$$

is an isomorphism.

Proof. By the Top Dimension-de Rham-Integration Proposition **??** the map is well-defined and it is an epimorphism. On the other hand, by the Top-Cohomology Proposition 110.14 and Theorem 118.9 we know that $\mathrm{H}^n_{\mathrm{dR}}(M) \cong \mathbb{R}$, which implies that the map is an isomorphism. ■

118.5. **Realizing cohomology classes by maps to S^1.** We start out this section with the following definition.

Definition. Let G be a group, let Y be a topological space and let $y_0 \in Y$ be a point with an identification $\pi_1(Y, y_0) = G$. Given another topological space X and $x_0 \in X$ we say that a homomorphism $\varphi\colon \pi_1(X, x_0) \to G = \pi_1(Y, y_0)$ is realized by a map $g\colon X \to Y$ if $g(x_0) = y_0$ and if $g_* = \varphi$. If there is no danger of confusion we will drop the base points from the notation.

In the following we will mostly discuss the case $G = \mathbb{Z}$ and $Y = S^1$. Before we can continue we introduce the following conventions.

Convention.

(1) We use the diffeomorphism $\mathbb{R}/\mathbb{Z} \to S^1$ given by $[t] \mapsto \exp(2\pi i t)$ to make the identification $\mathbb{R}/\mathbb{Z} = S^1$.
(2) By Proposition 48.17 we have an isomorphism $\mathbb{Z} \to \pi_1(S^1, 1)$ given by sending 1 to the equivalence class of the loop $t \mapsto \exp(2\pi i t)$. We use this isomorphism to make the identification $\pi_1(\mathbb{R}/\mathbb{Z}, 0) = \pi_1(S^1, 1) = \mathbb{Z}$.
(3) Let X be a path-connected topological space and let $x_0 \in X$. As shown in the First Cohomology Proposition 110.17 we can use the Hurewicz homomorphism and the evaluation homomorphism to make the identifications $\mathrm{H}^1(X;\mathbb{Z}) = \mathrm{Hom}(\mathrm{H}_1(X);\mathbb{Z}) = \mathrm{Hom}(\pi_1(X, x_0), \mathbb{Z})$.

Now we can formulate the following question.

Question 118.12. Let X be a path-connected topological space that is "reasonable", e.g. that is a smooth manifold or a CW-complex. Let $x_0 \in X$ be a base point. Can every homomorphism $\varphi\colon \pi_1(X, x_0) \to \mathbb{Z} = \pi_1(S^1, 1)$ be realized by a map $(X, x_0) \to (S^1, 1)$?

The following proposition gives an affirmative answer if X is a smooth manifold.

Proposition 118.13. (Maps-to-S^1 Proposition) Let M be a connected smooth manifold (here for once we allow M to have boundary) and let P be a base point.

(1) Every class $\phi \in \mathrm{H}^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M, P), \mathbb{Z})$ can be realized by a smooth map $g\colon M \to S^1$ with $g(P) = 1$.
(2) We suppose that $\partial M = \varnothing$. If $f, g\colon M \to S^1$ are two smooth maps with $g(P) = 1$, such that $f_* = g_*\colon \pi_1(M, P) \to \pi_1(S^1, 1)$, then f and g are homotopic.
(3) Let $\phi \in \mathrm{H}^1(M;\mathbb{Z})$ and let $P \in \partial M$. If $f\colon \partial M \to S^1$ is a smooth map that realizes $\phi|_{\partial M}$, then there exists a smooth map $g\colon M \to S^1$ with $g|_{\partial M} = f$ that realizes ϕ.

Remark.

(1) Sometimes the following reformulation of the Maps-to-S^1 Proposition 118.13 is quite useful: Let M be a connected smooth manifold and let P be a base point. Given any $\phi \in \mathrm{H}^1(M;\mathbb{Z})$ there exists a smooth map $g\colon M \to S^1$ and a generator $\psi \in \mathrm{H}^1(S^1;\mathbb{Z})$ with $\phi = g^*(\psi)$.
(2) the Maps-to-S^1 Proposition 118.13 (2) can be viewed as a partial converse to the Homotopic Maps-De Rham Cohomology Proposition **??**.
(3) Later on, in the Maps-to-K(π, n)-Proposition 163.15 we will prove an analogue of the Maps-to-S^1 Proposition 118.13 for CW-complexes. This will allows in Proposition 163.20 to give a new proof of the Maps-to-S^1 Proposition 118.13.

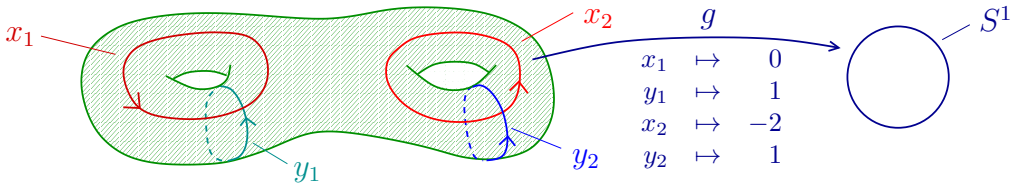
Examples.

(1) We consider the torus $T = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$ and we make the usual identification $\mathrm{H}_1(T;\mathbb{Z}) = \mathbb{Z}^2$. We consider the cohomology classes in $\mathrm{H}^1(T;\mathbb{Z}) = \mathrm{Hom}(\mathrm{H}_1(T), \mathbb{Z})$ that are given by

$$\begin{array}{rcl} \phi\colon \mathrm{H}_1(T;\mathbb{Z}) = \mathbb{Z}^2 & \to & \mathbb{Z} \\ (m, n) & \mapsto & m \end{array} \quad \text{and} \quad \begin{array}{rcl} \psi\colon \mathrm{H}_1(T;\mathbb{Z}) = \mathbb{Z}^2 & \to & \mathbb{Z} \\ (m, n) & \mapsto & n. \end{array}$$

These two cohomology classes are evidently realized by the projection $S^1 \times S^1 \to S^1$ onto the first, respectively the second factor. In Exercise 118.3 the goal will be to give an example of an explicit map $g\colon T \to S^1$ that realizes the homomorphism $\varphi\colon \mathrm{H}_1(T;\mathbb{Z}) = \mathbb{Z}^2 \to \mathbb{Z}$ given by $\varphi(m, n) = 6m + 4n$.

(2) In the figure below we consider the surface Σ of genus 2. As we saw on page 1781, the oriented curves x_1, y_1, x_2, y_2 represent a basis for $\mathrm{H}_1(\Sigma)$. We consider some random epimorphism $\phi\colon \mathrm{H}_1(\Sigma) \to \mathbb{Z}$, e.g. we could consider the epimorphism $\phi\colon \mathrm{H}_1(\Sigma) \to \mathbb{Z}$ that is given by $\phi(x_1) = 0$, $\phi(y_1) = 1$, $\phi(x_2) = -2$ and $\phi(y_2) = 1$. By the Maps-to-S^1 Proposition 118.13 we know that ϕ can be realized by some map $g\colon \Sigma \to S^1$. But can one "visualize" such a map? We will return to this question on page 2840.



We break the proof of the Maps-to-S^1 Proposition 118.13 into three parts.

Proof of the Maps-to-S^1 Proposition 118.13 (1). Let M be a connected smooth manifold. We first assume that M has no boundary. We start out with the following definition. Let ω be a smooth differential 1-form on M and let $\gamma\colon [a,b] \to M$ be a piecewise smooth path. We pick $a = t_0 < t_1 < \cdots < t_n = b$ such that the restriction of γ to each subinterval $[t_i, t_{i+1}]$ is smooth. We define

$$\int\limits_{[a,b]} \gamma^*\omega \;=\; \sum_{i=0}^{n-1} \int\limits_{[t_i,t_{i+1}]} \gamma^*\omega.$$

It follows immediately from the Form Integration Proposition **??** (6) that this definition does not depend on the choice of the subdivision.

After this preparation we start with the actual proof of the proposition. We fix a base point $P \in M$. Furthermore let $\phi \in \mathrm{H}^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M,P),\mathbb{Z})$. We denote by u the natural isomorphism from de Rham cohomology to singular cohomology with real coefficients from Theorem 118.9. We pick a smooth closed differential 1-form ω on M with $u([\omega]) = \phi$.

Claim. Let $\delta\colon [0,1] \to M$ be a piecewise smooth loop in P. Then

$$\int\limits_{[0,1]} \delta^*\omega \;=\; \phi\Big(\underbrace{[\delta]}_{\in \pi_1(M,P)}\Big) \;\in\; \mathbb{R}.$$

Proof. Let $\delta\colon [0,1] \to M$ be a piecewise smooth loop in P. One can show easily that after possibly performing an orientation-preserving reparametrization we can write $\delta = \delta_1 * \cdots * \delta_k$ as a product of smooth paths $\delta_i\colon [0,1] \to M$, $i = 1, \ldots, k$. As in the definition of the Hurewicz homomorphism we use the map $t \mapsto (1-t, t)$ to make the identification $[0,1] = \Delta^1$. Using this identification we view each δ_i also as a smooth 1-simplex. In particular we have $\delta_1 + \cdots + \delta_k \in \mathrm{C}_1^{\mathrm{smooth}}(M)$. Then

$$\int\limits_{[0,1]} \delta^*\omega \;\underset{\text{by the definitions and by the Form Integration Proposition ?? (1) and (6)}}{=}\; \int\limits_{\delta_1+\cdots+\delta_k} \omega \;\underset{\text{by Theorem 118.9 since } u([\omega]) = \phi}{=}\; \langle \phi, [\delta_1 + \cdots + \delta_k] \rangle \;\underset{\text{via the identification } \mathrm{H}^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M,P),\mathbb{Z})}{=}\; \phi([\delta]).$$

⊞

We define

$$\begin{array}{rcl} g\colon M & \to & S^1 = \mathbb{R}/\mathbb{Z} \\ Q & \mapsto & \left[\int\limits_{[0,1]} \gamma^*\omega\right] \quad \text{where } \gamma \text{ is a smooth path}^{1448} \text{ from } P \text{ to } Q. \end{array}$$

We claim that the map g has the required properties. It remains to prove the following four statements:

(1) The map g is well-defined.
(2) We have $g(P) = [0] \in \mathbb{R}/\mathbb{Z}$.
(3) The map g is smooth.
(4) We have $g_* = \phi\colon \pi_1(M,P) \to \pi_1(S^1,1) = \mathbb{Z}$.

Now we turn to the proofs of the statements.

(1) Let Q in M. Suppose we are given two piecewise smooth paths β and γ from the base point P to Q. (See the left of the figure below for an illustration.) Then

$$\int\limits_{[0,1]} \beta^*\omega - \int\limits_{[0,1]} \gamma^*\omega \quad \underset{\text{follows from the definitions and the Form Integration Proposition ?? (1) and (6)}}{=} \quad \int\limits_{[0,1]} (\beta * \overline{\gamma})^*\omega \quad \underset{\text{by the above claim, since } \beta * \overline{\gamma} \text{ is a piecewise smooth loop}}{=} \quad \phi([\beta * \overline{\gamma}]) \ \in \ \mathbb{Z}.$$

This shows that g is indeed well-defined.

(2) We determine $g(P)$ by taking the constant path from P to P. It follows immediately from the definition of g that $g(P) = [0] \in \mathbb{R}/\mathbb{Z}$.

(3) Let $Q \in M$. We need to show that g is smooth in an open neighborhood of Q. We pick a path γ from P to Q. We pick a chart $\Phi\colon U \to B_r(0)$ around Q with $\Phi(Q) = 0$. Given a point $R \in U$ we denote by δ_R the piecewise smooth path that is given by the concatenation of the following two paths:
   (a) the path γ from P to Q,
   (b) the "radial path" from Q to R that corresponds, under the diffeomorphism Φ, to the radial path from 0 to $\Phi(R) \in B_r(0)$.

   (See the right of the figure below for an illustration.) Using these paths δ_R it is now relatively straightforward to show that g is smooth in U. We leave the verification of the details to the reader.

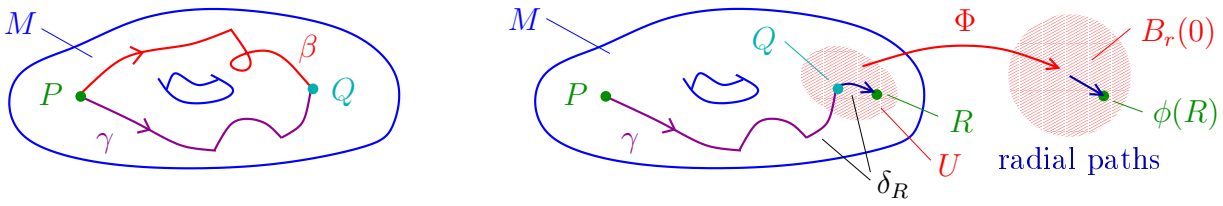
(4) Suppose we are given an element in $\pi_1(M,P)$. By the Smooth Path-Connectivity Proposition 19.29 (1) we can represent this element by a smooth loop $\delta\colon [0,1] \to M$. We have the following equality:

$$\begin{array}{rl} & \overset{\text{by the Fundamental Group-via-Actions Theorem 48.16}}{\downarrow} \\ g_*([\delta]) & = \text{endpoint of the lift of } g\circ\delta\colon [0,1] \to \mathbb{R} \text{ to the universal covering } \mathbb{R} \text{ of } \mathbb{R}/\mathbb{Z} \\ & = \text{endpoint of the lift of the path } t \mapsto \left[\int\limits_{[0,t]} \delta^*\omega\right] \text{ to the universal covering } \mathbb{R} \\ & = \text{endpoint of the path } t \mapsto \int\limits_{[0,t]} \delta^*\omega \ = \ \text{the point } \int\limits_{[0,1]} \delta^*\omega \ = \ \phi([\delta]). \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underset{\text{by the above claim}}{\uparrow} \end{array}$$

[1448] By the Smooth Path-Connectivity Proposition 19.29 (2) any two points on a connected smooth manifold.

We have thus shown that $g_* = \phi\colon \pi_1(M,P) \to \pi_1(S^1,1) = \mathbb{Z}$.

This concludes the proof of the theorem in the case that M has empty boundary.



Finally let M be a smooth manifold with non-empty boundary. To deal with this case we consider $W := M \cup (\partial M \times [0,1))$ where we make the obvious identification of $P = P \times \{0\}$ for $P \in \partial M$. Using the Smooth Collar Neighborhood Theorem 28.3 one can easily verify that W is a smooth manifold such that M is a submanifold and such that W has no boundary. Evidently M is a deformation retract of W. It follows from the Deformation Retract-Homotopy Equivalence Lemma 15.5 and the Homotopy-π_1-Proposition 50.3 (2) that the induced map $i_*\colon \pi_1(M,P) \to \pi_1(W,P)$ is an isomorphism. Now let $\phi \in \operatorname{Hom}(\pi_1(M,P),\mathbb{Z})$. By the previously considered case we can realize $\phi \circ (i_*)^{-1}\colon \pi_1(W,P) \to \mathbb{Z}$ by a smooth map $g\colon W \to S^1$. The restriction of g to M has all the desired properties. ■

Proof of the Maps-to-S^1 Proposition 118.13 (2). We suppose that $\partial M = \varnothing$. In the following let $f, g\colon M \to S^1 = \mathbb{R}/\mathbb{Z}$ be two smooth maps with $g(P) = 1$, such that we have the equality $f_* = g_*\colon \pi_1(M,P) \to \pi_1(S^1,1)$. We need to show that f and g are homotopic. Let ω be the standard closed differential 1-form on $S^1 = \mathbb{R}/\mathbb{Z}$. We set $\alpha := f^*\omega$ and $\beta := g^*\omega$.

Claim. There exists a closed differential 1-form ψ on $M \times [0,1]$ such that the restriction to $M \times \{0\} = M$ equals α and such that the restriction to $M \times \{1\} = M$ equals β.

Proof. First note that it follows from the fact that $f_* = g_*$ and from the natural isomorphisms $\mathrm{H}^1_{\mathrm{dR}}(M) = \mathrm{H}^1(M;\mathbb{R}) = \operatorname{Hom}(\pi_1(M,P),\mathbb{R})$ that $[\alpha] = f^*([\omega]) = g^*([\omega]) = [\beta] \in \mathrm{H}^1_{\mathrm{dR}}(M)$. By definition this means that there exists a 0-form φ on M with $d\varphi = \alpha - \beta$. Next let $p\colon M \times [0,1] \to M$ and $q\colon M \times [0,1] \to [0,1]$ be the natural projection. Furthermore let $h\colon [0,1] \to [0,1]$ be a map that is zero close to 0 and one close to 1. On $M \times [0,1]$ we consider the 1-form that given by $\psi := p^*\alpha - d(q^*h \cdot \varphi)$. It follows from the Differentials-on-Smooth Manifolds Proposition **??** (2) that ψ is closed. We calculate that

$$\psi = p^*\alpha - d(q^*h \cdot \varphi) \underset{\text{by Proposition ?? (1)}}{=} p^*\alpha - (q^*(dh) \cdot \varphi + q^*h \cdot d\varphi) = p^*\alpha - \overbrace{q^*(dh)}^{=0 \text{ on } M\times\{0\} \text{ and } M\times\{1\}} \cdot \varphi - \underbrace{q^*h \cdot d\varphi}_{= 0 \text{ on } M\times\{0\} \text{ and } = p^*\alpha - p^*\beta \text{ on } M\times\{1\}}.$$

⊞

Next note that, as in the proof of Statement (1), the closed 1-form ψ gives rise to a map $F\colon M \times [0,1] \to S^1$. It follows fairly easily from the definitions that $F_0 = f$ and $F_1 = g$. We leave it to the reader to fill in the details. ■

Proof of the Maps-to-S^1 Proposition 118.13 (3). Let $\phi \in \mathrm{H}^1(M;\mathbb{Z})$, let $P \in \partial M$ and let $f\colon \partial M \to S^1$ be a smooth map that realizes $\phi|_{\partial M}$. By the Smooth Collar Neighborhood Theorem 28.3 we can pick a collar neighborhood $[0,1] \times \partial M$. We set

$W := M \setminus ([0,1) \times \partial M)$ and we equip it with the base point $(P,1)$. By the Basics-of-Collars Lemma 28.4 (1) we know that W is an n-dimensional smooth manifold with $\partial W = \{1\} \times M$. By the Maps-to-$S^1$ Proposition 118.13 (1) there exists a smooth map $h\colon W \to S^1$ which realizes $\phi|_W$. It is now straightforward to see that under the obvious isomorphism $\{0\} \times M \to \{1\} \times M$ the maps $f_*\colon \mathrm{H}_1(M;\mathbb{Z}) \to \mathrm{H}_1(S^1;\mathbb{Z})$ and $h_*\colon \mathrm{H}_1(M;\mathbb{Z}) \to \mathrm{H}_1(S^1;\mathbb{Z})$ agree. At this point we can conclude from the Maps-to-S^1 Proposition 118.13 (2) that there exists a smooth map $F\colon [0,1] \times \partial M \to S^1$ such that $F_0 = f$ and $F_1 = h|_{\{1\}\times M}$.

Note that by definition of a collar neighborhood and by the Basics-of-Collars Lemma 28.4 we know that $[0,1] \times \partial M$ and W are closed subsets of M. Thus we obtain from the Pasting Proposition 2.6 (2) that the two maps F and g combine to a continuous map $h\colon M \to S^1$.

Since h is smooth on $[0,1] \times \partial M$ we obtain from the Whitney Approximation Theorem 29.1 that h is homotopic rel $\{0\} \times \partial M$ to a smooth map $g\colon M \to S^1$. This map now has all the desired properties. ■

Of course one can ask many variations on Question 118.12. Since arbitrary topological spaces can be quite weird we restrict ourselves in the following questions to CW-complexes.

Question 118.14. Let X be a connected CW-complex.

(1) Let $g \geq 1$ and let Σ_g be the surface of genus $g \geq 1$. Can every homomorphism $\pi_1(X) \to \pi_1(\Sigma_g)$ be realized by a map $X \to \Sigma_g$?

(2) Can every homomorphism $\pi_1(X) \to \pi_1\Big(\bigvee_{i=1}^{k} S^1\Big) = \langle x_1, \ldots, x_k \rangle$ be realized by a map $X \to \bigvee_{i=1}^{k} S^1$?

(3) Can every homomorphism $\pi_1(X) \to \mathbb{Z}_2 = \pi_1(\mathbb{R}\mathrm{P}^2)$ be realized by a map $X \to \mathbb{R}\mathrm{P}^2$?

118.6. **Rewriting products $M \times S^1$.** the Maps-to-S^1 Proposition 118.13 (1) has many consequences. For example in the Codimension-One Submanifold Existence Proposition 130.4 we will use it to show that for a given compact orientable n-dimensional smooth manifold M any class in $\mathrm{H}_{n-1}(M;\mathbb{Z})$ can be represented by a submanifold. In this section we will give a different application, namely we will see that in many situations one can find interesting ways to rewrite a product $M \times S^1$ where M is a closed connected smooth manifold.
To formulate that application it is convenient to recall the following definition from page 467.

Definition. Let X be a topological space and let $f\colon X \to X$ be a map. We refer to

$$\mathrm{Tor}(X,f) \;:=\; (X \times [0,1])/\!\sim \qquad \text{where } (x,0) \sim (f(x),1) \text{ for all } x \in X,$$

as the mapping torus of (X,f). We refer to the map

$$\begin{array}{rccc} q\colon \mathrm{Tor}(X,f) = (X \times [0,1])/(x,0) \sim (f(x),1) & \to & S^1 \\ [(x,t)] & \mapsto & \exp(2\pi i t) \end{array}$$

as the natural projection onto S^1.

Throughout this section we also use the following notation.

Notation. We denote by $\Theta\colon \mathrm{H}_1(S^1) \to \mathbb{Z}$ the isomorphism given by $\Theta([S^1]) = 1$.

Now we can state the main result of this section.

Proposition 118.15. Let M be a closed connected manifold and let $\phi\colon \mathrm{H}_1(M\times S^1;\mathbb{Z})\to\mathbb{Z}$ be an epimorphism. We set $k = k(\phi) := \phi([\{*\}\times S^1])$. If $k\neq 0$, then there exists a $|k|$-fold covering $p\colon \widetilde{M}\to M$ and a deck transformation $f\colon \widetilde{M}\to\widetilde{M}$ together with a diffeomorphism[1449] $\Xi\colon \mathrm{Tor}(\widetilde{M},f)\to M\times S^1$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_1(\mathrm{Tor}(\widetilde{M},f);\mathbb{Z}) & \xrightarrow{\quad\Xi_*\quad} & \mathrm{H}_1(M\times S^1;\mathbb{Z}) \\ & {\scriptstyle \Theta\circ q_*}\searrow \quad \swarrow{\scriptstyle \phi} & \\ & \mathbb{Z}. & \end{array}$$

Here the diagonal map to the left is induced by the natural projection.

Remark.

(1) Proposition 118.15 can be viewed as a "manifold analogue" of the group-theoretic statement of Exercise 50.19.

(2) One mathematician once called products "slippery". Indeed, as we will see in the examples below, Proposition 118.15 shows that many products have alternative descriptions.

Examples.

(1) Let M be a closed connected non-empty smooth manifold and let $\psi\colon \mathrm{H}_1(M;\mathbb{Z})\to\mathbb{Z}$ be a homomorphism. We apply Proposition 118.15 to the epimorphism

$$\begin{array}{rcl} \phi\colon \mathrm{H}_1(M\times S^1;\mathbb{Z}) = \mathrm{H}_1(M;\mathbb{Z})\oplus \mathrm{H}_1(S^1;\mathbb{Z}) & \to & \mathbb{Z} \\ (\sigma, n\cdot[S^1]) & \mapsto & \psi(\sigma)+n. \end{array}$$

Note that we have $k = k(\phi) = 1$. This implies that the map $p\colon \widetilde{M}\to M$ is a covering of degree 1. By the Covering Uniqueness Theorem 61.9 we can assume that $\widetilde{M} = M$ and that $p = \mathrm{id}$. Note that the only deck transformation of the 1-fold covering $\mathrm{id}\colon M\to M$ is the identity. Thus we obtain from Proposition 118.15 that there exists a diffeomorphism $\Xi\colon M\times S^1\to M\times S^1$ such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}_1(M\times S^1;\mathbb{Z}) & \xrightarrow{\quad\Xi_*\quad} & \mathrm{H}_1(M\times S^1;\mathbb{Z}) \\ & {\scriptstyle \Theta\circ q_*}\searrow \quad \swarrow{\scriptstyle \phi} & \\ & \mathbb{Z}, & \end{array}$$

where $q\colon M\times S^1\to S^1$ denotes the natural projection. Note that if $\psi\colon \mathrm{H}_1(M;\mathbb{Z})\to\mathbb{Z}$ is non-zero, then Ξ_* is not the identity. Summarizing, we have now shown that if $\mathrm{H}^1(M;\mathbb{Z}) = \mathrm{Hom}(\mathrm{H}_1(M;\mathbb{Z}),\mathbb{Z})\neq 0$, then $M\times S^1$ can be decomposed as a product in more than one way.

(2) Let Σ be the surface of some genus $g\in\mathbb{N}_{\geq 2}$ and let $\phi\colon \mathrm{H}_1(\Sigma\times S^1;\mathbb{Z})\to\mathbb{Z}$ be an epimorphism such that $k = k(\phi)\neq 0$. It follows from Proposition 118.15, together with the Subgroups-of-Surface Groups Proposition 64.6 (1) or alternatively with Lemma 87.9, that $\Sigma\times S^1$ is diffeomorphic to a mapping torus of the form $\mathrm{Tor}(\widetilde{\Sigma},f)$ where $\widetilde{\Sigma}$ is a surface of genus $|k|\cdot(g-1)+1$. In particular we obtain the

[1449] By the Manifold-Mapping Torus Lemma 16.25 the mapping torus $\mathrm{Tor}(\widetilde{M},f)$ naturally has the structure of a smooth manifold.

slightly mind boggling statement that the product $\Sigma \times S^1$ can also be viewed as a mapping torus of a self-diffeomorphism of a surface of arbitrarily high genus.

Proof. First we pick points $* \in M$ and $* \in S^1$ and we denote by $j_1 \colon M \xrightarrow{x \mapsto (x,*)} M \times S^1$ and $j_2 \colon S^1 \xrightarrow{z \mapsto (*,z)} M \times S^1$ the corresponding inclusions. Note that by the Maps-to-S^1 Proposition 118.13 (1) there exists a smooth map $g \colon M \to S^1$ such that the map $\mathrm{H}_1(M) \xrightarrow{g_*} \mathrm{H}_1(S^1) \xrightarrow{\Theta} \mathbb{Z}$ equals $\phi \circ j_{1*} \colon \mathrm{H}_1(M) \to \mathbb{Z}$. Now let us consider the following pullback diagram:

$$\begin{array}{ccc} \widetilde{M} := \{(x,z) \in M \times S^1 \mid g(x) = z^k\} & \xrightarrow{(x,z) \mapsto z} & S^1 \\ \Big\downarrow{\scriptstyle (x,z) \mapsto x} & & \Big\downarrow{\scriptstyle z \mapsto z^k} \\ M & \xrightarrow{\qquad g \qquad} & S^1. \end{array}$$

By the Topological-Pullback Lemma 56.16 we know that $p \colon \widetilde{M} \to M$ is a k-fold covering. The map

$$\begin{array}{rcl} f \colon \widetilde{M} & \to & \widetilde{M} \\ (x,z) & \mapsto & (x, \exp(-2\pi\,\mathrm{i}/k) \cdot z) \end{array}$$

is evidently a deck transformation. Next we consider the following diagram:

$$\begin{array}{ccc} \widetilde{M} \times S^1 & \xrightarrow{((x,z),w) \mapsto ((x,z), z^{-1} \cdot w)} & \widetilde{M} \times S^1 \\ \Big\downarrow{\scriptstyle ((x,z),\exp(2\pi\,\mathrm{i}\,t)) \mapsto [((x,z), k \cdot t \bmod \mathbb{Z})]} & & \Big\downarrow{\scriptstyle ((x,z),w) \mapsto (x,w)} \\ \mathrm{Tor}(\widetilde{M}, f) & \xrightarrow[\Xi]{[((x,z),t)] \mapsto (x, z^{-1} \cdot \exp(2\pi\,\mathrm{i}\,t/k))} & M \times S^1. \end{array}$$

We make the following observations:

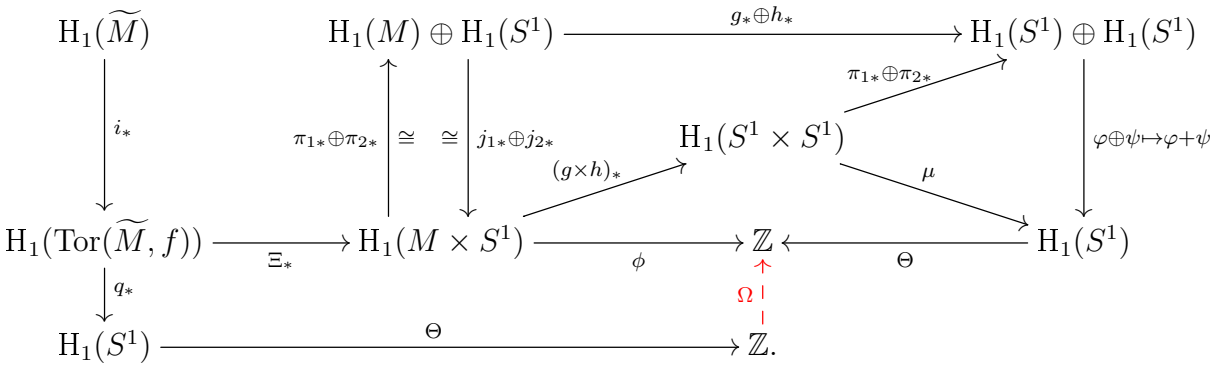
(1) One can easily verify that the bottom map is in fact well-defined and that the diagram commutes.
(2) It is pretty clear that the top map is a diffeomorphism.[1450]
(3) It is straightforward to show that the vertical maps are local diffeomorphisms and they are degree k coverings.

It follows from the above together with Exercise 49.4 that the bottom map is also a diffeomorphism. It remains to prove the following claim.

Claim. We have $\phi \circ \Xi_* = \Theta \circ q_* \colon \mathrm{H}_1(\mathrm{Tor}(\widetilde{M}, f)) \to \mathbb{Z}$.

[1450] The reader might note that the top map is the restriction of the map $(M \times S^1) \times S^1 \to (M \times S^1) \times S^1$ given by the same formula, which is clearly a diffeomorphism, to the submanifold $\widetilde{M} \times S^1$ of $(M \times S^1) \times S^1$.

Proof. We consider the following diagram:



We make the following clarifications and observations:

(1) The map $i\colon \widetilde{M} \to \mathrm{Tor}(\widetilde{M}, f))$ is the natural inclusion given by $x \mapsto [(x, 0)]$.
(2) The map $\mu\colon S^1 \times S^1 \to S^1$ is the multiplication map.
(3) In both settings we denote by π_1 and π_2 the projection onto the first respectively second factor.
(4) The map $h\colon S^1 \to S^1$ is given by $z \mapsto z^k$. Note that by Lemma 76.10 we know that $h_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(S^1)$ is given by multiplication by k.
(5) It follows from Exercise 84.5, or alternatively from the discussion on page 2499, that the vertical maps in the middle are isomorphisms and inverses of one another.
(6) The upper left triangle commutes by the functoriality of homology groups.
(7) The triangle to the right commutes by Exercise 84.4.
(8) The big rectangle to the right commutes by the choice of g and h.
(9) It follows from (5), (6), (7) and (8) that the triangle at the bottom commutes.
(10) Recall that any point $(x, z) \in \widetilde{M}$ satisfies by definition $g(x) = z^k$. It now follows immediately that the map $(g \times h) \circ \mu \circ \Xi \circ i\colon \widetilde{M} \to S^1$ is the constant map.
(11) It follows from (9) and (10) that $\phi \circ \Xi_* \circ i_*\colon \mathrm{H}_1(\widetilde{M}) \to \mathbb{Z}$ is the zero map.
(12) By the Mapping Torus-Homology Lemma 78.22 we know that the vertical sequence to the left is exact.
(13) It follows from (10) and (11) that the vertical map Ω exists.
(14) We pick $(x, z) \in \widetilde{M}$. We use the homeomorphism from the Mapping Torus-of-Homeomorphism Lemma 16.28 to make the identification $\mathrm{Tor}(\widetilde{M}, f) = (\widetilde{M} \times \mathbb{R})/(w, t) \sim (f(w), t + 1)$. Now we consider the singular 1-cycle

$$\begin{array}{rcl} \sigma\colon \Delta^1 & \to & \mathrm{Tor}(\widetilde{M}, f) = (\widetilde{M} \times \mathbb{R})/(w, t) \sim (f(w), t + 1) \\ (1 - t, t) & \mapsto & [((x, z), kt)]. \end{array}$$

It follows immediately from the definitions that $(\phi \circ \Xi_*)([\sigma]) = k$ and $(\Theta \circ q_*)([\sigma]) = k$. Thus we have $\Omega(k) = k$. Since $k \neq 0$ we see that $\Omega = \mathrm{id}$. ∎

118.7. **The wedge product of differential forms** (∗). For a real vector space V we introduced in the Wedge Product the Wedge Product Proposition **??** the bilinear wedge

product

$$\wedge^k V^* \times \wedge^l V^* \ \to \ \wedge^{k+l} V^*.$$

Given a smooth manifold M we can define this wedge product for each tangent space $V = \mathrm{T}_P M$. As we saw in Lemma **??** we have a bilinear map

$$\wedge\colon \Omega^k(M) \times \Omega^l(M) \ \to \ \Omega^{k+l}(M).$$

Let M be a smooth manifold. We recall that in Proposition **??** we saw that the map

$$\begin{array}{rcl} \wedge\colon \mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^l_{\mathrm{dR}}(M) & \to & \mathrm{H}^{k+l}_{\mathrm{dR}}(M) \\ ([\omega],[\sigma]) & \mapsto & [\omega] \wedge [\sigma] := [\omega \wedge \sigma] \end{array}$$

is well-defined and that it has many nice properties.
At this point it is helpful to introduce the following definition.

Definition.

(1) A graded ring is a ring[1451] R together with a direct sum decomposition[1452] $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ into subgroups such that for any $m, n \in \mathbb{N}_0$ the multiplication map restricts to a map

$$R_m \times R_n \ \to \ R_{m+n}.$$

(2) A graded ring $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ such that for any two homogeneous elements a and b the equality $a \cdot b = (-1)^{\deg(a) \cdot \deg(b)} \cdot b \cdot a$ holds is called a superalgebra.[1453]

(3) Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ and $S = \bigoplus_{n \in \mathbb{N}_0} S_n$ be two graded rings. A morphism between the two graded rings is a ring homomorphism $\varphi\colon R \to S$ such that for any $n \in \mathbb{N}_0$ we have $\varphi(R_n) \subset S_n$.

(4) We define the category $\mathcal{GrRing}$ of graded rings to be the category where the objects are graded rings and the morphisms are morphisms of graded rings. Similarly we define the category of graded superalgebras.

Example. We consider the polynomial ring $R = \mathbb{Q}[x]$. We can view this ring as a graded ring by defining $R_n := \mathbb{Q} \cdot x^n$ for $n \in \mathbb{N}_0$.

Recall that for a given smooth manifold M we write

$$\mathrm{H}^*_{\mathrm{dR}}(M) \ = \ \bigoplus_{i=0}^{\infty} \mathrm{H}^i_{\mathrm{dR}}(M).$$

We consider the following map

$$\begin{array}{rcl} \wedge\colon \mathrm{H}^*_{\mathrm{dR}}(M) \times \mathrm{H}^*_{\mathrm{dR}}(M) & \to & \mathrm{H}^*_{\mathrm{dR}}(M) \\ \left(\sum_{i=0}^{\infty} x_i, \sum_{i=0}^{\infty} y_i\right) & \mapsto & \sum_{i,j=0}^{\infty} x_i \wedge y_j. \end{array}$$

where it is understood that each x_i and y_i is homogeneous of degree i.

[1451] Here we mean by a ring an abelian group together with a multiplicative structure which is associative, distributive and that admits a multiplicatively neutral element. We do not demand that a ring is commutative.

[1452] In particular, R viewed as an abelian group is $\mathbb{N}_0$-graded.

[1453] The word "algebra" in our context is justified by the fact that any ring is a $\mathbb{Z}$-algebra. We could also have referred to such an object as a superring, but this terminology seems to be uncommon.

The following proposition summarizes the key features of the wedge product on de Rham cohomology.

Proposition 118.16. The maps

$$\begin{array}{rcl} M & \mapsto & (\mathrm{H}^*_{\mathrm{dR}}(M), \wedge) \\ (f\colon M \to N) & \mapsto & (f^*\colon \mathrm{H}^*_{\mathrm{dR}}(N) \to \mathrm{H}^*_{\mathrm{dR}}(M)) \end{array}$$

define a contravariant functor from the category of smooth manifolds to the category of graded superalgebras.

Proof.

(1) First note that it follows almost immediately from Proposition **??** that for a given smooth manifold M the wedge product turns $\mathrm{H}^*_{\mathrm{dR}}(M)$ into a graded superalgebra. It follows immediately from the definitions that the multiplicatively neutral element in de Rham cohomology is hereby the 0-dimensional cohomology class that is represented by the constant function $f\colon M \to \mathbb{R}$, $x \mapsto 1$.

(2) If $f\colon M \to N$ is a smooth map between smooth manifolds, then it follows immediately from the definitions that for any choice of $\omega \in \Omega^k(N)$ and $\sigma \in \Omega^l(N)$ the following equality holds:

$$(f^*\omega \wedge f^*\sigma) \;=\; f^*(\omega \wedge \sigma).$$

Therefore the smooth map f induces a morphism $f^*\colon \mathrm{H}^*_{\mathrm{dR}}(N) \to \mathrm{H}^*_{\mathrm{dR}}(M)$ of graded rings. ∎

Example. We consider the differential 1-forms dx and dy on $\mathbb{R}^2$ which are given by[1454] $dx(v,w) = v$ and $dy(v,w) = w$. Furthermore we identify the 2-torus T with the quotient space $\mathbb{R}^2/\mathbb{Z}^2$ and we denote the projection $\mathbb{R}^2 \to T = \mathbb{R}^2/\mathbb{Z}^2$ by p. It is straightforward to see that dx and dy induce differential 1-forms on T.[1455] We denote these also by dx and dy.[1456] The 1-forms dx and dy are closed, i.e. $d(dx) = d(dy) = 0$.[1457] Thus they define cohomology classes $[dx], [dy] \in \mathrm{H}^1_{\mathrm{dR}}(T)$. A straightforward calculation, see e.g. [**Frie2016a**, p. 248], shows that if we integrate the 2-form $dx \wedge dy$ on T with the standard orientation, then we obtain 1. It follows from the Top Dimension-de Rham-Integration Proposition **??** that $[dx \wedge dy] = [dx] \wedge [dy]$ is a non-trivial element in $\mathrm{H}^2_{\mathrm{dR}}(T)$.

From the fact that $\mathrm{H}^*_{\mathrm{dR}}(T)$ is a superalgebra we know that

$$[dx] \wedge [dy] \;=\; (-1)^{1\cdot 1} \cdot [dy] \wedge [dx] \;=\; -[dy] \wedge [dx].$$

Since this wedge product is non-zero we see that $(\mathrm{H}^*_{\mathrm{dR}}(T), \wedge)$ is a non-commutative ring.

This discovery of a ring structure on de Rham cohomology raises the question whether one can also define such a ring structure using singular cohomology and whether such a ring structure exists on singular cohomology for more general spaces. We will answer these questions in Chapter 113.

[1454] As always we identify $\mathrm{T}_P\mathbb{R}^2$ with $\mathbb{R}^2$.

[1455] Details are given in [**Frie2016a**, p. 248].

[1456] More precisely, we showed that there exists a unique 1-form dx on T such that for each point $P \in \mathbb{R}^2$ and $v \in \mathrm{T}_P\mathbb{R}^2$ we have $dx_P(v) = dx_{p(P)}(p_*(v))$. Of course the same statement also holds for dy.

[1457] This is clear for dx and dy viewed as differential 1-forms on $\mathbb{R}^2$. But the projection $p\colon \mathbb{R}^2 \to T = \mathbb{R}^2/\mathbb{Z}^2$ is a local diffeomorphism. Since the differential commutes with smooth maps, see the Differentials-on-Smooth Manifolds Proposition **??** (2), we also see that $d(dx) = d(dy) = 0$ on $\mathbb{R}^2/\mathbb{Z}^2$.

Exercises for Chapter 118.

Exercise 118.1. Let $\mathcal{H}^*$ be a smooth cohomology theory with $\mathcal{H}^n(\star) = 0$ for $n \geq 1$. We denote by $f\colon S^1 \to S^1$ the map that is given by $(x, y) \mapsto (x, -y)$. Show, only using the axioms of a smooth cohomology theory, that $f^*\colon \mathcal{H}^1(S^1) \to \mathcal{H}^1(S^1)$ is given by multiplication by -1.

Exercise 118.2. Let M be a smooth manifold. Show that if A and B are subsets of M such that $\overline{A}$ and $\overline{B}$ are compact, then $\overline{A \cap B}$ and $\overline{A \cup B}$ are also compact.

Exercise 118.3. We make the usual identification $\pi_1(\mathbb{R}^2/\mathbb{Z}^2) = \mathbb{Z}^2$. We consider the homomorphism $\phi\colon \mathbb{Z}^2 = \pi_1(\mathbb{R}^2/\mathbb{Z}^2) \to \mathbb{Z}$ given by $\phi((1,0)) = 6$ and $\phi((0,1)) = 4$. Provide an *explicit* map $f\colon \mathbb{R}^2/\mathbb{Z}^2 \to S^1$ such that $f_* = \phi$.

Exercise 118.4. Let M be a smooth manifold and let $\phi \in \mathrm{H}^1(M; \mathbb{Z})$. Show that $\phi \cup \phi = 0$.

Exercise 118.5. Let M be a smooth manifold and let A be a union of components of ∂M. We define

$$\mathrm{C}^k_{\mathrm{dr}}(M, A) \;:=\; \{\varphi \in \Omega^k(M) \,|\, \text{there exists a neighborhood } U \text{ of } A \text{ such that } \varphi|_U \equiv 0\}.$$

(a) Show that the usual differential $d\colon \mathrm{C}^k_{\mathrm{dR}}(M) \to \mathrm{C}^{k+1}_{\mathrm{dR}}(M)$ restricts to a coboundary map $d\colon \mathrm{C}^k_{\mathrm{dR}}(M, A) \to \mathrm{C}^{k+1}_{\mathrm{dR}}(M, A)$. We refer to resulting cohomology groups by $\mathrm{H}^k_{\mathrm{dR}}(M, A)$.

(b) Show that there exists a natural connecting homomorphism $\delta\colon \mathrm{H}^k_{\mathrm{dR}}(A) \to \mathrm{H}^{k+1}_{\mathrm{dR}}(X, A)$ which gives rise to a long exact sequence

$$\ldots \longrightarrow \mathrm{H}^k_{\mathrm{dR}}(M, A) \longrightarrow \mathrm{H}^k_{\mathrm{dR}}(M) \longrightarrow \mathrm{H}^k_{\mathrm{dR}}(A) \xrightarrow{\delta} \mathrm{H}^{k+1}_{\mathrm{dR}}(M, A) \longrightarrow \ldots$$

(c) Show that there exists a natural isomorphism $u\colon \mathrm{H}_k(M, A) \to$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & \mathrm{H}^k(M, A; \mathbb{R}) & \longrightarrow & \mathrm{H}^k(M; \mathbb{R}) & \longrightarrow & \mathrm{H}^k(A; \mathbb{R}) & \xrightarrow{\delta} & \mathrm{H}^{k+1}(M, A; \mathbb{R}) & \longrightarrow \ldots \\
& \cong \downarrow u & & \cong \downarrow u & & \cong \downarrow u & & \cong \downarrow u & \\
\ldots \longrightarrow & \mathrm{H}^k_{\mathrm{dR}}(M, A) & \longrightarrow & \mathrm{H}^k_{\mathrm{dR}}(M) & \longrightarrow & \mathrm{H}^k_{\mathrm{dR}}(A) & \xrightarrow{\delta} & \mathrm{H}^{k+1}_{\mathrm{dR}}(M, A) & \longrightarrow \ldots
\end{array}$$

(d) Let $\varphi \in \mathrm{H}^1(M, A; \mathbb{Z})$. Show that there exists a smooth map $f\colon M \to S^1$ that vanishes on an open neighborhood of U and which has the property that the induced map $f_*\colon \mathrm{H}_1(M, A; \mathbb{Z}) \to \mathrm{H}_1(S^1) = \mathbb{Z}$ equals the given cohomology class $\varphi \in \mathrm{H}^1(M, A; \mathbb{Z}) = \mathrm{Hom}(\mathrm{H}_1(M, A; \mathbb{Z}), \mathbb{Z})$.
Remark. This statement is the analogue of the Maps-to-S^1 Proposition 118.13 (1).

Part XII

Poincaré Duality

119. The Topological Poincaré Duality Theorem

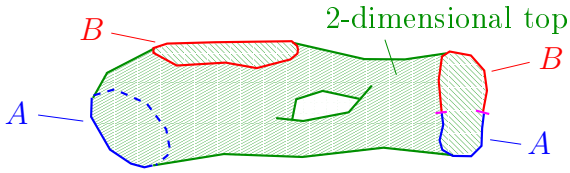
In Chapter 103 we stated and proved the Simplicial Poincaré Duality Theorem 103.4 and we saw that it gives rise to the Poincaré Duality Theorem 103.5 for closed smooth manifolds. As discussed on page 2268, that approach has its limitations. We will rectify this by proving in this chapter the Poincaré Duality Theorem 119.1, which in particular has the advantage that it also applies to topological manifolds.

119.1. **The Poincaré Duality Theorem.** The following theorem is by far the most important theorem regarding homology and cohomology groups of topological manifolds.

Theorem 119.1. (Poincaré Duality Theorem) Let M be a compact, non-empty n-dimensional topological manifold and let R be a commutative ring. We suppose that M is R-oriented. We denote by $[M] \in \mathrm{H}_n(M, \partial M; R)$ the R-fundamental class. Furthermore suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. Then for each $k \in \mathbb{N}_0$ the map[1458]

$$\begin{array}{rcl} \mathrm{H}^k(M, A; R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M, B; R) \\ \sigma & \mapsto & \sigma \cap [M] \end{array}$$

is an isomorphism.



for all commutative rings R we have $\mathrm{H}^k(M, A; R) \xrightarrow{\cong} \mathrm{H}_{2-k}(M, B; R)$

We proceed as follows: in this section and the next section we discuss several special cases and applications of the Poincaré Duality Theorem 119.1. In the final two sections of the chapter we will then provide a rigorous, but arguably less intuitive proof, of the theorem. In the coming chapters we will then use the Poincaré Duality Theorem 119.1 to prove many more statements.

Notation. The isomorphism of the Poincaré Duality Theorem 119.1 is called the Poincaré Duality isomorphism. We denote it and its inverse both by PD_M, i.e. with the notation of the Poincaré Duality Theorem 119.1 we have the isomorphism[1459]

$$\begin{array}{rcl} \mathrm{PD}_M \colon \mathrm{H}^k(M, A; R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M, B; R) \\ \sigma & \mapsto & \sigma \cap [M] \end{array}$$

together with the inverse

$$\mathrm{PD}_M \colon \mathrm{H}_{n-k}(M, B; R) \xrightarrow{\cong} \mathrm{H}^k(M, A; R).$$

[1458] Recall that the Excisive Triad-Proposition 109.12 (5) implies that (M, A, B) is an excisive triad. In particular, following the definition of the relative cap product on page 2557, we obtain a relative cap product

$$\mathrm{H}^k(M, A; R) \times \underbrace{\mathrm{H}_n(M, \overbrace{A \cup B}^{=\partial M}; R)}_{\text{contains } [M]} \xrightarrow{\cap} \mathrm{H}_{n-k}(X, B; R).$$

This shows that the map in the Poincaré Duality Theorem is in fact defined.

The isomorphisms PD_M thus allow us to go back and forth between homology and cohomology of an R-oriented topological manifold.

Remark. Let M be a compact n-dimensional topological manifold.

(1) the $\mathbb{F}_2$-Orientability Proposition 105.13 says that M is $\mathbb{F}_2$-orientable, i.e. the duality statement holds with $\mathbb{F}_2$-coefficients without any further assumptions on M.
(2) If M is in fact orientable, then it follows from Lemma 105.14 that M is R-orientable for any commutative ring R. In particular the duality statement holds for any commutative ring.
(3) The two most important instances of A and B are that we have either $A = \varnothing$ or $A = \partial M$. Under the hypothesis of the Poincaré Duality Theorem 119.1 we then get isomorphisms

$$\begin{array}{ccc} \mathrm{H}^k(M;R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M,\partial M;R) \\ \sigma & \mapsto & \sigma \cap [M] \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{H}^k(M,\partial M;R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M;R) \\ \sigma & \mapsto & \sigma \cap [M]. \end{array}$$

(4) For topological manifolds with non-empty boundary the Poincaré Duality Theorem is sometimes also called the *Poincaré-Lefschetz Duality Theorem*.

Examples.

(1) Let P be a point in M and suppose that M is connected. As we pointed out on page 2558, we have $[M]^* \cap [M] = [P] \in \mathrm{H}_0(M;\mathbb{Z})$. Therefore we obtain that $\mathrm{PD}_M([P]) = [M]^*$.
(2) Earlier we had already performed several explicit calculations which can be viewed as special case of the Poincaré Duality Theorem 119.1:
   (a) On page 2555 we saw explicitly that for any surface Σ the map
   $$- \cap [\Sigma] \colon \mathrm{H}^1(\Sigma;\mathbb{Z}) \to \mathrm{H}_1(\Sigma;\mathbb{Z})$$
   is an isomorphism. On page 2555 we had obtained similar results for $\mathbb{R}\mathrm{P}^2$ with $\mathbb{F}_2$-coefficients.
   (b) On page 2559 we saw explicitly that for the annulus $X = S^1 \times [0,1]$ the map
   $$- \cap [X] \colon \mathrm{H}^1(X;\mathbb{Z}) \to \mathrm{H}_1(X,\partial X;\mathbb{Z})$$
   is an isomorphism.
(3) It is instructive to consider the case $k = 0$ of the Poincaré Duality Theorem 119.1. We obtain for a compact connected non-empty n-dimensional topological manifold that
$$\mathrm{H}_n(M,\partial M;\mathbb{Z}) \cong \mathrm{H}^0(M;\mathbb{Z}) \underset{\uparrow}{\cong} \mathbb{Z} \quad \& \quad \mathrm{H}_n(M;\mathbb{Z}) \cong \mathrm{H}^0(M,\partial M;\mathbb{Z}) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } \partial M = \varnothing, \\ 0, & \text{if } \partial M \neq \varnothing. \end{cases}$$
H^0-Proposition 108.11 — see the H^0-Proposition 108.11 and page 2408
   This calculation is consistent with the results from the Orientation-Top Homology Theorem 106.1 and the Top-Homology Theorem 106.3.

We formulate our last example as a proposition.

[1459] When M is understood, then we drop it from the notation.

Proposition 119.2. (Poincaré Duality-Products Proposition) Let M and N be two compact oriented connected topological manifolds. Furthermore let $x_0 \in M$ and $y_0 \in N$.

(a) We denote by $p\colon M \times N \to M$ and $q\colon M \times N \to N$ the obvious projections.

(b) We denote by $i\colon M \xrightarrow{x \mapsto (x,y_0)} M \times N$ and $j\colon N \xrightarrow{y \mapsto (x_0,y)} M \times N$ the obvious inclusions.

(c) We equip $M \times N$ with the product orientation introduced in the Topological Product Orientation Proposition 113.10.

(d) We work with the decomposition $\partial(M \times N) = (\partial M \times N) \cup (M \times \partial N)$.

(e) We write $m = \dim(M)$ and $n = \dim(N)$.

The following equalities hold:

$$\begin{aligned} \mathrm{PD}_{M\times N}(i_*([M])) &= (-1)^{mn} \cdot q^*([N]^*) &&\in \mathrm{H}^m(M\times N, M\times\partial N;\mathbb{Z}) \\ \mathrm{PD}_{M\times N}(j_*([N])) &= p^*([M]^*) &&\in \mathrm{H}^n(M\times N, M\times\partial N;\mathbb{Z}) \\ \mathrm{PD}_{M\times N}(i_*([M])) \cup \mathrm{PD}_{M\times N}(j_*([N])) &= [M\times N]^* &&\in \mathrm{H}^{m+n}(M\times N, \partial(M\times N);\mathbb{Z}). \end{aligned}$$

Proof. By the Fundamental Class-Product Proposition 117.2 (2) we have

$$p^*([M]^*) \cap [M\times N] = i_*([M]) \quad \text{and} \quad q^*([N]^*) \cap [M\times N] = (-1)^{mn} \cdot j_*([N]).$$

The first two promised equalities follow from these two calculations and the definition of the Poincaré Duality isomorphisms. The third equality is verified in the following little calculation:

$$\begin{aligned} \mathrm{PD}_{M\times N}(i_*([M])) \cup \mathrm{PD}_{M\times N}(j_*([N])) &\overset{\text{by the first two equalities}}{=} (-1)^{mn} \cdot q^*([N]^*) \cup p^*([M]^*) \\ &= (-1)^{mn} \cdot (-1)^{mn} \cdot p^*([M]^*) \cup q^*([N]^*) \\ &\overset{\text{by the Cup Product-Commutativity Property}}{=} [M\times N]^*. \end{aligned}$$

by the Fundamental Class-Product Proposition

119.2. **Consequences of the Poincaré Duality Theorem 119.1.** In this section we give several more or less immediate corollaries to the Poincaré Duality Theorem 119.1. If we restrict ourselves to closed smooth manifolds, then the statements below can also be deduced from the Poincaré Duality Theorem 103.5 that we proved in the last chapter. We start out with the following proposition.

Proposition 119.3. Let M be a compact orientable n-dimensional topological manifold and suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. For every $k \in \mathbb{N}_0$ there exists an isomorphism[1460]

$$\mathrm{H}_{n-k}(M, A;\mathbb{Z}) \cong \underset{\uparrow}{\mathrm{FH}_k}(M, B;\mathbb{Z}) \oplus \underset{\uparrow}{\mathrm{Tor}}(\mathrm{H}_{k-1}(M, B;\mathbb{Z})).$$

given an abelian group G we denote by $\mathrm{Tor}(G)$ its torsion-subgroup and we denote by $\mathrm{F}G := G/\mathrm{Tor}(G)$ the maximal torsion-free quotient

Proof. We have the following isomorphisms

$$\begin{aligned} \mathrm{H}_{n-k}(M, A;\mathbb{Z}) &\overset{\text{Poincaré Duality Theorem 119.1}}{\cong} \mathrm{H}^k(M, B;\mathbb{Z}) \overset{\text{Universal Coefficient Theorem 110.12}}{\cong} \mathrm{Hom}(\mathrm{H}_1(M, B),\mathbb{Z}) \oplus \mathrm{Ext}(\mathrm{H}_{k-1}(M, B;\mathbb{Z}),\mathbb{Z}) \\ &\cong \mathrm{FH}_k(M, B;\mathbb{Z}) \oplus \mathrm{Tor}(\mathrm{H}_{k-1}(M, B;\mathbb{Z})). \end{aligned}$$

follows from the Ext-Calculation Lemma 110.5 (5) and the fact that by Lemma 96.8 together with Topological Manifolds-Invariants Proposition 104.14 (4) we know that the homology groups $\mathrm{H}_i(M, B;\mathbb{Z})$ are finitely generated abelian groups

[1460] It is worth stressing that this isomorphism is not natural.

■

Example. Let $p, q \in \mathbb{N}$ be coprime. In this example we consider the corresponding lens space $L(p,q) = S^3/\mathbb{Z}_p$ as defined and discussed on page 1075. Note that $L(p,q)$ is a closed orientable 3-dimensional smooth manifold. In the Lens Space Homology Lemma 98.9 we computed the following homology groups:

$$\begin{array}{rcl} \mathrm{H}_3(L(p,q);\mathbb{Z}) & \cong & \mathbb{Z} \\ \mathrm{H}_2(L(p,q);\mathbb{Z}) & \cong & 0 \\ \mathrm{H}_1(L(p,q);\mathbb{Z}) & \cong & \mathbb{Z}_p \\ \mathrm{H}_0(L(p,q);\mathbb{Z}) & \cong & \mathbb{Z}. \end{array}$$

Fortunately this data is indeed consistent with Proposition 119.3.

The following straightforward corollary to the previous proposition gives in particular a new proof to the Smooth Manifold-Top Homology Theorem 98.6 (2) and Proposition 106.22.

Corollary 119.4. Let M be a compact orientable n-dimensional topological manifold and suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. Then the homology groups $\mathrm{H}_{n-1}(M, A; \mathbb{Z})$ and $\mathrm{H}_{n-1}(M, B; \mathbb{Z})$ are free abelian groups.

Proof. We have

$$\mathrm{H}_{n-1}(M,A;\mathbb{Z}) \overset{\text{Proposition 119.3}}{\cong} \underbrace{\mathrm{FH}_1(M,B)}_{\text{free abelian group}} \oplus \underbrace{\mathrm{Ext}(\overbrace{\mathrm{H}_0(M,B)}^{\text{free abelian group}},\mathbb{Z})}_{=0 \text{ by the Ext-Calculation Lemma 110.5}}$$

■

Let X be a topological space X and let $n \in \mathbb{N}_0$. Recall that on page 1882 we defined the n-th Betti number of X as $b_n(X) := \mathrm{rank}(\mathrm{H}_n(X))$. With this definition we have the following immediate corollary to Proposition 119.3.

Corollary 119.5. (Betti Number Symmetry Corollary) Let M be a closed orientable n-dimensional topological manifold. Given any $k \in \mathbb{N}_0$ we have $b_{n-k}(M) = b_k(M)$.

Remark. Henri Poincaré [**Poi1895, Poi2010, PGL1953**, Chapter 9], formulated the conclusion of the Betti Number Symmetry Corollary 119.5 in a distinctly more elegant way: in his seminal paper "analysis situs", published in 1895, he wrote

> "[...] pour une variété fermé les nombres de Betti
> également distants des extrêmes sont égaux".

In fact when Henri Poincaré first developed what became known as Poincaré Duality he thought that for any closed orientable connected n-dimensional smooth manifold the groups $\mathrm{H}_k(M;\mathbb{Z})$ and $\mathrm{H}_{n-k}(M;\mathbb{Z})$ are isomorphic. A Danish mathematician, Poul Heegaard [**Hee1898**], pointed out to him that this is not true in general, for example as we just saw, for the lens space $L(p,q)$ we have by the Lens Space Homology Lemma 98.9 that $\mathrm{H}_1(L(p,q);\mathbb{Z}) \cong \mathbb{Z}_p$ but $\mathrm{H}_2(L(p,q);\mathbb{Z}) = 0$. Shortly afterwards Henri Poincaré [**Poi1900**] came up with the correct formulation of Poincaré Duality, namely the one given in Proposition 119.3.

Next let us consider (co-) homology groups with field coefficients. In this context we have the following almost immediate consequence of the Poincaré Duality Theorem 119.1.

Theorem 119.6. (Co-Homology Symmetry Theorem) Let M be a compact n-dimensional topological manifold and let $k \in \mathbb{N}_0$.

(1) We have[1461]

$$\begin{aligned} \dim_{\mathbb{F}_2}(\mathrm{H}_k(M;\mathbb{F}_2)) &= \dim_{\mathbb{F}_2}(\mathrm{H}_{n-k}(M,\partial M;\mathbb{F}_2)) \\ \text{and} \quad \dim_{\mathbb{F}_2}(\mathrm{H}^k(M;\mathbb{F}_2)) &= \dim_{\mathbb{F}_2}(\mathrm{H}^{n-k}(M,\partial M;\mathbb{F}_2)). \end{aligned}$$

(2) If M is orientable, then for any field $\mathbb{F}$ we have

$$\begin{aligned} \dim_{\mathbb{F}}(\mathrm{H}_k(M;\mathbb{F})) &= \dim_{\mathbb{F}}(\mathrm{H}_{n-k}(M,\partial M;\mathbb{F})) \\ \text{and} \quad \dim_{\mathbb{F}}(\mathrm{H}^k(M;\mathbb{F})) &= \dim_{\mathbb{F}}(\mathrm{H}^{n-k}(M,\partial M;\mathbb{F})). \end{aligned}$$

Remark. In the De Rham Cohomology Symmetry Corollary **??** we proved the analogous statement in terms of dimensions of de Rham cohomology groups.

Proof. Let M be a compact n-dimensional topological manifold. We have

$$\begin{aligned} \dim_{\mathbb{F}_2}(\mathrm{H}_k(M;\mathbb{F}_2)) &\underset{\text{Theorem 119.1}}{=} \dim_{\mathbb{F}_2}(\mathrm{H}^{n-k}(M,\partial M;\mathbb{F}_2)) \\ &\underset{\text{Cohomology-Field Coefficients Proposition 110.18}}{=} \dim_{\mathbb{F}_2}(\mathrm{Hom}(\mathrm{H}_{n-k}(M,\partial M;\mathbb{F}_2),\mathbb{F}_2)) \\ &\underset{\substack{\text{since Propositions 104.14 (4) and 75.2 (3) and (4) together with Lemma 96.8} \\ \text{imply that } \mathrm{H}_{n-k}(M,\partial M;\mathbb{F}_2) \text{ is finite-dimensional}}}{=} \dim_{\mathbb{F}_2}(\mathrm{H}_{n-k}(M,\partial M;\mathbb{F}_2)). \end{aligned}$$

The proofs of all other statements are basically the same. ■

Example. We consider the real projective space $\mathbb{R}\mathrm{P}^n$. On page 1930 we had determined that

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } k = 0,\ldots,n, \\ 0, & \text{otherwise.} \end{cases}$$

So the homology groups with $\mathbb{F}_2$-coefficients are in particular symmetric, which is exactly what the above Co-Homology Symmetry Theorem 119.6 with $\mathbb{F}_2$-coefficients predicts.

Remark. Let M be a compact n-dimensional topological manifold and let $\mathbb{F}$ be a field. In the Topological Manifolds-Invariants Proposition 104.14 (4) we had mentioned that all homology groups $\mathrm{H}_k(M;\mathbb{F})$ are finite-dimensional. But this proof relied on the slightly mysterious Topological Manifold-Simplicial Complex Retract Theorem 104.13. Now we can give an alternative argument if M is closed and orientable. Indeed, we have

$$\mathrm{H}_k(M;\mathbb{F}) \underset{\text{this follows from the Poincaré Duality Theorem 119.1}}{\cong} \mathrm{H}^{n-k}(M;\mathbb{F}) \overset{\text{Cohomology-Field Coefficients Proposition 110.18}}{\cong} \mathrm{H}_{n-k}(M;\mathbb{F})^* \underset{}{\cong} \mathrm{H}^k(M;\mathbb{F})^* \overset{}{\cong} (\mathrm{H}_k(M;\mathbb{F})^*)^*.$$

(Here, as always given an $\mathbb{F}$-vector space V we denote by $V^* = \mathrm{Hom}_{\mathbb{F}}(V,\mathbb{F})$ its dual vector space.) But by [**Jacb1953**, p. 244-248] (see also [**DF2004**, Chapter 11.4]) a vector space V is isomorphic to its double dual $(V^*)^*$ if and only if the vector space is finite-dimensional.

Using the Co-Homology Symmetry Theorem Theorem 119.6 we obtain the following pretty proposition.

[1461]The statement implies of course that $\mathrm{H}^k(M;\mathbb{F})$ and $\mathrm{H}^{n-k}(M,\partial M;\mathbb{F})$ are isomorphic. We prefer to state the result in terms of dimensions to stress that the isomorphism is not natural.

Proposition 119.7. (Odd-Dimensional Manifold-Euler Characteristic Proposition) For every compact odd-dimensional topological manifold M we have

$$\chi(M) \ = \ \tfrac{1}{2}\chi(\partial M).$$

In particular if M is closed, then $\chi(M) = 0$.

Remark.

(1) In the Odd-Dimensional Manifold-Euler Characteristic Lemma 45.4 we showed that the smooth Euler characteristic of a closed orientable odd-dimensional smooth manifold is zero.
(2) In Corollary 140.5 we will use handle decompositions to give an alternative proof of the Odd-Dimensional Manifold-Euler Characteristic Proposition 119.7 in the setting of closed smooth manifolds.

Proof. Let M be a compact topological manifold of dimension $2n+1$. Then

$$\begin{array}{lll}
\chi(M) & = \sum_{i=0}^{2n+1}(-1)^i\cdot\dim_{\mathbb{F}_2}(\mathrm{H}_i(M;\mathbb{F}_2)) & = \sum_{i=0}^{2n+1}(-1)^i\cdot\dim_{\mathbb{F}_2}(\mathrm{H}_{2n+1-i}(M,\partial M;\mathbb{F}_2)) \\
\uparrow & & \uparrow \\
\text{Manifold-}\chi\text{-Proposition 104.36 (4)} & & \text{by the Co-Homology Symmetry Theorem 119.6} \\
= -\sum_{j=0}^{2n+1}(-1)^j\cdot\dim_{\mathbb{F}_2}(\mathrm{H}_j(M,\partial M;\mathbb{F}_2)) & = -\chi(M,\partial M) & = -\chi(M)+\chi(\partial M). \\
\uparrow & \uparrow & \uparrow \\
\text{follows from the substitution } j=2n+1-i & \text{Manifold-}\chi\text{-Proposition 104.36 (4)} & \\
\text{together with the fact that } (-1)^{2n+1}=-1 & &
\end{array}$$

Summarizing we have shown that $\chi(M) = -\chi(M) + \chi(\partial M)$. The desired equality follows from solving for $\chi(M)$. ■

the Odd-Dimensional Manifold-Euler Characteristic Proposition 119.7 raises the following question: what can we say about Euler characteristics of closed even-dimensional topological manifolds. By the classification of closed oriented connected 2-dimensional topological manifolds, see the Surface Classification Theorem 55.4, we know that the Euler characteristic of such a topological manifold is even. In higher dimensions there is evidently much more freedom for constructing topological manifolds.

Question 119.8. Let $n \geq 4$ be even. Is every integer the Euler characteristic of a closed orientable connected n-dimensional topological manifold?

Using the Odd-Dimensional Manifold-Euler Characteristic Proposition 119.7 we easily obtain the following surprising (?) corollary.

Corollary 119.9.

(1) If M is a closed even-dimensional topological manifold that is homeomorphic to the boundary of a compact topological manifold, then $\chi(M)$ is even.
(2) Let n be even. The topological manifolds $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ are not homeomorphic to the boundary of a compact topological manifold.

Remark. In Exercise 119.5 we will study the question, whether the Klein bottle is homoemorphic to the boundary of a compact 3-dimensional topological manifold M. It follows from the CW-structure on page 1471 that the Euler characteristic of the Klein bottle K is zero, therefore Corollary 119.9 (1) does not obstruct the existence of such an M.

Proof.

(1) Let M is a closed even-dimensional topological manifold that is homeomorphic to the boundary of a compact topological manifold W. Since W is odd-dimensional we obtain from the Odd-Dimensional Manifold-Euler Characteristic Proposition 119.7 that $\chi(M) = \chi(\partial W) = 2 \cdot \chi(W) \in 2 \cdot \mathbb{Z}$.

(2) This statement follows from Statement (1) together with the fact, shown on page 1879, that $\chi(\mathbb{R}\mathrm{P}^n) = \frac{1}{2}(1 + (-1)^n)$ and that $\chi(\mathbb{C}\mathrm{P}^n) = n + 1$. ■

Note that in the Non-Orientable $\mathbb{Z}_2$-Epimorphism Corollary 49.8 we showed that if N is a non-orientable topological manifold, then $\pi_1(N)$ admits an epimorphism onto $\mathbb{Z}_2$.[1462] It follows from the Hurewicz Theorem 84.5 that there exists also an epimorphism $\mathrm{H}_1(N;\mathbb{Z}) \to \mathbb{Z}_2$. For closed 3-dimensional topological manifolds the following Lemma 119.10 is thus a considerable strengthening of the Non-Orientable $\mathbb{Z}_2$-Epimorphism Corollary 49.8.

Lemma 119.10. Let M be a connected closed 3-dimensional topological manifold. If M is non-orientable, then $\mathrm{H}_1(M;\mathbb{Z})$ is infinite.

Proof. Let M be a connected closed 3-dimensional topological manifold. We have

$$0 \overset{\text{Odd-Dimensional Manifold-Euler Characteristic Proposition 119.7}}{=} \chi(M) \overset{\text{see page 2315}}{=} \sum_{n\in\mathbb{N}_0} (-1)^n \cdot b_n(M) \overset{\text{Theorems 106.1 and 106.3}}{=} b_0(M) - b_1(M) + b_2(M) \overset{\text{by the } \mathrm{H}_0\text{-Proposition 72.5}}{=} 1 - b_1(M) + b_2(M).$$

This equality can only hold if $b_1(M) > 0$, i.e. if $\mathrm{H}_1(M;\mathbb{Z})$ is infinite. ■

Definition. Let $f\colon M \to N$ be a map between closed, oriented connected non-empty n-dimensional topological manifolds. We consider the following diagram of maps:

$$\begin{array}{ccc} \mathrm{H}_k(M;\mathbb{Z}) & \underset{\cap[M]}{\overset{\mathrm{PD}_M}{\rightleftarrows}} & \mathrm{H}^{n-k}(M;\mathbb{Z}) \\ \downarrow{\scriptstyle f_*} & & \uparrow{\scriptstyle f^*} \\ \mathrm{H}_k(N;\mathbb{Z}) & \underset{\cap[N]}{\overset{\mathrm{PD}_N}{\rightleftarrows}} & \mathrm{H}^{n-k}(N;\mathbb{Z}). \end{array}$$

We refer to

$$\begin{array}{rcl} f^!\colon \mathrm{H}_k(N;\mathbb{Z}) & \to & \mathrm{H}_k(M;\mathbb{Z}) \\ \sigma & \mapsto & f^*(\mathrm{PD}_N(\sigma)) \cap [M] \end{array}$$

as the Umkehr map of f.[14631464]

The following lemma shows that in some circumstances the Umkehr map is just the transfer map that we introduced on page 1952.

[1462] If in the proof of the Non-Orientable $\mathbb{Z}_2$-Epimorphism Corollary 49.8 we replace the Orientation Covering-Proposition 49.5 by the Orientation Covering-Proposition 105.15 we see that the statement also holds for non-orientable topological manifolds.

[1463] This map is indeed called "Umkehr map" in English. The name stems from the fact that it is a map on homology that "goes the wrong way".

[1464] Wikipedia writes: "In category theory [...] certain unusual functors are denoted $f^!$ and $f_!$, with the exclamation mark used to indicate that they are exceptional in some way. They are thus accordingly sometimes called shriek maps, with "shriek" being slang for an exclamation mark [...]."

Lemma 119.11. Let $p\colon \widetilde{N} \to N$ be a finite cover of a closed, oriented, non-empty n-dimensional topological manifold. If we equip $\widetilde{N}$ with the orientation given by the Topological Manifold-Induced Orientation Lemma 105.17 (3), then for any $k \in \mathbb{N}_0$ the Umkehr map $p^!\colon \mathrm{H}_k(N;\mathbb{Z}) \to \mathrm{H}_k(\widetilde{N};\mathbb{Z})$ equals the transfer map p^*.

Proof. We start our proof with the following claim.

Claim 1. Let $r \leq s$. Given any singular simplex $\sigma\colon \Delta^s \to N$ and any cochain $\phi \in \mathrm{C}^r(N;\mathbb{Z})$ we have

$$p^*(\phi) \cap p^*(\sigma) \;=\; p^*(\phi \cap \sigma).$$

Proof. We write $m = [\widetilde{N} : N]$. We denote by $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_m$ the lifts of σ. Then

$$\begin{aligned} p^*(\phi) \cap p^*(\sigma) \;&=\; p^*(\phi) \cap \sum_{i=1}^m \widetilde{\sigma}_i \;=\; \sum_{i=1}^m p^*(\phi)(\widetilde{\sigma}_i \circ [v_0,\ldots,v_r]) \cdot \widetilde{\sigma}_i \circ [v_r,\ldots,v_s] \\ &= \sum_{i=1}^m \phi((p_* \circ \widetilde{\sigma}_i) \circ [v_0,\ldots,v_r]) \cdot \widetilde{\sigma}_i \circ [v_r,\ldots,v_s] \\ &\underset{\uparrow}{=} p^*\big(\phi(\sigma \circ [v_0,\ldots,v_r]) \cdot \sigma \circ [v_r,\ldots,v_s]\big) \;=\; p^*(\phi \cap \sigma) \end{aligned}$$

since $p_* \circ \widetilde{\sigma}_i = \sigma$ and since $\widetilde{\sigma}_i \circ [v_0,\ldots,v_r]$, $i = 1,\ldots,m$ are the lifts of $\sigma \circ [v_0,\ldots,v_r]$ ⊞

Now we turn to the actual proof of the lemma. We pick an n-cycle τ that is a representative for the fundamental class $[N] \in \mathrm{H}_n(N;\mathbb{Z})$ of N. By the Fundamental Class-Covering Proposition 106.25 we know that $p^*(\tau)$ represents the fundamental class of $\widetilde{N}$. By definition of the Umkehr map it suffices to prove the following claim.

Claim 2. For any cocycle $\phi \in \mathrm{C}^l(N;\mathbb{Z})$ we have

$$f^*(\phi) \cap p^*(\tau) \;=\; p^*(\phi \cap \tau).$$

Proof. This claim is an immediate consequence of the first claim. ■

We continue with the following useful and typical application of the Poincaré Duality Theorem 119.1.

Proposition 119.12. (Homotopy Sphere-Characterization Proposition) Let $n \geq 2$. Give a closed connected n-dimensional topological manifold X the following statements are equivalent:

(1) X is $\lfloor \frac{n}{2} \rfloor$-connected,
(2) X is $(n-1)$-connected,
(3) $\pi_1(X)$ is trivial and $\mathrm{H}_i(X;\mathbb{Z}) = 0$ for $i = 2,\ldots,n-1$,
(4) $\pi_1(X)$ is trivial and $\mathrm{H}_i(X;\mathbb{Z}) = 0$ for $i = 2,\ldots,\lfloor \frac{n}{2} \rfloor$.

Definition. A closed connected n-dimensional topological manifold M that satisfies any of the four equivalent statements in Proposition **??** is called a homotopy n-sphere. If M is actually a smooth manifold, then we refer to M as a smooth homotopy n-sphere.

Remark. In Corollaries 153.15 and Proposition **??** we will give alternative characterizations of homotopy spheres.

Proof. We have (1) ⇒ (4) by the Hurewicz Corollary 85.7. We can easily show that (4) ⇒ (3) by applying the Poincaré Duality Theorem 119.1 (here we use that X is orientable by

the Non-Orientable $\mathbb{Z}_2$-Epimorphism Corollary 105.16) and the Universal Coefficient Theorem 110.12 for Cohomology Groups.[1465] The conclusion (3) $\Rightarrow$ (2) is again the Hurewicz Corollary 85.7 and finally (2) $\Rightarrow$ (1) is trivial. ■

Finally we want to study degree-one maps between topological manifolds. We recall that on page 2379 we defined the degree of a map $f\colon M \to N$ between closed, oriented connected non-empty n-dimensional topological manifolds to be the unique integer $\deg(f) \in \mathbb{Z}$ with $f_*([M]) = \deg(f) \cdot [N]$. We had obtained the following results:

(1) In the Degree Realization Proposition 38.13 we showed that for any closed oriented non-empty n-dimensional topological manifold M there exists a degree-one map $M \to S^n$.[1466]
(2) In the Degree One-Epimorphism Proposition 101.8 we saw that if $\deg(f) = \pm 1$, then $f_*\colon \pi_1(M) \to \pi_1(N)$ is an epimorphism.
(3) In Lemma 117.5 we had used cup products to show that there does not exist a map $f\colon S^4 \to S^2 \times S^2$ of non-zero degree.

Now we add the following proposition to our list on results on the non-existence of degree-one maps.

Proposition 119.13. (Umkehr Map Proposition) Let $f\colon M \to N$ be a map between closed oriented non-empty n-dimensional topological manifolds. For every $k \in \mathbb{N}_0$ the following statements hold:

(1) The map $f_* \circ f^!$ is multiplication by $\deg(f)$.
(2) Suppose that f is a map of degree ± 1.
  (a) The map $f^!\colon \mathrm{H}_k(N;\mathbb{Z}) \to \mathrm{H}_k(M;\mathbb{Z})$ splits.
  (b) The image $f^!(\mathrm{H}_k(N;\mathbb{Z}))$ is a subsummand of $\mathrm{H}_k(M;\mathbb{Z})$, i.e. there exists a subgroup Q of $\mathrm{H}_k(M;\mathbb{Z})$ such that $\mathrm{H}_k(M;\mathbb{Z}) = f^!(\mathrm{H}_k(N;\mathbb{Z})) \oplus Q$.
  (c) There exists an isomorphism $\mathrm{H}_k(M;\mathbb{Z}) \cong \mathrm{H}_k(N;\mathbb{Z}) \oplus Q$ for some finitely generated abelian group Q.

Proof. Let $f\colon M \to N$ be a between closed oriented non-empty n-dimensional topological manifolds. We write $d = \deg(f)$. Let $\sigma \in \mathrm{H}_k(N;\mathbb{Z})$. We have

$$(f_* \circ f^!)(\sigma) \underset{\text{definition of } f^!}{=} f_*\big(f^*(\mathrm{PD}_N(\sigma)) \cap [M]\big) \underset{\substack{\text{Cap Product-Naturality} \\ \text{Lemma 116.8}}}{=} \mathrm{PD}_N(\sigma) \cap f_*([M]) \underset{\text{by definition of } d=\deg(f)}{=} d \cdot \mathrm{PD}_N(\sigma) \cap [N] \underset{\text{definition of } \mathrm{PD}_N}{=} d \cdot \sigma.$$

The second statement of the lemma follows from combining the first statement with the Splitting Lemma 78.2. ■

It follows from remark (2) above that there is no degree-one map from $\mathbb{R}\mathrm{P}^3$ to $S^1 \times S^2$. In Question 101.11 we had asked whether there exists a degree-one map from $S^1 \times S^2$ to $\mathbb{R}\mathrm{P}^3$. Now we can give a negative answer.

Corollary 119.14. There is no degree-one map from $S^1 \times S^2$ to $\mathbb{R}\mathrm{P}^3$.

[1465] Note that the proof of (4) $\Rightarrow$ (3) is almost identical to the proof of Lemma 148.1.

[1466] Basically the same proof, together with say the Orientation-Top Homology Theorem 106.1 (4) and an excision argument, shows that the same statement also holds for topological manifolds. We leave it to the reader to fill in the details.

Proof. The corollary follows immediately from the Umkehr Map Proposition 119.13 and the observation that the group $\mathrm{H}_1(\mathbb{R}\mathrm{P}^3;\mathbb{Z}) \cong \mathbb{Z}_2$ is not a subsummand of $\mathrm{H}_1(S^1 \times S^2;\mathbb{Z}) \cong \mathbb{Z}$. ■

We conclude this long list of applications of the Poincaré Duality Theorem 119.1 with the following result.

Corollary 119.15. Let $n \in \mathbb{N}_{\geq 2}$ and let M be a closed oriented connected n-dimensional topological manifold. If there exists a degree one map $f\colon S^n \to M$, then M is a homotopy n-sphere.

Proof. Let $f\colon S^n \to M$ be a degree one map. Recall that by the Degree One-Epimorphism Proposition 101.8 we know that $f_*\colon \pi_1(S^n) \to \pi_1(M)$ is an epimorphism. Thus we see that M is simply connected. Furthermore it follows from the Umkehr Map Proposition 119.13 that for $i = 2, \ldots, n-1$ the homology group $\mathrm{H}_i(M;\mathbb{Z})$ is isomorphic to a subgroup of $\mathrm{H}_i(S^n;\mathbb{Z})$. Since the latter is zero we obtain from the Homotopy Sphere-Characterization Proposition 119.12 that M is indeed a homotopy sphere. ■

119.3. **The proof of the Poincaré Duality Theorem 119.1 for topological manifolds without boundary.** In this section we will prove the Poincaré Duality Theorem 119.1 for closed topological manifolds. More precisely, we will prove the following theorem.

Theorem 119.16. Let M be a closed, non-empty n-dimensional topological manifold and let R be a commutative ring. We suppose that M is R-oriented. We denote by $[M] \in \mathrm{H}_n(M;R)$ the R-fundamental class. Then for each $k \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{PD}_M\colon \mathrm{H}^k(M;R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M;R) \\ \sigma & \mapsto & \sigma \cap [M] \end{array}$$

is an isomorphism.

In order to simplify the notation a little bit we will work throughout the section with $R = \mathbb{Z}$. The proof for other commutative rings is verbatim the same.

The key idea is somewhat similar to the strategy of the proof of the Homology-Section Theorem 106.10 . More precisely, using a Mayer–Vietoris argument and a limit argument we want to reduce the proof eventually to the fairly trivial case of an open ball in $\mathbb{R}^n$.
To carry out the above strategy we need a version of Poincaré Duality for every topological manifold (not necessarily compact) with empty boundary. Formulating this generalized Poincaré Duality Theorem requires some preparations. First we recall some definitions and results from earlier on:

(1) Given a topological space X we denote by $\mathcal{K}(X)$ the set of all compact subsets of X. Let $k \in \mathbb{N}_0$. By the $\mathrm{H}^n_{\mathrm{c}}$-via-Direct Limits Proposition 112.4 we have an identification

$$\varinjlim_{K \in \mathcal{K}(X)} \mathrm{H}^k(X, X \setminus K) \;=\; \mathrm{H}^k_{\mathrm{c}}(X).$$

(2) Let M be an n-dimensional topological manifold with empty boundary. If M is equipped with an orientation $\{\mu_x\}_{x \in M}$, then we saw in the Homology-Section Theorem 106.10 (i) that given any compact subset K of M there exists a unique class

$\mu_K^M \in \mathrm{H}_n(M, M \setminus K)$ such that for any $x \in K$ the image of μ_K^M under the map

$$\mathrm{H}_n(M, M \setminus K) \;\to\; \mathrm{H}_n(M, M \setminus \{x\})$$

equals the given orientation μ_x at x. If M is understood, then we drop it from the notation, i.e. we just write μ_K instead of μ_K^M.

Lemma 119.17. Let M be an oriented n-dimensional topological manifold with empty boundary.

(1) Let $K \subset L \subset M$ be compact subsets. We denote by $f_{KL} \colon (M, M \setminus L) \to (M, M \setminus K)$ the inclusion map of pairs.
   (a) The map $f_{KL*} \colon \mathrm{H}_n(M, M \setminus L) \to \mathrm{H}_n(M, M \setminus K)$ sends μ_L to μ_K.
   (b) For any $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k(M, M \setminus K) & \xrightarrow{\cap \mu_K} & \mathrm{H}_{n-k}(M). \\ \downarrow{\scriptstyle f_{KL}^*} & & \\ \mathrm{H}^k(M, M \setminus L) & \xrightarrow{\cap \mu_L} & \end{array}$$

(2) For any $k \in \mathbb{N}_0$ there exists a unique map

$$\mathrm{PD}_M \colon \mathrm{H}_{\mathrm{c}}^k(M) \;\to\; \mathrm{H}_{n-k}(M)$$

such that for each compact subset $K \subset M$ the following diagram commutes[1467]

$$\begin{array}{ccc} \mathrm{H}^k(M, M \setminus K) & \xrightarrow{\cap \mu_K} & \mathrm{H}_{n-k}(M). \\ & \searrow \quad \nearrow_{\mathrm{PD}_M} & \\ & \mathrm{H}_{\mathrm{c}}^k(M) & \end{array}$$

Proof. Let M be an oriented n-dimensional topological manifold with empty boundary.

(1) (a) This is precisely Corollary 106.12 (1).
   (b) Let $\varphi \in \mathrm{H}^k(M, M \setminus K)$. We see that

$$\varphi \cap \mu_K \underset{\text{part (1a)}}{=} \varphi \cap f_{KL*}(\mu_L) \underset{\text{Cap Product-Naturality Lemma 116.8}}{=} f_{KL}^*(\varphi) \cap \mu_L.$$

   But this means precisely that the diagram under consideration commutes.

(2) This statement follows immediately from (1b) and the universal property of the direct limit. ∎

We will prove the following generalization of Theorem 119.16.[1468] This more general version of Poincaré Duality will not only be useful for the divide-et-impera approach to the proof of Theorem 119.16, but it will also be of interest in its own right later on.

Theorem 119.18. (Non-Compact PD Theorem) Let M be an oriented n-dimensional topological manifold with empty boundary. Then for each $k \in \mathbb{N}_0$ the map

$$\mathrm{PD}_M \colon \mathrm{H}_{\mathrm{c}}^k(M) \;\xrightarrow{\cong}\; \mathrm{H}_{n-k}(M)$$

[1467] Note that for a class $\varphi \in \mathrm{H}^k(M, M \setminus K)$ and $\mu_K \in \mathrm{H}_n(M, M \setminus K)$ the cap product $\varphi \cap \mu_K$ does indeed lie in $\mathrm{H}_{n-k}(M)$.

[1468] Why is it a generalization of Theorem 119.16?

from Lemma 119.17 is an isomorphism.

Example. Let M be an oriented connected, non-empty n-dimensional topological manifold with empty boundary. Then

$$\begin{array}{ccc} \mathrm{H}_n(M;\mathbb{Z}) & \cong & \mathrm{H}^0_{\mathrm{c}}(M;\mathbb{Z}) \\ & & \uparrow \\ & & \text{the Non-Compact PD Theorem 119.18 see page 2480} \end{array} \qquad \begin{array}{c} \cong \\ \uparrow \end{array} \begin{cases} \mathbb{Z}, & \text{if } M \text{ is compact,} \\ 0, & \text{if } M \text{ is non-compact.} \end{cases}$$

This calculation is consistent with the results obtained in Theorems 106.1 and 106.3.

Remark.

(1) Let M be an oriented connected, non-empty n-dimensional topological manifold with empty boundary. Following the convention introduced on page 2618 we denote the inverse of the isomorphism $\mathrm{PD}_M \colon \mathrm{H}^k_{\mathrm{c}}(M) \xrightarrow{\cong} \mathrm{H}_{n-k}(M)$ also by PD_M.

(2) Given a possibly non-compact oriented topological manifold without boundary the Non-Compact PD Theorem 119.18 allows us to relate singular homology to cohomology with compact support. There is also a different version of Poincaré Duality that relates singular cohomology to a different flavor of homology, namely Borel-Moore homology. We refer to [**Greenb1967**], [**Bre1997**, Chapter V.9] and [**PS2008**, p. 332] for details.

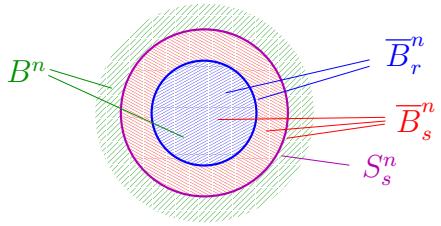
As a warm up for the proof of the Non-Compact PD Theorem 119.18 we will deal with the case $M = B^n$ by hand.[1469] More precisely, we will prove the following lemma.

Lemma 119.19. For each $k \in \mathbb{N}_0$ the map

$$\mathrm{PD}_{B^n} \colon \mathrm{H}^k_{\mathrm{c}}(B^n) \xrightarrow{\cong} \mathrm{H}_{n-k}(B^n)$$

is an isomorphism.

Proof. Let $k \in \mathbb{N}_0$. Given any $s \in (0,1)$ we write $\overline{B}^n_s := \{x \in \mathbb{R}^n \mid \|x\| \leq s\}$ and we also write $S^{n-1}_s := \partial \overline{B}^n_s$. For the remainder of the proof we fix $0 < r < s < 1$. We denote by $i \colon (\overline{B}^n_s, S^n_s) \to (B^n, B^n \setminus \overline{B}^n_r)$ the inclusion map of pairs.



We consider the following diagram:

$$\begin{array}{ccccc} \mathrm{H}^k\big(\overline{B}^n_s, S^n_s\big) & \xleftarrow{i^*} & \mathrm{H}^k\big(B^n, B^n \setminus \overline{B}^n_r\big) & \longrightarrow & \mathrm{H}^k_{\mathrm{c}}(B^n) \\ \downarrow{\scriptstyle \cap[\overline{B}^n_s]} & & \downarrow{\scriptstyle \cap \mu_{\overline{B}^n_r}} & & \downarrow{\scriptstyle \mathrm{PD}_{B^n}} \\ \mathrm{H}_{n-k}\big(\overline{B}^n_s\big) & \xrightarrow{i_*} & \mathrm{H}_{n-k}(B^n) & \xrightarrow{\mathrm{id}} & \mathrm{H}_{n-k}(B^n). \end{array}$$

[1469] We use the convention that any n-dimensional submanifold of $\mathbb{R}^n$ is equipped with the orientation induced from the standard orientation of $\mathbb{R}^n$, see the Topological Manifold-Induced Orientation Lemma 105.17.

The lemma is an immediate consequence of the following claim.

Claim.

(1) The diagram commutes,
(2) the left vertical map is an isomorphism, and
(3) all horizontal maps are isomorphisms.

Proof.

(1) The square on the right commutes by definition of PD_{B^n}. So it remains to show that the square on the left commutes. Thus let $\varphi \in \mathrm{H}^k(B^n, B^n \setminus \overline{B}^n_r)$. We have

$$i_*\big(i^*\varphi \cap [\overline{B}^n_s]\big) \;=\; \varphi \cap i_*([\overline{B}^n_s]) \qquad\qquad = \;\varphi \cap \mu_{\overline{B}^n_r} \;\in\; \mathrm{H}_{n-k}(B^n).$$

$\uparrow$ Cap Product-Naturality Lemma 116.8 $\qquad\qquad$ $\uparrow$ since the images of $[\overline{B}^n_s]$ and $\mu_{\overline{B}^n_r}$ in $\mathrm{H}_n(B^n, B^n \setminus \{0\})$ agree

(2) We consider the left vertical map of the diagram. Clearly both groups are zero for $k \neq n$. Evidently a generator for $\mathrm{H}^n(\overline{B}^n_s, S^n_s)$ is given by the dual fundamental class $[\overline{B}^n_s]^*$. Since $\overline{B}^n_s$ is path-connected we see, following the discussion on page 2558, that $[\overline{B}^n_s]^* \cap [\overline{B}^n_s] = \langle [\overline{B}^n_s]^*, [\overline{B}^n_s] \rangle = 1 \in \mathrm{H}_0(\overline{B}^n_s) = \mathbb{Z}$. In particular the left vertical map is an isomorphism.

(3) It follows easily from Corollary 74.16 and the Homotopic Maps-H*-Lemma 108.13 that the left horizontal maps are isomorphisms. It remains to show that the top right horizontal map is an isomorphism. We consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^k(B^n, B^n \setminus \overline{B}^n_r) & \longrightarrow & \varinjlim_{K \in \mathcal{K}(B^n)} \mathrm{H}^k(B^n, B^n \setminus K) = \mathrm{H}^k_{\mathrm{c}}(B^n). \\ & \searrow \quad \nearrow & \\ & \varinjlim_{t \in (0,1)} \mathrm{H}^k(B^n, B^n \setminus \overline{B}^n_t) & \end{array}$$

We make the following two observations:

(a) Every compact subset of B^n is contained in a closed ball $\overline{B}^n_t$ with $t \in (0,1)$. Therefore it follows from the Limit-over-Subset Lemma 111.3 that the right diagonal map is an isomorphism.
(b) It follows easily from Lemmas 108.13 and 111.2 (4) that the left diagonal map is an isomorphism.

These two observations together imply that the horizontal map is also an isomorphism. ∎

In the proof of the Non-Compact PD Theorem 119.18 we will need the following two lemmas.

Lemma 119.20. (Compact Subset Decomposition Lemma) Let M be a topological manifold and let U, V be open subsets such that $M = U \cup V$. Then given any compact subset $C \subset M$ there exist compact subsets $K \subset U$ and $L \subset V$ such that $C = K \cup L$.



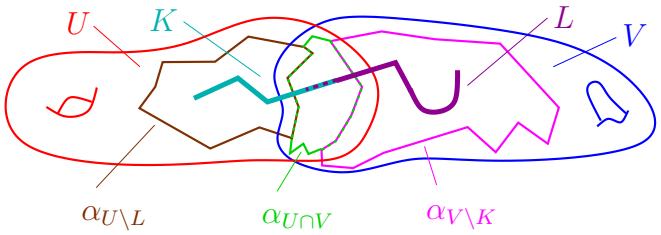
Proof. Let M be an n-dimensional topological manifold and let U, V be open subsets with $M = U \cup V$. Furthermore let C be a compact subset of M. Using charts one can easily show that given any point $x \in C$ there exists a compact set A_x with the following properties:

(1) the point x lies in the interior of A_x,
(2) if $x \in U$ then the set A_x is contained in U,
(3) if $x \in V$ then the set A_x is contained in V.

It follows from (1), the fact that the sets A_x cover all of C together with the hypothesis that C is compact that there exist finitely many $y_1, \ldots, y_r \in C$ with $C \subset \mathring{A}_{y_1} \cup \cdots \cup \mathring{A}_{y_r}$. One can now easily verify that $K := \bigcup_{y_i \in U} (A_{y_i} \cap C)$ and $L := \bigcup_{y_i \in V} (A_{y_i} \cap C)$ have the desired properties. ■

Lemma 119.21. (Fundamental Cycle Decomposition Lemma) Let M be a compact oriented n-dimensional topological manifold and let U, V be open subsets of M such that $M = U \cup V$ and let $K \subset U$ and $L \subset V$ be compact subsets. Then there exist singular n-chains $\alpha_{U\setminus L}$, $\alpha_{U\cap V}$ and $\alpha_{V\setminus K}$ in $U \setminus L$, $U \cap V$ and $V \setminus K$ such that the following three equalities hold:

(a)	$\mu^M_{K\cup L}$	$=$	$[\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}]$	in $\mathrm{H}_n(M, M \setminus (K \cup L))$
(b)	$\mu^{U\cap V}_{K\cap L}$	$=$	$[\alpha_{U\cap V}]$	in $\mathrm{H}_n(U \cap V, U \cap V \setminus (K \cap L))$
(c)	μ^U_K	$=$	$[\alpha_{U\setminus L} + \alpha_{U\cap V}]$	in $\mathrm{H}_n(U, U \setminus K)$.



Proof. We pick a representative $\alpha \in \mathrm{C}_n(M)$ for $\mu^M_{K\cup L}$. Note that the open sets $U \setminus L, U \cap V$ and $V \setminus K$ cover M. It follows from Lemmas 74.27 and 74.30 that there exist singular chains $\alpha_{U\setminus L}$, $\alpha_{U\cap V}$ and $\alpha_{V\setminus K}$ in $U \setminus L$, $U \cap V$ and $V \setminus K$ such that α is homologous in $\mathrm{C}_k(M)$ to $\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}$. The chains are in particular homologous in $\mathrm{C}_k(M, M \setminus (K \cup L))$ which implies that (a) holds.[1470] Next note that we have

$$\mu^M_{K\cap L} \underset{\text{Lemma 119.17}}{\uparrow\!=} \text{image of } \mu^M_{K\cup L} = [\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}] \underset{\text{since } \alpha_{U\setminus L}, \alpha_{V\setminus K} \text{ lie in } \mathrm{C}_n(M \setminus (K \cap L))}{\uparrow\!=} [\alpha_{U\cap V}] \in \mathrm{H}_n(M, M \setminus (K \cap L)).$$

[1470] Why is $\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}$ a cycle in $\mathrm{C}_*(M, M \setminus (K \cup L))$?

But $\alpha_{U\cap V}$ lies in $U\cap V$, therefore it follows from Corollary 106.12 (2) that $\mu_{K\cap L}^{U\cap V} = [\alpha_{U\cap V}]$ in $\mathrm{H}_n(U\cap V, U\cap V\setminus(K\cap L))$. We have thus verified (b).

Finally the proof of equality (c) is basically the same as the proof of equality (b). ■

Throughout the remainder of this section we will make use of the following notation.

Notation. Let $k\in\mathbb{N}_0$ and let X be a topological space.

(1) Given any subset A of X we denote by $i_*\colon \mathrm{H}_k(A)\to\mathrm{H}_k(X)$ and $i^*\colon \mathrm{H}_k(X)\to\mathrm{H}_k(A)$ the inclusion induced map.

(2) Suppose that X is Hausdorff and that U is an open subset of X. Then we denote by $i\colon \mathrm{H}^k_{\mathrm{c}}(U)\to\mathrm{H}^k_{\mathrm{c}}(X)$ the map defined by

$$\mathrm{H}^k_{\mathrm{c}}(U) = \varinjlim_{K\in\mathcal{K}(U)} \mathrm{H}^k(U, U\setminus K) \xleftarrow{\cong} \varinjlim_{K\in\mathcal{K}(U)} \mathrm{H}^k(X, X\setminus K) \to \varinjlim_{K\in\mathcal{K}(X)} \mathrm{H}^k(X, X\setminus K) = \mathrm{H}^k_{\mathrm{c}}(X).$$

We refer to Lemma 112.5 for the proof that these inclusion maps are covariantly functorial with respect to compositions of inclusions.

We recall the following notation and convention:

Convention. Let X be a topological space, let A be a subset and let $k\in\mathbb{N}_0$. As on page 2534 we make the identification

$$\mathrm{C}^k(X,A) \xrightarrow{=} \text{ all homomorphisms } \mathrm{C}_k(X)\to\mathbb{Z} \text{ that vanish on } \mathrm{C}_k(A).$$

Furthermore, given another subset B of X we write

$$\mathrm{C}^k(X,\{A,B\}) := \text{ all homomorphisms } \mathrm{C}_k(X)\to\mathbb{Z} \text{ that vanish on } \mathrm{C}_k(A) \text{ and on } \mathrm{C}_k(B).$$

If A and B are open subsets of X then we use the isomorphism

$$\mathrm{H}^k(X, A\cup B) \xrightarrow{\cong} \mathrm{H}^k(X,\{A,B\})$$

from Lemma 115.1 to identify the two groups.

In the proof of the Non-Compact PD Theorem 119.18 we will also need the following "inverted" Mayer–Vietoris Theorem, which is very similar to Theorem 78.7.

Theorem 119.22. (Inverted Mayer–Vietoris Theorem for Cohomology groups) Let X be a topological space and let A and B be open subsets.

(1) For every $n\in\mathbb{N}_0$ and every cochain $\varphi\in\mathrm{C}^n(X, A\cup B)$ there exist $\varphi_A\in\mathrm{C}^n(X,A)$ and $\varphi_B\in\mathrm{C}^n(X,B)$ such that $\varphi_A+\varphi_B=\varphi$.

(2) For all $n\in\mathbb{N}_0$ there exists a unique homomorphism

$$\delta\colon \mathrm{H}^n(X, A\cap B) \to \mathrm{H}^{n+1}(X, A\cup B)$$

which has the following property: for every $\varphi_A\in\mathrm{C}^n(X,A)$ and $\varphi_B\in\mathrm{C}^n(X,B)$ such that $\varphi_A+\varphi_B$ is a cocycle in $\mathrm{C}^n(X, A\cap B)$ we have

$$\delta([\varphi_A+\varphi_B]) = [\delta\varphi_A] = -[\delta\varphi_B] \in \mathrm{H}^{n+1}(X,\{A,B\}) = \mathrm{H}^{n+1}(X, A\cup B).$$

(3) The sequence

$$\begin{aligned}\ldots \to \mathrm{H}^n(X, A\cup B) &\xrightarrow{i^*\oplus -i^*} \mathrm{H}^n(X,A)\oplus\mathrm{H}^n(X,B) \xrightarrow{i^*\oplus i^*} \mathrm{H}^n(X, A\cap B)\\ &\xrightarrow{\delta} \mathrm{H}^{n+1}(X, A\cup B) \to \ldots\end{aligned}$$

is exact.

Proof. We start out with the following two observations:

(1) It is straightforward to verify that the following sequence of cochain maps between cochain complexes is exact:

$$0 \to \mathrm{C}^*(X, \{A, B\}) \xrightarrow{i^* \oplus -i^*} \mathrm{C}^*(X, A) \oplus \mathrm{C}^*(X, B) \xrightarrow{i^* \oplus i^*} \mathrm{C}^*(X, A \cap B) \to 0.$$

(2) Since the subsets A and B are open we know from the Excisive Triad-Proposition 109.12 (4) that the triad (X, A, B) is excisive. Therefore it follows from Lemma 115.1 that the obvious map $\mathrm{C}^*(X, A \cup B) \to \mathrm{C}^*(X, \{A, B\})$ induces for every $n \in \mathbb{N}_0$ an isomorphism $\mathrm{H}^n(X, A \cup B) \xrightarrow{\cong} \mathrm{H}^n(X, \{A, B\})$.

With these two observations it is straightforward to modify the proof of Theorem 78.5 to obtain the theorem. We leave it to the reader to fill in the details. ■

The key to the proof of the Non-Compact PD Theorem 119.18 is the following proposition.

Proposition 119.23. Let M be an oriented n-dimensional topological manifold with empty boundary that is the union of two open sets U and V. Then there exists a diagram

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & \mathrm{H}^k_{\mathrm{c}}(U \cap V) & \xrightarrow{i \oplus -i} & \mathrm{H}^k_{\mathrm{c}}(U) \oplus \mathrm{H}^k_{\mathrm{c}}(V) & \xrightarrow{i+i} & \mathrm{H}^k_{\mathrm{c}}(M) & \longrightarrow & \mathrm{H}^{k+1}_{\mathrm{c}}(U \cap V) & \longrightarrow \ldots \\
& \downarrow{\scriptstyle \mathrm{PD}_{U \cap V}} & & \downarrow{\scriptstyle \mathrm{PD}_U \oplus \mathrm{PD}_V} & & \downarrow{\scriptstyle \mathrm{PD}_M} & & \downarrow{\scriptstyle \mathrm{PD}_{U \cap V}} & \\
\ldots \longrightarrow & \mathrm{H}_{n-k}(U \cap V) & \xrightarrow{i \oplus -i} & \mathrm{H}_{n-k}(U) \oplus \mathrm{H}_{n-k}(V) & \xrightarrow{i+i} & \mathrm{H}_{n-k}(M) & \longrightarrow & \mathrm{H}_{n-k-1}(U \cap V) & \longrightarrow \ldots
\end{array}$$

with the following properties:

(1) the first two squares commute, the third square commutes up to the sign $(-1)^{k+1}$,
(2) the lower exact sequence is the long exact Mayer–Vietoris sequence for $M = U \cup V$,
(3) the upper horizontal sequence is also exact.

Proof. Let M be an oriented n-dimensional topological manifold with empty boundary. Throughout the proof, given any two subsets $X \subset Y$ and $i \in \mathbb{N}_0$ we write as before $\mathrm{H}^i(Y|X) := \mathrm{H}^i(Y, Y \setminus X)$. We note that if U is open and $K \subset U$ is compact we obtain a map

$$- \cap \mu^U_K \colon \mathrm{H}^k(U|K) \to \mathrm{H}_{n-k}(U).$$

Now let let U and V be open subsets with $M = U \cup V$. Furthermore let $K \subset U$ and $L \subset V$ be compact subsets. We consider the following diagram

$$\begin{array}{ccccccccc}
\ldots \to & \mathrm{H}^k(M|K \cap L) & \to & \mathrm{H}^k(M|K) \oplus \mathrm{H}^k(M|L) & \longrightarrow & \mathrm{H}^k(M|K \cup L) & \xrightarrow{\delta} & \mathrm{H}^{k+1}(M|K \cap L) & \to \ldots \\
& \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & & & \downarrow{\scriptstyle \cong} & \\
& \mathrm{H}^k(U \cap V|K \cap L) & & \mathrm{H}^k(U|K) \oplus \mathrm{H}^k(V|L) & & \downarrow{\scriptstyle \cap \mu^M_{K \cup L}} & & \mathrm{H}^{k+1}(U \cap V|K \cap L) & \\
& \downarrow{\scriptstyle \cap \mu^{U \cap V}_{K \cap L}} & & \downarrow{\scriptstyle \cap \mu^U_K \oplus \cap \mu^V_L} & & & & \downarrow{\scriptstyle \cap \mu^{U \cap V}_{K \cap L}} & \\
\ldots \to & \mathrm{H}_{n-k}(U \cap V) & \longrightarrow & \mathrm{H}_{n-k}(U) \oplus \mathrm{H}_{n-k}(V) & \longrightarrow & \mathrm{H}_{n-k}(M) & \xrightarrow{\partial} & \mathrm{H}_{n-k-1}(U \cap V) & \to \ldots
\end{array}$$

which we now explain in more detail:

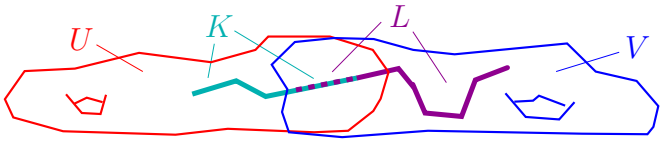
(1) We write $A = M \setminus K$ and $B = M \setminus L$. The upper horizontal sequence is the inverted Mayer–Vietoris Sequence for cohomology groups from Theorem 119.22.
(2) The lower horizontal sequence is the Mayer-Vietoris sequence coming from Theorem 78.5, corresponding to the decomposition $M = U \cup V$.

(3) The three vertical maps on the top are induced by inclusions of pairs of topological spaces, they are isomorphisms by the Excision Theorem 109.1 for Cohomology Groups.

The following is the key claim of the proof.

Claim. The first two rectangles commute and the third rectangle commutes up to the sign $(-1)^{k+1}$.

$M = U \cup V$



Before we prove the claim we show that it implies our lemma. For each group appearing in the above diagram we take the direct limit over the directed set[1471]

$$\mathcal{K}(U,V) \;=\; \{(K,L) \mid K \text{ compact subset of } U \text{ and } L \text{ compact subset of } V\}$$

and we consider the induced maps of direct limits that were introduced on page 1753. We make the following remarks regarding this new diagram:

(1) Using the Cofinal-Isomorphism Lemma 111.4 one can show that the groups that we obtain taking the direct limit over $\mathcal{K}(U,V)$ are naturally isomorphic to the corresponding cohomology groups with compact support. We carry out the argument for two groups:

$$\varinjlim_{(K,L)\in\mathcal{K}(U,V)} \mathrm{H}^k(U\cap V|K\cap L) \;=\; \varinjlim_{C\in\mathcal{K}(U\cap V)} \mathrm{H}^k(U\cap V|C) \;=\; \mathrm{H}^k_{\mathrm{c}}(U\cap V).$$

$\uparrow$ we apply the Cofinal-Isomorphism Lemma 111.4 to the cofinal morphism of directed sets given by $(K,L)\mapsto K\cap L$

and

$$\varinjlim_{(K,L)\in\mathcal{K}(U,V)} \mathrm{H}^k(M|K\cup L) \;=\; \varinjlim_{C\in\mathcal{K}(M)} \mathrm{H}^k(M|C) \;=\; \mathrm{H}^k_{\mathrm{c}}(M).$$

$\uparrow$ we apply the Cofinal-Isomorphism Lemma 111.4 to the morphism of directed sets given by $(K,L)\mapsto K\cup L$, this morphism is cofinal by the Compact Subset Decomposition Lemma 119.20

All other cases are treated in a very similar fashion.

(2) By the functoriality of direct limits, and since the signs in the commutativity do not depend on K and L, the new diagram also commutes (up to the same signs).

(3) Using (2) and the definition of the "PD-maps" in Lemma 119.17 one can easily verify that the direct limit of the diagram does indeed give rise to the diagram specified in the statement of the lemma.

(4) the Direct Limit-Exactness Proposition 79.2 says that the direct limit of exact sequences is again exact.

All these remarks put together show that the direct limit of the above diagram gives rise precisely to the diagram that we claimed exists and that all claimed statements are satisfied. This concludes the proof that the claim implies the lemma.

[1471] We view $\mathcal{K}(U,V)$ as a directed set given by the relation $(K,L)\leq(K',L')$ if $K\subset K'$ and $L\subset L'$. It is clear that each of the groups in the diagram forms a direct system with respect to this directed set.

Proof. Now we turn to the actual proof of the claim. We show in three separate subclaims that the three rectangles commute. The proof of the first two subclaims consists of a clever usage of the Cap Product-Naturality Lemma 116.8. The proof of the third subclaim is much more sophisticated since it relies on understanding the connecting homomorphisms in the Mayer-Vietoris sequences in (co-) homology.

Subclaim 1. The first rectangle of the diagram on page 2633 commutes.

So we need to show that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{H}^k(M, M\setminus(K\cap L)) & \xrightarrow{i\oplus -i} & \mathrm{H}^k(M, M\setminus K)\oplus \mathrm{H}^k(M, M\setminus L) \\
\downarrow\cong & \searrow^{i\oplus -i} & \downarrow\cong \\
\mathrm{H}^k(U\cap V, (U\cap V)\setminus(K\cap L)) & & \mathrm{H}^k(U, U\setminus K)\oplus \mathrm{H}^k(V, V\setminus L) \\
\downarrow\cap\mu^{U\cap V}_{K\cap L} & & \downarrow\cap\mu^U_K\oplus\cap\mu^V_L \\
\mathrm{H}_{n-k}(U\cap V) & \xrightarrow{i\oplus -i} & \mathrm{H}_{n-k}(U)\oplus \mathrm{H}_{n-k}(V).
\end{array}$$

Evidently the triangle commutes. We note that the remaining pentagon breaks up into the "direct sum" of two diagrams. By symmetry reasons it suffices to consider the left-hand summands. So let $\phi\in \mathrm{H}^k(M, M\setminus(K\cap L))$. We consider the following diagram of maps between pairs of topological spaces

$$\begin{array}{ccc}
(U\cap V, (U\cap V)\setminus(K\cap L)) & \xrightarrow{\alpha} & (U, U\setminus(K\cap L)) \\
\downarrow{\scriptstyle f\circ\alpha} & \swarrow_{f} & \uparrow{\scriptstyle \beta} \\
(M, M\setminus(K\cap L)) & \xleftarrow[f\circ\beta]{} & (U, U\setminus K).
\end{array}$$

With this notation we need to show that

$$\alpha_*\big((f\circ\alpha)^*(\phi)\cap\mu^{U\cap V}_{K\cap L}\big) \;=\; (f\circ\beta)^*(\phi)\cap\mu^U_K.$$

But this is indeed the case, since

$$\begin{array}{ccccc}
\alpha_*\big(\underbrace{(f\circ\alpha)^*}_{=\alpha^*\circ f^*}(\phi)\cap\mu^{U\cap V}_{K\cap L}\big) & = & f^*(\phi)\cap\alpha_*(\mu^{U\cap V}_{K\cap L}) & = & f^*(\phi)\cap\mu^U_{K\cap L} \\
& & \uparrow & & \uparrow \\
& & \text{Cap Product-Naturality Lemma 116.8} & & \text{by Corollary 106.12 we have } \alpha_*(\mu^{U\cap V}_{K\cap L})=\mu^U_{K\cap L} \\
& & = \quad \beta_*\big((f\circ\beta)^*(\phi)\cap\mu^U_K\big) & = & (f\circ\beta)^*(\phi)\cap\mu^U_K \\
& & \uparrow & & \uparrow \\
& & \text{same argument backwards with the roles of } \alpha \text{ and } \beta \text{ exchanged} & & \text{since } \beta \text{ is on the first coordinate the identity}
\end{array}$$

This concludes the proof of Subclaim 1.

Subclaim 2. The second rectangle of the diagram on page 2633 commutes.

The proof of that subclaim is very similar to the proof of the previous subclaim. Thus we leave it to the reader to fill in the details.

Subclaim 3. The third rectangle of the diagram on page 2633 commutes up to the sign $(-1)^{k+1}$.

As a reminder, we need to show that the following diagram commutes up to the sign $(-1)^{k+1}$:

$$\begin{array}{ccccc} \mathrm{H}^k(M, M\setminus(K\cup L)) & \xrightarrow{\delta} & \mathrm{H}^{k+1}(M, M\setminus(K\cap L)) & \xrightarrow{\cong} & \mathrm{H}^{k+1}(U\cap V, (U\cap V)\setminus(K\cap L)) \\ \downarrow{\scriptstyle \cap\mu_{K\cup L}} & & & & \downarrow{\scriptstyle \cap\mu_{K\cap L}} \\ \mathrm{H}_{n-k}(M) & & \xrightarrow{\qquad\qquad\partial\qquad\qquad} & & \mathrm{H}_{n-k-1}(U\cap V). \end{array}$$

So let $[\varphi] \in \mathrm{H}^k(M, M\setminus(K\cup L)) = \mathrm{H}^k(M, A\cap B)$. We start out with the following preparations:

(1) By Theorem 119.22 (1) we can find $\varphi_A \in \mathrm{C}^k(M, A)$ and $\varphi_B \in \mathrm{C}^k(M, B)$ such that $\varphi = \varphi_A + \varphi_B$.

(2) By the Fundamental Cycle Decomposition Lemma 119.21 there exist singular chains $\alpha_{U\setminus L}$, $\alpha_{U\cap V}$ and $\alpha_{V\setminus K}$ in $U\setminus L$, $U\cap V$ and $V\setminus K$ such that the following three equalities hold:

(2a) $\mu_{K\cup L} = [\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}]$ in $\mathrm{H}_n(M, M\setminus(K\cup L))$

(2b) $\mu_{K\cap L}^{U\cap V} = [\alpha_{U\cap V}]$ in $\mathrm{H}_n(U\cap V, (U\cap V)\setminus(K\cap L))$

(2c) $\mu_K^U = [\alpha_{U\setminus L} + \alpha_{U\cap V}]$ in $\mathrm{H}_n(U, U\setminus K)$.

After these preparations we obtain the following equalities in $\mathrm{H}_{n-k-1}(U\cap V)$:[1472]

$$\begin{array}{l} \text{maps in the rectangle applied clockwise} \\ \text{to some } [\varphi]\in \mathrm{H}^k(M, M\setminus(K\cup L)) \end{array} = \delta[\varphi]\cap\mu_{K\cap L}^{U\cap V} \overset{\text{Theorem 119.22 and (2b)}}{=} [\delta\varphi_A \cap \alpha_{U\cap V}]$$

$$= [\varphi_A\cap\partial\alpha_{U\cap V}] = [\varphi_A\cap\partial\alpha_{U\cap V} - \underbrace{\varphi_A\cap\partial(\alpha_{U\cap V}+\alpha_{U\setminus L})}_{=0}] = [-\varphi_A\cap\partial\alpha_{U\setminus L}]$$

Lemma 116.1; note that (2c) implies that $\partial(\alpha_{U\cap V}+\alpha_{U\setminus L})$ is a singular chain in $U\setminus K$, but φ_A vanishes on singular chains in $A = M\setminus K$

$$= [-\varphi_A\cap\partial\alpha_{U\setminus L} - \underbrace{\varphi_B\cap\partial\alpha_{U\setminus L}}_{=0}] = [-(\varphi_A+\varphi_B)\cap\partial\alpha_{U\setminus L}] = [-\varphi\cap\partial\alpha_{U\setminus L}]$$

since φ_B vanishes on singular chains in $B = M\setminus L$; since $\varphi = \varphi_A + \varphi_B$

$$= [(-1)^{k+1}\cdot\partial(\varphi\cap\alpha_{U\setminus L})] = (-1)^{k+1}\cdot\partial[\underbrace{(\varphi\cap\alpha_{U\setminus L})}_{\in \mathrm{C}_{n-k}(U)} + \underbrace{(\varphi\cap\alpha_{U\cap V}+\varphi\cap\alpha_{V\setminus K})}_{\in \mathrm{C}_{n-k}(V)}]$$

by Lemma 116.1 and since $\delta\varphi = 0$; by the definition of the connecting homomorphism ∂ in the Mayer–Vietoris sequence, see Theorem 78.5 (1)

$$= (-1)^{k+1}\cdot\partial\big([\varphi]\cap\mu_{K\cup L}^M\big) = (-1)^{k+1}\cdot \text{maps in the rectangle applied counterclockwise to } [\varphi].$$

by equality (2a)

This concludes the proof of Subclaim 3, thus of the claim and thus also of the lemma. ■

Proposition 119.23 actually contains an interesting and useful statement which is worth jotting down separately.

[1472] The maps δ and ∂ written in blue are the connecting homomorphisms in the Mayer–Vietoris sequences.

Lemma 119.24. Let M be an oriented n-dimensional topological manifold with empty boundary and let U be an open subset of M. We denote by $i\colon U \to M$ the inclusion. For any $k \in \mathbb{N}_0$ the following two diagrams commute:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{c}}(U) & \xrightarrow{i_*} & \mathrm{H}^k_{\mathrm{c}}(M) \\ \downarrow{\scriptstyle \mathrm{PD}_U} & & \downarrow{\scriptstyle \mathrm{PD}_M} \\ \mathrm{H}_{n-k}(U) & \xrightarrow{i_*} & \mathrm{H}_{n-k}(M) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{H}_k(U) & \xrightarrow{i_*} & \mathrm{H}_k(M) \\ \downarrow{\scriptstyle \mathrm{PD}_U} & & \downarrow{\scriptstyle \mathrm{PD}_M} \\ \mathrm{H}^{n-k}_{\mathrm{c}}(U) & \xrightarrow{i_*} & \mathrm{H}^{n-k}_{\mathrm{c}}(M). \end{array}$$

Proof. The statement that the diagram on the left commutes for any k is contained in Proposition 119.23 (1). Since the maps PD_U and PD_M are isomorphisms the statement that the diagram on the right commutes is just a reformulation of the fact that the diagrams on the left commutes. ■

Now we turn to the actual proof of the Non-Compact PD Theorem 119.18.

Proof of the Non-Compact PD Theorem 119.18. We fix a dimension n. Throughout this proof by a "manifold" we mean an oriented n-dimensional topological manifold without boundary. We say *Poincaré Duality holds for a manifold M* if for each $k \in \mathbb{N}_0$ the map

$$\mathrm{PD}_M\colon \mathrm{H}^k_{\mathrm{c}}(M) \;\to\; \mathrm{H}_{n-k}(M)$$

from Lemma 119.17 is an isomorphism. Our goal is to show that Poincaré Duality holds for all manifolds.

We start out with the following claim.

Claim 1. Let W be a manifold.

(A) If Poincaré Duality holds for W equipped with an orientation, then it also holds for any other orientation of W.
(B) If W is homeomorphic to a manifold for which Poincaré Duality holds, then Poincaré Duality also holds for W.
(C) Suppose that W is the union of two open subsets U and V. If Poincaré Duality holds for U, V and $U \cap V$, then it also holds for W.
(D) If W is the union of a nested sequence $U_1 \subset U_2 \subset \ldots$ of open subsets such that Poincaré Duality holds for each U_i, then it also holds for W.

Proof. So let W be a manifold.

(A) Clearly it suffices to consider the case that W is path-connected. By Lemma 105.10 the path-oriented connected manifold W admits only one other orientation, namely the reverse orientation. Evidently Poincaré Duality also holds for that orientation.
(B) If we have an orientation-preserving homeomorphism, then this statement is obvious. For a non-orientation-preserving homeomorphism this statement follows from (A).
(C) If all maps $\mathrm{PD}_U, \mathrm{PD}_V$ and $\mathrm{PD}_{U\cap V}$ are isomorphisms, then it follows immediately from Proposition 119.23 and the Five Lemma 74.10 that also each PD_W is an isomorphism, i.e. Poincaré Duality holds for $W = U \cup V$.
(D) Let $k \in \mathbb{N}_0$. We suppose that for every $i \in \mathbb{N}$ the map $\mathrm{PD}_{U_i}\colon \mathrm{H}^k_{\mathrm{c}}(U_i) \to \mathrm{H}_{n-k}(U_i)$ is an isomorphism. Using Lemmas 116.8 and 119.17 it is straightforward to verify that

for any $i \leq j$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_{\mathrm{c}}^{k}(U_i) & \xrightarrow{\mathrm{PD}_{U_i}} & \mathrm{H}_{n-k}(U_i) \\ \downarrow & & \downarrow \\ \mathrm{H}_{\mathrm{c}}^{k}(U_j) & \xrightarrow{\mathrm{PD}_{U_j}} & \mathrm{H}_{n-k}(U_j). \end{array}$$

We obtain isomorphisms

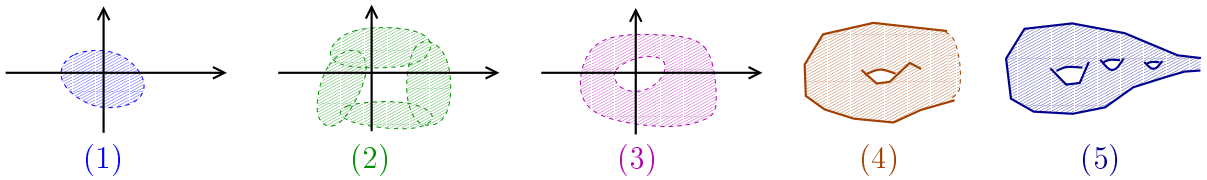
$$\begin{array}{ccccccc} \mathrm{H}_{\mathrm{c}}^{k}(W) & = & \varinjlim \mathrm{H}_{\mathrm{c}}^{k}(U_i) & \xrightarrow{\varinjlim \mathrm{PD}_{U_i}} & \varinjlim \mathrm{H}_{n-k}(U_i) & = & \mathrm{H}_{n-k}(W). \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{Exhaustion-H}_{\mathrm{c}}^{*}\text{-Proposition 112.6} & & \text{isomorphism by our hypothesis and since the above diagram commutes} & & & & \text{Exhaustions-H}_{*}\text{-Proposition 79.4} \end{array}$$

We leave it to the reader to verify that the map from left to right is indeed given by PD_W. ⊞

The following claim completes the proof of the theorem.

Claim 2.

(1) Poincaré Duality holds for every open bounded convex subset of $\mathbb{R}^n$.
(2) For every $m \in \mathbb{N}$ Poincaré Duality holds for the union of m open bounded convex subsets of $\mathbb{R}^n$.
(3) Poincaré Duality holds for every open subset of $\mathbb{R}^n$.
(4) For every $m \in \mathbb{N}$ Poincaré Duality holds for all manifolds that admit an atlas with m charts.
(5) Poincaré Duality holds for all manifolds.



Proof.

(1) Let M be an open bounded convex subset of $\mathbb{R}^n$. By the Convex-to-Ball Proposition 2.20 (1) there exists a homeomorphism $M \to B^n$. It follows from Lemma 119.19 and (B) that Poincaré Duality also holds for M.
(2) We prove the claim by induction on m. By (1) we know that the statement holds for $m = 1$. Now suppose it holds for all unions of $m-1$ open bounded convex subsets of $\mathbb{R}^n$. Let M be the union of m open bounded convex subsets $U_1, \ldots, U_m$. We set $V = U_1 \cup \cdots \cup U_{m-1}$. By our hypothesis and by induction the statement holds for U_m, for V and also for $V \cap U_m = (U_1 \cap U_m) \cup \cdots \cup (U_{m-1} \cap U_m)$, which is the union of $(m-1)$ open bounded convex subsets.[1473] Hence by (C) the statement also holds for

$$M \;=\; V \cup U_m \;=\; (U_1 \cup \cdots \cup U_{m-1}) \cup U_m.$$

[1473]Here we once again use the convenient fact that the intersection of two convex sets is convex.

(3) Let M be an arbitrary open subset of $\mathbb{R}^n$. By the Rational Open Balls Lemma 118.4 we can write M as the union of countably many open balls $B_i, i \in \mathbb{N}$. For $i \in \mathbb{N}$ we set $U_i := \bigcup_{j \leq i} B_j$. By (2) Poincaré Duality holds for each $U_i \in \mathbb{N}$. But then it follows from (D) that Poincaré Duality also holds for M.

(4) We prove this statement by induction on m. The case $m = 1$ is taken care of by (3) and (B). Now suppose Poincaré Duality holds for all manifolds that admit an atlas with $m-1$ charts. Let M be a manifold that admits an atlas $\{\phi_i \colon U_i \to V_i\}_{i=1,\ldots,m}$ with m charts. We write $U = U_1 \cup \cdots \cup U_{m-1}$ and $V = U_m$. Both are open subsets of a manifold, hence they are also manifolds. Furthermore U and V are manifolds that admit an atlas with $m-1$ charts respectively one chart. Similarly $U \cap V = (U_1 \cap U_m) \cup \cdots \cup (U_{m-1} \cap U_m)$ is a manifold that admits an atlas with $m-1$ charts. By induction Poincaré Duality holds for U, V and $U \cap V$. Thus by (C) Poincaré Duality also holds for M itself.

(5) Finally let M be any manifold. By the Countable Covering Lemma 9.2 we know that M admits a countable atlas $\{\Phi_j \colon U_j \to V_j\}_{j \in \mathbb{N}}$. For $i \in \mathbb{N}$ we set $X_i := \bigcup_{j \leq i} U_j$. By (4) Poincaré Duality holds for each X_i. It follows from (D) that Poincaré Duality also holds for M. ■

119.4. **The proof of Poincaré Duality for topological manifolds with boundary.** Now we will finally prove the general statement of the Poincaré Duality Theorem 119.1. For the reader's convenience we recall the statement.

Theorem 119.1. (Poincaré Duality Theorem) Let M be a compact, non-empty n-dimensional topological manifold and let R be a commutative ring. We suppose that M is R-oriented. We denote by $[M] \in \mathrm{H}_n(M, \partial M; R)$ the R-fundamental class. Furthermore suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. Then for each $k \in \mathbb{N}_0$ the map

$$\begin{array}{ccc} \mathrm{H}^k(M, A; R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M, B; R) \\ \sigma & \mapsto & \sigma \cap [M] \end{array}$$

is an isomorphism.

Proof of Theorem 119.1 for $A = \partial M$ and $B = \varnothing$. In this proof we consider the case that $A = \partial M$ and $B = \varnothing$. To simplify the notation we will once again only consider the case that $R = \mathbb{Z}$. The proof of the general case is basically the same. We introduce the following objects:

(i) From the Topological Collar Neighborhood Theorem 75.5 we obtain a collar neighborhood $[-1, 0] \times \partial M$ of $\partial M = \{0\} \times M$. We refer to it as the "internal collar".

(ii) We define

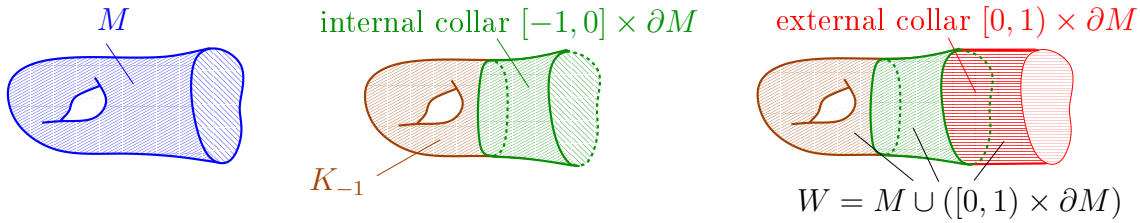
$$W := M \cup ([0, 1) \times \partial M)/\sim \qquad \text{with } x \sim (x, 0) \text{ for every } x \in \partial M.$$

Sometimes $[0, 1) \times \partial M$ is called an *external collar of* M. Furthermore, given any $t \in [-1, 1)$ we write

$$K_t := (M \setminus ([-1, 0] \times \partial M)) \cup ([-1, t] \times \partial M)/\sim .$$

(iii) We denote by $i\colon X \to W$ the obvious map. By the Topological Pushout-Maps Lemma 5.34 (3) and (4) this map is a closed embedding. We will use this to view X as a closed subset of W. Note that it follows easily from the Homotopy Pushout Lemma 14.9 that $X = i(X)$ is a deformation retract of W.

(iv) We denote by $i\colon (X, \partial X) \to (W, W \setminus K_{-1})$ the inclusion map.



By elementary arguments and using the Topological Manifold Boundary Proposition 75.2 (1) one can easily show the following statements:

(A) W is an n-dimensional topological manifold with empty boundary.

(B) W admits an orientation such that M is a submanifold and such that $M \to W$ is orientation-preserving.

(C) Every K_t is a compact subset of W and given any compact subset K of W there exists a $t \in [-1, 1)$ with $K \subset K_t$.

We denote by $i\colon (M, \partial M) \to (W, W \setminus K_{-1})$ the obvious inclusion map. We obtain the following diagram:

$$\begin{array}{ccccc} \mathrm{H}^k(M, \partial M) & \xleftarrow{i^*} & \mathrm{H}^k(W, W \setminus K_{-1}) & \longrightarrow & \mathrm{H}^k_{\mathrm{c}}(W) \\ \downarrow{\scriptstyle \cap [M]} & & \downarrow{\scriptstyle \cap \mu_{K_t}} & & \downarrow{\scriptstyle \mathrm{PD}_W} \\ \mathrm{H}_{n-k}(M) & \xrightarrow{i_*} & \mathrm{H}_{n-k}(W) & \xrightarrow{\mathrm{id}} & \mathrm{H}_{n-k}(W). \end{array}$$

It remains to prove the following claim which resembles closely the claim in the proof of Lemma 119.19.

Claim. The following three statements hold:

(1) the diagram commutes,

(2) the right vertical map is an isomorphism, and

(3) all horizontal maps are isomorphisms.

Proof.

(1) The right-hand square commutes by definition of PD_W. Using (B) we can apply the same argument as in the proof of the claim in Lemma 119.19 to show that the left-hand square commutes.

(2) the Non-Compact PD Theorem 119.18 implies that the right vertical map is an isomorphism.

(3) It follows easily from the discussion in (iii) together with Corollary 74.16 and the Homotopic Maps-H*-Lemma 108.13 that the left horizontal maps are isomorphisms. Note that the bottom right horizontal map is the identity. Thus it remains to show that the top right horizontal map is an isomorphism. Similar to the proof

of Lemma 119.19 we consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^k(W, W \setminus K_{-1}) & \longrightarrow & \varinjlim_{K \in \mathcal{K}(W)} \mathrm{H}^k(W, W \setminus K) = \mathrm{H}^k_{\mathrm{c}}(W). \\ & \searrow \quad \nearrow & \\ & \varinjlim_{t \in [-1,1)} \mathrm{H}^k(W, W \setminus K_t) & \end{array}$$

We make the following two observations:

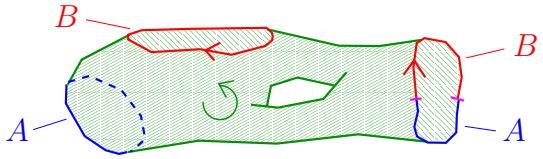
(a) By (C) every compact subset of W is contained in some K_t. Therefore it follows from the Limit-over-Subset Lemma 111.3 that the right diagonal map is an isomorphism.

(b) It follows easily from Lemmas 108.13 and 111.2 (4) that the left diagonal map is an isomorphism.

These two observations together imply that the horizontal map is also an isomorphism.

This concludes the proof of the claim and it concludes the proof of the Poincaré Duality Theorem 119.1 in the special case that $A = \partial M$ and $B = \varnothing$. ■

Proof of Theorem 119.1 for the general case. Now we finally consider the general case of Theorem 119.1. So we suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. We recall that by the Excisive Triad-Proposition 109.12 (5) we know that the triad (M, A, B) is excisive. In this proof, once again we simplify the notation and we only work with $\mathbb{Z}$-coefficients.



We equip ∂M and then also B with the orientations coming from the Topological Manifold-Boundary Orientation Proposition 105.19 and the Topological Manifold-Induced Orientation Lemma 105.17. We consider the following diagram

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \mathrm{H}^k(M, \partial M) & \longrightarrow & \mathrm{H}^k(M, A) & \longrightarrow & \mathrm{H}^k(\partial M, A) & \xrightarrow{\delta} & \mathrm{H}^{k+1}(M, \partial M) & \longrightarrow & \ldots \\ & & \cong \downarrow -\cap[M] & & \downarrow -\cap[M] & & \cong \downarrow i^* & & \cong \downarrow -\cap[M] & & \\ & & & & & & \mathrm{H}^k(B, \partial B) & & & & \\ & & & & & & \cong \downarrow -\cap[B] & & & & \\ \ldots & \longrightarrow & \mathrm{H}_{n-k}(M) & \longrightarrow & \mathrm{H}_{n-k}(M, B) & \xrightarrow{\partial} & \mathrm{H}_{n-k-1}(B) & \longrightarrow & \mathrm{H}_{n-k-1}(M) & \longrightarrow & \ldots \end{array}$$

We make the following observations:

(1) The top horizontal sequence is the long exact sequence in cohomology of the triple $(M, \partial M, A)$ provided by the Cohomological LES-Lemma 108.15.

(2) The bottom horizontal sequence is the long exact sequence in homology of the pair (M, B) provided by the Topological LES-Corollary 74.14.

(3) The above vertical map $i^*\colon \mathrm{H}^k(\partial M, A) \to \mathrm{H}^k(B, \partial B)$ is induced by the inclusion $i\colon (B, \partial B) \to (\partial M, A)$. It is an isomorphism by the Excisive Triad-Proposition 109.12 (5) together with the Excisive Characterization Lemma 109.13 (3).
(4) The other maps decorated with "$\cong$" are isomorphisms by the special case of the Poincaré Duality Theorem 119.1 that we have already dealt with.

By the Five Lemma 74.10 it remains to prove the following claim.

Claim. The above diagram commutes up to signs, more precisely:
(1) the first rectangle commutes,
(2) the second rectangle commutes up to the sign $(-1)^k$,
(3) the third rectangle commutes.

Proof. We consider the three rectangles separately.

(1) We consider the following diagram:

$$\begin{array}{ccc}
\mathrm{H}^k(M, \partial M) \times \mathrm{H}_n(M, \partial M) & \xrightarrow{\cap} & \mathrm{H}_{n-k}(M) \\
{\scriptstyle =}\uparrow \quad \downarrow{\scriptstyle =} & & \downarrow \\
\mathrm{H}^k(M, \partial M) \times \mathrm{H}_n(M, \partial M) & \xrightarrow{\cap} & \mathrm{H}_{n-k}(M, B) \\
\downarrow \quad \uparrow{\scriptstyle =} & & {\scriptstyle =}\uparrow \\
\mathrm{H}^k(M, A) \times \mathrm{H}_n(M, \partial M) & \xrightarrow{\cap} & \mathrm{H}_{n-k}(M, B).
\end{array}$$

We can convince ourselves that the diagram commutes by applying the Cap Product-Naturality Lemma 116.8 to the maps $(M, \partial M, \varnothing) \to (M, \partial M, B)$ and $(M, A, B) \to (M, \partial M, B)$ of excisive triads. It follows easily from this observation that the first rectangle of the claim commutes.

Before we continue with the discussion of the other two rectangles we need to make one remark. We denote by $j\colon (B, \partial B) \to (\partial M, A)$ the inclusion map and we denote by $\partial\colon \mathrm{H}_n(M, \partial M) \to \mathrm{H}_{n-1}(\partial M, A)$ the connecting homomorphism in the long exact sequence of the triple $(M, \partial M, A)$. It follows from the Fundamental Class-of-Boundary Proposition 106.27 (1), the Codimension Zero-Fundamental Class Lemma 106.24 and the remark on page 1629 that the following equality holds

$$(*) \qquad j_*([B]) \;=\; \partial[M] \in \mathrm{H}_{n-1}(\partial M, A).$$

Now we continue with the proof of the claim.

(2) We apply the Cap Product-Connecting Homomorphism Lemma 116.10 (2) to the excisive triad (M, A, B). (Recall that $A \cup B = \partial M$ and $A \cap B = \partial B$.) We obtain that for any $k \in \mathbb{N}_0$ the following diagram commutes up to the sign $(-1)^k$:

$$\begin{array}{ccc}
\mathrm{H}^k(M, A) \times \mathrm{H}_n(M, \partial M) & \xrightarrow{\cap} & \mathrm{H}_{n-k}(M, B) \\
{\scriptstyle i^*}\downarrow \quad {\scriptstyle \partial}\downarrow & & \\
\mathrm{H}^k(\partial M, A) \times \mathrm{H}_{n-1}(\partial M, A) & & \Big\downarrow \partial \\
{\scriptstyle j^*}\downarrow \cong \quad {\scriptstyle (j_*)^{-1}}\downarrow \cong & & \\
\mathrm{H}^k(B, \partial B) \times \mathrm{H}_{n-1}(B, \partial B) & \xrightarrow{\cap} & \mathrm{H}_{n-k-1}(B).
\end{array}$$

Together with $(*)$ we obtain that the second rectangle of the claim does indeed commute up to the sign $(-1)^k$.

(3) We consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}^k(B,\partial B) \times \mathrm{H}_{n-1}(B,\partial B) & \xrightarrow{\cap} & \mathrm{H}_{n-k-1}(B) \\
\uparrow{\scriptstyle i^*} & \downarrow{\scriptstyle i_*} & \downarrow{\scriptstyle i_*} \\
\mathrm{H}^k(\partial M,A) \times \mathrm{H}_{n-1}(\partial M,A) & \xrightarrow{\cap} & \mathrm{H}_{n-k-1}(\partial M) \\
\downarrow{\scriptstyle \delta} & \uparrow{\scriptstyle \partial} & \downarrow{\scriptstyle i_*} \\
\mathrm{H}^{k+1}(M,\partial M) \times \mathrm{H}_n(M,\partial M) & \xrightarrow{\cap} & \mathrm{H}_{n-k-1}(M).
\end{array}$$

The upper part of this diagram commutes by the Cap Product-Naturality Lemma 116.8 applied to the map $(B,\partial B,\varnothing) \to (\partial M,A,\varnothing)$ between excisive triads. The lower part commutes by the Cap Product-Connecting Homomorphism Lemma 116.10 (1) applied to the triple $(M,\partial M,A)$. This discussion, together with $(*)$, implies that the third rectangle of the diagram in the claim commutes.

This concludes the proof of the claim and thus also of the Poincaré Duality Theorem 119.1. ■

Remark. It follows from the Poincaré Duality Theorem 119.1 and Corollary 114.12 that if X is a topological space that is homotopy equivalent to a closed orientable n-dimensional topological manifold, then there exists a $\phi \in \mathrm{H}_n(X;\mathbb{Z})$ such that for any $k \in \mathbb{N}_0$ the map

$$- \cap \phi \colon \mathrm{H}^k(X;\mathbb{Z}) \ \to \ \mathrm{H}_{n-k}(X;\mathbb{Z})$$

is an isomorphism. One can ask to what degree the converse holds. Namely, if say a CW-complex X admits such a ϕ, is X homotopy equivalent to a closed orientable topological manifold? We will discuss this question on page **??**.

We conclude this section with two propositions that relate Poincaré Duality on a manifold to Poincaré Duality on its boundary.

Proposition 119.25. Let M be a compact n-dimensional topological manifold with non-empty boundary. Let R be a commutative ring. We suppose that M is R-oriented. We equip the boundary ∂M with the R-orientation given by the Topological Manifold-Boundary Orientation Proposition 105.19. As usual we denote by $[M] \in \mathrm{H}_n(M,\partial M;R)$ and $[\partial M] \in \mathrm{H}_{n-1}(\partial M;R)$ the corresponding R-fundamental classes. We consider the following diagram:

$$\begin{array}{ccccccccc}
\ldots \to \mathrm{H}^k(M,\partial M;R) & \longrightarrow & \mathrm{H}^k(M;R) & \longrightarrow & \mathrm{H}^k(\partial M;R) & \xrightarrow{\delta} & \mathrm{H}^{k+1}(M,\partial M;R) & \to \ldots \\
\cong\downarrow{\scriptstyle -\cap[M]} & & \cong\downarrow{\scriptstyle -\cap[M]} & & \cong\downarrow{\scriptstyle -\cap[\partial M]} & & \cong\downarrow{\scriptstyle -\cap[M]} & \\
\ldots \longrightarrow \mathrm{H}_{n-k}(M;R) & \longrightarrow & \mathrm{H}_{n-k}(M,\partial M;R) & \to & \mathrm{H}_{n-k-1}(\partial M;R) & \longrightarrow & \mathrm{H}_{n-k-1}(M;R) & \xrightarrow{\partial} \ldots
\end{array}$$

where the horizontal sequences are the long exact sequences in (co-) homology of the pair $(M,\partial M)$. (Note that the vertical maps are isomorphisms by Theorem 119.1.) The first and the third square commute, furthermore the second square commutes up to the sign $(-1)^k$.

Proof. We had implicitly proved the proposition in the above proof of the Poincaré Duality Theorem 119.1 applied to the special case that $A = \varnothing$ and $B = \partial M$. ■

Finally we have the following proposition.

Proposition 119.26. Let M be a compact, non-empty n-dimensional topological manifold and let R be a commutative ring. We suppose that M is R-oriented. We denote by $[M] \in \mathrm{H}_n(M, \partial M; R)$ the R-fundamental class. Furthermore suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. We consider the following two diagrams:

$$\begin{array}{ccc} \mathrm{H}_k(M, A; R) & \xrightarrow{\partial} & \mathrm{H}_k(A; R) \\ {\scriptstyle \mathrm{PD}_M}\downarrow & & \downarrow{\scriptstyle \mathrm{PD}_A} \\ \mathrm{H}^{n-k}(M, B; R) & \xrightarrow{w^*} & \mathrm{H}^{n+1-k}(A, \partial A; R) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{H}_k(A, \partial A; R) & \xrightarrow{w_*} & \mathrm{H}_k(M, B; R) \\ {\scriptstyle \mathrm{PD}_A}\downarrow & & \downarrow{\scriptstyle \mathrm{PD}_M} \\ \mathrm{H}^{n-k}(A; R) & \xrightarrow{\delta} & \mathrm{H}^{n+1-k}(M, A; R). \end{array}$$

The first diagram commutes up to multiplication by $(-1)^k$ whereas the second commutes as it is.

Proof. We already showed on page 2642 that the first diagram commutes up to multiplication by $(-1)^k$. The fact that the second diagram commutes is a fairly straightforward consequence of the Cap Product-Connecting Homomorphism Lemma 116.10. We leave it to the reader to fill in the details. ■

Exercises for Chapter 119.

Exercise 119.1. Let M be a closed 0-connected n-dimensional topological manifold. We suppose there exists an $(n+1)$-dimensional topological manifold N such that for every $i \in \{1, \ldots, n\}$ we have $\widetilde{\mathrm{H}}_i(N) \cong \mathrm{H}_{i-1}(M)$. Show that M is a homology n-sphere, i.e. show that $\mathrm{H}_i(M) \cong \mathrm{H}_i(S^n)$ for all $i \in \mathbb{N}_0$.
Remark. In view of the Suspension-H_*-Proposition 76.7 this exercise can be viewed as a variation on Exercise 76.8.

Exercise 119.2. Let M and N be two closed orientable connected non-empty 3-dimensional topological manifolds such that $\pi_1(M)$ is isomorphic to $\pi_1(N)$. Show that for all $i \in \mathbb{N}_0$ there exists an isomorphism $\mathrm{H}_i(M; \mathbb{Z}) \cong \mathrm{H}_i(N; \mathbb{Z})$.
Remark. This exercise is almost identical to Exercise 103.4.

Exercise 119.3.

(a) Let M be an n-dimensional closed orientable connected topological manifold such that $\pi_1(M)$ is isomorphic to the permutation group S_5. What can you say about $\mathrm{H}^{n-1}(M; \mathbb{Z})$? Does the answer depend on whether or not M has boundary?
(b) Let M be a non-compact orientable topological manifold. Show that the group $\mathrm{H}_{n-1}(M, \partial M; \mathbb{F}_2)$ is non-zero.

Exercise 119.4. Let $p\colon \widetilde{N} \to N$ be a finite covering of a closed, oriented n-dimensional topological manifold. We equip $\widetilde{N}$ with the orientation given by the Topological Manifold-Induced Orientation Lemma 105.17

(a) Let $r \leq s$. Show that given any singular simplex $\sigma\colon \Delta^s \to N$ and any cochain $\phi \in \mathrm{C}^r(N;\mathbb{Z})$ we have

$$p^*(\phi) \cap p^*(\sigma) \;=\; p^*(\phi \cap \sigma).$$

Here p^* for homology classes denotes the transfer map.

(b) Show that for any $k \in \mathbb{N}_0$ the Umkehr map $p^!\colon \mathrm{H}_k(N;\mathbb{Z}) \to \mathrm{H}_k(\widetilde{N};\mathbb{Z})$ equals the transfer map p^*.
Hint. the Fundamental Class-Covering Proposition 106.25 might be useful.

Exercise 119.5. Is the Klein bottle the boundary of a compact 3-dimensional topological manifold?

Exercise 119.6.

(a) Let M be a topological manifold and let U, V be open subsets such that $M = U \cup V$. Show that given any compact subset $C \subset M$ there exist compact subsets $K \subset U$ and $L \subset V$ such that $C = K \cup L$.

(b) Does statement (a) hold for any topological space?

Exercise 119.7. Let M be a closed connected 4-dimensional topological manifold that is simply connected and with $\chi(M) = 2$. What can you say about the homeomorphism type of M?

Exercise 119.8.

(a) Let W be a closed orientable connected 4-dimensional topological manifold with $\chi(W) = -3$. Show that W admits a finite connected covering $\widetilde{W}$ with $\chi(\widetilde{W}) < -27$.

(b) Is the hypothesis in (a) that W is orientable necessary?

Remark. This exercise is almost identical to Exercise 103.5.

Exercise 119.9. Let M be a closed 4-dimensional topological manifold.

(a) Show that if M is simply connected, then there exist at most finitely many isomorphism classes of finite groups which can act discretely on M.

(b) Is the hypothesis in (a) that M is simply connected necessary?

Exercise 119.10. Let M be a compact orientable 4-dimensional topological manifold with $\partial M = A \sqcup B$ where A and B are two connected 3-dimensional topological manifolds with $\mathrm{H}_1(A;\mathbb{Z})$ finite. We suppose that the inclusion induced map $\mathrm{H}_1(A;\mathbb{Z}) \to \mathrm{H}_1(M;\mathbb{Z})$ is an epimorphism. Show that the inclusion induced map $\mathrm{H}_1(B;\mathbb{Z}) \to \mathrm{H}_1(M;\mathbb{Z})$ is a monomorphism.

Exercise 119.11. Let M be a closed orientable connected odd-dimensional topological manifold and let $f\colon M \to M$ be a map such that the induced map $f_*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M)$ is an isomorphism. Show that the Lefschetz number

$$\Lambda(\varphi) \;=\; \sum_{n \in \mathbb{N}_0} (-1)^n \cdot \mathrm{tr}(\varphi_*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M)),$$

equals zero.
Remark. This exercise is almost identical to Exercise 103.6.

Exercise 119.12. Let M be a closed oriented connected n-dimensional topological manifold. Furthermore let $i \in \{0, \dots, n\}$ and let $v_1, \dots, v_m$ be a basis of $\mathrm{FH}^i(M;\mathbb{Z})$. Show that there exists a basis $\widetilde{v}_1, \dots, \widetilde{v}_m$ of $\mathrm{FH}^{n-i}(M;\mathbb{Z})$ such that for any $k, l \in \{1, \dots, m\}$ we have $\langle v_k \cup \widetilde{v}_l, [M]\rangle = \delta_{kl}$.

Exercise 119.13. Let M be a compact oriented connected n-dimensional topological manifold. We say M is an *n-dimensional homotopy ball* if $\pi_i(M) = 0$ for $i \geq 1$ and if $\pi_1(\partial M) = \{e\}$. Show that if M is an n-dimensional homotopy ball, then ∂M is an homotopy $(n-1)$-sphere, as defined in Exercise 119.13.

Exercise 119.14. Let $n \in \mathbb{N}_{\geq 2}$ and let M be a compact oriented connected n-dimensional topological manifold. Suppose that there exists a degree one map $f\colon (\overline{B}^n, S^{n-1}) \to (M, \partial M)$. Show that M is an n-dimensional homotopy ball.

Exercise 119.15. Let M be a compact oriented connected n-dimensional topological manifold. Let Y be a boundary component of M which is homeomorphic to S^{n-1}. Show that either $[Y] \neq 0 \in \pi_{n-1}(M)$ or M is a homotopy ball, as defined in Exercise 119.13.
Hint. Show that if $[Y] = 0$, then there exists a degree one map $(\overline{B}^n, S^{n-1}) \to (M, Y)$. Now use Exercise 119.14.

Exercise 119.16. Let $f\colon M \to N$ be a homeomorphism between two compact oriented connected n-dimensional topological manifolds and let A, B be two compact $(n-1)$-dimensional submanifolds of ∂M with $A \cup B = \partial M$, $A \cap B = \partial A = \partial B$. Show that for any $\sigma \in \mathrm{H}_k(M, A)$ we have

$$f^*(\mathrm{PD}_N(f_*(\sigma)) \;=\; \begin{cases} \mathrm{PD}_M(\sigma), & \text{if } \sigma \text{ is orientation-preserving,} \\ -\mathrm{PD}_M(\sigma), & \text{if } \sigma \text{ is orientation-reversing.} \end{cases}$$

Remark. This exercise consists mostly of collecting fairly standard facts.

Exercise 119.17. Let M be an oriented connected non-empty n-dimensional topological manifold. As in Exercise 106.5 we consider the group $\overleftarrow{\mathrm{H}}_n(M;\mathbb{Z}) := \varprojlim \mathrm{H}_n(M, M \setminus K; G)$. In Exercise 106.7 we showed that $\overleftarrow{\mathrm{H}}_n(M;\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and we showed that there exists a natural generator $[M]$. Let $k \in \mathbb{N}_0$.

(a) Let $\cap\colon \mathrm{H}_{\mathrm{c}}^{n-k}(M;\mathbb{Z}) \times \overleftarrow{\mathrm{H}}_n(M;\mathbb{Z}) \to \mathrm{H}_k(M;\mathbb{Z})$ be the natural cap product that we introduced in Exercise 116.4. Show that the map $\mathrm{H}_{\mathrm{c}}^{n-k}(M;\mathbb{Z}) \to \mathrm{H}_k(M;\mathbb{Z})$ given by capping with $[M]$ defines an isomorphism.
Remark. Use Lemma 119.17.

(b) Now let $f\colon M \to N$ be a proper map between two connected orientable non-empty n-dimensional topological manifolds. Since f is proper we obtain by Exercise 106.5 an induced map $f_*\colon \overleftarrow{\mathrm{H}}_k(M;\mathbb{Z}) \to \overleftarrow{\mathrm{H}}_k(N;\mathbb{Z})$. We define $\deg(f) \in \mathbb{Z}$ via the equality $f_*([M]) = \deg(f) \cdot [N]$. Furthermore, using (a), we can define the Umkehr map $f^!\colon \overleftarrow{\mathrm{H}}_k(N;\mathbb{Z}) \to \overleftarrow{\mathrm{H}}_k(M;\mathbb{Z})$. Show that the map $f^! \circ f_*\colon \mathrm{H}_k(M;\mathbb{Z}) \to \mathrm{H}_k(M;\mathbb{Z})$ is given by multiplication by $\deg(f)$.

Remark. The goal of this exercise is to generalize the ideas and concepts behind the Umkehr Map Proposition 119.13 to the setting of non-compact topological manifolds.

Exercise 119.18. Recall that given a topological space X we defined on page 2547 its *cup length* $\mathrm{cl}(X) \in \mathbb{N}_0 \cup \{\infty\}$ as follows:

$$\mathrm{cl}(X) \;:=\; \max\left\{ n \in \mathbb{N}_0 \;\middle|\; \begin{array}{c} \text{there exists a commutative ring } R \text{ and } R\text{-cohomology} \\ \text{classes } x_1, \ldots, x_n \text{ in degrees} \geq 1 \text{ with } x_1 \cup \cdots \cup x_n \neq 0 \end{array} \right\}.$$

Let M and N be two closed oriented connected n-dimensional topological manifolds. Show that if there exists a map $f\colon M \to N$ of degree one, then $\mathrm{cl}(M) \geq \mathrm{cl}(N)$.
Remark. At least in spirit this exercise is related to the Rudyak Conjecture 115.16.

Exercise 119.19. Let M be a closed simply connected 4-dimensional topological manifold. Show that $\mathrm{H}_1(M;\mathbb{Z}) = 0$, $\mathrm{H}_3(M;\mathbb{Z}) = 0$ and show that $\mathrm{H}_2(M;\mathbb{Z})$ is torsion-free.

Exercise 119.20. Let X and Y be compact oriented connected non-empty n-dimensional topological manifolds and let $f\colon X \to Y$ be a map of degree one. Does it follow that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_k(X;\mathbb{Z}) & \xrightarrow{\quad f_* \quad} & \mathrm{H}_k(Y;\mathbb{Z}) \\ {\scriptstyle \mathrm{PD}_X}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \mathrm{PD}_Y} \\ \mathrm{H}^{n-k}(X;\mathbb{Z}) & \xleftarrow{\quad f^* \quad} & \mathrm{H}^{n-k}(Y;\mathbb{Z})\,? \end{array}$$

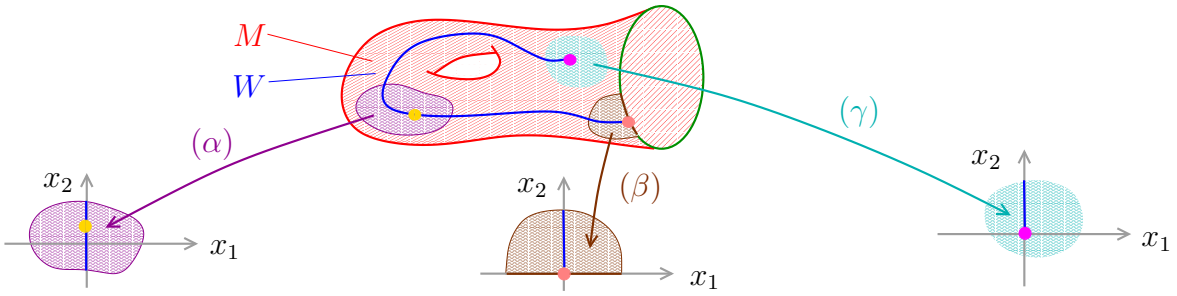
Exercise 119.21. Let M be a compact orientable n-dimensional topological manifold. Determine the homology groups of the double $\mathrm{D}M$ in terms of the homology groups of M.
Remark. It is tempting to use Mayer–Vietoris sequences, but this will most likely not lead very far. At some stage in the calculation it will be helpful to use the Poincaré Duality Theorem 119.1.

Exercise 119.22. Let M be a compact orientable 3-dimensional topological manifold such that $\mathrm{H}_1(M;\mathbb{Z})$ is torsion-free. We denote by $i\colon \partial M \to M$ the inclusion. Show that there exists a subgroup G of $\mathrm{H}_1(M;\mathbb{Z})$ such that $\mathrm{H}_1(M;\mathbb{Z}) = i_*(\mathrm{H}_1(\partial M;\mathbb{Z})) \oplus G$.

120. Poincaré Duality and codimension-zero submanifolds

In this short chapter we will study the interplay between Poincaré Duality on a given compact topological manifold M and on a given codimension-zero submanifold W of M. Later on, in the applications of this chapter's results, we will also deal with submanifolds with corner, so we will use this opportunity to finally introduce the notion of a submanifold with corner. The results and concepts of this chapter we will be used on several occasions. Nonetheless it is safe to skip this chapter and to move on to the next, significantly more entertaining, chapter.

120.1. **Submanifolds with corner.** Recall that on page 524 we defined what it means for a subset W of a manifold M to be a submanifold, namely it means that given any $P \in W$ there is a chart for M of one of the three types (α), (β) and (γ) that are defined properly on page 524 and that are sketched in the figure below.



Now we introduce the more general notion of a submanifold with corner.

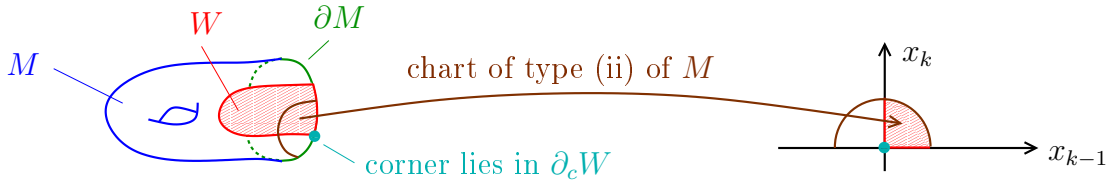
Definition. Let M be an n-dimensional topological manifold. We say a subset $W \subset M$ is a k-dimensional submanifold with corner[1474] if for every $P \in W$ there exists a chart of the type (α), (β) and (γ) from page 524 or if

(δ) there exists a chart $\Phi\colon U \to V$ of type (ii)[1475] for M such that

$$\Phi(U \cap W) \subset \{(0,\dots,0,x_1,\dots,x_k)\,|\,x_1,\dots,x_k \in \mathbb{R} \text{ with } x_{k-1} \geq 0 \text{ and } x_k \geq 0\}$$

and with $\Phi(P) \in \{(0,\dots,0,x_1,\dots,x_{k-2},0,0)\,|\,x_1,\dots,x_{k-2} \in \mathbb{R}\}$.

If M is an n-dimensional smooth manifold, then we demand that the charts come from the maximal smooth atlas. We refer to the set of points in W that admit a chart of type (δ) as the corner set of W and we denote it by $\partial_c W$.



Remark. Throughout these lecture notes we will only consider submanifolds with corner of codimension *zero*. But the concept can of course also play a role for other codimensions.

[1474] Often in the literature such objects are called "submanifold with corners". Following [**Wall2016**, p. 30] we write "submanifold with corner" instead.

[1475] See page 511 for the definition of a chart of type (ii).

The following lemma also introduces a notation that we will use throughout these notes.

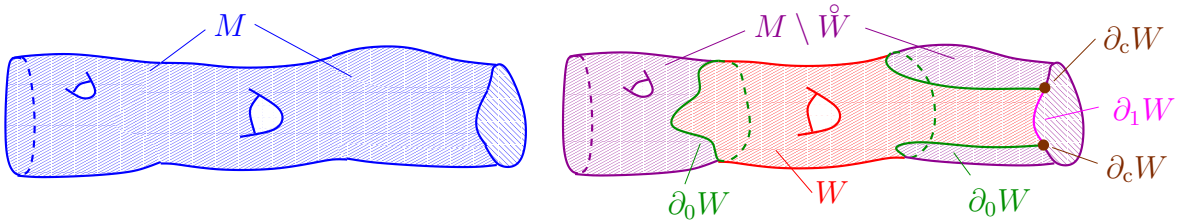
Lemma 120.1. Let M be an n-dimensional topological manifold and let W be a compact codimension-zero submanifold with corner. We write

$$\partial_0 W \;:=\; W \cap \overline{M \setminus W} \quad \text{and} \quad \partial_1 W \;:=\; W \cap \partial M.$$

The following five statements hold:

(1) The subset W itself is an n-dimensional topological manifold.
(2) The boundary of the topological manifold W equals $\partial_0 W \cup \partial_1 W$. Furthermore $\partial_0 W$ and $\partial_1 W$ are $(n-1)$-dimensional submanifolds of the boundary ∂W such that
$$\partial_0 W \cap \partial_1 W \;=\; \partial(\partial_0 W) \;=\; \partial(\partial_1 W) \;=\; \partial_{\mathrm{c}} W.$$
(3) The set $\mathring{W} := W \setminus \partial_0 W$ is the interior of W in M, in particular $\mathring{W}$ is an open subset of M.
(4) The subset $M \setminus \mathring{W}$ is a codimension-zero submanifold of M with corner given by $\partial_{\mathrm{c}}(M \setminus \mathring{W}) = \partial_{\mathrm{c}} W$. Furthermore $\partial_0(M \setminus \mathring{W}) = \partial_0 W$ and $\partial_1(M \setminus \mathring{W}) = \overline{\partial M \setminus \partial_1 W}$.
(5) $\partial_0 W$ is a proper submanifold of M, in the sense of the definition on page 525.

The analogous statements also hold if we replace "topological" by "smooth".



Remark. Let M be a topological manifold and let W be a submanifold in the sense of the definition on page 524. Evidently it is also a submanifold with corner. Furthermore note that it follows from Proposition 75.3 (1) together with Lemma 120.1 that $\partial_{\mathrm{c}} W = \varnothing$.

Sketch of proof. In the proof that W itself is a (topological) manifold we only have to worry about the corners. But for these we can use the approach taken in Proposition **??** to find a (smooth) atlas. The remaining statements follow reasonably easily from the Topological Manifold Boundary Proposition 75.2 and the definitions. We leave the slightly annoying task of filling in the details to the reader. ■

In the following we will generalize a few results on codimension-zero submanifolds to codimension-zero submanifolds with corner. It is unlikely that the reader's pulse will quicken while reading the remainder of this section.
First we have the following variation on the Excision Theorem 75.10 for Topological Manifolds.

Theorem 120.2. (Excision Theorem for Submanifolds with Corner) Let M be an n-dimensional topological manifold and let W be a compact codimension-zero submanifold of M with corner. We write $\mathring{W} := W \setminus \partial_0 W$. For every $k \in \mathbb{N}_0$ and every abelian group G

the inclusion induced maps

$$\begin{array}{lrcl} \text{(a)} & \mathrm{H}_k(W, \partial_0 W; G) & \xrightarrow{\cong} & \mathrm{H}_k(M, M \setminus \mathring{W}; G) \\ \text{(b)} & \mathrm{H}_k(W, \partial W; G) & \xrightarrow{\cong} & \mathrm{H}_k(M, \underbrace{(M \setminus \mathring{W}) \cup \partial M}_{(M \setminus \mathring{W}) \cup \partial_1 W}; G) \\ \text{(c)} & \mathrm{H}_k(W, \partial_1 W; G) & \xrightarrow{\cong} & \mathrm{H}_k(W \cup \partial M, \partial M; G) \end{array}$$

are isomorphisms. Similarly, all maps in cohomology "in the other direction" are also isomorphisms.

Proof. We will prove the statements on homology groups with $\mathbb{Z}$-coefficients in Exercise 120.2. The statement for homology with G-coefficients and the statement for cohomology with G-coefficients are an immediate consequence of the above together with Corollaries 89.18 and 108.20. ■

We turn to the topic of orientations and fundamental classes. First we have the following lemma which generalizes the Topological Manifold-Induced Orientation Lemma 105.17 (2).

Lemma 120.3. Let R be a commutative ring and let M be a topological manifold that is equipped with an R-orientation. If W is a codimension-zero submanifold of M with corner, then W admits a unique R-orientation such that the inclusion map $N \to M$ is orientation-preserving.

Proof. This statement follows immediately from the Topological Manifold-Induced Orientation Lemma 105.17 (1). ■

Convention. Let M be a topological manifold and let W be a codimension-zero submanifold with corner. If M is oriented, then we equip W with the orientation coming from Lemma 120.3.

The following lemma is a generalization of the Codimension Zero-Fundamental Class Lemma 106.24.

Lemma 120.4. Let M be a compact oriented n-dimensional topological manifold. Furthermore let $W \subset M$ be a compact non-empty codimension-zero submanifold *with corner*. We write $\mathring{W} = W \setminus \partial W$. The following two statements hold:

(1) The inclusion induced map $\mathrm{H}_n(W, \partial W; \mathbb{Z}) \to \mathrm{H}_n(M, (M \setminus \mathring{W}) \cup \partial_1 W; \mathbb{Z})$ is an isomorphism.

(2) The image of the fundamental classes $[M] \in \mathrm{H}_n(M, \partial M; \mathbb{Z})$ and $[W] \in \mathrm{H}_n(W, \partial W; \mathbb{Z})$ under the maps

$$\mathrm{H}_n(M, \partial M; \mathbb{Z}) \longrightarrow \mathrm{H}_n(M, (M \setminus \mathring{W}) \cup \partial_1 W; \mathbb{Z}) \underset{\substack{\uparrow \\ \text{isomorphism by the Excision Theorem 120.2 (2)}}}{\longleftarrow} \mathrm{H}_n(W, \partial W; \mathbb{Z})$$

agree.

Proof. The proof of Lemma 120.4 is basically identical to the proof of the Codimension Zero-Fundamental Class Lemma 106.24. ■

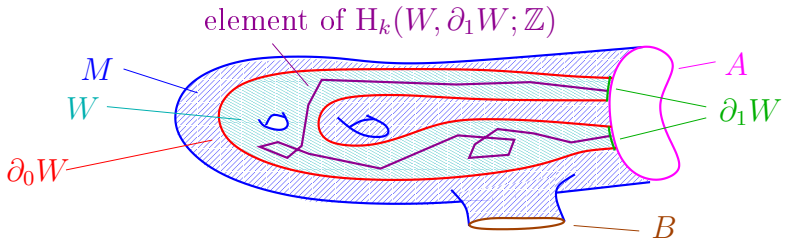
120.2. **Poincaré Duality and codimension-zero submanifolds.** The following lemma says that given a homology class in a codimension-zero submanifold W of some topological manifold M we can apply Poincaré Duality in M or W and essentially we end up with the same result.

Lemma 120.5. (Codimension-Zero Submanifold-PD Lemma) Let M be a compact oriented n-dimensional topological manifold and let W be a compact codimension-zero with corner. We equip W with the induced orientation from the Topological Manifold-Induced Orientation Lemma 105.17. We write $\mathring{W} = W \setminus \partial_0 W$.[1476] Furthermore let A and B be disjoint unions of components of ∂M with $\partial M = A \cup B$ and $\partial_1 W \subset A$. We fix the following notation:

(1) We denote by w the obvious map $(W, \partial_1 W) \to (M, A)$ and also the obvious map $(W, \partial_0 W) \to (M, M \setminus \mathring{W})$,
(2) we denote by $p\colon (M, B) \to (M, M \setminus \mathring{W})$ the obvious map.

Then for every $k \in \mathbb{N}_0$ the following diagram commutes[14771478]

$$\begin{array}{ccccc} \mathrm{H}_k(W, \partial_1 W; \mathbb{Z}) & \xrightarrow{\mathrm{PD}_W} & \mathrm{H}^{n-k}(W, \partial_0 W; \mathbb{Z}) & \xrightarrow[\cong]{(w^*)^{-1}} & \mathrm{H}^{n-k}(M, M \setminus \mathring{W}; \mathbb{Z}) \\ \downarrow{\scriptstyle w_*} & & & & \downarrow{\scriptstyle p^*} \\ \mathrm{H}_k(M, A; \mathbb{Z}) & & \xrightarrow{\qquad\qquad\mathrm{PD}_M\qquad\qquad} & & \mathrm{H}^{n-k}(M, B; \mathbb{Z}). \end{array}$$



In the proof of Lemmas 120.5 and 120.6 will use the full force of the Cap Product-Naturality Lemma 116.8 applied to excisive triads. Since this can be seriously confusing we recall the general statement of the lemma, before we provide the proof of the Codimension-Zero Submanifold-PD Lemma 120.5.

Lemma 116.8. Let $f\colon (X, A, B) \to (\widetilde{X}, \widetilde{A}, \widetilde{B})$ be a map between two excisive triads. Then for any choice of $\varphi \in \mathrm{H}^{n-k}(\widetilde{X}, \widetilde{A}; \mathbb{Z})$ and $\sigma \in \mathrm{H}_m(X, A \cup B; \mathbb{Z})$ we have

$$\begin{array}{c} f_*\colon \mathrm{H}_m(X, A \cup B) \to \mathrm{H}_m(\widetilde{X}, \widetilde{A} \cup \widetilde{B}) \qquad f^*\colon \mathrm{H}^{n-k}(\widetilde{X}, \widetilde{A}) \to \mathrm{H}^{n-k}(X, A) \\ \varphi \cap \underset{\downarrow}{f_*}(\sigma) \;=\; \underset{\uparrow}{f_*}(\underset{\downarrow}{f^*}(\varphi) \cap \sigma) \;\in\; \mathrm{H}_k(\widetilde{X}, \widetilde{B}; \mathbb{Z}). \\ f_*\colon \mathrm{H}_k(X, B) \to \mathrm{H}_k(\widetilde{X}, \widetilde{B}) \end{array}$$

Proof of the Codimension-Zero Submanifold-PD Lemma 120.5. By definition of the isomorphisms PD_W and PD_M, proving the lemma is equivalent to proving that the

[1476] Recall that by Proposition 75.3 the set $\mathring{W} = W \setminus \partial_0 W$ is indeed the interior of W in M.
[1477] Here we use that by the Excision Theorem 120.2 we know that $w\colon (W, \partial_0 W) \to (M, M \setminus \mathring{W})$ induces an isomorphism on cohomology groups.
[1478] Note that by Lemma 120.1 we can apply the Poincaré Duality Theorem 119.1 to obtain that $\cap[W]\colon \mathrm{H}^{n-k}(W, \partial_1 W; \mathbb{Z}) \to \mathrm{H}_k(W, \partial_0 W; \mathbb{Z})$ is an isomorphism. We denote its inverse by PD_W.

following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}_k(W,\partial_1 W;\mathbb{Z}) & \xleftarrow[\cong]{\cap[W]} & \mathrm{H}^{n-k}(W,\partial_0 W;\mathbb{Z}) & \xleftarrow[w^*]{\cong} & \mathrm{H}^{n-k}(M,M\setminus \mathring{W};\mathbb{Z}) \\ \downarrow{\scriptstyle w_*} & & & & \downarrow{\scriptstyle p^*} \\ \mathrm{H}_k(M,A;\mathbb{Z}) & \xleftarrow[\cong]{\cap[M]} & & & \mathrm{H}^{n-k}(M,B;\mathbb{Z}). \end{array}$$

In the following we use the following notation:

(1) We write $\widehat{W} := W \setminus \partial W$.
(2) We denote the map $(W,\partial W) \to (M, M\setminus \widehat{W})$ by w as well
(3) We denote the map $(M,\partial M) \to (M, M\setminus \widehat{W})$ by p as well.

Now let $\beta \in \mathrm{H}^{n-k}(M, M\setminus \mathring{W};\mathbb{Z})$. We have the following equality in $\mathrm{H}_k(M,A;\mathbb{Z})$:

$$\begin{array}{ccccccc} w_*(w^*(\beta)\cap[W]) & \underset{\uparrow}{=} & \beta\cap w_*([W]) & \underset{\uparrow}{=} & \beta\cap p_*([M]) & \underset{\uparrow}{=} & p_*(\overbrace{p^*(\beta)\cap[M]}^{\in \mathrm{H}_k(M,A)}) \underset{\uparrow}{=} p^*(\beta)\cap[M]. \\ \text{Cap Product-Naturality} & & \text{by Lemma 120.4 we have } p_*([M]) = w_*([W]) \in \mathrm{H}_m(M, M\setminus\widehat{W}) & & \text{Cap Product-Naturality} & & \text{since } p\colon (M,A)\to(M,A) \\ \text{Lemma 116.8 applied to the map} & & \downarrow & & \text{Lemma 116.8 applied to the map} & & \text{is id we have } p_* = \mathrm{id}_* = \mathrm{id} \\ w\colon (W,\partial_0 W,\partial W)\to(M, M\setminus\mathring{W}, M\setminus\widehat{W}) & & & & p\colon (M,B,A)\to(M,M\setminus\mathring{W},A) & & \end{array}$$

We have thus shown that the above diagram does indeed commute. ■

We conclude this short introduction to submanifolds with corner with the following lemma that also relates Poincaré Duality in a topological manifold M to Poincaré Duality in a compact submanifold W with corner.

Lemma 120.6. Let M be a compact oriented n-dimensional topological manifold and let W be a compact codimension-zero submanifold with corner. Furthermore let A and B be disjoint unions of components of ∂M with $\partial M = A \cup B$. We fix the following notation:

(1) we set $C := \partial_1 W \cap A$ and $D := (\partial_1 W \cap B) \cup \partial_0 W$,
(2) we set $\mathring{W} := W \setminus \partial_0 W$,
(3) the obvious maps $(W,C)\to(M,A)$ and $(W,D)\to(M,(M\setminus\mathring{W})\cup B)$ are denoted by w, and
(4) we denote by $p\colon (M,B)\to(M,M\setminus\mathring{W})$ the obvious map.

Then for every $k\in\mathbb{N}_0$ the following diagram commutes [1479]

$$\begin{array}{ccccc} \mathrm{H}_k(M,B;\mathbb{Z}) & & \xrightarrow{\mathrm{PD}_M} & & \mathrm{H}^{n-k}(M,A;\mathbb{Z}) \\ \downarrow{\scriptstyle p_*} & & & & \downarrow{\scriptstyle w^*} \\ \mathrm{H}_k(M,(M\setminus\mathring{W})\cup B;\mathbb{Z}) & \xrightarrow[\cong]{(w_*)^{-1}} & \mathrm{H}_k(W,D;\mathbb{Z}) & \xrightarrow{\mathrm{PD}_W} & \mathrm{H}^{n-k}(W,C;\mathbb{Z}). \end{array}$$

Example. We continue with the notation from the previous lemma. If we set $A := \partial M$ and $B := \varnothing$, then $C = \partial_1 W$ and $D = \partial_0 W$ and we obtain that the following diagram

[1479] We use that $w\colon (W,D)\to(M,(M\setminus\mathring{W})\cup B)$ induces an isomorphism on cohomology. This follows from a slight generalization of the Excision Theorem 120.2.

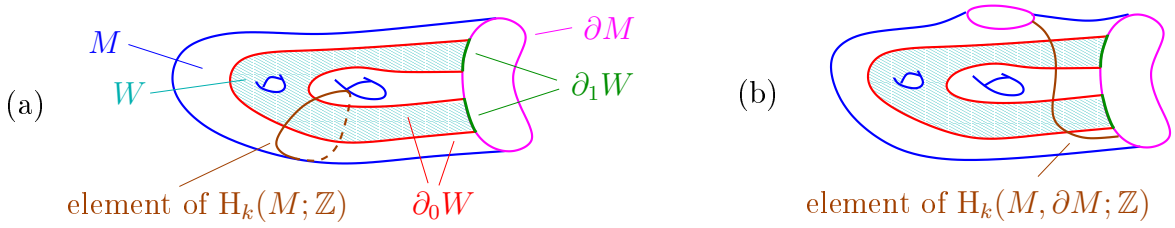
commutes:

$$\text{(a)}\qquad \begin{array}{ccccc} \mathrm{H}_k(M;\mathbb{Z}) & & \xrightarrow{\quad\mathrm{PD}_M\quad} & & \mathrm{H}^{n-k}(M,\partial M;\mathbb{Z}) \\ \downarrow{\scriptstyle p_*} & & & & \downarrow{\scriptstyle w^*} \\ \mathrm{H}_k(M, M\setminus \mathring{W};\mathbb{Z}) & \xrightarrow[\cong]{(w_*)^{-1}} & \mathrm{H}_k(W,\partial_0 W;\mathbb{Z}) & \xrightarrow{\mathrm{PD}_W} & \mathrm{H}^{n-k}(W,\partial_1 W;\mathbb{Z}). \end{array}$$

Furthermore, if we set $A := \varnothing$ and $B := \partial M$, then $C = \varnothing$ and $D = \partial W$, and we see that the following diagram commutes:

$$\text{(b)}\qquad \begin{array}{ccccc} \mathrm{H}_k(M,\partial M;\mathbb{Z}) & & \xrightarrow{\quad\mathrm{PD}_M\quad} & & \mathrm{H}^{n-k}(M;\mathbb{Z}) \\ \downarrow{\scriptstyle p_*} & & & & \downarrow{\scriptstyle w^*} \\ \mathrm{H}_k(M,(M\setminus \mathring{W})\cup\partial M;\mathbb{Z}) & \xrightarrow[\cong]{(w_*)^{-1}} & \mathrm{H}_k(W,\partial W;\mathbb{Z}) & \xrightarrow{\mathrm{PD}_W} & \mathrm{H}^{n-k}(W;\mathbb{Z}). \end{array}$$

These two important special cases are illustrated in the figure below.



Proof. We set $\widehat{W} := W \setminus (\partial_0 W \cup C)$. Note that $M \setminus \widehat{W} = (M \setminus \mathring{W}) \cup B$. By definition of the isomorphisms PD_W and PD_M, proving the lemma is equivalent to proving that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}_k(M,B;\mathbb{Z}) & & \xleftarrow[\cong]{\cap[M]} & & \mathrm{H}^{n-k}(M,A;\mathbb{Z}) \\ \downarrow{\scriptstyle p_*} & & & & \downarrow{\scriptstyle w^*} \\ \mathrm{H}_k(M, M\setminus \widehat{W};\mathbb{Z}) & \xleftarrow[\cong]{w_*} & \mathrm{H}_k(W,D;\mathbb{Z}) & \xleftarrow{\cap[W]} & \mathrm{H}^{n-k}(W,C;\mathbb{Z}). \end{array}$$

So let $\beta \in \mathrm{H}^{n-k}(M,A;\mathbb{Z})$. We have the following equalities in $\mathrm{H}_k(M, M\setminus \mathring{W};\mathbb{Z})$:

$$\begin{array}{ccccc} p_*(\beta\cap[M]) & = & \beta\cap p_*([M]) & = & \beta\cap w_*([W]) \\ \uparrow & & \uparrow & & \uparrow \\ \text{Cap Product-Naturality Lemma 116.8 applied to the map } p\colon (M,A,B)\to(M,A,M\setminus\widehat{W}) \text{ of excisive triads} & & \text{by Lemma 120.4 we have } p_*([M])=w_*([W]) \text{ in } \mathrm{H}_n(M,(M\setminus\mathring{W})\cup\partial M) & & \text{Cap Product-Naturality Lemma 116.8 applied to the map } w\colon (W,C,D)\to(M,A,M\setminus\widehat{W}) \text{ of excisive triads} \end{array} = w_*(w^*(\beta)\cap[W]).$$

■

Exercises for Chapter 120.

Exercise 120.1. Let M be an n-dimensional topological manifold and let W be a compact codimension-zero submanifold of M with corner. Show that there exists an embedding $\varphi\colon [0,1]\times\partial_0 W \to W$ with the following two properties:

(1) For every $x \in \partial_0 W$ we have $\varphi(0,x) = x$.

(2) For every $t \in [0,1]$ and every $x \in \partial_c W = \partial(\partial_0 W)$ we have $\varphi(t,x) \in \partial_1 W$.

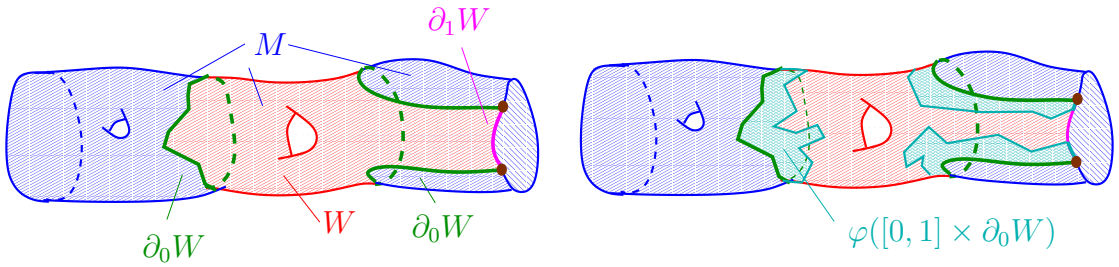
Hint. Adapt the proof of the Topological Collar Neighborhood Theorem 75.5.

Exercise 120.2. Let M be an n-dimensional topological manifold and let W be a compact codimension-zero submanifold of M with corner. We write $\mathring{W} := W \setminus \partial_0 W$. Show that for every $k \in \mathbb{N}_0$ the inclusion induced maps

$$\begin{array}{lrcl} \text{(a)} & \mathrm{H}_k(W, \partial_0 W) & \xrightarrow{\cong} & \mathrm{H}_k(M, M \setminus \mathring{W}) \\ \text{(b)} & \mathrm{H}_k(W, \partial W) & \xrightarrow{\cong} & \mathrm{H}_k(M, (M \setminus \mathring{W}) \cup \partial M) \\ \text{(c)} & \mathrm{H}_k(W, \partial_1 W) & \xrightarrow{\cong} & \mathrm{H}_k(W \cup \partial M, \partial M) \end{array}$$

are isomorphisms.

Remark. This exercise is a variation on the Excision Theorem 75.10 for Topological Manifolds. You might want to use Exercise 120.1.



Part XIII

Applications of Poincaré Duality

121. The cup product of projective spaces

In this chapter we will use the Poincaré Duality Theorem 119.1 to determine the cup product on $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ and $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$. These calculations will allow us to prove several interesting consequences which at first glance have very little to do with cup products.

121.1. **Non-singular pairings and the cup product.** Before we start drawing more conclusions from the Poincaré Duality Theorem 119.1 we need to introduce some purely algebraic definitions.

Definition. Let R be a commutative ring and let V, W and S be R-modules.

(1) A pairing on $V \times W$ is a map
$$\varphi\colon V \times W \ \to \ S$$
such that for all $v, v' \in V$ and $W, w' \in W$ we have $\varphi(v+v', w) = \varphi(v,w) + \varphi(v',w)$ and $\varphi(v, w+w') = \varphi(v,w) + \varphi(v,w')$.[1480]

(2) A pairing is called R-bilinear if it is R-linear in V and if it is R-linear in W.[1481]

(3) We say that an R-bilinear pairing $\langle\ ,\ \rangle\colon V \times W \to S$ is non-singular if the induced maps
$$\begin{array}{ccc} V & \to & \mathrm{Hom}_R(W,S) \\ v & \mapsto & \begin{pmatrix} W & \to & S \\ w & \mapsto & \langle v,w\rangle \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{ccc} W & \to & \mathrm{Hom}_R(V,S) \\ w & \mapsto & \begin{pmatrix} V & \to & S \\ v & \mapsto & \langle v,w\rangle \end{pmatrix} \end{array}$$
are isomorphisms. Otherwise we say that the R-bilinear pairing is singular.

We start out with the following elementary lemma.

Lemma 121.1. (Non-Singularity Lemma) Let R be a commutative ring. Let V and W be free R-modules of the same finite rank and let
$$\langle\ ,\ \rangle\colon V \times W \to R$$
be an R-bilinear pairing.

(1) We denote by n the rank of V. Let $v_1, \ldots, v_n$ be a basis for V and let $w_1, \ldots, w_n$ be a basis for W. Then the following statement holds:
$$\text{the pairing } \langle\ ,\ \rangle \text{ is non-singular} \quad \Longleftrightarrow \quad \det(\langle v_i, w_j\rangle)_{i,j=1,\ldots,n}) \text{ is a unit of } R.$$

(2) Suppose V and W are of rank one. If v and w are generators of V respectively W, then the following statement holds:
$$\text{the pairing } \langle\ ,\ \rangle \text{ is non-singular} \quad \Longleftrightarrow \quad \langle v,w\rangle \text{ is a unit of the ring } R.$$

Proof. The second statement is evidently just a special case of the first statement. So it suffices to prove the first statement. Let $v_1, \ldots, v_n$ be a basis for V and let $w_1, \ldots, w_n$ be a basis for W. We denote by $v_1^*, \ldots, v_n^* \in \mathrm{Hom}(V,R)$ and $w_1^*, \ldots, w_n^* \in \mathrm{Hom}(W,R)$ the dual bases. We consider the maps
$$\begin{array}{ccc} V & \to & \mathrm{Hom}_R(W,R) \\ v & \mapsto & \begin{pmatrix} W & \to & R \\ w & \mapsto & \langle v,w\rangle \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{ccc} W & \to & \mathrm{Hom}_R(V,R) \\ w & \mapsto & \begin{pmatrix} V & \to & R \\ v & \mapsto & \langle v,w\rangle \end{pmatrix} \end{array}.$$

[1480] In most examples we will be interested in $S = R$, but later on we will, oddly enough, also develop a fondness for $R = \mathbb{Z}$ and $S = \mathbb{Q}/\mathbb{Z}$.

[1481] Note that a pairing is by definition always $\mathbb{Z}$-bilinear.

Since $\langle\ ,\ \rangle$ is R-bilinear both maps are R-homomorphisms. Furthermore, it is straightforward to show that with respect to these bases both the R-homomorphisms are represented by the transpose of the matrix $(\langle v_i, w_j\rangle_{i,j=1,\dots,n})$. The lemma is an immediate consequence of this observation. ■

Remark. The proof of the Non-Singularity Lemma 121.1 shows that if V and W are free R-modules of finite rank, then the pairing is non-singular if and only if either of the two maps $V \to \mathrm{Hom}(W, R)$ or $W \to \mathrm{Hom}(V, R)$ is an isomorphism.[1482]

Now we return to topology.

Proposition 121.2. (Manifold-Non Singularity Proposition) Let M be a closed n-dimensional topological manifold and let $k \in \mathbb{N}_0$.

(1) Suppose M is oriented with fundamental class $[M]$. If $\mathrm{H}_{k-1}(M;\mathbb{Z})$ and $\mathrm{H}_{n-k-1}(M;\mathbb{Z})$ are torsion-free, then the cup product pairing
$$\begin{array}{rcl} \mathrm{H}^k(M;\mathbb{Z}) \times \mathrm{H}^{n-k}(M;\mathbb{Z}) & \to & \mathbb{Z} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [M]\rangle \end{array}$$
is non-singular.

(2) Let $\mathbb{F}$ be a field. If M is $\mathbb{F}$-oriented with fundamental class $[M]$, then for any $k \in \mathbb{N}_0$ the cup product pairing
$$\begin{array}{rcl} \mathrm{H}^k(M;\mathbb{F}) \times \mathrm{H}^{n-k}(M;\mathbb{F}) & \to & \mathbb{F} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [M]\rangle \end{array}$$
is non-singular.

Proof. Let M be a closed n-dimensional topological manifold and let $k \in \mathbb{N}_0$.

(1) We suppose that M is oriented with fundamental class $[M]$ and we suppose that $\mathrm{H}_{k-1}(M;\mathbb{Z})$ and $\mathrm{H}_{n-k-1}(M;\mathbb{Z})$ are torsion-free. It follows from the Topological Manifolds-Invariants Proposition 104.14 (4), our hypothesis on $\mathrm{H}_{k-1}(M;\mathbb{Z})$ and $\mathrm{H}_{n-k-1}(M;\mathbb{Z})$, the classification of finitely generated abelian groups, the Ext-Calculation Lemma 110.5 (3) and the Universal Coefficient Theorem 110.12 that $\mathrm{H}_{k-1}(M;\mathbb{Z})$, $\mathrm{H}_{n-k-1}(M;\mathbb{Z})$, $\mathrm{H}^k(M;\mathbb{Z})$ and $\mathrm{H}^{n-k}(M;\mathbb{Z})$ are finitely generated free abelian groups. Therefore by the remark on page 2657 it suffices to show that the map
$$\begin{array}{rcl} \Omega\colon \mathrm{H}^{n-k}(M;\mathbb{Z}) & \to & \mathrm{Hom}(\mathrm{H}^k(M;\mathbb{Z}),\mathbb{Z}) \\ \psi & \mapsto & (\varphi \mapsto \langle \varphi \cup \psi, [M]\rangle) \end{array}$$
is an isomorphism. We start out with the following claim.

Claim. The following diagram commutes:[1483]
$$\begin{array}{ccc} \mathrm{H}^{n-k}(M;\mathbb{Z}) & \xrightarrow{\quad\Omega\quad} & \mathrm{Hom}(\mathrm{H}^k(M;\mathbb{Z}),\mathbb{Z}). \\ & {\scriptstyle \mathrm{ev}}\searrow \quad \nearrow{\scriptstyle (\cap[M])^*} & \\ & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_{n-k}(M;\mathbb{Z}),\mathbb{Z}) & \end{array}$$

[1482] Is the hypothesis on V and W necessary to draw this conclusion?

[1483] The right diagonal is the dual map corresponding to the map $\mathrm{H}^k(M;\mathbb{Z}) \xrightarrow{\cap[M]} \mathrm{H}_{n-k}(M;\mathbb{Z})$.

Proof. Note that for any $\psi \in \mathrm{H}^{n-k}(M;\mathbb{Z})$ and any $\varphi \in \mathrm{H}^k(M;\mathbb{Z})$ the following equalities hold in $\mathbb{Z}$:

$$\Omega(\psi)(\varphi) \underset{\substack{\uparrow \\ \text{by definition of } \Omega \text{ and the} \\ \text{Cap Product-Kronecker Lemma 116.4}}}{=} (\varphi \cup \psi) \cap [M] \overset{\substack{\downarrow \\ \text{Cup-Cap Lemma 116.7}}}{=} \psi \cap (\varphi \cap [M]) \underset{\substack{\uparrow \\ \text{Cap Product-Kronecker Lemma 116.4}}}{=} \langle \psi, \varphi \cap [M] \rangle = \mathrm{ev}(\psi)(\varphi \cap [M]).$$

But this observation immediately shows that the diagram commutes. ⊞

We need to show that in the above diagram the horizontal map is an isomorphism. By the claim it suffices to show that the two diagonal maps are isomorphisms. The left diagonal map is an isomorphism by the Universal Coefficient Theorem 110.12 for Cohomology Groups together with the Ext-Calculation Lemma 110.5 (3) and the fact, pointed out above, that $\mathrm{H}_{n-k-1}(M;\mathbb{Z})$ is free abelian. Furthermore the second map is an isomorphism by the Poincaré Duality Theorem 119.1.

(2) Now let $\mathbb{F}$ be a field. We suppose that M is $\mathbb{F}$-oriented with corresponding fundamental class $[M]$. By the Cohomology-Field Coefficients Proposition 110.18 the map

$$\begin{array}{rcl} \mathrm{ev}\colon \mathrm{H}^k(M;\mathbb{F}) & \to & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_k(M;\mathbb{F}),\mathbb{F}) \\ \varphi & \mapsto & \left(\begin{array}{rcl} \mathrm{H}_k(M;\mathbb{F}) & \to & \mathbb{F} \\ \sigma & \mapsto & \langle \varphi, \sigma \rangle \end{array} \right) \end{array}$$

is a natural isomorphism of $\mathbb{F}$-vector spaces. The proof of (2) is now basically the same as the proof of (1). ■

The following corollary will be surprisingly effective in determining cup products on topological manifolds.

Corollary 121.3. (Rank One-PD Corollary) Let M be a closed connected n-dimensional topological manifold and let $k \in \mathbb{N}_0$.

(1) Suppose that M is oriented and that $\mathrm{H}_{k-1}(M;\mathbb{Z})$ and $\mathrm{H}_{n-k-1}(M;\mathbb{Z})$ are torsion-free. Furthermore suppose that $\mathrm{H}^k(M;\mathbb{Z}) \cong \mathbb{Z}$ and $\mathrm{H}^{n-k}(M;\mathbb{Z}) \cong \mathbb{Z}$. If $\varphi \in \mathrm{H}^k(M;\mathbb{Z})$ and $\psi \in \mathrm{H}^{n-k}(M;\mathbb{Z})$ are generators, then

$$\varphi \cup \psi \;=\; \epsilon \cdot [M]^* \;\in\; \mathrm{H}^n(M;\mathbb{Z})$$

for some $\epsilon \in \{-1, 1\}$.

(2) Let $\mathbb{F}$ be a field such that M is $\mathbb{F}$-orientable. Suppose that $\mathrm{H}^k(M;\mathbb{F}) \cong \mathbb{F}$ and $\mathrm{H}^{n-k}(M;\mathbb{F}) \cong \mathbb{F}$. If $\varphi \in \mathrm{H}^k(M;\mathbb{F})$ and $\psi \in \mathrm{H}^{n-k}(M;\mathbb{F})$ are non-zero, then $\varphi \cup \psi$ is non-zero in $\mathrm{H}^n(M;\mathbb{F}) \cong \mathbb{F}$.

Proof. Let M be a closed connected n-dimensional topological manifold and let $k \in \mathbb{N}_0$.

(1) By our hypothesis and by the Manifold-Non Singularity Proposition 121.2 (1) the cup product pairing

$$\begin{array}{rcl} \mathrm{H}^k(M;\mathbb{Z}) \times \mathrm{H}^{n-k}(M;\mathbb{Z}) & \to & \mathbb{Z} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [M] \rangle \end{array}$$

is non-singular. It follows from this fact and the Non-Singularity Lemma 121.1 (2) that $\langle \varphi \cup \psi, [M] \rangle = \pm 1$. But by the Dual Fundamental Class Lemma 110.15 this just means that $\varphi \cup \psi \in \mathrm{H}^n(M;\mathbb{Z})$ is a generator.

(2) This proof is almost identical to the proof of (1), we just need to replace the Manifold-Non Singularity Proposition 121.2 (1) by the Manifold-Non Singularity Proposition 121.2 (2). ■

Example. Let $m \neq n \in \mathbb{N}$. As on many earlier occasions we denote by $p\colon S^m \times S^n \to S^m$ and by $q\colon S^m \times S^n \to S^n$ the two natural projections. Let $1 \in \mathrm{H}^0(S^m \times S^n;\mathbb{Z})$ be the neutral element from the Cup Product-Ring Proposition 114.7. In Lemma 117.3 we showed that the cohomology of $S^m \times S^n$ is given as follows:

$$\mathrm{H}^*(S^m \times S^n;\mathbb{Z}) \;=\; \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot p^*([S^m]^*) \oplus \mathbb{Z} \cdot q^*([S^n]^*) \oplus \mathbb{Z} \cdot [S^m \times S^n]^*.$$

It follows from the Rank One-PD Corollary 121.3 (1)[1484] that $p^*([S^m]^*) \cup q^*([S^n]^*) = \pm[S^m \times S^n]^*$. This result recovers (up to the sign indeterminacy) the calculation of the non-trivial cup product on $S^m \times S^n$ that we had initially performed in Lemma 117.3 using the somewhat mysterious Product Theorem 117.1. In Exercise 121.4 we will see, that with slightly more effort, one can also prove the same type of result for $m = n$.

Next we consider the topological spaces $S^2 \vee S^4$ and $\mathbb{C}\mathrm{P}^2$. It follows from Propositions 47.13 and 52.3 and the discussion on page 1533 that $S^2 \vee S^4$ and $\mathbb{C}\mathrm{P}^2$ are both simply connected. Furthermore it follows from Propositions 74.4 and 79.8, the calculation on page 1772 and the Universal Coefficient Theorem 110.12 for Cohomology Groups that for any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(S^2 \vee S^4;\mathbb{Z}) \cong \mathrm{H}^k(S^2 \vee S^4;\mathbb{Z}) \cong \mathrm{H}^k(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) \cong \mathrm{H}_k(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0,2,4, \\ 0, & \text{otherwise.} \end{cases}$$

We can thus view $S^2 \vee S^4$ as a (co-) homological doppelgänger to $\mathbb{C}\mathrm{P}^2$.
Using the Rank One-PD Corollary 121.3 (1) we can easily prove the following lemma which implies in particular that $S^2 \vee S^4$ is not homotopy equivalent to $\mathbb{C}\mathrm{P}^2$.

Lemma 121.4. The topological space $S^2 \vee S^4$ is not homotopy equivalent to any closed topological manifold.

Proof. Let X be a closed topological manifold whose homology and cohomology groups are isomorphic to the corresponding groups of $S^2 \vee S^4$. It follows from the above calculation of homology groups, our hypothesis that X is compact, from Propositions 72.5 and 106.22 and Theorems 106.1 and 106.3 that X is a closed oriented closed connected 4-dimensional topological manifold. But then the Rank One-PD Corollary 121.3 (1) implies that the cup product

$$\mathrm{H}^2(X;\mathbb{Z}) \times \mathrm{H}^2(X;\mathbb{Z}) \;\to\; \mathrm{H}^4(X;\mathbb{Z})$$

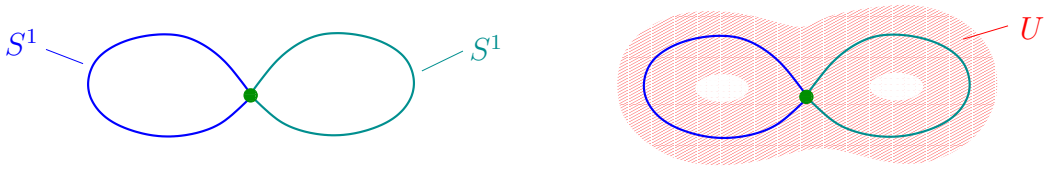
is non-zero. On the other hand in Lemma 114.11 we saw that the cup product

$$\cup\colon \mathrm{H}^2(S^2 \vee S^4;\mathbb{Z}) \times \mathrm{H}^2(S^2 \vee S^4;\mathbb{Z}) \;\to\; \mathrm{H}^4(S^2 \vee S^4;\mathbb{Z})$$

is the zero-map. By Corollary 114.12 and the above calculation of the cup product of X this implies that $S^2 \vee S^4$ and X are not homotopy equivalent. ■

[1484] Of course here we also use the fact, with which we are very familiar by now, that the homology groups of $S^m \times S^n$ are torsion-free.

Remark. One can ask whether the adjective "closed" in Lemma 121.4 is necessary. It turns out that this is the case. In fact every wedge of finitely many spheres can be viewed as a deformation retract of an open subset of a sufficiently high-dimensional $\mathbb{R}^n$. For the wedge $S^1 \vee S^1$ we sketch the argument in the figure below. With slightly more care one can also show that this wedge of finitely many spheres is also a deformation retract of some compact n-dimensional smooth manifold of $\mathbb{R}^n$. We leave it to the reader to provide a formal proof for these statements.



open neighborhood U admits a deformation retraction onto $S^1 \vee S^1$

We conclude this section with the following application to de Rham cohomology. We will not make use of it later on.

Corollary 121.5. If M is a closed oriented n-dimensional smooth manifold, then for any $k \in \mathbb{N}_0$ the $\mathbb{R}$-bilinear pairing

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^{n-k}_{\mathrm{dR}}(M) & \to & \mathbb{R} \\ ([\varphi],[\psi]) & \mapsto & \displaystyle\int_M \varphi \wedge \psi \end{array}$$

is non-singular.

Since there is no "de Rham homology" the above corollary is usually called the Poincaré Duality Theorem for de Rham cohomology.

Remark. Evidently there are also more direct proofs of Corollary 121.5 that take place within the setting of de Rham cohomology. We refer to [**BoT1982**, p. 44] or alternatively [**War1983**, Theorem 6.13] for such a proof.

Proof. It is straightforward to see that it suffices to deal with the case that M is connected. Theorem **??** and Proposition 118.10 give us the following commutative diagram with vertical isomorphisms:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^{n-k}_{\mathrm{dR}}(M) & \xrightarrow{(\varphi,\psi)\mapsto \int_M \varphi\wedge\psi} & \mathbb{R} \\ \cong\downarrow \quad \downarrow\cong & & \downarrow= \\ \mathrm{H}^k(M;\mathbb{R}) \times \mathrm{H}^{n-k}(M;\mathbb{R}) & \xrightarrow{(\varphi,\psi)\mapsto\langle\varphi\cup\psi,[M]\rangle} & \mathbb{R}. \end{array}$$

By the Manifold-Non Singularity Proposition 121.2 (2) the $\mathbb{R}$-bilinear pairing on the bottom is non-singular. Thus the $\mathbb{R}$-bilinear pairing on the top is also non-singular. ■

121.2. **The cup product of $\mathbb{C}\mathrm{P}^n$ and applications.** In this section we apply our acquired wisdom to the complex projective spaces $\mathbb{C}\mathrm{P}^n$. Recall that in the Complex Projective Space-is-Complex Manifold Lemma 32.16 we showed that $\mathbb{C}\mathrm{P}^n$ is a closed $2n$-dimensional smooth manifold that comes with a natural orientation. Thus we will always view $\mathbb{C}\mathrm{P}^n$ as

an *oriented* smooth manifold. In particular $\mathbb{C}\mathrm{P}^n$ comes with a canonical fundamental class $[\mathbb{C}\mathrm{P}^n] \in \mathrm{H}_{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ and a canonical dual fundamental class $[\mathbb{C}\mathrm{P}^n]^* \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$.
An astute observer might already have noticed that the Rank One-PD Corollary 121.3 (1) can be used to compute the cup product on $\mathrm{H}^*(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$. In fact in this section we will use the Rank One-PD Corollary 121.3 (1) and the naturality of the cup product to compute the cohomology ring of any complex projective space $\mathbb{C}\mathrm{P}^n$. Throughout this section, given $m \leq n$ we will view $\mathbb{C}\mathrm{P}^m$ as a submanifold of $\mathbb{C}\mathrm{P}^n$ via the smooth embedding

$$\begin{array}{rcl} i\colon \mathbb{C}\mathrm{P}^m & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \cdots : z_m] & \mapsto & [z_0 : \cdots : z_m : 0 : \cdots : 0]. \end{array}$$

We prepare the ground with the following elementary lemma.

Lemma 121.6.

(1) For every $n \in \mathbb{N}$ and every $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \;\cong\; \mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \ldots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Let $m \leq n$. We denote by $i\colon \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ the inclusion map. For any $k \leq 2m$ the induced maps $i_*\colon \mathrm{H}_k(\mathbb{C}\mathrm{P}^m;\mathbb{Z}) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ and $i^*\colon \mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \to \mathrm{H}^k(\mathbb{C}\mathrm{P}^m;\mathbb{Z})$ are isomorphisms.

Proof. First we recall that in Lemma 68.1 we showed that we can view $\mathbb{C}\mathrm{P}^n$ as a CW-complex which admits exactly one cell in dimensions $0, 2, \ldots, 2n$ and which admits no cells in any other dimension. Furthermore we saw that the CW-structure can be chosen in such a way that for $k \in \{0, \ldots, n\}$ the $(2k+1)$-skeleton on $\mathbb{C}\mathrm{P}^n$ equals $\mathbb{C}\mathrm{P}^k$.

Furthermore recall that by Propositions 80.4 we know that singular and cellular homology are *naturally* isomorphic. Together with the above discussion of the CW-structures it follows that for every $n \in \mathbb{N}$ and every $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \;\xrightarrow{\cong}\; \mathrm{H}_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \ldots, 2n, \\ 0, & \text{otherwise} \end{cases}$$

and that for every $m \leq n$ the inclusion induced maps $i_*\colon \mathrm{H}_k(\mathbb{C}\mathrm{P}^m;\mathbb{Z}) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ are isomorphisms for $k \leq 2m$.

In the Singular-Cellular H^*-Isomorphism Proposition 109.4 we had also seen that singular and cellular cohomology are naturally isomorphic. So the same logic applies to cohomology groups. ∎

We also introduce the following notation.

Notation.

(1) Let X be a topological space, let R be a commutative ring and let $\varphi \in \mathrm{H}^k(X;R)$. We define $\varphi^0 = 1_X$ and given any $n \in \mathbb{N}$ we define

$$\varphi^n \;=\; \underbrace{\varphi \cup \cdots \cup \varphi}_{n \text{ times}} \;\in\; \mathrm{H}^{kn}(X;R).$$

(2) Recall that we denote by $[\mathbb{C}\mathrm{P}^1]^* \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^1;\mathbb{Z})$ the dual fundamental class of $\mathbb{C}\mathrm{P}^1$. For any $n \geq 1$ we denote by $i\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^n$ the above inclusion map. By Lemma 121.6

we can define $x := (i^*)^{-1}([\mathbb{C}\mathrm{P}^1]^*) \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$. We refer to $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ as the standard generator.

Now we have the tools to determine the cup product of all complex projective spaces.

Proposition 121.7. (Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition) Let $n \in \mathbb{N}$.

(1) For every $k \in \{0,\dots,n\}$ the k-th power $x^k \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ is a generator.
(2) If we view $\mathbb{Z}[x]/(x^{n+1})$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 2$, then the obvious map
$$\mathbb{Z}[x]/(x^{n+1}) \;\to\; (\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}),\cup)$$
is an isomorphism of superalgebras.

Proof. It follows immediately from the definitions and the Cup Product-Commutativity Proposition 114.8 that the map in (2) is a homomorphism of superalgebras. This observation, together with Lemma 121.6, shows that (2) is a consequence of (1). Thus it remains to prove (1)

We prove (1) by induction on n. Evidently the statement holds for $n = 1$. Now suppose it holds for some $n-1$ with $n \geq 2$. We denote by $i\colon \mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^n$ the inclusion map. Note that $i^*(x) = x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^{n-1};\mathbb{Z})$. It follows easily from Lemma 121.6 together with the Cup Product-Naturality Lemma 114.10 and our induction hypothesis that for each $k \in \{1,\dots,n-1\}$ the cohomology class $x^k \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is a generator. It remains to show that $x^n \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is also a generator. This is an immediate consequence of the Rank One-PD Corollary 121.3 (1), applied to the generator $\varphi := x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ and the generator $\psi := x^{n-1} \in \mathrm{H}^{2(n-1)}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$. ■

The previous proof leaves it open whether nor not $x^n \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ equals the dual fundamental class $[\mathbb{C}\mathrm{P}^n]^*$. We formulate this as a question which we will answer later in the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 126.11, when we will have obtained more tools for calculating cup products.

Question 121.8. Let $n \in \mathbb{N}$. Does the n-th power $x^n \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ equal the dual fundamental class $[\mathbb{C}\mathrm{P}^n]^*$?

In Lemma 121.4 we had in particular shown that $\mathbb{C}\mathrm{P}^2$ is not homotopy equivalent to its (co-) homological doppelgänger $S^2 \vee S^4$. On page 1949 we pointed out that $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ is another example of a pair of topological spaces that cannot be distinguished using the fundamental group and using homology groups. This motivated us to ask in Question 90.9 whether the topological spaces $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ are homeomorphic. Now we can answer the question in the negative, in fact we can prove an even stronger result.

Corollary 121.9. The topological spaces $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ are not homotopy equivalent.

Proof. In Lemma 117.3 we saw that the cup product
$$\mathrm{H}^2(S^2 \times S^4;\mathbb{Z}) \times \mathrm{H}^2(S^2 \times S^4;\mathbb{Z}) \;\to\; \mathrm{H}^4(S^2 \times S^4;\mathbb{Z})$$
is zero, whereas we saw in the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 121.7 that the cup product
$$\mathrm{H}^2(\mathbb{C}\mathrm{P}^3;\mathbb{Z}) \times \mathrm{H}^2(\mathbb{C}\mathrm{P}^3;\mathbb{Z}) \;\to\; \mathrm{H}^4(\mathbb{C}\mathrm{P}^3;\mathbb{Z})$$
is non-zero. It follows from Corollary 114.12 that $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ are not homotopy equivalent. ■

We recall the following definition from page 2388.

Definition. Let M be a compact oriented topological manifold. We say that M is amphichiral if M admits an orientation-reversing self-homeomorphism, otherwise we say that M is chiral.

On page 2391 we saw that many smooth manifolds are amphichiral, for example we showed that all compact oriented 2-dimensional topological manifolds are amphichiral.
In Question 107.5 we had asked whether there exist compact oriented topological manifolds that are chiral. Now we will see that the complex projective space $\mathbb{C}\mathrm{P}^2$ is chiral. This gives an affirmative answer to Question 107.5. Note that the following corollary also gives a negative answer to Question 101.12 (3).

Corollary 121.10. The smooth manifold $\mathbb{C}\mathrm{P}^2$ is chiral, i.e. there does not exist a self-homeomorphism $f\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ with $f_*([\mathbb{C}\mathrm{P}^2]) = -[\mathbb{C}\mathrm{P}^2]$.

Remark. In Exercise 121.5 we will discuss for which $n \in \mathbb{N}$ the complex projective space $\mathbb{C}\mathrm{P}^n$ is chiral.

Proof. As usual we denote by $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ the standard generator. By the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 121.7 we know that $x \cup x \in \mathrm{H}^4(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ is a generator. This implies that $x \cup x = \epsilon \cdot [\mathbb{C}\mathrm{P}^2]^*$ for some $\epsilon \in \{-1, 1\}$.

Now let $f\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ be a self-homeomorphism. Since we want to show that $\mathbb{C}\mathrm{P}^2$ is chiral we need to show that $f_*([\mathbb{C}\mathrm{P}^2]) = [\mathbb{C}\mathrm{P}^2] \in \mathrm{H}_4(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$, or equivalently, that $f^*([\mathbb{C}\mathrm{P}^2]^*) = [\mathbb{C}\mathrm{P}^2]^* \in \mathrm{H}^4(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$.

Now we see that

$$\begin{aligned}
f^*([\mathbb{C}\mathrm{P}^2]^*) &= \epsilon \cdot f^*(x \cup x) \underset{\text{Cup Product-Naturality Lemma 114.10}}{=} \epsilon \cdot (f^*(x) \cup f^*(x)) \underset{\substack{f^*(x) \text{ is also a generator of } \mathrm{H}^2(\mathbb{C}\mathrm{P}^4;\mathbb{Z}),\\ \text{hence } f^*(x) = \eta \cdot x \text{ for some } \eta \in \{-1,1\}}}{=} \epsilon \cdot (\eta \cdot x \cup \eta \cdot x) \\
&\underset{\cup \text{ is bilinear}}{=} \epsilon \cdot \eta^2 \cdot (x \cup x) \underset{\text{since } \eta \in \{-1,1\}}{=} \epsilon \cdot (x \cup x) = [\mathbb{C}\mathrm{P}^2]^*.
\end{aligned}$$

■

Next we turn to the calculation of the cohomology ring of the infinite-dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty = \varinjlim \mathbb{C}\mathrm{P}^n$.

Lemma 121.11.

(1) For every $k \in \mathbb{N}$ we have

$$\mathrm{H}^k(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \cong \mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

(2) Let $n \in \mathbb{N}$. We denote by $i\colon \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^\infty$ the inclusion map. For any $k \leq 2n$ the induced maps $i_*\colon \mathrm{H}_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$ and $i^*\colon \mathrm{H}^k(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \to \mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ are isomorphisms.

Proof. The statements in (1) and (2) regarding homology groups follow from the discussion on page 1772. The statements in (1) and (2) regarding cohomology groups then follow from the Universal Coefficient Theorem 110.12 for Cohomology Groups. ■

Remark. One could also prove the lemma using Lemma 121.6 and a suitable generalization of the Cohomology-via-Exhaustions Proposition 111.12.

Notation. We denote by $i\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^\infty$ the inclusion map. By Lemma 121.11 we can set $x := (i^*)^{-1}([\mathbb{C}\mathrm{P}^1]^*) \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$. We refer to x as the standard generator of $\mathrm{H}^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$.

Proposition 121.12. (Cup Product-$\mathbb{C}\mathrm{P}^\infty$-Proposition)

(1) For every $k \in \mathbb{N}$ the k-th power $x^k \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \cong \mathbb{Z}$ is a generator.

(2) If we view $\mathbb{Z}[x]$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 2$, then the obvious map

$$\mathbb{Z}[x] \;\to\; (\mathrm{H}^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}), \cup)$$

is an isomorphism of superalgebras.

Proof. Exactly as in the proof of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 121.7 we see that (2) is a consequence of (1). Thus it remains to prove (1) So let $k \in \mathbb{N}$. We consider the inclusion map $j\colon \mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^\infty$. By the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 121.7 and the Cup Product-Naturality Lemma 114.10 we know that $j^*(x^k) = j^*(x)^k = x^k \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^k;\mathbb{Z})$ is a generator. It follows from Lemma 121.11 that $x^k \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$ is also a generator. ■

Remark. It follows from the Cohomology Ring-of-Product Proposition 117.21 and the discussion on page 2586 that for each $k \in \mathbb{N}$ we also have an isomorphism

$$(\mathrm{H}^*(\underbrace{\mathbb{C}\mathrm{P}^\infty \times \ldots \times \mathbb{C}\mathrm{P}^\infty}_{k \text{ copies}};\mathbb{Z}), \cup) \;\cong\; \mathbb{Z}[x_1, \ldots, x_k]$$

of superalgebras. Here $x_1, \ldots, x_k$ are given by preimages of the standard generator of the group $\mathrm{H}^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$ under the k natural projections.

The fact that so far we worked with $\mathbb{Z}$-coefficients was mostly irrelevant. In fact most of the results carry over, with obvious modifications, to the cohomology groups $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;R)$ where R is an arbitrary commutative ring. For future reference we state the following proposition.

Proposition 121.13. Let $n \in \mathbb{N}$.

(1) For every every $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}_2) \;\cong\; \mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}_2) \;\cong\; \begin{cases} \mathbb{Z}_2, & \text{if } k = 0, 2, 4, \ldots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) We denote by $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}_2)$ the unique non-trivial element. If we view the ring $\mathbb{Z}_2[x]/(x^{n+1})$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 2$, then the obvious map

$$\mathbb{Z}_2[x]/(x^{n+1}) \;\to\; (\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}_2), \cup)$$

is an isomorphism of superalgebras.

The obvious analogues for $\mathbb{C}\mathrm{P}^\infty$ also hold.

Proof. This proposition can be proved in an almost identical way as the Cup Product-$\mathbb{C}\mathrm{P}^\infty$-Proposition 121.12. Alternatively we can deduce the proposition from the Cup Product-$\mathbb{C}\mathrm{P}^\infty$-Proposition 121.12, the Universal Coefficient Theorem 110.25, Lemma 110.29 and the Cup Product-Naturality Proposition 115.4 (2). We leave it to the impeccably conscientious reader to fill in the details. ■

Towards the end of this section we recall the definition of a retract. Thus let A be a subset of a topological space X. We denote by $i\colon A \to X$ the inclusion map. Recall that we say that A is a *retract* of X if there exists a map $r\colon X \to A$ such that $r \circ i = \mathrm{id}_A$. It follows immediately from the functoriality of homology groups and Splitting Lemma 78.2 that for each k the group $\mathrm{H}_k(A;\mathbb{Z})$ is a subsummand of $\mathrm{H}_k(X;\mathbb{Z})$.
Let $m < n$. In Question 85.4 we had asked whether $\mathbb{C}\mathrm{P}^m \subset \mathbb{C}\mathrm{P}^n$ is a retract of $\mathbb{C}\mathrm{P}^n$. Note that Lemma 121.6 shows that a study of homology groups does not preclude that for $m < n$ the complex projective space $\mathbb{C}\mathrm{P}^m$ is a retract of $\mathbb{C}\mathrm{P}^n$. Nonetheless, the final result of the section gives a negative answer to Question 85.4, i.e. we show that $\mathbb{C}\mathrm{P}^m$ is not a retract of $\mathbb{C}\mathrm{P}^n$.

Proposition 121.14. If $1 \leq m < n$, then $\mathbb{C}\mathrm{P}^m \subset \mathbb{C}\mathrm{P}^n$ is not a retract of $\mathbb{C}\mathrm{P}^n$.

Proof. Let $1 \leq m < n$. We denote by $i\colon \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ the inclusion map. Suppose there exists a retraction $r\colon \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^m$. We denote by $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ the standard generator. We have

$$x^n \underset{\text{by Lemma 121.6 the map } i^*\colon \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \to \mathrm{H}^2(\mathbb{C}\mathrm{P}^m;\mathbb{Z}) \text{ is an isomorphism, together with } i^* \circ r^* = (r \circ i)^* = \mathrm{id} \text{ we obtain that } r^* \circ i^* = \mathrm{id}}{=} (r^*(i^*(x)))^n \overset{\text{Cup Product-Naturality Lemma 114.10}}{=} r^*(i^*(x)^n) \underset{\text{note that } n > m \text{ implies that } i^*(x)^n \text{ lies in the trivial group } \mathrm{H}^n(\mathbb{C}\mathrm{P}^m;\mathbb{Z}) = 0}{=} r^*(0) = 0 \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}).$$

But this calculation contradicts the fact shown in the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 121.7 that $x^n \neq 0$. ∎

121.3. **The cup product of $\mathbb{R}\mathrm{P}^n$ and applications.** In this section we intend to determine the cup product of the real projective space $\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients. Evidently the approach will be very similar to the one taken in the previous section. First we recall that on page 564 we saw that the real projective space $\mathbb{R}\mathrm{P}^n$ is a closed n-dimensional smooth manifold. Given $m \leq n$ we will view $\mathbb{R}\mathrm{P}^m$ as a submanifold of $\mathbb{R}\mathrm{P}^n$ via the smooth embedding

$$\begin{array}{rcl} i\colon \mathbb{R}\mathrm{P}^m & \to & \mathbb{R}\mathrm{P}^n \\ [x_0 : \cdots : x_m] & \mapsto & [x_0 : \cdots : x_m : 0 : \cdots : 0]. \end{array}$$

The following lemma plays the role of Lemma 121.6.

Lemma 121.15.

(1) For every $n \in \mathbb{N}$ we have

$$\mathrm{H}^k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \cong \mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } k = 0, 1, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Let $m \leq n$. For any $k \leq m$ the induced maps $i_*\colon \mathrm{H}_k(\mathbb{R}\mathrm{P}^m;\mathbb{F}_2) \to \mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ and $i^*\colon \mathrm{H}^k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \to \mathrm{H}^k(\mathbb{R}\mathrm{P}^m;\mathbb{F}_2)$ are isomorphisms.

Proof. The calculation of the isomorphism types of the (co-) homology groups and of the induced maps on homology is basically the same as in the proof of Lemma 121.15. The argument that determines the induced map on $\mathbb{F}_2$-cohomology is slightly different. This time we use the Cohomology-Field Coefficients Proposition 110.18 which gives us the

following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) & \xrightarrow[\cong]{\mathrm{ev}} & \mathrm{Hom}(\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2),\mathbb{F}_2) \\ \downarrow i^* & & \downarrow i^* \\ \mathrm{H}^k(\mathbb{R}\mathrm{P}^m;\mathbb{F}_2) & \xrightarrow[\cong]{\mathrm{ev}} & \mathrm{Hom}(\mathrm{H}_k(\mathbb{R}\mathrm{P}^m;\mathbb{F}_2),\mathbb{F}_2). \end{array}$$

As we had just pointed out, the maps on the right are isomorphisms for $k \leq m$. It follows that the maps on the left are also isomorphisms for $k \leq m$. ■

Notation. Given $n \in \mathbb{N}$ we denote by $x \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ the unique non-trivial element.

The following proposition is the analogue of the Cup Product-$\mathbb{C}\mathrm{P}^n$ Proposition 121.7 and the Cup Product-$\mathbb{C}\mathrm{P}^\infty$ Proposition 121.12.

Proposition 121.16. (Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition) Let $n \in \mathbb{N}$.

(1) For every $k \in \{0,\ldots,n\}$ the k-th power $x^k \in \mathrm{H}^k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \cong \mathbb{F}_2$ is non-trivial.
(2) If we view $\mathbb{F}_2[x]/(x^{n+1})$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 1$, then the obvious map
$$\mathbb{F}_2[x]/(x^{n+1}) \;\to\; (\mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2),\cup)$$
is an isomorphism of superalgebras.
(3) If we view $\mathbb{F}_2[x]$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 1$, then the obvious map
$$\mathbb{F}_2[x] \;\to\; (\mathrm{H}^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2),\cup)$$
is an isomorphism of superalgebras.

Proof. First recall that the $\mathbb{F}_2$-Orientability Proposition 105.13 implies that any topological manifold, in particular $\mathbb{R}\mathrm{P}^n$, is $\mathbb{F}_2$-orientable. With this observation the proof of the proposition is basically identical to the proofs of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 121.7 and 121.12. We just need to replace Lemma 121.6 by Lemma 121.15 and we need to replace the Rank One-PD Corollary 121.3 (1) by the Rank One-PD Corollary 121.3 (2). ■

Remark.

(1) In principle one could have tried to compute the cup product on $\mathbb{R}\mathrm{P}^n$ using the same approach that we had taken in Section 114.5. More precisely, suppose one has found an explicit $\mathbb{F}_2$-fundamental class σ for $\mathbb{R}\mathrm{P}^n$. We denote by $\varphi \in \mathrm{C}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ the 1-cocycle from page 2530. Now we know from the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 121.16 that $\langle \varphi^n, \sigma\rangle = 1$. But more interestingly, if one shows with an explicit calculation that $\langle \varphi^n, \sigma\rangle = 1$, then this gives a new and elementary proof of the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 121.16.
(2) We have just computed the cup product on $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$. We leave it as a highly instructive exercise to use the same approach to determine the cup product on $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{Z})$. This cup product is also determined, using a somewhat different approach, in [**Hat2002**, p. 214].
(3) In Corollary 119.14 we showed that there is no map $S^1 \times S^2$ to $\mathbb{R}\mathrm{P}^3$ of degree one, thus giving a negative answer to Question 101.11. On page 2569 and in the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 121.16 we had determined the cup product of $S^1 \times S^2$ and

$\mathbb{R}\mathrm{P}^3$ on cohomology with $\mathbb{F}_2$-coefficients. Using an argument similar to the proof of Lemma 117.5 one can easily give an alternative proof of Corollary 119.14.

For completeness' sake we mention the following analogue of Proposition 121.14.

Proposition 121.17. If $1 \leq m < n$, then $\mathbb{R}\mathrm{P}^m \subset \mathbb{R}\mathrm{P}^n$ is not a retract of $\mathbb{R}\mathrm{P}^n$.

Proof. The proof is totally analogous to the proof of Proposition 121.14. In the proof we just need to replace Lemma 121.6 by Lemma 121.15. ■

Remark. We can also give an alternative proof of Proposition 121.17 using higher homotopy groups. More precisely, let $1 \leq m < n$. By Proposition 71.10 together with the Coverings-π_n-Proposition 71.13 we know that $\pi_m(S^n) = 0$. Furthermore by Lemma 85.2 together with the Coverings-π_n-Proposition 71.13 we know that $\pi_m(S^m) \neq 0$. Since higher homotopy groups are functorial, see Proposition 71.6, we can use the standard argument, see e.g. the proof of Lemma 47.16, to show that $\mathbb{R}\mathrm{P}^m$ is not a retract of $\mathbb{R}\mathrm{P}^n$.

One of the many consequences of our calculations from this chapter is that we can now calculate the Lusternik-Schnirelmann category for $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$. More precisely, we have the following proposition that gives a complete answer to Question 115.10.

Proposition 121.18. For any $n \in \mathbb{N}_0$ we have $\mathrm{cat}(\mathbb{R}\mathrm{P}^n) = \mathrm{cat}(\mathbb{C}\mathrm{P}^n) = n$.

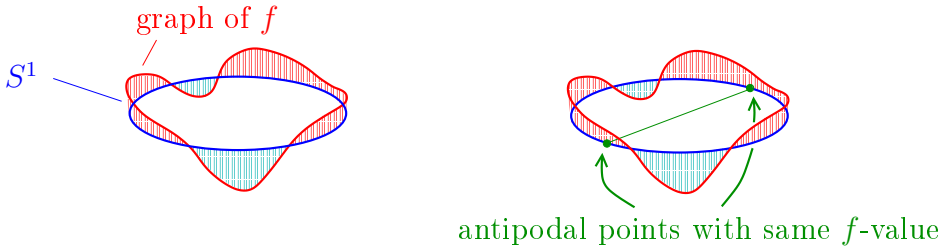
Proof. Given any $n \in \mathbb{N}_0$ we have

$$n \underset{\text{see page 2544}}{\geq} \mathrm{cat}(\mathbb{R}\mathrm{P}^n) \underset{\text{Proposition 115.14}}{\geq} \overbrace{\text{cup length cl}(\mathbb{R}\mathrm{P}^n)}^{\text{see page 2547 for the definition}} \underset{\text{Cup Product-}\mathbb{R}\mathrm{P}^n\text{-Proposition 121.16}}{\geq} n.$$

The same argument also works for $\mathbb{C}\mathrm{P}^n$, we just need to replace Propositions 121.16, by Propositions 121.7. ■

We conclude this section with an alternative proof for the Borsuk-Ulam Theorem. First we recall the statement of the Borsuk-Ulam Theorem.

Theorem 39.8. (Borsuk-Ulam Theorem) For every map $f\colon S^n \to \mathbb{R}^n$ there exists a pair of antipodal points x and $-x$ on S^n with $f(x) = f(-x)$.



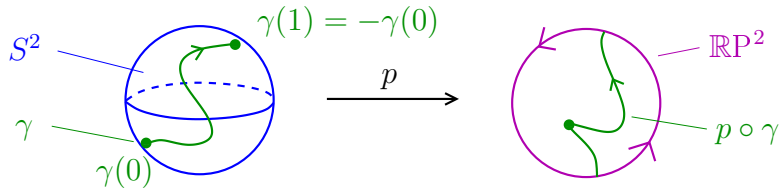
Remark. As we have pointed out on numerous occasions, the Borsuk-Ulam Theorem 39.8 has many beautiful applications, in particular the Lusternik-Schnirelmann Theorem 40.1, the Ham-Sandwich Theorem 40.2, Pfister's Level Theorem 40.9 and the Necklace Theorem 40.11.

In the new proof of the Borsuk-Ulam Theorem below we will need the following Lemma.

Lemma 121.19.

(1) Let $k \geq 2$. As usual we denote by $p\colon S^k \to \mathbb{R}\mathrm{P}^k$ the natural projection. If $\gamma\colon [0,1] \to S^k$ is a path with $\gamma(1) = -\gamma(0)$, then $p \circ \gamma$ represents the unique non-trivial element in $\pi_1(\mathbb{R}\mathrm{P}^k)$.

(2) Let $k, l \geq 2$. Furthermore let $h\colon \mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^l$ be a map. If $h_*\colon \pi_1(\mathbb{R}\mathrm{P}^k) \to \pi_1(\mathbb{R}\mathrm{P}^l)$ has non-trivial image, then the induced maps $h_*\colon \mathrm{H}_1(\mathbb{R}\mathrm{P}^k; \mathbb{F}_2) \to \mathrm{H}_1(\mathbb{R}\mathrm{P}^l; \mathbb{F}_2)$ and $h^*\colon \mathrm{H}^1(\mathbb{R}\mathrm{P}^l; \mathbb{F}_2) \to \mathrm{H}^1(\mathbb{R}\mathrm{P}^k; \mathbb{F}_2)$ are also isomorphisms.



Proof of Lemma 121.19. First we recall that for $k \geq 2$ we know from Corollary 48.18 that $\pi_1(\mathbb{R}\mathrm{P}^k) \cong \mathbb{Z}_2$.

(1) The first statement is an immediate consequence of Lemma 48.15 (1) and the fact that $\pi_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}_2$.

(2) Since $k, l \geq 2$ both fundamental groups are isomorphic to $\mathbb{Z}_2$. In particular if $h_*\colon \pi_1(\mathbb{R}\mathrm{P}^k) \to \pi_1(\mathbb{R}\mathrm{P}^l)$ has non-trivial image, then this map is already an isomorphism. It follows from the naturality of the Hurewicz homomorphism, see Proposition 84.2 (5), together with the Hurewicz Theorem 84.5 that the induced map $h_*\colon \mathrm{H}_1(\mathbb{R}\mathrm{P}^k; \mathbb{F}_2) \to \mathrm{H}_1(\mathbb{R}\mathrm{P}^l; \mathbb{F}_2)$ is also an isomorphism. Finally note that it follows from the naturality of the isomorphisms in the First Cohomology Proposition 110.17 that the induced map $h^*\colon \mathrm{H}^1(\mathbb{R}\mathrm{P}^l; \mathbb{F}_2) \to \mathrm{H}^1(\mathbb{R}\mathrm{P}^k; \mathbb{F}_2)$ is also an isomorphism. ■

Proof of the Borsuk-Ulam 39.8 for $n \geq 2$. Let us suppose that there exists a map $f\colon S^n \to \mathbb{R}^n$ such that for any $x \in S^n$ we have that $f(x) \neq f(-x)$. We consider the map

$$\begin{array}{rcl} g\colon S^n & \to & S^{n-1} \\ x & \mapsto & \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}. \end{array}$$

By our hypothesis this map is defined. It is clear that $g(-x) = -g(x)$ for all $x \in S^n$. Therefore this map g descends to a map

$$\begin{array}{rcl} h\colon \mathbb{R}\mathrm{P}^n & \to & \mathbb{R}\mathrm{P}^{n-1} \\ [x] & \mapsto & [g(x)] \end{array}$$

i.e. we get the following commutative diagram

$$\begin{array}{ccc} S^n & \xrightarrow{\quad g \quad} & S^{n-1} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ \mathbb{R}\mathrm{P}^n & \xrightarrow{[x]\mapsto h(x):=[g(x)]} & \mathbb{R}\mathrm{P}^{n-1} \end{array}$$

where p and q denote the natural projections.[1485]

[1485]The map $q \circ g\colon S^n \to \mathbb{R}\mathrm{P}^{n-1}$ is continuous. It follows from the Topological-Quotient Proposition 5.15 (1b) that the map h is also continuous.

Claim. The map $h^*\colon \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2) \to \mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ is an isomorphism.

Proof. By Lemma 121.19 (2) we only have to show that $h_*\colon \pi_1(\mathbb{R}\mathrm{P}^n) \to \pi_1(\mathbb{R}\mathrm{P}^{n-1})$ has non-trivial image. Let $\gamma\colon [0,1] \to S^n$ be a path with $\gamma(1) = -\gamma(0)$. Note that $p\circ\gamma\colon [0,1] \to \mathbb{R}\mathrm{P}^n$ is a loop. We calculate that

$$h_*([p\circ\gamma]) \;=\; [h\circ p\circ\gamma] \;\overset{\text{by the above commutative diagram}}{\overset{\downarrow}{=}}\; [q\circ(g\circ\gamma)] \qquad \underset{\uparrow}{\neq}\; e \;\in \pi_1(\mathbb{R}\mathrm{P}^{n-1}).$$

by Lemma 121.19 (1) since $(g\circ\gamma)(1) = g(\gamma(1)) = g(-\gamma(0)) = -g(\gamma(0)) = -(g\circ\gamma)(0)$ ⊞

We denote by $y \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$ and $x \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ the unique non-trivial elements. We have the following equality in $\mathrm{H}^n(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$:[1486]

$$0 \;\overset{\text{Cup Product-}\mathbb{R}\mathrm{P}^n\text{-Proposition 121.16}}{\overset{\downarrow}{\neq}}\; x^n \;\underset{\text{by the claim}}{\underset{\uparrow}{=}}\; (h^*(y))^n \;\overset{\text{Cup Product-Naturality Lemma 114.10}}{\overset{\downarrow}{=}}\; h^*(y^n) \;\underset{\text{Cup Product-}\mathbb{R}\mathrm{P}^n\text{-Proposition 121.16}}{\underset{\uparrow}{=}}\; h^*(0) \;=\; 0.$$

We have thus obtained the desired contradiction. ■

121.4. **The cup product of quaternionic and octonionic projective spaces.** In this very short section we discuss the cup product of the quaternionic projective spaces $\mathbb{H}\mathrm{P}^n$ that we introduced on page 1970. We will also calculate the cup product of the octonionic projective plane $\mathbb{O}\mathrm{P}^2$ that we defined on page 1972. These calculations might not sound all that interesting, but later in Chapter 122 we will use them to show that certain homotopy groups of spheres are non-trivial.

The following proposition is the only, and thus also the most interesting, result of this section. The statement is of course, in light of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 121.7, entirely predictable.

Proposition 121.20.

(1) Given any $n \in \mathbb{N}$ there exists an isomorphism

$$\mathbb{Z}[z]/(z^{n+1}) \;\to\; (\mathrm{H}^*(\mathbb{H}\mathrm{P}^n;\mathbb{Z}),\cup)$$

of superalgebras, where z is an element of degree 4.

(2) There exists an isomorphism

$$\mathbb{Z}[z]/(z^3) \;\to\; (\mathrm{H}^*(\mathbb{O}\mathrm{P}^2;\mathbb{Z}),\cup)$$

of superalgebras, where z is an element of degree 8.

In both cases the obvious analogues with $\mathbb{Z}$ replaced by $\mathbb{Z}_2$ also hold.

Proof.

(1) Let $n \in \mathbb{N}_0$. By the Quaternion-Projective Space Lemma 92.9 we know that $\mathbb{H}\mathrm{P}^n$ is a $4n$-dimensional closed orientable smooth manifold which admits a CW-structure that has exactly one cell in the dimensions $0, 4, 8, \ldots, 4n$. The proof of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 121.7 can be used, with very minor modifications, to prove the desired result.

[1486] The argument is very similar to the argument in the proof of Proposition 121.14.

(2) In the Octonion Plane Proposition 92.10 (3) we showed that the octonionic projective plane $\mathbb{O}\mathrm{P}^2$ is a closed orientable 16-dimensional smooth manifold which admits a CW-structure with precisely one cell in dimensions 0, 8 and 16, and which has no other cells. Once again the proof of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 121.7 clinches the deal. ■

Propositions 121.7 and 121.20 do raise a few questions, which can be viewed as a more specific versions of Question 117.19. The first question we formulate asks whether we can realize polynomial rings in other degrees.

Question 121.21. Let m be an even number and $n \in \mathbb{N}$. Does there exist a topological space X such that $(\mathrm{H}^*(X;\mathbb{Z}), \cup)$ is isomorphic to $\mathbb{Z}[z]/(z^{n+1})$ where z is now an element of degree m?

The following question is even more specific, it asks whether there exists a "cohomological $\mathbb{O}\mathrm{P}^3$.

Question 121.22. Does there exist a topological space X such that $(\mathrm{H}^*(X;\mathbb{Z}), \cup)$ is isomorphic to $\mathbb{Z}[z]/(z^4)$ where z is now an element of degree 8?

It will not take the reader very long to see that a negative answer to Question 121.22 would also tragically give a negative answer to Question 92.11.

121.5. **Division algebras.** We recall that on page 1959 we defined a *real division algebra* to be an algebra over $\mathbb{R}$ such that for all $a \neq 0$ and $b \in A$ there exists an $x \in A$ with $a \cdot x = b$ and a $y \in A$ with $y \cdot a = b$. In Section 92.1 we gave several examples of real division algebras, namely $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ (which are not commutative) and the Cayley Octonions $\mathbb{O}$ (which are neither commutative nor associative). These real division algebras are defined over $\mathbb{R}^n$ with $n = 1, 2, 4$ and 8.
These dimensions are powers of 2 and the following theorem, which was first proved by Heinz Hopf [**Hopf1940a**] in 1940, shows that this is not a coincidence.

Theorem 121.23. (Hopf 1940) If $\mathbb{R}^n$ admits the structure of a real division algebra, then n is a power of 2.

In the proof of Theorem 121.23 we will need the following lemma which is interesting in its own right.

Lemma 121.24. Let $k \in \mathbb{N}$. We denote by $\varphi \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^k;\mathbb{F}_2)$ the unique non-trivial element. We denote by p and q the two natural projections $\mathbb{R}\mathrm{P}^k \times \mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^k$ and we write $\alpha = p^*(\varphi)$ and $\beta = q^*(\varphi)$. Then the obvious homomorphism

$$\mathbb{F}_2[\alpha,\beta]/(\alpha^{k+1},\beta^{k+1}) \;\to\; \mathrm{H}^*(\mathbb{R}\mathrm{P}^k \times \mathbb{R}\mathrm{P}^k;\mathbb{F}_2)$$

of superalgebras is an isomorphism.

Proof. The lemma is an immediate consequence of Propositions 121.16 and 117.24 and the discussion on page 2586. ■

Proof of Theorem 121.23. Let "$*$" be a real division algebra structure on $\mathbb{R}^n$. We want to show that n is a power of 2. Since 1 and 2 are powers of 2 we can assume that $n \geq 3$.

As in the proof of Theorem 92.4 we consider the map

$$\begin{array}{rcl} \widetilde{f}\colon S^{n-1}\times S^{n-1} & \to & S^{n-1} \\ (x,y) & \mapsto & \widetilde{f}(x,y) := \dfrac{x*y}{\|x*y\|}. \end{array}$$

We make the following observations:

(1) In the proof of Theorem 92.4 we already showed, using the hypothesis that "$*$" defines a division algebra, that this map is well-defined, i.e. that we never divide by zero.
(2) In the proof of Theorem 92.4 we had also shown that the map $\widetilde{f}$ is continuous.
(3) Since "$*$" defines in particular an algebra we have

$$\widetilde{f}(-x,y) = -\widetilde{f}(x,y) \quad \text{and} \quad \widetilde{f}(x,-y) = -\widetilde{f}(x,y) \quad \text{for any } x,y\in S^{n-1}.$$

From (3) it follows that the above map $\widetilde{f}$ descends to a map[1487]

$$\begin{array}{rcl} f\colon \mathbb{R}\mathrm{P}^{n-1}\times\mathbb{R}\mathrm{P}^{n-1} & \to & \mathbb{R}\mathrm{P}^{n-1} \\ [\widetilde{x}],[\widetilde{y}] & \mapsto & \big[\widetilde{f}(x,y)\big]. \end{array}$$

We pick a point $z\in S^{n-1}$. We consider the inclusion maps

$$\begin{array}{rcl} i\colon \mathbb{R}\mathrm{P}^{n-1} & \to & \mathbb{R}\mathrm{P}^{n-1}\times\mathbb{R}\mathrm{P}^{n-1} \\ x & \mapsto & (x,[z]) \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon \mathbb{R}\mathrm{P}^{n-1} & \to & \mathbb{R}\mathrm{P}^{n-1}\times\mathbb{R}\mathrm{P}^{n-1} \\ x & \mapsto & ([z],x). \end{array}$$

Our first goal is to prove the following claim.

Claim 1. The map

$$(f\circ i)_*\colon \mathrm{H}_1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2) \to \mathrm{H}_1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$$

is an isomorphism. The same conclusion holds with i_* replaced by j_*.

Proof. We provide the proof for the first statement. The proof with i_* replaced by j_* is basically the same. By Lemma 121.19 (2) it suffices to show that the map

$$(f\circ i)_*\colon \pi_1(\mathbb{R}\mathrm{P}^{n-1}) \to \pi_1(\mathbb{R}\mathrm{P}^{n-1})$$

has non-trivial image. We introduce the following notation:

(1) We denote by $\pi\colon S^{n-1}\to\mathbb{R}\mathrm{P}^{n-1}$ the natural projection.
(2) We pick a path $\gamma\colon[0,1]\to S^{n-1}$ with $\gamma(1)=-\gamma(0)$. Note that $\pi\circ\gamma\colon[0,1]\to\mathbb{R}\mathrm{P}^{n-1}$ is a loop.
(3) We denote by $\widetilde{i}$ the inclusion map $S^{n-1}\to S^{n-1}\times S^{n-1}$ that is given by $\widetilde{i}(x)=(x,z)$.

Note that we get the following commutative diagram

$$\begin{array}{ccccccc} [0,1] & \xrightarrow{\gamma} & S^{n-1} & \xrightarrow{\widetilde{i}} & S^{n-1}\times S^{n-1} & \xrightarrow{\widetilde{f}} & S^{n-1} \\ & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi\times\pi} & & \downarrow{\scriptstyle\pi} \\ & & \mathbb{R}\mathrm{P}^{n-1} & \xrightarrow{i} & \mathbb{R}\mathrm{P}^{n-1}\times\mathbb{R}\mathrm{P}^{n-1} & \xrightarrow{f} & \mathbb{R}\mathrm{P}^{n-1}. \end{array}$$

Finally we note that

$$(f\circ i)_*([\pi\circ\gamma]) = [f\circ i\circ\pi\circ\gamma] \overset{\text{by the above commutative diagram}}{=} \big[\pi\circ(\widetilde{f}\circ\widetilde{i}\circ\gamma)\big] \neq e \in \pi_1(\mathbb{R}\mathrm{P}^{n-1}).$$

this follows from Lemma 121.19 (1) since the following equality holds:
$(\widetilde{f}\circ\widetilde{i}\circ\gamma)(1)=\widetilde{f}(\gamma(1),z)=\widetilde{f}(-\gamma(0),z)=-\widetilde{f}(\gamma(0),z)=-(\widetilde{f}\circ\widetilde{i}\circ\gamma)(0)$

[1487] The argument of Footnote 1485 shows that f is continuous.

⊞

We denote by $\varphi \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$ the unique non-trivial element. Furthermore, as so often we denote by p and q the two natural projections $\mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}$. Finally we write $\alpha = p^*(\varphi)$ and $\beta = q^*(\varphi) \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$.

Claim 2. The following equality holds

$$f^*(\varphi) \;=\; \alpha + \beta \;\in\; \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2).$$

Proof. We start out the proof of the claim with the following three observations:

(1) The Cohomology-Field Coefficients Proposition 110.18 says that for any topological space W the map

$$\begin{array}{rcl} \mathrm{ev}\colon \mathrm{H}^1(W;\mathbb{F}_2) & \to & \mathrm{Hom}(\mathrm{H}_1(W;\mathbb{F}_2),\mathbb{F}_2) \\ \psi & \mapsto & (\sigma \mapsto \langle \psi, \sigma\rangle) \end{array}$$

is an isomorphism.

(2) Let $x \in \mathrm{H}_1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$ be the unique non-trivial element. By the discussion on page 2499 the map

$$\begin{array}{rcl} \mathrm{H}_1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2) \oplus \mathrm{H}_1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2) & \to & \mathrm{H}_1(\mathbb{R}\mathrm{P}^{n-1}\times\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2) \\ a \oplus b & \mapsto & i_*(a) \oplus j_*(b) \end{array}$$

is an isomorphism. In particular we see that $\mathrm{H}_1(\mathbb{R}\mathrm{P}^{n-1}\times\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$ is generated by $i_*(x)$ and $j_*(x)$.

(3) Since $\mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$ and $\mathrm{H}_1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$ both contain a unique non-trivial element, namely φ and x, we obtain from Observation (1) that $\mathrm{ev}(\varphi)(x) = \langle \varphi, x\rangle = 1$.

It follows from (1) and (2) that it suffices to show that $\mathrm{ev}(f^*(\varphi))$ and $\mathrm{ev}(\alpha+\beta)$ agree on $i_*(x)$ and $j_*(y)$. In fact we have

$$\mathrm{ev}(f^*(\varphi))(i_*(x)) \underset{\text{by definition}}{=} \langle f^*(\varphi), i_*(x)\rangle \underset{\text{Lemma 109.6 (3)}}{=} \langle \varphi, f_*(i_*(x))\rangle \underset{\substack{\text{by Claim 1 since } x \text{ is unique non-zero element of } \mathrm{H}_1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)}}{=} \langle \varphi, x\rangle \underset{\text{observation (3)}}{=} 1.$$

We also calculate that

$$\mathrm{ev}(\alpha+\beta)(i_*(x)) \underset{\text{by definition}}{=} \langle p^*\varphi + q^*\varphi, i_*(x)\rangle \underset{\text{Lemma 109.6 (3)}}{=} \langle \varphi, p_*(i_*(x))\rangle + \langle \varphi, q_*(i_*(x))\rangle \underset{\text{since } p\circ i = \mathrm{id} \text{ and } q\circ i \text{ is a constant map}}{=} \langle \varphi, x\rangle \;=\; 1.$$

We have thus shown that $\mathrm{ev}(f^*(\varphi))$ and $\mathrm{ev}(\alpha+\beta)$ agree on $i_*(x)$. The same argument also works with $i_*(x)$ replaced by $j_*(y)$. ⊞

Using Lemma 121.24 we can and will make the identification

$$\mathrm{H}^*(\mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2) \;=\; \mathbb{F}_2[\alpha,\beta]/(\alpha^n,\beta^n).$$

Using this identification we see that

$$0 = f^*(0) \underset{\substack{\text{by the Cup Product-}\mathbb{R}\mathrm{P}^n\text{-Proposition 121.16,} \\ \text{since } \varphi \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)}}{=} f^*(\varphi^n) \overset{\substack{\text{by Cup Product-Naturality Lemma 114.10} \\ \downarrow}}{=} (f^*(\varphi))^n \underset{\text{by the previous claim}}{=} (\alpha+\beta)^n = \sum_{k=0}^{n}\binom{n}{k}\cdot\alpha^k\beta^{n-k} \underset{\text{since } \alpha^n=\beta^n=0}{=} \sum_{k=1}^{n-1}\binom{n}{k}\cdot\alpha^k\beta^{n-k}.$$

This equality holds in the polynomial ring $\mathbb{F}_2[\alpha,\beta]/(\alpha^n,\beta^n)$. In this ring the $n-1$ monomials $\alpha\beta^{n-1}, \alpha^2\beta^{n-2}, \dots, \alpha^{n-1}\beta$ are linearly independent over the field $\mathbb{F}_2$. This implies that for $k = 1, \dots, n-1$ we have $\binom{n}{k} \equiv 0 \bmod 2$. Thus it remains to prove the following purely algebraic claim.

Claim 3. If $n \in \mathbb{N}$ satisfies that $\binom{n}{k} \equiv 0 \bmod 2$ for $k = 1, \dots, n-1$, then n is a power of 2.

Proof. We can and will write n as a sum $n = n_1 + \cdots + n_r$ where each n_i is a power of 2 and with $n_1 < \cdots < n_r$. In the polynomial ring $\mathbb{F}_2[x]$ we obtain that

$$1 + x^n \overset{\text{by our hypothesis}}{\downarrow} = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n \overset{\text{since } n = n_1 + \cdots + n_r}{\downarrow} = (1+x)^{n_1} \cdot \cdots \cdot (1+x)^{n_r}$$
$$= (1+x^{n_1}) \cdot \cdots \cdot (1+x^{n_r}) \overset{\text{since } n_1 < \cdots < n_r}{\uparrow} = 1 + x^{n_1} + \text{higher powers.}$$
$$\overset{\text{since each } n_i \text{ is a power of 2 and since in } \mathbb{F}_2[x] \text{ we have } (a+b)^2 = a^2 + b^2}{\uparrow}$$

But this is only possible if $n = n_1$, i.e. if n itself is a power of 2. ■

In fact, with more fancy topological methods one can actually strengthen the previous theorem to obtain the following theorem which gives a complete answer to Question 92.7. Unfortunately the proof is well beyond the scope of these modest lecture notes.

Theorem 121.25. (Bott-Kervaire-Milnor 1958) Every finite-dimensional division algebra over $\mathbb{R}$ is of dimension 1,2, 4 or 8.

Proof. The theorem was first proved using topological methods by Michel Kervaire [**Kerv1958**] and John Milnor [**Miln1958a**, Corollary 1], building on the "Bott Periodicity Theorem" proved by Raoul Bott [**Bot1957, Bot1959**].[1488] Full details for the proof can also be found in [**Hat2**, Theorem 2.16]. As of right now there is no purely algebraic proof for the statement. ■

Remark. One might ask whether any finite-dimensional division algebra over $\mathbb{R}$ is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$. The statement holds under the stronger hypothesis that the algebra is "normed" or "alternative". We refer to [**BD1973**, p. 73], [**EHHKMNPR1991**, p. 262] and [**Baez2001**] for results in this direction. As is shown in [**Hat2002**, p. 59], [**Bruc1944, Osb1962**], without any extra hypotheses one cannot expect a uniqueness statement.

Exercises for Chapter 121.

Exercise 121.1. Let R be a commutative ring, let V be an R-module and let $\varphi\colon V \times V \to R$ be a symmetric R-bilinear pairing. Given a submodule $W \subset V$ we obtain a symmetric R-bilinear pairing $\varphi\colon W \times W \to R$ by restricting φ to $W \times W$.

(a) Show that if φ is non-degenerate, then it does not necessarily follow that $\varphi|_W$ is non-degenerate.

[1488] We will discuss the statement of the Bott Periodicity Theorem in Section 156.4.

(b) Let $\langle -,-\rangle\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the usual Euclidean pairing. Show that for every $\mathbb{R}$-vector subspace $W \subset \mathbb{R}^n$ the corresponding pairing $W \times W \to \mathbb{R}$ is non-singular.

(c) Let $\langle -,-\rangle\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ be the usual Euclidean pairing. Show that for a $\mathbb{Z}$-submodule $W \subset \mathbb{Z}^n$ the corresponding pairing $W \times W \to \mathbb{Z}$ is non-degenerate and show that in general it is not non-singular.

Exercise 121.2. Let V and W be free $\mathbb{Z}$-modules of finite rank, i.e. V and W are free abelian groups. Furthermore let $\lambda\colon V \times W \to \mathbb{Z}$ be a pairing. Recall that we say that λ is *non-singular* if the induced maps

$$\begin{array}{ccc} V & \to & \mathrm{Hom}_{\mathbb{Z}}(W,\mathbb{Z}) \\ v & \mapsto & \left(\begin{array}{ccc} W & \to & \mathbb{Z} \\ w & \mapsto & \lambda(v,w) \end{array}\right) \end{array} \quad \text{and} \quad \begin{array}{ccc} W & \to & \mathrm{Hom}_{\mathbb{Z}}(V,\mathbb{Z}) \\ w & \mapsto & \left(\begin{array}{ccc} V & \to & \mathbb{Z} \\ v & \mapsto & \lambda(v,w) \end{array}\right) \end{array}$$

are isomorphisms. We say that the pairing is *non-degenerate*, if the two maps are monomorphisms.

(a) Let $v_1,\dots,v_n$ be a basis for V. Show that the following two statements are equivalent:
  (i) There exists some basis $w_1,\dots,w_n$ for W such that $\lambda(v_i,w_j) = \delta_{ij}$ for every choice of $i,j \in \{1,\dots,n\}$.[1489]
  (ii) The pairing λ is non-singular.

(b) Suppose that λ is non-degenerate. Show that V and W have the same rank.

(c) Suppose that V and W are of rank n and suppose we are given $v_1,\dots,v_n \in V$ and $w_1,\dots,w_n \in W$ such that $\det(\lambda(v_i,w_j)) = \pm 1$. Show that $v_1,\dots,v_n$ is a basis for V and that $w_1,\dots,w_n$ is a basis for W.

Exercise 121.3. Let V be a free $\mathbb{Z}$-module of finite rank, i.e. V is a free abelian group. Let $\varphi\colon V \times V \to \mathbb{Z}$ be a form. Given a subgroup $U \subset V$ we write

$$U^{\perp} \;=\; \{v \in V \mid \varphi(u,v) = 0 \text{ for all } u \in U\}.$$

We suppose that the form φ is non-singular. Show that for any subgroup $U \subset V$ we have $\mathrm{rank}(U^{\perp}) = \mathrm{rank}(V) - \mathrm{rank}(U)$.

Exercise 121.4. Let $m,n \in \mathbb{N}$. We denote by $p\colon S^m \times S^n \to S^m$ and by $q\colon S^m \times S^n \to S^n$ the two natural projections. Use Poincaré Duality to show that

$$p^*([S^m]^*) \cup q^*([S^n]^*) \;=\; \pm[S^m \times S^n]^*.$$

Put differently, you should show the equality without using the Product Theorem 117.1.

Exercise 121.5. For which $n \in \mathbb{N}$ is the complex projective space $\mathbb{C}\mathrm{P}^n$ chiral, i.e. for which $n \in \mathbb{N}$ does $\mathbb{C}\mathrm{P}^n$ admit an orientation-reversing self-homeomorphism?

Exercise 121.6. Does there exist a degree one map from $\mathbb{C}\mathrm{P}^2$ to $S^2 \times S^2$?

Exercise 121.7. Is $S^3 \vee \mathbb{R}\mathrm{P}^2$ homotopy equivalent to a smooth manifold?

Exercise 121.8. Show that there exists a connected CW-complex X with base point x_0 and a homomorphism $\varphi\colon \pi_1(X,x_0) \to \mathbb{Z}_2 = \pi_1(\mathbb{R}\mathrm{P}^2,*)$ which cannot be realized by a map $g\colon X \to \mathbb{R}\mathrm{P}^2$, i.e. such that there is no map $f\colon X \to \mathbb{R}\mathrm{P}^2$ with $f(x_0) = *$ such that $f_* = \varphi$. *Remark.* This gives a negative answer to Question 118.14 (3).

[1489] The basis $w_1,\dots,w_n$ is in fact unique.

Exercise 121.9.

(a) Let $n \in \mathbb{N}$ and let M be a closed orientable $2n$-dimensional smooth manifold. Suppose that M admits a CW-structure with precisely three cells. Show that M has precisely one cell in dimensions 0, n and $2n$.

(b) Show that there is no closed orientable 6-dimensional smooth manifold that has only three cells.

(c) In (a) and (b), do we really need to assume that M is orientable?

Exercise 121.10. Let $n \in \mathbb{N}_0$ and let $\varphi\colon \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ be map.

(a) Show that there exists a $d \in \mathbb{Z}$ such that

$$\underbrace{\sum_{k=0}^{2n} (-1)^k \cdot \operatorname{tr}(\varphi_*\colon \mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^n))}_{\text{Lefschetz number } \Lambda(\varphi)} \;=\; 1 + \sum_{l=1}^{n} d^l.$$

Hint. Use the cohomological reformulation of the Lefschetz number given in Exercise 110.12.

(b) Suppose n is odd. Use the Lefschetz Fixed Point Theorem 97.9 to show that φ admits a fixed point.

Remark. In Exercise 97.8 we showed that for any even $n \in \mathbb{N}_0$ there exists a self-map of $\mathbb{C}\mathrm{P}^n$ with no fixed points.

Exercise 121.11. Let $n \in \mathbb{N}_{\geq 2}$ and let $\varphi\colon S^2 \to S^2$ be a map of degree n. We consider $X := \operatorname{Cone}(S^2 \xrightarrow{\varphi} S^2 = \mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^2)$ and $Y := \operatorname{Cone}(S^2 \xrightarrow{\varphi} S^2) \vee S^4$.

(a) Show that the cohomology rings of X and Y with $\mathbb{Z}$-coefficients are isomorphic.

(b) Show that the cohomology rings of X and Y with $\mathbb{Z}$-coefficients are not isomorphic.

122. The Hopf invariant

In previous chapters we have seen that the introduction and the calculation of the cup product allowed us to answer many questions on topological spaces that we could not deal with before. In this section we want to use the cup product to address the following question that we formulated a long time ago.

Question 71.18. Does the Hopf map $H\colon S^3 \to S^2$ represent a non-trivial element in the homotopy group $\pi_3(S^2)$?

In this chapter we will use the calculation of the cup product of $\mathbb{C}\mathrm{P}^2$ to give an affirmative answer to this question. Furthermore we will use the quaternions and octonions to define higher dimensional analogues of the Hopf map which also represent non-trivial elements in homotopy groups of spheres.

122.1. The Hopf map is not null-homotopic.

First we recall the definition of the Hopf map. On page 258 we gave an explicit identification $\mathbb{C}\mathrm{P}^1 = S^2$. Furthermore on page 1576 we made the following definition.

Definition. We refer to the map[1490]

$$\begin{array}{rcl} H\colon S^3 = \{(z_1,z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^2 = \mathbb{C}\mathrm{P}^1 \\ (z_1,z_2) & \mapsto & [z_1 : z_2] \end{array}$$

as the Hopf map.

Since the Hopf map is so much fun we will introduce two higher-dimensional analogues of the Hopf map. More precisely, we will repeat the trick with $\mathbb{C}$ replaced by the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. First we point out that we can use the homeomorphisms from the Quaternion-Projective Space Lemma 92.9 (2) and the Octonion Plane Proposition 92.10 (2) to make the identifications $\mathbb{H}\mathrm{P}^1 = S^4$ and $\mathbb{O}\mathrm{P}^1 = S^8$. Now we can introduce the quaternionic and octonionic Hopf maps.

Definition. We refer to

$$\begin{array}{rcl} H_{\mathbb{H}}\colon S^7 = \{(z_1,z_2) \in \mathbb{H}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^4 = \mathbb{H}\mathrm{P}^1 \\ (z_1,z_2) & \mapsto & [z_1 : z_2] \end{array}$$

as the quaternionic Hopf map and we refer to

$$\begin{array}{rcl} H_{\mathbb{O}}\colon S^{15} = \{(z_1,z_2) \in \mathbb{O}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^8 = \mathbb{O}\mathrm{P}^1 \\ (z_1,z_2) & \mapsto & [z_1 : z_2] \end{array}$$

as the octonionic Hopf map.

The idea for answering Question 71.18 is to translate this question about maps to a question about topological spaces for which we can then bring our new invariants to bear. To carry out this approach we recall the following definitions from earlier on:

Definition.

(1) On page 450 we defined the cone of a topological space A to be the topological space $\mathrm{Cone}(A) := (A \times [0,1])/\sim$ where $(a,0) \sim (a',0)$ for all $a, a' \in A$.

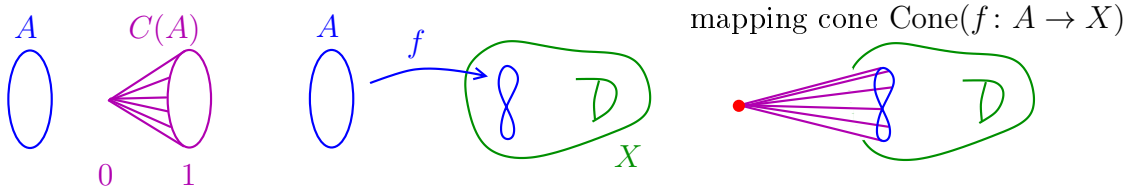
[1490] See also the figure on page 1577 for an illustration of the map.

(2) On page 457 we defined the mapping cone of a map $f\colon A \to X$ between topological spaces as

$$\text{Cone}(f\colon A \to X) \;:=\; (\text{Cone}(A) \cup X)/\!\sim$$

where $(a,1) \sim f(a)$ for all $a \in A$.

Both definitions are illustrated in the figure below.



Examples.

(1) It follows from the Compact-Hausdorff Proposition 2.17 (3) and the Topological-Quotient Proposition 5.15 (1b) that the map

$$\begin{array}{rcl} \text{Cone}(S^{n-1}) & \to & \overline{B}^n \\ [(P,t)] & \mapsto & t \cdot P \end{array}$$

is a homeomorphism. We use this homeomorphism to identify $\text{Cone}(S^{n-1})$ with $\overline{B}^n$. In particular, if X is a topological space and if $f\colon S^{n-1} \to X$ is a map, then we get an identification

$$\text{Cone}(f\colon S^{n-1} \to X) \;=\; \overline{B}^n \cup_f X.$$

It follows immediately from this observation that any finite CW-complex can be viewed as an iterated mapping cone (where we start out with a finite discrete set).

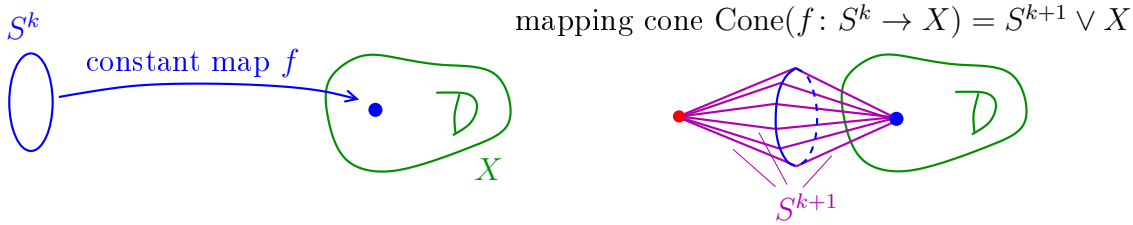
(2) In Exercise 122.1 we will see that the mapping cone of the map $f\colon S^1 \to S^1$ that is given by $f(z) = z^2$ is a topological space that we are very familiar with.

The next lemma gives us another elementary but oddly useful example.

Lemma 122.1. (Mapping Cone-of-Constant Map Lemma) If $f\colon S^k \to X$ is a constant map, then there exists a homeomorphism

$$\text{Cone}(f\colon S^k \to X) \;\xrightarrow{\cong}\; S^{k+1} \vee X,$$

where we perform the wedge for suitable points in S^{k+1} and X.



Proof. Suppose $f\colon S^k \to X$ is a constant map. It follows easily from the example on page 2677 and the homeomorphism $\overline{B}^{k+1}/S^k \cong S^{k+1}$ from the Ball-Quotient Sphere Lemma 5.20 that $\text{Cone}(f\colon S^k \to X)$ is homeomorphic to a wedge $S^{k+1} \vee X$. We leave it to the reader to fill in the details. ∎

The following lemma gives a particularly interesting example of a mapping cone.

Lemma 122.2. (Mapping Cone-of-Hopf Map Lemma) There exist homeomorphisms

$$\begin{array}{lrcl} (1) & \mathrm{Cone}(H\colon S^3 \to S^2) & \cong & \mathbb{C}\mathrm{P}^2 \\ (2) & \mathrm{Cone}(H_{\mathbb{H}}\colon S^7 \to S^4) & \cong & \mathbb{H}\mathrm{P}^2 \\ (3) & \mathrm{Cone}(H_{\mathbb{O}}\colon S^{15} \to S^8) & \cong & \mathbb{O}\mathrm{P}^2. \end{array}$$

Proof.

(1) In Lemma 68.1 we saw that $\mathbb{C}\mathrm{P}^2$ is a CW-complex that is obtained from the CW-complex $\mathbb{C}\mathrm{P}^1 = S^2$ by attaching a 4-cell. The attaching map $S^3 \to \mathbb{C}\mathrm{P}^1$ of the 4-cell is precisely the Hopf map. This observation, together with the discussion on page 2677, already implies the statement. For completeness' sake we give the argument more explicitly and in more detail. We consider the maps[1491]

$$\begin{array}{rcl} f\colon \mathrm{Cone}(S^3) & \to & \mathbb{C}\mathrm{P}^2 \\ [(z_0,z_1),r] & \mapsto & [r\cdot z_0 : r\cdot z_1 : 1-r] \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \mathbb{C}\mathrm{P}^1 & \to & \mathbb{C}\mathrm{P}^2 \\ [z_0:z_1] & \mapsto & [z_0:z_1:0]. \end{array}$$

It follows immediately from the definitions together with the Topological-Quotient Proposition 5.15 (1b) that these two maps define a continuous map

$$\underbrace{\mathrm{Cone}(H\colon S^3 \to S^2)}_{=(\mathrm{Cone}(S^3)\cup\mathbb{C}\mathrm{P}^1)/\sim} \to \mathbb{C}\mathrm{P}^2.$$

One can easily verify that this map is a bijection. Furthermore, using the Compact-Hausdorff Proposition 2.17 (3) and the Projective Spaces-Compact+Hausdorff Proposition 5.30 one sees that the map is in fact a homeomorphism.

(2) The proof of (2) is basically identical to the proof of (1).

(3) Implicitly we mostly proved this statement in the proof of the Octonion Plane Proposition 92.10. Alternatively we also refer to [**Lackm2012**, p. 4] for details. ■

Now we can finally answer Question 71.18 in the affirmative. More precisely we have the following theorem. The statement regarding the "classical" Hopf map was first proved by Heinz Hopf [**Hopf1931**] in 1931.

Theorem 122.3.

(1) The Hopf map $H\colon S^3 \to S^2$ represents a non-trivial element in the third homotopy group $\pi_3(S^2)$.

(2) We have $[H_{\mathbb{H}}] \neq 0 \in \pi_7(S^4)$.

(3) We have $[H_{\mathbb{O}}] \neq 0 \in \pi_{15}(S^8)$.

Proof. First we prove (1). Recall that for two topological spaces X and Y we write $X \simeq Y$ if they are homotopy equivalent. We have

$$\begin{array}{ccccccc} \mathrm{Cone}(\text{Hopf map } H) & \cong & \mathbb{C}\mathrm{P}^2 & \not\simeq & S^2 \vee S^4 & \cong & \mathrm{Cone}(\text{constant map}). \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Mapping Cone-of-Hopf Map Lemma 122.2 (1)} & & \text{Lemma 121.4} & & \text{Mapping Cone-of-Constant Map Lemma 122.1} & \end{array}$$

But by the Mapping Cone Inclusion Lemma 16.16 this means that the Hopf map is not homotopic to the constant map. In particular the Hopf map $H\colon S^3 \to S^2$ represents a non-trivial element in the homotopy group $\pi_3(S^2)$.

[1491] It follows from the Topological-Quotient Proposition 5.15 (1b) and the Topological-Quotient Proposition 5.15 (1a) that these two maps are continuous.

The proof of the other two statements is quite similar. We again use the Mapping Cone-of-Hopf Map Lemma 122.2 and we use the calculations of the cup products of $\mathbb{H}\mathrm{P}^2$ and $\mathbb{O}\mathrm{P}^2$, see Proposition 121.20, to conclude that the mapping cones of the quaternionic and octonionic Hopf maps are not homotopy equivalent to a wedge of spheres. ■

122.2. **The Hopf invariant.** It is natural to ask how far we can push the methods of Theorem 122.3 to gain more insight into homotopy groups of spheres. In this section, given $n \geq 2$ we will introduce the Hopf invariant

$$\mathrm{Hopf}\colon \pi_{2n-1}(S^n) \;\to\; \mathbb{Z}$$

which satisfies $\mathrm{Hopf}(H) = \pm 1$, $\mathrm{Hopf}(H_{\mathbb{H}}) = \pm 1$ and $\mathrm{Hopf}(H_{\mathbb{O}}) = \pm 1$.
The idea behind the definition of the Hopf invariant is straightforward. Namely, given $n \in \mathbb{N}_{\geq 2}$ and given a map $f\colon S^{2n-1} \to S^n$ we consider again the corresponding mapping cone $\mathrm{Cone}(f\colon S^{2n-1} \to S^n)$. We will see in Lemma 122.5 that it follows from the fact that $n \geq 2$ that we have $\mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z}) \cong \mathbb{Z}$ and $\mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z}) \cong \mathbb{Z}$. We will equip $\mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z})$ with a "preferred generator". We pick a generator $\varphi \in \mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z})$ and we define $\mathrm{Hopf}(f)$ to be the unique integer that satisfies

$$\varphi \cup \varphi \;=\; \mathrm{Hopf}(f) \cdot \text{preferred generator} \;\in\; \mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z}).$$

To carry out this program we need to carefully study the cohomology groups of a mapping cone. Recall that in the Mapping Cone-H_*-Lemma 78.17 we saw that the homology groups of a mapping cone fit naturally into a long exact sequence of homology groups. The following lemma shows that the analogous statement also holds for the cohomology groups of a mapping cone.

Lemma 122.4. (Mapping Cone-H^*-Lemma) Let R be a commutative ring.

(1) Let $f\colon A \to X$ be a map between topological spaces.
   (a) We denote by $j(f)\colon X \to \mathrm{Cone}(f) = (\mathrm{Cone}(A) \sqcup X)/\!\sim$ the obvious inclusion map.
   (b) We denote by U the image of $A \times [0, \frac{3}{4})$ in $\mathrm{Cone}(f)$ and we denote by V the complement of the image of $A \times [0, \frac{1}{4}]$. (See also the figure below.) We consider the following diagram

$$\begin{array}{ccccccc} \ldots \longrightarrow \mathrm{H}^{k-1}(U\cap V;R) & \xrightarrow{\delta} & \mathrm{H}^k(\mathrm{Cone}(f);R) & \longrightarrow & \mathrm{H}^k(U;R)\oplus \mathrm{H}^k(V;R) & \longrightarrow & \ldots \\ \downarrow{=} & \nearrow_{=:\delta^{k-1}(f)} & & & & & \\ \mathrm{H}^{k-1}(A;R). & & & & & & \end{array}$$

   Here the top sequence is the long exact sequence arising from the Mayer–Vietoris Theorem 109.15 for Cohomology Groups[1492] and the vertical map comes from the obvious identifications $\mathrm{H}^{k-1}(U\cap V;R) = \mathrm{H}^{k-1}(A\times\frac{1}{2};R) = \mathrm{H}^{k-1}(A;R)$. From the above diagram we obtain the map

$$\delta^{k-1}(f)\colon \mathrm{H}^{k-1}(A;R) \;\to\; \mathrm{H}^k(\mathrm{Cone}(f);R).$$

The following sequence is exact

$$\ldots \to \widetilde{\mathrm{H}}^k(\mathrm{Cone}(f);R) \xrightarrow{j(f)^*} \mathrm{H}^k(X;R) \xrightarrow{f^*} \mathrm{H}^k(A;R) \xrightarrow{\delta^k(f)} \widetilde{\mathrm{H}}^{k+1}(\mathrm{Cone}(f);R) \to \ldots$$

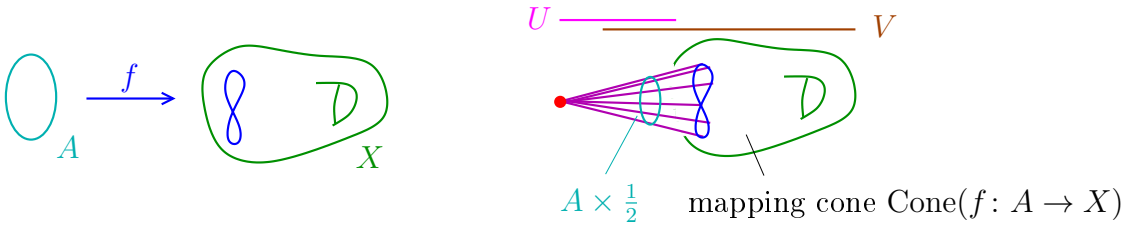
(2) The connecting homomorphism in (1) is natural. In particular given any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \widetilde{A} & \xrightarrow{\widetilde{f}} & \widetilde{X} \end{array}$$

of maps between topological spaces the following diagram commutes:

$$\begin{array}{ccccccccc} \ldots \to \widetilde{\mathrm{H}}^k(\mathrm{Cone}(f);R) & \xrightarrow{j(f)^*} & \mathrm{H}^k(X;R) & \xrightarrow{f^*} & \mathrm{H}^k(A;R) & \xrightarrow{\delta^k(f)} & \widetilde{\mathrm{H}}^{k+1}(\mathrm{Cone}(f);R) & \to \ldots \\ \uparrow & & {\scriptstyle\psi^*}\uparrow & & {\scriptstyle\varphi^*}\uparrow & & \uparrow & \\ \ldots \to \widetilde{\mathrm{H}}^k(\mathrm{Cone}(\widetilde{f});R) & \xrightarrow{j(\widetilde{f})^*} & \mathrm{H}^k(\widetilde{X};R) & \xrightarrow{\widetilde{f}^*} & \mathrm{H}^k(\widetilde{A};R) & \xrightarrow{\delta^k(\widetilde{f})} & \widetilde{\mathrm{H}}^{k+1}(\mathrm{Cone}(\widetilde{f});R) & \to \ldots \end{array}$$

Here the vertical maps between the cohomology groups of the mapping cones are induced by the maps from the Mapping Cone Functor Lemma 16.14 (3).



Proof. The proof is basically identical to the proof of the Mapping Cone-H_*-Lemma 78.17. The meticulous reader who filled in the details of the proof of the Mapping Cone-H_*-Lemma 78.17 will have no troubles working out the details of the present proof. ■

Lemma 122.5. Let $f\colon S^m \to S^n$ be a map. If $m \geq n+1$ and $n \geq 1$, then the maps

$$\mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z}) \xrightarrow{j(f)^*} \mathrm{H}^n(S^n;\mathbb{Z})$$

and

$$\mathrm{H}^m(S^m;\mathbb{Z}) \xrightarrow{\delta^m(f)} \mathrm{H}^{m+1}(\mathrm{Cone}(f);\mathbb{Z})$$

are isomorphisms and all other reduced cohomology groups of $\mathrm{Cone}(f)$ are zero. An analogous statement holds for homology groups. In particular we have isomorphisms

$$\mathrm{H}_k(\mathrm{Cone}(f\colon S^m \to S^n);\mathbb{Z}) \cong \mathrm{H}^k(\mathrm{Cone}(f\colon S^m \to S^n);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, n, m+1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The lemma is an almost immediate consequence of the Mapping Cone-H^*-Lemma 122.4 and the calculation of the cohomology groups of spheres, see page 2422. For $n = 1$ one also needs to invoke Exercise 108.6. ■

[1492]Note that U and V are open subsets of $\mathrm{Cone}(f)$. Thus it follows from the Excisive Triad-Proposition 109.12 that we can in fact apply the theorem.

Throughout this section we use the notation that we had already introduced in the formulation of the Mapping Cone-H*-Lemma 122.4. More precisely, given a map $f\colon A \to X$ between two topological spaces we denote by $j(f)\colon X \to \mathrm{Cone}(f) = (\mathrm{Cone}(A) \cup X)/\sim$ the obvious inclusion map and furthermore we denote by $\delta^{k-1}(f)\colon \mathrm{H}^{k-1}(A;\mathbb{Z}) \to \mathrm{H}^k(\mathrm{Cone}(f);\mathbb{Z})$ the coboundary map in the long exact sequence in cohomology of the Mapping Cone-H*-Lemma 122.4.
Now we are finally in a position to turn the idea from page 2679 into a proper definition.

Definition. Let $f\colon S^{2n-1} \to S^n$ be a map with $n \geq 2$. The Hopf invariant $\mathrm{Hopf}(f)$ is defined to be the unique integer that satisfies[1493]

$$\underbrace{(j(f)^*)^{-1}([S^n]^*)}_{\substack{\text{generator of}\\ \mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z})}} \cup \underbrace{(j(f)^*)^{-1}([S^n]^*)}_{\substack{\text{generator of}\\ \mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z})}} = \mathrm{Hopf}(f) \cdot \underbrace{\delta^{2n-1}(f)([S^{2n-1}]^*)}_{\substack{\text{generator of}\\ \mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z})}}$$

in $\mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z}) \cong \mathbb{Z}$.

Remark.

(1) In the definition of the Hopf invariant we could have squared any of the two generators of $\mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z}) \cong \mathbb{Z}$, since the sign ambiguity goes away with the square.
(2) It follows immediately from the (anti-) commutativity of the cup product, see the Cup Product-Commutativity Proposition 114.8, that the Hopf invariant is zero for all odd n.

Example.

(1) As usual we denote by $H\colon S^3 \to S^2$ the Hopf map. In the Mapping Cone-of-Hopf Map Lemma 122.2 we showed that the mapping cone $\mathrm{Cone}(H)$ is homeomorphic to $\mathbb{C}\mathrm{P}^2$. By Lemma 122.5 we know that $(j(H)^*)^{-1}([S^2]^*) \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ and $\delta^3(f)([S^3]^*) \in \mathrm{H}^4(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ are generators. It follows from the calculation of the cup product on $\mathbb{C}\mathrm{P}^2$, see the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 121.7, that $\mathrm{Hopf}(H) = -1$ or $\mathrm{Hopf}(H) = 1$.[1494]
(2) Using Proposition 121.20 one can show, by the same argument as in (1), that the Hopf invariants of the quaternionic Hopf map $H_{\mathbb{H}}\colon S^7 \to S^4$ and the octonionic Hopf map $H_{\mathbb{O}}\colon S^{15} \to S^8$ are equal to ± 1.

Lemma 122.6. Let $n \geq 2$ and let $f_0, f_1\colon S^{2n-1} \to S^n$ be two maps. If f_0 and f_1 are homotopic, then $\mathrm{Hopf}(f_0) = \mathrm{Hopf}(f_1)$.

Proof. Let $H\colon S^{2n-1} \times [0,1] \to S^n$ be a homotopy between f_0 and f_1. We denote by $i_0, i_1\colon S^{2n-1} \to S^{2n-1} \times [0,1]$ the two obvious inclusion maps. We obtain the following two commutative diagrams

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{f_0} & S^n \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle =} \\ S^{2n-1}\times[0,1] & \xrightarrow{H} & S^n \end{array} \qquad \text{and} \qquad \begin{array}{ccc} S^{2n-1} & \xrightarrow{f_1} & S^n \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle =} \\ S^{2n-1}\times[0,1] & \xrightarrow{H} & S^n. \end{array}$$

[1493] Here we implicitly use that in Lemma 122.5 we showed that $j(f)^*$ and $\delta^{2n-1}(f)$ are isomorphisms.
[1494] With more care one can show that in fact $\mathrm{Hopf}(H) = 1$, but this is irrelevant for our purpose.

By the Mapping Cone Functor Lemma 16.14 (3) the diagrams give rise to inclusion maps $\Xi_0\colon \operatorname{Cone}(f_0) \to \operatorname{Cone}(H)$ and $\Xi_1\colon \operatorname{Cone}(f_1) \to \operatorname{Cone}(H)$. From the Mapping Cone-H*-Lemma 122.4 we obtain the following commutative diagram of long exact sequences:

$$\begin{array}{ccccccccc}
\ldots \to \widetilde{\mathrm{H}}^k(\operatorname{Cone}(f_0)) & \xrightarrow{j(f_0)^*} & \mathrm{H}^k(S^n) & \xrightarrow{f_0^*} & \mathrm{H}^k(S^{2n-1}) & \xrightarrow{\delta} & \widetilde{\mathrm{H}}^{k+1}(\operatorname{Cone}(f_0)) \to \ldots \\
\uparrow{\scriptstyle \Xi_0^*} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \cong\, i_0^*} & & \uparrow{\scriptstyle \Xi_0^*} \\
\ldots \to \widetilde{\mathrm{H}}^k(\operatorname{Cone}(H)) & \xrightarrow{j(H)^*} & \mathrm{H}^k(S^n) & \xrightarrow{\mathrm{H}^*} & \mathrm{H}^k(S^{2n-1}\times[0,1]) & \xrightarrow{\delta} & \widetilde{\mathrm{H}}^{k+1}(\operatorname{Cone}(H)) \to \ldots \\
\downarrow{\scriptstyle \Xi_1^*} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \cong\, i_1^*} & & \downarrow{\scriptstyle \Xi_1^*} \\
\ldots \to \widetilde{\mathrm{H}}^k(\operatorname{Cone}(f_1)) & \xrightarrow{j(f_1)^*} & \mathrm{H}^k(S^n) & \xrightarrow{f_1^*} & \mathrm{H}^k(S^{2n-1}) & \xrightarrow{\delta} & \widetilde{\mathrm{H}}^{k+1}(\operatorname{Cone}(f_1)) \to \ldots
\end{array}$$

Note that the vertical maps between the cohomology groups of the mapping cones are isomorphisms. This can be shown using the Five Lemma 74.10, or alternatively, in a more fancy way using the Mapping Cone Inclusion Lemma 16.16.

Next we consider the generator $\Theta := (i_0^*)^{-1}([S^{2n-1}]^*) = (i_1^*)^{-1}([S^{2n-1}]^*)$ of the group $\mathrm{H}^{2n-1}(S^{2n-1}\times[0,1];\mathbb{Z})$. We define a number $\operatorname{Hopf}(H)$ via the equality

$$(j(H)^*)^{-1}([S^n]^*) \cup (j(H)^*)^{-1}([S^n]^*) \;=\; \operatorname{Hopf}(H)\cdot\delta^{2n-1}(H)(\Theta)$$

in $\mathrm{H}^{2n}(\operatorname{Cone}(H);\mathbb{Z}) \cong \mathbb{Z}$. Finally we apply the above commutative diagram and we apply the Cup Product-Naturality Lemma 114.10 to the maps $\Xi_0\colon \operatorname{Cone}(f_0) \to \operatorname{Cone}(H)$ and $\Xi_1\colon \operatorname{Cone}(f_1) \to \operatorname{Cone}(H)$ and we see that $\operatorname{Hopf}(f_0) = \operatorname{Hopf}(H) = \operatorname{Hopf}(f_1)$. ■

Definition. It follows from Lemma 122.6 that for each $n \geq 2$ the map

$$\begin{array}{rcl} \pi_{2n-1}(S^n,*) & \to & \mathbb{Z} \\ [f] & \mapsto & \operatorname{Hopf}(f) \end{array}$$

is well-defined. This map is called the Hopf invariant.

Our next proposition says that the Hopf invariant is a homomorphism. This is arguably somewhat surprising since at first glance the definition of the Hopf variant, given by squaring an element, does not look particularly like a linear map.

Proposition 122.7. For any $n \geq 2$ the Hopf invariant $\operatorname{Hopf}\colon \pi_{2n-1}(S^n,*) \to \mathbb{Z}$ is a homomorphism.

Together with the calculation of the Hopf invariant for the Hopf maps H, $H_{\mathbb{H}}$ and $H_{\mathbb{O}}$ on page 2681 we obtain the following immediate corollary to Proposition 122.7.

Corollary 122.8. The Hopf invariants

$$\begin{array}{lrcl} (1) & \operatorname{Hopf}\colon \pi_3(S^2) & \to & \mathbb{Z} \\ (2) & \operatorname{Hopf}\colon \pi_7(S^4) & \to & \mathbb{Z} \\ (3) & \operatorname{Hopf}\colon \pi_{15}(S^8) & \to & \mathbb{Z} \end{array}$$

are epimorphisms.

The technically slightly dicey proof of Proposition 122.7 will require the remainder of this section. The reader is cordially invited to skip the argument and to jump to the much more entertaining content of the next section.

Our proof[1495] of Proposition 122.7 will require a few preparations. First of all, on several occasions we will need the following construction.

Construction 122.9. Suppose we are given the following commutative diagram of topological spaces:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y}. \end{array}$$

Then, as discussed in the Mapping Cone Functor Lemma 16.14 (3), we obtain an induced map

$$\begin{array}{rcl} \mathrm{Cone}(f\colon X \to Y) & \to & \mathrm{Cone}(f\colon \widetilde{X} \to \widetilde{Y}) \\ [P] & \mapsto & \begin{cases} [(\varphi(x), t)], & \text{if } P = (x,t) \text{ with } x \in X,\ t \in [0,1], \\ [\psi(y)], & \text{if } P = y \text{ with } y \in Y \end{cases} \end{array}$$

By the Mapping Cone Functor Lemma 16.14 (4) the map fits into the following commutative diagram:

$$\begin{array}{ccccc} Y & \xhookrightarrow{j} & \mathrm{Cone}(f\colon X \to Y) & \xrightarrow{p} & \Sigma(X) \\ {\scriptstyle\psi}\downarrow & & \downarrow & & \downarrow{\scriptstyle\Sigma(\varphi)} \\ \widetilde{Y} & \xhookrightarrow{j} & \mathrm{Cone}(\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}) & \xrightarrow{p} & \Sigma(\widetilde{X}) \end{array}$$

where the horizontal maps to the left are the inclusions maps i that we introduced in the Mapping Cone Basics Lemma 16.13 (2) and where the horizontal maps p on the right are given by the projection and the homeomorphism from the Mapping Cone Basics Lemma 16.13 (4).

Finally we will need the following slight reinterpretation of the Hopf invariant.

Lemma 122.10. Let $f\colon S^{2n-1} \to S^n$ be a map. We denote by $j\colon S^n \to \mathrm{Cone}(f)$ the inclusion map and we denote by $p\colon \mathrm{Cone}(f) \to \mathrm{Cone}(f)/S^n = \Sigma(S^{2n-1})$ the map given by the projection and homeomorphism from the Mapping Cone Basics Lemma 16.13 (4). Then

$$(j^*)^{-1}([S^n]^*) \cup (j^*)^{-1}([S^n]^*) \;=\; \mathrm{Hopf}(f) \cdot p^*(\Sigma([S^{2n-1}]^*)) \;\in\; \mathrm{H}^{2n}(\mathrm{Cone}(f); \mathbb{Z}).$$

Proof of Lemma 122.10. The statement follows basically immediately from the discussion on page 2435[1496] and the cohomological analogue of Lemma 78.19. ∎

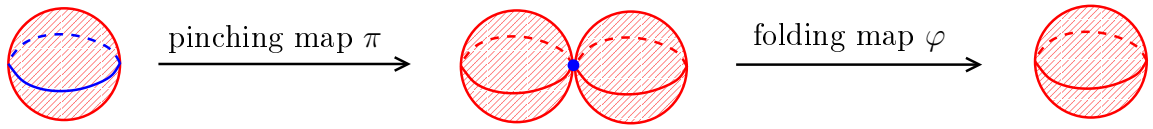
Proof of Proposition 122.7. We start out with a few preparations.

(1) Given a pointed topological space (X, x_0) we define $X \vee X$ by identifying two copies of X along the base point. We refer to the map $\varphi\colon X \vee X \to X$ which is given by the identity on either of the two copies as the *folding map*.
(2) Let $k \in \mathbb{N}_{\geq 2}$. On page 1565 we introduced a pinching map $\pi\colon S^k \to S^k \vee S^k$. The precise definition is rather irrelevant to us right now. What matters is that we saw on page 1566 that given any pointed topological space (X, x_0) and given any two

[1495] Our proof is based on the proof in [**MTa1968**, p. 35]. An alternative proof is sketched in [**Stro2011**, p. 557].
[1496] Note that here we use the convenient fact that $(-1)^{(2n-1)+1} = +1$. In reality the sign is largely irrelevant, as long as we can be sure that the sign depends only on n.

maps $f_1, f_2 \colon (S^k, *) \to (X, x_0)$ we have

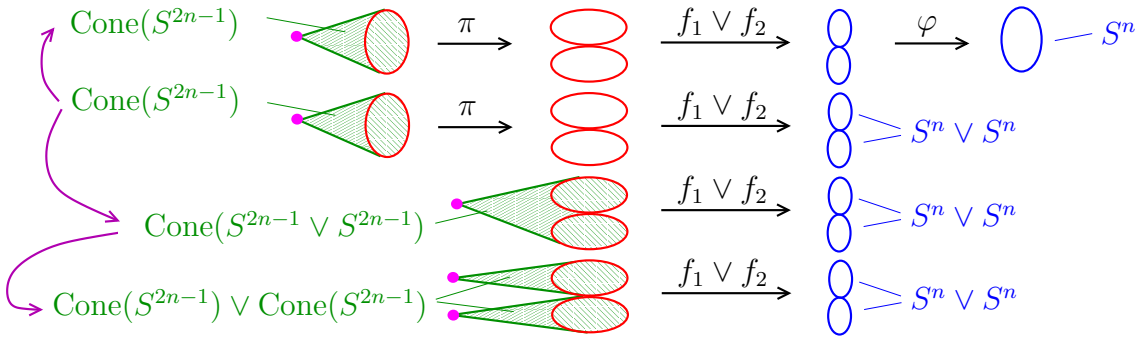
$$[f_1 * f_2] \;=\; \left[S^k \xrightarrow{\pi} S^k \vee S^k \xrightarrow{f_1 \vee f_2} X \vee X \xrightarrow{\varphi} X\right] \;\in\; \pi_k(X, x_0).$$



Now we turn to our actual situation. Let $n \in \mathbb{N}_{\geq 2}$. Suppose we are given two maps $f_1, f_2 \colon (S^{2n-1}, *) \to (S^n, *)$. For $i = 1, 2$ we consider the following diagram

$$\begin{array}{ccccc}
S^n & \xrightarrow{j} & \mathrm{Cone}(\varphi \circ (f_1 \vee f_2) \circ \pi \colon S^{2n-1} \to S^n) & \xrightarrow{p} & \Sigma S^{2n-1} \\
{\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle=} \\
S^n \vee S^n & \xrightarrow{j} & \mathrm{Cone}((f_1 \vee f_2) \circ \pi \colon S^{2n-1} \to S^n \vee S^n) & \xrightarrow{p} & \Sigma S^{2n-1} \\
{\scriptstyle=}\downarrow & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\Sigma(\pi)} \\
S^n \vee S^n & \xrightarrow{j} & \mathrm{Cone}(f_1 \vee f_2 \colon S^{2n-1} \vee S^{2n-1} \to S^n \vee S^n) & \xrightarrow{p} & \Sigma(S^{2n-1} \vee S^{2n-1}) \\
{\scriptstyle=}\uparrow & & \uparrow{\scriptstyle\Xi} & & \uparrow{\scriptstyle\Omega} \\
S^n \vee S^n & \xrightarrow{j \vee j} & \mathrm{Cone}(f_1 \colon S^{2n-1} \to S^n) \vee \mathrm{Cone}(f_2 \colon S^{2n-1} \to S^n) & \xrightarrow{p \vee p} & \Sigma S^{2n-1} \vee \Sigma S^{2n-1} \\
{\scriptstyle q_i}\downarrow & & \downarrow{\scriptstyle q_i} & & \downarrow{\scriptstyle q_i} \\
S^n & \xrightarrow{j} & \mathrm{Cone}(f_i \colon S^{2n-1} \to S^n) & \xrightarrow{p} & \Sigma S^{2n-1}.
\end{array}$$

The top three vertical maps are illustrated in the figure below.



We make the following clarifications:

(1) The horizontal maps to the left are the inclusions maps j introduced in the Mapping Cone Basics Lemma 16.13 (2) and the horizontal maps p on the right are given by the projection and the homeomorphism from the Mapping Cone Basics Lemma 16.13 (4).

(2) The vertical maps between the first and the second and between the second and the third row are obtained from Construction 122.9 applied to the following two commutative diagrams

$$\begin{array}{ccc}
S^{2n-1} & \xrightarrow{(f_1 \vee f_2) \circ \pi} & S^n \vee S^n \\
{\scriptstyle=}\downarrow & & \downarrow{\scriptstyle\varphi} \\
S^{2n-1} & \xrightarrow{\varphi \circ (f_1 \vee f_2) \circ \pi} & S^n
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
S^{2n-1} & \xrightarrow{(f_1 \vee f_2) \circ \pi} & S^n \vee S^n \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle=} \\
S^{2n-1} \vee S^{2n-1} & \xrightarrow{f_1 \vee f_2} & S^n \vee S^n.
\end{array}$$

In particular we see that the four upper squares commute.

(3) Next we consider the two diagrams

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{\quad f_i \quad} & S^n \\ \downarrow & & \downarrow \\ S^{2n-1} \vee S^{2n-1} & \xrightarrow{f_1 \vee f_2} & S^n \vee S^n \end{array}$$

which are given by the injections into the first respectively second factor. By Construction 122.9 these diagrams give rise to two sequences of maps. We "wedge" the two sequences and we obtain the vertical maps from the fourth to the third row.

(4) At the bottom, in each column the map q_i is the projections onto the i-th factor.

Next we apply the cohomology functor $X \mapsto \mathrm{H}^k(X) = \mathrm{H}^k(X;\mathbb{Z})$ with $k = n$ and $k = 2n$ to the above diagram (except that at the bottom we consider both maps q_1 and q_2). We obtain the following commutative diagram:[1497]

$$\begin{array}{ccccc}
\mathrm{H}^k(S^n) & \xleftarrow[\cong]{j^*} & \mathrm{H}^k(\mathrm{Cone}(\varphi\circ(f_1\vee f_2)\circ\pi : S^{2n-1}\to S^n)) & \xleftarrow[\cong]{p^*} & \mathrm{H}^k(\Sigma S^{2n-1}) \\
{\scriptstyle\varphi^*}\uparrow & & \downarrow{\scriptstyle\varphi^*} & & \downarrow{\scriptstyle =} \\
\mathrm{H}^k(S^n\vee S^n) & \xleftarrow[\cong]{j^*} & \mathrm{H}^k(\mathrm{Cone}((f_1\vee f_2)\circ\pi : S^{2n-1}\to S^n\vee S^n)) & \xleftarrow[\cong]{p^*} & \mathrm{H}^k(\Sigma S^{2n-1}) \\
{\scriptstyle =}\uparrow & & \uparrow{\scriptstyle\psi^*} & & \uparrow{\scriptstyle\Sigma(\pi)^*} \\
\mathrm{H}^k(S^n\vee S^n) & \xleftarrow[\cong]{j^*} & \mathrm{H}^k(\mathrm{Cone}(f_1\vee f_2 : S^{2n-1}\vee S^{2n-1}\to S^n\vee S^n)) & \xleftarrow[\cong]{p^*} & \mathrm{H}^k(\Sigma(S^{2n-1}\vee S^{2n-1})) \\
{\scriptstyle =}\downarrow & & \cong\downarrow{\scriptstyle\Xi^*} & & \cong\downarrow{\scriptstyle\Omega^*} \\
\mathrm{H}^k(S^n\vee S^n) & \xleftarrow[\cong]{(j\vee j)^*} & \mathrm{H}^k(\mathrm{Cone}(f_1)\vee\mathrm{Cone}(f_2)) & \xleftarrow[\cong]{(p\vee p)^*} & \mathrm{H}^k(\Sigma S^{2n-1}\vee\Sigma S^{2n-1}) \\
{\scriptstyle q_1^*\oplus q_2^*}\uparrow\cong & & \cong\uparrow{\scriptstyle q_1^*\oplus q_2^*} & & \cong\uparrow{\scriptstyle q_1^*\oplus q_2^*} \\
\mathrm{H}^k(S^n)\oplus\mathrm{H}^k(S^n) & \xleftarrow[\cong]{j_1^*\oplus j_2^*} & \mathrm{H}^k(\mathrm{Cone}(f_1))\oplus\mathrm{H}^k(\mathrm{Cone}(f_2)) & \xleftarrow[\cong]{p_1^*\oplus p_2^*} & \mathrm{H}^k(\Sigma S^{2n-1})\oplus\mathrm{H}^k(\Sigma S^{2n-1}).
\end{array}$$

We make the following observations:

(1) It follows almost immediately from the Mapping Cone-H*-Lemma 122.4 that the horizontal maps to the left are isomorphisms in degree n. Since these isomorphisms are only in degree n they are colored in red.

(2) It follows easily from the Mapping Cone-H*-Lemma 122.4 and the cohomological analogue of Lemma 78.19 that the horizontal maps to the right are isomorphisms for $k = 2n$. Since these isomorphisms are only in degree $2n$ they are colored in green.

(3) Similar to the calculation in Exercise 79.1 we see that the map Ω^* is an isomorphism.

(4) It follows from (1), (2) and (3) that the map Ξ^* is an isomorphism.

(5) The vertical maps at the very bottom are induced by the natural projections. The maps are isomorphisms by the Wedge-H*-Proposition 109.17.

Next we denote by $\alpha = [S^n]^* \in \mathrm{H}^n(S^n)$ the dual fundamental class. Furthermore we write $\beta = ([\Sigma([S^{2n-1}]^*)]) \in \mathrm{H}^{2n}(\Sigma S^{2n-1})$.

Claim 1. We have $\varphi^*(\alpha) = q_1^*(\alpha) + q_2^*(\alpha)$ and $((\Sigma(\pi)^* \circ (\Omega^*)^{-1})(q_1^*(\beta) + q_2^*(\beta)) = \beta$.

Proof. We leave it to the reader to verify this claim.

[1497] Any reader who knows of a program which can automatically turn the latex-code for a diagram of topological spaces into the latex-code for the corresponding diagram of cohomology groups should contact me urgently.

The key calculation takes place in the fourth row. More precisely, we have the following claim.

Claim 2. The following equality holds in $\mathrm{H}^{2n}(\mathrm{Cone}(f_1) \vee \mathrm{Cone}(f_2))$:

$$\big((j \vee j)^*\big)^{-1}(q_1^*(\alpha) + q_2^*(\alpha))\big)^2 \;=\; (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot (p \vee p)^*(q_1^*(\beta) + q_2^*(\beta)).$$

Proof. We calculate that

$$\begin{array}{l}
\big((j \vee j)^{*-1}(q_1^*\alpha + q_2^*\alpha)\big)^2 \overset{\text{since } j_i \circ q_i = q_i \circ (j \vee j)\colon S^n \vee S^n \to \mathrm{Cone}(f_i\colon S^{2n-1} \to S^n)}{=} \big(q_1^*(j_1^{*-1}\alpha) + q_2^*(j_2^{*-1}\alpha)\big) \cup \big(q_1^*(j_1^{*-1}\alpha) + q_2^*(j_2^{*-1}\alpha)\big) \\
\underset{\text{by the Suspension-Cup Product Lemma 115.7 the ``mixed'' terms are zero}}{=} q_1^*(j_1^{*-1}\alpha) \cup q_1^*(j_1^{*-1}\alpha) + q_2^*(j_2^{*-1}\alpha) \cup q_2^*(j_2^{*-1}\alpha) \underset{\text{Cup Product-Naturality Lemma 114.10}}{=} q_1^*(j_1^{*-1}\alpha \cup j_1^{*-1}\alpha) + q_2^*(j_2^{*-1}\alpha \cup j_2^{*-1}\alpha) \\
\underset{\text{by the reinterpretation of } \mathrm{Hopf}(f_1) \text{ and } \mathrm{Hopf}(f_2) \text{ given in Lemma 122.10}}{=} q_1^*(\mathrm{Hopf}(f_1) \cdot p_1^*(\beta)) + q_2^*(\mathrm{Hopf}(f_2) \cdot p_2^*(\beta)) \underset{\text{since } p_i \circ q_i = q_i \circ (p \vee p)\colon\ \mathrm{Cone}(f_1) \vee \mathrm{Cone}(f_2) \to \Sigma S^{2n-1}}{=} (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot (p \vee p)^*(q_1^*(\beta) + q_2^*(\beta)).
\end{array}$$

⊞

It follows from $[f_1 * f_2] = [\varphi \circ (f_1 \vee f_2) \circ \pi]$, the definition of α and β and the reinterpretation of the Hopf invariant given in Lemma 122.10, that it remains to prove the following claim.

Claim 3. The following equality holds in the first row:

$$((j^*)^{-1}(\alpha))^2 \;=\; (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot p^*(\beta).$$

Proof. The idea is of course to push the equality from Claim 2 through the above commutative diagram towards the top. The key tool hereby is the naturality of the cup product, see the Cup Product-Naturality Lemma 114.10. First we note that it follows from the Cup Product-Naturality Lemma 114.10 and the fact that φ^* is an isomorphism in degree $2n$ that it remains to prove that the following equality holds in the second row:

$$(*) \qquad (j^*)^{-1}(\varphi^*(\alpha))^2 \;=\; (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot p^*(\beta).$$

To prove this equality we note that it follows from Claims 1 and 2 together with the Cup Product-Naturality Lemma 114.10 that the following equality holds in the third row:

$$((j^*)^{-1}(\varphi^*(\alpha)))^2 \;=\; (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot p^*((\Omega^*)^{-1}(q_1^*(\beta) + q_2^*(\beta))).$$

Again, using the Cup Product-Naturality Lemma 114.10 we see that the following equality holds in the second row:

$$((j^*)^{-1}(\varphi^*(\alpha)))^2 \;=\; (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot p^*(\underbrace{\Sigma(\pi)^*((\Omega^*)^{-1}(q_1^*(\beta) + q_2^*(\beta)))}_{=\beta \text{ by Claim 1}}).$$

But using Claim 1 we see that this is precisely the desired equality that we formulated in $(*)$. ■

122.3. **The Hopf invariant and multiplication maps.** By a *multiplication map* we mean any map of the form $f\colon S^k \times S^k \to S^k$. In this section we will see that such multiplication maps give rise to potentially interesting maps $\Theta_f\colon S^{2k+1} \to S^k$. In the key calculation of this section we show how we can determine the Hopf invariant $\mathrm{Hopf}(\Theta_f)$ directly in terms of homological data of the original map f.

We start out with recalling the following not terribly inspired definition from Exercise 76.20.

Definition. Let $n \in \mathbb{N}$ and let $f\colon S^n \times S^n \to S^n$ be a map. We pick $x_0, y_0 \in S^n$. We define the bidegree of f as[1498]

$$\operatorname{bideg}(f) := (p,q) \text{ where } p := \deg\begin{pmatrix} S^n & \to & S^n \\ x & \mapsto & f(x,y_0) \end{pmatrix} \text{ and } q := \deg\begin{pmatrix} S^n & \to & S^n \\ y & \mapsto & f(x_0,y) \end{pmatrix}.$$

Examples.

(1) The map $f\colon S^1 \times S^1 \to S^1$ that is given by complex multiplication has bidegree $(1,1)$. This can be seen as follows: taking $x_0 = y_0 = 1$, we see that the two maps $S^1 \to S^1$ given by $z \mapsto f(z,1)$ and $z \mapsto f(1,z)$ are in fact the identity.
(2) The same argument as in (1) shows that the quaternion multiplication $S^3 \times S^3 \to S^3$ is a map of bidegree $(1,1)$ and that the octonion multiplication $S^7 \times S^7 \to S^7$ is also a map of bidegree $(1,1)$.
(3) Let $n \in \mathbb{N}$ and let $f\colon S^n \times S^n \to S^n$ be a map of bidegree (p,q). Furthermore let $\varphi, \psi\colon S^n \to S^n$ two maps. It follows almost immediately from the definitions that the bidegree of the map $(x,y) \mapsto f(\varphi(x), \psi(y))$ equals $(p \cdot \deg(\varphi), q \cdot \deg(\psi))$.

The previous example shows that the division algebras $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ give rise to maps of bidegree $(1,1)$. This result can be generalized to all division algebras.

Lemma 122.11. Let $n \in \mathbb{N}_{\geq 2}$. If $\mathbb{R}^n$ admits the structure of a division algebra[1499], then there exists a map $S^{n-1} \times S^{n-1} \to S^{n-1}$ of bidegree (p,q) where both p and q are odd.

Proof. Let $*$ be the multiplication map of a division algebra on $\mathbb{R}^n$. As in the proof of Theorem 92.4 we consider the map

$$\begin{array}{rcl} f\colon S^{n-1} \times S^{n-1} & \to & S^{n-1} \\ (x,y) & \mapsto & f(x,y) := \dfrac{x*y}{\|x*y\|}. \end{array}$$

In the proof of Theorem 92.4 we already showed, using the hypothesis that "$*$" defines a division algebra, that this map is well-defined and that it is continuous. Now we pick a point $\star \in S^{n-1}$ and we consider the maps

$$\begin{array}{rcl} \varphi\colon S^{n-1} & \to & S^{n-1} \\ x & \mapsto & f(x,\star) \end{array} \quad \text{and} \quad \begin{array}{rcl} \psi\colon S^{n-1} & \to & S^{n-1} \\ y & \mapsto & f(\star,y). \end{array}$$

It follows easily from the fact that $*$ is the multiplication of a real algebra that for all $z \in S^{n-1}$ we have $\varphi(-z) = -\varphi(z)$ and $\psi(-z) = -\psi(z)$. By the Odd Degree Proposition 91.3 this implies that $\deg(\varphi)$ and $\deg(\psi)$ are odd. Conveniently enough that is exactly what he had set out to show. ∎

[1498] In Exercise 76.20 (a) we saw that the definition does not depend on the choice of x_0 and y_0.

[1499] Recall that by the definition on page 1959 a division algebra is an algebra that has left and right inverses. We do not assume that division algebras are associative and we do not assume that they preserve the scalar product.

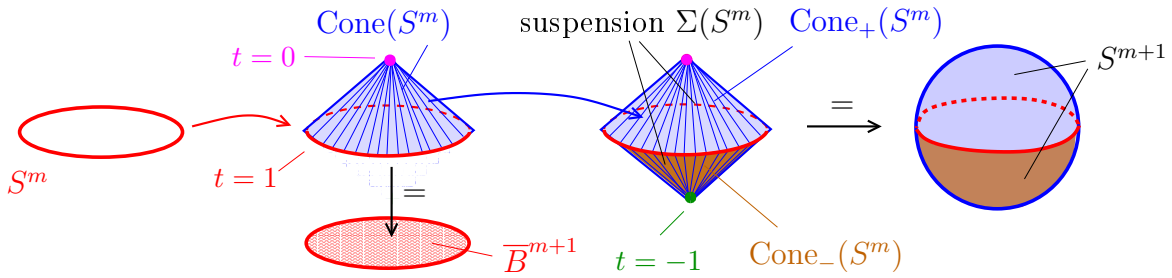
Before we continue we recall a few conventions that we had introduced along the way. Most of them are also illustrated in the figure below.

(1) Given a topological space we consider the cone $\mathrm{Cone}(X)$, the suspension $\Sigma(X)$ and we consider the two copies $\mathrm{Cone}_{\pm}(X)$ of the cone in the suspension.[1500]

(2) Let $m \in \mathbb{N}_0$. Note that by the discussion on page 450 we can make the identification $\mathrm{Cone}(S^m) = \overline{B}^{m+1}$ and by the discussion on page 452 we can make the identification $\Sigma(S^m) = S^{m+1}$.

(3) Let $m, n \in \mathbb{N}_0$. By the discussion on page 263 we can make the identification

$$(S^m \times \overline{B}^{n+1}) \cup_{\mathrm{id}_{S^m \times S^n}} (\overline{B}^{m+1} \times S^n) \xrightarrow{=} S^{m+n+1}.$$



Now we can move on to the promised construction of potentially interesting maps between spheres.

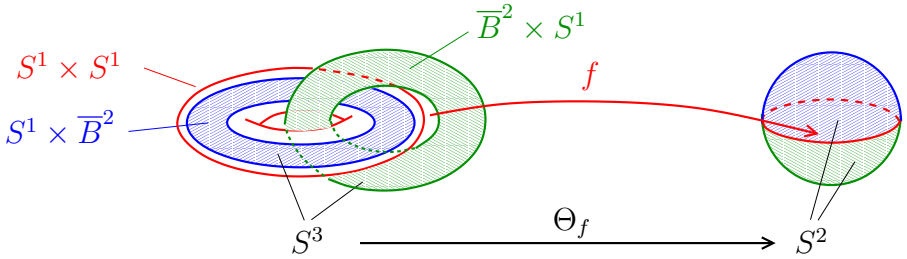
Definition. Let $f\colon S^p \times S^q \to S^r$ be a map. We consider the maps

$$\begin{array}{rcl} S^p \times \mathrm{Cone}(S^q) & \to & \mathrm{Cone}_+(S^r) \\ (P, [(Q,t)]) & \mapsto & [(f(P,Q), 1-t)] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{Cone}(S^p) \times S^q & \to & \mathrm{Cone}_-(S^r) \\ ([(P,t)], Q) & \mapsto & [(f(P,Q), t-1)]. \end{array}$$

We will see in Exercise 122.2 that these two maps are continuous. We leave the task of showing that these two maps are continuous to the reader. Evidently the two maps agree on the overlap $S^p \times S^q$. Thus by the Topological Pushout Proposition 5.33 (1b) these two maps define a continuous map

$$\Theta_f\colon \underbrace{(S^p \times \mathrm{Cone}(S^q)) \cup_{S^p \times S^q} (\mathrm{Cone}(S^p) \times S^q) = S^{p+q+1}}_{=(S^p \times \overline{B}^{q+1}) \cup_{S^p \times S^q} (\overline{B}^{p+1} \times S^q)} \to \underbrace{\Sigma(S^r)}_{=S^{r+1}}.$$

We refer to the figure below for a moderately successful attempt at illustrating the map Θ_f.



Example.

[1500] Here we implicitly use the Cone-Suspension Basics Lemma 16.8.

(1) Again we consider the map $f\colon S^1 \times S^1 \to S^1$ that is given by complex multiplication. We obtain by the above construction a map $\Theta_f\colon S^3 \to S^2$. The heroically inclined reader will surely rise to the challenge of showing that this map is homotopic to the Hopf map.[1501]
(2) Similarly to (1) one can show that the quaternion multiplication $S^3 \times S^3 \to S^3$ gives rise to the quaternionic Hopf map $S^7 \to S^4$ and that the octonion multiplication $S^7 \times S^7 \to S^7$ gives rise to the quaternionic Hopf map $S^{15} \to S^8$.

Given a map $f\colon S^{n-1} \times S^{n-1} \to S^{n-1}$ the next proposition computes the Hopf invariant of Θ_f in terms of the bidegree of f.

Proposition 122.12. Let $n \in \mathbb{N}_{\geq 2}$. If $f\colon S^{n-1} \times S^{n-1} \to S^{n-1}$ is a map of bidegree (r, s), then the induced map $\Theta_f\colon S^{2n-1} \to S^n$ has Hopf invariant $\pm rs$.[1502]

Proof. We make the following preparations:

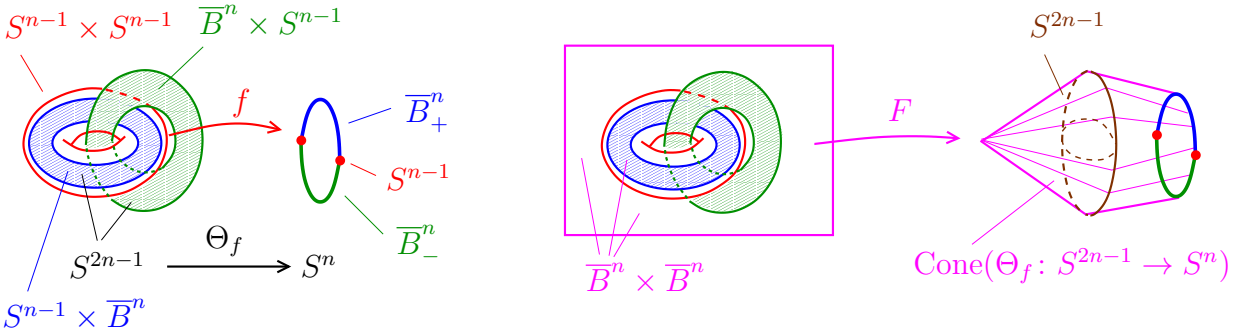
(1) Throughout the proof all cohomology groups are understood to be with $\mathbb{Z}$-coefficients.
(2) To simplify the notation we write $\mathrm{C}(\Theta_f)$ instead of $\mathrm{Cone}(\Theta_f)$.
(3) By the Mapping Cylinder Basics Lemma 16.10 we know that the inclusion $S^n \to \mathrm{C}(\Theta_f\colon S^{2n-1} \to S^n)$ is actually an embedding, thus we can identify S^n with its image in $\mathrm{C}(\Theta_f)$.
(4) We denote by $\overline{B}^n_+$ the upper hemisphere of S^n and we denote by $\overline{B}^n_-$ the lower hemisphere of S^n.
(5) By the discussion on page 450 we can make the identification

$$\overline{B}^n \times \overline{B}^n \;=\; \overline{B}^{2n} \;=\; \mathrm{Cone}\Big(\underbrace{(S^{n-1} \times \overline{B}^n) \cup_{S^{n-1}\times S^{n-1}} (\overline{B}^n \times S^{n-1})}_{=S^{2n-1}}\Big).$$

We denote by

$$F\colon \overline{B}^n \times \overline{B}^n \;\to\; \mathrm{C}(\Theta_f\colon S^{2n-1} \to S^n) \;=\; \underbrace{\mathrm{Cone}(S^{2n-1})}_{=\overline{B}^n\times\overline{B}^n=\overline{B}^{2n}} \cup_{\Theta_f} S^n$$

the obvious map which is given by the identification $\overline{B}^n \times \overline{B}^n = \mathrm{Cone}(S^{2n-1})$ followed by the projection.



[1501] To be on the safe side, let us more cautiously say that the map Θ_f is either homotopic to the Hopf map or that it is homotopic to the composition of the Hopf map with a reflection in a hyperplane.
[1502] We leave it to the sign aficionados to determine the correct sign.

Next we pick generators for various cohomology groups:

(1) We denote by $x \in \mathrm{H}^n(\mathrm{C}(\Theta_f))$ and $z \in \mathrm{H}^{2n}(\mathrm{C}(\Theta_f))$ the generators that are given in the definition of the Hopf invariant, see page 2681.
(2) The long exact sequence in cohomology of the pair $(\mathrm{C}(\Theta_f), \overline{B}^n_\pm)$ shows that the maps $\mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_\pm) \to \mathrm{H}^n(\mathrm{C}(\Theta_f))$ are isomorphisms. Let $x_\pm \in \mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_\pm)$ be the generators that correspond to x under the given isomorphisms.
(3) We denote by $[S^{n-1}]^* \in \mathrm{H}^{n-1}(S^{n-1})$ and $[\overline{B}^n]^* \in \mathrm{H}^n(\overline{B}^n, S^{n-1})$ the standard dual generators. For $i = 1, 2$ we denote by $p_i \colon \overline{B}^n \times \overline{B}^n \to \overline{B}^n$ the projection onto the first respectively second coordinate. We write

$$\begin{aligned} y_1 &= p_1^*([\overline{B}^n]^*) \in \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n) \quad \text{and} \\ y_2 &= p_2^*([\overline{B}^n]^*) \in \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, \overline{B}^n \times S^{n-1}). \end{aligned}$$

The following claim contains the key calculation of the proof.

Claim. The map F induces the following maps on relative cohomology:

$$\begin{array}{lrcl} \text{(i)} & \mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_+) & \xrightarrow{F^*} & \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n) \\ & x_+ & \mapsto & \pm r \cdot y_1 \\ \text{(ii)} & \mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_-) & \xrightarrow{F^*} & \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, \overline{B}^n \times S^{n-1}) \\ & x_- & \mapsto & \pm s \cdot y_2. \end{array}$$

Proof. Before we turn to the actual calculation we make the following two preparations:

(a) We point out that the restrictions of F to various subsets gives us the following commutative diagram

$$\begin{array}{ccc} \overline{B}^n \times \overline{B}^n & \xrightarrow{F} & \mathrm{C}(\Theta_f) \\ \uparrow & & \uparrow \\ \overline{B}^n \times S^{n-1} & \xrightarrow{\Theta_f} & \overline{B}^n_- \\ \uparrow & & \uparrow \\ S^{n-1} \times S^{n-1} & \xrightarrow{f} & S^{n-1}. \end{array}$$

(b) We pick a point $* \in S^{n-1}$.

To provide the actual proof of the claim we consider the following diagram:

$$\begin{array}{ccccc} \mathrm{H}^n(\mathrm{C}(\Theta_f)) & \xleftarrow{\cong} & \mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_+) & \xrightarrow{F^*} & \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n) \\ \downarrow \cong & & \downarrow & & \downarrow \cong \\ \mathrm{H}^n(S^n) & \xleftarrow{\cong} & \mathrm{H}^n(S^n, \overline{B}^n_+) \xrightarrow[\text{exc.}]{\cong} & \mathrm{H}^n(\overline{B}^n_-, S^{n-1}) & \xrightarrow{\Theta_f^*} \mathrm{H}^n(\overline{B}^n \times *, S^{n-1} \times *) \\ & & \cong \uparrow \delta & & \delta \uparrow \cong \\ & & \mathrm{H}^{n-1}(S^{n-1}) & \xrightarrow[[S^{n-1}]^* \mapsto r \cdot [S^{n-1} \times *]^*]{f^*} & \mathrm{H}^{n-1}(S^{n-1} \times *). \end{array}$$

We make the following clarifications and observations:

(1) The maps that are not specified are induced by the obvious inclusions of (pairs) of topological spaces.

(2) The inclusion induced maps that are decorated by "$\cong$" are isomorphisms for the following reasons:
   (a) by the long exact sequence in cohomology for the corresponding pair,
   (b) the maps are induced by a homotopy equivalence,
   (c) by the long exact sequence provided by the Mapping Cone-H*-Lemma 122.4,
   (d) by an appropriate version of the Excision Theorem, see the remark on page 2418.
(3) The quadrilateral to the left commutes since all maps are induced by inclusions.
(4) It follows from (2) and (3) that the diagonal map is an isomorphism.
(5) The triangle towards the center commutes since all maps are induced by inclusions.
(6) It follows from (2), (4) and (5) that the middle vertical map is an isomorphism.
(7) The squares to the right commute by the commutative diagram shown in (a) and by the naturality of the connecting homomorphisms.
(8) The connecting homomorphisms δ are easily seen to be isomorphisms since the other groups in the corresponding long exact sequence are zero.
(9) Basically by the definition of the bidegree of the map f the bottom horizontal map is given by $[S^{n-1}]^* \mapsto r \cdot [S^{n-1} \times *]^*$.

The above arguments show that the vertical maps of the blue part of the diagram are isomorphisms. Since the bottom map is given by multiplication by r we see that the top map, with whatever choice of generators is also given by multiplication by $\pm r$. This concludes the proof of part (i) of the claim. The proof of part (ii) is evidently basically the same. ⊞

Finally we consider the following diagram

$$\begin{array}{ccccc}
\mathrm{H}^n(\mathrm{C}(\Theta_f)) & \xleftarrow[\cong]{\substack{x \leftarrow\!\shortmid x_- \\ x \leftarrow\!\shortmid x_+}} & \mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_-) & \xrightarrow[F^*]{\substack{x_- \mapsto \pm r \cdot y_1 \\ x_+ \mapsto \pm s \cdot y_2}} & \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n) \\
\times & & \times & & \times \\
\mathrm{H}^n(\mathrm{C}(\Theta_f)) & & \mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_+) & & \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, \overline{B}^n \times S^{n-1}) \\
\cup \downarrow {\scriptstyle (x,x) \mapsto x^2 = \mathrm{Hopf}(\Theta_f) \cdot z} & & \downarrow \cup & & \cup \downarrow {\scriptstyle (y_1, y_2) \mapsto y_1 \cup y_2 = [\overline{B}^n \times \overline{B}^n]^*} \\
\mathrm{H}^{2n}(\mathrm{C}(\Theta_f)) & \xleftarrow[\cong]{} & \mathrm{H}^{2n}(\mathrm{C}(\Theta_f), S^n) & \xrightarrow[\cong]{F^*} & \mathrm{H}^{2n}(\overline{B}^n \times \overline{B}^n, \overline{B}^n \times S^{n-1} \cup S^{n-1} \times \overline{B}^n).
\end{array}$$

$$\pm z \leftarrow\!\shortmid [\overline{B}^n \times \overline{B}^n]^*$$

We make the following comments:

(1) It follows almost immediately from the Excisive Triad-Proposition 109.12 (5) that $(\mathrm{C}(\Theta_f), \overline{B}^n_-, \overline{B}^n_+)$ and $(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n, \overline{B}^n \times S^{n-1})$ are excisive triads. Thus we see that the relative cup product in the center and to the right are actually defined.
(2) It follows from the naturality of the cup product, see the Cup Product-Naturality Proposition 115.4, that the two squares commute.
(3) By the Fundamental Class-Product Proposition 117.2 (1) we have $y_1 \cup y_2 = p_1^*([\overline{B}^n]) \cup p_2^*([\overline{B}^n]) = [\overline{B}^n \times \overline{B}^n]^*$.
(4) By the above claim we know that the top right horizontal map is in fact given by $x_- \mapsto \pm r \cdot y_1$ and $x_+ \mapsto \pm s \mapsto s \cdot y_2$.
(5) Since the bottom horizontal maps are isomorphisms we see that going right to left the generator $[\overline{B}^n \times \overline{B}^n]^*$ gets sent to $\pm z$.

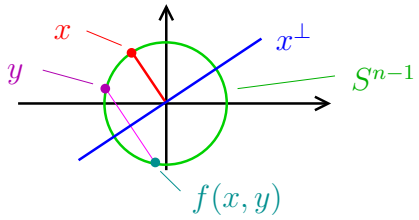
It follows immediately from the above comments that $\mathrm{Hopf}(\Theta_f) = \pm r \cdot s$. ■

Since the Hopf invariant gives us a homomorphism $\pi_{2n-1}(S^n) \to \mathbb{Z}$ we are now interested in finding interesting maps $S^{n-1} \times S^{n-1} \to S^{n-1}$. Perhaps somewhat surprisingly it is not that hard to find interesting maps.

Proposition 122.13. Given any *even* $n \in \mathbb{N}$ the map

$$\begin{aligned} f\colon S^{n-1} \times S^{n-1} &\to S^{n-1} \\ (x,y) &\mapsto y - 2 \cdot \langle x,y\rangle \cdot x = \text{reflection of } y \text{ in the hyperplane } x^\perp \end{aligned}$$

has bidegree $(2,-1)$.



Proof. We need to show that the given map $f\colon S^{n-1} \times S^{n-1} \to S^{n-1}$ has bidegree $(2,-1)$. The proof of this statement naturally breaks up into two parts.

(1) We take $y_0 = (1,0,\ldots,0)$. We consider the map $\varphi\colon S^{n-1} \to S^{n-1}$ that is given by $x \mapsto (x,y_0)$. Note that $\varphi^{-1}((1,0,\ldots,0)) = (\pm 1,0,\ldots,0)$. Using this observation together with the Local-Global Degree Proposition 76.21 and Proposition 76.20 one can easily show that the degree of the map $x \mapsto (x,y_0)$ equals $1+(-1)^{n-2}$. Since n is by hypothesis even we see that the degree equals two.

(2) We take $x_0 = (1,0,\ldots,0)$. The corresponding $S^{n-1} \to S^{n-1}$, $y \mapsto (x_0,y)$ is the reflection in a hyperplane, hence it follows from the Sphere Degree Lemma 76.11 that the map has degree -1. ∎

The combination of Propositions 122.7 and 122.13 gives us the following interesting corollary.

Corollary 122.14. Given any even $n \in \mathbb{N}$ there exists an epimorphism $\pi_{2n-1}(S^n) \to \mathbb{Z}$.

As always in mathematics, every result opens up lots of new questions. Here are a few:

Question 122.15.

(1) We have now seen that there exists an epimorphism $\pi_3(S^2) \to \mathbb{Z}$. Is this map an isomorphism?

(2) We have now seen that there exists an epimorphism $\pi_7(S^4) \to \mathbb{Z}$. Is this map an isomorphism?

We will provide the answer to Question 122.15 (1) in Theorem 155.10 and we will see the answer to Question 122.15 (2) in Theorem 159.10 (1).

Question 122.15 points to a problem: we have now developed some tools for showing that certain homotopy groups of spheres are non-trivial, but we have no tools yet for computing any interesting homotopy groups. We will partially rectify this problem in Chapter 159.

Here is another question. We delay the answer to Theorem 160.1.

Question 122.16. Given any $n \in \mathbb{N}$ we know by Lemma 85.2 and Corollary 122.14 that $\pi_n(S^n)$ and $\pi_{4n-1}(S^{2n})$ admit an epimorphism onto $\mathbb{Z}$. Are there are any other homotopy groups of spheres that admit an epimorphism onto $\mathbb{Z}$?

In light of Lemma 122.11 and Proposition 122.13 the following question becomes interesting:

Question 122.17. For which $n \in \mathbb{N}_{\geq 2}$ does there exist a map $f\colon S^{2n-1} \to S^n$ such that $\mathrm{Hopf}(f) = 1$?[1503]

We will see the complete answer to Question 122.17 in Theorems 152.10 and 152.11. Initially one might not appreciate Question 122.17. But rather mysteriously it is not that hard to come up with maps of Hopf degree 2, but it is strikingly difficult to come up with maps of Hopf degree 1. To underline our point we conclude this chapter with another construction of a map of Hopf degree 2.

Example. Let $n \in \mathbb{N}$. We start out with the following two reminders:

(1) We can view S^n as a CW-complex with one 0-cell and one n-cell. In particular, if we denote by S^n_a and S^n_b two copies of S^n then we view S^n_a as a CW-complex with one 0-cell P and one n-cell that we denote by S^n_a. Similarly we view S^n_b as a CW-complex with one 0-cell Q and one n-cell S^n_b.
(2) Given two finite CW-complexes X and Y we introduced on page 1497 the product CW-structure on $X \times Y$. Recall that the n-cells of $X \times Y$ correspond precisely to products $e_x \times f_y$ where e_x is a p-cell of X and f_y is an $(n-p)$-cell of Y.

It follows from (1) and (2) that we can view $S^n_a \times S^n_b$ as a CW-complex with one 0-cell (P, Q), two n-cells $S^n_a \times \{Q\}$ and $\{P\} \times S^n_b$ and one $2n$-cell $S^n_a \times S^n_b$. We consider the maps

$$S^{2n-1} \xrightarrow{\varphi} \overbrace{S^n_a \times \{Q\} \cup \{P\} \times S^n_b}^{=S^n_a \vee S^n_b} \xrightarrow{p} S^n.$$

φ: attaching map of the $2n$-cell to the $(2n-1)$-skeleton of $S^n_a \times S^n_b$ which is precisely $S^n_a \times \{Q\} \cup \{P\} \times S^n_b$

p: obvious projection on $S^n_a \times \{Q\}$ and $\{P\} \times S^n_b$

The composition of the maps φ and p defines a map $f\colon S^{2n-1} \to S^n$. It is clear from the

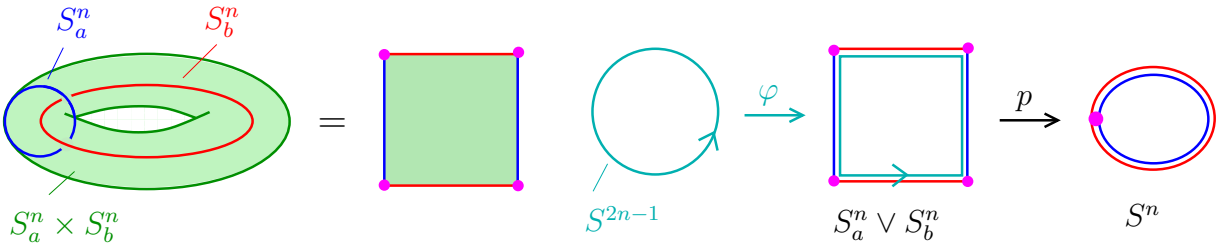


figure above that this map is not very interesting for $n = 1$. But using fairly elementary arguments one can show that for all even n we have $\mathrm{Hopf}(f) = \pm 2$. We refer to [**Hat2002**, p. 428] or [**Vic1994**, p. 138] for details. Alternatively we leave it as a brain teaser for the reader to perform this calculation.

[1503]Note that Propositions 122.7 and 122.13 tell us that this question is equivalent to the question, for which $n \in \mathbb{N}$ does there exist a map $f\colon S^{2n-1} \to S^n$ such that $\mathrm{Hopf}(f)$ is odd?

Exercises for Chapter 122.

Exercise 122.1. What is the homeomorphism type of the mapping cone of the map $S^1 \to S^1$ given by $z \mapsto z^2$?
Hint. It is a topological space that we are quite familiar with.

Exercise 122.2. Let $f\colon S^p \times S^q \to S^r$ be a map. (As always, f is understood to be continuous.) Show that the map

$$\begin{array}{rcl} S^p \times \mathrm{Cone}(S^q) & \to & \mathrm{Cone}(S^r) \\ (P, [(Q,t)]) & \mapsto & [(f(P,Q),t)] \end{array}$$

is continuous.
Hint. Use the Whitehead Quotient Theorem 13.4 and Lemma 13.5 (2).

Exercise 122.3. Let $f\colon S^k \times S^k \to S^k$ be the projection onto the first factor. Show that the corresponding map $\Theta_f\colon S^{2k+1} \to S^{k+1}$, that is defined on page 2688, is homotopic to a constant map.
Hint. The exercise is very easy.

Exercise 122.4. Let $n \in \mathbb{N}$ be even and let $a, b \in \mathbb{Z}$. Show that if at least one of a or b is even, then there exists a map $S^{n-1} \times S^{n-1} \to S^{n-1}$ of bidegree (a, b).

123. Regular neighborhoods and the Basic Alexander Duality Theorem

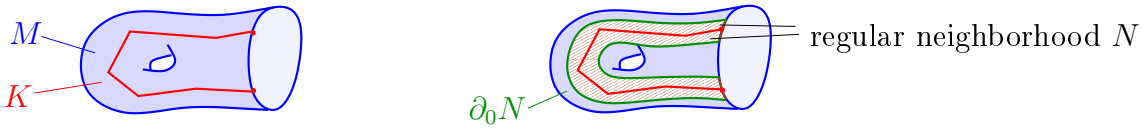
In this chapter we will study the Alexander Duality Theorem which, given a "reasonable" subset X of S^n relates the homology groups of $S^n \setminus X$ to the cohomology groups of X.

123.1. **Regular neighborhoods.** In this section we will introduce the notion of a regular neighborhood of a compact subset of a topological manifold. This notion is a variation on the eponymous concept that we defined on page 2096 and it is a variation on the notion of a tubular neighborhood that we encountered on page **??**.

Definition. Let M be an m-dimensional topological manifold and let K be a compact subset. A regular neighborhood of K is an m-dimensional compact submanifold N with corner of M with the following properties:

(1) K is contained in the interior $\mathring{N}$ of N,
(2) K is a deformation retract of N,
(3) $\partial_0 N$ is a deformation retract of $N \setminus K$.

We refer to $\mathring{N}$ as an open regular neighborhood of K.



Example. Let M be a PL-manifold and let $K \subset M \setminus \partial M$ be a subcomplex. In Section 96.3 we defined the regular neighborhood $N_K(M)$ of K in M. By the Simplicial Regular Neighborhood Proposition 96.13 we know that $N_K(M)$ does indeed have the properties demanded in the above definition of a regular neighborhood. In other words, by the Simplicial Regular Neighborhood Proposition 96.13 we know that K has a regular neighborhood in the above sense.

We continue this section with the following lemma which can be viewed as a generalization of Lemma 100.16 (3).

Lemma 123.1. Let K be a compact subset of a topological manifold M. If N is a regular neighborhood of K, then the following statements hold:

(1) $M \setminus \mathring{N}$ is a deformation retract of $M \setminus K$.
(2) The inclusion $M \setminus \mathring{N} \to M \setminus K$ is a homotopy equivalence, in particular it induces an isomorphism of homotopy groups (for suitable base points) and homology groups.
(3) $M \setminus \mathring{N}$ is path-connected if and only if $M \setminus K$ is path-connected.
(4) The inclusion map $M \setminus \mathring{N} \to M \setminus K$ induces isomorphisms of fundamental groups and (co-) homology groups.

Proof.

(1) By definition of a regular neighborhood there exists a deformation retraction r from $N \setminus K$ to $\partial_0 N$. We extend this deformation retraction to all of $M \setminus K$ in the obvious

way. More precisely, we consider the map

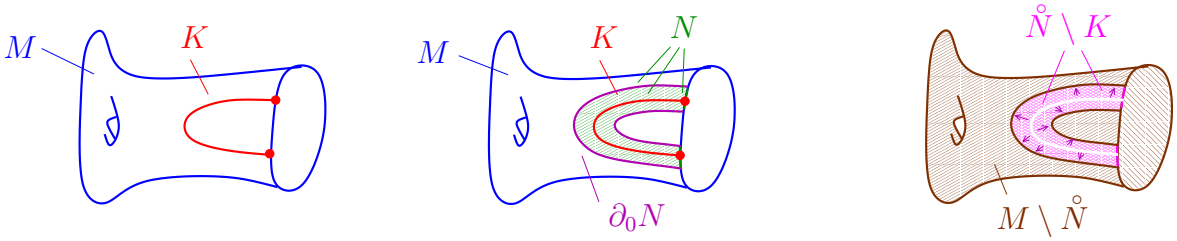
$$\begin{aligned} f\colon (M \setminus K) \times [0,1] &\to M \setminus K \\ (P,t) &\mapsto \begin{cases} P, & \text{if } P \notin N, \\ r(P,t), & \text{if } P \in N \setminus K. \end{cases} \end{aligned}$$

Using the Homotopy Stacking Lemma 14.3 and the Codimension-Zero Smooth Embedding Proposition 24.16 one can easily show that this map is indeed continuous. Once continuity is verified it is straightforward to show that f is indeed a deformation retraction from $M \setminus K$ to $M \setminus \mathring{N}$.

(2) It follows from (1) and the Deformation Retract-Homotopy Equivalence Lemma 15.5 that the inclusion is a homotopy equivalence. The statement regarding homotopy groups and homology groups follow from the Homotopy-π_n-Proposition 71.7 (2) and the Homotopy Equivalence-H_*-Corollary 73.9.

(3) This statement follows from (2) and the π_0-Homotopy Equivalence Lemma 15.3.

(4) This last statement follows from (2) together with the Homotopy-π_1-Proposition 50.3 (2), the Homotopy Equivalence-H_*-Corollary 73.9 and the Homotopic Maps-H^*-Lemma 108.13 (3). ■



The following corollary, which in the literature is often used implicitly or explicitly, gives one application of the theory of regular neighborhoods.

Lemma 123.2. Let M be a compact topological manifold and let $K \subset M$ be a compact subset. If K admits a regular neighborhood, then the following two statements hold:

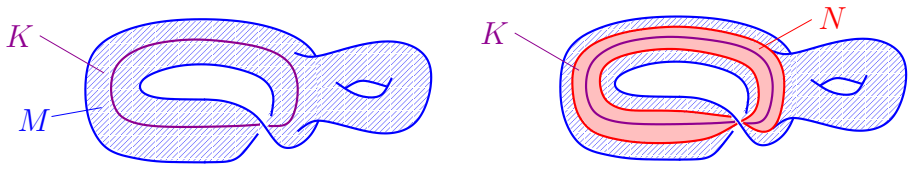
(1) the fundamental group $\pi_1(M \setminus K)$ is finitely presented, and
(2) for each $k \in \mathbb{N}_0$ the homology group $\mathrm{H}_k(M \setminus K; \mathbb{Z})$ is finitely generated.

Proof. Note that by Lemma 123.1 the topological space $M \setminus K$ is homotopy equivalent to $M \setminus \mathring{N}$. Since M is compact it follows from Lemma 120.1 (3) that $M \setminus \mathring{N}$ is also compact. Furthermore, by Lemma 120.1 (4) we know that $M \setminus \mathring{N}$ is an n-dimensional topological manifold. Summarizing we have seen that $M \setminus \mathring{N}$ is a compact n-dimensional topological manifold. The lemma now follows from the Topological Manifolds-Invariants Proposition 104.14 and Lemma 123.1 (4). ■

In the Tubular Neighborhood Theorem **??** we showed that if M is an orientable smooth manifold, then every closed 1-dimensional submanifold admits a trivial tubular neighborhood, which is easily seen to be a regular neighborhood. It is thus natural to guess that something similar should also hold for higher-dimensional submanifolds. In fact we have the following theorem.

Theorem 123.3. (Regular Neighborhood Theorem) Let M be a smooth manifold. Every compact proper submanifold of M admits a regular neighborhood.

An outline of the proof will be sketched in the next section.



The following proposition finally implies Theorem 123.3.

Proposition 123.4. Let M be a smooth manifold and let K be a compact proper submanifold of M. Every tubular neighborhood of K is also a regular neighborhood of K.

Proof. Let M be an n-dimensional smooth manifold and let K be a compact proper k-dimensional submanifold of M. Let $(N, p\colon N \to K)$ be a tubular neighborhood of K. Recall that this means that the following three statements are satisfied:

(1) N is a compact codimension-zero submanifold with corner of M that is a neighborhood of K.
(2) The map $p\colon N \to K$ is a linear $\overline{B}^{n-k}$-bundle such that for all $x \in K$ we have $p(x) = x$. We denote by $\{(U_i, \Phi_i\colon p^{-1}(U_i) \to \overline{B}^{n-k} \times U_i)\}_{i\in I}$ the corresponding family of trivializations.
(3) We have $\partial_1 N = p^{-1}(\partial K)$, put differently, we have $\partial M \cap N = p^{-1}(\partial M \cap K)$.

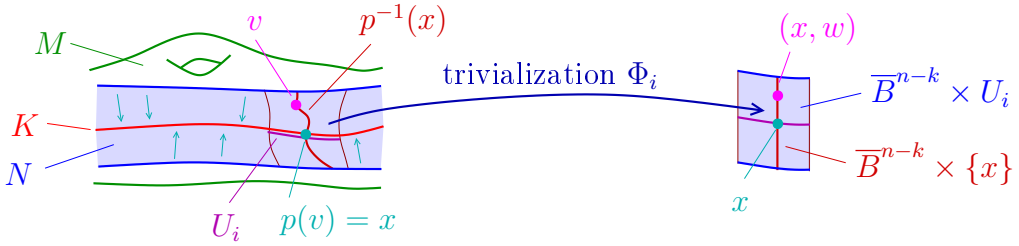
We start out the actual proof of the proposition with the following definition and construction. Let $v \in N$. We write $x := p(v)$. We pick an $i \in I$ with $p(v) \in U_i$ and we write $\Phi_i(v) = (w, x)$ where $w \in \overline{B}^m$.

(1) Given $s \in [0,1]$ we write $\varphi_s(v) := \Phi_i^{-1}(s \cdot w, x)$.
(2) Suppose $v \notin K$. This is equivalent to saying that $w \neq 0$. Then given $t \in [0,1]$ we write $\psi_t(v) := \Phi_i^{-1}\big((1-t)\cdot w + t \cdot \frac{w}{\|w\|}, x\big)$.

We leave it to the reader to verify that these definitions are independent of the choice of the trivialization Φ_i.[1504] From this independence it also follows easily that the maps

$$\begin{array}{rcl} \Phi\colon N \times [0,1] & \to & N \\ (v,s) & \mapsto & \varphi_s(v) \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi\colon (N \setminus K) \times [0,1] & \to & N \setminus K \\ (v,t) & \mapsto & \psi_t(v) \end{array}$$

are continuous.[1505] We refer to the figure below for an illustration of the definition of Φ and Ψ. Finally recall that by definition of a regular neighborhood we have to prove the



following claim.

[1504] At this point it is essential that we work with a linear $\overline{B}^{n-k}$-bundle.

[1505] Evidently we only have to show continuity locally. So let $v \in N$. We can pick a trivialization that contains x and then it follows immediately from the definitions that the maps Φ and Ψ are continuous in the neighborhood of x corresponding to the trivialization.

Claim.

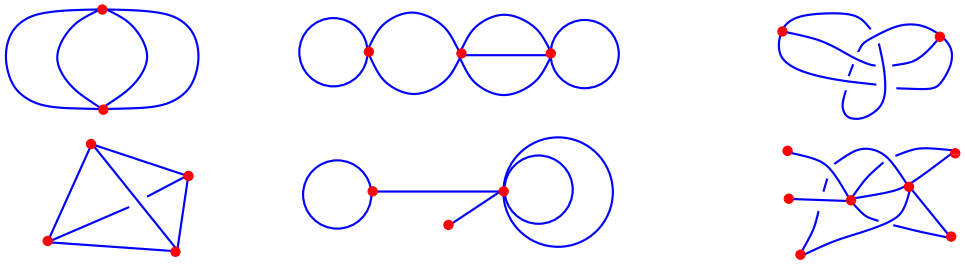
(1) K is contained in the interior of N,
(2) K is a deformation retract of N,
(3) $\partial_0 N$ is a deformation retract of $N \setminus K$.

Proof.

(1) By definition of a neighborhood we know that K is contained in the interior of N.
(2) The above map $\Phi\colon N \times [0,1] \to N$ is easily seen to be a deformation retraction from N to K.
(3) The above map $\Psi\colon (N \setminus K) \times [0,1] \to N \setminus K$ is easily seen to be a deformation retraction from $N \setminus K$ to $\partial_0 N$.

This concludes the proof of the claim and thus also of the proposition. ■

123.2. **Spatial graphs.** Note that in the definition of a regular neighborhood we did not demand that K is a submanifold, in particular K could a priori be any subset. As we will see in Exercise 123.1 it is not difficult to come up with compact subsets of even just $\mathbb{R}$ which do not admit regular neighborhoods. On the other hand one might hope that reasonable subsets of some $\mathbb{R}^n$ admit regular neighborhoods. For example consider the "spatial graphs" shown in the figure below. It seems reasonable to expect that these actually admit regular neighborhoods.



To make sense of this hunch we first need a definition of a "spatial graph" which captures the above examples. A first attempt might be to attempt to define a "a spatial" as the image of an embedding of a (say finite) topological into $\mathbb{R}^3$. But ever since on page 2293 and in Exercise 104.4 we learned of wild knots and its strange behaviors we know that we need some control over such embeddings to end up with reasonable objects.
Some leisurely thinking leads us to the following definition is modelled on the notion of a topological graph that we introduced on page 310.

Definition. A spatial graph is a triple $(G = (V, E, i, t), \Theta\colon |G| \to \mathbb{R}^3)$ where the following holds:

(1) The quadruple $G = (V, E, i, t)$ is a finite abstract graph.
(2) The map $\Theta\colon |G| \to \mathbb{R}^3$ is an embedding of the topological realization $|G|$ into $\mathbb{R}^3$.
(3) For every edge e with characteristic map $\Phi_e\colon [0,1] \to |G|$ the following statements hold:
   (a) The map $f := \Theta \circ \Phi_e\colon [0,1] \to \mathbb{R}^3$ is smooth. Note that by the Smooth Functions on Compact Interval Proposition 19.5 this means that the derivative f' is defined on all of $(0,1)$ and that it extends continuously to a map $f'\colon [0,1] \to \mathbb{R}^3$.

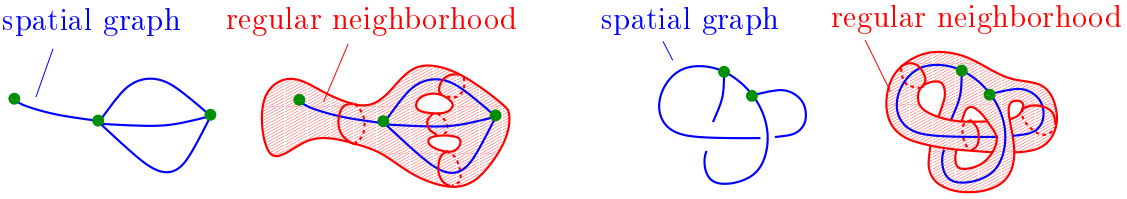
(b) The first derivative $f'(t)$ is non-zero for all $t \in [0,1]$,

Sometimes we just refer to $\Theta(|G|)$ as a "spatial graph" and we just ignore the extra data involved in the proper definition of a spatial graph.

Example. It should hopefully be clear that the examples sketched in the figure on page 2698 are spatial graphs.

Now we can formulate the following proposition.

Proposition 123.5. Every finite spatial graph G in $\mathbb{R}^3$ admits a regular neighborhood.



Proof. This proposition is proved in [**FrH2018, FrH2019**]. Note that in [**FrH2018, FrH2019**] it is also shown that there is a type of regular neighborhood which is unique in an appropriate sense. ■

123.3. **The Basic Alexander Duality Theorem.** We start out with the basic Alexander Duality Theorem which will be surprisingly easy to prove and which is already very helpful.

Theorem 123.6. (Alexander Duality Theorem) Let K be a compact non-empty subset of S^n that admits a regular neighborhood in the sense of the definition on page 2695. For any $i \in \mathbb{N}_0$ there exist isomorphisms

(a) $\quad \widetilde{\mathrm{H}}_i(S^n \setminus K;\mathbb{Z}) \cong \widetilde{\mathrm{H}}^{n-i-1}(K;\mathbb{Z})$

and

(b) $\quad \widetilde{\mathrm{H}}^i(S^n \setminus K;\mathbb{Z}) \cong \widetilde{\mathrm{H}}_{n-i-1}(K;\mathbb{Z}).$

By the Regular Neighborhood Theorem 123.3 this conclusion holds for any closed non-empty submanifold of S^n. In this particular case the isomorphisms are natural.

Proof. Let K be a compact non-empty subset of S^n that admits a regular neighborhood N. We first consider statement (a) and the case that $i > 0$. In this case we have the following isomorphisms:

$$\begin{array}{ccccc}
\widetilde{\mathrm{H}}_i(S^n \setminus K;\mathbb{Z}) & \xleftarrow{\cong} & \widetilde{\mathrm{H}}_i(S^n \setminus \mathring{N};\mathbb{Z}) & \xleftarrow{\cong} & \mathrm{H}^{n-i}(S^n \setminus \mathring{N}, \partial N;\mathbb{Z}) \xleftarrow{\cong} \mathrm{H}^{n-i}(S^n, N;\mathbb{Z}) \\
 & & \uparrow & & \\
 & & \widetilde{\mathrm{H}}^{n-i-1}(N;\mathbb{Z}) & \xrightarrow{\cong} & \widetilde{\mathrm{H}}^{n-i-1}(K;\mathbb{Z}).
\end{array}$$

Poincaré Duality Theorem 119.1 applied to $S^n \setminus \mathring{N}$, which is a compact smooth manifold by Lemma 120.1 (4)

Lemmas 123.1 and 108.16 (8)

Excision Theorem 74.18

by the long exact sequence in reduced cohomology of the pair (S^n, N), see Lemma 108.17 here we use that $i > 0$

by the Reduced Cohomology Lemma 108.16 (8) since K is a deformation retract of N

The case $i = 0$ is proved almost the same way. Thus we leave it to the reader to verify this case. This concludes the proof of (a). The proof of the existence of the isomorphism (b) is basically identical.

Finally suppose that K is a closed non-empty submanifold of S^n. It follows from the General Tubular Neighborhood Theorem **??** and Proposition 123.4 that we can take a tubular neighborhood as a regular neighborhood. By Proposition **??** tubular neighborhoods are unique in an appropriate sense which then allows us to get a natural isomorphism. ■



Remark.

(1) The argument in the proof of the Alexander Duality Theorem 123.6 implies in particular that given any compact, non-empty n-dimensional submanifold W of S^n there exists an isomorphism
$$\widetilde{\mathrm{H}}_i(S^n \setminus \mathring{W}; \mathbb{Z}) \cong \widetilde{\mathrm{H}}^{n-i-1}(W; \mathbb{Z}).$$

(2) In Alexander Duality Theorem 123.6 we explicitly do not write "natural" isomorphism. The isomorphism that we construct in the proof depends on the choice of a regular neighborhood. It is not clear to me whether two regular neighborhoods are necessarily in any way "equivalent".

In practice we often like to consider objects that lie in $\mathbb{R}^n$ instead of S^n. The following variation on the Alexander Duality Theorem 123.6 deals with this situation.

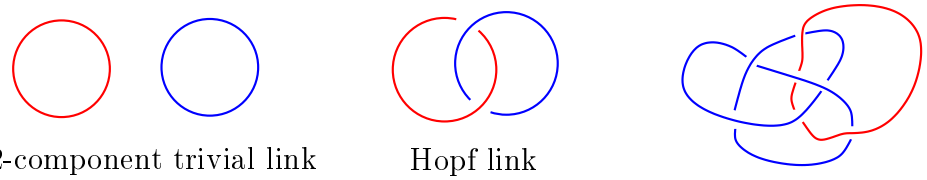
Corollary 123.7. Let K be a compact non-empty subset of $\mathbb{R}^n$. If K admits a regular neighborhood, then for any $i \in \mathbb{N}_0$ we have an isomorphism
$$\widetilde{\mathrm{H}}_i(\mathbb{R}^n \setminus K) \cong \mathrm{H}^{n-i-1}(K).$$
By the Regular Neighborhood Theorem 123.3 this conclusion holds for any compact submanifold of $\mathbb{R}^n$.

Proof. Let K be a compact non-empty subset of $\mathbb{R}^n$ that admits a regular neighborhood N in $\mathbb{R}^n$. Recall that N is by definition compact. It follows easily that N is also a regular neighborhood for K in S^n. Now let $i \in \mathbb{N}_0$. Then
$$\widetilde{\mathrm{H}}^{n-i-1}(K) \underset{\substack{\uparrow \\ \text{Alexander Duality Theorem 123.6 which we can apply since } K \text{ admits a regular neighborhood in } S^n}}{\cong} \widetilde{\mathrm{H}}_i(S^n \setminus K) \underset{\substack{\uparrow \\ \text{see Exercise 78.12}}}{\cong} \begin{cases} \widetilde{\mathrm{H}}_i(\mathbb{R}^n \setminus K), & \text{if } i \neq n-1, \\ \widetilde{\mathrm{H}}_{n-1}(\mathbb{R}^n \setminus K) \oplus \mathbb{Z}, & \text{if } i = n-1. \end{cases}$$

The corollary now follows from Lemmas 108.16 and 51.9 (1). ■

In the remainder of this section we discuss a few applications of the Alexander Duality Theorem 123.6. The following definition generalizes the notion of a knot.

Definition. Let $m \in \mathbb{N}_0$. An m-component link is a 1-dimensional submanifold of S^3 that is diffeomorphic to the disjoint union of m circles.[1506]



2-component trivial link Hopf link

The following lemma computes the homology groups of a link complement. It can be viewed as a generalization of Lemma 100.16 (4).

Lemma 123.8. For any m-component link L in S^3 we have

$$\mathrm{H}_i(S^3 \setminus L; \mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}^{m-1}, & \text{if } i = 2, \\ \mathbb{Z}^m, & \text{if } i = 1, \\ \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let L be an m-component link. We calculate that

$$\widetilde{\mathrm{H}}_i(S^3 \setminus L; \mathbb{Z}) \;\underset{\substack{\uparrow \\ \text{Alexander Duality Theorem 123.6}}}{\cong}\; \widetilde{\mathrm{H}}^{2-i}(L; \mathbb{Z}) \;\cong\; \widetilde{\mathrm{H}}^{2-i}(\underbrace{S^1 \sqcup \cdots \sqcup S^1}_{m\text{-copies}}; \mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}^{m-1}, & \text{if } i = 2, \\ \mathbb{Z}^m, & \text{if } i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

The lemma follows from this calculation together with the Reduced Cohomology Lemma 108.16 (5). ■

For any 2-component link $L = K_1 \cup K_2$, for example for any of the links shown in the figure on page 2701, we see that

$$\mathrm{H}_1(S^3 \setminus L; \mathbb{Z}) \;\cong\; \mathbb{Z}^2 \quad \text{and} \quad \mathrm{H}_2(S^3 \setminus L; \mathbb{Z}) \;\cong\; \mathbb{Z}.$$

The problem with this calculation is that we do not obtain any understanding of what the homology classes "look like". It seems reasonable to guess, along the lines of Lemma 100.16, that the first homology group is generated by the two meridians. But (a), we do not know how to prove it, and (b), we have no idea what a generator of the second homology group looks like. So the following question arises:

Question 123.9. Given a link L, what are explicit bases for the homology groups of $S^3 \setminus L$?

We will return to this question in Lemma 131.19. We continue with another example.

Examples.

(1) We consider the connected topological graph G with two vertices and three edges that is illustrated in the figure below on the left. On the right of the figure we show a spatial graph G' in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ that is homeomorphic to G. We have the

[1506] Sometimes, to distinguish this notion from the higher-dimensional analogue that we define on page ?? we refer to such a link in S^3 as a *classical link*.

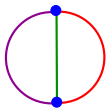
following isomorphisms:

$$\widetilde{\mathrm{H}}_i(S^3 \setminus G'; \mathbb{Z}) \;\cong\; \widetilde{\mathrm{H}}^{2-i}(G'; \mathbb{Z}) \;\cong\; \widetilde{\mathrm{H}}^{2-i}(G; \mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}^2, & \text{if } i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

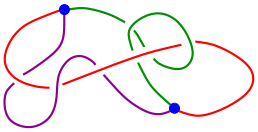
by Proposition 123.5 we can apply Alexander Duality Theorem 123.6

similar to Lemma 87.12

In this case it is also rather unclear what a basis of $\mathrm{H}_1(S^3 \setminus G; \mathbb{Z})$ should look like. Later, on page 2873, we will find an explicit basis for $\mathrm{H}_1(S^3 \setminus G; \mathbb{Z})$.

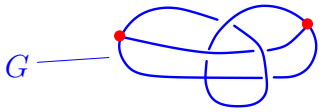


graph G



the spatial graph G' in $S^3 = \mathbb{R}^3 \cup \{\infty\}$

(2) Let G be the spatial graph that is shown in the figure below and that we already considered in Exercise 123.3.



The same argument as in (1) shows that $\mathrm{H}_1(S^3 \setminus G; \mathbb{Z}) \cong \mathbb{Z}^2$. Together with the Hurewicz Theorem 84.5 and Exercise 78.12 we see that $\pi_1(\mathbb{R}^3 \setminus G)_{\mathrm{ab}} \cong \mathbb{Z}^2$. This gives an answer to Exercise 123.3 (b) is $\mathbb{Z}^2$.

We conclude this section with the following variation on the above Alexander Duality Theorem 123.6. It will come in handy at times. But arguably it is best to study it only when it is actually needed.

Theorem 123.10. (Alexander Duality Theorem) Let $n \in \mathbb{N}_0$ and let $j \in \{0, \dots, n\}$. Given any compact oriented n-dimensional smooth manifold X which has the property that $\mathrm{H}_j(X; \mathbb{Z}) = \mathrm{H}_{j+1}(X; \mathbb{Z}) = 0$ and given any compact proper submanifold G of X there exist natural isomorphisms

$$\alpha_G \colon \mathrm{H}_j(X \setminus G; \mathbb{Z}) \xrightarrow{\cong} \mathrm{H}^{n-j-1}(G, \partial G; \mathbb{Z}) \quad \text{and} \quad \alpha_G \colon \mathrm{H}^j(X \setminus G; \mathbb{Z}) \xrightarrow{\cong} \mathrm{H}_{n-j-1}(G, \partial G; \mathbb{Z})$$

such that the following conditions are satisfied:

(1) If $X = S^n$, then the isomorphisms are precisely the isomorphisms from the Alexander Duality Theorem 123.6.
(2) The isomorphisms decompose in the obvious way if X is disconnected.
(3) If W is a compact oriented n-dimensional smooth manifold with

$$\mathrm{H}_j(W; \mathbb{Z}) \;=\; \mathrm{H}_{j+1}(W; \mathbb{Z}) \;=\; \mathrm{H}_j(\partial W; \mathbb{Z}) \;=\; \mathrm{H}_{j+1}(\partial W; \mathbb{Z}) \;=\; 0,$$

and if F is a compact proper submanifold of W, then we have commutative diagrams

$$\begin{array}{ccc} \mathrm{H}_j(W\setminus F;\mathbb{Z}) & \xrightarrow[\alpha_F]{\cong} & \mathrm{H}^{n-j-1}(F,\partial F;\mathbb{Z}) \\ {\scriptstyle i_*}\uparrow & & \uparrow{\scriptstyle \delta} \\ \mathrm{H}_j(\partial W\setminus \partial F;\mathbb{Z}) & \xrightarrow[\alpha_{\partial F}]{\cong} & \mathrm{H}^{n-j-2}(\partial F;\mathbb{Z}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{H}^j(W\setminus F;\mathbb{Z}) & \xrightarrow[\alpha_F]{\cong} & \mathrm{H}_{n-j-1}(F,\partial F;\mathbb{Z}) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle \partial} \\ \mathrm{H}^j(\partial W\setminus \partial F;\mathbb{Z}) & \xrightarrow[\alpha_{\partial F}]{\cong} & \mathrm{H}_{n-j-2}(\partial F;\mathbb{Z}) \end{array}$$

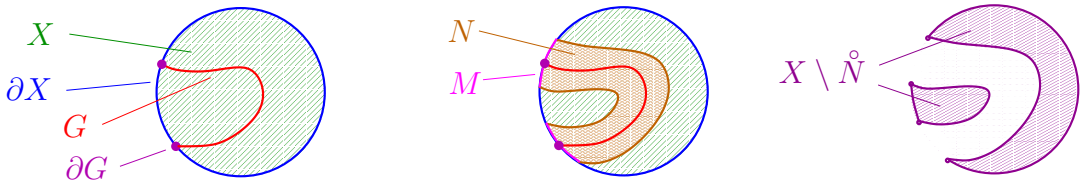
where we denote by $i\colon \partial W\setminus\partial F \to W\setminus F$ the inclusion map and where the right-hand vertical maps are the connecting homomorphisms of the long exact sequences in (co-) homology of the pair $(F,\partial F)$.

Example. The above Alexander Duality Theorem 123.10 will be applied later on to the smooth manifolds $W=[0,1]\times S^n$ and to $W=\overline{B}^n$. For example, suppose that F is a connected properly embedded surface in $\overline{B}^4$ such that $\partial F = K$ is a knot. We can apply the Alexander Duality Theorem 123.10 for $j=1$. Together with Proposition 119.25 we obtain the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{H}_1(\overline{B}^4\setminus F;\mathbb{Z}) & \xrightarrow[\cong]{\alpha_F} & \mathrm{H}^2(F,K;\mathbb{Z}) & \xrightarrow[\cong]{-\cap[F]} & \mathrm{H}_0(F;\mathbb{Z}) \\ {\scriptstyle i_*}\uparrow & & \uparrow{\scriptstyle \delta} & & \downarrow{\scriptstyle i_*} \\ \mathrm{H}_1(S^3\setminus K;\mathbb{Z}) & \xrightarrow[\cong]{\alpha_K} & \mathrm{H}^1(K;\mathbb{Z}) & \xrightarrow[\cong]{-\cap[K]} & \mathrm{H}_0(K;\mathbb{Z}). \end{array}$$

The right hand vertical map is of course an isomorphism and both groups are isomorphic to $\mathbb{Z}$. Now we see that both groups on the left are isomorphic to $\mathbb{Z}$ and that the inclusion induced map is an isomorphism.

Sketch of proof. Let $n\in\mathbb{N}_0$ and let $j\in\{0,\dots,n\}$. Let X be a compact orientable n-dimensional smooth manifold with $\mathrm{H}_j(X;\mathbb{Z})=\mathrm{H}_{j+1}(X;\mathbb{Z})=0$ and let G be a compact proper submanifold of X. We pick a tubular neighborhood N for G as provided by the General Tubular Neighborhood Theorem **??**. Recall that N is a submanifold with corner. We define $\mathring{N}$ as in Lemma 120.1. We write $M := N\cap\partial X$. Note that M is a tubular neighborhood for $\partial F\subset\partial X$. We consider the following diagram



$$\begin{array}{ccccccc} \mathrm{H}_j(X\setminus\mathring{N}) & \xleftarrow[\mathrm{PD}]{\cong} & \mathrm{H}^{n-j}(X\setminus\mathring{N},\partial(X\setminus\mathring{N})) & \xleftarrow[\mathrm{exc}]{\cong} & \mathrm{H}^{n-j}(X,N\cup\partial X) & \xleftarrow[\delta]{\cong} & \mathrm{H}^{n-j-1}(N\cup\partial X,\partial X) \\ \cong\downarrow & & & & & & {\scriptstyle \mathrm{exc}}\uparrow{\scriptstyle \cong} \\ & & & & & & \mathrm{H}^{n-j-1}(N,M) \\ & & & & & & \downarrow\cong \\ \mathrm{H}_j(X\setminus G) & & \xrightarrow{\quad =:\alpha_G \quad} & & & & \mathrm{H}^{n-j-1}(G,\partial G). \end{array}$$

We make the following comments:

(a) All maps where no map is specified are induced by inclusions. It follows from Proposition 123.4, together with the Homotopy Equivalence-H_*-Corollary 73.9 and the Homotopic Maps-H^*-Lemma 108.13 (4) that the vertical inclusion maps to the left and bottom right are isomorphisms.

(b) The two maps that are decorated with "exc" are inclusion induced maps that are isomorphisms by "excision". We leave it to the reader to verify that these inclusion maps are indeed isomorphisms. As an example, if $\partial X = \varnothing$, then it follows from the Excision Theorem 74.18 that the top "exc"-map is an isomorphism.

(c) The map on the top right indicated by "δ" is the connecting homomorphism of the long exact sequence in cohomology of the triple $(X, N \cup \partial X, \partial X)$. By our hypothesis we have $\mathrm{H}_j(X;\mathbb{Z}) = \mathrm{H}_{j+1}(X;\mathbb{Z}) = 0$. By the Poincaré Duality Theorem 119.1 this implies that $\mathrm{H}^{n-j}(X, \partial X;\mathbb{Z}) = \mathrm{H}^{n-j-1}(X, \partial X;\mathbb{Z}) = 0$, which in turn implies that the connecting homomorphism $\delta\colon \mathrm{H}^{n-j-1}(N \cup \partial X, \partial X;\mathbb{Z}) \to \mathrm{H}^{n-j}(X, N \cup \partial X;\mathbb{Z})$ is an isomorphism.

We have now seen that all maps in the diagram are isomorphisms. Thus we can define α_G to be the map on the bottom. A priori the definition of α_G depends on the choice of the tubular neighborhood N. But by Proposition **??** tubular neighborhoods are unique in an appropriate sense. This implies that isomorphisms do not depend on the choice of the tubular neighborhood.

First note that Statement (1) is basically clear once one compares the definitions. Next note that Statement (2) follows from all the usual results for the behavior of invariants under the disjoint union. Finally we turn to the proof of Statement (3). Let W be a compact oriented n-dimensional smooth manifold with

$$(*) \qquad \mathrm{H}_j(W;\mathbb{Z}) = \mathrm{H}_{j+1}(W;\mathbb{Z}) = \mathrm{H}_j(\partial W;\mathbb{Z}) = \mathrm{H}_{j+1}(\partial W;\mathbb{Z}) = 0.$$

Furthermore let F be a compact proper submanifold of W. We pick a tubular neighborhood N for F. We define $\mathring{N}$ as above. We write $M := N \cap \partial W$. Note that M is a tubular neighborhood for ∂F. We consider the following diagram

$$\begin{array}{ccccccc}
\mathrm{H}_j(W\backslash\mathring{N}) & \xleftarrow[\mathrm{PD}]{\cong} & \mathrm{H}^{n-j}(W\backslash\mathring{N}, \partial(W\backslash\mathring{N})) & \xleftarrow[\mathrm{exc}]{\cong} & \mathrm{H}^{n-j}(W, N\cup\partial W) & \xleftarrow[\delta]{\cong} & \mathrm{H}^{n-j-1}(N\cup\partial W, \partial W) \\
\downarrow \cong & & & & & & \uparrow \cong \\
\mathrm{H}_j(W\backslash F) & & & & & & \mathrm{H}^{n-j-1}(N, M) \\
& & & & & & \downarrow \cong \\
\mathrm{H}_j(W\backslash F) & & \xrightarrow{\qquad\qquad\alpha_F\qquad\qquad} & & & & \mathrm{H}^{n-j-1}(F, \partial F) \\
i_* \uparrow & & & & & & \delta \uparrow \\
\mathrm{H}_j(\partial W\backslash F) & & \xrightarrow{\qquad\qquad\alpha_{\partial F}\qquad\qquad} & & & & \mathrm{H}^{n-j-2}(\partial F) \\
\cong \uparrow & & & & & & \uparrow \cong \\
\mathrm{H}_j(\partial W\backslash\mathring{M}) & \xleftarrow[\mathrm{PD}]{\cong} & \mathrm{H}^{n-j}(\partial W\backslash\mathring{M}, \partial M) & \xleftarrow[\mathrm{exc}]{\cong} & \mathrm{H}^{n-j-1}(\partial W, M) & \xleftarrow[\delta]{\cong} & \mathrm{H}^{n-j-2}(M).
\end{array}$$

We make the following observations:

(1) By our hypothesis $(*)$ and the discussion above the horizontal maps α_G and $\alpha_{\partial G}$ are defined.
(2) One can easily see that the bottom "$\cup$" of the diagram is indeed the map $\alpha_{\partial F}$.
(3) If one ignores the blue arrows, then the brave reader can verify that the diagram does indeed commute.

This concludes the proof of Statement (3) for the diagram to the left. Basically the same argument also takes care of the proof of the diagram to the right. ■

123.4. **Immersions and smooth embeddings of smooth manifolds into $\mathbb{R}^n$ II.** We can now provide a new proof of the following proposition.

Proposition 43.12. (Smooth Codimension-One Orientability Proposition) Let N be a closed n-dimensional connected smooth manifold. If N is non-orientable, then there is no smooth embedding of N into $\mathbb{R}^{n+1}$.

Proof. Let N be a closed n-dimensional connected non-orientable smooth manifold. Suppose that N can be embedded into $\mathbb{R}^{n+1}$. We can thus view N as a submanifold of $\mathbb{R}^{n+1}$. We obtain that

$$\mathbb{Z}_2 \overset{\text{Top-Cohomology Proposition 110.14}}{\underset{\cong}{\downarrow}} \mathrm{H}^n(N;\mathbb{Z}) \underset{\text{Alexander Duality Theorem 123.7}}{\underset{\uparrow}{\cong}} \widetilde{\mathrm{H}}_0(\mathbb{R}^{n+1}\setminus N;\mathbb{Z}) \underset{\text{H}_0\text{-Proposition 72.5}}{\underset{\uparrow}{\cong}} \mathbb{Z}^{\text{number of path-components of } \mathbb{R}^{n+1}\setminus N \text{ minus } 1}.$$

But this is absurd. We have thus obtained the desired contradiction. ■

It is worth recalling the following almost immediately corollary to Proposition 43.12. This corollary gives in particular a negative answer to Question 24.13 (2).

Corollary 43.13. Neither $\mathbb{R}\mathrm{P}^2$ nor the Klein bottle admit a smooth embedding into $\mathbb{R}^3$.

Proof. By the Surface Non-Orientability Lemma 25.20 we know that $\mathbb{R}\mathrm{P}^2$ and the Klein bottle are both non-orientable. Since both are closed 2-dimensional smooth manifolds it follows from the Smooth Codimension-One Orientability Proposition 43.12 that neither $\mathbb{R}\mathrm{P}^2$ nor the Klein bottle admit a smooth embedding into $\mathbb{R}^3$. ■

We recall the following related question.

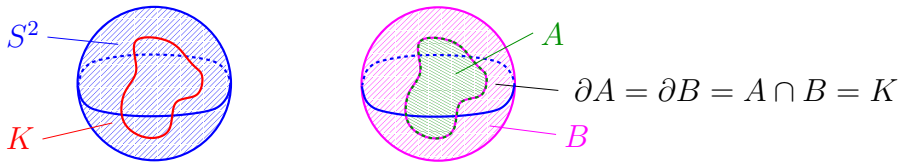
Question 27.5. Given $n \in \mathbb{N}$, what is the minimal $k \in \mathbb{N}$ such that $\mathbb{R}\mathrm{P}^n$ admits a (topological) smooth embedding into $\mathbb{R}^k$?

Recall that it follows from the Real Projective Space Orientability Lemma 25.21, or alternatively from the Orientation-Top Homology Theorem 106.1 and the calculation of the homology groups of $\mathbb{R}\mathrm{P}^n$ in Proposition 80.11, that $\mathbb{R}\mathrm{P}^n$ is orientable if and only if n is odd. In particular it follows from the Smooth Codimension-One Orientability Proposition 43.12 that for n even it is not possible to embed $\mathbb{R}\mathrm{P}^n$ into S^{n+1}. But this raises the question, whether for n odd one can embed $\mathbb{R}\mathrm{P}^n$ into S^{n+1}. It turns out that this is also not the case:

Proposition 123.11. If $n \in \mathbb{N}_{\geq 2}$, then there is no smooth embedding from $\mathbb{R}\mathrm{P}^n$ into S^{n+1}.

In the proof of Proposition 123.11 we will need the following lemma which is interesting in its own right.

Lemma 123.12. If K is a closed orientable connected $(n-1)$-dimensional smooth submanifold of the smooth manifold S^n, then there exist two compact n-dimensional submanifolds A and B of S^n such that $A \cap B = \partial A = \partial B = K$.



Proof. First we note that by Proposition **??** the submanifold K admits a tubular neighborhood $[-1,1] \times K$. Our hypothesis that K is connected implies that $[-1,0] \times K$ and $[0,1] \times K$ are also connected.

We first calculate that

$$\mathrm{H}_0(S^n \setminus K;\mathbb{Z}) \underset{\text{Reduced Homology Lemma 74.1 (4a)}}{\cong} \widetilde{\mathrm{H}}_0(S^n \setminus K;\mathbb{Z}) \oplus \mathbb{Z} \underset{\text{Alexander Duality Theorem 123.6}}{\cong} \mathrm{H}^{n-1}(K;\mathbb{Z}) \oplus \mathbb{Z} \underset{\text{Orientation-Top Homology Theorem 106.1}}{\cong} \mathbb{Z} \oplus \mathbb{Z}.$$

It follows from the H_0-Proposition 72.5 that $S^n \setminus K$ has precisely two path-components A' and B'. Since $[-1,0] \times K$ is connected and since it is contained in $S^n \setminus K = A' \cup B'$ it follows that $[-1,0] \times K$ is either contained in A' or it is contained in B'. Without loss of generality we can assume that $[-1,0] \times K$ is contained in A'.

Claim. The subset $[0,1] \times K$ is contained in B'.

Proof. We consider the set $A'' := A' \cup [-1,1] \times K$. Note that $S^n = A'' \cup B'$. Since both A'' and B' are open and since S^n is connected it follows that $A'' \cap B' \neq \varnothing$. But this implies that $[0,1] \times K \cap B' \neq \varnothing$. But since $[0,1] \times K$ is connected we see that $[0,1] \times K$ is contained in B'. ⊞

We set $A := A' \cup K = A' \cup [-1,0] \times K$ and $B := B' \cup K = B' \cup [0,1] \times K$. It is easy to show that A and B are compact submanifolds of S^n such that $A \cap B = \partial A = \partial B = K$. ■

Proof of Proposition 123.11. As we had discussed just before the proposition, it remains to deal with the case that n is odd. So suppose that $n \geq 3$ is odd and that there exists a smooth embedding of $\mathbb{R}\mathrm{P}^n$ into S^{n+1}. We identify $\mathbb{R}\mathrm{P}^n$ with its image under the smooth embedding, i.e. we view $\mathbb{R}\mathrm{P}^n$ as a submanifold of S^{n+1}. Note it follows from $n \geq 3$ and the discussion on page 2419 that $\mathrm{H}^1(S^n;\mathbb{F}_2) = \mathrm{H}^2(S^n;\mathbb{F}_2) = 0$.

Since n is odd we know that $\mathbb{R}\mathrm{P}^n$ is orientable. Thus we can apply Lemma 123.12 which tells us that there exist two compact $(n+1)$-dimensional submanifolds A and B of S^{n+1} with $A \cap B = \partial A = \partial B = K$. We denote by $i\colon \mathbb{R}\mathrm{P}^n \to A$ and by $j\colon \mathbb{R}\mathrm{P}^n \to B$ the inclusion maps. As usual we denote by $x \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \cong \mathbb{F}_2$ the unique generator.

By the Excisive Triad-Proposition 109.12 (5) we can apply the Mayer–Vietoris Theorem 109.15 for Cohomology Groups to $S^{n+1} = A \cup B$ and we obtain the long exact sequence

$$\cdots \to \underbrace{\mathrm{H}^1(S^{n+1};\mathbb{F}_2)}_{=0 \text{ since } n \geq 3} \to \mathrm{H}^1(A;\mathbb{F}_2) \oplus \mathrm{H}^1(B;\mathbb{F}_2) \xrightarrow{i^*+j^*} \underbrace{\mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)}_{=\mathbb{F}_2 \cdot x} \to \underbrace{\mathrm{H}^2(S^{n+1};\mathbb{F}_2)}_{=0 \text{ since } n \geq 3} \to \dots$$

Thus we see that the map $i^* + j^* \colon \mathrm{H}^1(A;\mathbb{F}_2) \oplus \mathrm{H}^1(B;\mathbb{F}_2) \to \mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \cong \mathbb{F}_2 \cdot x$ is an isomorphism. Without loss of generality we can assume that i^* is non-trivial which means that there exists an $a \in \mathrm{H}^1(A;\mathbb{F}_2)$ with $i^*(a) = x$. We have

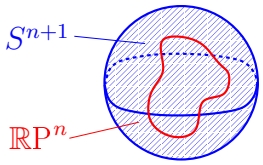
$$i^*(a^n) \underset{\text{Cup Product-Naturality Lemma 114.10}}{\overset{\uparrow}{=}} (i^*(a))^n = x^n \underset{\text{Cup Product-}\mathbb{R}\mathrm{P}^n\text{-Proposition 121.16}}{\overset{\uparrow}{\neq}} 0 \in \mathrm{H}^n(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2).$$

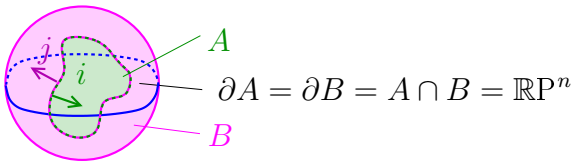
In particular the map $i^* \colon \mathrm{H}^n(A;\mathbb{F}_2) \to \mathrm{H}^n(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ is non-zero. Now we consider the other end of the above Mayer-Vietoris sequence and we obtain the exact sequence

$$\mathrm{H}^n(A;\mathbb{F}_2) \oplus \mathrm{H}^n(B;|!\mathbb{F}_2) \xrightarrow{i^*+j^*} \overbrace{\mathrm{H}^n(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)}^{=\mathbb{F}_2 \cdot x^n} \to \overbrace{\mathrm{H}^{n+1}(S^{n+1};\mathbb{F}_2)}^{\cong \mathbb{F}_2} \to \underbrace{\mathrm{H}^{n+1}(A;\mathbb{F}_2) \oplus \mathrm{H}^{n+1}(B;\mathbb{F}_2)}_{\substack{\text{both are zero by the Top-Cohomology Proposition 110.14,} \\ \text{since } A \text{ and } B \text{ are compact, connected} \\ (n+1)\text{-dimensional smooth manifolds} \\ \text{with non-empty boundary}}}.$$

$$a^n \oplus 0 \qquad \mapsto \quad x^n$$

Thus we see that the map on the left is an epimorphism and that the group on the right is trivial. But then the fact that $\mathrm{H}^{n+1}(S^{n+1};\mathbb{F}_2) \neq 0$ shows that the sequence cannot possibly be exact. We have thus obtained the desired contradiction. ■





Remark. The above proposition does not give a complete answer to Question 27.5, we have only shown that $\mathbb{R}\mathrm{P}^n$ cannot embed into S^{n+1}. More lower bounds on the minimal dimension of some $\mathbb{R}^k$ into which we can embed $\mathbb{R}\mathrm{P}^n$ have been obtained by Michael Atiyah [**At1961**].

Exercises for Chapter 123.

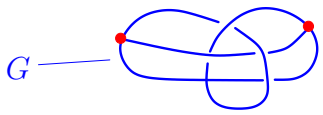
Exercise 123.1. Show that the subset $K := \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ of $\mathbb{R}$ does not admit a regular neighborhood.

Exercise 123.2. Let G and G' be two spatial graphs in $\mathbb{R}^3$ with regular neighborhoods $N(G)$ and $N(G')$. Which of the following two implications is correct?

$$G \text{ and } G' \text{ are isotopic} \iff N(G) \text{ and } N(G') \text{ are isotopic?}$$

Exercise 123.3. We consider the spatial graph G shown in the figure below.

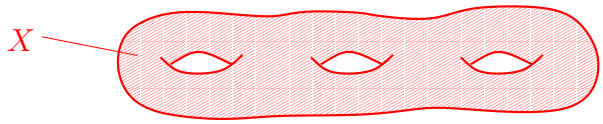
(a) Determine a presentation for $\pi_1(\mathbb{R}^3 \setminus G)$.
(b) What is the abelianization of $\pi_1(\mathbb{R}^3 \setminus G)$?



Exercise 123.4. Let $X \subset S^3$ be a closed orientable connected 2-dimensional submanifold. We view S^3 as a subset of S^4 in the usual way.

(a) Determine the homology groups of $S^4 \setminus X$.

(b) Now let $X \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ be the surface shown in the figure below. Give an example of a closed curve in $S^4 \setminus X$ that is not null-homotopic.



Exercise 123.5. This exercise is a continuation of Exercise 123.4.

(a) Let $X \subset S^n$ be a closed submanifold. We view S^n as a subset of S^{n+1} in the usual way. Show that $S^{n+1} \setminus X$ is homotopy equivalent to $\Sigma(S^n \setminus X)$. (Here $\Sigma(S^n \setminus X)$ denotes the suspension of $S^n \setminus X$.)
Remark. You will need to use at some point that X is a "reasonable subset", e.g. that X is a closed submanifold. The statement does not hold for all subsets of S^n.

(b) Again let $X \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ be the surface shown in the figure above. As above we view it as a submanifold of S^4. Determine $\pi_1(S^4 \setminus X)$.

Exercise 123.6.

(a) Let F be an orientable connected 2-dimensional smooth manifold with non-empty boundary and let $i\colon F \to \mathbb{R}^3$ be an injective map. Is the complement $\mathbb{R}^3 \setminus F$ necessarily path-connected?

(b) Let F be an orientable connected surface non-compact 2-dimensional smooth manifold with empty boundary and let $i\colon F \to S^3$ be an injective map. Is the complement $S^3 \setminus F$ necessarily path-connected?

Exercise 123.7. Let M be a closed orientable n-dimensional smooth manifold.

(a) Let $A \subset M$ be a proper submanifold. Show that for each $k \in \mathbb{N}_0$ we have an isomorphism
$$\mathrm{H}^k(M \setminus A) \;\cong\; \mathrm{H}_{n-k}(M, A).$$

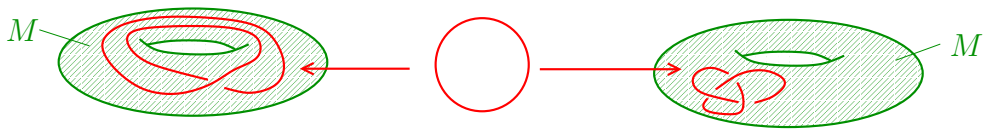
(b) Let $A \subset B \subset M$ be proper submanifolds. Show that for each $k \in \mathbb{N}_0$ we have an isomorphism
$$\mathrm{H}^k(M \setminus A, M \setminus B) \;\cong\; \mathrm{H}_{n-k}(B, A).$$
Hint. Use long exact sequences of triples.

Exercise 123.8. Let M be an n-dimensional smooth manifold and let $\varphi\colon S^1 \to M$ be a smooth embedding. By the Smooth Embedding Theorem 24.10 we know that $\varphi(S^1)$ is a smooth submanifold that is diffeomorphic to S^1.

(a) We suppose that $\mathrm{H}_2(M) = 0$ and that $\mathrm{H}_1(M)$ is torsion-free. (For example M could be the solid torus $\overline{B}^2 \times S^1$.) Compute the isomorphism types of the homology groups of $M \setminus \varphi(S^1)$.

(b) Under the hypothesis of (a) show that the isomorphism type of $\mathrm{H}_*(M \setminus \varphi(S^1))$ does not depend on the choice of the smooth embedding φ.

(c) Provide examples that show that in (b) the hypothesis $\mathrm{H}_2(M) = 0$ cannot be dropped and provide examples that show that the hypothesis that $\mathrm{H}_1(M)$ is torsion-free cannot be dropped.



Exercise 123.9. Let $n \in \mathbb{N}$. Show that there is no smooth embedding of the complex projective space $\mathbb{C}\mathrm{P}^n$ into $\mathbb{R}^{2n+1}$.

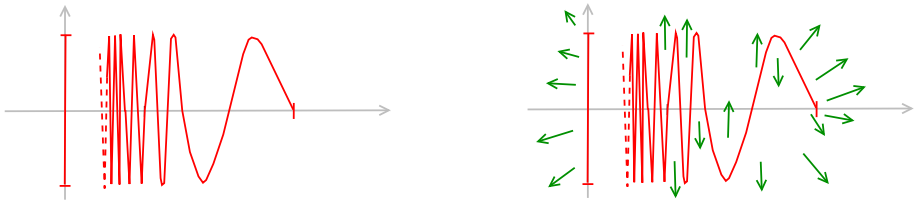
124. The Generalized Alexander Duality Theorem

In this chapter we will formulate and proof a signification generalization of the Alexander Duality Theorem 123.6. Note that the proof of the Generalized Alexander Duality Theorem makes use of Borsuk's Theorem 104.3.

124.1. **The generalized Alexander Duality Theorem.** In a bout of optimism one might hope that the statement of Alexander Duality Theorem 123.6 holds for any (compact) subset of S^n. But this is not the case. More precisely, once again we consider the Topologist's Sine Curve

$$X \;=\; \big\{(0,y)\,\big|\,y\in[-1,1]\big\}\,\cup\,\big\{(x,\sin(\tfrac{1}{x}))\,\big|\,x\in(0,\tfrac{1}{\pi}]\big\}\;\subset\;\mathbb{R}^2$$

that we had initially introduced on page 164. It is fairly straightforward to see that X is a closed subset of $\mathbb{R}^2$. Evidently X is bounded. Thus the Heine-Borel Theorem 1.24 implies that X is in fact a compact subset of $\mathbb{R}^2$ and thus of $S^2=\mathbb{R}^2\cup\{\infty\}$. Back on page 164 we saw that X is not path-connected, which by the H^0-Proposition 108.11 and Lemma 108.14 implies that $\widetilde{\mathrm{H}}^0(X;\mathbb{Z})\neq 0$. On the other hand, with some effort one can show that there exists a deformation retraction from $S^2\setminus X=(\mathbb{R}^2\setminus X)\cup\{\infty\}$ to ∞, which implies that $\mathrm{H}_1(S^2\setminus X;\mathbb{Z})=0$. But this shows that $\widetilde{\mathrm{H}}_1(S^2\setminus X;\mathbb{Z})\ncong\widetilde{\mathrm{H}}^0(X;\mathbb{Z})$.



X is not path-connected $\qquad$ $S^2\setminus X$ is contractible

We formulated Alexander Duality Theorem 123.6 for smooth submanifolds of S^n. We will now generalize this statement in two ways:

(1) Instead of only considering S^n we will now also allow topological homology spheres.
(2) Instead of smooth submanifolds we will now consider compact weakly locally contractible subsets.

We now recall these definitions in reverse order. First we recall the following definition from page 2290.

Definition. Let X be a topological space. We say X is weakly locally contractible if given any $Q\in X$ and given any open neighborhood U of Q there exists an open neighborhood V of Q that is contained in U such that the inclusion $V\to U$ is homotopic to a constant map.

Next we recall the following definition from page 2159.

Definition. We say that an n-dimensional topological manifold X is a topological homology n-sphere if for every $k\in\mathbb{N}_0$ we have $\mathrm{H}_k(X;\mathbb{Z})\cong\mathrm{H}_k(S^n;\mathbb{Z})$.

The following lemma is a generalization of Lemma 98.10.

Lemma 124.1. Let M be a topological manifold. If M is a topological homology n-sphere, then M is closed, orientable and connected.

Proof. The lemma is an immediate consequence of the H_0-Proposition 72.5 and the Top-Homology Theorem 106.3. ■

Now we can formulate our main theorem of the chapter which, by Lemma 104.2, is a significant generalization of the earlier Alexander Duality Theorem 123.6 (a).

Theorem 124.2. (Generalized Alexander Duality Theorem) Let $n \geq 2$, let Σ be a topological homology n-sphere and let $X \subsetneq \Sigma$ be a weakly locally contractible, non-empty compact subset of Σ. Given any $j \in \mathbb{N}_0$ and given any abelian group G we have a natural isomorphism

$$\Theta_X \colon \widetilde{\mathrm{H}}_j(\Sigma \setminus X; G) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{n-j-1}(X; G).$$

Here "natural" means, that if $X \subset Y \subsetneq \Sigma$ are two weakly locally contractible, non-empty compact subset of Σ, then given any $j \in \mathbb{N}_0$ and given any abelian group G we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_j(\Sigma \setminus X; G) & \xrightarrow[\cong]{\Theta_X} & \widetilde{\mathrm{H}}^{n-j-1}(X; G) \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{H}}_j(\Sigma \setminus Y; G) & \xrightarrow[\cong]{\Theta_Y} & \widetilde{\mathrm{H}}^{n-j-1}(Y; G), \end{array}$$

here the left vertical map is induced by the inclusion $\Sigma \setminus X \to \Sigma \setminus Y$ and the right vertical map is induced by the inclusion $X \to Y$.

Example. In the figure below we show a subset $K \subset S^3$ that is, as we saw on page 659, homeomorphic to S^1. In Proposition 124.17 we will show that K does not admit a regular neighborhood in the sense of the definition on page 2695.[1507] In particular Alexander Duality Theorem 123.6 does not apply to $K \subset S^3$.



On the other hand K is homoemorphic to S^1, thus it is weakly locally contractible and we see that Generalized Alexander Duality Theorem 124.2 applies. We see that

$$\widetilde{\mathrm{H}}_j(S^3 \setminus K; \mathbb{Z}) \underset{\uparrow}{\xrightarrow{\cong}} \widetilde{\mathrm{H}}^{2-j}(K; \mathbb{Z}) \cong \widetilde{\mathrm{H}}^{2-j}(S^1; \mathbb{Z}) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

isomorphism Θ_K from the Generalized Alexander Duality Theorem 124.2

by the calculation on page 2419 together with the Reduced Cohomology Lemma 108.16

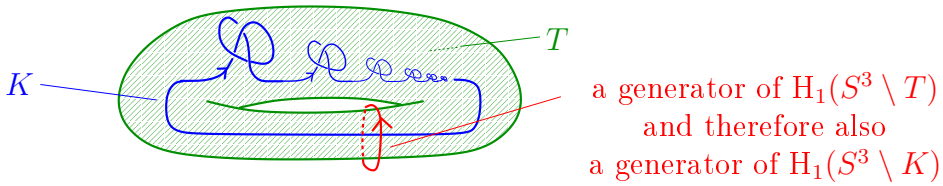
We have thus shown that $\mathrm{H}_1(S^3 \setminus K; \mathbb{Z}) \cong \mathbb{Z}$. But from the above isomorphism it is difficult to specify an explicit generator for $\mathrm{H}_1(S^3 \setminus K; \mathbb{Z})$. We will do so using the naturality of the isomorphism Θ_K. Namely, we consider the "standard solid torus" $T \subset S^3$ that is shown in the figure below. We denote by $i \colon K \to T$ the inclusion map. By the naturality of the isomorphism of the Generalized Alexander Duality Theorem 124.2 we obtain the following

[1507] In Theorem 124.13 we will show the related result, that K is not a topological submanifold of S^3.

commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_j(\Sigma \setminus K;\mathbb{Z}) & \xrightarrow[\cong]{\Theta_K} & \widetilde{\mathrm{H}}^{n-j-1}(K;\mathbb{Z}) \\ \downarrow & & \downarrow i^* \\ \widetilde{\mathrm{H}}_j(\Sigma \setminus T;\mathbb{Z}) & \xrightarrow[\cong]{\Theta_T} & \widetilde{\mathrm{H}}^{n-j-1}(T;\mathbb{Z}). \end{array}$$

Each map $i_*\colon \mathrm{H}_*(K;\mathbb{Z}) \to \mathrm{H}_*(T;\mathbb{Z})$ is an isomorphism.[1508] It follows from Corollary 108.20 that the map $i^*\colon \mathrm{H}^1(K;\mathbb{Z}) \to \mathrm{H}^1(T;\mathbb{Z})$ is also an isomorphism. It follows from the above commutative diagram that the inclusion induced map $\widetilde{\mathrm{H}}_1(\Sigma \setminus K;\mathbb{Z}) \to \mathrm{H}_1(\Sigma \setminus T;\mathbb{Z})$ is an isomorphism. But it is straightforward to specify a generator for $\widetilde{\mathrm{H}}_1(\Sigma \setminus K;\mathbb{Z})$.



Remark.

(1) We leave it to the reader to determine – say using search engines provided by California based companies – whether the obvious generalization of Alexander Duality Theorem 123.6 (b) holds.

(2) The Generalized Alexander Duality Theorem 124.2 can be generalized even further. More precisely, an analogous statement also holds for any compact subset K of S^n, e.g. the Topologist's Sine Curve. But hereby one needs to replace the usual singular cohomology on the right-hand side by the *Čech cohomology* of K. We refer to [**Hat2002**, p. 256], [**Dol1980**, Theorem VIII.7.2], [**Bre1993**, Corollary VI.8.7], [**Mass1981b**] or [**Pra2007**, p. 271] for more details.

We will provide the proof of the Generalized Alexander Duality Theorem 124.2 in the next section. For the time being we discuss some special cases and consequences.
For most applications of Alexander Duality we will actually need the following straightforward consequence of the above theorem.

Theorem 124.3. (Alexander Duality Theorem for Topological Manifolds) Let $n \geq 2$, let Σ be a topological homology n-sphere and let X be a compact, non-empty k-dimensional topological manifold. If $f\colon X \to \Sigma$ is an injective map, then for any $j \in \mathbb{N}_0$ and any abelian group G we have a natural isomorphism

$$\widetilde{\mathrm{H}}_j(\Sigma \setminus f(X);G) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{n-j-1}(X;G),$$

which is natural in sense of the previous theorem.

[1508] There are many ways of seeing this. For example let $f\colon S^1 \to K$ be a homeomorphism and let $p\colon T \to S^1$ be the obvious homotopy equivalence. We only need to worry about homology in dimension one since all other reduced homology groups vanish. One can easily show, say using Proposition 76.20, that $(p \circ i \circ f)_*\colon \mathrm{H}_*(K;\mathbb{Z}) \to \mathrm{H}_*(T;\mathbb{Z})$ is an isomorphism. But then $i_*\colon \mathrm{H}_*(K;\mathbb{Z}) \to \mathrm{H}_*(T;\mathbb{Z})$ is also an isomorphism.

Proof. Let $f\colon X \to \Sigma$ be an injective map from a compact, non-empty k-dimensional topological manifold to a topological homology n-sphere Σ with $k < n$. It is a consequence of the Compact-Hausdorff Proposition 2.17 (3) that $f\colon X \to f(X)$ is a homeomorphism. From $k < n$ and the Topological Manifold-Dimension Proposition 75.4 it follows that $f(X) \neq \Sigma$. We have the following isomorphisms:

$$\widetilde{\mathrm{H}}_j(\Sigma \setminus f(X); G) \xrightarrow[\uparrow]{\cong} \widetilde{\mathrm{H}}^{n-j-1}(f(X); G) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{n-j-1}(X; G).$$

we can apply the Generalized Alexander Duality Theorem 124.2 since by the Topological Manifolds-Local Properties Lemma 18.8 we know that $f(X)$ is weakly locally contractible, and since $f(X) \neq \Sigma$ ■

Examples.

(1) Let $n \geq 2$ and let $k < n$. Furthermore let $f\colon S^k \to S^n$ be an injective map. We obtain that

$$\widetilde{\mathrm{H}}_j\big(S^n \setminus f(S^k); \mathbb{Z}\big) \underset{\text{Theorem 124.3}}{\cong} \widetilde{\mathrm{H}}^{n-j-1}(S^k; \mathbb{Z}) \underset{\text{see page 2422 and the Reduced Cohomology Lemma 108.16}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } j = n-k-1, \\ 0, & \text{otherwise.} \end{cases}$$

This calculation gives a new proof for the Sphere Complement-H_*-Theorem 82.1 (2). Basically the same approach also gives a new proof of the Sphere Complement-H_*-Theorem 82.1 (1). Finally note that this calculation gives an answer to Exercise 104.3 (b).

(2) Let $n \geq 2$, let X be a compact connected non-empty $(n-1)$-dimensional topological manifold with $\partial X \neq \varnothing$ and let $f\colon X \to S^n$ be an injective map. We see that

$$\widetilde{\mathrm{H}}_0\big(S^n \setminus f(X); \mathbb{Z}_2\big) \underset{\substack{\text{Alexander Duality} \\ \text{Theorem 124.3}}}{\cong} \widetilde{\mathrm{H}}^{n-1}(X; \mathbb{Z}_2) \underset{\text{since } n \geq 2}{\cong} \mathrm{H}^{n-1}(X; \mathbb{Z}_2) \underset{\substack{\text{Poincaré Duality} \\ \text{Theorem 119.1}}}{\cong} \mathrm{H}_0(X, \partial X; \mathbb{Z}_2) \underset{\substack{\text{since } X \text{ is connected} \\ \text{and since } \partial X \neq \varnothing}}{=} 0.$$

It follows from the analogue of the Reduced Homology Lemma 74.1 (5) for $\mathbb{Z}_2$-coefficients that $S^n \setminus f(X)$ is path-connected.

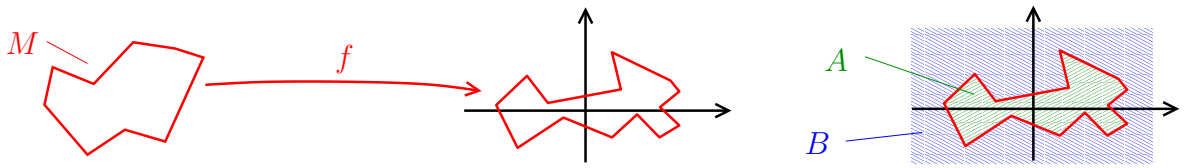
Corollary 124.4. (Generalized Jordan–Brouwer Separation Theorem) Let $n \in \mathbb{N}_{\geq 2}$, let M be an $(n-1)$-dimensional closed connected non-empty topological manifold and let $f\colon M \to \mathbb{R}^n$ be an injective map.

(1) The complement $\mathbb{R}^n \setminus f(M)$ has precisely two path-components, one of which is bounded and one which is unbounded.
(2) If A and B are these path-components, then $\partial A = \partial B = A \cap B = f(M)$.

Remark. The Generalized Jordan–Brouwer Separation Theorem 124.4 we first proved by James Alexander [**Al1922**, p. 348] in 1922. It is a generalization of the Generalized Smooth Jordan-Brouwer Separation Theorem 43.10 (1) and the Jordan–Brouwer Separation Theorem 82.2.

Proof. [1509] Let $n \in \mathbb{N}_{\geq 2}$, let M be an $(n-1)$-dimensional closed connected non-empty topological manifold and let $f\colon M \to \mathbb{R}^n$ be an injective map.

[1509] The proof of the Generalized Jordan–Brouwer Separation Theorem 124.4 is very similar to the Jordan–Brouwer Separation Theorem 82.2. The biggest difference is that we need to replace the Sphere



(1) We use the embedding $\mathbb{R}^n \to \mathbb{R}^n \cup \{\infty\} = S^n$ to view $\mathbb{R}^n$ as a subspace of S^n. Now we see that

$$\begin{array}{ccccccc} & & \text{see Exercise 78.12} & & \text{Alexander Duality Theorem 124.3} & & \text{since } n \geq 2 \\ & & \downarrow & & \downarrow & & \downarrow \\ \widetilde{\mathrm{H}}_0(\mathbb{R}^n \setminus f(M); \mathbb{Z}_2) & \cong & \widetilde{\mathrm{H}}_0(S^n \setminus f(M); \mathbb{Z}_2) & \cong & \widetilde{\mathrm{H}}^{n-1}(M; \mathbb{Z}_2) & \cong & \mathrm{H}^{n-1}(M; \mathbb{Z}_2) \\ & \cong & \mathrm{H}_0(M; \mathbb{Z}_2) & \cong & \mathbb{Z}_2. & & \\ & & \uparrow & & \uparrow & & \end{array}$$

by the Poincaré Duality Theorem 119.1 which we can apply since M is closed

by the analogue of the H_0-Proposition 72.5, since M connected and non-empty

It follows from the analogue of the Reduced Homology Lemma 74.1 (5) for $\mathbb{Z}_2$-coefficients that $\mathbb{R}^n \setminus f(M)$ has precisely two path-components. It remains to show that precisely one of them is unbounded.

Since M is compact there exists an $r \in \mathbb{R}$ such that $f(M) \subset B^n_r$. It is elementary to show that $\mathbb{R}^n \setminus B^n_r$ is path-connected. It follows that $\mathbb{R}^n \setminus B^n_r$ is contained in one of the two path-components. This shows that precisely one path-component is unbounded.

(2) First note that it follows from the Compact Image Lemma 2.13 and 1.21 that $\mathbb{R}^n \setminus f(M)$ is open. In particular, as we remarked on page 176, $\mathbb{R}^n \setminus f(M)$ is locally path-connected. It follows easily that A and B are both open subsets of $\mathbb{R}^n$.

Claim. We have $\overline{A} = A \cup f(M)$.

Proof. Since B is open we know that $A \cup f(M) = \mathbb{R}^n \setminus B$ is closed. By definition of $\overline{A}$ this implies that $\overline{A} \subset A \cup f(M)$. Now let $P \in A \cup f(M)$. We need to show that $P \in \overline{A}$. Suppose that this is not the case. By the Interior-Closure Lemma 1.9 we know that $\overline{A}$ is closed. Thus our assumption that $P \notin \overline{A}$ implies that there exists an open subset $U \subset \mathbb{R}^n$ with $P \in U \subset \mathbb{R}^n \setminus \overline{A}$. Using a chart one can show that there exists a subspace $D \subset M$, which is a compact $(n-1)$-dimensional topological manifold, such that $M \setminus D \subset f^{-1}(U)$, i.e. such that $f(M \setminus D) \subset U$. We now obtain the following equalities:

$$\begin{array}{ccccc} \underbrace{\mathbb{R}^n \setminus f(D)} & = & A \sqcup B \sqcup (f(M) \setminus f(D)) & \overset{\downarrow}{=} & A \sqcup (B \cup (U \cap \mathbb{R}^n \setminus f(D)). \\ \text{path-connected by the discussion on page 2713 and Exercise 78.12} & & & & \uparrow \\ & & & & \text{disjoint union since } U \cap A = \varnothing \end{array}$$

since $f(M) \setminus f(D) = f(M \setminus D) \subset U$

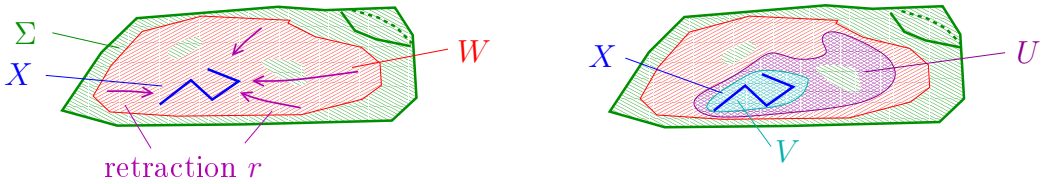
We have thus written the path-connected space $\mathbb{R}^n \setminus f(D)$ as the disjoint union of the two open non-empty sets A and $B \cup (U \cap \mathbb{R}^n \setminus f(D))$. But by the Path-Connected Implies Connected Corollary 2.23 this is not possible. We have thus obtained a contradiction. ∎

Complement-H_*-Theorem 82.1 by the Alexander Duality Theorem for Topological Manifolds 124.3. There are a few other subtle changes though.

124.2. **Proof of the Generalized Alexander Duality Theorem 124.2.** Before we can provide the proof of the Generalized Alexander Duality Theorem 124.2. let us recall the following proposition which is a consequence of Borsuk's Theorem 104.3. The proposition will be crucial input in our proof of the Generalized Alexander Duality Theorem 124.2.

Proposition 104.4. Let Σ be a topological manifold and let $X \subset \Sigma$ be a closed subset that is weakly locally contractible.

(1) There exists an open neighborhood $W \subset \Sigma$ of X for which there exists a retraction $r\colon W \to X$.
(2) We can find W and r such that furthermore the following condition is satisfied: given any neighborhood U of X with $X \subset U \subset W$ there exists an open neighborhood V with $X \subset V \subset U$ such that the inclusion $i\colon V \to U$ and the retraction $r\colon V \to X \subset U$ are homotopic maps from V to U.



We now turn to the proof of the Generalized Alexander Duality Theorem 124.2. We first deal with the case $j \neq 0$. Afterwards we will deal with the case $j = 0$, which is quite similar, but slightly different.

Proof of the Generalized Alexander Duality Theorem 124.2 for $j \neq 0$. Let $n \geq 2$ and let Σ be a topological homology n-sphere. Furthermore let $X \subsetneq \Sigma$ be a weakly locally contractible, non-empty compact subset of Σ. Note that to simplify the notation we work throughout the proof with the abelian group $G = \mathbb{Z}$.

We recall the following definition:

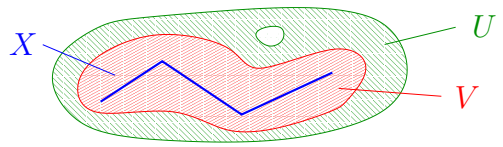
(1) Given a topological space Y we denote by $\mathcal{K}(Y)$ the set of all compact subsets of Y. Given $K, L \in \mathcal{K}(Y)$ we write $K \leq L$ if $K \subset L$. The set $\mathcal{K}(X)$ together with the relation "$\leq$" is a directed set since the union of two compact subsets is again compact.

We also introduce a new definition and notation:

(2) We denote by $\mathcal{N}(X)$ the set of all open neighborhoods of X. Given $U, V \in \mathcal{N}(X)$ we write $U \leq V$ if $V \subset U$. (Note the reversal of V and U.) The set $\mathcal{N}(X)$ together with the relation "$\leq$" is a directed set since the intersection of two open neighborhoods of X is again an open neighborhood of X.[1510]
(3) Given any subsets $A \subset B$ of Σ we denote by $i_{A,B}\colon A \to B$ the inclusion map.
(4) It follows from our definition of "$\leq$" on $\mathcal{N}(X)$ that for any $m \in \mathbb{N}_0$ the cohomology groups $\{\widetilde{\mathrm{H}}^m(U)\}_{U \in \mathcal{N}(X)}$ together with the inclusion induced maps form a *direct* system of abelian groups.

We continue with the key technical part of the proof of the theorem.

[1510] Here the idea is that the neighborhoods "approximate" X, in particular $U \leq V$ means that V is a "better approximation of X", which just means that $V \subset U$.



we have $U \leq V$ and $i^*_{V,U} \colon \widetilde{\mathrm{H}}^m(U) \to \widetilde{\mathrm{H}}^m(V)$

Claim. For every $m \in \mathbb{N}_0$ the map

$$\Phi \colon \varinjlim_{U \in \mathcal{N}(X)} \widetilde{\mathrm{H}}^m(U) \to \widetilde{\mathrm{H}}^m(X),$$

that is induced by the restriction maps $i^*_{X,U} \colon \widetilde{\mathrm{H}}^m(U) \to \widetilde{\mathrm{H}}^m(X)$, is an isomorphism.

Proof. By our hypothesis we know that X is compact and weakly locally contractible. Since Σ is a topological manifold we know by Proposition 104.4 that there exists an open neighborhood $W \subset \Sigma$ of $X \subset \Sigma$ and a retraction $r \colon W \to X$ such that furthermore the following condition is satisfied:

(∗) Given any neighborhood U of X with $X \subset U \subset W$ there exists an open neighborhood V with $X \subset V \subset U$ such that the inclusion map $i_{V,U} \colon V \to U$ and the retraction $r \colon V \to X \subset U$ are homotopic maps from V to U.

Next we consider the following commutative diagram

$$\begin{array}{ccc} & \widetilde{\mathrm{H}}^m(W) & \\ {\scriptstyle i^*_{X,W}} \swarrow & & \searrow {\scriptstyle r^*} \\ \varinjlim_{U \in \mathcal{N}(X)} \widetilde{\mathrm{H}}^m(U) & \xrightarrow[\Phi]{} & \widetilde{\mathrm{H}}^m(X). \end{array}$$

From $(i^*_{X,W} \circ r^*) = (r \circ i_{X,W})^* = \mathrm{id}^*_X$ we deduce that the map $i^*_{X,W}$ is an epimorphism. It follows that the horizontal map Φ is also an epimorphism. It remains to show that Φ is a monomorphism.

By Lemma 111.2 (3) it remains to show that for any $U \in \mathcal{N}(X)$ there exists a $V \in \mathcal{N}(X)$ with $U \leq V$, i.e. with $V \subset U$, such that

$$\ker\left(i^*_{X,U} \colon \widetilde{\mathrm{H}}^m(U) \to \widetilde{\mathrm{H}}^m(X)\right\} \subset \ker\left(i^*_{V,U} \colon \widetilde{\mathrm{H}}^m(U) \to \widetilde{\mathrm{H}}^m(V)\right).$$

So let $U \in \mathcal{N}(X)$. After possibly replacing U by $U \cap W$ we can without loss of generality assume that $U \subset W$.

By (∗) there exists an open neighborhood V with $X \subset V \subset U$ such that the inclusion $i_{V,U} \colon V \to U$ and the retraction $r \colon V \to X \subset U$ are homotopic maps from V to U.

We consider the following diagram

$$\begin{array}{ccc} \widetilde{\mathrm{H}}^m(U) & \xrightarrow{\quad i^*_{X,U} \quad} & \widetilde{\mathrm{H}}^m(X) \\ & {\scriptstyle r^*} \searrow \quad \swarrow {\scriptstyle r^*} & \\ {\scriptstyle i^*_{V,U}} \searrow & & \swarrow \\ & \widetilde{\mathrm{H}}^m(V). & \end{array}$$

Since the two maps $i_{V,U} \colon V \to U$ and $r \colon V \to X \subset U$ are homotopic it follows from the Homotopic Maps-H*-Lemma 108.13 that the two maps $i^*_{V,U}$ and r^* from $\widetilde{\mathrm{H}}^m(U)$ to

$\widetilde{\mathrm{H}}^m(V)$ agree. Since r is the identity on X we see that $r\colon U \to X \subset V$ equals the map $i_{X,U} \circ r\colon U \to X$. This shows that the diagram commutes. It follows immediately from these observations that V has the desired property. ⊞

Now we can finally turn to the actual proof of the theorem. First let $j \neq 0$. The desired isomorphism will be given by a sequence of natural isomorphisms as follows:

$$\begin{aligned}\mathrm{H}_j(\Sigma\backslash X) \underset{(1)}{\overset{\cong}{\leftarrow}} \mathrm{H}_{\mathrm{c}}^{n-j}(\Sigma\backslash X) &\underset{(2)}{\overset{\cong}{\leftarrow}} \varinjlim_{K\in\mathcal{K}(\Sigma\backslash X)} \mathrm{H}^{n-j}(\Sigma\backslash X, \overbrace{(\Sigma\backslash X)\backslash K}^{=(\Sigma\backslash K)\backslash X}) \underset{(3)}{\overset{\cong}{\leftarrow}} \varinjlim_{U\in\mathcal{N}(X)} \mathrm{H}^{n-j}(\Sigma\backslash X, U\backslash X)\\ &\underset{(4)}{\overset{\cong}{\leftarrow}} \varinjlim_{U\in\mathcal{N}(X)} \mathrm{H}^{n-j}(\Sigma, U) \underset{(5)}{\overset{\cong}{\leftarrow}} \varinjlim_{U\in\mathcal{N}(X)} \widetilde{\mathrm{H}}^{n-j-1}(U) \underset{(6)}{\overset{\cong}{\rightarrow}} \widetilde{\mathrm{H}}^{n-j-1}(X).\end{aligned}$$

We explain the natural isomorphisms in more detail:

(1) Since X is compact we know from the Compact-Closed Lemma 1.21 (2) that $\Sigma \setminus X$ is an open subset of $\Sigma \setminus X$. In particular $\Sigma \setminus X$ is also an n-dimensional topological manifold and via the Topological Manifold-Induced Orientation Lemma 105.17 it inherits an orientation from Σ. Also note that it follows from the Topological Manifold Boundary Proposition 75.2 that $\Sigma \setminus X$ has no boundary. After this preamble the first isomorphism is the Poincaré Duality isomorphism from the Non-Compact PD Theorem 119.18 corresponding to the oriented topological manifold $\Sigma \setminus X$. By Lemma 119.24 this isomorphism is natural.

(2) The second isomorphism is provided by the $\mathrm{H}_{\mathrm{c}}^n$-via-Direct Limits Proposition 112.4.

(3) Recall that the X is by hypothesis a compact subset of Σ. By the Compact-Closed Lemma 1.21 (2) this implies that X is a closed subset of S^3. This observation implies that the map $\mathcal{N}(X) \to \mathcal{K}(\Sigma \setminus X)$ given by $U \mapsto \Sigma \setminus U$ is an isomorphism of directed sets which gives us an isomorphism of direct systems. The third isomorphisms is an immediate consequence of this fact.

(4) For any open neighborhoods U and V of X with $U \leq V$, i.e. with $V \subset U$, we have a commutative diagram

$$\begin{array}{ccc}\mathrm{H}^{n-j}(\Sigma\backslash X, V\backslash X) & \longleftarrow & \mathrm{H}^{n-j}(\Sigma, V)\\ \uparrow & & \uparrow\\ \mathrm{H}^{n-j}(\Sigma\backslash X, U\backslash X) & \longleftarrow & \mathrm{H}^{n-j}(\Sigma, U)\end{array}$$

where all the maps are induced by the obvious inclusions of pairs of topological spaces. It follows from the Excision Theorem 109.1 for Cohomology Groups that the horizontal maps are isomorphisms. The desired isomorphism of direct limits is now an immediate consequence of the Cofinal-Isomorphism Lemma 111.4.

(5) For any open neighborhoods U and V of X with $U \leq V$, i.e. with $V \subset U$, we have a commutative diagram

$$\begin{array}{ccc}\mathrm{H}^{n-j}(\Sigma, V) & \xleftarrow{\delta} & \widetilde{\mathrm{H}}^{n-j-1}(V)\\ \uparrow & & \uparrow\\ \mathrm{H}^{n-j}(\Sigma, U) & \xleftarrow{\delta} & \mathrm{H}^{n-j-1}(U)\end{array}$$

where the vertical maps are induced by the obvious inclusions and where the horizontal maps are the connecting homomorphisms of the long exact sequences in reduced cohomology provided by Lemma 108.17. Since $j \geq 1$ we obtain from these long exact sequences and the calculation of the cohomology groups of Σ that the horizontal maps are in fact isomorphisms. As in (4) we appeal to the Cofinal-Isomorphism Lemma 111.4 to obtain the desired isomorphism of direct limits.

(6) The last map is the isomorphism of the above claim.

This concludes the proof of the construction of an isomorphism

$$\Theta_X \colon \widetilde{\mathrm{H}}_j(\Sigma \setminus X; G) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{n-j-1}(X; G).$$

Finally let $X \subset Y \subsetneq \Sigma$ be two weakly locally contractible, non-empty compact subset of Σ. As we more or less discussed, the isomorphisms (1)–(6) that we just constructed are all natural with respect to inclusions. Thus we obtain the following commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_j(\Sigma \setminus X) & \xrightarrow[\cong]{\Theta_X} & \widetilde{\mathrm{H}}^{n-j-1}(X) \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{H}}_j(\Sigma \setminus Y) & \xrightarrow[\cong]{\Theta_Y} & \widetilde{\mathrm{H}}^{n-j-1}(Y), \end{array}$$

We leave it to the reader to double-check this claim. ■

Proof of the Generalized Alexander Duality Theorem 124.2 for $j = 0$. It still remains to deal with the case $j = 0$. The argument is almost entirely the same as for $j \neq 0$. In particular we can again construct the maps (1)–(6). In the case $= 0$ all of the above maps (1)–(6) are still isomorphisms, except for map (5). To address this issue we consider the following diagram:

$$\begin{array}{ccccccc} \mathrm{H}_0(\Sigma\setminus X) & \xrightarrow[\cong]{(1)} & \mathrm{H}^n_{\mathrm{c}}(\Sigma\setminus X) & \xrightarrow[\cong]{(2),(3)} & \varinjlim_{U\in\mathcal{N}(X)} \mathrm{H}^n(\Sigma\setminus X, U\setminus X) & \xrightarrow[\cong]{(4)} & \varinjlim_{U\in\mathcal{N}(X)} \mathrm{H}^n(\Sigma, U) \\ \downarrow & & \downarrow & & \cong\downarrow & & \downarrow \\ \mathrm{H}_0(\Sigma) & \xrightarrow[\cong]{(1)} & \mathrm{H}^n_{\mathrm{c}}(\Sigma) & \xrightarrow[\cong]{(2),(3)} & \varinjlim_{U\in\mathcal{N}(X)} \mathrm{H}^n(\Sigma, U) & \xrightarrow[\cong]{(4)} & \varinjlim_{U\in\mathcal{N}(X)} \mathrm{H}^n(\Sigma). \end{array}$$

Here the second vertical map is the inclusion induced map defined by Lemma 112.5. The first and the fourth vertical maps are the obvious inclusion induced maps. Furthermore the third vertical map is the inverse of the isomorphism which is induced by the isomorphisms $\mathrm{H}^n(\Sigma, U) \to \mathrm{H}^n(\Sigma\setminus X, U\setminus X)$ that are given by the Excision Theorem 109.1 for Cohomology Groups. The top horizontal maps are precisely the above maps (1), (2), (3) and (4). The bottom horizontal maps are defined completely analogously with $X = \varnothing$. Since the maps defined in (1), (2), (3) and (4) are natural we see that the diagram commutes. It follows

that

induced by the natural map $\varkappa\colon \widetilde{\mathrm{H}}_0(\Sigma\backslash X) \to \widetilde{\mathrm{H}}_0(\Sigma\backslash X)$ it follows from Lemma 74.2 that this map is in fact an isomorphism — by the above commutative diagram

$$\begin{array}{ccccc} \widetilde{\mathrm{H}}_0(\Sigma\backslash X) & \overset{\cong}{\to} & \ker(\mathrm{H}_0(\Sigma\backslash X) \to \mathrm{H}_0(\Sigma)) & \overset{\cong}{\to} & \ker\left(\varinjlim \mathrm{H}^n(\Sigma, U) \to \varinjlim \mathrm{H}^n(\Sigma)\right) \\ & \overset{\cong}{\leftarrow} & \varinjlim \ker(\mathrm{H}^n(\Sigma, U) \to \mathrm{H}^n(\Sigma)) & \overset{\cong}{\leftarrow} & \varinjlim \widetilde{\mathrm{H}}^{n-1}(U) \quad \overset{\cong}{\to} \quad \widetilde{\mathrm{H}}^{n-1}(X). \\ & & \uparrow & & \uparrow \qquad\qquad \uparrow \end{array}$$

Direct Limit-(Co-)-Kernel Corollary 79.3 — by the long exact sequence of Lemma 108.17 and since $n \geq 2$ — by the claim

This concludes the proof of the construction of an isomorphism

$$\Theta_X\colon \widetilde{\mathrm{H}}_j(\Sigma \setminus X; G) \overset{\cong}{\to} \widetilde{\mathrm{H}}^{n-j-1}(X; G).$$

We leave the task of verifying naturality to the reader. ■

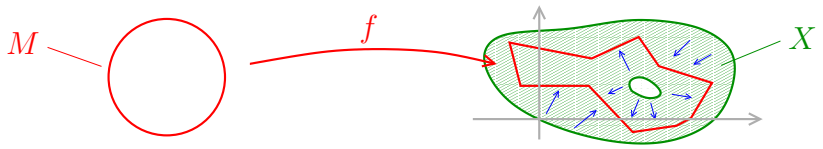
124.3. **Consequences of the Generalized Alexander Duality Theorem 124.2.** The following proposition is the topological analogue of the Smooth Codimension-One Orientability Proposition 43.12 which we proved on pages 1002 and 2705.

Proposition 124.5. (Topological Codimension-One Orientability Proposition) Let M be a closed connected n-dimensional topological manifold. If M is non-orientable, then there is no *topological* embedding of M into $\mathbb{R}^{n+1}$.

Proof. The proof of the Topological Codimension-One Orientability Proposition 124.5 is verbatim the same as the proof of the Smooth Codimension-One Orientability Proposition 43.12, we only need to replace the original Alexander Duality Theorem 123.6 by the Alexander Duality Theorem 124.3 for Topological Manifolds.

We also give an alternative argument that does not require the slightly obscure Generalized Alexander Duality Theorem 124.2. Thus let M be a closed connected n-dimensional topological manifold which admits an embedding $f\colon M \to \mathbb{R}^{n+1}$. We have to show that M is orientable. As usual we view $\mathbb{R}^{n+1}$ as a subset of S^{n+1}. We make the following observations:

(1) It follows from Borsuk's Theorem 104.6 together with Proposition 104.8 (1) that there exists a compact $(n+1)$-dimensional smooth submanifold X of $\mathbb{R}^{n+1}$ that contains $f(M)$ and such that there exists a retraction $r\colon X \to f(M)$. In the following we denote by $i\colon f(M) \to X$ the inclusion map.
(2) Since $r \circ i = \mathrm{id}_{f(M)}$ we see that the map $r^*\colon \mathrm{H}^n(f(M); \mathbb{Z}) \to \mathrm{H}^n(X; \mathbb{Z})$, induced by the retraction, is a monomorphism.
(3) By the remark on page 2700 there is an isomorphism $\mathrm{H}^n(X; \mathbb{Z}) \cong \mathrm{H}_0(S^{n+1} \setminus X; \mathbb{Z})$. It follows in particular from the H_0-Proposition 72.5 that $\mathrm{H}^n(X; \mathbb{Z})$ is a free abelian group.
(4) It follows from (2) and (3) that $\mathrm{H}^n(f(M); \mathbb{Z})$ is a torsion-free abelian group.
(5) Since $f\colon M \to f(M)$ is a homeomorphism we obtain from (4) that $\mathrm{H}^n(M; \mathbb{Z})$ is a torsion-free abelian group.
(6) We deduce from (5) and the Top-Cohomology Proposition 110.14 that the closed connected n-dimensional topological manifold M is orientable. ■



In Question 24.13 (3) we asked whether $\mathbb{R}\mathrm{P}^2$ or the Klein bottle admit an embedding into $\mathbb{R}^3$. We can now give a negative answer.

Corollary 124.6. Neither $\mathbb{R}\mathrm{P}^2$ nor the Klein bottle admit an embedding into $\mathbb{R}^3$.

Proof. By the Surface Non-Orientability Lemma 25.20 we know that $\mathbb{R}\mathrm{P}^2$ and the Klein bottle are both non-orientable as smooth manifolds. It follows from the Smooth-Topological Orientation Proposition 105.11 that both are also non-orientable as topological manifolds. Since both are closed 2-dimensional topological manifolds it follows from the Topological Codimension-One Orientability Proposition 124.5 that neither $\mathbb{R}\mathrm{P}^2$ nor the Klein bottle admit an embedding into $\mathbb{R}^3$. ■

The following proposition is a strengthening of Proposition 123.11.

Proposition 124.7. If $n \in \mathbb{N}_{\geq 2}$, then there is no embedding of $\mathbb{R}\mathrm{P}^n$ into S^{n+1}.

Proof. First let n be even. In this case we know by the Real Projective Space Orientability Lemma 25.21 and the Smooth-Topological Orientation Proposition 105.11 that $\mathbb{R}\mathrm{P}^n$ is non-orientable. Thus the result is an immediate consequence of the Topological Codimension-One Orientability Proposition 124.5.

Now let n be odd. This case is significantly harder and it was first proved by René Thom [**Tho1952**, p. 180] in 1952. We cannot provide the argument. But it seems like the argument of Thom is somewhat reminiscent of the argument in the proof of Proposition 123.11. So suppose there exists an embedding $f\colon \mathbb{R}\mathrm{P}^n \to S^{n+1}$. It follows from the H_0-Proposition 72.5, the Generalized Alexander Duality Theorem 124.2 and the fact that $\mathrm{H}^n(\mathbb{R}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ (here we use that n is odd), that $S^{n+1} \setminus f(\mathbb{R}\mathrm{P}^n)$ consists of two path components. We denote the closures of the path components by A and B. The problem is that the triad (S^{n+1}, A, B) is in general not excisive so we cannot apply the Mayer–Vietoris Theorem 109.15 for Cohomology Groups. The idea in the proof is to replace usual singular cohomology by Alexander-Spanier cohomology for which the above triad is excisive. We refer to [**Mass1978a**, Chapter 8] or [**Mass1978b**] for an introduction to Alexander-Spanier cohomology and we refer to [**Mass1978a**, Theorem II.8.15] for the proof that in this setting the above triad is excisive. ■

124.4. **Generalized Alexander Duality Theorem for Homology Balls.** We have the following obvious analogue of the definition of a homology sphere.

Definition. We say that an n-dimensional topological manifold X is a topological homology n-ball if X is compact and if for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X;\mathbb{Z}) \cong \mathrm{H}_k(\overline{B}^n;\mathbb{Z})$.

Lemma 124.8. If X is a topological homology n-ball, then ∂X is a topological homology $(n-1)$-sphere.

Proof. Let X be a topological homology n-ball. Note that this implies in particular that $\mathrm{H}_1(X;\mathbb{Z}) = 0$. Therefore it follows from the Non-Orientable $\mathbb{Z}_2$-Epimorphism Corollary 105.16 together with the Hurewicz Theorem 85.5 that X is orientable.

At this state this lemma follows easily from the Poincaré Duality Theorem 119.1 and the long exact sequence in homology of the pair $(X, \partial X)$. We will fill in the details in Exercise 124.2. ■

We can now formulate the following variation on the Generalized Alexander Duality Theorem 124.2.

Theorem 124.9. (Generalized Alexander Duality Theorem for Balls) Let $n \geq 2$, let B be an n-dimensional homology ball and let $X \neq B$ be a non-empty compact subset of B. We write $\partial X := X \cap \partial B$. If X and ∂X are weakly locally contractible, then given any $j \in \mathbb{N}_0$ we have a natural isomorphism

$$\Theta_X \colon \widetilde{\mathrm{H}}_j(B \setminus X;\mathbb{Z}) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{n-j-1}(X, \partial X;\mathbb{Z}).$$

Here natural means that if we are given $X \subset Y \subset B$ that both satisfy the above conditions, then the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_j(B \setminus Y;\mathbb{Z}) & \xrightarrow[\cong]{\Theta_Y} & \mathrm{H}^{n-j-1}(Y, \partial Y;\mathbb{Z}) \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{H}}_j(B \setminus X;\mathbb{Z}) & \xrightarrow[\cong]{\Theta_X} & \mathrm{H}^{n-j-1}(X, \partial X;\mathbb{Z}), \end{array}$$

where the vertical maps are induced by the inclusions $B \setminus Y \to B \setminus X$ and $X \to Y$.

Proof. One could try to imitate the proof of the Generalized Alexander Duality Theorem 124.2, but instead we will do a little trick to reduce most of the statement to the Generalized Alexander Duality Theorem 124.2.

Let $X \subsetneq B$ be a non-empty compact subset of B such that X and $\partial X := X \cap \partial B$ are both weakly locally contractible.

Let $X \subsetneq B$ be a weakly locally contractible compact non-empty subset. We consider

$$K := (X \times \{0\}) \cup \underbrace{(\partial X \times [0,1]) \cup (B \times \{1\})}_{=:Y} \subset \underbrace{(B \times \{0\}) \cup (\partial B \times [0,1]) \cup (B \times \{1\})}_{=\partial(B \times [0,1])}.$$

Claim 1. The topological space K is again weakly locally contractible.

Proof. This claim can be proved easily by considering the following cases:

(1) The points $(x,0)$ with $x \notin \partial X$ and $(x,0)$ with $x \in \partial X$.
(2) The points (x,t) with $t \in (0,1)$.
(3) The points $(x,1)$ with $x \in B \setminus \partial B$ and $(x,1)$ with $x \in \partial B$.

The first cases can be dealt with easily using the fact that X is weakly locally contractible. For the second case we need to use that ∂X is weakly locally contractible. The third case can be dealt with without any assumptions on X. ⊞

Now we consider the following maps

$$\begin{array}{ccccc}
\widetilde{\mathrm{H}}_j(B\setminus X) & \xrightarrow[\cong]{\text{(a)}} & \widetilde{\mathrm{H}}_j((B\times 0)\setminus(X\times\{0\})) & \xrightarrow[\cong]{\text{(b)}} & \widetilde{\mathrm{H}}_j(\partial(B\times[0,1])\setminus K) \\
\downarrow\cong & & \downarrow\cong & & \text{(c)}\downarrow\cong \\
\mathrm{H}^{n-j-1}(X,\partial X) & \xleftarrow[\text{(e)}]{\cong} & \mathrm{H}^{n-j-1}(K,Y) & \xleftarrow[\text{(d)}]{\cong} & \widetilde{\mathrm{H}}^{n-j-1}(K).
\end{array}$$

We make the following observations regarding the various maps:

(a) This map is induced by the obvious homeomorphism.

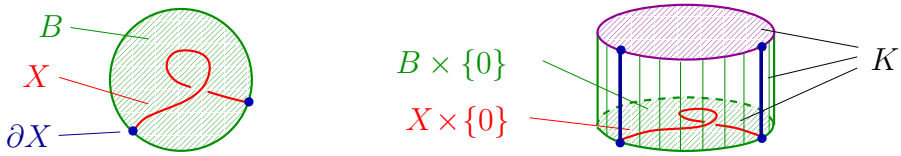
(b) This map is an isomorphism since $(B\times 0)\setminus(X\times\{0\})$ is a deformation retract of $\partial(B\times[0,1])\setminus K$.

(c)
- It follows from the Künneth Theorem 90.8 together with the Topological Manifold Product Proposition 18.4 that $B^n\times[0,1]$ is a topological homology $(n+1)$-ball with boundary $(\partial B^n\times[0,1])\cup(B^n\times\{0,1\})$. It follows from Lemma 124.8 that $\partial(B^n\times[0,1])$ is a topological homology n-sphere.
- In Claim 1 we just showed that K is again weakly locally contractible.
- By the above we can appeal to the Generalized Alexander Duality Theorem 124.2 to obtain a natural isomorphism.

(d) This map is an isomorphism since Y is contractible.

(e) This map is an isomorphism by a suitable variation on the Excision Theorem 109.1.

The combination of the above gives us a natural vertical isomorphism Θ_X to the left. The proof of the naturality of Θ_X follows from the fact that all the isomorphisms we considered are natural under inclusions.



■

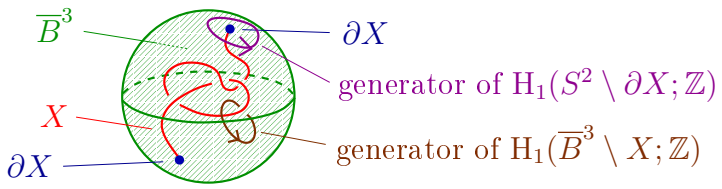
Theorem 124.10. (Generalized Alexander Duality Theorem for Balls) Let $n\geq 2$, let B be an n-dimensional homology ball and let $X\neq B$ be a non-empty compact subset of B. We write $\partial X := X\cap\partial B$. If X and ∂X are weakly locally contractible, non-empty compact subset of B, then given any $j\in\mathbb{N}_0$ we have a natural isomorphism

$$\Theta_X\colon \widetilde{\mathrm{H}}_j(B\setminus X;\mathbb{Z}) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{n-j-1}(X,\partial X;\mathbb{Z}).$$

such that the following diagram commutes

$$\begin{array}{ccc}
\widetilde{\mathrm{H}}_{j-1}(\partial B\setminus\partial X;\mathbb{Z}) & \xrightarrow[\cong]{\Theta_{\partial X}} & \widetilde{\mathrm{H}}^{n-j-1}(\partial X;\mathbb{Z}) \\
{\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle \delta} \\
\widetilde{\mathrm{H}}_j(B\setminus X;\mathbb{Z}) & \xrightarrow[\cong]{\Theta_X} & \mathrm{H}^{n-j-1}(X,\partial X;\mathbb{Z}),
\end{array}$$

up to a sign that depends only on j and n.[1511] Here the bottom map is the isomorphism given in the proof of the Generalized Alexander Duality Theorem 124.2.



Proof. We consider

$$\widetilde{K} := (B \times \{0\}) \cup \underbrace{(\partial X \times [0,1]) \cup (B \times \{1\})}_{=:Y} \subset \underbrace{(B \times \{0\}) \cup (\partial B \times [0,1]) \cup (B \times \{1\})}_{=\partial(B \times [0,1])}.$$

By the naturality of the isomorphism from the Generalized Alexander Duality Theorem 124.2 we obtain the following commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_j(\partial(B \times [0,1]) \setminus \widetilde{K}) & \longrightarrow & \widetilde{\mathrm{H}}_j(\partial(B \times [0,1]) \setminus K) \\ {\scriptstyle (c)}\downarrow\cong & & {\scriptstyle (c)}\downarrow\cong \\ \widetilde{\mathrm{H}}^{n-j-1}(\widetilde{K}) & \longrightarrow & \widetilde{\mathrm{H}}^{n-j-1}(K). \end{array}$$

■

In the next section we will make use of the following lemma.

Corollary 124.11. Let $n \geq 2$ and let B be a topological n-dimensional homology ball and let $f\colon \overline{B}^k \to B$ be a proper embedding, i.e. f is an embedding with $f^{-1}(\partial B) = S^{k-1}$. We denote by $i\colon \partial B \setminus f(S^{k-1}) \to B \setminus f(\overline{B}^k)$ the inclusion map. The following two statements hold:

(1) The maps $i_*\colon \mathrm{H}_j(\partial B \setminus f(S^{k-1})) \to \mathrm{H}_j(B \setminus f(\overline{B}^k))$ are all isomorphisms.
(2) For each j we have

$$\mathrm{H}_j(B \setminus f(\overline{B}^k)) \cong \begin{cases} \mathbb{Z}, & \text{if } j = 0, n-k-1, \\ 0, & \text{else.} \end{cases}$$

Proof. By hypothesis $f\colon \overline{B}^k \to B$ is a proper embedding. Since $\overline{B}^k$ is evidently weakly locally contractible we see that $f(\overline{B}^k)$ is also weakly locally contractible. Thus we obtain from the Generalized Alexander Duality Theorem 124.10 for Balls that for any $j \in \mathbb{N}_0$ we have an isomorphism

$$\widetilde{\mathrm{H}}_j(B \setminus f(\overline{B}^k); \mathbb{Z}) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{n-j-1}(\overline{B}^k, S^{k-1}; \mathbb{Z}).$$

such that the following diagram commutes

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_{j-1}(\partial B \setminus f(S^{k-1}); \mathbb{Z}) & \xrightarrow{\cong} & \widetilde{\mathrm{H}}^{n-j-1}(S^{k-1}); \mathbb{Z}) \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle \delta} \\ \widetilde{\mathrm{H}}_j(B \setminus f(\overline{B}^k); \mathbb{Z}) & \xrightarrow{\cong} & \mathrm{H}^{n-j-1}(\overline{B}^k, S^{k-1}; \mathbb{Z}), \end{array}$$

up to a sign. We make the following observations:

[1511] I will not embarrass myself by trying to guess the sign.

(a) It follows from the Single Point-Cohomology Lemma 108.12 and 108.13 that the reduced cohomology groups of $\overline{B}^k$ vanish.
(b) It follows from (a) and from the long exact sequence of reduced cohomology groups, see Lemma 108.17, that the vertical maps to the right are isomorphisms. Together with the commutativity of the diagram this implies Statement (1).
(c) On page 2419 we calculated the cohomology groups of S^{k-1}. Together with Lemma 108.17 and the above isomorphisms we obtain a proof for Statement (2). ■

Theorem 124.12. (Generalized Alexander Duality Theorem for Balls) Let $n \geq 2$, let B be an n-dimensional homology ball and let $X \neq B$ be a non-empty compact subset of B. We write $\partial X := X \cap \partial B$. We suppose that X and ∂X are weakly locally contractible, non-empty compact subsets of B. If ∂X is a smooth submanifold of ∂B and it has a collar in B, then the following diagram commutes

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_{j-1}(\partial B \setminus \partial X;\mathbb{Z}) & \xrightarrow[\cong]{\Theta_{\partial X}} & \widetilde{\mathrm{H}}^{n-j-1}(\partial X;\mathbb{Z}) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \delta} \\ \widetilde{\mathrm{H}}_{j}(B \setminus X;\mathbb{Z}) & \xrightarrow[\cong]{\Theta_{X}} & \mathrm{H}^{n-j-1}(X,\partial X;\mathbb{Z}), \end{array}$$

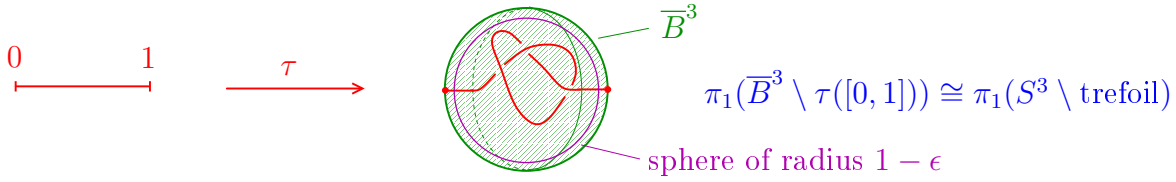
up to a sign that depends only on j and n.[1512] Here the bottom map is the isomorphism given in the proof of the Generalized Alexander Duality Theorem 124.2.

124.5. **Wild arcs and wild knots.** In this section we will use the Generalized Alexander Duality Theorem for Balls 124.12 to prove the following theorem.

Theorem 124.13. There exists an embedding $\gamma\colon S^1 \to S^3$ such that $\gamma(S^1) \subset S^3$ is not a topological manifold.

The construction of such a map $\gamma\colon S^1 \to S^3$ requires some preparations. We start out with the following lemma.

Lemma 124.14. There exists a proper smooth embedding $\tau\colon [0,1] \to \overline{B}^3$ with the following two properties:
(1) there exists an $\epsilon > 0$ such that $\tau(t) = (2t-1, 0, 0)$ for $t \in [0, \frac{\epsilon}{2}]$ and $t \in [1-\frac{\epsilon}{2}, 1]$ and such that $\tau^{-1}(\{x \in \overline{B}^3 \mid \|x\| > 1-\epsilon\}) = [0, \frac{\epsilon}{2}] \cup [1-\frac{\epsilon}{2}, 1]$,
(2) $\pi_1(\overline{B}^3 \setminus \tau([0,1]))$ is non-abelian.



Sketch of proof. Let $T \subset S^3$ be the trefoil, as defined on page 1312. In Proposition 59.6 we showed that $\pi_1(S^3 \setminus T)$ is non-abelian. In the proof of Proposition 59.14 we showed

[1512] I will not embarrass myself by trying to guess the sign.

that T gives rise to a desired proper smooth embedding $\tau\colon [0,1] \to \overline{B}^3$ that satisfies (a) and such that $\pi_1(\overline{B} \setminus \tau([0,1])) \cong \pi_1(S^3 \setminus T)$ ■

Our goal is to construct an embedding $f\colon S^1 \to S^3$ such that $f(S^1)$ is not a topological submanifold of S^3. To simplify the notation we will use the homeomorphisms from the Ball-Quotient Sphere Lemma 5.20 and pages261 to make the identifications $[0,2]/0 \sim 2 = S^1$ and $\overline{B}^3_1 \cup_{S^2_1 = S^2_2} \overline{B}^3_2 = S^3$.

Notation.

(1) For the remainder of this section we fix a proper smooth embedding $\tau\colon [0,1] \to \overline{B}^3$ as in Lemma 124.14.

(2) Given $n \in \mathbb{N}$ we consider the smooth embedding

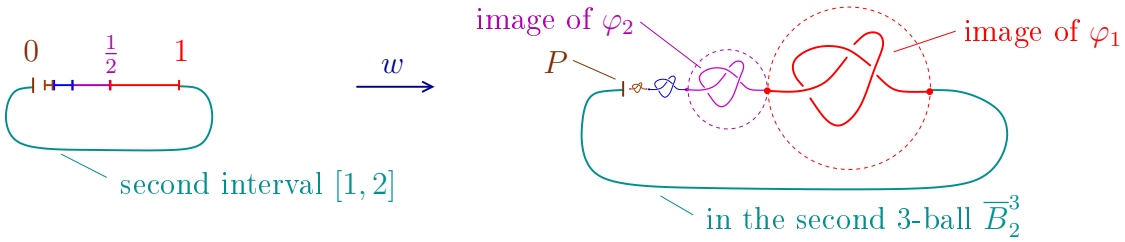
$$\begin{array}{rcl} \varphi_n\colon \overline{B}^3 & \to & B^3 \subset \overline{B}^3 \\ x & \mapsto & (-1 + \frac{3}{2^n}, 0, 0) + \frac{1}{2^n} \cdot x. \end{array}$$

(3) We consider the map

$$w\colon \overbrace{[0,2]/0 \sim 2}^{=S^1} \to \overbrace{\overline{B}^3_1 \cup_{S^2_1 = S^2_2} \overline{B}^3_2}^{=S^3}$$

$$t \mapsto \begin{cases} (-1,0,0) \in \overline{B}^3_1, & \text{if } t = 0, \\ \varphi_n(\tau(\underbrace{2^{n-1} \cdot (t - \frac{1}{2^n})}_{\in (0,1]})) \in \overline{B}^3_1, & \text{if } t \in (\frac{1}{2^n}, \frac{1}{2^{n-1}}] \text{ for some } n \in \mathbb{N}, \\ (3 - 2t, 0, 0) \in \overline{B}^3_2, & \text{if } t \in [1,2]. \end{cases}$$

(4) We set $W := w(S^1)$ and we write $P := w([0])$.



The following proposition immediately implies Theorem 124.13.

Proposition 124.15. Let $w\colon S^1 \to S^3$ be the above map.

(1) The map $w\colon S^1 \to S^3$ is an embedding.

(2) There is no submanifold chart, in the sense of the definition on page 524, for the subset $W := w(S^1) \subset S^3$ around $P = w([0])$. In particular $W = w(S^1)$ is not a submanifold of S^3.

Our proof of Proposition 124.15 crucially relies on the following lemma, which is a direct descendant of the Generalized Alexander Duality Theorem 124.2.

Lemma 124.16. For any $n \in \mathbb{N}$ the inclusion induced map

$$\mathrm{H}_1(\varphi_n(S^2) \setminus W) \to \mathrm{H}_1(S^3 \setminus W)$$

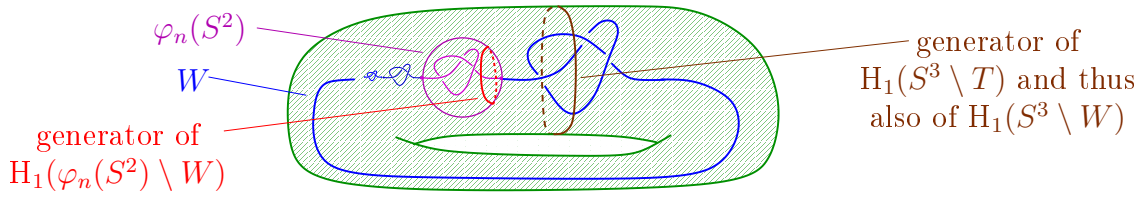
is an isomorphism.

Proof. As the reader will see short, the proof of Lemma 124.16 is almost identical to the argument on page 2711 where we identified a generator for $\mathrm{H}_1(S^3 \setminus W)$. For peace of mind we provide the argument.

Let $n \in \mathbb{N}$. We denote by T the "standard solid torus" in $S^3 = \overline{B}^3_1 \cup_{S^2_1 = S^2_2} \overline{B}^3_2$ that is given by the points $(x, y, z) \in \overline{B}_i$ in each ball with $y^2 + z^2 \leq \frac{1}{2}$. Note that $W \subset T$. We denote by $i \colon W \to T$ the inclusion map. By the naturality of the isomorphism of the Generalized Alexander Duality Theorem 124.2 we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_j(\Sigma \setminus X; G) & \xrightarrow[\cong]{\Theta_X} & \widetilde{\mathrm{H}}^{n-j-1}(X; G) \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{H}}_j(\Sigma \setminus Y; G) & \xrightarrow[\cong]{\Theta_Y} & \widetilde{\mathrm{H}}^{n-j-1}(Y; G), \end{array}$$

here the left vertical map is induced by the inclusion $\Sigma \setminus X \to \Sigma \setminus Y$ and the right vertical map is induced by the inclusion $X \to Y$. ■



Our proof of Proposition 124.15 relies also on the following earlier proposition.

Proposition 53.24. (Amalgamated Product-Monomorphism Proposition) Let $\alpha \colon G \to A$ and $\beta \colon G \to B$ be two group homomorphisms. We denote by $A *_G B$ the corresponding amalgamated product as defined on page 1209. If α and β are both monomorphisms, then the natural homomorphisms $A \to A *_G B$ and $B \to A *_G B$ are also both monomorphisms.

Proof of Proposition 124.15. First we provide the proof of Statement (1). Using the ϵ-δ-criterion from the Metric Continuity Proposition 2.8 one can easily show that the map $[0, 1] \to S^3$, induced by f, is continuous. It is clear that the corresponding map $[1, 2] \to S^3$ is continuous. It follows from the Pasting Pasting Proposition 2.6 together with the Topological-Quotient Proposition 5.15 (1b) that the given map $f \colon S^1 \to S^3$ is continuous. It now follows from the Compact-Hausdorff Proposition 2.17 (2) that $f \colon S^1 \to S^3$ is actually an embedding.

Now we turn to the proof of Statement (2). We start out with the following harmless general claim.

Claim 1. Let X be an n-dimensional topological manifold, let $A \subset X$ be a subset and let $Q \in A$. If there exists a submanifold chart of type (α) for $A \subset X$ and $Q \in A$, then given any open neighborhood U of $Q \in A$ there exists a neighborhood V with the following properties:

(1) $A \in V \subset U$,

(2) V is homeomorphic to $\overline{B}^3$,

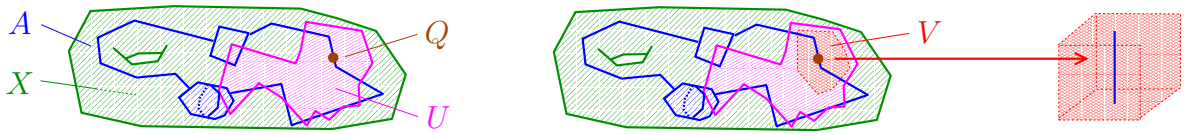
(3) either $V = A \cap V$ or $V \setminus (A \cap V)$ is non-empty and path-connected with abelian $\pi_1(V \setminus (A \cap V))$.[1513]

Proof. It follows almost immediately from the definition of a chart for A and Q that given any open neighborhood U of $Q \in A$ there exists a neighborhood V with $A \in V \subset U$ such that the pair $(V, A \cap V)$ is homeomorphic to the pair $([-1,1]^n, \{0\} \times [-1,1]^k)$. If $n = k$, then $V = A \cap V$. If $k < n$, then we see that

$$\pi_1(V \setminus (A \cap V)) \;\cong\; \pi_1([-1,1]^n \setminus (\{0\} \times [-1,1]^k) \;\underset{\uparrow}{\cong}\; \pi_1(S^{n-k-1}) \;\underset{\uparrow}{\cong}\; \begin{cases} \mathbb{Z}, & \text{if } n-k=2, \\ 0, & \text{otherwise.} \end{cases}$$

topological spaces are homotopy equivalent — Propositions 48.17 and 47.13

By the Convex-to-Ball Proposition 2.20 we know that $[-1,1]^n$ is homeomorphic to $\overline{B}^n$. Thus V has indeed all the desired properties. ⊞



Note that S^1 and S^3 are topological manifolds with empty boundary. It follows from the Topological Manifold Boundary Proposition 75.2 that neither $\mathbb{R}$ nor $\mathbb{R}^3$ admits a chart of type (i) in the sense of the definition on page 511. This implies that if there exists a submanifold, it has to be a chart of type (α). It follows from this discussion, together with Claim 1, that it suffices to prove the following claim.

Claim 2. For every path-connected neighborhood V of $P \in S^3$, that is homeomorphic to $\overline{B}^3$, the fundamental group $\pi_1(V \setminus (W \cap V))$ is non-abelian.

Proof. Let V be a path-connected open neighborhood of $P \in S^3$ that is homeomorphic to $\overline{B}^3$. Since V is a neighborhood of $P = w([0])$ there exists an $n \in \mathbb{N}$ such that $\varphi_n(\overline{B}^3) \subset V$. We pick an $\epsilon > 0$ as in Lemma 124.14.

(1) By the Topological Invariance of Domain Theorem 82.7 we know that $\varphi_n(B^3)$ is an open subset of S^3.
(2) Note that $S^2_{1-\epsilon}$ is a submanifold of B^3 that intersects $\tau([0,1])$ transversally in two points.
(3) Note that $\overline{B}^3_{1-\epsilon}$ is a codimension-zero submanifold of B^3.
(4) It follows from (a) and (c) that $\varphi_n(\overline{B}^3_{1-\epsilon})$ is a codimension-zero submanifold of B^3.
(5) It follows from Proposition 75.3 and (d) that $S^3 \setminus \varphi_n(\overline{B}^3_{1-\epsilon})$ is also a codimension-zero submanifold of B^3. In particular it is a topological manifold in its own right.

We introduce the following notation:

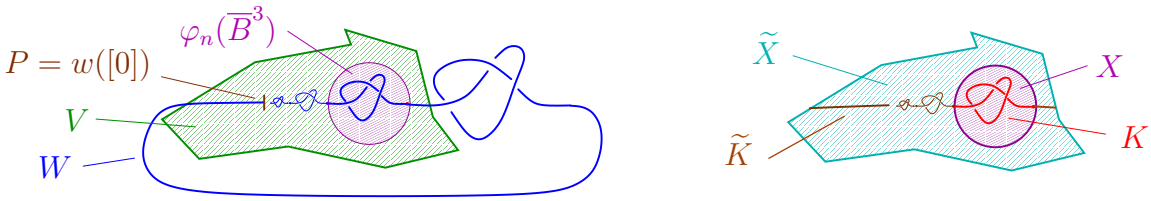
(1) We write $X := \varphi_n(\overline{B}^3_{1-\epsilon})$, $\widetilde{X} := V \setminus \varphi(B^3_{1-\epsilon})$ and $Y := \varphi_n(S^2_{1-\epsilon})$,
(2) We write $K := W \cap X$, $\widetilde{K} := K \cap \widetilde{X}$ and $L := K \cap Y$.

[1513] Since $V \setminus (A \cap V)$ is path-connected we know by the Base Point Independence Corollary 47.11 that the isomorphism type of the fundamental group does not depend on the choice of the base point.

We now see that

follows from the Seifert–van Kampen Theorem for Topological Manifolds 54.4

$$\pi_1(V \setminus (W \cap V)) \overset{\cong}{\downarrow} \pi_1(X \setminus K) *_{\pi_1(Y \setminus L)} \pi_1(\widetilde{X} \setminus \widetilde{K}).$$



We know that $\pi_1(X \setminus K)$ is non-abelian. To conclude that $\pi_1(V \setminus (W \cap V))$ is non-abelian, it suffices by the Amalgamated Product-Monomorphism Proposition 53.24 to show that the two maps $\pi_1(Y \setminus L) \to \pi_1(X \setminus K)$ and $\pi_1(Y \setminus L) \to \pi_1(\widetilde{X} \setminus \widetilde{K})$ are monomorphisms. Since $\pi_1(Y \setminus L) \cong \mathbb{Z}$ it suffices to show that the maps $\mathrm{H}_1(Y \setminus L) \to \mathrm{H}_1(X \setminus K)$ and $\mathrm{H}_1(Y \setminus L) \to \mathrm{H}_1(\widetilde{X} \setminus \widetilde{K})$ are monomorphisms. In the former case that follows easily from Corollary 124.11.

Now consider $\widetilde{X}$. As discussed above, we know that $\widetilde{X}$ is a compact 3-dimensional topological manifold in its own right. Furthermore, by the Generalized Alexander Duality Theorem 124.12 for Balls we know that $\widetilde{X}$ is a homology 3-ball. (In all likelihood $\widetilde{X}$ is in fact homeomorphic to $\overline{B}^3$, in fact we even know that $\pi_1(\widetilde{X}) = 0$, but from what we know we cannot conclude so.)

But now it follows again from Corollary 124.11 that the map $\mathrm{H}_1(Y \setminus L) \to \mathrm{H}_1(\widetilde{X} \setminus \widetilde{K})$ is an isomorphism. ■

Proposition 124.17. The subset $W := w(S^1) \subset S^3$ does not admit a regular neighborhood in the sense of the definition on page 2695.

Sketch of proof. $\pi_1(S^3 \setminus W)$ is infinitely generated. One can see this by exhausting $S^3 \setminus W$ by sets where swallow all but finitely many balls and then do a limit argument. But this contradicts Lemma 123.1. ■

Exercises for Chapter 124.

Exercise 124.1. The Generalized Alexander Duality Theorem 124.2 was formulated for $n \geq 2$. What is the correct statement for $n = 1$?

Exercise 124.2. Let X be a topological homology n-ball. Show that ∂X is a topological homology $(n-1)$-sphere.

Exercise 124.3. We say an embedding $\varphi\colon [0,1] \to \mathbb{R}^3$ is tame if there exists a homeomorphism $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(\varphi(t)) = (t,0,0)$ for all $t \in [0,1]$. In the figure below we show the images of three embeddings $\alpha, \beta, \gamma\colon [0,1] \to \mathbb{R}^3$.

(a) Show that α is tame.
(b) Show that β is tame.
(c) Show that γ is not tame.

(d) Let $\delta\colon [0,1] \to \mathbb{R}^3$ be an embedding that is also a smooth map. Is δ tame?
Remark. Note that a smooth embedding, as defined on page 652, is more than just an embedding that is a smooth map.



Exercise 124.4. In the figure below we show two subsets X and Y of $\mathbb{R}^3$ that are homeomorphic to the compact interval $[0,1]$. Which of the fundamental groups $\pi_1(\mathbb{R}^3 \setminus X)$ and $\pi_1(\mathbb{R}^3 \setminus Y)$ is abelian and which is non-abelian?



125. Applications of cohomology groups

In this chapter we give two applications of cohomology groups, namely we prove the table theorem and we address two special cases of the square peg problem. Neither of these applications can count as a "real life" application, but at least in both cases cohomology, Poincaré Duality and Alexander duality get used to prove statements which initially have absolutely no connection to cohomology theory at all.

125.1. **The Table Theorem ❀.** As a warm-up we start out with the fairly elementary but charming Wobbly Table Theorem. In the vernacular it says that given a wobbly table we can rotate it around its center, by an angle of at most $\frac{\pi}{2}$, such that eventually all four legs touch the ground. Slightly more precisely, it can be formulated as follows:

Theorem 125.1. (Wobbly Table Theorem) Let $f\colon \mathbb{C} = \mathbb{R}^2 \to \mathbb{R}$ be a function, let $x \in \mathbb{C}$ and let $z \in \mathbb{C} \setminus \{0\}$. There exists a $\theta \in [0, \frac{\pi}{2}]$ such that

$$f(x + e^{\mathrm{i}\theta} \cdot z) \;=\; f(x + e^{\mathrm{i}(\theta+\frac{\pi}{2})} \cdot z) \;=\; f(x + e^{\mathrm{i}(\theta+\pi)} \cdot z) \;=\; f(x + e^{\mathrm{i}(\theta+\frac{3\pi}{2})} \cdot z).$$

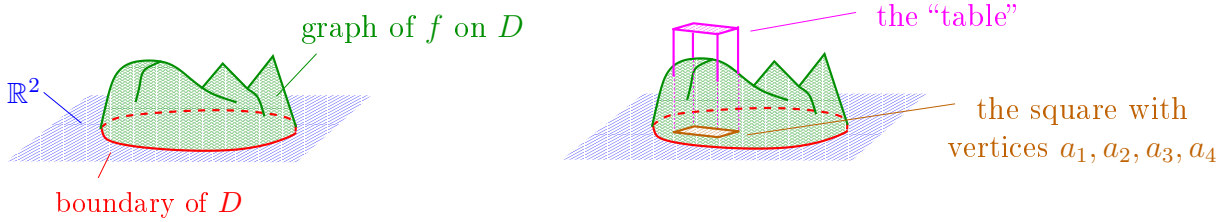
Proof. The theorem is a fairly elementary consequence of the Intermediate Value Theorem from real analysis. Filling in the details is your task in Exercise 125.1. Alternatively the reader can look at [**Loy2017**, p. 289] for a proof. ■

Our goal in this section is to prove a much more fancy theorem. More precisely we will prove the following "Table Theorem" which was first formulated and proved by Roger Fenn [**Fen1970**] in 1970.

Theorem 125.2. (Table Theorem) Let $D \subset \mathbb{R}^2$ be a compact convex non-empty subset and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a map such that $f(x) \geq 0$ for all $x \in D$ and such that $f(x) = 0$ for all $x \notin D$. Given any $d \in \mathbb{R}_{>0}$ there exist four points $a_1, a_2, a_3, a_4 \in \mathbb{R}^2$ with the following properties:

(1) the points form a square of side length d,
(2) the center of the square lies in D,
(3) we have $f(a_1) = f(a_2) = f(a_3) = f(a_4)$.

In other words, the Table Theorem 125.2 says that any given square table can be placed on the ground defined by the graph of f, such that all four legs lie on the ground, such that the table is horizontal and such that the center of the table is located in D.



Remark. In [**Meye1981**] it is shown that in the formulation of the Table Theorem 125.2 one cannot drop the hypothesis that D is convex.

The proof of the Table Theorem 125.2 will require the remainder of this section. Throughout the section we fix a compact convex non-empty subset $D \subset \mathbb{R}^2$ and we fix a function

$f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f(x) \geq 0$ for all $x \in D$ and such that $f(x) = 0$ for all $x \notin D$. Let $d \in \mathbb{R}_{>0}$. We make the following elementary observations and preparations:

(i) Since we can rescale D we can and will without loss of generality assume that $d = 1$.
(ii) If the interior of D is the empty set, then the function f is necessarily the zero function[1514]. Thus the conclusion of the Table Theorem 125.2 holds trivially.
(iii) It follows from (ii) that henceforth we can assume that the interior of D is non-empty. Note that the Convex-to-Ball Proposition 2.20 implies that there exists a homeomorphism of pairs $(D, \partial D) \xrightarrow{\cong} (\overline{B}^2, S^1)$.
(iv) After a translation we can and will assume that the origin $(0, 0)$ is contained in the interior of D.
(v) As always we will make the identification $\mathbb{R}^2 = \mathbb{C}$, in particular we will work a lot with the multiplication "$\cdot\, i$", i.e. with "rotation by $\frac{\pi}{2}$".
(vi) If there exists a square in $\mathbb{R}^2$ of side length $d = 1$ with center in D, such that the f-value at all of the four vertices is zero, then we have evidently found the desired table. Thus we can henceforth assume that for any square of side length $d = 1$ with center in D the f-value of at least one vertex is non-zero.

In the following discussion we will at times take a few liberties with mathematical language, but these liberties will not infringe on the rigor of the argument.

We start out with introducing the following notation, which gives us a convenient way for parameterizing the set of all squares with side length $d = 1$ and whose center lies in D.

Notation.

(1) We consider the map[1515]

$$\begin{array}{rcl} a\colon D \times S^1 & \to & \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \\ & & \text{the four vertices } a_1, a_2, a_3, a_4 \text{ of the unique square} \\ (x, \gamma) & \mapsto & \text{with center } x \text{ such that } a_2 - a_1 = \gamma \text{ and} \\ & & \text{such that } a_1, a_2, a_3, a_4 \text{ are ordered in the} \\ & & \text{counterclockwise direction.} \end{array}$$

This definition is illustrated in the figure below.

(2) We also consider the closely related map

$$\begin{array}{rcl} Q\colon D \times S^1 & \to & \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \\ (x, \gamma) & \mapsto & \text{the points } Q_1, Q_2, Q_3, Q_4 \text{ on the graph corresponding to } a(x, \gamma). \end{array}$$

When $(x, \gamma) \in D \times S^1$ is understood from the context, then we drop it at times from the notation. In other words, often we write a_i instead of $a_i(x, \gamma)$ and we write Q_i instead of $Q_i(x, \gamma)$.

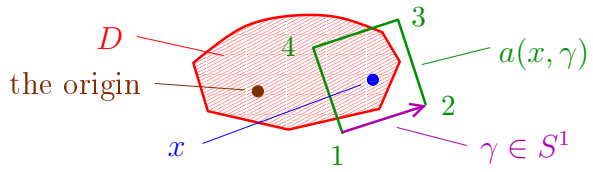
One of the key ideas behind the proof of the Table Theorem 125.2 is to split Condition (3) of the Table Theorem 125.2 into two separate statements. Namely we note that the condition

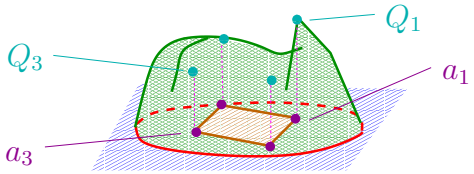
(3) $f(a_1) = f(a_2) = f(a_3) = f(a_4)$,

is satisfied precisely if the following two conditions are satisfied:

[1514] Why is that?

[1515] We leave the task of writing down an explicit formula to the reader. Once the formula is written down it is clear that a is actually continuous.





(3a) the points $\underbrace{(a_1, f(a_1))}_{=Q_1}, \ldots, \underbrace{(a_4, f(a_4))}_{=Q_4}$ lie on a plane,

(3b) the plane from (3a) is horizontal.

The trick now is to encode these two statements in a clever way. We will do so by introducing two maps.

Definition.

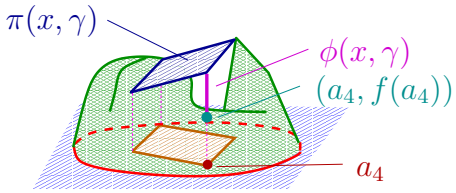
(1) Given $(x,\gamma) \in D^2 \times S^1$ we denote by $\pi(x,\gamma)$ the plane through the three points $Q_1(x,\gamma), Q_2(x,\gamma)$ and $Q_3(x,\gamma)$.

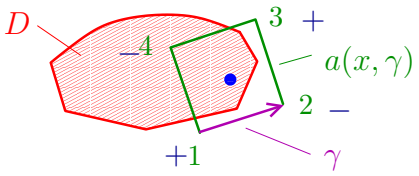
(2) We consider the map[1516]

$$\begin{array}{rcl} \phi\colon D \times S^1 & \to & \mathbb{R} \\ (x,\gamma) & \mapsto & \underbrace{x_3\text{-value of the plane } \pi(x,\gamma) \text{ at } a_4(x,\gamma) \text{ minus } f(a_4(x,\gamma))}_{=f(a_1)+f(a_3)-f(a_2)-f(a_4)} \end{array}$$

The definition of this map is illustrated in the figure below. Note that $\phi(x,\gamma) = 0$ if and only if the corresponding points Q_1, Q_2, Q_3, Q_4 lie on a plane.

(3) We set $A := \{(x,\gamma) \in D \times S^1 \mid \phi(x,\gamma) = 0\}$.





Definition.

(1) Given $(x,\gamma) \in D^2 \times S^1$ we denote by $n(x,\gamma) \in \mathbb{R}^3$ the unique vector that is normal to $\pi(x,\gamma)$, that has unit length and that has positive x_3-coordinate. This definition is illustrated in the figure below on the left.

(2) We consider the map

$$\begin{array}{rcl} \psi\colon D \times S^1 & \to & \mathbb{R}^2 \\ (x,\gamma) & \mapsto & \text{projection of } n(x,\gamma) \text{ onto the } x_1x_2\text{-plane.} \end{array}$$

[1516] Note that the equality

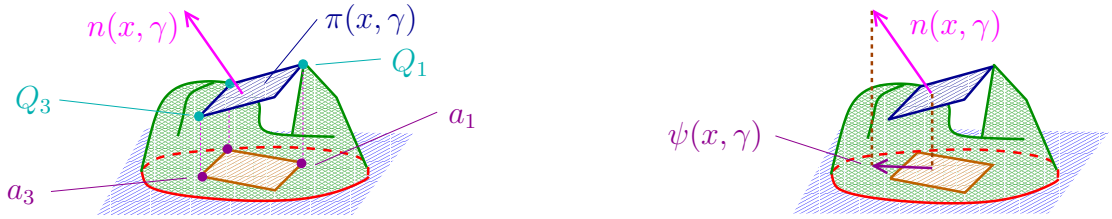
$$a_4 = a_2 + (a_3 - a_2) + \underbrace{(a_4 - a_3)}_{=a_1-a_2} = a_2 + (a_1 - a_2) + (a_3 - a_2) = a_1 + a_3 - a_2,$$

together with the fact that the x_3-coordinate of the plane $\pi(x,\gamma)$ is an affine function on $\mathbb{R}^2$, implies that

$$x_3\text{-coordinate of the plane } \pi(x,\gamma) \text{ at } a_4 \;=\; f(a_1) + f(a_3) - f(a_2).$$

The definition of this map is illustrated on the right hand side of the figure below. Note that $\psi(x,\gamma)=0$ if and only if the plane $\pi(x,\gamma)$ is horizontal.

(3) We set $B := \{(x,\gamma)\in D\times S^1 \,|\, \psi(x,\gamma)=0\}$.



Remark. The Wobbly Table Theorem 125.1 implies that given any $x\in D$ there exist at least four $\gamma\in S^1$ with $(x,\gamma)\in A$. In particular the set A is more complicated than in our subsequent illustrations.

We make the following elementary observation.

Observation 125.3. To prove the Table Theorem 125.2 it suffices to show that $A\cap B\neq\varnothing$.

In the following two lemmas we collect some properties of the two maps ϕ and ψ.

Lemma 125.4. The map $\phi\colon D\times S^1\to\mathbb{R}$ has the following properties:

(1) The map $\phi\colon D\times S^1\to\mathbb{R}$ is continuous.
(2) For any $(x,\gamma)\in D\times S^1$ we have $\phi(x,i\cdot\gamma)=-\phi(x,\gamma)$.
(3) Given $(x,\gamma)\in D\times S^1$ the following statements are equivalent:
  (a) $\phi(x,\gamma)=0$,
  (b) $\phi(x,i\cdot\gamma)=0$,
  (c) (x,γ) lies in A,
  (d) the corresponding points Q_1,Q_2,Q_3,Q_4 lie on a plane.

Proof.

(1) As we pointed out above, we have

$$\phi(x,\gamma) \;=\; f(\underbrace{a_1(x,\gamma)}_{\text{odd vertex}}) + f(\underbrace{a_3(x,\gamma)}_{\text{odd vertex}}) - f(\underbrace{a_2(x,\gamma)}_{\text{even vertex}}) - f(\underbrace{a_4(x,\gamma)}_{\text{even vertex}}).$$

It follows from the continuity of f and the continuity of $(x,\gamma)\mapsto a_i(x,\gamma)$ that ϕ is indeed continuous.

(2) Note that for any $(x,y)\in D\times S^1$ and any $j\in\{1,2,3,4\}$ we have the equality $a_j(x,i\cdot\gamma)=a_{j+1}(x,\gamma)$. In particular the odd/even vertices corresponding to $(x,i\cdot\gamma)$ are precisely the even/odd vertices corresponding to (x,γ). It follows immediately from this observation and the explicit formula for ϕ given in (1) that the equality $\phi(x,i\cdot\gamma)=-\phi(x,\gamma)$ holds.



(3) This statement follows immediately from (2) and the definition of A. ∎

Next we turn to the study of the map $\psi\colon D \times S^1 \to \mathbb{R}^2$. In the formulation of the next lemma we need the following definition.

Definition. We say that $v, w \in \mathbb{R}^2$ are parallel if one of v, w is zero or if there exists an $\lambda \in \mathbb{R}_{>0}$ such that $v = \lambda w$.

Lemma 125.5. The map $\psi\colon D \times S^1 \to \mathbb{R}^2$ has the following properties:

(1) The map $\psi\colon D \times S^1 \to \mathbb{R}$ is continuous.
(2) If $(x, \gamma) \in A$, then $\psi(x, \gamma) = \psi(x, i^k \cdot \gamma)$ for any $k \in \mathbb{Z}$.
(3) Let $x \in \partial D$ and let $\gamma \in S^1$. If $(x, \gamma) \in A$, then the vector $\psi(x, \gamma)$ is not parallel to $-x$.

Sketch of proof.

(1) Let $(x, \gamma) \in D \times S^1$. Basic linear algebra shows that

$$\psi(x,\gamma) = \begin{array}{l}\text{the vector in } \mathbb{R}^3 \text{ that is obtained as follows:}\\ (1)\ \text{perform the cross product } (Q_3(x,\gamma)-Q_2(x,\gamma))\times(Q_1(x,\gamma)-Q_2(x,\gamma))\\ (2)\ \text{normalize the vector from (1) to length one}^{1517}\\ (3)\ \text{project the vector from (2) onto the } xy\text{-plane.}\end{array}$$

It follows immediately from this observation that ψ is continuous.

(2) Let $(x, \gamma) \in A$. As mentioned in Lemma 125.4 (3), this means that the four points $Q_1(x, \gamma)$, $Q_2(x, \gamma)$, $Q_3(x, \gamma)$ and $Q_4(x, \gamma)$ lie on a plane. But this immediately implies that for any $k \in \mathbb{Z}$ we have $\pi(x, \gamma) = \pi(x, i^k \cdot \gamma)$ which then gives us the equality $\psi(x, \gamma) = \psi(x, i^k \cdot \gamma)$.

(3) The final statement is a little more interesting. So let $x \in \partial D$ and let $\gamma \in S^1$. We assume that $(x, \gamma) \in A$. We write $a_i = a_i(x, \gamma)$, $i = 1, 2, 3, 4$.

Claim. There exist two adjacent vertices which lie outside of $\mathring{D}$ and there are two adjacent vertices which have non-zero f-value.

Proof. We make the following observations:
(a) Since $(x, \gamma) \in A$ we have $f(a_1) + f(a_3) - f(a_2) - f(a_4) = 0$.
(b) By choice of f we know that all f-values are non-negative.
(c) By the preliminary remark (vi) on page 2731 we know that not all of the f-values are zero.
(d) Since x lies on ∂D and since D is convex we see that at most two f-values can be non-zero.[1518]

The claim follows easily from the above considerations and the elementary observation that every odd vertex is adjacent to every even vertex. ⊞

To simplify the notation let us now assume that the adjacent vertices a_1 and a_2 do *not* lie in the interior of D. Consequently $f(a_3) > 0$ and $f(a_4) > 0$. A straightforward

[1517] Since $a_3(x, \gamma) - a_2(x, \gamma)$ and $a_1(x, \gamma) - a_2(x, \gamma)$ are linearly independent we see that the two vectors in the cross product are linearly independent, in particular the cross product is indeed non-zero. Thus it makes sense to normalize the vector to length one.

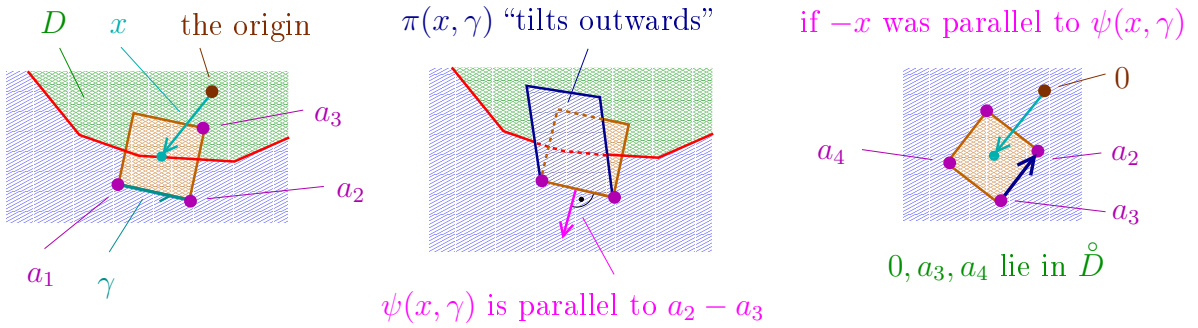
[1518] Any vertex with non-zero f-value lies in the interior of D. So if three vertices have non-zero f, then the triangle spanned by these vertices lies in the interior, in particular x would lie in the interior of D.

calculation shows that

$$\psi(x,\gamma) \;=\; \begin{array}{c}\text{projection of}\\ \text{normalization of}\end{array} \begin{pmatrix} a_3 - a_2 \\ f(a_3)\end{pmatrix} \times \begin{pmatrix} a_1 - a_2 \\ 0\end{pmatrix} \text{ into the } xy\text{-plane} \;=\; \lambda\cdot(a_2 - a_3)$$

for some $\lambda > 0$.

Now suppose that $\psi(x,\gamma)$ is parallel to $-x$. This means, by the above calculation, that $a_3 - a_2 = -\mu \cdot x$ for some $\mu > 0$. It is now straightforward to verify that x lies in the interior of the convex hull of 0, a_3 and a_4. Since 0, a_3 and a_4 lie in the interior of D we see that x also lies in the interior of D. But that is a contradiction to our hypothesis that $x \in \partial D$. ∎



Lemma 125.6. We set

$$T \;:=\; (D \times S^1)/\!\sim \;\text{ where } (x,\gamma) \sim (x, i\cdot\gamma)$$

and we consider the following diagram

$$\begin{array}{ccc} D \times S^1 & \xrightarrow{\quad\psi\quad} & \mathbb{R}^2 \\ {\scriptstyle =:p}\big\downarrow{\scriptstyle (x,\gamma)\mapsto[(x,\gamma)]} & & \big\downarrow \\ T & \xrightarrow[{[(x,\gamma)]\mapsto \frac{1}{4}\sum_{k=0}^{3}\psi(x,i^k\gamma)}]{=:\psi'} & \mathbb{R}^2. \end{array}$$

The following statements hold:

(1) The map $p\colon D \times S^1 \to T$ is a 4-fold covering and the topological space T is homeomorphic to the solid torus $D^2 \times S^1$.
(2) The map ψ' is continuous.
(3) The restriction of the diagram to $A \subset D \times S^1$ is commutative.
(4) If we set
$$A' \;:=\; p(A) \quad \text{and} \quad B' \;:=\; \{[(x,\gamma)] \in T \,|\, \psi'([x,\gamma]) = 0\}$$
then we have $p(A \cap B) = A' \cap B'$.

Proof.

(1) For the experienced reader it should be clear that $p\colon D \times S^1 \to T = (D \times S^1)/\!\sim$ is a 4-fold covering and that $T \cong D \times (S^1/\!\sim) \cong D \times S^1$. On page 2731 we remarked that D is homeomorphic to $\overline{B}^2$, thus we see that T is indeed homeomorphic to the solid torus $\overline{B}^2 \times S^1$.

(2) The map $D^2 \times S^1 \to \mathbb{R}^2$ given by $(x, \gamma) \to \frac{1}{4}\sum_{k=0}^{3} \psi(x, i^k\gamma)$ is continuous. It descends to our map $\psi' \colon T = (D \times S^1)/\sim \to \mathbb{R}^2$. Thus it follows from the Topological-Quotient Proposition 5.15 (1b) that ψ' is continuous.
(3) It follows immediately from Lemma 125.5 (2) that for any $(x, \gamma) \in A$ we have the equality $\psi'([x, \gamma]) = \psi(x, \gamma)$.
(4) The equality $p(A \cap B) = A' \cap B'$ follows easily from (3). ■

From Lemma 125.6 (3) and Observation 125.3 we obtain the following observation.

Observation 125.7. To prove the Table Theorem 125.2 it suffices to show that $A' \cap B' \neq \varnothing$.

After spending so much time on preparations we can now finally apply the machinery from algebraic topology to tackle the problem. We introduce the following convention.

Convention. By Lemma 125.6 (1) we know that T is a solid torus. Given any $x_0 \in T$ we make an identification $\pi_1(T, x_0) = \mathbb{Z}$.[1519]

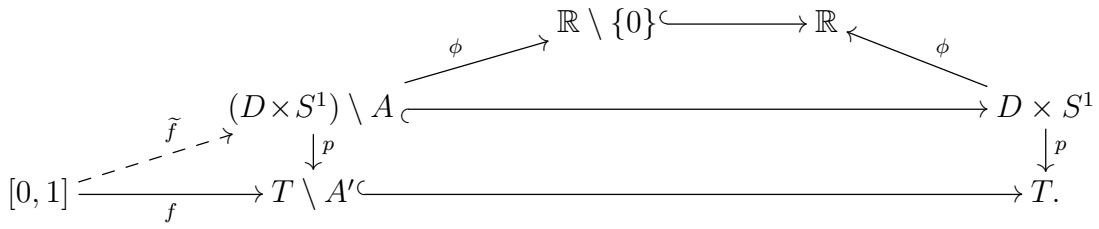
The following lemma turns our data into algebraic-topological information.

Lemma 125.8. We continue with the above notation. Let $x_0 \in T \setminus A'$. The image of $\pi_1(T \setminus A', x_0)$ in $\pi_1(T, x_0) = \mathbb{Z}$ is contained in the subgroup $2 \cdot \mathbb{Z}$.

Proof. Let $x_0 = [(x, \gamma)] \in T \setminus A'$. Note that by definition of A' we have $\phi(x, \gamma) \neq 0$.

(∗) Without loss of generality we can assume that $\phi(x, \gamma) > 0$.

Now let $[f] \in \pi_1(X \setminus A', x_0)$. In other words, $f \colon [0, 1] \to T \setminus A'$ is a loop in $x_0 = [(x, \gamma)]$. We need to show that $[f] \in 2 \cdot \mathbb{Z}$. By Proposition 48.11 we know that there exists a unique lift $\widetilde{f} \colon [0, 1] \to (D \times S^1) \setminus A$ with $\widetilde{f}(0) = (x, \gamma)$. We consider the following diagram:



We make the following observations:

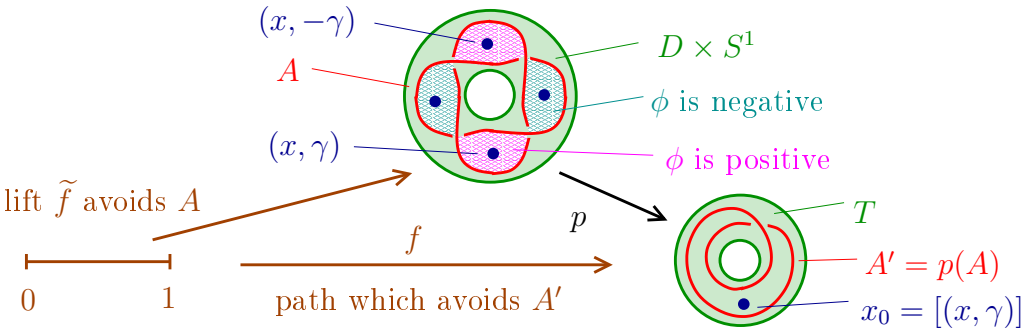
(1) By definition of a lift we know that $\widetilde{f}(1)$ lies in the subset

$$p^{-1}(f(1)) = p^{-1}(f(0)) = p^{-1}(x_0) = \{(x, \gamma), (x, i \cdot \gamma), (x, -\gamma), (x, -i \cdot \gamma)\}.$$

(2) By Lemma 125.4 (2) and (∗) we know that $\phi(x, \gamma) = \phi(x, -\gamma) > 0$ and we know that $\phi(x, i \cdot \gamma) = \phi(x, -i \cdot \gamma) < 0$.
(3) Recall that by definition $A' = p(A)$. Therefore it follows from Lemma 125.4 (3) that $A = p^{-1}(A')$. In particular we have $(D \times S^1) \setminus A = p^{-1}(T \setminus A')$.
(4) It follows from (3) and the above diagram that the image of $\phi \circ \widetilde{f} \colon [0, 1] \to \mathbb{R}$ is contained in $\mathbb{R} \setminus \{0\}$.

[1519] In the following we do not care about the precise isomorphism, in particular we do not care about any sign ambiguity.

(5) Since $[0,1]$ is connected and since $\phi(\widetilde{f}(0)) = \phi(x,\gamma) > 0$ we obtain from (4) that $\phi(\widetilde{f}(1)) > 0$.
(6) It follows from (1), (2) and (5) that $\widetilde{f}(1) = (x,\gamma)$ or $\widetilde{f}(1) = (x,-\gamma)$.
(7) It follows easily[1520] from (6) that the loop $f * f$ lifts to a loop in $(D \times S^1, (x,\gamma))$.
(8) It follows from (7) and Lemma 48.15 (1) that $2 \cdot [f] = [f] \cdot [f] = [f * f]$ actually lies in the subgroup $p_*(\pi_1(D \times S^1, (x,\gamma)))$.
(9) Since $p\colon D \times S^1 \to T$ is a 4-fold connected covering we obtain from Lemma 48.15 (3) that $p_*(\pi_1(D \times S^1, (x,\gamma))) = 4 \cdot \mathbb{Z}$.
(10) The combination of (8) and (9) shows that $[f]$ lies in $2 \cdot \mathbb{Z}$. ■



The next convention is basically self-explanatory.

Convention. Any map denoted by i is given by an inclusion of the corresponding (pairs) of topological spaces.

The following lemma consists mostly of repackaging the result of Lemma 125.8.

Lemma 125.9. We continue with the above notation. Suppose that we are given a compact submanifold with corner $L \subset T$ such that $A' \subset \mathring{L} := L \setminus \partial_0 L$.[1521] The map

$$i_*\colon \mathrm{H}_2(L, L \cap \partial T; \mathbb{Z}_2) \to \mathrm{H}_2(T, \partial T; \mathbb{Z}_2)$$

is an epimorphism.



Proof. Let $L \subset T$ be a compact submanifold with corner such that $A' \subset \mathring{L} := L \setminus \partial_0 L$. If $L = T$ then there is nothing to show. So we assume that $L \neq T$. This implies in particular that $T \setminus \mathring{L} \neq \varnothing$.

We start out with the following claim which is basically a rewriting of Lemma 125.8.

Claim. For any homomorphism $\varphi\colon \mathrm{H}_1(T;\mathbb{Z}) \to \mathbb{Z}_2$ the composition

$$\mathrm{H}_1(T \setminus \mathring{L}; \mathbb{Z}) \xrightarrow{i_*} \mathrm{H}_1(T;\mathbb{Z}) \xrightarrow{\varphi} \mathbb{Z}_2$$

is the zero map.

[1520] How does this work?
[1521] Here we use the notation for submanifolds with corner that we introduced in Lemma 120.1.

Proof. It suffices to show the analogous statement for any path-component Y of $T \setminus \mathring{L}$. So let Y be a path-component of $T \setminus \mathring{L}$. We pick a point $x_0 \in Y$. We consider the following commutative diagram

$$\begin{array}{ccccccccc}
\pi_1(Y, x_0) & \longrightarrow & \pi_1(T \setminus \mathring{L}, x_0) & \longrightarrow & \pi_1(T \setminus A', x_0) & \overset{=0 \text{ by } (4)}{\longrightarrow} & \pi_1(T, x_0) & \overset{\eta}{\twoheadrightarrow} & \mathbb{Z}_2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \cong & & \\
\mathrm{H}_1(Y; \mathbb{Z}) & \longrightarrow & \mathrm{H}_1(T \setminus \mathring{L}; \mathbb{Z}) & \longrightarrow & \mathrm{H}_1(T \setminus A'; \mathbb{Z}) & \longrightarrow & \mathrm{H}_1(T; \mathbb{Z}). & &
\end{array}$$

We make the following clarifications and observations:

(1) The map $\eta\colon \pi_1(T, x_0) \to \mathbb{Z}_2$ is the unique epimorphism.
(2) All the vertical maps are given by the corresponding Hurewicz homomorphism. The diagram commutes by the naturality of the Hurewicz homomorphisms.
(3) The Hurewicz homomorphism to the left is an epimorphism by the Hurewicz Theorem 84.5 since Y is path-connected. The Hurewicz homomorphism to the right is an isomorphism by the Hurewicz Theorem 84.5 since T is path-connected and since $\pi_1(T, x_0) \cong \mathbb{Z}$ is abelian.
(4) It follows from Lemma 125.8 that the map $\pi_1(T \setminus A', x_0) \to \pi_1(T, x_0) \xrightarrow{\eta} \mathbb{Z}_2$ is the trivial map.
(5) The claim follows immediately from combining (1) to (4). ⊞

Now we turn to the actual proof of the desired statement. We write $W := T \setminus \mathring{L}$. We consider the following diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}(\mathrm{H}_1(W; \mathbb{Z}), \mathbb{Z}_2) & \overset{\text{ev}}{\underset{\cong}{\leftarrow}} & \mathrm{H}^1(W; \mathbb{Z}_2) & \overset{\cap[W]}{\underset{\cong}{\longrightarrow}} & \mathrm{H}_2(W, \partial W; \mathbb{Z}_2) & \overset{i_*}{\underset{\cong}{\longrightarrow}} & \mathrm{H}_2(T, \partial T \cup L; \mathbb{Z}_2) \\
\uparrow{\scriptstyle i^*} & & \uparrow{\scriptstyle i^*} & & & & \uparrow{\scriptstyle i_*} \\
\mathrm{Hom}(\mathrm{H}_1(T; \mathbb{Z}), \mathbb{Z}_2) & \overset{\text{ev}}{\underset{\cong}{\leftarrow}} & \mathrm{H}^1(T; \mathbb{Z}_2) & & \overset{\cap[T]}{\underset{\cong}{\longrightarrow}} & & \mathrm{H}_2(T, \partial T; \mathbb{Z}_2) \\
 & & & \nearrow{\scriptstyle i_*} & & & \uparrow{\scriptstyle i_*} \\
 & & & & \mathrm{H}_2(L, L \cap \partial T; \mathbb{Z}_2) & \overset{i_*}{\underset{\cong}{\longrightarrow}} & \mathrm{H}_2(\partial T \cup L, \partial T; \mathbb{Z}_2).
\end{array}$$

We make the following comments and clarifications:

(1) The vertical maps are induced by the various inclusions.
(2) The square to the left commutes by the naturality of the evaluation homomorphism. Furthermore, the horizontal maps to the left are isomorphisms by the First Cohomology Proposition 110.17.
(3) By Lemma 120.1 we know that W is a compact 3-dimensional smooth manifold. We equip it with the orientation coming from T.
(4) The horizontal maps "$\cap[W]$" and "$\cap[T]$" in the center are given by capping with the fundamental classes $[W]$ and $[T]$. These maps are isomorphisms by the Poincaré Duality Theorem 119.1.
(5) The horizontal map to the top right is an isomorphism by the Excision Theorem 120.2 (b).

(6) The vertical maps to the right are a piece of the long exact sequence of the triple $(T, \partial T \cup L, \partial T)$, see the Topological LES-Proposition 74.13.
(7) The rectangle to the right commutes by Lemma 120.6, see in particular the example (a) on page 2653.
(8) By the above claim we know that the vertical map to the left is zero. Since all horizontal maps are isomorphisms and since the upper part of the diagram commutes we see that the vertical map to the top right is also the zero map.
(9) It follows from (6) and (8) that the vertical map to the bottom right is an epimorphism.
(10) The bottom triangle commutes since all maps are inclusion induced maps.
(11) The horizontal map at the very bottom is an isomorphism by the Excision Theorem 120.2 (c).
(12) Finally we note that it follows immediately from (9), (10) and (11) that the diagonal map $i_*\colon \mathrm{H}_2(L, L \cap \partial T; \mathbb{Z}_2) \to \mathrm{H}_2(T, \partial T; \mathbb{Z}_2)$ is an epimorphism. Fortunately that is precisely what we wanted to show. ■

Lemma 125.10. Let X be a topological space and let $f, g\colon X \to S^1$ be maps. If for every $x \in X$ we have $f(x) \neq -g(x) \in S^1$, then f and g are homotopic.

Sketch of proof. We consider the map

$$\begin{array}{ccccc} X & \xrightarrow{f \times g} & S^1 \times S^1 & \to & S^1 \\ & & (z, w) & \mapsto & z \cdot w. \end{array}$$

It follows from our hypothesis that this map takes values in $S^1 \setminus \{-1\} \cong (-\pi, \pi)$. Since $(-\pi, \pi)$ admits a deformation retraction to 0 one can now easily show that f and g are homotopic. We are fully confident that the reader did not fail to fill in all details when solving Exercise 14.3. ■

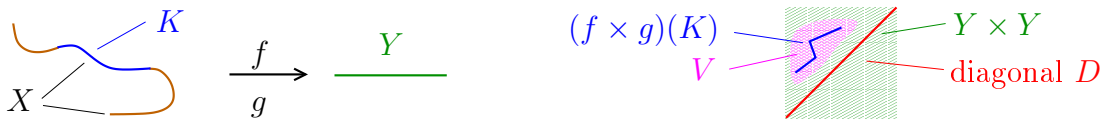
Lemma 125.11. Let $f, g\colon X \to Y$ be two maps between topological spaces. Suppose that there exists a compact subset $K \subset X$ such that $f(x) \neq g(x)$ for all $x \in K$. If Y is Hausdorff, then there exists an open neighborhood U of K such that $f(x) \neq g(x)$ for all $x \in U$.

Proof. We consider the map

$$\begin{array}{rccc} f \times g\colon X & \to & Y \times Y \\ x & \mapsto & (f(x), g(x)). \end{array}$$

By the Topological-Product Proposition 5.1 (1b) we know that $f \times g$ is continuous. Since K is compact we obtain from the Compact Image Lemma 2.13 (1) that $(f \times g)(K)$ is a compact subset of $Y \times Y$. Let $D := \{(y, y) \mid y \in Y\}$ be the "diagonal" in $Y \times Y$. Since Y is Hausdorff we obtain from Exercise 5.23 that D is a closed subset of $Y \times Y$. Now, using our hypothesis that Y is Hausdorff we obtain from Exercise 1.31 (which is a variation on the Compact Hausdorff Spaces-Normal Lemma 1.16) an open neighborhood V of $(f \times g)(K)$ with $V \cap D = \varnothing$. We set $U := (f \times g)^{-1}(V)$. Since $f \times g$ is continuous we see that U is an open neighborhood of K. Evidently for any $x \in U$ we have $(f \times g)(x) \notin D$, i.e. we have $f(x) \neq g(x)$. ■

Now we head towards the denouement.



Proof of the Table Theorem 125.2. First recall that by Observation 125.7 it suffices to show that $A' \cap B' \neq \varnothing$. We do a proof by contradiction. So we make the following assumption:

(∗) We assume that $A' \cap B' = \varnothing$.

In other words, we assume that ψ' never vanishes on A'. Throughout the proof the following two maps will play a major role:

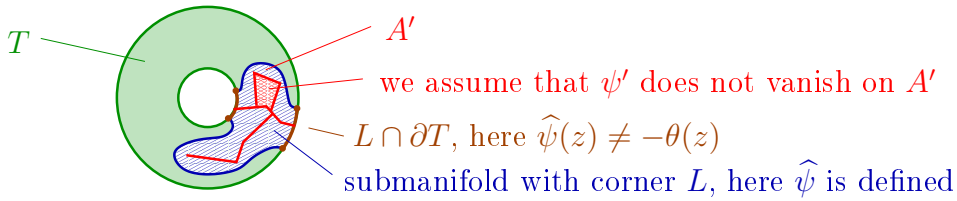
$$\widehat{\psi}\colon \{z \in T \,|\, \psi'(z) \neq 0\} \to S^1, \quad z \mapsto \tfrac{1}{\|\psi'(z)\|}\cdot\psi'(z) \qquad \text{and} \qquad \theta\colon \partial T = \partial D \times (S^1/\!\sim) \to \partial D \to S^1, \quad z \mapsto \tfrac{1}{\|z\|}\cdot z.$$

Note that A' has the following two properties:

(α) ψ' is non-zero on A' (in particular $\widehat{\psi}$ is defined on A'), and
(β) By Lemma 125.5 (3) we know that for any $z \in A' \cap \partial T$ we have $\widehat{\psi}(z) \neq -\theta(z)$.

The following claim says that we can enlarge A' to a submanifold with corner which still has properties (α) and (β). More precisely, we have the following claim.

Claim. The smooth manifold T admits a compact submanifold with corner L with the following properties:
(i) $A' \subset \mathring{L}$,
(ii) ψ' is non-zero on L (in particular $\widehat{\psi}$ is defined on L), and
(iii) for any $z \in L \cap \partial T$ we have $\widehat{\psi}(z) \neq -\theta(z)$.

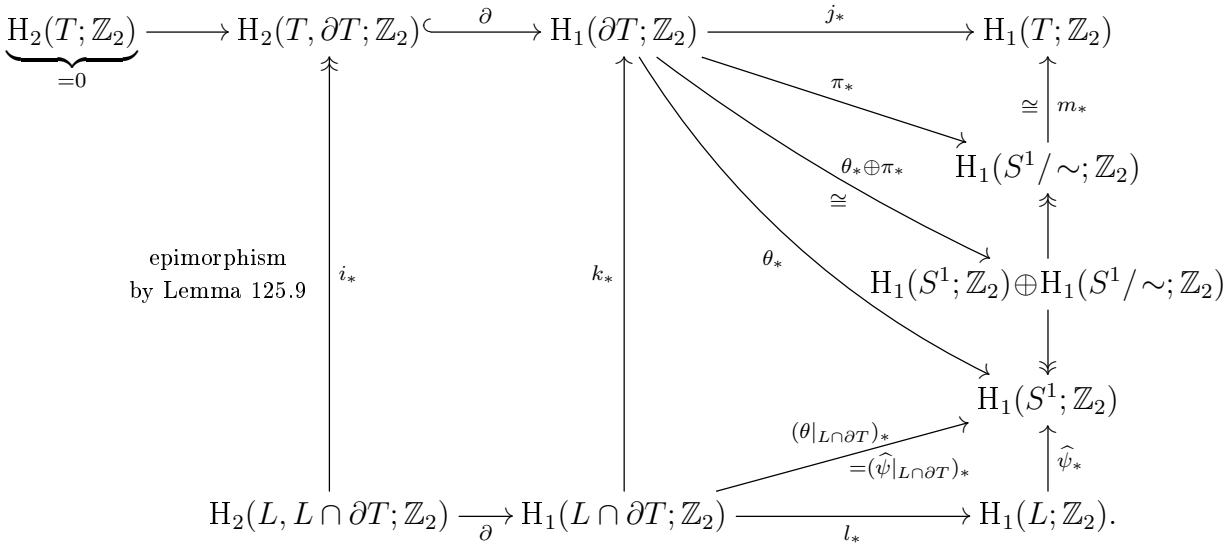


Proof. Recall that T is a solid torus, in particular T is compact. Since A is a closed subset of $D \times S^1$ we obtain from the Compact-Closed Lemma 1.21 (1) and the Compact Image Lemma 2.13 that $A' = p(A)$ is compact.

(1) It follows from (α), the fact that A' is compact and Lemma 125.11 that there exists an open neighborhood U of A' such that ψ' vanishes nowhere on U.
(2) It follows from (β), Exercise 1.22 (which is an immediate consequence of the Compact-Closed Lemma 1.21 (1)) and Lemma 125.11 that there exists an open neighborhood V of $A' \cap \partial T$ such that $\widehat{\psi}(z) \neq -\theta(z)$ for all $z \in V$.

It follows from Lemma **??** (2) that there exists a submanifold L with corner of T such that $A' \subset \mathring{L} \subset L \subset U$ and such that $L \cap \partial T \subset V$. In particular we see that ψ' is non-zero on L and that for any $z \in L \cap \partial T$ we have $\widehat{\psi}(z) \neq -\theta(z)$. ⊞

We consider the following diagram:



We make the following comments, clarifications and observations:

(1) The horizontal maps at the top respectively the bottom are extracts of the long exact sequence in homology of the pair $(T,\partial T)$ respectively of the pair $(L,L\cap\partial T)$.

(2) The maps i,j,k,l indicate the obvious inclusion maps.

(3) We denote by $m\colon S^1/\!\sim \xrightarrow{\cong} \{0\}\times S^1/\!\sim \to D\times(S^1/\!\sim) = T$ the obvious map. Since D is convex we see that m is a homotopy equivalence, hence we obtain from the Homotopy Equivalence-H_*-Corollary 73.9 that m_* is an isomorphism.

(4) The map $\pi\colon \partial T = \partial D\times(S^1/\!\sim)\to S^1/\!\sim$ is the natural projection. It is elementary to show that the maps $m\circ\pi$ and j from ∂T to T are homotopic. Thus we obtain from the Homotopic Maps-and-Homology Proposition 73.6 that they induce the same map on homology. In other words, the triangle to the top right commutes.

(5) It follows from the naturality of the connecting homomorphism, see the Topological LES-Proposition 74.13, that the rectangle to the left commutes. All the remaining triangles to the right commute basically by definition or by the functoriality of homology groups.

(6) By the discussion on page 159 the map $\theta\colon \partial D\to S^1$ is a homeomorphism. It follows easily from this observation, together with the fact that $\partial T = \partial D\times(S^1/\!\sim)$, that the map $\theta_*\oplus\pi_*\colon \mathrm{H}_1(\partial T;\mathbb{Z}_2)\to \mathrm{H}_1(S^1;\mathbb{Z}_2)\oplus\mathrm{H}_1(S^1/\!\sim;\mathbb{Z}_2)$ is an isomorphism.

(7) It follows from (ii) and Lemma 125.10 that the restrictions of $\widehat{\psi}$ and θ to $L\cap\partial T$ are homotopic. We deduce from the Homotopic Maps-and-Homology Proposition 73.6 that the induced maps on homology are the same. Put differently, the two maps indicated at the upward-pointing diagonal arrow are actually the same.

(8) By Lemma 125.9 the left vertical map is an epimorphism. Since $\mathrm{H}_2(T,\partial T;\mathbb{Z}_2)\neq 0$ there exists a $\xi\in\mathrm{H}_1(L,L\cap\partial T;\mathbb{Z}_2)$ with $i_*(\xi)\neq 0$.

(9) It follows from the long exact sequence on top that $\partial i_*(\xi)\neq 0$ and that $j_*\partial i_*(\xi)=0$.

(10) It follows from (3), (4) and (9) that $\pi_*\partial i_*(\xi)=0$.

(11) By (6) we know that the map $\theta_*\oplus\pi_*$ is an isomorphism. Therefore we obtain from (8) and (10) that $\theta_*\partial i_*(\xi)\neq 0\in\mathrm{H}_1(S^1;\mathbb{Z}_2)$.

(12) Since the diagram commutes we obtain from (11) that $l_*\partial\xi \neq 0$ in $\mathrm{H}_1(L;\mathbb{Z}_2)$. But that is absurd, since the horizontal sequence at the bottom is exact. ■

125.2. **The square-peg problem ❀.** In this section we will give another amusing, but unfortunately even less practical application of cohomology theory. To introduce the topic we need the following definition.

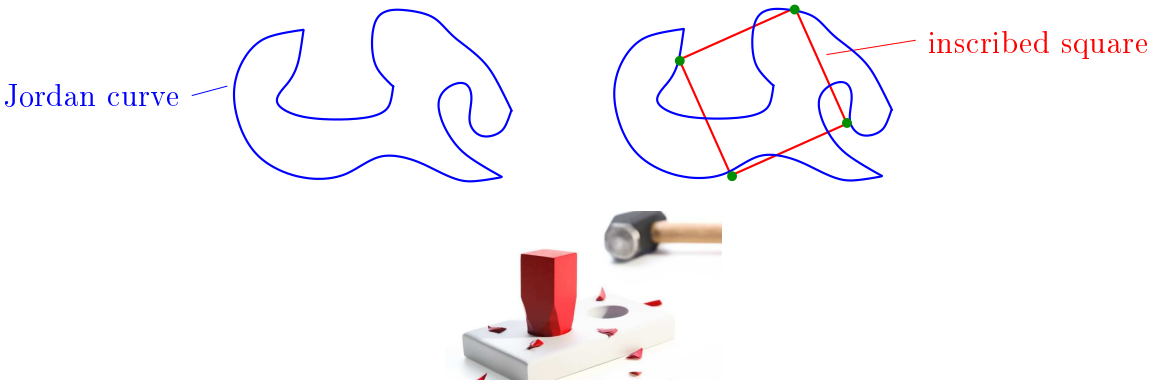
Definition.

(1) The image of an injective map $S^1 \to \mathbb{R}^2$ is calleda Jordan curve.[1522]
(2) Given a Jordan curve C we say that a rectangle R is inscribed in C if all vertices of R lie on C.

The following question was asked by Otto Toeplitz [**Toep1911**] in 1911.[1523]

Question 125.12. Does every Jordan curve admit an inscribed square?

This question is known as the "inscribed square problem" or the "square-peg problem". Lev Schnirelmann [**Schni1944**][1524] gave a positive answer if the curve is smooth, in fact he gave a proof if the curve is C^2, i.e. if the map $S^1 \to \mathbb{R}^2$ is twice continuously differentiable. But the general case is still open.[1525] A detailed discussion of the question and its history is given in [**Matsc2014**].



In this section we will provide two partial answers to Question 125.12.

(1) We will use the Table Theorem 125.2 to deal with the case that the Jordan curve is the boundary of a compact convex obtuse subset of $\mathbb{R}^2$.
(2) We will show that we can inscribe a rectangle into every Jordan curve.

Let us start with the first result.

Proposition 125.13. Let J be a Jordan curve which is the boundary of a compact convex subset D. If D is obtuse, then J admits an inscribed square.

[1522]Note that this definition is slightly different from the definition we gave on page 795 where we defined a Jordan curve to be an injective map $S^1 \to \overline{\mathbb{C}} = S^2$.

[1523]Otto Toeplitz (1881-1940) was a Jewish German mathematician who was forced to emigrate in 1939. He is mainly known for his contributions to analysis.

[1524]We had already encountered Lev Schnirelmann in Theorem 40.1.

[1525]Recently the Fields medalist Terence Tao [**Tao2017**] wrote a long paper in which he outlined an approach to the square-peg problem. See also the following video:

https://www.youtube.com/watch?v=RGxnWgy5i80

Remark. This proposition also admits proofs that do not rely on topology, see for example **[Zi1921, Chr1950]**.

Proof. Let $D \subset \mathbb{R}^2$ be a compact convex obtuse subset. We need to show that there exist four points $a_1, a_2, a_3, a_4 \in \partial D$ which form a square. After a translation we can assume that the origin 0 is contained in the interior of D. Given $x \in D \setminus \{0\}$ we set

$$\rho(x) \;:=\; \sup\big\{\|rx\| \,\big|\, r \in \mathbb{R}_{>0} \text{ and } r \cdot x \in D\big\} \;\in \mathbb{R}_{>0}.$$

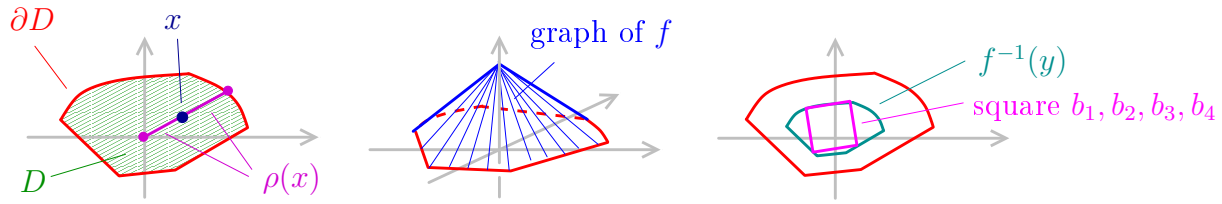
It follows almost immediately from Lemma 2.21[1526] that the map

$$\begin{array}{rcl} f\colon \mathbb{R}^2 & \to & [0,1] \\ x & \mapsto & \begin{cases} 1 - \|x\| \cdot \frac{1}{\rho(x)}, & \text{if } x \in D \setminus \{0\}, \\ 1, & \text{if } x = 0, \\ 0, & \text{if } x \notin D \end{cases} \end{array}$$

is continuous. Since D is obtuse we obtain from Lemma **??** that there exists a $d > 0$ such that D is d-non trivial. By the Table Theorem 125.2 there exist four points $b_1, b_2, b_3, b_4 \in \mathbb{R}^2$ with the following properties:

(1) the points form a square of side length d,
(2) the center of the square lies in D,
(3) we have $f(b_1) = f(b_2) = f(b_3) = f(b_4)$.

Since D is d-non trivial we see that at least one vertex lies inside of $\mathring{D}$. But since f is non-zero on $\mathring{D}$ we see that the common f-value, let's call it y, lies in the open interval $(0,1)$. It follows immediately from the definition of f that the four points b_1, b_2, b_3, b_4 lie on the subset $(1-y) \cdot \partial D$. In other words, if we multiply these four points by $\frac{1}{1-y}$, then we obtain the desired four points a_1, a_2, a_3, a_4 on ∂D. ∎



We conclude this chapter with the second contribution to the square-peg problem.

Proposition 125.14. Every Jordan curve admits an inscribed rectangle.

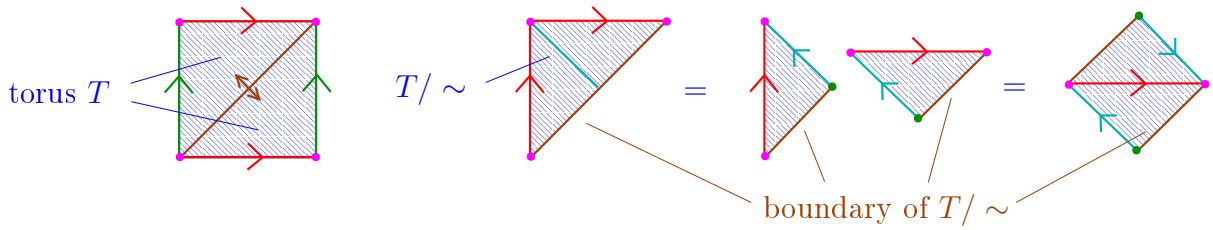
In **[Meye1982]** the subsequent ingenious proof of Proposition 125.14 is attributed to Herbert Vaughan. In preparation for the proof we need the following elementary lemma.

Lemma 125.15. Let $T = S^1 \times S^1$ be the torus. Let $\sim$ be the equivalence relation on T generated by $(z,w) \sim (w,z)$. The quotient space $T/\!\sim$ is homeomorphic to the Möbius band and

$$\partial(T/\!\sim) \;=\; \{[(z,z)] \,|\, z \in S^1\}.$$

Proof. The homeomorphism is illustrated in the figure below. ∎

[1526] Well-trained eyes will not fail to spot that implicitly we also make use of the Interior-Closure Lemmas 1.9, the Pasting Proposition 2.6 and the Compact-Closed Lemma 1.21 to show that f is indeed continuous.



Now we are already ready to prove Proposition 125.14.

Proof of Proposition 125.14. Let $\gamma\colon S^1 \to \mathbb{R}^2$ be an injective map. We have to show that $\gamma(S^1)$ admits an inscribed rectangle. We consider the map

$$\begin{array}{rcl} \Phi\colon T := S^1 \times S^1 & \to & \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \\ (z,w) & \mapsto & \Big(\frac{\gamma(z)+\gamma(w)}{2}, |\gamma(z)-\gamma(w)|\Big). \end{array}$$

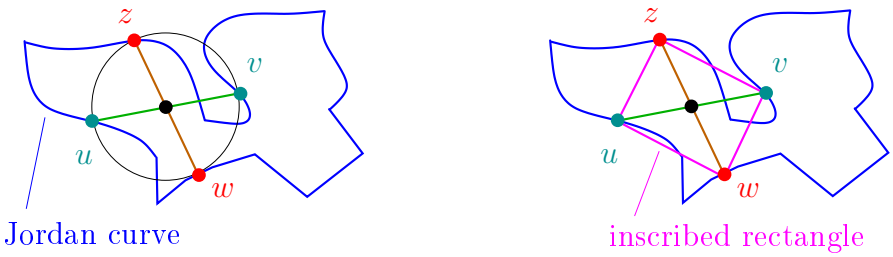
Since for all $z, w \in S^1$ we have $\Phi(z,w) = \Phi(w,z)$ we see that Φ descends to a map

$$\begin{array}{rcl} \Psi\colon T/\sim & \to & \mathbb{R}^3 \end{array}$$

where $\sim$ denotes of course the equivalence relation generated by $(z,w) \sim (w,z)$.

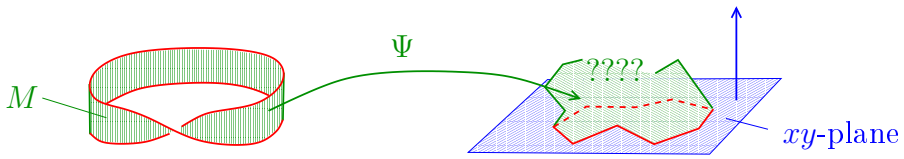
Claim. The map $\Psi\colon T/\sim \to \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ is not injective.

We first show that the claim implies the proposition. So now that we know that Ψ is not injective we know that there exist distinct pairs of points $\{z,w\}$ and $\{u,v\}$ with $\Phi(z,w) = \Phi(u,v)$. We consider the quadrilateral given by the points z,u,v,w. It follows from $\Phi(z,w) = \Phi(u,v)$ that the diagonals bisect each other and that the diagonals have the same length. But such a quadrilateral is necessarily a rectangle.[1527] We refer to the figure below for an illustration.



Proof. Let us assume that $\Psi\colon T/\sim \to H := \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ is injective. By Lemma 125.15 the quotient $M := T/\sim$ is homeomorphic to the Möbius band and the boundary ∂M is given by $\{[(z,z)] \,|\, z \in S^1\}$. We see immediately from the definition of Ψ that for a point $[(z,w)] \in M = T/\sim$ we have $\Psi([(z,w)]) \in \partial H = \mathbb{R} \times \mathbb{R} \times \{0\}$ if and only if $[(z,w)] \in \partial M$. We denote by M_1 and M_2 two copies of M and we denote by $\mathrm{D}M = M_1 \cup_{\partial M_1 = \partial M_2} M_2$ the double of M. It follows from the proof of Lemma **??** (1) that $\mathrm{D}M$ is in fact homeomorphic to the Klein bottle, in particular $\mathrm{D}M$ is a closed, non-orientable 2-dimensional smooth manifold.

[1527] Hmm, why is that?



We denote by $\rho\colon \mathbb{R}^3 \to \mathbb{R}^3$ the reflection in the xy-plane, i.e. ρ is the map that is given by $\rho(x,y,z) = (x,y,-z)$. We consider the map

$$\begin{array}{rcll} \mathrm{D}M = M_1 \cup_{\partial M_1 = \partial M_2} M_2 & \to & \mathbb{R}^3 & \\ P & \mapsto & \Psi(P), & \text{if } P \in M_1, \\ P & \mapsto & \rho(\Psi(P)), & \text{if } P \in M_2. \end{array}$$

that is given by

It follows immediately from the above discussion that this map is well-defined and injective. Furthermore it follows from the Topological-Quotient Proposition 5.15 (1b) that the map is in fact continuous. But now we have constructed an injective map from the Klein bottle to $\mathbb{R}^3$. Since the Klein bottle is compact we obtain from the Compact-Hausdorff Proposition 2.17 that such a map is in fact an embedding. But we had just proved in the Topological Codimension-One Orientability Proposition 124.5 that such an embedding cannot exist. ■

Remark.

(1) The above proof of Proposition 125.14 is also explained in the following video:

`https://www.youtube.com/watch?v=AmgkSdhK4K8`

(2) There are many variations on Question 125.12. For example, given $r \neq 1$ one can ask whether every Jordan curve admits an inscribed rectangle with aspect ratio r. Even for smooth curves this had been an open problem for a long time. The first real progress since the work by Schnirelmann [**Schni1944**] happened in 2018 when Cole Hugelmeyer [**Hug2018**] showed that a deep result in knot theory, due to Josh Batson [**Bats2014**], implies that every smooth Jordan curve admits an inscribed rectangle of aspect ratio $\sqrt{3}$. The key idea of [**Hug2018**] is rather similar to the proof of Proposition 125.14. Even more impressively, an affirmative answer for smooth curves and any aspect ratio was given in 2020 by Josh Greene and Andrew Lobb [**GrL2020**].

Exercises for Chapter 125.

Exercise 125.1. Let $f\colon \mathbb{C} = \mathbb{R}^2 \to \mathbb{R}$ be a function, let $x \in \mathbb{C}$ and let $z \in \mathbb{C} \setminus \{0\}$. Show that there exists a $\theta \in [0, \frac{\pi}{2}]$ such that

$$f(x + e^{\mathrm{i}\theta} \cdot z) \;=\; f(x + e^{\mathrm{i}(\theta+\frac{\pi}{2})} \cdot z) \;=\; f(x + e^{\mathrm{i}(\theta+\pi)} \cdot z) \;=\; f(x + e^{\mathrm{i}(\theta+\frac{3\pi}{2})} \cdot z).$$

Exercise 125.2. Let X be a non-empty compact subset of $\mathbb{R}^2$ of diameter d. Does there exist a $P \in \mathbb{R}^2$ such that $X \subset \overline{B}^2_{\frac{d}{2}}(P)$?

Exercise 125.3. We consider the solid torus $T = \overline{B}^2 \times S^1$. Let $f\colon T \to \mathbb{R}$ be a map such that for any $(x, \gamma) \in T$ we have $f(x, -\gamma) = -f(x, \gamma)$. We write $A := \{(x, \gamma) \in T \mid f(x, \gamma) = 0\}$.

(a) Show that the inclusion induced map $\mathrm{H}_1(T \setminus A; \mathbb{Z}_2) \to \mathrm{H}_1(T; \mathbb{Z}_2)$ is the zero map.

(b) Give an example of a map $f\colon T \to \mathbb{R}$ as above such that the inclusion induced map $\mathrm{H}_1(T \setminus A; \mathbb{Z}) \to \mathrm{H}_1(T; \mathbb{Z})$ is *not* the zero map.

126. The cup product and algebraic intersection numbers

In Chapter 43 we introduced the algebraic intersection number $X \cdot Y$ of two oriented submanifolds X and Y of complementary dimensions of a given oriented smooth manifold M. The main technical result that we proved back then is the Intersection Number Homotopy Invariance Theorem 43.5 which says that the algebraic intersection numbers is unchanged under smooth isotopies of $X \sqcup Y$.

In this chapter we first recall the definition of the algebraic intersection number. In the Intersection Number-via-Cup Product Theorem 126.5 we relate this algebraic intersection number to the cup product of suitable cohomology classes of M. The fact that algebraic intersection numbers correspond to the cup product can be used to compute cup products in a geometric and often entertaining way. Conversely this result can also be used to prove deep statements about intersections of submanifolds. For example we can give a new and very short proof of the Intersection Number Homotopy Invariance Theorem.

This chapter is written in a way such that it can be read independently of Chapter 43. In particular we will motivate some questions and results, even though we might have solved some of the issues already in Chapter 43.

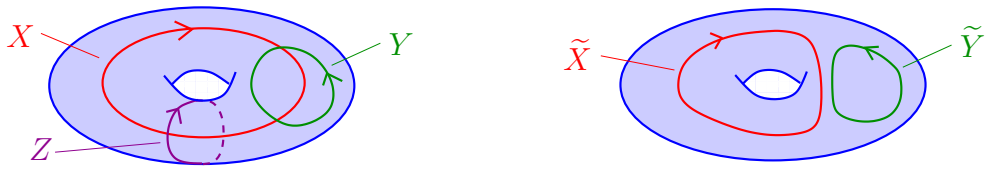
126.1. **Algebraic intersection numbers of submanifolds.** Before we (re-) introduce the algebraic intersection number of submanifolds we want to recall a definition from page 817 and we want to give a motivating question.

Definition. Let M be a smooth manifold.

(1) We say that two oriented proper submanifolds Z_0 and Z_1 of M are properly smoothly isotopic if there exists a proper smooth isotopy between Z_0 and Z_1, i.e. if there exists a proper smooth isotopy $F\colon Z_0 \times [0,1] \to M$ with $F_0 = \mathrm{id}_{Z_0}$ and such that $F_1\colon Z_0 \to Z_1$ is an orientation-preserving diffeomorphism.

(2) Let X and Y be two oriented proper submanifolds of M. We say that X and Y can be made disjoint if there exist oriented proper disjoint submanifolds $\widetilde{X}$ and $\widetilde{Y}$ of M such that X and $\widetilde{X}$ and also Y and $\widetilde{Y}$ are properly smoothly isotopic.

Example. We consider the three submanifolds X, Y and Z of the torus that are illustrated in the figure below on the left. As is shown in the same figure on the right, one can make X and Y disjoint.



It does not look like we can make the submanifolds X and Z disjoint. But how can we prove it? We will record this question for later.

Question 126.1. Can we make the submanifolds X and Z, that are illustrated in the figure above, disjoint?

Our approach to addressing Question 126.1 is to introduce the "algebraic intersection number". Before we can introduce this notion we need to recall the following definition from page 968.

Definition. Let M be an m-dimensional smooth manifold. Let X and Y be proper submanifolds of M.

(1) Let $P \in X \cap Y$. We say P is a transverse intersection point if $P \notin \partial M$ and if X and Y intersect transversally in P, i.e. if the equality $\mathrm{T}_P X + \mathrm{T}_P Y = \mathrm{T}_P M$ holds.
(2) We say that X and Y intersect transversally if each intersection point is a transverse intersection point.

Lemma 126.2. Let M be an m-dimensional smooth manifold, let $X \subset M$ be a proper submanifold of dimension k and let $Y \subset M$ be a proper submanifold of the complementary dimension $m - k$. If X and Y intersect transversally and if X and Y are both compact, then the intersection $X \cap Y$ consists of finitely many points.

Proof. By the Submanifold Transversal Intersection Theorem 42.1 we know that $X \cap Y$ is a 0-dimensional submanifold of M. It follows easily from the definition of a 0-dimensional submanifold that $X \cap Y$ is a discrete subset of M. Since X and Y are compact we know by the Compact-Union-Intersection Lemma 1.20 (2) that $X \cap Y$ is also compact. It now follows from the Compact-Discrete Lemma 1.22 that $X \cap Y$ is finite. ■

Next we recall the following definition from page 989.

Definition. Let $m \in \mathbb{N}$ and let $k \in \{1, \ldots, m-1\}$. Furthermore let M be an m-dimensional oriented smooth manifold, let $X \subset M$ be a compact oriented proper submanifold of dimension k and let $Y \subset M$ be a compact oriented proper submanifold of the complementary dimension $m - k$. Suppose that X and Y intersect transversally.

(1) Let P be an intersection point of X and Y. We set $\operatorname{sign}(P) = 1$, if

$$(\text{positive basis of } \mathrm{T}_P X, \text{positive basis of } \mathrm{T}_P Y) \;=\; \text{positive basis of } \mathrm{T}_P M.$$

Otherwise we set $\operatorname{sign}(P) = -1$.[1528]
(2) We define[1529]

$$\text{geometric intersection number of } X \text{ and } Y \;:=\; \#(X \cap Y)$$

and

$$\text{algebraic intersection number of } X \text{ and } Y \;:=\; X \cdot Y \;:=\; \sum_{P \in X \cap Y} \operatorname{sign}(P) \in \mathbb{Z}.$$

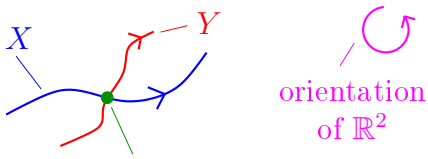
If X and Y do not intersect, then the algebraic intersection number of X and Y is of course zero.

Remark. It follows from Lemma 43.1 that the above definition of the algebraic intersection number of two submanifolds agrees with the definition on page 989.

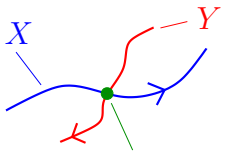
Examples.

(1) In the figure below we show pieces of oriented submanifolds of the oriented smooth manifold $\mathbb{R}^2$. On the left the sign of the intersection point is positive, on the right it is negative.

[1528] It is a straightforward exercise in linear algebra to show that the definition of $\operatorname{sign}(P)$ does not depend on the choice of the positive bases of $\mathrm{T}_P X$ and $\mathrm{T}_P Y$.
[1529] It follows from Lemma 126.2 that the geometric intersection number is finite and that the algebraic intersection number is given by a finite sum.



sign of the intersection point of the submanifolds X and Y in $\mathbb{R}^2$ equals $+1$



sign of the intersection point of the submanifolds X and Y in $\mathbb{R}^2$ equals -1

(2) We consider the surface of genus 2 with the orientation where the normal vector "sticks outward". Furthermore we consider the oriented submanifolds A, B, C, D and E that are illustrated in the figure below. For the submanifolds A and B we have

$$\begin{aligned} \text{geometric intersection number of } A \text{ and } B &= 1, \\ \text{the sign of the intersection point is negative, hence} & \\ \text{algebraic intersection number of } A \text{ and } B &= -1. \end{aligned}$$

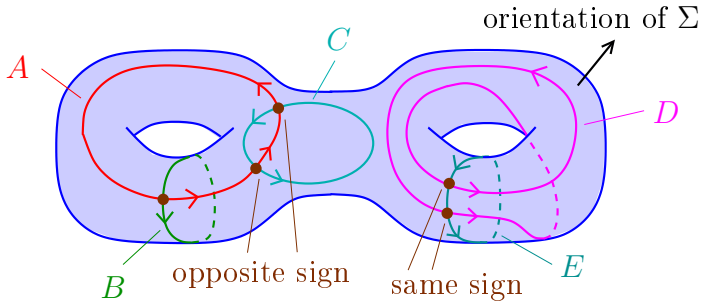
For the submanifolds A and C we have

$$\begin{aligned} \text{geometric intersection number of } A \text{ and } C &= 2, \\ \text{the intersection points have opposite signs, therefore} & \\ \text{algebraic intersection number of } A \text{ and } C &= 0. \end{aligned}$$

Finally for the submanifolds D and E we have

$$\begin{aligned} \text{geometric intersection number of } D \text{ and } E &= 2, \\ \text{here both intersection points have negative sign, thus} & \\ \text{algebraic intersection number of } D \text{ and } E &= -2. \end{aligned}$$

It is particularly instructive to consider the submanifolds A and C. They intersect geometrically in two points, but the signs are opposite. Note that these two intersection points also "vanish" if we "pull A and C apart" by a smooth isotopy.



(3) Now suppose that M and N are oriented smooth manifolds of dimensions m respectively n. We saw in Propositions 19.31 (1) and 20.6 that $M \times N$ is naturally[1530] an $(m+n)$-dimensional smooth manifold (possibly with corner). It is straightforward to show that for each $P \in M \setminus \partial M$ and $Q \in N \setminus \partial N$ the natural map[1531]

$$\mathrm{T}_P M \oplus \mathrm{T}_Q N \;\to\; \mathrm{T}_{(P,Q)}(M \times N)$$

[1530] What does he mean by "naturally"?

[1531] What is the natural map?

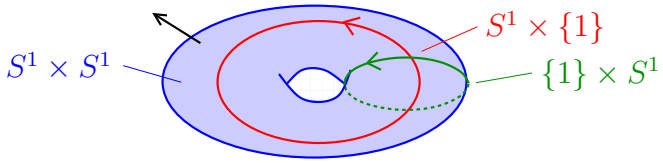
is an isomorphism. If $\{v_1, \ldots, v_m\}$ is a positive basis for $\mathrm{T}_P M$ and $\{w_1, \ldots, w_n\}$ is a positive basis for $\mathrm{T}_Q N$, then we endow $\mathrm{T}_{(P,Q)}(M \times N)$ with the orientation that makes the above isomorphism orientation-preserving.[1532]

Given $P \in M$ and $Q \in N$ it follows easily from the definitions that

$$\text{algebraic intersection number of } M \times \{Q\} \text{ and } \{P\} \times N \;=\; 1.$$

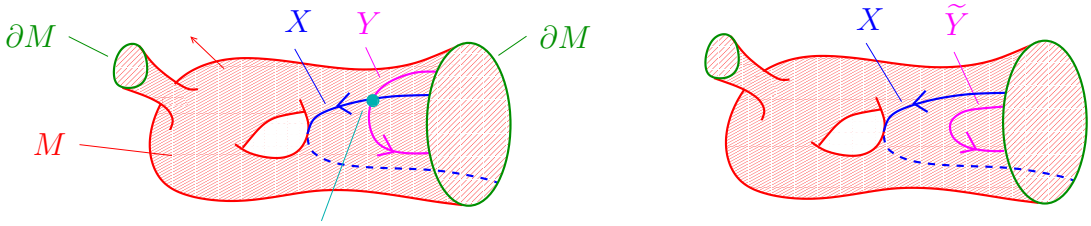
In the figure below we illustrate the statement that in the torus $S^1 \times S^1$ we have

$$\text{algebraic intersection number of } S^1 \times \{*\} \text{ and } \{*\} \times S^1 \;=\; 1.$$



(4) In Section 126.4 we will study the algebraic intersection number of complex submanifolds of a given complex manifold.

(5) The above example (2) suggests that if two submanifolds have non-zero algebraic intersection number, then they cannot be made disjoint. As we will see in the Intersection Number Homotopy Invariance Theorem 126.6, this hunch is basically correct. But if we deal with submanifolds with boundary one needs to think carefully about the correct setting. For example in the figure below we show on the left a smooth manifold M and we show two proper submanifolds X and Y such that the boundaries ∂X and ∂Y lie on the same component of ∂M. The algebraic intersection number is $+1$, but as we see in the figure below on the right, these two submanifolds can be made disjoint. Therefore in the following discussion we will always assume that the boundary components of X and Y lie in different boundary components of M.



the algebraic intersection number of X and Y equals $+1$

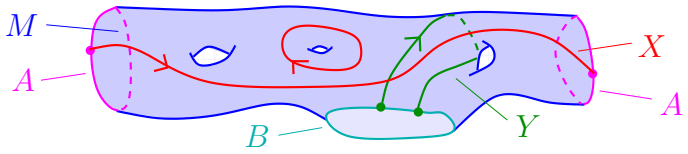
Before we continue, and in light of the last example, we introduce the following definition.

Definition. Let M be an oriented smooth manifold.

(1) A boundary decomposition for M is a decomposition $\partial M = A \cup B$ where A and B are unions of components of ∂M and $A \cap B = \varnothing$.

[1532] One can easily verify that these orientations on the tangent spaces of $M \times N$ define an orientation on $M \times N$. In fact it follows fairly easily from Proposition 113.11 that this definition of the product orientation agrees with the definition we gave on page 2502.

(2) Let X and Y be two submanifolds of M. We say X and Y are complementary if the following five conditions are satisfied:
   (a) X and Y are proper submanifolds,
   (b) there exists a boundary decomposition $\partial M = A \cup B$ with $\partial X \subset A$ and $\partial Y \subset B$,
   (c) both X and Y are oriented,
   (d) we have $\dim(X) + \dim(Y) = \dim(M)$ and we have $\dim(X), \dim(Y) \geq 1$,
   (e) the submanifolds X and Y intersect transversally.

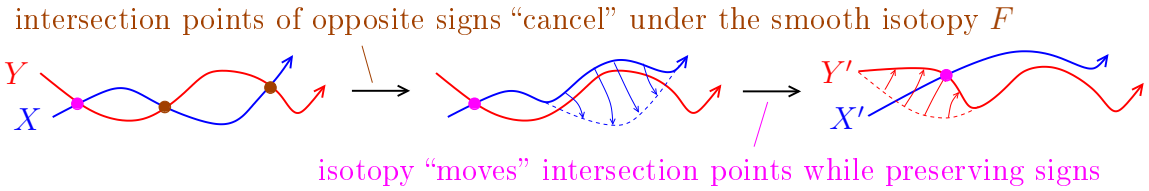


We would be able to answer Question 126.1 if we could prove the following statement.

Statement 126.3. Let M be a compact oriented smooth manifold and let $\{X, Y\}$ and $\{X', Y'\}$ be two pairs of complementary submanifolds of M. Then the following holds:

$$\begin{array}{c} X, X' \text{ are properly smoothly isotopic and} \\ Y, Y' \text{ are properly smoothly isotopic} \end{array} \quad \Longrightarrow \quad X \cdot Y \;=\; X' \cdot Y'.$$

But how can we prove such a statement? In the figure below we sketch an idea: We break the smooth isotopies from X to X' and from Y to Y' into "smaller isotopies" such that each smaller isotopy either does not change the type of intersection points, or two intersection points of opposite sign get created or erased. Even though the idea is fairly convincing, it is very hard to turn this idea into a proper proof.



In the Intersection Number Homotopy Invariance Theorem 43.5 we gave a proof of the above statement, but if one goes through all the results that go into the proof of that theorem one sees that this approach is certainly not elementary.

We will provide a proof of the statement later in this section, once we have explained the relationship between algebraic intersection numbers and the cup product.

We conclude this introduction to algebraic intersection numbers with the following elementary lemma, which is basically the same as Lemma 43.2.

Lemma 126.4. Let M be a compact oriented smooth manifold. If X and Y are complementary submanifolds of M, then

$$Y \cdot X \;=\; (-1)^{\dim(X)\cdot\dim(Y)} \cdot X \cdot Y.$$

Sketch of proof. The statement follows easily from the definitions and the observation that the matrix that represents the linear homeomorphism that swaps the first k vectors with the last $(m-k)$ vectors of a basis of an m-dimensional vector space is represented by

the matrix $\begin{pmatrix} 0 & \mathrm{id}_k \\ \mathrm{id}_{m-k} & 0 \end{pmatrix}$. One easily verifies that the determinant of this matrix equals $(-1)^{k(m-k)}$. ■

126.2. **The cup product and algebraic intersection numbers.** We recall the following notation.

Notation.

(1) Given a compact oriented n-dimensional smooth manifold W we denote as usual by $[W] \in \mathrm{H}_n(W, \partial W)$ the fundamental class that we introduced on page 2199 or alternatively on page 2351. If W is a proper submanifold of some topological manifold M, then sometimes we denote by $[W] \in \mathrm{H}_n(M, \partial M)$ the image of $[W]$ under the inclusion induced map $\mathrm{H}_n(W, \partial W) \to \mathrm{H}_n(M, \partial M)$.

(2) Let M be a compact oriented n-dimensional smooth manifold and suppose that we are given a boundary decomposition $\partial M = A \cup B$. By the Poincaré Duality Theorem 119.1 we know that for each $k \in \mathbb{N}_0$ the map

$$\mathrm{H}^k(M, A) \xrightarrow{\cap [M]} \mathrm{H}_{n-k}(M, B)$$

is an isomorphism. We denote by

$$\mathrm{PD}_M \colon \mathrm{H}_{n-k}(M, B) \longrightarrow \mathrm{H}^k(M, A)$$

its inverse.[1533] If M is understood from the context then often we just write PD instead of PD_M.

The following theorem relates the cup product on a smooth manifold to algebraic intersection numbers. Or conversely, it expresses the algebraic intersection number of submanifolds as a cup product.

Theorem 126.5. (Intersection Number-via-Cup Product Theorem) Let M be a compact oriented m-dimensional smooth manifold together with a boundary decomposition $\partial M = A \cup B$. Let X and Y be a complementary pair of submanifolds of M with $\partial X \subset A$ and $\partial Y \subset B$. We write $k = \dim(X)$ and we denote by $i \colon X \to M$ and $j \colon Y \to M$ the obvious inclusion maps. Furthermore we denote by $[X] \in \mathrm{H}_k(X, \partial X)$ and $[Y] \in \mathrm{H}_{m-k}(Y, \partial Y)$ the fundamental classes of X and Y. Then

$$\underbrace{X \cdot Y}_{\substack{\text{defined} \\ \text{on page 2748}}} = \Big\langle \overbrace{\underbrace{\mathrm{PD}_M(\underbrace{i_*([X])}_{\in \mathrm{H}_k(M,A)})}_{\in \mathrm{H}^{m-k}(M,B)} \cup \underbrace{\mathrm{PD}_M(\underbrace{j_*([Y])}_{\in \mathrm{H}_{m-k}(M,B)})}_{\in \mathrm{H}^k(M,A)}}^{\in \mathrm{H}^m(M,\partial M)}, \overbrace{[M]}^{\in \mathrm{H}_m(M,\partial M)} \Big\rangle \in \mathbb{Z}.$$

In particular with the abbreviation $[X] = i_*([X])$ and $[Y] := j_*([Y])$ we have the slightly more readable equality

$$X \cdot Y = \langle \mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y]), [M] \rangle.$$

Remark.

(1) As a reality check, note that a short moment's thought shows that the (anti-) commutativity of the algebraic intersection number of Lemma 126.4 does indeed correspond

[1533] If M is closed this statement is also contained in the Poincaré Duality Theorem 103.5.

precisely to the (anti-) commutativity of the cup product from the Cup Product-Commutativity Proposition 115.3.

(2) The following citation from [**Kir1989**, p. 21] encapsulates the roles of algebraic intersection numbers and cup products:

> "Think with [algebraic] intersection [numbers], prove with cup products."

More precisely, algebraic intersection numbers are very intuitive and often they can be used to guess a statement. But when it comes to proving statements, e.g. why does the intersection number only depend on the smooth isotopy type of a submanifold?, it is usually much easier to work with cup products.

(3) Alternative proofs of the Intersection Number-via-Cup Product Theorem 126.5 are given in [**GrH1981**, Proposition 31.7], [**Bre1993**, Theorem VI.11.9] and [**DaK2001**, Theorem 10.32]. Furthermore a quick outline of a proof is given in [**GoS1999**, Proposition 1.2.5].

(4) One could use the right-hand side of the Intersection Number-via-Cup Product Theorem 126.5 as the definition of the algebraic intersection number. We will discuss this idea in Exercise 126.3.

(5) If we drop the condition that M, X and Y are orientable, then we can still define an algebraic intersection number $X \cdot_{\mathbb{F}_2} Y$ and we can formulate Poincaré Duality with $\mathbb{F}_2$-coefficients. This allows us to formulate a variation on the Intersection Number-via-Cup Product Theorem 126.5. To save the trees we will not spell out the details at this point, instead we refer to the more general Theorem 127.15 for details.

Before we move on to the somewhat lengthy proof of the Intersection Number-via-Cup Product Theorem 126.5 we want to convince ourselves that proving the theorem is worth the effort.

One application of the Intersection Number-via-Cup Product Theorem 126.5 is that now we can provide a proof of a strengthened version of Statement 126.3. More precisely, we have the following theorem, which is basically the same as the earlier Intersection Number Homotopy Invariance Theorem .

Theorem 126.6. (Intersection Number Homotopy Invariance Theorem) Let M be a compact oriented smooth manifold.

(1) Let $\{X, Y\}$ and $\{X', Y'\}$ be pairs of complementary submanifolds of M. If X, X' are properly smoothly isotopic and if Y, Y' are also properly smoothly isotopic, then $X \cdot Y = X' \cdot Y'$.

(2) If X and Y are two complementary submanifolds of M that can be made disjoint, then $X \cdot Y = 0$.

Example. Now we can answer Question 126.1. It follows from the Intersection Number Homotopy Invariance Theorem 126.6 that the submanifolds X and Z, that are illustrated in the figure on page 2747, cannot be made disjoint since their algebraic intersection number is non-zero.

Proof. Let M be a compact oriented smooth manifold. We start out with the following claim.

Claim. Let Z and Z' be oriented proper s-dimensional submanifolds of M. If Z and Z' are properly smoothly isotopic, then $[Z] = [Z'] \in \mathrm{H}_s(M, \partial M)$.

Proof. We denote by $i\colon Z \to M$ and $i'\colon Z' \to M$ the inclusion maps. Since Z and Z' are properly smoothly isotopic there exists in particular an orientation-preserving diffeomorphism $f\colon Z \to Z'$ such that the maps $i, i' \circ f\colon (Z, \partial Z) \to (M, \partial M)$ are homotopic. We obtain that

$$[Z] \;=\; i_*([Z]) \;\underset{\text{Proposition 74.15}}{\uparrow}{=}\; (i' \circ f)_*([Z]) = \; i'_*(f_*([Z])) \;\underset{\substack{\text{by Proposition 107.2 since } f \text{ is an} \\ \text{orientation-preserving homeomorphism}}}{\uparrow}{=}\; i'_*([Z']) \;=\; [Z']$$

⊞

Now let $\{X, Y\}$ and $\{X', Y'\}$ be two pairs of complementary submanifolds of M. We assume that the submanifolds X, X' are properly smoothly isotopic and that the smooth manifolds Y, Y' are smoothly isotopic. We obtain that

$$X \cdot Y \;\underset{\text{Theorem 126.5}}{\uparrow}{=}\; \langle \mathrm{PD}_M([X]), \mathrm{PD}_M([Y]), [M] \rangle \;\underset{\text{by the claim}}{\uparrow}{=}\; \langle \mathrm{PD}_M([X']), \mathrm{PD}_M([Y']), [M] \rangle \;\underset{\text{Theorem 126.5}}{\uparrow}{=}\; X' \cdot Y'.$$

This concludes the proof of the first statement. The second statement is an immediate consequence of the first statement. ■

Now we will use the Intersection Number-via-Cup Product Theorem 126.5 "the other way", i.e. we will apply the theorem to use the algebraic intersection number to draw conclusions on (co-) homology classes and to determine cup products using our geometric understanding of submanifolds.

Corollary 126.7. Let M be a compact oriented connected m-dimensional smooth manifold together with a boundary decomposition $\partial M = A \cup B$. Let X be a k-dimensional proper submanifold of M with $\partial X \subset A$. If there exists a proper submanifold Y with $\partial Y \subset B$ such that X and Y form a complementary pair and if $X \cdot Y = \pm 1$, then we can write $\mathrm{H}_k(M, A) = \mathbb{Z} \cdot [X] \oplus K$ for some abelian group K. In particular we see that $[X]$ is a primitive element of $\mathrm{H}_k(M, A)$.

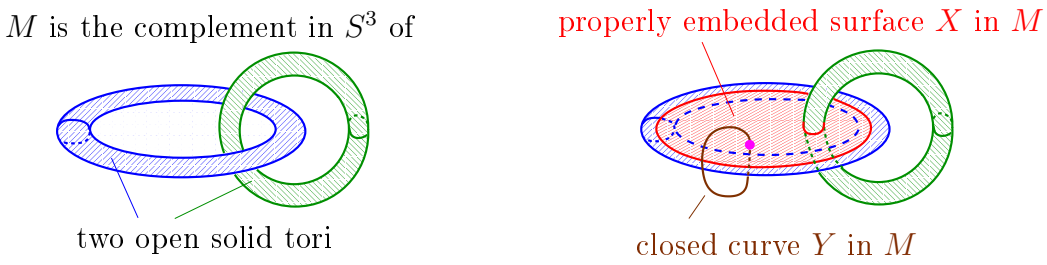
Proof. We consider the homomorphism

$$\begin{aligned} \Theta\colon \mathrm{H}_k(M, A) &\to \mathbb{Z} \\ \varphi &\mapsto \langle \mathrm{PD}_M(\varphi) \cup \mathrm{PD}_M([Y]), [M] \rangle. \end{aligned}$$

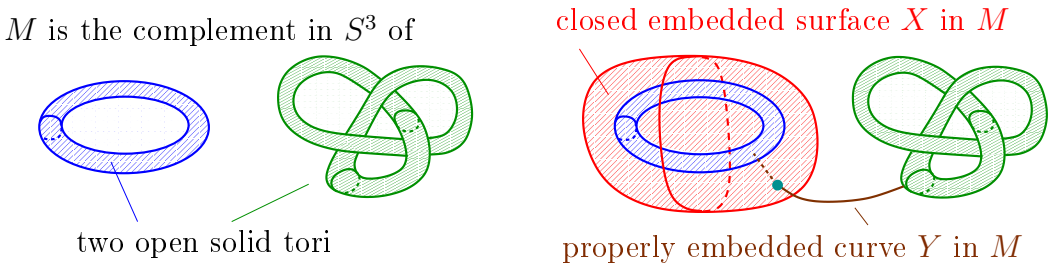
It follows from our hypothesis and the Intersection Number-via-Cup Product Theorem 126.5 that $\Theta([X]) = \pm 1$. The corollary now follows from the elementary discussion on page 1718. ■

Examples.

(1) Let M be the complement in S^3 of the two open 3-dimensional tori shown in the figure below on the left. On the right of that figure we show an annulus X that is properly embedded in M. On the right we also see a closed 1-dimensional submanifold Y of M. The two submanifolds X and Y are clearly transverse and they intersect in one point. So with whatever orientation, we have $X \cdot Y = \pm 1$. By Corollary 126.7 this implies in particular that $[X] \in \mathrm{H}_2(M, \partial M)$ is a primitive element.



(2) This time let M be the complement in S^3 of the two open 3-dimensional tori shown in the figure below on the left. Almost the same argument as in (1) shows that the surface X defines a primitive element in $\mathrm{H}_2(M)$.



In the next example we will show how to use the Intersection Number-via-Cup Product Theorem 126.5 to compute cup products.

Example. We consider the torus $T = S^1 \times S^1$ together with the submanifolds $X = S^1 \times \{*\}$ and $Y = \{*\} \times S^1$ which we endow with the usual orientations. By now we have many ways[1534] for showing that $\mathrm{PD}([X])$ and $\mathrm{PD}([Y])$ form a basis for $\mathrm{H}^1(T)$. Thus if we want to understand the cup product on the torus, it suffices to understand the cup products involving these two cohomology classes. We calculate that

$$-\langle \mathrm{PD}([Y]) \cup \mathrm{PD}([X]), [S^1 \times S^1] \rangle \underset{\uparrow}{=} \langle \mathrm{PD}([X]) \cup \mathrm{PD}([Y]), [S^1 \times S^1] \rangle \underset{\uparrow}{=} X \cdot Y \underset{\uparrow}{=} +1.$$

Cup Product-Commutativity Proposition 114.8 — Theorem 126.5 — see the figure on page 2756

Next we want to calculate the "square" $\mathrm{PD}([X]) \cup \mathrm{PD}([X])$. This requires a little trick since evidently X is not transverse to itself. To deal with this issue we consider a "parallel copy" of X, namely we consider $X' = S^1 \times \{z\}$ with $z \neq 1$. We calculate that

$$\langle \mathrm{PD}([X]) \cup \mathrm{PD}([X]), [S^1 \times S^1] \rangle \underset{\uparrow}{=} \langle \mathrm{PD}([X]) \cup \mathrm{PD}([X']), [S^1 \times S^1] \rangle \underset{\uparrow}{=} X \cdot X' \underset{\uparrow}{=} 0.$$

by the claim in the proof of the Intersection Number Homotopy Invariance Theorem 126.6 — by Theorem 126.5, since X and X' are disjoint, in particular transverse

and with the same argument we show that $\langle \mathrm{PD}([Y]) \cup \mathrm{PD}([Y]), [S^1 \times S^1] \rangle = 0$. We will now convince ourselves that these calculations match our previous calculations. To do so we denote by p_1 respectively p_2 the projection $S^1 \times S^1 \to S^1$ onto the first respectively the second factor and we denote by $\theta := \theta_{\mathbb{Z}} \in \mathrm{C}^1; \mathbb{Z})$ the singular 1-cocycle that we introduced

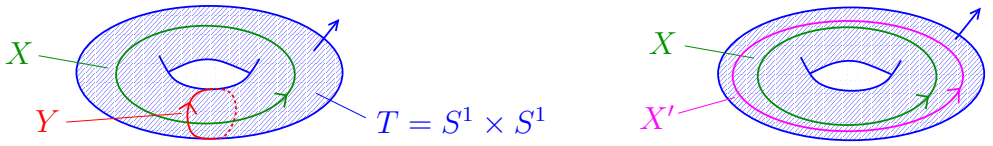
[1534] So, how many ways do you know?

on page 2404. Now we see that

$$\begin{pmatrix} \mathrm{PD}([X])\cup\mathrm{PD}([X]) & \mathrm{PD}([X])\cup\mathrm{PD}([Y]) \\ \mathrm{PD}([Y])\cup\mathrm{PD}([X]) & \mathrm{PD}([Y])\cup\mathrm{PD}([Y]) \end{pmatrix} = \begin{pmatrix} [p_2^*\theta]\cup[p_2^*\theta] & -[p_2^*\theta]\cup[p_1^*\theta] \\ -[p_1^*\theta]\cup[p_2^*\theta] & [p_1^*\theta]\cup[p_1^*\theta] \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

on page 2555 we already showed that $[p_1^*\theta]\cap[T] = [1\times S^1] = [Y]$ and $[p_2^*\theta]\cap[T] = -[S^1\times 1] = -[X]$

see page 2555



Since this was so much fun, let us consider one more example.

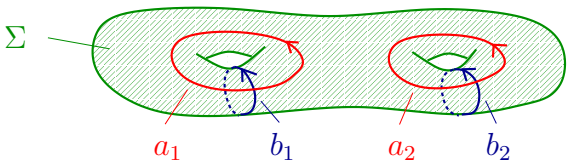
Example. Let Σ be the surface of genus two. We consider the following bilinear pairing:[1535]

$$\begin{aligned} Q\colon \mathrm{H}_1(\Sigma)\times\mathrm{H}_1(\Sigma) &\to \mathbb{Z} \\ (\sigma,\tau) &\mapsto \langle \mathrm{PD}_\Sigma(\sigma)\cup\mathrm{PD}_\Sigma(\tau),[\Sigma]\rangle. \end{aligned}$$

Recall that by Proposition 80.10 we know that $\mathrm{H}_1(\Sigma)\cong\mathbb{Z}^4$. Next we consider the closed oriented curves a_1, b_1, a_2, b_2 that are shown in the figure below. It follows easily from the Intersection Number-via-Cup Product Theorem 126.5 that[1536]

$$\begin{pmatrix} Q([a_1],[a_1]) & Q([a_1],[b_1]) & Q([a_1],[a_2]) & Q([a_1],[b_2]) \\ Q([b_1],[a_1]) & Q([b_1],[b_1]) & Q([b_1],[a_2]) & Q([b_1],[b_2]) \\ Q([a_2],[a_1]) & Q([a_2],[b_1]) & Q([a_2],[a_2]) & Q([a_2],[b_2]) \\ Q([b_2],[a_1]) & Q([b_2],[b_1]) & Q([b_2],[a_2]) & Q([b_2],[b_2]) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Note that the determinant of the matrix to the right is one. Since Q is bilinear we obtain from Lemma 109.8 a new proof that $[a_1]$, $[b_1]$, $[a_2]$, $[b_2]$ is a basis for $H_1(\Sigma)$.[1537] By the Poincaré Duality Theorem 119.1 together with the Intersection Number-via-Cup Product Theorem 126.5 we have now also computed the cup product $\mathrm{H}^1(\Sigma;\mathbb{Z})\times\mathrm{H}^1(\Sigma;\mathbb{Z})\to\mathrm{H}^2(\Sigma;\mathbb{Z})$.



We will do one more calculation of the cup product right after the proof of the Intersection Number-via-Cup Product Theorem 126.5.

We conclude this lengthy discussion of the merits of the Intersection Number-via-Cup Product Theorem 126.5 with the following definition that generalizes the concept introduced on page

[1535] Later on, on page 2800 we will refer to this map as the *intersection form of* Σ and we will introduce the intersection form of any closed oriented even-dimensional topological manifold.

[1536] For the diagonal terms we use the "little trick" from the previous example, namely we use a "parallel copy" of the curves a_1, b_1, a_2, b_2.

[1537] We gave a different argument on page 1781.

Definition. Let $m \in \mathbb{N}$ and let M be a compact oriented m-dimensional smooth manifold together with a boundary decomposition $\partial M = A \cup B$. Furthermore let $k \in \{1, \dots, m-1\}$, let X be a compact oriented proper k-dimensional submanifold of M with $\partial X \subset A$ and let Y be a compact oriented proper $(m-k)$-dimensional submanifold of M with $\partial Y \subset B$. We denote by $i\colon X \to M$ and $j\colon Y \to M$ the inclusion maps. Then we refer to

$$\Big\langle \overbrace{\mathrm{PD}_M(\underbrace{i_*([X])}_{\in \mathrm{H}_k(M,A)}) \cup \mathrm{PD}_M(\underbrace{j_*([Y])}_{\in \mathrm{H}_{m-k}(M,B)})}^{\in \mathrm{H}^m(M,\partial M)}, \overbrace{[M]}^{\in \mathrm{H}_m(M,\partial M)} \Big\rangle \in \mathbb{Z}$$

$\in \mathrm{H}^{m-k}(M,B)$ $\quad$ $\in \mathrm{H}^k(M,A)$

as the algebraic intersection number of X and Y. By the Intersection Number-via-Cup Product Theorem 126.5 this definition is consistent with the earlier definition on page 2748.

126.3. **Proof of the Intersection Number-via-Cup Product Theorem 126.5.** In this section we will prove the Intersection Number-via-Cup Product Theorem 126.5. Our proof of the Intersection Number-via-Cup Product Theorem 126.5 relies heavily on Lemmas 120.5 and 120.6 that relate Poincaré Duality in an oriented topological manifold M to Poincaré Duality in a compact codimension 0 submanifold W. For the reader's convenience we recall the Codimension-Zero Submanifold-PD Lemma 120.5. This lemma says that given a homology class in W we can apply Poincaré Duality in M or W and essentially we end up with the same result:

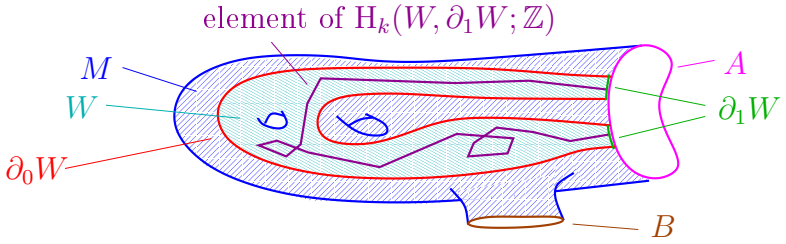
Lemma 120.5. (Codimension-Zero Submanifold-PD Lemma) Let M be a compact oriented m-dimensional topological manifold and let W be a compact m-dimensional submanifold with $\partial M \cap \partial W = \varnothing$. We write $\mathring{W} = W \setminus \partial W$. We fix the following notation:

(1) We denote by w the inclusion map $W \to M$ and also the inclusion map of the pairs $(W, \partial W) \to (M, M \setminus \mathring{W})$.

(2) We denote by $p\colon (M, \partial M) \to (M, M \setminus \mathring{W})$ the inclusion map of pairs.

For every $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}_k(W;\mathbb{Z}) & \xrightarrow{\mathrm{PD}_W} & \mathrm{H}^{m-k}(W,\partial W;\mathbb{Z}) & \xrightarrow[\cong]{(w^*)^{-1}} & \mathrm{H}^{m-k}(M, M\setminus \mathring{W};\mathbb{Z}) \\ \downarrow{\scriptstyle w_*} & & & & \downarrow{\scriptstyle p^*} \\ \mathrm{H}_k(M;\mathbb{Z}) & & \xrightarrow{\qquad\qquad \mathrm{PD}_M \qquad\qquad} & & \mathrm{H}^{m-k}(M,\partial M;\mathbb{Z}). \end{array}$$



We take a short rest from more technicalities and enjoy the fact that we can already prove the Intersection Number-via-Cup Product Theorem 126.5 in the special case that M is closed and that the submanifolds *do not intersect*.

Proof of Theorem 126.5 for disjoint submanifolds. Let M be a closed oriented m-dimensional smooth manifold and let X and Y be a complementary pair of *disjoint* submanifolds of M. Since the submanifolds are disjoint it suffices to show that the cup product of the Poincaré duals of $[X]$ and $[Y]$ is also zero.

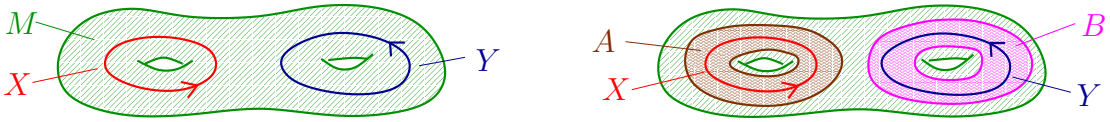
We write $k = \dim(X)$. We apply the General Tubular Neighborhood Theorem **??** to the submanifold $X \cup Y$ of M. We obtain disjoint tubular neighborhoods A for X and B for Y. We denote by $p\colon (M, \varnothing) \to (M, M \setminus \mathring{A})$ and $p\colon (M, \varnothing) \to (M, M \setminus \mathring{B})$ the obvious inclusion maps of pairs of topological spaces.

We apply the Codimension-Zero Submanifold-PD Lemma 120.5 to the codimension-zero submanifold A and the homology class $[X] \in \mathrm{H}_k(A;\mathbb{Z})$. We obtain an $\alpha \in \mathrm{H}^{m-k}(M, M \setminus \mathring{A};\mathbb{Z})$ with $p^*(\alpha) = \mathrm{PD}_M([X])$. Similarly we see that there exists a $\beta \in \mathrm{H}^k(M, M \setminus \mathring{B};\mathbb{Z})$ with $p^*(\beta) = \mathrm{PD}_M([Y])$.

We consider the following diagram

$$\begin{array}{ccc}
\mathrm{H}^{m-k}(M;\mathbb{Z}) \times \mathrm{H}^k(M;\mathbb{Z}) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^m(M;\mathbb{Z}) \\
\uparrow{\scriptstyle p^*\times p^*} \quad \begin{array}{c}(\mathrm{PD}_M([X]), \mathrm{PD}_M([Y])) \\ \updownarrow \\ (\alpha,\beta)\end{array} & & \uparrow \\
\mathrm{H}^{m-k}(M, M\setminus\mathring{A};\mathbb{Z}) \times \mathrm{H}^k(M, M\setminus\mathring{B};\mathbb{Z}) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^m(M, \underbrace{(M\setminus\mathring{A}) \cup (M\setminus\mathring{B})}_{=M};\mathbb{Z}).
\end{array}$$

The diagram commutes by the Cup Product-Naturality Proposition 115.4 (1). It follows from the fact that the lower right group is zero that the cup product of $\mathrm{PD}_M([X])$ and $\mathrm{PD}_M([Y])$ is zero. ∎



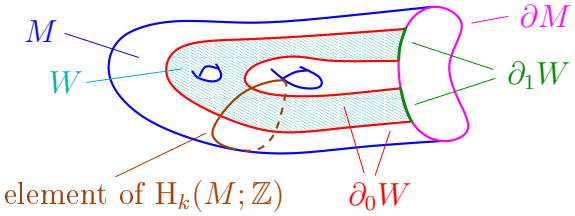
After this refreshing proof of the special case we continue with trying to prove the general case of the Intersection Number-via-Cup Product Theorem 126.5. We will need Lemma 120.6. Since the statement is rather technical, we recall the statement of the special case that is discussed in example (a) on page 2653.

Lemma 120.6. Let M be a compact oriented m-dimensional smooth manifold and let W be a compact m-dimensional submanifold with corner. We fix the following notation:

(1) We write $\partial_1 W = W \cap \partial M$ and $\partial_0 W = \overline{\partial W \setminus \partial_1 W}$,
(2) We write $\mathring{W} = W \setminus \partial_0 W$.
(3) The inclusion maps $(W, \partial_0 W) \to (M, M\setminus\mathring{W})$ and $(W, \partial_1 W) \to (M, \partial M)$ are denoted by w.
(4) Finally we denote by $p\colon (M, \varnothing) \to (M, M \setminus \mathring{W})$ the obvious map.

For every $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}_k(M;\mathbb{Z}) & & \xrightarrow{\quad\mathrm{PD}_M\quad} & & \mathrm{H}^{m-k}(M,\partial M;\mathbb{Z}) \\ \downarrow{\scriptstyle p_*} & & & & \downarrow{\scriptstyle w^*} \\ \mathrm{H}_k(M, M\setminus \mathring{W};\mathbb{Z}) & \xrightarrow[\cong]{(w_*)^{-1}} & \mathrm{H}_k(W,\partial_0 W;\mathbb{Z}) & \xrightarrow{\mathrm{PD}_W} & \mathrm{H}^{m-k}(W,\partial_1 W;\mathbb{Z}). \end{array}$$



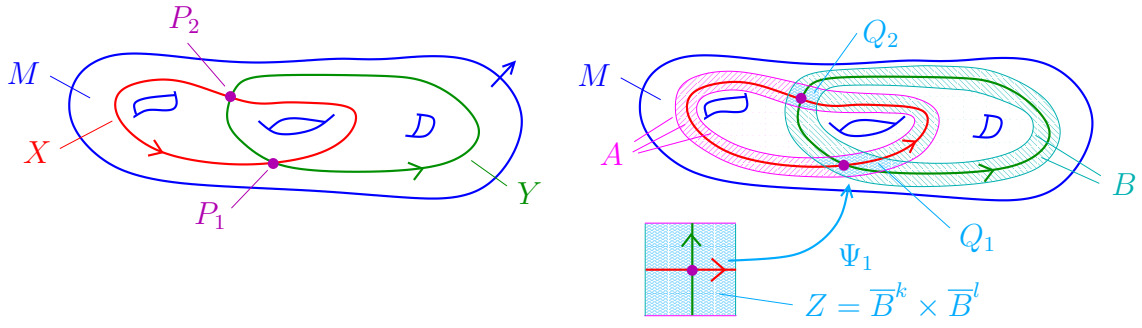
In the proof of the Intersection Number-via-Cup Product Theorem 126.5 we will also need the following proposition.

Proposition 126.8. Let M be a compact oriented smooth manifold and let X and Y be two complementary proper submanifolds of dimensions k and l. We denote by $P_1, \dots, P_s$ the intersection points of X and Y. Then there exist

(1) compact submanifolds with corner A and B such that A is a neighborhood for X and B is a neighborhood for Y, and
(2) injective maps $\Psi_i \colon \overline{B}^k \times \overline{B}^l \to M$, $i = 1, \dots, s$

such that the following statements hold:

(a) For each $i \in \{1, \dots, s\}$ we have $\Psi_i((0,0)) = P_i$.
(b) The images $Q_i := \Psi_i\big(\overline{B}^k \times \overline{B}^l\big)$, $i = 1, \dots, s$ are disjoint.
(c) We have $Q_1 \cup \dots \cup Q_s = A \cap B$.
(d) For each $i \in \{1, \dots, s\}$ the following statements hold:
  (d1) the restriction of Ψ_i to $B^k \times B^l$ is a diffeomorphism onto its image,
  (d2) we have $Q_i \cap X = \Psi_i(\overline{B}^k \times \{0\})$ and the map $\Psi_i \colon \overline{B}^k \times \{0\} \to X$ is an orientation-preserving smooth embedding,
  (d3) we have $Q_i \cap Y = \Psi_i(\{0\} \times \overline{B}^l)$ and the map $\Psi_i \colon \{0\} \times \overline{B}^l \to Y$ is an orientation-preserving smooth embedding.

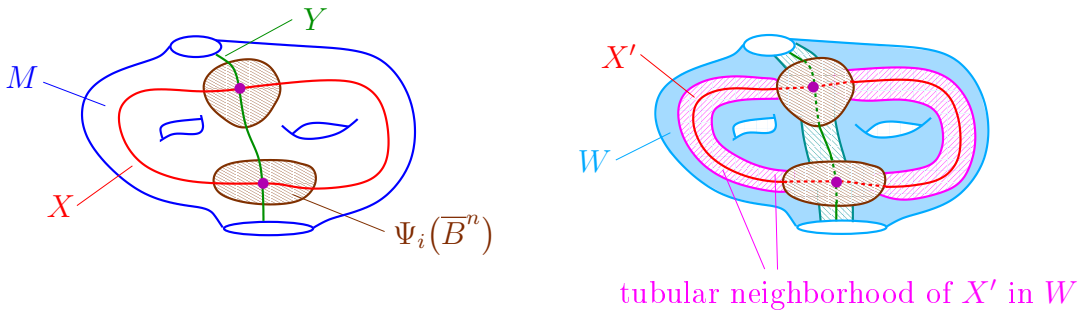


Sketch of a proof. Let M be a compact oriented n-dimensional smooth manifold and let X and Y be two complementary proper submanifolds of dimensions k and l. We denote

by $P_1, \dots, P_s$ the intersection points of X and Y. For $i = 1, \dots, s$ we can and will pick smooth embeddings $\Psi_i \colon \overline{B}^n \to M \setminus \partial M$ such that the following conditions are satisfies:

(1) the images are disjoint,
(2) for each $i \in \{1, \dots, s\}$ the map Ψ_i restricts to orientation-preserving diffeomorphisms $B^k \times \{0\} \to \Psi_i(B^n) \cap X$ and $\{0\} \times B^l \to \Psi_i(B^n) \cap Y$.

We consider $W := M \setminus \bigcup_{i=1}^{s} \Psi_i(B^n)$. Note that $X' := W \cap X$ and $Y' := W \cap Y$ are proper submanifolds of the compact smooth manifold W. Thus we can apply the General Tubular Neighborhood Theorem **??** to obtain tubular neighborhoods for X' and Y'. Using the Connected Sum-Proposition **??** we can arrange the intersections of the tubular neighborhoods with $\Psi_i(S^{n-1})$ to be "standard $(n-1)$-dimensional balls". At this stage it is fairly straightforward to extend the tubular neighborhoods to the interiors of the $\Psi_i(B^n)$ and to obtain all the desired objects. We have faith in the unwavering enthusiasm of our readers for filling in the details. ■



Remark. An alternative approach to Proposition 126.8 would be to use tubular neighborhoods provided by Lemma **??**.

Given the situation of Proposition 126.8 we can make one more statement.

Lemma 126.9. We use the notation of Proposition 126.8. Let $i \in \{1, \dots, s\}$. We equip the topological manifold Q_i, which is a codimension-zero submanifold of M, with the induced orientation coming from the Topological Manifold-Induced Orientation Lemma 105.17. Furthermore we equip the topological manifold $\overline{B}^k \times \overline{B}^l$ with the orientation coming from the standard orientation of $\mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^{k+l}$. With these choices the following statement holds:

(f) If we denote by σ_i the sign of the intersection point P_i, then

$$\Psi_{i*}\big([\overline{B}^k \times \overline{B}^l]\big) \;=\; \sigma_i \cdot [Q_i].$$

Proof. Let $i \in \{1, \dots, s\}$. It follows from (d2) and (d3) that at the point $(0,0)$ the diffeomorphism $\Psi_i \colon B^k \times B^l \to \mathring{Q}_i$ is orientation-preserving if and only if $\sigma_i = 1$. Now the statement follows easily from the definition of σ_i, Proposition 107.2 and Lemma 105.12. ■

The last preparation we need before we can provide the actual proof of the Intersection Number-via-Cup Product Theorem 126.5 is the Poincaré Duality-Products Proposition 119.2, which for the reader's convenience we recall using our present notation.

Proposition 119.2. Let X be a compact oriented connected m-dimensional topological manifold and let Y be a compact oriented connected n-dimensional topological manifold. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the natural projections. We equip $X \times Y$ with the product orientation introduced in the Topological Product Orientation Proposition 113.10. Given any points $P \in X$ and $Q \in Y$ we have the following equality in $\mathrm{H}^{m+n}(X \times Y, \partial(X \times Y); \mathbb{Z})$:

$$\mathrm{PD}_{X\times Y}([X \times \{Q\}]) \cup \mathrm{PD}_{X\times Y}([\{P\} \times Y]) \;=\; [X \times Y]^*.$$

We turn to the actual proof of the Intersection Number-via-Cup Product Theorem 126.5.

Proof of the Intersection Number-via-Cup Product Theorem 126.5. Let M be a compact oriented connected m-dimensional smooth manifold. To simplify the notation we assume that M is closed. The case that M has boundary is proved in a rather similar fashion.[1538]

Let X and Y be a complementary pair of submanifolds of M. We introduce the following notation:

(1) We write $k = \dim(X)$ and $l = \dim(Y)$. Note that $l = m - k$.
(2) As usual we denote by $[X] \in \mathrm{H}_k(X)$ and $[Y] \in \mathrm{H}_l(Y)$ the fundamental classes of X and Y. By a slight abuse of notation we denote by $[X]$ respectively $[Y]$ also the images of these homology classes in any subset of M that contains X respectively Y.
(3) We denote the intersection points of X and Y by $P_1, \ldots, P_s$ and we denote the corresponding signs by $\sigma_1, \ldots, \sigma_s$.
(4) As usual, given a topological space X we denote by $\epsilon_X = \epsilon\colon \mathrm{H}_0(X) \to \mathbb{Z}$ the augmentation map.

With the notation that we just introduced, and keeping in mind the Cap Product-Kronecker Lemma 116.4 (1), the statement of the Intersection Number-via-Cup Product Theorem 126.5 can be reformulated as the following statement.

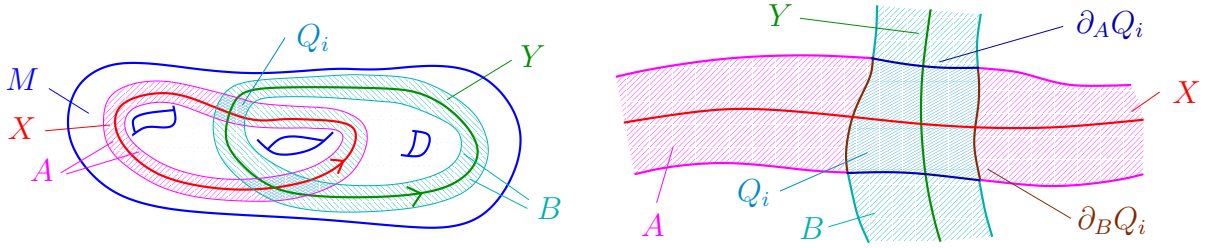
Statement.
$$\epsilon_M\big((\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y])) \cap [M]\big) \;=\; \sum_{i=1}^{s} \sigma_i.$$

We introduce many more objects, notations and conventions:

(5) We apply Proposition 126.8 to the submanifolds X and Y. In the remainder of the proof we use the notation that we introduced in that proposition. As a reminder, most of the relevant objects are illustrated in the figure on page 2759.
(6) We write $\mathring{A} = A \setminus \partial A$ and $\mathring{B} = B \setminus \partial B$.
(7) We denote all of the inclusion maps $(M, \varnothing) \to (M, M \setminus \mathring{A})$, $(M, \varnothing) \to (M, M \setminus \mathring{B})$ and $(M, \varnothing) \to (M, M \setminus (\mathring{A} \cap \mathring{B}))$ by p.
(8) We write $C = A \cap B = Q_1 \cup \cdots \cup Q_s$. Furthermore we write $\partial_A C = \partial A \cap B$ and $\partial_B C = A \cap \partial B$. Note that $\partial C = \partial_A C \cup \partial_B C$. Also note that $(M \setminus \mathring{A}) \cup (M \setminus \mathring{B}) = M \setminus \mathring{C}$.
(9) We denote by $c\colon C \to M$ the inclusion map and we denote by c also all the corresponding inclusion maps of pairs where the first terms are C and M.

[1538] Admittedly, the case that M has boundary also requires the generalization of the Codimension-Zero Submanifold-PD Lemma 120.5 to the case that W is a codimension-zero submanifold with corner. But that is not really a problem.

(10) Given $i \in \{1, \ldots, s\}$ we write $\partial_A Q_i = \partial_A C \cap Q_i$ and $\partial_B Q_i = \partial_B C \cap Q_i$. Note that $\partial Q_i = \partial_A Q_i \cup \partial_B Q_i$.
(11) Given $i \in \{1, \ldots, s\}$ we denote by $q_i \colon Q_i \to C$ the inclusion map and we denote by q_i also all the corresponding inclusion maps of pairs where the first terms are Q_i and C.
(12) Note that C and each Q_i are codimension-zero submanifolds of M. We equip these submanifolds with the corresponding orientation introduced in the Topological Manifold-Induced Orientation Lemma 105.17.



Now we consider the following diagram:[1539]

$$\begin{array}{ccccccc}
\mathrm{H}^l(M) \times \mathrm{H}^k(M) & \xrightarrow{\cup} & \mathrm{H}^m(M) & \xrightarrow{\cap[M]} & \mathrm{H}_0(M) & \xrightarrow{\epsilon_M} & \mathbb{Z} \\
\uparrow{\scriptstyle p^*\times p^*} & & \uparrow{\scriptstyle p^*} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\
\mathrm{H}^l(M, M\setminus \mathring{A}) \times \mathrm{H}^k(M, M\setminus \mathring{B}) & \xrightarrow{\cup} & \mathrm{H}^m(M, M\setminus \mathring{C}) & \xrightarrow{\cap p_*([M])} & \mathrm{H}_0(M) & \xrightarrow{\epsilon_M} & \mathbb{Z} \\
\downarrow{\scriptstyle c^*\times c^*} & & \downarrow{\scriptstyle c^*} & & \uparrow{\scriptstyle c_*} & & \uparrow{\scriptstyle =} \\
\mathrm{H}^l(C, \partial_A C) \times \mathrm{H}^k(C, \partial_B C) & \xrightarrow{\cup} & \mathrm{H}^m(C, \partial C) & \xrightarrow{\cap[C]} & \mathrm{H}_0(C) & \xrightarrow{\epsilon_C} & \mathbb{Z} \\
\downarrow{\scriptstyle \bigoplus_{i=1}^s q_i^*\times q_i^*} & & \downarrow{\scriptstyle \bigoplus_{i=1}^s q_i^*} & & \uparrow{\scriptstyle \sum_{i=1}^s q_{i*}} & & \uparrow{\scriptstyle \mathrm{id}} \\
\bigoplus_{i=1}^s \mathrm{H}^l(Q_i, \partial_A Q_i) \times \mathrm{H}^k(Q_i, \partial_B Q_i) & \xrightarrow{\oplus\cup} & \bigoplus_{i=1}^s \mathrm{H}^m(Q_i, \partial Q_i) & \xrightarrow{\bigoplus_{i=1}^s \cap[Q_i]} & \bigoplus_{i=1}^s \mathrm{H}_0(Q_i) & \xrightarrow{\sum \epsilon_{Q_i}} & \mathbb{Z}.
\end{array}$$

Claim 1. The above diagram commutes.

Proof. In the following we deal with all of the nine squares of the diagram.

(a) The left squares in the first and second row commute by the Cup Product-Naturality Proposition 115.4 (1).
(b) The left square in the third row commutes by the Cup-Cap Disjoint Union Lemma 116.9 (1) together with the aforementioned the Cup Product-Naturality Proposition 115.4 (1).
(c) The middle square in the first row commutes by the Cap Product-Naturality Lemma 116.8 (1).
(d) It follows from our orientation conventions and the Codimension Zero-Fundamental Class Lemma 106.24 that $c_*([C]) = p_*([M])$. It follows from the Cap Product-Naturality Lemma 116.8 (1) that the middle square in the second row commutes.

[1539] Many of the vertical maps in the diagram are isomorphisms by excision, see page 2418 for details. But this is in fact irrelevant for our purpose.

(e) It follows from the remark on page 2351 that $\sum_{i=1}^{s} q_{i*}([Q_i]) = [C]$. Now we obtain from the Cap Product-Naturality Lemma 116.8 (1) and the Cup-Cap Disjoint Union Lemma 116.9 (2) that the middle square in the third row commutes.

(f) The squares that involve the augmentation maps commute by the discussion on page 1594. ⊞

We introduce a few last pieces of notation:

(13) We denote by $a\colon (A, \partial A) \to (M, M \setminus \mathring{A})$ and $b\colon (B, \partial B) \to (M, M \setminus \mathring{B})$ the inclusion maps, note that a and b induce isomorphisms on cohomology by excision, see page 2418 for details.

(14) We write

$$\alpha = (a^*)^{-1}(\underbrace{\mathrm{PD}_A([X])}_{\in \mathrm{H}^l(A,\partial A)}) \in \mathrm{H}^l(M, M\setminus\mathring{A}) \text{ and } \beta = (b^*)^{-1}(\underbrace{\mathrm{PD}_B([Y])}_{\in \mathrm{H}^k(B,\partial B)}) \in \mathrm{H}^k(M, M\setminus\mathring{B}).$$

(15) Given $i \in \{1, \dots, s\}$ we write $X_i = X \cap Q_i$ and $Y_i = Y \cap Q_i$. It follows from Proposition 126.8 (e) and (f) that X_i and Y_i are codimension zero submanifolds of X and Y. Since X and Y are oriented we can view X_i and Y_i also as oriented topological manifolds. As usual we denote by $[X_i] \in \mathrm{H}_k(X_i, \partial X_i)$ and $[Y_i] \in \mathrm{H}_k(Y_i, \partial Y_i)$ the corresponding fundamental classes. With the same convention as before we denote by $[X_i]$ also the image under the inclusion induced map $\mathrm{H}_k(X_i, \partial X_i) \to \mathrm{H}_k(Q_i, \partial_B Q_i)$ and similarly we denote by $[Y_i]$ also the image under the inclusion induced map $\mathrm{H}_l(Y_i, \partial Y_i) \to \mathrm{H}_l(Q_i, \partial_A Q_i)$.

Claim 2. We have

(a) We have $p^*(\alpha) = \mathrm{PD}_M([X]) \in \mathrm{H}^l(M)$ and $p^*(\beta) = \mathrm{PD}_M([Y]) \in \mathrm{H}^k(M)$.

(b) For any $i \in \{1, \dots, s\}$ we have

$$(q_i^* \circ c^*)(\alpha) = \mathrm{PD}_{Q_i}([X_i]) \in \mathrm{H}^l(Q_i, \partial_A Q_i) \text{ and } (q_i^* \circ c^*)(\beta) = \mathrm{PD}_{Q_i}([Y_i]) \in \mathrm{H}^k(Q_i, \partial_B Q_i).$$

(c) For any $i \in \{1, \dots, s\}$ we have

$$\epsilon_{Q_i}\big((\mathrm{PD}_{Q_i}([X_i]) \cup \mathrm{PD}_{Q_i}([Y_i])) \cap [Q_i]\big) = \sigma_i.$$

Proof.

(a) By the Codimension-Zero Submanifold-PD Lemma 120.5 the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}_k(A) \arrow[r, "\mathrm{PD}_A"'] \arrow[rr, bend left, "{[X]\mapsto\alpha}"] \arrow[d, "{[X] \mapsto [X]}"'] & \mathrm{H}^l(A, \partial A) \arrow[r, "\cong", "(a^*)^{-1}"'] & \mathrm{H}^l(M, M\setminus\mathring{A}) \arrow[d, "p^*"] \\
\mathrm{H}_k(M) \arrow[rr, "\mathrm{PD}_M"] & & \mathrm{H}^l(M).
\end{tikzcd}$$

We apply this commutative diagram to $[X] \in \mathrm{H}_k(A)$ and we obtain immediately that $p^*(\alpha) = \mathrm{PD}_M([X])$. The same way one shows that $p^*(\beta) = \mathrm{PD}_M([Y])$.

(b) Let $i \in \{1, \dots, s\}$. We denote by $r_i \colon Q_i \to A$ the inclusion map. We consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}_k(A) & & \xrightarrow{\mathrm{PD}_A} & & \mathrm{H}^l(A, \partial A) \xleftarrow[\cong]{a^*} \mathrm{H}^l(M, M \setminus \mathring{A}) \\
\downarrow & & & & \downarrow r_i^* \quad \swarrow (q_i \circ c)^* \\
\mathrm{H}_k(A, A \setminus \mathring{Q}_i) & \xleftarrow[\cong]{r_{i*}} & \mathrm{H}_k(Q_i, \partial_B Q_i) & \xrightarrow{\mathrm{PD}_{Q_i}} & \mathrm{H}^l(Q_i, \partial_A Q_i).
\end{array}$$

We make the following observations:

(α) The triangle on the right commutes since $a \circ r_i = c \circ q_i \colon Q_i \to M$ are both the inclusion map.

(β) It follows from Lemma 120.6, applied to the codimension-zero submanifold Q_i of A, that the rectangle commutes.[1540]

(γ) By the Codimension Zero-Fundamental Class Lemma 106.24 we have $r_{i*}([X_i]) = [X] \in \mathrm{H}_k(A, A \setminus \mathring{Q}_i)$.

(δ) By the Excision Theorem 120.2 the map $r_{i*} \colon \mathrm{H}_k(Q_i, \partial_B Q_i) \to \mathrm{H}_k(A, A \setminus \mathring{Q}_i)$ is an isomorphism.

The desired equality $(q_i^* \circ c^*)(\alpha) = \mathrm{PD}_{Q_i}([X_i])$ follows from the above commutative diagram applied to $[X] \in \mathrm{H}_k(A)$. Exactly the same way we can also show that $(q_i^* \circ c^*)(\beta) = \mathrm{PD}_{Q_i}([Y_i])$.

(c) Let $i \in \{1, \dots, s\}$. We set $Z = \overline{B}^k \times \overline{B}^l$. We consider the following diagram:

$$\begin{array}{ccccccc}
\mathrm{H}^l(Q_i, \partial_A Q_i) \times \mathrm{H}^k(Q_i, \partial_B Q_i) & \xrightarrow{\cup} & \mathrm{H}^m(Q_i, \partial Q_i) & \xrightarrow{\cap [Q_i]} & \mathrm{H}_0(Q_i) & \xrightarrow{\epsilon_{Q_i}} & \mathbb{Z} \\
\downarrow \Psi_i^* \times \Psi_i^* & & \downarrow \Psi_i^* & & \uparrow \Psi_i^* & & \uparrow \mathrm{id} \\
\mathrm{H}^l(Z, \overline{B}^k \times \partial \overline{B}^l) \times \mathrm{H}^k(Z, \partial \overline{B}^k \times \overline{B}^l) & \xrightarrow{\cup} & \mathrm{H}^m(Z, \partial Z) & \xrightarrow{\cap \sigma_i \cdot [Z]} & \mathrm{H}_0(Z) & \xrightarrow{\epsilon_Z} & \mathbb{Z}.
\end{array}$$

As in the previous diagram, the squares to the left and right commute. The middle square is slightly more interesting. Lemma 126.9 says that $\Psi_{i*}([Q_i]) = \sigma_i \cdot [Z]$. Using this observation we obtain once again from the Cap Product-Naturality Lemma 116.8 (1) that the middle square commutes.

We calculate that

$$\begin{aligned}
\epsilon_{Q_i}\big((\mathrm{PD}_{Q_i}([X_i]) \cup \mathrm{PD}_{Q_i}([Y_i])) \cap [Q_i]\big) &\overset{\text{by the above commutative diagram}}{=} \sigma_i \cdot \epsilon_Z\big((\Psi_i^*(\mathrm{PD}_{Q_i}([X_i])) \cup \Psi_i^*(\mathrm{PD}_{Q_i}([Y_i]))) \cap [Z]\big) \\
&= \sigma_i \cdot \epsilon_Z\big((\mathrm{PD}_Z([\overline{B}^k \times \{0\}]) \cup \mathrm{PD}_Z([\{0\} \times \overline{B}^l])) \cap [Z]\big) \\
&\overset{\text{by Proposition 126.8 (d)}}{=} \sigma_i \cdot \epsilon_{\overline{B}^k \times \overline{B}^l}\big([\overline{B}^k \times \overline{B}^l]^* \cap [\overline{B}^k \times \overline{B}^l]\big) \overset{\text{definition of the dual fundamental class}}{=} \sigma_i.
\end{aligned}$$

the Poincaré Duality-Products Proposition 119.2

$\boxplus$

[1540] One can easily convince oneself that $\partial_A Q_i = \partial_1 Q_i$ and $\partial_B Q_i = \partial_0 Q_i$ for Q_i viewed as a submanifold with corner of A.

After all these preparations we can prove the desired equality:

$$\epsilon_M\big((\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y])) \cap [M]\big) \overset{\text{by the first row of the commutative diagram from Claim 1 and by Claim 2 (a)}}{\downarrow} = \epsilon_M\big((\alpha \cup \beta) \cap p_*([M])\big)$$
$$= \sum_{i=1}^{s} \epsilon_{Q_i}\big((\mathrm{PD}_{Q_i}([X_i]) \cup \mathrm{PD}_{Q_i}([Y_i])) \cap [Q_i]\big) \overset{\text{by Claim 2 (c)}}{=\!=\!\uparrow} \sum_{i=1}^{s} \sigma_i.$$
$$\uparrow \text{ by the second and third row of commutative diagram from Claim 1 and by Claim 2 (b)}$$

This concludes the proof of the theorem. ■

Remark. Towards the end of the proof of the theorem we marked in red color that we used the Poincaré Duality-Products Proposition 119.2. Most of the other statements that we used in the proof were fairly straightforward lemmas regarding cup and cap products. But arguably the key ingredient in the proof is the Poincaré Duality-Products Proposition 119.2 which was a variation on the Fundamental Class-Product Proposition 117.2 and which in turn was a fairly straightforward consequence of the Product Theorem 117.1.

126.4. **Algebraic intersection numbers of complex submanifolds.** In the following section we will apply the Intersection Number-via-Cup Product Theorem 126.5 to the study of the complex projective spaces $\mathbb{C}\mathrm{P}^n$. Before we do so it is convenient to study the algebraic intersection numbers of complex submanifolds of any complex manifold.
First let us recall the following definition that goes all the way back to pages 797.

Definition.

(1) An n-dimensional complex manifold is a $2n$-dimensional smooth manifold M without boundary together with a smooth atlas $\{\phi_j \colon U_j \to V_j\}_{j\in J}$ such that all transition maps are biholomorphisms.
(2) We define the maximal holomorphic atlas of a complex manifold the same way as we defined the maximal smooth atlas of a smooth manifold, see page 542.
(3) We say $W \subset M$ is a k-dimensional complex submanifold if for every point $P \in W$ there exists a chart $\phi_j \colon U_j \to V_j$ from the maximal holomorphic atlas with $P \in U_j$ and which satisfies that $\phi_j(U_j \cap W) \subset \mathbb{C}^k \times \{0\}$.

Remark.

(1) Let M be a complex manifold. It follows easily from the definitions that every complex submanifold of M is a complex manifold in its own right.
(2) Let V be an n-dimensional complex vector space. In the Complex Vector Space-Orientation Lemma 32.22 we showed that if $(v_1, \ldots, v_n)$ is a basis for V as a complex vector space, then the ordered basis $(v_1, \mathrm{i}v_1, \ldots, v_n, \mathrm{i}v_n)$ gives a well-defined orientation of the $2n$-dimension real vector space V, called the *natural orientation*.
(3) Let M be an n-dimensional complex manifold. In the Complex Manifold-Orientation Proposition 32.23 we showed that $\mathrm{T}_P M$ is canonically a complex vector space and we showed that the corresponding natural orientations define a natural orientation of M, viewed as a $2n$-dimensional smooth manifold.

Now we can formulate this elementary but rather useful lemma.

Lemma 126.10. Let M be a closed complex manifold and let X and Y be two closed complex submanifolds of M with $\dim(X) + \dim(Y) = \dim(M)$. Then the following hold:

(1) The sign at every transverse intersection point of X and Y is equal to $+1$.
(2) If X and Y intersect transversally, then the algebraic intersection number equals the geometric intersection number.

Proof.

(1) We write $k = \dim(X)$ and $l = \dim(Y)$. Suppose that P is a transverse intersection point of the submanifolds X and Y. We pick a basis $(v_1, \ldots, v_k)$ for the complex vector space $\mathrm{T}_P X$ and we pick a basis $(w_1, \ldots, w_l)$ for the complex vector space $\mathrm{T}_P Y$. Then $(v_1, \ldots, v_k, w_1, \ldots, w_l)$ is evidently a basis for the complex vector space $\mathrm{T}_P X \oplus \mathrm{T}_P Y = \mathrm{T}_P M$. We have
$$\underbrace{v_1, \mathrm{i}v_1, \ldots, v_k, \mathrm{i}v_k}_{\text{positive basis for } \mathrm{T}_P X}, \underbrace{w_1, \mathrm{i}w_1, \ldots, w_l, \mathrm{i}w_l}_{\text{positive basis for } \mathrm{T}_P Y} = \underbrace{v_1, \mathrm{i}v_1, \ldots, v_k, \mathrm{i}v_k, w_1, \mathrm{i}w_1, \ldots, w_l, \mathrm{i}w_l}_{\text{positive basis for } \mathrm{T}_P M}.$$
This shows that $\mathrm{sign}(P) = +1$. We have concluded the proof of the first statement.
(2) The second statement is an immediate consequence of the first statement. ∎

Example. We consider the complex projective space $\mathbb{C}\mathrm{P}^n$. In the Complex Projective Space-is-Complex Manifold Lemma 32.16 we saw that $\mathbb{C}\mathrm{P}^n$ is a complex manifold of complex dimension n. It follows easily from the definitions that for any $k \leq n$ the subset
$$X_k := \{[z_0 : \ldots : z_k : 0 : \ldots : 0] \mid (z_0, \ldots, z_k) \in \mathbb{C}^{k+1} \setminus \{0\}\}$$
is a k-dimensional complex submanifold. Similarly we see that for any $l \leq n$ the subset
$$Y_l := \{[0 : \ldots : 0 : z_{n-l} : \ldots : z_n] \mid (z_{n-l}, \ldots, z_n) \in \mathbb{C}^{l+1} \setminus \{0\}\}$$
is an l-dimensional complex submanifold. Evidently X_k and Y_l are biholomorphic to $\mathbb{C}\mathrm{P}^k$ respectively $\mathbb{C}\mathrm{P}^l$. Furthermore given any $k \leq n$ we have
$$X_k \cap Y_{n-k} = [0 : \ldots : 0 : \underset{\substack{\uparrow \\ (k+1)\text{-st entry}}}{1} : 0 : \ldots : 0]$$
consists of a single point P. We leave it as an illuminating exercise to verify that X_k and Y_{n-k} intersect transversally in P. It follows from Lemma 126.10 that $X_k \cdot Y_{n-k} = 1$. The same way we also see that $Y_k \cdot X_{n-k} = 1$.

126.5. **Applications to complex projective spaces.** In this section we will give two more applications of the Intersection Number-via-Cup Product Theorem 126.5.

Notation. Let $n \in \mathbb{N}$ and let $m \in \{0, \ldots, n\}$.

(1) We consider the inclusion map
$$\begin{aligned} i_{m,n} \colon \mathbb{C}\mathrm{P}^m &\to \mathbb{C}\mathrm{P}^n \\ [z_0 : \ldots : z_m] &\mapsto [z_0 : \ldots : z_m : 0 : \cdots : 0]. \end{aligned}$$
When m and n are understood we write i instead of $i_{m,n}$. Recall that in Lemma 121.6 we showed that for any $k \leq 2m$ the induced maps $i_* \colon \mathrm{H}_k(\mathbb{C}\mathrm{P}^m; \mathbb{Z}) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ and $i^* \colon \mathrm{H}^k(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \to \mathrm{H}^k(\mathbb{C}\mathrm{P}^m; \mathbb{Z})$ are isomorphisms.

(2) By a slight abuse of notation we abbreviate $(i_{m,n})_*([\mathbb{C}\mathrm{P}^m])$ to $[\mathbb{C}\mathrm{P}^m] \in \mathrm{H}_{2m}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$. Furthermore, again by a slight abuse of notation, we write $[\mathbb{C}\mathrm{P}^m]^* \in \mathrm{H}^{2m}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ for the unique element with $\langle [\mathbb{C}\mathrm{P}^m]^*, [\mathbb{C}\mathrm{P}^m]\rangle = 1$. Finally we write $x = [\mathbb{C}\mathrm{P}^1]^*$.

Now we can formulate the main result of this section.

Proposition 126.11. (Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition) Let $n \in \mathbb{N}$. For every $k \in \{0,\ldots,n\}$ we have

$$(\underbrace{[\mathbb{C}\mathrm{P}^1]^*}_{=x})^k \;=\; [\mathbb{C}\mathrm{P}^k]^* \;\in\; \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}).$$

Remark. In the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 121.7 we had used the Poincaré Duality Theorem 119.1 to show that for given $n \in \mathbb{N}$ and $k \in \{0,\ldots,n\}$ the k-th power $x^k \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ is a generator. In other words, we had proved that $x^k = \pm[\mathbb{C}\mathrm{P}^k]^*$. the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 126.11 is a much better result since it allows us to nail down the cohomology class x^k. In particular we see that the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 126.11 gives an affirmative answer to Question 121.8.

The key to the proof of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 126.11 is the following lemma.

Lemma 126.12. Let $n \in \mathbb{N}$. For any $k \in \{0,\ldots,n\}$ we have

$$\mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^k]) \cup \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^{n-k}]) \;=\; [\mathbb{C}\mathrm{P}^n]^* \;\in\; \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}).$$

Proof. Let $n \in \mathbb{N}$. For $k \in \{0,\ldots,n\}$ we consider the inclusion map

$$\begin{array}{rcl} t_{k,n}\colon \mathbb{C}\mathrm{P}^k & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \ldots : z_k] & \mapsto & [0 : \ldots : 0 : z_0 : \ldots : z_k]. \end{array}$$

We also consider the map[1541]

$$\begin{array}{rcl} \Phi\colon \mathbb{C}\mathrm{P}^k \times [0,1] & \to & \mathbb{C}\mathrm{P}^n \\ ([z],t) & \mapsto & \big[\underbrace{\big(z\cdot\cos^2\big(t\cdot\frac{\pi}{2}\big),0\big)}_{\in\mathbb{C}^{n+1}} + \underbrace{\big(0,z\cdot\sin^2\big(t\cdot\frac{\pi}{2}\big)\big)}_{\in\mathbb{C}^{n+1}}\big]. \end{array}$$

Then

$$\begin{array}{rcll} \langle \mathrm{PD}([\mathbb{C}\mathrm{P}^k]) \cup \mathrm{PD}([\mathbb{C}\mathrm{P}^{n-k}]), [\mathbb{C}\mathrm{P}^n]\rangle & = & \langle \mathrm{PD}(i_{k,n*}([\mathbb{C}\mathrm{P}^k])) \cup \mathrm{PD}(i_{n-k,n*}([\mathbb{C}\mathrm{P}^{n-k}])), [\mathbb{C}\mathrm{P}^n]\rangle & \\ & = & \langle \mathrm{PD}(t_{k,n*}([\mathbb{C}\mathrm{P}^k])) \cup \mathrm{PD}(i_{n-k,n*}([\mathbb{C}\mathrm{P}^{n-k}])), [\mathbb{C}\mathrm{P}^n]\rangle & \\ & & \uparrow & \\ & & \text{by the Homotopic Maps-and-Homology Proposition 73.6, since } \Phi \text{ is a homotopy from } i_{k,n} \text{ to } t_{k,n} & \\ & = & \langle \mathrm{PD}([t_{k,n}(\mathbb{C}\mathrm{P}^k)]) \cup \mathrm{PD}([i_{n-k,n}(\mathbb{C}\mathrm{P}^{n-k})]), [\mathbb{C}\mathrm{P}^n]\rangle & \\ & = & t_{k,n}(\mathbb{C}\mathrm{P}^k)\cdot i_{n-k,n}(\mathbb{C}\mathrm{P}^{n-k}) \;=\; 1. & \\ & & \uparrow \qquad\qquad\qquad\qquad \uparrow & \\ & & \text{by the Intersection Number-via-Cup Product Theorem 126.5 since submanifolds } t_{k,n}(\mathbb{C}\mathrm{P}^k) \text{ and } i_{n-k,n}(\mathbb{C}\mathrm{P}^{n-k}) \text{ are transverse} \quad \text{see page 2766 for details} & \end{array}$$

By definition of the dual fundamental class this implies the desired result. ∎

We record the following fairly immediate corollary.

[1541] It is quite straightforward to verify that this map is well-defined, i.e. it does not depend on the choice of the representative $z \in \mathbb{C}^{k+1}\setminus\{0\}$ and it does indeed take values in $\mathbb{C}\mathrm{P}^n$ since for any $t \in [0,1]$ we have $\big(z\cdot\cos^2\big(t\cdot\frac{\pi}{2}\big),0\big) + \big(0,z\cdot\sin^2\big(t\cdot\frac{\pi}{2}\big)\big) \neq 0$. Also note that it follows fairly easily from the Homotopy Quotient Theorem 14.7 that the map is continuous.

Corollary 126.13. Let $n \in \mathbb{N}$. For any $k \in \{0, \dots, n\}$ we have
$$\mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^k]) = [\mathbb{C}\mathrm{P}^{n-k}]^* \in \mathrm{H}^{2n-2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z}).$$

Proof. Let $n \in \mathbb{N}$ and let $k \in \{0, \dots, n\}$. Then

$$\begin{array}{rcl} \langle \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^k]), [\mathbb{C}\mathrm{P}^{n-k}] \rangle & \overset{\text{by definition of } \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^{n-k}])}{\downarrow =} & \langle \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^k]), \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^{n-k}]) \cap [\mathbb{C}\mathrm{P}^n] \rangle \\ & \underset{\text{by the Cup-Cap Lemma 116.7 and the Cup Product-Commutativity Proposition 114.8}}{\uparrow =} & \langle \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^k]) \cup \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^{n-k}]), [\mathbb{C}\mathrm{P}^n] \rangle \underset{\text{Lemma 126.12}}{\uparrow =} 1. \end{array}$$

By definition of the dual fundamental class this implies the desired result. ■

Proof of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 126.11. Recall that we need to show that for every $n \in \mathbb{N}$ and every $k \in \{0, \dots, n\}$ we have $([\mathbb{C}\mathrm{P}^1]^*)^k = [\mathbb{C}\mathrm{P}^k]^* \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$. We start out with the following observation that follows immediately from the discussion on page 2766 and the Cup Product-Naturality Lemma 114.10.

Observation. Let $m \in \mathbb{N}$ and let $k \in \{1, \dots, m\}$. If $([\mathbb{C}\mathrm{P}^1]^*)^k = [\mathbb{C}\mathrm{P}^k]^* \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^m; \mathbb{Z})$, then for any $n \geq m$ we also have $([\mathbb{C}\mathrm{P}^1]^*)^k = [\mathbb{C}\mathrm{P}^k]^* \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$

By the observation it suffices to prove the following claim.

Claim. For any $n \in \mathbb{N}$ we have $([\mathbb{C}\mathrm{P}^1]^*)^n = [\mathbb{C}\mathrm{P}^n]^* \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$.

Proof. We prove the claim by induction on n. For $n = 1$ the statement holds by definition. So suppose that the statement holds for some $n-1$ with $n \geq 2$. Then the following equality holds in $\mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$:

$$([\mathbb{C}\mathrm{P}^1]^*)^n = ([\mathbb{C}\mathrm{P}^1]^*)^{n-1} \cup [\mathbb{C}\mathrm{P}^1]^* \underset{\text{by induction and the observation}}{\uparrow =} [\mathbb{C}\mathrm{P}^{n-1}]^* \cup [\mathbb{C}\mathrm{P}^1]^* \underset{\text{Corollary 126.13}}{\uparrow =} \mathrm{PD}([\mathbb{C}\mathrm{P}^1]) \cup \mathrm{PD}([\mathbb{C}\mathrm{P}^{n-1}]) \underset{\text{Lemma 126.12}}{\uparrow =} [\mathbb{C}\mathrm{P}^n]^*.$$

This concludes the proof of the claim and thus of the proposition. ■

Now we can answer Question **??**.

Proposition 126.14. Every tubular neighborhood of $\mathbb{C}\mathrm{P}^1 = i_{1,2}(\mathbb{C}\mathrm{P}^1)$ in $\mathbb{C}\mathrm{P}^2$ is non-trivial.[1542]

In the figure on page 571 we had implicitly seen that the tubular neighborhood of $\mathbb{R}\mathrm{P}^1$ in $\mathbb{R}\mathrm{P}^2$ is a Möbius band, and thus non-trivial. The above proposition is thus an analogue of this simpler, and more visual, observation.

Proof. Suppose that $\mathbb{C}\mathrm{P}^1$ admits a trivial tubular neighborhood in $\mathbb{C}\mathrm{P}^2$. By definition this would mean that there exists a smooth embedding $\Phi\colon \mathbb{C}\mathrm{P}^1 \times \overline{B}^2 \to \mathbb{C}\mathrm{P}^2$ with $\Phi(P, 0) = P$ for all $P \in \mathbb{C}\mathrm{P}^1$. We denote by $i = i_{1,2}\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2$ the usual inclusion map and we denote

[1542] We refer to page **??** for the definition of a trivial tubular neighborhood.

by $f\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2$ the smooth embedding that is given by $P \mapsto \Phi(P,1)$. Then

$$\pm 1 \underset{\uparrow}{=} \langle \mathrm{PD}(i_*([\mathbb{C}\mathrm{P}^1]) \cup \mathrm{PD}(i_*([\mathbb{C}\mathrm{P}^1])), [\mathbb{C}\mathrm{P}^2]\rangle \underset{\uparrow}{=} \langle \mathrm{PD}(i_*([\mathbb{C}\mathrm{P}^1]) \cup \mathrm{PD}(f_*([\mathbb{C}\mathrm{P}^1])), [\mathbb{C}\mathrm{P}^2]\rangle$$

by the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 126.11 or by the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 121.7

by the Homotopic Maps-and-Homology Proposition 73.6 since $i, f\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2$ are homotopic via the homotopy $(P,t) \mapsto \Phi(P,t)$

$$\underset{\uparrow}{=} i(\mathbb{C}\mathrm{P}^1) \cdot f(\mathbb{C}\mathrm{P}^1) \underset{\uparrow}{=} 0.$$

by Theorem 126.5 since $i(\mathbb{C}\mathrm{P}^1)$ and $f(\mathbb{C}\mathrm{P}^1)$ are disjoint

We have thus obtained a contradiction. ∎

126.6. **Higher dimensional analogue of Theorem 126.5.** Let M be a compact oriented m-dimensional smooth manifold, let $X \subset M$ be a closed oriented submanifold of dimension k and let Y be a compact oriented proper submanifold of the complementary dimension $m-k$. We suppose that X and Y are transverse. By assigning to each intersection point in $X \cap Y$ the sign introduced on page 2748 we can view $X \cap Y$ as a 0-dimensional oriented submanifold of M.[1543] With this convention we can reformulate the conclusion of the Intersection Number-via-Cup Product Theorem 126.5 as follows:

Theorem 126.15. With the above notation we have

$$\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y]) \;=\; \mathrm{PD}_M([X \cap Y]) \;\in\; \mathrm{H}^m(M, \partial M; \mathbb{Z}).$$

Proof. Using the Cup-Cap Disjoint Union Lemma 116.9 one can easily verify that it suffices to prove the theorem for each connected component of M separately. Put differently, without loss of generality we can assume that M is connected. We denote by $P_1, \ldots, P_n$ the points in $X \cap Y$ and we denote by $\sigma_1, \ldots, \sigma_n \in \{\pm 1\}$ the corresponding signs. Now we calculate that

$$\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y]) \underset{\uparrow}{=} (X \cdot Y) \cdot [M]^* = \sum_{i=1}^n \sigma_i \cdot [M]^* \underset{\uparrow}{=} \sum_{i=1}^n \mathrm{PD}_M(\sigma_i \cdot [\{P_i\}]) = \mathrm{PD}_M([X \cap Y]).$$

by Theorem 126.5 and since M is connected

see page 2619

∎

The formulation of Theorem 126.15 has the nice feature that with some effort one can also generalize it to submanifolds that are not of complementary dimensions. Before we can formulate this generalization we need to formulate the following lemma.

Lemma 126.16. Let M be an oriented m-dimensional smooth manifold, let $X \subset M$ be an oriented proper submanifold and let Y be an oriented proper submanifold. Suppose that X and Y intersect transversally. By the Submanifold Transversal Intersection Theorem 42.1 we know that $X \cap Y$ is a submanifold. Given $P \in X \cap Y$ we have a natural short exact sequence induced by the various inclusion maps:

$$0 \;\to\; \mathrm{T}_P(X \cap Y) \;\to\; \mathrm{T}_P X \oplus \mathrm{T}_P Y \;\xrightarrow{\rho}\; \mathrm{T}_P M \;\to\; 0.$$

We pick a set of vectors $\mathcal{C}$ in $\mathrm{T}_P X \oplus \mathrm{T}_P Y$ such that $\rho(\mathcal{C})$ is a positive basis for $\mathrm{T}_P M$. We say that a basis $\mathcal{B}$ for $\mathrm{T}_P(X \cap Y)$ is *positive* if the basis $\{\mathcal{B}, \mathcal{C}\}$ is a positive basis for $\mathrm{T}_P X \oplus \mathrm{T}_P Y$. This definition is independent of the choices and it turns $X \cap Y$ into an oriented submanifold of M.

[1543] Recall that on page 675 we remarked that the orientation of a 0-dimensional manifold M is the same as a map $M \to \{\pm 1\}$.

Proof. We leave the elementary proof to the reader. ■

Example. Let $n \in \mathbb{N}$ and let $0 \leq k \leq l \leq n$. We consider $\mathbb{C}\mathrm{P}^n$ with the complex submanifolds

$$\begin{aligned} A &= \{[z_0 : \cdots : z_l : 0 : \cdots : 0] \,|\, (z_0, \ldots, z_l) \in \mathbb{C}^{l+1} \setminus \{0\}\} \\ B &= \{[0 : \cdots : 0 : z_k : \cdots : z_n] \,|\, (z_k, \ldots, z_n) \in \mathbb{C}^{n+1-k} \setminus \{0\}\} \end{aligned}$$

and

$$C = \{[0 : \cdots : 0 : z_k : \cdots : z_l : 0 : \cdots : 0] \,|\, (z_k, \ldots, z_l) \in \mathbb{C}^{l+1-k} \setminus \{0\}\}.$$

As usual we equip $\mathbb{C}\mathrm{P}^n$ and A, B, C with the natural orientations coming from the Complex Manifold-Orientation Proposition 32.23. Then $A \cap B = C$ and one can easily show that the orientation of $A \cap B$ equals the orientation of C.

The following theorem is a generalization of Theorem 126.15 and thus also of the Intersection Number-via-Cup Product Theorem 126.5.

Theorem 126.17. Let M be a compact oriented m-dimensional smooth manifold. Furthermore let $X \subset M$ be a closed oriented submanifold of codimension k and let Y be a compact oriented proper submanifold of codimension l. Then[1544]

$$\underbrace{\mathrm{PD}_M([X])}_{\in \mathrm{H}^k(M,\partial M)} \cup \underbrace{\mathrm{PD}_M([Y])}_{\in \mathrm{H}^l(M)} = \mathrm{PD}_M([X \cap Y]) \in \mathrm{H}^{k+l}(M, \partial M; \mathbb{Z}).$$

Proof. A proof is given in [**Bre1993**, Theorem VI.11.9]. ■

Remark.

(1) Theorem 126.17 can be summarized as saying that cup products correspond to taking intersections of submanifolds. With this point of view, the symbol "$\cup$" for the cup product does not seem to be a great choice.
(2) It is conceivable that the conclusion of Theorem 126.17 also holds if we have a decomposition $\partial M = A \cup B$ of the boundary and we have proper submanifold X and Y with $\partial X \subset A$ and $\partial Y \subset B$. We leave the task of verifying such a generalization to other authors.

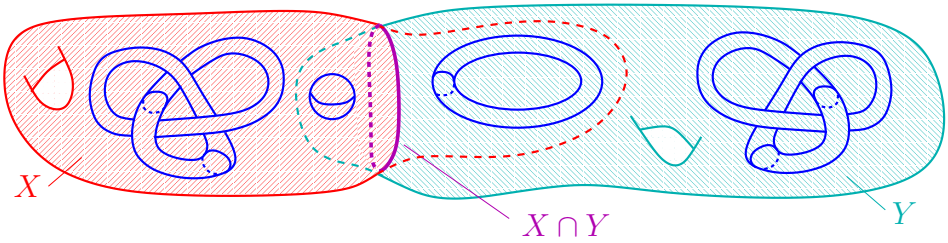
Examples.

(1) Let M be the smooth manifold that is the complement in S^3 of the three open solid tori and the open ball that are shown in the figure below. Furthermore let X and Y be the surfaces shown in that figure. Basically the same argument as on page 2755 shows that $[X]$ and $[Y]$ are non-trivial in $\mathrm{H}_2(M; \mathbb{Z})$. On the other hand the class $[X \cap Y] \in \mathrm{H}_1(M; \mathbb{Z})$ is clearly trivial. Therefore we deduce from Theorem 126.17 that $\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y]) = 0$.
(2) Using Theorem 126.17 and using the example preceding that theorem one can fairly easily give a completely geometric calculation of the cup product of $\mathbb{C}\mathrm{P}^n$.

Remark. Using Theorem 126.17 one can easily modify the proof of Proposition 126.14 to show that given any $n \geq 2$ every tubular neighborhood of $\mathbb{C}\mathrm{P}^{n-1}$ in $\mathbb{C}\mathrm{P}^n$ is non-trivial.

[1544]The sign should be taken with a slight grain of salt. It is conceivable that the sign depends on the various dimensions.

M is the complement in S^3 of three open solid tori and an open ball



Exercises for Chapter 126.

Exercise 126.1. Let M be a smooth manifold that admits two compact codimension-zero submanifolds A and B such that $A \cap B$ is a union of boundary components of A and a union of boundary components of B. By the Mayer–Vietoris Theorem 78.9 for Manifolds we obtain a long exact sequence

$$\cdots \to \mathrm{H}_n(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \xrightarrow{i_A + i_B} \mathrm{H}_n(M) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A \cap B) \to \dots$$

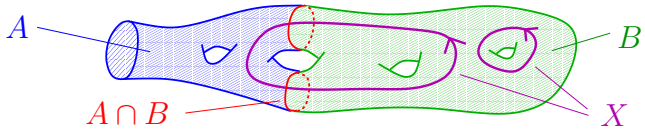
Let X be a closed oriented n-dimensional submanifold of M that intersects $A \cap B$ transversally.

(a) Show that $X \cap (A \cap B)$ is a closed $(n-1)$-dimensional submanifold of $A \cap B$.

We equip $X \cap (A \cap B)$ with the orientation coming from $X \cap (A \cap B) = \partial(X \cap A)$. We view the fundamental class $[X]$ as an element in $\mathrm{H}_n(M)$ and we view the fundamental class $[X \cap (A \cap B)]$ as an element in $\mathrm{H}_{n-1}(A \cap B)$.

(b) Show that $\partial_n([X]) = \pm[X \cap (A \cap B)] \in \mathrm{H}_{n-1}(A \cap B)$.

(c) Determine the correct sign in (b).



Exercise 126.2. Let A, B be two 1-dimensional oriented submanifolds of S^2 with $A \cdot B = 0$. Sketch a proof for the claim that A and B can be made disjoint via isotopies of A and B.

Exercise 126.3. Let M be a compact oriented m-dimensional smooth manifold. Furthermore let $k \in \{1, \dots, m-1\}$, let X be a proper oriented k-dimensional submanifold of M and let Y be a proper oriented $(m-k)$-dimensional submanifold of M. We denote by $i\colon X \to M$ and $j\colon Y \to M$ the inclusion maps. We could define the algebraic intersection number of X and Y via the formula in the Intersection Number-via-Cup Product Theorem 126.5, i.e. we could define

$$X \cdot Y \;:=\; \langle \mathrm{PD}_M(i_*([X]) \cup \mathrm{PD}_M(j_*([Y])), [M] \rangle.$$

What, if any, conditions do you need to impose on the boundaries of X and Y for this definition to make any sense?

Remark. This exercise can be viewed as a continuation of the discussion on page 2750.

Exercise 126.4. Let M be a closed oriented connected m-dimensional smooth manifold. Furthermore let $p\colon \widetilde{M} \to M$ be a finite covering.

(a) Let X, Y be a complementary pair of submanifolds of M. We write $\widetilde{X} := p^{-1}(X)$ and $\widetilde{Y} := p^{-1}(Y)$. What is the relationship between $X \cdot Y$ and $\widetilde{X} \cdot \widetilde{Y}$? It suffices to give a short justification.

(b) Let X be a closed oriented submanifold of M. What is the relationship between $\mathrm{PD}_{\widetilde{M}}([\widetilde{X}])$ and $p^*(\mathrm{PD}_M([X])$?

(c) Reprove (a) using (b) and the Intersection Number-via-Cup Product Theorem 126.5.

Exercise 126.5. Let M be an oriented m-dimensional smooth manifold, let X be a k-dimensional oriented proper submanifold and let Y be an $(m-k)$-dimensional oriented proper submanifold of M. We suppose that $X \cap Y \cap \partial M = \varnothing$. What is the relationship between the algebraic intersection number $X \cdot Y$ in M and $DX \cdot DY$ in $\mathrm{D}M$?
Here, as always, given a smooth manifold N we denote by $\mathrm{D}M = N \cup_{\partial N = \partial N'} N'$ the double of N. We view $\mathrm{D}M$ as an oriented smooth manifold where the orientation is chosen such that the orientation of $\mathrm{D}M$ agrees with the orientation on N.

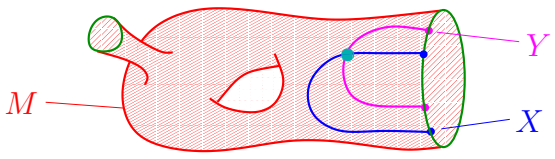
Exercise 126.6. Let $n \in \mathbb{Z}$. In Exercise 32.12 we showed that

$$W_n \;:=\; \{[x:y:z] \in \mathbb{C}\mathrm{P}^2 \,|\, x^n + y^n + z^n = 0\}$$

is a closed 1-dimensional complex submanifold of the complex projective space $\mathbb{C}\mathrm{P}^2$. Show that $[W_n] = n \cdot [\mathbb{C}\mathrm{P}^1] \in \mathrm{H}_2(\mathbb{C}\mathrm{P}^2; \mathbb{Z})$.
Hint. Use Lemma 126.10.

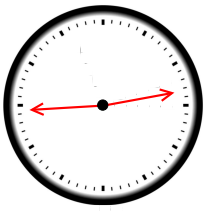
Exercise 126.7. Let $\Sigma \subset S^2 \times (-1,1)$ be a closed surface. Suppose that there exists a $P \in S^2$ such that Σ intersects $\{P\} \times [-1,1]$ in a single point. Show that if this intersection point is a transversal intersection point, then Σ intersects every path $\gamma\colon [0,1] \to S^2 \times [-1,1]$ from a point in $S^2 \times \{-1\}$ to a point in $S^2 \times \{1\}$.

Exercise 126.8. We consider the 2-dimensional smooth manifold M that is shown together with the two proper submanifolds X and Y that are shown in the figure below. Show that X and Y cannot be made disjoint by a smooth isotopy that keeps the boundaries of X and Y fixed.

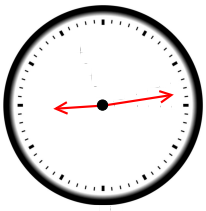


Exercise 126.9. By accident you ended up buying a clock where both hands have the same length, so you cannot distinguish the hand which shows the hour from the hand that indicates the minute. Fortunately, very often you can still tell the precise time. For example, in the configuration in the figure below it does not make sense for the hand that is indicating the hour to be just *before* 9 whereas the hand showing the minute is at 15.

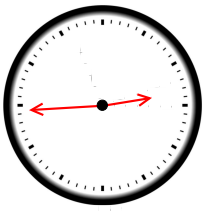
Of course there will be some configurations when it is indeed impossible to tell which hand is which. How many such confusions moments will there be in a 12 hour period?
Hint. Consider a suitable map $\gamma\colon S^1 \to S^1 \times S^1$.



clock where both hands have the same length



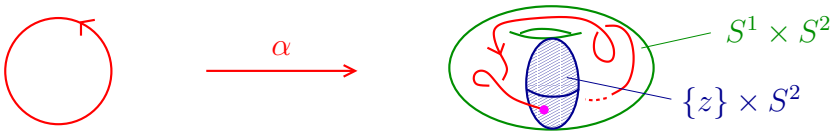
this makes no sense



the time is 2:44

Exercise 126.10. Let C and D be closed curves in $\mathbb{R}^2$ that intersect transversally. Give a rigorous proof that the number of intersection points between C and D is an even number.

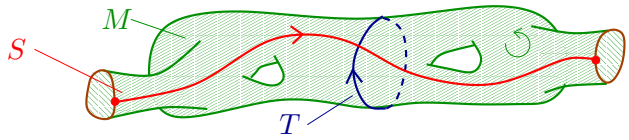
Exercise 126.11. Let $\alpha\colon S^1 \to S^1 \times S^2$ be a smooth embedding. We assume that there exists a $z \in S^1$ such that $\gamma(S^1)$ intersects the submanifold $\{z\} \times S^2$ in a single point P and we assume that the intersection point is transversal intersection point of sign $+1$. Show that $\alpha\colon S^1 \to S^1 \times S^2$ is smoothly isotopic to the embedding $\beta\colon S^1 \to S^1 \times S^2$ that is given by $t \mapsto (t, P)$.
Remark. This statement is sometimes referred to as the *light bulb trick*.



Exercise 126.12. Let M be a compact oriented 2-dimensional smooth manifold. We consider the following construction:

$$\begin{array}{rcl} \mathrm{H}_1(M, \partial M) \times \mathrm{H}_1(M, \partial M) & \to & \mathbb{Z} \\ (\sigma, \tau) & \mapsto & \text{algebraic intersection number } S \cdot T \text{ of proper transverse} \\ & & \text{submanifolds } S \text{ and } T \text{ of } M \text{ with } [S] = \sigma \text{ and } [T] = \tau. \end{array}$$

Does this give a well-defined map?



Exercise 126.13. We consider the surface Σ of genus 2 together with the four oriented curves a_1, b_1, a_2, b_2 that are shown in the figure below.

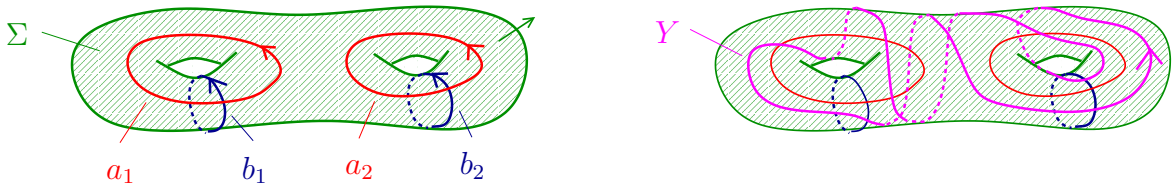
(a) Let X be a closed oriented 1-dimensional submanifold of Σ that intersects a_1, b_1, a_2, b_2 transversally. Show that

$$[X] \;=\; (X \underset{\uparrow}{\cdot} b_1) \cdot [a_1] \;-\; (X \underset{\uparrow}{\cdot} a_1) \cdot [b_1] \;+\; (Y \underset{\uparrow}{\cdot} b_2) \cdot [a_2] \;-\; (X \underset{\uparrow}{\cdot} a_2) \cdot [b_2] \;\in\; \mathrm{H}_1(\Sigma).$$

these are the algebraic intersection numbers of the given oriented curves

(b) Let X and Y be two closed oriented 1-dimensional submanifolds of Σ that are both transverse to a_1, a_2, b_1, b_2. Show that if X and Y have the same algebraic intersection numbers with a_1, a_2, b_1, b_2, then $[X] = [Y] \in \mathrm{H}_1(\Sigma)$.

Remark. The above gives a practical way, for Σ and similarly for many other smooth manifolds, to determine precisely the homology class represented by a submanifold. In particular it can be used to deal with the type of problems as in Exercise 80.16.



127. Intersections of cycles and the cup product

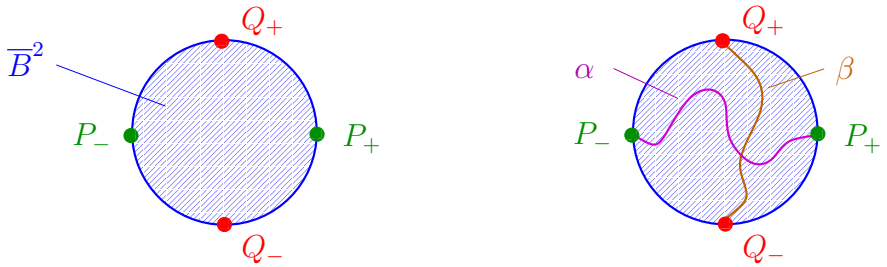
In this chapter we generalize some of the results from Chapter 126. In particular we will define the "algebraic intersection number" of two maps of topological manifolds into a given topological manifold, as long as the maps intersect "nicely". This notion generalizes the algebraic intersection number of two transverse submanifolds of a given smooth manifold that we introduced on page 2748.

Furthermore we introduce the algebraic intersection number of two singular cycles that intersect "nicely". Afterwards we will see in Theorems 127.7 and 127.9 how these algebraic intersection numbers relate to the cup product. These theorems can be viewed as a generalization of the Intersection Number-via-Cup Product Theorem 126.5.

For many applications the results from Chapter 126 are good enough and for many readers it might make sense to move on to the next chapter. The main application of the results of this chapter will be the Lefschetz-Hopf Theorem 128.9 and the Poincaré-Hopf Theorem 129.7.

Since much of this chapter is quite technical we start out with the following motivating question.

Question 127.1. As shown in the figure below we consider the disk $\overline{B}^2$ together with the four points $P_\pm = (\pm 1, 0)$ and $Q_\pm = (0, \pm 1)$. Do there exist disjoint paths α and β such that α connects P_- to P_+ and β connects Q_- to Q_+?



It is pretty clear that the answer should be no. In fact once we have developed all the tools we will find it quite easy to answer Question 127.1 in the negative. We refer to Lemma 127.8 for the details.

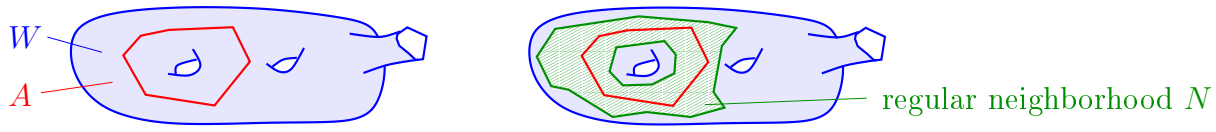
127.1. The algebraic intersection number of maps of topological manifolds.

In this section we introduce the algebraic intersection number of two maps of topological manifolds into a given topological manifold. The precise setup and the definitions require some preparations. First we recall the following definition from page 2695.

Definition. Let W be a compact topological manifold and let $A \subset W \setminus \partial W$ be a subset. A regular neighborhood of A is a compact codimension-zero submanifold N of W with the following properties:

(1) N is contained in $W \setminus \partial W$ and A is contained in the interior of N.
(2) A is a deformation retract of N.
(3) ∂N is a deformation retract of $N \setminus A$.

The following proposition can be viewed as a variation on the Poincaré Duality Theorem 119.1.



Proposition 127.2. Let Z be a compact oriented m-dimensional topological manifold and let A be a closed submanifold of ∂Z.

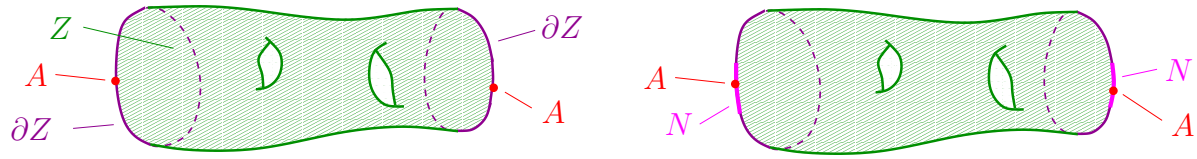
(1) Let $N \subset \partial Z$ be a regular neighborhood of A. We denote by $j\colon A \to N$ the inclusion map. Given any $l \in \mathbb{N}_0$ the following two maps are isomorphisms:[1545]

$$\mathrm{H}^l(Z, \partial Z \setminus A) \xrightarrow[\cong]{\cap[Z]} \mathrm{H}_{m-l}(Z, N) \xleftarrow[\cong]{j_*} \mathrm{H}_{m-l}(Z, A).$$

(2) We make the following technical hypothesis:
(∗) Given any neighborhood U of $A \subset \partial Z$ there exists a regular neighborhood N of $A \subset \partial Z$ with $N \subset U$.
Under this hypothesis the isomorphism

$$\mathrm{PD}_Z\colon \mathrm{H}^l(Z, \partial Z \setminus A) \xrightarrow{\cong} \mathrm{H}_{m-l}(Z, A)$$

given in (1) does not depend on the choice of the regular neighborhood N. We denote the inverse of this map by PD_Z as well.



Proof.

(1) Let $N \subset \partial Z$ be a regular neighborhood of A. We denote by $i\colon \partial Z \setminus \mathring{N} \to \partial Z \setminus A$ and $j\colon A \to N$ the inclusion maps. We consider the following diagram:

$$\begin{array}{ccccc} \mathrm{H}^l(Z, \partial Z \setminus A) & & \xrightarrow[\cong]{\cap[Z]} & & \mathrm{H}_{m-l}(Z, N) \xleftarrow[\cong]{j_*} \mathrm{H}_{m-l}(Z, A). \\ & \searrow^{i_*} & & \nearrow_{\cong}^{\cap[Z]} & \\ & & \mathrm{H}^l(Z, \partial Z \setminus \mathring{N}) & & \end{array}$$

We make the following observations:
(a) The diagonal cap product exists by Proposition 75.3 (4) together with the Excisive Triad-Proposition 109.12 (4).
(b) It follows from the Cap Product-Naturality Lemma 116.8 (1) that the triangle commutes.
(c) It follows from the hypothesis that N is a regular neighborhood, together with Lemma 123.1 (1), that the maps i and j are both homotopy equivalences. Thus

[1545] It follows from the Excisive Triad-Proposition 109.12 (3) that the cap product $\cap[Z]\colon \mathrm{H}^l(Z, \partial Z \setminus A) \to \mathrm{H}_{m-l}(Z, N)$ is actually defined. It is also worth pointing out that it is not possible to go directly from $\mathrm{H}^l(Z, \partial Z \setminus A)$ to $\mathrm{H}_{m-l}(Z, A)$ via capping with $[Z]$, since $(\partial Z, \partial Z \setminus A, A)$ is not an excisive triad.

we obtain from Corollary 74.16 and the Homotopic Maps-H*-Lemma 108.13 that the induced maps i^* and j_* are isomorphisms.

(d) It follows from the Poincaré Duality Theorem 119.1, which we can apply by Proposition 75.3 (4), that the right diagonal map is an isomorphism.

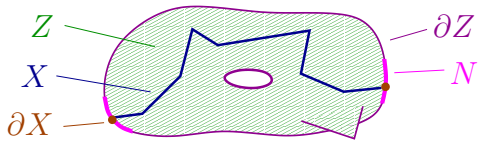
The above argument shows that the two horizontal maps are both isomorphisms.

(2) We need to show that the map in (1) does not depend on the choice of the regular neighborhood N. Thus let M and N be regular neighborhoods of $A \subset \partial Z$. It follows immediately from our hypothesis $(*)$ that it suffices to consider the case that $M \subset N$. We consider the following diagram:

$$\begin{array}{ccccc} & & \mathrm{H}_{m-l}(Z,M) & & \\ & \overset{\cap[Z]}{\nearrow}\cong & \downarrow & \overset{\cong}{\nwarrow} & \\ \mathrm{H}^l(Z,\partial Z\setminus A) & & & & \mathrm{H}_{m-l}(Z,A). \\ & \underset{\cap[Z]}{\searrow}\cong & \downarrow & \underset{\cong}{\swarrow} & \\ & & \mathrm{H}_{m-l}(Z,N) & & \end{array}$$

Here all unmarked maps are induced by inclusions. The triangle to the left commutes by the Cap Product-Naturality Lemma 116.8 and the triangle to the right commutes by the functoriality of cohomology groups. But this shows that the upper and the lower path from the left to the right give exactly the same map. ■

Definition. Let Z be a topological manifold. We say a proper submanifold X of Z is well-behaved if given any open neighborhood U of $\partial X \subset \partial Z$ the submanifold ∂X of ∂Z admits a regular neighborhood N with $N \subset U$.



Remark. Almost any proper submanifold X of some topological manifold Z that appears in nature is well-behaved. For example it follows from the Simplicial Regular Neighborhood Proposition 96.13 and Theorem 123.3 that X is well-behaved if ∂X is a PL-submanifold of ∂Z or if ∂X is a smooth submanifold of ∂Z.

In the following we will study maps between topological manifolds. To simplify the discussion we use the following convention.

Convention. Let M be a topological manifold and let X be a submanifold of M. In our notation and our language often we do not distinguish between X as a submanifold and between the embedding $X \to M$.

Next we fix a convenient notation.

Notation. Given a map $f\colon Y \to X$ between topological spaces we write $|f| := f(Y)$.

Now we can give the first key definition of this chapter.

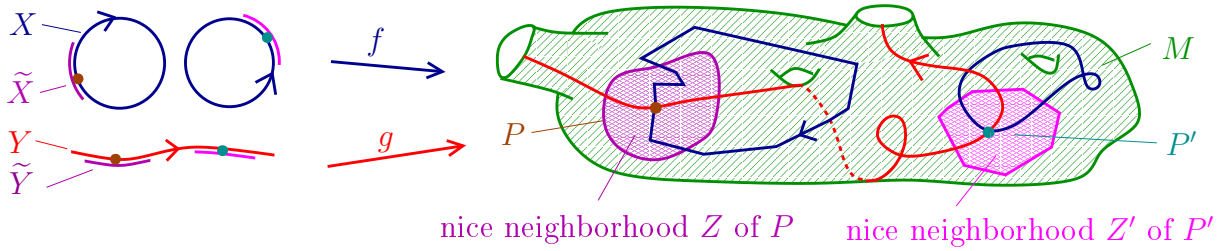
Definition. Let M be an m-dimensional topological manifold, let X be a k-dimensional topological manifold and let Y be an $(m-k)$-dimensional topological manifold. Furthermore let $f\colon X \to M$ and $g\colon Y \to M$ be maps.

(1) We say a point $P \in |f| \cap |g|$ is a nice intersection point of f and g if given any neighborhood U of P there exists a nice neighborhood Z, this means a submanifold $Z \subset M \setminus \partial M$ such that the following holds:
   (a) Z is homeomorphic to $\overline{B}^m$.
   (b) P is contained in[1546] $\mathring{Z} = Z \setminus \partial Z$ and Z is contained in U.
   (c) $|f| \cap |g| \cap Z = \{P\}$.

   Before we continue we set $\widetilde{X} := f^{-1}(|f| \cap Z)$ and we set $\widetilde{Y} := g^{-1}(|g| \cap Z)$. With this notation we also demand the following:
   (d) $\widetilde{X}$ is a compact codimension-zero submanifold of X and $\widetilde{Y}$ is a compact codimension-zero submanifold of Y.
   (e) The maps $f\colon \widetilde{X} \to Z$ and $g\colon \widetilde{Y} \to Z$ are both proper embeddings.
   (f) $f(\widetilde{X})$ and $g(\widetilde{Y})$ are both well-behaved submanifolds of Z.
   (g) $f(\widetilde{X})$ is homeomorphic to $\overline{B}^k$ and $g(\widetilde{Y})$ is homeomorphic to $\overline{B}^{m-k}$.

   In the figure below we illustrate the definition for two different nice intersection points.

(2) We say that f and g intersect nicely, if all points in $|f| \cap |g|$ are nice intersection points.



Remark. By the above convention we now also defined the notion of a nice intersection point of two submanifolds and we defined the notion that two submanifolds intersect nicely.

Convention. Let W be an oriented topological manifold. We equip any codimension-zero submanifold of W with the orientation coming from the Topological Manifold-Induced Orientation Lemma 105.17.

Now we are ready to give the second key definition of this chapter.

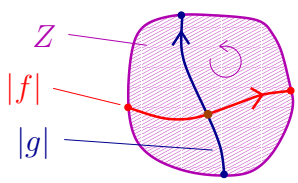
Definition. Let M be an oriented m-dimensional topological manifold, let X be an oriented k-dimensional topological manifold and let Y be an oriented $(m-k)$-dimensional topological manifold. Furthermore let $f\colon X \to M$ and $g\colon Y \to M$ be maps and let P be a nice intersection point of f and g. We pick a nice neighborhood Z of P. We define the

[1546] Note that by (a) we know that Z is a codimension-zero submanifold of M. Since $Z \subset M \setminus \partial M$ we know by Proposition 75.3 that the boundary of Z as a topological manifold agrees with the boundary of Z as a subset of M.

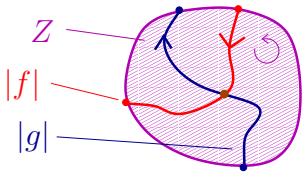
index of f and g at P as follows:[1547]

$$\operatorname{index}(f,g,P) \;=\; \Big\langle \overbrace{\underbrace{\mathrm{PD}_Z(\underbrace{f_*([\widetilde{X}])}_{\in \mathrm{H}_k(Z,\partial f(\widetilde{X}))})}_{\in \mathrm{H}^{m-k}(Z,\partial Z\setminus \partial f(\widetilde{X}))} \cup \underbrace{\mathrm{PD}_Z(\underbrace{g_*([\widetilde{Y}])}_{\in \mathrm{H}_{m-k}(Z,\partial(g(\widetilde{Y})))})}_{\in \mathrm{H}^{k}(Z,\partial Z\setminus \partial g(\widetilde{Y}))}}^{\in \mathrm{H}^m(Z,\partial Z)}, \overbrace{[Z]}^{\in \mathrm{H}_m(Z,\partial Z)} \Big\rangle \in \mathbb{Z}.$$

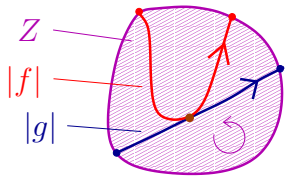
Example. In the figure below we show, in three cases, the indices of nice intersection points. The first two indices can be determined with some effort using the Poincaré Duality-Products Proposition 119.2 together with Proposition 113.11. Alternatively see Proposition 127.5 below. Finally we outsource the calculation of the third index to Exercise 127.2.



$\operatorname{index}(f,g,P) = +1$



$\operatorname{index}(f,g,P) = -1$



$\operatorname{index}(f,g,P) = 0$

Next we need to show that the index of a nice intersection point is well-defined.

Lemma 127.3. Let M, $f\colon X \to M$ and $g\colon Y \to M$ as in the previous definition and let P be a nice intersection point of f and g. The corresponding index $\operatorname{index}(f,g,P)$ is well-defined, i.e. it is independent of the choice of the nice neighborhood Z of P.

Sketch of a proof. By the "given any neighborhood U of P" aspect of the definition of a nice intersection point it suffices to deal with nice neighborhoods Z_1 and Z_2 with $Z_1 \subset \mathring{Z}_2$. We define $\widetilde{X}_1$, $\widetilde{X}_2$, $\widetilde{Y}_1$ and $\widetilde{Y}_2$ in the obvious way. We pick regular neighborhoods A_1 of $\partial X_1 \subset \partial Z_1$ and B_1 of $\partial Y_1 \subset \partial Z_1$. Similarly we pick A_2 and B_2 with the obvious meaning.

Let $i \in \{1,2\}$. We write $C_i := \partial Z_i \setminus \mathring{A}_i$ and $D_i := \partial Z_i \setminus \mathring{B}_i$. Similar to the proof of the Excision Theorem for Topological Manifolds 75.10 we use the Topological Collar Neighborhood Theorem 75.5 to find open subsets $\widetilde{A}_i$, $\widetilde{B}_i$, $\widetilde{C}_i$ and $\widetilde{D}_i$ of ∂Z_i that admit deformation retractions to A_i, B_i, C_i respectively D_i. Now we consider the following maps

[1547] Here are a few clarifications:

(1) By hypothesis we know that Z, $\widetilde{X}$ and $\widetilde{Y}$ are compact codimension-zero submanifolds of M, X and Y. As we just remarked, by the Topological Manifold-Induced Orientation Lemma 105.17 they inherit an orientation from the given orientations on M, X and Y. As always we denote by $[Z]$, $[\widetilde{X}]$ and $[\widetilde{Y}]$ the corresponding fundamental classes.

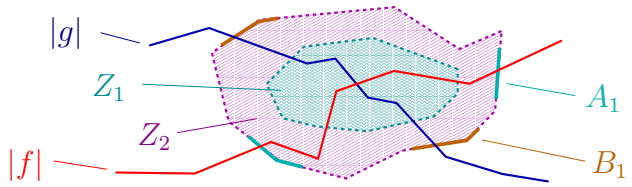
(2) By hypothesis $f(\widetilde{X})$ and $g(\widetilde{Y})$ are well-behaved submanifolds of Z. Thus we obtain from Proposition 127.2 the corresponding maps PD_Z.

(3) By hypothesis $f(\widetilde{X})$ and $g(\widetilde{Y})$ are compact. It follows from the Topological Manifold Boundary Proposition 75.2 (4b) that $\partial f(\widetilde{X})$ and $\partial g(\widetilde{Y})$ are compact. By the Compact-Closed Lemma 1.21 (2) this implies that $\partial Z \setminus \partial f(\widetilde{X})$ and $\partial Z \setminus \partial g(\widetilde{Y})$ are open subsets of ∂Z. Since P is the only intersection point of f and g in Z we see that these two open subsets cover all of ∂Z. It now follows from the Excisive Triad-Proposition 109.12 (4) that the cup product is actually defined.

of triads of topological spaces:

$$\begin{array}{ccccccccc} Z_1 & & Z_1 & & Z_2 & & Z_2 & & Z_2 \\ A_1 & \to & \widetilde{A}_1 & \to & \widetilde{A}_1 \cup ((\mathring{Z}_2 \setminus Z_1) \cup \widetilde{A}_2) & \leftarrow & \widetilde{A}_2 & \leftarrow & A_2 \\ \partial Z_1 \setminus A_1 & & \partial Z_1 \setminus A_1 & & (\partial Z_1 \setminus A_1) \cup (\mathring{Z}_2 \setminus Z_1) \cup (\partial Z_2 \setminus A_2) & & \partial Z_2 \setminus A_2 & & \partial Z_1 \setminus A_2. \end{array}$$

It follows from the Excisive Triad-Proposition 109.12 (4) that all these triads are excisive. Evidently the analogous statement also holds if we replace the A's by B's. Now the lemma can be deduced from the Cup Product-Naturality Proposition 115.4 and the Cap Product-Naturality Lemma 116.8 together with the Codimension Zero-Fundamental Class Lemma 106.24. We leave it to the reader to fill in the details. ■



The following lemma is a generalization of Lemma 126.2.

Lemma 127.4. Let M be an m-dimensional topological manifold, let X be a compact k-dimensional topological manifold and let Y be a compact $(m-k)$-dimensional topological manifold. Furthermore let $f\colon X \to M$ and $g\colon Y \to M$ be maps. If f and g intersect nicely, then $|f| \cap |g|$ is finite.

Proof. Let P be a nice intersection point of f and g. We pick a nice neighborhood Z for P. Note that this means in particular that Z is a codimension-zero submanifold of M with $Z \subset M \setminus \partial M$, $P \in Z \setminus \partial Z$ and with $|f| \cap |g| \cap Z = \{P\}$. By Proposition 75.3 (3) we know that $Z \setminus \partial Z$ is an open subset of M. We have thus shown that P is an isolated point, in the sense of the definition on page 235, of the subspace $|f| \cap |g|$. Since by hypothesis every intersection point is nice we see that $|f| \cap |g|$ is a discrete subset of M. Since X and Y are compact we know by the Compact Image Lemma 2.13 together with the Compact-Union-Intersection Lemma 1.20 (2) that $|f| \cap |g|$ is also compact. Thus we see that $|f| \cap |g|$ is a discrete compact space. By the Compact-Discrete Lemma 1.22 (1) this implies that $|f| \cap |g|$ is a finite set. ■

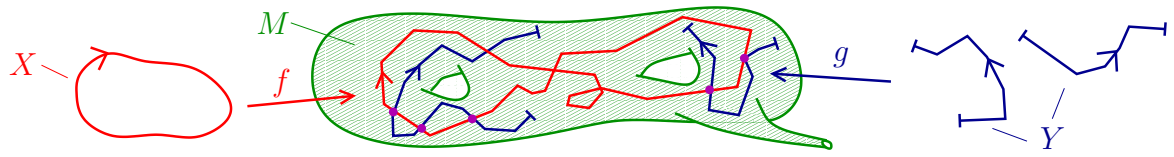
By Lemma 127.4 it now makes sense to introduce the following definition which can be viewed as an analogue of the definition on page 2748.

Definition. Let M be an oriented m-dimensional topological manifold. Let X be a compact oriented k-dimensional topological manifold and let Y be a compact oriented $(m-k)$-dimensional topological manifold. Furthermore let $f\colon X \to M$ and $g\colon Y \to M$ be maps. If f and g intersect nicely, then we define the algebraic intersection number of f and g to be

$$f \cdot g \;:=\; \sum_{P \in |f| \cap |g|} \operatorname{index}(f, g, P).$$

↑ finite sum by Lemma 127.4

Remark. Recall that by our convention on page 2777 we do not distinguish between a submanifold X of M and the actual embedding $X \to M$. In particular in the above definition



we also introduced the algebraic intersection number $X \cdot Y$ of two suitable submanifolds of M.

127.2. **Transverse intersection points of maps.** In this section we will see that the notion of a nice intersection points incorporates the notion of a transverse intersection point that we introduced on pages 2748. We will also see that the concept of the algebraic intersection number that we defined in the previous section is a generalization of the algebraic intersection number that we introduced on page 2748.
The following definition can be viewed as a generalization of the concepts we introduced on pages 968 and 2748.

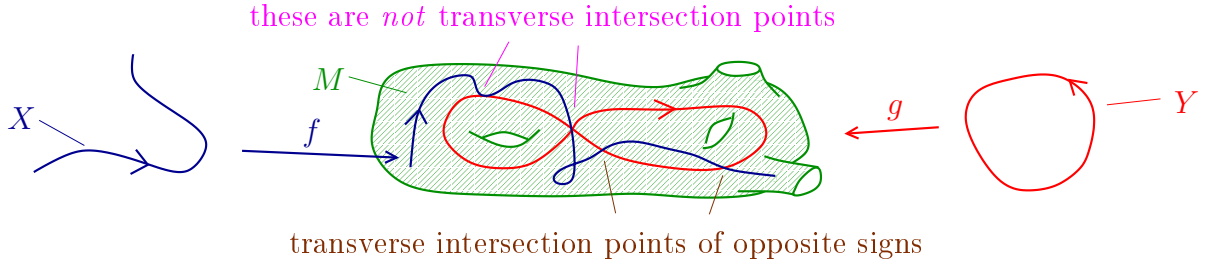
Definition. Let M be an m-dimensional smooth manifold and let $k \in \{1, \dots, m-1\}$. Furthermore let X be a k-dimensional smooth manifold and let Y be an $(m-k)$-dimensional smooth manifold. Furthermore let $f\colon X \to M$ and $g\colon Y \to M$ be smooth maps.

(1) We say a point $P \in |f| \cap |g|$ is a transverse intersection point of f and g if $P \in M \setminus \partial M$, if $f^{-1}(P)$ contains a single point x_0, if $g^{-1}(P)$ contains a single point y_0, if $x_0 \in X \setminus \partial X$ and $y_0 \in Y \setminus \partial Y$, and if $f_*(\mathrm{T}_{x_0}X) + g_*(\mathrm{T}_{y_0}Y) = \mathrm{T}_P M$.

(2) Now we assume that M, X and Y are oriented. Let P be a transverse intersection point of f and g. We set $\operatorname{sign}(f, g, P) := +1$ if

$$(f_*(\text{positive basis of } \mathrm{T}_{x_0}X), g_*(\text{positive basis of } \mathrm{T}_{y_0}Y)) \;=\; \text{positive basis of } \mathrm{T}_P M,$$

and otherwise we set $\operatorname{sign}(f, g, P) := -1$.



Remark. If X and Y are compact proper submanifolds of M and if f and g are just the inclusion maps, then the above definitions are precisely the definitions provided on pages 968 and 2748.

Proposition 127.5. Let M be an m-dimensional smooth manifold and let $k \in \{1, \dots, m-1\}$. Furthermore let X be a k-dimensional smooth manifold and let Y be an $(m-k)$-dimensional smooth manifold. Finally let $f\colon X \to M$ and $g\colon Y \to M$ be smooth maps.

(1) Let $P \in |f| \cap |g|$. If P is a transverse intersection point of f and g, then P is a nice intersection point of f and g.

(2) If M, X and Y are oriented, then for any transverse intersection point P of f and g we have $\text{index}(f,g,P) = \text{sign}(f,g,P)$.

Proof. The alert reader will doubtlessly notice that the proof of the proposition mostly recycles ideas from the proof of Proposition 126.8 and Lemma 126.9.

Let U be an open neighborhood of P. It follows from the Local Smooth Embedding Theorem 24.8 that there exists an open neighborhood A of x_0 such that the restriction of f to $f\colon A \to M$ is a smooth embedding and such that $f(A) \subset U$. Furthermore, using the hypothesis that $f^{-1}(P) = \{x_0\}$ and using Exercise 18.13 we see that, after possibly shrinking A, we can assume that $f^{-1}(f(A)) = A$.

The same way we can evidently find an open neighborhood B of y_0 such that $g\colon B \to M$ is a smooth embedding, such that $g(B) \subset U$ and such that $g^{-1}(g(B)) = B$.

By the Smooth Embedding Theorem 24.10 we know that $f(A)$ and $g(B)$ are smooth submanifolds of M. By our hypothesis we know that P is a transverse intersection point of $f(A)$ and $g(B)$. It follows from the Submanifold Transversal Intersection Theorem 42.1 that there exists a chart $\Phi\colon V \to W$ for M around P with

$$\Phi(V \cap f(X)) \;=\; W \cap (\mathbb{R}^k \times \{0\}) \quad \text{and} \quad \Phi(V \cap g(Y)) \;=\; W \cap (\{0\} \times \mathbb{R}^{m-k}).$$

Next note that after possibly rescaling W we can assume that $\overline{B}^k \times \overline{B}^{m-k} \subset W$. We set $Z := \Phi^{-1}(\overline{B}^k \times \overline{B}^{m-k})$ and $\widetilde{X} := f^{-1}(Z)$ and $\widetilde{Y} := g^{-1}(Z)$. Note that by construction the maps $\Phi \circ f\colon \widetilde{X} \to \overline{B}^k \times \{0\}$ and $\Phi \circ g\colon \widetilde{Y} \to \{0\} \times \overline{B}^{m-k}$ are diffeomorphisms. It follows easily from the construction that Z has all the properties stated in the definition of a nice intersection point. This concludes the proof of Statement (1).

Now assume that M, X and Y are oriented. We set $\sigma := \text{sign}(f,g,P)$. We make the following preparations:

(1) We equip each $\overline{B}^n$ with the usual orientation coming from $\mathbb{R}^n$.

(2) After possibly composing $\Phi\colon Z \to \overline{B}^k \times \overline{B}^{m-k}$ with reflections in some $(x_i = 0)$-hyperplanes we can assume that the diffeomorphisms $\Phi \circ f\colon f^{-1}(Z) = \widetilde{X} \to \overline{B}^k \times \{0\}$ and $\Phi \circ g\colon g^{-1}(Z) = \widetilde{Y} \to \{0\} \times \overline{B}^{m-k}$ are both orientation-preserving.

(3) We equip the topological manifold $\overline{B}^k \times \overline{B}^{m-k}$ with the product orientation as defined on page 2502. It follows from Proposition 113.11 that with this orientation the inclusion map $\overline{B}^k \times \overline{B}^{m-k} \to \mathbb{R}^m$ is orientation-preserving.

(4) Elementary linear algebra shows that (2) implies that $\Phi\colon Z \to \overline{B}^k \times \overline{B}^{m-k}$ is orientation-preserving if and only if $\sigma = +1$.

(5) It follows from Proposition 107.2 and Lemma 105.12 that $\Phi_*([Z]) = \sigma \cdot [\overline{B}^k \times \overline{B}^{m-k}]$.

Now we perform the following calculation:

$$\overbrace{\big\langle \text{PD}_Z(f_*[\widetilde{X}]) \cup \text{PD}_Z(g_*[\widetilde{Y}]), [Z] \big\rangle}^{=\text{index}(f,g,P),\ \text{by definition}} \overset{\text{by Lemma 109.6 (3) and the Cup Product-Naturality Lemma 114.10}}{\downarrow}$$

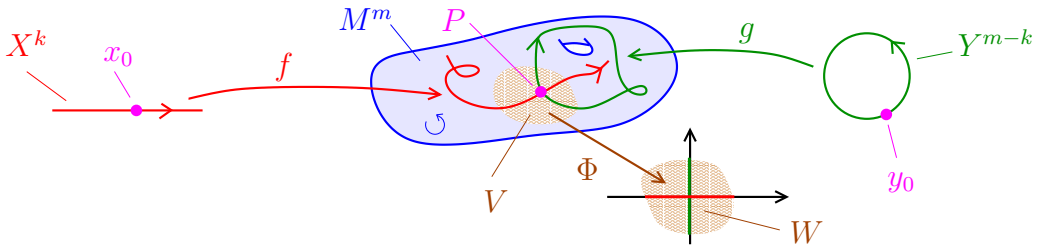
$$= \big\langle \Phi^*(\text{PD}_Z(f_*[\widetilde{X}])) \cup \Phi^*(\text{PD}_Z(g_*[\widetilde{Y}])), \Phi_*([Z]) \big\rangle$$

$$= \big\langle \sigma \cdot \text{PD}_{\overline{B}^k \times \overline{B}^{m-k}}([\overline{B}^k \times 0]) \cup \sigma \cdot \text{PD}_{\overline{B}^k \times \overline{B}^{m-k}}([0 \times \overline{B}^{m-k}])), \sigma \cdot [\overline{B}^k \times \overline{B}^{m-k}] \big\rangle$$

$$\overset{\text{by (4) and the pretty elementary Exercise 119.16}}{\uparrow}$$

$$= \sigma^3 \cdot \underbrace{\big\langle \text{PD}_{\overline{B}^k \times \overline{B}^{m-k}}([\overline{B}^k \times 0]) \cup \text{PD}_{\overline{B}^k \times \overline{B}^{m-k}}([0 \times \overline{B}^{m-k}])), [\overline{B}^k \times \overline{B}^{m-k}] \big\rangle}_{=1 \text{ by the Poincaré Duality-Products Proposition 119.2}} = \underbrace{\text{sign}(f,g,P)}_{=\sigma}.$$

■



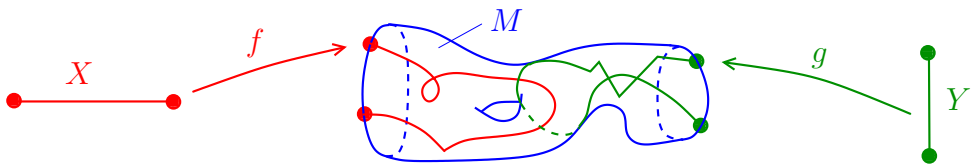
The following is an immediate corollary to Proposition 127.5.

Corollary 127.6. Let M be an oriented m-dimensional smooth manifold, let X be an oriented k-dimensional submanifold and let Y be an oriented $(m-k)$-dimensional submanifold of M. We denote by $i\colon X \to M$ and $j\colon Y \to M$ the inclusion maps. If the submanifolds are transverse in the sense of the definition on page 968, then the maps $i\colon X \to M$ and $j\colon Y \to M$ are intersect nicely and the two definitions of the algebraic intersection given on pages 2748 and 2780 agree.

By Corollary 127.6 the following theorem is a generalization of the Intersection Number-via-Cup Product Theorem 126.5.

Theorem 127.7. Let M be a compact oriented connected m-dimensional topological manifold. Furthermore let A and B be two compact $(m-1)$-dimensional submanifolds of ∂M with $\partial A = \partial B = A \cap B$ and with $\partial M = A \cup B$. Let X be a compact oriented k-dimensional topological manifold and let Y be a compact oriented $(m-k)$-dimensional topological manifold. Finally let $f\colon X \to M$ and $g\colon Y \to M$ be two proper maps with $f(\partial X) \subset A$ and $g(\partial Y) \subset B$. If f and g intersect nicely, then

$$f \cdot g = \Big\langle \overbrace{\underbrace{\mathrm{PD}_M(\underbrace{f_*([X])}_{\in \mathrm{H}_k(M,A)})}_{\in \mathrm{H}^{m-k}(M,B)} \cup \underbrace{\mathrm{PD}_M(\underbrace{g_*([Y])}_{\in \mathrm{H}_{m-k}(M,B)})}_{\in \mathrm{H}^{k}(M,A)}}^{\in \mathrm{H}^m(M,\partial M)}, \overbrace{[M]}^{\in \mathrm{H}_m(M,\partial M)} \Big\rangle \in \mathbb{Z}.$$



Remark. It follows immediately from Proposition 127.5 that Theorem 127.7 contains our earlier the Intersection Number-via-Cup Product Theorem 126.5 as a special case.

In Theorem 127.14 below we will give a result which takes more time to formulate and which implies the above Theorem 127.7. To keep at least the flow of the definitions and statements coherent we decided to formulate Theorem 127.14 in this section. But rather awkwardly this means that we now provide a proof of Theorem 127.7 that relies on the future Theorem 127.14.

Proof of Theorem 127.7 using Theorem 127.14. It follows easily from Lemma 127.12 that there exists a k-cycle $\sigma \in \mathrm{C}_k(M, A)$ and an $(m-k)$-cycle $\tau \in \mathrm{C}_{m-k}(M, B)$ with the following properties:

(1) we have $[\sigma] = f_*([X])$ and $[\tau] = g_*([Y])$,
(2) the cycles σ and τ are transverse,
(3) the equality $f \cdot g = \sigma \cdot \tau$ holds.

It follows that

$$f \cdot g \;=\; \sigma \cdot \tau \;\underset{\substack{\uparrow \\ \text{Theorem 127.14}}}{=}\; \langle \mathrm{PD}_M([\sigma]) \cup \mathrm{PD}_M([\tau]), [M] \rangle \;\underset{\substack{\uparrow \\ \text{since } [\sigma] = f_*([X]) \text{ and } [\tau] = g_*([Y])}}{=}\; \langle \mathrm{PD}_M(f_*[X]) \cup \mathrm{PD}_M(g_*[Y]), [M] \rangle.$$

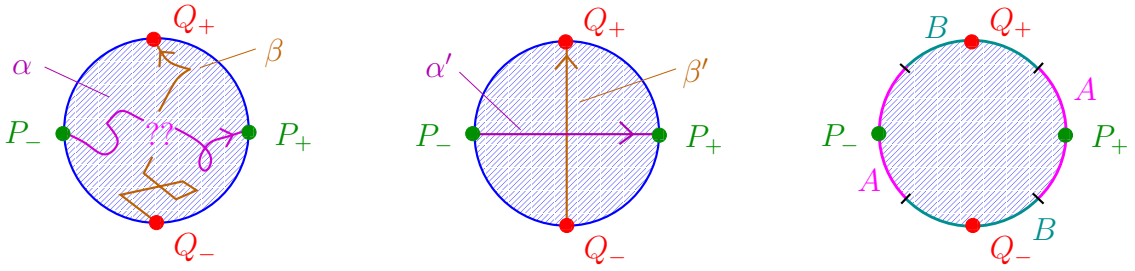
■

As promised, we now find it quite easy to answer Question 127.1 in the negative.

Lemma 127.8. We consider the disk $\overline{B}^2$ together with the four points $P_\pm = (\pm 1, 0)$ and $Q_\pm(0, \pm 1)$. If $\alpha\colon [-1,1] \to \overline{B}^2$ is a proper map with $\alpha(\pm 1) = P_\pm$ and if $\beta\colon [-1,1] \to \overline{B}^2$ is a proper map with $\beta(\pm 1) = Q_\pm$, then α and β intersect in at least one point.

Proof. We write $I = [-1, 1]$. Suppose that there exist proper maps $\alpha\colon I \to \overline{B}^2$ and $\beta\colon I \to \overline{B}^2$ with $\alpha(\pm 1) = P_\pm$ and with $\beta(\pm 1) = Q_\pm$ such that $|\alpha| \cap |\beta| = \varnothing$. We introduce the following notation:

(1) We consider the "straight paths" $\alpha'\colon I \to \overline{B}^2$ given by $\alpha'(t) = (t, 0)$ and $\beta'\colon I \to \overline{B}^2$ given by $\beta'(t) = (0, t)$.
(2) We write $A = \left\{ \exp(\mathrm{i}t) \,\middle|\, t \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right] \right\}$ and $B = \left\{ \exp(\mathrm{i}t) \,\middle|\, t \in \left[-\frac{3\pi}{4}, -\frac{\pi}{4}\right] \cup \left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \right\}$.
(3) We denote by $\mathrm{PD}_{\overline{B}^2}$ the isomorphisms from the Poincaré Duality Theorem 119.1 corresponding to the decomposition $\partial \overline{B}^2 = A \cup B$.



We perform the following calculation:

$$0 \;\underset{\substack{\downarrow \\ \text{since } |\alpha| \cap |\beta| = \varnothing}}{=}\; \alpha \cdot \beta \;\underset{\substack{\downarrow \\ \text{Theorem 127.7}}}{=}\; \left\langle \mathrm{PD}_{\overline{B}^2}(\alpha_*([I])) \cup \mathrm{PD}_{\overline{B}^2}(\beta_*([I]), [\overline{B}^2] \right\rangle$$

$$\underset{\substack{\uparrow \\ \text{by considering the long exact sequences in homology} \\ \text{of the pairs } (\overline{B}^2, A) \text{ and } (\overline{B}^2, B) \text{ we see that} \\ \alpha_*([I]) = \alpha'_*([I]) \text{ and } \beta_*([I]) = \beta'_*([I])}}{=}\; \left\langle \mathrm{PD}_{\overline{B}^2}(\alpha'_*([I])) \cup \mathrm{PD}_{\overline{B}^2}(\beta'_*([I]), [\overline{B}^2] \right\rangle \;\underset{\substack{\uparrow \\ \text{Theorem 127.7}}}{=}\; \alpha' \cdot \beta' \;\underset{\substack{\uparrow \\ \alpha', \beta' \text{ intersect} \\ \text{nicely in one} \\ \text{point}}}{=}\; \pm 1.$$

We have thus obtained a contradiction. ■

127.3. **Intersections of cycles.** Now we turn to the study of singular chains in a topological manifold. To be on the safe side in our subsequent definitions we introduce the following convention.

Convention.

(1) Let X be a topological space. When we write "let $\sigma = a_1 \cdot \sigma_1 + \cdots + a_r \cdot \sigma_r$ be a singular m-chain in X" it is understood that the σ_i are pairwise different and that all the a_i are non-zero. We write $|\sigma| = |\sigma_1| \cup \cdots \cup |\sigma_r|$.

(2) Let M be an oriented m-dimensional topological manifold M and let $\sigma\colon \Delta^k \to M$ and $\tau\colon \Delta^{m-k} \to M$ be singular simplices. Since each Δ^n is in particular a topological manifold it makes sense to say that the singular simplices intersect nicely. Furthermore, throughout this chapter we equip the topological manifold Δ^n with the orientation given by the discussion on page 1581 together with the Smooth-Topological Orientation Proposition 105.11. In particular, if $\sigma\colon \Delta^k \to M$ and $\tau\colon \Delta^{m-k} \to M$ intersect nicely, then we obtain an algebraic intersection number $\sigma \cdot \tau$.

Definition. Let M be an m-dimensional topological manifold and let $\sigma = a_1 \cdot \sigma_1 + \cdots + a_r \cdot \sigma_r$ be a singular k-chain and let $\tau = b_1 \cdot \tau_1 + \cdots + b_s \cdot \tau_s$ be a singular $(m-k)$-chain in M. We say that σ and τ intersect nicely if the following two conditions are satisfied:

(1) For all $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s\}$ the maps $\sigma_i\colon \Delta^k \to M$ and $\tau_j\colon \Delta^{m-k} \to M$ intersect nicely.

(2) For all $(i, j) \neq (i', j') \in \{1, \ldots, r\} \times \{1, \ldots, s\}$ we have

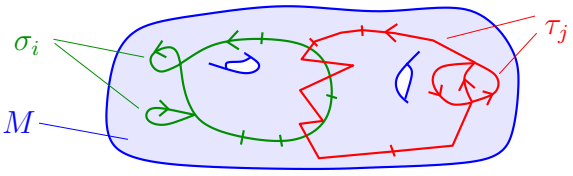
$$(|\sigma_i| \cap |\tau_j|) \cap (|\sigma_{i'}| \cap |\tau_{j'}|) \;=\; \varnothing.$$

Put differently, any intersection point is the intersection point of a unique pair of singular simplices.

If this is the case and if M is oriented, then we define the algebraic intersection number of σ and τ to be

$$\sigma \cdot \tau \;:=\; \sum_{i,j} (a_i \cdot b_j) \cdot \underbrace{(\sigma_i \cdot \tau_j)}_{\uparrow}.$$

algebraic intersection number of $\sigma_i\colon \Delta^k \to M$ and $\tau_j\colon \Delta^{m-k} \to M$



Before we can state the main theorem of this chapter we recall the formulation of the Poincaré Duality theorem for (not necessarily compact) topological manifolds without boundary. Thus let W be an oriented m-dimensional topological manifold without boundary and let $K \subset W$ be a compact subset. By the Homology-Section Theorem 106.10 there exists a unique lift of the orientation classes of W to a class in $\mathrm{H}_m(W, W \setminus K)$. Given any $k \in \mathbb{N}_0$ we denote by

$$- \cap W\colon \mathrm{H}^k(W, W \setminus K) \;\to\; \mathrm{H}_{m-k}(W)$$

the map that is given by capping with this unique lift. Lemma 119.17 says that these maps define a well-defined map

$$- \cap W \colon \mathrm{H}^k_{\mathrm{c}}(W) \ \to \ \mathrm{H}_{m-k}(W).$$

Furthermore the Poincaré Duality the Non-Compact PD Theorem 119.18 says that this map is an isomorphism. We denote by

$$\mathrm{PD}_W \colon \mathrm{H}_{m-k}(W) \ \to \ \mathrm{H}^k_{\mathrm{c}}(W)$$

the inverse map. Finally we remind the reader that given a topological space X we denote by $\epsilon_X = \epsilon \colon \mathrm{H}_0(X) \to \mathbb{Z}$ the augmentation map.
Now we can formulate the main theorem of this chapter.

Theorem 127.9. Let M be an oriented m-dimensional topological manifold without boundary. Let $\sigma \in \mathrm{C}_k(M)$ and $\tau \in \mathrm{C}_{m-k}(M)$ be two cycles. If σ and τ intersect nicely, then[1548]

$$\sigma \cdot \tau \ = \ \epsilon_M\Big(\big(\overbrace{\mathrm{PD}_M([\sigma])}^{\in \mathrm{H}^{m-k}_{\mathrm{c}}(M)} \underbrace{\cup \overbrace{\mathrm{PD}_M([\tau])}^{\in \mathrm{H}^{k}_{\mathrm{c}}(M)}}_{\in \mathrm{H}^{m}_{\mathrm{c}}(M)}\big) \cap M\Big).$$

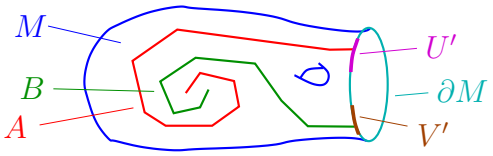
127.4. **Proof of Theorem 127.9.** Before we can provide the proof of Theorem 127.9 we need to recall some definitions and we need to prove several lemmas. First let us recall the following definition from page 123.

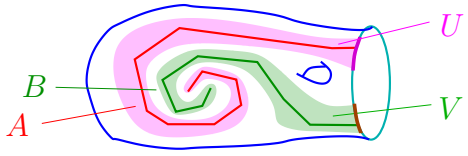
Definition. Let X be a topological space.

(1) Let A and B be disjoint subsets of X. We say U and V are separating open neighborhoods for A and B if U and V are disjoint, if U is an open neighborhood of A and if V is an open neighborhood of B.
(2) A topological space X is called normal if any two disjoint closed subsets A and B admit separating open neighborhoods.

Proposition 127.10.

(1) Every topological manifold is normal.
(2) Let M be a topological manifold and let A and B be two closed disjoint subsets of M. If $U' \subset \partial M$ and $V' \subset \partial M$ are separating open neighborhoods for $A \cap \partial M$ and $B \cap \partial M$, then there exist separating open neighborhoods U and V of A and B such that $U \cap \partial M = U'$ and $V \cap \partial M = V'$.





Sketch of proof.

(1) We had proved this statement in the Manifold Paracompact-Normal Proposition 18.9 (2).

[1548] We refer to page 2539 for the definition of the cup product on cohomology with compact support.

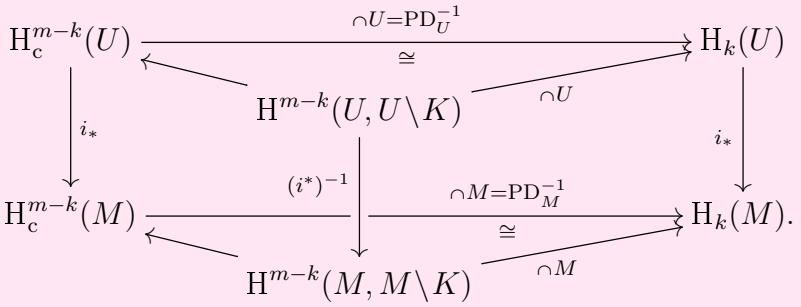
(2) Let M be a topological manifold and let A and B be two closed disjoint subsets of M. We pick a collar neighborhood $\partial M \times [0,1]$ as given by the Topological Collar Neighborhood Theorem 75.5.

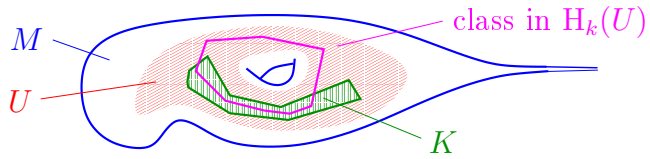
Suppose that we are given separating neighborhoods $U' \subset \partial M$ and $V' \subset \partial M$ for $A \cap \partial M$ and $B \cap \partial M$. By the Topological Manifold Boundary Proposition 75.2 the complement $M \setminus \partial M$ is again a smooth manifold. By (1) it is normal. Thus there exist separating open neighborhoods $\widetilde{U}$ and $\widetilde{V}$ for the closed subsets $A \cap (M \setminus \partial M)$ and $B \cap (M \setminus \partial M)$. It follows from the Manifold Boundary-Closed Subset Lemma 18.6 (1) that $\widetilde{U}$ and $\widetilde{V}$ are open subsets of M. It is now straightforward to verify that $U := \widetilde{U} \cup ([0,1) \times U')$ and $V := \widetilde{V} \cup ([0,1) \times V')$ have all the desired properties. ■

Lemma 127.11. Let M be an oriented m-dimensional topological manifold with $\partial M = \varnothing$ and let $U \subset M$ be an open subset. Let $i\colon U \to M$ be the inclusion map. For any compact subset $K \subset U$ and any $k \in \mathbb{N}_0$ the map

$$i^*\colon \mathrm{H}^{m-k}(M, M \setminus K) \;\to\; \mathrm{H}^{m-k}(U, U \setminus K)$$

is an isomorphism and the following diagram commutes:





Proof. We make the following comments and observations:

(i) By excision, see page 2418, the map $i^*\colon \mathrm{H}^{m-k}(M, M \setminus K) \to \mathrm{H}^{m-k}(U, U \setminus K)$ is an isomorphism.

(iii) The triangles on the top and the bottom commute by definition of PD_U and PD_M, see Lemma 119.17.

(iv) The square to the left commutes by definition of the map $i_*\colon \mathrm{H}_{\mathrm{c}}^{m-k}(U) \to \mathrm{H}_{\mathrm{c}}^{m-k}(M)$, see page 2484.

(v) It follows from the Cap Product-Naturality Lemma 116.8 and Corollary 106.12 (2) that the square to the right commutes.

(vi) The rectangle in the back commutes by Lemma 119.24. ■

As in the case of the Intersection Number-via-Cup Product Theorem 126.5 it is perhaps instructive to first prove Theorem 127.9 in the special case that the cycles do not intersect.

Proof of Theorem 127.9 for disjoint cycles. Let M be an oriented m-dimensional topological manifold with $\partial M = \varnothing$. Let $\sigma \in \mathrm{C}_k(M)$ and $\tau \in \mathrm{C}_{m-k}(M)$ be two cycles such that $|\sigma| \cap |\tau| = \varnothing$. We have to show that $\mathrm{PD}_M([\sigma]) \cup \mathrm{PD}_M([\tau]) = 0$.

By Proposition 127.10 we can find separating open neighborhoods U and V for $|\sigma|$ and $|\tau|$. Given a compact subset J of M and given $i \in \mathbb{N}_0$ we denote by $\varphi_J \colon \mathrm{H}^i(M, M \setminus J) \to \mathrm{H}^i_{\mathrm{c}}(M)$ the natural map.

Claim.

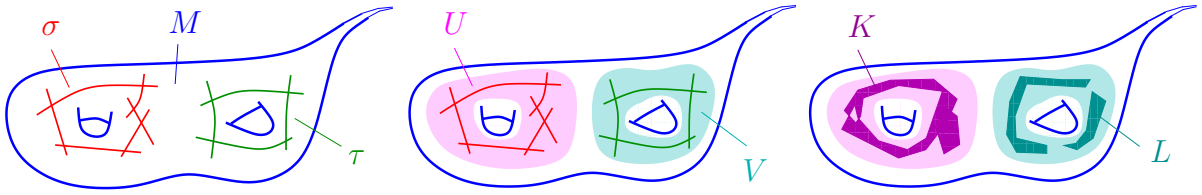
(1) There exists a compact subset $K \subset U$ and a cohomology class $\alpha \in \mathrm{H}^{m-k}(M, M \setminus K)$ such that $\varphi_K(\alpha) = \mathrm{PD}_M([\sigma])$.
(2) There exists a compact subset $L \subset V$ and a cohomology class $\beta \in \mathrm{H}^k(M, M \setminus L)$ such that $\varphi_L(\beta) = \mathrm{PD}_M([\tau])$.

Proof. We consider $\mathrm{PD}_U([\sigma]) \in \mathrm{H}^{m-k}_{\mathrm{c}}(U)$. By definition of cohomology with compact support there exists a compact subset K of U such that $\mathrm{PD}_U([\sigma])$ lies in the image of the natural map $\mathrm{H}^{m-k}(U, U \setminus K) \to \mathrm{H}^{m-k}_{\mathrm{c}}(U)$. The first statement of the claim now follows from Lemma 127.11. The proof of the second statement is evidently basically the same. ⊞

We consider the following diagram

$$\begin{array}{ccc}
\mathrm{H}^{m-k}_{\mathrm{c}}(M) \times \mathrm{H}^k_{\mathrm{c}}(M) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^m_{c}(M) \\
\Big\uparrow{\scriptstyle \varphi_K \times \varphi_L} \quad \Big\uparrow{\scriptstyle (\mathrm{PD}_M(\alpha), \mathrm{PD}_M(\beta)) \atop \updownarrow \atop (\alpha, \beta)} & & \Big\uparrow{\scriptstyle \varphi_M} \\
\mathrm{H}^{m-k}(M, M \setminus K) \times \mathrm{H}^k(M, M \setminus L) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^m(M, \underbrace{(M \setminus K) \cup (M \setminus L)}_{=M}).
\end{array}$$

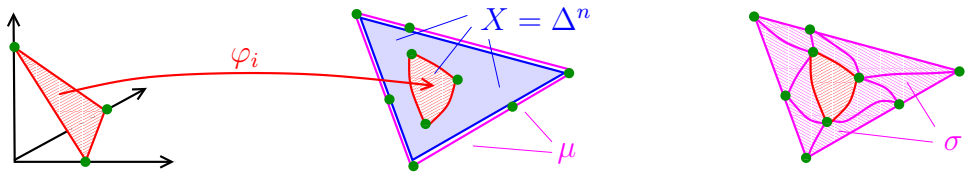
The diagram commutes by the definition of the cup product on cohomology with compact support, see page 2539. It follows from the observation that the lower right group is zero that the cup product of $\mathrm{PD}_M([\sigma])$ and $\mathrm{PD}_M([\tau])$ is zero. ■



Lemma 127.12. Let X be a compact oriented, n-dimensional topological manifold. Furthermore let $\varphi_i \colon \Delta^n \to X \setminus \partial X$, $i = 1, \ldots, r$, be orientation-preserving embeddings such that the images are disjoint. We write $W = X \setminus (\varphi_1(\mathring{\Delta}^n) \cup \cdots \cup \varphi_r(\mathring{\Delta}^n))$. Given any representative $\mu \in \mathrm{C}_{n-1}(\partial X)$ of the fundamental class of ∂X there exists a singular n-chain $\sigma \in \mathrm{C}_n(W)$ such that

$$\partial(\sigma + \varphi_1 + \cdots + \varphi_r) = \mu \quad \text{and with} \quad [\sigma + \varphi_1 + \cdots + \varphi_r] \;=\; [X].$$

Example. In the proof below of Theorem 127.9 we will apply Lemma 127.12 to the case that $X = \Delta^n$ itself is the standard n-simplex. This particular situation is illustrated in the figure below.



Proof. By Proposition 75.3 we know that W is a compact n-dimensional topological manifold with $\partial W = \varphi_1(\partial\Delta^n) \cup \cdots \cup \varphi_r(\partial\Delta^n) \cup \partial X$. Since the φ_i are orientation-preserving we obtain from the Fundamental Class-of-Boundary Proposition 106.27 that $-\partial\varphi_1 - \cdots - \partial\varphi_r + \mu \in \mathrm{C}_{n-1}(\partial W)$ represents the fundamental class of ∂W. Again by the Fundamental Class-of-Boundary Proposition 106.27 we know that there exists a singular n-chain $\sigma \in \mathrm{C}_n(W)$ with $\partial\sigma = -\partial\varphi_1 - \cdots - \partial\varphi_r + \mu$. It follows from the Fundamental Class-Criterion Proposition 100.4 that $\sigma + \varphi_1 + \cdots + \varphi_r$ represents the fundamental class of X. ■

The formulation of the following, rather technical lemma, might initially be somewhat intimidating. We hope that the figure below helps somewhat with keeping track of the various objects. Also note that the lemma can be viewed as a cousin of first degree to the Codimension-Zero Submanifold-PD Lemma 120.5.

Lemma 127.13. Let M be an oriented m-dimensional topological manifold and let Z be a compact codimension-zero submanifold of M. Furthermore let C and D be two compact codimension-zero submanifolds of ∂Z with $C \cup D = \partial Z$ and with $\partial C = \partial D = C \cap D$.[1549] Let $U \subset M$ be an open subset with $U \cap \partial Z \subset C$ and let $K \subset U$ be a compact subset. We consider the three inclusion maps $p\colon (U, \varnothing) \to (U, U \setminus \mathring{Z})$, $g\colon (Z \cap U, C \cap U) \to (Z, C)$ and $z\colon (Z \cap U, C \cap U) \to (U, U \setminus \mathring{Z})$. We note that the induced map

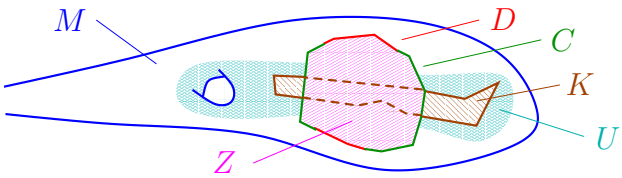
$$z_*\colon \mathrm{H}_k(Z \cap U, C \cap U) \;\to\; \mathrm{H}_k(U, U \setminus \mathring{Z})$$

is an isomorphism by the Excision Theorem 75.10 for topological manifolds. With this preparation we can now formulate the conclusion of the lemma. Namely, the following diagram commutes:

$$\begin{array}{ccccc}
 & & \mathrm{H}_k(M) & \xleftarrow{\cap M} & \mathrm{H}^{m-k}(M, M\setminus K) \\
 & \nearrow & & \swarrow & \\
\mathrm{H}_k(U) & & \xleftarrow{\cap U} & & \mathrm{H}^{m-k}(U, U\setminus K) \\
\downarrow{\scriptstyle p_*} & & & & \\
\mathrm{H}_k(U, U\setminus\mathring{Z}) & \xrightarrow{(z_*)^{-1}} & \mathrm{H}_k(Z\cap U, C\cap U) & \xrightarrow{g_*} & \mathrm{H}_k(Z, C) \xleftarrow{\cap[Z]} \mathrm{H}^{m-k}(Z, D).
\end{array}$$

Here all the unmarked maps are induced by the obvious inclusion maps of (pairs) of topological spaces. Also note that the bottom cap product $\cap[Z]\colon \mathrm{H}^{m-k}(Z, D) \to \mathrm{H}_k(Z, C)$ is defined since our hypothesis on C and D together with the Excisive Triad-Proposition 109.12 (5) imply that (Z, C, D) is an excisive triad.

[1549] In fact in the proof we only need that we have a decomposition $\partial Z = C \cup D$ such that (Z, C, D) is an excisive triad.



Proof. We make the following preparations:

(a) We pick a representative $\nu_M \in \mathrm{H}_m(M, M \setminus K)$ for the lift of the orientation of M and similarly we pick $\nu_U \in \mathrm{H}_m(U, U \setminus K)$.

(b) We consider the two obvious inclusion maps $f\colon (Z, \partial Z) \to (Z, \overbrace{Z \setminus (\mathring{Z} \cap K)}^{=(Z\setminus K)\cup C})$ and $g\colon (Z \cap U, (Z \cap U) \setminus (\mathring{Z} \cap K)) \to (Z, Z \setminus (\mathring{Z} \cap K))$. It follows from the Excision Theorem 74.18 that the map $g_*\colon \mathrm{H}_m(Z \cap U, (Z \cap U) \setminus (\mathring{Z} \cap K)) \to \mathrm{H}_m(Z, Z \setminus (\mathring{Z} \cap K))$ is an isomorphism. We write $\nu_Z := g_*^{-1}(f_*([Z]))$.

Next we consider the following expanded version of the diagram shown in the lemma:

$$\begin{array}{ccccc}
 & & \mathrm{H}_k(M) & \xleftarrow{\cap \nu_M} & \mathrm{H}^{m-k}(M, M \setminus K) \\
 & \nearrow & & \swarrow & \\
\mathrm{H}_k(U) & \xleftarrow{\cap \nu_U} & \mathrm{H}^{m-k}(U, U \setminus K) & & \\
\downarrow & & \downarrow & & \\
\mathrm{H}_k(U, U \setminus \mathring{Z}) & \xrightarrow{(z_*)^{-1}} & \mathrm{H}_k(Z \cap U, C \cap U) & \xleftarrow{\cap \nu_Z} & \mathrm{H}^{m-k}(Z \cap U, (Z \cap U) \setminus K). \\
 & & & \searrow & \\
 & & \mathrm{H}_k(Z, C) & \xleftarrow[\cap f_*([Z])]{} & \mathrm{H}^{m-k}(Z, Z \setminus K) \\
 & & \downarrow = & & \downarrow \\
 & & \mathrm{H}_k(Z, C) & \xleftarrow[\cap [Z]]{} & \mathrm{H}^{m-k}(Z, D).
\end{array}$$

It suffices to show that all the five regions of the diagram commute:

(1) The parallelogram on the top of the diagram commutes by the Cap Product-Naturality Lemma 116.8 applied to the map $h\colon (U, U \setminus K, \varnothing) \to (M, M \setminus K, \varnothing)$, together with the observation that the Codimension Zero-Fundamental Class Lemma 106.24 implies that $h_*(\nu_U) = \nu_M$.

(2) The 4-gon on the right commutes since all maps are induced by inclusions.

(3) For the lower parallelogram we obtain commutativity from applying the Cap Product-Naturality Lemma 116.8 to the inclusion map[1550] $g\colon (Z \cap U, Z \cap U \setminus K, C \cap U) \to (Z, Z \setminus K, C)$.

(4) To show commutativity of the lower rectangle we apply the Cap Product-Naturality Lemma 116.8 to the inclusion map $f\colon (Z, D, C) \to (Z, Z \setminus K, C)$.

[1550]Here we use the symbol for this map of triples of topological spaces. We feel that recycling the name g is justified since this map of triples "contains" our original map $g\colon (Z \cap U, C \cap U) \to (Z, C)$ of pairs of topological spaces.

Finally we turn to the upper rectangle. We need to show that the following rectangle commutes:

$$\begin{array}{ccccc} \mathrm{H}_k(U) & \xleftarrow{\qquad\qquad\cap\nu_U\qquad\qquad} & & & \mathrm{H}^{m-k}(U, U\setminus K) \\ \downarrow{\scriptstyle p_*} & & & & \downarrow{\scriptstyle z^*} \\ \mathrm{H}_k(U, U\setminus \mathring{Z}) & \xleftarrow{z_*} & \mathrm{H}_k(Z\cap U, C\cap U) & \xleftarrow{\cap\nu_Z} & \mathrm{H}^{m-k}(Z\cap U, (Z\cap U)\setminus K). \end{array}$$

Now let $\beta \in \mathrm{H}^{m-k}(U, U\setminus K)$. We have the following equalities in $\mathrm{H}_k(U, U\setminus \mathring{Z})$:

$$\begin{array}{ccccc} p_*(\beta\cap\nu_U) & = & \beta\cap p_*(\nu_U) & & \\ & & \uparrow & & \\ & & \text{Cap Product-Naturality Lemma 116.8 applied to the map } p\colon (U, U\setminus K, \varnothing)\to(U, U\setminus K, U\setminus \mathring{Z}) & & \end{array}$$

$$\begin{array}{ccccc} & & \text{it follows from Lemma 106.31 that we have } p_*(\nu_U) = z_*(\nu_Z) \text{ in } \mathrm{H}_m(U, U\setminus K\cup U\setminus \mathring{Z}) & & \\ & & \downarrow & & \\ \beta\cap z_*(\nu_Z) & = & z_*(z^*(\beta)\cap\nu_Z). & & \\ \uparrow & & & & \\ \text{Cap Product-Naturality Lemma 116.8 applied to the map } z\colon (Z\cap U, Z\cap U\setminus K, C\cap U)\to(U, U\setminus K, U\setminus \mathring{Z}) & & & & \end{array}$$

We have thus shown that the diagram commutes. ■

After all these preparations we can finally attack Theorem 127.9.

Proof of Theorem 127.9. Let M be an oriented m-dimensional topological manifold with $\partial M = \varnothing$. Also let $\sigma = a_1\cdot\sigma_1 + \cdots + a_r\cdot\sigma_r \in \mathrm{C}_k(M)$ and $\tau = b_1\cdot\tau_1 + \cdots + b_s\cdot\tau_s \in \mathrm{C}_{m-k}(M)$ be two singular cycles that intersect nicely.

Claim 1. We can assume that we are in the following setting: There exists an $n\in\mathbb{N}_0$, there exist distinct points $P_1,\ldots,P_n\in M$ and there exist disjoint m-dimensional submanifolds $Z_1,\ldots,Z_n$ of M such that the following hold:

(1) For $i\neq j$ we have $|\sigma_i|\cap|\tau_j| = \varnothing$.
(2) For each $i\in\{1,\ldots,n\}$ we have $|\sigma_i|\cap|\tau_i| = \{P_i\}$.
(3) For $i = n+1,\ldots,\min\{r,s\}$ we have $|\sigma_i|\cap|\tau_i| = \varnothing$.
(4) For $i\in\{1,\ldots,n\}$ the following hold:[1551]
  (a) P_i is contained in $Z_i\setminus\partial Z_i$.
  (b) Z_i is homeomorphic to $\overline{B}^m$.
  (c) The map $\sigma_i\colon\Delta^k\to M$ is actually a proper embedding $\sigma_i\colon\Delta^k\to Z_i$ such that $X_i := \sigma_i(\Delta^k)$ is a well-behaved submanifold of Z_i. By the definition, see page 2777 this means that the submanifold $\partial X_i = \sigma_i(\partial\Delta^k)$ of ∂Z_i admits a regular neighborhood A_i.
  (d) Similarly to (c) the map $\tau_i\colon\Delta^{m-k}\to M$ is a proper embedding $\tau_i\colon\Delta^{m-k}\to Z_i$ such that $Y_i := \tau_i(\Delta^{m-k})$ is a well-behaved submanifold of Z_i. This means that the submanifold $\partial Y_i = \tau_i(\partial\Delta^k)$ of ∂Z_i admits a regular neighborhood B_i.
  (e) We have $A_i\cap B_i = \varnothing$.
  (f) For $j\neq i$ we have $|\sigma_j|\cap Z_i = \varnothing$ and $|\tau_j|\cap Z_i = \varnothing$.

Proof. The proof of the claim is built out of the following ingredients:

(i) The fact, established in Exercise 18.13, that given any map $f\colon\Delta^l\to M$ and given any $P\in M\setminus f(\Delta^l)$ there exists an open neighborhood U of P with $f(\Delta^l)\cap U = \varnothing$.
(ii) The "given any neighborhood U" feature of the definition of a nice intersection point.

[1551] These statements imply in particular that P_i is a nice intersection point of σ_i and τ_i.

(iii) By the Standard Simplex Homeomorphism Lemma 72.1 we know that any Δ^l is homeomorphic to $\overline{B}^l$.

(iv) We use that it follows from (iii) and the definition of a nice intersection point that any $f(\widetilde{X})$ is homeomorphic to Δ^k and that any $g(\widetilde{Y})$ is homeomorphic to Δ^{m-k}. (Here we use the notation from the definition of a nice intersection point, i.e. $\widetilde{X}$ and $\widetilde{Y}$ are defined on page 2778.) We combine this piece of information with Lemma 127.12.

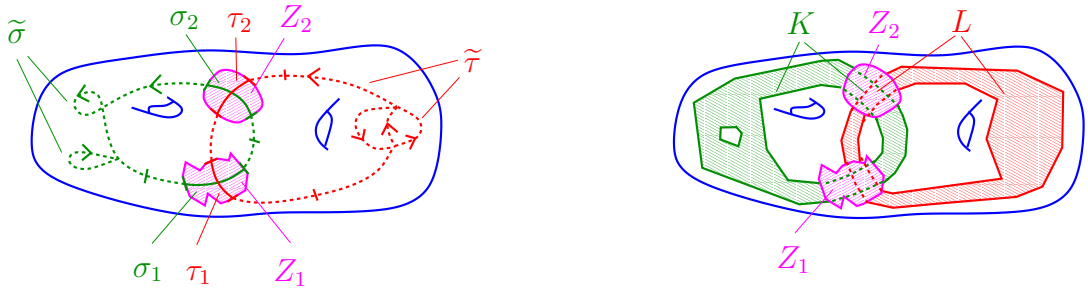
We leave it to the reader to work out the elementary albeit slightly messy details. ⊞

Next we introduce the following notation:

(5) We write $Z = Z_1 \cup \cdots \cup Z_n$.

(6) We write $\widetilde{\sigma} = a_{n+1} \cdot \sigma_{n+1} + \cdots + a_r \cdot \sigma_r$ and we denote by $|\widetilde{\sigma}|$ the union of the images of $\sigma_{n+1}, \ldots, \sigma_r$. Similarly we define $\widetilde{\tau}$ and $|\widetilde{\tau}|$.

We refer to the figure below on the left for an illustration.



Claim 2. There exist compact subsets K and L of M with the following properties:

(a) There exists a cohomology class $\varphi \in \mathrm{H}^{m-k}(M, M \setminus K)$ such that the image of φ in $\mathrm{H}^{m-k}_{\mathrm{c}}(M)$ equals $\mathrm{PD}_M([\sigma])$.

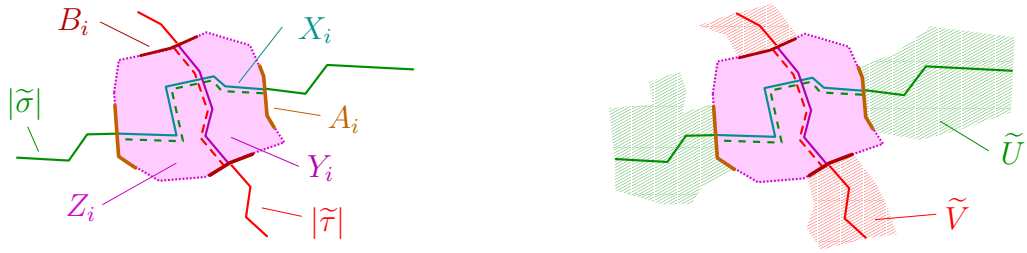
(b) There exists a cohomology class $\psi \in \mathrm{H}^k(M, M \setminus L)$ such that the image of ψ in $\mathrm{H}^k_{\mathrm{c}}(M)$ equals $\mathrm{PD}_M([\tau])$.

(c) We have $K \cap L \cap (M \setminus Z) = \varnothing$.

(d) For each $i \in \{1, \ldots, n\}$ we have $K \cap \partial Z_i \subset A_i$ and $L \cap \partial Z_i \subset B_i$.

We refer to the figure above on the right for an illustration.

Proof. By Proposition 75.3 we know that $W := M \setminus \mathring{Z}$ is an m-dimensional topological manifold. It follows from Proposition 127.10 (2) that there exist *disjoint* open neighborhoods $\widetilde{U} \subset W$ of $|\widetilde{\sigma}|$ and $\widetilde{V} \subset W$ of $|\widetilde{\tau}|$ such that for each $i \in \{1, \ldots, n\}$ we have $\widetilde{U} \cap \partial Z_i = \mathring{A}_i$ and $\widetilde{V} \cap \partial Z_i = \mathring{B}_i$. Next we write $U = \widetilde{U} \cup \bigcup_{i=1}^{n} \mathring{Z}_i$ and $V = \widetilde{V} \cup \bigcup_{i=1}^{n} \mathring{Z}_i$. Note that it follows from the elementary Lemma 1.8 (2) that U and V are open subsets of M. We refer to the figure below for an illustration.



We denote by $i\colon U \to M$ the inclusion map. Evidently we have $\sigma \in \mathrm{C}_k(U)$. Since U is an open subset of M we can thus consider $\mathrm{PD}_U([\sigma]) \in \mathrm{H}_{\mathrm{c}}^{m-k}(U)$. By Lemma 112.2 there exists a compact subset K of U and a cohomology class $\widetilde{\varphi} \in \mathrm{H}^{m-k}(U, U \setminus K)$ such that $\mathrm{PD}_U([\sigma])$ equals the image of $\widetilde{\varphi}$ under the natural map $\mathrm{H}^{m-k}(U, U \setminus K) \to \mathrm{H}_{\mathrm{c}}^{m-k}(U)$. By excision, see page 2418, the map $i^*\colon \mathrm{H}^{m-k}(M, M \setminus K) \to \mathrm{H}^{m-k}(U, U \setminus K)$ is an isomorphism. We set $\varphi := (i^*)^{-1}(\widetilde{\varphi})$. By Lemma 127.11 we know that the image of φ in $\mathrm{H}_{\mathrm{c}}^{m-k}(M)$ equals $\mathrm{PD}_M([\sigma])$.

By precisely the same argument we can also find a compact subset L of V and some cohomology class $\psi \in \mathrm{H}^k(M, M \setminus L)$ such that the image of ψ in $\mathrm{H}_{\mathrm{c}}^k(M)$ equals $\mathrm{PD}_M([\tau])$.

Finally note that by construction K and L, and also φ and ψ satisfy all the desired conditions set out in (a), (b), (c) and (d). ⊞

We continue with some more observations and notation.

(7) For $i = 1, \ldots, n$ we write $K_i = K \cap Z_i$ and $L_i = L \cap Z_i$. Note that by construction we have $K_i \cap L_i \subset \mathring{Z}_i = Z_i \setminus \partial Z_i$.

(8) Given any compact subset T of M we denote by $o\colon (M, \varnothing) \to (M, M \setminus T)$ the obvious map. Also we denote by $o^*\colon \mathrm{H}^*(M, M \setminus T) \to \mathrm{H}_{\mathrm{c}}^*(M)$ the obvious map.

(9) We denote by $z\colon Z \to M$ the inclusion map and we denote by z also all the inclusion maps of pairs of topological spaces where the total spaces are Z and M.

(10) Given $i \in \{1, \ldots, n\}$ we denote by $z_i\colon Z_i \to Z$ the inclusion map. We adopt the same convention as in (9) for pairs of topological spaces.

(11) We view Z as an m-dimensional submanifold of M and we equip Z with the orientation coming from the Topological Manifold-Induced Orientation Lemma 105.17. Given any subset $W \subset Z$ with $\partial Z \subset W$ we denote by $[Z]$ also the image of the fundamental class of Z under the map $\mathrm{H}_m(Z, \partial Z) \to \mathrm{H}_m(Z, W)$. We adopt the same convention for the $[Z_i]$.

(12) Recall that for each $i \in \{1, \ldots, n\}$ we have $K \cap \partial Z_i \subset A_i$ and $L \cap \partial Z_i \subset B_i$. This implies that we obtain the maps of pairs $(Z_i, \partial Z_i \setminus A_i) \to (Z_i, Z_i \setminus K_i)$ and $(Z_i, \partial Z_i \setminus B_i) \to (Z_i, Z_i \setminus L_i)$. We refer to each of these maps as the canonical map c.

(13) For each $i \in \{1, \ldots, n\}$ we have maps of pairs $d\colon (Z_i, \partial Z_i \setminus A_i) \to (Z_i, \partial Z_i \setminus \partial X_i)$ and $d\colon (Z_i, \partial Z_i \setminus B_i) \to (Z_i, \partial Z_i \setminus \partial Y_i)$. Since A_i is a regular neighborhood of ∂X_i and since B_i is a regular neighborhood of ∂Y_i we obtain from the Homotopic Maps-H*-Lemma 108.13 (4) that these maps induce isomorphisms of relative cohomology groups.

(14) By the Homology-Section Theorem 106.10 there exists a unique lift $\mu \in \mathrm{H}_m(M, M \setminus (K \cap L))$ of the orientation classes of M.

With the above definitions and observations it now makes sense to consider the following diagram:

$$
\begin{array}{ccccccc}
\mathrm{H}_{\mathrm{c}}^{m-k}(M)\times\mathrm{H}_{\mathrm{c}}^{k}(M) & \xrightarrow{\cup} & \mathrm{H}_{\mathrm{c}}^{m}(M) & \xrightarrow{\cap M} & \mathrm{H}_0(M) & \xrightarrow{\epsilon_M} & \mathbb{Z} \\
\uparrow{\scriptstyle o^*\times o^*} & & \uparrow{\scriptstyle o^*} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{H}^{m-k}(M,M\backslash K)\times\mathrm{H}^{k}(M,M\backslash L) & \xrightarrow{\cup} & \mathrm{H}^{m}(M,M\backslash(K\cap L)) & \xrightarrow{\cap\mu} & \mathrm{H}_0(M) & \xrightarrow{\epsilon_M} & \mathbb{Z} \\
\downarrow{\scriptstyle z^*\times z^*} & & \downarrow{\scriptstyle z^*} & & \uparrow{\scriptstyle z_*} & & \uparrow{\scriptstyle \mathrm{id}} \\
\mathrm{H}^{m-k}(Z,Z\backslash K)\times\mathrm{H}^{k}(Z,Z\backslash L) & \xrightarrow{\cup} & \mathrm{H}^{m}(Z,Z\backslash K\cap L) & \xrightarrow{\cap[Z]} & \mathrm{H}_0(Z) & \xrightarrow{\epsilon_Z} & \mathbb{Z} \\
\downarrow{\scriptstyle \bigoplus_{i=1}^{n} z_i^*\times z_i^*} & & \downarrow{\scriptstyle \bigoplus_{i=1}^{n} z_i^*} & & \uparrow{\scriptstyle \sum_{i=1}^{n} z_{i*}} & & \uparrow{\scriptstyle \mathrm{id}} \\
\bigoplus_{i=1}^{n}\mathrm{H}^{m-k}(Z_i,Z_i\backslash K_i)\times\mathrm{H}^{k}(Z_i,Z_i\backslash L_i) & \xrightarrow{\oplus\cup} & \bigoplus_{i=1}^{n}\mathrm{H}^{m}(Z_i,Z_i\backslash K_i\cap L_i) & \xrightarrow{\oplus\cap[Z_i]} & \bigoplus_{i=1}^{n}\mathrm{H}_0(Z_i) & \xrightarrow[\sum\epsilon_{Z_i}]{} & \mathbb{Z} \\
\downarrow{\scriptstyle \bigoplus_{i=1}^{n} c^*\times c^*} & & \downarrow{\scriptstyle \bigoplus_{i=1}^{n} c^*} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} \\
\bigoplus_{i=1}^{n}\mathrm{H}^{m-k}(Z_i,\partial Z_i\backslash \mathring{A}_i)\times\mathrm{H}^{k}(Z_i,\partial Z_i\backslash \mathring{B}_i) & \xrightarrow{\oplus\cup} & \bigoplus_{i=1}^{n}\mathrm{H}^{m}(Z_i,\partial Z_i) & \xrightarrow{\oplus\cap[Z_i]} & \bigoplus_{i=1}^{n}\mathrm{H}_0(Z_i) & \xrightarrow[\sum\epsilon_{Z_i}]{} & \mathbb{Z} \\
\cong\uparrow{\scriptstyle \bigoplus_{i=1}^{n} d^*\times d^*} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} \\
\bigoplus_{i=1}^{n}\mathrm{H}^{m-k}(Z_i,\partial Z_i\backslash \partial X_i)\times\mathrm{H}^{k}(Z_i,\partial Z_i\backslash \partial Y_i) & \xrightarrow{\oplus\cup} & \bigoplus_{i=1}^{n}\mathrm{H}^{m}(Z_i,\partial Z_i) & \xrightarrow{\oplus\cap[Z_i]} & \bigoplus_{i=1}^{n}\mathrm{H}_0(Z_i) & \xrightarrow[\sum\epsilon_{Z_i}]{} & \mathbb{Z}.
\end{array}
$$

Claim 3. The above diagram commutes.

Proof. In the following we deal with all of the fifteen squares of the diagram.

(a) The square on the top left commutes by definition of the cup product on cohomology with compact support, see page 2539.

(b) The left squares in the second row and also in the fourth row commute by the Cup Product-Naturality Proposition 115.4 (1).

(c) The left square in the third row commutes by the Cup-Cap Disjoint Union Lemma 116.9 (1) together with the aforementioned the Cup Product-Naturality Proposition 115.4 (1).

(d) The middle square in the first row commutes by definition of the map $\cap M$.

(e) It follows from Corollary 106.12 that $z_*([Z]) = \mu \in \mathrm{H}_m(M, M \setminus (K \cap L))$. Therefore we obtain from the Cap Product-Naturality Lemma 116.8 (1) that the middle square in the second row commutes.

(f) It follows from the remark on page 2351 that $\sum_{i=1}^{n} z_{i*}([Z_i]) = [Z]$. Now we obtain from the Cap Product-Naturality Lemma 116.8 (1) and the Cup-Cap Disjoint Union Lemma 116.9 (2) that the middle square in the third row commutes.

(g) The squares that involve the augmentation maps commute by the discussion on page 1594.

(h) It follows from the Cap Product-Naturality Lemma 116.8 (1) and our conventions regarding the meaning of $[Z_i]$ that the middle square in the fourth row commutes.

(i) It follows from the Cup Product-Naturality Proposition 115.4 that the bottom left square commutes.

(j) The remaining squares, that we did not discuss, commute for the trivial reason that the opposite maps in the square are actually on the nose the same map. ⊞

We introduce one last piece of notation:

(15) For each $i \in \{1, \dots, n\}$ we denote by $\mathrm{PD}_{Z_i} \colon \mathrm{H}_k(Z_i, \partial X_i) \to \mathrm{H}^{m-k}(Z_i, \partial Z_i \setminus \partial X_i)$ the isomorphism from Proposition 127.2.

Claim 4. Let $i \in \{1, \dots, n\}$.

(i) For the above $\varphi \in \mathrm{H}^{m-k}(M, M \setminus K)$ we have $(c^* \circ z_i^* \circ z^*)(\varphi) = a_i \cdot d^*(\mathrm{PD}_{Z_i}([X_i]))$.

(ii) For the above $\psi \in \mathrm{H}^k(M, M \setminus L)$ we have $(c^* \circ z_i^* \circ z^*)(\psi) = b_i \cdot d^*(\mathrm{PD}_{Z_i}([Y_i]))$.

(iii) We have
$$\epsilon_{Z_i}\Big(\big((d^*(\mathrm{PD}_{Z_i}([X_i])) \cup d^*(\mathrm{PD}_{Z_i}([Y_i]))\big) \cap [Z_i]\Big) \;=\; \mathrm{index}(\sigma_i, \tau_i, P_i).$$

Proof.

(i) We introduce the following notation:

(α) We set $U := (M \setminus Z) \cup \bigcup_{i=1}^{n} \mathring{Z}_i \cup \bigcup_{i=1}^{n} \mathring{A}_i = M \setminus \bigcup_{i=1}^{n} (\partial Z_i \setminus \mathring{A}_i)$. It is elementary to see that U is in fact an open subset of M.

(β) We write $C := A_i$ and $D := \partial Z_i \setminus \mathring{A}_i$. Note that it follows from Proposition 75.3 that C and D are compact codimension-zero submanifolds of ∂Z_i with $C \cup D = \partial Z_i$ and with $C \cap D = \partial C = \partial D$.

(γ) We denote by $p \colon (U, \varnothing) \to (U, U \setminus \mathring{Z}_i)$, $\widetilde{z} \colon (Z_i \cap U, C \cap U) \to (U, U \setminus \mathring{Z}_i)$ and $g \colon (Z_i \cap U, C \cap U) \to (Z_i, C)$ the obvious inclusion maps.

We perform the following calculation in $\mathrm{H}_k(Z_i, A_i)$:

$$\underbrace{(c^* \circ z_i^* \circ z^*)(\varphi)}_{\in \mathrm{H}^{m-k}(Z_i, \partial Z_i \setminus \mathring{A}_i)} \cap [Z_i] \;\underset{\substack{\uparrow \\ \text{this follows from Lemma 127.13, applied to the above choices} \\ \text{of } C \text{ and } D\text{, and the observation that } \varphi \cap M = [\sigma] \text{ with } |\sigma| \subset U, \\ \text{note that here we use that we have } U \cap \partial Z_i \subset C \text{ and } K \subset U}}{=}\; g_*(\widetilde{z}_*^{-1}(p_*([\sigma]))) \;\overset{\substack{\text{since } \sigma = a_1 \cdot \sigma_1 + \cdots + a_r \cdot \sigma_r\text{, since } p_*(\sigma_j) = 0 \text{ for } j \neq i \text{ and since } [\sigma_i] = [X_i] \\ \downarrow}}{=}\; g_*(\widetilde{z}_*^{-1}(p_*(a_i \cdot [X_i]))) \;\underset{\substack{\uparrow \\ \text{by the Codimension Zero-} \\ \text{Fundamental Class Lemma 106.24}}}{=}\; a_i \cdot [X_i].$$

Now recall that in the proof of Proposition 127.2 we saw that the isomorphism PD_{Z_i} is given as follows:

$$\begin{tikzcd} \mathrm{H}^{m-k}(Z_i, \partial Z_i \setminus \partial X_i) \arrow[r, "d^*", "\cong"'] & \mathrm{H}^{m-k}(Z_i, \partial Z_i \setminus \mathring{A}_i) \arrow[r, "\cap [Z_i]", "\cong"'] & \mathrm{H}_k(Z_i, A_i) & \mathrm{H}_k(Z_i, \partial X_i). \arrow[l, "j_*"', "\cong"] \arrow[lll, bend left, "=:\mathrm{PD}_{Z_i}"] \end{tikzcd}$$

Since the inclusion induced map $j_* \colon \mathrm{H}_k(Z_i, \partial X_i) \to \mathrm{H}_k(Z_i, A_i)$ is an isomorphism we now see that $(c^* \circ z_i^* \circ z^*)(\varphi) = a_i \cdot d^*(\mathrm{PD}_{Z_i}([X_i]))$.

(ii) The proof of this statement is evidently the same as the proof of (2).

(iii) This statement follows immediately from the definitions together with the fact, obtained in Claim 3, that the bottom squares in the above diagram commute. ⊞

After all these preparations we can prove the desired equality:

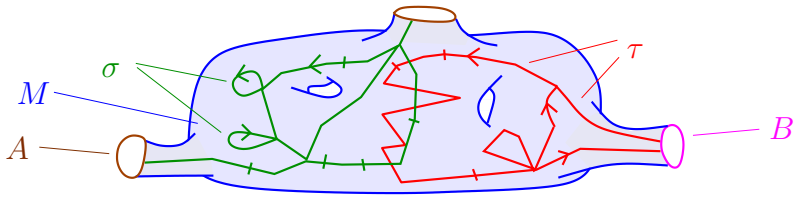
$$\begin{aligned}
\epsilon_M\big((\mathrm{PD}_M([\sigma]) \cup \mathrm{PD}_M([\tau])) \cap M\big) &\overset{\text{by construction of } \varphi \text{ and } \psi \text{ in Claim 1 and the first row of the commutative diagram from Claim 2}}{=} \epsilon_M((\varphi \cup \psi) \cap \mu) \\
&\overset{\text{by the second, third and fourth row of the commutative diagram from Claim 2}}{=} \sum_{i=1}^{n} \epsilon_{Z_i}\big(\big((c^* \circ z_i^* \circ z^*)(\varphi) \cup (c^* \circ z_i^* \circ z^*)(\psi)\big) \cap [Z_i]\big) \\
&\overset{\text{by Claim 3}}{=} \sum_{i=1}^{n} a_i \cdot b_i \cdot \mathrm{index}(\sigma_i, \tau_i, P_i) \overset{\text{by the setting from page 2791}}{=} \sigma \cdot \tau.
\end{aligned}$$

This concludes the proof of the theorem. ■

We conclude this section with the following variation on Theorem 127.9.

Theorem 127.14. Let M be a compact orientable connected m-dimensional topological manifold. Furthermore let A and B be compact $(m-1)$-dimensional submanifolds of ∂M with $\partial A = \partial B = A \cap B$ and with $\partial M = A \cup B$. Finally let $\sigma \in \mathrm{C}_k(M, A)$ and $\tau \in \mathrm{C}_{m-k}(M, B)$ be two cycles. If σ and τ intersect nicely, then

$$\sigma \cdot \tau = \big\langle \underbrace{\overbrace{\mathrm{PD}_M([\sigma])}^{\in \mathrm{H}^{m-k}(M,B)} \cup \overbrace{\mathrm{PD}_M([\tau])}^{\in \mathrm{H}^{k}(M,A)}}_{\in \mathrm{H}^{m}(M,\partial M)}, [M] \big\rangle.$$



Sketch of proof. The proof of the theorem follows along the lines of the proof of Theorem 127.9. The key difference is that we need to use Poincaré Duality for non-compact topological manifolds with boundary. More precisely, let U be an oriented m-dimensional topological manifold with boundary. By Theorem 106.30, given any compact set K of U the orientation defines a natural homology class μ_K in $\mathrm{H}_m(U, (U \setminus K) \cup \partial U)$. The same construction as in Lemma 119.17, this time using Lemma 106.31 instead of Corollary 106.12, shows that capping with these homology classes defines a well-defined map

$$\mathrm{PD}_U \colon \mathrm{H}^k_{\mathrm{c}}(U) \to \mathrm{H}_{m-k}(U, \partial U).$$

A moderately heroic effort shows that the proof of the Poincaré Duality Theorem 119.1, as provided in Section 119.4 can be modified to show that the above map PD_U is an isomorphism. Once this is out of the way it is fairly straightforward to modify the proof of Theorem 127.9 to obtain a proof of Theorem 127.14. ■

127.5. **The algebraic $\mathbb{F}_2$-intersection number** $(*)$**.** In this section we quickly discuss the "$\mathbb{F}_2$-analogue" of the results of the previous sections.

Definition. Let M be an m-dimensional topological manifold, let X be a k-dimensional topological manifold and let Y be an $(m-k)$-dimensional topological manifold. Furthermore let $f\colon X \to M$ and $g\colon Y \to M$ be maps. If f and g intersect nicely, then we define

$$\text{the algebraic } \mathbb{F}_2\text{-intersection number of } f \text{ and } g \;:=\; f \pitchfork_{\mathbb{F}_2} g \;:=\; \#(|f| \cap |g|) \bmod 2.$$

If X and Y are in fact submanifolds and f and g are the inclusion map, then we write $X \pitchfork_{\mathbb{F}_2} Y := f \pitchfork_{\mathbb{F}_2} g$.

We recall the following notation.

Notation.

(1) In the $\mathbb{F}_2$-Fundamental Class Corollary 106.5 we saw that any compact n-dimensional topological manifold W comes with its $\mathbb{F}_2$-fundamental class $[W]_{\mathbb{F}_2} \in \mathrm{H}_n(W, \partial W; \mathbb{F}_2)$. If W is in fact a proper submanifold of some other topological manifold M, then frequently we denote by $[W]_{\mathbb{F}_2} \in \mathrm{H}_n(M, \partial M; \mathbb{F}_2)$ the image of $[W]_{\mathbb{F}_2}$ under the inclusion induced map $\mathrm{H}_n(W, \partial W; \mathbb{F}_2) \to \mathrm{H}_n(M, \partial M; \mathbb{F}_2)$.

(2) In this section, given a compact topological manifold we denote by PD_M the Poincaré Duality isomorphisms with $\mathbb{F}_2$-coefficients provided by the Poincaré Duality Theorem 119.1.

Now we can formulate the following theorem, which is a variation on Theorem 127.7.

Theorem 127.15. Let M be a compact connected m-dimensional topological manifold. Furthermore let A and B be compact $(m-1)$-dimensional submanifolds of ∂M such that $\partial A = \partial B = A \cap B$ and with $\partial M = A \cup B$. Let X be a compact k-dimensional topological manifold and let Y be a compact $(m-k)$-dimensional topological manifold. Finally let $f\colon X \to M$ and $g\colon Y \to M$ be two proper maps with $f(\partial X) \subset A$ and $g(\partial Y) \subset B$. If f and g intersect nicely, then

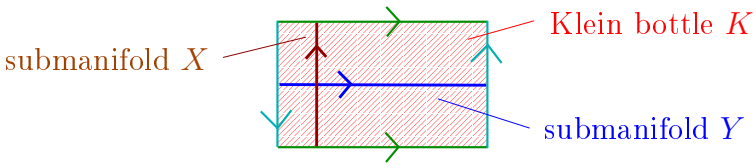
$$f \pitchfork_{\mathbb{F}_2} g \;=\; \Big\langle \overbrace{\mathrm{PD}_M\big(\underbrace{f_*([X]_{\mathbb{F}_2})}_{\in \mathrm{H}_k(M,A;\mathbb{F}_2)}\big)}^{\in \mathrm{H}^{m-k}(M,B;\mathbb{F}_2)} \cup \overbrace{\mathrm{PD}_M\big(\underbrace{g_*([Y]_{\mathbb{F}_2})}_{\in \mathrm{H}_{m-k}(M,B;\mathbb{F}_2)}\big)}^{\in \mathrm{H}^{k}(M,A;\mathbb{F}_2)}, \overbrace{[M]_{\mathbb{F}_2}}^{\in \mathrm{H}_m(M,\partial M;\mathbb{F}_2)} \Big\rangle \;\in\; \mathbb{F}_2^{\in \mathrm{H}^m(M,\partial M;\mathbb{F}_2)}.$$

In particular, if X and Y are already submanifolds of M, then

$$X \pitchfork_{\mathbb{F}_2} Y \;=\; \langle \mathrm{PD}_M([X]_{\mathbb{F}_2}) \cup \mathrm{PD}_M([Y]_{\mathbb{F}_2}), [M]_{\mathbb{F}_2} \rangle.$$

Proof. By the $\mathbb{F}_2$-Orientability Proposition 105.13 every topological manifold is $\mathbb{F}_2$-orientable. With this observation the proof of the theorem is a variation, and basically a simplification, of the proof of the Intersection Number-via-Cup Product Theorem 126.5. We leave it to the skeptical reader to fill in the details. ■

Example. Similar to the discussion on page 2754 we can use Theorem 127.15 to show that some homology classes are in fact non-zero. For example in the figure below we show the Klein bottle K together with two closed 1-dimensional submanifolds. These intersect in precisely one point, so their algebraic $\mathbb{F}_2$-intersection number is necessarily non-zero. Theorem 127.15 implies that $[X]_{\mathbb{F}_2}$ and $[Y]_{\mathbb{F}_2}$ are non-zero in $\mathrm{H}_1(K; \mathbb{F}_2)$.

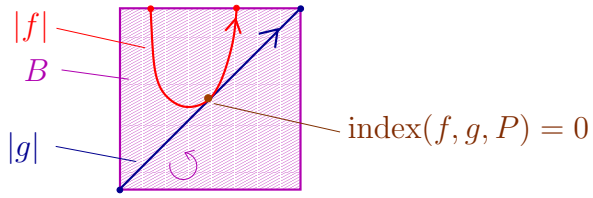


Remark. A discussion of intersection numbers in topological manifolds is also given in [**Dol1980**, Chapter VIII.13].

Exercises for Chapter 127.

Exercise 127.1. Is every topological space that is Hausdorff also normal?

Exercise 127.2. We consider the manifold $M = Z = [0,1]^2$ and the two embeddings $f, g\colon [0,1] \to Z$ that are shown in the figure below. Note that $P = (0,0)$ is the unique intersection point of $|f|$ and $|g|$. Show that $\operatorname{index}(f,g,P) = 0$.



Exercise 127.3. Can one use the Generalized Alexander Duality Theorem 124.2 to answer Question 127.1? Or some other of the earlier theorems? If yes, how would you do it? You could also try to restrict yourselves to the case that the map α is injective.

Exercise 127.4. Let M be a compact oriented connected n-dimensional topological manifold and let $N \subset M$ be a compact oriented proper submanifold of codimension one. Finally let $\gamma\colon S^1 \to M$ be a map that intersects N nicely. Show that

$$\langle \mathrm{PD}_M([N]), \gamma_*([S^1]) \rangle \;=\; \gamma \cdot N.$$

Exercise 127.5. In the figure on page 2779 we saw that $-1, 0, 1$ can be indices of nice intersection points of oriented submanifolds. Can any other numbers occur as indices of nice intersection points?

Exercise 127.6. Let $n \in \mathbb{N}$. By the discussion on page 2423 we know that for any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}^k(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } k = 0, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

Use Theorem 127.15 to show that the obvious map

$$\mathbb{F}_2[x]/(x^{n+1}) \;\to\; (\mathrm{H}^*(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2), \cup)$$

is a ring isomorphism.

Remark. In the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 121.16 we used a rather different approach to calculating the cup product on $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2)$.

128. The Lefschetz-Hopf Theorem

First we recall the following definition from page 2116.

Definition. Let M be a compact topological manifold and let $f\colon M \to M$ be a map. We define the Lefschetz number of f to be

$$\Lambda(f) := \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \operatorname{tr}(f_*\colon \mathrm{H}_n(M;\mathbb{Z}) \to \mathrm{H}_n(M;\mathbb{Z})).$$

The following is one of our main results on the Lefschetz number.

. Theorems 97.10 and 104.12 (Lefschetz Fixed Point Theorem). Let M be a compact smooth or topological manifold and let $f\colon M \to M$ be a map. If the Lefschetz number $\Lambda(f)$ is non-zero, then f has a fixed point.

In this chapter we will prove the Lefschetz-Hopf Theorems 128.5 and 128.9. These allow us to compute the Lefschetz number $\Lambda(f)$ as the sum of "indices" of the fixed points of a given map $f\colon M \to M$. In particular, if there are no fixed points, then this sum is zero. Thus the Lefschetz-Hopf Theorems are generalizations of the above version of the Lefschetz Fixed Point Theorem.

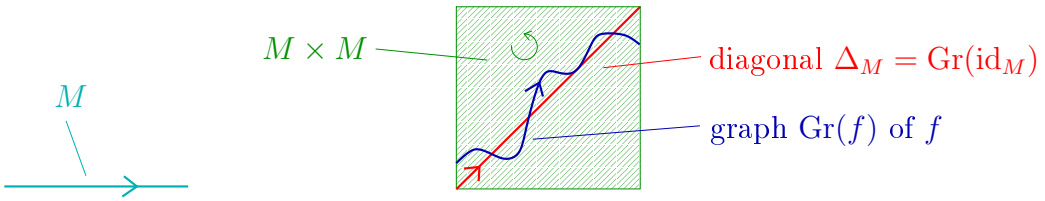
Recall that in the Euler Characteristic-Vector Field Theorem 97.15 we used the Lefschetz Fixed Point Theorem 97.10 to show show that if a closed smooth manifold has non-zero Euler characteristic, then every vector field on M has at least one zero. In the following chapter we will use the Lefschetz-Hopf Theorems 128.5 and 128.9 to prove the Poincaré-Hopf Theorem 129.7 which is a significant generalization of the Euler Characteristic-Vector Field Theorem 97.15.

128.1. **The Lefschetz number as an intersection number.** We start out with recalling and introducing some notation. We will make use of this notation throughout this chapter.

Notation. Let M be an n-dimensional topological respectively smooth manifold.

(1) It follows from Propositions 18.4 and 20.6 that we can view $M \times M$ as an $2n$-dimensional topological respectively smooth manifold (possibly with corner).
(2) We denote by $p_1, p_2\colon M \times M \to M$ the projection onto the first respectively second factor.
(3) We write $\Delta_M := \{(x,x) \in M \times M \mid x \in M\}$.
(4) Let $f\colon M \to M$ be a map. We denote by $\mathrm{id} \times f\colon M \to M \times M$ the map that is given by $x \mapsto (x, f(x))$. We write $\operatorname{Gr}(f) := (\mathrm{id} \times f)(M)$. Note that $\operatorname{Gr}(\mathrm{id}) = \Delta_M$.
(5) The map $\mathrm{id} \times f\colon M \to M$ is easily seen to be a (smooth) proper embedding. Thus we can view $\operatorname{Gr}(f)$ as a proper n-dimensional submanifold of $M \times M$.
(6) Now assume that M is oriented.
    (a) We equip $M \times M$ with the product orientation that we defined on pages 695 and 2502.
    (b) We equip $\operatorname{Gr}(f)$ with the unique orientation that turns $\mathrm{id} \times f$ into an orientation-preserving map.
    (c) If M is furthermore closed, then we denote as usual by $[\operatorname{Gr}(f)] \in \mathrm{H}_n(\operatorname{Gr}(f);\mathbb{Z})$ the fundamental class of $\operatorname{Gr}(f)$. By the usual slight abuse of notation we denote

by $[\mathrm{Gr}(f)] \in \mathrm{H}_n(M \times M; \mathbb{Z})$ the homology class represented by this submanifold. Note that $[\mathrm{Gr}(f)] = (\mathrm{id} \times f)_*([M]) \in \mathrm{H}_n(M \times M; \mathbb{Z})$.



To formulate the main result of this section we will also need the following definition.

Definition.

(1) Given a finitely generated abelian group we write $\mathrm{F}H := H/\operatorname{Tor}(H)$, where $\operatorname{Tor}(H)$ denotes the torsion subgroup of H.

(2) Let W be a closed oriented $2n$-dimensional topological manifold. We refer to the map

$$\begin{array}{rccl} Q_W\colon & \mathrm{FH}_n(W;\mathbb{Z}) \times \mathrm{FH}_n(W;\mathbb{Z}) & \to & \mathbb{Z} \\ & (a,b) & \mapsto & \langle \mathrm{PD}_W(a) \cup \mathrm{PD}_W(b), [W] \rangle \end{array}$$

as the intersection form of W.

Remark. We will discuss the intersection form in great detail in Chapter 132.

Now we can formulate the following, potentially quite counterintuitive proposition.

Proposition 128.1. Let M be a closed oriented connected n-dimensional topological manifold.

(1) We have $Q_{M\times M}([\Delta_M], [\Delta_M]) = \chi(M)$.

(2) Given any map $f\colon M \to M$ we have $Q_{M\times M}([\mathrm{Gr}(f)], [\Delta_M]) = \Lambda(f)$.

Remark. Sometimes in the literature, in line with the notation introduced on page 2781, the statement of Proposition 128.1 (1) gets shortened to the potentially ambiguous statement "$M \cdot M = \chi(M)$".

Examples.

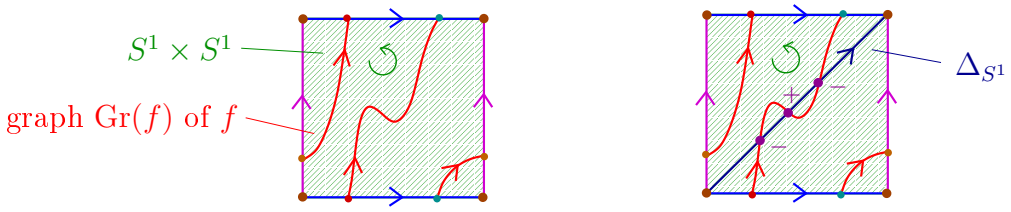
(1) Suppose that in Proposition 128.1 we actually have a smooth manifold M and a smooth map $f\colon M \to M$. In this case we can view Δ_M and $\mathrm{Gr}(f)$ as oriented n-dimensional submanifolds of $M \times M$. If these two submanifold are transverse, then it follows from the Intersection Number-via-Cup Product Theorem 126.5 that we can calculate $Q_{M\times M}$ geometrically, more precisely, we have the following equality:

$$Q_{M\times M}([\mathrm{Gr}(f)], [\Delta_M]) \;=\; \begin{array}{c}\text{algebraic intersection number } \mathrm{Gr}(f)\cdot\Delta_M \text{ of the oriented} \\ \text{submanifolds } \Delta_M \text{ and } \mathrm{Gr}(f), \text{ as defined on page 2748.}\end{array}$$

For example, let us consider the map $f\colon S^1 \to S^1$ that is illustrated in the figure below. We see that

$$\mathrm{Gr}(f)\cdot\Delta_M \underset{\substack{\uparrow\\ \text{there are three intersection points}\\ \text{with signs } -1,\ +1 \text{ and } -1}}{=} -1+1-1 \;=\; -1 \underset{\substack{\uparrow\\ \text{since } \mathrm{tr}(f \circlearrowright \mathrm{H}_0(S^1;\mathbb{Z})) = 1\\ \text{and } \mathrm{tr}(f \circlearrowright \mathrm{H}_1(S^1;\mathbb{Z})) = 2}}{=} (-1)^0\cdot 1 + (-1)^1 \cdot 2 \;=\; \Lambda(f).$$

Fortunately this is consistent with Proposition 128.1 (2).



(2) Let $n \in \mathbb{N}$. We have

$$Q_{S^n \times S^n}([\Delta_{S^n}], [\Delta_{S^n}]) \underset{\uparrow}{=} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ (-1)^{n^2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 + (-1)^{n^2} = \chi(S^n).$$

see Lemma 117.3

Again this is consistent with Proposition 128.1 (1).

The proof of Proposition 128.1 requires some preparations. In particular we will need the following lemma.

Lemma 128.2. Let M be a closed oriented connected n-dimensional topological manifold, let $i \in \{0, \ldots, n\}$ and let $v_1, \ldots, v_m$ be a basis of $\mathrm{FH}^i(M; \mathbb{Z})$. There exists a basis $\widetilde{v}_1, \ldots, \widetilde{v}_m$ of $\mathrm{FH}^{n-i}(M; \mathbb{Z})$ such that for any $k, l \in \{1, \ldots, m\}$ we have $\langle v_k \cup \widetilde{v}_l, [M] \rangle = \delta_{kl}$.

Proof of Lemma 128.2. The statement of the lemma is precisely the content of Exercise 119.12. For the reader's convenience we nonetheless provide the proof. For $j = 1, \ldots, m$ we set $\sigma_j := v_j \cap [M] \in \mathrm{FH}_{n-i}(M; \mathbb{Z})$. By the Poincaré Duality Theorem 119.1 we know that $\sigma_1, \ldots, \sigma_m$ is a basis for $\mathrm{FH}_{n-i}(M; \mathbb{Z})$. Next note that it follows from the Universal Coefficient Theorem 110.12 for Cohomology Groups together with the Ext-Calculation Lemma 110.5 (5) and the Topological Manifolds-Invariants Proposition 104.14 (4) that the map

$$\begin{array}{ccc} \mathrm{FH}^{n-i}(M; \mathbb{Z}) & \to & \mathrm{Hom}(\mathrm{FH}_{n-i}(M; \mathbb{Z}), \mathbb{Z}) \\ \varphi & \mapsto & (\sigma \mapsto \langle \varphi, \sigma \rangle) \end{array}$$

is an isomorphism. We denote by $\widetilde{v}_1, \ldots, \widetilde{v}_m$ the cohomology classes with $\langle \widetilde{v}_k, \sigma_l \rangle = \delta_{kl}$. By the above $\widetilde{v}_1, \ldots, \widetilde{v}_m$ are a basis for $\mathrm{FH}^{n-i}(M; \mathbb{Z})$. Now we see that for any $k, l \in \{1, \ldots, m\}$ we have

■

Lemma 128.2 allows us to formulate our next lemma, which will be a key ingredient in the proof of Proposition 128.1.

Lemma 128.3. (Dual-of-Diagonal Lemma) Let M be a closed oriented connected n-dimensional topological manifold. For $i = 0, \ldots, n$ we pick a basis $v_1^i, \ldots, v_{m_i}^i$ of $\mathrm{FH}^i(M; \mathbb{Z})$. By Lemma 128.2 there exists a basis $\widetilde{v}_1^{n-i}, \ldots, \widetilde{v}_{m_i}^{n-i}$ of $\mathrm{FH}^{n-i}(M; \mathbb{Z})$ such that for any $k, l \in \{1, \ldots, m_i\}$ we have $\langle v_k^i \cup \widetilde{v}_l^{n-i}, [M] \rangle = \delta_{kl}$.

(1) We have the following equality in $\mathrm{H}^n(M \times M; \mathbb{Z})$:

$$\mathrm{PD}_{M \times M}([\Delta_M]) = \sum_{i=0}^{n} (-1)^{i \cdot (n-i-1)} \cdot \sum_{r=1}^{m_i} p_1^*(\widetilde{v}_r^{n-i}) \cup p_2^*(v_r^i).$$

(2) Given any map $f\colon M \to M$ we have the following equality in $\mathrm{H}^n(M \times M; \mathbb{Z})$:

$$\mathrm{PD}_{M\times M}([\mathrm{Gr}(f)]) \;=\; \sum_{i=0}^{n} (-1)^{i\cdot(n-i-1)} \cdot \sum_{k=1}^{m_i}\sum_{l=1}^{m_i} \underbrace{\langle v_k^i \cup f^*(\widetilde{v}_l^{n-i}), [M]\rangle}_{\in\mathbb{Z}} \cdot p_1^*(\widetilde{v}_k^{n-i}) \cup p_2^*(v_l^i).$$

Proof of the Dual-of-Diagonal Lemma 128.3. First we show that Statement (1) follows almost immediately from Statement (2). Indeed, we have

$$\mathrm{PD}_{M\times M}(\overbrace{[\mathrm{Gr}(\mathrm{id}_M)]}^{=\Delta_M}) \;=\; \sum_{i=0}^{n} (-1)^{i\cdot(n-i-1)} \cdot \sum_{r=1}^{m_i}\sum_{s=1}^{m_i} \underbrace{\langle v_r^i \cup \widetilde{v}_s^{n-i}, [M]\rangle}_{=\delta_{rs}} \cdot p_1^*(\widetilde{v}_r^{n-i}) \cup p_2^*(v_s^i).$$

$\uparrow$ Statement (2) applied to $f = \mathrm{id}_M$

Thus it remains to prove Statement (2) of the Dual-of-Diagonal Lemma 128.3. We start out the proof of Statement (2) with the following claim.

Claim. For $j \in \{0, \ldots, n\}$ and $r, s \in \{1, \ldots, m_j\}$ there exist a unique $a_{j,r,s} \in \mathbb{Z}$ such that

$$\mathrm{PD}_{M\times M}([\mathrm{Gr}(f)]) \;=\; \sum_{j=0}^{n}\sum_{r=1}^{m_j}\sum_{s=1}^{m_j} a_{j,r,s} \cdot p_1^*(\widetilde{v}_r^{n-j}) \cup p_2^*(v_s^j) \;\in\; \mathrm{H}^n(M \times M; \mathbb{Z}).$$

Proof. Recall that by the Topological Manifolds-Invariants Proposition 104.14 (4) we know that the homology groups of M are all finitely generated. Thus it follows from the Künneth Theorem 117.16 for Cohomology Groups that there exists a short exact sequence of the following form:

$$0 \to \bigoplus_{j=0}^{n} \mathrm{H}^{n-j}(M;\mathbb{Z}) \otimes \mathrm{H}^j(M;\mathbb{Z}) \to \mathrm{H}^n(M\times M;\mathbb{Z}) \to \bigoplus_{j=0}^{n+1} \overbrace{\mathrm{Tor}(\mathrm{H}^{n+1-j}(M;\mathbb{Z}), \mathrm{H}^j(M;\mathbb{Z}))}^{\substack{\text{torsion group by the}\\ \text{Tor-Properties Lemma 89.15 (8)}}} \to 0.$$

$$\varphi \otimes \psi \mapsto p_1^*(\varphi) \cup p_2^*(\psi)$$

The Künneth Theorem 117.16 also says that the short exact sequence splits. Since the group on the right is torsion it follows easily, using some of the elementary properties of tensor product that we listed in the Tensor Lemma 89.3, that the map

$$\bigoplus_{j=0}^{n} \underbrace{\mathrm{FH}^{n-j}(M;\mathbb{Z})}_{=\bigoplus_r \mathbb{Z}\cdot\widetilde{v}_r^{n-j}} \otimes \underbrace{\mathrm{FH}^j(M;\mathbb{Z})}_{=\bigoplus_s \mathbb{Z}\cdot v_s^j} \xrightarrow{\varphi\otimes\psi\mapsto p_1^*(\varphi)\cup p_2^*(\psi)} \mathrm{FH}^n(M\times M;\mathbb{Z})$$

is an isomorphism. The claim follows immediately from this fact. ⊞

First we remind the reader of a few facts on the cup, cap and cross product. To shorten the discussion we leave it to the reader to figure out what the various symbols stand for.

(a) Given $\varphi \in \mathrm{H}^i(M;\mathbb{Z})$ and $\sigma \in \mathrm{H}_i(M;\mathbb{Z})$ we have $\varphi \cap \sigma = \langle \varphi, \sigma\rangle \in \mathrm{H}_0(M;\mathbb{Z}) = \mathbb{Z}$.
(b) By the Cup-Cap Lemma 116.7 we have $(\psi \cup \varphi) \cap \sigma = \varphi \cap (\psi \cap \sigma)$.
(c) By the Cup Product-Commutativity Proposition 114.8 we have $\varphi \cup \psi = (-1)^{\deg(\varphi)\cdot\deg(\psi)} \cdot \psi \cup \varphi$.
(d) Given $g\colon X \to Y$ we know by Lemma 109.6 (3) we have $\langle g^*(\varphi), \sigma\rangle = \langle \varphi, g_*(\sigma)\rangle$.
(e) Given $g\colon X \to Y$ we know by the Cup Product-Naturality Lemma 114.10 that $g^*(\varphi) \cup g^*(\psi) = g^*(\varphi \cup \psi)$.

(f) By the Topological Product Orientation Proposition 113.10 we have $[M \times M] = [M] \times [M]$. Furthermore, by the Fundamental Class-Product Proposition 117.2 we have $p_1^*([M]^*) \cup p_2^*([M]^*) = [M \times M]^*$.

(g) By the Product Theorem 117.1 we have

$$\big(p_1^*(\alpha) \cup p_2^*(\beta)\big) \cap (\mu \times \nu) = (-1)^{\deg(\alpha)\cdot\deg(\beta)+\deg(\beta)\cdot\deg(\mu)} \cdot (\alpha \cap \mu) \times (\beta \cap \nu).$$

(h) By Exercise 113.2 the cross product on H_0 corresponds to multiplication under the augmentation maps.

(i) Given any $k \in \mathbb{Z}$ we have $(-1)^k = (-1)^{k^2}$.

Given $i \in \{0, \dots, n\}$ and $k, l \in \{1, \dots, m_i\}$ we calculate that

$$\begin{aligned}
&\langle p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i}) \cup \mathrm{PD}_{M\times M}([\mathrm{Gr}(f)]), [M\times M]\rangle \\
&\quad = (-1)^n \cdot \langle p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i}), \mathrm{PD}_{M\times M}([\mathrm{Gr}(f)]) \cap [M\times M]\rangle && \text{by (a), (b), (c) and (i)} \\
&\quad = (-1)^n \cdot \langle p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i}), [\mathrm{Gr}(f)]\rangle && \text{by definition of } \mathrm{PD}_{M\times M} \\
&\quad = (-1)^n \cdot \langle p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i}), (\mathrm{id}\times f)_*([M])\rangle && \text{by definition of } [\mathrm{Gr}(f)] \\
&\quad = (-1)^n \cdot \langle (\mathrm{id}\times f)^*(p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i})), [M]\rangle && \text{by (d)} \\
&\quad = (-1)^n \cdot \langle (p_1\circ(\mathrm{id}\times f))^*(v_k^i) \cup (p_2\circ(\mathrm{id}\times f))^*(\widetilde{v}_l^{n-i}), [M]\rangle && \text{by (e)} \\
&\quad = (-1)^n \cdot \langle v_k^i \cup f^*(\widetilde{v}_l^{n-i}), [M]\rangle && \text{since } p_1\circ(\mathrm{id}\times f) = \mathrm{id} \\
&&& \text{and } p_2\circ(\mathrm{id}\times f) = f.
\end{aligned}$$

Next we perform an alternative calculation:

$$\begin{aligned}
&\langle p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i}) \cup \mathrm{PD}_{M\times M}([\mathrm{Gr}(f)]), [M\times M]\rangle \\
&\quad = \Big\langle (p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i})) \cup \sum_{j=0}^{n}\sum_{r=1}^{m_j}\sum_{s=1}^{m_j} a_{j,r,s}\cdot p_1^*(\widetilde{v}_r^{n-j}) \cup p_2^*(v_s^j), [M\times M]\Big\rangle && \text{by the claim} \\
&\quad = (-1)^{(n-i)(n-j)} \cdot \sum_{j=0}^{n}\sum_{r=1}^{m_j}\sum_{s=1}^{m_j} a_{j,r,s} \cdot \langle p_1^*(\overbrace{v_k^i \cup \widetilde{v}_r^{n-j}}^{\in \mathrm{H}^{i+n-j}(M)}) \cup p_2^*(\overbrace{\widetilde{v}_l^{n-i} \cup v_s^j}^{\in \mathrm{H}^{n-i+j}(M)}), [M\times M]\rangle && \text{by (c) and (e)} \\
&\qquad \underbrace{\qquad\qquad}_{=0, \text{ for } i+n-j>n} \quad \underbrace{\qquad\qquad}_{=0, \text{ for } n-i+j>n} \\
&\qquad \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{=0, \text{ for } i\neq j} \\
&\quad = (-1)^{n-i} \cdot \sum_{r=1}^{m_i}\sum_{s=1}^{m_i} a_{i,r,s} \cdot \langle p_1^*(v_k^i \cup \widetilde{v}_r^{n-i}) \cup p_2^*(\widetilde{v}_l^{n-i} \cup v_s^i), [M]\times[M]\rangle && \text{by (f) and (i)} \\
&\quad = (-1)^{n-i} \cdot \sum_{r=1}^{m_i}\sum_{s=1}^{m_i} a_{i,r,s} \cdot \langle v_k^i \cup \widetilde{v}_r^{n-i}, [M]\rangle \cdot \langle \widetilde{v}_l^{n-i} \cup v_s^i, [M]\rangle && \text{by (a), (g) and (h)} \\
&\quad = (-1)^{n-i} \cdot \sum_{r=1}^{m_i}\sum_{s=1}^{m_i} a_{i,r,s} \cdot (-1)^{i\cdot(n-i)} \cdot \underbrace{\langle v_k^i \cup \widetilde{v}_r^{n-i}, [M]\rangle}_{=\delta_{kr}} \cdot \underbrace{\langle v_s^i \cup \widetilde{v}_l^{n-i}, [M]\rangle}_{=\delta_{ls}} && \text{by (c)} \\
&\quad = (-1)^{n-i} \cdot \sum_{r=1}^{m_i}\sum_{s=1}^{m_i} a_{i,r,s} \cdot (-1)^{i\cdot(n-i)} \cdot \delta_{kr} \cdot \delta_{ls} = (-1)^{n-i+i\cdot(n-i)} \cdot a_{i,k,l}.
\end{aligned}$$

Comparing these two calculations of $\langle p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i}) \cup \mathrm{PD}_{M\times M}([\mathrm{Gr}(f)]), [M\times M]\rangle$ we see that $(-1)^{i\cdot(n-i-1)} \cdot \langle v_k^i \cup f^*(\widetilde{v}_l^{n-i}), [M]\rangle = a_{i,k,l}$. But that is exactly what we needed to show. ∎

Now we are finally ready to provide the proof of Proposition 128.1 (1). Note, that as we pointed out in Lemma 97.6, we have $\Lambda(\mathrm{id}) = \chi(M)$. Thus we see that Proposition 128.1 (1) is just a special case of Proposition 128.1 (2). Nonetheless, to reduce the notational mess we first prove Proposition 128.1 (1) separately.

Proof of Proposition 128.1 (1). We start out with the following claim which gives us a description of $\mathrm{PD}_{M\times M}([\Delta_M])$ that differs slightly from the one provided by the Dual-of-Diagonal Lemma 128.3 (1).

Claim. $$\mathrm{PD}_{M\times M}([\Delta_M]) \;=\; \sum_{j=0}^{n} (-1)^j \cdot \sum_{s=1}^{m_j} p_1^*(v_s^j) \cup p_2^*(\widetilde{v}_s^{n-j}).$$

Proof. Let $j \in \{0,\dots,n\}$. Recall that we have $v_r^{n-j} \cup \widetilde{v}_s^j = \delta_{rs}$. From the Cup Product-Commutativity Proposition 114.8 we obtain that $\widetilde{v}_s^j \cup v_r^{n-j} = (-1)^{j\cdot(n-j)} \cdot \delta_{rs}$. In other words, we have $\widetilde{v}_s^j \cup (-1)^{j\cdot(n-j)} \cdot v_r^{n-j} = \delta_{rs}$. Now we see that

$$\mathrm{PD}_{M\times M}([\Delta_M]) \;\underset{\uparrow}{=}\; \sum_{j=0}^{n} \overbrace{(-1)^{j\cdot(n-j-1)} \cdot (-1)^{j\cdot(n-j)}}^{=(-1)^j} \cdot \sum_{s=1}^{m_j} p_1^*(v_s^j) \cup p_2^*(\widetilde{v}_s^{n-j}).$$

by the Dual-of-Diagonal Lemma 128.3 (1), applied to the bases $\{\widetilde{v}_r^j\}_{r=1,\dots,m_j}$ and the corresponding dual bases $\{(-1)^{j\cdot(n-j)} \cdot v_s^j\}_{s=1,\dots,m_j}$

⊞

Now we calculate that

$$\begin{aligned}
&\mathrm{PD}_{M\times M}([\Delta_M]) \cup \mathrm{PD}_{M\times M}([\Delta_M]) = \\
&\quad\underset{\uparrow}{=}\; \Big(\sum_{i=0}^{n} (-1)^{i\cdot(n-i-1)} \cdot \sum_{r=1}^{m_i} p_1^*(\widetilde{v}_r^{n-i}) \cup p_2^*(v_r^i)\Big) \cup \Big(\sum_{j=0}^{n} (-1)^j \cdot \sum_{s=1}^{m_j} p_1^*(v_s^j) \cup p_2^*(\widetilde{v}_s^{n-j})\Big)\\
&\qquad \text{by the Dual-of-Diagonal Lemma 128.3 (1) and the claim}\\
&\quad= \sum_{i=0}^{n}\sum_{j=0}^{n} (-1)^{i\cdot(n-i-1)+j} \cdot \sum_{r=1}^{m_i}\sum_{s=1}^{m_j} p_1^*(\widetilde{v}_r^{n-i}) \cup p_2^*(v_r^i) \cup p_1^*(v_s^j) \cup p_2^*(\widetilde{v}_s^{n-j})\\
&\quad\underset{\uparrow}{=}\; \sum_{i=0}^{n}\sum_{j=0}^{n} (-1)^{i\cdot(n-i-1)+j} \cdot \sum_{r=1}^{m_i}\sum_{s=1}^{m_j} (-1)^{i\cdot j + j\cdot(n-i)} \cdot \underbrace{p_1^*(\overbrace{v_s^j \cup \widetilde{v}_r^{n-i}}^{\in \mathrm{H}^{j+n-i}(M)}) \cup p_2^*(\overbrace{v_r^i \cup \widetilde{v}_s^{n-j}}^{\in \mathrm{H}^{i+n-j}(M)})}_{=0,\text{ for } i\neq j}\\
&\qquad \underbrace{\qquad\qquad}_{=0 \text{ if } j+n-i>n} \qquad\qquad \underbrace{\qquad\qquad}_{=0 \text{ if } i+n-j>n}\\
&\qquad \text{by the Cup Product-Commutativity Proposition 114.8 and the Cup Product-Naturality Lemma 114.10}\\
&\quad= \sum_{i=0}^{n} (-1)^{i\cdot(n-i-1)} \cdot \sum_{r=1}^{m_i}\sum_{s=1}^{m_i} (-1)^{i\cdot(n-i)} \cdot p_1^*(\underbrace{v_s^i \cup \widetilde{v}_r^{n-i}}_{=\delta_{sr}\cdot[M]^*}) \cup p_2^*(\underbrace{v_r^i \cup \widetilde{v}_s^{n-i}}_{=\delta_{rs}\cdot[M]^*})\\
&\quad\underset{\uparrow}{=}\; \sum_{i=0}^{n} (-1)^{i\cdot(n-i-1)} \cdot \sum_{r=1}^{m_i}\sum_{s=1}^{m_i} (-1)^{i\cdot(n-i)} \cdot p_1^*(\underbrace{v_s^i \cup \widetilde{v}_r^{n-i}}_{=\delta_{sr}\cdot[M]^*}) \cup p_2^*(\underbrace{v_r^i \cup \widetilde{v}_s^{n-i}}_{=\delta_{rs}\cdot[M]^*})\\
&\qquad \text{since } i+i^2 \text{ is even}\\
&\quad= \sum_{i=0}^{n} (-1)^i \cdot \sum_{r=1}^{m_i} \underbrace{p_1^*([M]^*) \cup p_2^*([M]^*)}_{=[M\times M]^* \text{ by the Fundamental Class-Product Proposition 117.2}} \;=\; \chi(M)\cdot[M\times M]^*.
\end{aligned}$$

The desired statement now follows from the observation that by definition of $[M\times M]^*$ we have $\langle [M\times M]^*, [M\times M]\rangle = 1$. ■

Proof of Proposition 128.1 (2). The proof of Proposition 128.1 (2) is a generalization of the proof of Proposition 128.1 (1). It turns out that this argument is even less readable. In the following we quickly run through the argument and we only explain the steps that

differ from the proof of Proposition 128.1 (2). We calculate that

$$\begin{aligned}
&\mathrm{PD}_{M\times M}([\mathrm{Gr}(f)]) \cup \mathrm{PD}_{M\times M}([\Delta_M]) = \\
&= \Big(\sum_{i=0}^{n} (-1)^{i\cdot(n-i-1)} \cdot \sum_{r=1}^{m_i} \sum_{s=1}^{m_i} \langle v_r^i \cup f^*(\widetilde{v}_s^{n-i}), [M]\rangle \cdot p_1^*(\widetilde{v}_r^{n-i}) \cup p_2^*(v_s^i) \Big) \cup \Big(\sum_{j=0}^{n} (-1)^j \cdot \sum_{t=1}^{m_j} p_1^*(v_t^j) \cup p_2^*(\widetilde{v}_t^{n-j}) \Big) \\
&\quad \uparrow \text{ by the Dual-of-Diagonal Lemma 128.3 (2) and the rewriting procedure as in the claim in the proof of Proposition 128.1 (1)} \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{i\cdot(n-i-1)+j} \cdot \sum_{r=1}^{m_i} \sum_{s=1}^{m_i} \sum_{t=1}^{m_j} \langle v_r^i \cup f^*(\widetilde{v}_s^{n-i}), [M]\rangle \cdot p_1^*(\widetilde{v}_r^{n-i}) \cup p_2^*(v_s^i) \cup p_1^*(v_t^j) \cup p_2^*(\widetilde{v}_t^{n-j}) \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{i\cdot(n-i-1)+j} \cdot \sum_{r=1}^{m_i} \sum_{s=1}^{m_i} \sum_{t=1}^{m_j} (-1)^{i\cdot j + j\cdot(n-i)} \cdot \langle v_r^i \cup f^*(\widetilde{v}_s^{n-i}), [M]\rangle \cdot \underbrace{p_1^*(v_t^j \cup \widetilde{v}_r^{n-i}) \cup p_2^*(v_s^i \cup \widetilde{v}_t^{n-j})}_{=0 \text{ unless } i=j} \\
&= \sum_{i=0}^{n} (-1)^{i\cdot(n-i-1)+i} \cdot \sum_{r=1}^{m_i} \sum_{s=1}^{m_i} \sum_{t=1}^{m_i} (-1)^{i\cdot i + i\cdot(n-i)} \cdot \langle v_r^i \cup f^*(\widetilde{v}_s^{n-i}), [M]\rangle \cdot p_1^*(\underbrace{v_t^i \cup \widetilde{v}_r^{n-i}}_{=\delta_{tr}\cdot[M]^*}) \cup p_2^*(\underbrace{v_s^i \cup \widetilde{v}_t^{n-i}}_{=\delta_{st}\cdot[M]^*}) \\
&= \sum_{i=0}^{n} (-1)^i \cdot \sum_{r=1}^{m_i} \langle v_r^i \cup f^*(\widetilde{v}_r^{n-i}), [M]\rangle \cdot [M\times M]^* = \sum_{i=0}^{n} (-1)^i \cdot \sum_{r=1}^{m_i} \Big\langle v_r^i \cup \sum_{s=1}^{m_i} a_{rs}^i \cdot \widetilde{v}_s^{n-i}, [M] \Big\rangle \cdot [M\times M]^* \\
&\quad \uparrow \text{ we define } a_{rs}^i \text{ in the obvious way such that this equality holds} \\
&= \sum_{i=0}^{n} (-1)^i \cdot \sum_{r=1}^{m_i} \sum_{s=1}^{m_i} a_{rs}^i \cdot \underbrace{\langle v_r^i \cup \widetilde{v}_s^{n-i}, [M]\rangle}_{=\delta_{rs}} \cdot [M\times M]^* = \sum_{i=0}^{n} (-1)^i \cdot \underbrace{\sum_{r=1}^{m_i} a_{rr}^i}_{=\mathrm{tr}(f^* \circlearrowleft \mathrm{H}^i(M;\mathbb{Z}))} \cdot [M\times M]^* = \underset{\uparrow}{\Lambda(f)} \cdot [M\times M]^*. \\
&\qquad\qquad \text{by Exercise 110.12}
\end{aligned}$$

As in the proof of Proposition 128.1 (1) we are now done. ■

128.2. **The Lefschetz-Hopf Theorem I.** In Proposition 128.1 we saw that we can express the Lefschetz number of a self-map of a manifold as an intersection number. Furthermore in the Intersection Number-via-Cup Product Theorem 126.5 and in the more fancy Theorem 127.9 we learned how to determine intersection numbers using signs and indices of intersection points. The combination of the results, together with some fairly standard arguments, allows to prove two versions of the Lefschetz-Hopf Theorem. In this section we will use the more elementary the Intersection Number-via-Cup Product Theorem 126.5 to prove the more basic version of the Lefschetz-Hopf Theorem.

Definition.

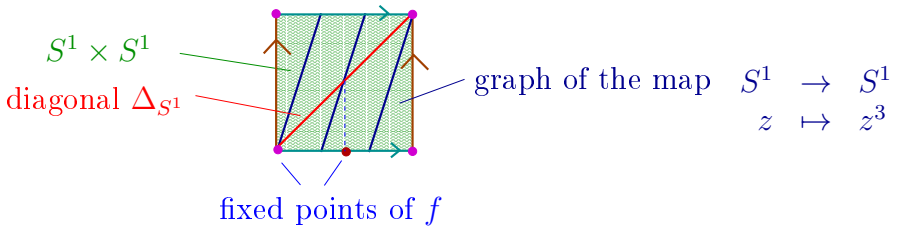
(1) Let M be a smooth manifold and let $f\colon M \to M$ be a smooth map.
   (a) Let $x \in M$ be a fixed point of f. Note that the differential $\mathrm{D}f_x$ is an endomorphism $\mathrm{D}f_x\colon \mathrm{T}_xM \to \mathrm{T}_xM$. We say that the fixed point x is non-degenerate if the endomorphism $\mathrm{id}_{\mathrm{T}_xM} - \mathrm{D}f_x$ of T_xM is invertible. Otherwise we say that the fixed point is degenerate.
   (b) Given a non-degenerate fixed point $x \in M$ we define

$$\mathrm{index}(f,x) \;:=\; \mathrm{sign}(\det(\mathrm{id} - \mathrm{D}f_x)) \;=\; \frac{\det(\mathrm{id} - \mathrm{D}f_x)}{\|\det(\mathrm{id} - \mathrm{D}f_x)\|}.$$

(2) Let X be a topological space and let $f\colon X \to X$ be a map. We say a fixed point x of f is isolated if there exists a neighborhood U of x which does not contain another fixed point.

Example. Let $n \in \mathbb{Z}$. We consider the map $f\colon S^1 \to S^1$ that is given by $f(z) = z^n$. An elementary calculation shows that for each fixed point $x \in S^1$ the differential $\mathrm{D}f_x$ is given by multiplication by n.

(1) If $n = 1$, then every point on S^1 is a fixed point of f, and each fixed point is degenerate.
(2) If $n \neq 1$, then the fixed points of f are $x_j = \exp(2\pi \mathrm{i} j/(n-1))$, $j = 1, \ldots, |n-1|$. If $n \geq 1$, then the index of every fixed point is $\operatorname{sign}(1-n) = -1$, and if $n \leq 0$, then the index of every fixed point is $\operatorname{sign}(1-n) = +1$.



Lemma 128.4.

(1) Let M be a smooth manifold and let $f\colon M \to M$ be a smooth map. Every non-degenerate fixed point is also an isolated fixed point.
(2) Let X be a topological space and let $f\colon X \to X$ be a map. We assume that X is compact and Hausdorff. In this setting every fixed point of f is isolated if and only if f has only finitely many fixed points.

Proof.

(1) Using charts we see that we only need to consider the case that M is an open subset of some $\mathbb{R}^n$ and we can assume that the fixed point is $x = 0$. We consider the map $g\colon M \to \mathbb{R}^n$ that is given by $x \mapsto x - f(x)$. By our hypothesis we know that $\mathrm{D}g_0 = \mathrm{id}_{\mathrm{T}_0 M} - \mathrm{D}f_0$ is invertible. It follows from the Inverse Function Theorem 19.12 that the map $x \mapsto g(x) = x - f(x)$ is a local diffeomorphism. In particular there exists an open neighborhood U of 0 such that the map $x \mapsto g(x) = x - f(x)$ is injective. But since $g(0) = 0$ this implies that U does not contain any other fixed point of f.
(2) This statement is the content of Exercise 5.20. For the reader's peace of mind we provide the proof. Thus let X be a topological space that is compact and Hausdorff. Furthermore let $f\colon X \to X$ be map. We denote by $F \subset X$ be the fixed point set of f.

If F is finite, then it follows easily from the hypothesis that X is Hausdorff that F is a discrete subset of X. Conversely, suppose that F is a discrete subset of X. Note that it follows from the Compact Image Lemma 2.13 and the Compact-Closed Lemma 1.21 together with the Product Topology Properties Proposition 5.6 that Δ_X and $\mathrm{Gr}(f)$ are closed subsets of $X \times X$. Thus $\Delta_X \cap \mathrm{Gr}(f)$ itself is a closed subset of $X \times X$, in particular it is a closed subset of Δ_X. We denote by $p\colon \Delta_X \to X$ the projection onto the first factor. Evidently this map is a homeomorphism and $F = p(\Delta_X \cap \mathrm{Gr}(f))$. Thus we have shown that F is a closed subset of X and by

hypothesis we know that F is discrete. Since X is compact we obtain from the Compact-Discrete Lemma 1.22 (1) that F is indeed finite. ■

Now we can formulate the first version of the Lefschetz-Hopf Theorem.

Theorem 128.5. (Lefschetz-Hopf Theorem I) Let M be a compact orientable smooth manifold and let $f\colon M \to M$ be a smooth map. If every fixed point of f is non-degenerate and if there is no fixed point on ∂M, then[1552]

$$\sum_{\substack{x \text{ a fixed} \\ \text{point of } f}} \operatorname{index}(f,x) \;=\; \Lambda(f).$$

Remark. If a smooth map has no fixed points, then the left hand side of the Lefschetz-Hopf Theorem 128.5 is evidently zero. Thus we see that the Lefschetz-Hopf Theorem 128.5 is a generalization of the Lefschetz Fixed Point Theorem 97.10, applied to the field $\mathbb{F} = \mathbb{Q}$.

Our first example hopefully puts any lingering worries about signs to rest.

Example. Let $n \in \mathbb{Z} \setminus \{1\}$. As on page 2806 we consider the map $f\colon S^1 \to S^1$ that is given by $f(z) = z^n$. We perform the following calculation:

$$\sum_{\substack{x \text{ a fixed} \\ \text{point of } f}} \operatorname{index}(f,x) \underset{\uparrow}{=} \sum_{j=1}^{|n-1|} \operatorname{index}(f, e^{2\pi \mathrm{i} j/(n-1)}) \underset{\uparrow}{=} \sum_{j=1}^{|n-1|} -\operatorname{sign}(n-1) \;=\; -(n-1).$$

both equalities follow from the discussion on page 2806

Fortunately this is consistent with the Lefschetz-Hopf Theorem 128.5 and the following calculation:

$$\Lambda(f) \;=\; \underbrace{\operatorname{tr}(f_*\colon \mathrm{H}_0(S^1;\mathbb{Z}) \to \mathrm{H}_0(S^1;\mathbb{Z}))}_{=1, \text{ see page 1594}} - \underbrace{\operatorname{tr}(f_*\colon \mathrm{H}_1(S^1;\mathbb{Z}) \to \mathrm{H}_1(S^1;\mathbb{Z}))}_{=n, \text{ by Lemma 76.10}} \;=\; 1-n.$$

Just for fun let us consider a second example.

Example. Let $n \in \mathbb{N}$. We consider $M = \mathbb{R}^n \cup \{\infty\}$. We denote by 0 the origin of $\mathbb{R}^n$. In Exercise 23.13 we showed that M is an n-dimensional smooth manifold with the atlas given by the obvious chart for $\mathbb{R}^n$ together with the chart

$$\begin{array}{rcl} M \setminus \{0\} & \to & \mathbb{R}^n \\ x & \mapsto & \begin{cases} \frac{1}{\|x\|^2} \cdot x, & \text{if } x \neq \infty, \\ 0, & \text{if } x = \infty. \end{cases} \end{array}$$

Furthermore we showed that the map

$$\begin{array}{rcl} f\colon M & \to & M \\ x & \mapsto & \begin{cases} 3x, & \text{if } x \neq \infty, \\ \infty, & \text{if } x = \infty \end{cases} \end{array}$$

is smooth, that it has two fixed points, namely the origin 0 and ∞, and that $\mathrm{D}f_0 = 3 \cdot \mathrm{id}$ and $\mathrm{D}f_\infty = \frac{1}{3} \cdot \mathrm{id}$. Thus we see

$$\sum_{\substack{x \text{ a fixed} \\ \text{point of } f}} \operatorname{index}(f,x) = \operatorname{index}(f,0) + \operatorname{index}(f,\infty) = \operatorname{sign}(\underbrace{(1-3)^n}_{\det(\mathrm{id} - \mathrm{D}f_0)}) + \operatorname{sign}(\underbrace{(1-\tfrac{1}{3})^n}_{\det(\mathrm{id} - \mathrm{D}f_\infty)}) = (-1)^n + 1.$$

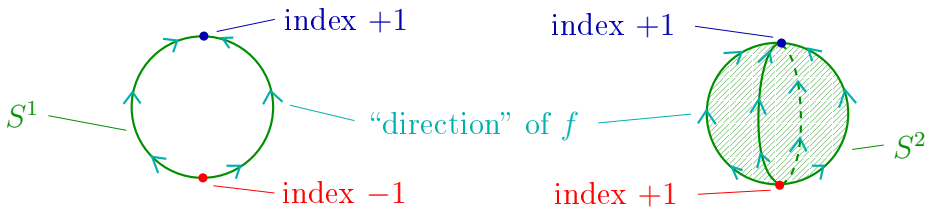
Conveniently enough this is consistent with the Lefschetz-Hopf Theorem 128.5 and the following calculation:

$$\Lambda(f) \underset{\uparrow}{=} \Lambda(\mathrm{id}) \;=\; \chi(S^n) \;=\; 1 + (-1)^n.$$

follows from Lemma 97.7, since f is easily seen to be homotopic to id

[1552] Note that by Lemma 128.4 we know that the left-hand side is a finite sum.

In Lemma 2.18 we gave an explicit diffeomorphism between $\Phi\colon S^n \to M = \mathbb{R}^n \cup \{\infty\}$ that sends the South Pole to the origin and that sends the North Pole to ∞. For $n = 1$ and $n = 2$ we illustrate the map $\Phi^{-1} \circ f \circ \Phi\colon S^n \to S^n$ in the figure below.



Proof of the Lefschetz-Hopf Theorem 128.5. Let M be a compact orientable n-dimensional smooth manifold and let $f\colon M \to M$ be a smooth map such that every fixed point is non-degenerate. We pick an orientation. First we assume that M is closed.

We pick an orientation for M. We equip $\mathrm{Gr}(f) \subset M \times M$ and $\Delta_M \subset M \times M$ with the orientation which turns the projection onto the first factor into an orientation-preserving diffeomorphism.

Claim. Let x be a fixed point of f.[1553]

(1) The point (x, x) is a transverse intersection point of the submanifolds $\mathrm{Gr}(f)$ and Δ_M if and only if x is non-degenerate.

(2) If (x, x) is a transverse intersection point of $\mathrm{Gr}(f)$ and Δ_M, then the sign of the intersection point is given by $\mathrm{index}(f, x) = \frac{\det(\mathrm{id} - \mathrm{D}f_x)}{\|\det(\mathrm{id} - \mathrm{D}f_x)\|}$.

Proof. Note that we have a natural identification $\mathrm{T}_{(x,x)}(M \times M) = \mathrm{T}_x M \times \mathrm{T}_x M$. We pick a positive basis $v_1, \ldots, v_n$ for $\mathrm{T}_x M$. By the definition of the product orientation on $M \times M$, see page 695, we see that the basis $(v_1, 0), \ldots, (v_n, 0), (0, v_1), \ldots, (0, v_n)$ is a positive basis for $\mathrm{T}_{(x,x)}(M \times M)$. Next note that with respect to this basis we have

$$\underbrace{\begin{pmatrix} \mathrm{id}_n \\ \mathrm{D}f_x \end{pmatrix}}_{\substack{\text{positive basis} \\ \text{for } \mathrm{T}_{(x,x)}\mathrm{Gr}(f)}} \text{ followed by } \underbrace{\begin{pmatrix} \mathrm{id}_n \\ \mathrm{id}_n \end{pmatrix}}_{\substack{\text{positive basis} \\ \text{for } \mathrm{T}_{(x,x)}\Delta_M}} \text{ which gives us the matrix } \begin{pmatrix} \mathrm{id}_n & \mathrm{id}_n \\ \mathrm{D}f_x & \mathrm{id}_n \end{pmatrix} \text{ which has the same determinant as } \begin{pmatrix} \mathrm{id}_n & 0 \\ \mathrm{id}_n & \mathrm{id}_n - \mathrm{D}f_x \end{pmatrix}.$$

The desired statements follow immediately from this observation and the definitions. ⊞

Now we perform the following calculation:

$$\Lambda(f) \overset{\text{Proposition 128.1}}{=} \mathrm{Gr}(f) \cdot \Delta_M \overset{\substack{\text{by the Intersection Number-via-Cup Product} \\ \text{Theorem 126.5 and Claim (1)}}}{=} \sum_{\substack{(x,x) \text{ an intersection point} \\ \text{of } \mathrm{Gr}(f) \text{ and } \Delta_M}} \text{sign of } (x,x) \overset{\text{by Claim (2)}}{=} \sum_{\substack{x \text{ a fixed} \\ \text{point of } f}} \mathrm{index}(f, x).$$

We have thus proved the theorem in the special case that M is closed and orientable.

Now we allow that the smooth manifold M has non-empty boundary. As on page 1667 we consider the double $\mathrm{D}M := (M \times \{1\}) \sqcup_{\partial M \times \{1\} = \partial M \times \{2\}} (M \times \{2\})$. By the Doubling-Manifold Lemma 75.12 we know that $\mathrm{D}M$ is a closed orientable smooth manifold. We also

[1553] We refer to pages 2748 and 2748 for the definition of a transverse intersection point and the sign of a transverse intersection point.

consider the "double of f", i.e. we consider the map

$$\begin{array}{rcl} \psi\colon \mathrm{D}M & \to & \mathrm{D}M \\ [(P,\epsilon)] & \mapsto & [(f(P),\epsilon)]. \end{array}$$

Let $x_1,\ldots,x_k$ be the fixed points of $f\colon M\to M$. Recall that we demand that no fixed point of f lies on ∂M. It follows immediately that the fixed points of ψ are precisely $(x_1,1),\ldots,(x_k,1)$, $(x_1,2),\ldots,(x_k,2)$. Now we see that

$$\begin{aligned} 2\cdot\sum_{i=1}^{k}\operatorname{index}(f,x_i) &\underset{\uparrow}{=} \sum_{i=1}^{k}\operatorname{index}(\psi,(x_i,1))+\sum_{i=1}^{k}\operatorname{index}(\psi,(x_i,2)) \underset{\uparrow}{=} \Lambda(\psi\colon \mathrm{D}M\to\mathrm{D}M) \\ &\underset{\uparrow}{=} 2\cdot\Lambda(f\colon M\to M)+\Lambda(f\colon \partial M\to\partial M) \underset{\uparrow}{=} 2\cdot\Lambda(f). \end{aligned}$$

it is elementary to see that $\operatorname{index}(f,x_i)=\operatorname{index}(\psi,(x_i,1))=\operatorname{index}(\psi,(x_i,2))$

by the above, since $\mathrm{D}M$ is a closed orientable smooth manifold

follows from Exercise 97.15

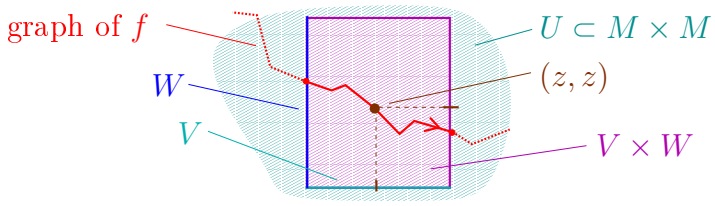
since there are no fixed points on ∂M and since ∂M is a closed orientable smooth manifold we know by the above that $\Lambda(f\colon \partial M\to\partial M)=0$

Dividing both sides by 2 gives us the desired equality. ■

128.3. **The Lefschetz-Hopf Theorem II.** In this section we will formulate and prove an advanced version of the Lefschetz-Hopf Theorem 128.5 which puts fewer restrictions on the fixed points. The down-side is that in general we need to work with a more tricky definition of the index of a fixed point.
We start out with the following not overly exciting lemma.

Lemma 128.6. Let M be an n-dimensional topological manifold and let $f\colon M\to M$ be a map. Let $z\in M\setminus\partial M$ be an isolated fixed point of $f\colon M\to M$. Given any neighborhood U of $(z,z)\in M\times M$ there exists a neighborhood V of $z\in M$ and a neighborhood W of $z\in M$ with the following properties:

(1) We have $V\times W\subset U$.
(2) z is the only fixed point of f contained in V.
(3) V and W are both submanifolds of M and both are homeomorphic to $\overline{B}^n$.
(4) We have $V\subset\mathring{W}$ and $f(V)\subset\mathring{W}$.



Proof. This lemma follows fairly easily from Exercise 18.6. We leave it to the reader to fill in the details. ■

The lemma allows us to introduce the following definition.

Definition. Let M be an n-dimensional topological manifold and let $f\colon M\to M$ be a map. Furthermore let $z\in M\setminus\partial M$ be an isolated fixed point of f. We pick neighborhoods V and W of z that satisfy Properties (2) and (4) of Lemma 128.6. Furthermore we pick an embedding $\Phi\colon W\to\mathbb{R}^n$ with $\Phi(z)=0$. We define the index $\operatorname{index}(f,z)$ of the fixed

point z as the degree of the following homomorphism:

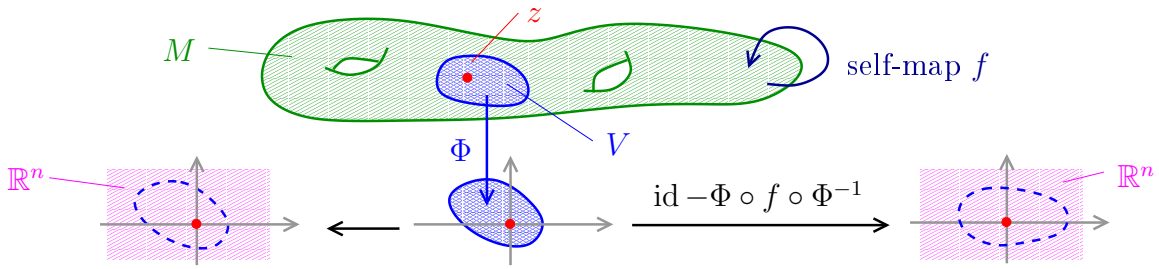
$\cong \mathbb{Z}$ by the Manifold-Local Homology Lemma 75.1 (2)

$$\overbrace{\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})} \quad \xleftarrow[\uparrow]{\cong} \quad \mathrm{H}_n(\Phi(V), \Phi(V) \setminus \{0\}) \quad \xrightarrow[\uparrow]{\mathrm{id} - (\Phi \circ f \circ \Phi^{-1})_*} \quad \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}).$$

this map is induced by the inclusion $\Phi(V) \subset \mathbb{R}^n$, it is an isomorphism by the Excision Theorem 74.18, here we also use that we obtain from the Topological Invariance of Domain Theorem 82.7 that $\Phi(V)$ is a neighborhood of 0

map defined since $f(V) \subset W$ and since x is only fixed point of f contained in V

We illustrate this definition in the figure below.



Example. Let $n \in \mathbb{N}$. We consider the map

$$\begin{aligned} f \colon \mathbb{R}^2 = \mathbb{C} &\to \mathbb{C} = \mathbb{R}^2 \\ z &\mapsto z + z^n. \end{aligned}$$

The point $z = 0$ is evidently an isolated fixed point. Note that $f'(z) = 1 + nz^{n-1}$. It follows that for $n \geq 2$ we have $\mathrm{id} - \mathrm{D}f_0 = \mathrm{id} - \mathrm{id} = 0$. Thus we we see that for $n \geq 2$ the fixed point 0 is degenerate. Since 0 is an isolated fixed point we can nonetheless define the index using the approach above. We set $V = \overline{B}^2$ and $W = \overline{B}^2_3$ and $\Phi = \mathrm{id}$. Since

$$\mathrm{id} - (\Phi \circ f \circ \Phi^{-1}) \;=\; \mathrm{id} - f \;=\; \mathrm{id} - (\mathrm{id} + (z \mapsto z^n)) \;=\; -(z \mapsto z^n)$$

we obtain easily from Lemma 76.10 that $\mathrm{index}(f, 0) = -n$.

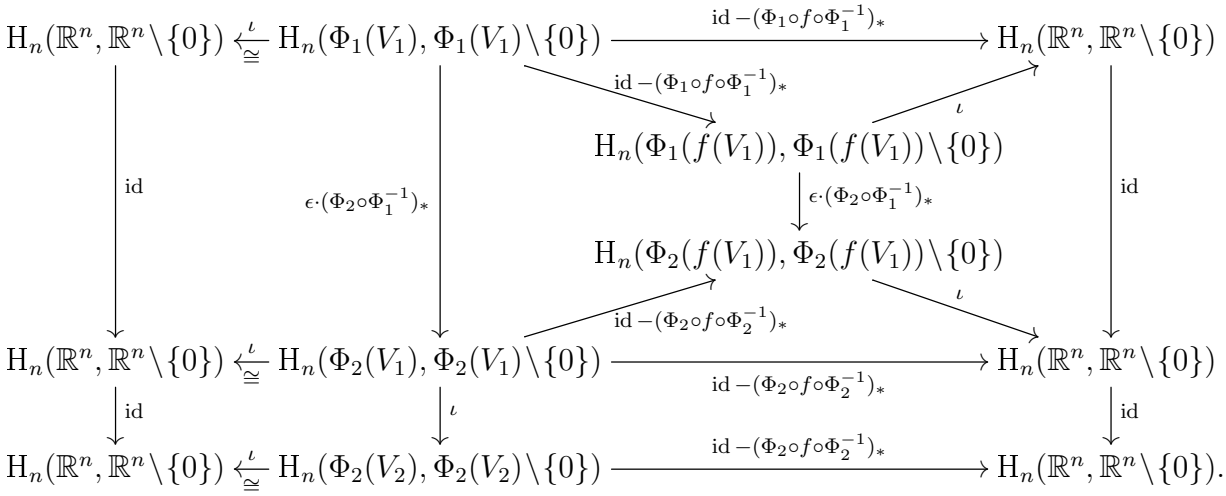
The following lemma says that the above index is well-defined.

Lemma 128.7. Let M be an n-dimensional topological manifold and let $f \colon M \to M$ be a map. For an isolated fixed point $z \in M \setminus \partial M$ of f the above definition of $\mathrm{index}(f, z)$ does not depend on any choices.

Proof. Let V_1, W_1 and V_2, W_2 be neighborhoods of z that satisfy the properties stated in Lemma 128.6 (2) and (4). We pick embeddings $\Phi_1 \colon W_1 \to \mathbb{R}^n$ and $\Phi_2 \colon W_2 \to \mathbb{R}^n$ as in the definition of the index. We need to show that the choices lead to the same index.

First note that by the "given any neighborhood" feature of Lemma 128.6 we can assume that $V_1 \times W_1 \subset V_2 \times W_2$. Next we set $\epsilon := +1$ if $\Phi_2 \circ \Phi_1^{-1}$ is orientation-preserving, otherwise

we set $\epsilon := -1$. Next we consider the following diagram:



We make the following clarifications and observations:

(1) The maps indicated by ι are the natural inclusion induced maps.
(2) Since $\Phi_1\colon W_1 \to \mathbb{R}^n$ is an embedding and since $V_1 \subset W_1$ we can consider the inverse $\Phi_1^{-1}\colon \Phi_1(V_1) \to V_1$. Furthermore, since $V_1 \subset V_2$ we can consider $\Phi_2(V_1)$ and we have $\Phi_2(V_1) \subset \Phi_2(V_2)$.
(3) It follows fairly immediately from the definition of ϵ that the diagram commutes.
(4) The left horizontal maps are isomorphisms by the Excision Theorem 74.18.

It follows from the above that the top and the bottom horizontal maps from left to right agree. But this means that we obtain the same degree. ■

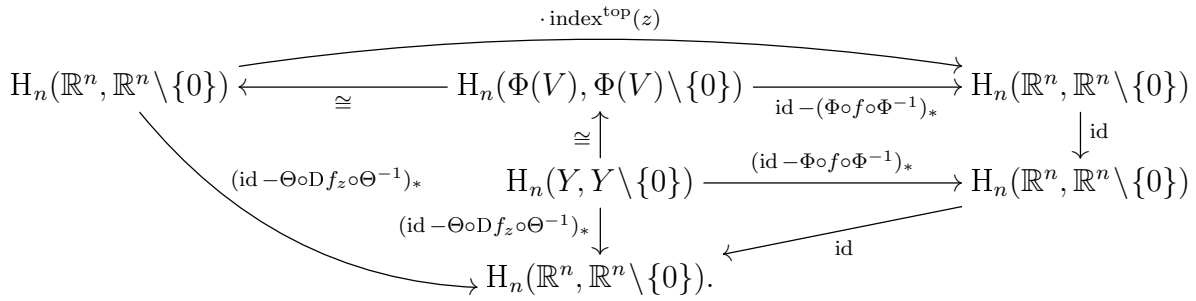
The following lemma shows that the two notions of the index of a fixed point of a smooth map, introduced on pages 2805 and 2810 are consistent.

Lemma 128.8. Let M be a smooth manifold, let $f\colon M \to M$ be a smooth map and let $z \in M \setminus \partial M$ be a non-degenerate fixed point of f. The fixed point z is isolated, and the index of z, as defined on page 2805, agrees with the index of z as defined on page 2810.

Proof. Let M be a smooth manifold, let $f\colon M \to M$ be a smooth map and let $z \in M\setminus\partial M$ be a non-degenerate fixed point of f. For the purpose of this proof we denote by $\mathrm{index}^{\mathrm{sm}}(z)$ the index of z as defined on page 2805 and we denote by $\mathrm{index}^{\mathrm{top}}(z)$ the index of z as defined on page 2810.

First note that we already showed in Lemma 128.4 that z is an isolated fixed point. Thus it remains to show that $\mathrm{index}^{\mathrm{top}}(z) = \mathrm{index}^{\mathrm{sm}}(z)$. From our smooth atlas for M we pick a chart $\Phi\colon W \to \mathbb{R}^n$ around z with $\Phi(z) = 0$. It follows fairly easily from the hypothesis that z is a non-degenerate fixed point that there exists an open neighborhood V around z such that $V \subset W$, such that $\Phi(V) \subset W$, and such that V contains no other fixed point of f. We set $\Theta := \mathrm{D}\Phi_z\colon \mathrm{T}_z M \to \mathrm{T}_0\mathbb{R}^n = \mathbb{R}^n$.

Now we consider the following diagram:



We make the following clarifications and observations:

(1) By construction V and W satisfy the properties of Lemma 128.6 (2) and (4). This implies that the composition of the top two horizontal maps is given by multiplication by $\operatorname{index}^{\mathrm{top}}(z)$.
(2) It follows immediately from Lemma 76.15 that there exists an open neighborhood Y of 0, contained in $\Phi(V)$, such that the map $\mathrm{id} - \Phi \circ f \circ \Phi^{-1}$ is homotopic, as a map of pairs $(\Phi(V), \Phi(V) \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$, to the map $\mathrm{D}(\mathrm{id} - \Phi \circ f \circ \Phi^{-1})_0$.
(3) By the Chain Rule 19.10 we have $\mathrm{D}(\mathrm{id} - \Phi \circ f \circ \Phi^{-1})_0 = \mathrm{id} - \Theta \circ \mathrm{D}f_z \circ \Theta^{-1}$.
(4) It follows from (2) together with Proposition 74.15 and some elementary considerations that the diagram commutes.
(5) It follows from the Excision Theorem 74.18 that the two maps decorated with "$\cong$" are indeed isomorphisms.

We see that

$$\begin{array}{ccc} & \overset{\text{by the above diagram and the above discussion}}{\downarrow} & \overset{\text{by Lemma 76.16}}{\downarrow} \\ \operatorname{index}^{\mathrm{top}}(z) & = \deg((\mathrm{id}_{\mathbb{R}^n} - \Theta \circ \mathrm{D}f_z \circ \Theta^{-1})_*) & = \operatorname{sign}(\det(\mathrm{id}_{\mathbb{R}^n} - \Theta \circ \mathrm{D}f_z \circ \Theta^{-1})) \\ & = \operatorname{sign}(\det(\mathrm{id}_{\mathrm{T}_z M} - \mathrm{D}f_z)) & = \operatorname{index}^{\mathrm{sm}}(z). \\ & \underset{\text{basic linear algebra}}{\uparrow} & \underset{\text{by the definition on page 2805}}{\uparrow} \end{array}$$

■

By Lemma 128.8 the following theorem is a generalization of our previous Lefschetz-Hopf Theorem 128.9.

Theorem 128.9. (Lefschetz-Hopf Theorem II) Let M be a compact orientable topological manifold and let $f\colon M \to M$ be a map which has no fixed points on ∂M. If f has finitely many fixed points, then

$$\sum_{\substack{x \text{ a fixed} \\ \text{point of } f}} \operatorname{index}(f, x) \;=\; \Lambda(f).$$

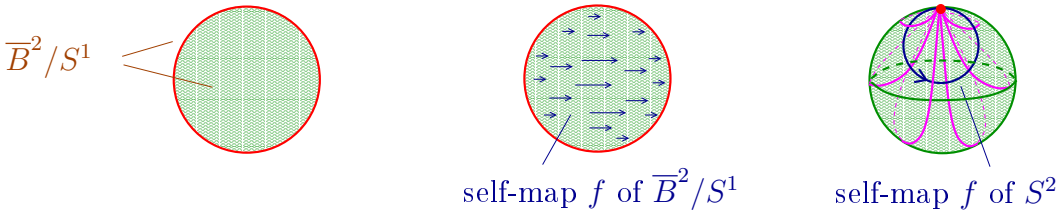
Remark.

(1) The Lefschetz-Hopf Theorem has its origin in the work of Heinz Hopf [**Hopf1929**, Section 8]. Various variations on the Lefschetz-Hopf Theorem 128.9 can be found in [**GD2003**, Chapter V.16] and [**JM2006**, Chapter 2.4].
(2) As on page 2807 we see that the Lefschetz-Hopf Theorem 128.9 is a generalization of the Lefschetz Fixed Point Theorem 104.12 for compact orientable topological manifolds.

Example. This time we consider S^2 with the avatar given by $\overline{B}^2/S^1$. We consider the map $f\colon \overline{B}^2/S^1 \to \overline{B}^2/S^1$ that is illustrated, to the best of the author's abilities, in the figure below.[1554] It has precisely one fixed point, namely $P := [S^1]$. We see that

$$\operatorname{index}(f,P) \underset{\text{Lefschetz-Hopf Theorem 128.9}}{\overset{\uparrow}{=}} \Lambda(f) \underset{\text{since } f \text{ is homotopic to id}}{\overset{\uparrow}{=}} \Lambda(\mathrm{id}) = \chi(S^2) = 2.$$

Note that with a little bit of care, say using the Smooth Transition Function Lemma 19.4, one can arrange that f is smooth. Also note that $\mathrm{D}f = \mathrm{id}$. Thus $P = [S^1]$ is a degenerate fixed point of f.[1555]



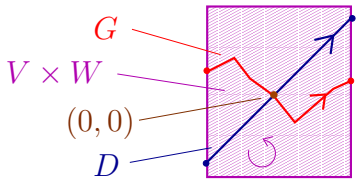
The proof of the Lefschetz-Hopf Theorem 128.9 rests heavily on the shoulders of the following two lemmas.

Lemma 128.10. Let $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^n$ be two compact connected n-dimensional topological submanifolds of $\mathbb{R}^n$ that contain 0 in the interior. We assume that $V \subset \mathring{W}$. Let $f\colon V \to \mathbb{R}^n$ be a map with $f(V) \subset \mathring{W}$ and which has a unique fixed point, namely 0. We set $G := \{(v, f(v)) \,|\, v \in V\}$ and we set $D := \{(v,v) \,|\, v \in V\}$.

(1) G and D are well-behaved proper submanifolds of $V \times W$, in the sense of the definition on page 2777.

(2) The following equality holds:[1556] [1557]

$$\Big\langle \underbrace{\mathrm{PD}_{V\times W}([G])}_{\in \mathrm{H}^n((V\times W)\setminus \partial G)} \cup \underbrace{\mathrm{PD}_{V\times W}([D])}_{\in \mathrm{H}^n((V\times W)\setminus \partial D)}, [V\times W] \Big\rangle = \deg \begin{pmatrix} \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\ \uparrow \cong \\ \mathrm{H}_n(V, V\setminus\{0\}) \\ \downarrow (\mathrm{id}-f)_* \\ \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \end{pmatrix}.$$



[1554] An alternative description of this map is given by considering $S^2 = \mathbb{R}^2 \cup \{\infty\}$. In this setting our map corresponds to the map $f\colon \mathbb{R}^2 \cup \{\infty\} \to \mathbb{R}^2 \cup \{\infty\}$ that is given by $f(x) = x + (1,0)$ for all $x \in \mathbb{R}^2$ and $f(\infty) = \infty$.

[1555] Incidentally this example gives an affirmative answer to the question asked in Exercise 2.12.

[1556] By (1) we know that G is a well-behaved proper submanifold of $V \times W$. Thus we can consider the isomorphism $\mathrm{PD}_{V\times W}\colon \mathrm{H}_n(V\times W, \partial G) \to \mathrm{H}^n(V\times W, (V\times W)\setminus \partial G)$ that we introduced in Proposition 127.2. Similarly we consider the isomorphism $\mathrm{PD}_{V\times W}$ with G replaced by D.

[1557] Since 0 is the only fixed point of f we see that $(\mathrm{id}-f)(V\setminus\{0\}) \subset \mathbb{R}^n \setminus \{0\}$.

Sketch of proof. First we prove Statement (1). More precisely, we will show that G is a well-behaved proper submanifold of $V \times W$. The proof that D is also a well-behaved submanifold is basically identical. Note that by Exercise 18.8 we know that G is a proper submanifold of $V \times W$. Next note by definition of a well-behaved submanifold we need to show that given any open neighborhood U of $\partial G \subset \partial(V \times W) = (\partial V \times W) \cup (V \times \partial W)$ the submanifold ∂G of $\partial(V \times W)$ admits a regular neighborhood N with $N \subset U$. Note that using the fact that ∂V is compact and using the hypothesis that $f(V) \subset \mathring{W}$ it is fairly elementary to see that there exists an $\epsilon > 0$ such that for every $x \in \partial G$ we have $\{x\} \times \overline{B}^n_\epsilon(f(x)) \subset U$. It is straightforward to verify that given any $\nu \in (0, \epsilon)$ the set

$$N_\nu \;:=\; \{(x,y) \in \partial V \times W \,|\, x \in \partial V \text{ and } y \in \overline{B}^n_\nu(f(x))\} \;\subset\; \partial(V \times W)$$

is a regular neighborhood of ∂G with $N_\nu \subset U$. This concludes the proof of Statement (1).

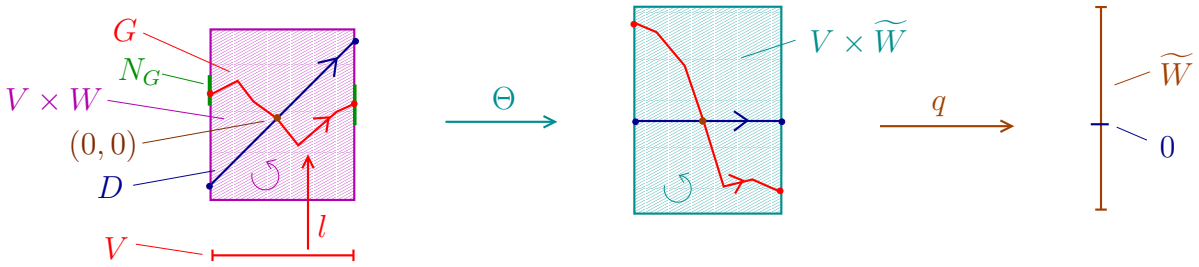
Next we turn to the proof of Statement (2). The key idea is to "tilt" the situation to turn the diagonal D into a "horizontal submanifold". Afterwards we can use the Poincaré Duality-Products Proposition 119.2 to perform the desired calculation.

Now we carry out the details. To do so we introduce the following objects:

(a) We pick a compact submanifold $\widetilde{W}$ of $\mathbb{R}^n$ with $W \subset \widetilde{W}$ and $(\mathrm{id} - f)(W) \subset \widetilde{W}$.
(b) We consider the maps

$$\begin{array}{rcl} l\colon V & \to & G \\ v & \mapsto & (v, f(v)) \end{array} \qquad \begin{array}{rcl} \Theta\colon V \times W & \to & V \times \widetilde{W} \\ (v,w) & \mapsto & (v, w - f(v)) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} q\colon V \times \widetilde{W} & \to & \widetilde{W} \\ (v,w) & \mapsto & w. \end{array}$$

(c) Note that by (1) we can pick a regular neighborhood N_G for $\partial G \subset \partial(V \times W)$ with $N_G \subset \partial V \times W$ and with $N_G \cap \partial D = \varnothing$.
(d) We denote by $\mu_{\widetilde{W},0} \in \mathrm{H}_n(\widetilde{W}, \widetilde{W} \setminus \{0\}) \cong \mathbb{Z}$ the standard generator, as defined on page 2328. Furthermore we denote by $\mu^*_{\widetilde{W},0} \in \mathrm{H}^n(\widetilde{W}, \widetilde{W} \setminus \{0\}; \mathbb{Z}) \cong \mathbb{Z}$ the corresponding dual.



Now we can perform the desired calculation:[1558]

$$\begin{array}{l}
\text{by Lemmas 116.4 and 116.7, under the natural identification } \mathrm{H}_0(V\times W)=\mathbb{Z} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big\downarrow \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =[G]\in \mathrm{H}_n(V\times W,N_G) \text{ by definition of } \mathrm{PD}_{V\times W} \\
\big\langle \mathrm{PD}_{V\times W}([G])\cup \mathrm{PD}_{V\times W}([D]),[V\times W]\big\rangle \;=\; \mathrm{PD}_{V\times W}([D])\cap\big(\overbrace{\mathrm{PD}_{V\times W}([G])\cap[V\times W]}\big) \\
\quad = \mathrm{PD}_{\Theta(V\times W)}(\Theta_*([D]))\cap\Theta_*([G]) \qquad\quad = \mathrm{PD}_{V\times\widetilde{W}}(\Theta_*([D]))\cap\Theta_*([G]) \\
\quad\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\
\text{by the Cap Product-Naturality Lemma 116.8 and since the map } \Theta\colon V\times W\to\Theta(V\times W)\subset V\times\widetilde{W} \text{ is an } \textit{orientation-preserving} \text{ homeomorphism} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{follows from an argument similar to the one provided in the Codimension-Zero Submanifold-PD Lemma 120.5} \\
\quad = \mathrm{PD}_{V\times\widetilde{W}}([V\times\{0\}])\cap\Theta_*(l_*([V])) \quad = (-1)^n\cdot q^*(\mu^*_{\widetilde{W},0})\cap\Theta_*(l_*([V])) \\
\quad\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\
\text{since } \Theta(D)=V\times\{0\} \text{ and } l(V)=G \text{ and since both maps are orientation-preserving} \qquad \text{this can be deduced fairly easily from the Poincaré Duality-Products Proposition 119.2} \\
\quad = (-1)^n\cdot\big\langle\mu^*_{\widetilde{W},0},q_*(\Theta_*(l_*([V])))\big\rangle \quad = \big\langle\mu^*_{\widetilde{W},0},(\mathrm{id}-f)_*([V])\big\rangle. \\
\quad\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\
\text{by Lemmas 116.4 and 109.6 (3)} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{since } q\circ\Theta\circ l=f-\mathrm{id}\colon V\to\widetilde{W} \text{ and since we know from Lemma 76.16 that } (f-\mathrm{id})_*=(-1)^n\cdot(\mathrm{id}-f)_* \\
\quad = \deg\big(\mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\})\xleftarrow{\cong}\mathrm{H}_n(V,V\setminus\{0\})\xrightarrow{(\mathrm{id}-f)_*}\mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\})\big). \\
\quad\uparrow \\
\text{this follows from the following commutative diagram}
\end{array}$$

$$\begin{array}{ccccc}
\mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) & \xleftarrow{\cong} & \mathrm{H}_n(V,V\setminus\{0\}) & \xrightarrow{(\mathrm{id}-f)_*} & \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) \\
 & & \uparrow & & \uparrow \\
 & & \mathrm{H}_n(V,\partial V) & \xrightarrow{(\mathrm{id}-f)_*} & \mathrm{H}_n(\widetilde{W},\widetilde{W}\setminus\{0\})
\end{array}$$

together with the fact that $[V]$ and $\mu_{\widetilde{W},0}$ are both mapped to the standard generator of $\mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\})$

Thus we have proved the desired equality. ∎

Lemma 128.11. Let M be a closed oriented n-dimensional topological manifold. Furthermore let $f\colon M\to M$ be a map. Finally let $z\in M$ be an isolated fixed point of f. We consider the two submanifolds $\mathrm{Gr}(f)$ and Δ_M of the $2n$-dimensional topological manifold $M\times M$. The following two statements hold:

(1) The point $(z,z)\in M\times M$ is a nice intersection point of the two submanifolds $\mathrm{Gr}(f)$ and Δ_M, in the sense of the definition on page 2778.
(2) We have the equality

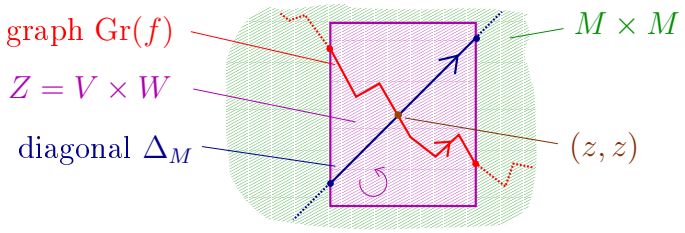
$$\mathrm{index}(f,z) \;=\; \underbrace{\mathrm{index}(\mathrm{Gr}(f),\Delta_M,(z,z))}_{\substack{\text{index of the nice intersection point } (z,z),\\ \text{as defined on page 2779}}}.$$

Remark. On page 2813 we gave an example of a map $f\colon S^2\to S^2$ that has a unique fixed point P, which has index 2. It follows from Lemma 128.11 that (P,P) is a nice intersection point of the submanifolds $\mathrm{Gr}(f)$ and Δ_{S^2} of $S^2\times S^2$ of index 2. This gives an affirmative answer to the question posed in Exercise 127.5.

[1558] Careful reading of the equalities shows that it is in fact at times difficult to determine which relative (co-) homology groups the objects lie in. In particular on several occasions we implicitly use the Cap Product-Naturality Lemma 116.8 and Lemma 109.6 (3) to go from one relative homology group to another. Writing down the calculation in full detail would make the proof even more unreadable that it already is.

Proof.

(1) Let $z \in M$ be an isolated fixed point of $f\colon M \to M$. We need to show that the point $(z,z) \in M \times M$ is a nice intersection point of the two submanifolds $\mathrm{Gr}(f)$ and Δ_M. Thus let U be an open neighborhood of (z,z) in $M \times M$. By Lemma 128.6 there exists a neighborhood V of $z \in M$ and a neighborhood W of $z \in M$ with the following properties:
(i) We have $V \times W \subset U$.
(ii) z is the only fixed point of f contained in V.
(iii) V and W are both submanifolds of M and both are homeomorphic to $\overline{B}^n$.
(iv) We have $V \subset \mathring{W}$ and $f(V) \subset \mathring{W}$.
By (i) it remains to show that $Z := V \times W$ is a nice neighborhood of (z,z) in the sense of the definition on page 2778. Most of the conditions are easily verified. For example by (iii) we know that $V \times W$ is homeomorphic to $\overline{B}^{2n}$. Furthermore, by (ii) we know that $\mathrm{Gr}(f) \cap \Delta_M \cap (V \times W) = \{(z,z)\}$. Finally note that it follows almost immediately from Lemma 128.10 (1) that $\mathrm{Gr}(f)$ and Δ_M are both well-behaved proper submanifolds of $V \times W$.



(2) To simplify the notation we can and will assume that $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^n$, $z = 0$ and $\Phi = \mathrm{id}$. But now in this setting the desired equality is an immediate consequence of Lemma 128.10 (2). ■

Now that we have Lemma 128.11 under our belt the proof of the Lefschetz-Hopf Theorem 128.9 is formally very similar to the proof of the Lefschetz-Hopf Theorem 128.5.

Proof of the Lefschetz-Hopf Theorem 128.9. Let M be a compact orientable topological manifold and let $f\colon M \to M$ be a map which has only finitely many fixed points $z_1, \dots, z_k$, and all of them lie in $M \setminus \partial M$. We pick an orientation for M.

First we assume that M is actually closed. Now we easily obtain the desired equality:

$$\Lambda(f) \underset{\uparrow}{=} Q_{M\times M}([\mathrm{Gr}(f)],[\Delta_M]) \underset{\uparrow}{=} \mathrm{Gr}(f)\cdot\Delta_M \underset{\uparrow}{=} \sum_{i=1}^{k} \mathrm{index}(\mathrm{Gr}(f),\Delta_M,(z_i,z_i)) \underset{\uparrow}{=} \sum_{i=1}^{k} \mathrm{index}(f,z_i).$$

by Proposition 128.1 — by Lemma 128.11 (1) and Theorem 127.7 — by definition, see page 2780 — by Lemma 128.11 (2)

Now we assume that M has non-empty boundary. Once again we consider the double $\mathrm{D}M := (M \times \{1\}) \sqcup_{\partial M \times \{1\} = \partial M \times \{2\}} (M \times \{2\})$. By the Doubling-Manifold Lemma 75.12 we know that $\mathrm{D}M$ is a closed orientable topological manifold. We consider the map

$$\begin{array}{rcl} \psi\colon \mathrm{D}M & \to & \mathrm{D}M \\ [(P,\epsilon)] & \mapsto & [(f(P),\epsilon)]. \end{array}$$

Since all of the fixed points of f lie in $M \setminus \partial M$ we see that the fixed points of ψ are $(z_1, 1), \ldots, (z_k, 1), (z_1, 2), \ldots, (z_k, 2)$. Now we see that

$$2 \cdot \sum_{i=1}^{k} \operatorname{index}(f, z_i) \underset{\uparrow}{=} \sum_{i=1}^{k} \operatorname{index}(\psi, (z_i, 1)) + \sum_{i=1}^{k} \operatorname{index}(\psi, (z_i, 2)) \underset{\uparrow}{=} \Lambda(\psi \colon \mathrm{D}M \to \mathrm{D}M)$$

it follows immediately from the definitions that $\operatorname{index}(f, z_i) = \operatorname{index}(\psi, (z_i, 1)) = \operatorname{index}(\psi, (z_i, 2))$

by the above, since $\mathrm{D}M$ is a closed orientable topological manifold

$$\underset{\uparrow}{=} 2 \cdot \Lambda(f \colon M \to M) + \Lambda(f \colon \partial M \to \partial M) \underset{\uparrow}{=} 2 \cdot \Lambda(f).$$

follows from Exercise 97.15

since there are no fixed points on ∂M and since ∂M is a closed orientable topological manifold we know by the above that $\Lambda(f \colon \partial M \to \partial M) = 0$

Dividing both sides by 2 yields the desired result. ■

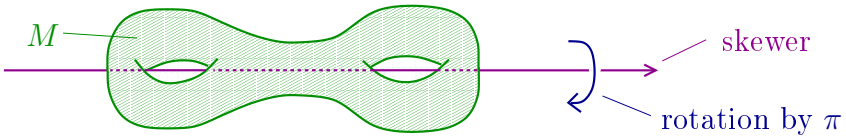
Exercises for Chapter 128.

Exercise 128.1. We consider the surface $M \subset \mathbb{R}^3$ as shown in the figure below. Furthermore we consider the map $\varphi \colon M \to M$ which is given by "rotating M around the skewer by an angle of π".

(a) For $i = 0, 1, 2$ determine $\operatorname{tr}(\varphi_* \colon \mathrm{H}_i(M; \mathbb{Z}) \to \mathrm{H}_i(M; \mathbb{Z}))$.

(b) For each fixed point x of φ determine $\operatorname{index}(\varphi, x)$.

Remark. Evidently you should use the Lefschetz-Hopf Theorem 128.9 to check your calculations in (a) and (b) for consistency.



Exercise 128.2.

(a) Let $k, l \in \mathbb{Z}$. We consider the sphere $S^3 = \{(z, w) \in \mathbb{C}^2 \,|\, |z|^2 + |w^2| = 1\}$ and the self-map

$$\begin{array}{rcl} f \colon S^3 & \to & S^3 \\ (z, w) & \mapsto & (z^k, w^l). \end{array}$$

(b) We view S^2 as a subset of $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. We consider the map

$$\begin{array}{rcl} g \colon S^2 & \to & S^2 \\ (z, t) & \mapsto & (z^2, t) \end{array}$$

(c) Let $n \in \mathbb{N}$. We consider the maps

$$h \colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}, \quad z \mapsto \begin{cases} z^n, & \text{if } z \in \mathbb{C}, \\ \infty, & \text{if } z = \infty \end{cases} \qquad \text{and} \qquad k \colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}, \quad z \mapsto \begin{cases} z + z^n, & \text{if } z \in \mathbb{C}, \\ \infty, & \text{if } z = \infty. \end{cases}$$

In each case determine all fixed points, determine whether they are non-degenerate and determine the index of each fixed point.

129. The Poincaré-Hopf Theorem

Now we return to the study of vector fields on smooth manifolds.[1559] Recall that the Hairy Ball Theorem 38.12 says that there does not exist a nowhere-vanishing vector field on any even-dimensional sphere. Also recall that later on we proved a significant generalization of this result. Namely, using the Lefschetz Fixed Point Theorem 97.9 we proved the following proposition.

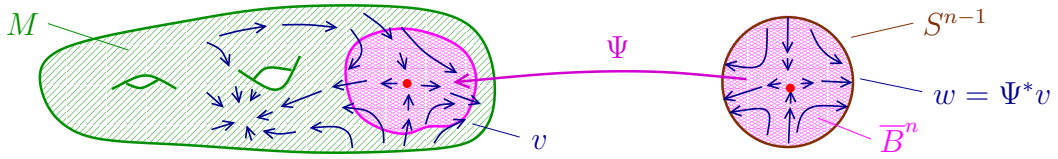
Theorem 97.15. Let M be a closed smooth manifold. If M admits a nowhere-vanishing vector field, then $\chi(M) = 0$.

In Chapter 127 we proved the Lefschetz-Hopf Theorem 128.9 which is a significant generalization of the Lefschetz Fixed Point Theorem 97.10 that we had just mentioned. In this chapter we will use this recently acquired theorems to prove the Poincaré-Hopf Theorem 129.7 which is a significant generalization of the above the Euler Characteristic-Vector Field Theorem 97.15.

129.1. **The index of a zero of a vector field.** In the following discussion it will be helpful to study zeros of vector fields in more detail. As we will, not all zeros are created equal but they actually come in different flavors. More precisely, we have the following definition.

Definition. Let M be an n-dimensional smooth manifold, let v be a vector field on M and let $x \in M \setminus \partial M$ be an isolated zero of v. We pick a smooth embedding $\Psi\colon \overline{B}^n \to M \setminus \partial M$ with $\Psi(0) = x$ and such that x is the only zero of v on $\Psi(\overline{B}^n)$. We denote by $w = \Psi^*(v)$ the pullback vector field on $\overline{B}^n$ as defined on page 641. In other words, given $y \in \overline{B}^n$ we write $w(y) := \mathrm{D}\Psi^{-1}_{\Psi(y)}(v(\Psi(y)))$. We define the index of v at x as follows:

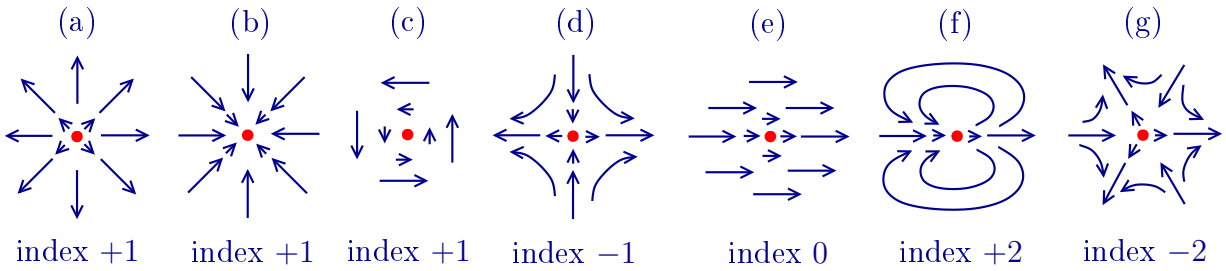
$$\begin{array}{rcccc} \mathrm{index}(v,x) & := & \text{degree of the map} & \Theta_{\Psi,v}\colon S^{n-1} & \to & S^{n-1} \\ & & & y & \mapsto & \dfrac{w(y)}{\|w(y)\|}. \end{array}$$



Example. We consider the vector fields on $\mathbb{R}^2 = \mathbb{C}$ with isolated zero at the origin that are shown in the figure below. In the following we use the identity chart $\Psi = \mathrm{id}$ to determine the various indices. In fact, using Lemmas 76.10 and 76.11, we obtain the same table as on page 1022:

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
vector field v at $(x,y)=z$	(x,y)	$(-x,-y)$	$(-y,x)$	$(x,-y)$	$(\lvert x\rvert+\lvert y\rvert,0)$	z^2	z^{-2}
degree of $\Theta_{\Psi,v}\colon S^1 \to S^1$	$+1$	$+1$	$+1$	-1	0	$+2$	-2.

[1559] We refer to page 639 for the definition of a vector field on a given smooth manifold M.



Lemma 129.1. Let M be an n-dimensional smooth manifold.

(1) Let v be a vector field on M. The definition of the index $\operatorname{index}(v,x)$ of an isolated zero $x \in M \setminus \partial M$ of a vector field v does not depend on the choice of the smooth embedding $\Psi\colon \overline{B}^n \to M$.
(2) If v and w are two vector fields on M that agree on an open subset $U \subset M$ and if z is an isolated zero of v in U, then $\operatorname{index}(v,z) = \operatorname{index}(w,z)$.

Proof. The second statement is a straightforward consequence of the first statement. Thus it remains to prove the first statement. Let $\Psi\colon \overline{B}^n \to M$ be a smooth embedding as in the definition of $\operatorname{index}(v,x)$. We need to show that any other such smooth embedding leads to the same index. The proof is performed in two stages.

(a) Let $A \in \mathrm{O}(n)$ be an orthogonal matrix. We denote by $\rho_A\colon \mathbb{R}^n \to \mathbb{R}^n$ the map that is given by multiplication by the matrix A. Next we consider the smooth embedding $\Psi \circ \rho_A\colon \overline{B}^n \to M$. A straightforward calculation using the Chain Rule 19.10, shows that $\Theta_{\Psi\circ\rho_A,v} = \rho_A^{-1} \circ \Theta_{\Psi,v} \circ \rho_A\colon S^{n-1} \to S^{n-1}$. It follows from the Sphere Degree Lemma 76.11 that $\deg(\Theta_{\Psi\circ\rho_A,v}) = \deg(\Theta_{\Psi,v})$.

(b) Now assume we are given some other smooth embedding $\Xi\colon \overline{B}^n \to M$. We consider

$$N := \underbrace{\{y \in M \mid v(y) \neq 0\}}_{\text{open subset by Exercise 97.16}} \cup \{x\}.$$

It follows from Exercise 97.16 and the fact that x is an isolated zero that N is an open subset of M. In particular we see that N is an n-dimensional submanifold of M. Since x is the only zero of v in $\Psi(\overline{B}^n)$ and $\Xi(\overline{B}^n)$ we see that $\Psi\colon \overline{B}^n \to M$ and $\Xi\colon \overline{B}^n \to M$ are tubular maps, in the sense of the definition on page **??**, for the 0-dimensional submanifold $\{x\}$ of N. By Proposition **??** there exists a diffeotopy $H\colon N \times [0,1] \to N$ rel $\{x\}$ and a matrix $A \in \mathrm{O}(n)$ such that $H_0 = \mathrm{id}$ and such that $H_1 \circ \Xi = \Psi \circ \rho_A\colon \overline{B}^n \to N$. Now we see that

$$\deg(\Theta_{\Xi,v}) = \deg(\Theta_{H_0\circ\Xi,v}) \underset{\uparrow}{=} \deg(\Theta_{H_1\circ\Xi,v}) = \deg(\Theta_{\Psi\circ\rho_A,v}) \underset{\uparrow}{=} \deg(\Theta_{\Psi,v}).$$

maps $\Theta_{H_t\circ\Xi,v}$ with $t \in [0,1]$ define a homotopy between $\Theta_{H_0\circ\Xi,v}$ and $\Theta_{H_1\circ\Xi,v}$, thus by the Sphere Degree Lemma 76.11 (3) the maps have the same degree

by (1)

■

The next example shows that any integer can occur as the index of an isolated zero of a vector field.

Example. Let $n \in \mathbb{N}_{\geq 2}$ and let $d \in \mathbb{Z}$. It follows from Lemma 76.10 and the Suspension-Degree Lemma 76.12 that there exists a self-map $\varphi\colon S^{n-1} \to S^{n-1}$ of degree d. Let $\lambda \in \mathbb{R}_{>0}$.

It is straightforward to verify that

$$\begin{array}{rcl} v(\varphi)\colon \mathbb{R}^n & \to & \mathbb{R}^n \\ x & \mapsto & \begin{cases} 0, & \text{if } x = 0, \\ \lambda \cdot x \cdot \varphi(\frac{x}{\|x\|}), & \text{if } x \neq 0 \end{cases} \end{array}$$

is a continuous map, in particular it defines a vector field on $\mathbb{R}^n$. Evidently it has a single isolated zero at the origin and the corresponding index is precisely $d = \deg(\varphi)$.

The next lemma can be convenient for calculating the index of a zero of a vector field.

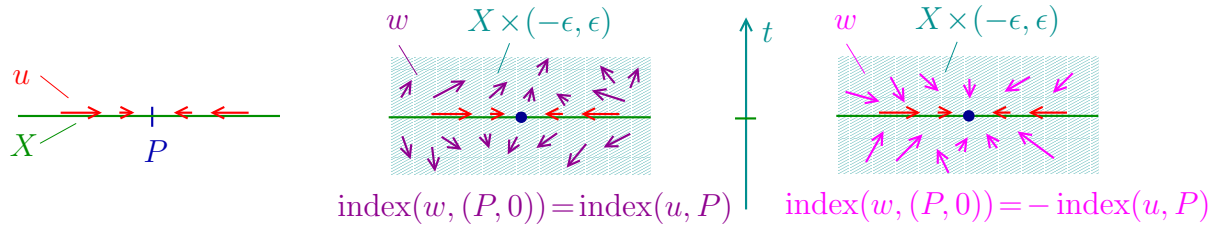
Lemma 129.2. Let $X \subset \mathbb{R}^n$ be an open subset and let $\epsilon > 0$. Furthermore suppose we are given a vector field $w = (u, v)\colon X \times (-\epsilon, \epsilon) \to \mathbb{R}^n \times \mathbb{R}$ with an isolated zero at $(P, 0)$.

(1) If $v(x,t) > 0$ whenever $t > 0$ and if $v(x,t) < 0$ whenever $t < 0$, then

$$\operatorname{index}(w, (P,0)) \;=\; \operatorname{index}(u, P).$$

(2) If $v(x,t) < 0$ whenever $t > 0$ and if $v(x,t) > 0$ whenever $t < 0$, then

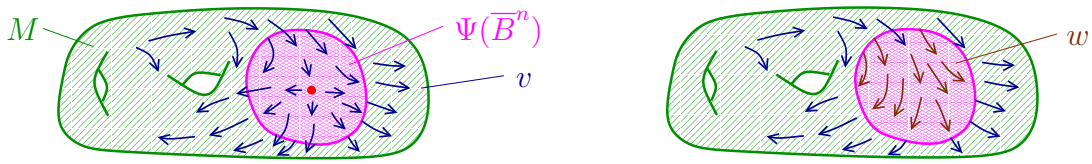
$$\operatorname{index}(w, (P,0)) \;=\; -\operatorname{index}(u, P).$$



Proof. We leave it to the reader to verify that the two statements follow easily from the definitions together with Exercise 76.13. ■

The following lemma says that isolated zeros of a vector field of index 0 can be eliminated by adjusting the vector field in the neighborhood of the zero.

Lemma 129.3. Let M be an n-dimensional smooth manifold, let v be a vector field on M and let x be an isolated zero of v with $\operatorname{index}(v, x) = 0$. Let $\Psi\colon \overline{B}^n \to M \setminus \partial M$ be a smooth embedding with $\Psi(0) = x$ and such that x is the only zero of v in $\Psi(\overline{B}^n)$. There exists a vector field w on M that agrees with v on $M \setminus \Psi(B^n)$ and which has no zeros on $\Psi(\overline{B}^n)$.



Sketch of proof. By Corollary 85.6 we know that the map $\deg\colon \pi_{n-1}(S^{n-1}) \to \mathbb{Z}$ is an isomorphism. In particular this implies, using our hypothesis that $\operatorname{index}(v, x) = 0$, that there exists a homotopy $H\colon S^{n-1} \times [0,1] \to S^{n-1}$ from the map $\Theta_{\Psi,v}\colon S^{n-1} \to S^{n-1}$ to a

constant map. Using the homotopy H it is now straightforward to write down an explicit vector field w on M with the following properties:

(1) On $M \setminus \Psi(B^n)$ the vector field w equals v.
(2) On $\Psi(\overline{B}^n_{\frac{1}{2}})$ the vector field w is "constant" and non-zero.
(3) On $\Psi(\overline{B}^n \setminus B_{\frac{1}{2}})$ the vector field interpolates between (1) and (2).

We leave it to the reader to fill in the details. ■

We continue with the following refinement of the Non-Degenerate Vector Field Existence Proposition 45.15.

Proposition 129.4. (Non-Degenerate Vector Field Existence Proposition II) Every closed smooth manifold M admits a smooth vector field v such that all zeros are non-degenerate and such that

$$\sum_{x \text{ a zero of } v} \operatorname{index}(v, x) \;=\; \chi(M).$$

Remark. On page 2983 we will give an alternative proof of the Non-Degenerate Vector Field Existence Proposition 129.4.

Before we turn to the proof of the Non-Degenerate Vector Field Existence Proposition 129.4 let us first state and prove the following corollary.

Corollary 129.5. For every closed smooth manifold M the Euler characteristic $\chi(M)$ equals the smooth Euler characteristic $\chi^{\mathrm{sm}}(M)$ of closed orientable smooth manifolds that we had introduced on page 1019.

Proof of Corollary 129.5. Let v be a smooth vector on M as in the Non-Degenerate Vector Field Existence Proposition 129.4. Then we see that

$$\chi(M) \;\underset{\substack{\uparrow\\ \text{choice of } v}}{=}\; \sum_{x \text{ a zero of } v} \operatorname{index}(v, x) \;\underset{\substack{\uparrow\\ \text{Smooth Poincaré–Hopf Theorem 45.17}}}{=}\; \chi^{\mathrm{smooth}}(M).$$

■

Proof of Proposition 129.4. Let M be a closed connected smooth manifold. First note that by the Smooth Manifold-Simplicial Structure Theorem 96.2 we know that M admits a smooth simplicial structure $(K = (V, S), \Theta\colon |K| \to M)$. Inductively we can define a smooth vector field u on M with the following properties:

(1) The vector field u is defined to be the zero vector field on the vertices of the triangulation.
(2) Suppose the vector field u is defined on the k-skeleton of M, then on every $(k+1)$-simplex s we extend the vector field u in the following way:
   (a) the vector field vanishes on the barycenter $\underline{s}$ of s,
   (b) for all other points in the open simplex the vector field "points towards" the barycenter $\underline{s}$ of s.

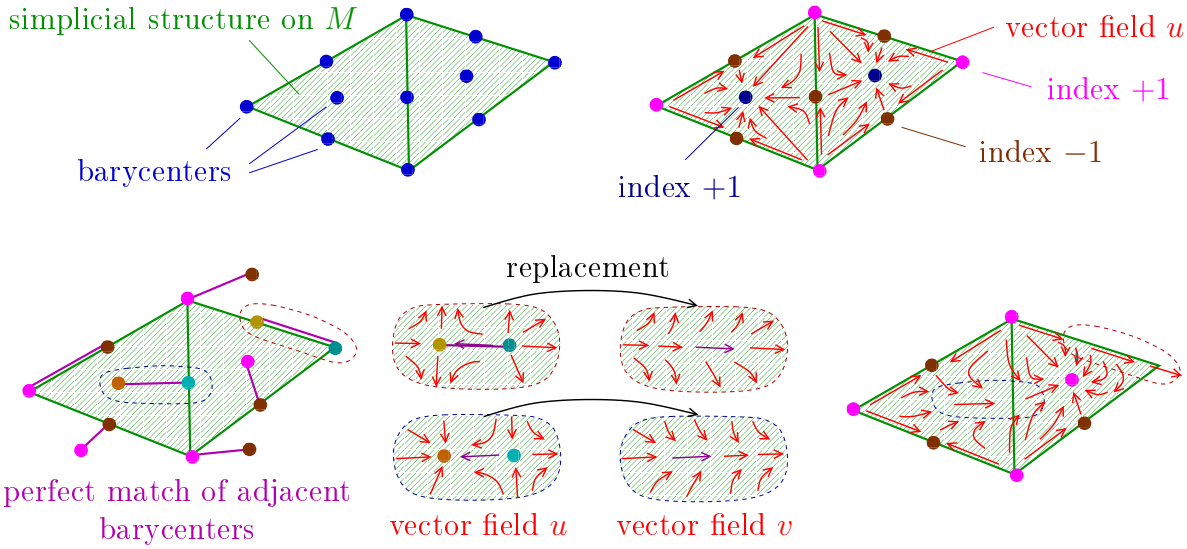
   This can be done explicitly.[1560]

[1560]For the resilient reader we provide the gory details. Suppose we are given a vector field v on $\partial\Delta^{k+1}$. We denote by $b := (\frac{1}{k+1}, \ldots, \frac{1}{k+1})$ the barycenter of Δ^{k+1}. We denote by $r\colon \Delta^{k+1} \setminus \{b\} \to \partial\Delta^{k+1}$ the map that is given by sending x to the unique intersection point of the ray $b + \mathbb{R}_{\geq 0} \cdot (x - b)$ with $\partial\Delta^{k+1}$. Next

We sketch the construction of the vector field u in the figure below. Note that the zeros of the vector field u are precisely the barycenters of the original simplices. Furthermore, one can fairly easily verify that given a k-simplex s we have[1561] $\operatorname{index}(u, \underline{s}) = (-1)^k$. Thus we see that

$$\sum_{x \text{ a zero of } u} \operatorname{index}(u, x) = \sum_{s \text{ a } k\text{-simplex}} \operatorname{index}(u, \underline{s}) = \sum_{s \text{ a } k\text{-simplex}} (-1)^k \underset{\text{Corollary 93.25 (5)}}{=} \chi(M).$$

This argument is also sketched in [**Stee1951**, p. 202]. ■



We conclude this chapter with the following converse to the Smooth Euler Characteristic-Vector Field Theorem 45.22 and the Euler Characteristic-Vector Field Theorem 97.15.

Theorem 129.6. (Nowhere Vanishing Vector Field Existence Theorem) Let M be a closed connected smooth manifold. If $\chi(M) = 0$, then M admits a nowhere-vanishing smooth vector field.

Sketch of proof. Let M be a closed connected smooth manifold with $\chi(M) = 0$. We need to show that M admits a nowhere-vanishing smooth vector field. This statement is proved in [**Bre1993**, Corollary VII.14.5], [**Stee1951**, Corollary 39.8], [**Hirs1976**, Theorem 5.2.10] and alternatively also in [**Tur1990**, Lemma 6.3.4]. In the following we provide a terse sketch

---

we consider the map

$$\begin{array}{rcl} \Delta^{k+1} & \to & \mathbb{R}^{k+2} \\ x & \mapsto & \begin{cases} \frac{\|x-b\|}{\|r(x)-b\|} \cdot (v(r(x)) + (b-x) \cdot \|x - r(x)\|), & \text{if } x \neq b, \\ 0, & \text{if } x = b. \end{cases} \end{array}$$

It is now fairly elementary to see that this vector field, transported back to M, has the desired properties. As always, with a little more care, using say the Smooth Transition Function Lemma 19.4, one can arrange that the vector field is actually smooth.

[1561]The fact that $\operatorname{index}(u, \underline{s}) = (-1)^k$ can be seen as follows. First note that the vector field u points in k directions towards $\underline{s}$ and in $(n-k)$ directions away from $\underline{s}$. Using a suitable embedding $\Psi\colon \overline{B}^m \to M$ with $\Psi(0) = \underline{s}$ we see that the corresponding map $\Theta_{\Psi,v}\colon S^{n-1} \to S^{n-1}$ is given by the diagonal matrix with k entries equal to -1 and $(n-k)$-entries equal to $+1$. Thus it follows from the Sphere Degree Lemma 76.11 that the degree of this map equals $(-1)^k$.

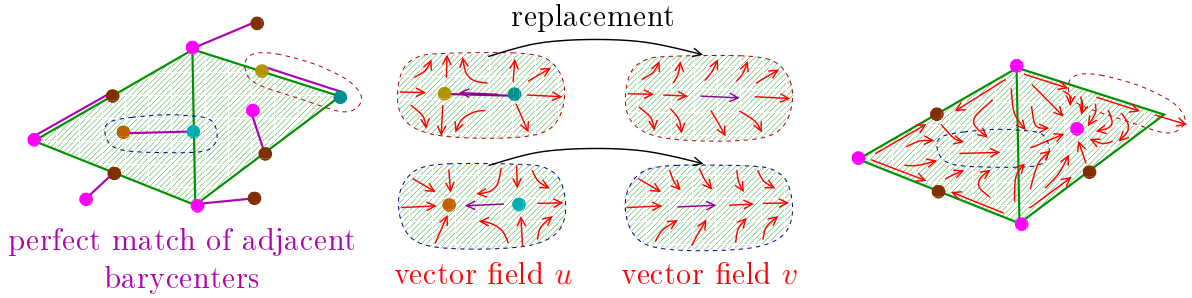
of the proof given in [**Tur1990**, Lemma 6.3.4]. We continue with the discussion of the above proof of Proposition 129.4 and we make the following observations and constructions.

(1) Some thought shows that it follows from our hypothesis that M is connected and that $\chi(M) = 0$ that we can find a perfect match between adjacent barycenters. More precisely, we can find a decomposition $S = \bigsqcup_{i \in I} \{s_i, t_i\}$ of the simplex set, such that each s_i is a codimension one face of t_i.

(2) Given each pair $\{s_i, t_i\}$ we consider a small suitable neighborhood of the 1-simplex $\{\underline{s_i}, \underline{t_i}\}$ of the barycentric subdivision. This neighborhood contains two critical points. As of right now, on this 1-simplex the vector field points from $\underline{s_i}$ to $\underline{t_i}$.

We can replace the vector field u on this neighborhood by a different vector field v that agrees with u on the boundary, which on the 1-simplex now points from $\underline{t_i}$ to $\underline{s_i}$ and which has no critical points. This replacement is illustrated in the figure below.

Hopefully this outline is convincing. Evidently filling in all details is a nightmarish task. Thankfully the details are provided in [**Tur1990**, Lemma 6.3.4]. ■



129.2. **The Poincaré-Hopf Theorem.** Now we can formulate the namesake of this chapter.

Theorem 129.7. (Poincaré-Hopf Theorem) Let M be compact smooth manifold. Let v be a vector field[1562] on M with finitely many zeros. If M has boundary, then we insist that along the boundary ∂M the vector field v is pointing outward. Then we have the formula

$$\sum_{x \text{ a zero of } v} \text{index}(v, x) \;=\; \chi(M).$$

Remark.

(1) In the 2-dimensional setting the Poincaré-Hopf Theorem was first proved by Henri Poincaré [**Poi1885**] in the late 19th century. The generalization to the higher-dimensional case was first proved by Heinz Hopf [**Hopf1927b**] in 1927. There are many different proofs of the Poincaré-Hopf Theorem, sometimes with somewhat different hypotheses, in the literature, see e.g. [**BoT1982**, Theorem 11.25], [**Vic1994**, Theorem 7.27], [**Bre1993**, Theorem VI.12.13], [**Miln1965a**, p. 35], [**DFN1985**, Theorem 15.3.2], [**BaT2003**, Theorem 7.6.5] and [**MTo1997**, Theorem 12.1]. See

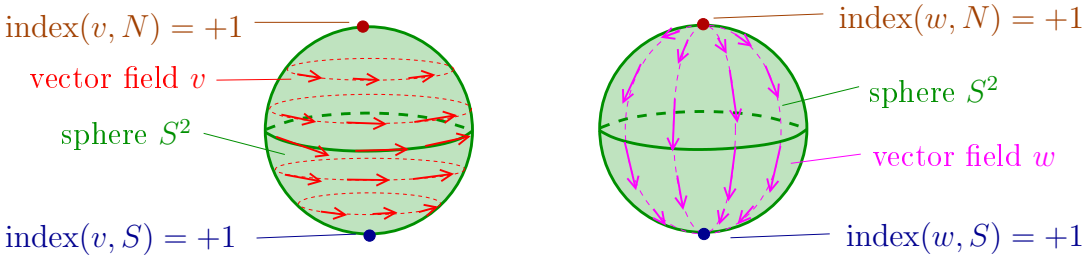
[1562]It is perhaps worth recalling that for us vector fields are assumed to be continuous. We do not make any assumptions on differentiability.

also the sketch in [**Bra2007**, p. 69]. The proof we provide is partly inspired by the proofs in [**DFN1985**] and [**Vic1994**].

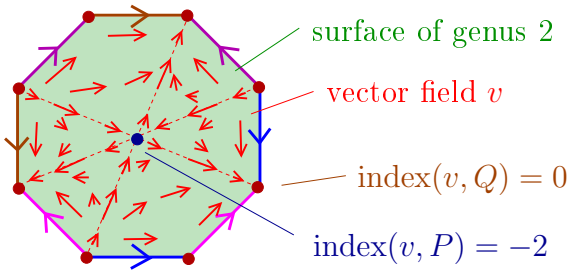
(2) The proof of the Poincaré-Hopf Theorem 129.7 relies very much on the Lefschetz-Hopf Theorem 128.9. For readers who prefer to work with the weaker Lefschetz-Hopf Theorem 128.5 we leave it as a little challenge to deduce from it the conclusion of the Poincaré-Hopf Theorem 129.7 under the stronger hypotheses that the vector field is smooth and that the indices of the zeros are ± 1.

Examples.

(1) In the figure below we see two vector fields v and w on S^2. Each of these vector fields has precisely two zeros, namely the North Pole N and the South Pole S. Furthermore, considering the table on page 1022, it it pretty clear that for both vector fields both zeros have index $+1$. In particular we see that for both vector fields these two indices add up reassuringly to $\chi(S^2) = 2$.



(2) In the figure below we see a vector field on the surface Σ of genus two with two zeros P and Q. The table on page 1022 strongly suggests that $\text{index}(v, P) = -2$. By the discussion on page 1879 we also know that $\chi(\Sigma) = -2$. Thus it follows from the Poincaré-Hopf Theorem 129.7 that $\text{index}(v, Q) = 0$. By Lemma 129.3 this means that we can modify the vector field v to obtain a vector field which has no zero at Q. Thus we have found a vector field on Σ with a single zero, namely at P. This gives a belated solution to Exercise 97.12.



It is perhaps worth stating the following immediate corollary to the Poincaré-Hopf Theorem 129.7. Note that this corollary generalized the statement of the Euler Characteristic-Vector Field Theorem 97.15 to smooth manifolds with non-empty boundary.

Corollary 129.8. Let M be a compact smooth manifold. If M admits a nowhere-vanishing vector field v that either points outward along all of ∂M or that points inward along all of ∂M, then $\chi(M) = 0$.

Proof. First note that after possibly replacing the vector field v by $-v$ we can assume that v points outward along ∂M. If v has no zeros, then the sum over the indices of the zeros is also zero. Thus it follows from the Poincaré-Hopf Theorem 129.7 that $\chi(M) = 0$. ∎

In preparation for the proof of the Poincaré-Hopf Theorem 129.7 let us first let us make a little remark and prove a useful lemma.

Remark. Let M be a topological manifold. Furthermore let $U \subset M$ be an open subset and let $f\colon U \to M$ be map. If x is an isolated fixed point of f, then we can define $\operatorname{index}(f, x) \in \mathbb{Z}$ the same way as we did for the case $U = M$ on page 2810.

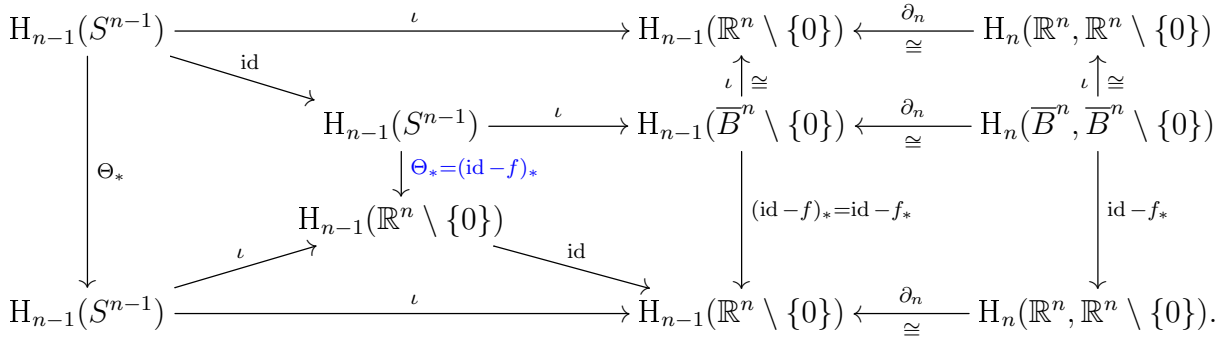
Lemma 129.9. Let U be an open neighborhood of the origin $0 \in \mathbb{R}^n$ which contains the closed ball $\overline{B}^n$. Furthermore let $v\colon U \to \mathbb{R}^n$ be a vector field on U which has a single zero, namely at 0. We consider the map

$$\begin{aligned} f\colon U &\to \mathbb{R}^n \\ x &\mapsto x - v(x). \end{aligned}$$

The point $x = 0$ is an isolated fixed point of f and we have the following equality:

$$\underbrace{\operatorname{index}(v, 0)}_{\substack{\text{index of a zero of a} \\ \text{vector field, see page 2818}}} = \underbrace{\operatorname{index}(f, 0).}_{\substack{\text{index of a fixed} \\ \text{point, see page 2810}}}$$

Proof. Since v has a single zero at $x = 0$ we see that $x = 0$ is the only fixed point of f, in particular it is an isolated fixed point. We consider the embedding $\Psi = \mathrm{id}\colon \overline{B}^n \to \mathbb{R}^n$ to compute the indices $\operatorname{index}(v, 0)$ and $\operatorname{index}(f, 0)$. We consider the map $\Theta\colon S^{n-1} \to S^{n-1}$ that is given by $x \mapsto \frac{v(x)}{\|v(x)\|}$. Furthermore we consider the following diagram:



We make the following observations and clarifications:

(1) The maps ι are the various inclusion induced maps.

(2) The map

$$\begin{aligned} H\colon S^{n-1} \times [0, 1] &\to \mathbb{R}^n \setminus \{0\} \\ (x, t) &\mapsto \frac{v(x)}{(1-t)\cdot\|v(x)\| + t} \end{aligned}$$

is a homotopy between $H_0 = \Theta$ and $H_1 = v = \mathrm{id} - f$. It follows from the existence of such a homotopy together with the Homotopic Maps-and-Homology Proposition 73.6 that we have the equality $\Theta_* = (\mathrm{id} - f)_*\colon \mathrm{H}_{n-1}(S^{n-1}) \to \mathrm{H}_{n-1}(\mathbb{R}^n \setminus \{0\})$.

(3) It follows from the naturality of the connecting homomorphisms, see the Topological LES-Corollary 74.14 that the squares to the right commute. All the remaining parts of the diagram commute by the functoriality of homology groups.

It follows from the above discussion that

$$\operatorname{index}(v,0) \;=\; \deg\begin{pmatrix} \mathrm{H}_{n-1}(S^{n-1}) \\ \downarrow \Theta_* \\ \mathrm{H}_{n-1}(S^{n-1}) \end{pmatrix} \;=\; \deg\begin{pmatrix} \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\ {\scriptstyle \iota}\uparrow\cong \\ \mathrm{H}_n(\overline{B}^n, \overline{B}^n \setminus \{0\}) \\ \downarrow {\scriptstyle \mathrm{id} - f_*} \\ \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \end{pmatrix} \;=\; \operatorname{index}(f,0).$$

$\uparrow$ by the definition on page 2818 — $\uparrow$ by the above discussion — $\uparrow$ see definition on page 2810 ■

Now we can provide the proof of the Poincaré-Hopf Theorem 129.7. To simplify the discussion we first deal with the special case that the smooth manifold is closed and oriented. We will deal with the general case right afterwards.

Proof of the Poincaré-Hopf Theorem for closed orientable M. Let M be a closed orientable n-dimensional smooth manifold. Let v be a vector field on M with finitely many zeros $P_1, \ldots, P_k$. The idea now is similar to the idea behind the proof of the Euler Characteristic-Vector Field Theorem 97.15. Namely we intend to use the vector field v to produce a self-map f of M. We will then roll out the Lefschetz-Hopf Theorem 128.5 to obtain the desired result.

As the reader surely remembers, in Proposition 97.16 we gave two different constructions of a self-map f from a given vector field v. The first approach used more machinery whereas the second proof was much more hands-on. It is technically slightly tricky to adapt the first approach to our current needs. Thus in the following we present a slight refinement of the second construction of the self-map f. We refer to the proof of Proposition 97.16 for details.

(1) By Exercise 27.6, which is a variation on Theorem 27.1, we can assume the following:
   (a) M is a submanifold of some $\mathbb{R}^m$.
   (b) For each zero P_i there exists an open neighborhood U_i of P_i that is "flat", in the sense that there exists a $y_i \in \mathbb{R}^{m-n}$ such that $U_i \subset \mathbb{R}^n \times \{y_i\}$.

(2) We consider the map

$$\begin{array}{rcl} F\colon M \times \mathbb{R}_{\geq 0} & \to & \mathbb{R}^m \\ (P,t) & \mapsto & P - t \cdot v(P). \end{array}$$

(3) There exists an $\epsilon > 0$ such that $Z := \{P + w \,|\, P \in M \text{ and } w \in (\widetilde{\mathrm{T}}_P M)^\perp \text{ with } \|w\| \leq \epsilon\}$ is a neighborhood of M and such that the map

$$\begin{array}{rcll} q\colon Z & \to & M & \\ P + w & \mapsto & P & \text{where } P \in M \text{ and } w \in (\widetilde{\mathrm{T}}_P M)^\perp \text{ with } \|w\| \leq \epsilon \end{array}$$

   is well-defined and continuous.

(4) There exists a $\mu > 0$ such that $F(M \times [0,\mu]) \subset Z$.

(5) We set $f := q \circ F_\mu\colon M \to M$.

(6) The map $q \circ F\colon M \times [0,\mu] \to M$ defines a homotopy between the identity and f.

Claim. Let $P \in M$.

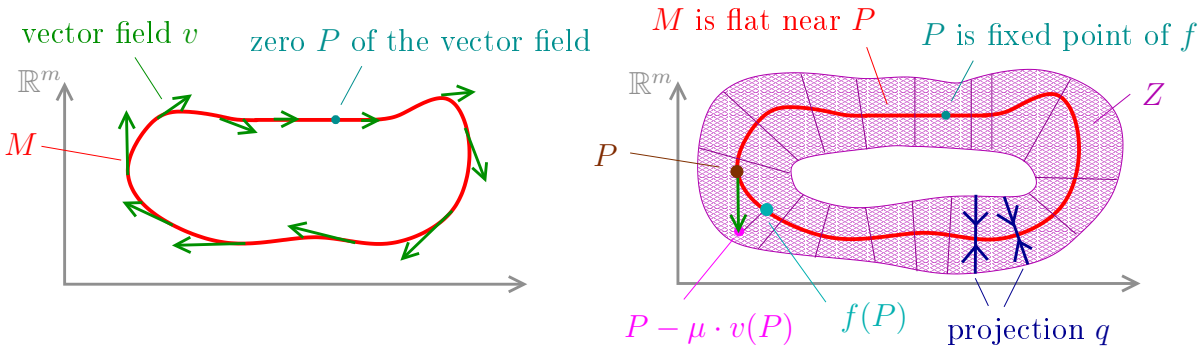
(i) P is a zero of v if and only if P is a fixed point of f.

(ii) If P is a zero of v, then $\operatorname{index}(v, P) = \operatorname{index}(f, P)$.

Proof.

(i) Evidently if P is a zero of v, then it is a fixed point of f. Now suppose that P is not a zero of v. We can write $F_\mu(P) = P - \mu \cdot v(P) =: Q + w$ with $Q \in M$ and $w \in (\widetilde{\mathrm{T}}_Q M)^\perp$ with $\|w\| \leq \epsilon$. Since $\widetilde{\mathrm{T}}_P M \cap \widetilde{\mathrm{T}}_P M^\perp = \{0\}$ and since $v(P)$ is a non-zero vector in $\widetilde{\mathrm{T}}_P M$ we see that $P \neq Q$. By the definition in (5) we have $f(P) = Q$. Thus we have shown that $f(P) \neq P$.

(ii) It follows from (1b) and our explicit construction of f that we are essentially in the setting of Lemma 129.9, which then provides us with the desired equality. ⊞



Now we see that

$$\sum_{P \text{ a zero of } v} \operatorname{index}(v, P) \underset{\text{by the claim}}{=} \sum_{P \text{ a fixed point of } f} \operatorname{index}(f, P) \underset{\substack{\text{by the Lefschetz-Hopf Theorem 128.9} \\ \text{note that here we use that } M \text{ is orientable}}}{=} \Lambda(f) \overset{\text{follows from Lemma 97.7 and the fact, mentioned in (6), that } f \text{ is homotopic to id}}{=} \Lambda(\mathrm{id}) \underset{\text{see Lemma 97.6}}{=} \chi(M).$$

■

Next we drop the hypothesis that M needs to be closed.

Proof of the Poincaré-Hopf Theorem 129.7 for compact orientable M. In the following let M be a compact orientable n-dimensional smooth manifold and let v be a vector field on M that points outward on ∂M and that has only finitely many zeros $P_1, \ldots, P_k$.

We perform the following steps:

(1) By the Smooth Collar Neighborhood Theorem 28.3 we know that there exists a collar neighborhood $[-1, 0] \times \partial M$ for $\partial M = \{0\} \times \partial M$. Possibly after rescaling the collar neighborhood we can arrange that none of the points $P_1, \ldots, P_k$ lies in $[-1, 0] \times \partial M$.

(2) Given $(t, x) \in [-1, 0] \times \partial M$ we can use the isomorphism from the Smooth Manifold Product Proposition 19.31 (3) to make the identification $\mathrm{T}_{(t,x)}([-1, 0] \times \partial M) = (\mathrm{T}_t \mathbb{R}) \times \mathrm{T}_x \partial M = \mathbb{R} \times \mathrm{T}_x \partial M$. In particular on $[-1, 0] \times \partial M$ we can write the vector field v as $v(t, x) = (v_1(t, x), v_2(t, x))$ with $v_1(t, x) \in \mathbb{R}$ and $v_2(t, x) \in \mathrm{T}_x \partial M$.

(3) It follows easily from the hypothesis that v points outward, see page 634 for the definition of an outward pointing tangent vector, and the fact that ∂M is compact that there exists an $\epsilon \in [-1, 0)$ such that $v_1(t, x) > 0$ for every $x \in \partial M$ and every $t \in [\epsilon, 0]$.

(4) After further rescaling the collar neighborhood we might as well assume that $\epsilon = -1$.
(5) By the Non-Degenerate Vector Field Existence Proposition 129.4 we know that there exists a vector field u on ∂M with finitely many zeros $Q_1, \ldots, Q_l$ and such that the sum of the indices of the zeros of u equals precisely $\chi(\partial M)$.
(6) We consider the double $\mathrm{D}M := M \sqcup_{\partial M = \partial M'} M'$ where M' is a copy of M. By the Doubling-Manifold Lemma 75.12 we know that $\mathrm{D}M$ is a closed orientable n-dimensional smooth manifold.
(7) We denote by $\rho\colon \mathrm{D}M \to \mathrm{D}M$ the natural involution swapping the two copies of M.
(8) Our next goal is to construct a vector field w on M that equals v outside of the collar neighborhood, that equals u on ∂M and that interpolates between these vector fields on the collar neighborhood without creating new zeros. In fact the following vector field has precisely those properties:

$$w(x) := \begin{cases} v(x), & \text{if } x \in M \setminus ([-1,0] \times \partial M), \\ (\underbrace{|t| \cdot v_1(y,t)}_{\in \mathbb{R}}, \underbrace{|t| \cdot v_2(y,t) + (1+t) \cdot u(y)}_{\in \mathrm{T}_y \partial M}), & \text{if } x = (t,y) \text{ with } t \in [-1,0] \ \&\ y \in \partial M. \end{cases}$$

We use the natural involution $\rho\colon \mathrm{D}M \to \mathrm{D}M$ to extend this vector field to $\mathrm{D}M$. More precisely, given $x' \in M'$ we set $w(x') = \mathrm{D}\rho_{\rho(x')}(w(\rho(x))$. Note that w is indeed a vector field on $\mathrm{D}M$, i.e. there are no continuity issues along $\partial M = \partial M'$.
(9) The zeros of w are $P_1, \ldots, P_k, \rho(P_1), \ldots, \rho(P_k)$ and $Q_1, \ldots, Q_l$. We make the following three observations:
  (a) We have $\mathrm{index}(w, P_i) = \mathrm{index}(v, P_i)$, since w equals v in a neighborhood of P_i.
  (b) We have $\mathrm{index}(w, \rho(P_i)) = \mathrm{index}(w, P_i)$, since ρ is a diffeomorphism.
  (c) It follows from our hypothesis that v points *outward* together with Lemma 129.2 that we have $\mathrm{index}(w, Q_j) = -\mathrm{index}(u, Q_j)$.

Now we have the following equality:

since $\mathrm{D}M$ is a *closed* orientable smooth manifold we already know from the above that the conclusion of the Poincaré-Hopf Theorem 129.7 holds for $\mathrm{D}M$

$$\sum_{i=1}^{k} \underbrace{\mathrm{index}(w, P_i)}_{\mathrm{index}(v,P_i) \text{ by (8a)}} + \sum_{i=1}^{k} \underbrace{\mathrm{index}(w, \rho(P_i))}_{= \mathrm{index}(v,P_i) \text{ by (8b) together with (8a)}} + \sum_{j=1}^{l} \underbrace{\mathrm{index}(w, Q_j)}_{= -\mathrm{index}(u,Q_j) \text{ by (8c)}} \overset{\downarrow}{=} \chi(\mathrm{D}M).$$

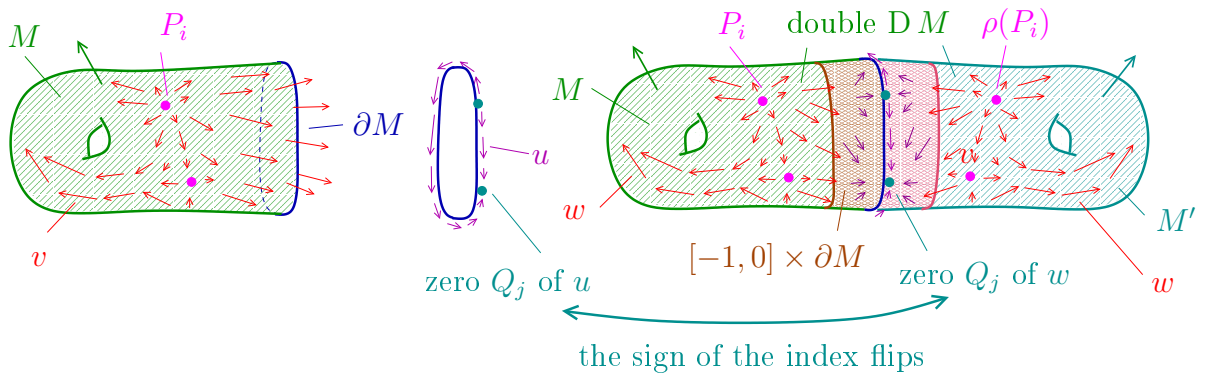
From this equality, together with (8a), (8b) and (8c) and the Euler Characteristic-Addition Lemma 87.5 we obtain the following equality:

$$2 \cdot \sum_{i=1}^{k} \mathrm{index}(v, P_i) - \underbrace{\sum_{j=1}^{l} \mathrm{index}(u, Q_j)}_{= \chi(\partial M) \text{ by (5)}} = \underbrace{2 \cdot \chi(M) - \chi(\partial M).}_{= \chi(\mathrm{D}M) \text{ by Lemma 87.5 and the Smooth Manifold-CW Structure Theorem 96.5}}$$

Cleaning up the bottom equality we obtain the desired equality. ∎

Finally we drop the hypothesis that M needs to be orientable.

Proof of the Poincaré-Hopf Theorem. Let M be a compact n-dimensional smooth manifold and let v be a vector field on M that points outward on ∂M and that has only finitely many zeros $P_1, \ldots, P_k$. Evidently we only need to consider the case that M is



connected. Furthermore, by the previous discussions it remains to consider the case that M is non-orientable.

By the Orientation Covering-Proposition 49.5 there exists a 2-fold covering $p\colon \widetilde{M} \to M$ where $\widetilde{M}$ is a compact orientable smooth manifold and p is a local diffeomorphism. We equip $\widetilde{M}$ with the vector field p^*v that is the pullback of v under p, see the Vector Field-Pullback Lemma 23.21. In other words, p^*v is the unique vector field on $\widetilde{M}$ that satisfies $\mathrm{D}p_x(p^*v(x)) = v(p(x))$ for every $x \in \widetilde{M}$. We see that

$$2 \cdot \sum_{i=1}^{k} \operatorname{index}(v, P_i) \underset{\uparrow}{=} \sum_{i=1}^{k} \sum_{Q \in p^{-1}(P_i)} \operatorname{index}(p^*v, Q) \underset{\uparrow}{=} \chi(\widetilde{M}) \underset{\uparrow}{=} 2 \cdot \chi(M).$$

each P_i has two preimages and indices of the preimages agree with the index of P_i

by the above discussion, since $\widetilde{M}$ is orientable

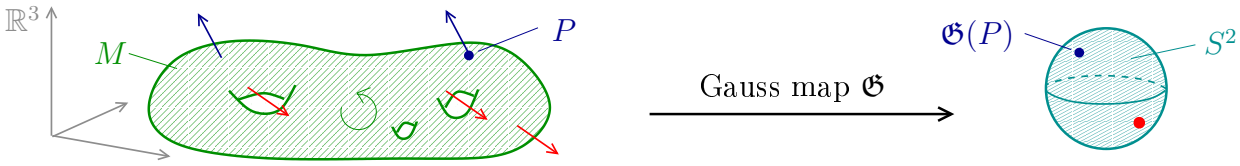
by Proposition 69.4 and the Smooth Manifold-CW Structure Theorem 96.5 ■

129.3. **The Gauß map II.** In this section we return to the study of the Gauß map which we had introduced in Section 45.7. For the reader's convenience we recall the definition.

Definition. Let $n \in \mathbb{N}_0$ and let $M \subset \mathbb{R}^{n+1}$ be a closed oriented smooth submanifold of codimension one. We refer to the map

$$\begin{aligned} \mathfrak{G}\colon M &\to S^n \\ P &\mapsto \text{the unique } \mathfrak{G}(P) \in \mathbb{R}^{n+1} \text{ that has the following properties:} \\ &\quad (1)\ \mathfrak{G}(P) \text{ is normal to the physical tangent space } \widetilde{\mathrm{T}}_P M, \\ &\quad (2)\ \mathfrak{G}(P) \text{ has length one, i.e. } \mathfrak{G}(P) \in S^n, \\ &\quad (3)\ \mathfrak{G}(P) \text{ followed by a positive basis for } \widetilde{\mathrm{T}}_P M \text{ is a positive basis for } \mathbb{R}^{n+1} \end{aligned}$$

as the Gauß map of M.[1563] [1564]



Our goal now is to give another proof of the following theorem from Section 45.7.

[1563] A variation on the Gauß map appeared also in the definition of the rotation number of an immersion $S^1 \to \mathbb{R}^2$ on page 955.

[1564] As discussed in the Orientations-via-Normal Vector Fields Proposition 25.11, this map is smooth.

Theorem 45.24. (Gauß Map-Degree Theorem) Let $M \subset \mathbb{R}^{n+1}$ be a closed oriented connected smooth submanifold of codimension one. If n is even, then

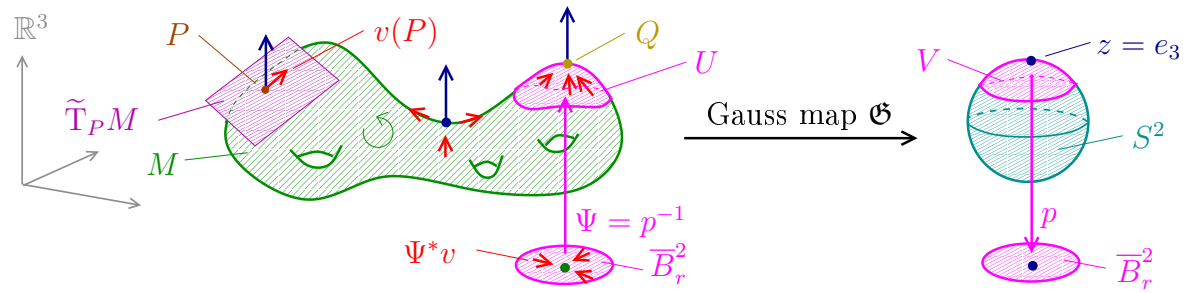
$$\deg(\text{Gauß map } \mathfrak{G}\colon M \to S^n) \;=\; \tfrac{1}{2} \cdot \chi(M).$$

Remark. We refer to Section 45.7 for a discussion of consequences and alternative proofs of the Gauß Map-Degree Theorem 45.24.

Proof. Let $n \in \mathbb{N}_0$ and let $M \subset \mathbb{R}^{n+1}$ be a closed oriented connected smooth submanifold of codimension one. The proof we will now provide is a variation on the proof in Section 45.7. We refer to the earlier proof for details of the common aspects of the proofs. By Sard's Theorem 31.1 there exists some $z \in S^n$ such that z and $-z$ are both regular values of the Gauß map $\mathfrak{G}$. After a rotation, we can assume that $z = e_{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. Next we consider the vector field

$$\begin{array}{rcl} v\colon M & \to & \mathbb{R}^{n+1} \\ Q & \mapsto & \underbrace{z - \langle z, \mathfrak{G}(Q)\rangle \cdot \mathfrak{G}(Q)}_{\text{orthogonal projection of } z \text{ to } \widetilde{\mathrm{T}}_Q M}. \end{array}$$

Clearly we have $v(Q) = 0$ if and only if $\mathfrak{G}(Q) = z$ or $\mathfrak{G}(Q) = -z$.



Let $Q \in M$ with $\mathfrak{G}(Q) = \pm z$. By design we know that Q is a regular value of $\mathfrak{G}$. By the Stack of Records Lemma 26.2 (1) we know that $\mathfrak{G}\colon M \to S^n$ is a local diffeomorphism at Q, thus, as on page 898, it makes sense to introduce the following definition:

$$\deg(\mathfrak{G}, Q) \;=\; \begin{cases} +1, & \text{if } \mathfrak{G} \text{ is orientation-preserving at } Q \\ -1, & \text{if } \mathfrak{G} \text{ is orientation-reversing at } Q. \end{cases}$$

Claim. Let $Q \in M$.
(1) If $\mathfrak{G}(Q) = z$, then $\operatorname{index}(v, Q) = (-1)^n \cdot \deg(\mathfrak{G}, Q)$.
(2) If $\mathfrak{G}(Q) = -z$, then $\operatorname{index}(v, Q) = \deg(\mathfrak{G}, Q)$.

Proof. First we consider the case that $\mathfrak{G}(Q) = z$. We make the following observations and preparations.

(a) We denote by $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ the projection onto the first n coordinates.
(b) It follows fairly easily from the Inverse Mapping Theorem 24.1 that there exists a neighborhood U of $Q \in M$, a neighborhood V of $z \in S^n$ and an $r \in \mathbb{R}_{>0}$ with the following properties:
 (i) U contains no other points x with $\mathfrak{G}(x) = z$.
 (ii) We have $\mathfrak{G}(U) \subset V$.

(iii) The restriction of the projection $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ to U and V defines diffeomorphisms $p\colon U \to \overline{B}^n_r$ and $p\colon V \to \overline{B}^n_r$. It follows immediately from the orientation conventions and some elementary linear algebra that both maps are orientation-preserving if n is even and both maps are orientation-reversing if n is odd.

(c) We set $\Psi := p^{-1}\colon \overline{B}^n_r \to U$.

(d) Recall that by definition we have $v(x) = z - \langle z, \mathfrak{G}(x)\rangle \cdot \mathfrak{G}(x)$. As we discussed on page 1038 one can easily show, using the Product Rule for partial differentials and using that $\mathfrak{G}(Q) = e$, that

$$\mathrm{D}(p \circ (x \mapsto -v(x)) \circ \Psi)_{p(Q)} \;=\; \mathrm{D}(p \circ \mathfrak{G} \circ \Psi)_{p(Q)}.$$

(e) We denote by $\rho\colon \overline{B}^n_r \to \overline{B}^n_r$ the map given by $\rho(x) = -x$.

We now see that

$$\begin{aligned}
\mathrm{index}(v,Q) &\overset{\substack{\text{definition of index}(v,Q)\text{, see page 2818,}\\ \text{applied with our given }\Psi}}{=} \deg\begin{pmatrix} S^{n-1}_r \to S^{n-1}_r \\ x \mapsto \frac{p(v(\Psi(x)))}{\|p(v(\Psi(x)))\|}\end{pmatrix} \overset{\substack{\text{by Exercises}\\ \text{14.4 and 76.4}}}{=} \deg\big((p\circ v\circ\Psi)_* \circlearrowleft \mathrm{H}_n(\overline{B}^n_r, \overline{B}^n_r\setminus\{0\})\big) \overset{\substack{\text{by (d) together with the Local}\\ \text{Diffeomorphism-Degree Proposition 76.14}}}{=} \\
&= \deg\big((\rho\circ p\circ\mathfrak{G}\circ\Psi)_* \circlearrowleft \mathrm{H}_n(\overline{B}^n_r, \overline{B}^n_r\setminus\{0\})\big) \\
&\overset{\text{by Lemma 76.16}}{=} (-1)^n \cdot \deg\big((p\circ\mathfrak{G}\circ\Psi)_* \circlearrowleft \mathrm{H}_n(\overline{B}^n_r, \overline{B}^n_r\setminus\{0\})\big) \\
&\underset{\substack{\text{this follows from the definition of deg}(\mathfrak{G},Q)\text{ together with the Local}\\ \text{Diffeomorphism-Degree Proposition 76.14 and the fact that we know that }p\text{ and }\Psi\text{ are}\\ \text{either both orientation-preserving or they are both orientation-reversing}}}{=} (-1)^n \cdot \deg(\mathfrak{G}, Q).
\end{aligned}$$

Now we consider the case that $\mathfrak{G}(Q) = -z$. The calculation is basically identical to the above. The only difference is that in this setting the map p is orientation-preserving if and only if n is odd, whereas before p was orientation-preserving if and only if n is even. On the other hand the behavior of the orientation of Ψ does not change, i.e. Ψ is orientation-preserving if and only if n is even. Thus we obtain the same result as above, except that by Lemma 76.16 we pick up a sign $(-1)^n$. ⊞

Now finally we assume that n is even. We perform the following calculation:

$$\begin{aligned}
2\cdot\deg(\text{Gauß map } \mathfrak{G}\colon M \to S^n) &\overset{\substack{\text{since }z\text{ and }-z\text{ are regular values we can apply the Local-Global Degree Proposition 101.6 to }z\text{ and to }-z}}{=} \sum_{\substack{Q\in M\text{ with}\\ \mathfrak{G}(Q)=z}} \deg(\mathfrak{G},Q) + \sum_{\substack{Q\in M\text{ with}\\ \mathfrak{G}(Q)=-z}} \deg(\mathfrak{G},Q) \\
&\underset{\substack{\text{since }v(Q)=0\text{ if and only if}\\ \mathfrak{G}(Q)=-z\text{ or }\mathfrak{G}(Q)=z}}{=} \sum_{\substack{Q\in M\text{ with}\\ v(Q)=0}} \deg(\mathfrak{G},Q) \underset{\substack{\text{by the claim and}\\ \text{since }n\text{ is even}}}{=} \sum_{\substack{Q\in M\text{ with}\\ v(Q)=0}} \mathrm{index}(v,Q) \underset{\substack{\text{by the Poincaré-Hopf}\\ \text{Theorem 129}}}{=} \chi(M).
\end{aligned}$$

■

Exercises for Chapter 129.

Exercise 129.1. We consider the two vector fields v and w that are shown in the figure below with a single zero at P. On page 1022 we already saw that $\mathrm{index}(v,P) = +1$. What is $\mathrm{index}(w,P)$?

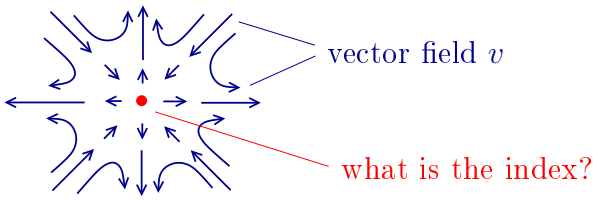


Exercise 129.2.

(a) We consider the vector field v on $\mathbb{R}^2$ shown in the figure below. What is the index of v at the unique zero?

(b) Sketch a vector field v on $\mathbb{R}^2$ with a zero at some point P, such the corresponding index equals $+3$.

(c) Let $n \geq 2$. Show that for every $k \in \mathbb{Z}$ there exists a vector field v on $\mathbb{R}^n$ with a single zero at the origin 0 such that $\operatorname{index}(v, 0) = k$.



Exercise 129.3. Let M be a closed oriented n-dimensional smooth manifold. We consider the tangent bundle

$$\mathrm{T}M \;:=\; \bigsqcup_{x \in M} (\mathrm{T}_x M \times \{x\}).$$

(a) Show that $\mathrm{T}M$ is naturally an oriented $2n$-dimensional smooth manifold such that the map $\varphi\colon M \to \mathrm{T}M$ given by $x \mapsto (0, x) \in \mathrm{T}_x M \times \{x\}$ is a smooth embedding.

(b) Show that there exists a smooth embedding $\psi\colon M \to \mathrm{T}M$, that is homotopic to φ, such that the submanifolds $\varphi(M)$ and $\psi(M)$ of $\mathrm{T}M$ intersect transversally and such that $\varphi(M) \cdot \psi(M) = \chi(M)$. Here the left-hand side is the algebraic intersection number as defined on page 2748.
Hint. Use the Non-Degenerate Vector Field Existence Proposition 129.4.
Remark. In the literature the statement we just proved sometimes gets shortened to "$M \cdot M = \chi(M)$".

Exercise 129.4. We say that two vector fields v and w on a smooth manifold M are equivalent if there exists a diffeotopy $H\colon M \times [0,1] \to M$ with $H_0 = \mathrm{id}$ and such that $(H_1)_*(v) = w$, i.e. such that for every $x \in M$ we have $(\mathrm{D}H_1)_x(v(x)) = w_{H_1(x)}$. Are all nowhere-vanishing smooth vector fields on the torus $S^1 \times S^1$ equivalent?

130. Homology, cohomology and submanifolds

In this chapter, given an oriented, compact smooth manifold M we will relate the following objects:

(1) Codimension one submanifolds of M,
(2) cohomology classes in $\mathrm{H}^1(M;\mathbb{Z})$ of dimension one,
(3) homology classes in $\mathrm{H}_{n-1}(M,\partial M;\mathbb{Z})$ of codimension one,
(4) maps $M \to S^1$.

130.1. **Codimension one submanifolds.** Let M be a compact oriented connected n-dimensional topological manifold. Every compact oriented proper submanifold $Y \subset M$ of *codimension* k gives rise to a homomorphism

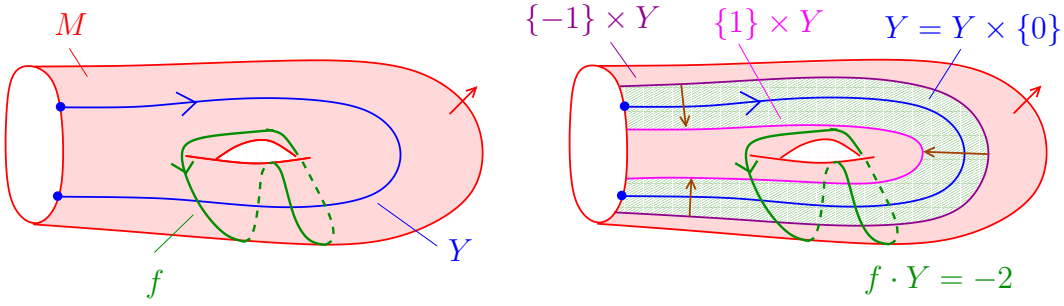
$$\begin{array}{rcl} \mathrm{H}_k(M;\mathbb{Z}) & \to & \mathbb{Z} \\ \sigma & \mapsto & \langle \underbrace{\mathrm{PD}_M([Y])}_{\in \mathrm{H}^k(M;\mathbb{Z})}, \sigma \rangle. \end{array}$$

In many circumstances the next proposition gives an explicit description of this homomorphism $\mathrm{H}_k(M;\mathbb{Z}) \to \mathbb{Z}$:

Proposition 130.1. Let M be a compact oriented connected n-dimensional topological manifold and let $Y \subset M$ be a compact oriented proper submanifold of *codimension* k. Let $f\colon X \to M$ be a map from a closed k-dimensional topological manifold X to M. If f is transverse to Y, then

$$\langle \underbrace{\mathrm{PD}_M([Y])}_{\in \mathrm{H}^k(M;\mathbb{Z})}, \underbrace{f_*([X])}_{\in \mathrm{H}_k(M;\mathbb{Z})} \rangle \;=\; f \cdot Y.$$

Remark. Let M be a compact oriented connected n-dimensional topological manifold and let $Y \subset M$ be a compact oriented proper submanifold of codimension one. In Proposition **??** we saw that Y admits a tubular neighborhood of the form $[-1,1]\times Y$ such that the product orientation of $[-1,1] \times Y$ matches the orientation of M. If $f\colon S^1 \to M$ is a map, then loosely speaking, the algebraic intersection number $f \cdot Y$ counts (with signs) how often f goes from $Y \times \{\mp 1\}$ to $Y \times \{\pm 1\}$.



Proof. We have

$$f \cdot Y \underset{\text{by Theorem 127.7 since } f \text{ and } Y \text{ are transverse}}{=} \epsilon_M\big((\mathrm{PD}_M(f_*([X])) \cup \mathrm{PD}_M([Y])) \cap [M]\big) \underset{\text{by the Cup-Cap Lemma 116.7 and the discussion on page 2558}}{=} \epsilon_M\big(\mathrm{PD}_M([Y]) \cap (\mathrm{PD}_M(f_*([X]) \cap [M]))\big)$$

$$\underset{\text{definition of } \mathrm{PD}_M(f_*([X]))}{=} \epsilon_M(\mathrm{PD}_M([Y]) \cap f_*([X])) \qquad \underset{\text{Cap Product-Kronecker Lemma 116.4}}{=} \langle \mathrm{PD}_M([Y]), f_*([X]) \rangle.$$

■

Now we can prove the following proposition which gives an easy criterion for showing that a codimension-one submanifold represents a non-trivial class in homology.

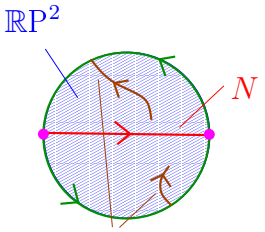
Proposition 130.2. (Non-Separating Codimension-One Submanifold Proposition) Let M be a compact oriented n-dimensional smooth manifold and let N be a compact orientable non-empty proper submanifold of codimension one. If the complement $M \setminus N$ is connected, then $[N] \in \mathrm{H}_{n-1}(M, \partial M; \mathbb{Z})$ is a primitive[1565] element of infinite order.

Example. The Non-Separating Codimension-One Submanifold Proposition 130.2 can in particular be seen as a generalization of the argument given on page 2754.

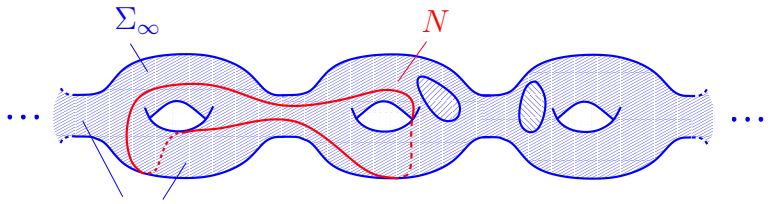
Remark. In any statement of a lemma, proposition or theorem it is always a good exercise to see whether the hypothesis stated are actually necessary. In the following we will discuss whether the conditions in the Non-Separating Codimension-One Submanifold Proposition 130.2 that M needs to be compact and oriented are necessary.

(1) We consider the real projective plane $\mathbb{R}\mathrm{P}^2$ and the 1-dimensional submanifold N that is illustrated in the figure below to the left. We see that $\mathbb{R}\mathrm{P}^2 \setminus N$ is connected, but since $\mathrm{H}_1(\mathbb{R}\mathrm{P}^2; \mathbb{Z}) \cong \mathbb{F}_2$ we see that N cannot possibly represent an element of infinite order. On the other hand, an "$\mathbb{F}_2$-version" of the Non-Separating Codimension-One Submanifold Proposition 130.2, which by the $\mathbb{F}_2$-Orientability Proposition 105.13 also applies to non-orientable smooth manifolds, shows that N represents a non-trivial element in $\mathrm{H}_1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$.

(2) Let M be a non-compact oriented n-dimensional smooth manifold and let N be a compact orientable non-empty proper submanifold of codimension one. If ∂M is compact, then we will see in Exercise 130.1 using the Compact Submanifold Exhaustion Proposition 31.2 and 130.2, that the homology class $[N] \in \mathrm{H}_{n-1}(M; \mathbb{Z})$ is still an element of infinite order.



path in $\mathbb{R}\mathrm{P}^2 \setminus N$ shows that $\mathbb{R}\mathrm{P}^2 \setminus N$ is connected

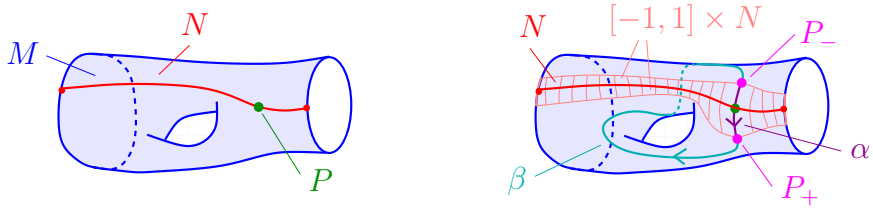


$\Sigma_\infty \setminus N$ is connected

[1565] Recall that an element g of an additive abelian group G is called *primitive* if there does not exist an $h \in G$ and an $n \geq 2$ with $n \cdot h = g$.

Proof. Let M be a compact oriented n-dimensional smooth manifold and let N be a compact orientable proper submanifold of codimension one such that $M \setminus N$ is connected. By Proposition **??** we can find a trivial tubular neighborhood $[-1,1] \times N$ in M. By hypothesis $M \setminus N$ is connected. It follows from Proposition 123.4, Lemma 123.1 and the Topological Manifolds-Local Properties Lemma 18.8 (3) that $M \setminus (-1,1) \times N$ is path-connected.

It follows from our hypothesis that N is non-empty, together with the Topological Manifold Boundary Proposition 75.2 (5), that we can pick a point $P \in N \setminus \partial N$. We write $P_\pm = P \times \{\pm 1\} \in [-1,1] \times N$. We denote by $\alpha\colon [0, \frac{1}{2}] \to [-1,1] \times N$ the "horizontal path" in $[-1,1] \times N$ from P_- to P_+. Since $M \setminus (-1,1) \times N$ is path-connected we can connect the two points P_+ to P_- via a path $\beta\colon [\frac{1}{2}, 1] \to M \setminus (-1,1) \times N$.



We consider the map $\gamma\colon S^1 = [0,1] \to M$ that is defined by α on $[0, \frac{1}{2}]$ and that is given by β on $[\frac{1}{2}, 1]$. We consider the following homomorphism

$$\begin{aligned} \Psi\colon \mathrm{H}_{n-1}(M, \partial M; \mathbb{Z}) &\to \mathbb{Z} \\ \sigma &\mapsto \langle \underbrace{\mathrm{PD}_M(\sigma)}_{\in \mathrm{H}^1(M;\mathbb{Z})}, \underbrace{\gamma_*([S^1])}_{\in \mathrm{H}_1(M;\mathbb{Z})} \rangle. \end{aligned}$$

We want to show that $[N] \in \mathrm{H}_{n-1}(M, \partial M; \mathbb{Z})$ is a primitive element of infinite order. It suffices to show that $\Psi([N]) = \pm 1 \in \mathbb{Z}$. In fact we have

$$\Psi([N]) \underset{\text{by definition}}{=} \langle \mathrm{PD}_M([N]), \gamma_*([S^1]) \rangle \underset{\substack{\text{by Proposition 130.1 since} \\ \gamma \text{ and } N \text{ are clearly transverse}}}{=} \gamma \cdot N \underset{\text{by construction of } \gamma}{=} \pm 1,$$

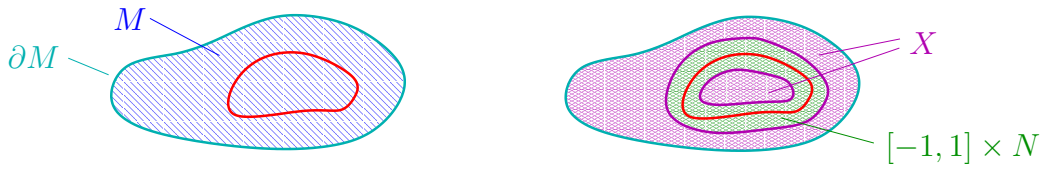
as desired. ∎

We conclude this section with the following lemma.

Lemma 130.3. Let M be a compact orientable n-dimensional smooth manifold with at most one boundary component and which furthermore satisfies $\mathrm{H}_1(M; \mathbb{Z}) = 0$. (For example we could take $M = \overline{B}^n$ or $M = S^n$). Then for every connected submanifold N of codimension one with $N \cap \partial M = \varnothing$ there exists a compact n-dimensional submanifold W of M with $\partial W = N$.

Proof. By Proposition **??** we can find a trivial tubular neighborhood $[-1,1] \times N$ in M. It follows from the argument of the proof of the Non-Separating Codimension-One Submanifold Proposition 130.2 and our assumption that M is orientable and the fact that M is compact that $X := M \setminus (-1,1) \times N$ is disconnected. Furthermore it follows from the Codimension-Zero Submanifold Proposition 19.38 that X is a submanifold of M with $\partial X = \partial M \cup (\{-1\} \times N) \cup (\{1\} \times N)$. Since ∂M consists of at most one component, since M is connected and since X is disconnected we see that there exists a component Y of X

with $\partial X = \{\epsilon\} \times N$ for some $\epsilon \in \{-1, 1\}$. By adding "half" of the tubular neighborhood to Y we obtain a submanifold with $\partial Y = M$. ■



130.2. **Representing homology classes by submanifolds II.** First we recall the following definition from page 2246.

Definition. Let M be a smooth manifold and let A be a union of components of ∂M.

(1) If N is a compact oriented k-dimensional submanifold of M with $\partial N \subset A$, then we denote by $[N]$ the image of the fundamental class $[N]$ in $\mathrm{H}_k(M, A; \mathbb{Z})$ under the inclusion induced map $\mathrm{H}_k(N, \partial N; \mathbb{Z}) \to \mathrm{H}_k(M, A; \mathbb{Z})$.
(2) We say that a homology class $z \in \mathrm{H}_k(M, A; \mathbb{Z})$ is represented by a submanifold if there exists a compact oriented proper k-dimensional submanifold N with $\partial N \subset A$ such that $[N] = z$.

Let M be an orientable, compact n-dimensional smooth manifold. We saw in the Intersection Number-via-Cup Product Theorem 126.5 that we can calculate the cup product of $\alpha \in \mathrm{H}^k(M, \partial M; \mathbb{Z})$ and $\beta \in \mathrm{H}^{n-k}(M; \mathbb{Z})$ if the Poincaré duals $\mathrm{PD}_M(\alpha) \in \mathrm{H}_{n-k}(M; \mathbb{Z})$ and $\mathrm{PD}_M(\beta) \in \mathrm{H}_k(M, \partial M; \mathbb{Z})$ are represented by transverse submanifolds. This rekindles our interest in the following question.

Question 102.11. Let M be an n-dimensional smooth manifold and let A be a union of components of ∂M. For which $k \in \mathbb{N}_0$ can homology classes in $\mathrm{H}_k(M, A; \mathbb{Z})$ be represented by submanifolds?

We already studied Question 102.11 in great detail in Section 102.4. For example in Corollary 102.14 and Corollary 102.17 we proved the following:

(1) If $n \geq 2$, then every class in $\mathrm{H}_1(M, A; \mathbb{Z})$ can be represented by a submanifold.
(2) If $n \geq 4$, then every class in $\mathrm{H}_2(M, A; \mathbb{Z})$ can be represented by a connected submanifold.

Instead of moving on to the next highest dimension, we now actually consider homology classes of *codimension one*. More precisely, we want to prove the following proposition.

Proposition 130.4. (Codimension-One Submanifold Existence Proposition) Let M be a compact orientable n-dimensional smooth manifold and let $A \subset \partial M$ be a union of boundary components. Every class in $\mathrm{H}_{n-1}(M, A; \mathbb{Z})$ can be represented by a submanifold.

Remark.

(1) In Proposition 130.6 we will prove a refinement of the Codimension-One Submanifold Existence Proposition 130.4.
(2) In Proposition **??** we will prove the codimension-two analogue of the Codimension-One Submanifold Existence Proposition 130.4. Furthermore in Proposition **??** we will prove an "$\mathbb{F}_2$-coefficient"-analogue of the Codimension-One Submanifold Existence Proposition 130.4.

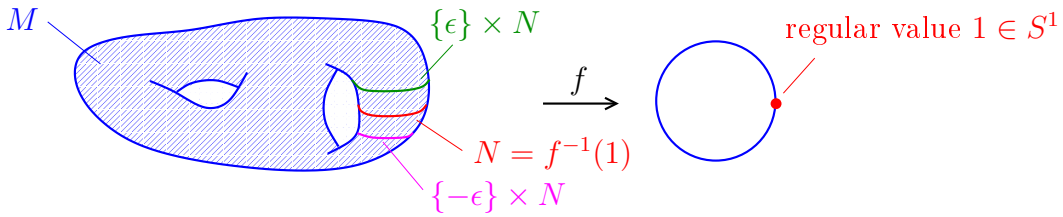
Proof. Let M be a compact orientable n-dimensional smooth manifold. In the following we deal with the case that $A = \partial M$. The general case is proved in a similar fashion, but now one needs to use Exercise 118.5 (4) instead of the Maps-to-S^1 Proposition 118.13. We leave it to the reader to work out the details.

Now let $\sigma \in \mathrm{H}_{n-1}(M, \partial M; \mathbb{Z})$. We write $\phi = \mathrm{PD}_M(\sigma) \in \mathrm{H}^1(M; \mathbb{Z})$. By the Maps-to-$S^1$ Proposition 118.13 or alternatively by Proposition 163.20 there exists a smooth map $f\colon M \to S^1$ such that $\phi = f^*([S^1]^*)$ where $[S^1]^* \in \mathrm{H}^1(S^1; \mathbb{Z})$ is the dual fundamental class.

By Sard's Theorem 31.1 and the Full Measure-Density Proposition 30.3 (2) there exists a regular value $z \in S^1$. To simplify the notation we assume that in fact $z = 1$ is a regular value. We write $N := f^{-1}(1)$. By the Regular Value Theorem **??** (1), (2), (3) we know that N is a compact orientable $(n-1)$-dimensional smooth manifold. Furthermore we know from Theorem **??** (4) that there exists an $\epsilon \in (0, \pi)$ and a tubular neighborhood $[-\epsilon, \epsilon] \times N$ of N such that the following diagram commutes

$$\begin{array}{ccc} [-\epsilon, \epsilon] \times N & \hookrightarrow & M \\ \downarrow{\scriptstyle (s,x)\mapsto s} & & \downarrow{\scriptstyle f} \\ [-\epsilon, \epsilon] & \xrightarrow{s\mapsto \exp(2\pi \mathrm{i} s)} & S^1. \end{array}$$

We pick the orientation for N such that the inclusion map $[-\epsilon, \epsilon] \times N \to M$ is orientation-preserving. It remains to prove that $[N] = \sigma \in \mathrm{H}_{n-1}(M, \partial M; \mathbb{Z})$. Recall that $\phi = \mathrm{PD}_M(\sigma)$, i.e. $\sigma = \phi \cap [M]$. Therefore it suffices to show that $[N] = \phi \cap [M] \in \mathrm{H}_{n-1}(M, \partial M; \mathbb{Z})$.



We introduce the following notation:

(1) We write $I = [-\epsilon, \epsilon]$ and $\mathring{I} = (-\epsilon, \epsilon)$,
(2) we denote by $i\colon I \times N \to M$ the inclusion map and we denote by i also all maps of pairs where the first map is given by the inclusion map,
(3) we denote by q all maps of pairs of topological spaces where the first map is just the identity on M,
(4) finally we denote by $p\colon I \times N \to I$ the projection .

We consider the following diagram[1566]

$$\begin{array}{ccccc}
\mathrm{H}^1(M) & \times & \mathrm{H}_n(M,\partial M) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{n-1}(M,\partial M) \\
{\scriptstyle q^*}\uparrow & & \downarrow{\scriptstyle q_*} & & {\scriptstyle q_*}\downarrow{\scriptstyle =} \\
\mathrm{H}^1(M, M\setminus(\mathring{I}\times N)) & \times & \mathrm{H}_n(M, M\setminus(\mathring{I}\times N)\cup\partial M) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{n-1}(M,\partial M) \\
{\scriptstyle i^*}\downarrow{\scriptstyle \cong} & & \uparrow{\scriptstyle i_*} & & {\scriptstyle i_*}\uparrow \\
\mathrm{H}^1(I\times N,\partial I\times N) & \times & \mathrm{H}_n(I\times N,\partial(I\times N)) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{n-1}(I\times N, I\times\partial N) \\
{\scriptstyle p^*}\uparrow & & \uparrow{\scriptstyle -\times[N]} & & {\scriptstyle -\times[N]}\uparrow \\
\mathrm{H}^1(I,\partial I) & \times & \mathrm{H}_1(I,\partial I) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_0(I).
\end{array}$$

The proposition follows from the following claim.

Claim.

(1) The diagram commutes.[1567]
(2) The map $i^*\colon \mathrm{H}^1(M, M\setminus(\mathring{I}\times N)) \to \mathrm{H}^1(I\times N, \partial I\times N)$ is an isomorphism.
(3) We have $(q^*\circ i_*^{-1}\circ p^*)([I]^*) = \phi$.
(4) We have $i_*([I]\times[N]) = q_*([M])$.

Proof.

(1) The top two squares commute by the Cap Product-Naturality Lemma 116.8 applied to the two maps

$$\begin{aligned} q\colon (M,\varnothing,\partial M) &\to (M, M\setminus(\mathring{I}\times N),\partial M) \\ \text{and} \\ i\colon (I\times N,\partial I\times N, I\times\partial N) &\to (M, M\setminus(\mathring{I}\times N),\partial M). \end{aligned}$$

Finally regarding the bottom square, note that

$$p^*([I]^*)\cap([I]\times[N]) \underset{\text{Topological Product Orientation Proposition 113.10}}{=} p^*([I]^*)\cap([I\times N]) \underset{\text{Fundamental Class-Product Proposition 117.2 (2)}}{=} [\{0\}\times N] = (\overbrace{[I]^*\cap[I]}^{=[\{0\}]})\times[N].$$

Since $[I]^*\in\mathrm{H}^1(I,\partial I)$ and $[I]\in\mathrm{H}_1(I,\partial I)$ are generators it follows from this calculation that the bottom square commutes.

(2) It follows from excision for cohomology groups, see page 2418, that i^* is indeed an isomorphism. We leave it to the reader to verify the details.

(3) We identify the interval I with its image in S^1 under the orientation-preserving map defined by $s\mapsto\exp(2\pi\mathrm{i}s)$. We denote by $j\colon I\to S^1$ and $j\colon (I,\partial I)\to(S^1, S^1\setminus\mathring{I})$ the inclusion maps. Furthermore we denote by $r\colon (S^1,\varnothing)\to(S^1, S^1\setminus\mathring{I})$ the obvious

[1566]For completeness' sake we recall that the relative cap product is only defined for excisive triads, see page 2536. So one should think for a second about why the second cap product is in fact defined.
[1567]Note that here for the first three rows we use the convention of page 2538 for what "commutativity" actually means.

map of pairs of topological spaces. We consider the following diagram

$$\begin{array}{ccc}
\mathrm{H}^1(S^1) & \xrightarrow{\quad f^* \quad} & \mathrm{H}^1(M) \\
{\scriptstyle r^*}\uparrow & & \uparrow{\scriptstyle q^*} \\
\mathrm{H}^1(S^1, S^1 \setminus \mathring{I}) & \xrightarrow{\quad f^* \quad} & \mathrm{H}^1(M, M \setminus (\mathring{I} \times N)) \\
{\scriptstyle \cong}\downarrow{\scriptstyle j^*} & & {\scriptstyle i^*}\downarrow{\scriptstyle \cong} \\
\mathrm{H}^1(I, \partial I) & \xrightarrow{\quad f^*=p^* \quad} & \mathrm{H}^1(I \times N, \partial I \times N).
\end{array}$$

Here the map j^* is an isomorphism by excision for cohomology groups. Also note that at the bottom horizontal map we have $f^* = p^*$ by the discussion on page 2837. Furthermore we note that by the functoriality of cohomology groups both squares commute. Finally we point out that it follows immediately from the Codimension Zero-Fundamental Class Lemma 106.24 that $r^*((j^*)^{-1}([I]^*)) = [S^1]^*$. The desired statement is a consequence of the observation that $\phi = f^*([S^1]^*)$.

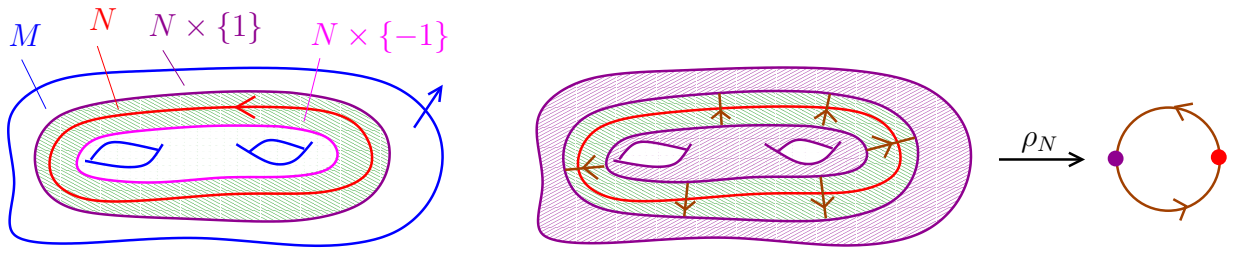
(4) This statement follows immediately from the Codimension Zero-Fundamental Class Lemma 106.24. ■

130.3. **The basic Thom-Pontryagin Theory.** In the proof of the Codimension-One Submanifold Existence Proposition 130.4 we saw that a map from a smooth manifold N to S^1 gives rise to a codimension-one submanifold. In this section we will go in the opposite direction.

Definition. Let M be a compact oriented n-dimensional smooth manifold and let N be a proper oriented $(n-1)$-dimensional submanifold. By Proposition **??** there exists a trivial tubular neighborhood of the form $[-1,1]\times N$ such that the product orientation of $[-1,1]\times N$ matches the orientation of $[-1,1] \times N$, viewed as a codimension-zero submanifold of M. We refer to the map[1568]

$$\begin{array}{rccl}
\rho_N\colon & M & \to & S^1 \\
& P & \mapsto & \begin{cases} \exp(\pi i t), & \text{if } P = (t, Q) \in [-1,1] \times N, \\ -1, & \text{otherwise} \end{cases}
\end{array}$$

as the Thom-Pontryagin map.[1569] The definition of ρ_N is illustrated in the figure below.



[1568] It follows easily from the Codimension-Zero Smooth Embedding Proposition 24.16 and the Homotopy Stacking Lemma 14.3 that this map is indeed continuous.

[1569] Lev Pontryagin (1908-1988) was a Soviet topologist, he made many major contributions to topology, despite having lost his eyesight in an accident when he was 14.

Remark. The Thom-Pontryagin map ρ_N that we had just constructed depends on the choice of a tubular neighborhood. But it follows from the uniqueness of tubular neighborhoods, see Proposition **??**, together with Lemma **??** (2) that any other choice of a tubular neighborhood $[-1,1] \times N$ leads to a map $M \to S^1$ that is homotopic to the original map.

Lemma 130.5. Let M be a compact oriented n-dimensional smooth manifold and let N be a proper oriented proper $(n-1)$-dimensional submanifold. We have

$$\rho_{N*} \;=\; \mathrm{PD}_M([N]) \;\in\; \mathrm{Hom}(\mathrm{H}_1(N;\mathbb{Z}),\mathbb{Z}) \;\underset{\uparrow}{=}\; \mathrm{H}^1(N;\mathbb{Z}).$$

identification from the First Cohomology Proposition 110.17

Proof. We had implicitly verified this lemma in the above proof of the Codimension-One Submanifold Existence Proposition 130.4. ■

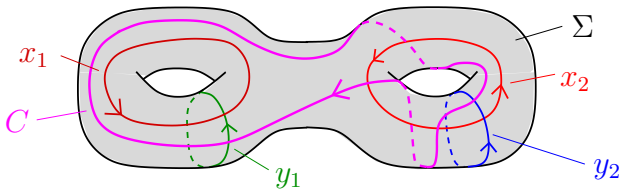
Example. We return to the example that we had discussed on page 2607. More precisely, let Σ be the surface of genus 2. The closed curves x_1, y_1, x_2, y_2 shown in the figure below represent a basis for $\mathrm{H}_1(\Sigma;\mathbb{Z})$. We consider the epimorphism $\phi\colon \mathrm{H}_1(\Sigma) \to \mathbb{Z}$ that is given by $\phi(x_1) = 0$, $\phi(y_1) = 1$, $\phi(x_2) = -2$ and $\phi(y_2) = 1$. By the Maps-to-S^1 Proposition 118.13 we know that ϕ can be realized by some map $g\colon \Sigma \to S^1$. Now we will construct such a map explicitly.

We consider the closed curve C that is illustrated in the figure below. We claim that the corresponding map $\rho_C\colon \Sigma \to S^1$ has the desired property. Put differently, we claim that $\rho_{C*} = \phi \in \mathrm{Hom}(\mathrm{H}_1(N;\mathbb{Z}),\mathbb{Z})$. It suffices to show that the two homomorphisms agree on the basis $[x_1], [y_1], [x_2], [y_2]$ of $\mathrm{H}_1(\Sigma;\mathbb{Z})$. For example we have

$$\rho_{C*}([x_2]) \underset{\uparrow}{=} \langle \mathrm{PD}_M([C]), [x_2]\rangle \underset{\uparrow}{=} x_2 \cdot C \underset{\uparrow}{=} -2 \;=\; \phi(x_2).$$

Lemma 130.5 — Proposition 130.1 — see the figure below

It is straightforward to verify that the same equalities hold for all the other basis elements.



Now we have the tools to prove the following refinement of Proposition 130.6.

Proposition 130.6. Let M be a compact orientable n-dimensional smooth manifold and let $\phi \in \mathrm{H}_{n-1}(M, \partial M;\mathbb{Z})$. Let $\partial\phi \in \mathrm{H}_{n-2}(\partial M)$ be the image of ϕ under the connecting homomorphism in the long exact sequence of the pair $(M, \partial M)$. If $F \subset \partial M$ is a submanifold that represents $\partial\phi \in \mathrm{H}_{n-2}(A)$, then ϕ can be represented by a submanifold N of M with $\partial N = F$.

Proof. Let M be a compact oriented n-dimensional smooth manifold. Suppose we are given a homology class $\phi \in \mathrm{H}_{n-1}(M, \partial M;\mathbb{Z})$ and a submanifold $F \subset \partial M$ that represents $\partial\phi \in \mathrm{H}_{n-2}(\partial M)$. As above we pick a trivial tubular neighborhood $[-1,1] \times F$ in ∂M such that the product orientation of $[-1,1] \times F$ matches the orientation of ∂M. Next we

consider the corresponding Thom-Pontryagin map

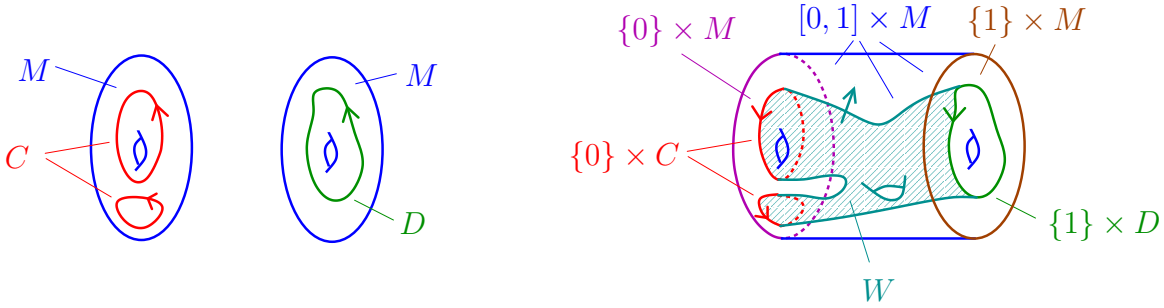
$$\begin{array}{rcl} \rho_F \colon \partial N & \to & S^1 \\ P & \mapsto & \begin{cases} \exp(\pi \mathrm{i} t), & \text{if } P = (t, Q) \in [-1,1] \times F, \\ -1, & \text{otherwise} \end{cases} \end{array}$$

Note that ρ_F is smooth on the closed subset $[-\frac{1}{2}, \frac{1}{2}] \times F$. It follows quite easily from the Whitney Approximation Theorem 29.1 (5) that ρ_F is homotopic rel $[-\frac{1}{2}, \frac{1}{2}] \times F$ to a smooth map $f \colon \partial N \to S^1$ which is "close to ρ_F" in the sense that it has the property that for $P \notin [-\frac{1}{2}, \frac{1}{2}] \times F$ we have $f(P) \notin \{\exp(\pi \mathrm{i}) \,|\, t \in [-\frac{1}{4}, \frac{1}{4}]\}$. Now note that it follows from Proposition 119.25 together with the Maps-to-S^1 Proposition 118.13 (3) that there exists a smooth map $g \colon M \to S^1$ that represents $\mathrm{PD}_M(\phi) \in \mathrm{H}^1(M, \partial M; \mathbb{Z})$ and that agrees with f on ∂M.

By Sard's Theorem 31.1 and the Full Measure-Density Proposition 30.3 (2) there exists an $s \in (-\frac{1}{4}, \frac{1}{4})$ such that $\exp(\pi \mathrm{i} s)$ is a regular value of g. We write $N := g^{-1}(z)$. By the proof of the Codimension-One Submanifold Existence Proposition 130.4 (1) we know that N is a smooth proper submanifold that represents ϕ. Note that by construction we have $\partial N = N \cap \partial M = \{s\} \times F$. Using a collar neighborhood of ∂M it is straightforward to find an isotopy that arranges that $\partial N = \{0\} \times F = F$. We leave it to the reader to fill in the few remaining details. ∎

Later on we will make use of the following useful corollary.

Corollary 130.7. (Cobounding Submanifolds Corollary) Let M be a closed orientable n-dimensional smooth manifold and let C and D be two closed oriented $(n-1)$-dimensional submanifolds. If $[C] = [D] \in \mathrm{H}_n(M; \mathbb{Z})$, then there exists a compact oriented proper n-dimensional submanifold $W \subset [0,1] \times M$ such that $\partial W = (\{0\} \times C) \cup (\{1\} \times -D)$.



Proof. We consider the following diagram:

$$\begin{array}{ccccccccc} \ldots \longrightarrow & \mathrm{H}_n([0,1] \times M) & \xrightarrow{\partial} & \mathrm{H}_{n-1}((\{0\} \times M) \cup (\{1\} \times M)) & \xrightarrow{i_*} & \mathrm{H}_{n-1}([0,1] \times M) & \longrightarrow \ldots \\ & & & & & \downarrow{\scriptstyle p_*}\cong & \\ & & & & & \mathrm{H}_n(M) & \end{array}$$

Here the horizontal sequence is the long exact sequence in homology corresponding to the pair $([0,1] \times M, (\{0\} \times M) \cup (\{1\} \times M))$. Furthermore $p \colon [0,1] \times M \to M$ denotes the obvious projection. Our hypothesis that $[C] = [D] \in \mathrm{H}_n(M; \mathbb{Z})$ implies immediately that $p_*([\{0\} \times C]) = p_*([\{1\} \times D])$. Since p_* is an isomorphism and since the sequence is exact we see that there exists a homology class $\varphi \in \mathrm{H}_n([0,1] \times M)$ with $\partial \varphi = [\{0\} \times C] - [\{1\} \times D]$. Finally

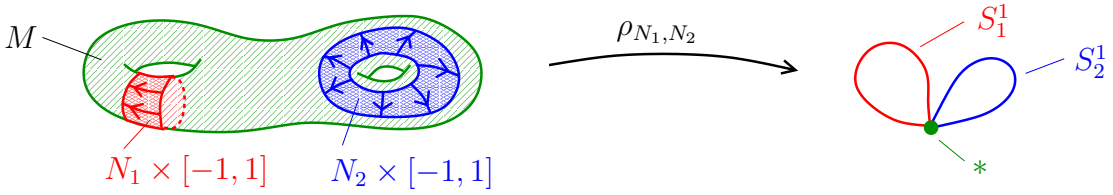
note that it follows immediately from Proposition 130.6 that there exists in particular a compact oriented proper n-dimensional submanifold W of $[0,1] \times M$ with $[W] = \varphi$ and with $\partial W = (\{0\} \times C) \cup (\{1\} \times -D)$. ■

In the remainder of this chapter we study an amusing generalization of the above Thom-Pontryagin map.

Definition. Let M be a compact oriented n-dimensional smooth manifold. Furthermore let $N_1, \ldots, N_k$ be disjoint proper oriented $(n-1)$-dimensional submanifolds. As on page 2839 we pick a trivial tubular neighborhood of the form $[-1,1] \times (N_1 \cup \cdots \cup N_k)$. We consider the wedge $\bigvee_{j=1}^{k} S^1_j$ given by identifying the points $-1 \in S^1_j$ to a single point $*$. We again refer to the map

$$\begin{aligned} \rho_{N_1,\ldots,N_k} \colon M &\to \bigvee_{j=1}^{k} S^1_j \\ P &\mapsto \begin{cases} \exp(\pi i t) \in S^1_j, & \text{if } P = (t,Q) \in [-1,1] \times N_j, \\ *, & \text{otherwise} \end{cases} \end{aligned}$$

as the Thom-Pontryagin map.



Before we continue we recall that in the Fundamental Group-Wedge-of-Circles Lemma 52.4 we gave an identification

$$\langle t_1, \ldots, t_k \rangle \;=\; \pi_1\Big(\bigvee_{j=1}^{k} S^1_j, * \Big)$$

where each t_j is represented by a loop that goes once around the j-th circle S^1_j in a positive direction.

Proposition 130.8. (Epimorphism-onto-Free Group Proposition) Let M be a compact oriented connected n-dimensional smooth manifold and let $N_1, \ldots, N_k$ be disjoint proper oriented non-empty $(n-1)$-dimensional submanifolds. If $M \setminus (F_1 \cup \cdots \cup F_k)$ is connected, then for any $x_0 \in M$ the map

$$\rho_{N_1,\ldots,N_k} \colon \pi_1(M, x_0) \;\to\; \pi_1\Big(\bigvee_{j=1}^{k} S^1_j, * \Big) \;=\; \langle t_1, \ldots, t_k \rangle$$

is an epimorphism.

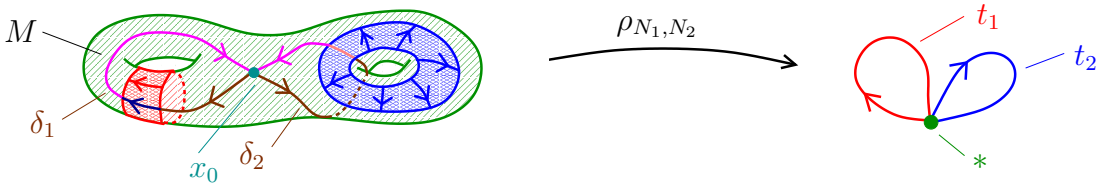
Proof. We write $N = N_1 \cup \cdots \cup N_k$ and $\rho = \rho_{N_1,\ldots,N_k}$. We pick a trivial tubular neighborhood of the form $[-1,1] \times N$. It follows from our hypothesis that $M \setminus N$ is connected, together with Proposition 123.4 and Lemma 123.1 that $M \setminus (-1,1) \times N$ is also connected. Since M is connected it follows from Proposition 47.20 that it suffices to consider a base point x_0 that lies in $M \setminus (-1,1) \times N$.

For $j = 1, \ldots, k$ we do the following:

(1) we pick a point $P_i \in N_i$,

(2) since $M \setminus (-1,1) \times N$ is connected we can pick a path α_j from x_0 to $(-1, P_j)$ and we can pick a path γ_j from $(1, P_j)$ to x_0,
(3) we denote by β_j the "horizontal" path in $[-1,1] \times \mathbb{N}_j$ from $(-1, P_j)$ to $(1, P_j)$,
(4) we write $\delta_j := \alpha_j * \beta_j * \gamma_j$.

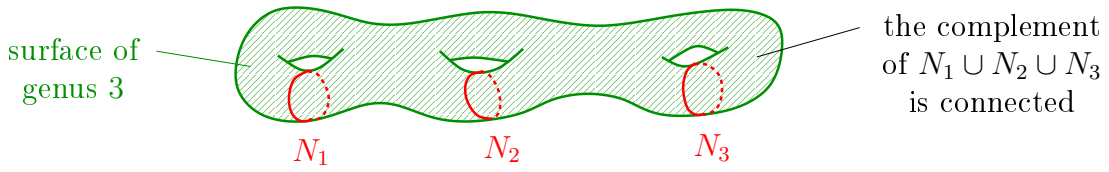
It follows easily from the definitions that for $j = 1, \ldots, k$ we have $\rho_*([\delta_j]) = t_j$. We have thus shown that the generating set $\{t_1, \ldots, t_k\}$ lies in the image of ρ_*. In other words, ρ_* is indeed an epimorphism. ■



Example. Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g. As should be pretty clear from considering the figure below, the surface Σ_g contains g disjoint curves $N_1, \ldots, N_g$ such that the complement is connected. Thus it follows from the Epimorphism-onto-Free Group Proposition 130.8 that there exists an epimorphism

$$\pi_1(\Sigma_g) \ \to \ \langle x_1, \ldots, x_g \rangle.$$

In fact, as we will see in Exercise 130.2, it is not difficult to show the existence of such an epimorphism on a more algebraic level.



The previous example raises the following thought-provoking question.

Question 130.9. Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g.
(1) What is the maximal number $k = k(g) \in \mathbb{N}_0$ for which we can find k disjoint curves $F_1, \ldots, F_k$ on Σ_g such that $\Sigma_g \setminus (F_1 \cup \cdots \cup F_k)$ is connected?
(2) What is the maximal $l = l(g) \in \mathbb{N}_0$ for which there exists an epimorphism from $\pi_1(\Sigma_g)$ onto a free group of k generators?

In summary, as of right now we know the following:

$$g \ \underset{\text{by the above example}}{\overset{\leq}{\uparrow}} \ k(g) \ \underset{\text{Epimorphism-onto-Free Group Proposition 130.8}}{\overset{\leq}{\uparrow}} \ l(g) \ \underset{\text{Abelianization Proposition 53.20}}{\overset{\leq}{\uparrow}} \ 2g.$$

130.4. **Cyclic coverings of manifolds.** Our goal in this section is to give explicit constructions of cyclic coverings of manifolds. To simplify the language will use the following conventions throughout this section.

Convention.

(1) We will not distinguish between two coverings $p\colon X \to B$ and $q\colon T \to B$ that are equivalent in the sense of the definition on page 1368.
(2) We will use the Change-of-Base Point Proposition 47.10 as an excuse to suppress all base points from our notation.[1570]

The following proposition combines several results on coverings that we proved a while ago.

Proposition 130.10. Let M be a connected smooth manifold and let $\varphi\colon \pi_1(M) \to G$ be an epimorphism onto an abelian group. There exists a unique covering $p\colon \widetilde{M} \to M$ such that $p_*(\pi_1(\widetilde{M})) = \ker(\varphi)$. Furthermore this covering has the following properties:
(1) The covering is regular and the deck transformation group is isomorphic to G.
(2) The index of the covering is precisely $|G|$.

Proof. The proposition follows immediately from the Covering Existence Theorem 61.5 and Proposition 62.7 together with the Covering Uniqueness Theorem 61.9 and the Regular Covering-Characterization Corollary 62.8. ■

We sort-of introduced the following definition on page 1383.

Definition. We say a covering $p\colon X \to Y$ is (infinite) cyclic if p is a regular covering and if the deck transformation group is an (infinite) cyclic group.

Let M be a smooth manifold M. Given any primitive $\phi \in \mathrm{H}^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M),\mathbb{Z})$ and given $n \in \mathbb{N}$ we know by Proposition 130.10 that there exist corresponding (infinite) cyclic covers of M. Our goal is to a very explicit constructions of these (infinite) cyclic covers for many smooth manifolds.
In the remainder of this section we will use the following notation:

Notation. Let M be a connected smooth manifold.
(1) We use the First Cohomology Proposition 110.17 to make the identification $\mathrm{H}^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M),\mathbb{Z})$. It follows almost immediately from the definitions that under this identification the primitive elements in $\mathrm{H}^1(M;\mathbb{Z})$ correspond to the epimorphisms $\pi_1(M) \to \mathbb{Z}$.
(2) Let $\phi \in \mathrm{H}^1(M;\mathbb{Z})$ again be a primitive element. Furthermore let $n \in \mathbb{N}$. In the following we denote by $\phi_n\colon \pi_1(M) \to \mathbb{Z}_n$ the epimorphism that is given by the composition $\pi_1(M) \xrightarrow{\phi} \mathbb{Z} \to \mathbb{Z}_n$.

At this stage it is worth recalling the following definitions from page **??**.

Definition. Let M be an oriented smooth manifold and let $F \subset M$ be a compact oriented proper submanifold of codimension one. We pick an orientation-preserving tubular map $\Phi\colon [-1,1] \times F \to M$. We define

$$M \setminus\!\setminus F \;:=\; (M \setminus F) \sqcup (F \times \{\pm 0\})$$

[1570] More precisely, using the Change-of-Base Point Proposition 47.10 it is not difficult to show that for any topological space X, any abelian group G and any two paths $p, q\colon [0,1] \to X$ from a point x_0 to a point x_1 we have $p_* = q_*\colon \mathrm{Hom}(\pi_1(X,x_0),G) \to \mathrm{Hom}(\pi_1(X,x_1),G)$. In particular, given a path-connected topological space X and $x_0, x_1 \in X$ the groups $\mathrm{Hom}(\pi_1(X,x_0),G)$ and $\mathrm{Hom}(\pi_1(X,x_0),G)$ are canonically isomorphic.

with the topology that is generated by the following sets:

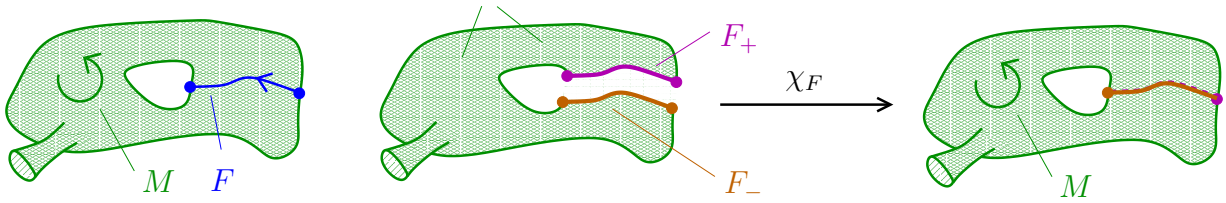
(1) the usual open subset of $M \setminus F$,
(2) the sets of the form $\Phi(U \times (0,\epsilon)) \cup (U \times \{+0\})$ where U is an open subset of F and where $\epsilon \in (0,1)$,
(3) the sets of the form $\Phi(U \times (-\epsilon,0)) \cup (U \times \{-0\})$ where U is an open subset of F and where $\epsilon \in (0,1)$.

We refer to $M \setminus\setminus F$ as M cut along F. This definition is illustrated in the figure below. We also introduce the following extra notation:

(4) We denote by $F_\pm$ the images of $F \times \{\pm 0\}$ in $M \setminus\setminus F$. Furthermore we denote by $i_\pm \colon F_\pm \to M \setminus\setminus F$ the obvious inclusion maps.
(5) We denote by $\sim_F$ the equivalence relation on the set $M \setminus\setminus F$ that is generated by $(P,+0) \sim (P,-0)$ for $P \in F$.
(6) We denote by χ_F the map

$$\begin{array}{rcl} \chi_F \colon M \setminus\setminus F & \to & M \\ P & \mapsto & \begin{cases} P, & \text{if } P \in M \setminus F, \\ Q, & \text{if } P = (Q, \pm 0). \end{cases} \end{array}$$

slightly incorrect picture of M cut along F



Now we can give the explicit construction of the cyclic coverings.

Proposition 130.11. (Cyclic-Covering Proposition) Let M be compact oriented connected smooth manifold. Furthermore let $\phi \in \mathrm{H}^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M),\mathbb{Z})$ be a primitive element. Finally let $F \subset M$ be a compact oriented proper codimension-one submanifold[1571] that is dual to ϕ in the sense that $\mathrm{PD}_M([F]) = \phi$. We denote by $i_\pm \colon F \to F_\pm \subset M \setminus\setminus F$ the two obvious inclusion maps.

(1) We write

$$\widetilde{M}_\infty := \Big(\bigsqcup_{k \in \mathbb{Z}} (M \setminus\setminus F) \times \{k\} \Big) / \sim \qquad \begin{array}{c} \text{where } (i_-(P),k) \sim (i_+(P),k+1) \\ \text{for } P \in F \text{ and } k \in \mathbb{Z}. \end{array}$$

The map

$$\begin{array}{rcl} p \colon \widetilde{M}_\infty & \to & M \\ [(P,k)] & \mapsto & [\chi_F(P)] \end{array}$$

is the infinite cyclic covering corresponding to ϕ.

(2) Let $n \in \mathbb{N}$. We write

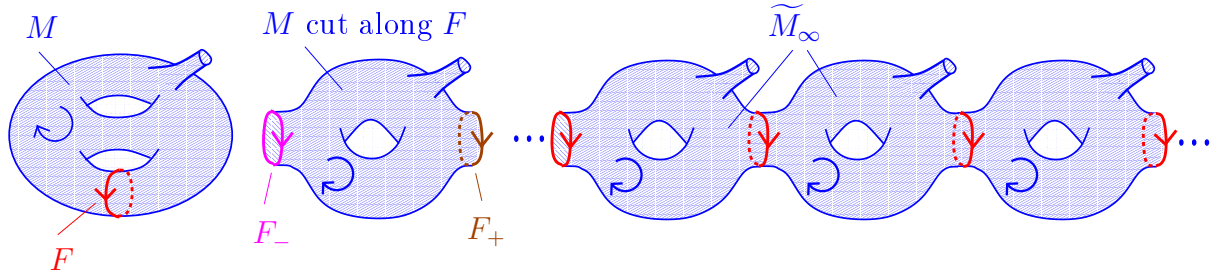
$$\widetilde{M}_n := \Big(\bigsqcup_{k \in \mathbb{Z}_n} (M \setminus\setminus F) \times \{k\} \Big) / \sim \qquad \begin{array}{c} \text{where } (i_-(P),k) \sim (i_+(P),k+1) \\ \text{for } P \in F \text{ and } k \in \mathbb{Z}_n. \end{array}$$

The map

$$\begin{array}{rcl} p \colon \widetilde{M}_n & \to & M \\ [(P,k)] & \mapsto & [\chi_F(P)] \end{array}$$

is the finite cyclic covering corresponding to ϕ_n.

Example. We consider the 2-dimensional smooth manifold M together with the submanifold F that are shown in the figure below. We illustrate the explicit construction of the infinite cyclic covering $\widetilde{M}_\infty$. The reader will not fail to spot the uncanny similarities with Lemma 56.12 and the figure on page 1272.



Proof.

(1) We pick a tubular map $\Phi\colon [-1,1]\times F \to M$ for F. It follows almost immediately from the definitions that the open sets $M \setminus F$ and $\Phi((-1,1)\times F)$ are uniformly covered in the sense of the definition on page 1067. Since these two open subsets cover all of M we see that $p\colon \widetilde{M}_\infty \to M$ is indeed a covering.

In the following we will show that $p\colon \widetilde{M}_\infty \to M$ is indeed the covering corresponding to $\ker(\phi)$.[1572] We consider the following map:

$$\begin{array}{rcl} \alpha\colon M & \to & S^1 \\ P & \mapsto & \begin{cases} \exp(\pi \mathrm{i} t), & \text{if } P = \Phi(t,Q) \text{ with } t \in [-1,1] \text{ and } Q \in F, \\ -1, & \text{otherwise} \end{cases} \end{array}$$

This is of course the Thom-Pontryagin map corresponding to the tubular map that we introduced on page 2839. Let $q\colon \mathbb{R} \to S^1$ be the universal covering that is given by $t \mapsto \exp(\pi \mathrm{i} t)$. Furthermore let

$$\begin{array}{rcl} r\colon \alpha^*\mathbb{R} = \{(P,t) \in M \times \mathbb{R} \mid \alpha(P) = p(t)\} & \to & M \\ (P,t) & \mapsto & P \end{array}$$

be the pullback of the covering q. Note that by the Topological-Pullback Lemma 56.16 (3) we know that $r\colon \alpha^*\mathbb{R} \to M$ is also a covering.

Claim. The covering $p\colon \widetilde{M}_\infty \to M$ is equivalent to the covering $r\colon \alpha^*\mathbb{R} \to M$.

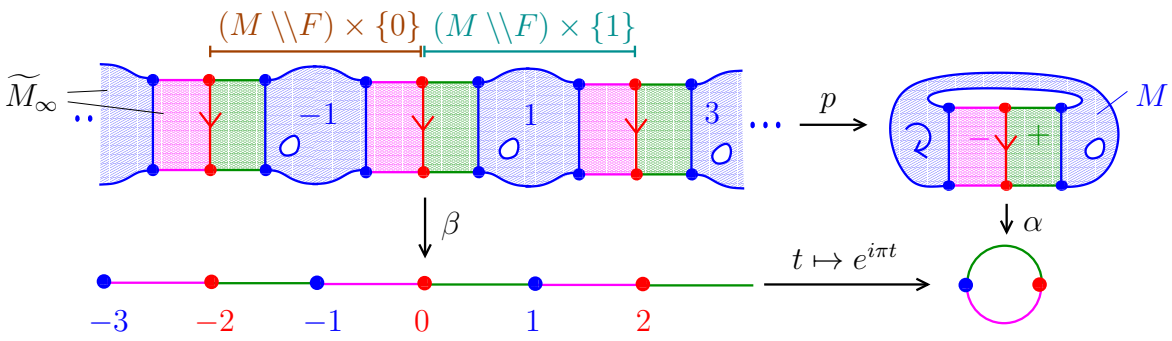
Proof. The key to proving the claim is to consider the map

$$\begin{array}{rcl} \beta\colon \widetilde{M}_\infty & \to & \mathbb{R} \\ [P] & \mapsto & \begin{cases} 2i-1, & \text{if } P \in (M \setminus \Phi([-1,1]\times F)) \times \{i\}, \\ 2i+t, & \text{if } P \in \Phi(\{t\}\times F) \times \{i\} \text{ for some } t \in [-1,0), \\ 2i+t-2, & \text{if } P \in \Phi(\{t\}\times F) \times \{i\} \text{ for some } t \in (0,1], \\ 2i, & \text{if } P \in F_- \times \{i\} \text{ or } P \in F_+ \times \{i+1\}. \end{cases} \end{array}$$

Using Pasting Proposition 2.6 (2) one easily verify that this map is continuous. Next

[1571] Note that we do not assume that F is connected.

[1572] An alternative proof of this statement is given in [**Herr2019**, Lemma 1.31].



we consider the map

$$\begin{array}{rccc} \Theta\colon & \widetilde{M}_\infty & \to & \alpha^*\mathbb{R} \\ & P & \mapsto & (p(P), \beta(P)). \end{array}$$

It is straightforward to see that Θ is a bijection. We leave it to the reader to show that Θ is actually a homeomorphism. One can easily verify that $\Theta \circ r = p$. In summary we conclude that the two coverings q and r are indeed equivalent. ⊞

Now we see that

$$\begin{array}{ccccccccc} q_*(\pi_1(\widetilde{M}_\infty)) & = & r_*(\pi_1(\alpha^*\mathbb{R})) & \overset{\text{Topological-Pullback Lemma 56.16}}{=} & \alpha_*^{-1}(\pi_1(\mathbb{R})) & = & \ker(\alpha_*) & = & \ker(\phi). \\ & \uparrow & & & & \uparrow & & \uparrow & \\ & \text{since } q \text{ and } r \text{ are equivalent coverings} & & & & \text{since } \pi_1(\mathbb{R}) = 0 & & \text{Lemma 130.5} & \end{array}$$

We have thus shown that $q_*\colon \widetilde{M}_\infty \to M$ is indeed the covering corresponding to $\ker(\phi)$.

(2) The proof of this statement is very similar to the proof of (1). We just need to replace the covering $q\colon \mathbb{R} \to S^1$ by the covering $q\colon S^1 \to S^1$ given by $z \mapsto z^n$. We leave it to the reader to once again lessen the burden on the author by filling in the details. ■

130.5. **Curves on 2-dimensional smooth manifolds** (∗). We conclude this chapter with a discussion of curves on 2-dimensional smooth manifolds. The results of this section are nice-to-have, but they will not be used in the remainder of these notes. Thus it is safe to skip this section in the unlikely event that the reader is in a hurry.

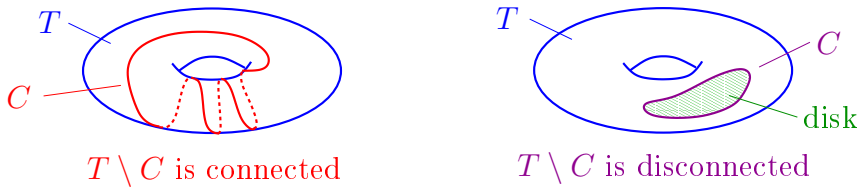
Let us now recall the following definition from page 1123: A *closed oriented curve* C in a smooth manifold M is a closed oriented connected 1-dimensional submanifold of M or of ∂M. In this section we want to study curves on 2-dimensional smooth manifolds. Our main focus will be curves on the torus.

Before we state the results we recall that on page 1733 we saw that a closed oriented curve C in a smooth manifold M defines a homology class $[C] \in \mathrm{H}_1(M;\mathbb{Z})$. We start out with the following lemma.

Lemma 130.12. Let $T = S^1 \times S^1$ be the torus and let C be a closed oriented curve on T.

(1) If $T \setminus C$ is connected, then $[C]$ is a primitive element of $\mathrm{H}_1(T;\mathbb{Z})$.

(2) If $T \setminus C$ is disconnected, then C is the boundary of an embedded disk.



Proof. If $T \setminus C$ is connected, then it follows from the Non-Separating Codimension-One Submanifold Proposition 130.2 that $[C]$ is a primitive element of $\mathrm{H}_1(T;\mathbb{Z})$. Now suppose that $T \setminus C$ is disconnected. By the Tubular Neighborhood Theorem **??** there exists a tubular neighborhood $[-1,1] \times C$. From Lemma 123.1 and Proposition 123.4 we obtain that $T \setminus (-1,1) \times C$ is also disconnected. Since C is connected this implies that $T \setminus (-1,1) \times C$ has two components A and B, one of which contains $\{-1\} \times C$ and one which contains $\{1\} \times C$.[1573] By the Codimension-Zero Submanifold Proposition 19.38 we know that A and B are compact 2-dimensional smooth manifolds, in particular they are finite CW-complexes. We have

$$\chi(A) + \chi(B) \;=\; \underbrace{\chi(T)}_{=0} + \underbrace{\chi(\{-1\} \times C)}_{=0} + \underbrace{\chi(\{1\} \times C)}_{=0} \;=\; 0.$$

Euler Characteristic-Addition Lemma 87.5

Since A and B have boundary it follows from the classification of compact 2-dimensional smooth manifolds and the calculation of Euler characteristics, see Lemma 87.6, that one of A or B is diffeomorphic to the disk $\overline{B}^2$. ■

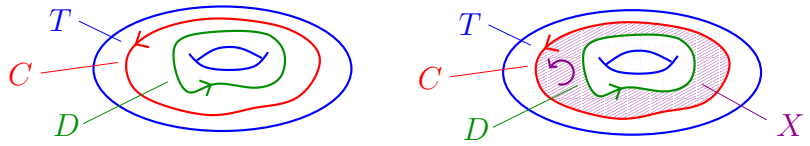
Lemma 130.13. Let $T = S^1 \times S^1$ be the torus. Let C and D be two closed oriented curves on T. If C and D are disjoint and if $[C]$ and $[D]$ are both non-zero[1574] in $\mathrm{H}_1(T;\mathbb{Z})$, then $[C] = \pm[D]$.

Proof. From the Intersection Number-via-Cup Product Theorem 126.5 together with our hypothesis that $C \cap D = \varnothing$ we obtain that $\mathrm{PD}_T([C]) \cup \mathrm{PD}_T([D]) = 0$. It follows easily from the calculation of the cup product of the torus, see Lemma 114.14, that $\mathrm{PD}_T([C]) = k \cdot \mathrm{PD}_T([D])$ for some $k \in \mathbb{Z}$. We obtain from Lemma 130.12 that $k \in \{-1, 1\}$. ■

Lemma 130.14. Let $T = S^1 \times S^1$ be the torus. Let C and D be two closed oriented curves on T. Suppose that $[C] = [D] \neq 0$, then the following hold:

(1) The two closed oriented curves are smoothly isotopic.
(2) If C and D are furthermore disjoint, then there exists an embedded annulus $X \subset T$ with $\partial X = C \cup -D$.

Furthermore, the same statements also hold if we replace the torus $T = S^1 \times S^1$ by the annulus $S^1 \times [-1,1]$.



[1573] Why is that?
[1574] Is this condition that $[C]$ and $[D]$ are both non-zero necessary?

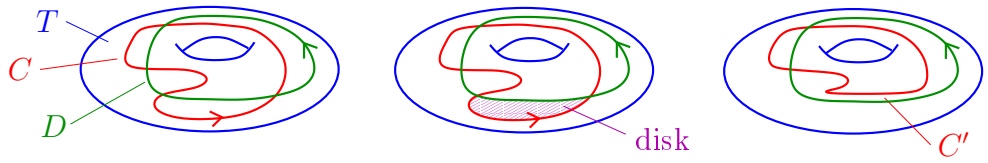
Sketch of proof. We provide a sketch of a proof for the various statements:

(1) By the Transversality Theorem 42.4 (2) we can assume that C and D are already transverse. It follows from the calculation of the cup product of the torus and the Intersection Number-via-Cup Product Theorem 126.5 that the algebraic intersection number is 0. The argument is now similar to the argument of the proof of Theorem **??**, but this time we use Lemma 130.12 to find the desired embedded disks. We refer to the figure below for an illustration.

It is a good, and not entirely trivial, mental exercise to fill in the details. The argument is sketched in more detail in [**Rolf1990**, Lemma 5 of Chapter 2.C].

(2) The proof of this statement is similar to the proof of Lemma 130.12 (2). We leave it to the reader to fill in the details.

The proof in the case that $T = S^1 \times [-1, 1]$ is easier. We will discuss it in Exercise 131.1. ■



For completeness sake' we state the following proposition, even though we will not make use of it.

Proposition 130.15. Let F be a 2-dimensional smooth manifold and let C and D be two closed oriented curves on F.

(1) If C and D are homotopic, then C and D are smoothly isotopic.

(2) If C and D are furthermore disjoint and if they are not null-homotopic, then there exists an embedded annulus $X \subset F$ with $\partial X = C \cup -D$.

Proof.

(1) This statement is [**FaM2011**, Proposition 1.10]. (Closely related statements are also proved in [**Baer1928**], [**Eps1966**, Theorem 2.1].) Unfortunately all the references are somewhat cavalier when it comes to distinguishing continuous and smooth maps, but with some effort one can see that all the references provide smooth isotopies.

(2) If F is orientable, then we will provide the proof in Exercise 130.7. The proof of the non-orientable case is not much harder. ■

Exercises for Chapter 130.

Exercise 130.1. Let M be a *non-compact* oriented n-dimensional smooth manifold with compact boundary. Furthermore let N be a compact orientable non-empty proper submanifold of codimension-one such that $M \setminus N$ is connected. Show that $[N] \in \mathrm{H}_{n-1}(M; \mathbb{Z})$ is an element of infinite order.
Hint. Use Propositions 31.2 and 130.2.

Exercise 130.2. Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g. By the Fundamental Group-of-Surfaces Proposition 54.5 we know that there exists an isomorphism

$$\pi_1(\Sigma_g) \;\cong\; \langle x_1, y_1, \ldots, x_g, y_g \mid [x_1, y_1] \cdot \cdots \cdot [x_g, y_g] \rangle.$$

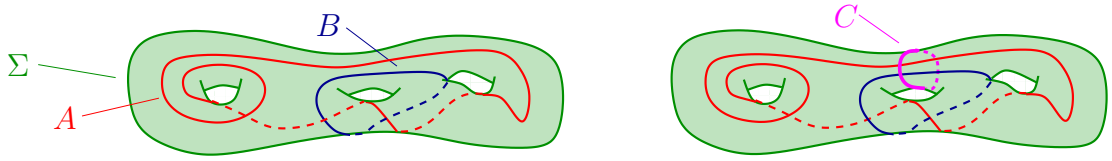
Use this isomorphism to give a purely algebraic proof that $\pi_1(\Sigma_g)$ admits an epimorphism onto a free group on g generators.

Exercise 130.3. Here is one possible approach to answering Question 130.9 (1). Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g. Let $F_1, \dots, F_k$ be disjoint curves on Σ_g such that $\Sigma_g \setminus (F_1 \cup \dots \cup F_k)$ is connected. By the Tubular Neighborhood Theorem **??** we can pick a tubular neighborhood $\bigcup_{i=1}^{k} [-1,1] \times F_i$ for the submanifold $F_1 \cup \dots \cup F_k \subset \Sigma_g$. We consider the compact orientable 2-dimensional smooth manifold $X := \Sigma_g \setminus \bigcup_{i=1}^{k} (-1,1) \times F_i$. What can you say about the Euler characteristic of X? What does the classification of compact orientable 2-dimensional smooth manifolds tell you?

Exercise 130.4. Let Σ be a compact orientable connected 2-dimensional smooth manifold and let $A, B \subset \Sigma$ be two disjoint non-separating curves.

(a) Show that there exists a closed 1-dimensional smooth manifold $C \subset \Sigma$ that intersects each of A and B transversally in a single point.
Remark. Exercise 80.7 might come in handy.

(b) Can one also arrange that C is connected?



Exercise 130.5. Let M be a compact oriented connected smooth manifold and let F be a compact oriented proper submanifold. We use the notation from page 2845. In particular we denote by $i_\pm \colon F \to M \setminus\setminus F$ the two inclusion maps and we denote by $\chi_F \colon M \setminus\setminus F \to M$ the natural projection. Show that there exists a long exact sequence of the form

$$\dots \to \mathrm{H}_i(F) \xrightarrow{i_- - i_+} \mathrm{H}_i(M \setminus\setminus F) \xrightarrow{\chi_F} \mathrm{H}_i(M) \xrightarrow{\partial} \mathrm{H}_{i-1}(F) \to \dots$$

Exercise 130.6. Let (X, x_0) be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $\phi \colon \pi_1(X, x_0) \to \langle t \rangle$ be an epimorphism. We denote by $p \colon \widetilde{X}_\infty \to X$ the covering corresponding to $\ker(\phi)$. By Proposition 62.7 we have an explicit identification of the group $\langle t \rangle$ with the deck transformation group of $p \colon \widetilde{X}_\infty \to X$. Given $\sigma \in \mathrm{C}_n(\widetilde{X}_\infty)$ and given $k \in \mathbb{Z}$ we write $t^k \cdot \sigma := (t^k)_*(\sigma)$.

(a) Let $n \in \mathbb{N}_0$ and let $\sigma, \mu \in \mathrm{C}_n(\widetilde{X}_\infty)$. Show that $p_*(\sigma) = p_*(\mu) \in \mathrm{C}_n(X)$ if and only if there exists a $k \in \mathbb{Z}$ with $\sigma = t^k \cdot \mu$.

(b) Let $n \in \mathbb{N}_0$. Show that the following sequence is exact:

$$0 \to \mathrm{C}_n(\widetilde{X}_\infty) \xrightarrow{t - \mathrm{id}} \mathrm{C}_n(\widetilde{X}_\infty) \xrightarrow{p_*} \mathrm{C}_n(X) \to 0.$$

Remark. By the LES Proposition 74.9 we obtain a corresponding long exact sequence

$$\dots \to \mathrm{H}_{n+1}(X) \xrightarrow{\partial} \mathrm{H}_n(\widetilde{X}_\infty) \xrightarrow{t - \mathrm{id}} \mathrm{H}_n(\widetilde{X}_\infty) \xrightarrow{p_*} \mathrm{H}_n(X) \xrightarrow{\partial} \mathrm{H}_{n-1}(\widetilde{X}_\infty) \to \dots.$$

(c) As a reality check, what does the long exact sequence in (b) look like for the covering $p \colon \mathbb{R} \to S^1$?

(d) Let $r \in \mathbb{N}$. We denote by $\widetilde{X}_r$ the covering of X corresponding to the epimorphism $\ker(\phi\colon \pi_1(X, x_0) \to \langle t \rangle = \mathbb{Z} \to \mathbb{Z}_r)$. Show that there exists an exact sequence relating the homology groups of $\widetilde{X}_\infty$ and $\widetilde{X}_r$.

Exercise 130.7. Let F be an oriented 2-dimensional smooth manifold and let C and D be two closed oriented *disjoint* curves on F that are not null-homotopic. Assume that C and D are homotopic. Show that there exists an embedded annulus $X \subset F$ with $\partial X = C \cup -D$.
Hint. Use Exercise 131.2. You might also want to use some of the ideas of the proof of Lemma 130.12.

Exercise 130.8. Let $T = S^1 \times S^1$ be the torus and let C and D be two closed oriented 1-dimensional submanifolds of T with $[C] = [D] \in \mathrm{H}_1(T; \mathbb{Z})$. Use the results from Section 130.5 to give a down-to-earth proof of the statement that there exists a compact oriented proper 2-dimensional submanifold Ω of $T \times [0, 1]$ such that

$$\partial\Omega \;=\; (\{0\}\times) \cup (\{1\} \times -D).$$

Remark. Evidently this is just a special case of the Cobounding Submanifolds Corollary 130.7. But it is fun to give a bare-hands proof in this special case.

Exercise 130.9. Let Σ be the surface of some genus $g \in \mathbb{N}_0$ and let $f\colon \Sigma \to \Sigma$ be some orientation-preserving self-diffeomorphism. We consider the corresponding mapping torus

$$\mathrm{Tor}(\Sigma, f) \;:=\; (\Sigma \times [0, 1])\,/\!\sim \qquad \text{where } (x, 0) \sim (f(x), 1) \text{ for all } x \in \Sigma.$$

Let F be a surface of some genus k. Suppose that there exists a map $\varphi\colon F \to \mathrm{Tor}(\Sigma, f)$ such that $\varphi_*([F]) = [\{0\} \times \Sigma] \in \mathrm{H}_2(\mathrm{Tor}(\Sigma, f))$. Show that $k \geq g$.
Hint. Make clever use of the Cyclic-Covering Proposition 130.11 and Exercise 102.3.

Exercise 130.10. Let M be a closed oriented n-dimensional smooth manifold. Furthermore let $\phi \in \mathrm{H}_{n-1}(M; \mathbb{Z})$, let $k \in \mathbb{N}$ and let F be a closed oriented $(n-1)$-dimensional submanifold that represents $k \cdot \phi$. Show that we can decompose $F = F_1 \sqcup \cdots \sqcup F_k$ where each F_i represents ϕ.
Hint. Consider the Thom-Pontryagin map $\rho = \rho_F\colon M \to S^1$ that we introduced on page 2839. Note that $\rho^{-1}(1) = F$. Let $p\colon S^1 \to S^1$ be the covering map given by $z \mapsto z^k$. Show that there exists a map $\widetilde{\rho}\colon M \to S^1$ such that $\rho = p \circ \widetilde{\rho}$.

Exercise 130.11. Given a closed oriented 2-dimensional smooth manifold S with components $S_1, \ldots, S_m$ we define

$$\chi_-(S) \;:=\; \sum_{i=1}^{m} \max\{-\chi(S_i), 0\}.$$

Let M be a closed oriented 3-dimensional smooth manifold. We consider the map

$$\begin{aligned} x_M\colon \mathrm{H}_2(M; \mathbb{Z}) &\to \mathbb{N}_0 \\ \sigma &\mapsto \min\left\{ \chi_-(S) \;\middle|\; \begin{array}{l} S \text{ is a closed oriented 2-dimensional} \\ \text{submanifold of } M \text{ with } [S] = \sigma \end{array} \right\}. \end{aligned}$$

Show that the map x_M is a seminorm, i.e. show that it has the following two properties:

(a) Given any $\sigma \in \mathrm{H}_2(M; \mathbb{Z})$ and $k \in \mathbb{Z}$ we have $x_M(k \cdot \sigma) = |k| \cdot x_M(\sigma)$.
Hint. Use Exercise 130.10.

(b) Given $\mu, \nu \in \mathrm{H}_2(M;\mathbb{Z})$ we have $x_M(\mu+\nu) \leq x_M(\mu) + x_N(\nu)$.
Hint. Use the Transversality Theorem 42.4 and Proposition 102.21.

(c) Show that there exists a unique seminorm $x_M \colon \mathrm{H}_2(M;\mathbb{R}) \to \mathbb{R}_{\geq 0}$ such that the map $\mathrm{H}_2(M;\mathbb{Z}) \xrightarrow{[\sigma] \mapsto [\sigma \otimes 1_{\mathbb{R}}]} \mathrm{H}_2(M;\mathbb{R}) \to \mathbb{R}_{\geq 0}$ agrees with the above map x_M.

Remark. This seminorm $x_M \colon \mathrm{H}_2(M;\mathbb{Z}) \to \mathbb{R}_{\geq 0}$ is usually called the *Thurston norm of* M. It was introduced by William Thurston [**Thu1986**] in 1976 and, as is shown in [**Thu1986**] and [**Gab1983**], it has many remarkable properties.

Exercise 130.12. We continue with the definitions introduced in Exercise 130.11. Let Σ be a closed connected 2-dimensional smooth manifold with $\chi(\Sigma) \leq 0$.

(1) Let $f \colon \Sigma \to \Sigma$ be a diffeomorphism. We consider the mapping torus $\mathrm{Tor}(\Sigma, f)$. Show that $x([\Sigma]) = \chi_-(\Sigma)$.
Hint. Let F be any closed oriented 2-dimensional submanifold of $\mathrm{Tor}(\Sigma, f)$ with $[F] = [\Sigma]$. Using the Cyclic-Covering Proposition 130.11 we obtain a map $F \to \Sigma \times \mathbb{R}$. Consider the map $F \to \Sigma \times \mathbb{R} \to \Sigma$ and use Exercise 101.4.

(2) We can to consider the Thurston norm of $N \times S^1$.
(a) Show that $x_{\Sigma^1 \times S^1}([\Sigma \times \{*\}] = \chi_-(\Sigma)$.
(b) Show that for any $\sigma \in \mathrm{H}_1(\Sigma)$ we have $x_{\Sigma^1 \times S^1}([S^1] \times \sigma) = 0$.
(c) Show that for any $\phi \in \mathrm{H}_2(\Sigma \times S^1;\mathbb{R})$ we have

$$x_{\Sigma \times S^1}(\phi) \;=\; \langle \underbrace{\mathrm{PD}_{\Sigma \times S^1}}_{\in \mathrm{H}^1(M;\mathbb{R})}, \underbrace{[\{*\} \times S^1]}_{\in \mathrm{H}_1(M;\mathbb{Z})} \rangle.$$

Hint. Either use Proposition 118.15 and (1). Alternatively use (2a) and (2b) together with Exercise 130.11 (b).

Exercise 130.13. Let M be a closed oriented n-dimensional smooth manifold. Given $\phi \in \mathrm{H}^1(M;\mathbb{Z})$ we define

$$x_M(\sigma) \;=\; \inf\{\|N\|_{\mathbb{R}} \mid N \text{ is a closed oriented submanifold of } M \text{ with } \mathrm{PD}_M([N]) = \phi\}.$$

($\|N\|_{\mathbb{R}}$: simplicial volume as defined in Exercise 101.9)

Let N be a closed oriented connected $(n-1)$-dimensional smooth manifold. Given any cohomology class $\phi \in \mathrm{H}^1(N \times S^1)$ determine $x_{N \times S^1}(\phi)$ in terms of $\|N\|_{\mathbb{R}}$ and $\phi(\{*\} \times S^1)$.
Hint. Use Proposition 118.15 and (a).

131. Applications to knot theory

After so much theory we want to practice what we have learned by applying our recent results to knot theory.

131.1. **The longitude of a knot.** We start out our discussion of knots with the following definition.

Definition. Let $K \subset S^3$ be an oriented knot.

(1) A thickening of K is an orientation-preserving smooth embedding[1575] $F\colon \overline{B}^2 \times K \to S^3$ such that $F(0,P) = P$ for all $P \in K$.

(2) Given a thickening F of K we refer to $X_K := S^3 \setminus F(B^2 \times K)$ as the knot exterior. We equip the codimension-zero submanifold X_K of S^3. with the corresponding standard orientation that we introduced on page 684.

(3) Given a thickening F of K and given $* \in K$ we refer to the oriented submanifold $\mu_K := F(S^1 \times \{*\})$ of $S^3 \setminus K$ as the meridian of K.

(4) By Lemma 100.16 (5) the inclusion induced map $\mathrm{H}_1(X_K;\mathbb{Z}) \xrightarrow{\cong} \mathrm{H}_1(S^3 \setminus K;\mathbb{Z})$ is an isomorphism and we have $\mathrm{H}_1(X_K;\mathbb{Z}) = \mathbb{Z} \cdot [\mu_K]$. We denote by ϕ_K the isomorphism

$$\begin{array}{rccc} \phi_K\colon \mathrm{H}_1(S^3 \setminus K;\mathbb{Z}) & \xleftarrow{\cong} & \mathrm{H}_1(X_K;\mathbb{Z}) & \xrightarrow{\cong} \mathbb{Z} \\ & & [\mu_K] & \to 1. \end{array}$$

(5) Given a thickening $F\colon \overline{B}^2 \times K \to S^3$ we define

$$\text{self-linking number of } F \;:=\; \phi_K(\underbrace{F_*([\{*\} \times K])}_{\in \mathrm{H}_1(X_K;\mathbb{Z})}),$$

where $* \in \overline{B}^2$ is a random point.

Remark. It follows from the uniqueness of tubular neighborhoods, see Proposition **??**, that the exterior X_K and the meridian μ_K are, in an appropriate sense, independent of the choice of the thickening. Therefore often we just say "the exterior" and "the meridian".

Lemma 131.1. (Knot Thickening Lemma) Let K be an oriented knot in S^3.

(1) Let $F\colon \overline{B}^2 \times K \to S^3$ be a thickening of K and let $s \in \mathbb{Z}$. Let $\gamma\colon S^1 \to K$ be an orientation-preserving diffeomorphism. Similar to Lemma **??** we consider the diffeomorphism

$$\begin{array}{rccc} \Psi\colon \overline{B}^2 \times K & \to & \overline{B}^2 \times K \\ (w, \gamma(z)) & \mapsto & (\underbrace{w \cdot_{\mathbb{C}} z^s}_{\substack{\text{multiplication in } \mathbb{C}, \text{ where} \\ w \in \overline{B}^2 \subset \mathbb{C} \text{ and } z^s \in \mathbb{C}}}, \gamma(z)). \\ \uparrow & & \\ \text{where } z \in S^1 & & \end{array}$$

The map $F \circ \Psi$ is a thickening with

$$\text{self-linking number of } F \circ \Psi \;=\; \text{self-linking number of } F + s.$$

(2) Given any $n \in \mathbb{Z}$ there exists a thickening of K with self-linking number n.

[1575] We equip S^3 and $\overline{B}^2 \times S^1$ with the standard orientation that we introduced on page 681 and on page 2502.

(3) If $F, F' \colon \overline{B}^2 \times K \to S^3$ are two thickenings of K with self-linking number n, then there exists a diffeotopy $H \colon S^3 \times [0,1] \to S^3$ rel K with $H_0 = \mathrm{id}$ and $H_1 \circ F = F'$.

Given $K \subset \mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ the analogous statements hold if we consider thickenings that are contained in $\mathbb{R}^3$. Furthermore, in this setting the diffeotopy in (3) of $\mathbb{R}^3$ can be assumed to have compact support.

Remark. Given an oriented knot K and $n \in \mathbb{Z}$ sometimes we use the uniqueness statement of the Knot Thickening Lemma 131.1 to talk about *the thickening of K with self-linking number n*. Sometimes we shorten this cumbersome expression to *the n-thickening of K*.

Proof.

(1) We denote by m the self-linking number of F. We write $\lambda := F(\{1\} \times K)$ and $\mu := F(S^1 \times \{1\})$ and we view λ and μ as oriented 1-dimensional submanifolds with the orientations given by K and S^1. We calculate that

$$\text{self-linking number of } F \circ \Psi = \phi_K(F_*(\Psi_*([\{1\} \times K]))) \underset{\uparrow}{=} \phi_K(s \cdot [\mu] + [\lambda]) \underset{\uparrow}{=} s + m.$$

follows from the Action-on-Torus Homology Lemma 84.7 (2) — since $\phi_K(\mu) = 1$ and by definition of m

(2) From the Tubular Neighborhood Theorem **??** we know that there exists a thickening $F \colon \overline{B}^2 \times K \to S^3$. We denote by m the self-linking number of F. The desired statement now follows from (1) applied to $n - m$.

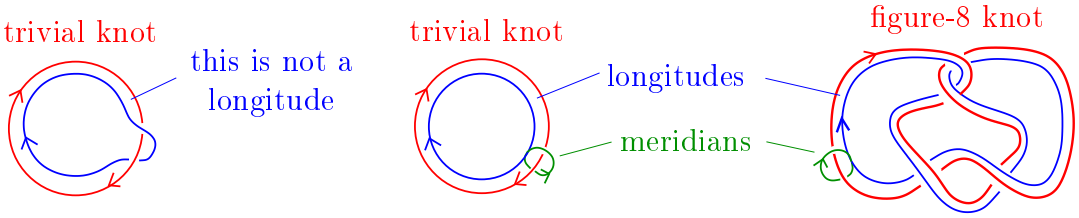
(3) The uniqueness statement is an immediate consequence of (1) together with Proposition **??** and Lemma **??** (3).

The same arguments as above also apply if we replace S^3 by $\mathbb{R}^3$. ■

Definition. Let $K \subset S^3$ be an oriented knot. We pick a thickening $F \colon \overline{B}^2 \times K \to S^3$ of self-linking number zero. We refer to any $\lambda_K := F(\{*\} \times K)$ as a longitude of K.[1576]

Examples.

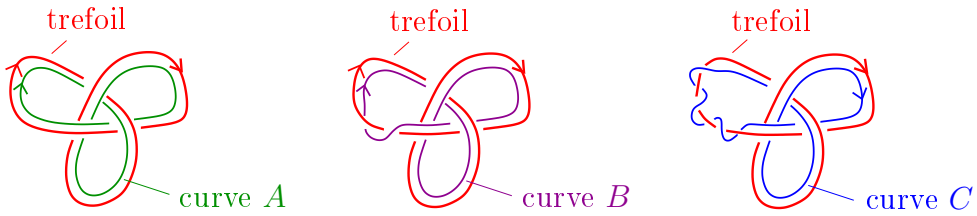
(1) In the figure below we show the longitude and the meridian for the trivial knot and the figure-8 knot.



(2) In the figure below we show the trefoil knot with three wannabe longitudes. It is not entirely clear which one is the right longitude. We will soon develop a tool with which it will be straightforward to settle this issue.

On several occasions the following lemma comes in handy.

[1576] It follows from the Knot Thickening Lemma 131.1 (3) that the longitude is in an appropriate way uniquely determined.



Lemma 131.2. (Longitude-Meridian Lemma) Let K be an oriented knot and let $F\colon \overline{B}^2 \times K \to S^3$ be a thickening with self-linking number zero. We denote by $X_K = S^3 \setminus F(B^2 \times K)$ the corresponding knot exterior. Furthermore we denote as usual by $\mu_K = F(S^1 \times \{*\})$ and $\lambda_K = F(\{*\} \times K)$ the corresponding meridian and longitude.

(1) The classes $[\mu_K]$ and $[\lambda_K]$ form a basis for $\mathrm{H}_1(\partial X_K;\mathbb{Z})$.

(2) There exists a short exact sequence of the following form

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{H}_2(X_K,\partial X_K;\mathbb{Z}) & \to & \overbrace{\mathrm{H}_1(\partial X_K;\mathbb{Z})}^{=\mathbb{Z}\cdot[\mu_K]\oplus\mathbb{Z}\cdot[\lambda_K]} & \to & \overbrace{\mathrm{H}_1(X_K;\mathbb{Z})}^{=\mathbb{Z}\cdot[\mu_K]} & \to & 0 \\ & & & & [\mu_K] & \mapsto & [\mu_K] & & \\ & & & & [\lambda_K] & \mapsto & 0. & & \end{array}$$

(3) The longitude λ_K is the unique, up to smooth isotopy, closed oriented curve on the torus $F(S^1 \times K)$ that is homologous to $K = \{0\} \times K$ in $F(\overline{B}^2 \times K)$ and that is null-homologous in X_K.

(4) If we view μ_K and λ_K as closed oriented curves on the oriented torus $F(S^1 \times K)$, then the algebraic intersection number equals $\mu_K \cdot \lambda_K = +1$.

(5) If denote by $i\colon \mu_K \to \partial X_K$ the inclusion map, then[15771578]

$$i^*([\mu_K]^*) \cap [\partial X_K] \;=\; -[\lambda_K] \;\in\; \mathrm{H}_1(\partial X_K;\mathbb{Z}).$$

(6) Recall that given an oriented curve C we denote by $-C$ the curve with the opposite orientation. With this notation we have $\mu_{-K} = -\mu_K$ and $\lambda_{-K} = -\lambda_K$.

Proof.

(1) This statement follows from Lemma 78.11 (2).

(2) We consider the long exact sequence of homology groups of the pair $(X_K, \partial X_K)$:

$$\underbrace{\mathrm{H}_2(X_K;\mathbb{Z})}_{=0 \text{ by Lemma 100.16 (4)}} \to \mathrm{H}_2(X_K,\partial X_K;\mathbb{Z}) \to \mathrm{H}_1(\partial X_K;\mathbb{Z}) \xrightarrow{\phi} \mathrm{H}_1(X_K;\mathbb{Z}).$$

By definition we have $\phi([\lambda_K]) = 0$. Furthermore we know from Lemma 100.16 that the right-hand map is an epimorphism. The statement follows immediately from these observations.

(3) It follows from (1) that the kernel of $\mathrm{H}_1(\partial X_K;\mathbb{Z}) \to \mathrm{H}_1(X_K;\mathbb{Z})$ equals $\mathbb{Z}\cdot[\lambda_K]$. Since $\mathrm{H}_1(F(\overline{B}^2 \times K);\mathbb{Z}) = \mathbb{Z}\cdot[K] = \mathbb{Z}\cdot[\lambda_K]$ we see that there exists a unique homology

[1577] We denote by $[\mu_K]^* \in \mathrm{H}^1(\mu_K;\mathbb{Z})$ the dual fundamental class as defined on page 2451, i.e. the unique cohomology class with $\langle [\mu_K]^*, [\mu_K] \rangle = 1$.

[1578] As we mentioned above, we equip X_K with a standard orientation. We equip ∂X_K with the corresponding boundary orientation as defined on page 689.

class that has the properties we are interested in. The desired statement now follows from Lemma 130.14 (1).

(4) This statement follows from the definitions and the easy observation that the oriented submanifolds $S^1 \times \{*\}$ and $\{*\} \times S^1$ of the oriented smooth manifold $S^1 \times S^1$ have algebraic intersection number one.

(5) We calculate that

$$\begin{array}{lll} i^*([\mu_K]^*) \cap [\partial X_K] & = i^*([\mu_K]^*) \cap F_*(-[S^1 \times K]) & = F_*\big([S^1 \times \{*\}]^* \cap (-[S^1 \times K])\big) \\ & \uparrow & \uparrow \\ & \text{using that } F \text{ is orientation-preserving} & \text{Cap Product-Naturality Lemma 116.8} \\ & \text{and using Exercise 25.19 we see} & \\ & \text{that } F_*([S^1 \times S^1]) = [\partial F(\overline{B}^2 \times K)] = -[\partial X_K] & \\ & = -F_*([\{*\} \times K]) & = -[\lambda_K]. \\ & \uparrow & \uparrow \\ & \text{see page 2555} & \text{by definition of } \lambda_K \end{array}$$

(6) This statement follows easily from the definitions. We leave it to the reader to provide the details. ■

131.2. **Detecting non-invertible and chiral knots.** Now that we have the notion of the self-linking number of thickening a knot we can also outline how to answer several questions on symmetries of knots we had asked a long time ago.

We start out with the following definition.

Definition. Let K be an oriented knot.

(1) The mirror K^{mir} is defined as the reflection of K in any hyperplane of $\mathbb{R}^4$.[1579] We give the mirror K^{mir} the orientation that turns the reflection into an orientation-preserving diffeomorphism $K \to K^{\text{mir}}$.

(2) The reverse K^{rev} is defined as the knot K with the reverse orientation.

(3) The inverse K^{inv} is defined as the mirror of the reverse of K.

Remark. The names "reverse" and "inverse" at times also get used with different meanings. We follow [**Conw1970**, p. 336] for our naming convention.



knot K | the mirror K^{mir} | the reverse K^{rev} | the inverse K^{inv}

Now the question arises whether an oriented knot is smoothly isotopic to any of its doppelganger. This leads us to the following definition.

Definition. Let K be an oriented knot.

(1) We say that K is amphichiral if it is smoothly isotopic to its mirror image. Otherwise we call the knot K chiral.[1581]

[1579] Note that it follows from Exercise 59.1 that, up to a smooth isotopy, the definition of K^{mir} does not depend on the choice of the hyperplane.

[1580] Note that on page 2387 we introduced the mirror of an unoriented knot. For the purpose of this section it is better to work with oriented knots.

(2) We say K is reversible if it is smoothly isotopic to its reverse.
(3) We say K is invertible if it is smoothly isotopic to its inverse.

Example. As we saw in the figure on page 1303, it is basically clear that the trivial knot is amphichiral and reversible. Furthermore, in the figure on page 1316 we showed that the figure-8 knot J is amphichiral. Also note that in Exercise 34.7 we showed that the trefoil and the figure-8 knot are reversible.

This lead us to the following two questions.

Question 58.2. Is every knot reversible?

Question 59.9. Is the trefoil chiral?

Before we show how to answer both questions let us introduce the following simple-minded definition.

Definition.

(1) A group system is a system (G, g_1, g_2) consisting of a group G together with two elements $g_1, g_2 \in G$.
(2) A isomorphism between group systems (G, g_1, g_2) and (H, h_1, h_2) is a group isomorphism $\varphi\colon G \to H$ with $\varphi(g_1) = h_1$ and $\varphi(g_2) = h_2$.

The following proposition explains why we care about group systems.

Proposition 131.3. (Knot-Group System Proposition)

(1) Let $K \subset S^3$ be an oriented knot. We pick a thickening $F\colon \overline{B}^2 \times K \to S^3$ of K with self-linking number zero.[1582] Furthermore we pick an orientation-preserving diffeomorphism $\gamma\colon S^1 \to K$. We write

$$x_{K,F} := F(1, \gamma(1)), \quad \mu_{K,F}\colon \begin{array}{ccc} S^1 & \to & S^3 \setminus K \\ z & \mapsto & F(z, \gamma(1)), \end{array} \quad \lambda_{K,F,\gamma}\colon \begin{array}{ccc} S^1 & \to & S^3 \setminus K \\ z & \mapsto & F(1, \gamma(z)). \end{array}$$

The isomorphism type of the group system

$$(\pi_1(S^3 \setminus K, x_{K,F}), [\mu_{K,F}], [\lambda_{K,F,\gamma}])$$

depends only on the choice of the oriented knot K. Any group system that is isomorphic to the above group system is called a *group system for* K.
(2) Let K and J be two oriented knots. If K and J are smoothly isotopic, then the corresponding group systems are isomorphic.
(3) Given an oriented knot K with a group system $(\pi_K, \mu_K, \lambda_K)$ the following hold:
   (a) a group system for the mirror K^{mir} is given by $(\pi_K, \mu_K^{-1}, \lambda_K)$
   (b) a group system for the reverse K^{rev} is given by $(\pi_K, \mu_K^{-1}, \lambda_K^{-1})$
   (c) a group system for the inverse K^{inv} is given by $(\pi_K, \mu_K\ , \lambda_K^{-1})$.

Remark. Friedhelm Waldhausen [**Wald1968b**, Corollary 6.5] (see also [**BZH2014**, Theorem 3.16]) showed in 1968 that the converse to the Knot-Group System Proposition 131.3 (2) holds. Thus we are certainly on the right track.

Proof.

[1581] On page 1316 we introduced these notions of unoriented knots.
[1582] We know by the Knot Thickening Lemma 131.1 (2) that such a thickening always exists.

(1) This statement follows easily from the Knot Thickening Lemma 131.1 (3).
(2) This statement follows equally easily from the Isotopy Extension Theorem 36.1.
(3) Let $F\colon \overline{B}^2 \times K \to S^3$ be a thickening of K with self-linking number zero and let $\gamma\colon S^1 \to K$ be an orientation-preserving diffeomorphism. Throughout the proof of (3) we work with the group system $(\pi_1(S^3 \setminus K, x_{K,F}), [\mu_{K,F}], [\lambda_{K,F,\gamma}])$ for K.

In the following we denote by ρ the reflection in a hyperplane of $\mathbb{R}^4$ and we set $K^{\text{mir}} = \rho(K)$. Finally we denote by $\varphi\colon S^1 \to S^1$ and $\psi\colon \overline{B}^2 \to \overline{B}^2$ the maps given by complex conjugation.

(a) First note that the map[1583]

$$\begin{aligned} F^{\text{mir}}\colon \overline{B}^2 \times K^{\text{mir}} &\to S^3 \\ (x,y) &\mapsto \rho(F(\psi(x), \rho(y))) \end{aligned}$$

is a thickening of K^{mir} with self-linking number zero. Furthermore note that $\gamma^{\text{mir}} := \rho \circ \gamma\colon S^1 \to K^{\text{mir}}$ is orientation-preserving. Thus we see that ρ_* induces an isomorphism between the group system $(\pi_1(S^3 \setminus K, x_{K,F}), [\mu_{K,F}]^{-1}, [\lambda_{K,F,\gamma}])$ and the group system $(\pi_1(S^3 \setminus K^{\text{mir}}), x_{K^{\text{mir}},F^{\text{mir}}}, [\mu_{K^{\text{mir}},F^{\text{mir}}}], [\lambda_{K^{\text{mir}},F^{\text{mir}},\gamma^{\text{mir}}}])$.

(b) First note that the map

$$\begin{aligned} F^{\text{rev}}\colon \overline{B}^2 \times K^{\text{rev}} &\to S^3 \\ (x,y) &\mapsto F(\psi(x), y) \end{aligned}$$

is a thickening of K^{rev} with self-linking number zero. Furthermore note that $\gamma^{\text{rev}} := \gamma \circ \varphi\colon S^1 \to K^{\text{rev}}$ is orientation-preserving. Finally note that the group system $(\pi_1(S^3 \setminus K^{\text{rev}}, x_{K^{\text{rev}},F^{\text{rev}}}), [\mu_{K^{\text{rev}},F^{\text{rev}}}], [\lambda_{K^{\text{rev}},F^{\text{rev}},\gamma^{\text{rev}}}])$ equals the group system $(\pi_1(S^3 \setminus K, x_{K,F}), [\mu_{K,F}^{-1}], [\lambda_{K,F,\gamma}^{-1}])$.

(c) This statement follows immediately from (a) and (b). ■

On page 1334 we saw that looking at finite quotients of groups can be a convenient tool for distinguishing groups. We now apply the same idea to group systems. There are various ways of making this idea precise. We will work with the following approach.

Definition. Let G be a *finite* group and let $x \in G$.

(1) Given a group system (π, g_1, g_2) where π is a finitely generated group we define

$$P_G^x(\pi, g_1, g_2) \;:=\; \overset{\text{finite sum by Exercise 53.3}}{\overset{\downarrow}{\sum_{\{\varphi \in \text{Hom}(\pi,G)\,|\,\varphi(g_1)=x\}}}} \varphi(g_2) \;\in\; \overset{\text{group ring as defined on page 1872}}{\overset{\downarrow}{\mathbb{Z}[G]}}.$$

It is clear that isomorphic group systems give the same element in $\mathbb{Z}[G]$.

(2) Let K be a knot. Following [**Eis2007**, Definition 1.2] we refer to

$$P_G^x(K) \;:=\; P_G^x(\text{group system of } K) \;\in\; \mathbb{Z}[G],$$

as the (G,x)-coloring polynomial of K.

The art now is to find, given a knot, a finite group G and element $x \in G$ such that the (G,x)-coloring polynomial has content.

[1583] In particular note that the "extra ψ" ensures this map F^{mir} is indeed orientation-preserving.

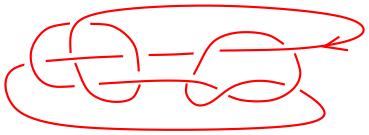
Example. We consider the Kinoshita-Terasaka knot and the Conway knot that are shown in the figure below. This pair of knots is notoriously hard to distinguish. Now let G be the Mathieu group M_{11}, this is the unique simple group of order $7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$ and the smallest of the sporadic simple groups, see [**CCNPW1985**] for details. As is explained in [**Eis2007**, p. 315], one way of describing $G = M_{11}$ is to note that it is the subgroup of the alternating group A_{11} that is generated by

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 3 & 5 & 7 & 10 & 1 & 8 & 6 & 11 & 9 & 4 \end{pmatrix}.$$

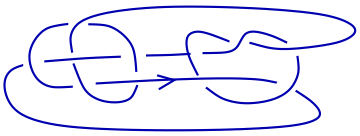
Note that x has order 11 in A_{11} and thus also in $G = M_{11}$. By [**Eis2007**, p. 317] we have the following table:

$$\begin{array}{lcllcl} P^x_G(K) & = & 1 + 11x^3 + 11x^7, & P^x_G(C) & = & 1 + 11x^3 + 11x^7, \\ P^x_G(K^{\mathrm{mir}}) & = & 1 + 11x^4 + 11x^8, & P^x_G(C^{\mathrm{mir}}) & = & 1 + 11x^4 + 11x^8, \\ P^x_G(K^{\mathrm{rev}}) & = & 1 + 11x^4 + 22x^8, & P^x_G(C^{\mathrm{rev}}) & = & 1 + 11x^4 + 11x^6 + 11x^8, \\ P^x_G(K^{\mathrm{inv}}) & = & 1 + 22x^3 + 11x^7, & P^x_G(C^{\mathrm{inv}}) & = & 1 + 11x^3 + 11x^5 + 11x^7. \end{array}$$

Since x has order 11 in G we see that all these elements of $\mathbb{Z}[G]$ are different. This shows that these eight oriented knots are pairwise non-smoothly isotopic. In particular this shows that the Kinoshita-Terasaka knot and the Conway knot are neither amphichiral, nor reversible nor invertible.



Kinoshita-Terasaka knot



Conway knot

For the record let us formulate the following proposition.

Proposition 131.4. There exists an oriented knot that is neither amphichiral, nor reversible nor invertible.

We can now finally answer Question 59.9 in the affirmative.

Proposition 131.5. The trefoil is chiral.

Proof. We equip the trefoil K with either orientation. In Proposition 59.6 we showed that there exists an epimorphism $\pi_1(S^3 \setminus K) \to S_3$. For a suitable choice of $x \in S_3$ one sees, using the Knot-Group System Proposition 131.3 and using the knot coloring polynomial $P^x_{S_3}$ that K is neither smoothly isotopic to it mirror nor to its inverse. Thus the unoriented trefoil is chiral. We will work out the details in Exercise 131.4. ∎

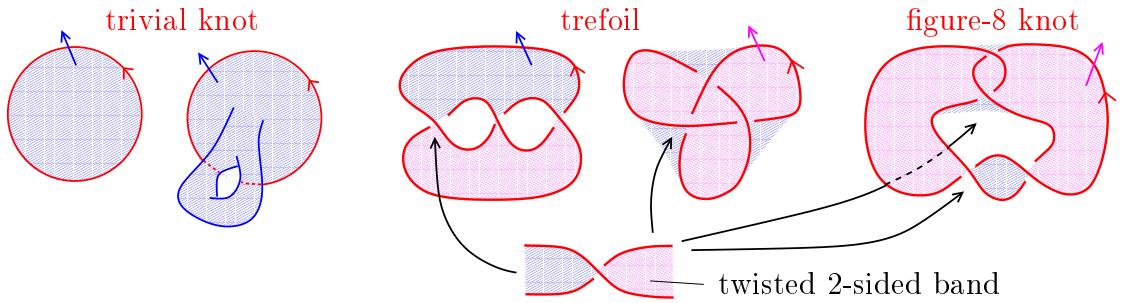
131.3. **Seifert surfaces.** The following is the key object in this section.

Definition. Let $K \subset S^3$ be a knot. A Seifert surface[1590] for K is a compact orientable connected 2-dimensional submanifold F of S^3 such that $\partial F = K$. If K itself is oriented, then we equip a Seifert surface F with the orientation such that $\partial F = K$ as oriented smooth manifolds.

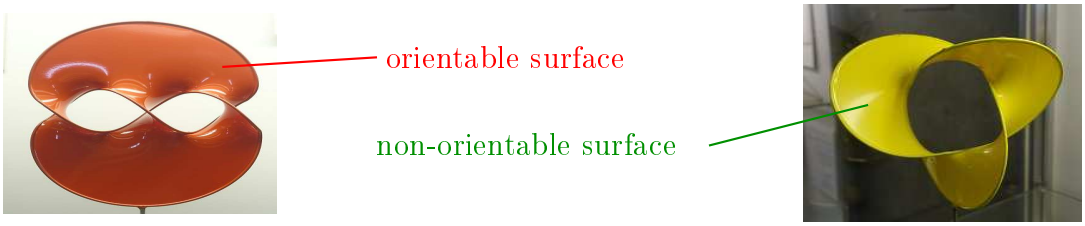
[1590] Seifert surfaces are named after Herbert Seifert (1907-1996) whom we had already encountered at the Mayer–Vietoris Theorem 52.2.

Examples.

(1) In the figure below we show Seifert surfaces for the trivial knot, the trefoil and the figure-8 knot.



(2) In the figure below we see two surfaces of minimal surface area that bound the trefoil, both were manufactured by Jürgen Neukirch. The one on the left is orientable, thus it is a Seifert surface. The one on the right is diffeomorphic to the Möbius band, thus it is not a Seifert surface.



The following proposition is arguably one of the most useful results in knot theory.

Proposition 131.6. (Seifert Surface-Existence Proposition) Every knot $K \subset S^3$ admits a Seifert surface.

Remark.

(1) The proposition was first proved in 1930 by F. Frankl and Lev Pontryagin [**FPo1930**] in 1930. Shortly afterwards Herbert Seifert [**Sei1934**] gave a different proof. More precisely Seifert found an algorithm that given a diagram for a knot produces an explicit Seifert surface for the knot. Seifert's algorithm is also described in most books on knot theory, see e.g. [**Rolf1990**, p. 120] or [**Lic1997a**, Theorem 2.2].

(2) A slight generalization of the argument we give proves the following stronger statement: given any closed orientable n-dimensional smooth manifold $V \subset S^{n+2}$ there exists a compact orientable $(n+1)$-dimensional submanifold $W \subset S^{n+2}$ such that $\partial W = V$. We refer to [**Bre1993**, Theorem VII.14.7], [**Er1969**, Lemma 2.2] and [**BDR2017**] for details. The proof in these two references is close to the approach taken in the proof of the Seifert Surface-Existence Proposition 131.6, but it circumvents the use of the Codimension-One Submanifold Existence Proposition 130.4 by a more formal argument.

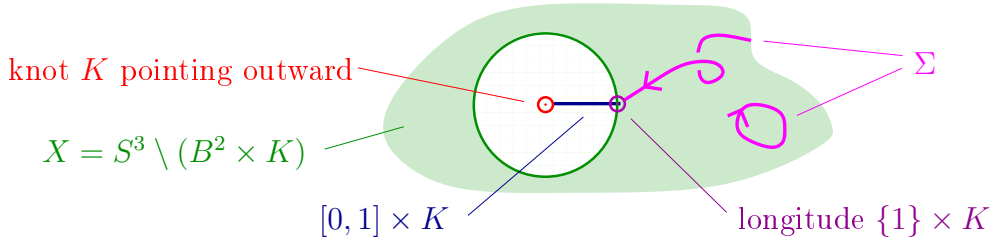
Proof. Let K be a knot. We need to show that there exists a compact orientable connected 2-dimensional submanifold F of S^3 such that $\partial F = K$.

First note that by the Knot Thickening Lemma 131.1 (2) we can pick a thickening $\overline{B}^2 \times K$ of the knot K with self-linking number zero. We denote by $X := S^3 \setminus (B^2 \times K)$ the knot

exterior. Note that by the Longitude-Meridian Lemma 131.2 (2) we know that there exists a unique homology class $\sigma \in \mathrm{H}_2(X, \partial X; \mathbb{Z})$ such that $\partial\sigma = [\{*\} \times K] \in \mathrm{H}_1(\partial X; \mathbb{Z})$. It follows from Proposition 130.6 that there exists a compact oriented proper 2-dimensional submanifold Σ of X such that $[\Sigma] = \sigma$ and such that $\partial\Sigma = \{*\} \times K$. We apply Proposition **??** to

$$([0,1] \times K) \cup \Sigma \ \subset \ (\overline{B}^2 \times K) \cup X \ = \ S^3$$

and we obtain a compact orientable 2-dimensional submanifold Z whose boundary equals K. This submanifold might not be connected, but the component of Z that contains K is the desired Seifert surface. ■

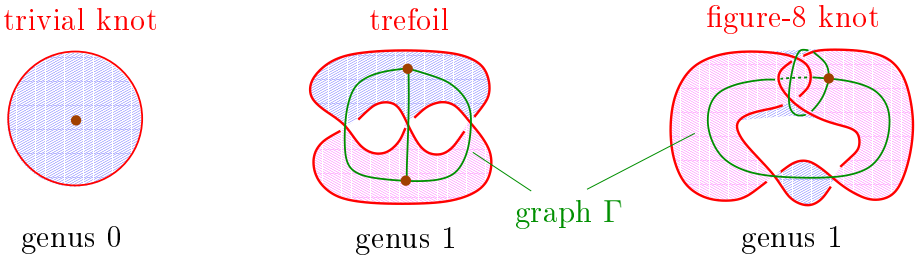


Definition. Let F be a compact orientable connected 2-dimensional smooth manifold with m boundary components. By the Surface Classification Theorem 55.4 there exists a unique $g \in \mathbb{N}_0$ such that F is homeomorphic to the surface of genus g minus m open disks. We refer to g as the genus of F.

Given a Seifert surface it is sometimes not entirely obvious how to determine the genus. The following lemma gives an easy criterion to determine the genus of a Seifert surface.

Lemma 131.7. Let F be a compact orientable connected 2-dimensional submanifold with one boundary component. If $\Gamma \subset F$ is a topological graph that is a deformation retract of F, then

$$\text{genus of } F \ = \ \tfrac{1}{2} \cdot (1 - \chi(F)) \ = \ \tfrac{1}{2} \cdot (1 - \#\text{vertices of } \Gamma + \#\text{edges of } \Gamma).$$

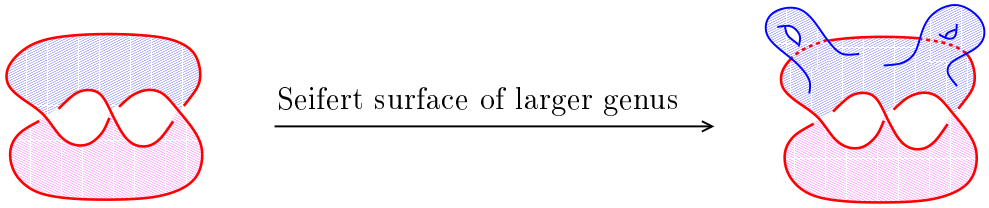


Proof. We calculate

$$\text{genus of } F \ \underset{\text{by Proposition 80.10 we have } \chi(F) = 1-2g}{=} \ \tfrac{1}{2}(1 - \chi(F)) \ \underset{\text{by Corollary 87.2 since } \Gamma \text{ is a deformation retract of } F}{=} \ \tfrac{1}{2} \cdot (1 - \chi(\Gamma)).$$

■

If a knot K bounds a Seifert surface of genus g, then the construction sketched in the figure below shows pretty convincingly that K also bounds Seifert surfaces of any genus $h \geq g$. So the interesting question is, what is the minimal genus of a Seifert surface for a given knot? This leads us to the following definition.



Definition. Given a knot $K \subset S^3$ we define

$$\text{genus}(K) := \text{minimal genus of a Seifert surface of } K.$$

Examples.

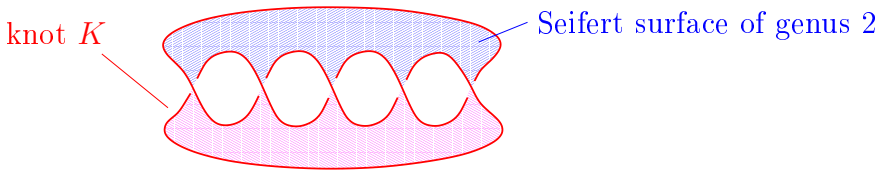
(1) As we have seen in the figure on page 2861, the trivial knot has genus 0. More interestingly, we showed in the Genus-Zero Knots Proposition 58.8 that if a knot has genus 0, then it is smoothly isotopic to the trivial knot.

(2) We had also just seen that the trefoil admits a Seifert surface of genus 1. Furthermore in Proposition 59.6 (2) we saw that the trefoil is not equivalent to the trivial knot. It follows from (1) that

$$\text{genus(trefoil)} \;=\; 1.$$

We conclude this section with the following question.

Question 131.8. We consider the knot K in the figure below. It admits a Seifert surface of genus two. Does it also admit a Seifert surface of genus one?



131.4. **Linking numbers.** We recall the following definition from page 2700.

Definition. An m-component link is a closed 1-dimensional submanifold of S^3.

Remark. It follows from the Smooth 1-Dimensional Manifold Classification Theorem 22.7 that a link is the same as a 1-dimensional submanifold of S^3 that is diffeomorphic to the disjoint union of finitely many copies of S^1.

Convention. When we write, "let $L = L_1 \sqcup \cdots \sqcup L_m$ be an m-component link", then it is understood that each L_i is non-empty. In particular each L_i is a knot in its own right.

The following definition is the obvious generalization of the definition on page 1302.

Definition. We say that two m-component links $L = L_1 \sqcup \cdots \sqcup L_m$ and $J = J_1 \sqcup \cdots \sqcup J_m$ are smoothly isotopic if there exists a smooth map

$$\begin{array}{rccc} F\colon (S^1_1 \sqcup \cdots \sqcup S^1_m) \times [0,1] & \to & S^3 \\ (z,t) & \mapsto & F(z,t) \end{array}$$

such that the following hold:

(1) for each $i \in \{1, \dots, m\}$ we have $F(S^1_i \times \{0\}) = K_i$ and $F(S^1_i \times \{1\}) = J_i$, and
(2) for each $t \in [0,1]$ the map $F_t \colon S^1 \sqcup \cdots \sqcup S^1 \to S^3$ is a smooth embedding.

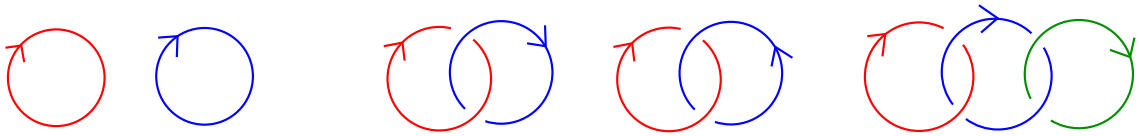
If L and J are oriented links, then we also demand that the isotopy is orientation-preserving, i.e. the map $F_1 \circ F_0^{-1} \colon K \to J$ is orientation-preserving

Examples.

(1) We say an m-component link $L \subset S^3$ is a *trivial link* if it bounds m disjoint disks in S^3. It follows immediately from the Lower-Dimensional Smooth Ball Embedding Theorem 37.10 that all m-component trivial links are in fact smoothly isotopic.
(2) In the figure below we show four examples of oriented links: the 2-component trivial link, two versions $\mathrm{H}^{\pm}$ of the Hopf link and one further link. The Hopf link is in fact related to the Hopf map

$$\begin{aligned} H \colon S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} &\to S^2 = \mathbb{C}\mathrm{P}^1 \\ (z_1, z_2) &\mapsto [z_1 : z_2] \end{aligned}$$

from page 1576. More precisely, if P, Q are two distinct points in S^2, then the preimage $\mathrm{H}^{-1}(\{P, Q\})$ is a 2-component link that is smoothly isotopic to the Hopf links without orientation. This can be seen as follows: for $P = [1:0]$ and $Q = [0:1]$ one can verify "by hand" that the preimage is smoothly isotopic to a Hopf link. Given any two other distinct points P' and Q' in S^2 we can find disjoint smooth paths from P to P' and Q to Q', the preimages of these paths give rise to a smooth isotopy of links.



2-component trivial link H^- Hopf links H^+ chain link

Our goal now is to distinguish links that are not smoothly isotopic. It follows immediately from the definition that if two links $L = L_1 \sqcup \cdots \sqcup L_m$ and $J = J_1 \sqcup \cdots \sqcup J_m$ are smoothly isotopic, then the components are smoothly isotopic knots. Therefore we are mostly interested in distinguishing links for which the components are smoothly isotopic. The following proposition generalizes Proposition 58.4. The proof is verbatim the same.

Proposition 131.9. Let $L = L_1 \sqcup \cdots \sqcup L_m$ and $J = J_1 \sqcup \cdots \sqcup J_m$ be two m-component links. If the links are smoothly isotopic, then the following two statements hold:

(1) There exists an orientation-preserving diffeomorphism $\Phi \colon S^3 \to S^3$ with $\Phi(L_i) = J_i$ for $i = 1, \dots, m$. If L and J are oriented, then there exists such a diffeomorphism Φ which for each $i \in \{1, \dots, m\}$ restricts to an orientation-preserving diffeomorphism $L_i \to J_i$.
(2) The link complements $S^3 \setminus L$ and $S^3 \setminus J$ are diffeomorphic, in particular homeomorphic.

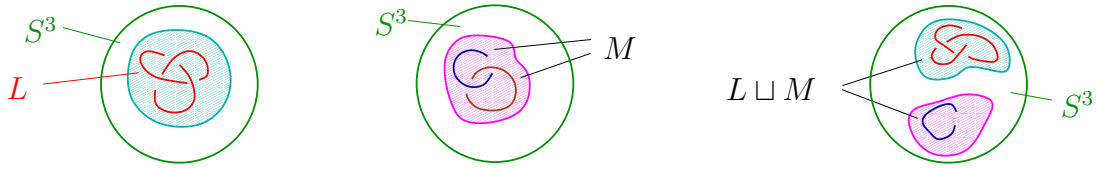
Remark. By the Gordon-Luecke Theorem 58.7 we pointed out that for knots the converse to Proposition 58.4 (2) holds. More precisely, if K and J are two knots such that there

exists an orientation-preserving diffeomorphism between the knot complements $S^3 \setminus K$ and $S^3 \setminus J$, then K and J are smoothly isotopic. The analogous statement is no longer true for links. In the figure below we show two links $L = K \sqcup J$ and $L' = K' \sqcup J'$. In Exercise 131.7 we will see that $S^3 \setminus L$ and $S^3 \setminus L'$ are diffeomorphic. Even though we cannot prove it at the moment, it should be fairly clear that L and L' are not smoothly isotopic. This shows in particular that the problem of classifying links cannot be reduced to the problem of classifying the complements of links.



The following lemma gives a general construction of new links out of given ones.

Lemma 131.10. Let L and M be two links in S^3. We pick an orientation-preserving embedding $\varphi\colon \overline{B}^3 \to S^3$ with $L \subset \varphi(B^3)$ and we pick an orientation-preserving embedding $\psi\colon \overline{B}^3 \to S^3$ with $M \subset \psi(B^3)$. Finally we pick two orientation-preserving embeddings $\alpha, \beta\colon \overline{B}^3 \to S^3$ with disjoint images. The smooth isotopy type of $\alpha(\varphi^{-1}(L)) \cup \beta(\psi^{-1}(M))$ is well-defined and we refer to it as the split union $L \sqcup M$ of the links L and M.



Proof. We start out with the following statement which might actually be of independent interest.

Claim. Let X be a compact subset of an n-dimensional smooth manifold M and let $\varphi, \widetilde{\phi}\colon \overline{B}^n \to M$ be two smooth embeddings with $X \subset \varphi(B^n)$ and $X \subset \widetilde{\varphi}(B^n)$. There exists a smooth isotopy $\Theta\colon \overline{B}^n \times [0,1] \to M$ such that $\Theta_0 = \varphi$, $\Theta_1 = \widetilde{\varphi}$ and such that for each $t \in [0,1]$ we have $X \subset \Theta_t(B^n)$.

Proof. If X is empty, then there is nothing to show. So now assume that X is non-empty. We pick $P \in X$. It follows from the Smooth Isotopy Transitivity Proposition 33.3 and Proposition 36.4 that we can assume without loss of generality that $\varphi(0) = \widetilde{\varphi}(0) = P$. Note that this means that we can view φ and $\widetilde{\varphi}$ as tubular maps for the 0-dimensional submanifold $\{P\} \subset M$. It follows from Proposition **??** that there exists a smooth isotopy $\Theta\colon \overline{B}^n \times [0,1] \to M$ such that $\Theta_0 = \varphi$, $\Theta_1 = \widetilde{\varphi}$ and such that for each $t \in [0,1]$ we have $\Theta_t(0) = P$.

It follows from Exercise 14.8 that there exists an open neighborhood U of P such that for each $t \in [0,1]$ we have $U \subset \Theta_t(B^n)$. If $X \subset U$, then we are done. Otherwise we use Exercise 33.2 and the fact that X is by hypothesis compact, together with the Smooth Isotopy Transitivity Proposition 33.3, to reduce the general case to the case that $X \subset U$. ⊞

Now we go back the actual statement of the lemma. First note that it follows from the claim and Exercise 33.3 that the smooth isotopy type of $\alpha(\varphi^{-1}(L)) \cup \beta(\psi^{-1}(M))$ does not depend on the choice of φ and ψ. Finally note that it follows from the Smooth Ball Embedding Theorem 37.8 that the smooth isotopy type of $\alpha(\varphi^{-1}(L)) \cup \beta(\psi^{-1}(M))$ does not depend on the choice of α and β. ■

To distinguish links with components that are smoothly isotopic we will now introduce an invariant of oriented links.

Definition. Let $K \subset S^3$ be an oriented knot.

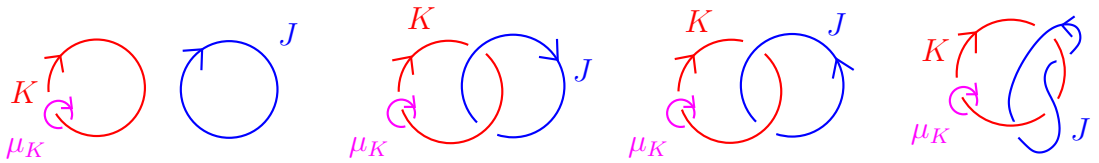
(1) As before we denote by μ_K the meridian as defined on page 1305 and as above we denote by ϕ_K the epimorphism $\phi_K \colon \mathrm{H}_1(S^3 \setminus K; \mathbb{Z}) \to \mathbb{Z}$ that is given by $\phi_K([\mu_K]) = 1$.
(2) We denote by $[\mu_K]^* \in \mathrm{H}^1(S^3 \setminus K; \mathbb{Z})$ the unique generator with $\langle [\mu_K]^*, [\mu_K] \rangle = 1$.
(3) If $J \subset S^3$ is an oriented knot that is disjoint from K, then we define the linking number of K and J to be

$$\mathrm{lk}(K, J) \;:=\; \phi_K([J]) \;\underset{\uparrow}{=}\; \langle [\mu_K]^*, [J] \rangle$$

equality holds by the above definitions

where $[J] \in \mathrm{H}_1(S^3 \setminus K; \mathbb{Z})$ denotes the image of the fundamental class of the oriented smooth manifold J under the inclusion $J \to S^3 \setminus K$.

Example. If K is the trivial knot, then we saw in Proposition 59.4 that $S^3 \setminus K$ is diffeomorphic to the open solid torus $B^2 \times S^1$ and it is fairly straightforward to determine $\phi_K \colon \mathrm{H}_1(S^3 \setminus K; \mathbb{Z}) \to \mathbb{Z}$ applied to a given curve. This allows us to determine the linking numbers of the oriented links shown in the figure below.



$\mathrm{lk}(K, J) = 0$ $\mathrm{lk}(K, J) = -1$ $\mathrm{lk}(K, J) = 1$ $\mathrm{lk}(K, J) = 2$

Proposition 131.11.

(1) Let $L = L_1 \sqcup \cdots \sqcup L_m$ and $J = J_1 \sqcup \cdots \sqcup J_m$ be two m-component oriented links. If the links L and J are smoothly isotopic, then for all $r, s \in \{1, \ldots, m\}$ we have

$$\mathrm{lk}(L_r, L_s) \;=\; \mathrm{lk}(J_r, J_s).$$

(2) Let $K \sqcup J$ be an oriented 2-component link. If we change the orientation of one of the components of the link, then the sign of the linking number flips.

Proof. The first statement is an immediate consequence of Proposition 131.9 (1). The second statement follows easily from the definitions. ■

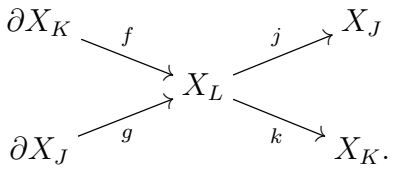
Example. It follows from Proposition 131.11 together with the calculation of the linking numbers in the figure on page 2865 that the two Hopf links H^- and H^+ are not smoothly isotopic.

The following proposition says that the linking number $\mathrm{lk}(K,J)$ is symmetric in the oriented knots K and J.

Proposition 131.12. (Linking Number-Symmetry Proposition) For any oriented 2-component link $J \sqcup K$ we have

$$\mathrm{lk}(J,K) \;=\; \mathrm{lk}(K,J).$$

Proof. We pick a thickening for the link $L = J \sqcup K$ and use it to define the exteriors X_L, X_K and X_J. Throughout the proof we consider the following inclusion maps



Furthermore, throughout the proof we use the isomorphisms from the H_0-Proposition 72.5 and page 1594 to make the identifications $\mathrm{H}_0(\partial X_J;\mathbb{Z}) = \mathrm{H}_0(X_L;\mathbb{Z}) = \mathrm{H}_0(X_J;\mathbb{Z}) = \mathbb{Z}$. We start out with the following claim.

Claim. We have

$$\mathrm{lk}(K,J) \;=\; \big(\underbrace{k^*([\mu_K]^*)}_{\in \mathrm{H}^1(X_L;\mathbb{Z})} \cup \underbrace{j^*([\mu_J]^*)}_{\in \mathrm{H}^1(X_L;\mathbb{Z})}\big) \cap \underbrace{g_*([\partial X_J])}_{\in \mathrm{H}_2(X_L;\mathbb{Z})}.$$

Proof. We denote by $[J] \in \mathrm{H}_1(J;\mathbb{Z})$ the fundamental class. We have the following equalities:

$$\begin{array}{ccccc}
 & & & & \text{Cup Product-Commutativity Proposition 115.3} \\
 & & & & \downarrow \\
(k^*([\mu_K]^*) \cup j^*([\mu_J]^*)) \cap g_*([\partial X_J]) & & = & & -(j^*([\mu_J]^*) \cup k^*([\mu_K]^*)) \cap g_*([\partial X_J]) \\
= & & & & = \\
\uparrow & & & & \uparrow \\
-k^*([\mu_K]^*) \cap (j^*([\mu_J]^*) \cap g_*([\partial X_J])) & & = & & -k^*([\mu_K]^*) \cap g_*(g^*j^*[\mu_J]^* \cap [\partial X_J]) \\
\text{Cup-Cap Lemma 116.7} & & & & \text{Cup Product-Naturality Lemma 114.10 applied to } g \\
= & & = & & = \\
\uparrow & & \uparrow & & \uparrow \\
k^*([\mu_K]^*) \cap g_*([\lambda_J]) & & \langle [\mu_K]^*, k_*g_*([\lambda_J])\rangle & & \langle [\mu_K]^*, [J]\rangle \\
\text{Longitude-Meridian Lemma 131.2 (5)} & & \text{Lemmas 109.6 (3) and 116.4} & & \text{since } [\lambda_J] = [J] \in \mathrm{H}_1(X_K)
\end{array} \quad = \quad \mathrm{lk}(K,J).$$

see page 2865

⊞

Now we conclude that

$$\begin{array}{ccccc}
\mathrm{lk}(K,J) & = & (j^*([\mu_J]^*) \cup k^*([\mu_K]^*)) \cap g_*([\partial X_J]) & = & (-k^*([\mu_K]^*) \cup j^*([\mu_J]^*)) \cap (-f_*([\partial X_K])) \\
 & \uparrow & & \uparrow & \\
 & \text{by the claim} & \text{by the Cup Product-Commutativity Proposition 115.3 and since by Corollary 106.28 we have } g_*([\partial X_J]) + f_*([\partial X_K]) = 0 \in \mathrm{H}_2(X_L;\mathbb{Z}) & &
\end{array}$$

$$= \mathrm{lk}(J,K).$$

$$\uparrow$$

by the claim

■

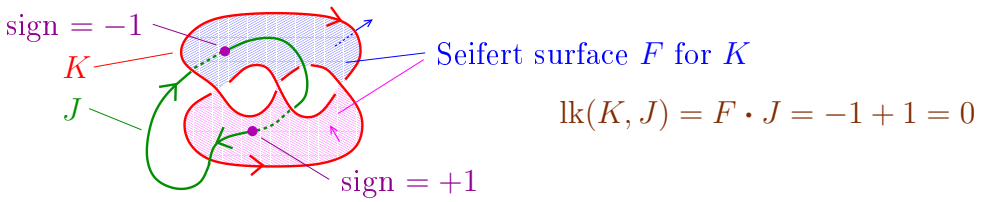
To determine the linking number in more complicated cases it is helpful to have different ways to calculate the linking number.

Proposition 131.13. (Linking Numbers-via-Intersection Numbers Proposition) Let $K \sqcup J$ be an oriented 2-component link. If F is a Seifert surface for K that is transverse to J, then

$$\mathrm{lk}(K,J) \;=\; F \cdot J$$

where $F \cdot J$ denotes the algebraic intersection number of F and J.

Example. In the figure below we show an oriented link $K \sqcup J$ and a Seifert surface F for K. Clearly we have $F \cdot J = 0$, thus it follows from the Linking Numbers-via-Intersection Numbers Proposition 131.13 that $\mathrm{lk}(K, J) = 0$.



Remark. The combination of the Linking Numbers-as-Intersection Numbers Proposition **??** and the Intersection Number-via-Cup Product Theorem 126.5, and some mild massaging, generalizes the statement of the Linking Numbers-via-Intersection Numbers Proposition 131.13 to the higher-dimensional setting and it gives an alternative proof of Proposition 131.13.

Proof. Let $K \sqcup J$ be an oriented 2-component link. We pick a tubular neighborhood $\overline{B}^2 \times K$ for K. After possibly shrinking the tubular neighborhood we can and will assume that $S^1 \times K$ intersects F transversally in a single curve[1591] and that $\overline{B}^2 \times K$ does not intersect J. We write $X := S^3 \setminus (B^2 \times K)$ and we write $F' := F \cap X$. It follows from our choice of tubular neighborhood and from the Submanifold Transversal Intersection Theorem 42.1 that F' is a compact proper submanifold of F.

We start out with the following claim.

Claim. We have $\mathrm{PD}_X([F']) = [\mu_K]^* \in \mathrm{H}^1(X; \mathbb{Z})$.

Proof. By definition of the Poincaré Duality isomorphism PD_X the claim is equivalent to showing that $[\mu_K]^* \cap [X] = [F']$. From the Longitude-Meridian Lemma 131.2 (2) we know that the connecting homomorphism $\partial\colon \mathrm{H}_2(X, \partial X; \mathbb{Z}) \to \mathrm{H}_1(\partial X; \mathbb{Z})$ is a monomorphism. Therefore it suffices to show that $\partial([\mu_K]^* \cap [X]) = \partial[F']$. We denote by $i\colon \partial X \to X$ the inclusion map. We calculate that

$$\partial([\mu_K]^* \cap [X]) \underset{\substack{\uparrow \\ \text{Proposition 119.25}}}{=} -i^*([\mu_K]^*) \cap [\partial X] \overset{\substack{\text{Longitude-Meridian Lemma 131.2 (5) and (4)} \\ \downarrow}}{=} [\lambda_K] \overset{\downarrow}{=} [\partial F'] \underset{\substack{\uparrow \\ \text{Fundamental Class-of-Boundary Proposition 106.27}}}{=} \partial([F']).$$

⊞

Now we calculate that

$$F \cdot J \overset{\substack{\text{Lemma 126.4} \\ \downarrow}}{=} J \cdot F \overset{\substack{\text{since } J \subset X \\ \downarrow}}{=} J \cdot F' \overset{\substack{\text{Intersection Number-via-Cup} \\ \text{Product Theorem 126.5} \\ \downarrow}}{=} (\mathrm{PD}_X([J]) \cup \mathrm{PD}_X([F'])) \cap [X] \overset{\substack{\text{Cup-Cap Lemma 116.7} \\ \downarrow}}{=} \mathrm{PD}_X([F']) \cap (\mathrm{PD}_X([J]) \cap [X])$$
$$\underset{\substack{\uparrow \\ \text{by the claim we have } \mathrm{PD}_X([F']) = [\mu_K]^* \text{ and} \\ \text{by definition we have } \mathrm{PD}_X([J]) \cap [X] = [J]}}{=} [\mu_K]^* \cap [J] = \langle [\mu_K]^*, [J] \rangle \underset{\substack{\uparrow \\ \text{see page 2865}}}{=} \mathrm{lk}(K, J).$$

We have thus proved the desired equality. ■

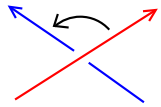
[1591] This last statement requires a little bit of thought. We leave the technical details to the reader.

Next we will formulate a proposition that in many "real-life" situations is often the most efficient way to compute linking numbers. Before we formulate the proposition, recall that on page 1322 we introduced the notion of a knot diagram and the notion of an over- and undercrossing. These notion can be generalized in a pretty obvious way to the notion of a link diagram.
To enhance readability we will formulate the next proposition in a slightly informal way.

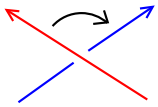
Proposition 131.14. (Linking Number-via-Signed Crossings Proposition) Let $K \sqcup J \subset S^3$ be an oriented 2-component link that is given by a diagram in the xy-plane. Then

$$\mathrm{lk}(K,J) \;=\; \begin{array}{l} \frac{1}{2} \cdot \text{number of positive crossings between } K \text{ and } J \\ - \;\; \frac{1}{2} \cdot \text{number of negative crossings between } K \text{ and } J \end{array}$$

where a crossing P is called a positive (respectively negative) crossing if the direction of the upper strand together with the direction of the lower strand give a positive (respectively negative) basis for $\mathbb{R}^2$.



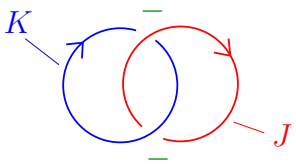
positive crossing



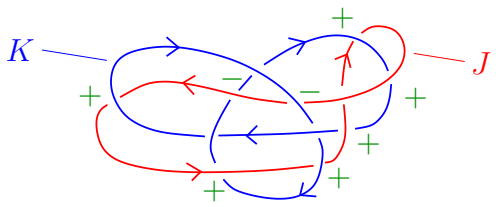
negative crossing

Examples.

(1) In the figure below we show two oriented links and we compute the linking number using Linking Number-via-Signed Crossings Proposition 131.14 by considering the signs of the crossings between K and J. As a reality check, on the left we consider the Hopf link that we had already considered in the figure on page 2865. In both cases we obtain that the linking number equals -1.

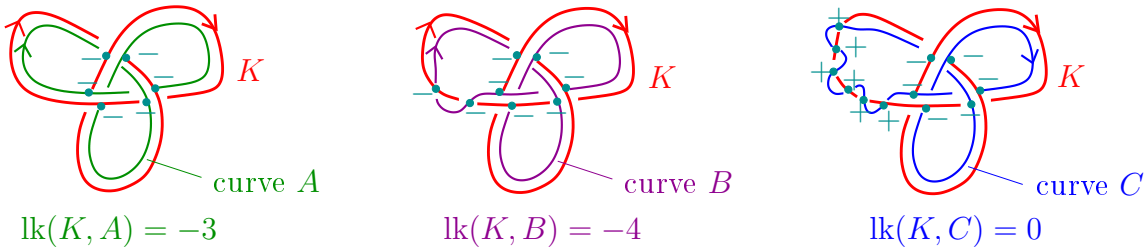


linking number $= -1$



linking number $= 2$

(2) Now we can settle the question from page 2854, which of the three closed curves shown in the figure on page 2869 is actually the longitude of the trefoil. More precisely, by definition the longitude λ_K satisfies $\phi_K([\lambda_K]) = 0$, but again by definition this means that $\mathrm{lk}(K, \lambda_K) = 0$. Conversely, loosely speaking any curve that runs "parallel to K" and that has linking number zero with K, is a longitude. Using Linking Number-via-Signed Crossings Proposition 131.14 we calculate the linking numbers of the oriented knot K with the oriented curves A, B and C. We obtain that C is the only curve with linking number zero. Thus C is the longitude.



Remark. Let $K \sqcup J \subset S^3$ be an oriented 2-component link that is given by a diagram in the xy-plane. An amusing exercise shows that

$$\mathrm{lk}(K,J) \;=\; \begin{array}{l} \text{number of positive crossings where } K \text{ crosses over } J \\ - \;\text{number of negative crossings where } K \text{ crosses over } J. \end{array}$$

Sketch of proof. [1592] We introduce the following notation:

(1) Given $t \in \mathbb{R}$ we consider the map

$$\begin{array}{rcl} \lambda_t \colon \mathbb{R}^3 & \to & \mathbb{R}^3 \\ (x,y,z) & \mapsto & (x,y,z+t). \end{array}$$

We extend this definition to a homeomorphism of S^3 by setting $\lambda_t(\infty) = \infty$.[1593]

(2) By the Seifert Surface-Existence Proposition 131.6 the knot J admits a Seifert surface F.

(3) We pick $r \in \mathbb{R}$ such that the minimal z-coordinate of any point on $\lambda_r(F)$ is greater than the maximal z-coordinate of any point in K.

(4) We write $F' = ((J \times [0,1]) \cup F)/\!\sim$ where we identify any point in $J \subset F$ with the corresponding point in $J \times \{1\}$. Note that F' is a compact 2-dimensional smooth manifold that inherits an orientation from F.

(5) We consider the map[1594]

$$\begin{array}{rcl} \Psi \colon F' = ((J \times [0,1]) \cup F)/\!\sim & \to & \mathbb{R}^3 \\ Q & \mapsto & \begin{cases} \lambda_{t\cdot r}(P) & \text{if } Q = (P,t) \in J \times [0,1], \\ \lambda_r(Q), & \text{if } Q \in F. \end{cases} \end{array}$$

Then

$$\mathrm{lk}(K,J) \underset{\substack{\uparrow \\ \text{by the same argument} \\ \text{as in Proposition 131.13}}}{=} K \cdot \Psi \underset{\substack{\uparrow \\ \text{by choice of } r \text{ we have} \\ K \cap \Psi(F) = \varnothing}}{=} K \cdot \Psi|_{J \times [0,1]}.$$

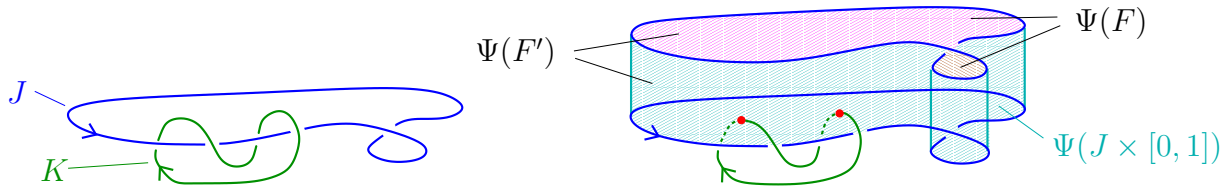
The right-hand side can be computed easily, in fact the points in $K \cap \Psi(J \times [0,1])$ correspond precisely to the crossings of the diagrams where K crosses over J and one can easily convince oneself that the signs agree. The proposition now follows from the reformulation of the linking number in the remark preceding the proof. ∎

In the following lemma we give a 4-dimensional interpretation of the linking number.

[1592] A proof can evidently only be as precise as the input. Since we did not bother to completely rigorously formulate the statement we can also not expect a completely rigorous proof. But with some effort one can formulate both the statement and the proof rigorously.

[1593] Why is this map continuous?

[1594] Note that the map $\Psi \colon F' \to S^3$ is not necessarily injective. In particular the image is not necessarily a Seifert surface for J.



Lemma 131.15. Let K, K' be two disjoint oriented knots.

(1) There exist oriented proper 2-dimensional submanifolds S and S' of $\overline{B}^4$ with $\partial S = K$ and $\partial S' = K'$ such that S and S' are transverse.
(2) Let S, S' be two oriented proper 2-dimensional submanifolds of $\overline{B}^4$ with $\partial S = K$ and $\partial S' = K'$. If S and S' are transverse, then

$$\mathrm{lk}(K, K') \;=\; S \cdot_{\overline{B}^4} S'.$$

Remark. The combination of the Linking Numbers-as-Intersection Numbers Proposition **??** and the Intersection Number-via-Cup Product Theorem 126.5 generalizes the statement of Lemma 131.15 to the higher-dimensional setting and it gives an arguably mathematically purer proof of Lemma 131.15.

Sketch of proof. Let K, K' be two disjoint oriented knots.

(1) By the Seifert Surface-Existence Proposition 131.6 and the Transversality Theorem 42.4 (2) there exists a Seifert surface F for K that is transverse to K'. Furthermore let F' be a Seifert surface for K'. We pick a collar neighborhood $[0,2] \times S^3$ for $S^3 = \partial \overline{B}^4$. We use the collar neighborhood $[0,1] \times S^3$ to define the push-in[1595] $\widetilde{F}$ of F and we use the collar neighborhood $[0,2] \times S^3$ to define the push-in $\widetilde{F}'$ of F'. It follows easily from Lemma **??** that $\widetilde{F}$ and $\widetilde{F}'$ have the desired properties.
(2) First we note that

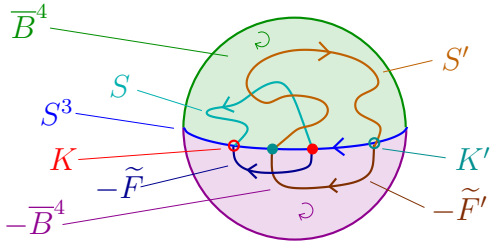
$$\mathrm{lk}(K, K') \underset{\text{Proposition 131.13}}{=} F \cdot_{S^3} K' \underset{\text{by inspection of the intersection points}}{=} \widetilde{F} \cdot_{\overline{B}^4} \widetilde{F}' \underset{\text{follows easily from the definitions}}{=} -(-\widetilde{F}) \cdot_{-\overline{B}^4} (-\widetilde{F}').$$

Now let S, S' be two oriented proper 2-dimensional submanifolds of $\overline{B}^4$ with $\partial S = K$ and $\partial S' = K'$. We think of $S^4 = \overline{B}^4 \cup (-\overline{B}^4)$ as the union of two copies of $\overline{B}^4$ with opposite orientations, the first copy contains S, S' and the second contains $-\widetilde{F}, -\widetilde{F}'$. (See the figure below for an illustration.) It follows fairly easily from the Smooth Collar Neighborhood Theorem **??** that $S \cup -\widetilde{F}$ and $S' \cup -\widetilde{F}'$ are oriented submanifolds of S^4.[1596] We calculate that

$$S \cdot_{\overline{B}^4} S' - \mathrm{lk}(K, K') \underset{\text{by the above}}{=} S \cdot_{\overline{B}^4} S' + (-\widetilde{F}) \cdot_{-\overline{B}^4} (-\widetilde{F}') \underset{\text{since } S \cap \widetilde{F}' = S' \cap \widetilde{F} = \varnothing}{=} (S \cup -\widetilde{F}) \cdot_{S^4} (S' \cup -\widetilde{F}') \underset{\text{by Theorem 127.7 and since } \mathrm{H}^2(S^4; \mathbb{Z}) = 0}{=} 0.$$

[1595] We refer to page **??** for the definition of push-ins.
[1596] The word "submanifold of S^4" is used in a slightly ambiguous way since it depends on whether we view S^4 as a topological manifold or a smooth manifold. In this case we view S^4 only as a topological manifold.

We have thus shown the desired equality $S \cdot_{\overline{B}^4} S' = \mathrm{lk}(K, K')$. ■



Given two disjoint knots K and J in $\mathbb{R}^3$ we now have different ways to define and think about the linking number. The following proposition, which mirrors [**Rolf1990**, Theorem 5.D.5], shows that they are all the same.

Proposition 131.16. (Linking Number Reformulation Proposition) Let $K \sqcup J$ be an oriented 2-component link in $\mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ and let F be a Seifert surface for J that intersects K transversally. The following numbers are all equal:[1597]

(1) $\phi_K([J])$ (definition of $\mathrm{lk}(K, J)$ page 2865)

(2) $K \cdot F$

(3) the signed count of crossings as in Proposition 131.14

(4) $-\deg \begin{pmatrix} K \times J & \to & S^2 \\ (x, y) & \mapsto & \frac{x-y}{\|x-y\|} \end{pmatrix}$ (definition of $\mathrm{lk}(K, J)$ on page 1007)

(5) $\frac{1}{4\pi} \int_{s=0}^{s=2\pi} \int_{t=0}^{t=2\pi} \frac{\det \overbrace{\begin{pmatrix} (f(s) - g(t)) & f'(s) & g'(t) \end{pmatrix}}^{3 \times 3\text{–matrix}}}{\|f(s) - g(t)\|^3} \, dt \, ds$

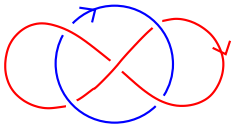
Proof. The equality (1)=(2) is the content of the above Linking Numbers-via-Intersection Numbers Proposition 131.13 and the equality (1)=(3) is the content of the above Linking Number-via-Signed Crossings Proposition 131.14. The equality (3)=(4) was shown, with slightly different language, in Proposition 44.1. We proved (2)=(4) in the Linking Numbers-via-Intersection Numbers Proposition 44.3 The equality (4)=(5) was proved in the Gauß Linking Number Theorem **??**. ■

Example. Given any topological invariant the question that arises is, how good is it at distinguishing objects. It is quite easy to find examples of links which show that the linking number is far from being a complete invariant. In the figure below we show the Whitehead link and the Borromean rings. Both are links with zero linking numbers and unknotted components but which are appear to be non-trivial. The Whitehead link is named after John Henry Constantine Whitehead who also introduced the concept of CW-complexes. The Borromean rings are named after the Borromeo family that carries the Borromean rings in its coat of arms since medieval days.

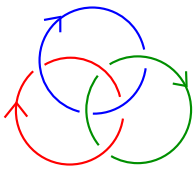
We record the following question for later on.

Question 131.17. How can we show that neither the Whitehead link nor the Borromean rings are smoothly isotopic to the trivial link?

[1597] Here we use the notion of "degree" that we introduced on page 899 or on page **??**, which in turn equals the degree defined on page 2379 or alternatively on pages 2222.



Whitehead link



Borromean rings



coat of arms of the Borromeo family

Remark. We can of course try to determine the fundamental groups of the complements of the trivial links and the Whitehead link and the Borromean rings. But then the question arises, how can we distinguish these groups?

131.5. **Alexander duality and linking numbers** (∗). Let Y be a compact non-empty subset of S^3 that admits a regular neighborhood. (For example, by Propositions 123.5, **??** and 123.4 the subset Y could be a link or a spatial graph.) We can compute the isomorphism types of the first homology group of the complement $S^3 \setminus Y$ as follows:

$$\begin{array}{ccccccc} \mathrm{H}_1(S^3 \setminus Y;\mathbb{Z}) & \overset{\cong}{\to} & \mathrm{H}^1(Y;\mathbb{Z}) & \overset{\cong}{\to} & \mathrm{Hom}(\mathrm{H}_1(Y;\mathbb{Z}),\mathbb{Z}) & \cong & \mathrm{H}_1(Y;\mathbb{Z})/\text{torsion}. \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{Alexander Duality} & & \text{Universal Coefficient} & & & & \text{non-natural isomorphism} \\ \text{Theorem 123.6} & & \text{Theorem 110.12} & & & & \end{array}$$

The catch with the Alexander Duality Theorem 123.6 is that it is quite hard to "see" the isomorphism. For example, suppose we are given a basis for $\mathrm{H}_1(Y;\mathbb{Z})/\text{torsion}$, how do we get a basis for $\mathrm{H}_1(S^3 \setminus Y;\mathbb{Z})$? In many situations the following lemma resolves that issue.

Lemma 131.18. Let Y be a compact non-empty subset of S^3 that admits a regular neighborhood $N(Y)$. We suppose that $\mathrm{H}_1(Y;\mathbb{Z})$ is a free abelian group[1598] of rank k. Let $v(x), C_n$ be oriented curves in $N(Y)$ and let $D_1,\dots,D_k$ be oriented curves in $S^3 \setminus N(Y)$. If

$$\det\Big(\underbrace{(\mathrm{lk}(C_i,D_j)_{i,j=1,\dots,k})}_{k\times k\text{–matrix}}\Big) \;=\; \pm 1,$$

then $[D_1],\dots,[D_k]$ form a basis of $\mathrm{H}_1(S^3 \setminus Y;\mathbb{Z})$.

Proof. Let Y be a compact non-empty subset of S^3 that admits a regular neighborhood $N = N(Y)$. We denote by $\mathring{N}$ the interior of N. We consider the homomorphisms

$$\begin{array}{rcl} \Omega\colon \mathbb{Z}^k & \to & \mathrm{H}_1(S^3\setminus\mathring{N};\mathbb{Z}) \\ (a_1,\dots,a_k) & \mapsto & \sum_{i=1}^{k} a_i\cdot[D_i] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \Theta\colon \mathrm{H}_1(S^3\setminus\mathring{N};\mathbb{Z}) & \to & \mathbb{Z}^k \\ \sigma & \mapsto & (\phi_{C_1}(\sigma),\dots,\phi_{C_k}(\sigma)). \end{array}$$

It follows from our hypothesis and the definition of linking numbers that $\Theta\circ\Omega$ is an epimorphism. Furthermore we know from the Alexander Duality Theorem 123.6 and the Universal Coefficient Theorem 110.12 that $\mathrm{H}_1(S^3 \setminus \mathring{N};\mathbb{Z})$ is also isomorphic to $\mathbb{Z}^k$. But by Lemma 51.9 (5) an epimorphism between two free abelian groups of the same rank is an isomorphism. Thus we see that Θ and Ω are isomorphisms.

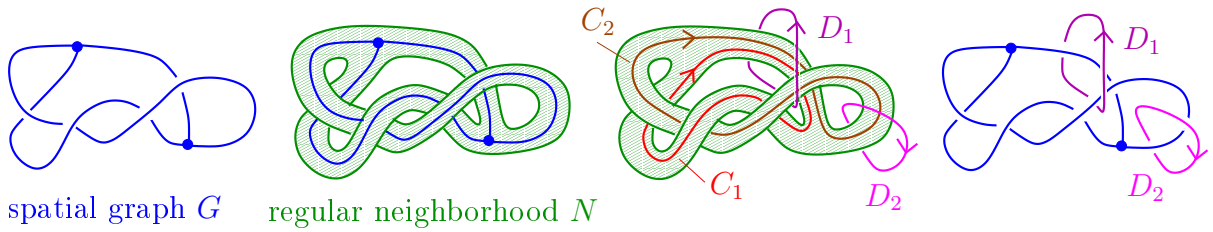
[1598] Is it possible that $\mathrm{H}_1(Y;\mathbb{Z})$ is not a free abelian group?

Since $\{[D_1], \dots, [D_k]\}$ is the image under Ω of the standard basis of $\mathbb{Z}^k$ we see that $[D_1], \dots, [D_k]$ form a basis for $\mathrm{H}_1(S^3 \setminus \mathring{N}; \mathbb{Z})$. But by Lemma 123.1 the inclusion induced map $\mathrm{H}_1(S^3 \setminus Y; \mathbb{Z}) \to \mathrm{H}_1(S^3 \setminus \mathring{N}; \mathbb{Z})$ is an isomorphism. Thus we see that $[D_1], \dots, [D_k]$ form a basis of $\mathrm{H}_1(S^3 \setminus Y; \mathbb{Z})$.[1599] ■

Example. In the figure below we consider a spatial graph G together with a regular neighborhood N. By Lemma 87.12 we know that $\mathrm{H}_1(G; \mathbb{Z}) \cong \mathbb{Z}^2$. In the figure below we also see two oriented curves C_1 and C_2 in N and two oriented curves D_1 and D_2 in $S^3 \setminus \mathring{N}$. Using Linking Number-via-Signed Crossings Proposition 131.14 one can easily calculate that

$$\begin{pmatrix} \mathrm{lk}(C_1, D_1) & \mathrm{lk}(C_1, D_2) \\ \mathrm{lk}(C_2, D_1) & \mathrm{lk}(C_2, D_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

It follows from Lemma 131.18 that $[D_1], [D_2]$ form a basis for $\mathrm{H}_1(S^3 \setminus G; \mathbb{Z})$.



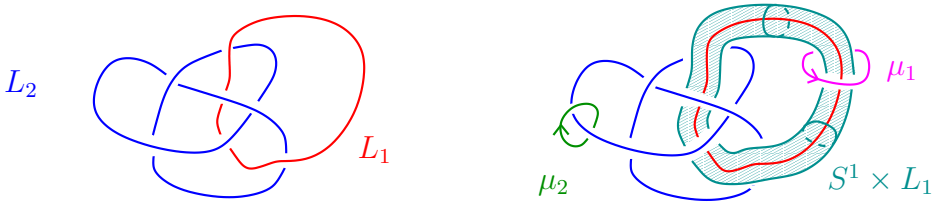
In Lemma 123.8 we computed the homology groups of the complement of a link, but we could not determine explicit generators. The following lemma now solves that problem. In particular it gives an answer to Question 123.9.

Lemma 131.19. Let $L = L_1 \sqcup \cdots \sqcup L_m$ be an m-component link. We pick a thickening $\overline{B}^2 \times L$ for L.

(1) A basis for $\mathrm{H}_1(S^3 \setminus L; \mathbb{Z}) \cong \mathbb{Z}^m$ is given by the homology classes that are represented by the meridians $\mu_1, \dots, \mu_m$.

(2) A basis for $\mathrm{H}_2(S^3 \setminus L; \mathbb{Z}) \cong \mathbb{Z}^{m-1}$ is given by considering the fundamental classes of all but one of the tori $S^1 \times L$, $i \in \{1, \dots, m\}$. More precisely, for any $j \in \{1, \dots, m\}$ a basis for $\mathrm{H}_2(S^3 \setminus L; \mathbb{Z}) \cong \mathbb{Z}^{m-1}$ is given by

$$[S^1 \times L_1], \dots, [S^1 \times L_{i-1}], [S^1 \times L_{i+1}], \dots, [S^1 \times L_m].$$



Sketch of proof. Let $L = L_1 \sqcup \cdots \sqcup L_m$ be an m-component link. We know from Lemma 123.8 that $\mathrm{H}_1(S^3 \setminus L; \mathbb{Z}) \cong \mathbb{Z}^m$ and that $\mathrm{H}_2(S^3 \setminus L; \mathbb{Z}) \cong \mathbb{Z}^{m-1}$. We pick a thickening $\overline{B}^2 \times L$ for L. By the Smooth Collar Neighborhood Theorem 28.3 we can extend this thickening to a thickening $2\overline{B}^2 \times L$.

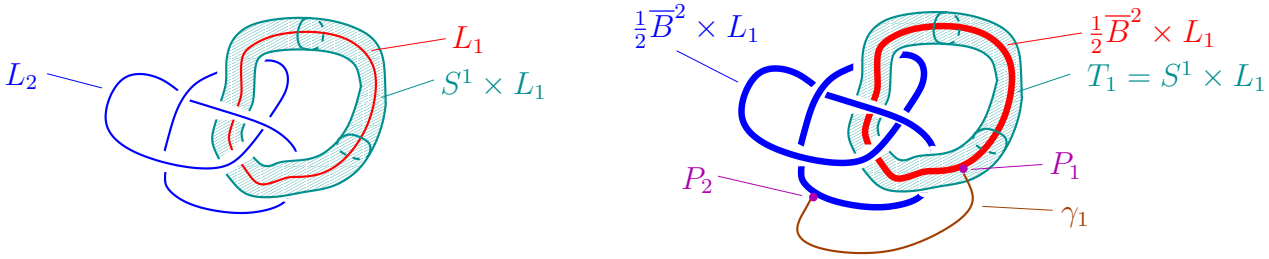
[1599] The alert reader will have noticed that the proof resembles the argument given in the Kronecker-Basis Lemma 109.7.

(1) The first statement follows from Lemma 131.18 applied to $\mathrm{C}_i = \{0\} \times L_i$, $i = 1, \ldots, m$ and $D_i := \mu_i$, $i = 1, \ldots, m$.

(2) To simplify the notation we consider the case $j = m$. We write $X' := S^3 \setminus \frac{1}{2}B^2 \times L$. For $i = 1, \ldots, m$ we write $T_i := S^1 \times L_i$. For $i = 1, \ldots, m$ we pick a point P_i on $\frac{1}{2}S^1 \times L_i \subset \partial X'$. Since $S^3 \setminus (2B^2 \times L)$ is path-connected we can pick for each $i \in \{1, \ldots, m-1\}$ a path γ_i from P_i to P_m that intersects T_i transversally in a single point and that does not intersect any of the T_j with $j \neq i, m$. Now we consider the map

$$\begin{aligned} \Psi \colon \mathrm{H}_1(X', \partial X'; \mathbb{Z}) \times \mathrm{H}_2(X'; \mathbb{Z}) &\to \mathbb{Z} \\ (\sigma, \tau) &\mapsto \langle \mathrm{PD}_{X'}(\sigma) \cup \mathrm{PD}_{X'}(\tau), [X'] \rangle. \end{aligned}$$

It follows easily from Theorem 127.7 that for $i, j \in \{1, \ldots, m-1\}$ we have the equality $\Psi([\gamma_i], [T_j]) = \delta_{ij}$. By Lemma 109.8 this implies that $[T_1], \ldots, [T_{m-1}]$ do indeed form a basis of $\mathrm{H}_2(X'; \mathbb{Z})$. ■



Exercises for Chapter 131.

Exercise 131.1. Let $A = S^1 \times [-1, 1]$ be the annulus. Let C and D be two closed oriented curves on A. Suppose that $[C] = [D] \neq 0$. Show that the following two statements hold:

(a) the two closed oriented curves are smoothly isotopic,

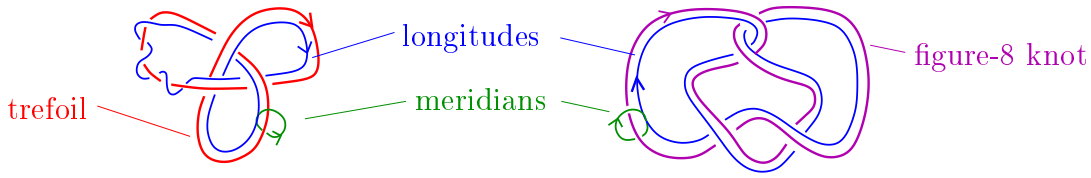
(b) if C and D are furthermore disjoint, then there exists an embedded annulus $X \subset A$ with $\partial A = C \cup -D$.

Hint. Using the fact that A is an annulus you can find a smooth isotopy that makes the curves disjoint. At that point it is perhaps better to first prove (2).

Exercise 131.2. Let F be an oriented connected 2-dimensional smooth manifold and let C and D be distinct boundary curves. Suppose that C and D are homotopic. Show that F is diffeomorphic to the annulus $S^1 \times [-1, 1]$.

Exercise 131.3. Let K be a knot together with a diagram. In Section 59.6 we gave an explicit algorithm that determines the Wirtinger presentation for $\pi_1(S^3 \setminus K)$. Explain what element in this presentation corresponds to the longitude. Alternatively, for readers who like more explicit questions, determine the group system for the trefoil and the figure-8 knot.

Exercise 131.4. Show that the trefoil K is chiral.
Hint. In Proposition 59.6 we showed that there exists an epimorphism $\pi_1(S^3 \setminus K) \to S_3$.



Exercise 131.5. Find explicit Seifert surfaces for the knots K_1 and K_2 shown in the figure below.
Remark. The answer to a question can only be as rigorous as the input. Since the input is a picture it suffices here of course to draw a convincing picture of a Seifert surface.



Exercise 131.6. Let $\varphi\colon \overline{B}^3 \to S^3$ and $\psi\colon \overline{B}^3 \to S^3$ be two smooth embeddings with disjoint images. Let $K \subset \varphi(\overline{B}^3)$ and $J \subset \psi(\overline{B}^3)$ be oriented knots. Show that $\operatorname{lk}(K, J) = 0$.

Exercise 131.7. We consider the two links shown in the figure below. Show that the link complements are diffeomorphic.



Exercise 131.8. Is every knot $K \subset S^3$ the boundary of a *non-orientable* compact surface that is smoothly embedded in S^3?

Exercise 131.9. The Borromean rings, shown in the figure on page 2872, have the property that if you remove one component, then the remaining two knots form a trivial link. Does there exist a four component link with the property that if you remove any of the four components, then you get a three component trivial link?

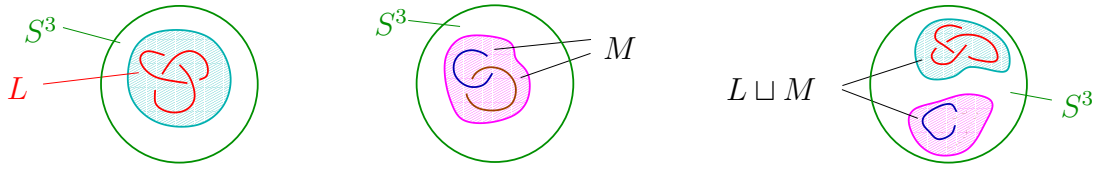
Exercise 131.10. Let $L = K \sqcup J$ and $L' = K' \sqcup J'$ be two oriented links such that the complements are diffeomorphic. Does it follow that $\operatorname{lk}(K, J) = \pm \operatorname{lk}(K', J')$?

Exercise 131.11. An m-component link $L = L_1 \sqcup \cdots \sqcup L_m \subset S^3$ is called *split link* if there exist disjoint closed smoothly embedded balls $B_1, \ldots, B_m \subset S^3$ such that each L_i is contained in B_i. Furthermore L is called a *boundary link* if there exist m disjoint Seifert surfaces $F_1, \ldots, F_m$.

(a) A split link is easily seen to be a boundary link. Do there exist boundary links that are not split?

(b) Show that if $L = L_1 \sqcup \cdots \sqcup L_m$ is a boundary link, then there exists an epimorphism from $\pi_1(S^3 \setminus L)$ onto a free group of rank m.
Hint. You could try to find an interesting map from $S^3 \setminus L$ onto the wedge of m circles.

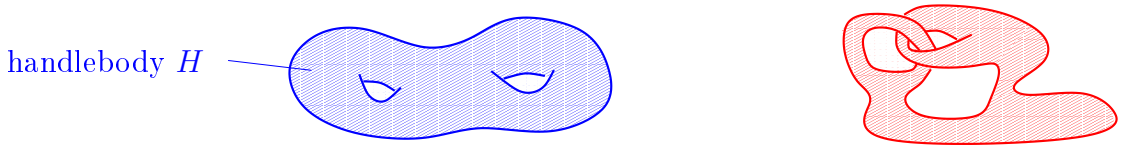
Exercise 131.12. Let L and M be two links in S^3. We pick an orientation-preserving embedding $\varphi\colon \overline{B}^3 \to S^3$ with $L \subset \varphi(B^3)$ and we pick an orientation-preserving embedding $\psi\colon \overline{B}^3 \to S^3$ with $M \subset \psi(B^3)$. Finally we pick two orientation-preserving embeddings $\alpha, \beta\colon \overline{B}^3 \to S^3$ with disjoint images. We refer to $\alpha(\varphi^{-1}(L)) \cup \beta(\psi^{-1}(M))$ as the *split union* $L \sqcup M$ *of the links* L *and* M. The goal of this exercise is to show that the smooth isotopy class of the split union $L \sqcup M$ is independent of all the choices, in other words, show that the smooth isotopy class of the split union only depends on the smooth isotopy type of L and M.
Remark. Use Exercise **??**.



Exercise 131.13. Let $K \sqcup J$ be an oriented link in $\mathbb{R}^3$ and let $K^* \sqcup J^*$ be a mirror image as defined on page 1315. As usual we view $\mathbb{R}^3$ as a subset of $\mathbb{R}^3 \cup \{\infty\} = S^3$. Determine the relationship between $\mathrm{lk}(K, J)$ and $\mathrm{lk}(K^*, J^*)$.

Exercise 131.14. Let $n \geq 4$ and let $K \subset S^n$ be a closed orientable smooth submanifold of codimension 2. Show that there exists a compact orientable submanifold W of S^n with $\partial W = K$.
Remark. Note that in general the analogous statement is not true if we do not impose any conditions on the codimension. For example by the Euclidean Embedding Theorem 27.1 and the Smooth Embedding Theorem 24.10 we can view $\mathbb{C}\mathrm{P}^2$ as a smooth submanifold of some $S^n = \mathbb{R}^n \cup \{\infty\}$. But by the proof of Corollary 144.7 we know that $\mathbb{C}\mathrm{P}^2$ is not the boundary of any compact orientable smooth manifold, let alone one which is embedded in S^n.

Exercise 131.15. In the figure below we show two smooth embeddings of the genus two handlebody into $\mathbb{R}^3$. Are these two embeddings smoothly isotopic?



132. The intersection form I: Definition and calculations

In this chapter we will introduce and study the intersection form of compact oriented even-dimensional topological manifolds. Before we do so it is convenient to remind ourselves of a few basic notions and results in algebra.

132.1. **Pairings and forms.** On page 2656 we introduced the notion of a pairing over a commutative ring R. We recall the definition and we introduce some new notions.

Definition. Let R be a commutative ring.

(1) A pairing on $V \times W$ is a map
$$\varphi\colon V \times W \ \to \ S$$
such that for all $v, v' \in V$ and $W, w' \in W$ we have $\varphi(v+v', w) = \varphi(v,w) + \varphi(v',w)$ and $\varphi(v, w+w') = \varphi(v,w) + \varphi(v,w')$.[1600]

(2) A pairing is called R-bilinear if it is R-linear in V and if it is R-linear in W.

(3) We say that an R-bilinear pairing $\varphi\colon V \times V \to S$ is symmetric if $\varphi(v,w) = \varphi(w,v)$ for all $v, w \in V$. Furthermore we call the pairing anti-symmetric if $\varphi(v,w) = -\varphi(w,v)$ for all $v, w \in V$.

(4) We say that two R-bilinear pairings (V, φ) and (W, ψ) are isometric, written as $(V,\varphi) \cong (W,\psi)$, if there exists an isometry, i.e. an isomorphism $f\colon V \to W$ of R-modules such that $\varphi(v_1, v_2) = \psi(f(v_1), f(v_2))$ for all $v_1, v_2 \in V$.

(5) Given an R-bilinear pairing $\varphi\colon V \times V \to S$ we denote by $-\varphi$ the R-bilinear pairing that is given by $(-\varphi)(v,w) := -\varphi(v,w)$ for all $v, w \in V$.

(6) Given two R-bilinear pairings $\varphi\colon V \times V \to S$ and $\psi\colon W \times W \to S$ we define the direct sum of φ and ψ to be the R-bilinear pairing
$$\begin{array}{rcl} \varphi \oplus \psi\colon (V \oplus W) \times (V \oplus W) & \to & S \\ (v \oplus w, v' \oplus w') & \mapsto & \varphi(v,v') + \psi(w,w'). \end{array}$$

(7) Given an R-bilinear pairing (V,φ) and $n \in \mathbb{Z}$ we define
$$n \cdot (V,\varphi) \ := \ \begin{cases} \overbrace{(V,\varphi) \oplus \cdots \oplus (V,\varphi)}^{n\text{-summands}}, & \text{if } n \geq 0, \\ \underbrace{(V,-\varphi) \oplus \cdots \oplus (V,-\varphi)}_{|n|\text{-summands}}, & \text{if } n \leq 0. \end{cases}$$

(8) Given an R-bilinear pairing $(V, \varphi\colon V \times V \to S)$ and a submodule W of V we denote by $\varphi|_W$ the R-bilinear pairing that is given by restricting the map $\varphi\colon V \times V \to S$ to $W \times W \to S$.

In this chapter we will mostly be interested in the following special type of an R-bilinear pairing.

Definition. Let R be a commutative ring. A form[1601] is a finitely generated free R-module V together with an R-bilinear pairing $\varphi\colon V \times V \to R$ which is either symmetric or anti-symmetric. We refer to the rank of V also as the rank of the form.[1602]

For the reader's convenience we recall the following definition from page 2656.

Definition. Let R be a commutative ring. A form $\varphi\colon V \times V \to R$ is called non-singular if the map

$$\begin{array}{ccc} V & \to & \mathrm{Hom}_R(V,R) \\ v & \mapsto & \left(\begin{array}{ccc} V & \to & R \\ w & \mapsto & \varphi(v,w) \end{array} \right) \end{array}$$

is an isomorphisms.[1602]

In many situations it is convenient to represent a form by a matrix. We recall the relevant definition from the linear algebra courses.

Definition. Let R be a commutative ring and let $\varphi\colon V \times V \to R$ be a form. Given any basis $\{v_1, \ldots, v_k\}$ of the free R-module V we refer to the $(k \times k)$–matrix

$$(\varphi(v_i, v_j))_{i,j=1,\ldots,k} \ \in \ \mathrm{M}(k \times k, R)$$

as a matrix representing the form φ. Note that by the Non-Singularity Lemma 121.1 the form is non-singular if and only if the matrix representing the form is invertible.

The following elementary lemma is usually proved in a standard course on linear algebra. It is also straightforward to verify it using only the relevant definitions.[1603]

Lemma 132.1. Let R be a commutative ring and let $\varphi\colon V \times V \to R$ be a form. Let $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$ be bases of the free R-module V. We write

$$A \ := \ (\varphi(v_i, v_j))_{i,j=1,\ldots,k} \quad \text{and} \quad B \ := \ (\varphi(w_i, w_j))_{i,j=1,\ldots,k}.$$

If $P = (p_{ij})_{i,j=1,\ldots,k} \in \mathrm{GL}(k, R)$ denotes the transition matrix that is uniquely defined by the property that $v_i = \sum_{j=1}^{k} p_{ij} w_j$ for $i = 1, \ldots, k$, then

$$A \ = \ P^T B P.$$

The following lemma is even more elementary, it follows basically immediately from the definitions.

Lemma 132.2.

(1) Let $\varphi\colon V \times V \to R$ be a form. If A is a matrix that represents φ, then the matrix $-A$ represents the form $-\varphi$.

[1600] The various definitions of "form" in the literature vary wildly. It is always a good idea, reading a paper or a book, to look up the precise definition of "form" that is used by the author.

[1601] The rank of a finitely generated free module over a commutative ring is well-defined, see e.g. [**Lam1999**, p. 4]. Note though that the rank of a finitely generated free module over a non-commutative ring is not necessarily well-defined. More precisely, there exists a non-commutative ring R and $m \neq n \in \mathbb{N}$ such that R^m is isomorphic to R^n. We refer to [**Lam1999**, p. 4] for such examples. WE will discuss this issue in Section **??**.

[1602] Note that a form is by definition (anti-) symmetric. It follows from this fact that the given condition is satisfied if and only if the condition with the two entries of φ swapped is satisfied. Thus the definition we give here is indeed consistent with the definition from page 2656.

[1603] It is easy though to get confused.

(2) Let $\varphi\colon V\times V\to R$ and $\psi\colon W\times W\to R$ be two forms. If A is a matrix that represents φ and if B is a matrix that represents ψ, then the matrix $A\oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ represents the form $\varphi\oplus\psi$.
(3) Let $\varphi\colon V\times V\to R$ be a form and let $n\in\mathbb{N}_0$. If A is a matrix that represents φ, then the matrix $n\cdot A := \underbrace{A\oplus\cdots\oplus A}_{n\text{-times}}$ represents the form $n\cdot\varphi$.

Before we can turn to topology we need to recall the following definition from page 2114.

Definition. Given a finitely generated abelian group H we refer to

$$\mathrm{F}H \;:=\; H/\operatorname{Tor}(H) \quad \text{where } \operatorname{Tor}(H) \text{ denotes the torsion subgroup of } H$$

as the maximal torsion-free quotient of H.

The following lemma summarizes everything we need to know about maximal torsion-free quotients.

Lemma 132.3.

(1) The maps

$$H \;\mapsto\; \mathrm{F}H \qquad \text{and} \qquad (\alpha\colon G\to H) \;\mapsto\; \begin{pmatrix} \alpha\colon \mathrm{F}G & \to & \mathrm{F}H \\ [g] & \mapsto & [\alpha(g)] \end{pmatrix}$$

define a covariant functor from the category of finitely generated abelian groups to the category of finitely generated free abelian groups.
(2) For every homomorphism $\alpha\colon A\to B$ between two finitely generated abelian groups the following diagram commutes:[1604]

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle a\mapsto[a]}\downarrow & & \downarrow{\scriptstyle b\mapsto[b]} \\ \mathrm{F}A & \xrightarrow{\alpha} & \mathrm{F}B. \end{array}$$

(3) Any (anti-) symmetric pairing

$$\varphi\colon H\times H \;\to\; \mathbb{Z}$$

over a finitely generated abelian group descends to a well-defined form

$$\begin{aligned} \mathrm{F}H\times\mathrm{F}H &\to \mathbb{Z} \\ ([g],[h]) &\mapsto \varphi(g,h). \end{aligned}$$

Proof. In Lemma 97.4 we already showed that if H is a finitely generated abelian group, then $\mathrm{F}H$ is a finitely generated free abelian group. The remaining statements of the lemma are basically elementary. ∎

Convention. On many occasions we do not distinguish in the notation between an element in an abelian group H and the element it represents in the maximal torsion-free quotient $\mathrm{F}H$. Furthermore, given a homomorphism $\alpha\colon A\to B$ between two finitely generated abelian groups we often denote the induced map $\mathrm{F}A\to\mathrm{F}B$ by α as well.

[1604] In fancy speak this means that the projections $H\to\mathrm{F}H$ define a natural transformation from the identity functor to the functor $H\mapsto\mathrm{F}H$.

132.2. **The definition of the intersection form of topological manifolds.** Let M be a compact oriented m-dimensional topological manifold and let A and B be compact $(m-1)$-dimensional submanifolds of ∂M with $A \cup B = \partial M$ and with $A \cap B = \partial A = \partial B$. Recall that by the Poincaré Duality Theorem 119.1 we know that for any $k \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{H}^{m-k}(M, A; \mathbb{Z}) & \to & \mathrm{H}_k(M, B; \mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

is an isomorphism. As on page 2618 we denote by

$$\mathrm{PD}_M \colon \mathrm{H}_k(M, A; \mathbb{Z}) \to \mathrm{H}^{m-k}(M, B; \mathbb{Z})$$

the inverse of this isomorphism. This leads us to the following definition.

Definition. Let M be a compact oriented topological manifold of the even dimension $2n$. We refer to the map[1605][1606]

$$\begin{array}{rcl} Q_M \colon \mathrm{FH}_n(M; \mathbb{Z}) \times \mathrm{FH}_n(M; \mathbb{Z}) & \to & \mathbb{Z} \\ (a, b) & \mapsto & \big\langle \underbrace{\underbrace{\mathrm{PD}_M(a)}_{\in \mathrm{FH}^n(M, \partial M; \mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(b)}_{\in \mathrm{FH}^n(M, \partial M; \mathbb{Z})}}_{\in \mathrm{FH}^{2n}(M, \partial M; \mathbb{Z})}, [M] \big\rangle \end{array}$$

as the intersection form of M.

Remark. Since $\mathrm{PD}_M \colon \mathrm{H}_n(M; \mathbb{Z}) \to \mathrm{H}^n(M, \partial M; \mathbb{Z})$ is an isomorphism it is clear that the intersection form is basically just a reformulation of the cup product pairing

$$\begin{array}{rcl} \mathrm{H}^n(M, \partial M; \mathbb{Z}) \times \mathrm{H}^n(M, \partial M; \mathbb{Z}) & \to & \mathrm{H}^{2n}(M, \partial M; \mathbb{Z}) \\ (\alpha, \beta) & \mapsto & \alpha \cup \beta \end{array}$$

on cohomology groups in the middle dimension that we introduced in the Manifold-Non Singularity Proposition 121.2. The big psychological advantage of homology is that very often one can "see homology classes". For example, in many cases homology classes are represented by submanifolds and we can then use the Intersection Number-via-Cup Product Theorem 126.5 to compute the intersection form of a smooth manifold. Sometimes we can also represent homology classes by explicit cycles and we can use Theorem 127.9 to compute the intersection form.

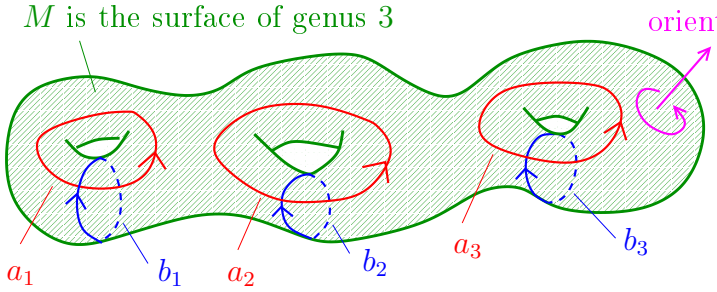
Examples.

(1) First we consider the surface M of genus 3. In principle we could obtain the intersection form from the calculation of the cup product that we had performed in Lemma 114.16. But it is much more fun and instructive to use the Intersection Number-via-Cup Product Theorem 126.5 instead. More precisely, we consider the surface M that is illustrated in the figure below. We give it the usual orientation. We already saw in the discussion on page 1781 that the closed oriented curves $a_1, b_1, a_2, b_2, a_3, b_3$ illustrated in the figure below represent a basis for $\mathrm{H}_1(M; \mathbb{Z})$. It follows easily from the Intersection Number-via-Cup Product Theorem 126.5 that

[1605] Note that the homology group $\mathrm{H}_n(M; \mathbb{Z})$ in the middle dimension is not necessarily torsion-free. For example $M = S^1 \times \mathbb{R}\mathrm{P}^3$ is orientable, but it follows easily from the Künneth Formula 90.8 and the calculation of the homology groups of $\mathbb{R}\mathrm{P}^3$ in Proposition 80.11 that $\mathrm{H}_2(S^1 \times \mathbb{R}\mathrm{P}^3; \mathbb{Z}) \cong \mathbb{Z}_2$.

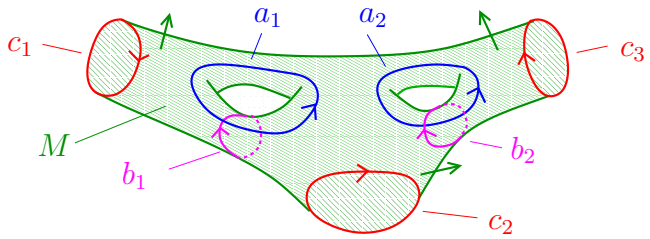
[1606] As mentioned above, in the notation usually we do not distinguish between an element in $\mathrm{H}_n(M; \mathbb{Z})$ and the element it represents in $\mathrm{FH}_n(M; \mathbb{Z})$.

with respect to this basis the intersection form is represented by the matrix shown in the figure below. An analogous statement holds of course for any surface of genus $g \geq 1$.[1607]



$$Q_M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

(2) Next we let M be the surface of genus two with three disks removed that is shown in the figure below. By the discussion on page 1784 we know that a basis for $\mathrm{H}_1(M;\mathbb{Z})$ is represented by the curves a_1, b_1, a_2, b_2 together with two out of the three boundary curves c_1, c_2, c_3. It follows from the Intersection Number-via-Cup Product Theorem 126.5 that with respect to the basis $[a_1], [b_1], [a_2], [b_2], [c_1], [c_2]$ the intersection form is represented by the matrix given in the figure below.



$$Q_M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(3) Next let us consider the $2n$-dimensional torus $M = S^n \times S^n$. By the Künneth Formula 90.8 we know that a basis for $\mathrm{H}_n(S^n \times S^n;\mathbb{Z})$ is given by $a = [S^n \times \{*\}]$ and $b = [\{*\} \times S^n]$. We have two approaches for determining the intersection with respect to this basis:

(a) We can use the Product Theorem 117.1 and its corollaries. More precisely, we first determine the Poincaré duals of $[S^n \times \{0\}]$ and $[\{0\} \times S^n]$ as in the Fundamental Class-Product Proposition 117.2 (2) and then we can apply Lemma 117.3.

(b) As on page 2755 we can use algebraic intersection numbers and the Intersection Number-via-Cup Product Theorem 126.5 to determine the intersection form.

In both cases we see that with respect to this basis the intersection form of $S^n \times S^n$ is represented by the matrix

$$\begin{pmatrix} Q_M(a,a) & Q_M(a,b) \\ Q_M(b,a) & Q_M(b,b) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}.$$

(4) We consider the complex manifold $\mathbb{C}\mathrm{P}^2$. As always we equip $\mathbb{C}\mathrm{P}^2$ with the natural orientation coming from the Complex Manifold-Orientation Proposition 32.23. It

[1607] What do the curves a_i and b_i look like in the more precise description of the surface of genus 3 as a dodecagon with identified sides?

follows from the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 126.11 that the intersection form

$$Q_{\mathbb{C}\mathrm{P}^2}\colon \mathrm{H}_2(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) \times \mathrm{H}_2(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) \;\to\; \mathbb{Z}$$

is represented by the 1×1–matrix (1).

(5) Let M be a closed oriented $2n$-dimensional topological manifold. As usual we denote by $-M$ the same topological manifold with the opposite orientation. It follows immediately from the definitions together with Proposition 107.2 that $Q_{-M} = -Q_M$.

(6) For the complex projective space $\mathbb{C}\mathrm{P}^2$ we use the common notation that we write $\overline{\mathbb{C}\mathrm{P}}^2$ instead of $-\mathbb{C}\mathrm{P}^2$. With this notation it follows from (4) and (5) that the intersection form

$$Q_{\overline{\mathbb{C}\mathrm{P}}^2}\colon \mathrm{H}_2(\overline{\mathbb{C}\mathrm{P}}^2;\mathbb{Z}) \times \mathrm{H}_2(\overline{\mathbb{C}\mathrm{P}}^2;\mathbb{Z}) \;\to\; \mathbb{Z}$$

is represented by the 1×1–matrix (-1).

132.3. **Properties of the intersection form.** In the following proposition we summarize some of the key properties of the intersection form. The proposition shows in particular that the intersection form is a form in the sense of the definition on page 2877.

Proposition 132.4. (Intersection Form Proposition) Let M be a compact oriented $2n$-dimensional topological manifold. Then the following hold:

(1) The group $\mathrm{H}_n(M;\mathbb{Z})$ is finitely generated and $\mathrm{FH}_n(M;\mathbb{Z})$ is a finitely generated free abelian group.

(2) If n is even, then the intersection form Q_M is symmetric, otherwise it is anti-symmetric.

(3) If M is closed, then the intersection form is non-singular.

The proof of the Intersection Form Proposition 132.4 (3) relies on the following reformulation of the intersection form, which will also be useful later, on a separate occasion.

Lemma 132.5. Let M be a compact oriented connected $2n$-dimensional topological manifold. We denote by $i\colon (M,\varnothing) \to (M,\partial M)$ the inclusion map. We write

$$\begin{array}{rccccccc} \Phi\colon \mathrm{H}_n(M;\mathbb{Z}) & \xrightarrow{i_*} & \mathrm{H}_n(M,\partial M;\mathbb{Z}) & \xrightarrow{\mathrm{PD}_M} & \mathrm{H}^n(M;\mathbb{Z}) & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}(\mathrm{FH}_n(M;\mathbb{Z}),\mathbb{Z}). \\ & & & & \gamma & \mapsto & (a \mapsto \langle \gamma, a)\end{array}$$

For any $a, b \in \mathrm{FH}_n(M;\mathbb{Z})$ we have

$$Q_M(a,b) \;=\; \Phi(b)(a).$$

Proof of Lemma 132.5. First we consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_n(M;\mathbb{Z}) \times \mathrm{H}_n(M;\mathbb{Z}) & \xrightarrow{\mathrm{id}\times i_*} & \mathrm{H}_n(M;\mathbb{Z}) \times \mathrm{H}_n(M,\partial M;\mathbb{Z}) \\ \downarrow{\scriptstyle \mathrm{PD}_M\times\mathrm{PD}_M} & & \downarrow{\scriptstyle \mathrm{PD}_M\times\mathrm{PD}_M} \\ \mathrm{H}^n(M,\partial M;\mathbb{Z}) \times \mathrm{H}^n(M,\partial M;\mathbb{Z}) & \xrightarrow{\mathrm{id}\times i^*} & \mathrm{H}^n(M,\partial M;\mathbb{Z}) \times \mathrm{H}^n(M;\mathbb{Z}) \\ \downarrow{\scriptstyle \cup} & & \downarrow{\scriptstyle \cup} \\ \mathrm{H}^{2n}(M,\partial M;\mathbb{Z}) & \xrightarrow{\mathrm{id}} & \mathrm{H}^{2n}(M,\partial M;\mathbb{Z}). \end{array}$$

By Proposition 119.25 we know that the upper square commutes. We obtain from the Cup Product-Naturality Proposition 115.4, applied to the inclusion map $f\colon (M,\varnothing,\partial M) \to (M,\partial M,\partial M)$, that the lower square also commutes.

Now let $a, b \in \mathrm{FH}_n(M;\mathbb{Z})$. We calculate that

$$\begin{array}{lll}
 & \overset{\text{definition of } Q_M}{\downarrow} & \overset{\text{Cap Product-Kronecker Lemma 116.4 (1)}}{\downarrow} \\
Q_M(b,a) & = \langle \mathrm{PD}_M(b) \cup \mathrm{PD}_M(a), [M] \rangle & = (\mathrm{PD}_M(b) \cup \mathrm{PD}_M(a)) \cap [M] \\
 & = (\mathrm{PD}_M(b) \cup \mathrm{PD}_M(i_*(a))) \cap [M] & = (\mathrm{PD}_M(i_*(a)) \cap (\mathrm{PD}_M(b) \cap [M]) \\
 & \overset{\text{by the above commutative diagram}}{\uparrow} & \overset{\text{Cup-Cap Lemma 116.7}}{\uparrow} \\
 & = \mathrm{PD}_M(i_*(a)) \cap b & = \langle \mathrm{PD}_M(i_*(a)), b \rangle \quad = \Phi(a)(b). \\
 & \overset{\text{definition of } \mathrm{PD}_M(b)}{\uparrow} & \overset{\text{Cap Product-Kronecker Lemma 116.4 (1)}}{\uparrow} \qquad \overset{\text{definition of } \Phi \text{ and ev}}{\uparrow}
\end{array}$$

■

Proof of the Intersection Form Proposition 132.4.

(1) We proved this statement in the Topological Manifolds-Invariants Proposition 104.14.

(2) Given $v, w \in \mathrm{FH}_n(M;\mathbb{Z})$ we have the following equality:

$$\overbrace{\langle \underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{H}^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{H}^n(M,\partial M;\mathbb{Z})}, [M] \rangle}^{=Q_M(v,w)} = (-1)^{n\cdot n} \cdot \overbrace{\langle \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{H}^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{H}^n(M,\partial M;\mathbb{Z})}, [M] \rangle}^{=Q_M(w,v)}.$$

$$\uparrow$$

Cup Product-Commutativity Proposition 115.3

The statement follows from the observation that $(-1)^{n\cdot n} = (-1)^n$.

(3) Now suppose that M is closed. We consider the following diagram:

$$\begin{array}{ccc}
\mathrm{FH}_n(M;\mathbb{Z}) & \xrightarrow{b \mapsto (a \mapsto Q_M(b,a))} & \mathrm{Hom}(\mathrm{FH}_n(M;\mathbb{Z}),\mathbb{Z}). \\
 & \overset{\cong}{\searrow}^{\mathrm{PD}_M} \quad \overset{\mathrm{ev}}{\nearrow}_{\mathrm{ev}} & \\
 & \mathrm{FH}^n(M;\mathbb{Z}) &
\end{array}$$

We make the following observations:

(a) Since M is closed the map PD_M does indeed take values in the cohomology of M. Furthermore, by the Poincaré Duality Theorem 119.1 the map PD_M is an isomorphism.

(b) It follows from the Universal Coefficient Theorem 110.12 for Cohomology Groups together with the Ext-Calculation Lemma 110.5 (5) and the Topological Manifolds-Invariants Proposition 104.14 (4) that the evaluation map

$$\mathrm{ev}\colon\ \mathrm{FH}^n(M;\mathbb{Z}) \ \to\ \mathrm{Hom}(\mathrm{FH}_n(M;\mathbb{Z}),\mathbb{Z})$$

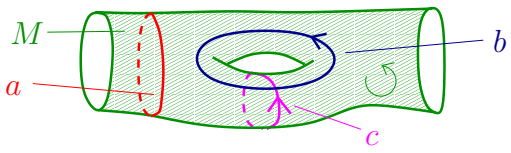
is an isomorphism.

(c) The diagram commutes by Lemma 132.5.

(d) We had just seen that the two diagonal maps are isomorphisms and that the diagram commutes. Thus we obtain that the top map is an isomorphism. By the definition on page 2878 this means precisely that the intersection form is non-singular. ■

Remark. Let M be a compact oriented $2n$-dimensional topological manifold. In the Intersection Form Proposition 132.4 we just saw that if M is closed, then the intersection form is non-singular. If M is not closed, then this conclusion does not necessarily hold. For example in the figure below we let M be the torus minus two open disks. We show a basis $\{a, b, c\}$ for $\mathrm{H}_1(M;\mathbb{Z})$. Using the Intersection Number-via-Cup Product Theorem 126.5 one can easily

see that the intersection form of M with respect to the given basis is represented by the matrix given in the figure. We see that the intersection form is anything but non-singular, in fact it is not even non-degenerate.[1608] Note though that in the Asymmetric Intersection



$$Q_M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Pairing Proposition 132.15 and the Reduced Intersection Form Proposition 132.21 we will see that even to a topological manifold that is not closed we can associate two pairings that are non-singular or at least non-degenerate.

In the next lemma we want to see to what degree the isometry type of the intersection form is an invariant of the homeomorphism type of the topological manifold.

Lemma 132.6. Let M and N be compact oriented connected $2n$-dimensional topological manifolds. Let $f\colon (M,\partial M) \to (N,\partial N)$ be a homotopy equivalence of pairs of topological spaces.[1609]

(1) If $f_*([M]) = \quad [N]$, then f induces an isometry $Q_M \cong \quad Q_N$.
(2) If $f_*([M]) = -[N]$, then f induces an isometry $Q_M \cong -Q_N$.

Note that if f is a homeomorphism, then it follows from Proposition 107.2 that the first case occurs precisely when f is orientation-preserving and the second case occurs precisely when f is orientation-reversing.

Proof. [1610] Let $f\colon (M,\partial M) \to (N,\partial N)$ be a homotopy equivalence of pairs of topological spaces. It follows from Corollary 74.16 together with the Fundamental Class-Corollary 106.4 (3) that there exists an $\varepsilon \in \{-1,1\}$ which satisfies $f_*([M]) = \varepsilon \cdot [N]$.

Claim. For any $c \in \mathrm{H}_n(M;\mathbb{Z})$ we have $\mathrm{PD}_N(f_*(c)) = \varepsilon \cdot (f^{-1})^*(\mathrm{PD}_M(c)) \in \mathrm{H}^n(N,\partial N;\mathbb{Z})$.

Proof. So let $c \in \mathrm{H}_n(M;\mathbb{Z})$. We have

$$(f^{-1})^*(\mathrm{PD}_M(c)) \cap [N] \underset{\text{by the definition of } \varepsilon}{=} (f^{-1})^*(\mathrm{PD}_M(c)) \cap \varepsilon \cdot f_*([M]) \underset{\text{Cap Product-Naturality Lemma 116.8}}{=} \varepsilon \cdot f_*(\mathrm{PD}_M(c) \cap [M]) \underset{\text{definition of } \mathrm{PD}_M}{=} \varepsilon \cdot f_*(c).$$

It follows from the definition of PD_N that we get the desired equality. ⊞

[1608] In fact the annulus $[0,1] \times S^1$ also works as an example, but it is less fun.
[1609] Note that by Corollary 74.16 we know that the induced map $f_*\colon \mathrm{H}_{2n}(M,\partial M;\mathbb{Z}) \to \mathrm{H}_{2n}(N,\partial N;\mathbb{Z})$ is an isomorphism. By the Fundamental Class-Corollary 106.4 the fundamental classes are generators of the homology groups. Thus one of the two cases has to occur.
[1610] The proof is perhaps slightly longer than one might expect. In a sense the statement of the lemma is basically trivial, but we have to pay a price for using the slightly awkward Poincaré Duality isomorphism $\mathrm{PD}_M\colon \mathrm{H}_n(M;\mathbb{Z}) \to \mathrm{H}^{2n-n}(M,\partial M;\mathbb{Z})$.

Now let $a, b \in \mathrm{FH}_n(M;\mathbb{Z})$. Then

$$\begin{array}{lll} & \overset{\text{by definition of } Q_N}{\downarrow} & \overset{\text{by the claim}}{\downarrow} \\ Q_N(f_*(a), f_*(b)) & = \langle \mathrm{PD}_N(f_*(a)) \cup \mathrm{PD}_N(f_*(b)), [N]\rangle & = \\ & = \langle \varepsilon \cdot (f^{-1})^*(\mathrm{PD}_M(a)) \cup \varepsilon \cdot (f^{-1})^*(\mathrm{PD}_M(b)), \varepsilon \cdot f_*([M])\rangle & \\ & = \varepsilon \cdot \langle (f^{-1})^*(\mathrm{PD}_M(a) \cup \mathrm{PD}_M(b)), f_*([M])\rangle & = \varepsilon \cdot Q_M(a,b). \\ & \uparrow & \uparrow \\ & \text{by Proposition 115.4 (1) and since } \varepsilon^3 = \varepsilon & \text{by Lemma 109.6 (3) and the definition of } Q_M \end{array}$$

The lemma now follows from the observation that it follows from the hypothesis that f is a homotopy equivalence together with the Homotopy Equivalence-H_*-Corollary 73.9 that $f_*\colon \mathrm{FH}_n(M;\mathbb{Z}) \to \mathrm{FH}_n(N;\mathbb{Z})$ is an isomorphism. ■

132.4. **The intersection form and degree-one maps.** As we remarked on page 2880, the intersection form of a compact oriented $2n$-dimensional topological manifold contains basically the same information as the cup product pairing

$$\begin{array}{rcl} \mathrm{H}^n(M, \partial M;\mathbb{Z}) \times \mathrm{H}^n(M, \partial M;\mathbb{Z}) & \to & \mathrm{H}^{2n}(M, \partial M;\mathbb{Z}) \\ (\alpha, \beta) & \mapsto & \alpha \cup \beta. \end{array}$$

Given a map $f\colon X \to Y$ between topological spaces it follows from the Cup Product-Naturality Lemma 114.10 that the cup product behaves well under the induced maps f^* on cohomology groups. It stands to reason that the intersection form should also behave well with respect to induced maps on homology.

So suppose we are given a degree-one map $f\colon X \to Y$ between closed oriented connected $2n$-dimensional topological manifolds. By the aforementioned the Cup Product-Naturality Lemma 114.10 we know that for any $\alpha \in \mathrm{FH}^i(Y;\mathbb{Z})$ and $\beta \in \mathrm{FH}^j(Y;\mathbb{Z})$ we have $f^*\alpha \cup f^*\beta = f^*(\alpha \cup \beta)$. It seems reasonable to expect that given $\alpha \in \mathrm{H}_n(X;\mathbb{Z})$ and $\beta \in \mathrm{H}_n(X;\mathbb{Z})$ the following equality holds:

$$(*) \qquad Q_X(\alpha, \beta) \;=\; Q_Y(f_*(\alpha), f_*(\beta)).$$

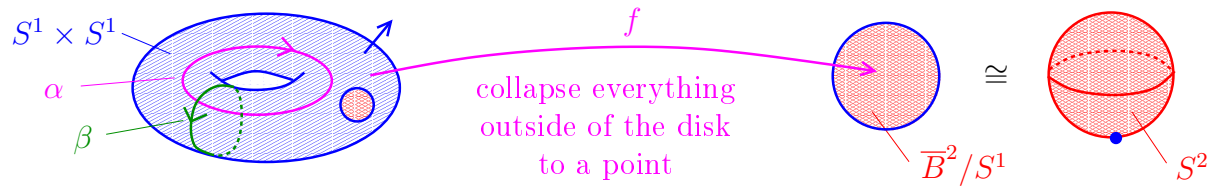
In fact here is a diagram that looks awfully relevant:

$$\begin{array}{ccccccc} \mathrm{H}_n(X;\mathbb{Z}) \times \mathrm{H}_n(X;\mathbb{Z}) & \xrightarrow[\cong]{\mathrm{PD}\times\mathrm{PD}} & \mathrm{H}^n(X;\mathbb{Z}) \times \mathrm{H}^n(X;\mathbb{Z}) & \xrightarrow{\cup} & \mathrm{H}^{2n}(X;\mathbb{Z}) & \xrightarrow{\langle -,[X]\rangle} & \mathbb{Z} \\ \downarrow{\scriptstyle f_*\times f_*} & & {\scriptstyle f^*\times f^*}\uparrow & & {\scriptstyle f^*}\uparrow{\scriptstyle \cong} & & \mathrm{id}\downarrow \\ \mathrm{H}_n(Y;\mathbb{Z}) \times \mathrm{H}_n(Y;\mathbb{Z}) & \xrightarrow[\cong]{\mathrm{PD}\times\mathrm{PD}} & \mathrm{H}^n(Y;\mathbb{Z}) \times \mathrm{H}^n(Y;\mathbb{Z}) & \xrightarrow{\cup} & \mathrm{H}^{2n}(Y;\mathbb{Z}) & \xrightarrow{\langle -,[Y]\rangle} & \mathbb{Z}. \end{array}$$

If this diagram did commute, then $(*)$ would hold. One could now waste a lot of time trying to convince oneself that the diagram commutes. Or alternatively one might notice, as we do in the following example, that $(*)$ just does not hold in general.

Example. We consider the map $f\colon S^1 \times S^1 \to S^2$ that is illustrated in the figure below. It follows easily from Lemma 107.3 that this is a degree-one map. We consider the two homology classes α and β on the torus that are sketched in the figure below. Note that it follows easily from the Intersection Number-via-Cup Product Theorem 126.5 that $Q_{S^1\times S^1}(\alpha, \beta) = +1$. On the other hand note that we have $f_*(\alpha) = f_*(\beta) = 0 \in \mathrm{H}_1(S^2;\mathbb{Z})$, which evidently implies that $Q_{S^2}(f_*(\alpha), f_*(\beta)) = 0$. Thus in this case the equality $(*)$ does not hold.[1611]

[1611] This still begs the question, in the above diagram, which of the squares does not commute?



Remark. With this discussion we also learned an interesting lesson:

Not every diagram commutes, not even up to sign.

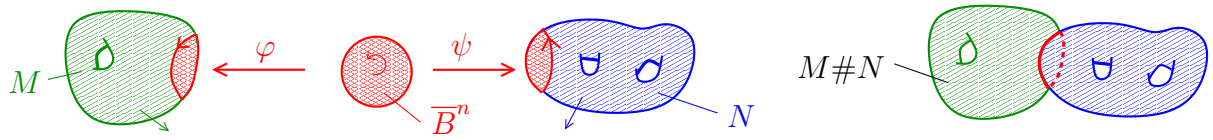
132.5. **The connected sum of topological manifolds.** To get more examples of interesting intersection forms we want to study the intersection forms of the connected sum of smooth and topological manifolds. Recall that on page **??** we introduced the connected sum of two *smooth* manifolds. In this section we will now also define the connected sum of topological manifolds. The reader who is not that interested in topological manifolds can safely move on to the next section.

For those who are staying, let us recall the definition of the connected sum of two *oriented* smooth manifolds.

Definition. Let $n \in \mathbb{N}$ and let M and N be two connected oriented non-empty n-dimensional smooth manifolds. Furthermore let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\psi\colon \overline{B}^n \to N \setminus \partial N$ be smooth embeddings such that φ is orientation-preserving and such that ψ is orientation-reversing. We define the connected sum of the oriented smooth manifolds M and N as

$$M\#N := \big(M \setminus \varphi(B^n)\big) \sqcup \big(N \setminus \psi(B^n)\big)/\!\sim \text{ where } \varphi(P) \sim \psi(P) \text{ for all } P \in S^{n-1}.$$

In the Connected Sum-Proposition **??** we showed that the connected sum of two oriented smooth manifolds is again a smooth manifold.



Now we intend to generalize this notion to the setting of topological manifolds. The idea is to suitably adapt the concepts used in the above definition. Browsing through earlier chapters we see the following:

(1) On page 152 we defined an embedding to be a map $f\colon X \to Y$ between two topological spaces that is a homeomorphism onto its image.
(2) On page 2335 we introduced the notion of an orientation of a topological manifold and of an orientation-preserving map.

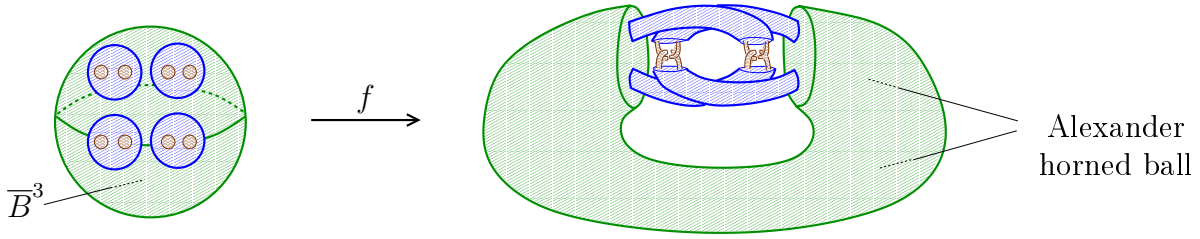
So are we done? Can we just copy-paste the above definition?

As we had just mentioned, in the Connected Sum-Proposition **??** we showed that the connected sum of two oriented smooth manifolds is again a smooth manifold. Recall that the proof of the Connected Sum-Proposition **??** relied heavily on the Smooth Embedding Theorem 24.10 which says that the image of a smooth embedding $f\colon X \to Y \setminus \partial Y$ is a smooth submanifold of the smooth manifold Y.

The question arises, whether the analogous statement holds for topological manifolds. In other words, we have the following question.

Question 132.7. Let X be an n-dimensional topological manifold. Is the image of an embedding $f\colon \overline{B}^n \to X \setminus \partial X$ necessarily a topological submanifold of X?

For better or worse the answer to Question 132.7 is no.[1612] Namely consider the embedding $f\colon \overline{B}^3 \to \mathbb{R}^3$ that is illustrated in the figure below. We refer to the image $f(\overline{B}^3)$ as the *Alexander horned ball*.[1613] It is clear that this is a weird embedding. We leave it to the reader to make the embedding precise and to show that $f(\overline{B}^3)$ is not a topological submanifold of $\mathbb{R}^3$. But even without filling in all details this map should shatter the reader's confidence that Question 132.7 has a positive answer.



This stupefying example leads us to the following definition.

Definition. Let X and Y be topological manifolds. We say an embedding $f\colon X \to Y$ is locally flat if $f(X)$ is a submanifold of Y.

Remark. Let X and Y be a smooth manifolds. It follows from the Smooth Embedding Theorem 24.10 that every smooth embedding $X \to Y$ is actually a locally flat embedding.

With this definition we can now finally give a reasonable definition of the connected sum of two *topological* manifolds. For completeness' sake we now also define the connected sum if one or both topological manifolds involved are non-orientable.

Definition. Let $n \in \mathbb{N}$. Let M and N be two connected non-empty n-dimensional topological manifolds.

(1) Suppose that at least one of M or N is non-orientable. We pick two locally flat embeddings $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\psi\colon \overline{B}^n \to N \setminus \partial N$ and we define the connected sum of M and N as

$$M\#N := \big(M \setminus \varphi(B^n)\big) \sqcup \big(N \setminus \psi(B^n)\big)/\!\sim \text{ where } \varphi(P) \sim \psi(P) \text{ for all } P \in S^{n-1}.$$

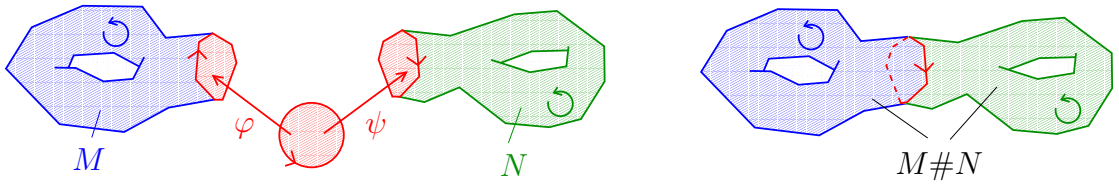
(2) If both M and N are orientable, then we pick orientations for M and N. Let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\psi\colon \overline{B}^n \to N \setminus \partial N$ be locally flat embeddings such that φ is orientation-preserving and such that ψ is orientation-reversing. As before we define the connected sum of the oriented topological manifolds M and N as

$$M\#N := \big(M \setminus \varphi(B^n)\big) \sqcup \big(N \setminus \psi(B^n)\big)/\!\sim \text{ where } \varphi(P) \sim \psi(P) \text{ for all } P \in S^{n-1}.$$

[1612] That is not really a surprise, on page 659 we already saw wild knots in $\mathbb{R}^3$ and on page 1814 we saw the Alexander horned sphere. Both give examples of embeddings of topological manifolds such that the image is not a submanifold.

[1613] On page 1814 we referred to $f(S^2)$ as the Alexander horned sphere.

We refer to the figure below for an illustration. For completeness we define the connected sum of an empty manifold with another connected topological manifold M to be M, i.e. we set $\varnothing \# M := M \# \varnothing := M$.



The following proposition is the analogue of the Connected Sum-Proposition **??** that we formulated in the context of smooth manifolds.

Proposition 132.8. (Connected Sum-Proposition) Let $n \in \mathbb{N}$. Let M and N be two connected n-dimensional topological manifolds. (If both M and N are orientable, then we demand that M and N are oriented.) Let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\psi\colon \overline{B}^n \to N \setminus \partial N$ be two locally flat embeddings. (If M and N are orientable, then we demand that φ is orientation-preserving and that ψ is orientation-reversing.) The following five statements hold:

(1) The resulting connected sum $M \# N$ is an n-dimensional topological manifold such that $M \setminus \varphi(B^n)$ and $N \setminus \psi(B^n)$ are submanifolds.
(2) If M and N are closed, then $M \# N$ is also a closed topological manifold.
(3) If $n \geq 2$ or if at least one of M or N is closed, then $M \# N$ is also connected.
(4) The homeomorphism type of $M \# N$ does not depend on the choice of φ and ψ.
(5) If M and N are both oriented, then the topological manifold $M \# N$ admits a unique orientation that coincides with the orientations of the submanifolds $M \setminus \varphi(B^n)$ and $N \setminus \psi(B^n)$.

Convention. We always equip the connected sum of two oriented connected n-dimensional topological manifolds with the orientation from the Connected Sum-Proposition 132.8 (5).

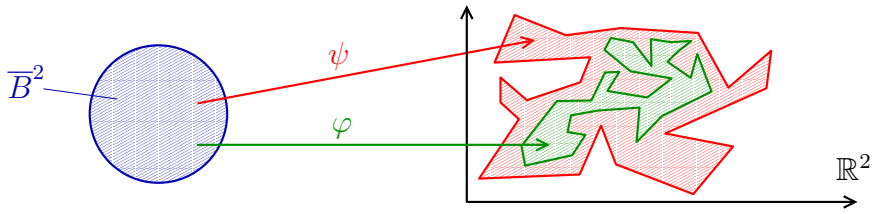
Most of the statements of the Connected Sum-Proposition 132.8 are easy to prove.

Proof of the Connected Sum-Proposition 132.8 (1), (2), (3) and (5). The proof is basically identical to the proofs of the corresponding statements in the smooth setting that we proved in the Connected Sum-Proposition **??**. There are just a few minor modifications that we need to make:

(1) We used the Codimension-Zero Smooth Embedding Proposition 24.16 to show that the image of a smooth embedding is a submanifold. This feature is now built into the definition of a locally flat embedding.
(2) We need to replace Propositions 24.16 and **??** by Propositions 75.3 and 75.8. ■

The fact that the connected sum operation does not really depend on the choice of the locally flat topological embeddings is much more difficult to prove. The key to doing so is the following difficult theorem.

Theorem 132.9. (Annulus Theorem) Let $n \in \mathbb{N}$ and let $\varphi, \psi\colon \overline{B}^n \to \mathbb{R}^n$ be two locally flat embeddings. If $\varphi(\overline{B}^n) \subset \psi(B^n)$, then $\psi(\overline{B}^n) \setminus \varphi(B^n)$ is homeomorphic to $[0,1] \times S^{n-1}$.



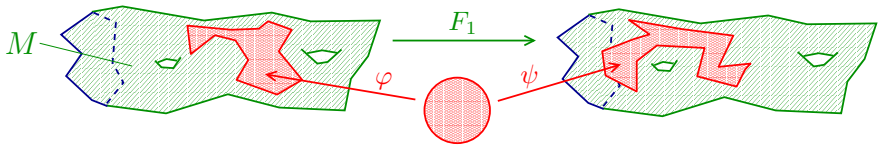
Proof. The case $n = 1$ is trivial. For $n = 2$ and $n = 3$ the Annulus Theorem follows from the work of Tibor Radó [**Rad1926**] and Edwin Moise [**Moi1952, Moi1977**]. The Annulus Theorem was proved for dimensions $n \geq 5$ by Rob Kirby [**Kir1969**] and in dimension 4 by Frank Quinn [**Qu1982**, p. 506]. We also refer to [**Edw1984**, p. 247] for more information. ■

Using the Annulus Theorem 132.9 one can now prove the following theorem which can be viewed as the topological analogue of the Smooth Ball Embedding Theorem 37.8.

Theorem 132.10. (Topological Ball Embedding Theorem) Let M be an n-dimensional topological manifold. (If M is orientable, then we pick an orientation for M.) In the following let $\varphi_1, \dots, \varphi_m \colon \overline{B}^n \to M \setminus \partial M$ and $\psi_1, \dots, \psi_m \colon \overline{B}^n \to M \setminus \partial M$ be two sets of m locally flat embeddings of $\overline{B}^n$ with disjoint images. (If M is orientable, then we demand that all φ_i and ψ_i are orientation-preserving or that all are orientation-reversing.) If $n \geq 2$ or if $m = 1$, then there exists a homeotopy

$$F \colon M \times [0,1] \;\to\; M$$

rel some neighborhood of ∂M from the identity $F_0 = \mathrm{id}_M$ to a homeomorphism $F_1 \colon M \to M$ such that for each $i \in \{1, \dots, m\}$ we have $F_1 \circ \varphi_i = \psi_i$. (Note that if M is orientable, then the homeomorphism F_1 is in fact orientation-preserving by Exercise 105.3).



Remark. We consider $\overline{B}^3$ as a submanifold of $\mathbb{R}^3$. Since the image of the embedding $\varphi \colon \overline{B}^3 \to \mathbb{R}^3$ that is given by the Alexander horned ball is not a submanifold of the topological manifold we see that there is no homeotopy $F \colon \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ with $F_1 \circ \varphi = \mathrm{id}$. In particular we see that in the formulation of Theorem 132.10 we cannot drop the hypothesis that the embeddings are locally flat.

Proof of Theorem 132.10. This theorem follows from the Annulus Theorem 132.9 together with reasonably elementary arguments. The details are worked out in [**FNOP2019**, Proof of Theorem 5.11]. Alternatively the reader is encouraged to assemble the argument by using the obvious analogues Exercise 49.7 and Exercise **??**, which were both formulated in the setting of smooth manifolds, in the setting of topological manifolds. ■

Proof of the Connected Sum-Proposition 132.8 (4). Thanks to our preparations the proof is basically identical to the proof of the corresponding statement in the smooth setting that we proved in the Connected Sum-Proposition **??**. All we need to is to replace the Smooth Ball Embedding Theorem 37.8 by Theorem 132.10. ■

132.6. **The intersection form and the connected sum operation.** First let use see how the homology groups and fundamental groups behave under the connected sum operation.

Proposition 132.11. (Connected Sum-Invariants Proposition) Let $n \in \mathbb{N}$ and let M and N be two closed oriented connected n-dimensional topological manifolds. Furthermore let $\varphi\colon \overline{B}^n \to M$ be an orientation-preserving locally flat embedding and let $\psi\colon \overline{B}^n \to N$ be an orientation-reversing locally flat embedding. Then $\mathrm{H}_0(M\#N;R) \cong \mathrm{H}_n(M\#N;R) \cong R$. Furthermore, for any $i \neq 0, n$ and any commutative ring R the following inclusion induced maps are isomorphisms

$$\mathrm{H}_i(M\#N;R) \xleftarrow{\cong} \mathrm{H}_i(M\backslash\varphi(B^n);R) \oplus \mathrm{H}_i(N\backslash\psi(B^n);R) \xrightarrow{\cong} \mathrm{H}_i(M;R) \oplus \mathrm{H}_i(N;R).$$

Furthermore if $n \geq 3$, then the following inclusion induced maps are isomorphisms

$$\pi_1(M\#N) \xleftarrow{\cong} \pi_1(M \setminus \varphi(B^n)) \ * \ \pi_1(N \setminus \psi(B^n)) \xrightarrow{\cong} \pi_1(M) * \pi_1(N).$$

Proof. We proved the statements in the Connected Sum-π_1-Proposition 52.12 and the Connected Sum-H_*-Proposition 100.13 for smooth manifolds. The proof for topological manifolds is very similar, we just need to note that it follows from by the Excisive Triad-Proposition 109.12 (5) and the Mayer–Vietoris Theorem 109.14 that the Mayer–Vietoris argument in the proof of Lemma 100.14 works for topological manifolds. ■

The next proposition says that the isomorphism of homology groups from the Connected Sum-Invariants Proposition 132.11 induces an isometry of intersection forms.

Proposition 132.12. (Intersection Form-Additivity Proposition) Let $n \geq 1$ and let M and N be compact oriented $2n$-dimensional topological manifolds. Furthermore let $\varphi\colon \overline{B}^n \to M$ be an orientation-preserving locally flat embedding and let $\psi\colon \overline{B}^n \to N$ be an orientation-reversing locally flat embedding. The isomorphism $\mathrm{H}_n(M\#N;\mathbb{Z}) \xrightarrow{\cong} \mathrm{H}_n(M;\mathbb{Z}) \oplus \mathrm{H}_n(N;\mathbb{Z})$ from the Connected Sum-Invariants Proposition 132.11 induces an isometry

$$Q_{M\#N} \xrightarrow{\cong} Q_M \oplus Q_N$$

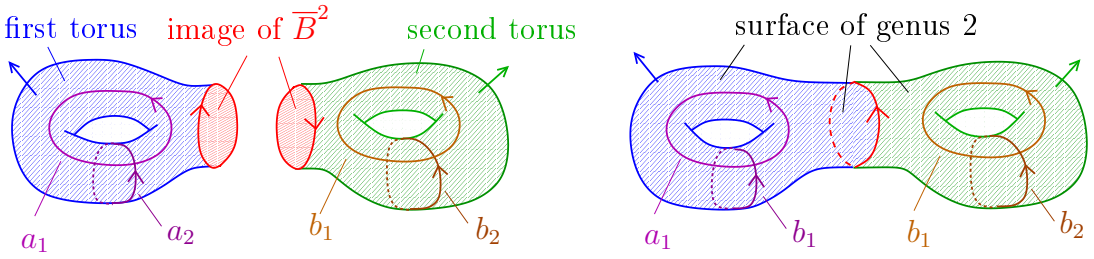
of forms.

Example. We consider the surface Σ of genus two. In the figure below we recall that we can view Σ as the connected sum of two tori. It thus follows from the Intersection Form-Additivity Proposition 132.12, the discussion on page 2881 and Lemma 132.2 (2) that the intersection form on $\mathrm{H}_1(\Sigma;\mathbb{Z}) \cong \mathbb{Z}^4$ is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Not surprisingly this recovers the result from page 2881.

Remark. Let $n \geq 1$ and let M and N be closed oriented $2n$-dimensional topological manifolds. Furthermore let $\varphi\colon \overline{B}^n \to M$ be an orientation-preserving locally flat embedding and



let $\psi\colon \overline{B}^n \to N$ be an orientation-reversing locally flat embedding. the figure above suggests the following approach to proving the Intersection Form-Additivity Proposition 132.12:

(1) First we pick n-dimensional cycles $a_1, \dots, a_k$ in $M \setminus \varphi(\overline{B}^{2n})$ such that $[a_1], \dots, [a_k]$ form a basis of $\mathrm{FH}_n(M;\mathbb{Z})$ and which intersect nicely, in the sense of the definition on page 2785.

(2) Similarly we pick transverse n-dimensional cycles $b_1, \dots, b_l$ in $N \setminus \psi(\overline{B}^{2n})$ such that $[b_1], \dots, [b_l]$ form a basis of $\mathrm{FH}_n(N;\mathbb{Z})$ and which intersect nicely.

(3) It follows immediately from the Connected Sum-Invariants Proposition 132.11 that $[a_1], \dots, [a_k], [b_1], \dots, [b_l]$ form a basis for $\mathrm{FH}_n(M\#N;\mathbb{Z})$.

(4) In Theorem 127.9 we saw that the intersection form of cycles that intersect nicely in terms of the algebraic intersection number that we introduced on page 2785. We use this fact together with the basis from (3) to determine the intersection form of $M\#N$. Here we make the following observations:
   (a) By the orientation conventions from the Connected Sum-Proposition 132.8 (3) the algebraic intersection numbers of the a_i in M are the same as in $M\#N$.
   (b) Similarly the algebraic intersection numbers of the b_i in N are the same as in $M\#N$,
   (c) Since the a_i do not intersect the b_j we see that the algebraic intersection number $a_i \cdot b_j = 0$ for any i, j.

   Summarizing we obtain immediately from Theorem 127.9 that $Q_M \oplus Q_N \cong Q_{M\#N}$.

The above argument sounds very convincing. But there is a major catch: how can we find such $a_1, \dots, a_k$ and $b_1, \dots, b_l$? The problem is not that we need to avoid $\varphi(\overline{B}^{2n})$ and $\psi(\overline{B}^{2n})$.[1614] The problem is to find such cycles that are transverse. Even though we have some tools, e.g. we have Theorem **??** and the Transversality Theorem 42.4, it is very hard to turn the above approach into a rigorous proof. Furthermore, any attempt to make this approach work would inevitably end up with an unreadable mess.

After the discussion in the previous remark we recall the advice from Rob Kirby [**Kir1989**, p. 21], see page 2753:

> "Think with [algebraic] intersection [numbers], prove with cup products."

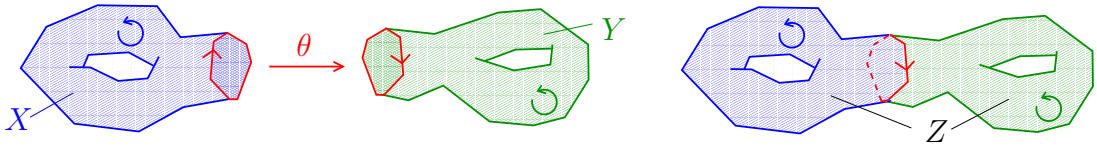
Thus we will use cup products to prove the Intersection Form-Additivity Proposition 132.12.

Proof of the Intersection Form-Additivity Proposition 132.12. Throughout the proof let $n \geq 1$. In the following we consider the case that the topological manifolds M and N

[1614] Why is this not a problem?

are closed. We leave it to the reader to make the slight adjustments needed to deal with the case that M and N are compact with possibly non-empty boundary.

We start out with a lengthy preparation. Thus let X and Y be two oriented connected $2n$-dimensional topological manifolds such that ∂X and ∂Y are both homeomorphic to S^{2n-1}. We equip ∂X and ∂Y with the orientation coming from the Topological Manifold-Boundary Orientation Proposition 105.19. Let $\theta\colon \partial X \to \partial Y$ be an orientation-reversing homeomorphism. We write $Z = X \cup_\theta Y$. In the following we view X and Y as subsets of Z. By the Gluing-Topological Manifolds-Proposition 75.8 we know that X and Y are in fact codimension-zero submanifolds of Z and that we can equip Z with an orientation such that the induced orientations on X and Y are the given orientations.



Claim. The inclusions $i\colon X \to Z$ and $j\colon Y \to Z$ induce an isometry

$$Q_X \oplus Q_Y \xrightarrow{\cong} Q_Z.$$

Proof. As in the Intersection Form-Additivity Proposition 132.12 we see that the two inclusion maps i and j induce an isomorphism $\mathrm{H}_n(X;\mathbb{Z}) \oplus \mathrm{H}_n(Y;\mathbb{Z}) \to \mathrm{H}_n(Z;\mathbb{Z})$. Evidently this isomorphism descends to an isomorphism $\mathrm{FH}_n(X;\mathbb{Z}) \oplus \mathrm{FH}_n(Y;\mathbb{Z}) \to \mathrm{FH}_n(Z;\mathbb{Z})$. Therefore it suffices to prove the following three statements.

(1) for $a, b \in \mathrm{FH}_n(X;\mathbb{Z})$ we have $Q_X(a,b) = Q_Z(i_*(a), i_*(b))$,
(2) for $a, b \in \mathrm{FH}_n(Y;\mathbb{Z})$ we have $Q_Y(a,b) = Q_Z(j_*(a), j_*(b))$,
(3) for $a \in \mathrm{FH}_n(X;\mathbb{Z})$ and $b \in \mathrm{FH}_n(Y;\mathbb{Z})$ we have $Q_Z(i_*(a), j_*(b)) = 0$.

We start out with the proof of the first statement. Thus let $a, b \in \mathrm{FH}_n(X;\mathbb{Z})$. We write $\mathring{X} = X \setminus \partial X$. We denote by $p\colon (Z, \varnothing) \to (Z, Z \setminus \mathring{X})$ and $i\colon (X, \partial X) \to (Z, Z \setminus \mathring{X})$ the two obvious inclusion maps of pairs of topological spaces. It is a consequence of the Excision Theorem 74.18 for Topological Manifolds that $i_*\colon \mathrm{FH}_k(X, \partial X;\mathbb{Z}) \to \mathrm{FH}_k(Z, Z \setminus \mathring{X};\mathbb{Z})$ is an isomorphism for every $k \in \mathbb{N}_0$. Then

$$\begin{aligned} Q_X(a,b) &\underset{\uparrow}{=} \langle \mathrm{PD}_X(a) \cup \mathrm{PD}_X(b), [X] \rangle \\ &\quad \text{by definition of } Q_X \\ &\underset{\uparrow}{=} \big\langle (i^*)^{-1}(\mathrm{PD}_X(a)) \cup (i^*)^{-1}(\mathrm{PD}_X(b)), i_*([X]) \big\rangle \\ &\quad \text{Lemma 109.6 (3) applied to } i\colon (X,\partial X) \to (Z, Z \setminus \mathring{X}), \\ &\quad \text{together with the Cup Product-Naturality Proposition 115.4 (1)} \\ &\underset{\uparrow}{=} \big\langle p^*((i^*)^{-1}(\mathrm{PD}_X(a))) \cup p^*((i^*)^{-1}(\mathrm{PD}_X(b))), [Z] \big\rangle \\ &\quad \text{first note that by the Codimension Zero-Fundamental Class Lemma 106.24 we have, } i_*([X]) = p_*([Z]), \text{ next} \\ &\quad \text{apply Lemma 109.6 (3) to } p, \text{ finally apply the Cup Product-Naturality Proposition 115.4 (1) to } p \\ &\underset{\uparrow}{=} \big\langle \mathrm{PD}_Z(i_*(a)) \cup \mathrm{PD}_Z(i_*(b)), [Z] \big\rangle \underset{\uparrow}{=} Q_Z(i_*(a), i_*(b)). \\ &\quad \text{by the Codimension-Zero Submanifold-PD} \quad \text{definition of } Q_Z \\ &\quad \text{Lemma 120.5 we know} \\ &\quad \text{that } p^*(i^*)^{-1}(\mathrm{PD}_X(c)) = \mathrm{PD}_Z(i_*(c)) \end{aligned}$$

The second statement is of course proved the same way. We turn to the proof of the third statement. So let $a \in \mathrm{FH}_n(X;\mathbb{Z})$ and $b \in \mathrm{FH}_n(Y;\mathbb{Z})$. The above argument shows that we can compute $Q_Z(i_*(a), j_*(b))$ in terms of the cup product in Z of an element in the image

$\mathrm{H}^n(Z, Z \setminus \mathring{X}) \to \mathrm{H}^n(Z)$ and an element in the image of $\mathrm{H}^n(Z, Z \setminus \mathring{Y}; \mathbb{Z}) \to \mathrm{H}^n(Z; \mathbb{Z})$. But $(Z \setminus \mathring{X}) \cup (Z \setminus \mathring{Y}) = Z$. So it follows from the Cup Product-Naturality Proposition 115.4 (1) that this cup product is zero. ⊞

After these preparations it is very easy to prove the actual statement of the proposition. Thus let M and N be closed oriented $2n$-dimensional topological manifolds and let $\varphi\colon \overline{B}^{2n} \to M$ be an orientation-preserving locally flat embedding and let $\psi\colon \overline{B}^{2n} \to N$ be an orientation-reversing locally flat embedding. We write $X = M \setminus \varphi(B^{2n})$ and $Y = N \setminus \psi(B^{2n})$. It follows from the definition of a locally flat embedding together with Proposition 75.3 that X and Y are compact topological manifolds. With this notation we can write $M = X \cup \overline{B}^{2n}$ and $N = Y \cup \overline{B}^{2n}$. It follows from the claim and the fact that $\mathrm{H}_n(\overline{B}^{2n}; \mathbb{Z}) = 0$ that the inclusions induce isometries

$$Q_{M\#N} \xleftarrow{\cong} Q_X \oplus Q_Y \xrightarrow{\cong} Q_M \oplus Q_N.$$

We have thus proved the desired statement. ■

We recall the following question.

Question 100.15. Are the topological spaces $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$, $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$ and $S^2 \times S^2$ homeomorphic?

These topological spaces are hard to distinguish since we saw that they are simply connected and that their (co-) homology groups agree. Now we want to look at the intersection forms to see whether they can help us in distinguishing these three smooth manifolds.

Example. On page 2882 we saw that the intersection form of $\mathbb{C}\mathrm{P}^2$ is represented by the 1×1–matrix (1) and we saw that the intersection form of $\overline{\mathbb{C}\mathrm{P}}^2$ is represented by the 1×1–matrix (-1). From the Intersection Form-Additivity Proposition 132.12 we thus obtain:

(1) the intersection form of $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$ is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, whereas

(2) the intersection form of $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$ is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Finally we recall that on page 2881 we showed that

(3) the intersection form of $S^2 \times S^2$ is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Motivated by Question 100.15 and Lemma 132.6 the following question arises.

Question 132.13. Are the forms over $\mathbb{Z}^2$ represented by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

isometric (possibly up to a sign reversal)?

In Chapter 133 we will develop the theory of non-singular forms to answer Question 132.13.

132.7. **The asymmetric intersection pairings.** On page 2883 we saw that the intersection form of a topological manifold with non-empty boundary can be degenerate. In this and the following section we will associate to a compact oriented topological manifold the asymmetric intersection pairings which are non-singular and the reduced intersection form which is at least non-degenerate.

Before we provide the definitions let us recall and introduce a new definition.

Definition. Let R be a commutative ring and let V, W and S be R-modules. Recall that according to the definition on page 2656 we say that an R-bilinear pairing $\langle\ ,\ \rangle\colon V\times W\to R$ is non-singular if the induced maps

$$\begin{array}{ccc} V & \to & \mathrm{Hom}_R(W,S) \\ v & \mapsto & \left(\begin{array}{ccc} W & \to & S \\ w & \mapsto & \langle v,w\rangle \end{array}\right) \end{array} \quad \text{and} \quad \begin{array}{ccc} W & \to & \mathrm{Hom}_R(V,S) \\ w & \mapsto & \left(\begin{array}{ccc} V & \to & S \\ v & \mapsto & \langle v,w\rangle \end{array}\right) \end{array}$$

are isomorphisms. We now say that the R-bilinear pairing is non-degenerate if the above two maps are both monomorphisms. Otherwise we say that the R-bilinear pairing is degenerate.

Example. Let R be a commutative domain and let $A\in \mathrm{M}(n\times n;R)$. Elementary algebra, similar to the proof of the Non-Singularity Lemma 121.1, shows that the form

$$\begin{array}{ccc} R^n\times R^n & \to & R \\ (v,w) & \mapsto & v^TAw \end{array}$$

is non-singular if and only if $\det(A)$ is a unit in R and it is non-singular if and only if $\det(A)$ is non-zero.

For the record let us state the following lemma that follows almost immediately from the definitions.

Lemma 132.14. Let R be a commutative domain. Let V, W and S be R-modules and let

$$\langle\ ,\ \rangle\colon V\times W\to S$$

be an R-bilinear pairing. The following two statements are equivalent:

(1) The R-bilinear pairing is non-degenerate.
(2) For every $v\in V$ there exists a $w\in W$ with $\langle v,w\rangle\neq 0$ and for every $w\in W$ there exists a $v\in V$ with $\langle v,w\rangle\neq 0$.

In the Intersection Form Proposition 132.4 we saw that if we are given a *closed* oriented $2n$-dimensional topological manifold M, then the intersection form

$$\begin{array}{rccl} Q_M\colon & \mathrm{FH}_n(M;\mathbb{Z})\times \mathrm{FH}_n(M;\mathbb{Z}) & \to & \mathbb{Z} \\ & (a,b) & \mapsto & \Big\langle \underbrace{\underbrace{\mathrm{PD}_M(a)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(b)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})}}_{\in \mathrm{FH}^{2n}(M,\partial M;\mathbb{Z})}, [M] \Big\rangle. \end{array}$$

is non-singular. On page 2883 we saw that if M is *not closed*, then this conclusion does not necessarily hold. More precisely we gave an explicit example of a compact orientable smooth manifold, namely the torus minus two open disks, such that the intersection form is degenerate, in particular it is singular.

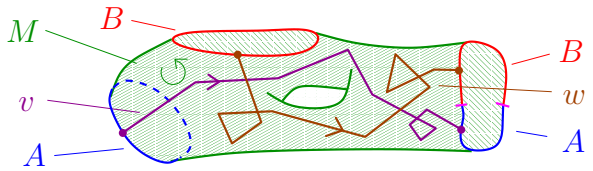
We move on to the definition of asymmetric intersection pairings on topological manifolds.

Definition. Let M be a compact oriented n-dimensional topological manifold and suppose that we are given two compact $(n-1)$-dimensional submanifolds A and B of ∂M. We suppose that $A\cap B=\partial A=\partial B$. We write $X:=\partial M\setminus \mathring{A}$ and $Y:=\partial M\setminus \mathring{B}$. For each

$k \in \mathbb{N}_0$ we refer to the pairing[1615]

$$\begin{aligned} Q^{\text{as}}_{M,A,B}\colon \operatorname{FH}_k(M,A;\mathbb{Z}) \times \operatorname{FH}_{n-k}(M,B;\mathbb{Z}) &\to \mathbb{Z} \\ (v,w) &\mapsto \Big\langle \underbrace{\underbrace{\operatorname{PD}_M(v)}_{\in \operatorname{H}^{n-k}(M,X;\mathbb{Z})} \cup \underbrace{\operatorname{PD}_M(w)}_{\in \operatorname{H}^{k}(M,Y;\mathbb{Z})}}_{\in \operatorname{H}^{n}(M,\partial M;\mathbb{Z})}, [M] \Big\rangle \end{aligned}$$

as an asymmetric intersection pairing of M. When A and B are clear from the context we write Q^{as}_M instead of the cumbersome $Q^{\text{as}}_{M,A,B}$.



$$Q^{\text{as}}_{M,A,B}(v,w) = +1 - 1 = 0$$

Remark. In the previous definition, if we take $A = B = \varnothing$, then we obtain precisely the intersection form as introduced on page 2800, in other words, $Q^{\text{as}}_{M,\varnothing,\varnothing} = Q_M$.

Example. Let M be the torus minus two open balls. We are interested in the (asymmetric) intersection pairings

$$Q_M\colon \operatorname{H}_1(M;\mathbb{Z}) \times \operatorname{H}_1(M;\mathbb{Z}) \to \mathbb{Z} \quad \text{and} \quad Q^{\text{as}}_{M,\varnothing,\partial M}\colon \operatorname{H}_1(M;\mathbb{Z}) \times \operatorname{H}_1(M,\partial M;\mathbb{Z}).$$

We consider the four curves a, b, c, d on M that are shown in the figure below. Note that the curves a, b, c represent a basis for $\operatorname{H}_1(M;\mathbb{Z})$, whereas the curves a, b, d represent a basis for $\operatorname{H}_1(M,\partial M;\mathbb{Z})$.[1616] Using these bases and using the Intersection Number-via-Cup Product Theorem 126.5 we see that

Q_M is represented by $\begin{array}{c} \\ a \\ b \\ c \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$ and $Q^{\text{as}}_{M,\varnothing,\partial M}$ is represented by $\begin{array}{c} \\ a \\ b \\ c \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a & b & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$.

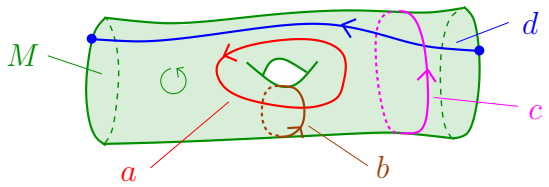
Note that the former matrix has determinant zero whereas the latter matrix has determinant 1. This calculation, together with the Non-Singularity Lemma 121.1 shows that the former pairing is degenerate whereas the latter pairing is non-singular.[1617] As we will see in the next proposition, it is not a coincidence that in this particular example the asymmetric intersection pairing is non-singular.

[1614] In most cases we will have $A \cup B = \partial M$. But the case $A = B = \varnothing$ can also be interesting.

[1615] Note that by Proposition 75.3 we know that X and Y are also compact submanifolds of ∂M with $\partial X = \partial A = A \cap X$ and $\partial Y = \partial B = B \cap Y$. Furthermore note that it follows from $A \cap B = \partial A = \partial B$ that $\partial M = X \cup Y$ and that $\partial M = (X \setminus \partial X) \cup (Y \setminus \partial Y) \cup C$ where each component of C is a component of ∂X and a component of ∂Y. Thus it follows from a slight generalization of the Excisive Triad-Proposition 109.12 (5), see also Exercise 109.5, that (M, X, Y) is an excisive triad. In particular it follows, see the definition on page 2536, that the cup product is actually defined.

[1616] Why is that?

[1617] In fact, using this observation and Exercise 121.2 (c) one can easily answer the previous footnote.



Proposition 132.15. (Asymmetric Intersection Pairing Proposition) Let M be a compact oriented n-dimensional topological manifold. Furthermore suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$.

(1) Given any $k \in \mathbb{N}_0$ the following pairing is non-singular:[1618]

$$Q^{\text{as}}_{M,A,B} \colon \mathrm{FH}_k(M, A; \mathbb{Z}) \times \mathrm{FH}_{n-k}(M, B; \mathbb{Z}) \to \mathbb{Z}.$$

For the remaining two statements of the proposition we assume that $n = 2k$ is even. We denote by $p_* \colon \mathrm{H}_k(M; \mathbb{Z}) \to \mathrm{H}_k(M, \partial M; \mathbb{Z})$ the obvious map.

(2) The following diagram commutes:

$$\begin{array}{ccc} \mathrm{FH}_k(M;\mathbb{Z}) \times \mathrm{FH}_k(M;\mathbb{Z}) & \xrightarrow{Q_M} & \mathbb{Z} \\ \downarrow{\scriptstyle \mathrm{id} \times p_*} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{FH}_k(M;\mathbb{Z}) \times \mathrm{FH}_k(M, \partial M;\mathbb{Z}) & \xrightarrow{Q^{\text{as}}_{M,\varnothing,\partial M}} & \mathbb{Z}. \end{array}$$

(3) Let $v_1, \ldots, v_m$ be a basis of the free abelian group $\mathrm{FH}_k(M; \mathbb{Z})$. By (1) together with Exercise 121.2 (a) there exists a unique "dual" basis $v_1^*, \ldots, v_m^*$ of the free abelian group $\mathrm{FH}_k(M, \partial M; \mathbb{Z})$ with $Q^{\text{as}}_{M,\varnothing,\partial M}(v_i, v_j^*) = \delta_{ij}$ for all $i, j \in \{1, \ldots, m\}$. With this notation the following diagram commutes:

$$\begin{array}{ccc} \mathrm{FH}_k(M;\mathbb{Z}) & \xrightarrow{p_*} & \mathrm{FH}_k(M, \partial M;\mathbb{Z}) \\ \cong \uparrow{\scriptstyle e_i \mapsto v_i} & & {\scriptstyle e_i \mapsto v_i^*} \uparrow \cong \\ \mathbb{Z}^m & \xrightarrow[\text{matrix } (Q_M(v_i, v_j))_{i,j=1,\ldots,m}]{\text{multiplication by}} & \mathbb{Z}^m. \end{array}$$

Put differently, let $(Q_M(v_i, v_j))_{i,j=1,\ldots,m}$ be the $(m \times m)$–matrix that represents the intersection form Q_M with respect to the basis $v_1, \ldots, v_m$. This matrix represents the map $p_* \colon \mathrm{FH}_k(M; \mathbb{Z}) \to \mathrm{FH}_k(M, \partial M; \mathbb{Z})$ with respect to the bases $v_1, \ldots, v_m$ and $v_1^*, \ldots, v_m^*$.

Proof.

(1) We consider the map

$$\Psi \colon \mathrm{FH}_k(M, A; \mathbb{Z}) \xrightarrow{\mathrm{PD}_M} \mathrm{FH}^{n-k}(M, B; \mathbb{Z}) \xrightarrow{\mathrm{ev}} \mathrm{Hom}(\mathrm{FH}_{n-k}(M, B; \mathbb{Z}), \mathbb{Z}).$$

First note that almost the same argument as in the proof of Lemma 132.5 shows that for every $a \in \mathrm{FH}_k(M, A; \mathbb{Z})$ and every $b \in \mathrm{FH}_{n-k}(M, B; \mathbb{Z})$ we have $Q^{\text{as}}_{M,A,B}(a, b) = \Psi(b)(a)$. Furthermore note that as in the proof of the Intersection Form Proposition 132.4 (3) we see that the two maps PD_M and $\mathrm{ev} \colon \mathrm{FH}^{n-k}(M, B; \mathbb{Z}) \to \mathrm{Hom}(\mathrm{FH}_{n-k}(M, B; \mathbb{Z}), \mathbb{Z})$

[1618] This statement is a generalization of the Intersection Form Proposition 132.4 (3).

are isomorphisms. In particular $\Psi = \mathrm{ev} \circ \mathrm{PD}$ is an isomorphism. Together with the remark on page 2657 this discussion implies that $Q^{\mathrm{as}}_{M,A,B}$ is non-singular.

(2) This statement follows easily from the definitions together with the Cap Product-Naturality Lemma 116.8 and the Cup Product-Naturality Proposition 115.4. In fact we will work out the details in the proof of Lemma 132.16 (2).

(3) Let $i \in \{1, \ldots, m\}$. We write $p_*(v_i) = \sum_{j=1}^{m} b_{ji} \cdot v_j^*$. Let $j \in \{1, \ldots, m\}$. We see that

$$b_{ji} \underset{\substack{\uparrow \\ \text{since } Q^{\mathrm{as}}_{M,\varnothing,\partial M}(v_j, v_k^*) = \delta_{jk}}}{=} Q^{\mathrm{as}}_{M,\varnothing,\partial M}\Big(v_j, \sum_{k=1}^{m} b_{ki} \cdot v_k^*\Big) = Q^{\mathrm{as}}_{M,\varnothing,\partial M}(v_j, p_*(v_i)) \underset{\substack{\uparrow \\ \text{follows from (2)}}}{=} Q_M(v_j, v_i).$$

Since by definition the matrix (b_{ij}) represents the map p_* with respect to the given bases we obtain the desired result. ∎

In the remainder of this section we prove several properties of the asymmetric intersection pairings.

Lemma 132.16. Let M be a compact oriented connected n-dimensional topological manifold and suppose that we are given two compact $(n-1)$-dimensional submanifolds A and B of ∂M with $A \cap B = \partial A = \partial B$. The following two statements hold.

(1) If we are given $v \in \mathrm{H}_k(M, A; \mathbb{Z})$ and $w \in \mathrm{H}_{n-k}(M; \mathbb{Z})$, then

$$Q^{\mathrm{as}}_{M,A,B}(v, w) = (-1)^{k\cdot(n-k)} \cdot Q^{\mathrm{as}}_{M,B,A}(w, v).$$

(2) Suppose we are given two compact $(n-1)$-dimensional submanifolds A' and B' of ∂M that also satisfy (i) or (ii) and that furthermore satisfy $A \subset A'$ and $B \subset B'$. We denote by $p\colon (M, A) \to (M, A')$ and $q\colon (M, B) \to (M, B')$ the obvious maps. For any $v \in \mathrm{H}_k(M, A'; \mathbb{Z})$ and $w \in \mathrm{H}_{n-k}(M, B'; \mathbb{Z})$ we have the following equality:

$$Q^{\mathrm{as}}_{M,A',B'}(p_*(v), q_*(w)) = Q^{\mathrm{as}}_{M,A,B}(v, w).$$

Proof. We write $X = \partial M \setminus \mathring{A}$ and $Y = \partial M \setminus \mathring{B}$.

(1) Given $v \in \mathrm{H}_k(M, A; \mathbb{Z})$ and $w \in \mathrm{H}_{n-k}(M, B; \mathbb{Z})$ we have the following equality:

$$\overbrace{\Big\langle \underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{H}^{n-k}(M,X;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{H}^{k}(M,Y;\mathbb{Z})}, [M] \Big\rangle}^{= Q^{\mathrm{as}}_{M,A,B}(v,w)} \underset{\substack{\uparrow \\ \text{by the Cup Product-Commutativity Proposition 115.3}}}{=} (-1)^{(n-k)\cdot k} \cdot \overbrace{\Big\langle \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{H}^{k}(M,Y;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{H}^{n-k}(M,X;\mathbb{Z})}, [M] \Big\rangle}^{= Q^{\mathrm{as}}_{M,B,A}(w,v)}.$$

(2) Not surprisingly we write $X' = \partial M \setminus \mathring{A}'$ and $Y' = \partial M \setminus \mathring{B}'$. Note that by hypothesis we have $X' \subset X$ and $Y' \subset Y$. Let $f\colon (M, X') \to (M, X)$ and $g\colon (M, Y') \to (M, Y)$ be the obvious maps. Given $v \in \mathrm{H}_k(M, A'; \mathbb{Z})$ and $w \in \mathrm{H}_{n-k}(M, B'; \mathbb{Z})$ we have the

following equality:

$$\overbrace{\Big\langle \underbrace{\mathrm{PD}_M(p_*(v))}_{\in \mathrm{H}^{n-k}(M,X';\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(q_*(w))}_{\in \mathrm{H}^{k}(M,Y';\mathbb{Z})}, [M] \Big\rangle}^{=Q^{\mathrm{as}}_{M,A',B'}(p_*(v),q_*(w))} \quad \begin{array}{l}\text{follows from the Cap Product-Naturality Lemma 116.8}\\ \downarrow \\ = \Big\langle \underbrace{f^*(\mathrm{PD}_M(v))}_{\in \mathrm{H}^{n-k}(M,X';\mathbb{Z})} \cup \underbrace{g^*(\mathrm{PD}_M(w))}_{\in \mathrm{H}^{k}(M,Y';\mathbb{Z})}, [M] \Big\rangle\end{array}$$

$$\begin{array}{l}\text{Cup Product-Naturality Proposition 115.4}\\ \downarrow \\ = \Big\langle \underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{H}^{n-k}(M,X;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{H}^{k}(M,Y;\mathbb{Z})}, [M] \Big\rangle \end{array} \quad = \quad Q^{\mathrm{as}}_{M,A,B}(v,w).$$

■

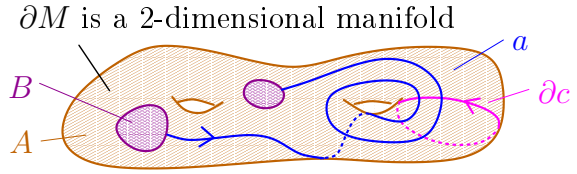
Our next result on asymmetric intersection pairing deals with the interplay between the asymmetric intersection pairing on a topological manifold and the asymmetric intersection pairings on its boundary.

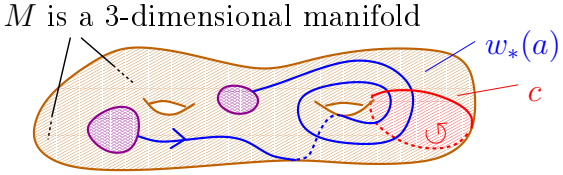
Proposition 132.17. (Interaction Pairing-Boundary Proposition) Let M be a compact oriented n-dimensional topological manifold. Suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. Let $a \in \mathrm{H}_k(A, \partial A; \mathbb{Z})$ and let $c \in \mathrm{H}_{n-k+1}(M, A; \mathbb{Z})$. We denote by $w\colon (A, \partial A) \to (M, B)$ the obvious map and we denote by $\partial\colon \mathrm{H}_{n-k+1}(M, A; \mathbb{Z}) \to \mathrm{H}_{n-k}(A; \mathbb{Z})$ the connecting homomorphism of the long exact sequence in homology of the pair (M, A). In this setting the following equality holds:

$$Q^{\mathrm{as}}_{A,\varnothing,\partial A}(\partial c, a) \;=\; Q^{\mathrm{as}}_{M,A,B}(c, w_*(a)).$$

In other words, the following diagram commutes in a suitable sense:

$$\begin{array}{ccc} \mathrm{H}_{n-k+1}(M, A; \mathbb{Z}) \times \mathrm{H}_k(M, B; \mathbb{Z}) & & \\ \Big\downarrow \partial \qquad \Big\uparrow w_* & \xrightarrow{\;Q^{\mathrm{as}}_{M,A,B}\;} & \mathbb{Z}. \\ \mathrm{H}_{n-k}(A; \mathbb{Z}) \times \mathrm{H}_k(A, \partial A; \mathbb{Z}) & \xrightarrow[\;Q^{\mathrm{as}}_{A,\varnothing,\partial A}\;]{} & \end{array}$$





In the proof of the Interaction Pairing-Boundary Proposition 132.17 we will need the following generalization of the Cup Product-Connecting Homomorphism Lemma 115.6.

Lemma 132.18. Let (X, A, B) be an excisive triad of topological spaces and let R be a commutative ring. We denote by $i\colon (A, A \cap B) \to (X, B)$ and $j\colon (A, A \cap B) \to (A \cup B, B)$ the inclusion maps and we denote by δ the connecting homomorphisms in the long exact sequence in cohomology corresponding to the pair (X, A) and the triple $(X, A \cup B, B)$. Note that it follows from the Excisive Characterization Lemma 109.13 (b) that $j^*\colon \mathrm{H}^{p+q}(A \cup$

$B, B; R) \to \mathrm{H}^{p+q}(A, A \cap B; R)$ is an isomorphism. The following diagram commutes

$$\begin{array}{ccc}
\mathrm{H}^p(A, A\cap B; R) \times \mathrm{H}^q(A; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(A, A\cap B; R) \\
\uparrow{\scriptstyle i^*} & & \cong\downarrow{\scriptstyle (j^*)^{-1}} \\
 & \downarrow{\scriptstyle \delta} & \mathrm{H}^{p+q}(A\cup B, B; R) \\
 & & \downarrow{\scriptstyle \delta} \\
\mathrm{H}^p(X, B; R) \times \mathrm{H}^{q+1}(X, A; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q+1}(X, A\cup B; R)
\end{array}$$

up to the sign $(-1)^p$. Recall that according to the convention on page 2538 "commutes" means that if we start with an element in $\mathrm{H}^p(X, B; R)$ and an element in $\mathrm{H}^q(A; R)$, then the two routes to $\mathrm{H}^{p+q+1}(X, A \cup B; \mathbb{Z})$ lead to the same element up to multiplication by $(-1)^p$.

Proof of Lemma 132.18. We leave it to the long-suffering reader to go through the slightly painful task of modifying the proof of the Cup Product-Connecting Homomorphism Lemma 115.6 to get the desired result. ■

Proof of the Interaction Pairing-Boundary Proposition 132.17. We consider the following diagram of maps

$$\begin{array}{ccccc}
\mathrm{H}^k(A, \partial A; \mathbb{Z}) \times \mathrm{H}^{n-k}(A; \mathbb{Z}) & \xrightarrow{\cup} & \mathrm{H}^n(A, \partial A; \mathbb{Z}) & \xrightarrow{\langle -, [A]\rangle} & \\
\uparrow{\scriptstyle w^*} & & \cong\downarrow{\scriptstyle (u^*)^{-1}} & & \\
 & \downarrow{\scriptstyle \delta} & \mathrm{H}^n(\partial M, B; \mathbb{Z}) & \xrightarrow{\langle -, p_*([\partial M])\rangle} & \mathbb{Z}. \\
 & & \downarrow{\scriptstyle \delta} & & \\
\mathrm{H}^k(M, B; \mathbb{Z}) \times \mathrm{H}^{n-k+1}(M, A; \mathbb{Z}) & \xrightarrow{\cup} & \mathrm{H}^{n+1}(M, \partial M) & \xrightarrow{\langle -, [M]\rangle} &
\end{array}$$

We make the following clarifications and observations:

(1) By the Excisive Triad-Proposition 109.12 we know that (M, A, B) is an excisive triad.
(2) We denote by $u\colon (A, \partial A) \to (\partial M, B)$ the inclusion map. It follows from (1) and the Excisive Characterization Lemma 109.13 (b) that the map u induces an isomorphism on cohomology groups.
(3) We denote by $\delta\colon \mathrm{H}^n(A; \mathbb{Z}) \to \mathrm{H}^{n+1}(M, A; \mathbb{Z})$ and $\delta\colon \mathrm{H}^n(\partial M, B; \mathbb{Z}) \to \mathrm{H}^{n+1}(M, \partial M; \mathbb{Z})$ the connecting homomorphisms in cohomology of the pair (M, A) and of the triple $(M, \partial M, B)$.
(4) It follows from (1) and Lemma 132.18 that the left part of the diagram commutes up to the sign $(-1)^k$.
(5) We denote by $p\colon (\partial M, \varnothing) \to (\partial M, B)$ the obvious map. By the Codimension Zero-Fundamental Class Lemma 106.24 we have $p_*([\partial M]) = u_*([A]) \in \mathrm{H}_n(\partial M, B; \mathbb{Z})$. Together with Lemma 109.6 (3) it follows that the top right triangle commutes.
(6) Note that by the Fundamental Class-of-Boundary Proposition 106.27 we know that $\partial([M]) = [\partial M] \in \mathrm{H}_n(\partial M; \mathbb{Z})$. Together with the Cap Product-Connecting Homomorphism Lemma 116.10 this implies that the bottom right triangle commutes.

Next we consider the following two diagrams:

$$\begin{array}{ccc} \mathrm{H}_k(M,A;\mathbb{Z}) & \xrightarrow{\partial} & \mathrm{H}_k(A;\mathbb{Z}) \\ {\scriptstyle \mathrm{PD}_M}\downarrow & & \downarrow{\scriptstyle \mathrm{PD}_A} \\ \mathrm{H}^{n-k}(M,B;\mathbb{Z}) & \xrightarrow{w^*} & \mathrm{H}^{n+1-k}(A,\partial A;\mathbb{Z}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{H}_k(A,\partial A;\mathbb{Z}) & \xrightarrow{w_*} & \mathrm{H}_k(M,B;\mathbb{Z}) \\ {\scriptstyle \mathrm{PD}_A}\downarrow & & \downarrow{\scriptstyle \mathrm{PD}_M} \\ \mathrm{H}^{n-k}(A;\mathbb{Z}) & \xrightarrow{\delta} & \mathrm{H}^{n-k+1}(M,A;\mathbb{Z}). \end{array}$$

By Proposition 119.26 we know that the diagram to the left commutes up to the factor $(-1)^k$ whereas the diagram to the right commutes as it is. With all these preparations we can now easily carry out the desired calculation:

$$\begin{aligned} Q_{A,\varnothing,\partial A}(\partial c, a) &\overset{\text{by definition of } Q_A}{\downarrow} \\ &= \langle \mathrm{PD}_A(\partial c) \cup \mathrm{PD}_A(a), [A] \rangle \\ &= \langle \mathrm{PD}_M(c) \cup \delta(\mathrm{PD}_A(a)), [M] \rangle \\ &\overset{\text{since the first diagram commutes up to multiplication by } (-1)^k}{\uparrow} \\ &= Q^{\mathrm{as}}_{M,A,B}(c, w_*(a)). \end{aligned} \qquad \begin{aligned} &\overset{\text{since the second diagram commutes up to multiplication by } (-1)^k}{\downarrow} \\ &= (-1)^k \cdot \langle w^*(\mathrm{PD}_M(c)) \cup \mathrm{PD}_A(a), [A] \rangle \\ &= \langle \mathrm{PD}_M(c) \cup \mathrm{PD}_M(w_*(a)), [M] \rangle \\ &\overset{\text{since the third diagram commutes}}{\uparrow} \end{aligned}$$

■

The following proposition relates the asymmetric intersection pairing on a topological manifold to the asymmetric intersection pairing on a codimension-zero submanifold.

Proposition 132.19. (Intersection Form-Submanifold Proposition) Let M be a compact oriented m-dimensional topological manifold and let $W \subset M$ be a compact codimension-zero submanifold with corner.[1619] Let A and B be disjoint unions of components of ∂M with $\partial_1 W \subset A$. We denote by $i\colon (W, \partial_1 W) \to (M, A)$, $w\colon (W, \partial_0 W) \to (M, M \setminus \mathring{W})$ and $p\colon (M, B) \to (M, M \setminus \mathring{W})$ the obvious maps. For every $\alpha \in \mathrm{H}_k(W, \partial_1 W)$ and $\beta \in \mathrm{H}_{n-k}(M, B)$ we have the following equality:[1620]

$$Q^{\mathrm{as}}_{W,\partial_1 W,\partial_0 W}(\alpha, (w_*)^{-1}(p_*(\beta))) \;=\; Q^{\mathrm{as}}_{M,A,B}(i_*(\alpha), \beta).$$

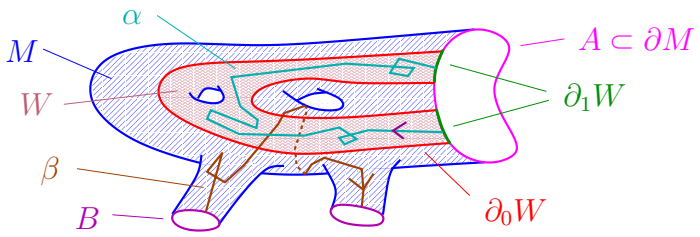
In other words, the following diagram commutes in a suitable sense:

$$\begin{array}{ccc} \mathrm{H}_k(W,\partial_1 W;\mathbb{Z}) \times \mathrm{H}_{n-k}(W,\partial_0 W;\mathbb{Z}) & & \\ \Big\downarrow{\scriptstyle i_*} & {\scriptstyle w_*}\downarrow\cong & \searrow^{Q^{\mathrm{as}}_{W,\partial_1 W,\partial_0 W}} \\ & \mathrm{H}_{n+k}(M, M \setminus \mathring{W};\mathbb{Z}) & \to \mathbb{Z}. \\ & {\scriptstyle p_*}\uparrow & \nearrow_{Q^{\mathrm{as}}_{M,A,B}} \\ \mathrm{H}_k(M,A;\mathbb{Z}) \times \mathrm{H}_{n-k}(M,B;\mathbb{Z}) & & \end{array}$$

The analogous statement also holds with the roles of the first and the second entry swapped.

[1619] We refer to Lemma 120.1 for the definition of $\partial_0 W$ and $\partial_1 W$.

[1620] Here we use that we know by the Excision Theorem 120.2 that the map $w\colon (W, \partial W) \to (M, M \setminus \mathring{W})$ induces an isomorphism on homology groups.



Proof. We perform the following calculation:

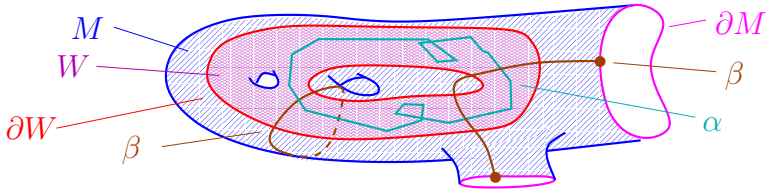
$$\begin{aligned}
Q^{\mathrm{as}}_{W,\partial_1 W,\partial_0 W}(\alpha,(w_*)^{-1}(p_*(\beta))) &= \langle \mathrm{PD}_W(\alpha)\cup \mathrm{PD}_W((w_*)^{-1}(p_*(\beta))),[W]\rangle \\
&= \langle w^*((w^*)^{-1}(\mathrm{PD}_W(\alpha))\cup w^*(\mathrm{PD}_M(\beta)),[W]\rangle \underset{\uparrow}{=} \langle (w^*)^{-1}(\mathrm{PD}_W(\alpha))\cup \mathrm{PD}_M(\beta), w_*([W])\rangle \\
&\quad \uparrow \text{ by Lemma 120.6} \qquad\qquad \uparrow \text{ by Lemma 109.6 (3)} \\
&= \langle (w^*)^{-1}(\mathrm{PD}_W(\alpha))\cup \mathrm{PD}_M(\beta), p_*([M])\rangle \underset{\uparrow}{=} \langle p^*((w^*)^{-1}(\mathrm{PD}_W(\alpha)))\cup p^*(\mathrm{PD}_M(\beta)),[M]\rangle \\
&\quad \uparrow \text{ by the Codimension Zero-Fundamental Class Lemma 106.24 we have } w_*([W])=p_*([M]) \qquad \uparrow \text{ Lemma 109.6 (3)} \\
&= \langle \mathrm{PD}_M(i_*(\alpha)),\mathrm{PD}_M(\beta),[M]\rangle \;=\; Q^{\mathrm{as}}_{M,A,B}(i_*(\alpha),\beta). \\
&\quad \uparrow \text{ by the Codimension-Zero Submanifold-PD Lemma 120.5}
\end{aligned}$$

■

The case we will use most often is the special case that $\partial_0 W = \partial W$, $\partial_1 W = \varnothing$, $A = \varnothing$ and $B = \partial M$. In this case Proposition 132.20 specializes to the following slightly more readable proposition.

Proposition 132.20. Let M be a compact oriented m-dimensional topological manifold and let $W \subset M\backslash\partial M$ be a compact m-dimensional submanifold. We denote by $i\colon W \to M$, $w\colon (W,\partial W) \to (M, M\setminus \mathring{W})$ and $p\colon (M,\partial M) \to (M, M\setminus \mathring{W})$ the obvious maps. For every $\alpha \in \mathrm{H}_k(W)$ and $\beta \in \mathrm{H}_{n-k}(M,\partial M)$ we have the following equality:

$$Q^{\mathrm{as}}_{W,\varnothing,\partial W}(\alpha,(w_*)^{-1}(p_*(\beta))) \;=\; Q^{\mathrm{as}}_{M,\varnothing,\partial M}(i_*(\alpha),\beta).$$



132.8. **The reduced intersection form.** After the exhaustive and exhausting discussion of the asymmetric intersection pairings in the previous section, let us now move on to our second variation on the intersection form.

Proposition 132.21. (Reduced Intersection Form Proposition) Let M be a compact oriented $2n$-dimensional topological manifold. We denote by $i\colon \partial M \to M$ the inclusion map.

(1) Let $\alpha \in \mathrm{FH}_n(M;\mathbb{Z})$. The following two statements are equivalent:
   (a) $\alpha \in \mathrm{im}(i_*\colon \mathrm{FH}_n(\partial M;\mathbb{Z}) \to \mathrm{FH}_n(M;\mathbb{Z}))$.
   (b) For every $\beta \in \mathrm{FH}_n(M;\mathbb{Z})$ we have $Q_M(\alpha,\beta) = 0$.

(2) We write $\widehat{\mathrm{H}}_n(M;\mathbb{Z}) := \operatorname{coker}(i_*\colon \mathrm{H}_n(\partial M;\mathbb{Z}) \to \mathrm{H}_n(M;\mathbb{Z}))$. The form

$$\begin{array}{rccc} Q_M^{\mathrm{red}}\colon & \mathrm{F}\widehat{\mathrm{H}}_n(M;\mathbb{Z}) \times \mathrm{F}\widehat{\mathrm{H}}_n(M;\mathbb{Z}) & \to & \mathbb{Z} \\ & ([\varphi],[\psi]) & \mapsto & Q_M(\varphi,\psi) \end{array}$$

is well-defined and non-degenerate. We refer to this form as the reduced intersection form of M.

(3) If the intersection form Q_M is non-degenerate, then the obvious map $\mathrm{FH}_n(M;\mathbb{Z}) \to \mathrm{FH}_n(M;\mathbb{Z})$ is an isomorphism and it induces an isometry $Q_M \to Q_M^{\mathrm{red}}$.

Proof. We denote by $i\colon \partial M \to M$ and $j\colon (M,\varnothing) \to (M,\partial M)$ the obvious maps. Before we do anything else we recall that we have a long exact sequence

$$\ldots \to \mathrm{H}_n(\partial M;\mathbb{Z}) \xrightarrow{i_*} \mathrm{H}_n(M;\mathbb{Z}) \xrightarrow{j_*} \mathrm{H}_n(M,\partial M;\mathbb{Z}) \xrightarrow{\partial} \mathrm{H}_{n-1}(\partial M) \to \ldots$$

Now we turn to the actual proofs of the statements.

(1) Let $\alpha \in \mathrm{H}_n(M;\mathbb{Z})$. First we prove "(a)⇒(b)". Thus we assume that $\alpha = i_*(\gamma)$ for some $\gamma \in \mathrm{H}_n(\partial M;\mathbb{Z})$. Let $\beta \in \mathrm{H}_n(M;\mathbb{Z})$. We perform the following calculation:

$$\begin{aligned} Q_M(\alpha,\beta) &\underset{\text{by definition of } Q_M}{=} \langle \mathrm{PD}(i_*(\gamma)) \cup \mathrm{PD}(\beta), [M]\rangle \underset{\text{Cup-Cap Lemma 116.7 (3)}}{=} \langle \mathrm{PD}(\beta), j_*(\underbrace{\mathrm{PD}(i_*(\gamma)) \cap [M]}_{= i_*(\gamma) \text{ by definition of PD}})\rangle \\ &= \langle \mathrm{PD}(\beta), j_*(i_*(\gamma))\rangle \underset{\text{since } j_* \circ i_* = 0}{=} 0. \end{aligned}$$

Now we prove "(b)⇒(a)". Thus we assume that $Q_M(\alpha,\beta) = 0$ for all $\beta \in \mathrm{FH}_n(M;\mathbb{Z})$. This implies for any $\beta \in \mathrm{FH}_n(M;\mathbb{Z})$ we have the following equalities:

$$Q^{\mathrm{as}}_{M,\partial M,\varnothing}(j_*(\alpha),\beta) \underset{\substack{\text{by the Asymmetric Intersection} \\ \text{Pairing Proposition 132.15 (2)}}}{=} Q_M(\alpha,\beta\rangle \underset{\text{by assumption}}{=} 0.$$

By the Asymmetric Intersection Pairing Proposition 132.15 (1) we know that the pairing $Q^{\mathrm{as}}_{M,\partial M,\varnothing}$ is non-singular, in particular it is non-degenerate. Thus it follows from the above, together with Lemma 132.14, that $j_*(\alpha) = 0 \in \mathrm{FH}_n(M;\mathbb{Z})$. It follows from the above long exact sequence, together with Exercise 89.2, that $\operatorname{im}(i_*\colon \mathrm{FH}_n(\partial M;\mathbb{Z}) \to \mathrm{FH}_n(M;\mathbb{Z}))$.

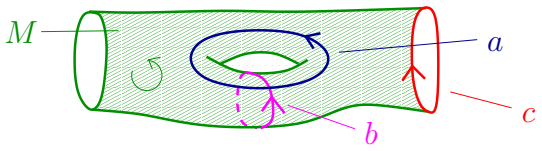
(2) It follows immediately from (1) "(a)⇒(b)" that the form Q_M^{red} is well-defined. Furthermore it follows from (1) "(b)⇒(a)" and Lemma 132.14 that Q_M^{red} is non-degenerate.

(3) This statement follows immediately from (1) and Lemma 132.14. ∎

Examples.

(1) One last time we consider again our favorite example M, namely the torus minus two open disks. We use the notation from the figure on page 2903. A basis for $\mathrm{H}_1(M;\mathbb{Z})$ is given by a,b,c and the image of $\mathrm{H}_1(\partial M;\mathbb{Z}) \to \mathrm{H}_1(M;\mathbb{Z})$ is given by $\mathbb{Z}\cdot c$. With respect to these bases we see that

$$Q_M \text{ is represented by } \begin{array}{c} \\ a \\ b \\ c \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \text{ and } Q_M^{\mathrm{red}} \text{ is represented by } \begin{array}{c} \\ a \\ b \end{array}\!\!\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array}.$$



(2) In Exercise 132.7 we will provide an example of a compact orientable 4-dimensional smooth manifold M for which the reduced intersection form is not non-singular. We will obtain many more such examples later on in Proposition **??**.

Exercises for Chapter 132.

Exercise 132.1. We equip $\mathrm{H}_1(S^1 \times S^1; \mathbb{Z})$ with the standard basis $\mu = [S^1 \times \{*\}]$ and $\lambda = [\{*\} \times S^1]$. Show that

$$Q_{S^1 \times S^1}(a \cdot \mu + b \cdot \lambda, c \cdot \mu + d \cdot \lambda) \;=\; \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Remark. The exercise is really quite easy, but it does produce a neat formula.

Exercise 132.2. Let M be a compact oriented topological manifold. Recall that we say that M is *amphichiral* if M admits an orientation-reversing self-homeomorphism, otherwise we say that M is *chiral*. For which $k, l \in \mathbb{N}_0$ is the connected sum $k \cdot \mathbb{CP}^2 \# l \cdot \overline{\mathbb{CP}}^2$ chiral? More precisely:

(a) for the values that you exclude, show that $k \cdot \mathbb{CP}^2 \# l \cdot \overline{\mathbb{CP}}^2$ is not chiral,
(b) for all the other values show explicitly that $k \cdot \mathbb{CP}^2 \# l \cdot \overline{\mathbb{CP}}^2$ is chiral.

Exercise 132.3. Let $n \in \mathbb{N}$ and let M and M' be two closed oriented connected non-empty n-dimensional topological manifolds. Let $X \subset M$ and $X' \subset M'$ be submanifolds that are homeomorphic to $\overline{B}^n$ and let $\varphi\colon \partial X \to \partial X'$ be an orientation-reserving homeomorphism. Show that $(M \setminus \mathring{X}) \cup_{\varphi\colon \partial X \to \partial X'} (M \setminus \mathring{X}')$ is homeomorphic to the connected sum $M \# N$.
Hint. Use the Alexander trick from Exercise 5.35.
Remark. Initially it might sound like this statement holds "by definition". But in fact in Exercise **??** we will see that the obvious analogue in the setting of smooth manifolds does *not* hold.

Exercise 132.4. Let R be a field and let $\varphi\colon V \times V \to R$ be a form. The vector subspace $R := \{v \in V \,|\, \varphi(v, w) = 0 \text{ for all } w \in V\}$ is called the *radical* of the form φ.

(a) Show that the map

$$\begin{array}{rcl} \overline{\varphi}\colon V/R \times V/R & \to & R \\ ([v], [w]) & \mapsto & \varphi(v, w) \end{array}$$

is well-defined and non-singular.
(b) Let S be a vector subspace of V with $V = R \oplus S$. Show that $(V/R, \overline{\varphi})$ is isometric to $(S, \varphi|_S)$.
(c) Show that (V, φ) is isometric to $(R, 0) \oplus (V/R, \overline{\varphi})$.

Remark. Recall that by our definition on page 2877 a form (V, φ) is definition symmetric or anti-symmetric and V is a finite-dimensional free module over the ring R.

Exercise 132.5. Let H be a finite-dimensional real vector space and let $\lambda\colon H \times H \to \mathbb{R}$ be a non-singular form. Furthermore let $A_1, A_2 \subset H$ be two vector subspaces with $A_1^\perp = A_2$ and $A_2^\perp = A_1$. Show that $(A_1 \cap A_2)^\perp = A_1 + A_2$.

Exercise 132.6. Let M be a compact oriented 4-dimensional topological manifold and let $i\colon \partial M \to M$ be the inclusion map.

(a) Show that if $i_*\colon \mathrm{H}_1(\partial M;\mathbb{Z}) \to \mathrm{H}_1(M;\mathbb{Z})$ is a monomorphism, then the intersection form Q_M is non-singular.

(b) Show that if $i_*\colon \mathrm{H}_1(\partial M;\mathbb{Q}) \to \mathrm{H}_1(M;\mathbb{Q})$ is a monomorphism, then the intersection form Q_M is non-degenerate.

Exercise 132.7. Let $n \in \mathbb{N}$. We consider the map

$$\begin{array}{rcl} \varphi\colon \overline{B}^2 \times S^1 & \to & \overline{B}^2 \times S^1 \subset \partial(\overline{B}^2 \times \overline{B}^2) \\ (z,w) & \mapsto & (z, w \cdot z^n). \end{array}$$

Furthermore we consider [1621]

$$M := (\overline{B}^2 \times \overline{B}^2) \underset{\substack{\overline{B}^2 \times S^1 \to \partial(\overline{B}^2 \times \overline{B}^2) \\ (z,w) \mapsto \varphi(z,w)}}{} (\overline{B}^2 \times \overline{B}^2).$$

One can show by hand that M is a compact orientable 4-dimensional smooth manifold.[1622]

(a) Show that $\pi_1(M) = 0$, $\mathrm{H}_2(M;\mathbb{Z}) \cong \mathbb{Z}$ and $\mathrm{H}_2(M, \partial M;\mathbb{Z}) \cong \mathbb{Z}$.

(b) Show that $\mathrm{H}_1(\partial M;\mathbb{Z}) \cong \mathbb{Z}_n$ and $\mathrm{H}_2(\partial M;\mathbb{Z}) = 0$. (In fact very well-trained eyes will spot that ∂M is diffeomorphic to the lens space $L(n,1)$.)

(c) Show that if $n \geq 2$, then the intersection form Q_M is represented by one of the two 1×1-matrices $(-n)$ or (n).
Hint. Use the Asymmetric Intersection Pairing Proposition 132.15 (3).
Remark. Note that it follows from (b) that the intersection form equals the reduced intersection form. Thus we have shown, as promised on page 2903, that there exists a compact orientable 4-dimensional smooth manifold M for which the reduced intersection form is not non-singular.

Exercise 132.8. Let M be a compact oriented n-dimensional topological manifold and let A, B be disjoint unions of components of ∂M. Furthermore let V and W be compact codimension-zero submanifolds of M as in the figure below and let $\alpha \in \mathrm{H}_k(M, A)$ and $\beta \in \mathrm{H}_{n-k}(X, B)$.

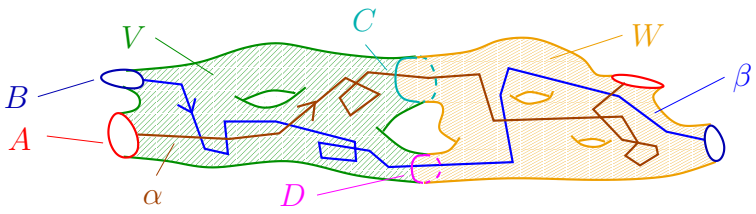
(a) Inspired by the figure below state a formula which computes $Q_M(\alpha, \beta)$ in terms of suitable intersection numbers of V and W.
Hint. What is the role of C and D in the figure below?

(b) Prove (a).

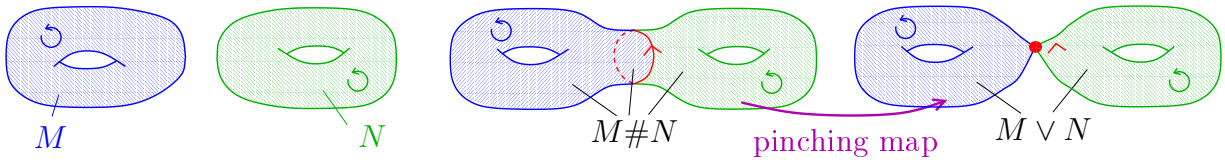
Exercise 132.9. Let $n \in \mathbb{N}$ and let M and N be two compact oriented connected non-empty n-dimensional smooth manifolds. In Exercise 100.16 we introduced the pinching map $p\colon M \# N \to M \vee N$ and studied the induced map on homology groups. Use the pinching

[1621] Later on page 2953 we will say that M is obtained from the 4-ball $\overline{B}^2 \times \overline{B}^2$ by attaching a 2-handle.

[1622] We will carry out the details in the Handle Attachment Proposition 136.1, but this is irrelevant for the present exercise.



map and the Wedge-Cup Product Lemma 115.8 to give a new proof of the Intersection Form-Additivity Proposition 132.12 for smooth manifolds. In other words, show that the intersection form $Q_{M\#N}$ is isometric to $Q_M \# Q_N$.



133. The intersection form II: Algebra and some topology

In Chapter 132 we introduced the intersection form of compact oriented even-dimensional topological manifolds. On page 2893 we considered the intersection forms of $S^2 \times S^2$, $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$. At that point it became clear that we have to invest some time to study (anti-) symmetric forms as purely algebraic objects.
We start out with the following lemma that summarizes all the basic facts about the interplay between forms and matrices that we will need in this chapter.

Lemma 133.1. (Forms-Matrices Lemma) Let R be a commutative ring.

(1) Let A be a (anti-) symmetric[1623] $n \times n$–matrix over R. We consider the map

$$\begin{array}{rccc} \varphi(A)\colon & R^n \times R^n & \to & R \\ & (v,w) & \mapsto & v^T A w. \end{array}$$

The following statements hold:
(a) $\varphi(A)$ is an (anti-) symmetric form over R.
(b) The form $\varphi(A)$ is non-singular, in the sense of the definition on page 2878, if and only if A is invertible, i.e. if $\det(A)$ is a unit in R
(c) If $P \in \mathrm{GL}(n,R)$ is an invertible matrix, then the forms $\varphi(P^T A P)$ and $\varphi(A)$ are isometric.
(d) If B is a (anti-) symmetric $n \times n$-matrix over R such that $\varphi(A)$ and $\varphi(B)$ are isometric, then there exists a $P \in \mathrm{GL}(n,R)$ with $B = P^T A P$.

(2) Let (V,ψ) be an (anti-) symmetric form over R of rank n. If A is a matrix that represents ψ, then $(V,\psi) \cong (R^n, \varphi(A))$. In particular every (anti-) symmetric form over R is isometric to $\varphi(A)$ for some (anti-) symmetric $n \times n$–matrix A over R.

Convention. The Forms-Matrices Lemma 133.1 allows us to go back and forth between (anti-) symmetric forms and (anti-) symmetric matrices. Thus in the following discussion we will blur the distinction between these two concepts. For example, given a (anti-) symmetric matrix A over a commutative ring we will often just refer to the (anti-) symmetric form $\varphi(A)$ as the form A.

Proof.

(1) (a) This statement follows almost immediately from the definitions and the observation that for any $v, w \in R^n$ we have

$$\varphi(A)(w,v) \;=\; w^T A v \;\underset{\uparrow}{=}\; (w^T A^T v)^T \;=\; v^T A^T w \;=\; \varphi(A^T)(v,w).$$

for a (1×1)–matrix (x) we have $(x)^T = (x)$

(b) The proof of this statement is an instructive exercise in linear algebra. We will not rob the reader of the joy of doing this exercise.

(c) We claim that the isomorphism $v \mapsto Pv$ defines an isometry from the form $(R^n, \varphi(P^T A P))$ to the form $(R^n, \varphi(A))$. Indeed, for any $v, w \in R^n$ we have

$$\varphi(A)(Pv, Pw) \;=\; (Pv)^T A P w \;=\; v^T (P^T A P) w \;=\; \varphi(P^T A P)(v,w).$$

(d) Let B be a (anti-) symmetric $n \times n$-matrix over R and let $f\colon R^n \to R^n$ be an isometry between $\varphi(A)$ and $\varphi(B)$, i.e. let f be an isomorphism such that for all

[1623] Recall that a matrix A is called *anti-symmetric* if $A^T = -A$.

$v, w \in R^n$ we have $v^T A w = \varphi(A)(v,w) = \varphi(B)(f(v), f(w)) = f(v)^T A f(w)$. Let P be the matrix representing f with respect to the standard basis of R^n. Since f is an isomorphism we know that $P \in \mathrm{GL}(n,R)$. Finally a straightforward calculation shows that $B = P^T A P$.

(2) Let $\psi\colon V \times V \to R$ be an (anti-) symmetric form over R. Let $v_1, \dots, v_n$ be a basis of V. We denote by $A = (\psi(v_i, v_j))_{i,j=1,\dots,n}$ the corresponding matrix representing the form ψ. We consider the isomorphism

$$\begin{array}{rcl} \Theta\colon R^k & \to & V \\ (\lambda_1, \dots, \lambda_n) & \mapsto & \lambda_1 v_1 + \cdots + \lambda_n v_n. \end{array}$$

We claim that Θ is an isometry from the form $(R^n, \varphi(A))$ to the form (V, ψ). Indeed, if we denote by $e_1, \dots, e_n$ the standard basis for R^n, then for any $i, j \in \{1, \dots, n\}$ we have

$$\psi(\Theta(e_i), \Theta(e_j)) \;=\; \psi(v_i, v_j) \;=\; ij\text{-entry of } A \;=\; e_i^T A e_j \;=\; \varphi(A)(e_i, e_j).$$

By bilinearity of forms we obtain the corresponding statement for all pairs of elements in R^n. ■

The Forms-Matrices Lemma 133.1 (1b), (1d) and (2) shows that the following definition makes sense.

Definition. Let R be a commutative ring. We denote by R^* the multiplicative units of R. Let (V, ψ) be a non-singular (anti-) symmetric form over R of rank n. We pick a basis $v_1, \dots, v_n$ of V. We refer to

$$\det(\psi) \;:=\; \det(\psi(v_i, v_j)_{i,j=1,\dots,n}) \;\in\; R^*/\{a^2 \mid a \in R^*\}$$

as the determinant of the form (V, ψ). Note that for $R = \mathbb{Z}$ the determinant is a well-defined element of $\{\pm 1\}$.

Example.

(1) For a matrix $A \in \mathrm{GL}(n,R)$ we have of course $\det(\varphi(A)) = \det(A) \in R^*/\{a^2 \mid a \in R^*\}$.

(2) We work with $R = \mathbb{Z}$. We set $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since $\det(A) = 1$ and $\det(A) = -1$ we see that $\varphi(A)$ and $\varphi(B)$ are non-isometric forms over $\mathbb{Z}^2$.

133.1. **Non-singular anti-symmetric forms.** In this section we will completely classify non-singular anti-symmetric forms over the real numbers and over the integers.
First we consider the matrix

$$S \;:=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is anti-symmetric and it has determinant 1. From the Forms-Matrices Lemma 133.1 we obtain an anti-symmetric form $\varphi(S)$. The following proposition shows that $\varphi(S)$ is in some sense the only example of a non-singular anti-symmetric form over the real numbers or over the integers.

Proposition 133.2. (Anti-Symmetric Forms Proposition) If (V, φ) is a non-singular anti-symmetric form over $\mathbb{R}$ or over $\mathbb{Z}$, then there exists a $k \in \mathbb{N}_0$ such that[1624] $(V, \varphi) \cong$

$k \cdot \varphi(S)$, i.e. φ is represented by the $2k \times 2k$–matrix

$$\underbrace{\begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & -1 & 0 \end{pmatrix}}_{k \text{ diagonal blocks of } 2 \times 2\text{-matrices}}.$$

In particular the rank of V is even.

Remark. Propositions 132.4 and 133.2 imply in particular that the isometry type of the intersection of a surface of genus g is determined by the formal properties of an intersection form. In particular we did not need to go through the discussion on page 2881 if we just need to know the *isometry type* of the intersection form.

Sketch of proof. First we remark that throughout the proof on many occasions we use the Forms-Matrices Lemma 133.1 without saying so explicitly.

After this preamble, let (V, φ) be a non-singular anti-symmetric form over $\mathbb{R}$ of rank n. We pick an $(n \times n)$–matrix A that represents φ. Since φ is anti-symmetric we see that all terms of A on the diagonal are zero. In the following we consider the usual three types of elementary matrices:

$$D_\lambda := \begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}, \lambda \neq 0, \quad P := \begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix} \quad \text{and} \quad Q_\lambda := \begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \lambda & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}.$$

We make the following elementary observations:

(i) Replacing a matrix B by $D_\lambda^T B D_\lambda$ means that first we multiply a column by λ, and then we multiply the corresponding row by λ.

(ii) Replacing a matrix B by $P^T B P$ means that first we exchange two columns, and then we exchange the corresponding two rows.

(iii) Replacing a matrix B by $Q_\lambda^T B Q_\lambda$ means that first we add λ-times one column to another column, and then we do the same operation for the corresponding rows.

Now we can perform the following sequence of operations to A:

$$A \overset{(1)}{=} \begin{pmatrix} 0 & * & * & \dots & * \\ * & 0 & * & \dots & * \\ * & * & 0 & * & * \\ \vdots & \vdots & * & \ddots & * \\ * & * & * & * & 0 \end{pmatrix} \overset{(2)}{\to} \begin{pmatrix} 0 & 1 & * & \dots & * \\ -1 & 0 & * & \dots & * \\ * & * & 0 & * & * \\ \vdots & \vdots & * & \ddots & * \\ * & * & * & * & 0 \end{pmatrix} \overset{(3)}{\to} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & * & \dots & * \\ 0 & * & 0 & * & * \\ \vdots & \vdots & * & \ddots & * \\ 0 & * & * & * & 0 \end{pmatrix} \overset{(4)}{\to} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & * & * \\ \vdots & \vdots & * & \ddots & * \\ 0 & 0 & * & * & 0 \end{pmatrix}$$

where we perform the following steps:

(1) We write out A, here and throughout the steps we use the convention that $*$ can mean any real number, the only condition being that all matrices are anti-symmetric.

(2) Since $\det(A) \neq 0$ there exists at least one entry on the first row which is non-zero. After operations of type (ii) and (i) we can arrange that the $(1,2)$-entry equals 1. By the anti-symmetry we see that the $(2,1)$-entry now equals -1.

[1624] Here, when we write $k \cdot \varphi(S)$ we use the definition from page 2877.

(3) Using operations of type (iii) we can "clean up" the remaining terms on the first row and column.
(4) Similarly, we can use operations of type (iii) to "clean up" the remaining terms on the second row and column.

At this stage we can continue with the same process, applied to the $(n-2)\times(n-2)$–matrix at the right bottom. Eventually we will have turned A into a matrix of the promised type.

Now suppose that (V,φ) is a non-singular anti-symmetric form over $\mathbb{Z}$. In Exercise 133.7 we will modify the above proof to deal with this new situation. ■

In a sense the Anti-Symmetric Forms Proposition 133.2 is rather disappointing. We had hoped that intersection forms will give us new meaningful invariants. But the Anti-Symmetric Forms Proposition 133.2 together with the Intersection Form Proposition 132.4 shows that intersection forms of closed even-dimensional topological manifolds do not give us any extra information that would allow us to distinguish. manifolds.

Nonetheless, the Anti-Symmetric Forms Proposition 133.2 has just enough content to answer Question 119.8 where we had asked which integers appear as Euler characteristics of closed orientable connected even-dimensional topological manifolds.[1625] Now we can provide a complete answer to this question.

Proposition 133.3.

(1) Let $k \in \mathbb{N}$. Given any $n \in \mathbb{Z}$ there exists a closed orientable connected $4k$-dimensional smooth manifold with Euler characteristic n.
(2) (a) The Euler characteristic of any closed orientable connected topological manifold of dimension 2 is even and ≤ 2. Conversely, any even number ≤ 2 is the Euler characteristic of such a smooth manifold.
(b) Let $k \in \mathbb{N}$. The Euler characteristic of any closed orientable connected topological manifold of dimension $4k+2$ is even. Conversely, any even number is the Euler characteristic of such a smooth manifold.

Remark. In Exercise 140.2 we will deal with the possible Euler characteristics of smooth manifolds with possibly non-empty boundary.

Proof. We start out with two claims that give us the stated restrictions on Euler characteristics of closed orientable topological manifolds.

Claim. Let $k \in \mathbb{N}_0$. The Euler characteristic of every closed orientable topological manifold M of dimension $4k+2$ is even.

Proof. We have

$$\chi(M) \underset{\uparrow}{=} \sum_{i=0}^{4k+2}(-1)^i\cdot\dim_{\mathbb{R}}(\mathrm{H}_i(M;\mathbb{R})) \underset{\uparrow}{=} \dim_{\mathbb{R}}(\mathrm{H}_{2k+1}(M;\mathbb{R})) + \sum_{i=0}^{2k}(-1)^i\cdot\overbrace{2\cdot\dim_{\mathbb{R}}(\mathrm{H}_i(M;\mathbb{R}))}^{\text{even}}$$

Manifold-χ-Proposition 104.36 (4); by the Co-Homology Symmetry Theorem Theorem 119.6 we have $\dim_{\mathbb{R}}(\mathrm{H}_i(M;\mathbb{R})) = \dim_{\mathbb{R}}(\mathrm{H}_{4k+2-i}(M;\mathbb{R}))$

$$\underset{\uparrow}{\equiv} 0 \mod 2.$$

by the Intersection Form Proposition 132.4 and the facts that M is closed and that $2k+1$ is odd we know that the intersection form on $\mathrm{H}_{2k+1}(M;\mathbb{R})$ is anti-symmetric and non-singular, by the Anti-Symmetric Forms Proposition 133.2 we thus know that $\dim_{\mathbb{R}}(\mathrm{H}_{2k+1}(M;\mathbb{R}))$ is even

[1625] This question arose from the fact, shown in the Odd-Dimensional Manifold-Euler Characteristic Proposition 119.7, that the Euler characteristic of every closed odd-dimensional topological manifold is necessarily zero. So it was natural to wonder what is happening for closed even-dimensional topological manifolds.

⊞

Claim. The Euler characteristic of every every closed orientable connected 2-dimensional topological manifold M is ≤ 2.

Proof. Of course this statement follows from the Surface Classification Theorem 55.4 together with the calculation of the Euler characteristics on page 1879. But one can prove the claim directly. First the case that M is empty is trivial. Now assume that M is non-empty. In this case we have the following (in-) equalities:

$$\chi(M) \underset{\substack{\uparrow \\ \text{by the Euler Characteristic-H}_* \\ \text{Proposition 87.1}}}{=} \sum_{i\in\mathbb{N}_0} (-1)^i \cdot \operatorname{rank}(\mathrm{H}_i(M)) \underset{\substack{\uparrow \\ \text{by the Top-Homology Theorem 106.3}}}{=} \overbrace{\operatorname{rank}(\mathrm{H}_0(M))}^{=1 \text{ by the H}_0\text{-Proposition 72.5}} - \operatorname{rank}(\mathrm{H}_1(M)) + \overbrace{\operatorname{rank}(\mathrm{H}_2(M))}^{=1 \text{ by the Orientation-Top Homology Theorem 106.1}} \leq 2.$$

⊞

It remains to show that besides these two restrictions on the possible values of Euler characteristics, all other values can be realized. Our arguments in all of the three cases rely on the following observation.

Observation.

(i) On page 1879 we saw that for every $m \in \mathbb{N}_0$ we have $\chi(\mathbb{C}\mathrm{P}^m) = m+1$.

(ii) the Connected Sum-χ-Lemma 87.7 says that if M and N are two compact oriented connected *even-dimensional* smooth manifolds, then $\chi(M\#N) = \chi(M)+\chi(N)-2$.

(iii) There is any number of reasons, e.g. we could use the Product-χ-Lemma 87.8, why given any $m \in \mathbb{N}$ the Euler characteristic of the m-dimensional torus $(S^1)^m$ is zero.

Proof.

(1) Let $k \in \mathbb{N}$ and let $n \in \mathbb{Z}$. Note that $\chi(\mathbb{C}\mathrm{P}^k) = 2k+1$ is odd and ≥ 3. From the above observations it follows that

$$\chi(n \cdot \mathbb{C}\mathrm{P}^{2k} \# nk \cdot (S^1)^{4k}) \;=\; n\cdot(2k+1) - nk\cdot 2 \;=\; n.$$

(2) (a) Evidently we have $\chi(S^2) = 2$. Note let $n = 2m$ be an even integer with $n \leq 0$. It follows from the above observations that

$$\chi((m+1)\cdot(S^1\times S^1)) \;=\; (m+1-1)\cdot 2 \;=\; n.$$

(b) Let $k \in \mathbb{N}$ and let n be an even integer. From the above observations it follows that

$$\chi\big(n \cdot \mathbb{C}\mathrm{P}^{2k+1} \# \tfrac{n}{2}(2k+1)\cdot(S^1)^{4k+2}\big) \;=\; n\cdot(2k+2) - \tfrac{n}{2}(2k+1)\cdot 2 \;=\; n. \qquad \blacksquare$$

133.2. **Interlude: Sphere-packing.** In this short section we want to consider sphere packings. The relationship with our actual goal of dealing with symmetric forms is arguably tenuous but nonetheless fascinating.

Definition. Let $v_1, \ldots, v_n$ be linearly independent vectors in $\mathbb{R}^n$. We refer to the group

$$\mathbb{Z}\cdot v_1 + \cdots + \mathbb{Z}\cdot v_n \;=\; \left\{ \sum_{i=1}^n a_i \cdot v_i \,\middle|\, a_1, \ldots, a_n \in \mathbb{Z} \right\}$$

as the lattice spanned by $v_1, \ldots, v_n$.[1626]

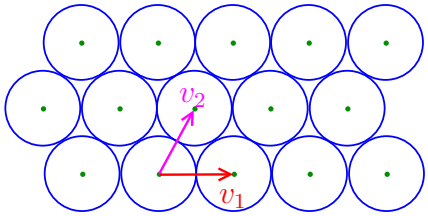
We start out with the following theorem. The fact that strictly speaking we did not define all the terms will not bother us.

Theorem 133.4.

(1) A densest packing of disks of radius $\frac{1}{2}$ in $\mathbb{R}^2$ is given by setting the centers on the "hexagonal" lattice spanned by $v_1 = (1,0)$ and $v_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ shown in the figure below on the left. The density equals $\frac{\pi}{\sqrt{12}}$.

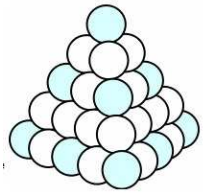
(2) No packing of balls of the same radius in $\mathbb{R}^3$ has density greater than that of the "face-centered cubic packing" shown in the figure below in the middle and on the right. The density equals $\frac{\pi}{\sqrt{18}}$.

Proof. The first statement was proved by L. Fejes Tóth [**FT1943, FT1972**] in 1943. It is much harder to prove the second statement. This statement had been conjectured by Kepler in 1613 and it was finally proved by Thomas Hales [**Hale20000, Hale2005, Hale2017**] in 2005. ■



densest disk packing





ball packings of highest density

For a mathematician it is natural to wonder what are the highest densities of packings in higher dimensions. Not surprisingly this is a very hard problem. Astonishingly the next case that was dealt with is the 8-dimensional setting. More precisely, the following theorem was proved by Maryna Viazovska [**Via2017**] in 2016.

Theorem 133.5. We consider the following matrix

$$E_8 := \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

No packing of unit balls in $\mathbb{R}^8$ has density greater than that given by the lattice spanned by the eight column vectors of the E_8–matrix. Furthermore, this packing has density $\frac{\pi^4}{384}$.

Remark. We also refer to [**Baez2018**] for other interesting appearances of the E_8–matrix, for example its relationship to the regular icosahedron

[1626] This is perhaps not the standard definition of a lattice, but who cares?

Shortly afterwards the 24-dimensional case was settled in [**CKMRV2017**], in this case a densest packing is given by the "Leech lattice". OK, that was fun, but now let us return to our actual problem, and let us see whether this section was of any use.

133.3. **Non-singular symmetric forms over the real numbers.** In an earlier section we saw that non-singular *anti-symmetric* forms are almost depressingly boring mathematical objects. In this and the next section we will consider non-singular *symmetric* forms. These will, fortunately, turn out to be much more interesting.

Examples. We list some symmetric integral matrices with determinant equal to ± 1. By the Forms-Matrices Lemma 133.1 these define in particular non-singular forms over $\mathbb{R}$ and also over $\mathbb{Z}$.

(1) The 1×1-matrices $(+1)$ and (-1).
(2) The slightly mysterious 8×8–matrix E_8 from the previous section.[1627]
(3) The matrix
$$H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
is symmetric and it has determinant -1.

It is natural to ask whether there is some type of analogue of the Anti-Symmetric Forms Proposition 133.2 for symmetric forms. In fact using the methods of the proof of the Anti-Symmetric Forms Proposition 133.2 one can prove the following statement.

Proposition 133.6. (Symmetric Forms-over-$\mathbb{R}$-Proposition) Let (V, φ) be a symmetric form over $\mathbb{R}$ of rank n.

(1) There exist $k_+, k_- \in \mathbb{N}_0$ such that $(V, \varphi) \cong k_+ \cdot (+1) \oplus k_- \cdot (-1) \oplus (n - k_- - k_+) \cdot (0)$.
(2) If the symmetric form (V, φ) is nonsingular, then there exists a $k \in \mathbb{N}_0$ such that $(V, \varphi) \cong k \cdot (+1) \oplus (n - k) \cdot (-1)$.

Remark. Recall that the Anti-Symmetric Forms Proposition 133.2 dealt with anti-symmetric forms over $\mathbb{R}$ and also with anti-symmetric forms over $\mathbb{Z}$. In contrast the Symmetric Forms-over-$\mathbb{R}$-Proposition 133.6 only deals with symmetric forms over $\mathbb{R}$. It is a very instructive exercise to run through the proof of the Symmetric Forms-over-$\mathbb{R}$-Proposition 133.6 and to determine where it breaks down if we work over $\mathbb{Z}$.

Sketch of proof. First let (V, φ) be a non-singular symmetric form over $\mathbb{R}$ of rank n. In the subsequent arguments below we use the conventions and notions we introduced in the Anti-Symmetric Forms Proposition 133.2. In particular we will also use the operations of type (i), (ii) and (iii) that we introduced in the proof of the Anti-Symmetric Forms Proposition 133.2.

We pick an $(n \times n)$–matrix A that represents φ. Now we can perform the following sequence of operations to A:

$$A \overset{(1)}{=} \begin{pmatrix} * & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & * \end{pmatrix} \overset{(2)}{\longrightarrow} \begin{pmatrix} \epsilon_1 & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & * \end{pmatrix} \overset{(3)}{\longrightarrow} \begin{pmatrix} \epsilon_1 & 0 & \dots & 0 \\ 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \dots & * \end{pmatrix}$$

[1627] A miracle! The matrix E_8 actually has determinant one.

where we perform the following steps:

(1) We write out A. Here and throughout the steps we use the convention that $*$ can mean anything, the only condition being that all matrices are symmetric.
(2) If Since $\det(A) \neq 0$ there exists at least one entry on the first row which is non-zero. After operations of type (iii) and (i) we can arrange that the $(1,1)$-entry equals some $\epsilon_1 \in \{-1, 1\}$.
(3) Using operations of type (iii) we can "clean up" the remaining terms on the first row and column.

At this stage we can continue with the same process, applied to the $(n-1)\times(n-1)$–matrix at the right bottom. Eventually we will have turned A into a diagonal matrix where all entries lie in $\{-1, 0, 1\}$. After operations of type (ii) we can arrange that the first diagonal entries are equal to $+1$ and the final ones are equal to -1.

Now suppose that (V, φ) is any symmetric form over $\mathbb{R}$. As in Exercise 132.4 we consider the radical

$$R \ := \ \{v \in V \,|\, \varphi(v, w) = 0 \text{ for all } w \in V\}.$$

Note that R is a vector subspace of V. We pick a vector subspace S of V with $R \oplus S = V$. As in Exercise 132.4 we see that (V, φ) is isometric to $(R, 0) \oplus (S, \varphi)$, and $(S, \varphi|_S)$ is non-singular. Thus the desired statement now follows from the above discussion of non-singular symmetric forms. ■

Let $n \in \mathbb{N}_0$. the Symmetric Forms-over-$\mathbb{R}$-Proposition 133.6 raises the question whether or not the symmetric forms defined by $k \cdot (+1) \oplus (n-k) \cdot (-1)$ are isometric for different values of $k \in \{0, \ldots, n\}$. To answer this question it is helpful to introduce the following definition.

Definition. Let (V, φ) be a symmetric form over the real numbers.

(1) We say φ is positive-definite if $\varphi(v, v) > 0$ for all $v \neq 0$.
(2) We say φ is negative-definite if $\varphi(v, v) < 0$ for all $v \neq 0$.
(3) We say φ is indefinite if φ is neither positive-definite nor negative-definite.

Examples.

(1) The form $(+1)$ is positive-definite and the form (-1) is negative-definite.
(2) The form E_8 is positive-definite. Evidently one can show this by a brute force calculation. A more conceptual proof is given in [**HGK2010**, Theorem 1.16.14].
(3) The form H is indefinite, indeed, we have

$$\varphi(H)\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right) \ = \ (1 \ \ 1)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \ = \ 2$$

and

$$\varphi(H)\left(\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right) \ = \ (1 \ \ -1)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} \ = \ -2.$$

To distinguish non-isometric forms we need to introduce some invariants of forms.

Definition. Let (V, φ) be a symmetric form over the real numbers. We define

$b^+(\varphi) :=$ maximal dimension of a vector subspace U of V s. th. $\varphi|_U$ is positive-definite,
$b^-(\varphi) :=$ maximal dimension of a vector subspace U of V s. th. $\varphi|_U$ is negative-definite.

Furthermore we define the signature $\operatorname{sign}(\varphi)$ as

$$\operatorname{sign}(\varphi) \;:=\; b^+(\varphi) - b^-(\varphi).$$

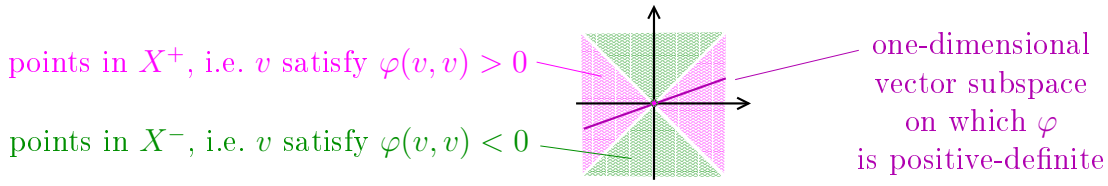
Example. We consider the symmetric form

$$\begin{array}{rcl} \varphi\colon \mathbb{R}^2 \times \mathbb{R}^2 & \to & \mathbb{R} \\ \left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) & \mapsto & (v_1 \;\; v_2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \end{array}$$

In the figure below we show the subsets

$$X^+ := \{v \in V \,|\, \varphi(v,v) > 0\} \cup \{0\} \quad \text{and} \quad X^- := \{v \in V \,|\, \varphi(v,v) < 0\} \cup \{0\}.$$

These subsets are evidently *not* vector subspaces of $V = \mathbb{R}^2$. The maximal dimension of a vector subspace contained in X^+ and X^- is in both cases equal to one. Thus we see that $b^+(\varphi) = b^-(\varphi) = 1$. In particular we see that $\operatorname{sign}(\varphi) = 0$.



The following lemma is an immediate consequence of the definitions.

Lemma 133.7. Let (V,φ) and (W,ψ) be two symmetric forms over $\mathbb{R}$. If (V,φ) and (W,ψ) are isometric, then

$$\dim_{\mathbb{R}}(V) = \dim_{\mathbb{R}}(W), \quad b^{\pm}(\varphi) = b^{\pm}(\psi) \quad \text{and} \quad \operatorname{sign}(\varphi) = \operatorname{sign}(\psi).$$

In the following proposition we determine $b^{\pm}$ and the signature for a non-singular form that is given by a diagonal matrix.

Proposition 133.8. (Signature-Eigenvalues Proposition) If D is a diagonal $(n \times n)$–matrix where the diagonal entries $d_1, \ldots, d_n$ are non-zero real numbers, then

$$b^+(\varphi(D)) \;=\; \text{number of positive } d_i\text{'s} \quad \text{and} \quad b^-(\varphi(D)) \;=\; \text{number of negative } d_i\text{'s}.$$

In particular

$$\operatorname{sign}(\varphi(D)) \;=\; \text{number of positive } d_i\text{'s} \;-\; \text{number of negative } d_i\text{'s}.$$

Proof. After reordering the diagonal entries we can assume that the first k entries of D are positive and that the remaining $(n-k)$ entries are negative. We write

$$\begin{array}{rcll} V^+ & := & \{(x_1, \ldots, x_k, 0, \ldots, 0) \in \mathbb{R}^n \,|\, x_1, \ldots, x_k \in \mathbb{R}\} & \text{and} \\ V^- & := & \{(0, \ldots, 0, y_1, \ldots, y_{n-k}) \in \mathbb{R}^n \,|\, y_1, \ldots, y_{n-k} \in \mathbb{R}\} & \end{array}$$

It is clear that $\varphi(D)|_{V^+}$ is positive-definite and that $\varphi(D)|_{V^-}$ is negative-definite. Thus $b^+(\varphi(D)) \geq k$ and $b^-(\varphi(D)) \geq n-k$. If remains to prove the following claim.

Claim. For any symmetric form (W,ψ) over $\mathbb{R}$ we have $b^+(\psi) + b^-(\psi) \leq \dim_{\mathbb{R}}(W)$.

Proof. Thus let W^+ be a positive-definite subspace of W and let W^- be a negative-definite subspace of W. The intersection $W^+ \cap W^-$ is a subspace that is positive-definite and negative-definite. So for any non-zero vector $v \in W^+ \cap W^-$ we have $\psi(v,v) > 0$ and

$\psi(v,v) < 0$. This is only possible if no such v exists, i.e. we have shown that the intersection $W^+ \cap W^- = \{0\}$. Now we see that

$$\dim(W^+) + \dim(W^-) \underset{\text{linear algebra}}{\overset{\uparrow}{=}} \dim(W^+ + W^-) + \dim(W^+ \cap W^-) \underset{W^+\cap W^- = \{0\}}{\overset{\uparrow}{=}} \dim(W^+ + W^-) \underset{\text{since } W^+ + W^- \subset W}{\overset{\uparrow}{\leq}} \dim(W). \qquad \blacksquare$$

Corollary 133.9.

(1) Let (V,φ) be a symmetric form over $\mathbb{R}$. We write $n = \dim_{\mathbb{R}}(V)$. The following statements hold:
   (a) We have $b^{\pm}(-\varphi) = b^{\mp}(\varphi)$, in particular we have $\operatorname{sign}(-\varphi) = -\operatorname{sign}(\varphi)$.
   (b) If (V,φ) is non-singular, then we have $b^+(\varphi) + b^-(\varphi) = \dim_{\mathbb{R}}(V)$.
   (c) If (V,φ) is non-singular, then we have $b^+(\varphi) = \frac{1}{2}(n + \operatorname{sign}(\varphi))$.
   (d) If we write $k_{\pm} = b^{\pm}(\varphi)$, then the given form (V,φ) is isometric to the form $k_+ \cdot (+1) \oplus k_- \cdot (-1) \oplus (n - k_- - k_+) \cdot (-1)$.
   (e) If (V,φ) is non-singular, then we have $\operatorname{sign}(\varphi) \equiv n \bmod 2$.

(2) If (V,φ) and (W,ψ) are two symmetric forms over $\mathbb{R}$, then

$$\operatorname{sign}(\varphi \oplus \psi) \;=\; \operatorname{sign}(\varphi) + \operatorname{sign}(\psi).$$

Proof.

(1) (a) This statement follows immediately from the definitions.
   (b) By the Symmetric Forms-over-$\mathbb{R}$-Proposition 133.6 and Lemma 133.7 it suffices to prove the statement for non-singular symmetric forms that are represented by diagonal matrices. But for those forms the statement follows immediately from the Signature-Eigenvalues Proposition 133.8.
   (c) This statement is an immediate consequence of (b) and the definition of the signature.
   (d) This statement follows from the Symmetric Forms-over-$\mathbb{R}$-Proposition 133.6, Lemma 133.7 and the Signature-Eigenvalues Proposition 133.8.
   (e) We have $\operatorname{sign}(\varphi) = b^+(\varphi) - b^-(\varphi) \equiv b^+(\varphi) + b^-(\varphi) = n \bmod 2$.

(2) This statement follows immediately from Statement (1d), the Symmetric Forms-over-$\mathbb{R}$-Proposition 133.6 applied to φ and ψ and the Signature-Eigenvalues Proposition 133.8. $\blacksquare$

Example. Given a non-singular symmetric form φ on $\mathbb{R}^3$ one can visualize it by considering the "unit-sphere"

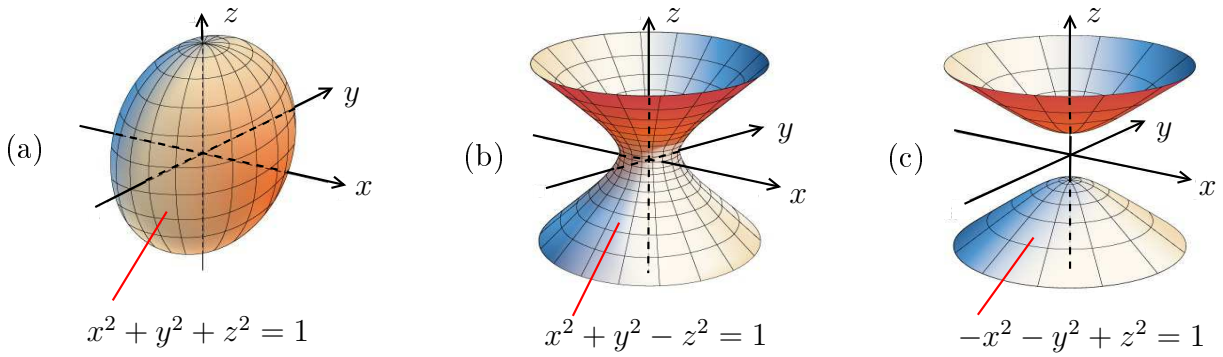
$$S(\varphi) \;:=\; \{v \in \mathbb{R}^3 \,|\, \varphi(v,v) = 1\}.$$

Corollary 133.9 (3) implies in particular that every non-singular symmetric form on $\mathbb{R}^3$ is isometric to one of the following four forms

(a) $3 \cdot (1)$, (b) $2 \cdot (1) \oplus (-1)$
(c) $2 \cdot (-1) \oplus (1)$ (d) $3 \cdot (-1)$.

In the last case the unit-sphere is the empty set. In the other cases the unit-spheres are sketched in the figure below.

The following theorem says that the dimension and the signature are a complete invariant for isometry types of non-singular symmetric forms over $\mathbb{R}$.



Theorem 133.10. (Sylvester's Theorem a. k. a. Sylvester's law of inertia) Let (V, φ) and (W, ψ) be two non-singular symmetric forms over $\mathbb{R}$. Then

$$(V,\varphi) \text{ and } (W,\psi) \text{ are isometric} \iff \dim_{\mathbb{R}}(V) = \dim_{\mathbb{R}}(W) \text{ and } \operatorname{sign}(\varphi) = \operatorname{sign}(\psi).$$

Proof. The "$\Rightarrow$"-direction is precisely Lemma 133.7. It remains to prove the "$\Leftarrow$"-direction. So suppose we are given two non-singular symmetric forms (V, φ) and (W, ψ) over $\mathbb{R}$ with same dimension and same signature. It follows from our hypothesis together with Corollary 133.9 (c) and (d) that there exist $n, k \in \mathbb{N}_0$ such that (V, φ) and (W, ψ) are both isometric to $k \cdot (+1) \oplus (n-k) \cdot (-1)$. But this implies that (V, φ) and (W, ψ) are themselves isometric. ■

Remark. It is also very easy to classify symmetric forms over $\mathbb{R}$ that are not necessarily non-singular. More precisely, given a symmetric form (V, φ) over $\mathbb{R}$ we define the *nullity* $\sigma(\varphi)$ as

$$\nu(\varphi) \ := \ \text{maximal dimension of a subspace } W \text{ of } V \text{ such that } \varphi|_W \text{ is the zero form.}$$

It is an amusing little exercise to modify the proof of Sylvester's Theorem 133.10 to show that two symmetric forms over $\mathbb{R}$ are isometric if and only if their dimensions, signatures and nullities agree.

Now it is time to apply our recently acquired knowledge to topology.

Definition.

(1) If (V, φ) is a symmetric form over the integers, then we define

$$\begin{array}{rccl} g \otimes \mathbb{R} \colon & V \otimes \mathbb{R} \times V \otimes \mathbb{R} & \to & \mathbb{R} \\ & (v \otimes r, w \otimes s) & \mapsto & r \cdot \varphi(v,w) \cdot s. \end{array}$$

Furthermore we define

$$\operatorname{sign}(\varphi) \ := \ \operatorname{sign}(\varphi \otimes \mathbb{R}).$$

(2) Let $k \geq 1$ and let M be a compact oriented $4k$-dimensional topological manifold. We define the signature of M to be[1628]

$$\operatorname{sign}(M) \ := \ \operatorname{sign}(Q_M).$$

For convenience we define the signature of the empty topological manifold to be zero.

Remark. The signature of a manifold was first introduced in 1923 by Hermann Weyl [**Wey1923**, p. 432] in a little known paper written in Spanish. As is explained in [**Eckm2006**], Hermann Weyl deliberately chose this obscurity since at this point, according to Hermann Weyl, algebraic topology was not considered "a respectable field of mathematics".

Lemma 133.11. (Manifold-Signature Lemma) Let $k \in \mathbb{N}$.

(1) Let M be a compact oriented $4k$-dimensional topological manifold. The form $Q_M \otimes \mathbb{R}$ is isometric to the form

$$\begin{array}{rcl} \mathrm{H}^{2k}(M, \partial M; \mathbb{R}) \times \mathrm{H}^{2k}(M, \partial M; \mathbb{R}) & \to & \mathbb{R} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [M] \rangle. \end{array}$$

(2) The signature is additive under the connected sum operation and under disjoint union.

(3) For every compact oriented $4k$-dimensional topological manifold we have

$$\operatorname{sign}(-M) \;=\; -\operatorname{sign}(M).$$

(4) Let M and N be two compact oriented $4k$-dimensional topological manifolds.
   (a) If M and N are homeomorphic, then $\operatorname{sign}(M) = \pm \operatorname{sign}(N)$.
   (b) If the pairs $(M, \partial M)$ and $(N, \partial N)$ are homotopy equivalent, then we have the equality $\operatorname{sign}(M) = \pm \operatorname{sign}(N)$.

Proof.

(1) This statement follows easily from the various definitions and Corollary 89.19.

(2) It follows easily from the Cup-Cap Disjoint Union Lemma 116.9 that the signature is additive under disjoint union. If we are given compact topological manifolds, then it follows immediately from the Intersection Form-Additivity Proposition 132.12 and Corollary 133.9 that the signature is additive under the connected sum operation.

(3) The third statement is an immediate consequence of Lemma 132.6 and Corollary 133.9.

(4) The last statement is also an immediate consequence of Lemma 132.6 and Corollary 133.9. ■

Examples. We have

$$\begin{array}{ccccccccccc} \operatorname{sign}(\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2) & = & \operatorname{sign}(\mathbb{C}\mathrm{P}^2) + \operatorname{sign}(\mathbb{C}\mathrm{P}^2) & = & \operatorname{sign}(1) + \operatorname{sign}(1) & = & 1+1 & = & 2 \\ & & \uparrow & & \uparrow & & & & \\ & & \text{Manifold-Signature Lemma 133.11 (2) and (3)} & & \text{see page 2882} & & & & \\ & & \downarrow & & \downarrow & & & & \\ \operatorname{sign}(\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2) & = & \operatorname{sign}(\mathbb{C}\mathrm{P}^2) - \operatorname{sign}(\mathbb{C}\mathrm{P}^2) & = & \operatorname{sign}(1) - \operatorname{sign}(1) & = & 1-1 & = & 0. \end{array}$$

Furthermore we have

$$\operatorname{sign}(S^2 \times S^2) \underset{\text{see page 2881}}{\overset{\uparrow}{=}} \operatorname{sign} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underset{\text{Forms-Matrices Lemma 133.1}}{\overset{\uparrow}{=}} \operatorname{sign} \left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right) = \operatorname{sign} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \underset{\text{Signature-Eigenvalues Proposition 133.8}}{\overset{\uparrow}{=}} 0.$$

The above examples, together with Lemmas 132.6 and 133.11 (4) now allow us to give a partial answer to Question 100.15.

[1628] Recall that on page 2800 we introduced the intersection form $Q_M \colon \mathrm{FH}_{2k}(M; \mathbb{Z}) \times \mathrm{FH}_{2k}(M; \mathbb{Z}) \to \mathbb{Z}$. Furthermore, recall that by the Intersection Form Proposition 132.4 the intersection form Q_M is indeed symmetric, so it makes sense to consider its signature.

Corollary 133.12. The topological space $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ is neither homotopy equivalent (in particular it is not homeomorphic) to $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ nor to $S^2 \times S^2$.

Remark. Corollary 133.12 shows in particular that in the definition of the connected sum of two orientable (topological) smooth manifolds we needed to work with *oriented* (topological) smooth manifolds.

Unfortunately we are not yet able to distinguish the two smooth manifolds $S^2 \times S^2$ and $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$. In fact Sylvester's Theorem 133.10 might suggest that we are again at a dead end. But this is not the case since Sylvester's Theorem is a statement about *real* symmetric forms, but we are really dealing with symmetric forms defined over the integers. We will exploit this extra piece of information in the next section.

133.4. **Non-singular symmetric forms over the integers.** We consider the matrices

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and } D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It follows from Sylvester's Theorem 133.10 that the matrices give rise to *isometric forms over the real numbers*. The question arises, whether the forms on $\mathbb{Z}^2$ defined by these matrices are isometric. To get an idea let us consider what we can say about the "squares $\varphi(v, v)$" defined by these forms. Note that given any $v = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ we have

$$\varphi(H)(v,v) = \begin{pmatrix} a & b \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2ab \quad \text{and} \quad \varphi(H)(v,v) = \begin{pmatrix} a & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = a^2 - b^2.$$

We see that for the form to the left we only get even integers whereas to the right any integer can occur. This already shows that the two forms are not isometric over the integers. It follows immediately from this discussion and the earlier calculations of the intersection forms that the intersection forms of $S^2 \times S^2$ and $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ are not isometric, not even up to sign. This allows us to settle the remaining case of Question 100.15. More precisely, by Lemma 132.6 we have the following proposition.

Proposition 133.13. The topological space $S^2 \times S^2$ is not homotopy equivalent (in particular it is not homeomorphic) to $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$.

This encouraging discussion leads us to the following definition.

Definition. A symmetric form (V, φ) over the integers is called even if

$$\varphi(v, v) \equiv 0 \mod 2 \quad \text{for all } v \in V,$$

otherwise the intersection form is called odd. We refer to the property of φ being even or odd as the parity of the form.

The following lemma summarizes some basic facts about the parity of symmetric forms.

Lemma 133.14. (Parity Lemma)

(1) Isometric forms have the same parity.
(2) For a symmetric form φ the parities of φ and $-\varphi$ agree.
(3) The direct sum of an odd form with any other form is also odd.

(4) If A is a symmetric square matrix over the integers, then

$$\varphi(A) \text{ is even} \iff \text{all diagonal entries of } A \text{ are even.}$$

Examples. It follows from the Parity Lemma 133.14 that the forms $(+1)$, (-1) and $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ are odd and that the forms E_8 and $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are even.

Proof. The only statement which requires a little thought is the fact that if $A = (a_{ij})$ is a symmetric $(n \times n)$–matrix with even diagonal entries, then $\varphi(A)$ is even. Thus let $v = (v_1, \dots, v_n) \in \mathbb{Z}^n$. We calculate that

$$\varphi(A)(v,v) = v^T A v = \sum_{i=1}^{n}\sum_{j=1}^{n} v_i v_j \cdot a_{ij} \underset{\text{since } A \text{ is symmetric}}{\overset{\uparrow}{=}} 2\sum_{i=1}^{n}\sum_{j=1}^{i-1} v_i v_j \cdot a_{ij} + \sum_{i=1}^{n} v_i^2 a_{ii} \underset{\text{since the } a_{ii} \text{ are even}}{\overset{\uparrow}{\equiv}} 0 \bmod 2.$$

■

We just saw that the two matrices

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and } D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

give rise to *isometric forms over the real numbers* and that the two matrices give rise to *non-isometric forms over the integers*. The question arises, how much more complicated is the classification of non-singular symmetric forms over the integers than the classification of non-singular forms over the real numbers. In particular the following question arises.

Question 133.15. Can we classify non-singular symmetric forms over the integers?

The following is the key theorem in the classification of non-singular *indefinite* symmetric forms over $\mathbb{Z}$.

Theorem 133.16. (Indefinite Forms Proposition) If φ_1 and φ_2 are two non-singular *indefinite* symmetric forms over the integers, then the following holds:

$$\varphi_1 \text{ and } \varphi_2 \text{ are isometric} \iff \varphi_1 \text{ and } \varphi_2 \text{ have the same rank, signature and parity.}$$

Example. Using the parity we had just seen that the intersection forms of the closed oriented 4-dimensional smooth manifolds $M = S^2 \times S^2$ and $N = \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$ are not isometric, even up to sign. Now we consider the closed oriented 4-dimensional smooth manifolds that we obtain from M and N by performing the connected sum with $\mathbb{C}\mathrm{P}^2$. In other words, we are considering $M \# \mathbb{C}\mathrm{P}^2 = (S^2 \times S^2) \# \mathbb{C}\mathrm{P}^2$ and $N \# \mathbb{C}\mathrm{P}^2 = \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2 \# \mathbb{C}\mathrm{P}^2$. It follows from the Intersection Form-Additivity Proposition 132.12 that the intersection forms of these two smooth manifolds are represented by the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Both forms are non-singular indefinite and odd. Thus it follows from the Indefinite Forms Theorem 133.16, or alternatively from Exercise 133.12, that the intersection forms are isometric. Thus we cannot distinguish $M \# \mathbb{C}\mathrm{P}^2 = (S^2 \times S^2) \# \mathbb{C}\mathrm{P}^2$ and $N \# \mathbb{C}\mathrm{P}^2 = \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2 \# \mathbb{C}\mathrm{P}^2$ using their intersection forms. In fact, it turns out that there is a good reason why we have

a hard time distinguishing them: as is shown in [**GoS1999**, p. 151] the two smooth manifolds $(S^2 \times S^2)\#\mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2\#\mathbb{C}\mathrm{P}^2$ are in fact diffeomorphic, in fact with good eyes one can write down an explicit diffeomorphism.

Proof. The proof of the theorem is provided in [**MH1973**, Theorem II.5.3] (see also [**GoS1999**, Theorem 1.2.14] for an outline of the proof). The proof relies crucially on a significant input from algebraic number theory, namely the Hasse-Minkowski theorem. This theorem says that given $a_1, \ldots, a_n \in \mathbb{Q} \setminus \{0\}$ the equation

$$x_1^2 \cdot a_1 + \cdots + x_n^2 \cdot a_n \;=\; 0$$

has a non-trivial integral solution if and only if it has a non-trivial real solution and if for each prime p the equation has also has a non-trivial solution over the p-adic numbers $\mathbb{Q}_p$. A proof of the Hasse-Minkowski theorem is given in [**Ser1973**, Chapter IV Theorem 8], [**BSh1966**, Section 7.1] or [**Neuk1999**, p. 385]. ■

As an immediate corollary to the Indefinite Forms Theorem 133.16 we obtain the classification of non-singular *indefinite odd* forms. More precisely, we have the following corollary.

Corollary 133.17. (Indefinite Odd Forms Lemma) If φ is a non-singular *indefinite* odd symmetric form over the integers, then

$$\varphi \;\cong\; b^+(\varphi) \cdot (1) \,\oplus\, b^-(\varphi) \cdot (-1).$$

Proof. Let φ be a non-singular indefinite odd symmetric form. It follows from the Signature-Eigenvalues Proposition 133.8 that the rank, signature and parity of the two non-singular indefinite odd forms in the statement of the corollary agree. So these two forms are isometric by the Indefinite Forms Theorem 133.16. ■

Now we turn to the study of even forms. We have the following, perhaps somewhat surprising proposition.

Proposition 133.18. (Even Forms Proposition) Let φ be a non-singular symmetric form over the integers. If φ is even, then $\mathrm{sign}(\varphi) \equiv 0 \bmod 8$.

Example. As a reality check, the two even non-singular symmetric forms that we are acquainted with are $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and E_8. These forms a non-singular even forms and the signatures are 0 and 8, so that matches the Even Forms Proposition 133.18.

Proof. Given a non-singular symmetric form (W, ψ) over $\mathbb{Z}$ we say that an element $v \in W$ is *characteristic* if for all $w \in W$ we have

$$\varphi(v, w) \;\equiv\; \varphi(w, w) \mod 2.$$

We make the following observations:

(1) It follows immediately from the definitions that (W, ψ) is even if and only if 0 is a characteristic element.

(2) If $W = k \cdot (1) \oplus l \cdot (-1)$, then it follows from a straightforward calculation as in the proof of the Parity Lemma 133.14 that an element $u \in \mathbb{Z}^{k+l}$ is characteristic if and only if all its entries are odd.

It follows from Observation (1) that the following claim implies in particular the desired statement of the Even Forms Proposition 133.18.

Claim. Let (V, φ) be a non-singular symmetric form over $\mathbb{Z}$. If $v \in V$ is a characteristic element, then
$$\varphi(v, v) \equiv \operatorname{sign}(\varphi) \mod 8.$$

Proof. Let $v \in V$ be a characteristic element of (V, φ). It is straightforward to verify that $v + e_- + e_+$ is a characteristic element in $(V \oplus \mathbb{Z} \oplus \mathbb{Z}, \varphi \oplus (1) \oplus (-1))$, where e_- and e_+ are generators of the first respectively the second $\mathbb{Z}$-summand. Note that, regardless of φ, the form $\varphi \oplus (1) \oplus (-1))$ is indefinite and odd. Thus by the Indefinite Odd Forms Corollary 133.17 there exists an isometry[1629]
$$\Phi\colon \varphi \oplus (1) \oplus (-1) \xrightarrow{\cong} (b^+(\varphi) + 1) \cdot (1) \oplus (b^-(\varphi) + 1) \cdot (-1).$$
We write $u := \Phi(v + e_- + e_+)$. It is clear that u is a characteristic element for the form on the right-hand side. The above Observation (2) tells us that each entry of u is an odd integer. Now we calculate that

$$\begin{aligned}
\varphi(v,v) &\overset{\text{straightforward calculation}}{=} (\varphi \oplus (1) \oplus (-1))(v + e_- + e_+, v + e_- + e_+) \overset{\text{since } \Phi \text{ is an isometry}}{=} \\
&= ((b^+(\varphi) + 1) \cdot (1) \oplus (b^-(\varphi) + 1) \cdot (-1))(u, u) \\
&= \sum_{i=1}^{b^+(\varphi)+1} (\underbrace{i\text{-th entry of } u}_{\text{odd integer}})^2 + \sum_{i=b^+(\varphi)+2}^{b^+(\varphi)+b^-(\varphi)+2} (-1) \cdot (\underbrace{i\text{-th entry of } u}_{\text{odd integer}})^2 \\
&\equiv \sum_{i=1}^{b^+(\varphi)+1} 1 + \sum_{i=b^+(\varphi)+2}^{b^+(\varphi)+b^-(\varphi)+2} (-1) = (b^+(\varphi) + 1) - (b^-(\varphi) + 1) = \operatorname{sign}(\varphi) \bmod 8.
\end{aligned}$$
$\uparrow$ since for each odd number k we have $k^2 \equiv 1 \bmod 8$[1630]

■

Now we can provide the classification of non-singular indefinite *even* symmetric forms.

Corollary 133.19. (Indefinite Even Forms Corollary) If φ is a non-singular *indefinite* even symmetric form over the integers, then
$$\varphi \cong \tfrac{\operatorname{sign}(\varphi)}{8} \cdot E_8 \oplus \tfrac{\operatorname{rank}(\varphi) - |\operatorname{sign}(\varphi)|}{2} \cdot H.$$

Proof. First let φ be a non-singular even symmetric form over the integers. Note that it follows from the Even Forms Proposition 133.18 that $\operatorname{sign}(\varphi)$ is indeed divisible by 8. Furthermore it follows from $\operatorname{rank}(\varphi) = b^+(\varphi) + b^-(\varphi)$ and $\operatorname{sign}(\varphi) = b^+(\varphi) - b^-(\varphi)$ that $\operatorname{rank}(\varphi) - |\operatorname{sign}(\varphi)|$ is divisible by 2. So it makes sense to write down the right-hand side.

Next we recall that the two non-singular symmetric forms E_8 and H are even, that $\operatorname{rank}(E_8) = \operatorname{sign}(E_8) = 8$ and that $\operatorname{rank}(H) = 2$ and $\operatorname{sign}(H) = 0$. Now assume that φ is furthermore indefinite. It follows from the above observation regarding E_8 and H that the rank, signature and parity of the two non-singular indefinite even forms in the statement of the corollary agree. So these two forms are isometric by the Indefinite Forms Theorem 133.16. ■

[1629] Note that both sides are non-singular odd indefinite forms, regardless of what non-singular symmetric form φ we actually consider.

[1630] Why is it true that for each odd number k we have $k^2 \equiv 1 \bmod 8$?

Now we turn to the study of non-singular *definite* symmetric forms over the integers. We start out with some more examples of such forms. The following lemma is proved in [**MH1973**, Lemmas 6.1 and 6.2] and [**Ser1973**, p. 51].

Lemma 133.20. We denote by $e_1, \dots, e_{4m}$ the standard basis for $\mathbb{Z}^{4m}$. We denote by Γ_{4m} the submodule of $\mathbb{Z}^{4m}$ spanned by the vectors $\{e_i + e_j\}_{i,j=1,\dots,4m}$ and $\frac{1}{2}(e_1 + \dots + e_{4m})$.

(1) The restriction of the usual form $\langle\ ,\ \rangle$ on $\mathbb{Z}^{4m}$ to Γ_{4m} is a non-singular positive-definite symmetric form.
(2) The form Γ_{4m} is odd if and only if m is odd.
(3) The form Γ_8 is isometric to E_8.
(4) The form Γ_{16} is not isometric to $E_8 \oplus E_8$.

It turns out that there are many more *definite* symmetric forms than indefinite symmetric forms and that there is no known classification of such forms. For example, by [**MH1973**, p. 28] and [**Ser1973**, p. 55] we have the following table

dimension	8	16	24	32	40
number of isometry types of non-singular positive-definite *even* symmetric forms	1	2	24	$\geq 10^7$	$\geq 10^{51}$

Here in dimension 8, up to isometry, the unique example is $\Gamma_8 = E_8$ whereas in dimension 16 the unique examples are $\Gamma_8 \oplus \Gamma_8 = E_8 \oplus E_8$ and Γ_{16}.
Similar results are also known for the number of isometry classes of non-singular positive-definite *odd* symmetric forms of a given rank. We refer to [**CSl1999**, p. 416] and [**Kin2003**, Table 2] for details.

133.5. **Forms over $\mathbb{F}_2$.** Considering the power of the intersection form of a closed *oriented* even-dimensional topological manifold it is natural to ask, whether one can obtain similar results for closed (possibly) *non-orientable* even-dimensional topological manifolds. In fact we have the following definition.

Definition. Let M be a closed $2n$-dimensional topological manifold M (not necessarily orientable). We denote by $[M]_{\mathbb{F}_2} \in \mathrm{H}_{2n}(M;\mathbb{F}_2)$ the $\mathbb{F}_2$-fundamental class given by the $\mathbb{F}_2$-Fundamental Class Corollary 106.5. Given $k \in \mathbb{N}_0$ we denote by

$$\mathrm{PD}_M^{\mathbb{F}_2} \colon \mathrm{H}_k(M;\mathbb{F}_2) \ \to\ \mathrm{H}^{2n-k}(M;\mathbb{F}_2)$$

the inverse of the isomorphism given by the Poincaré Duality Theorem 119.1. We can now define the $\mathbb{F}_2$-valued intersection form as follows:

$$\begin{aligned} Q_M^{\mathbb{F}_2} \colon \mathrm{H}_n(M;\mathbb{F}_2) \times \mathrm{H}_n(M;\mathbb{F}_2) &\to \mathbb{F}_2 \\ (a,b) &\mapsto \left\langle \mathrm{PD}_M^{\mathbb{F}_2}(a) \cup \mathrm{PD}_M^{\mathbb{F}_2}(b), [M]_{\mathbb{F}_2} \right\rangle. \end{aligned}$$

The above definition is completely analogous to the definition of the $\mathbb{Z}$-valued intersection form on page 2800. Not surprisingly the $\mathbb{F}_2$-valued intersection forms shares many properties of the $\mathbb{Z}$-valued intersection form.

Proposition 133.21. ($\mathbb{F}_2$-Intersection Form Proposition)

(1) The $\mathbb{F}_2$-valued intersection form $Q_M^{\mathbb{F}_2}$ on a closed $2n$-dimensional topological manifold M is symmetric and non-singular.

(2) The $\mathbb{F}_2$-valued intersection forms of homeomorphic closed topological manifolds are isometric.

Proof. The proof is a very mild variation on the proofs of the Intersection Form Proposition 132.4 and Lemma 132.6. ■

Examples.

(1) Similar to the discussion on page 2881 one can show that the $\mathbb{F}_2$-valued intersection form of the torus $S^1 \times S^1$ is represented by the matrx $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(2) The Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 121.16, together with the obvious analogue of the Intersection Form-Additivity Proposition 132.12, implies that the $\mathbb{F}_2$-valued intersection form of $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The question arises, whether we can classify non-singular symmetric forms over $\mathbb{F}_2$. For better or worse it turns out that there is a straightforward classification. To state the classification we need the following definition which is inspired by our definition on page 2918.

Definition. Let V be a finite-dimensional $\mathbb{F}_2$-vector space. We say a form $\lambda\colon V \times V \to \mathbb{F}_2$ is even if $\lambda(v, v) = 0 \in \mathbb{F}_2$ for all $v \in V$. Otherwise we say that the symmetric form is odd. We call the property of a form to be even or odd to be the parity of the form.

Example. As we had just seen, the $\mathbb{F}_2$-valued intersection form of the torus $S^1 \times S^1$ is even whereas the $\mathbb{F}_2$-valued intersection form of the torus $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ is odd. Thus we have found a new, and particularly complicated, proof that these two 2-dimensional smooth manifolds are not homeomorphic.

The following proposition gives a complete classification of non-singular symmetric forms over $\mathbb{F}_2$ up to isometry.

Proposition 133.22. (Non-Singular Forms-over-$\mathbb{F}_2$-Proposition) Two non-singular symmetric forms (V, κ) and (W, λ) over $\mathbb{F}_2$ are isometric if and only if $\dim(V) = \dim(W)$ and if κ and λ have the same parity.

Proof. We will prove the Non-Singular Forms-over-$\mathbb{F}_2$-Proposition 133.22 in Exercise 133.14. For peace of mind we point out that the proposition is proved in [**Hopk2015**, Theorem 2.7]. It can also be deduced from [**Szy1997**, Corollary 6.3.1]. ■

The proposition shows that, unfortunately, we extract only very little information from the $\mathbb{F}_2$-valued intersection form.

Remark. Let V be a finite-dimensional vector space over $\mathbb{F}_2$ and let $\langle\ ,\ \rangle\colon V \times V \to \mathbb{F}_2$ be an even form. It follows from the Non-Singular Forms-over-$\mathbb{F}_2$-Proposition 133.22 that V admits a *symplectic basis*, i.e. there exists a basis $a_1, \dots, a_k, b_1, \dots, b_k$ for V such that the following conditions are satisfied:

(1) For every $i, j \in \{1, \dots, k\}$ we have $\langle a_i, a_j \rangle = \langle b_i, b_j \rangle = 0$.
(2) For every $i, j \in \{1, \dots, k\}$ we have $\langle a_i, a_j \rangle = \delta_{ij}$.

This observation will come in handy later on.

Exercises for Chapter 133.

Exercise 133.1. Let φ be a symmetric form on a finite-dimensional real vector space V. Prove the following statement: If φ is positive-definite, then φ is non-singular.

Exercise 133.2. Let (V, g) be a non-singular symmetric form over the reals. We suppose that there exists a subspace W with $g|_W = 0$ and with $\dim(W) = \frac{1}{2}\dim(V)$. Show that $\operatorname{sign}(g) = 0$.

Exercise 133.3. Let R be a commutative ring. In this exercise we will see that non-singular symmetric forms over R give rise to a group. (We refer to page 2877 for the definition of a non-singular symmetric form.) Note that the set of symmetric non-singular forms $\varphi\colon R^k \times R^k \to R$ with the direct sum operation is a monoid.

We say a symmetric form φ is *hyperbolic* if it is isometric to a form of the form

$$\begin{array}{rcl} \psi\colon (V \oplus V) \times (V \oplus V) & \to & \mathbb{R} \\ (v_1 \oplus v_2), (w_1 \oplus w_2) & \mapsto & (v_1^T\, v_2^T) \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \end{array}$$

(a) Show that the monoid of symmetric non-singular forms $\varphi\colon R^k \times R^k \to R$ modulo the submonoid generated by hyperbolic forms is an abelian group. We refer to this group as the *Witt group* $W(R)$.

(b) Show that the maps

$$\begin{array}{rcl} W(\mathbb{R}) & \to & \mathbb{Z} \\ \varphi & \mapsto & \operatorname{sign}(\varphi) \end{array} \quad \text{and} \quad \begin{array}{rcl} W(\mathbb{Z}) & \to & \mathbb{Z} \\ \varphi & \mapsto & \operatorname{sign}(\varphi \otimes \mathbb{R}) \end{array}$$

are isomorphisms.

(c) Show that $W(\mathbb{Q})$ admits an epimorphism onto $\mathbb{Z} \oplus (\mathbb{Z}_2)^\infty$.
Hint. Consider determinants of matrices representing a given form.
Remark. The group $W(\mathbb{Q})$ is computed in [**MH1973**, Chapter IV.2] or alternatively in [**Lam2005**, Chapter VI.4].

Exercise 133.4. Let M be a closed oriented 3-dimensional smooth manifold. Show that $\operatorname{sign}(M \times [0,1]) = 0$.

Exercise 133.5.

(a) Let M be a closed oriented connected $4k$-dimensional topological manifold and let N be a closed oriented $4l$-dimensional topological manifold. Show that

$$\operatorname{sign}(M \times N) \;=\; \operatorname{sign}(M) \cdot \operatorname{sign}(N).$$

(b) We extend the definition of the signature from page 2916 by defining $\operatorname{sign}(M) = 0$ for any closed oriented connected topological manifold whenever $\dim(M) \not\equiv 0 \bmod 4$. Show that the conclusion of (a) holds without any restrictions on the dimensions of M and N.

Remark. This exercise is carried out in [**Brun2005**, Lemma 2.6].

Exercise 133.6. Let M be a closed non-orientable $4k$-dimensional smooth manifold. Let $p\colon \widetilde{M} \to M$ be the orientation covering that we introduced on page 1109. Show that $\operatorname{sign}(\widetilde{M}) = 0$.

Exercise 133.7.

(a) Explain how to modify the proof of the Anti-Symmetric Forms Proposition 133.2 so that it also holds for the integers.

(b) Explain why the proof of the Symmetric Forms-over-$\mathbb{R}$-Proposition 133.6 cannot be generalized to the integers.

Exercise 133.8. Let A and C be invertible real $k \times k$-matrices. We assume that C is symmetric.

(a) Show that the signature of the matrix $\begin{pmatrix} 0 & A \\ A^T & C \end{pmatrix}$ is zero.

(b) Does the conclusion of (a) still hold if A and C are not invertible?

Exercise 133.9. For which $k \in \mathbb{Z}$ does there exist a closed oriented 3-dimensional smooth manifold N with $\operatorname{sign}(S^1 \times N) = k$?

Exercise 133.10. Let $A \in \mathrm{GL}(n, \mathbb{R})$. Show that

$$b^+(\varphi(A)) \;=\; \max\left\{ k \in \mathbb{N}_0 \;\middle|\; \begin{array}{l} \text{there exists a } k\text{-dimensional subspace } V \\ \text{of } \mathbb{R}^n \text{ with } v^T A v > 0 \text{ for all } v \in V \cap S^{n-1} \end{array} \right\}.$$

Exercise 133.11. Show that the map

$$\begin{array}{rcl} \{A \in \mathrm{GL}(n, \mathbb{R}) \,|\, A = A^T\} & \to & \mathbb{Z} \\ A & \mapsto & \operatorname{sign}(A) \end{array}$$

is continuous.

Hint. You can work either with the definition of the signature (or better, with the reformulation given in Exercise 133.10) or you could use Proposition 5.42.

Exercise 133.12. Show "by hand" that the forms on $\mathbb{Z}^3$ defined by the matrices

$$A \;=\; \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are isometric.

Exercise 133.13. Let $\varphi\colon V \times V \to \mathbb{Z}$ be a form over $\mathbb{Z}$. Show that there exists a non-degenerate form $\psi\colon W \times W \to \mathbb{Z}$ such that φ is isometric to the orthogonal sum of ψ with a form that is identically zero.

Exercise 133.14. Show that two non-singular symmetric forms (V, κ) and (W, λ) over $\mathbb{F}_2$ are isometric if and only if $\dim(V) = \dim(W)$ and if κ and λ have the same parity.

134. The Intersection Form III: The Novikov Additivity Theorem

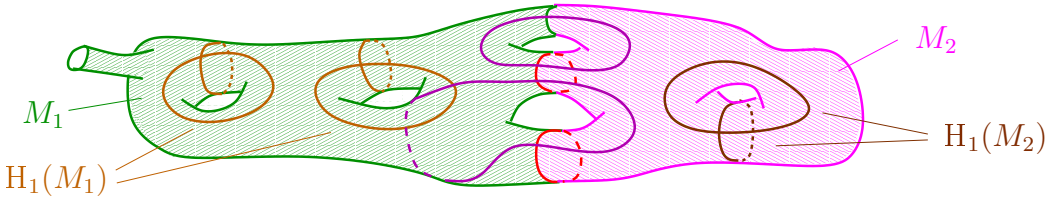
134.1. **Statement of the Novikov Additivity Theorem.** Let M be a compact oriented $4k$-dimensional topological manifold M. On page 2800 we introduced the intersection form

$$\begin{array}{rcl} Q_M\colon \mathrm{H}_{2k}(M)\times \mathrm{H}_{2k}(M) & \to & \mathbb{Z}. \\ (\varphi,\psi) & \mapsto & \langle \mathrm{PD}(\varphi)\cup \mathrm{PD}(\psi),[M]\rangle. \end{array}$$

On page 2916 we defined the signature $\mathrm{sign}(M)$ of a M as the signature of the intersection form Q_M.

In the Intersection Form-Additivity Proposition 132.12 we saw that the intersection form is additive under the connected sum operation. In the Manifold-Signature Lemma 133.11 we pointed out that this implies that the signature is additive under the connected sum operation.

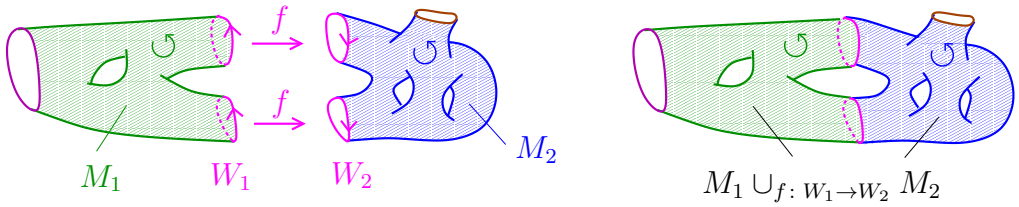
In general, as we can see in the example below, if we glue two compact oriented $4k$-dimensional topological manifolds M_1 and M_2 along arbitrary boundary components, then the middle homology is in general not additive. In particular the intersection form is certainly in no way additive.



Rather surprisingly, the following theorem, which was first stated by Sergej Novikov [**Nov1970**, p. 154] in the mid 1960s, shows that the signature is additive under gluing along boundary components.

Theorem 134.1. (Novikov Additivity Theorem) Let M_1 and M_2 be two compact oriented $4k$-dimensional topological manifolds, let W_1 be a union of components of ∂M_1 and let W_2 be a union of components of ∂M_2. Finally let $f\colon W_1\to W_2$ be an orientation-reversing homeomorphism. Then

$$\mathrm{sign}\Big(\underbrace{M_1\cup_{f\colon W_1\to W_2} M_2}_{\substack{\text{by the Gluing-Topological}\\ \text{Manifolds-Proposition 75.8 this is an}\\ \text{oriented topological manifold}}}\Big) = \mathrm{sign}(M_1)+\mathrm{sign}(M_2).$$



Example.

(1) In Exercise 134.1 we will use the Novikov Additivity Theorem 134.1 to give a new proof that the signature is additive under the connected sum operation. This gives an alternative proof of the Manifold-Signature Lemma 133.11 (2).

(2) In Exercise 134.2 we will use the Novikov Additivity Theorem 134.1 to show that the signature of the double of a compact oriented $4k$-dimensional topological manifold is necessarily zero.

Remark.

(1) If one glues two $4k$-manifolds along submanifolds of the boundary that are not union of components, then in general the conclusion of the Novikov Additivity Theorem 134.1 no longer holds.

For example, by the discussion on page 2882 we know that $\text{sign}(\mathbb{C}\text{P}^2) = 1$. On the other hand the Handle Decomposition Theorem 138.5 says that every closed smooth manifold, in particular $\mathbb{C}\text{P}^2$, can be obtained iteratively by handle attachments.[1631] Each handle has vanishing homology, in particular vanishing intersection form and thus signature zero.

(2) In this more general setting, where we glue smooth manifolds along submanifolds of the boundary, C. T. C. Wall [**Wall1969**] proved a refinement of the Novikov Additivity Theorem 134.1. This refinement is usually referred to as the Wall Non-Additivity Theorem. A detailed discussion is given in [**Brun2005**, Chapter 5].

Before we move on to the long and exhausting proof of the Novikov Additivity Theorem 134.1 let us state the general theorem for general culture. It says that the signature is essentially the only invariant of compact oriented smooth manifolds that has the additivity property stated in the Novikov Additivity Theorem 134.1.

Theorem 134.2. (Jänich Theorem) Let $k \in \mathbb{N}$. If φ is a real valued orientation-preserving diffeomorphism[1632] invariant for compact oriented $4k$-dimensional smooth manifolds that satisfies the additivity property stated in the Novikov Additivity Theorem 134.1, then there exists a $C \in \mathbb{R}$ such that for every closed oriented k-dimensional smooth manifold we have $\varphi(M) = C \cdot \text{sign}(M)$.

Proof. The theorem was proved by Klaus Jänich [**Jä1968, Jä1969**] in 1968. ■

The remainder of this chapter is dedicated to the proof of the Novikov Additivity Theorem 134.1. Except for readers who have an insatiable appetite for technical proofs it is perhaps best to just move on to the next chapter, which makes for much more pleasant reading.

134.2. **Proof of the Novikov Additivity Theorem I.** We start out with the following blanket convention that applies to the whole chapter.

Convention. Throughout the remainder of this chapter we will work for (co-) homology groups with real coefficients. To simplify the notation we drop the "; $\mathbb{R}$" throughout.

By definition, and if you want by the Cup Product-Naturality Proposition 115.4, the signature of an oriented 4-dimensional topological manifold Z is defined in terms of the intersection form

$$Q_Z \colon \text{H}_{2k}(Z) \times \text{H}_{2k}(Z) \ \to \ \mathbb{R}.$$

[1631] It follows from Exercise 137.3 that $\mathbb{C}\text{P}^2$ has a handle decomposition consisting of a single 0-handle, a single 2-handle and a single 4-handle.

[1632] It is not clear to me whether the analogous statement also holds for topological manifolds.

For our purpose it is more useful to completely bypass homology, despite our emotional attachment to it, and to work with cohomology throughout. This leads us to the following definition.

Definition. Given a compact oriented $4k$-dimensional topological manifold Z we consider the form

$$\begin{aligned} I_Z\colon \mathrm{H}^{2k}(Z,\partial Z)\times \mathrm{H}^{2k}(Z,\partial Z) &\to \mathbb{R}\\ (\varphi,\psi) &\mapsto \langle \varphi\cup\psi,[Z]\rangle. \end{aligned}$$

Note that by the Cup Product-Commutativity Proposition 115.3 we know that this form is symmetric.

Next recall that in the Reduced Intersection Form Proposition 132.21 we introduced the reduced intersection form and we showed that it is non-singular. The following proposition gives us the cohomological analogue of the reduced intersection form.

Proposition 134.3. Let Z be a compact oriented $4k$-dimensional topological manifold. We denote by $\delta\colon \mathrm{H}^{2k-1}(\partial Z)\to \mathrm{H}^{2k}(Z,\partial Z)$ the connecting homomorphism of the long exact sequence of the pair $(Z,\partial Z)$.

(1) The following equality of vector subspaces holds:[1633]

$$\mathrm{im}(\delta\colon \mathrm{H}^{2k-1}(\partial Z)\to \mathrm{H}^{2k}(Z,\partial Z)) \;=\; \mathrm{H}^{2k}(Z,\partial Z)^{\perp},$$

where we take "$\perp$" with respect to the pairing I_Z

(2) We write $\widehat{\mathrm{H}}^{2k}(Z,\partial Z) := \mathrm{coker}(\delta\colon \mathrm{H}^{2k-1}(\partial Z)\to \mathrm{H}^{2k}(Z,\partial Z))$. The form

$$\begin{aligned} I_Z^{\mathrm{red}}\colon \widehat{\mathrm{H}}^{2k}(Z,\partial Z)\times \widehat{\mathrm{H}}^{2k}(Z,\partial Z) &\to \mathbb{R}\\ ([\varphi],[\psi]) &\mapsto \langle \varphi\cup\psi,[Z]\rangle \end{aligned}$$

is well-defined and non-singular.

(3) We have $\mathrm{sign}(Z)=\mathrm{sign}(I_Z)=\mathrm{sign}(I_Z^{\mathrm{red}})$.

The proof of Proposition 134.3 will require the remainder of this section. In the proof of Proposition 134.3, and also later on in this chapter, we will need to make use of the Cup Product-Naturality Proposition 115.4. To facilitate reading the coming proofs it is perhaps convenient to have the statement of the Cup Product-Naturality Proposition 115.4 at hand:

Proposition 115.4. (Cup Product-Naturality Proposition) Let (X,A,B) and (Y,C,D) be two excisive triads of topological spaces. Let $f\colon (X,A,B)\to(Y,C,D)$ be a map. For all $p,q\in\mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p(Y,C)\times \mathrm{H}^q(Y,D) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(Y,C\cup D)\\ \downarrow{\scriptstyle f^*\times f^*} & & \downarrow{\scriptstyle f^*}\\ \mathrm{H}^p(X,A)\times \mathrm{H}^q(X,B) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(X,A\cup B). \end{array}$$

Next we introduce some notation which we will use throughout this chapter.

Notation.

(1) Given a real vector space V we denote by $\overline{V}=\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$ the dual vector space.

[1633] In the words of the definition in Exercise 132.4 this means that $\mathrm{im}(\delta\colon \mathrm{H}^{2k-1}(\partial Z)\to \mathrm{H}^{2k}(Z,\partial Z))$ is the radical of the form I_Z.

(2) Given a homomorphism $f\colon V \to W$ between two real vector spaces we denote by $\overline{f}\colon \overline{W} \to \overline{V}$ the dual homomorphism.
(3) Given a compact oriented m-dimensional topological manifold Z we consider the maps

$$\begin{array}{rcl} \mathrm{D}\colon \mathrm{H}^r(Z) & \to & \overline{\mathrm{H}^{m-r}(Z,\partial Z)} \\ \varphi & \mapsto & (\psi \mapsto \langle \varphi \cup \psi, [Z]\rangle) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{D}\colon \mathrm{H}^r(Z,\partial Z) & \to & \overline{\mathrm{H}^{m-r}(Z)} \\ \varphi & \mapsto & (\psi \mapsto \langle \varphi \cup \psi, [Z]\rangle). \end{array}$$

It follows from the Manifold-Non Singularity Proposition 121.2, which is a consequence of the Poincaré Duality Theorem 119.1, that these two maps are isomorphisms.[1634]
(4) Given a compact oriented m-dimensional topological manifold Z we also consider the maps

$$\begin{array}{rcl} \widetilde{\mathrm{D}}\colon \mathrm{H}^r(Z,\partial Z) & \to & \overline{\mathrm{H}^{m-r}(Z,\partial Z)} \\ \varphi & \mapsto & (\psi \mapsto \langle \varphi \cup \psi, [Z]\rangle = I_Z(\varphi,\psi) \end{array}.$$

(5) We denote by

$$\begin{array}{rcl} i\colon \partial Z & \to & Z \\ j\colon (Z,\varnothing) & \to & (Z,\partial Z) \end{array}$$

the obvious maps.

The following lemma is closely related to Proposition 119.25.

Lemma 134.4. Let Z be a compact oriented m-dimensional topological manifold. For any $r \in \mathbb{N}_0$ we consider the following diagram:

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & \mathrm{H}^{r-1}(\partial Z) & \xrightarrow{\delta} & \mathrm{H}^r(Z,\partial Z) & \xrightarrow{j^*} & \mathrm{H}^r(Z) & \xrightarrow{i^*} & \ldots \\ & & \cong\downarrow \mathrm{D} & & \cong\downarrow \mathrm{D} & \overset{\widetilde{\mathrm{D}}}{\searrow} & \cong\downarrow \mathrm{D} & & \\ \ldots & \longrightarrow & \overline{\mathrm{H}^{m-r}(\partial Z)} & \xrightarrow{\overline{i^*}} & \overline{\mathrm{H}^{m-r}(Z)} & \xrightarrow[\overline{j^*}]{} & \overline{\mathrm{H}^{m-r}(Z,\partial Z)} & \xrightarrow{\overline{\delta}} & \ldots \end{array}$$

The following statements hold:
(1) The top and bottom horizontal sequences are exact.
(2) The diagram commutes.

Proof. The top horizontal sequence is just the long exact sequence in cohomology of the pair $(Z,\partial Z)$, hence it is exact. It follows by the same argument as in Lemma 108.4 that applying the $\mathrm{Hom}_{\mathbb{R}}(-,\mathbb{R})$ functor preserves exactness. This shows that the bottom horizontal sequence is also exact.

We turn to the proof that the diagram commutes. First we show that the square to the left commutes. Recall that given a topological space X we denote by $\epsilon_X\colon \mathrm{H}_0(X) \to \mathbb{R}$ the augmentation map that we introduced on pages 1590 and 1914. Let $\alpha \in \mathrm{H}^{r-1}(\partial Z)$ and let

[1634] Note that this statement only holds since we work with $\mathbb{R}$-coefficients.

$\beta \in \mathrm{H}^{m-r}(Z)$. Then

$$\begin{array}{lllll}
(\mathrm{D}\circ\delta)(\alpha)(\beta) & \overset{\text{by definition}}{=} \langle \delta(\alpha)\cup\beta, [Z]\rangle & \overset{\text{Cap Product-Kronecker Lemma 116.4}}{=} \epsilon_Z((\delta(\alpha)\cup\beta)\cap[Z]) & & \\
& = \epsilon_Z(\beta\cap(\delta(\alpha)\cap[Z])) & = \epsilon_Z(\beta\cap i_*(\alpha\cap[\partial Z])) & & \\
& \overset{\text{Cup-Cap Lemma 116.7}}{=} \epsilon_{\partial Z}(i^*(\beta)\cap(\alpha\cap[\partial Z])) & \overset{\text{by the Cap Product-Connecting Homomorphism Lemma 116.10 and the Fundamental Class-of-Boundary Proposition 106.27}}{=} \langle \alpha\cup i^*(\beta), [\partial Z]\rangle & \overset{\text{by definition}}{=} (\overline{i^*}\circ\mathrm{D})(\alpha)(\beta).\\
& \overset{\text{Cap Product-Naturality Lemma 116.8 and the discussion on page 1594}}{=} & \overset{\text{by the above arguments run backwards}}{=} & &
\end{array}$$

We turn to the two triangles on the right. Let $\alpha \in \mathrm{H}^r(Z, \partial Z)$ and let $\beta \in \mathrm{H}^{m-r}(Z, \partial Z)$. We perform the following calculation:

$$(\mathrm{D}\circ j^*)(\alpha)(\beta) \overset{\text{by definition of D}}{=} \langle j^*(\alpha)\cup\beta, [Z]\rangle = \overbrace{\langle \alpha\cup\beta, [Z]\rangle}^{=\widetilde{\mathrm{D}}(\alpha)(\beta)} = \langle \alpha\cup j^*(\beta), [Z]\rangle \overset{\text{by definition of D}}{=} (\overline{j^*}\circ\mathrm{D})(\alpha)(\beta).$$

Cup Product-Naturality Proposition 115.4 applied to the map $(Z, \varnothing, \partial Z) \to (Z, \partial Z, \partial Z)$ and map $(Z, \partial Z, \varnothing) \to (Z, \partial Z, \partial Z)$

Conveniently enough this calculation takes care of both triangles. ■

Proof. Proof of Proposition 134.3] The lemma can be deduced, with a little bit of jiu-jitsu from the Reduced Intersection Form Proposition 132.21. But esthetically it is more appealing to give a direct proof. So let us do so.

(1) We have the equalities

$$\begin{array}{lll}
\mathrm{im}(\delta\colon \mathrm{H}^{2k-1}(\partial Z) \to \mathrm{H}^{2k}(Z, \partial Z)) & \overset{\text{by Lemma 134.4}}{=} & \ker\left(\mathrm{D}\circ\overline{j^*}\colon \mathrm{H}^{2k}(Z,\partial Z) \to \overline{\mathrm{H}^{2k}(Z,\partial Z)}\right)\\
& = \ker\left(\widetilde{\mathrm{D}}\colon \mathrm{H}^{2k}(Z,\partial Z) \to \overline{\mathrm{H}^{2k}(Z,\partial Z)}\right) & = \mathrm{H}^{2k}(Z,\partial Z)^{\perp}.\\
& \overset{\text{by Lemma 134.4 we have } \mathrm{D}\circ\overline{j^*} = \widetilde{\mathrm{D}}}{=} & \overset{\text{by definition of D and } I_Z}{=}
\end{array}$$

(2) It follows immediately from (1) "$\subset$", together with the symmetry of the cup product, that the form I_Z^{red} is well-defined. Furthermore it follows from (1) (b)"$\supset$" and some elementary algebra, see Exercise 132.4, that I_Z^{red} is non-singular.

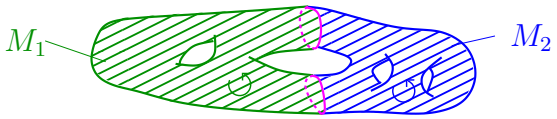
(3) Note that

$$\mathrm{sign}(Z) \overset{\text{Manifold-Signature Lemma 133.11}}{=} \mathrm{sign}(I_Z) \overset{\text{by Exercise 132.4}}{=} \mathrm{sign}\begin{pmatrix} 0 & 0\\ 0 & I_Z^{\mathrm{red}}\end{pmatrix} \overset{\text{Corollary 133.9}}{=} \mathrm{sign}(I_Z^{\mathrm{red}}).$$

■

134.3. **Proof of the Novikov Additivity Theorem II.** In this section we deal with the Novikov Additivity Theorem 134.1 in the special case that we glue the whole boundary of M_1 to the whole boundary of M_2.

Proof of Theorem 134.1 if $W_1 = \partial M_1$ and $W_2 = \partial M_2$. Let M_1 and M_2 be two compact oriented $4k$-dimensional topological manifolds and let $f\colon \partial M_1 \to \partial M_2$ be an orientation-reversing homeomorphism. We set $N := M_1 \cup_{f\colon \partial M_1 \to \partial M_2} M_2$. We will use the Gluing-Topological Manifolds-Proposition 75.8 to view M_1 and M_2 as codimension-zero submanifolds of N. We need to show that $\mathrm{sign}(N) = \mathrm{sign}(M_1) + \mathrm{sign}(M_2)$.
For $s = 1, 2$ we denote by

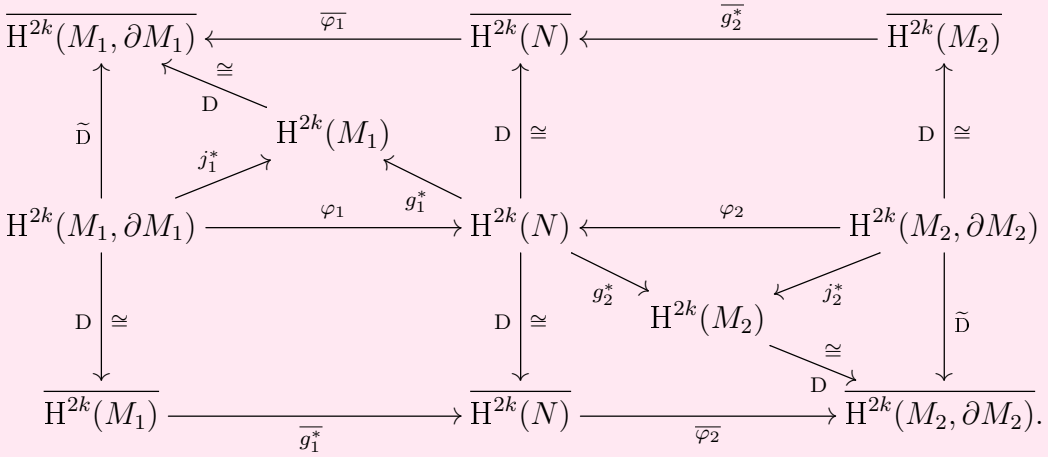


$$\begin{aligned} a_s\colon (M_s, \partial M_s) &\to (N, M_{3-s}),\\ b_s\colon (N, \varnothing) &\to (N, M_{3-s}),\\ j_s\colon (M_s, \varnothing) &\to (M_s, \partial M_s)\\ g_s\colon M_s &\to N \end{aligned}$$

the obvious maps. Note that the induced map $a_s^*\colon \mathrm{H}^{2k}(N, M_{3-s}) \to \mathrm{H}^{2k}(M_s, \partial M_s)$ is an isomorphism by the Excision Theorem 109.1. Therefore we can define

$$\varphi_s := b_s^* \circ (a_s^*)^{-1}\colon \mathrm{H}^{2k}(M_s, \partial M_s) \to \mathrm{H}^{2k}(N).$$

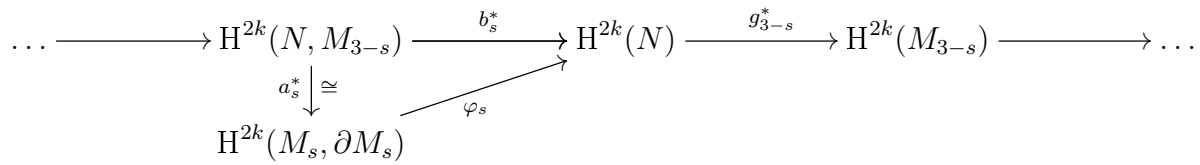
Claim 1. We consider the following diagram:



This diagram has the following two properties:

(1) The top and bottom horizontal sequences are exact.
(2) The diagram commutes.

Proof. First we show that the top and bottom horizontal sequences are exact. For $s \in \{1, 2\}$ we consider the following diagram



The top horizontal sequence is just a segment of the long exact sequence in cohomology of the pair (N, M_{3-s}). As discussed above, the vertical map is an isomorphism by the Excision Theorem 109.1. By definition of φ_s the diagram commutes. It follows that the sequence with the $\mathrm{H}^{2k}(M_s, \partial M_s)$-term is also exact. It follows by the same argument as in Lemma 108.4 that applying the $\mathrm{Hom}_{\mathbb{R}}(-, \mathbb{R})$ functor preserves exactness. This concludes the proof that the top and bottom horizontal sequences in the big diagram are exact.

We turn to the proof of the commutativity of the diagram. By symmetry it suffices to show that the left-hand side commutes.

(a) First we consider the big triangle. The argument that this triangle commutes is quite similar to the proof of the Codimension-Zero Submanifold-PD Lemma 120.5. But for completeness' sake we carry out the details. Thus we now let $\alpha \in \mathrm{H}^{2k}(N)$ and we let $\beta \in \mathrm{H}^{2k}(M_1, \partial M_1)$. Next we set $\gamma := (a_1^*)^{-1}(\beta) \in \mathrm{H}^{2k}(N, M_2)$. We perform the following calculation:

$$\begin{array}{lll}
\overbrace{\langle g_1^*(\alpha) \cup \beta, [M_1]\rangle}^{=(D \circ g_1^*)(\alpha)(\beta)} & & \\
= \langle g_1^*(\alpha) \cup a_1^*(\gamma), [M_1]\rangle & \overset{\text{Cup Product-Naturality Proposition 115.4 applied to the map } (M_1, \varnothing, \partial M_1) \to (N, \varnothing, M_2)}{=} \langle g_1^*(\alpha \cup \gamma), [M_1]\rangle & \overset{\text{Lemma 109.6 (3)}}{=} \langle \alpha \cup \gamma, a_{1*}([M_1])\rangle \\
= \langle \alpha \cup \gamma, b_{1*}([N])\rangle & = \langle b_1^*(\alpha \cup \gamma), [N]\rangle & \\
\overset{\text{Codimension Zero-Fundamental Class Lemma 106.24}}{=} \langle \alpha \cup b_1^*(\gamma), [N]\rangle = \langle \alpha \cup (b_1 \circ (a_1^*)^{-1})(\beta), [N]\rangle = \underbrace{\langle \alpha \cup \varphi_1(\beta), [N]\rangle}_{=(\overline{\varphi_1} \circ D)(\alpha)(\beta)}. & & \\
\text{Cup Product-Naturality Proposition 115.4 applied to the map } (N, M_1, \varnothing) \to (N, M_1, M_1) & &
\end{array}$$

(b) We showed in Lemma 134.4 that the small triangle to the left commutes.

(c) We turn to the small triangle for which $\mathrm{H}^{2k}(N)$ is a vertex. First note that it follows immediately from the definitions that $b_1 \circ g_1 = a_1 \circ j_1 \colon (M_1, \varnothing) \to (N, M_2)$. This implies that $g_1^* \circ b_1^* = j_1^* \circ a_1^*$. Since $\varphi_1 = b_1^* \circ (a_s^*)^{-1}$ we see that $g_1^* \circ \varphi_1 = j_1^*$.

(d) Finally we consider the square at the bottom left. In (a) we just showed that

$$D \circ g_1^* = \overline{\varphi_1} \circ D \colon \mathrm{H}^{2k}(N) \ \to \ \overline{\mathrm{H}^{2k}(M_1, \partial M_1)}.$$

We take the $\mathrm{Hom}(-, \mathbb{R})$ dual of this statement. Using the symmetry of the cup product, see the Cup Product-Commutativity Proposition 115.3, we see that $\overline{\mathrm{D}} = \mathrm{D}$. Thus we obtain, as desired, that

$$\overline{g_1^*} \circ D = D \circ \varphi_1 \colon \mathrm{H}^{2k}(M_1, \partial M_1) \ \to \ \overline{\mathrm{H}^{2k}(N)}. \qquad \boxplus$$

Claim 2. For $s = 1, 2$ we set

$$A_s \ := \ \mathrm{im}(\varphi_s \colon \mathrm{H}^{2k}(M_s, \partial M_s) \to \mathrm{H}^{2k}(N)).$$

With respect to the form $I_N \colon \mathrm{H}^{2k}(N) \times \mathrm{H}^{2k}(N) \to \mathbb{R}$ we have the following equalities:

$$(1) \quad A_s^\perp \ = \ A_{3-s} \qquad \text{and} \qquad (2) \quad (A_1 \cap A_2)^\perp \ = \ A_1 + A_2.$$

Proof. First we prove Equality (1). This equality follows immediately from the commutative diagram in Claim 1, the fact that the bottom sequence is exact and one minute's worth of concentration. More precisely, we consider the following diagram, which is just given by a slight reorganization of the bottom part of the commutative diagram in Claim 1:

$$\begin{array}{ccccc}
\mathrm{H}^{2k}(M_1, \partial M_1) & \xrightarrow{\varphi_1} & \mathrm{H}^{2k}(N) & \xrightarrow{g_2^*} & \mathrm{H}^{2k}(M_2) \\
\mathrm{D} \downarrow \cong & & \mathrm{D} \downarrow \cong & & \mathrm{D} \downarrow \cong \\
\overline{\mathrm{H}^{2k}(M_1)} & \xrightarrow{\overline{g_1^*}} & \overline{\mathrm{H}^{2k}(N)} & \xrightarrow{\overline{\varphi_2}} & \overline{\mathrm{H}^{2k}(M_2, \partial M_2)}.
\end{array}$$

Since the diagram commutes, since the bottom sequence is exact and since the vertical maps are isomorphisms we see that

$$\underbrace{\operatorname{im}\left(\varphi_1\colon \mathrm{H}^{2k}(M_1,\partial M_1)\to \mathrm{H}^{2k}(N)\right)}_{=A_1} \;=\; \underbrace{\ker\left(\overline{\varphi_2}\circ \mathrm{D}\colon \mathrm{H}^{2k}(N)\to \overline{\mathrm{H}^{2k}(M_2,\partial M_2)}\right)}_{=A_2^\perp}.$$

Thus we have shown that $A_1 = A_2^\perp$. The same argument shows that $A_2 = A_1^\perp$. This concludes the proof of Equality (1).

Next note that Equality (2) follows from Equality (1) together with some elementary algebra, see Exercise 132.5. For the reader's convenience we provide the argument. First note that it follows immediately from (1) that $A_1 + A_2 \subset (A_1 \cap A_2)^\perp$. To show equality it suffices to show that both vector subspaces have the same dimension. As the following calculation shows, this is indeed the case:

since N is closed we know by the Manifold-Non Singularity Proposition 121.2 that I_N is non-singular, hence for any vector subspace $B \subset \mathrm{H}^{2k}(N)$ we have $\dim(B^\perp) = \dim(\mathrm{H}^{2k}(N)) - \dim(B)$

$$\begin{array}{ccc} \dim((A_1\cap A_2)^\perp) & \overset{\downarrow}{=} & \dim(\mathrm{H}^{2k}(N))-\dim(A_1\cap A_2) \\ & = & \dim(A_1)+\dim(A_2)-\dim(A_1\cap A_2) \\ & \overset{\uparrow}{=} & \text{by (1)} \end{array} \qquad \begin{array}{ccc} & \overset{\downarrow}{=} & \dim(A_1)+\dim(A_1^\perp)-\dim(A_1\cap A_2) \\ & = & \dim(A_1+A_2). \\ & \overset{\uparrow}{=} & \text{since } 0\to A_1\cap A_2\to A_1\oplus A_2\to A_1+A_2\to 0 \text{ is exact} \end{array}$$ ⊞

Claim 3. Let $s \in \{1,2\}$. As in Proposition 134.3 we write

$$\widehat{\mathrm{H}}^{2k}(M_s,\partial M_s) \;:=\; \operatorname{coker}(\delta\colon \mathrm{H}^{2k-1}(\partial M_s)\to \mathrm{H}^{2k}(M_s,\partial M_s)).$$

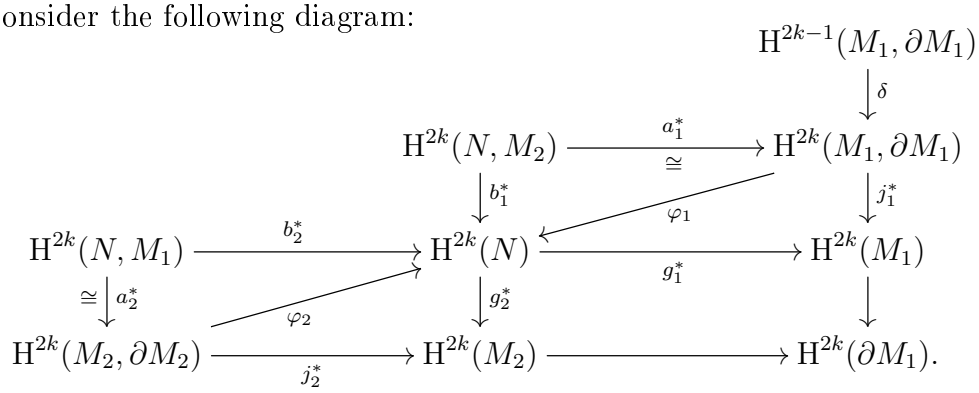
The map

$$\varphi_s\colon \mathrm{H}^{2k}(M_s,\partial M_s) \;\to\; \mathrm{H}^{2k}(N)$$

induces an isomorphism

$$\begin{array}{rcl} \widehat{\varphi}_s\colon \widehat{\mathrm{H}}^{2k}(M_s,\partial M_s) & \to & A_s/(A_1\cap A_2) \\ [\sigma] & \mapsto & [\varphi_s(\sigma)]. \end{array}$$

Proof. We consider the following diagram:



We make the following clarifications and observations:

- All undecorated maps are induced by the obvious maps.
- The vertical sequences and the horizontal sequences are long exact sequences of pairs.
- All parts of the diagram commute. This basically follows from the definition of φ_s and φ_{3-s} and the fact that all the remaining maps are induced by maps of pairs of topological spaces. Note that the fact that the triangles involving φ_s, j_s^* and g_s^* commute was already discussed in Claim 1.

A little diagram chase now shows that each map $\widehat{\varphi}_s\colon \widehat{\mathrm{H}}^{2k}(M_s, \partial M_s) \to A_s/(A_1 \cap A_2)$, that is given by $[\sigma] \mapsto [\varphi_s(\sigma)]$, is well-defined and an isomorphism. ⊞

Claim 4. We consider the projections

$$p_s\colon \mathrm{H}^{2k}(M_s, \partial M_s) \to \widehat{\mathrm{H}}^{2k}(M_s, \partial M_s) \quad \text{and} \quad q\colon \mathrm{H}^{2k}(N) \to \mathrm{H}^{2k}(N)/(A_1 + A_2)$$

and we pick corresponding left-inverses

$$t_s\colon \widehat{\mathrm{H}}^{2k}(M_s, \partial M_s) \to \mathrm{H}^{2k}(M_s, \partial M_s) \quad \text{and} \quad u\colon \mathrm{H}^{2k}(N)/(A_1 + A_2) \to \mathrm{H}^{2k}(N).$$

Next we consider the following vector subspaces of $\mathrm{H}^{2k}(N)$:

$$\begin{aligned} H_1 &:= \operatorname{im}(\varphi_1 \circ t_1\colon \widehat{\mathrm{H}}^{2k}(M_1, \partial M_1) \to \mathrm{H}^{2k}(N)) \\ H_2 &:= \operatorname{im}(\varphi_2 \circ t_2\colon \widehat{\mathrm{H}}^{2k}(M_2, \partial M_2) \to \mathrm{H}^{2k}(N)) \\ D &:= A_1 \cap A_2 \\ E &:= \operatorname{im}(u\colon \mathrm{H}^{2k}(N)/(A_1 + A_2) \to \mathrm{H}^{2k}(N)). \end{aligned}$$

The following two statements hold:

(1) We have $\dim(D) = \dim(E)$.

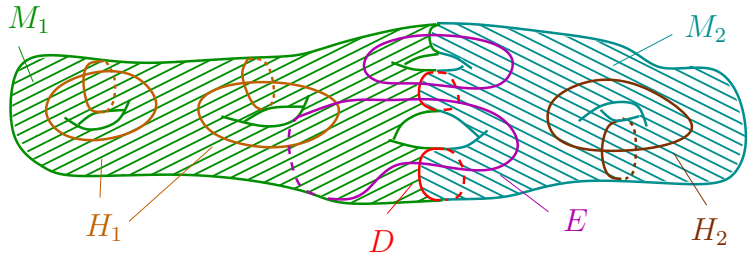
(2) The various inclusions induce an isomorphism

$$H_1 \oplus H_2 \oplus D \oplus E \xrightarrow{\cong} \mathrm{H}^{2k}(N).$$

(3) With respect to this decomposition of $\mathrm{H}^{2k}(N)$ given in (1) the form I_N can be written as follows:

$$I_N = \begin{array}{c} \\ H_1 \\ H_2 \\ D \\ E \end{array} \begin{array}{c} \begin{array}{cccc} H_1 & H_2 & D & E \end{array} \\ \begin{pmatrix} I^{\mathrm{red}}_{M_1} & 0 & 0 & * \\ 0 & I^{\mathrm{red}}_{M_2} & 0 & * \\ 0 & 0 & 0 & X \\ * & * & X^T & * \end{pmatrix} \end{array}$$

where X is represented by an invertible square matrix.

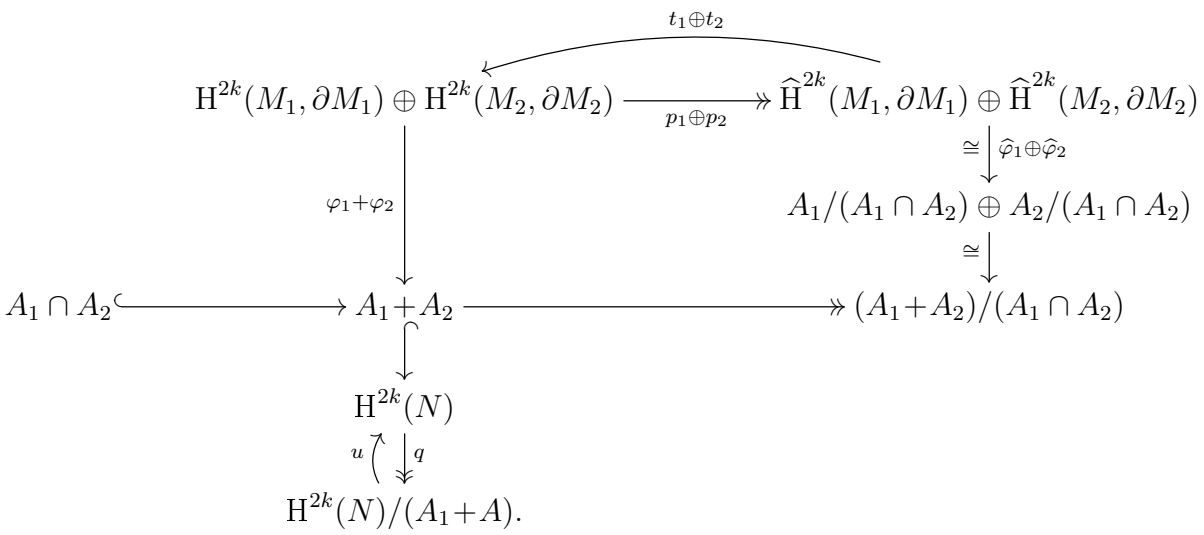


in each case the cohomology classes are the Poincaré duals of the homology classes

Proof.

(1) Recall that in the proof of Claim 2 (2) we already showed that we have the equality $\dim(A_1 \cap A_2) = \dim(N) - \dim(A_1 + A_2)$. But this means precisely that we have the promised equality $\dim(D) = \dim(E)$.

(2) We consider the following diagram:

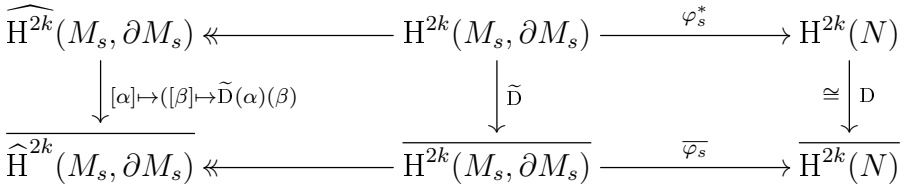


The desired statement follows easily from the following observations:

- The central horizontal sequence is exact and the vertical sequence starting with $A_1 + A_2$ are exact.
- One can easily verify that the diagram commutes.

(3) The statement follows from the following three bullet points:

- The three zeros in the "D-column" and the "D-row" follow immediately from the fact, shown in Claim 3, that $(A_1 \cap A_2)^\perp = A_1 + A_2$.
- We consider the following diagram:



The square to the left commutes by definition and the square to the right commutes since it equals the upper left respectively the lower right quadrant of the diagram in Claim 1. But this means precisely that the "H_s-diagonal term" is given by $I^{\mathrm{red}}_{M_s}$.

- By (1) we know that X is a square matrix. The fact that X is invertible now follows from the fact, shown in the Manifold-Non Singularity Proposition 121.2, that I_N is non-singular. ⊞

Now we see that

$$\operatorname{sign}(N) = \operatorname{sign}(I_N) = \operatorname{sign}\begin{pmatrix} I^{\mathrm{red}}_{M_1} & 0 & 0 & * \\ 0 & I^{\mathrm{red}}_{M_2} & 0 & * \\ 0 & 0 & 0 & X \\ * & * & X^T & * \end{pmatrix} = \operatorname{sign}\begin{pmatrix} I^{\mathrm{red}}_{M_1} & 0 & 0 & 0 \\ 0 & I^{\mathrm{red}}_{M_2} & 0 & 0 \\ 0 & 0 & 0 & X \\ 0 & 0 & X^T & * \end{pmatrix}$$
$$= \operatorname{sign}(I^{\mathrm{red}}_{M_2}) + \operatorname{sign}(I^{\mathrm{red}}_{M_2}) + \underbrace{\operatorname{sign}\begin{pmatrix} 0 & X \\ X^T & * \end{pmatrix}}_{=0 \text{ by Exercise 133.2, (or Lemma 144.8), since } X \text{ is invertible}} = \operatorname{sign}(M_1) + \operatorname{sign}(M_2).$$

Proposition 134.3 (3) — since X is invertible there exists an isometry between the two forms — by Corollary 133.9 (2) — Proposition 134.3 (3)

■

134.4. **Proof of the Novikov Additivity Theorem III.** Before we formulate the next special case of Theorem 134.1 we introduce some definitions.

Definition.

(1) We say that a closed oriented n-dimensional topological manifold S is nullbordant if there exists a compact oriented $(n+1)$-dimensional topological manifold T such that $\partial T = S$ as oriented topological manifolds. Such a T is called a nullbordism for S.

(2) In the setting of the Novikov Additivity Theorem 134.1 we refer to each of $\partial M_1 \setminus W_1$ and $\partial M_2 \setminus W_2$ as the unglued boundary

With this definition we can formulate the next special case of the Novikov Additivity Theorem 134.1 that we can deal with.

Proof of Theorem 134.1 if the unglued boundaries are nullbordant. We assume that the unglued boundary $\partial M_1 \setminus W_1$ and $\partial M_2 \setminus W_2$ are nullbordant. By the above definition this means that there exist compact oriented $4k$-dimensional topological X_1 and X_2 with $\partial X_s = \partial M_s \setminus W_s$ as oriented topological manifolds. Using the homeomorphism $f\colon W_1 \to W_2$ we can write $W := W_1 = W_2$. We perform the following calculation:

$$\operatorname{sign}(-X_1 \cup_{\partial M_1 \setminus W_1} M_1 \cup_W M_2 \cup_{\partial M_2 \setminus W_2} -X_2) \overset{\text{special case applied to } M_1 \sqcup -X_2 \text{ glued to } M_2 \sqcup -X_1 \text{ glued along the common boundary}}{=} \operatorname{sign}(M_1 \sqcup -X_2) + \operatorname{sign}(M_2 \sqcup -X_1)$$
$$= \operatorname{sign}(M_1) + \operatorname{sign}(M_2) - \operatorname{sign}(X_1) - \operatorname{sign}(X_2).$$

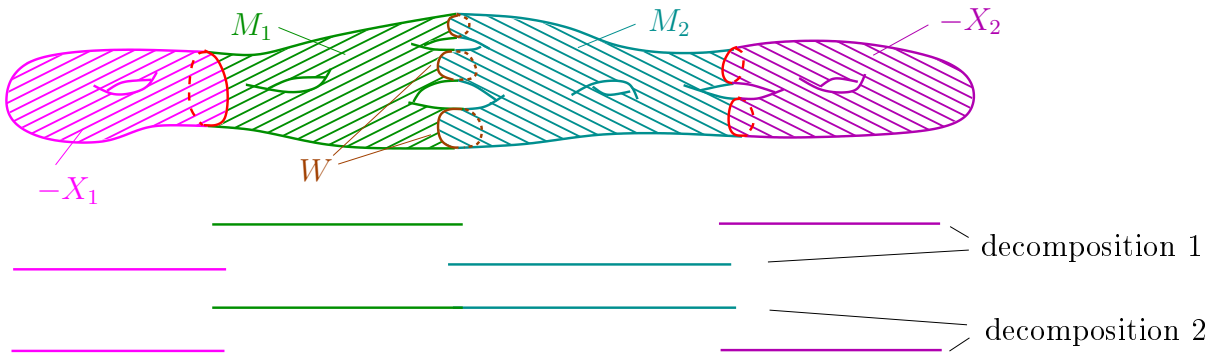
by the Manifold-Signature Lemma 133.11 we have $\operatorname{sign}(A \sqcup B) = \operatorname{sign}(A) + \operatorname{sign}(B)$ and we have $\operatorname{sign}(-A) = -\operatorname{sign}(A)$

Since this was so much fun we compute the signature of the glued manifold once again with a different decomposition:

$$\operatorname{sign}(-X_1 \cup_{\partial M_1 \setminus W_1} M_1 \cup_W M_2 \cup_{\partial M_2 \setminus W_2} -X_2) \overset{\text{special case applied to } M_1 \cup_W M_2 \text{ glued to } -X_2 \sqcup -X_1 \text{ glued along the common boundary}}{=} \operatorname{sign}(M_1 \cup_W M_2) + \operatorname{sign}(-X_1 \sqcup -X_2)$$
$$= \operatorname{sign}(M_1 \cup_W M_2) - \operatorname{sign}(X_1) - \operatorname{sign}(X_2).$$

Comparing these two calculations we obtain the desired result. ■

Now we can finally turn to the proof of the general case of the Novikov Additivity Theorem 134.1.



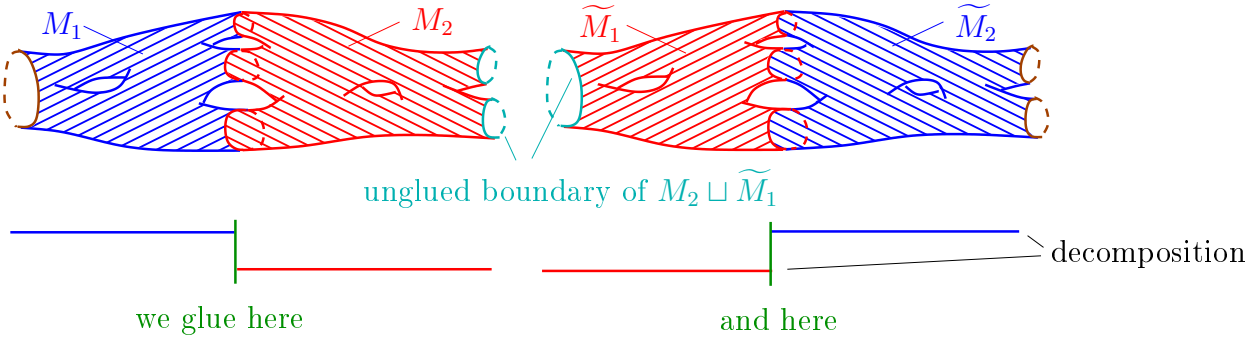
Proof of the Novikov Additivity Theorem 134.1. Let $\widetilde{M}_1$ and $\widetilde{M}_2$ be copies of M_1 and M_2. Furthermore we denote by $\widetilde{W} = \widetilde{W}_1 = \widetilde{W}_2$ a copy of $W = W_1 = W_2$. We perform the following calculation:

$$\begin{aligned} 2 \cdot \operatorname{sign}(M_1 \cup_W M_2) &= \operatorname{sign}\big((M_1 \cup_W M_2) \sqcup (\widetilde{M}_1 \cup_{\widetilde{W}} \widetilde{M}_2)\big) \\ &= \operatorname{sign}\big((M_1 \sqcup \widetilde{M}_2)_{W \sqcup \widetilde{W}} (\widetilde{M}_1 \sqcup M_2)\big) \\ &= \operatorname{sign}\big(M_1 \sqcup \widetilde{M}_2\big) + \operatorname{sign}\big(\widetilde{M}_1 \sqcup M_2\big) \;=\; 2 \cdot (\operatorname{sign}(M_1) + \operatorname{sign}(M_2)). \end{aligned}$$

follows from the previous special case, which we can apply for the following reason:
unglued boundary of $M_1 \sqcup \widetilde{M}_2$ is the boundary of $\widetilde{M}_1 \cup_{\widetilde{W}_1 = W_2} M_2$, in particular it is nullbordant,
unglued boundary of $\widetilde{M}_1 \sqcup M_2$ is the boundary of $M_1 \cup_{W_1 = \widetilde{W}_2} \widetilde{M}_2$, in particular it is nullbordant

the Manifold-Signature Lemma 133.11

Dividing both sides by 2 gives us the desired result. ∎



Exercises for Chapter 134.

Exercise 134.1. Use the Novikov Additivity Theorem 134.1 to show that the signature is additive under the connected sum operation.
Remark. This gives a new proof of the Manifold-Signature Lemma 133.11 (2).

Exercise 134.2. Let M be a compact oriented $4k$-dimensional topological manifold and let W be the union of some components of ∂M. As on page 1667 we refer to

$$\mathrm{D}_W M \;:=\; (M \times \{1\}) \cup_{W \times \{1\} = W \times \{2\}} (M \times \{2\})$$

as the *double of M along W*. By Lemmas 75.12 and 105.18 we know that $\mathrm{D}_W M$ is a compact orientable topological manifold. Show that the signature of $\mathrm{D}_W M$ is zero.

Exercise 134.3. Let $k \in \mathbb{N}$. Give an example of two compact oriented 4-dimensional topological manifolds M_1 and M_2 together with a homeomorphism $f\colon \partial M_1 \to \partial M_2$ such that the following statements are satisfied:

(a) $b_2(M_1) = k$ and $b_2(M_2) = k$.
(b) The inclusion induced maps $\mathrm{H}_2(M_1) \to \mathrm{H}_2(M_1 \cup_f M_2)$ and $\mathrm{H}_2(M_2) \to \mathrm{H}_2(M_1 \cup_f M_2)$ are both the zero map.

135. The intersection form IV: 4-dimensional manifolds

We have now introduced lots of invariants of topological spaces, namely the fundamental group, higher homotopy groups, homology groups and the cohomology groups together with the cup product. The question arises, whether we are done at some point, or whether this list of ever new invariants continues forever.
In an attempt to respond to this question we want to consider closed oriented simply connected 4-dimensional (topological) smooth manifolds and their intersection forms. What follows is a long survey on some amazing results. Unfortunately the proofs of any of these statements goes well beyond what we can cover in these introductory notes.
The following question will guide us through the chapter.

Question 135.1.

(1) Is the intersection form a complete invariant of closed oriented simply connected 4-dimensional (topological) smooth manifolds?[1635]
(2) Which isometry types of non-singular symmetric forms are realized by closed oriented simply connected 4-dimensional (topological) smooth manifolds?

135.1. **The K3-surface and Rokhlin's Theorem.** In the following we will give a mostly chronological account of the developments regarding the two (or possibly four?) questions posed in Question 135.1.
On page 2911 we introduced the initially rather unexpected form E_8. One might wonder whether a form as strange as E_8 can ever appear "in nature". As we will now see, somewhat surprisingly this is indeed the case.

Definition. We refer to

$$\mathrm{K3} := \left\{[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}\mathrm{P}^3 \,\middle|\, z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\right\}$$

as the K3-surface. Sometimes the K3-surface is also called the Kummer surface.

Remark. The odd name "K3-surface" comes about as follows.

(1) First of all, as we will see in Proposition 135.2, K3 is a *2-dimensional* complex manifold, i.e. it is a complex *surface*.
(2) According to [**OTL2016**] the definition of the K3-surface is already implicit in the work of Srinivasa Ramanujan in the 1910s. The K3-surface was (re-) discovered by André Weil [**Weil1958**] who named it in honor of three algebraic geometers, Kummer, Kähler and Kodaira, and also the mountain K2 in Pakistan which is the second highest mountain on earth. More precisely, André Weil wrote: "Dans la seconde partie de mon rapport, il s'agit des varietés kählériennes dites K3, ainsi nommées en l'honneur de Kummer, Kähler, Kodaira et de la belle montagne K2 au Cachemire".

The following proposition summarizes some of the key aspects of the K3-surface.

[1635]Note that for a simply connected 4-dimensional topological manifold M we have $\mathrm{H}_1(M;\mathbb{Z}) = 0$ by the Hurewicz Theorem 84.5. If M is closed and orientable, then we have $\mathrm{H}_3(M;\mathbb{Z}) = 0$ by the Poincaré Duality Theorem 119.1. Thus we see that for closed oriented simply connected 4-dimensional topological manifolds the intersection form contains all the (co-) homological data we could ask for.



the mountain K2, almost as pretty as the K3-surface

Proposition 135.2. (K3-Intersection Form Proposition) The K3-surface is a closed simply connected complex manifold of complex dimension 2 or equivalently of real dimension 4. Furthermore we have

$$\text{intersection form of the K3-surface} \;\cong\; 2 \cdot E_8 \oplus 3 \cdot H.$$

Proof. All the statements are proved in [**GoS1999**, Theorem 1.3.8] or alternatively [**McS2016**, p. 176]. The intersection form is also calculated in [**Huy2016**, Proposition 3.5]. The calculation of the intersection form, perhaps somewhat disappointingly, makes use of the Indefinite Even Forms Corollary 133.19. ■

Remark. The K3-surface, albeit with a somewhat different definition as ours, plays a big role in (algebraic) geometry, as is indicated by the fact that a whole book [**Huy2005**] is dedicated to the study of the K3-surface(s). Note that it follows from [**Huy2016**, Theorem 2.4.2 and 7.1.1] that the K3-surface as defined above, viewed as a smooth manifold, admits many non-biholomorphic complex structures.[1636]

In the following we want to discuss one particular aspect of the K3-surface. Namely, the calculation of the intersection form of the K3-surface, of the intersection form of $S^2 \times S^2$ and the Intersection Form-Additivity Proposition 132.12 raises the question whether one can split off copies of $S^2 \times S^2$ form the K3-surface. More precisely the following question arose almost immediately after the introduction of the K3-surface.

Question 135.3.

(1) Does there exist a closed oriented 4-dimensional topological manifold M such that the K3-surface is homeomorphic to $M \# S^2 \times S^2$?

(2) Does there exist a closed oriented 4-dimensional smooth manifold M such that the K3-surface is diffeomorphic to $M \# S^2 \times S^2$?

We will return to the question shortly. But before we do so, we continue with our chronological discussion. Next in line is a result proved by Vladimir Rokhlin in 1952. To formulate Rokhlin's Theorem we need the notion of a "spin structure". The definition of a "spin structure" is for example given in [**GoS1999**, Chapter 1.4.2], but the precise definition is of no concern to us and we will use this notion as a black box. We just note that in the following we say that a smooth manifold is *spin*, if it admits a spin structure. We now state the key result on the existence of spin structures in our context.

Proposition 135.4. Let X be a closed oriented 4-dimensional smooth manifold.

(1) If X is spin, then the intersection form Q_X is even.

[1636] We had seen already in Corollary 63.2 that there exist complex manifolds that are diffeomorphic but not biholomorphic.

(2) If the intersection form Q_X is even and if $\mathrm{H}_1(X;\mathbb{Z})$ has no 2-torsion, then the smooth manifold X is spin.

Proof. The proposition is the content of [**GoS1999**, Corollary 5.7.6], which in turn is proved using the "Wu formula". The proposition is also discussed in [**Scor2005**, p. 163]. ■

Now let us state Rokhlin's celebrated theorem.

Theorem 135.5. (Rokhlin 1952) Let X be a closed oriented 4-dimensional smooth manifold. If X is spin, then $\mathrm{sign}(X)$ is divisible by 16.

Proof. This theorem was proved by Vladimir Rokhlin [**Rok1952**][**Rok1986a**, p. 21] in 1952. More modern proofs of Rokhlin's Theorem are given in [**FK1978**], [**LaM1989**, Theorem II.2.13 and Corollary IV.1.2] and [**Sav2002**, Theorem 2.1]. ■

Remark. We will now see that the converse to neither of the two statements in Proposition 135.4 holds.

(1) We consider the *Enriques surfaces*

$$X := \underbrace{\left\{[z_0:z_1:z_2:z_3] \in \mathbb{C}\mathrm{P}^3 \;\middle|\; z_0^4+z_1^4+z_2^4+z_3^4=0\right\}}_{=\text{K3-surface}} \Big/ (z_0,z_1,z_2,z_3) \sim (\overline{z}_1,-\overline{z}_0,\overline{z}_3,-\overline{z}_2).$$

We make the following observations:

(a) It is elementary to show that the equivalence relation $\sim$ on K3 defines a smooth free orientation-preserving action by the group $\mathbb{Z}_2$.

(b) It follows from (a) and Propositions 19.32 and 25.19 that the Enriques surface is a closed oriented connected 4-dimensional smooth manifold.

(c) Since the K3-surface is simply connected it follows from (a), together with Lemma 48.5, the Actions-Covering Proposition 48.9 and the Fundamental Group-via-Actions Theorem 48.16, that $\pi_1(X) \cong \mathbb{Z}_2$. In particular we obtain from the Hurewicz Theorem 84.5 that $\mathrm{H}_1(X;\mathbb{Z})$ has 2-torsion.

(d) As is shown in [**Hab1982**], the intersection form Q_X is isometric to $E_8 \oplus H$, in particular Q_X is even and $\mathrm{sign}(X) = 8$.

Note that it follows from (d) together with Rokhlin's Theorem 135.5 that X cannot be spin. In particular we see that the converse to Proposition 135.4 (1) does not hold. We refer to [**Hab1982**], [**FS1984**], [**GoS1999**, p. 93] and [**LaM1989**, p. 90] for more details on the Enriques surface.

(2) We consider the smooth manifold $S^1 \times \mathbb{R}\mathrm{P}^3$. It follows from the discussion on page 3066 that $S^1 \times \mathbb{R}\mathrm{P}^3$ is parallelizable, i.e. its tangent bundle is a trivial bundle. This implies that $S^1 \times \mathbb{R}\mathrm{P}^3$ admits a spin structure. Since $\mathrm{H}_2(S^1 \times \mathbb{R}\mathrm{P}^3;\mathbb{Z}) = 0$ we see that its intersection form is even. But $\mathrm{H}_1(S^1 \times \mathbb{R}\mathrm{P}^3) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ has 2-torsion. Thus we see that the converse to Proposition 135.4 (2) does not hold.

The combination of Proposition 135.4 and Rokhlin's Theorem 135.5 gives us the following variation on Rokhlin's Theorem. This was historically the first result to address Question 135.1 (2).

Theorem 135.6. (Rokhlin 1952) Let X be a closed oriented 4-dimensional smooth manifold. If the intersection form Q_X is even and if $\mathrm{H}_1(X;\mathbb{Z})$ has no 2-torsion (e.g. if X is simply connected), then $\mathrm{sign}(X)$ is divisible by 16.[1637]

In a different direction the following theorem was the first noteworthy result towards an affirmative answer to Question 135.1 (1).

Theorem 135.7. (Milnor 1958) Let X and Y be two closed oriented simply connected 4-dimensional topological manifolds. Then

$$X \text{ and } Y \text{ are homotopy equivalent} \iff Q_X \text{ is isometric to } \pm Q_Y.$$

Proof. This theorem was proved for smooth manifolds by John Milnor [**Miln1958b**, Theorem 3] in 1958, building on work of John Whitehead [**WhdJ1949a**]. Furthermore [**GoS1999**, Theorem 1.2.25] and [**Kir1989**, Theorem 2.1] sketch a proof that also works for topological manifolds. ■

135.2. **The work of Michael Freedman and Simon Donaldson.** After the somewhat preliminary results by Vladimir Rokhlin and John Milnor the early 1980s saw two major breakthroughs by Michael Freedman and Simon Donaldson.
First, in 1981 Michael Freedman [**Fre1982**] gave a complete answer to Question 135.1 in the category of topological manifolds.[1638] More precisely, Michael Freedman [**Fre1982**] proved the following theorem.

Theorem 135.8. (Freedman 1981) Let Q be a nonsingular symmetric form over the integers.

(1) If Q is even, then up to an orientation-preserving homeomorphism, there exists a unique closed oriented simply connected topological manifold whose intersection form is isometric to Q.
(2) If Q is odd, then up to orientation-preserving homeomorphisms, there exist precisely two closed oriented simply connected topological manifolds whose intersection forms are isometric to Q. These two topological manifolds are distinguished by the Kirby-Siebenmann invariant which is discussed in the subsequent remark.

Remark.

(1) Let X be a simply connected closed topological manifold. By the Hurewicz Theorem 85.5 we know that the Hurewicz homomorphism $\pi_2(X) \to \mathrm{H}_2(X;\mathbb{Z})$ is an isomorphism. In particular, if $\pi_2(X)$ vanishes, then $\mathrm{H}_2(X;\mathbb{Z}) = 0$. This fact shows that Theorem 135.8, applied to the form on the trivial group $\{0\}$, implies that every homotopy 4-sphere is homeomorphic to S^4. We will discuss this in greater detail in Chapter **??**.
(2) The two homeomorphism types of Theorem 135.8 are distinguished by the Kirby-Siebenmann invariant[1639] $\mathrm{ks}(M) \in \mathrm{H}^4(M;\mathbb{Z}_2) = \mathbb{Z}_2$ of a closed connected non-empty 4-dimensional topological manifold M. This invariant is defined in [**KSi1977**, p. 318]

[1637] Recall that from the Even Forms Proposition 133.18 we know, for purely algebraic reasons, that the signature of a non-singular even symmetric form is divisible by 8.
[1638] Michael Freedman (*1951) is an American mathematician, in 1986 he was awarded the Fields medal for the proof of Theorem 135.8.
[1639] Here it is worth recalling that the Top-Cohomology Proposition 110.14 (3) says that for a closed connected non-empty 4-dimensional topological manifold we have $\mathrm{H}^4(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$.

or alternatively in [**FQ1990**, Section 10.2B] and [**Rudy2016**, Chapter 3.4]. The Kirby-Siebenmann invariant has the following properties:

(a) By [**Rudy2016**, Theorem 3.4.3] this invariant vanishes if X admits a PL-structure.[1640] Recall that in Theorem 96.15 we saw that every smooth manifold admits a PL-structure. In particular we see that the Kirby-Siebenmann invariant vanishes for smooth manifolds. We refer to [**Cer1968**, p. IX] and [**Scor2005**, p. 220] for more details.

(b) By [**FNOP2019**, Theorem 7.3] the Kirby-Siebenmann invariant is additive under the connected sum operation.

(3) Freedman's Theorem also answers Question 135.3 (1) in the affirmative. More precisely, by the existence statement of Theorem 135.8 (1) there exists a closed oriented simply connected 4-dimensional topological manifold X with $Q_X \cong E_8$. It follows from Propositions 132.11 and 132.12, together with the uniqueness statement of Theorem 135.8 (1), there exists an orientation-preserving homeomorphism of the K3-surface to $X\#X\#3\cdot(S^2\times S^2)$.

(4) We conclude this remark with a quick foray into the high-dimensional world. For $n\geq 3$ Wall [**Wall1962**] (see also [**Ran2002**, p. 269]) classified diffeomorphism classes of pairs $(M,\partial M)$ where M is an orientable $(n-1)$-connected $2n$-dimensional smooth manifold and where the boundary ∂M is a homotopy sphere.[1641] The classification goes via the intersection form and one extra invariant which lies in the homotopy group $\pi_{n-1}(\mathrm{SO}(n))$.

Example. On page 2882 we saw that the intersection form of $\mathbb{C}\mathrm{P}^2$ is represented by the (1×1)–matrix (1). By (2b) we know that the Kirby-Siebenmann invariant of $\mathbb{C}\mathrm{P}^2$ vanishes. Theorem 135.8 (2) together with Remark (2) says that there exists a closed oriented simply connected 4-dimensional topological manifold, which usually is denoted by $*\mathbb{C}\mathrm{P}^2$, with the same intersection form as $\mathbb{C}\mathrm{P}^2$ but with non-zero Kirby-Siebenmann invariant. Theorem 135.8 (2) also says that $*\mathbb{C}\mathrm{P}^2$ is unique up to an orientation-preserving homeomorphism. The topological manifolds $\mathbb{C}\mathrm{P}^2$ and $*\mathbb{C}\mathrm{P}^2$ are not homeomorphic since the Kirby-Siebenmann invariants differ. But note that by Theorem 135.7 the topological manifolds $\mathbb{C}\mathrm{P}^2$ and $*\mathbb{C}\mathrm{P}^2$ are homotopy equivalent.

Among many other consequences we obtain the following rather surprising result.

Theorem 135.9. There exist closed orientable 4-dimensional topological manifolds that do not admit a smooth structure.

Proof. At this point we can combine the above results to give two types of examples of such smooth manifolds:

(1) It follows from Theorem 135.8 (2) and the above Remark (3) that given any non-singular odd symmetric form Q there exists a closed oriented 4-dimensional topological manifold X with $Q_X\cong Q$ and non-trivial Kirby-Siebenmann invariant. As we pointed out above in Remark (2a), this topological manifold does not admit a smooth structure.

[1640] We refer to page 2093 for the definition of a PL-structure.

[1641] See page 2625 for the definition of a homotopy sphere

(2) If we apply Freedman's Theorem 135.8 (1) to the non-singular positive-definite even form E_8 we obtain a certain closed oriented simply connected 4-dimensional topological manifold which, by Rokhlin's Theorem 135.6, cannot admit a smooth structure. ■

In a remarkable twist, shortly after Freedman proved his results on the intersection forms of 4-dimensional topological manifolds, Simon Donaldson[1642] "In 1982, when he was a second-year graduate student, Simon Donaldson proved a result that stunned the mathematical world." Simon Donaldson was awarded the Fields medal in 1986. , see [**Don1983**, Theorem A] and [**Don1987a**, Theorem 1], proved the following result regarding intersection forms of 4-dimensional smooth manifolds.[1643]

Theorem 135.10. (Donaldson 1982) Let X be a closed oriented connected 4-dimensional smooth manifold. If Q_X is positive-definite, then Q_X is represented by the identity matrix.

Example. We consider the positive-definite symmetric form $E_8 \oplus E_8$. It has signature 16, hence Rokhlin's Theorem 135.6 has no objections. But Donaldson's Theorem 135.10 implies that this form cannot be the intersection form of a closed oriented connected 4-dimensional smooth manifold.

To understand the significance of Donaldson's Theorem, often referred to as *Donaldson's Theorem A*, recall that on page 2922 we saw that there are more than 10^{51} isometry classes of non-singular positive-definite symmetric forms of rank 40. By Freedman's Theorem 135.8 all of them arise as intersection forms of closed oriented connected 4-dimensional topological manifolds. By Donaldson's Theorem 135.10 the only one which appears as the intersection form of a closed oriented connected 4-dimensional smooth manifold is the one represented by the identity matrix.[1644]



Remark. It is difficult to exaggerate the importance of Donaldson's Theorem in mathematics.[1645] For example it is at the heart of the proof of Theorem **??** which says that $\mathbb{R}^4$ admits uncountably many smooth structures. But Donaldson's Theorem can also appear in totally unexpected situations. For example in 2017 Duncan McCoy [**McCoy2017**] used

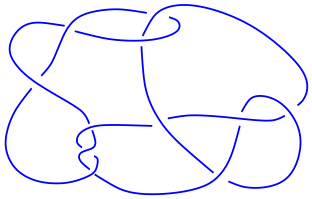
[1642]Simon Donaldson (*1957) is an English mathematician. He proved Theorem 135.10 when he was a second year PhD student. The Fields medalist Michael Atiyah described in [**At1987**] the work of Simon Donaldson as follows:

[1643]An exposition of the proof is also given in [**Law1985**] and a different proof, using Seiberg-Witten invariants is given in [**Nic2000**, Theorem 2.4.18] and a proof using Heegaard Floer homology is given in [**OS2003**, Theorem 9.1].

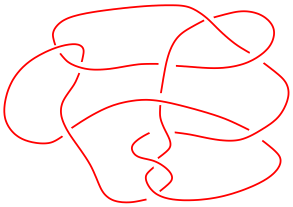
[1644]The identity matrix represents, by the Intersection Form-Additivity Proposition 132.12 and the discussion on page 2882, the intersection form of the connected sum of 40 copies of $\mathbb{C}P^2$.

[1645]Admittedly it is also hard to underestimate the role of Donaldson's Theorem outside of mathematics.

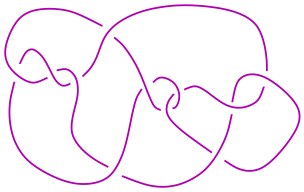
Donaldson's Theorem, and many other ingredients, to show that if K is an alternating knot with unknotting number one[1646], then such a crossing change can already be performed in an alternating diagram. For example, according to **[LiM]** the three alternating knots shown in the figure below have unknotting number one. By the aforementioned result we now know that we can turn these three knots into the trivial knot by changing a crossing in the alternating diagrams shown. Which crossings do you need to change?



the knot $11a_{12}$



the knot $11a_{41}$



the knot $11a_{56}$

Now that we have digested Donaldson's Theorem 135.10, the question arises, what can we say about *indefinite* intersection forms of closed oriented 4-dimensional smooth manifolds? First recall that in the Indefinite Odd Forms Corollary 133.17 we saw that every non-singular indefinite odd symmetric form is isometric to $m \cdot (1) \oplus n \cdot (-1)$ for some $m, n \in \mathbb{N}$. Next note that according to the Intersection Form-Additivity Proposition 132.12 and the discussion on page 2882 every such form is realized by the smooth manifold $m \cdot \mathbb{C}\mathrm{P}^2 \# n \cdot \overline{\mathbb{C}\mathrm{P}}^2$. Therefore, in the following we only need to discuss the realizability of non-singular indefinite *even* symmetric forms. Recall that in the Indefinite Even Forms Corollary 133.19 we showed that every non-singular even indefinite symmetric form is isometric to $m \cdot H \oplus n \cdot E_8$ for some $m \in \mathbb{N}_0$ and $n \in \mathbb{Z} \setminus \{0\}$. The following theorem, proved by Mikio Furuta **[Furu2001]** in 2001 gives some restrictions on the possible values of m and n.

Theorem 135.11. (Furuta's 10/8-Theorem) If M is a closed oriented connected 4-dimensional smooth manifold with *indefinite* even intersection form, then

$$b_2(M) \;\geq\; \tfrac{10}{8} \cdot \mathrm{sign}(M) + 2.$$

In particular[1647] $Q_M \cong m \cdot H \oplus n \cdot E_8$ with $m \geq |n| + 1$.

Remark. Furuta's 10/8-Theorem allows us to answer Question 135.3 (2) in the negative. More precisely, suppose that we have a decomposition of the K3-surface as a connected sum $M \# N$. We make the following four observations:

(1) Without loss of generality we have $\mathrm{sign}(M) \geq \mathrm{sign}(N)$.
(2) It follows from Proposition 135.2, the Parity Lemma 133.14 (3) and the Intersection Form-Additivity Proposition 132.12 that Q_M and Q_N are even.
(3) From the Manifold-Signature Lemma 133.11 (2) and Proposition 135.2 we obtain that $\mathrm{sign}(M) + \mathrm{sign}(N) = 16$.

[1646] We refer to page 1343 for the definition of the unknotting number of a knot. A knot is called *alternating* if it admits a diagram that is alternating, i.e. which has the property that each undercrossing is followed by an overcrossing and vice versa.

[1647] This second statement follows from the first statement together with the Indefinite Even Forms Corollary 133.19 and the calculation of the rank and signature of H and E_8.

(4) By Proposition 135.2 and the Connected Sum-Invariants Proposition 132.11 we know that $b_2(M) + b_2(N) = 22$.

If we combine these facts with Rokhlin's Theorem 135.6 we obtain easily that we have $\text{sign}(M) = 16$. Furthermore, it follows from Donaldson's Theorem 135.10 that Q_M is indefinite. Now Furuta's 10/8-Theorem implies that $b_2(M) \geq \frac{10}{8} \cdot 16 + 2 = 22$, but by Proposition 135.2 this equals $b_2(\text{K3})$. Thus we obtain from the Connected Sum-Invariants Proposition 132.11 that $b_2(N) = 0$.[1648] We can conclude that we cannot split off a copy of $S^2 \times S^2$ from K3. This answers Question 135.3 (2) in the negative.

Furuta's 10/8 Theorem is not yet optimal since it does not quite close the gap between the forms which we can realize by smooth manifolds and the forms which we can exclude. More precisely it follows from the Intersection Form-Additivity Proposition 132.12, the calculation of the intersection forms of the K3-surface, see Proposition 135.2, and of the intersection form of $S^2 \times S^2$ that for any $m \in \mathbb{Z} \setminus \{0\}$ and any $n \geq 3|m|$ there exists a closed oriented simply connected 4-dimensional smooth manifold with intersection form isometric to $m \cdot E_8 \oplus n \cdot H$. More precisely, we have

$$\text{intersection form of } m \cdot \text{K3} \,\#\, (n - 3|m|) \cdot (S^2 \times S^2) \;\cong\; m \cdot E_8 \oplus n \cdot H.$$

The following conjecture predicts that this result is optimal.

Conjecture 135.12. (11/8-Conjecture) If M is a closed oriented 4-dimensional smooth manifold with indefinite even intersection form, then

$$b_2(M) \;\geq\; \tfrac{11}{8} \cdot \text{sign}(M).$$

Equivalently, if $Q_M \cong m \cdot H \oplus n \cdot E_8$ with $n \neq 0$, then $m \geq \frac{3}{2}|n|$.[1649]

Remark.

(1) A proof of the 11/8-Conjecture would imply, by Freedman's Theorem 135.8, that any closed oriented simply connected 4-dimensional smooth manifold is *homeomorphic* to either a connected sum of the form $m \cdot \mathbb{C}\text{P}^2 \# n \cdot \overline{\mathbb{C}\text{P}}^2$ or alternatively to a connected sum of the form $m \cdot \text{K3} \# n \cdot (S^2 \times S^2)$.

(2) For simply connected smooth manifolds currently the best result is [**HLSX2018**, Corollary 1.13] which says that if M is a closed oriented simply connected 4-dimensional smooth manifold that is not homeomorphic to S^4, $S^2 \times S^2$ or the K3-surface and whose intersection form is indefinite and even, then $b_2(M) \geq \frac{10}{8} \cdot |\,\text{sign}(M)| + 4$.

The other question which arises in this context is whether closed orientable 4-dimensional smooth manifolds that are homeomorphic are also necessarily diffeomorphic. It turns out that in many cases the answer is very much no. For example the following theorem was proved in 1998 by Ron Fintushel and Ron Stern [**FS1998**].

[1648] By the Connected Sum-Invariants Proposition 132.11 and Proposition 135.2 we also know that $\pi_1(N) = 0$. Recall that by the Hurewicz Theorem 85.5 we know that if $\pi_1(N) = 0$ and $\text{H}_2(N; \mathbb{Z}) = 0$, then $\pi_2(N) = 0$. Thus it follows from Theorem **??** that N is homeomorphic to S^4. Since Question **??** is still open, we do not know whether N is diffeomorphic to S^4.

[1649] As before, the fact that these two statements are equivalent follows from the Indefinite Even Forms Corollary 133.19 and the calculation of the rank and signature of H and E_8.

Theorem 135.13. There exist closed orientable 4-dimensional smooth manifolds X_s, $s \in \mathbb{N}$ such that each X_s is homeomorphic to the K3-surface, but such that the X_s are pairwise non-diffeomorphic.

Similar results have been obtained for many other 4-dimensional smooth manifolds, e.g. for $\mathbb{C}\mathrm{P}^2 \# k\overline{\mathbb{C}\mathrm{P}}^2$ with $k = 9$ by Simon Donaldson [**Don1987a**] in 1987, $k = 8$ by Dieter Kotschick [**Kot1989**] in 1989 and $k = 7$ by Jongil Park [**Park2005**] in 2005, and since then also for $k = 6, 5, 4, 3, 2$, see e.g. [**AP2008, AP2010**] for more details on the history of this problem.

The following question is surely one of the most interesting problem in the theory of 4-dimensional smooth manifolds.

Question 135.14. Does every closed oriented connected 4-dimensional smooth manifold M admit infinitely many pairwise non-diffeomorphic smooth structures? Put differently, given any such M, do there always exist infinitely many smooth manifolds M_s, $s \in \mathbb{N}$ which are homeomorphic to M but which are pairwise non-diffeomorphic?

Note that an affirmative answer to Question 135.14 would also give a positive answer to Question **??**.

The following theorem shows that smooth manifolds that are homeomorphic are in fact "stably diffeomorphic".

Theorem 135.15. (Stable Diffeomorphism Theorem) Let M and N be two compact 4-dimensional smooth manifolds that are homeomorphic.[1650]

(1) If M and N are orientable, then there exists a $k \in \mathbb{N}$ such that $M \# k \cdot (S^2 \times S^2)$ and $N \# k \cdot (S^2 \times S^2)$ are diffeomorphic.
(2) If M and N are non-orientable, then there exists a $k \in \mathbb{N}$ such that $M \# k \cdot (S^2 \widetilde{\times} S^2)$ and $N \# k \cdot (S^2 \widetilde{\times} S^2)$ are diffeomorphic.[1651]

Proof.

(1) The theorem was first proved by Robert Gompf [**Gom1984**, p. 116] in 1984. An alternative proof, using Matthias Kreck's modified surgery [**Kre1999**] is given in [**Tei1992**, Theorem 5.1.1] and [**FNOP2019**, Theorem 12.2].
(2) The case that the smooth manifolds are closed and simply connected was first proved by C. T. C. Wall [**Wall1964**, Theorem 3] in 1964 (see also [**Scor2005**, p. 149] and [**Qu1983**, Theorem 1.1]). ■

This raises the question, whether in the above theorem $k = 1$ suffices. More precisely the following question is still open.

Question 135.16. Let M and N be two closed oriented simply connected 4-dimensional topological manifolds that homeomorphic. Is $M \# (S^2 \times S^2)$ diffeomorphic to $N \# (S^2 \times S^2)$?

There are some indications, see [**JZ2018**], that the answer to the above question should be no.

[1650] By the Smooth-Topological Orientation Proposition 105.11 we know that being orientable is a homeomorphism invariant.

[1651] Here $S^2 \widetilde{\times} S^2$ denotes the total space of the non-trivial S^2-bundle over S^2 that we introduced on page **??**.

Exercises for Chapter 135.

Exercise 135.1. As we discussed on 2943, it follows from Theorem 135.8 that there exists a closed orientable simply connected 4-dimensional topological manifold $*\mathbb{C}\mathrm{P}^2$ whose intersection form is represented by the (1×1)-matrix (1) and with $\mathrm{ks}(*\mathbb{C}\mathrm{P}^2)=1$.

Let M and N be two closed oriented simply connected 4-dimensional topological manifolds. Show that there exist $m,m',n,n'\in\mathbb{N}_0$ and $\epsilon\in\{0,1\}$ such that the topological manifolds $M\# m\cdot\mathbb{C}\mathrm{P}^2\# m'\cdot\overline{\mathbb{C}\mathrm{P}^2}\#\epsilon\cdot *\mathbb{C}\mathrm{P}^2$ and $N\# n\cdot\mathbb{C}\mathrm{P}^2\# n'\cdot\overline{\mathbb{C}\mathrm{P}^2}$ are homeomorphic.
Hint. Make use of the properties of the Kirby-Siebenmann invariant discussed on page 2943. Furthermore it is strongly advisable to use the Indefinite Forms Theorem 133.16.

Exercise 135.2. Let M and N be two closed oriented simply connected 4-dimensional smooth manifolds. Show that there exist $m,m',n,n'\in\mathbb{N}_0$ such that $M\# m\cdot\mathbb{C}\mathrm{P}^2\# m'\cdot\overline{\mathbb{C}\mathrm{P}^2}$ is diffeomorphic to $N\# n\cdot\mathbb{C}\mathrm{P}^2\# n'\cdot\overline{\mathbb{C}\mathrm{P}^2}$.
Remark. Make use of Exercise 135.1 and use the fact from page 2920 that the two smooth manifolds $(S^2\times S^2)\#\mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2\#\mathbb{C}\mathrm{P}^2$ are diffeomorphic.

Part XIV

Handle Structures and Morse Theory

136. Handle decompositions of smooth manifolds

As we have seen, CW-complexes and simplicial complexes have turned out to be useful tools in topology since often they make it possible to break a given topological space into smaller, more manageable pieces. In Theorems 96.2 and 96.5 we saw that every smooth manifold admits a simplicial structure, in particular it admits a CW-structure. Thus in principle we can study a smooth manifold by starting with a 0-dimensional CW-complex and attaching cells of dimensions $1, 2, \ldots$. The catch is that the inbetween objects are usually not smooth manifolds.

In this chapter we will introduce handle decompositions of smooth manifolds. These are in many ways quite similar to CW-structures, except that now all intermediate objects are smooth manifolds. In Chapter 138 we will use Morse theory to show that every compact smooth manifold admits a handle decomposition.

136.1. **Handle attachments.** The following definition is inspired by the definition of a thickened knot that we gave on page 2853.

Definition. Let W be an m-dimensional smooth manifold. A thickened l-sphere is a smooth embedding $\overline{B}^{m-l} \times S^l \to W$.

Our first proposition introduces the notion of a handle attachment and it summarizes some of the key properties.

Proposition 136.1. (Handle Attachment Proposition) Let M be a compact n-dimensional smooth manifold. Furthermore suppose that we are given a thickened $(k-1)$-sphere in the boundary ∂M, i.e. suppose we are given a smooth embedding $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$. We refer to[16521653]

$$M \cup_\varphi h^k \;:=\; M \cup_{\partial M \xleftarrow{\varphi} \overline{B}^{n-k} \times S^{k-1}} \left(\overline{B}^{n-k} \times \overline{B}^k\right)$$

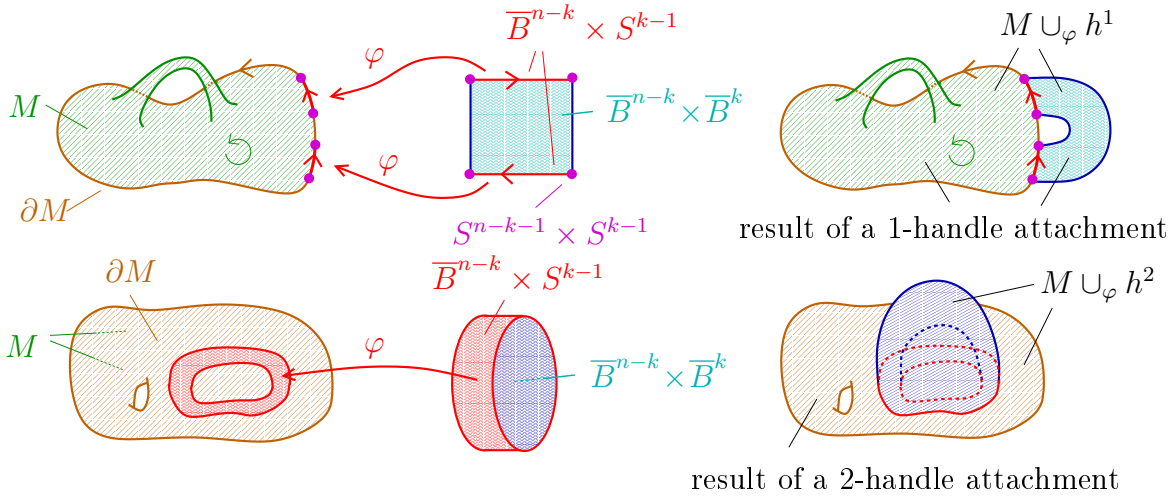
as the result of a k-handle attachment to M. The following statements hold:

(1) (a) The topological space $M \cup_\varphi h^k$ is an n-dimensional topological manifold.
(b) Every choice of a collar[1654] $\Theta\colon [0,1] \times \partial M \to M$ equips $M \cup_\varphi h^k$ canonically with the structure of an n-dimensional smooth manifold. This smooth structure has the property that M and $\overline{B}^{n-k} \times \overline{B}^k$ are both submanifolds with corner, where the corner set in both cases is given by $\varphi(S^{n-k-1} \times S^{k-1})$.
(c) Let Θ and Ξ be two collars. By (1b) these two collars give rise to two smooth structures $\mathcal{A}_\Theta$ and $\mathcal{A}_\Xi$ on the topological manifold $M \cup_\varphi h^k$. Given any neighborhood U of the image of $\varphi(\overline{B}^{n-k} \times S^{k-1})$ in $M \cup_\varphi h^k$ there exists a homeotopy $H\colon (M \cup_\varphi h^k) \times [0,1] \to M \cup_\varphi h^k$ with the following four properties:
(i) $H_0 = \mathrm{id}$.
(ii) $H_1\colon (M \cup_\varphi h^k, \mathcal{A}_\Theta) \to (M \cup_\varphi h^k, \mathcal{A}_\Xi)$ is a diffeomorphism.
(iii) For each $t \in [0,1]$ the map h_t restricts to a diffeomorphism of M and it restricts to a diffeomorphism of $\overline{B}^{n-k} \times \overline{B}^k$.
(iv) $H(x,t) = x$ for all $x \notin U$ and all $t \in [0,1]$.
(d) Any two choices of collar neighborhoods give rise to diffeomorphic smooth structures on M.

(2) $M \cup_\varphi h^k$ is compact.
(3) The boundary of the smooth manifold $M \cup_\varphi h^k$ is given by

$$\partial(M \cup_\varphi h^k) \;=\; (\partial M \setminus \varphi(B^{n-k} \times S^{k-1})) \cup_{\varphi|_{S^{n-k-1}\times S^{k-1}}} (S^{n-k-1} \times \overline{B}^k).$$

(4) If $\varphi, \psi \colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ are two smooth embeddings that are smoothly isotopic, then $M \cup_\varphi h^k$ and $M \cup_\psi h^k$ are diffeomorphic. Furthermore, given any neighborhood U of ∂M one can find a diffeomorphism that is the identity on the common subset $M \setminus U$.



Remark.

(1) The fact that the precise smooth structure on $M \cup_\varphi h^k$ depends on a choice of a collar is a nuisance. Usually we use (1c) as an excuse to ignore this issue, albeit with a slightly bad conscience.
(2) The modification of the boundary that we saw in the Handle Attachment Proposition 136.1 (5) is called a *surgery along* $\varphi(\{0\} \times S^{k-1}) \subset \partial M$. We will study surgeries in greater detail in Chapter **??**.
(3) We deal with the question of whether $M \cup_\varphi h^k$ is orientable respectively connected in the Handle-Connected Lemma 136.8 and the Handles-Orientability Proposition 136.9.

Sketch of proof.

(1) The proof of Statement (1) is quite similar to the proof of the Gluing-Smooth Manifolds-Proposition **??** (1).
   (a) Let $\Theta \colon [0,1] \times \partial M \to M$ be a collar. We use this collar and the standard product neighborhood $[0,1] \times \overline{B}^{n-k} \times S^{k-1} \subset \overline{B}^{n-k} \times \overline{B}^k$ together with the argument from the proof of the Gluing-Smooth Manifolds-Proposition **??** (1) to find suitable

[1652] Recall that by the definition on page 262 this means that

$$M \cup_\varphi h^k \;:=\; \big(M \sqcup (\overline{B}^{n-k} \times \overline{B}^k)\big)/\!\sim \quad \text{where } \varphi(P) \sim P \text{ for } P \in \overline{B}^{n-k} \times S^{k-1}.$$

[1653] The fact that in the notation the map φ now goes from right to left is a little awkward, but all other conventions lead sooner or later also to slightly unnatural results.
[1654] Recall that by the Smooth Collar Neighborhood Theorem 28.3 a collar $[0,1] \times \partial M \to M$ always exists.

canonical charts for all points that do not lie on $\varphi(S^{n-k-1} \times S^{k-1})$. Furthermore, for points on $\varphi(S^{n-k-1} \times S^{k-1})$ one can easily find "charts" such that the image equals $\{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_{n-1} \leq 0 \text{ or } x_n \geq 0\}$. Similar to the proof of Proposition **??** we compose those charts with the map

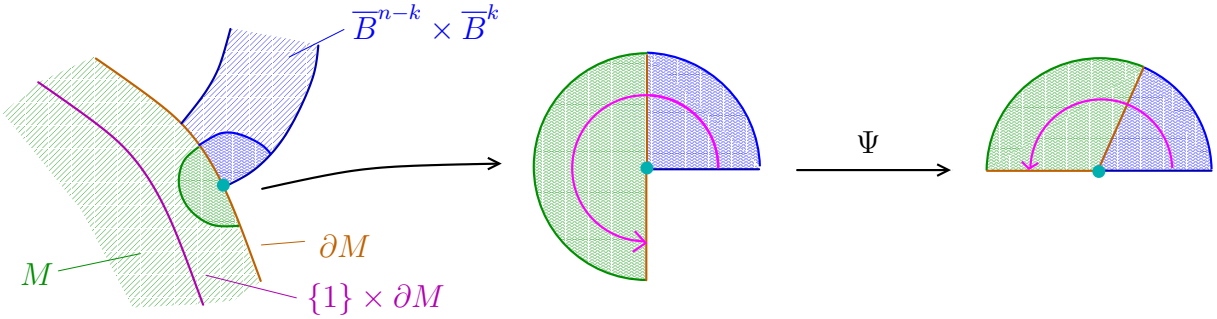
$$\begin{array}{rcl} \Psi : \mathbb{R}^{n-2} \times \{(x,y) \in \mathbb{R}^2 \,|\, x \leq 0 \text{ or } y \geq 0\} & \to & \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R}_{\geq 0} \\ (x_1, \dots, x_{n-2}, \underbrace{r\cos(\varphi), r\sin(\varphi)}_{r \in \mathbb{R}_{\geq 0} \text{ and } \varphi \in [0, \frac{3}{2}\pi]}) & \mapsto & (x_1, \dots, x_{n-2}, r\cos(\tfrac{2}{3}\varphi), r\sin(\tfrac{2}{3}\varphi)). \end{array}$$

If one interprets the above discussion with good will, then one can see that this defines a smooth atlas on $M \cup_\varphi h^k$ which has all the desired properties.

(b) We proved this statement already in (a).

(c) As in the proof of the Gluing-Smooth Manifolds-Proposition **??** (1c) this statement follows without too much effort from the uniqueness statement for collars, see the Uniqueness of Collar-Proposition **??**.

(d) This statement is an immediate consequence of (c).



(2) It follows from the compactness of M and $\overline{B}^{n-k} \times \overline{B}^k$ together with the Quotient-Compact Connected Lemma 5.16 (1) that $M \cup_\varphi h^k$ is compact.

(3) Similar to the proof of the Gluing-Smooth Manifolds-Proposition **??** (4) one can prove the statement about the description of $\partial(M \cup_\varphi h^k)$ quite easily using the Smooth Manifold Boundary Proposition 19.26.

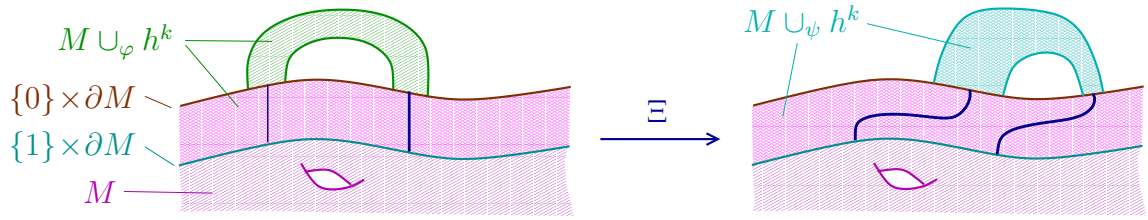
(4) It follows from (1c) that throughout the proof of (4) we can work with the same collar neighborhood $[0,1] \times \partial M$. Now let $\varphi, \psi \colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be two smooth embeddings. We suppose that they are smoothly isotopic. Let $F \colon \overline{B}^{n-k} \times S^{k-1} \times [0,1] \to \partial M$ be a smooth homotopy with $F_0 = \varphi$ and $F_1 = \psi$. By the Isotopy Extension Theorem 36.1 together with the Trivial-Near-The-Ends-Lemma 33.4 there exists a diffeotopy $G \colon \partial M \times [0,1] \to \partial M$ with $G_t = \mathrm{id}$ for $t \in [0, \frac{1}{4}]$ and with $G_t \circ \varphi = \psi$ for all $t \in [\frac{3}{4}, 1]$. One can now easily verify that the map

$$\begin{array}{rcl} \Xi \colon M \cup_\varphi h^k & \to & M \cup_\psi h^k \\ [x] & \mapsto & \begin{cases} [x], & \text{if } x \in M \setminus ([0,1] \times \partial M), \\ [(t, G(1-t, y))], & \text{if } x = (t,y) \in [0,1] \times \partial M, \\ [x], & \text{if } x \in \overline{B}^{n-k} \times \overline{B}^k \end{cases} \end{array}$$

is a diffeomorphism. Finally suppose that we are given a neighborhood U of ∂M. Evidently we can pick a collar neighborhood $[0,1] \times \partial M$ that is contained in $M \setminus U$.

The diffeomorphism that we just constructed is now the identity on the common subset $M \setminus U$.

As on so many other occasions we leave it to the reader to fill in the annoying details. ■



Remark. The approach in the proof of the Handle Attachment Proposition 136.1 for defining charts for $M \cup_\varphi h^k$ at the points in $\varphi(S^{n-k-1} \times S^{k-1})$ is sometimes called *rounding corners* or *smoothing corners*.

The Handle Attachment Proposition 136.1 motivates the following definition.

Definition. Let M be a compact n-dimensional smooth manifold. Suppose we are given a thickened $(k-1)$-sphere $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$.

(1) We say

$$M \cup_\varphi h^k \;:=\; M \cup_{\partial M \overset{\varphi}{\leftarrow} \overline{B}^{n-k} \times S^{k-1}} \left(\overline{B}^{n-k} \times \overline{B}^k\right)$$

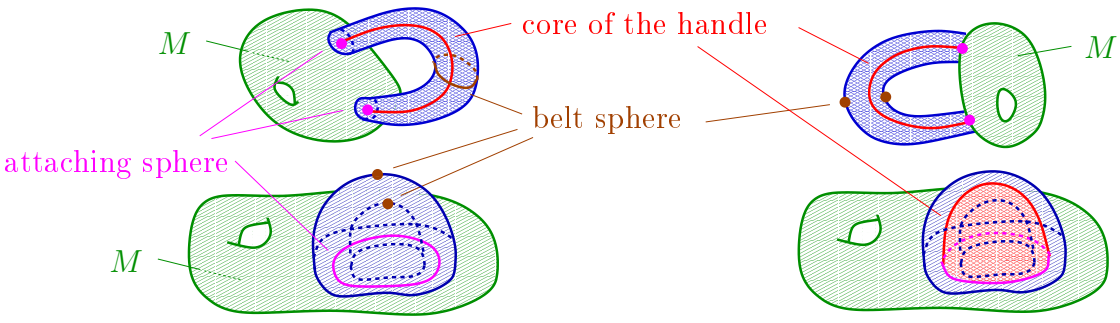
is obtained from M by attaching the k-handle $\overline{B}^{n-k} \times \overline{B}^k$ along φ. In particular we refer to the image of $\overline{B}^{n-k} \times \overline{B}^k$ in $M \cup_\varphi h^k$ as a k-handle or as a handle of index k.

(2) (a) We refer to the image of $\{0\} \times S^{k-1}$ in ∂M as the attaching sphere of the handle.
(b) We refer to the image of $\{0\} \times \overline{B}^k$ in $M \cup_\varphi h^k$ as the core of the handle.
(c) We refer to the image of $S^{n-k} \times \{0\}$ in $\partial(M \cup_\varphi h^k)$ as the belt sphere of the handle.

We refer to the figure below for an illustration of these definitions.

attachment of 1-handle to 3-dimensional smooth manifold

attachment of 1-handle to 2-dimensional smooth manifold

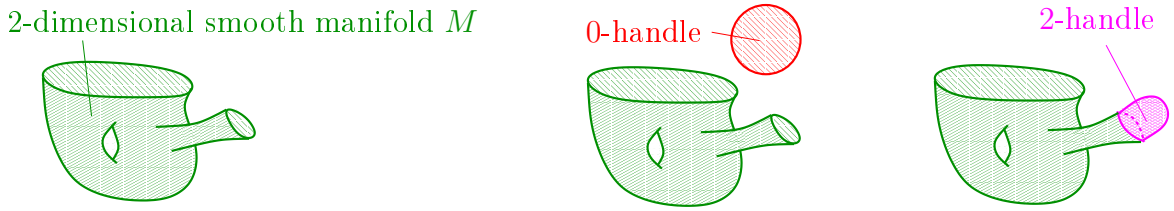


attachment of 2-handle to 3-dimensional smooth manifold

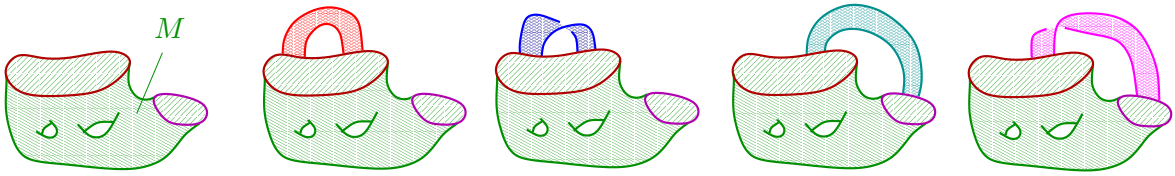
Example. Let M be a compact n-dimensional smooth manifold.

(1) Let us consider the extreme cases of attaching a 0-handle and attaching an n-handle.
   (a) It follows basically from the definitions that attaching a 0-handle is the same as taking the disjoint union with $\overline{B}^n$.
   (b) It follows immediately from the Codimension-Zero Smooth Embedding Proposition 24.16 (3) that any smooth embedding $\varphi\colon \overline{B}^0 \times S^{n-1} \to \partial M$ is a diffeomorphism onto a component of ∂M. Thus an n-handle attachment is the same as gluing an n-ball to a boundary component that is diffeomorphic to S^{n-1} via the diffeomorphism φ. Note that in Proposition **??** we will see that in general the result depends on the choice of φ.

   We illustrate these two examples in the figure below.



(2) Next we turn to attachments of 1-handles. As is shown in the figure below, the effect of attaching a 1-handle depends on the choice of the attaching map $\varphi\colon \overline{B}^{n-1} \times S^0 \to \partial M$. In the 1-Handle Attachment Lemma 136.10 we will see that the four examples of 1-handle attachments that we see in the figure below exhaust in a suitable sense all possibilities for 1-handle attachments.



four types of 1-handle attachments

(3) For handle attachments that are not of index 0, 1 and $\dim(M)$ there is no straightforward classification of all possibilities. For example we consider the smooth manifold $M = \overline{B}^4$. Let $\gamma\colon S^1 \to S^3$ be a smooth embedding. Note that $K = \gamma(S^1)$ is a knot. In the Knot Thickening Lemma 131.1 we saw that for each $d \in \mathbb{Z}$ there exists an essentially unique tubular map $\overline{B}^2 \times K \to S^3$ of self-linking number d. Any such tubular map describes a 2-handle attachment to $\overline{B}^4$. There is no reason to believe that all these 2-handle attachments can be classified easily.

136.2. **Handle decompositions.** The following is a good example of a self-explanatory definition.

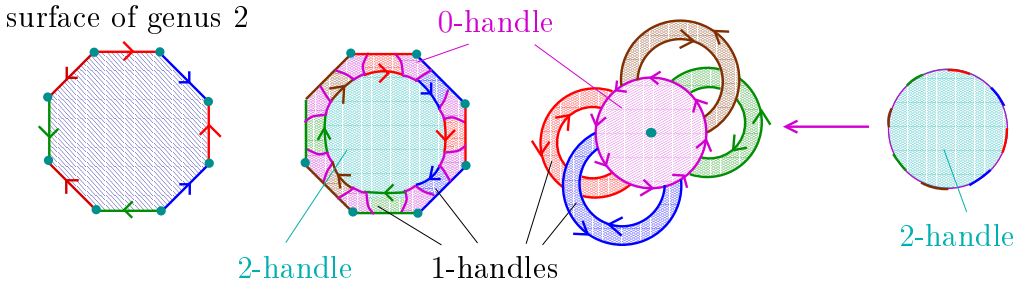
Definition. Let M be a compact n-dimensional smooth manifold. A handle decomposition for M is a diffeomorphism

$$\Theta\colon M \xrightarrow{\cong} \varnothing \cup_{\varphi_1} h^{k_1} \cup \cdots \cup_{\varphi_m} h^{k_m}$$

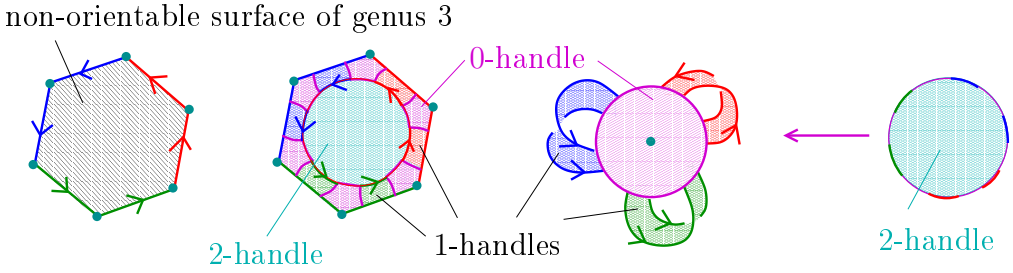
where for every $i = 1, \ldots, m$ the map φ_i is a thickened $(k_i - 1)$-sphere in the boundary of the smooth manifold $\varnothing \cup_{\varphi_1} h^{k_1} \cup \cdots \cup_{\varphi_{i-1}} h^{k_{i-1}}$. Usually we suppress the name of the diffeomorphism, in this case Θ, from the notation.

Examples.

(1) In the figure below we show a handle decomposition for the surface of genus $g = 2$. This handle decomposition consists of one 0-handle, $2g$ 1-handles and one 2-handle.



(2) In the figure below we show a handle decomposition for the non-orientable surface of genus $k = 3$. This handle decomposition consists of one 0-handle, k 1-handles and one 2-handle.



We formulate the next two types of examples as a lemma.

Lemma 136.2. (Handle Example Lemma)

(1) Let $n \in \mathbb{N}$. We consider the thickened $(n-1)$-sphere that is given by the identity

$$\mathrm{id}\colon \overline{B}^0 \to S^{n-1} \;\to\; \partial(\overline{B}^n).$$

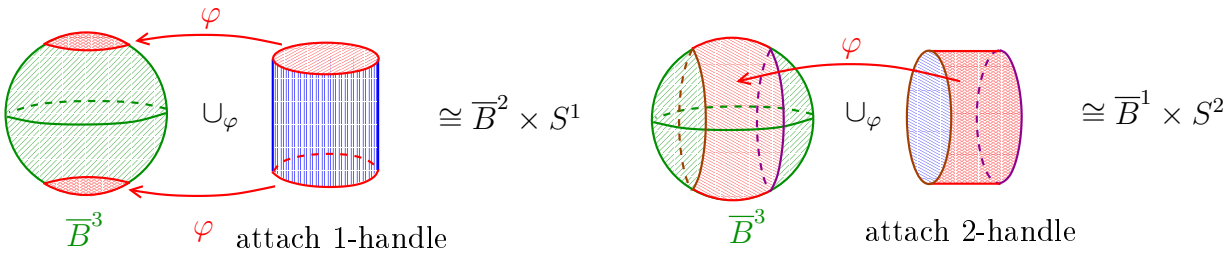
If we attach an n-handle to $\overline{B}^n$ along id we obtain a smooth manifold that is diffeomorphic to S^n. In particular we see that S^n admits a handle decomposition with a single 0-handle and a single n-handle.

(2) Let $m, k \in \mathbb{N}$. We consider the thickened $(k-1)$-sphere

$$\begin{aligned} \varphi\colon \overline{B}^m \times S^{k-1} &\to S^{m+k-1} = \partial(\overline{B}^{m+k}) \\ (x, y) &\mapsto \frac{(\frac{1}{2} \cdot x, y)}{\|(\frac{1}{2} \cdot x, y)\|} \\ &\uparrow \\ &\text{with } (x, y) \in \overline{B}^m \times S^{k-1} \subset \mathbb{R}^m \times \mathbb{R}^k = \mathbb{R}^{m+k} \end{aligned}$$

If we attach a k-handle to the 0-handle $\overline{B}^{m+k}$ along the thickened $(k-1)$-sphere φ, then we obtain a smooth manifold that is diffeomorphic to $\overline{B}^m \times S^k$. In particular

we see that $\overline{B}^m \times S^k$ admits a handle decomposition with a single 0-handle and a single k-handle.



Proof.

(1) Let $n \in \mathbb{N}$. As on page 261 we consider the maps

$$\begin{aligned} g_\pm\colon \overline{B}^n = \overline{B}^n_\pm &\to S^n \\ (x_1, \dots, x_n) &\mapsto \big((-1)^n \cdot x_1, \dots, x_n, \pm\sqrt{1 - x_1^2 - \cdots - x_n^2}\big). \end{aligned}$$

We leave it to the reader to verify that the map[1655]

$$\begin{aligned} \overline{B}^n_+ \cup_{S^{n-1} \xleftarrow{\mathrm{id}} S^{n-1}} \overline{B}^n_- &\to S^n \\ [P] &\mapsto \begin{cases} g_+(P), & \text{if } P \in \overline{B}^n_+, \\ g_-(P), & \text{if } P \in \overline{B}^n_- \end{cases} \end{aligned}$$

is an orientation-preserving diffeomorphism.

(2) We will prove this statement in Exercise 136.2. ■

In the above we have now seen that some of our favorite smooth manifolds admit a handle decomposition. In Chapter 138 we will prove the Handle Decomposition Theorem 138.5 which states that in fact every compact smooth manifold admits a handle decomposition. In the remainder of this chapter we perform the less exciting task of proving several basic facts about handle attachments and handle decompositions.

136.3. **Handle rearrangements.** The following proposition shows that when we attach several handles, then we can rearrange the handle attachments so that they are done with increasing indices.

Proposition 136.3. (Handle Swapping Proposition) Let M be a compact n-dimensional smooth manifold. Suppose that for $i = 1, \dots, m$ we are given smooth embeddings

$$\varphi_i\colon \overline{B}^{n-k_i} \times S^{k_i-1} \to \partial(M \cup_{\varphi_1} h^{k_1} \cdots \cup_{\varphi_{i-1}} h^{k_i}).$$

Furthermore let

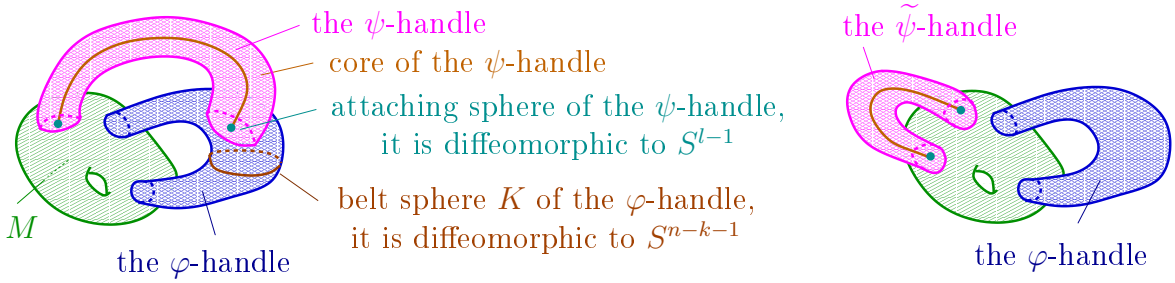
$$\psi\colon \overline{B}^{n-l} \times S^{l-1} \to \partial(M \cup_{\varphi_1} h_1^{k_1} \cdots \cup_{\varphi_m} h_m^{k_m})$$

[1655] As we mentioned in the formulation of the Handle Attachment Proposition 136.1, the precise smooth structure on $\overline{B}^n_+ \cup_{S^{n-1} \xleftarrow{\mathrm{id}} S^{n-1}} \overline{B}^n_-$ depends on the choice of a collar. We follow the convention from page 737, namely we consider $\overline{B}^n_\pm$ with the standard collar that we introduced on page 737. With this convention the given map is indeed an honest-to-God diffeomorphism.

be a smooth embedding. If for $i = 1, \dots, m$ we have $l \leq k_i$, then there exists a smooth embedding

$$\widetilde{\psi} \colon \overline{B}^{n-l} \times S^{l-1} \;\to\; \text{interior of } (\partial M \cap \partial(M \cup_{\varphi_1} h_1^{k_1} \cdots \cup_{\varphi_m} h_m^{k_m})) \text{ in } \partial M$$

such that $M \cup_{\varphi_1} h_1^{k_1} \cdots \cup_{\varphi_m} h_m^{k_m} \cup_{\psi} h^l$ is diffeomorphic to[1656] $M \cup_{\widetilde{\psi}} h^l \cup_{\varphi_1} h_1^{k_1} \cdots \cup_{\varphi_m} h_m^{k_m}$. Furthermore, given any neighborhood U of ∂M one can find a diffeomorphism that is the identity on the common subset $M \setminus U$.



Remark. Colloquially replacing the ψ-handle by the $\widetilde{\psi}$-handle in (the proof of) the Handle Swapping Proposition 136.3 is called a *handle slide*

Proof. To simplify the notation we prove the proposition for the special case $m = 1$. The proof of the general case is not much harder and we leave it to the reader to fill in the details.

Thus suppose that we are given a smooth embedding $\varphi \colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ and suppose that we are given a smooth embedding $\psi \colon \overline{B}^{n-l} \times S^{l-1} \to \partial(M \cup_{\varphi} h^k)$ with $l \leq k$. To simplify the notation we make the obvious identification $S^{l-1} = \{0\} \times S^{l-1} \subset \overline{B}^{n-l} \times S^{l-1}$. We write $f := \psi|_{S^{l-1}}$. Note that $f(S^{l-1})$ is precisely the attaching sphere of the ψ-handle. We denote by $K = S^{n-k-1} \times \{0\} \subset \partial(M \cup_{\varphi} h^k)$ the belt sphere of the φ-handle.

Claim 1. There exists a smooth homotopy $F \colon S^{l-1} \times [0,1] \to \partial(M \cup_{\varphi} h^k)$ such that $F_0 = f$ and $F_1(S^{l-1}) \subset \partial(M \cup_{\varphi} h^k) \setminus K$.

Proof. Note that we know by the Smooth Embedding Theorem 24.10 and the Handle Attachment Proposition 136.1 (1) that $f(S^{l-1})$ and K are both submanifolds of $\partial(M \cup_{\varphi} h^k)$. By the Transversality Theorem 42.4 there exists a smooth homotopy $F \colon S^{l-1} \times [0,1] \to \partial(M \cup_{\varphi} h^k)$ with $F_0 = f$ such that $F_1(S^{l-1})$ is transverse to the belt sphere K. Note that

$$\dim(F_1(S^{l-1})) + \dim(K) \;=\; l - 1 + n - k - 1 \;\underset{\uparrow}{\leq}\; n - 2 \;<\; n - 1 \;=\; \dim(\partial(M \cup_{\varphi} h^k)).$$

by hypothesis we have $l \leq k$

It follows immediately from the definition of transverse intersection and this inequality of dimensions that $F_1(S^{l-1})$ is actually disjoint from the belt sphere K. ⊞

We continue with the following technical enhancement of the previous claim.

Claim 2. There exists a smooth homotopy $\Psi \colon (\overline{B}^{n-l} \times S^{l-1}) \times [0,1] \to \partial(M \cup_{\varphi} h^k)$ with $\Psi_0 = \psi$ and such that $\Psi_1(\overline{B}^{n-l} \times S^{l-1}) \subset \partial M \setminus \varphi(\overline{B}^{n-k} \times S^{k-1})$.

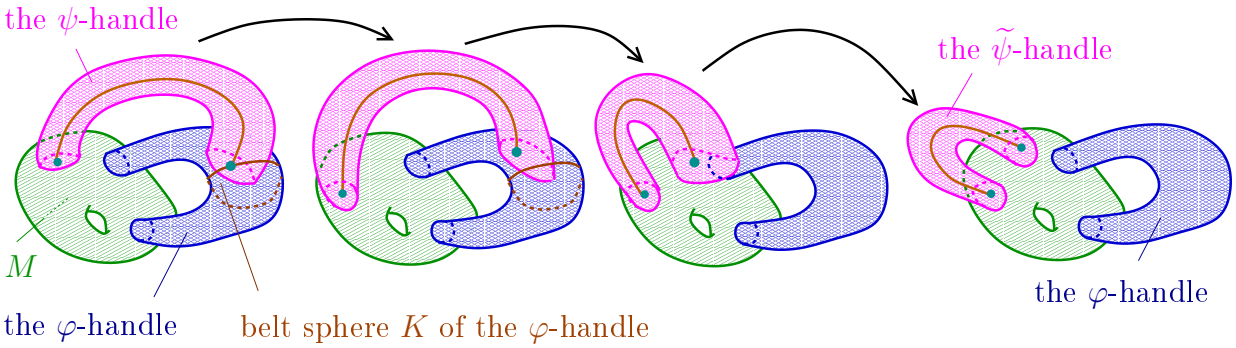
[1656] Note that it follows from the fact that ψ takes values in the interior of $\partial M \cap \partial(M \cup_{\varphi_1} h_1^{k_1} \cdots \cup_{\varphi_m} h_m^{k_m}))$ in ∂M that the maps $\varphi_1, \dots, \varphi_m$ can also be used to attach successively handles to $M \cup_{\widetilde{\psi}} h^l$.

Proof. It follows from the Handle Attachment Proposition 136.1 (1) that $\varphi(S^{n-k-1} \times \overline{B}^k)$ is a tubular neighborhood for the belt sphere $K = S^{n-k-1} \times \{0\} \subset \partial(M \cup_\varphi h^k)$. We obtain from the previous claim together with Exercise **??** and the Smooth Isotopy Transitivity Proposition 33.3 a smooth homotopy $G\colon f(S^{l-1}) \times [0,1] \to \partial(M \cup_\varphi h^k)$ such that $G_0 = \mathrm{id}$ and such that $G_1(f(S^{l-1})) \subset \partial(M \cup_\varphi h^k) \setminus \varphi(\overline{B}^{n-k} \times S^{k-1})$. Next note that using the Isotopy Extension Theorem 36.1 we can extend G to a smooth homotopy $G\colon (\psi(\overline{B}^{n-l} \times S^{l-1})) \times [0,1] \to \partial(M \cup_\varphi h^k)$. Unfortunately a priori we do not know that $G_1(\psi(\overline{B}^{n-l} \times S^{l-1})) \subset \partial M \setminus \varphi(\overline{B}^{n-k} \times S^{k-1})$. But it follows immediately from the "Tube Lemma", see Exercise (a), that there exists an $\epsilon > 0$ such that the map

$$\begin{array}{rcl} \Psi\colon (\overline{B}^{n-l} \times S^{l-1}) \times [0,1] & \to & \partial(M \cup_\varphi h^k) \\ ((v,P),t) & \mapsto & G(\psi(\underbrace{((1-t)+\epsilon \cdot t) \cdot v, P)}_{\in \overline{B}^{n-l} \times S^{l-1}}, t) \end{array}$$

has all the desired properties. ⊞

It follows easily from the Handle Attachment Proposition 136.1 (5) that $\widetilde{\psi} := \Psi_1$ has all the desired properties. ■



The Handle Swapping Proposition 136.3 leads us to the following definition.

Definition. Let M be a compact n-dimensional smooth manifold. Suppose we are given a handle decomposition where the handles are attached with non-decreasing indices, i.e. we have a handle decomposition of the following form:

$$M = \varnothing \underbrace{\cup_{\varphi_{0,1}} h^0 \cdots \cup_{\varphi_{0,r_0}} h^0}_{0\text{-handles}} \cup \ldots \underbrace{\cup_{\varphi_{n,1}} h^n \cdots \cup_{\varphi_{n,r_n}} h^n}_{n\text{-handles}}.$$

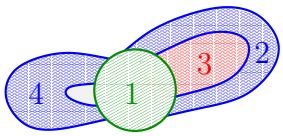
We introduce the following two concepts:

(1) Given $i \in \{0,\ldots,n\}$ we refer to

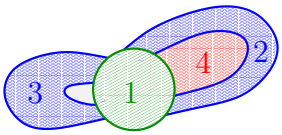
$$M^i := \varnothing \underbrace{\cup_{\varphi_{0,1}} h^0 \cdots \cup_{\varphi_{0,r_0}} h^0}_{0\text{-handles}} \cup \ldots \underbrace{\cup_{\varphi_{i,1}} h^i \cdots \cup_{\varphi_{i,r_i}} h^i}_{i\text{-handles}}.$$

as the i-skeleton of the handle decomposition.

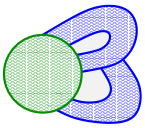
(2) We say that a handle decomposition is standard if it is of the above form and if for each $i \in \{0,\ldots,n\}$ the attaching maps $\varphi_{i,1},\ldots,\varphi_{i,r_i}$ take values in ∂M^{i-1}.



non-standard
handle decomposition



standard
handle decomposition



non-standard
handle decomposition

Corollary 136.4. (Standard Handle Decomposition Corollary) Let M be a compact n-dimensional smooth manifold. If M admits a handle decomposition, then it also admits a handle decomposition with the same number of handles of each index, but which is now standard.

Proof. This follows immediately from iteratively applying the Handle Swapping Proposition 136.3 to reorder the handles. ■

Remark. In Chapter ?? we will see several other ways for modifying a given handle decomposition. But for the time being the above rearranging of handle attachments is good enough.

136.4. **Dual handle decomposition.** In this section we will see that we can associate to each handle decomposition a "dual" handle decomposition. More precisely, we have the following proposition.

Proposition 136.5. (Dual Handle Proposition) Let M be a closed n-dimensional smooth manifold together with a handle decomposition

$$M = \varnothing \cup_{\varphi_0} \underbrace{h^{k_0}}_{\text{0-th handle}} \cup \cdots \cup_{\varphi_m} \underbrace{h^{k_m}}_{m\text{-th handle}}.$$

There exists canonical handle decomposition

$$M = \varnothing \cup_{\psi_m} \underbrace{\widetilde{h}^{n-k_m}}_{\text{0-th dual handle}} \cup \cdots \cup_{\psi_0} \underbrace{\widetilde{h}^{n-k_0}}_{m\text{-th dual handle}},$$

called dual handle decomposition, with the following properties:

(1) For each $i \in \{0, \ldots, m\}$ the following statements hold:
   (a) The i-th handle of the dual handle decomposition equals, as a subset of M, the $(m-i)$-th handle of the original handle decomposition.
   (b) The indices of the handles are reversed, in the sense that

$$\begin{array}{c}\text{index of the } i\text{-th handle of}\\ \text{the dual handle decomposition}\end{array} = n - \begin{array}{c}\text{index of the } (m-i)\text{-th handle of}\\ \text{the original handle decomposition.}\end{array}$$

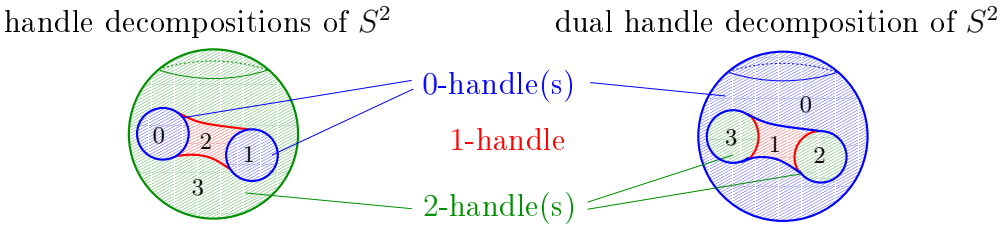
   In particular the number i-handles in the dual handle decomposition equals the number of $(n-i)$-handles in the original decomposition.
   (c) The roles of attaching spheres and belt spheres is reversed in the sense that

$$\begin{array}{c}\text{belt sphere of the } i\text{-th handle of}\\ \text{the dual handle decomposition}\end{array} = \begin{array}{c}\text{attaching sphere of the } (m-i)\text{-th handle}\\ \text{of the original handle decomposition.}\end{array}$$

   Furthermore the same statement, with the roles of attaching and belt spheres reversed, also holds.

(2) If the original handle decomposition is standard, then so is the dual handle decomposition.
(3) The dual of the dual handle decomposition is the original handle decomposition.

Example. In the figure below we show to the left a handle decomposition of S^2. But as we see, the decomposition of the sphere into subsets can also be read the other way and we obtain the dual handle decomposition.



Remark.

(1) It follows immediately from the Dual Handle Proposition 136.5 that every handle decomposition of a closed connected non-empty n-dimensional smooth manifold admits at least one n-handle. This gives in particular a solution to Exercise 136.3 (a).
(2) The handle decomposition that we construct in the proof of the Dual Handle Proposition 136.5 is sometimes called the *dual handle decomposition*.

The following lemma is the key ingredient in the proof of the Dual Handle Proposition 136.5.

Lemma 136.6. Let X be an n-dimensional smooth manifold and let $\varphi\colon \overline{B}^{n-k}\times S^{k-1}\to\partial X$ be a smooth embedding. Let $X\cup_\varphi h^k := X\cup_{\partial X\overset{\varphi}{\leftarrow}\overline{B}^{n-k}\times S^{k-1}} (\overline{B}^{n-k}\times\overline{B}^k)$ be the result of attaching the corresponding k-handle to X. Furthermore let Y be an n-dimensional smooth manifold and let $\gamma\colon \partial Y\to\partial(X\cup_\varphi h^k)$ be a diffeomorphism. We consider the maps

$$\psi\colon \overline{B}^k\times S^{n-k-1}\to\partial Y,\quad (x,y)\mapsto\gamma^{-1}(y,x) \qquad \text{and} \qquad \delta\colon \partial X\to\partial(Y\cup_\psi h^k),\quad P\mapsto\begin{cases}\gamma^{-1}(P), & \text{if } P\in\partial X\setminus\varphi(B^{n-k}\times S^{k-1}),\\ \varphi^{-1}(P), & \text{if } P\in\varphi(B^{n-k}\times S^{k-1}).\end{cases}$$

The obvious map from

$$\underbrace{\big(X\cup_{\partial X\overset{\varphi}{\leftarrow}\overline{B}^{n-k}\times S^{k-1}}(\overline{B}^{n-k}\times\overline{B}^k)\big)}_{=X\cup_\varphi h^k}\cup_{\partial(X\cup_\varphi h^k)\overset{\gamma}{\leftarrow}\partial Y} Y$$

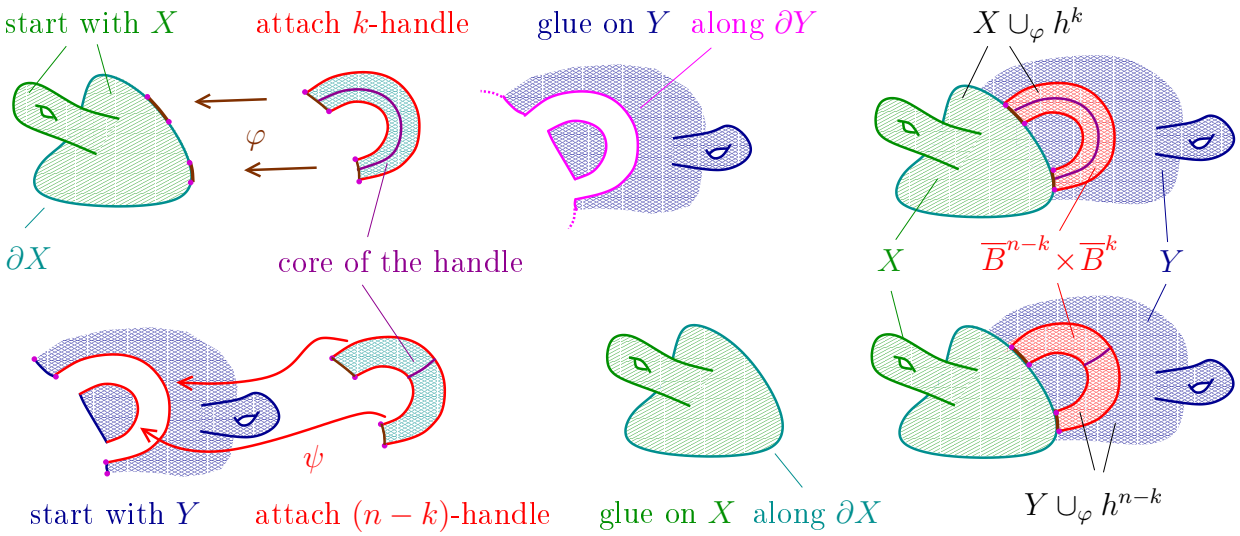
to

$$\underbrace{\big(Y\cup_{\partial Y\overset{\psi}{\leftarrow}\overline{B}^{k}\times S^{n-k-1}}(\overline{B}^{n-k}\times\overline{B}^k)\big)}_{=Y\cup_\psi h^k}\cup_{\partial(Y\cup_\psi h^k)\overset{\delta}{\leftarrow}\partial X} X$$

is a diffeomorphism.

Lemma 136.6 can be summarized as follows: If a smooth manifold W is obtained from X by attaching a k-handle and then gluing on Y, then we can also obtain W from Y by attaching an $(n-k)$-handle and gluing on X.

Sketch of proof of Lemma 136.6. Once one has digested the statement of the lemma, one realizes, that the lemma comes close to being a tautology. One can easily verify that the "obvious" maps in both directions are well-defined and inverses of one another.



Furthermore using the Topological Pushout Proposition 5.33 (1b) it is straightforward to verify that these maps are continuous, thus both maps are homeomorphisms. One still needs to show that both maps are in fact smooth. This joyless task is best left to the reader's office mate. ■

Proof of the Dual Handle Proposition 136.5. Let $M = \varnothing \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_m} h^{k_m}$ be a handle decomposition. Given $i = 0, \ldots, m$ we write $X_i := \varnothing \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_{i-1}} h^{k_{i-1}}$, $M_i := X_i \cup_{\varphi_i} h^{k_i}$ and $Y_i := M \setminus \mathring{M}_i$. Note that Y_i is an n-dimensional smooth manifold by the Codimension-Zero Submanifold Proposition 19.38. Therefore note that $Y_m = M \setminus M = \varnothing$. Since M is closed we can apply Lemma 136.6 iteratively altogether $(m+1)$ times. We see that $M = \varnothing \cup_{\psi_m} h^{n-k_m} \cup \cdots \cup_{\psi_0} h^{n-k_0}$ where for $i = 0, \ldots, m$ the map ψ_i is given by

$$\begin{array}{rcl} \psi_i \colon \overline{B}^{k_i} \times S^{n-k_i-1} & \to & \partial(\varnothing \cup_{\psi_m} h^{n-k_m} \cup \cdots \cup_{\psi_{i+1}} h^{n-k_{i+1}}) \\ (x, y) & \mapsto & [(y, x)]. \end{array}$$

It is straightforward to verify that this dual handle decomposition has all the properties stated in the proposition. ■

We conclude this section with the following perhaps slightly surprising corollary.

Corollary 136.7. Let M be a *closed* connected non-empty n-dimensional smooth manifold. For every handle decomposition of M we have the inequality:

$$1 + \#(n-1)\text{-handles} - \#n\text{-handles} \;\geq\; \text{minimal size of a generating set of } \pi_1(M).$$

Proof. Let M be a closed connected non-empty n-dimensional smooth manifold that is equipped with a handle decomposition of M. We have the following (in-) equalities:

$$\overbrace{\begin{array}{l} 1 + \#(n-1)\text{-handles} \\ \quad - \#n\text{-handles} \end{array}}^{\text{original handle decomposition}} \underset{\uparrow}{=} \overbrace{\begin{array}{l} 1 + \#1\text{-handles} \\ \quad - \#0\text{-handles} \end{array}}^{\text{dual handle decomposition}} \underset{\uparrow}{\geq} \begin{array}{l} \text{minimal size of a} \\ \text{generating set of } \pi_1(M). \end{array}$$

Dual Handle Proposition 136.5 Corollary 136.12

■

136.5. **Handle attachments: Connectedness and orientability.** In this section we will see how handle attachments affect connectedness and orientability. In the subsequent section we will see how handle attachments affect various algebraic invariants. First let us deal with the basic concept of connectedness.

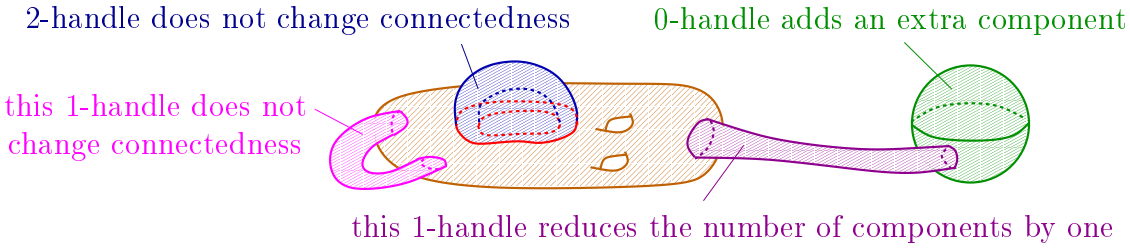
Lemma 136.8. (Handle-Connected Lemma) Let M be a compact n-dimensional smooth manifold. Suppose that we are given a thickened $(k-1)$-sphere in the boundary ∂M, i.e. suppose we are given a smooth embedding $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$. The following statements hold:

($k>1$) The inclusion induced map $\pi_0(M) \to \pi_0(M \cup_\varphi h^k)$ is a bijection.

($k=1$) (a) If the attaching map $\varphi\colon \overline{B}^{n-1} \times S^0 \to \partial M$ takes values in a single component of ∂M, then the inclusion induced map $\pi_0(M) \to \pi_0(M \cup_\varphi h^k)$ is a bijection.

(b) If the attaching map $\varphi\colon \overline{B}^{n-1} \times S^0 \to \partial M$ takes values in two components of ∂M, then the number of components goes down by one.

($k=0$) $M \cup_\varphi h^0$ has one extra component, namely the 0-handle.



Proof. The statements for $k > 1$ and $k = 1$ follow easily from the Union Connected Lemma 2.28. The statement for $k = 0$ follows from the fact that adding a 0-handle is just taking the disjoint union with the 0-handle. ■

Next we study the slightly subtle question how attaching a handle affects orientability.

Proposition 136.9. (Handles-Orientability Proposition) Let M be a compact oriented n-dimensional smooth manifold. Suppose that we are given a thickened $(k-1)$-sphere in the boundary ∂M, i.e. suppose we are given a smooth embedding $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$. The following statements hold:

(i) Let $\tau\colon S^{k-1} \to S^{k-1}$ be a reflection in some hyperplane of $\mathbb{R}^k$. The corresponding map $\varphi \circ (\mathrm{id}_{\overline{B}^{n-k}} \times \tau)\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ is also a thickened $(k-1)$-sphere and the smooth manifold $M \cup_{\varphi\circ(\mathrm{id}\times\tau)} h^k$ is diffeomorphic to $M \cup_\varphi h^k$.

(ii) If $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ is orientation-reversing, then there exists an orientation-reversing smooth embedding $\psi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ such that $M \cup_\psi h^k$ is diffeomorphic to $M \cup_\varphi h^k$.

Now we consider the various dimensions separately:

($k\geq 2$) $M \cup_\varphi h^k$ admits an orientation which agrees with the given orientation on $M \setminus \partial M$.

($k=1$) (a) If[1657] $\varphi\colon \overline{B}^{n-1} \times S^0 \to \partial M$ is orientation-preserving or orientation-reversing, then $M \cup_\varphi h^k$ admits an orientation that agrees with the given orientation on $M \setminus \partial M$.

(b) We suppose that the image of φ lies in a single component of M. If (a) is not satisfied, then $M \cup_\varphi h^k$ is non-orientable.

($k=0$) $M \cup_\varphi h^0 = M \sqcup \overline{B}^n$ is again orientable.

Proof.

(i) The reflection $\tau\colon S^{k-1} \to S^{k-1}$ extends in an obvious way to a diffeomorphism $\tau\colon \overline{B}^k \to \overline{B}^k$. The diffeomorphism from $M \cup_{\varphi\circ(\mathrm{id}\times\tau)} h^k$ to $M \cup_\varphi h^k$ is given by the identity on M and by $\mathrm{id}_{\overline{B}^{n-k}} \times \tau$ on $\overline{B}^{n-k} \times \overline{B}^k$.

(ii) This statement follows immediately from (1).

After these preparations we turn to the proofs of the main statements of the proposition.

($k \geq 2$) This statement follows immediately from (ii) together with the Gluing-Smooth Manifolds-Proposition **??** (6).

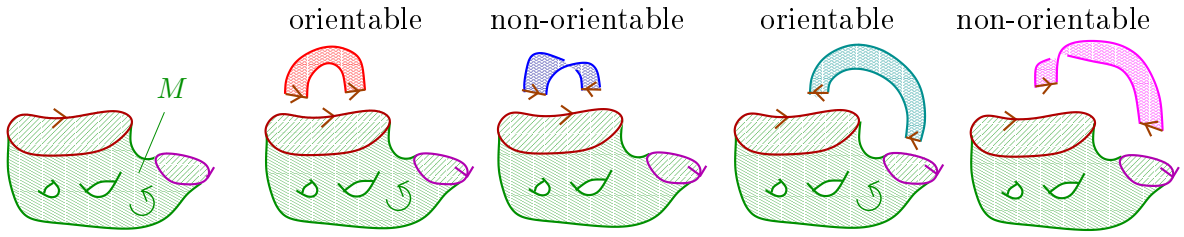
($k=1$) (a) This statement follows immediately from (ii) together with the Gluing-Smooth Manifolds-Proposition **??** (6).

(b) This statement is basically a special case of Exercise **??**.

($k=0$) This case is trivial. ■

The Handles-Orientability Proposition 136.9 (3) motivates the following definition.

Definition. Let M be a compact oriented n-dimensional smooth manifold. We say a thickened 0-sphere $\varphi\colon \overline{B}^n \times S^0 \to \partial M$ is orientable if φ is orientation-preserving or if it is orientation-reversing. Otherwise we say that the thickened 0-sphere is non-orientable.



The following lemma basically shows that there are not that many different possibilities for attaching 1-handles.

Lemma 136.10. (1-Handle Attachment Lemma) Let M be a compact oriented n-dimensional smooth manifold, let F be a component of ∂M and let $\varphi, \psi\colon \overline{B}^{n-1} \times S^0 \to F$ be two thickened 0-spheres. If φ and ψ are both orientable or if they are both non-orientable, then $M \cup_\varphi h^1$ and $M \cup_\psi h^1$ are diffeomorphic.



Proof. Let $\varphi, \psi\colon \overline{B}^{n-1} \times S^0 \to F$ be two thickened 0-spheres. First assume that $n \geq 3$. As in the Handles-Orientability Proposition 136.9 (1) let $\tau\colon S^0 \to S^0$ be the map that swaps

[1657] Recall that the orientation on S^0 is given by $-\{-1\} \cup +\{1\}$. By definition of the product orientation, see page 695, $\overline{B}^{n-1} \times S^0 \to -\overline{B}^{n-1} \sqcup \overline{B}^{n-1}$.

the two points ± 1. By the Handles-Orientability Proposition 136.9 we can, if convenient, replace φ by $\varphi \circ (\mathrm{id} \times \tau)$ and we can replace ψ by $\psi \circ (\mathrm{id} \times \tau)$. It follows from this observation that we can arrange that the maps $\varphi\colon \overline{B}^{n-1} \times \{1\} \to \partial M$ and $\psi\colon \overline{B}^{n-1} \times \{1\} \to \partial M$ are both orientation-preserving. It follows from our orientation hypothesis on φ and ψ that $\varphi\colon \overline{B}^{n-1} \times \{-1\} \to \partial M$ and $\psi\colon \overline{B}^{n-1} \times \{-1\} \to \partial M$ are both orientation-preserving or they are both orientation-reversing.

Since $n = \dim(M) \geq 3$ we have $\dim(\partial M) \geq 2$. Thus we can apply the Smooth Ball Embedding Theorem 37.8 which implies that the two maps φ and ψ are smoothly isotopic. Thus it follows from the Handle Attachment Proposition 136.1 that $M \cup_\varphi h^1$ and $M \cup_\psi h^1$ are diffeomorphic.

It remains to deal with the cases $n = 1$ and $n = 2$. The case $n = 1$ is basically trivial. Finally we will deal with the case $n = 2$ in Exercise 136.1. ■

136.6. **Handle attachments: Algebraic invariants.** In this section we will discuss how a handle attachment alters some of our favorite algebraic invariants of topological spaces, namely the fundamental group and the homology groups. The statements below are basically the same as for cell attachments that we gave in Propositions 69.11 and the Cell Attachment-H_*-Lemma 78.18. First we have the following analogue of the Cell Attachment-π_1-Proposition 69.11.

Proposition 136.11. (Handle Attachment-π_1-Proposition) Let M be a compact n-dimensional smooth manifold, let $k \in \mathbb{N}$ and let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere. We pick $(x_0, y_0) \in S^{n-k-1} \times S^{k-1}$ and we set $P := \varphi(x_0, y_0) \in \partial M$. The following statements hold:

($k>2$) The inclusion induced map $\pi_1(M, P) \to \pi_1(M \cup_\varphi h^k, P)$ is an isomorphism.

($k=2$) Let $\beta\colon S^1 \to M$ be the loop that is given by $z \mapsto \varphi(x_0, z)$. The inclusion induced map $\pi_1(M, P) \to \pi_1(M \cup_\varphi h^2, P)$ descends to an isomorphism

$$\pi_1(M, P)/\langle\langle [\beta] \rangle\rangle \xrightarrow{\cong} \pi_1(M \cup_\varphi h^2, P).$$

($k=1$) We set $P' := \varphi(x_0, -y_0) \in \partial M$.

(a) Suppose that P and P' lie in the same component of M. Let γ be a loop in $M \cup_\varphi h^1$ that is given by the concatenation of the obvious path in the 1-handle from P to P' and some path in M from P' to P. With this notation the map

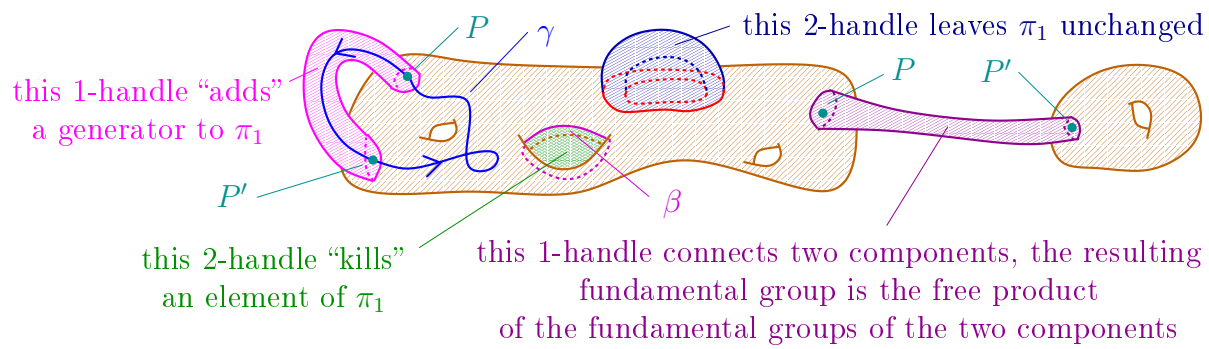
$$\pi_1(M, P) * \langle t \rangle \to \pi_1(M \cup_\varphi h^1, P)$$

that is given by the inclusion induced map $\pi_1(M, P) \to \pi_1(M \cup_\varphi h^1, P)$ and given by $t \mapsto [\gamma]$ is an isomorphism.

(b) Suppose that P and P' lie in two different components of M. Then we get an isomorphism

$$\pi_1(M, P) * \pi_1(M, P') \to \pi_1(M \cup_\varphi h^1, P).$$

Proof. The statements follow fairly easily from a slight generalization of the Seifert–van Kampen Theorem 54.4 and the HNN-Seifert–van Kampen Theorem 57.3 (b). Overall the proof is very similar to the proof of the Cell Attachment-π_1-Proposition 69.11 and we leave it to the reader to make the obvious alterations to obtain a proof for the present proposition. ■



Corollary 136.12. Let M be a compact connected non-empty smooth manifold. For every handle decomposition of M we have the inequality:

$$1 + \#\text{1-handles} - \#\text{0-handles} \;\geq\; \text{minimal size of a generating set of } \pi_1(M).$$

Proof. Let M be a compact connected non-empty smooth manifold that is equipped with a handle decomposition. We denote by r the number of 0-handles and we denote by s the number of 1-handles. First note, that by the Standard Handle Decomposition Corollary 136.4 we can assume that the handle decomposition is standard. We pick a base point P in one of the 0-handles. By the definition of a generating set, see page 1159, it suffices to prove the following claim.

Claim. There exists an epimorphism from a group of rank $1 + s - r$ onto $\pi_1(M, P)$.

Proof. Given $k \in \mathbb{N}_0$ we denote by M^k the k-skeleton of M. It follows fairly easily from our hypothesis that M is connected, together with the Handle-Connected Lemma 136.8 and the Handle Attachment-π_1-Proposition 136.11, that $\pi_1(M^1, P)$ is a free group on $1 + s - r$ generators. Next note that it follows from the Handle Attachment-π_1-Proposition 136.11 that the inclusion induced map $\pi_1(M^1, P) \to \pi_1(M, P)$ is an epimorphism. Thus we have shown, as desired, that there exists an epimorphism from a group of rank $1 + s - r$ onto $\pi_1(M, P)$. ∎

Next we study the effect of a handle attachment to homology groups. The following lemma is an analogue of the Cell Attachment-H_*-Lemma 78.18.

Lemma 136.13. (Handle Attachment-H_*-Lemma) Let M be a compact n-dimensional smooth manifold, let $k \in \mathbb{N}_0$, let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere and let $x \in \overline{B}^{n-k}$. We denote by $i\colon M \to M \cup_\varphi h^k$ the inclusion map. There exists a natural exact sequence of the form[1658]

$$0 \to \mathrm{H}_k(M) \xrightarrow{i_*} \mathrm{H}_k(M \cup_\varphi h^k) \to \underbrace{\widetilde{\mathrm{H}}_{k-1}(S^{k-1})}_{\cong \mathbb{Z}} \xrightarrow[\text{induced by the map } y \mapsto \varphi(x,y)]{} \mathrm{H}_{k-1}(M) \xrightarrow{i_*} \mathrm{H}_{k-1}(M \cup_\varphi h^k) \to 0.$$

Furthermore, for any $l \neq k-1, n$ the inclusion induced map

$$i_*\colon \mathrm{H}_l(M) \xrightarrow{\cong} \mathrm{H}_l(M \cup_\varphi h^k)$$

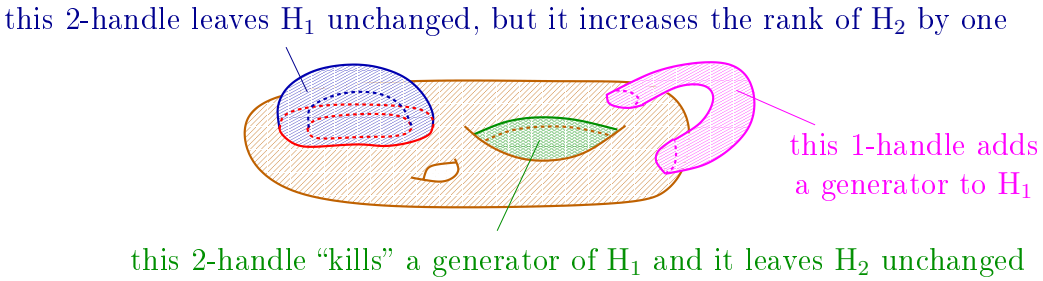
is an isomorphism.

Remark. By considering the 0-th homology groups one can deduced the Handle-Connected Lemma 136.8 from the Handle Attachment-H_*-Lemma 136.13.

Proof. We consider the following diagram:

$$\mathrm{H}_j(M\cup_\varphi h^k) \xrightarrow{\partial} \widetilde{\mathrm{H}}_{j-1}(\varphi(\overline{B}^{n-k}\times S^{k-1})) \to \mathrm{H}_{j-1}(M)\oplus \overbrace{\widetilde{\mathrm{H}}_{j-1}(\varphi(\overline{B}^{n-k}\times \overline{B}^{k}))}^{=0} \to \mathrm{H}_{j-1}(M\cup_\varphi h^k) \to$$
$$\cong\uparrow \qquad\qquad \nearrow$$
$$\widetilde{\mathrm{H}}_{j-1}(S^{k-1})$$

Here the upper sequence is the long exact sequence coming from a slight generalization of the Mayer-Vietoris Theorem 78.9 for Smooth Manifolds. The maps emanating from below are the obvious maps. The statements of the lemma follow from staring at this diagram for a few short minutes. ■



136.7. **Handle attachments: Invariants of the boundary.** In this section, given a handle attachment $M\cup_\varphi h^k$ we will study the interaction between algebraic invariants of $M\cup_\varphi h^k$ and its boundary $\partial(M\cup_\varphi h^k)$. Again we first consider fundamental groups.

Lemma 136.14. Let M be a connected n-dimensional smooth manifold such that ∂M is also connected. Let $\varphi\colon \overline{B}^{n-k}\times S^{k-1}\to \partial M$ be a thickened $(k-1)$-sphere. The following statements hold:

(1) If $1\le k<n-1$, then the boundary $\partial(M\cup_\varphi h^k)$ is connected and non-empty.
(2) (a) If $1\le k\le n-3$ and if $\pi_1(\partial M)\to\pi_1(M)$ is an isomorphism, then the induced map $\pi_1(\partial(M\cup_\varphi h^k))\to\pi_1(M\cup_\varphi h^k)$ is also an isomorphism.
(b) If $1\le k\le n-2$ and if $\pi_1(\partial M)\to\pi_1(M)$ is an epimorphism, then the induced map $\pi_1(\partial(M\cup_\varphi h^k))\to\pi_1(M\cup_\varphi h^k)$ is also an epimorphism.

Proof. First we prove Statement (1). By the Handle Attachment Proposition 136.1 we have

$$\partial(M\cup_\varphi h^k) \;=\; \underbrace{(\partial M\setminus\varphi(B^{n-k}\times S^{k-1}))}_{\substack{\text{connected by the Complement}\\ \text{Connected Lemma 2.29,}\\ \text{since } \partial M \text{ is connected}}} \cup_{\varphi|\underbrace{S^{n-k-1}\times S^{k-1}}_{\neq\varnothing \text{ since } 1\le k<n-1}} \underbrace{(S^{n-k-1}\times \overline{B}^{k})}_{\substack{\text{connected since}\\ k<n-1}}.$$

It follows from the Complement Connected Lemma 2.29 that $\partial(M\cup_\varphi h^k)$ is indeed connected. Since $n-k-1\ge 0$ we see that the boundary is furthermore non-empty.

[1658] Note that this statement also holds for $k=0$, we just need to remember that by the Reduced Homology Lemma 74.1 we have $\widetilde{\mathrm{H}}_{-1}(\varnothing)=\mathbb{Z}$.

We turn to the proof of Statement (2). We pick and immediately ignore a base point in $\varphi(S^{n-k-1} \times S^{k-1}$. We consider the following diagram:

$$\begin{array}{ccc} \pi_1\big(\overbrace{(\partial M \backslash \varphi(B^{n-k} \times S^{k-1})) \underset{S^{n-k-1}\times S^{k-1}}{\cup} (S^{n-k-1} \times \overline{B}^k)}^{=\partial(M \cup_\varphi h^k)}\big) & \longrightarrow & \pi_1\big(\overbrace{M \underset{\overline{B}^{n-k}\times S^{k-1}}{\cup} (\overline{B}^{n-k} \times \overline{B}^k)}^{=M \cup_\varphi h^k}\big) \\ & \searrow & \uparrow \\ & & \pi_1\big(\partial M \underset{S^{n-k-1}\times S^{k-1}}{\cup} (S^{n-k-1} \times \overline{B}^k)\big). \end{array}$$

All the maps are induced by inclusions, hence the diagram commutes. First we consider the case that $1 \leq k \leq n-3$. In this setting we need to show that the top map is an isomorphism. This statement is a consequence of the following two observations:

(i) Note that it follows from Lemma 54.13 (1) and our hypothesis that $k \leq n-3$ that the map $\pi_1(\partial M \backslash \varphi(B^{n-k} \times S^{k-1})) \to \pi_1(\partial M)$ is an isomorphism. Next note that it follows from this observation together with the Seifert–van Kampen Theorem 54.4 and the Amalgamated Product Lemma 53.22 (1) that the diagonal map is an isomorphism.

(ii) By hypothesis we know that $\pi_1(\partial M) \to \pi_1(M)$ is an isomorphism. It follows from this observation together with the Seifert–van Kampen Theorem 54.4 and the Amalgamated Product Lemma 53.22 (1) that the vertical map is an isomorphism.

Finally we consider the case that $k \leq n-2$. In this setting we perform the same argument as above, except that need to replace Lemma 54.13 (1) by Lemma 54.13 (2) and that we need to replace the Amalgamated Product Lemma 53.22 (1) by the Amalgamated Product Lemma 53.22 (2). ■

Corollary 136.15. Let $n \in \mathbb{N}$ with $n \geq 4$. Let M be a compact connected non-empty n-dimensional smooth manifold that is equipped with a handle decomposition.

(1) If all handles have index $\leq n-3$, then the boundary ∂M is connected and the inclusion induced map $\pi_1(\partial M) \to \pi_1(M)$ is an isomorphism.

(2) If all handles have index $\leq n-2$, then the boundary ∂M is connected and the inclusion induced map $\pi_1(\partial M) \to \pi_1(M)$ is an epimorphism.

Proof. Let M be a compact connected non-empty n-dimensional smooth manifold that is equipped with a handle decomposition. If M has a single 0-handle, then the statement follows immediately from Lemma 136.14.

Now suppose that M has in fact more than one 0-handle. In Corollary **??** we will see that we can arrange that the handle decomposition has a single 0-handle. Alternatively we can argue directly as follows. Let $A_1, \ldots, A_k$ be the 0-handles. It follows easily from the hypothesis that M is connected that there exist 1-handles $B_1, \ldots, B_{k-1}$ such that $W := (A_1 \cup \cdots \cup A_k) \cup (B_1 \cup \cdots \cup B_{k-1})$ is connected. It is straightforward to verify that $\pi_1(W)$ is trivial and that ∂W is connected. We obtain M from W by attaching handles of index ≥ 1. Thus the desired statement again follows from Lemma 136.14. ■

We continue with the "homological analogue" of Lemma 136.14.

Lemma 136.16. Let M be an n-dimensional smooth manifold. Suppose we are given a thickened $(k-1)$-sphere $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$. Let $j \in \mathbb{N}_0$. The following statements hold:

(1) If $j \leq n-k-2$ and if $\mathrm{H}_i(\partial M) \to \mathrm{H}_i(M)$ is an isomorphism for $i = 0, \ldots, j$, then $\mathrm{H}_j(\partial(M \cup_\varphi h^k)) \to \mathrm{H}_j(M \cup_\varphi h^k)$ is also an isomorphism for $i = 0, \ldots, j$.

(2) If $j \leq n-k-1$ and if $\mathrm{H}_i(\partial M) \to \mathrm{H}_i(M)$ is an isomorphism for $i = 0, \ldots, j-1$ and if it is an epimorphism for $i = j$, then the analogous statements hold for the maps $\mathrm{H}_j(\partial(M \cup_\varphi h^k)) \to \mathrm{H}_j(M \cup_\varphi h^k)$.

Proof. The lemma follows from applying the Mayer-Vietoris Theorem 78.9 to the decomposition $\partial M = (\partial M \setminus \varphi(B^{n-k} \times S^{k-1})) \cup (\overline{B}^{n-k} \times S^{k-1})$, to $M \cup_\varphi h^k$ and finally to $\partial(M \cup_\varphi h^k) = (\partial M \setminus \varphi(B^{n-k} \times S^{k-1})) \cup (S^{n-k-1} \times \overline{B}^k)$. Since we will not make use of the lemma we leave the pleasure of filling in the details to the reader. ■

The following is an immediate corollary to Lemma 136.16.

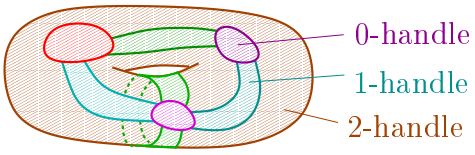
Corollary 136.17. Let M be a compact non-empty n-dimensional smooth manifold that is equipped with a handle decomposition and let $j \in \mathbb{N}_0$.

(1) If all handles have index $\leq n-j-2$, then for $i = 0, \ldots, j$ the inclusion induced map $\mathrm{H}_j(\partial M) \to \mathrm{H}_j(M)$ is an isomorphism.

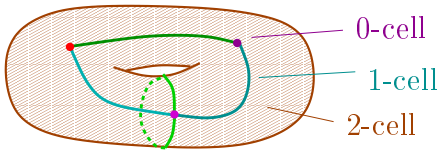
(2) If all handles have index $\leq n-j-1$, then for $i = 0, \ldots, j-1$ the inclusion induced map $\mathrm{H}_i(\partial M) \to \mathrm{H}_i(M)$ is an isomorphism and the inclusion induced map $\mathrm{H}_j(\partial M) \to \mathrm{H}_j(M)$ is an epimorphism.

136.8. **Handle decompositions and CW-structures.** We conclude this chapter with a proposition that shows that a handle decomposition of a smooth manifold M gives rise to a CW-complex that is *homotopy equivalent* to the smooth manifold M. We will not really make use of this result, but it is good to know it exists, and in principle it can be used to deduce some results on handle decompositions from our long experience with CW-structures.

Proposition 136.18. (Handles-Cells Proposition) Every compact smooth manifold M that is equipped with a handle decomposition is homotopy equivalent to a CW-complex X such that for each $k \in \mathbb{N}_0$ the number of k-handles of M equals the number of k-cells of X.



handle decomposition of the torus



CW-complex

Remark. We will prove the Handles-Cells Proposition 136.18 using the Homotopy Pushout Theorem 17.19, which comes with a rather intimidating proof. A slightly more direct proof for the Handles-Cells Proposition 136.18 is given in [**Miln1963a**, Theorem 3.4].

As we mentioned in the remark, the key to proving the Handles-Cells Proposition 136.18 is the following theorem whose easy formulation belies the difficulty of the proof.

Theorem 17.19. (Homotopy Pushout Theorem) Suppose we are given the following commutative diagram of maps between topological spaces:

$$\begin{array}{ccccc} Y & \xleftarrow{f} & A & \xhookrightarrow{i} & X \\ {\scriptstyle\varphi_Y}\downarrow & & \downarrow{\scriptstyle\varphi_A} & & \downarrow{\scriptstyle\varphi_X} \\ Y' & \xleftarrow{f'} & A' & \xhookrightarrow{i'} & X'. \end{array}$$

If the vertical maps are homotopy equivalences and if the maps i and i' to the right are closed cofibrations[1659], then the induced map

$$Y \cup_A X \;\to\; Y' \cup_{A'} X'$$

between the pushouts is a homotopy equivalence.

Proof of the Handles-Cells Proposition 136.18. Let M be a compact smooth manifold with a given handle decomposition. By the Standard Handle Decomposition Corollary 136.4 we can arrange, without changing the number of handles of any index, that the handle decomposition is standard.

Given $k \in \mathbb{N}_0$ we denote, as on page 2958, by M^k the k-skeleton of the handle decomposition. In other words, M^k is the union of all handles of dimension $\leq k$. Furthermore, given any CW-complex X we denote by X^k its k-skeleton, i.e. X^k is the union of cells of dimension $\leq k$. We will prove the following slight refinement of the original statement of the Handles-Cells Proposition 136.18.

Statement. There exists a CW-complex X and a homotopy equivalence $g\colon M \to X$ with the following two properties:

(1) for each $k \in \mathbb{N}_0$ we have $g(M^k) \subset X^k$,
(2) for each $k \in \mathbb{N}_0$ the number of k-cells of X equals the number of k-handles of M.

By an elementary induction argument it suffices to prove the following claim.

Claim. Let N be a compact n-dimensional smooth manifold that is equipped with a standard handle decomposition. We suppose that there exists a homotopy equivalence $f\colon N \to X$ to some CW-complex X such that for every $k \in \mathbb{N}_0$ we have $f(N^k) \subset X^k$. Let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial N$ be a smooth embedding. There exists a cellular map[1660] $\psi\colon S^{k-1} \to X$ and a homotopy equivalence[1661]

$$g\colon \underbrace{\big(N \sqcup \big(\overline{B}^{n-k} \times \overline{B}^{k}\big)\big)/\!\sim}_{=N\cup_\varphi h^k} \;\to\; \underbrace{\big(X \sqcup \overline{B}^{k}\big)/\!\sim}_{\psi(x)\sim x \text{ for all } x\in S^{k-1}}.$$

Proof. We consider the map $\eta\colon S^{k-1} \to X$ that is given by $\eta(x) := (f \circ \varphi)(0, x)$. By the Cellular Approximation Theorem 70.15 there exists a homotopy $F\colon S^{k-1} \times [0,1] \to X$ from $F_0 = \eta\colon S^{k-1} \to X$ to a cellular map $F_1 = \psi\colon S^{k-1} \to X$. We obtain the following

[1659] We refer to page 478 for the definition of a closed cofibration. To get the hang of the argument it is not that important though to recall the definition of a closed cofibration.
[1660] We equip S^{k-1} with the canonical CW-structure that we introduced on page 1472.
[1661] Note that it follows from the CW-Complex Construction Lemma 68.32 and the fact that ψ is cellular that $(X \sqcup \overline{B}^k)/\!\sim$ is a CW-complex with one extra k-cell.

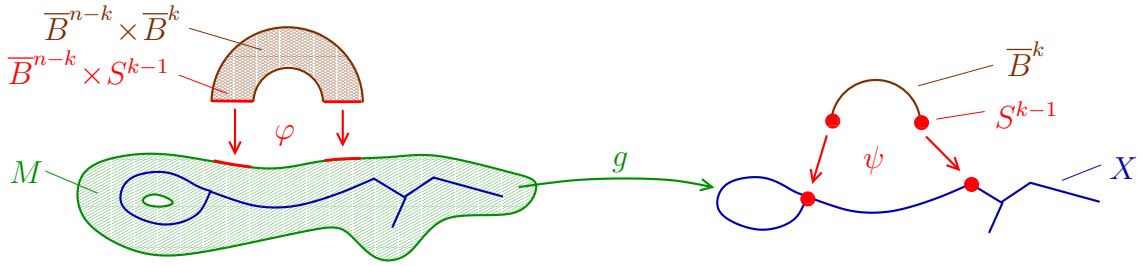
commutative diagram:

$$
\begin{array}{ccccc}
N & \hookleftarrow & \partial N & \xleftarrow{\varphi} & \overline{B}^{n-k} \times S^{k-1} \hookrightarrow \overline{B}^{n-k} \times \overline{B}^{k} \\
{\scriptstyle \mathrm{id}}\uparrow & & & & \uparrow{\scriptstyle x \mapsto (0,x)} \\
N & \xleftarrow{x \mapsto \varphi(0,x)} & & & S^{k-1} \hookrightarrow \overline{B}^{k} \\
{\scriptstyle f}\downarrow & & & & \downarrow \\
X & \xleftarrow{\eta} & & & S^{k-1} \hookrightarrow \overline{B}^{k} \\
{\scriptstyle \mathrm{id}}\downarrow & & & & \downarrow{\scriptstyle x \mapsto (x,0)} \\
X & \xleftarrow{F} & & & S^{k-1} \times [0,1] \hookrightarrow \overline{B}^{k} \times [0,1] \\
{\scriptstyle \mathrm{id}}\uparrow & & & & \uparrow{\scriptstyle x \mapsto (x,1)} \\
X & \xleftarrow{\psi} & & & S^{k-1} \hookrightarrow \overline{B}^{k}.
\end{array}
$$

It is clear that all of the vertical maps are homotopy equivalences. Furthermore note that it follows immediately from the Manifold Boundary-Cofibration Proposition 17.8 and the Product-Cofibration Proposition 17.11 that all of the right horizontal maps are closed cofibrations. Thus it follows from the Homotopy Pushout Theorem 17.19, applied altogether four times, that the induced maps between the various pushouts have homotopy inverses. Combining these four homotopy equivalences, respectively their homotopy inverses, we obtain the promised homotopy equivalence

$$
\underbrace{N \cup_{\overline{B}^{n-k} \times S^{k-1}} (\overline{B}^{n-k} \times \overline{B}^{k})}_{= (N \sqcup (\overline{B}^{n-k} \times \overline{B}^{k}))/\sim} \quad \to \quad \underbrace{X \cup_{S^{k-1}} \overline{B}^{k}}_{= (X \sqcup \overline{B}^{k})/\sim}.
$$

■



Exercises for Chapter 136.

Exercise 136.1. Let M be a compact oriented 2-dimensional smooth manifold and let F be a component of ∂M. Show that any two orientation-preserving 1-handle attachments to F lead to diffeomorphic smooth manifolds.

Exercise 136.2. Let $m, k \in \mathbb{N}$. We consider the thickened $(k-1)$-sphere

$$
\begin{array}{rcl}
\varphi \colon \overline{B}^{m} \times S^{k-1} & \to & S^{m+k-1} = \partial(\overline{B}^{m+k}) \\
(x, y) & \mapsto & \frac{(\frac{1}{2} \cdot x, y)}{\|(\frac{1}{2} \cdot x, y)\|}.
\end{array}
$$

(a) Show that $\overline{B}^{m+k} \cup_{\varphi} h^{k}$ is homeomorphic to $\overline{B}^{m} \times S^{k}$.

(b) Show that $\overline{B}^{m+k} \cup_\varphi h^k$ is diffeomorphic to $\overline{B}^m \times S^k$.
Remark. This exercise is meant only for the heroically inclined reader.
Remark. The full details for a very similar statement are provided in [**Wall2016**, Lemma 5.4.2].

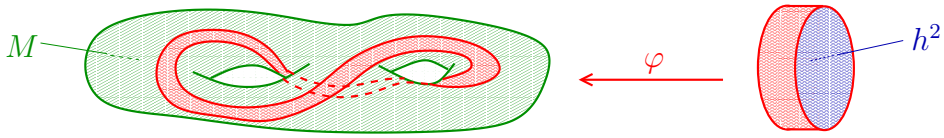
Exercise 136.3. Let M be a closed connected non-empty n-dimensional smooth manifold.
(a) Show that every handle decomposition of M admits at least one n-handle.
(b) Suppose we are given a handle decomposition of M with a unique n-handle. Show that this n-handle has non-empty intersection with every other handle.

Exercise 136.4. On page 2955 we saw that the surface of genus 2 admits a handle decomposition with four 1-handles. Does it also admit a handle decomposition with less than four 1-handles?

Exercise 136.5. Let M be a compact oriented n-dimensional smooth manifold and let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere. Show that if $k \geq 2$ and $n \geq k+3$, then there exists an epimorphism $\pi_1(\partial M) \to \pi_1(\partial(M \cup_\varphi h^k))$.

Exercise 136.6. Let $K \subset S^3$ be a knot. We pick a tubular neighborhood $\overline{B}^2 \times K$ and as on page 2213 we consider the knot exterior $X_K := S^3 \setminus (B^2 \times K)$. Can we add handles to X_K to obtain a smooth manifold that is diffeomorphic to S^3?

Exercise 136.7. We consider the 3-dimensional smooth manifold M shown in the figure below. Furthermore we consider the thickened 1-sphere $\varphi\colon \overline{B}^1 \times S^1 \to \partial M$ shown in the figure below. Can you recognize the diffeomorphism type of $M \cup_\varphi h^2$? Can you prove that it is what you think it is?



Exercise 136.8. Let M be a connected n-dimensional smooth manifold. Furthermore let $\varphi\colon \overline{B}^{n-1} \times S^0 \to \partial M$ be a thickened 0-sphere in ∂M. Let $\tau\colon \overline{B}^{n-1} \to \overline{B}^{n-1}$ be the reflection in a hyperplane. We consider the thickened 0-sphere

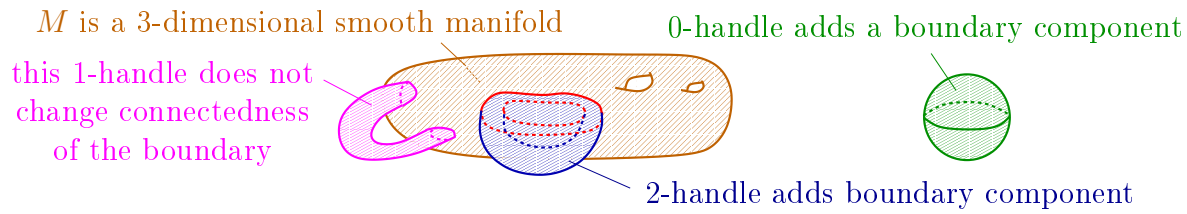
$$\begin{array}{rcl} \overline{B}^n \times S^0 & \to & \partial M \\ (P,\epsilon) & \mapsto & \begin{cases} \varphi(\tau(P),-1), & \text{if } \epsilon = -1, \\ \varphi(P,1), & \text{if } \epsilon = 1. \end{cases} \end{array}$$

Is it possible that $M \cup_\varphi h^1$ and $M \cup_\psi h^1$ are diffeomorphic?

Exercise 136.9. Let M be a compact n-dimensional smooth manifold such that ∂M is connected. Let $k \in \mathbb{N}_0$ and let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere. Show that if $\in \{1, \ldots, n-1\}$, then $\partial(M \cup_\varphi h^k)$ is also connected.

Exercise 136.10. Let M be a compact n-dimensional smooth manifold.
(a) Let $k \in \mathbb{N}_0$ and let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere. Prove that

$$\chi(M \cup_\varphi h^k) \;=\; \chi(M) + (-1)^k.$$



(b) Show that given any handle decomposition for M the following equality holds:

$$\sum_{k\in\mathbb{N}_0} (-1)^k \cdot \text{number of } k\text{-handles} \;=\; \chi(M).$$

Exercise 136.11. Let $k \in \mathbb{N}$ and let M be a compact orientable $2k$-dimensional smooth manifold which admits a handle decomposition such that all handles have index $\leq k$. Is the inclusion induced map $\mathrm{H}_k(\partial M) \to \mathrm{H}_k(M)$ necessarily an epimorphism?

Exercise 136.12. Let M be a compact smooth that is equipped with a standard handle decomposition. Given $k \in \mathbb{N}_0$ we denote by M^k the k-skeleton of the handle decomposition, i.e. M^k is the union of all handles up to dimension $\leq k$. Let P be a point in the 0-skeleton and let $i \in \mathbb{N}$.

(a) Show that for any $k > i$ the inclusion induced map $\pi_i(M^k, P) \to \pi_i(M, P)$ is an isomorphism.

(b) Show that the inclusion induced map $\pi_k(M^k, P) \to \pi_k(M, P)$ is an epimorphism.

Hint. Use the Whitney Approximation Theorem 29.1, the Replace-by-Smooth Homotopy Proposition 29.5 and Transversality Theorem 42.4 applied to a suitable choice of proper submanifolds.

Remark. This exercise can be viewed as a handle analogue of the corresponding statement for CW-complexes that we proved in the CW-Skeleton-π_n-Proposition 71.9.

Exercise 136.13. Let M be a compact oriented n-dimensional smooth manifold, let F be a component of ∂M and let $\varphi, \psi\colon \overline{B}^{n-1} \times S^0 \to F$ be two thickened 0-spheres. Show that if F is non-orientable, then then $M \cup_\varphi h^1$ and $M \cup_\psi h^1$ are diffeomorphic.

Remark. This exercise is a variation on the 1-Handle Attachment Lemma 136.10.

137. Morse functions

In the last chapter we introduced handle decompositions and showed explicitly that some of our favorite compact smooth manifolds admit handle decompositions. Our next goal is to show that every compact smooth manifold admits a handle decomposition. We will do so via Morse theory, which builds on the fairly straightforward concept of a Morse function. In this chapter we will introduce Morse functions, we will show that they always exist and we will study their gradient vector fields.

137.1. **Non-degenerate critical points.** The following definition is just a special case of the definition on page 701.

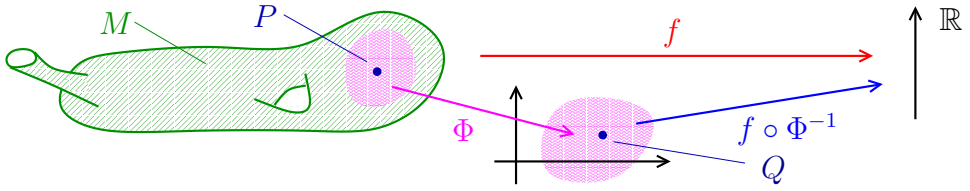
Definition. Let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. We say $P \in M$ is a regular point if $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathbb{R}$ is non-zero, otherwise we say P is a critical point.

In our previous discussions we were trying hard to avoid critical points. In the following we will see that the study of critical points can actually be quite fruitful. In particular, as we will see shortly, critical points come in different flavors which have a great influence on the topology of the smooth manifold.

Lemma 137.1. Let M be an n-dimensional smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Let $P \in M \setminus \partial M$ be a critical point. We pick a chart $\Phi\colon U \to V$ for P. We write $Q := \Phi(P)$. The congruence[1662] class of the Hessian matrix[1663]

$$\mathrm{Hess}(f\circ\Phi^{-1})_Q \;:=\; \left(\frac{\partial^2}{\partial x_i \partial x_j}\Big|_Q f\circ\Phi^{-1}\right)_{i,j=1,\dots,n} \;\in\; \mathrm{M}(n\times n,\mathbb{R})$$

is independent of the choice of the chart.



Proof. There are two approaches to proving the lem:

(1) Let $\Psi\colon X \to Y$ be some other chart around P. We set $R := \Psi(Q)$ and we set $Z := \mathrm{D}(\Psi\circ\Phi^{-1})_P$. A slightly heroic calculation, see [**Gaul1982**, p. 51], shows that $Z^T \cdot \mathrm{Hess}(f\circ\Psi^{-1})_R \cdot Z = \mathrm{Hess}(f\circ\Phi^{-1})_P$.

(2) The statement can also be proved more elegantly by defining intrinsically a symmetric form $\mathrm{H}_P f\colon \mathrm{T}_P M \times \mathrm{T}_P M \to \mathbb{R}$ which now happens to be represented by the matrix $\mathrm{Hess}(f\circ\Phi^{-1})_P$. At this point the lemma is a consequence of the purely algebraic the Forms-Matrices Lemma 133.1. This approach is taken in [**Miln1963a**, p. 4] and [**Nic2011**, p. 6]. ∎

[1662] Here recall that two real $n\times n$-matrices G and H are called *congruent* if there exists a matrix $Z \in \mathrm{GL}(n,\mathbb{R})$ such that $Z^T G Z = H$.

[1663] Implicitly we use throughout that the Hessian matrix is symmetric. The fact that given a smooth function the partial derivatives commute is variously known (depending on the geographic location of the author) as the Schwarz Theorem, the Young Theorem or the Clairaut Theorem.

It follows from Lemma 137.1, common sense and the combination of the Forms-Matrices Lemma 133.1 (1c) and the Signature-Eigenvalues Proposition 133.8 that the following definition is independent of any choices.

Definition. Let M be an n-dimensional smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Let $P \in M \setminus \partial M$ be a critical point. We pick a chart $\Phi\colon U \to V$ for P and we write $Q := \Phi(P)$.

(1) We say the critical point P is non-degenerate if $\det(\mathrm{Hess}(f \circ \Phi^{-1})_Q) \neq 0$. Otherwise we call the critical point degenerate.

(2) We refer to[1664]

$$\mathrm{index}(f, P) \;:=\; \text{number of negative eigenvalues of } \mathrm{Hess}(f \circ \Phi^{-1})_Q$$

as the index of the critical point P.

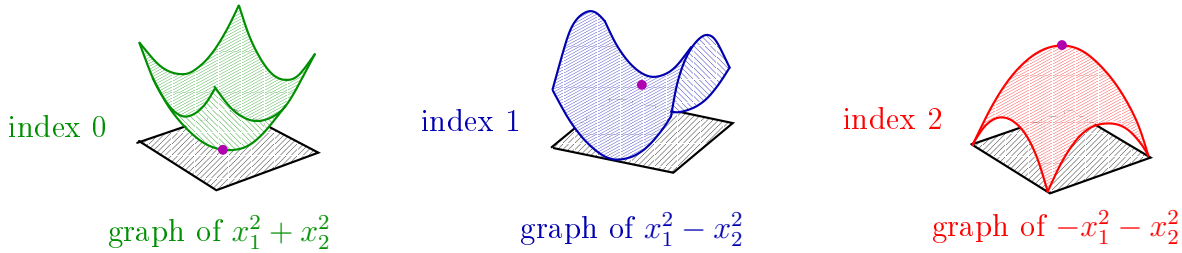
Examples.

(1) The "mother of all examples" are the functions

$$\begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R} \\ (x_1, \ldots, x_n) & \mapsto & c + \sum_{i=1}^{n} a_i x_i^2 \end{array}$$

with $c, a_1, \ldots, a_n \in \mathbb{R}$. Evidently the origin 0 is a critical point. With respect to the identity chart the Hessian $\mathrm{Hess}(f)_0$ is the diagonal matrix with diagonal entries $a_1, \ldots, a_n$. Thus the origin is a non-degenerate critical point if and only if all a_i are non-zero. Finally, we have the following equality:

$$\mathrm{index}(f, 0) \;=\; \text{number of negative } a_i.$$



graph of $x_1^2 + x_2^2$ graph of $x_1^2 - x_2^2$ graph of $-x_1^2 - x_2^2$

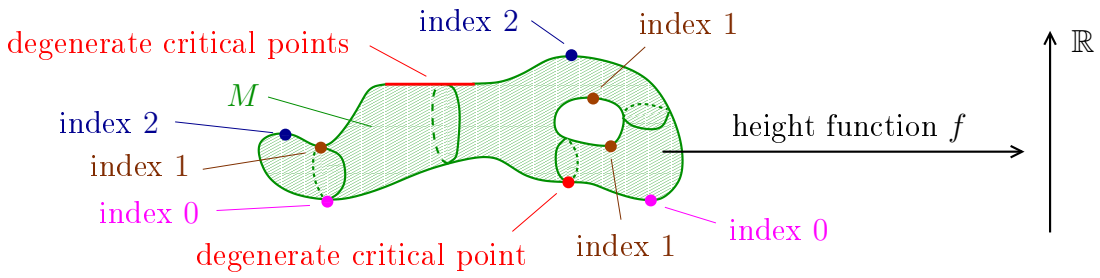
(2) We consider the smooth manifold $M \subset \mathbb{R}^3$ that is shown in the figure below together with the "height function" $f\colon M \to \mathbb{R}$ that is defined by $f(x, y, z) = z$. In the figure below we indicate all critical points of f and we show their indices.

The following result says that, up to taking an appropriate chart, the above mother of all examples is the only example of a non-degenerate critical point. The result was first proved by Marston Morse [**MorsM1934**] in 1934.

[1664] Alternatively one could use the Signature-Eigenvalues Proposition 133.8 to define

$$\mathrm{index}(f, P) \;:=\; b^-(\mathrm{H}_P f\colon \mathrm{T}_P M \times \mathrm{T}_P M \to \mathbb{R})$$

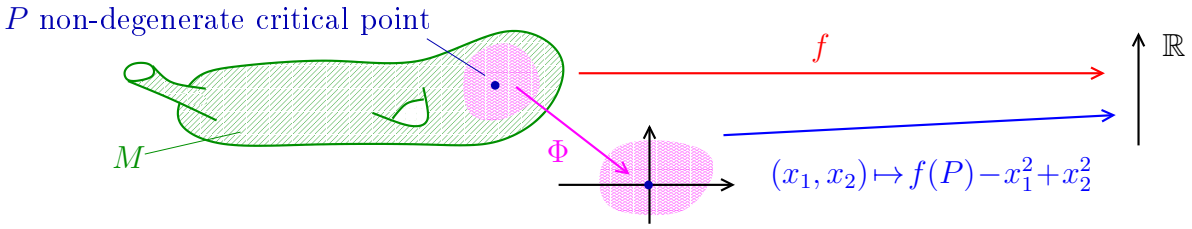
where $\mathrm{H}_P f$ is the intrinsic Hessian mentioned above and where $b^-(\varphi)$ of a symmetric form $\varphi\colon V \times V \to \mathbb{R}$ is defined, see page 2913, as the maximal dimension of a vector subspace W of V for which the symmetric form $\varphi|_W$ is negative-definite.



Theorem 137.2. (Morse Lemma) Let M be an n-dimensional smooth manifold. Furthermore let $f\colon M \to \mathbb{R}$ be a smooth function and let $P \in M \setminus \partial M$ be a critical point. If P is non-degenerate, then there exists a chart $\Phi\colon U \to V$ around P with $U \subset M \setminus \partial M$ and with $\Phi(P) = 0$ such that for every $(x_1, \ldots, x_n) \in V$ we have

$$(f \circ \Phi^{-1})(x_1, \ldots, x_n) \;=\; f(P) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$$

where $k = \operatorname{index}(f, P)$.



Remark.

(1) Recall that the Inverse Mapping Theorem 24.1 says that if the differential $\mathrm{D}f_P$ of a given map $f\colon M \to N$ between two smooth manifolds at a point $P \in M \setminus \partial M$ is invertible, then the map is locally a diffeomorphism. The Morse lemma is a similar statement: at a non-degenerate critical point the local behavior of a smooth map $f\colon M \to \mathbb{R}$ is determined by the second partial derivatives.

(2) Our proof of the Morse Lemma 137.2 is basically the same proof as in [**Miln1963a**, Lemma 2.2]. Somewhat different proofs are given in [**Nic2011**, Theorem 1.12] and [**Lau2011**, Section 3.3.2].

(3) Let M be an n-dimensional smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Furthermore let $P \in M \setminus \partial M$ be a non-degenerate critical point. It follows easily from the Morse Lemma 137.2 that
   (a) P is a local maximum if and only if $\operatorname{index}(f, P) = n$.
   (b) P is a local minimum if and only if $\operatorname{index}(f, P) = 0$.

One of the key ingredients to the proof of the Morse Lemma 137.2 is the following proposition.

Proposition 19.14 (Map-as-Matrix Multiplication Proposition). Let $U \subset \mathbb{R}^n$ be an open convex subset that contains the origin and let $f\colon U \to \mathbb{R}$ be a smooth function with $f(0) = 0$. There exist smooth maps $a_1, \ldots, a_n\colon U \to \mathbb{R}$ such that for $i = 1, \ldots, n$ we

have $a_i(0) = \frac{\partial f}{\partial x_i}(0)$ and such that

$$f(x_1, \dots, x_n) \;=\; \sum_{i=1}^{n} a_i(x_1, \dots, x_n) \cdot x_i \quad \text{for all } x = (x_1, \dots, x_n) \in U.$$

This proposition gives us the following corollary, which is actually the content of Exercise 19.13.

Corollary 137.3. (Function-near-Critical Point Corollary) Let $U \subset \mathbb{R}^n$ be an open convex subset that contains the origin and let $f\colon U \to \mathbb{R}$ be a smooth function with $f(0) = 0$. If 0 is a critical point of f, then there exist smooth functions $h_{ij}\colon U \to \mathbb{R}$, $i, j = 1, \dots, n$ such that $h_{ij} = h_{ji}$ and such that

$$f(x) \;=\; \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij}(x) \cdot x_i \cdot x_j \quad \text{for all } x \in U.$$

Proof of Corollary 137.3. First note that by the Map-as-matrix multiplication Proposition 19.14 there exist smooth maps $a_1, \dots, a_n\colon U \to \mathbb{R}$ such that for $i = 1, \dots, n$ we have $a_i(0) = \frac{\partial f}{\partial x_i}(0) = 0$ and such that

$$f(x) \;=\; \sum_{i=1}^{n} a_i(x) \cdot x_i \quad \text{for all } x = (x_1, \dots, x_n) \in U.$$

Since 0 is a critical point we know that $a_1(0) = \cdots = a_n(0) = 0$. Thus, given $i \in \{1, \dots, n\}$ we can once again apply the ever-useful the Map-as-matrix multiplication Proposition 19.14 to the smooth map a_i and we obtain smooth maps $b_{i1}, \dots, b_{in}\colon U \to \mathbb{R}$ with $b_{ij}(0) = (\frac{\partial}{\partial x_j} a_i)(0) = (\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} f)(0)$ and such that

$$a_i(x) \;=\; \sum_{j=1}^{n} b_{ij}(x) \cdot x_j \quad \text{for all } x \in U.$$

It now clear that the maps $h_{ij} := \frac{1}{2}(b_{ij} + b_{ji})$ have the desired properties. ■

Proof of the Morse Lemma 137.2. We make a few preparations.

(1) We can assume that M is an open convex neighborhood U of $0 \in \mathbb{R}^n$, that $P = 0$ is the non-degenerate critical point of $f\colon U = M \to \mathbb{R}$ and that $f(0) = 0$.
(2) Throughout the proof we use the usual notation that given $x \in \mathbb{R}^n$ we denote its coordinates by $x_1, \dots, x_n$.
(3) Let V and W be open neighborhoods of $0 \in \mathbb{R}^n$. We say two smooth maps $g\colon V \to \mathbb{R}$ and $h\colon W \to \mathbb{R}$ are *equivalent*, if there exist open sets $0 \in V' \subset V$ and $0 \in W' \subset W$ and a diffeomorphism $\Psi\colon V' \to W'$ with $\Psi(0) = 0$ such that $(g \circ \Psi^{-1})(y) = h(y)$ for all $y \in W'$. Note that this is indeed an equivalence relation.

With these preparations we can reformulate our goal as follows.

Goal. We want to show that $f\colon U \to \mathbb{R}$ is equivalent to a map that is of the form

$$(x_1, \dots, x_n) \;\mapsto\; \epsilon_1 \cdot x_1^2 + \cdots + \epsilon_n \cdot x_n^2$$

for some $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$.

Note that once we have proved this goal we are done, since it follows from Lemma 137.1 that the number of negative ϵ_i's in a map as above equals precisely $\operatorname{index}(f, P)$.

Now let us turn to the task of achieving this goal. First note that by the Function-near-Critical Point Corollary 137.3 we know that there exist smooth functions $h_{ij}\colon U \to \mathbb{R}$,

$i, j = 1, \ldots, n$ such that $h_{ij} = h_{ji}$ and such that f is given by

$$(x_1, \ldots, x_n) \;\mapsto\; \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij}(x) \cdot x_i \cdot x_j.$$

The idea now is to "diagonalize" the matrices $(h_{ij}(x))_{i,j=1,\ldots,n}$. At this point it is worth recalling that in the Symmetric Forms-over-$\mathbb{R}$-Proposition 133.6 we gave a constructive proof of the fact that given any real symmetric $(n \times n)$-matrix $A = (a_{ij})$ with $\det(A) \neq 0$ there exists a matrix $P \in \mathrm{GL}(n, \mathbb{R})$ and $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$ such that $P^T A P$ is the diagonal matrix with entries $\epsilon_1, \ldots, \epsilon_n$. The idea now is to imitate the proof of the Symmetric Forms-over-$\mathbb{R}$-Proposition 133.6. As in the proof of the Symmetric Forms-over-$\mathbb{R}$-Proposition 133.6 we proceed iteratively. This leads us to the following claim.

Claim. Given any $s \in \{0, \ldots, n\}$ the map $f \colon U \to \mathbb{R}$ is equivalent to a map of the form

$$(x_1, \ldots, x_n) \;\mapsto\; \sum_{i=1}^{s} \sum_{j=1}^{s} \mu_{ij}(x) \cdot x_i \cdot x_j \;+\; \sum_{i=s+1}^{n} \epsilon_i \cdot x_i^2,$$

where the μ_{ij} are smooth functions with $\mu_{ij} = \mu_{ji}$ and with $\epsilon_i \in \{\pm 1\}$.

Proof. Recall that our goal was precisely to prove the claim for $s = 0$. As we remarked above, for $s = n$ there is nothing left to show, since we already saw that it follows from the Function-near-Critical Point Corollary 137.3 that f is of the required form. Next suppose that the statement holds for some $s \in \{1, \ldots, n\}$. We need to show that the statement also holds for $s - 1$. We might as well assume that f itself is already of the form

$$(x_1, \ldots, x_n) \;\mapsto\; \sum_{i=1}^{s} \sum_{j=1}^{s} \mu_{ij}(x) \cdot x_i \cdot x_j \;+\; \sum_{i=s+1}^{n} \epsilon_i \cdot x_i^2 \qquad \text{with } \mu_{ij} = \mu_{ji}.$$

Since 0 is a non-degenerate critical point we know that the matrix $(\mu_{ij}(0))_{i,j=1,\ldots,s}$ is invertible. As we saw in the proof of the Symmetric Forms-over-$\mathbb{R}$-Proposition 133.6, it is an elementary linear algebra fact that there exists a matrix $A \in \mathrm{GL}(s, \mathbb{R})$ such that the ss-entry of $A^T \cdot (\mu_{ij}(0))_{i,j=1,\ldots,s} \cdot A$ is non-zero. By applying the diffeomorphism Φ that is given by the invertible matrix $\begin{pmatrix} A & 0 \\ 0 & \mathrm{id}_{n-s} \end{pmatrix}$ we see that we might as well assume that $\mu_{ss}(0) \neq 0$. After possibly shrinking U we can assume that $\mu_{ss}(x) \neq 0$ for all $x \in U$. Now we consider the map

$$\begin{aligned} \Phi \colon U &\to \mathbb{R}^n \\ x = (x_1, \ldots, x_n) &\mapsto \Big(x_1, \ldots, x_{s-1}, \sqrt{|\mu_{ss}(x)|} \cdot \Big(x_s + \sum_{i=1}^{s-1} \tfrac{\mu_{is}(x)}{|\mu_{ss}(x)|} \cdot x_i\Big), x_{s+1}, \ldots, x_n\Big). \end{aligned}$$

Evidently $\Phi(0) = 0$. A straightforward calculation shows that $\det(\mathrm{D}\Phi_0) = \sqrt{|\mu_{ss}(0)} \neq 0$. It follows from the Inverse Function Theorem 19.12 that, possibly after shrinking U even further, the map $\Phi \colon U \to \mathbb{R}^n$ is a diffeomorphism onto its image. It remains to show that $f \circ \Phi^{-1}$ is of the form

$$y = (y_1, \ldots, y_n) \;\mapsto\; \sum_{i=1}^{s-1} \sum_{j=1}^{s-1} \nu_{ij}(y) \cdot y_i \cdot y_j \;+\; \sum_{i=s}^{n} \epsilon_i \cdot y_i^2 \qquad \text{with } \nu_{ij} = \nu_{ji}.$$

Thus let $y = (y_1, \dots, y_n) \in \Phi(U)$. We write $(x_1, \dots, x_n) = x := \Phi^{-1}(y)$. Note that by definition of Φ we have

$$(y_1, \dots, y_n) \;=\; \Big(x_1, \dots, x_{s-1}, \sqrt{|\mu_{ss}(x)|} \cdot \Big(x_s + \sum_{i=1}^{s-1} \frac{\mu_{is}(x)}{|\mu_{ss}(x)|} \cdot x_i\Big), x_{s+1}, \dots, x_n\Big).$$

We perform the following little calculation:

$$\begin{aligned} y_s^2 \;&=\; |\mu_{ss}(x)| \cdot \Big(x_s + \sum_{i=1}^{s-1} \frac{\mu_{is}(x)}{|\mu_{ss}(x)|} \cdot x_i\Big) \cdot \Big(x_s + \sum_{j=1}^{s-1} \frac{\mu_{js}(x)}{|\mu_{ss}(x)|} \cdot x_j\Big) \\ &=\; |\mu_{ss}(x)| \cdot \Big(x_{ss}^2 + \sum_{i=1}^{s-1} \sum_{j=1}^{s-1} \frac{\mu_{is}(x) \cdot \mu_{js}(x)}{|\mu_{ss}(x)|^2} \cdot x_i \cdot x_j + 2 \cdot \sum_{i=1}^{s-1} \frac{\mu_{is}(x)}{|\mu_{ss}(x)|} \cdot x_i \cdot x_s\Big). \end{aligned}$$

Finally note that

$$\begin{aligned} (f \circ \Phi^{-1})(y) \;=\; f(x) \;&\overset{\text{follows from the above description of } f}{=}\; \sum_{i=1}^{s-1} \sum_{j=1}^{s-1} \mu_{ij}(x) \cdot x_i \cdot x_j \;+\; 2 \cdot \sum_{i=1}^{s} \mu_{is} \cdot x_i \cdot s_s \;+\; \sum_{i=s+1}^{n} \epsilon_i \cdot x_i^2 \;\overset{\text{follows immediately from the above calculation of } y_s^2}{=} \\ &=\; \sum_{i=1}^{s-1} \sum_{j=1}^{s-1} \underbrace{\Big(\mu_{ij}(x) - \frac{\mu_{is}(x) \cdot \mu_{js}(x)}{\mu_{ss}(x)}\Big)}_{=:\widetilde{\mu}_{ij}(x)} \cdot x_i \cdot x_j + \frac{\mu_{ss}(x)}{|\mu_{ss}(x)|} \cdot y_s^2 + \sum_{i=s+1}^{n} \epsilon_i \cdot x_i^2 \\ &=\; -\sum_{i=1}^{s-1} \sum_{j=1}^{s-1} \widetilde{\mu}_{ij}(\Phi^{-1}(y)) \cdot y_i \cdot y_j \;+\; \underbrace{\frac{\mu_{ss}(0)}{|\mu_{ss}(0)|}}_{\in\{-1,1\}} \cdot y_s^2 + \sum_{i=s+1}^{n} \epsilon_i \cdot y_i^2. \end{aligned}$$

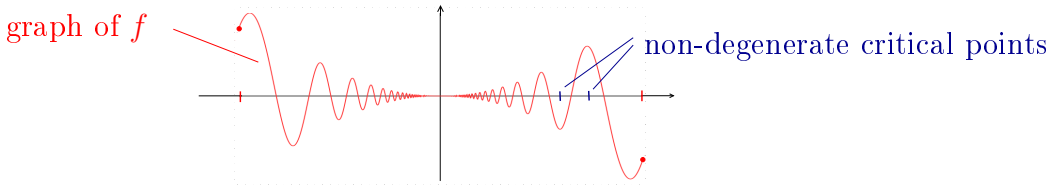
(the last step follows from $x_i = y_i$ for $i \neq s$, from $x = \Phi^{-1}(y)$ and from the fact that the sign of $\mu_{ss}(x)$ is constant on U)

This concludes the induction step. ∎

Corollary 137.4. Let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function such that no critical point is contained in ∂M.

(1) The set of non-degenerate critical points of f is a discrete subset of M.
(2) If M is compact and if all critical points are non-degenerate, then there exist only finitely many non-degenerate critical points of f.

Remark. As shown in the figure below, the set of non-degenerate critical points of a smooth function is not necessarily a closed subset. This shows in particular that in Corollary 137.4 (2) we really need to assume that all critical points are non-degenerate.



Proof.

(1) It follows immediately from the Morse Lemma 137.2 that every non-degenerate critical point admits an open neighborhood which contains no other critical point. But this means, by definition, that the non-degenerate critical points of f form a discrete subset of M.

(2) In the Regular Value-Open Subset Lemma 26.1 we showed that the set of critical points of f is a closed subset of M. Since we assume that all critical points are non-degenerate we see that the set of non-degenerate critical points is closed. By (1) we also know that this set is discrete. It follows from the hypothesis that M is compact, together with the Compact-Discrete Lemma 1.22 (2), that the set of non-degenerate critical points is indeed finite. ■

137.2. **Existence of Morse functions.** The following definition introduces the key character of the reminder of this chapter.

Definition. Let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. We say f is a Morse function if all critical points are non-degenerate and if no critical point lies on ∂M.

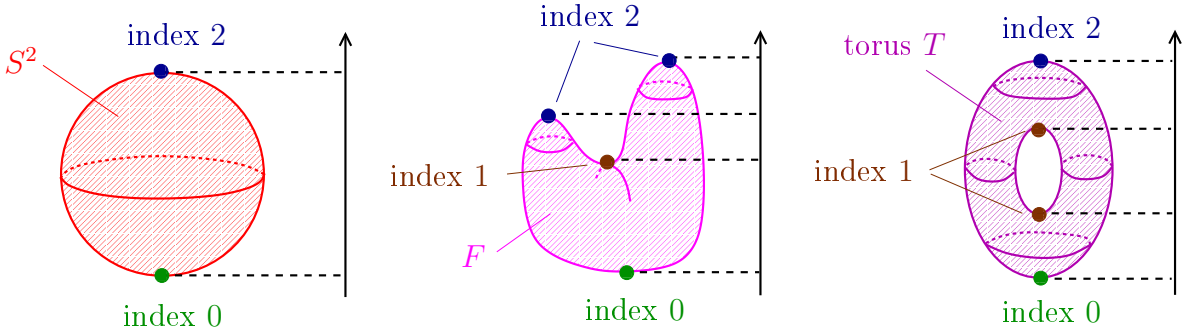
Before we discuss the general existence of Morse functions on smooth manifolds let us consider some explicit examples of Morse functions.

Examples.

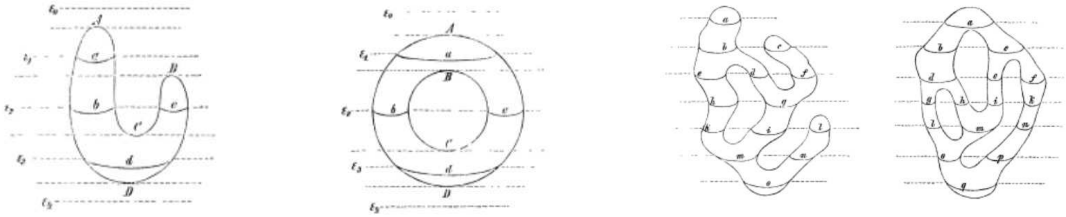
(1) In the figure below we show three surfaces S^2, F and T in $\mathbb{R}^3$. In each case the "height function"

$$\begin{array}{rcl} h\colon \mathbb{R}^3 & \to & \mathbb{R} \\ (x,y,z) & \mapsto & z \end{array}$$

is a Morse function. Evidently S^2 and F are diffeomorphic, but the numbers of critical points differ. Just for fun we point out that August Möbius [**Möb1863**] in



1863 drew basically the same pictures, see the figure below.



(2) Let $n \in \mathbb{N}_0$. We pick real numbers $c_0 < c_1 < \cdots < c_n$. We consider the following functions on $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$:

$$\begin{array}{rcl} f\colon \mathbb{R}\mathrm{P}^n & \to & \mathbb{R} \\ [x_0 : \cdots : x_n] & \mapsto & \dfrac{\sum_{j=0}^{n} c_j \cdot x_j^2}{x_0^2 + \cdots + x_n^2} \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \mathbb{C}\mathrm{P}^n & \to & \mathbb{R} \\ [z_0 : \cdots : z_n] & \mapsto & \dfrac{\sum_{j=0}^{n} c_j \cdot |z_j|^2}{|z_0|^2 + \cdots + |z_n|^2}. \end{array}$$

Using the explicit charts given in Exercise 19.16 and the Complex Projective Space-is-Complex Manifold Lemma 32.16 we will verify in Exercise 137.3 that both maps are Morse functions, that both maps have precisely $n+1$ critical points and that these are of index $0, 1, \ldots, n$ respectively $0, 2, \ldots, 2n$.

(3) Let $n \in \mathbb{N}$. In the matrix Group-Smooth Manifold Proposition 26.7 we showed that $\mathrm{SO}(n)$ is a closed smooth manifold of dimension $\frac{1}{2}(n-1)$. Let $1 < c_1 < \cdots < c_n$ be real numbers. In [**Mata2002**, Lemma 3.12] it is shown that the map

$$\begin{array}{rcl} \mathrm{SO}(n) & \to & \mathbb{R} \\ A = (a_{ij}) & \mapsto & \sum_{j=1}^{n} c_j \cdot a_{jj} \end{array}$$

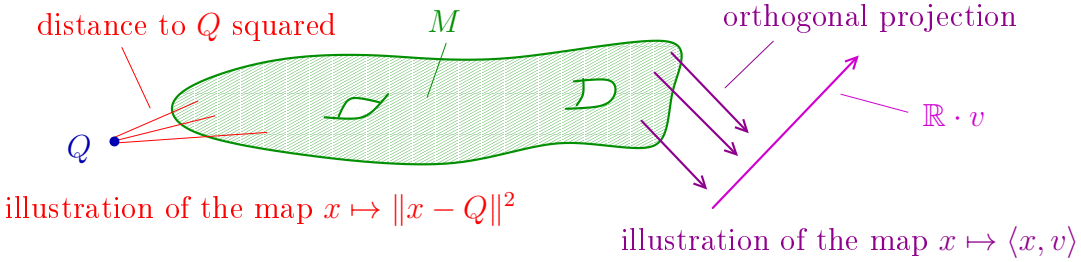
is a Morse function. Furthermore it is shown that the critical points are precisely the diagonal matrices in $\mathrm{SO}(n)$. With very similar ideas one can also obtain explicit Morse functions on the smooth manifolds $\mathrm{U}(n)$ and $\mathrm{SU}(n)$. We refer to [**Mata2002**, Examples 3.13 and 3.14] for details.

The following proposition provides a rich source of Morse functions.

Proposition 137.5. (Morse Function Existence Proposition) Let M be a closed smooth submanifold of some $\mathbb{R}^n$.

(1) The set of $Q \in \mathbb{R}^n$ for which the map $M \to \mathbb{R}$ given by $x \mapsto \|x - Q\|^2$ is a Morse function is a set of full measure.[1665]

(2) The set of $v \in \mathbb{R}^n$ for which the map $M \to \mathbb{R}$ given by $x \mapsto \langle x, v \rangle$ is a Morse function is a set of full measure.



Sketch of proof. Let M be a closed smooth submanifold of some $\mathbb{R}^n$.

(1) Given $Q \in \mathbb{R}^n$ we consider the map

$$\begin{array}{rcl} f_Q \colon M & \to & \mathbb{R} \\ x & \mapsto & \|x - Q\|^2. \end{array}$$

Furthermore we consider

$$N := \{(p, w) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, p \in M \text{ and } w \in (\widetilde{\mathrm{T}}_P M)^{\perp}\}.$$

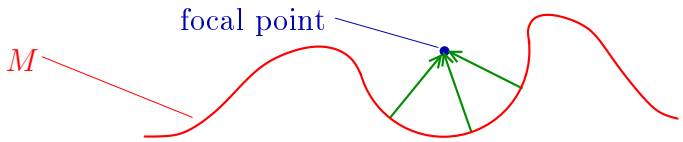
It is not particularly difficult to show that N is an n-dimensional submanifold of $\mathbb{R}^{2n}$. Next we consider the "endpoint map"

$$\begin{array}{rcl} E \colon N & \to & \mathbb{R}^n \\ (p, w) & \mapsto & p + w \end{array}$$

[1665] We refer to page 762 for the definition of a subset of full measure.

which is evidently smooth. We say $Q \in \mathbb{R}^n$ is a *focal point* if Q is a critical value of E. A slightly heroic calculation, see [**Miln1963a**, Lemma 6.5], shows that Q is a focal point if and only if the map $f_Q \colon M \to \mathbb{R}$ has non-degenerate critical points. The desired statement now follows from the above discussion together with Sard's Theorem 31.1 which tells us that the set of critical values of E is of measure zero in $\mathbb{R}^n$.

(2) This statement is proved in [**Nic2011**, Corollary 1.25]. ■



Proposition 137.6. (Morse Function-Existence Proposition) Every closed smooth manifold admits a Morse function.

Remark.

(1) There are many refinements of the Morse Function-Existence Proposition 137.6 in the literature. For example, by [**Mata2002**, Theorem 2.20] every smooth function on a closed smooth manifold can be approximated by Morse functions.

(2) Using a doubling argument and collar neighborhoods it is a routine exercise to show, using the Morse Function-Existence Proposition 137.6, that also every compact smooth manifold admits a Morse function. But as we will see in Chapter 139, for compact smooth manifolds it is much more convenient to work with Morse functions that "respect the boundary". In the Morse Function-Existence Proposition 139.4 we will provide such a refinement of the Morse Function-Existence Proposition 137.6 for smooth manifolds with boundary.

(3) In Proposition 139.10 we will see that one can always find "self-indexing" Morse functions.

Proof. Let M be a closed smooth manifold. It follows from the Euclidean Embedding Theorem 27.1 together with the Smooth Embedding Theorem 24.10 that we can assume that M is a submanifold of some $\mathbb{R}^n$. It now follows from the Morse Function Existence Proposition 137.5 that M admits a Morse function. ■

137.3. **The gradient vector field.** In this section we will study the gradient vector field ∇f of a smooth function $f \colon M \to \mathbb{R}$ on a smooth submanifold $M \subset \mathbb{R}^n$. We will see that the gradient vector field of a Morse function has particularly nice properties. In particular we will obtain a new proof of the existence of non-degenerate vector fields on smooth manifolds.

Before we can introduce the gradient vector field we need to recall the following definition from page 2656.

Definition. Let V be a real vector space and let $\varphi\colon V \times V \to \mathbb{R}$ be a symmetric $\mathbb{R}$-bilinear pairing. We say φ is non-singular if the induced map

$$\begin{array}{rcl} V & \to & \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \\ v & \mapsto & (w \mapsto \varphi(v, w)) \end{array}$$

is an isomorphism.

The one example we care about is provided by the following lemma.

Lemma 137.7. Let $\langle -, - \rangle\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the usual Euclidean pairing and let $V \subset \mathbb{R}^n$ be a vector subspace. The restriction of $\langle -, - \rangle\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ to $V \times V \to \mathbb{R}$ is non-singular.

Proof. Since V is finite-dimensional it suffices to show that $V \to \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ is a monomorphism. Thus let $v \in V$ be non-zero. Then $\langle v, v \rangle \neq 0$, which shows that the homomorphism $w \mapsto \langle v, w \rangle$ is not the zero homomorphism. ■

Definition. Let M be a smooth submanifold of some $\mathbb{R}^n$ and let $f\colon M \to \mathbb{R}$ be a smooth function. We refer to the vector field

$$\begin{array}{rcl} \nabla f\colon M & \to & \mathbb{R}^n \\ P & \mapsto & \text{the unique vector}^{1666}\ \nabla f(P) \in \widetilde{\mathrm{T}}_P M \text{ such that for} \\ & & \text{all } w \in \widetilde{\mathrm{T}}_P M \text{ we have } \langle \nabla f(P), w \rangle = \mathrm{D}f_P(w). \end{array}$$

as the gradient vector field of f.

Before we discuss gradient vector fields in detail it is worth recalling the following definition from page 1021.

Definition. Let M be a smooth manifold and let v be a smooth vector field on M. Let $P \in M \setminus \partial M$ be a zero of v. We pick a chart $\Phi\colon U \to V$ with $\Phi(P) = 0$. We write $\Psi := \Phi^{-1}$. We consider the map

$$\begin{array}{rcl} \Psi^* v\colon V & \to & \mathbb{R}^n \\ x & \mapsto & (\Psi^* v)(x) = \mathrm{D}\Phi_{\Psi(x)} v(\Psi(x)). \end{array}$$

(1) If $\mathrm{D}(\Psi^* v)_0$ is an invertible matrix, then we say that the zero is non-degenerate. Otherwise we call the zero degenerate.
(2) If the zero is non-degenerate, then we define

$$\mathrm{index}(v, x) := \begin{cases} +1, & \text{if } \det(\mathrm{D}(\Psi^* v)_0) > 0, \\ -1, & \text{if } \det(\mathrm{D}(\Psi^* v)_0) < 0. \end{cases}$$

In Lemma 45.6 we showed that these notions do not depend on the choice of the chart Φ.

Proposition 137.8. (Gradient Vector Field Proposition) Let M be a smooth submanifold of some $\mathbb{R}^n$ and let $f\colon M \to \mathbb{R}$ be a smooth function. The gradient vector field ∇f is smooth and given any $P \in M$ the following statements hold:

(1) P is a critical point of f if and only if P is a zero of ∇f.
(2) If P is a critical point of f, then P is a non-degenerate critical point of f if and only if P is a non-degenerate zero of ∇f.
(3) If P is a non-degenerate critical point of f, then $\mathrm{index}(\nabla f, P) = (-1)^{\mathrm{index}(f,P)}$.

[1666] The existence and uniqueness of $v(P)$ is of course guaranteed by Lemma 137.7.

Before we get to the slightly messy proof of the Gradient Vector Field Proposition 137.8 let us first point out that we can now give a new proof of the Non-Degenerate Vector Field Existence Proposition 45.15.

Proposition 129.4. (Non-Degenerate Vector Field Existence Proposition) Every closed smooth manifold M admits a smooth vector field v such that all zeros are non-degenerate.

Proof of the Non-Degenerate Vector Field Existence Proposition 129.4. Let M be a closed smooth manifold. By the Euclidean Embedding Theorem 27.1 we can assume that M is a smooth submanifold of some $\mathbb{R}^n$. By the Morse Function Existence Proposition 137.5 we know that M admits a Morse function $f\colon M \to \mathbb{R}$. It follows from the Gradient Vector Field Proposition 137.8 that the gradient vector field ∇f is a vector field such that all zeros are non-degenerate. ■

The proof of the Gradient Vector Field Proposition 137.8 rests on the following two purely algebraic lemmas.

Auxiliary Lemma 137.9. Let T be a k-dimensional real vector space and let $\alpha\colon T\times T \to \mathbb{R}$ be a symmetric $\mathbb{R}$-bilinear pairing. Furthermore let $t_1,\dots,t_k$ be a basis for T. We set $\Theta := (\alpha(t_i,t_j))_{i,j=1,\dots,k} \in \mathrm{M}(k\times k,\mathbb{R})$.

(1) If α is non-singular, then $\det(\Theta)\neq 0$.
(2) If α is positive-definite, then $\det(\Theta) > 0$.

Proof. The first Statement follows from the Forms-Matrices Lemma 133.1 (1b) and the second statement follows from the Signature-Eigenvalues Proposition 133.8 and the fact, that very symmetric real matrix is diagonalizable. ■

Auxiliary Lemma 137.10. Let $\Psi\colon \mathbb{R}^k \to T$ be an isomorphism between real vector spaces. We suppose that T is equipped with a non-singular symmetric $\mathbb{R}$-bilinear pairing α. Let $f\colon T\to\mathbb{R}$ be a linear map. Given $i,j\in\{1,\dots,k\}$ we set $\theta_{ij} := \alpha(\Psi(e_i),\Psi(e_j))$ and we set $\Theta := (\theta_{ij})_{i,j=1,\dots,k} \in \mathrm{M}(k\times k,\mathbb{R})$. If $\widetilde{v}\in\mathbb{R}^k$ satisfies $\langle \widetilde{v}, v\rangle = (f\circ\Psi)(v)$ for all $v\in\mathbb{R}^k$, then[1667]

$$\widetilde{t} \;:=\; \sum_{j=1}^{k}\sum_{i=1}^{k} (\Theta^{-1})_{ij}\cdot\langle\widetilde{v},e_i\rangle\cdot\Psi(e_j)$$

has the property that

$$\alpha(\widetilde{t},t) \;=\; f(t) \qquad\qquad \text{for all } t\in T.$$

Proof of Lemma 137.10. It suffices to show that $\widetilde{t}$ satisfies the condition for the basis vectors $t=\Psi(e_r)$. For $r\in\{1,\dots,k\}$ we calculate

$$\begin{aligned}\alpha(\widetilde{u},\Psi(e_r)) &\overset{\text{definition of } \widetilde{t} \text{ and bilinearity of } \alpha}{\downarrow} \sum_{j=1}^{k}\sum_{i=1}^{k}(\Theta^{-1})_{ij}\cdot\langle\widetilde{v},e_i\rangle\cdot\underbrace{\alpha(\Psi(e_j),\Psi(e_r))}_{=\theta_{jr}} \;=\; \sum_{i=1}^{k}\underbrace{\sum_{j=1}^{k}(\Theta^{-1})_{ij}\cdot\theta_{jr}}_{=\delta_{ir}\text{ by definition of }\Theta^{-1}}\cdot\langle\widetilde{v},e_i\rangle \\ &= \langle\widetilde{v},e_r\rangle \underset{\text{by the defining property of } \widetilde{v}}{\uparrow} (f\circ\Psi)(e_r) \;=\; f(\Psi(e_r)).\end{aligned}$$

■

[1667] By Lemma 137.9 we know that Θ is invertible.

Auxiliary Lemma 137.11. Let $V \subset \mathbb{R}^k$ be an open subset and let $\Xi\colon V \to \mathrm{M}(k \times k, \mathbb{R})$ and $u\colon A \to \mathbb{R}^k$ be smooth maps. Let $Q \in A$. If $u(Q) = 0$, then

$$\mathrm{D}(\underbrace{x \mapsto \Xi(x) \cdot u(x)}_{\text{map } A \to \mathbb{R}^n})_S \;=\; \Xi(S) \cdot \mathrm{D}u_S \quad \in \mathrm{M}(n \times m, \mathbb{R}).$$

Proof of Lemma 137.11. The lemma follows immediately from the Product Rule applied to each entry and the fact that $u(Q) = 0$. ■

Proof of the Gradient Vector Field Proposition 137.8. We start out with a proof of a special case.

Claim 1. Let V be an open subset of some $\mathbb{R}^k$ or of some $\mathbb{H}^k$ and let $g\colon V \to \mathbb{R}$ be a smooth function. For any $Q \in V$ we have

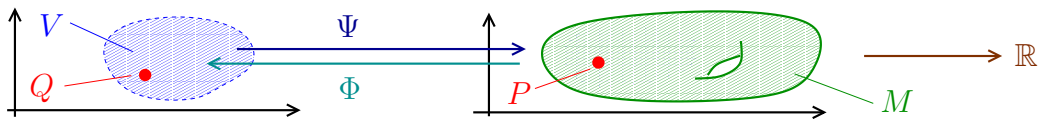
$$\nabla g(Q) \;=\; \begin{pmatrix} \frac{\partial g}{\partial x_1}(Q) \\ \vdots \\ \frac{\partial g}{\partial x_k}(Q) \end{pmatrix}$$

Proof. By the differential-from-Partial Derivatives Proposition 19.9 we know that for each $Q \in V$ we have $\mathrm{D}g_Q = (\frac{\partial g}{\partial x_1}(Q) \dots \frac{\partial g}{\partial x_k}(Q)) \in \mathrm{M}(1 \times k, \mathbb{R})$. It follows easily from this observation and the definitions that the gradient vector field ∇g is indeed given by the map

$$\begin{array}{rcl} \nabla g\colon V & \to & \mathbb{R}^k \\ Q & \mapsto & \begin{pmatrix} \frac{\partial g}{\partial x_1}(Q) \\ \vdots \\ \frac{\partial g}{\partial x_k}(Q) \end{pmatrix}. \end{array}$$

Note that it follows from this description that the gradient vector field is smooth. Furthermore note that for any $Q \in V$ we have $\mathrm{D}(\nabla g)_Q = \mathrm{Hess}(g)_Q$. Thus it follows immediately from the definition of non-degenerate zeros and the index of a non-degenerate zero, using the identity chart, that the gradient vector field satisfies Statements (1)–(3). The case that V is an open subset of $\mathbb{H}^k$ is dealt with in the same way. ⊞

Now let M be a k-dimensional smooth submanifold of some $\mathbb{R}^n$ and let $f\colon M \to \mathbb{R}$ be a smooth function. The statements of the proposition are all local. Thus, after possibly restricting ourselves to the domain of a chart, we can assume that there exists a diffeomorphism $\Phi\colon M \to V$ where V is an open subset of $\mathbb{R}^k$ or $\mathbb{H}^k$. We write $\Psi := \Phi^{-1}\colon V \to M$. Given $i, j \in \{1, \dots, k\}$ and given $Q \in V$ we set $\theta_{ij}(Q) := \langle \mathrm{D}\Psi_Q(e_i), \mathrm{D}\Psi_Q(e_j) \rangle$. Furthermore we write $\Theta(Q) := (\theta_{ij}(Q))_{i,j=1,\dots,k} \in \mathrm{M}(k \times k, \mathbb{R})$. The following claim now gives a calculation of Δf.



Claim 2.

(1) Let $P \in M$. We set $Q := \Phi(P)$. Then

$$\nabla f(P) \;:=\; \sum_{j=1}^{k}\sum_{i=1}^{k} \underbrace{(\Theta(Q)^{-1})_{ij} \cdot \frac{\partial(f\circ\Psi)}{\partial x_i}(Q)}_{\in\mathbb{R}} \cdot \underbrace{\mathrm{D}\Psi_Q(e_j)}_{\in\widetilde{\mathrm{T}}_P M} \quad \in \widetilde{\mathrm{T}}_P M.$$

(2) For any $Q \in V$ we have

$$(\Psi^*(\nabla f))(Q) \;=\; (\Theta(Q)^{-1})^T \cdot (\nabla(f\circ\Psi))(Q) \quad \in \mathbb{R}^k.$$

Proof.

(1) This statement follows immediately from Claim 1, applied to the smooth function $g = f \circ \Psi \colon V \to \mathbb{R}$, and Lemma 137.10, applied to the isomorphism $\mathrm{D}\Psi_Q \colon \mathbb{R}^k \to \widetilde{\mathrm{T}}_P M$.

(2) For any any $Q \in V$ we have

$$\begin{aligned}
(\Psi^*(\nabla f))(Q) &\overset{\text{by definition of } \Psi^*}{=} \mathrm{D}\Phi_{\Psi(Q)}((\nabla f)(\Psi(Q))) \\
&= (\Theta(Q)^{-1})^T \cdot \begin{pmatrix} \frac{\partial(f\circ\Psi)}{\partial x_1}(Q) \\ \vdots \\ \frac{\partial(f\circ\Psi)}{\partial x_k}(Q) \end{pmatrix} \\
&\overset{\text{follows from Statement (1), the linearity of } \mathrm{D}\Phi_{\Psi(Q)} \text{ and fact that } \mathrm{D}\Phi_{\Psi(Q)}(\mathrm{D}\Psi_Q(e_j)) = e_j}{=} \sum_{j=1}^{k}\sum_{i=1}^{k} (\Theta(Q)^{-1})_{ij} \cdot \frac{\partial(f\circ\Psi)}{\partial x_i}(Q) \cdot e_j \\
&= (\Theta(Q)^{-1})^T \cdot (\nabla(f\circ\Psi))(Q). \qquad \text{by Claim 1}
\end{aligned}$$

⊞

It follows immediately from Claim 2 (a) that $\nabla f \colon M \to \mathbb{R}^n$ is smooth. It remains to prove Statements (1), (2) and (3). Since the various notions are defined via charts it suffices to prove the following claim.

Claim 3. Given any $Q \in V$ the following statements hold:

(1) Q is a critical point of the smooth map $f \circ \Psi \colon V \to \mathbb{R}$ if and only if Q is a zero of the vector field $\Psi^*(\nabla f)$ on V.

(2) If Q is a critical point of $f \circ \Psi$, then Q is a non-degenerate critical point of the map $f \circ \Psi \colon V \to \mathbb{R}$ if and only if Q is a non-degenerate zero of the vector field $\Psi^*(\nabla f)$ on V.

(3) If Q is a non-degenerate critical point of the smooth map $f \circ \Psi \colon V \to \mathbb{R}$, then $\mathrm{index}(\Psi^*(\nabla f), Q) = (-1)^{\mathrm{index}(f\circ\Psi, Q)}$.

Proof. Let $\Theta \in V$. We start out with the following two observations:

(a) By Lemma 137.9 we know that $\det(\Theta(Q)) > 0$, which implies also that $\det((\Theta^{-1})^T) > 0$.

(b) Let Q be a critical point of $f \circ \Psi$. Then

$$\begin{aligned}
\mathrm{D}(x \mapsto (\Theta(x)^{-1})^T \cdot (\nabla(f\circ\Psi))(x))_Q &\overset{\text{by Lemma 137.11, since } \nabla(f\circ\Psi)(Q) = 0}{=} (\Theta(Q)^{-1})^T \cdot \mathrm{D}_Q(\nabla(f\circ\Psi)) \\
&= (\Theta(Q)^{-1})^T \cdot \mathrm{Hess}_Q(f\circ\Psi) \;\in\; \mathrm{M}(k\times k, \mathbb{R}). \\
&\qquad \text{follows immediately from Claim 1 and the definitions}
\end{aligned}$$

(c) Let $H \in \mathrm{M}(k \times k, \mathbb{R})$ be a symmetric matrix with non-zero determinant. Since the determinant of H is the product of the eigenvalues of H we see that

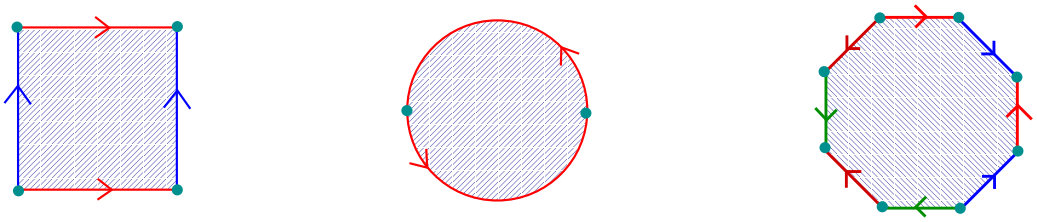
$$(-1)^{\text{number of negative eigenvalues of } H} = \begin{cases} -1, & \text{if } \det(H) < 0, \\ 1, & \text{if } \det(H) > 0. \end{cases}$$

All three desired statements follow almost immediately from Claim 2 (b) together with (a), (b) and (c). ■

Exercises for Chapter 137.

Exercise 137.1. Show that every Morse function is a height function. More precisely, let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a Morse function. Show that there exists a smooth embedding $\varphi\colon M \to \mathbb{R}^n$ such that $f = h \circ \varphi$ where $h\colon \mathbb{R}^n \to \mathbb{R}$ is the map given by $(x_1, \ldots, x_n) \mapsto x_n$.

Exercise 137.2. Sketch Morse functions on the torus $T = [0,1]^2/\!\sim$, the real projective plane $\mathbb{R}\mathrm{P}^2 = \overline{B}^2/x \sim -x$ and the surface of genus two $E_8/\!\sim$ such that each Morse function has a single local maximum and a single local minimum.



Exercise 137.3. Let $n \in \mathbb{N}_0$. We pick real numbers $c_0 < c_1 < \cdots < c_n$. We consider the following functions on $\mathbb{R}\mathrm{P}^n = S^n/\{\pm \mathrm{id}\}$ and $\mathbb{C}\mathrm{P}^n = S^{2n+1}/S^1$:

$$\begin{array}{rcl} f\colon \mathbb{R}\mathrm{P}^n & \to & \mathbb{R} \\ [x_0 : \cdots : x_n] & \mapsto & \sum_{j=0}^{n} c_j \cdot x_j^2 \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \mathbb{C}\mathrm{P}^n & \to & \mathbb{R} \\ [z_0 : \cdots : z_n] & \mapsto & \sum_{j=0}^{n} c_j \cdot |z_j|^2. \end{array}$$

(a) Show that f is a Morse function, that f has precisely $n+1$ critical points and that these are of index $0, 1, \ldots, n$.

(b) Show that g is a Morse function, that g has precisely $n+1$ critical points and that these are of index $0, 2, \ldots, 2n$.

Remark. You might want to use the explicit charts given in Exercise 19.16 and the Complex Projective Space-is-Complex Manifold Lemma 32.16.
Remark. This exercise is basically a reformulation of Exercise 45.5.

Exercise 137.4. Let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a Morse function. Furthermore let $\lambda\colon M \to \mathbb{R}$ be a smooth function such that $\mathrm{D}\lambda_P \neq 0$ for every $P \in M$. Show that $\lambda \cdot f\colon M \to \mathbb{R}$ is also a Morse function and that the critical points, and the corresponding indices, of $\lambda \cdot f$ agree with the critical points, and the corresponding indices, of λ.

Exercise 137.5. Let M be a nd N be two closed smooth manifolds and let $f\colon M \to \mathbb{R}$ and $g\colon N \to \mathbb{R}$ be Morse functions.

(a) Let $a, b \in \mathbb{R}$ with $a > \max(|f|)$ and $b > \max(|g|)$. Show that the map

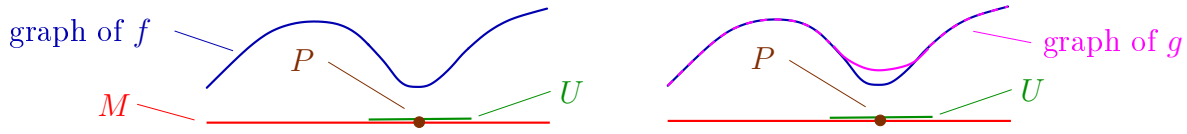
$$\begin{array}{rccc} f \cdot g\colon & M \times N & \to & \mathbb{R} \\ & (x, y) & \mapsto & (a + f(x)) \cdot (b + g(y)) \end{array}$$

is a Morse function.

(b) Determine the critical points, and their indices, of the Morse function of $f \cdot g$ in terms of the critical points, and their indices, of f and g.

Exercise 137.6. Let M be a smooth manifold, let $f\colon M \to \mathbb{R}$ be a smooth function and let $P \in M \setminus \partial M$ be a non-degenerate critical point of f. Let U be a compact neighborhood of $P \in M \setminus \partial M$ which contains no other critical point. Show that there exists an $\epsilon > 0$ such that for every $\mu \in [0, \epsilon]$ there exists a smooth function $g\colon M \to \mathbb{R}$ with the following four properties:

(1) The function g agrees with f outside of U.
(2) P is a critical point of g with $\operatorname{index}(g, P) = \operatorname{index}(f, P)$.
(3) $g(P) = f(P) + \mu$.
(4) P is the only critical point of g in U.



138. Morse theory I: Closed smooth manifolds

In the last chapter we showed that every compact smooth manifolds admits Morse functions. In this chapter we will put Morse functions to good use, namely we will use them to show that every compact smooth manifold admits a handle decomposition.

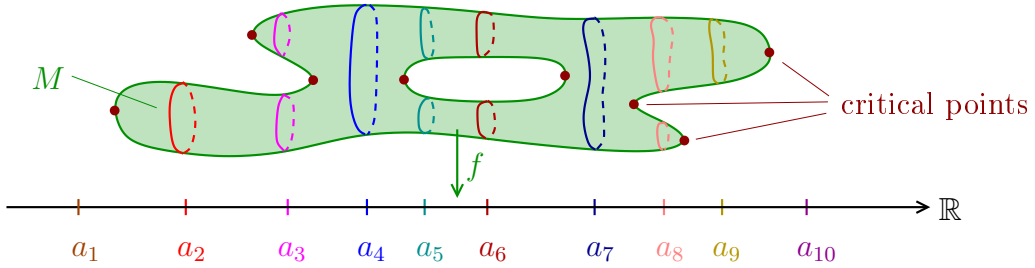
138.1. **The Smooth Manifold-Product Theorem.** In this section and the following section we will see that there is a close relationship between the topology of a smooth manifold and the non-degenerate critical points of a Morse function.

Notation. Let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Given $a \in \mathbb{R}$ we write

$$M_a := f^{-1}((-\infty, a]) = \{x \in M \mid f(x) \leq a\}.$$

Note that if a is a regular value and if $\partial M \cap f^{-1}(a) = \varnothing$, then it follows from the Regular Value Theorem 26.4 that $f^{-1}(a)$ is a closed submanifold of M and it follows from Exercise 26.6 that M_a is a submanifold of M with boundary $f^{-1}(a) \cup (\partial M \cap f^{-1}((-\infty, a]))$.

Example. In the figure below we show a smooth manifold M together with the "length" function $f\colon M \to \mathbb{R}$ given by $(x, y, z) \mapsto x$. For various $a_i \in \mathbb{R}$ we show the corresponding preimage $f^{-1}(a_i)$. The submanifold M_{a_i} is of course everything "to the $\leq$-left" of $f^{-1}(a_i)$.



Theorem 138.1. (No Critical Point-Diffeomorphism Theorem) Let M be a compact smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Let $a < b \in \mathbb{R}$ such that $f^{-1}([a, b])$ does not contain a critical point of f. If $f^{-1}([a, b]) \cap \partial M = \varnothing$, then there exists a diffeomorphism $M_a \to M_b$. Furthermore, given any $\epsilon > 0$ one can find diffeomorphism $M_a \to M_b$ that is the identity on the common subset $M_{a-\epsilon}$.

Example. In the figure above there is no critical point in $f^{-1}([a_5, a_6])$. Thus we obtain from the No Critical Point-Diffeomorphism Theorem 138.1 that the smooth manifolds M_{a_5} and M_{a_6} are diffeomorphic. Furthermore, we can find a diffeomorphism that is the identity on "most of M_{a_5}".

Our proof of the No Critical Point-Diffeomorphism Theorem 138.1 rests on the following result, that will turn out to be of independent interest later on.

Theorem 138.2. (Smooth Manifold-Product Theorem) Let M be a compact smooth manifold, let $f\colon M \to \mathbb{R}$ be a smooth function and let $a < b \in \mathbb{R}$. We suppose that for every $P \in f^{-1}([a, b])$ the differential $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathbb{R}$ is non-zero.[1668] Furthermore we suppose that the following conditions are satisfied:

(1) $f^{-1}(a)$ is a closed submanifold of M or it is a union of components of ∂M.
(2) $f^{-1}(b)$ is a closed submanifold of M or it is a union of components of ∂M.

Then $f^{-1}([a,b])$ is diffeomorphic to $[a,b] \times f^{-1}(a)$.[1669]

Sketch of a proof of Theorem 138.2. Note that using the Euclidean Embedding Theorem 27.1 and the Smooth Embedding Theorem 24.10 we can assume that M is a submanifold of some $\mathbb{R}^n$. Given $P \in M$ we denote by[1670] $\operatorname{grad} f_P \in \widetilde{\mathrm{T}}_P M$ the gradient of f at P.[1671] Note that our hypothesis on f just says that given any $P \in f^{-1}([a,b])$ the gradient is non-zero. Thus given any $P \in f^{-1}([a,b])$ we can define

$$v(P) := \frac{\operatorname{grad} f_P}{\|\operatorname{grad} f_P\|^2} \in \widetilde{\mathrm{T}}_P M.$$

Claim 1. If $\gamma\colon [0,\epsilon] \to f^{-1}([a,b])$ is a solution to the differential equation $\gamma'(t) = v(\gamma(t))$, then $(f \circ \gamma)'(t) \equiv 1$.

Proof. We perform the following calculation:

$$\begin{array}{rcccccl} & & \text{Chain Rule 19.10} & & \text{definition of gradient, here we set } P{:=}\gamma(t) & & \\ (f\circ\gamma)'(t) & = & \mathrm{D}(f\circ\gamma)_t & \overset{\downarrow}{=} & \mathrm{D}f_{\gamma(t)}(\gamma'(t)) & \overset{\downarrow}{=} & \langle \gamma'(t), \operatorname{grad} f_P\rangle \\ & & & = & \langle v(\gamma(t)), \operatorname{grad} f_P\rangle & = & \left\langle \frac{1}{\|\operatorname{grad} f_P\|^2}\cdot \operatorname{grad} f_P, \operatorname{grad} f_P\right\rangle = 1. \\ & & & \overset{\uparrow}{} & & \overset{\uparrow}{} & \end{array}$$

by hypothesis we have $\gamma'(t) = v(\gamma(t))$ since $P = \gamma(t)$ and by definition of v ⊞

We set $W := f^{-1}(a)$. It remains to prove the following claim.

Claim 2.

(1) Given any $P \in W$ there exists a unique smooth map $\gamma_P\colon [a,b] \to f^{-1}([a,b])$ such that $\gamma_P(a) = P$ and such that for each $t \in [a,b]$ we have $\gamma_P'(t) = v(\gamma_P(t))$.

(2) The map

$$\begin{array}{rccc} \Theta\colon & [a,b]\times W & \to & M \\ & (t,P) & \mapsto & \gamma_P(t) \end{array}$$

is a smooth embedding with image $f^{-1}([a,b])$.

[1668] Let $P \in M$. We recall the following definition from page 701:

(1) If $P \notin M \setminus \partial M$, then P is a regular point if $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathbb{R}$ is non-zero.
(2) If $P \in \partial M$, then P is a regular point if the map $\mathrm{D}f_P\colon \mathrm{T}_P \partial M \to \mathbb{R}$ is non-zero.

Thus we see that for points in $M \setminus \partial M$ our condition just says that we demand that P is a regular point. For points on the boundary our condition is more relaxed.

[1669] To make sense of the proposition we need to make the following two observations:

(i) Our hypotheses (1) and (2), together with Exercise 26.6, imply that the Submanifold Smooth Structure Lemma 19.24 that $f^{-1}([a,b])$ is naturally a smooth manifold.
(ii) Our hypothesis (1), together with the combination of the Submanifold Smooth Structure Lemma 19.24 and the Smooth Manifold Boundary Proposition 19.26 shows that $f^{-1}(a)$ is a closed smooth manifold in it own right. It follows from the Smooth Manifold product Proposition 19.31 that the product $[a,b] \times f^{-1}(a)$ is naturally a smooth manifold.

Thus it actually does make sense to say that $f^{-1}([a,b])$ is diffeomorphic to $[a,b] \times f^{-1}(a)$.

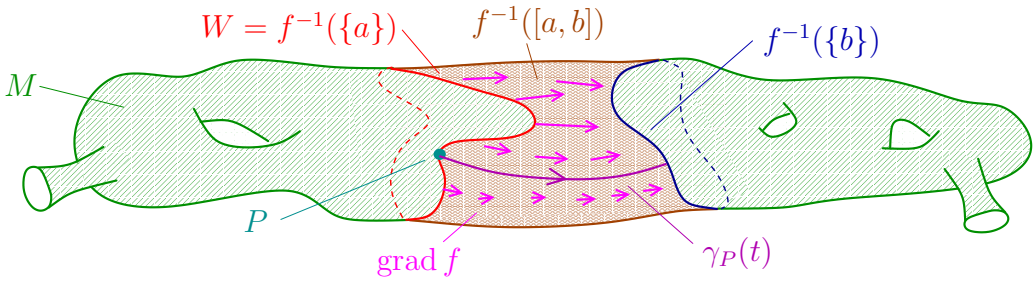
[1670] Recall that on page 620 we introduced the physical tangent space $\widetilde{\mathrm{T}}_P M$ which, by the Tangent Space-Natural isomorphism Proposition 23.16, is naturally isomorphic to $\mathrm{T}_P M$.

[1671] The gradient $\operatorname{grad} f_P \in \widetilde{\mathrm{T}}_P M$ of f at $P \in M$ is defined as the unique vector in $\widetilde{\mathrm{T}}_P M$ which satisfies $\langle w, \operatorname{grad} f_P\rangle = \mathrm{D}f_P(w) \in \mathbb{R}$ for all $w \in \widetilde{\mathrm{T}}_P M$. It is fairly elementary to show that $P \mapsto \operatorname{grad} f_P$ is a smooth map $M \to \mathbb{R}^n$.

Proof.

(1) This statement follows from the previous claim and standard theory of differential equations.
(2) It follows from (1) that the map $\Theta\colon [a,b]\times W\to M$ is well-defined. Furthermore it follows again from standard theory of differential equations that the map is smooth and that for each $t\in[a,b]$ the map $W\to M$ given by $P\mapsto\gamma_P(t)$ is a smooth embedding with image $f^{-1}(t)$. It follows easily from this observation and the previous claim that the map $\Theta\colon [a,b]\times W\to M$ is actually a smooth embedding with image $f^{-1}([a,b])$. ⊞

We leave it to the reader to dig out their favorite books on differential equations to fill in the details. Very similar statements are also proved in [**Miln1963a**, p. 10] and [**Lee2002**, Theorem 9.12, Corollary 9.17]. ■

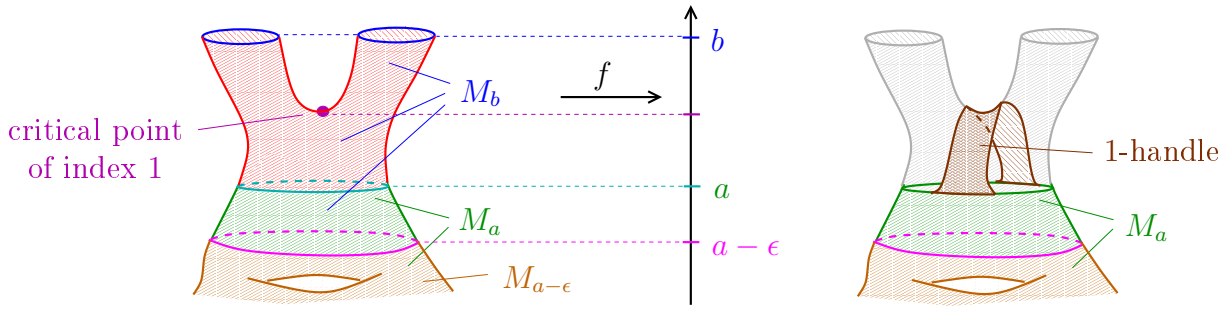


Now we have no troubles with providing the desired proof of the No Critical Point-Diffeomorphism Theorem 138.1.

Proof of the No Critical Point-Diffeomorphism Theorem 138.1. Let M be a compact smooth manifold and let $f\colon M\to\mathbb{R}$ be a smooth function. Let $a<b\in\mathbb{R}$ such that $f^{-1}([a,b])$ does not contain a critical point of f. We suppose that $f^{-1}([a,b])\cap\partial M=\varnothing$. By the Regular Value Theorem 26.4 we know that $W:=f^{-1}(a)$ is a closed smooth submanifold of M. By Theorem 138.2 we know that $f^{-1}([a,b])$ is diffeomorphic to $[a,b]\times W=[a,b]\times f^{-1}(a)$. It now follows from Exercise **??** (or alternatively from Lemma 139.2) that $M_b=M_a\cup f^{-1}([a,b])$ is diffeomorphic to M_a. Furthermore, if we are given some $\epsilon>0$, then it follows from Exercise **??** that we can find a diffeomorphism that is the identity on $M_{a-\epsilon}$. ■

138.2. **The Handle Addition Theorem.** Let M be a compact smooth manifold and let $f\colon M\to\mathbb{R}$ be a smooth function. Furthermore let $a<b\in\mathbb{R}$ such that a and b are regular values and such that $f^{-1}([a,b])\cap\partial M=\varnothing$. Note that the Regular Value Theorem 26.4 implies that $f^{-1}(a)$ and $f^{-1}(b)$ are closed submanifolds of M. Note that if the stronger condition holds that $f^{-1}([a,b])$ does not contain a critical point of f, then the Smooth Manifold-Product Theorem 138.2 says that M_b is diffeomorphic to M_a. But what happens if $f^{-1}((a,b))$ contains a critical point?
On the left of the figure below we show a 2-dimensional smooth manifold M together with a smooth function $f\colon M\to\mathbb{R}$ such that there exists a unique critical point of index $k=1$ in $f^{-1}((a,b))$. On the left of the figure below we show the result of attaching a handle of index $k=1$ to M_a. Some goodwill shows that M_b is diffeomorphic to the result of attaching

this handle of index $k = 1$ to M_a. In fact given any $\epsilon > 0$ one can find a diffeomorphism that is the identity on the common subset $M_{a-\epsilon}$.



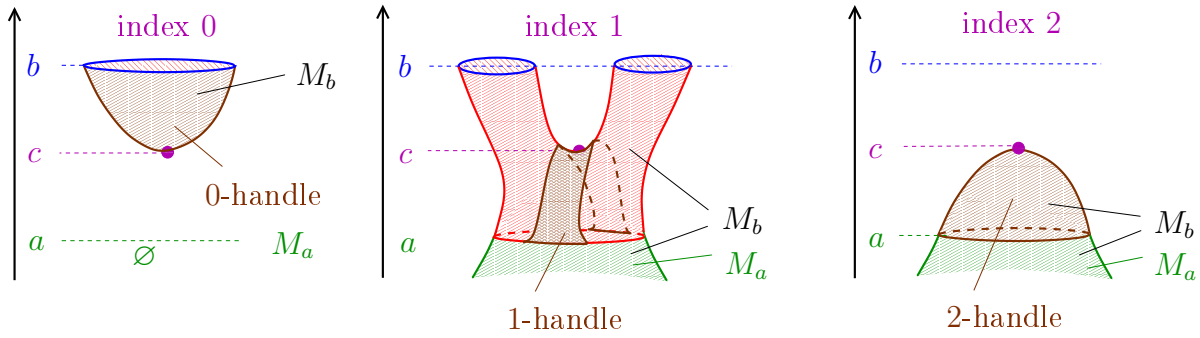
The following all-important theorem shows that a suitable generalization of the above observation holds in all reasonable settings.

Theorem 138.3. (Handle Addition Theorem) Let M be a compact smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Furthermore let $a < b \in \mathbb{R}$ such that a and b are regular values and such that $f^{-1}([a,b]) \cap \partial M = \varnothing$.

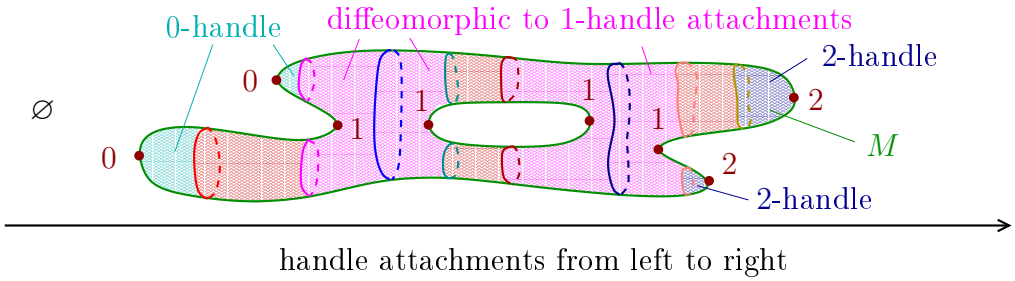
(1) Suppose there exists a single critical point $P \in f^{-1}([a,b])$. We set $k := \text{index}(f,P)$. If P is non-degenerate, then M_b is diffeomorphic to the result of attaching a k-handle to M_a.

(2) Suppose there exists a single critical value $c \in [a,b]$. Let $P_1, \ldots, P_m$ be the preimages of c. We suppose that $P_1, \ldots, P_m$ are non-degenerate critical points. Let $k_1, \ldots, k_m$ be the indices of the critical points. Then M_b is diffeomorphic to the result of attaching m handles of index k_i, $i = 1, \ldots, m$ to M_a along thickened spheres with disjoint images.

Furthermore, in both settings, given any $\epsilon > 0$ there exists an diffeomorphism as above which is the identity on the common subset $M_{a-\epsilon}$.



Example. In the figure below we return to the setting of the figure on page 2988. We show all the critical points and we indicate the indices. Furthermore, reading from left to right we show how M gets built out of the empty manifold by various handle attachments. To simplify our pictures we do not distinguish between a smooth manifold that is the result of attaching a k-handle and a smooth manifold that is diffeomorphic to the result of attaching a k-handle.



Remark. We provide an almost complete proof of Theorem 138.3. Only at the very end of the proof, on page 2995, when we need to prove that a certain map is a diffeomorphism, we not feel like dealing with the technical challenge of verifying that a given homeomorphism is in fact a diffeomorphism. We refer to our usual sources, namely [**Nic2011**, Theorem 2.7 and Remark 2.9], [**Miln1963a**, Theorem 3.2, Remark on p. 17, Remark 3.3] and [**Kos1993**, Chapter VII.2] for more details.

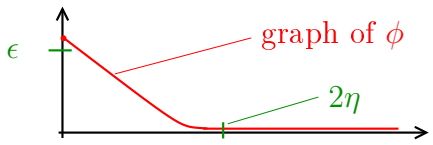
Proof. Let M be a compact smooth manifold, let $f\colon M \to \mathbb{R}$ be a smooth function and let $a < b \in \mathbb{R}$ be regular values such that $f^{-1}([a,b]) \cap \partial M = \varnothing$. In the following we will only prove Statement (1). Statement (2) can be proved either by generalizing the proof of Statement (1) or by using Exercise 137.6. We leave it to the reader to fill in the details for the proof of Statement (2).

Thus we suppose there exists a single critical point $P \in f^{-1}([a,b])$ and we suppose that P is non-degenerate. We set $c := f(P)$ and we set $k := \operatorname{index}(f,P)$. We need to show that M_b is diffeomorphic to the result of attaching a k-handle to M_a.

We make the following preparations:

(1) We set $l := n - k$.
(2) We make the identification $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$. Given $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$ we write $x^2 = x_1^2 + \cdots + x_k^2$ and similarly given $y = (y_1, \ldots, y_l) \in \mathbb{R}^l$ we write $y^2 = y_1^2 + \cdots + y_l^2$.
(3) By the Morse Lemma 137.2 we can identify a neighborhood of P in $M \setminus \partial M$ with a neighborhood V of $(0,0) \in \mathbb{R}^k \times \mathbb{R}^l$ such that $P = (0,0)$ and such that for every $(x,y) \in V$ we have $f(x,y) = c - x^2 + y^2$.
(4) Since a and b are regular values we know that $c \in (a,b)$. Thus we can pick an $\eta > 0$ such that $a \leq c - \eta < c + \eta \leq b$ and such that $\overline{B}^n_{\sqrt{2}\eta}(0) \subset V$. Note that by our hypothesis there are no critical points in $f^{-1}([a, c-\eta])$ and there are no critical points in $f^{-1}([c+\eta, b])$. Thus it follows from the No Critical Point-Diffeomorphism Theorem 138.1 that it suffices to show that $M_{c+\eta}$ is diffeomorphic to the result of attaching a k-handle to $M_{c-\eta}$.

Note that only in the neighborhood V do we have completely control over f. Unfortunately a priori the difference $M_b \setminus \mathring{M}_a$ is not contained in the neighborhood V. To deal with this issue we introduce an auxiliary function. First note that by the Smooth Transition Function Lemma 19.4 there exists a smooth map $\phi\colon [0,\infty) \to \mathbb{R}_{\geq 0}$ with $\phi(0) > \eta$, $\phi(t) = 0$ for $t \geq 2\eta$ and such that $\phi'(t) \in (-1, 0]$ for all $t \in [0,\infty)$. We refer to the figure below for an illustration.



Next we consider the map

$$F\colon M \to \mathbb{R}$$
$$P \mapsto \begin{cases} f(x,y) - \phi(x^2+2y^2) = c - x^2 + y^2 - \phi(x^2+2y^2), & \text{if } P=(x,y) \in \overline{B}^n_{\sqrt{2}\eta}(0), \\ f(P), & \text{otherwise.} \end{cases}$$

The following claim summarizes some of the key properties of F.

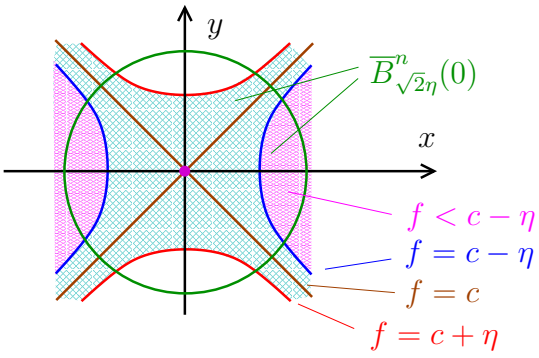
Claim 1.

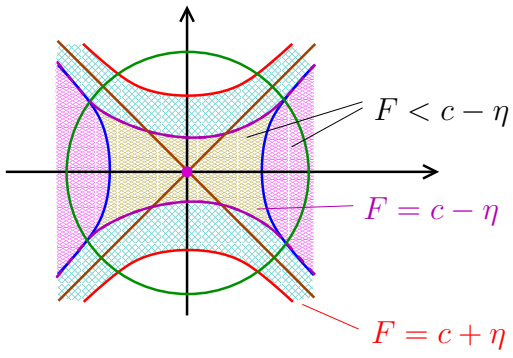
(1) The map F is smooth.
(2) The critical points of F are precisely the critical points of f.
(3) Given any $r \in \mathbb{R}$ we have $f^{-1}((-\infty, r]) \subset F^{-1}((-\infty, r])\}$.
(4) We consider the ellipsoid $E := \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + 2y^2 \leq 2\eta\}$. Given any $r \in \mathbb{R}$ we have $F^{-1}((-\infty, r]) \setminus f^{-1}((-\infty, r]) \subset E \subset \overline{B}^n_{\sqrt{2}\eta}(0)$.
(5) The smooth manifold $F^{-1}((-\infty, c-\eta])$ is diffeomorphic to $F^{-1}((-\infty, c+\eta])$.
(6) We have $F^{-1}((-\infty, c+\eta]) = f^{-1}((-\infty, c+\eta])$.

In particular we have

$$M_{c-\eta} = f^{-1}((-\infty, c-\eta]) \overset{(3)}{\subseteq} F^{-1}((-\infty, c-\eta]) \cong F^{-1}((-\infty, c+\eta]) \overset{(6)}{=} f^{-1}((-\infty, c+\eta]) = M_{c+\eta}.$$

by (4) the difference is contained in $E \subset \overline{B}^n_{\sqrt{2}\eta}(0)$ — diffeomorphic by (5)





Proof. We prove the statements in a somewhat different order.

(1) Note that ϕ is zero outside of $\overline{B}^n_{\sqrt{2}\eta}(0)$. It follows from this observation together with the fact $\overline{B}^n_{\sqrt{2}\eta}(0)$ is contained in V that F is smooth.
(2) It is clear that we only need to consider the points (x,y) with $x^2 + 2y^2 \leq 2\eta$. For those points we compute, using the Chain Rule 19.10 that

$$\mathrm{D}f_{(x,y)} = (2x, 2y) \text{ and } \mathrm{D}F_{(x,y)} = (2x \cdot \overbrace{(-1 - \underbrace{\phi'(x^2+2y^2)}_{>-1 \text{ since } \phi'(t)\in(-1,0]})}^{<0}, 2y \cdot \overbrace{(1 - 2\cdot\underbrace{\phi'(x^2+2y^2)}_{\leq 0 \text{ since } \phi'(t)\in(-1,0]})}^{\geq 1}).$$

We see that $\mathrm{D}f_{(x,y)} = (0,0)$ if and only if $\mathrm{D}F_{(x,y)} = (0,0)$.

(3) Since $\phi(x,y) \geq 0$ for all $(x,y) \in \overline{B}^n_{\sqrt{2}\eta}(0)$ we see that $F(P) \leq f(P)$ for all $P \in M$. But this implies that for any $r \in \mathbb{R}$ we have $f^{-1}((-\infty, r]) \subset F^{-1}((-\infty, r])$.

(4) Note that the two maps $f\colon M \to \mathbb{R}$ and $F\colon M \to \mathbb{R}$ agree outside of $E \subset \overline{B}^n_{\sqrt{2}\eta}(0)$. The desired statement is an immediate consequence of this observation.

(6) Note that given any $(x,y) \in E$ we have

$$\begin{array}{ccccccc} F(x,y) & \underset{\uparrow}{\leq} & f(x,y) & \underset{\uparrow}{=} & c - x^2 + y^2 & \leq & c + \frac{1}{2}x^2 + y^2 \quad \underset{\uparrow}{\leq} \quad c + \eta. \\ \text{since } \phi(x,y) \geq 0 & & \text{since } E \subset \overline{B}^n_{\sqrt{2}\eta}(0) \subset V & & & & \text{since } (x,y) \in E = \{(x,y) \mid x^2 + 2y^2 \leq 2\eta\} \end{array}$$

The desired equality follows from this observation together with (4).

(5) We need to show that the smooth manifold $F^{-1}((-\infty, c-\eta])$ is diffeomorphic to $F^{-1}((-\infty, c-\eta])$. By the No Critical Point-Diffeomorphism Theorem 138.1 together with (2) it suffices to prove that the preimage $F^{-1}([c-\eta, c+\eta])$ does not contain a critical point of f. This statement follows from the following inclusions and statements:

$$\overbrace{F^{-1}([c-\eta, c+\eta])}^{\substack{\text{by construction we have } \phi(0) > \eta \\ \text{thus } F(0,0) = c - \phi(0) < c - \eta \\ \text{thus this set does not contain } (0,0)}} = F^{-1}((-\infty, c+\eta]) \setminus F^{-1}((-\infty, c-\eta))$$
$$\subset f^{-1}((-\infty, c+\eta]) \setminus f^{-1}((-\infty, c-\eta)) = \underbrace{f^{-1}([c-\eta, c+\eta])}_{\substack{\text{by hypothesis this contains} \\ \text{a single critical point of } f, \\ \text{namely } (0,0)}}.$$
$$\uparrow$$
follows from (3) and (6)

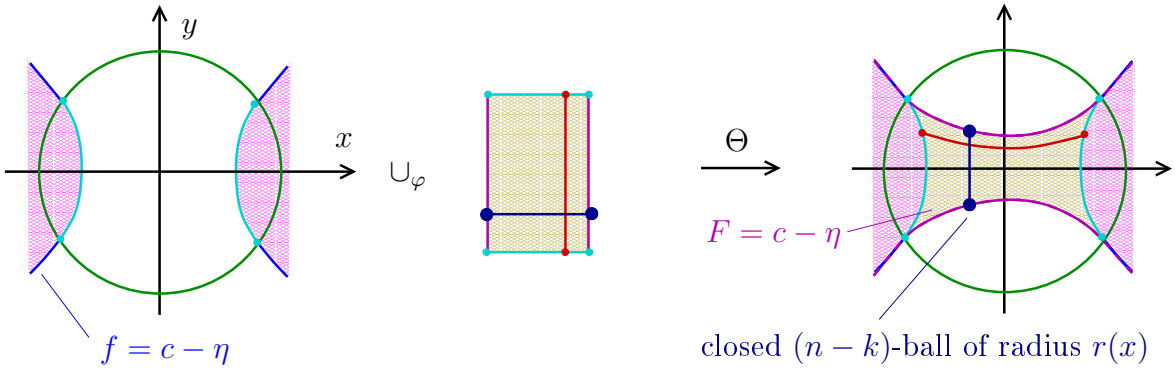
⊞

By Claim 1 it suffices to prove that the smooth manifold $F^{-1}((-\infty, c-\eta])$ is diffeomorphic to the result of attaching a k-handle to $M_{c-\eta} = f^{-1}((-\infty, c-\eta])$. This step is performed in the following claim.

Claim 2. We consider the thickened $(k-1)$-sphere

$$\begin{array}{rcl} \varphi\colon \overline{B}^{n-k} \times S^{k-1} & \to & f^{-1}(c-\eta) \subset \partial M_{c-\eta} \\ (x,y) & \mapsto & \left(\sqrt{2 \cdot \eta \cdot x^2 + \eta} \cdot y, \sqrt{2\eta} \cdot x\right). \end{array}$$

The smooth manifold $M_{c-\varphi} \cup_\varphi h^k = f^{-1}((-\infty, c-\eta]) \cup_\varphi h^k$ is diffeomorphic to the smooth manifold $F^{-1}((-\infty, c-\eta])$.



Proof. We only provide a sketch of the proof of Claim 2. As so often trying to show that a given map is really a diffeomorphism quickly leads to unreadable arguments. The full details for the proof are provided in [**Kos1993**, p. 129]. First note that a fairly straightforward argument shows that for any $x \in \overline{B}^k_{\sqrt{\eta}}(0)$ there exists an $r(x) \in \mathbb{R}_{>0}$ such that

$$\{y \in \overline{B}^l \mid F(x,y) \leq c + \eta\} \;=\; \overline{B}^{n-k}_{r(x)}(0).$$

Now the desired diffeomorphism is given by

$$\begin{aligned} f^{-1}((-\infty, c-\eta]) \cup_\varphi (\overline{B}^l \times \overline{B}^k) &\to F^{-1}((-\infty, c-\eta]) \\ P &\mapsto \begin{cases} P, & \text{if } P \in f^{-1}((-\infty, c-\eta]), \\ \left(\sqrt{2\eta x^2 + \eta}\cdot y, x \cdot r(\sqrt{2\eta x^2 + \eta})\right), & \text{if } P = (x,y) \in \overline{B}^l \times \overline{B}^k. \end{cases} \end{aligned}$$

This completes the sketch of the proof of the claim. ⊞

Finally suppose we are given an $\epsilon > 0$. We need to show that there exists an diffeomorphism as above which is the identity on the common subset $M_{a-\epsilon}$. As the reader will surely have noticed, this is an immediate consequence of our construction together with the "furthermore" statement of the No Critical Point-Diffeomorphism Theorem 138.1. ■

138.3. **The Handle Decomposition Theorem.** With the results from the previous section it is now pretty straightforward to prove the following result.

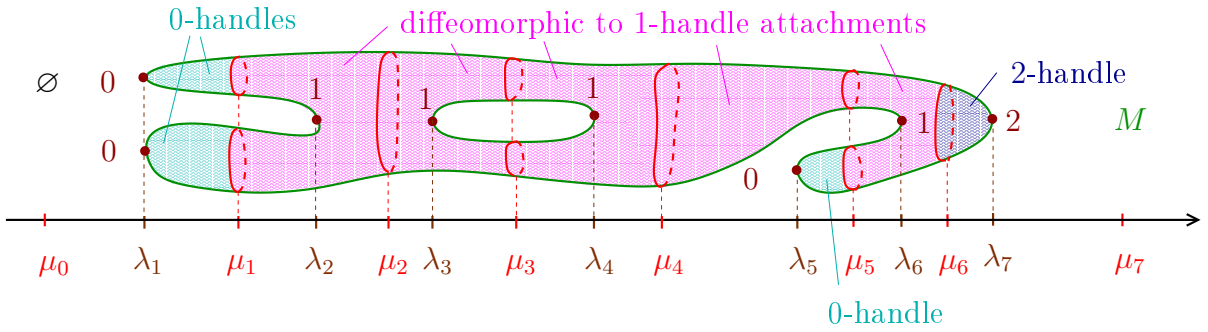
Proposition 138.4. (Morse Function-Handle Decomposition Proposition) Let M be a closed smooth manifold. If $f\colon M \to \mathbb{R}$ is a Morse function, then there exists a standard handle decomposition of M such that for each $k \in \mathbb{N}_0$ the number of k-handles equals the number of critical points of f of index k.

Proof. Let M be a closed smooth manifold and furthermore let $f\colon M \to \mathbb{R}$ be a Morse function. Recall that by Corollary 137.4 (2) we know that f has only finitely many critical points. We order the corresponding critical values by $\lambda_1 < \cdots < \lambda_s$. Next we pick real numbers $\mu_0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \mu_{s-1} < \lambda_s$ and furthermore we pick some $\mu_s \geq \max(f\colon M \to \mathbb{R})$. Note that $M_{\mu_0} = \varnothing$ and $M_{\mu_s} = M$. It follows from Theorem 138.3 that each M_{μ_i} is diffeomorphic to the result of attaching finitely many handles to $M_{\mu_{i-1}}$.[1672] In fact Theorem 138.3 says that we can arrange that the handles are in one-to-one correspondence to the critical points with critical value λ_i and that the indices of the handles correspond precisely to the indices of the critical points. This shows that $M = M_{\mu_s}$ admits a handle decomposition with the predicted number of handles. Finally we apply the Standard Handle Decomposition Corollary 136.4 to conclude that M actually admits a standard handle decomposition with the same number of handles with the same indices. ■

Remark.

(1) In the Handles-to-Morse Function Proposition 139.9 we will prove a converse to the Morse Function-Handle Decomposition Proposition 138.4, namely we will see that handle decompositions give rise to Morse functions.

[1672]Note that implicitly we use our hypothesis that M is closed, since to apply Theorem 138.3 we need to ensure that $f^{-1}([\mu_{i-1}, \mu_i]) \cap \partial M = \varnothing$.



(2) Let M be a closed smooth manifold and let $f\colon M \to \mathbb{R}$ is a Morse function. In the proof of the Morse Function-Handle Decomposition Proposition 138.4 we just saw that f gives rise to a handle decomposition of M. Evidently $-f\colon M \to \mathbb{R}$ is also a Morse function. The corresponding handle decomposition is for all intents and purposes the same as the dual handle decomposition that we introduced in the Morse Function-Handle Decomposition Proposition 138.4. Since we will not make use of this statement we will also not bother with providing a proof.

Now we can easily prove the main theorem of this chapter.

Theorem 138.5. (Handle Decomposition Theorem) Every closed smooth manifold admits a standard handle decomposition.

Remark.

(1) In the literature the Handle Decomposition Theorem is often called the Handle Presentation Theorem.
(2) We will deal with compact smooth manifolds with non-empty boundary in the following chapter. In particular we will prove the Relative Handle Decomposition Theorem 139.6 that says that every compact smooth manifold admits a handle structure rel a given union of boundary components. As we point out in Theorem 139.7, this implies in particular that every compact smooth manifold admits a standard handle decomposition.

Proof. Let M be a closed smooth manifold. By the Morse Function-Existence Proposition 137.6 we know that there exists a Morse function $f\colon M \to \mathbb{R}$. It follows from the Morse Function-Handle Decomposition Proposition 138.4 that M admits a standard handle decomposition. ■

We can now give a new proof for some of the statements of the Smooth Manifolds-Invariants Proposition 96.6.

Proposition 96.6. (Smooth Manifolds-Invariants Proposition) Let M be a closed n-dimensional smooth manifold.

(1) For every $k > n$ we have $\mathrm{H}_k(M;\mathbb{Z}) = 0$.
(3) Given any $P \in M$ the fundamental group $\pi_1(M,P)$ is finitely presented.
(4) All homology groups of M are finitely generated abelian groups.

Remark.

(1) Morse theory can also be used to prove the above formulation of the Smooth Manifolds-Invariants Proposition 96.6 for compact smooth manifolds with non-empty boundary. In fact there are at least two approaches to achieving this goal:
   (a) Either one applies the above result to the double DM of a compact smooth manifold with boundary and one applies a little bit of algebra, e.g. the Finite Presentation-Retract Lemma 53.16 .
   (b) Alternatively one can wait for Theorem 139.7 where we prove the generalization of the Handle Decomposition Theorem 138.5 to compact smooth manifolds with non-empty boundary.
(2) As we saw in the proof of Proposition 96.7, one can use the Smooth Manifolds-Invariants Proposition 96.6 to show that homology groups and fundamental groups of arbitrary, not necessarily compact, smooth manifolds are countable.
(3) In the Morse Inequalities Theorem 140.12 and Proposition 140.11 we will give explicit upper bounds on the "size" of the homology groups of a closed smooth manifold in terms of the number of critical points of a Morse function respectively the number of handles in a handle decomposition.

Sketch of proof. Let M be a closed n-dimensional smooth manifold. By the Handle Decomposition Theorem 138.5 we know that M admits a handle decomposition. It follows easily from a straightforward induction argument using the Handle Attachment-π_1-Proposition 136.11 that for any $P \in M$ the fundamental group $\pi_1(M, P)$ is finitely presented. Similarly it follows from a straightforward induction argument using the Handle Attachment-H_*-Lemma 136.13 that for every $k > n$ we have $\mathrm{H}_k(M; \mathbb{Z}) = 0$ and, using the elementary group-theoretic the Rank Additivity Lemma 51.7 (1), we see that all homology groups of M are finitely generated abelian groups. ■

The next proposition is also a straightforward consequence of our results.

Proposition 138.6. (Morse Functions-Cells Proposition) Let M be a closed smooth manifold.

(1) If f is a Morse function for M, then M is homotopy equivalent to a CW-complex where the number of k-cells equals precisely the number of critical points of index k.
(2) The smooth manifold is homotopy equivalent to a finite CW-complex.

Example. Let $n \in \mathbb{N}_0$. Recall that on page 2979 we showed that there exists a Morse function $f\colon \mathbb{R}\mathrm{P}^n \to \mathbb{R}$ with precisely one critical point for each index $0, \ldots, n$. It follows from the Morse Functions-Cells Proposition 138.6 that $\mathbb{R}\mathrm{P}^n$ is homotopy equivalent to a CW-complex with precisely one cell of dimension $0, \ldots, n$ and no other cells. In the same vein the above results show that $\mathbb{C}\mathrm{P}^n$ is homotopy equivalent to a CW-structure with precisely one cell of degree $0, 2, \ldots, 2n$ and no other cells. This discussion almost recovers the result of Lemma 68.1.

Proof.

(1) This statement follows immediately from Corollary 137.4 (2), Theorem 138.3 and the Handles-Cells Proposition 136.18.
(2) This statement is a consequence of (1) and the Morse Function-Existence Proposition 137.6. Put differently, it is a consequence of the Handle Decomposition Theorem 138.5 and the Handles-Cells Proposition 136.18. ■

Remark.

(1) There are various other proofs that Morse functions give rise to CW-complexes that are homotopy equivalent to M, see e.g. [**BaH2004**, Theorem 3.28], [**Miln1965b**, Chapter 3] and [**Mata2002**, Theorem 4.18]. Terse argument are also given in [**Nic2011**, Theorem 2.10] and [**Wall2016**, Proposition 5.3.1]. Furthermore an even terser argument is given in [**Hirs1976**, Theorem 6.4.1].

(2) Given a closed smooth manifold there is a notion of a more refined version of a Morse function, namely of a Morse-Smale function. We will not provide the definition, instead we refer to [**BaH2004**, p. 7] for the definition. But we try to pique the reader's interest with the following two statements:
   (a) The Kupka-Smale Theorem, proved independently by Ivan Kupka [**Kupk1963**] and Steven Smale [**Sma1963**], says that every closed smooth manifold admits a Morse-Smale function. Alternatively see [**BaH2004**, Theorem 6.6] for a proof.
   (b) In [**Lau1992**, p. 222] (see also [**BaH2004**, p. 179]) it is shown that a Morse-Smale function gives rise to a CW-complex that is actually *homeomorphic* to M and such that for each $k \in \mathbb{N}_0$ the critical points of index k are in one-to-one correspondence to the k-cells.

(3) The Morse Functions-Cells Proposition 138.6 (2) is related to several earlier results:
   (a) In Theorems 96.2 we showed that every compact smooth manifold M admits a finite simplicial structure. By the Smooth Manifold-CW Structure Theorem 96.5 this implies in particular that M admits a finite CW-structure.
   (b) In Corollary 95.37 we used the Nerve Theorem 95.32 to show that every compact smooth manifold is weakly homotopy equivalent to a finite CW-complex.
   (c) Finally in Corollary 104.20 we used Borsuk's Theorem 104.3 to show that every topological manifold is homotopy equivalent to a CW-complex.

138.4. **Reeb's Theorem.** Theorem 138.3 also allows us to prove the following theorem which was first proved by Georges Reeb [**Reeb1946**] in 1946.

Theorem 138.7. (Reeb's Theorem) Let M be a closed n-dimensional smooth manifold. If M admits a smooth function $f\colon M \to \mathbb{R}$ with precisely two critical points which are both non-degenerate, then M is *homeomorphic* to S^n.

Remark. For the record we mention that by [**Miln1964b**, Theorem 1'] the conclusion of Theorem 138.7 also holds if the critical points are degenerate.

Reeb's Theorem 138.7 is an immediate consequence of Theorem 138.3 together with the following proposition.

Proposition 138.8. Let $n \in \mathbb{N}$.

(1) If $\varphi\colon S^{n-1} \to S^{n-1}$ is a homeomorphism, then $\overline{B}^n \cup_\varphi \overline{B}^n$ is homeomorphic to S^n.

(2) Let M be a closed n-dimensional smooth manifold. If M admits a handle decomposition consisting of a single 0-handle and a single n-handle, then M is *homeomorphic* to S^n.

Proof.

(1) Let $\varphi\colon S^{n-1} \to S^{n-1}$ be a homeomorphism. We apply the Alexander trick to get an extension of $\varphi\colon S^{n-1} \to S^{n-1}$ to the ball $\overline{B}^n$. More precisely we consider the following

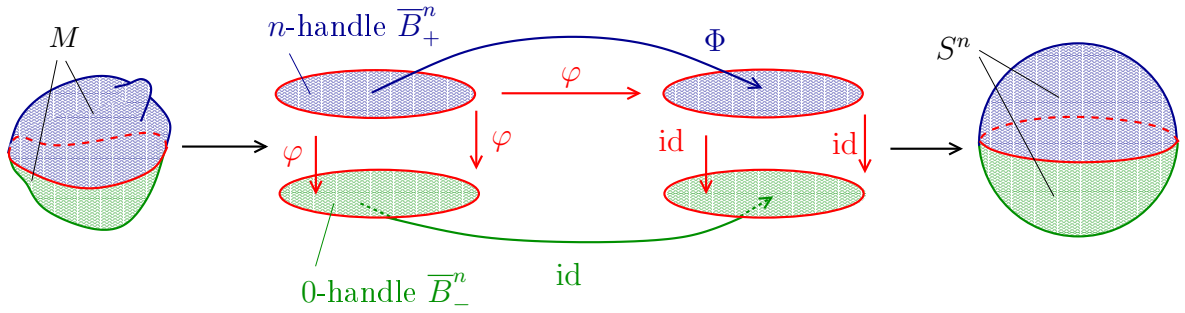
map:

$$\begin{array}{rcl} \Phi\colon \overline{B}^n & \to & \overline{B}^n \\ x & \mapsto & \begin{cases} \|x\| \cdot \varphi(\frac{x}{\|x\|}), & \text{if } x \neq 0, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

By Exercise 5.35 we know that this map Φ is a homeomorphism. Finally we consider the following sequence of homeomorphisms:

$$\begin{array}{rcl} \overline{B}^n_+ \cup_{\varphi\colon S^{n-1}\to S^{n-1}} \overline{B}^n_- & \overset{\cong}{\to} & \overline{B}^n_+ \cup_{\mathrm{id}\colon S^{n-1}\to S^{n-1}} \overline{B}^n_- \xrightarrow[\text{from page 261}]{\text{homeomorphism}} S^n. \\ [P] & \mapsto & \begin{cases} [\Phi(P)], & \text{if } P \in \overline{B}^n_+, \\ [P], & \text{if } P \in \overline{B}^n_-. \end{cases} \end{array}$$

(2) Let M be a closed n-dimensional smooth manifold that M admits a handle decomposition consisting of a single 0-handle and a single n-handle. This means that there exists a diffeomorphism $\varphi\colon S^{n-1} \to S^{n-1}$ such that M is diffeomorphic to[1673] $\overline{B}^n \cup_\varphi \overline{B}^n$. It now follows from (1) that M is homeomorphic to S^n. ■



Reeb's Theorem 138.7 and Proposition 138.8 raise several fantastic questions. First of all, it is natural to ask whether the analogue of Proposition 138.8 (1) holds if we replace "homeomorphism" by "diffeomorphism". Put differently, we have the following question.

Question 138.9. Let $\varphi\colon S^{n-1} \to S^{n-1}$ be a diffeomorphism. Is the smooth manifold $\overline{B}^n \cup_\varphi \overline{B}^n$ *diffeomorphic* to S^n?

Next let us reflect whether we can upgrade the composition of maps towards the end of the proof Proposition 138.8 into a diffeomorphism. The good news is that, by Exercise **??**, we can replace the map on the right-hand side by a diffeomorphism. The bad news is that, as we saw in Exercise 5.35, the map $\Phi\colon B^n \to \overline{B}^n$ given by the Alexander trick is not smooth at the origin, in particular Φ is not diffeomorphism. But perhaps we can pick Φ more cleverly? This leads us to our second question.

Question 138.10. Let $n \in \mathbb{N}$. Does every diffeomorphism $\varphi\colon S^n \to S^n$ extend to a diffeomorphism of $\overline{B}^{n+1}$?

In Propositions **??** and **??** we will provide answers to Questions 138.10 and 138.9.

[1673] Note that by the Gluing-Smooth Manifolds-Proposition **??** we can equip $\overline{B}^n \cup_\varphi \overline{B}^n$ with a canonical smooth structure.

Exercises for Chapter 138.

Exercise 138.1. Let M be a nd N be two closed n-dimensional smooth manifolds and let $f\colon M \to \mathbb{R}$ and $g\colon N \to \mathbb{R}$ be Morse functions with unique critical points of index 0 and n. Show that the connected sum $M\#N$ admits a Morse function φ with unique critical points of index 0 and n and such that for $k \in \{1, \dots, n-1\}$ we have

$$\begin{aligned}\text{number of critical points of } \varphi\colon M\#N \to \mathbb{R} = \quad & \text{number of critical points of } f\colon M \to \mathbb{R} \\ & + \text{number of critical points of } g\colon N \to \mathbb{R}.\end{aligned}$$

Exercise 138.2. Use handle decomposition to show that every closed n-dimensional smooth manifold admits an atlas with $n+1$ charts.
Remark. This exercise gives a new proof for some of the statements of Proposition 27.8.

139. Morse theory II: The Relative Handle Decomposition Theorem

In the Morse Function-Existence Proposition 137.6 we showed that every *closed* smooth manifold admits a Morse function. Using Theorem 138.3 this allowed us to prove the Handle Decomposition Theorem 138.5 which says that every *closed* smooth manifold admits a handle decomposition.

As we remarked on page 2981, it is not difficult to prove, using the Morse Function-Existence Proposition 137.6, that in fact every *compact* smooth manifold M admits a Morse function $f\colon M \to \mathbb{R}$. But this is not enough to prove the Handle Decomposition Theorem 138.5 for compact smooth manifolds with *non-empty boundary* since Theorem 138.3 has the hypothesis that $f^{-1}([a,b]) \cap \partial M = \varnothing$.

In this chapter we will see that compact smooth manifolds have in fact a handle decomposition. In fact we will do more, namely we will develop the theory of relative handle decompositions and we will study more refined versions of Morse functions. As we will see on several occasions, this extra effort for smooth manifolds with boundary will also come in handy in the study of *closed* smooth manifolds.

139.1. **Relative Handle Decompositions.** For smooth manifolds with non-empty boundary it will be convenient to study relative handle decompositions. This concept generalizes the notion of a handle decomposition that we introduced on page 2954.

Definition. Let M be a compact n-dimensional smooth manifold and let A be a (possibly empty) union of components of ∂M.

(1) A handle decomposition for M rel A is a diffeomorphism

$$\Theta\colon M \xrightarrow{\cong} ([0,1] \times A) \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_m} h^{k_m}$$

such that for $i = 0, \ldots, m$ the map φ_i is a thickened $(k_i - 1)$-sphere that takes values in

$$\partial\big(([0,1] \times A) \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_{i-1}} h^{k_{i-1}}\big) \setminus (\{0\} \times A)$$

and such that for each $P \in A$ we have $\Theta(P) = (0, P)$.[1674]

In the following we will mostly suppress Θ from the notation. With this convention we introduce the following definitions:

(2) Given $i \in \{0, \ldots, n\}$ we refer to

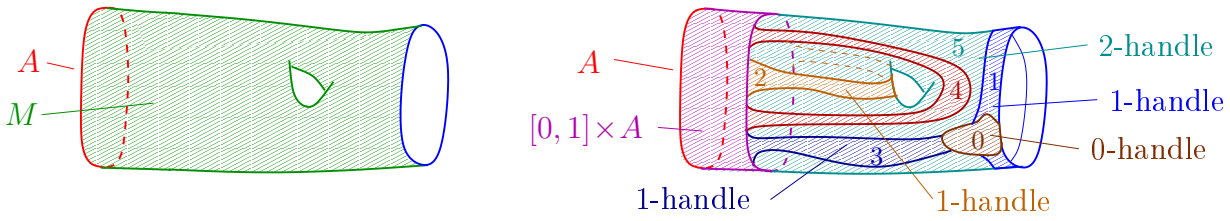
$$M^i := ([0,1] \times A) \underbrace{\cup_{\varphi_{0,1}} h^0 \cdots \cup_{\varphi_{0,r_0}} h^0}_{0\text{-handles}} \cup \ldots \underbrace{\cup_{\varphi_{i,1}} h^i \cdots \cup_{\varphi_{i,r_i}} h^i}_{i\text{-handles}}.$$

as the i-skeleton of the relative handle decomposition.

(3) We say that a relative handle decomposition is standard if it is of the above form and if for each $i \in \{0, \ldots, n\}$ the attaching maps $\varphi_{i,1}, \ldots, \varphi_{i,r_i}$ take values in ∂M^{i-1}.

The following proposition is basically a generalization of the Dual Handle Proposition 136.5.

Proposition 139.1. (Relative Dual Handle Proposition) Let M be a compact n-dimensional smooth manifold. Furthermore let $\partial M = A \sqcup B$ be a decomposition of ∂M into two disjoint unions of components A and B. Finally let



standard handle decomposition of M rel A

$$M \;=\; ([0,1]\times A)\cup_{\varphi_0} \underbrace{h^{k_0}}_{\text{0-th handle}} \cup\cdots\cup_{\varphi_m} \underbrace{h^{k_m}}_{m\text{-th handle}}$$

be a handle decomposition of M rel A. We consider

$$M' \;:=\; (M\setminus([0,1)\times A))\cup_{B=B\times\{0\}} ([0,1]\times B)$$

and we set $A' := \{1\}\times A \subset M'$ and $B' := \{1\}\times B \subset M'$.

(0) There exists a diffeomorphism from M to M' that restricts to the obvious diffeomorphism $A \to A'$ and $B \to B'$.

There exists canonical handle decomposition

$$M' \;=\; ([0,1]\times B')\cup_{\psi_m} \underbrace{\widetilde{h}^{n-k_m}}_{\text{0-th dual handle}} \cup\cdots\cup_{\psi_0} \underbrace{\widetilde{h}^{n-k_0}}_{m\text{-th dual handle}},$$

of M' rel B' called dual handle decomposition, with the following properties:

(1) For each $i \in \{0,\dots,m\}$ the following statements hold:

(a) The i-th handle of the dual handle decomposition equals, as a subset of M', the $(m-i)$-th handle of the original handle decomposition.

(b) The indices of the handles are reversed, in the sense that

$$\begin{array}{c}\text{index of the } i\text{-th handle of}\\ \text{the dual handle decomposition}\end{array} \;=\; n - \begin{array}{c}\text{index of the } (m-i)\text{-th handle of}\\ \text{the original handle decomposition.}\end{array}$$

In particular the number of i-handles in the dual handle decomposition equals the number of $(n-i)$-handles in the original decomposition.

(c) The roles of attaching spheres and belt spheres is reversed in the sense that

$$\begin{array}{c}\text{belt sphere of the } i\text{-th handle of}\\ \text{the dual handle decomposition}\end{array} \;=\; \begin{array}{c}\text{attaching sphere of the } (m-i)\text{-th handle}\\ \text{of the original handle decomposition.}\end{array}$$

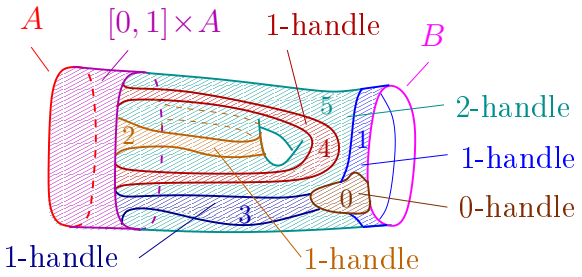
Furthermore the same statement, with the roles of attaching and belt spheres reversed, also holds.

(2) If the original handle decomposition is standard, then so is the dual handle decomposition.

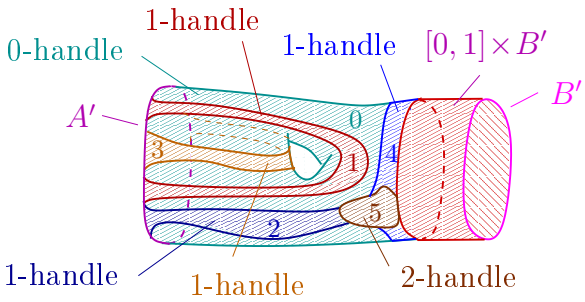
(3) The dual of the dual handle decomposition is the original handle decomposition.

Example. In the figure below we show again M, the torus minus two open disks, with two boundary components A and B. To the left we show the above handle decomposition of M rel A. To the right we see the corresponding dual handle decomposition of M' rel B'.

In the proof of the Relative Dual Handle Proposition 139.1 we will make use of the following lemma, which also doubles as Exercise **??**.



standard handle decomposition of M rel A



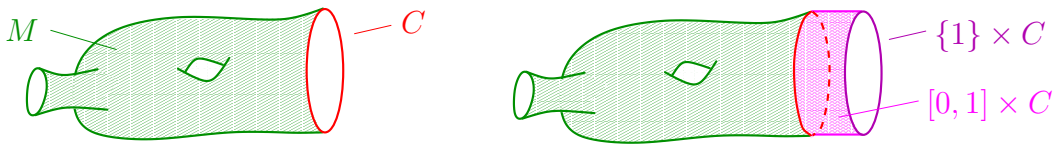
dual handle decomposition of M' rel B'

Lemma 139.2. Let M be a compact smooth manifold and let C be a union of components of ∂M. Let $\varphi\colon C \to C$ be a diffeomorphism. Given any neighborhood U of $C \subset M$ there exists a diffeomorphism

$$M \;\to\; \underbrace{(M \sqcup ([0,1]\times C))/\!\sim}_{\substack{\text{smooth manifold by the}\\ \text{Gluing-Smooth Manifolds-Proposition ??}}} \qquad \text{where } x \sim \varphi(x,0) \text{ for } x \in C$$

that is the identity on the common subset $M \setminus U$ and that restricts to the obvious diffeomorphism from C to $\{1\}\times C$.

Remark. In the setting of Lemma 139.2, we sometimes refer to $[0,1]\times C$ as an *external collar*.



Proof of Lemma 139.2. As is explained in the Gluing-Smooth Manifolds-Proposition **??** (1), to say that the topological space $(M \sqcup ([0,1]\times C))/\!\sim$ is a smooth manifold we actually need to pick a collar $[-1,0]\times C$ with $C = \{0\}\times C$ to define the actual smooth atlas. Since U is a neighborhood of C and since C is compact there exists an $\epsilon > 0$ such that $[-\epsilon, 0]\times C \subset U$. Using the Smooth Transition Function Lemma 19.4 one can easily show that there exists a smooth function $\eta\colon [-\epsilon,0] \to [-\epsilon,1]$ with $\eta(t) = t$ for $t \in [-\epsilon, -\frac{\epsilon}{2}]$, $\eta(0) = 1$ and such that $\eta'(t) > 0$ for all t. The desired diffeomorphism is now given by the identity on $M \setminus ([-\epsilon,0]\times C)$ and by stretching $[-\epsilon,0]\times C$ to $[-\epsilon,1]\times C$ via the map $(t,P) \mapsto (\eta(t),P)$. ∎

Proof of the Relative Dual Handle Proposition 139.1.

(0) It follows from the Codimension-Zero Smooth Embedding Proposition 24.16 that $W := M \setminus ([0,1)\times A)$ is a smooth manifold. It follows from Lemma 139.2 that both $M = ([0,1]\times A)\cup W$ and $M' = W \cup ([0,1]\times B)$ are diffeomorphic to W.

(1)-(3) The proof of the remaining statements is basically identical to the proof of the Dual Handle Proposition 136.5. We only need a minuscule generalization of Lemma 136.6, namely instead of considering diffeomorphisms $\gamma\colon \partial Y \to \partial(X \cup_\varphi h^k)$ we also need to allow diffeomorphisms $\gamma\colon C \to D$ where C is a component of ∂Y

and D is a component of $\partial(X \cup_\varphi h^k)$ that contains $h^k \cap \partial(X \cup_\varphi h^k)$. We leave it to the reader to fill in the details. ■

The following proposition says that we can "stack" relative handle decompositions.

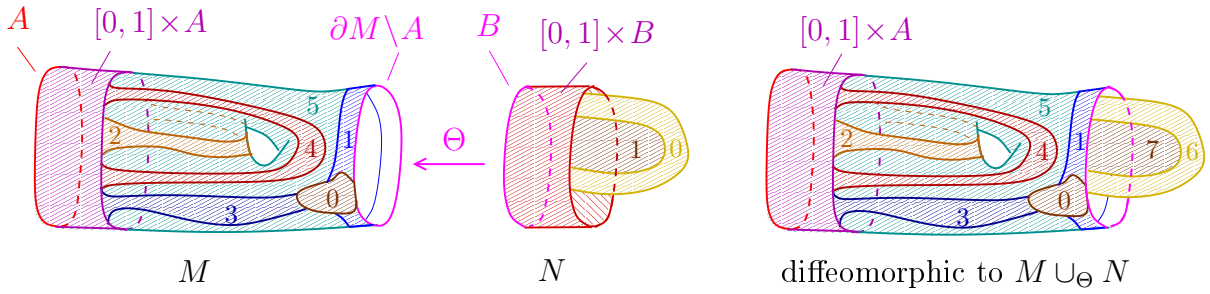
Proposition 139.3. (Stacking Proposition) Let M be a compact n-dimensional smooth manifold and let N be a compact n-dimensional smooth manifold. Furthermore let A be a union of boundary components of M and let B be a union of boundary components of N. Next let

$$\begin{aligned} M &= ([0,1]\times A)\cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_r} h^{k_r} \\ \text{and} \qquad N &= ([0,1]\times B)\cup_{\psi_0} h^{l_0} \cup \cdots \cup_{\psi_s} h^{l_s} \end{aligned}$$

be handle decompositions rel A respectively rel B. Finally let $\Theta\colon B \to \partial M \setminus A$ be a diffeomorphism. We consider the smooth manifold $M \cup_\Theta N$. There exists a handle decomposition of the form

$$M \cup_\Theta N \xrightarrow{\cong} ([0,1]\times A)\cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_r} h^{k_r} \cup_{\widetilde{\psi}_0} h^{l_0} \cup \cdots \cup_{\widetilde{\psi}_s} h^{l_s}$$

rel A, where the attaching maps $\widetilde{\psi}_0, \ldots, \widetilde{\psi}_s$ are obtained in a hopefully obvious way from the attaching maps $\psi_0, \ldots, \psi_s$.



Proof. We have the following diffeomorphisms:

$$\begin{aligned} M \cup_\Theta N &= M \cup_\Theta (([0,1]\times B)\cup_{\psi_0} h^{l_0} \cup \cdots \cup_{\psi_s} h^{l_s}) = (M \cup_\Theta ([0,1]\times B))\cup_{\psi_0} h^{l_0} \cup \cdots \cup_{\psi_s} h^{l_s} \\ &\cong M\cup_{\widetilde{\psi}_0} h^{l_0} \cup \cdots \cup_{\widetilde{\psi}_s} h^{l_s} = ((([0,1]\times A)\cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_r} h^{k_r}) \cup_{\widetilde{\psi}_0} h^{l_0} \cup \cdots \cup_{\widetilde{\psi}_s} h^{l_s}. \end{aligned}$$

$\uparrow$ Lemma 139.2

■

Examples.

(1) Let M be a compact smooth manifold with a handle decomposition

$$M = \varnothing \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_r} h^{k_r}.$$

We can view this as a handle decomposition rel $\varnothing$. By the Relative Dual Handle Proposition 139.1 we can equip M with the dual handle decomposition, which is de facto a handle decomposition rel ∂M. If we glue two copies of M along the common boundary, i.e. if we consider the double $\mathrm{D}M$, then it follows from the Stacking Proposition 139.3 that we can stack the original handle decomposition and the dual decompositions. In summary we obtain a handle decomposition for the double $\mathrm{D}M$ of the form

$$\mathrm{D}M = \varnothing \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_r} h^{k_r} \cup_{\widetilde{\psi}_r} h^{n-k_r} \cup \cdots \cup_{\widetilde{\psi}_0} h^{n-k_0}.$$

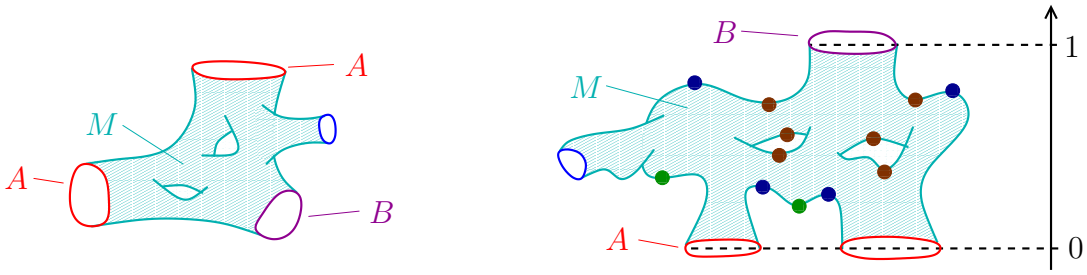
(2) First let $k, l \in \mathbb{N}$.
   (a) In Exercise 136.2 we showed that $\overline{B}^k \times S^l$ admits a handle decomposition rel $\varnothing$ consisting of one 0-handle and one l-handle.
   (b) By (a) together with the Relative Dual Handle Proposition 139.1 we see that $\overline{B}^k \times S^l$ admits a handle decomposition rel $\partial(\overline{B}^k \times S^l)$ consisting of one $(k+l)$-handle and one k-handle.

Now let $m, n \in \mathbb{N}$. In Lemma 59.1 we showed that S^{m+n-1} is diffeomorphic to $(S^{m-1} \times \overline{B}^n) \cup_{S^{m-1}\times S^{n-1}} (\overline{B}^m \times S^{n-1})$. Note that by the Stacking Proposition 139.3 we can stack the handle decomposition of $S^{m-1} \times \overline{B}^n$ rel $\varnothing$ with the handle decomposition of $\overline{B}^m \times S^{n-1}$ rel $\partial(\overline{B}^m \times S^{n-1})$. Thus we see that S^{m+n-1} admits a handle decomposition consisting of four handles of index 0, $m-1$, m and $m+n-1$.

139.2. **Morse functions for smooth manifolds with boundary.** Our goal will be to show the existence of suitable relative handle decompositions. The key tool to doing so is the following generalization of the Morse Function-Existence Proposition 137.6.

Proposition 139.4. (Morse Function-Existence Proposition) Let M be a compact smooth manifold and let A and B be two disjoint unions of components of ∂M. There exists a Morse function $f\colon M \to [0,1]$ such that $A = f^{-1}(0)$ and such that $B = f^{-1}(1)$.

Example. In the figure below we show a surface M with A and B as in the Morse Function-Existence Proposition 139.4. We also show a different embedding of M into $\mathbb{R}^3$ such that the height function $(x, y, z) \mapsto z$ has the desired properties.

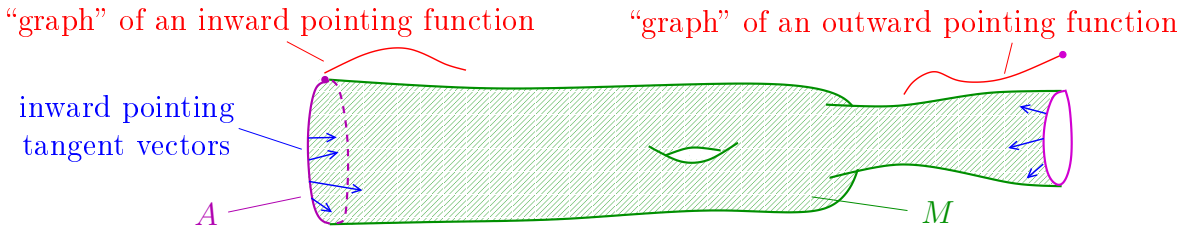


The proof of the Morse Function-Existence Proposition 139.4 requires some technical preparations. First let us introduce the following definition.

Definition. Let M be a smooth manifold, let $f\colon M \to \mathbb{R}$ be a smooth function and let A be a union of components of ∂M. We say f is pointing inward on A if for every $P \in A$ and every inward pointing tangent vector[1675] $P \in \mathrm{T}_P M$ we have $\mathrm{D}f_P(v) > 0$. Similarly we are defining that f is pointing outward on A.

Lemma 139.5. Let N be a closed smooth manifold and let $f\colon [0,1] \times N \to \mathbb{R}$ be a smooth function that is pointing inward on $\{0\} \times N$ and that has no critical points. There exists a smooth function $g\colon [0,1] \times N \to \mathbb{R}$ with the following properties:
(1) The map g agrees with f on $[\frac{1}{2}, 1] \times N$.
(2) The map g has no critical points and it is pointing inward on $\{0\} \times N$.

[1675] We refer to page 634 for the definition of an inward pointing tangent vector.



(3) The map g is constant on $\{0\} \times N$ and the value on $\{0\} \times N$ equals the minimum of g on $[0,1] \times N$.

Sketch of a proof of Lemma 139.5. We pick some $s \in \mathbb{R}$ that is less than the minimum of f on $[0,1] \times N$. Using the Smooth Transition Function Lemma 19.4 it is pretty straightforward to see that there exist smooth maps $\alpha, \beta\colon [0,1] \to \mathbb{R}$ with $\alpha + \beta \equiv 1$, $\alpha(t) = 1$ for sufficiently small t and $\beta(t) = 1$ for $t \in [\frac{1}{2}, 1]$ such that the map

$$\begin{array}{rcl} g\colon [0,1] \times N & \to & \mathbb{R} \\ (t, P) & \mapsto & \alpha(t) \cdot (s+t) + \beta(t) \cdot f(P,t) \end{array}$$

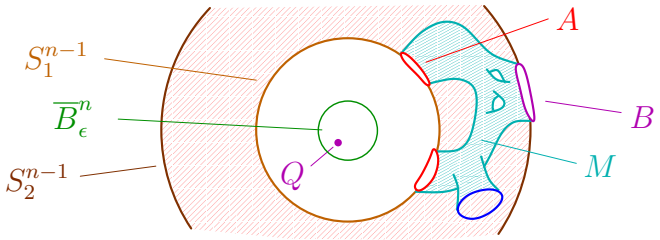
has all the desired properties. We leave it to the reader to fill in the dull technical details. ■

Sketch of a proof of the Morse Function-Existence Proposition 139.4. Let M be a compact smooth manifold and let A and B be two disjoint unions of components of ∂M. A slight refinement of the Euclidean Embedding Theorem 27.1, similar to Exercise 27.3, shows, together with the Smooth Embedding Theorem 24.10, that we can view M as a submanifold of $\{x \in \mathbb{R}^n \mid \|x\| \in [1,2]\}$ such that $\varphi(A) \subset S_1^{n-1}$ and such that $\varphi(B) \subset S_2^{n-1}$. Using the Smooth Collar Neighborhood Theorem 28.3 we obtain collar neighborhoods $[0,1] \times A$ and $[0,1] \times B$.

As in the proof of the Morse Function Existence Proposition 137.5 one can show that given any $\epsilon > 0$ there exists a $Q \in \overline{B}_\epsilon^n$ such that

$$\begin{array}{rcl} f_Q\colon M & \to & \mathbb{R} \\ x & \mapsto & \|x - Q\|^2 \end{array}$$

is a Morse function. It is straightforward to see that by picking a sufficiently small $\epsilon > 0$ we can arrange that $f_Q\colon M \to \mathbb{R}$ is pointing inward on A and that it is pointing outward on B. Thus, Using Lemma 139.5, applied to $[0,1] \times A$ and $[0,1] \times B$ we can modify f_Q to obtain a Morse function on M that is constant on A and that is constant on B and such that all other values lie between the values on A and B. After rescaling we have obtained the desired Morse function.



A more detailed and authoritative proof of the Morse Function-Existence Proposition 139.4 is given in [**Miln1965b**, Theorem 2.5]. ■

139.3. **The Relative Handle Decomposition Theorem.** Our goal in this section is to prove the following refinement of the Handle Decomposition Theorem 138.5.

Theorem 139.6. (Relative Handle Decomposition Theorem) Let M be a compact n-dimensional smooth manifold. Given any union A of components of ∂M there exists a standard handle decomposition of M rel A.

If we apply the Relative Handle Decomposition Theorem 139.6 to $A = \varnothing$ we obtain for free the following generalization of the Handle Decomposition Theorem 138.5.

Theorem 139.7. (Handle Decomposition Theorem) Every compact smooth manifold admits a standard handle decomposition.

Proof of the Relative Handle Decomposition Theorem 139.6. Let M be a compact smooth manifold and let A be a union of components of ∂M. We denote by B the union of the remaining components of ∂M. By the Morse Function-Existence Proposition 139.4 there exists a Morse function $f\colon M \to [0,1]$ such that $A = f^{-1}(0)$ and such that $B = f^{-1}(1)$. We make the following two observations:

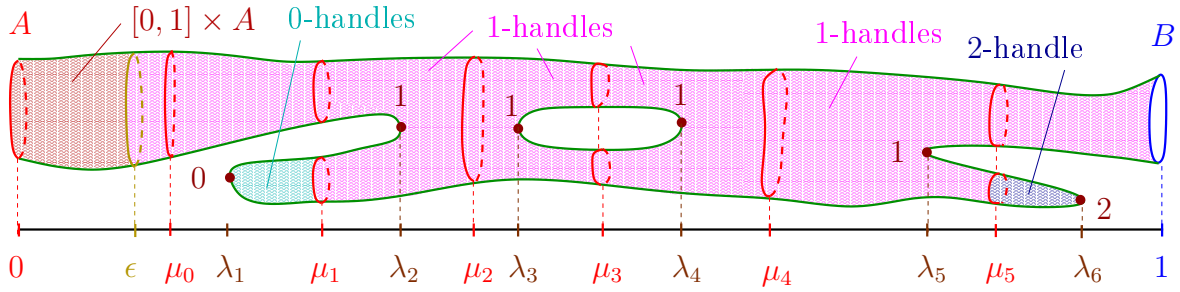
(a) Since $\partial M = A \cup B$ we have $f^{-1}((0,1)) \cap \partial M = \varnothing$.
(b) Note that $A \cup B = \partial M$. Thus, by definition of a Morse function, A and B do not contain critical points, in other words, 0 is not a critical value and 1 is not a critical value.

By Corollary 137.4 (2) the Morse function has only finitely many critical points. We order the critical values by $0 < \lambda_1 < \cdots < \lambda_s < 1$. (Note that by (b) we know that $0 < \lambda_1$ and $\lambda_s < 1$.) We pick real numbers $0 < \epsilon < \mu_0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \mu_{s-1} < \lambda_s < 1 = \mu_s$.

Note that it follows from (a) and the Regular Value Theorem 26.4 that $f^{-1}(\epsilon)$ is a closed submanifold of M. Thus it follows from the Smooth Manifold-Product Theorem 138.2 that $M_\epsilon = f^{-1}([0,\epsilon])$ is diffeomorphic to $[0,1] \times A$.

Next note that it follows from Theorem 138.3 and (a) that for each $i \in \{1,\dots,s\}$ there exists a diffeomorphism from M_{μ_i} to the result of attaching finitely many handles to $M_{\mu_{i-1}}$. Furthermore, Theorem 138.3 says that we can find such a diffeomorphism that is the identity on the common subset M_ϵ. This shows that $M = M_{\mu_s}$ admits a handle decomposition rel A.

Finally note that it follows immediately from iteratively applying the Handle Swapping Proposition 136.3 that we can rearrange our handle attachments to obtain a standard handle decomposition of M rel A. ∎



We conclude this section with the following generalization of Proposition 138.6.

Proposition 139.8. (Smooth Manifold-CW-Homotopy Type Proposition) Every compact smooth manifold is homotopy equivalent to a finite CW-complex.

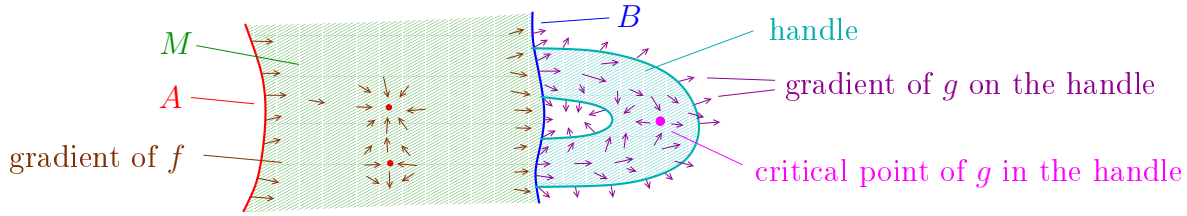
Proof. This statement follows immediately from the Handle Decomposition Theorem 139.7 together with the Handles-Cells Proposition 136.18. ■

139.4. **Constructing Morse functions from handle decompositions.** By the Handle Addition Theorem 138.3 we know that Morse functions give rise to handle decompositions. In this section we will see that we can also go backwards. For example we have the following converse to Theorem 138.3.

Proposition 139.9. (Handles-to-Morse Function Proposition) Let M be a compact smooth manifold and suppose we are given a decomposition $\partial M = A \sqcup B$ where A and B are unions of components of ∂M.

(1) Let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to B$ be a thickened $(k-1)$-sphere and let $f\colon M \to \mathbb{R}$ be a map that points inward on A and that points outward on B. Then there exists a smooth map $g\colon M \cup_\varphi h^k \to \mathbb{R}$ with the following properties:
   (a) The restriction of g to M equals f.
   (b) g points outward on $\partial(M \cup_\varphi h^k) \setminus A$.
   (c) In the handle $\overline{B}^{n-k} \times \overline{B}^k$ the map g has precisely one critical point, and this critical point has index k.

(2) If M is equipped with a handle decomposition $M \cong ([0,1] \times A) \cup_{\varphi_1} h^{k_1} \cup \cdots \cup_{\varphi_m} h^{m_k}$ rel A, then there exists a smooth map $g\colon M \to \mathbb{R}$ with the following properties:
   (a) On $[0,1] \times A$ the map g equals the projection onto $[0,1]$,
   (b) The map g has precisely m critical points $P_1, \ldots, P_m$, each of which is non-degenerate and such that for $i = 1, \ldots, m$ the index of the critical point P_i equals the index of the handle φ_i.



Proof.

(1) This statement is proved by running the proof of Theorem 138.3 "backwards". In particular the idea is, to define g on the k-handle to be a suitable modification of the function

$$\begin{array}{rcl} \overline{B}^k \times \overline{B}^{n-k} & \to & \mathbb{R} \\ (x_1, \ldots, x_k, y_1, \ldots, y_{n-k}) & \mapsto & -x_1^2 - \cdots - x_k^2 + y_1^2 + \cdots + y_{n-k}^2. \end{array}$$

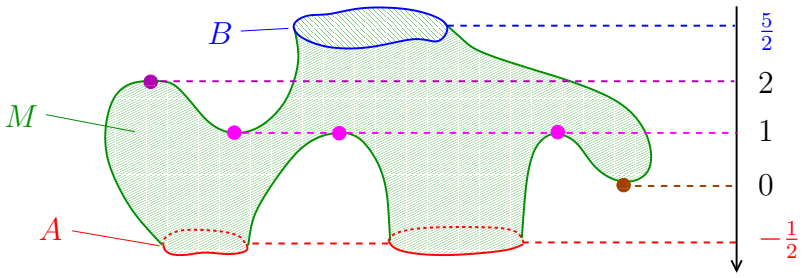
   We refer to [**Kos1993**, p. 130] for the details.

(2) This statement follows immediately from applying (1) altogether m times. ■

We conclude this section with a neat refinement of the Morse Function-Existence Proposition 137.6. To formulate the result we introduce the following definition.

Definition. Let M be a compact n-dimensional smooth manifold and suppose we are given a decomposition $\partial M = A \sqcup B$ where A and B are unions of components of ∂M. A Morse function $f\colon M \to \mathbb{R}$ is called self-indexing with respect to A and B, if the following conditions are satisfied.

(1) We have $f|_A \equiv -\frac{1}{2}$ and $f|_B \equiv n + \frac{1}{2}$.
(2) If P is a critical point of f, then $f(P) = \operatorname{index}(f, P)$.



Proposition 139.10. Let M be a compact n-dimensional smooth manifold and suppose we are given a decomposition $\partial M = A \sqcup B$ where A and B are unions of components of ∂M. There exists a self-indexing Morse function $f\colon M \to \mathbb{R}$ with respect to A and B.

Sketch of proof. Let M be a compact n-dimensional smooth manifold and suppose we are given a decomposition $\partial M = A \sqcup B$ where A and B are unions of components of ∂M. By the Handle Decomposition Theorem 138.5 there exists a standard handle decomposition

$$M \;=\; ([0,1] \times A) \underbrace{\cup_{\varphi_{0,1}} h^0 \cdots \cup_{\varphi_{0,r_0}} h^0}_{0\text{-handles}} \cup \ldots \underbrace{\cup_{\varphi_{n,1}} h^n \cdots \cup_{\varphi_{n,r_n}} h^n}_{n\text{-handles}}$$

rel A. Given $i \in \{-1, \ldots, n\}$ we denote by M^i the i-skeleton of the relative handle decomposition and we set $B_i := \partial M^i \setminus A$. It suffices to prove the following claim.

Claim. For $i = -1, \ldots, n$ there exists a Morse function f_i on M^i with the following properties:

(1) We have $f|_A \equiv -\frac{1}{2}$ and $f|_{B_i} \equiv i + \frac{1}{2}$.
(2) If P is a critical point of f_i, then $f_i(P) = \operatorname{index}(f_i, P)$.

Proof. For $i = -1$ we have $M^i = [0,1] \times A$ and we choose the map that is given by $(t, a) \mapsto t - \frac{1}{2}$. Now suppose the claim holds for some $i \in \{-1, \ldots, n-1\}$. By the Handles-to-Morse Function Proposition 139.9 we can extend the map $f_i\colon M^i \to \mathbb{R}$ to a Morse function on $g\colon M^{i+1} \to \mathbb{R}$ such that g has precisely r_{i+1} extra critical points, and each of these critical points has index $i+1$. Note that the value of g on these critical points is necessarily greater than $i + \frac{1}{2}$ and that the value of g on each $S^{n-k-1} \times B^k$ is greater than $i + \frac{1}{2}$. By rescaling, see Exercise 137.4, we can arrange that the value of g at each of these critical points is precisely $i + 1$ and we can arrange that the value of g on each $S^{n-k-1} \times B^k$ is greater than $i + \frac{1}{2}$. The values of g on B_i now lie in $[i + \frac{1}{2}, i + \frac{3}{2})$ and g points outward on B_i. Using a collar neighborhood for B_i together with Lemma 139.5 it is now fairly straightforward to arrange that the map g actually equals $i + \frac{3}{2}$ on B_i. As always, the adventurous reader is warmly invited to fill in the details. ⊞

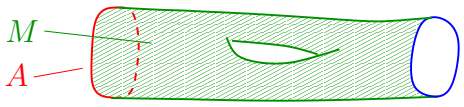
For full details we refer to the original proof by Stephen Smale [**Sma1961b**, Theorem C]. Alternatively the statement is also proved in [**Miln1965b**, p. 45] and [**Nic2011**, Chapter 2.4]. ■

139.5. **Handle decompositions of topological manifolds.** The notion of a handle decomposition also makes sense, with a few suitable modifications in the definition, for compact topological manifolds. So the question arises, does every compact topological manifold admit a handle decomposition? It turns out that this is a subtle question.

(1) By Theorem **??** every compact topological manifold of dimension ≤ 3 admits a smooth structure. Thus it follows from Theorem 139.7 that every such topological manifold admits a handle decomposition.
(2) Rob Kirby and Larry Siebenmann [**KSi1977**, Essay III.2, Theorem 2.1] (see also [**Kupe2017b**, Theorem 1.22]) showed that every compact topological manifold of dimension ≥ 6 admits a topological handle decomposition. Later Frank Quinn [**Qu1982**, Theorem 2.3.1] extended this result to the 5-dimensional setting.
(3) It follows from Proposition **??** that every closed 4-dimensional topological manifold that admits a topological handle decomposition also admits a smooth structure. But in Theorem 135.9 we saw that there exist closed 4-dimensional topological manifolds that do not admit a smooth structure. In particular there exist closed 4-dimensional topological manifolds that do not admit a handle decomposition.

Exercises for Chapter 139.

Exercise 139.1. In the figure below we show once again M, the torus with two disks removed. Let A be one of the two boundary components. In the figure below we showed that M admits a handle decomposition rel A with one 0-handle, three 1-handles and one 2-handle. Does there exist a handle decomposition of M rel A with fewer handles?



Exercise 139.2. Let M be a compact connected non-empty smooth manifold that is equipped with a handle decomposition. On page 3004 we saw that the double $\mathrm{D}M$ comes with a canonical handle decomposition. For which dimensions of M can you use this fact to argue that the inclusion induced map $\pi_1(M) \to \pi_1(\mathrm{D}M)$ is an isomorphism?

Exercise 139.3. Give an explicit self-indexing Morse function for the torus $S^1 \times S^1$ and the Klein bottle.



140. Handle homology and the Morse inequalities

In this chapter we will see that a handle decomposition of a compact smooth manifold gives rise to a chain complex, namely the handle chain complex. This chain complex can be viewed as an analogue of the cellular chain complex of a CW-complex. We will see that the corresponding homology groups, called handle homology groups, are naturally isomorphic to the usual singular homology groups. We will use this isomorphism to prove the Morse inequalities, which show that the "size" of singular homology groups gives a lower bound on the number of singularities of a Morse function on a given smooth manifold.

140.1. **Handle homology.** In Chapter 80 we introduced the cellular chain complex of a CW-complex X. Recall that this chain complex has the following two features:

(1) The chain groups are defined to be $\mathrm{C}_k^{\mathrm{CW}}(X) := \mathrm{H}_k(X^k, X^{k-1})$. By the Skeleton-Homology Lemma 80.1 this chain group can be viewed as the free abelian groups on the set of k-cells.
(2) The corresponding cellular homology groups are naturally isomorphic to the singular homology groups.

In this section we will outline how one can perform something very similar in the setting of smooth manifolds with a given handle decomposition. The following lemma is an analogue of the Skeleton-Homology Lemma 80.1.

Lemma 140.1. (Handle Skeleton-Homology Lemma) Let M be a compact n-dimensional smooth manifold together with a standard handle decomposition

$$M \cong \varnothing \underbrace{\cup_{\varphi_{0,1}} h^0 \cdots \cup_{\varphi_{0,r_0}} h^0}_{0\text{-handles}} \cup \ldots \underbrace{\cup_{\varphi_{n,1}} h^n \cdots \cup_{\varphi_{n,r_n}} h^n}_{n\text{-handles}}.$$

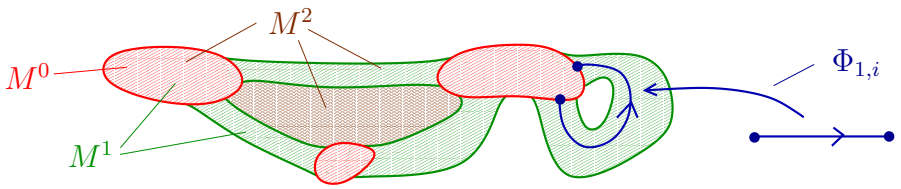
Let $k \in \mathbb{N}_0$. As on page 2958 we denote by M^k the k-skeleton of the handle decomposition, i.e. M^k is union of all handles of index $\leq k$.[1676]

(1) Let $k \in \{0, \ldots, n\}$. Given $i \in \{1, \ldots, r_i\}$ we denote by $\Phi_{k,i} \colon \overline{B}_i^k \to \{0\} \times \overline{B}_i^k \to M^k$ the map that is given by the inclusion map of the core of the i-th k-handle. For every $k, j \in \mathbb{N}_0$ the following map is an isomorphism:[1677]

$$\bigoplus_{i=1}^{r_i} \Phi_{k,i*} \colon \bigoplus_{i=1}^{r_i} \mathrm{H}_j(\overline{B}_i^k, \partial \overline{B}_i^k) \longrightarrow \mathrm{H}_j(M^k, M^{k-1}).$$

In particular the following holds:
   (a) For $i \neq k$ we have $\mathrm{H}_i(M^k, M^{k-1}) = 0$.
   (b) The group $\mathrm{H}_k(M^k, M^{k-1})$ is a free abelian group where the rank equals the number of k-handles.
(2) For $i > k$ we have $\mathrm{H}_i(M^k) = 0$.
(3) For $i < k$ the inclusion $j \colon M^k \to M$ induces an isomorphism $j_* \colon \mathrm{H}_i(M^k) \xrightarrow{\cong} \mathrm{H}_i(M)$.
(4) The inclusion $j \colon M^k \to M$ induces an epimorphism $j_* \colon \mathrm{H}_k(M^k) \twoheadrightarrow \mathrm{H}_k(M)$.

[1676] Note that this definition makes sense since the handle decomposition is standard.
[1677] Note that by definition of a standard handle decomposition, see page 2958, the restriction of $\Phi_{k,i}$ to $\partial \overline{B}_i^k$ does indeed take values in M^{k-1}.



Remark. The statement of the Handle Skeleton-Homology Lemma 140.1 (3) and (4) is very similar in spirit to the content of Exercise 136.12.

Proof. Not surprisingly the proof of the lemma is very similar to the proof of the Skeleton-Homology Lemma 80.1. But completeness sake let us go through the motions.

(1) We consider the following diagram

$$\begin{array}{ccc} \bigoplus_{i=1}^{r_i} \mathrm{H}_k(\overline{B}_i^k, \partial \overline{B}_i^k) & \xrightarrow{\oplus \Phi_{k,i*}} & \mathrm{H}_k(M^k, M^{k-1}) \\ \downarrow & & \uparrow \\ \bigoplus_{i=1}^{r_i} \mathrm{H}_k(\overline{B}^{n-k} \times \overline{B}_i^k, \overline{B}^{n-k} \times \partial \overline{B}_i^k) & \xrightarrow{\oplus \Phi_{k,i*}} & \bigoplus_{i=1}^{r_i} \mathrm{H}_k(\Phi_{k,i}(\overline{B}^{n-k} \times \overline{B}_i^k), \Phi_{k,i}(\overline{B}^{n-k} \times \partial \overline{B}_i^k)). \end{array}$$

The left vertical map is induced by the inclusions $x \mapsto (0, x)$. It is an isomorphism by Corollary 74.16. The bottom horizontal map is induced by homeomorphisms, thus it is also an isomorphism. Next note that the right vertical map is induced by the obvious inclusions. It follows almost immediately from the Excision Theorem 120.2 that the right vertical map is an isomorphism. Clearly he diagram clearly commutes. Hence we get that the top horizontal map is indeed an isomorphism. Finally the "in particular" statements now follow immediately from Lemma 76.1.

(2) We fix $i \in \mathbb{N}$. We want to show that $\mathrm{H}_i(M^k) = 0$ for $k = 0, \ldots, i-1$. First of all we have $\mathrm{H}_i(M^0) = 0$ since M^0 is diffeomorphic to a disjoint union of n-balls. Now assume that we already know that $\mathrm{H}_i(M^k) = 0$ for some $k < i-1$. We want to show that $\mathrm{H}_i(M^{k+1}) = 0$. We consider the long exact sequence of the pair (M^{k+1}, M^k):

$$\ldots \to \mathrm{H}_{i+1}(M^{k+1}, M^k) \to \underbrace{\mathrm{H}_i(M^k)}_{=0} \to \mathrm{H}_i(M^{k+1}) \to \underbrace{\mathrm{H}_i(M^{k+1}, M^k)}_{=0 \text{ by (1) since } i>k+1} \to \ldots$$

Thus we obtain that indeed $\mathrm{H}_i(M^{k+1}) = 0$.

(3) We fix $i \in \mathbb{N}_0$. Let $m > i$. This time we consider the following excerpt from the long exact sequence of the pair (M^{m+1}, M^m):

$$\ldots \to \underbrace{\mathrm{H}_{i+1}(M^{m+1}, M^m)}_{=0 \text{ by (1) since } i<m} \to \mathrm{H}_i(M^m) \to \mathrm{H}_i(M^{m+1}) \to \underbrace{\mathrm{H}_i(M^{m+1}, M^m)}_{=0 \text{ by (1) since } i<m} \to \ldots$$

This discussion implies in particular that we have inclusion induced isomorphisms $\mathrm{H}_i(M^k) \xrightarrow{\cong} \mathrm{H}_i(M^{k+1}) \xrightarrow{\cong} \ldots \xrightarrow{\cong} \mathrm{H}_i(M^n) = \mathrm{H}_i(M)$.

(4) In (3) we showed that the inclusion induced map $\mathrm{H}_k(M^{k+1}) \to \mathrm{H}_k(M)$ is an isomorphism. Thus it suffices to show that the inclusion induced map $\mathrm{H}_k(M^k) \to \mathrm{H}_k(M^{k+1})$ is an epimorphism. To show this we consider the following excerpt from long exact

sequence of the pair (M^{k+1}, M^k):

$$\ldots \to \mathrm{H}_{k+1}(M^{k+1}, M^k) \to \mathrm{H}_k(M^k) \to \mathrm{H}_k(M^{k+1}) \to \underbrace{\mathrm{H}_k(M^{k+1}, M^k)}_{=0 \text{ by (1)}} \to \ldots$$

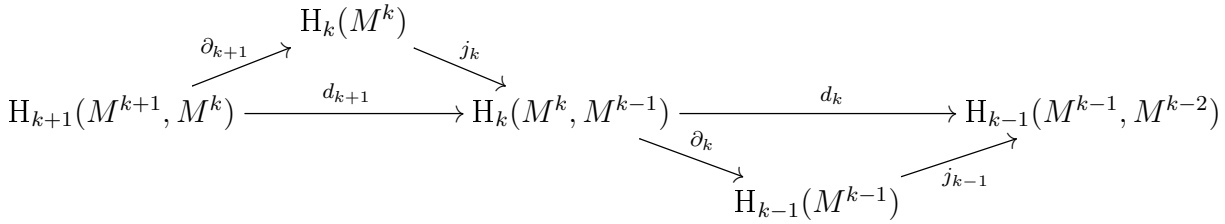
We see that the inclusion induced map $\mathrm{H}_k(M^k) \to \mathrm{H}_k(M^{k+1})$ is indeed an epimorphism. ■

Lemma 140.2. Let M be compact smooth manifold together with a standard handle decomposition. Given $k \in \mathbb{N}_0$ we denote by $d = d_k$ the map

$$\mathrm{H}_k(M^k, M^{k-1}) \xrightarrow{\partial_k} \mathrm{H}_{k-1}(M^{k-1}) \xrightarrow{j_{k-1}} \mathrm{H}_{k-1}(M^{k-1}, M^{k-2})$$
$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{=:d=d_k}$$

where ∂_k is the connecting map in the long exact sequence of the pair (M^k, M^{k-1}) and where j_{k-1} is the map induced by the projection $\mathrm{C}_{k-1}(M^{k-1}) \to \mathrm{C}_{k-1}(M^{k-1}, M^{k-2})$. These maps have the property that for every $k \in \mathbb{N}_0$ we have $d_k \circ d_{k+1} = 0$.

Proof. This proof is easy. We just need to copy-paste the proof of Lemma 80.2. Thus we consider the following commutative diagram of maps



The map $\partial_k \circ j_k \colon \mathrm{H}_k(M^k) \to \mathrm{H}_{k-1}(M^{k-1})$ is the composition of two successive maps in the long exact sequence of the pair (M^k, M^{k-1}), i.e. the map $\partial_k \circ j_k$ is the zero map. Since the diagram commutes we obtain that $d_k \circ d_{k+1} = 0$. ■

Definition. Let M be compact smooth manifold together with a standard handle decomposition.

(1) We refer to the map d_k from Lemma 140.2 as the handle boundary map.
(2) We write $\mathrm{C}_k^{\mathrm{han}}(M) := \mathrm{H}_k(M^k, M^{k-1})$ and we refer to $(\mathrm{C}_*^{\mathrm{han}}(M), d_*)$ as the handle chain complex.[1678]
(3) We denote the homology groups of the handle chain complex by $\mathrm{H}_k^{\mathrm{han}}(M)$ and we refer to these groups as the handle homology groups of the smooth manifold M with the given handle decomposition.

In the Singular-Cellular H_*-Isomorphism Proposition 80.4 we showed that for a CW-complex the singular and cellular homology groups are naturally isomorphic. The following proposition proves the analogous statement in our present setting.

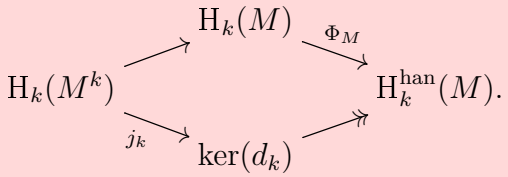
Proposition 140.3. (Singular-Handle H_*-Isomorphism Proposition) Let M be compact smooth manifold together with a standard handle decomposition. Given any

[1678] It follows from Lemma 140.2 that $(\mathrm{H}_k(M^k, M^{k-1}), d_k)_{k \in \mathbb{N}_0}$ is indeed a chain complex.

$k \in \mathbb{N}_0$ there exists a uniquely determined natural[1679] isomorphism

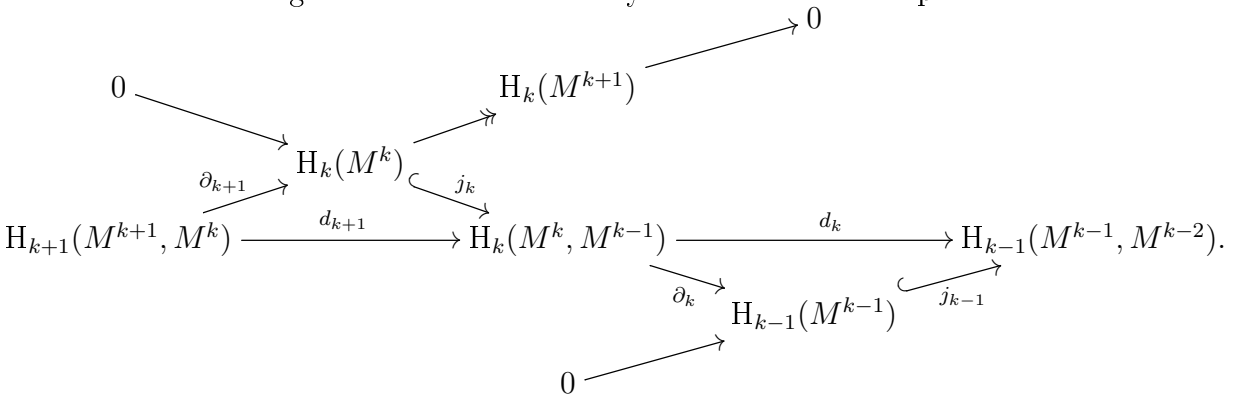
$$\Phi_M \colon \mathrm{H}_k(M) \xrightarrow{\cong} \mathrm{H}_k^{\mathrm{han}}(M)$$

with the property that the following diagram commutes:[1680]

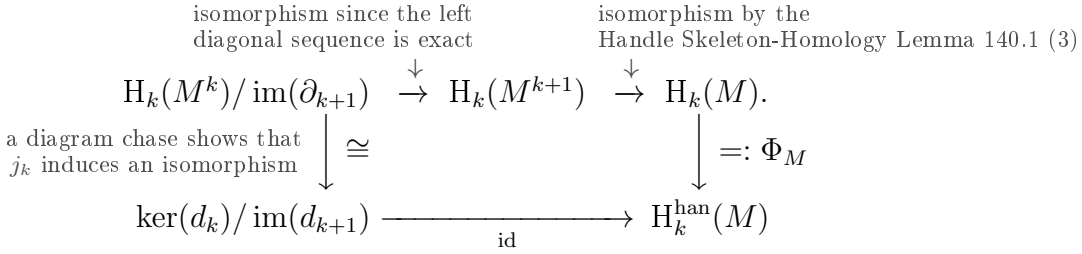


In particular, in light of the above and the Homotopy Equivalence-H_*-Corollary 73.9, the isomorphism type of the handle homology groups depends only on the homotopy type of M.

Proof of the Singular-Handle H_*-Isomorphism Proposition 140.3. The proof of the Singular-Handle H_*-Isomorphism Proposition 140.3 is almost embarrassingly similar to the proof of the Singular-Cellular H_*-Isomorphism Proposition 80.4 (1). For the reader's convenience we nonetheless provide an abbreviated version of the argument. As in the proof of the Singular-Cellular H_*-Isomorphism Proposition 80.4 (1) we add a couple of maps to the commutative diagram that we had already encountered in the proof of Lemma 140.2:



Recall that by the Handle Skeleton-Homology Lemma 140.1 (1) and (2) we know that $\mathrm{H}_k(M^{k+1}, M^k) = 0$, $\mathrm{H}_k(M^{k-1}) = 0$ and $\mathrm{H}_{k-1}(M^{k-2}) = 0$. It follows almost immediately from these facts that the diagonal sequences in the above diagram are exact. Next we consider the diagram:



[1679] We leave it to the reader to figure out what "natural" should mean in this context.

[1680] Here we again denote by $j_k \colon \mathrm{H}_k(M^k) \to \mathrm{H}_k(M^k, M^{k-1})$ the obvious map. In the long exact sequence of the pair (M^k, M^{k-1}) we see that the image of $\mathrm{H}_k(M^n)$ in $\mathrm{H}_k(M^k, M^{k-1})$ lies in the kernel of the connecting homomorphism ∂_k. This implies that the map $j_k \colon \mathrm{H}_k(M^k) \to \mathrm{H}_k(M^k, M^{k-1})$ takes values in $\ker(d_k)$.

Next note that by construction the isomorphism Φ_M has the property that the following diagram commutes:

$$\begin{array}{ccccc} & & \mathrm{H}_k(M) & & \\ & \nearrow\!\!\!\!\!\rightarrow & & \searrow^{\Phi_M} & \\ \mathrm{H}_k(M^k) & & & & \mathrm{H}_k^{\mathrm{han}}(M). \\ & \searrow_{j_k} & & \nearrow\!\!\!\!\!\rightarrow & \\ & & \ker(d_k) & & \end{array}$$

Note that the top-left diagonal map is an epimorphism by the Handle Skeleton-Homology Lemma 140.1 (4). This implies immediately that Φ_M is uniquely determined.

Finally we leave it to the reader who figured out what it means for the isomorphism to be "natural" to actually show that it is natural. ■

In the next section we will discuss a practical approach to calculating the boundary maps in the handle chain complex. But even without this knowledge we can already obtain interesting consequences of the Singular-Handle H_*-Isomorphism Proposition 140.3. For example we can prove the following corollary.[1681]

Corollary 140.4. (Handles-Euler Characteristic Corollary) Let M be a compact smooth manifold. Given any handle decomposition for M the following equality holds:

$$\sum_{k\in\mathbb{N}_0} (-1)^k \cdot \text{number of } k\text{-handles} \;=\; \chi(M).$$

Proof. Let M be a compact smooth manifold together with a handle decomposition. Given $k \in \mathbb{N}_0$ we denote by c_k the number of k-handles. By the Standard Handle Decomposition Corollary 136.4 we can find a *standard* handle structure on M such that for every $k \in \mathbb{N}_0$ the number of k-handles also equals c_k. Using this new handle decomposition we perform the following calculation:

$$\begin{array}{rcccl} & \text{see definition on page 1882} & & \text{Singular-Handle } \mathrm{H}_*\text{-Isomorphism Proposition 140.3} & \\ \chi(M) & \overset{\downarrow}{=} & \displaystyle\sum_{k\in\mathbb{N}_0} (-1)^k \cdot \mathrm{rk}(\mathrm{H}_k(M)) & \overset{\downarrow}{=} & \displaystyle\sum_{k\in\mathbb{N}_0} (-1)^k \cdot \mathrm{rk}(\mathrm{H}_k^{\mathrm{han}}(M)) \\ & \overset{\uparrow}{=} & \displaystyle\sum_{k\in\mathbb{N}_0} (-1)^k \cdot \mathrm{rk}(\mathrm{C}_k^{\mathrm{han}}(M)) & \overset{\uparrow}{=} & \displaystyle\sum_{k\in\mathbb{N}_0} (-1)^k \cdot c_k. \\ & \text{Lemma 87.3} & & \text{Handle Skeleton-Homology Lemma 140.1} & \end{array}$$

■

Next we obtain the following corollary.

Corollary 140.5. (Odd-Dimensional Manifold-Euler Characteristic Corollary) The Euler characteristic of every closed odd-dimensional smooth manifold is zero.

Remark.

(1) In the Odd-Dimensional Manifold-Euler Characteristic Lemma 45.4 we showed that the smooth Euler characteristic of a closed orientable odd-dimensional smooth manifold is zero.
(2) In the Odd-Dimensional Manifold-Euler Characteristic Proposition 119.7 we used Poincaré Duality to prove a generalization of the Odd-Dimensional Manifold-Euler Characteristic Corollary 140.5 to compact odd-dimensional topological manifolds.

[1681] Admittedly we could already have proved that particular corollary earlier on, namely using the Handle Attachment-H_*-Lemma 136.13. This was precisely the content of Exercise 136.10.

Proof. Let M be a closed $(2n+1)$-dimensional smooth manifold. By the Handle Decomposition Theorem 138.5 we can equip M with a handle decomposition. By the Dual Handle Proposition 136.5 we can also consider the corresponding dual handle decomposition. Now we see that

$$\begin{aligned}\chi(M) &\overset{\text{Handles-Euler Characteristic Corollary 140.4}}{\downarrow} \sum_{k=0}^{2n+1}(-1)^k \cdot \begin{array}{c}\text{number of } k\text{-handles in}\\ \text{the handle decomposition}\end{array} \overset{\text{by definition of the dual handle decomposition}}{\downarrow} \sum_{k=0}^{2n+1}(-1)^k \cdot \begin{array}{c}\text{number of } (2n+1-k)\text{-handles in}\\ \text{the dual handle decomposition}\end{array}\\ &= \underset{\text{substitution } m=2n+1-k}{\uparrow} -\sum_{m=0}^{2n+1}(-1)^m \cdot \begin{array}{c}\text{number of } m\text{-handles in the}\\ \text{dual handle decomposition}\end{array} = \underset{\text{Handles-Euler Characteristic Corollary 140.4}}{\uparrow} -\chi(M).\end{aligned}$$

Thus we see that $\chi(M) = 0$. ■

We continue with an even more pleasing corollary.

Corollary 140.6. (Euler Characteristic-via-Critical Points Corollary) Let M be a closed smooth manifold. For any Morse function $f\colon M \to \mathbb{R}$ we have

$$\sum_{k\in\mathbb{N}_0}(-1)^k \cdot \#\text{critical points of } f \text{ of index } k \;=\; \chi(M).$$

Remark.

(1) For $M = S^2$ the statement of the Euler Characteristic-via-Critical Points Corollary 140.6 implicitly appears already in a paper written by Clerk Maxwell [**Max1870**] in 1870 with the beautiful title "on hills and dales".
(2) In Exercise 140.13 we will see that the Euler Characteristic-via-Critical Points Corollary 140.6 can also be deduced from the Poincaré-Hopf Theorem 129.7.

Proof. Let $f\colon M \to \mathbb{R}$ be a Morse function on a closed smooth manifold. It follows from the Morse Function-Handle Decomposition Proposition 138.4 that M admits a handle decomposition such that for each $k \in \mathbb{N}_0$ the number of k-handles equals the number of critical points of index k. The desired equality now follows immediately from the Handles-Euler Characteristic Corollary 140.4. ■

Corollary 129.5. For every closed smooth manifold M the Euler characteristic $\chi(M)$ equals the smooth Euler characteristic $\chi^{\mathrm{sm}}(M)$ of closed orientable smooth manifolds that we had introduced on page 1019.

Proof. Let M be a closed smooth manifold. In the following we first recall the proof of the Non-Degenerate Vector Field Existence Proposition 129.4 that we gave on page 2983. By the Euclidean Embedding Theorem 27.1 we can assume that M is a smooth submanifold of some $\mathbb{R}^n$. By the Morse Function Existence Proposition 137.5 we know that M admits a Morse function $f\colon M \to \mathbb{R}$. It follows from the Gradient Vector Field Proposition 137.8 that the gradient vector field ∇f, that we defined on page 2982, is a vector field such that all zeros are non-degenerate.[1682]

[1682] We refer to page 2982 for the definition of a non-degenerate zero of a vector field and of the index of a non-degenerate zero of a vector field.

Now that we have repeated the argument on page 2983 we can easily prove the corollary, namely we see that

$$\begin{array}{ccccccc} \chi^{\mathrm{sm}}(M) & = & \sum\limits_{x \text{ a zero of } \nabla f} \operatorname{index}(\nabla f, x) & = & \sum\limits_{\substack{x \text{ a critical} \\ \text{point of } f}} \operatorname{index}(f, x) & = & \chi(M). \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{Smooth Poincaré–Hopf} & & \text{Gradient Vector Field} & & \text{Euler Characteristic-via-Critical} & & \\ \text{Theorem 45.17} & & \text{Proposition 137.8} & & \text{Points Corollary 140.6} & & \end{array}$$

∎

Recall that we had formulated the Smooth Poincaré–Hopf Theorem 45.17 in terms of the smooth Euler characteristic. We can now formulate it in terms of the usual Euler characteristic:

Theorem 140.7. (Smooth Poincaré–Hopf Theorem) Let M be a closed orientable smooth manifold and let v be a smooth vector field on M such that all zeros of v are non-degenerate. Then we have the equality

$$\sum_{x \text{ a zero of } v} \operatorname{index}(v, x) \;=\; \chi(M).$$

Proof. As we just mentioned in the proof of Corollary 129.5, the Smooth Poincaré–Hopf Theorem 45.17 contains precisely the desired statement, except that it is formulated with the smooth Euler characteristic $\chi^{\mathrm{sm}}(M)$. Fortunately, thanks to Corollary 129.5 we now know that the smooth Euler characteristic $\chi^{\mathrm{sm}}(M)$ equals the usual Euler characteristic $\chi(M)$. ∎

140.2. **The boundary maps in the handle chain complex.** The following proposition is a useful tool for calculating the boundary maps in a given handle chain complex.

Proposition 140.8. (Handle Chain Complex-Boundary Map Proposition) Let M be an oriented n-dimensional smooth manifold together with a standard handle decomposition

$$M \cong \varnothing \underbrace{\cup_{\varphi_{0,1}} h^0 \cdots \cup_{\varphi_{0,r_0}} h^0}_{0\text{-handles}} \cup \ldots \underbrace{\cup_{\varphi_{n,1}} h^n \cdots \cup_{\varphi_{n,r_n}} h^n}_{n\text{-handles}}.$$

We assume that the inclusion maps of all the handles are orientation-preserving. Given $k \in \mathbb{N}_0$ and given $i \in \{1, \ldots, r_i\}$ we denote by $\Phi_{k,i} \colon \{0\} \times \overline{B}^k_i \to M^k$ the map that is given by the inclusion map of the core of the handle. By the Handle Skeleton-Homology Lemma 140.1 we know that

$$\mathcal{B}_k \;:=\; \{\Phi_{k,1*}([\{0\} \times \overline{B}^k]), \ldots, \Phi_{k,r_k*}([\{0\} \times \overline{B}^k])\}$$

is a basis of $\mathrm{H}^{\mathrm{han}}_k(M) := \mathrm{H}_k(M^k, M^{k-1})$. Given $j \in \{1, \ldots, r_{k+1}\}$ and $i \in \{1, \ldots, r_k\}$ we have the following equality:[1683]

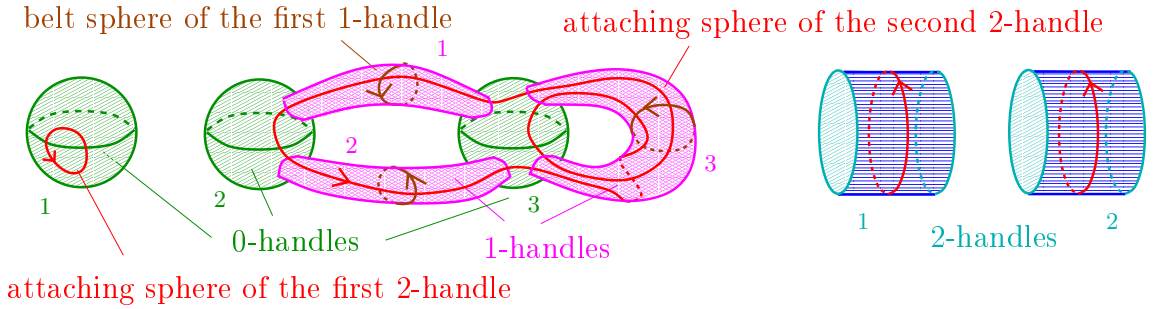
$$\begin{array}{ccc} (i,j)\text{-entry of the matrix representing} \\ d_{k+1} \colon \mathrm{H}^{\mathrm{han}}_{k+1}(M) \to \mathrm{H}^{\mathrm{han}}_k(M) \\ \text{with respect to the bases } \mathcal{B}_{k+1} \text{ and } \mathcal{B}_k \end{array} \quad = \quad \begin{array}{c} \text{algebraic intersection number in } \partial M^k \text{ of} \\ \text{the attaching sphere of the } (k+1,j)\text{-handle} \\ \text{with the belt sphere of the } (k,i)\text{-handle.} \end{array}$$

[1683] For the right-hand side note that ∂M^k is a compact oriented $(n-1)$-dimensional smooth manifold, the attaching sphere of the $(k+1,j)$-handle is an oriented k-dimensional sphere in ∂M^k and the belt sphere of the (k,i)-handle is an oriented $(n-k-1)$-dimensional sphere in ∂M^k. So, according to the definition on page 2757, it makes sense to consider in ∂M^k the algebraic intersection number of the two spheres.

Example. We consider the 3-dimensional smooth manifold M which is shown in the figure below. It has three 0-handles, three 1-handles and two 2-handles. Using the Handle Chain Complex-Boundary Map Proposition 140.8 one sees that the corresponding handle chain complex is isomorphic to the following chain complex:

$$0 \to \begin{matrix}\mathbb{Z}\\ \mathbb{Z}\end{matrix} \xrightarrow{\begin{pmatrix}0 & -1\\ 0 & 1\\ 0 & 2\end{pmatrix}} \begin{matrix}\mathbb{Z}\\ \mathbb{Z}\\ \mathbb{Z}\end{matrix} \xrightarrow{\begin{pmatrix}0 & 0 & 0\\ -1 & 1 & 0\\ 1 & -1 & 0\end{pmatrix}} \begin{matrix}\mathbb{Z}\\ \mathbb{Z}\\ \mathbb{Z}\end{matrix} \to 0. \qquad \text{Thus we obtain that} \qquad \mathrm{H}_k^{\mathrm{han}}(M) \cong \begin{cases} 0, & \text{if } k \geq 3, \\ \mathbb{Z}, & \text{if } k = 2, \\ \mathbb{Z}, & \text{if } k = 1, \\ \mathbb{Z}^2, & \text{if } k = 0. \end{cases}$$

By the Handle Example Lemma 136.2 (2) we know that the component of M that is given by gluing the first 2-handle to the first 0-handle is diffeomorphic to $\overline{B}^1 \times S^2$. But what we can say about the other component of M? We will answer this question on page **??**.



Remark. Let M be an oriented smooth manifold that is equipped with a standard handle decomposition.

(1) If the various attaching spheres and the belt spheres intersect transversally, then we know by the Intersection Number-via-Cup Product Theorem 126.5 that the algebraic intersection number of the attaching spheres and the belt spheres can be determined in a pleasingly geometric way. If they do not intersect transversally, do no despair, since we can iteratively apply the Transversality Theorem 42.4 to find an isotopy of the attaching spheres so that the resulting attaching spheres intersect the belt spheres transversally. By the Handle Chain Complex-Boundary Map Proposition 140.8 (4) these isotopies do not change the diffeomorphism type of the smooth manifold. Put differently, after possibly fiddling a little bit with the handle decomposition we can compute the handle homology groups using intersection numbers of transverse spheres.

(2) In (1) we just argued that the boundary maps in the handle chain complex can be determined geometrically. In [**DGK2019**] this approach is used to define handle homology without referring to singular homology. Furthermore in [**DGK2019**] it is shown using the Cerf Theorem **??**, but again without making use of singular homology groups, that the isomorphism types of these geometrically defined handle homology groups only depend on the diffeomorphism type of the underlying smooth manifold. An outline of the argument is also given in [**GoS1999**, Chapter 4.2].

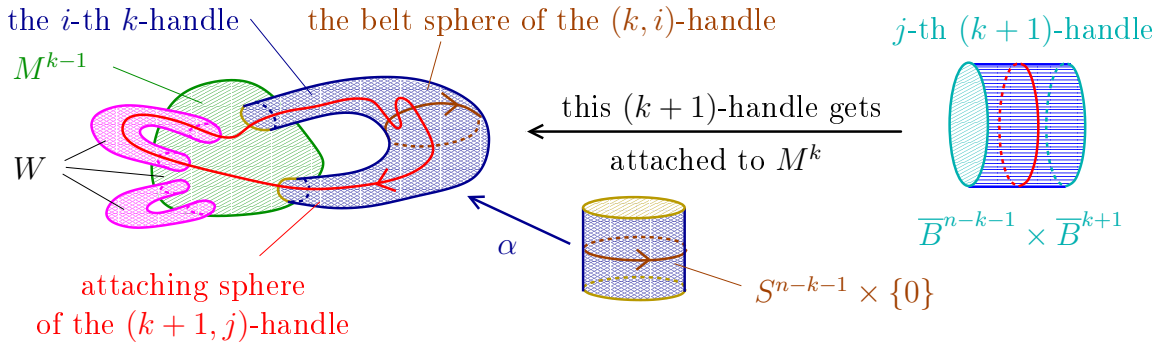
In fact this approach, together with Lemma 136.6, can be used to give an alternative proof of some of the statements of the Poincaré Duality Theorem 119.1. We refer to [**Kos1993**, Theorem VII.5.1] and [**DGK2019**] for details.

Proof. Let M be an oriented n-dimensional smooth manifold together with a standard handle decomposition

$$M \cong \varnothing \underbrace{\cup_{\varphi_{0,1}} h^0 \cdots \cup_{\varphi_{0,r_0}} h^0}_{0\text{-handles}} \cup \ldots \underbrace{\cup_{\varphi_{n,1}} h^n \cdots \cup_{\varphi_{n,r_n}} h^n}_{n\text{-handles}}$$

which has the property that the inclusion maps of all the handles are orientation-preserving. Let $k \in \mathbb{N}_0$, let $j \in \{1, \ldots, r_{k+1}\}$ and let $i \in \{1, \ldots, r_k\}$. We introduce the following notation:

(1) We denote by W the result of attaching to M^{k-1} all k-handles *except* for the i-th k-handle.
(2) We denote by $\alpha\colon \overline{B}^{n-k} \times \overline{B}^k \to M^k$ the embedding corresponding to the (k,i)-handle. Since the handle decomposition is standard we see that $\alpha(\overline{B}^{n-k} \times S^{k-1}) \subset M^{k-1} \subset W$. Furthermore note that $\alpha(S^{n-k-1} \times \{0\})$ is the belt sphere of the (k,i)-handle.
(3) Let $\beta\colon \overline{B}^{n-k-1} \times \overline{B}^{k+1} \to M^{k+1}$ be the embedding corresponding to the $(k+1,j)$-handle. Note that $\beta(\{0\} \times S^k)$ is the attaching sphere of the $(k+1,j)$-handle. Since we are dealing with a standard handle decomposition we see that $\beta(\{0\} \times S^k) \subset \partial M^k$.



Next we consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}_{k+1}(\{0\}\times\overline{B}^{k+1}, \{0\}\times S^k) & \xrightarrow{\beta_*} & \mathrm{H}_{k+1}(M^{k+1}, M^k) & \xrightarrow{\partial_{k+1}} & \mathrm{H}_k(M^k) \\
\downarrow{\scriptstyle \partial_{k+1}} & & & & \downarrow \\
\mathrm{H}_k(\{0\}\times S^k) & \xrightarrow{\beta_*} & \mathrm{H}_k(M^k, W) & \longleftarrow & \mathrm{H}_k(M^k, M^{k-1}) \\
\downarrow{\scriptstyle \beta_*} & \nearrow & \uparrow{\scriptstyle h_*}\ \cong\ \downarrow{\scriptstyle h_*^{-1}} & & \\
\mathrm{H}_k(\partial M^k, \partial M^k \setminus \alpha(S^{k-1}\times B^{n-k})) & & & & \\
\uparrow{\scriptstyle h_*}\ \cong & & & & \\
\mathrm{H}_k(\alpha(S^{n-k-1}\times\overline{B}^k), \alpha(S^{n-k-1}\times S^{k-1})) & \longrightarrow & \mathrm{H}_k(\alpha(\overline{B}^{n-k}\times\overline{B}^k), \alpha(\overline{B}^{n-k}\times S^{k-1})) & & \\
\uparrow{\scriptstyle \alpha_*}\ \cong & & \downarrow{\scriptstyle \alpha_*^{-1}} & & \\
\mathrm{H}_k(S^{n-k-1}\times\overline{B}^k, S^{n-k-1}\times S^{k-1}) & \longrightarrow & \mathrm{H}_k(\overline{B}^{n-k}\times\overline{B}^k, \overline{B}^{n-k}\times S^{k-1}) & & \\
\downarrow{\scriptstyle \sigma\mapsto Q([S^{n-k-1}\times\{0\}],\sigma)} & & \downarrow{\scriptstyle \sigma\mapsto \widetilde{Q}([\overline{B}^{n-k}\times\{0\}],\sigma)} & \searrow{\scriptstyle n\cdot[\{0\}\times\overline{B}^k]\mapsto n} & \\
\mathbb{Z} & \xrightarrow[\mathrm{id}]{} & \mathbb{Z} & \xrightarrow[\mathrm{id}]{} & \mathbb{Z}.
\end{array}$$

We make the following clarifications and observations:

(1) By a slight abuse of notation we denote all of the following inclusion maps by h:

$$\begin{aligned} h\colon \alpha(\overline{B}^{n-k}\times\overline{B}^{k}) &\to M^k \\ h\colon (\alpha(\overline{B}^{n-k}\times\overline{B}^{k}),\alpha(\overline{B}^{n-k}\times S^{k-1})) &\to (M^k,W) \\ h\colon (\alpha(S^{n-k-1}\times\overline{B}^{k}),\alpha(S^{n-k-1}\times S^{k-1})) &\to (\partial M^k,\partial M^k\setminus\alpha(S^{k-1}\times B^{n-k})). \end{aligned}$$

(2) It follows from the Excision Theorem 120.2 that the maps

$$\begin{aligned} h_*\colon \mathrm{H}_k(\alpha(\overline{B}^{n-k}\times\overline{B}^{k}),\alpha(\overline{B}^{n-k}\times S^{k-1})) &\to \mathrm{H}_k(M^k,W) \quad \text{and} \\ h_*\colon \mathrm{H}_k(\alpha(S^{n-k-1}\times\overline{B}^{k}),\alpha(S^{n-k-1}\times S^{k-1})) &\to \mathrm{H}_k(\partial M^k,\partial M^k\setminus\alpha(S^{k-1}\times B^{n-k})) \end{aligned}$$

are isomorphisms.

(3) We consider the two asymmetric intersection pairings

$$\begin{aligned} Q\colon \mathrm{FH}_{n-k-1}(S^{n-k-1}\times\overline{B}^{k})\times\mathrm{FH}_k(S^{n-k-1}\times\overline{B}^{k},S^{n-k-1}\times S^{k-1}) &\to \mathbb{Z} \\ \widetilde{Q}\colon \mathrm{FH}_{n-k}(\overline{B}^{n-k}\times\overline{B}^{k},S^{n-k-1}\times\overline{B}^{k})\times\mathrm{FH}_k(\overline{B}^{n-k}\times\overline{B}^{k},\overline{B}^{n-k}\times S^{k-1}) &\to \mathbb{Z} \end{aligned}$$

as defined on page 2895.

(4) The undecorated maps are the obvious maps induced by inclusions.

(5) It follows from the naturality of the connecting homomorphism that the top rectangle in the diagram commutes.

(6) The regions between the second and the fifth row commute by the functoriality of relative homology groups.

(7) It follows from the Interaction Pairing-Boundary Proposition 132.17 that the bottom left square commutes.

(8) Finally note that it follows from the Poincaré Duality-Products Proposition 119.2 that $Q([\overline{B}^{n-k}\times\{0\}],[\{0\}\times\overline{B}^{k}])=1$. It follows from this observation that the bottom right triangle commutes.

We denote by $\Xi\colon \mathrm{H}_{k+1}(\{0\}\times\overline{B}^{k+1},\{0\}\times S^k)\to\mathbb{Z}$ the composition of the blue maps, i.e. the composition of the maps on right-hand path. We can now prove the desired equality:

$$\begin{array}{ll} & \text{this equality holds basically by definition} \\ (i,j)\text{-entry of the matrix representing } d_{k+1} & \downarrow = \Xi([\{0\}\times\overline{B}^{k+1}]) \\ \text{intersection form of } S^{n-k-1}\times\overline{B}^{k} & \text{intersection form of } \alpha(S^{n-k-1}\times\overline{B}^{k}) \\ \downarrow = Q([S^{n-k-1}\times\{0\}],\alpha_*^{-1}(h_*^{-1}([\beta(\{0\}\times S^k)]))) & \downarrow = Q([\alpha(S^{n-k-1}\times\{0\})],h_*^{-1}([\beta(\{0\}\times S^k)])) \\ \uparrow & \uparrow \\ \text{by the commutative diagram and since by the Fundamental Class-of-Boundary Proposition 106.27 we have } \partial([\{0\}\times\overline{B}^{k+1}])=[\{0\}\times S^k] & \text{since } \alpha \text{ is an orientation-preserving embedding} \\ = Q_{\partial M^k}([\underbrace{\alpha(S^{n-k-1}\times\{0\})}_{\text{belt sphere}}],[\underbrace{\beta(\{0\}\times S^k)}_{\text{attaching sphere}}]). & \\ \uparrow & \\ \text{by the Intersection Form-Submanifold Proposition 132.19} & \end{array}$$

This is precisely the equality that we had intended to show. ■

140.3. **The Morse inequalities.** Given a topological space X or a smooth manifold M we show in this section that the "size" of homology groups gives rise to lower bounds on the number of cells of a CW-structure for X or number of handles in a handle decomposition of M. Furthermore we will see that the homology groups give lower bounds on the number of non-degenerate critical points of a Morse function on M.
The key to proving these results is some charming algebra. To formulate the main algebraic statement we need to introduce the following notation.

Notation. Given a chain complex C_* and given $j \in \mathbb{N}_0$ we denote by $b_j(C) \in \mathbb{N}_0 \cup \{\infty\}$ the rank of $\mathrm{H}_j(C)$ and we denote by $q_j(C) \in \mathbb{N}_0 \cup \{\infty\}$ the minimal number of generators of the torsion subgroup $\operatorname{Tor}\mathrm{H}_j(C)$.

Proposition 140.9. Let C_* be a chain complex[1684] of finitely generated free abelian groups.

(a) Given any $k \in \mathbb{N}_0$ we have the inequality

$$\sum_{j=0}^{k}(-1)^j \cdot \operatorname{rank}(\mathrm{C}_{k-j}) \;\geq\; q_k(C) + \sum_{j=0}^{k}(-1)^j \cdot b_{k-j}(C).$$

(b) Given any $k \in \mathbb{N}_0$ we have the inequality

$$\operatorname{rank}(\mathrm{C}_k) \;\geq\; q_k(C) + q_{k-1}(C) + b_k(C).$$

Remark.

(α) (a) Let $k \in \mathbb{N}_0$. If we add Inequality (a) for k and $k-1$, then we obtain precisely Inequality (b). In other words, Proposition 140.9 (b) is a consequence of Proposition 140.9 (a). In Exercise 140.3 we will see that the converse is not true. Put differently, in general the inequalities one obtains from (a) are stronger than the inequalities from (b).

(b) By adding and subtracting equations one can also show easily that the following two statements are equivalent:

(i) For every $k \in \mathbb{N}_0$ the inequality in (a) is an equality.

(ii) For every $k \in \mathbb{N}_0$ the inequality in (b) is an equality.

(β) We refer to [**Far2004**, Chapter 1] for an alternative proof of Proposition 140.9 and for a generalization to chain complexes over more general rings.

The proof of Proposition 140.9 relies on the following elementary lemma.

Lemma 140.10. If $F \xrightarrow{\varphi} G \to H \to 0$ is an exact sequence where F and G are finitely generated free abelian groups, then

$$\operatorname{rank}(F) \;\geq\; \text{minimal number of generators of the torsion subgroup of } H.$$

Proof of Lemma 140.10. We write $m = \operatorname{rank}(F)$ and $n = \operatorname{rank}(G)$. Furthermore we set $k := \min\{m, n\}$. It follows from the Smith Normal form Theorem 51.5 that we can find bases for F and G such that the only entries of the corresponding $(n \times m)$-matrix A representing φ are on the diagonal. We denote by $a_1, \ldots, a_k$ these diagonal entries. Then

[1684] Note that according to our convention on page 1591 a chain complex "starts" at degree 0.

we see that

$$\text{torsion subgroup of } H \;\cong\; \text{torsion subgroup of } \mathbb{Z}^n/A\mathbb{Z}^m \;\cong\; \bigoplus_{a_i\neq 0}\mathbb{Z}_{a_i}.$$

Thus we see that the torsion subgroup of H admits a generating set with at most k elements. Since $k \leq m$ we see that the minimal number of generators of the torsion subgroup of H is $\leq \operatorname{rank}(F)$. ■

Proof of Proposition 140.9.

(a) Let (C_*, ∂_*) be a chain complex of finitely generated free abelian groups. For each $k \in \mathbb{N}_0$ we write $Z_k := \ker(\partial_k \colon C_k \to C_{k-1})$, $B_k := \operatorname{im}(\partial_{k+1} \colon C_{k+1} \to C_k)$ and $H_k = Z_k/B_k = \mathrm{H}_k(C_*)$. Basically by definition we have for each $k \in \mathbb{N}_0$ the short exact sequences

$$\text{(a)}\quad 0 \to B_k \to Z_k \to H_k \to 0 \qquad \text{and} \qquad \text{(b)}\quad 0 \to Z_k \to \mathrm{C}_k \xrightarrow{\partial} B_{k-1} \to 0.$$

Given $k \in \mathbb{N}_0$ we perform the following calculation:

$$\begin{aligned}\sum_{j=0}^{k}(-1)^j\cdot\operatorname{rank}(\mathrm{C}_{k-j}) &\overset{\text{follows from the Rank Additivity Lemma 51.7 (2) applied to the above two short exact sequences}}{=} \sum_{j=0}^{k}(-1)^j\cdot(\operatorname{rank}(H_{k-j})+\operatorname{rank}(B_{k-j})+\operatorname{rank}(B_{k-j-1}))\\ &\underset{\text{all the other terms cancel}}{=} \operatorname{rank}(B_k)+\sum_{j=0}^{k}(-1)^j\cdot\operatorname{rank}(H_{k-j}) \underset{\text{by Lemma 140.10 applied to the exact sequence (a)}}{\geq} q_k(C)+\sum_{j=0}^{k}(-1)^j\cdot\overbrace{\operatorname{rank}(H_{k-j})}^{=b_{k-j}(C)}.\end{aligned}$$

(b) As mentioned above, this inequality follows immediately from adding the inequality (a) for k and $k-1$. Indeed, one sees that almost all of the terms cancel and thus one obtains the promised inequality. ■

Evidently the idea now is to apply Proposition 140.9 to topology. To do so it is convenient to introduce the following definition.

Notation. Given a topological space X and given $j \in \mathbb{N}_0$ we write[1685]

$$b_j(X) := b_j(\mathrm{C}_*(X)) = \operatorname{rank}(\mathrm{H}_j(X)) \qquad \text{and} \qquad q_j(X) := q_j(\mathrm{C}_*(X)).$$

In other words, $q_j(X)$ is the minimal number of generators of the torsion subgroup $\operatorname{Tor}\mathrm{H}_j(X)$.

Proposition 140.11.

(1) Let M be a compact smooth manifold that is equipped with a handle decomposition. Given $k \in \mathbb{N}_0$ we denote by d_k the number of k-handles in the handle decomposition.

(a) Given any $k \in \mathbb{N}_0$ we have the inequality

$$\sum_{j=0}^{k}(-1)^j\cdot d_{k-j} \;\geq\; q_k(M)+\sum_{j=0}^{k}(-1)^j\cdot b_{k-j}(M).$$

(b) Given any $k \in \mathbb{N}_0$ we have the inequality

$$d_k \;\geq\; q_k(M)+q_{k-1}(M)+b_k(M).$$

[1685] On page 1882 we called $b_j(X)$ the j-th Betti number of X.

(2) Let X be a CW-complex with finitely many cells in each dimension. Given $k \in \mathbb{N}_0$ we denote by d_k the number of k-cells. The same inequalities as in (1) hold.

Example. Let $n \in \mathbb{N}_0$. By Proposition 80.11 we know that given any $j \in \mathbb{N}_0$ we have

$$\mathrm{H}_j(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } j = 0 \text{ or if } j = n \text{ and } n \text{ is odd,} \\ \mathbb{Z}_2, & \text{if } j \text{ is odd and } j < n, \\ 0, & \text{if } j \text{ is even and } 0 < j \neq 0 \text{ or if } j > n. \end{cases}$$

It follows almost immediately from this calculation that for any $j \in \{0, \dots, n\}$ we have $q_j(\mathbb{R}\mathrm{P}^n) + q_{j-1}(\mathbb{R}\mathrm{P}^n) + q_{j-1}(\mathbb{R}\mathrm{P}^n) = 1$. By Proposition 140.11 (1b) this implies that any handle decomposition of $\mathbb{R}\mathrm{P}^n$ has at least one handle of each index $0, \dots, n$. Furthermore by Proposition 140.11 (2b) this calculation implies that any CW-decomposition of $\mathbb{R}\mathrm{P}^n$ has at least one cell of each index $0, \dots, n$. In particular this shows that the CW-structure of $\mathbb{R}\mathrm{P}^n$ that we introduced in Lemma 68.1 is "minimal". This observation also gives a solution to Exercise 80.9.

Proof.

(1) Let M be a compact smooth manifold that is equipped with a handle decomposition. Given $k \in \mathbb{N}_0$ we denote by d_k the number of k-handles in the handle decomposition. It follows from the Standard Handle Decomposition Corollary 136.4 that without loss of generality we can assume that the handle decomposition is standard. This allows us to consider the handle chain complex $\mathrm{C}_*^{\mathrm{han}}(M)$. Now we can prove Inequality (a):

$$\sum_{j=0}^{k} (-1)^j \cdot d_{k-j} \overset{\text{by the Handle Skeleton-Homology Lemma 140.1 we have } d_i = \mathrm{rank}(\mathrm{C}_i^{\mathrm{han}}(M))}{\downarrow} = \sum_{j=0}^{k} (-1)^j \cdot \mathrm{rank}(\mathrm{C}_{k-j}^{\mathrm{han}}(M)) \overset{\text{Proposition 140.11}}{\downarrow} \geq q_k(\mathrm{C}_*^{\mathrm{han}}(M)) + \sum_{j=0}^{k} (-1)^j \cdot b_{k-j}(\mathrm{C}_{k-j}^{\mathrm{han}}(M))$$
$$= q_k(M) + \sum_{j=0}^{k} (-1)^j \cdot b_{k-j}(M).$$
$$\uparrow$$
by the Singular-Handle H_*-Isomorphism Proposition 140.3 we know that $\mathrm{H}_j(\mathrm{C}_*^{\mathrm{han}}(M)) \cong \mathrm{H}_j(M)$

Finally, as we pointed out above, Inequality (b) is an immediate consequence of Inequality (a).

(2) Let X be a CW-complex with finitely many cells in each dimension. The proof in this setting is almost identical to the proof in (1), we just need to work with the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(X)$ and we make use of the fact that by the Skeleton-Homology Lemma 80.1 we know that $d_i = \mathrm{rank}(\mathrm{C}_i^{\mathrm{CW}}(X))$ and we use that by the Singular-Cellular H_*-Isomorphism Proposition 80.4 we know that $\mathrm{H}_j(\mathrm{C}_*^{\mathrm{CW}}(X)) \cong \mathrm{H}_j(X)$. ■

Now we can formulate what is arguably the most interesting result of this section, since it connects two initially rather disparate objects.

Theorem 140.12. (Morse Inequalities Theorem) Let M be a closed smooth manifold together with a Morse function $f\colon M \to \mathbb{R}$. Given $k \in \mathbb{N}_0$ we write

$$d_k := \text{number of critical points of index } k.$$

The following two inequalities hold:

(a) Given any $k \in \mathbb{N}_0$ we have the inequality

$$\sum_{j=0}^{k}(-1)^j \cdot d_{k-j} \;\geq\; q_k(M) + \sum_{j=0}^{k}(-1)^j \cdot b_{k-j}(M).$$

(b) Given any $k \in \mathbb{N}_0$ we have the inequality

$$d_k \;\geq\; q_k(M) + q_{k-1}(M) + b_k(M).$$

Examples. Let $n \in \mathbb{N}_0$.

(1) Recall that on page 3023 we just showed that for every $j \in \{0, \ldots, n\}$ we have $q_j(\mathbb{R}\mathrm{P}^n) + q_{j-1}(\mathbb{R}\mathrm{P}^n) + q_{j-1}(\mathbb{R}\mathrm{P}^n) = 1$. By the Morse Inequalities Theorem 140.12 (b) this implies that any Morse function on $\mathbb{R}\mathrm{P}^n$ has at least one critical point of each index $0, \ldots, n$. In particular the Morse function on $\mathbb{R}\mathrm{P}^n$ that we gave on page 2979 has the minimal number of critical points.

(2) On page 1772 we showed that given any $j \in \mathbb{N}_0$ we have

$$\mathrm{H}_j(\mathbb{C}\mathrm{P}^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } j = 0, 2, 4, \ldots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

As in (1) it follows that the Morse function on $\mathbb{C}\mathrm{P}^n$ that we gave on page 2979 has the minimal number of critical points.

(3) Let M be a closed connected non-empty n-dimensional smooth manifold. By the Smooth Manifold-Top Homology Theorem 98.6, or alternatively by the Orientation-Top Homology Theorem 106.1 and Proposition 106.22 we know the following:
(a) If M is orientable, then $\mathrm{H}_n(M) \cong \mathbb{Z}$ and $\mathrm{Tor}(\mathrm{H}_{n-1}(M)) = 0$.
(b) If M is non-orientable, then $\mathrm{H}_n(M) \cong \mathbb{Z}$ and $\mathrm{Tor}(\mathrm{H}_{n-1}(M)) \cong \mathbb{Z}_2$.
Rather charmingly, in either case we see that $q_n(M) + q_{n-1}(M) + b_n(M) = 1$, which matches the fact that any Morse function has to have at least one global maximum, i.e. one critical point of index n.

Proof of Theorem 140.12. Let M be a closed smooth manifold together with a Morse function $f\colon M \to \mathbb{R}$. By the Morse Function-Handle Decomposition Proposition 138.4 we know that M admits a standard handle decomposition of M such that for each $k \in \mathbb{N}_0$ the number of k-handles equals the number of critical points of f of index k. The inequalities now follow from the inequalities of Proposition 140.11 (1). ■

Remark. Let M be a closed smooth manifold and let $f\colon M \to \mathbb{R}$ be a Morse function. Given $j \in \mathbb{N}_0$ we denote by d_j the number of critical points of f of index j. It follows from the Morse Inequalities Theorem 140.12, together with the de Rham Theorem 118.9 and Corollary 110.19 (2), that for every $k \in \mathbb{N}_0$ we have the inequality

$$\sum_{j=0}^{k}(-1)^j \cdot d_{k-j} \;\geq\; \sum_{j=0}^{k}(-1)^j \cdot \dim_{\mathbb{R}}(\underbrace{\mathrm{H}_{\mathrm{dR}}^{k-j}(M)}_{\uparrow}).$$

de Rham cohomology groups of M

This inequality can also proved directly using purely analytic methods, we refer to [**Chan1993**, Appendix] and [**Wit1982**] for details.

The question arises how "sharp" the Morse inequalities are. This leads us to the following definition.

Definition. Let M be a closed smooth manifold.

(1) A Morse function $f\colon M \to \mathbb{R}$ is called perfect if for every $k \in \mathbb{N}_0$ we have

$$\text{number of critical points of index } k \;=\; q_k(M) + q_{k-1}(M) + b_k(M).$$

(2) A handle decomposition is called minimal if for every $k \in \mathbb{N}_0$ we have

$$\text{number of handles of index } k \;=\; q_k(M) + q_{k-1}(M) + b_k(M).$$

Remark. Let M be a closed smooth manifold.

(1) Recall that by Propositions 138.4 and 139.9 we can go back and forth between Morse functions and handle decompositions. In particular we see that M admits a perfect Morse function if and only if M admits a minimal handle decomposition.
(2) Note that by the remark on page 3021 we know that if a Morse function is perfect, then the inequalities stated in the Morse Inequalities Theorem 140.12 (a) are also equalities. The same logic applies to the inequalities on the number of handles stated in Proposition 140.11 (1a).

It is now natural to ask which closed smooth manifolds admit perfect Morse functions. It follows from Corollaries 136.12 and 136.7 that the fundamental group gives additional lower bounds on the number of critical points of a Morse function. Thus in the following discussion we restrict ourselves to the case of simply connected smooth manifolds. So now we have a reasonable question:

Question 140.13. Does every closed simply connected smooth manifold admit a perfect Morse function?

In fact in 1962 Steven Smale [**Sma1962**, Theorem 6.1] proved the following theorem which gives an affirmative answer to Question 140.13 for high-dimensional smooth manifolds.[1686]

Theorem 140.14. Let M be a closed smooth manifold. If M is simply connected and if $\dim(M) \geq 6$, then f admits a perfect Morse function.

Remark. In [**Shark1993**, Chapter VII] Theorem 140.14 gets generalized, in a suitable sense, to certain closed non-simply connected smooth manifolds.

Thus let us turn our gaze towards the low-dimensional case. It follows from the classification of 2-dimensional smooth manifolds, see Theorem 55.4, and the resolution of the Poincaré Conjecture in dimension 3, see Theorem **??**, that Question 140.13 has an affirmative answer in dimensions 2 and 3 for the silly reason that the spheres are the only closed simply connected smooth manifolds in those dimensions. Furthermore for 5-dimensional smooth manifolds it follows from the classification of closed simply connected 5-dimensional smooth manifolds given by Denis Barden in 1965, see [**Bard1965**, Corollary 2.2.2], that Question 140.13 has an affirmative answer.
Finally we turn to the 4-dimensional setting. Since low-dimensional topologists usually prefer to think about handles we switch from Morse functions to handles.

[1686] We will sketch a proof in Exercise **??**.

Lemma 140.15. Let M be a closed orientable simply connected 4-dimensional smooth manifold. A handle decomposition is minimal if and only if it has no 1-handles and no 3-handles.

Proof. Let M be a closed orientable simply connected 4-dimensional smooth manifold. Note that in Exercise 119.19 we saw that it follows easily from the Hurewicz Theorem 84.5, the Non-Orientable $\mathbb{Z}_2$-Epimorphism Corollary 49.8, the Poincaré Duality Theorem 119.1 and the Universal Coefficient Theorem 110.12 for Cohomology Groups that $\mathrm{H}_1(M) = 0$, $\mathrm{H}_3(M) = 0$ and that $\mathrm{H}_2(M)$ is torsion-free. In particular we see that $q_k(M) + q_{k-1}(M) + b_k(M) = 0$ for $k = 1$ and $k = 3$.

Now we turn to the actual proof of the lemma. First note that the above calculation shows that a minimal handle decomposition has no 1-handles and no 3-handles. Conversely, suppose that we have a handle decomposition with no 1-handles and no 3-handles. It follows easily from the Handle-Connected Lemma 136.8 that the handle decomposition contains precisely one 0-handle and, say by considering the dual handle decomposition, we see that the handle decomposition contains precisely one 4-handle. Finally note that

$$\#\text{2-handles} \underset{\substack{\uparrow\\ \text{by the Handles-Euler}\\ \text{Characteristic Corollary 140.4}}}{=} \chi(M) - 2 \underset{\substack{\uparrow\\ \text{since } \mathrm{H}_1(M)=\mathrm{H}_3(M)=0\\ \text{and since } \mathrm{H}_0(M)=\mathrm{H}_4(M)\cong\mathbb{Z}}}{=} b_2(M) \underset{\substack{\uparrow\\ \text{since } q_1(M)=q_2(M)=0}}{=} q_2(M) + q_1(M) + b_2(M).$$

We have thus shown that the handle decomposition is minimal. ■

This leads us to following open problem, see [**Kir1997**, Propblem 4.18].

Question 140.16.

(1) Does every M closed simply connected 4-dimensional smooth manifold admit a handle decomposition without 1-handles?
(2) Does every M closed simply connected 4-dimensional smooth manifold admit a handle decomposition without 1- and 3-handles?

Note that a positive answer to Question 140.16 (2) would imply, by Reeb's Theorem 138.7 that every closed simply connected 4-dimensional smooth manifold M with $\mathrm{H}_2(M) = 0$ is homeomorphic to S^4. By Freedman's Theorem **??** this statement is correct. But this indicates that in all likelihood it will be very difficult to show that Question 140.16 (2) has a positive answer.

Remark. Chapters 138, 139 and 140 contain almost everything that we want to say about Morse theory. More detailed accounts of Morse theory are given in [**Miln1963a, Nic2011, Mata2002, Lau2011, Kos1993, Knu2015**]. There are also many variations on standard Morse theory. We want to mention two of them:

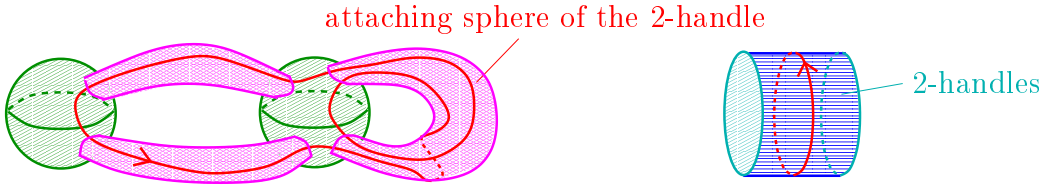
(1) Instead of smooth maps $M \to \mathbb{R}$ one can also consider smooth maps $M \to S^1$. The notion of a Morse function also makes sense, with basically the same definition, in this setting. The corresponding theory is developed in [**Nov1981, Nov1982, Paj2006, Far2004**]. In particular one can use "twisted homology" groups to give lower bounds on the number of critical points of a Morse function $f\colon M \to S^1$. We refer to [**Far2004**, Corollary 3.3] and [**Nov1981, Nov1982**] for details.
(2) Given a CW-complex or an (abstract) simplicial complex Robin Forman [**Form1995, Form1998**] developed the, initially surely counterintuitive, notion of "discrete Morse

theory". We refer to [**Form02, Knu2015, Scov2019**] and [**Koz2008**, Chapter 11] for details.

Exercises for Chapter 140.

Exercise 140.1. Consider the 3-dimensional smooth manifold that is sketched in the figure below.

(a) Determine $\pi_1(M)$.
(b) Make an educated guess what well-known smooth manifold M is diffeomorphic to.



Exercise 140.2.

(a) Let M be a compact orientable connected 2-dimensional topological manifold. Show that $\chi(M) \leq 2$.
Remark. Evidently this follows from the Surface Classification Theorem 55.4. But it would be more elegant to have a proof that does not rely on this result.
(b) Show that given any $k \in \mathbb{Z}_{\leq -2}$ there exists a compact orientable connected 2-dimensional smooth manifold M with $\chi(M) = k$.
(c) Let $n \in \mathbb{N}_{\geq 3}$. Show that given any $k \in \mathbb{Z}$ there exists a compact orientable connected n-dimensional smooth manifold M with $\chi(M) = k$.

Exercise 140.3. Give an example of a chain complex of finitely generated free abelian groups where the inequality of Proposition 140.9 (b) holds for all $k \in \mathbb{N}_0$, but such that the inequality of Proposition 140.9 (a) does not hold for all $k \in \mathbb{N}_0$.

Exercise 140.4. Given a closed connected non-empty smooth manifold M and given a Morse function $f\colon M \to \mathbb{R}$ we denote by $d_k(M, f)$ the number of critical points of f of index k. From the Morse Inequalities Theorem 140.12 and from Corollaries 136.12 and 136.7 we obtain lower bounds on the number of critical points $d_k(M, f)$ of a Morse function f in terms of the homology groups and the fundamental group of M. Show that these lower bounds are not optimal, i.e. show that there exists a closed connected non-empty smooth manifold M that does not admit a Morse function $f\colon M \to \mathbb{R}$ such that these lower bounds are equalities.
Hint. Apply the Morse Inequalities Theorem 140.12 to finite covers of a suitable M. For example you could take M to be the mapping torus $\mathrm{Tor}((S^1)^m, \varphi)$ for a suitable diffeomorphism of a high-dimensional torus $(S^1)^m$.

Exercise 140.5. Let M be a smooth homology 3-sphere, i.e. let M be a 3-dimensional smooth manifold such that for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(M; \mathbb{Z}) \cong \mathrm{H}_k(S^3; \mathbb{Z})$. Show that if $\pi_1(M)$ is non-trivial, then every Morse function on M has at least six critical points.
Remark. In Proposition 98.12 we saw that the Poincaré Homology Sphere is an example of such M.

Exercise 140.6. Let M be a closed smooth homology 5-sphere, i.e. let M be a 5-dimensional smooth manifold such that for ever $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(M;\mathbb{Z}) \cong \mathrm{H}_k(S^5;\mathbb{Z})$. Show that if $\pi_1(M)$ is non-trivial, then every Morse function on M has at least ten critical points.
Remark. In Exercise 140.5 we dealt with the case of smooth homology 3-spheres. Furthermore note that in Exercise 98.13 and in Proposition 141.5 we give examples of such M.

Exercise 140.7. Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. We pick $r, s \in \mathbb{Z}$ such that $qr - ps = -1$. We write

$$A = \begin{pmatrix} q & p \\ s & r \end{pmatrix} \qquad \text{and we consider the orientation-reversing diffeomorphism} \qquad \begin{array}{rcl} \varphi\colon \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ [v] & \mapsto & [Av]. \end{array}$$

Let $\theta\colon \mathbb{R}^2/\mathbb{Z}^2 \to (S^1)^2$ be the usual diffeomorphism that is given by $[(x,y)] \mapsto (e^{2\pi \mathrm{i} x}, e^{2\pi \mathrm{i} y})$. We set

$$M := ((\overline{B}^2 \times S^1) \sqcup (\overline{B}^2 \times S^1))/\sim \qquad \text{where } (x,y) \sim (\theta \circ \varphi \circ \theta^{-1})(x,y) \text{ for } x, y \in S^1 \times S^1.$$

By the Gluing-Smooth Manifolds-Proposition **??** we know that M is a closed 3-dimensional smooth manifold.

(a) Show that M admits a handle decomposition with a unique handle in dimensions 0,1,2 and 3.
(b) Show that for every $n \in \mathbb{N}_0$ we have

$$\mathrm{H}_n^{\mathrm{hand}}(M) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 3, \\ 0, & \text{if } n = 2 \text{ or } n > 3, \\ \mathbb{Z}_p, & \text{if } n = 1. \end{cases}$$

Remark. In the Lens Space-Splitting Lemma 146.2 we will see that M is diffeomorphic to the lens space $L(p,q)$. Thus this exercise together with the Singular-Handle H_*-Isomorphism Proposition 140.3 gives us a new calculation of the singular homology groups of lens spaces.

Exercise 140.8. Let $g, n \in \mathbb{N}_0$. As usual we denote by $\Sigma_{g,n}$ the surface of genus g minus n open disks. What is the minimal number of handles in a handle decomposition of $\Sigma_{g,n}$?

Exercise 140.9. Let $n \in \mathbb{N}$. What is the minimal number k of critical points of a Morse function on the n-dimensional torus $(S^1)^n$? You need to show that there exists a Morse function such that the number of critical points equals k and you need to show that every Morse function has at least k critical points.

Exercise 140.10. Let M be a compact smooth manifold and let A be a union of components of ∂M. Suppose we are given a handle decomposition of M rel A. Define a relative handle chain complex $\mathrm{C}_*^{\mathrm{han}}(M,A)$ such that the corresponding homology groups $\mathrm{H}_*^{\mathrm{han}}(M,A)$ are isomorphic to the relative singular homology groups of (M,A).

Exercise 140.11. Let M be a compact smooth manifold and let A be two a union of components of ∂M. Furthermore let $f\colon M \to [0,1]$ be a Morse function such that $A = f^{-1}(0)$ and such that $\partial M = f^{-1}(1)$. formulate a result that gives a lower bound on the number of critical points of such f in terms of the relative homology groups $\mathrm{H}_*(M,A)$.
Remark. In the Morse Function-Existence Proposition 139.4 we saw that such a Morse function always exists.
Hint. Make use of Exercise 140.10.

Exercise 140.12. Let M be a closed smooth manifold and let $f\colon M \to \mathbb{R}$ be a Morse function. Given $k \in \mathbb{N}_0$ we denote by d_k the number of critical points of index k. The Morse function is called *lacunary* if for every $k \in \mathbb{N}_0$ at least one of d_k and d_{k+1} is zero. Show that every lacunary Morse function is perfect.
Remark. This statement is sometimes called the *Morse lacunary principle*.

Exercise 140.13. Let M be a closed smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth map. Furthermore let g be a Riemannian structure on M, as defined on page 1426. Given $P \in M$ we define the *gradient of f at P* to be the unique vector $\operatorname{grad} f_P \in \mathrm{T}_P M$ that satisfies

$$g_P(\operatorname{grad} f_P, w) \;=\; \mathrm{D}f_P(w) \;\in\; \mathrm{T}_{f(P)}\mathbb{R} = \mathbb{R} \qquad \text{for all } w \in \mathrm{T}_P M.$$

One can easily show that $P \mapsto \operatorname{grad} f_P$ defines a vector field on M in the sense of the definition on page 638.

(a) We suppose that f is a Morse function. Show that every zero of the vector field $\operatorname{grad} f$ is isolated.
(b) Let P be a critical point of f. Show that

$$\underbrace{\operatorname{index}(\operatorname{grad} f, P)}_{\text{see page 2818 for the definition}} \;=\; (-1)^{\operatorname{index}(f,P)}.$$

Remark. This exercise shows that the Euler Characteristic-via-Critical Points Corollary 140.6 can also be deduced from the Poincaré-Hopf Theorem 129.7.

141. Constructing smooth manifolds using handles

In this chapter we will do the following:

(1) We will use handles to construct interesting compact and closed smooth manifolds out of finite presentations of groups.
(2) We will show that every finite CW-complex is homotopy equivalent to a compact smooth manifold.

141.1. Finite presentations of groups and handle decompositions.

Proposition 141.1. Let $\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle$ be a finite presentation. Given any $n \geq 4$ there exists a compact orientable connected non-empty n-dimensional smooth manifold with the following properties:

(1) $\pi_1(M) \cong \pi$.
(2) M admits a standard handle decomposition with precisely one 0-handle, k 1-handles, l 2-handles and no handles of index > 2.

Proof. Let $\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle$ be a finite presentation and let $n \geq 4$. We start with the 0-handle $\overline{B}^n$. Since $n \geq 2$ we can pick k *orientation-preserving* thickened 0-spheres $\varphi_1, \dots, \varphi_k \colon \overline{B}^{n-1} \times S^0 \to \partial(\overline{B}^n)$ with disjoint images and we denote by W the result of attaching the corresponding k 1-handles to the 0-handle $\overline{B}^n$. Note that W, and thus also ∂W, is orientable by the Handles-Orientability Proposition 136.9.

We pick a base point $P \in \partial W$. It follows from the Handle Attachment-π_1-Proposition 136.11 that we can make an identification $\pi_1(W, P) = \langle g_1, \dots, g_k \rangle$. Since $n \geq 3$ we know by Lemma 136.14 that the inclusion induced map $\pi_1(\partial W, P) \to \pi_f 1(W, P)$ is an epimorphism. Therefore we can pick loops $\alpha_1, \dots, \alpha_l \colon (S^1, *) \to (\partial W, P)$ that represent $r_1, \dots, r_l \in \pi_1(W, P) = \langle g_1, \dots, g_k \rangle$.

By hypothesis we have $\dim(\partial W) = n - 1 \geq 3$. Thus we can apply Theorem **??** and we obtain that the map $\alpha_1 \sqcup \dots \sqcup \alpha_l \colon S^1 \sqcup \dots \sqcup S^1 \to \partial W$ is in fact homotopic to a smooth embedding $\beta_1 \sqcup \dots \sqcup \beta_l \colon S^1 \sqcup \dots \sqcup S^1 \to \partial W$. It follows from the Smooth Embedding Theorem 24.10 and the Tubular Neighborhood Theorem **??**[1687] that we can find smooth embeddings $\psi_1, \dots, \psi_l \colon \overline{B}^{n-2} \times S^1 \to \partial W$ with disjoint images such that for each $i \in \{1, \dots, l\}$ and each $x \in S^1$ we have $\psi_i((0, x)) = \beta_i(x)$. Finally we consider

$$M \;:=\; W \cup_{\psi_1} h^2 \cdots \cup_{\psi_l} h^2 \;=\; \varnothing \cup \overline{B}^n \cup_{\varphi_1} h^1 \cup \cdots \cup_{\varphi_k} h^1 \cup_{\psi_1} h^2 \cdots \cup_{\psi_l} h^2.$$

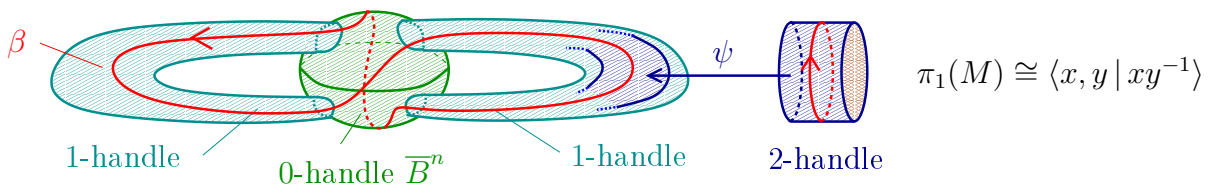
The proposition follows from the following calculation:

$$\begin{array}{ccccc} & & & & \\ \pi_1(M, P) & \overset{\text{by the Handle Attachment-}\pi_1\text{-Proposition 136.11, note that the } x_i \text{ are necessary since we lost track of the base points}}{\cong} & \langle g_1, \dots, g_k \rangle / \langle\langle x_1 r_1 x_1^{-1}, \dots, x_l r_l x_l^{-1} \rangle\rangle & & \\ & \overset{\text{by Lemma 53.2 (5)}}{\cong} & \langle g_1, \dots, g_k \rangle / \langle\langle r_1, \dots, r_l \rangle\rangle & \overset{\text{by definition}}{=} & \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle. \end{array}$$

■

Now we can provide a new proof of the following proposition.

[1687] Note that we can apply the Tubular Neighborhood Theorem **??** since ∂W is orientable.



Proposition 54.12. (Fundamental Group-Manifold Realization Proposition) Let $\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle$ be a finite presentation. Given any $n \geq 4$ there exists a closed orientable connected non-empty n-dimensional smooth manifold W with $\pi_1(W) \cong \pi$.

Remark. The Fundamental Group-Manifold Realization Proposition 54.12 is a key ingredient of the proof of the fact, stated in Theorem 60.4, that closed smooth manifolds of a fixed dimension ≥ 4 cannot be classified.

Proof. Let $\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle$ be a finite presentation and let $n \geq 4$. By Proposition 141.1 there exists a compact orientable connected non-empty n-dimensional smooth manifold with the following properties:

(1) $\pi_1(M) \cong \pi$.
(2) M admits a handle decomposition $M = \varnothing \cup h^0 \cup_{\varphi_1} h^1 \cdots \cup_{\varphi_k} h^1 \cup_{\psi_1} h^2 \cdots \cup_{\psi_l} h^2$.

By the Doubling-Manifold Lemma 75.12 the double $W := \mathrm{D}M$ is a closed orientable connected non-empty n-dimensional smooth manifold. Thus it suffices to prove the following claim.

Claim. There exists an isomorphism $\pi_1(\mathrm{D}M) \cong \pi$.

Proof. For fun we provide two different proofs.

(1) Note that by Corollary 136.15 we know that the boundary ∂M is connected. Therefore it follows from the Seifert-van Kampen Theorem 54.4 we have an isomorphism $\pi_1(\mathrm{D}M) \cong \pi_1(M) *_{\pi_1(\partial M)} \pi_1(M)$. Since $n \geq 4$ and since M has only handles of index ≤ 2 we obtain from Corollary 136.15 that the inclusion induced map $\pi_1(\partial M) \to \pi_1(M)$ is an epimorphism. It follows from the group-theoretic Lemma 75.15 that the map $\pi_1(M) \to \pi_1(\mathrm{D}M)$ is actually an isomorphism.
(2) On page 3004 we pointed out that it is a consequence of Propositions 139.1 and 139.3 that the double admits a handlebody presentation of the following form:

$$\mathrm{D}M = \underbrace{\varnothing \cup h^0 \cup_{\varphi_1} h^1 \cdots \cup_{\varphi_k} h^1 \cup_{\psi_1} h^2 \cdots \cup_{\psi_l} h^2}_{=M} \cup_{\alpha_l} h^{n-2} \cdots \cup_{\alpha_1} h^{n-2} \cup_{\beta_k} h^{n-1} \cdots \cup_{\beta_1} h^{n-1} \cup h^n,$$

where the $(n-2)$-handles corresponding to $\alpha_l, \dots, \alpha_1$ are dual to the 2-handles given by $\psi_l, \dots, \psi_1$ and the $(n-1)$-handles corresponding to $\beta_k, \dots, \beta_1$ are dual to the 1-handles given by $\varphi_k, \dots, \varphi_1$.
(a) First let us deal with the case that $n \geq 5$. In this case $\mathrm{D}M$ is obtained from M by attaching handles of index ≥ 3. But by the Handle Attachment-π_1-Proposition 136.11 we know that handle attachments of index ≥ 3 do not affect the fundamental group. Thus we see that $\pi_1(\mathrm{D}M) \cong \pi_1(M) \cong \pi$.

(b) The case $n = 4$ is a little bit more interesting. We have

$$\begin{array}{rcl} & & \text{by the Handle Attachment-}\pi_1\text{-Proposition 136.11} \\ & & \downarrow \\ \pi_1(\mathrm{D}M) & = & \pi_1(M)/\langle\langle \text{attaching circles of the dual 2-handles } \alpha_l, \dots, \alpha_1 \rangle\rangle \\ & = & \pi_1(M)/\langle\langle \text{belt circles of the 2-handles } \psi_l, \dots, \psi_1 \rangle\rangle \quad \cong \quad \pi_1(M) \cong \pi. \\ & & \uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\ & & \text{Relative Dual Handle Proposition 139.1 (1c)} \qquad \text{since belt circles are zero-homotopic} \end{array}$$ ■

This is a good moment to reminder the reader of the following question.

Question 54.15. Is every finitely presented group the fundamental group of a closed (or at least compact) orientable connected 3-dimensional smooth manifold?

We will answer Question 54.15 in Theorem 146.5.

141.2. **Integral homology spheres and homotopy spheres.** We remind the reader of the following definitions from page 2159 and page 2625.

Definition. Let M be an n-dimensional smooth manifold M.

(1) If for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(M) \cong \mathrm{H}_k(S^n)$, then we refer to M as a smooth homology n-sphere.
(2) If M is a homology sphere and if M is simply connected, then we refer to M as a smooth homotopy n-sphere.

Our goal in this section is to apply the construction used in the Fundamental Group-Manifold Realization Proposition 54.12 to construct smooth homology and potentially interesting smooth homotopy spheres.

We start out with the following proposition, which the reader might recognize as Exercise 119.21.

Proposition 141.2. (Homology-of-Double Proposition) Let M be a compact orientable n-dimensional topological manifold. For every $j \in \mathbb{N}_0$ there exists an isomorphism

$$\mathrm{H}_j(\mathrm{D}M) \cong \mathrm{H}_j(M) \oplus \mathrm{Hom}(\mathrm{H}_{n-j}(M), \mathbb{Z}) \oplus \mathrm{Tor}(\mathrm{H}_{n-j-1}(M)).$$

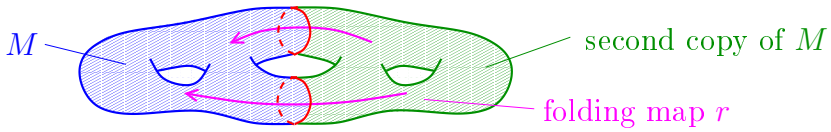
Proof. Let $i\colon M \to \mathrm{D}M$ be the natural inclusion and let $r\colon \mathrm{D}M \to M$ be the folding map that we introduced on page 1668. Note that by definition we have $r \circ i = \mathrm{id}_M$. We consider the long exact sequence in homology of the pair $(\mathrm{D}M, M)$:

$$\dots \to \mathrm{H}_j(M) \xrightarrow{i_*} \mathrm{H}_j(\mathrm{D}M) \to \mathrm{H}_j(\mathrm{D}M, M) \to \dots$$

Now we see that we have the following isomorphisms:

$$\begin{array}{ccccc} & & & & \text{Excision Theorem 75.10, here the blue } M \text{ stands for the second copy of } M \text{ in the double } \mathrm{D}M \\ & & & & \downarrow \\ \mathrm{H}_j(\mathrm{D}M) & \cong & \mathrm{H}_j(M) \oplus \mathrm{H}_j(\mathrm{D}M, M) & \cong & \mathrm{H}_j(M) \oplus \mathrm{H}_j(M, \partial M) \\ & & \uparrow & & \\ & & \text{since } r \circ i = \mathrm{id}_M \text{ we see that the above long exact sequence breaks up into short exact sequences,} & & \\ & & \text{furthermore it follows from Lemma 78.2 that each short exact sequence splits and} & & \\ & & \text{that the middle term is isomorphic to the direct sum of the two other terms} & & \\ & \cong & \mathrm{H}_j(M) \oplus \mathrm{H}^{n-j}(M) & \cong & \mathrm{H}_j(M) \oplus \mathrm{Hom}(\mathrm{H}_{n-j}(M), \mathbb{Z}) \oplus \mathrm{Tor}(\mathrm{H}_{n-j-1}(M)). \\ & & \uparrow & & \uparrow \\ & & \text{Poincaré Duality Theorem 119.1} & & \text{Universal Coefficient Theorem 110.12} \end{array}$$ ■

Now we can prove the following refinement of the Fundamental Group-Manifold Realization Proposition 54.12.



Corollary 141.3. Let $\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle$ be a finite presentation.

(1) Given any $n \geq 5$ there exists a closed orientable connected non-empty n-dimensional smooth manifold W with $\pi_1(W) \cong \pi$ and such that[1688]

$$\mathrm{H}_i(W) \;\cong\; \begin{cases} \pi_{\mathrm{ab}}, & \text{if } i = 1, \\ \mathbb{Z}^{l-k+\mathrm{rk}(\pi_{\mathrm{ab}})}, & \text{if } i = 2, \\ \mathbb{Z}^{l-k+\mathrm{rk}(\pi_{\mathrm{ab}})} \oplus \mathrm{Tor}(\pi_{\mathrm{ab}}), & \text{if } i = n-2, \\ \mathbb{Z}^{\mathrm{rk}(\pi_{\mathrm{ab}})}, & \text{if } i = n-1, \\ \mathbb{Z}, & \text{if } i = 0 \text{ or if } i = n, \\ 0, & \text{else.} \end{cases}$$

(2) There exists a closed orientable connected non-empty smooth manifold W of dimension $n = 4$ with $\pi_1(W) \cong \pi$ and such that the homology groups are as in (1), except that for $i = 2$ we take the direct sum of the terms that we wrote down for $i = 2$ and $i = n - 2$.

Proof. As in the proof of the Fundamental Group-Manifold Realization Proposition 54.12 that we just gave on page 3031 we use Proposition 141.1 to obtain a compact orientable connected non-empty n-dimensional smooth manifold with the following properties:

(1) $\pi_1(M) \cong \pi$.
(2) M admits a handle decomposition with precisely one 0-handle, k 1-handles, l 2-handles and no handles of index > 2.

In the above proof of the Fundamental Group-Manifold Realization Proposition 54.12 we saw that the fundamental group of $W := \mathrm{D}M$ is isomorphic to π and that W is a closed orientable connected non-empty n-dimensional smooth manifold. It remains to show that the homology groups of $W = \mathrm{D}M$ are as stated. In fact it follows from the Homology-of-Double Proposition 141.2 that it suffices to prove the following claim.

Claim.

$$\mathrm{H}_0(M) \cong \mathbb{Z}, \quad \mathrm{H}_1(M) \cong \pi_{\mathrm{ab}}, \quad \mathrm{H}_2(M) \cong \mathbb{Z}^{l-k+\mathrm{rk}(\pi_{\mathrm{ab}})} \quad \text{and} \quad \mathrm{H}_i(M) = 0 \text{ for } i \geq 3.$$

Proof. First note that the statements for $\mathrm{H}_0(M)$ and $\mathrm{H}_1(M)$ follow immediately from the H_0-Proposition 72.5 and the Hurewicz Theorem 84.5. Next note that

$$\mathrm{H}_i(M) \;\underset{\text{Singular-Handle } \mathrm{H}_*\text{-Isomorphism Proposition 140.3}}{\overset{\uparrow}{\cong}}\; \mathrm{H}_i^{\mathrm{han}}(M) \;\underset{\text{since } M \text{ has no handles of index} \geq 3}{\overset{\uparrow}{=}}\; \begin{cases} 0, & \text{if } i > 2, \\ \ker(\mathrm{C}_2^{han}(M) \to \mathrm{C}_1^{\mathrm{han}}(M)), & \text{if } i = 2. \end{cases}$$

[1688] We denote by π_{ab} the abelianization of π, as defined on page 1203.

Note that this observation for $i = 2$, together with Lemma 51.9, shows that $\mathrm{H}_2(M)$ is a free abelian group. Finally note that

$$\mathrm{rk}(\mathrm{H}_2(M)) \underset{\text{by definition of } \chi(M)}{\uparrow} \chi(M) + \sum_{i\neq 2} (-1)^{i+1} \cdot \mathrm{rk}(\mathrm{H}_i(M)) \underset{\text{by the Handles-Euler Characteristic Corollary 140.4 and above calculations}}{\uparrow} (1-k+l) - 1 + \mathrm{rk}(\pi_{\mathrm{ab}}) = l - k + \mathrm{rk}(\pi_{\mathrm{ab}}).$$

■

As we mentioned, our goal is to construct smooth homology spheres and smooth homotopy spheres. Corollary 141.3 leads us to the following group theoretic definitions.

Definition.

(1) The deficiency of a finite presentation $\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ is defined to be $k - l$.
(2) A finite presentation is called balanced, if its deficiency equals zero.
(3) A group π is called perfect if its abelianization π_{ab} is trivial.

The following proposition gives us some interesting examples of (presentations of) groups.

Proposition 141.4.

(1) For every closed connected non-empty 3-dimensional smooth manifold M the fundamental group $\pi_1(M)$ admits a balanced presentation.
(2) The binary icosahedral group, that we introduced on page 2160, is a perfect group of order 120 that admits a balanced presentation.
(3) For every odd prime p the group $\mathrm{SL}(2, \mathbb{F}_p)$ is perfect and it admits a balanced presentation.

Proof.

(1) We will prove this statement in Exercise 141.2. Alternatively the statement is proved in Theorem 146.5.
(2) Let π be the binary icosahedral group. In Proposition 98.12 we showed that π is a perfect group of order 120. It remains to show that π admits a balanced presentation. There are two possible approaches:
   (a) In [**KSc1979**, Chapter II] it is shown that the binary icosahedral group is isomorphic to $\langle x, y \,|\, x^3 = y^5, x^3 = (xy)^2 \rangle$.
   (b) Recall that by definition $\pi \subset \mathrm{SU}(2)$. Thus it makes sense to consider the quotient $M := S^3/\pi$, which we called the Poincaré Homology Sphere. In Proposition 98.12 we showed that M is a closed orientable 3-dimensional smooth manifold. Thus it follows from (1) that $\pi \cong \pi_1(M)$ admits a balanced presentation.
(3) In Exercise 141.3 we will see that $\mathrm{SL}(2, \mathbb{F}_p)$ is perfect. The much more subtle fact that $\mathrm{SL}(2, \mathbb{F}_p)$ admits a balanced presentation is shown in [**CR1980**]. ■

In Proposition 98.11 we showed that there exist smooth homology 3-spheres. The combination of Corollary 141.3 and Proposition 141.4 gives us the high-dimensional analogue.

Proposition 141.5. Given any $n \geq 4$ there exists a smooth homology n-sphere with non-trivial fundamental group.

Remark.

(1) In Exercise 98.13 we used a different construction to obtain high-dimensional smooth homology spheres.

(2) Michel Kervaire [**Kerv1969**, Theorem 1] completely determined which finitely presented groups can occur as fundamental groups of smooth homology n-spheres for $n \geq 4$.

The construction that we used in the proof of Proposition 141.1 and that we used in the above proof of Propositions 54.12 is a never-ending source of joy, since even for presentations of the trivial group one obtains interesting examples of smooth manifolds:

Example. Let $k \in \mathbb{Z}$. We consider the presentation

$$\langle x, y \,|\, \underbrace{xyx = yxy}_{=(1)}, \underbrace{x^{k+1} = y^k}_{=(2)} \rangle.$$

We will now see that this is a presentation of the trivial group. Indeed, we have

(a) $y = x^{-1}y^{-1}yxy = (yx)^{-1}yxy \underset{\text{by (1)}}{=} (yx)^{-1}x(yx)$

(b) $y^{k+1} \underset{\substack{\text{follows from taking} \\ (k+1)\text{-st power of (a)}}}{=} (yx)^{-1}x^{k+1}(yx) \underset{\text{by (2)}}{=} (yx)^{-1}y^k(yx) = x^{-1}y^kx \underset{\text{by (2)}}{=} x^{-1}x^{k+1}x = x^{k+1} \underset{\text{by (2)}}{=} y^k.$

Thus $y^{k+1} = y^k$ which implies that $y = 1$. From (1) it follows that also $x = 1$. We have thus shown that this is indeed a presentation of the trivial group.

Let $n \geq 4$. We apply the construction of the proof of Propositions 54.12 and 141.1 to the above balanced presentation of the trivial group. By Corollary 141.3 we obtain a smooth homotopy n-sphere.

The following question arises:

Question 141.6. Are the smooth homotopy n-spheres that we just constructed[1689] diffeomorphic to S^n?

For $n = 4$ and some special cases Question 141.6 was answered in the affirmative [**Gom1991**]. In general though, at least for $n = 4$ Question 141.6 seems to be wide open. We refer to [**GoS1999**, p. 149] and [**Kir1989**, p. 18] for more information.

We have now seen that finite presentations of groups give rise to interesting smooth manifolds. The question arises, to what degree different presentations give rise to different smooth manifolds. It is worth recalling the following theorem.

Theorem 53.7. (Tietze Theorem) Any two finite presentations for some given group are related by a finite sequence of Tietze transformations, i.e. by a sequence of the following two transformations:

(a) $\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle \quad \leftrightsquigarrow \quad \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l, s \rangle$

where s is a word in in $r_1, \dots, r_l$, and

(b) $\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle \quad \leftrightsquigarrow \quad \langle x_1, \dots, x_k, x \,|\, r_1, \dots, r_l, s \cdot x \cdot t \rangle$

where $x \neq x_1, \dots, x_k$ and where s, t are words in $x_1, \dots, x_k$.

In our context, where we study balanced presentations of the trivial group, we could like to go from one presentation to another while keeping the deficiency fixed. This leads us to the following definition.

[1689] Evidently there are various choices involved in these constructions.

Definition. Let $\langle x_1, \dots, x_k \mid r_1, \dots, r_l \rangle$ be a presentation. An Andrews-Curtis transformation is one of the following ways to obtain a new presentation:

(a) $\langle x_1, \dots, x_k \mid r_1, \dots, r_i, \dots, r_l \rangle \leftrightsquigarrow \langle x_1, \dots, x_k \mid r_1, \dots, x_j^{-\epsilon} \cdot r_i \cdot x_j^{\epsilon}, \dots r_l \rangle$
where $j \in \{1, \dots, k\}$ and $\epsilon \in \{\pm 1\}$

(b) $\langle x_1, \dots, x_k \mid r_1, \dots, r_i, \dots, r_l \rangle \leftrightsquigarrow \langle x_1, \dots, x_k, \mid r_1, \dots, r_i \cdot r_j, \dots, r_l \rangle$
where $i \neq j \in \{1, \dots, l\}$

(c) $\langle x_1, \dots, x_k \mid r_1, \dots, r_l \rangle \leftrightsquigarrow \langle x_1, \dots, x_k, x \mid r_1, \dots, r_l, x \rangle$
where $x \neq x_1, \dots, x_k$.

The following is now a slight reformulation of the Andrews-Curtis Conjecture which we first encountered in Section 53.4.

Conjecture 53.18. (Andrews-Curtis Conjecture) Any balanced presentation of the trivial group can be turned into the empty presentation by a finite sequence of Andrews-Curtis transformations.

In the Handle Cancellation Theorem **??** and the Handle Addition Theorem **??** we will see that the Andrews-Curtis transformations have analogues for handle decompositions. In particular, as is argued on [**GoS1999**, p. 149], an affirmative answer to the Andrews-Curtis Conjecture 53.18 would imply that Question 141.6 has a positive answer.

For better or worse the evidence towards the Andrews-Curtis Conjecture 53.18 is rather shaky. For example, let $k \in \mathbb{Z}$. As above we consider the following balanced presentation of the trivial group:

$$\langle x, y \mid xyx = yxy, x^{k+1} = y^k \rangle.$$

For most k it is unknown whether the presentation can be turned into the empty presentation by a finite sequence of Andrews-Curtis transformations. In fact the consensus seems to be that the Andrews-Curtis Conjecture 53.18 is most likely false.

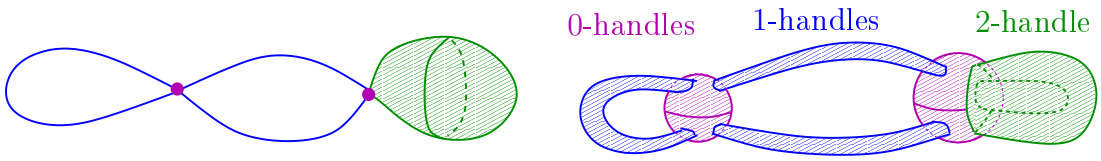
Remark.

(1) More information and some partial results regarding the Andrews-Curtis Conjecture 53.18 can be found in [**Barm2011**, Chapter 11].
(2) Sergey Matveev [**Matv1987**, p. 917] (see also [**Kupe2017a**, Proposition 4.2] and [**Barm2011**, Chapter 11]) showed that the Andrews-Curtis Conjecture 53.18 is equivalent to a special case of the Zeeman Conjecture 162.15.

141.3. **Handle decompositions and CW-structures.** In the Handles-Cells Proposition 136.18 we saw that every compact smooth manifold M that is equipped with a handle decomposition is homotopy equivalent to a CW-complex X such that for each $k \in \mathbb{N}_0$ the number of k-handles of M equals the number of k-cells of X. In this section we will prove the following converse to this statement.

Proposition 141.7. (Manifold Construction Proposition) Given any finite CW-complex X there exists a compact orientable smooth manifold M that is homotopy equivalent to X and which admits a handle decomposition such that for each $k \in \mathbb{N}_0$ the number of k-cells of X equals the number of k-handles of M.

Remark.



CW-complex X thickening M of X

(1) In the Manifold Construction Proposition 141.7 we do not make any claims about the dimension of M. We leave it to the reader to figure out what is the lowest dimension for M that one can achieve with our proof.
(2) Let X be a finite CW-complex. Any compact smooth manifold M that is homotopy equivalent to X is called a *thickening of* X. the Manifold Construction Proposition 141.7 thus says in particular that every finite CW-complex admits a thickening.
(3) The theory of thickenings of finite CW-complexes is developed in detail in [**Maz1963**] and [**Wall1966b**]. In particular given a finite k-dimensional CW-complex it follows from [**Wall1966b**, p. 76] (see also [**BCFHLN2019**, Theorem 4.1] that the following statements hold:
   (a) Given any $n \geq 2k$ there exists an n-dimensional thickening of X.
   (b) If $n \geq 2k+1$ and if $n \geq 6$, then any two n-dimensional thickenings of X are actually diffeomorphic.

   We refer to [**BCFHLN2019**, Chapter 4] for more information on thickenings.
(4) The topic of finding smooth manifolds that "look" like a given topological space is also discussed in [**BuS2010**, Chapter 2].

The proof of the Manifold Construction Proposition 141.7 rests on the following lemma.

Lemma 141.8. If $N \geq 2r+2$, then for every smooth embedding $\varphi\colon S^r \to \mathbb{R}^N$ there exists an orientation-preserving smooth embedding $\Phi\colon \overline{B}^{N-r} \times S^r \to \mathbb{R}^N$ such that $\Phi(0,P) = \varphi(P)$ for all $P \in S^r$.

Proof. The statement of the lemma is basically the content of the Tubular Neighborhood Theorem **??** (5). For the reader's convenience we recall the proof. First we consider the standard smooth embedding $\theta\colon S^r \to \mathbb{R}^{r+1} \times \mathbb{R}^{N-r-1} = \mathbb{R}^N$ which is given by $x \mapsto (x,0)$. This smooth embedding can be "thickened" up. For example we could consider the smooth embedding

$$\begin{array}{rcl} \Theta\colon \overline{B}^{N-r} \times S^r & \to & \mathbb{R}^{r+1} \times \mathbb{R}^{N-r-1} = \mathbb{R}^N \\ (\underbrace{(x,y)}_{\substack{\uparrow \\ \text{with } x \in \mathbb{R} \text{ and } y \in \mathbb{R}^{N-r-1}}}, P) & \mapsto & ((1+\frac{x}{2}) \cdot P, y). \end{array}$$

After possibly composing Θ with a reflection in the last coordinate we see that Θ is orientation-preserving. By hypothesis we have $N \geq 2r+2$. Thus we obtain from Proposition **??** that there exists a smooth homotopy $F\colon S^r \times [0,1] \to \mathbb{R}^N$ with $F_0 = \theta$ and $F_1 = \varphi$. By the Isotopy Extension Theorem 36.1 there exists a diffeotopy $G\colon \mathbb{R}^N \times [0,1] \to \mathbb{R}^N$ with $G_1 \circ \theta = \varphi$. The map $\Phi := G_1 \circ \Theta\colon \overline{B}^{N-r} \times S^r \to \mathbb{R}^N$ has all the desired properties. ∎

The heart of the proof of the Manifold Construction Proposition 141.7 is contained in the following lemma.

Lemma 141.9. Let X be a finite CW-complex and let $\alpha\colon S^{k-1} \to X$ be a map. If X is homotopy equivalent to a compact orientable smooth manifold M, then there exists a compact orientable smooth manifold N and an orientation-preserving thickened $(k-1)$-sphere $\varphi\colon \overline{B}^{\dim(N)-k} \times S^{k-1} \to \partial N$ such that the following are satisfied:

(1) N is also homotopy equivalent to X.
(2) $N \cup_\varphi h^k$ is homotopy equivalent to $X \cup_\alpha \overline{B}^k$.
(3) If M is equipped with a handle decomposition, then we can equip N with a handle decomposition such that for each $k \in \mathbb{N}_0$ the number of k-handles of M equals the number of k-handles of N.

Proof of the Manifold Construction Proposition 141.7. Note that in the setting of Lemma 141.9 we know by Propositions 136.1 and 136.9 that $N \cup_\varphi h^k$ is again a compact orientable smooth manifold with one extra k-handle. Thus we see that the Manifold Construction Proposition 141.7 follows immediately from Lemma 141.9 and an elementary induction argument. ■

Proof of Lemma 141.9. Let $f\colon X \to M$ be a homotopy equivalence between a finite CW-complex X and a compact orientable m-dimensional smooth manifold M. Furthermore let $\alpha\colon S^{k-1} \to X$ be a map.

We pick some $r \geq \max\{k, 2k - \dim(M)\}$. We write $\widetilde{M} := M \times \overline{B}^r$ and we equip the product $\widetilde{M}$ with the smooth atlas provided by Proposition **??**. Note that the map $i\colon M \to \widetilde{M}$ given by $x \mapsto (x, 0)$ is a homotopy equivalence.

Claim. We write $\widetilde{f} := i \circ f\colon X \to \widetilde{M}$. There exists a homotopy $F\colon S^{k-1} \times [0,1] \to \widetilde{M}$ from the map $\widetilde{f} \circ \alpha\colon S^{k-1} \to \widetilde{M} = M \times \overline{B}^r$ to a smooth embedding $\theta\colon S^{k-1} \to \partial\widetilde{M}$.

Proof. By the combination of the Whitney Approximation Theorem 29.1 and the Transversality Theorem 42.4 we know that the map $\widetilde{f} \circ \alpha\colon S^{k-1} \to \widetilde{M}$ is homotopic to a smooth map that is transverse to the proper submanifold $M \times \{0\}$ of $\widetilde{M} = M \times \overline{B}^r$. Note that

$$\dim(S^{k-1}) + \dim(M \times \{0\}) \;=\; k - 1 + \dim(M) \underset{\substack{\uparrow \\ \text{since } r \geq k}}{<} r + \dim(M) \;=\; \dim(M \times \overline{B}^r).$$

Thus the image of a map $S^{k-1} \to \widetilde{M}$, which is transverse to $M \times \{0\}$, actually misses $M \times \{0\}$. Therefore we now know that $\widetilde{f} \circ \alpha\colon S^{k-1} \to \widetilde{M}$ is homotopic to a map that misses $M \times \{0\}$. Next, using the deformation retraction from $\overline{B}^k \setminus \{0\}$ to S^{k-1} we see that $\widetilde{f} \circ \alpha\colon S^{k-1} \to \widetilde{M}$ is actually homotopic to a map $\varphi\colon S^{k-1} \to M \times S^{k-1} \subset \partial\widetilde{M}$. Finally, since

$$2(k-1) \underset{\substack{\uparrow \\ \text{since } r \geq 2k - \dim(M)}}{<} \dim(M) + r - 1 \;=\; \dim(\partial\widetilde{M})$$

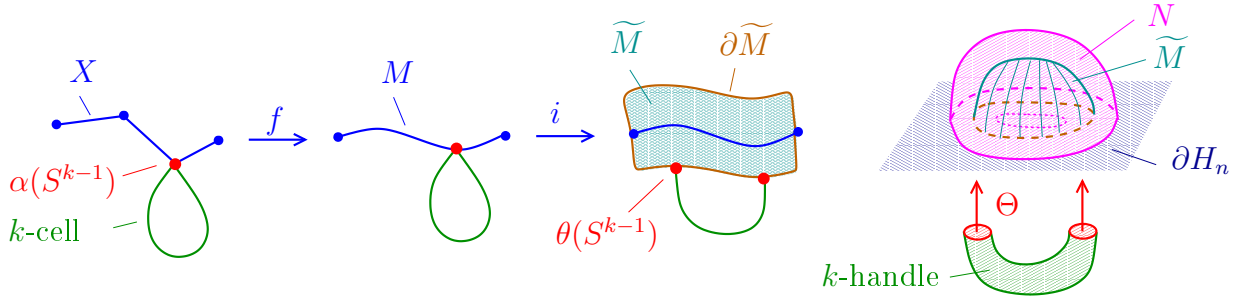
we obtain from the Whitney Approximation Theorem 29.1 together with Theorem **??** (1) that $\varphi\colon S^{k-1} \to \partial\widetilde{M}$ is homotopic to a smooth embedding $\theta\colon S^{k-1} \to \partial\widetilde{M}$. ⊞

By the Euclidean Embedding Theorem 27.1 there exists an $n \geq 2k+1$ such that there exists a proper smooth embedding $\widetilde{M} \to \mathbb{H}^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_n \geq 0\}$. In particular,

by the Smooth Embedding Theorem 24.10 we can assume that $\widetilde{M}$ is a proper smooth submanifold of $\mathbb{H}^n$ for some $n \geq 2k$.

By the General Tubular Neighborhood Theorem **??** the proper submanifold $\widetilde{M}$ of $\mathbb{H}^n$ admits a tubular neighborhood $N \subset H_n$. By definition of a tubular neighborhood, see page **??**, we know that N is a compact codimension-zero submanifold of $\mathbb{H}^n$ with corner. Furthermore, by Proposition 123.4 we know that the inclusion $j\colon \widetilde{M} \to N$ is a homotopy equivalence. As we mentioned in Lemma 120.1, by rounding corners as in the proof of Proposition **??** we can view N as an n-dimensional smooth manifold.

The map $j \circ \theta\colon S^{k-1} \to \partial\widetilde{M} \subset \partial N$ takes values in $\partial\,\mathbb{H}^n$. Since $N \cap \partial\,\mathbb{H}^n$ is a codimension-zero submanifold of $\partial\,\mathbb{H}^n$ we see that there exists an open neighborhood U of $\theta(S^{k-1})$ in $\partial\,\mathbb{H}^n$ that is contained in N. Since $\partial\,\mathbb{H}^n$ is diffeomorphic to $\mathbb{R}^{n-1}$ and since, by our choice of n, we have $n-1 \geq 2(k-1)+2$, we obtain from Lemma 141.8 that there exists an orientation-preserving smooth embedding $\Theta\colon \overline{B}^{n-k} \times S^{k-1} \to \partial\,\mathbb{H}^n$ such that $\Theta(0,P) = \theta(P)$ for all $P \in S^{k-1}$. After possibly "shrinking" the balls we can arrange that the image of Θ actually lies in $U \subset N \cap \partial\,\mathbb{H}^n$.

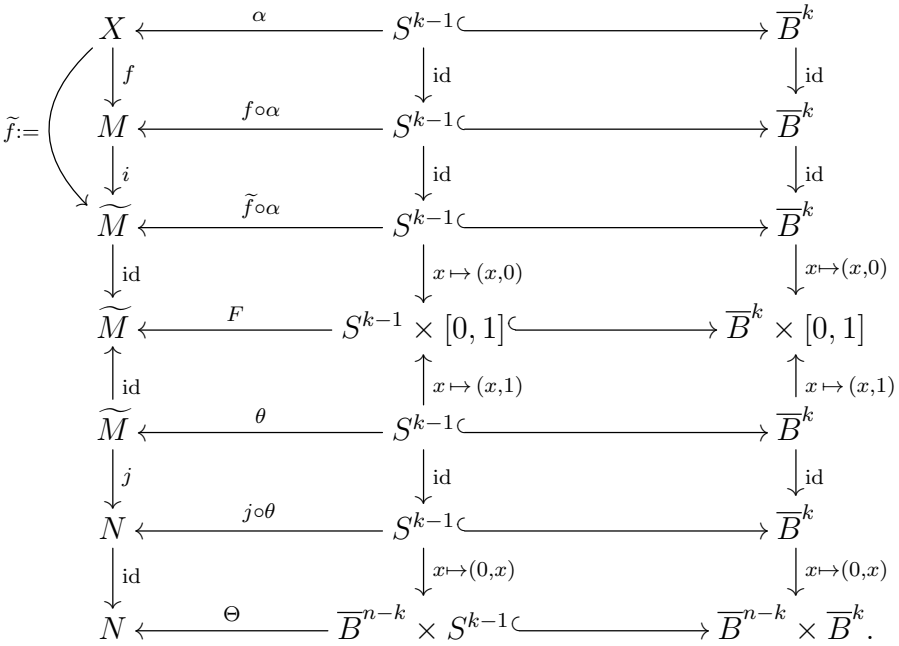


Statement (2) of the lemma is a consequence of the following claim.

Claim. The topological space $X \cup_\alpha \overline{B}^k$ is homotopy equivalent to the smooth manifold $N \cup_\Theta h^k$.

Proof. The experienced reader will doubtlessly spot that this is the right moment to roll out the Homotopy Pushout Theorem 17.19. To apply this great theorem let us consider

the following commutative diagram:



All of the vertical maps are clearly homotopy equivalences. Furthermore note that it follows immediately from the Manifold Boundary-Cofibration Proposition 17.8 and the Product-Cofibration Proposition 17.11 that all of the right horizontal maps are closed cofibrations. Thus it follows from the Homotopy Pushout Theorem 17.19, applied altogether six times, that the induced maps between the various pushouts have homotopy inverses. Combining these six homotopy equivalences, respectively their homotopy inverses, we obtain the promised homotopy equivalence from $X \cup_\alpha \overline{B}^k$ to $N \cup_\Theta h^k$. ⊞

Finally assume that M is equipped with a handle decomposition. We sketch the proof how we can find a handle decomposition for N with the same number of handles. First, except for a slight sense of uneasy about rounding corners, it is pretty clear that we can equip $\widetilde{M} = M \times \overline{B}^r$ with a handle decomposition that is given by taking the product of each handle of M with $\overline{B}^r$. Now we consider the actual smooth manifold N which we are interested in. Note that N is by definition of a tubular neighborhood the total space of a linear $\overline{B}^s$-bundle $p\colon N \to \widetilde{M}$. This bundle might well be non-trivial, thus there is no reason to assume that N is actually a product $\widetilde{M} \times \overline{B}^s$. But, it follows from the same argument as in the Vector Bundles-Homotopies Proposition 143.13 and the fact that each handle is contractible that the restriction of p to each handle is actually a trivial linear $\overline{B}^s$-bundle. Thus we see that the handle structure on $\widetilde{M}$ does give rise to a handle structure on N with exactly the same number of handles of each index. ■

The next proposition explains what type of closed smooth manifolds one can obtain by the ever-popular doubling procedure to the smooth manifolds constructed in the proof of the Manifold Construction Proposition 141.7.

Proposition 141.10. Let X be a finite k-dimensional CW-complex. There exists an $N \geq 2k+1$ such that for every $n \geq N$ there exists a closed orientable n-dimensional

smooth manifold M which admits a standard handle decomposition and which admits a homotopy equivalence

$$\varphi\colon X \to M^{n-k-1}$$

from X to the $(n-k-1)$-skeleton of M.

Remark. Note that in the setting of Proposition 141.10 we obtain from Exercise 136.12 and the Handle Skeleton-Homology Lemma 140.1 (3) and (4), together with the Homotopy-π_n-Proposition 71.7 and the Homotopy Equivalence-H_*-Corollary 73.9, that the map $X \to M^{n-k-1} \to M$ induces an isomorphism of homotopy groups and homology groups up to dimension $n-k-2$.

Proof. Let X be a finite k-dimensional CW-complex. By the Manifold Construction Proposition 141.7 there exists a homotopy equivalence $\psi\colon X \to V$ from X to a compact smooth manifold V which has the nice property that it admits a handle decomposition such that all handles are of index $\leq k$. By the Standard Handle Decomposition Corollary 136.4 we can arrange that this handle decomposition is standard. After possibly replacing V by some $\overline{B}^s \times V$ we can arrange that $\dim(V) \geq 2k+1$. We set $N := \dim(V)$. Now let $n \geq N$. We set $W := \overline{B}^{n-N} \times V$ and we set $M := \mathrm{D}W$. We equip the double $M = \mathrm{D}W$ with the handle decomposition that we described on page 3004, namely which is given by the handle decomposition on M and the dual handle decomposition on the second copy of M. In particular we see that we can equip M with a handle decomposition such that the $(n-k-1)$-skeleton equals W. The map $\varphi\colon X \to M^{n-k-1} = V \times \overline{B}^{n-N}$ given by $P \mapsto (\psi(P), 0)$ is now the promised homotopy equivalence from X to the $(n-k-1)$-skeleton of M. ■

In the Manifold Construction Proposition 141.7 we just showed that every finite CW-complex X is homotopy equivalent to a compact smooth manifold X. The following question, which implicitly we had already asked on page 2643, arises.

Question 141.11. Given a finite CW-complex X, what extra conditions do we need to impose on X to ensure that X is homotopy equivalent to a *closed* smooth or topological manifold?

We make a couple of initial observations. Let X be a CW-complex.

(1) If X is homotopy equivalent to a closed orientable n-dimensional topological manifold, then it follows from the Poincaré Duality Theorem 119.1, together with the Homotopic Maps-and-Homology Proposition 73.6, Lemmas 108.13 and 116.8, that there exists a class $\sigma \in \mathrm{H}_n(X;\mathbb{Z})$ such that for each $k \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \cap\sigma\colon \mathrm{H}^{n-k}(X;\mathbb{Z}) & \to & \mathrm{H}_k(X;\mathbb{Z}) \\ \psi & \mapsto & \psi \cap \sigma \end{array}$$

is an isomorphism.

(2) If X is homotopy equivalent to a closed *non-orientable* topological manifold, then it follows from the Orientation Covering-Proposition 105.15 that X admits a 2-fold covering $\widetilde{X}$ which is homotopy equivalent to a closed orientable topological manifold.

(3) If X is homotopy equivalent to a closed (orientable) n-dimensional topological manifold, then every finite covering of X is also homotopy equivalent to a closed (orientable) n-dimensional topological manifold. In particular given any finite covering $\widetilde{X}$ the properties stated in (1) and (2) need to hold.

(4) These observations in (1), (2) and (3) already give lots of restrictions. We will generalize these conditions on page **??**. There we will show that any CW-complex X which is homotopy equivalent to a closed orientable n-dimensional topological manifold, is in fact a "Poincaré Duality space".

The extra conditions stated in (2), (3) and (4) vanish, if X is a simply connected CW-complex. Thus we have arrived at the following subtle variation on Question 141.11

Question 141.12. Let X be a finite simply connected CW-complex X which admits a class $\sigma \in \mathrm{H}_n(X;\mathbb{Z})$ as in (1). Is X homotopy equivalent to a closed n-dimensional smooth or topological manifold?

It turns out that the answer to Question 141.12 is no. Namely, there are other, more subtle bundle-theoretic obstructions to being homotopy equivalent to a closed smooth or topological manifold. Slightly more precisely, the obstruction is given by showing that the "Spivak Normal Fibration" of a given finite CW-complex has the structure of a vector bundle. In fact in [**DaP2015**, p. 21] (see also [**MadM1979**, p. 32]) it is shown that there exists a finite simply connected 5-dimensional CW-complex X such that the cohomology ring is isomorphic to the cohomology ring of $S^2 \times S^3$ but such that X is not homotopy equivalent to a closed topological manifold. Similar examples of such simply connected CW-complexes are given in [**GiS1965**, Theorem 5.2]. We also refer to [**CLM2021**, Chapter 5.5] and to [**CW2021**, Chapter 1.1.2] for details.

Exercises for Chapter 141.

Exercise 141.1. In the proof of the Fundamental Group-Manifold Realization Proposition 54.12 that we gave on page 3031 we used the following statement. Let M be a compact connected non-empty smooth manifold that admits a handle decomposition such that all handles have index ≤ 2. If $\dim(M) \geq 4$, then the inclusion induced map $\pi_1(M) \to \pi_1(\mathrm{D}M)$ is an isomorphism. Does this statement also hold if $\dim(M) = 3$?

Exercise 141.2. Let M be a closed connected 3-dimensional smooth manifold. Show that $\pi_1(M)$ admits a balanced presentation.
Hint. By the Handle Decomposition Theorem 138.5 we know that M admits a handle decomposition. By the Handle-Euler Characteristic Corollary 140.4 and the Odd-Dimensional Manifold-Euler Characteristic 140.5 we know that

$$\sum_{k=0}^{3}(-1)^k \cdot \text{number of } k\text{-handles} \;=\; 0.$$

You could first consider the case that the handle decomposition contains precisely one 0-handle and one 3-handle. Afterwards try to tackle the general case.

Exercise 141.3. Let k be a field with at least three elements. Show that $\mathrm{SL}(2,k)$ is perfect, i.e. show that the abelianization is the trivial group.
Hint. Use that for any non-zero $d \in k$ and $a \in k$ we have

$$\begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} \;=\; \begin{pmatrix} 1 & (d^2-1)a \\ 0 & 1 \end{pmatrix}.$$

Exercise 141.4. Give an example of a finite CW-complex X which is not homotopy equivalent to a closed topological manifold but which admits a class $\sigma \in \mathrm{H}_n(X;\mathbb{Z})$ such that for each $k \in \mathbb{N}_0$ the map

$$\begin{array}{rccc} \cap\sigma\colon \mathrm{H}^{n-k}(X;\mathbb{Z}) & \to & \mathrm{H}_k(X;\mathbb{Z}) \\ \psi & \mapsto & \psi \cap \sigma \end{array}$$

is an isomorphism.

Hint. You could consider the wedge of a sphere with a suitable CW-complex with non-trivial fundamental group.

142. The classification of compact 2-dimensional smooth manifolds II

In this fairly short chapter we will finally complete the proof of the classification of compact 2-dimensional smooth manifolds. But first, as a warm-up, let us go one dimension lower, namely let us give a proof for half of the content of the Smooth 1-dimensional Manifold Classification Theorem 22.7.

Proposition 142.1. Every compact connected non-empty 1-dimensional smooth manifold is diffeomorphic either to S^1 or to $[0,1]$.

Remark. In Exercise 142.2 the reader is invited to show, using Propositions 31.2 and 142.1, that every non-compact connected 1-dimensional smooth manifold is diffeomorphic either to $[0,1)$ or to $\mathbb{R}$. This then completes the reproof of the Smooth 1-dimensional Manifold Classification Theorem 22.7.

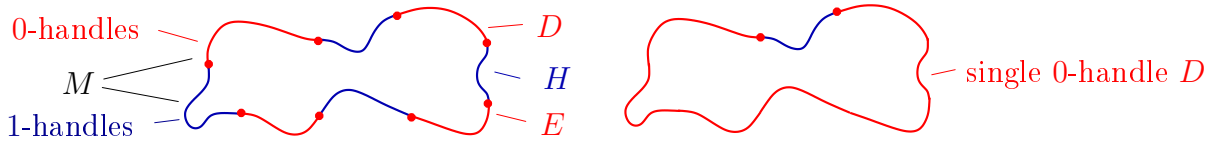
Proof. Let M be a compact connected non-empty 1-dimensional smooth manifold. By the Handle Decomposition Theorem 138.5 we can equip M with a standard handle decomposition.

Claim. There exists a standard handle decomposition for M with a single 0-handle.

Proof. Since M is non-empty we know that M has at least one 0-handle. Now suppose that our handle decomposition has more than one 0-handle. Since M is connected and since the handle decomposition is standard it follows from the Handle-Connected Lemma 136.8 that there exists a 1-handle H that gets glued to two different 0-handles D and E. It is elementary to see that $D \cup E \cup H$ is diffeomorphic to $[-1,1]$. Thus we can combine $D \cup E \cup H$ to a single 0-handle. Note that the resulting handle decomposition is still standard. We now proceed iteratively. ⊞

We now fix a handle decomposition for M with a single 0-handle D. If the handle decomposition does not contain a 1-handle, then $M = D$, in particular M is diffeomorphic to $[0,1]$.

Now suppose that the handle decomposition contains at least one 1-handle. Since the handle decomposition is standard we know that all attaching maps of the 1-handles take values in $\partial M^1 = \partial D$. Now let $\varphi\colon \{0\} \times S^0 \to \partial D$ be an attaching map of a 1-handle. Note that the map is injective. Since both sets consist of precisely two elements we see that the attaching map is a bijection. In particular we see that $D \cup_\varphi (\{0\} \times \overline{B}^1)$ is a closed 1-dimensional smooth manifold. It is elementary to see that this smooth manifold is diffeomorphic to S^1. Since this smooth manifold is closed we see that there are no other 1-handles. Thus we have shown that M is diffeomorphic to S^1. ■



142.1. **Handle decompositions of 2-dimensional smooth manifolds.** In this section we will study handle decompositions of some of our favorite 2-dimensional smooth manifolds. We will also learn how to modify handle decompositions of 2-dimensional smooth manifolds. In the next section we will use the results from this section to give a surprisingly short proof of the statement that any compact connected non-empty 2-dimensional smooth

manifold is diffeomorphic to one of the standard examples.

We start out with the following lemma which is very specific to the 2-dimensional setting.

Lemma 142.2. Let M be a 2-dimensional smooth manifold, let C be a boundary component and let $\varphi\colon S^1 \to C$ be a diffeomorphism. The diffeomorphism type of $M \cup_\varphi h^2$ does not depend on the choice of φ.

Remark. The proof of Lemma 142.2 rests on the fact, proved in the diffeomorphisms-of-S^1-Extension Corollary 34.6, that for $n = 2$ every diffeomorphism of S^{n-1} extends to a diffeomorphism of $\overline{B}^n$. Note that by Theorem 146.7 and the Cerf Theorem 58.5 the analogous statement also holds for $n = 3$ and $n = 4$. In particular the obvious analogue of Lemma 142.2 also holds in those dimensions. But we will see in Proposition **??** that the analogue of Lemma 142.2 does *not* hold for $n = 7$.

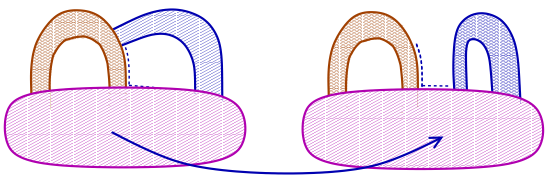
Proof. Let M be a 2-dimensional smooth manifold, let C be a boundary component of M and let $\varphi, \psi\colon S^1 \to C$ be two diffeomorphisms. We consider the self-diffeomorphism $f := \varphi \circ \psi^{-1}\colon S^1 \to S^1$. By the diffeomorphisms-of-S^1-Extension Corollary 34.6 the self-diffeomorphism f of S^1 extends to a self-diffeomorphism F of the disk $\overline{B}^2$. Now we obtain from the Gluing-Smooth Manifolds-Proposition **??** (7) that $M \cup_\varphi h^2 = M \cup_{C \overset{\varphi}{\leftarrow} S^1} \overline{B}^2$ is diffeomorphic to $M \cup_\psi h^2 = M \cup_{C \overset{\psi}{\leftarrow} S^1} \overline{B}^2$. ■

In the following we will state and prove a few results which are not germane to the 2-dimensional setting, but which also have analogues in the higher-dimensional setting, which we will eventually discuss in Chapter **??**. In this and the following section we will use slightly informal language and we will partially work with figures. Nonetheless the mathematically mature reader should have no troubles with making the whole discussion completely and utterly rigorous.

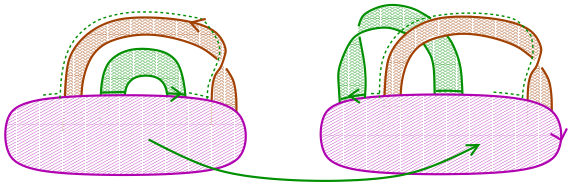
First we formulate the following special case of the Handle Attachment Proposition 136.1 (4).

Lemma 142.3. Let M be a 2-dimensional smooth manifold. Given any smooth homotopy $F\colon (\overline{B}^1 \times S^0) \times [0,1] \to \partial M$ there exists a diffeomorphism $M \cup_{F_0} h^1 \to M \cup_{F_1} h^1$. Any such replacement of the 1-handle given by F_0 by the 1-handle given by F_1 is called a *handle slide*.

Example. For what follows it will be very helpful to get a little experience with what handle slides can do.



sliding the blue handle
off the brown handle



sliding the green handle
over the brown handle

Lemma 142.4. Every closed connected non-empty 2-dimensional smooth manifold admits a handle decomposition with the following properties:

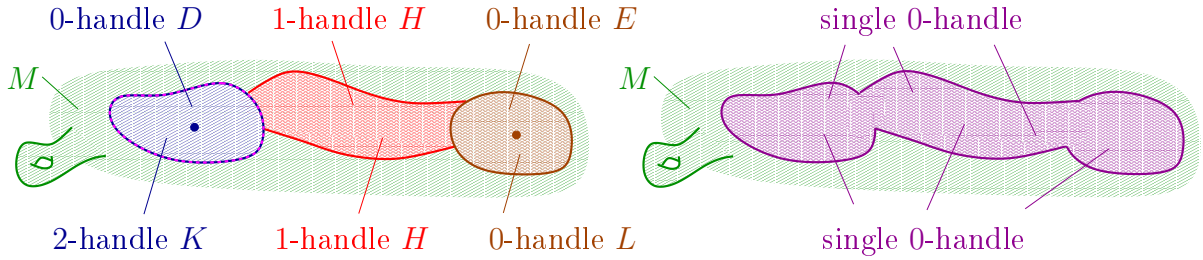
(1) There exists a single 0-handle.
(2) The attaching maps of all of the 1-handles take values in the boundary of the 0-handle.[1690]
(3) There exists a single 2-handle.

Remark. In Corollary **??** we will see that the conclusion of Lemma 142.4 in fact holds in all dimensions.

Sketch of proof. By the Handle Decomposition Theorem 138.5 we can equip M with a standard handle decomposition. Since M is non-empty we know that there exists at least one 0-handle. Suppose that M has more than one 0-handle. Since M is connected it follows from the fact that the handle decomposition is standard together with the Handle-Connected Lemma 136.8 that there exists a 1-handle H that gets glued to two different 0-handles D and E. In Exercise 142.3 we will see that $D \cup E \cup H$ is again diffeomorphic to a disk. Thus we can combine $D \cup E \cup H$ to a single 0-handle. It is straightforward to verify that the new handle decomposition is again standard and that the number of 2-handles is unaffected. We now proceed iteratively till we get down to a single 0-handle.

Next we apply the above procedure to the dual handle decomposition and we can arrange that the dual handle decomposition has a single 0-handle. Furthermore, as mentioned above, we can do this without changing the fact that the dual handle decomposition is standard and without changing the fact that the dual handle decomposition has a single 2-handle.

In summary, we have found a standard handle decomposition with a single 0-handle and a single 2-handle. ■

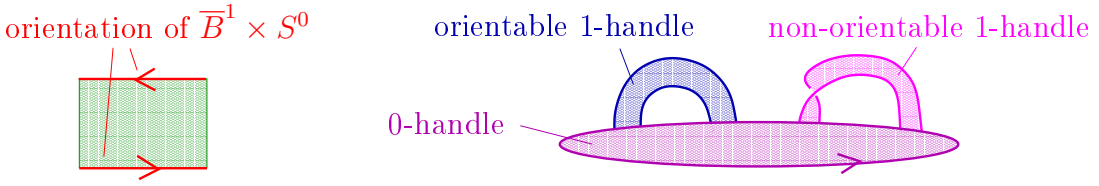


Definition.

(1) We say a handle decomposition of 2-dimensional smooth manifold is completely standard, if it is of the form described in Lemma 142.4.
(2) Let M be a 2-dimensional smooth manifold that is equipped with a completely standard handle decomposition. We equip $\overline{B}^1 \times \overline{B}^1$ with the product orientation and we equip $\overline{B}^1 \times S^0 = \overline{B}^1 \times \partial\overline{B}^1 \subset \overline{B}^1 \times \overline{B}^1$ with the boundary orientation. As on page 2963 we say that a 1-handle with attaching map $\varphi\colon \overline{B}^1 \times S^0 \to \partial\overline{B}^2$ is orientable if the attaching map is either orientation-preserving on both components

[1690] In fact this statement is in this context equivalent to saying that the handle decomposition is standard in the sense of the definition on page 2958.

of $\overline{B}^1 \times S^0$ or if it is orientation-reversing on both components of $\overline{B}^1 \times S^0$. Otherwise we say that the 1-handle is non-orientable.



Lemma 142.5. Let M be a 2-dimensional smooth manifold M with a completely standard handle decomposition. The following two statements are equivalent:

(1) M is orientable.
(2) All 1-handles are orientable.

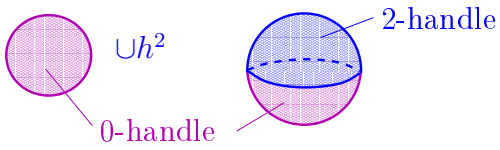
Proof. In the Gluing-Smooth Manifolds-Proposition **??** (6) we dealt with the orientability of smooth manifolds that are given by gluing oriented smooth manifolds along unions of boundary components. In our setting we only glue along codimension-zero submanifolds of the boundary. But the same statements hold and we see that our lemma is basically a special case of the Gluing-Smooth Manifolds-Proposition **??** (6). ■

Convention.

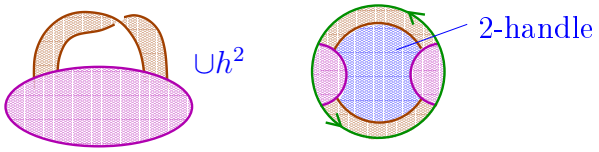
(1) Let M be a 2-dimensional smooth manifold M with a single 0-handle $\overline{B}^2$. If M is orientable, then we equip M with the orientation such that the natural map $\overline{B}^2 \to M$ given by the 0-handle is orientation-preserving.
(2) Let M be a closed 2-dimensional smooth manifold. If we are given a handle decomposition for M with a single 2-handle, then in the figures we show the 0-handles and the 1-handle attachments and we indicate with a discrete $\cup h^2$ the 2-handle. Note that by Lemma 142.2 the 2-handle attachment is basically unambiguous.

Examples.

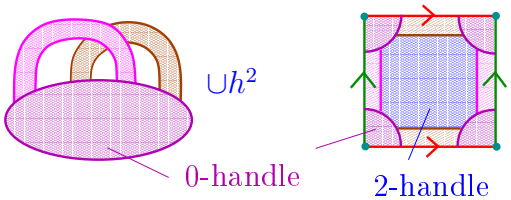
(1) In the figure below we show several completely standard handle decompositions for several of our favorite surfaces.



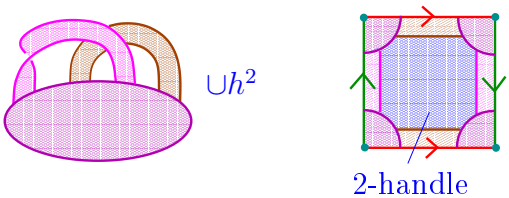
handle decomposition for the sphere



handle decomposition for $\mathbb{R}\mathrm{P}^2$

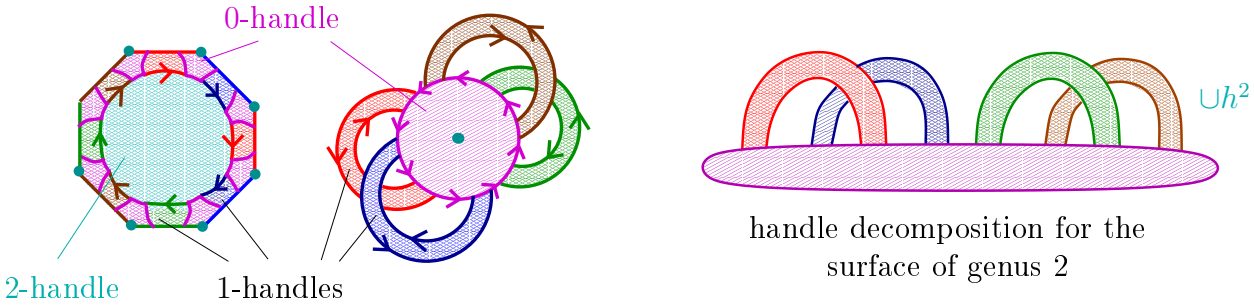


handle decomposition for the torus



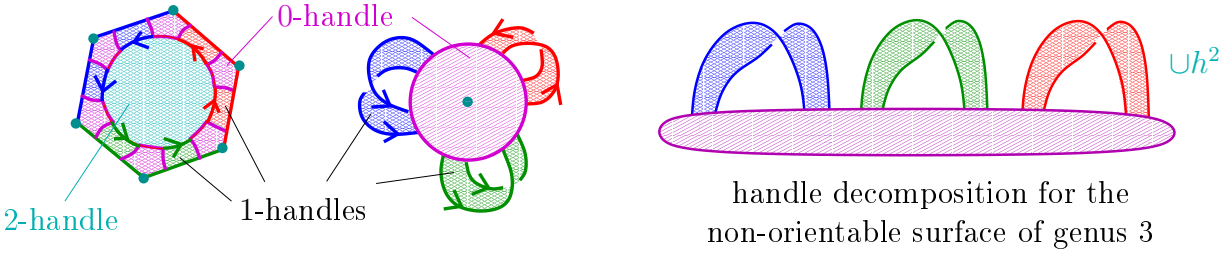
handle decomposition for the Klein bottle

(2) (a) In the figure below we show a completely standard handle decomposition for the surface of genus two. This handle decomposition has an obvious generalizations to higher genera.



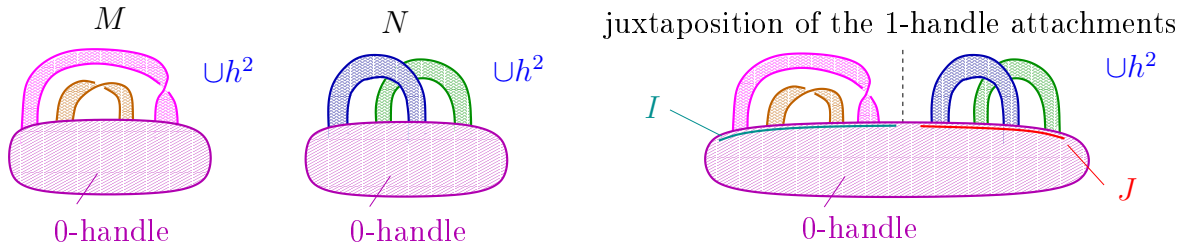
handle decomposition for the surface of genus 2

(b) In the figure below we show a completely standard handle decomposition for the non-orientable surface of genus 3, as defined on page 266. Again this example has obvious generalizations to higher genera.



handle decomposition for the non-orientable surface of genus 3

Lemma 142.6. Let M be a nd N be two closed 2-dimensional smooth manifolds. We assume that both are equipped with a completely standard handle decomposition. Then $M\#N$ has a completely standard handle decomposition such that the 1-handles are given by "juxtaposing" the 1-handle attachments for M and the 1-handle attachments for N. We will not give a formal definition of "juxtaposing", instead we refer to the figure below for a hopefully unambiguous "definition by illustration".



handle decomposition for $M\#N$

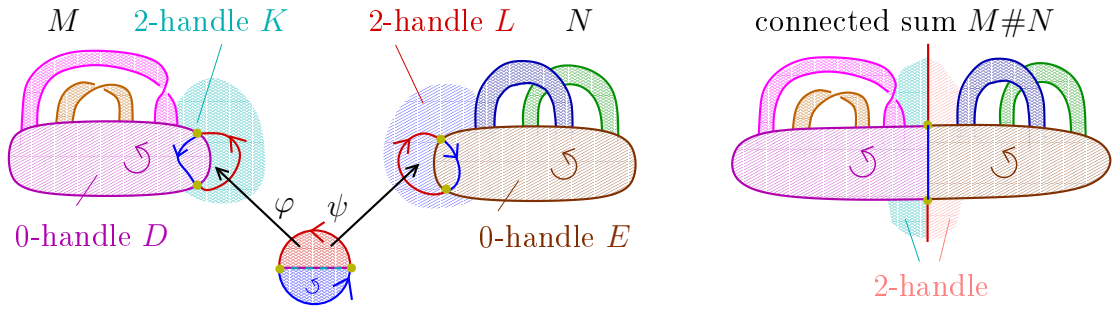
Proof. We denote by $D = \overline{B}^2$ the unique 0-handle of M and we denote by K the unique 2-handle of M. Furthermore we denote by $E = \overline{B}^2$ the unique 0-handle of N and we denote by L the unique 2-handle of N. It follows almost immediately from Exercise 136.3 that we can pick a smooth embedding $\varphi\colon \overline{B}^2 \to M$ such that $\varphi(\overline{B}^2_{\leq 0})$ is contained in the 0-handle D, such that $\varphi\colon \overline{B}^2_{\leq 0} \to D = \overline{B}^2$ is orientation-preserving and such that $\varphi(\overline{B}^2_{\geq 0})$ is contained in the 2-handle K. Similarly we pick a smooth embedding $\varphi\colon \overline{B}^2 \to N$, with the only real difference being, that we demand that the map $\psi\colon \overline{B}^2_{\leq 0} \to E$ is orientation-reversing. Now

we have

$$M\#N \overset{\text{by definition of the connected sum, note that if } M \text{ and } N \text{ are oriented, then the orientation conventions work out}}{\overset{\downarrow}{=}} (M \setminus \varphi(B^2)) \sqcup (N \setminus \psi(B^2))/\varphi(x) \sim \psi(x) \text{ for } x \in S^1$$

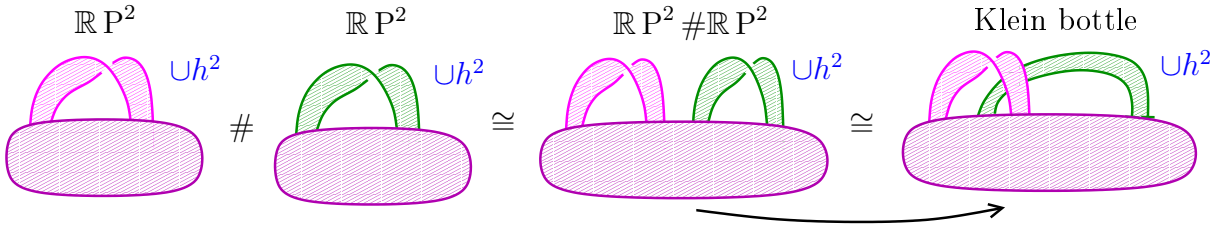
$$= \overbrace{(D \setminus \varphi(B^2_{\leq 0})) \cup (E \setminus \psi(B^2_{\leq 0}))}^{\text{diffeomorphic to } \overline{B}^2} \cup \text{ 1-handles of } M \cup \text{ 1-handles of } N \cup \underbrace{(K \setminus \varphi(B^2_{\geq 0})) \cup (L \setminus \psi(B^2_{\geq 0}))}_{\text{diffeomorphic to } \overline{B}^2}.$$

Thus we have obtained a handle decomposition for $M\#N$ with a single 0-handle, a single 2-handle, and such that the handles of M and N get attached to disjoint semicircles in the boundary of the 0-handle. In other words, the 1-handle attachments are juxtaposed. ■



Examples.

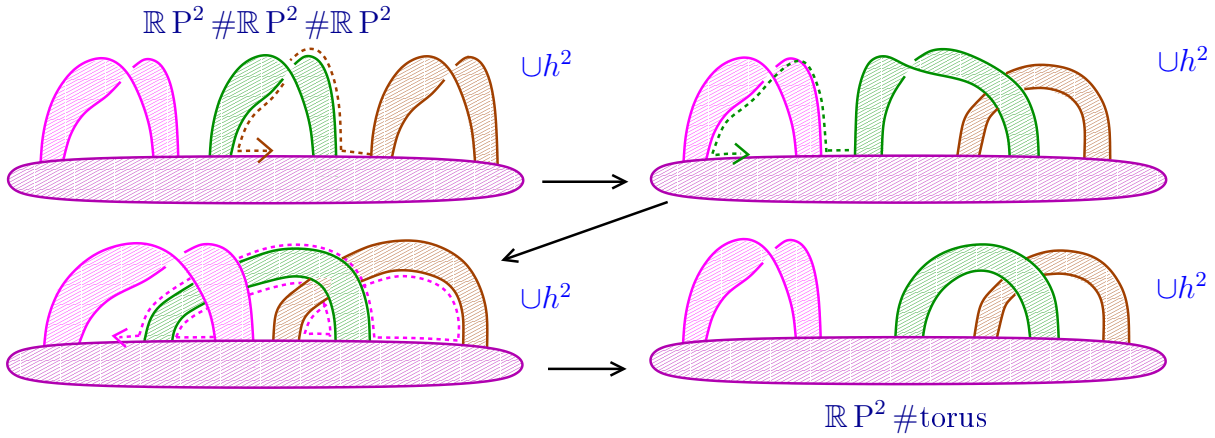
(1) In the figure below we use Lemma 142.6 to give a handle decomposition of the connected sum $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ of two copies of the real projective plane $\mathbb{R}\mathrm{P}^2$. Furthermore we show a handle slide that results in the handle decomposition of the Klein bottle that we already saw in the figure on page 3047. This argument shows in particular that $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ is diffeomorphic to Klein bottle. Thus we have obtained a new proof of Lemma **??**.



(2) (a) In the figure below we show, using altogether three handle slides, that the connected sum $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ is diffeomorphic to the connected sum of $\mathbb{R}\mathrm{P}^2$ with the torus.

(b) Now let $g \in \mathbb{N}_0$. We consider the connected sum of $2g+1$ copies of $\mathbb{R}\mathrm{P}^2$ as given by attaching $2g+1$ non-orientable 1-handles to a single 0-handle. We iteratively apply the procedure in the figure below to the three right-most non-orientable 1-handles. We turn the two right-most non-orientable 1-handles into a pair of

interlocking orientable 1-handles. Eventually we see that the connected sum of $2g+1$ copies of $\mathbb{R}\mathrm{P}^2$ is diffeomorphic to the connected sum of $\mathbb{R}\mathrm{P}^2$ with the surface of genus g.



142.2. **The classification of compact 2-dimensional smooth manifolds.** We recall the following notation that we introduced on page 1234.

Notation. Given $g, n \in \mathbb{N}_0$ we write[1691]

$$\Sigma_{g,n} := \text{the surface of genus } g \text{ minus } n \text{ open disks.}$$

Furthermore for $g \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we write

$$N_{g,n} := \text{the non-orientable surface of genug } g \text{ minus } n \text{ open disks.}$$

In the Surfaces-Non Homeomorphic Theorem 55.6 we showed that the above surfaces are pairwise non-homeomorphic, in particular we showed they are pairwise non-diffeomorphic.[1692] In this section we prove that any compact connected 2-dimensional smooth manifold is diffeomorphic to one of the above surfaces. This finally completes the classification of compact surfaces that we started in Chapter 55.

Here is the precise theorem we intend to prove.

Theorem 142.7. (Diffeomorphism-to-Standard Surfaces Theorem) Let Σ be a compact connected non-empty 2-dimensional smooth manifold.

(1) If Σ is orientable, then Σ is diffeomorphic to the surface $\Sigma_{g,n}$ for some $g, n \in \mathbb{N}_0$.
(2) If Σ is non-orientable, then Σ is diffeomorphic to the surface $N_{k,n}$ for some $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.

Remark.

(1) Our proof of the Diffeomorphism-to-Standard Surfaces Theorem 142.7 is inspired by the ambitious [**GoS1999**, Exercise 4.2.4]. The key idea was communicated to us by Robert Gompf. The proof we provide is also outlined in [**Mart2016**, Theorem 6.1.6].
(2) In Exercise **??** we will also sketch an alternative approach to proving the Diffeomorphism-to-Standard Surfaces Theorem 142.7.

[1691] We refer to page 1233 for the definition of "minus n open disks".
[1692] In the Surfaces-Non diffeomorphic Corollary 45.21 we proved state the surfaces Σ_g are pairwise non-diffeomorphic.

(3) One can prove the Diffeomorphism-to-Standard Surfaces Theorem 142.7 through Morse theory, without using handles. This approach is taken in [**Mata2002**, Chapter 5.2] and [**Hirs1976**, Chapter 3].
(4) On page 1240 we discuss several other approaches to proving the analogue of the Diffeomorphism-to-Standard Surfaces Theorem 142.7 in the setting of topological manifolds.

To increase readability we split the proof of the Diffeomorphism-to-Standard Surfaces Theorem 142.7 into two parts. First we treat closed manifolds, afterwards we will prove the general case.

Proof of Theorem 142.7 for closed manifolds. By the Handle Decomposition Theorem 138.5 together with Lemma 142.4 it suffices to prove the proposition for a closed 2-dimensional smooth manifold that is equipped with a completely standard handle decomposition, i.e. that is equipped with a handle decomposition with the following properties:

(1) There exists a single 0-handle D.
(2) The attaching maps of all of the 1-handles take values in the boundary of the 0-handle.
(3) There exists a single 2-handle.

We prove this statement by induction on the number of 1-handles. If we have zero 1-handles, then M is given by gluing a 2-handle to a 0-handle via some attaching map $\varphi\colon S^1 \to S^1$. Now we see that we have the following diffeomorphisms:

$$M = \overline{B}^2 \cup_\varphi h^2 = \overline{B}^2 \cup_{S^1 \xleftarrow{\varphi} S^1} \overline{B}^2 \underset{\uparrow}{\cong} \overline{B}^2 \cup_{S^1 \xleftarrow{\mathrm{id}} S^1} \overline{B}^2 \underset{\uparrow}{\cong} S^2.$$

by Lemma 142.2 all attaching maps give same diffeomorphism type

Handle Example Lemma 136.2

Next suppose that the claim holds whenever the number of 1-handles is $\leq k$. Thus let M be a smooth manifold as above with $k+1$ 1-handles.

Perhaps slightly surprisingly the case that the smooth manifold is non-orientable is slightly easier. So let us deal with this case initially. Since M is now assumed to be non-orientable we obtain from Lemma 142.5 that there exists a non-orientable 1-handle. We denote the corresponding attaching map by α. We denote by $\psi_1, \ldots, \psi_k$ the attaching maps of the remaining 1-handles. Let $I \subset \partial D$ be a compact interval that contains the image of the attaching map of the chosen 1-handle and let $J \subset \partial D$ be the compact interval with $\partial D = I \cup J$ and $I \cap J = \partial I = \partial J$. Here comes the key observation: *Since the 1-handle is non-orientable we see that $\partial(\overline{B}^2 \cup_\alpha h^1)$ is connected. Thus we can iteratively slide all handle attachments $\psi_1, \ldots, \psi_k$ that end up on I across the chosen 1-handle so that they take values in the interval J.* (We refer to the figure below for an illustration.) We denote the new handle attachment maps by $\widetilde{\psi}_1, \ldots, \widetilde{\psi}_k$.
We now see that

follows from Lemmas 142.2 and 142.3

$$M \cong \overline{B}^2 \cup_\alpha h^1 \cup_{\psi_1} h^1 \cdots \cup_{\psi_k} h^1 \cup h^2 \cong \overline{B}^2 \cup_\alpha h^1 \cup_{\widetilde{\psi}_1} h^1 \cdots \cup_{\widetilde{\psi}_k} h^1 \cup h^2$$

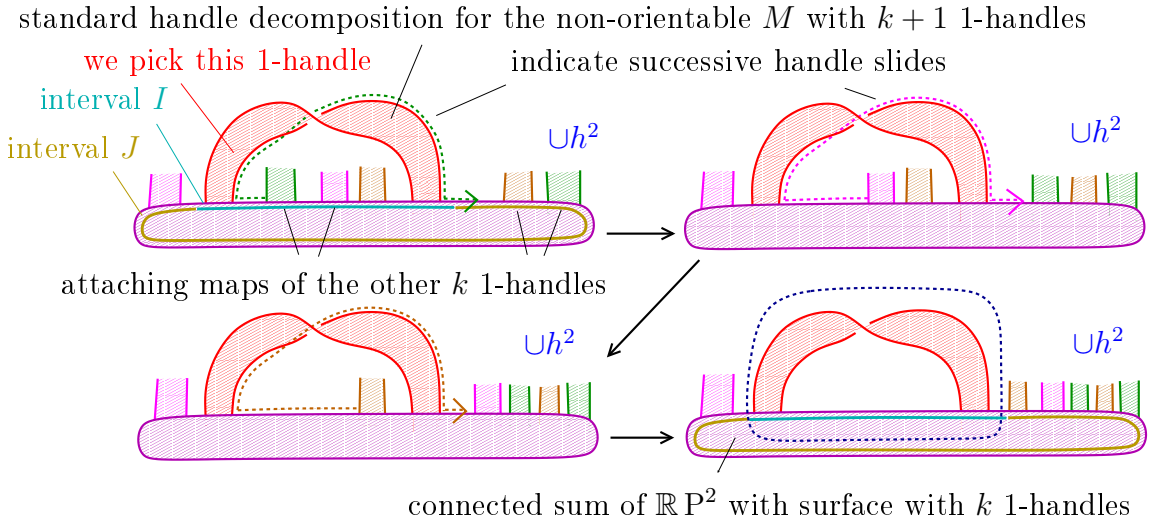
$$\cong \underbrace{(\overline{B}^2 \cup_\alpha h^1 \cup h^2)}_{\cong \mathbb{R}\mathrm{P}^2, \text{ see the figure below}} \# \underbrace{(\overline{B}^2 \cup_{\widetilde{\psi}_1} h^1 \ldots \cup_{\widetilde{\psi}_k} h^1 \cup h^2)}_{\text{by induction } \cong k \cdot \mathbb{R}\mathrm{P}^2 \text{ or } \cong \Sigma_{\frac{k}{2}}} \cong (k+1)\cdot\mathbb{R}\mathrm{P}^2 \cong \text{non-orientable surface of genus } k+1.$$

by Lemma 142.6

see the figure below where we see that $\mathbb{R}\mathrm{P}^2 \# \Sigma_l \cong (2l+1)\cdot\mathbb{R}\mathrm{P}^2$

by Lemma ?? (2)

We have thus proved the induction step if M is non-orientable.



Now we turn to the case that M is orientable. We pick any 1-handle. Since M is orientable the chosen 1-handle is orientable. Note that this implies that the boundary of $\overline{B}^2 \cup_\alpha h^1$ is disconnected. But since we work with a completely standard handle decomposition we know that, once we attached all 1-handles, the boundary is connected. This implies that there exists at least one other 1-handle that connects the two components of the boundary of $\overline{B}^2 \cup_\alpha h^1$. We pick such a 1-handle and we call it the *dual handle*. We denote its attaching map by β. We denote by $\psi_1, \ldots, \psi_{k-1}$ the attaching maps of the remaining 1-handles. Let $I \subset \partial D$ be a compact interval that contains the image of the attaching maps α and β and let $J \subset \partial D$ be the compact interval with $\partial D = I \cup J$ and $I \cap J = \partial I = \partial J$. Here comes the key observation: *It is straightforward to see that* $\partial(\overline{B}^2 \cup_\alpha h^1 \cup_\beta h^1)$ *is connected. Thus we can iteratively slide all handle attachments* $\psi_1, \ldots, \psi_k$ *that end up on* I *across the two 1-handles so that they take values in the interval* J. (We refer to the figure below for an illustration.) We denote the new handle attachment maps by $\widetilde{\psi}_1, \ldots, \widetilde{\psi}_{k-1}$.

We now see that

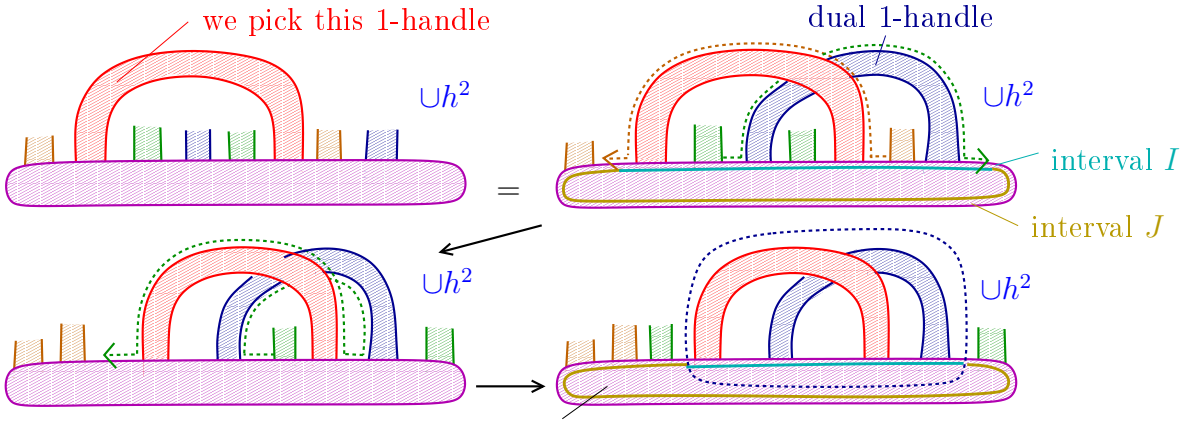
$$M \cong \overline{B}^2 \cup_\alpha h^1 \cup_\beta h^1 \cup_{\psi_1} h^1 \cdots \cup_{\psi_{k-1}} h^1 \cup h^2 \overset{\text{follows from Lemmas 142.2 and 142.3}}{\cong} \overline{B}^2 \cup_\alpha h^1 \cup_\beta h^1 \cup_{\widetilde{\psi}_1} h^1 \cdots \cup_{\widetilde{\psi}_{k-1}} h^1 \cup h^2$$

$$\cong \underbrace{(\overline{B}^2 \cup_\alpha h^1 \cup_\beta h^1 \cup h^2)}_{\cong \text{ torus, see the figure on page 3047}} \# \underbrace{(\overline{B}^2 \cup_{\widetilde{\psi}_1} h^1 \ldots \cup_{\widetilde{\psi}_{k-1}} h^1 \cup h^2)}_{\text{since all bands are orientable we know that this is an orientable surface, thus by induction we know that it is diffeomorphic to } \Sigma_{\frac{k-1}{2}}} \overset{\text{follows from Lemma ?? (1) or alternatively the discussion on page 3049}}{\cong} \text{surface of genus } \tfrac{k+1}{2}.$$

by Lemma 142.6

We have thus now proved the induction step also in the case that M is orientable. ∎

standard handle decomposition for the orientable M with $k+1$ 1-handles



connected sum of torus with surface with $k-1$ orientable 1-handles

In the following we will finally deal with 2-dimensional smooth manifolds with non-empty boundary.

Proof of Theorem 142.7 for all manifolds. Let M be a compact connected non-empty 2-dimensional smooth manifold. By the Smooth Manifold Boundary Proposition 19.26 the boundary ∂M is a closed 1-dimensional smooth manifold. Since ∂M is compact it has finitely many components that we denote by $C_1, \ldots, C_n$. (Note that we allow the case that $n = 0$.) Each component C_i is compact and has no boundary. Thus it follows from Proposition 142.1 that for each $i = 1, \ldots, n$ there exists a diffeomorphism $f_i \colon S^1 \to C_i$.

We denote by $\widehat{M}$ the result of gluing a disk to each boundary component. More precisely, we consider

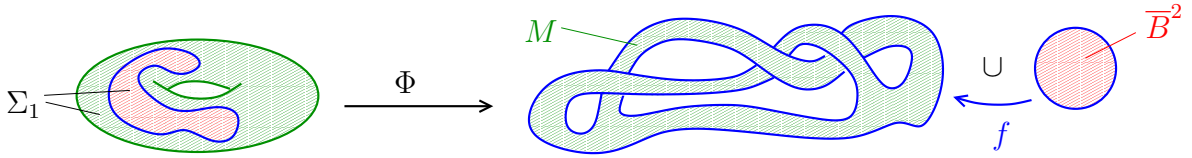
$$\widehat{M} := \left(M \sqcup \bigsqcup_{i=1}^{n} (\overline{B}^2 \times \{i\}) \right) / \sim \quad \text{where } f_i(P) \sim (P, i) \text{ for } P \in S^1 \text{ and } i \in \{1, \ldots, n\}.$$

By the Gluing-Smooth Manifolds-Proposition **??** we know that $\widehat{M}$ is a closed connected non-empty 2-dimensional smooth manifold that contains M as a smooth submanifold and which has the property that the obvious maps $\overline{B}^2 \times \{i\} \to \widehat{M}$ are all smooth embeddings. Furthermore it follows from the Gluing-Smooth Manifolds-Proposition **??** (6) and the observation that $\overline{B}^2$ admits an orientation-reversing diffeomorphism, that M is orientable if and only if $\widehat{M}$ is orientable.

Now suppose that M is orientable. By the above $\widehat{M}$ is also orientable. Since $\widehat{M}$ is closed orientable connected and non-empty we know by the above that there exists a diffeomorphism $\Phi\colon \Sigma_g \to \widehat{M}$ from the surface of some genus g to $\widehat{M}$. Then Φ restricts to a diffeomorphism

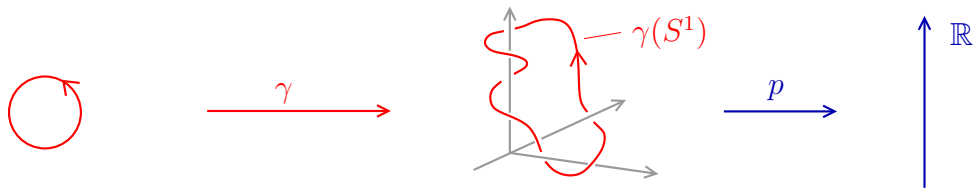
$$\Phi\colon \underbrace{\Sigma_g \setminus \bigcup_{i=1}^{n} \Phi^{-1}\big(B^2 \times \{i\}\big)}_{= \Sigma_{g,n} \text{ since the maps } \overline{B}\times\{i\}\to\Sigma_g \text{ are all smooth embeddings}} \to \; M.$$

If M is non-orientable, then $\widehat{M}$ is closed connected non-orientable and non-empty, thus we know that it is diffeomorphic to the non-orientable surface N_g for some g. As above we see that M is diffeomorphic to $N_{g,n}$. ■



Exercises for Chapter 142.

Exercise 142.1. We denote by $p\colon \mathbb{R}^3 \to \mathbb{R}$ the projection onto the third coordinate. Let $\gamma\colon S^1 \to \mathbb{R}^3$ be a smooth embedding such that the map $p \circ \gamma\colon S^1 \to \mathbb{R}$ has precisely two critical points. Show that $\gamma(S^1)$, viewed as a submanifold of $\mathbb{R}^3 \cup \{\infty\} = S^3$ is smoothly isotopic to the trivial knot.



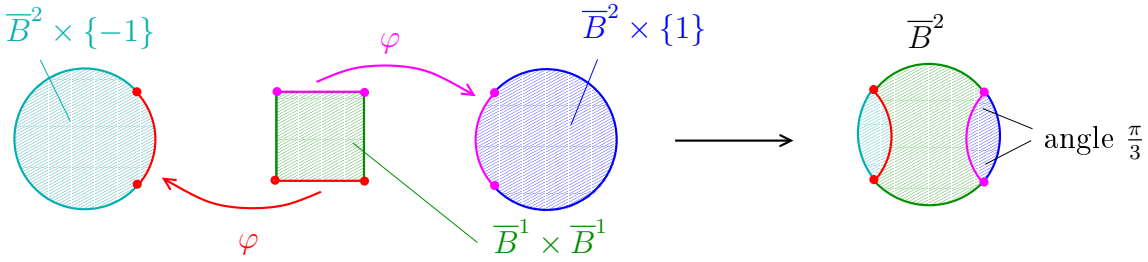
Exercise 142.2. Show that every non-compact connected 1-dimensional smooth manifold is diffeomorphic either to $[0,1)$ or to $\mathbb{R}$.
Hint. Use Propositions 31.2 and 142.1.

Exercise 142.3. We consider the attaching map

$$\begin{aligned} \varphi\colon \overline{B}^1 \times S^0 &\to \overline{B}^2 \times \{\pm 1\} \\ (y,\epsilon) &\mapsto \Big(\big(\epsilon \cdot \sqrt{1-\tfrac{y^2}{4}}, \tfrac{y}{2}\big), \epsilon\Big). \end{aligned}$$

Show that $(\overline{B}^1 \times \overline{B}^1) \cup_{\varphi\colon \overline{B}^1\times S^0 \to S^1 \times \{\pm 1\}} (\overline{B}^2 \times \{\pm 1\})$ is diffeomorphic to $\overline{B}^2$.
Remark.

(1) It follows easily from this exercise together with the Smooth Ball Embedding Theorem 37.8 that any attaching of a 1-handle to two disks is diffeomorphic to $\overline{B}^2$.
(2) By "rotating around the x-axis" one sees that a similar statement also holds in the higher-dimensional setting.


$\overline{B}^2 \times \{-1\}$
φ
$\overline{B}^2 \times \{1\}$
$\overline{B}^2$
angle $\frac{\pi}{3}$
φ
$\overline{B}^1 \times \overline{B}^1$

Part XV

Bundles

143. Vector Bundles

In this chapter we introduce vector bundles and in the following chapter we will introduce their Pontryagin classes. Both chapters are only intended as a quick introduction. Therefore on many occasions we will only provide sketches of proofs or we will refer to the literature for the proofs. Furthermore we will be less worried about why maps are actually continuous. The best reference for these two chapters is the classic book by Milnor-Stasheff [**MiS1974**] on characteristic classes. Other references are [**Miln1957a, Hat2**] and [**Hus1993**].

143.1. **Vector bundles.** In this section we introduce vector bundles and we give several constructions and examples. The definition of a vector bundle is fairly similar to the definition of a smooth $\overline{B}^m$-bundle that we gave on page **??** and the definition of a linear $\overline{B}^m$-bundle that we gave on page **??**.
Loosely speaking, a vector bundle over a topological space X consists of a vector space for each $x \in X$, where the "vector spaces vary continuously with x". The following definition makes this idea precise.

Definition. An n-dimensional real vector bundle over a topological space X is a topological space E together with a map $p\colon E \to X$ and a real vector space structure on each $p^{-1}(x)$, $x \in X$ such that for each $x \in X$ there exists an open neighborhood U of x and a homeomorphism $\varphi\colon p^{-1}(U) \to U \times \mathbb{R}^n$ such that the following two conditions are satisfied:

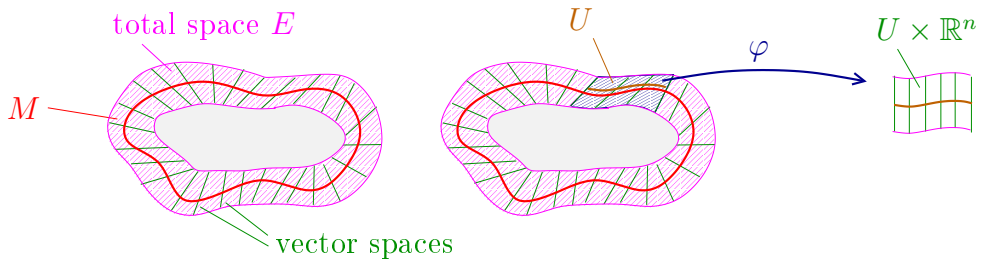
(1) The following diagram commutes

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\varphi} & U \times \mathbb{R}^n \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (y,v)\mapsto y} & \\ & U. & \end{array}$$

(2) For each $y \in U$ the map $\varphi\colon p^{-1}(y) \to \{y\} \times \mathbb{R}^n = \mathbb{R}^n$ is an isomorphism of vector spaces.

(The definition is illustrated in the figure below.) Exactly the same way we also define the notion of an n-dimensional complex vector bundle. When we are agnostic about whether we consider real or complex vector bundles, then sometimes we just say vector bundle. We introduce the following notation and language:

(a) We refer to the topological space E as the total space of the vector bundle and we refer to X as the base space of the vector bundle. Furthermore we refer to the map $p\colon E \to X$ as the projection.
(b) A 1-dimensional real (respectively complex) vector bundle is often called a real (respectively complex) line bundle.
(c) Given a vector bundle $p\colon E \to X$ and given $x \in X$ we write $E_x := p^{-1}(x)$. Sometimes we refer to E_x as the fiber of E over x.
(d) We refer to a homeomorphism $\varphi\colon p^{-1}(U) \to U \times \mathbb{R}^n$ which satisfies the above conditions (1) and (2) as a trivialization of the vector bundle.



Examples. In each of the following examples it is fairly elementary to verify that these are indeed vector bundles. For the sake of brevity[1693] we do not give all the details.

(1) For each topological space X the product $X \times \mathbb{R}^n$ together with the projection $p\colon X \times \mathbb{R}^n \to X$ is an n-dimensional vector bundle over X. We refer to this vector bundle as the *trivial n-dimensional real vector bundle over X*. The same way we also define the *trivial n-dimensional complex vector bundle over X*.

(2) Let M be an m-dimensional submanifold of $\mathbb{R}^k$.

(a) The map[1694]

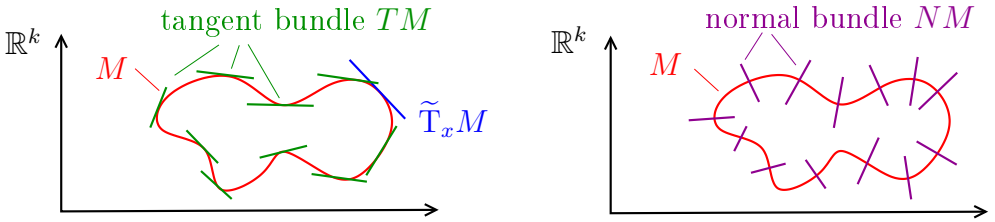
$$\begin{array}{rcl} \mathrm{T}M := \{(x,v) \in M \times \mathbb{R}^k \,|\, x \in M \text{ and } v \in \widetilde{\mathrm{T}}_x M\} & \to & M \\ (x,v) & \mapsto & x \end{array}$$

is an m-dimensional real vector bundle over M. We refer to it as the *tangent bundle* of the submanifold M.

(b) The map

$$\begin{array}{rcl} NM := \{(x,v) \in M \times \mathbb{R}^k \,|\, x \in M \text{ and } v \in \mathbb{R}^k \text{ with } v \perp \widetilde{\mathrm{T}}_x M\} & \to & M \\ (x,v) & \mapsto & x \end{array}$$

is a $(k-m)$-dimensional real vector bundle over M. We call it the *normal bundle* of M in $\mathbb{R}^k$.



(3) Given any vector bundle $p\colon E \to X$ and given any subset $W \subset X$ the map $p\colon p^{-1}(W) \to W$ is evidently also a vector bundle. We refer to it as the restriction of p to W.

(4) Recall that given $n \in \mathbb{N}_0$ we defined on page 251 the real projective space to be

$$\mathbb{R}\mathrm{P}^n \;:=\; (\mathbb{R}^{n+1} \setminus \{0\})/\!\sim$$

[1693] No snickering please.

[1694] We refer to page 620 for the definition of the "physical tangent spaces" $\widetilde{\mathrm{T}}_x M \subset \mathbb{R}^k$ of a submanifold $M \subset \mathbb{R}^k$.

where $\sim$ denotes the equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$ given by multiplication by an element in $\mathbb{R} \setminus \{0\}$. The map

$$\begin{array}{rcl} X = \{([v], \lambda v) \in \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1} \,|\, [v] \in \mathbb{R}\mathrm{P}^n, \lambda \in \mathbb{R}\} & \to & \mathbb{R}\mathrm{P}^n \\ ([v], w) & \mapsto & [v] \end{array}$$

defines a one-dimensional vector bundle over $\mathbb{R}\mathrm{P}^n$. We refer to it as the tautological real line bundle over $\mathbb{R}\mathrm{P}^n$. The same way we also define the tautological complex line bundle over $\mathbb{C}\mathrm{P}^n$ and the same way we also define the tautological line bundles over $\mathbb{R}\mathrm{P}^\infty$ and $\mathbb{C}\mathrm{P}^\infty$.

Before we introduce more examples we give the following, basically self-evident definition.

Definition. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to X$ and $q\colon F \to X$ be two $\mathbb{K}$-vector bundles over a topological space X. An isomorphism of the $\mathbb{K}$-vector bundles over X is a map $\Phi\colon E \to F$ that satisfies the following two conditions:

(1) The following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X. & \end{array}$$

(2) For each $x \in X$ the map $\Phi|_{E_x}\colon E_x \to F_x$ is an isomorphism of $\mathbb{K}$-vector spaces.

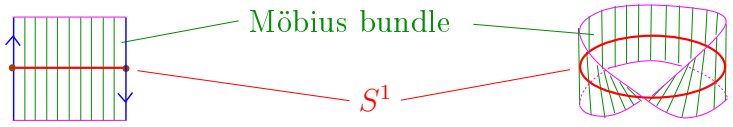
We continue with a few more examples of real bundles.

Examples.

(1) The map

$$\begin{array}{rcl} ([0,1] \times \mathbb{R})/(0,v) \sim (1,-v) & \to & S^1 \\ [(t,v)] & \mapsto & \exp(2\pi\,\mathrm{i}t) \end{array}$$

is a line bundle over S^1. The discussion on page 264 shows that the total space is homeomorphic to the open Möbius band. Therefore we refer to this bundle as the *Möbius bundle*. This bundle, and a bundle that is isomorphic to it, are illustrated in the figure below.



(2) Let M be an n-dimensional smooth manifold. (Note that in contrast to the discussion on page 3059 we do not assume that M is a submanifold of some $\mathbb{R}^k$.) We consider the map

$$\begin{array}{rcl} p\colon \mathrm{T}M := \{(x,v) \,|\, x \in M \text{ and } v \in \mathrm{T}_x M\} & \to & M \\ (x,v) & \mapsto & x. \end{array}$$

We would like to show that $p\colon \mathrm{T}M \to M$ is an n-dimensional real vector bundle. For each $x \in M$ the preimage $p^{-1}(x) = \{x\} \times \mathrm{T}_x M$ admits an obvious real vector space structure. Our next goal is to equip $\mathrm{T}M$ with a reasonable topology. For this we

note that any chart $\Phi\colon U \to V$ for M induces a bijection

$$\begin{array}{rcl} \widetilde{\Phi}\colon p^{-1}(U) & \to & V \times \mathbb{R}^n \\ (x, v) & \mapsto & (\Phi(x), \underbrace{\mathrm{D}\Phi_x(v)}_{\in \mathrm{T}_{\Phi(x)}V = \mathbb{R}^n}). \end{array}$$

We say that a subset $W \subset \mathrm{T}M$ is *open* if for every chart $\Phi\colon U \to V$ for M the image $\widetilde{\Phi}(W) \subset V \times \mathbb{R}^n$ is open. With this definition it is now reasonably easy to show that $p\colon \mathrm{T}M \to M$ is an n-dimensional real vector bundle over M. We refer to it as the *tangent bundle of* M.[1695]

If M is an n-dimensional submanifold of $\mathbb{R}^k$, then using the usual isomorphism $\widetilde{\mathrm{T}}_P M \cong \mathrm{T}_P M$ from the Tangent Space-Isomorphism Proposition 23.16 one can show fairly easily that the two tangent bundles that we defined above are isomorphic. In practice we will not distinguish between these two notions.

(3) Let M be an n-dimensional *complex* manifold. Since M is also a $2n$-dimensional smooth manifold we can consider the tangent bundle $p\colon \mathrm{T}M \to M$ as in (2). Using charts from the holomorphic atlas one can endow each tangent space $\mathrm{T}_x M$ with the structure of an n-dimensional complex vector space. This shows that the tangent bundle $\mathrm{T}M \to M$ can be viewed as a $\mathbb{C}$-vector bundle where the complex structure varies "smoothly". A "smooth" $\mathbb{C}$-vector bundle structure on the tangent space $\mathrm{T}M$ of a $2n$-dimensional smooth manifold is called an *almost complex structure*. The Newlander–Nirenberg Theorem, see [**NN1957**] and [**Hör1990b**, Section 5.7], gives a complete criterion for whether or not an almost complex structure on $\mathrm{T}M$ arises from a complex structure on M.

Proposition 143.1. (Vector Bundles-from-Representation) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let M be a connected locally contractible topological space, let $x_0 \in M$ be a point and furthermore let $\varphi\colon \pi_1(M, x_0) \to \mathrm{GL}(n, \mathbb{K})$ be a representation. We denote by $p\colon \widetilde{M}_{x_0} \to M$ the explicit universal covering that we constructed in the proof of the Covering Existence Theorem 61.5. The group $\pi_1(M, x_0)$ acts on $\widetilde{M}_{x_0}$ via the action provided by Proposition 61.14. We consider[1696]

$$\mathbb{K}^n \times_{\pi_1(M,x_0)} \widetilde{M}_{x_0} \ := \ (\mathbb{K}^n \times \widetilde{M}_{x_0})/\sim \qquad \text{where } (\varphi(g^{-1}) \cdot v, p) \sim (v, g \cdot p).$$

We equip $\mathbb{K}^n \times \widetilde{M}_{x_0}$ with the product topology and we equip $(\overline{M} \times \widetilde{M}_{x_0})/\sim$ with the quotient topology.

(1) The map

$$\begin{array}{rcl} q\colon \mathbb{K}^n \times_{\pi_1(M,x_0)} \widetilde{M}_{x_0} & \to & M \\ [(m, x)] & \mapsto & p(x) \end{array}$$

is continuous.

[1695] It does not take too much effort to show that the total space $\mathrm{T}M$ of the tangent bundle is naturally a $2n$-dimensional smooth manifold. (A related statement for physical tangent spaces of smooth submanifolds of $\mathbb{R}^n$ is proved in Exercise 19.29.) We will not make use of this neat fact.

(2) Let $x \in M$. We have $q^{-1}(x) = M \times_{\pi_1(M,x_0)} p^{-1}(x)$. We pick a path $\mu\colon [0,1] \to M$ from x to x_0. The map

$$\begin{array}{rcl} q^{-1}(x) = \mathbb{K}^n \times_{\pi_1(M,x_0)} p^{-1}(x) & \to & \mathbb{K}^n \\ [(m,\alpha)] & \mapsto & \varphi([\mu * \overline{\alpha}])(m) \end{array}$$

α is a path from x_0 to x

is well-defined.

(3) The map from (2) is a bijection.

(4) We use the $\mathbb{K}$-vector space structure on $\mathbb{K}^n$ and the bijection from (3) to define a group structure on $q^{-1}(x)$. This $\mathbb{K}$-vector space structure does not depend on the choice of μ.

All this data combined defines an n-dimensional $\mathbb{K}$-vector bundle over M.

Proof. Let $\varphi\colon \pi_1(M,x_0) \to \mathrm{GL}(n,\mathbb{K})$ be a representation.

(1) It follows immediately from the Topological-Product Proposition 5.1 (1a) and the Topological-Quotient Proposition 5.15 (1b) that the given map $q\colon \mathbb{K}^n \times_{\pi_1(M,x_0)} \widetilde{M}_{x_0} \to M$ is continuous.

(2) Let $x \in M$. Evidently we have $q^{-1}(x) = \mathbb{K}^n \times_{\pi_1(M,x_0)} p^{-1}(x)$. We pick a path $\mu\colon [0,1] \to M$ from x to x_0. We consider the map

$$\begin{array}{rcl} F\colon q^{-1}(x) = \mathbb{K}^n \times_{\pi_1(M,x_0)} p^{-1}(x) & \to & \mathbb{K}^n \\ [(m,\alpha)] & \mapsto & \varphi([\mu * \overline{\alpha}])(m) \end{array}$$

We need to show that the map is well-defined. Thus let $[\gamma] \in \pi_1(X,x_0)$. Then

$$\begin{array}{rcl} F(\varphi([\gamma]^{-1}) \cdot v, [\alpha]) & = & \varphi(\mu * \overline{\alpha}) \cdot \varphi([\gamma]^{-1}) \cdot v \\ & = & \varphi(\mu * \overline{\gamma * \alpha}) \cdot v \end{array} \begin{array}{rcl} & = & \varphi(\mu * \overline{\alpha} * \overline{\gamma}) \cdot v \\ & = & F([(v, [\gamma * \alpha])]) \end{array} = F([(v, [\gamma] \cdot [\alpha])]).$$

(3) By the Regular Covering-Characterization Corollary 62.8 and the Deck Group-Free Action Lemma 62.2 we know that the action of $\pi_1(M,x_0)$ on $p^{-1}(x)$ is free and transitive. It follows easily that the above map is a bijection.

(4) We need to show that the resulting $\mathbb{K}$-vector space structure on $q^{-1}(x)$ does not depend on the choice of μ. Thus let $\nu\colon [0,1] \to M$ be another path from x to x_0. We see that

$$\varphi([\nu * \overline{\alpha}])(m) \;=\; \varphi([\nu * \overline{\mu} * \mu * \overline{\alpha}])(m) \;=\; \varphi([\nu * \overline{\mu}]) \cdot \varphi([\mu * \overline{\alpha}])(m).$$

Thus we see that the maps $[(m,\alpha)] \mapsto \varphi([\nu * \overline{\alpha}])(m)$ and $[(m,\alpha)] \mapsto \varphi([\mu * \overline{\alpha}])(m)$ differ by multiplication by the invertible matrix $\varphi([\nu * \overline{\mu}])$. Thus both induce the same $\mathbb{K}$-vector space structure on $q^{-1}(x)$.

It remains to show that the above data defines an n-dimensional $\mathbb{K}$-vector bundle over M. Thus let $x \in M$. We pick an open neighborhood U of x such that $p^{-1}(U)$ is uniformly covered in the sense of the definition on page 1067. Furthermore we pick a path $\mu\colon [0,1] \to$

[1696] The perhaps slightly odd way of defining $\mathbb{K}^n \times_{\pi_1(M,x_0)} \widetilde{M}_{x_0}$ has, for the author, the origin from the analogous situation when in Proposition 174.4 we introduce local coefficient systems corresponding to representations.

M from x_0 to x. Similar to the above we consider the map

$$\begin{aligned} F\colon q^{-1}(U) = \mathbb{K}^n \times_{\pi_1(M,x_0)} p^{-1}(U) &\to \mathbb{K}^n \times U \\ [(v,[\alpha])] &\mapsto (\varphi(\mu * \overline{\alpha}) \cdot v, [\alpha]). \end{aligned}$$

Using our discussion above one can easily verify that this map is a trivialization. ■

Definition. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let M be a connected locally contractible topological space, let $x_0 \in M$ be a point and let $\varphi\colon \pi_1(M, x_0) \to \mathrm{GL}(n, \mathbb{K})$ be a representation. We refer to the vector bundle that we defined in the Vector Bundles-from-Representation 143.1 as the $\mathbb{K}$-vector bundle defined by φ.

Example. It is fairly straightforward to show that the real vector bundle over S^1 corresponding to the representation $\pi_1(S^1, 1) \to \mathrm{GL}(1, \mathbb{R})$ given by sending the standard generator of $\pi_1(S^1, 1)$ to (-1) is isomorphic to the Möbius bundle.

The following definition is basically self-explanatory.

Definition. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $p\colon E \to X$ be an n-dimensional $\mathbb{K}$-vector bundle.

(1) We say p is trivializable if p is isomorphic to the trivial n-dimensional $\mathbb{K}$-vector bundle $X \times \mathbb{K}^n \to X$.
(2) Any choice of an isomorphism $E \to X \times \mathbb{K}^n$ is called a trivialization of the vector bundle p.

The following lemma is basically trivial, but it simplifies discussions later on.

Lemma 143.2. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $p\colon E \to X$ be an n-dimensional $\mathbb{K}$-vector bundle.

(1) The $\mathbb{K}$-vector bundle is trivializable if and only if there exists a map $\varphi\colon E \to \mathbb{K}^n$ such that $p \times \varphi$ is a trivialization.
(2) If p is trivializable, then for any subset $W \subset X$ the restriction of p to W is also trivializable.
(3) If W is a subset of X such that the restriction of p to each component of W is trivializable, then the restriction of p to W is also trivial. ■

Proof. We leave it to the reader to provide the elementary proofs of the statements of the lemma. ■

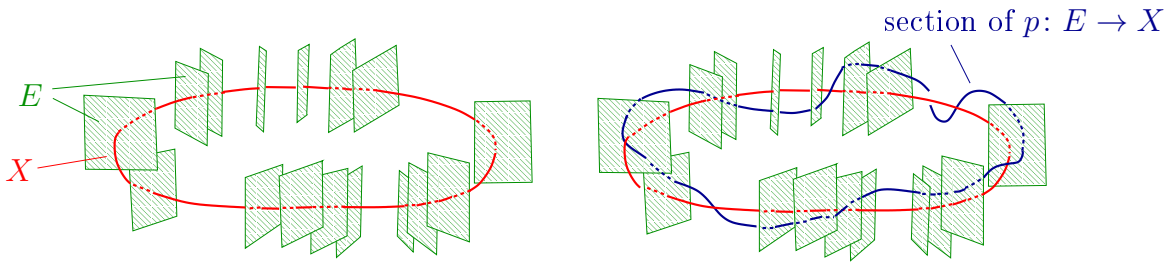
Now we move on to two more definitions.

Definition.

(1) We say that a smooth manifold M is parallelizable if its tangent bundle $\mathrm{T}M$ is trivializable.
(2) A section of a vector bundle $p\colon E \to X$ is a continuous map $s\colon X \to E$ such that $p \circ s = \mathrm{id}_X$. We say a section is non-vanishing if $s(x) \neq 0$ for all $x \in X$.

Lemma 143.3. Let $p\colon E \to X$ be a vector bundle.

(1) The map $X \to E$ that sends each $x \in X$ to the unique zero element of $p^{-1}(x)$ is a section.
(2) If $s, t\colon X \to E$ are two sections, then the map $x \mapsto s(x) + t(x)$ is also a section.
(3) If $s\colon X \to E$ is a section and $\lambda \in \mathbb{R}$, then the map $x \mapsto \lambda \cdot s(x)$ is also a section.



Proof. We need to show that the given maps $X \to E$ are continuous. By the Pasting Proposition 2.6 we have to verify continuity only locally. But locally the statement follows easily from the definition of a vector bundle. ■

Examples.

(1) Given a vector bundle $p\colon E \to X$ a section $s\colon X \to E$ is given by sending each $y \in E$ to the zero-element of the vector space $p^{-1}(y)$.[1697] This section is usually called the zero-section of p.

(2) Let M be a smooth manifold. A section of the tangent bundle is usually called a vector field on M. A little bit of thought shows that this is is precisely the notion that we introduced on page 639 and that, given a submanifold M of $\mathbb{R}^n$, this definition of a vector field corresponds in a natural way to the notion that we introduced on page 638.

Lemma 143.4. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

(1) Given any n-dimensional $\mathbb{K}$-vector bundle $p\colon E \to X$ the following two statements are equivalent:
   (a) The $\mathbb{K}$-vector bundle is trivializable.
   (b) There exist n sections $s_1, \dots, s_n$ such that for every point $x \in X$ the vectors $s_1(x), \dots, s_n(x) \in E_x := p^{-1}(x)$ form a basis of the $\mathbb{K}$-vector space $E_x = p^{-1}(x)$.

(2) A line bundle is trivializable if and only if it admits a non-vanishing section.

(3) If M is an orientable 2-dimensional smooth manifold, then its tangent bundle is trivializable if and only if it admits a non-vanishing vector field.

Sketch of proof.

(1) Let $p\colon E \to X$ be an n-dimensional $\mathbb{K}$-vector bundle. Suppose that there exists an isomorphism $\varphi\colon X \times \mathbb{K}^n \to E$ of $\mathbb{K}$-vector bundles over X. It is clear that in this case the maps $s_i\colon x \mapsto \varphi(x, e_i)$, $i = 1, \dots, n$, define sections such that for each $x \in X$ the vectors $s_1(x), \dots, s_n(x) \in E_x = p^{-1}(x)$ form a basis of $E_x = p^{-1}(x)$. Conversely, if we are given such sections, then the map

$$\begin{array}{rcl} X \times \mathbb{K}^n & \to & E \\ (x, (\lambda_1, \dots, \lambda_n)) & \mapsto & \sum_{i=1}^{n} s_i(x) \cdot \lambda_i \end{array}$$

   is easily seen to be an isomorphism of $\mathbb{K}$-vector bundles over X.

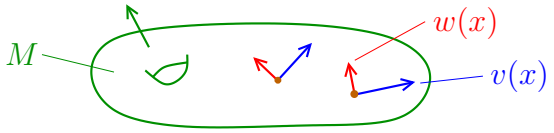
(2) This statement is an immediate consequence of (1).

[1697] It is a good exercise in practicing the definitions to convince oneself that this map $s\colon X \to E$ is indeed continuous.

(3) Let M be an orientable 2-dimensional smooth manifold. By (1) we only have to show the following statement: if M admits a non-vanishing vector field v, then the tangent bundle is trivializable. So suppose that v is a non-vanishing vector field on M. We pick an orientation for M. By [**Lee2000**, Proposition 13.3] the smooth manifold M admits a Riemannian structure g.[1698] Since M is 2-dimensional and oriented we can now define

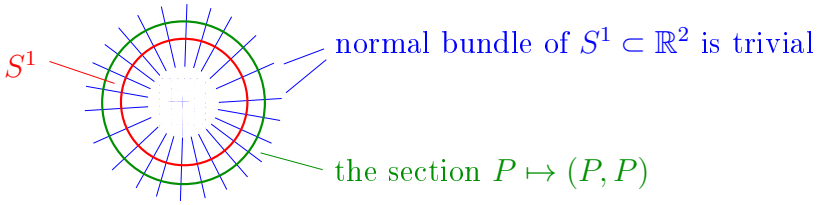
$$\begin{array}{rcl} w\colon M & \to & \mathrm{T}M \\ x & \mapsto & \begin{array}{l}\text{unique vector } w(x) \in \mathrm{T}_x M \text{ that has the following three properties:} \\ \text{(a)} \quad g(v(x), w(x)) = 0, \text{ i.e. } v(x) \text{ and } w(x) \text{ are orthogonal,} \\ \text{(b)} \quad g(w(x), w(x)) = 1, \text{ i.e. } w(x) \text{ has length one,} \\ \text{(c)} \quad \text{the vectors } v(x) \text{ and } w(x) \text{ form a positive basis for } \mathrm{T}_x M\end{array} \end{array}$$

(We refer to the figure below for an illustration of the definition of $w(x)$.) It is fairly easy to see that this is a section of the tangent bundle. It has the property that for each $x \in M$ the vectors $v(x)$ and $w(x)$ form a basis for $\mathrm{T}_x M$. Thus it follows from (1) that $\mathrm{T}M$ is a trivial bundle. ■



Examples.

(1) By Lemma 143.4 (2) the normal bundle of $S^n \subset \mathbb{R}^{n+1}$ is trivializable since it admits the non-vanishing section $P \mapsto (P, P)$. We illustrate this statement for $n = 1$ in the figure below.



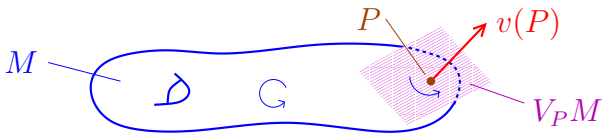
(2) We can easily generalize the previous example. More precisely, let $n \in \mathbb{N}$ and let M be a closed oriented submanifold of $\mathbb{R}^n$ of codimension one. The map

$$\begin{array}{rcl} v\colon M & \to & NM \\ P & \mapsto & \begin{array}{c}\text{the unique vector } v(P) \in \mathbb{R}^n \text{ of length one that is normal to } \widetilde{\mathrm{T}}_P M \\ \text{and which, followed by a positive basis for } \widetilde{\mathrm{T}}_P M, \\ \text{forms a positive basis for } \mathbb{R}^n\end{array} \end{array}$$

is easily seen to be a section of the normal bundle. Since the normal bundle is one-dimensional we obtain from Lemma 143.4 (2) that the normal bundle of $M \subset \mathbb{R}^n$ is trivializable. We refer to the figure below for an illustration.

(3) In Exercise 143.6 we will show that the Möbius bundle does not admit a non-vanishing section. It follows from Lemma 143.4 (2) that the Möbius bundle is non-trivial.

[1698] We refer to page 1426 for the definition of a Riemannian structure.



(4) It follows from the Hairy Ball Theorem 38.12 and Lemma 143.4 (1) that even-dimensional spheres are not parallelizable.
(5) For any $n \in \mathbb{N}$ the n-dimensional torus $(S^1)^n = \mathbb{R}^n/\mathbb{Z}^n$ is parallelizable, in fact if we denote by $q\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ the projection , then an isomorphism of vector bundles over $\mathbb{R}^n/\mathbb{Z}^n$ is given by

$$\begin{aligned} \varphi\colon (\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n &\to \mathrm{T}(\mathbb{R}^n/\mathbb{Z}^n) \\ (q(x), v) &\mapsto (q(x), \underset{\uparrow}{\mathrm{D}q_x}(v)) \end{aligned}$$

$$\mathrm{D}q_x\colon \mathbb{R}^n = \mathrm{T}_x\mathbb{R}^n \to \mathrm{T}_{q(x)}(\mathbb{R}^n/\mathbb{Z}^n)$$

(6) Let $g \in \mathbb{N}_0$. On page 1879 we calculated that the Euler characteristic of the surface of genus g equals $\chi(\Sigma_g) = 2 - 2g$. Thus if $g \neq 1$ we obtain from the Euler Characteristic-Vector Field Theorem 97.15, which is a generalization of the Hairy Ball Theorem 38.12, together with Lemma 143.4 (1) that the surface of genus g is not parallelizable.
(7) If a smooth manifold is parallelizable, then evidently it is also orientable in the sense of Section 25.2. Put differently, a non-orientable smooth manifold is not parallelizable.
(8) Every closed orientable 3-manifold is parallelizable. This was first proved by Eduard Stiefel [**Stie1935**] in 1935. An elementary proof for this statement is given in [**BeL18**]. Alternative proofs can be found in [**DGGK2020**] and [**Kir1989**, p. 46].
(9) Let M and N be closed smooth manifolds. If M and N are parallelizable, then one can easily show that $M \times N$ is also parallelizable.

143.2. **Constructing vector bundles via transition maps.** The starting point of this section is the following lemma which says that we can construct vector bundles by specifying "transition maps".

Lemma 143.5. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let X be a topological space and let $\{U_i\}_{i\in I}$ be family of subsets of X such that $\bigcup_{i\in I} \mathring{U}_i = X$. Furthermore let $n \in \mathbb{N}_0$. Suppose that for every $i, j \in I$ we have a map $\varphi_{ij}\colon U_i \cap U_j \to \mathrm{GL}(n, \mathbb{K})$ such that the following conditions are satisfied:

(1) For every $i, j, k \in I$ we have $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}\colon U_i \cap U_j \cap U_k \to \mathrm{GL}(n, \mathbb{K})$.
(2) For every $i, j \in I$ and any $x \in U_i \cap U_j$ we have $\varphi_{ij}(x) = \varphi_{ji}(x)^{-1}$.

The map

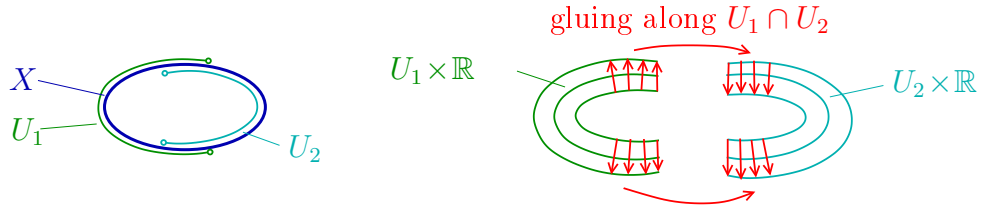
$$\begin{aligned} p\colon \Big(\bigsqcup_{i\in I}(U_i \times \mathbb{K}^n)\Big)\Big/\underset{\downarrow}{(x, v) \sim (x, \varphi_{ij}(x)\cdot v)} &\to X \\ [(x, v)] &\mapsto x, \end{aligned}$$

$$\text{where } x \in U_i \cap U_j$$

together with the obvious $\mathbb{K}$-vector space structure on each $p^{-1}(x)$, is an n-dimensional $\mathbb{K}$-vector bundle over X.

Example. We consider $X = S^1$ and two open overlapping intervals $U_1, U_2 \subset X = S^1$. Let $\varphi\colon U_1 \cap U_2 \to \mathrm{GL}(1, \mathbb{R})$ be the map that equals (1) on one component of $U_1 \cap U_2$ and that

equals (-1) on the other component of $U_1 \cap U_2$. It is fairly straightforward to see that the resulting real vector bundle is isomorphic to the Möbius bundle. We refer to the figure below for an illustration.



Proof. Let $i \in I$. It follows fairly easily from (1) and (2) that the map

$$\begin{array}{ccc} U_i \times \mathbb{K}^n & \xrightarrow{(x,v)\mapsto[(x,v)]} & p^{-1}(U_i) \\ & {\scriptstyle (x,v)\mapsto x} \searrow \quad \swarrow {\scriptstyle p} & \\ & U_i & \end{array}$$

is a well-defined isomorphism of $\mathbb{K}$-vector bundles over U_i. Since the interiors of the U_i cover all of X we see that p is indeed an n-dimensional $\mathbb{K}$-vector bundle over X. ∎

We also have the following converse to Lemma 143.5.

Lemma 143.6. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Every $\mathbb{K}$-vector bundle $p\colon E \to X$ admits an isomorphism to a $\mathbb{K}$-vector bundle of the type constructed in Lemma 143.5.

Proof. Let $p\colon E \to X$ be an n-dimensional $\mathbb{K}$-vector bundle. Note that by definition of a vector bundle we know that there exists an open cover $\{U_i\}_{i\in I}$ such that the restriction of $p\colon E \to X$ to each U_i is trivializable. It follows from Lemma 143.2 that given any $i \in I$ there exists a map $\Theta_i\colon p^{-1}(U_i) \to \mathbb{K}^n$ such that $p \times \Theta_i\colon p^{-1}(U_i) \to U_i \times \mathbb{K}^n$ is a trivialization. Given $x \in U_i$ we write $\Theta_{i,x} := \Theta_i|_{E_x}$. Next, given $i, j \in I$ we consider the map

$$\begin{array}{rcl} \varphi_{ij}\colon U_i \cap U_j & \to & \mathrm{GL}(n,\mathbb{K}) = \mathrm{Aut}(\mathbb{K}^n) \\ x & \mapsto & \Theta_{i,x} \circ \Theta_{j,x}^{-1} = \left(\mathbb{K}^n \xrightarrow{\Theta_{j,x}^{-1}} E_x \xrightarrow{\Theta_{i,x}} \mathbb{K}^n\right) \end{array}$$

One can now easily verify that the map

$$\begin{array}{rcll} p\colon \left(\bigsqcup_{i\in I}(U_i \times \mathbb{K}^n)\right)/(x,v) \sim (x, \varphi_{ij}(x)\cdot v) & \to & E & \\ [(x,v)] & \mapsto & (p \times \Theta_i)^{-1}(x,v) & \text{if } x \in U_i \end{array}$$

is an isomorphism of $\mathbb{K}$-bundles over X. ∎

Corollary 143.7. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let $n \in \mathbb{N}_0$ and let X be a topological space. The isomorphism classes of n-dimensional $\mathbb{K}$-vector bundles over X form a set.

Proof. This statement follows immediately from Lemma 143.6, modulo the minor nuisance that we never said what a set is. The less this particular author says about set theory, the better. ∎

Corollary 143.7 leads us to the following notation.

Notation. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let X be a topological space. Given $n \in \mathbb{N}_0$ we denote by $\mathrm{Vect}^n_{\mathbb{K}}(X)$ the set of isomorphism classes of n-dimensional $\mathbb{K}$-vector bundles over X.

143.3. **The pullback of vector bundles.** In this section and the coming sections we will introduce several constructions for obtaining new vector bundles out of given vector bundles. Before we head to the constructions it is helpful to introduce the following definition.

Definition. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A bundle map between $\mathbb{K}$-vector bundles $p\colon E \to X$ and $q\colon F \to Y$ is a pair of maps $\Phi\colon E \to F$ and $\varphi\colon X \to Y$ such that the following two conditions are satisfied:

(1) The following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{\varphi} & Y. \end{array}$$

(2) For each $x \in X$ the map $\Phi|_{E_x}\colon E_x \to F_{\varphi(x)}$ is a homomorphism of $\mathbb{K}$-vector spaces.

One can easily verify that the class of $\mathbb{K}$-vector bundles together with the above notion of morphisms and the obvious composition of bundle maps form a category. We refer to it as the category $\mathcal{VectBund}_{\mathbb{K}}$ of $\mathbb{K}$-vector bundles.

Remark. The author now faces a dilemma: given two vector bundles $p\colon E \to X$ and $q\colon E \to X$ over the same topological space X we now have two conflicting definitions for p and q to be "isomorphic". We have the definition on page 3060 which agrees with the above definition as long as $\varphi = \mathrm{id}_X$. In the discussion below we will almost invariably work with the former definition. Hopefully it is always clear from the context what is meant.

The following lemma, which is just a special case of Lemma **??** introduces the pullback of vector bundles.

Lemma 143.8. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let $g\colon X \to Y$ be a map between topological spaces and let $q\colon F \to Y$ be an n-dimensional $\mathbb{K}$-vector bundle. As in the Topological-Pullback Lemma 56.16 (1) we consider the pullback

$$g^*F \;:=\; \{(x,v)\,|\,x \in X \text{ and } v \in q^{-1}(g(x))\} \;\subset\; X \times F,$$

which is equipped with the subspace topology.

(1) The map
$$\begin{array}{rcl} p\colon g^*F \;:=\; \{(x,v)\,|\,x \in X \text{ and } v \in q^{-1}(g(x))\} & \to & X \\ (x,v) & \mapsto & x \end{array}$$
is also an n-dimensional $\mathbb{K}$-vector bundle where for each $x \in X$ the $\mathbb{K}$-vector space structure on $p^{-1}(x) = \{x\} \times q^{-1}(g(x)) \subset X \times F$ is the obvious $\mathbb{K}$-vector space structure coming from $\{x\} \times q^{-1}(g(x)) = q^{-1}(g(x))$.

(2) The following diagram commutes
$$\begin{array}{ccc} g^*F & \xrightarrow{(x,v)\mapsto v} & F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{g} & Y, \end{array}$$
in other words, the horizontal maps define a bundle map.

(3) If we are given a bundle map

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{\varphi} & Y \end{array}$$

such that for each $x \in X$ the map $\Phi|_{E_x} \colon E_x \to F_{\varphi(x)}$ is an isomorphism, then the $\mathbb{K}$-vector bundle $E \to X$ is naturally isomorphic to the pullback $\varphi^* F \to X$.

Proof.

(1) Let $x \in X$. By Lemma 143.2 there exists a neighborhood $V \subset Y$ of $g(x)$ and a map $\varphi \colon q^{-1}(V) \to \mathbb{K}^n$ such that $q \times \varphi \colon q^{-1}(V) \to V \times \mathbb{K}^n$ is a trivialization. We consider the open neighborhood $U := g^{-1}(V) \subset X$ of x. It is now straightforward to verify that the map

$$\begin{array}{rcl} \psi \colon p^{-1}(U) & \to & U \times \mathbb{K}^n \\ (z, v) & \mapsto & (p(z), \varphi(v)) \end{array}$$

note that $v \in q^{-1}(g(z))$ and that $g(z) \in g(U) = V$

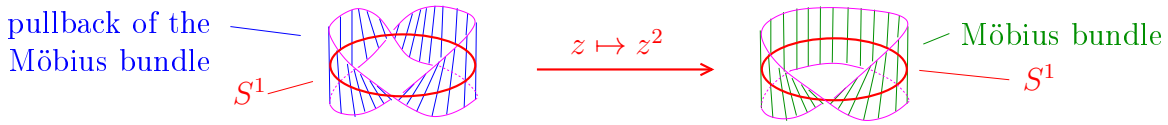
is a trivialization of the restriction of p to U.

(2) It is straightforward to verify that the diagram commutes.

(3) We consider the map $E \to g^* F$ given by $e \mapsto (\Phi(e), \varphi(p(e)))$. We leave it to the reader to verify that this map is a bundle isomorphism. ∎

Definition. Using the notation from Lemma 143.8 we refer to $g^* F \to X$ as the pullback of the $\mathbb{K}$-vector bundle $q \colon F \to Y$ under the map $g \colon X \to Y$.

Example. Let $p \colon M \to S^1$ be the Möbius bundle. We denote by $g \colon S^1 \to S^1$ the map that is given by $g(z) = z^2$. In Exercise 143.9 we will see that the pullback of the Möbius bundle under g is a trivial bundle. We refer to the figure below for an illustration of the pullback of the Möbius bundle.[1699]



The following lemma, which is basically just a slight variation on Lemma **??**, summarizes a few properties of the pullback of vector bundles.

Lemma 143.9. (Vector Bundle Pullback Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

(1) Let $g \colon X \to Y$ be a map between two topological spaces and let $E \to Y$ and $F \to Y$ be two $\mathbb{K}$-vector bundles on Y. If E and F are isomorphic, then so are $g^* E$ and $g^* F$.

(2) The pullback of a trivializable $\mathbb{K}$-vector bundle is again a trivializable $\mathbb{K}$-vector bundle.

(3) If $q \colon F \to Y$ is a $\mathbb{K}$-vector bundle and if X is a subset of Y with inclusion map $i \colon X \to Y$, then

[1699] The reader who is blessed with a good memory might recognize that this statement is quite similar to the discussion on page 1070 where we saw that the annulus is a 2-fold covering of the Möbius band.

$$\begin{array}{rcl} i^*F & \to & q^{-1}(X) \\ (x,v) & \mapsto & v \end{array}$$

is a natural isomorphism from the $\mathbb{K}$-vector bundle $i^*F \to X$ to the restriction of F to the $\mathbb{K}$-vector bundle $q^{-1}(X) \to X$.

Proof. The proof of this lemma is basically identical to the not very substantial proof of Lemma **??**. We just need to replace Lemma **??** (3) by Lemma 143.8 (3). ■

The following lemma, which is basically just a slight variation on Lemma **??**, shows that the pullback of vector bundles is covariantly functorial in a suitable sense.

Lemma 143.10. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

(1) Let $q\colon F \to Z$ be a $\mathbb{K}$-vector bundle. Then

$$\begin{array}{rcl} \mathrm{id}_Z^* F & \to & F \\ (x,v) & \mapsto & v \end{array}$$

is a natural isomorphism from the pullback $\mathbb{K}$-vector bundle $\mathrm{id}_Z^* F \to Z$ to the original $\mathbb{K}$-vector bundle $q\colon F \to Z$.

(2) Let $q\colon F \to Z$ be a $\mathbb{K}$-vector bundle. If $f\colon X \to Y$ and $g\colon Y \to Z$ are maps between topological spaces, then

$$\begin{array}{rcl} f^*(g^*F) & \to & (g \circ f)^*F \\ (x,(y,v)) & \mapsto & (x,v) \end{array}$$

is a natural isomorphism from the double pullback $\mathbb{K}$-vector bundle $f^*(g^*F) \to X$ to the pullback $\mathbb{K}$-vector bundle $(g \circ f)^*F \to X$.

Proof. The proof of this lemma is basically identical to the not very substantial proof of Lemma **??**. Again we just need to replace Lemma **??** (3) by Lemma 143.8 (3). ■

Corollary 143.11. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

(1) Given any map $g\colon X \to Y$ between two topological spaces the pullback of $\mathbb{K}$-vector bundles induces a well-defined map

$$\begin{array}{rcl} f^*\colon \mathrm{Vect}_{\mathbb{K}}^n(Y) & \to & \mathrm{Vect}_{\mathbb{K}}^n(X) \\ [q\colon F \to Y] & \mapsto & [g^*F \to X]. \end{array}$$

(2) The maps

$$\begin{array}{rcl} X & \to & \mathrm{Vect}_{\mathbb{K}}^n(X) \\ (g\colon X \to Y) & \to & (g^*\colon \mathrm{Vect}_{\mathbb{K}}^n(Y) \to \mathrm{Vect}_{\mathbb{K}}^n(X)) \end{array}$$

define a contravariant functor from the category $\mathcal{T}op$ of topological spaces to the category of sets $\mathcal{S}et$.

Remark. By using "linear algebra of vector bundles", we will see that $\bigcup_{n \in \mathbb{N}_0} \mathrm{Vect}_{\mathbb{K}}^n(X)$ is "more than a set", namely that we can endow it with two monoid structures corresponding to the direct sum and tensor product. Thus we will upgrade Corollary 143.11 to a functor from the category of topological spaces to an algebraic category.

Proof. The first statement is an immediate consequence of the Vector Bundle Pullback Lemma 143.9 (0), whereas The first statement is an immediate consequence of the Vector Bundle Pullback Lemma 143.9 (3) and (4). ■

Later on we will need the following lemma.

Lemma 143.12.

(1) If $f\colon M \to N$ is a local diffeomorphism between smooth manifolds, then the map

$$\begin{array}{ccc} \mathrm{T}M & \to & f^*\mathrm{T}N \\ (x,v) & \mapsto & (x, \underbrace{(f(x), \mathrm{D}f_x(v))}_{\in \mathrm{T}N}) \end{array}$$

is a natural isomorphism of real vector bundles.

(2) If $p\colon \widetilde{M} \to M$ is a covering of a smooth manifold, then $p^*\mathrm{T}M$ is naturally isomorphic to $T\widetilde{M}$.

Proof. We rely on the never-ending energy of the reader to provide the proof of the first statement. The second statement is an immediate consequence of the first statement. ■

In the remainder of this chapter we will use on several occasions the notion of a "paracompact" topological space that we introduced on page 354. We will use the explicit definition only in the proof of the Vector Bundles-Homotopies Proposition 143.13. Otherwise it is enough to know that most topological spaces that we are interested in are paracompact. For example we established in the Manifold Paracompact-Normal Proposition 18.9 (1) and the CW-Complex Paracompactness Proposition 68.13 that all topological manifolds and all CW-complexes are paracompact.

Remark. Some books, see e.g. [**Hat2**, p. 77], make at some point the blanket assumption that all base spaces of vector bundles are paracompact.

The following proposition is one of the key statements in the theory of vector bundles.

Proposition 143.13. (Vector Bundles-Homotopies Proposition) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

(1) Let $q\colon F \to Y$ be a $\mathbb{K}$-vector bundle, let X be a topological space and finally let $f_0, f_1\colon X \to Y$ be two maps. If f_0 and f_1 are homotopic and if X is paracompact, then the $\mathbb{K}$-vector bundles $f_0^*F \to X$ and $f_1^*F \to X$ are isomorphic. In fact any choice of a homotopy induces naturally an isomorphism between the two $\mathbb{K}$-vector bundles.

(2) Let X and Y be two paracompact spaces. If $f\colon X \to Y$ is a homotopy equivalence, then the map

$$f^*\colon \mathrm{Vect}^n_{\mathbb{K}}(Y) \to \mathrm{Vect}^n_{\mathbb{K}}(X),$$

that we introduced in Corollary 143.11, is a bijection.

(3) Let X be a topological space. If X is contractible and if X is paracompact (e.g. if X compact), then every $\mathbb{K}$-vector bundle over X is trivializable.

Example. It follows immediately from the aforementioned fact that topological manifolds are paracompact, see the Manifold Paracompact-Normal Proposition 18.9 (1), together with the Vector Bundles-Homotopies Proposition 143.13 (3) that every $\mathbb{K}$-vector bundle over the topological spaces $\overline{B}^n, B^n$ and $\mathbb{R}^n$ is trivializable.

The key to proving the Vector Bundles-Homotopies Proposition 143.13 is the following lemma.

Lemma 143.14. Let X be a paracompact space and let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We denote by $i_0 \colon X \xrightarrow{x \mapsto (x,0)} X \times [0,1]$ and $i_1 \colon X \xrightarrow{x \mapsto (x,1)} X \times [0,1]$ the obvious embeddings. Given any $\mathbb{K}$-vector bundle $p \colon E \to X \times [0,1]$ the pullbacks $i_0^* E$ and $i_1^* E$ are isomorphic.

Remark. Note that Lemma 143.14 does not say that the $i_0^* E$ and $i_1^* E$ are *naturally* isomorphic. In fact, as we will see in Exercise 143.15, the isomorphism we provide depends on choices.

Before we provide the proof of Lemma 143.14 let us first show that the Vector Bundles-Homotopies Proposition 143.13 is a pretty straightforward consequence of Lemma 143.14.

Proof of the Vector Bundles-Homotopies Proposition 143.13 assuming Lemma 143.14.

(1) Let $q \colon F \to Y$ be a $\mathbb{K}$-vector bundle, let X be a topological space that is paracompact and let $H \colon X \times [0,1] \to Y$ be a homotopy between two maps $f_0, f_1 \colon X \to Y$. We denote by $i_0, i_1 \colon X \to X \times [0,1]$ the two obvious inclusions. Now we see that

$$\begin{array}{ccccccccc} f_0^* F & = & (H \circ i_0)^* F & \overset{\text{by the Vector Bundle Pullback Lemma 143.9}}{\cong} & \mathrm{H}^*(i_0^* F) & \underset{\text{Lemma 143.14}}{\cong} & \mathrm{H}^*(i_1^* F) & \overset{\text{by the Vector Bundle Pullback Lemma 143.9}}{\cong} & (H \circ i_1)^* F & = & f_1^* F. \\ & \uparrow & & & & & & & & \uparrow & \\ & \text{since } f_0 = H \circ i_0 & & & & & & & & \text{since } f_1 = H \circ i_1 & \end{array}$$

(2) Let $g \colon Y \to X$ be a homotopy inverse to f. Recall that this means that $f \circ g$ is homotopic to id_Y and that $g \circ f$ is homotopic to id_X. We consider the induced maps $f^* \colon \mathrm{Vect}_{\mathbb{K}}^n(Y) \to \mathrm{Vect}_{\mathbb{K}}^n(X)$. We have

$$\begin{array}{ccccc} g^* \circ f^* & = & (f \circ g)^* & = & \mathrm{id}_Y^* & = & \mathrm{id}_{\mathrm{Vect}_{\mathbb{K}}^n(Y)}. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Corollary 143.11 (2)} & & \underset{\text{since } f \circ g \simeq \mathrm{id}_Y}{\text{by Statement (1)}} & & \text{Corollary 143.11 (2)} & \end{array}$$

Similarly we see that $f^* \circ g^* = \mathrm{id}_{\mathrm{Vect}_{\mathbb{K}}^n(X)}$.

(3) Let $F \to X$ be a $\mathbb{K}$-vector bundle over a topological space that is paracompact and contractible. By definition of being contractible there exists a homotopy equivalence from the topological space X to the topological space $\{*\}$ consisting of a single point. Since every $\mathbb{K}$-vector bundle over $\{*\}$ is trivializable it follows from (2) and the Vector Bundle Pullback Lemma 143.9 (1) that every $\mathbb{K}$-vector bundle over X is trivializable. ∎

We conclude this section with the proof of Lemma 143.14.

Proof of Lemma 143.14. [1700] Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Claim 1. Let $p \colon F \to Y \times [a,c]$ be an n-dimensional $\mathbb{K}$-vector bundle and let $b \in [a,c]$. If the restrictions of p to $Y \times [a,b]$ and to $Y \times [b,c]$ are both trivializable, then so is the restriction to $Y \times [a,c]$.

Proof. Since the restrictions of p to $Y \times [a,b]$ and to $Y \times [b,c]$ are both trivializable we can pick trivializations $\Phi \colon p^{-1}(Y \times [a,b]) \to \mathbb{K}^n \times Y \times [a,b]$ and $\Psi \colon p^{-1}(Y \times [b,c]) \to \mathbb{K}^n \times Y \times [b,c]$. We denote by $q \colon \mathbb{K}^n \times Y \times [a,c] \to \mathbb{K}^n$ the projection. It follows almost immediately from Proposition 143.18 that there exists a continuous map $\Theta \colon Y \times \{b\} \to \mathrm{GL}(\mathbb{K}, n)$ such that

[1700] The structure of the proof is somewhat similar to the idea behind the proof of the Topological Collar Neighborhood Theorem 75.5.

for all $v \in p^{-1}(Y \times \{b\})$ we have $q(\Phi(v)) = \Theta(p(v)) \cdot q(\Psi(v)) \in \mathbb{K}^n$. It is straightforward to verify that

$$\begin{array}{rcl} p^{-1}(Y \times [a,c]) & \to & \mathbb{K}^n \times Y \times [a,c] \\ v & \mapsto & \begin{cases} \Phi(v), & \text{if } p(v) \in Y \times [a,b], \\ (p(v), \Theta(p(y,b)) \cdot q(\Psi(v))), & \text{if } p(v) = (y,t) \in Y \times [b,c] \end{cases} \end{array}$$

is an isomorphism of $\mathbb{K}^n$-vector bundles. ⊞

Now let X be a paracompact space and $p\colon E \to X \times [0,1]$ be an n-dimensional $\mathbb{K}$-vector bundle.

Claim 2. There exists an open cover $\{U_i\}_{i \in I}$ of X such that the restriction of the $\mathbb{K}$-vector bundle $p\colon E \to X \times [0,1]$ to each $U_i \times [0,1]$ is trivializable.

Proof. Let $x \in X$. By definition of a vector bundle and by compactness of $[0,1]$ there exist neighborhoods $V_1, \dots, V_m$ of x and $0 = t_0 < \cdots < t_m = 1$ such that the restriction of $p\colon E \to X \times [0,1]$ to each $V_i \times [t_{i-1}, t_i]$ is trivial. It follows from Claim 1 and Lemma 143.2 (2) that the restriction of p to $(V_1 \cap \cdots \cap V_m) \times [0,1]$ is also trivial. Now we set $U_x := V_1 \cap \cdots \cap V_m$. The resulting sets $\{U_x\}_{x \in X}$ have the desired properties. ⊞

By Claim 2 there exists an open cover $\{U_i\}_{i \in I}$ of X such that the restriction of p to any $U_i \times [0,1]$ is trivializable. By the Countable Cover Proposition 10.11 there exists a *countable* locally finite open cover $\{V_k\}_{k \in \mathbb{N}_0}$ such that each component of any V_k is contained in some U_i. Note that it follows from Lemma 143.2 that the restriction of p to any $V_k \times [0,1]$ is also trivial and that for each $k \in \mathbb{N}_0$ there exists a map $\Phi_k\colon p^{-1}(V_k \times [0,1]) \to \mathbb{K}^n$ such that $p \times \Phi_k$ is a trivialization of the vector bundle over V_k.

Next note that it follows from the Paracompact-Partition of Unity Theorem 10.9 that there exists a countable partition of unity $\{f_k\colon X \to [0,1]\}_{k \in \mathbb{N}_0}$ such that for each $k \in \mathbb{N}_0$ we have $\operatorname{supp}(f_k) \subset V_k$. Given $i \in \mathbb{N}_0$ we set $g_i := f_0 + \cdots + f_{i-1}$ and we set

$$X_i \;:=\; \text{graph of } g_i\colon X \to [0,1] \;:=\; \{(x, g_i(x)) \,|\, x \in X\} \;\subset\; X \times [0,1].$$

Note that $X_0 = X \times \{0\}$. We also set $X_\infty = X \times \{1\}$. Given $i \in \mathbb{N}_0 \cup \{\infty\}$ we write $E_i := p^{-1}(X_i)$. Next we consider the map

$$\begin{array}{rcl} \Psi_i\colon E_i & \to & E_{i+1} \\ v & \mapsto & \begin{cases} v, & \text{if } p(v) \notin U_i \times [0,1], \\ (p \times \Phi_i)^{-1}\underbrace{(x, g_{i+1}(x), \Phi_i(v))}_{\in U_i \times [0,1] \times \mathbb{K}^n}, & \text{if } p(v) = (x, g_i(x)) \in U_i \times [0,1]. \end{cases} \end{array}$$

We combine these maps to obtain the map

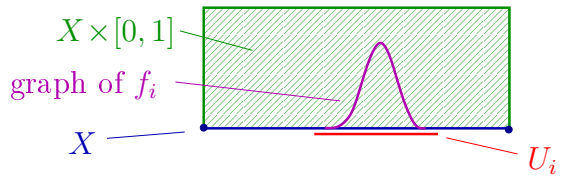
$$\begin{array}{rcll} \Theta\colon E_0 & \to & E_\infty & \\ x & \mapsto & \Psi_k(\dots(\Psi_0(x))) & \text{if } k \in \mathbb{N}_0 \text{ with } g_k(x) = 1. \end{array}$$

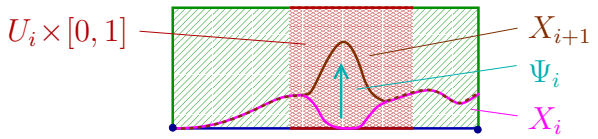
It follows immediately from the definitions that this map is well-defined, i.e. it does not depend on the choice of $k \in \mathbb{N}_0$ for a given x. Using this observation it is straightforward to verify that Ψ is continuous. Finally note that, basically by construction, the following

diagram commutes and that it defines an isomorphism of $\mathbb{K}$-vector bundles.

$$\begin{array}{ccc} E_0 = p^{-1}(X\times\{0\}) & \xrightarrow{\quad\Theta\quad} & E_\infty = p^{-1}(X\times\{1\}) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X\times\{0\} & \xrightarrow{\quad(x,0)\mapsto(x,1)\quad} & X\times\{1\}. \end{array}$$

Together with the Vector Bundle Pullback Lemma 143.9 (2) this implies that the pullbacks i_0^*E and i_1^*E are isomorphic. ■





143.4. **The Grassmannian manifolds.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. On page 3060 we saw that the projective spaces $\mathbb{K}\mathrm{P}^n$ come with a tautological line bundle. The projective space $\mathbb{K}\mathrm{P}^n$ admits an obvious bijection with the set of all 1-dimensional subspaces of $\mathbb{K}^{n+1}$. With this view point the fiber of each point in $\mathbb{K}\mathrm{P}^n$ is precisely this 1-dimensional subspace. In the following we generalize this construction to higher-dimensional subspaces to obtain interesting higher-dimensional vector bundles.

Definition. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Given natural numbers $n, k \in \mathbb{N}$ we consider

$$G(n,k,\mathbb{K}) \;:=\; \text{the set of all } n\text{-dimensional linear subspaces of } \mathbb{K}^{n+k}.$$

To equip the set $G(n,k,\mathbb{K})$ with a topology we first note that we have a surjection[1701]

$$\begin{array}{rcl} \overbrace{\left\{\begin{array}{c}\text{all } n\text{-tuples } (v_1,\dots,v_n) \text{ of vectors in } \mathbb{K}^{n+k} \\ \text{that are linearly independent}\end{array}\right\}}^{\text{open subset of } \mathbb{K}^{n(n+k)}} & \to & G(n,k,\mathbb{K}) \\ (v_1,\dots,v_n) & \mapsto & \begin{array}{c}\text{subspace of } \mathbb{K}^{n+k} \\ \text{spanned by } v_1,\dots,v_n.\end{array} \end{array}$$

We equip $G(n,k,\mathbb{K})$ with the quotient topology. We refer to $G(n,k,\mathbb{K})$ as a real Grassmannian respectively complex Grassmannian.[1702] Next note that by definition we have $G(1,n,\mathbb{K}) = \mathbb{K}\mathrm{P}^n$. With some modest effort one can show that

$$\begin{array}{rcl} T := \overbrace{\{(V,w)\,|\,V\in G(n,k,\mathbb{K}) \text{ and } w\in V\}}^{\subset G(n,k,\mathbb{K})\times\mathbb{K}^n} & \to & G(n,k,\mathbb{K}) \\ (V,w) & \mapsto & V \end{array}$$

is an n-dimensional $\mathbb{K}$-vector bundle. With a little bit of thought one can show that the maps $\mathbb{K}^{n+k} \xrightarrow{v\mapsto(v,0)} \mathbb{K}^{n+k}\times\{0\} \subset \mathbb{K}^{n+k+1}$ induce embeddings $G(n,k,\mathbb{K}) \to G(n,k+1,\mathbb{K})$ and we refer to the direct limit as infinite real (respectively complex) Grassmannian $G(n,\mathbb{K})$. The n-dimensional $\mathbb{K}$-vector bundles over $G(n,k,\mathbb{K})$ give rise to an n-dimensional $\mathbb{K}$-vector bundle over $G(n,\mathbb{K})$. We refer to this $\mathbb{K}$-vector bundle as the tautological $\mathbb{K}$-vector bundle over the $\mathbb{K}$-Grassmannian.

Remark.

(1) A little bit of thought shows that we can identify $G(n, \mathbb{K})$ with the set of n-dimensional linear subspaces of $\mathbb{K}^\infty$.
(2) It is shown in [**MiS1974**, p. 56] (see also [**Wel2008**, Example 1.8] and [**Hat2**, p. 28 and 34]) that one can equip $G(n, k, \mathbb{K})$ with a smooth atlas such that $G(n, k, \mathbb{K})$ becomes a closed orientable nk-dimensional smooth (complex) manifold. We will not make use of this pretty fact.

Initially these tautological vector bundles might appear only mildly interesting. But rather stunningly they can be viewed as the "mother of all vector bundles". More precisely, the following proposition says that every n-dimensional $\mathbb{K}$-vector bundle is the pullback of the tautological bundle over the infinite Grassmannian $G(n, \mathbb{K})$ in an "essentially unique way".

Proposition 143.15. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We denote by $T \to G(n, \mathbb{K})$ the tautological n-dimensional $\mathbb{K}$-vector bundle over the infinite Grassmannian $G(n, \mathbb{K})$ that we introduced on page 3074. Let X be a paracompact space. The following two statements hold:

(1) Given any n-dimensional $\mathbb{K}$-vector bundle $p\colon E \to X$ over X there exists a map $g\colon X \to G(n, \mathbb{K})$ such that g^*T is isomorphic to E.
(2) If $g_0, g_1\colon X \to G(n, \mathbb{K})$ are maps such that the $\mathbb{K}$-vector bundles g_0^*T and g_1^*T over X are isomorphic, then g_0 and g_1 are homotopic.

Before we turn to the proof of Proposition 143.15 let us formulate the following almost immediate corollary.

Corollary 143.16. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let X be a topological space that is paracompact. Given any $n \in \mathbb{N}_0$ we have a bijection

$$\begin{array}{rcl} [X, G(n, \mathbb{K})] & \to & \text{isomorphism classes of } n\text{-dimensional} \\ & & \mathbb{K}\text{-vector bundles over } X \\ [f\colon X \to G(n, \mathbb{K})] & \mapsto & f^*(\text{tautological vector bundle over } G(n, \mathbb{K})). \end{array}$$

Proof. This corollary is an immediate consequence of the Vector Bundles-Homotopies Proposition 143.13 (1) together with Proposition 143.15. ∎

In the following we will sketch the proof of Proposition 143.15. Full details for the proof are given in [**MiS1974**, Theorems 5.6 and 5.7]. For better readability we break the proof into two parts.

Sketch of proof of Proposition 143.15 (1). Let $p\colon E \to X$ be an n-dimensional $\mathbb{K}$-vector bundle over a paracompact space X.

Claim. There exists a map $\Theta\colon E \to \mathbb{K}^\infty$ such that for any $x \in X$ the restriction to the fiber E_x is a monomorphism of $\mathbb{K}$-vector spaces.

Suppose for a second that we have proved the claim. In this case we consider the map

$$\begin{array}{rcl} g\colon X & \to & G(n, \mathbb{K}) \\ x & \mapsto & [\ \underbrace{\Theta(E_x)}_{n\text{-dimensional subspace of } \mathbb{K}^\infty}\ ]. \end{array}$$

[1701] The left-hand side is in fact an open subset of $\mathbb{K}^{n(n+k)}$. Why is that?
[1702] These objects are named after the German mathematician Hermann Grassmann (1809-1877).

Note that we have the following bundle map:

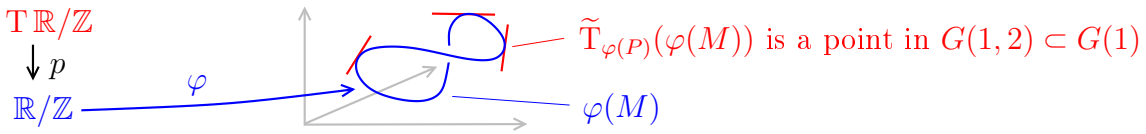
$$\begin{array}{ccc} E & \xrightarrow{v\mapsto(g(p(v)),\Theta(v))} & T \\ \downarrow{\scriptstyle p} & & \downarrow \\ X & \xrightarrow{g} & G(n,\mathbb{K}) \end{array}$$

Note that it follows easily from Lemma 143.8 (3) that the $\mathbb{K}$-vector bundles g^*T and E over X are isomorphic.

Proof.

(1) First, just for fun, we consider the important special case that we are dealing with the tangent bundle $p\colon \mathrm{T}M \to M$ over some closed n-dimensional smooth manifold M. By the Euclidean Embedding Theorem 27.1 we know that there exists a $k \in \mathbb{N}$ and a smooth embedding $\varphi\colon M \to \mathbb{R}^{n+k}$.[1703] The following map has the desired properties:

$$\begin{array}{rcl} g\colon M & \to & G(n,n+k,\mathbb{K}) \subset G(n,\mathbb{K}) \\ P & \mapsto & \underbrace{V_{\varphi(P)}(\varphi(M))}_{\substack{\text{physical tangent space of} \\ \text{the submanifold } \varphi(M)}}. \end{array}$$



(2) Now we consider the general case, i.e. let $p\colon E \to X$ be any n-dimensional $\mathbb{K}$-vector bundle over a paracompact space X. By definition of a vector bundle there exists an open cover $\{U_i\}_{i\in I}$ of X such that the restriction of p to each U_i is trivializable. Since X is paracompact we know by the Countable Cover Proposition 10.11 that there exists a *countable* open cover $\{V_j\}_{j\in\mathbb{N}}$ of X such that each component of any V_j is contained in some U_i. It follows from Lemma 143.2 that the restriction of p to each V_j is also trivial. By Lemma 143.2 this means that for each $j \in \mathbb{N}$ there exists a map $\Phi_j\colon p^{-1}(V_j) \to \mathbb{K}^n$ such that $p \times \Phi_j$ is a trivialization. Next, since X is paracompact we know by the Paracompact-Partition of Unity Theorem 10.9 that there exists a family of maps $\{f_j\colon X \to [0,1]\}_{j\in\mathbb{N}}$ with the following properties:

(a) For each $x \in X$ there exists an open neighborhood W of x such that f_j vanishes on W for all but finitely many j.

(b) For every $x \in X$ we have $\sum_{j\in\mathbb{N}} f_j(x) = 1$.

(c) For each $j \in \mathbb{N}$ we have $\mathrm{supp}(f_j) \subset V_j$.

Now we consider the map

$$\begin{array}{rcl} E & \to & \mathbb{K}^\infty = (\mathbb{K}^n)^\infty \\ v & \mapsto & \displaystyle\sum_{j\in J \text{ with } f_j(p(v))\neq 0} f_j(p(v)) \cdot \underbrace{\Phi_j(v)}_{\substack{\uparrow \\ \text{viewed as a vector in the } j\text{-th copy of } \mathbb{K}^n \text{ in } \mathbb{K}^\infty}} \end{array}$$

[1703] Note that in the proof of that proposition we used implicitly that smooth manifolds are paracompact.

We leave it to the reader to verify that this map has the desired properties. ■

Sketch of proof of Proposition 143.15 (2). Let $g_0, g_1 \colon X \to G(n, \mathbb{K})$ be two maps such that the $\mathbb{K}$-vector bundles g_0^*T and g_1^*T over X are isomorphic. This means that there exists a continuous map of the form

$$\begin{array}{ccc} g_0^*T & \to & g_1^*T \\ (x, v) & \mapsto & (x, \Phi_x(v)) \end{array}$$

where for each $x \in X$ the map $\Phi_x \colon g_0(x) \to g_1(x)$ is an isomorphism of $\mathbb{K}$-vector spaces. We need to show g_0 and g_1 are homotopic.

First suppose that for any $(x, v) \in g_0^*T$ and any $\lambda \in \mathbb{R}_{<0}$ we have $\Phi_x(v) \neq \lambda v$. In this case we consider the map

$$\begin{array}{rcl} H \colon X \times [0,1] & \to & G(n, \mathbb{K}) \\ (x, t) & \mapsto & \underbrace{\{(1-t) \cdot v + t \cdot \Phi_x(v) \mid v \in g_0(x)\}}_{\substack{n\text{-dimensional linear subspace of } \mathbb{K}^\infty \text{ since for} \\ \text{any } v \in g_0(x) \text{ and any } \lambda \in \mathbb{R}_{<0} \text{ we have } \Phi_x(v) \neq \lambda v}} \end{array} .$$

On [**MiS1974**, p. 68] it is shown that this map H is indeed continuous. Once continuity is out of the way it is clear that H is a homotopy between g_0 and g_1.

The general case can be reduced easily to the above special case. We consider the maps

$$\begin{array}{rcl} h_{\text{even}} \colon \mathbb{K}^\infty & \to & \mathbb{K}^\infty \\ (t_1, t_2, \dots) & \mapsto & (0, t_1, 0, t_2, 0, \dots) \end{array} \quad \text{and} \quad \begin{array}{rcl} h_{\text{odd}} \colon \mathbb{K}^\infty & \to & \mathbb{K}^\infty \\ (t_1, t_2, \dots) & \mapsto & (t_1, 0, t_2, 0, \dots) \end{array}$$

Note that both maps are homotopic to the identity.[1704] Now see that

$$g_0 \underset{\uparrow}{\sim} h_{\text{even}} \circ g_0 \overset{\downarrow}{\sim} h_{\text{odd}} \circ g_1 \underset{\uparrow}{\sim} g_1.$$

(now the previous special case applies; since h_{even} and h_{odd} are homotopic to the identity)

This shows that g_0 is homotopic to g_1. ■

143.5. **Trivializations of a vector bundle.** We recall the following definition from page 3063.

Definition. A trivialization of an n-dimensional $\mathbb{K}$-vector bundle $p \colon E \to X$ is an isomorphism $\Phi \colon E \to X \times \mathbb{K}^n$ of vector bundles.

Lemma 143.17. trivializations naturally give rise to orientations

trivializations correspond to choice of linearly independent sections

Proposition 143.18. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let $p \colon E \to X$ be an n-dimensional $\mathbb{K}$-vector bundle $p \colon E \to X$ over a topological space X and let $\Phi \colon E \to X \times \mathbb{K}^n$ be a trivialization.

[1704] For h_{even} the homotopy to the identity map is given by the map $\mathbb{K}^\infty \times [0,1] \to \mathbb{K}^\infty$ which is given by $(x, t) \mapsto h_{\text{even}}(x) \cdot (1-t) + x \cdot t$. Using the Direct Limits-and-Products Lemma 56.4 one can show that this map is actually continuous.

We denote by $q\colon X \times \mathbb{K}^n \to \mathbb{K}^n$ the projection. We equip $\mathrm{GL}(n,\mathbb{K}) \subset \mathrm{M}(n\times n,\mathbb{K}) = \mathbb{K}^{n^2}$ with the usual topology that we introduced on page 171.

(1) If $f\colon E \to \mathrm{GL}(n,\mathbb{K})$ is a continuous map, then the map

$$\begin{array}{rcl} E & \to & X \times \mathbb{K}^n \\ e & \mapsto & (p(e), \underbrace{f(p(e))}_{\in \mathrm{GL}(n,\mathbb{K})} \cdot \underbrace{q(\Phi(e))}_{\in \mathbb{K}^n}) \end{array}$$

is also a trivialization.

(2) Every trivialization of $p\colon E \to X$ is of the form described in (1).

Proof. The first statement is basically clear. For the second statement we just need to consider self-automorphisms of the trivial bundle $X \times \mathbb{K}^n$. If $\Phi\colon X \times \mathbb{K}^n \to X \times \mathbb{K}^n$ is a self-automorphism, then we define

$$\begin{array}{rcl} f\colon X & \to & \mathrm{GL}(n,\mathbb{K}) \\ x & \mapsto & \text{the unique matrix } A(x) \in \mathrm{GL}(n,\mathbb{K}) \text{ such that on the fiber } \{x\}\times\mathbb{K}^n \text{ the} \\ & & \text{endomorphism } f\colon x \times \mathbb{K}^n \to \{x\} \times \mathbb{K}^n \text{ is given by multiplication by } A(x). \end{array}$$

We leave it to the reader to verify that this map is continuous. ■

Lemma 143.19. Let $n \in \mathbb{N}$ and let $p\colon E \to \overline{B}^1$ be an oriented trivializable n-dimensional $\mathbb{K}$-vector bundle. Suppose we are given linearly independent vectors $v_1,\dots,v_n \in p^{-1}(-1)$ and $w_1,\dots,w_n \in p^{-1}(-1)$. The following two statements are equivalent:

(1) There exists a trivialization of E that restricts to the given trivializations on the endpoints.

(2) $v_1,\dots,v_n \in p^{-1}(-1)$ and $w_1,\dots,w_n \in p^{-1}(-1)$ are either both positive or both negative bases.

Proof. First note that the (1) $\Rightarrow$ (2)-implication is an immediate consequence of Lemma 143.17 and **??**.

Now we turn to the proof of the (2) $\Rightarrow$ (1)-implication. Without loss of generality we can assume that both $v_1,\dots,v_n \in p^{-1}(-1)$ and $w_1,\dots,w_n \in p^{-1}(-1)$ are positive. Since p is trivializable we know that $p\colon E \to \overline{B}^1$ admits a trivialization. It follows from Proposition 143.18 that the two trivializations on the endpoints differ by a map $\varphi\colon \{\pm 1\} \to \mathrm{GL}(n,\mathbb{R})$. It follows from Lemma **??** that $\det(\varphi(-1)) = \det(\varphi(1))$. By the Matrix Group Path-Component Proposition 2.37 this implies that there exists a map $\varphi[-1,1] \to \mathrm{GL}(n,\mathbb{R})$ that extends φ. But then φ times the trivialization has the desired property. ■

143.6. **Classifying space.** Given a topological group (perhaps with CW-structure) G need to construct classifying space BG. Get a fiber bundle $G \to EG \to BG$. Since EG is contractible we see that $\pi_i(BG) \cong \pi_{i-1}(G)$.

In Proposition 156.20 we calculated $\pi_i(\mathrm{O})$ etc. Note that for $[f\colon S^k \to \mathrm{BO}] \in \pi_k(\mathrm{BO})$ we get a vector bundle (well-defined up to stabilization), thus we can define $w_0([f]) \in \mathbb{Z}_2$, $w_1([f]) \in \mathbb{Z}_2$ and $p_1([f]) \in \mathbb{Z}$. According to PT these give you the isomorphisms in Proposition 156.20.

Also get map $\pi_i(\mathrm{BO}) \to \pi_i(\mathrm{BG})$ (which is more or less stable homotopy groups of spheres), which is called the J-homomorphism. For $i = 1, 2$ this is an isomorphism from $\mathbb{Z}_2 \to \mathbb{Z}_2$.

talk about $BTop/BG$, $BPl/BTop$ and BO/BPl. Some of these are Eilenberg-Maclane spaces, see Fnop?

143.7. **The Whitney sum of vector bundles.** In this section we will generalize the notion of the direct sum of two vector spaces to the notion of the direct sum of two vector bundles over a given topological space.

Definition. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to X$ be an m-dimensional $\mathbb{K}$-vector bundle and let $q\colon F \to X$ be an n-dimensional $\mathbb{K}$-vector bundle over a topological space X. The map

$$\begin{array}{rcl} E \oplus F := \{(v,w) \in E \times F \,|\, p(v) = q(w)\} & \to & X \\ (v,w) & \mapsto & p(v) \end{array}$$

is easily seen to be an $(m+n)$-dimensional $\mathbb{K}$-vector bundle. It is called the direct sum of the $\mathbb{K}$-vector bundles E and F.[1705]

Examples.

(1) Let $M \subset \mathbb{R}^n$ be a submanifold. Then for any point $P \in M$ we evidently have $\widetilde{\mathrm{T}}_P M \oplus (\widetilde{\mathrm{T}}_P M)^\perp = \mathbb{R}^n$. With this observation it follows that the map

$$\begin{array}{rcl} \mathrm{T}M \oplus NM & \to & M \times \mathbb{R}^n \\ ((P,v),(P,w)) & \mapsto & (P, v+w) \end{array}$$

is an isomorphism of real vector bundles. Put differently, the direct sum of the tangent bundle and the normal bundle is a trivial real bundle.

(2) We consider the sphere $S^2 \subset \mathbb{R}^3$. On page 3066 we saw that the tangent bundle $\mathrm{T}M$ is non-trivial and on page 3065 we saw that the normal bundle is trivializable. Thus we see that the direct sum of a non-trivial real vector bundle with a trivial real vector bundle can be trivial.

We state the last example as a lemma.

Lemma 143.20. Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold. We denote by $p\colon M \times N \to M$ and $q\colon M \times N \to N$ the natural projections. The map[1706]

$$\begin{array}{rcl} \mathrm{T}(N \times N) & \to & p^*(\mathrm{T}M) \oplus q^*(\mathrm{T}N) \\ ((x,y),v) & \mapsto & \big(((x,y),(x,p_*(v))),\, ((x,y),(y,q_*(v)))\big) \end{array}$$

is an isomorphism of $(m+n)$-dimensional real vector bundles over $M \times N$.

Proof. The lemma is fairly elementary, a detailed proof is given in [**Tu2011**, p. 371]. ■

143.8. **Linear algebra on vector bundles.** In this section we will see that the same way that we could obtain new vector spaces out of given vector spaces we can also obtain new vector bundles out of given vector bundles. We had seen one instance already in the last section, namely the "bundle-fication" of the direct sum of vector spaces. In this section we will see several other constructions. These construction will not play a big role in this

[1705] The direct sum of two $\mathbb{K}$-vector bundles is often also called the Whitney sum.

[1706] The map might appear slightly intimidating, but it is "the obvious" map. It looks somewhat complicated since the pullback vector bundles $p^*(\mathrm{T}M)$ and $q^*(\mathrm{T}M)$ are both defined to be pairs, where the first entry is a point in $M \times N$ and the second entry is also a pair.

chapter, but, surprisingly, they will play a very useful role later on when compute homotopy groups of spheres. First we give (or recall) the definition of the tensor product of two vector spaces. The definition is a slight variation on the definition of the tensor products of two abelian groups that we gave in Section 89.1.

Definition. Let $\mathbb{K}$ be a field and let V and W be two $\mathbb{K}$-vector spaces. We define the tensor product $V \otimes_{\mathbb{K}} W$ to be the following $\mathbb{K}$-vector space:

$$V \otimes_{\mathbb{K}} W := \left\{ \begin{array}{l} \mathbb{K}\text{-vector spaces which is generated by all} \\ \text{symbols of the form } v \otimes w \text{ with } v \in V,\ w \in W \end{array} \right\} \Big/ N(V,W)$$

where $N(V,W)$ is the linear subspace generated by the elements of the form

$$(\mu v + \mu' v') \otimes \nu w - (\mu v \otimes \nu w + \mu' v' \otimes \nu w) \quad \text{and} \quad \mu v \otimes (\nu w + \nu' w') - (\mu v \otimes \nu w + \mu v \otimes \nu' w')$$

with $v, v' \in V$, $w, w' \in W$ and with $\mu, \mu', \nu, \nu' \in \mathbb{K}$. For $v \in V$ and $w \in W$ we denote, by a slight abuse of notation, the image of $v \otimes w$ in $V \otimes W$ again by $v \otimes w$.

Example. Let $\mathbb{K}$ be a field, let V be a $\mathbb{K}$-vector space with basis $v_1, \ldots, v_m$ and let W be a $\mathbb{K}$-vector space with basis $w_1, \ldots, w_n$. We leave it to the reader to verify that the tensor product $V \otimes_{\mathbb{K}} W$ is a $\mathbb{K}$-vector space with basis $\{v_i \otimes w_j\}_{i=1,\ldots,m, j=1,\ldots,n}$.

Note that the tensor product of vector spaces satisfies the obvious analogues of the tensor product of abelian groups, see Lemma 89.1, 89.2 and 89.3. Surely there is no need to spell these statements out in detail.
Next we define the tensor product of vector bundles.

Definition. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to X$ be an m-dimensional $\mathbb{K}$-vector bundle and let $q\colon F \to X$ be an n-dimensional $\mathbb{K}$-vector bundle with the same base space X. We consider the set

$$E \otimes_{\mathbb{K}} F := \bigsqcup_{x \in X} \underbrace{E_x \otimes_{\mathbb{K}} F_x}_{\mathbb{K}\text{-vector space}} \qquad \text{with the obvious map } r\colon E \otimes_{\mathbb{K}} F \to X.$$

We equip $E \otimes_{\mathbb{K}} F$ with the smallest topology that has the property that for any open subset $U \subset X$ and any bundle isomorphisms $\phi\colon p^{-1}(U) \to U \times \mathbb{K}^m$ and $\psi\colon q^{-1}(U) \to U \times \mathbb{K}^n$ the map

$$\begin{array}{ccc} r^{-1}(U) & \to & U \times \overbrace{(\mathbb{K}^m \otimes_{\mathbb{K}} \mathbb{K}^n)}^{\text{topology is defined on page 153}} \\ v_x \otimes w_x & \mapsto & (x, \phi_x(v_x) \otimes \psi_x(w_x)) \end{array}$$

is a homeomorphism. One can now easily show that $r\colon E \otimes_{\mathbb{K}} F \to X$ is an $m \cdot n$-dimensional $\mathbb{K}$-vector bundle. It is called the tensor product of the $\mathbb{K}$-vector bundles E and F.

The tensor product gives us a functorial way to turn real vector spaces and bundles into complex vector spaces and bundles. We summarize this construction in the following lemma.

Lemma 143.21.

(1) Let V be a real vector space. The tensor product $V \otimes_{\mathbb{R}} \mathbb{C}$ with the complex multiplication induced by $z \cdot (v \otimes w) := v \otimes (z \cdot w)$ is a complex vector space with $\dim_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C}) = \dim_{\mathbb{R}}(V)$.

(2) Let $p\colon E \to X$ be a real vector bundle. We denote by $q\colon X \times \mathbb{C} \to X$ the trivializable 1-dimensional complex vector bundle. The vector bundle $E \otimes_{\mathbb{R}} \mathbb{C} := E \otimes_{\mathbb{R}} (X \times \mathbb{C})$, with the complex vector space structure on each fiber defined as in (1), is a complex vector bundle.
(3) The map $V \mapsto V \otimes_{\mathbb{R}} \mathbb{C}$ defines a covariant functor from the category $\mathcal{V}ec_{\mathbb{R}}$ to the category $\mathcal{V}ec_{\mathbb{C}}$ and the map $E \mapsto E \otimes_{\mathbb{R}} \mathbb{C}$ defines a covariant functor from $\mathcal{V}ect\mathcal{B}und_{\mathbb{R}}$ to $\mathcal{V}ect\mathcal{B}und_{\mathbb{C}}$.

Proof. We leave it to the reader to make the statements more precise and to prove them. ■

We can extend several other constructions for vector spaces to the setting of vector bundles.

Definition. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $p\colon E \to X$ be an m-dimensional $\mathbb{K}$-vector bundle. We denote by $E^* \to X$ the $\mathbb{K}$-vector bundle where for each $x \in X$ the corresponding fiber is given by the dual vector space $\mathrm{Hom}_{\mathbb{K}}(E_x, \mathbb{K})$. It should be clear how to fill in the precise details for the definitions. We refer to $E^* \to X$ as the dual vector bundle of $p\colon E \to X$.

The next lemma brings together the tensor product of two vector bundles and the dual of a vector bundle in an elegant way.

Lemma 143.22. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $p\colon E \to X$ be an n-dimensional $\mathbb{K}$-vector bundle. There is a unique map

$$\Theta\colon E \otimes E^* \ \to \ X \times \mathrm{M}(n \times n, \mathbb{K})$$

of $\mathbb{K}$-vector bundles with the property that for any $x \in X$ and any basis $v_1, \ldots, v_n$ of the fiber $E_x = p^{-1}(x)$ with corresponding dual basis $v_1^*, \ldots, v_n^*$ of the dual vector space E_x^* we have $\Theta(v_i \otimes v_j^*) = (x, e_{ij})$ for any choice of $i, j \in \{1, \ldots, n\}$. Furthermore this map is in fact an isomorphism of $\mathbb{K}$-vector bundles over X.

Sketch of proof. In Exercise 143.10 we will see that the map on each fiber does not depend on the choice of the basis $v_1, \ldots, v_n$. With this observation it is then fairly easy to see that the map is an isomorphism of $\mathbb{K}$-vector bundles. We leave all further details to the reader. Over all this lemma is a good example of the magic of clever definitions. ■

It turns out that out of the blue we now have a new, and potentially interesting, invariant for CW-complexes. More precisely, we have the following proposition.

Proposition 143.23. Let $\mathbb{K} = \mathbb{R}$ or let $\mathbb{K} = \mathbb{C}$. The maps

$$\begin{array}{rcl} X & \mapsto & (\mathrm{Vect}^1_{\mathbb{K}}(X), \otimes) \\ (f\colon X \to Y) & \mapsto & \left(\begin{array}{rcl} f^*\colon \mathrm{Vect}^1_{\mathbb{K}}(Y) & \to & \mathrm{Vect}^1_{\mathbb{K}}(X) \\ E & \mapsto & f^*E \end{array} \right) \end{array}$$

define a contravariant functor from the category $\mathcal{T}op$ of topological space to the category $\mathcal{A}b\mathcal{G}roup$ of abelian groups. Here, in the group $(\mathrm{Vect}^1_{\mathbb{K}}(X), \otimes)$ the neutral element is given by the trivial 1-dimensional $\mathbb{K}$-vector bundle and the inverse of $[E \to X]$ is given by $[E^* \to X]$.

Sketch of proof. Let X be a topological space. First of all, note that by Corollary 143.7 we know that $\mathrm{Vect}^1_{\mathbb{K}}(X)$ is a set. Next, it is straightforward to show that "$\otimes$" defines an associative and commutative product on $\mathrm{Vect}^1_{\mathbb{K}}(X)$ with neutral element given by the 1-dimensional trivial bundle. Finally note that given a 1-dimensional $\mathbb{K}$-vector bundle

$E \to X$ we have

$$[E \to X] \otimes [E^* \to X] \;=\; [E \otimes E^* \to X] \underset{\uparrow}{=} [X \times \mathrm{M}(1 \times 1, \mathbb{K})] \;=\; [\mathbb{K} \times X].$$

Lemma 143.22

This shows that $\mathrm{Vect}^1_{\mathbb{K}}(X)$ is indeed an abelian group. The remaining statements of the proposition follow easily from the definitions and are left to the reader. ■

The proposition raises the following question.

Question 143.24. Is the contravariant functor $X \mapsto \mathrm{Vect}^1_{\mathbb{K}}(X)$ a "new" contravariant functor or is it a well-known functor in slight disguise?

We will give a complete answer to Question 143.24 in Theorem **??**. Now we provide two more definitions.

Definition. Let $p\colon E \to X$ be a real vector bundle.

(1) An orientation of $E \to X$ is a choice of an orientation for each E_x which is compatible with all trivializations.
(2) Given $k \in \mathbb{N}_0$ we denote by $\wedge^k E \to X$ the real vector bundle where for each $x \in X$ the fiber is given by the vector space $\wedge^k E_x^*$ that we introduced on page **??**.

We have enough faith in the reader to be confident that the reader can turn these slogans into proper rigorous definitions.

Examples.

(1) For any smooth manifold M an orientation of M is, almost by definition, the same as an orientation of the tangent bundle $\mathrm{T}M$.
(2) Similar to the proof of the Complex Manifold-Orientation Proposition 32.23 one can show that every k-dimensional complex vector bundle is an orientable $2k$-dimensional real vector bundle. In Exercise 143.12 we will prove a partial converse to this statement.
(3) Given a smooth manifold M a section of the bundle $\wedge^k \mathrm{T}M \to M$ is basically the same as a continuous k-form on M as defined on page **??**.

We conclude this section with the following lemma.

Lemma 143.25. Let M be an n-dimensional smooth manifold. The smooth manifold M is orientable if and only if the vector bundle $\wedge^n \mathrm{T}M^*$ is a trivial line bundle.[1707]

Proof. The bored reader will surely be pleased to hear that the proof is outsourced as Exercise 143.11. ■

Lemma 143.25 motivates the following definition which will come in handy later on.

Definition. Given an n-dimensional smooth manifold M we refer to the vector bundle $\wedge^n \mathrm{T}M^*$ as the orientation bundle.

143.9. **Vector bundles over spheres.** We conclude this chapter with a discussion of vector bundles over spheres.

[1707] Note that we know by the Alternating Forms-Basis Proposition **??** that $\wedge^n \mathrm{T}M^*$ is a line bundle.

Lemma 143.26. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let $k \in \mathbb{N}$ and let $n \in \mathbb{N}$.

(1) Let $\varphi\colon S^{k-1} \to \mathrm{GL}(n,\mathbb{K})$ be a map. The map

$$\begin{array}{rcl} \Theta(\varphi)\colon ((S^k_{\geq 0} \times \mathbb{K}^n) \sqcup (S^k_{\leq 0} \times \mathbb{K}^n))/(x,v) \sim (x, \varphi(x)\cdot v) & \to & S^k \\ [(x,v)] & \mapsto & x \end{array}$$

is an n-dimensional $\mathbb{K}$-vector bundle over S^k. (The map $\varphi\colon S^{k-1} \to \mathrm{GL}(n,\mathbb{K})$ is called the clutching map and this construction of $\mathbb{K}$-vector bundles is called the clutching construction.)

(2) The map

$$\begin{array}{rcl} \overbrace{[S^{k-1}, \mathrm{GL}(n,\mathbb{K})]}^{\text{set of homotopy classes of maps } S^{k-1} \to \mathrm{GL}(n,\mathbb{K})} & \to & \mathrm{Vect}^n_{\mathbb{K}}(S^k) \\ [\varphi] & \mapsto & [\Theta(\varphi)] \end{array}$$

is well-defined.

Example. Let $k = 1$. We consider the map $\varphi\colon S^0 \to \mathrm{GL}(1,\mathbb{R})$ which is conveniently given by $\pm 1 \to (\pm 1)$. It is not difficult to see that the resulting real 1-dimensional vector bundle over S^1 is isomorphic to the Möbius bundle.

Sketch of proof. For Statement (1) we just note that, using collar neighborhoods of S^{k-1} in $S^k_{\geq 0}$ and $S^k_{\leq 0}$ it is straightforward to verify by hand that the clutching construction gives rise to a $\mathbb{K}$-vector bundle. We leave the details to the reader.

We turn to the proof of Statement (2). Let $f_0, f_1\colon S^{k-1} \to \mathrm{GL}(n,\mathbb{K})$ be two maps that are homotopic. This implies that there exists a homotopy $F\colon S^{k-1} \times [0,1] \to \mathrm{GL}(n,\mathbb{K})$ with $F_0(x) = f_1(x) \cdot f_0^{-1}(x)$ and $F_1 = \mathrm{id}$. By picking a suitable collar neighborhood of $S^{k-1} = \partial S^k_{\geq 0}$ we obtain an identification $S^k_{\geq 0} = (S^{k-1} \times [0,1]) \cup S^k_{\geq \frac{1}{2}}$. One can now easily verify that the following map is an isomorphism of $\mathbb{K}$-vector bundles over S^k:

$$\begin{array}{rcl} (S^k_{\geq 0} \times \mathbb{K}^n \sqcup S^k_{\leq 0} \times \mathbb{K}^n)/(x,v) \sim (x, f_0(x)\cdot v) & \to & (S^k_{\geq 0} \times \mathbb{K}^n \sqcup S^k_{\leq 0} \times \mathbb{K}^n)/(x,v) \sim (x, f_1(x)\cdot v) \\ (x,v) & \mapsto & \begin{cases} (x,v), & \text{if } x \in S^k_{\leq 0} \text{ or if } x \in S^k_{\geq \frac{1}{2}}, \\ (x, F(y,t)\cdot v), & \text{if } x = (y,t) \in S^{k-1} \times [0,1]. \end{cases} \end{array}$$

This concludes the proof of Statement (2). ∎

The following proposition gives a convenient description of $\mathrm{Vect}^n_{\mathbb{C}}(S^k)$.

Proposition 143.27. The map

$$\begin{array}{rcl} \Theta\colon [S^{k-1}, \mathrm{GL}(n,\mathbb{C})] & \to & \mathrm{Vect}^n_{\mathbb{C}}(S^k) \\ [\varphi] & \mapsto & [\Theta(\varphi)] \end{array}$$

is a well-defined bijection.

Proof. Let $\overline{B}^k_{\pm}$ be two copies of $\overline{B}^k$. In the following we make the hopefully obvious identification $S^k = S^k_{\geq 0} \cup_{S^{k-1}} S^k_{\leq 0} = \overline{B}^k_+ \cup_{S^{k-1}} \overline{B}^k_-$ and henceforth we work with the topological space to the right.

Now let $p\colon E \to S^k = \overline{B}^k_+ \cup_{S^{k-1}} \overline{B}^k_-$ be an n-dimensional complex vector bundle. We write $E^{\pm} := p^{-1}(\overline{B}^k_{\pm})$. Since $\overline{B}^k_+$ and $\overline{B}^k_-$ are compact and contractible we know by the Vector Bundles-Homotopies Proposition 143.13 that the bundles $E^{\pm} \to \overline{B}^k_{\pm}$ are trivial. By

Lemma 143.2 this means that there exist maps $\Phi^{\pm}\colon E^{\pm} \to \mathbb{C}^n$ such that $p \times \Phi^{\pm}\colon E^{\pm} \to \overline{B}^k_{\pm} \times \mathbb{C}^n$ are trivializations of $E^{\pm} \to \overline{B}^k_{\pm}$. Given $x \in \overline{B}^n_{\pm}$ we write $\Phi^{\pm}_x := \Phi^{\pm}|_{E_x}\colon E_x \xrightarrow{\cong} \mathbb{C}^n$. Similar to the proof of Lemma 143.6 we consider the map

$$\begin{aligned} \Omega(E)\colon S^{k-1} &\to \mathrm{GL}(n,\mathbb{C}) = \mathrm{Aut}(\mathbb{C}^n) \\ x &\mapsto \Phi^-_x \circ (\Phi^+_x)^{-1} = \left(\mathbb{C}^n \xrightarrow{(\Phi^+_x)^{-1}} E_x \xrightarrow{\Phi^-_x} \mathbb{C}^n\right) \end{aligned}$$

Claim. The homotopy class $[\Omega(E)] \in [S^{k-1}, \mathrm{GL}(n,\mathbb{C})]$ does not depend on the choice of trivializations $\Phi^{\pm}$.

Proof. Suppose we are given two other maps $\Psi^{\pm}\colon E^{\pm} \to \mathbb{C}^n$ such that $p \times \Psi^{\pm}\colon E^{\pm} \to \overline{B}^k_{\pm} \times \mathbb{C}^n$ are trivializations of $E^{\pm} \to \overline{B}^k_{\pm}$. By the Matrix Group Path-Component Proposition 2.37 we know that $\mathrm{GL}(n,\mathbb{C})$ is path-connected. This implies that there exists a path $\gamma\colon [1,2] \to \mathrm{GL}(n,\mathbb{C})$ with $\gamma(1) = \Phi^-_0 \circ (\Phi^+_0)^{-1}$ and $\gamma(2) = \Psi^-_0 \circ (\Psi^+_0)^{-1}$. The map

$$\begin{aligned} S^{k-1} \times [0,3] &\to \mathrm{GL}(n,\mathbb{C}) \\ (x,t) &\mapsto \begin{cases} \Phi^-_{x\cdot(1-t)} \circ (\Phi^+_{x\cdot(1-t)})^{-1}, & \text{if } t \in [0,1], \\ \gamma(t), & \text{if } t \in [1,2], \\ \Psi^-_{x\cdot(t-2)} \circ (\Psi^+_{x\cdot(t-2)})^{-1}, & \text{if } t \in [2,3] \end{cases} \end{aligned}$$

is now the desired homotopy. ⊞

We leave it to the reader to verify that the maps $\Theta\colon [S^{k-1}, \mathrm{GL}(n,\mathbb{C})] \to \mathrm{Vect}^n_{\mathbb{C}}(S^k)$ and $\Omega\colon \mathrm{Vect}^n_{\mathbb{C}}(S^k) \to [S^{k-1}, \mathrm{GL}(n,\mathbb{C})]$ are actually inverses of one-another. ■

Note that the group structure on $\mathrm{GL}(n,\mathbb{C})$ endows $[S^{k-1}, \mathrm{GL}(n,\mathbb{C})]$ with a group structure. Furthermore note that by Proposition 143.23 we know that $\mathrm{Vect}^1_{\mathbb{C}}(S^k)$ comes with a group structure. The following lemma relates these two group structures.

Lemma 143.28. Let $k \in \mathbb{N}$. The map

$$\begin{aligned} \Theta\colon [S^{k-1}, \mathrm{GL}(1,\mathbb{C})] &\to \mathrm{Vect}^1_{\mathbb{C}}(S^k) \\ [\varphi] &\mapsto [\Theta(\varphi)] \end{aligned}$$

is a group homomorphism, in other words, given $\varphi, \psi\colon S^{k-1} \to \mathrm{GL}(1,\mathbb{C})$ we have the equality $[\Theta(\varphi \cdot \psi)] = [\Theta(\varphi)] \otimes [\Theta(\psi)]$.

Sketch of proof. At the end of the day the proof boils down to the following observation. Given $a, b \in \mathbb{C} \setminus \{0\}$ the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} & \xrightarrow{x\otimes y \mapsto x\cdot y} & \mathbb{C} \\ {\scriptstyle (x\mapsto a\cdot x)\otimes(y\mapsto b\cdot y)}\downarrow & & \downarrow{\scriptstyle z \mapsto a\cdot b\cdot z} \\ \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} & \xrightarrow{x\otimes y \mapsto x\cdot y} & \mathbb{C}. \end{array}$$

We leave it to the reader to assemble a proof out of this observation. ■

Remark. We consider 1-dimensional complex vector bundles over S^2. We have the following isomorphisms of abelian groups:

$$\mathrm{Vect}^1_{\mathbb{C}}(S^2) \xleftarrow[\uparrow]{\cong} [S^1, \mathrm{GL}(1,\mathbb{C})] \xleftarrow[\uparrow]{\cong} [S^1, U(2)] = [S^1, S^1] \xrightarrow[\uparrow]{[f]\mapsto \deg(f)} \mathbb{Z}.$$

group isomorphism by Proposition 143.27 — group isomorphism by Lemmas ?? and 15.12 (2) — group isomorphism by the Degree Isomorphism Proposition 86.8

We have thus completely classified the group of complex line bundles over S^2. Later on, in Theorem **??** we will generalize this calculation to give a complete calculation of $\mathrm{Vect}^1_{\mathbb{C}}(X)$ for any CW-complex X.

The reader might wonder why we formulated Proposition 143.27 for complex vector bundles instead of real vector bundles. The reason is that in the proof of Proposition 143.27 we used that $\mathrm{GL}(n,\mathbb{C})$ is path-connected. In the Matrix Group Path-Component Proposition 2.37 we saw that in contrast $\mathrm{GL}(n,\mathbb{R})$ has two path-components, one of which is $\mathrm{GL}_+(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid \det(A) > 0\}$. For formulate a "real" analogue of Proposition 143.27 it is best to introduce the following notation.

Notation. Let X be a topological space.

(1) We say two oriented vector bundles $p\colon E \to X$ and $q\colon F \to X$ are isomorphic if there exists an isomorphism $\Phi\colon E \to F$ of vector bundles as defined on page 3060, such that for each $x \in X$ the map $E_x \to F_x$ is an orientation-preserving isomorphism.

(2) Let $n \in \mathbb{N}_0$. We denote by $\mathrm{Vect}^{n,+}_{\mathbb{R}}(X)$ the set of isomorphism classes of n-dimensional oriented $\mathbb{R}$-vector bundles over X.

Now we can formulate the following analogue of Proposition 143.27.

Proposition 143.29. The map

$$\begin{array}{rcl} \Theta\colon [S^{k-1}, \mathrm{GL}_+(n,\mathbb{R})] & \to & \mathrm{Vect}^{n,+}_{\mathbb{R}}(S^k) \\ [\varphi] & \mapsto & [\Theta(\varphi)] \end{array}$$

is a well-defined bijection.

Proof. The proof if is quite similar to the proof of Proposition 143.27. We leave it to the reader to verify that the statement regarding the orientations works out. ∎

Remark. Similar to the above we have the following bijections:

$$\mathrm{Vect}^{2,+}_{\mathbb{R}}(S^2) \xleftarrow[\uparrow]{\cong} [S^1, \mathrm{GL}_+(2,\mathbb{R})] \xleftarrow[\uparrow]{\cong} [S^1, O(2)] = [S^1, S^1] \xrightarrow[\uparrow]{[f]\mapsto \deg(f)} \mathbb{Z}.$$

bijection given by Proposition 143.29

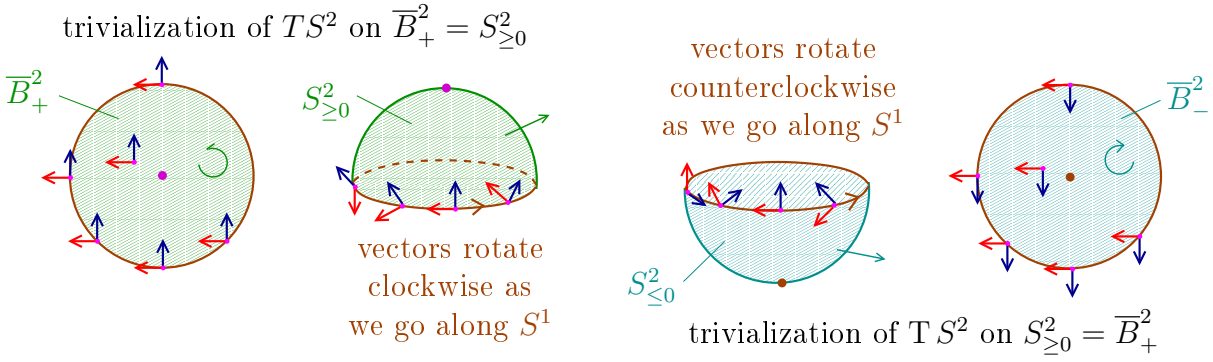
group isomorphism by Proposition 15.7 and Lemma 15.12 (2)

group isomorphism by the

Given an oriented 2-dimensional real vector bundle $p\colon E \to S^2$ we refer to the corresponding integer on the right as the *Euler number of the vector bundle* $p\colon E \to S^2$.

Example. We just introduced the Euler number of an oriented 2-dimensional real vector bundle over S^2. In this final example we will try to understand the Euler number of the tangent bundle $\mathrm{T}S^2$ of S^2. In the figure below we try our best to illustrate the situation. We show two trivializations of the $\mathrm{T}S^2$ over $\overline{B}^2_+ = S^2_{\geq 0}$ and $S^2_{\leq 0} = \overline{B}^2_-$. We can restrict the two trivializations of $\mathrm{T}S^2$ to S^1.[1708] As we go along S^1 we see that at $z \in S^1$ we have to rotate the trivialization coming from $S^2_{\geq 0}$ by z^2 to obtain the trivialization coming from $S^2_{\leq 0}$. This implies that the Euler number of $\mathrm{T}S^2$ equals $+2$. The author is full aware that the figure below is only mildly helpful and we encourage the readers to draw pictures themselves.

[1708] Perhaps it is worth stressing, we consider the restriction of $\mathrm{T}S^2$ to S^1, we do not consider $\mathrm{T}S^1$.



Remark. Propositions 143.27 and 143.29, together with the discussion in Section 70.3, tell us that the key to understanding vector bundles over spheres lies in the understanding of the homotopy groups of $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{GL}_+(n,\mathbb{R})$. It turns out that this is a difficult problem. We will obtain some partial calculations in Chapter 156.

Exercises for Chapter 143.

Exercise 143.1. Let $p\colon E \to X$ be a vector bundle and let $s\colon X \to E$ be a section. Show that the zero set $\{x \in X \,|\, s(x) = 0\}$ is a closed subset of X.
Hint. Use Lemma 1.8 (2b).

Exercise 143.2. Does there exist a smooth vector field on S^2 with a single zero?

Exercise 143.3. Let W be a smooth manifold and let M be a submanifold of W. We denote by $i\colon M \to W$ the inclusion map. Show that there exists a vector bundle $E \to M$ such that $\mathrm{T}M \oplus E$ is isomorphic to $i^*(\mathrm{T}M)$.

Exercise 143.4. Show that S^3 and S^7 are parallelizable.

Exercise 143.5. Let $z \in \mathbb{R} \setminus \{0\}$. We denote by $\rho_z\colon \pi_1(S^1,1) \to \mathrm{GL}(1,\mathbb{R})$ the representation that sends the standard generator of $\pi_1(S^1,1) \cong \mathbb{Z}$ to the matrix (z). We denote by $E_z \to S^1$ the line bundle corresponding to this representation, see the definition on page 3063. For which z is this line bundle isomorphic to the trivial bundle and for which z is it isomorphic to the Möbius bundle?

Exercise 143.6. Show by hand that the Möbius bundle, as defined on page 3060, does not admit a section which is nowhere-vanishing.

Exercise 143.7. By Exercise 5.25 we can identify $\mathbb{R}\mathrm{P}^1$ with S^1. Let $p\colon E \to \mathbb{R}\mathrm{P}^1 = S^1$ be the tautological line bundle over $\mathbb{R}\mathrm{P}^1 = S^1$ from page 3060. Is this line bundle trivial? Is it isomorphic to the Möbius bundle?

Exercise 143.8. Let $g \geq 1$ and let Σ be the surface of genus g. Show that Σ admits a non-trivial real line bundle.

Exercise 143.9. Let $p\colon M \to S^1$ be the Möbius bundle and let $g\colon S^1 \to S^1$ be the map that is given by $g(z) = z^2$. Show that the pullback of the Möbius bundle under g is a trivial bundle.

Exercise 143.10. Let $\mathbb{K}$ be a field and let V be an n-dimensional $\mathbb{K}$-vector space. We write $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$. Let $v_1, \ldots, v_n$ be a basis of V. We denote by $v_1^*, \ldots, v_n^*$ the corresponding dual basis of V^*. We consider the $\mathbb{K}$-homomorphism

$$\begin{array}{rcl} V \otimes V^* & \to & \mathrm{M}(n \times n, \mathbb{K}) \\ v_i \otimes v_j^* & \mapsto & e_{ij} := \begin{array}{c}(n \times n)\text{-matrix with a single non-zero entry,} \\ \text{namely } 1 \in \mathbb{K} \text{ in the } (i,j)\text{-entry.}\end{array} \end{array}$$

Show that this homomorphism does not depend on the choice of the basis $v_1, \ldots, v_n$.

Exercise 143.11. Let M be an n-dimensional smooth manifold. Show that M is orientable if and only if the orientation bundle $\wedge^n \mathrm{T}M^*$ is a trivial bundle.

Exercise 143.12. Let $p\colon E \to X$ be an oriented 2-dimensional real vector bundle over a paracompact space X. We want to show that E can be equipped with the structure of a 1-dimensional complex vector bundle.

(a) Show that given any $x \in X$ we can find a non-singular positive symmetric form $\langle\, , \,\rangle_x$ on E_x, such that the form varies "continuously with x". (Here you need that X is paracompact.)

(b) Given $x \in X$ we consider the map

$$\begin{array}{rcl} J\colon E_x \setminus \{0\} & \to & E_x \setminus \{0\} \\ v & \mapsto & \begin{array}{c}\text{the unique vector } w \in E_x \text{ with } \langle v, w\rangle_x = 0 \\ \text{and such that } v, w \text{ is a positive basis for } E_x.\end{array} \end{array}$$

Show that $J^2 = -\mathrm{id}_{E_x}$.

(c) Show that E can be equipped with the structure of a 1-dimensional complex vector bundle.

Exercise 143.13. Let M be the Möbius band. Show that there is no immersion $f\colon M \to \mathbb{R}^2$.

Hint. Suppose there was an immersion $f\colon M \to \mathbb{R}^2$. What can you say about the pull back of the tangent bundle of $\mathbb{R}^2$?

Exercise 143.14. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of vector bundles over a topological space X. Show that the vector bundle $B \to X$ is isomorphic to the vector bundle $A \oplus C \to X$.

Exercise 143.15. Let X be a paracompact space and let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We denote by $i_0\colon X \xrightarrow{x \mapsto (x,0)} X \times [0,1]$ and $i_1\colon X \xrightarrow{x \mapsto (x,1)} X \times [0,1]$ the obvious embeddings. In Lemma 143.14 we showed that given any $\mathbb{K}$-vector bundle $p\colon E \to X \times [0,1]$ the pullbacks i_0^*E and i_1^*E are isomorphic. Show in an example that the isomorphism that we constructed in the proof of Lemma 143.14 depends in general on the choices we make in the proof.

144. Cobordisms

One could ask whether every closed smooth manifold is the boundary of a compact smooth manifold. In fact in Corollary 119.9 we already saw that the answer is very much no. In this chapter we will "quantify" to what degree the answer is no by introducing the "cobordism group", which is an interesting object in its own right.

144.1. **The cobordism rings.** Given two closed n-dimensional smooth manifolds M and N we obtain a new closed n-dimensional smooth manifold by taking the disjoint union $M \sqcup N$. This operation has a neutral element, given by the empty smooth manifold, and it looks abelian and associative.[1709] Thus it looks that all closed n-dimensional smooth manifolds together with the operation "$\sqcup$" form a monoid.
For the reader's convenience we recall the definition of a monoid.

Definition. A monoid is a set S together with a map

$$\begin{array}{ccc} S \times S & \to & S \\ (x,y) & \mapsto & x \cdot y \end{array}$$

such that the following two conditions are satisfied:[1710]

(1) there exists a neutral element $e \in S$, i.e. an element which satisfies $e \cdot x = x \cdot e = x$ for all $x \in S$,
(2) the operation "$\cdot$" is associative, i.e. for all $x, y, z \in S$ we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Careful comparison between the definition of a monoid and the discussion of smooth manifolds above reveals a subtle problem: "all closed n-dimensional smooth manifolds" do not form a set. As most mathematicians I have no intention of getting into the debate about what is a set. Instead we refer to [**FPr1985**] and [**Halm1960**] for details. Also, at this stage reading "Logicomix" [**DoP2009**], see the figure below, is almost a must.
Fortunately, we are saved by the following lemma.

Lemma 144.1. Let $n \in \mathbb{N}_0$.

(1) The homeomorphism classes of n-dimensional topological manifolds form a set.
(2) The diffeomorphism classes of n-dimensional smooth manifolds form a set.

Remark. Note that here it is essential that smooth manifolds are assumed to be second-countable. If we did not make this assumption, then any set with the discrete topology would be a 0-dimensional smooth manifold, but the bijection classes of sets do not form a set.

Sketchy proof.

(1) It follows from the Euclidean Embedding Theorem 27.1 (1) that every n-dimensional topological manifold is homeomorphic to a subspace of some $\mathbb{R}^k$. Thus we can identify "homeomorphism classes of n-dimensional topological manifolds" with a subset of the power set of $\mathbb{R}^\infty$, which is a set.

[1709] Very well-trained eyes will spot that purely set theoretically, $M \sqcup N$ and $N \sqcup M$ are not exactly the same, see the definition on page 88.
[1710] A *semi-group* is defined similarly, except that we do not demand the existence of a neutral element.



(2) As in (1) we see, using the Euclidean Embedding Theorem 27.1 (2) and using the Smooth Embedding Theorem 24.10 we see that it suffices to show that the diffeomorphism classes of n-dimensional manifolds, where the underlying topological space is a subset of some $\mathbb{R}^k$, form a set. But this follows from the fact that given a subset of some $\mathbb{R}^k$ the possible atlases form a set.

We leave it to our friends the logicians to turn the above sloppy argument into a proper proof. ■

Next we introduce a notation and we recall a definition from page **??**.

Definition.

(1) Given $n \in \mathbb{N}_0$ we denote by $\mathcal{M}_n$ the set of all diffeomorphism classes of closed oriented n-dimensional smooth manifolds.[17111712]

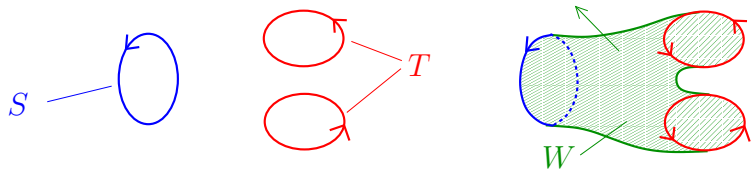
(2) Given $S, T \in \mathcal{M}_n$ we say that S and T are cobordant if there exists a cobordism between S and T, i.e. a compact oriented $(n+1)$-dimensional smooth manifold W such that[1713] ∂W admits an orientation-preserving diffeomorphism to $S \sqcup -T$. If S and T are cobordant, then we write $S \sim T$.

(3) We say $S \in \mathcal{M}_n$ is nullbordant if S is cobordant to the empty n-dimensional manifold, in other words, S is nullbordant if there exists a compact oriented $(n+1)$-dimensional smooth manifold W such that ∂W admits an orientation-preserving diffeomorphism to S.

[1711] Recall that the empty set is a closed oriented n-dimensional smooth manifold.

[1712] Also recall that an orientation for a 0-dimensional smooth manifold M consists of a choice of a sign in $\{\pm 1\}$ for each point in M, see the remark on page 675.

[1713] Recall that each component of ∂W inherits an orientation from the orientation of W.



Convention. In the following we will often be slightly sloppy in our notation, meaning that we will not distinguish between a smooth manifold and its diffeomorphism class. For example, given oriented smooth manifolds W, S and T we often write $\partial M = S \sqcup T$, even though strictly speaking we have $\partial M = S' \sqcup T'$ where S' and T' are oriented smooth manifolds that admit an orientation-preserving diffeomorphism to S respectively T.

Remark. Let $n \in \mathbb{N}_{\geq 2}$ and let $S, T \in \mathcal{M}_n$. We suppose that S and T are cobordant. Using the connected sum operation, in particular using the Connected Sum-Proposition **??**, one can easily show that there exists a connected cobordism between S and T.

Lemma 144.2. Let $n \in \mathbb{N}_0$.

(1) The relation $\sim$ on $\mathcal{M}_n$ is an equivalence relation.

(2) The map
$$\begin{array}{rcl} \mathcal{M}_n \times \mathcal{M}_n & \to & \mathcal{M}_n \\ ([S],[T]) & \mapsto & [S \sqcup T] \end{array}$$
is well-defined and it descends to an abelian group structure on the set $\mathcal{M}_n/\sim$[1714] where the neutral element is given by the empty smooth manifold.

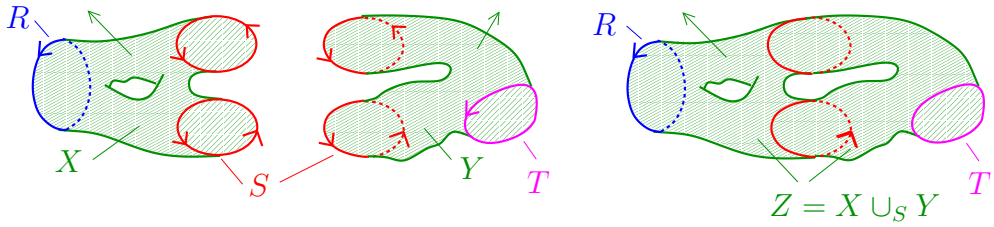
Proof.

(1) We first show that the relation $\sim$ on $\mathcal{M}_n$ is an equivalence relation.

(a) Given $M \in \mathcal{M}_n$ we have $M \sim M$ since, by the discussion on page 696, we have $\partial([0,1] \times M) = \{0\} \times (-M) \cup \{1\} \times M$.

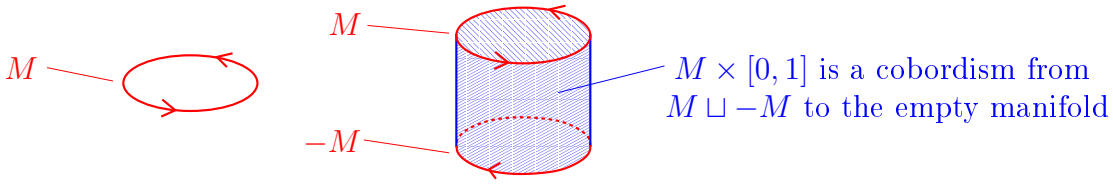
(b) Suppose that $S \sim T$, i.e. there exists a compact oriented $(n+1)$-dimensional smooth manifold W with $\partial W = S \sqcup -T$. Then $\partial(-W) = -S \sqcup T = T \sqcup (-S)$ which shows that $T \sim S$.

(c) Finally suppose that we have $R, S, T \in \mathcal{M}_n$ with $R \sim S$ and $S \sim T$. We showed already in Lemma 144.2 that R is also cobordant to T. But there is no harm done in repeating the simple argument. This means that there exist compact oriented $(n+1)$-dimensional smooth manifolds X and Y with $\partial X = R \sqcup -S$ and $\partial Y = S \sqcup -T$. We set $Z := X \cup_S Y$. It follows from the Gluing-Smooth Manifolds-Proposition **??** that Z is a compact orientable $(n+1)$-dimensional smooth manifold with $\partial Z = R \sqcup -T$. We refer to the figure below for an illustration.



[1714] By Lemma 144.1 we know that $\mathcal{M}_n$ and thus also $\mathcal{M}_n/\sim$ are a set.

(2) This statement follows from the following five observations:
   (a) One can easily verify that the definition does not depend on the choice of the representatives in $\mathcal{M}_n$.
   (b) It is clear that the empty smooth manifold is a neutral element for the above operation.
   (c) We need to show that any $[M] \in \mathcal{M}_n$ has an inverse element. In fact this is represented by the oriented smooth manifold $-M$, since by the discussion on page 696, we have $\partial([0,1] \times M) = \{0\} \times (-M) \cup \{1\} \times M$, i.e. the smooth manifold $[0,1] \times M$ defines a cobordism from $M \sqcup -M$ to the empty smooth manifold. We refer to the figure below for an illustration.



   (d) The product structure is commutative since $A \sqcup B$ and $B \sqcup A$ are evidently naturally diffeomorphic.
   (e) The product structure is associative since $(A \sqcup B) \sqcup C$ and $A \sqcup (B \sqcup C)$ are evidently naturally diffeomorphic. ∎

Definition. Let $n \in \mathbb{N}_0$. We refer to

$$\Omega_n \ := \ (\mathcal{M}_n/\sim, \sqcup)$$

as the n-th cobordism group.

We have another natural operation on oriented connected smooth manifolds, namely the connected sum operation. The following lemma says that up to cobordisms the connected sum operation is the same as the disjoint union.

Lemma 144.3. Let $n \in \mathbb{N}$ and let M and N be two closed oriented connected n-dimensional smooth manifolds, then

$$M \# N \ = \ M \sqcup N \ \in \ \Omega_n.$$

Sketch of proof. A proof of this lemma is provided by the combination of Lemma ?? and of the Trace-of-Surgery Proposition ??. For the reader's convenience we also provide a direct proof.

Thus let M and N be two closed oriented connected n-dimensional smooth manifolds. Let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\psi\colon \overline{B}^n \to N \setminus \partial N$ be smooth embeddings such that φ is orientation-preserving and such that ψ is orientation-reversing.

We consider $[0,1] \times M$ and $[0,1] \times N$. Note that by the discussion on page 696, we have $\partial([0,1] \times M) = (\{0\} \times (-M)) \cup (\{1\} \times M)$ and $\partial([0,1] \times N) = (\{0\} \times (-N)) \cup (\{1\} \times N)$. By a slight abuse of notation we denote by φ and ψ the obvious smooth embeddings of $\overline{B}^n$ into $\{1\} \times M$ and $\{1\} \times N$. We use these two smooth embeddings to form the boundary connected sum

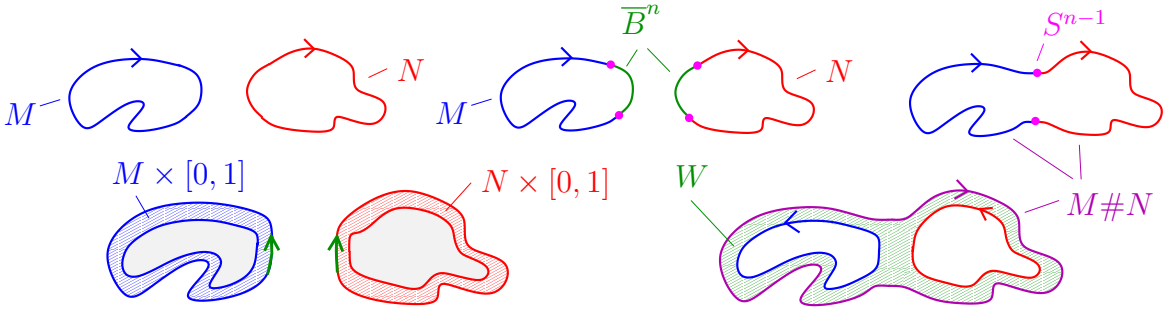
$$W \ := \ ([0,1] \times M) \#_b ([0,1] \times N) \ := \ ([0,1] \times M) \sqcup ([0,1] \times N) / \varphi(x) = \psi(x) \text{ for } x \in S^{n-1}$$

that we introduced on page ??.

By Proposition ?? we know that W admits the structure of an $(n+1)$-dimensional orientable smooth manifold such that

$$\partial W \;=\; \underbrace{(\{1\}\times M)\#(\{1\}\times N)}_{=M\#N} \sqcup \underbrace{(-\{0\}\times M)}_{=-M} \sqcup \underbrace{(-\{0\}\times N)}_{=-N}.$$

We have thus found the desired cobordism. ■



Definition. If we use exactly the same definitions as before except that we drop the condition that the smooth manifolds are orientable and we drop the condition that the cobordisms are orientable, then given $n\in\mathbb{N}_0$ we obtain the unoriented cobordism group

$$\mathcal{N}_n \;:=\; \{\text{closed } n\text{-dimensional smooth manifolds}\}/\text{unoriented cobordisms}.$$

The following lemma summarizes a few basic facts about unoriented cobordism groups.

Lemma 144.4.

(1) Let $n\in\mathbb{N}_0$. Every non-trivial element in $\mathcal{N}_n$ is of order two, i.e. given any $M\in\mathcal{N}_n$ we have $M\sqcup M=0\in\mathcal{N}_n$.

(2) Let $n\in\mathbb{N}$. Let M and N be two closed connected n-dimensional smooth manifolds.[1715] Then

$$M\#N \;=\; M\sqcup N \;\in\; \mathcal{N}_n.$$

Proof. Let $n\in\mathbb{N}_0$.

(1) Let $M\in\mathcal{N}_n$. We have

$$2\cdot M \;=\; M\sqcup M \;=\; (\{0\}\times M)\sqcup(\{1\}\times M) \;\underset{\uparrow}{=}\; \partial([0,1]\times M).$$

by the Smooth Manifold Product Proposition 19.31 we have this equality of unoriented smooth manifolds

(2) The proof of this statement is verbatim the same as in Lemma 144.3. ■

In the following proposition we collect the computations of the low-dimensional (unoriented) cobordism groups.

[1715] If M and N are orientable, then the connected sum $M\#N$ is only well-defined once we pick an orientation for M and N. If M and N are orientable, then the equality is understood as holding for the definition of the connected sum for any choice of orientation for M and N.

Proposition 144.5. We have the following table of isomorphism types of the (unoriented) cobordism groups for $n = 0, 1, 2, 3$:

	$n=0$	$n=1$	$n=2$	$n=3$
Ω_n	$\mathbb{Z}$	0	0	0
$\mathcal{N}_n$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 \cdot [\mathbb{R}\mathrm{P}^2]$	0

Proof.

($n=0$) Using the Smooth 1-Dimensional Manifold Classification Theorem 22.7 one can easily see that the maps

$$\begin{array}{rcl} \Omega_0 & \to & \mathbb{Z} \\ \bigsqcup_{i=1}^{k} \{(P_i, \epsilon_i)\} & \mapsto & \sum_{i=1}^{k} \epsilon_i \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathcal{N}_0 & \to & \mathbb{Z}_2 \\ \bigsqcup_{i=1}^{k} \{P_i\} & \mapsto & k \bmod 2 \end{array}$$

are isomorphisms.

($n=1$) In the Smooth 1-Dimensional Manifold Classification Theorem 22.7 we saw that every closed connected 1-dimensional smooth manifold is diffeomorphic to S^1. Evidently $S^1 = \partial \overline{B}^{2}$[1716], hence every generator, i.e. every (oriented) compact 1-dimensional smooth manifold represents the trivial element in Ω_1 and in $\mathcal{N}_1$. Therefore we see that $\Omega_1 = 0$ and $\mathcal{N}_1 = 0$.

($n=2$) In the Surface Classification Theorem 55.4 we saw that every closed orientable 2-dimensional smooth manifold is diffeomorphic to the surface of genus g for some g. By Proposition **??** (3) the surface of genus g is the boundary of the genus g handlebody. Thus we see that $\Omega_2 = 0$.

We turn to the calculation of $\mathcal{N}_2$. We want to prove the following claim.

Claim. The map

$$\begin{array}{rcl} \varphi\colon \mathbb{Z}_2 & \to & \mathcal{N}_2 \\ n & \mapsto & n \cdot \mathbb{R}\mathrm{P}^2 \end{array}$$

is an isomorphism.

Proof. We prove the claim in several steps:

(a) First we show that φ is a homomorphism. We denote by $\psi\colon \mathbb{Z} \to \mathcal{N}_2$ the map that is given by $n \mapsto n \cdot \mathbb{R}\mathrm{P}^2$. Since $\mathbb{Z}$ is a free group this is well-defined homomorphism. It remains to show that ψ in fact factors through the projection $\mathbb{Z} \to \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. But that is an immediate consequence of Lemma 144.4 (1).

(b) Next we show that $\varphi\colon \mathbb{Z}_2 \to \mathcal{N}_2$ is a monomorphism. We only need to show that $\varphi(1) = \mathbb{R}\mathrm{P}^2$ represents a non-trivial element in $\mathcal{N}_2$. In Corollary 119.9 we already saw, using the Euler characteristic and Poincaré Duality, that there is no compact 3-dimensional smooth manifold whose boundary is homeomorphic to $\mathbb{R}\mathrm{P}^2$. Put differently, $\mathbb{R}\mathrm{P}^2$ represents a non-trivial element in $\mathcal{N}_2$.

(c) It remains to show that φ is an epimorphism. So let M be a compact 2-dimensional smooth manifold representing an element in $\mathcal{N}_2$. Lemma 144.4 (2) shows that without loss of generality we can assume that M is connected. If M is orientable, then we already saw above that M is the boundary of a compact 3-dimensional smooth manifold, hence $[M] = 0 \in \mathcal{N}_2$.

[1716] This is an equality of oriented smooth manifolds if we equip S^1 and $\overline{B}^2$ with the usual orientation and also if we equip both smooth manifolds with the opposite orientation.

Finally we consider the case that M is non-orientable. We have

$$[M] \underset{\uparrow}{=} [\text{connected sum of } n \text{ copies of } \mathbb{R}\mathrm{P}^2] \underset{\uparrow}{=} n\cdot[\mathbb{R}\mathrm{P}^2] \underset{\uparrow}{=} (n \bmod 2)\cdot[\mathbb{R}\mathrm{P}^2].$$

by Theorem 55.4 that there exists an $n \in \mathbb{N}_0$ such that M is diffeomorphic to the connected sum of n copies of $\mathbb{R}\mathrm{P}^2$ — Lemma 144.4 (2) — Lemma 144.4 (1)

This completes the proof that φ is an epimorphism.

($n=3$) In the 3-Manifolds Bound Theorem 147.3 we already proved that $\Omega_3 = 0$. The fact that $\mathcal{N}_3 = 0$ was first shown by Albrecht Dold [**Dol1956**] in 1956. An alternative proof is given in [**Lic1963**, Thoerem 4.]. ■

We have thus shown that the (unoriented) cobordism groups for $n = 0, 1, 2, 3$ are not particularly exciting. Our next goal is to study Ω_4. By the discussion on page 1879 we know that $\chi(\mathbb{C}\mathrm{P}^2) = 3$. Thus the argument of Corollary 119.9 shows that $\mathbb{C}\mathrm{P}^2$ is a non-trivial element[1717] in Ω_4, but from Corollary 119.9 we cannot conclude whether or not $\mathbb{C}\mathrm{P}^2$ is an element of infinite order in Ω_4.

The following proposition is the key to showing that $\mathbb{C}\mathrm{P}^2$ is in fact an element of infinite order in Ω_4.

Proposition 144.6. (Boundary-Signature Theorem) Let M be a closed oriented $4k$-dimensional topological manifold with $k \geq 0$.[1718] If there exists a compact orientable topological manifold X with $\partial X = M$, then $\operatorname{sign}(M) = 0$.[1719]

It will take us a while to prove the Boundary-Signature Theorem 144.6, therefore we first provide the following corollary.

Corollary 144.7. For any $k \in \mathbb{N}$ the map

$$\begin{array}{rcl} \operatorname{sign}\colon \Omega_{4k} & \to & \mathbb{Z} \\ [M] & \mapsto & \operatorname{sign}(M) \end{array}$$

is an epimorphism.

Proof of Corollary 144.7. First we need to show that the map is well-defined. Thus suppose we are given closed oriented $4k$-dimensional smooth manifolds M and N with $[M] = [N] \in \Omega_{4k}$. By definition there exists a compact oriented smooth manifold X with $\partial X = M \sqcup -N$. We have

$$0 \underset{\uparrow}{=} \operatorname{sign}(\partial X) = \operatorname{sign}(M \sqcup (-N)) \underset{\uparrow}{=} \operatorname{sign}(M) + \operatorname{sign}(-N) \underset{\uparrow}{=} \operatorname{sign}(M) - \operatorname{sign}(N).$$

Boundary-Signature Theorem 144.6 — Manifold-Signature Lemma 133.11 (2) and (3)

This shows that $\operatorname{sign}\colon \Omega_{4k} \to \mathbb{Z}$ is well-defined. It follows from the Manifold-Signature Lemma 133.11 (2) that this map is a homomorphism. Finally it follows immediately from the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 121.7 that $\operatorname{sign}(\mathbb{C}\mathrm{P}^{2k}) = \pm 1$,[1720] which shows that $\operatorname{sign}\colon \Omega_{4k} \to \mathbb{Z}$ is an epimorphism. ■

The proof of the Boundary-Signature Theorem 144.6 will rely on a purely algebraic lemma. Before we can formulate that lemma we have to introduce one more definition.

[1717] In fact Corollary 119.9 also implies that $\mathbb{C}\mathrm{P}^2$ is a non-trivial element in $\mathcal{N}_4$.

[1718] Does the statement also hold for 0-dimensional smooth manifolds?

[1719] We refer to page 2916 for the definition of the signature of M.

[1720] In fact the more fancy Lemma 126.12 says that $\operatorname{sign}(\mathbb{C}\mathrm{P}^{2k}) = +1$, but this is irrelevant for the proof.

Definition. Given a form (V, g) and a subset $W \subset V$ we write

$$W^{\perp} := \{v \in V \mid g(v, w) = 0 \text{ for all } w \in W\}.$$

Example. We consider the form on $\mathbb{R}^2$ that is defined by the symmetric matrix

$$P := \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}.$$

In this case for $H = \mathbb{R} \cdot (1, 0)$ we have $\mathrm{H}^{\perp} = H$.

The following lemma gives the solution to Exercise 133.2.

Lemma 144.8. Let (V, g) be a non-singular symmetric form over the reals. If there exists a subspace W with $g|_W = 0$ and with $\dim(W) = \frac{1}{2}\dim(V)$, then $\mathrm{sign}(g) = 0$.

Proof of Lemma 144.8. It follows from our hypothesis that $\dim(W) = \frac{1}{2}\dim(V)$ that we can find a basis $\{a_1, \ldots, a_k, b_1, \ldots, b_k\}$ for V such that $\{a_1, \ldots, a_k\}$ is a basis for W. Since $g|_W = 0$ and since g is symmetric we see that with respect to this basis the form g is given by a matrix of the form

$$\begin{pmatrix} 0 & P \\ P^T & Q \end{pmatrix}$$

where Q is a symmetric matrix. It follows from the hypothesis that g is non-singular that $\det(P) \neq 0$, see e.g. the Non-Singularity Lemma 121.1. Now we consider the following three transformations

$$\begin{pmatrix} P^{-1} & 0 \\ 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & P \\ P^T & Q \end{pmatrix} \begin{pmatrix} (P^T)^{-1} & 0 \\ 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & Q \end{pmatrix},$$

$$\begin{pmatrix} \mathrm{id} & 0 \\ -\frac{1}{2}Q & \mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & Q \end{pmatrix} \begin{pmatrix} \mathrm{id} & -\frac{1}{2}Q \\ 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \mathrm{id} & \mathrm{id} \\ \mathrm{id} & -\mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{id} & \mathrm{id} \\ \mathrm{id} & -\mathrm{id} \end{pmatrix} = \begin{pmatrix} 2\,\mathrm{id} & 0 \\ 0 & -2\,\mathrm{id} \end{pmatrix}.$$

It follows from the Forms-Matrices Lemma 133.1 (1) that the three matrices to the right all define isometric forms. Considering the last matrix we obtain from the Signature-Eigenvalues Proposition 133.8 that the signature is zero. ∎

The next result is the key ingredient in the proof of. As the name suggests, it is interesting in its own right.

Theorem 144.9. (Half-Live Half-Die Theorem) Let X be a compact orientable topological manifold of dimension $2n + 1$. We denote by $i\colon \partial X \to X$ the inclusion map.

(1) For any field $\mathbb{F}$ we have the following equalities:

(a) $$\dim\left(\ker(i_*\colon \mathrm{H}_n(\partial X; \mathbb{F}) \to \mathrm{H}_n(X; \mathbb{F}))\right) = \frac{1}{2}\dim(\mathrm{H}_n(\partial X; \mathbb{F}))$$

and

(b) $$\dim\left(\mathrm{im}(i^*\colon \mathrm{H}^n(X; \mathbb{F}) \to \mathrm{H}^n(\partial X; \mathbb{F}))\right) = \frac{1}{2}\dim(\mathrm{H}^n(\partial X; \mathbb{F})).$$

The same statement also holds, regardless of whether or not X is orientable, if we restrict ourselves to $\mathbb{F}_2$-coefficients.

(2) We suppose that $n \geq 1$. For any $a, b \in \ker(i_*\colon \mathrm{H}_n(\partial X;\mathbb{Z}) \to \mathrm{H}_n(X;\mathbb{Z}))$ we have $Q_{\partial X}(a,b) = 0$.[1721] The same statement also holds, regardless of whether or not X is orientable, if we work with $\mathbb{F}_2$-coefficients and the $\mathbb{F}_2$-valued intersection form $Q^{\mathbb{F}_2}_{\partial X}$.

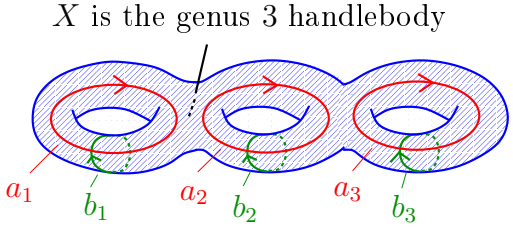
Remark. Given any homomorphism $\varphi\colon V \to W$ between finite-dimensional vector spaces the equality $\dim(\ker(\varphi))+\dim(\mathrm{im}(\varphi)) = \dim(V)$ holds. Thus statement (1a) of the theorem can be reformulated as saying that

$$\dim\big(\ker(i_*\colon \mathrm{H}_n(\partial X;\mathbb{F}) \to \mathrm{H}_n(X;\mathbb{F}))\big) \;=\; \dim\big(\mathrm{im}(i_*\colon \mathrm{H}_n(\partial X;\mathbb{F}) \to \mathrm{H}_n(X;\mathbb{F}))\big).$$

This explains why the theorem is called the "Half-Live Half-Die Theorem".

Examples.

(1) We consider the genus 3 handlebody X. In the figure below we show the usual basis $a_1, a_2, a_3, b_1, b_2, b_3$ for $\mathrm{H}_1(\partial X;\mathbb{R})$. In this case the kernel of the inclusion induced map $\mathrm{H}_1(\partial X;\mathbb{R}) \to \mathrm{H}_1(X;\mathbb{R})$ is the three dimensional subspace generated by b_1, b_2 and b_3.



(2) Let $K \subset S^3$ be a knot, we denote by X_K the knot exterior, see page 2213. The boundary of X_K is a torus. By the Longitude-Meridian Lemma 131.2 (1) the meridian μ and the longitude λ form a basis for $\mathrm{H}_1(X_K;\mathbb{R})$. have

$$\dim\Big(\underbrace{\ker(\mathrm{H}_1(\partial X_K;\mathbb{R}) \to \mathrm{H}_1(X_K;\mathbb{R}))}_{=\mathbb{R}\cdot\lambda \text{ by the Longitude-Meridian Lemma 131.2 (1)}}\Big) = 1 \;=\; \tfrac{1}{2}\cdot 2 \;=\; \tfrac{1}{2}\cdot\dim(\mathrm{H}_1(\partial X_K;\mathbb{R})).$$

For clarity we prove the two parts of the Half-Live Half-Die Theorem 144.9 separately.

Proof of the Half-Live Half-Die Theorem 144.9 (1). Let X be a compact oriented $(2n+1)$-dimensional topological manifold. We denote by $i\colon \partial X \to X$ the inclusion map and we write $M = \partial X$. Let $\mathbb{F}$ be any field. We first prove the first equality. Until further notice, we henceforth work with $\mathbb{F}$-coefficients for all (co-) homology groups. For space reasons we do not mention the $\mathbb{F}$-coefficients explicitly. We consider the long exact sequence in homology of the pair (X, M):

$$\to \mathrm{H}_{n+1}M \to \mathrm{H}_{n+1}X \to \mathrm{H}_{n+1}(X,M) \xrightarrow{\partial} \mathrm{H}_nM \xrightarrow{i_*} \mathrm{H}_nX \to \mathrm{H}_n(X,M) \to \mathrm{H}_{n-1}M \to$$

The key observation is that this long exact sequence is "symmetric" around $\mathrm{H}_n(M;\mathbb{F})$. More precisely, by the Co-Homology Symmetry Theorem Theorem 119.6 (2)[1722] we have

[1721] Here $Q_{\partial X}$ denotes the $\mathbb{Z}$-valued intersection form that we introduced on page 2800. Also recall that on page 2922 we introduced the $\mathbb{F}_2$-valued intersection form.

[1722] Even though it is irrelevant we would like to stress that the isomorphisms $\mathrm{H}_k(M;\mathbb{F}) \cong \mathrm{H}_{2n-k}(M;\mathbb{F})$, $\mathrm{H}_k(X;\mathbb{F}) \cong \mathrm{H}_{2n+1-k}(X,M;\mathbb{F})$ and $\mathrm{H}_k(X,M;\mathbb{F}) \cong \mathrm{H}_{2n+1-k}(X;\mathbb{F})$ are not natural.

the following isomorphisms:

$$\to \overbrace{\mathrm{H}_{n+1}M}^{\cong \mathrm{H}_{n-1}M} \to \overbrace{\mathrm{H}_{n+1}X}^{\cong \mathrm{H}_n(X,M)} \to \overbrace{\mathrm{H}_{n+1}(X,M)}^{\cong \mathrm{H}_nX} \xrightarrow{\partial} \mathrm{H}_nM \xrightarrow{i_*} \mathrm{H}_nX \to \mathrm{H}_n(X,M) \to \mathrm{H}_{n-1}M \to$$

Therefore it suffices to prove the following purely algebraic claim.

Claim. If

$$0 \to A_k \xrightarrow{\beta_k} A_{k-1} \to \cdots \to A_2 \to A_1 \to B \xrightarrow{\gamma} A_1 \to A_2 \to \cdots \to A_{k-1} \xrightarrow{\alpha_k} A_k \to 0$$

is an exact sequence of finite-dimensional real vector spaces,[1723] then

$$\dim(\ker(\gamma\colon B \to A_1)) \;=\; \tfrac{1}{2} \cdot \dim B.$$

Proof. We prove the claim by induction on k. The case $k = 0$ is clear, since in this case $B = 0$. Now consider a symmetric sequence

$$0 \to A_k \xrightarrow{\beta_k} A_{k-1} \to \cdots \to A_2 \to A_1 \to B \xrightarrow{\gamma} A_1 \to A_2 \to \cdots \to A_{k-1} \xrightarrow{\alpha_k} A_k \to 0.$$

as above. It follows from the Break-into-SES Lemma 78.4 that the following sequence is also exact

$$0 \to A_{k-1}/\beta_k(A_k) \to \cdots \to A_2 \to A_1 \to B \xrightarrow{\gamma} A_1 \to A_2 \to \cdots \to \ker(A_{k-1} \xrightarrow{\alpha_k} A_k) \to 0.$$

But we have

$$\dim(A_{k-1}/\beta_k(A_k)) \underset{\substack{\uparrow \\ \text{since } \beta_k \text{ is injective}}}{=} \dim(A_{k-1}) - \dim(A_k) \underset{\substack{\uparrow \\ \text{since } \alpha_k \text{ is surjective}}}{=} \dim(\ker(A_{k-1} \xrightarrow{\alpha_k} A_k)).$$

So the new shorter sequence is also symmetric. So we can apply the induction hypothesis to get the desired equality. This concludes the proof of the claim and thus of the first equality of the lemma.

The proof of the second equality of the lemma is almost identical to the proof of the first equality. This concludes the proof of the lemma regarding $\mathbb{F}$-valued (co-) homology groups.

Now let X be any compact $(2n+1)$-dimensional topological manifold, without any assumptions on orientability. In this case verbatim the same argument as above works, we just need to replace $\mathbb{F}$-coefficients by $\mathbb{F}_2$-coefficients and we need to replace the Co-Homology Symmetry Theorem Theorem 119.6 (2) by the Co-Homology Symmetry Theorem Theorem 119.6 (1). ∎

Proof of the Half-Live Half-Die Theorem 144.9 (2). In the following we will only prove the statement regarding $\mathbb{Z}$-coefficients. The reader will have no troubles with making the straightforward adjustments needed to obtain a proof for $\mathbb{F}_2$-coefficients. Now let $a, b \in \ker(\mathrm{H}_n(M;\mathbb{Z}) \to \mathrm{H}_n(X;\mathbb{Z}))$.

Claim. There exist cohomology classes $\varphi, \psi \in \mathrm{H}^n(X;\mathbb{Z})$ such that $\mathrm{PD}_M(a) = i^*(\varphi)$ and $\mathrm{PD}_M(b) = i^*(\psi)$.

[1723] Recall that by the Topological Manifolds-Invariants Proposition 104.14 (4) together with Lemma 96.8 (1) we know that the real homology groups of X, M and (X, M) are finite-dimensional vector spaces.

Proof. We consider the following diagram

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & \mathrm{H}^n(X,M;\mathbb{Z}) & \longrightarrow & \mathrm{H}^n(X;\mathbb{Z}) & \xrightarrow{i^*} & \mathrm{H}^n(M;\mathbb{Z}) & \xrightarrow{\delta} & \mathrm{H}^{n+1}(X,M;\mathbb{Z}) & \longrightarrow \ldots \\
& \cong\downarrow{\scriptstyle -\cap[X]} & & \cong\downarrow{\scriptstyle -\cap[X]} & & \cong\downarrow{\scriptstyle -\cap[M]} & & \cong\downarrow{\scriptstyle -\cap[X]} & \\
\ldots \longrightarrow & \mathrm{H}_{n+1}(X;\mathbb{Z}) & \longrightarrow & \mathrm{H}_{n+1}(X,M;\mathbb{Z}) & \longrightarrow & \mathrm{H}_n(M;\mathbb{Z}) & \xrightarrow{i_*} & \mathrm{H}_n(X;\mathbb{Z}) & \xrightarrow{\partial} \ldots
\end{array}$$

where the horizontal sequences of the pair (X,M). By Theorem 119.1 and Proposition 119.25 we know that the vertical maps are isomorphisms and we know that the diagram commutes up signs. The claim follows almost immediately from the above discussion. ⊞

Now we calculate that

$$\begin{array}{rcccccl}
Q_M(a,b) & = & \langle \mathrm{PD}_M(a) \cup \mathrm{PD}_M(b), [M]\rangle & = & \langle i^*(\varphi)\cup i^*(\psi), [M]\rangle_M & & \\
& = & \langle i^*(\varphi\cup\psi), [M]\rangle_M & = & \langle \varphi\cup\psi, i_*([M])\rangle_X & = & \langle \varphi\cup\psi, 0\rangle_X \;=\; 0. \\
& \uparrow & & \uparrow & & \uparrow & \\
& \text{Cup Product-Naturality Lemma 114.10} & & \text{Lemma 109.6 (3)} & & \text{Corollary 106.28} &
\end{array}$$

■

Finally we can turn to the actual proof of the Boundary-Signature Theorem 144.6.

Proof of the Boundary-Signature Theorem 144.6. Let X be a compact oriented $(4k+1)$-dimensional topological manifold with $k \geq 1$. We write $M = \partial X$. We want to show that $\mathrm{sign}(M) = 0$. If $M = \varnothing$, then there is nothing to show. So suppose that $M \neq \varnothing$.

Recall that it follows from the Intersection Form Proposition 132.4 that $Q_M \colon \mathrm{FH}_n(M;\mathbb{Z}) \times \mathrm{FH}_n(M;\mathbb{Z}) \to \mathbb{Z}$ is symmetric and non-singular. It follows easily that the form $Q_M \otimes \mathbb{R}$ is also symmetric and non-singular. Furthermore note that it follows almost immediately from the Half-Live Half-Die Theorem 144.9 (2) and Corollary 89.19 that $Q_M \otimes \mathbb{R}$ vanishes on $P := \ker(\mathrm{H}_{2k}(M;\mathbb{R}) \to \mathrm{H}_{2k}(X;\mathbb{R}))$.

Next we point out that by Half-Live Half-Die Theorem 144.9 (1b) together with Corollary 89.19 we know that $\dim(P) = \frac{1}{2}\dim(\mathrm{H}_{2k}(M;\mathbb{R}))$. Since, as we discussed, $Q_M \otimes \mathbb{R}$ is non-singular and symmetric and since it vanishes on the half-dimensional subspace P we now finally obtain from Lemma 144.8 that $\mathrm{sign}(M) = \mathrm{sign}(Q_M \otimes \mathbb{R}) = 0$. ■

Before we continue with the determination of cobordism groups we prove the following elementary lemma.

Lemma 144.10. For any $m, n \in \mathbb{N}_0$ the maps

$$\begin{array}{ccc} \Omega_m \times \Omega_n & \to & \Omega_{m+n} \\ ([M],[N]) & \mapsto & [M\times N] \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathcal{N}_m \times \mathcal{N}_n & \to & \mathcal{N}_{m+n} \\ ([M],[N]) & \mapsto & [M\times N] \end{array}$$

are well-defined.

Proof. We prove the statement regarding the Ω_k's, the proof for the $\mathcal{N}_k$'s is verbatim the same. So suppose that we have $[M] = [M'] \in \Omega_m$ and $[N] \in \Omega_n$. This means that there exists a compact oriented $(m+1)$-dimensional smooth manifold W with $\partial W = M \sqcup (-M')$. But then we also have

$$\begin{array}{rcccl} \partial(W\times N) & = & \partial W \times N \;=\; (M \sqcup (-M')) \times N & = & (M\times N) \sqcup -(M'\times N). \\ & \uparrow & & \uparrow & \\ & \text{by the Smooth Manifold Product Proposition 19.31 (3) since } N \text{ is closed} & & \text{since } (-M')\times N = -(M'\times N) & \end{array}$$

This shows that $[M\times N] = [M'\times N]$. The same argument shows that the map does not depend on the representative of $[N] \in \Omega_n$. ■

Definition. We define

$$\Omega_* := \bigoplus_{n\in\mathbb{N}_0} \Omega_n \quad \text{and} \quad \mathcal{N}_* := \bigoplus_{n\in\mathbb{N}_0} \mathcal{N}_n.$$

Together with the maps defined in Lemma 144.10 we can view these groups as commutative rings. We refer to Ω_* as the cobordism ring and we refer to $\mathcal{N}_*$ as the unoriented cobordism ring.

The following theorem summarizes some of the most important results on the cobordism ring Ω_*.

Theorem 144.11. (Thom Cobordism Theorem)

(1) The ring homomorphism[1724][1725][1726]

$$\begin{aligned} \mathbb{Q}[\{x_m\}_{m\in\mathbb{N}}] &\to \Omega_* \otimes \mathbb{Q} \\ x_m &\mapsto \mathbb{C}\mathrm{P}^{2m} \end{aligned}$$

that is determined by

is an isomorphism.

(2) For small n the isomorphism types of Ω_n are given by

n	0	1	2	3	4	5	6	7	8	9	10	11
Ω_n	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	0	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Hereby a generator for $\Omega_4 \cong \mathbb{Z}$ is given by $\mathbb{C}\mathrm{P}^2$ and a basis for $\Omega_8 \cong \mathbb{Z}^2$ is given by $\{\mathbb{C}\mathrm{P}^4, \mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2\}$.[1727]

(3) The group Ω_n is non-trivial for every $n \geq 8$.

(4) Every torsion element in Ω_n has order 2.

Proof.

(1) The first statement was proved by René Thom [**Tho2007**, Chapter IV], see also [**Stong1968**, p. 177]. An outline of the proof is presented in [**Kupe2017c**].

(2) The calculation of the cobordism groups $\Omega_0, \ldots, \Omega_3$ was given in Proposition 144.5, the values $\Omega_4, \ldots, \Omega_7$ are given by René Thom in [**Tho2007**, Theorem IV.13]. For $\Omega_8, \ldots, \Omega_{11}$ we refer to [**MiS1974**, p. 203].

(3) This statement is proved in [**Dol1956**, p. 34].

(4) The last statement was proved by Terrence Wall [**Wall1960**], see also [**Stong1968**, p. 182]. ■

The following theorem, which is due to [**Tho2007**, Theorem IV.12] and [**Dol1956**, p. 32], gives the complete calculation of the unoriented cobordism ring $\mathcal{N}_*$.

[1724] On the left-hand side $\mathbb{Q}[\{x_m\}_{m\in\mathbb{N}}]$ denotes the polynomial ring over $\mathbb{Q}$ with variables $\{x_m\}_{m\in\mathbb{N}}$.

[1725] Note that the tensor product of two commutative rings is again a commutative ring with the "naive multiplication". In particular the right-hand side $\Omega_* \otimes \mathbb{Q}$ is a commutative ring.

[1726] If we give each variable x_m the grading $4m$, and if we equip $\Omega_* \otimes \mathbb{Q}$ with the obvious grading, then this is an isomorphism of graded rings.

[1727] Note that in the ring homomorphism from (1) we have $x_1 \mapsto [\mathbb{C}\mathrm{P}^2]$ and $x_1^2 \mapsto [\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2]$ and also $x_2 \mapsto [\mathbb{C}\mathrm{P}^4]$.

Theorem 144.12. (Thom-Dold Cobordism Theorem) We let $I = \{n \in \mathbb{N} \,|\, n \neq 2^i - 1\} = \{2, 4, 5, 6, 8, 9, \dots\}$. There exists a ring isomorphism

$$\mathbb{Z}_2[\{x_i\}_{i \in I}] \;\to\; \mathcal{N}_*$$

which is given by sending x_{2k} to $\mathbb{R}\mathrm{P}^k$ and given by sending x_i for odd i to an appropriate i-dimensional Dold smooth manifold. Here a Dold smooth manifold is a smooth manifold of the form[1728]

$$P(m,n) \;:=\; (S^m \times \mathbb{C}\mathrm{P}^n)/(x,z) \sim (-x,\overline{z}) \quad \text{for } x \in S^m \text{ and } z \in \mathbb{C}\mathrm{P}^n$$

for some $n, m \in \mathbb{N}_0$.

Exercises for Chapter 144.

Exercise 144.1. We consider the Klein bottle

$$K \;=\; (S^1 \times [0,1])/\!\sim \quad \text{where } (z,0) \;\sim\; (\overline{z},1) \text{ for all } z \in S^1.$$

In Lemma **??** (1) we showed that K is diffeomorphic to $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$. Thus it follows from Lemma 144.4 that $[K] = [\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2] = [\mathbb{R}\mathrm{P}^2 \sqcup \mathbb{R}\mathrm{P}^2] = 2 \cdot [\mathbb{R}\mathrm{P}^2] = 0 \in \mathcal{N}_2$. Write down an explicit compact 3-dimensional smooth manifold M with $\partial M = K$.

Exercise 144.2. Provide a proof for Lemma 144.8 that does not uses bases.[1729] Let us elaborate. Let (V, g) be a non-singular symmetric form over $\mathbb{R}$ and suppose that there exists a subspace W of V with $g|_W = 0$ and with $\dim(W) = \frac{1}{2}\dim(V)$. Show that given any subspace A of V such that $g|_A$ is positive (respectively negative) definite there exists a subspace B of V such that $g|_B$ is negative (respectively positive) definite and which satisfies $\dim(A) = \dim(B)$. A proof of this statement implies that $\mathrm{sign}(g) = 0$.

Exercise 144.3. Let $L = L_1 \cup \dots \cup L_m$ be an m-component link in S^3. We denote by X_L its exterior which is defined the same way as we defined the exterior of a knot on page 2213. The boundary of X_L consists of m tori $T_1, \dots, T_m$. Let $k \in \{1, \dots, m\}$. We denote by $i\colon T_k \to X_L$ the inclusion map.

(a) Show that $\dim(\mathrm{im}(i_*\colon \mathrm{H}_1(T_k;\mathbb{R}) \to \mathrm{H}_1(X_L;\mathbb{R}))) \geq 1$.

(b) Do we necessarily have $\dim(\mathrm{im}(i_*\colon \mathrm{H}_1(T_k;\mathbb{R}) \to \mathrm{H}_1(X_L;\mathbb{R}))) = 1$?

Exercise 144.4. Provide a proof for Lemma 150.1.

Exercise 144.5.

(a) Let M be an orientable n-dimensional smooth manifold with $n \geq 4$ and let $C \cong S^1$ be a closed connected one-dimensional submanifold. By the Tubular Neighborhood Theorem **??** there exists a tubular neighborhood $\overline{B}^{n-1} \times C = \overline{B}^{n-1} \times S^1$. By Propositions 24.16 and **??**

$$W \;:=\; (M \setminus (C \times B^{n-1})) \cup_{C \times S^{n-2} = S^1 \times S^{n-2}} (\overline{B}^2 \times S^{n-2})$$

is an n-dimensional smooth manifold. Compute $\pi_1(W)$ in terms of $\pi_1(M)$ and C. (This approach to modifying the smooth manifold M is sometimes called a *surgery*.)

[1728] It follows from the Manifold Quotient-by-Group Action Proposition 19.32 that each $P(m,n)$ is in fact a closed smooth manifold.

[1729] As the reader surely knows, introducing bases is morally wrong and should be avoided at all costs.

(b) Let $n \geq 3$ and let X be a closed n-dimensional smooth manifold. Suppose that there exists a compact orientable $(n+1)$-dimensional M with $\partial M = X$. Show that there exists also a compact orientable *simply connected* $(n+1)$-dimensional smooth manifold N with $\partial N = X$.
(c) Does (b) also hold for $n = 2$?

Exercise 144.6. Let V be an $\mathbb{F}_2$-vector space of dimension $2k$ and let $\langle\ ,\ \rangle\colon V \times V \to \mathbb{F}_2$ be a symmetric non-singular form. Suppose we are given vectors $a_1, \ldots, a_k \in V$ such that $\langle a_i, a_j \rangle = 0$ for all $i, j \in \{1, \ldots, k\}$. Show that there exist vectors $b_1, \ldots, b_k \in V$ such that the following conditions are satisfied:

(1) The vectors $a_1, \ldots, a_k, b_1, \ldots, b_k$ are a basis for V.
(2) For each $i, j \in \{1, \ldots, k\}$ we have $\langle a_i, b_j \rangle = \delta_{ij}$.
(3) For each $i, j \in \{1, \ldots, k\}$ we have $\langle b_i, b_j \rangle = 0$.

Remark. Such a basis for V is often called a *symplectic basis*.

145. Pontryagin classes and the Hirzebruch Signature Theorem

In this chapter we will introduce the Pontryagin classes of vector bundles. Afterwards we will formulate the Hirzebruch Signature Theorem, which amazingly, shows that the Pontryagin classes of the tangent bundle of a smooth manifold determine the signature.

145.1. **Pontryagin classes.** Given a topological space X and $i \in \mathbb{N}_0$ we define

$$\overline{\mathrm{H}}^i(X;\mathbb{Z}) \ := \ \mathrm{H}^i(X;\mathbb{Z})/\{a \in \mathrm{H}^i(X;\mathbb{Z}) \,|\, 2a = 0\}.$$

It should be clear that these cohomology groups $\overline{H}$ are also contravariantly functorial and that all the results on usual cohomology groups have obvious analogues.
The following result is one of the key theorems of this chapter. It was proved initially in about 1940 by Lev Pontryagin.

Theorem 145.1. For each $i \in \mathbb{N}_0$ we can assign to every vector bundle $p\colon E \to X$ over a paracompact space a cohomology class $p_i(E) \in \overline{\mathrm{H}}^{4i}(X;\mathbb{Z})$, called *$i$-th Pontryagin class*, such that the following conditions are satisfied:

(1) The 0-th Pontryagin class of any vector bundle over a topological space X is the unit element in $\overline{\mathrm{H}}^0(X;\mathbb{Z})$.
(2) For any trivial vector bundle all Pontryagin classes p_i with $i \geq 1$ are zero.
(3) The Pontryagin class behaves well under pullbacks, i.e. if we are given a vector bundle $q\colon F \to Y$ and a map $g\colon X \to Y$ of topological spaces, then for any $i \in \mathbb{N}_0$ we have
$$p_i(g^*F) \ = \ g^*(p_i(F)) \ \in \ \overline{\mathrm{H}}^{4i}(X;\mathbb{Z}).$$
(4) If $p\colon E \to X$ is an n-dimensional vector bundle, then $p_i(E) = 0$ for $i > \frac{n}{2}$.
(5) We denote by $x \in \overline{\mathrm{H}}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ the standard generator as introduced on page 2662 and in the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 126.11. Then for $i = 0, \ldots, n+1$ we have[1730]
$$p_i(T\mathbb{C}\mathrm{P}^n) \ = \ \binom{n+1}{i} \cdot x^{2i} \ \in \ \overline{\mathrm{H}}^{4i}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \ = \ \mathbb{Z}[x]/(x^{n+1})$$
and all other Pontryagin classes are zero.

Given a vector bundle $p\colon E \to X$ we refer to
$$p(E) \ := \ \underbrace{\bigoplus_{i=0}^{\infty} p_i(E)}_{\text{finite sum by Theorem 145.1 (4)}} \ \in \ \overline{\mathrm{H}}^*(X;\mathbb{Z}) \ = \ \bigoplus_{i=0}^{\infty} \overline{\mathrm{H}}^i(X;\mathbb{Z})$$
as the *total Pontryagin class*. With this definition we can state the last property of Pontryagin classes:

(6) The total Pontryagin class is multiplicative with respect to the direct sum of vector bundles *up to 2-torsion*. More precisely, if $e\colon E \to X$ and $f\colon F \to X$ are two vector bundles, then
$$p(E \oplus F) \ = \ p(E) \cup p(F) \ \in \ \overline{\mathrm{H}}^*(X;\mathbb{Z}).$$

[1730] Note that for $i > \lfloor \frac{n}{2} \rfloor$ we have $x^{2i} = 0 \in \mathrm{H}^{4i}(\mathbb{C}\mathrm{P}^n) = 0$. Nonetheless, for later it is convenient to write down this expression for all i up to n.

Remark.

(1) The above introduction to Pontryagin classes differs somewhat from the usual approach taken in [**MiS1974, Hat2**]. In particular Pontryagin classes are usually defined as elements in the usual cohomology groups.
(2) It is not entirely clear whether classes $p_i(E)$ are uniquely defined by the properties stated in Theorem 145.1. Similar uniqueness statements hold for "Stiefel-Whitney classes" and "Chern classes", see [**Hat2**, Theorems 3.1 and 3.2], but it is nonetheless not clear to me how to prove uniqueness for Pontryagin classes. Fortunately that is of no concern to us.

For category aficionados we make the following remark.

Remark. We can squeeze some of the statements of Theorem 145.1 into the language of category theory:

(1) Let $\mathcal{Top}$ be the category of topological spaces and let $\mathcal{Set}$ be the category of sets, i.e. the objects are sets and the morphisms are just maps.
(2) As on page 3067 we denote by $\mathrm{Vect}^n_{\mathbb{R}}(X)$ the set of isomorphism classes of n-dimensional real vector bundles over X. We write $\mathrm{Vect}_{\mathbb{R}}(X) = \bigcup_{n\in\mathbb{N}} \mathrm{Vect}^n_{\mathbb{R}}(X)$.
(3) For each $i \in \mathbb{N}_0$ we consider the following two contravariant functors:

$$\begin{array}{c} \mathcal{Top} \to \mathcal{Set} \\ X \mapsto \mathrm{H}^{4i}(X;\mathbb{Z}) \\ \big(X \xrightarrow{f} Y\big) \mapsto \big(\overline{\mathrm{H}}^{4i}(Y) \xrightarrow{f^*} \overline{\mathrm{H}}^{4i}(X)\big) \end{array} \quad \& \quad \begin{array}{c} \mathcal{Top} \to \mathcal{Set} \\ X \mapsto \mathrm{Vect}_{\mathbb{R}}(X) \\ \big(X \xrightarrow{f} Y\big) \mapsto \begin{pmatrix} \mathrm{Vect}_{\mathbb{R}}(Y) \to \mathrm{Vect}_{\mathbb{R}}(X) \\ [G \xrightarrow{p} Y] \mapsto [f^*G \to X] \end{pmatrix}. \end{array}$$

(4) In this language we can say that Pontryagin classes are a natural transformation between these two contravariant functors. This sentence now models property (3) of Pontryagin classes.

Sketchy sketch of a proof. Given $n \in \mathbb{N}$ we denote by $G(n)$ the infinite Grassmannian that we introduced on page 3074. In [**Hat2**, Proposition 3.16] it is shown that[1731][1732]

$$(\overline{\mathrm{H}}^*(G(n);\mathbb{Z}), \cup) \;\cong\; \mathbb{Z}[q_1, \ldots, q_k] \text{ for } n = 2k \text{ or } n = 2k+1$$

where each q_i has degree $4i$. Now let $p\colon E \to X$ be an n-dimensional vector bundle. We denote by $F \to G(n)$ the n-dimensional vector bundle over the infinite Grassmannian $G(n)$ that we introduced on page 3074. By Proposition 143.15 there exists a map $g\colon X \to G(n)$ such that g^*F is isomorphic to E. We define $p_i(E) := g^*(q_i) \in \overline{\mathrm{H}}^{4i}(X;\mathbb{Z})$. It follows from Proposition 143.15 that this definition does not depend on the choice of the map g.

The proof that the Pontryagin classes have all the stated properties is given in [**MiS1974**, Chapter 15]. ■

[1731] The fact that the cohomology ring $(\overline{\mathrm{H}}^*(G(n);\mathbb{Z}), \cup)$ is a polynomial algebra should ring a bell: When we introduced the (infinite) Grassmannian $G(n)$ we were saying that it should be viewed as a generalization of the (infinite) real projective spaces. Thus the calculation of the cup product mirrors our calculation that $(\mathrm{H}^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}), \cup) \cong \mathbb{Z}[x]$, see the Cup Product-$\mathbb{C}\mathrm{P}^\infty$-Proposition 121.12, and our calculation that $(\mathrm{H}^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2), \cup) \cong \mathbb{F}_2[x]$, see the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 121.16 (3).

[1732] The calculation of the cohomology ring over the usual cohomology groups $\mathrm{H}^*(G(n);\mathbb{Z})$ is somewhat more involved, see [**Hat2**, Theorem 3.9] for more information.

Example. Let $n \in \mathbb{N}$. Let M be a closed orientable submanifold of $\mathbb{R}^n$ of codimension one. For example we could take $M = S^{n-1} \subset \mathbb{R}^n$ or M could be the surface of some genus g which, by the discussion on page 708, we can view as a submanifold of $\mathbb{R}^3$.[1733] We want to determine the total Pontryagin class of the tangent bundle $\mathrm{T}M$. As before we denote by NM the normal bundle of $M \subset \mathbb{R}^n$. Then

$$p(\mathrm{T}M) \underset{\uparrow}{=} p(\mathrm{T}M) \cup p(NM) \underset{\uparrow}{=} p(\mathrm{T}M \oplus NM) \underset{\uparrow}{=} p(\text{trivial bundle } M \times \mathbb{R}^n) \underset{\uparrow}{=} 1.$$

by Theorem 145.1 (1) and (2) since the vector bundle NM is trivial, see page 3065 — by Theorem 145.1 (6) — see page 3079 — Theorem 145.1 (2)

In the remainder of this chapter we are mostly interested in tangent bundles of smooth manifolds, which leads us to the following definition.

Definition. Given a smooth manifold M we define its Pontryagin classes as the Pontryagin classes of its tangent bundle, i.e. for $i \in \mathbb{N}_0$ we write $p_i(M) := p_i(\mathrm{T}M) \in \overline{\mathrm{H}}^{4i}(M;\mathbb{Z})$.

Example. With these definitions the statement of Theorem 145.1 (5) is equivalent to

$$p(\mathbb{C}\mathrm{P}^n) \underset{\uparrow}{=} \sum_{i=0}^{n} \binom{n+1}{i} \cdot x^{2i} \underset{\uparrow}{=} \overbrace{(1+x^2)^{n+1}}^{\text{higher terms are zero in the ring}} \in \overline{\mathrm{H}}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \underset{\uparrow}{=} \mathbb{Z}[x]/(x^{n+1}).$$

by Theorem 145.1 (2) and (5) since $T\mathbb{C}\mathrm{P}^n$ is $2n$-dimensional — general binomial formula, together with the fact that $x^j = 0$ for $j \geq n+1$ — Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 126.11

For example we have

$$p(\mathbb{C}\mathrm{P}^1) = 1, \quad p(\mathbb{C}\mathrm{P}^2) = 1 + 3x^2, \quad p(\mathbb{C}\mathrm{P}^3) = 1 + 4x^2, \quad p(\mathbb{C}\mathrm{P}^4) = 1 + 5x^2 + 10x^4.$$

145.2. **Pontryagin classes as invariants of smooth manifolds.** So far, throughout these lectures we have basically only studied invariants of topological spaces. Even the de Rham cohomology groups that we introduced in Section **??** were shown in Proposition 118.6 to be topological invariants, i.e. their isomorphism types depend only on the homeomorphism type of the underlying topological space.
The Pontryagin classes of a smooth manifold M turn out to be more subtle invariants: they are defined using the tangent bundle of M. The definition of the tangent bundle in turn relies on the choice of the smooth structure. Put differently, a priori the Pontryagin classes $p_i(M) \in \overline{\mathrm{H}}^{4i}(M;\mathbb{Z})$ depend on the smooth structure of the smooth manifold.
The following lemma shows how diffeomorphisms interact with Pontryagin classes.

Lemma 145.2. If $f\colon M \to N$ is a local diffeomorphism between two smooth manifolds, then for any $i \in \mathbb{N}_0$ we have

$$p_i(\mathrm{T}M) = f^*(p_i(\mathrm{T}N)) \in \overline{\mathrm{H}}^{4i}(M;\mathbb{Z}).$$

Proof. The lemma is an immediate consequence of Lemma 143.12 (1) and Theorem 145.1 (3). ∎

[1733] The Pontryagin classes for surfaces are zero for the stupid reason that they live in dimensions $4i$, but that is besides the point for this example.

In [**Sharm1985**, Theorem 1.6] it is shown that the first Pontryagin class cannot detect the difference between different smooth structures. But the following theorem demonstrates that the higher Pontryagin classes contain information on smooth structures of smooth manifolds.

Theorem 145.3. For any $n \geq 2$ there exist two closed smooth manifolds M and M' with the following properties:

(1) the smooth manifolds M and M' are homeomorphic,
(2) the n-th Pontryagin class of M is trivial, whereas
(3) the n-th Pontryagin class of M' is non-zero.

In particular it follows from Lemma 145.2 that the smooth manifolds M and M' are not diffeomorphic.

Proof. This theorem is proved in [**KrL2005**, Theorem 4.8], see also [**KrL2005**, Proof of Proposition 4.7]. Examples of non-compact smooth manifolds M and M' with the same properties (1), (2) and (3) were given before by John Milnor [**Miln1964a**, p. 78] in 1964, see also [**Rudy2016**, p. 87]. Furthermore, similar ideas had already been used before by John Milnor [**Miln1956a**] in the proof of Theorem **??** where he showed that there exists a smooth manifold that is homeomorphic but not diffeomorphic to S^7. ■

In fact the non-zero Pontryagin classes of the smooth manifold M' in [**Miln1964a**, p. 78] and [**KrL2005**, Theorem 4.8] happen to be torsion elements, so the Pontryagin classes of M and M' differ by some torsion element in cohomology. This leads us to the definition of the *rational Pontryagin* classes of a smooth manifold.

Definition. Given a smooth manifold M and $i \in \mathbb{N}_0$ we define the i-th rational Pontryagin class $p_i(M;\mathbb{Q})$ of M to be the image of $p_i(M)$ under the map $\overline{\mathrm{H}}^{4i}(M;\mathbb{Z}) \to \mathrm{H}^{4i}(M;\mathbb{Q})$.

The following theorem says that rational classes can *not* be used to find examples of smooth manifolds that are homeomorphic but non-diffeomorphic. The theorem was proved by Sergei Novikov [**Nov1965**] in 1965.[1734]

Theorem 145.4. Let $f\colon M \to M'$ be a homeomorphism between closed smooth manifolds. We denote by $f^*\colon \mathrm{H}^*(M';\mathbb{Q}) \to \mathrm{H}^*(M;\mathbb{Q})$ the induced map on rational cohomology. Then for any $i \in \mathbb{N}_0$ we have

$$p_i(M;\mathbb{Q}) \;=\; f^*(p_i(M';\mathbb{Q})).$$

145.3. **Pontryagin numbers and the signature.** We start out with the definition of the Pontryagin numbers of a smooth manifold.

Definition.

(1) Let $n \in \mathbb{N}_0$. A partition of n is a set $I = \{i_1, \ldots, i_r\}$ of numbers in $\mathbb{N}$ such that $n = i_1 + \cdots + i_r$. Often we just write, $I = i_1 + \cdots + i_r$ is a partition of n.[1735]

[1734] Sergei Novikov (*1938) is a Russian mathematician who obtained a Fields medal in 1970.

(2) Let $n \in \mathbb{N}_0$ and let $I = i_1 + i_2 + \cdots + i_r$ be a partition of n. Given a closed oriented $4n$-dimensional smooth manifold we define its I-Pontryagin class as

$$p_I(M) \;:=\; p_{i_1}(M) \cup \cdots \cup p_{i_r}(M) \;\in\; \overline{\mathrm{H}}^{4n}(M;\mathbb{Z})$$

and we define the corresponding I-Pontryagin number as

$$\widetilde{p}_I(M) \;:=\; \langle p_I(M), [M]\rangle \;\in\; \mathbb{Z}.$$

Remark. Let M and M' be two closed oriented $4n$-dimensional smooth manifolds. If there exists an orientation-preserving diffeomorphism from M to M', then it follows easily from Lemma 145.2 that all of the Pontryagin numbers of M and M' agree.

The following proposition relates Pontryagin classes to Chapter 144 which dealt with cobordisms. The proposition was first proved by Pontryagin.

Proposition 145.5. Let $n \in \mathbb{N}_0$ and let $I = i_1 + i_2 + \cdots + i_r$ be a partition of n.

(1) Let M be a closed oriented $4n$-dimensional smooth manifold. If M is the boundary of a compact orientable smooth manifold W, then $\widetilde{p}_I(M) = 0$.

(2) The map

$$\begin{array}{rccc} \widetilde{p}_I\colon & \Omega_{4n} & \to & \mathbb{Z} \\ & M & \mapsto & \widetilde{p}_I(M) \end{array}$$

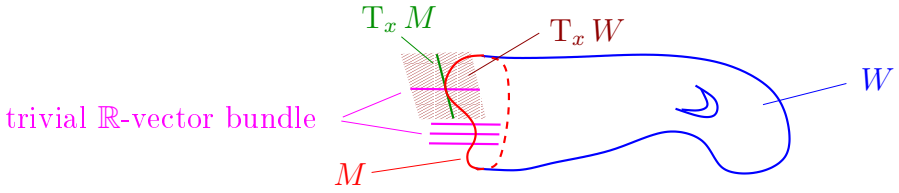
is a well-defined homomorphism.

Proof. Let W be a compact oriented $(4n+1)$-dimensional smooth manifold. We set $M = \partial W$ and we denote by $i\colon M \to W$ the inclusion map. The subsequent calculation relies on the following claim.

Claim. If we denote by $\mathrm{T}M \oplus \mathbb{R}$ the direct sum of $\mathrm{T}M$ and the one-dimensional trivial bundle over M, then $i^*(\mathrm{T}(M) \cong \mathrm{T}M \oplus \mathbb{R}$.

Proof. By the Smooth Collar Neighborhood Theorem 28.3 we can find a neighborhood $[0,1] \times M \subset W$ for $M = \{0\} \times M$. We have

$$\begin{array}{ccccc} i^*(\mathrm{T}(M) & \cong & i^*(\mathrm{T}N \times [0,1])) & \cong & \mathrm{T}M \oplus i^*(T[0,1]) \;\cong\; \mathrm{T}M \oplus \mathbb{R}. \\ \uparrow & & \uparrow & & \uparrow \\ \text{Lemma 143.12 (1)} & & \text{Lemma 143.20} & & \text{Vector Bundle Pullback Lemma 143.9 (1)} \end{array}$$ ⊞



Let $I = i_1 + i_2 + \cdots + i_r$ be a partition of n. We turn to the proof of the first statement. We calculate that

$$\begin{array}{ccccccc} \widetilde{p}_I(M) & = & \langle p_I(\mathrm{T}M), [M]\rangle & = & \langle p_I(\mathrm{T}M \oplus \mathbb{R}), [M]\rangle & = & \langle p_I(i^*\mathrm{T}(M), [M]\rangle \\ & & & & \uparrow & & \uparrow \\ & & & & \text{Theorem 145.1 (2) and (6)} & & \text{by the claim} \\ & = & \langle i^* p_I(\mathrm{T}(M), [M]\rangle & = & \langle p_I(\mathrm{T}(M), i_*([M])\rangle & = & 0. \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \text{Theorem 145.1 (3)} & & \text{Lemma 109.6 (3)} & & \text{Corollary 106.28} \end{array}$$

[1735] Note that a partition is unordered, hence we view $4 = 3+1$ and $4 = 1+3$ as the same partition.

Finally the second statement is an immediate consequence of the first statement and the trivial[1736] observation that for two closed oriented $4n$-dimensional smooth manifolds M and N we have

$$\widetilde{p}_I(M \sqcup N) \;=\; \widetilde{p}_I(M) + \widetilde{p}_I(N). \qquad \blacksquare$$

The following theorem gives the converse to Proposition 145.5.

Theorem 145.6. Let $n \in \mathbb{N}_0$. We denote by $\mathcal{I}$ the set of all partitions of n.

(1) The map

$$\bigoplus_{I\in\mathcal{I}} \widetilde{p}_I \colon \Omega_{4n} \otimes \mathbb{Q} \;\to\; \bigoplus_{i\in\mathcal{I}} \mathbb{Q} \cdot i$$

is an isomorphism.

(2) Let M be a closed oriented $4n$-dimensional smooth manifold. If all Pontryagin numbers of M vanish, then $[M] = 0 \in \Omega_{4n}/\text{torsion}$.

Sketch of proof. We consider the following diagram of homomorphisms:

$$\begin{array}{ccc}
\Omega_{4n}/\text{torsion} & \xrightarrow{\bigoplus_{I\in\mathcal{I}} \widetilde{p}_I} & \bigoplus_{i\in\mathcal{I}} \mathbb{Z} \cdot i \\
\downarrow & & \downarrow \\
\Omega_{4n} \otimes \mathbb{Q} & \xrightarrow{\bigoplus_{I\in\mathcal{I}} \widetilde{p}_I} & \bigoplus_{i\in\mathcal{I}} \mathbb{Q} \cdot i. \\
= \uparrow & \nearrow_{\Psi} & \\
\bigoplus_{i_1+\cdots+i_r=n\in\mathcal{I}} \mathbb{Q} \cdot (\mathbb{C}\mathrm{P}^{2i_1} \times \ldots \times \mathbb{C}\mathrm{P}^{2i_r}) & &
\end{array}$$

The left upper vertical map is a monomorphism by the discussion on page 1910. The right upper vertical map is evidently a monomorphism. It is clear that the diagram commutes. Thus we see that Statement (2) of the theorem is an immediate consequence of Statement (1).

Put differently, it remains to show that the bottom horizontal map is an isomorphism. Note that the lower vertical map is the identity by the Thom Cobordism Theorem 144.11. Thus it suffices to show that the diagonal map is an isomorphism. As a reality check, note that the dimensions agree, in both cases the dimension is just given by the cardinality of $\mathcal{I}$.

In the following claim we carry out the arguments for $n = 1$ and $n = 2$ "by hand". The general case needs more clever arguments, for which we refer to [**Tho2007**, Corollary IV.16].

Claim.

(1) Let $n = 1$. With respect to the bases $\{\mathbb{C}\mathrm{P}^2\}$ and $\{p_1\}$ the diagonal map Ψ is represented by the matrix (3).

(2) Let $n = 2$. With respect to the bases $\{\mathbb{C}\mathrm{P}^4, \mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2\}$ and $\{p_2, p_{1+1}\}$ the diagonal map Ψ is represented by the matrix $\begin{pmatrix} 10 & 9 \\ 25 & 18 \end{pmatrix}$.

In particular in both cases the matrix is invertible.

[1736] Why is this "trivial"? How can one prove this "obvious" equality?

Proof. The case $n = 1$ is easy: We make the usual identification $p\mathrm{H}^*(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) = \mathbb{Z}[x]/(x^3)$, and we calculate that

$$\widetilde{p}_1(\mathbb{C}\mathrm{P}^2) \quad = \quad \langle p_1(\mathbb{C}\mathrm{P}^2), [\mathbb{C}\mathrm{P}^2]\rangle \quad \underset{\text{Theorem 145.1 (5)}}{=} \quad \langle 3x^2, [\mathbb{C}\mathrm{P}^2]\rangle \quad \underset{\text{Cup Product-}\mathbb{C}\mathrm{P}^n\text{-Proposition 126.11}}{=} \quad 3.$$

The case $n = 2$ is slightly more involved. First note that by Theorem 145.1 (5) we have $p(\mathbb{C}\mathrm{P}^4) = 1 + 5x^2 + 10x^4 \in \mathrm{H}^*(\mathbb{C}\mathrm{P}^4;\mathbb{Z}) = \mathbb{Z}[x]/(x^5)$. Thus as above we see that $\widetilde{p}_2(\mathbb{C}\mathrm{P}^4) = 10$. Furthermore, with similar arguments as above we have

$$\widetilde{p}_{1+1}(\mathbb{C}\mathrm{P}^4) \;=\; \langle p_1(\mathbb{C}\mathrm{P}^4) \cup p_1(\mathbb{C}\mathrm{P}^4), [\mathbb{C}\mathrm{P}^4]\rangle \;=\; \langle 5x^2 \cdot 5x^2, [\mathbb{C}\mathrm{P}^4]\rangle \;=\; 25.$$

This concludes the calculation of the first column of the matrix.

It remains to consider $\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2$. By the Cohomology Ring-of-Product Proposition 117.21 we can make the identification $\mathrm{H}^*(\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2;\mathbb{Z}) = \mathbb{Z}[x,y]/(x^3,y^3)$ where x and y are the pullbacks of the standard generator of $\mathrm{H}^2(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ under the two natural projections. Using Lemma 143.20 and Theorem 145.1 (3), (5), (6) we see that $p(\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2) = (1+3x^2)\cdot(1+3y^2)$. In particular $p_1(\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2) = 3x^2 + 3y^2$ and $p_2(\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2) = 9x^2y^2$. Furthermore note that this implies that $p_{1+1}(\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2) = (3x^2+3y^2)^2 = 18x^2y^2$. The claim for $n = 2$ now follows from the observation that the Fundamental Class-Product Proposition 117.2 tells us that $\langle x^2 \cdot y^2, [\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2]\rangle = 1$. ∎

Our next goal is to use Theorem 145.6 to prove the following proposition. It says that the signature of a closed oriented smooth manifold can be determined from its Pontryagin numbers.

Proposition 145.7. Let $n \in \mathbb{N}_0$. We denote by $\mathcal{I}$ the set of all partitions of n. For each $I \in \mathcal{I}$ there exists $a_I \in \mathbb{Q}$ such that for any closed oriented $4n$-dimensional smooth manifold M we have

$$\mathrm{sign}(M) \;=\; \sum_{I\in\mathcal{I}} a_I \cdot \widetilde{p}_I(M).$$

Proof. Let $n \in \mathbb{N}_0$. We denote by $\mathcal{I}$ the set of all partitions of n. We consider the following diagram

$$\begin{array}{ccccc}
\Omega_{4n} & \xrightarrow{[M]\mapsto[M]\otimes 1} & \Omega_{4n}\otimes\mathbb{Q} & \xrightarrow[\cong]{\bigoplus_{I\in\mathcal{I}} \widetilde{p}_I} & \bigoplus_{i\in\mathcal{I}}\mathbb{Q}\cdot i \\
\Big\downarrow{\scriptstyle \mathrm{sign}} & & \Big\downarrow & \swarrow{\scriptstyle \Phi} & \\
\mathbb{Z} & \hookrightarrow & \mathbb{Q} & &
\end{array}$$

$$[M]\otimes r \;\mapsto\; \mathrm{sign}(M)\cdot r$$

Here the left-hand map is a homomorphism by Corollary 144.7. By Proposition 145.5 we know that the top right map is an isomorphism. Therefore there exists a diagonal homomorphism Φ that makes the diagram commute. Given $i \in I$ we set $a_i \in \mathbb{Q}$ to be the value of Φ at $1 = 1\cdot i \in \mathbb{Q}\cdot i \subset \bigoplus_{i\in I}\mathbb{Q}\cdot i$. In other words, the map

$$\begin{array}{rcl}
\bigoplus_{i\in\mathcal{I}}\mathbb{Q}\cdot i & \to & \mathbb{Q} \\
\bigoplus_{i\in\mathcal{I}} x_i\cdot i & \mapsto & \sum_{i\in\mathcal{I}} x_i\cdot a_i.
\end{array}$$

is given by

The desired equality $\operatorname{sign}(M) = \sum_{j=1}^{r} a_j \cdot \widetilde{p}_{I_j}(M)$ now follows immediately from the commutativity of the previous diagram. ■

The coefficients in Proposition 145.7 had been completely determined by Friedrich Hirzebruch [**Hirz1953**]. The low-dimensional terms are given in the following theorem.

Theorem 145.8. (Hirzebruch Signature Theorem) Let $n \in \{1,2,3,4\}$ and let M be a closed oriented $4n$-dimensional smooth manifold. Given $i \in \mathbb{N}_0$ we write $p_i := p_i(M) \in \overline{\mathrm{H}}^{4i}(M;\mathbb{Z})$. The signature of M can be determined as follows:

n	$\operatorname{sign}(M)$ equals the Kronecker pairing of $[M]$ with the cohomology class
1	$\frac{1}{3}p_1$
2	$\frac{1}{45}(7p_2 - p_1^2)$
3	$\frac{1}{945}(62p_3 - 13p_2 \cup p_1 + 2p_1^3)$
4	$\frac{1}{14175}(381p_4 - 71p_3 \cup p_1 - 19p_2^2 + 22p_2 \cup p_1^2 - 3p_1^4)$.

Proof. We carry out the case $n = 2$. We adopt the notation from the proof of Proposition 145.7. By the Thom Cobordism Theorem 144.11 we have $\Omega_8 \otimes \mathbb{Q} = \mathbb{Q} \cdot [\mathbb{C}\mathrm{P}^4] \oplus \mathbb{Q} \cdot [\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2]$. Now let M be a closed oriented 8-dimensional smooth manifold. By the proof of Proposition 145.7 we have $\operatorname{sign}(M) = a_2 \cdot \widetilde{p}_2(M) + a_{1+1} \cdot \widetilde{p}_{1+1}(M)$ where

$$\begin{pmatrix} a_2 \\ a_{1+1} \end{pmatrix} = \begin{array}{c}\text{matrix of sign applied to}\\ \text{the basis } \mathbb{C}\mathrm{P}^4 \text{ and } \mathbb{C}\mathrm{P}^2\times\mathbb{C}\mathrm{P}^2\end{array} \cdot \begin{array}{c}\text{inverse of the matrix that expresses } p_2, p_{1+1}\\ \text{for the smooth manifolds } \mathbb{C}\mathrm{P}^4 \text{ and } \mathbb{C}\mathrm{P}^2\times\mathbb{C}\mathrm{P}^2\end{array}$$

$$= \underset{\uparrow}{} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 10 & 9 \\ 25 & 18 \end{pmatrix}^{-1} = \tfrac{-1}{45} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 18 & -9 \\ -25 & 10 \end{pmatrix} = \tfrac{1}{45} \begin{pmatrix} 7 \\ -1 \end{pmatrix}.$$

by Lemma 126.12 and Exercise 133.5 we know that the signatures of these two smooth manifolds are $+1$ for the second matrix we use the calculation from the proof of Theorem 145.6

This gives precisely the desired result for $n = 2$. The case $n = 1$ is evidently even easier.

The higher-dimensional cases are dealt with, more cleverly, in [**Hirz1953**] or alternatively in [**MiS1974**, p. 224]. These references also contain a precise formula for all $n \in \mathbb{N}_0$. ■

Remark.

(1) The expressions in Theorem 145.8 might seem random, but they do arise in a cogent way from the coefficients of the power series $\frac{\sqrt{t}}{\tanh(t)}$. This connection is explained in [**MiS1974**, Chapter 19].

(2) The signature of a smooth manifold is of course an integer. Thus we obtain from Theorem 145.8 the slightly mysterious statement that for any closed oriented 4-dimensional smooth manifold the integer $\langle p_1(M), [M] \rangle$ is divisible by 3.

The Hirzebruch Signature Theorem 145.8 has many fascinating consequences. Let us start out with the following.

Proposition 145.9. Let M be a closed orientable 12-dimensional smooth manifold. If $b_4(M) = 0$, then $b_6(M)$ is even.

Proof. Let M be a closed orientable 12-dimensional smooth manifold with $b_4(M) = 0$. It follows from the Universal Coefficient Theorem 110.12 that $\mathrm{H}^4(M;\mathbb{Z})$ is torsion. Thus we

see that modulo 2 we have the following equalities:

$$b_6(M) \underset{\uparrow}{\overset{\downarrow}{\equiv}} \operatorname{sign}(M) \underset{\uparrow}{=} \frac{1}{945}(62p_3(M) - 13p_2(M) \cup p_1(M) + 2p_1(M)^3) \underset{\uparrow}{=} 2 \cdot \frac{31}{945} \cdot p_3(M).$$

Intersection Form Proposition 132.4 and Corollary 133.9

by the Hirzebruch Signature Theorem 145.8

since $b_4(M) = 0$ we know by the Universal Coefficient Theorem 110.12 that $\mathrm{H}^4(M;\mathbb{Z})$ is torsion, hence we have $p_1^3(M) = p_2(M) \cup p_1(M) = 0$

Since $b_6(M)$ and $p_3(M)$ are integers we obtain from the equality $b_6(M) = 2 \cdot \frac{31}{945} \cdot p_3(M)$ that $b_6(M)$ is necessarily even. ■

In Proposition 69.4 we saw that the Euler characteristic of a finite CW-complex is multiplicative under finite covers. In fact we saw that this statement follows almost immediately from the definition of a CW-complex. The following theorem gives the analogous statement that the signature of a closed oriented $4n$-dimensional smooth manifold is multiplicative under finite covers.

Theorem 145.10. (Signature-Multiplicativity Theorem) Let $n \in \mathbb{N}_0$. Let M be a closed oriented $4n$-dimensional smooth manifold and let $\widetilde{M} \to M$ be a finite covering. Then

$$\operatorname{sign}\left(\widetilde{M}\right) = \left[\widetilde{M} : M\right] \cdot \operatorname{sign}(M).$$

Remark.

(1) Perhaps it is initially difficult to appreciate the significance of the Signature-Multiplicativity Theorem 145.10. If we only consider (co-) homological invariants, then there are basically only two statements that relate the invariants of a (reasonable) topological space X and an n-fold covering $p\colon \widetilde{X} \to X$ thereof:
   (a) as pointed out above, if X is a finite CW-complex, then $\chi(\widetilde{X}) = n \cdot \chi(X)$,
   (b) the Transfer Map Proposition 91.1 shows that $p_*\colon \mathrm{H}_*(\widetilde{X};\mathbb{Q}) \to \mathrm{H}_*(X;\mathbb{Q})$ is an epimorphism.

   Besides these observations, (co-) homological invariants behave fairly erratically under finite covers. One example of such "random" behavior is given in Exercise 145.2.

(2) One might think that one should be able to prove the Signature-Multiplicativity Theorem 145.10 in a naive way. But this is not the case. This can be seen from the fact that our sketch of the proof of the Signature-Multiplicativity Theorem 145.10 works only for smooth manifolds, the proof that we will sketch does not apply to topological manifolds.

(3) The statement of the Signature-Multiplicativity Theorem 145.10 does in fact hold for topological manifolds, but the proof is even harder, see e.g. [**SchafJ1970**, Theorem 8].

(4) The fact that there is no proof of the Signature-Multiplicativity Theorem 145.10 that does not in an essential way make use of the fact that M is a (topological) smooth manifold is highlighted by [**Wall1967**, Corollary 5.4.1].

(5) For compact $4n$-dimensional smooth manifolds with non-empty boundary the signature is in general *not* multiplicative under finite covers. We refer to [**APS1975**] for details.

Proof. Let $n \in \mathbb{N}_0$. By Proposition 145.7 there exist partitions $I_1, \ldots, I_r$ of $4n$ and rational numbers $a_1, \ldots, a_r$ such that for any closed oriented $4n$-dimensional smooth manifold X the following equality holds:

$$(*) \qquad \operatorname{sign}(X) \;=\; \sum_{j=1}^{r} a_j \cdot \langle p_{I_j}(X), [X] \rangle_X .$$

Now let M be a closed oriented $4n$-dimensional smooth manifold and let $q\colon \widetilde{M} \to M$ be a finite covering. We have

$$\operatorname{sign}\big(\widetilde{M}\big) \underset{\text{by } (*)}{=} \sum_{j=1}^{r} a_j \cdot \big\langle p_{I_j}\big(\widetilde{M}\big), \big[\widetilde{M}\big] \big\rangle_{\widetilde{M}} \underset{\substack{\text{by Theorem 145.1 (3), the Cup Product-Naturality Lemma 114.10 and} \\ \text{since } q^*\mathrm{T}M = T\widetilde{M}, \text{ see Lemma 143.12 (2)}}}{=} \sum_{j=1}^{r} a_j \cdot \big\langle q^*\big(p_{I_j}(M)\big), \big[\widetilde{M}\big] \big\rangle_{\widetilde{M}}$$

$$\underset{\text{Lemma 109.6 (3)}}{=} \sum_{j=1}^{r} a_j \cdot \big\langle p_{I_j}(M), q_*\big[\widetilde{M}\big] \big\rangle_M \underset{\text{Fundamental Class-Covering Proposition 106.25}}{=} \big[\widetilde{M}{:}M\big] \cdot \sum_{j=1}^{r} a_j \cdot \langle p_{I_j}(M), [M] \rangle_M \underset{\text{by } (*)}{=} \big[\widetilde{M}{:}M\big] \cdot \operatorname{sign}(M).$$

■

We conclude this chapter with a short discussion of other characteristic classes.

(1) Loosely speaking, a *characteristic class* is a map that associates to a vector bundle a cohomology class in the base space. We had just discussed the Pontryagin classes in some detail. There are several other characteristic classes:
   (a) To a vector bundle $p\colon E \to X$ one can also associate the *Stiefel-Whitney classes* $w_i(E) \in \overline{\mathrm{H}}^i(X; \mathbb{F}_2)$, $i \in \mathbb{N}_0$.
   (b) If $p\colon E \to X$ is an "oriented" n-dimensional vector bundle, then there exists the *Euler class* $e(E) \in \overline{\mathrm{H}}^n(X; \mathbb{Z})$.
   (c) Finally, if $p\colon E \to X$ is a complex vector bundle, then the corresponding *Chern classes* $c_i(E) \in \overline{\mathrm{H}}^{2i}(X; \mathbb{Z})$, $i \in \mathbb{N}_0$ are defined.

   The properties of Stiefel-Whitney classes and Chern classes are very similar to the properties stated in Theorem 145.1, in fact these characteristic classes are completely determined by the corresponding properties. Finally note that there are some relations between the various characteristic classes, for example given a vector bundle $p\colon E \to X$ the i-th Pontryagin class equals[1737]

$$p_i(E) \;:=\; c_{2i}(\text{the "complexification" } E \otimes \mathbb{C}) \;\in\; \overline{\mathrm{H}}^{4i}(X; \mathbb{Z}).$$

   We refer to **[MiS1974]** and **[Hat2]** for more details on these characteristic classes.
(2) For complex vector bundles over a smooth manifold M the images of the Chern classes in $\overline{\mathrm{H}}^{2i}(M; \mathbb{R})$ can also be defined in the context of de Rham cohomology. Since Pontryagin classes are defined via Chern classes this shows in particular that the images of the Pontryagin classes in real cohomology can be defined via de Rham cohomology. We refer to **[Che1979, MiS1974]** for details.

[1737] In fact, this usually the way Pontryagin classes are defined.

Exercises for Chapter 145.

Exercise 145.1. Let M be a closed oriented smooth manifold of dimension $4k+3$. Show that $\operatorname{sign}(S^1 \times M) = 0$.

Exercise 145.2. Show that given any $n \in \mathbb{N}$ and $k \in \mathbb{N}_{\geq 2}$ there exists a CW-complex X with $\mathrm{H}_k(X;\mathbb{Q}) = 0$ and that admits a 2-fold covering $\widetilde{X}$ with $\mathrm{H}_k(\widetilde{X};\mathbb{Q}) \cong \mathbb{Q}^n$.
Hint. You can take X to be a mapping torus of some high-dimensional torus. Use the Mapping Torus-Homology Proposition 78.20 to compute homology groups.

Part XVI

3-dimensional Manifolds

146. Handlebodies and Heegaard splittings of 3-dimensional smooth manifolds

Given coprime $p, q \in \mathbb{N}$ we introduced the lens spaces $L(p,q)$ on page 1075. Using the fundamental groups it was very easy to see that homeomorphic lens spaces have the same p-parameter. But so far the more advanced invariants, i.e. higher homotopy groups, homology groups and cohomology groups failed miserably at providing us with more information.
In the Chapter 148 we will cleverly apply cup products to obtain information on lens spaces $L(p,q)$ which goes beyond detecting the first parameter p. But before we do so we find it convenient to introduce several more general definitions regarding smooth manifolds.

146.1. Heegaard splittings of 3-dimensional smooth manifolds.

Definition. Let M be a closed oriented connected 3-dimensional smooth manifold.

(1) A splitting of M consists of a triple (X, Y, F) where X and Y are compact 3-dimensional submanifolds of M with $M = X \cup Y$ and such that $F = X \cap Y = \partial X = \partial Y$. Given such a splitting we endow X and Y with the orientation coming from M and we endow F with the orientation coming from $F = \partial X$.[1738]

(2) A splitting (X, Y, F) of M is called a Heegaard splitting if X and Y are both diffeomorphic to handlebodies. We refer to the genus of the surface F as the genus of the Heegaard splitting.

Often, when there is no danger of confusion, we say "$M = X \cup Y$ is a splitting".

Remark. Heegaard splittings are named after the Danish mathematician Poul Heegaard (1871–1948) who proved the first important theorem regarding Heegaard splittings, see below. We had encountered Poul Heegaard before on page 2621.

Examples.

(1) Let $n \in \mathbb{N}$. On page 152 we saw that we can write the n-sphere as the union of the upper hemisphere $S^n_{\geq 0}$ and the lower hemisphere $S^n_{\leq 0}$, both of which are diffeomorphic[1739] to $\overline{B}^n$ and which intersect in an $(n-1)$-sphere. This observation gives us two examples of Heegaard splittings at once:
   (a) We obtain a genus 0 Heegaard splitting $S^3 = S^3_{\geq 0} \cup S^3_{\leq 0}$.
   (b) We also obtain a genus 1 Heegaard splitting

$$S^1 \times S^2 \;=\; \underbrace{(S^1 \times S^2_{\geq 0})}_{\text{solid torus}} \cup \underbrace{(S^1 \times S^2_{\leq 0}).}_{\text{solid torus}}$$

(2) A closed 3-dimensional smooth manifold can have many different Heegaard splittings. For example we consider $S^3 = \left\{(z,w) \in \mathbb{C}^2 \,\middle|\, |w|^2 + |z|^2 = 1\right\}$ and the following two

[1738] Note that the orientation coming from $F = \partial Y$ would be the opposite orientation.

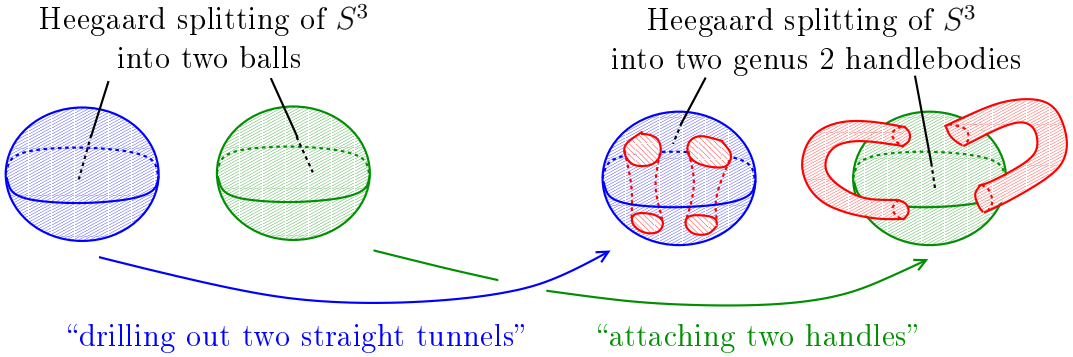
[1739] Actually one has to think for a minute about why the hemispheres are *diffeomorphic* to $\overline{B}^n$. On page 152 we saw that the projection from the hemispheres to $\overline{B}^n$ is a homeomorphism. But this is not a diffeomorphism (why not?). Fortunately stereographic projections, as defined in Lemma 2.18, can be used to give an explicit diffeomorphism.

subspaces of S^3:

$$X := \{(z,w) \in \mathbb{C}^2 \,|\, |w|^2 + |z|^2 = 1 \text{ and } |z|^2 \geq \tfrac{1}{2}\}$$
and
$$Y := \{(z,w) \in \mathbb{C}^2 \,|\, |w|^2 + |z|^2 = 1 \text{ and } |w|^2 \geq \tfrac{1}{2}\}.$$

In Lemma 59.1 we showed that both X and Y are diffeomorphic to the solid torus $S^1 \times \overline{B}^2$. In particular Lemma 59.1 shows that $(X, Y, \partial X = \partial Y)$ is a genus 1 Heegaard splitting of S^3.

(3) The sphere S^3 admits in fact a Heegaard splitting of arbitrarily high genus. This can be seen as follows: we start out with the genus 0 Heegaard decomposition given by decomposing S^3 into two closed 3-balls X and Y. Now we "drill g unknotted open tunnels" out of X and we add (the closure of) what we just drilled out to Y, which has the effect of "adding g handles to Y". In this construction we turned X and Y into genus g handlebodies. This example is illustrated in the figure below.



In fact we have now found all Heegaard splittings of S^3. More precisely, given a genus g Heegaard splitting (X, Y, F) of S^3 there exists a diffeomorphism f of S^3 which turns the Heegaard splitting into the Heegaard splittings of the same genus listed above. For $g = 0$ this statement is equivalent to the Generalized Smooth Schönflies Theorem 82.5 in dimension 3. For $g \geq 1$ the uniqueness of Heegaard splittings was shown in 1968 by Friedhelm Waldhausen **[Wald1968a, Schle2017]**.

(4) We consider the 3-dimensional cube $[0,1]^3$. We consider the equivalence relation $\sim$ that identifies the opposite sides of the cube. More precisely, on $[0,1]^3$ we denote by $\sim$ the equivalence relation that is generated by

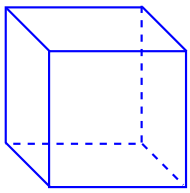
$$(0,a,b) \sim (1,a,b), \quad (a,0,b) \sim (a,1,b) \quad \text{and} \quad (a,b,0) \sim (a,b,1) \quad \text{for } a,b \in [0,1].$$

It follows easily from the Compact-Hausdorff Proposition 2.17 (3) that the map $[0,1]^3/\sim \to S^1 \times S^1 \times S^1$ given by $(x,y,z) \mapsto (\exp(2\pi \mathrm{i} x), \exp(2\pi \mathrm{i} y), \exp(2\pi \mathrm{i} z))$ is a homeomorphism. We use this homeomorphism to identify $[0,1]^3/\sim$ with the 3-dimensional torus $S^1 \times S^1 \times S^1$. In the figure below we show the cube $[0,1]$ and we show the subset

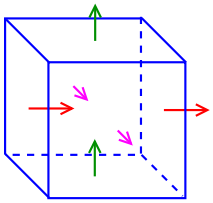
$$X := \{(x_1,x_2,x_3) \in [0,1]^3 \,|\, \text{at least two coordinates satisfy } |x_i - \tfrac{1}{2}| \leq \tfrac{1}{4}\}.$$

We leave it to the reader to show that $X/\sim$ is homeomorphic to the genus 3 handlebody and that $Y := \overline{([0,1]^3 \setminus X)/\sim}$ is also homeomorphic to the genus 3 handlebody. This shows that the 3-dimensional torus admits a genus 3 Heegaard splitting. It is

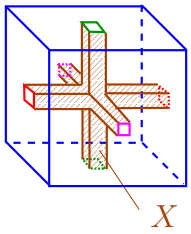
natural to ask whether the 3-dimensional torus admits a Heegaard splitting of smaller genus. We study this question in Exercise 146.3



the cube $[0,1]^3$



opposite sides get identified



(5) Let F be the surface of genus g and let $\varphi\colon F \to F$ be an orientation-preserving self-diffeomorphism. By the Manifold-Mapping Torus Lemma 16.25 the mapping torus

$$\mathrm{Tor}(F,\varphi) \;:=\; F \times [0,1] \,/\, (x,0) \sim (\varphi(x),1)$$

is a closed orientable 3-dimensional smooth manifold. In Exercise 146.4 we will see that $\mathrm{Tor}(F,\varphi)$ admits a Heegaard splitting of genus $2g+1$.

The following lemma gives us a slightly different point of view regarding Heegaard splittings.

Lemma 146.1.

(1) Let X and Y be two oriented handlebodies of the same genus. Given any diffeomorphism $\varphi\colon \partial X \to \partial Y$ we write

$$M(X,Y,\varphi) \;:=\; (X \sqcup Y)/\!\sim \qquad \text{with } x \sim \varphi(x) \text{ for } x \in X.$$

(a) $M(X,Y,\varphi)$ is a closed 3-dimensional smooth manifold.
(b) $M(X,Y,\varphi)$ admits a unique orientation such that the orientation of X, viewed as a codimension zero submanifold of $M(X,Y,\varphi)$, agrees with the orientation of $M(X,Y,\varphi)$.
(c) The images of X and Y in $M(X,Y,\varphi)$ define a Heegaard splitting.
(d) If $\varphi\colon \partial X \to \partial Y$ is an orientation-preserving diffeomorphism, then there exists an orientation-reversing diffeomorphism $\psi\colon \partial X \to \partial Y$ such that there exists an orientation-preserving diffeomorphism from $M(X,Y,\varphi)$ to $M(X,Y,\psi)$.
(e) Let $\varphi,\psi\colon \partial X \to \partial Y$ be two diffeomorphisms. If φ and ψ are diffeotopic, then there exists an orientation-preserving diffeomorphism from $M(X,Y,\varphi)$ to $M(X,Y,\psi)$.

(2) Let M be a closed oriented connected 3-dimensional smooth manifold and let (X,Y,F) be a Heegaard splitting of M. There exists an orientation-reversing diffeomorphism $\varphi\colon \partial X \to \partial Y$ such that the smooth embeddings $X \to M$ and $Y \to M$ induce an orientation-preserving diffeomorphism $M(X,Y,\varphi) \to M$.

Proof.

(1) Statements (a), (b) and (c) follow immediately from the Gluing-Smooth Manifolds-Proposition **??**. Statement (e) follows almost immediately from Lemma **??** and the Gluing-Smooth Manifolds-Proposition **??** (7). Finally Statement (d) follows from Lemma **??**.
(2) Let $\varphi = \mathrm{id}\colon \partial X \to \partial Y$. It follows immediately from the orientation conventions that φ is orientation-preserving. ■

In the following we want to consider Heegaard splittings of lens spaces in detail. Recall that given coprime $p, q \in \mathbb{N}$ the map

$$\begin{array}{rcl} \mathbb{Z}_p \times S^3 & \to & S^3 \\ (k + p\mathbb{Z}, (z, w)) & \mapsto & (\exp(2\pi \mathrm{i} k/p) \cdot z, \exp(2\pi \mathrm{i} kq/p) \cdot w) \end{array}$$

defines a free orientation-preserving action of the group $\mathbb{Z}_p$ on S^3. By definition we have $L(p, q) = S^3/\mathbb{Z}_p$ and, following the convention established on page 1075, we equip $L(p, q)$ with the obvious orientation coming from S^3. It follows immediately from the definitions that this action by $\mathbb{Z}_p$ on $S^3 = X \cup Y$ preserves the solid tori X and Y from page 3115 setwise. In fact the quotients $X/\mathbb{Z}_p$ and $Y/\mathbb{Z}_p$ are again diffeomorphic to the solid torus. More precisely, we pick $r \in \mathbb{N}$ with $r \cdot q \equiv -1 \bmod p$. Similar to the proof of Lemma 59.1 it is fairly straightforward to verify that the maps

$$\begin{array}{rcl} \Phi\colon X/\mathbb{Z}_p = \left\{(z, w) \in \mathbb{C}^2 \,\middle|\, |w|^2 + |z|^2 = 1 \text{ and } |z|^2 \geq \frac{1}{2}\right\}/\mathbb{Z}_p & \to & S^1 \times \overline{B}^2 \\ [(z, w)] & \mapsto & \left(\frac{z^p}{|z^p|}, \frac{z^{-q} w}{|z^{1-q}|}\right) \end{array}$$

and

$$\begin{array}{rcl} \Psi\colon Y/\mathbb{Z}_p = \left\{(z, w) \in \mathbb{C}^2 \,\middle|\, |w|^2 + |z|^2 = 1 \text{ and } |w|^2 \geq \frac{1}{2}\right\}/\mathbb{Z}_p & \to & S^1 \times \overline{B}^2 \\ [(z, w)] & \mapsto & \left(\frac{w^p}{|w^p|}, \frac{z w^r}{|w^{1+r}|}\right) \end{array}$$

are well-defined diffeomorphisms and that Φ is orientation-preserving.[1740] This shows that

$$L(p, q) \;=\; X/\mathbb{Z}_p \cup Y/\mathbb{Z}_p$$

is a genus 1 Heegaard decomposition of the lens space $L(p, q)$. In the following lemma we describe this Heegaard splitting in greater detail.

Lemma 146.2. (Lens Space-Splitting Lemma) Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. We pick $r, s \in \mathbb{Z}$ such that $qr - ps = -1$. We write

$$A \;=\; \begin{pmatrix} q & p \\ s & r \end{pmatrix}$$

and we consider the orientation-reversing diffeomorphism

$$\begin{array}{rcl} \varphi\colon \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ [v] & \mapsto & [Av]. \end{array}$$

If we make the usual identification $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$, then there exists an orientation-preserving diffeomorphism

$$L(p, q) \;\xrightarrow{\cong}\; M(S^1 \times \overline{B}^2, S^1 \times \overline{B}^2, \varphi).$$

Proof. In this proof we continue to use the notation that we introduced before stating the lemma. Now we write $U = X/\mathbb{Z}_p$ and $V = Y/\mathbb{Z}_p$ and as above we consider the two diffeomorphisms $\Phi\colon U \to S^1 \times \overline{B}^2$ and $\Psi\colon V \to S^1 \times \overline{B}^2$. We denote by $\gamma\colon S^1 \times S^1 \to S^1 \times S^1$ the diffeomorphism that is given by

$$S^1 \times S^1 \;=\; \partial(S^1 \times \overline{B}^2) \;\xleftarrow{\Phi}\; \partial U \;=\; \partial V \;\xrightarrow{\Psi}\; \partial(S^1 \times \overline{B}^2) \;=\; S^1 \times S^1.$$

[1740] To show that Φ is orientation-preserving it suffices by Proposition 107.2 to show that Φ is orientation-preserving at a single point. Using that $p > 0$ one can calculate explicitly that Φ is orientation-preserving at the point $[(1, 0)] \in X/\mathbb{Z}_p$.

Since Φ and Ψ are diffeomorphisms the two maps together give rise to a well-defined diffeomorphism

$$\begin{array}{rcl} L(p,q) = U \cup V & \to & M(S^1 \times \overline{B}^2, S^1 \times \overline{B}^2, \gamma) = (S^1 \times \overline{B}^2) \cup_\gamma (S^1 \times \overline{B}^2) \\ P & \mapsto & \begin{cases} \Phi(P) \text{ in the first solid torus,} & \text{if } P \in U, \\ \Psi(P) \text{ in the second solid torus,} & \text{if } P \in V. \end{cases} \end{array}$$

It follows from the orientation convention in Lemma 146.1 (1), together with the fact that Φ is orientation-preserving, that this diffeomorphism is in fact orientation-preserving.

We consider the following diagram of diffeomorphisms:

$$\begin{array}{ccccccccccc} S^1 \times S^1 & \xleftarrow{=} & \partial(S^1 \times \overline{B}^2) & \xleftarrow{\Phi} & \partial U & \xrightarrow{=} & \partial V & \xrightarrow{\Psi} & \partial(S^1 \times \overline{B}^2) & \xrightarrow{=} & S^1 \times S^1 \\ \downarrow{=} & & & & & & & & & & \downarrow{=} \\ \mathbb{R}^2/\mathbb{Z}^2 & & & & & & \xrightarrow{\qquad\qquad\varphi\qquad\qquad} & & & & \mathbb{R}^2/\mathbb{Z}^2. \end{array}$$

Starting from ∂U, it is straightforward to verify that the diagram commutes. But this just shows that, under the identification $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$, the two self-diffeomorphisms γ and φ of $S^1 \times S^1$ agree. ■

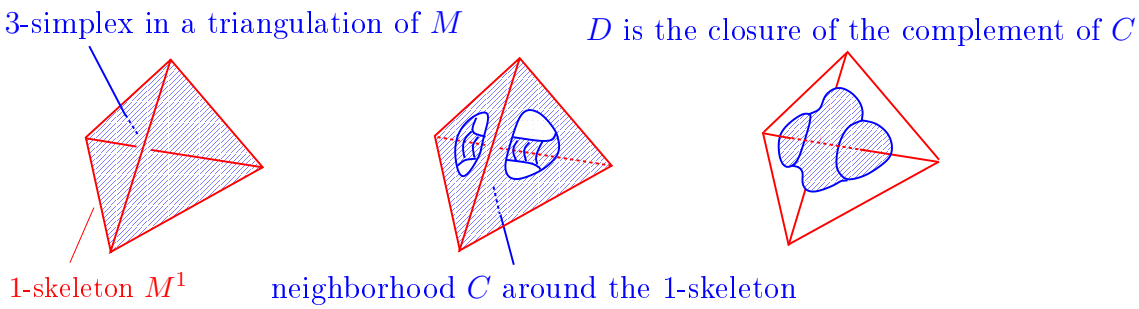
We have now given explicit Heegaard splittings of some basic examples of closed 3-manifolds. Perhaps somewhat surprisingly it is relatively straightforward to generalize these examples to all closed orientable 3-dimensional smooth manifolds. More precisely, in 1898 Poul Heegaard proved the following theorem in his PhD thesis [**Hee1898**]. This result is one of the earliest results in the topology of 3-dimensional smooth manifolds.

Theorem 146.3. Every closed orientable connected 3-dimensional smooth manifold admits a Heegaard splitting.

Remark. As we had mentioned on page 3115, for each $g \geq 1$ the 3-sphere S^3 admits (up to diffeomorphism) a unique Heegaard splitting. In general though closed orientable connected 3-dimensional smooth manifolds can admit inequivalent Heegaard splittings of a fixed genus, see e.g. [**Engm1970, BGAM1976**].

Sketch of proof. The idea of the proof of the theorem is quite simple. Let M be a closed orientable connected 3-dimensional smooth manifold. By the Smooth Manifold-Simplicial Structure Theorem 96.2 the smooth manifold M admits a triangulation, i.e. a simplicial structure. As usual we denote by M^1 the 1-skeleton of M. We pick a closed neighborhood C around the 1-skeleton M^1 such that the intersection of C with each 3-simplex is of the form shown in the middle of the figure below. Furthermore we write $D = \overline{M \setminus C}$. The claim is that both C and D are handlebodies. The idea is that C and D are built out of pieces as sketched in the figure below and thus C and D are "obviously" handlebodies. But if one looks more carefully into the argument one sees that one needs the hypothesis that M is orientable to conclude that C and D are indeed handlebodies. We will not fill in the details of the argument. Unfortunately the proofs in the literature tend to be equally short on details, in particular the role of the orientability condition gets all too often swiped under the carpet. We refer to [**Sav2012**, Theorem 1.1], [**Schar2002**, p. 924] [**Hem1976**, Theorem 2.5] for proofs of the theorem which follow the above line of argumentation.

As so often in topology, there is another proof of the theorem, which might be less intuitive and less visual, but which is much more rigorous. This proof uses Morse theory.



We refer to [**JohJ200707**, Theorem 4.29] or [**Mata2002**, Chapter 5.2] for details on how to obtain Heegaard splittings from Morse theory. ■

146.2. **Fundamental groups of 3-dimensional smooth manifolds.** Let $n \geq 4$. In the Fundamental Group-Manifold Realization Proposition 54.12 we saw that given any finitely presented group π there exists a closed orientable connected n-dimensional smooth manifold with fundamental group isomorphic to π. Using Theorem 146.3 we will shortly see that the class of finitely presented groups that appear as fundamental groups of closed orientable connected 3-dimensional smooth manifolds is much more restricted. Before we can state the result in detail we need to introduce the following group theoretic notion.

Definition. The deficiency of a finite presentation $\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$ is defined to be $k - l$.

Examples.

(1) The free group $\langle x_1, \ldots, x_k \rangle$ on k generators admits, by definition, a presentation of deficiency k.

(2) Let $m \in \mathbb{N}$. In Lemma 53.6 (2) we saw that there exists an isomorphism

$$\langle x_1, \ldots, x_m \mid [x_i, x_j] = e \text{ for all } i < j \in \{1, \ldots, m\} \rangle \xrightarrow{\cong} \mathbb{Z}^m.$$

This shows that $\mathbb{Z}^m$ admits a presentation of deficiency $m - \frac{m(m-1)}{2}$.

(3) Let $r \in \mathbb{N}$ and $d_1, \ldots, d_s \in \mathbb{N}$. It follows basically immediately from the above isomorphism in (2) that there exists an isomorphism

$$\left\langle x_1, \ldots, x_{r+s} \,\middle|\, \begin{array}{c} [x_i, x_j] = e \text{ for all } i < j \in \{1, \ldots, r+s\} \\ \text{and } x_1^{d_1} = \cdots = x_s^{d_s} = e \end{array} \right\rangle \xrightarrow{\cong} \mathbb{Z}^r \oplus \bigoplus_{i=1}^{s} \mathbb{Z}_{d_i}.$$

This shows that $\mathbb{Z}^r \oplus \bigoplus_{i=1}^{s} \mathbb{Z}_{d_i}$ admits a presentation of deficiency $r - \frac{(r+s)(r+s-1)}{2}$.

Given a finite presentation for a group π it is straightforward to find a finite presentation of smaller deficiency: we just add trivial relations to a given presentation. This observation naturally leads us to the following definition:

Definition. Let π be a finitely presented group. We define the deficiency $\mathrm{def}(\pi)$ of π as the maximal deficiency of a finite presentation for π.

The following lemma shows, in particular, that the deficiency is in fact a finite natural number.

Lemma 146.4. (Deficiency Lemma)

(1) If π is a finitely presented group, then[1741]

$$\mathrm{def}(\pi) \;\leq\; \mathrm{rank}(\pi_{\mathrm{ab}}).$$

(2) The deficiency of the free group $\langle x_1, \ldots, x_k \rangle$ equals k.

(3) The deficiency of $\mathbb{Z}^m$ equals $m - \frac{m(m-1)}{2}$.

(4) The deficiency of an abelian group is at most one. Furthermore, up to isomorphism, we have that

(a) the only abelian groups of deficiency 1 are $\mathbb{Z}$ and $\mathbb{Z}^2$, and

(b) the only abelian groups of deficiency 0 are $\mathbb{Z}_n$ and $\mathbb{Z} \oplus \mathbb{Z}_n$ with $n \neq 0$ and $\mathbb{Z}^3$.

Proof.

(1) Let π be a finitely presented group and let $\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$ be a finite presentation. It follows from the Abelianization Proposition 53.20 (5) that there exists a short exact sequence of the form

$$0 \;\to\; A \;\to\; \mathbb{Z}^k \;\to\; \pi_{\mathrm{ab}} \;\to\; 0$$

where A is an abelian group that admits an epimorphism $\mathbb{Z}^l \to A$. We obtain that

$$\begin{array}{ccccc} k - l & \underset{\uparrow}{\leq} & k - \mathrm{rank}(A) & \underset{\uparrow}{=} & \mathrm{rank}(\pi_{\mathrm{ab}}). \\ & \text{Lemma 51.9 (6)} & & \text{Rank Additivity Lemma 51.7 (2)} & \end{array}$$

It follows that $\mathrm{def}(\pi) \leq \mathrm{rank}(\pi_{\mathrm{ab}})$.

(2) Let $F = \langle x_1, \ldots, x_k \rangle$ be the free group on k generators. Evidently its deficiency is at least k. By the Abelianization Proposition 53.20 (4) we know that $F_{\mathrm{ab}} \cong \mathbb{Z}^k$. Thus it follows from (1) that the deficiency of F is also at most k.

(3) We had just seen that $\mathrm{def}(\mathbb{Z}^m) \geq m - \frac{m(m-1)}{2}$. It follows from [**Eps1961**, Lemma 1.3] that this inequality is in fact an equality.

(4) This statement also follows from the above examples of presentations and [**Eps1961**, Lemma 1.3]. ∎

Now we have all the tools to state, and partially prove, the following theorem, which gives a negative answer to Question 54.15.

Theorem 146.5. Let M be a closed orientable connected 3-dimensional smooth manifold.

(1) We have $\mathrm{def}(\pi_1(M)) \geq 0$,

(2) If $\pi_2(M) = 0$, then $\mathrm{def}(\pi_1(M)) = 0$.

(3) If $\pi_1(M)$ is abelian, then $\pi_1(M)$ is either isomorphic to $\mathbb{Z}^3$ or to $\mathbb{Z}_n$ for some $n \in \mathbb{N}_0$.

Remark. The list in (3) is complete, in the sense that all these abelian groups are realized by fundamental groups of closed orientable connected 3-dimensional smooth manifolds. Indeed, given $n \in \mathbb{N}$, we obtain from Corollary 48.18 and the Fundamental Group-of-Product Proposition 48.20 that

$$\pi_1(S^1 \times S^1 \times S^1) \cong \mathbb{Z}^3 \quad \pi_1(L(n,1)) \cong \mathbb{Z}_n, \qquad \text{and } \pi_1(S^1 \times S^2) \cong \mathbb{Z}.$$

[1741] Here we denote by π_{ab} the abelianization of π that we introduced on page 1203. Furthermore, given a finitely generated abelian group H we denote by $\mathrm{rank}(H)$ the rank that we introduced on page 1152.

Proof. Let M be a closed orientable connected 3-dimensional smooth manifold.

(1) By Theorem 146.3 there exists a genus g Heegaard splitting $M = X \cup_F Y$ for M. We pick a base point $x_0 \in F = X \cap Y$. By Lemma **??** and the Fundamental Group-of-Surfaces Proposition 54.5 we can make identifications $\pi_1(X, x_0) = \langle x_1, \dots, x_g \rangle$, $\pi_1(Y, x_0) = \langle y_1, \dots, y_g \rangle$ and

$$\pi_1(F, x_0) \;=\; \langle a_1, \dots, a_{2g} \,|\, [a_1, a_2] \cdot \dots \cdot [a_{2g-1}, a_{2g}] \rangle.$$

We denote by $\varphi\colon \pi_1(F, x_0) \to \pi_1(X, x_0)$ and $\psi\colon \pi_1(F, x_0) \to \pi_1(Y, x_0)$ the inclusion induced maps. Now we have

$$\begin{array}{rcl} & & \text{Seifert–van Kampen Theorem 54.4 for smooth manifolds} \\ & & \downarrow \\ \pi_1(M, x_0) & \cong & \pi_1(X, x_0) *_{\pi_1(F, x_0)} \pi_1(Y, x_0) \\ & = & \langle x_1, \dots, x_g, y_1, \dots, y_g \,|\, \varphi(a_1) = \psi(a_1), \dots, \varphi(a_{2g}) = \psi(a_{2g}) \rangle. \\ & & \uparrow \\ & & \text{Lemma 53.23} \end{array}$$

We have thus shown that $\pi_1(M)$ admits a presentation of deficiency zero.

(2) This statement is proved in [**Eps1961**, Lemma 3.1].

(3) It follows from (1) and the Deficiency Lemma 146.4 (4), that the only isomorphism types of abelian groups that can occur are $\mathbb{Z}_n$ for some $n \in \mathbb{N}_0$, $\mathbb{Z}^2$, $\mathbb{Z} \oplus \mathbb{Z}_n$ for some $n \geq 2$ and $\mathbb{Z}^3$. With somewhat more effort, see e.g. [**Eps1961**, Theorem 3.3] or [**Rei1936**, p. 28] one can also exclude the groups $\mathbb{Z}^2$ and $\mathbb{Z} \oplus \mathbb{Z}_n$ with $n \geq 2$ from the list. ■

We conclude this discussion of Heegaard splittings with a particularly fascinating aspect of Heegaard splittings: they can be used to give a purely group theoretic formulation of the Poincaré Conjecture in dimension three. More precisely, the following theorem was proved by John Stallings [**Stal1965a**, Chapter 3] and William Jaco [**Jaco1969**, Theorem 6.3]. The origin of the theorem lies in the description of the fundamental group of a closed oriented connected 3-dimensional smooth manifold in terms of a Heegaard splitting that we gave in the proof of Theorem 146.5.

Theorem 146.6. We introduce the following notation and language:

(a) Given $g \in \mathbb{N}_0$ we denote by π_g the fundamental group of the surface of genus g and we denote by F_g the free group of rank g.

(b) We say a homomorphism $\beta\colon G \to A * B$ from a group G to the free product of two groups A and B is *essential*, if there is no $h \in A * B$ such that $h\beta(G)h^{-1}$ is contained in A or B.

The following two statements are equivalent:

(1) Every homotopy 3-sphere[1742] is diffeomorphic to S^3.

(2) For any $g \in \mathbb{N}$ any epimorphism $\alpha\colon \pi_g \to F_g \times F_g$ factors through an essential monomorphism $\beta\colon \pi_g \to A * B$ from π_g to the free product of two groups A and B.

Put differently, by Perelman's proof of Theorem **??** we now know that the first statement of Theorem 146.6 holds, thus the second statement also holds. This is the only known proof of that purely group theoretic statement. Conversely, any proof of the second statement

[1742] See page 2625 for the definition of a homotopy sphere.

of Theorem 146.6 would immediately reprove Theorem **??**. Surely that is a quick and easy way to mathematical glory.

Remark.

(1) Perhaps by now it is no longer surprising that there is a huge literature on Heegaard splittings, see e.g. [**Schle2017**, Chapter 6] for an introduction.
(2) Recently the notion of a Heegaard splitting has been modified by David Gay and Rob Kirby [**GK2016**] to give the notion of a *trisection* of a closed orientable connected 4-dimensional smooth manifold. One remarkable feature of trisections is that by recent work of Aarons Abrams, David Gay and Rob Kirby [**AGK2018**] they allow for a purely group theoretic reformulation of the smooth Poincaré Conjecture in dimension 4 that we had formulated as Question **??**. This can be viewed as an analogue of Theorem 146.6.

146.3. **The mapping class group of 2-dimensional smooth manifolds.** In Section **??** we introduced the mapping class group of smooth manifolds. In this section we recall the definition and we discuss the mapping class groups of surfaces in more detail. We will use this discussion to show that every 3-dimensional smooth manifold with a genus 1 Heegaard decomposition is in fact a lens space.
First we recall the following definition from page 812.

Definition. Let M and N be two smooth manifolds. Two diffeomorphisms $f, g\colon M \to N$ are called diffeotopic, if there exists a diffeotopy from f to g, i.e. a smooth map[1743]

$$\begin{array}{rcl} H\colon M \times [0,1] & \to & N \\ (x,t) & \mapsto & H(x,t) \end{array}$$

such that the following hold:

(1) for all $x \in M$ we have $H(x,0) = f(x)$ and $H(x,1) = g(x)$, and
(2) for each $t \in [0,1]$ the map $M \to N$ given by $x \mapsto H(x,t)$ is a diffeomorphism.

Remark. In the following we will implicitly use on several occasions the fact, shown in the Smooth Isotopy Transitivity Proposition 33.3, that being diffeotopic is an equivalence relation.

We move on to introducing some notation.

Notation. Let M be an oriented smooth manifold.

(1) We write $\mathrm{Diff}(M)$ for the group of all self-diffeomorphisms of M.
(2) We denote by $\mathrm{Diff}_0(M)$ the set of all self-diffeomorphisms of M that are diffeotopic to the identity.
(3) We write $\mathrm{Diff}^+(M)$ for the group of all orientation-preserving self-diffeomorphisms of M.

We recall the following definition from page **??**.

Definition. Let M be a smooth manifold. The mapping class group $\mathrm{MCG}(M)$ of M is defined as

$$\mathrm{MCG}(M) \;:=\; \mathrm{Diff}(M)/\,\mathrm{Diff}_0(M).$$

[1743] See page 811 for the definition of what it means that a map on $M \times [0,1]$ is smooth.

If M is oriented, then the orientation-preserving mapping class group $\mathrm{MCG}^+(M)$ of M is defined as

$$\mathrm{MCG}^+(M) := \mathrm{Diff}^+(M)/\mathrm{Diff}_0(M).$$

Example.

(1) Let $n \in \mathbb{N}$. In the Diffeotopies-of-$\mathbb{R}^n$-Classification Proposition 34.3 and Exercise 34.1 we showed, with different language, that $\mathrm{MCG}^+(\mathbb{R}^n)$ and $\mathrm{MCG}^+(\overline{B}^1)$ and $\mathrm{MCG}^+([0,1)$ are the trivial group.
(2) We now consider the orientation-preserving mapping class group of 1-dimensional smooth manifolds. By the Smooth 1-Dimensional Manifold Classification Theorem 22.7 we know that we only need to consider the smooth manifolds $\mathbb{R}$, $[0,1)$, $[0,1]$ and S^1. In the Diffeotopies-of-$\mathbb{R}^n$-Classification Proposition 34.3, Exercise 34.4 and the Diffeotopies-of-S^1-Classification Proposition 34.5 we showed that $\mathrm{MCG}^+(\mathbb{R})$, $\mathrm{MCG}^+([0,1))$, $\mathrm{MCG}^+([0,1])$, and $\mathrm{MCG}^+(S^1)$ are the trivial group.
(3) Note that by the Smooth Isotopy-Orientation Proposition 34.2 an orientation-reversing self-diffeomorphism of a smooth manifold M always represents a non-trivial element of the mapping class group $\mathrm{MCG}(M)$.

For surfaces of genus 0 and 1, i.e. for the sphere S^2 and the torus $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$ the following theorem gives a complete description of the mapping class groups.

Theorem 146.7.

(1) $\mathrm{MCG}^+(S^2)$ is the trivial group.
(2) The group $\mathrm{MCG}(S^2)$ contains two elements, one is represented by the identity and the other one is represented by any reflection in a hyperplane.
(3) Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-dimensional torus. We use the generators from Lemma 78.11 (2) to make the usual identification $\mathrm{H}_1(T;\mathbb{Z}) = \mathbb{Z}^2$ and we use this identification to make the corresponding identifications $\mathrm{Aut}(\mathrm{H}_1(T;\mathbb{Z})) = \mathrm{Aut}(\mathbb{Z}^2) = \mathrm{GL}(2,\mathbb{Z})$ and $\mathrm{SAut}(\mathrm{H}_1(T;\mathbb{Z})) = \mathrm{SAut}(\mathbb{Z}^2) = \mathrm{SL}(2,\mathbb{Z})$. The maps

$$\begin{aligned} \Phi\colon \mathrm{MCG}(T) &\mapsto \mathrm{GL}(2,\mathbb{Z}) \\ [f\colon T \to T] &\mapsto \underbrace{(f_*\colon \mathrm{H}_1(T) \to \mathrm{H}_1(T))}_{\in \mathrm{Aut}(\mathrm{H}_1(T;\mathbb{Z}))=\mathrm{GL}(2,\mathbb{Z})} \end{aligned} \quad \text{and} \quad \begin{aligned} \Psi\colon \mathrm{GL}(2,\mathbb{Z}) &\mapsto \mathrm{MCG}(T) \\ A &\mapsto \begin{bmatrix} \mathbb{R}^2/\mathbb{Z}^2 \mapsto \mathbb{R}^2/\mathbb{Z}^2 \\ [v] \mapsto [Av] \end{bmatrix} \end{aligned}$$

are inverses of one another.
(4) The same statement as in (3) holds if we replace $\mathrm{MCG}^+(\mathbb{R}^2/\mathbb{Z}^2)$ by $\mathrm{MCG}(\mathbb{R}^2/\mathbb{Z}^2)$ and if we replace $\mathrm{SL}(2,\mathbb{Z})$ by $\mathrm{GL}(2,\mathbb{Z})$.[1744]

Proof. The first two statements were proved in 1959 by James Munkres [**Mun1960b**] and independently by Stephen Smale [**Sma1959**].[1745] It is an amusing exercise to show that $\Phi \circ \Psi = \mathrm{id}$ on $\mathrm{GL}(2,\mathbb{Z})$. The fact that $\Psi \circ \Phi = \mathrm{id}$ on $\mathrm{MCG}(\mathbb{R}^2/\mathbb{Z}^2)$ is somewhat harder. We refer to [**FaM2011**, Theorem 2.5] or alternatively to [**Rolf1990**, Chapter 2.D] for proofs. The claim regarding the orientation-preserving mapping class group follows from the above result together with Lemma 114.15. ■

Remark.

[1744] We showed in Lemma 114.15 that the map $\Phi\colon \mathrm{MCG}^+(T) \to \mathrm{GL}(2,\mathbb{Z})$ takes values in $\mathrm{SL}(2,\mathbb{Z})$.
[1745] We had already encountered Stephen Smale in the proof of the Poincaré Conjecture in dimensions ≥ 5, see Theorem **??**.

(1) We will discuss the higher-dimensional analogues of Theorem 146.7 (1) in Theorem **??**.
(2) The obvious analogue of Theorem 146.7 (1) in higher dimensions does not hold. More precisely, Allen Hatcher and John Wagoner [**HatW1973**] and Kiyoshi Igusa [**Ig1979, Ig1984**] proved that given any $n \geq 5$ there exists a short exact sequence of the form:

$$0 \ \to \ \mathbb{Z}_2^\infty \ \to \ \mathrm{MCG}((S^1)^n) \ \to \ \mathrm{GL}(n,\mathbb{Z}) \ \to \ 1.$$

We refer to [**Hat1978**, Theorem 4.1] for more details and to [**HsSh1976**, Theorem 2.5] for an alternative proof if $n \geq 6$.

Now we can give the following characterization of lens spaces.

Corollary 146.8. If M is a closed orientable connected 3-dimensional smooth manifold, then the following two statements are equivalent:

M admits a genus 1 Heegaard splitting $\iff$ M is a lens space or $M \cong S^1 \times S^2$.

Proof. By the Lens Space-Splitting Lemma 146.2 and the discussion on page 3114 we know that lens spaces and $S^1 \times S^2$ admit genus 1 Heegaard splittings.

Now suppose that M is a closed orientable connected 3-dimensional smooth manifold that admits a genus 1 Heegaard splitting $M = (S^1 \times \overline{B}^2) \cup_\varphi (S^1 \times \overline{B}^2)$ where φ is an orientation-reversing self-diffeomorphism of $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$. It is an immediate consequence of Theorem 146.7 (3), Lemma 114.15 and Lemma **??** that, after a diffeotopy, we can assume that φ is given by multiplication by a matrix $\begin{pmatrix} q & p \\ s & r \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$ with $qr - ps = -1$.

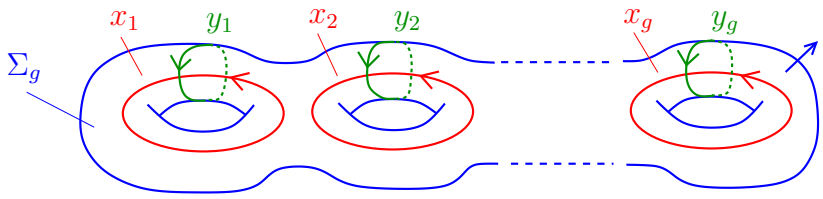
First assume that $p = 0$. We leave it as an amusing exercise to show that in this case M is diffeomorphic to $S^1 \times S^2$. If $p > 0$, then it follows from the Lens Space-Splitting Lemma 146.2 that M is diffeomorphic to $L(p,q)$. Finally we consider the case that $p < 0$. We denote by ψ the self-diffeomorphism of $\mathbb{R}^2/\mathbb{Z}^2$ that is given by multiplication by the matrix $\begin{pmatrix} q & -p \\ -s & r \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$. The map

$$(S^1 \times \overline{B}^2) \cup_\varphi (S^1 \times \overline{B}^2) \ \to \ (S^1 \times \overline{B}^2 \cup_\psi S^1 \times \overline{B}^2),$$

given by reflection on both of the S^1 factors is easily seen to be a well-defined diffeomorphism. But as we had just shown, the right-hand side is diffeomorphic to the lens space $L(-p,q)$. ∎

We had just seen that the mapping class group of the 2-dimensional sphere and the torus are groups we are very familiar with. It is natural to ask, whether we can also understand the mapping class groups of higher genus surfaces.

Throughout the remainder of this section, given $g \geq 1$ we denote by Σ_g the surface of genus g. To facilitate the discussion, we equip $\mathrm{H}_1(\Sigma_g;\mathbb{Z})$ with the standard basis $x_1, \ldots, x_g, y_1, \ldots, y_g$ shown in the figure below and we use this basis to make the identification $\mathrm{H}_1(\Sigma_g;\mathbb{Z}) = \mathbb{Z}^{2g}$.
Here is a somewhat optimistic interpretation of the above question.



Question 146.9. Let $g \geq 2$. Is the map

$$\begin{array}{rcl} \Phi_g\colon \mathrm{MCG}(\Sigma_g) & \mapsto & \mathrm{GL}(2g,\mathbb{Z}) \\ [f\colon \Sigma_g \to \Sigma_g] & \mapsto & \underbrace{\left(f_*\colon \mathrm{H}_1(\Sigma_g;\mathbb{Z}) \to \mathrm{H}_1(\Sigma_g;\mathbb{Z})\right)}_{\in \mathrm{Aut}(\mathrm{H}_1(\Sigma_g;\mathbb{Z}))=\mathrm{GL}(2g,\mathbb{Z})} \end{array}$$

an isomorphism?

It turns out that the answer to Question 146.9 is no. In fact we have already developed all the tools to show that Φ_g is not an epimorphism. At this point it is useful to introduce the following definition.

Definition.

(1) Given $g \in \mathbb{N}$ we write $J_g := \begin{pmatrix} 0 & \mathrm{id}_g \\ -\mathrm{id}_g & 0 \end{pmatrix}$.

(2) We say a matrix $A \in \mathrm{GL}(2g,\mathbb{Z})$ is symplectic[1746] if $A^T J_g A = J_g$.

(3) We define the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$ to be the group[1747] of all symplectic matrices.

Now we can formulate the following proposition.

Proposition 146.10. Let $g \in \mathbb{N}$. The homomorphism $\Phi_g\colon \mathrm{MCG}^+(\Sigma_g) \to \mathrm{GL}(2g,\mathbb{Z})$ takes values in $\mathrm{Sp}(2g,\mathbb{Z})$.

Proof. Recall that as above we use the homology classes represented by the curves $x_1,\dots,x_g$ and $y_1,\dots,y_g$ that are shown in the figure on page 3125 to make the identification $\mathbb{Z}^{2g} = \mathrm{H}_1(\Sigma_g;\mathbb{Z})$. It follows from the calculation of the intersection form of Σ_g on page 2881 that we have a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}^{2g} \times \mathbb{Z}^{2g} & \xrightarrow{(v,w)\mapsto v^T J w} & \mathbb{Z} \\ \downarrow{=} & & \downarrow{=} \\ \mathrm{H}_1(\Sigma_g;\mathbb{Z}) \times \mathrm{H}_1(\Sigma_g;\mathbb{Z}) & \xrightarrow{Q_{\Sigma_g}} & \mathbb{Z}. \end{array}$$

Now let f be an orientation-preserving self-homeomorphism of Σ_g. We denote by A the matrix that represents the induced map $f_*\colon \mathrm{H}_1(\Sigma_g;\mathbb{Z}) \to \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ with respect to the above basis.

It is a consequence of Lemma 132.6 that the induced map f_* preserves the intersection form of Σ_g. Therefore we obtain from the commutative diagram that the induced map

[1746] The origin of the word "symplectic" is explained in the following webpage

`http://jeff560.tripod.com/mathword.html`

which explores the earliest known uses of some of the words of mathematics.

[1747] It is straightforward to verify that the set of all symplectic $(2g \times 2g)$-matrices is in fact a subgroup of $\mathrm{GL}(2g,\mathbb{Z})$.

on $\mathrm{H}_1(\Sigma_g;\mathbb{Z}) = \mathbb{Z}^{2g}$ also preserves the form on the top. But this means that for any two vectors $v, w \in \mathbb{Z}^g$ we have $v^T A^T J A w = (Av)^T J A w = v^T J w$. This in turn implies that $A^T J A = J$. ■

The question now arises, how big is the symplectic group? The following proposition summarizes some known results:

Proposition 146.11. (Symplectic Group Proposition) Let $g \in \mathbb{N}$.

(1) We have $\mathrm{Sp}(2g,\mathbb{Z}) \subset \mathrm{SL}(2g,\mathbb{Z})$.
(2) We have $\mathrm{Sp}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})$.
(3) For $g \geq 2$ the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$ is a proper subgroup of $\mathrm{SL}(2g,\mathbb{Z})$.

Proof.

(1) The fact that every symplectic matrix has determinant equal to $+1$ is a somewhat non-trivial exercise in linear algebra. A proof is for example given in [**Serd2010**, Corollary 10.2] or [**McS2016**, Lemma 1.1.15].
(2) A short and amusing calculation (worth doing!) shows that every matrix in $\mathrm{SL}(2,\mathbb{Z})$ is in fact symplectic.
(3) Let $g \geq 2$. Unless one is very unlucky, almost any randomly picked matrix in $\mathrm{SL}(2g,\mathbb{Z})$ will not lie in $\mathrm{Sp}(2g,\mathbb{Z})$. For example an elementary calculation shows that the matrix

$$A := \begin{pmatrix} 2 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

does not satisfy $A^T J A = J$, i.e. A does not lie in $\mathrm{Sp}(4,\mathbb{Z})$. A straightforward modification of this example shows that given any $g \geq 2$ there exist matrices in $\mathrm{SL}(2g,\mathbb{Z})$ that do not lie in $\mathrm{Sp}(2g,\mathbb{Z})$. ■

In a last ditch effort to salvage Question 146.9 one could ask whether perhaps the map $\Phi_g \colon \mathrm{MCG}(\Sigma_g)^+ \to \mathrm{Sp}(2g,\mathbb{Z})$ is an isomorphism.
Now is perhaps a good moment to recall that we had already constructed diffeomorphisms of Σ_g. More precisely we recall the following definition from page 471.

Definition. Let Σ be an oriented surface and let $C \subset \Sigma$ be a closed curve on Σ, i.e. C is a submanifold of Σ diffeomorphic to S^1. We pick an orientation for C. By the Tubular Neighborhood Theorem **??** there exists an orientation-preserving smooth embedding $\Phi \colon [-1,1] \times S^1 \to \Sigma$ such that $\Phi \colon \{0\} \times S^1 \to C$ is an orientation-preserving diffeomorphism. By the Smooth Transition Function Lemma 19.4 we can pick a smooth map $\gamma \colon [-1,1] \to [0,2\pi]$ and with $\gamma(-1) = 0$ and $\gamma(1) = 2\pi$ that is constant close to -1 and 1.

(1) We refer to map

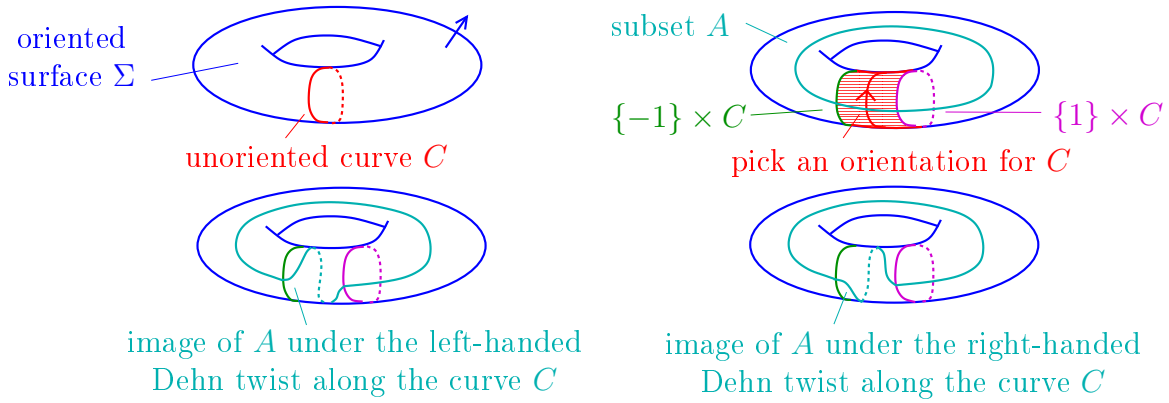
$$\begin{array}{ccc} \Sigma & \to & \Sigma \\ P & \mapsto & \begin{cases} P, & \text{if } P \notin \Phi([-1,1] \times S^1), \\ \Phi(t, z \cdot \exp(\gamma(t) \cdot \mathrm{i})) & \text{if } P = \Phi(t,z) \text{ for some } t \in [-1,1] \text{ and } z \in S^1, \end{cases} \end{array}$$

as a left-handed Dehn twist along the curve C.

(2) We refer to the orientation-preserving diffeomorphism

$$\begin{array}{ccc} \Sigma & \to & \Sigma \\ P & \mapsto & \begin{cases} P, & \text{if } P \notin \Phi([-1,1] \times S^1), \\ \Phi(t, z \cdot \exp(-\gamma(t) \cdot \mathrm{i})) & \text{if } P = \Phi(t,z) \text{ for some } t \in [-1,1] \text{ and } z \in S^1, \end{cases} \end{array}$$

as a right-handed Dehn twist along the curve C.

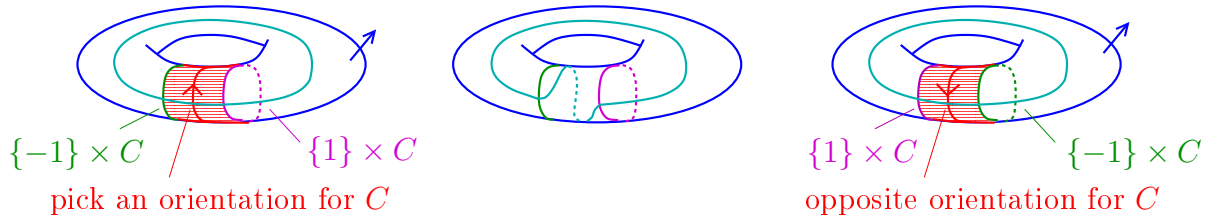


Remark.

(1) Dehn twists were introduced by Max Dehn [**Deh1938**, Chapter 2] in 1938.
(2) Our conventions of right-handed and left-handed Dehn twists match the definitions in [**BZH2014**, Definition 11.6] and in [**FaM2011**, Chapter 3.1.1]. At first glance our definition might look slightly different, but the only difference is that we prefer to work with $[-1,1] \times S^1$ instead of $S^1 \times [-1,1]$, since we always want to put the factor with non-empty boundary first.

Lemma 146.12. (Dehn Twist Lemma) Let Σ be an oriented surface and let C be a submanifold of Σ diffeomorphic to S^1.

(1) Both Dehn twists are orientation-preserving diffeomorphisms and the left and right-handed Dehn twists are inverses of one another.
(2) The diffeotopy class of the left-handed Dehn twist along C neither depends on the choice of the orientation nor does it depend on the choice of the tubular neighborhood. In other words, the unoriented curve C gives rise to a well-defined element of $\mathrm{MCG}^+(\Sigma)$. The same statement also holds for right-handed Dehn twists.



Proof.

(1) It follows easily from the fact that γ is constant near the ends and the Smooth Pasting Proposition 19.23 that both Dehn twists are smooth. Furthermore it follows

from the Basics-of-Orientations Lemma 25.13 (3) that each Dehn twist is orientation-preserving. Finally note one can easily verify that the left-handed and right-handed Dehn twists are inverses of one another.

(2) In the following we will deal with left-handed Dehn twists. The argument for right-handed Dehn twists is evidently almost the same.

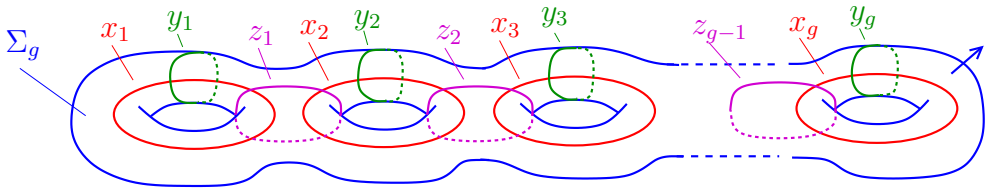
First note that by the usual interpolation one can show easily that the definition does not depend on the choice of γ. Next we show that the definition does not depend on the choice of the orientation of C. Indeed, if we pick the opposite orientation for C, then we need to replace the map $\Phi\colon [-1,1]\times S^1 \to \Sigma$ by the map $\widetilde{\Phi}$ that is given by $\widetilde{\Phi}(t,z) = \Phi(-t,\overline{z})$ and we need to replace γ by the map that is given by $\widetilde{\gamma}(t) = 2\pi - \gamma(-t)$. We claim that Φ, $\widetilde{\Phi}$ and $\widetilde{\gamma}$ give rise to the same self-diffeomorphism of Σ. Thus let $P \in M$. Since Φ and $\widetilde{\Phi}$ have the same image we only need to consider the case that $P = \Phi(t,z)$ for some $t \in [-1,1]$ and $z \in S^1$. Note that $P = \widetilde{\Phi}(-t,\overline{z})$. Then we have indeed that

$$\underbrace{\Phi(t, z\cdot\exp(\gamma(t)\cdot \mathrm{i}))}_{\text{definition using } \Phi} \;=\; \Phi(t, z\cdot\overline{\exp(\widetilde{\gamma}(-t))\cdot \mathrm{i})}) \;=\; \underbrace{\widetilde{\Phi}(-t, \overline{z}\cdot\exp(\widetilde{\gamma}(-t)\cdot \mathrm{i}))}_{\text{definition using } \widetilde{\Phi}}.$$

Finally note that it follows easily from the uniqueness of tubular neighborhoods, see Proposition **??**, that any choice of another map $\Phi\colon [-1,1]\times S^1 \to \Sigma$ leads to a diffeotopic self-diffeomorphism of Σ. ■

The following theorem shows that, in some sense, left-handed Dehn twists are all orientation-preserving diffeomorphisms we ever need.

Theorem 146.13. (Lickorish Twist Theorem) The left-handed Dehn twists along the $3g-1$ curves shown in the figure below form a generating set for $\mathrm{MCG}^+(\Sigma_g)$.



Remark. It is significantly harder to show that $\mathrm{MCG}^+(\Sigma_g)$ admits a finite presentation, see [**HT1980**] and [**Waj1983**] for details.

Proof. This proposition proved in 1963 by Raymond Lickorish [**Lic1964**], see also [**FaM2011**, Theorem 4.13] or [**Bir1974**, Theorem 4.3]. ■

In fact this source of orientation-preserving diffeomorphisms is rich enough to prove the following proposition.

Proposition 146.14. For any $g \geq 1$ the above map $\Phi_g\colon \mathrm{MCG}^+(\Sigma_g) \to \mathrm{Sp}(2g,\mathbb{Z})$ is an epimorphism.

Proof. The result is proved in [**FaM2011**, Theorem 6.4]. ■

This leads us to the following definition.

Definition. Given $g \geq 1$ we refer to

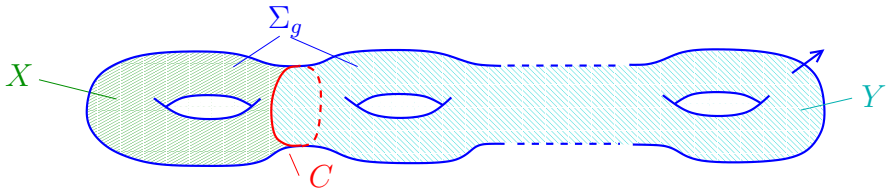
$$T_g := \ker(\Phi_g \colon \mathrm{MCG}(\Sigma_g)^+ \to \mathrm{Sp}(2g, \mathbb{Z}))$$

as the Torelli group of Σ_g.[1748]

The question now becomes, whether the Torelli group T_g is non-trivial. In fact the next proposition shows precisely that.

Proposition 146.15. (Torelli Group Proposition) Let $g \in \mathbb{N}$.

(1) If C is a closed curve on Σ_g that is not null-homotopic, then the left-handed Dehn twist t_C defines a non-trivial element in $\mathrm{MCG}(\Sigma_g)$.
(2) If C is a closed curve on Σ_g that is null-homologous, then the left-handed Dehn twist t_C defines an element of the Torelli group T_g.
(3) For $g \geq 2$ the left-handed Dehn twist along the curve C shown in the figure below defines a non-trivial element in the Torelli group T_g.



Proof.

(1) The first statement is proved in [**FaM2011**, Proposition 3.1].
(2) This statement follows from [**FaM2011**, Proposition 3.2]. With some effort one could also prove this statement "by hand".
(3) By (1) and (2) it suffices to show that C is null-homologous but not null-homotopic. Note that C is the boundary of a submanifold X of Σ_g. We obtain from Corollary 106.28 (1) that $[C] = 0 \in \mathrm{H}_1(X; \mathbb{Z})$, which implies that C is also null-homologous in Σ_g. On the other hand we know by Proposition 55.18 that C is not null-homotopic. ■

There are still lots of open questions regarding mapping class groups of surfaces. For example the following question is an interesting open problem.

Question 146.16. Let Σ_g be the surface of genus $g \in \mathbb{N}_0$. Is the mapping class group $\mathrm{MCG}(\Sigma_g)$ linear, i.e. is it isomorphic to a subgroup of $\mathrm{GL}(n, \mathbb{C})$ for some large enough n?

We had just seen in Theorem 146.7 that the answer is yes for $g = 0$ and $g = 1$. Furthermore Stephen Bigelow and Ryan Budney [**BB2001**] showed in 2001 that the answer is yes for $g = 2$. The case $g \geq 3$ is still wide open.

As in almost any topic we have touched so far, there is again a tremendous amount of literature and work on mapping class groups. We refer to [**Bir1974, FaM2011, Hat2012**] for more information regarding mapping class groups and diffeomorphism groups of smooth manifolds in dimension 2 and in higher dimensions.

[1748] For slightly mysterious reasons this group is named after the Italian mathematician Ruggiero Torelli (1884–1915) who died in World War I.

Exercises for Chapter 146.

Exercise 146.1. Let H_g be the genus g handlebody and let $W = \mathrm{H}_g \cup \mathrm{H}_g$ be the double of H_g. Show that W is diffeomorphic to the connected sum of g copies of $S^1 \times S^2$.

Exercise 146.2. Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. We consider the corresponding lens space $L(p,q)$. In the Lens Space Homology Lemma 98.9 we used a roundabout way to obtain the following result:

$$\mathrm{H}_n(L(p,q)) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n = 0, 3, \\ 0, & \text{if } n = 2 \text{ or } n > 3, \\ \mathbb{Z}_p, & \text{if } n = 1. \end{cases}$$

Use the Lens Space-Splitting Lemma 146.2 and the Mayer–Vietoris Theorem 78.9 for Manifolds to give an alternative, and arguably more insightful, calculation of the homology groups of $L(p,q)$.
Remark. You might want to use the Action-on-Torus Homology Lemma 84.7.

Exercise 146.3. Let M be a closed oriented connected 3-dimensional smooth manifold. We define its *Heegaard genus* as follows:

$$g(M) \;:=\; \min\{g \in \mathbb{N}_0 \,|\, \text{there exists a Heegaard splitting of } M \text{ of genus } g\}.$$

(a) Show that $g(M)$ is greater of equal than the minimal number of generators of $\mathrm{H}_1(M;\mathbb{Z})$.
Remark. Note that it purposely fully says $g(M)$ instead of the less interesting bound $2g(M)$.
(b) Determine the Heegaard genus the 3-dimensional torus $S^1 \times S^1 \times S^1$.
(c) Given $k \in \mathbb{N}_0$ determine the Heegaard genus of the connected sum of k copies of $S^1 \times S^2$.

Exercise 146.4. Let F be the surface of genus g and let $\varphi\colon F \to F$ be an orientation-preserving self-diffeomorphism. By the Gluing-Smooth Manifolds-Proposition **??** we know that the mapping torus

$$\mathrm{Tor}(F,\varphi) \;:=\; F \times [0,1] \,/\, (x,0) \sim (\varphi(x),1)$$

is a closed orientable connected 3-dimensional smooth manifold. Show that $\mathrm{Tor}(F,\varphi)$ admits a Heegaard splitting of genus $2g+1$.

Exercise 146.5. Let $k, q \in \mathbb{N}$. Show that there exists a smooth embedding of a closed connected 2-dimensional smooth manifold F with $\chi(F) = 2-k$ into the lens space $L(2k,q)$.
Hint. Consider the Heegaard decomposition $L(2k,q) \cong M(S^1 \times \overline{B}^2, S^1 \times \overline{B}^2, \varphi)$ from the Lens Space-Splitting Lemma 146.2. Consider a disk $\{*\} \times \overline{B}^2$ in the first solid torus. Show that the intersection of this disk with the boundary of the second torus can be viewed as a torus knot. (We refer to page 1312 for the definition of a torus knot.)

Exercise 146.6. Let Σ be an oriented surface and let C and D be two closed curves on Σ which intersect transversally. We denote by T_C and T_D corresponding left-handed Dehn twists. Show that in the mapping class group $\mathrm{MCG}(\Sigma)$ the following equality holds:

$$T_C \circ T_D \circ T_C \;=\; T_D \, T_C \, T_D.$$

147. Surgeries on 3-dimensional smooth manifolds

In Chapter **??** we introduced the general setup for surgeries along thickened spheres in smooth manifolds. In this chapter we will study the particularly interesting case of surgery along thickened knots and links in 3-dimensional smooth manifolds.

147.1. **The Lickorish-Wallace Theorem.** We kick off our discussion with the following slight variations on the definition of a thickening of an oriented knot that we gave on page 2853 and of the notion of a surgery along a thickened sphere that we gave on page **??**.

Definition. Let $L \subset S^3$ be an oriented m-component link.

(1) A thickening of the oriented link L is defined as an orientation-preserving smooth embedding $F\colon \bigsqcup_{i=1}^{m}(\overline{B}^2 \times S^1) \to S^3$ that restricts to an orientation-preserving diffeomorphism $\bigsqcup_{i=1}^{m}(\{0\}^2 \times S^1) \to L$.

(2) We define the surgery along a thickened link as the simultaneous surgery along all the thickened components.

The following theorem was first proved independently by William Lickorish [**Lic1962**] and Andrew Wallace [**Walla1960**] in 1960.

Theorem 147.1. (Lickorish-Wallace Theorem) Every closed orientable connected 3-dimensional smooth manifold can be obtained through surgery along a thickened link.

The following lemma contains one of the key geometric ingredients in the proof of the Lickorish-Wallace Theorem 147.1.

Lemma 147.2. Let Σ be the surface of some genus g and let C be an oriented curve on Σ. We denote by $\tau\colon \Sigma \to \Sigma$ a corresponding Dehn twist along C. There exists a fattened 1-sphere $\varphi\colon \overline{B}^2 \times S^1 \to (-1,1) \times \Sigma$ with $\varphi(\{0\} \times S^1) = C$ and a diffeomorphism

$$\Psi\colon ([-1,0] \times \Sigma) \cup_\tau ([0,1] \times \Sigma) \;\to\; ([-1,1] \times \Sigma)(\varphi)$$

which is the identity in a neighborhood of $\{-1,1\} \times \Sigma$.

Remark. As on many other occasions we provide a formal proof of Lemma 147.2 with the full gory details. Less formal, but perhaps more readable proofs, are given in [**Rolf1990**, p. 275] and [**Sav2012**, p. 34].

Proof. To simplify the notation we will identify C with S^1 using an orientation-preserving diffeomorphism. Let $[-1,1] \times S^1 \subset \Sigma$ be a tubular neighborhood of $C = S^1$ and let $\gamma\colon [-1,1] \to [0,2\pi]$ a map as such that the Dehn twist τ is given by the map

$$\begin{array}{rccc} \tau\colon & [-1,1] \times S^1 & \to & [-1,1] \times S^1 \\ & (t,z) & \mapsto & (t, z \cdot \exp(\mathrm{i}\gamma(t))) \end{array}$$

We think of $\overline{B}^2$ and S^1 as subspaces of $\mathbb{C}$. We consider $B^2_+ = \{x + \mathrm{i}y \in \overline{B}^2 \,|\, y \geq 0\}$ and $B^2_- = \{x + \mathrm{i}y \in \overline{B}^2 \,|\, y \leq 0\}$. A variation on the tubular neighborhood theorem shows there exists a tubular neighborhood $\overline{B}^2 \times C \subset (-1,1) \times \Sigma$ such that $\overline{B}^2_- \times C \subset [-1,0] \times \Sigma$ and $\overline{B}^2_+ \times C \subset [0,1] \times \Sigma$ and such that $[-1,1] \times C$ is precisely the tubular neighborhood of C

in Σ. Next we consider the following composition of maps:

$$\begin{array}{rcl}\Theta\colon S^1\times\overline{B}^2 \xrightarrow{(a,b)\mapsto(b,a)} \overline{B}^2\times S^1 \xrightarrow{(c,d)\mapsto(cd^{-1},d)} \overline{B}^2\times S^1 & \to & (\overline{B}^2_+\times S^1)\cup_\tau(\overline{B}^2_-\times S^1)\\ (x{+}iy,z) & \mapsto & \begin{cases}(x{+}iy,z), & \text{if } y\geq 0,\\ (x{+}iy,z\cdot\exp(\mathrm{i}\gamma(x))), & \text{if } y\leq 0.\end{cases}\end{array}$$

Note that Θ is a diffeomorphism, in fact it is straightforward to write down an inverse. Note that Θ restricts to a self-diffeomorphism of the boundary torus $S^1\times S^1$.

Claim. The restriction of Θ to a map $\Theta\colon S^1\times S^1\to S^1\times S^1$ can be extended to a diffeomorphism $\varphi\colon\overline{B}^2\times S^1\to\overline{B}^2\times S^1$.[1749]

Proof. A charming little calculation shows that the induced map on $\mathrm{H}_1(S^1\times S^1)$ is represented by the matrix $\begin{pmatrix}-1 & 1\\ 0 & 1\end{pmatrix}$. Another map that induces this map on $\mathrm{H}_1(S^1\times S^1)$ is given by $\Xi(a,b)=(b\cdot\overline{a},b)$. By Theorem 146.7 we know that there exists a diffeotopy $F\colon S^1\times S^1\times[0,1]\to S^1\times S^1$ with $F_0=\Xi\circ\Theta^{-1}$ and $F_1=\mathrm{id}$. The map

$$\begin{array}{rcl}\varphi\colon\overline{B}^2\times S^1 & \to & \overline{B}^2\times S^1\\ (z,w) & \mapsto & \begin{cases}|z|\cdot F_{2\cdot|z|-1}(\Theta(\frac{z}{|z|},w)), & \text{if } |z|\geq\frac{1}{2}\\ (w\cdot\overline{z},w), & \text{if } |z|\leq\frac{1}{2}.\end{cases}\end{array}$$

is the desired extension. ⊞

We will now see that the map $\varphi\colon\overline{B}^2\times S^1\to[-1,2]\times\Sigma$ is the desired framing. To do so we set $W:=([-1,1]\times\Sigma)\setminus(\overline{B}^2\times C$. It follows from the above that the following map is a diffeomorphism:

$$\begin{array}{rcl}\Theta\colon\overbrace{((\overline{B}^2_+\times S^1)\cup_\tau(\overline{B}^2_-\times S^1))\cup_{\mathrm{id}\colon S^1\times S^1\to W}W}^{=([-1,0]\times\Sigma)\cup_\tau([0,1]\times\Sigma)} & \to & \overbrace{(S^1\times\overline{B}^2)\cup_{\varphi\colon S^1\times S^1\to W}W}^{=([-1,1]\times\Sigma)(\varphi)}\\ P & \mapsto & \begin{cases}P, & \text{if } P\in W,\\ \psi(P), & \text{otherwise.}\end{cases}\end{array}$$ ■

Now we have all the ingredients to provide the proof of the Lickorish-Wallace Theorem 147.1.

Proof of the Lickorish-Wallace Theorem 147.1. Let M be a closed orientable connected 3-dimensional smooth manifold. The first three steps of the proof are straightforward.

(1) By Theorem 146.3 we know that M admits a Heegaard splitting, as defined on page 3114. By Lemma 146.1 this means that there exist two handlebodies H and H' of some genus g and an orientation-reversing diffeomorphism $\varphi\colon\partial H\to\partial H'$ such that M is diffeomorphic to $H\cup_\varphi H'$.

(2) By the discussion on page 3115 we know that S^3 admits Heegaard splittings of any genus. It follows from Lemma 146.1 (1d) that there exists an orientation-reversing diffeomorphism $\psi\colon\partial H\to\partial H'$ such that S^3 is diffeomorphic to $H\cup_\psi H'$.

(3) We write $\Sigma=\partial H$.

[1749] Note the all important change in the order of S^1 and $\overline{B}^2$.

(4) By the Lickorish Twist Theorem 146.13 we know that the mapping class group $\mathrm{MCG}^+(\Sigma)$ is generated by Dehn twists. Note that the inverse of a Dehn twist along an oriented curve c is given by a Dehn twist along c with the opposite orientation. This implies that there exist oriented curves $C_1, \dots, C_k$ on ∂H such that the composition of Dehn twists along $C_1, \dots, C_k$ is diffeotopic to $\psi^{-1} \circ \varphi \colon \Sigma \to \Sigma$.
(5) We pick orientation-preserving tubular neighborhoods $[-1,1] \times C_i$. Furthermore we pick a smooth map $\gamma \colon [-1,1] \to [0, 2\pi]$ with $\gamma(-1) = 0$ and $\gamma(1) = 2\pi$ which is constant close to -1 and 1.
(6) For $i = 1, \dots, k$ we denote by t_i the corresponding Dehn twist along C_i.
(7) By the Smooth Collar Neighborhood Theorem 28.3 we can pick a Smooth Collar Neighborhood $\Sigma \times [0, k+1] \times \Sigma$ of $\Sigma = \partial H$.

The desired statement now follows from the above, together with Lemma **??** and the following claim.

Summarizing we have the following sequence of orientation-preserving diffeomorphisms.

$$
\begin{array}{l}
\text{follows from a collar neighborhood argument} \qquad\qquad \text{follows easily from } \psi^{-1} \circ \varphi = t_k \circ \cdots \circ t_1 \\
M \cong H \cup_{\varphi \colon \partial H \to \partial H'} H' \overset{\downarrow}{\cong} H \cup_{\mathrm{id}} ([0,k+1] \times \Sigma) \cup_{\varphi} H' \overset{\downarrow}{\cong} H \cup_{\mathrm{id}} ([0,k+1] \times \Sigma) \cup_{\psi \circ t_k \circ \cdots \circ t_1} H' \\
\cong H \cup_{\mathrm{id}} ([0,1] \times \Sigma) \cup_{t_1} ([1,2] \times \Sigma) \cup_{t_2} ([2,3] \times \Sigma) \cdots \cup_{t_k} ([k,k+1] \times \Sigma) \cup_{\psi} H' \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}^{\cong H \cup_{\mathrm{id}} ([0,k+1] \times \Sigma) \cup_{\psi} H' \cong H \cup_{\psi} H' \cong S^3} \\
\cong \text{surgery of } \overbrace{H \cup_{\mathrm{id}} ([0,1] \times \Sigma) \cup_{\mathrm{id}} ([1,2] \times \Sigma) \cup_{\mathrm{id}} ([2,3] \times \Sigma) \cdots \cup_{\mathrm{id}} \cup ([k,k+1] \times \Sigma) \cup_{\psi} H'}^{\text{along thickenings of the oriented curves } \{1\} \times C_1, \dots, \{k\} \times C_k.} \\
\uparrow \\
\text{by Lemma 147.2}
\end{array}
$$

We have thus shown that M is diffeomorphic to the surgery of S^3 along a thickened link. ∎

Theorem 147.3. (3-Manifolds Bound Theorem) Every closed orientable connected 3-dimensional smooth manifold is the boundary of a compact orientable connected 4-dimensional smooth manifold.

Remark.

(1) The theorem was first proved by Vladimir Rokhlin [**Rok1951**] in 1951. The argument is also sketched in [**GoS1999**, p. 159]. Alternative proofs are given by René Thom [**Tho1954b, Tho2007**], also particularly short proof that $\Omega_3 = 0$ is given by Colin Rourke [**Rou1985**].
(2) We will prove the 3-Manifolds Bound Theorem 147.3 via the Lickorish-Wallace Theorem 147.1. In Exercise 147.1 we will see that one can also go easily the other way, namely we will deduce the Lickorish-Wallace Theorem 147.1 from the 3-Manifolds Bound Theorem 147.3.

Proof. Let M be a closed orientable connected 3-dimensional smooth manifold. By the Lickorish-Wallace Theorem 147.1 we know that M can be obtained as the surgery of S^3 along a thickened link. This means that there exist orientation-preserving embeddings $\varphi_1, \dots, \varphi_m \colon \overline{B}^2 \times S^1 \to S^3$ with disjoint images such that M is the diffeomorphic to the surgery of S^3 along these thickened knots. As in the proof of the Handle-Surgery Proposition **??** we see that M is the boundary of the result of attaching 2-handles to $\overline{B}^4$ along these thickened curves. ∎

147.2. **Dehn surgery along knots.** In the remainder of this chapter we will only discuss the case of surgery along knots and links in S^3. An argument as in Exercise **??** shows that we do not loose anything if we restrict ourselves to orientation-preserving thickenings. To simplify the discussion we henceforth say "thickening" instead off "orientation-preserving thickening".
In Section 131.1 we discussed in detail the thickenings of knots. In the following we recall the main definitions and results from that section.

Definition. Let $K \subset S^3$ be an oriented knot.

(1) A thickening of K is an orientation-preserving smooth embedding $F\colon \overline{B}^2 \times S^1 \to S^3$ that restricts to an orientation-preserving diffeomorphism $\{0\} \times S^1 \to K$.[1750]
(2) Given a thickening F of K we refer to $X_K := S^3 \setminus F(B^2 \times S^1)$ as the knot exterior.
(3) Given a thickening F of K and given $* \in K$ we refer to the oriented submanifold $\mu_K := F(S^1 \times \{*\})$ of $S^3 \setminus K$ as the meridian of K.
(4) By Lemma 100.16 (5) the inclusion induced map $\mathrm{H}_1(X_K;\mathbb{Z}) \xrightarrow{\cong} \mathrm{H}_1(S^3 \setminus K;\mathbb{Z})$ is an isomorphism and we have $\mathrm{H}_1(X_K;\mathbb{Z}) = \mathbb{Z} \cdot [\mu_K]$. As on page 2853 we denote by ϕ_K the isomorphism
$$\begin{array}{rcl} \phi_K\colon \mathrm{H}_1(S^3 \setminus K;\mathbb{Z}) = \mathrm{H}_1(X_K;\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z} \\ [\mu_K] & \to & 1. \end{array}$$
(5) Suppose we are given a thickening F for K. As on page 2853 we define
$$\text{self-linking number of } F \ := \ \phi_K(\underbrace{F_*([\{1\} \times S^1])}_{\in \mathrm{H}_1(X_K;\mathbb{Z})}).$$

Lemma 131.1. (Knot Thickening Lemma) Let K be an oriented knot in S^3.

(1) Let $F\colon \overline{B}^2 \times S^1 \to S^3$ be a thickening of K and let $s \in \mathbb{Z}$. If we set
$$\begin{array}{rcl} \Psi\colon \overline{B}^2 \times S^1 & \to & \overline{B}^2 \times S^1 \\ (z,w) & \mapsto & (z \cdot w^s, w), \end{array}$$
then the map $F \circ \Psi$ is a thickening with
$$\text{self-linking number of } F \circ \Psi \ = \ \text{self-linking number of } F + s.$$
(2) Given any $n \in \mathbb{Z}$ there exists a thickening of K with self-linking number n.
(3) If $F, F'\colon \overline{B}^2 \times S^1 \to S^3$ are two thickenings with self-linking number n, then there exists a diffeotopy $H\colon S^3 \times [0,1] \to S^3$ rel K with $H_0 = \mathrm{id}$ and $H_1 \circ F = F'$.

Remark. Given an oriented knot K and $n \in \mathbb{Z}$ sometimes we use the uniqueness statement of the Knot Thickening Lemma 131.1 to talk about *the thickening of K with self-linking number n*. Sometimes we shorten this cumbersome expression to *the n-thickening of K*.

Now we can also recall the following definition from page 2854.

Definition. Let $K \subset S^3$ be an oriented knot. We pick a thickening $F\colon \overline{B}^2 \times K \to S^3$ of self-linking number zero. We refer to $\lambda_K := F(\{1\} \times K)$ as longitude of K.

[1750] Readers who live on a carrot-rich diet will spot that the definition differs slightly from the definition on page 2853. But the difference is immaterial.

Let $K \subset S^3$ be an oriented knot and let $p, q \in \mathbb{N}_0$ be coprime. Our goal now is to define the "p/q-surgery $S^3_{p/q}(K)$ of S^3 along K" as the result of gluing the solid torus $S^1 \times \overline{B}^2$ in such a way to the knot exterior X_K such that boundary of $\overline{B}^2$ corresponds to $p \cdot \mu + q \cdot \lambda$. The following proposition turns this idea into a valid concept.

Proposition 147.4. (Dehn Surgery Proposition) Let $K \subset S^3$ be an oriented knot and let $p, q \in \mathbb{Z}$ be coprime. We pick a thickening $\Phi\colon \overline{B}^2 \times S^1 \to S^3$ of K of self-linking number zero.

(1) There exists an orientation-preserving diffeomorphism $\Theta\colon S^1 \times S^1 \to S^1 \times S^1$ such that $(\Phi \circ \Theta)_*([\{*\} \times S^1]) = p \cdot [\mu] + q \cdot [\lambda]$. We set
$$S^3_{p/q}(K) \;:=\; (S^1 \times \overline{B}^2) \cup_{\Phi \circ \Theta|_{S^1 \times S^1}} (\underbrace{S^3 \setminus \Phi(B^2 \times S^1)}_{=X_K}).$$
(2) $S^3_{p/q}(K)$ is a closed 3-dimensional smooth manifold which admits a unique orientation that restricts to the usual orientation of $X_K = S^3 \setminus \Phi(B^2 \times S^1)$.
(3) The oriented diffeomorphism type $of S^3_{p/q}(K)$ depends neither on the choice of the thickening of self-linking number zero nor on the choice of the orientation-preserving diffeomorphism Θ with $(\Phi \circ \Theta)_*([\{*\} \times S^1]) = p \cdot [\mu] + q \cdot [\lambda]$.
(4) There exists an orientation-preserving diffeomorphism $S^3_{p/q}(K) \to S^3_{-p/-q}(K)$.
(5) If $-K$ denotes the knot K with the opposite orientation, then $S^3_{p/q}(K)$ admits an orientation-preserving diffeomorphism to $S^3_{p/q}(-K)$.
(6) We have $\pi_1(S^3_{p/q}(K)) \cong \pi_1(X_K)/\langle\langle \mu_K^p \rangle\rangle$ and $\mathrm{H}_1(S^3_{p/q}(K)) \cong \mathbb{Z}_q \cdot [\mu_K]$.
(7) If $q = 1$, then $S^3_{p/1}(K)$ is the surgery on S^3 along the thickening of K of self-linking number p.

Proof.

(1) Since $p, q \in \mathbb{N}_0$ are coprime there exist $r, s \in \mathbb{N}_0$ with $qr - ps = 1$. We consider the map
$$\begin{aligned} \Theta\colon S^1 \times S^1 &\to S^1 \times S^1 \\ (z, w) &\mapsto (z^r \cdot w^p, z^s \cdot w^q). \end{aligned}$$
It follows from $qr - ps = 1$ and the Maps-on-Torus Lemma 19.37 that this map is an orientation-preserving diffeomorphism. Furthermore it follows easily from the Action-on-Torus Homology Lemma 84.7 (2) that $(\Phi \circ \Theta)_*([\{*\} \times S^1]) = p \cdot [\mu] + q \cdot [\lambda]$.
(2) This statement follows immediately from the Gluing-Smooth Manifolds-Proposition **??**.
(3) It follows easily from the Knot Thickening Lemma 131.1 that the definition does not depend on the choice of the thickening of self-linking number zero.

It remains to show that the choice of Θ is irrelevant. This proof consists of two steps. First we show that choice of r and s in (1) is irrelevant. Indeed, suppose we are given $\widetilde{r}, \widetilde{s} \in \mathbb{N}_0$ with $q\widetilde{r} - p\widetilde{s} = 1$. It follows that $qr - ps = q\widetilde{r} - p\widetilde{s}$, i.e. $q(\widetilde{r} - r) = p(\widetilde{s} - s)$. Since q and p are coprime we see that $\widetilde{r} = r + p \cdot m$ and $\widetilde{s} = s + q \cdot m$ for some $m \in \mathbb{Z}$. We define $\Theta_{r,s}$ and $\Theta_{\widetilde{r},\widetilde{s}}$ in the obvious way. It is

straightforward to verify that the map

$$\begin{array}{rcl}(S^1\times\overline{B}^2)\cup_{\Theta_{r,s}\circ\Phi|_{S^1\times S^1}}(S^3\setminus\Phi(B^2\times S^1)) & \to & (S^1\times\overline{B}^2)\cup_{\Theta_{\tilde{r},\tilde{s}}\circ\Phi|_{S^1\times S^1}}(S^3\setminus\Phi(B^2\times S^1))\\ P & \mapsto & \begin{cases} P, & \text{if } P\in S^3\setminus\Phi(B^2\times S^1),\\ (z,wz^m), & \text{if } P=(z,w)\in S^1\times\overline{B}^2\end{cases}\end{array}$$

is a diffeomorphism.

Now let $\Xi\colon S^1\times S^1\to S^1\times S^1$ be any orientation-preserving diffeomorphism $\Xi\colon S^1\times S^1\to S^1\times S^1$ with $(\Phi\circ\Xi)_*([\{*\}\times S^1])=p\cdot[\mu]+q\cdot[\lambda]$. We write $(\Phi\circ\Xi)_*([\{*\}\times S^1])=r\cdot[\mu]+s\cdot[\lambda]$. Since Θ is orientation-preserving we know by Lemma 114.15 that $qr-ps=1$. We use r and s to define the orientation-preserving diffeomorphism $\Theta\colon S^1\times S^1\to S^1\times S^1$ as above. Since Ξ and Θ are both orientation-preserving diffeomorphisms of $S^1\times S^1$ that induce the same map on $\mathrm{H}_1(S^1\times S^1)$ we know by Theorem 146.7 we know that there exists a diffeotopy $F\colon S^1\times S^1\times[0,1]\to S^1\times S^1$ with $F_0=\Xi\circ\Theta^{-1}$ and $F_1=\mathrm{id}$. It is straightforward to verify that the map

$$\begin{array}{rcl}(S^1\times\overline{B}^2)\cup_{\Theta\circ\Phi|_{S^1\times S^1}}(S^3\setminus\Phi(B^2\times S^1)) & \to & (S^1\times\overline{B}^2)\cup_{\Xi\circ\Phi|_{S^1\times S^1}}(S^3\setminus\Phi(B^2\times S^1))\\ P & \mapsto & \begin{cases} P, & \text{if } P\in S^3\setminus\Phi(B^2\times S^1),\\ |w|\cdot F_{2\cdot|w|-1}(z,\frac{w}{|w|}), & \text{if } P=(z,w) \text{ with } |w|\geq\frac{1}{2}\\ (z,w), & \text{if } P=(z,w) \text{ with } |w|\leq\frac{1}{2}.\end{cases}\end{array}$$

is a diffeomorphism.

(4) We consider the map

$$\begin{array}{rcl}\Xi\colon S^1\times S^1 & \to & S^1\times S^1\\ (z,w) & \mapsto & (\overline{z}^r\cdot\overline{w}^p,\overline{z}^s\cdot\overline{w}^q).\end{array}$$

It is straightforward to see that $(\Phi\circ\Xi)_*([\{*\}\times S^1])=-p\cdot[\mu]-q\cdot[\lambda]$. The map $\Xi\circ\Theta^{-1}\colon S^1\times S^1\to S^1\times S^1$ is given by $(z,w)\mapsto(\overline{z},\overline{w})$. Since this map extends to a diffeomorphism of $S^1\times\overline{B}^2$ in the obvious way. This observation now gives us the following orientation-preserving diffeomorphism

$$\begin{array}{rcl}\overbrace{(S^1\times\overline{B}^2)\cup_{\Theta\circ\Phi|_{S^1\times S^1}}(S^3\setminus\Phi(B^2\times S^1))}^{=S^3_{p/q}(K)} & \to & \overbrace{(S^1\times\overline{B}^2)\cup_{\Xi\circ\Phi|_{S^1\times S^1}}(S^3\setminus\Phi(B^2\times S^1))}^{=S^3_{-p/-q}(K)}\\ P & \mapsto & \begin{cases} P, & \text{if } P\in S^3\setminus\Phi(B^2\times S^1),\\ (\overline{z},\overline{w}), & \text{if } P=(z,w)\in S^1\times\overline{B}^2.\end{cases}\end{array}$$

(5) In the Longitude-Meridian Lemma 131.2 (6) we showed that swapping the orientation of K has the effect of swapping the orientation of the meridian and the longitude. Thus we see that this statement follows immediately from Statement (4).

(6) We have the following isomorphisms:

$$\begin{array}{ccccccc}
 & & \text{by the Seifert–van Kampen Theorem 54.4, where} & & & & \\
 & & \text{map } \pi_1(S^1\times S^1)\to\pi_1(X_K) \text{ is induced by the map } \varphi & & & & \\
 & & \downarrow & & & & \\
\pi_1(S^3_{p/q}(K)) & \cong & \pi_1(S^1\times\overline{B}^2)*_{\pi_1(S^1\times S^1)}\pi_1(X_K) & \cong & \langle s\rangle *_{\langle s\rangle\times\langle t\rangle}\pi_1(X_K) & & \\
 & \cong & \{e\}*_{\langle\langle t\rangle\rangle}\pi_1(X_K) & \cong & \pi_1(X_K)/\langle\langle\varphi_*(t)\rangle\rangle & \cong & \pi_1(X_K)/\langle\langle\mu_K^p\rangle\rangle. \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \text{by the Amalgamated Product} & & \text{by the Amalgamated Product} & & \text{by definition of } \varphi \\
 & & \text{Lemma 53.22 (5) and (6)} & & \text{Lemma 53.22 (2)} & &
\end{array}$$

The statement regarding $\mathrm{H}_1(S^3_{p/q}(K))$ can be deduced easily from the above calculation together with the Hurewicz Theorem 84.5. Alternatively it is of course also a great pleasure to prove the statement directly using the Mayer-Vietoris Theorem 78.9.

(7) Suppose $q=1$. Then we can set $r=1$ and $s=0$. In this case the above map Θ extends to the diffeomorphism

$$\begin{array}{rcl}\Theta\colon \overline{B}^2\times S^1 & \to & \overline{B}^2\times S^1\\ (z,w) & \mapsto & (z\cdot w^p, w^q).\end{array}$$

Note that by the Knot Thickening Lemma 131.1 (1) we know that $\Phi\circ\Theta$ is a thickening of K of self-linking number p. We have

$$\begin{array}{ccccccc}
(S^3)(\Phi\circ\Theta) & = & (S^1\times\overline{B}^2)\cup_{\Theta\circ\Phi|_{S^1\times S^1}}(S^3\setminus\overbrace{(\Phi\circ\Theta)(B^2\times S^1)}^{=\Phi(B^2\times S^1)}) & = & S^3_{p/q}(K). & & \blacksquare \\
\uparrow & & & & \uparrow & & \\
\text{see definition on page ??} & & & & \text{by definition of } S^3_{p/q}(K) & &
\end{array}$$

Definition. Let $K\subset S^3$ be an knot and let $r\in\mathbb{Q}\cup\{\infty\}$. We pick an orientation for K.

(1) If $r\in\mathbb{Q}$, then we pick coprime $p,q\in\mathbb{Z}$ with $r=\frac{p}{q}$ and we define $S^3_r(K):=S^3_{p/q}(K)$.
(2) For $r=\infty$ we define $S^3_r(K):=S^3_{1/0}(K)$.

It follows from the Dehn Surgery Proposition 147.4 (4) and (5) that, up to an orientation-preserving diffeomorphism, the definition of $S^3_r(K)$ does not depend on any of the choices we just made. In either case we refer to $S^3_r(K)$ as the r-Dehn surgery along K. Sometimes we refer to this procedure as a rational Dehn surgery instead of just Dehn surgery. If $r\in\mathbb{Z}$, then we refer to this Dehn surgery as an integral Dehn surgery.

The following lemma gives us a few basic examples.

Lemma 147.5.

(1) Let $p,q\in\mathbb{N}$ be coprime. The $\frac{p}{q}$-Dehn surgery along the unknot admits an orientation-preserving diffeomorphism to the lens space $L(p,q)$.
(2) Given any $n\in\mathbb{Z}$ the $\frac{1}{n}$-Dehn surgery along the unknot is diffeomorphic to S^3.
(3) Given any knot $K\in S^3$ the ∞-Dehn surgery is diffeomorphic S^3.

Proof.

(1) Let $p\in\mathbb{N}$ and let $q\in\mathbb{Z}$ be coprime to p. We pick $r,s\in\mathbb{Z}$ such that $qr-ps=-1$. We consider the map

$$\begin{array}{rcl}\Theta\colon S^1\times S^1 & \to & S^1\times S^1\\ (z,w) & \mapsto & (z^r\cdot w^p, z^s\cdot w^q).\end{array}$$

We write

$$A = \begin{pmatrix} q & p \\ s & r \end{pmatrix} \quad \text{and we consider the orientation-reversing diffeomorphism} \quad \begin{array}{rcl} \varphi\colon \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ [v] & \mapsto & [Av]. \end{array}$$

If we make the usual identification $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$, then there exists an orientation-preserving diffeomorphism

$$L(p,q) \xrightarrow{\cong} M(S^1 \times \overline{B}^2, S^1 \times \overline{B}^2, \varphi).$$

This statement follows immediately from the Lens Space-Splitting Lemma 146.2.

(2) This statement follows immediately from (1). For entertainment we will also prove this statement in Exercise 147.2.

(3) Let K be a knot. We pick an orientation and a thickening $\Phi\colon \overline{B}^2 \times S^1 \to S^3$ of self-linking number 0. We consider the map $\Theta\colon S^1 \times S^1 \to S^1 \times S^1$ that is given by swapping the coordinates, i.e. it is given by $\Theta(z,w) = (w,z)$. It follows almost immediately from the definitions that $(\Phi \circ \Theta)_*([\{*\} \times S^1]) = 1 \cdot [\mu] + 0 \cdot [\lambda]$. Thus we see that

$$\begin{array}{rcl} S^3_\infty(K) = S^3_{1/0}(K) & \cong & (S^1 \times \overline{B}^2) \cup_{\Phi\circ\Theta|_{S^1\times S^1}} (S^3 \setminus \Phi(B^2 \times S^1)) \\ & \cong & (\overline{B}^2 \times S^1) \cup_{\Phi|_{S^1\times S^1}} (S^3 \setminus \Phi(B^2 \times S^1)) \cong S^3. \\ & \uparrow & \end{array}$$

diffeomorphism given by the obvious extension of Θ to a diffeomorphism $S^1 \times \overline{B}^2 \to \overline{B}^2 \times S^1$ ■

The following proposition shows that Dehn surgeries can behave rather unpredictably.

Proposition 147.6. Let $r, s \in \mathbb{N}$ be coprime and let $T(r,s)$ be the (r,s)-torus knot as defined on page 1312. Furthermore let $p, q \in \mathbb{Z}$.

(1) If $|rsp+q| = 1$, then the p/q-Dehn surgery along $T(r,s)$ is diffeomorphic to the lens space $L(|q|, ps^2)$.

(2) If $|rsp+q| = 0$, then the p/q-Dehn surgery along $T(r,s)$ is diffeomorphic to the connected sum $L(r,s)\#L(s,r)$.

Proof. The proof is basically elementary and can be performed as a challenging exercise. Alternatively the proposition is proved in [**Mos1971**, Propositions 3.2 and 4]. ■

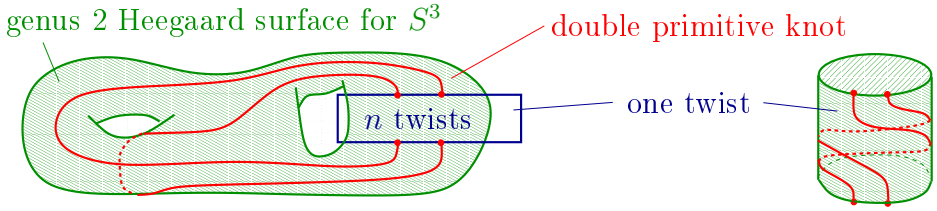
Now the following subtle questions arises: which manifolds can arise through Dehn surgeries along knots? We state two very difficult sample theorems in this direction before we state two open problems, to give the reader a flavor of what is know and what is still left.

Theorem 147.7. (Property P Theorem) Let $K \subset S^3$ be a knot and let $r \in \mathbb{Q}$. If $\pi_1(S^3_r(K))$ is trivial, then K is smoothly isotopic to the unknot.

Proof. Let $K \subset S^3$ be a knot and let $r \in \mathbb{Q}$. Cameron Gordon and John Luecke [**GL1989**, Theorem 2] showed in the 1980s that if $S^3_r(K)$ is diffeomorphic to S^3, then K is the unknot. Thus the Property P Theorem follows from this fact together with the proof of the Poincaré Conjecture by Grigori Perelman in 2004, see Theorem **??**. An alternative proof of the Property P Theorem was given by Peter Kronheimer and Tomasz Mrowka [**KrM2004**], building on the Cyclic Surgery Theorem [**CGLS1987**]. ■

We just saw that all lens spaces arise as Dehn surgery along the unknot. The question arises, which knots admit a surgery that is diffeomorphic to a lens space? In 1990 John Berge [**Berge1990**] proved the following result.

Proposition 147.8. Let $K \subset S^3$ be a knot that lies on a Heegaard surface of genus two, such that for each of the two corresponding handlebody the knot K represents a primitive element in the fundamental group of the handlebody. (Such knots are called *double primitive*). Then K admits a Dehn surgery that is a lens space.



Rather surprisingly it is not known whether the converse to the proposition holds. More precisely, the following conjecture, which goes back to [**Berge1990**] and [**Gor1972**, Question 5.5] is still open.

Conjecture 147.9. (Berge Conjecture) If K is a knot that admits a Dehn surgery that is diffeomorphic to a lens space, then K is double primitive.

Even though there is no really good reason why the conjecture should indeed be true there is by now an impressive literature, see e.g. [**Ni2007, Greene2013**] supporting the conjecture.

147.3. **Dehn surgery along links.** We now generalize the above setup from the previous section to links in the obvious way.

Definition.

(1) A framed (oriented) link consists of an (oriented) link $L = L_1 \cup \cdots \cup L_m$ in S^3 together with am m-tuple $r = (r_1, \ldots, r_m)$ where each r_i lies in $\mathbb{Q} \cup \{\infty\}$.
(2) Give an framed link $(L = L_1 \cup \cdots \cup L_m, r = (r_1, \ldots, r_m))$ we define the corresponding Dehn surgery $S^3_r(L)$ by doing the r_i-Dehn surgeries along the components L_i simultaneously, using disjoint thickenings. If all the r_i are integers, then we refer to the Dehn surgery as an integral Dehn surgery along L. As for knots we sometimes refer to a Dehn surgery as a rational Dehn surgery to differentiate it from the special case of an integral Dehn surgery
(3) Let M be a closed smooth manifold. A surgery description for M is a framed link (L, r) such that M admits an orientation-preserving diffeomorphism to $S^3_r(L)$. We say the surgery description is integral if all the r_i lie in $\mathbb{Z}$.

Lemma 147.10. Let (L, r) and $(\widetilde{L}, \widetilde{r})$ be two framed links in S^3 that are separated by a smoothly embedded sphere, i.e. we assume that there exists a smooth embedding $\varphi\colon S^2 \to S^3$ such that L is contained in one component of $S^3 \setminus \varphi(S^2)$ and $\widetilde{L}$ is contained in the other of $S^3 \setminus \varphi(S^2)$. Then $S^3_{r \cup \widetilde{r}}(L \cup \widetilde{L})$ admits an orientation-preserving diffeomorphism to the connected sum $S^3_r(L) \# S^3_{\widetilde{r}}(\widetilde{L})$.

Proof. It follows from the Generalized Smooth Schönflies Theorem 82.5 in dimension 3 that without loss of generality we can assume that $\varphi(S^2)$ is the "equatorial sphere" $S^3_{=0} = \{(x_1, x_2, x_3, x_4) \in S^3 \,|\, x_4 = 0\}$ and that L is contained in $S^3_{\geq 0}$ and that $\widetilde{L}$ is contained in $S^3_{\leq 0}$. Now let $f\colon \overline{B}^3 \to S^3_{\leq 0}$ be an orientation-preserving embedding and let $\rho\colon S^3 \to S^3$ be

the reflection in the $(x_4 = 0)$-hyperplane. We see that

$$S^3_{r\cup\widetilde{r}}(L \cup \widetilde{L}) \underset{\text{since } S^3 = S^3_{\geq 0} \cup_{S^3_{=0}} S^3_{\leq 0}}{\cong} (S^3_r(L) \setminus f(B^3)) \cup_{S^3_{=0}} (S^3_{\widetilde{r}}(\widetilde{L}) \setminus (\rho \circ f)(B^3)) \underset{\text{by the definition of connected sum}}{\cong} S^3_r(L) \# S^3_{\widetilde{r}}(\widetilde{L}).$$

■

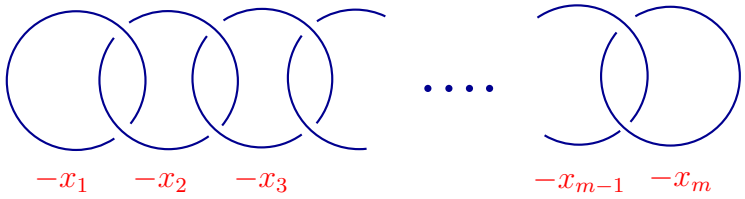
We can now give the following reformulation of the Lickorish-Wallace Theorem 147.1.

Theorem 147.11. (Lickorish-Wallace Theorem) Every closed orientable connected 3-dimensional smooth manifold M can be obtained through integral Dehn surgery along a link. In other words, M admits an integral surgery description.

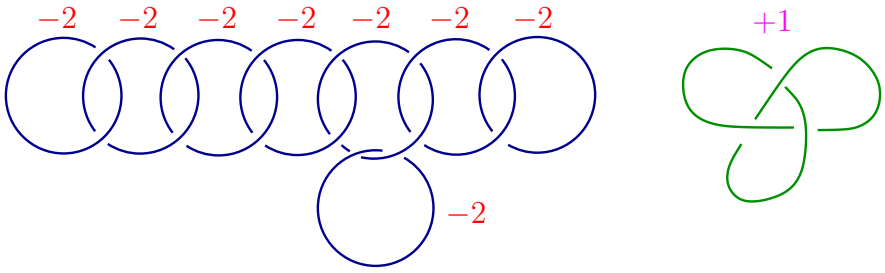
Proof. Let M be a closed orientable connected 3-dimensional smooth manifold. In the proof of the Lickorish-Wallace Theorem 147.1 we showed that there exists a link L together with a thickening such that M is the surgery along this thickened link. But any such surgery is an integral Dehn surgery. In fact reading the proof carefully one sees that all the surgery coefficients are in fact equal to $+1$. ■

Examples.

(1) The Lickorish-Wallace Theorem 147.11 implies in particular that any lens space $L(p, q)$ can be obtained from integral Dehn surgery along a link. One possible way of doing this is as follows: Let $\frac{p}{q} = x_1 - \cfrac{1}{x_2 - \cfrac{1}{x_3 - \cfrac{1}{\cdots - \cfrac{1}{x_n}}}}$ be a continued fraction expansion with $x_1, \ldots, x_n \in \mathbb{Z}$. By [**Sav2012**, Theorem 2.3] we know that the figure below gives a surgery description for the lens space $L(p, q)$.



(2) By [**Sav2012**, p. 49] or [**KSc1979**] we know that the figure below gives two surgery descriptions for the Poincaré Homology Sphere that we introduced on page 2161.



This short discussion surely wets the reader's appetite for more. We refer the curious reader to [**Sav2012, GoS1999, Rolf1990**] for more examples.

Before we can state our next proposition we need to make a few preparations.

Definition. Let K be an oriented knot and let $\varphi\colon \overline{B}^2 \times S^1 \to S^3 = \partial \overline{B}^4$ be a framing for K. Since $\mathrm{H}_2(\overline{B}^4;\mathbb{Z}) = \mathrm{H}_1(\overline{B}^4;\mathbb{Z}) = 0$ we obtain from the Handle Attachment-H_*-Lemma 136.13 a natural isomorphism $\mathrm{H}_2((\overline{B}^2 \times \overline{B}^2) \cup_{\varphi|_{\overline{B}^2 \times S^1}} \overline{B}^4;\mathbb{Z}) \to \mathrm{H}_1(S^1;\mathbb{Z})$. There exists a unique class in $\mathrm{H}_2((\overline{B}^2 \times \overline{B}^2) \cup_{\varphi|_{\overline{B}^2 \times S^1}} \overline{B}^4;\mathbb{Z})$ that gets sent to $-[S^1]$. We refer to this class as the canonical generator.

Lemma 147.12. Let K be an oriented knot and let $\varphi\colon \overline{B}^2 \times S^1 \to S^3 = \partial \overline{B}^4$ be a framing for K. Let Σ be a Seifert surface for K and let $\widetilde{\Sigma}$ be a push-in of the Seifert surface into $\overline{B}^4$. The submanifold $\widetilde{\Sigma} \cup (\{0\} \times \overline{B}^2)$, with the orientation coming from the Seifert surface, represents the canonical generator of $\mathrm{H}_2((\overline{B}^2 \times \overline{B}^2) \cup_{\varphi|_{\overline{B}^2 \times S^1}} \overline{B}^4;\mathbb{Z})$.

Proposition 147.13. Let $(L = L_1 \cup \cdots \cup L_m, r = (r_1, \ldots, r_m))$ be a framed oriented link. For $i = 1, \ldots, m$ we pick a thickening $\varphi_i\colon \overline{B}^2 \times S^1 \to S^3$ for L_i. We assume that the images are disjoint. Let W be the compact oriented 4-dimensional smooth manifold that is obtained from $\overline{B}^4$ by attaching handles along $\varphi_1, \ldots, \varphi_m$. This smooth manifold has the following properties:

(1) $\partial(W) = S^3_r(L)$.
(2) For $i = 1, \ldots, m$ there exists a unique homology class $\alpha_i \in \overline{H}_2(W;\mathbb{Z})$ that lies in the image of $\mathrm{H}_2(\overline{B}^4 \cup_{\varphi(\overline{B})}$

Exercises for Chapter 147.

Exercise 147.1.

(a) Let X be a compact orientable 4-dimensional smooth manifold that has a handle decomposition consisting of a single 0-handle and k 1-handles. Show that there exists a compact orientable 4-dimensional smooth manifold $\widetilde{X}$ that has a handle decomposition consisting of a single 0-handle and k 2-handles with $\partial \widetilde{X} = \partial X$.
Hint. $\partial(S^1 \times \overline{B}^3) = \partial(\overline{B}^2 \times S^2)$.
(b) Let X be a compact orientable 4-dimensional smooth manifold. Show that there exists a compact orientable 4-dimensional smooth manifold $\widetilde{X}$ without 1-handles which satisfies $\partial X = \partial X$.
(c) Let M be a closed orientable connected 3-manifold that is the boundary a compact orientable 4-dimensional smooth manifold X. Show that there exists a link $L \subset S^3$ such that M is diffeomorphic to surgery along L.

Remark. This exercise shows that one can deduce the Lickorish-Wallace Theorem 147.1 from the 3-Manifolds Bound Theorem 147.3.

Exercise 147.2. Let K be a knot. Show "by hand" that the ∞-Dehn surgery along K is diffeomorphic to S^3.

Exercise 147.3. Let $K \subset S^3$ be a non-trivial knot and let $r \in \mathbb{Q} \cup \{\infty\}$. By the Property P Theorem 147.7 we know in particular that if $S^3_r(K)$ is diffeomorphic to S^3, then $r = \infty$.

Now let K and L be two knots such that the knot exteriors X_K and X_L are diffeomorphic. Show that K is smoothly isotopic to L or to the mirror image of L.
Hint. Show that a knot is determined by its knot exterior X_K *together* with the meridian $\mu_K \subset X_K$.

Exercise 147.4. Let M be a closed oriented 3-dimensional smooth manifold and let be a thickened link. What is surgery presentation for $-M$?

Exercise 147.5. Effect of surgery in the complement of a knot, i.e. get crossing change

148. The linking pairing on rational homology spheres

148.1. **Rational homology spheres.** We recall the following lemma.

Lemma 48.6. Let $p \in \mathbb{N}$ and furthermore let $q, r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$. If $q \equiv \pm r^{\pm 1} \bmod p$,[1751] then $L(p,q)$ and $L(p,r)$ are diffeomorphic.

In Question 48.7 we had asked whether the converse to the Lens Spaces-Diffeomorphisms Lemma 48.6 holds. So far we have made little progress. On page 1089 we saw, using fundamental groups, that if two lens spaces $L(p,q)$ and $L(\widetilde{p},\widetilde{q})$ are homeomorphic, then $p = \widetilde{p}$. Afterwards, on page 1574 we saw that higher homotopy groups do not carry any additional information. In the Lens Space Homology Lemma 98.9 we had determined the homology groups of lens spaces, more precisely we showed that

$$\mathrm{H}_n(L(p,q);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0,3, \\ \mathbb{Z}_p, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

From the Universal Coefficient Theorem for Cohomology Groups 110.12 it follows that

$$\mathrm{H}^n(L(p,q);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0,3, \\ \mathbb{Z}_p, & \text{if } n = 2, \\ 0, & \text{otherwise.} \end{cases}$$

This shows that homology groups and cohomology groups cannot be used to distinguish the lens spaces $L(p,q)$ and $L(p,q')$.

In Chapter 132 we had successfully applied the cup product to distinguish several smooth manifolds that could not be distinguished just using cohomology groups. It is therefore tempting to try to use cup products to distinguish lens spaces. But here we immediately encounter the problem, that all non-trivial cup products are trivially trivial. More precisely, all cup products of cohomology classes in degree ≥ 1 are zero, since too many cohomology groups are zero. For example, the cup product on the non-trivial group $\mathrm{H}^2(L(p,q);\mathbb{Z})$ with itself is zero since it takes values in $\mathrm{H}^4(L(p,q);\mathbb{Z}) = 0$.

In this chapter we will develop an analogue of the intersection form, namely the *linking pairing*. This is an invariant defined for lens spaces, or more generally, for odd-dimensional rational homology spheres, which are defined as follows:

Definition. Let R be a commutative ring and let M be an n-dimensional topological manifold.

(1) We say M is a topological R-homology sphere, if for each $k \in \mathbb{N}_0$ we have an isomorphism $\mathrm{H}_k(M;R) \cong \mathrm{H}_k(S^n;R)$, i.e. we have

$$\mathrm{H}_k(M;R) \cong \begin{cases} R, & \text{if } k = 0,n, \\ 0, & \text{otherwise.} \end{cases}$$

Sometimes we also just say R-homology sphere.

(2) Instead of $\mathbb{Z}$-homology sphere we usually say integral homology sphere and instead of $\mathbb{Q}$-homology sphere we usually say rational homology sphere.

[1751] By our hypothesis we have $\gcd(p,r) = 1$. This implies that there exists a multiplicative inverse of r modulo p, i.e. there exists an s with $r \cdot s \equiv 1 \bmod p$. We write $r^{-1} := s$.

(3) If M is actually a smooth manifold, then we say smooth R-homology sphere.

Examples.

(1) Integral homology spheres are precisely the homology spheres that we introduced on pages 2159 and 2710.
(2) It follows from the aforementioned calculation of the homology groups of lens spaces and Corollary 89.19 (which is a straightforward consequence of the Universal Coefficient Theorem 89.16), that lens spaces are smooth rational homology spheres.
(3) It follows from the Connected Sum-Invariants Proposition 132.11 that the connected sum of two n-dimensional R-homology spheres is again an n-dimensional R-homology sphere.

The following lemma gives an alternative description of rational homology spheres.

Lemma 148.1. Let M be an n-dimensional topological manifold M. Then the following holds:[1752]

$$M \text{ is a rational homology sphere} \iff \begin{array}{c} M \text{ is closed connected and orientable and} \\ \mathrm{H}_i(M;\mathbb{Z}) \text{ is finite for } i = 1, \dots, \lfloor \frac{n-1}{2} \rfloor. \end{array}$$

Proof. The proof of the lemma is basically an exercise in combining all the standard facts on homology groups of topological manifolds that we have gathered so far. More precisely, the "⇒" direction follows from Corollary 89.19, the Topological Manifolds-Invariants Proposition 104.14 (4), the H_0-Proposition 72.5 and Theorems 106.1 and 106.3. The proof of the "⇐" direction also requires the above ingredients, furthermore it also requires the Poincaré Duality Theorem 119.1. We leave it to the reader to connect the dots. ■

148.2. **The Bockstein homomorphisms.** Before we can introduce the linking pairing of rational homology spheres we need to give some harmless extensions of our definitions and results from (co-) homology theory. We will do so in the coming two sections.
Let X be a topological space and let $f\colon A \to B$ be a homomorphism of abelian groups. We recall that in Lemma 89.7 (2) we saw that the maps

$$\begin{array}{rcl} f_*\colon \mathrm{C}_*(X;A) & \to & \mathrm{C}_*(X;B) \\ \left[\sum_{i=1}^{k} \sigma_i \otimes a_i\right] & \mapsto & \left[\sum_{i=1}^{k} \sigma_i \otimes f(a_i)\right] \end{array}$$

form a chain map, in particular for each $n \in \mathbb{N}_0$ we obtain an induced map

$$f_*\colon \mathrm{H}_n(X;A) \to \mathrm{H}_n(X;B).$$

Similarly, it follows from the discussion on page 2401 that the maps

$$\begin{array}{rcl} f_*\colon \mathrm{C}^n(X;A) = \mathrm{Hom}(\mathrm{C}_n(X), A) & \to & \mathrm{C}^n(X;B) = \mathrm{Hom}(\mathrm{C}_n(X), B) \\ \varphi & \mapsto & f \circ \varphi \end{array}$$

define a cochain map, in particular we get for each $n \in \mathbb{N}_0$ an induced map

$$f_*\colon \mathrm{H}^n(X;A) \to \mathrm{H}^n(X;B)$$

In the following lemma we discuss one subtle example.

[1752]Here recall that given a real number r we denote by $\lfloor r \rfloor = \max\{n \in \mathbb{Z} \mid n \leq r\}$ the result of rounding r down.

Lemma 148.2. Let $k \in \mathbb{Z}$. We denote by $f\colon \mathbb{Z} \to \mathbb{Z}$ the map that is given by multiplication by k. Given any pair (X, A) of topological spaces and given any $n \in \mathbb{N}_0$ the maps $f_*\colon \mathrm{H}_n(X, A; \mathbb{Z}) \to \mathrm{H}_n(X, A; \mathbb{Z})$ and $f_*\colon \mathrm{H}^n(X, A; \mathbb{Z}) \to \mathrm{H}^n(X, A; \mathbb{Z})$ are given by multiplication by m.

Proof. It is arguable whether this lemma actually needs a proof. Nonetheless, since at one point we will make use of the statement for cohomology, we provide the proof for the induced maps on cohomology. Thus let $\gamma\colon \mathrm{C}_k(X, A) \to \mathbb{Z}$ be a cochain. We see that

$$f_*([\gamma]) = [f \circ \gamma] = \left[\begin{array}{r} \mathrm{C}_k(X, A) \to \mathbb{Z} \\ \sigma \mapsto \underbrace{n \cdot f(\sigma)}_{=(f\circ\gamma)(\sigma)} \end{array}\right] = \left[\begin{array}{r} \mathrm{C}_k(X, A) \to \mathbb{Z} \\ \sigma \mapsto (n \cdot f)(\sigma) \end{array}\right] = [n \cdot f] = n \cdot [\gamma].$$

this is the point where something happens ↑

■

These observations lead us to the definitions of the Bockstein homomorphisms.

Definition. Let X be a topological space and let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of abelian groups. Let $n \in \mathbb{N}_0$. It follows from the fact that $\mathrm{C}_n(X)$ is a free abelian group and from the natural isomorphisms of the Tensor Lemma 89.3 that

$$0 \;\to\; \mathrm{C}_n(X; A) \;\xrightarrow{i_*}\; \mathrm{C}_n(X; B) \;\xrightarrow{p_*}\; \mathrm{C}_n(X; C) \;\to\; 0$$

is also exact. Thus we obtain a short exact sequence

$$0 \;\to\; \mathrm{C}_*(X; A) \;\xrightarrow{i_*}\; \mathrm{C}_*(X; B) \;\xrightarrow{p_*}\; \mathrm{C}_*(X; C) \;\to\; 0$$

of chain complexes. Furthermore, from the LES Proposition 74.9 we obtain a long exact sequence

$$\ldots \xrightarrow[=:\beta_{n-1}]{\partial_{n-1}} \mathrm{H}_n(X; A) \xrightarrow{i_*} \mathrm{H}_n(X; B) \xrightarrow{p_*} \mathrm{H}_n(X; C) \xrightarrow[=:\beta_n]{\partial_n} \mathrm{H}_{n-1}(X; A) \to \ldots$$

which is called the Bockstein sequence in homology.[1753] Furthermore for each $n \in \mathbb{N}_0$ the connecting homomorphism $\beta = \partial_n\colon \mathrm{H}_n(X; C) \to \mathrm{H}_{n-1}(X; A)$ is called the n-th Bockstein homomorphism in homology.

The same setup also works in cohomology.

Definition. We continue with the notation from the previous definition. Since each $\mathrm{C}_n(X)$ is a free abelian group one sees easily that

$$0 \;\to\; \mathrm{C}^*(X; A) \;\xrightarrow{i}\; \mathrm{C}^*(X; B) \;\xrightarrow{p_*}\; \mathrm{C}^*(X; C) \;\to\; 0$$

is a short exact sequence of cochain complexes. From the Connecting Homomorphism Proposition 108.6 we obtain a natural long exact sequence of cohomology groups

$$\ldots \xrightarrow[=:\beta_{n-1}]{\delta_{n-1}} \mathrm{H}^n(X; A) \xrightarrow{i_*} \mathrm{H}^n(X; B) \xrightarrow{p_*} \mathrm{H}^n(X; C) \xrightarrow[=:\beta_n]{\delta_n} \mathrm{H}^{n+1}(X; A) \to \ldots$$

which is called the Bockstein sequence in cohomology. Furthermore for each $n \in \mathbb{N}_0$ the connecting homomorphism $\beta = \delta_n\colon \mathrm{H}^n(X; C) \to \mathrm{H}^{n+1}(X; A)$ is called the n-th Bockstein homomorphism in cohomology.

[1753] The Bockstein homomorphism is named after the Russian mathematician Meyer Bockstein (1913-1990).

Examples. Let X be a topological space. The following two instances of the Bockstein homomorphisms are arguably the ones which are most frequently used:

(1) Let $k \in \mathbb{N}$. We consider the short exact sequence $0 \to \mathbb{Z}_k \xrightarrow{\cdot k} \mathbb{Z}_{k^2} \xrightarrow{p} \mathbb{Z}_k \to 0$ where p stands for the natural projection. Given a topological space X we obtain Bockstein homomorphisms $\beta_n \colon \mathrm{H}_n(X;\mathbb{Z}_k) \to \mathrm{H}_{n-1}(X;\mathbb{Z}_k)$, $n \in \mathbb{N}_0$. In Exercise 148.1 we will see that $\beta_{n-1} \circ \beta_n \colon \mathrm{H}_n(X;\mathbb{Z}_k) \to \mathrm{H}_{n-2}(X;\mathbb{Z}_k)$ is the zero map.
(2) In Section 148.4 we will consider the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. It gives rise to Bockstein homomorphisms $\beta_n \colon \mathrm{H}_n(X;\mathbb{Q}/\mathbb{Z}) \to \mathrm{H}_{n-1}(X;\mathbb{Z})$ and $\beta_n \colon \mathrm{H}^n(X;\mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^{n+1}(X;\mathbb{Z})$.

For completeness' sake we mention the following lemma.

Lemma 148.3. Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of abelian groups.

(1) Let $n \in \mathbb{N}_0$. The n-th Bockstein homomorphisms β_n corresponding to the short exact sequence of abelian groups form a natural transformation from the contravariant functor $X \mapsto \mathrm{H}^n(X;C)$ to the contravariant functor $X \mapsto \mathrm{H}^{n+1}(X;A)$.
(2) Given a map $f \colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccccccccccc}
\ldots & \xrightarrow{\beta_{n-1}} & \mathrm{H}^n(Y;A) & \xrightarrow{i_*} & \mathrm{H}^n(Y;B) & \xrightarrow{p_*} & \mathrm{H}^n(Y;C) & \xrightarrow{\beta_n} & \mathrm{H}^{n+1}(Y;A) & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \\
\ldots & \xrightarrow{\beta_{n-1}} & \mathrm{H}^n(X;A) & \xrightarrow{i_*} & \mathrm{H}^n(X;B) & \xrightarrow{p_*} & \mathrm{H}^n(X;C) & \xrightarrow{\beta_n} & \mathrm{H}^{n+1}(X;A) & \longrightarrow & \ldots
\end{array}$$

Proof. The first statement is essentially a consequence of the Connecting Homomorphism Lemma 74.8 (2). The second statement follows from the first statement together with Lemma 108.9. ■

148.3. **The cup product and the cap product with coefficients.** On page 2513, given a topological space X, we introduced the cup product for the R-cohomology groups $\mathrm{H}^*(X;R)$ provided the coefficients are given by a commutative ring R. In this section we will generalize the notion of the cup product where we no longer demand that the coefficients lie in a commutative ring R. Furthermore we also define a slight variation on the cap product. It is advisable to quickly look at the definitions and then move on to the next section. All the statements proved in this section are basically the obvious generalizations of earlier results.
Before we introduce the more general cup product, we recall the following notation from page 2503.

Notation. Given any $i \in \{0,\ldots,n\}$ we write $v_i := (0,\ldots,1,0,\ldots,0) \in \mathbb{R}^{n+1}$ and secondly, given $a_0,\ldots,a_s \in \Delta^n$ we denote by $[a_0,\ldots,a_s]$ the unique affine linear map $\Delta^s \to \Delta^n$ that sends v_i to a_i for $i = 0,\ldots,s$.

Now we can give the quite simple-minded definition of the more general cup product.

Definition. Let X be a topological space and let A and B be abelian groups. We define

$$\begin{aligned} \mathrm{C}^p(X;A)\times\mathrm{C}^q(X;B) &\to \mathrm{C}^{p+q}(X;A\otimes B) \\ (\varphi,\psi) &\mapsto \left(\begin{array}{l} \varphi\cup_\otimes\psi\colon \mathrm{C}_{p+q}(X)\to A\otimes B \\ \quad (\sigma\colon \Delta^{p+q}\to X)\mapsto \underbrace{\varphi(\sigma\circ[v_0,\dots,v_p])}_{\in A}\otimes\underbrace{\psi(\sigma\circ[v_p,\dots,v_{p+q}])}_{\in B} \end{array} \right). \end{aligned}$$

We have the following elementary lemma. The proof is basically verbatim the same as the proof of Coboundary-Cup Product Lemma 114.2.

Lemma 148.4. Let X be a topological space and let A and B be two abelian groups. If $\varphi\in\mathrm{C}^p(X;A)$ and $\psi\in\mathrm{C}^q(X;B)$, then

$$\delta(\varphi\cup_\otimes\psi) \;=\; \delta\varphi\cup_\otimes\psi+(-1)^p\cdot\varphi\cup_\otimes\delta\psi \;\in\; \mathrm{C}^{p+q+1}(X;A\otimes B).$$

Now we can easily prove the following lemma. The proof again is basically the same as the proof of Lemma 114.3, we just need to replace Coboundary-Cup Product Lemma 114.2 by Lemma 148.4.

Lemma 148.5. Let X be a topological space and let A and B be abelian groups. Then the map

$$\begin{aligned} \cup_\otimes\colon \mathrm{H}^p(X;A)\times\mathrm{H}^q(X;B) &\to \mathrm{H}^{p+q}(X;A\otimes B) \\ ([\varphi],[\psi]) &\mapsto [\varphi\cup_\otimes\psi] \end{aligned}$$

is well-defined.

We refer to the pairing from Lemma 148.5 again as a *cup product*. The following lemma is an immediate consequence of the definitions.

Lemma 148.6. Let X be a topological space.

(1) If $\alpha\colon A\to A'$ and $\beta\colon B\to B'$ are homomorphisms between two abelian groups, then for any $k,l\in\mathbb{N}_0$ the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}^k(X;A)\times\mathrm{H}^l(X;B) & \xrightarrow{\cup_\otimes} & \mathrm{H}^{k+l}(X;A\otimes B) \\ \downarrow{\scriptstyle \alpha_*\times\beta_*} & & \downarrow{\scriptstyle (\alpha\otimes\beta)_*} \\ \mathrm{H}^k(X;A')\times\mathrm{H}^l(X;B') & \xrightarrow{\cup_\otimes} & \mathrm{H}^{k+l}(X;A'\otimes B'). \end{array}$$

(2) If $f\colon X\to X'$ is a map to another topological space X' and if A and B are two abelian groups, then for any $k,l\in\mathbb{N}_0$ the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}^k(X';A)\times\mathrm{H}^l(X';B) & \xrightarrow{\cup_\otimes} & \mathrm{H}^{k+l}(X';A\otimes B) \\ \downarrow{\scriptstyle f^*\times f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{H}^k(X;A)\times\mathrm{H}^l(X;B) & \xrightarrow{\cup_\otimes} & \mathrm{H}^{k+l}(X;A\otimes B). \end{array}$$

To formulate the relationship to the earlier definition of the cup product we need to introduce the following definition.

Definition. Let R be a commutative ring. We refer to the homomorphism[17541755]

$$\begin{aligned} \mu\colon R\otimes R &\to R \\ \sum_{i=1}^n \underbrace{a_i}_{\in\mathbb{Z}}\cdot\underbrace{r_i}_{\in R}\otimes\underbrace{s_i}_{\in R} &\mapsto \sum_{i=1}^n a_i\cdot(\underbrace{r_i\cdot s_i}_{\in R}) \end{aligned}$$

as the multiplication induced map.

Now we can formulate the relationship between the two cup products.

Lemma 148.7. Let X be a topological space. If R is a commutative ring, then we have the following commutative diagram

$$\begin{array}{ccc} & \mathrm{H}^p(X;R)\times\mathrm{H}^q(X;R) & \\ {\scriptstyle\cup_\otimes}\swarrow & & \searrow{\scriptstyle\cup} \\ \mathrm{H}^{p+q}(X;R\otimes R) & \xrightarrow{\quad\mu_*\quad} & \mathrm{H}^{p+q}(X;R) \end{array}$$

where $\mu\colon R\otimes R\to R$ denotes the multiplication induced map.

The proof of the following proposition is almost identical to the proof of the earlier the Cup Product-Commutativity Proposition 114.8.

Proposition 148.8. Let X be a topological space and let A and B be abelian groups. Furthermore let $\varphi\in\mathrm{H}^p(X;A)$ and $\psi\in\mathrm{H}^q(X;B)$. If we denote by $\Theta\colon A\otimes B\to B\otimes A$ the natural isomorphism induced by $a\otimes b\mapsto b\otimes a$, see the Tensor Lemma 89.3 (1), then

$$\Theta_*\big(\underbrace{a\cup_\otimes b}_{\in\mathrm{H}^{p+q}(X;A\otimes B)}\big) \;=\; (-1)^{p\cdot q}\;\cdot\;\underbrace{b\cup_\otimes a}_{\in\mathrm{H}^{p+q}(X;B\otimes A)}.$$

We also need a barely visible variation on the cap product.

Definition. Let X be a topological space, let U be a subset of X and let G be an abelian group. Furthermore let $k\le l$. We define the cap product

$$\begin{array}{rcl} \mathrm{C}^k(X;G)\times\mathrm{C}_l(X,U) & \to & \mathrm{C}_{l-k}(X,U;G)=\mathrm{C}_{l-k}(X,U)\otimes G \\ (\varphi,[\sigma\colon\Delta^l\to X]) & \mapsto & \varphi\cap\sigma := \underbrace{\sigma\circ[v_k,\dots,v_l]}_{\in\mathrm{C}_{l-k}(X,U)}\otimes\underbrace{\varphi(\sigma\circ[v_0,\dots,v_k])}_{\in G}. \end{array}$$

For $k>l$ we define the cap product to be the zero map.

The following lemma summarizes the key properties of the more general cap product.

Lemma 148.9. Let X be a topological space, let U be a subset of X, let G be an abelian group and let $k,l\in\mathbb{N}_0$.

(1) The map

$$\begin{array}{rcl} \mathrm{H}^k(X;G)\times\mathrm{H}_l(X,U;\mathbb{Z}) & \to & \mathrm{H}_{l-k}(X,U;G) \\ ([\varphi],[\sigma]) & \mapsto & [\varphi]\cap[\sigma] := [\varphi\cap\sigma] \end{array}$$

is well-defined.

(2) For any $\varphi\in\mathrm{H}^k(X;G)$ and any $\sigma\in\mathrm{H}_k(X;\mathbb{Z})$ we have

$$\epsilon_X(\varphi\cap\sigma) \;=\; \langle\varphi,\sigma\rangle \;\in\; A,$$

where $\epsilon_X\colon\mathrm{H}_0(X;G)\to G$ is the augmentation map and $\langle\ ,\ \rangle$ denotes the Kronecker pairing that we introduced in Lemma 109.6.

(3) If $\rho\colon G\to B$ is a homomorphism of abelian groups, then for any $\varphi\in\mathrm{H}^k(X;G)$ and any $\sigma\in\mathrm{H}_l(X,U;\mathbb{Z})$ we have

$$\rho_*(\varphi)\cap\sigma \;=\; \rho_*(\varphi\cap\sigma) \;\in\; \mathrm{H}_{l-k}(X,U;B).$$

[1754] Note that this is indeed a homomorphism of abelian groups.

[1755] Here on the right-hand side we implicitly use that any abelian group, in particular R, is a $\mathbb{Z}$-module.

(4) If $f\colon (X,U) \to (Y,V)$ is a map between pairs of topological spaces, then for any $\varphi \in \mathrm{H}^k(Y;G)$ and any $\sigma \in \mathrm{H}_l(X,U;\mathbb{Z})$ we have
$$f_*(f^*(\varphi) \cap \sigma) \;=\; \varphi \cap f_*(\sigma) \;\in\; \mathrm{H}_{l-k}(Y,V;G).$$
Note that here f^* denotes the induced map $f^*\colon \mathrm{H}^k(Y;G) \to \mathrm{H}^k(X;G)$ on absolute cohomology, whereas f_* denotes the induced maps $\mathrm{H}_l(X,U;\mathbb{Z}) \to \mathrm{H}_l(Y,V;\mathbb{Z})$ and $\mathrm{H}_{l-k}(X,U;A) \to \mathrm{H}_{l-k}(Y,V;A)$ on relative homology groups.

(5) Let $\varphi \in \mathrm{H}^k(X;\mathbb{Z})$, $\psi \in \mathrm{H}^l(X;G)$ and furthermore let $\sigma \in \mathrm{H}_n(X;\mathbb{Z})$. We denote by $c\colon G \otimes \mathbb{Z} \to G$ and $c\colon \mathbb{Z} \otimes G \to G$ the obvious isomorphisms given by the Tensor Lemma 89.3. Then
$$\varphi \cap (\psi \cap \sigma) \;=\; \underbrace{c_*(\overbrace{\psi \cup_\otimes \varphi}^{\in \mathrm{H}^{k+l}(X;G\otimes\mathbb{Z})})}_{\in \mathrm{H}^{k+l}(X;G)} \cap \sigma \;=\; (-1)^{kl} \cdot \underbrace{c_*(\overbrace{\varphi \cup_\otimes \psi}^{\in \mathrm{H}^{k+l}(X;\mathbb{Z}\otimes G)})}_{\in \mathrm{H}^{k+l}(X;G)} \cap \sigma \;\in\; \mathrm{H}_{n-k-l}(X;G).$$

Proof. The proof of the first statement is entirely analogous to the proof of Lemma 116.2 or Lemma 90.6. The second statement and the third statement are an immediate consequence of the definitions. The fourth statement is a mild generalization of the Cap Product-Naturality Lemma 116.8 and it is also a straightforward consequence of the definitions. Furthermore the first equality of the last statement is an equally mild generalization of the Cup-Cap Lemma 116.7. Finally the last equality of the last statement is a consequence of Proposition 148.8. ■

Later on we will also need the following slight variation on the Poincaré Duality Theorem 119.1. The proof is verbatim the same as the proof of the Poincaré Duality Theorem 119.1.

Proposition 148.10. Let M be a compact oriented non-empty n-dimensional topological manifold and let G be an abelian group. For each $k \in \mathbb{N}_0$ the map
$$\begin{array}{ccc} \mathrm{H}^k(M;G) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M,\partial M;G) \\ \sigma & \mapsto & \sigma \cap [M] \end{array}$$
is an isomorphism.

The above proposition leads us to the following definition.

Definition. Let M be a closed oriented non-empty n-dimensional topological manifold and let G be an abelian group. We denote by
$$\mathrm{PD}_M^G\colon \mathrm{H}_k(M;G) \xrightarrow{\cong} \mathrm{H}^{n-k}(M;G)$$
the inverse of the isomorphism of the Poincaré Duality isomorphism given by Proposition 148.10.

The following lemma is an immediate consequence of the relevant definitions together with Lemma 148.9 (3).

Lemma 148.11. Let M be a closed oriented non-empty n-dimensional topological manifold. If $\rho\colon G \to H$ is a homomorphism of abelian groups, then for every $k \in \mathbb{N}_0$ the

following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_k(M;G) & \xrightarrow{\mathrm{PD}_M^G} & \mathrm{H}^{n-k}(M;G) \\ \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \rho_*} \\ \mathrm{H}_k(M;H) & \xrightarrow{\mathrm{PD}_M^H} & \mathrm{H}^{n-k}(M;H). \end{array}$$

148.4. **The linking pairing on odd-dimensional rational homology spheres.** Now we return to the study of rational homology spheres. We start out with the following elementary lemma.

Lemma 148.12. Let $n \geq 1$. For every $(2n+1)$-dimensional rational homology sphere M the Bockstein homomorphism $\beta\colon \mathrm{H}^n(M;\mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^{n+1}(M;\mathbb{Z})$ corresponding to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ is an isomorphism.

Proof. It follows from our hypothesis together with the Cohomology-Field Coefficients Proposition 110.18 (or alternatively together with the Co-Homology Symmetry Theorem Theorem 119.6), that for the rational homology sphere M we have $\mathrm{H}^n(M;\mathbb{Q}) = \mathrm{H}^{n+1}(M;\mathbb{Q}) = 0$. Thus it follows from the Bockstein sequence in cohomology that the Bockstein homomorphism $\mathrm{H}^n(M;\mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^{n+1}(M;\mathbb{Z})$ is an isomorphism. ■

We continue with the following purely algebraic definition.

Definition. A linking pairing on an abelian group H is a pairing that takes values in $\mathbb{Q}/\mathbb{Z}$, i.e. it is a bilinear map $\lambda\colon H \times H \to \mathbb{Q}/\mathbb{Z}$.

Example. Let $n \in \mathbb{N}$ and let $k \in \mathbb{Z}$. The map

$$\begin{array}{rcl} \lambda\colon \mathbb{Z}_n \times \mathbb{Z}_n & \to & \mathbb{Q}/\mathbb{Z} \\ (a,b) & \mapsto & \frac{k}{n} \cdot a \cdot b \end{array}$$

is a symmetric linking pairing. It is an amusing exercise to show that this linking pairing is non-singular, in the sense of the definition on page 2656, if and only if k is coprime to n.

The following lemma is a variation on the Non-Singularity Lemma 121.1. It gives another rich source of linking pairings.

Lemma 148.13. (Linking Form-Matrix Lemma) If $A \in \mathrm{M}(n \times n, \mathbb{Z})$ is a symmetric matrix with $\det(A) \neq 0$, then the map

$$\begin{array}{rcl} \lambda(A)\colon \mathbb{Z}^n/A\mathbb{Z}^n \times \mathbb{Z}^n/A\mathbb{Z}^n & \to & \mathbb{Q}/\mathbb{Z} \\ ([v],[w]) & \mapsto & v^T A^{-1} w \end{array}$$

is well-defined and it defines a non-singular symmetric linking pairing.

Since we will not make use of the Linking Form-Matrix Lemma 148.13 we will only provide a sketch of a proof.

Sketch of proof. It follows easily from the symmetry of A that the linking pairing is well-defined, i.e. that it is independent of the choice of the representatives v and w. Furthermore it is straightforward to see that the linking pairing is symmetric. It remains to show that the linking pairing is non-singular. By the symmetry of the linking pairing it suffices to

show that the map

$$\begin{array}{rcl}\Phi\colon \mathbb{Z}^n/A\mathbb{Z}^n & \to & \mathrm{Hom}(\mathbb{Z}^n/A\mathbb{Z}^n, \mathbb{Q}/\mathbb{Z}) \\ [v] & \mapsto & ([w] \mapsto v^T A^{-1} w)\end{array}$$

is an isomorphism. We make the following three observations:

(1) From Lemma 51.8 and $\det(A) \neq 0$ we know that $\mathbb{Z}^n/A\mathbb{Z}^n$ is a torsion-group.
(2) It follows fairly easily from the Finitely Generated Abelian Group Classification see Theorem 51.4 that for any finite abelian group H the groups $\mathrm{Hom}(H, \mathbb{Q}/\mathbb{Z})$ and H have the same number of elements.
(3) By (1) and (2) the map Φ is a homomorphism between finite abelian groups with the same number of elements. Thus it suffices to show that Φ is a monomorphism.

Now let $v \in \mathbb{Z}^n$ such that $\Phi([v]) = 0$. This implies that for the i-th standard basis vector e_i the product $v^T A^{-1} e_i =: \mu_i$ lies in $\mathbb{Z}$. But this means that $v^T A^{-1} = (\mu_1 \ \dots \ \mu_n) \in \mathbb{Z}^n$, i.e. using the symmetry of A, we obtain that

$$v \ = \ A^T(\mu_1 \ \dots, \mu_n)^T \ = \ A(\mu_1 \ \dots \ \mu_n)^T \ \in \ A \cdot \mathbb{Z}^n. \qquad \blacksquare$$

Before we continue we fix some notation that we will use throughout the remainder of this chapter.

Notation. Let S be a commutative ring and let R be a subring of S. We adopt the following notations:

(1) We denote by $i\colon R \to S$ the inclusion map and we denote by $p\colon S \to R/S$ the projection .
(2) Given a topological space X we denote by β the Bockstein homomorphism corresponding to the short exact sequence $0 \to R \xrightarrow{i} S \xrightarrow{p} S/R \to 0$.
(3) Similar to the discussion on page 3147 we can consider the multiplication induced homomorphisms

$$\begin{array}{rcl} R \otimes S/R & \to & S/R \\ \sum_{i=1}^k r_i \otimes [s_i] & \mapsto & \sum_{i=1}^k [r_i \cdot s_i] \end{array} \quad \text{and} \quad \begin{array}{rcl} S/R \otimes R & \to & S/R \\ \sum_{i=1}^k [s_i] \otimes r_i & \mapsto & \sum_{i=1}^k [s_i \cdot r_i]. \end{array}$$

We denote both homomorphisms by ν.

The following definition can be viewed as an odd-dimensional analogue of the definition of the intersection form on a compact oriented even-dimensional smooth manifold that we gave on page 2800.

Definition. Let M be an oriented rational homology sphere of dimension $2n+1$ with $n \geq 1$. We refer to the map

$$\begin{array}{rcl}\lambda_M\colon \mathrm{H}_n(M;\mathbb{Z}) \times \mathrm{H}_n(M;\mathbb{Z}) & \to & \mathbb{Q}/\mathbb{Z} \\ (a,b) & \mapsto & \Big\langle \nu_*\big(\overbrace{\beta^{-1}(\mathrm{PD}(a))}^{\in \mathrm{H}^{n+1}(M;\mathbb{Q}/\mathbb{Z})} \cup_{\otimes} \overbrace{\mathrm{PD}(b)}^{\in \mathrm{H}^{n+1}(M;\mathbb{Z})}\big), [M] \Big\rangle \end{array}$$

$\in \mathrm{H}^{2n+1}(M;\mathbb{Q}/\mathbb{Z}\otimes\mathbb{Z})$

as the linking pairing λ_M of M.

Remark. With very minor efforts one can generalize the previous definition. More precisely, let M be a closed oriented $(2n+1)$-dimensional smooth manifold. We denote by

$\operatorname{Tor} \mathrm{H}_n(M;\mathbb{Z})$ the torsion subgroup of $\mathrm{H}_n(M;\mathbb{Z})$. In an almost identical fashion to the above one can define a pairing

$$\operatorname{Tor} \mathrm{H}_n(M;\mathbb{Z}) \times \operatorname{Tor} \mathrm{H}_n(M;\mathbb{Z}) \ \to \ \mathbb{Q}/\mathbb{Z}.$$

Furthermore, with similar proofs one can generalize the subsequent Propositions 148.16 and Proposition 148.17 to this more general setup. But since we will not make use of this generalization we leave it to the reader to fill in the details.

Lemma 148.14. Let $n \geq 1$ and let $f\colon M \to N$ be a homotopy equivalence between oriented $(2n+1)$-dimensional rational homology spheres. By the Homotopic Maps-and-Homology Proposition 73.6 there exists an $\epsilon \in \{-1,1\}$ with $f_*([M]) = \epsilon\cdot[N] \in \mathrm{H}_{2n+1}(N;\mathbb{Z})$. Then for any $x, y \in \mathrm{H}_n(M;\mathbb{Z})$ the following equality holds:

$$\lambda_N(f_*(x), f_*(y)) \ = \ \epsilon \cdot \lambda_M(x,y).$$

Proof. In the proof of Lemma 132.6 we had implicitly proved the following statement.[1756]

Statement 148.15. For any $c \in \mathrm{H}_k(M;\mathbb{Z})$ we have

$$f^*(\mathrm{PD}^{\mathbb{Z}}_N(f_*(c))) \ = \ \varepsilon \cdot \mathrm{PD}^{\mathbb{Z}}_M(c) \in \mathrm{H}^{2n+1-k}(M;\mathbb{Z}).$$

Now we can prove the desired equality. More precisely, we have the following equalities in $\mathbb{Q}/\mathbb{Z}$:

$$\begin{aligned}
\lambda_M(x,y) &\overset{\text{by definition of the linking pairing } \lambda_N}{\downarrow} \\
&= \left\langle \beta^{-1}(\mathrm{PD}^{\mathbb{Z}}_M(x)) \cup_{\otimes} \mathrm{PD}^{\mathbb{Z}}_M(y), [M] \right\rangle \\
&\overset{\text{by the above statement, the convenient truth that } \epsilon^2 = 1 \text{ and the definition of } \epsilon, \text{ we also used the naturality of } \beta, \text{ which is a consequence of the Connecting Homomorphism Proposition 108.6 (2)}}{=} \left\langle f^*(\beta^{-1}(\mathrm{PD}^{\mathbb{Z}}_N(f_*(x)))) \cup_{\otimes} f^*(\mathrm{PD}^{\mathbb{Z}}_N(f_*(y))), \epsilon\cdot(f_*)^{-1}([N]) \right\rangle \\
&\overset{\text{by Lemma 109.6 (3) and Lemma 148.6 (2)}}{=} \left\langle \beta^{-1}(\mathrm{PD}^{\mathbb{Z}}_N(f_*(x))) \cup_{\otimes} \mathrm{PD}^{\mathbb{Z}}_M(f_*(y)), \epsilon\cdot[N] \right\rangle \ \overset{\text{definition of } \lambda_N}{=} \ \epsilon \cdot \lambda_N(f_*(x), f_*(y)).
\end{aligned}$$

■

In the Intersection Form Proposition 132.4 we saw that the intersection form of a closed oriented even-dimensional topological manifold is (anti-) symmetric and non-singular. The following two propositions give us similar statements for linking pairings.

Proposition 148.16. (Linking Pairing-Symmetry Proposition) Let $n \geq 1$ and let M be an oriented $(2n+1)$-dimensional rational homology sphere. If n is odd, then the linking pairing λ_M on $\mathrm{H}_n(M;\mathbb{Z})$ is symmetric otherwise it is anti-symmetric.

Remark. Linking pairings were initially introduced in 1933 by Herbert Seifert [**Sei1933**, p. 814] for oriented 3-dimensional rational homology spheres. He gave a somewhat informal argument why the linking pairing is symmetric. The fact that linking pairings are (anti-) symmetric is well-known and implicitly or implicitly used many times. But published proofs of this statement are few and far between. The subsequent proof of the Linking Pairing-Symmetry Proposition 148.16 first appeared in [**CFH2016**].

[1756] Strictly speaking, in the proof of Lemma 132.6 the map f was a homeomorphism, but that makes no difference.

Proposition 148.17. Let $n \geq 1$. The linking pairing λ_M of any oriented $(2n+1)$-dimensional rational homology sphere is non-singular.

In these modest lecture notes we will not make use of Propositions 148.16 and 148.17. Since the proofs are somewhat technical we hide them in the next section.

148.5. **Proofs of Propositions 148.16 and 148.17.** We will see that the Linking Pairing-Symmetry Proposition 148.16 is a fairly straightforward consequence of the next lemma.

Lemma 148.18. Let S be a commutative ring and let R be a subring of S. Let X be a topological space. For any $x \in \mathrm{H}^k(X; S/R)$ and any $y \in \mathrm{H}^l(X; S/R)$ we have

$$\nu_*\big(\underbrace{\beta(x) \cup_\otimes y}_{\in \mathrm{H}^{k+l+1}(X; R\otimes S/R)}\big) \;=\; (-1)^{k+1} \cdot \nu_*\big(\underbrace{x \cup_\otimes \beta(y)}_{\in \mathrm{H}^{k+l+1}(X; S/R\otimes R)}\big) \;\in\; \mathrm{H}^{k+l+1}(X; S/R).$$

Proof of Proposition 148.16 assuming Lemma 148.18. Let $n \geq 1$ and let M be an oriented $(2n+1)$-dimensional rational homology sphere. Furthermore let $a, b \in \mathrm{H}_n(M; \mathbb{Z})$. We write $x = \beta^{-1}(\mathrm{PD}(a)) \in \mathrm{H}^n(M; \mathbb{Q}/\mathbb{Z})$ and $y = \beta^{-1}(\mathrm{PD}(b)) \in \mathrm{H}^n(M; \mathbb{Q}/\mathbb{Z})$. Then

$$\begin{aligned}
\lambda_M(a,b) &\overset{\text{by definition of } \lambda_M}{=} \langle \nu_*(x \cup_\otimes \beta(y)), [M]\rangle \overset{\text{by Lemma 148.18}}{=} (-1)^{n+1} \cdot \langle \nu_*(\beta(x) \cup_\otimes y), [M]\rangle \\
&= (-1)^{n+1} \cdot \langle \nu_*(y \cup_\otimes \beta(x)), [M]\rangle \;=\; (-1)^{n+1} \cdot \lambda_M(b,a).
\end{aligned}$$

(by Proposition 148.8 since $n(n+1)$ is even; by definition of λ_M) ∎

Proof of Lemma 148.18. First recall that we have a short exact sequence of cochain complexes:

$$0 \;\to\; \mathrm{C}^*(X; R) \;\xrightarrow{i_*}\; \mathrm{C}^*(X; S) \;\xrightarrow{p_*}\; \mathrm{C}^*(X; S/R) \;\to\; 0.$$

In particular we can and will pick $f \in \mathrm{C}^k(X; S)$ and $g \in \mathrm{C}^l(X; S)$ such that $[p_*(f)] = x$ and such that $[p_*(g)] = y$. Next recall that the Bockstein homomorphisms in cohomology are the connecting homomorphisms corresponding to the above short exact sequence of cochain complexes. It follows immediately from the definition of the connecting homomorphism, see page 2399 and the Connecting Homomorphism Proposition 108.6, that there exist unique cocycles $\beta(f) \in \mathrm{C}^{k+1}(X; R)$ and $\beta(g) \in \mathrm{C}^{l+1}(X; R)$ such that $i_*(\beta(f)) = \delta^k(f) \in C^{k+1}(X; S)$ and also such that $i_*(\beta(g)) = \delta^l(g) \in \mathrm{C}^{l+1}(X; S)$. We have the following equality in $\mathrm{H}^{k+l+1}(X; S \otimes S)$:

$$\begin{aligned}
(*) \qquad 0 \;&=\; [\delta(f \cup_\otimes g)] \overset{\text{see Lemma 148.4}}{=} [\delta(f) \cup_\otimes g \,+\, (-1)^k \cdot f \cup_\otimes \delta(g)] \\
&\uparrow \text{by cocycle condition} \qquad = \big[\underbrace{i_*(\beta(f)) \cup_\otimes g}_{\in \mathrm{C}^{k+l+1}(X; R\otimes S)} + (-1)^k \cdot \underbrace{f \cup_\otimes i_*(\beta(g))}_{\in \mathrm{C}^{k+l+1}(X; S\otimes R)}\big].
\end{aligned}$$

We consider the following diagram of group homomorphisms

$$\begin{array}{ccccc}
R \otimes S & \xrightarrow{i\otimes \mathrm{id}} & S \otimes S & \xleftarrow{\mathrm{id}\otimes i} & S \otimes R \\
\downarrow{\scriptstyle \mathrm{id}\otimes p} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle p\otimes \mathrm{id}} \\
R \otimes S/R & & S & & S/R \otimes R \\
& {\scriptstyle \nu}\searrow & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle \nu} & \\
& & S/R & &
\end{array}$$

where, as usual, ν and μ stand for the obvious maps that are induced by multiplication in the ring S. It is straightforward to see that the diagram commutes. We have the following equalities in $\mathrm{H}^{k+l+1}(X;S/R)$:

$$\begin{aligned}
&\text{by the above equality } (*)\\
0 \;&\overset{\downarrow}{=}\; (p\circ\mu)_*\big(\big[i_*(\beta(f))\cup_\otimes g \;+\; (-1)^k\cdot f\cup_\otimes i_*(\beta(g))\big]\big)\\
&=\; \big[\nu_*\big(\beta(f)\cup_\otimes p_*(g)\big) \;+\; (-1)^k\cdot\nu_*\big(p_*(f)\cup_\otimes\beta(g)\big)\big]\\
&\text{by the above commutative diagram of group homomorphisms and some basic observations regarding the functoriality of the cup product in the coefficients}\\
&\overset{\uparrow}{=}\; \nu_*\big([\beta(f)]\cup_\otimes[p_*(g)]\big) \;+\; (-1)^k\cdot\nu_*\big([p_*(f)]\cup_\otimes[\beta(g)]\big)\\
&\text{by definition of } \cup_\otimes \text{ and since } \beta(f), p_*(g), p_*(f) \text{ and } \beta(g) \text{ are cocycles}\\
&\overset{\uparrow}{=}\; \nu_*(\beta(x)\cup_\otimes y) \;+\; (-1)^k\cdot\nu_*(x\cup_\otimes\beta(y)).\\
&\text{since } [\beta(f)]=\beta([p_*(f)])=:\beta(x) \text{ and } [p_*(g)]=y \text{ and similar for the second summand}
\end{aligned}$$

This of course implies the desired statement. ■

Before we give the proof of Proposition 148.17 we prove the following lemma which gives a slight reformulation of the linking pairing.

Lemma 148.19. Let $n\geq 1$ and let M be an oriented $(2n+1)$-dimensional rational homology sphere. For any choice of $a,b\in\mathrm{H}_n(M;\mathbb{Z})$ we have

$$\lambda_M(a,b) \;=\; \epsilon_M\big(\beta^{-1}(\mathrm{PD}_M^{\mathbb{Z}}(a))\cap b\big) \;=\; \big\langle\beta^{-1}(\overbrace{\mathrm{PD}_M^{\mathbb{Z}}(a)}^{\in\mathrm{H}^{n+1}(M;\mathbb{Z})}),b\big\rangle.$$

Proof. Let $a,b\in\mathrm{H}_n(M;\mathbb{Z})$. We have the following equalities in $\mathbb{Q}/\mathbb{Z}$:

$$\begin{aligned}
&\text{by definition and Lemma 148.9 (2)}\\
\lambda_M(a,b) \;&\overset{\downarrow}{=}\; \epsilon_M\big(\overbrace{\nu_*\big(\underbrace{\beta^{-1}(\mathrm{PD}_M^{\mathbb{Z}}(a))}_{\in\mathrm{H}^{n+1}(M;\mathbb{Q}/\mathbb{Z})}\cup_\otimes\underbrace{\mathrm{PD}_M^{\mathbb{Z}}(b)}_{\in\mathrm{H}^n(M;\mathbb{Z})}\big)}^{\in\mathrm{H}^{2n+1}(M;\mathbb{Q}/\mathbb{Z})}\cap[M]\big)\\
&=\; \epsilon_M\big(\underbrace{\beta^{-1}(\mathrm{PD}_M^{\mathbb{Z}}(a))}_{\in\mathrm{H}^n(M;\mathbb{Q}/\mathbb{Z})}\cap\underbrace{(\mathrm{PD}_M^{\mathbb{Z}}(b)\cap[M])}_{\in\mathrm{H}_n(M;\mathbb{Z})}\big) \;=\; \epsilon_M\big(\beta^{-1}(\mathrm{PD}_M^{\mathbb{Z}}(a))\cap b\big).\\
&\quad\uparrow\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\uparrow\\
&\text{by Lemma 148.9 (5) and the convenient fact that } n(n+1) \text{ is always even}\qquad\qquad\text{definition of } \mathrm{PD}_M^{\mathbb{Z}}(b)\\
&\overset{\uparrow}{=}\; \big\langle\beta^{-1}(\mathrm{PD}_M^{\mathbb{Z}}(a)),b\big\rangle.\\
&\text{Lemma 148.9 (2)}
\end{aligned}$$

We have thus verified both desired equalities. ■

We conclude this technical section with the proof of Proposition 148.17.

Proof of Proposition 148.17. Let M be an oriented $(2n+1)$-dimensional rational homology sphere. We need to show that the linking pairing $\lambda_M\colon\mathrm{H}_n(M;\mathbb{Z})\times\mathrm{H}_n(M;\mathbb{Z})\to\mathbb{Q}/\mathbb{Z}$ is non-singular. To do so we consider the map

$$\Phi_M\colon\mathrm{H}_n(M;\mathbb{Z})\xrightarrow{\mathrm{PD}_M^{\mathbb{Z}}}\mathrm{H}^{n+1}(M;\mathbb{Z})\xrightarrow[\cong]{\beta^{-1}}\mathrm{H}^n(M;\mathbb{Q}/\mathbb{Z})\xrightarrow{\mathrm{ev}}\mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_n(M;\mathbb{Z}),\mathbb{Q}/\mathbb{Z}).$$

Here the first map is given by Poincaré Duality, the second map is the inverse of the Bockstein isomorphism given by Lemma 148.12 and the third map is given by the evaluation map.

In the Linking Pairing-Symmetry Proposition 148.16 we showed that the linking pairing is (anti-) symmetric. Therefore it suffices to prove that the map

$$\begin{array}{rcl} \mathrm{H}_n(M;\mathbb{Z}) & \to & \mathrm{Hom}(\mathrm{H}_n(M;\mathbb{Z}),\mathbb{Q}/\mathbb{Z}) \\ a & \mapsto & (b \mapsto \lambda_M(a,b)) \end{array}$$

is an isomorphism. But this in turn follows immediately from the following claim.

Claim.

(1) The map Φ_M is an isomorphism.
(2) For any $a, b \in \mathrm{H}_n(M;\mathbb{Z})$ we have $\lambda_M(a,b) = \Phi_M(a)(b)$.

Proof. First note that the second statement is an immediate consequence of Lemma 148.19. For the first statement it remains to show that the third map in the definition of Φ_M is an isomorphism. Recall that by the Universal Coefficient Theorem 110.12 for Cohomology Groups we have a short exact sequence

$$0 \to \mathrm{Ext}(\mathrm{H}_{n-1}(M;\mathbb{Z}),\mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^n(M;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{ev}} \mathrm{Hom}(\mathrm{H}_n(M;\mathbb{Z}),\mathbb{Q}/\mathbb{Z}) \to 0.$$

On page 2441 we remarked that the abelian group $\mathbb{Q}/\mathbb{Z}$ is divisible. By Proposition 110.6 this implies that the Ext-term to the left is zero. But this shows that the evaluation homomorphism to the right is an isomorphism. ■

148.6. **Linking pairings of lens spaces.** In the next section we will prove the following proposition.

Proposition 148.20. (Lens Space-Linking Pairing Proposition) Let $p, q \in \mathbb{N}$ be coprime. The linking pairing of the lens space $L(p,q)$ is isometric to[17571758]

$$\begin{array}{rcl} \mathbb{Z}_p \times \mathbb{Z}_p & \to & \mathbb{Q}/\mathbb{Z} \\ (x,y) & \mapsto & -\frac{q}{p} \cdot x \cdot y \mod \mathbb{Z}. \end{array}$$

Remark. There are many different approaches to calculating the linking pairing of lens spaces. The first calculation was given by Seifert [**Sei1933**, p. 824], see also [**ST1934**, p. 290] and [**ST1934**, p. 279]. Alternative calculations are given in [**Lic1985**], [**Hat2002**, p. 306], [**Mun1984**, Chapter 69] and [**Bre1993**, p. 364]. The proof of the Lens Space-Linking Pairing Proposition 148.20 that we give is rather different from the previous proofs. The approach taken below has the advantage that it can also be used to determine the linking pairing of any oriented 3-dimensional rational homology sphere in terms of a Heegaard splitting. We refer to [**CFH2016**] for details.

[1759] Recall that we have $L(p,q) = S^3/\mathbb{Z}_p$ and, that following the convention established on page 1075, we equip $L(p,q)$ with the obvious orientation coming from S^3.

[1760] By now I am fairly confident that the sign in the calculation of the linking pairing of the lens space $L(p,q)$ is correct. But the fact that, in my attempts to calculate the linking pairing, the sign flipped on numerous occasions means that one should take the result with a grain of salt.

Before we turn to the somewhat lengthy, and not overly enjoyable proof, we want to use this section to state and prove several corollaries of the Lens Space-Linking Pairing Proposition 148.20.

Corollary 148.21. Let $L(p,q)$ and $L(p,\widetilde{q})$ be lens spaces. Suppose there exists a homotopy equivalence $f\colon L(p,q) \to (p,\widetilde{q})$. By the Homotopic Maps-and-Homology Proposition 73.6 there exists an $\epsilon \in \{-1,1\}$ such that $f_*([L(p,q)]) = \epsilon \cdot [L(p,\widetilde{q})]$. Then there exists $r \in \mathbb{Z}$, coprime to p, such that

$$q \;\equiv\; \epsilon \cdot r^2 \cdot \widetilde{q} \bmod p.$$

Proof. We pick isomorphisms $\varphi\colon \mathbb{Z}_p \to \mathrm{H}_1(L(p,q);\mathbb{Z})$ and $\widetilde{\varphi}\colon \mathbb{Z}_p \to \mathrm{H}_1(L(p,\widetilde{q});\mathbb{Z})$ as provided by the Lens Space-Linking Pairing Proposition 148.20. The map $\widetilde{\varphi}^{-1} \circ f_* \circ \varphi\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is an isomorphism, hence it is given by multiplication by some $r \in \mathbb{Z}$ that is coprime to p. Together with Lemma 148.14 we see that the following diagram commutes

$$\begin{array}{ccccc}
 & & \overset{(x,y)\mapsto -\frac{q}{p}\cdot x\cdot y}{\frown} & & \\
\mathbb{Z}_p \times \mathbb{Z}_p & \xrightarrow{\varphi\times\varphi} & \mathrm{H}_1(L(p,q);\mathbb{Z}) \times \mathrm{H}_1(L(p,q);\mathbb{Z}) & \xrightarrow{\lambda_{L(p,q)}} & \mathbb{Q}/\mathbb{Z} \\
\downarrow{\scriptstyle (x,y)\mapsto(rx,ry)} & & \downarrow{\scriptstyle f_*\times f_*} & & \downarrow{\scriptstyle \cdot(-1)} \\
\mathbb{Z}_p \times \mathbb{Z}_p & \xrightarrow[\widetilde{\varphi}\times\widetilde{\varphi}]{} & \mathrm{H}_1(L(p,\widetilde{q});\mathbb{Z}) \times \mathrm{H}_1(L(p,\widetilde{q});\mathbb{Z}) & \xrightarrow{\lambda_{L(p,\widetilde{q})}} & \mathbb{Q}/\mathbb{Z}. \\
 & & \underset{(x,y)\mapsto -\frac{\widetilde{q}}{p}\cdot x\cdot y}{\smile} & &
\end{array}$$

If we start out with $(1,1) \in \mathbb{Z}_p \times \mathbb{Z}_p$ on the upper left corner and we compare the different routes to the lower right corner we obtain that $-\epsilon \cdot \frac{q}{p} = r^2 \cdot \frac{\widetilde{q}}{p}$. Multiplying by p we obtain the desired equality. ■

This is a good moment to state the classification of lens spaces up to homotopy equivalences:

Theorem 148.22. (Lens Space-Homotopy Classification Theorem) Let $p \in \mathbb{N}$. If $q, \widetilde{q} \in \mathbb{N}$ are coprime to p, then

$$\begin{array}{c}\text{the lens spaces } L(p,q) \text{ and } L(p,\widetilde{q}) \\ \text{are homotopy equivalent}\end{array} \quad\Longleftrightarrow\quad \begin{array}{c}\text{there exists an } r \in \mathbb{Z}, \text{ coprime to } p, \\ \text{such that } q \equiv \pm r^2 \cdot \widetilde{q} \bmod p.\end{array}$$

Proof. The "⇒"-direction is precisely the statement of Corollary 148.21. The "⇐"-direction was first shown by John Whitehead [**WhdJ1941b**, Theorem 10]. A direct proof is also given in [**CoheM1973**, Chapter 29]. We will prove the "⇐"-direction later in the Lens Space-Homotopy Equivalence Proposition 162.4, using the Whitehead Theorem 162.1. ■

So, after a very long wait we have now finally made some serious progress towards answering Question 48.7. For example, one can easily verify, using Theorem 148.22 that the lens spaces $L(5,1)$ and $L(5,2)$ are not homotopy equivalent. Nonetheless, we have not yet completely answered Question 48.7. For example the lens spaces $L(7,1)$ and $L(7,4)$ are homotopy equivalent by Theorem 148.22. But it is not clear whether these two lens spaces are homeomorphic. We will solves this conundrum in the Lens Space-Homeomorphism Classification Theorem **??**.

Remark. For those who might get tired of chasing down lens spaces, I would like to quote from John F. Kennedy's speech at Rice University on September 12 1962, where he announced that the United States would attempt to land a man on the moon by the end of the decade:

> *But why, some say, the Moon? Why choose this as our goal? And they may well ask, why climb the highest mountain? Why, 35 years ago, fly the Atlantic? Why does Rice play Texas?*
>
> *We choose to go to the Moon! We choose to go to the Moon in this decade and do the other things, not because they are easy, but because they are hard; because that goal will serve to organize and measure the best of our energies and skills, because that challenge is one that we are willing to accept, one we are unwilling to postpone, and one we intend to win.*



John F. Kennedy speaking at Rice University in September 1962

In Question 107.7 we had asked whether the lens space $L(3,1)$ is amphichiral, i.e. we had asked $L(3,1)$ admits an orientation-reversing self-homeomorphism. Now we have the tools to address this question.

Proposition 148.23. If there exists an *orientation-reversing* homeomorphism between two lens spaces $L(p,q)$ and $L(p,\widetilde{q})$, then there exists an $r \in \mathbb{Z}$ such that $q = -r^2 \cdot \widetilde{q}^{\pm 1} \bmod p$.

Example. Note that $1 \neq -r^2 \bmod 3$ for any $r \in \mathbb{Z}$. Thus it follows from Proposition 148.23 that there is no orientation-reversing homeomorphism from the lens space $L(3,1)$ to itself, i.e. the lens space $L(3,1)$ is chiral. This now answers Question 107.7.

Proof. If $f\colon L(p,q) \to L(p,\widetilde{q})$ is an orientation-reversing homeomorphism, then we have $f_*([L(p,q)]) = -[L(p,\widetilde{q})]$, i.e. we have $\epsilon = -1$ in the statement of Corollary 148.21. The proposition now follows immediately from Corollary 148.21. ■

148.7. **The proof of the Lens Space-Linking Pairing Proposition 148.20.** This section is almost entirely dedicated to the proof of the Lens Space-Linking Pairing Proposition 148.20. Unless the reader suffers from an insatiable appetite for cup and cap products, it is recommended to skip this section.

We start out with the following lemma that relates the Bockstein homomorphisms in homology and cohomology via the Poincaré Duality

Lemma 148.24. Let M be a closed oriented m-dimensional topological manifold. If we are given a short exact sequence $0 \to P \xrightarrow{i} Q \xrightarrow{p} R \to 0$ of abelian groups, then for every

$k \in \{0, \dots, m\}$ the two diagrams

$$\begin{array}{ccc} \mathrm{H}^k(M;R) & \xrightarrow{\cap[M]} & \mathrm{H}_{m-k}(M;R) \\ \downarrow\beta & & \downarrow\beta \\ \mathrm{H}^{k+1}(M;P) & \xrightarrow{\cap[M]} & \mathrm{H}_{m-k-1}(M;P) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{H}_{m-k}(M;R) & \xrightarrow{\mathrm{PD}_M^R} & \mathrm{H}^k(M;R) \\ \downarrow\beta & & \downarrow\beta \\ \mathrm{H}_{m-k-1}(M;P) & \xrightarrow{\mathrm{PD}_M^P} & \mathrm{H}^{k+1}(M;P) \end{array}$$

commute up to the sign $(-1)^{k+1}$.

Proof. We pick a cycle $\sigma \in \mathrm{C}_m(M;\mathbb{Z})$ that represents the fundamental class. Furthermore let $\psi \in \mathrm{C}^k(M;R)$ be a cocycle. We pick $\varphi \in \mathrm{C}^k(M;Q)$ with $p_*(\varphi) = \psi$. Then we have the following equalities in $\mathrm{H}_{m-k-1}(M;P)$:[1761]

$$\beta([\psi]) \cap [M] \overset{\text{by definition of the connecting homomorphism on page 2399}}{=} [\delta(\varphi) \cap \sigma] = [\partial(\varphi \cap \sigma)] \overset{\text{by definition of the connecting homomorphism on page 1620}}{=} \beta([p_*(\varphi \cap \sigma)]) = \beta([\psi] \cap [M]).$$

(the middle equality: by Lemma 116.1 since σ is a cycle)

But this implies that the left-hand diagram does indeed commute. The right-hand diagram is, by definition of the Poincaré Duality isomorphisms, basically the same diagram. ■

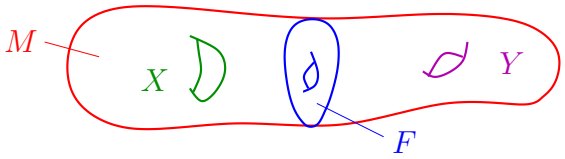
We continue with the following definition.

Definition. Let M be a 3-dimensional rational-homology sphere. Given a splitting (X, Y, F) of M, as defined on page 3114, we denote by $\Theta(X, Y, F)$ the homomorphism

$$\mathrm{H}_1(M;\mathbb{Z}) \xrightarrow{\beta^{-1}} \mathrm{H}_2(M;\mathbb{Q}/\mathbb{Z}) \to \mathrm{H}_2(M, X;\mathbb{Q}/\mathbb{Z}) \leftarrow \mathrm{H}_2(Y, F;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\partial} \mathrm{H}_1(F;\mathbb{Q}/\mathbb{Z}),$$

($\leftarrow$: isomorphism by the Excision Theorem 74.18 for smooth manifolds)

where "∂" denotes the connecting homomorphism of the long exact sequence of $\mathbb{Q}/\mathbb{Z}$-homology groups corresponding to the pair (Y, F). If the splitting (X, Y, F) is understood, then we just write Θ instead of $\Theta(X, Y, F)$.



awfully schematic picture of a splitting of M

The following lemma will allow us to compute the linking pairing of an oriented 3-dimensional rational homology sphere in terms of Poincaré Duality on the surface F.

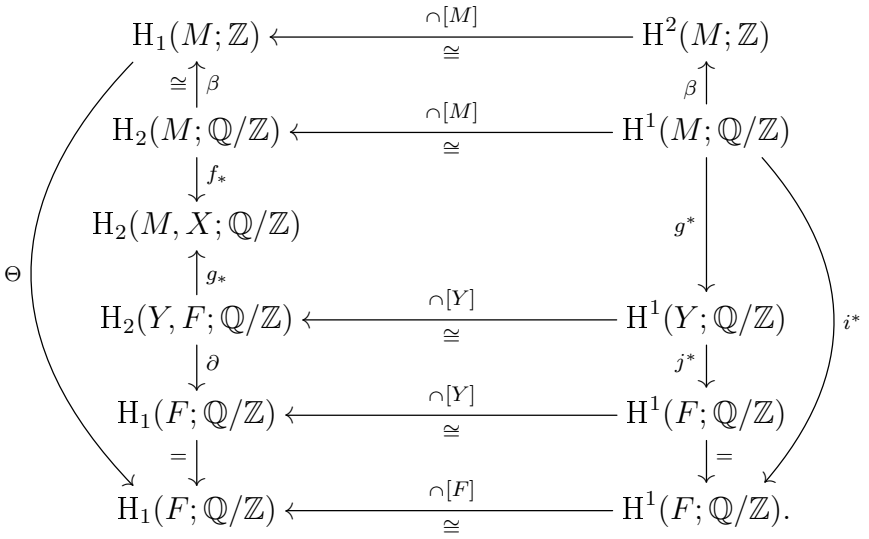
Lemma 148.25. Let M be an oriented 3-dimensional rational homology sphere and let (X, Y, F) be a splitting of M, as defined on page 3114. We denote by $i\colon F \to M$ the

[1761] The perspicacious reader will have noticed that the commutativity of the left-hand diagram has little to do with the hypothesis that we are capping with the fundamental class of a closed oriented topological manifold. Since we do not need the more general statements we leave the lemma as it is.

inclusion map. Then the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}_1(M;\mathbb{Z}) & \xrightarrow{\mathrm{PD}_M^{\mathbb{Z}}} & \mathrm{H}^2(M;\mathbb{Z}) & \xrightarrow{\beta^{-1}} & \mathrm{H}^1(M;\mathbb{Q}/\mathbb{Z}) \\ \downarrow{\scriptstyle \Theta} & & & & \downarrow{\scriptstyle i^*} \\ \mathrm{H}_1(F;\mathbb{Q}/\mathbb{Z}) & & \xrightarrow[\cong]{\mathrm{PD}_F^{\mathbb{Q}/\mathbb{Z}}} & & \mathrm{H}^1(F;\mathbb{Q}/\mathbb{Z}). \end{array}$$

Proof. First of all, we denote the inclusion map $F \to X$ by j. Furthermore we denote by $f\colon (M,\varnothing) \to (M,X)$ and $g\colon (Y,F) \to (M,X)$ the obvious maps between pairs of topological spaces. We write also $g\colon Y \to M$ for the inclusion of Y into M. We consider the following diagram:



We consider the commutativity of the four rectangles in the diagram separately.

(1) The top rectangle commutes by Lemma 148.24. (Strictly speaking Lemma 148.24 says that the diagram commutes up to the sign $(-1)^{1+1} = 1$.)

(2) We turn to the second rectangle (or equivalently the first and only pentagon). Let $\varphi \in \mathrm{H}^1(M;\mathbb{Q}/\mathbb{Z})$. We have the following equalities in $\mathrm{H}_2(M,X;\mathbb{Q}/\mathbb{Z})$:

$$\begin{array}{ccccc} f_*(\varphi \cap [M]) & \underset{\uparrow}{=} & \varphi \cap f_*([M]) & \overset{\downarrow}{=} & \varphi \cap g_*([Y]) & \underset{\uparrow}{=} & g_*(g^*(\varphi) \cap [Y]). \\ \text{Lemma 148.9 (4) applied to } f\colon (M,\varnothing) \to (M,X) \text{ where we use that } f^* = \mathrm{id}\colon \mathrm{H}^1(M;\mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(M;\mathbb{Q}/\mathbb{Z}) & & & \text{by the Codimension Zero-Fundamental Class Lemma 106.24 we have } f_*([M]) = g_*([Y]) & & \text{Lemma 148.9 (4) applied to } g\colon (Y,F) \to (M,X) & & \end{array}$$

(3) The third rectangle anti-commutes[1762] by the obvious extension of Proposition 119.25 from $\mathbb{Z}$-coefficients to $\mathbb{Q}/\mathbb{Z}$-coefficients.

(4) It follows from our orientation convention from page 3114 that $F = -\partial Y$ as oriented smooth manifolds. By the discussion on page 2373 this implies $[F] = -\partial[Y]$. We obtain from this observation that the bottom rectangle anti-commutes.

[1762]Here by "anti-commutes" we mean that the left-down path agrees with the down-left path except for a minus sign.

Summarizing we see that the two approaches of going from $\mathrm{H}^2(M;\mathbb{Z})$ to $\mathrm{H}_1(F;\mathbb{Q}/\mathbb{Z})$ agree. The lemma follows from the observation, that by definition PD_M is the inverse of the top horizontal map and that similarly $\mathrm{PD}_F^{\mathbb{Q}/\mathbb{Z}}$ is the inverse of the bottom horizontal map. ■

Now we turn to lens spaces. In the following we will use the description of lens spaces that we had obtained in the Lens Space-Splitting Lemma 146.2. At this point it is convenient to recall and introduce some notation:

Notation.

(1) We write $F = S^1 \times S^1$. We write $a = [S^1 \times 1]$ and $b = [1 \times S^1]$ and we use this ordered basis of $\mathrm{H}_1(F;\mathbb{Z})$ to make the usual identification $\mathrm{H}_1(F;\mathbb{Z}) = \mathbb{Z}^2$. By a slight abuse of notation we also denote by a and b the corresponding elements in $\mathrm{H}_1(F;\mathbb{Q}) = \mathrm{H}_1(F;\mathbb{Z}) \otimes \mathbb{Q}$.

(2) Let $p, q \in \mathbb{N}$ be coprime. We pick $r, s \in \mathbb{N}$ such that $ps - qr = 1$. We write

$$A = \begin{pmatrix} q & p \\ s & r \end{pmatrix} \quad \text{and we consider the diffeomorphism} \quad \begin{array}{rcl} \varphi\colon F = \mathbb{R}^2/\mathbb{Z}^2 & \to & F = \mathbb{R}^2/\mathbb{Z}^2 \\ [v] & \mapsto & [Av]. \end{array}$$

By the Action-on-Torus Homology Lemma 84.7 (2) the induced map φ_* on the homology group $\mathrm{H}_1(F;\mathbb{Z}) = \mathbb{Z}^2$ is given by multiplication by the matrix A.

(3) We write $X = S^1 \times \overline{B}^2$ and we denote by Y another disjoint copy of the solid torus $S^1 \times \overline{B}^2$. We equip X and Y with the usual product orientation. In the Lens Space-Splitting Lemma 146.2 we gave an explicit orientation-preserving diffeomorphism

$$L(p,q) \xrightarrow{\cong} (X \sqcup Y)/x \sim \varphi(x) \quad \text{for } x \in F = \partial X$$

that we will use to identify $L(p,q)$ with the topological space to the right. By an abuse of notation we denote the images of the X and Y in $L(p,q)$ again by X and Y. Note that (X, Y, F) is a Heegaard splitting for $L(p,q)$.

(4) As usual, for want of a better notation, we denote by i the various inclusion maps $F \to X$, $X \to L(p,q)$ and $F \to L(p,q)$.

(5) We denote by $\rho\colon \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ the natural projection.

(6) Finally we denote by $\Theta\colon \mathrm{H}_1(L(p,q);\mathbb{Z}) \to \mathrm{H}_1(F;\mathbb{Q}/\mathbb{Z})$ the map that we introduced on page 3158.

Lemma 148.26. The following statements hold:

(1) $i_*(a) \in \mathrm{H}_1(L(p,q);\mathbb{Z})$ is a generator,

(2) the equality $\Theta(i_*(a)) = \rho_*(a - \frac{q}{p} \cdot b)$ holds in $\mathrm{H}_1(F;\mathbb{Q}/\mathbb{Z})$.

Proof. By the Excisive Triad-Proposition 109.12 we can apply the Mayer–Vietoris Theorem 109.14 to the decomposition $L(p,q) = X \cup Y$ and we obtain the following long exact sequence

$$\mathrm{H}_1(F;\mathbb{Z}) \xrightarrow{\substack{i_* \\ -i_*}} \begin{matrix} \mathrm{H}_1(X;\mathbb{Z}) \\ \mathrm{H}_1(Y;\mathbb{Z}) \end{matrix} \to \mathrm{H}_1(L(p,q);\mathbb{Z}) \to 0.$$

Recall that we denote by $a, b \in \mathrm{H}_1(F;\mathbb{Z}) = \mathrm{H}_1(S^1 \times S^1;\mathbb{Z})$ the standard basis. Since $S^1 \times S^1$ is the boundary of $S^1 \times \overline{B}^2$ we denote by a also the corresponding generator of $\mathrm{H}_1(X;\mathbb{Z}) = \mathrm{H}_1(S^1 \times \overline{B}^2;\mathbb{Z})$. Furthermore we denote by $a' \in \mathrm{H}_1(Y;\mathbb{Z}) = \mathrm{H}_1(S^1 \times \overline{B}^2;\mathbb{Z})$ the obvious generator. It follows from the above discussion and conventions, in particular

the above description of φ_*, that with respect to our bases the above long exact sequence Mayer–Vietoris sequence becomes

$$\begin{matrix}\mathbb{Z}\cdot a\\ \mathbb{Z}\cdot b\end{matrix} \xrightarrow{\begin{pmatrix}1 & 0\\ q & p\end{pmatrix}} \begin{matrix}\mathbb{Z}\cdot a\\ \mathbb{Z}\cdot a'\end{matrix} \;\to\; \mathrm{H}_1(L(p,q);\mathbb{Z}) \;\to\; 0.$$

At this stage it is an elementary linear algebra exercise to show that the image of a in $\mathrm{H}_1(L(p,q);\mathbb{Z})$ is a generator.[1763]

Now we turn to the proof of the second statement. In the following we denote by $h\colon (L(p,q),\varnothing) \to (L(p,q),X)$ and $j\colon (Y,F) \to (L(p,q),X)$ the obvious maps of pairs of topological spaces. We recall that Θ is defined as the composition of the following maps:

$$\begin{array}{ccccccc}\mathrm{H}_1(L(p,q);\mathbb{Z}) & \xrightarrow{\beta^{-1}} & \mathrm{H}_2(L(p,q);\mathbb{Q}/\mathbb{Z}) & \xrightarrow{h_*} & \mathrm{H}_2(L(p,q),X;\mathbb{Q}/\mathbb{Z}) & \underset{\cong}{\xleftarrow{j_*}} & \mathrm{H}_2(Y,F;\mathbb{Q}/\mathbb{Z})\\ & & & & & & \downarrow\ \partial\\ & & & & & & \mathrm{H}_1(F;\mathbb{Q}/\mathbb{Z}).\end{array}$$

As before we denote by $i\colon F \to L(p,q)$ the inclusion map. For clarity we recall in the following diagram all the maps that play a role in the definition of the Bockstein homomorphism $\beta\colon \mathrm{H}_2(L(p,q);\mathbb{Q}/\mathbb{Z}) \to \mathrm{H}_1(L(p,q);\mathbb{Z})$:

$$\begin{array}{ccccccc} & & & & \mathrm{C}_2(L(p,q);\mathbb{Q}) & \xrightarrow{\rho_*} & \mathrm{C}_2(L(p,q);\mathbb{Q}/\mathbb{Z}) \longrightarrow 0\\ & & & & \big\downarrow\partial & & \\ 0 \longrightarrow & \mathrm{C}_1(L(p,q);\mathbb{Z}) & \longrightarrow & \mathrm{C}_1(L(p,q);\mathbb{Q}). & & & \end{array}$$

Finally, we denote by $\widetilde{a}$ and $\widetilde{b}$ the obvious singular chains in F representing the homology classes a and b.

Now we turn to the actual proof of the equality $\Theta(i_*(a)) = \rho_*(a - \frac{q}{p}\cdot b)$ in $\mathrm{H}_1(F;\mathbb{Q}/\mathbb{Z})$. We consider $a - \frac{q}{p}\cdot b \in \mathrm{H}_1(F;\mathbb{Q})$. Note that

$$\varphi_*\big(a - \tfrac{q}{p}\cdot b\big) \underset{\uparrow}{=} \begin{pmatrix}q & p\\ s & r\end{pmatrix}\cdot\begin{pmatrix}1\\ -\frac{q}{p}\end{pmatrix} = \begin{pmatrix}0\\ *\end{pmatrix} \in \mathrm{H}_1(F;\mathbb{Q}) = \begin{matrix}\mathbb{Q}\cdot a\\ \mathbb{Q}\cdot b\end{matrix},$$

by the Action-on-Torus Homology Lemma 84.7 and the naturality of the Universal Coefficient Theorem 89.17

i.e. the image of $\varphi_*\big(a - \frac{q}{p}\cdot b\big)$ in $\mathrm{H}_1(Y;\mathbb{Q})$ is zero.

This implies that there exists a singular 2-chain $\sigma \in \mathrm{C}_2(Y;\mathbb{Q})$ with $\partial(\sigma) = \widetilde{a} - \frac{q}{p}\cdot\widetilde{b}$. Furthermore b represents the zero element in $\mathrm{H}_1(X;\mathbb{Z})$, hence $\widetilde{b}$ is null-homologous in X,

[1763]This can be seen as follows. We consider the map

$$\begin{array}{rcccc}\operatorname{coker}\left\{\begin{pmatrix}1 & 0\\ q & p\end{pmatrix}\colon \begin{matrix}\mathbb{Z}\cdot a\\ \mathbb{Z}\cdot b\end{matrix} \to \begin{matrix}\mathbb{Z}\cdot a\\ \mathbb{Z}\cdot a'\end{matrix}\right\} & \to & \mathbb{Z}_p\\ ra + sa' & \mapsto & qr - s.\end{array}$$

It is straightforward to see that this map is well-defined. It follows from Lemma 51.8 that the group on the left has order p. Since the map is an epimorphism and since both groups have the same order, we see that the map is in fact an isomorphism. Since the image of a on the right-hand side is a generator (here we use that q is coprime to p) we see that a also represents a generator on the left.

which implies that there exists a singular 2-chain $\mu \in \mathrm{C}_2(X;\mathbb{Z})$ with $\partial(\mu) = \widetilde{b}$. Now we calculate that

$$\beta\big(\underbrace{\big[\rho_*\big(\overbrace{\sigma + \tfrac{q}{p}\cdot\mu}^{\in \mathrm{C}_2(L(p,q);\mathbb{Q})}\big)\big]}_{\text{cycle in } \mathrm{C}_2(L(p,q);\mathbb{Q}/\mathbb{Z})}\big) \;=\; \underset{\text{definition of } \beta}{\uparrow}\; \big[\partial\big(\sigma + \tfrac{q}{p}\cdot\mu\big)\big] \;=\; \underset{\text{since } \partial(\sigma) = a - \frac{q}{p}\cdot b}{\uparrow}\; \big[\widetilde{a} - \tfrac{q}{p}\cdot\widetilde{b} + \tfrac{q}{p}\cdot\widetilde{b}\big] \;=\; i_*(a) \in \mathrm{H}_1(L(p,q);\mathbb{Q}/\mathbb{Z}).$$

Since $\mu \in \mathrm{C}_2(X;\mathbb{Z})$ and since $\sigma \in \mathrm{C}_2(Y;\mathbb{Q})$ we have the equality

$$h_*\big(\rho_*\big(\sigma + \tfrac{q}{p}\mu\big)\big) \;=\; j_*(\rho_*(\sigma)) \in \mathrm{C}_2(L(p,q), X;\mathbb{Q}/\mathbb{Z}).$$

We conclude that we have the following equalities in $\mathrm{H}_1(F;\mathbb{Q}/\mathbb{Z})$:

$$\begin{aligned}\Theta(i_*(a)) \;&\underset{\text{since } \beta([\rho_*(\sigma + \frac{q}{p}\cdot\mu)]) = i_*(a)}{=}\; \partial\big(j_*^{-1}\big(h_*\big(\big[\rho_*\big(\sigma + \tfrac{q}{p}\mu\big)\big]\big)\big)\big) \;\underset{\text{since } h_*(\rho_*(\sigma + \frac{q}{p}\mu)) = j_*(\rho_*(\sigma))}{=}\; \partial([\rho_*(\sigma)]) \\ &\underset{\text{since } \rho_*(\sigma) \text{ lives in } \mathrm{C}_2(Y;\mathbb{Q}/\mathbb{Z}) \text{ we can calculate the connecting homomorphism of } [\rho_*(\sigma)] \text{ and we obtain } [\rho_*(\partial(\sigma))]}{=}\; [\rho_*(\partial(\sigma))] \;\underset{\text{by the above}}{=}\; \big[\rho_*\big(\widetilde{a} - \tfrac{q}{p}\cdot\widetilde{b}\big)\big] \\ &=\; \rho_*\big(a - \tfrac{q}{p}\cdot b\big).\end{aligned}$$

■

Now it is straightforward to provide the proof of the Lens Space-Linking Pairing Proposition 148.20.

Proof of the Lens Space-Linking Pairing Proposition 148.20. Let $p, q \in \mathbb{N}$ be coprime. We use the notation F, X, Y and a, b as above. We write $L = L(p,q)$. We denote by $\rho\colon \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ the natural projection. To simplify the notation we use the convention that given any abelian group A we make the usual identification $\mathrm{H}_0(L;A) = A$ and $\mathrm{H}_0(F;A) = A$ from page 1914.

We start out this proof with the following claim.

Claim. If $x, y \in \mathrm{H}_1(F;\mathbb{Z})$, then

$$\mathrm{PD}_F^{\mathbb{Z}}(x) \cap y \;=\; Q_F(y,x) \;\in\; \mathrm{H}_0(F;\mathbb{Z}) = \mathbb{Z}.$$

Proof. This claim follows from a calculation which by now is standard fare for us. For completeness' sake we carry out the details. We have

$$\begin{aligned}\mathrm{PD}_F^{\mathbb{Z}}(x) \cap y \;&\underset{\text{by definition of } \mathrm{PD}_F^{\mathbb{Z}}}{=}\; \mathrm{PD}_F^{\mathbb{Z}}(x) \cap \big(\mathrm{PD}_F^{\mathbb{Z}}(y) \cap [F]\big) \;\underset{\text{Cup-Cap Lemma 116.7}}{=}\; \big(\mathrm{PD}_F^{\mathbb{Z}}(y) \cup \mathrm{PD}_F^{\mathbb{Z}}(x)\big) \cap [F] \\ &\underset{\text{Cap Product-Kronecker Lemma 116.4}}{=}\; \big\langle \mathrm{PD}_F^{\mathbb{Z}}(y) \cup \mathrm{PD}_F^{\mathbb{Z}}(x), [F]\big\rangle \;\underset{\text{by definition of } Q_F}{=}\; Q_F(y,x).\end{aligned}$$

⊞

From Lemma 148.26 we know that the map

$$\begin{array}{ccc} \mathbb{Z}_p & \to & \mathrm{H}_1(L(p,q);\mathbb{Z}) \\ x & \mapsto & x \cdot i_*(a) \end{array}$$

is an isomorphism. Thus it remains to prove the following claim.

Claim.

$$\lambda_L(i_*(a), i_*(a)) \;=\; \frac{q}{p} \in \mathbb{Q}/\mathbb{Z}.$$

Proof. We calculate that

$$\begin{array}{lll}
\lambda_L(i_*(a), i_*(a)) & = \overbrace{\beta^{-1}(\mathrm{PD}_L^{\mathbb{Z}}(i_*(a)))}^{\in \mathrm{H}^1(L;\mathbb{Q}/\mathbb{Z})} \cap \overbrace{i_*(a)}^{\in \mathrm{H}_1(L;\mathbb{Z})} & = \overbrace{i^*\big(\mathrm{PD}_F^{\mathbb{Q}/\mathbb{Z}}(\Theta(i_*(a)))\big)}^{\in \mathrm{H}^1(L;\mathbb{Q}/\mathbb{Z})} \cap \overbrace{i_*(a)}^{\in \mathrm{H}_1(L;\mathbb{Z})} \\
\uparrow & & \uparrow \\
\text{Lemma 148.19} & & \text{Lemma 148.25 applied to } i_*(a) \in \mathrm{H}_1(L;\mathbb{Z}) \\
= \mathrm{PD}_F^{\mathbb{Q}/\mathbb{Z}}\big(\rho_*\big(a - \tfrac{q}{p}\cdot b\big)\big) \cap a & & = \rho_*\big(\mathrm{PD}_F^{\mathbb{Q}}\big(a - \tfrac{q}{p}\cdot b\big)\big) \cap a \\
\uparrow & & \uparrow \\
\text{Lemmas 148.9 (4) and 148.26 (2)} & & \text{Lemma 148.11} \\
= \rho\big(\mathrm{PD}_F^{\mathbb{Q}}\big(a - \tfrac{q}{p}\cdot b\big) \cap a\big) & & = \rho\big(\mathrm{PD}_F^{\mathbb{Z}}(a) \cap a - \tfrac{q}{p}\cdot\big(\mathrm{PD}_F^{\mathbb{Z}}(b) \cap a\big)\big) \\
\uparrow & & \uparrow \\
\text{Lemma 148.9 (3)} & & \text{Lemma 148.11} \\
= \rho\big(Q_F(a,a) - \tfrac{q}{p}Q_F(a,b)\big) & & = -\rho\big(\tfrac{q}{p}\big). \\
\uparrow & & \uparrow \\
\text{by the claim} & & \text{by page 2881 we have } Q_F(a,a) = 0 \text{ and } Q_F(a,b) = 1
\end{array}$$

∎

148.8. **Computing the linking pairing by intersecting singular simplices.** Many working topologists would rather forgo the austere beauty of cup and cap products and they would rather work and calculate by "intersecting simplices". The following proposition shows that in many (most? all?) situations one can calculate linking pairings geometrically by determining algebraic intersection numbers of singular chains.

Proposition 148.27. Let $n \geq 1$, let M be an oriented $(2n+1)$-dimensional rational homology sphere and let $x, y \in \mathrm{H}_n(M;\mathbb{Z})$. Let σ and τ be cycles in $\mathrm{C}_n(M;\mathbb{Z})$ that represent x and y. From Lemma 148.1 we know that $\mathrm{H}_n(M;\mathbb{Z})$ is torsion, hence there exists a $d \in \mathbb{N}$ such that $d \cdot x = 0 \in \mathrm{H}_n(M;\mathbb{Z})$ which means that $d \cdot \sigma$ is null-homologous. Suppose $\Sigma \in \mathrm{C}_{n+1}(M;\mathbb{Z})$ is a singular chain with $\partial\Sigma = d \cdot \sigma$. If Σ and τ intersect transversally in the sense of the definition on page 2785, then

$$\lambda_M(x,y) \;=\; (-1)^n \cdot \frac{1}{d} \cdot \underbrace{\Sigma \cdot \tau}_{\substack{\text{algebraic intersection} \\ \text{number as defined on page 2785}}} \;\in \mathbb{Q}/\mathbb{Z}.$$

Remark. Alas, all too often in the literature the right-hand side of Proposition 148.27 gets confused with the actual definition of the linking pairing, see e.g. [**Frie2004**, p. 899] for a typical example.

The proof of the proposition rests on a variation on Theorem 127.9. Before we can formulate that variation we make the following observation.

Observation. Let M be an oriented m-dimensional topological manifold and let A and B be abelian groups. Basically the same definitions as on page 2785 allow us to say that chains $\sigma \in \mathrm{C}_k(M;A)$ and $\tau \in \mathrm{C}_{m-k}(M;B)$ are *transverse* and they allow us to define the *algebraic intersection invariant* $\sigma \cdot_{\otimes} \tau \in A \otimes B$.

Now we can formulate the following slight variation on Theorem 127.9.

Proposition 148.28. Let M be a closed oriented m-dimensional topological manifold, let A and B be abelian groups and let $\sigma \in \mathrm{C}_k(M;A)$ and $\tau \in \mathrm{C}_{m-k}(M;B)$ be two cycles. If σ and τ are transverse, then

$$\epsilon_M\Big(\big(\overbrace{\mathrm{PD}_M^A([\sigma])}^{\in \mathrm{H}^{m-k}(M;A)} \underbrace{\cup_\otimes}_{\in \mathrm{H}^m(M;A\otimes B)} \overbrace{\mathrm{PD}_M^B([\tau])}^{\in \mathrm{H}^k(M;B)}\big) \cap [M]\Big) \;=\; \sigma \cdot_\otimes \tau \;\in\; A \otimes B.$$

Proof. The proof of Theorem 127.9 generalizes without a hitch to give us this statement with more fancy coefficients. ∎

Proof of Proposition 148.27. As before we denote by $\nu\colon \mathbb{Q}/\mathbb{Z}\otimes\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ the multiplication induced isomorphism. We perform the following calculation:

$$\begin{aligned}
\lambda_M(x,y) &\overset{\text{by the definition on page 3151, Lemma 148.9 (2) and Lemma 148.9 (3) applied to } \nu}{\downarrow} \\
&= \nu_*\big(\epsilon_M\big(\big(\beta^{-1}(\mathrm{PD}_M^{\mathbb{Z}}(x)) \cup_\otimes \mathrm{PD}_M^{\mathbb{Z}}(y)\big) \cap [M]\big)\big) \\
&= (-1)^n \cdot \nu_*\big(\epsilon_M\big(\big(\mathrm{PD}_M^{\mathbb{Q}/\mathbb{Z}}(\beta^{-1}(x)) \cup_\otimes \mathrm{PD}_M^{\mathbb{Z}}(y)\big) \cap [M]\big)\big) \\
&\overset{\text{Lemma 148.24}}{\uparrow} \\
&= (-1)^n \cdot \nu_*\big(\big(\tfrac{1}{d}\otimes\Sigma\big) \cdot_\otimes \tau\big) \;\underset{\text{follows immediately from the definitions}}{=}\; (-1)^n \cdot \tfrac{1}{d}\cdot\Sigma\cdot\tau \;\in\; \mathbb{Q}/\mathbb{Z}.
\end{aligned}$$

$\uparrow$ by Proposition 148.28 and the observation that $\frac{1}{d}\otimes\Sigma$ represents $\beta^{-1}(x)$

∎

148.9. **Linking pairings and intersection forms.** Let M be a compact oriented connected 4-dimensional topological manifold. We suppose that $Y = \partial M$ is a rational homology sphere. In this section we want to study, in favorable situations, the relationship between the intersection form Q_M on M and the linking pairing λ_Y on Y. Before we can state the relevant result let us recall the following definitions from pages 2800 and 2895.

Definition. Let M be a compact oriented connected $2n$-dimensional topological manifold M. The intersection form of M is defined as the form[1764]

$$\begin{aligned}
Q_M\colon \mathrm{FH}_n(M;\mathbb{Z}) \times \mathrm{FH}_n(M;\mathbb{Z}) &\to \mathbb{Z} \\
(a,b) &\mapsto \Big\langle \underbrace{\underbrace{\mathrm{PD}_M(a)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(b)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})}}_{\in \mathrm{FH}^{2n}(M,\partial M;\mathbb{Z})}, [M] \Big\rangle.
\end{aligned}$$

Furthermore the asymmetric intersection pairing of M is defined as the map

$$\begin{aligned}
\widetilde{Q}_M\colon \mathrm{FH}_n(M;\mathbb{Z}) \times \mathrm{FH}_n(M,\partial M;\mathbb{Z}) &\to \mathbb{Z} \\
(v,w) &\mapsto \Big\langle \underbrace{\underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{FH}^n(M;\mathbb{Z})}}_{\in \mathrm{FH}^{2n}(M,\partial M;\mathbb{Z})}, [M] \Big\rangle
\end{aligned}$$

For the reader's convenience we recall the following proposition.

[1764] Here, given an abelian group H we write $\mathrm{F}H := H/\text{torsion subgroup}$ for the maximal torsion-free quotient of H.

Proposition 132.15 (Reduced Intersection Form Proposition). Let M be a compact oriented connected $2n$-dimensional topological manifold. We denote by $p_*\colon \mathrm{H}_n(M;\mathbb{Z}) \to \mathrm{H}_n(M,\partial M;\mathbb{Z})$ the obvious map.

(1) The pairing $Q_M^{\mathrm{as}}\colon \mathrm{FH}_n(M;\mathbb{Z}) \times \mathrm{FH}_n(M,\partial M;\mathbb{Z}) \to \mathbb{Z}$ is non-singular.

(2) The following diagram commutes:

$$\begin{array}{ccc} \mathrm{FH}_n(M;\mathbb{Z}) \times \mathrm{FH}_n(M;\mathbb{Z}) & \xrightarrow{\quad Q_M \quad} & \mathbb{Z} \\ \downarrow{\scriptstyle \mathrm{id}\times p_*} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{FH}_n(M;\mathbb{Z}) \times \mathrm{FH}_n(M,\partial M;\mathbb{Z}) & \xrightarrow{\quad Q_M^{\mathrm{as}} \quad} & \mathbb{Z}. \end{array}$$

(3) Let $v_1,\dots,v_m$ be a basis of the free abelian group $\mathrm{FH}_n(M;\mathbb{Z})$. By (1) together with Exercise 121.2 (a) there exists a unique basis $w_1,\dots,w_m$ of the free abelian group $\mathrm{FH}_k(M,\partial M;\mathbb{Z})$ with $Q_M^{\mathrm{as}}(v_i,w_j)=\delta_{ij}$ for all $i,j\in\{1,\dots,m\}$. We denote by A the $(m\times m)$–matrix that represents the intersection form Q_M with respect to the basis $v_1,\dots,v_n$, i.e. we have $A=(a_{ij})$ with $a_{ij}=Q_M(v_i,v_j)$. This matrix represents the map $p_*\colon \mathrm{FH}_n(M;\mathbb{Z}) \to \mathrm{FH}_n(M,\partial M;\mathbb{Z})$ with respect to the bases $v_1,\dots,v_m$ and $w_1,\dots,w_m$.

The following proposition is the main result of this section.

Proposition 148.29. (Linking Pairing-via-Intersection Form Proposition) Let M be a compact oriented connected 4-dimensional topological manifold such that $Y=\partial M$ is a rational homology sphere. We suppose that $\mathrm{H}_1(M;\mathbb{Z})=0$. The following statements hold:

(1) The groups $\mathrm{H}_2(M,Y;\mathbb{Z})$ and $\mathrm{H}_2(M;\mathbb{Z})$ are free abelian groups of finite rank.

(2) By (1) we can pick a basis $v_1,\dots,v_m$ for the free abelian group $\mathrm{H}_2(M;\mathbb{Z})$. By the Asymmetric Intersection Pairing Proposition 132.15 together with Exercise 121.2 (a) there exists a unique basis $w_1,\dots,w_m$ for the free abelian group $\mathrm{H}_2(M,Y;\mathbb{Z})$ with $Q_M^{\mathrm{as}}(v_i,w_j)=\delta_{ij}$. We denote by $A=(Q_M(v_i,v_j))$ the $(m\times m)$–matrix that represents the intersection form with respect to the basis $v_1,\dots,v_m$. The following three statements hold:

(a) If we denote by $\Psi\colon \mathbb{Z}^m \to \mathrm{H}_2(M,Y;\mathbb{Z})$ the isomorphism that is given by the basis $w_1,\dots,w_m$, then the map

$$\begin{array}{rcl} \Theta\colon \mathbb{Z}^m/A\mathbb{Z}^m & \to & \mathrm{H}_1(Y;\mathbb{Z}) \\ [u] & \mapsto & \partial(\Psi(u)) \end{array}$$

∂: connecting homomorphism $\partial\colon \mathrm{H}_2(M,Y;\mathbb{Z}) \to \mathrm{H}_1(Y;\mathbb{Z})$ of the pair (M,Y)

is well-defined and it is an isomorphism.

(b) The matrix A is symmetric and it satisfies $\det(A)\neq 0$.

(c) The isomorphism Θ from (a) is in fact an isometry from[1765]

$$\begin{array}{rcl} \mathbb{Z}^m/A\mathbb{Z}^m \times \mathbb{Z}^m/A\mathbb{Z}^m & \xrightarrow{\lambda(A)} & \mathbb{Q}/\mathbb{Z} \\ ([v],[w]) & \longmapsto & v^T A^{-1} w \end{array} \quad \text{to} \quad \underbrace{\mathrm{H}_1(Y;\mathbb{Z}) \times \mathrm{H}_1(Y;\mathbb{Z}) \xrightarrow{\lambda_Y} \mathbb{Q}/\mathbb{Z}.}_{\text{linking pairing}}$$

Remark.

[1765] It follows from (b) and the Linking Form-Matrix Lemma 148.13 that the form $\lambda(A)$ is in fact defined.



(1) We leave it to the reader to figure out what is a correct higher-dimensional analogue of the Linking Pairing-via-Intersection Form Proposition 148.29. The reader will found out, that determining the correct sign is a never ending source of hilarious entertainment.
(2) Often it is (psychologically) easier to compute intersection forms of smooth manifolds, thus on many occasions in the literature linking pairings are computed by calculating the intersection form of a suitable bounding smooth manifold.

The following lemma is the key technical ingredient in the proof of the Linking Pairing-via-Intersection Form Proposition 148.29.

Lemma 148.30. Let M be a compact oriented connected 4-dimensional topological manifold such that $Y = \partial M$ is a rational homology sphere. We denote by $i\colon Y \to M$ the inclusion and we denote by $p\colon \mathrm{C}_*(M) \to \mathrm{C}_*(M,Y)$ the projection . The following diagram commutes up to a minus sign:

$$\begin{array}{ccccc} \mathrm{H}^2(M;\mathbb{Z}) & \xrightarrow{i^*} & \mathrm{H}^2(Y;\mathbb{Z}) & \xrightarrow[\cong]{\beta^{-1}} & \mathrm{H}^1(Y;\mathbb{Q}/\mathbb{Z}) \\ \downarrow & & & & \downarrow{\scriptstyle \delta} \\ \mathrm{H}^2(M;\mathbb{Q}) & \xrightarrow[\cong]{(p^*)^{-1}} & \mathrm{H}^2(M,Y;\mathbb{Q}) & \longrightarrow & \mathrm{H}^2(M,Y;\mathbb{Q}/\mathbb{Z}). \end{array}$$

Remark. The encyclopedic reader will not have failed to notice that Lemma 148.30 is a special case of Exercise 74.13.

Proof of Lemma 148.30. Throughout the proof we adopt the following conventions:

(1) Given a cochain ψ that takes values in $\mathbb{Q}$ we denote by $\overline{\psi}$ the corresponding cochain that takes values in $\mathbb{Q}/\mathbb{Z}$.
(2) If ψ is a cochain that takes values in $\mathbb{Z}$, then we view it also as a cochain with values in $\mathbb{Q}$.

Now let $\phi \in \mathrm{C}^2(M;\mathbb{Z}) = \mathrm{Hom}(\mathrm{C}_2(M),\mathbb{Z})$ be a cochain. Note that the Poincaré Duality Theorem 119.1 implies that $\mathrm{H}^i(Y;\mathbb{Q}) \cong \mathrm{H}_i(Y;\mathbb{Q}) = 0$ for $i = 1,2$. Thus we obtain from the long exact sequence of the pair (M,Y) that the map $p^*\colon \mathrm{H}^2(M,Y;\mathbb{Q}) \to \mathrm{H}^2(M;\mathbb{Q})$ is an isomorphism. In particular we see that there exists a cochain $\psi \in \mathrm{C}^2(M,Y;\mathbb{Q})$ such that $[\phi] = [p^*\psi] \in \mathrm{H}^2(M;\mathbb{Q})$. This means that there exists some $\alpha \in \mathrm{C}^1(M;\mathbb{Q})$ such that $p^* \circ \psi - \phi = \delta\alpha$. We write $\varphi := \alpha|_{\mathrm{C}_1(Y)}$. We calculate that

$$\delta\overline{\varphi} \underset{\text{restriction is a cochain map}}{\overset{\uparrow}{=}} \overline{\delta\alpha}|_{\mathrm{C}_2(Y)} = \overline{p^*\circ\psi - \phi}|_{\mathrm{C}_2(Y)} \underset{\text{since } p^*\circ\psi \text{ vanishes on } \mathrm{C}_2(Y)}{\overset{\uparrow}{=}} \overline{\phi}|_{\mathrm{C}_2(Y)} \underset{\text{since } \phi \text{ takes integral values.}}{\overset{\uparrow}{=}} 0 \in \mathrm{C}^2(Y;\mathbb{Q}/\mathbb{Z}).$$

In other words, $\overline{\varphi}$ represents an element in $\mathrm{H}^1(Y;\mathbb{Q}/\mathbb{Z})$. The following claim computes the two possible connecting homomorphisms.

Claim. We have $\delta([\overline{\varphi}]) = [\overline{\psi}]$ and $\beta([\overline{\varphi}]) = -[i^*(\phi)]$.

Proof. The claim follows from considering the following two sequences of maps and elements:

$$\underbrace{\begin{array}{ccc} \overset{(4)\ \overline{\psi}}{\mathrm{C}^2(M,Y;\mathbb{Q}/\mathbb{Z})} & \overset{p^*}{\to} & \overset{(3)\ \overline{p^*\psi-\phi=p^*\psi}}{\mathrm{C}^2(M;\mathbb{Q}/\mathbb{Z})} \\ & & \delta\uparrow \\ & & \mathrm{C}^1(M;\mathbb{Q}/\mathbb{Z}) \overset{i^*}{\to} \underset{(1)\ \overline{\varphi}}{\mathrm{C}^1(Y;\mathbb{Q}/\mathbb{Z})} \end{array}}_{\text{defines the connecting homomorphism } \delta} \quad \& \quad \underbrace{\begin{array}{ccc} \overset{(4)\ -i^*\phi}{\mathrm{C}^2(Y;\mathbb{Z})} & \to & \overset{(3)\ \overbrace{i^*p^*\psi}^{=0}-i^*\phi}{\mathrm{C}^2(Y;\mathbb{Q})} \\ & & \delta\uparrow \\ & & \underset{(2)\ \varphi=i^*\alpha}{\mathrm{C}^1(Y;\mathbb{Q})} \longrightarrow \underset{(1)\ \overline{\varphi}}{\mathrm{C}^1(Y;\mathbb{Q}/\mathbb{Z})}. \end{array}}_{\text{defines the Bockstein homomorphism } \beta}$$

⊞

Now we are basically done. We consider $[\phi] \in \mathrm{H}^2(M;\mathbb{Z})$ and we consider its two possible journeys to $\mathrm{H}^2(M,Y;\mathbb{Q}/\mathbb{Z})$ through the following diagram:

$$\begin{array}{ccccc} \mathrm{H}^2(M;\mathbb{Z}) & \xrightarrow{i^*} & \mathrm{H}^2(Y;\mathbb{Z}) & \xrightarrow[\cong]{\beta^{-1}} & \mathrm{H}^1(Y;\mathbb{Q}/\mathbb{Z}) \\ \downarrow\Theta & & & & \downarrow\delta \\ \mathrm{H}^2(M;\mathbb{Q}) & \xrightarrow[\cong]{(p^*)^{-1}} & \mathrm{H}^2(M,Y;\mathbb{Q}) & \xrightarrow[[\omega]\mapsto[\overline{\omega}]]{\Omega} & \mathrm{H}^2(M,Y;\mathbb{Q}/\mathbb{Z}). \end{array}$$

First we go down and right. We see that

$$\Omega((p^*)^{-1}(\Theta([\phi\colon \mathrm{C}_2(M) \to \mathbb{Z}]))) \underset{\text{definition of } \Theta}{=} \Omega((p^*)^{-1}([\phi\colon \mathrm{C}_2(M) \to \mathbb{Q}])) \underset{\text{since } [\phi]=[p^*\psi]}{=} \Omega([\psi]) = [\overline{\psi}]$$

Next we go right and down. We see that

$$\delta(\beta^{-1}(i^*([\phi]))) = \delta(\beta^{-1}([i^*(\phi)])) \underset{\text{by the claim}}{=} -\delta([\overline{\varphi}]) \underset{\text{by the claim}}{=} -[\overline{\psi}]$$

This shows that the two possible outcomes differ by a minus sign. ■

The following lemma relates the linking pairing of a rational homology sphere to the intersection form of a bounding smooth manifold. The lemma is of interest also outside of the application to the Linking Pairing-via-Intersection Form Proposition 148.29.

Lemma 148.31. Let M be a compact oriented connected 4-dimensional topological manifold such that its boundary $Y = \partial M$ is a rational homology sphere. There exists a pairing $\chi\colon \mathrm{H}_2(M,Y;\mathbb{Z}) \times \mathrm{H}_2(M,Y;\mathbb{Z}) \to \mathbb{Q}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_2(M;\mathbb{Z}) \times \mathrm{H}_2(M;\mathbb{Z}) & \xrightarrow{Q_M} & \mathbb{Z} \\ \downarrow{\scriptstyle p_*\times p_*} & & \downarrow \\ \mathrm{H}_2(M,Y;\mathbb{Z}) \times \mathrm{H}_2(M,Y;\mathbb{Z}) & \xrightarrow{\chi} & \mathbb{Q} \\ \downarrow{\scriptstyle \partial\times\partial} & & \downarrow \\ \mathrm{H}_1(Y;\mathbb{Z}) \times \mathrm{H}_1(Y;\mathbb{Z}) & \xrightarrow[\lambda_Y]{} & \mathbb{Q}/\mathbb{Z}. \end{array}$$

Proof. Let M be a compact oriented connected 4-dimensional topological manifold such that its boundary $Y = \partial M$ is a rational homology sphere. In the following we denote by $\varphi\colon \mathbb{Z} \to \mathbb{Q}$ the inclusion. Next we consider the following diagram:

$$\begin{array}{ccccc} \mathrm{H}_2(M;\mathbb{Z}) & \xrightarrow{p_*} & \mathrm{H}_2(M,Y;\mathbb{Z}) & \xrightarrow{\partial} & \mathrm{H}_1(Y;\mathbb{Z}) \\ \cong\downarrow \mathrm{PD}_M & & \cong\downarrow \mathrm{PD}_M & & \cong\downarrow \mathrm{PD}_Y \\ \mathrm{H}^2(M,Y;\mathbb{Z}) & \xrightarrow{p^*} & \mathrm{H}^2(M;\mathbb{Z}) & \xrightarrow{i^*} & \mathrm{H}^2(Y;\mathbb{Z}). \end{array}$$

By Proposition 119.25 the diagram commutes.[1766] The diagram with the vertical isomorphisms allows us to translate our above problem into a different task. Namely it follows from the diagram and the definition of the intersection form Q_M that it suffices to show that there exists a pairing $\chi\colon \mathrm{H}^2(M;\mathbb{Z}) \times \mathrm{H}^2(M;\mathbb{Z}) \to \mathbb{Q}$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}^2(M,Y\,\mathbb{Z}) \times \mathrm{H}^2(M,Y;\mathbb{Z}) & \xrightarrow{\cup} & \mathrm{H}^4(M,Y;\mathbb{Z}) & \xrightarrow{\langle -,[M]\rangle} & \mathbb{Z} \\ \downarrow p^*\times p^* & & & & \downarrow \varphi \\ \mathrm{H}^2(M;\mathbb{Z}) \times \mathrm{H}^2(M;\mathbb{Z}) & & \xrightarrow{\chi} & & \mathbb{Q} \\ \downarrow \mathrm{PD}_Y^{-1}\circ i^* \times \mathrm{PD}_Y^{-1}\circ i^* & & & & \downarrow \\ \mathrm{H}_1(Y;\mathbb{Z}) \times \mathrm{H}_1(Y;\mathbb{Z}) & & \xrightarrow[\lambda_Y]{} & & \mathbb{Q}/\mathbb{Z}. \end{array}$$

The key to proving the existence of such a pairing χ is the following daunting[1767] diagram:

$$\begin{array}{ccccccc}
\mathrm{H}^2(M,Y;\mathbb{Z})\times\mathrm{H}^2(M,Y;\mathbb{Z}) & & \xrightarrow{\cup} & & \mathrm{H}^4(M,Y;\mathbb{Z}) & \xrightarrow{\langle -,[M]\rangle} & \mathbb{Z} \\
\downarrow & \searrow^{\varphi_*\times\mathrm{id}} & \text{(I)} & & \downarrow\varphi_* & \text{(II)} & \downarrow\varphi \\
 & \mathrm{H}^2(M,Y;\mathbb{Q})\times\mathrm{H}^2(M,Y;\mathbb{Z}) & \xrightarrow[\cup]{} & \mathrm{H}^4(M,Y;\mathbb{Q}) & & \xrightarrow[\langle -,[M]\rangle]{} & \mathbb{Q} \\
p^*\times p^* \quad \text{(III)} & \mathrm{id}\downarrow \quad \downarrow p^* & \text{(IV)} & \downarrow = & & \text{(V)} & \downarrow = \\
 & \mathrm{H}^2(M,Y;\mathbb{Q})\times\mathrm{H}^2(M;\mathbb{Z}) & \xrightarrow[\cup]{} & \mathrm{H}^4(M,Y;\mathbb{Q}) & & \xrightarrow[\langle -,[M]\rangle]{} & \mathbb{Q} \\
 & p^*\downarrow \quad \downarrow \mathrm{id} & & & & & \\
\boxed{\mathrm{H}^2(M;\mathbb{Z})\times\mathrm{H}^2(M;\mathbb{Z})} & \xrightarrow{\varphi_*\times\mathrm{id}} & \boxed{\mathrm{H}^2(M;\mathbb{Q})\times\mathrm{H}^2(M;\mathbb{Z})} & & \text{(VI)} & & = \\
\downarrow & & (p^*)^{-1}\downarrow \quad \downarrow \mathrm{id} & & & & \downarrow \\
 & & \boxed{\mathrm{H}^2(M,Y;\mathbb{Q})\times\mathrm{H}^2(M;\mathbb{Z})} & \xrightarrow{\cup} & \boxed{\mathrm{H}^4(M,Y;\mathbb{Q})} & \xrightarrow{\langle -,[M]\rangle} & \boxed{\mathbb{Q}} \\
i^*\times i^* \quad \text{(VII)} & & \downarrow \quad \downarrow \mathrm{id} & \text{(VIII)} & \downarrow & \text{(IX)} & \downarrow \\
 & & \mathrm{H}^2(M,Y;\mathbb{Q}/\mathbb{Z})\times\mathrm{H}^2(M;\mathbb{Z}) & \xrightarrow{\cup} & \mathrm{H}^4(M,Y;\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\langle -,[M]\rangle} & \mathbb{Q}/\mathbb{Z} \\
 & & \delta\uparrow \quad \downarrow i^* & \text{(X)} & \delta\uparrow & \text{(XI)} & \downarrow = \\
\mathrm{H}^2(Y;\mathbb{Z})\times\mathrm{H}^2(Y;\mathbb{Z}) & \xrightarrow{\beta^{-1}\times\mathrm{id}} & \mathrm{H}^1(Y;\mathbb{Q}/\mathbb{Z})\times\mathrm{H}^2(Y;\mathbb{Z}) & \xrightarrow{\cup} & \mathrm{H}^3(Y;\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\langle -,[Y]\rangle} & \mathbb{Q}/\mathbb{Z} \\
\uparrow \mathrm{PD}_Y\times\mathrm{PD}_Y & & \text{(XII)} & & & & \downarrow = \\
\mathrm{H}_1(Y;\mathbb{Z}) \times \mathrm{H}_1(Y;\mathbb{Z}) & & \xrightarrow[\lambda_Y]{} & & & & \mathbb{Q}/\mathbb{Z}.
\end{array}$$

[1766] To be more precise, the square on the right commutes up to the $(-1)^2$, which happens to be $+1$.

[1767] As a wise person once said: if you want to do topology rigorously, then the diagrams become large.

The promised pairing χ is given by the traveling along the route that is traced out by the boxes. Thus it remains to prove that the diagram commutes. We make the following clarifications and observations:

(I) The lower cup product is the composition of the cup product as defined in Lemma 148.5 with the obvious map $\mathbb{Z} \otimes \mathbb{Q} \xrightarrow{\cong} \mathbb{Q}$. It follows basically immediately from the definitions that the square commutes, see also the Cup Product-Naturality Proposition 115.4 (2).
(II) This square commutes by Lemma 109.6 (2).
(III) This square commutes since changing coefficients is a natural transformation, see Lemma 108.9.
(IV) This square commutes by a mild generalization of the Cup Product-Naturality Proposition 115.4 (1).
(V) What a relief, there is nothing to show.
(VI) Note that the two vertical maps to the left of the square are inverses of one another. So we see that nothing happens in this square.
(VII) This rectangle commutes by Lemma 148.30.
(VIII) Similar to (I) the cup products are the composition of the cup product as defined in Lemma 148.5 with the obvious maps $\mathbb{Z} \otimes \mathbb{Q} \xrightarrow{\cong} \mathbb{Q}$ and $\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z} \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z}$. As in (I) we see that the square commutes.
(IX) By the same argument as in (II) we see that the square commutes.
(X) A slight generalization of the Cup Product-Connecting Homomorphism Lemma 115.6 to more general coefficients shows that the square commutes up to the sign $(-1)^2 = 1$.
(XI) By the Fundamental Class-of-Boundary Proposition 106.27 we have $\partial([M]) = [Y]$. It follows from a minute generalization of the Cap Product-Connecting Homomorphism Lemma 116.10 (1) that this square commutes.
(XII) This rectangle commutes by definition of the linking pairing. ■

Proof of the Linking Pairing-via-Intersection Form Proposition 148.29. We consider the following maps:

$$\begin{array}{ccccccccc}
\mathrm{H}_2(M;\mathbb{Z}) & \xrightarrow{p_*} & \mathrm{H}_2(M,Y;\mathbb{Z}) & \xrightarrow{\partial} & \mathrm{H}_1(Y;\mathbb{Z}) & \to & \overbrace{\mathrm{H}_1(M;\mathbb{Z})}^{=0 \text{ by hypothesis}} & \to & \mathrm{H}_1(M,Y;\mathbb{Z}) \to 0. \\
\cong\downarrow \mathrm{PD}_M & & \cong\downarrow \mathrm{PD}_M & & & & & & \\
\mathrm{H}^2(M,Y;\mathbb{Z}) & & \mathrm{H}^2(M;\mathbb{Z}) & & & & & & \\
\cong\downarrow \mathrm{ev} & & \cong\downarrow \mathrm{ev} & & & & & & \\
\mathrm{Hom}(\mathrm{H}_2(M,Y;\mathbb{Z}),\mathbb{Z}) & & \mathrm{Hom}(\mathrm{H}_2(M;\mathbb{Z}),\mathbb{Z}). & & & & & &
\end{array}$$

We make the following observations:

(a) The horizontal sequence is the long exact sequence in reduced homology of the pair (M,Y).
(b) By hypothesis we have $\mathrm{H}_1(M;\mathbb{Z}) = 0$.
(c) It follows from (a) and (b) that $\mathrm{H}_1(M,Y;\mathbb{Z}) = 0$.
(d) The vertical maps PD_M are the Poincaré Duality isomorphisms given by the Poincaré Duality Theorem 119.1.

(e) It follows from (b), (c) and the Universal Coefficient Theorem 110.12 that both evaluation homomorphisms ev are isomorphisms.

(f) We obtain from the above discussion that $\mathrm{H}_2(M;\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_2(M,Y;\mathbb{Z}),\mathbb{Z})$ and that $\mathrm{H}_2(M,Y;\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_2(M;\mathbb{Z}),\mathbb{Z})$. It follows from the Topological Manifolds-Invariants Proposition 104.14 (4) and Lemma 96.8 (1) together with the Finitely Generated Abelian Group Classification Theorem 51.4 that $\mathrm{H}_2(M;\mathbb{Z})$ and $\mathrm{H}_2(M,Y;\mathbb{Z})$ are free abelian groups. In particular we have now shown Statement (1) of the proposition.

We pick a basis $v_1,\dots,v_m$ for the free abelian group $\mathrm{H}_2(M;\mathbb{Z})$. We denote by $w_1,\dots,w_m$ the unique basis for the free abelian group $\mathrm{H}_2(M,Y;\mathbb{Z})$ with $Q_M^{\mathrm{as}}(v_i,w_j)=\delta_{ij}$. We denote by $A=(a_{ij})$ the $(m\times m)$–matrix that is given by $a_{ij}=Q_M(v_i,v_j)$. Note that it follows from the Intersection Form Proposition 132.4 (2) that A is symmetric.

We denote by $\Phi\colon \mathbb{Z}^m \xrightarrow{\cong} \mathrm{H}_2(M;\mathbb{Z})$ and $\Psi\colon \mathbb{Z}^m \xrightarrow{\cong} \mathrm{H}_2(M,Y;\mathbb{Z})$ the isomorphisms that are given by the above bases. From the above long exact sequence together with the Asymmetric Intersection Pairing Proposition 132.15 we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
\mathbb{Z}^m & \xrightarrow{u\mapsto A\cdot u} & \mathbb{Z}^m & \longrightarrow & \mathbb{Z}^m/A\mathbb{Z}^m & \longrightarrow & 0 \\
\downarrow{\scriptstyle\Phi}\cong & & \cong\downarrow{\scriptstyle\Psi} & & \vdots{\scriptstyle\Theta} & & \\
\mathrm{H}_2(M;\mathbb{Z}) & \xrightarrow{p_*} & \mathrm{H}_2(M,Y;\mathbb{Z}) & \xrightarrow{\partial} & \mathrm{H}_1(Y;\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

Note that the lower sequence is exact by the discussion above. Furthermore note that the diagram commutes by the Asymmetric Intersection Pairing Proposition 132.15 (2). It follows immediately that the map $\partial\circ\Psi\colon \mathbb{Z}^m \to \mathrm{H}_1(Y;\mathbb{Z})$ descends to an isomorphism $\mathbb{Z}^m/A\mathbb{Z}^m \to \mathrm{H}_1(Y;\mathbb{Z})$. This concludes the proof of statement (2a).

Next note that it follows from statement (2a), the fact that $\mathrm{H}_1(Y;\mathbb{Z})$ is finite and Lemma 51.8 that $\det(A)\neq 0$. This concludes the proof of statement (2b).

It remains to prove statement (2c). By Lemma 148.31 we know that there exists a pairing $\chi\colon \mathrm{H}_2(M,Y;\mathbb{Z})\times\mathrm{H}_2(M,Y;\mathbb{Z})\to\mathbb{Q}$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{H}_2(M;\mathbb{Z})\times\mathrm{H}_2(M;\mathbb{Z}) & \xrightarrow{Q_M} & \mathbb{Z} \\
\downarrow{\scriptstyle p_*\times p_*} & & \downarrow \\
\mathrm{H}_2(M,Y;\mathbb{Z})\times\mathrm{H}_2(M,Y;\mathbb{Z}) & \xrightarrow{\chi} & \mathbb{Q} \\
\downarrow{\scriptstyle\partial\times\partial} & & \downarrow \\
\mathrm{H}_1(Y;\mathbb{Z})\times\mathrm{H}_1(Y;\mathbb{Z}) & \xrightarrow[\lambda_Y]{} & \mathbb{Q}/\mathbb{Z}.
\end{array}$$

If we equip $\mathrm{H}_2(M;\mathbb{Z})$ and $\mathrm{H}_2(M,Y;\mathbb{Z})$ with the above bases, then we know by the Asymmetric Intersection Pairing Proposition 132.15 (3) that the map p_* is represented by the matrix

A. Furthermore, by definition A represents the intersection form Q_M. Now, for purely algebraic reasons we see that χ is represented by the rational matrix A^{-1}.[1768] Statement (2c) follows immediately from these observations and the above commutative diagram. ■

Exercises for Chapter 148.

Exercise 148.1. Let $k \in \mathbb{N}$. We consider the short exact sequence

$$0 \;\to\; \mathbb{Z}_k \;\xrightarrow{[a]\mapsto[a\cdot k]}\; \mathbb{Z}_{k^2} \;\xrightarrow{p([a]=[a]}\; \mathbb{Z}_k \;\to\; 0.$$

Given a topological space X we obtain Bockstein homomorphisms $\beta_n\colon \mathrm{H}_n(X;\mathbb{Z}_k) \to \mathrm{H}_{n-1}(X;\mathbb{Z}_k)$, $n \in \mathbb{N}_0$. Show that

$$\beta_{n-1} \circ \beta_n\colon \mathrm{H}_n(X;\mathbb{Z}_k) \;\to\; \mathrm{H}_{n-2}(X;\mathbb{Z}_k)$$

is the zero map.

Exercise 148.2. Let H be a finite abelian group and let $\lambda\colon H \times H \to \mathbb{Q}/\mathbb{Z}$ be a non-singular symmetric linking pairing. Given a subgroup $P \subset H$ we define

$$P^\perp \;:=\; \{v \in H \,|\, \lambda(v,w) = 0 \text{ for all } w \in P\}.$$

We say that a subgroup P is a *metabolizer of* λ if $P = P^\perp$. Now let $P \subset H$ be a subgroup with $P \subset P^\perp$. Prove the following equivalence of statements:

$$|P^2| = |H| \quad\Longleftrightarrow\quad P \text{ is a metabolizer.}$$

Exercise 148.3. Let $A \in \mathrm{M}(n \times n, \mathbb{Z})$ be a symmetric matrix with $\det(A) \neq 0$. We consider the pairings

$$\begin{array}{rcl} \mu(A)\colon \mathbb{Z}^n \times \mathbb{Z}^n & \to & \mathbb{Z} \\ ([v],[w]) & \mapsto & v^T A w \end{array} \quad \text{and} \quad \begin{array}{rcl} \lambda(A)\colon \mathbb{Z}^n/A\mathbb{Z}^n \times \mathbb{Z}^n/A\mathbb{Z}^n & \to & \mathbb{Q}/\mathbb{Z} \\ ([v],[w]) & \mapsto & v^T A^{-1} w. \end{array}$$

Given $n \in \{0,1,2,3\}$ we consider the matrices $A_n = \begin{pmatrix} 0 & 3 \\ 3 & n \end{pmatrix}$.

(a) For which $m, n \in \{0,1,2,3\}$ are the pairings $\mu(A_m)$ and $\mu(A_n)$ isometric?
(b) For which $m, n \in \{0,1,2,3\}$ are the pairings $\lambda(A_m)$ and $\lambda(A_n)$ isometric?

Exercise 148.4. Let M and N be oriented rational homology spheres of dimension $2n+1$ with $n \geq 1$. Show that linking pairing of the connected sum $M \# N$ is isometric to the orthogonal sum of the linking pairing of M and the linking pairing of N.

Exercise 148.5.

(a) Show that every lens space admits a handle decomposition with precisely one handle of index 0, 1,2 and 3.

[1768] The purely algebraic reason is the following. Suppose we are given a commutative diagram of the form

$$\begin{array}{ccc} \mathbb{Z}^m \times \mathbb{Z}^m & \xrightarrow{(x,y)\mapsto x^T P y} & \mathbb{Z} \\ \downarrow{\scriptstyle (x,y)\mapsto(Cx,Dx)} & & \downarrow \\ \mathbb{Z}^m \times \mathbb{Z}^m & \xrightarrow{(x,y)\mapsto x^T Q y} & \mathbb{Q}. \end{array}$$

Then $P = C^T Q D$. If C and D are invertible over $\mathbb{Q}$, then $Q = (C^T)^{-1} P D^{-1}$. In our case we have $C = D = P = A$. Furthermore A is symmetric and we know that $\det(A) \neq 0$. Thus we see that $Q = A^{-1} A A^{-1} = A^{-1}$.

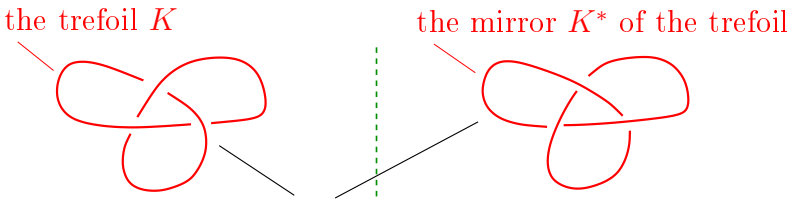
(b) Let X and Y be two CW-complexes with precisely one cell in dimension 0,1,2 and no other cell. Show that X and Y are homotopy equivalent if and only if $\pi_1(X)$ is isomorphic to $\pi_1(Y)$.
Hint. What choices do you have for the attaching maps of the cells? Make use of the Mapping Cone Homotopy Equivalent Proposition 16.15.

(c) Let $p \in \mathbb{N}$ and $q, q' \in \mathbb{N}$. Furthermore let $\Phi\colon \overline{B}^3 \to L(p,q)$ and $\Psi\colon \overline{B}^3 \to L(p,q')$ be smooth embeddings. Show that $L(p,q) \setminus \Phi(B^3)$ and $L(p,q') \setminus \Psi(B^3)$ are homotopy equivalent.
Remark. Use the Handles-Cells Proposition 136.18.

149. Branched coverings and chiral knots

After this long and technical, and basically picture free chapter, it is time to return to some more concrete topological questions. We recall the following question from some time ago.

Question 59.9. Is the trefoil chiral, i.e. can the mirror of the trefoil be deformed into the trefoil itself? Put differently, are the two knots shown in the figure below smoothly isotopic in the sense of the definition on page 817?



are these two knots smoothly isotopic?

In the last chapters we had developed several tools for showing that certain closed smooth manifolds, e.g. the complex projective space $\mathbb{C}\mathrm{P}^2$ and the lens space $L(3,1)$, are chiral. In this section we will attack Question 59.9 by reducing it to a question that we had already dealt with. More precisely, the idea is to associate to a knot in S^3 an oriented rational homology sphere that retains at least some information on the knot. Then we will use the linking pairing of this rational homology sphere to draw conclusions on the chirality of the knot.

149.1. **Branched coverings.** In this section we will introduce branched coverings. We start out by recalling the following definition from page **??**.

Definition. Let M be a smooth manifold and let $W \subset M$ be a proper codimension two submanifold. A trivial tubular neighborhood for W is a smooth embedding

$$F\colon \overline{B}^2 \times W \;\to\; M$$

such that $F(0,P) = P$ for all $P \in W$ and such that $F(\overline{B}^2 \times \partial W) = \partial M \cap F(\overline{B}^2 \times W)$.

Examples.

(1) It is clear that every 0-dimensional submanifold of a 2-dimensional smooth manifold admits a trivial tubular neighborhood.
(2) It follows from the Tubular Neighborhood Theorem **??** that every closed 1-dimensional submanifold of a closed orientable 3-dimensional smooth manifold admits a trivial tubular neighborhood.
(3) In Proposition 126.14 we saw that the 2-dimensional submanifold $W = \mathbb{C}\mathrm{P}^1$ of the 4-dimensional smooth manifold $M = \mathbb{C}\mathrm{P}^2$ does not admit a trivial tubular neighborhood.

Convention. Let $\overline{B}^2 \times W \to M$ be a smooth embedding as above. On many occasions, by an abuse of notation, we will identify the image of the smooth embedding $\overline{B}^2 \times W \to M$ with $\overline{B}^2 \times W$. Put differently, we often use the phrase let "$\overline{B}^2 \times W$ be a trivial tubular neighborhood".

Definition. Let M be a smooth manifold and let $W \subset M$ be a compact proper submanifold of codimension two. We say a map $p\colon \widetilde{M} \to M$ is a k-fold branched covering with branch locus W if the following two conditions are satisfied:

(1) The restriction $p\colon \widetilde{M} \setminus p^{-1}(W) \to M \setminus W$ is a k-fold covering map in the usual sense.

(2) There exists a trivial tubular neighborhood $\overline{B}^2 \times W$ of W with the following properties:

(a) For each component X of $p^{-1}(\overline{B}^2 \times W)$ there exists an $m \in \mathbb{N}$ and a diffeomorphism $\Psi\colon X \to \overline{B}^2 \times W$ such that the following diagram commutes[1769]

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & \overline{B}^2 \times W \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (z,w)\mapsto(z^m,w)} & \\ & \overline{B}^2 \times W. & \end{array}$$

We refer to m as the branching degree of X.

(b) There exists at least one component of $p^{-1}(\overline{B}^2 \times W)$ with branching degree ≥ 2.

We say the branched covering is simple if $m = k$ throughout. Furthermore we say that the branched covering is regular if the covering map $p\colon \widetilde{M} \setminus p^{-1}(W) \to M \setminus W$ is regular in the sense of the definition on page 1383. Otherwise we say that the branched covering is irregular.

Remark. The notion of a branched covering makes sense in a much more general context, not necessarily restricted to smooth manifolds, see e.g. [**Fox1957**].

Examples.

(1) The map $p\colon \widetilde{M} = \mathbb{C} \to M = \mathbb{C}$ given by $z \mapsto z^k$ is a simple k-fold branched covering with branch locus $W = \{0\}$. This is basically clear:

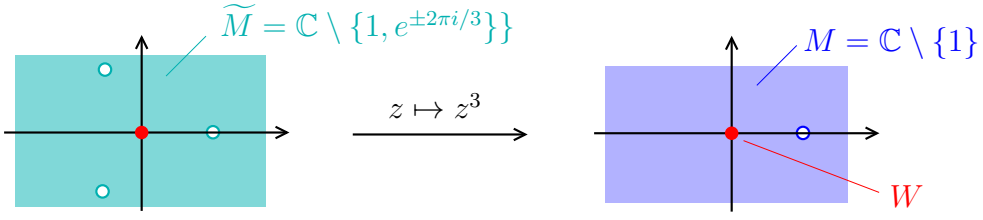
(a) On page 1070 we saw that the restriction of p to a map $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is a k-fold covering map.

(b) We take the obvious trivial tubular neighborhood $\overline{B}^2$ for 0 and we denote by Ψ the identity map. It is obvious that the above diagram commutes.

(2) The same argument as in (1) shows that the map

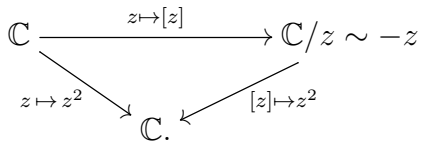
$$\begin{array}{rcl} \widetilde{M} = \mathbb{C} \setminus \{\{\exp(2\pi \mathrm{i} k/3) \,|\, k = 0,1,2\} & \to & \mathbb{C} \setminus \{1\} \\ z & \mapsto & z^3 \end{array}$$

that is illustrated in the figure below is a simple 3-fold branched covering with branch locus $W = \{0\}$.



[1769] Here we view $\overline{B}^2 = \{z \in \mathbb{C} \,|\, |z| \leq 1\}$ as a subset of $\mathbb{C}$.

(3) The projection $\widetilde{M} = \mathbb{C} \to M := \mathbb{C}/z \sim -z$ is a simple 2-fold branched covering with branch locus $W = \{0\}$. To verify this statement we consider the following diagram



The diagram is clearly commutative and the right-hand diagonal map is easily seen to be a homeomorphism. We use this homeomorphism to equip $\mathbb{C}/z \sim -z$ with a smooth structure. In (1) we saw that the left diagonal map is a branched covering, thus the top horizontal map is also a branched covering.

(4) Let Σ be a closed 1-dimensional complex manifold (i.e. Σ is a closed 2-dimensional smooth manifold with a complex structure) and let $f\colon \Sigma \to \mathbb{C}$ be a meromorphic non-constant function. We consider the map

$$\begin{array}{rcl} \Sigma & \to & \mathbb{C}\mathrm{P}^1 = S^2 \\ z & \mapsto & \begin{cases} [f(z):1], & \text{if } f(z) \neq \infty \\ [1:0], & \text{otherwise.} \end{cases} \end{array}$$

It is straightforward to see that this map is holomorphic. Using [**Jä2011**, Satz 10] one can easily show that $f\colon \Sigma \to \mathbb{C}\mathrm{P}^1 = S^2$ is a branched covering.

Before we give more interesting examples of branched coverings we state and prove the following lemma that, given a simple branched covering $p\colon \widetilde{M} \to M$, relates the Euler characteristic of $\widetilde{M}$ to the Euler characteristic of M, the Euler characteristic of the branch locus and the degree of the branched covering.
The lemma carries a fancy name, even though it is quite elementary.

Lemma 149.1. (Riemann-Hurwitz Formula) Let M be a compact smooth manifold and let W be a compact proper codimension 2 submanifold of M. If $p\colon \widetilde{M} \to M$ is a simple k-fold branched covering with branch locus W, then

$$\chi\big(\widetilde{M}\big) \;=\; k \cdot \chi(M) + (1-k) \cdot \chi(W).$$

A similar statement also holds for non-simple branched coverings. The formulation becomes more messy, but the proof in this more general setting is equally trivial. Since we will not make use of this more general formula we decline to formulate it.

Proof. As usual, statements about Euler characteristics can be proved quite easily. To prove the Riemann-Hurwitz Formula we pick a tubular neighborhood $\overline{B}^2 \times W$ and a diffeomorphism $\Psi\colon p^{-1}(\overline{B}^2 \times W) \to \overline{B}^2 \times W$ as in the definition of a branched covering. Now

we calculate that

$$\begin{array}{l}\text{Manifold-}\chi\text{-Proposition 104.36 (3) applied to } \widetilde{M} = \big(\widetilde{M} \setminus \Psi(B^2 \times W)\big) \cup \Psi\big(\overline{B}^2 \times W\big) \\ \chi\big(\widetilde{M}\big) \overset{\downarrow}{=} \chi\big(\widetilde{M} \setminus \Psi(B^2 \times W)\big) + \chi\big(\Psi(\overline{B}^2 \times W)\big) - \chi\big(\Psi(S^1 \times W)\big) \\ \overset{\uparrow}{=} k{\cdot}\chi(M \setminus (B^2 \times W)) + \chi(W) \\ \text{by the Manifold-}\chi\text{-Proposition 104.36 (5) since } p\colon \widetilde{M} \setminus \Psi(B^2 \times W) \to M \setminus (B^2 \times W) \text{ is a } k\text{-fold covering} \\ \text{and using that the Manifold-}\chi\text{-Proposition 104.36 (2) implies that } \chi(\Psi(\overline{B}^2 \times W)) = \chi(W) \text{ and } \chi(\Psi(S^1 \times W)) = 0 \\ \overset{\uparrow}{=} k{\cdot}\chi(M) - k{\cdot}\chi(\overline{B}^2 \times W) + k{\cdot}\chi(W \times S^1) + \chi(W) \overset{\uparrow}{=} k{\cdot}\chi(M) + (1-k){\cdot}\chi(W). \\ \text{by the Manifold-}\chi\text{-Proposition 104.36 (3) applied to } M = M \setminus (B^2 \times W) \cup \overline{B}^2 \times W \qquad \text{again by the Manifold-}\chi\text{-Proposition 104.36 (2)}\end{array}$$

■

Remark. The concept of a map that is a covering outside of a sufficiently small subset also appears in other branches of mathematics. It is often also referred to as a "branched covering" or "ramified covering" or "ramification". In the figure below we show an extract of Jürgen Neukirch's book on algebraic number theory [**Neuk1999**].

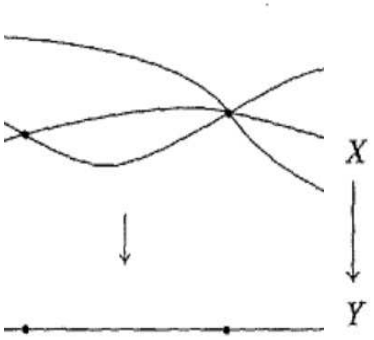
6. Extensions. Let $\mathfrak{o}$ be a Dedekind domain with field of fractions K. Let $L|K$ be a finite separable extension, and $\mathcal{O}$ the integral closure of $\mathfrak{o}$ in L. Let $Y = \operatorname{Spec}(\mathfrak{o})$, $X = \operatorname{Spec}(\mathcal{O})$, and

$$f : X \longrightarrow Y$$

the morphism induced by the inclusion $\mathfrak{o} \hookrightarrow \mathcal{O}$. If $\mathfrak{p}$ is a maximal ideal of $\mathfrak{o}$ and

$$\mathfrak{p}\mathcal{O} = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}$$

the prime decomposition of $\mathfrak{p}$ in $\mathcal{O}$, then $\mathfrak{P}_1, \ldots, \mathfrak{P}_r$ are the different points of X which are mapped to $\mathfrak{p}$ by f. The morphism f is a "ramified covering." It is graphically represented by the following picture:



149.2. **The hyperelliptic involution and branched coverings of surfaces.** Before we can provide the reader with new examples of branched coverings we need to introduce the following fun object.

Definition. As on page 1880 we consider the surface Σ_g of genus g as the quotient of the regular $(4g+2)$-gon E_{4g+2} where we identify opposite sides by a translation. The map

$$\begin{array}{rccc}\Theta\colon & E_{4g+2}/\!\sim & \to & E_{4g+2}/\!\sim \\ & [z] & \mapsto & [-z]\end{array}$$

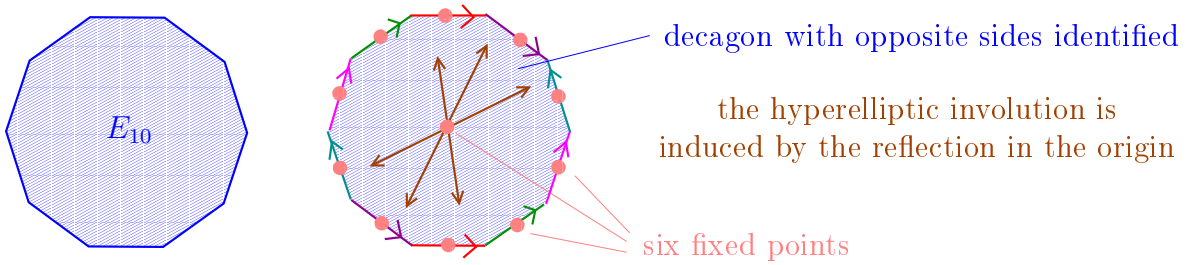
is called the hyperelliptic involution.[17701771] For $g = 2$ the surface $\Sigma_2 = E_{10}/\!\sim$ and the hyperelliptic involution are illustrated in the figure below.

The following lemma summarizes some properties of the hyperelliptic involution.

Lemma 149.2. Let $g \in \mathbb{N}_0$.

(1) The hyperelliptic involution $\Theta\colon \Sigma_g \to \Sigma_g$ is orientation-preserving.
(2) The hyperelliptic involution $\Theta\colon \Sigma_g \to \Sigma_g$ has precisely $2g+2$ fixed points.
(3) The map $\Theta_*\colon \mathrm{H}_1(\Sigma_g;\mathbb{Z}) \to \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ equals $-\mathrm{id}_{\mathrm{H}_1(\Sigma_g;\mathbb{Z})}$.

[1770] Recall that a map $f\colon X \to X$ is called an *involution* if $f \circ f = \mathrm{id}_X$.
[1771] I do not know where the fancy name "hyperelliptic" comes from.



(4) The quotient space $\Sigma_g/x \sim \Theta(x)$ is homeomorphic to the 2-sphere and it can be given a smooth structure such that the quotient map $\Sigma_g \to \Sigma_g/x \sim \Theta(x)$ is a 2-fold branched covering where the branch locus consists of $2g+2$ points.

Sketch of proof.

(1) It follows easily from Proposition 107.2, applied to the origin, that the hyperelliptic involution Θ is orientation-preserving.

(2) It is straightforward to show that the fixed points of the hyperelliptic involution are given by the origin and the points that are represented by the $4g+2$ midpoints of the sides of the $(4g+2)$-gon. Since opposite midpoints get identified we see that we have precisely $1+\frac{1}{2}(4g+2) = 2g+2$ fixed points.

(3) We equip E_{4g+2} with the obvious CW-structure with $2g+2$ 0-cells, $2g+2$ 1-cells and one 2-cell. Since the hyperelliptic involution identifies cells with cells we can equip $\Sigma_g = E_{4g+2}/\!\sim$ with a corresponding quotient CW-structure. The hyperelliptic involution is a cellular map and one can easily compute that the induced map on $\mathrm{H}_1^{\mathrm{CW}}(\Sigma_g;\mathbb{Z})$ is $-\mathrm{id}$. By the Singular-Cellular H_*-Isomorphism Proposition 80.4 we know that singular homology is *naturally* isomorphic to cellular homology. In our setting this means that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_1(\Sigma_g;\mathbb{Z}) & \longrightarrow & \mathrm{H}_1^{\mathrm{CW}}(\Sigma_g;\mathbb{Z}) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{H}_1(\Sigma_g;\mathbb{Z}) & \longrightarrow & \mathrm{H}_1^{\mathrm{CW}}(\Sigma_g;\mathbb{Z}), \end{array}$$

where the horizontal maps are the same. Since the right vertical map is $-\mathrm{id}$ we now see that the left vertical map is also $-\mathrm{id}$.

(4) We leave it to the conscientious reader to show that the quotient space $\Sigma_g/x \sim \Theta(x)$ is a closed orientable connected 2-dimensional smooth manifold such that the quotient map $\Sigma_g \to \Sigma_g/x \sim \Theta(x)$ is a 2-fold branched covering with branch locus given by the images of the fixed points.[1772] Once we believe this fact we can calculate that

$$2-2g \;=\; \chi(\Sigma_g) \underset{\uparrow}{=} 2\cdot\chi(\Sigma_g/\!\sim) - \chi(\text{branch locus}) \underset{\uparrow}{=} 2\cdot\chi(\Sigma_g/\!\sim) - (2g+2).$$

Lemma 149.1 — by (1) there exist $2g+2$ fixed points

[1772] In the verification of this statement it can be useful to keep the second example on page 3174 in mind.

Thus we see that $\chi(\Sigma_g/\!\sim) = 2$. It follows from Theorem 55.4 that $\Sigma_g/\!\sim$ is indeed diffeomorphic to the 2-dimensional sphere.[1773] ■

More interestingly, the hyperelliptic involution is in some sense fully determined by being an orientation-preserving involution and satisfying Statement (2) or Statement (3) of Lemma 149.2. More precisely, the following proposition holds:

Proposition 149.3. Let $g \in \mathbb{N}_0$ and let Ξ be an orientation-preserving involution of Σ_g. Suppose one of the following two conditions is satisfied:

(1) The map Ξ has precisely $2g+2$ fixed points.
(2) The map $\Xi_*\colon \mathrm{H}_1(\Sigma_g;\mathbb{Z}) \to \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ equals $-\mathrm{id}$.

Then Ξ and the hyperelliptic involution Θ represent conjugate elements in the orientation-preserving mapping class group $\mathrm{MCG}^+(\Sigma_g)$.

Proof. This statement is proved in Proposition 7.15 and the discussion on page 216 of [**FaM2011**]. ■

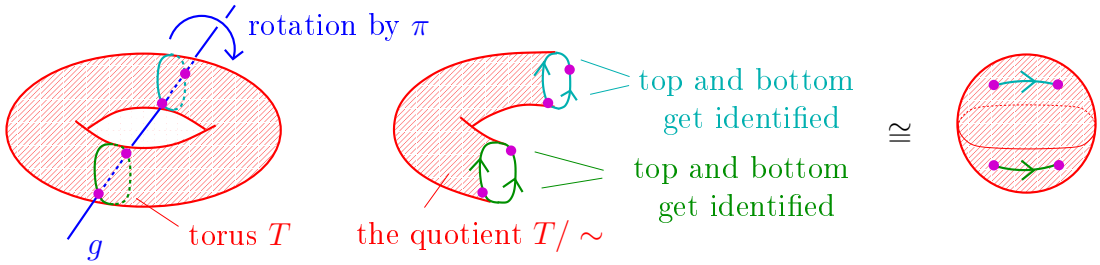
Examples.

(1) Using Proposition 149.3 one can show easily that the maps

$$\begin{array}{ccc} \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ [(x,y)] & \mapsto & [(-x,-y)] \end{array} \quad \text{and} \quad \begin{array}{ccc} S^1 \times S^1 & \to & S^1 \times S^1 \\ (z,w) & \mapsto & (z^{-1},w^{-1}), \end{array}$$

under a suitable diffeomorphism between $\mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1$ and $E_6/\!\sim$, correspond to the hyperelliptic involution.[1774] In Lemma 5.7 we gave an explicit diffeomorphism between $S^1 \times S^1$ and a specific submanifold of $\mathbb{R}^3$. Under this diffeomorphism the above hyperelliptic involution corresponds to rotation by π around a line in the xy-plane through the origin.

In the figure below we show the torus as a submanifold of $\mathbb{R}^3$ together with the hyperelliptic involution. We do our best to visualize that the quotient of the torus is indeed given by the sphere.

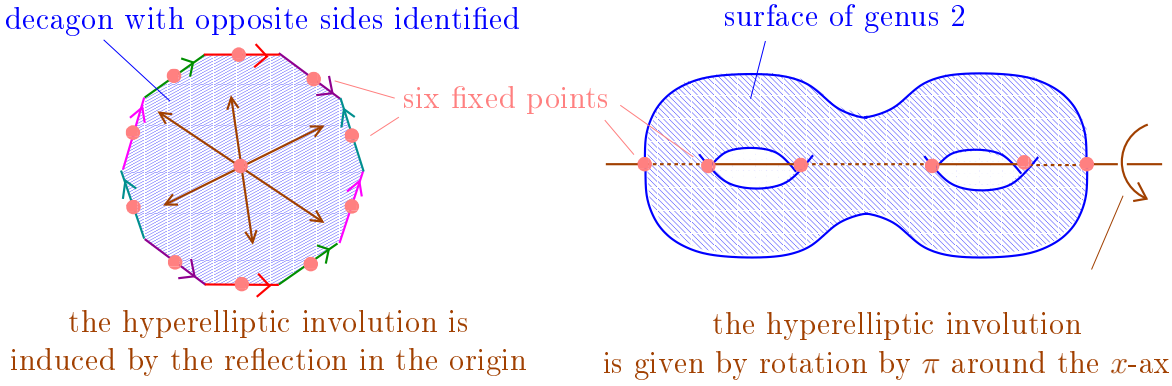


(2) Using Proposition 149.3 we can now also visualize the hyperelliptic involution in our favorite "physical" model of the surface of genus g. More precisely, in the figure below on the right we show the surface of genus 2 as a submanifold of $\mathbb{R}^3$. Rotation by π around the x-axis is an orientation-preserving involution which has 6 fixed points.

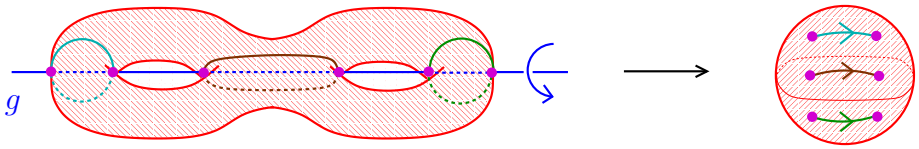
[1773] It always feels like cheating to use the calculation of Euler characteristics together with the Surface Classification Theorem 55.4 to determine the diffeomorphism type of a given closed orientable connected 2-dimensional smooth manifold. Can you find a more direct proof that $\Sigma_g/\!\sim$ is diffeomorphic to S^2?

[1774] Clearly both maps are orientation-preserving involutions. Thus it remains to show that these involutions of $\mathbb{R}^2/\mathbb{Z}^2$ and $S^1 \times S^1$ have precisely four fixed points. What are they?

It follows from Proposition 149.3 that, under a suitable diffeomorphism to Σ_g, this involution corresponds to the hyperelliptic involution. For convenience we refer to this involution also as the hyperelliptic involution.



Finally, in the figure below we indicate why the quotient is indeed the sphere.



149.3. **Branched coverings of 3-dimensional smooth manifolds.** In this section we consider branched coverings of 3-dimensional smooth manifolds. Before we start out with definitions and examples we provide a word of caution. Whenever one wants to deal with "real-life topological spaces" it is very hard to write down everything absolutely rigorously. In this chapter we will therefore take some liberties and we will not write down every argument in full detail.

As a balm to our mathematical conscience we recall that we pointed out on page **??**, every 3-dimensional topological manifold admits a unique smooth structure. Hence we do not need to distinguish between the concepts of being "homeomorphic" and "diffeomorphic".[1775]

Before we give a 3-dimensional example of a branched covering we need to introduce one more definition.

Definition.

(1) An arc in a 3-dimensional smooth manifold M is a proper submanifold of M that is diffeomorphic to the interval $[0,1]$.

(2) We say a finite set of arcs $A_1, \dots, A_m$ in $\overline{B}^3$ is unknotted if there exists a homeomorphism $f\colon \overline{B}^3 \to \overline{B}^2 \times [0,1]$ and distinct points $P_1, \dots, P_m$ such that for $i = 1, \dots, m$ we have $f(A_i) = P_i \times [0,1]$.

Example. In the figure below we show to the left an arc in the solid torus. To the right we show two arcs in $\overline{B}^3$ that are unknotted.

[1775]This fact that by smoothing we can always replace a homeomorphism by a diffeomorphism should not be overgeneralized. For example, Bing [**Bin1952**] gave an example of an involution f on S^3 that is not isotopic to a smooth involution. More precisely, we can isotope f to a smooth map, but we can no longer arrange that that map is an involution.



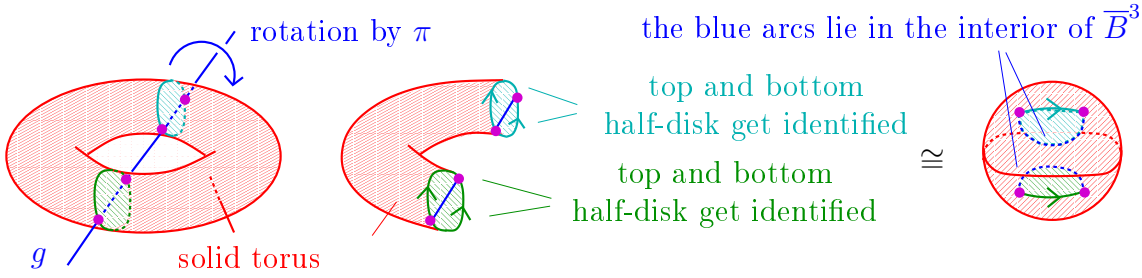
Now we can give the promised examples of branched coverings of 3-dimensional smooth manifolds.

Examples.

(1) We consider the map

$$\begin{array}{rcl} \Theta\colon \overline{B}^2 \times S^1 & \to & \overline{B}^2 \times S^1 \\ (r \cdot \exp(\mathrm{i}\varphi), z) & \mapsto & (r \cdot \exp(-\mathrm{i}\varphi), z^{-1}). \end{array}$$

The map Θ is an involution that restricts to the hyperelliptic involution on the boundary. In Lemma 5.7 we had identified the solid torus $\overline{B}^2 \times S^1$ with a submanifold of $\mathbb{R}^3$. Under this identification the above involution Θ corresponds to the rotation by π around a line in the xy-plane. This map restricts to an involution on the solid torus $\overline{B}^2 \times S^1 \subset \mathbb{R}^3$. As is illustrated in the figure below, the quotient $\overline{B}^2 \times S^1 / x \sim \Theta(x)$ is the 3-ball and the quotient map $p\colon \overline{B}^2 \times S^1 \to \overline{B}^3$ is a 2-fold branched covering where the branch locus is given by two unknotted arcs.[1776]



(2) Above we saw that the surface of genus g is the branched covering of S^2 with branch locus given by $2g+2$ points. Similarly to (1) we can now replace the surface of genus g by the handlebody H_g of genus g and we obtain a branched covering of H_g over the 3-ball $\overline{B}^3$ where the branch locus is given by $g+1$ unknotted arcs.

It follows from the classification of closed connected 2-dimensional smooth manifolds, see the Surface Classification Theorem 55.4, together with discussion on page 3179 that every closed orientable connected 2-dimensional smooth manifold Σ admits a simple branched covering over the 2-dimensional sphere. The following theorem says that a similar statement also holds in the 3-dimensional setting.

[1776] It is painful, but not impossible to prove this statement rigorously. Indeed, first of all we can replace $\overline{B}^2 \times S^1$ by $([-1,1]/-1 \sim 1) \times [0,2]^2$ where the involution is given by $(s,x,y) \sim (-s,x,2-y)$. The quotient space is then homeomorphic to $[0,1] \times [0,2]^2$ where $(0,x,y) \sim (0,x,2-y)$ and $(1,x,y) \sim (1,x,2-y)$. Using this description it is no longer that difficult to show that the quotient space is homeomorphic to $\overline{B}^2 \times [0,1]$ where the fixed point set is given by two arcs of the form $P_1 \times [0,1]$ and $P_2 \times [0,1]$. We leave it to the reader to fill in the details.

Theorem 149.4. (Hilden-Montesinos) Every closed orientable connected 3-dimensional smooth manifold M admits a map $p\colon M \to S^3$ which is a simple 3-fold branched covering (not necessarily regular) with branch locus a knot.

Remark. The theorem has its origins in the work of James Alexander [**Al1920**] from 1920. He had claimed that every closed orientable 3-dimensional smooth manifold admits a map $p\colon M \to S^3$ which is a branched covering with branch locus a link. But the definition of a branched covering map in [**Al1920**] is weaker than our notion and few details were provided in [**Al1920**].

Proof. The theorem was independently proved by Hugh Hilden [**Hild1976**] and José Montesinos [**Mon1976**] in the early 1970s. An exposition of the proof is also given in [**PrS1997**, Theorem 23.1]. The definition of a branched covering used in the above references is slightly less restrictive, but it is clear from the proofs that they provide branched coverings in our sense. ■

149.4. **Branched coverings defined by knots.** In the previous sections we saw that several maps $p\colon \widetilde{M} \to M$ between smooth manifolds are branched coverings and we had determined the branch locus of those branched coverings.
In this section we go the other way round. We set $M = S^3$, we prescribe the branch locus as a given knot K and we describe a degree m. Given this data we will construct an essentially unique m-fold branched covering $p\colon \Sigma_m(K) \to S^3$ with branch locus K.

Definition. Let K be a knot in S^3 and let $m \in \mathbb{N}$.

(1) By the Knot Thickening Lemma 131.1 (2) we can pick a thickening $F\colon \overline{B}^2 \times K \to S^3$ with self-linking number 0.
   (a) As usual we write $X_K := S^3 \setminus F(B^2 \times K)$.
   (b) We equip X_K with the natural orientation coming from S^3.
   (c) We denote by $f\colon S^1 \times K \to \partial X_K = \partial F(\overline{B}^2 \times K)$ the diffeomorphism that is given by restricting F to the boundary torus $S^1 \times K$.
   (d) We denote by $\phi_K\colon \pi_1(X_K) \to \mathrm{H}_1(X_K;\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$ one of the two epimorphisms. (It will not matter which one we take.)

(2) By the Covering Existence Theorem 61.5 there exists a connected covering $p\colon \widetilde{X_K} \to X_K$ that corresponds to $\ker\big(\pi_1(X_K) \xrightarrow{\phi_K} \mathbb{Z} \to \mathbb{Z}_m\big)$, i.e. that satisfies

$$p_*(\pi_1(\widetilde{X_K})) \;=\; \ker\big(\pi_1(X_K) \xrightarrow{\phi_K} \mathbb{Z} \to \mathbb{Z}_m\big).$$

   Furthermore, by the Covering Uniqueness Theorem 61.9 this covering is in an appropriate sense unique. We equip $\widetilde{X_K}$ with the orientation coming from the Manifold Covering Proposition 49.2 and coming from the above orientation on X_K.

(3) We consider the branched covering

$$\begin{array}{rcl} q\colon \overline{B}^2 \times K & \to & \overline{B}^2 \times K \\ (z,p) & \mapsto & (z^m,p). \end{array}$$

   Since the self-linking number of F was chosen to be zero, and by the uniqueness statement of the Covering Uniqueness Theorem 61.9, we see that the restriction of

q to the boundary $S^1 \times K$ can be identified with the m-fold covering corresponding to

$$\ker\left(\pi_1(S^1 \times K) \xrightarrow{f_*} \pi_1(X_K) \to \mathbb{Z} \to \mathbb{Z}_m\right).$$

(4) It follows from (3) that f lifts to a diffeomorphism $\widetilde{f}\colon S^1 \times K \to \partial\widetilde{X_K}$ such that the following diagram commutes

$$\begin{array}{ccc} S^1 \times K & \xrightarrow{\widetilde{f}} & \partial\widetilde{X_K} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ S^1 \times K & \xrightarrow{f} & \partial X_K. \end{array}$$

(5) We define the m-fold branched covering of K to be the topological space

$$\Sigma_m(K) \ := \ \left(\overline{B}^2 \times K\right) \cup_{\widetilde{f}} \widetilde{X_K}.$$

The following proposition summarizes some basic properties of branched coverings of knots.

Proposition 149.5. (Knot-Branched Covering Proposition) Let K be a knot and let $m \in \mathbb{N}$.

(1) The m-fold branched covering $\Sigma_m(K)$ is a closed orientable 3-dimensional topological manifold. It admits a unique smooth structure and a unique orientation such that $\widetilde{X_K}$ is a submanifold and such that the orientation on the submanifold $\widetilde{X_K}$ agrees with the orientation coming from the Manifold Covering Proposition 49.2 and the natural orientation on $X_K \subset S^3$.

(2) The oriented diffeomorphism type of $\Sigma_m(K)$ is independent of all of our choices.

(3) The map

$$\begin{array}{ccccc} \Sigma_m(K) & = & \overline{B}^2 \times K & \cup_{\widetilde{f}} & \widetilde{X_K}. \\ \downarrow & & \downarrow q & & \downarrow p \\ S^3 & = & \overline{B}^2 \times K & \cup_f & X_K \end{array}$$

is well-defined and it is a regular simple m-fold branched covering with branch locus given by the knot K.

(4) If two knots K and J are smoothly isotopic, then there exists an orientation-preserving diffeomorphism $\Sigma_m(K) \to \Sigma_m(J)$.

(5) Let K be a knot in $\mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$. Let $K^{\text{mir}} \subset \mathbb{R}^3$ be a mirror image, i.e. $K^{\text{mir}} = \rho(K)$ where ρ is the reflection in a hyperplane of $\mathbb{R}^3$. There exists an *orientation-reversing* diffeomorphism from $\Sigma_m(K)$ to $\Sigma_m(K^{\text{mir}})$.

(6) If K is amphichiral, then $\Sigma_m(K)$ is amphichiral.

(7) We have isomorphisms

$$\mathrm{H}_1(\Sigma_m(K);\mathbb{Z}) \oplus \mathbb{Z} \cong \mathrm{H}_1(\widetilde{X}_K;\mathbb{Z}) \cong \mathrm{H}_1\left(\begin{matrix}\text{covering of } S^3 \setminus K \text{ corresponding to} \\ \text{the epimorphism } \pi_1(S^3 \setminus K) \to \mathbb{Z}_m\end{matrix};\mathbb{Z}\right).$$

(8) If p is a prime number and $r \in \mathbb{N}$, then the homology group $\mathrm{H}_1(\Sigma_{p^r}(K);\mathbb{Z})$ is finite.

Sketch of proof.

(1) The first statement follows from the Gluing-Smooth Manifolds-Proposition **??**.

(2) The second statement follows fairly easily from the Covering Uniqueness Theorem 61.9 and Proposition **??**.

(3) The third statement follows easily from the definitions.

(4) Suppose that K and J are smoothly isotopic. This means that there exists a smooth isotopy from the knot K to the knot J. As we pointed out in the proof of Proposition 58.4, the smooth isotopy can be extended to a smooth isotopy of S^3 carrying K to J. It now follows easily from the definitions that there exists an orientation-preserving diffeomorphism $\Sigma_m(K) \to \Sigma_m(J)$.
(5) Let ρ be a reflection in a hyperplane of $\mathbb{R}^3$. From Exercise 2.46 we obtain that, by setting $\rho(\infty) = \infty$, the map ρ extends to a homeomorphism of $S^3 = \mathbb{R}^3 \cup \{\infty\}$. We denote this homeomorphism again by ρ. It follows from Proposition 107.2, say applied to the origin, that $\rho\colon S^3 \to S^3$ is orientation-reversing. It follows again easily from the definitions that ρ induces an orientation-reversing homeomorphism $\Sigma_m(K) \to \Sigma_m(K^{\mathrm{mir}})$.
(6) This statement is an immediate consequence of (4) and (5).
(7) An amusing Mayer–Vietoris argument shows that there exists an isomorphism as stated on the left. The inclusion of the knot exterior into the knot complement is a homotopy equivalence, therefore it induces an isomorphism as stated on the right.
(8) The last statement is the only statement that is non-trivial. The case $p = 2$ and $r = 1$ is proved for example on [**Rolf1990**, p. 213]. The general case is proved in [**Gor1972**, p. 362]. ■

Remark. One could also define the m-fold branched covering of K as the set $K \cup \widetilde{Y}$, where $\widetilde{Y}$ is the m-fold cyclic covering of $Y = S^3 \setminus K$, together with a suitable topology. This approach, which is closer in spirit to discussion on page **??**, has the advantage that the definition of the m-fold branched covering involves fewer choices and gives an object that is more naturally defined. For didactical reasons we stuck to the more conventional definition given above.

Now we can consider some examples of branched coverings of knots.

Examples.

(1) The 1-fold branched covering of a knot K is, basically by definition, given by S^3.
(2) For K the trivial knot it is an amusing exercise to verify that all branched coverings are again diffeomorphic to S^3.
(3) The following converse to (2) holds: If K is a knot and $m \in \mathbb{N}_{\geq 2}$ such that $\Sigma_m(K)$ is diffeomorphic to S^3, then K is smoothly isotopic to the trivial knot. This is a consequence of the *Smith Conjecture* which was proved in 1984, see [**MB1984**].[1777]
(4) More generally one can ask to what degree branched coverings remember the original knot K. The bad news is that by [**Nak1981, Saku1981**] there are many examples of triples (K, K', m) where K and K' are "prime" knots which are not smoothly isotopic and where $m \in \mathbb{N}_{\geq 2}$ such that $\Sigma_m(K)$ is diffeomorphic to $\Sigma_m(K')$. The good news is that it is shown in [**Koj1986**] that if K and K' are two "prime" knots such that $\Sigma_m(K)$ and $\Sigma_m(K')$ are diffeomorphic for infinitely many $m \in \mathbb{N}$, then K and K' are smoothly isotopic.

[1777]The Smith Conjecture, which was originally formulated in 1949 [**Eil1949**, Problem 36], says that if f is a diffeomorphism of S^3 of finite order, i.e. such that there exists an $n \in \mathbb{N}$ with $f^n = \mathrm{id}_{S^3}$, then the fixed point set of f cannot be a non-trivial knot. The conjecture was proved in 1984 [**MB1984**]. By [**Gif1966**] the analogous statement is wrong for 4-dimensional smooth manifolds.

The idea now is of course to apply the Knot-Branched Covering Proposition 149.5 to settle Question 59.9, namely to show that the trefoil is not amphichiral. Therefore we need to determine at least one interesting branched covering of the trefoil.

Definition. Let $J \subset S^3$ be a knot.

(1) We write

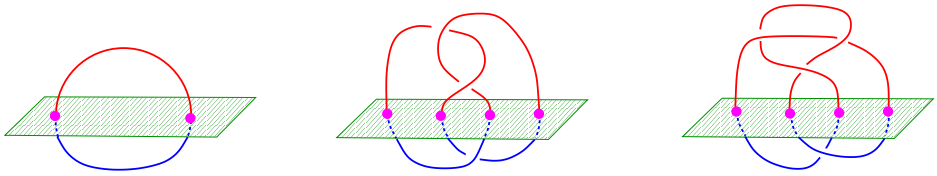
$$\begin{array}{rcl} S^3_{\geq 0} & := & \{(x_1,x_2,x_3,x_4) \in S^3 \,|\, x_4 \geq 0\} \quad \text{and} \quad J_{\geq 0} := J \cap S^3_{\geq 0} \\ S^3_{\leq 0} & := & \{(x_1,x_2,x_3,x_4) \in S^3 \,|\, x_4 \leq 0\} \quad \text{and} \quad J_{\leq 0} := J \cap S^3_{\leq 0} \end{array}$$

and

$$S^3_{=0} := \{(x_1,x_2,x_3,x_4) \in S^3 \,|\, x_4 = 0\} \quad \text{and} \quad J_{=0} := J \cap S^3_{=0}.$$

(2) We say J is in m-bridge position if the following three conditions are satisfied:
   (a) J intersects the sphere $S^3_{=0} = \text{xy-plane} \cup \{\infty\}$ transversally,
   (b) the intersection $J_{\geq 0} = J \cap S^3_{\geq 0}$ consists of m trivial arcs in the 3-ball $S^3_{\geq 0}$,
   (c) the intersection $J_{\leq 0} = J \cap S^3_{\leq 0}$ consists of m trivial arcs in the 3-ball $S^3_{\leq 0}$.

Example. In the figure below we show three knots which are isotopic to the trivial knot, the trefoil and the figure-8 knot.[1778] The first knot is in 1-bridge position whereas the latter two knots are in 2-bridge position.



trivial knot | the trefoil | figure-8 knot

Lemma 149.6. Every knot K in S^3 is smoothly isotopic to a knot J that is in m-bridge position for some m.

Sketch of proof. If a knot K is represented by a diagram with m crossings, then it is already in an m-bridge position. We refer to [**BZH2014**, Chapter 2.D] for more details. ■

Definition. Given a knot $K \subset S^3$ we define its bridge number as

$$b(K) := \min\{m \in \mathbb{N} \,|\, K \text{ is smoothly isotopic to a knot in } m\text{-bridge position}\}.$$

If $b(K) = m$, then we also just say that K is an m-bridge knot.

Examples.

(1) If K is a knot with bridge number 1, then K is smoothly isotopic to the trivial knot. This sounds pretty convincing, but as so often, it is rather painful to write down a completely rigorous argument.
(2) It follows from (1) and the figure above that the trefoil and the figure-8 knot are 2-bridge knots.

[1778] We leave it to the reader to verify that the knots pictured in the figure below are in fact isotopic to the knots shown in the figure on page 1302.

Proposition 149.7. Let K be a knot. The 2-fold branched covering $\Sigma_2(K)$ admits a Heegaard splitting of genus $b(K)-1$.

Sketch of proof. By the Knot-Branched Covering Proposition 149.5(4) we can assume that K itself is already in m-bridge position. We denote by $p\colon \Sigma_2(K) \to S^3$ the branched covering with branch locus K. The restriction of p to $p^{-1}(S^3_{\geq 0}) \to S^3_{\geq 0}$ is a 2-fold branched covering with branch locus $K_{\geq 0}$. Since $K_{\geq 0}$ are unknotted arcs this branched covering is essentially the same as the branched covering considered on page 3180.[1779] But this implies that $p^{-1}(S^3_{\geq 0})$ is a genus $m-1$ handlebody. Similarly we see that $p^{-1}(S^3_{\leq 0})$ is a genus $m-1$ handlebody and that the intersection of these two handlebodies, which is given by $p^{-1}(S^3_{=0})$ is a surface of genus $m-1$. But that means precisely that $\Sigma_2(K)$ admits a genus $m-1$ Heegaard splitting. ■

Corollary 149.8. If K is a knot with bridge number $b(K)=2$, then $\Sigma_2(K)$ is diffeomorphic to a lens space.

Proof. Let K be a knot with bridge number $b(K)=2$. It follows from Proposition 149.7 that the 2-fold branched covering $\Sigma_2(K)$ admits a Heegaard splitting of genus one. It follows from the Knot-Branched Covering Proposition 149.5 (1) and Corollary 146.8 that $\Sigma_2(K)$ is either diffeomorphic to a lens space or to $S^1\times S^2$. In the Knot-Branched Covering Proposition 149.5 (8) we saw that $\mathrm{H}_1(\Sigma_2(K);\mathbb{Z})$ is finite, thus we can exclude the possibility that $\Sigma_2(K)$ is diffeomorphic to $S^1\times S^2$. ■

We had just seen that the trefoil and the figure-8 knot are 2-bridge knots. So their 2-fold branched coverings are lens spaces. The following proposition tells us what these 2-fold branched coverings are.

Proposition 149.9. [1780]

(1) The 2-fold branched covering of the trefoil is diffeomorphic to the lens space $L(3,2)$.
(2) The 2-fold branched covering of the figure-8 knot is diffeomorphic to the lens space $L(5,2)$.

Remark. Horst Schubert [**SchubH1956**] showed in 1956 that every lens space is the 2-fold branched covering of a 2-bridge knot.

Proof. Let K be a 2-bridge knot. There is a general algorithm that determines the lens space which equals the 2-fold branched covering $\Sigma_2(K)$, see e.g. [**BZH2014**, Proposition 12.3]. The statement about the trefoil is then implied by [**BZH2014**, Example 12.16]. The statement about the figure-8 knot is implicit in [**BZH2014**, Section 12].

There are also several other proofs for the first statement:

(1) Let K be the trefoil. It follows from the calculation on page 1336, the Hurewicz Theorem 84.5, the Knot-Branched Covering Proposition 149.5 and Lemma 51.9 (1) that $\mathrm{H}_1(\Sigma_2(K);\mathbb{Z}) \cong \mathbb{Z}_3$. Furthermore, by Corollary 149.8 we know that $\Sigma_2(K)$ is a

[1779] An argument similar to the proof of the Knot-Branched Covering Proposition 149.5 (2) shows that all 2-fold branched coverings of the 3-ball with branch locus given by two unknotted arcs are homeomorphic.
[1780] Since it is exceedingly painful to keep track of orientation conventions we will not attempt to determine whether there exist orientation-preserving or orientation-reversing diffeomorphisms between the branched coverings and the lens spaces.

lens space. Thus we see that $\Sigma_2(K)$ is diffeomorphic to $L(3,1)$ or to $L(3,2)$. But these two lens spaces are diffeomorphic by the Lens Spaces-Diffeomorphisms Lemma 48.6.

(2) We also refer to [**Sti1993**, Section 7.2.2] and [**Rolf1990**, p. 304] for two alternative proofs of the first statement. ■

Finally we can answer Question 59.9 in the negative.

Proposition 149.10. The trefoil is chiral.

Remark. In Proposition 131.5 we gave an alternative proof of Proposition 149.10 which is more elementary, but less conceptual.

Proof. Let K be the trefoil. By Proposition 149.9 (9) we know that $\Sigma_2(K)$ is diffeomorphic to the lens space $L(3,2)$. We know from Proposition 148.23 that $L(3,2)$ is chiral. But by the Knot-Branched Covering Proposition 149.5 (6) this implies that K is also chiral. ■

We have now shown that the trefoil is not smoothly isotopic to its mirror image. Therefore one often distinguishes in the literature between the left-handed trefoil (which equals our trefoil from before) and the right-handed trefoil. The names stem from the observation that if we equip the right-handed trefoil with either of the two orientations the following holds: if the thumb of the right-hand points in the direction of the overcrossing-strand, then the fingers point in the direction of the undercrossing-strand. (It is a fun exercise to verify this statement.)



left-handed trefoil



right-handed trefoil

Remark. On page 2392 we saw that $L(5,2)$ is amphichiral. This is consistent with the fact, illustrated in the figure on page 1316, that the figure-8 knot is amphichiral and Propositions 149.9 and 149.5 (6).

Remark. In this chapter we studied questions about smooth embeddings of a given closed k-dimensional smooth manifold W into a given closed $(k+2)$-dimensional smooth manifold M, namely of smooth embeddings of $W = S^1$ into $M = S^3$, by associating to a smooth embedding a branched covering. The same approach can also be used in many other contexts, for example it can and is being used to study smooth embeddings of surfaces into closed 4-dimensional smooth manifolds, see e.g. [**HsSz1971**].

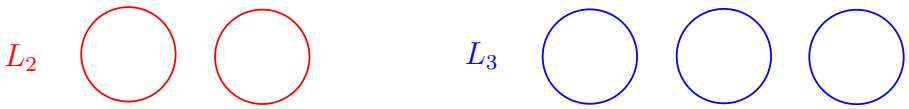
Exercises for Chapter 149.

Exercise 149.1. The definition of the m-fold branched cover of a knot easily in S^3 generalizes to the notion of an m-fold branched cover of a link in S^3.

(a) We consider the trivial 2-component link L_2. What is the diffeomorphism type of the corresponding 2-fold branched cover?

(b) We consider the trivial 2-component link L_2. What is the diffeomorphism type of the corresponding 5-fold branched cover?

(c) We consider the trivial 2-component link L_3. What is the diffeomorphism type of the corresponding 2-fold branched cover?



Exercise 149.2. Let K and J be two oriented knots in S^3. We denote by $K\#J$ the connected sum of the two oriented knots as defined on page 1319. Show that for every $n \in \mathbb{N}$ the branched cover $\Sigma_n(K\#J)$ is diffeomorphic to the connected sum $\Sigma_n(K)\#\Sigma_n(J)$.

Exercise 149.3. Let K be the trefoil.

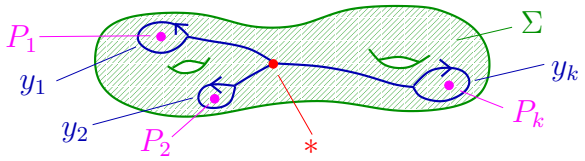
(a) Show that the square knot $K\#K$ is not smoothly isotopic to its mirror image.
(b) Show that the granny knot $K\#K^{\mathrm{mir}}$ is neither smoothly isotopic to the square knot $K\#K$ nor to its mirror image.

Hint. Use Exercise 149.2 and 148.4.

Exercise 149.4. Let Σ be the surface of genus g and let $P_1, \dots, P_k$ be disjoint points in Σ. We set $F := \Sigma \setminus \{P_1, \dots, P_k\}$. We pick a base point $* \in F$.

(a) Let $\Gamma \subset \pi_1(F, *)$ be a subgroup of finite index. By the Covering Existence Theorem 61.5 we know that there exists a finite covering $p\colon \widetilde{F} \to F$ with $p_*(\pi_1(\widetilde{F}), *) = \Gamma$. Show that the covering $p\colon \widetilde{F} \to F$ can be extended to a branched covering $q\colon \widetilde{\Sigma} \to \Sigma$.
(b) Let $\varphi\colon \pi_1(F, *) \to \mathbb{Z}_m$ be an epimorphism. We pick loops $y_1, \dots, y_k$ as illustrated in. We apply (a) to $\Gamma = \ker(\varphi)$ and we obtain a branched covering $q\colon \widetilde{\Sigma} \to \Sigma$. What is the genus of $\widetilde{\Sigma}$ in terms of g and the orders of the images of $\varphi([y_1]), \dots, \varphi([y_k])$?
(c) How does (b) generalize to the situation where $\varphi\colon \pi_1(F, *) \to G$ is an epimorphism onto some finite group?

Remark. The Preimage-of-Covering Proposition 63.5 might come in handy.



Exercise 149.5. Let $q(z) =$ be a non-constant complex polynomial. Show that the map

$$\begin{array}{rcl} \mathbb{C}\mathrm{P}^1 & \to & \mathbb{C}\mathrm{P}^1 \\ [z:w] & \mapsto & \begin{cases} [q(a):1], & \text{if } [z:w] = [a:1] \text{ for some } a \in \mathbb{C}, \\ [1:0], & \text{if } [z:w] = [1:0] \end{cases} \end{array}$$

is a branched covering. In particular determine the branch locus.
Hint. Use the Locally-a-Power Theorem 32.8.

look at proof of Proposition 150.10 First we recall the definition of an R-homology sphere from page 3143 and we introduce some, almost self-explanatory, related notions.

Definition. Let R be a commutative ring and let M be a compact orientable n-dimensional topological manifold. We introduce the following notions:

M is an R-homology sphere,	if for each $k \in \mathbb{N}_0$ we have	$\mathrm{H}_k(M;R) \cong \mathrm{H}_k(S^n;R)$
M is an R-homology ball,	if for each $k \in \mathbb{N}_0$ we have	$\mathrm{H}_k(M;R) \cong \mathrm{H}_k(\overline{B}^n;R)$
M is an R-cylinder,	if for each $k \in \mathbb{N}_0$ we have	$\mathrm{H}_k(M;R) \cong \mathrm{H}_k(S^{n-1}\times[0,1];R)$.

Instead of $\mathbb{Z}$-homology sphere we usually say integral homology sphere and instead of $\mathbb{Q}$-homology sphere we usually say rational homology sphere. The same convention holds also for balls and cylinders.

Remark. On pages 2159, 2710 and 3143 we defined homology spheres without the condition that M is compact and orientable. In fact we saw in Lemma 124.1 that the purely homological definition already implies that M is closed and orientable. A similar discussion applies to homology balls that we first introduced on page 2720.

Lemma 150.1. Let M be an n-dimensional topological manifold.

(1) If M is a rational homology n-ball, then ∂M is a rational homology $(n-1)$-sphere.[1781]
(2) Let M be a rational homology cylinder.
   (a) The boundary ∂M has precisely two components X and Y, each of which is a rational homology $(n-1)$-sphere.
   (b) The inclusion induced maps $\mathrm{H}_*(X;\mathbb{Q}) \to \mathrm{H}_*(M;\mathbb{Q})$ and $\mathrm{H}_*(Y;\mathbb{Q}) \to \mathrm{H}_*(M;\mathbb{Q})$ are isomorphisms.
(3) Let M be a rational homology ball. If $\varphi\colon \overline{B}^n \to M \setminus \partial M$ is a smooth embedding, then $M \setminus \varphi(B^n)$ is a rational homology cylinder.

The same statements also hold if we replace "rational" by "integral".

Proof. The lemma will be proved in Exercise 144.4. ■

Now we can modify the cobordism groups Ω_n to obtain new and interesting groups.

Definition. Let $n \in \mathbb{N}$. We define

$\Theta^n_{\mathbb{Q}} := \{\text{diffeomorphism classes of oriented } n\text{-dimensional rational homology spheres}\}/\!\sim_{\mathbb{Q}}$

where we write $M \sim_{\mathbb{Q}} N$ if there exists an oriented cobordism W between M to N that is a rational homology cylinder. Similarly we define

$\Theta^n_{\mathbb{Z}} := \{\text{diffeomorphism classes of oriented } n\text{-dimensional integral homology spheres}\}/\!\sim_{\mathbb{Z}}$

where we write $M \sim_{\mathbb{Z}} N$ if there exists an oriented cobordism W between M and N that is an integral homology cylinder. Furthermore we define $\Theta^{\mathrm{top},n}_{\mathbb{Q}}$ and $\Theta^{\mathrm{top},n}_{\mathbb{Z}}$ the same way as above except that we "throw over board" all smooth structures. More precisely, we make the following two changes:

(1) We consider homeomorphism classes of oriented n-dimensional rational homology spheres.

[1781] This statement is closely related to Lemma 124.8.

(2) We no longer demand that the cobordism W is a smooth manifold, we only assume that it is a topological manifold.

To simplify the discussion, and out of personal preference, henceforth we only consider the case $n = 3$ and we drop the decoration "3" from the notation. Recall that Theorem **??** says that in dimension *three* any topological manifold admits a smooth structure and two smooth manifolds are homeomorphic if and only if they are diffeomorphic. Put differently, there is basically no difference between the study of 3-dimensional topological manifolds and the study of 3-dimensional smooth manifolds. On the other hand, we saw in Theorem 135.9 that in dimension *four* there is a big difference between the theory of topological manifolds and the theory of smooth manifolds. In particular the above notions of cobordisms might change depending on whether or not we consider 4-dimensional topological manifolds or 4-dimensional smooth manifolds.

Proposition 150.2. In all four cases the connected sum operation defines an abelian group structure where the neutral element is given by S^3 and where the inverse of a homology sphere Y is given by $-Y$.

Example. Let Y be a 3-dimensional rational homology sphere. If Y is amphichiral, then we have the following equality in $\Theta_{\mathbb{Q}}$:

$$[Y]\#[Y] \underset{\substack{\uparrow \\ \text{since } Y \text{ amphichiral}}}{=} [Y]\#[-Y] \underset{\substack{\uparrow \\ \text{Proposition 150.2}}}{=} S^3 \in \Theta_{\mathbb{Q}}$$

This implies that an amphichiral 3-dimensional rational sphere defines either the trivial element in $\Theta_{\mathbb{Q}}$ or an element of order 2.

Sketch of proof. In the following steps we verify all the statements that are implicit in the formulation of the proposition for $\Theta_{\mathbb{Q}}$.

(1) It follows from the Connected Sum-H_*-Proposition 100.13 that the connected sum of two rational homology spheres is again a rational homology sphere.

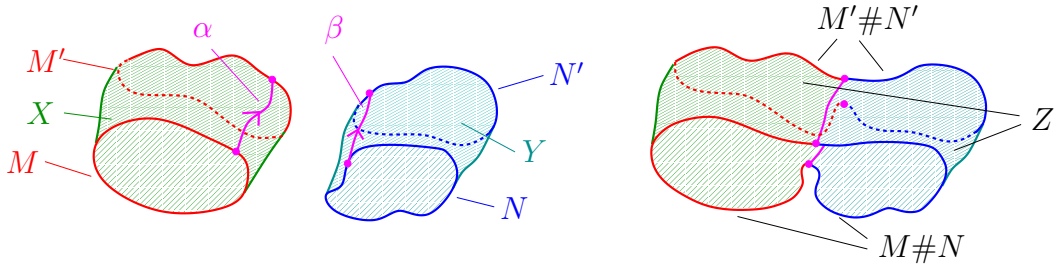
(2) We need to show that the connected sum operation descends to a well-defined operation on $\Theta_{\mathbb{Q}}$. Thus suppose that we are given oriented, rational homology spheres M and M' and also N and N' such that $M = M' \in \Theta_{\mathbb{Q}}$ and $N = N' \in \Theta_{\mathbb{Q}}$. We need to show that $M\#N = M'\#N' \in \Theta_{\mathbb{Q}}$. We pick compact oriented 4-dimensional smooth manifolds X and Y with $\partial X = M \sqcup -M'$ and $\partial Y = N \sqcup -N'$ as in the definition of $\sim_{\mathbb{Q}}$.

By Corollary 102.14 we can pick an oriented proper connected 1-dimensional submanifold $[0,1] \cong \alpha \subset X$ which connects a point on M to a point on M'. Similarly we can pick a proper connected 1-dimensional submanifold $[0,1] \cong \beta \subset Y$ which connects a point on N to a point on N'. By the Tubular Neighborhood Theorem **??** we can thicken the submanifolds α and β to submanifolds $[0,1] \times \overline{B}^3 = \alpha \times \overline{B}^3$ and $[0,1] \times \overline{B}^3 = \beta \times \overline{B}^3$. We can and will pick the former map to be orientation-preserving and the latter orientation-reversing. We define

$$Z := \big((X \setminus (\alpha \times B^3)) \sqcup (Y \setminus (\beta \times B^3))\big)/\sim$$

where we identify a point on $\alpha \times S^2 = [0,1] \times S^2$ with the corresponding point on $\beta \times S^2 = [0,1] \times S^2$. After smoothing corners, see [**Wall2016**, Proposition 2.6.2],

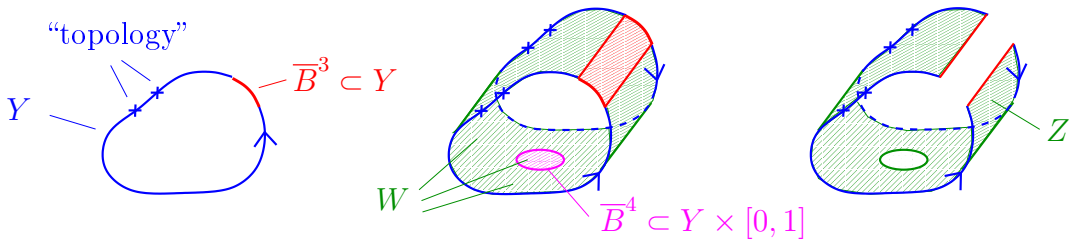
we can view Z as a compact oriented smooth manifold such that $\partial Z = (M\#N) \sqcup -(M'\#N')$. (This construction is illustrated in the figure below.) Using that X and Y are rational homology cylinders one can fairly easily show that Z itself is also a rational homology cylinder.



(3) It follows from Lemma **??** that the connected sum operation is commutative and associative. Furthermore it follows from Lemma **??** that S^3 is the neutral element with respect to the connected sum operation.

(4) Let Y be a rational homology sphere. We have to show that $(Y\# - Y) \sim_{\mathbb{Q}} S^3$. We pick a closed 3-ball $\overline{B}^3$ in Y. We consider $W = ([0,1] \times Y) \setminus ([0,1] \times B^3)$. After smoothing corners we have $\partial W = Y\# - Y$. Finally we remove an open 4-ball from W and we obtain a 4-manifold Z with $\partial Z = (Y\# - Y) \sqcup S^3$. We leave it to the reader to verify that Z is a rational homology cylinder. We illustrate this construction in the figure below.

The proof that $\Theta_{\mathbb{Z}}$ forms a group is basically identical to the above proof. The proof that $\Theta^{\text{top}}_{\mathbb{Q}}$ and $\Theta^{\text{top}}_{\mathbb{Z}}$ form groups is in principle the same, except that we need to replace the references dealing with smooth manifolds by references that take care of topological manifolds. For example, we need to replace the Tubular Neighborhood Theorem **??** by the fact that compact 1-dimensional submanifolds of compact topological manifolds admit trivial tubular neighborhoods, see the discussion on page **??**. ■



The following lemma gives a characterization of smooth manifolds representing the trivial element in the various groups that we just introduced.

Lemma 150.3. Let Y be a 3-dimensional rational homology sphere. Then

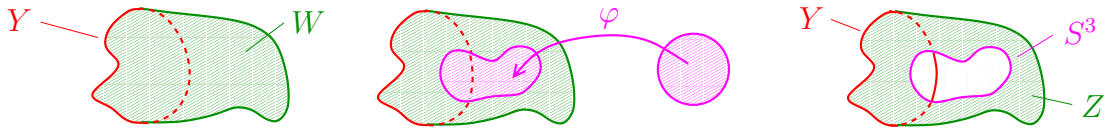
$$[Y] = 0 \in \Theta_{\mathbb{Q}} \iff \text{there exists a 4-dimensional smooth manifold } W \text{ that is a rational homology ball } W \text{ with } \partial W = Y.$$

There are obvious analogues to this statement for the groups $\Theta_{\mathbb{Z}}$, $\Theta^{\text{top}}_{\mathbb{Q}}$ and $\Theta^{\text{top}}_{\mathbb{Z}}$.

Proof. Let Y be a 3-dimensional rational homology sphere.

"⇒" If Y represents the trivial element in $\Theta_{\mathbb{Q}}$, then by definition there exists a 4-dimensional rational homology cylinder Z with $\partial Z = Y \sqcup -S^3$. A straightforward argument, using the Gluing-Smooth Manifolds-Proposition **??**, Lemma 150.1 and the Mayer–Vietoris Theorem 78.9 for Manifolds shows that $W = Z \cup_{\partial Z = S^3} \overline{B}^4$ is a rational homology ball. Evidently we have $\partial W = Y$.

"⇐" Now suppose that there exists a 4-dimensional smooth manifold W that is a rational homology ball W with $\partial W = Y$. We pick a smooth embedding $\varphi\colon \overline{B}^4 \to W \setminus \partial W$. It follows from Lemma 150.1 (3) that $Z = Y \setminus \varphi(B^4)$ is a rational cylinder which then shows that $[Y] = [S^3] = 0 \in \Theta_{\mathbb{Q}}$. We refer to the figure below for an illustration. ■



We start out with the investigation of the group $\Theta_{\mathbb{Q}}^{\text{top}}$.

Proposition 150.4. Let M be a 3-dimensional rational homology sphere. If $M = 0 \in \Theta_{\mathbb{Q}}^{\text{top}}$, then $|\mathrm{H}_1(M;\mathbb{Z})|$ is a square.

As we will see, Proposition 150.4 is an almost immediate consequence of Statement (1) of the following lemma.

Lemma 150.5. Let M be a 3-dimensional rational homology sphere and let W a rational homology ball W with $\partial W = M$.

(1) We have[17821783]
$$|\ker(\mathrm{H}_1(M;\mathbb{Z}) \to \mathrm{H}_1(W;\mathbb{Z}))|^2 \;=\; |\mathrm{H}_1(M;\mathbb{Z})|.$$

(2) [1784] The linking pairing λ_M vanishes on $\ker(\mathrm{H}_1(M;\mathbb{Z}) \to \mathrm{H}_1(W;\mathbb{Z}))$, more precisely, for any $x, y \in \ker(\mathrm{H}_1(M;\mathbb{Z}) \to \mathrm{H}_1(W;\mathbb{Z}))$ we have $\lambda_M(x,y) = 0$.

(3) The subgroup $P := \ker(\mathrm{H}_1(M;\mathbb{Z}) \to \mathrm{H}_1(W;\mathbb{Z}))$ is a metabolizer of the linking pairing. This means, according to the definition on page 3171, that $P = P^{\perp}$ where $P^{\perp} := \{v \in \mathrm{H}_1(M;\mathbb{Z}) \,|\, \lambda(v,w) = 0 \text{ for all } w \in P\}$.

Proof of Proposition 150.4 assuming Lemma 150.5. Let M be a 3-dimensional rational homology sphere which represents the trivial element in $\Theta_{\mathbb{Q}}^{\text{top}}$. By Lemma 150.3 there exists a 4-dimensional smooth manifold W that is a rational homology ball W with $\partial W = M$. Thus the proposition follows from Lemma 150.5 (1). ■

Proof of Lemma 150.5. Let M be a 3-dimensional rational homology sphere and let W a rational homology ball W with $\partial W = M$. First we note that the fact that M is a rational homology sphere and that W is a rational homology ball implies, by the argument of

[1784] This claim is precisely the content of Exercise 150.5.

[1783] Put differently, we can say that $|\ker(\mathrm{H}_1(M;\mathbb{Z}) \to \mathrm{H}_1(W;\mathbb{Z}))| = \sqrt{|\mathrm{H}_1(M;\mathbb{Z})|}$. With this formulation the statement of the claim is quite similar to the statement of the Half-Live Half-Die Theorem 144.9. In fact we will see that the proof of the claim is a variation on the proof of the Half-Live Half-Die Theorem 144.9.

[1784] We will not make use of statements (1) and (2). But these two statements play an important ingredient in some branches of topology, for example it plays a crucial role in the seminal paper [**CaG1986**].

Lemma 148.1, that the homology groups $\mathrm{H}_1(M;\mathbb{Z})$, $\mathrm{H}_2(M;\mathbb{Z})$ and also $\mathrm{H}_1(W;\mathbb{Z})$, $\mathrm{H}_2(W;\mathbb{Z})$ are finite.

(1) We consider the long exact sequence in homology groups corresponding to the pair (W, M) with $\mathbb{Z}$-coefficients:

$$\begin{aligned}\ldots \quad &\to \overbrace{\mathrm{H}_2(M;\mathbb{Z})}^{\substack{\text{isomorphic to } \mathrm{H}^1(M;\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_1(M;\mathbb{Z}),\mathbb{Z}) \text{ by the Poincaré Duality Theorem 119.1,}\\ \text{and the First Cohomology Proposition 110.17, therefore zero since } \mathrm{H}_1(M;\mathbb{Z}) \text{ is finite}\\ \downarrow}} \to \mathrm{H}_2(W;\mathbb{Z}) \to \mathrm{H}_2(W,M;\mathbb{Z})\\ &\to \mathrm{H}_1(M;\mathbb{Z}) \to \mathrm{H}_1(W;\mathbb{Z}) \to \mathrm{H}_1(W,M;\mathbb{Z}) \to \mathrm{H}_0(M;\mathbb{Z}) \underset{\substack{\uparrow\\ \text{isomorphism by the discussion on page 1594}}}{\to} \mathrm{H}_0(W;\mathbb{Z}) \to 0.\end{aligned}$$

By the Break-into-SES Lemma 78.4 we now get the following exact sequence:

$$0 \to \mathrm{H}_2(W;\mathbb{Z}) \to \mathrm{H}_2(W,M;\mathbb{Z}) \to \mathrm{H}_1(M;\mathbb{Z}) \to \mathrm{H}_1(W;\mathbb{Z}) \to \mathrm{H}_1(W,M;\mathbb{Z}) \to 0.$$

We consider the group on the left. We have

$$\mathrm{H}_2(W,M;\mathbb{Z}) \underset{\substack{\uparrow\\ \text{Poincaré Duality}\\ \text{Theorem 119.1}}}{\cong} \mathrm{H}^2(W;\mathbb{Z}) \underset{\substack{\uparrow\\ \text{Universal Coefficient}\\ \text{Theorem 110.12}}}{\cong} \overbrace{\mathrm{Hom}(\mathrm{H}_2(W;\mathbb{Z}),\mathbb{Z})}^{=0 \text{ since } \mathrm{H}_2(W;\mathbb{Z}) \text{ is finite}} \oplus \underbrace{\mathrm{Ext}(\mathrm{H}_1(W;\mathbb{Z}),\mathbb{Z})}_{\substack{\text{by the Ext-Calculation Lemma 110.5 (5)}\\ \text{since } \mathrm{H}_1(W;\mathbb{Z}) \text{ is finite}}} \underset{\substack{\uparrow\\ \text{by the Ext-Calculation Lemma 110.5 (5)}\\ \text{since } \mathrm{H}_1(W;\mathbb{Z}) \text{ is finite}}}{\cong} \mathrm{H}_1(W;\mathbb{Z}).$$

Almost the same argument shows that $\mathrm{H}_2(W;\mathbb{Z}) \cong \mathrm{H}_1(W,M;\mathbb{Z})$. This shows that the above exact sequence is an exact sequence of finite abelian groups that is "symmetric" around $\mathrm{H}_1(M;\mathbb{Z})$. Therefore it remains to prove the following claim.

Claim. If

$$0 \to A_k \to \ldots \to A_2 \to A_1 \to B \xrightarrow{\gamma} A_1 \to A_2 \to \ldots \to A_k \to 0$$

is an exact sequence of finite abelian groups, then

$$|\ker(\gamma\colon B \to A_1)|^2 = |B|.$$

Proof. The claim is proved by induction on k, exactly the same way as we proved the claim on page 3097. We leave the details to the reader.

(2) By Lemma 148.31 we know that exists a pairing $\chi\colon \mathrm{H}_2(W,M;\mathbb{Z}) \times \mathrm{H}_2(W,M;\mathbb{Z}) \to \mathbb{Q}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_2(W,M;\mathbb{Z}) \times \mathrm{H}_2(W,M;\mathbb{Z}) & \xrightarrow{\quad\chi\quad} & \mathbb{Q} \\ \downarrow{\scriptstyle \partial\times\partial} & & \downarrow \\ \mathrm{H}_1(M;\mathbb{Z}) \times \mathrm{H}_1(M;\mathbb{Z}) & \xrightarrow[\lambda_M]{} & \mathbb{Q}/\mathbb{Z}. \end{array}$$

We make the following observations:

(a) By the long exact sequence in homology of the pair (W, M) we have

$$\ker(\mathrm{H}_1(M;\mathbb{Z}) \to \mathrm{H}_1(W;\mathbb{Z})) = \mathrm{im}(\partial\colon \mathrm{H}_2(W,M;\mathbb{Z}) \to \mathrm{H}_1(M;\mathbb{Z})).$$

(b) By the long exact sequence in homology and the above information on the homology groups of W and M we know that $\mathrm{H}_2(W,M;\mathbb{Z})$ is finite.

(c) Since $\mathrm{H}_2(W,M;\mathbb{Z})$ is finite we see that the pairing χ is the zero-pairing.

(d) It follows from (1), (3) and the commutative diagram that λ_M does indeed vanish for any $x, y \in \ker(\mathrm{H}_1(M;\mathbb{Z}) \to \mathrm{H}_1(W;\mathbb{Z}))$.

(3) It follows from (1), (2) and Exercise 148.2 that $\ker(\mathrm{H}_1(M;\mathbb{Z}) \to \mathrm{H}_1(W;\mathbb{Z}))$ is a metabolizer for the linking pairing. ■

Corollary 150.6. Let M and N be two 3-dimensional rational homology spheres. If we have $[M] = [N] \in \Theta^{\mathrm{top}}_{\mathbb{Q}}$, then $|\mathrm{H}_1(M;\mathbb{Z})| \cdot |\mathrm{H}_1(N;\mathbb{Z})|$ is a square.

Proof. Let M and N be two 3-dimensional rational homology spheres which satisfy the equality $[M] = [N] \in \Theta^{\mathrm{top}}_{\mathbb{Q}}$. From Proposition 150.2 we obtain immediately the equality $[M\#(-N)] = [M]-[N] = 0 \in \Theta^{\mathrm{top}}_{\mathbb{Q}}$. Now Proposition 150.4 implies that $|\mathrm{H}_1(M\#(-N);\mathbb{Z})|$ is a square. The corollary follows from the Connected Sum-H_*-Proposition 100.13 which states that $\mathrm{H}_1(M\#(-N);\mathbb{Z}) \cong \mathrm{H}_1(M;\mathbb{Z}) \oplus \mathrm{H}_1(N;\mathbb{Z})$. ■

The following proposition shows that the groups $\Theta^{\mathrm{top}}_{\mathbb{Q}}$ and $\Theta_{\mathbb{Q}}$ are very much non-trivial.

Proposition 150.7.

(1) The groups $\Theta^{\mathrm{top}}_{\mathbb{Q}}$ and $\Theta_{\mathbb{Q}}$ admit an epimorphism onto $\mathbb{Z}_2^\infty$.

(2) The groups $\Theta^{\mathrm{top}}_{\mathbb{Q}}$ and $\Theta_{\mathbb{Q}}$ admit a subsummand isomorphic to $\mathbb{Z}_2^\infty$.[1785]

Remark. It follows from [**KK1980**, Theorems 5.2 and 6.1] and a generalization of the arguments used in the proof of Proposition 150.7 (1) that the group $\Theta^{\mathrm{top}}_{\mathbb{Q}}$ admits an epimorphism onto $\mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty \oplus \mathbb{Z}_4^\infty$.

The proof of Proposition 150.7 (2) relies on the following theorem from number theory. We refer to [**IR1990**, p. 251] for a proof.

Theorem 150.8. (Dirichlet's Theorem on Arithmetic Progressions) For any coprime $a, n \in \mathbb{N}$ there exist infinitely many primes of the form $a + kn$ with $k \in \mathbb{N}$.

Proof of Proposition 150.7.

(1) We have an obvious epimorphism $\Theta_{\mathbb{Q}} \to \Theta^{\mathrm{top}}_{\mathbb{Q}}$, so it suffices to show that the group $\Theta^{\mathrm{top}}_{\mathbb{Q}}$ admits an epimorphism onto $\mathbb{Z}_2^\infty$. For each prime p we consider the map

$$\begin{array}{rccl} \alpha_p\colon \Theta^{\mathrm{top}}_{\mathbb{Q}} & \to & \mathbb{Z}_2 \\ M & \mapsto & \max\{n \in \mathbb{N}_0 \,|\, p^n \text{ divides } |\mathrm{H}_1(M;\mathbb{Z})|\} \mod 2. \end{array}$$

It follows easily from Corollary 150.6, together with the uniqueness of the prime power decomposition of natural numbers and the fact that p is a prime, that this map is well-defined. Furthermore it follows from the Connected Sum-H_*-Proposition 100.13 that α_p is a group homomorphism.

It remains to show that the homomorphism

$$\begin{array}{rccl} \alpha\colon \Theta^{\mathrm{top}}_{\mathbb{Q}} & \to & \bigoplus\limits_{\text{prime } p} \mathbb{Z}_2 \\ M & \mapsto & \bigoplus\limits_{\text{prime } p} \alpha_p(M) \end{array}$$

[1785] Here we say that an abelian group G is a subsummand of an abelian group H if there exists an abelian group A such that $H \cong G \oplus A$.

is an epimorphism. But this follows immediately from considering lens spaces. More precisely, let p be a prime. By Corollary 84.6 we have $\mathrm{H}_1(L(p,1);\mathbb{Z}) \cong \mathbb{Z}_p$, i.e. we have $\alpha_p(L(p,1)) = 1$ and $\alpha_q(L(p,1)) = 0$ for every prime $q \neq p$.

(2) By Dirichlet's Theorem 150.8 there exist infinitely many primes $p_i, i \in \mathbb{N}$ such that $p_i \equiv 1 \bmod 4$. By the calculation of the Legendre symbol, see e.g. [**IR1990**, p. 52], given any such prime p_i there exists an $a_i \in \{1, \dots, p_i - 1\}$ such that $a_i^2 \equiv -1 \bmod p_i$. It follows from Lemma 107.6 that the lens spaces $L(p_i, a_i)$ are amphichiral. It follows from the discussion on page 3189 and from (1) that we have well-defined homomorphisms

$$\begin{array}{ccccc} \mathbb{Z}_2^\infty & \to & \Theta_{\mathbb{Q}}^{\mathrm{top}} & \to & \mathbb{Z}_2^\infty \\ & & M & \mapsto & \sum\limits_i \alpha_{p_i}(M) \\ (x_1, x_2, \dots) & \mapsto & \#\limits_{\substack{i \text{ with} \\ x_i = 1}} L(p_i, a_i), & & \end{array}$$

furthermore we see easily that the composition of these two maps is the identity. This shows that the right-hand map splits. Therefore it follows from Splitting Lemma 78.2 that $\mathbb{Z}_2^\infty$ is in fact a subsummand of $\Theta_{\mathbb{Q}}^{\mathrm{top}}$. Exactly the same argument applies to $\Theta_{\mathbb{Q}}$. ■

By definition any 3-dimensional integral homology sphere lies in the kernel of the epimorphism $\Theta_{\mathbb{Q}}^{\mathrm{top}} \to \mathbb{Z}_2^\infty$ that we had just constructed in the proof of Proposition 150.7.
In fact it was proved by Michael Freedman [**Fre1982**, Theorem 1.4'] in 1981 that every integral homology 3-dimensional sphere bounds a simply connected integral homology 4-dimensional ball.[1786] This result implies in particular the following theorem.

Theorem 150.9. The group $\Theta_{\mathbb{Z}}^{\mathrm{top}}$ is trivial.

It turns out that the situation is quite different if we consider the group $\Theta_{\mathbb{Z}}$ instead of the group $\Theta_{\mathbb{Z}}^{\mathrm{top}}$. In fact we have the following proposition.

Proposition 150.10. Given a 3-dimensional integral homology sphere M we define

$$\mu(M) \ := \ \tfrac{1}{8}\operatorname{sign}(W) \ \bmod 2$$

where W is a compact oriented 4-dimensional smooth manifold with $\partial W = M$, with $\mathrm{H}_1(W;\mathbb{Z}) = 0$ and with even intersection form.

(1) The number $\mu(M)$ is well-defined and it lies in $\mathbb{Z}_2$.

(2) The map

$$\begin{array}{rccc} \mu\colon \Theta_{\mathbb{Z}} & \to & \mathbb{Z}_2 \\ M & \mapsto & \mu(M), \end{array}$$

is a well-defined homomorphism, it is called the *Rokhlin homomorphism*.

(3) The Rokhlin homomorphism $\mu\colon \Theta_{\mathbb{Z}} \to \mathbb{Z}_2$ is an epimorphism, in particular $\Theta_{\mathbb{Z}}$ is non-trivial.

Remark. The Rokhlin homomorphism is the starting point for many developments in the study of 3-dimensional and 4-dimensional smooth manifolds. For example Andrew Casson

[1786] The methods that Michael Freedman used in the proof of this result are closely related to the methods in Freedman's proof of the topological Poincaré Conjecture in dimension 4, see Theorem **??**.

showed in the mid 1980s that there exists a map

$$\lambda\colon \left\{ \begin{array}{c} \text{diffeomorphism classes of oriented} \\ \text{3-dimensional integral homology spheres} \end{array} \right\} \to \mathbb{Z}$$

that has the following properties:

(1) It is additive in the sense that for two 3-dimensional integral homology spheres M and N we have $\lambda(M\#N) = \lambda(M) + \lambda(N)$.
(2) For every 3-dimensional integral homology sphere we have $\mu(M) = \lambda(M) \bmod 2$.

More information on the Casson invariant λ can be found in [**Sav2002, AkM1990**].

Sketch of the proof of Proposition 150.10.

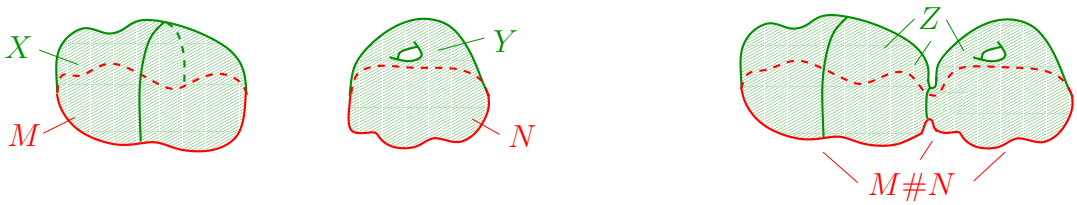
(1) Let M be a 3-dimensional integral homology sphere. The following arguments show that $\mu(M)$ is indeed defined, that it is well-defined and that it lies in $\mathbb{Z}_2$.
   (a) By [**Kap1979**], see also [**Sav2002**, p. 38], there exists a compact oriented 4-dimensional smooth manifold with $\partial W = M$, with $\mathrm{H}_1(W;\mathbb{Z}) = 0$ and with even intersection form.[1787]
   (b) Since ∂W is an integral homology sphere and since $\mathrm{H}_1(W;\mathbb{Z}) = 0$ one can easily adjust the argument of the proof of the Intersection Form Proposition 132.4 to show that the intersection form Q_W is non-singular.
   (c) Since the intersection form Q_W is even and non-singular it follows from the Even Forms Proposition 133.18 that $\mathrm{sign}(W) = \mathrm{sign}(Q_W)$ is divisible by 8, i.e. $\frac{1}{8}\mathrm{sign}(W) \in \mathbb{Z}$.
   (d) Now suppose that we have another compact oriented 4-dimensional smooth manifold W' with $\mathrm{H}_1(W';\mathbb{Z}) = 0$, with $\partial W' = M$ and with even intersection form. We need to show that $\mathrm{sign}(W) \equiv \mathrm{sign}(W') \bmod 16$.
   We consider the closed oriented 4-dimensional smooth manifold[1788]
   $$X \;:=\; W \cup_{\partial W = \partial W'} (-W')$$
   that we obtain from gluing W and $-W'$ along the common boundary given by M. Since $\mathrm{H}_1(W;\mathbb{Z}) = \mathrm{H}_1(W';\mathbb{Z}) = 0$ and since M is an integral homology sphere one can easily adapt the proof of the Connected Sum-H_*-Proposition 100.13 to show that $\mathrm{H}_1(X;\mathbb{Z}) = 0$ and one can modify the proof of the Intersection Form-Additivity Proposition 132.12 to conclude that $Q_X \cong Q_W \oplus -Q_{W'}$, in particular Q_X is even. Now we can apply Rokhlin's Theorem 135.5 to conclude that $\mathrm{sign}(X) \equiv 0 \bmod 16$. But by the isomorphism $Q_X \cong Q_W \oplus -Q_{W'}$ this implies that $\mathrm{sign}(W) - \mathrm{sign}(W') \equiv 0 \bmod 16$, which gives us the desired result.
(2) Now we have to show that μ defines a well-defined homomorphism $\Theta_{\mathbb{Z}} \to \mathbb{Z}_2$.
   (a) First we show that μ is additive. So let M and N be two 3-dimensional integral homology spheres. We pick 4-dimensional smooth manifolds X and Y such that $\partial X = M$ and $\partial Y = N$ and which have the extra properties that are required in the definition of $\mu(M)$ and $\mu(N)$. We denote by $Z = X\#_b Y$ the boundary connected sum that we introduced on page **??**. It follows from Proposition **??**

[1787] In fact in [**Kap1979**] it is shown that W can be chosen to be simply connected, which implies by the Hurewicz Theorem 84.5 that the first homology group vanishes.
[1788] Here we use the Gluing-Smooth Manifolds-Proposition **??** to conclude that X is indeed a closed oriented 4-dimensional smooth manifold.

that Z is a 4-dimensional smooth manifold with $\partial Z = M\#N$ and such that $\mathrm{H}_1(Z;\mathbb{Z}) = 0$. Furthermore a modest modification of the proof of the Intersection Form-Additivity Proposition 132.12 shows that $Q_Z \cong Q_X \oplus Q_Y$. This means that we can use Z to calculate $\mu(M\#N)$ and we obtain that

$$\mu(M\#N) \;=\; \tfrac{1}{8}\operatorname{sign}(Z) \;=\; \tfrac{1}{8}\operatorname{sign}(X) + \tfrac{1}{8}\operatorname{sign}(Y) \;=\; \mu(M) + \mu(N).$$

We refer to the figure below for an illustration of the argument.



(b) It follows easily from the definitions that for any integral homology sphere M we have $\mu(-M) = \mu(M)$.

(c) It remains to show that μ is well-defined. By (a) and (b) it suffices to show that if M represents the trivial element in $\Theta_{\mathbb{Z}}$, then $\mu(M) = 0$. Let such M be given. It follows from Lemma 150.3 that M is the boundary of an integral homology sphere W. Since $\mathrm{H}_2(W;\mathbb{Z}) = 0$ the intersection form is zero, in particular even. Since $\mathrm{H}_1(W;\mathbb{Z}) = 0$ we can use W to determine $\mu(M)$ and we get immediately that $\mu(M) = \frac{1}{8}\operatorname{sign}(W) = 0$.

(3) Finally we want to show that there exists a 3-dimensional integral homology sphere M with $\mu(M) = 1$. Let M be the Poincaré Homology Sphere that we introduced on page 2161. In [**Sav2002**, p. 38] it is shown that M is the boundary of a compact oriented simply connected 4-dimensional smooth manifold W with intersection form isometric to $-E_8$. As we saw on page 2913, the form $-E_8$ of rank 8 is negative-definite, hence its signature is -8. By the definition of the Rokhlin homomorphism this implies that $\mu(M) = 1$. ■

In fact a much stronger statement than Proposition 150.10 holds. Building on the techniques developed by Simon Donaldson [**Don1983, Don1987a**] Mikio Furuta [**Furu1990**] proved in 1990 the following theorem.

Theorem 150.11. The group $\Theta_{\mathbb{Z}}$ contains a subgroup that is isomorphic to $\mathbb{Z}^\infty$.

Among the four groups that we have studied in this section, the trivial group $\Theta^{\mathrm{top}}_{\mathbb{Z}}$ is the only one which has been fully calculated. At the moment we still know depressingly little about the remaining three groups. For example the following three questions are still open.

Question 150.12.

(1) Is the group $\Theta^{\mathrm{top}}_{\mathbb{Q}}$ isomorphic to $\mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty \oplus \mathbb{Z}_8^\infty$?

(2) Does the group $\Theta^{\mathrm{top}}_{\mathbb{Q}}$ contain elements of infinite order?

(3) Let $n \neq 1, 2$ be a natural number. Does there exist an element in the groups $\Theta_{\mathbb{Z}}$ or $\Theta_{\mathbb{Q}}$ of order n?

Exercises for Chapter 150.

Exercise 150.1. Let M be an oriented 3-dimensional rational homology sphere. We denote by $\lambda_M\colon \mathrm{H}_1(M;\mathbb{Z})\times \mathrm{H}_1(M;\mathbb{Z})\to \mathbb{Q}/\mathbb{Z}$ the linking pairing that we introduced on page 3151. Let W be a 4-dimensional rational homology ball with $\partial W = M$. We denote by $i\colon M\to W$ the inclusion and we write $K := \ker(i_*\colon \mathrm{H}_1(M;\mathbb{Z})\to \mathrm{H}_1(W;\mathbb{Z}))$. Show that λ_M vanishes on $K\times K$.

Exercise 150.2. Let Y_1 and Y_2 be two compact 3-dimensional smooth manifolds with one torus boundary component. Let $Y = Y_1\cup Y_2$ be the result of gluing Y_1 and Y_2 along the boundary. Show that if $b_1(Y)=0$, then $b_1(Y_1)=1$ and $b_1(Y_2)=1$.

Exercise 150.3. Let W be compact orientable 3-dimensional smooth manifold such that $\mathrm{H}_1(W;\mathbb{Z})\cong\mathbb{Z}$ and such that the boundary ∂W is diffeomorphic to the torus. Show that there exists a homology 3-sphere Y and a knot $K\subset Y$ such that W is diffeomorphic to the exterior of K.
Remark. Recall that the exterior X_K of the knot K in Y is defined as $X_K = Y\setminus(B^2\times K)$ where $\overline{B}^2\times K$ is a tubular neighborhood of K in Y.

Exercise 150.4. Does there exist a compact simply connected 3-dimensional smooth manifold M such that ∂M has at least one boundary component that is a torus?

Exercise 150.5. Let Y be a 3-dimensional rational homology sphere. Now we suppose that there exists a compact orientable 4-dimensional topological manifold M such that $\mathrm{H}_*(M;\mathbb{Q})\cong \mathrm{H}_*(\overline{B}^4;\mathbb{Q})$ (sometimes such M is called a 4-dimensional rational homology ball) with $Y=\partial M$. Show that

$$|\ker(\mathrm{H}_1(Y;\mathbb{Z})\to \mathrm{H}_1(M;\mathbb{Z}))| \;=\; |\mathrm{H}_1(Y;\mathbb{Z})|.$$

Remark. The statement and the proof of the exercise are similar in spirit to the Half-Live Half-Die Theorem 144.9.

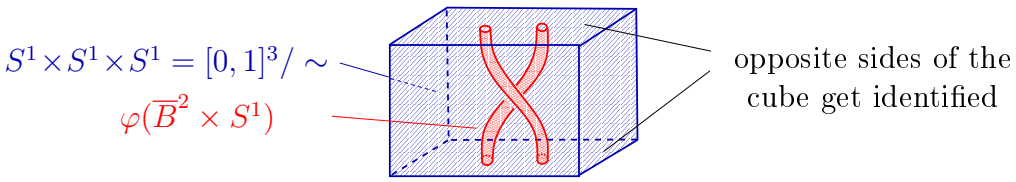
Exercise 150.6. We consider the 3-dimensional torus $S^1\times S^1\times S^1 = [0,1]^3/\!\sim$ and we consider the embedding $\varphi\colon \overline{B}^2\times S^1\to S^1\times S^1\times S^1 = [0,1]^3/\!\sim$ shown in the figure below. We set $M := (S^1\times S^1\times S^1)\setminus \varphi(B^2\times S^1)$.

(a) We denote by $i\colon \partial M\to M$ the inclusion map. Show that there exists isomorphisms $\varphi\colon \mathrm{H}_1(\partial M;\mathbb{Z})\to\mathbb{Z}^2$ and $\psi\colon \mathrm{H}_1(M;\mathbb{Z})\to\mathbb{Z}^3\oplus\mathbb{Z}_2$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_1(\partial M;\mathbb{Z}) & \longrightarrow & \mathrm{H}_1(M;\mathbb{Z}) \\ \downarrow\cong & & \downarrow\cong \\ \mathbb{Z}^2 & \xrightarrow{(a,b)\mapsto(2a,0,0,b)} & \mathbb{Z}^3\oplus\mathbb{Z}_2. \end{array}$$

(b) How does the calculation in (a) fit with Half-Live Half-Die Theorem 144.9 for the fields $\mathbb{F}=\mathbb{Q}$, $\mathbb{F}=\mathbb{F}_3$ and $\mathbb{F}=\mathbb{F}_2$?

Remark. We constructed an example with similar properties in Exercise 90.3.


$S^1 \times S^1 \times S^1 = [0,1]^3/\sim$
$\varphi(\overline{B}^2 \times S^1)$
opposite sides of the
cube get identified

Part XVII

Steenrod operations

151. Steenrod Operations

In this chapter we introduce the Steenrod operations. These are cohomology operations which satisfy several axioms. In the first three sections we will state the axioms and draw several interesting conclusions. Afterwards we will sketch a proof for the existence of the Steenrod operations.

151.1. The axioms of Steenrod operations.

Definition.

(1) Let A and B be abelian groups and let $m, n \in \mathbb{N}_0$. Any natural transformation from the contravariant functor

$$\mathrm{H}^m(-;A)\colon \text{category of topological spaces} \to \text{category of sets}$$

to the contravariant functor

$$\mathrm{H}^n(-;B)\colon \text{category of topological spaces} \to \text{category of sets}$$

is called a cohomology operation of degree $n-m$. In other words, a cohomology operation consists for any topological space X of a map $\Phi_X\colon \mathrm{H}^m(X;A) \to \mathrm{H}^n(X;B)$ of sets such that for any map $f\colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^m(Y;A) & \xrightarrow{f^*} & \mathrm{H}^m(X;A) \\ {\scriptstyle \Phi_Y}\downarrow & & \downarrow{\scriptstyle \Phi_X} \\ \mathrm{H}^n(Y;B) & \xrightarrow{f^*} & \mathrm{H}^n(X;B). \end{array}$$

(2) A cohomology operation is called additive if each map $\mathrm{H}^m(X;A) \to \mathrm{H}^n(X;B)$ is a homomorphism.

(3) A cohomology operation is called non-trivial if there exists a topological space X such that the map $\mathrm{H}^m(X;A) \to \mathrm{H}^n(X;B)$ is non-trivial in the sense that the image is not just the trivial element.

With the obvious modifications the same definitions also apply to relative cohomology viewed as functors from the category of pairs of topological spaces to the category of sets.

Perhaps somewhat surprisingly we have already assembled a substantial list of non-trivial cohomology operations.

Examples.

(1) Let $n \in \mathbb{N}_0$ and let G be an abelian group. Given a pair of topological spaces (X,A) we consider the connecting homomorphism $\mathrm{H}^n(X,A;G) \to \mathrm{H}^{n+1}(X;G)$. It follows from the Cohomological LES-Lemma 108.15 (3) that these homomorphisms define an additive cohomology operation of degree one.

(2) Let $\varphi\colon A \to B$ be a homomorphism between two abelian groups and let $n \in \mathbb{N}_0$. On page 2401 we saw that given any topological space X we obtain an induced homomorphism $\varphi_*\colon \mathrm{H}^n(X;A) \to \mathrm{H}^n(X;B)$. By Lemma 108.9 we know that these maps define an additive cohomology operation of degree zero.

(3) Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of abelian groups. On page 3145 we introduced the n-th Bockstein homomorphism $\beta_n\colon \mathrm{H}^n(X;C) \to \mathrm{H}^{n+1}(X;A)$. In

Lemma 148.3 we saw that the n-th Bockstein homomorphisms define an additive cohomology operation of degree one.

(4) Let $k \in \mathbb{N}$. It follows from the Cup Product-Naturality Proposition 115.4 (1) that for any $m \in \mathbb{N}_0$ and any commutative ring R the maps[1789]

$$\begin{array}{rcl} \mathrm{H}^m(X,A;R) & \to & \mathrm{H}^{km}(X,A;R) \\ \varphi & \mapsto & \varphi^k = \underbrace{\varphi \cup \cdots \cup \varphi}_{k \text{ times}} \end{array}$$

define a cohomology operation of degree $km - m$. Note that this type of a cohomology operation is in general not additive. For example, if $k \geq 2$, then it follows easily from the calculation of the cup product of $\mathbb{C}\mathrm{P}^k$, see the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 121.7, that the map $\mathrm{H}^2(\mathbb{C}\mathrm{P}^k;\mathbb{Z}) \to \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^k;\mathbb{Z})$ given by $\varphi \mapsto \varphi^k$ is not a homomorphism.

(5) If R is a commutative ring with $1+1=0$, then the cohomology operation $\varphi \mapsto \varphi^2$ is in fact additive. Indeed for any $\varphi, \psi \in \mathrm{H}^m(X,A;R)$ we have[1790]

$$\begin{array}{rcl} (\varphi+\psi)^2 & = & \varphi \cup \varphi + \varphi \cup \psi + \psi \cup \varphi + \psi \cup \psi \\ & \underset{\uparrow}{=} & \varphi \cup \varphi + (1+(-1)^{m^2}) \cdot \varphi \cup \psi + \psi \cup \psi \ \underset{\uparrow}{=}\ \varphi \cup \varphi + \psi \cup \psi \ = \ \varphi^2 + \psi^2. \end{array}$$

by the Cup Product-Commutativity Proposition 114.8 we have $\psi \cup \varphi = (-1)^{m^2} \cdot (\varphi \cup \psi)$

since $1-1=0=1+1$ in R

Note that all of the cohomology operations that we have introduced above are of non-negative degree. This raises the following amusing question.

Question 151.1. Does there exist a non-trivial cohomology operation of negative degree?

We will come back to this question in Proposition **??**.

The following theorem gives us new and interesting cohomology operations. The theorem was first proved by Norman Steenrod [**Stee1947**] in 1947.

Theorem 151.2. (Steenrod Theorem) There exists a family

$$\mathrm{Sq}^i \colon \mathrm{H}^n(X,A;\mathbb{Z}_2) \ \to \ \mathrm{H}^{n+i}(X,A;\mathbb{Z}_2) \quad \text{with } i \in \mathbb{N}_0 \text{ and } n \in \mathbb{N}_0$$

of additive cohomology operations which satisfies the following axioms:

(1) $\mathrm{Sq}^0 = \mathrm{id}$.

(2) For any $\varphi \in \mathrm{H}^i(X,A;\mathbb{Z}_2)$ we have $\mathrm{Sq}^i(\varphi) = \varphi^2 = \varphi \cup \varphi \in \mathrm{H}^{2i}(X,A;\mathbb{Z}_2)$.

(3) For any $\varphi \in \mathrm{H}^j(X,A;\mathbb{Z}_2)$ with $i > j$ we have $\mathrm{Sq}^i(\varphi) = 0$.

(4) Let (X,A,B) be an excisive triad of topological spaces. For any $\varphi \in \mathrm{H}^r(X,A;\mathbb{Z}_2)$ any $\psi \in \mathrm{H}^s(X,B;\mathbb{Z}_2)$ and any $k \in \mathbb{N}_0$ we have[1791]

$$\mathrm{Sq}^k(\varphi \cup \psi) \ = \ \sum_{i=0}^{k} \underbrace{\mathrm{Sq}^i(\varphi)}_{\in \mathrm{H}^{r+i}(X,A;\mathbb{Z}_2)} \ \cup \ \underbrace{\mathrm{Sq}^{k-i}(\psi)}_{\in \mathrm{H}^{s+k+i}(X,B;\mathbb{Z}_2)} \ \in \ \mathrm{H}^{r+s+k}(X, A \cup B; \mathbb{Z}_2).$$

This equality is often called the Cartan formula.

[1789] We refer to page 2536 for the definition of the relative cup product.

[1790] This can be viewed as the cohomological "freshman's dream". Perhaps it should called the incoming PhD student's dream.

In the next two sections we will discuss some properties and applications of these cohomology operations. Only after these two sections, once we are convinced that Steenrod operations are indeed useful and interesting, will we provide a proof for the Steenrod Theorem 151.2.

Remark.

(1) The first three axioms of the Steenrod Theorem 151.2 can be summarized as follows: Given any $\varphi \in \mathrm{H}^k(X, A; \mathbb{Z}_2)$ we obtain the following table

$\mathrm{Sq}^0(\varphi)$	$\mathrm{Sq}^1(\varphi)$	$\dots$	$\mathrm{Sq}^{k-1}(\varphi)$	$\mathrm{Sq}^k(\varphi)$	$\mathrm{Sq}^{k+1}(\varphi)$	$\dots$
φ	?	$\dots$	?	φ^2	0	$\dots$

(2) Throughout the remainder of this chapter we will almost invariably consider cohomology with $\mathbb{Z}_2$-coefficients. We point out that by the Cup Product-Commutativity Proposition 114.8 the cup product on the $\mathbb{Z}_2$-cohomology $\mathrm{H}^*(X, A; \mathbb{Z}_2)$ of a given pair of topological spaces (X, A) is actually *commutative*.

Theorem 151.3. Any two families of additive cohomology operations that satisfy Axioms (1) to (4) are identical.

Proof. This theorem is proved in [**SE1962**, Theorem VIII.3.10]. ■

This theorem allows us to give a name to the above cohomology operations.

Definition. We refer to the cohomology operations from the Steenrod Theorem 151.2 as the Steenrod operations.[1792]

The following theorem shows that the first Steenrod operation Sq^1 is actually an old acquaintance of ours.

Theorem 151.4. The first Steenrod operation Sq^1 is the Bockstein homomorphism corresponding to the short exact sequence $0 \to \mathbb{Z}_2 \xrightarrow{[a]\mapsto[2a]} \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$.

Proof. We will not make use of this theorem, thus we decline to provide a proof. The theorem is proved in [**Bre1993**, p. 418] or alternatively in [**MTa1968**, p. 23] or [**Hat2002**, Theorem 4.L.12]. ■

Evidently our goal now is to compute Steenrod operations in specific examples. We start out with the following slightly disappointing lemma.

Lemma 151.5. For any wedge of finitely many spheres all Steenrod operations in degrees ≥ 1 are zero.

Proof. We will prove the lemma in Exercise 151.2. ■

In remainder of this section we intend to compute the Steenrod operations for the real projective spaces $\mathbb{R}\mathrm{P}^n$ with $n \in \mathbb{N}_0$. Before we consider the real projective spaces in detail it is convenient to reformulate the Cartan formula.

[1791] We refer to page 2429 for the definition of an excisive triad. By the Excisive Triad-Proposition 109.12 most "reasonable" triads are in fact excisive. In fact we will only use the special cases that $A = B$, or that at least one of A or B is the empty set. We refer again to page 2536 for the definition of the relative cup product.

[1792] Sometimes they are also called Steenrod squares, and the term Steenrod operations is used for a more general concept.

For the reformulation we need the following notation:

Notation. Let (X, A) be a pair of topological spaces. Given $\varphi \in \mathrm{H}^k(X, A; \mathbb{Z}_2)$ we write

$$\mathrm{Sq}(\varphi) \ := \ \mathrm{Sq}^0(\varphi) + \mathrm{Sq}^1(\varphi) + \cdots + \mathrm{Sq}^k(\varphi) \ \in \ \mathrm{H}^*(X, A; \mathbb{Z}_2) = \bigoplus_{n \in \mathbb{N}_0} \mathrm{H}^n(X, A; \mathbb{Z}_2).$$

Furthermore we extend the homomorphism Sq in the obvious way to a homomorphism $\mathrm{Sq}\colon \mathrm{H}^*(X; \mathbb{Z}_2) \to \mathrm{H}^*(X; \mathbb{Z}_2)$.

Remark. Let (X, A) be a pair of topological spaces and let $\varphi \in \mathrm{H}^k(X, A; \mathbb{Z}_2)$. Recall that by Axiom (3) given $\varphi \in \mathrm{H}^k(X, A; \mathbb{Z}_2)$ we have $\mathrm{Sq}^i(\varphi) = 0$ for $i > k$. Thus we can write

$$\mathrm{Sq}(\varphi) \ = \ \mathrm{Sq}^0(\varphi) + \mathrm{Sq}^1(\varphi) + \cdots + \mathrm{Sq}^k(\varphi) \ = \ \sum_{i \in \mathbb{N}_0} \mathrm{Sq}^i(\varphi).$$

With this notation we can formulate the following lemma.

Lemma 151.6. Let (X, A) be a pair of topological spaces.

(1) Given any $\varphi, \psi \in \mathrm{H}^*(X, A; \mathbb{Z}_2)$ we have

$$\mathrm{Sq}(\varphi \cup \psi) \ = \ \mathrm{Sq}(\varphi) \cup \mathrm{Sq}(\psi).$$

(2) Given any $\varphi \in \mathrm{H}^*(X, A; \mathbb{Z}_2)$ and given any $k \in \mathbb{N}_0$ we have $\mathrm{Sq}(\varphi^k) = \mathrm{Sq}(\varphi)^k$.

Proof. The first statement follows easily from the Cartan formula. Indeed, we see that

$$\begin{aligned} \mathrm{Sq}(\varphi \cup \psi) = \sum_{k \in \mathbb{N}_0} \mathrm{Sq}^k(\varphi \cup \psi) \ &\overset{\text{Cartan formula, i.e. Axiom (4)}}{\downarrow} \\ &= \sum_{k \in \mathbb{N}_0} \sum_{i=0}^{k} \mathrm{Sq}^i(\varphi) \cup \mathrm{Sq}^{k-i}(\psi) \\ &= \sum_{r \in \mathbb{N}_0} \sum_{s \in \mathbb{N}_0} \mathrm{Sq}^r(\varphi) \cup \mathrm{Sq}^s(\psi) = \Big(\sum_{r \in \mathbb{N}_0} \mathrm{Sq}^r(\varphi) \Big) \cup \Big(\sum_{s \in \mathbb{N}_0} \mathrm{Sq}^s(\psi) \Big) = \mathrm{Sq}(\varphi) \cup \mathrm{Sq}(\psi). \end{aligned}$$

The second statement is an immediate consequence of the first statement. ∎

Before we state a corollary we introduce the following semi-obvious definition.

Definition. Let $r, s \in \mathbb{N}_0$. If $0 \leq r \leq s$, then we set $\binom{s}{r} = \frac{s!}{r!(s-r)!}$, otherwise we set $\binom{s}{r} = 0$.

Now we can state the promised corollary.

Corollary 151.7. Let X be a topological space. For any cohomology class $\varphi \in \mathrm{H}^1(X; \mathbb{Z}_2)$ of degree *one*, any $k \in \mathbb{N}$ and any $i \in \mathbb{N}$ we have

$$\mathrm{Sq}^i(\varphi^k) \ = \ \binom{k}{i} \cdot \varphi^{k+i} \ \in \ \mathrm{H}^{k+i}(X; \mathbb{Z}_2).$$

Examples.

(1) By the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 121.16 we know that given any $n \in \mathbb{N}_0$ we have a ring isomorphism $(\mathrm{H}^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2), \cup) \cong \mathbb{Z}_2[x]/(x^{n+1})$, where $x \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$, and we have a ring isomorphism $(\mathrm{H}^*(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2), \cup) \cong \mathbb{Z}_2[x]$, where $x \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2)$. Since Steenrod operations are additive we see that the result of Corollary 151.7 gives us the promised calculations of all Steenrod operations for the real projective spaces.

(2) In Exercise 151.3 we will use a similar argument to calculate the Steenrod operations on $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}_2)$.

Proof. We calculate that

$$\mathrm{Sq}(\varphi^k) \underset{\text{Lemma 151.6 (2)}}{=} \mathrm{Sq}(\varphi)^k \underset{\substack{\text{by Axiom (3)}\\ \text{since } \varphi\in \mathrm{H}^1(X;\mathbb{Z}_2)}}{=} (\mathrm{Sq}^0(\varphi)+\mathrm{Sq}^1(\varphi))^k \underset{\text{Axioms (1) and (2)}}{=} (\varphi+\varphi^2)^k \underset{\substack{\text{since the ring } (\mathrm{H}^*(X;\mathbb{Z}_2),\cup)\\ \text{is commutative}}}{=} \varphi^k\cup(1+\varphi)^k = \sum_{j=0}^{k}\binom{k}{j}\varphi^{k+j}.$$

The desired conclusion follows from comparing the terms in degree $k+i$ on both sides. ■

151.2. **Steenrod operations and connecting homomorphisms.** This section is completely dedicated to the proof of the following proposition. In the subsequent section we will see that the proposition has powerful applications.

Proposition 151.8. (Steenrod-Coboundary Proposition) If (X,A) is a pair of topological spaces, then the Steenrod operations commute with the connecting homomorphisms in cohomology. More precisely, given any $i,n\in\mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^n(A;\mathbb{Z}_2) & \xrightarrow{\quad\delta\quad} & \mathrm{H}^{n+1}(X,A;\mathbb{Z}_2) \\ \downarrow{\scriptstyle \mathrm{Sq}^i} & & \downarrow{\scriptstyle \mathrm{Sq}^i} \\ \mathrm{H}^{n+i}(A;\mathbb{Z}_2) & \xrightarrow{\quad\delta\quad} & \mathrm{H}^{n+i+1}(X,A;\mathbb{Z}_2). \end{array}$$

In the proof of the Steenrod-Coboundary Proposition 151.8 we will make use of the cross product on cohomology which we now introduce.

Definition. Let R be a commutative ring, let X be a topological space and let (Y,B) be a pair of topological spaces. Furthermore let $\varphi\in\mathrm{H}^m(X;R)$ and $\psi\in\mathrm{H}^n(Y,B;R)$. We denote by $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ the projections. We define the cross product[1793]

$$\varphi\times\psi := \underbrace{p^*(\varphi)}_{\text{lies in } \mathrm{H}^m(X\times Y;R)} \cup \underbrace{q^*(\psi)}_{\text{lies in } \mathrm{H}^n(X\times Y,X\times B;R)} \in \mathrm{H}^{m+n}(X\times Y, X\times B;R).$$

The following proposition basically says that in Axiom (4), i.e. in the Cartan formula, we can replace the cup product by the cross product.

Proposition 151.9. Let X be a topological space and let (Y,B) be a pair of topological spaces. For any $\varphi\in\mathrm{H}^r(X;\mathbb{Z}_2)$, $\psi\in\mathrm{H}^s(Y,B;\mathbb{Z}_2)$ and $n\in\mathbb{N}_0$ the following equality holds:

$$\mathrm{Sq}^n(\varphi\times\psi) = \sum_{i=0}^{n}\mathrm{Sq}^i(\varphi)\times\mathrm{Sq}^{n-i}(\psi) \in \mathrm{H}^{r+s+n}(X\times Y, X\times B;\mathbb{Z}_2).$$

Proof. Let X be a topological space and let (Y,B) be a pair of topological spaces. Furthermore let $\varphi\in\mathrm{H}^r(X;\mathbb{Z}_2)$ and $\psi\in\mathrm{H}^s(Y,B;\mathbb{Z}_2)$. We denote by $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ the projections. We start out with the following claim.

[1793]Recall that on page 2498, given two *homology* classes $\mu\in\mathrm{H}_m(X;\mathbb{Z})$ and $\nu\in\mathrm{H}_n(Y;\mathbb{Z})$ we defined the *cross product* $\mu\times\nu\in\mathrm{H}_{m+n}(X\times Y;\mathbb{Z})$. It should usually be clear from the context whether in a given situation we mean the cross product on homology or on cohomology.

Claim. We have the following equalities:[1794]

(a) $p^*(\varphi) = \varphi \times 1_Y \in \mathrm{H}^r(X \times Y; \mathbb{Z}_2)$ and $q^*(\psi) = 1_X \times \psi \in \mathrm{H}^s(X \times Y; \mathbb{Z}_2)$.

(b) $\varphi \times \psi = (\varphi \times 1_Y) \cup (1_X \times \psi) \in \mathrm{H}^{r+s}(X \times Y; \mathbb{Z}_2)$.

Proof.

(a) We perform the following easy calculation:

$$\varphi \times 1_Y = p^*(\varphi) \cup q^*(1_Y) \underset{\uparrow}{=} p^*(\varphi) \cup 1_{X\times Y} \underset{\uparrow}{=} p^*(\varphi).$$

by definition we have $p^*(1_X) = q^*(1_Y) = 1_{X\times Y}$; $1_{X\times Y}$ is multiplicatively neutral, see the Cup Product-Ring Proposition 114.7

Evidently the same argument also shows that $q^*(\psi) = 1_X \times \psi$.

(b) This statement follows from (a) and the definition of the cross product.

⊞

Now we turn to the proof of the actual statement of the proposition. We have the following equalities in $\mathrm{H}^{r+s+n}(X \times Y; \mathbb{Z}_2)$:

$$\begin{aligned}
\mathrm{Sq}^n(\varphi \times \psi) &\underset{\downarrow}{=} \mathrm{Sq}^n\big((\varphi \times 1_Y) \cup (1_X \times \psi)\big) && \text{by the Cartan formula, i.e. by Axiom (4)} \\
&= \sum_{i=0}^{n} \mathrm{Sq}^i(\varphi \times 1_Y) \cup \mathrm{Sq}^{n-i}(1_X \times \psi) \\
&\underset{\uparrow}{=} \sum_{i=0}^{n} \mathrm{Sq}^i(p^*(\varphi)) \cup \mathrm{Sq}^{n-i}(q^*(\psi)) && \text{since } \mathrm{Sq}^i \text{ and } \mathrm{Sq}^{n-i} \text{ are natural transformations} \\
&\underset{\uparrow}{=} \sum_{i=0}^{n} p^*(\mathrm{Sq}^i(\varphi)) \cup q^*(\mathrm{Sq}^{n-i}(\psi)) && \text{part (a) of the claim} \\
&\underset{\uparrow}{=} \sum_{i=0}^{n} \mathrm{Sq}^i(\varphi) \times \mathrm{Sq}^{n-i}(\psi). && \text{by definition of the cross product}
\end{aligned}$$

We have thus obtained the desired equality. ■

The proof of the Steenrod-Coboundary Proposition 151.8 rests on the following lemma.

Lemma 151.10. Let X be a topological space, let (Y, B) be a pair of topological spaces and let R be a commutative ring. If $\varphi \in \mathrm{H}^m(X; R)$ and $\psi \in \mathrm{H}^n(B; R)$, then

$$\underset{\uparrow}{\delta}(\overbrace{\varphi \times \psi}^{\in \mathrm{H}^{m+n}(X\times B;R)}) = (-1)^m \cdot \varphi \times \overbrace{\underset{\uparrow}{\delta\psi}}^{\in \mathrm{H}^{n+1}(Y,B;R)} \in \mathrm{H}^{m+n+1}(X \times Y, X \times B; R).$$

connecting homomorphism in cohomology for the pair $(X \times Y, X \times B)$; connecting homomorphism in cohomology for the pair (Y, B)

Proof of Lemma 151.10. This equality follows fairly easily from Coboundary-Cup Product Lemma 114.2, the definition of the cross product via the cup product and the explicit description of the connecting homomorphism on page 2399. We leave it to the reader to fill in the details. ■

Proof of the Steenrod-Coboundary Proposition 151.8. We start out with a warm-up case.

[1794] Recall that given a topological space W we denote by $1_Z \in \mathrm{H}^0(W; \mathbb{Z}_2)$ the cohomology class that is represented by the cocycle that assigns to each singular 0-simplex the value $1 \in \mathbb{Z}_2$.

Claim 1. Let Z be a topological space and let $a < b$ be two real numbers. The conclusion of the proposition holds for the pair $(Z \times [a,b], Z \times \partial([a,b]))$ of topological spaces.

Proof. Let $\theta \in \mathrm{H}^n(Z \times \partial([a,b]); \mathbb{Z}_2) = \mathrm{H}^n(Z \times \{a,b\}; \mathbb{Z}_2)$. Recall that we need to show that $\mathrm{Sq}^i(\delta\theta) = \delta(\mathrm{Sq}^i(\theta))$. First note that θ can be written as $\theta = \varphi \times \psi$ for some $\varphi \in \mathrm{H}^n(Z; \mathbb{Z}_2)$ and some $\psi \in \mathrm{H}^0(\partial([a,b]); \mathbb{Z}_2) = \mathrm{H}^0(\{a,b\}; \mathbb{Z}_2)$.[1795] Note that we have the following equalities in $\mathrm{H}^{n+i+1}(Z \times [a,b], Z \times \partial([a,b]); \mathbb{Z}_2)$:

since $\theta = \varphi \times \psi$ — by Lemma 151.10, we also use that $(-1)^n = 1 \in \mathbb{Z}_2$ — Proposition 151.9 — since $\delta\psi \in \mathrm{H}^1([a,b], \partial([a,b]); \mathbb{Z}_2)$ and since $\mathrm{H}^i([a,b], \partial([a,b]); \mathbb{Z}_2) = 0$ for every $i \geq 2$ we know that $\mathrm{Sq}^j(\delta\psi) = 0$ for $j \geq 1$

$$\begin{array}{lllll} \mathrm{Sq}^i(\delta\theta) & = \mathrm{Sq}^i(\delta(\varphi\times\psi)) & = \mathrm{Sq}^i(\varphi \times \delta\psi) & = \sum_{j=0}^{i} \mathrm{Sq}^{i-j}(\varphi)\times\mathrm{Sq}^j(\delta\psi) & = \mathrm{Sq}^i(\varphi)\times\mathrm{Sq}^0(\delta\psi) \\ & = \mathrm{Sq}^i(\varphi) \times \delta\psi & = \delta(\mathrm{Sq}^i(\varphi) \times \psi) & = \delta(\mathrm{Sq}^i(\varphi\times\psi)) & = \delta(\mathrm{Sq}^i(\theta)). \end{array}$$

Axiom (1) says that $\mathrm{Sq}^0 = \mathrm{id}$ — Lemma 151.10 — by Proposition 151.9 together with Axioms (1) and (3) and the fact that $\psi \in \mathrm{H}^0(\partial([a,b]); \mathbb{Z}_2)$ ⊞

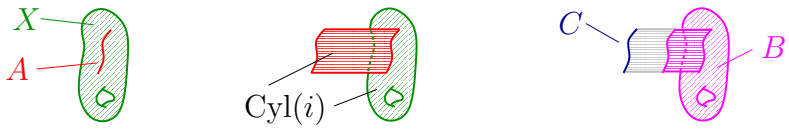
The remainder of the proof of the proposition consists in cleverly reducing the general case to the above special case. Thus let (X, A) be a pair of topological spaces. We denote by $i\colon A \to X$ the inclusion map. We set

$$\mathrm{Cyl}(i) = \begin{array}{c}\text{mapping cylinder of the} \\ \text{inclusion } i\colon A \to X\end{array} := ((A \times [0,1]) \sqcup X)/\!\sim \quad \text{where } (a,1) \sim a \text{ for all } a \in A.$$

We point out that by Exercise 16.12 and the Mapping Cylinder Basics Lemma 16.10 we know that the obvious inclusions $A \times [0,1] \to \mathrm{Cyl}(i)$ and $X \to \mathrm{Cyl}(i)$ are embeddings. Thus we can view $A \times [0,1]$ and X as subsets of $\mathrm{Cyl}(i)$. Now we consider the following subsets of the mapping cylinder $\mathrm{Cyl}(i)$:

$$B := (A \times [\tfrac{1}{2}, 1]) \cup X \quad \text{and} \quad C := A \times \{0\}.$$

We refer to the figure below for an illustration. The five statements in the following claim

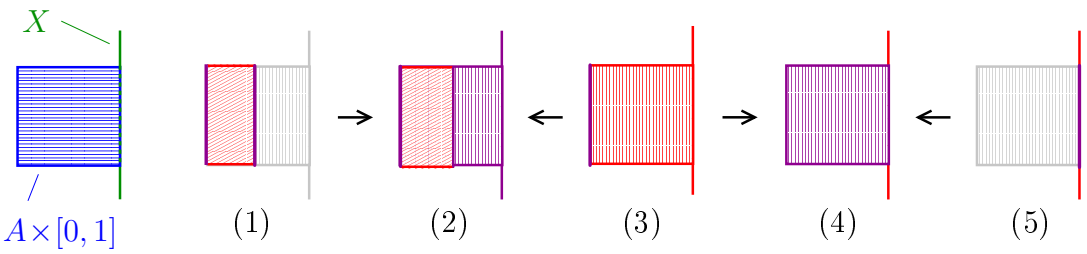


break the proof of the proposition into five manageable steps.

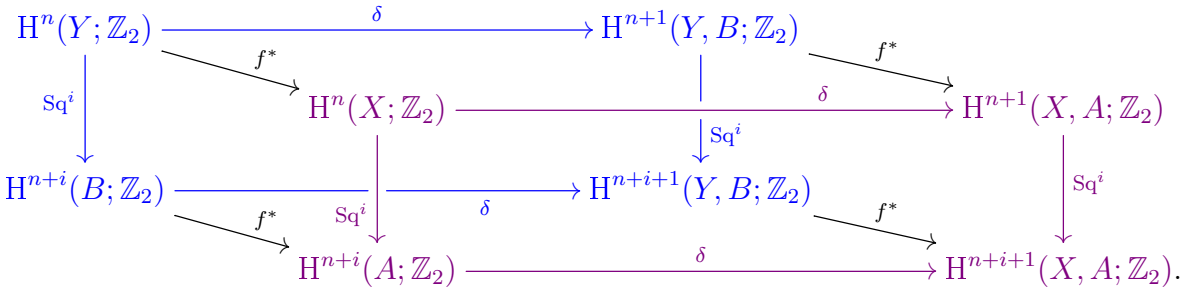
Claim 2.

(1) The conclusion of the proposition holds for $(A \times [0, \frac{1}{2}], A \times \partial([0, \frac{1}{2}]))$.
(2) The conclusion of the proposition holds for $(\mathrm{Cyl}(i), B \cup C)$.
(3) The conclusion of the proposition holds for $(\mathrm{Cyl}(i), C)$.
(4) The conclusion of the proposition holds for $(\mathrm{Cyl}(i), A \times [0,1])$.
(5) The conclusion of the proposition holds for (X, A).

[1795] This statement follows quite easily from the fact that $Z \times \partial([a,b]) = (Z \times \{a\}) \sqcup (Z \times \{b\})$, the observation that $\mathrm{H}^0(\{a,b\}) = \mathbb{Z} \cdot 1_{\{a\}} \oplus \mathbb{Z} \cdot 1_{\{b\}}$, the Cup Product-Ring Proposition 114.7 and the Cup-Cap Disjoint Union Lemma 116.9. We leave it to the assiduous reader to assemble these steps in the right logical order.



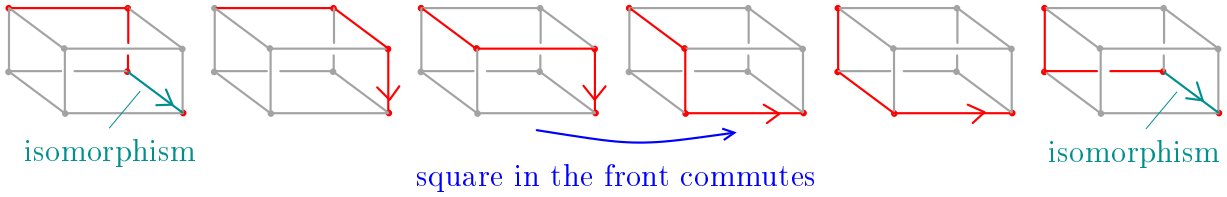
Proof. Before we work our way through the five steps we need to make two related observations. Let $f\colon (X, A) \to (Y, B)$ be a map of pairs of topological spaces. We obtain the following cubical diagram:



It follows from the naturality of the Steenrod operations that the parallelograms to the left and right commute. Furthermore, it follows from the naturality of the connecting homomorphism, see the Cohomological LES-Lemma 108.15, that the top and bottom parallelograms commute. Using these facts we can make the following two observations:

(i) If the square in the front commutes and if the map f^* to the bottom right is an isomorphism, then it follows from a stroll through the cube, as indicated in the figure below, that the square in the back also commutes.

(ii) If the square in the back commutes and if the map f^* to the top left is an isomorphism, then it follows as in (i) that the square in the front also commutes.



Now we turn to the actual proof of the five steps in the claim.

(1) This statement follows immediately from the previous claim.

(2) We consider the inclusion map $f\colon (A \times [0, \frac{1}{2}], A \times \partial([0, \frac{1}{2}])) \to (\mathrm{Cyl}(i), B \cup C)$ and we consider the corresponding cube. By (1) we know that the square in the front commutes. We leave it to the reader to cleverly apply the Excision Theorem for Cohomology Groups 109.1 to show that the map f^* to the bottom right of the cube is an isomorphism. Thus we obtain from (i) that the square in the back commutes.

(3) We consider the inclusion map $f\colon (\mathrm{Cyl}(i), C) \to (\mathrm{Cyl}(i), B \cup C)$ and we consider the corresponding cube. By (2) we know that the square in the back commutes. Since

on the first entry f is the identity we see that the top left map f^* is an isomorphism. Thus we obtain from (ii) that the square in the front also commutes.

(4) We consider the inclusion map $f\colon (\mathrm{Cyl}(i), A\times\{0\}) \to (\mathrm{Cyl}(i), \overbrace{A\times[0,1]}^{=C})$. By (3) we know that the square in the front commutes. Since $A\times\{0\}$ is a deformation retract of $A\times[0,1]$ we obtain from the Homotopic Maps-H*-Lemma 108.13 (4) that the map f^* to the bottom right of the cube is an isomorphism. Thus we obtain from (i) that the square in the back also commutes.

(5) We consider the obvious inclusion

$$\begin{array}{rcl} f\colon X & \to & \mathrm{Cyl}(i) \\ x & \mapsto & [(x,1)]. \end{array}$$

It also defines a map $f\colon (X,A) \to (\mathrm{Cyl}(i), A\times[0,1])$ of pairs of topological spaces. We consider the corresponding cube. By (4) we know that the square in the back commutes. By the Mapping Cylinder Basics Lemma 16.12 (1) the map $f\colon X \to \mathrm{Cyl}(i)$ is in fact a homotopy equivalence. Thus it follows from the Homotopic Maps-H*-Lemma 108.13 that the top left f^* is an isomorphism. Therefore we obtain from (ii) that the square in the front commutes.

■

151.3. **Steenrod operations and suspensions.** In the Steenrod-Coboundary Proposition 151.8 we saw that Steenrod operations commute with connecting homomorphisms in cohomology. In this section we will use this fact to show that Steenrod operations behave well under taking suspensions. This fact will help us to obtain new information on homology groups of spheres.

First recall that in the Suspension-H$_*$-Proposition 76.7 we saw that taking the suspension $\Sigma(X)$ of a topological space shifts the reduced homology groups "upwards by one". The following proposition says that the same phenomenon occurs when we consider cohomology groups.

Proposition 151.11. Let X be a non-empty topological space, let $n \in \mathbb{N}$[1796] and let G be an abelian group. As on page 1681 we denote by C_-X and C_+X the "lower and upper half" of the suspension $\Sigma(X)$. We refer to the figure below for an illustration.

(1) The following maps are isomorphisms:[1797]

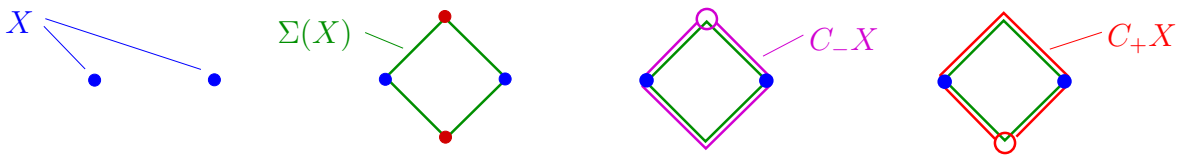
$$\mathrm{H}^n(X;G) \xrightarrow{\delta} \mathrm{H}^{n+1}(C_+X, X; G) \leftarrow \mathrm{H}^{n+1}(\Sigma(X), C_-X; G) \to \mathrm{H}^{n+1}(\Sigma(X); G).$$

Here the map to the left is the connecting homomorphism of the pair (C_+X, X) from the Cohomological LES-Lemma 108.15 The remaining two horizontal maps are induced by the obvious inclusion maps.

(2) The isomorphism

$$\Sigma_X\colon \mathrm{H}^n(X;G) \xrightarrow{\cong} \mathrm{H}^{n+1}(\Sigma(X);G)$$

that is given by the maps in (1) is in fact a natural isomorphism.



Proof. The proof is a slight variation on the proof of the Suspension-H_*-Proposition 76.7. We leave it to the reader to fill in the details. ■

The following proposition shows that Steenrod operations behave well under taking suspensions.

Proposition 151.12. (Steenrod-Suspension Proposition) Given any topological space X, any $n \in \mathbb{N}$ and any $i \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^n(X;\mathbb{Z}_2) & \xrightarrow[\cong]{\Sigma_X} & \mathrm{H}^{n+1}(\Sigma(X);\mathbb{Z}_2) \\ \downarrow{\scriptstyle \mathrm{Sq}^i} & & \downarrow{\scriptstyle \mathrm{Sq}^i} \\ \mathrm{H}^{n+i}(X;\mathbb{Z}_2) & \xrightarrow[\cong]{\Sigma_X} & \mathrm{H}^{n+i+1}(\Sigma(X);\mathbb{Z}_2). \end{array}$$

Remark. The proposition says in particular that if there exists a non-trivial Steenrod operation on a topological space X, then its suspension inherits a non-trivial Steenrod operation. This is in so far interesting as we know by the Suspension-Cup Product Lemma 115.7 that the cup product vanishes on suspensions. The fact that suspensions can have non-trivial Steenrod operations will allow us to prove several hitherto inaccessible results.

Proof. We consider the following diagram

$$\begin{array}{ccccccc} & & & \overset{\Sigma_X}{\frown} & & & \\ \mathrm{H}^n(X;\mathbb{Z}_2) & \xrightarrow[\delta]{} & \mathrm{H}^{n+1}(C_+X,X;\mathbb{Z}_2) & \xleftarrow[\cong]{} & \mathrm{H}^{n+1}(\Sigma(X),C_-X;\mathbb{Z}_2) & \xrightarrow[\cong]{} & \mathrm{H}^{n+1}(\Sigma(X);\mathbb{Z}_2) \\ \downarrow{\scriptstyle \mathrm{Sq}^i} & & \downarrow{\scriptstyle \mathrm{Sq}^i} & & \downarrow{\scriptstyle \mathrm{Sq}^i} & & \downarrow{\scriptstyle \mathrm{Sq}^i} \\ \mathrm{H}^{n+i}(X;\mathbb{Z}_2) & \xrightarrow{\delta} & \mathrm{H}^{n+1+i}(C_+X,X;\mathbb{Z}_2) & \xleftarrow{\cong} & \mathrm{H}^{n+1+i}(\Sigma(X),C_-X;\mathbb{Z}_2) & \xrightarrow{\cong} & \mathrm{H}^{n+1+i}(\Sigma(X);\mathbb{Z}_2). \\ & & & \underset{\Sigma_X}{\smile} & & & \end{array}$$

The first square commutes by the Steenrod-Coboundary Proposition 151.8. The two other squares commute by the naturality of the Steenrod operation Sq^i. Thus we see that, as desired, the diagram commutes. ■

Next we recall that by the definitions on pages 1576 and 2676 we have three different types of Hopf maps:

[1796] Basically the same statement works for $n = 0$, the only nuisance is that in that case we need to work with reduced cohomology groups.

[1797] Here we use the Cone-Suspension Basics Lemma 16.8 (3) to identify X with the subset $C_-X \cap C_+X$ of $\Sigma(X)$.

Definition. We consider the following maps

$$\underbrace{\underbrace{\begin{array}{rcl} H\colon S^3 & \to & S^2 = \mathbb{C}\mathrm{P}^1 \\ \underbrace{(z_1, z_2)}_{\in \mathbb{C}^2} & \mapsto & [z_1 : z_2] \end{array}}_{\text{the Hopf map}} \quad \underbrace{\begin{array}{rcl} H_{\mathbb{H}}\colon S^7 & \to & S^4 = \mathbb{H}\mathrm{P}^1 \\ \underbrace{(z_1, z_2)}_{\in \mathbb{H}^2} & \mapsto & [z_1 : z_2] \end{array}}_{\text{quaternionic Hopf map}} \quad \text{and} \quad \underbrace{\begin{array}{rcl} H_{\mathbb{O}}\colon S^{15} & \to & S^8 = \mathbb{O}\mathrm{P}^1 \\ \underbrace{(z_1, z_2)}_{\in \mathbb{O}^2} & \mapsto & [z_1 : z_2]. \end{array}}_{\text{octonionic Hopf map}}}$$

In Theorem 122.3 we showed that the Hopf map $H\colon S^3 \to S^2$ is not null-homotopic. The key to doing so were the following two observations:

(1) The mapping cone $\mathrm{Cone}(H\colon S^3 \to S^2)$ is homeomorphic to $\mathbb{C}\mathrm{P}^2$.
(2) The cup product on $\mathbb{C}\mathrm{P}^2$ has the property that for a generator $\varphi \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^2; \mathbb{Z})$ the square φ^2 is non-zero.

Since the Steenrod operations are a generalization of the mod 2 "cup product square" and since Steenrod operations behave well with respect to suspensions it seems reasonable that we can use Steenrod operations to show that the suspensions of the Hopf maps are not null-homotopic.
Indeed, we can prove the following theorem.

Theorem 151.13.

(1) For any $n \geq 2$ the $(n-2)$-fold suspension

$$\Sigma^{n-2}(H)\colon \underbrace{\Sigma^{n-2}(S^3)}_{=S^{n+1}} \to \underbrace{\Sigma^{n-2}(S^2)}_{=S^n}$$

of the Hopf map defines a non-trivial element in $\pi_{n+1}(S^n)$. In particular for any $n \geq 2$ we have $\pi_{n+1}(S^n) \neq 0$.

(2) For any $n \geq 4$ the $(n-4)$-fold suspension

$$\Sigma^{n-4}(H_{\mathbb{H}})\colon \underbrace{\Sigma^{n-4}(S^7)}_{=S^{n+3}} \to \underbrace{\Sigma^{n-4}(S^4)}_{=S^n}$$

of the quaternionic Hopf map defines a non-trivial element in $\pi_{n+3}(S^n)$. In particular for any $n \geq 4$ we have $\pi_{n+3}(S^n) \neq 0$.

(3) For any $n \geq 8$ the $(n-8)$-fold suspension

$$\Sigma^{n-8}(H_{\mathbb{O}})\colon \underbrace{\Sigma^{n-8}(S^{15})}_{=S^{n+7}} \to \underbrace{\Sigma^{n-8}(S^8)}_{=S^n}$$

of the octonionic Hopf map defines a non-trivial element in $\pi_{n+7}(S^n)$. In particular for any $n \geq 8$ we have $\pi_{n+7}(S^n) \neq 0$.

Before we turn to the proof of Theorem 151.13 we recall, for the reader's convenience, the following, frequently overlooked, proposition which says that taking suspensions and taking mapping cones commutes.

Proposition 16.19. (Mapping Cone-Suspension Proposition) If $f\colon X \to Y$ is a map between topological spaces, then there exists a homeomorphism

$$g\colon \mathrm{Cone}(\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)) \xrightarrow{\cong} \Sigma(\mathrm{Cone}(f\colon X \to Y)).$$

We also state the following lemma which we are already dimly aware of.

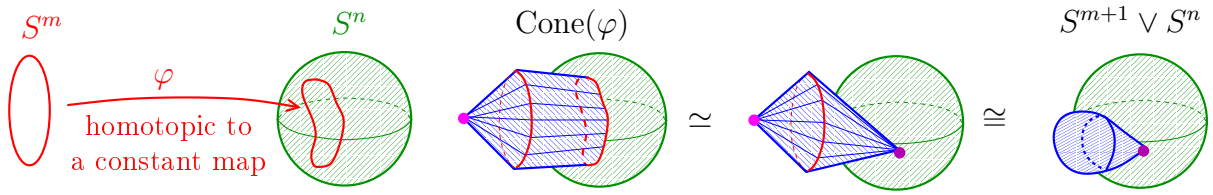
Lemma 151.14. If a map $\varphi\colon S^m \to S^n$ is homotopic to a constant map, then the corresponding mapping cone $\mathrm{Cone}(\varphi\colon S^m \to S^n)$ is homotopy equivalent to $S^{m+1} \vee S^n$.

Proof. We have

$$\mathrm{Cone}(\varphi\colon S^m \to S^n) \underset{\uparrow}{\simeq} \mathrm{Cone}(\text{constant map}\colon S^m \to S^n) \underset{\uparrow}{\cong} S^{m+1} \vee S^n.$$

by the Mapping Cone Inclusion Lemma 16.16, since φ is homotopic to a constant map

Mapping Cone-of-Constant Map Lemma 122.1 ∎



Now we turn to the proof of Theorem 151.13. For clarity's sake we first provide a detailed proof of Theorem 151.13 (1). Afterwards we explain how to modify that proof to obtain statements (2) and (3) of Theorem 151.13.

Proof of Theorem 151.13 (1). In the following we will show that $\Sigma(H)\colon S^4 \to S^3$ is not homotopic to a constant map. The general statement regarding the $(n-2)$-fold suspension $\Sigma^{n-2}(H)$ follows easily from iterating the argument below.

For once we do a proof by contradiction. So let us assume that the map $\Sigma(H)\colon S^4 \to S^3$ is in fact homotopic to a constant map. We make the following preparations:

(1) To simplify the notation we denote the mapping cone of a map $f\colon X \to Y$ by $\mathrm{C}(f)$ instead of $\mathrm{Cone}(f)$. Furthermore we write $\mathbb{F} = \mathbb{Z}_2$.
(2) By the Mapping Cone-of-Hopf Map Lemma 122.2 there exists a homeomorphism $f\colon \mathrm{C}(H\colon S^3 \to S^2) \xrightarrow{\cong} \mathbb{C}\mathrm{P}^2$.
(3) We denote by $g\colon \Sigma(\mathrm{C}(H)) \xrightarrow{\cong} \mathrm{C}(\Sigma(H))$ the homeomorphism from the Mapping Cone-Suspension Proposition 16.19.
(4) Since we assume that the map H is homotopic to a constant map we obtain from Lemma 151.14 that there exists a homotopy equivalence $h\colon \mathrm{C}(\Sigma(H)) \to S^3 \vee S^5$.

Now we consider the following diagram:

$$\begin{array}{ccccccccc}
\mathrm{H}^2(\mathbb{C}\mathrm{P}^2;\mathbb{F}) & \xrightarrow[\cong]{f^*} & \mathrm{H}^2(\mathrm{C}(H);\mathbb{F}) & \xrightarrow[\cong]{\Sigma} & \mathrm{H}^3(\Sigma(\mathrm{C}(H));\mathbb{F}) & \xleftarrow[g^*]{\cong} & \mathrm{H}^3(\mathrm{C}(\overbrace{\Sigma(H)}^{S^4\to S^3});\mathbb{F}) & \xleftarrow[\cong]{h^*} & \mathrm{H}^3(S^5\vee S^3;\mathbb{F}) \\
{\scriptstyle x\mapsto x\cup x}\downarrow\ \downarrow{\scriptstyle \mathrm{Sq}^2} & & \downarrow{\scriptstyle \mathrm{Sq}^2} & & \downarrow{\scriptstyle \mathrm{Sq}^2} & & \downarrow{\scriptstyle \mathrm{Sq}^2} & & \downarrow{\scriptstyle \mathrm{Sq}^2} \\
\mathrm{H}^4(\mathbb{C}\mathrm{P}^2;\mathbb{F}) & \xrightarrow[\cong]{f^*} & \mathrm{H}^4(C(H);\mathbb{F}) & \xrightarrow[\cong]{\Sigma} & \mathrm{H}^5(\Sigma(\mathrm{C}(H));\mathbb{F}) & \xleftarrow[g^*]{\cong} & \mathrm{H}^5(\mathrm{C}(\Sigma(H));\mathbb{F}) & \xleftarrow[\cong]{h^*} & \mathrm{H}^5(S^5\vee S^3;\mathbb{F}).
\end{array}$$

We make the following observations:

(1) The maps Σ are the suspension isomorphisms given by Proposition 151.11.
(2) It follows from Axiom (2) that the two vertical maps on the left are actually the same.
(3) It follows from the naturality of Sq^2 that the first, third and the fourth square commute.

(4) It follows from the Steenrod-Suspension Proposition 151.12 that the second square commutes.
(5) Since f and g are homeomorphisms we see that the induced maps f^* and g^* are isomorphisms. Furthermore, since h is a homotopy equivalence we obtain from the Homotopic Maps-H^*-Lemma 108.13 (3) that the horizontal maps h^* are isomorphisms.
(6) By Proposition 121.13 we know that there exists a unique non-trivial $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^2;\mathbb{F})$ and that this element satisfies $x \cup x \neq 0 \in \mathrm{H}^4(\mathbb{C}\mathrm{P}^2;\mathbb{F}) \cong \mathbb{F}$. In particular we see that the left vertical homomorphism is an isomorphism and that it is non-trivial.
(7) By Lemma 151.5 we know that the vertical homomorphism to the right is trivial.

In summary we have shown that the above diagram commutes, we have shown that all horizontal maps are isomorphisms, we have seen that the left vertical map is non-trivial and have pointed out that the right vertical map is trivial. But this is absurd. In other words, we have obtained a contradiction to our hypothesis that the map $\Sigma(H)\colon S^4 \to S^3$ is null-homotopic. ■

Proof of Theorem 151.13 (2) and (3). As we will see shortly, the proof for the quaternionic and octonionic Hopf maps is basically the same as the proof for the "classical" Hopf map.

Let us first consider the quaternionic Hopf. By the Mapping Cone-of-Hopf Map Lemma 122.2 (2) there exists a homeomorphism $f\colon \mathrm{C}(H_{\mathbb{H}}) \xrightarrow{\cong} \mathbb{H}\mathrm{P}^2$. If $\Sigma(H_{\mathbb{H}})\colon S^8 \to S^5$ was null-homotopic we would, as in the proof of (1), obtain a commutative diagram of the following form:

$$\begin{array}{ccccccccc}
\mathrm{H}^4(\mathbb{H}\mathrm{P}^2;\mathbb{F}) & \xrightarrow[\cong]{f^*} & \mathrm{H}^4(\mathrm{C}(H_{\mathbb{H}});\mathbb{F}) & \xrightarrow{\Sigma} & \mathrm{H}^5(\Sigma(\mathrm{C}(H_{\mathbb{H}}));\mathbb{F}) & \xleftarrow[g^*]{\cong} & \mathrm{H}^5(\mathrm{C}(\overbrace{\Sigma(H_{\mathbb{H}})}^{S^8\to S^5});\mathbb{F}) & \xleftarrow[\cong]{h^*} & \mathrm{H}^5(S^9\vee S^5;\mathbb{F}) \\
{\scriptstyle x\mapsto x\cup x}\downarrow & \downarrow{\scriptstyle \mathrm{Sq}^4} & \downarrow{\scriptstyle \mathrm{Sq}^4} & & \downarrow{\scriptstyle \mathrm{Sq}^4} & & \downarrow{\scriptstyle \mathrm{Sq}^4} & & \downarrow{\scriptstyle \mathrm{Sq}^4} \\
\mathrm{H}^8(\mathbb{H}\mathrm{P}^2;\mathbb{F}) & \xrightarrow[\cong]{f^*} & \mathrm{H}^8(C(H_{\mathbb{H}});\mathbb{F}) & \xrightarrow{\Sigma} & \mathrm{H}^9(\Sigma(\mathrm{C}(H_{\mathbb{H}}));\mathbb{F}) & \xleftarrow[g^*]{\cong} & \mathrm{H}^9(\mathrm{C}(\Sigma(H_{\mathbb{H}}));\mathbb{F}) & \xleftarrow[\cong]{h^*} & \mathrm{H}^9(S^9\vee S^5;\mathbb{F}).
\end{array}$$

By Proposition 121.20 (1) the cup product on the left is non-trivial. Thus the same logic as in the proof of (1) leads to a contradiction.

The argument for the octonionic Hopf map is almost identical to the argument for the quaternionic Hopf map. We just need to redo the above argument with the Mapping Cone-of-Hopf Map Lemma 122.2 (3) and with Proposition 121.20 (2). ■

For the record we jot down the following proposition that we proved in the proof of Theorem 151.13. We will make use of it later on.

Proposition 151.15. Given any $n \in \mathbb{N}_0$ the maps

$$\begin{array}{lrcl}
(1) & \mathrm{Sq}^2\colon \mathrm{H}^{2+n}(\mathrm{Cone}(\Sigma^n(H));\mathbb{Z}_2) & \to & \mathrm{H}^{4+n}(\mathrm{Cone}(\Sigma^n(H));\mathbb{Z}_2) \\
(2) & \mathrm{Sq}^4\colon \mathrm{H}^{4+n}(\mathrm{Cone}(\Sigma^n(H_{\mathbb{H}}));\mathbb{Z}_2) & \to & \mathrm{H}^{8+n}(\mathrm{Cone}(\Sigma^n(H_{\mathbb{H}}));\mathbb{Z}_2) \\
(3) & \mathrm{Sq}^8\colon \mathrm{H}^{8+n}(\mathrm{Cone}(\Sigma^n(H_{\mathbb{O}}));\mathbb{Z}_2) & \to & \mathrm{H}^{16+n}(\mathrm{Cone}(\Sigma^n(H_{\mathbb{O}}));\mathbb{Z}_2)
\end{array}$$

are isomorphisms.

151.4. **Proof of the Steenrod Theorem 151.2 I.** In the final two sections we will give a partial proof of the Steenrod Theorem 151.2. More precisely, we will construct the Steenrod

operations and we will provide the proof of some of the properties. The proof is split into two parts:

(1) The goal of this section is to construct a certain interesting sequence of natural maps $\Delta_n\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$, $n \in \mathbb{N}_0$, where the zeroth map Δ_0 is a diagonal approximation.
(2) In the subsequent section we will show how this sequence of maps can be used to define the Steenrod operations.

In preparation for the construction of the sequence Δ_n we recall some definitions and results from earlier on. First recall that on page 1940 we introduced the tensor product $A_* \otimes B_*$ of two chain complexes A_* and B_*. We will make use of the following harmless lemma.

Lemma 117.7. Let A_* and B_* be chain complexes. The unique map

$$T\colon A_* \otimes B_* \quad \to \quad B_* \otimes A_*$$

that for $a \in A_k$ and $b \in B_l$ is given by

$$a \otimes b \quad \mapsto \quad (-1)^{kl} \cdot b \otimes a$$

is a chain map.

Since we will make use of this later on, we point out that T is natural in A_* and B_* and we point out that $T^2 = \mathrm{id}$. Next we recall the following definition from page 2517.

Definition. A diagonal approximation consists of a natural chain map

$$\Phi\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$$

for each topological space X, such that for each $x \in X$, viewed as a singular 0-simplex, the following equality holds in $(\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_0 = \mathrm{C}_0(X) \otimes \mathrm{C}_0(X)$:

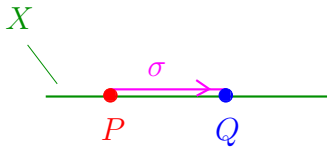
$$\Phi(x) = x \otimes x.$$

Example. We will try to rise to the challenge of finding a suitable illustration for a diagonal approximation Φ. We start out with the following observation: if $\sigma\colon \Delta^1 \to X$ is a singular 1-simplex from a point P to a point Q, then $\Phi(\sigma) \in (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_1$ needs to satisfy

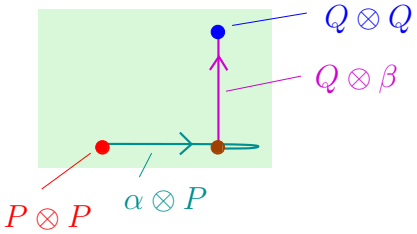
$$\partial\Phi(\sigma) \underset{\uparrow}{=} \Phi(\partial\sigma) = \Phi(Q - P) = \Phi(Q) - \Phi(P) \underset{\uparrow}{=} Q \otimes Q - P \otimes P.$$

since Φ is a chain map; since Φ is determined on singular 0-simplices

For example $\Phi(\sigma) = (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_1 = \mathrm{C}_1(X) \otimes \mathrm{C}_0(X) \oplus \mathrm{C}_0(X) \otimes \mathrm{C}_1(X)$ could be of the form $\alpha \otimes P + Q \otimes \beta$ where $\alpha\colon \Delta^1 \to X$ and $\beta\colon \Delta^1 \to X$ are singular simplices from P to Q. This situation is illustrated, with artistic liberties, in the figure below.



diagonal approximation
$\Delta_0(\sigma) = \alpha \otimes P + Q \otimes \beta$



The following proposition contains the two key results on diagonal approximations.

Proposition 114.6.

(1) There exist diagonal approximations.

(2) Given any two diagonal approximations Φ and Ψ there exists a natural chain homotopy equivalence between Φ and Ψ.

Proof. We gave an explicit example of a diagonal approximation in the Alexander-Whitney Lemma 114.5. In the Diagonal Approximation Uniqueness Proposition 113.17 we gave a proof of (2) using the Acyclic Model Theorem 113.16 (2). ■

The following theorem shows that any diagonal approximation can be extended to a sequence of maps.

Theorem 151.16.

(1) Given any topological space X there exists a sequence of natural maps

$$\Delta_n \colon \mathrm{C}_*(X) \ \to \ \mathrm{C}_*(X) \otimes \mathrm{C}_*(X), \quad n \in \mathbb{N}_0$$

of degree n[1798] such that the following two conditions are satisfied:

(a) The map Δ_0 is a diagonal approximation.

(b) For every $n \in \mathbb{N}_0$ the following equality holds:[1799]

$$\partial \circ \Delta_{n+1} + (-1)^n \cdot \Delta_{n+1} \circ \partial \ = \ (T + (-1)^{n+1} \cdot \mathrm{id}) \circ \Delta_n.$$

(2) Given two sequences Δ_n, Δ'_n, $n \in \mathbb{N}_0$, as in (1) there exists a sequence of natural maps

$$H_n \colon \mathrm{C}_*(X) \ \to \ \mathrm{C}_*(X) \otimes \mathrm{C}_*(X), \quad n \in \mathbb{N}_0$$

of degree n such that the following two conditions are satisfied:

(a) H_0 is the zero map.

(b) For every $n \in \mathbb{N}_0$ the following equality holds:[1800]

$$\partial \circ H_{n+1} + (-1)^n \cdot H_{n+1} \circ \partial \ = \ \Delta_n - \Delta'_n - (T + (-1)^n \cdot \mathrm{id}) \circ H_n.$$

In the proof of Theorem 151.16 we will need the following innocuous variation on the definition of a chain map that we had given on page 1592.

Definition.

(1) A chain map $f \colon C_* \to D_*$ of degree d between two chain complexes consists of a family $\{f_n \colon \mathrm{C}_n \ \to \ D_{n+d}\}_{n \in \mathbb{N}_0}$ of maps such that for each $n \in \mathbb{N}_0$ the following equality holds:

$$f_{n-1} \circ \partial_n \ = \ \partial_{n+d} \circ f_n.$$

(2) An anti-chain map $f \colon C_* \to D_*$ of degree d is basically the same data as a chain map of degree d, except that for each $n \in \mathbb{N}_0$ the following equality holds:

$$f_{n-1} \circ \partial_n \ = \ -\partial_{n+d} \circ f_n.$$

Remark. Given an anti-chain map $\{f_n\}_{n \in \mathbb{N}_0}$ of degree d one can easily see that the maps $\{(-1)^n \cdot f_n\}_{n \in \mathbb{N}_0}$ form a chain map of degree d. This little trick makes it possible to

[1798] By a map of degree n between chain complexes C_* and D_* we mean a collection of maps $\mathrm{C}_k \to D_{k+n}$, $k \in \mathbb{N}_0$.

[1799] Note that both sides are maps $\mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ of degree n.

[1800] In other words, the map H_{n+1} is a chain homotopy between Δ_n and Δ'_n up to the "error term" $-(T + (-1)^n \cdot \mathrm{id}) \circ H_n$.

translate all kinds of statements about chain maps to anti-chain maps. For example, as in Lemma 72.7 we see that an anti-chain map induces a map on homology groups.

In the proof of Theorem 151.16 we will need the following consequence of the Acyclic Model Theorem 113.13.

Theorem 151.17. (Acyclic Model Theorem) Let $\mathcal{T}op$ be the category of topological spaces and let $\mathcal{C}$ be the category of generalized[1801] chain complexes.

(1) Let $S\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ be a natural chain map of degree d. If we have $d \geq 1$, then for any topological space X and any $k \in \mathbb{N}_0$ there exists a natural map $P_X\colon \mathrm{C}_k(X) \to (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_{k+d+1}$ such that for any $k \in \mathbb{N}_0$ we have
$$\partial \circ P_X + P_X \circ \partial \;=\; S_X\colon \mathrm{C}_k(X) \;\to\; (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_{k+d}.$$

(2) For a natural anti-chain map $S\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ of degree d the analogous data exists except that now we have
$$\partial \circ P_X - P_X \circ \partial \;=\; S_X\colon \mathrm{C}_k(X) \;\to\; (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_{k+d}.$$

Proof of the Acyclic Model Theorem 151.17.

(1) Given any topological space X we consider $G(X) := (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))[-d]$. Here we use the notation introduced on page 2571, namely given a chain complex D_* we denote by $D_*[-d]$ the chain complex D_* "shifted by d to the right". This shift turns $G(X)$ into a generalized chain complex. Furthermore, this shift turns $S\colon \mathrm{C}_*(X) \to G(X)$ into a natural chain map of degree 0.

Next note that for any $k \in \mathbb{N}_0$ and any $i \geq 0$ we have
$$\begin{array}{rcccl} \mathrm{H}_i(G(\Delta^k)) & = & \mathrm{H}_i((\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))[-d]) & = & \mathrm{H}_{i+d}(\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)) \\ & = & \bigoplus_{p+q=i+d} \mathrm{H}_p(\Delta^k) \otimes \mathrm{H}_q(\Delta^k) & = & 0. \\ & \uparrow & & \uparrow & \end{array}$$
by the Künneth Theorem 90.7 for chain complexes; by the calculation of $\mathrm{H}_*(\Delta^k)$ and since $d \geq 1$

Thus we see that all the hypotheses of the Acyclic Model Theorem 113.13 are satisfied. Therefore we see that for any topological space X and any $k \in \mathbb{N}_0$ there exists a natural map $P_X\colon \mathrm{C}_k(X) \to G_{k+1}(X)$ such that for any $k \in \mathbb{N}_0$ we have
$$\partial \circ P_X + P_X \circ \partial \;=\; S_X\colon \mathrm{C}_k(X) \;\to\; G_k(X).$$

(2) Suppose we are given a natural anti-chain map S of degree d. As in the remark preceding the theorem we can turn S into a natural chain map T by setting $T_k = (-1)^k \cdot S_k$, $k \in \mathbb{N}_0$. We apply (1) and we obtain natural maps Q_X. Now we define $P_k := (-1)^k \cdot (Q_X)_k$, $k \in \mathbb{N}_0$. One easily verifies that the desired equality holds. ∎

Now, finally, we can turn to the actual proof of Theorem 151.16.

Proof of Theorem 151.16.

(1) We construct the maps $\Delta_n\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$, $n \in \mathbb{N}_0$ iteratively.

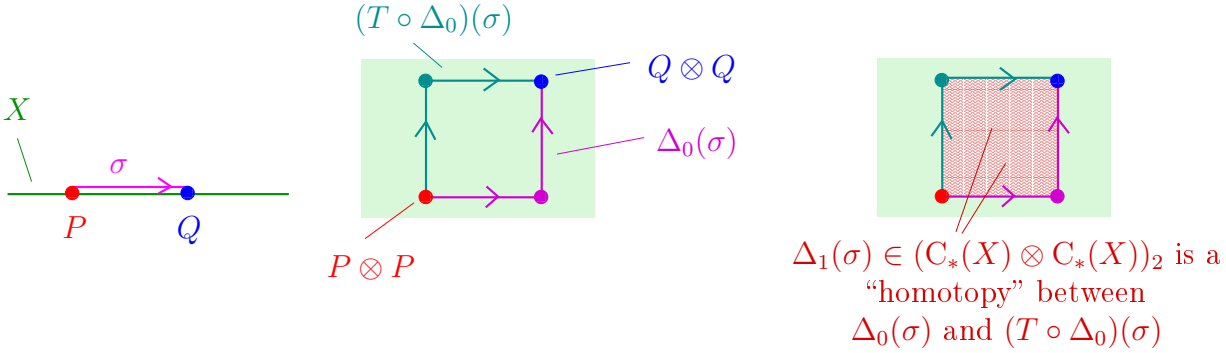
(i) First of all, we know by the Proposition 114.6 (1) above that there exists a diagonal approximation $\Delta_0\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$.

[1801] Recall that a generalized chain complex is like a chain complex except that we allow chain groups in negative degrees.

(ii) Note that $T \circ \Delta_0 \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ is another diagonal approximation. It follows from Proposition 114.6 (2) that there exists a natural chain homotopy $\Delta_1 \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ between Δ_0 and $T \circ \Delta_0$. In other words, there exists a natural map $\Delta_1 \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ of degree one such that

$$\partial \circ \Delta_1 + \Delta_1 \circ \partial = \underbrace{T \circ \Delta_0 - \Delta_0}_{=(T-\mathrm{id}) \circ \Delta_0}.$$

Thus we are done with the construction of the map Δ_1. This step is illustrated in the figure below.



(iii) The starting point for defining the next map Δ_2 is the following amusing little calculation:[1802]

$$\begin{array}{ccccccc} & & \text{since } T^2 = \mathrm{id} & & & & \text{by (ii)} \\ & & \downarrow & & & & \downarrow \\ 0 & = & (T^2 - \mathrm{id}) \circ \Delta_0 & = & (T + \mathrm{id}) \circ (T - \mathrm{id}) \circ \Delta_0 & = & (T + \mathrm{id}) \circ (\partial \circ \Delta_1 + \Delta_1 \circ \partial) \\ & & & & = \partial \circ (T + \mathrm{id}) \circ \Delta_1 + (T + \mathrm{id}) \circ \Delta_1 \circ \partial. & & \\ & & & & \uparrow & & \\ & & & & \text{since } T \text{ and id are chain maps they commute with the boundary maps} & & \end{array}$$

This shows that the map $(T + \mathrm{id}) \circ \Delta_1 \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ is a natural anti-chain map of degree one. By the Acyclic Model Theorem 151.17 (2) there exists a natural map $\Delta_2 \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ of degree two such that

$$\partial \circ \Delta_2 - \Delta_2 \circ \partial = (T + \mathrm{id}) \circ \Delta_1.$$

This conclude the construction of the map Δ_2.

(iv) Just for kicks we do the calculation for the next stage as well. We perform the following calculation:

$$\begin{array}{ccccccc} & & \text{since } T^2 = \mathrm{id} & & & & \text{by (iii)} \\ & & \downarrow & & & & \downarrow \\ 0 & = & (T^2 - \mathrm{id}) \circ \Delta_1 & = & (T - \mathrm{id}) \circ (T + \mathrm{id}) \circ \Delta_1 & = & (T - \mathrm{id}) \circ (\partial \circ \Delta_2 - \Delta_2 \circ \partial) \\ & & & & = \partial \circ (T - \mathrm{id}) \circ \Delta_2 - (T - \mathrm{id}) \circ \Delta_2 \circ \partial. & & \\ & & & & \uparrow & & \\ & & & & \text{since } T \text{ and id are chain maps they commute with the boundary maps} & & \end{array}$$

This shows that the map $(T - \mathrm{id}) \circ \Delta_2 \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ is a natural chain map of degree two. By the Acyclic Model Theorem 151.17 (1) there exists a natural map $\Delta_3 \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ of degree three such that

$$\partial \circ \Delta_3 + \Delta_3 \circ \partial = (T - \mathrm{id}) \circ \Delta_2.$$

[1802] These are equalities of maps $\mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$.

This concludes the construction of the map Δ_3.

(iv) Now we iterate this procedure and we obtain the desired sequence of natural maps Δ_n.

(2) Given two sequences Δ_n, Δ'_n as in (1) we define the desired maps H_n, $n \in \mathbb{N}_0$ in a fashion that is quite similar to the approach taken in (1).

(i) We set $H_0 = 0$.

(ii) Next we point out that by definition Δ_0 and Δ'_0 are diagonal approximations. Thus it follows from Proposition 114.6 (2) that there exist natural maps H_1 such that $\partial \circ H_1 + H_1 \circ \partial = \Delta_0 - \Delta'_0$.

(iii) We calculate that

$$\begin{aligned}
&\partial \circ (\Delta_1 - \Delta'_1 - (T - \mathrm{id}) \circ H_1) + (\Delta_1 - \Delta'_1 - (T - \mathrm{id}) \circ H_1) \circ \partial \\
&\quad = (\partial \circ \Delta_1 + \Delta_1 \circ \partial) - (\partial \circ \Delta'_1 + \Delta'_1 \circ \partial) - \partial \circ (T - \mathrm{id}) \circ H_1 - (T - \mathrm{id}) \circ H_1 \circ \partial \\
&\quad = \underset{\substack{\uparrow \\ \text{since } \partial\circ\Delta_1 + \Delta_1\circ\partial = (T-\mathrm{id})\circ\Delta_0 \text{ (same for } \Delta'_0, \Delta'_1) \\ \text{and since } T \text{ and id are chain maps}}}{(T - \mathrm{id}) \circ (\Delta_0 - \Delta'_0)} - \underset{\substack{\uparrow \\ \text{since } \partial\circ H_1 + H_1\circ\partial = \Delta_0 - \Delta'_0}}{(T - \mathrm{id}) \circ (\partial \circ H_1 + H_1 \circ \partial)} = 0.
\end{aligned}$$

This shows that $\Delta_1 - \Delta'_1 - (T - \mathrm{id}) \circ H_1$ is an anti-chain map of degree one. By the Acyclic Model Theorem 151.17 (2) there exists a natural map H_2 with

$$H_2 \circ \partial - \partial \circ H_2 = -(T - \mathrm{id}) \circ H_1 + \Delta_1 - \Delta'_1.$$

(iv) Now we iterate this procedure to obtain the desired sequence of natural maps. One easily sees that we get the correct pattern for the signs involved. ∎

Remark. If there did exist a diagonal approximation $\Delta_0 \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ that is symmetric in the sense that $T \circ \Delta_0 = \Delta_0$, then we could take $\Delta_1 = 0$ and we could also take all higher Δ_n to be zero. Later, in Chapter 151, we will see that this is in general not the case. So even though diagonal approximations are symmetric on the degree 0 level, they cannot be chosen to be symmetric in higher degrees.

151.5. **Proof of the Steenrod Theorem 151.2 II.** In this section we will use the sequence of natural maps $\Delta_n \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$, $n \in \mathbb{N}_0$, from Theorem 151.16 to define the Steenrod operations.

To get an inspiration for how to define the Steenrod operations we first recall the definition of the cup product. To do so we need the following lemma which is inspired by the definitions on from page 2512.

Lemma 151.18. Let $\mathcal{C}_*$ and $\mathcal{D}_*$ be chain complexes and let R be a commutative ring. Given $p, q \in \mathbb{N}_0$ we consider the map[1803]

$$\begin{aligned}
\Xi_{p,q} \colon \mathrm{Hom}(\mathcal{C}_p, R) \otimes \mathrm{Hom}(\mathcal{D}_q, R) &\to \mathrm{Hom}((\mathcal{C} \otimes \mathcal{D})_{p+q}, R) \\
\varphi \otimes \psi &\mapsto \left((\mathcal{C}_* \otimes \mathcal{D}_*)_{p+q} \xrightarrow{\text{projection}} \mathcal{C}_p \otimes \mathcal{D}_q \xrightarrow{\varphi\otimes\psi} R \otimes R \to R, \; a \otimes b \mapsto a \cdot b \right).
\end{aligned}$$

The corresponding map

$$\Xi := \bigoplus_{p,q \in \mathbb{N}_0} \Xi_{p,q} \colon \mathrm{Hom}(\mathcal{C}_*, R) \otimes \mathrm{Hom}(\mathcal{D}_*, R) \to \mathrm{Hom}(\mathcal{C}_* \otimes \mathcal{D}_*, R)$$

is a cochain map.

Proof. The statement follows basically immediately from the definitions. ■

The following lemma gives us a slight reformulation of the original definition of the cup product.

Lemma 151.19. Let X be a topological space and let R be a commutative ring. Furthermore let $\Delta_0\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ be a diagonal approximation. We consider the cochain map

$$\begin{tikzcd} \mathrm{Hom}(\mathrm{C}_*(X), R) \otimes \mathrm{Hom}(\mathrm{C}_*(X), R) \arrow[r, "\Xi"] \arrow[rr, bend right, "=:h_0"'] & \mathrm{Hom}(\mathrm{C}_*(X) \otimes \mathrm{C}_*(X), R) \arrow[r, "\Delta_0^*"] & \mathrm{Hom}(\mathrm{C}_*(X), R). \end{tikzcd}$$

The corresponding map

$$\begin{array}{rcl} \mathrm{H}^p(X;R) \times \mathrm{H}^q(X;R) & \to & \mathrm{H}^{p+q}(X;R) \\ ([\varphi],[\psi]) & \mapsto & [h_0(\varphi \otimes \psi)] \end{array}$$

equals the cup product as defined on page 2513.

Proof. We denote by $d\colon X \to X \times X$ the diagonal map that is given by $d(x) = (x, x)$. Let $\Theta\colon \mathrm{C}_*(X \times X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ denote a Eilenberg-Zilber map from Theorem 113.1. For the purpose of this proof the only thing we need to know about the Eilenberg-Zilber map Θ is that, basically by definition, the map $\Theta \circ d\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X \times X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ is a diagonal approximation. We consider the following diagram:

$$\begin{tikzcd} \mathrm{Hom}(\mathrm{C}_*(X), R) \otimes \mathrm{Hom}(\mathrm{C}_*(X), R) \arrow[r, "\Xi"] \arrow[rr, bend left, "=:h_0"] & \mathrm{Hom}(\mathrm{C}_*(X) \otimes \mathrm{C}_*(X), R) \arrow[r, "\Delta_0^*"] \arrow[d, "\Theta^*"] & \mathrm{Hom}(\mathrm{C}_*(X), R) \\ & \mathrm{Hom}(\mathrm{C}_*(X \times X), R). \arrow[ur, "d^*"'] & \end{tikzcd}$$

The lower sequence of maps is at the heart of the definition of the cup product on page 2513. Since $\Theta \circ d$ is a diagonal approximation we obtain from Proposition 114.6 that Δ_0 and $\Theta \circ d$ are chain homotopic. But by the Dual Cochain Map Lemma 108.8 (4) this means that the two maps induce the same map on cohomology. The lemma follows almost immediately from the above observations. ■

Now we turn to the actual construction of the Steenrod operations. To do so we pick a sequence of natural maps

$$\Delta_n\colon \mathrm{C}_*(X) \;\to\; \mathrm{C}_*(X) \otimes \mathrm{C}_*(X), \quad n \in \mathbb{N}_0$$

which satisfies the conditions set out in Theorem 151.16 (1). In light of Lemma 151.19 it is clear what we want to do next. Namely we want to replace Δ_0 by the "higher maps" Δ_n. This leads us straight to the following definition.

[1803] The tensor product of cochain complexes is defined basically the same way as we defined the tensor product of chain complexes, see page 1940.

Definition. Let X be a topological space and let R be a commutative ring. Given $i \in \mathbb{N}_0$ we define

$$\mathrm{Hom}(\mathrm{C}_*(X), R) \otimes \mathrm{Hom}(\mathrm{C}_*(X), R) \xrightarrow{\Xi} \mathrm{Hom}(\mathrm{C}_*(X) \otimes \mathrm{C}_*(X), R) \xrightarrow{\Delta_i^*} \mathrm{Hom}(\mathrm{C}_*(X), R).$$

$$=: h_i$$

Recall that Δ_i is a map of degree i. Since we dualize we see that h_i is a map of degree $-i$. For convenience we define $h_i := 0$ for $i \in \mathbb{Z}_{<0}$.

Now we can finally define the Steenrod operations.

Theorem 151.20. Let X be a topological space, let $A \subset X$ be a subset and let R be a commutative ring. If $1 + 1 = 0$ in R, then for any $q \in \mathbb{N}_0$ and any $i \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{Sq}^i \colon \mathrm{H}^q(X, A; R) & \to & \mathrm{H}^{q+i}(X, A; R) \\ [\underbrace{\varphi}_{\in \mathrm{C}^q(X;R)}] & \mapsto & [h_{q-i}(\underbrace{\varphi \otimes \varphi}_{\in \mathrm{C}^q \otimes \mathrm{C}^q})] \end{array}$$

is a well-defined natural homomorphism.

Remark. The proof of Theorem 151.20 that we give below is a long and winding calculation, with few interesting view points. Perhaps more interesting than the question, why the definition works, is the question, why does the map

$$\begin{array}{rcl} \mathrm{C}^q(X, A; R) \otimes \mathrm{C}^r(X, A; R) & \to & \mathrm{C}^{q+r+j}(X, A; R) \\ \varphi \otimes \psi & \mapsto & h_j(\underbrace{\varphi \otimes \psi}_{\in \mathrm{C}^q \otimes \mathrm{C}^r}) \end{array}$$

not descend to a map on cohomology?

The proof of Theorem 151.20 requires some preparations. First of all, note that the equalities in Theorem 151.16 translate into the following lemma.

Lemma 151.21. Let X be a topological space. Furthermore let R be a commutative ring with $1_R + 1_R = 0_R$. For any $\alpha, \beta \in \mathrm{C}^*(X; R)$ we have the following equality:

$$h_{n+1}(\delta(\alpha \otimes \beta)) + \delta(h_{n+1}(\alpha \otimes \beta)) \;=\; h_n(\alpha \otimes \beta + \beta \otimes \alpha) \;\in\; \mathrm{C}^*(X; R).$$

Proof. Recall, see Theorem 151.16, that one of the defining properties of the sequence Δ_n is that for every $n \in \mathbb{N}_0$ the following equality holds:

$$\partial \circ \Delta_{n+1} + (-1)^n \cdot \Delta_{n+1} \circ \partial \;=\; (T + (-1)^{n+1} \cdot \mathrm{id}) \circ \Delta_n.$$

We dualize this equality, we use the equality $1 + 1 = 0$, we use the fact, shown in Lemma 151.18 and that Ξ is a cochain map. After performing all these steps one obtains the desired equality. We leave it to the reader to fill in the few remaining details. ■

For elements of the form $\alpha \otimes \alpha$ the following lemma shows how the maps h_n relate to coboundaries.

Lemma 151.22. Let X be a topological space and let R be a commutative ring. If $1 + 1 = 0$ in R, then for any $\alpha \in \mathrm{C}^q(X; R)$ we have

$$h_{n+1}(\delta\alpha \otimes \delta\alpha) \;=\; \delta h_{n+1}(\alpha \otimes \delta\alpha) + \delta h_n(\alpha \otimes \alpha) \;\in\; \mathrm{C}^{n+1}(X; R).$$

Proof. Since we assume that $1+1=0$ in R we can ignore conveniently enough ignore signs throughout the proof. Thus we compute

$$\begin{array}{rcl} h_{n+1}(\delta\alpha\otimes\delta\alpha) & \overset{\text{since } \delta(c\otimes d)=\delta c\otimes d+c\otimes\delta d}{\downarrow} & \overset{\text{Lemma 151.21}}{\downarrow} \\ & h_{n+1}(\delta(\alpha\otimes\delta\alpha)) & = \delta h_{n+1}(\alpha\otimes\delta\alpha)+h_n(\alpha\otimes\delta\alpha+\delta a\otimes\alpha) \\ & = \delta h_{n+1}(\alpha\otimes\delta\alpha)+h_n(\delta(\alpha\otimes\alpha)) & = \delta h_{n+1}(\alpha\otimes\delta\alpha)+\delta h_n(\alpha\otimes\alpha). \\ & \uparrow & \uparrow \\ & \text{definition of } \delta \text{ on tensor product} & \text{by Lemma 151.21 and since } h_{n-1}(\underbrace{\alpha\otimes\alpha+\alpha\otimes\alpha}_{=0 \text{ since } 1+1=0})=0 \end{array}$$

■

In the proof of Theorem 151.20 we will need the following little lemma.

Lemma 151.23. Let X be a topological space and let R be a commutative ring with $1+1=0$. For any $\alpha,\beta\in \mathrm{C}^q(X;R)$ the following equality holds:

$$h_n((\alpha+\beta)\otimes(\alpha+\beta))=h_n(\alpha\otimes\alpha)+h_n(\beta\otimes\beta)+h_{n+1}(\delta(\alpha\otimes\beta))+\delta(h_{n+1}(\alpha\otimes\beta)).$$

Proof of Lemma 151.23. This follows from "multiplying out", the linearity of h_n and the equality

$$h_n(\alpha\otimes\beta+\beta\otimes\alpha) \underset{\text{by Lemma 151.21}}{\overset{\uparrow}{=}} h_{n+1}(\delta(\alpha\otimes\beta))+\delta(h_{n+1}(\alpha\otimes\beta)).$$

■

Now, after these lengthy preparations we can provide the proof of Theorem 151.20.

Proof of Theorem 151.20. Let X be a topological space, let $A\subset X$ be a subset and let R be a commutative ring. First we consider the case that $A=\varnothing$. Let $\varphi\in \mathrm{C}^q(X;R)$ be a representative of some element in $\mathrm{H}^q(X;R)$.

(1) Note that $\varphi\otimes\varphi$ is of degree $2q$. Thus $h_{q-i}(\varphi\otimes\varphi)$ is of degree $2q-(q-i)=q+i$.
(2) Let $\varphi\in \mathrm{C}^q(X;R)$ be a cocycle. We calculate that

$$\delta h_{q-i}(\varphi\otimes\varphi) \underset{\text{Lemma 151.21}}{\overset{\uparrow}{=}} h_{q-i}(\delta(\varphi\otimes\varphi))+h_{q-i-1}(\overbrace{\varphi\otimes\varphi+\varphi\otimes\varphi}^{=0 \text{ since } 1+1=0}) \underset{\text{definition of } \delta \text{ on tensor product}}{\overset{\uparrow}{=}} h_{q-i}(\delta\varphi\otimes\varphi+\varphi\otimes\delta\varphi) \underset{\text{since } \varphi \text{ is a cocycle}}{\overset{\uparrow}{=}} 0.$$

Thus we have shown that $h_{q-i}(\varphi\otimes\varphi)$ is a cocycle. In other words, we have shown that it defines an element in cohomology.
(3) Let $\psi\in \mathrm{C}^{q-1}(X;R)$. We have the following equality in $\mathrm{H}^{q+i}(X;R)$:

$$[h_{q-i}((\varphi+\delta\psi)\otimes(\varphi+\delta\psi))] \underset{\text{by Lemma 151.23, here we use that } \delta(\varphi\otimes\delta\psi)=\delta\varphi\otimes\delta\psi+\varphi\otimes\delta\delta\psi=0}{\overset{\uparrow}{=}} [h_{q-i}(\varphi\otimes\varphi)+h_{q-i}(\delta\psi\otimes\delta\psi)] \underset{\text{Lemma 151.22}}{\overset{\uparrow}{=}} [h_{q-i}(\varphi\otimes\varphi)].$$

This shows that the map Sq^i is well-defined, i.e. it does not depend on the choice of the representative of the cohomology class.
(4) It follows quite easily from Lemma 151.23 that Sq^i is in fact a homomorphism.

Finally we consider the case that $A\neq\varnothing$. Using the observation on page 2407 we make the identification

$$\mathrm{C}^q(X,A;R) \;=\; \{\varphi\in \mathrm{C}^q(X;R)=\mathrm{Hom}(\mathrm{C}_q(X),R)\,|\,\varphi \text{ vanishes on } \mathrm{C}_q(A)\subset \mathrm{C}_q(X)\}.$$

It follows from the naturality of h_n that if $\varphi\in \mathrm{C}^q(X)$ vanishes on $\mathrm{C}_q(A)$, then $h_{q-i}(\varphi\otimes\varphi)$ vanishes on the subgroup $\mathrm{C}_*(A)\otimes\mathrm{C}_*(X)+\mathrm{C}_*(X)\otimes\mathrm{C}_*(A)$ of $\mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$. Thus we

see that the map

$$\begin{array}{rcl} \mathrm{Sq}^i\colon \mathrm{H}^q(X,A;R) & \to & \mathrm{H}^{q+i}(X,A;R) \\ [\underbrace{\varphi}_{\in \mathrm{C}^q(X;R)}] & \mapsto & [h_{q-i}(\varphi\otimes\varphi)] \end{array}$$

is well-defined. ■

We continue with the following proposition.

Proposition 151.24. The definition of the maps Sq^i does not depend on the choice of the natural maps Δ_n.

Proof. Suppose we are given another sequence of maps $\widetilde{\Delta}_n$ that have the properties described in Theorem 151.16 (1). We define the maps $\widetilde{h}_i$ in the obvious way. By Theorem 151.16 (2) there exists a sequence of natural maps

$$\mathrm{H}_n\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X)\otimes\mathrm{C}_*(X), \quad n\in\mathbb{N}_0$$

of degree n such that $H_0 = 0$ and such that for every $n\in\mathbb{N}_0$ the following equality holds:

$$(*) \qquad \partial\circ \mathrm{H}_{n+1} + (-1)^n\cdot \mathrm{H}_{n+1}\circ\partial \;=\; \Delta_n - \widetilde{\Delta}_n - (T+(-1)^n\cdot\mathrm{id})\circ \mathrm{H}_n.$$

Given a cocycle $\varphi\in \mathrm{C}^q(X,A;\mathbb{Z}_2)$ and $i\in\mathbb{N}_0$ we see that we have the following equality in $\mathrm{H}^{2q-i}(X,A;R)$:

$$\begin{array}{rcl} [h_i(\varphi\otimes\varphi)] - [\widetilde{h}_i(\varphi\otimes\varphi)] & \overset{\text{definition of } h_i,\widetilde{h}_i}{=} & [\Delta_i^*(\Xi(\varphi\otimes\varphi)) - \widetilde{\Delta}_i^*(\Xi(\varphi\otimes\varphi))] \\ & = & [\mathrm{H}^*_{i+1}(\delta(\Xi(\varphi\otimes\varphi)) + \underbrace{\delta(\mathrm{H}^*_{i+1}(\varphi\otimes\varphi))}_{\text{coboundary}} + \mathrm{H}^*_i((T+\mathrm{id})^*(\Xi(\varphi\otimes\varphi)))] \\ & \overset{\text{by } (*) \text{ and since } 1+1=0}{=} & [\mathrm{H}^*_{i+1}(\Xi(\underbrace{\delta(\varphi\otimes\varphi)}_{=0 \text{ since } \varphi \text{ cocycle}})) + \mathrm{H}^*_i(\Xi(\underbrace{\varphi\otimes\varphi+\varphi\otimes\varphi}_{=0 \text{ since } 1+1=0}))] \;=\; 0. \\ & \overset{\text{since } \Xi \text{ is a cochain map and since } \Xi \text{ commutes with } T}{=} & \end{array}$$

We have thus shown that the two definitions using Δ_n and $\widetilde{\Delta}_n$ agree. ■

By Proposition 151.24 the maps Sq^i are unambiguously defined. Thus we can make the following definition.

Definition. Given a pair of topological spaces (X,A) and $i\in\mathbb{N}_0$ we refer to each map

$$\mathrm{Sq}^i\colon \mathrm{H}^q(X,A;\mathbb{Z}_2) \to \mathrm{H}^{q+i}(X,A;\mathbb{Z}_2)$$

as the i-th Steenrod operation.

By Theorem 151.20 the Steenrod operations are additive cohomology operations. To complete the proof of the Steenrod Theorem 151.2 we need to verify that the maps Sq^i have all the other desired properties. In other words we need to prove the following theorem.

Theorem 151.25. The Steenrod operations Sq^i have the following properties:

(1) $\mathrm{Sq}^0 = \mathrm{id}$.
(2) For any $\varphi\in\mathrm{H}^i(X,A;\mathbb{Z}_2)$ we have $\mathrm{Sq}^i(\varphi) = \varphi^2$.
(3) For any $\varphi\in\mathrm{H}^j(X,A;\mathbb{Z}_2)$ with $j<i$ we have $\mathrm{Sq}^i(\varphi) = 0$.

(4) For any $\varphi \in \mathrm{H}^r(X, A; \mathbb{Z}_2)$ and $\psi \in \mathrm{H}^s(X, A; \mathbb{Z}_2)$ we have

$$\mathrm{Sq}^k(\varphi \cup \psi) \;=\; \sum_{i=0}^{k} \mathrm{Sq}^i(\varphi) \cup \mathrm{Sq}^{k-i}(\psi).$$

Proof.

(1) The notation might suggest that this statement is surely trivial, but that is not the case. For time, space and mental energy reasons we will not give the proof, which at first glance does not seem terribly enlightening anyway. Instead we refer to [**Spa1995**, p. 274] for details.

(2) First we deal with the absolute case. Thus let $\varphi \in \mathrm{C}^i(X; R)$ be a cocycle. We calculate

$$\mathrm{Sq}^i(\varphi) \underset{\text{by definition}}{\overset{\uparrow}{=}} [h_{i-i}(\varphi \otimes \varphi)] \;=\; [h_0(\varphi \otimes \varphi)] \underset{\text{Lemma 151.19}}{\overset{\uparrow}{=}} [\varphi] \cup [\varphi].$$

The case that we are dealing with a cocycle $\varphi \in \mathrm{C}^i(X, A; R)$ is proved basically the same way.

(3) Let $\varphi \in \mathrm{C}^j(X, A; \mathbb{Z}_2)$ be a cocycle and suppose that $i > j$. We have

$$\mathrm{Sq}^i([\varphi]) \underset{\text{by definition}}{\overset{\uparrow}{=}} [h_{j-i}(\varphi)] \underset{\text{since } j-i<0 \text{ we have } h_{j-i}=0}{\overset{\uparrow}{=}} [0].$$

(4) This statement is proved in [**Spa1995**, p. 274]. ∎

Exercises for Chapter 151.

Exercise 151.1.

(a) Show that for any $k \in \mathbb{N}_{\geq 3}$ and $m \in \mathbb{N}$ the map $\mathrm{H}^m(X; \mathbb{Z}_2) \to \mathrm{H}^{mk}(X; \mathbb{Z}_2)$ given by $\varphi \mapsto \varphi^k$ is not an additive cohomology operation.

(b) Is the map $\mathrm{H}^1(X; \mathbb{Z}_3) \to \mathrm{H}^3(X; \mathbb{Z}_3)$ given by $\varphi \mapsto \varphi^3$ an additive cohomology operation?

Exercise 151.2. Let $X = S^m \vee S^n$ be the wedge of two spheres. Show that for $i \geq 1$ all Steenrod operations $\mathrm{Sq}^i \colon \mathrm{H}^k(X; \mathbb{Z}_2) \to \mathrm{H}^k(X; \mathbb{Z}_2)$ are zero.
Hint. Reread the proof of Lemma 114.11.

Exercise 151.3. Find a closed formula for the i-th Steenrod operation on $\mathrm{H}^*(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}_2)$.
Hint. Use Proposition 121.13 (2).

152. The Adem relations

The following theorem was first proved by José Ádem [**Ade1952, Ade1957**] in 1952.

Theorem 152.1. (Adem Relations Theorem) For any $0 < a < 2b$ we have the following equality of cohomology operations:[1804]

$$\mathrm{Sq}^a \circ \mathrm{Sq}^b \;=\; \sum_{j=0}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \cdot \mathrm{Sq}^{a+b-j} \circ \mathrm{Sq}^j \,.$$

These equalities are called the *Adem relations*.

Remark. As is remarked on [**SE1962**, p. 2], the Adem relations are a consequence of the properties of Steenrod operations that were stated in the Steenrod Theorem 151.2. This observation also follows from the uniqueness result stated in Theorem 151.3.

Proof. Since the proof would consume too much of our valuable time we will not provide it. Instead we refer to the original paper by Adem [**Ade1952, Ade1957**] or alternatively to [**BuM1982**], [**SE1962**, Theorem VIII.1.5], [**MTa1968**, p. 23], or [**Hat2002**, Theorem 4.L.13] for a proof. ■

Even though we will not prove the Adem relations, we will shamelessly exploit them in this chapter. Afterwards the conscientious reader will surely feel compelled to read the proof of the Adem relations.
Just to get a feeling for the Adem relations we determine for small values of a what the relations really look like.

Example. For $a = 1$ and for any $b \in \mathbb{N}$ we have

$$\mathrm{Sq}^1 \circ \mathrm{Sq}^b \;=\; \underbrace{\sum_{j=0}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \cdot \mathrm{Sq}^{a+b-j} \circ \mathrm{Sq}^j}_{\text{with } a=1} \;=\; \underbrace{\binom{b-1}{1} \cdot \mathrm{Sq}^{1+b} \circ \mathrm{Sq}^0}_{j=0} \;=\; \underset{\text{since } \mathrm{Sq}^0 = \mathrm{id}}{\uparrow} \;(b-1) \cdot \mathrm{Sq}^{1+b} \,.$$

Furthermore, for $a = 2$ and any $b \in \mathbb{N}_{\geq 2}$ we have

$$\mathrm{Sq}^2 \circ \mathrm{Sq}^b \;=\; \underbrace{\binom{b-1}{2} \cdot \mathrm{Sq}^{2+b} \circ \mathrm{Sq}^0}_{j=0} + \underbrace{\binom{b-2}{0} \cdot \mathrm{Sq}^{1+b} \circ \mathrm{Sq}^1}_{j=1} \;=\; \underset{\text{since } \mathrm{Sq}^0 = \mathrm{id}}{\uparrow} \;\binom{b-1}{2} \cdot \mathrm{Sq}^{2+b} + \mathrm{Sq}^{1+b} \circ \mathrm{Sq}^1 \,.$$

In particular setting $b = 2$ we see that

$$\mathrm{Sq}^2 \circ \mathrm{Sq}^2 \;=\; \mathrm{Sq}^3 \circ \mathrm{Sq}^1 \,.$$

152.1. **Application of the Adem relations to homotopy groups of spheres.** Recall that in Theorem 151.13 we used Steenrod operations to show that the three Hopf maps $H\colon S^3 \to S^2$, $H_{\mathbb{H}}\colon S^7 \to S^4$ and $H_{\mathbb{O}}\colon S^{15} \to S^8$ and their suspensions define non-trivial elements in the corresponding homotopy groups of spheres.

[1804] These are equalities of cohomology operations $\mathrm{H}^n(X, A; \mathbb{Z}_2) \to \mathrm{H}^{n+a+b}(X, A; \mathbb{Z}_2)$. In particular, since we work over $\mathbb{Z}_2$-coefficients, we only care about the binomial coefficient modulo 2. We will discuss the mod 2 reductions of binomial coefficients in Lemma 152.3.

In this section we will use the Adem relations to show that the 2-fold iteration of these Hopf maps also define non-trivial elements in the homotopy groups of spheres. More precisely, we have the following theorem.

Theorem 152.2.

(1) For any $n \geq 2$ the map

$$\Sigma^{n-2}\Big(\underbrace{\overbrace{H}^{S^3\to S^2} \circ \overbrace{\Sigma(H)}^{S^4\to S^3}}_{\text{map } S^4\to S^2}\Big)\colon S^{n+2} \to S^n$$

defines a non-trivial element in $\pi_{n+2}(S^n)$. In particular for any $n \geq 2$ we have $\pi_{n+2}(S^n) \neq 0$.

(2) For any $n \geq 4$ the map

$$\Sigma^{n-4}\Big(\underbrace{\overbrace{H_{\mathbb{H}}}^{S^7\to S^4} \circ \overbrace{\Sigma^3(H_{\mathbb{H}})}^{S^{10}\to S^7}}_{\text{map } S^{10}\to S^4}\Big)\colon S^{n+6} \to S^n$$

defines a non-trivial element in $\pi_{n+6}(S^n)$. In particular for any $n \geq 4$ we have $\pi_{n+6}(S^n) \neq 0$.

(3) For any $n \geq 8$ the map

$$\Sigma^{n-8}\Big(\underbrace{\overbrace{H_{\mathbb{O}}}^{S^{15}\to S^8} \circ \overbrace{\Sigma^7(H_{\mathbb{O}})}^{S^{22}\to S^{15}}}_{\text{map } S^{22}\to S^8}\Big)\colon S^{n+14} \to S^n$$

defines a non-trivial element in $\pi_{n+14}(S^n)$. In particular for any $n \geq 8$ we have $\pi_{n+14}(S^n) \neq 0$.

For the reader's convenience we recall the two lemmas which will play a key role in the proof of Theorem 152.2.

Lemma 122.4. Let $f\colon A \to X$ be a map between two topological spaces. We denote by $j\colon X \to \mathrm{Cone}(f\colon A \to X)$ the obvious inclusion. Then there exists a natural long exact sequence of the form

$$\ldots \to \mathrm{H}^{k-1}(A;\mathbb{Z}_2) \xrightarrow{\delta} \widetilde{\mathrm{H}}^k(\mathrm{Cone}(f\colon A \to X);\mathbb{Z}_2) \xrightarrow{j^*} \mathrm{H}^k(X;\mathbb{Z}_2) \xrightarrow{f^*} \mathrm{H}^k(A;\mathbb{Z}_2) \xrightarrow{\delta} \ldots$$

As an example for the Mapping Cone-H*-Lemma 122.4 let us consider the mapping cone of the iterated suspension of the Hopf map.

Example. Let $k \in \mathbb{N}_0$. We consider $\Sigma^k(H)\colon S^{k+3} \to S^{k+2}$, i.e. we consider the k-fold suspension of the Hopf map $H\colon S^3 \to S^2$. As we saw in Lemma 122.5, it follows almost immediately from the long exact sequence of the Mapping Cone-H*-Lemma 122.4 that the maps

$$\widetilde{\mathrm{H}}^{k+2}(\mathrm{Cone}(\Sigma^k(H)\colon S^{k+3} \to S^{k+2});\mathbb{Z}_2) \xrightarrow[\cong]{j^*} \mathrm{H}^{k+2}(S^{k+2};\mathbb{Z}_2)$$

and

$$\mathrm{H}^{k+3}(S^{k+3};\mathbb{Z}_2) \xrightarrow[\cong]{\delta} \widetilde{\mathrm{H}}^{k+4}(\mathrm{Cone}(\Sigma^k(H)\colon S^{k+3} \to S^{k+2});\mathbb{Z}_2)$$

are isomorphisms, and that all the other reduced cohomology groups of the mapping cone vanish.

In the following we recall three statements of the Mapping Cone Basics Lemma 16.13 which are at the moment relevant to us. It might be helpful to have a peek at the figure below.

Lemma 16.13. Let $f\colon A \to X$ be a map between topological spaces.

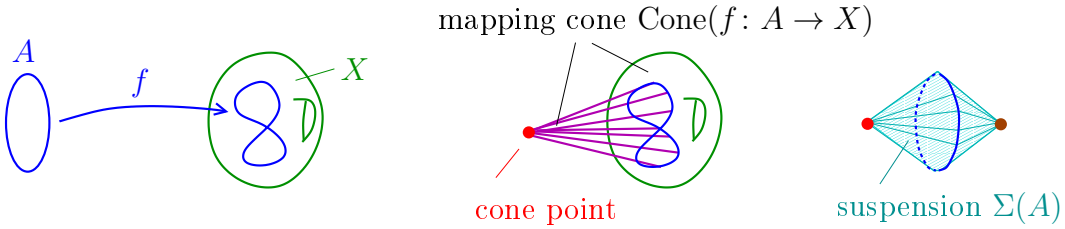
(1) The inclusion $X \to \mathrm{Cone}(f\colon A \to X)$ is an embedding.

(2) The map

$$\begin{array}{rccc} h\colon & \mathrm{Cone}(f\colon A \to X)/X & \to & \Sigma(A) \\ & [(a,t)] & \mapsto & [(a,1-2t)] \end{array}$$

is a homeomorphism.

(4) The subset X is a deformation retract of the open neighborhood that is given by the complement of the cone point.



Now we turn to the proof of Theorem 152.2. As in the proof of Theorem 151.13 we first consider the case of the classical Hopf map, i.e. we first prove Statement (1).

Proof of Theorem 152.2 (1). Let $n \in \mathbb{N}_{\geq 2}$. We write

$$\Sigma^{n-2}(H \circ \Sigma(H)) \;=\; \overbrace{\underbrace{\Sigma^{n-2}(H)}_{=:g}}^{S^{n+1}\to S^n} \circ \overbrace{\underbrace{\Sigma^{n-1}(H)}_{=:f}}^{S^{n+2}\to S^{n+1}} : S^{n+2} \;\to\; S^n.$$

$\uparrow$ Suspension Functor Lemma 16.9 (2)

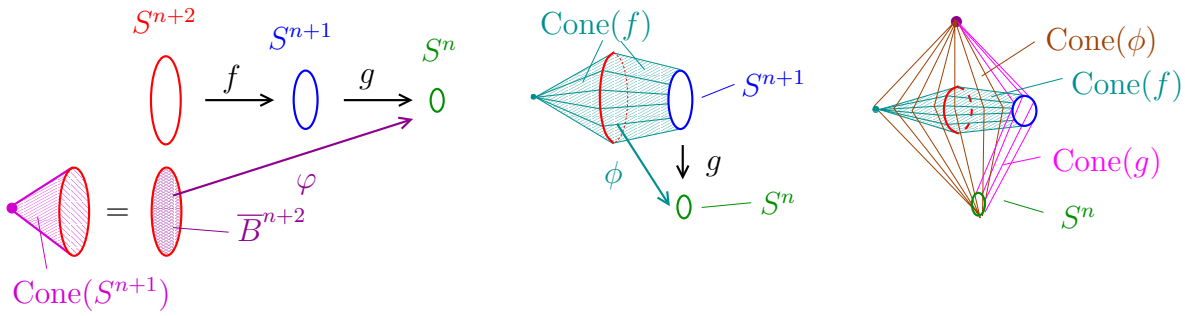
We need to show that $g\circ f\colon S^{n+2} \to S^n$ defines a non-trivial element in $\pi_{n+2}(S^n)$. As in the proof of Theorem 151.13 we do a proof by contradiction. Thus suppose that there exists a homotopy from $g \circ f\colon S^{n+2} \to S^n$ to a constant map. By the Trivial-in-π_n-Lemma 71.4 we can use this homotopy to construct a map[1805] $\varphi\colon \mathrm{Cone}(S^{n+2}) = \overline{B}^{n+3} \to S^n$ such that $\varphi|_{S^{n+2}} = g \circ f$. We consider the map[1806]

$$\begin{array}{rccl} \phi\colon & \overbrace{(\mathrm{Cone}(S^{n+2}) \sqcup S^{n+1})/\!\sim}^{=\mathrm{Cone}(f\colon S^{n+2}\to S^{n+1})} & \to & S^n \\ & [P] & \mapsto & \begin{cases} g(P), & \text{if } P \in S^{n+1}, \\ \varphi(P), & \text{if } P \in \mathrm{Cone}(S^{n+2}) = \overline{B}^{n+3}. \end{cases} \end{array}$$

Our main object of interest is now, perhaps rather confusingly, the mapping cone of ϕ. This mapping cone comes with two interesting maps:

[1805] As we discussed on page 450, given any $m \in \mathbb{N}_0$ the map $\mathrm{Cone}(S^m) \to \overline{B}^{m+1}$ given by $[(v,t)] \mapsto v \cdot t$ is a homeomorphism. We will use this map to make the identification $\overline{B}^{m+1} = \mathrm{Cone}(S^m)$.

[1806] It follows from $\varphi|_{S^{n+2}} = g \circ f$ that this map is well-defined. Furthermore we obtain immediately from the Topological Pushout Proposition 5.33 (1b) that this map is continuous.



(1) Since we have an inclusion $S^{n+1} \to \mathrm{Cone}(f)$ and since $\phi|_{S^{n+1}} = g$ we obtain, see the Mapping Cone Functor Lemma 16.14 (3), also an inclusion map

$$i\colon \mathrm{Cone}(g\colon S^{n+1} \to S^n) \;\to\; \mathrm{Cone}(\phi\colon \mathrm{Cone}(f) \to S^n).$$

(2) We also consider the projection

$$p\colon \mathrm{Cone}(\phi\colon \mathrm{Cone}(f) \to S^n) \;\to\; \mathrm{Cone}(\phi\colon \mathrm{Cone}(f) \to S^n)/S^n.$$

In the following two claims we will study the maps on cohomology induced by the inclusion i and the projection p.

Claim 1.

(a) The induced map

$$i^*\colon \mathrm{H}^k(\mathrm{Cone}(\phi);\mathbb{Z}_2) \;\to\; \mathrm{H}^k(\mathrm{Cone}(g);\mathbb{Z}_2)$$

is an isomorphism for $k \neq n+1$.

(b) We have $\mathrm{H}^{n+1}(\mathrm{Cone}(\phi);\mathbb{Z}_2) = 0$.

Proof. We consider the following diagram:

$$\begin{array}{ccccccccccc}
 & & \mathrm{H}^{k-2}(S^{n+2};\mathbb{Z}_2) & & & & & & \mathrm{H}^{k-1}(S^{n+2};\mathbb{Z}_2) & & \\
 & & \downarrow & & & & & & \downarrow & & \\
\to & \mathrm{H}^{k-1}(\mathrm{Cone}(f);\mathbb{Z}_2) & \to & \mathrm{H}^k(\mathrm{Cone}(\phi\colon \mathrm{Cone}(f) \to S^n);\mathbb{Z}_2) & \to & \mathrm{H}^k(S^n;\mathbb{Z}_2) & \xrightarrow{\phi^*} & \mathrm{H}^k(\mathrm{Cone}(f);\mathbb{Z}_2) & \to & & \\
 & \downarrow & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & & \\
\longrightarrow & \mathrm{H}^{k-1}(S^{n+1};\mathbb{Z}_2) & \longrightarrow & \mathrm{H}^k(\mathrm{Cone}(g\colon S^{n+1} \to S^n);\mathbb{Z}_2) & \longrightarrow & \mathrm{H}^k(S^n;\mathbb{Z}_2) & \xrightarrow{f^*} & \mathrm{H}^k(S^{n+1};\mathbb{Z}_2) & \longrightarrow & & \\
 & \downarrow{\scriptstyle f^*} & & & & & & \downarrow{\scriptstyle f^*} & & & \\
 & \mathrm{H}^{k-1}(S^{n+2};\mathbb{Z}_2) & & & & & & \mathrm{H}^k(S^{n+2};\mathbb{Z}_2) & & &
\end{array}$$

Here the two horizontal and the two vertical sequences are obtained from the Mapping Cone-H^*-Lemma 122.4. The diagram commutes by the naturality of the long exact sequence. The two statements of the claim follow easily from the above diagram (see also the discussion on page 3224) and the calculation of the cohomology groups of spheres with $\mathbb{Z}_2$-coefficients. ⊞

Claim 2. The map

$$p^*\colon \mathrm{H}^k(\mathrm{Cone}(\phi)/S^n;\mathbb{Z}_2) \;\to\; \mathrm{H}^k(\mathrm{Cone}(\phi);\mathbb{Z}_2)$$

is an isomorphism for $k \neq n, n+1$.

Proof. We consider the following diagram:

$$\begin{array}{ccccccccc} \dots \longrightarrow \mathrm{H}^{k-1}(S^n;\mathbb{Z}_2) & \xrightarrow{\delta} & \mathrm{H}^k(\mathrm{Cone}(\phi), S^n;\mathbb{Z}_2) & \longrightarrow & \mathrm{H}^k(\mathrm{Cone}(\phi);\mathbb{Z}_2) & \longrightarrow & \mathrm{H}^k(S^n;\mathbb{Z}_2) & \longrightarrow & \dots \\ & & \downarrow & \nearrow_{p^*} & & & & & \\ & & \mathrm{H}^k(\mathrm{Cone}(\phi)/S^n;\mathbb{Z}_2) & & & & & & \end{array}$$

Here on top we see the long exact sequence in cohomology of the pair $(\mathrm{Cone}(\phi), S^n)$. Furthermore the vertical map is the natural map from the Cohomology-of-Quotients Proposition 109.2 which has in particular the property that it makes the diagram commute. Furthermore, it follows from the Cohomology-of-Quotients Proposition 109.2 (2) and the Mapping Cone Basics Lemma 16.13 (6) that the vertical map is an isomorphism. The desired statement follows immediately from this discussion and the calculation of the cohomology groups of S^n. ⊞

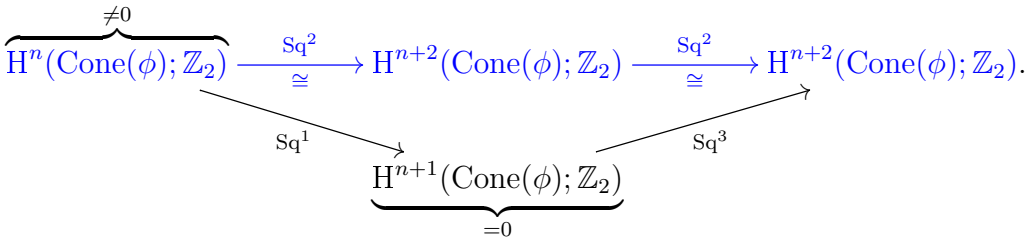
Next we consider the following diagram.

$$\begin{array}{ccccccc} \mathrm{H}^n(\mathrm{Cone}(\Sigma^{n-2}(H));\mathbb{Z}_2) & \xrightarrow[\cong]{\mathrm{Sq}^2} & \mathrm{H}^{n+2}(\mathrm{Cone}(\Sigma^{n-2}(H));\mathbb{Z}_2) & & & & \\ \downarrow{=} & & {=}\downarrow & & & & \\ \mathrm{H}^n(\mathrm{Cone}(g);\mathbb{Z}_2) & \xrightarrow[\cong]{\mathrm{Sq}^2} & \mathrm{H}^{n+2}(\mathrm{Cone}(g);\mathbb{Z}_2) & & & & \\ {i^*}\uparrow{\cong} & & {\cong}\uparrow{i^*} & & & & \\ \mathrm{H}^n(\mathrm{Cone}(\phi);\mathbb{Z}_2) & \xrightarrow{\mathrm{Sq}^2} & \mathrm{H}^{n+2}(\mathrm{Cone}(\phi);\mathbb{Z}_2) & \xrightarrow{\mathrm{Sq}^2} & \mathrm{H}^{n+4}(\mathrm{Cone}(\phi);\mathbb{Z}_2) & & \\ & & {p^*}\uparrow{\cong} & & {\cong}\uparrow{p^*} & & \\ & & \mathrm{H}^{n+2}(\mathrm{Cone}(\phi)/S^n;\mathbb{Z}_2) & \xrightarrow{\mathrm{Sq}^2} & \mathrm{H}^{n+4}(\mathrm{Cone}(\phi)/S^n;\mathbb{Z}_2) & & \\ & & {h^*}\uparrow{\cong} & & {\cong}\uparrow{h^*} & & \\ & & \mathrm{H}^{n+2}(\Sigma(\mathrm{Cone}(f));\mathbb{Z}_2) & \xrightarrow[\cong]{\mathrm{Sq}^2} & \mathrm{H}^{n+4}(\Sigma(\mathrm{Cone}(f));\mathbb{Z}_2) & & \\ & & {\theta^*}\downarrow{\cong} & & {\cong}\downarrow{\theta^*} & & \\ & & \mathrm{H}^{n+2}(\mathrm{Cone}(\Sigma(f));\mathbb{Z}_2) & \xrightarrow[\cong]{\mathrm{Sq}^2} & \mathrm{H}^{n+4}(\mathrm{Cone}(\Sigma(f));\mathbb{Z}_2) & & \\ & & {=}\downarrow & & \downarrow{=} & & \\ & & \mathrm{H}^{n+2}(\mathrm{Cone}(\Sigma^n(H));\mathbb{Z}_2) & \xrightarrow[\cong]{\mathrm{Sq}^2} & \mathrm{H}^{n+4}(\mathrm{Cone}(\Sigma^n(H));\mathbb{Z}_2). & & \end{array}$$

We make the following clarifications and observations:

(1) At the top and bottom we use that by definition $f = \Sigma^{n-1}(H)\colon S^{n+2} \to S^{n+1}$ and $g = \Sigma^{n-2}(H)\colon S^{n+1} \to S^n$.
(2) We obtain from Proposition 151.15 (1) that the top and bottom horizontal maps Sq^2 are isomorphisms.
(3) It follows from the above two claims that the vertical maps i^* and p^* are isomorphisms.
(4) The map $h\colon \mathrm{Cone}(\phi\colon \mathrm{Cone}(f) \to S^n)/S^n \to \Sigma(\mathrm{Cone}(f))$ is the homeomorphism from the Mapping Cone Basics Lemma 16.13 (4).
(5) The map $\theta\colon \mathrm{Cone}(\Sigma(f)\colon \Sigma(S^{n+2}) \to \Sigma(S^{n+1})) \to \Sigma(\mathrm{Cone}(f\colon S^{n+2} \to S^{n+1}))$ is the homeomorphism from the Mapping Cone-Suspension Proposition 16.19.
(6) It follows from the naturality of the Steenrod operations that all the squares commute.
(7) The combination of the above arguments implies that both of the maps Sq^2 in the third row are isomorphisms.

Finally we consider the following diagram.



We make the following observations:

(1) By the discussion preceding the diagram we know that the two maps on the top are isomorphisms.

(2) As we saw on page 3223, we have the Adem relation $\mathrm{Sq}^2 \circ \mathrm{Sq}^2 = \mathrm{Sq}^3 \circ \mathrm{Sq}^1$. This implies that the diagram commutes.

(3) We have isomorphisms

$$\mathrm{H}^n(\mathrm{Cone}(\phi);\mathbb{Z}_2) \underset{\text{Claim 1 (a)}}{\cong} \mathrm{H}^n(\mathrm{Cone}(\overbrace{\Sigma^{n-2}(H)}^{=g});\mathbb{Z}_2) \underset{\text{see the example on page 3224}}{\cong} \mathrm{H}^n(S^n;\mathbb{Z}_2) \cong \mathbb{Z}_2.$$

(4) In Claim 1 (b) we had shown that the group $\mathrm{H}^{n+1}(\mathrm{Cone}(\phi);\mathbb{Z}_2)$ is trivial.

By (1) and (3) the two horizontal maps in the diagram give us an isomorphism of a non-trivial group. On the other hand, by (2) and (4) this isomorphism factors through a trivial group. This is absurd. We have thus obtained a contradiction. ■

Now we turn to the proof of the remaining two statements of Theorem 152.2.

Proof of Theorem 152.2 (2) and (3). The proof for the quaternionic Hopf map $H_{\mathbb{H}}$ and the octonion Hopf map $H_{\mathbb{O}}$ is almost the same as the above proof that deals with the classical Hopf map H. We just need to replace Proposition 151.15 (1) by Proposition 151.15 (2) and (3) and we need to use the following Adem relations:

$$\begin{aligned} \mathrm{Sq}^4 \circ \mathrm{Sq}^4 &= \mathrm{Sq}^6 \circ \mathrm{Sq}^2 + \mathrm{Sq}^7 \circ \mathrm{Sq}^1 \\ \mathrm{Sq}^8 \circ \mathrm{Sq}^8 &= \mathrm{Sq}^{12} \circ \mathrm{Sq}^4 + \mathrm{Sq}^{14} \circ \mathrm{Sq}^2 + \mathrm{Sq}^{15} \circ \mathrm{Sq}^1 . \end{aligned}$$

These relations can be obtained easily by hand from the Adem Relation Theorem 152.1. Alternatively one can use the Adem relations calculator:

`https://math.berkeley.edu/~kruckman/adem/`

As in the proof for the Hopf maps one sees that the initial Steenrod operations on the right hand side all end up in a cohomology group which is zero. Thus one obtains again a contradiction to the hypothesis that the maps $\Sigma^{n-4}(H_{\mathbb{H}} \circ \Sigma^3(H_{\mathbb{H}})) \colon S^{n+6} \to S^n$ and $\Sigma^{n-8}(H_{\mathbb{O}} \circ \Sigma^7(H_{\mathbb{O}})) \colon S^{n+14} \to S^n$ are null-homotopic. ■

152.2. **Decomposing Steenrod operations and the Hopf invariant revisited.** The key to extracting extra mileage out of Steenrod operations is the following innocuous lemma from elementary number theory.

Lemma 152.3. Let $n, k \in \mathbb{N}_0$. If we write $n = \sum_{i \in \mathbb{N}_0} n_i \cdot 2^i$ and $k = \sum_{i \in \mathbb{N}_0} k_i \cdot 2^i$ with $n_i, k_i \in \{0, 1\}$, then

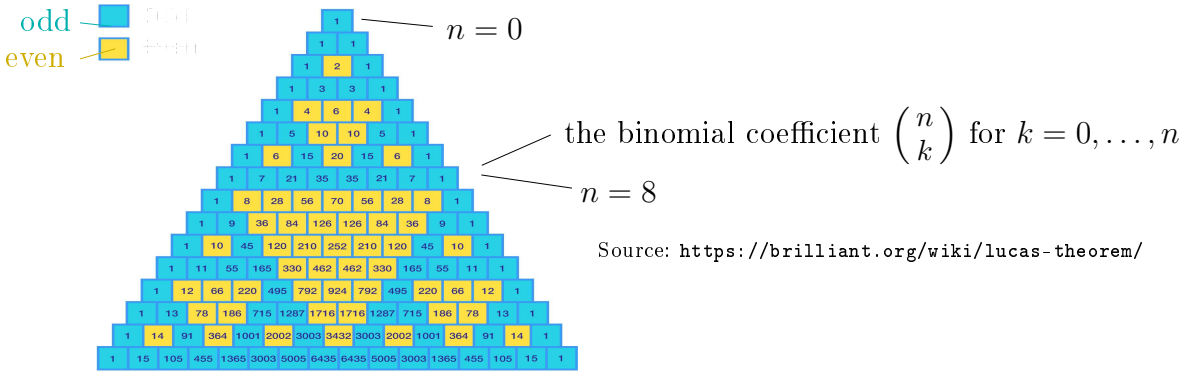
$$\binom{n}{k} = \prod_{i \in \mathbb{N}_0} \binom{n_i}{k_i} \mod 2.$$

Remark.

(1) Note that for $r, s \in \{0, 1\}$ it follows easily from the definition on page 3203 that

$$\binom{s}{r} = \begin{cases} 1, & \text{if } r \leq s, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Let $n \in \mathbb{N}$ and let $k \in \{0, \ldots, d\}$. We define n_i, k_i as above. It follows from Lemma 152.3 and (1) that $\binom{n}{k}$ is odd if and only if for each i we have $k_i \leq n_i$. If we draw the usual Pascal triangle, then by shading odd and even terms in different colors we obtain a Sierpinski triangle. We refer to the figure below for an illustration.[1807]



(3) In fact the obvious analogue of Lemma 152.3 holds with the prime 2 replaced by any prime p. This more general statement is known as Lucas' Theorem, a proof is for example provided in [**Fi1947**].

Proof. Let $n, k \in \mathbb{N}_0$. We write $n = \sum_{i \in \mathbb{N}_0} n_i \cdot 2^i$ and $k = \sum_{i \in \mathbb{N}_0} k_i \cdot 2^i$ with $n_i, k_i \in \{0, 1\}$ In the polynomial ring $\mathbb{Z}_2[x]$ we have the following equalities[1808]

$$\sum_{i=0}^{n} \binom{n}{i} x^i \underset{\text{binomial formula}}{\uparrow} = (1+x)^n = (1+x)^{\sum n_i 2^i} = \prod_{i \in \mathbb{N}_0} (1+x)^{2^i n_i} \underset{\text{since in } \mathbb{Z}_2[x] \text{ we have } (a+b)^2 = a^2 + b^2}{\uparrow} = \prod_{i \in \mathbb{N}_0} (1+x^{2^i})^{n_i} \underset{\text{binomial formula}}{\uparrow} = \prod_{i \in \mathbb{N}_0} \sum_{j_i=0}^{n_i} \binom{n_i}{j_i} x^{j_i \cdot 2^i}.$$

[1807] A more elaborate version of this figure can be found here:

https://upload.wikimedia.org/wikipedia/commons/8/87/Sierpinski_Pascal_triangle.svg

[1808] The argument has some similarities with the proof of the claim on page 2673.

Now we compare the x^k-coefficients, we see that[1809]

$$\binom{n}{k} = x^k\text{-coefficient of } \sum_{i=0}^{n}\binom{n}{i}x^i \underset{\text{by the above equality}}{=} x^k\text{-coefficient of } \prod_{i\in\mathbb{N}_0}\sum_{j_i=0}^{n_i}\binom{n_i}{j_i}x^{j_i\cdot 2^i} \underset{\substack{\text{by the uniqueness of the 2-adic description}\\ \text{of a natural number we only get } x^k\\ \text{if each variable } j_i \text{ equals } k_i}}{=} \prod_{i\in\mathbb{N}_0}\binom{n_i}{k_i}.$$

We have thus proved the desired equality. ■

This lemma allows us to prove the following theorem.

Theorem 152.4. Let $i \in \mathbb{N}_0$. If i is not a power of 2, then Sq^i is decomposable, in the sense that it can be written as the sum of compositions of Steenrod operations of smaller degree.

Example. By the discussion on page 3223 we know that for any $k \in \mathbb{N}_0$ we have

$$\mathrm{Sq}^{2k+1} = \mathrm{Sq}^1 \circ \mathrm{Sq}^{2k} \quad \text{and} \quad \mathrm{Sq}^{4k+2} = \mathrm{Sq}^2 \circ \mathrm{Sq}^{4k} + \mathrm{Sq}^{4k+1} \circ \mathrm{Sq}^1 .$$

In particular we see that

$$\mathrm{Sq}^3 = \mathrm{Sq}^1 \circ \mathrm{Sq}^2, \quad \mathrm{Sq}^5 = \mathrm{Sq}^1 \circ \mathrm{Sq}^4, \quad \mathrm{Sq}^6 = \mathrm{Sq}^2 \circ \mathrm{Sq}^4 + \mathrm{Sq}^5 \circ \mathrm{Sq}^1 \quad \text{and} \quad \mathrm{Sq}^7 = \mathrm{Sq}^1 \circ \mathrm{Sq}^6 .$$

Proof. Suppose that we are given $i \in \mathbb{N}_0$ that is not a power of two. This means that we can write $i = a + b$ where $b = 2^k$ and $a \in \{1, \dots, 2^k - 1\}$. Since $0 < a < 2b$ we obtain an Adem relation for $\mathrm{Sq}^a \circ \mathrm{Sq}^b$ from the Adem Relations Theorem 152.1. This relation can be rewritten as follows:

$$\binom{b-1}{a} \cdot \mathrm{Sq}^{a+b} = \underbrace{\mathrm{Sq}^a \circ \mathrm{Sq}^b + \sum_{j=1}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \cdot \mathrm{Sq}^{a+b-j} \circ \mathrm{Sq}^j}_{\text{compositions of Steenrod operations of degree} < a+b} .$$

It remains to show that $\binom{b-1}{a} = 1 \bmod 2$. We calculate

$$\binom{b-1}{a} \underset{\substack{\text{by Lemma 152.3 where } b-1 = \sum_{i\in\mathbb{N}_0} d_i\cdot 2^i \text{ and } a = \sum_{i\in\mathbb{N}_0} c_i\cdot 2^i \text{ with } c_i, d_i \in \{0,1\}\\ \downarrow}}{=} \prod_{i\in\mathbb{N}_0}\binom{d_i}{c_i} \underset{\uparrow}{=} \prod_{i=0}^{k-1}\binom{1}{c_i}\cdot\prod_{i=k}^{\infty}\binom{0}{0} = 1 \mod 2.$$

since $b-1 = 2^k - 1 = 1 + 2 + \cdots + 2^{k-1}$ we have $d_0 = \cdots = d_{k-1} = 1$ and $d_k = d_{k+1} = \cdots = 0$
furthermore, since $a < 2^k - 1$ we have $c_k = c_{k+1} = \cdots = 0$ ■

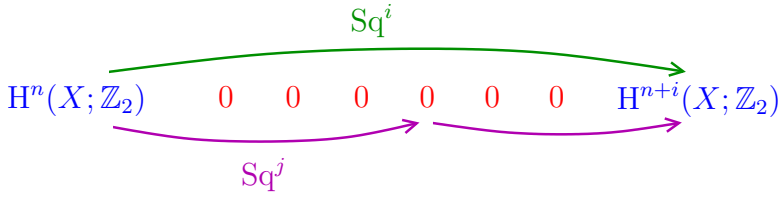
Corollary 152.5. Let X be a topological space and let $n \in \mathbb{N}_0$. We consider the Steenrod operation

$$\mathrm{Sq}^i \colon \mathrm{H}^n(X;\mathbb{Z}_2) \to \mathrm{H}^{n+i}(X;\mathbb{Z}_2).$$

If the cohomology groups inbetween vanish, i.e. if $\mathrm{H}^{n+1}(X;\mathbb{Z}_2) = \ldots = \mathrm{H}^{n+i-1}(X;\mathbb{Z}_2) = 0$, and if i is not a power of 2, then the Steenrod operation is the zero map.

[1809] Note that the equalities also make sense for $k > n$. In this setting one sees easily that all terms are in fact zero.

Proof. Since i is not a power of 2 we know by Theorem 152.4 that Sq^i can be written as the sum of compositions of Steenrod operations of smaller degree. But by our hypothesis for any $j \in \{1, \ldots, i-1\}$ we have $\mathrm{H}^{n+j}(X; \mathbb{Z}_2) = 0$. But this implies in particular that for any such j we have $\mathrm{Sq}^j \colon \mathrm{H}^n(X; \mathbb{Z}_2) \to \mathrm{H}^{n+j}(X; \mathbb{Z}_2) = 0$. ■



In Question 117.19 we asked whether all (reasonable) superalgebras can be realized as cohomology rings. We will now see that this is not the case. Before we can state the corresponding proposition we need to introduce the following definition.

Definition. Let $m \in \mathbb{N}$ be an even number and let $s \in \mathbb{Z}$. We define $R_m(s)$ to be the superalgebra which is given by the groups[1810]

$$R_m(s) = \begin{cases} \mathbb{Z} \cdot 1, & \text{in degree } 0, \\ \mathbb{Z} \cdot x, & \text{in degree } m, \\ \mathbb{Z} \cdot y, & \text{in degree } 2m, \\ 0, & \text{in all other degrees} \end{cases}$$

and where the multiplication is given by $x \cdot x = s \cdot y$. In other words, the multiplication table is given as follows

	1	x	y
1	1	x	y
x	x	$s \cdot y$	0
y	y	0	0.

Example.

(1) For any m the superalgebra $R_m(1)$ is evidently isomorphic to $\mathbb{Z}[x]/(x^3)$ where x is of degree m.

(2) By (1) together with the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 121.7 and Proposition 121.20 we have isomorphisms

$$(\mathrm{H}^*(\mathbb{C}\mathrm{P}^2; \mathbb{Z}), \cup) \cong R_2(1), \ (\mathrm{H}^*(\mathbb{H}\mathrm{P}^2; \mathbb{Z}), \cup) \cong R_4(1) \text{ and } (\mathrm{H}^*(\mathbb{O}\mathrm{P}^2; \mathbb{Z}), \cup) \cong R_8(1).$$

Next we prove the following proposition which gives a negative answer to Questions 117.19 and 121.21.

Proposition 152.6. Let m be an even number and let s be an odd number. If m is not a power of 2, then there is no topological space X such that $(\mathrm{H}^*(X; \mathbb{Z}), \cup)$ is isomorphic to $R_m(s)$.

In the proof of Proposition 152.6 we will need the following lemma.

[1810] It follows from the fact that m is even that this is indeed a superalgebra.

Lemma 152.7. Let $m \in \mathbb{N}$. We denote by $\varphi\colon \mathbb{Z} \to \mathbb{Z}_m$ the canonical map. Given any topological space X there exists a long exact sequence

$$\ldots \to \mathrm{H}^k(X;\mathbb{Z}) \xrightarrow{\cdot m} \mathrm{H}^k(X;\mathbb{Z}) \xrightarrow{\varphi_*} \mathrm{H}^k(X;\mathbb{Z}_m) \xrightarrow{\delta} \mathrm{H}^{k+1}(X;\mathbb{Z}) \to \ldots$$

Proof of Lemma 152.7. We denote by $\psi\colon \mathbb{Z} \to \mathbb{Z}$ the map that is given by multiplication by m. We have the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\psi} \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}_m \to 0.$$

By the discussion on page 3145 we have the corresponding Bockstein sequence in cohomology which looks as follows:

$$\ldots \xrightarrow{\delta} \mathrm{H}^k(X;\mathbb{Z}) \xrightarrow{\psi_*} \mathrm{H}^k(X;\mathbb{Z}) \xrightarrow{\varphi_*} \mathrm{H}^k(X;\mathbb{Z}_2) \xrightarrow{\delta} \mathrm{H}^{k+1}(X;\mathbb{Z}) \to \ldots$$

By the almost, but not entirely trivial Lemma 148.2 we know that ψ_* is indeed multiplication by m. ∎

Proof of Proposition 152.6. Let m be an even number. Furthermore let X be a topological space such that $(\mathrm{H}^*(X;\mathbb{Z}), \cup) = R_m(s)$ for some odd number s. We need to show that m is a power of 2. We denote by $\varphi\colon \mathbb{Z} \to \mathbb{Z}_2$ the canonical map. We consider the following diagram

$$\begin{array}{ccc}
\mathrm{H}^m(X;\mathbb{Z}) & \xrightarrow{x \mapsto x^2 = x \cup x = s \cdot y} & \mathrm{H}^{2m}(X;\mathbb{Z}) \\
\downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\
\mathrm{H}^m(X;\mathbb{Z}_2) & \xrightarrow{z \mapsto z^2 = z \cup z} & \mathrm{H}^{2m}(X;\mathbb{Z}_2) \\
\downarrow & & \downarrow \\
\mathrm{H}^m(X;\mathbb{Z}_2) & \xrightarrow{z \mapsto \mathrm{Sq}^m(z)} & \mathrm{H}^{2m}(X;\mathbb{Z}_2).
\end{array}$$

We make the following clarifications and observations:

(1) It follows from the Cup Product-Naturality Proposition 115.4 (2) that the upper square commutes.
(2) It follows from Lemma 152.7 and our hypothesis on the topological space X that the map $\varphi_*\colon \mathrm{H}^{2m}(X;\mathbb{Z}) \to \mathrm{H}^{2m}(X;\mathbb{Z}_2)$ is an epimorphism and that $\mathrm{H}^{2m}(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$. In particular we see that $\varphi_*(y) \in \mathrm{H}^{2m}(X;\mathbb{Z}_2)$ is a non-trivial element of order 2.
(3) Since s is odd we obtain from (2) that $\varphi_*(s \cdot y) = s \cdot \varphi_*(y)$ is a non-trivial element of $\mathrm{H}^{2m}(X;\mathbb{Z}_2)$.
(4) The lower square commutes by Axiom (2) of Steenrod operations, see the Steenrod Theorem 151.2 (2).
(5) By our hypothesis on X we know that the intermediate cohomology groups vanish, i.e. we have $\mathrm{H}^{m+1}(X;\mathbb{Z}_2) = \cdots = \mathrm{H}^{2m-1}(X;\mathbb{Z}_2) = 0$.

It follows from (3) and (4) that the bottom horizontal map has non-trivial image. But by Corollary 152.5 together with (5) this is only possible if m is indeed a power of 2. ∎

As promised in the title of the section, we now return to the Hopf invariant. Let $f\colon S^{2n-1} \to S^n$ be a map where $n \geq 2$. Recall that on page 2681 we introduced the Hopf invariant $\mathrm{Hopf}(f) \in \mathbb{Z}$. The following lemma translates the Hopf invariant into the language of the above discussion. The lemma can also be viewed as an alternative definition of the Hopf invariant, at least up to sign.

Lemma 152.8. Let $n \geq 2$. If $f\colon S^{2n-1} \to S^n$ is a map, then there exists an isomorphism

$$(\mathrm{H}^*(\mathrm{Cone}(f);\mathbb{Z}),\cup) \;\cong\; R_n(\mathrm{Hopf}(f))$$

of superalgebras.

Proof. This statement follows immediately from Lemma 122.5 and the definition of the Hopf invariant, see page 2681. ■

The following proposition is the summary of our previous discussion of the Hopf invariant.

Proposition 152.9. Let $n \geq 2$.

(1) The Hopf invariant $\mathrm{Hopf}\colon \pi_{2n-1}(S^n) \to \mathbb{Z}$ is a homomorphism.
(2) The Hopf invariant $\mathrm{Hopf}\colon \pi_{2n-1}(S^n) \to \mathbb{Z}$ is zero for odd n.
(3) If $\mathbb{R}^n$ can be equipped with the structure of a division algebra, then there exists a map $f\colon S^{2n-1} \to S^n$ with $\mathrm{Hopf}(f) = 1$.
(4) For $k = 1, 2, 4$ the Hopf maps $H, H_{\mathbb{H}}$ and $H_{\mathbb{O}}$ give elements in $\pi_{4k-1}(S^{2k})$ with Hopf invariant 1.
(5) Given any $k \in \mathbb{N}$ there exists a map $f\colon S^{4k-1} \to S^{2k}$ with $\mathrm{Hopf}(f) = 2$.

Proof. The proof follows from Proposition 122.7, Corollary 122.8, Lemma 122.11 and Propositions 122.13 and 122.12. ■

We recall the following question that we asked a while ago.

Question 122.17. For which $n \in \mathbb{N}_{\geq 2}$ does there exist a map $f\colon S^{2n-1} \to S^n$ with $\mathrm{Hopf}(f) = 1$?

The following theorem is an immediate consequence of Proposition 152.6 and Lemma 152.8.

Theorem 152.10. Let $n \geq 2$. If there exists a map $f\colon S^{2n-1} \to S^n$ with $\mathrm{Hopf}(f) = 1$, then n is a power of 2.

The combination of Proposition 152.9 (3) and Theorem 152.10 gives us an alternative proof of the following theorem.

Theorem 121.23. (Hopf 1940) If $\mathbb{R}^n$ admits the structure of a real division algebra, then n is a power of 2.

It is an amusing mental exercise, left to the reader, to compare the two proofs of Theorem 121.23. The first one relies on the calculation of the cup product of $\mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n$ with $\mathbb{Z}_2$-coefficients and a fact in elementary number theory that is proved on page 2673. The second proof uses Steenrod operations and a different input from elementary number theory, namely Lemma 152.3. Note that in some sense the former proof is more satisfactory since we could provide full details, whereas the latter proof relies on the Adem relations that we did not prove.

For completeness we remark that in 1960 Frank Adams [**Adam1960**] (see also [**AdA1966**], [**May1999a**, Chapter 24.6] and [**Hus1993**, Theorem 15.4.3]) proved the following strengthening of Theorem 152.10. Note that this theorem finally gives a complete answer to Question 122.17.

Theorem 152.11. Let $n \geq 2$. If there exists a map $f\colon S^{2n-1} \to S^n$ with $\mathrm{Hopf}(f) = 1$, then $n = 2, 4$ or $n = 8$.

The combination of Proposition 152.9 (3) and Theorem 152.11 gives us an alternative proof of the following beautiful theorem.

Theorem 121.25. Every finite-dimensional division algebra over $\mathbb{R}$ is of dimension 1,2, 4 or 8.

152.3. **The Steenrod cyclic reduced powers.** The Steenrod operations Sq^i are cohomology operations on cohomology groups with $\mathbb{Z}_2$-coefficients. It turns out that there exist also analogous cohomology operations for cohomology with $\mathbb{Z}_p$-coefficients. More precisely, the following theorem holds:

Theorem 152.12. (Steenrod Theorem) Let p be an odd prime. There exists a family[1811]

$$P^i\colon \mathrm{H}^n(X,A;\mathbb{Z}_p) \;\to\; \mathrm{H}^{n+2i(p-1)}(X,A;\mathbb{Z}_p) \quad \text{with } i\in\mathbb{N}_0 \text{ and } n\in\mathbb{N}_0$$

of additive cohomology operations which satisfies the following axioms:

(1) $P^0 = \mathrm{id}$.
(2) For any $\varphi\in\mathrm{H}^{2i}(X,A;\mathbb{Z}_p)$ we have $P^i(\varphi)=\varphi^p\in\mathrm{H}^{2pi}(X,A;\mathbb{Z}_p)$.
(3) For any $\varphi\in\mathrm{H}^j(X,A;\mathbb{Z}_p)$ with $i>j$ we have $P^i(\varphi)=0$.
(4) Let (X,A,B) be an excisive triad of topological spaces. For any $\varphi\in\mathrm{H}^r(X,A;\mathbb{Z}_p)$ any $\psi\in\mathrm{H}^s(X,B;\mathbb{Z}_p)$ and any $k\in\mathbb{N}_0$ we have[1812]

$$P^k(\varphi\cup\psi) \;=\; \sum_{i=0}^{k} \underbrace{P^i(\varphi)}_{\in\mathrm{H}^{r+2i(p-1)}(X,A;\mathbb{Z}_p)} \cup \underbrace{P^{k-i}(\psi)}_{\in\mathrm{H}^{s+2(k-i)(p-1)}(X,B;\mathbb{Z}_p)} \in \mathrm{H}^{r+s+2k(p-1)}(X,A\cup B;\mathbb{Z}_p).$$

This equality is often called the *Cartan formula*.

Proof. This theorem is proved in [**SE1962**, Chapter VI]. A very different proof is given in [**Hat2002**, Theorem 4.L.16]. ■

The following theorem plays the role of the Adem Relations Theorem 152.1.

Theorem 152.13. (Adem Relations Theorem) Let p be an odd prime.

(1) For any $0<a<pb$ we have the following equality of cohomology operations:

$$P^a\circ P^b \;=\; \sum_{j=0}^{\left\lfloor\frac{a}{p}\right\rfloor} (-1)^{a+j}\cdot\binom{(p-1)(b-j)-1}{a-pj}\cdot P^{a+b-j}\circ P^j.$$

(2) Given $i\in\mathbb{N}_0$ we denote by $\beta\colon \mathrm{H}^i(X;\mathbb{Z}_p)\to\mathrm{H}^{i+1}(X;\mathbb{Z}_p)$ the Bockstein homomorphism that corresponds to the short exact sequence $0\to\mathbb{Z}_p\to\mathbb{Z}_{p^2}\to\mathbb{Z}_p\to 0$. For

[1811] Note that if for argument's sake one plugs $p=2$ into the statements of the Steenrod Theorem 152.12, then one does *not* get the statements of the Steenrod Theorem 151.2 since the degree formulas are different. In other words there is no easy way to combine Theorems 151.2 and 152.12 into a single statement.

[1812] We refer to page 2429 for the definition of an excisive triad. By the Excisive Triad-Proposition 109.12 most "reasonable" triads are in fact excisive. In fact we will only use the special cases that $A=B$, or that at least one of A or B is the empty set. We refer again to page 2536 for the definition of the relative cup product.

any $0 < a \leq pb$ the following mystifying equality holds:

$$P^a \circ \beta \circ P^b = \sum_{j=0}^{\left\lfloor \frac{a}{p} \right\rfloor} (-1)^{a+j} \cdot \binom{(p-1)(b-j)}{a-pj} \cdot \beta \circ P^{a+b-j} \circ P^j - \sum_{j=0}^{\left\lfloor \frac{a-1}{p} \right\rfloor} (-1)^{a+j} \cdot \binom{(p-1)(b-j)-1}{a-pj-1} \cdot P^{a+b-j} \circ \beta \circ P^j.$$

These equalities are called the *Adem relations*.

Example. Let us consider the case of the prime $p = 3$. In this case the corresponding reduced Steenrod operations are additive cohomology operations of the form

$$P^i \colon \mathrm{H}^n(X, A; \mathbb{Z}_3) \to \mathrm{H}^{n+4i}(X, A; \mathbb{Z}_3).$$

For $a = 1$ and $b = 3$ we obtain the Adem relation

$$P^1 \circ P^3 \underset{\text{Adem Relations Theorem 152.13 (1)}}{=} \sum_{j=0}^{\left\lfloor \frac{1}{3} \right\rfloor} (-1)^{1+j} \binom{2 \cdot (3-j) - 1}{1 - 3 \cdot j} \cdot P^{4-j} \circ P^j \underset{\text{only one summand, namely for } j=0}{=} (-1) \cdot \binom{5}{1} \cdot P^4 = -5 \cdot P^4 \underset{\text{we work in } \mathbb{Z}_3}{=} P^4.$$

The Steenrod cyclic reduced powers can, not surprisingly, be used to give more information on homotopy groups of spheres. We refer to [**SE1962**, Chapter VI.5] and [**Hat2002**, Example 4.L.6] for details.

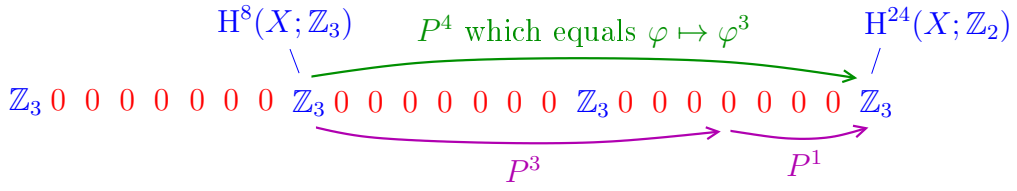
Instead of providing more results on homotopy groups of spheres we want to give a different application of the Steenrod cyclic reduced powers, namely we want to prove the following proposition which gives a negative answer to Question 121.22.

Proposition 152.14. There is no topological space X such that $\mathrm{H}^*(X; \mathbb{Z}_3)$ is isomorphic to $\mathbb{Z}_3[\varphi]/(\varphi^4)$ with $\varphi \in \mathrm{H}^8(X; \mathbb{Z}_3)$.

Proof. We provide a proof by contradiction. So suppose there exists a topological space X such that $\mathrm{H}^*(X; \mathbb{Z}_3)$ is isomorphic to $\mathbb{Z}_3[\varphi]/(\varphi^4)$ with $\varphi \in \mathrm{H}^8(X; \mathbb{Z}_3)$. We see that

$$0 \underset{\substack{\text{since we assume that} \\ \mathrm{H}^*(X;\mathbb{Z}_3) \cong \mathbb{Z}_3[\varphi]/(\varphi^4)}}{\neq} \varphi^3 \overset{\text{Steenrod Theorem 152.12 (2)}}{=} P^4(\varphi) \underset{\text{since } P^4 = P^1 \circ P^3}{=} (P^1 \circ P^3)(\varphi) = P^1(\overbrace{P^3(\varphi)}^{\in \mathrm{H}^{8+3\cdot 4}(X;\mathbb{Z}_3)}) \underset{\text{since } \mathrm{H}^{20}(X;\mathbb{Z}_3) = 0}{=} 0.$$

We have thus obtained a contradiction. ∎



Remark. In Exercise 152.2 we will use Proposition 152.14 to give a negative answer to Question 92.11. So with most reasonable interpretations it is fair to say that there is no smooth manifold that deserves the name $\mathbb{O}\mathrm{P}^n$ with $n \geq 3$.

Exercises for Chapter 152.

Exercise 152.1. Let X be a topological space and let $x \in \mathrm{H}^1(X;\mathbb{Z}_2)$ be a cohomology class of degree *one*. Show that for any $k \in \mathbb{N}$ and any $i \in \mathbb{N}$ we have

$$\mathrm{Sq}^i(x^{2^k}) \;=\; \begin{cases} x^{2^k}, & \text{if } i = 0, \\ x^{2^{k+1}}, & \text{if } i = 2^k, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 152.2. Let $n \geq 3$. Show that there is no closed orientable $8n$-dimensional smooth manifold that admits a CW-structure with precisely one cell in dimensions $0, 8, 16, \ldots, 8n$.
Remark. This exercise gives a negative answer to Question 92.11.

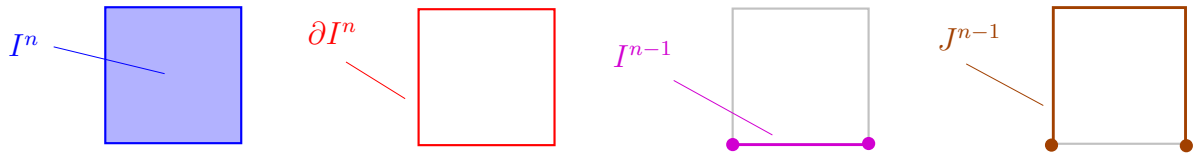
Part XVIII

Homotopy Theory and Eilenberg-Maclane Spaces

153.1. **The definition of relative homotopy groups.** While studying homology groups we saw at some point that it was convenient to introduce the relative homology groups $\mathrm{H}_k(X, A)$ corresponding to a pair of topological spaces (X, A). In our latest attempt to understand homotopy groups we will now, perhaps not surprisingly, introduce relative homotopy groups. We start out with introducing the following notation.

Notation. Throughout this chapter we write $I = [0, 1]$ for the unit interval. Furthermore, given $n \in \mathbb{N}$ we use the following notation:

(1) we denote by $I^n \subset \mathbb{R}^n$ the cube in $\mathbb{R}^n$,
(2) we denote by ∂I^n the boundary of I^n viewed as a subset of $\mathbb{R}^n$,
(3) by a slight abuse of notation we make the identification $I^{n-1} = I^{n-1} \times \{0\}$,
(4) we write $J^{n-1} := \overline{\partial I^n \setminus I^{n-1}}$.

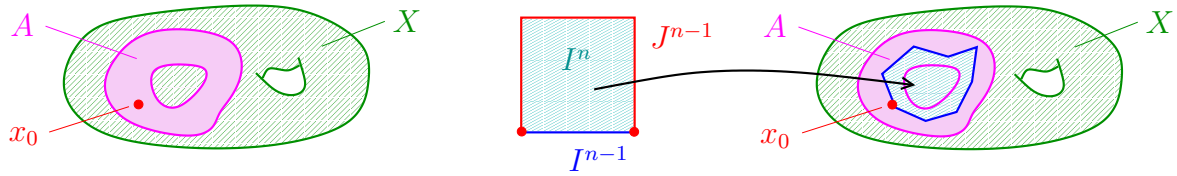


The following definition is a variation on the notion of a pointed topological space that we introduced on page 1061.

Definition. A pointed pair of topological spaces is a triple (X, A, x_0), where X is a topological space, A is a subset of X and x_0 is a point in A. A morphism between pointed pairs of topological spaces (X, A, x_0) and (Y, B, y_0) is a map $f \colon X \to Y$ with $f(A) \subset B$ and $f(x_0) = y_0$. We refer to the corresponding category as the category of pointed pairs of topological spaces.

Definition. Let (X, A, x_0) be a pointed pair of topological spaces. Given $n \geq 1$ we define the n-th relative homotopy group of (X, A, x_0) to be[1813]

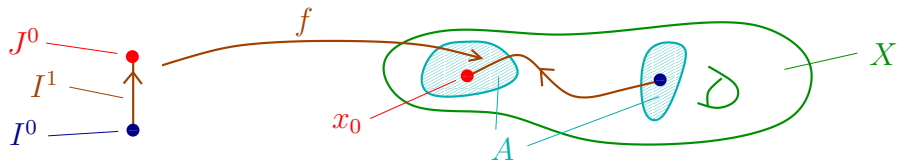
$$\pi_n(X, A, x_0) \ := \ \text{homotopy classes of maps } (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\}).$$

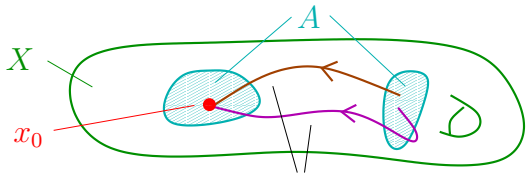


Examples.

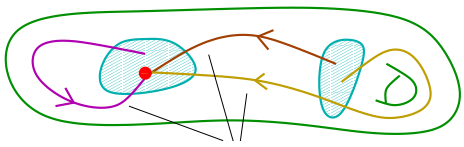
(1) Let (X, A, x_0) be a pointed pair of topological spaces. We want to get a better understanding of elements in $\pi_1(X, A, x_0)$. For $n = 1$ we have $I^1 = [0, 1]$, $\partial I^1 = \{0, 1\}$ and $J^0 = \{1\}$. A map $(I^1, \partial I^1, J^0) \to (X, A, \{x_0\})$ is thus the same as a path in X that starts at a point in A and that ends in x_0. We refer to the figures below for an illustration.

[1813] On page 439 we defined the notion of maps between pairs of topological spaces and we defined the notion of a homotopy between such maps. These notions extend in an obvious way to triples of topological spaces.





the maps define the same element in $\pi_1(X, A, x_0)$



the maps define (presumably) three different elements in $\pi_1(X, A, x_0)$

(2) Let (X, x_0) be a pointed topological space. For any $n \geq 1$ the map

$$\begin{array}{rcl} \pi_n(X, x_0) & \to & \pi_n(X, \{x_0\}, x_0) \\ [f\colon (I^n, \partial I^n) \to (X, \{x_0\})] & \mapsto & [f\colon (I^n, \partial I^n, J^{n-1}) \to (X, \{x_0\}, \{x_0\})] \end{array}$$

defines, basically by definition, a natural bijection. In the following we will use this natural bijection to make the identification $\pi_n(X, x_0) = \pi_n(X, \{x_0\}, x_0)$.

(3) Let X be a topological space, let A be a subset of X and let $x_0 \in A$ be a point. For any $n \geq 1$ the map

$$\begin{array}{rcl} \pi_n(X, x_0) & \to & \pi_n(X, A, x_0) \\ [f\colon (I^n, \partial I^n) \to (X, \{x_0\})] & \mapsto & [f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})] \end{array}$$

is a natural map from absolute homotopy groups to relative homotopy groups.

In the previous discussion we already referred to $\pi_n(X, A, x_0)$ as a "relative homotopy group". This was slightly premature, but now we will actually deliver the goods and provide a group structure.

Definition. Let (X, A, x_0) be a pointed pair of topological spaces and let $n \geq 2$.[1814] Given two maps $f, g\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})$ we define[1815]

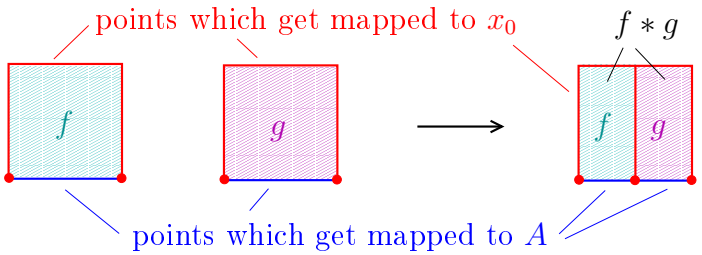
$$\begin{array}{rcl} f * g\colon I^n & \to & X \\ (t_1, \ldots, t_n) & \mapsto & \begin{cases} f(2t_1, t_2, \ldots, t_n), & \text{if } t_1 \in [0, \frac{1}{2}], \\ g(2t_1 - 1, t_2, \ldots, t_n) & \text{if } t_1 \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

Note that $f * g$ is again a map $(I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})$ of triples. The definition of $f * g$ is illustrated in the figure below.

The following proposition finally gives the justification for the name *relative homotopy group*.

[1814] What, if anything, goes wrong for $n = 1$?

[1815] This map is continuous by the Pasting Proposition 2.6 (2).



Proposition 153.1. (Relative Homotopy Group Proposition) Let (X, A, x_0) be a pointed pair of topological spaces.

(1) Let $n \geq 2$. The map

$$\begin{array}{rcl} \pi_n(X,A,x_0) \times \pi_n(X,A,x_0) & \to & \pi_n(X,A,x_0) \\ ([f],[g]) & \mapsto & [f*g] \end{array}$$

is well-defined and it defines a group structure on $\pi_n(X, A, x_0)$. Furthermore the following two statements hold:

(a) the trivial element in $\pi_n(X, A, x_0)$ is represented by the constant map $I^n \to X$ that sends every point in I^n to x_0,

(b) the inverse of the relative homotopy class $[f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})]$ is represented by the map

$$\begin{array}{rcl} (I^n, \partial I^n, J^{n-1}) & \to & (X, A, \{x_0\}) \\ (t_1, t_2, \ldots, t_n) & \mapsto & (1 - t_1, t_2, \ldots, t_n). \end{array}$$

(2) For $n \geq 2$ the above natural bijection $\pi_n(X, x_0) \xrightarrow{=} \pi_n(X, \{x_0\}, x_0)$ is in fact a group isomorphism.

(3) For $n \geq 2$ the above natural map $\pi_n(X, x_0) \to \pi_n(X, A, x_0)$ is a group homomorphism.

(4) For $n \geq 3$ the group $\pi_n(X, A, x_0)$ is abelian.

(5) Let $n \in \mathbb{N}$. Given a morphism $f\colon (X, A, x_0) \to (Y, B, y_0)$ of pointed pairs of topological spaces the map

$$\begin{array}{rcl} f_*\colon \pi_n(X,A,x_0) & \to & \pi_n(Y,B,y_0) \\ [\varphi] & \mapsto & [f \circ \varphi] \end{array}$$

is well-defined. Furthermore the following four statements hold:

(a) For $n \geq 2$ the induced map f_* is a group homomorphism.

(b) The maps f_* define a functor from the category of pointed pairs of topological spaces to the category of sets (respectively groups for $n \geq 2$).

(c) Let $n \geq 1$ and let $f, g\colon (X, A) \to (Y, B)$ be two maps between pairs of topological spaces. If $f_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, f(x_0))$ is a bijection for all $x_0 \in A$ and if g is homotopic[1816] to f, then $g_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, g(x_0))$ is also a bijection for all $x_0 \in A$.

(d) Let $f\colon (X, A) \to (Y, B)$ be a map between pairs of topological spaces and let $x_0 \in A$. If f is a homotopy equivalence, then the map

$$f_*\colon \pi_n(X,A,x_0) \to \pi_n(Y,B,f(x_0))$$

is a bijection (in particular it is an isomorphism if $n \geq 2$).

Sketch of proof.

(1) The proof of the first statement is quite similar to the proof of Proposition 71.1. We leave it to the reader to fill in the details.
(3) This statement requires just slightly more thought than one might think. More precisely, given two maps f and g from $(I^n, \partial I^n, J^{n-1})$ to $(X, \{x_0\}, \{x_0\})$, the "product maps" $f * g$ considered on page 1561 for $\pi_n(X, \{x_0\})$ (and illustrated in the figure on page 1561) and in the above proposition for $\pi_n(X, A, x_0)$ are not identical. Nonetheless, using Lemma 71.2 or the Addition-in-π_n-Proposition 85.11 it is straightforward to show that they are homotopic. This observation shows that the above identification of sets is in fact an identification of groups.
(2) We already know that the map is a bijection. Thus it suffices to show that it is a group homomorphism. But this follows from (3) applied to $A = \{x_0\}$.
(4) The proof of this statement is an entertaining variation on the proof of the Homotopy Group-Abelian Proposition 71.3. The proof is the content of Exercise 153.2.
(5) Statements (a) and (b) follow basically immediately from the definitions. Statement (c) and (d) are proved in a similar fashion to the Homotopy-π_n-Proposition 71.7. ■

Remark.

(1) We will see shortly that the relative homotopy group $\pi_2(X, A, x_0)$ is in general not abelian.
(2) For want of a better idea, given a pointed pair of topological spaces (X, A, x_0) we refer to $\pi_1(X, A, x_0)$ also as a *relative homotopy group* even though strictly speaking we did not define a group structure on $\pi_1(X, A, x_0)$.

In light of the Relative Homotopy Group Proposition 153.1 we adopt the following convention:

Convention. For $n \geq 3$ we usually use additive notation for the group structure on the relative homotopy groups $\pi_n(X, A, x_0)$.

Let X be a topological space and let $A \subset B \subset X$ be two subsets. In the Topological LES-Proposition 74.13 we saw that this data gives rise to a long exact sequence of homology groups. Shortly we will see that a similar statement holds for relative homotopy groups. We did not define a group structure on $\pi_1(X, A, x_0)$ and we did not define a group structure on $\pi_0(X, x_0)$. (In fact we will see in Exercise 153.3 that there is no natural way to turn these sets into groups.) Nonetheless we would like to include these objects at the lower end of the long exact sequence of (relative) homotopy groups. In fact, as it turns out, $\pi_1(X, A, x_0)$ and $\pi_0(X, x_0)$ have ever so slightly more structure than just the structure of a set.

Definition.

(1) A pointed set is a pair (S, s_0) where S is a set and $s_0 \in S$ is a point. We call s_0 the trivial element in S.

[1816] Here we mean of course, "homotopic as a map between pairs of topological spaces".

(2) A morphism from a pointed set (S, s_0) to a pointed set (T, t_0) is a map $f\colon S \to T$ with $f(s_0) = t_0$. Furthermore we adopt the following language:
   (a) Given a morphism $f\colon (S, s_0) \to (T, t_0)$ we refer to $\ker(f) := f^{-1}(t_0)$ as the kernel of f.
   (b) We say a morphism $f\colon (S, s_0) \to (T, t_0)$ is trivial if $\ker(f) = S$, put differently, if all elements in S get sent to t_0.
   (c) We say a morphism $f\colon (S, s_0) \to (T, t_0)$ is a monomorphism if $\ker(f) = \{s_0\}$.
   (d) We say a morphism $f\colon (S, s_0) \to (T, t_0)$ is an epimorphism if f is surjective.

(3) We say that a sequence

$$\ldots \xrightarrow{f_{n+2}} (S_{n+1}, s_{n+1}) \xrightarrow{f_{n+1}} (S_n, s_n) \xrightarrow{f_n} (S_{n-1}, s_{n-1}) \xrightarrow{f_{n-1}} \ldots$$

of morphisms of pointed sets is exact if for every $n \in \mathbb{N}_0$ we have the usual equality $\ker(f_n) = \operatorname{im}(f_{n+1})$.

Examples.

(1) Any group G gives rise to the pointed set $(G, \{e\})$ where $e \in G$ evidently denotes the trivial element. We will thus always view a group as a pointed set. Any group homomorphism is also a morphism of the corresponding pointed sets. Furthermore an exact sequence of homomorphisms of groups is also an exact sequence of pointed sets.

(2) If G is a group and H is a subgroup, then the set of right cosets G/H is a pointed set with trivial element given by $H \in G/H$. The natural projection $G \to G/H$ is a morphism of pointed sets.

Lemma 153.2. Let $\varphi\colon G \to H$ be a group homomorphism. Furthermore let (S, s_0) be a pointed set which is equipped with an H-action. Suppose that we have an exact sequence

$$\{e\} \to G \xrightarrow{\varphi} H \xrightarrow{\psi} (S, s_0) \to \{e\}$$

where ψ is a morphism of pointed sets that preserves the H-action. Then $\varphi(G)$ is a subgroup of H whose index equals the cardinality of the set S.

Proof. We will prove the lemma in Exercise 153.4. ■

Example. The previous lemma might sound rather artificial. But we have seen it before in group theory: suppose that H is a group and G is a subgroup. As above we view the set of right cosets G/H as a pointed set. Left-multiplication by G defines an action on G/H. Evidently the projection $G \to G/H$ preserves the right-action and we get a short exact sequence $\{e\} \to G \to H \to G/H \to \{e\}$ as in Lemma 153.2.

We continue with pointed sets that arise in topology.

Examples.

(1) Let (X, x_0) be a pointed topological space. Recall that on page 1560 we defined $\pi_0(X, x_0)$ to be the set of homotopy classes of maps from the point I^0 to X. In other words, $\pi_0(X, x_0)$ is the set of path-components of X. We view $\pi_0(X, x_0)$ as a pointed set where the trivial element is given by the map that sends the unique point in I^0 to x_0. A map $f\colon X \to Y$ between topological spaces defines a morphism

$f_*\colon \pi_0(X, x_0) \to \pi_0(Y, f(x_0))$ of pointed sets in the obvious way. Sometimes, by an unfortunate abuse of language, we refer to $\pi_0(X, x_0)$ as a homotopy group of X.

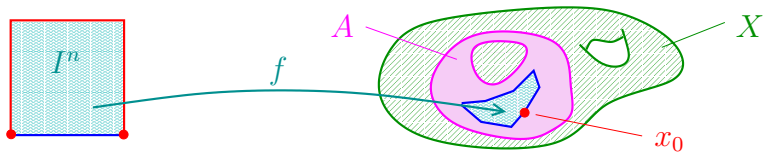
(2) Similarly to the above, given a pointed pair of topological spaces (X, A, x_0) we view $\pi_1(X, A, x_0)$ as pointed set where the trivial element is once again represented by the constant path $I \to X$ given by sending all points to x_0.

We conclude this section with the following elementary lemma.

Lemma 153.3. Let (X, A, x_0) be a pointed pair of topological spaces and let $n \geq 1$. Let $f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})$ be a map. If $f(I^n) \subset A$, then f represents the trivial element in $\pi_n(X, A, x_0)$.[1817][1818]

Proof. The lemma will be proved in Exercise 153.6. ■

Since it is much easier to convince oneself of a statement if one sees an explicit example, we show such a map with $f(I^n) \subset A$ in the figure below.



153.2. **The long exact homotopy sequence of a pair.** In the Topological LES-Proposition 74.13 we saw that we can associate to a pair (X, A) of topological spaces a long exact sequence of (relative) homology groups. The goal of this section is to do the same for homotopy groups. More precisely we will associate to a pointed pair (X, A, x_0) of topological spaces a long exact sequence of (relative) homotopy groups.

Proposition 153.4. (Homotopy Groups-LES Proposition) Let (X, A, x_0) be a pointed pair of topological spaces.

(1) For any $n \geq 1$ the map

$$\begin{array}{rcl} \partial_n\colon \pi_n(X, A, x_0) & \to & \pi_{n-1}(A, x_0) \\ [f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})] & \mapsto & \left[\begin{array}{rcl} (I^{n-1}, \partial I^{n-1}) & \to & (A, \{x_0\}) \\ (x_1, \ldots, x_{n-1}) & \mapsto & f(x_1, \ldots, x_{n-1}, 0) \end{array}\right] \end{array}$$

is well-defined. (See the figure below.) We refer to it as the *connecting morphism*. It has the following two properties:

(a) The connecting morphism is a morphism of pointed sets and it is a group homomorphism for $n \geq 2$.

(b) The connecting morphism is a natural transformation from the covariant functor $\pi_n(X, A, x_0)$ to the covariant functor $\pi_{n-1}(A, x_0)$. In everyday language this just simply means that for a map $f\colon (X, A, x_0) \to (Y, B, y_0)$ between pointed pairs

[1817] Put differently, as for relative homology groups, all objects that "land in A" are in fact trivial in the relative homotopy group.

[1818] Note that by the above discussion of pointed sets the statement of Lemma 153.3 is also meaningful for $n = 1$.

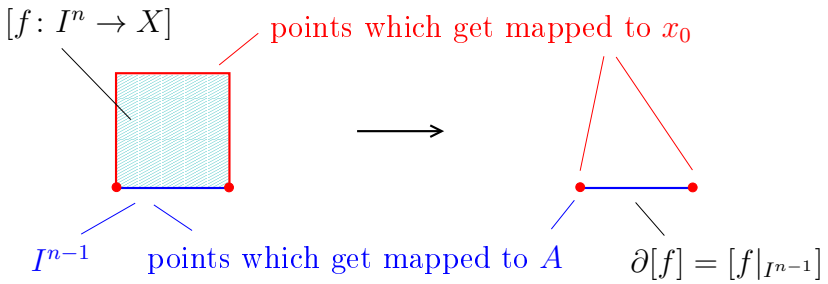
of topological spaces the following diagram commutes:

$$\begin{array}{ccc} \pi_n(X, A, x_0) & \xrightarrow{\partial} & \pi_{n-1}(A, x_0) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \pi_n(Y, B, y_0) & \xrightarrow{\partial} & \pi_{n-1}(B, y_0). \end{array}$$

(2) The following sequence is exact

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \to \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0) \to \cdots \to \pi_0(X, x_0).$$

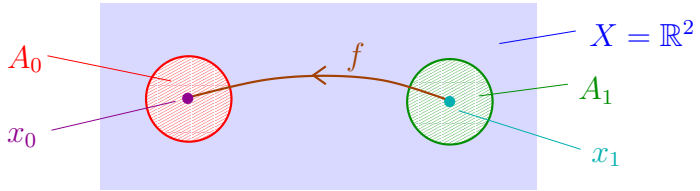
Here $i\colon A \to X$ is the inclusion map and the map $\pi_n(X, x_0) \to \pi_n(X, A, x_0)$ is the natural map introduced on page 3239.



Examples.

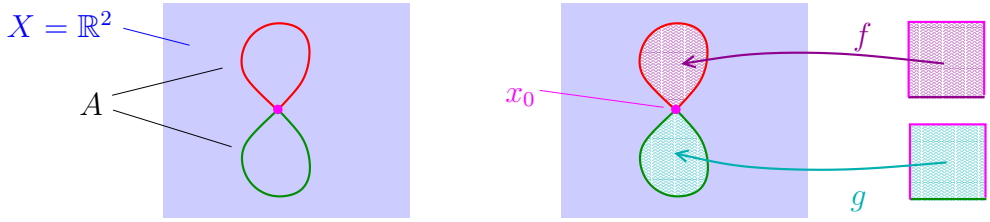
(1) We consider $X = \mathbb{R}^2$ and we consider $A = A_0 \sqcup A_1$ which is the union of disjoint closed disks A_0 and A_1 in $X = \mathbb{R}^2$. We denote by $x_0 \in A_0$ and $x_1 \in A_1$ the centers of the two disks. Furthermore we denote by $f\colon I = [0, 1] \to X$ a map with $f(0) = x_1$ and $f(1) = x_0$. (We refer to the figure below for an illustration.) Finally, for any $n \in \mathbb{N}_0$ and any $i \in \{0, 1\}$ we denote by $c_{x_i}\colon I^n \to A_i$ the constant map that sends all points to x_i. It follows from the Deformation Retract-Homotopy Equivalence Lemma 15.5 together with the π_0-Homotopy Equivalence Lemma 15.3 and Propositions 50.3 (2) and 71.7 (2) that all the homotopy groups of A and X are generated by $[c_{x_0}]$ except for $\pi_0(A, x_0)$ that equals the set $\{[c_{x_0}], [c_{x_1}]\}$.

As discussed on page 3238, f defines an element in $\pi_1(X, A, x_0)$. Using the definitions one sees almost immediately that $\partial[f] = [c_{x_1}] \in \pi_0(A, x_0)$. It follows easily from this data and the Homotopy Groups-LES Proposition 153.4 (2) that $\pi_1(X, A, x_0)$ consists of precisely two elements, namely $[c_{x_0}]$ and $[f]$.



(2) Let $X = \mathbb{R}^2$. We denote by A the wedge of two circles, viewed as a subset of $X = \mathbb{R}^2$. Furthermore we denote by x_0 the wedge point. It follows from the Homotopy Groups-LES Proposition 153.4 (2) that $\pi_2(X, A, x_0) \xrightarrow{\cong} \pi_1(A, x_0)$. In Lemma 54.6 we saw that $\pi_1(A, x_0)$ is non-abelian, thus $\pi_2(X, A, x_0)$ is also non-abelian.

In fact we can identify two elements in $\pi_2(X, A, x_0)$ that do not commute. We consider the two maps $f, g\colon (I^2, \partial I^2, J^1) \to (X, A, \{x_0\})$ that are illustrated in the figure below. It follows easily from the definitions that $\partial[f], \partial[g]$ are the standard generators of the free group[1819] $\pi_1(S^1 \vee S^1, x_0)$. Since these do not commute we see that $[f]$ and $[g]$ do not commute either.



Proof. We start out with the proof of Statement (1). In fact this statement follows mostly easily from the definitions. One only needs to think for a minute about the question why the connecting homomorphism is in fact a homomorphism.

We turn to the proof of Statement (2). In fact we prove a slight variation on the second statement. More precisely, let X be a topological space, let A be a subset of X and let B be a subset of A. Finally let $x_0 \in B$. For any $n \geq 2$ the map

$$\begin{array}{rcl} \partial_n\colon \pi_n(X, A, x_0) & \to & \pi_{n-1}(A, B, x_0) \\ [f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})] & \to & \left[\begin{array}{rcl} (I^{n-1}, \partial I^{n-1}, J^{n-2}) & \to & (A, B, \{x_0\}) \\ (x_1, \ldots, x_{n-1}) & \mapsto & f(x_1, \ldots, x_{n-1}, 0) \end{array}\right] \end{array}$$

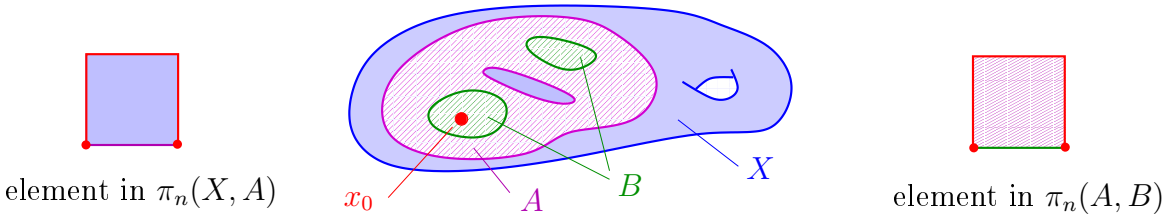
is easily seen to be a well-defined morphism of pointed sets. Our goal is to prove the following claim.

Claim. The following sequence is exact[1820]

$$\ldots \to \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, B, x_0) \to \ldots \to \pi_1(X, A, x_0).$$

where $i\colon (A, B) \to (X, B)$ and $j\colon (X, B) \to (X, A)$ are the obvious morphisms of pointed pairs of topological spaces.

Note that this claim, applied to $B = \{x_0\}$ together with the Relative Homotopy Group Proposition 153.1 (2) almost implies that the sequence of the proposition is exact. The only problem is that the sequence of the claim stops slightly earlier, more precisely it stops at the subindex one instead of subindex zero. Thus the only piece missing is the exactness of the sequence of the proposition at the last three pointed sets. We leave the verification of exactness at these three spots as an elementary exercise to the conscientious reader.



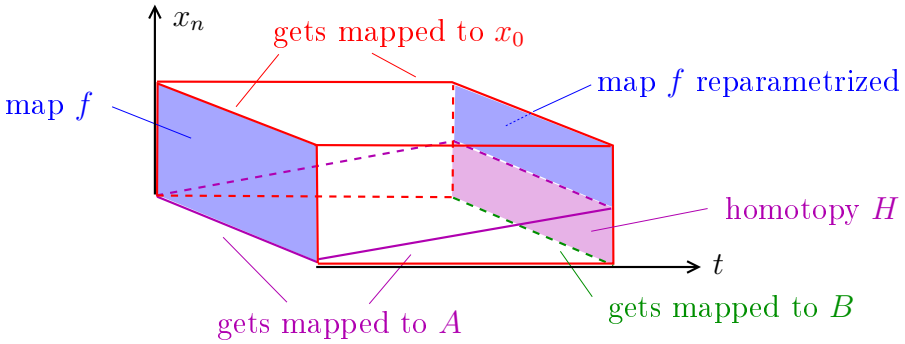
[1819] We refer to page 1173 for the calculation of the fundamental group of $S^1 \vee S^1$.

[1820] This sequence plays of course the role of the long exact sequence in homology groups of a triple of topological spaces, see the Topological LES-Proposition 74.13.

Proof. Not surprisingly the proof of the claim breaks up into three separate proofs, namely we need to verify exactness at the three different groups.
Exactness at $\pi_n(X, B, x_0)$. It follows easily from Lemma 153.3 that $j_* \circ i_* = (j \circ i)_*$ is the trivial morphism. We leave it to the reader to verify that $\ker(j_*) \subset \operatorname{im}(i_*)$.
Exactness at $\pi_n(X, A, x_0)$. It follows again easily from Lemma 153.3 and the definition of the connecting homomorphism ∂ that $\partial \circ j_*$ is the trivial morphism. Now suppose we are given $f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})$ such that the restriction of f to $I^{n-1} \times \{0\}$ represents the trivial element in $\pi_{n-1}(A, B, x_0)$. This means that there exists a homotopy $H\colon I^{n-1} \times I \to A$ (with all the required restrictions) from $f|_{I^n \times \{0\}}$ to the constant map. The map

$$\begin{array}{rcl} I^n \times [0,1] & \to & X \\ (x_1, \ldots, x_n, t) & \mapsto & \begin{cases} f\big(x_1, \ldots, x_{n-1}, \frac{t}{2} \cdot x_n + (1 - x_n)\big), & \text{if } x_n \in \left[\frac{t}{2}, 1\right] \\ H\big(x_1, \ldots, x_{n-1}, 2x_n - t\big), & \text{if } x_n \in \left[0, \frac{t}{2}\right] \end{cases} \end{array}$$

is continuous by the Homotopy Stacking Lemma 14.3 and it defines a homotopy (again with all the desired extra properties) from f to a map that represents an element in $\pi_n(X, B, x_0)$. The definition of the homotopy is sketched, to the best of my abilities, in the figure below.



Exactness at $\pi_n(A, B, x_0)$. First note that $i_* \circ \partial$ is the trivial morphism. Indeed, if $f\colon (I^{n+1}, \partial I^{n+1}, J^n) \to (X, A, \{x_0\})$ is a map, then $f\colon (I^n \times [0,1], \partial I^n \times [0,1]) \to (X, x_0)$ defines a homotopy from $\partial f = f|_{I^n \times \{0\}}$ to the constant map c_{x_0}. Thus $(i_* \circ \partial)[f] = [i_* \circ \partial f]$ is the trivial element in $\pi_n(X, B, x_0)$.

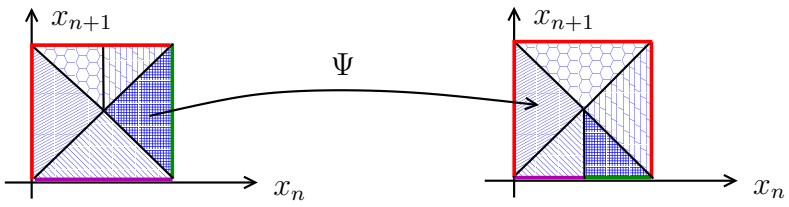
Now suppose that we are given a map $f\colon (I^n, \partial I^n, J^{n-1}) \to (A, B)$ such that $j \circ f$ represents the trivial element in $\pi_n(X, B, x_0)$. This means that there exists a homotopy H (with all the obvious conditions) from $j \circ f$ to the constant map c_{x_0}. We define

$$\begin{array}{rcl} g\colon I^n & \to & B \\ (x_1, \ldots, x_n) & \mapsto & H(x_1, \ldots, x_{n-1}, 0, x_n). \end{array}$$

We pick a homeomorphism $\Psi\colon I^2 \to I^2$ with the properties as illustrated in the figure below.[1821] (Here it is psychologically convenient to think of I^2 as the square in the plane with x_n and x_{n+1}-coordinates.) It follows immediately from the definitions that the map

$$\begin{array}{rcl} I^{n+1} & \to & X \\ (x_1, \ldots, x_n, x_{n+1}) & \to & H(x_1, \ldots, x_{n-1}, \Psi(x_n, x_{n+1})) \end{array}$$

[1821] The triangles in the figure below indicate how one could write down an explicit homeomorphism.



defines an element in $\pi_{n+1}(X, A, x_0)$ such that the connecting morphism sends it to the element in $\pi_n(A, B, x_0)$ represented by the map[1822]

$$\begin{array}{rcl} f * g \colon I^n & \to & X \\ (t_1, \ldots, t_n) & \mapsto & \begin{cases} f(t_1, \ldots, t_{n-1}, 2t_n), & \text{if } t_n \in [0, \frac{1}{2}], \\ g(t_1, \ldots, t_{n-1}, 2t_n - 1), & \text{if } t_n \in [\frac{1}{2}, 1] \end{cases} \end{array}$$

It remains to show that $f * g$ and f represent the same element in $\pi_n(A, B, x_0)$. This can be done in a fashion similar to the above proof of exactness. More precisely, a homotopy between f and $f * g$, again with all the desired properties, is given by

$$\begin{array}{rcl} I^n \times [0,1] & \to & X \\ (x_1, \ldots, x_n, t) & \mapsto & \begin{cases} f\big(x_1, \ldots, x_{n-1}, \frac{t}{2} \cdot x_n + (1 - x_n)\big), & \text{if } x_n \in \big[\frac{t}{2}, 1\big] \\ H\big(x_1, \ldots, x_{n-1}, 2x_n - t\big), & \text{if } x_n \in \big[0, \frac{t}{2}\big] \end{cases} \end{array}$$

■

153.3. **Paths acting on relative homotopy groups.** Let (X, x_0) be a pointed topological space. In Section 86.3 we saw that the fundamental group $\pi_1(X, x_0)$ of X acts on any homotopy group $\pi_n(X, x_0)$. In this section we will discuss an analogous construction for relative homotopy groups. In particular, given a pointed pair (X, A, x_0) of topological spaces we will see that the fundamental group $\pi_1(A, x_0)$ of A (let us stress that here we write A instead of X) acts on any relative homotopy group $\pi_n(X, A, x_0)$.
The following definition is quite similar to the definition on page 1869.

Definition. Let X be a topological space, let A be a subset of X and let $n \in \mathbb{N}_{\geq 2}$.[1823] Furthermore let $\gamma \colon [0,1] \to A$ be a path in the subset A[1824] from a point x_0 to a point x_1. Finally let

$$f \colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_1\})$$

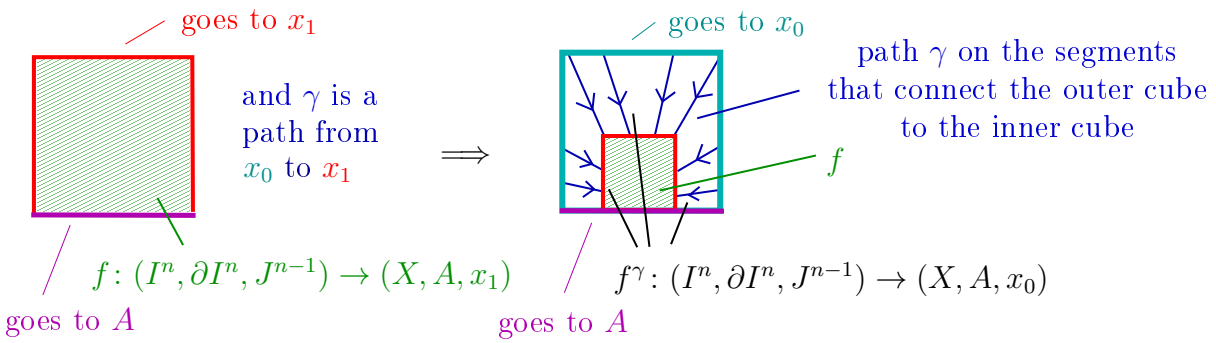
be a map. We define

$$f^\gamma \colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})$$

to be the map that is illustrated in the figure below. For our particularly pedantic readers we now also give a super-rigorous definition of f^γ: First of all, given $t \in [0,1]$ we define $\mathrm{C}_t = \big[\frac{t}{4}, \frac{4-t}{4}\big]^{n-1} \times [0, 1 - \frac{t}{2}]$. Now we define

$$\begin{array}{rcl} f^\gamma \colon (I^n, \partial I^n, J^{n-1}) & \to & (X, A, x_0) \\ (x_1, \ldots, x_n) & \to & \begin{cases} f\big(2x_1 - \frac{1}{2}, \ldots, 2x_n\big), & \text{if } (x_1, \ldots, x_n) \in \mathrm{C}_1 = \big[\frac{1}{4}, \frac{3}{4}\big]^{n-1} \times [0, \frac{1}{2}], \\ \gamma(t), & \text{if } (x_1, \ldots, x_n) \in \partial \mathrm{C}_t \text{ for } t \in [0,1]. \end{cases} \end{array}$$

[1822] Strictly speaking this definition of $f * g$ differs from the definition on page 3239 since we use a different coordinate to perform the product. But it is convenient to use the same notation.



The following proposition can be viewed as the "relative analogue" of the Change-of-Base Point Proposition 71.5.

Proposition 153.5. (Change-of-Base Point Proposition) Let X be a topological space, let A be a subset of X and let $n \in \mathbb{N}_{\geq 2}$.

(1) If $\gamma\colon [0,1] \to A$ is a path from x_0 to x_1, then the map

$$\begin{array}{rcl} \gamma_*\colon \pi_n(X, A, x_1) & \to & \pi_n(X, A, x_0) \\ [f] & \mapsto & [f^\gamma] \end{array}$$

is a well-defined map that is a group homomorphism.

(2) If x_0 and x_1 are two points in A which lie in the same path-component of A, then $\pi_n(X, A, x_0)$ is isomorphic to $\pi_n(X, A, x_1)$.

(3) Let $\varphi\colon (X, A) \to (Y, B)$ be a map between pairs of topological spaces.

(a) If $\gamma\colon [0,1] \to A$ is a path from x_0 to x_1, then the following diagram commutes

$$\begin{array}{ccc} \pi_n(X, A, x_1) & \xrightarrow{\gamma_*} & \pi_n(X, A, x_0) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \pi_n(Y, B, \varphi(x_1)) & \xrightarrow{(\varphi\circ\gamma)_*} & \pi_n(Y, B, \varphi(x_0)). \end{array}$$

(b) If $\varphi_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, \varphi(x_0))$ is an isomorphism (epimorphism, monomorphism) for some base point $x_0 \in A$, then for any other point x_1 in the same path component of A the map $\varphi_*\colon \pi_n(X, A, x_1) \to \pi_n(Y, B, \varphi(x_1))$ is also an isomorphism (epimorphism, monomorphism).

(4) Let γ and δ be two paths in A from x_0 to x_1. If γ and δ are path-homotopic in A, then the corresponding maps γ_* and δ_* from $\pi_n(X, A, x_1)$ to $\pi_n(X, A, x_0)$ agree.

(5) If γ is a path in A from x_0 to x_1 and if δ is a path in A from x_1 to x_2, then[1825]

$$(\underbrace{\gamma * \delta}_{\text{path in } A \text{ from } x_0 \text{ to } x_2})_* = \gamma_* \circ \delta_*\colon \pi_n(X, A, x_2) \to \pi_n(X, A, x_0).$$

(6) The map

$$\begin{array}{rcl} \pi_1(A, x_0) \times \pi_n(X, A, x_0) & \to & \pi_n(X, A, x_0) \\ ([\gamma], [f]) & \mapsto & [\gamma] \cdot [f] := [f^\gamma] \end{array}$$

[1823] What is wrong with $n = 1$?

[1824] Note that in contrast to the definition on page 1869 we only consider paths in A. At this point we do not allow paths in X.

defines an action (in the sense of the definition on page 248) of the group $\pi_1(A, x_0)$ on the group $\pi_n(X, A, x_0)$.

(7) We denote by $i\colon A \to X$ the inclusion map and by $\varphi\colon \pi_n(X, x_0) \to \pi_n(X, A, x_0)$ the natural map. Then, with the convention from page 2538, the following diagram commutes

$$\begin{array}{ccc} \pi_1(X, x_0) \times \pi_n(X, x_0) & \longrightarrow & \pi_n(X, x_0) \\ {\scriptstyle i_*}\uparrow \quad \downarrow{\scriptstyle \varphi_*} & & {\scriptstyle \varphi_*}\downarrow \\ \pi_1(A, x_0) \times \pi_n(X, A, x_0) & \longrightarrow & \pi_n(X, A, x_0). \end{array}$$

(8) If $\gamma\colon [0,1] \to A$ is a path from x_0 to x_1, then the following diagram commutes:

$$\begin{array}{ccc} \pi_n(X, A, x_1) & \xrightarrow{[f]\mapsto[f^\gamma]} & \pi_n(X, A, x_0) \\ \downarrow{\scriptstyle \partial} & & {\scriptstyle \partial}\downarrow \\ \pi_{n-1}(A, x_1) & \xrightarrow{[f]\mapsto[f^\gamma]} & \pi_{n-1}(A, x_0). \end{array}$$

Proof. The proof of the first six statements of the proposition is basically identical to the proof of the Change-of-Base Point Proposition 86.9. The last two statements follow easily from the definitions. To protect the trees we will not carry out the details. ■

Remark. Let X be a topological space, let A be a non-empty subset and let $n \in \mathbb{N}_{\geq 2}$. If A is path-connected, then it follows from the Change-of-Base Point Proposition 153.5 (2) that the isomorphism type of the relative higher homotopy groups $\pi_n(X, A, x_0)$ does not depend on the choice of a base point in A. Thus if we are only interested in the isomorphism type of the n-th relative homotopy group of (X, A), then sometimes we just write $\pi_n(X, A)$, i.e. we suppress the base point from the notation.

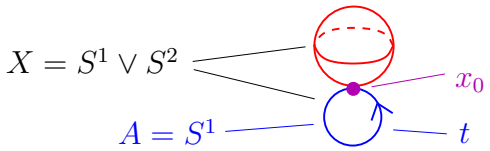
Recall that for the relative homotopy groups $\pi_n(X, A, x_0)$ with $n \geq 3$ we use the additive notation for the group structure. Thus Lemma 86.13, Propositions 71.3, 153.1 (4) and 153.5 lead us to the following observation:

Observation. Let (X, A, x_0) be a pointed pair of topological spaces. If $n \geq 3$, then the abelian group $\pi_n(X, A, x_0)$ admits a natural $\mathbb{Z}[\pi_1(A, x_0)]$-left module structure.

Example. We consider the wedge $X = S^1 \vee S^2$ where we identify $i \in S^1$ with $(0, 0, -1) \in S^2$. We denote by x_0 the wedge point. Furthermore let $A = S^1 \subset X$. We denote by $t \in \pi_1(S^1, i)$ the standard generator. (We refer to the figure below for an illustration.) By the Wedge-π_1-Proposition 52.3 we know that the inclusion $A \to X$ induces an isomorphism $\pi_1(A, x_0) \to \pi_1(X, x_0)$. We use this isomorphism to make the identification $\pi_1(A, x_0) = \pi_1(X, x_0) = \langle t \rangle$. It follows from the Homotopy Groups-LES Proposition 153.4 and the Change-of-Base Point Proposition 153.5 (7) that the natural map $\pi_2(X, x_0) \to \pi_2(X, A, x_0)$ is an isomorphism of $\mathbb{Z}[t^{\pm 1}]$-modules. In particular, by Proposition 86.15, we have

$$\pi_2(X, A, x_0) \overset{\cong}{\leftarrow} \pi_2(S^1 \vee S^2) \overset{\cong}{\rightarrow} \mathbb{Z}[t^{\pm 1}].$$

[1825]Here it is worth remembering that by definition, see page 1052, the product path $\gamma * \delta$ is given by first traversing γ and then traversing δ.



153.4. **Alternative description of relative homotopy groups.** Let (X, x_0) be a pointed topological space. Recall that on page 1565 we had in particular introduced the following identifications:

$$\pi_n(X, x_0) \quad = \quad \begin{array}{c}\text{set (of homotopy classes)}\\ \text{of maps } (I^n, \partial I^n) \to (X, x_0)\end{array} \quad = \quad \begin{array}{c}\text{set (of homotopy classes)}\\ \text{of maps } (S^n, *) \to (X, x_0).\end{array}$$

These identifications allowed us to go back and forth between different view points regarding homotopy groups.

Before we continue with our discussion of relative homotopy groups we also want to introduce a second view point regarding relative homotopy groups. The key to doing so is the following elementary lemma.

Lemma 153.6. Let $n \in \mathbb{N}$. There exists an explicit[1826] map $\varphi\colon I^n \to \overline{B}^n$ with the following three properties:

(1) we have $\varphi(J^{n-1}) = * := (0, \ldots, 0, 1)$ and $\varphi(I^{n-1}) = S^{n-1}$,
(2) the map φ descends to a homeomorphism $(I^n/J^{n-1}, I^{n-1}/\partial I^{n-1}) \xrightarrow{\cong} (\overline{B}^n, S^{n-1})$,
(3) the restriction of φ to the map $I^n \setminus \partial I^n \to B^n$ is an orientation-preserving diffeomorphism.



Sketch of a proof for Lemma 153.6. We write $Q = (\frac{1}{2}, \ldots, \frac{1}{2}, 0) \in I^n$. Given any point $x \in I^n \setminus \{Q\}$ we define

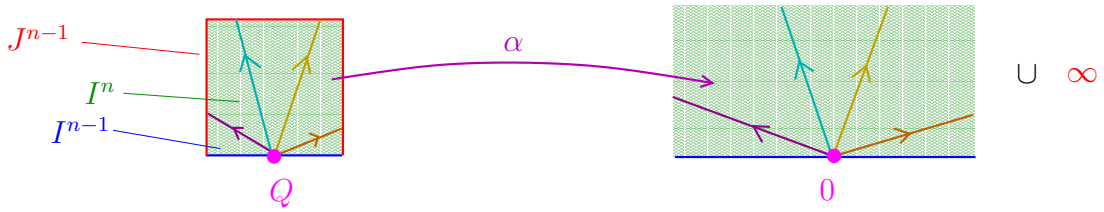
$$\rho(x) \;:=\; \sup\big\{\|rx\| \,\big|\, r \in \mathbb{R}_{>0} \text{ and } r(x-Q)+Q \in I^n\big\}.$$

We consider the map

$$\begin{array}{rcl} \alpha\colon I^n/J^{n-1} & \to & \overbrace{\{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_n \geq 0\}}^{=H_n} \cup \{\infty\} \\ [x] & \mapsto & \begin{cases} \frac{x-Q}{\|x-Q\|} \cdot \tan\left(\frac{\pi}{2} \cdot \frac{\|x-Q\|}{\rho(x)}\right), & \text{if } x \neq Q, \\ 0, & \text{if } x = Q. \end{cases} \end{array}$$

This map stretches each of the segments from Q to a point on J^{n-1} to an infinite ray emanating from the origin. In particular all points in J^{n-1} get sent to infinity, i.e. the map is well-defined. We refer to the figure below for an illustration. Similar to the proof of the Convex-to-Ball Proposition 2.20 we see that the map α is a homeomorphism that restricts to a homeomorphism $I^{n-1}/\partial I^{n-1} \to (\mathbb{R}^{n-1} \times \{0\}) \cup \{\infty\}$. Let $\beta\colon \mathbb{R}^n \cup \{\infty\} \to S^n$

[1826] "Explicit" means that in the proof we explicitly write down such a map that does not depend on any choices. No reader will ever want to see the precise definition, but the fact that the map is explicitly given implies that there is no ambiguity involved when we write "let φ be the map from Lemma 153.6."



be the inverse of the stereographic projection that we introduced in Lemma 2.18. Note that $\beta(\infty) = (0, \ldots, 0, 1)$. Furthermore note that the homeomorphism β restricts to a homeomorphism[1827]

$$\beta\colon H^n \cup \{\infty\} \;\to\; S^n_{\geq 0} := \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_n \geq 0\}.$$

Finally we consider the following map:

$$\begin{array}{ccccccc} I^n & \to & I^n/J^{n-1} & \xrightarrow{\alpha} & H_n \cup \{\infty\} & \xrightarrow{\beta} & S^n_{\geq 0} \\ & & & & (x_1, \ldots, x_{n-1}, x_n, x_{n+1}) & \mapsto & (x_1, \ldots, x_{n-1}, x_{n+1}). \end{array} \quad \to \quad \overline{B}^n$$

Using Exercise 25.9 one can easily verify that this map has all the desired properties.[1828] ■

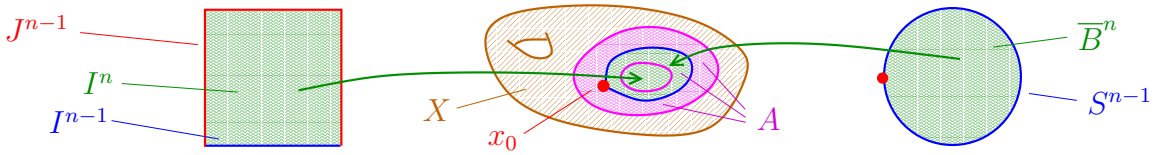
Convention. Using the map

$$\varphi\colon (I^n, \partial I^n, J^{n-1}) \;\to\; (\overline{B}^n, S^{n-1}, *)$$

from Lemma 153.6 we make the following identifications

$$\pi_n(X, A, x_0) \;=\; \begin{array}{c}\text{set (of homotopy classes) of} \\ \text{maps } (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)\end{array} \;=\; \begin{array}{c}\text{set (of homotopy classes) of} \\ \text{maps } (\overline{B}^n, S^{n-1}, *) \to (X, A, x_0).\end{array}$$

We refer to the figure below for an illustration.



Remark. Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}$. As above we make the identifications

$$\pi_n(X, A, x_0) = \left\{ \begin{array}{c}\text{homotopy classes of maps} \\ (\overline{B}^n, S^{n-1}, *) \to (X, A, x_0)\end{array} \right\} \text{ and } \pi_{n-1}(A, x_0) = \left\{ \begin{array}{c}\text{homotopy classes of maps} \\ (S^{n-1}, *) \to (A, x_0)\end{array} \right\}.$$

We leave it to the interested reader to verify that the connecting homomorphism $\pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ from the Homotopy Groups-LES Proposition 153.4 is given by the map

$$\begin{array}{ccc} \partial_n\colon \pi_n(X, A, x_0) & \to & \pi_{n-1}(A, x_0) \\ [f\colon (\overline{B}^n, S^{n-1}, *) \to (X, A, x_0)] & \mapsto & [f|_{S^{n-1}}\colon (S^{n-1}, *) \to (A, x_0)]. \end{array}$$

[1827] Note that the condition $x_n \geq 0$ is not a typo.

[1828] More precisely, the author is somewhat optimistic that the signs are right and that the map is indeed orientation-preserving.

153.5. **Connected (pairs of) topological spaces.** For convenience we recall the following definition from page 1845.

Definition. Let $k \in \mathbb{N}_0$. We say a topological space X is k-connected, if $\pi_i(X) = 0$ for $i = 0, \ldots, k$.

This definition has the following obvious "relative" analogue.

Definition. Let X be a topological space, let $A \subset X$ and let $k \in \mathbb{N}$. We say (X, A) is k-connected if every path-component of X contains a point of A and if for every $i \in \{1, \ldots, k\}$ and every $x_0 \in A$ we have $\pi_i(X, A, x_0) = 0$.

The following lemma provides a convenient reformulation of the previous definition.

Lemma 153.7. Let X be a topological space, let $A \subset X$ and let $k \in \mathbb{N}$. The following two statements are equivalent:

(1) The pair (X, A) is k-connected.
(2) Every path-component of X contains a point of A, furthermore for every $a \in A$ and every $j \in \{1, \ldots, k-1\}$ the inclusion induced map $\pi_j(A, a) \to \pi_j(X, a)$ is an isomorphism and the inclusion induced map $\pi_k(A, a) \to \pi_k(X, a)$ is an epimorphism.

Example. Let X be a topological space and let $A \subset X$ be a deformation retract. It follows from the Homotopy-π_n-Proposition 71.7 (2) that for any $x_0 \in A$ and any $i \in \mathbb{N}_0$ the inclusion induced map $\pi_i(A, x_0) \to \pi_i(X, x_0)$ is an isomorphism. It follows from Lemma 153.7 that (X, A) is k-connected for every $k \in \mathbb{N}$.

Proof. We denote by $i\colon A \to X$ the inclusion map. Let $a \in A$. By the Homotopy Groups-LES Proposition 153.4 we have the following long exact sequence of homotopy groups:

$$\ldots \to \pi_k(A, a) \xrightarrow{i_*} \pi_k(X, a) \to \pi_k(X, A, a) \xrightarrow{\partial} \pi_{k-1}(A, a) \xrightarrow{i_*} \pi_{k-1}(X, a) \to \ldots$$

The lemma is an immediate consequence of the existence of this long exact sequence. ∎

The following proposition gives a convenient characterization of the vanishing relative homotopy groups.

Proposition 153.8. Let X be a topological space and let $A \subset X$. For any $n \in \mathbb{N}$ the following two statements are equivalent:

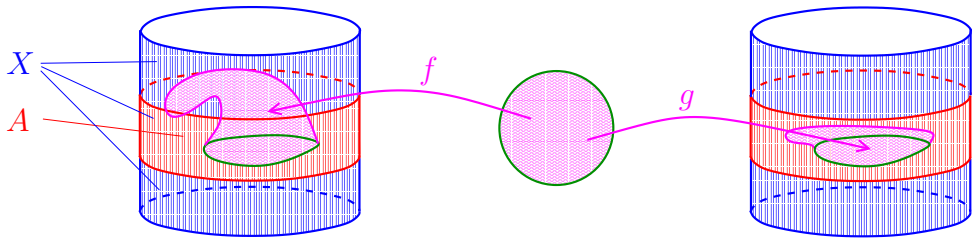
(1) Every map $(\overline{B}^n, S^{n-1}) \to (X, A)$ is homotopic rel S^{n-1} to a map $\overline{B}^n \to A$,
(2) for every $x_0 \in A$ we have $\pi_n(X, A, x_0) = 0$.

Furthermore, for $n = 0$ the above statement (1) is equivalent to

(2′) every path-component of X contains a point of A.

Example. We consider the infinite cylinder $X = S^1 \times \mathbb{R}$ and the annulus $A = S^1 \times [-1, 1]$. Evidently A is a deformation retract of X. Thus, by the previous example the pair (X, A) is k-connected for every $k \in \mathbb{N}_0$. In the figure below we show a map $f\colon (\overline{B}^2, S^1) \to (X, A)$. Furthermore we show a map $g\colon \overline{B}^2 \to A$ that is homotopic, rel S^1, to f.

Proof of Proposition 153.8. First note that it is elementary to see that for $n = 0$ the statements (1) and (2′) are equivalent. Now let $n \geq 1$. The implication (1) $\Rightarrow$ (2) is an immediate consequence of Lemma 153.3 and the by now well-known fact that there exists a homeomorphism $(I^n, \partial I^n) \xrightarrow{\cong} (\overline{B}^n, S^{n-1})$.



The most interesting statement of the proposition is the implication (2) $\Rightarrow$ (1). Thus we now assume that $\pi_n(X, A, x_0) = 0$ for every $x_0 \in A$. Let $f\colon (\overline{B}^n, S^{n-1}) \to (X, A)$ be a map. We need to show that f is homotopic rel S^{n-1} to a map that takes values only in A. Let $\varphi\colon I^n \to \overline{B}^n$ be the map from Lemma 153.6 with $\varphi(J^{n-1}) = (0, \ldots, 0, 1)$. We set $x_0 := f(0, \ldots, 0, 1)$. Note that the map $f \circ \varphi\colon I^n \to X$ is actually a map

$$f \circ \varphi\colon (I^n, \partial I^n, J^{n-1}) \;\to\; (X, A, \{x_0\})$$

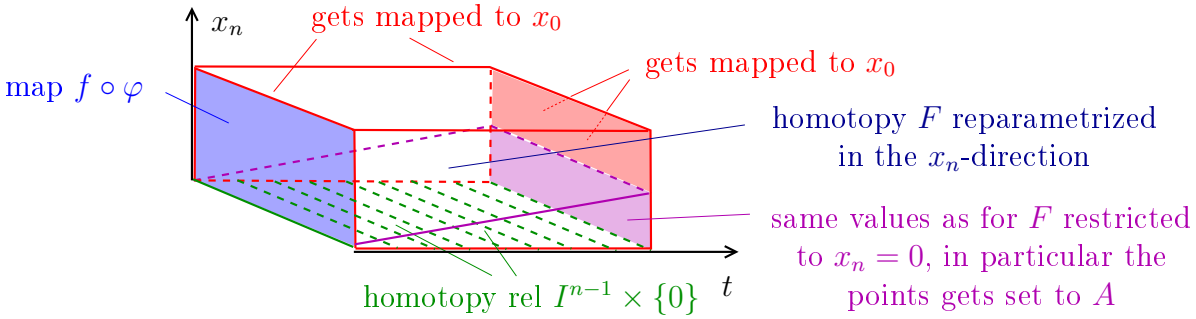
of triples. Since $\pi_n(X, A, x_0) = 0$ we see that there exists a homotopy

$$F\colon (I^n, \partial I^n, J^{n-1}) \times [0,1] \;\to\; (X, A, \{x_0\})$$

which satisfies $F_0 = f \circ \varphi$ and $F_1 = c_{x_0}$. Now we consider the map

$$\begin{aligned} I^n \times [0,1] &\to X \\ (x_1, \ldots, x_n, t) &\mapsto \begin{cases} F\big(x_1, \ldots, x_{n-1}, \frac{1}{1-\frac{t}{2}} \cdot (x_n - \frac{t}{2}), t\big), & \text{if } x_n \in \big[\frac{t}{2}, 1\big] \\ F\big(x_1, \ldots, x_{n-1}, 0, 2 \cdot x_n\big), & \text{if } x_n \in \big[0, \frac{t}{2}\big]. \end{cases} \end{aligned}$$

The definition of the homotopy is sketched, with moderate success, in the figure below. This map has the following properties:



(0) the map is continuous by the Homotopy Stacking Lemma 14.3,
(1) the map is a homotopy rel $I^{n-1} \times \{0\}$,
(2) the map is a homotopy from $f \circ \varphi$ to a map that takes values in A,
(3) the map sends $J^{n-1} \times [0,1]$ to x_0.

It follows from Property (3), and using the above map $\varphi\colon (I^n, J^{n-1}) \to (\overline{B}^n, (0, \ldots, 0, 1))$, that F defines a homotopy $G\colon \overline{B}^n \times [0,1] \to X$. Property (1) implies that G is a homotopy rel S^{n-1}. Furthermore Property (2) says that G is a homotopy from f to a map that takes values in A. Thus G is the desired a homotopy. ∎

The following proposition gives us an important example of a k-connected pair of topological spaces.

Proposition 153.9. Let X be a CW-complex and let $k \in \mathbb{N}_0$.

(1) If A is a subcomplex such that all cells in $X \setminus A$ have dimension greater than k, then the pair (X, A) is k-connected.
(2) The pair (X, X^k) is k-connected.[1829]

Proof.

(1) Let X be a CW-complex, let A be a subcomplex and let $k \in \mathbb{N}_0$. Suppose that all cells in $X \setminus A$ have dimension greater than k. This implies in particular that every path-component of X contains at least one point in A. The remainder of the proof is almost the same as the proof of the CW-Skeleton-π_n-Proposition 71.9. The key difference is that we need to replace the Cellular Approximation Theorem 70.15 by the Cellular Approximation Theorem 70.16 for Pairs of CW-complexes and that now we use the (1) $\Rightarrow$ (2) implication of Proposition 153.8. We leave it to the reader to carry out the modest modifications of the proof of the CW-Skeleton-π_n-Proposition 71.9.
(2) This statement is of course an immediate consequence of the first statement. ■

The following corollary is an extremely useful consequence of Lemma 153.7 and Proposition 153.9 (2). We had proved it earlier in the guise of the CW-Skeleton-π_n-Proposition 71.9.

Corollary 153.10. Let X be a CW-complex, let $k \in \mathbb{N}_0$ and let $x \in X^k$. The following two statements hold:

(1) For $j = 1, \ldots, k-1$ the inclusion induced map $\pi_j(X^k, x) \to \pi_j(X, x)$ is an isomorphism.
(2) The inclusion induced map $\pi_k(X^k, x) \to \pi_k(X, x)$ is an epimorphism.

153.6. **The relative Hurewicz Theorem.** In this section we state the "Relative Hurewicz Theorem" that generalizes the original Hurewicz Theorem 85.5 to the relative setting.

Definition. Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}$. We denote by $[I^n] \in \mathrm{H}_n(I^n, \partial I^n; \mathbb{Z})$ the fundamental class of I^n.[1830] We refer to the map

$$\begin{array}{rcl} \Phi_{(X,A,x_0)} \colon \pi_n(X, A, x_0) & \to & \mathrm{H}_n(X, A; \mathbb{Z}) \\ [f \colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})] & \mapsto & f_*([I^n]) \end{array}$$

as the relative Hurewicz homomorphism.

The following lemma summarizes some elementary properties of the relative Hurewicz homomorphism.

Lemma 153.11.

(1) Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}_{\geq 2}$. The map

$$\begin{array}{rcl} \Phi_{(X,A,x_0)} \colon \pi_n(X, A, x_0) & \to & \mathrm{H}_n(X, A; \mathbb{Z}) \\ [f \colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})] & \mapsto & f_*([I^n]) \end{array}$$

is well-defined and it is a homomorphism.

[1829] Recall that X^k denotes the k-skeleton of X, i.e. the union of all cells of dimension $\leq k$.
[1830] Here we endow $I^n \subset \mathbb{R}^n$ with the usual orientation coming from $\mathbb{R}^n$.

(2) Let (X, x_0) be a pointed topological space and let $n \geq 2$. The following diagram commutes

$$\begin{array}{ccc} \pi_n(X, x_0) & \xrightarrow{\Phi_{(X,x_0)}} & \mathrm{H}_n(X; \mathbb{Z}) \\ \cong\downarrow & & \downarrow\cong \\ \pi_n(X, \{x_0\}, x_0) & \xrightarrow{\Phi_{(X,\{x_0\},x_0)}} & \mathrm{H}_n(X, \{x_0\}; \mathbb{Z}). \end{array}$$

Here the left-hand vertical map is given by the identification from page 3239 and the right-hand vertical map is the obvious map from absolute to relative homology groups.

(3) Let (X, A, x_0) be a pointed pair of topological spaces. The following diagram commutes

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \pi_n(A, x_0) & \longrightarrow & \pi_n(X, x_0) & \longrightarrow & \pi_n(X, A, x_0) & \xrightarrow{\partial} & \pi_{n-1}(A, x_0) & \longrightarrow \ldots \to & \pi_1(X, x_0) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \ldots & \longrightarrow & \mathrm{H}_n(A; \mathbb{Z}) & \longrightarrow & \mathrm{H}_n(X; \mathbb{Z}) & \longrightarrow & \mathrm{H}_n(X, A; \mathbb{Z}) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(A; \mathbb{Z}) & \longrightarrow \ldots \to & \mathrm{H}_1(X; \mathbb{Z}). \end{array}$$

Here the horizontal sequences are the long exact sequences given by the Topological LES-Corollary 74.14 and by the Homotopy Groups-LES Proposition 153.4 and the vertical maps are the Hurewicz homomorphisms.

Sketch of proof.

(1) The proof of the first statement is evidently just a straightforward variation on the proof of Lemma 85.1. We refer to [**Bre1993**, Lemma VII.10.2] or [**Hat2002**, Proposition 4.36] for details.

(3) We have to show that each of the three squares commutes.
   (a) The first square commutes basically by definition, see also Lemma 85.1 (3).
   (b) The fact that the second square commutes is an immediate consequence of the definitions of the "absolute" and the "relative" Hurewicz homomorphisms, see above and page 1842.
   (c) Finally it follows fairly easily from the Codimension Zero-Fundamental Class Lemma 106.24 and the Fundamental Class-of-Boundary Proposition 106.27 that the third square commutes.

(2) This is just a special case of the second square considered in (3). ∎

Again the following question arises: to what degree is the relative Hurewicz homomorphism in fact an isomorphism? One might hope that if a pair (X, A) is $(n-1)$-connected, then the Hurewicz homomorphism $\Phi_{(X,A,x_0)} \colon \pi_n(X, A, x_0) \to \mathrm{H}_n(X, A; \mathbb{Z})$ is an isomorphism. But the following example shows that this is *not* the case.

Example. We consider again $X = S^1 \vee S^2$ and $A = S^1 \subset X$. We denote by x_0 the wedge point. It follows easily from the long exact sequence of homotopy groups provided by the Homotopy Groups-LES Proposition 153.4 that (X, A) is 1-connected.[1831] We see that in

[1831] Alternatively we could view X as a CW-complex in the usual way. Then we obtain from Proposition 153.9 that the pair (X, A) is 1-connected.

this case the map

$$\Phi_{(X,A,x_0)}\colon \underbrace{\pi_2(X,A,x_0)}_{\cong \mathbb{Z}^\infty, \text{ see page 3249}} \to \underbrace{\mathrm{H}_2(X,A;\mathbb{Z})}_{\cong \mathbb{Z}}$$

is *not* an isomorphism.

The following lemma is the key to understanding why the relative Hurewicz homomorphism fails to be a monomorphism.

Lemma 153.12. Let (X,A,x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}_{\geq 2}$. Furthermore let $\sigma \in \pi_n(X,A,x_0)$. For any $g \in \pi_1(A,x_0)$ we have

$$\Phi_{(X,A,x_0)}(g \cdot \sigma) \;=\; \Phi_{(X,A,x_0)}(\sigma).$$

Proof. Let (X,A,x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}_{\geq 2}$. Furthermore let $[f] \in \pi_n(X,A,x_0)$ and let $[\gamma] \in \pi_1(A,x_0)$. As in the Change-of-Base Point Proposition 86.9 (7) we see that the maps $f\colon (I^n,\partial I^{n-1}) \to (X,A)$ and $f^\gamma\colon (I^n,\partial I^{n-1}) \to (X,A)$ are homotopic. We obtain that

$$\Phi_{(X,A,x_0)}([\gamma]\cdot[f]) \;=\; \Phi_{(X,A,x_0)}([f^\gamma]) \;=\; (f^\gamma)_*([I^n]) \underset{\uparrow}{=} f_*([I^n]) \;=\; \Phi_{(X,A,x_0)}([f]).$$

by Proposition 74.15, since f and f^γ are homotopic as maps $(I^n,\partial I^n) \to (X,A)$ ∎

The above lemma leads us to the following definition.

Definition. Let (X,A,x_0) be a pointed pair of topological spaces.[1832]

(1) For $n \in \mathbb{N}_{\geq 3}$ we define $\pi'_n(X,A,x_0)$ to be the quotient group[1833]

$$\pi'_n(X,A,x_0) \;:=\; \pi_n(X,A,x_0)\big/\langle\{g\cdot\sigma-\sigma \,|\, g \in \pi_1(A,x_0) \text{ and } \sigma \in \pi_n(X,A,x_0)\}\rangle.$$

(2) We define[1834]

$$\pi'_2(X,A,x_0) \;:=\; \pi_2(X,A,x_0)\big/\langle\!\langle\{(g\cdot\sigma)\cdot\sigma^{-1} \,|\, g \in \pi_1(A,x_0), \sigma \in \pi_2(X,A,x_0)\}\rangle\!\rangle.$$

(3) By Lemma 153.12 the Hurewicz homomorphism

$$\Phi_{(X,A,x_0)}\colon \pi_n(X,A,x_0) \;\to\; \mathrm{H}_n(X,A;\mathbb{Z})$$

descends to a homomorphism

$$\Phi_{(X,A,x_0)}\colon \pi'_n(X,A,x_0) \;\to\; \mathrm{H}_n(X,A;\mathbb{Z}).$$

We refer to this homomorphism again as the Hurewicz homomorphism.

Remark. Let (X,A,x_0) be a pointed pair of topological spaces. It is a fun exercise to show "by hand" that $\pi'_2(X,A,x_0)$ is an abelian group. We refer to [**Hat2002**, Lemma 4.39] for details.

Example. Once again we consider $X = S^1 \vee S^2$ and $A = S^1 \subset X$. We denote by x_0 the wedge point. As usual we write $\pi_1(A,x_0) = \langle t\rangle$. On page 3249 we already saw that

[1833] Here we use the common, but sometimes confusing, convention that we write the group structure on the abelian group $\pi_n(X,A,x_0)$, $n \geq 3$, additively whereas we write the group structure on $\pi_2(X,A,x_0)$ multiplicatively. Evidently there is no rule without an exception, hence, if $\pi_2(X,A,x_0)$ happens to be abelian after all, then we also like to use additive notation for $\pi_2(X,A,x_0)$.

[1834] Recall that given a group π and a subset $X \subset \pi$ we denote by $\langle\!\langle X\rangle\!\rangle$ the subgroup of π normally generated by X. We refer to page 1192 for details.

$\pi_2(X, A, x_0) \cong \mathbb{Z}[t^{\pm 1}]$ as $\mathbb{Z}[\pi_1(A, x_0)] = \mathbb{Z}[t^{\pm 1}]$-modules. Thus we have an isomorphism

$$\pi_2'(X, A, x_0) \;\cong\; \mathbb{Z}[t^{\pm 1}]/\{(t^k - 1) \cdot f \mid f \in \mathbb{Z}[t^{\pm 1}] \text{ and } k \in \mathbb{Z}\}.$$

It is an amusing algebra exercise to show that the group on the right-hand side is isomorphic to $\mathbb{Z}$.

The following theorem is a generalization of the Hurewicz Theorem 85.5 to the relative case.

Theorem 153.13. (Relative Hurewicz Theorem) Let (X, A, x_0) be a pointed pair of topological spaces such that X and A are path-connected. Let $n \geq 2$. If the pair (X, A) is $(n-1)$-connected, then the following two statements hold:

(1) We have $\mathrm{H}_k(X, A; \mathbb{Z}) = 0$ for $k = 1, \ldots, n-1$.

(2) The Hurewicz homomorphism

$$\Phi_{(X,A,x_0)} \colon \pi_n'(X, A, x_0) \;\to\; \mathrm{H}_n(X, A; \mathbb{Z})$$

is an isomorphism. In particular, if $\pi_1(A, x_0) = 0$, then the Hurewicz homomorphism

$$\Phi_{(X,A,x_0)} \colon \pi_n(X, A, x_0) \;\to\; \mathrm{H}_n(X, A; \mathbb{Z})$$

is an isomorphism.

Proof. We will not provide a proof of the Relative Hurewicz Theorem. A full proof is given in [**Bre1993**, Theorem VII.10.7] or alternatively in [**Hat2002**, Theorem 4.37] and [**Spa1995**, Proposition 7.5.1]. If A is simply connected, then it is a great exercise to figure out to what degree the proof of the Hurewicz Theorem 85.5 can be modified to deal with the relative case. ■

The following theorem, which can be viewed as a corollary to the Relative Hurewicz Theorem 153.13, is a nice illustration of the usefulness of relative homotopy groups.

Theorem 153.14. Let X and Y be two topological spaces and furthermore let $f \colon X \to Y$ be a map. We pick a base point $x_0 \in X$ and we write $y_0 = f(x_0)$.

(1) If for every $i \in \mathbb{N}_{\geq 2}$ the induced map

$$f_* \colon \pi_i(X, x_0) \;\xrightarrow{\cong}\; \pi_i(Y, y_0)$$

is an isomorphism, then for every $i \in \mathbb{N}_{\geq 2}$ the induced map

$$f_* \colon \mathrm{H}_i(X; \mathbb{Z}) \;\xrightarrow{\cong}\; \mathrm{H}_i(Y; \mathbb{Z})$$

is also an isomorphism.

(2) If X and Y are simply connected, then the converse to (1) holds. More precisely, in this case, if for every $i \in \mathbb{N}_{\geq 2}$ the induced map

$$f_* \colon \mathrm{H}_i(X; \mathbb{Z}) \;\xrightarrow{\cong}\; \mathrm{H}_i(Y; \mathbb{Z})$$

is an isomorphism, then for every $i \in \mathbb{N}_{\geq 2}$ the induced map

$$f_* \colon \pi_i(X, x_0) \;\xrightarrow{\cong}\; \pi_i(Y, y_0)$$

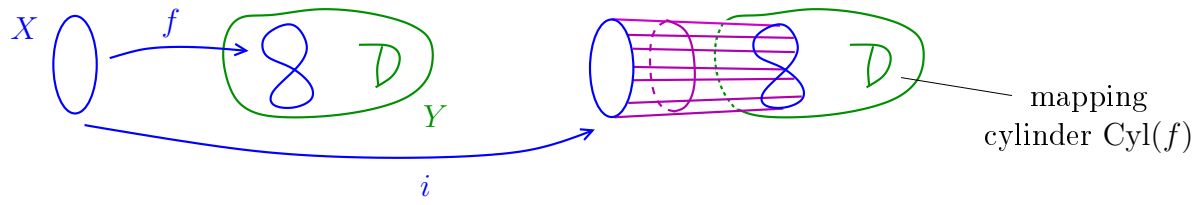
is also an isomorphism.

Remark. In some textbooks, see e.g. [**Bre1993**, p. 481], [**DaK2001**, p. 162] and [**Spa1995**, p. 399], this theorem is referred to as the "Whitehead Theorem". We will reserve the name "Whitehead Theorem" for a much more interesting later result, namely Theorem 162.1.

Proof. The idea is to "replace" the map $f\colon X \to Y$ by an inclusion map so that we can apply the commutative diagram from Lemma 153.11. The key to doing so is the notion of the mapping cylinder that we introduced on page 454. Recall that the mapping cylinder corresponding to f is defined as the topological space

$$\mathrm{Cyl}(f) \;:=\; ((X \times [0,1]) \sqcup Y)/\!\sim \;\text{ where } (x,1) \sim f(x) \text{ for all } x \in X.$$

We consider the maps



$$i\colon X \to \mathrm{Cyl}(f),\; x \mapsto (x,0) \quad \text{respectively} \quad r\colon \mathrm{Cyl}(f) \to Y,\; P \mapsto \begin{cases} P, & \text{if } P \in Y, \\ f(Q), & \text{if } P = [(Q,t)] \text{ with } Q \in X, t \in [0,1]. \end{cases}$$

Note that we have $r \circ i = f\colon X \to Y$. Also note that in the Mapping Cylinder Basics Lemma 16.12 (2a) we showed that the retraction $r\colon \mathrm{Cyl}(f) \to Y$ is a homotopy equivalence. We use the inclusion $i\colon X \to \mathrm{Cyl}(f)$ to identify X with $i(X)$, i.e. we view X as a subset of $\mathrm{Cyl}(f)$.

In the following we prove Statement (2) of the theorem. Thus we assume that X and Y are simply connected and we assume that for every $i \in \mathbb{N}_{\geq 2}$ the induced map

$$f_*\colon \mathrm{H}_i(X;\mathbb{Z}) \;\xrightarrow{\cong}\; \mathrm{H}_i(Y;\mathbb{Z})$$

is an isomorphism. We need to show that for every $i \in \mathbb{N}_{\geq 2}$ the induced map

$$f_*\colon \pi_i(X,x_0) \;\xrightarrow{\cong}\; \pi_i(Y,y_0)$$

is also an isomorphism. We consider the following maps

$$\begin{array}{ccccccccccc}
 & & & & & & \pi_2(Y,y_0) & & & & \\
 & & & & & \overset{f_*}{\nearrow} & \uparrow{\scriptstyle r_*} & & & & \\
\ldots & \longrightarrow & \pi_3(\mathrm{Cyl}(f),X,x_0) & \xrightarrow{\partial} & \pi_2(X,x_0) & \xrightarrow{i_*} & \pi_2(\mathrm{Cyl}(f),x_0) & \longrightarrow & \pi_2(\mathrm{Cyl}(f),X,x_0) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & \mathrm{H}_3(\mathrm{Cyl}(f),X;\mathbb{Z}) & \xrightarrow{\partial} & \mathrm{H}_2(X;\mathbb{Z}) & \xrightarrow{i_*} & \mathrm{H}_2(\mathrm{Cyl}(f);\mathbb{Z}) & \longrightarrow & \mathrm{H}_2(\mathrm{Cyl}(f),X;\mathbb{Z}) & \longrightarrow & 0 \\
 & & & & & \underset{f_*}{\searrow} & \downarrow{\scriptstyle r_*} & & & & \\
 & & & & & & \mathrm{H}_2(Y;\mathbb{Z}) & & & &
\end{array}$$

where the vertical maps between the second and the third row are the Hurewicz homomorphisms. The desired statement follows from the following observations:

(a) The triangles commute since $f = r \circ i$. Furthermore the squares commute by Lemma 153.11.
(b) By our hypothesis the maps $f_*\colon \mathrm{H}_*(X;\mathbb{Z}) \to \mathrm{H}_*(Y;\mathbb{Z})$ are isomorphisms. Furthermore, since r is a deformation retraction we obtain from the Homotopy Equivalence-H_*-Corollary 73.9 that the maps $r_*\colon \mathrm{H}_*(\mathrm{Cyl}(f);\mathbb{Z}) \to \mathrm{H}_*(Y;\mathbb{Z})$ are isomorphisms.
(c) Since $f_* = (r \circ i)_* = r_* \circ i_*$ we obtain immediately from our result in (b) that the maps $i_*\colon \mathrm{H}_*(X;\mathbb{Z}) \to \mathrm{H}_*(\mathrm{Cyl}(f);\mathbb{Z})$ are also isomorphisms.
(d) It follows from (c) and the bottom long exact sequence that all the relative homology groups $\mathrm{H}_*(\mathrm{Cyl}(f), X;\mathbb{Z})$ vanishes.
(e) It follows easily from (d), the hypothesis that X is simply connected and the Relative Hurewicz Theorem 153.13 that all the relative homotopy groups $\pi_*(\mathrm{Cyl}(f), X, x_0)$ vanish.
(f) Using (e) we obtain immediately from the top long exact sequence that the maps $i_*\colon \pi_*(X, x_0) \to \pi_*(\mathrm{Cyl}(f), x_0)$ are isomorphisms.
(g) By the Homotopy-π_n-Proposition 71.7 (2) we know that the maps $r_*\colon \pi_*(\mathrm{Cyl}(f), x_0) \to \pi_*(Y, y_0)$ are isomorphisms.
(h) Finally note that it follows from $f_* = r_* \circ i_*$ together with (f) and (g) that the maps $f_*\colon \pi_*(X, x_0) \to \pi_*(Y, y_0)$ are indeed isomorphisms.

The proof of Statement (1) is very similar to the above argument. We leave it to the reader to fill in the details. ■

We conclude this section with a discussion of homotopy spheres. Recall that by the definition on page 2625 a homotopy n-sphere Σ is a closed n-dimensional topological manifold that is $(n-1)$-connected, i.e. the homotopy groups of Σ agree with the homotopy groups of S^n up to degree $n-1$.
The following pleasant corollary shows that in fact all homotopy groups of Σ are isomorphic to the corresponding homotopy groups of S^n.

Corollary 153.15. Let $n \in \mathbb{N}_{\geq 2}$. If Σ is a homotopy n-sphere, then for every $i \in \mathbb{N}_0$ we have $\pi_i(\Sigma) \cong \pi_i(S^n)$.

Proof. Let Σ be a homotopy n-sphere. Note that by Theorem 153.14 (2) it suffices to find a map $f\colon S^n \to \Sigma$ such that for every $i \in \mathbb{N}_{i\geq 2}$ the induced map $f_*\colon \mathrm{H}_i(S^n;\mathbb{Z}) \to \mathrm{H}_i(\Sigma;\mathbb{Z})$ is an isomorphism.

So let us try to find such a map f. We pick a base point $x_0 \in \Sigma$. By the definition and Proposition **??** we know that Σ is a closed connected non-empty n-dimensional topological manifold with $\mathrm{H}_i(\Sigma;\mathbb{Z}) \cong \pi_i(\Sigma, x_0) = 0$ for $i = 1, \ldots, n-1$. It follows from the Hurewicz Theorem 85.5 that the Hurewicz homomorphism

$$\begin{array}{rcl} \Phi\colon \pi_n(\Sigma, x_0) & \to & \mathrm{H}_n(\Sigma;\mathbb{Z}) \\ [f\colon (S^n, *) \to (\Sigma, x_0)] & \mapsto & f_*([S^n]) \end{array}$$

is an isomorphism. Note that the fact that $\pi_1(\Sigma, x_0) = 0$ together with the Non-Orientable $\mathbb{Z}_2$-Epimorphism Corollary 105.16 implies that Σ is orientable. Therefore by the Orientation-Top Homology Theorem 106.1 we know that $\mathrm{H}_n(\Sigma;\mathbb{Z}) \cong \mathbb{Z}$. Since the Hurewicz homomorphism is an isomorphism we can pick a map $f\colon S^n \to \Sigma$ such that $f_*([S^n]) \in \mathrm{H}_n(\Sigma;\mathbb{Z}) \cong \mathbb{Z}$ is a generator.

We claim that f has the desired properties. By choice of f we know that the map $f_*\colon \mathrm{H}_n(S^n;\mathbb{Z}) \to \mathrm{H}_n(\Sigma;\mathbb{Z})$ is an epimorphism. But since both groups are isomorphic to $\mathbb{Z}$ we see that the map is in fact an isomorphism. Since S^n and Σ are path-connected we obtain from the discussion on page 1594 that $f_*\colon \mathrm{H}_0(S^n;\mathbb{Z}) \to \mathrm{H}_0(\Sigma;\mathbb{Z})$ is also an isomorphism. Furthermore for all other dimensions f_* induces an isomorphism of homology groups for the rather banal reasons that all other homology groups of S^n and Σ are zero. (Indeed, we saw above that the homology groups in dimension less than n vanish and it follows from the Top-Homology Theorem 106.3 that the homology groups vanish in dimension greater than n.) ■

153.7. **Excision for relative homotopy groups?** We have now seen that the (relative) homotopy groups and (relative) homology groups have some properties in common:

(1) both invariants are functorial,
(2) in both cases homotopic maps induce the same maps on the invariants,
(3) a (pointed) pair of topological spaces gives rise to a long exact sequence.

These formal similarities between (relative) homotopy and (relative) homology groups makes one wonder whether an analogue of the Excision Theorem 74.17 holds for relative homotopy groups.
For better or worse this is not the case. More precisely, the next lemma shows that a naive analogue of the Excision Theorem 74.17 fails even in very simple situations.

Lemma 153.16. We consider the 2-dimensional sphere $X = S^2$ together with the subset $A := S^2_{\leq 0} = \{(x,y,z) \in S^2 \,|\, z \leq 0\}$ given by the lower hemisphere and together with the subset $Z = \{(0,0,-1)\}$ that is given by the South Pole. Then[1835]

$$\pi_3(X,A) \;\ncong\; \pi_3(X \setminus Z, A \setminus Z).$$



Proof. We start out with the following observations:

(1) On page 152 we saw that $A = S^2_{\leq 0}$ is homeomorphic to $\overline{B}^2$. In particular A is contractible which implies by the Homotopy-π_n-Proposition 71.7 (2) that $\pi_2(A) = \pi_3(A) = 0$.
(2) The above homeomorphism also shows that $A \setminus Z$ is homeomorphic to $\overline{B}^2 \setminus \{(0)\}$ which is homotopy equivalent to S^1. This implies that $\pi_2(A \setminus Z) = \pi_3(A \setminus Z) = 0$.
(3) Finally in Lemma 2.18 we saw that $X \setminus Z = S^2 \setminus \{(0,0,-1)\}$ is homeomorphic to $\mathbb{R}^2$ which implies that $\pi_2(X \setminus Z) = \pi_3(X \setminus Z) = 0$.

It follows from these observations and the long exact sequence of homotopy groups, provided by the Homotopy Groups-LES Proposition 153.4, that $\pi_3(X \setminus Z, A \setminus Z) \cong \pi_2(A \setminus Z) \cong \pi_2(S^1) = 0$ and furthermore that $\pi_3(X,A) \cong \pi_3(S^2)$. But in Theorem 122.3 we saw that

[1835] Here evidently the closure of Z is contained in the interior of A, i.e. the hypotheses for the Excision Theorem 74.17 for homology groups are satisfied.

$\pi_3(S^2)$ is non-trivial. We have thus shown that $\pi_3(X, A)$ is not isomorphic to $\pi_3(X \setminus Z, A \setminus Z)$. ■

Remark. We just used the fact that $\pi_3(S^2) \neq 0$ to show that relative homotopy groups do not satisfy excision. It is a good moment to remind ourselves of the Eilenberg-Steenrod Uniqueness Theorem 89.25. The theorem suggests that if relative homotopy groups did satisfy excision, then it would follow from the Homotopy Groups-LES Proposition 153.4 that, at least morally, (abelianized) homotopy groups would agree with homology groups.[1836] But the calculations $\pi_3(S^2) \neq 0$ and $\mathrm{H}_3(S^2) = 0$ show that this is not the case.

In much more restricted circumstances there is in fact an excision theorem. For the record we state the following theorem. We will not make use of it and therefore we will not provide the proof (which by all accounts seems to be somewhat messy.)

Theorem 153.17. (Blakers-Massey Theorem)

(1) Let X be a topological space and let A and B be subsets such that $X = \mathring{A} \cup \mathring{B}$. If $(A, A \cap B)$ is n-connected and if $(B, A \cap B)$ is m-connected, then the inclusion induced map
$$\pi_k(A, A \cap B) \ \to \ \pi_k(X, B)$$
is an isomorphism for $k < m + n$ and it is an epimorphism for $k = m + n$.

(2) The conclusion of (1) also holds if X is a CW-complex and A and B are two subcomplexes of X with $X = A \cup B$ such that $A \cap B$ is non-empty and connected.

Proof. The first statement was originally proved in 1952 by Albert Blakers and William Massey [**BlM1952**]. Proofs can also be found in [**Gra1975**, Corollary 16.27], [**tD2008**, Theorem 6.4.1], [**tDKP1970**, p. 211] or [**WhdG1978**, Chapter VII.7]. The second statement can presumably be deduced from the first statement. Alternatively see [**Hat2002**, Theorem 4.23]. ■

Exercises for Chapter 153.

Exercise 153.1. Show that the three paths to the right of the figure on page 3239 define three different elements in $\pi_1(X, A, x_0)$.

Exercise 153.2. Let (X, A, x_0) be a pointed pair of topological spaces. Show that if $n \geq 3$, then the relative homotopy group $\pi_n(X, A, x_0)$ is abelian.

Exercise 153.3. Let X be a topological space and $x_0 \in X$. We consider the set $\pi_0(X, x_0)$. We all know that $\pi_0(X, x_0)$ is "not a group". But why not? After all, every non-empty set can be equipped with a group structure (why is that?). So why is not possible to equip $\pi_0(X, x_0)$ with a group structure in a "meaningful way"? More precisely, if we could equip it with a group structure in a reasonable sense, what properties would we demand? Why can these not be satisfied?

Exercise 153.4. Let $\varphi\colon G \to H$ be a group homomorphism. Furthermore let (S, s_0) be a pointed set which is equipped with an H-action. Suppose that we have an exact sequence
$$\{e\} \ \to \ G \ \xrightarrow{\varphi} \ H \ \xrightarrow{\psi} \ (S, s_0) \ \to \ \{e\}$$

[1836] To turn this logic into a proper argument might be a little tricky because of base point issues.

where ψ is a morphism of pointed sets that preserves the H-action. Show that $\varphi(G)$ is a subgroup of H whose index equals the cardinality of the set S.

Exercise 153.5. Is a monomorphism of pointed sets necessarily injective?

Exercise 153.6. Provide a proof for Lemma 153.3.

Exercise 153.7. Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}$. By the convention of page 3251 we can view elements in $\pi_n(X, A, x_0)$ as homotopy classes of maps $(\overline{B}^n, S^{n-1}, *) \to (X, A, \{x_0\})$. Using this point of view, how can we visualize the action of $\pi_1(A, x_0)$ on $\pi_n(X, A, x_0)$?

Exercise 153.8. Give an example of a connected CW-complex X, a connected subcomplex $A \subset X$ and an $n \in \mathbb{N}$ such that the homotopy group $\pi_n(X/A)$ is not isomorphic to the relative homotopy group $\pi_n(X, A)$.

154. Fibrations

In this chapter we introduce fibrations and we show that fibrations give rise to a long exact sequence of homotopy groups. In the subsequent two chapters we will give many more examples of fibrations and we will see that fibrations allow us to compute various interesting higher homotopy groups.

154.1. **Definition and examples of fibrations.** The following is the key definition of this chapter.

Definition. Let $p\colon Y \to B$ be a map between topological spaces.

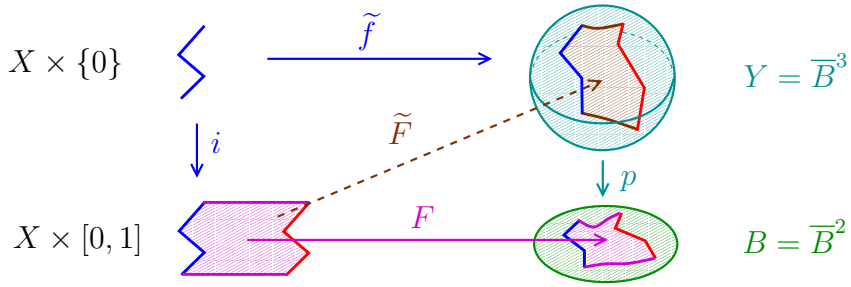
(1) Let X be a topological space. We say that the map $p\colon Y \to B$ has the homotopy lifting property with respect to X if given any map $F\colon X \times [0,1] \to B$ and given any map $\widetilde{f}\colon X \times \{0\} \to Y$ with $p \circ \widetilde{f} = F \circ i$ there exists a map $\widetilde{F}\colon X \times [0,1] \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\widetilde{f}} & Y \\ \downarrow{\scriptstyle i} & \overset{\widetilde{F}}{\nearrow}{\scriptstyle ??} & \downarrow{\scriptstyle p} \\ X \times [0,1] & \xrightarrow[F]{} & B. \end{array}$$

Sometimes we refer to $\widetilde{F}\colon X \times [0,1] \to Y$ as a lift of $F\colon X \times [0,1] \to B$. (Note that we do *not* demand that the lift $\widetilde{F}$ is any way unique.) We refer to the figure below for an illustration of the definition.

(2) We say that the map $p\colon Y \to B$ is a fibration if p has the homotopy lifting property with respect to all topological spaces.

(3) We say that the map $p\colon Y \to B$ is a Serre fibration if p has the homotopy lifting property with respect to all closed balls $\overline{B}^n$, $n \in \mathbb{N}_0$. So perhaps somewhat confusingly a Serre fibration is not necessarily a fibration.



Remark.

(1) Different textbooks mean different objects when they talk about "fibrations". Our definition coincides with the language used in [**Hat2002, Rot1988, DaK2001**]. In [**Bre1993**], what we call a "fibration" is called a "Hurewicz fibration".

(2) If $p\colon Y \to B$ is Serre fibration and if Y and B are CW-complexes, then it follows from [**StW1984, Cau1992**] that p is in fact a fibration. We will not make use of this result.

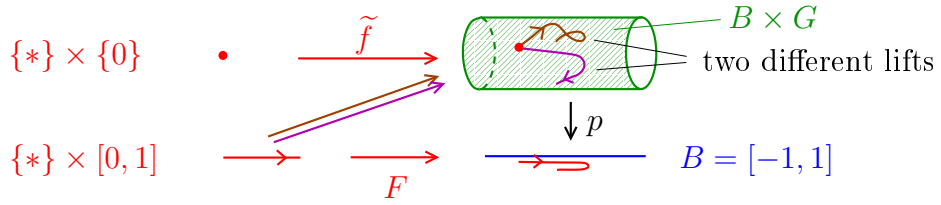
Examples.

(1) In the Path-Homotopy Lifting Proposition 48.12 we already showed that every covering $p\colon Y \to B$ has the homotopy lifting property (in fact with a unique lift) with respect to every topological manifold, in particular with respect to any closed ball $\overline{B}^n$. This implies in particular that every covering is a Serre fibration.

(2) Let B and G be topological spaces. We denote by $p\colon B\times G \to B$ and $q\colon B\times G \to G$ the two obvious projections. In the following we show that the projection $p\colon B\times G \to B$ is a fibration. Indeed, in the setting above the map

$$\begin{array}{rcl} \widetilde{F}\colon X\times[0,1] & \to & B\times G \\ (x,t) & \mapsto & \big(F(x,t), q\big(\underbrace{\widetilde{f}(x,0)}_{\in B\times G}\big)\big) \end{array}$$

makes the diagram commute. As is shown in the figure below, in contrast to the first example, the lift $\widetilde{F}$ is in general not unique.



(3) For every $n\in\mathbb{N}$ the map

$$\begin{array}{rcl} \overline{B}^{n+1} & \to & \overline{B}^n \\ (x_1,\dots,x_n,x_{n+1}) & \mapsto & (x_1,\dots,x_n) \end{array}$$

is a Serre fibration. With some effort this statement can be proved "by hand". The case $n=1$ is precisely the content of Exercise 154.1.

We conclude the list of examples with a "non-example". To make this example more readable we package it is as a lemma.

Lemma 154.1. The map $p\colon S^1 \to [-1,1]$ that is given by the projection onto the x-coordinate is not a Serre fibration.

Proof. We consider the topological space $X = \overline{B}^1 = [-1,1]$ together with the maps $\widetilde{f}\colon X\times\{0\}\to S^1$ and $F\colon [-1,1]\times[0,1]\to[-1,1]$ that are illustrated in the figure below. We claim that there is no lifting $\widetilde{F}\colon X\times[0,1]\to S^1$. More precisely, we prove the following claim.

Claim. If $\widetilde{F}\colon X\times[0,1]\to S^1$ is a map with $p\circ\widetilde{F}=F$, then $\widetilde{F}|_{X\times\{0\}}$ does not agree with $\widetilde{f}$.

Proof. Thus let $\widetilde{F}\colon X\times[0,1]\to S^1$ be a map with $p\circ\widetilde{F}=F$. We make the following observations:

(1) The preimage $F^{-1}(0)$ is connected by the definition of F.

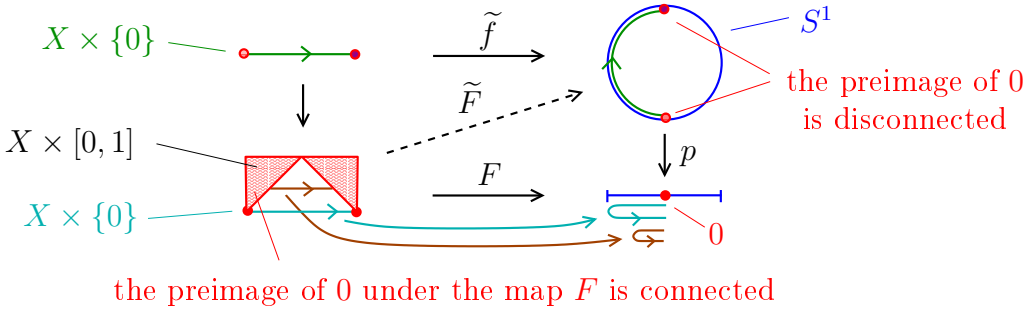
(2) We obtain from (1) and the Image-Connected Lemma 2.24 that the image $\widetilde{F}(F^{-1}(0))$ is also connected.

(3) The fact that $p\circ\widetilde{F}=F$ implies that $\widetilde{F}(F^{-1}(0))\subset p^{-1}(0)$.[1837]

[1837] Since I always find these arguments confusing let me give the argument: Let $y\in\widetilde{F}(F^{-1}(0))$. This means that there exists a $P\in F^{-1}(0)$ with $y=\widetilde{F}(P)$. We have $p(y)=p(\widetilde{F}(P))=F(P)$, but $P\in F^{-1}(0)$, thus $p(y)=0$. In other words, we have $y\in p^{-1}(0)$.

(4) By definition of the map p we have $p^{-1}(0) = \{(0, \pm 1)\}$.
(5) It follows from (2), (3) and (4) that $\widetilde{F}$ restricted to $F^{-1}(0)$ is constant.
(6) The map $\widetilde{f}$ is not constant on $F^{-1}(0)$.

The combination of (5) and (6) proves our claim. ■



We continue with introducing some language and notation.

Notation.

(1) Given a fibration or a Serre fibration $p\colon Y \to B$ we sometimes refer to B as the base space and to Y as the total space.[1838] Furthermore, given a point $* \in B$ we refer to $F := p^{-1}(*)$ as a fiber. The last definition is somewhat dangerous since, as we see in the last of the above examples, the homeomorphism type $p^{-1}(*)$ depends on the choice of $*$. We will shortly discuss "uniqueness" of the fiber. Finally throughout this chapter we denote by $i\colon F \to Y$ the inclusion map.
(2) Given $n \in \mathbb{N}_0$ we define I, I^n and J^n as on page 3238.

Before we can formulate the main theorem of this chapter we need to make some preparations.

Lemma 154.2.

(1) For each $n \in \mathbb{N}$ there exists an orientation-preserving homeomorphism

$$\big(\overline{B}^n \times [0,1], \overline{B}^n \times \{0\} \cup S^{n-1} \times [0,1]\big) \xrightarrow{\cong} \big(\overline{B}^n \times [0,1], \overline{B}^n \times \{0\}\big)$$

of pairs of topological spaces. (This statement is illustrated in the figure below on the left.)
(2) For each $n \in \mathbb{N}$ there exists a homeomorphism

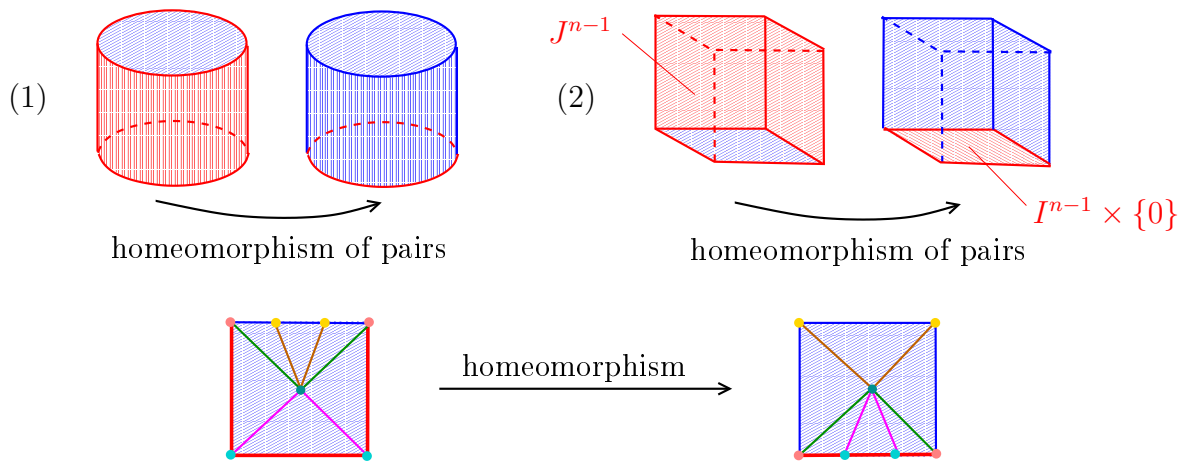
$$(I^n, J^{n-1}) \xrightarrow{\cong} (I^n, I^{n-1} \times \{0\}).$$

of pairs of topological spaces. (This statement is illustrated in the figure below on the right.)

Sketch of proof.

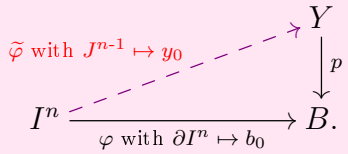
(1) For $n = 1$ such a homeomorphism is illustrated in the figure below. The general case is given by "rotating this homeomorphism around the $x_1 = 0$-axis in $\mathbb{R}^n$".
(2) Using the homeomorphism $\overline{B}^{n-1} \cong I^{n-1}$ from the Convex-to-Ball Proposition 2.20 (2) one can easily reduce the proof of (2) to the statement that we had just proved in (1). ■

[1838] This mirrors the language that we introduced on page 3058 for vector bundles.

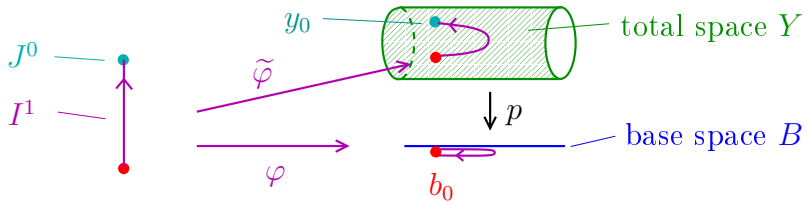


We will also soon need the following elementary lemma.

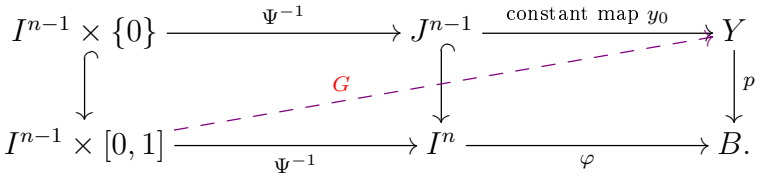
Lemma 154.3. Let $p\colon Y \to B$ be a Serre fibration. Let $y_0 \in Y$. We write $b_0 = p(y_0)$. Furthermore let $n \in \mathbb{N}$. Given any map $\varphi\colon (I^n, \partial I^n) \to (B, \{b_0\})$ there exists a map $\widetilde{\varphi}\colon (I^n, J^{n-1}) \to (Y, \{y_0\})$ such that the following diagram commutes:



We refer to the figure below for an illustration.



Proof. Let $\Psi\colon (I^n, J^{n-1}) \xrightarrow{\cong} (I^n, I^{n-1} \times \{0\})$ be a homeomorphism as in Lemma 154.2 (2). We consider the diagram



Note that the diagram without the red dashed arrow commutes. Since p is a Serre fibration and since I^{n-1} is homeomorphic to the $(n-1)$-ball, see the Convex-to-Ball Proposition 2.20 (2), it follows that a map $G\colon I^{n-1} \times I \to Y$ exists that makes the diagram commute. It is now straightforward to verify that $G \circ \Psi\colon I^n \to B$ has the desired properties. ∎

154.2. **The homotopy fiber of a fibration.** First we introduce the following, arguably long overdue notation.

Notation. Given two topological spaces X and Y we define $[X, Y]$ to be the set of homotopy equivalence classes of maps from X to Y.[1839]

The following lemma is a reformulation of Lemma 14.6.

Lemma 154.4. Let X, Y and Z be topological spaces. The map

$$\begin{array}{ccc} [Y,Z] \times [X,Y] & \mapsto & [X,Z] \\ ([g],[f]) & \mapsto & [g \circ f] \end{array}$$

is well-defined.

Now we can formulate the key technical proposition of this section.

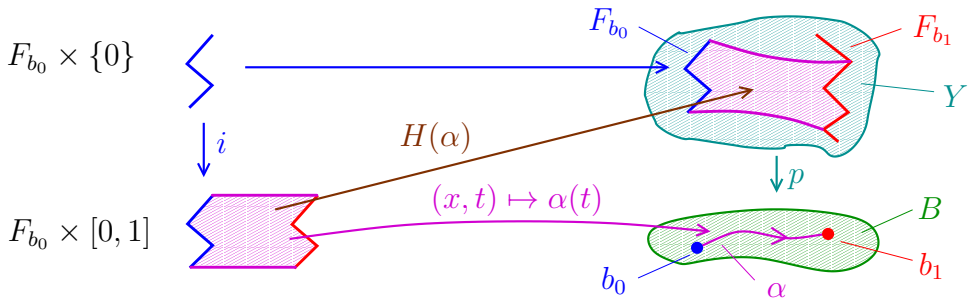
Proposition 154.5. Let $p\colon Y \to B$ be a fibration. Given $b \in B$ we write $F_b := p^{-1}(b)$. Let $\alpha\colon [0,1] \to B$ be a path from a point b_0 to a point b_1. We consider the following diagram

$$\begin{array}{ccc} F_{b_0} \times \{0\} & \xrightarrow{(x,0)\mapsto x} & Y \\ \downarrow{\scriptstyle i} & \nearrow{\scriptstyle H(\alpha)} & \downarrow{\scriptstyle p} \\ F_{b_0} \times [0,1] & \xrightarrow[(x,t)\mapsto \alpha(t)]{} & B. \end{array}$$

Since $p\colon Y \to B$ is a fibration there exists a diagonal map $H(\alpha)\colon F_{b_0} \times [0,1] \to Y$.[1840] (We refer to the figure below for an illustration.) We define $\Theta(\alpha) := H(\alpha)_1$. The following statements hold:

(1) The map $\Theta(\alpha)$ goes from F_{b_0} to F_{b_1}.
(2) The map $\Theta(\alpha)$, viewed as an element in $[F_{b_0}, F_{b_1}]$, is well-defined.
(3) The map $\Theta(\alpha)\colon F_{b_0} \to F_{b_1}$ is a homotopy equivalence.
(4) If two paths $\alpha, \beta\colon [0,1] \to B$ from b_0 to b_1 are path-homotopic, then $[\Theta(\alpha)] = [\Theta(\beta)]$.
(5) If c_b is the constant path at $b \in B$, then $[\Theta(c_b)] = [\mathrm{id}_{F_b}] \in [F_b, F_b]$.
(6) If α is a path in B from b_0 to b_1 and β is a bath in B from b_1 to b_2, then

$$\underbrace{[\Theta(\alpha * \beta)]}_{\in [F_{b_0}, F_{b_2}]} = \underbrace{[\Theta(\beta)]}_{\in [F_{b_1}, F_{b_2}]} \circ \underbrace{[\Theta(\alpha)]}_{\in [F_{b_0}, F_{b_1}]}.$$



[1839] Recall that in Lemma 14.5 we saw that "being homotopic" is an equivalence relation.

[1840] Recall that in general $H(\alpha)$ is not uniquely determined by the data, i.e. there might be many such maps.

Remark. The picture behind the statement of Proposition 154.5 is that while walking along B we can drag the fiber along. Thus any path defines a map between the corresponding fibers.

Proof. Let $p\colon Y \to B$ be a fibration over a topological space B. Given $b \in B$ we write $F_b := p^{-1}(b)$. Let $\alpha\colon [0,1] \to B$ be a path from a point b_0 to a point b_1. We consider the following diagram

$$\begin{array}{ccc} F_{b_0} \times \{0\} & \xrightarrow{(p,0)\mapsto p} & Y \\ \downarrow{\scriptstyle i} & \overset{H(\alpha)}{\nearrow} & \downarrow{\scriptstyle p} \\ F_{b_0} \times [0,1] & \xrightarrow[(x,t)\mapsto \alpha(t)]{} & B. \end{array}$$

Since $p\colon Y \to B$ is by hypothesis a fibration there exists in fact such a diagonal map $H(\alpha)\colon F_{b_0} \times [0,1] \to Y$. Note that the map $H(\alpha)_0\colon F_{b_0} \to B$ is just the obvious inclusion. We turn to the proofs of the statements, albeit perhaps in an unexpected order:

(1) Since the above diagram commutes we see immediately that the map $H(\alpha)_1$ is a map from $F_{b_0} \to F_{b_1}$.

(4) Let $\alpha, \beta\colon [0,1] \to B$ be two paths from b_0 to b_1. We pick maps $H(\alpha)$ and $H(\beta)$ as above, keeping in mind that $H(\alpha)$ and $H(\beta)$ are not uniquely determined. Now we assume that α and β are path-homotopic. We need to show the following claim.

Claim. The maps $H(\alpha)_1, H(\beta)_1\colon F_{b_0} \to F_{b_1}$ are homotopic.

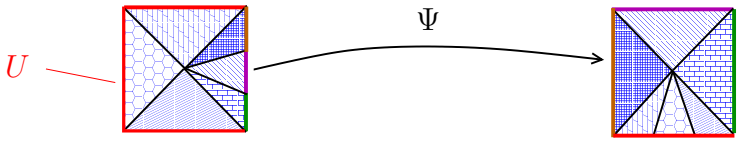
Proof. We start out with the following observation. It follows immediately from Lemma 154.2 (2), applied to $n = 2$, that there exists a self-homeomorphism

$$\Psi\colon [0,1] \times [0,1] \to [0,1] \times [0,1]$$

that restricts to a homeomorphism

$$\underbrace{([0,1] \times \{0,1\}) \cup (\{0\} \times [0,1])}_{=:U} \to [0,1] \times \{0\}.$$

For reassurance the construction of such a map is also sketched in the figure below. Now we pick a homotopy $G\colon [0,1] \times [0,1] \to B$ rel $\{0,1\}$ from α to β. We consider



the following diagram

$$\begin{array}{ccccc} F_{b_0} \times [0,1] \times \{0\} & \xrightarrow{\mathrm{id}_{F_{b_0}} \times \Psi^{-1}} & \overbrace{(F_{b_0} \times [0,1] \times \{0,1\}) \cup (F_{b_0} \times \{0\}) \times [0,1]}^{= F_{b_0} \times U} & \xrightarrow{\Omega} & Y \\ \downarrow & & \downarrow & \overset{\Xi}{\nearrow} & \downarrow{\scriptstyle p} \\ F_{b_0} \times [0,1] \times [0,1] & \xrightarrow[\mathrm{id}_{F_{b_0}} \times \Psi^{-1}]{} & F_{b_0} \times [0,1] \times [0,1] & \xrightarrow[(P,s,t)\mapsto G(s,t)]{} & B. \\ & \underset{\mathrm{id}_{F_{b_0}} \times \Psi}{\nwarrow} & & \overset{\Xi \circ (\mathrm{id}_{F_{b_0}} \times \Psi)}{\nearrow} & \end{array}$$

We make the following clarifications and observations:

(a) The map Ω is defined as follows:[1841]
  (i) on $F_{b_0} \times [0,1] \times \{0\} = F_{b_0} \times [0,1]$ the map Ω is given by $H(\alpha)$,
  (ii) on $F_{b_0} \times [0,1] \times \{1\} = F_{b_0} \times [0,1]$ the map Ω is given by $H(\beta)$,
  (iii) on $F_{b_0} \times \{0\} \times [0,1]$ the map Ω is given by the projection onto F_{b_0} followed by the inclusion into Y.

(b) It follows immediately from the definitions that the diagram commutes.

(c) Since $p\colon Y \to B$ is a fibration we see that there exists a map Ξ that makes the outer diagram commute.

(d) The map $\Xi \circ (\mathrm{id}_{F_{b_0}} \times \Psi)\colon F_{b_0} \times [0,1] \times [0,1] \to Y$ is a homotopy between $H(\alpha)$ and $H(\beta)$. In particular the map $(\Xi \circ (\mathrm{id}_{F_{b_0}} \times \Psi))_1\colon F_{b_0} \times [0,1] \to Y$ is the desired homotopy between $H(\alpha)_1$ and $H(\beta)_1$. ⊞

(5) This statement follows from the observation that for the constant path c_b at a point $b \in B$ we can choose $H(c_b)\colon F_b \times [0,1] \to F_b$ to be the projection. Evidently we obtain that $\Theta(c_b) = H(c_b)_1 = \mathrm{id}_{F_b}$.

(6) Let α be a path in B from b_0 to b_1 and let β be a bath in B from b_1 to b_2. Almost the same way that we define the product $\alpha * \beta$ we combine $H(\alpha)$ and $H(\beta)$. More precisely, we consider the map[1842]

$$\begin{array}{rcl} H(\alpha * \beta)\colon F_{b_0} \times [0,1] & \to & X \\ (P,t) & \mapsto & \begin{cases} H(\alpha)(P, 2t), & \text{if } t \in [0, \frac{1}{2}], \\ H(\beta)\big(\underbrace{H(\alpha(P),1)}_{\in F_{b_1}}, 2t-1\big), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

One can easily verify that this map has all the properties that we expect from a map that is called $H(\alpha * \beta)$. The desired statement is now an immediate consequence of the observation that $H(\alpha * \beta)_1 = H(\beta)_1 \circ H(\alpha)_1$.

(3) Let $\alpha\colon [0,1] \to B$ be a path from b_0 to b_1. We pick a map $H(\alpha)\colon F_{b_0} \times [0,1] \to X$ as above. We want to show that the corresponding map $H(\alpha)_1\colon F_{b_0} \to F_{b_1}$ is a homotopy equivalence. We denote by $\overline{\alpha}\colon [0,1] \to B$ the inverse path of α, i.e. the path from b_1 to b_0 that is given by $\overline{\alpha}(t) = \alpha(1-t)$. We pick a map $H(\overline{\alpha})\colon F_{b_1} \times [0,1] \to X$ as above. It remains to prove the following claim.

Claim.
(a) The map $H(\overline{\alpha})_1 \circ H(\alpha)_1\colon F_{b_0} \to F_{b_0}$ is homotopic to the identity of F_{b_0}.
(b) The map $H(\alpha)_1 \circ H(\overline{\alpha})_1\colon F_{b_1} \to F_{b_1}$ is homotopic to the identity of F_{b_1}.

Proof. We prove the first statement. The proof of the second statement is verbatim the same. We have

$$\begin{array}{ccccc} [\, H(\overline{\alpha})_1 \circ H(\alpha)_1 \,] & = & [\, H(\alpha * \overline{\alpha}) \,] & = & [\, H(c_{b_0}) \,] & = & [\, \mathrm{id}_{F_{b_0}} \,]. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{by (6)} & & \text{by (4) and Proposition 47.5 (3)} & & \text{by (5).} & \end{array}$$

⊞

[1841] As on so many other occasions we can use Pasting Proposition 2.6 (2) to show that the map Ω is indeed continuous.

[1842] It follows again from the Homotopy Stacking Lemma 14.3 that this map is continuous.

(2) We apply the argument from (4) to $\beta = \alpha$ together with two maps $H(\alpha)$ and $H(\beta)$ that make the diagram commute. We obtain by (3) that $H(\alpha)$ and $H(\beta)$ are homotopic, i.e. we see that $[H(\alpha)] \in [F_{b_0}, F_{b_1}]$ is well-defined. ■

Using Proposition 154.5, and the proof thereof, we almost immediately obtain the following proposition.

Proposition 154.6. (Fiber-Homotopy Type Proposition) Let $p\colon Y \to B$ be a map between topological spaces. We assume that B is path-connected.

(1) If $p\colon Y \to B$ is a fibration, then for any two points $b_0, b_1 \in B$ the fibers $p^{-1}(b_0)$ and $p^{-1}(b_1)$ are homotopy equivalent.
(2) If $p\colon Y \to B$ is a Serre fibration, then for any two points $b_0, b_1 \in B$ such that the fibers $p^{-1}(b_0)$ and $p^{-1}(b_1)$ are discrete subsets, the fibers $p^{-1}(b_0)$ and $p^{-1}(b_1)$ are homotopy equivalent.

Proof.

(1) This statement follows immediately from Proposition 154.5 (3) and the hypothesis that B is path-connected.
(2) Let $p\colon Y \to B$ be a Serre fibration with B a path-connected topological space. Let $b_0, b_1 \in B$ such that the fibers $F_{b_0} := p^{-1}(b_0)$ and $F_{b_1} := p^{-1}(b_1)$ are discrete subsets. In the proof of Proposition 154.5 (3) we used the homotopy lifting property for the topological spaces $X = F_{b_i} \times [0,1]^n$ with $i \in \{0,1\}$ and $n \in \{1,2\}$. Since F_{b_0} and F_{b_1} are discrete we see that each component of $X = F_{b_i} \times [0,1]^n$ is homeomorphic to $[0,1]^n$, i.e. each component is homeomorphic to $\overline{B}^n$. Since we can do the lifting component-wise it suffices in our context that we are given a Serre fibration. We leave it to the reader to fill in the details. ■

the Fiber-Homotopy Type Proposition 154.6 (1) allows us to make the following definition.

Definition. Let $p\colon Y \to B$ be a fibration over a path-connected topological space B. We refer to the homotopy type of the fibers as the homotopy fiber of the fibration.

Remark. A more precise statement of the Fiber-Homotopy Type Proposition 154.6 (1) holds: any path $\gamma\colon [0,1] \to B$ from b_0 to b_1 defines naturally a homotopy equivalence between the fibers $p^{-1}(b_0)$ and $p^{-1}(b_1)$. We refer to [**DaK2001**, Theorem 6.12] for the precise formulation and proof of this statement.

Examples.

(1) Let $p\colon \widetilde{X} \to X$ be a covering map over a path-connected topological space. As discussed on page 3264 such a map is a Serre fibration. It follows immediately from the definition of a covering map that for each $x \in X$ the fiber $p^{-1}(x)$ is a discrete subset of $\widetilde{X}$. Therefore it follows from the π_0-Homotopy Equivalence Lemma 15.3 and the Fiber-Homotopy Type Proposition 154.6 (2) that given any two points $x_0, x_1 \in X$ there exists a bijection $p^{-1}(x_0) \to p^{-1}(x_1)$. In other words, we have found a new proof for Lemma 48.1.
(2) Let us consider again the "non-example" from Lemma 154.1, i.e. we consider the map

$$\begin{array}{rcl} p\colon Y = S^1 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\} & \to & B = [-1,1] \\ (x,y) & \mapsto & x. \end{array}$$

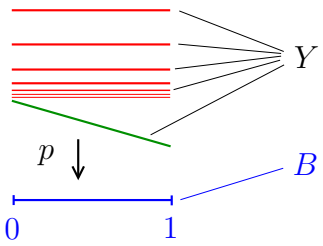
In this case $p^{-1}(-1) = \{(-1,0)\}$ and $p^{-1}(0) = \{(0,\pm 1)\}$ are discrete subsets, but the number of components is different. From the π_0-Homotopy Equivalence Lemma 15.3 we deduce that $p^{-1}(-1)$ is not homotopy equivalent to $p^{-1}(0)$. Thus the Fiber-Homotopy Type Proposition 154.6 (2) gives a new proof (after the more hands-on proof provided by Lemma 154.1) that p is not a Serre fibration.

(3) We consider the map $p\colon Y \to B$ given by the total space

$$Y := \{\underbrace{(x,y)}_{\in\mathbb{R}^2} \,|\, y = 1-x \text{ and } x \in [0,1]\} \cup \bigcup_{n\in\mathbb{N}} \{\underbrace{(x,y)}_{\in\mathbb{R}^2} \,|\, x \in [0,1] \text{ and } y = 1 + \tfrac{1}{n}\}$$

the base space $B = [0,1]$ and the natural projection $p\colon Y \to X$. (This example is illustrated in the figure below.) It is an amusing exercise to show that $p\colon Y \to B$ is a Serre fibration. We saw in Exercise 15.8 that the fibers $p^{-1}(0)$ and $p^{-1}(1)$ are not homotopy equivalent. It follows from the Fiber-Homotopy Type Proposition 154.6 (1) that $p\colon Y \to B$ is not a fibration.[1843]

We can also show more directly that p is not a fibration: We consider $X = p^{-1}(1)$, the usual projection $f\colon X \times [0,1] \to B = [0,1]$ together with the obvious inclusion $\widetilde{f}\colon X \times \{0\} \to Y$. It is once again an amusing elementary exercise to show that there is no map $\widetilde{F}\colon X \times [0,1] \to B$ with all the desired properties. This shows that $p\colon Y \to B$ is not a fibration.



154.3. **The long exact homotopy sequence of a Serre fibration.** For us the most interesting aspect of Serre fibrations is that they give rise to long exact sequences of homotopy groups. More precisely, we have the following theorem.

Theorem 154.7. (Fibration-LES Theorem) Let $p\colon Y \to B$ be a Serre fibration. Let $y_0 \in Y$. We write $b_0 = p(y_0)$ and $F = p^{-1}(b_0)$.

(1) Given any $n \in \mathbb{N}$ the following map is well-defined:

$$\begin{aligned} \partial_n\colon \pi_n(B,b_0) &\to \pi_{n-1}(F,y_0) \\ [\varphi\colon (I^n,\partial I^n) \to (B,b_0)] &\mapsto \left[\begin{array}{c} \text{pick a map } \widetilde{\varphi}\colon (I^n,J^{n-1}) \to (Y,\{y_0\}) \text{ with } p\circ\widetilde{\varphi} = \varphi \\ \text{(we know from Lemma 154.3 that such a map exists)} \\ \text{and then restrict the map } \widetilde{\varphi} \text{ to } I^{n-1} = I^{n-1}\times\{0\} \end{array}\right]. \end{aligned}$$

[1843] The discussion of this example would be more intelligible if we followed the convention of [**Bre1993**, p. 450] and if we referred to "fibrations" as "Hurewicz fibrations" to better distinguish them from "Serre fibrations".

(The definition of ∂_n is moderately successfully illustrated in the figure below.) Furthermore, these maps ∂_n have the following properties:

(a) The maps are morphisms of groups (if $n \geq 2$) and of pointed sets (if $n = 1$).

(b) The maps are natural in the following two senses: Suppose we are given another Serre fibration $p' \colon Y' \to B'$ and suppose we are given a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{p} & B \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ Y' & \xrightarrow{p'} & B'. \end{array}$$

If we write $y_0' = \Phi(y_0), b_0' = \phi(b_0)$ and $F' = (p')^{-1}(b_0')$, then for any $n \in \mathbb{N}$ the following diagram commutes

$$\begin{array}{ccc} \pi_n(B, b_0) & \xrightarrow{\partial_n} & \pi_{n-1}(F, y_0) \\ {\scriptstyle\phi_*}\downarrow & & \downarrow{\scriptstyle(\Phi|_F)_*} \\ \pi_n(B', b_0') & \xrightarrow{\partial_n} & \pi_{n-1}(F', y_0'). \end{array}$$

(c) Let $\alpha \colon [0,1] \to B$ be a path from b_0 to some point b_1.[1844] We write $F_1 = p^{-1}(b_1)$. If $\Theta(\alpha) \colon F \to F_1$ is a map as in Proposition 154.5, then the following diagram commutes

$$\begin{array}{ccc} \pi_n(B, b_0) & \xrightarrow{\partial_n} & \pi_{n-1}(F, y_0) \\ {\scriptstyle\alpha_*}\downarrow & & \downarrow{\scriptstyle\Theta(\alpha)_*} \\ \pi_n(B, b_1) & \xrightarrow{\partial_n} & \pi_{n-1}(F_1, y_1) \end{array}$$

where $y_1 = \Theta(\alpha)(y_0)$.

(2) The following sequence is exact:[1845]

$$\begin{aligned} \ldots \xrightarrow{\partial_{n+1}} \pi_n(F, y_0) \xrightarrow{i_*} \pi_n(Y, y_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial_n} \\ \xrightarrow{\partial_n} \pi_{n-1}(F, y_0) \xrightarrow{i_*} \ldots\ldots\ldots \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\partial_1} \\ \xrightarrow{\partial_1} \pi_0(F, y_0) \xrightarrow{i_*} \pi_0(Y, y_0) \xrightarrow{p_*} \twoheadrightarrow \pi_0(B, b_0). \end{aligned}$$

(3) If the inclusion map $i \colon F \to Y$ is actually homotopic rel y_0 to the constant map $c_{y_0} \colon F \to \{y_0\}$, then for any $n \in \mathbb{N}_{\geq 2}$ the above long exact sequence breaks into a short exact sequence

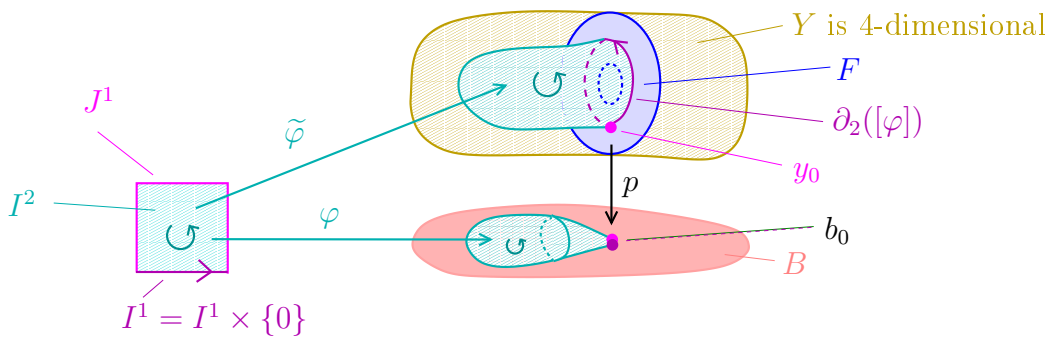
$$0 \;\to\; \pi_n(Y, y_0) \;\xrightarrow{p_*}\; \pi_n(B, b_0) \;\xrightarrow{\partial_n}\; \pi_{n-1}(F, y_0) \;\to\; 0.$$

Furthermore the short exact sequence splits, which implies that for any $n \in \mathbb{N}_{\geq 2}$ we have an isomorphism

$$\pi_n(B, b_0) \;\cong\; \pi_n(Y, y_0) \oplus \pi_{n-1}(F, y_0).$$

[1844] Here we allow of course that $b_1 = b_0$.

[1845] Here, at the lower end, we mean "exact" in the sense of the definition on page 3242.



Before we head towards the proof of the Fibration-LES Theorem 154.7 we want to see what the theorem tells us in two types of fibrations that we had considered above, namely for coverings and products.

Example. Let $p\colon Y \to B$ be a covering of path-connected topological spaces. As mentioned above, by the Path-Homotopy Lifting Proposition 48.12 we know that p is a Serre fibration. We pick $y_0 \in Y$ and we write $b_0 = p(y_0)$ and $F = p^{-1}(b_0)$. It follows immediately from the definition of a covering map that F, equipped with the subspace topology coming from Y, is a discrete space, in particular $\#\pi_0(F, y_0) = \#F$ and $\pi_i(F, y_0) = 0$ for $i \geq 1$. From the long exact sequence provided by the Fibration-LES Theorem 154.7 we obtain the following two pieces of information:

(1) At the lower end the long exact sequence looks as follows:

$$\underbrace{\pi_1(F, y_0)}_{=\{e\}} \to \pi_1(Y, y_0) \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\partial} \underbrace{\pi_0(F, y_0)}_{\#\pi_0(F,y_0)=\#F} \to \underbrace{\pi_0(Y, y_0)}_{=\{e\}} \to \underbrace{\pi_0(B, b_0)}_{=\{e\}}.$$

In other words we obtain the short exact sequence

$$\{e\} \to \pi_1(Y, y_0) \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\partial} \pi_0(F, y_0) \to \{e\}.$$

We draw the following two conclusions:

(a) The map $p_*\colon \pi_1(Y, y_0) \to \pi_1(B, b_0)$ is a monomorphism.

(b) With a little bit of extra thought[1846] we obtain from the short exact sequence, Proposition 154.5, the Fibration-LES Theorem 154.7 (1c) and Lemma 153.2 that $p_*(\pi_1(Y, y_0))$ is a subgroup of $\pi_1(B, b_0)$ of index $\#\pi_0(F, y_0) = \#F = [Y : B]$.

These two statements are precisely the content of Lemma 47.18 and Lemma 48.15 (3).

(2) Since all other homotopy groups of F vanish we see that for any $n \geq 2$ the map $p_*\colon \pi_n(Y, x_0) \to \pi_n(B, b_0)$ is an isomorphism. This result recovers the Coverings-π_n-Proposition 71.13.

We move on to the next application. More precisely we want to apply the Fibration-LES Theorem 154.7 to the case of a Serre fibration of the form $X \times Y \to X$. This leads to a new proof of Propositions 48.20 and 71.8.

[1846] We leave this extra thinking as a mental exercise to the reader.

Proposition 71.8. Let X and Y be two topological spaces and let $x_0 \in X$ and $y_0 \in Y$. We consider the maps

$$\begin{array}{rcl} i\colon X & \to & X \times Y \\ x & \mapsto & (x, y_0) \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon Y & \to & X \times Y \\ y & \mapsto & (x_0, y). \end{array}$$

For any $n \in \mathbb{N}$ the map[1847]

$$\begin{array}{rcl} \Phi\colon \pi_n(X, x_0) \times \pi_n(Y, y_0) & \to & \pi_n(X \times Y, (x_0, y_0)) \\ (\varphi, \psi) & \to & i_*(\varphi) \cdot j_*(\psi) \end{array}$$

is an isomorphism.

Proof. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the two obvious projections. On page 3264 we saw that both projections are fibrations. We concentrate on the first projection $p\colon X \times Y \to X$. It is a fibration with fiber $p^{-1}(x_0) = \{x_0\} \times Y$. Using the obvious homeomorphism $Y \to \{x_0\} \times Y$ we obtain from the Fibration-LES Theorem 154.7 the long exact sequence

$$\ldots \to \pi_n(Y, y_0) \xrightarrow{j_*} \pi_n(X \times Y, (x_0, y_0)) \xrightarrow{p_*} \pi_n(X, x_0) \to \pi_{n-1}(Y, y_0)) \xrightarrow{j_*} \ldots.$$

Evidently we have $q \circ j = \mathrm{id}_Y$ which implies that $q_* \circ j_* = \mathrm{id}_{\pi_1(Y, y_0)}$. It follows that for every $n \in \mathbb{N}_0$ the map $j_*\colon \pi_n(F, (x_0, y_0)) \to \pi_n(X \times Y, (x_0, y_0))$ is a monomorphism. Thus for each $n \in \mathbb{N}$ we see that

$$0 \to \pi_n(Y, y_0) \xrightarrow{j_*} \pi_n(X \times Y, (x_0, y_0)) \xrightarrow{p_*} \pi_n(X, x_0) \to 0$$

is exact. Once again using that $q_* \circ j_* = \mathrm{id}$ we obtain from Splitting Lemma 78.2 that the given map

$$\begin{array}{rcl} \Phi\colon \pi_n(X, x_0) \times \pi_n(Y, y_0) & \to & \pi_n(X \times Y, (x_0, y_0)) \\ (\varphi, \psi) & \to & i_*(\varphi) \cdot j_*(\psi) \end{array}$$

is indeed an isomorphism. ■

Even these modest applications should already convince the reader that it is worth proving the Fibration-LES Theorem 154.7. After the proof we will give many more examples of fibrations and we will find many more interesting applications.

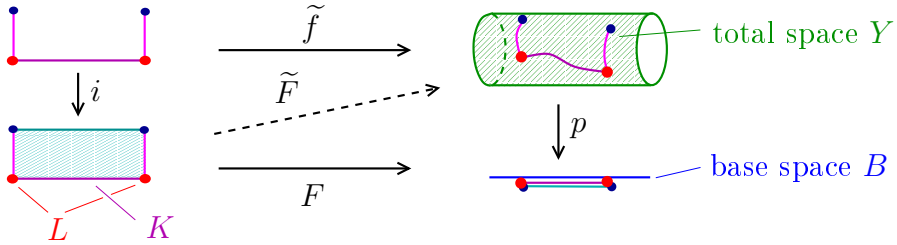
154.4. **Proof of the Fibration-LES Theorem 154.7.** The proof of the Fibration-LES Theorem 154.7 requires some preparations, i.e. we first need to prove some propositions.

Proposition 154.8. Let $p\colon Y \to B$ be a Serre fibration and let (K, L) be a pair of CW-complexes. We denote by $i\colon K \times \{0\} \cup L \times [0, 1] \to K \times [0, 1]$ the inclusion map. Given any map $F\colon K \times [0, 1] \to B$ and given any map $\widetilde{f}\colon K \times \{0\} \cup L \times [0, 1] \to Y$ with $p \circ \widetilde{f} = F \circ i$

[1847] Note that for $n \geq 2$ the map is a group homomorphism since in this case the group on the right-hand side is abelian. It is a priori not completely clear that the map is also a homomorphism for $n = 1$, but we will see in the proof that the map is indeed a homomorphism.

there exists a map $\widetilde{F}\colon K \times [0,1] \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} K \times \{0\} \cup L \times [0,1] & \xrightarrow{\widetilde{f}} & Y \\ \downarrow{\scriptstyle i} & \overset{\widetilde{F}}{\nearrow ??} & \downarrow{\scriptstyle p} \\ K \times [0,1] & \xrightarrow[F]{} & B. \end{array}$$



Proof. We start out with a reformulation of our task: Given $n = 0, 1, 2, 3, \ldots$ we need to define maps

$$\widetilde{F}^n\colon (K \times \{0\}) \cup (K^n \cup L) \times [0,1] \;\to\; Y$$

that make the above diagram commute and which have the property that for $n < m$ the maps $\widetilde{F}^n$ and $\widetilde{F}^m$ agree on the common domain.

We set $\widetilde{F}^{-1} = \widetilde{f}$. Suppose we have already defined maps $\widetilde{F}^{-1}, \ldots, \widetilde{F}^{n-1}$ as above. Let $\{\Phi_i\colon (\overline{B}^n, S^{n-1}) \to (K^n, L^{n-1})\}_{i \in I}$ be the characteristic maps of the n-cells of K that do not lie in L. Recall that by Lemma 154.2 (1) there exists a homeomorphism

$$\Psi\colon \big(\overline{B}^n \times [0,1], (\overline{B}^n \times \{0\}) \cup (S^{n-1} \times [0,1])\big) \;\overset{\cong}{\to}\; \big(\overline{B}^n \times [0,1], \overline{B}^n \times \{0\}\big).$$

For each $i \in I$ we consider the following diagram

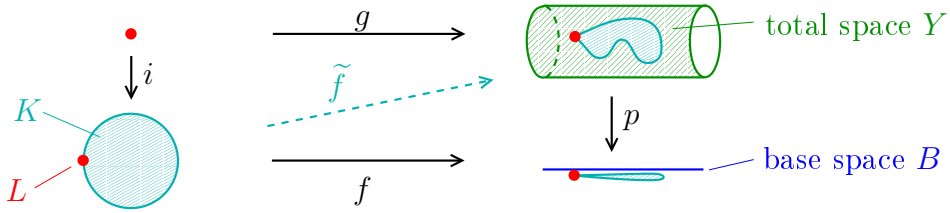
$$\begin{array}{ccccc} \overline{B}^n \times \{0\} & \xrightarrow{\Psi^{-1}} & \overline{B}^n \times \{0\} \cup S^{n-1} \times \{1\} & \xrightarrow{\widetilde{F}^{n-1} \circ \Phi_i} & Y \\ \downarrow & & \downarrow & \overset{\Xi_i}{\nearrow} & \downarrow{\scriptstyle p} \\ \overline{B}^n \times [0,1] & \xrightarrow[\Psi^{-1}]{} & \overline{B}^n \times [0,1] & \xrightarrow[F \circ \Phi_i]{} & B. \end{array}$$

One can easily verify that the two rectangles commute. Since $p\colon Y \to B$ is a Serre fibration there exists a map $\Xi_i\colon \overline{B}^n \times [0,1] \to Y$ that makes the diagram commute. These maps $\{\Xi_i\}_{i \in I}$ can now be used to extend $\widetilde{F}^{n-1}$ to the desired map $\widetilde{F}^n$. ∎

Proposition 154.9. Let $p\colon Y \to B$ be a Serre fibration and let (K, L) be a pair of CW-complexes such that L is a deformation retract of K. We denote by $i\colon L \to K$ the inclusion map. Given any map $f\colon K \to B$ and given any map $g\colon L \to Y$ with $f \circ i = p \circ g\colon L \to B$ there exists a map $\widetilde{f}\colon K \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{g} & Y \\ {\scriptstyle i}\downarrow & \overset{\widetilde{f}}{\nearrow} & \downarrow{\scriptstyle p} \\ K & \xrightarrow[f]{} & B. \end{array}$$

Example. Later on we will apply Proposition 154.9 to the special case that $K = \overline{B}^n$ and that $L = \{(0, \ldots, 0, 1)\}$. We refer to the figure below for an illustration.



Proof. We make the following preparations:

(1) Since L is a deformation retract of K there exists by definition of a deformation retraction, see page 433, a homotopy $H\colon K \times [0,1] \to K$ rel L such that $H_0 = \mathrm{id}_K$ and $H_1(K) \subset L$. We denote by $r = H_1\colon K \to L$ the corresponding retraction.

(2) We denote by $\mathrm{proj}\colon L \times [0,1] \to L$ the obvious projection.

Now we consider the following commutative diagram

$$\begin{array}{ccccccc} L \times \{0\} & \hookrightarrow & K \times \{1\} \cup L \times [0,1] & \xrightarrow{r \cup \mathrm{proj}} & L & \xrightarrow{g} & Y \\ \downarrow & & \downarrow & & \downarrow & & \downarrow p \\ K \times \{0\} & \hookrightarrow & K \times [0,1] & \xrightarrow[H]{} & K & \xrightarrow[f]{} & B. \end{array}$$

To be somewhat more precise, the map $r \cup \mathrm{proj}\colon K \times \{1\} \cup L \times [0,1]$ is defined via the retraction r on $K = K \times \{1\}$ and the projection $L \times [0,1] \to L$. The right-hand square commutes by our hypothesis. One easily verifies that the middle square commutes. It follows from Proposition 154.8 that there exists a map $\Phi\colon K \times [0,1] \to Y$ that makes the diagram commute. For any $k \in K$ we have

$$(p \circ \Phi)(k,0) \;=\; f(H(k,0)) \;=\; f(\mathrm{H}_0(k)) \;=\; f(k)$$

and for any $l \in L$ we have

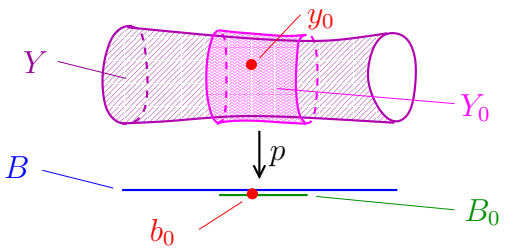
$$\Phi(l,0) \;=\; (g \circ \mathrm{proj})(l,0) \;=\; g(\mathrm{proj}(l,0)) \;=\; g(l).$$

This shows that the restriction of Φ to $K = K \times \{0\}$ has the desired properties. ∎

Proposition 154.10. Let $p\colon Y \to B$ be a Serre fibration. Let $y_0 \in Y$. We write $b_0 = p(y_0)$. Let $B' \subset B$ be any subset with $b_0 \in B'$. We set $Y' := p^{-1}(B')$. Then

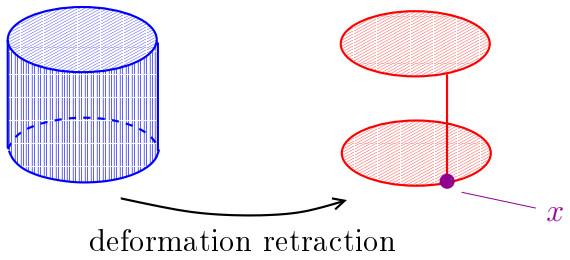
$$p_*\colon \pi_n(Y, Y', y_0) \;\to\; \pi_n(B, B', b_0)$$

is an isomorphism for all $n \in \mathbb{N}$. (We refer to the figure below for an illustration.)



In the proof of Proposition 154.10 we will need the following elementary lemma.

Lemma 154.11. Let $n \in \mathbb{N}$. If $x \in S^{n-1}$, then $(\overline{B}^n \times \partial[0,1]) \cup (\{x\} \times [0,1])$ is a deformation retract of $\overline{B}^n \times [0,1]$.



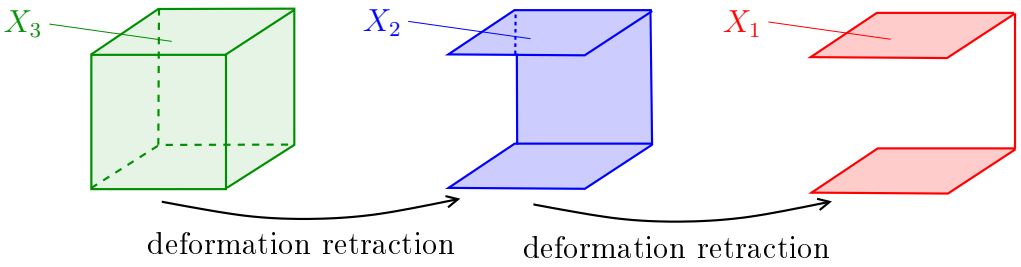
Proof of Lemma 154.11. As usual we write $I = [0,1]$. Given $k \in \{1, \ldots, n\}$ we consider

$$X_k \;:=\; (I^{n-1} \times \partial I) \;\cup\; (\{\underbrace{(1,1,\ldots,1)}_{\in \mathbb{R}^{n-k}}\} \times I^k).$$

(We refer to the figure below for an illustration.) Using the Convex-to-Ball Proposition 2.20 (2) and the Orthogonal Action Lemma 5.26 (3) one can show easily that there exists a homeomorphism

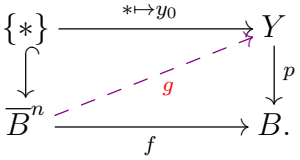
$$(\overline{B}^n \times [0,1], (\overline{B}^n \times \{0,1\}) \cup (\{x\} \times [0,1])) \;\overset{\cong}{\to}\; (X_n, X_1).$$

Thus it suffices to show that each X_i is a deformation retract of X_{i+1}. But such deformation retractions one can actually write down by hand. ∎



Proof of Proposition 154.10. Let $p\colon Y \to B$ be a Serre fibration and let $n \in \mathbb{N}$. Furthermore let $y_0 \in Y$. We write $b_0 = p(y_0)$. Finally let $B' \subset B$ be any subset with $b_0 \in B'$. We set $Y' := p^{-1}(B')$. Throughout this proof we use the identification from page 3251, i.e. we view elements in a relative homotopy group $\pi_n(X, A, x_0)$ as homotopy classes of maps $(\overline{B}^n, S^{n-1}, *) \to (X, A, x_0)$.

First we show that $p_*\colon \pi_n(Y, Y', y_0) \to \pi_n(B, B', b_0)$ is an epimorphism. Thus suppose we are given a map $f\colon (\overline{B}^n, S^{n-1}, *) \to (B, B', b_0)$. It follows from Proposition 154.9, applied to $(K, L) = (\overline{B}^n, *)$, that there exists a map $g\colon \overline{B}^n \to Y$ such that the following diagram commutes:



Basically by definition g is a map $(\overline{B}^n, S^{n-1}, *) \to (Y, Y', y_0)$. Since $f = p \circ g$ we have $[f] = p_*([g])$.

Next we show that $p_*\colon \pi_n(Y, Y', y_0) \to \pi_n(B, B', b_0)$ is also a monomorphism. Therefore suppose that we are given two maps $g_0, g_1\colon (\overline{B}^n, S^{n-1}, *) \to (Y, Y', y_0)$ such that the equality $p_*([g_0]) = p_*([g_1]) \in \pi_n(B, B', b_0)$ holds. This equality means that $p \circ g_0$ is homotopic to $p \circ g_1$ via a homotopy $F\colon (\overline{B}^n, S^{n-1}, *) \times [0,1] \to (B, B', b_0)$. We consider the following commutative diagram (first without the dashed arrow):

$$\begin{array}{ccc} (\overline{B}^n \times \{0,1\}) \cup (\{*\} \times [0,1]) & \xrightarrow{\text{via } g_0, g_1} & Y \\ \big\downarrow & \overset{\widetilde{F}}{\nearrow} & \big\downarrow p \\ \overline{B}^n \times [0,1] & \xrightarrow[F]{} & B. \end{array}$$

By Proposition 154.9 and Lemma 154.11 there exists a map $\widetilde{F}\colon \overline{B}^n \times [0,1] \to Y$ such that the above diagram commutes. This map $\widetilde{F}\colon \overline{B}^n \times [0,1] \to Y$ is easily seen to be a homotopy between g_0 and g_1 which shows that $[g_0] = [g_1] \in \pi_n(Y, Y', y_0)$. ■

With these preparations it is surprisingly easy to provide a proof of the Fibration-LES Theorem 154.7 (1) and (2).

Proof of the Fibration-LES Theorem 154.7 (1) and (2). Let $p\colon Y \to B$ be a Serre fibration. Furthermore let $y_0 \in Y$. We write $b_0 = p(y_0)$ and $F = p^{-1}(b_0)$. We denote by $i\colon F \to Y$ the inclusion map. We consider the following diagram of maps

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \pi_n(F, y_0) & \xrightarrow{i_*} & \pi_n(Y, y_0) & \longrightarrow & \pi_n(Y, F, y_0) & \xrightarrow{\partial_n} & \pi_{n-1}(F, y_0) & \xrightarrow{i_*} & \ldots \\ & & & & & & \cong \big\downarrow p_* & & & & \\ & & & & & & \pi_n(B, \{b_0\}, b_0) & & & & \\ & & & & & & \big\downarrow = & & & & \\ & & & & & & \pi_n(B, b_0). & & & & \end{array}$$

We make the following comments:

(a) The top sequence of maps is the long exact sequence of homotopy groups of the pointed pair of topological spaces (Y, F, y_0), see the Homotopy Groups-LES Proposition 153.4 for details.
(b) The top vertical map is an isomorphism by Proposition 154.10.
(c) The bottom vertical map is the identification given on page 3239.
(d) It follows immediately from the definitions that the left-hand triangle commutes.
(e) The right-hand triangle commutes by definition of the diagonal map.

It follows immediately from the definition of p_* that given any $n \in \mathbb{N}$ the map

$$\partial_n \circ p_*^{-1}\colon \pi_n(B, b_0) \to \pi_{n-1}(F, y_0)$$

is given by

$$[\varphi\colon (I^n, \partial I^n) \to (B, b_0)] \mapsto \left[\begin{array}{c} \text{the restriction of } \widetilde{\varphi} \text{ to } I^{n-1} = I^{n-1} \times \{0\} \\ \text{where } \widetilde{\varphi} \text{ is a map } (I^n, J^{n-1}) \to (Y, \{y_0\}) \\ \text{with } p \circ \widetilde{\varphi} = \varphi \end{array}\right].$$

This observation shows that the map given in Statement (1) of the theorem is well-defined and that it is a homomorphism (respectively a morphism of pointed sets). We leave it to the few readers who made it that far to verify that the maps ∂_n have the two properties stated in (1b) and (1c).

Furthermore, since the top sequence is exact and since the vertical maps are isomorphisms we see that the sequence given in Statement (2) of the theorem is indeed exact. ■

Proof of the Fibration-LES Theorem 154.7 (3). Let $p\colon Y \to B$ be a Serre fibration. Furthermore let $y_0 \in Y$. We write $b_0 = p(y_0)$ and $F = p^{-1}(b_0)$. We denote by $i\colon F \to Y$ the inclusion map. Now we suppose that the inclusion map $i\colon F \to Y$ is homotopic rel y_0 to the constant map $c_{y_0}\colon F \to \{y_0\}$. Put differently, we suppose that there exists a homotopy $G\colon F \times [0,1] \to Y$ rel y_0 such that $G_0 = i$ and $G_1(f) = y_0$ for every $f \in F$.

It follows almost immediately from Lemmas 78.2 and 78.4 that it suffices to show that for any $n \in \mathbb{N}_{\geq 2}$ there exists a homomorphism $\lambda\colon \pi_{n-1}(F, y_0) \to \pi_n(B, b_0)$ such that $\partial_n \circ \lambda = \mathrm{id}_{\pi_{n-1}(F,y_0)}$. We consider the map[1848]

$$\begin{aligned} \lambda\colon \pi_{n-1}(F, y_0) &\to \pi_n(Y, F, y_0) \xrightarrow{p_*} \pi_n(B, \{b_0\}, b_0) = \pi_n(B, b_0) \\ [\varphi\colon (I^{n-1}, \partial I^{n-1}) \to (F, y_0)] &\mapsto \left[\begin{array}{c}(I^n, \partial I^n, J^{n-1}) \to (Y, F, \{y_0\}) \\ (x,t) \mapsto G(\varphi(x), t)\end{array}\right] \end{aligned}$$

It is elementary to verify that λ is in fact a homomorphism. It follows almost immediately from the definitions that $\partial_n \circ \lambda = \mathrm{id}_{\pi_{n-1}(F,y_0)}$. ■

154.5. **The path space fibration.** We recall the following, long dormant, definition from page 392.

Definition. Let X and Y be topological spaces.

(1) We denote by $X^Y = C(Y, X)$ the set of all maps[1849] from Y to X.

(2) The compact-open topology on X^Y is the topology generated, in the sense of the definition on page 135, by all the sets of the form

$$M(K, U) := \{f \in X^Y \mid f(K) \subset U\}$$

where $K \subset Y$ is compact and where $U \subset X$ is open.

In the following we state slightly simplified versions of two technical results which we proved a long time ago.

Proposition 12.4. Let X be a topological space.

(1) The evaluation map

$$\begin{aligned} e\colon X^{[0,1]} \times [0,1] &\to X \\ ((f\colon [0,1] \to X), y) &\mapsto f(y) \end{aligned}$$

is continuous.

[1848] It takes a second to actually verify that the map $(x,t) \mapsto G(\varphi(x), t)$ defines in fact a map $(I^n, \partial I^n, J^{n-1}) \to (Y, F, \{y_0\})$.

[1849] Of course here we mean "all continuous maps".

(2) For each $y \in [0,1]$ the evaluation map

$$\begin{array}{rcl} e\colon X^{[0,1]} & \to & X \\ (f\colon [0,1] \to X) & \mapsto & f(y) \end{array}$$

is continuous.

Proposition 12.6. Let X and T be two topological spaces. Let $H\colon T \times [0,1] \to X$ be a map. Given $t \in T$ we denote by $H_t\colon [0,1] \to X$ the map defined by $H_t(y) = H(t,y)$. Then

$$H\colon T\times[0,1] \to X \text{ is continuous } \Leftrightarrow \begin{array}{l} (1)\ \text{each } H_t\colon [0,1] \to X \text{ is continuous, and} \\ (2)\ \text{the map } T \to X^{[0,1]} \text{ given by } t \mapsto H_t \text{ is continuous.} \end{array}$$

This reminder allows us to introduce our latest best friends.

Definition. Let X be a topological space.

(1) We define

$$\text{free path space } X^{[0,1]} \ := \ \text{set of all maps } [0,1] \to X$$

where $X^{[0,1]}$ is as usual equipped with the compact-open topology. Furthermore we introduce the

$$\text{free loop space } \Omega(X) \ := \ \{f \in X^{[0,1]} \mid f(0) = f(1)\}.$$

We equip the free loop space with the compact-open topology coming from $X^{[0,1]}$.

(2) Suppose we are also given a base point $x_0 \in X$. We introduce the

$$\begin{array}{rcl} \text{path space } P(X,x_0) & := & \{f \in X^{[0,1]} \mid f(0) = x_0\} \\ \text{loop space } \Omega(X,x_0) & := & \{f \in X^{[0,1]} \mid f(0) = f(1) = x_0\}. \end{array}$$

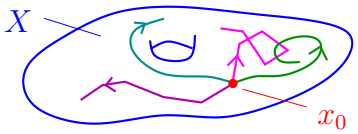
and the

We equip both sets with the compact-open topology coming from $X^{[0,1]}$.

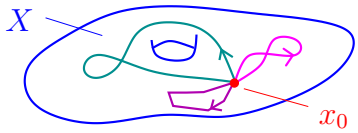
(3) We denote by $c_{x_0}\colon [0,1] \to X$ the constant path given by sending all t's to x_0.



elements in the free path space $X^{[0,1]}$



elements in the path space $P(X,x_0)$



elements in the loop space $\Omega(X,x_0)$

Examples.

(1) Let (X,x_0) be a pointed topological space and let $\alpha, \beta \in \Omega(X,x_0)$. It follows from the Compact-Open Exponential Proposition 12.6 that a path in $\Omega(X,x_0)$ from α to β is basically the same as a homotopy between the maps α and β viewed as loops in (X,x_0). Thus we see that we have a natural identification

$$\pi_0(\Omega(X,x_0)) \ = \ \pi_1(X,x_0).$$

(2) Let (X,x_0) be a pointed topological space. It follows from Proposition 12.4 that the map

$$\begin{array}{rcl} e\colon P(X,x_0) & \to & X \\ f & \mapsto & f(1) \end{array}$$

is continuous. We can ask whether this map has a left-inverse. More precisely, the question is whether there exists a continuous map $s\colon X \to P(X, x_0)$ such that $e(s(x)) = x$ for all $x \in X$. This sounds like a rather random question, so let us reformulate it. The question is, can we *continuously* assign to each $x \in X$ a path from x_0 to x? Clearly that is possible say for $(X, x_0) = (\mathbb{R}^2, 0)$, we just assign to each $x \in \mathbb{R}^2$ the direct path $e(x)$ that is given by $t \mapsto t \cdot x$. But can we do so say for $X = S^1$ or $X = S^2$?[1850] We will answer this question in Exercise 154.2.

For the record we state the following lemma.

Lemma 154.12. The maps

$$\begin{array}{rcl} (X, x_0) & \mapsto & (\Omega(X, x_0), c_{x_0}) \\ (f\colon (X, x_0) \to (Y, y_0)) & \mapsto & \left(\begin{array}{rcl} \Omega(X, x_0) & \to & \Omega(Y, y_0) \\ (\gamma\colon [0,1] \to X) & \mapsto & (f \circ \gamma\colon [0,1] \to Y) \end{array} \right) \end{array}$$

define a covariant functor from the category $\mathcal{PTop}$ of pointed topological spaces to the category $\mathcal{PTop}$ of pointed topological spaces. We refer to this functor as the *loop space functor*.

Proof. Let $f\colon (X, x_0) \to (Y, y_0)$ be a map between pointed topological spaces. It follows from the Compact-Open Functoriality Lemma 12.5 (1) that the map

$$\begin{array}{rcl} \Omega(X, x_0) & \to & \Omega(Y, y_0) \\ (\gamma\colon [0,1] \to X) & \mapsto & (f \circ \gamma\colon [0,1] \to Y) \end{array}$$

is actually continuous. With this observation out of the way it is straightforward to prove the lemma. ∎

The following lemma says that in some sense (free) path spaces, in isolation, are not particularly interesting examples of topological spaces.

Lemma 154.13.

(1) Let X be a topological space. We view X as a subset of the free path space $X^{[0,1]}$ via the inclusion $x \mapsto c_x$.[1851] Then X is a deformation retract of the free path space $X^{[0,1]}$.

(2) If (X, x_0) is a pointed topological space, then $\{c_{x_0}\}$ is a deformation retract of the path space $P(X, x_0)$, in particular the path space $P(X, x_0)$ is contractible.



Proof. We prove Statement (2). The proof of Statement (1) is basically identical. Thus let (X, x_0) be a pointed topological space. We consider the "obvious" deformation retraction given by

[1850] This question is somewhat similar to Question 83.7 and it also falls into the vaguely defined realm of "topological robotics".

[1851] Note that this inclusion $X \to X^{[0,1]}$ is by the Constant Map Lemma 12.3 an embedding.

$$\begin{array}{rcl} F\colon P(X,x_0)\times[0,1] & \to & P(X,x_0) \\ ((f\colon [0,1]\to X),t) & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & X \\ s & \mapsto & f(s\cdot(1-t)) \end{array}\right) \end{array}$$

The only reason why we are not already done is that we still need to verify that F is in fact continuous. As we will see this is slightly delicate. We consider the map

$$G\colon \underbrace{X^{[0,1]}\times[0,1]}_{=:T} \to X^{[0,1]}$$

that is defined in exactly the same way as F. Since $F = G|_{P(X,x_0)\times[0,1]}$ it suffices to show that G is continuous. By the "$\Rightarrow$"-direction of the Compact-Open Exponential Proposition 12.6 applied to $T = X^{[0,1]}\times[0,1]$ it suffices to show that the map

$$\begin{array}{rcl} H\colon [0,1]\times\overbrace{(X^{[0,1]}\times[0,1])}^{=T} & \to & X \\ (s,((g\colon [0,1]\to X),x)) & \mapsto & g(s\cdot x) \end{array}$$

is continuous. But the map H can be written as the composition

$$\begin{array}{rcccl} H\colon [0,1]\times X^{[0,1]}\times[0,1] & \to & X^{[0,1]}\times[0,1] & \to & X \\ (s,g,x) & \mapsto & (g,s\cdot(1-x)) & & \\ & & (g,t) & \mapsto & g(t). \end{array}$$

The first map is continuous by the Topological-Product Proposition 5.1 (1) and the second map is continuous by the Evaluation Map Proposition 12.4 (1). Thus we see that H itself is continuous. ■

In contrast to the previous proposition the following theorem shows that (free) path spaces give rise to interesting *maps*.

Theorem 154.14. (Path Space-Fibration Theorem)

(1) Let X be a topological space. The evaluation map

$$\begin{array}{rcl} p\colon X^{[0,1]} & \to & X \\ (f\colon [0,1]\to X) & \mapsto & f(1) \end{array}$$

is a fibration. Furthermore, given any $x_0\in X$ the fiber $p^{-1}(x_0)$ is the topological space of all paths $[0,1]\to X$ that *end* in x_0.

(2) Let (X,x_0) be a pointed topological space. The evaluation map

$$\begin{array}{rcl} p\colon P(X,x_0) & \to & X \\ (f\colon [0,1]\to X) & \mapsto & f(1) \end{array}$$

is a fibration with fiber $p^{-1}(x_0) = \Omega(X,x_0)$.

the Path Space-Fibration Theorem 154.14 leads us to the following definition.

Definition.

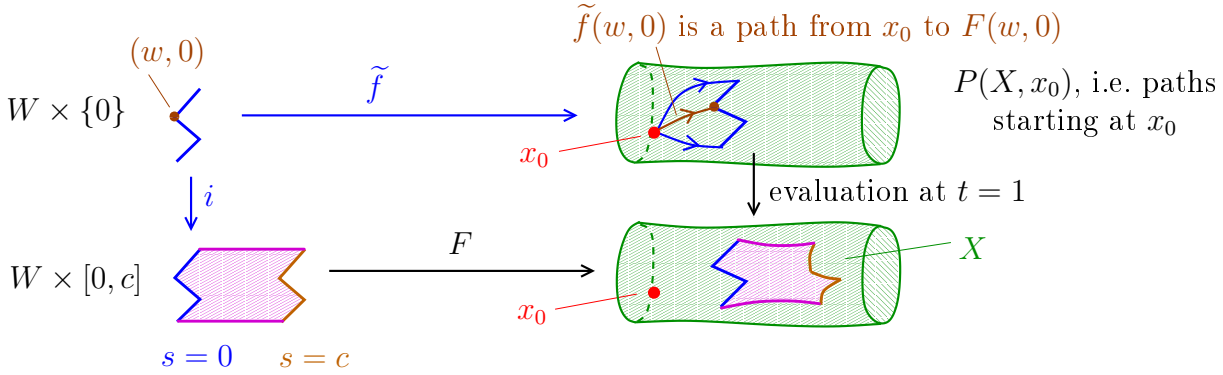
(1) Let X be a topological space. We refer to the above fibration $p\colon X^{[0,1]}\to X$ as the free path space fibration of X.

(2) Let (X,x_0) be a pointed topological space. In the following we refer to the above fibration $p\colon P(X,x_0)\to X$ as the path space fibration of (X,x_0).

Proof. We prove Statement (2). The proof of Statement (1) is basically identical and is thus left to the reader. Thus let (X, x_0) be a pointed topological space. First note that the map $p\colon P(X, x_0) \to X$ is continuous by the Evaluation Map Proposition 12.4 (2). Next recall that we need to solve the following type of extension problem for some topological space W and $c > 0$: [1852]

$$\begin{array}{ccc} W \times \{0\} & \xrightarrow{\widetilde{f}} & P(X, x_0) \subset X^{[0,1]} \\ \downarrow & \overset{\widetilde{F}}{\nearrow} & \Big\downarrow p \quad \Big\downarrow \gamma \mapsto \gamma(1) \\ W \times [0, c] & \xrightarrow[F]{} & X. \end{array}$$

We refer to the figure below for an illustration.



The idea for the proof is quite simple: Given $(w, s) \in W \times [0, c]$ we have to "systematically" find an element $\gamma \in P(X, x_0)$ with $p(\gamma) = F(w, s)$, i.e. we have to find a path $\gamma\colon [0, 1] \to X$ from x_0 to $F(w, s)$. But there is an obvious path, namely we first travel along $\widetilde{f}(w, 0)$ from x_0 to $F(w, 0)$ and then we use the path $t \mapsto F(w, t)$ to get from $F(w, 0)$ to $F(w, s)$. That is exactly what we will do, except that we need to be a little careful with our parametrizations.

Now we have to turn this idea into a proper proof. We consider the map

$$\begin{array}{rcl} \widetilde{F}\colon W \times [0, c] & \to & P(X, x_0) \\ (w, s) & \mapsto & \left(\begin{array}{rcl} [0, 1] & \to & X \\ t & \mapsto & \begin{cases} \widetilde{f}(w, 0)\big(\frac{t}{1 - \frac{s}{2c}}\big), & \text{if } t \in [0, 1 - \frac{s}{2c}], \\ F(w, 2 \cdot (t - 1 + \frac{s}{2})), & \text{if } t \in [1 - \frac{s}{2c}, 1] \end{cases} \end{array} \right). \end{array}$$

We claim that this map has all the desired properties:

(1) Let $(w, s) \in W \times [0, c]$. We make the following observations:
   (a) The given map $[0, 1] \to X$ starts at x_0.
   (b) It follows easily from Pasting Proposition 2.6 (2) that the given map $[0, 1] \to X$ is indeed continuous.

   In other words, the given map $[0, 1] \to X$ does indeed define an element in $P(X, x_0)$.

[1852] In the diagram we write $W \times [0, c]$ instead of the more customary $W \times [0, 1]$ for purely psychological reasons, it makes it easier to distinguish the two types of intervals which arise throughout the proof and which play very different roles.

(2) For each $(w,s) \in W \times [0,c]$ we have by construction that $\widetilde{F}(w,s)(t=1) = F(w,s)$. In other words, we have $p \circ \widetilde{F} = F$.

Thus it remains to prove the following claim:

Claim. The map $\widetilde{F}\colon W \times [0,c] \to P(X,x_0)$ is continuous.

Proof. We consider the map

$$\begin{array}{rcl} \widetilde{G}\colon W \times [0,c] \times [0,1] & \to & X \\ (w,s,t) & \mapsto & \widetilde{F}(w,s)(t) = \begin{cases} \widetilde{f}(w,0)\big(\frac{t}{1-\frac{s}{2c}}\big), & \text{if } t \in [0,1-\frac{s}{2c}], \\ F(w,2\cdot(t-1+\frac{s}{2})), & \text{if } t \in [1-\frac{s}{2c},1]. \end{cases} \end{array}$$

It follows easily from the fact that F and $\widetilde{f}$ are continuous that the restrictions of $\widetilde{G}$ to the closed subsets

$$\begin{array}{rcl} A_1 & := & W \times \{(s,t) \in [0,c] \times [0,1] \,|\, t \in [0,1-\frac{s}{2c}]\}, \\ \text{and} \quad A_2 & := & W \times \{(s,t) \in [0,c] \times [0,1] \,|\, t \in [1-\frac{s}{2c},1]\} \end{array}$$

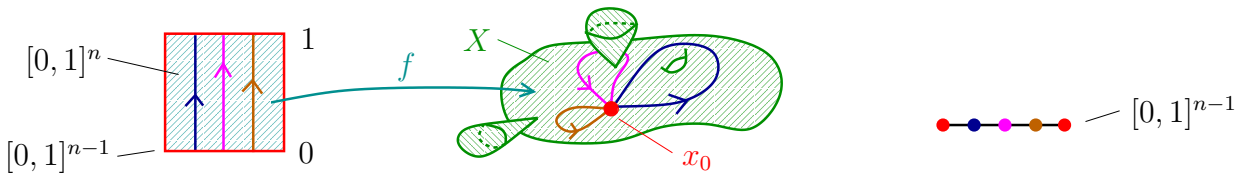
are continuous. Therefore it follows from the Pasting Proposition 2.6 (2) that $\widetilde{G}$ is continuous. Finally we deduce from the "$\Rightarrow$"-direction of the Compact-Open Exponential Proposition 12.6 applied to $T = W \times [0,c]$, $Y = [0,1]$ and $H = \widetilde{G}$ that $\widetilde{F}$ is also continuous. ∎

We obtain the following amusing corollary which later on will turn out to play an important role.

Corollary 154.15. (Loop Space-Homotopy Groups Corollary) Given any pointed topological space (X,x_0) and any $n \geq 1$ the map[1853]

$$\begin{array}{rcl} \partial_n\colon \pi_n(X,x_0) & \stackrel{\cong}{\to} & \pi_{n-1}(\Omega(X,x_0),c_{x_0}) \\ [f\colon ([0,1]^n,\partial([0,1]^n)) \to (X,x_0)] & \mapsto & \left[\begin{array}{r} ([0,1]^{n-1},\partial([0,1]^{n-1})) \to (\Omega(X,x_0),c_{x_0}) \\ x \mapsto \left(\begin{array}{r} [0,1] \to X \\ t \mapsto f(x,t) \end{array} \right) \end{array} \right] \end{array}$$

is a natural isomorphism.



Remark. For $n = 1$ the statement of the Loop Space-Homotopy Groups Corollary 154.15 recovers the result from page 3280 where we saw that there exists a natural identification

$$\pi_0(\Omega(X,x_0),c_{x_0}) \;=\; \pi_1(X,x_0).$$

Proof. There are two approaches to proving the corollary, namely one can prove it by "the machine" and one can prove it "by hand".

[1853] Using the Compact-Open Exponential Proposition 12.6 one can show directly, with some slight effort, that the map ∂_n is well-defined.

(1) Let $f\colon (X,x_0)\to(Y,y_0)$ be a map between pointed topological spaces. It is clear that we obtain a commutative diagram

$$\begin{array}{ccc} P(X,x_0) & \xrightarrow{\gamma\mapsto f\circ\gamma} & P(Y,y_0) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{\quad f\quad} & Y. \end{array}$$

It follows from Theorems 154.14 (1) and (2) and 154.7 that we have commutative diagram of long exact sequences

$$\begin{array}{ccccccccc} \ldots\to\pi_n(P(X,x_0),c_{x_0}) & \to & \pi_n(X,x_0) & \xrightarrow{\partial} & \pi_{n-1}(\Omega(X,x_0),c_{x_0}) & \to & \pi_{n-1}(P(X,x_0),c_{x_0}) & \to & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots\to\pi_n(P(Y,y_0),c_{y_0}) & \to & \pi_n(Y,y_0) & \xrightarrow{\partial} & \pi_{n-1}(\Omega(Y,y_0),c_{y_0}) & \to & \pi_{n-1}(P(Y,y_0),c_{y_0}) & \to & \ldots \end{array}$$

We obtain from Lemma 154.13 and the Homotopy-π_n-Proposition 71.7 (2) that the homotopy groups of the path spaces $P(X,x_0)$ and $P(Y,y_0)$ are trivial. Thus the connecting homomorphisms are *natural* isomorphisms. We leave it to the reader to verify that in our situation the connecting homomorphisms from the Fibration-LES Theorem 154.7 are precisely the maps that we wrote down. Incidentally this then also shows that the maps, as written down, actually do make sense.

(2) In Exercise 154.4 we will prove the corollary "by hand". ■

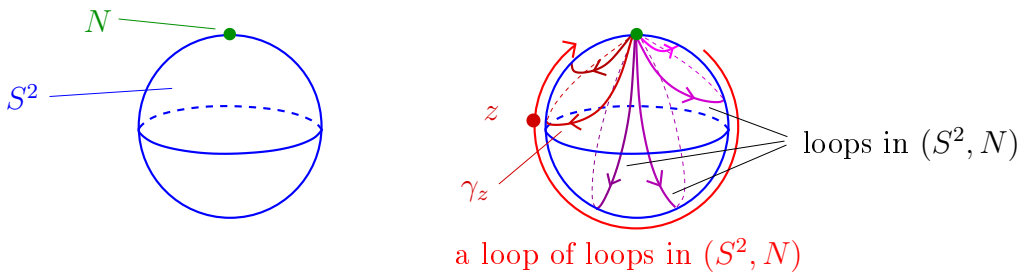
Example. We consider the loop space of the 2-dimensional sphere with the base point given by the North Pole $N=(0,0,1)$. We calculate that

$$\pi_1(\Omega(S^2,N),c_N) \quad \underset{\uparrow}{\overset{\cong}{\leftarrow}} \quad \pi_2(S^2,N) \quad \underset{\uparrow}{\overset{\cong}{\rightarrow}} \quad \mathbb{Z}.$$

Loop Space-Homotopy Groups Corollary 154.15 Corollary 85.6 (1)

In the figure below we see a loop $\{\gamma_z\}_{z\in S^1}$ of loops in the pointed topological space (S^2,N). Put differently, we see a loop in the pointed topological space $(\Omega(S^2,N),c_N)$. It is an amusing and instructive exercise to run through the above two isomorphisms to verify that our loop in the pointed topological space $(\Omega(S^2,N),c_N)$ is actually a generator of $\pi_1(\Omega(S^2,N),c_N)$.

So what does this example actually really mean? It means that even though each individual loop γ_z is null-homotopic in S^2, we cannot continuously deform all the loops γ_z altogether into the constant loop. It is a good idea to think about what it would mean for $[z\mapsto\gamma_z]\in\pi_1(\Omega(S^2,N),c_N)$ to be null-homotopic and why, pictorially, this is not the case in our situation.



Note that the statement of the Loop Space-Homotopy Groups Corollary 154.15 can be summarized as follows: The loop space functor

$$\begin{array}{rccc} \Omega\colon & \begin{array}{c}\text{category } \mathcal{PTop} \text{ of pointed}\\ \text{topological spaces}\end{array} & \to & \begin{array}{c}\text{category } \mathcal{PTop} \text{ of pointed}\\ \text{topological spaces}\end{array} \\ & (X,x_0) & \mapsto & (\Omega(X,x_0),c_{x_0}) \end{array}$$

shifts homotopy groups down by one degree, i.e. for every pointed topological space (X,x_0) and every $n\in\mathbb{N}$ we have a natural isomorphism

$$\pi_n(X,x_0) \xrightarrow{\cong} \pi_{n-1}(\Omega(X,x_0),c_{x_0}).$$

This should be compared to the fact that in the Suspension-H_*-Proposition 76.7 we saw that the suspension functor

$$\begin{array}{rccc} \Sigma\colon \text{category } \mathcal{Top} \text{ of topological spaces} & \to & \text{category } \mathcal{Top} \text{ of topological spaces} \\ X & \mapsto & \Sigma(X) \end{array}$$

shifts reduced homology groups up by one degree, i.e. for every topological space X and any $n\in\mathbb{N}$ we have a natural isomorphism

$$\widetilde{\mathrm{H}}_n(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_{n+1}(\Sigma(X)).$$

In Exercise 154.8 we will discuss whether we can also shift homotopy groups *upwards* and whether we can shift reduced homology groups *downwards* in a functorial way.

154.6. **Every map can be replaced by a fibration.** In this section we will see that one can "replace" any map $f\colon X\to Y$ between topological spaces by a fibration $P_f\to Y$ that has very similar properties to the map f.
We start out with the notion of the pullback of a fibration. Perhaps not surprisingly the definition is almost identical to the definition of the pullback of a vector bundle that we gave on page 3069.

Lemma 154.16. (Fibration-Pullback Lemma) Let $f\colon A\to B$ be a map between topological spaces and let $q\colon Z\to B$ be a fibration. As in the Topological-Pullback Lemma 56.16 (1) we consider the pullback

$$f^*Z := \{(a,z)\,|\,a\in A \text{ and } \overbrace{z\in q^{-1}(f(a))}^{\text{i.e. } q(z)=f(a)}\} \subset A\times Z.$$

The following statements hold:

(1) The pullback map

$$\begin{array}{rccc} p\colon f^*Z := \{(a,z)\,|\,a\in A \text{ and } z\in q^{-1}(f(a))\} & \to & A \\ (a,z) & \mapsto & a \end{array}$$

is also a fibration. Furthermore for any $a\in A$ the fiber $p^{-1}(a)$ is homeomorphic to the fiber $q^{-1}(f(a))$.

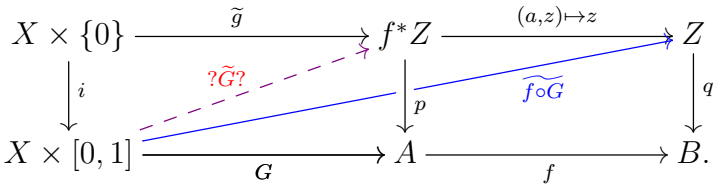
(2) The following diagram commutes

$$\begin{array}{ccc} f^*Z & \xrightarrow{(a,z)\mapsto z} & Z \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{f} & B. \end{array}$$

Proof.

(2) It is straightforward to verify that the diagram commutes.

(1) Let X be any topological space. We consider the following diagram where initially we ignore the diagonal arrows:



Here the maps $G\colon X \times [0,1] \to A$ and $\widetilde{g}\colon X \times \{0\} \to f^*Z$ are "the challenge" we have to deal with in the definition of a fibration. Since $q\colon Z \to B$ is a fibration there exists a map $\widetilde{f \circ G}\colon X \times [0,1] \to Z$ that makes the diagram commute. It is now straightforward to verify that the map

$$\begin{array}{rcl} X \times [0,1] & \to & f^*Z = \{(a,z) \,|\, a \in A \text{ and } z \in q^{-1}(f(a))\} \\ (x,t) & \mapsto & \big(G(x,t), \big(\widetilde{f \circ G}\big)(x,t)\big) \end{array}$$

makes the above diagram commute. We have thus shown that $p\colon f^*Z \to A$ is also a fibration.

Finally we have to prove the statement regarding the fibers. Thus let $a \in A$. It follows immediately from the definitions that $p^{-1}(a) = \{a\} \times q^{-1}(f(a)) \subset A \times Z$. It follows from the Product Topology-Basics Lemma 5.4 (2a) that the fiber $p^{-1}(a)$ is homeomorphic to the fiber $q^{-1}(f(a))$. ■

Definition. Using the notation from the Fibration-Pullback Lemma 154.16 we refer to $f^*Z \to A$ as the pullback of the fibration $q\colon Z \to B$ under the map $f\colon A \to B$.

Now we turn to the actual problem that we had stated in the beginning of this section.

Definition. Let $f\colon X \to Y$ be a map between topological spaces. Similar to the free path space fibration we now consider the fibration[1854]

$$\begin{array}{rcl} q\colon Y^{[0,1]} & \to & Y \\ \gamma & \mapsto & \gamma(0). \end{array}$$

We refer to the map

$$\begin{array}{rcl} p\colon P_f := f^*(Y^{[0,1]}) = \{(x, \gamma\colon [0,1] \to Y) \in X \times Y^{[0,1]} \,|\, \gamma(0) = f(x)\} & \to & Y \\ (x,\gamma) & \mapsto & \gamma(1) \end{array}$$

as the mapping path fibration.[1855]

In the Replace-by-Cofibration Proposition 17.5 we saw that any map $f\colon X \to Y$ between topological spaces can be "replaced" by a cofibration $X \to \mathrm{Cyl}(f)$. The following proposition is the analogue of the Replace-by-Cofibration Proposition 17.5 for fibrations.

Proposition 154.17. (Replace-by-Fibration Proposition) Let $f\colon X \to Y$ be a map between topological spaces. We denote by $p\colon P_f \to Y$ the corresponding mapping path fibration.

(1) The map

$$\begin{array}{rccl} h\colon & X & \to & P_f \\ & x & \mapsto & (x, \text{constant path } c_{f(x)}) \end{array}$$

is a homotopy equivalence and it has the property that it makes the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\quad h \quad} & P_f \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p} & \\ & Y. & \end{array}$$

(2) The map $p\colon P_f \to Y$ is a fibration.[1856] Given any $y \in Y$ the fiber is given by

$$\{(x,\gamma) \,|\, x \in X \text{ and } \gamma\colon [0,1] \to Y \text{ is a path from } f(x) \text{ to } y\}.$$

Definition. Let $f\colon X \to Y$ be a map between topological spaces. We suppose that Y is path-connected. We refer to the homotopy type of the fiber of the corresponding mapping path fibration $P_f \to Y$[1857] as the homotopy fiber of f.

Remark. If $f\colon X \to Y$ is a fibration, then it is shown in [**DaK2001**, Theorem 6.18] that the map $h\colon X \to P_f$ in the Replace-by-Fibration Proposition 154.17 (1) is a "fiber homotopy equivalence", thus, loosely speaking, the original fibration $f\colon X \to Y$ and the new fibration $p\colon P_f \to Y$ are essentially the same. We will not make use of this fact.

Example. the Replace-by-Fibration Proposition 154.17 thus says that any map $f\colon X \to Y$ between topological spaces can be "replaced" by a fibration. As a reality check let us consider again the "non-example" from Lemma 154.1, i.e. we consider the map $f\colon S^1 \to [-1,1]$ that is given by the projection onto the x-coordinate. Lemma 154.1 says that f is not a fibration. Let us see what the above construction does in this case.

As pointed out in the Replace-by-Fibration Proposition 154.17 (2), given any $y \in [-1,1]$ the preimage of y under the map $p\colon P_f \to Y = [-1,1]$ is given by

$$p^{-1}(y) \;=\; \{(x,\gamma) \,|\, x \in S^1 \text{ and } \gamma\colon [0,1] \to [-1,1] \text{ is a path from } f(x) \text{ to } y\}.$$

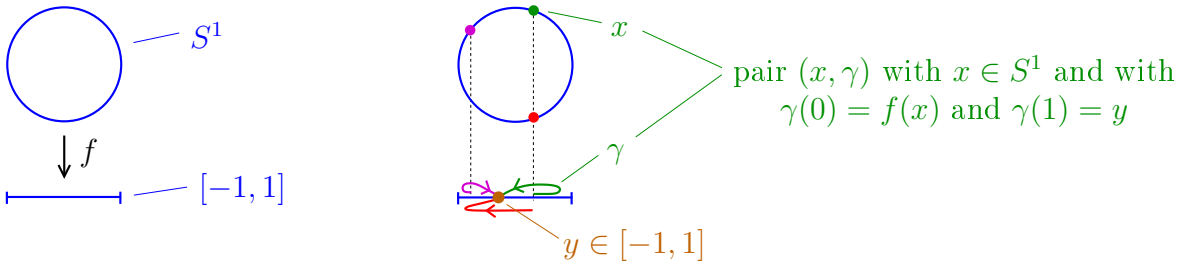
In the figure below we strive to illustrate three points in the preimage of a point $y \in [-1,1]$. In Exercise 154.3 we will see that for each $y \in [-1,1]$ the preimage $p^{-1}(y)$ is homotopy equivalent to S^1. In particular this shows that the homotopy fiber of $f\colon S^1 \to [-1,1]$ is S^1. This exercise does not quite show yet that the map $p\colon P_f \to [-1,1]$ is a fibration, but at the very least it is consistent with the Fiber-Homotopy Type Proposition 154.6 (1).

[1854] This map is basically the same as the free path space fibration except that now we consider the evaluation $\gamma(0)$ instead of the evaluation $\gamma(1)$.

[1855] Note that it follows from the Evaluation Map Proposition 12.4 (2) that this map is in fact continuous.

[1856] This statement justifies the name "mapping path *fibration*".

[1857] Recall that the Fiber-Homotopy Type Proposition 154.6 (1) says that the homotopy type of the fiber of a fibration over a path-connected topological space is well-defined.



Proof of the Replace-by-Fibration Proposition 154.17. Let $f\colon X \to Y$ be a map between topological spaces. We consider the map

$$\begin{array}{rcl} p\colon P_f = f^*(Y^{[0,1]}) = \{(x, \gamma\colon [0,1] \to Y) \in X \times Y^{[0,1]} \,|\, \gamma(0) = f(x)\} & \to & Y \\ (x,\gamma) & \mapsto & \gamma(1). \end{array}$$

We turn to the proof of the two statements:

(1) Given $y \in Y$ we denote, as usual, by $c_y\colon [0,1] \to Y$ the constant path given by $c_y(t) = y$ for all $t \in [0,1]$. We consider the map $h\colon X \to P_f$ that is given by $x \mapsto (x, c_{f(x)})$.[1858] We claim it has the desired properties. It is obvious that $p \circ h = f$, i.e. the diagram given in the proposition commutes. So it remains to show that h is a homotopy equivalence. We consider the map $g\colon P_f \to X$ given by $g(x,\gamma) = x$. Evidently we have $g \circ h = \mathrm{id}_X$. We are done once we have shown that $h \circ g$ is homotopic to id_{P_f}. In fact such a homotopy is given by[1859]

$$\begin{array}{rcl} H\colon P_f \times [0,1] & \to & P_f \\ ((x,\gamma),t) & \mapsto & \left(x, \left(\begin{array}{rcl} [0,1] & \to & Y \\ s & \mapsto & \gamma(s \cdot t) \end{array}\right)\right). \end{array}$$

$\uparrow$

$\gamma\colon [0,1] \to Y$ is a path with $\gamma(0) = f(x)$

One easily verifies that $H_0 = h \circ g$ and that $H_1 = \mathrm{id}_{P_f}$.

(2) We want to show that the map $p\colon P_f \to Y$ is a fibration. Thus let A be any topological space. Let $c > 0$. We consider the following "challenge diagram":[1860]

$$\begin{array}{ccc} A \times \{0\} & \xrightarrow{\widetilde{g}} & P_f = \{(x, \gamma\colon [0,1] \to Y) \in X \times Y^{[0,1]} \,|\, \gamma(0) = f(x)\} \\ \downarrow{\scriptstyle i} & \overset{\widetilde{G}}{\nearrow}{\scriptstyle ??} & \downarrow{\scriptstyle p} \quad (x,\gamma) \mapsto \gamma(1) \\ A \times [0,c] & \xrightarrow[G]{} & Y. \end{array}$$

To define $\widetilde{G}$ we first split up the map $\widetilde{g}$ into its components. More precisely, we define maps $\widetilde{g}_1\colon A \to X$ and $\widetilde{g}_2\colon A \to Y^{[0,1]}$ via the equality

$$\widetilde{g}(a,0) \;=\; \big(\widetilde{g}_1(a), \widetilde{g}_2(a)\big) \;\in\; P_f \subset X \times Y^{[0,1]} \qquad \text{for all } a \in A.$$

[1858] It follows from the Product Topology-Basics Lemma 5.4 (3) and the Constant Map Lemma 12.3 that this map is continuous.

[1859] An argument as in the proof of Lemma 154.13 (2) shows that the map H is indeed continuous.

[1860] As in the proof of the Path Space-Fibration Theorem 154.14 we prefer to work, for purely psychological reasons, with $A \times [0,c]$ instead of $A \times [0,1]$.

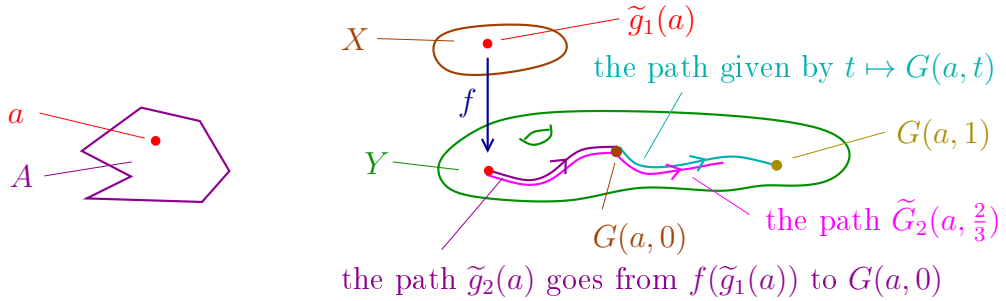
Note that by definition of P_f the path $\widetilde{g}_2(a) \in Y^{[0,1]}$ is a path from $f(\widetilde{g}_1(a))$ to $G(a,0)$. We need to define a suitable map $\widetilde{G}\colon A \times [0,c] \to P_f \subset X \times Y^{[0,1]}$. We consider the two factors separately:

(a) We already have the map $A \times [0,c] \to A \xrightarrow{\widetilde{g}_1} X$.

(b) To each $(a,s) \in A \times [0,c]$ we need to assign "systematically" a path $[0,1] \to Y$ from $f(x)$ to $G(a,s)$. The idea is quite simple: Given $(a,s) \in A \times [0,c]$ we consider the path $[0,1] \to Y$ that is given by first traveling along the path $\widetilde{g}_2(a)$ from $f(\widetilde{g}_1(a))$ to $G(a,0)$ and then we travel along the path $t \mapsto G(a,t)$ from $G(a,0)$ to $G(a,s)$. More precisely, similar to the proof of the Path Space-Fibration Theorem 154.14 we consider the map[1861]

$$\begin{array}{rcl} \widetilde{G}_2\colon A \times [0,c] & \to & Y^{[0,1]} \\ (a,s) & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & Y \\ t & \mapsto & \begin{cases} \widetilde{g}_2(a)\big(\frac{t}{1-\frac{s}{2c}}\big), & \text{if } t \in [0, 1-\frac{s}{2c}], \\ G(a, 2\cdot(t-1+\frac{s}{2})), & \text{if } t \in [1-\frac{s}{2c}, 1] \end{cases} \end{array} \right). \end{array}$$

We refer to the figure below for an illustration of the definition of $\widetilde{G}_2$.



Finally we consider the following map:

$$\begin{array}{rcl} \widetilde{G}\colon A \times [0,c] & \to & P_f \\ (a,s) & \mapsto & \big(\widetilde{g}_1(a), \widetilde{G}_2(a,s)\big). \end{array}$$

We make the following observations:

(a) It follows easily from the definitions that $\widetilde{G}$ takes values in P_f.

(b) It follows from the above and the Product Topology-Basics Lemma 5.4 (3) that $\widetilde{G}$ is continuous.

(c) It is straightforward to verify that $\widetilde{G}$ makes the diagram commute.

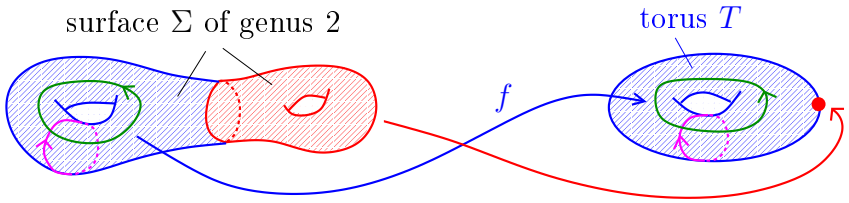
We have thus verified that $\widetilde{G}$ has all the desired properties. ■

The statement of the Replace-by-Fibration Proposition 154.17 that any map $f\colon X \to Y$ between topological spaces can be "replaced" by a fibration is quite surprising and perhaps also somewhat puzzling. Therefore we want to consider one more example.

Example. We denote by Σ the surface of genus 2 and we denote by T the torus. We consider the map $f\colon \Sigma \to T$ that is illustrated in the figure below. We denote by $p\colon P_f \to T$

[1861] The argument in the proof of the Path Space-Fibration Theorem 154.14 also shows that the map $\widetilde{G}_2$ does indeed take values in $Y^{[0,1]}$ and that the map $\widetilde{G}_2\colon A \times [0,c] \to Y^{[0,1]}$ is continuous.

the mapping path fibration and we denote by F the corresponding homotopy fiber. So what can we say about the mysterious homotopy fiber F?



We have the following diagram

$$\begin{array}{ccccccccccccc} \ldots & \longrightarrow & \pi_2(F) & \longrightarrow & \pi_2(P_f) & \xrightarrow{p_*} & \pi_2(T) & \xrightarrow{\partial} & \pi_1(F) & \longrightarrow & \pi_1(P_f) & \xrightarrow{p_*} & \pi_1(T) & \longrightarrow & \ldots \\ & & & & \uparrow{\scriptstyle h_*}\cong & \nearrow{\scriptstyle f_*} & & & & & \uparrow{\scriptstyle h_*}\cong & \nearrow{\scriptstyle f_*} & & & \\ & & & & \pi_2(\Sigma) & & & & & & \pi_1(\Sigma). & & & & \end{array}$$

We make the following explanations and observations:

(1) The top sequence is the long exact sequence coming from the Fibration-LES Theorem 154.7. The vertical maps are induced by the homotopy equivalence $h\colon \Sigma \to P_f$ from the Replace-by-Fibration Proposition 154.17 (1). In particular the vertical maps are isomorphisms by the Homotopy-π_n-Proposition 71.7 (2).
(2) Since $p \circ h = f$ we see that the diagram commutes.
(3) By the discussion on page 1575 we know that $\pi_n(T) = \pi_n(\Sigma) = 0$ for $n \geq 2$.
(4) It is pretty clear that the map $f_*\colon \pi_1(\Sigma) \to \pi_1(T)$ is an epimorphism.
(5) We obtain from the above discussion and from Proposition 64.8 that F is path-connected, that $\pi_n(F) = 0$ for $n \geq 2$ and that $\pi_1(F)$ is an *infinitely generated* free group.

Thus we see that the homotopy groups of F are isomorphic to the homotopy groups of an infinite wedge of circles.

154.7. **Adjoint functors.** In Chapter 17 we introduced the notion of a "cofibration" and we just got acquainted with the notion of a "fibration". In this section we will discuss the relationship between these two notions.
First we point out a formal similarity between fibrations and cofibrations, namely they both give rise to long exact sequences:

(1) Let $p\colon Y \to B$ be a fibration. Let $y_0 \in Y$. We write $b_0 = p(y_0)$ and we denote by $F = p^{-1}(b_0)$ the homotopy fiber of p. Let $i\colon F \to Y$ be the inclusion map. By the Fibration-LES Theorem 154.7 we obtain the following long exact sequence of homotopy groups:

$$\ldots \xrightarrow{\partial} \pi_n(F, y_0) \xrightarrow{i_*} \pi_n(Y, y_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, y_0) \xrightarrow{i_*} \ldots.$$

(2) Let $i\colon A \to X$ be a cofibration. We refer to the mapping cone $C := \mathrm{Cone}(i)$ as the *cofiber*. If $i(A)$ is a closed non-empty subset of X, then it follows easily from the Topological LES-Corollary 74.14 and Corollary 78.16 that we obtain a long exact sequence of reduced homology groups of the following form:

$$\ldots \xrightarrow{\partial} \widetilde{\mathrm{H}}_n(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_n(X) \xrightarrow{p_*} \widetilde{\mathrm{H}}_n(C) \xrightarrow{\partial} \widetilde{\mathrm{H}}_{n-1}(A) \xrightarrow{i_*} \ldots.$$

This basically recovers the statement of the Mapping Cone-H_*-Lemma 78.17.

Here is another similarity between fibrations and cofibrations:

(1) Let $p\colon Y \to B$ be a fibration and let $f\colon A \to B$ be a map. As in the Topological-Pullback Lemma 56.16 (1) we consider the pullback

$$\begin{array}{ccc} f^*Y & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[f]{} & B \end{array}$$

where $f^*Y = \{(a, y) \in A \times Y \mid f(a) = p(y)\}$. In the Fibration-Pullback Lemma 154.16 we saw that the pullback $f^*Y \to A$ is also a fibration.

(2) Now let $i\colon A \to X$ be a cofibration and let $f\colon A \to B$ be a map to another topological space. As on page 1260 we consider the pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow \\ X & \longrightarrow & f_*X \end{array}$$

where $f_*X = (X \sqcup B)/i(a) \sim f(a)$ for all $a \in A$. By the Opposite Map-Cofibration Proposition 17.18 (1) we will know that the pushout $B \to f_*X$ is again a cofibration.

So, to summarize: *the pullback of a fibration is again a fibration* and *the pushout of a cofibration is again a cofibration.*

The main goal of this section though is to formulate an arguably more direct relationship between the notions of a fibration and a cofibration. The key to doing so is the following definition from category theory.

Definition. We say that two covariant functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ between categories $\mathcal{C}$ and $\mathcal{D}$ form an adjoint pair if given any $c \in \mathrm{Ob}(\mathcal{C})$ and $d \in \mathrm{Ob}(\mathcal{D})$ there exists a bijection

$$\mathrm{Mor}_{\mathcal{D}}(F(c), d) \;\to\; \mathrm{Mor}_{\mathcal{C}}(c, G(d))$$

that is natural in c and d.[1862] We also say that the functor F is the left adjoint of G and that G is the right adjoint of F.

Examples.

(1) Let $\mathcal{S}$ be the category of sets and let $\mathcal{G}$ be the category of groups. We consider the two functors

$$\begin{array}{rcl} F\colon \mathcal{S} & \to & \mathcal{G} \\ S & \mapsto & \langle S \rangle \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G\colon \mathcal{G} & \to & \mathcal{S} \\ (\pi,\ \cdot\ ) & \mapsto & \pi. \end{array}$$

In other words, F associates to a set S the free group $\langle S \rangle$ generated by S that we defined on page 1158.[1863] Furthermore, G assigns to a group $(\pi,\ \cdot\ )$ the underlying

[1862] We leave it to our reader to figure out what "natural in c and d" means.

[1863] In what sense is this a functor?

set π.[1864] Using Lemma 51.15 one sees that for a set S and a group π the map

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{S}}(S, \overbrace{\pi}^{=G(\pi,\ \cdot\ )}) & \to & \mathrm{Mor}_{\mathcal{G}}(\overbrace{\langle S \rangle}^{=F(S)}, (\pi,\ \cdot\ )) \\ (\text{map } h\colon S \to \pi \text{ of sets}) & \mapsto & \begin{array}{l}\text{the unique homomorphism } \langle S \rangle \to (\pi,\ \cdot\ ) \\ \text{that restricts to } h \text{ on the subset } S \subset \langle S \rangle\end{array} \end{array}$$

is a natural bijection. Thus we see that the functors F and G are adjoint.

(2) Let $\mathcal{AbGroup}$ be the category of abelian groups and let A be an abelian group. We consider the two functors

$$\begin{array}{rcl} F\colon \mathcal{AbGroup} & \to & \mathcal{AbGroup} \\ B & \mapsto & A \otimes B \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon \mathcal{AbGroup} & \to & \mathcal{AbGroup} \\ B & \mapsto & \mathrm{Hom}(A, B). \end{array}$$

Now let $B, C \in \mathcal{AbGroup}$. We consider the map[1865]

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{AbGroup}}(\overbrace{A \otimes B}^{=F(B)}, C) & \to & \mathrm{Mor}_{\mathcal{A}}(B, \overbrace{\mathrm{Hom}(A, C)}^{=G(C)}) \\ (f\colon A \otimes B \to C) & \mapsto & \left(\begin{array}{rcl} B & \to & \mathrm{Hom}(A, C) \\ b & \mapsto & \left(\begin{array}{rcl} A & \to & Y \\ a & \mapsto & f(a \otimes b) \end{array} \right) \end{array} \right). \end{array}$$

Using Lemma 89.2 (3) one can easily show that this map is a bijection. This shows that F and G form an adjoint pair.

(3) Let $\mathcal{Top}$ be the category of topological spaces. Rather similarly to the previous example we consider the two functors[1866]

$$\begin{array}{rcl} F\colon \mathcal{Top} & \to & \mathcal{Top} \\ X & \mapsto & X \times [0,1] \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon \mathcal{Top} & \to & \mathcal{Top} \\ X & \mapsto & X^{[0,1]}. \end{array}$$

Now let $X, Y \in \mathrm{Ob}(\mathcal{Top})$, i.e. let X and Y be topological spaces. We consider the map

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Top}}(\overbrace{X \times [0,1]}^{=F(X)}, Y) & \to & \mathrm{Mor}_{\mathcal{Top}}(X, \overbrace{Y^{[0,1]}}^{=G(Y)}) \\ (f\colon X \times [0,1] \to Y) & \mapsto & \left(\begin{array}{rcl} X & \to & Y^{[0,1]} \\ x & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & Y \\ t & \mapsto & f(x,t) \end{array} \right) \end{array} \right). \end{array}$$

It follows from the Compact-Open Exponential Proposition 12.6 that this map is well-defined and that it is a bijection.[1867] We have thus shown that F and G form an adjoint pair.

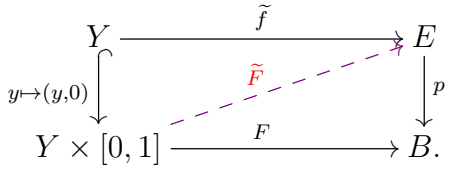
[1864] Such a functor, where we "forget" some of the given structure is often called a *forgetful functor*.

[1865] For abelian groups X and Y we have by definition that $\mathrm{Mor}_{\mathcal{AbGroup}}(X, Y) = \mathrm{Hom}(X, Y)$.

[1866] Here we equip $X \times [0,1]$ with the product topology and we equip $X^{[0,1]}$ with the compact-open topology.

[1867] More precisely, if we consider this map in the category of sets, then it is obviously a bijection. Note that the Compact-Open Exponential Proposition 12.6 shows that the map f on the left is continuous if and only if the corresponding map on the right is continuous. This shows that we also have a bijection between the sets of continuous maps, i.e. between the sets of morphisms in the category of topological spaces.

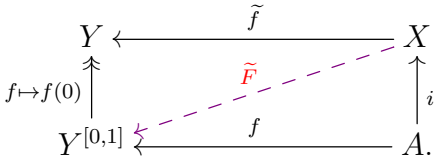
Now we can explain the formal relationship between fibrations and cofibrations. Let E and B be topological spaces and let $p\colon E \to B$ be a map. Recall that by definition $p\colon E \to B$ is a fibration if we can always complete the following diagram:



Now we "dualize" this diagram as follows:

(1) we reverse the orientation of each arrow,
(2) we replace monomorphisms by epimorphisms,
(3) we replace $Y \times [0,1]$ by $Y^{[0,1]}$,
(4) for purely psychological reasons we replace $p\colon E \to B$ by $i\colon A \to X$.

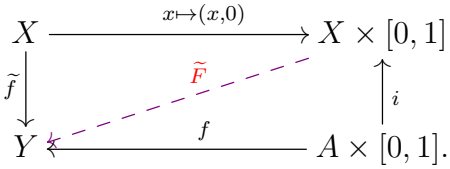
We obtain the following diagram:



Finally recall that we had just shown in Example (3) that given two topological spaces A and B we have a natural identification

$$\mathrm{Mor}(A \times [0,1], B) \;=\; \mathrm{Mor}(A, B^{[0,1]}).$$

Thus (with a few other more or less mysterious changes) the previous challenge diagram translates into the following challenge diagram:



But this challenge diagram looks familiar: it is precisely the diagram that appears in the definition of a cofibration.

We conclude this chapter with three quotes:

(1) "In the mathematical disciplines of algebraic topology and homotopy theory, Eckmann–Hilton duality in its most basic form, consists of taking a given diagram for a particular concept and reversing the direction of all arrows [...]."[1868] [1869]
(2) "Eckmann–Hilton duality can be extremely helpful as an organizational principle, reducing significantly what one has to remember, and providing valuable hints on how to proceed in various situations." [**Hat2002**, p. 464]

[1868] `https://en.wikipedia.org/wiki/Eckmann-Hilton_duality`
[1869] Note that the notion of pullback and pushout are related by Eckmann-Hilton duality.

(3) "In mathematics [...] currying[1870] is the technique of translating the evaluation of a function that takes multiple arguments into evaluating a sequence of functions, each with a single argument. For example, a function that takes two arguments, one from X and one from Y, and produces outputs in Z, by currying is translated into a function that takes a single argument from X and produces as outputs functions from Y to Z."[1871]

So we see that in the above discussion we performed Eckmann-Hilton duality combined with some currying.

Exercises for Chapter 154.

Exercise 154.1. Show that the projection

$$\begin{array}{rcl} \overline{B}^2 & \to & \overline{B}^1 = [-1,1] \\ (x,y) & \mapsto & x \end{array}$$

is a Serre fibration.

Exercise 154.2. . Let $n \in \mathbb{N}_0$ and let x_0 be a point on S^n. It follows from Proposition 12.4 that the map

$$\begin{array}{rcl} e\colon P(S^n, x_0) & \to & S^n \\ \{f\colon [0,1] \to S^n \mid f(0) = x_0\} & \mapsto & f(1) \end{array}$$

is continuous. For which $n \in \mathbb{N}_0$ does e admit a left-inverse?

Exercise 154.3. We consider the map $f\colon S^1 \to [-1,1]$ that is given by the projection onto the x-coordinate. We define the mapping path fibration $q\colon P_f \to [-1,1]$ as in the Replace-by-Fibration Proposition 154.17. Show that for each $y \in [-1,1]$ the preimage $q^{-1}(y)$ admits a deformation retraction to a circle.
Remark. See the figure on page 3289 for an illustration of the mapping path fibration.

Exercise 154.4. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}$. Show "with your bare hands" that the map

$$\begin{array}{rcl} \partial_n\colon \pi_n(X,x_0) & \overset{\cong}{\to} & \pi_{n-1}(\Omega(X,x_0), c_{x_0}) \\ [f\colon ([0,1]^n, \partial([0,1]^n)) \to (X,x_0)] & \mapsto & \left[\begin{array}{rcl} ([0,1]^{n-1}, \partial([0,1]^{n-1})) & \to & (\Omega(X,x_0), c_{x_0}) \\ x & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & X \\ t & \mapsto & f(x,t) \end{array} \right) \end{array} \right] \end{array}$$

is a natural isomorphism.
Remark. This is of course the statement of the Loop Space-Homotopy Groups Corollary 154.15.

[1870] This notion is named after the American mathematician Haskell Brooks Curry (1900-1982) who was an American mathematician and logician. There are three programming languages named after him, namely Haskell, Brook and Curry.
[1871] https://en.wikipedia.org/wiki/Currying#Algebraic_topology

Exercise 154.5. Let (X, x_0) be a pointed topological space. In the Loop Space-Homotopy Groups Corollary 154.15 we give for any $n \geq 1$ a natural isomorphism

$$\pi_n(X, x_0) \stackrel{\cong}{\to} \pi_{n-1}(\Omega(X, x_0), c_{x_0}).$$

What does this statement mean for $n = 1$? It seems to say that $\pi_0(\Omega(X, x_0), c_{x_0})$ admits a natural group structure. Can you define such a natural group structure without using the above isomorphism?

Exercise 154.6. Convince yourself, without using the Loop Space-Homotopy Groups Corollary 154.15, that the loop in the pointed topological space $(\Omega(S^2, N), c_N)$ that is shown in the figure on page 3285 is not null-homotopic.

Exercise 154.7. Let X be a topological space and let $x_0 \in X$ be a base point. Recall that the loop space $\Omega(X, x_0)$ is defined as the set $\Omega(X, x_0) = \{f \in X^{[0,1]} \mid f(0) = f(1) = x_0\}$ together with the compact-open topology. We denote by $\sim$ the equivalence relation on $\Omega(X, x_0)$ given by homotopy of paths. By definition we have $\pi_1(X, x_0) = \Omega(X, x_0)/\!\sim$. We equip $\pi_1(X, x_0)$ with the quotient topology. We can thus view $\pi_1(X, x_0)$ as a topological space. Show that $\pi_1(S^1, 1)$ is a discrete space.
Hint. It is possibly easier to show that for any pointed topological space (X, x_0) that admits a universal covering the fundamental group $\pi_1(X, x_0)$ is a discrete space.
Remark. There are examples of pointed topological spaces for which $\pi_1(X, x_0)$ is not a discrete space. For example the fundamental group of the Hawaiian Earrings, introduced in Exercise 56.1, is not a discrete space. We refer to [**Fab2005, Fab2011**] for details.

Exercise 154.8.

(a) Show that there is no functor

$$\Phi\colon \begin{array}{c}\text{category } \mathcal{PTop} \text{ of pointed}\\ \text{topological spaces}\end{array} \to \begin{array}{c}\text{category of } \mathcal{PTop} \text{ pointed}\\ \text{topological spaces}\end{array}$$

which admits for every $n \in \mathbb{N}$ a natural isomorphism

$$\pi_n(X, x_0) \stackrel{\cong}{\to} \pi_{n+1}(\Phi(X, x_0)).$$

(b) Show that there is no functor

$$\Psi\colon \text{category of topological spaces} \to \text{category of topological spaces}$$

which admits for every $n \in \mathbb{N}$ a natural isomorphism

$$\widetilde{\mathrm{H}}_n(X) \stackrel{\cong}{\to} \widetilde{\mathrm{H}}_{n-1}(\Psi(X)).$$

Hint. In Exercise 74.4 (b) we proved the following statement. If X is a topological space with $\widetilde{\mathrm{H}}_0(X) \cong \mathbb{Z}$, then given any map $f\colon X \to X$ the induced map f_* on $\widetilde{\mathrm{H}}_0(X)$ is given by multiplication by some $\epsilon \in \{-1, 0, 1\}$.

Exercise 154.9. Let X be a topological space. We consider the map

$$\begin{array}{rcl} p\colon X^{[0,1]} & \to & X \times X \\ (f\colon [0,1] \to X) & \mapsto & (f(0), f(1)) \end{array}$$

(a) Show that $p\colon X^{[0,1]} \to X \times X$ is a fibration.

(b) Show that there exists a left inverse to p, i.e. a map $s\colon X \times X \to X^{[0,1]}$ with $p \circ s = \mathrm{id}$, if and only if X is contractible.

155. Fiber bundles

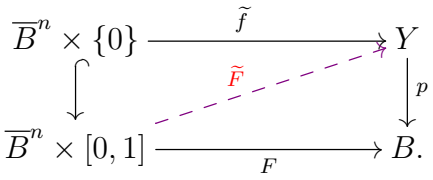
In this chapter we will introduce fiber bundles. For us they will be the most important source of fibrations. In this and the following chapter one of our favorite games will be to find more and more fiber bundles to obtain lots of non-trivial information on homotopy groups of various topological spaces.

155.1. **The definition of fiber bundles.** The following proposition says that being a Serre fibration is a "local property".

Proposition 155.1. Let $p\colon Y \to B$ be a map between topological spaces. If for every $b \in B$ there exists an open neighborhood U of b such that $p\colon p^{-1}(U) \to U$ is a Serre fibration, then $p\colon Y \to B$ itself is a Serre fibration.

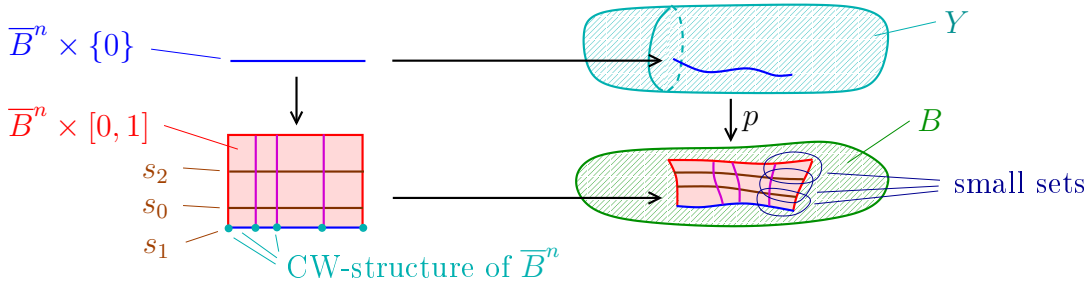
Proof. Let $p\colon Y \to B$ be a map between topological spaces. In this proof we say $U \subset B$ is *small* if $p\colon p^{-1}(U) \to U$ is a Serre fibration.

Now assume that $p\colon Y \to B$ has the property that B can be covered by small sets. We want to show that this hypothesis implies that the map $p\colon Y \to B$ is a Serre fibration. Therefore let $n \in \mathbb{N}_0$, let $F\colon \overline{B}^n \times [0,1] \to B$ and let $\widetilde{f}\colon \overline{B}^n \times \{0\} \to Y$ be maps such that the rectangle in the following diagram commutes:



We need to show that there exists a diagonal map $\widetilde{F}\colon \overline{B}^n \times [0,1] \to Y$ such that the above diagram commutes.

Claim. There exists a CW-structure X on $\overline{B}^n$ and real numbers $0 = s_0 < s_1 < \cdots < s_k = 1$ such that for each cell e of X and each $i \in \{0, \ldots, k-1\}$ the image $F(e \times [s_i, s_{i+1}])$ is small.



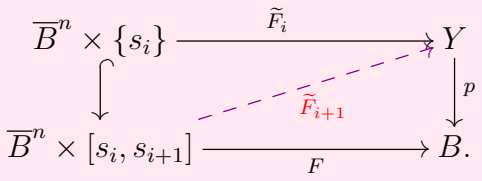
Proof. By the Standard Simplex Homeomorphism Lemma 72.1 we can identify $\overline{B}^n$ with $[0,1]^n$. Since B is by hypothesis covered by small open subsets and since any subset of a small set is again small we obtain from the Lebesgue Corollary 6.6 that there exists a $k \in \mathbb{N}$ such that each

$$F\Big(\underbrace{\left[\tfrac{a_1}{k}, \tfrac{a_1+1}{k}\right] \times \ldots \left[\tfrac{a_n}{k}, \tfrac{a_n+1}{k}\right]}_{\subset [0,1]^n = \overline{B}^n} \times \underbrace{\left[\tfrac{a_{n+1}}{k}, \tfrac{a_{n+1}+1}{k}\right]}_{\subset [0,1]}\Big)$$

is small for all $a_1, \dots, a_{n+1} \in \{0, \dots, k-1\}$. We equip $[0,1]^n = \overline{B}^n$ with the CW-structure defined on page 1499 given by the "subcubes" of side length $\frac{1}{k}$. Furthermore we set $s_i = \frac{i}{k}$, $i = 0, \dots, k$. By design this CW-structure and this choice of s_i has the desired properties. ⊞

We continue with the following claim.

Claim. Let $i \in \{0, \dots, k-1\}$. Given a map $\widetilde{F}_i \colon \overline{B}^n \times \{s_i\} \to Y$ with $p \circ \widetilde{F}_i = F|_{\overline{B}^n \times \{s_i\}}$ there exists a map $\widetilde{F}_{i+1} \colon \overline{B}^n \times [s_i, s_{i+1}] \to Y$ such that the following diagram commutes:



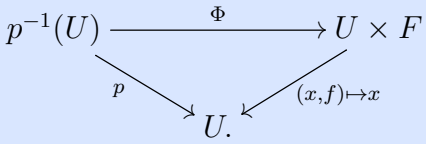
Proof. In fact the claim follows immediately from the argument of the proof of Proposition 154.8 applied to the CW-complex $K := X = \overline{B}^n$ and the subcomplex $L = \varnothing$. Indeed, in the proof of Proposition 154.8 we did not need to assume that $p \colon Y \to B$ is a Serre fibration, all we ever used is that for every cell e of B the image $F(e)$ is small in the above sense. But in our present setting this is guaranteed by our choice of the CW-structure on $\overline{B}^n$. ⊞

We set $\widetilde{F}_0 = \widetilde{f}_0 \colon \overline{B}^n \times \{0\} \to Y$. Iteratively applying the previous claim we obtain maps $\widetilde{F}_i \colon \overline{B}^n \times [s_i, s_{i+1}] \to Y$ that agree on the overlaps of the domains. By the Homotopy Stacking Lemma 14.3 all these maps together define a continuous map $\widetilde{F} \colon \overline{B}^n \times [0,1] \to Y$ which by construction has the desired properties. ■

As we saw on page 3264, every projection $p \colon B \times F \to B$ is a (Serre) fibration. This observation, together with Proposition 155.1 leads us naturally to the following definition.

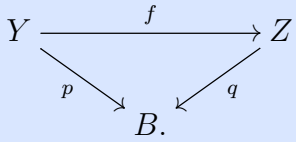
Definition. Let F be a topological space.

(1) A map $p \colon Y \to B$ between two topological spaces is called an F-bundle over B if given any $b \in B$ there exists an open neighborhood U of b and a homeomorphism $\Phi \colon p^{-1}(U) \to U \times F$ such that the following diagram commutes:



Alternatively we call such a map $p \colon Y \to B$ a bundle projection with fiber F. As for fibrations we sometimes refer to B as the base space and to Y as the total space. Furthermore we refer to the map $p \colon Y \to B$ as the projection.

(2) We say that two F-bundles $p \colon Y \to B$ and $q \colon Z \to B$ are equivalent if there exists a homeomorphism $f \colon Y \to Z$ such that the following diagram commutes:



The following proposition will be the incarnation of the Fibration-LES Theorem 154.7 that we will use most often.

Proposition 155.2. (Fiber-LES Proposition) Let $p\colon Y \to B$ be a fiber bundle with fiber F.

(1) The map $p\colon Y \to B$ is a Serre fibration.

(2) We pick $y_0 \in Y$ and we write $b_0 = p(y_0)$. We pick an identification $F \xrightarrow{\cong} p^{-1}(b_0)$. We denote by $i\colon F = p^{-1}(b_0) \to Y$ the inclusion. We use the fact from (1) that $p\colon Y \to B$ is a Serre fibration to define maps $\partial_n\colon \pi_n(B, b_0) \to \pi_{n-1}(F, y_0)$, $n \in \mathbb{N}$, as in the Fibration-LES Theorem 154.7. The following sequence is exact:

$$\begin{aligned}\ldots \xrightarrow{\partial_{n+1}} \pi_n(F, y_0) \xrightarrow{i_*} \pi_n(Y, y_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial_n} \\ \xrightarrow{\partial_n} \pi_{n-1}(F, y_0) \xrightarrow{i_*} \ldots\ldots\ldots \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\partial_1} \\ \xrightarrow{\partial_1} \pi_0(F, y_0) \xrightarrow{i_*} \pi_0(Y, y_0) \xrightarrow{p_*} \pi_0(B, b_0).\end{aligned}$$

(3) If we are given a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B \end{array}$$

where $p'\colon Y' \to B'$ is a fiber bundle with fiber F', then given any base point $y_0' \in Y'$ the obvious diagram of the long exact sequences of homotopy groups corresponding to the fiber bundles $p\colon Y \to B$ and $p'\colon Y' \to B'$ is commutative.

(4) If the fiber F and the base space B are path-connected, then the total space Y is also path-connected.

Proof.

(1) This part of the proposition is an immediate consequence of the definitions, Proposition 155.1 and the discussion on page 3264.

(2) The second statement follows immediately from (1) and the Fibration-LES Theorem 154.7 (2).

(3) This statement follows easily from the Fibration-LES Theorem 154.7 (1b).

(4) This statement follows immediately from (2). ■

the Fiber-LES Proposition 155.2 sparks our interest in fiber bundles. Fortunately, in the following discussion we will see that fiber bundles are ubiquitous and can be found throughout the notes.

Examples.

(1) For any two topological spaces B and F the projection $p\colon B \times F \to B$ onto the first factor is evidently a fiber bundle with fiber F. Any F-bundle that is equivalent to such a bundle is called *trivial*, otherwise it is called non-trivial.

(2) The "obvious projection" from the Möbius band to the circle S^1, i.e. the map

$$\begin{array}{rcl} Y = ([0,1] \times [-1,1])/(0,y) \sim (1,-y) & \to & S^1 \\ (x,y) & \mapsto & \exp(2\pi \mathrm{i} x) \end{array}$$

is easily seen to be a fiber bundle with fiber $[-1,1]$. Since the Möbius band is not homeomorphic to the trivial bundle $S^1 \times [-1,1]$ we see that the bundle is non-trivial.

(3) The same argument as in (2) shows that the Klein bottle is a non-trivial S^1-bundle over S^1. In Lemma **??** (1) we saw that the connected sum $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ is homeomorphic to the Klein bottle. We have thus shown that $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ is a non-trivial S^1-bundle over S^1. Amusingly an analogous statement holds in dimension four, namely the connected sum $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}^2}$ is a non-trivial S^2-bundle over S^2. It is a challenging exercise to prove this fact by hand, alternatively a proof can be found in [**GoS1999**, p. 107] or [**Scor2005**, p. 124]. Similarly, one can show that the connected sum $\mathbb{H}\mathrm{P}^2 \# \overline{\mathbb{H}\mathrm{P}^2}$ is a non-trivial S^4-bundle over S^4.

(4) An n-dimensional vector bundle $p\colon W \to B$, in the sense of the definition on page 3058, is by definition an $\mathbb{R}^n$-bundle over B.

We consider the next example in greater detail.

Lemma 155.3.

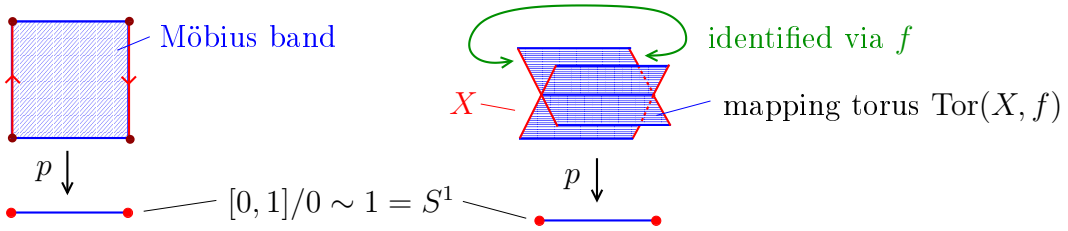
(1) Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. The natural projection

$$\begin{array}{rcl} \overbrace{\mathrm{Tor}(X,f) = ([0,1]\times X)/(0,x) \sim (1,f(x))}^{\text{basically the mapping torus of } (X,f)} & \to & [0,1]/0 \sim 1 = S^1 \\ [(t,x)] & \mapsto & [t] = \exp(2\pi\,\mathrm{i}t) \end{array}$$

is an X-bundle over S^1.

(2) Let X be a topological space and let $f_0, f_1\colon X \to X$ be two homeomorphisms. If f_0 and f_1 are isotopic, then the two X-bundles $\mathrm{Tor}(X,f_0) \to S^1$ and $\mathrm{Tor}(X,f_1) \to S^1$ are equivalent.

(3) Let V be an n-dimensional real vector space and let $\varphi\colon V \to V$ be an isomorphism of vector spaces. We equip V with the topology given on page 153. For each $t \in [0,1)$ we equip $\{t\} \times V \subset M(V,\varphi)$ with the obvious vector space structure. This turns the projection $M(V,\varphi) \to S^1$ into an n-dimensional vector bundle.



Proof.

(1) We denote by $\pi\colon [0,1] \to [0,1]/0 \sim 1$ the natural projection. We consider the two open sets $U_1 := \pi((\frac{1}{4},\frac{3}{4}))$ and $U_2 := \pi([0,\frac{1}{3}) \cup (\frac{2}{3},1])$ in $[0,1]/0 \sim 1$. Furthermore we consider the maps

$$\begin{array}{rcl} p^{-1}(U_1) & \xrightarrow{\Phi_1} & U_1 \times X \\ [(t,x)] & \mapsto & (t,x) \end{array} \quad \text{and} \quad \begin{array}{rcl} p^{-1}(U_2) & \xrightarrow{\Phi_2} & U_2 \times X \\ [(t,x)] & \mapsto & \begin{cases} (t,f(x)), & \text{if } t \in [0,\frac{1}{3}), \\ (t,x) & \text{if } t \in (\frac{2}{3},1]. \end{cases} \end{array}$$

Note that the map on the right is indeed well-defined. Using the Topological-Quotient Proposition 5.15 (1b) and using Lemma 14.3 one can show quite easily that both maps are homeomorphisms. Furthermore it is basically clear that Φ_1 and Φ_2 respect the projections onto U_1 and U_2. Therefore, since the two open subsets U_1 and U_2 cover all of $[0,1]/0 \sim 1$, we have shown that $p\colon \mathrm{Tor}(X,f) \to [0,1]/0 \sim 1$ is indeed an X-bundle.

(2) This statement will be proved in Exercise 155.3.

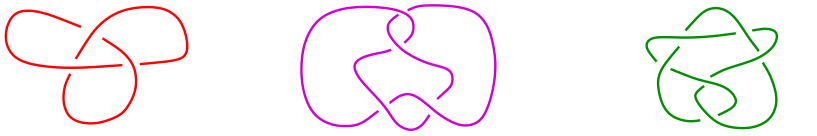
(3) The proof of this statement is basically the same as the proof of (1). We only have to make the extra observation that for each $i \in \{1,2\}$ and each $t \in U_i$ the map

$$\Phi_i\colon V = [\{t\} \times V] = p^{-1}(t) \;\to\; \{t\} \times V = V$$

is either given by id or ϕ. Thus in either case the map is given by an *isomorphism* of vector spaces. ■

Remark. The previous lemma shows that any self-homeomorphism of $\mathbb{R}^n$ gives rise to an $\mathbb{R}^n$-bundle over S^1, whereas any isomorphism of $\mathbb{R}^n$ gives rise to an n-dimensional vector bundle over S^1. The question arises whether every $\mathbb{R}^n$-bundle is secretly a vector bundle. More precisely, the question arises whether every $\mathbb{R}^n$-bundle is equivalent to an n-dimensional vector bundle. In Exercise 155.4 we will answer this question in the affirmative if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism.[1872]

It follows from the discussion on page 470 and Lemma 155.3 that the complement of the trefoil and the complement of the figure-8 knot are fiber bundles over S^1. Of course there are many other knots, for example consider the stevedore knot shown in the figure below. The following question arises:



trefoil figure-8 knot stevedore knot

Question 155.4. Is the complement of the stevedore knot a fiber bundle over S^1?

We continue with more examples of fiber bundles.

Examples.

(1) As we pointed out above, any n-dimensional vector bundle $p\colon W \to B$ is by definition an $\mathbb{R}^n$-bundle over B. In fact p gives rise to two other bundles which are much more interesting topological objects:

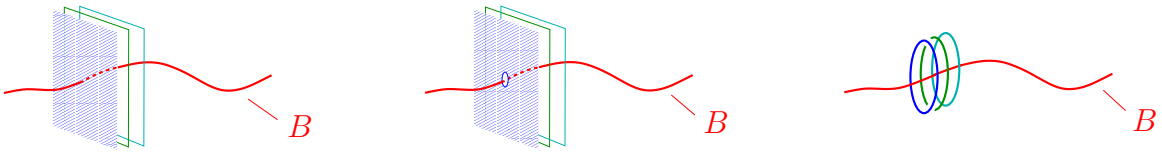
(a) We denote by $s\colon B \to W$ the zero-section, i.e. the map given by sending each $b \in B$ to the zero-element of the vector space $p^{-1}(y)$. We write $W^* := W \setminus s(B)$. It follows easily from the definitions that $p\colon W^* \to B$ is a fiber bundle with fiber $\mathbb{R}^n \setminus \{0\}$.

[1872] Frankly, I am not sure whether the same statement holds if f is a homeomorphism.

(b) Let $\lambda \in \mathbb{R}$. By definition of a vector bundle we have a well-defined multiplication by λ on each fiber. We define

$$SW := W^*/\sim \qquad \text{where } v \sim w \text{ if } p(v) = p(w) \text{ and if there exists a } \lambda > 0 \text{ with } \lambda v = w.$$

The map $p\colon W^* \to B$ descends to a map $p\colon SW \to B$. We leave it as an entertaining exercise to verify that $p\colon SW \to B$ is an S^{n-1}-bundle.



$\mathbb{R}^n$-bundle $p\colon W \to B$ $(\mathbb{R}^n\backslash\{0\})$-bundle $p\colon W^* \to B$ S^{n-1}-bundle $p\colon SW \to B$

(2) A smooth $\overline{B}^n$-bundle $p\colon W \to M$ over a smooth manifold M, in the sense of the definition on page **??**, is by definition an $\overline{B}^n$-bundle in the above sense. Slightly more interesting, if M has no boundary and if we restrict p to a map $\partial W \to M$, then it follows easily from Proposition 19.31 that the restriction of p to ∂W defines an S^{n-1}-bundle $p\colon \partial W \to M$.

The following lemma is almost verbatim the same as the Vector Bundle Pullback Lemma 143.9.

Lemma 155.5. Let $g\colon C \to B$ be a map between topological spaces and let $p\colon Y \to B$ be a fiber bundle with fiber F. As in the Topological-Pullback Lemma 56.16 (1) we consider the pullback

$$g^*Y := \{(c, f) \,|\, c \in C \text{ and } f \in p^{-1}(g(c))\} \subset C \times Y,$$

which is equipped with the subspace topology. Then the map

$$\begin{aligned} q\colon g^*Y := \{(c, f) \,|\, c \in C \text{ and } f \in p^{-1}(g(c))\} &\to C \\ (c, f) &\mapsto c \end{aligned}$$

is also an F-bundle.

Proof. The proof is, not surprisingly, also almost identical to the proof of the Vector Bundle Pullback Lemma 143.9. ∎

Remark. The pullback of bundles satisfies basically the same statements as the pullback of vector bundles, see the Vector Bundle Pullback Lemma 143.9. There is surely no need to spell out these properties again.

The following lemma classifies bundles over compact intervals $[a, b]$ and also over the circle S^1.

Lemma 155.6.

(1) Every bundle over a compact interval $[a, b]$ is trivial.
(2) If $p\colon Y \to S^1$ is a fiber bundle with fiber X, then there exists a homeomorphism $f\colon X \to X$ such that p is equivalent to the above fiber bundle $\mathrm{Tor}(X, f) \to S^1$.

Sketch of proof.

(1) This statement will be proved in Exercise 155.2.

(2) Suppose we are given a fiber bundle $p\colon Y \to S^1$ with fiber X. Let $\varphi\colon [0,2\pi] \to S^1$ be the map that is given by $\varphi(t) = \exp(\mathrm{i}t)$. It follows from (1) that the pullback bundle

$$q\colon \varphi^*Y = \{(t,v) \in [0,2\pi] \times Y \,|\, v \in p^{-1}(\varphi(t))\} \;\to\; [0,2\pi]$$

is trivial, i.e. there exists a homeomorphism $\Phi\colon [0,2\pi] \times X \to \varphi^*Y$ such that the following diagram commutes

$$\begin{array}{ccc} [0,2\pi] \times X & \xrightarrow{\quad\Phi\quad} & \varphi^*Y \\ & {\scriptstyle (t,x)\mapsto t} \searrow \quad \swarrow {\scriptstyle q} & \\ & [0,2\pi]. & \end{array}$$

We define $f\colon X \to X$ to be the homeomorphism that is given by

$$\begin{array}{ccccccccccc} & & & & & & \varphi(0) = 1 = \varphi(2\pi) & & & & \\ & & & & & & \downarrow & & & & \\ X & \xrightarrow{x\mapsto(0,x)} & \{0\}\times X & \xrightarrow{\Phi} & q^{-1}(0) & \to & p^{-1}(1) & \leftarrow & q^{-1}(2\pi) & \xleftarrow{\Phi} & \{2\pi\}\times X \xrightarrow{(2\pi,x)\mapsto x} X. \\ & & & & (0,v) & \mapsto & v & & & & \\ & & & & & & v & \mapsto & (2\pi,v). & & \end{array}$$

We leave it as a mildly interesting exercise to the reader to verify that $p\colon Y \to S^1$ is equivalent to the fiber bundle $\mathrm{Tor}(X,f) \to S^1$. ■

For the record we mention the following generalization of Lemma 155.6 (1).

Proposition 155.7. Let X be a topological space. If X is paracompact and contractible, then every bundle over X is trivial.

Proof. The statement of the proposition is almost the same as the statement of the Vector Bundles-Homotopies Proposition 143.13 (3) for vector bundles. The reference we gave for vector bundles, namely [**Hat2**, Theorem 1.6], also applies to general bundles. ■

We conclude this section with an example that shows how the above results can be used in practice. Thus let $n \in \mathbb{N}$ and K be a closed k-dimensional submanifold of the smooth manifold S^n. We calculate that for $i = 1, \ldots, n-k-2$ we have

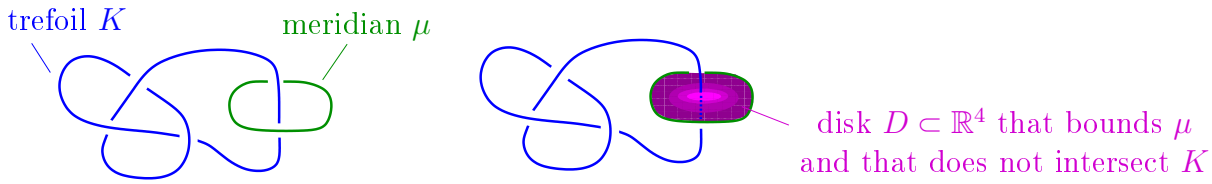
$$\begin{array}{ccccc} \mathrm{H}_i(S^n \setminus K;\mathbb{Z}) & \cong & \mathrm{H}^{n-i-1}(K;\mathbb{Z}) & = & 0. \\ & \uparrow & & \uparrow & \\ & \text{Alexander Duality} & & \text{by the Top-Cohomology Proposition 110.14 (5) and} & \\ & \text{Theorem 123.6} & & \text{the fact that } n-i-1 > k = \dim(K) & \end{array}$$

The following proposition shows that not only do the first $n-k-2$ homology vanish, but the first $n-k-2$ homotopy groups also vanish.

Proposition 155.8. Let $n \in \mathbb{N}$ and let K be a closed k-dimensional submanifold of the smooth manifold S^n. For $i = 1, \ldots, n-k-2$ we have $\pi_i(S^n \setminus K) = 0$.

Example. Let K be the trefoil in $\mathbb{R}^3$ and let $\mu \subset \mathbb{R}^3$ be a meridian as defined on page 1305. (See also the figure below to the left.) In Lemma 78.13 we saw that μ represents a non-zero element in $\mathrm{H}_1(S^3 \setminus K)$. It follows from the naturality of the Hurewicz homomorphism, see Proposition 84.2 (5), that μ also represents a non-trivial element in $\pi_1(S^3 \setminus K)$. Via the usual smooth embedding $\mathbb{R}^3 \to \mathbb{R}^3 \times \{0\} \to \mathbb{R}^4$ we can view K and μ as submanifolds of $S^4 = \mathbb{R}^4 \cup \{\infty\}$. By Proposition 155.8 we have $\pi_1(S^4 \setminus K) = 0$. In particular we obtain

that μ is trivial in $\pi_1(S^4 \setminus K)$. In fact in the figure below we show an embedded disk D in $\mathbb{R}^4 \subset S^4$ whose boundary is given by μ and that does not intersect K. Here we use the convention from page **??**, i.e. "pinkness" of a point measures how large is the fourth coordinate.



Proof. Let $n \in \mathbb{N}$ and K be a closed k-dimensional submanifold of the smooth manifold S^n. If $n-k-2 \leq 0$, then there is nothing to prove. Thus we assume that $n-k-2 \geq 1$, i.e. we assume that $n \geq k+3$.

We want to show that $\pi_i(S^n \setminus K) = 0$ for $i = 1, \dots, n-k-2$. We already know that the corresponding homology groups vanish. Thus by the Hurewicz Corollary 85.7 it suffices to show that $\pi_1(S^n \setminus K) = 0$. That is precisely what we will do in the remainder of this proof.

In the following we consider the case that K is connected. We leave it to the reader to modify the proof to deal with the case that K is disconnected. By the General Tubular Neighborhood Theorem **??** the closed submanifold K admits a tubular neighborhood $p\colon W \to K$. Recall that W is a compact n-dimensional submanifold of S^n and that p is in particular a $\overline{B}^{n-k}$-bundle over K. As we pointed out on page 3303, the projection $p\colon \partial W \to K$ is an S^{n-k-1}-bundle.

We write $X = S^n \setminus \mathring{W}$. By Proposition 123.4 and Lemma 123.1 we know that $\pi_1(S^n \setminus K)$ is isomorphic to $\pi_1(X)$. Thus it suffices to show that X is simply connected.

By the Codimension-Zero Submanifold Proposition 19.38 we know that X is a compact n-dimensional submanifold of S^n with $W \cap X = \partial W = \partial X$. From Proposition 155.2 (4) we know that ∂W is connected. We pick a base point $x_0 \in \partial W$. By the Seifert–van Kampen Theorem 54.4 for smooth manifolds we know that the inclusion induced map

$$\pi_1(X, x_0) *_{\pi_1(\partial W, x_0)} \pi_1(W, x_0) \ \to \ \underbrace{\pi_1(S^n, x_0)}_{=0}$$

is an isomorphism. We recall that the Amalgamated Product Lemma 53.22 says that $A *_G G \cong A$. Thus, if we want to show that $\pi_1(X, x_0) = 0$ it follows from the Amalgamated Product Lemma 53.22 that it suffices to prove the following claim.

Claim. The inclusion induced map $\pi_1(\partial W, x_0) \to \pi_1(W, x_0)$ is an isomorphism.

Proof. We denote both of the inclusions $\partial W \to W$ and $S^{n-k-1} \to \overline{B}^{n-k}$ by i. We have a commutative diagram

$$\begin{array}{ccc} \partial W & \xrightarrow{\ i\ } & W \\ & \searrow \quad \swarrow & \\ & K & \end{array}$$

of fiber bundles with fibers S^{n-k-1} and $\overline{B}^{n-k}$. From the Fiber-LES Proposition 155.2 (3) we obtain the following commutative diagram of long exact sequences of morphisms between

pointed sets:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & \pi_2(K) & \xrightarrow{\partial} & \pi_1(S^{n-k-1}) & \longrightarrow & \pi_1(\partial W) & \longrightarrow & \pi_1(K) & \xrightarrow{\partial} & \pi_0(S^{n-k-1}) & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle i_*} & & \\
\ldots & \longrightarrow & \pi_2(K) & \xrightarrow{\partial} & \pi_1(\overline{B}^{n-k}) & \longrightarrow & \pi_1(W) & \longrightarrow & \pi_1(K) & \xrightarrow{\partial} & \pi_0(\overline{B}^{n-k}) & \longrightarrow & \ldots
\end{array}$$

Since $n-k \geq 3$ we have $n-k-1 \geq 2$, thus $\pi_1(S^{n-k-1}) = 0$ by the Sphere-π_1-Proposition 47.13. It follows from a slight generalization[1873] of the Five Lemma 74.10 that the middle vertical map $i_*\colon \pi_1(\partial W) \to \pi_1(W)$ is an isomorphism. ■

Remark. We want to sketch an alternative approach to proving Proposition 155.8. Thus let $n \in \mathbb{N}$, let K be a closed k-dimensional submanifold of the smooth manifold S^n and finally let $i \in \{1, \ldots, n-k-2\}$. We want to show that $\pi_i(S^n \setminus K, x_0) = 0$ for some base point $x_0 \in S^n \setminus K$. Let $[\varphi\colon (S^i, *) \to (S^n \setminus K, x_0)] \in \pi_i(S^n \setminus K, x_0)$ be a random element. We make the following observations:

(1) By the Whitney Approximation Theorem 29.1 we can assume that φ is smooth.
(2) Since $i < n$ we know that $\pi_i(S^n, x_0) = 0$. It follows from the Trivial-in-π_n-Lemma 71.4 that there exists a map $\psi\colon \overline{B}^{i+1} \to S^n$ with $\psi|_{S^i} = \varphi$.
(3) Once again, by the Whitney Approximation Theorem 29.1 we can assume that ψ is also smooth.
(4) Since $k+i+1 < n$ one would like to argue, similar to the Transversality Theorem 42.4, that we can homotop ψ rel S^i to a map that does not intersect K, i.e. we would like to homotop ψ rel S^i to a map $\overline{B}^{i+1} \to S^n \setminus K$. By the Trivial-in-$\pi_n$-Lemma 71.4 this would then imply that $[\varphi] = 0 \in \pi_i(S^n \setminus K, x_0)$.

This approach can be turned into a proper proof, but we will not do so.

155.2. Hopf fibrations arising from complex and quaternionic projective spaces.

In this section we will see that the complex projective spaces $\mathbb{C}\mathrm{P}^n$ and the quaternionic projective spaces[1874] $\mathbb{H}\mathrm{P}^n$ give rise to interesting fiber bundles. In particular we will see that the Hopf map $S^3 \to S^2$ that we introduced on page 1576 is in fact an S^1-bundle.
In the following we introduce two straightforward higher-dimensional analogues of the original Hopf map.

Definition. Let $n \in \mathbb{N}$. We refer to the maps

$$\begin{array}{rcl}
p\colon S^{2n+1} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \,|\, |z_0|^2 + \cdots + |z_n|^2 = 1\} & \to & \mathbb{C}\mathrm{P}^n \\
(z_0, \ldots, z_n) & \mapsto & [z_0 : \cdots : z_n]
\end{array}$$

and[1875]

$$\begin{array}{rcl}
q\colon S^{4n+3} = \{(h_0, \ldots, h_n) \in \mathbb{H}^{n+1} \,|\, |h_0|^2 + \cdots + |h_n|^2 = 1\} & \to & \mathbb{H}\mathrm{P}^n \\
(h_0, \ldots, h_n) & \mapsto & [h_0 : \cdots : h_n]
\end{array}$$

as Hopf maps.

[1873] More precisely, if one goes through the proof of the Five Lemma 74.10, then one realizes that it suffices that the right most vertical map is an injective morphism of pointed sets, i.e. the right most vertical map does not need to be a homomorphism of groups.
[1874] We refer to page 1970 for the slightly subtle definition of the quaternionic projective spaces.

The following proposition, together with the Fiber-LES Proposition 155.2, explains our interest in the Hopf maps.[1876]

Proposition 155.9. (Hopf-Bundle Proposition) Let $n \in \mathbb{N}$.
(1) The Hopf map $p\colon S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ is an S^1-bundle.
(2) The Hopf map $q\colon S^{4n+3} \to \mathbb{H}\mathrm{P}^n$ is an S^3-bundle.

Proof. We only provide the proof of (2). The proof of (1) is a straightforward modification of the proof of (2). Given $j \in \{0, \dots, n\}$ we define

$$V_j \;:=\; \{[h_0 : \cdots : h_n] \in \mathbb{H}\mathrm{P}^n \,|\, h_j \neq 0\}.$$

The same argument as in the proof of the Complex Projective Space-is-Complex Manifold Lemma 32.16 shows that V_j is an open subset of $\mathbb{H}\mathrm{P}^n$.

Claim. The map

$$\begin{aligned}\Phi_j\colon q^{-1}(V_j) = \left\{(h_0, \dots, h_n) \in \mathbb{H}^{n+1} \;\middle|\; \sum_{i=0}^{n} |h_i|^2 = 1 \text{ and } h_j \neq 0\right\} &\to V_j \times S^3 \\ (h_0, \dots, h_n) &\mapsto \left([h_0 : \cdots : h_n], \tfrac{1}{|h_j|} \cdot h_j\right)\end{aligned}$$

is a homeomorphism.

Proof. We consider the map

$$\begin{aligned}\Psi_j\colon V_j \times S^3 &\to q^{-1}(V_j) \\ ([h_0 : \cdots : \underbrace{h_j}_{\neq 0} : \cdots : h_n], u) &\mapsto \text{the normalization to length one of the non-zero vector} \\ &\qquad \left(u \cdot h_j^{-1} \cdot h_0, \dots, \underbrace{u \cdot h_j^{-1} \cdot h_j}_{=u}, \dots, u \cdot h_j^{-1} \cdot h_n\right) \in \mathbb{H}^{n+1}.\end{aligned}$$

This map Ψ_j is well-defined. Indeed, a short calculation shows that if we pick a different representative for $[h_0 : \cdots : h_n]$, i.e. if we plug in $[\lambda \cdot h_0 : \cdots : \lambda \cdot h_n]$ for some $\lambda \in \mathbb{H} \setminus \{0\}$ then we obtain the same result.[1877] Using the Topological-Quotient Proposition 5.15 (1b) one can show that the map Ψ_j is continuous. One easily verifies that Ψ_j and Φ_j are inverses of one another. ⊞

It is clear that the open sets V_j cover all of $\mathbb{H}\mathrm{P}^n$ and that the triangles

$$\begin{array}{ccc} q^{-1}(V_j) & \xrightarrow{\Phi_j} & V_j \times S^3 \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle (v,h)\mapsto h} & \\ & S^3 & \end{array}$$

commute. This shows, according to the definition on page 3299, that $q\colon S^{4n+3} \to \mathbb{H}\mathrm{P}^n$ is an S^3-bundle. ■

The Hopf-Bundle Proposition 155.9, combined with the Fiber-LES Proposition 155.2 gives us two interesting theorems.

[1875] Recall that on page 1960 we defined the norm of a quaternion $h = a \cdot 1 + b \cdot i + c \cdot j + d \cdot k$ as $|h| = \sqrt{a^2 + b^2 + c^2 + d^2}$.

[1876] For mysterious reasons the Hopf maps are almost invariably called "Hopf fibrations", even though the name "Hopf fiber bundle" would be more precise.

[1877] Note though that the order of the factors in the definition of Ψ_j is important since $\mathbb{H}$ is not commutative.

Theorem 155.10.
(1) For all $n \geq 3$ we have $\pi_n(S^3) \cong \pi_n(S^2)$.
(2) We have $\pi_3(S^2) \cong \mathbb{Z}$ and $\pi_3(S^2)$ is generated by the Hopf map $S^3 \to S^2$.

The second statement gives in particular an answer to Question 122.15 (1).

Proof.

(1) On page 258 we gave an identification of $\mathbb{C}\mathrm{P}^1$ with S^2. Thus the Hopf-Bundle Proposition 155.9 (1), applied to $n = 1$, tells us that the Hopf map $p\colon S^3 \to S^2 = \mathbb{C}\mathrm{P}^1$ defines an S^1-bundle. the Fiber-LES Proposition 155.2 gives us the long exact sequence[1878]

$$\ldots \to \pi_n(S^1) \to \pi_n(S^3) \xrightarrow{p_*} \pi_n(S^2) \xrightarrow{\partial} \pi_{n-1}(S^1) \to \pi_{n-1}(S^3) \to \ldots$$

The desired statement now follows from the example on page 1573 where we saw that $\pi_k(S^1) = 0$ for $k \geq 2$.

(2) In Corollary 85.6 (1) we saw that $\pi_3(S^3) = \mathbb{Z} \cdot [\mathrm{id}_{S^3}]$. By (1) the Hopf map $p\colon S^3 \to S^2 = \mathbb{C}\mathrm{P}^1$ induces an isomorphism $p_*\colon \pi_3(S^3) \xrightarrow{\cong} \pi_3(S^2)$. The image of $[\mathrm{id}_{S^3}]$ under p_* is by definition precisely the class in $\pi_3(S^2)$ represented by the Hopf map. Thus we see that $\pi_3(S^2) \cong \mathbb{Z}$ and that $\pi_3(S^2)$ is generated by the Hopf map. ∎

Theorem 155.11.
(1) For all $n \geq 2$ we have $\pi_n(S^4) \cong \pi_n(S^7) \oplus \pi_{n-1}(S^3)$.
(2) For $n \in \{1, \ldots, 6\}$ we have $\pi_n(S^4) \cong \pi_{n-1}(S^3)$.

Remark. From Corollary 85.6 (1) we know that $\pi_7(S^7) \cong \mathbb{Z}$. Thus it follows from Theorem 155.11 (1) that $\pi_7(S^4) \cong \pi_7(S^7) \oplus \pi_6(S^3)$ admits an epimorphism onto $\mathbb{Z}$. In fact in Corollary 122.8 we already saw that the Hopf invariant is an epimorphism $\pi_7(S^4) \to \mathbb{Z}$.

Proof.

(1) By the Quaternion-Projective Space Lemma 92.9 (2) we can identify the quaternionic projective space $\mathbb{H}\mathrm{P}^1$ with S^4. Therefore if we apply the Hopf-Bundle Proposition 155.9 (2) with $n = 1$ we obtain an S^3-bundle $q\colon S^7 \to \mathbb{H}\mathrm{P}^1 = S^4$. We pick $y_0 \in S^7$ and we write $b_0 = q(y_0)$. We pick an identification $S^3 = q^{-1}(b_0)$ and we denote by $i\colon S^3 \to S^7$ the map that is given by the inclusion of the fiber $F = S^3$ into S^7. From the Fiber-LES Proposition 155.2 we obtain the following long exact sequence

$$\cdots \to \pi_n(S^3, y_0) \xrightarrow{i_*} \pi_n(S^7, y_0) \xrightarrow{q_*} \pi_n(S^4, b_0) \xrightarrow{\partial} \pi_{n-1}(S^3, y_0) \xrightarrow{i_*} \pi_{n-1}(S^7, y_0) \to \ldots.$$

By Proposition 71.10 we know that $\pi_3(S^7, y_0) = 0$. By definition this implies that the map $i\colon S^3 \to S^7$ is homotopic rel y_0 to the constant map $c_{y_0}\colon S^3 \to \{y_0\}$. By the

[1878] It is actually worth having a glance at the lower end of this sequence. We have the exact sequence

$$\pi_2(S^3) \xrightarrow{p_*} \pi_2(S^2) \xrightarrow{\partial} \pi_1(S^1) \to \pi_1(S^3).$$

From Proposition 71.10 we know that $\pi_1(S^3) = \pi_2(S^3) = 0$ and Proposition 48.17 we know that $\pi_1(S^1) \cong \mathbb{Z}$. Thus we obtain that $\pi_2(S^2) \cong \mathbb{Z}$. In Exercise 155.11 we will see that using this exact sequence we can obtain a new proof that id_{S^2} is a generator of $\pi_2(S^2)$.

Fibration-LES Theorem 154.7 (3) this implies that the above long exact sequence splits into short exact sequences

$$0 \to \pi_n(S^7, y_0) \xrightarrow{q_*} \pi_n(S^4, b_0) \xrightarrow{\partial} \pi_{n-1}(S^3, y_0) \to 0.$$

Furthermore the Fibration-LES Theorem 154.7 (3) says that these short exact sequences split, therefore we obtain the promised isomorphisms $\pi_n(S^4, b_0) \cong \pi_n(S^7, y_0) \oplus \pi_{n-1}(S^3, y_0)$.

(2) The statement holds trivially Proposition 71.10 for $n = 1$. For $n \geq 2$ this statement follows from (1) and Proposition 71.10 which says in this context that $\pi_n(S^7) = 0$ for $n = 1, \ldots, 6$. ■

As we will see, the Fiber-LES Proposition 155.2 is a goose that just cannot stop laying golden eggs. For example we have the following theorem.

Theorem 155.12. We use the homeomorphism from page 258 to make the identification $\mathbb{C}\mathrm{P}^1 = S^2$ and as on page 256 we view $\mathbb{C}\mathrm{P}^1$ as a subspace of each $\mathbb{C}\mathrm{P}^n$ and of $\mathbb{C}\mathrm{P}^\infty$.

(1) Let $n \in \mathbb{N}$. If we denote by $p\colon S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ the Hopf map from page 3306, then

$$\pi_k(\mathbb{C}\mathrm{P}^n) = \begin{cases} \mathbb{Z} \cdot [\mathrm{id}_{\mathbb{C}\mathrm{P}^1}], & \text{if } k = 2, \\ 0, & \text{if } k = 1, 3, 4, 5, \ldots, 2n, \\ \mathbb{Z} \cdot [p], & \text{if } k = 2n+1. \end{cases}$$

(2)
$$\pi_k(\mathbb{C}\mathrm{P}^\infty) = \begin{cases} \mathbb{Z} \cdot [\mathrm{id}_{\mathbb{C}\mathrm{P}^1}], & \text{if } k = 2, \\ 0, & \text{if } k = 1, 3, 4, 5, 6, \ldots. \end{cases}$$

Remark. Let $k < l$. In Question 85.4 we had asked whether the complex projective space $\mathbb{C}\mathrm{P}^k$ is a retract of $\mathbb{C}\mathrm{P}^l$. In Proposition 121.14 we used the calculation of the cup product to give a negative answer. Now we can also use homotopy groups for the same purpose. More precisely, from Theorem 155.12, we obtain that $\pi_{2k+1}(\mathbb{C}\mathrm{P}^k) \cong \mathbb{Z}$ but that $\pi_{2k+1}(\mathbb{C}\mathrm{P}^l) = 0$. The usual argument, see Lemma 47.16 implies that once again we obtain a negative answer to Question 85.4, i.e. $\mathbb{C}\mathrm{P}^k$ is not a retract of $\mathbb{C}\mathrm{P}^l$.

Proof.

(1) By the Hopf-Bundle Proposition 155.9 (1) we know that for every $n \in \mathbb{N}$ the Hopf map $p\colon S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ is an S^1-bundle. Thus we obtain from the Fiber-LES Proposition 155.2 the following long exact sequence of homotopy groups:

$$\ldots \to \pi_k(S^1) \to \pi_k(S^{2n+1}) \xrightarrow{p_*} \pi_k(\mathbb{C}\mathrm{P}^n) \xrightarrow{\partial} \pi_{k-1}(S^1) \to \ldots$$

We recall that $\pi_k(S^{2n+1}) = 0$ for $k = 1, \ldots, 2n$, that $\pi_{2n+1}(S^{2n+1}) \cong \mathbb{Z} \cdot [\mathrm{id}_{S^{2n+1}}]$ and that $\pi_k(S^1) = 0$ for $k \geq 2$. Putting everything together implies immediately that $\pi_k(\mathbb{C}\mathrm{P}^n) = 0$ for $k = 3, \ldots, 2n$ and that $\pi_{2n+1}(\mathbb{C}\mathrm{P}^n) \cong \mathbb{Z} \cdot [p]$.[1879] Only the lower end of the long exact sequence requires slightly more thought. More precisely, we have the exact sequence

$$\underbrace{\pi_2(S^{2n+1})}_{=0} \to \pi_2(\mathbb{C}\mathrm{P}^n) \xrightarrow{\partial} \pi_1(S^1) \to \underbrace{\pi_1(S^{2n+1})}_{=0} \to \pi_1(\mathbb{C}\mathrm{P}^n) \to \pi_0(S^1) \to \pi_0(S^{2n+1}).$$

[1879] Here we use the trivial, but useful observation that $p_*([\mathrm{id}_{S^n}]) = [p]$.

Since S^1 and S^{2n+1} are path-connected we see that the map on the right is an isomorphism of pointed sets. It follows that $\pi_2(\mathbb{C}\mathrm{P}^n) \xrightarrow{\cong} \pi_1(S^1) \cong \mathbb{Z}$ and that $\pi_1(\mathbb{C}\mathrm{P}^n) = 0$. (We had of course known the latter statement for a long time, see e.g. page 1533.) The only thing which is not entirely clearly from this argument is why $\mathrm{id}_{\mathbb{C}\mathrm{P}^1}$ represents a generator of $\pi_2(\mathbb{C}\mathrm{P}^n)$. One could either heroically fight one's way through the above isomorphism, in the same vein as in Exercise 155.11, and use that we know explicitly the generator of $\pi_1(S^1)$.

But fortunately we do not have to go down that route since in Corollary 85.6 (2) we already saw that $\mathrm{id}_{\mathbb{C}\mathrm{P}^1}$ represents a generator of $\pi_2(\mathbb{C}\mathrm{P}^n)$.

(2) We calculate that

$$\pi_k(\mathbb{C}\mathrm{P}^\infty) \;=\; \pi_k\Big(\bigcup_{k\in\mathbb{N}} \mathbb{C}\mathrm{P}^n\Big) \;\underset{\uparrow}{=}\; \varinjlim \pi_k(\mathbb{C}\mathrm{P}^n) \;\underset{\uparrow}{=}\; \begin{cases} \mathbb{Z}\cdot[\mathrm{id}_{\mathbb{C}\mathrm{P}^1}], & \text{if } k=2, \\ 0, & \text{if } k = 1,3,4,5,\dots. \end{cases}$$

by the Direct Limit-π_n-Proposition 71.11 since by Lemma 68.6 we can view $\mathbb{C}\mathrm{P}^\infty$ as a CW-complex where each k-skeleton is given by $\mathbb{C}\mathrm{P}^n \subset \mathbb{C}\mathrm{P}^\infty$

by (1)

■

155.3. **The Hopf fibration arising from the octonions** (∗). In the Hopf-Bundle Proposition 155.9 we had in particular seen that the Hopf map $p\colon S^3 \to \mathbb{C}\mathrm{P}^1 = S^2$ is an S^1-bundle and that the Hopf map $q\colon S^7 \to \mathbb{H}\mathrm{P}^1 = S^4$ is an S^3-bundle. We exploited these fiber bundles in Theorems 155.10 and 155.11 to obtain non-trivial information on higher homotopy groups of spheres.

In our discussion of division algebras in Section 92.1 we gave the quaternions as a first example of a non-commutative division algebra. The algebra structure on the quaternions is still nonetheless quite reasonable, in particular the algebra structure is associative which was enough to define the quaternionic projective spaces on page 1970.

Next on the list of ever weirder division algebras were the octonions $\mathbb{O}$ that we introduced on page 1963. Recall that the underlying vector space of the octonions is $\mathbb{R}^8$ but that this time the algebra structure is neither commutative nor associative. The only redeeming feature is that $\mathbb{O}$ is still a division algebra.

Since the octonions are no longer associative one cannot just copy-paste the definition of the quaternionic projective spaces on page 1970 to define octonionic projective spaces in any obvious way. But we can make a prediction: if there was a first octonionic projective space, then it would presumably give rise to a Hopf fibration $r\colon S^{15} \to \mathbb{O}P^1 = S^8$ with fiber S^7. Alas, octonionic projective spaces do not exist. But as we will see, we can still use octonions to produce such a fiber bundle. More precisely, our goal in this section is to prove the following proposition.

Proposition 155.13. There exists a fiber bundle $p\colon S^{15} \to S^8$ with fiber S^7.

The proof of Proposition 155.13 is rather longish, thus we will first point out that it allows us to prove the following theorem.

Theorem 155.14.

(1) For all $n \geq 2$ we have $\pi_n(S^8) \cong \pi_n(S^{15}) \oplus \pi_{n-1}(S^7)$.

(2) For $n \in \{1,\dots,14\}$ we have $\pi_n(S^8) \cong \pi_{n-1}(S^7)$.

Proof. The proof is basically the same as the proof of Theorem 155.11. We just need to replace the Hopf-Bundle Proposition 155.9 (2) by Proposition 155.13. ■

We can copy-paste the remark after Theorem 155.11:

Remark. From Corollary 85.6 (1) we know that $\pi_{15}(S^{15}) \cong \mathbb{Z}$. Thus it follows from Theorem 155.11 (1) that $\pi_{15}(S^8)$ admits an epimorphism onto $\mathbb{Z}$. In fact in Corollary 122.8 we already saw that the Hopf invariant is an epimorphism $\pi_{15}(S^8) \to \mathbb{Z}$.

The remainder of this chapter is occupied with providing a proof of Proposition 155.13. We will need the notion of the join of two topological spaces that we introduced on page 446.

Definition. Let X and Y be two non-empty topological spaces. We define the join $X * Y$ to be the topological space obtained from $X \times [0,1] \times Y$, equipped with the product topology, by performing the following two types of identifications:

(1) for every $x \in X$ we identify all points in $\{x\} \times \{0\} \times Y$ to a single point and
(2) for every $y \in Y$ we identify all points in $X \times \{1\} \times \{y\}$ to a single point.

Furthermore, if one of X or Y is the empty topological space, then we define $X * Y$ to be the other topological space.

We remind the reader of the following lemma.

Lemma 16.4. Given any $m, n \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} S^m * S^n & \to & S^{m+n+1} \\ [(x,t,y)] & \mapsto & \underbrace{\left(x \cdot \cos\left(\frac{\pi t}{2}\right), y \cdot \sin\left(\frac{\pi t}{2}\right)\right)}_{\in \mathbb{R}^{m+1} \times \mathbb{R}^{n+1} = \mathbb{R}^{m+n+2}} \end{array}$$

is a homeomorphism.

The following proposition explains our sudden interest in the join of two topological spaces.

Proposition 155.15. Let X be a topological space that is compact and Hausdorff. Furthermore let $\varphi\colon X \times X \to X$ be a map. Given $x \in X$ we consider the maps[1880]

$$\begin{array}{rcl} L_x\colon X & \to & X \\ a & \mapsto & \varphi(x,a) \end{array} \quad \text{and} \quad \begin{array}{rcl} R_x\colon X & \to & X \\ a & \mapsto & \varphi(a,x). \end{array}$$

If each map $L_x\colon X \to X$ and each map $R_x\colon X \to X$ is a homeomorphism, then the map

$$\begin{array}{rcl} h_\varphi\colon X * X & \to & \Sigma(X) \\ [(x,t,y)] & \mapsto & [(\varphi(x,y), 2t-1)] \end{array}$$

is a fiber bundle with fiber X.[1881]

Example. We write $S^3 = \{z \in \mathbb{H} \,|\, |z| = 1\}$. By the Quaternion Lemma 92.1 (7) we know that for any $w, z \in \mathbb{H}$ we have $|w \cdot z| = |w| \cdot |z|$. Thus multiplication of quaternions defines a map $\varphi\colon S^3 \times S^3 \to S^3$. Given any $z \in S^3$ the map L_z is a homeomorphism. In fact an inverse is given by $L_{z^{-1}}$ since for any $x \in S^3$ we have

$$(L_{z^{-1}} \circ L_z)(x) \;=\; z^{-1} \cdot (z \cdot x) \;\underset{\text{since } \mathbb{H} \text{ is associative}}{=}\; (z^{-1} \cdot z) \cdot x \;=\; x,$$

[1880] Here the mnemonic device is that "L" should suggest "left-multiplication" and "R" should suggest "right-multiplication".

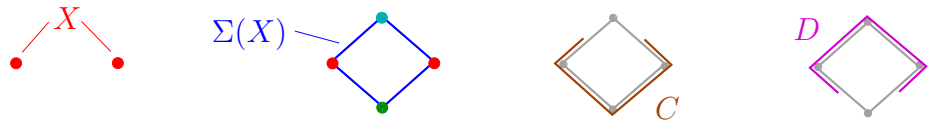
[1881] It is a good exercise to first verify that the map is actually well-defined.

and similarly we have $L_z \circ L_{z^{-1}} = \mathrm{id}_{S^3}$. The same argument shows that R_z is also a homeomorphism. Thus we obtain from Proposition 155.15 and the Join of Spheres Lemma 16.4 a fiber bundle $p\colon S^7 \to S^4$ with fiber S^3. We leave it to the heroically inclined reader to verify that this S^3-bundle is equivalent to the S^3-bundle $q\colon S^7 \to \mathbb{HP}^1 = S^4$ that we had constructed in the proof of the Hopf-Bundle Proposition 155.9.

Proof. We consider the following two subsets of $\Sigma(X)$:

$$C \;:=\; ([-1, \tfrac{1}{2}] \times X) \,/\, \{-1\} \times X \quad \text{and} \quad D \;:=\; ([-\tfrac{1}{2}, 1] \times X) \,/\, \{1\} \times X.$$

We refer to the figure below for an illustration. Since the interiors of these two sets cover $\Sigma(X)$ it suffices to find trivializations of the map $h_\varphi\colon X * X \to \Sigma(X)$ with fiber X over C and over D.



We start out with finding a trivialization over C.

Claim. The map
$$\begin{array}{rcl} \Psi\colon X \times C & \to & X * X \\ (x, [(y,t)]) & \mapsto & [(x, \frac{2}{3}(t+1), L_x^{-1}(y))] \end{array}$$
is continuous and it restricts to a homeomorphism onto $h_\varphi^{-1}(C)$.

Proof. By now the reader is surely aware of the fact that whenever we mix quotients with products we need to tread carefully. Thus let us first consider the maps

$$\begin{array}{rcl} \alpha\colon X \times X & \to & X \times X \\ (x,y) & \mapsto & (x, \varphi(x,y)) = (x, L_x(y)) \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta\colon X \times X & \to & X \times X \\ (x,y) & \mapsto & (x, L_x^{-1}(y)). \end{array}$$

Note that these two maps are inverses of one another. Also note that α is continuous by the Product Topology-Basics Lemma 5.4 (3). Furthermore note that α is a bijection, in fact the inverse is given by β. Since X is compact and Hausdorff we obtain from the Product Topology Properties Proposition 5.6 and the Compact-Hausdorff Proposition 2.17 (3) that α is a homeomorphism. In particular $\beta = \alpha^{-1}$ is also continuous.

Next we consider the map

$$\begin{array}{rcl} \psi\colon X \times ([-1, \frac{1}{2}] \times X) & \to & X \times [0,1] \times X \\ (x,t,y) & \mapsto & (x, \frac{2}{3}(t+1), L_x^{-1}(y)). \end{array}$$

Since $\beta(x,y) = (x, L_x^{-1}(y))$ and since β is continuous we see that ψ is also continuous. Finally we consider the following commutative diagram

$$\begin{array}{ccc} X \times [-1, \frac{1}{2}] \times X & \xrightarrow{(x,t,y)\mapsto(x,[(t,y)])} & \overbrace{X \times \big(([-1, \frac{1}{2}] \times X) \,/\, (\{-1\} \times X)\big)}^{= X \times C} \\ \Big\downarrow{\scriptstyle \psi} & \searrow^{f := p \circ \psi} & \Big\downarrow{\scriptstyle \Psi} \\ X \times [0,1] \times X & \xrightarrow[p(x,t,y) := [(x,t,y)]]{} & X * X. \end{array}$$

Claim. The map Ψ is continuous.

Proof. We had just seen that $f = p \circ \psi$ is continuous. Note that by the Quotient Map Lemma 13.3 (1) the projection $q\colon [-1, \frac{1}{2}] \times X \to ([-1, \frac{1}{2}] \times X) \,/\, (\{-1\} \times X)$ is a quotient map. Since X is compact and Hausdorff we know by Lemma 2.41 that X is regionally compact.[1882] Thus it follows from the Whitehead Quotient Theorem 13.4 that the top horizontal map $\mathrm{id}_X \times q$ is a quotient map. Thus it follows from Lemma 13.5 (2) that ψ is continuous. ⊞

One easily verifies that Ψ defines a bijection

$$\Psi\colon \underbrace{X \times C}_{\subset X * X} \to \underbrace{h_\varphi^{-1}(C)}_{\subset \Sigma(X)}.$$

Since X is compact we know by Lemma 16.2 (1) that the left-hand side is compact. By the Cone-Suspension Basics Lemma 16.8 (2) and our hypothesis that X is Hausdorff we know that the right-hand side is Hausdorff. Therefore it follows from the Compact-Hausdorff Proposition 2.17 (3) that the continuous bijection $\Psi\colon X \times C \to h_\varphi^{-1}(C)$ is indeed a homeomorphism. ⊞

We consider the following diagram

$$\begin{array}{ccc} X \times C & \xrightarrow{\quad\Psi\quad} & h_\varphi^{-1}(C) \\ & {\scriptstyle (x,c)\mapsto c} \searrow \quad \swarrow {\scriptstyle h_\varphi} & \\ & C. & \end{array}$$

It is straightforward to verify that this diagram commutes. Since the top map is homeomorphism by the previous claim we have found a trivialization of h_φ over C. We leave it to the reader to make the minor modifications needed to find a trivialization of h_φ over D.[1883] ■

Evidently the idea now is to replace the quaternions in the example on page 3312 by the octonions. In the following we recall the most salient bits from the discussion of octonions on page 1963. The octonions $\mathbb{O}$ are $\mathbb{R}^8$ together with an algebra structure over $\mathbb{R}$ that has the following properties:

(1) The octonions $\mathbb{O}$ are a division algebra, i.e. for all $a \neq 0$ and $b \in \mathbb{O}$ there exists an $x \in \mathbb{O}$ with $a \cdot x = b$ and a $y \in \mathbb{O}$ with $y \cdot a = b$.
(2) The algebra structure is neither commutative nor associative.
(3) Multiplication on $\mathbb{O}$ respects the euclidean norm, more precisely, given $a, b \in \mathbb{O}$ we have $\|a \cdot b\| = \|a\| \cdot \|b\|$.

It follows from (3) that the algebra multiplication induces a map $\varphi\colon S^7 \times S^7 \to S^7$.

Lemma 155.16. Given any $z \in S^7$ the maps $L_z\colon S^7 \to S^7$ and $R_z\colon S^7 \to S^7$, as defined in Proposition 155.15, are homeomorphisms.

Proof. First note that we can *not* recycle the argument from page 3311 since there we used that the quaternions are associative.

[1882] If the reader is unhappy about using Lemma 2.41 we point out that in our application of the proposition the topological space will be $X = S^n$ and thus obviously regionally compact.
[1883] Note that in this second part we will need to use that the maps $R_x\colon X \to X$ are homeomorphisms.

By Corollary 82.9 (2) it suffices to show that the maps L_z and R_z are injective. First we show that given any $z \in S^7$ the map $L_z \colon S^7 \to S^7$ is injective. So suppose that we have $a, b \in S^7$ with $L_z(a) = L_z(b)$. We obtain that

$$\|a-b\| \underset{\substack{\uparrow \\ \text{since } z \in S^7}}{=} \|z\| \cdot \|a-b\| \underset{\substack{\uparrow \\ \text{by (3)}}}{=} \|z \cdot (a-b)\| \underset{\substack{\uparrow \\ \text{since } \mathbb{O} \text{ is an algebra}}}{=} \|z \cdot a - z \cdot b\| \underset{\substack{\uparrow \\ \text{by definition of } L_z}}{=} \|L_z(a) - L_z(b)\| \underset{\substack{\uparrow \\ \text{since } L_z(a) = L_z(b)}}{=} 0.$$

Thus we see that $a = b$. Almost the same argument shows that R_z is injective.[1884] ■

Now we are basically done with the proof of Proposition 155.13.

Proof of Proposition 155.13. It follows immediately from Lemma 155.16, Proposition 155.15 and the Join of Spheres Lemma 16.4 that there exists a fiber bundle $p \colon S^{15} \to S^8$ with fiber S^7. ■

Exercises for Chapter 155.

Exercise 155.1. Let F be a topological space and let $p \colon Y \to B$ be an F-bundle. Suppose that the fiber F and the base space B are path-connected. Show directly, without making use of Proposition 155.2 (2), that the total space Y is also path-connected.

Exercise 155.2. Let $p \colon Y \to [0,1]$ be a bundle with fiber F. Show with your bare hands (i.e. without using Proposition 155.7) that p is trivial, i.e. show that p is equivalent to the trivial bundle $F \times [0,1] \to [0,1]$.

Exercise 155.3. Let X be a topological space and let $f_0, f_1 \colon X \to X$ be homeomorphisms. Suppose that f_0 and f_1 are isotopic. Show that the two X-bundles $\mathrm{Tor}(X, f_0) \to S^1$ and $\mathrm{Tor}(X, f_1) \to S^1$ are equivalent.

Exercise 155.4. Let $f \colon \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. Show that the corresponding $\mathbb{R}^n$-bundle $\mathrm{Tor}(\mathbb{R}^n, f) \to S^1$ is equivalent to an n-dimensional vector bundle.
Hint. Use Lemma 76.15 and Lemma 155.3 (2), the latter is a. k. a. Exercise 155.3.

Exercise 155.5. Use the Ehresmann Fibration Theorem, see Exercise **??**, to give an alternative proof of the fact that the Hopf map $S^3 \to \mathbb{C}\mathrm{P}^1$ defines an S^1-bundle over $\mathbb{C}\mathrm{P}^1 = S^2$.

Exercise 155.6. As on page 1576 we consider the Hopf map:

$$\begin{array}{rcl} H \colon S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^2 = \mathbb{C}\mathrm{P}^1 \\ (z_1, z_2) & \mapsto & [z_1 : z_2]. \end{array}$$

(a) Let $P, Q \in S^2$ be two distinct points. Show that $\mathrm{H}^{-1}(P) \cup \mathrm{H}^{-1}(Q)$ is isotopic to a Hopf link, that we defined in the figure on page 2863.
(b) Let $P, Q, R \in S^2$ be two distinct points. Sketch the link $\mathrm{H}^{-1}(P) \cup \mathrm{H}^{-1}(Q) \cup \mathrm{H}^{-1}(\{R\})$.

Exercise 155.7. Let $n \in \mathbb{N}$. We consider the map

$$\begin{array}{rcl} \varphi \colon \mathrm{U}(n) = \{A \in \mathrm{M}(n \times n, \mathbb{C}) \,|\, A \cdot \overline{A}^T = \mathrm{id}\} & \to & \mathbb{C} \setminus \{0\} \\ A & \mapsto & \det(A). \end{array}$$

[1884] At first glance it might appear odd that we never used that the octonions are a division algebra, we only used the innocuous looking third property. But it is clear that (3) already implies that $\mathbb{O}$ has no zero-divisors. By Lemma 92.5 this forces $\mathbb{O}$ to be a division algebra.

(a) Show that φ takes values in S^1.
(b) Show that the map $\varphi\colon \mathrm{U}(n) \to S^1$ is a fiber bundle.
Hint. You can use the Ehresmann Fibration Theorem from Exercise **??**.

Exercise 155.8. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. By the discussion on page 3301 and the Fiber-LES Proposition 155.2 we know that the map

$$\begin{array}{rcl} \mathrm{Tor}(X,f) = (X \times [0,1])/(x,0) \sim (f(x),1) & \to & S^1 \\ [(x,t)] & \mapsto & \exp(2\pi \mathrm{i} t) \end{array}$$

is a fibration. We make the obvious identification $X = X \times \{0\} \subset \mathrm{Tor}(X,f)$. Let $k \in \mathbb{N}_0$ and let $\alpha\colon [0,1] \to S^1$ be the usual loop given by $\alpha(t) = \exp(2\pi \mathrm{i} t)$. By Proposition 154.5 and Lemma 73.2 we get a well-defined map

$$\Theta(\alpha)_*\colon \mathrm{H}_k(X;\mathbb{Z}) \to \mathrm{H}_k(X;\mathbb{Z}).$$

Show that $\Theta(\alpha)_* = (f_*)^{-1}$.

Exercise 155.9. Let W be an n-dimensional smooth manifold and let K be a compact proper connected k-dimensional submanifold with $k \leq n-3$. Show that the inclusion induced map $\pi_1(W \setminus K) \to \pi_1(W)$ is an isomorphism.
Hint. Use the Amalgamated Product Lemma 53.22 (3).

Exercise 155.10. The same way that we defined the infinite complex projective space $\mathbb{C}\mathrm{P}^\infty$, see page 1480, we define the infinite quaternionic projective space $\mathbb{H}\mathrm{P}^\infty$ as the direct limit of the finite-dimensional quaternionic projective spaces $\mathbb{H}\mathrm{P}^k$. Show that for any $i \in \mathbb{N}_{\geq 2}$ we have an isomorphism $\pi_i(\mathbb{H}\mathrm{P}^\infty) \cong \pi_{i-1}(S^3)$.

Exercise 155.11. Let $p\colon S^3 \to S^2$ the Hopf fibration. In the Hopf-Bundle Proposition 155.9 we saw that it is an S^1-fiber bundle. We consider the connecting homomorphism $\partial\colon \pi_2(S^2) \to \pi_1(S^1)$ of the long exact sequence of homotopy groups from the Fiber-LES Proposition 155.2 (2). Show, just using the definition of the connecting homomorphism, that $\partial([\mathrm{id}_{S^2}]) = [\mathrm{id}_{S^1}]$.

Exercise 155.12. Show that the homotopy groups of S^2 and $S^3 \times \mathbb{C}\mathrm{P}^\infty$ are isomorphic.

Exercise 155.13. Let $m,k \in \mathbb{N}$ and let $p\colon M \to S^k$ be a fiber bundle with fiber S^m. We denote by $i\colon S^m \to M$ the homeomorphism to any of the fibers.

(a) Show that M is a closed $(m+k)$-dimensional topological manifold.
(b) We suppose that $m < k$ and we suppose that M is orientable. Show that the maps $p_*\colon \mathrm{H}_k(M) \to \mathrm{H}_k(S^k)$ and $i_*\colon \mathrm{H}_m(S^m) \to \mathrm{H}_m(M)$ are isomorphisms and show that $\mathrm{H}_i(M) = 0$ for $i \neq 0,k,m,m+n$.
Hint. Use Theorems 154.7 and 153.14 and use Poincaré Duality.
(c) We suppose again that M is orientable. Show that the conclusion in (b) does not necessarily hold if we drop the hypothesis that $m < k$.

156. Homotopy groups of Lie groups

In this chapter our focus will lie on the study of the homotopy groups of the matrix groups $\mathrm{O}(n)$ and $\mathrm{U}(n)$. Later on in the section we will see that we should also define a third type of groups, namely the groups $\mathrm{Sp}(n)$ which can be viewed as the quaternionic analogue of $\mathrm{O}(n)$ and $\mathrm{U}(n)$.
Since these groups are Lie groups and since this structure is helpful in our study of $\mathrm{O}(n)$ and $\mathrm{U}(n)$ we first give a short introduction to Lie groups.

156.1. **Topological groups and Lie groups.** First we recall the following definition from page 267.

Definition. A topological group is a topological space X together with a group structure such that the two maps

$$\begin{array}{ccc} X\times X & \to & X \\ (x,y) & \mapsto & x\cdot y \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \to & X \\ x & \mapsto & x^{-1} \end{array}$$

are continuous. A morphism between topological groups X and Y is a homomorphism $f\colon X\to Y$ that is furthermore continuous.

Examples.

(1) On page 268 we already saw that for any $n\in\mathbb{N}$ the matrix groups $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are topological groups.

(2) Let X be a topological space and let $x_0\in X$ be a base point. We consider the loop space $\Omega(X,x_0)$, i.e. the set

$$\Omega(X,x_0) \;=\; \{\gamma\colon [0,1]\to X \,|\, \gamma(0)=\gamma(1)=x_0\}/\text{homotopy}$$

equipped with the compact-open topology. As in Exercise 154.7 we equip the set $\pi_1(X,x_0)=\Omega(X,x_0)/\sim$ with the quotient topology, i.e. we view $\pi_1(X,x_0)$ as a topological space. Evidently $\pi_1(X,x_0)$ admits a group structure, but somewhat surprisingly, the multiplication map $\pi_1(X,x_0)\times\pi_1(X,x_0)\to\pi_1(X,x_0)$ is in general *not* continuous. In fact such an example is given by the Hawaiian Earrings, introduced in Exercise 56.1. We refer to [**Fab2011**] for details.[1885]

We leave the elementary proof of the following lemma to the reader.

Lemma 156.1. Let G be a topological group and let H be a subgroup of G. The map

$$\begin{array}{ccc} G\times H & \to & G \\ (g,h) & \mapsto & g\cdot h \end{array}$$

defines a right-action of H on G that is free and continuous.

To avoid any misunderstanding we introduce the following convention.

Convention. Let G be a topological group and let H be a subgroup. We denote by G/H the topological space that is given by the equivalence relation[1886]

$$g_1\sim g_2 \quad\Longleftrightarrow\quad \text{there exists an } h\in H \text{ with } g_1\cdot h=g_2.$$

[1885] In fact, as is pointed out in [**Fab2011**], the failure of $\pi_1(X,x_0)$ to be a topological group is another instance of the phenomenon, discussed in Section 13.1, that in general product and quotient topologies do not mix well.

The following definition of a Lie group is quite similar to the above definition of a topological group.

Definition. A Lie group is a smooth manifold X with empty boundary together with a group structure such that the two maps[1887]

$$\begin{array}{ccc} X \times X & \to & X \\ (x,y) & \mapsto & x \cdot y \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \to & X \\ x & \mapsto & x^{-1} \end{array}$$

are smooth.

Remark.

(1) It follows immediately from Lemma 19.22 that a Lie group is also a topological group.
(2) As we will see in Exercise 156.2, strictly speaking one does not need to demand that the map $x \mapsto x^{-1}$ is smooth, since it follows already from the other properties.

Examples.

(1) It is straightforward to see that the smooth manifold $\mathbb{R}$ with the group structure given by addition and the smooth manifold S^1 with the group structure given by multiplication are Lie groups.
(2) The product $G_1 \times G_2$ of two Lie groups G_1 and G_2 admits a canonical smooth manifold structure by the Smooth Manifold Product Proposition 19.31 and the group structures on G_1 and G_2 define a group structure on $G_1 \times G_2$. One can easily verify that $G_1 \times G_2$ is again a Lie group. In particular the torus $S^1 \times S^1$ is a Lie group.

As mentioned in the beginning of this chapter, we are now interested in the homotopy groups of the groups

$$\begin{aligned} \mathrm{O}(n) &= \{A \in \mathrm{M}(n \times n, \mathbb{R}) \,|\, A^T A = \mathrm{id}\} \\ \mathrm{U}(n) &= \{A \in \mathrm{M}(n \times n, \mathbb{C}) \,|\, \overline{A}^T A = \mathrm{id}\}. \end{aligned}$$

As we will see later on, even if we are only interested in these groups it makes sense to extend our gaze and to consider the quaternionic analogue of $\mathrm{O}(n)$ and $\mathrm{U}(n)$:

Definition. Let $n \in \mathbb{N}$. We set[1888]

$$\mathrm{Sp}(n) := \{A \in \mathrm{M}(n \times n, \mathbb{H}) \,|\, \overline{A}^T A = \mathrm{id}\}.$$

We will first discuss $\mathrm{Sp}(n)$ a little bit before we settle on a name for it:

Lemma 156.2.

(1) We have $\mathrm{Sp}(1) = S^3$.
(2) Let $n \in \mathbb{N}$. Matrix multiplication turns $\mathrm{Sp}(n)$ into a group.

[1886]The point of this convention is that in principle G/H could mean the quotient of the topological space G by the subset H.
[1887]Here we equip $X \times X$ with the product smooth manifold structure introduced in the Smooth Manifold Product Proposition 19.31.
[1888]Recall that the conjugate of a quaternion $h = a \cdot 1 + b \cdot \mathrm{i} + c \cdot \mathrm{j} + d \cdot \mathrm{k}$ is defined as $\overline{h} = a \cdot 1 - b \cdot \mathrm{i} - c \cdot \mathrm{j} - d \cdot \mathrm{k}$. Given a matrix $A = (a_{ij})$ over $\mathbb{H}$ we define $\overline{A} = (\overline{a}_{ij})$.

(3) Let $n \in \mathbb{N}$. We write $J = \begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 0 \end{pmatrix}$. The map

$$\begin{aligned} \Phi\colon \mathrm{Sp}(n) &\to \{X \in \mathrm{U}(2n) \mid JXJ^{-1} = \overline{X}\} \\ P &\mapsto \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \quad \text{where } A, B \in \mathrm{M}(n \times n, \mathbb{C}) \text{ with}^{1889}\ P = A + B\mathrm{j} \end{aligned}$$

is a group isomorphism.

Proof.

(1) In the Quaternion Lemma 92.1 (5) we saw that for any quaternion $h \in \mathbb{H}$ we have $h \cdot \overline{h} = \|h\|^2$. It follows immediately that $\mathrm{Sp}(1) = S^3$.

(2) We start out with the following observation: A straightforward calculation[1890] shows that for any two matrices $A, B \in \mathrm{M}(n \times n, \mathbb{H})$ we have $\overline{(AB)}^T = \overline{B}^T \overline{A}^T$. Now we turn to the proof that $\mathrm{Sp}(n)$ is a group:
   (a) The fact that $\mathbb{H}$ is associative implies immediately that the multiplication of matrices in $\mathrm{M}(n \times n, \mathbb{H})$ is associative.
   (b) The identity matrix is as usual a neutral element.
   (c) Given $A \in \mathrm{Sp}(n)$ we have by definition that $\overline{A}^T A = \mathrm{id}$, i.e. $\overline{A}^T$ is a left-inverse. The above observation implies that $\overline{A}^T$ is also a right-inverse.
   (d) It follows from the above observation that $\mathrm{Sp}(n)$ is closed under matrix multiplication and taking inverses.

   We have thus shown that $\mathrm{Sp}(n)$ is indeed a group.

(3) We will only use this isomorphism to justify the name for $\mathrm{Sp}(n)$, otherwise it will not play a role. Therefore we will not provide the fairly elementary proof of this statement, instead we refer to [**Bre1993**, p. 103] for the proof. ■

Not surprisingly we will now see that $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ are Lie groups.

Lemma 156.3.

(1) Let $n \in \mathbb{N}$.
   (a) The set $\mathrm{O}(n)$ is a submanifold of $\mathrm{M}(n \times n, \mathbb{R}) = \mathbb{R}^{n^2}$ of dimension $\frac{1}{2}n(n-1)$.
   (b) The set $\mathrm{U}(n)$ is a submanifold of $\mathrm{M}(n \times n, \mathbb{C}) = \mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$ of dimension n^2.
   (c) The set $\mathrm{Sp}(n)$ is a submanifold of $\mathrm{M}(n \times n, \mathbb{H}) = \mathbb{H}^{n^2} = \mathbb{R}^{4n^2}$ of dimension $2n^2 + n$.

   Furthermore, if we view $\mathrm{O}(n)$ as a subspace of $\mathrm{O}(n+1)$ in the obvious way, then $\mathrm{O}(n)$ is a submanifold of $\mathrm{O}(n+1)$. The same statements apply to $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$.

(2) $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$, viewed as smooth manifolds in their own right, are closed, in particular they are compact.

(3) The usual matrix multiplication turns $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ into Lie groups.

Remark.

[1889] It follows from $\mathrm{i} \cdot \mathrm{j} = \mathrm{k}$ that any quaternion $a \cdot 1 + b \cdot \mathrm{i} + c \cdot \mathrm{j} + d \cdot \mathrm{k}$ can be uniquely written as $(a \cdot 1 + b \cdot \mathrm{i}) + (c \cdot 1 + d \cdot \mathrm{i}) \cdot \mathrm{j}$ where $a \cdot 1 + b \cdot \mathrm{i}$ and $c \cdot 1 + d \cdot \mathrm{i}$ are complex numbers. The same statement applies to matrices.

[1890] In the calculation we need to use the fact, mentioned in the Quaternion Lemma 92.1 (3), that for quaternions $w, z \in \mathbb{H}$ we have $\overline{z \cdot w} = \overline{w} \cdot \overline{z}$.

(1) Recall that on page 3125 we defined the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$ as follows:

$$\mathrm{Sp}(2g,\mathbb{Z}) \;=\; \{A \in \mathrm{GL}(2g,\mathbb{Z}) \,|\, A^T J A = A\} \quad \text{where} \quad J = \begin{pmatrix} 0 & \mathrm{id}_g \\ -\mathrm{id}_g & 0 \end{pmatrix}.$$

This definition, together with Lemmas 156.2 and 156.3, leads us to calling $\mathrm{Sp}(n)$ the *n-th compact symplectic group*.

(2) The statements regarding the groups $\mathrm{O}(n)$ and $\mathrm{U}(n)$ are mostly already contained in the Matrix Group-Smooth Manifold Proposition 26.7.

Proof. In the Matrix Group-Smooth Manifold Proposition 26.7 we had basically proved statements (1) and (2) for the groups $\mathrm{O}(n)$ and $\mathrm{U}(n)$. The only aspect that we did not mention explicitly so far is that $\mathrm{O}(n)$ is a submanifold of $\mathrm{O}(n+1)$. We leave it to the reader to verify that this statement holds. The proof of (1) and (2) for $\mathrm{Sp}(n)$ is basically the same.

In Lemma 5.38 we showed that $\mathrm{O}(n)$ and $\mathrm{U}(n)$ are topological groups. Almost the same argument given in that proof shows that matrix multiplication and taking inverses are smooth maps. This implies that $\mathrm{O}(n)$ and $\mathrm{U}(n)$ are Lie groups. Furthermore it is clear that multiplication in $\mathrm{Sp}(n)$ is smooth. It remains to show that the map

$$\begin{array}{ccc} \mathrm{Sp}(n) & \to & \mathrm{Sp}(n) \\ A & \mapsto & A^{-1} \end{array}$$

is smooth. We will show this fact in Exercise 156.4. ■

Basically by definition we have $\mathrm{O}(1) = S^0$ and $\mathrm{U}(1) = S^1$, furthermore by Lemma 156.2 (1) we have $\mathrm{Sp}(1) = S^3$. Thus it follows from Lemma 156.3 that these three spheres can be viewed as Lie groups. The question arises, whether perhaps other (all?) spheres can be equipped with the structure of a Lie group. The following lemma gives an easy way to rule out the existence of Lie group structures on some smooth manifolds.

Lemma 156.4. Every Lie group is parallelizable.[1891]

Examples. Recall that on page 3066 we saw that $2n$-dimensional spheres with $n \geq 1$ are not parallelizable. Furthermore on page 3066 we showed that for $g \neq 1$ the surface of genus g is not parallelizable. Thus it follows from Lemma 156.4 that these spheres and surfaces cannot be equipped with a Lie group structure.

Sketch of proof. Let G be an n-dimensional Lie group with trivial element e. We denote by $R_g \colon G \to G$ the map that is given by right-multiplication by g. Since G is a Lie group this map is in fact a diffeomorphism. The map

$$\begin{array}{ccc} G \times T_eG & \to & TG = \{(g,h) \,|\, g \in G \text{ and } w \in T_hG\} \\ (g,v) & \mapsto & (g,(R_g)_*(v)) \end{array}$$

is an isomorphism of vector bundles. (This statement is basically obvious, except that perhaps it is not completely clear why the map is continuous, we refer to [**Lee2002**, Corollary 8.39] for more details.) But this shows that G is parallelizable. ■

The following lemma gives us more examples of Lie groups.

[1891] Recall that according to the definition on page 3063 a smooth manifold is called parallelizable if its tangent bundle is trivial.

Lemma 156.5. Let G be a Lie group and let K be a finite normal subgroup. The quotient G/K admits a unique smooth manifold structure such that the projection $G \to G/K$ is a local diffeomorphism. Furthermore the group structure on the quotient group G/K defines a Lie group structure on G/K.

Example. It follows from Lemma 156.5 that for any $n \in \mathbb{N}$ the quotients $\mathrm{SU}(n)/\{\pm \mathrm{id}\}$ and $(\mathrm{SU}(n) \times \mathrm{SU}(n))/\{\pm(\mathrm{id}, \mathrm{id})\}$ are Lie groups.

Proof. In Lemma 156.1 we already saw that the action of K on G is free and continuous. Since G is a Lie group one also sees that the action is smooth. Finally, since G is finite we get for free that the action is proper. Thus we obtain from the Manifold Quotient-by-Group Action Proposition 19.32 that G/K admits a unique smooth manifold structure (without boundary) such that the projection $G \to G/K$ is a local diffeomorphism. We leave it to the reader to verify that the induced multiplication map $G/K \times G/K \to G/K$ is again smooth. ■

Above we have seen that subgroups of matrix groups often give rise to Lie groups. Perhaps somewhat surprisingly, in a sense, all compact Lie groups arise that way. More precisely, the following theorem holds:[1892] Theorem 156.6 as an analogue of the Euclidean Embedding Theorem 27.1.

Theorem 156.6. Any compact Lie group is isomorphic (in the obvious sense) to a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$ for some $n \in \mathbb{N}$.

The theorem is a consequence of the so-called Peter-Weyl Theorem. We only stated the theorem for "general culture". We will not make use of it and thus we refer to [**BtD1995**, Theorem III.4.1] or alternatively to [**Sep2007**, Theorem 3.28] for a proof.

156.2. **The homotopy groups of $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$.** In this chapter we want to see what we can say about the homotopy groups of $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$.

Convention.

(1) Given $n \in \mathbb{N}$ we have the obvious monomorphism

$$\begin{array}{rcl} \mathrm{O}(n) & \to & \mathrm{O}(n+1) \\ A & \mapsto & \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

Often we will use it to identify $\mathrm{O}(n)$ with its image in $\mathrm{O}(n+1)$. Similarly we view $\mathrm{U}(n)$ as a subgroup of $\mathrm{U}(n+1)$ and we view $\mathrm{Sp}(n)$ as a subgroup of $\mathrm{Sp}(n+1)$.

(2) We will view any subgroup of $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$ or $\mathrm{GL}(n, \mathbb{H})$ as a topological space that is equipped with the base point given by the identity matrix. Thus in the remainder of this chapter we will suppress the base point in the discussion of fundamental groups.

The following lemma evidently uses the convention from page 3316.

[1892] One could view

Lemma 156.7. Let $n \in \mathbb{N}$. The maps

$$\begin{array}{rcl} \mathrm{O}(n+1)/\mathrm{O}(n) & \xrightarrow{p} & S^n \\ [(v_1 \ldots v_{n+1})] & \mapsto & v_{n+1} \end{array} \qquad \begin{array}{rcl} \mathrm{U}(n+1)/\mathrm{U}(n) & \xrightarrow{q} & S^{2n+1} \\ [(v_1 \ldots v_{n+1})] & \mapsto & v_{n+1} \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{Sp}(n+1)/\mathrm{Sp}(n) & \xrightarrow{r} & S^{4n+3} \\ [(v_1 \ldots v_{n+1})] & \mapsto & v_{n+1} \end{array}$$

are well-defined and they are homeomorphisms.

Proof.

(1) First we show that the given map $p\colon \mathrm{O}(n+1)/\mathrm{O}(n) \to S^n$ is well-defined. To do so we consider the map[1893]

$$\begin{array}{rcl} \mathrm{O}(n+1) & \to & S^n \\ V := (v_1 \ldots v_{n+1}) & \mapsto & v_{n+1} = V \cdot e_{n+1}. \end{array}$$

Note that if we multiply V on the right by a matrix of the form $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, then the multiplication with e_{n+1} is not affected. Therefore the above map $\mathrm{O}(n+1) \to S^n$ descends to a map $\mathrm{O}(n+1)/\mathrm{O}(n) \to S^n$ which is exactly the map p we consider in the lemma. This shows that p is well-defined.

Also note that basic linear algebra tells us that given any point $v_{n+1} \in S^n$ there exist vectors $v_1, \ldots, v_n \in \mathbb{R}^n$ such that $v_1, \ldots, v_n, v_{n+1}$ form an orthonormal basis $\mathbb{R}^{n+1}$. This shows that the given map p is a surjection. Similarly, elementary linear algebra shows that p is injective. Finally note that S^n is Hausdorff and note that Lemmas 156.3 and 2.13 imply that $\mathrm{O}(n+1)/\mathrm{O}(n)$ is compact. Therefore it follows from the Compact-Hausdorff Proposition 2.17 (3) that the map p is indeed a homeomorphism.

(2),(3) The proof for (2) and (3) is basically identical to the proof in (1), except that the linear algebra aspects require slightly more thought. ■

The following proposition gives us new interesting examples of fiber bundles.

Proposition 156.8. (Matrix Group-Fiber Proposition) Let $n \in \mathbb{N}$. The maps

$$\begin{array}{rcl} \mathrm{O}(n+1) & \to & S^n \\ (v_1 \ldots v_{n+1}) & \mapsto & v_{n+1} \end{array} \qquad \begin{array}{rcl} \mathrm{U}(n+1) & \to & S^{2n+1} \\ (v_1 \ldots v_{n+1}) & \mapsto & v_{n+1} \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{Sp}(n+1) & \to & S^{4n+3} \\ (v_1 \ldots v_{n+1}) & \mapsto & v_{n+1} \end{array}$$

are bundle projections with fibers given by $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ respectively.

Remark. Without too much effort one can show that the bundle projections $\mathrm{O}(n+1) \to S^n$ with fiber $\mathrm{O}(n)$ and $\mathrm{U}(n+1) \to S^{2n+1}$ with fiber $\mathrm{U}(n)$ from the proposition restrict to bundle projections $\mathrm{SO}(n+1) \to S^n$ with fiber $\mathrm{SO}(n)$ and $\mathrm{SU}(n+1) \to S^{2n+1}$ with fiber $\mathrm{SU}(n)$.

We will outline two approaches to proving the Matrix Group-Fiber Proposition 156.8. The first approach relies on the following proposition that shows more generally that Lie groups (and sometimes topological groups) give rise to fiber bundles.

Proposition 156.9. (Lie Group-Fiber Proposition)

(1) Let G be a Lie group. If H is a closed subgroup of G, then the projection $G \to G/H$ is a bundle projection with fiber H.

[1893] It follows easily from $A^T A = \mathrm{id}$ that every column v of A satisfies $\|v\| = 1$, i.e. every column lies in S^n.

(2) Let G be a topological group and H be a closed subgroup of G. If H admits the structure of a Lie group, then the projection $G \to G/H$ is a bundle projection with fiber H.

Remark. One might hope that the Lie Group-Fiber Proposition 156.9 (1) generalizes to all topological groups and subgroups. But as shown in [**Karu1958**, p. 347], there exists a topological group G and a closed subgroup H such that the projection $G \to G/H$ is not a bundle projection.

Proof of the Lie Group-Fiber Proposition 156.9.

(1) This statement is proved in [**Kiri2008**, Theorem 2.11]. Alternatively, if G is a closed subgroup of some $\mathrm{GL}(n, \mathbb{C})$, then the proposition is proved in [**Bre1993**, p. 110].
(2) This statement is proved in [**Pal1961**, Corollary 4.1]. ■

First proof of the Matrix Group-Fiber Proposition 156.8. These three statements are an immediate consequence of Lemma 156.7 and the Lie Group-Fiber Proposition 156.9 (1). ■

Second proof of the Matrix Group-Fiber Proposition 156.8. In the following we give an outline of a "bare hands only" proof that the first projection $p\colon \mathrm{O}(n+1) \to S^n$ is indeed a bundle projection with fiber $\mathrm{O}(n)$. We leave it to the reader to modify the proof to deal with the two other cases.[1894]

Thus let $n \in \mathbb{N}$. As usual we equip $\mathbb{R}^{n+1}$ with the euclidean inner product $\langle\,,\,\rangle$. In this proof we identify $\mathrm{O}(n+1)$ with the group of isometries of $(\mathbb{R}^{n+1}, \langle\,,\,\rangle)$. More precisely, we do not distinguish in the notation between an isometry of $(\mathbb{R}^{n+1}, \langle\,,\,\rangle)$ and the matrix it represents with respect to the standard basis of $\mathbb{R}^{n+1}$.

We consider the two compact subsets

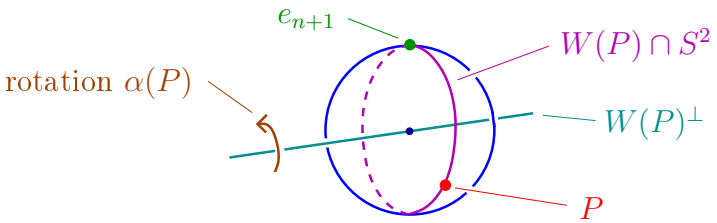
$$V \;=\; \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} \leq \tfrac{1}{2}\} \quad \text{and } W \;=\; \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} \geq -\tfrac{1}{2}\}.$$

We continue with the following claim.

Claim. Let $P \in V$. We write $W(P) := \mathrm{span}\{e_{n+1}, P\}$. There exists a unique orientation-preserving isometry $\alpha(P)$ of $\mathbb{R}^{n+1} = W(P) \oplus W(P)^\perp$ that satisfies the following two conditions:

(1) the restriction of $\alpha(P)$ to $W(P)$ is an orientation-preserving isometry of $W(P)$ that sends e_{n+1} to P,
(2) it equals the identity on $W(P)^\perp$.

Proof. We illustrate the definition of $\alpha(P)$ in the figure below. In the 3-dimensional setting the isometry $\alpha(P)$ is the unique rotation around "the $W(P)^\perp$-axis" that sends e_{n+1} to P.



[1894] Alternatively the reader can look up the proof of the Lie Group-Fiber Proposition 156.9 (1).

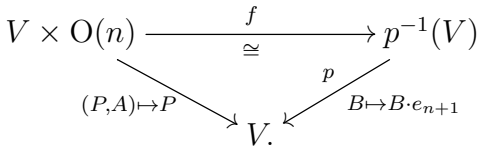
Note that $(W(P), \langle\,,\,\rangle)$ and $(\mathbb{R}^2, \langle\,,\,\rangle)$ are both positive-definite and symmetric forms. Thus it follows from Exercise 133.1 that both forms are also non-singular. Therefore we obtain from Sylvester's Theorem 133.10 that the two forms $(W(P), \langle\,,\,\rangle)$ and $(\mathbb{R}^2, \langle\,,\,\rangle)$ are isometric. In the latter case the orientation-preserving isometries are given by $\mathrm{SO}(2)$. Since $\mathrm{SO}(2) = S^1$ acts freely and transitively on S^1 we see that given any two points $X, Y \in \mathbb{R}^2$ with $\|X\| = \|Y\|$ there exists a unique orientation-preserving isometry β of $(\mathbb{R}^2, \langle\,,\,\rangle)$ with $\beta(X) = Y$. Since $(W(P), \langle\,,\,\rangle)$ is isometric to $(\mathbb{R}^2, \langle\,,\,\rangle)$ the same uniqueness statement applies to $W(P)$ and the two points $e_{n+1}, P \in W(P)$. The claim immediately follows from this observation. ⊞

We leave it to the reader to verify that the map $\alpha\colon V \to \mathrm{O}(n+1)$ that we had just defined is indeed continuous. Next we consider the map

$$\begin{array}{rcl} f\colon V \times \mathrm{O}(n) & \to & \mathrm{O}(n+1) \\ (P, A) & \mapsto & \alpha(P) \circ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}. \end{array}$$

An entertaining exercise in linear algebra shows that the map f restricts to a bijection $V \times \mathrm{O}(n) \to p^{-1}(V)$. Since V and $\mathrm{O}(n)$ are compact we obtain from the Compact-Hausdorff Proposition 2.17 (3) that f defines a homeomorphism $V \times \mathrm{O}(n) \to p^{-1}(V)$.

Now recall that for any $P \in V$ we have, by design, that $\alpha(P)(e_{n+1}) = P$. Using this observation we see almost immediately that the following diagram commutes:



This shows that we have found a trivialization of p over V. With very minor modifications one can also find a trivialization of p over W. Since the interiors of V and W cover all of S^n we see that these two trivializations show that $p\colon \mathrm{O}(n+1) \to S^n$ is indeed a fiber bundle with fiber $\mathrm{O}(n)$. ■

Now we can continue with the actual study of the homotopy groups of $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$.

Proposition 156.10. Let $n \in \mathbb{N}$.

(1) The inclusion induced map[1895]

$$\pi_k(\mathrm{O}(n)) \;\to\; \pi_k(\mathrm{O}(n+1)) \quad \text{is an} \quad \begin{cases} \text{isomorphism for } k < n-1, \\ \text{epimorphism for } k = n-1. \end{cases}$$

(2) The inclusion induced map

$$\pi_k(\mathrm{U}(n)) \;\to\; \pi_k(\mathrm{U}(n+1)) \quad \text{is an} \quad \begin{cases} \text{isomorphism for } k < 2n, \\ \text{epimorphism for } k = 2n. \end{cases}$$

(3) The inclusion induced map

$$\pi_k(\mathrm{Sp}(n)) \;\to\; \pi_k(\mathrm{Sp}(n+1)) \quad \text{is an} \quad \begin{cases} \text{isomorphism for } k < 4n+2, \\ \text{epimorphism for } k = 4n+2. \end{cases}$$

Example.

(1) We know that $\mathrm{U}(1) = S^1$. Thus we obtain from Proposition 156.10 (2) that all unitary groups $\mathrm{U}(n)$ are path-connected.
(2) Recall that in Lemma 156.2 (1) we pointed out that $\mathrm{Sp}(1) = S^3$. Thus we obtain from Proposition 156.10 (3) that all compact symplectic groups $\mathrm{Sp}(n)$ are simply connected.

Proof. We prove the first statement. The proofs of the remaining two statements are almost identical.[1896] By the Matrix Group-Fiber Proposition 156.8 we have a bundle projection $p\colon \mathrm{O}(n+1) \to S^n$ with fiber $\mathrm{O}(n)$. From the Fiber-LES Proposition 155.2 we obtain a long exact sequence of the form

$$\ldots \to \pi_{k+1}(S^n) \xrightarrow{\partial} \pi_k(\mathrm{O}(n)) \to \pi_k(\mathrm{O}(n+1)) \to \pi_k(S^n) \xrightarrow{\partial} \ldots$$

The desired statement now follows from the familiar fact that by Proposition 71.10 we have $\pi_i(S^k) = 0$ for $i < k$. ∎

As a corollary to Proposition 156.10 (1) we can now finally answer Question 83.5. More precisely, we have the following proposition.

Proposition 156.11.

(1) For any $n \in \mathbb{N}$ the inclusions $i\colon \mathrm{SO}(n) \to \mathrm{SL}(n,\mathbb{R})$ and $j\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{GL}_+(n,\mathbb{R})$[1897] are homotopy equivalences.
(2) Let $n \in \mathbb{N}_{\geq 2}$. We have isomorphisms

$$\pi_1(\mathrm{SO}(n)) \underset{i_*}{\xrightarrow{\cong}} \pi_1(\mathrm{SL}(n,\mathbb{R})) \underset{j_*}{\xrightarrow{\cong}} \pi_1(\mathrm{GL}_+(n,\mathbb{R})) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}, & \text{if } n \geq 3. \end{cases}$$

(3) In each of the three fundamental groups of (2) a generator of the group is represented by the loop

$$\begin{array}{ccc} S^1 & \to & \mathrm{SO}(n) \\ \exp(\mathrm{i}\varphi) & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-2} \end{pmatrix}. \end{array}$$

Proof.

(1) We proved this statement in the Matrix Groups Deformation Retract Proposition 15.7.
(2) First we consider the case $n = 2$. We have isomorphisms

$$\mathbb{Z} \underset{\uparrow}{\cong} \pi_1(S^1, \mathrm{id}) = \pi_1(\mathrm{SO}(2), \mathrm{id}) \underset{\uparrow}{\xrightarrow{\cong}} \pi_1(\mathrm{SL}(2,\mathbb{R}), \mathrm{id}) \underset{\uparrow}{\xrightarrow{\cong}} \pi_1(\mathrm{GL}_+(2,\mathbb{R}), \mathrm{id}).$$

Proposition 48.17 — by (1) and the Homotopy-π_1-Proposition 50.3 (1)

[1895] Recall that in the Matrix Group Path-Component Proposition 2.37 (3) we saw that $\mathrm{O}(n)$ is disconnected. Thus in principle it is not completely clear what we mean by $\pi_k(\mathrm{O}(n))$. But, as we pointed out on page 3320, we always consider $\mathrm{O}(n)$ as equipped with the base point that is given by the identity. Also note that it follows easily from Lemma 5.40 that for a topological group the isomorphism type of the homotopy groups does not depend on the choice of a base point.
[1896] The only catch is that one needs to be careful about the degrees involved.
[1897] Recall that we write $\mathrm{GL}_+(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid \det(A) > 0\}$.

Now we consider the case $n \geq 3$. We have isomorphisms

$$\begin{array}{ccccccc} \mathbb{Z}_2 & \cong & \pi_1(\mathrm{SO}(3), \mathrm{id}) & \cong & \pi_1(\mathrm{SO}(n), \mathrm{id}) & \xrightarrow{\cong} & \pi_1(\mathrm{SL}(n, \mathbb{R}), \mathrm{id}) \xrightarrow{\cong} \pi_1(\mathrm{GL}_+(n, \mathbb{R}), \mathrm{id}). \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{Corollary 83.4} & & \text{by Proposition 156.10 and the fact that by the Matrix Group Path-Component Proposition 2.37 (2) } \mathrm{SO}(k) \text{ is the path-component of } \mathrm{O}(k) \text{ that contains id} & & & & \text{by (1) and the Homotopy-}\pi_1\text{-Proposition 50.3 (1)} \end{array}$$

(3) We consider the loop

$$\begin{array}{rcl} \gamma\colon S^1 & \to & \mathrm{SO}(2) \\ \exp(\mathrm{i}\varphi) & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}. \end{array}$$

Since $\mathrm{SO}(2) = S^1$ we know by Proposition 48.17 that γ represents a generator of $\pi_1(\mathrm{SO}(2))$. By Proposition 156.10 we know that the inclusion $\mathrm{SO}(2) \to \mathrm{SO}(n)$ induces an epimorphisms of fundamental groups. It follows immediately that the loop given in (3) is indeed a generator of $\pi_1(\mathrm{SO}(n))$. Since the inclusions i and j induce isomorphisms of fundamental groups we see that the given loop also represents a generator of $\pi_1(\mathrm{SL}(n, \mathbb{R}))$ and $\pi_1(\mathrm{GL}(n, \mathbb{R}))$. ■

156.3. **Spin groups.** Let $n \geq 3$. Recall that in the Matrix Group Path-Component Proposition 2.37 (3) we showed that $\mathrm{SO}(n)$ is path-connected and in Corollary 156.11 (2) we had just seen that $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}_2$. By Lemma 48.15 this implies that the universal covering, as defined on page 1372, of $\mathrm{SO}(n)$ is a 2-fold covering of $\mathrm{SO}(n)$.[1898] For example in the SU(2)-SO(3)-Covering Theorem 83.2 we showed that the 2-fold covering of SO(3) is diffeomorphic to SU(2).
Thus the following question arises:

Question 156.12. Let $n \geq 3$.
(1) Is the universal covering of $\mathrm{SO}(n)$ a group?
(2) If yes, is it a group "we already know"?

It turns out that the answer to Question 156.12 (1) is yes. In fact, the following, initially perhaps somewhat surprising proposition holds. The proof turns out to be almost disappointingly easy.

Proposition 156.13. (Covering-of-Lie Group Proposition) Let G be a connected Lie group and let $p\colon \widetilde{G} \to G$ be some covering (e.g. it could be the universal covering). We denote by 1_G the neutral element in G. We pick a point $1_{\widetilde{G}} \in p^{-1}(1_G)$. Then $\widetilde{G}$ admits a unique Lie group structure such that $p\colon \widetilde{G} \to G$ is a smooth map and a homomorphism and such that $1_{\widetilde{G}}$ is the neutral element in $\widetilde{G}$.

Remark.
(1) Let G be a topological group that is connected and locally path-connected. The proof of the Covering-of-Lie Group Proposition 156.13 also shows that given any covering $p\colon \widetilde{G} \to G$ of G together with a choice of $\widetilde{e} \in \widetilde{G}$ with $p(\widetilde{e}) = e$ there exists a unique

[1898] Note that here we use that $\mathrm{SO}(n)$ is a smooth manifold, which implies by the discussion on page 1372 that the universal covering does indeed exist.

topological group structure on $\widetilde{G}$ with trivial element $\widetilde{e}$ which turns p into a group homomorphism.

(2) Let $k \in \mathbb{N}_{\geq 2}$. By the Covering-of-Lie Group Proposition 156.13 the universal covering $\widetilde{\mathrm{SL}(k,\mathbb{R})}$ of the Lie group $\mathrm{SL}(k,\mathbb{R})$ is again Lie group. In [**HallB2015**, Theorem C.12] it is shown that $\widetilde{\mathrm{SL}(k,\mathbb{R})}$ is *not* isomorphic to any subgroup of $\mathrm{GL}(n,\mathbb{R})$ for any $n \in \mathbb{N}$. This shows that Theorem 156.6 does not extend to non-compact Lie groups.

Sketch of the proof. First note that we saw in the Manifold Covering Proposition 49.2 that $\widetilde{G}$ is a topological manifold that admits a unique smooth structure such that $p\colon \widetilde{G} \to G$ is a smooth map. Now we turn towards defining the Lie group structure on the smooth manifold $\widetilde{G}$. In fact we will give two equivalent definitions of the group structure on $\widetilde{G}$:

(1) First note that by the Fundamental Group-of-Product Proposition 48.20 we know that $\widetilde{G} \times \widetilde{G}$ is also simply connected. We consider the maps

$$\begin{array}{ccc} & & \widetilde{G} \\ & & \big\downarrow p \\ \widetilde{G} \times \widetilde{G} & \xrightarrow[(x,y)\mapsto p(x)\cdot p(y)]{} & G. \end{array}$$

It follows from the Map Lifting Criterion 61.2 that there exists a unique lift

$$\begin{array}{ccc} & \overset{(1_{\widetilde{G}},1_{\widetilde{G}})\mapsto 1_{\widetilde{G}}}{\nearrow} & \widetilde{G} \\ & & \big\downarrow p \\ \widetilde{G} \times \widetilde{G} & \xrightarrow[(x,y)\mapsto p(x)\cdot p(y)]{} & G. \end{array}$$

(2) By the Covering Existence Theorem 61.5 and Corollary 61.11 we can assume that $\widetilde{G}$ equals the canonical universal covering, i.e. we can assume that

$$\widetilde{G} \;=\; \{\text{paths in } G \text{ starting at } 1_G\}/\ \text{homotopies that fix the starting point and the endpoint}$$

With that explicit description we define

$$\begin{array}{rcl} \widetilde{G} \times \widetilde{G} & \to & \widetilde{G} \\ ([a(t)],[b(t)]) & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & G \\ t & \mapsto & a(t)\cdot b(t) \end{array}\right]. \end{array}$$

Using the fact that the multiplication map $\widetilde{G} \times \widetilde{G} \to \widetilde{G}$ is continuous one can show fairly easily that this map is well-defined.

We leave it as a charming exercise to show that the two given definitions of the group structure agree and that in both cases we defined a Lie group structure on $\widetilde{G}$.[1899]

It remains to show that the group structure on $\widetilde{G}$ is the unique Lie group structure on the smooth manifold $\widetilde{G}$ for which $1_{\widetilde{G}}$ is the neutral element and that turns p into a homomorphism. Suppose we are given a smooth map $\varphi\colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ that has the same properties. We obtain the following commutative diagram:

$$\begin{array}{ccc} \widetilde{G} \times \widetilde{G} & \xrightarrow{\varphi} & \widetilde{G} \\ {\scriptstyle p\times p}\big\downarrow & \overset{(x,y)\mapsto p(x)\cdot p(y)}{\searrow} & \big\downarrow{\scriptstyle p} \\ G \times G & \xrightarrow[(x,y)\mapsto x\cdot y]{} & G. \end{array}$$

[1899] The smoothness of the group multiplication is perhaps easier to prove with the first point of view, the fact that it is a group structure is possibly less confusing to verify with the second point of view.

Since the diagram commutes we see that $\varphi\colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ is precisely a lift of the blue map $\widetilde{G} \times \widetilde{G} \to G$ that sends $(1_{\widetilde{G}}, 1_{\widetilde{G}})$ to $1_{\widetilde{G}}$. Thus by the uniqueness of the lift we see that the Lie group structure on $\widetilde{G}$ is uniquely determined by our wish list. ■

This proposition leads us to the following definition.

Definition. Let $n \geq 3$. We refer to the natural universal covering[1900] of the pointed topological space $(\mathrm{SO}(n), \mathrm{id})$ as the spin group $\mathrm{Spin}(n)$. By the Covering-of-Lie Group Proposition 156.13 there exists a unique Lie group structure on $\mathrm{Spin}(n)$ for which the canonical base point is the neutral element and which has the property that the projection $p\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a smooth map and a group homomorphism.

The combination of Propositions 156.11 (2) and 156.13 and Corollary 61.11 shows that $\mathrm{Spin}(n)$ is indeed a Lie group and that $p\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a 2-fold covering that is a group homomorphism.

Remark. By Corollary 61.11 the universal covering of a smooth manifold, even if it is equipped with a base point, is only well-defined up to "equivalence". This slightly awkward issue can be circumvented in two different ways:

(1) either we use the explicit model of the universal covering of a based smooth manifold in terms of equivalence classes of paths, or
(2) in this particular situation there is a more elegant solution, one can give a purely algebraic explicit description of $\mathrm{Spin}(n)$. We refer to [**Mein2013**, Chapter 3.1] for details.

It is harder to give an answer to Question 156.12 (2). For reasonably low dimensions the following proposition gives a positive answer.

Proposition 156.14.

(1) There exist Lie group isomorphisms[1901]

(a) $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\{\pm \mathrm{id}\}$
(b) $\mathrm{SO}(4) \cong (\mathrm{SU}(2) \times \mathrm{SU}(2))/\{\pm(\mathrm{id}, \mathrm{id})\}$
(c) $\mathrm{SO}(5) \cong \mathrm{Sp}(2)/\{\pm \mathrm{id}\}$.

(2) There exist Lie group isomorphisms

(a) $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$
(b) $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$
(c) $\mathrm{Spin}(5) \cong \mathrm{Sp}(2)$
(d) $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$.

Proposition 156.14 (b) thus says that for $n = 3, 4, 5, 6$ the spin group $\mathrm{Spin}(n)$ is isomorphic to Lie groups that we are already familiar with. But according to [**Mein2013**, p. 53] "for $n \geq 7$, there are no further accidental isomorphisms of this type." In other words, in general the answer to Question 156.12 (2) is negative.

Proof.

[1900] We refer to page 1372 for the definition of the canonical universal covering of a suitable pointed topological space and to the definition of the corresponding canonical base point.
[1901] Note that it follows from Lemma 156.5 that the groups on the right-hand side are indeed Lie groups.

(1) (a) This statement follows immediately from Lemma 83.1 and the SU(2)-SO(3)-Covering Theorem 83.2.

(b) In Proposition 92.2 we saw that the quaternion multiplication defines a group structure on $S^3 = \{h \in \mathbb{H} \,|\, |h| = 1\}$ where the trivial element is given by $1 \in \mathbb{H}$. Furthermore in Proposition 92.2 we had also seen that the Lie groups SU(2) and S^3 are isomorphic. Thus it suffices to show that the Lie groups $(S^3 \times S^3)/\{\pm(1,1)\}$ and SO(4) are diffeomorphic. We consider the following map:

$$\begin{array}{rcl} \Phi\colon S^3 \times S^3 & \to & \mathrm{SO}(4) \\ (q_1, q_2) & \mapsto & \left(\begin{array}{rcl} \mathbb{R}^4 = \mathbb{H} & \to & \mathbb{R}^4 = \mathbb{H} \\ z & \mapsto & q_1 z q_2^{-1} \end{array} \right). \end{array}$$

We make the following observations:

(i) In Exercise 92.3 we showed that the map $z \mapsto q_1 z q_2^{-1}$ does indeed define an element in SO(4).

(ii) One can easily verify that the map Φ is smooth and that it is a homomorphism.

(iii) We leave it to the energetic reader to verify that the differential of Φ at each point is invertible.

(iv) We note that the kernel of this homomorphism is given by $\{\pm(1,1)\}$. Indeed, the kernel of this homomorphism is given by the set of all $(q_1, q_2) \in S^3 \times S^3$ such that $q_1 z q_2^{-1} = z$ for all $z \in \mathbb{H}$. By setting $z = 1$ we obtain that $q_1 = q_2$. Furthermore, since $q_1 z q_1^{-1} = z$ for all $z \in S^3$ we see that $q_1 = q_2$ lies in the center of the group S^3. An elementary calculation shows that the center of S^3 is given by ± 1. Thus the kernel of Φ contains precisely the two elements $(1,1)$ and $(-1,-1)$.

It follows from the above, the Topological-Quotient Proposition 5.15 (1b) and Lemma 156.5 and elementary group theory[1902] that the above map Φ descends to a smooth and injective map

$$\overline{\Phi}\colon (S^3 \times S^3)/\{\pm(1,1)\} \to \mathrm{SO}(4)$$

such that the differential at each point is invertible. By Lemmas 156.3 and 156.5 both sides are closed connected 6-dimensional smooth manifolds. Thus it follows from Corollary 82.9 (2) that the map $\overline{\Phi}$ is also surjective. It follows from the above results and the Inverse Mapping Theorem 24.1 (2) that $\overline{\Phi}$ is in fact a diffeomorphism.

(c) Let V be the real vector space of quaternionic Hermitian 2×2-matrices with trace 0. In other words we consider

$$V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}(2 \times 2, \mathbb{H}) \,\middle|\, a + d = 0, a, d \in \mathbb{R} \text{ and } b = \overline{c} \right\}.$$

[1902] More precisely, we use the elementary fact that if $\alpha\colon G \to H$ is a group homomorphism, then the induced map $G/\ker(\alpha) \to H$ is injective.

Note that V is a 5-dimensional real vector space. The group $\mathrm{Sp}(2)$ acts by conjugation on V.[1903] One easily verifies that the kernel is given by $\{\pm \mathrm{id}\}$. Furthermore by [**Pro2007**, Chapter 4.5.3] there exists a positive-definite symmetric form on V that is preserved by this action. Thus, by Sylvester's Theorem 133.10 this action defines a homomorphism $\mathrm{Sp}(2) \to \mathrm{SO}(5)$. Similar to (b) we obtain a diffeomorphism $\mathrm{Sp}(2)/\{\pm \mathrm{id}\} \xrightarrow{\cong} \mathrm{SO}(5)$.

(2) (a) By (2a) we can make the identification $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm \mathrm{id}\}$ of Lie groups. By the SU(2)-SO(3)-Covering Theorem 83.2 we know that $\mathrm{SU}(2)$ is homeomorphic to S^3, in particular we see that $\mathrm{SU}(2)$ is simply connected. By Lemma 156.5 the quotient map $\mathrm{SU}(2) \to \mathrm{SU}(2)/\{\pm 1\}$ is a covering map, it is smooth and it is a homomorphism. Thus we see that $\mathrm{SU}(2)$ is in fact the universal covering of $\mathrm{SU}(2)/\{\pm \mathrm{id}\} = \mathrm{SO}(3)$ and we deduce from the Covering-of-Lie Group Proposition 156.13 that $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$.

(b) By the same argument as in (2a) this statement can be deduced from (1b).

(c) By the same argument as in (2a) this statement can be deduced from (1c).

(d) We will not make use of this results, hence we will not provide a proof, instead we refer to [**Mein2013**, p. 84] for details. ■

Remark. Initially the spin groups might appear as amusing curiosities. In fact they arise naturally in many parts of mathematics and physics. We refer to [**LaM1989, Morg1996, Mein2013**] for more details.

156.4. **The Bott Periodicity Theorem.** We continue with the following definition.

Definition. We define[1904]

$$\mathrm{O} := \varinjlim \mathrm{O}(n) = \left\{ A = (a_{ij})_{i,j\in\mathbb{N}} \;\middle|\; \begin{array}{l} \text{there exists an } n \in \mathbb{N} \\ \text{and a } B \in \mathrm{O}(n) \text{ with } \end{array} A = \begin{pmatrix} B & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \ddots \end{pmatrix} \right\}$$

where O has the obvious group structure and where O has the topology that is defined by the condition that subset $U \subset \mathrm{O}$ is open if and only if for each n the intersection $U \cap \mathrm{O}(n)$ is open. Similarly we define

$$\mathrm{U} := \varinjlim \mathrm{U}(n), \quad \mathrm{Sp} := \varinjlim \mathrm{Sp}(n)$$

In the Direct Limit Existence Proposition 56.1 we saw that every direct system $\{G_i\}_{i\in I}$ of groups admits a direct limit $\varinjlim G_i$. In fact for the above direct limit $\mathrm{O} = \varinjlim \mathrm{O}(n)$ the group structure is the "obvious" one given above. The following shows that these group structure are continuous, i.e. O, U and Sp are topological groups.

Lemma 156.15. The above group structures on O, U and Sp are continuous.

[1903] The group $\mathrm{Sp}(2)$ acts evidently by conjugation on all quaternionic 2×2-matrices, in principle one now needs to verify that this action restricts to an action on V.

[1904] When we talk about the direct limit we have to say what category we consider. Here we want to consider the category of groups and of topological spaces at the same time. Since we have obvious inclusions maps $\mathrm{O}(n) \to \mathrm{O}(n+1)$ we obtain from the Direct Limit-of-Topological Spaces Lemma 56.3 that in both settings we obtain the same underlying set, which we can then equip with a group structure and a topology.

Remark. Initially one might be tempted to dismiss Lemma 156.15 as obvious. But in general there is a subtle issue. If $\{G_i\}_{i\in I}$ is a direct system of topological groups, then the above multiplication on the direct limit $\varinjlim G_i$ might not be continuous. Such examples are for example provided in [**TSH1998**]. If one thinks about it for a while one realizes that this failure for the multiplication map to be continuous is another instance of the observation made in Section 13.1, that mixing product and quotient topologies is fraught with dangers.

Proof. This lemma can surely be proved, with a little effort, "by hand". Alternatively one can use [**TSH1998**, Theorem 2.7] which shows that for a direct system $\{X_i\}_{i\in\mathbb{N}}$ of compact topological groups the above multiplication on $\varinjlim X_i$ is indeed continuous. ■

For later usages of O, U and Sp we have to get the following technical lemma out of the way.

Lemma 156.16. The topological spaces O, U and Sp admit CW-structures such that each $\mathrm{O}(n) \subset \mathrm{O}$, $\mathrm{U}(n) \subset \mathrm{U}$ and $\mathrm{Sp}(n) \subset \mathrm{Sp}$ is a subcomplex.

Proof. In [**WhdJ1944**] it is shown that we can equip the $\mathrm{O}(n)$ with CW-structures such that for each n the subset $\mathrm{O}(n) \subset \mathrm{O}(n+1)$ is actually a subcomplex. Analogous statements for $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ are proved in [**Yok1955**, Theorem 5.3 and 5.5] or alternatively in [**Stro2011**, Chapter 9.3.3]. It follows from the Ascending Union-CW Complex Lemma 68.34 that O is a CW-complex. ■

Now we turn to the study of the homotopy groups of O, U and Sp.

Lemma 156.17. Given any $k \in \mathbb{N}$ we have

$$\begin{aligned}
&(1) \quad \pi_k(\mathrm{O}) = \varinjlim \pi_k(\mathrm{O}(n)) \xleftarrow{\cong} \pi_k(\mathrm{O}(N)) \quad \text{for any } N \in \mathbb{N} \text{ with } N > k+1,\\
&(2) \quad \pi_k(\mathrm{U}) = \varinjlim \pi_k(\mathrm{U}(n)) \xleftarrow{\cong} \pi_k(\mathrm{U}(N)) \quad \text{for any } N \in \mathbb{N} \text{ with } 2N > k,\\
&(3) \quad \pi_k(\mathrm{Sp}) = \varinjlim \pi_k(\mathrm{Sp}(n)) \xleftarrow{\cong} \pi_k(\mathrm{Sp}(N)) \quad \text{for any } N \in \mathbb{N} \text{ with } 4N > k-2.
\end{aligned}$$

Proof. We prove (1). The proofs of (2) and (3) are almost identical. We start out with the following claim.

Claim. Given any compact subset K of O there exists an $n \in \mathbb{N}$ with $K \subset \mathrm{O}(n)$.

Proof. Let K be a compact subset of O. Given $n \in \mathbb{N}$ we write $X_n := \{1,\ldots,n\}\times\{1,\ldots,n\}$. First recall that $\mathrm{O}(n)$ is equipped with the subspace topology coming from the inclusion map $g_n\colon \mathrm{O}(n) \subset \mathrm{M}(n\times n,\mathbb{R}) = \mathbb{R}^{X_n} = \mathbb{R}^{n^2}$. Given $n \in \mathbb{N}$ we denote by $f_n\colon \mathbb{R}^{X_n} \to \mathbb{R}^{X_{n+1}}$ the obvious inclusion map. These have the property that for any $n \in \mathbb{N}$ the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{O}(n) & \xrightarrow{\quad g_n \quad} & \mathbb{R}^{X_n} \\
\big\downarrow & & \big\downarrow{\scriptstyle f_n} \\
\mathrm{O}(n+1) & \xrightarrow{\quad g_{n+1} \quad} & \mathbb{R}^{X_{n+1}}.
\end{array}$$

The above inclusion maps $g_n\colon \mathrm{O}(n) \to \mathbb{R}^{X_n}$ induce a map $g\colon \mathrm{O} = \varinjlim \mathrm{O}(n) \hookrightarrow \varinjlim \mathbb{R}^{X_n}$. By our hypothesis that K is compact and the Compact Image Lemma 2.13 we know that $g(K) \subset \varinjlim \mathbb{R}^{X_n}$ is also compact. Basically the same argument as in the proof of

Lemma 68.3 shows that there exists an $n \in \mathbb{N}$ with $g(K) \subset \mathbb{R}^{X_n}$. But this implies that $K \subset \mathrm{O}(n)$. ⊞

Now let $N \in \mathbb{N}$ with $N > k+1$. We have

$$\begin{array}{ccccc} \pi_k(\mathrm{O}) & = & \varinjlim \pi_k(\mathrm{O}(n)) & = & \pi_k(\mathrm{O}(N)). \\ & & \uparrow & & \uparrow \\ & & \text{by the Direct Limit-}\pi_n\text{-Proposition 71.11} & & \text{by Proposition 156.10 (1) and the} \\ & & \text{and the claim} & & \text{Direct Limit Stabilization Lemma 56.2} \end{array}$$

■

We will now state two versions of the Bott Periodicity Theorem which implies in particular that the homotopy groups of O, U and Sp are periodic.
The first theorem requires just a little bit of preparation.

(1) As above we equip U, Sp and O with the base point given by the infinite identity matrix. We use these base points to form the loop spaces ΩU, $\Omega\,\mathrm{Sp}$ and ΩO.
(2) Given a point x in a topological space X we denote by c_x the corresponding constant path.
(3) We use notation (2) to equip each loop space $\Omega(X, x_0)$ with the base c_{x_0} and we can thus iterate the loop space construction.
(4) The obvious isomorphism $\mathbb{H} \xrightarrow{\cong} \mathbb{R}^4$ of real vector spaces defines inclusion maps $\mathrm{Sp}(n) \to \mathrm{O}(4n)$ and $\mathrm{Sp} \to \mathrm{O}$.
(5) The inclusion $\mathbb{R} \to \mathbb{H}$ defines inclusions $\mathrm{O}(n) \to \mathrm{Sp}(n)$ and thus $\mathrm{O} \to \mathrm{Sp}$.

Theorem 156.18. (Bott Periodicity Theorem) There exist homotopy equivalences

$$(1) \quad \mathrm{U} \to \Omega^2\,\mathrm{U} \qquad (2) \quad \mathrm{Sp} \to \Omega^4\,\mathrm{O} \qquad \text{and} \qquad (3) \quad \mathrm{O} \to \Omega^4\,\mathrm{Sp}.$$

Before we talk about the proof of the Bott Periodicity Theorem 156.18 let us move on to the following direct consequence.

Theorem 156.19. (Bott Periodicity Theorem) For any $i \in \mathbb{N}_0$ we have[1905]

$$\begin{array}{lll} (1) & \pi_{i+2}(\mathrm{U}) & \cong \pi_i(\mathrm{U}), \\ (2) & \pi_{i+4}(\mathrm{Sp}) & \cong \pi_i(\mathrm{O}), \\ (3) & \pi_{i+4}(\mathrm{O}) & \cong \pi_i(\mathrm{Sp}). \end{array}$$

In particular, combining (2) and (3) we have for any $i \in \mathbb{N}_0$ that

$$\begin{array}{lll} (4) & \pi_{i+8}(\mathrm{O}) & \cong \pi_i(\mathrm{O}), \\ (5) & \pi_{i+8}(\mathrm{Sp}) & \cong \pi_i(\mathrm{Sp}). \end{array}$$

Proof of the Bott Periodicity Theorems 156.18 and 156.19. First note that Theorem 156.19 follows immediately from Theorem 156.18 together with the Loop Space-Homotopy Groups Corollary 154.15. All the known proofs are non-trivial and go well-beyond what we can do in these modest notes.

Theorem 156.18 was first proved by Raoul Bott [**Bot1957, Bot1959, Bot1970**] in 1957. Another proof, using Morse theory, is given in [**Miln1963a**, p. 129, 142]. There explicit maps that give these homotopy equivalences, they are described well in [**DL61**]. Nonetheless the description is too long for us to write down at this point.

There are many proofs of Theorem 156.19 which use a slightly different language, in particular it is at first glance not entirely clear whether they also imply Theorem 156.18.

[1905] For $i = 0$ we actually have an isomorphism of pointed sets.

We refer to [**Stro2011**, Chapter 32.6], [**AGP2002**, Chapter 9.5], [**APr1999**] and [**Hat2**, Theorem 2.16] for a discussions and proofs of Theorem 156.19. ■

Using the Bott Periodicity Theorem 156.19 we can completely determine the homotopy groups of O, U and Sp.

Proposition 156.20. The homotopy groups of O, U and Sp are as follows:

$i \bmod 8$	0	1	2	3	4	5	6	7
$\pi_i(\mathrm{U})$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\pi_i(\mathrm{O})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\pi_i(\mathrm{Sp})$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$.

Proof. The homotopy groups of U are straightforward to determine. More precisely, let $i \in \mathbb{N}_0$. We have

$$\pi_i(\mathrm{U}) \underset{\text{Bott Periodicity Theorem 156.19}}{=} \pi_{i \bmod 2}(\mathrm{U}) \underset{\text{Lemma 156.17 (2)}}{\cong} \pi_{i \bmod 2}(\mathrm{U}(1)) \underset{\text{since } \mathrm{U}(1) = S^1}{=} \pi_{i \bmod 2}(S^1) \underset{\substack{\text{see the Matrix Group Path-Component} \\ \text{Proposition 2.37 and Proposition 48.17}}}{\cong} \begin{cases} 0, & \text{if } i = 0 \bmod 2, \\ \mathbb{Z}, & \text{if } i = 1 \bmod 2. \end{cases}$$

Now we consider the homotopy groups of O and Sp. By the Bott Periodicity Theorem 156.19 it suffices to compute the homotopy groups $\pi_i(\mathrm{O})$ and $\pi_i(\mathrm{Sp})$ for $i = 0, 1, 2, 3$.[1906] So let $i \in \{0, 1, 2, 3\}$. We have

$$\pi_i(\mathrm{Sp}) \underset{\text{Lemma 156.17 (3)}}{\cong} \pi_i(\mathrm{Sp}(1)) \underset{\text{since } \mathrm{Sp}(1) = S^3}{=} \pi_i(S^3) \underset{\text{Proposition 71.10 and Corollary 85.6 (1)}}{\cong} \begin{cases} 0, & \text{if } i = 0, 1, 2, \\ \mathbb{Z}, & \text{if } i = 3. \end{cases}$$

We turn to the calculation of the homotopy groups of O. First we note that for any $i \geq 1$ we have

$$\pi_i(\mathrm{O}) \underset{\text{Lemma 156.17 (3)}}{\cong} \pi_i(\mathrm{O}(i+2)) \underset{\substack{\text{Matrix Group Path-Component Proposition 2.37} \\ \text{and Lemma 5.40}}}{\cong} \pi_i(\mathrm{SO}(i+2)).$$

Thus it remains to prove the following claim.

Claim.
(a) $\pi_0(\mathrm{O}(2)) \cong \mathbb{Z}_2$
(b) $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$
(c) $\pi_2(\mathrm{SO}(4)) \cong 0$
(d) $\pi_3(\mathrm{SO}(5)) \cong \mathbb{Z}$.

Proof.

(a) By the Matrix Group Path-Component Proposition 2.37 (3) we know that $\mathrm{O}(2)$ consists of two path components, thus $\pi_0(\mathrm{O}(2))$ is by definition a pointed set with two elements.
(b) By Proposition 156.11 we know that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$.

[1906] We will see that $\pi_0(\mathrm{O})$ is the pointed set with two elements. It follows from the Bott Periodicity Theorem 156.19 that $\pi_{4+8k}(\mathrm{Sp})$ and $\pi_{8k}(\mathrm{O})$, $k \geq 1$, are also pointed sets with two elements. But these pointed sets are also groups. Since there is only one group structure on a pointed set with two elements we see that all these groups are isomorphic to $\mathbb{Z}_2$.

(c) We have the following isomorphisms:

$$\begin{array}{ccccccccc} \pi_2(\mathrm{SO}(4)) & \cong & \pi_2((S^3\times S^3)/\mathbb{Z}_2) & \cong & \pi_2(S^3\times S^3) & \cong & \pi_2(S^3)\oplus\pi_2(S^3) & = & 0. \\ \uparrow & & \uparrow & & \downarrow & & \uparrow & & \\ \text{by Proposition 156.14 (1b) and Proposition 92.2} & & \text{by Lemma 156.5 and the Coverings-}\pi_n\text{-Proposition 71.13} & & \text{by the Homotopy Groups-of-Product Proposition 71.8} & & \text{Proposition 71.10} & & \end{array}$$

(d) We have the following isomorphisms:

$$\begin{array}{ccccccccc} \pi_3(\mathrm{SO}(5)) & \cong & \pi_3(\mathrm{Sp}(2)/\{\pm\mathrm{id}\}) & \cong & \pi_3(\mathrm{Sp}(2)) & \cong & \pi_3(\mathrm{Sp}) & \cong & \mathbb{Z}. \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ \text{Proposition 156.14 (1c)} & & \text{by Lemma 156.5 and the Coverings-}\pi_n\text{-Proposition 71.13} & & \text{Proposition 156.10} & & \text{see above} & & \end{array}$$

■

156.5. **H-spaces.** In this final section of this chapter we want to state two general statements about homotopy groups of Lie groups. In fact these statements hold for a much wider class of topological spaces that we introduce now:

Definition. We say a topological space X is an H-space[1907] if there exists a map[1908]

$$\mu\colon X\times X\ \to\ X$$

and an element $e\in X$ such that $\mu(e,e)=e$ and such that the maps

$$\begin{array}{rcl}(X,e) & \mapsto & (X,e)\\ x & \mapsto & \mu(x,e)\end{array}\qquad\text{and}\qquad\begin{array}{rcl}(X,e) & \mapsto & (X,e)\\ x & \mapsto & \mu(e,x)\end{array}$$

of pairs of topological spaces are homotopic to the identity.

Examples.

(1) Evidently Lie groups and more generally, topological groups are H-spaces, where μ is given by the group structure and e is the trivial element of the group. In particular we see that the Lie groups S^1 and S^3 are H-spaces.
(2) The topological space S^7 together with the multiplication given by the octonion structure on $\mathbb{O}=\mathbb{R}^8$ and with $e=1=(1,0,\dots,0)\in S^7$ is an H-space.[1909] Since octonion multiplication is not associative we see that, at least with this multiplication, S^7 is not a topological group.

We have just seen that the spheres S^1, S^3 and S^7 are H-space. Before we continue with more examples we point out that we have all the tools to show that no other spheres have H-space structures.

Theorem 156.21. Let $n\in\mathbb{N}$. If S^n admits the structure of an H-space, then $n\in\{1,3,7\}$.

Proof. Let $\mu\colon S^n\times S^n\to S^n$ be the multiplication map of an H-space structure and let $e\in X$ be the "neutral" element. It follows immediately from the definition of the bidegree, see page 2687, together with the Homotopic Maps-and-Homology Proposition 73.6 and the

[1907] According to [**Hub1999**, p. 747] the term H-space was coined by Jean-Pierre Serre to honor the work of Heinz Hopf.
[1908] As usual, whenever we deal with topological spaces, all maps are understood to be continuous.
[1909] Recall that by the discussion on page 1963 we know that for octonions $w,z\in\mathbb{O}=\mathbb{R}^8$ we have $\|w\cdot z\|=\|w\|\cdot\|z\|$, in particular the multiplication restricts to a map on S^7. Also note that multiplication by $e=1$ is the identity.

existence of e, that the map μ has bidegree $(1,1)$. From Proposition 122.12 we obtain a map $S^{2n+1} \to S^{n+1}$ with Hopf invariant ± 1. By the Adams Theorem 152.11[1910] we know that $n+1 \in \{2,4,8\}$. ■

We continue with one more amusing example of an H-space.

Example. Given a non-zero polynomial $p(t) = \sum_{i=0}^{n} a_i t^i \in \mathbb{C}[t]$ we define

$$[p(t)] \;:=\; [a_0 : a_1 : \cdots : a_n : 0 : \ldots] \;\in\; \mathbb{C}\mathrm{P}^\infty.$$

Evidently every element in $\mathbb{C}\mathrm{P}^\infty$ is of that form. We consider the "multiplication" map

$$\begin{array}{rcl} \mu\colon \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty & \to & \mathbb{C}\mathrm{P}^\infty \\ ([p(t)],[q(t)]) & \mapsto & [\underbrace{p(t)\cdot q(t)}_{\in \mathbb{C}[t]}]. \end{array}$$

We leave it to the reader to verify that μ is continuous. It is clear that left and right "multiplication" by $e = [1] = [1:0:\ldots]$ is the identity. This shows that $\mathbb{C}\mathrm{P}^\infty$ together with μ and e is an H-space. Note that this multiplication μ is commutative and associative, but it does not have inverses[1911], thus μ does not define a group structure on $\mathbb{C}\mathrm{P}^\infty$.

We formulate the last example as a lemma.

Lemma 156.22. Let (X, x_0) be a pointed topological space. We denote by $\Omega(X,x_0)$ the loop space as defined on page 3280.

(1) The map

$$\begin{array}{rcl} \mu\colon \Omega(X,x_0) \times \Omega(X,x_0) & \to & \Omega(X,x_0) \\ (f\colon [0,1] \to X, g\colon [0,1] \to X) & \mapsto & (\underbrace{f * g}_{\substack{\text{multiplication of loops} \\ \text{as defined on page 1052}}} : [0,1] \to X) \end{array}$$

is continuous.

(2) The map $\mu\colon \Omega(X,x_0) \times \Omega(X,x_0) \to \Omega(X,x_0)$ from (1) together with the constant loop given by $e(t) \equiv x_0$, turns $\Omega(X,x_0)$ into an H-space.

Remark.

(1) Note that for the H-space $\Omega(X,x_0)$ "multiplication" by e is not the identity, it is only homotopy equivalent to the identity.

(2) We had hinted at the existence of a multiplication map on $\Omega(X,x_0)$ in Exercise 154.5.

Proof. Let (X,x_0) be a pointed topological space. To simplify the notation we write ΩX instead of $\Omega(X,x_0)$.

(1) We need to show that the map $\mu\colon \Omega X \times \Omega X \to \Omega X$ is in fact continuous. By the "⇒"-direction of the Compact-Open Exponential Proposition 12.6, applied to $T = \Omega X \times \Omega X$ and the regionally compact space $Y = [0,1]$, it suffices to show that the following map is continuous:

$$\Phi\colon (\Omega X \times \Omega X) \times [0,1] \xrightarrow{\mu\times \mathrm{id}} \Omega X \times [0,1] \xrightarrow{(f,t)\mapsto f(t)} X.$$

[1910] We partially proved the somewhat weaker Theorem 152.10.

[1911] Why are there elements in $\mathbb{C}\mathrm{P}^\infty$ which do not have multiplicative inverses?

On the subset $(\Omega X \times \Omega X) \times [0, \frac{1}{2}]$ this map is equal to

$$(\Omega X \times \Omega X) \times [0, \tfrac{1}{2}] \xrightarrow{((f,g)\mapsto f)\times(t\mapsto 2t))} \Omega X \times [0,1] \xrightarrow{(f,t)\mapsto f(t)} X.$$

The first map is evidently continuous. Furthermore, the second map is continuous by the Evaluation Map Proposition 12.4. Thus we have shown that the restriction of Φ to the subset $(\Omega X \times \Omega X) \times [0, \frac{1}{2}]$ is continuous. Basically the same argument shows that the restriction of Φ to $(\Omega X \times \Omega X) \times [\frac{1}{2}, 1]$ is continuous. it follows from the Homotopy Stacking Lemma 14.3 that Φ itself is continuous.

(2) Let $e \in \Omega(X, x_0)$ be the constant loop given by $e(t) \equiv x_0$. It is clear that $\mu(e, e) = e$. Thus it remains to show that the maps

$$\begin{array}{rcl} (\Omega(X,x_0),e) & \mapsto & (\Omega(X,x_0),e) \\ f & \mapsto & \mu(f,e) \end{array} \quad \text{and} \quad \begin{array}{rcl} (\Omega(X,x_0),e) & \mapsto & (\Omega(X,x_0),e) \\ f & \mapsto & \mu(e,f) \end{array}$$

of pairs of topological spaces are homotopic to the identity. We consider the map

$$\begin{array}{rcl} H\colon \Omega(X,x_0)\times[0,1] & \to & \Omega(X,x_0) \\ ((f\colon [0,1]\to X),t) & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & X \\ s & \mapsto & \begin{cases} f(\frac{2s}{t+1}), & \text{if } s \in [0, \frac{t+1}{2}], \\ x_0, & \text{otherwise.} \end{cases} \end{array} \right) \end{array}$$

We leave it to the underchallenged reader to show, using the "$\Rightarrow$"-direction of the Compact-Open Exponential Proposition 12.6, that this map H is in fact continuous. Note that we have $H(e,t) = e$ for all $t \in [0,1]$. With these observations it is basically clear that it is the desired homotopy between the map $f \mapsto \mu(f,e)$ and the identity.

Almost exactly the same way we deal with the second map $f \mapsto \mu(e,f)$. ■

The following lemma is the only statement on H-spaces that we can prove with our limited space resources.

Lemma 156.23. If X is a path-connected H-space, then $\pi_1(X)$ is abelian.

Remark. Let (Y, y_0) be a pointed topological space. We recall the following two facts:

(a) Recall that in the Loop Space-Homotopy Groups Corollary 154.15 we saw that given any $k \in \mathbb{N}$ there exists an isomorphism $\pi_{k+1}(Y,y_0) \xrightarrow{\cong} \pi_k(\Omega(Y,y_0))$.

(b) We pointed out above that the loop space $\Omega(Y,y_0)$ is an H-space.

The combination of (a), (b) and Lemma 156.23 implies that given any $k \in \mathbb{N}_{\geq 2}$ we have

$$\pi_k(Y,y_0) \underset{\text{by (a)}}{\cong} \pi_{k-1}(\Omega(Y,y_0)) \cong \ldots \underset{\text{by (a)}}{\cong} \pi_1(\overbrace{\Omega(\Omega\ldots\Omega}^{k \text{ times}}(Y,y_0))) \underset{\text{by (b) and since } k\geq 2}{=} \pi_1(\text{H-space}) \underset{\text{Lemma 156.23}}{=} \text{abelian}.$$

We have thus shown that $\pi_k(Y,y_0)$ is abelian for every $k \geq 2$. In other words, we have obtained a new proof for the Homotopy Group-Abelian Proposition 71.3.

Just for kicks we provide two different proofs for Lemma 156.23. The first proof, which we provide here is "down to earth". The second proof, which we will provide in the next section, is more structural.

First proof of Lemma 156.23. We denote by $\mu\colon X \times X \to X$ and $e \in X$ the objects in the definition of an H-space. Let $f, g\colon [0,1] \to X$ be loops in e. We need to show that the path $f * g$ is path-homotopic to the path $g * f$.

We denote by $\mu(f, e)$ the path $t \mapsto \mu(f(t), e)$ and similarly we define $\mu(e, g)$. We consider the map

$$\begin{array}{rccc} F\colon [0,1] \times [0,1] & \to & X \\ (s,t) & \mapsto & \mu(f(s), g(t)). \end{array}$$

Note that this map is continuous by the Product Topology-Basics Lemma 5.4 (3) and the fact that μ is continuous. Furthermore we consider the following path-homotopy:[1912]

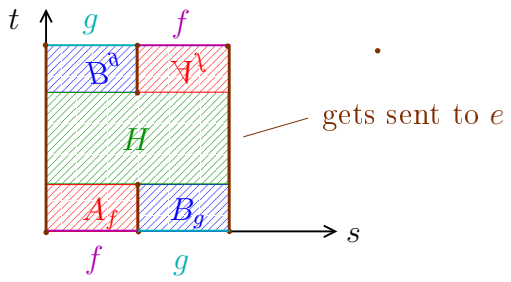
$$\begin{array}{rccl} H\colon [0,1] \times [0,1] & \to & X & \\ (s,t) & \mapsto & \begin{cases} F(2s \cdot (1-t,t)), & \text{if } s \leq \frac{1}{2} \\ F((2-2s) \cdot (1-t,t) + (2s-1) \cdot (1,1)), & \text{if } s > \frac{1}{2}. \end{cases} \end{array}$$

By carefully plugging in $t = 0$ and $t = 1$ we see that H is a path-homotopy from the loop $\mu(f, e) * \mu(e, g)$ to the loop $\mu(e, g) * \mu(f, e)$.

By definition of an H-space there exists a homotopy $A\colon X \times [0,1] \to X$ rel $\{e\}$ with $A_0 = \mathrm{id}$ and $A_1 = \mu(-, e)$ and there exists a homotopy $B\colon X \times [0,1] \to X$ rel $\{e\}$ with $B_0 = \mathrm{id}$ and $B_1 = \mu(e, -)$. We consider the maps

$$\begin{array}{rccc} A_f\colon [0,1] \times [0,1] & \to & X \\ (s,t) & \mapsto & A(\underbrace{f(s)}_{\in X}, t) \end{array} \quad \text{and} \quad \begin{array}{rccc} B_g\colon [0,1] \times [0,1] & \to & X \\ (s,t) & \mapsto & B(\underbrace{g(s)}_{\in X}, t). \end{array}$$

Note that A_f is a path-homotopy from the loop f to the loop $\mu(f, e)$ and B_g is a path-homotopy from the loop g to the loop $\mu(e, g)$. The path-homotopy between the loops $f * g$ and $g * f$ is given by combining H with A_f and B_g as indicated in the figure below. We leave it to the reader to conscientiously write down the precise homotopy. ■



The following theorem was proved by William Browder [**Browd1960**, Theorem 2] [**Browd1961**, Theorem 6.11] in 1961.

Theorem 156.24. Let X be a path-connected H-space. If the group $\bigoplus_{i=0}^{\infty} \mathrm{H}_i(X; \mathbb{Z})$ is finitely generated, then

$$\min\{m \in \mathbb{N}_{\geq 2} \,|\, \pi_m(X) \neq 0\}$$

is odd. In particular we have $\pi_2(X) = 0$.

Examples.

[1912] It follows easily from Pasting Proposition 2.6 (2′) that this map is continuous.

(1) On page 3334 we saw that $\mathbb{C}\mathrm{P}^\infty$ is an H-space. Furthermore in Corollary 85.6 (2) we saw that $\pi_2(\mathbb{C}\mathrm{P}^\infty) \neq 0$. Thus we see that in the statement of Theorem 156.24 we cannot drop the hypothesis that $\bigoplus_{i=0}^{\infty} \mathrm{H}_i(X;\mathbb{Z})$ is a finitely generated group.

(2) Let $k \in \mathbb{N}$. It follows from Theorem 156.24, together with Theorem 155.12 and the discussion on page 1772, that the finite complex projective space $\mathbb{C}\mathrm{P}^k$ does *not* admit an H-space structure.

We conclude this chapter with a corresponding theorem for Lie groups.

Theorem 156.25. If G is a path-connected Lie group, then

$$\min\{m \in \mathbb{N}_{\geq 2} \,|\, \pi_m(G) \neq 0\}$$

is odd. In particular $\pi_2(G) = 0$.

Remark. There are many direct proofs in the literature for the statement that given a path-connected Lie group G the second homotopy group $\pi_2(G)$ vanishes, see e.g. [**Carta1936**] or [**Miln1963a**, Theorem 21.7] or [**BtD1995**, Proposition V.7.5].

Note that Theorem 156.25 has no conditions on the Lie group except that it needs to be path-connected. The reason why we can get away with so few conditions is given by the following proposition.

Proposition 156.26. Let G be a Lie group. There exists a compact Lie subgroup K[1913] such that G is homeomorphic to $K \times \mathbb{R}^n$ for some $n \in \mathbb{N}_0$.

Proof of Proposition 156.26. Like many other foundational results on Lie groups this proposition has its origins in the work of Henri Cartan [**Carta1936**]. A full proof is given in [**Mal1945**, Theorem 11] or [**Mos1949**, Theorem 2]. A sketch of the proof is provided in [**Sam1952**, Chapter 7]. ■

Example. We consider the group $\mathrm{Isom}(\mathbb{R}^n)$ of isometries of the vector space $\mathbb{R}^n$ equipped with the usual inner product. It is an elementary, but not totally trivial fact, see [**Rees1983**, p. 11], that

$$\mathrm{Isom}(\mathbb{R}^n) \;=\; \{\text{all maps of the form } x \mapsto Ax + v \text{ where } A \in \mathrm{O}(n) \text{ and } v \in \mathbb{R}^n\}.$$

The maps

$$\begin{array}{rcl} \mathrm{O}(n) \times \mathbb{R}^n & \to & \mathrm{Isom}(\mathbb{R}^n) \\ (A, v) & \mapsto & (x \mapsto Ax + v) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathrm{Isom}(\mathbb{R}^n) & \to & \mathrm{O}(n) \times \mathbb{R}^n \\ f & \mapsto & (\underbrace{(x \mapsto (f(x) - f(0))}_{\text{represented by matrix in } \mathrm{O}(n)}, f(0)) \end{array}$$

are inverses of one another. Thus we can use the smooth manifold structure on $\mathrm{O}(n) \times \mathbb{R}^n$ to turn $\mathrm{Isom}(\mathbb{R}^n)$ into a smooth manifold and it is quasi-obvious to see that this turns $\mathrm{Isom}(\mathbb{R}^n)$ into a Lie group. We can view $\mathbb{R}^n$ and $\mathrm{O}(n)$ as subgroups of $\mathrm{Isom}(\mathbb{R}^n)$ in an obvious way. But one easily verifies that the map $\mathrm{O}(n) \times \mathbb{R}^n \to \mathrm{Isom}(\mathbb{R}^n)$ is not an isomorphism. In other words, as predicted by Proposition 156.26 the Lie group $\mathrm{Isom}(\mathbb{R}^n)$ splits topologically into a product. But the Lie group does not split as a Lie group.

[1913] In fact a Lie group contains a maximal compact Lie subgroup which is furthermore unique up to conjugation. We will not make use of this fact.

Proof of Theorem 156.25. Let G be a path-connected Lie group. By Proposition 156.26 we know that G is homotopy equivalent to a compact path-connected Lie group K. By the Homotopy-π_n-Proposition 71.7 (2) it suffices to prove the desired statement for K. Since K is in particular a compact smooth manifold we obtain from the Smooth Manifolds-Invariants Proposition 96.6 that $\bigoplus_{i=0}^{\infty} \mathrm{H}_i(X;\mathbb{Z})$ is a finitely generated abelian group. Thus we can conclude the proof by appealing to Theorem 156.24. ■

156.6. **The Eckmann-Hilton Theorem.** In this section we will state and prove a theorem that was provided by Bruno Eckmann and Peter Hilton [**EcH1962**] in 1962. The theorem is totally elementary, but at the same time it is surprisingly useful. In particular it will be the key to giving the second proof of Lemma 156.23.
The formulation of the Eckmann-Hilton Theorem requires the following harmless definition.

Definition. Let X be a set.
(1) A binary operation is nothing but a map $X \times X \to X$.
(2) A binary operation $\mu\colon X \times X \to X$ is called unital if there exists an element $1 \in X$ such that $\mu(1,x) = \mu(x,1) = x$ for all $x \in X$.

Example. The multiplication map on a group is evidently a unital binary operation.

The following theorem is in some sense quite elementary, but the cleverness lies in its formulation.

Theorem 156.27. (Eckmann-Hilton Theorem) Let X be a set equipped with two binary operations "$\circ$" and "$\otimes$". Suppose the following two conditions are satisfied:
(a) "$\circ$" and "$\otimes$" are both unital.
(b) For every $a,b,c,d \in X$ we have the equality

$$(a \otimes b) \circ (c \otimes d) \;=\; (a \circ c) \otimes (b \circ d).$$

Then "$\circ$" and "$\otimes$" are the same and they are commutative and associative.

Proof. First we show that the units of the two operations "$\circ$" and "$\otimes$" coincide:

$$1_\circ \;=\; 1_\circ \circ 1_\circ \;=\; (1_\otimes \otimes 1_\circ) \circ (1_\circ \otimes 1_\otimes) \underset{\text{by (b)}}{=} (1_\otimes \circ 1_\circ) \otimes (1_\circ \circ 1_\otimes) \;=\; 1_\otimes \otimes 1_\otimes \;=\; 1_\otimes.$$

Next we show that the two operations "$\circ$" and "$\otimes$" are the same and that they are commutative. Thus let $x,y \in X$. Then

$$x \circ y = (1 \otimes x) \circ (y \otimes 1) \underset{\text{by (b)}}{=} (1 \circ y) \otimes (x \circ 1) = y \otimes x = (y \circ 1) \otimes (1 \circ x) \underset{\text{again by (b)}}{=} (y \otimes 1) \circ (1 \otimes x) = y \circ x.$$

Finally we show that the operations are associative. Thus let $x,y,z \in X$. We have the following equality:

$$(x \otimes y) \otimes z \;=\; (x \otimes y) \otimes (1 \otimes z) \underset{\text{one last time by (b)}}{=} (x \otimes 1) \otimes (y \otimes z) \;=\; x \otimes (y \otimes z).$$

■

Now we can prove the following lemma, which in particular gives the promised second, more structural proof of Lemma 156.23.

Lemma 156.28. Let X be a path-connected H-space and let $x_0 \in X$. In the following we denote by $\mu\colon X \times X \to X$ the multiplication map of the H-space. We consider the following two binary operations on $\pi_1(X, x_0)$:

$$\begin{array}{rcl} \pi_1(X,x_0)\times\pi_1(X,x_0) & \xrightarrow{\circ} & \pi_1(X,x_0) \\ ([f],[g]) & \mapsto & [f * g] \end{array} \quad \text{and} \quad \begin{array}{rcl} \pi_1(X,x_0)\times\pi_1(X,x_0) & \xrightarrow{\otimes} & \pi_1(X,x_0) \\ ([f],[g]) & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & X \\ t & \mapsto & \mu(f(t),g(t)) \end{array}\right] \end{array}.$$

The following two statements hold:

(1) The two binary operations agree.
(2) The group $\pi_1(X, x_0)$ is abelian.

Proof. Let X be a path-connected H-space and let $x_0 \in X$. We denote by $\mu\colon X \times X \to X$ the multiplication map of the H-space. Since μ is an H-space structure one sees easily that "$\otimes$" is a unital binary operation. Furthermore we have known ever since Proposition 47.6 that "$\circ$" is a unital binary operation. In Exercise 156.9 we will show that Condition (b) of the Eckmann-Hilton Theorem 156.27 is satisfied. Thus we obtain from the Eckmann-Hilton Theorem 156.27 the amusing fact that the two operations agree and we obtain that both operations are commutative. In particular this means that $\pi_1(X, x_0)$ is abelian. ■

Exercises for Chapter 156.

Exercise 156.1. Let X be the torus minus one open disk. Does X admit the structure of a topological group?

Exercise 156.2. Let M be a smooth manifold without boundary and let $g\colon M \times M \to M$ be a map that is smooth and that defines a group structure on M. Show that (M, g) is a Lie group. In other words, show that the map $M \to M$ given by $x \mapsto x^{-1}$ is smooth.
Hint. Use the Implicit Function Theorem, see e.g. [**Lee2002**, p. 661].

Exercise 156.3. Let $n \in \mathbb{N}$. We define

$$\mathrm{GL}(n,\mathbb{H}) \;=\; \left\{ A \in \mathrm{M}(n\times n,\mathbb{H}) \;\middle|\; \begin{array}{c} \text{there exist matrices } X, Y \in \mathrm{M}(n\times n,\mathbb{H}) \\ \text{such that } AX = YA = \mathrm{id} \end{array} \right\}.$$

Is $\mathrm{GL}(n, \mathbb{H})$ an open subset of $\mathrm{M}(n \times n, \mathbb{H})$?

Exercise 156.4. Let $n \in \mathbb{N}$. In the proof of Lemma 156.3 we already saw that $\mathrm{Sp}(n)$ is a smooth manifold and that matrix multiplication turns $\mathrm{Sp}(n)$ into a group. Show that the map

$$\begin{array}{rcl} \mathrm{Sp}(n) & \to & \mathrm{Sp}(n) \\ A & \mapsto & A^{-1} \end{array}$$

is a smooth map.
Note: The argument in the proof of Lemma 5.38 can be used to show that taking inverses in $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ is smooth. Does this help us in any way?
Remark. In principle one could use Exercise 156.2. But what do readers do who are still traumatized by the Implicit Function Theorem?

Exercise 156.5. Use the Ehresmann Fibration Theorem that we had stated in Exercise **??** to give a third proof that the map $p\colon \mathrm{O}(n+1) \to S^n$ from the Matrix Group-Fiber Proposition 156.8 is a fiber bundle.

Exercise 156.6. Determine all closed non-orientable 2-dimensional smooth manifolds that admit a Lie group structure.

Exercise 156.7. We define $\mathrm{SU} = \varinjlim \mathrm{SU}(n)$ the same way as we defined U. Determine the homotopy groups of SU.
Remark. You might want to use Exercise 155.7.

Exercise 156.8.

(a) Let X be a CW-complex and let $e \in X^0$ be a point in the 0-skeleton. Suppose we are given a map $\mu\colon X \times X \to X$ such that the two maps

$$\begin{array}{ccc} X & \to & X \\ x & \mapsto & \mu(x,e) \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \to & X \\ x & \mapsto & \mu(e,x) \end{array}$$

are homotopic to the identity. Show that μ is homotopic to a map $\nu\colon X \times X \to X$ such that $\nu(x,e) = \nu(e,x) = x$ for all $x \in X$.

(b) Show that ν in (a) defines an H-space structure on X.

(c) Let X be a CW-complex. Suppose there exists a homotopy equivalence $f\colon X \to Y$ to an H-space Y. Show that X itself admits an H-space structure.

Exercise 156.9. Let (X, x_0) be a pointed path-connected H-space. Let $\mu\colon X \times X \to X$ be the multiplication of the H-space structure. We consider the following two binary operations on $\pi_1(X, x_0)$:

$$\begin{array}{rcl} \pi_1(X,x_0)\times\pi_1(X,x_0) & \xrightarrow{\circ} & \pi_1(X,x_0) \\ ([f],[g]) & \mapsto & [f * g] \end{array} \quad \text{and} \quad \begin{array}{rcl} \pi_1(X,x_0)\times\pi_1(X,x_0) & \xrightarrow{\otimes} & \pi_1(X,x_0) \\ ([f],[g]) & \mapsto & \begin{pmatrix} [0,1] \to X \\ t \mapsto \mu(f(t),g(t)) \end{pmatrix}. \end{array}$$

Show that "$\circ$" and "$\otimes$" satisfy Condition (b) of the Eckmann-Hilton Theorem 156.27.

Exercise 156.10. Let (X, x_0) be a pointed topological space and let $k \in \mathbb{N}_{\geq 2}$. Define the multiplication $*\colon \pi_k(X,x_0) \times \pi_k(X,x_0) \to \pi_k(X,x_0)$ as on page 1561. Provide a different multiplication $\otimes\colon \pi_k(X,x_0) \times \pi_k(X,x_0) \to \pi_k(X,x_0)$ such that "$*$" and "$\otimes$" satisfy the conditions of the Eckmann-Hilton Theorem 156.27. Conclude, once again, that $\pi_k(X,x_0)$ is abelian.

Exercise 156.11. Let $n \in \mathbb{N}_{\geq 3}$. In Proposition 156.11 we saw that $\pi_1(\mathrm{SO}(n), \mathrm{id}) \cong \mathbb{Z}_2$. Give an example of a loop that is a generator of $\pi_1(\mathrm{SO}(n), \mathrm{id})$ and show explicitly why it has order 2 in the fundamental group.

Exercise 156.12. Let $n \in \mathbb{N}$. Use the approach of the proof of Proposition 156.10 to show that $\mathrm{SO}(n)$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are path-connected.
Remark. This exercise provides a new and perhaps more satisfactory proof to some of the statements of the Matrix Group Path-Component Proposition 2.37.

157. Thom-Pontryagin Theory I: The basics

In this chapter we will introduce the Thom-Pontryagin approach to calculating homotopy groups of spheres. Furthermore we will see that it allows us to reprove several earlier results. In Chapters 158 and 159 we will use the Thom-Pontryagin approach to obtain new results.

Before we start with discussing the Thom-Pontryagin theory we fix some notations and conventions.

Notation. Let $n \in \mathbb{N}_0$.

(1) We denote by $*$ the point in $\overline{B}^n/S^{n-1}$ that is given by $[S^{n-1}]$.
(2) We denote by $*$ the "North Pole" $(0,\ldots,0,1) \in S^n$.

We make the following identifications:

(a) We use the homeomorphism $S^n \to \mathbb{R}^n \cup \{\infty\}$ from Lemma 2.18, which is given by stereographic projection, that sends the North Pole $*$ to ∞ to make the identification $S^n = \mathbb{R}^n \cup \{\infty\}$.

(b) In the Ball-Quotient Sphere Lemma 5.20 we wrote down an explicit homeomorphism $\overline{B}^n/S^{n-1} \to S^n$. This homeomorphism sends $*$ to $(0,\ldots,0,1)$. $(x_1,\ldots,x_{n+1}) \mapsto (x_{n+1},x_2,\ldots,x_n,-x_1)$. This way we obtain a homeomorphism $\overline{B}^n/S^{n-1} \to S^n$ that sends $*$ to $(0,\ldots,0,1)$.

(c) Putting the two homeomorphisms from (a) and (b) together we obtain a homeomorphism

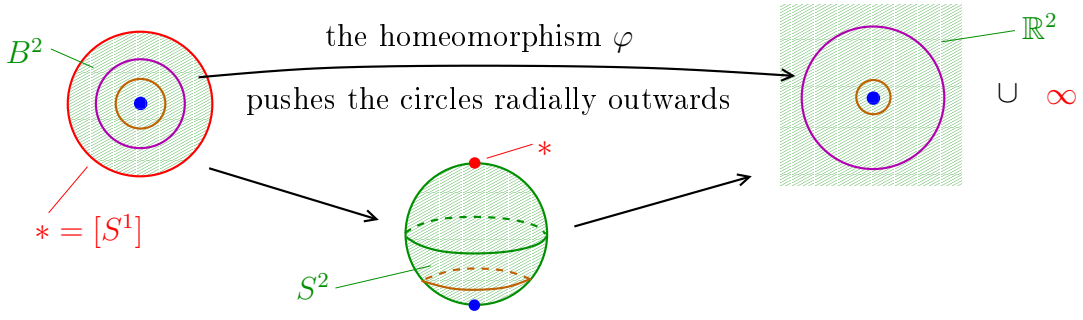
$$\varphi\colon \overline{B}^n/S^{n-1} \;\to\; \mathbb{R}^n \cup \{\infty\}.$$

It follows fairly easily from the definitions of the two maps in (a) and (b) that φ restricts to a diffeomorphism $B^n \to \mathbb{R}^n$ and that φ is of the form

$$x \;\mapsto\; \begin{cases} \eta(\|x\|)\cdot x, & \text{if } x \in B^n, \\ \infty, & \text{if } x = *, \end{cases}$$

where $\eta\colon [0,1) \to [0,\infty)$ is a strictly monotonously increasing function with $\eta(0)=0$ and $\lim\limits_{t\to 1}\eta(t) = \infty$.

Summarizing we have the identifications $S^n = \mathbb{R}^n \cup \{\infty\} = \overline{B}^n/S^{n-1}$ and under these identifications the points $* \in S^n$, ∞ and $* \in \overline{B}^n/S^{n-1}$ get identified. We refer to the figure below for an illustration.



157.1. **The generalized Thom-Pontryagin construction.** On page 2839 we saw that a proper oriented codimension-one submanifold of a compact oriented smooth manifold M

gives rise, via the Thom-Pontryagin construction, to a map $M \to S^1$. In this section we will generalize this construction.

The key objects in the more general construction of Thom-Pontryagin maps are "thickened" submanifolds:

Definition. Let M be an orientable n-dimensional smooth manifold.

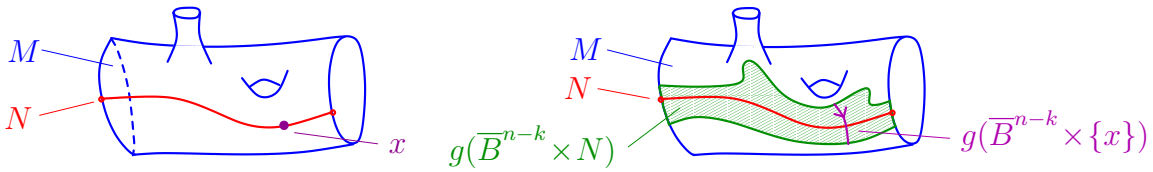
(1) Let N be a compact orientable proper k-dimensional submanifold of M. A thickening of N is an injective map $g\colon \overline{B}^{n-k} \times N \to M$ with the following properties:[1914][1915]
   (a) for all $P \in N$ we have $g(0, P) = P$,
   (b) the image $g(\overline{B}^{n-k} \times N)$ is a submanifold with corner of M,
   (c) we have $g(\overline{B}^{n-k} \times N) \cap \partial M = g(\overline{B}^{n-k} \times \partial N)$.
   We refer to the figure below for an illustration.

(2) A thickened submanifold of M is a pair (N, g) where N is a compact orientable proper[1917] submanifold of M and g is a thickening for N.

(3) Let $\varphi\colon M \to W$ be a smooth embedding of M into a smooth manifold W of the same dimension. Given a thickened k-dimensional submanifold $(N, g\colon \overline{B}^{n-k} \times N)$ of $M \setminus \partial M$ we equip $\varphi(N)$ with the "obvious thickening", namely with the thickening that is given by

$$\begin{array}{rcl} \varphi_*(g)\colon \overline{B}^{n-k} \times \varphi(N) & \to & W \\ (v, P) & \mapsto & g(v, \varphi^{-1}(P)). \end{array}$$

   Evidently $(\varphi(N), \varphi_*(g))$ is a thickened submanifold of W.



The following definition can be viewed as a generalization of the Thom-Pontryagin map introduced on page 2839.

Definition. Let M be an orientable n-dimensional smooth manifold. Given a thickened k-dimensional submanifold $(N, g\colon \overline{B}^{n-k} \times N \to M)$ we refer to the map[1918]

$$\begin{array}{rcl} \rho_{(N,g)}\colon M & \to & S^{n-k} = \overline{B}^{n-k}/S^{n-k-1} \\ P & \mapsto & \begin{cases} [x], & \text{if } P = g(x, y) \text{ with } (x, y) \in \overline{B}^{n-k} \times N, \\ *, & \text{otherwise} \end{cases} \end{array}$$

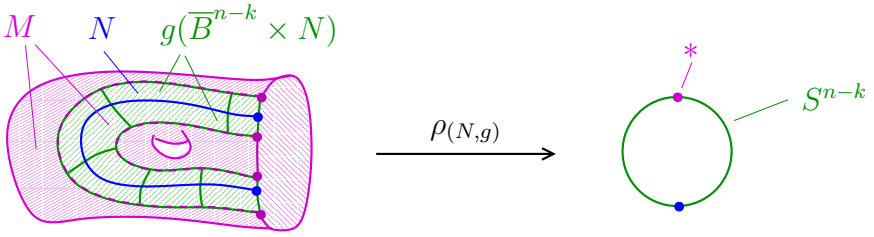
[1914] Here one really would like to say that $g\colon \overline{B}^{n-k} \times N \to M$ is an "smooth embedding", but if N has boundary, then it is not entirely clear in what sense $\overline{B}^{n-k} \times N$ is a smooth manifold and what "smooth embedding" is supposed to mean.

[1915] The definition looks very similar to the definition of a tubular neighborhood. In fact a thickening is the same data as a trivial tubular neighborhood, it is just phrased in slightly different words.

[1916] On page 2853 we introduced the thickening of an *oriented* knot. For knots the present definition is the same as before, except that we no longer demand that the knot is oriented.

[1917] Note that here, in the definition of a thickened submanifold, we sneaked in the adjectives "compact", "orientable" and "proper", so that we will not have to carry them around all the time.

as the Thom-Pontryagin map. (We refer to the figure below for an illustration.) Sometimes we refer to this approach of assigning to a thickened k-dimensional submanifold a map $M \to S^{n-k}$ as the Thom-Pontryagin construction.



Remark.

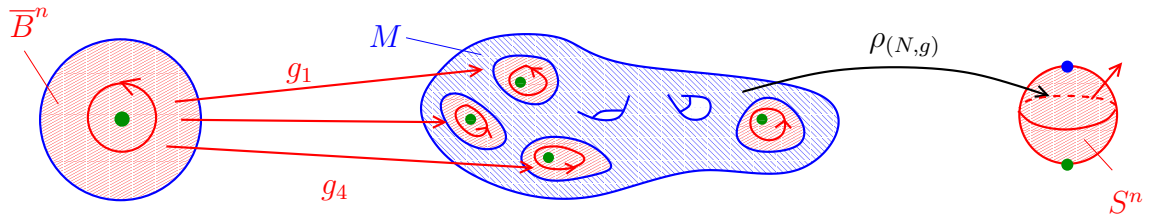
(1) Let N be a compact *oriented* proper codimension-one submanifold of an *oriented* smooth manifold M. It follows immediately from Proposition **??** that N admits a thickening $[-1,1] \times N$ such that the inclusion $(-1,1) \times N \to M$ is orientation-preserving.[1919] The map $\rho_N \colon M \to S^1$ that we defined on page 2839 is basically the same as the above map $\rho_{(N,g)} \colon M \to S^1$. The only difference is that the two maps differ by multiplication by $i \in S^1$.

(2) We saw in Proposition 126.14 that the submanifold $\mathbb{C}\mathrm{P}^1$ of $\mathbb{C}\mathrm{P}^2$ does *not* admit a trivial tubular neighborhood, in particular it does not admit a thickening.

Example. Let $n \in \mathbb{N}$ and let M be a closed orientable n-dimensional smooth manifold together with a 0-dimensional submanifold $N = \{P_1, \dots, P_m\}$. Suppose N is equipped with a thickening

$$g = g_1 \sqcup \cdots \sqcup g_m \colon (\overline{B}^n \times \{P_1\}) \sqcup \cdots \sqcup (\overline{B}^n \times \{P_m\}) \;\to\; M.$$

From the Thom-Pontryagin construction we obtain a map $\rho_{(N,g)} \colon M \to S^n$. It might not be clear what this is good for, but it is kind of fun. This example is illustrated in the figure below.



The following lemma gives us a first interesting example of a Thom-Pontryagin map.

Lemma 157.1. Let $n \in \mathbb{N}$ and let $P \in B^n$. We equip the 0-dimensional smooth manifold $\{P\}$ with any thickening $g \colon \overline{B}^n \times \{P\} \to B^n$ that is orientation-preserving[1920]. The corresponding map

$$\rho_{(\{P\},g)} \colon \overline{B}^n / S^{n-1} \;\to\; \overline{B}^n / S^{n-1}$$

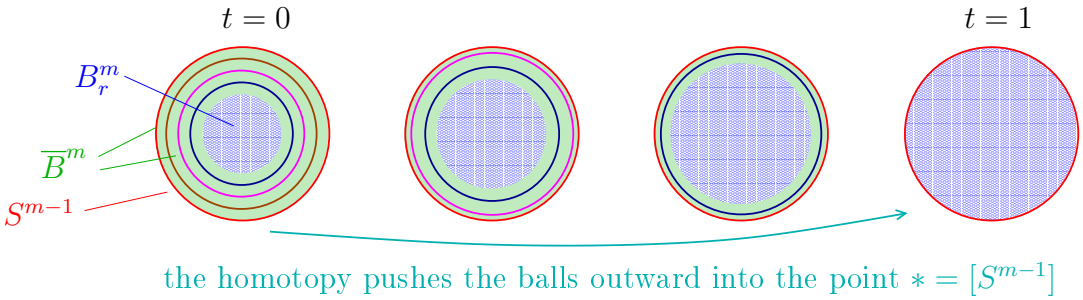
is homotopic rel $*$ to the identity map of $\overline{B}^n / S^{n-1}$.

[1918] It follows easily from Lemma 120.1 and Pasting Proposition 2.6 (2) that this map is indeed continuous.
[1919] Furthermore we saw in Proposition **??** that this thickening is essentially unique.

The proof of Lemma 157.1 relies on the following lemma which we will use on several occasions.

Lemma 157.2. Let $m \in \mathbb{N}$. Given any real number $r \in (0,1)$ there exists a homotopy $F\colon \overline{B}^m/S^{m-1} \times [0,1] \to \overline{B}^m/S^{m-1}$ rel $*$ with the following three properties:

(1) $F_0 = \mathrm{id}$,
(2) the restriction of F_1 to B^m_r is given by multiplication by $\frac{1}{r}$,
(3) for any point $x \notin B^m_r$ we have $F_1(x) = *$.



Proof. Let $r \in (0,1)$. We consider the map

$$\begin{array}{rcl} (\overline{B}^m/S^{m-1}) \times [0,1] & \to & \overline{B}^m/S^{m-1} \\ ([x],t) & \mapsto & \begin{cases} [x \cdot (1-t+t\cdot\frac{1}{r})], & \text{if } x \in B^m \text{ and } t < \frac{1-\|x\|}{(\frac{1}{r}-1)\cdot\|x\|} \\ *, & \text{otherwise.} \end{cases} \end{array}$$

We leave as to the engaged reader to verify that this map is indeed continuous. It is clear that F has the desired properties. ∎

Now we can easily prove Lemma 157.1.

Sketch of proof of Lemma 157.1. It follows quite easily[1921] from the Smooth Ball Embedding Theorem 37.8 that we can assume without loss of generality that $P = 0$ and that we can assume that g is the thickening $g\colon \overline{B}^n \times \{0\} \to B^n$ given by $g(v,0) = \frac{1}{2}v$. By Lemma 157.2 there exists a homotopy $F\colon (\overline{B}^n/S^{n-1}) \times [0,1] \to \overline{B}^n/S^{n-1}$ rel $*$ with the following three properties:

(1) $F_0 = \mathrm{id}$,
(2) the restriction of F_1 to $B^n_{\frac{1}{2}}$ is given by the map $x \mapsto 2 \cdot x$,
(3) for any point $x \notin B^n_{\frac{1}{2}}$ we have $F_1(x) = *$.

If one thinks about it for a second, then one realizes that F_1 equals $\rho_{(0,g)}$ "on the nose". Thus we have found the desired homotopy. ∎

157.2. **The Thom-Pontryagin construction and homotopy groups.** The following is the key observation which justifies our renewed interest in the Thom-Pontryagin construction.

[1920] We equip $\overline{B}^n$ and B^n with the usual orientation.
[1921] The slightly hesitant "quite easily" refers to the fact that, if one thinks about it, we need that the diffeotopy of $\mathbb{R}^{m+k}$ given by the Smooth Ball Embedding Theorem 37.8 has compact support.

Observation. Let $n, k \in \mathbb{N}_0$ and let $(N, g\colon \overline{B}^n \times N \to \mathbb{R}^{n+k})$ be a thickened k-dimensional submanifold of $\mathbb{R}^{n+k}$. We can view $(N, g\colon \overline{B}^n \times N \to \mathbb{R}^{n+k})$ also as a thickened submanifold of $\mathbb{R}^{n+k} \cup \{\infty\}$.[1922] Note that the corresponding Thom-Pontryagin map

$$\begin{array}{rcl} \rho_{(N,g)}\colon S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} & \to & \overline{B}^n / S^{n-1} = S^n \\ P & \mapsto & \begin{cases} [x], & \text{if } P = g(x,y) \text{ with } (x,y) \in \overline{B}^n \times N, \\ *, & \text{otherwise,} \end{cases} \end{array}$$

sends ∞ to $*$. In particular $\rho_{(N,g)}$ defines an element in $\pi_{n+k}(S^n, *)$. It is worth repeating this observation:

thickened k-dimensional submanifolds of $\mathbb{R}^{n+k}$ give rise to elements in $\pi_{n+k}(S^n, *)$.

The following questions immediately arise:

Question 157.3. Let $n, k \in \mathbb{N}_0$.

(1) Which elements in $\pi_{n+k}(S^n, *)$ can be realized by Thom-Pontryagin maps?
(2) When do two thickened k-dimensional submanifolds of $\mathbb{R}^{n+k}$ give rise to the same element in $\pi_{n+K}(S^n, *)$.

Before we address this question we intend to discuss two examples in greater detail:

(1) the case of thickened 0-dimensional submanifolds of some $\mathbb{R}^n$, and
(2) the case of thickened 1-dimensional submanifolds of $\mathbb{R}^3$.

We start out with the first case. Dealing with this case requires one definition.

Definition. Let $P \in \mathbb{R}^n$ be a point and let $g\colon \{P\} \times \overline{B}^n \to \mathbb{R}^n$ be a thickening of P. We equip $\{P\} \times \overline{B}^n = \overline{B}^n$ and $\mathbb{R}^n$ with the standard orientations. We define

$$\operatorname{sign}(g) := \begin{cases} +1, & \text{if } g \text{ is orientation-preserving,} \\ -1, & \text{if } g \text{ is orientation-reversing.} \end{cases}$$

Now we can discuss the case of thickened 0-dimensional submanifolds.

Example. Let $N = \{P_1, \dots, P_m\}$ be a 0-dimensional submanifold of some $\mathbb{R}^n$ and suppose N is equipped with a thickening

$$g = g_1 \sqcup \dots \sqcup g_m\colon (\overline{B}^n \times P_1) \sqcup \dots \sqcup (\overline{B}^n \times P_m) \to \mathbb{R}^n.$$

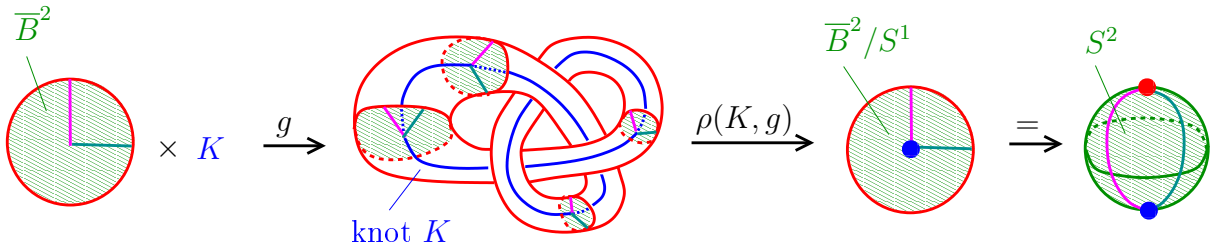
It follows almost immediately from the Addition-in-π_n-Proposition 85.11 that

$$[\rho_{(N,g)}] = \sum_{i=1}^{m} \operatorname{sign}(g_i) \cdot [\mathrm{id}_{S^n}] \in \pi_n(S^n, *).$$

We continue our discussion of examples of the Thom-Pontryagin construction with knots and links. By the above discussion every thickened knot or link in $\mathbb{R}^3$, i.e. every thickened 1-dimensional submanifold of $\mathbb{R}^3$, gives rise to an element in $\pi_3(S^2, *)$. For the trefoil together with some thickening we illustrate the corresponding map $S^3 \to S^2$ in the figure below.

Before we continue we need to introduce the following slight variation on a definition from page 2853.

[1922] Why can we do so?



Definition. Let K be a knot and let $g\colon \overline{B}^2 \times K \to S^3$ be a thickening. We pick an orientation for K and we equip $g(\{1\}\times K)$ with the corresponding definition. We define[1923]

$$\text{self-linking number of } g \;:=\; \mathrm{lk}(K, g(\{1\}\times K)).$$

It follows from Proposition 131.11 (2) that this definition does not depend on the choice of the orientation of K.

The following lemma deals with two basic, but important examples. Later on both examples will follow (at least up to sign) from a more general result, but for the geometrically inclined person it can be fun to consider the more direct arguments provided in the proof of the lemma.

Lemma 157.4. Let K be the trivial knot and let $g\colon \overline{B}^2 \times K \to \mathbb{R}^3$ be a thickening.

(1) If g is a thickening with self-linking number zero, then $[\rho_{(K,g)}] = 0 \in \pi_3(S^2, *)$.
(2) If g is a thickening with self-linking number one, then[1924] $[\rho_{(K,g)}] = [H] \in \pi_3(S^2, *)$ where $H\colon (S^3, *) \to (S^2, *)$ denotes the Hopf map, that we defined on page 1576.

Sketch of proof. It follows easily from the Knot Thickening Lemma 131.1[1925] that any two thickenings with the same self-linking number give rise to the same element in $\pi_3(S^2, *)$. Furthermore it follows almost immediately from Lemma 59.1 that we can make the identification

$$S^3 \;=\; (S^1 \times \overline{B}^2) \cup_{S^1\times S^1} (\overline{B}^2 \times S^1)$$

such that $K = \{0\}\times S^1$. Furthermore it follows from Lemma 59.2 that the obvious inclusion of $\overline{B}^2 \times S^1$ into the above decomposition of S^3 is a thickening of the trivial knot $K = S^1$ with self-linking number zero.

(1) Let g be the thickening of the trivial knot K with self-linking number zero. We want to show that $[\rho_{(K,g)}] = 0 \in \pi_3(S^2, *)$. By the Change-of-Base Point Proposition 86.9 it suffices to show that the map $\rho_{(K,g)}\colon S^3 \to S^2$ is homotopic to a constant map. (More precisely, the Change-of-Base Point Proposition 86.9 (7) says that in this context we do not need to worry about base points since S^2 is path-connected.) Using the above preparations we will prove in Exercise 157.1 that the map $\rho_{(K,g)}\colon S^3 \to S^2$ is indeed homotopic to a constant map.

[1923] We refer to page 2865 for the definition of the linking number $\mathrm{lk}(K, J)$ of two disjoint oriented knots K and J.
[1924] The sign should be correct. Hopefully. Perhaps.
[1925] Note that here we need that in the statement of the Knot Thickening Lemma 131.1 for $\mathbb{R}^3$ we stated that there exists a diffeotopy of $\mathbb{R}^3$ with compact support.

(2) First we note that under the above identification of S^3 the Hopf map is given as follows:[1926]

$$\begin{array}{rcl} H\colon \overbrace{(S^1\times\overline{B}^2)\cup_{S^1\times S^1}(\overline{B}^2\times S^1)}^{=S^3} & \to & \mathbb{C}\mathrm{P}^1 \\ [(z,w)] & \mapsto & \begin{cases} [\frac{w}{z}:1], & \text{if } z\in S^1 \text{ and } w\in\overline{B}^2, \\ [1:\frac{z}{w}], & \text{if } z\in\overline{B}^2 \text{ and } w\in S^1. \end{cases} \end{array}$$

Now note that it follows from the Knot Thickening Lemma 131.1 (1) that the map

$$\begin{array}{rcl} g\colon \overline{B}^2\times S^1 & \to & \overline{B}^2\times S^1 \subset S^3 = (S^1\times\overline{B}^2)\cup_{S^1\times S^1}(\overline{B}^2\times S^1) \\ (z,w) & \mapsto & (z\cdot w,w) \end{array}$$

is a thickening of the trivial knot of self-linking number one. Furthermore by definition the corresponding map $\rho_{(K,g)}$ is easily seen to be the map[1927]

$$\begin{array}{rcl} \rho_{(K,g)}\colon (S^1\times\overline{B}^2)\cup_{S^1\times S^1}(\overline{B}^2\times S^1) & \to & \overline{B}^2/S^1 \\ [(z,w)] & \mapsto & \begin{cases} \infty=[S^1], & \text{if } z\in S^1 \text{ and } w\in\overline{B}^2, \\ \left[\frac{z}{w}\right], & \text{if } z\in\overline{B}^2 \text{ and } w\in S^1.. \end{cases} \end{array}$$

Comparing the definitions of the Hopf map H and the map $\rho_{(K,g)}$ on $\overline{B}^2\times S^1$ we see that we are perhaps not that far off.

Next we need to relate the targets $\mathbb{C}\mathrm{P}^1$ and $\overline{B}^2/S^1$. To do so we consider the map $\Theta\colon \mathbb{C}\mathrm{P}^1\to\overline{B}^2/S^1$ that is given by $\Theta([1:z])=\frac{z}{1+|z|}$ for $z\in\mathbb{C}$ and that is given by $\Theta([0:1])=[S^1]=*\in\overline{B}^2/S^1$. It follows easily from the Compact-Hausdorff Proposition 2.17 (3) that Θ is a homeomorphism. The particularly audacious reader can verify that this map is homotopic to the "official" identification $\mathbb{C}\mathrm{P}^1=\overline{B}^2/S^1$ which is given by the identification $\mathbb{C}\mathrm{P}^1=S^2$ from page 258 and the above identification $S^2=\overline{B}^2/S^1$. Note that the image of the first solid torus $\overline{B}^2\times S^1$ under the map $\Theta\circ H$ equals the disk of radius $\frac{1}{2}$, i.e. it equals $\overline{B}^2_{\frac{1}{2}}\subset\overline{B}^2/S^1$.

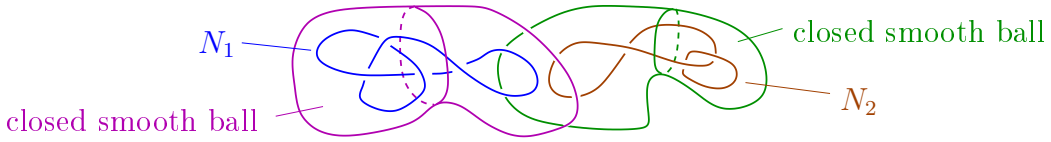
We apply Lemma 157.2 to obtain a homotopy F of $\overline{B}^2/S^1$ with the following properties:
(a) $F_0=\mathrm{id}$,
(b) the restriction of F_1 to $B^2_{\frac{1}{2}}$ is given by multiplication by 2,
(c) for any point $x\notin B^2_{\frac{1}{2}}$ we have $F_1(x)=*$.
The map $F_0\circ\Theta\circ H=\Theta\circ H$ is homotopic rel $*$ to $F_1\circ\Theta\circ H$. It is now straightforward to show that $F_1\circ\Theta\circ H$ is homotopic rel $*$ to $\rho_{(K,g)}$. We leave it to the reader to fill in the details. ■

We conclude this list of basic examples of the Thom-Pontryagin construction with the following lemma.

[1926]As a reality check one can verify that the map is well-defined, i.e. for $[(z,w)]$ with $(z,w)\in S^1\times S^1$ we get consistent answers.
[1927]Why does it say $[\frac{z}{w}]$ instead of $[z\cdot w]$, which might look more natural?

Lemma 157.5. Let $n, k \in \mathbb{N}_0$ with $n + k \geq 2$. Let (N_i, g_i), $i = 1, \ldots, m$ be thickened k-dimensional submanifolds of $\mathbb{R}^{n+k}$. If the images $g_i(\overline{B}^n \times N_i)$ are contained in m disjoint closed smooth $(n+k)$-dimensional balls[1928], then

$$\rho_{(N_1 \sqcup \cdots \sqcup N_m, g_1 \sqcup \cdots \sqcup g_m)} \;=\; \rho_{(N_1,g_1)} + \cdots + \rho_{(N_m,g_m)} \;\in\; \pi_{n+k}(S^n, *).$$



Proof. This lemma follows easily from the Addition-in-π_n-Proposition 85.11. We leave it to the reader to fill in the details. ■

The following proposition gives an affirmative answer to Question 157.3.

Proposition 157.6. Let $k, n \in \mathbb{N}_0$. Given any $\varphi \in \pi_{n+k}(S^n, *)$ there exists a thickened k-dimensional submanifold (N, g) of $\mathbb{R}^{n+k}$ with $[\rho_{(N,g)}] = \varphi$.

Proof. Throughout this proof we find it convenient to view each m-dimensional sphere as the quotient $\overline{B}^m/S^{m-1}$. So suppose we are given a map $\varphi\colon \overline{B}^{n+k}/S^{n+k-1} \to \overline{B}^n/S^{n-1}$ with $\varphi(*) = *$. As we pointed out in the proof of Proposition 71.10, it is a consequence of the Whitney Approximation Theorem 29.1 and the Replace-by-Smooth Homotopy Proposition 29.5 that we can arrange that φ is in fact a smooth map. It follows immediately from Sard's Theorem 31.1 and the Full Measure-Density Proposition 30.3 (2) that there exists a regular value $z \in \overline{B}^n/S^{n-1}$ such that $\varphi^{-1}(z)$ does not contain the point $*$. By Proposition 36.4 (2)[1929] we can without loss of generality assume that the origin 0 is a regular value. By the Regular Value Theorem **??** we know that the preimage $N := \varphi^{-1}(0)$ is a closed orientable k-dimensional submanifold of B^{n+k}. Furthermore by Statement (4) of the Regular Value Theorem **??** we know that there exists an $\epsilon > 0$ and a thickening $g\colon \overline{B}^n \times N \to B^{n+k}$ such that $\varphi^{-1}(\overline{B}^n_\epsilon) = g(\overline{B}^n \times N)$ and such that the following diagram commutes[19301931]

$$\begin{array}{ccc} \overline{B}^n \times N & \xrightarrow{\quad g \quad} & B^{n+k} \\ \downarrow{\scriptstyle (p,x)\mapsto \epsilon\cdot p} & & \downarrow{\scriptstyle \varphi} \\ \overline{B}^n_\epsilon & \hookrightarrow & \overline{B}^n/S^{n-1}. \end{array}$$

This thickening g allows us to consider the map $\rho_{(N,g)}\colon \overline{B}^{n+k}/S^{n+k-1} \to \overline{B}^n/S^{n-1}$. Note that in general φ and $\rho_{(N,g)}$ are quite different maps. (We refer to the figure below for an illustration of the two maps φ and $\rho_{(N,g)}$.) But that does not bother us too much, since it suffices to prove the following claim.

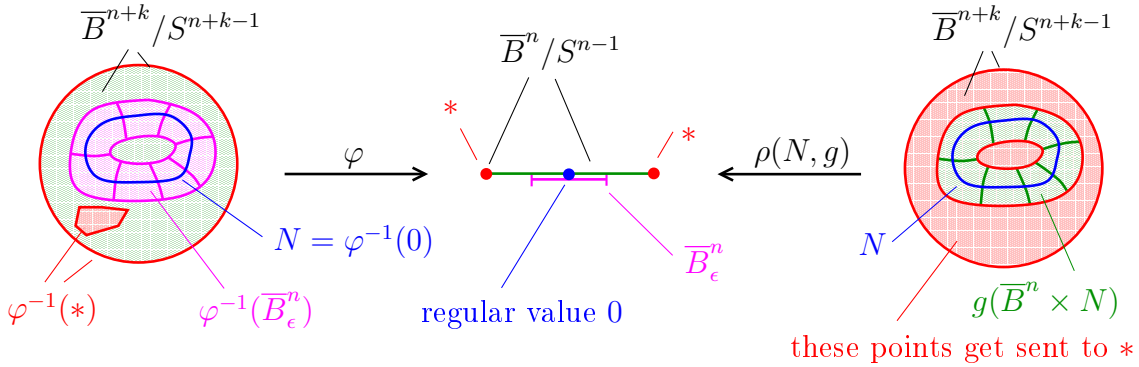
[1928] Here a "closed smooth $(n+k)$-dimensional ball" is shorthand for the image of a smooth embedding of $\overline{B}^{n+k}$ in $\mathbb{R}^{n+k}$.

[1929] To be more precise, we apply Proposition 36.4 to the smooth manifold $\overline{B}^n/S^{n-1} = \mathbb{R}^n \cup \{\infty\}$ with $P_1 = z$, $P_2 = \infty$ and $Q_1 = 0$ and $Q_2 = \infty$.

[1930] Note the discretely hidden multiplication by ϵ in the left-vertical map.

[1931] Note that the above information implies in particular that $\varphi^{-1}(B^n_\epsilon) = g(B^n \times N)$.

Claim. The maps φ and $\rho_{(N,g)}$ from $\overline{B}^{n+k}/S^{n+k-1}$ to $\overline{B}^{n}/S^{n-1}$ are homotopic rel $*$.



Proof. First we remind the reader that in Lemma 157.2 (1) we showed that there exists a homotopy $F\colon (\overline{B}^n/S^{n-1}) \times [0,1] \to \overline{B}^n/S^{n-1}$ rel $*$ with the following properties:

(1) $F_0 = \mathrm{id}$,
(2) the restriction of F_1 to B^n_ϵ is given by multiplication by $\frac{1}{\epsilon}$,
(3) for any point $x \notin B^n_\epsilon$ we have $F_1(x) = *$.

We define $\psi = F_1 \circ \varphi$. It follows from (1) that ψ is homotopic rel $*$ to φ. Therefore we are done once we have shown that ψ in fact equals $\rho_{(N,g)}$. As a reminder, see the definition on page 3343, the map $\rho_{(N,g)}$ is given as follows:[1932]

$$\begin{array}{rcl} \rho_{(N,g)}\colon \overline{B}^{n+k}/S^{n+k-1} & \to & S^n = \overline{B}^n/S^{n-1} \\ P & \mapsto & \begin{cases} x, & \text{if } P = g(x,y) \text{ with } (x,y) \in B^n \times N, \\ *, & \text{otherwise.} \end{cases} \end{array}$$

Now we verify that ψ does indeed agree with $\rho_{(N,g)}$. Thus first suppose we are given a point $P = g(x,y)$ with $(x,y) \in B^n \times N$. We have

$$\psi(P) \underset{\text{definition of } \psi}{\overset{\uparrow}{=}} F_1(\varphi(g(x,y))) \underset{\text{by the above commutative diagram}}{\overset{\uparrow}{=}} F_1(\epsilon \cdot x) \underset{\text{by Property (2) of } F_1 \text{ and since } \epsilon \cdot x \in B^n_\epsilon}{\overset{\uparrow}{=}} \tfrac{1}{\epsilon} \cdot \epsilon \cdot x = x = \rho_{(N,g)}(g(x,y)) = \rho_{(N,g)}(P).$$

Finally suppose that we are given a point $P \in \overline{B}^{n+k}/S^{n+k-1}$ that does not lie in the subset $g(B^n \times N)$. In this case we calculate that

$$\psi(P) \underset{\text{definition of } \psi}{\overset{\uparrow}{=}} F_1(\varphi(P)) \underset{\text{since } \varphi^{-1}(B^n_\epsilon) = g(B^n \times N) \text{ and by choice of } P}{\overset{\uparrow}{=}} F_1(\text{point that does not lie in } B^n_\epsilon) \underset{\text{by Property (3) of } F_1}{\overset{\uparrow}{=}} * = \rho_{(N,g)}(P).$$

■

[1932] Eyes which profited from a carrot-rich diet might notice that the description is not verbatim the same as on page 3343. But with either description of the map the points $P = g(x,y)$ with $x \in S^{n-1}$ and $y \in N$ get sent to $*$. So we really do consider the same map as before.

157.3. **The thickened cobordism groups and the Thom-Pontryagin Theorem.** Recall that the Thom-Pontryagin construction from page 3343 makes the assignment

$$\text{thickened } k\text{-dimensional submanifold of } \mathbb{R}^{n+k} \quad \rightsquigarrow \quad \text{map } (S^{n+k},*) \to (S^n,*).$$

Now we would like to answer Question 157.3 (2), i.e. we would like to determine when do two thickened k-dimensional submanifolds give rise to the same homotopy class of maps $(S^{n+k},*) \to (S^n,*)$. Loosely speaking we would like to fill in the left-hand side of

$$?? \quad \rightsquigarrow \quad \text{map } (S^{n+k}\times[0,1], *\times[0,1]) \to (S^n,*).$$

A quick look at the Thom-Pontryagin construction suggests that the left-hand side should be filled in with a suitable thickened $(k+1)$-dimensional submanifold of $\mathbb{R}^{n+k}\times[0,1]$. This makes sense, except that in the Smooth Manifold Product Proposition 19.31 we saw that for products of two smooth manifolds it is advisable to have the smooth manifold with boundary as a first factor. Thus we will consider submanifolds of $[0,1]\times\mathbb{R}^{n+k}$.[1933]
Before we continue we introduce the following straightforward convention.

Convention. Given a topological space X and an interval I and some $t \in I$ we make at times the identification $X = \{t\}\times X$ via the obvious bijection.

The above observation leads us to the following definition which looks awfully similar to the definition of the notion of a cobordism of smooth manifolds that we introduced on page 3089.

Definition. Let $k, m \in \mathbb{N}_0$.

(1) Let N_0 and N_1 be two closed orientable k-dimensional submanifolds of $\mathbb{R}^m$. A cobordism between N_0 and N_1 is a compact orientable proper $(k+1)$-dimensional submanifold W of $[0,1]\times\mathbb{R}^m$ with $\partial W = (\{0\}\times N_0)\cup(\{1\}\times N_1)$.[1934]

(2) Let (N_0,g_0) and (N_1,g_1) be two thickened k-dimensional submanifolds of $\mathbb{R}^m$. A thickened cobordism between (N_0,g_0) and (N_1,g_1) is a thickened $(k+1)$-dimensional submanifold (W,G) of $[0,1]\times\mathbb{R}^m$ where W is a cobordism between N_0 and N_1 and G is a thickening of W with the following two properties:[1935]
   (a) $G|_{\{0\}\times\overline{B}^m} = g_0$ under the identification $\{0\}\times\overline{B}^m = \overline{B}^m$,
   (b) $G|_{\{1\}\times\overline{B}^m} = g_1$ under the identification $\{1\}\times\overline{B}^m = \overline{B}^m$.
   If such a thickened cobordism exists, then we say that (N_0,g_0) and (N_1,g_1) are cobordant.[1936] If (N,g) is cobordant to the empty submanifold, then we say N is a thickened null-cobordism and we say that (N,g) is null-cobordant. The definition of a thickened cobordism is illustrated in the figure below.

The following lemma shows that diffeotopies give rise to thickened cobordisms.

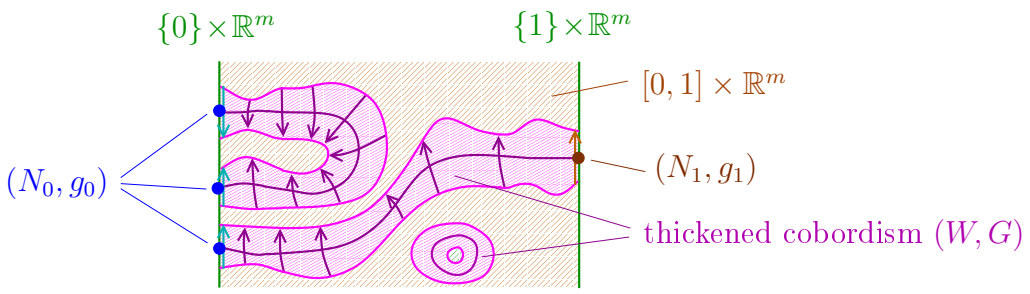
Lemma 157.7. Let (M,g) be a thickened submanifold of $\mathbb{R}^m$ and let φ be a diffeomorphism of $\mathbb{R}^m$. If φ is diffeotopic to the identity, then $(\varphi(M),\varphi_*(g))$ is cobordant to (M,g).

[1933] Perhaps in hindsight we should have defined homotopies as maps $[0,1]\times X \to Y$ instead of defining them as maps $X\times[0,1]\to Y$?

[1934] Note that we work with submanifolds that are orientable, but we do not assume that they are equipped with an orientation.

[1935] Note that it follows from property (d) of a thickened submanifold, see page 3342, that we know that in fact $G|_{\{0\}\times\overline{B}^m} \subset \{0\}\times\mathbb{R}^m$ and $G|_{\{1\}\times\overline{B}^m} \subset \{1\}\times\mathbb{R}^m$.

[1936] We spare the reader the even more logical name "thickened cobordant".



Example. Let $m \in \mathbb{N}$. Any map $\mathbb{R}^m \to \mathbb{R}^m$ of the form $x \mapsto r \cdot x + P$ for some $r > 0$ and $P \in \mathbb{R}^m$ is clearly diffeotopic to the identity. Thus Lemma 157.7 says that given a thickened submanifold (M, g) we can translate and shrink it without changing the thickened cobordism class.

Proof. Let $(M, g\colon \overline{B}^{m-k} \times M \to \mathbb{R}^m)$ be a thickened k-dimensional submanifold of $\mathbb{R}^m$ and furthermore let $F\colon \mathbb{R}^m \times [0,1] \to \mathbb{R}^m$ be a diffeotopy from the identity to a diffeomorphism φ. It is straightforward to verify that the map

$$\begin{array}{rcl} [0,1]\times M & \to & [0,1]\times\mathbb{R}^m \\ (t,P) & \mapsto & (t,F(P,t)) \end{array}$$

is a smooth embedding. We write $W := F([0,1]\times M)$. It is straightforward to verify that W together with the thickening given by

$$\begin{array}{rcl} \overline{B}^{m-k}\times W & \to & [0,1]\times\mathbb{R}^m \\ (v,(t,F(P,t))) & \mapsto & (t,F(g(v,P),t)) \end{array}$$

is a thickened cobordism between (M, g) and $(\varphi(M), \varphi_*(g))$. ■

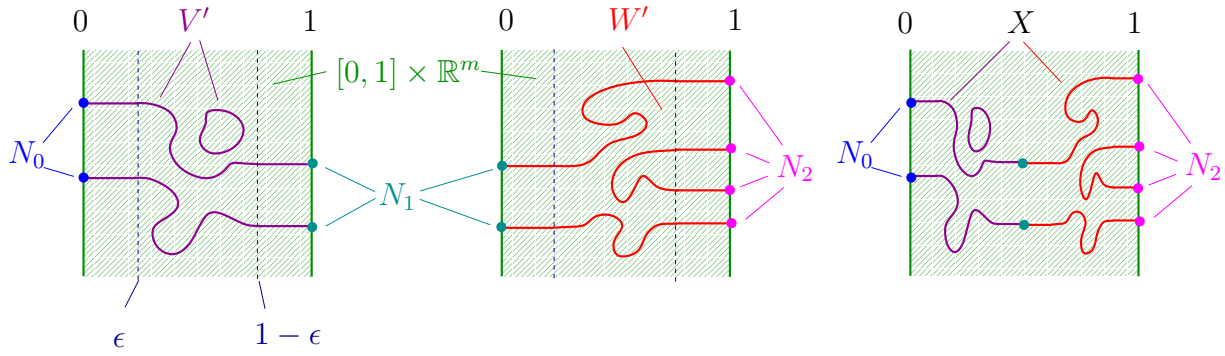
Lemma 157.8. "Being cobordant" is an equivalence relation on the set of thickened k-dimensional submanifolds of $\mathbb{R}^m$.

Sketch of proof. To simplify the discussion we ignore throughout the proof the thickenings. The reader should have no major troubles modifying the proof to also include the thickenings.

It is straightforward to see that "being cobordant" is reflexive and symmetric: Indeed, given a closed k-dimensional submanifold N a cobordism from N to itself is given by $[0,1]\times N \subset [0,1]\times\mathbb{R}^m$. Furthermore, suppose there exists a cobordism W between N_0 and N_1. "Flipping" W along $\mathbb{R}^m \times \{\frac{1}{2}\}$ we obtain a cobordism between N_1 and N_0.

The proof of transitivity takes more efforts. Initially the idea is quite simple. Given a cobordism V from N_0 to N_1 and another cobordism W from N_1 to N_2 one would like to stick them together to obtain a cobordism from N_0 to N_2. Unfortunately, as we had discussed on page **??**, the resulting object is in general not a (smooth) submanifold of the smooth manifold $\mathbb{R}^m$.

Fortunately we had already developed all the tools to deal with the situation. More precisely, in Proposition **??** we saw that we can modify the cobordisms V and W to obtain cobordisms V' and W' that are products on $[0,\epsilon]\times\mathbb{R}^m$ and $[1-\epsilon,1]\times\mathbb{R}^m$ for a suitable $\epsilon \in (0,\frac{1}{2})$. Now we can stick V' and W' together to obtain the desired cobordism X. We refer to the figure below for an illustration. ■



The previous lemma allows us to make the following definition.

Definition. We denote by $\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$ the set of thickened cobordism classes of thickened k-dimensional submanifolds of $\mathbb{R}^m$. In light of the subsequent Proposition 157.9 we refer to $\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$ as a thickened cobordism group.

The definition of $\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$ is reminiscent of the definition of the cobordism group Ω_k that we gave on page 3091. The group structure on Ω_k that we had introduced in Lemma 144.2 is given by "disjoint union". We will now see that $\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$ also admits a group structure that is defined quite similarly.

Proposition 157.9. Let $k\in\mathbb{N}_0$ and let $m\in\mathbb{N}_{\geq 2}$. Let (M,g) and (N,h) be two thickened k-dimensional submanifolds of $\mathbb{R}^m$. Since M and N are compact we can pick $r,s\in\mathbb{R}_{>0}$ with $g(\overline{B}^{m-k}\times M)\subset B_r^m(0)$ and $h(\overline{B}^{m-k}\times N)\subset B_s^m(0)$. Furthermore we pick two orientation-preserving smooth embeddings $\varphi\colon \overline{B}_r^m(0)\to\mathbb{R}^m$ and $\psi\colon \overline{B}_s^m(0)\to\mathbb{R}^m$ with disjoint images. We define

$$[(M,g)]+[(N,h)] \;:=\; [(\varphi(M)\sqcup\psi(N),\varphi_*(g)\sqcup\psi_*(h))] \;\in\; \Omega_k^{\mathrm{th}}(\mathbb{R}^m).$$

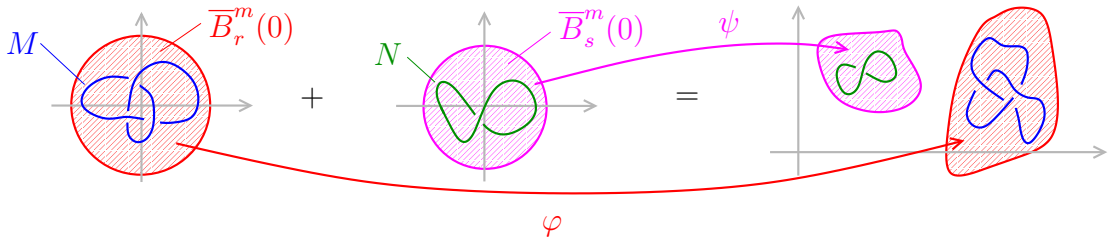
(The definition of the operation "+" is illustrated in the figure below.)

(1) The definition of "+" on $\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$ is well-defined and commutative.

(2) The operation "+" defines a group structure on $\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$. Here the following two statements hold:

(a) The neutral element is represented by the empty submanifold.[1937]

(b) Let $\rho\colon\mathbb{R}^m\to\mathbb{R}^m$ be the reflection in a hyperplane. The inverse of $[(M,g)]$ is given by $[(\rho(M),\rho_*(g))]$.



Proof.

[1937] It is worth remembering that the empty set is a submanifold of any dimension.

(1) It follows from the Smooth Ball Embedding Theorem 37.8 and Lemma 157.7 that "+" is well-defined and commutative. Note that here we use that $m \geq 2$.

(2) It is basically obvious that "+" is associative and that the empty submanifold is a neutral element. Thus it remains to prove the statement regarding the inverse. It follows from Lemmas 157.7 and 33.9 that without loss of generality we can assume that the reflection $\rho\colon \mathbb{R}^m \to \mathbb{R}^m$ is the reflection in the $(x_m = 0)$-hyperplane. So let $(M, g\colon \overline{B}^{m-k} \times M \to \mathbb{R}^m)$ be a thickened k-dimensional submanifold of $\mathbb{R}^m$. We set $\epsilon = \frac{1}{4}$. It follows immediately from the discussion on page 3351 that we can assume that $g(\overline{B}^{m-k} \times M)$ is contained in the ball $\overline{B}^m_\epsilon(\underbrace{(0,\dots,0,2\epsilon)}_{\in\mathbb{R}^m})$. We consider the map

$$\begin{array}{rcl} \varphi\colon \overline{B}^{m-k} \times (M \times [0,\pi]) & \to & [0,1] \times \mathbb{R}^m \\ (v,(P,\gamma)) & \mapsto & \begin{pmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & \mathrm{id}_{m-1} \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 0 \\ g(v,P) \end{pmatrix}}_{\in\mathbb{R}^{m+1}}. \end{array}$$

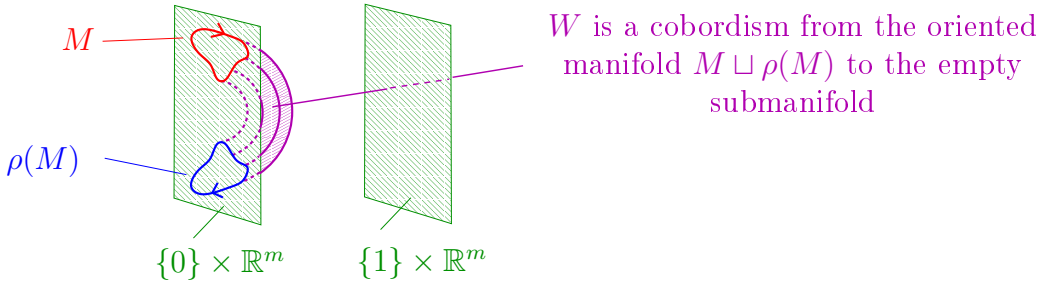
Using that $g(\overline{B}^{m-k} \times M)$ is contained $\overline{B}^m_\epsilon((0,\dots,0,2\epsilon))$ we can make the following three observations:

(a) The image of φ does indeed lie in $[0,1] \times \mathbb{R}^m$.

(b) The restriction to $\{0\} \times M \times [0,\pi]$ is a proper smooth embedding. Therefore we can now consider the compact orientable proper $(k+1)$-dimensional submanifold $W = \varphi(\{0\} \times M \times [0,\pi])$ of $[0,1] \times \mathbb{R}^m$.

(c) The map φ gives rise to a thickening for W.

It is now basically clear that we have constructed a thickened cobordism from the thickened submanifold $(M \sqcup \rho(M), g \sqcup \rho_*(g))$ to the empty submanifold. (We refer to the figure below for an illustration.) But by the definition of "+" this implies that $(\rho(M), \rho_*(g))$ represents the inverse of (M, g). ■



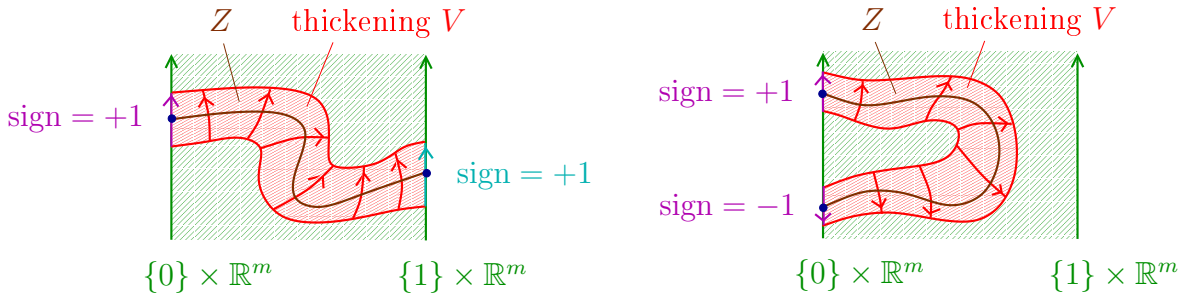
Evidently our challenge now is to compute the groups $\Omega^{\mathrm{th}}_k(\mathbb{R}^m)$ for suitable values of m and k. In general this might be quite difficult, but the following proposition gives a reasonably amusing calculation.

Proposition 157.10. Let $m \geq 2$. The map[1938]

$$\begin{array}{rcl} \varphi\colon \Omega^{\mathrm{th}}_0(\mathbb{R}^m) & \to & \mathbb{Z} \\ [(\{P_1\}, g_1) \sqcup \dots \sqcup (\{P_k\}, g_k)] & \mapsto & \sum_{i=1}^{k} \mathrm{sign}(g_i) \end{array}$$

is well-defined and it is an isomorphism.

Sketch of proof. First we need to show that $\varphi\colon \Omega_0^{\text{th}}(\mathbb{R}^m) \to \mathbb{Z}$ is well-defined. It suffices to show that for any *connected* thickened 1-dimensional cobordism $(Z, V) \subset [0,1] \times \mathbb{R}^m$ between two 0-dimensional thickened submanifolds (N_0, g_0) and (N_1, g_1) we have the equality $\varphi([(N_0, g_0)]) = \varphi([(N_1, g_1)])$. This statement follows from the Smooth 1-Dimensional Manifold Classification Theorem 22.7 and some orientation considerations. Since writing down the precise arguments regarding the orientations will only obfuscate the argument we refer to the figure below for convincing pictures, and we leave it to the reader to fill in the details.



Next note that the map $\varphi\colon \Omega_0^{\text{th}}(\mathbb{R}^m) \to \mathbb{Z}$ is, basically by definition, a homomorphism. Finally note that it is clear that φ is an epimorphism. Thus it remains to show that φ is a monomorphism. A straightforward induction argument shows that it suffices to prove the following claim.

Claim. For points $P, Q \in \mathbb{R}^m$ and thickenings g and h of P and Q with $\operatorname{sign}(g) = -\operatorname{sign}(h)$ we have $[(P, g)] + [(Q, h)] = 0 \in \Omega_0^{\text{th}}(\mathbb{R}^m)$.

Proof. In the following let $\rho\colon \mathbb{R}^m \to \mathbb{R}^m$ be the reflection in a hyperplane. By the discussion on page 3351 we can assume, after possibly applying a translation, that $Q = \rho(P)$. Since ρ is orientation-reversing we see that $\operatorname{sign}(\rho_*(g)) = -\operatorname{sign}(g) = \operatorname{sign}(h)$. Now we calculate that

$$[(P,g)] + [(Q,h)] = [(P,g)] + [(\rho(P),h)] \underset{\uparrow}{=} [(P,g)] + [(\rho(P),\rho_*(g))] \underset{\uparrow}{=} 0 \in \Omega_0^{\text{th}}(\mathbb{R}^m).$$

by $\operatorname{sign}(h) = \operatorname{sign}(\rho_*(g))$, by the Smooth Ball Embedding Theorem 37.8 and Lemma 157.7

by Proposition 157.9 (2b)

■

Considering the discussion on page 3350, the following theorem does not come as a big surprise. It justifies our interest in the groups $\Omega_k^{\text{th}}(\mathbb{R}^m)$.

Theorem 157.11. (Thom-Pontryagin Theorem) Let $n, k \in \mathbb{N}_0$ with $n + k \geq 2$. The map

$$\begin{aligned} \Psi\colon \Omega_k^{\text{th}}(\mathbb{R}^{n+k}) &\to \pi_{n+k}(S^n, *) \\ [(N, g)] &\mapsto [\rho_{(N,g)}] \end{aligned}$$

is well-defined and it is an isomorphism.

Sketch of proof. Fortunately we have already done basically all the work:

(1) First we argue that Ψ is well-defined. Thus suppose that we are given thickened k-dimensional submanifolds (M, g) and (N, h) of $\mathbb{R}^{n+k}$ that are cobordant. We pick

[1938] We refer to page 3345 for the definition of the sign of a thickening of a point.

a thickened cobordism $(W, F\colon \overline{B}^n \times W \to [0,1] \times \mathbb{R}^{n+k})$ in $[0,1] \times \mathbb{R}^{n+k}$. The corresponding map

$$\rho_{(W,F)}\colon [0,1] \times \mathbb{R}^{n+k} \to S^n$$

has the property that it sends a neighborhood of $[0,1] \times \{\infty\}$ to $*$. Therefore it follows easily from the Homotopy Stacking Lemma 14.3 that the following map is continuous:

$$\begin{array}{rcl} S^{n+k} \times [0,1] = (\mathbb{R}^{n+k} \cup \{\infty\}) \times [0,1] & \to & S^n \\ (P,t) & \mapsto & \begin{cases} \rho_{(W,F)}(t,P), & \text{if } P \in \mathbb{R}^{n+k}, \\ *, & \text{if } P = \infty. \end{cases} \end{array}$$

It follows from the above that this map is in fact a homotopy rel $*$ between the map $\rho_{(M,g)}\colon (S^{n+k}, *) \to (S^n, *)$ and the map $\rho_{(N,h)}\colon (S^{n+k}, *) \to (S^n, *)$. In particular we have $[\rho_{(M,g)}] = [\rho_{(N,h)}] \in \pi_{n+k}(S^n, *)$.

(2) It follows immediately from Lemma 157.5 that Ψ is a homomorphism.

(3) The statement of Proposition 157.6 is equivalent to saying that Ψ is an epimorphism.

(4) Finally it remains to show that Ψ is in fact a monomorphism. So suppose that we are given two thickened k-dimensional submanifolds (M,g) and (N,h) of $\mathbb{R}^{n+k}$ such that $[\rho_{(M,g)}] = [\rho_{(N,h)}] \in \pi_{n+k}(S^n, *)$, i.e. such that there exists a homotopy $F\colon S^{n+k} \times [0,1] \to S^n$ rel $*$ from $\rho_{(M,g)}$ to $\rho_{(N,h)}$.

The proof of Proposition 157.6 can be modified without too many problems to show that the homotopy F gives rise to a thickened cobordism between (M,g) and (N,h). We leave it to the reader to fill in the details. See also [**Kos1993**, Theorem 9.5.5] or [**Bre1993**, Chapter 16] for similar claims stated with more authority. ∎

From our above results we obtain the following corollary. Of course we proved this result earlier in Corollary 85.6, using very different methods.

Corollary 157.12. For any $m \geq 2$ the map

$$\begin{array}{rcl} \mathbb{Z} & \to & \pi_m(S^m, *) \\ n & \mapsto & n \cdot [\mathrm{id}_{S^m}] \end{array}$$

is an isomorphism.

Proof. Let $P \in \mathbb{R}^m$ and let g be a thickening for the submanifold $\{P\}$. We consider the two maps

$$\begin{array}{rcl} \mathbb{Z} & \to & \Omega_0^{\mathrm{th}}(\mathbb{R}^m) \\ n & \mapsto & n \cdot [(\{0\}, g \text{ with } \mathrm{sign}(g) = 1)] \end{array} \quad \text{and} \quad \begin{array}{rcl} \Omega_0^{\mathrm{th}}(\mathbb{R}^m) & \to & \pi_m(S^m, *) \\ [(N,g)] & \mapsto & [\rho_{(N,g)}]. \end{array}$$

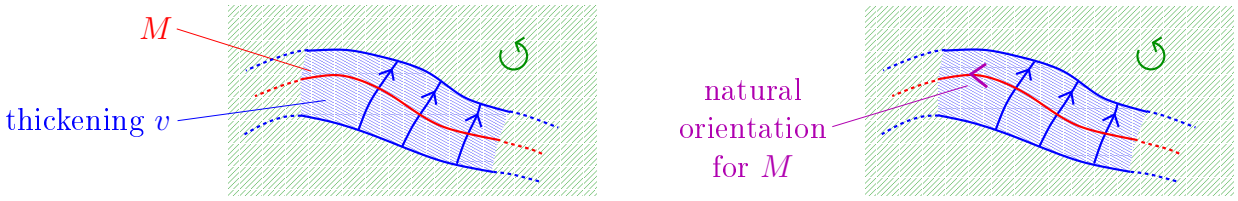
The first map is an isomorphism by Proposition 157.10 and the second map is an isomorphism by the Thom-Pontryagin Theorem 157.11, applied to $k = 0$ and $m \geq 2$. Thus the composition of the two maps is an isomorphism. Finally it follows from Lemma 157.1 that the composition of these two maps sends n to $n \cdot [\mathrm{id}_{S^m}]$. ∎

157.4. **The calculation of the thickened cobordism group $\Omega_1^{\mathrm{th}}(\mathbb{R}^3)$.** Surely the most interesting calculation of a higher homotopy group that we have performed so far is that in Theorem 155.10 (2) we saw $\pi_3(S^2) \cong \mathbb{Z}$. Our goal now is to give a new proof of this fact using the Thom-Pontryagin Theorem 157.11. Thus we need to show that $\Omega_1^{\mathrm{th}}(\mathbb{R}^3) \cong \mathbb{Z}$. We start out with the following slightly subtle definition.

Definition. Let $m \in \mathbb{N}$.

(1) Let $(M, v\colon \overline{B}^{m-k} \times \mathbb{R}^m)$ be a thickened $(m-k)$-dimensional submanifold of $\mathbb{R}^m$. The natural orientation of M is the unique orientation that turns v into an orientation-preserving map.[1939]

(2) Similarly we define the natural orientation of a thickened cobordism.



This definition leads us to the following convention.

Convention. In this section we equip any thickened submanifold and any thickened cobordism with the natural orientation.

Before we write down an explicit isomorphism we introduce the following definition which generalizes the definition of the self-linking number of a thickened knot that we introduced on page 2853.

Definition.

(1) Let $K = K_1 \sqcup \cdots \sqcup K_m$ and $L = L_1 \sqcup \cdots \sqcup L_n$ be two disjoint oriented links. We define the linking number of K and L as

$$\operatorname{lk}(K, L) \;:=\; \sum_{i=1}^{m} \sum_{j=1}^{n} \operatorname{lk}(K_i, L_j).$$

(2) Let (L, g) be a thickened link. We write $L' = g(\{1\} \times L)$. We define the self-linking number of (L, g) as $\operatorname{slk}(L, g) := \operatorname{lk}(L, L')$.[1940]

The following elementary lemma is at times useful for determining self-linking numbers of thickened links.

Lemma 157.13. If $(L, g) = (L_1 \sqcup \cdots \sqcup L_m, g_1 \sqcup \cdots \sqcup g_m)$ is a thickened link, then

$$\operatorname{slk}(L, g) \;:=\; \sum_{i \neq j} \operatorname{lk}(L_i, L_j) \;+\; \sum_{i=1}^{m} \operatorname{slk}(L_i, g_i).$$

Proof. We will prove the lemma in Exercise 157.2. ∎

Now we can give the promised isomorphism.

Theorem 157.14. The map

$$\begin{array}{rcl} \Psi\colon \Omega_1^{\text{th}}(\mathbb{R}^3) & \to & \mathbb{Z} \\ [(L, g)] & \mapsto & \text{self-linking number } \operatorname{slk}(L, g) \end{array}$$

is well-defined and it is an isomorphism. In particular we have $\pi_3(S^2) \cong \mathbb{Z}$.

[1939] Recall that M is assumed to be orientable, but a priori it does not come with an orientation. It is not difficult to see that there exists a unique orientation on M such that, if we equip $\overline{B}^{m-k} \times M$ with the product orientation, v becomes orientation-preserving.

[1940] Here we use the above convention to view L as an *oriented* submanifold of $\mathbb{R}^3$, and we equip $L' = g(\{1\} \times L)$ with the obvious corresponding orientation.

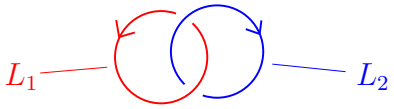
Examples.

(1) We consider the trivial knot K with a thickening g of self-linking number $+1$. By Theorem 157.14 the thickened knot (K, g) defines a generator of $\Omega_1^{\text{th}}(\mathbb{R}^3)$. It follows from the isomorphism provided by the Thom-Pontryagin Theorem 157.11 that $\rho_{(K,g)}\colon S^3 \to S^2$ represents a generator of $\pi_3(S^2, *)$. In Lemma 157.4 we saw that $\rho_{(K,g)}$ represents the same element of $\pi_3(S^2, *)$ as the Hopf map $H\colon S^3 \to S^2$. Thus we see once again that the Hopf map represents a generator of $\pi_3(S^2)$.

(2) We consider the Hopf link $L = L_1 \sqcup L_2$ with the orientations shown in the figure below. We equip $L = L_1 \sqcup L_2$ with a "trivial thickening" $g_1 \sqcup g_2$, i.e. where the thickening g_i of each component L_i has self-linking number 0. We see that

$$\operatorname{slk}(L_1 \sqcup L_2, g_1 \sqcup g_2) \underset{\text{Lemma 157.13}}{=} \operatorname{lk}(L_1, L_2) + \operatorname{lk}(L_2, L_1) + \overbrace{\operatorname{slk}(L_1, g_1)}^{=0} + \overbrace{\operatorname{slk}(L_2, g_2)}^{=0} \underset{\text{pick your favorite method from Section 131.4}}{=} 1 + 1 = 2.$$

This shows that the Hopf link, perhaps slightly counter-intuitively, gives rise to *twice* the generator of $\pi_3(S^2, *)$.



Since we already have an independent proof of Theorem 157.14 we will take the liberty to cut a few corners in the new proof of the theorem. The new proof of Theorem 157.14 will build on the following four lemmas. The first two lemmas generalize the Seifert Surface-Existence Proposition 131.6 and Lemma 131.15 from knots to links. They are perhaps interesting in their own right.

Lemma 157.15. Given any oriented link $L \subset S^3$ there exists a compact oriented connected surface $F \subset S^3$ such that $\partial F = L$.[1941]

Proof of Lemma 157.15. Let $L \subset S^3$ be an oriented link. A modest generalization of the Seifert Surface-Existence Proposition 131.6 shows that there exists a compact oriented surface $G \subset S^3$ with $\partial G = L$. In Exercise 157.5 we will show that we can turn G into a compact oriented *connected* surface F with $\partial F = L$.[1942] ■

Lemma 157.16. If $K = K_1 \sqcup \cdots \sqcup K_m$ and $L = L_1 \sqcup \cdots \sqcup L_n$ are two oriented links, then the following statements hold:

(1) There exist compact oriented proper 2-dimensional submanifolds S and T of $\overline{B}^4$ with $\partial S = K$ and $\partial T = L$ such that S and T are transverse.

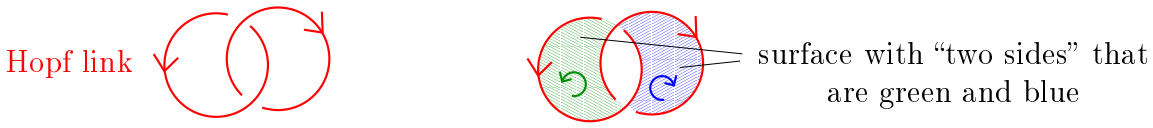
(2) Let S, T be two oriented proper 2-dimensional submanifolds of $\overline{B}^4$ with $\partial S = K$ and $\partial T = L$. If S and T are transverse, then[1943]

$$\operatorname{lk}(K, L) \;=\; S \cdot_{\overline{B}^4} T.$$

[1941] Here the equality $\partial F = L$ is understood to be an equality of *oriented* submanifolds.

[1942] Frankly it can be slightly annoying to give a completely rigorous proof that such a surface F exists. This is one of the main reasons why we made the caveat above that we are going to cut a few corners in the argument.

Example. As on page 2220 we consider the Hopf link that is illustrated in the figure below on the left. On the right of the figure below we show a compact oriented connected surface whose boundary equals the Hopf link.



Proof of Lemma 157.16.

(1) The proof of this statement is virtually the same as the proof of Lemma 131.15, we just need to replace the Seifert Surface-Existence Proposition 131.6 by Lemma 157.15.

(2) We leave it to the reader to verify that the proof of Lemma 131.15 (2) can be modified to deal not only with knots but also with links. ■

The following two lemmas that we need for the proof of Theorem 157.14 are arguably more technical.

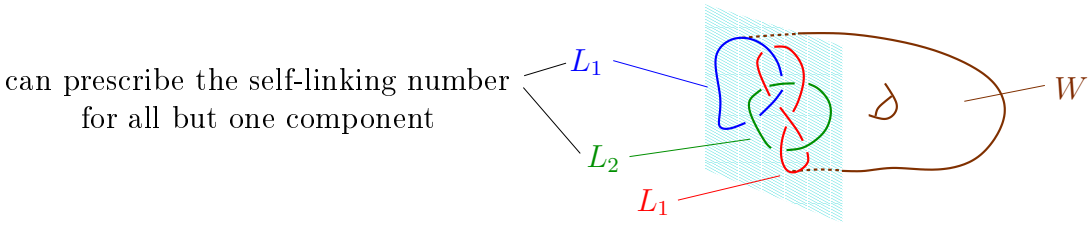
Lemma 157.17. Let K be an oriented *knot* in $\mathbb{R}^3$ and let g and h be two thickenings for K in $\mathbb{R}^3$. If $\operatorname{slk}(K,g) = \operatorname{slk}(K,h)$, then $[(K,g)] = [(K,h)] \in \Omega_1^{\text{th}}(\mathbb{R}^3)$.

Proof of Lemma 157.17. Since $\operatorname{slk}(K,g) = \operatorname{slk}(K,h)$ it follows from the Knot Thickening Lemma 131.1 that there exists a smooth isotopy $F\colon (\overline{B}^2 \times K) \times [0,1] \to \mathbb{R}^3$ between g and h which has compact support and which has the property that $F(0,k,t) = k$ for all $k \in K$ and all $t \in [0,1]$. We consider $W = [0,1] \times K$ with the thickening given by

$$\begin{array}{rcl} j\colon \overline{B}^2 \times ([0,1] \times K) & \to & [0,1] \times \mathbb{R}^3 \\ (v,(t,x)) & \mapsto & (t, F((v,x),t)). \end{array}$$

The thickened smooth manifold (W,j) is a thickened cobordism between (K,g) and (K,h). ■

Lemma 157.18. Let W be a compact oriented proper connected 2-dimensional submanifold of $\mathbb{R}^3 \times [0,1)$ with boundary components $L_1, \ldots, L_m$. Given $s_1, \ldots, s_{m-1} \in \mathbb{Z}$ there exists a thickening h for W such that for $i = 1, \ldots, m-1$ we have $\operatorname{slk}(L_i, h|_{L_i}) = s_i$.



Proof. Evidently we can assume that $m \geq 1$. In this case W is a compact oriented proper connected 2-dimensional submanifold with non-empty boundary. Thus by Proposition ?? (3) there exists a thickening $f\colon \overline{B}^2 \times W \to \mathbb{R}^3 \times [0,1)$ for W. For $i = 1, \ldots, m-1$ we write $r_i := \operatorname{slk}(L_i, g|_{L_i})$.

[1943] Here $S \cdot_{\overline{B}^4} T$ denotes the algebraic intersection number of the oriented transverse surfaces S and T in the oriented smooth manifold $\overline{B}^4$. We refer to page 2748 for the definition of the algebraic intersection number.

By Lemma 55.12 there exists a map $\varphi\colon W \to L_1 \vee \cdots \vee L_{m-1}$[1944] that is the identity on each L_i. We pick a map $\psi\colon L_1 \vee \cdots \vee L_{m-1} \to \mathrm{SO}(2) = S^1$ with the following property: under an orientation-preserving identification $L_i = S^1$ the restriction of ψ to L_i is given by $z \mapsto z^{s_i - r_i}$. Now we consider the thickening

$$\begin{aligned} h\colon \overline{B}^2 \times W &\to W \\ (v, P) &\mapsto f(\overbrace{(\psi \circ \varphi)(P)}^{\in \mathrm{SO}(2)} \cdot \underbrace{v, P)}_{\in \overline{B}^2 \times W}). \end{aligned}$$

By the Knot Thickening Lemma 131.1 (1) we know that for any $i \in \{1, \ldots, m-1\}$ we added $s_i - r_i$ to the self-linking number of L_i. Thus the thickening h has all the desired properties. ■

Now we are in a position to provide the proof of Theorem 157.14.

Sketch of proof of Theorem 157.14.

(1) First we need to show that $\Psi\colon \Omega_1^{\mathrm{th}}(\mathbb{R}^3) \to \mathbb{Z}$ is well-defined. In the proof of this fact we borrow some ideas from the proof of Lemma 131.15 (2). So let (W, F) be a thickened cobordism between (K, g) and (L, h). We write $W' = F(\{1\} \times W)$, $K' = g(\{1\} \times K)$ and $L' = h(\{1\} \times L)$. By Lemma 157.16 (1) there exist oriented proper 2-dimensional submanifolds S and S' of $\overline{B}^4$ with $\partial S = K$ and $\partial S' = K'$ such that S and S' are transverse. Similarly there exist T, T' for L and L'.

We can make the identification $S^4 = (-\overline{B}^4) \cup [0,1] \times S^3 \cup \overline{B}^4$[1945] (See the figure below for an illustration.) Using Proposition **??** we can arrange that both $(-S) \cup W \cup T$ and $(-S') \cup W' \cup T'$ are oriented submanifolds of the smooth manifold S^4. We calculate that

$$\begin{aligned} -\operatorname{slk}(K, g) + \operatorname{slk}(L, h) &\overset{\text{by definition of slk}}{=} -\operatorname{lk}(K, K') + \operatorname{lk}(L, L') \overset{\text{Lemma 157.16 (2)}}{=} (-S) \cdot_{\overline{B}^4} S' + T \cdot_{\overline{B}^4} T' \\ &= (-S) \cdot_{-\overline{B}^4} (-S') + T \cdot_{\overline{B}^4} T' \overset{\text{since the only intersections appear between } S, S' \text{ and } T, T', \text{ here we use that } W \text{ and } W' \text{ are disjoint}}{=} (-S \cup W \cup T) \cdot_{S^4} (-S' \cup W' \cup T') \overset{\text{by Theorem 127.7 and since } \mathrm{H}_2(S^4; \mathbb{Z}) = 0}{=} 0. \end{aligned}$$

We have thus shown that the self-linking numbers of (K, g) and (L, h) agree. This shows that the map $\Psi\colon \Omega_1^{\mathrm{th}}(\mathbb{R}^3) \to \mathbb{Z}$ is indeed well-defined.

(2) It follows easily from Exercise 131.6 that Ψ is a homomorphism.

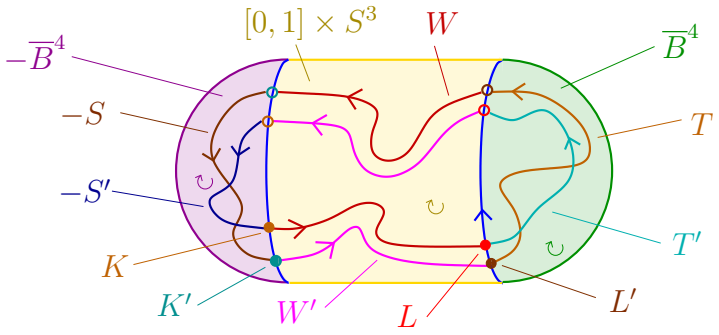
(3) By the Knot Thickening Lemma 131.1 there exists a knot in $\mathbb{R}^3$ with a thickening in $\mathbb{R}^3$ with self-linking number one, thus Ψ is an epimorphism.

(4) It remains to show that $\Psi\colon \Omega_1^{\mathrm{th}}(\mathbb{R}^3) \to \mathbb{Z}$ is a monomorphism. So suppose we are given a thickened m-component link $(L, g) = \bigsqcup_{i=1}^{m} (L_i, g_i)$ with $\Psi([(L, g)]) = 0$.

Note that by Lemma 157.15 there exists a compact oriented connected surface W in $\mathbb{R}^3 \times [0, 1)$ whose boundary equals $L \subset \mathbb{R}^3 = \mathbb{R}^3 \times \{0\}$. Next note that by

[1944] Here, as a reminder, $L_1 \vee \cdots \vee L_{m-1}$ denotes the wedge of the topological spaces $L_1, \ldots, L_{m-1}$.

[1945] If the reader does not feel like making this identification, then we point out that we only need that the right-hand side is a smooth manifold and that the second homology vanishes.



Lemma 157.18 we can equip W with a thickening h such that for $i = 1, \dots, m-1$ we have $\mathrm{slk}(L_i, h|_{L_i}) = \mathrm{slk}(L_i, g_i)$.

Claim. We also have $\mathrm{slk}(L_m, h|_{L_m}) = \mathrm{slk}(L_m, g_m)$.

Proof. Note that (W, h) is a thickened cobordism between $(L, h|_L)$ and the empty link. In particular we obtain from (1) that $\mathrm{slk}(L, h|_L) = 0$. By our hypothesis on (L, g) we also have $\mathrm{slk}(L, g) = 0$. We had already arranged that for $i = 1, \dots, m-1$ we have $\mathrm{slk}(L_i, h|_{L_i}) = \mathrm{slk}(L_i, g_i)$. It now follows immediately from Lemma 157.13 that $\mathrm{slk}(L_m, h|_{L_m}) = \mathrm{slk}(L_m, g_m)$. ⊞

Now we are done, since

$$[(L,g)] \underset{\text{Lemma 157.17}}{=} [(L, h|_g)] \underset{\text{since } (W,h) \text{ is a null-cobordism for } (L,h|_g)}{=} 0 \in \Omega_1^{\mathrm{th}}(\mathbb{R}^3).$$

■

157.5. **The linking number.** In the next section we will show that there exists an epimorphism $\Omega^{\mathrm{th}}_{2n-1}(\mathbb{R}^{4n-1}) \to \mathbb{Z}$. The key to doing so is the notion of the linking number which we discuss in this section.

Notation. Let $n \in \mathbb{N}$.

(1) Given a closed oriented m-dimensional submanifold K of S^{2m+1} we denote by

$$\beta_K \colon \mathrm{H}_m(K;\mathbb{Z}) \to \mathrm{H}^m(S^{2m+1} \setminus K;\mathbb{Z})$$

the inverse of the isomorphism given by the Alexander Duality Theorem 123.10.

(2) Given a compact oriented proper $(m+1)$-dimensional submanifold W of $[0,1]\times S^{2m+1}$ we denote by

$$\beta_W \colon \mathrm{H}_{m+1}(W, \partial W;\mathbb{Z}) \to \mathrm{H}^m(([0,1]\times S^{2m+1}) \setminus W;\mathbb{Z})$$

the inverse of the isomorphism given by the Alexander Duality Theorem 123.10.

With this notation we can introduce the following definition.

Definition. Let $m \in \mathbb{N}$ and let K and L be two disjoint closed oriented m-dimensional submanifolds of S^{2m+1}. We define the linking number of K and L as

$$\mathrm{lk}(K,L) := \langle \beta_K([K]), [L] \rangle \in \mathbb{Z}$$

where we view the fundamental class $[L]$ of L as a homology class in $\mathrm{H}_m(S^{2m+1} \setminus K;\mathbb{Z})$.

Remark. The vigilant reader will not have failed to notice that in the 3-dimensional setting it is a priori not clear that the above definition of the linking number agrees with

the definition given on page 2865. In Exercise 157.6 we will prove that in the 3-dimensional setting the two competing notions of linking numbers agree.

The following proposition is a generalization of the Linking Number-Symmetry Proposition 131.12.

Proposition 157.19. Let $m \in \mathbb{N}$. Given any two two disjoint closed oriented m-dimensional submanifolds K and L of S^{2m+1} we have the following equality

$$\mathrm{lk}(K,L) \;=\; (-1)^{m+1} \cdot \mathrm{lk}(L,K).$$

Proof. We leave it to the reader to modify the proof of the Linking Number-Symmetry Proposition 131.12 to obtain the desired statement. ■

157.6. **The Hopf invariant and the thickened cobordism group $\Omega^{\mathrm{th}}_{2n-1}(\mathbb{R}^{4n-1})$.** In Corollary 122.14 we had sketched a proof that given any $n \in \mathbb{N}$ there exists an epimorphism $\pi_{4n-1}(S^{2n}) \to \mathbb{Z}$. By the Thom-Pontryagin Theorem 157.11 this statement is equivalent to saying that there exists an epimorphism $\Omega^{\mathrm{th}}_{2n-1}(\mathbb{R}^{4n-1}) \to \mathbb{Z}$.
First of all we work with the same convention as in the last section.

Convention. In this section we equip any thickened submanifold and any thickened cobordism with the natural orientation that we introduced on page 3356.

Proposition 157.20. Let $n \in \mathbb{N}$ and let $* \in \overline{B}^{2n}$. The map

$$\begin{array}{rcl} \mathrm{Hopf}\colon \Omega^{\mathrm{th}}_{2n-1}(\mathbb{R}^{4n-1}) & \to & \mathbb{Z} \\ [(N, g\colon \overline{B}^{2n} \times N \to \mathbb{R}^{4n-1})] & \mapsto & \mathrm{lk}(N, g(* \times N)) \end{array}$$

is a well-defined homomorphism.

Remark. Let $n \in \mathbb{N}$. We recall that on page 2681 we used the cup product to define the Hopf invariant

$$\mathrm{Hopf}\colon \pi_{4n-1}(S^{2n}, *) \;\to\; \mathbb{Z}$$

which is, as shown in Proposition 122.7, in fact a homomorphism. The audacious reader can attempt to show that, under the above isomorphism $\Omega^{\mathrm{th}}_{2n-1}(\mathbb{R}^{4n-1}) \xrightarrow{\cong} \pi_{4n-1}(S^{2n}, *)$, the Hopf invariant and the homomorphism $\mathrm{Hopf}\colon \Omega^{\mathrm{th}}_{2n-1}(\mathbb{R}^{4n-1}) \to \mathbb{Z}$ from Proposition 157.20 agree up to sign.[1946]

Proof. In Exercise 157.7 we will verify that Hopf is a homomorphism. Now let us actually show that Hopf is well-defined. Thus suppose that we are given thickened $(2n-1)$-dimensional submanifolds (N_0, g_0) and (N_1, g_1) of $\mathbb{R}^{4n-1}$ that are cobordant via a thickened cobordism (W, G). We denote by $i_0\colon \{0\} \times S^{4n-1} \to [0,1] \times S^{4n-1}$ the inclusion. Similarly we define i_1. We also use i_0, i_1 in a rather obvious way for restrictions of the above inclusion maps to suitable subsets. It follows immediately from the Alexander Duality

[1946] Attempting to determine the correct sign with confidence might have negative effects on one's mental health.

Theorem 123.10 that we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}_{2n}(W, \partial W; \mathbb{Z}) & \xrightarrow[\beta_W]{\cong} & \mathrm{H}^{2n-1}(([0,1] \times S^{4n-1}) \setminus W; \mathbb{Z}) \\ \downarrow \partial & & i_0^* \\ & & i_1^* \downarrow \\ \mathrm{H}_{2n-1}(\{0\} \times N_0; \mathbb{Z}) & \xrightarrow[\cong]{\begin{pmatrix} \beta_{N_0} & 0 \\ 0 & \beta_{N_1} \end{pmatrix}} & \mathrm{H}^{2n-1}(\{0\} \times (S^{4n-1} \setminus N_0); \mathbb{Z}) \\ \mathrm{H}_{2n-1}(\{1\} \times N_1; \mathbb{Z}) & & \mathrm{H}^{2n-1}(\{1\} \times (S^{4n-1} \setminus N_1); \mathbb{Z}) \end{array}$$

where all horizontal maps are isomorphisms and where the left-hand vertical map is the connecting homomorphism of the long exact sequence in homology of the pair $(W, \partial W)$.

It follows that

$$\begin{array}{rcll} \mathrm{lk}(N_0, g_0(* \times N_0)) & = & \langle \beta_{N_0}([N_0]), [g_0(* \times N_0)] \rangle & \overset{\text{follows from the above diagram and the fact that by Fundamental Class-of-Boundary Proposition 106.27 we know that } [\partial W] = [\{0\} \times N_0] + [\{1\} \times N_1]}{=} \langle i_0^*(\beta_W([W])), [g_0(* \times N_0)] \rangle \\ & \overset{\text{Lemma 109.6 (3)}}{=} & \langle \beta_W([W]), (i_0)_*([g_0(* \times N_0)]) \rangle & \overset{\text{it follows from } \partial(G(* \times W)) = g_0(* \times N_0) \cup g_1(* \times N_1) \text{ together with Corollary 106.28 (2) that } (i_0)_*([g_0(* \times N_0)]) = (i_1)_*([g_1(* \times N_1)])}{=} \langle \beta_W([W]), (i_1)_*([g_1(* \times N_1)]) \rangle \\ & \overset{\text{same argument backwards}}{=} & \mathrm{lk}(N_1, g_1(* \times N_1)). & \end{array}$$

We have thus shown that Hopf is indeed well-defined. ∎

Of course an invariant is only interesting if we can show that it assumes non-trivial values. If $n = 1$, then we already know[1947] that the trivial knot K together with the thickening g of self-linking number one satisfies $\mathrm{Hopf}([(N, g)]) = 1$. In particular we see that $\mathrm{Hopf} \colon \Omega_1^{\mathrm{th}}(\mathbb{R}^3) \to \mathbb{Z}$ is an epimorphism.

Lemma 157.21. For any $n \in \mathbb{N}$ the Hopf invariant $\Omega_{2n}^{\mathrm{th}}(\mathbb{R}^{4n+1}) \to \mathbb{Z}$ is the zero map.

Proof. Let $(N, g \colon \overline{B}^{2n+1} \times N \to \mathbb{R}^{4n+1})$ be a thickened $2n$-dimensional submanifold of $\mathbb{R}^{4n+1}$. We calculate that

$$\overbrace{\mathrm{lk}(N, g(* \times N))}^{=\mathrm{Hopf}([(N,g)])} \overset{\text{Proposition 157.19}}{=} (-1)^{\dim(N)+1} \cdot \mathrm{lk}(g(* \times N), N) \overset{\text{since } N \sqcup g(* \times N) \text{ and } g(* \times N) \sqcup N \text{ are isotopic}}{=} -\overbrace{\mathrm{lk}(N, g(* \times N))}^{=\mathrm{Hopf}([(N,g)])}.$$

This calculation shows that $\mathrm{Hopf}([(N, g)]) = 0$. ∎

Recall that the combination of Propositions 122.7 and 122.13 shows that given any $n \in \mathbb{N}$ there exists a map $f \colon S^{4n-1} \to S^{2n}$ such that the Hopf invariant equals 2. The following proposition that can be viewed as an analogue of this fact.

Proposition 157.22. For any $n \in \mathbb{N}$ there exists a thickened $(2n-1)$-dimensional submanifold (N, g) in $\mathbb{R}^{4n-1}$ with $\mathrm{Hopf}([(N, g)]) = 2$.

Sketch of proof. We consider the "trivial $(2n-1)$-dimensional knot"

$$K \;=\; S^{2n-1} \times \{0\} \;=\; \{(z_1, \ldots, z_{2n}, 0, \ldots, 0) \in \mathbb{R}^{4n-1} \mid z_1^2 + \cdots + z_{2n}^2 = 1\}$$

[1947] Here we assume for convenience that we already know that the two different definitions of the linking pairing in the 3-dimensional setting agree.

together with the map[1948]

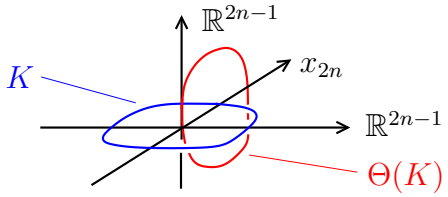
$$\begin{array}{rcl} g\colon \overline{B}^{2n} \times K & \to & \mathbb{R}^{4n-1} \\ \big((x, \underbrace{(y_1,\dots,y_{2n-1})}_{=:y\in\mathbb{R}^{2n-1}}), \underbrace{(z,0)}_{\in K}\big) & \mapsto & (z+\frac{1}{4}x\cdot z, y). \end{array}$$

One can easily verify that, with a suitable orientation for K, the pair (K,g) is a thickened $(2n-1)$-dimensional submanifold of $\mathbb{R}^{4n-1}$. It is not particularly difficult to show that $[g(*\times K)] = 0 \in \mathrm{H}_{2n-1}(\mathbb{R}^{4n-1}\setminus K)$. Unfortunately this implies that $\mathrm{lk}(K, g(*\times K)) = 0$.

The first idea would be to modify the thickening g in a similar vein as in the Knot Thickening Lemma 131.1. This might work in principle, but at least your author does not know how to make this work in general.[1949] Instead we will recycle the example from page 3357 where we saw that the Hopf link gives rise to a non-trivial element in $\Omega_1^{\mathrm{th}}(\mathbb{R}^3)$. More precisely, we consider the diffeomorphism

$$\begin{array}{rcl} \Theta\colon \mathbb{R}^{4n-1} & \to & \mathbb{R}^{4n-1} \\ (\underbrace{x_1,\dots,x_{2n}}_{=x}, \underbrace{y_1,\dots,y_{2n-1}}_{=y}) & \mapsto & (y,x) + (0,\dots,0,\underbrace{1}_{2n\text{-th coordinate}},0,\dots,0). \end{array}$$

Similar to the discussion on page 3357 one can now fairly easily show (voluntary exercise!) that $\mathrm{Hopf}([K\sqcup\Theta(K), g\sqcup\Theta_*(g)]) = \pm 2$. Since Hopf is a homomorphism we see that there exists a thickened $(2n-1)$-dimensional submanifold (N,g) in $\mathbb{R}^{4n-1}$ with $\mathrm{Hopf}([(N,g)]) = 2$. ■



Remark. In Theorem 152.11 we saw that, unless $n = 2, 4, 8$, there does not exist a map $f\colon S^{2n-1} \to S^n$ with $\mathrm{Hopf}(f) = 1$. If we believe for a second that the Hopf invariant corresponds to the above homomorphisms Hopf, then this shows that Proposition 157.22 is optimal for $n \notin \{1,2,4\}$.

In summary, the Thom-Pontryagin Theorem 157.11 together with Propositions 157.20 and 157.22 give us the following result, which is precisely the content of Corollary 122.14.

Proposition 157.23. Given any *even* $n \in \mathbb{N}$ there exists an epimorphism $\pi_{2n-1}(S^n) \to \mathbb{Z}$.

Exercises for Chapter 157.

Exercise 157.1. We consider the map

$$\begin{array}{rcl} f\colon (S^1\times\overline{B}^2)\cup_{S^1\times S^1}(\overline{B}^2\times S^1) & \to & \overline{B}^2/S^1 \\ P & \mapsto & \begin{cases} [Q], & \text{if } P=[(Q,z)] \text{ with } Q\in\overline{B}^2 \text{ and } z\in S^1, \\ [S^1], & \text{if } P=[(z,Q)] \text{ with } z\in S^1 \text{ and } Q\in\overline{B}^2. \end{cases} \end{array}$$

[1948] This map g is basically just the "obvious" thickening of $S^{2n-1}\subset\mathbb{R}^{2n}$ extended to higher dimensions.
[1949] Also the remark after the proof might temper one's optimism.

Show that f is null-homotopic.

Exercise 157.2. Let $(L = L_1 \sqcup \cdots \sqcup L_m, g = g_1 \sqcup \cdots \sqcup g_m)$ be a thickened link. Show that

$$\operatorname{slk}(L, g) \;:=\; \sum_{i \neq j} \operatorname{lk}(L_i, L_j) \;+\; \sum_{i=1}^{m} \operatorname{slk}(L_i, g_i).$$

Exercise 157.3. Let $m \geq 2$. The Thom-Pontryagin Theorem 157.11 implies in particular that $\Omega^{\text{th}}_{m-1}(\mathbb{R}^m) \cong \pi_1(S^m) = 0$. This implies that every thickened submanifold of $\mathbb{R}^m$ of codimension-one is null-cobordant. Prove this statement without making use of the Thom-Pontryagin Theorem 157.11.
Hint. Use Lemma 130.3.

Exercise 157.4. We consider the Hopf link H as shown in the figure below. Draw an explicit oriented connected compact surface whose boundary equals the oriented link H.
Remark. Note that the orientation of one of the components is different from the Hopf link that we had considered above in the figure on page 3358.



Exercise 157.5. Let $L \subset S^3$ be an oriented link and let $G \subset S^3$ be a compact oriented surface with $\partial G = L$. We assume that G has no closed components.

(a) Show that $S^3 \setminus G$ is path-connected.
(b) Give an outline of an argument why there exists a compact oriented *connected* surface F with $\partial F = L$.

Exercise 157.6. Let K and J be two oriented disjoint knots in S^3. Show that the two definitions of the linking number $\operatorname{lk}(K, J)$ given on pages 2865 and 3360 agree.

Exercise 157.7. Show that the map

$$\begin{array}{rcl} \Psi \colon \Omega^{\text{th}}_{2n-1}(\mathbb{R}^{4n-1}) & \to & \mathbb{Z} \\ [(N, g \colon \overline{B}^{2n} \times N \to \mathbb{R}^{4n-1})] & \mapsto & \operatorname{lk}(N, g(* \times N)) \end{array}$$

considered in Proposition 157.20 is a homomorphism.

Exercise 157.8. In Section 157.6 we had just seen that given any even $n \in \mathbb{N}$ there exists an epimorphism $\Omega^{\text{th}}_{n-1}(\mathbb{R}^{2n-1}) \to \mathbb{Z}$. What about the group $\Omega^{\text{th}}_{n-1}(\mathbb{R}^{2n-1})$ if n is odd? Can we use linking numbers to define a homomorphism to $\mathbb{Z}$? Can we use them to find an epimorphism from $\Omega^{\text{th}}_{n-1}(\mathbb{R}^{2n-1})$ to $\mathbb{Z}$?

158. Thom-Pontryagin Theory II: The Freudenthal Suspension Theorem

158.1. **The framed cobordism group.** In Chapter 157 we introduced the thickened cobordism groups $\Omega_k^{\text{th}}(\mathbb{R}^{n+k})$ and we showed that they are isomorphic to $\pi_{n+k}(S^n, *)$. Furthermore we computed that $\Omega_0^{\text{th}}(\mathbb{R}^n) \cong \mathbb{Z}$ and $\Omega_1^{\text{th}}(\mathbb{R}^3) \cong \mathbb{Z}$. Finally we showed that given any $n \in \mathbb{N}$ there exists an epimorphism $\Omega_{2n-1}^{\text{th}}(\mathbb{R}^{4n-1}) \to \mathbb{Z}$. These calculations gave us new proofs for Corollary 85.6, Theorem 155.10 (2) and Corollary 122.14.

Even though these new proofs were very pleasant, it is now time for some new results. In this section we will prove the Freudenthal Suspension Theorem and we will calculate the homotopy groups $\pi_{n+1}(S^n)$ for $n \geq 3$. For the subsequent calculations it is better though to work with the "framed cobordism groups" $\Omega_k^{\text{fr}}(\mathbb{R}^n)$ that we introduce in this section. As we will see, the framed cobordism groups are isomorphic to the thickened cobordism group, but at times they are easier to calculate since they allow us to reduce many issues to linear algebra.

First we introduce the following definitions in linear algebra.

Definition. Given $m, k \in \mathbb{N}_0$ we write

$$\begin{aligned} \mathrm{GL}(m,k) &:= \text{set of } k\text{-tuples of linearly independent vectors of } \mathbb{R}^m \\ \mathrm{O}(m,k) &:= \text{set of } k\text{-tuples of orthonormal vectors of } \mathbb{R}^m. \end{aligned}$$

We adopt the following conventions:

(1) Given $v \in \mathrm{GL}(m,k)$ we denote by $v_1, \ldots, v_k$ the corresponding column vectors.
(2) Throughout we view both $\mathrm{GL}(m,k)$ and $\mathrm{O}(m,k)$ as subsets of the smooth manifold $\mathrm{M}(m \times k, \mathbb{R}) = \mathbb{R}^{mk}$.

Now we can give the topological definition we are really interested in.

Definition. Let $m \in \mathbb{N}_0$ and let N be an m-dimensional submanifold of $\mathbb{R}^m$. (Later on we will only be interested in $N = \mathbb{R}^m$, $N = [0,1] \times \mathbb{R}^{m-1}$ and the case that $N = \overline{B}_r^m(P)$ is a closed m-dimensional ball.)

(1) Let M be a compact orientable proper k-dimensional submanifold of N.
   (a) Let $P \in M$. We say $v = (v_1, \ldots, v_{m-k}) \in \mathrm{GL}(m, m-k)$ is a frame of M at P if the vectors $v_1, \ldots, v_{m-k}$, together with $\mathrm{T}_P M$[1950] spans $\mathbb{R}^m$.
   (b) A framing for M is a smooth[1951] map $v\colon M \to \mathrm{GL}(m, m-k)$ such that for each $P \in M$ the vectors $v_1(P), \ldots, v_{m-k}(P)$ are a frame of M at P.
   (c) We say that a framing $v = (v_1, \ldots, v_{m-k})$ is perpendicular if for each $P \in M$ the vectors $v_1(P), \ldots, v_{m-k}(P)$ lie in the orthogonal complement of $\mathrm{T}_P M$, i.e. if they lie in the vector space

$$(\mathrm{T}_P M)^\perp = \{v \in \mathbb{R}^m \mid \langle v, w \rangle = 0 \text{ for all } w \in \mathrm{T}_P M\}.$$

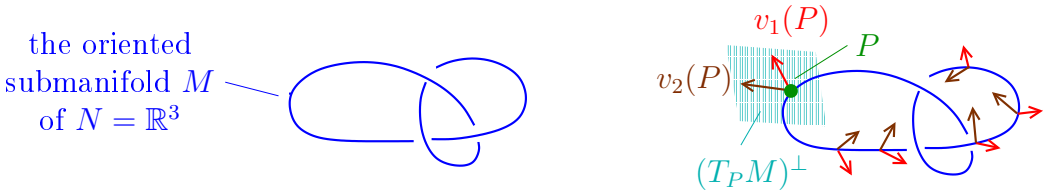
   If the framing is perpendicular and if all the vectors are orthonormal, then we that the framing is orthonormal .
(2) A framed k-dimensional submanifold of N is a pair (M, v) where M is a compact orientable proper k-dimensional submanifold of N and v is a framing for M.
(3) Let (M, v) be a framed k-dimensional submanifold of N and let $\varphi\colon N \to W$ be a smooth embedding of N into a smooth manifold of the same dimension. We consider

the map

$$\begin{array}{rccl} \varphi_*(v)\colon \varphi(M) & \to & \mathrm{GL}(m, m-k) \\ P & \mapsto & (\varphi_*(v_1(\varphi^{-1}(P))), \ldots, \varphi_*(v_{m-k}(\varphi^{-1}(P)))). \end{array}$$

The pair $(\varphi(M), \varphi_*(v))$ is a framed submanifold of W.

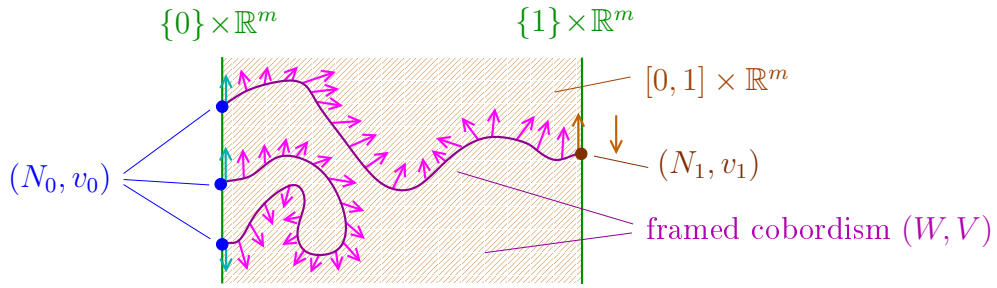
Some of the above definitions are illustrated in the figure below.



Definition. Let $k, m \in \mathbb{N}_0$ and let (N_0, v_0) and (N_1, v_1) be two framed k-dimensional submanifolds of $\mathbb{R}^m$. A framed cobordism between (N_0, v_0) and (N_1, v_1) is a framed $(k+1)$-dimensional submanifold (W, V) of $[0,1] \times \mathbb{R}^m$ such that the following three conditions are satisfied:

(1) $\partial W = (\{0\} \times N_0) \cup (\{1\} \times N_1)$,
(2) $V|_{\{0\}\times N_0} = v_0$ under the identification $\{0\} \times \mathbb{R}^m = \mathbb{R}^m$,
(3) $V|_{\{1\}\times N_1} = v_1$ under the identification $\{1\} \times \mathbb{R}^m = \mathbb{R}^m$,

If such a framed cobordism exists, then we say that (N_0, v_0) and (N_1, v_1) are cobordant. The definition of a framed cobordism is illustrated in the figure below.



Basically the same proof as in Lemma 157.8 gives us the following lemma.

Lemma 158.1. The property of "being cobordant" is an equivalence relation on the set of framed k-dimensional submanifolds of $\mathbb{R}^m$.

This lemma leads us, not surprisingly, to the following definition.

Definition. We denote by $\Omega_k^{\mathrm{fr}}(\mathbb{R}^m)$ the set of cobordism classes of framed k-dimensional submanifolds of $\mathbb{R}^m$. We refer to $\Omega_k^{\mathrm{fr}}(\mathbb{R}^m)$ as a framed cobordism group.

The following proposition, which is the analogue of Proposition 157.9, justifies the name "framed cobordism group".

[1950] Here we use the isomorphism from the Tangent Space-Isomorphism Proposition 23.16 to identify $\mathrm{T}_P M$ with the vector subspace $\widetilde{\mathrm{T}}_P M$ of $\mathbb{R}^m$.

[1951] Since we view $\mathrm{GL}(m, m-k)$ as a subset of the smooth manifold $\mathrm{M}(m \times (m-k), \mathbb{R}) = \mathbb{R}^{m\cdot(m-k)}$ it makes sense to say that a map $M \to \mathrm{GL}(m, m-k)$ is smooth.

Proposition 158.2. Let $k \in \mathbb{N}$ and let $m \in \mathbb{N}_{\geq 2}$. Let (M, v) and (N, w) be two framed k-dimensional submanifolds of $\mathbb{R}^m$. Since M and N are compact we can pick $r, s \in \mathbb{R}_{>0}$ with $M \subset B_r^m(0)$ and $N \subset B_s^m(0)$. Furthermore we pick two orientation-preserving smooth embeddings $\varphi\colon \overline{B}_r^m(0) \to \mathbb{R}^m$ and $\psi\colon \overline{B}_s^m(0) \to \mathbb{R}^m$ with disjoint images. We define

$$[(M, v)] + [(N, w)] := [(\varphi(M) \sqcup \psi(N), \varphi_*(v) \sqcup \psi_*(w))] \in \Omega_k^{\mathrm{fr}}(\mathbb{R}^m).$$

This operation "$+$" has the following properties:

(1) The definition of "$+$" on $\Omega_k^{\mathrm{fr}}(\mathbb{R}^m)$ is well-defined and commutative.
(2) The operation "$+$" defines a group structure on $\Omega_k^{\mathrm{fr}}(\mathbb{R}^m)$. The following statements hold:
   (a) The neutral element is represented by the empty submanifold.
   (b) Let $\rho\colon \mathbb{R}^m \to \mathbb{R}^m$ be the reflection in a hyperplane. The inverse of $[(M, v)]$ is given by $[(\rho(M), \rho_*(v))]$.

Proof. The proof is basically identical to the proof of Proposition 157.9. We leave it to the reader to carry out the minor adjustments that are necessary to make the transition from thickened cobordisms groups to framed cobordism groups. ■

Lemma 158.3. Let $m \in \mathbb{N}_{\geq 2}$ and $k \in \{0, \ldots, m\}$.

(1) Every framed k-dimensional submanifold (M, v) of $\mathbb{R}^m$ is cobordant to (M, w) where w is an orthonormal framing.
(2) Let (W, V) be a framed cobordism between two framed k-dimensional submanifolds (M_0, v_0) and (M_1, v_1) of $\mathbb{R}^m$. If v_0 and v_1 are orthonormal, then there exist an orthonormal framing V' for W such that (W, V') is a framed cobordism between (M_0, v_0) and (M_1, v_1).

Proof. We prove Statement (1) of the lemma. The proof of Statement (2) is a mild variation on the proof of Statement (1). Thus let (M, u) be a framed k-dimensional submanifold of $\mathbb{R}^m$. Using the fact that being cobordant is transitive, the proof of the lemma can be broken up into two stages:

(1) First we show that there exists a perpendicular framing v such that (M, u) is cobordant to (M, v).
(2) Then we show that there exists an orthonormal framing w such that (M, v) is cobordant to (M, w).

We carry out the two steps:

(1) Let $(M, u\colon M \to \mathrm{GL}(m, m-k))$ be a framed k-dimensional submanifold of $\mathbb{R}^m$. Given $P \in M$ we write $U_P M := \mathrm{span}\{u_1(P), \ldots, u_k(P)\}$ and furthermore we write $Z_P M := (\mathrm{T}_P M)^\perp$. For each $P \in M$ the decomposition $\mathbb{R}^m = \mathrm{T}_P M \oplus Z_P M$ defines a projection of $\mathbb{R}^m$ onto $Z_P M$. We denote the restriction of this projection to $U_P M$ by $\rho_P\colon U_P M \to Z_P M$. By definition of a framing this map is an isomorphism. We consider the product cobordism $W = [0, 1] \times M$ equipped with the framing[1952]

$$\begin{array}{rcl} V\colon W = [0,1] \times M & \to & \mathrm{GL}(m, m-k) \\ (t, P) & \mapsto & (u(P) \cdot (1-t) + \rho_P(u(P)) \cdot t). \end{array}$$

[1952]It takes a little bit of thought to show that this is indeed a smooth map.

Basically by definition this is a framed cobordism from (M,u) to (M,v) where $v = V_1$ is a perpendicular framing.

(2) In the Matrix Groups Smooth Deformation Retract Proposition 33.10 (1) we showed that there exists a smooth deformation retraction $R\colon \mathrm{GL}(m,m-k)\times[0,1]\to \mathrm{O}(m,m-k)$ from $\mathrm{GL}(m,m-k)$ to $\mathrm{O}(m,m-k)$ which has the following extra property:

($*$) For any $v=(v_1,\ldots,v_{m-k})\in \mathrm{GL}(m,m-k)$ and any $t\in[0,1]$ we have

$$\text{span of } R_t(v_1),\ldots,R_t(v_{m-k}) \;=\; \text{span of } v_1,\ldots,v_{m-k}.$$

Now let (M,v) be a framed k-dimensional submanifold. We consider the product cobordism $W=[0,1]\times M$ equipped with the framing

$$\begin{array}{rcl} V\colon W=[0,1]\times M & \to & \mathrm{GL}(m,m-k) \\ (t,P) & \mapsto & R_t(v(P)). \end{array}$$

It follows from the fact that R is smooth and that ($*$) holds that this is a framed cobordism from (M,v) to (M,w) where $w=V_1$ is an orthonormal framing. ■

The following proposition shows that the two types of cobordism groups that we introduced in this and the last chapter are in fact isomorphic.

Proposition 158.4. Let $m\in\mathbb{N}_{\geq 2}$ and let $k\in\{0,\ldots,m\}$. The map

$$\begin{array}{rcl} \Phi\colon \Omega_k^{\mathrm{th}}(\mathbb{R}^m) & \to & \Omega_k^{\mathrm{fr}}(\mathbb{R}^m) \\ [(N, g\colon \overline{B}^{m-k}\times N)] & \mapsto & \left[\begin{array}{l} N \text{ equipped with the framing that associates to } P\in N \\ \text{the frame } \mathrm{D}f_{(0,P)}(e_1),\ldots,\mathrm{D}f_{(0,P)}(e_{m-k}) \end{array}\right] \end{array}$$

is well-defined and it is an isomorphism.

Convention. Let $n,k\in\mathbb{N}_0$ with $n+k\geq 2$. We use the isomorphisms from the Thom-Pontryagin Theorem 157.11 and Proposition 158.4 to make the identifications

$$\pi_{n+k}(S^n,*) \;=\; \Omega_k^{\mathrm{th}}(\mathbb{R}^{n+k}) \;=\; \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}).$$

Sketch of proof.

(1) It follows basically from the definitions that the framing associated to a thickened submanifold is indeed a framing. Applying the same procedure to a thickened cobordism between thickened submanifolds we obtain a framed cobordism between framed submanifolds. This shows that Φ is well-defined.

(2) It follows immediately from the definitions of the group structures that Φ is a homomorphism.

(3) We sketch the construction of the inverse to Φ. Let $(N,\{v_1,\ldots,v_{m-k}\})$ be a framed k-dimensional submanifold of $\mathbb{R}^m$. We consider the map

$$\begin{array}{rcl} g(v)\colon \mathbb{R}^{m-k}\times N & \to & \mathbb{R}^m \\ ((t_1,\ldots,t_{m-k}),P) & \mapsto & P+t_1\cdot v_1(P)+\cdots+t_{m-k}\cdot v_{m-k}(P). \end{array}$$

The same argument as in the proof of Theorem **??** shows that there exists an $\epsilon>0$ such that $g(v)$ restricts to a smooth embedding $\overline{B}_\epsilon^k(0)\times N\to\mathbb{R}^m$. After rescaling the ball we thus get a thickening $\overline{B}^k\times N\to\mathbb{R}^m$ of N. It is straightforward to see

that the class $[(N, g(v)] \in \Omega_k^{\mathrm{th}}(\mathbb{R}^m)$ does not depend on the choice of $\epsilon > 0$. Since the same approach also works for cobordisms we obtain a well-defined map

$$\Psi\colon \Omega_k^{\mathrm{fr}}(\mathbb{R}^m) \ \to \ \Omega_k^{\mathrm{th}}(\mathbb{R}^m).$$

It remains to show that Φ and Ψ are inverses of one another. It follows basically immediately from the definitions and Lemma 158.3 (1) that $\Phi \circ \Psi$ is the identity on $\Omega_k^{\mathrm{fr}}(\mathbb{R}^m)$. The fact that $\Psi \circ \Phi$ is the identity on $\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$ follows fairly immediately from the uniqueness of tubular neighborhoods, see Proposition **??**, together with Lemma **??**. ■

Definition. Let N be an orientable k-dimensional submanifold of $\mathbb{R}^m$ and let $n \geq m$.

(1) We write $N_n := N \times \{0\} \subset \mathbb{R}^n$.

Now suppose we are given a framing $v\colon N \to \mathrm{GL}(m, m-k)$ for N.

(2) We define

$$\begin{array}{rcl} v_n\colon N_n & \to & \mathrm{GL}(n, n-k) \\ (P, 0) & \mapsto & \begin{pmatrix} v(P) & 0 \\ 0 & \mathrm{id}_{n-m} \end{pmatrix}. \end{array}$$

It is clear that v_n is a framing for N_n. Note that $N_m = N$ and $v_m = v$.

(3) We refer to (N_n, v_n) as the $(n-m)$-fold suspension of (N, v). In particular we refer to (N_{m+1}, v_{m+1}) as the suspension of (N, v).

The same notation applies in an obvious way to (framed) submanifolds of $[0,1] \times \mathbb{R}^m$.

The following lemma justifies the name "suspension" that we had used in the above definition.

Lemma 158.5. Let $m \in \mathbb{N}_{\geq 2}$ and let $k \in \{0, \dots, m\}$.

(1) The map

$$\begin{array}{rcl} \Omega_k^{\mathrm{fr}}(\mathbb{R}^m) & \to & \Omega_k^{\mathrm{fr}}(\mathbb{R}^{m+1}) \\ [(N, v)] & \mapsto & [\text{the suspension of } (N, v)] \end{array}$$

is well-defined and it is a homomorphism. We refer to it as the *suspension homomorphism*.

(2) Under the identification from the convention on page 3368 the suspension homomorphisms on $\pi_{n+k}(S^n, *)$ and $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k})$ agree up to sign,[1953] in other words, the following diagram commutes up to sign:

$$\begin{array}{ccc} \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) & \xrightarrow{\text{suspension}} & \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1}) \\ \downarrow \cong & & \downarrow \cong \\ \pi_{n+k}(S^n, *) & \xrightarrow{\text{suspension}} & \pi_{n+k+1}(S^{n+1}, *). \end{array}$$

Proof.

(1) The first statement is almost obvious.

(2) We leave it to the reader to go through all the definitions to verify this statement. ■

[1953] To be more precise, they might agree on the nose, but we will not spend valuable time on figuring out the sign.

158.2. **The Freudenthal Suspension Theorem.** In this section we will prove the following theorem that was first proved in 1937 by Hans Freudenthal [**Freu1937**, p. 300].

Theorem 158.6. (Freudenthal Suspension Theorem) Let $n, k \in \mathbb{N}_0$. We suppose that $n + k \geq 2$. The suspension homomorphism $\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ from the Suspension Homomorphism Lemma 71.19 is an isomorphism if $n > k + 1$ and it is an epimorphism if $n = k + 1$.

Remark.

(1) The Freudenthal Suspension Theorem 158.6 can be reformulated as follows: given $n \in \mathbb{N}$ the suspension homomorphism $\pi_m(S^n) \to \pi_{m+1}(S^{n+1})$ is an isomorphism if $m < 2n-1$ and that it is an epimorphism if $m = 2n-1$. We consider two situations in slightly more detail:
   (a) The Freudenthal Suspension Theorem 158.6 says that the suspension homomorphism $\pi_3(S^2) \to \pi_4(S^3)$ is an epimorphism and it says that for any $n \geq 3$ the suspension homomorphism $\pi_{n+1}(S^n) \to \pi_{n+2}(S^{n+1})$ is an isomorphism. This special case will play an important role shortly.
   (b) The Freudenthal Suspension Theorem 158.6 almost recovers Theorems 155.11 (2) and 155.14 (2). Recall that these theorems stated that for $m \in \{0, \ldots, 5\}$ we have $\pi_m(S^3) \cong \pi_{m+1}(S^4)$ and that for $m \in \{0, \ldots, 13\}$ we have $\pi_m(S^7) \cong \pi_{m+1}(S^8)$.

(2) Let $n \in \mathbb{N}_0$. Recall that on page 452 we saw that the suspension $\Sigma(S^n)$ is homeomorphic to S^{n+1}. Thus we see that the Freudenthal Suspension Theorem 158.6 is a consequence of the following stronger statement: Let X be an $(n-1)$-connected space and let $* \in X$ be a good base point.[1954] Then the suspension homomorphism $\pi_{n+k}(X, *) \to \pi_{n+k+1}(\Sigma(X), *)$ is an isomorphism for all $n > k + 1$ and it is an epimorphism for $n = k + 1$. This statement is proved in most textbooks on homotopy theory, see e.g. [**Gra1975**, Theorem 16.34], [**WhdG1978**, Theorem VII.7.13] or [**DaK2001**, Theorem 8.7 and Chapter 10.3]. If X is a CW-complex, then the statement is also proved in [**Hat2002**, Corollary 4.24]. The proof of [**Hat2002**, Corollary 4.24] is actually a corollary to the Blakers-Massey Theorem 153.17 that we had stated but not proved.

It follows from Lemma 158.5 that the Freudenthal Suspension Theorem 158.6 is equivalent to the following theorem.

Theorem 158.7. Let $n, k \in \mathbb{N}$.

(1) The suspension homomorphism $\Omega^{\mathrm{fr}}_k(\mathbb{R}^{n+k}) \to \Omega^{\mathrm{fr}}_k(\mathbb{R}^{n+k+1})$ is an epimorphism whenever $n \geq k + 1$.
(2) The suspension homomorphism $\Omega^{\mathrm{fr}}_k(\mathbb{R}^{n+k}) \to \Omega^{\mathrm{fr}}_k(\mathbb{R}^{n+k+1})$ is a monomorphism whenever $n > k + 1$.

The basic behind the proof of Theorem 158.7 (1) is quite simple: given a framed k-dimensional submanifold of $\mathbb{R}^{n+k+1}$ we want to use the "room provided by a large enough n" to show that it is cobordant to a framed submanifold coming from $\mathbb{R}^{n+k} = \mathbb{R}^{n+k} \times \{0\}$. This idea is for the most part carried out in the following two lemmas.

[1954] We refer to page 1171 for the definition of a good point.

Lemma 158.8. Let (M, v) be a framed k-dimensional submanifold of $\mathbb{R}^{n+k+1}$. If $n \geq k+1$, then there exists a framed cobordism from (M, v) to a framed k-dimensional submanifold (N, w) of $\mathbb{R}^{n+k+1}$ with $N \subset \mathbb{R}^{n+k} \times \{0\}$.

Proof. Let $(M, v\colon M \to \mathrm{GL}(n+k+1, n+1))$ be a framed k-dimensional submanifold of $\mathbb{R}^{n+k+1}$. We point out that by Lemma 158.3 (1) we can assume that v is an orthonormal framing.

It follows from $n \geq k+1$ and Lemma 30.8 that there exists a vector $\nu \in \mathbb{R}^{n+k+1}$ which satisfies the following condition:

(∗) the vector ν does not lie in any tangent space $\mathrm{T}_P M$[1955] and ν is not secant to M, i.e. for any two distinct points $x, y \in M$ the difference $x - y$ is not parallel to ν.

By the Orthogonal Action Lemma 5.26 (4) there exists a matrix $A \in \mathrm{SO}(n+k+1)$ with $A \cdot \nu = e_{n+k+1}$. By the Diffeotopies-via-Matrices Lemma 33.8 there exists a diffeotopy of $\mathbb{R}^{n+k+1}$ from $\mathrm{id}_{\mathbb{R}^{n+k+1}}$ to the diffeomorphism given by multiplying by A. Therefore, by the obvious analogue of Lemma 157.7 for the framed cobordism group, we can assume without loss of generality that $\nu = e_{n+k+1}$.

Claim. The map

$$\begin{array}{rcl} \Psi\colon [0,1] \times M & \to & [0,1] \times \mathbb{R}^{n+k+1} \\ (t, \underbrace{(x_1, \ldots, x_{n+k+1})}_{\in M}) & \mapsto & (t, \underbrace{x_1, \ldots, x_{n+k}, (1-t) \cdot x_{n+k+1}}_{\in \mathbb{R}^{n+k+1}}) \end{array}$$

is a proper smooth embedding. The definition of Ψ is illustrated in the figure below.

Proof. It follows from the fact that for any two distinct points $x, y \in M$ the difference $x - y$ is not parallel to $\nu = e_{n+k+1}$ that the map is injective. Furthermore note that the map Ψ is the composition of the inclusion $i\colon [0,1] \times M \to [0,1] \times \mathbb{R}^{n+k+1}$ with the map

$$\begin{array}{rcl} \Theta\colon [0,1] \times \mathbb{R}^{n+k+1} & \to & [0,1] \times \mathbb{R}^{n+k+1} \\ (t, (x_1, \ldots, x_{n+k+1})) & \mapsto & (t, x_1, \ldots, x_{n+k}, (1-t) \cdot x_{n+k+1}). \end{array}$$

A straightforward calculation shows that

$$D\Theta_{(t, x_1, \ldots, x_{n+k+1})} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{id}_{n+k} & 0 \\ -x_{n+k+1} & 0 & 1-t \end{pmatrix}.$$

It follows immediately from this calculation and the fact that e_{n+k+1} is not contained in any tangent space $\mathrm{T}_P M$ that $\Psi = \Theta \circ i$ is an immersion. Together with the injectivity of Θ, the fact that M is compact and the ever-popular the Compact-Hausdorff Proposition 2.17 (2) we obtain that $\Psi = \Theta \circ i$ is in fact a smooth embedding. It is also clear that the smooth embedding is proper. ⊞

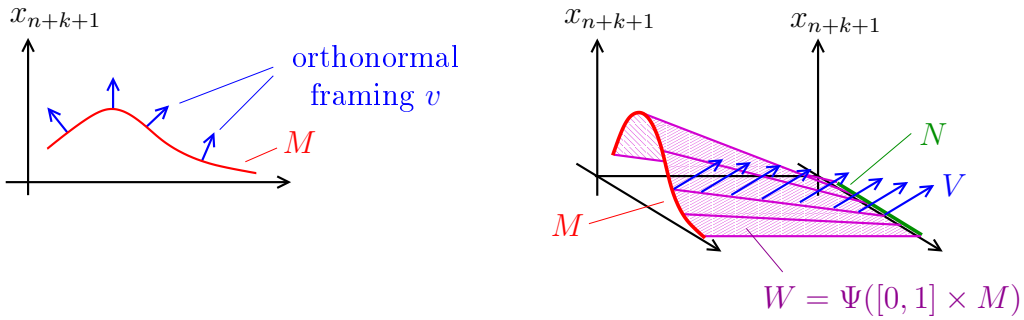
We set $W := \Psi([0,1] \times M)$. By the claim together with the Smooth Embedding Theorem 24.10 (2) we know that W is a proper $(k+1)$-dimensional submanifold of $[0,1] \times \mathbb{R}^{n+k+1}$. Basically by definition W is a cobordism between $M \subset \mathbb{R}^{n+k+1}$ and some submanifold $N \subset \mathbb{R}^{n+k} \times \{0\}$.

[1955] Here and throughout this proof we use the Tangent Space-Isomorphism Proposition 23.16 to view the tangent spaces of submanifolds of some $\mathbb{R}^m$ as vector subspaces of $\mathbb{R}^m$.

It remains to extend the orthonormal framing $v\colon M \to \mathrm{GL}(n+k+1, n+1)$ of M to a framing of W. In fact, we obtain a framing of $W = \Psi([0,1] \times M)$ by using the "same vectors" as for M. More precisely, we consider the map

$$\begin{array}{rcl} V\colon W = \Psi([0,1]\times M) & \to & \mathrm{GL}(n+k+2, n+1) \\ \Psi(t,P) & \mapsto & \begin{pmatrix} 0 \\ v(P) \end{pmatrix}. \end{array}$$

Evidently under the identification $\{0\} \times \mathbb{R}^{n+k+1} = \mathbb{R}^{n+k+1}$ we have $V|_M = v$. Thus it remains to show that V is indeed a framing. A short moment's thought shows that V is smooth. The definition of W and V are illustrated in the figure below.



Therefore it suffices to prove the following claim.

Claim. For any $t \in [0,1]$ and any $P \in M$ the matrix $V(\Psi(t,P)) \in \mathrm{GL}(n+k+2, n+1)$ is a frame for $W = \Psi([0,1] \times M)$ at $\Psi(t,P)$.

Proof. Let $t \in [0,1]$ and let $P \in M$. We point out that the columns of $V(\Psi(t,P))$ span an $(n+1)$-dimensional vector subspace of $\mathbb{R}^{n+k+2}$. Since $\mathrm{T}_{\Phi(t,P)}(W)$ is a vector subspace of the complementary dimension $k+1$ it suffices to show that the intersection of these two vector subspaces is trivial.

We note that

$$\underbrace{\mathrm{T}_{\Phi(t,P)}(W) \cap \begin{array}{c}\text{span of the}\\ \text{columns of}\\ V(\Psi(t,P))\end{array}}_{\subset \mathbb{R}^{1+n+k+1}} \overset{\text{by definition of } W = \Psi([0,1]\times M) = (\Theta \circ i)([0,1]\times M) \text{ and the calculation of } D\Theta}{\downarrow} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{id}_{n+k} & 0 \\ -x_{n+k+1} & 0 & 1-t \end{pmatrix} \cdot \begin{pmatrix} \mathbb{R} \\ \oplus \\ \mathrm{T}_P M \end{pmatrix} \cap \begin{array}{c}\text{span of the}\\ \text{columns of}\\ V(\Psi(t,P))\end{array}}_{\subset \mathbb{R}^{1+n+k+1}}$$

$$\cong \underbrace{\begin{pmatrix} \mathrm{id}_{n+k} & 0 \\ 0 & 1-t \end{pmatrix} \cdot \mathrm{T}_P(M) \cap \begin{array}{c}\text{span of the}\\ \text{columns of } v(P)\end{array}}_{\subset \mathbb{R}^{n+k+1}} = \begin{pmatrix} \mathrm{id}_{n+k} & 0 \\ 0 & 1-t \end{pmatrix} \cdot \mathrm{T}_P(M) \cap (\mathrm{T}_P M)^\perp.$$

$\uparrow$ since the first row of V is zero we can "ignore" the first coordinate

$\uparrow$ since v is an orthonormal framing

Now let $u \in \mathrm{T}_P M$ be a non-zero vector. We write $u = \begin{pmatrix} u' \\ u_{n+k+1} \end{pmatrix}$ with $u' \in \mathbb{R}^{n+k}$ and $u_{n+k+1} \in \mathbb{R}$. Note that u' is non-zero since we had arranged that $e_{n+k+1} \notin \mathrm{T}_P M$. Now we

calculate that

$$\left\langle \begin{pmatrix} \mathrm{id}_{n+k} & 0 \\ 0 & 1-t \end{pmatrix} \begin{pmatrix} u' \\ u_{n+k+1} \end{pmatrix}, \underbrace{\begin{pmatrix} u' \\ u_{n+k+1} \end{pmatrix}}_{=u\in \mathrm{T}_P M} \right\rangle = \langle u', u' \rangle + (1-t)\cdot u_{n+k+1}^2 > 0.$$

since $u' \neq 0$ and $(1-t) \geq 0$

But this shows that $\begin{pmatrix} \mathrm{id}_{n+k} & 0 \\ 0 & 1-t \end{pmatrix} \cdot u$ is not orthogonal to $\mathrm{T}_P M$, in other words, this vector does not lie in $(\mathrm{T}_P M)^\perp$. ■

Lemma 158.9. Let (M, v) be a framed k-dimensional submanifold in $\mathbb{R}^{n+k} \times \{0\} \subset \mathbb{R}^{n+k+1}$. If $n \geq k+1$, then there exists a framed cobordism to $(M, w = (w_1, \ldots, w_{n+1}))$ where we have $w_{n+1}(P) = e_{n+k+1}$ for all $P \in M$.

Proof. Let $(M, u = (u_1, \ldots, u_{n+1})\colon M \to \mathrm{GL}(n+k+1, n+1))$ be a framed k-dimensional submanifold that is contained in $\mathbb{R}^{n+k} \times \{0\} \subset \mathbb{R}^{n+k+1}$. By Lemma 158.3 (1) and Lemma 158.1 we can assume that u is orthonormal. We write $e := e_{n+k+1}$.

Claim 1. There exists an orthonormal framing $v = (v_1, \ldots, v_{n+1})$ for M such that (M, u) is cobordant to (M, v) and such that for every $P \in M$ we have $v_{n+1}(P) \neq -e$.

Proof. Let $x \in M$. Since M is contained in $\mathbb{R}^{n+k} \times \{0\}$ we know that e is orthogonal to $\mathrm{T}_x M$. Therefore we can uniquely write $e = \sum_{i=1}^{n+1} \lambda_i(x) \cdot u_i(P)$. Since $\|e\| = 1$ and since u is an orthonormal framing we actually have $\sum_{i=1}^{n+1} \lambda_i^2(P) = 1$. Thus we obtain a map

$$\begin{array}{rcl} \lambda\colon M & \to & S^n \\ x & \mapsto & (\lambda_1(x), \ldots, \lambda_{n+1}(x)) \end{array}$$

which is fairly easily seen to be smooth. If $(0, \ldots, 0, -1) \notin \lambda(M)$, then we can just take $u = v$ and we are done.[1956] If not we proceed as follows. Since $\dim(M) = k < n$ it follows from the Full Measure-Density Proposition 30.3 that there exists a $\nu \in S^n$ with $\nu \notin \lambda(M)$. By the Orthogonal Action Lemma 5.26 (4) there exists a matrix $A \in \mathrm{SO}(n+1)$ with $\nu = A \cdot (-e) \in S^n$. By the Matrix Group Path-Component Proposition 2.37 and the Smooth Path-Connectivity Proposition 19.29 there exists a smooth path $\gamma\colon [0,1] \to \mathrm{SO}(n+1)$ from id to A. The desired framed cobordism is now given by the smooth manifold $[0,1] \times M$ together with the framing[1957]

$$\begin{array}{rcl} [0,1] \times M & \to & \mathrm{GL}(n+k+2, n+1) \\ (t, x) & \mapsto & \underbrace{\begin{pmatrix} 0 \\ u(x) \end{pmatrix}}_{\in \mathrm{GL}(n+k+2, n+1)} \cdot \gamma(t). \end{array}$$

⊞

[1956] Indeed, if $v_{n+1}(x) = -e$, then $e = -v_{n+1}(x)$, i.e. $\lambda(x) = (0, \ldots, 0, -1)$.

[1957] That this framing has the correct properties can be seen as follows. Let v be the new framing at $\{1\} \times M$. If we denote by $\lambda_u, \lambda_v\colon M \to S^n$ the corresponding maps, then one easily sees that for each $x \in M$ we have $\lambda_v(x) = A^{-1} \cdot \lambda_u(x)$. Thus the fact that ν does not lie in the image of λ_u implies that $-e = A^{-1}\nu$ does not lie in the image of λ_v.

As a final step we want to "rotate v_{n+1} into e". To do this smoothly for every point we need to introduce two maps. Given $u \in \mathbb{R}^{n+1} \setminus \{0\}$ we write

$$\underbrace{\begin{array}{rcl} T_u \colon \mathbb{R}^{n+k+1} & \to & \mathbb{R}^{n+k+1} \\ y & \to & 2 \cdot \langle y, u \rangle \cdot u - y \end{array}}_{\text{reflection in the line } \mathbb{R} \cdot u} \quad \text{and} \quad \underbrace{\begin{array}{rcl} R_u \colon \mathbb{R}^{n+k+1} & \to & \mathbb{R}^{n+k+1} \\ y & \mapsto & T_e(T_u(y)). \end{array}}_{\text{orientation-preserving diffeomorphism}}$$

We will need the following elementary observation from linear algebra.

Claim 2. Let $w \in \mathbb{R}^{n+k+1}$ with $\|w\| = 1$. Let $u = \frac{w+e}{\|w+e\|}$ be the normalized midpoint between w and e. Then $R_u(w) = e$.

Proof. We have

$$R_u(w) \;=\; T_e(T_u(w)) \;=\; T_e\Big(2 \cdot \Big\langle w, \tfrac{w+e}{\|w+e\|} \Big\rangle \cdot \tfrac{w+e}{\|w+e\|} - w\Big) \underset{\text{short calculation using } \langle w,w\rangle = \langle e,e\rangle}{\uparrow} = T_e(e) \;=\; e.$$

⊞

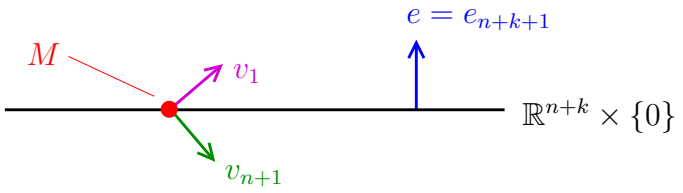
The above claim now allows us to come to the denouement of the argument. More precisely, given $t \in [0,1]$ and $x \in M$ we write

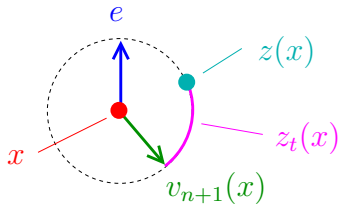
$$z(x) \;:=\; \underbrace{\frac{v_{n+1}(x)+e}{\|v_{n+1}(x)+e\|}}_{\text{note that } v_{n+1}(x)+e \neq 0} \quad \text{and} \quad z_t(x) \;=\; \underbrace{\frac{(1-t)\cdot e + t \cdot z(x)}{\|(1-t)\cdot e + t \cdot z(x)\|}}_{\text{note that } t \cdot e + (1\text{-}t) \cdot z(x) \neq 0}.$$

Note that $z_0 = e$ and $R_{z_0} = \mathrm{id}$. Furthermore note that $z_1 = z(x)$. Thus we obtain from the above claim that $R_{z_1(x)}(v_{n+1}(x)) = e$. Finally we consider the framed cobordism that is given by the submanifold $Z = [0,1] \times M$ of $[0,1] \times \mathbb{R}^{n+k+1}$ together with the framing

$$\begin{array}{rcl} W \colon Z = [0,1] \times M & \to & \mathrm{GL}(n+k+2, n+1) \\ (t,x) & \mapsto & \begin{pmatrix} 0 \\ R_{z_t(x)} \circ v(x) \end{pmatrix}. \end{array}$$

We leave it to the reader to verify that the map W is indeed a framing for the cobordism $Z = [0,1] \times M \subset [0,1] \times \mathbb{R}^{n+k+1}$. The discussion above shows that $(Z = [0,1] \times M, W)$ is a framed cobordism from (M, v) to a framed smooth manifold $(M, w = w_1, \ldots, w_n, e)$. ■





Now we turn to the actual proof of Theorem 158.7 and thus of the Freudenthal Suspension Theorem 158.6.

Proof. Let $n, k \in \mathbb{N}_0$ with $n + k \geq 2$.

(1) Suppose that $n \geq k+1$. We want to show that the suspension homomorphism $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1})$ is an epimorphism. Thus let (M, v) be a framed k-dimensional submanifold of $\mathbb{R}^{n+k+1}$. It follows immediately from Lemmas 158.8 and 158.9 that (M, v) is cobordant to a k-dimensional framed submanifold (N, w) such that $N \subset \mathbb{R}^{n+k} \times \{0\}$ and such that $w_{n+1}(P) = e_{n+k+1}$ for all $P \in N$. But this

means that $[(M,v])] = [(N,w)]$ lies in the image of the suspension homomorphism. Thus the suspension homomorphism $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1})$ is an epimorphism.

(2) Suppose that $n > k+1$. We want to show that the suspension homomorphism $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1})$ is a monomorphism. Therefore let (M,v) and (N,w) be two framed k-dimensional submanifolds of $\mathbb{R}^{n+k}$. Suppose there exists a framed cobordism (W,V) in $[0,1]\times\mathbb{R}^{n+k+1}$ between the suspensions of (M,v) and (N,w). Since $n > k+1$, an argument as in Lemmas 158.8 and 158.9 shows that we can turn (W,V) into a cobordism (W',V') with $W' \subset [0,1]\times\mathbb{R}^{n+k}\times\{0\}$ and such that $V'(P) = e_{n+k+1}$ for all $P \in W'$. But this shows that (M,v) and (N,w) are already cobordant. Thus we have shown that for $n > k+1$ the suspension homomorphism $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1})$ is a monomorphism. ■

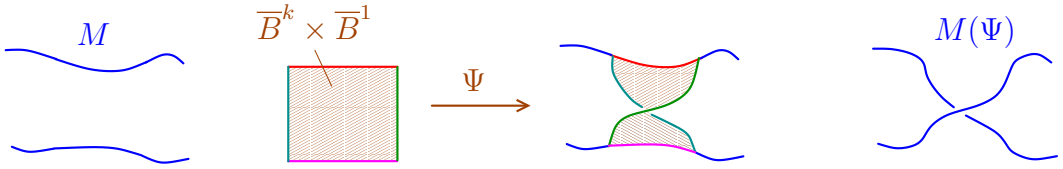
158.3. **Framed internal surgery.** We start out with the following general definition that is somewhat similar to the construction of the internal connected sum that we gave in Section 102.7.

Definition. Let M be a k-dimensional submanifold of $\mathbb{R}^{n+k}$. Let $\Psi\colon \overline{B}^k \times \overline{B}^1 \to \mathbb{R}^{n+k}$ be a smooth embedding with the following properties:

(1) $M \cap \Psi(\overline{B}^k \times \overline{B}^1) = \Psi(\overline{B}^k \times S^0)$,

(2) M intersects $\Psi(S^k \times \overline{B}^1)$ transversally.

We refer to
$$M(\Psi) := \underbrace{\Psi(S^k \times \overline{B}^1) \cup (M \setminus \Psi(B^k \times S^0))}_{\text{with corner } \Psi(S^{k-1}\times S^0) \text{ rounded as in Section 102.7}}$$
as the internal surgery along Ψ.



Remark. Note that $M(\Psi)$, viewed as a smooth manifold in its own right, is diffeomorphic to the surgery of M along $\Psi|_{\overline{B}^k\times S^0}$ as defined on page **??**.

Proposition 158.10. Let (M,v) be a framed k-dimensional submanifold of $\mathbb{R}^{n+k}$. Note that the framing and the orientation induced an orientation on M. Let $\Psi\colon \overline{B}^k\times\overline{B}^1 \to \mathbb{R}^{n+k}$ be a smooth embedding with the following properties:

(1) $M \cap \Psi(\overline{B}^k \times \overline{B}^1) = \Psi(\overline{B}^k \times S^0)$,

(2) M intersects $\Psi(S^k \times \overline{B}^1)$ transversally,

(3) the map
$$\Psi\colon \underbrace{\overline{B}^k \times S^0}_{\text{product orientation}} \to M$$
is orientation-preserving.

Then $M(\Psi)$ admits a framing w such that (M,v) and $(M(\Psi),w)$ are framed cobordant.

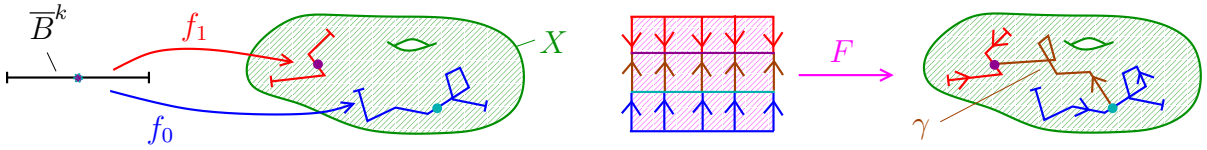
The proof of Proposition 158.10 makes use of the following lemma, which is basically the same statement as Exercise 14.2.

Lemma 158.11. Let X be a topological space and let $f_0, f_1 \colon \overline{B}^k \to X$ be two maps. If X is path-connected, then there exists a map $F \colon \overline{B}^k \times [0,1] \to X$ with $F_0 = f_0$ and with $F_1 = f_1$.

Proof of Lemma 158.11. Since X is path-connected there exists a map $\gamma \colon [\frac{1}{3}, \frac{2}{3}] \to X$ with $\gamma(\frac{1}{3}) = f_0(0)$ and with $\gamma(\frac{2}{3}) = f_1(0)$. Now we consider the map

$$\begin{array}{rcl} F \colon \overline{B}^k \times [0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} f_0(x \cdot (1-3t)), & \text{if } t \in [0, \frac{1}{3}], \\ \gamma(t), & \text{if } t \in [\frac{1}{3}, \frac{2}{3}], \\ f_1(x \cdot (3 \cdot t - 2)), & \text{if } t \in [\frac{2}{3}, 1]. \end{cases} \end{array}$$

It follows from the Homotopy Stacking Lemma 14.3 that F is continuous. It is clear that F has all the desired properties. ∎



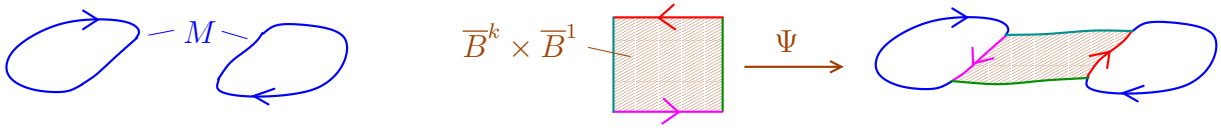
Sketch of a proof of Proposition 158.10. Let (M, v) be a framed k-dimensional submanifold of $\mathbb{R}^{n+k}$ and let $\Psi \colon \overline{B}^k \times \overline{B}^1 \to \mathbb{R}^{n+k}$ be a smooth embedding with the following properties:

(1) $M \cap \Psi(\overline{B}^k \times \overline{B}^1) = \Psi(\overline{B}^k \times S^0)$,
(2) M intersects $\Psi(S^k \times \overline{B}^1)$ transversally,
(3) the map

$$\Psi \colon \underbrace{\overline{B}^k \times S^0}_{\text{product orientation}} \to M$$

is orientation-preserving.

Note that by Lemma 158.3 we can assume that v is an orthonormal framing.

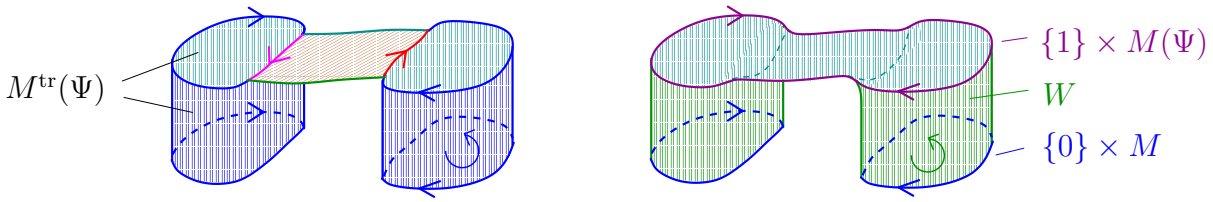


Similar to the discussion on page **??** and the proof of the Trace-of-Surgery Proposition **??** we consider the trace of the surgery

$$M^{\mathrm{tr}}(\Psi) \;:=\; ([0,1] \times M) \cup (\{1\} \times \Psi(\overline{B}^k \times \overline{B}^1)) \qquad \subset\; [0,1] \times \mathbb{R}^{n+k}.$$

In the Trace-of-Surgery Proposition **??** we saw that $M^{\mathrm{tr}}(\Psi)$ is a $(k+1)$-dimensional smooth manifold and we saw that we can equip $M^{\mathrm{tr}}(\Psi)$ with the orientation coming from the product orientation on $[0,1] \times M$. If we view $M^{\mathrm{tr}}(\Psi)$ as an oriented abstract smooth manifold, then its boundary is $(\{1\} \times M(\Psi)) \sqcup -(\{0\} \times M)$. Similar to the definition on

page ?? and Lemma ?? we can push $\Psi(B^k \times \overline{B}^1) \subset M^{\mathrm{tr}}(\Psi)$ into $(0,1) \times \mathbb{R}^{n+k}$ while keeping $\Psi(S^{k-1} \times \overline{B}^1)$ fixed. After rounding corners we obtain the desired cobordism.[1958]



For simplicity we now write $W = ([0,1] \times M) \cup \Psi(\overline{B}^k \times \overline{B}^1)$. Note that $\Psi(\overline{B}^k \times \overline{B}^1)$ is contractible and compact. Therefore it follows from the Vector Bundles-Homotopies Proposition 143.13 that the normal bundle of $\Psi(\overline{B}^k \times \overline{B}^1)$ is trivial. We pick a trivialization. Once we pick a trivialization an orthonormal framing is the same as a map $\Psi(\overline{B}^k \times \overline{B}^1) \to \mathrm{O}(n)$. Since $\Psi(\overline{B}^k \times \{0\})$ is connected we can as well assume that the map $\Psi(\overline{B}^k \times \{0\}) \to \mathrm{O}(n)$ takes values in $\mathrm{SO}(n)$. It follows from our hypothesis (3) that the map $\Psi(\overline{B}^k \times \{1\}) \to \mathrm{O}(n)$ also takes values in $\mathrm{SO}(n)$.

By Lemma 2.38 we know that $\mathrm{SO}(n)$ is path-connected. Thus it follows from Lemma 158.11 that the framing $\Psi(\overline{B}^k \times \{0,1\}) \to \mathrm{O}(n)$ extends to a framing $\Psi(\overline{B}^k \times [0,1]) \to \mathrm{O}(n)$. ■

158.4. **Suspension and the Hopf invariant.** The following theorem was also proved in 1937 by Hans Freudenthal [**Freu1937**, Satz I].

Theorem 158.12. (Freudenthal Suspension Theorem II) Let $n \in \mathbb{N}$. As above let $\Sigma\colon \pi_{2n}(S^n) \to \pi_{2n+1}(S^{n+1})$ be the suspension homomorphism and let $\mathrm{Hopf}\colon \pi_{2n+1}(S^{n+1}) \to \mathbb{Z}$ be Hopf invariant. With this notation we have

$$\mathrm{Im}(\Sigma\colon \pi_{2n}(S^n) \to \pi_{2n+1}(S^{n+1})) \;=\; \ker(\mathrm{Hopf}\colon \pi_{2n+1}(S^{n+1}) \to \mathbb{Z}).$$

Before we turn to the proof of the Freudenthal Suspension Theorem 158.12 let us state the following corollary.

Corollary 158.13. For every $n \in \mathbb{N}$ the suspension homomorphism

$$\Sigma\colon \pi_{4n}(S^{2n}) \;\to\; \pi_{4n+1}(S^{2n+1})$$

is an epimorphism.

Remark. In Theorem 159.9, when we show that $\pi_{n+2}(S^n) \cong \mathbb{Z}_2$ for $n \geq 2$, we will make essential use of the fact that by Corollary 158.13 we know that the suspension homomorphism $\Sigma\colon \pi_4(S^2) \to \pi_5(S^3)$ is an epimorphism.

Proof of Corollary 158.13. This corollary follows immediately from the Freudenthal Suspension Theorem 158.12 and the fact, shown in Lemma 157.21, that the Hopf invariant

$$\mathrm{Hopf}\colon \pi_{4n+1}(S^{2n+1}) \;\to\; \mathbb{Z}$$

is the zero map. ■

[1958] As the reader will have noticed, at this stage the author swiped some technicalities under a big carpet. This is way why we refer to this only as a "sketch of proof".

We now turn to the proof of the Freudenthal Suspension Theorem 158.12]. We follow the argument given in [**Put**, Chapter 9]. Let $n \in \mathbb{N}$. We need to show that

$$\mathrm{Im}(\Sigma\colon \pi_{2n}(S^n) \to \pi_{2n+1}(S^{n+1})) \;=\; \ker(\mathrm{Hopf}\colon \pi_{2n+1}(S^{n+1}) \to \mathbb{Z}).$$

We prove the "$\subset$"-inclusion and "$\supset$"-inclusion separately.

Proof of the Freudenthal Suspension Theorem 158.12 "$\subset$"-inclusion. Let $n \in \mathbb{N}$. We need to show that

$$\mathrm{Im}(\Sigma\colon \pi_{2n}(S^n) \to \pi_{2n+1}(S^{n+1})) \;\subset\; \ker(\mathrm{Hopf}\colon \pi_{2n+1}(S^{n+1}) \to \mathbb{Z}).$$

First let us show the "$\subset$"-inclusion. So let (M, v) be a framed n-dimensional submanifold of $\mathbb{R}^{2n}$. We write $\widehat{M} := M \times \{0\}$ and we denote by $\widehat{v})$ the framing on $\widehat{M}$ given by $P \mapsto (v(P), e_{2n+1})$. Then

$$\mathrm{Hopf}(\Sigma([M,v])) \;=\; \mathrm{Hopf}(\widetilde{M}, \widetilde{v}) \;=\; \mathrm{lk}(\widetilde{M}, \widetilde{M} + e_{n+1}) \;=\; 0.$$

$\uparrow$ since $[\widetilde{M} + e_{n+1}] = 0 \in \mathrm{H}_n(\mathbb{R}^{2n} \pm e_{n+1})$ and since $\mathbb{R}^{2n} + e_{n+1} \subset \mathbb{R}^{2n+1} \setminus M$

■

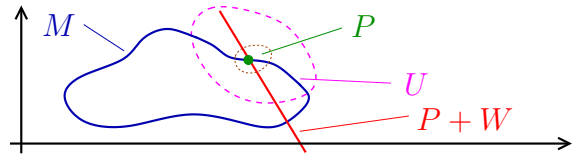
The proof of the reverse inclusion is much more interesting. Our proof relies on several preparations. We start out with the following proposition, that is mildly interesting in its own right.

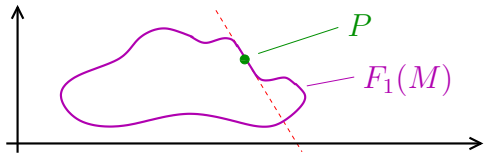
Proposition 158.14. Let M be an oriented n-dimensional submanifold of $\mathbb{R}^{2n+1}$. There exists an oriented cobordism from M to an oriented k-dimensional submanifold N of $\mathbb{R}^{2n+1}$ with $N \subset \mathbb{R}^{2n} \times \{0\}$.

Our proof of Proposition 158.14 relies on the Flattening Lemma 33.15, which we now recall for the reader's convenience.

Lemma 33.15. (Flattening Lemma) Let $M \subset \mathbb{R}^m$ be a smooth submanifold, let $P \in M$, let $W \subset \mathbb{R}^m$ be a vector subspace and let $\varphi\colon \mathbb{R}^m \to W$ be a homomorphism such that $\varphi\colon \widetilde{\mathrm{T}}_P M \to W$ is a monomorphism. Given any open neighborhood U of P there exists a smooth isotopy $F\colon M \times [0,1] \to \mathbb{R}^m$ rel $(M \setminus U) \cup \{P\}$ and an open neighborhood $\widetilde{U}$ of P with the following properties:

(1) $F_0 = \mathrm{id}$,
(2) $\widetilde{\mathrm{T}}_P F_1(M) = \varphi(\widetilde{\mathrm{T}}_P M)$,
(3) $F_1(\widetilde{U}) \subset P + W$,
(4) for each $x \in M$ and $t \in [0,1]$ the point $F(x,t)$ lies on the segment between x and $\varphi(x) + P$.





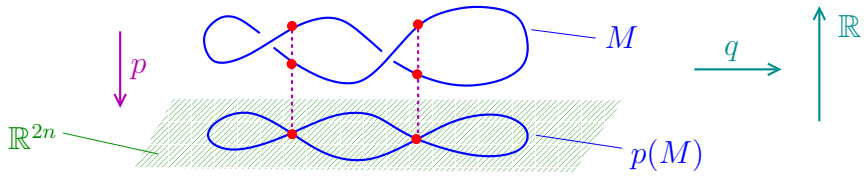
Proof. We consider the projections

$$\begin{array}{rcl} p\colon M & \to & \mathbb{R}^{2n} \\ (x_1, \ldots, x_{2n}, x_{2n+1}) & \to & (x_1, \ldots, x_{2n}) \end{array} \quad \text{and} \quad \begin{array}{rcl} p\colon M & \to & \mathbb{R}^{2n} \\ (x_1, \ldots, x_{2n}, x_{2n+1}) & \to & x_{2n+1}. \end{array}$$

By the same argument as in the proof of Lemma 59.20 we see that, after possibly applying a rotation to M, we can assume that the projection $p\colon M \to \mathbb{R}^{2n}$ has the following properties:

(1) the map $p\colon M \to \mathbb{R}^{2n}$ is an immersion,
(2) whenever $x \neq y$ in M satisfy $p(x) = p(y)$, then $\mathrm{D}p_x(\mathrm{T}_xM) \cap \mathrm{D}p_y(\mathrm{T}_yM) = \{0\}$,
(3) for every $z \in \mathbb{R}^{2n}$ the preimage $p^{-1}(z)$ contains at most two points in M.

As in Lemma 59.17 we see that it follows from the first two properties that there exist only finitely many $z_1, \ldots, z_m \in \mathbb{R}^{2n}$ whose preimage consists of more than one point.



Claim. There exist disjoint orientation-preserving embeddings $\Psi_i\colon \overline{B}^n \times \{0,1\} \to M$, $i = 1, \ldots, m$ with disjoint images such that the restriction of p to $M(\Psi_1, \ldots, \Psi_m)$ is an injective immersion.

Proof. To simplify we deal in the following with the case $m = 1$. The general case is done in almost the same way, we just need to consider all points $z_1, \ldots, z_m$ separately. Thus suppose that we have $z \in \mathbb{R}^{2n}$ such that $p^{-1}(z)$ consists of two points x and y.

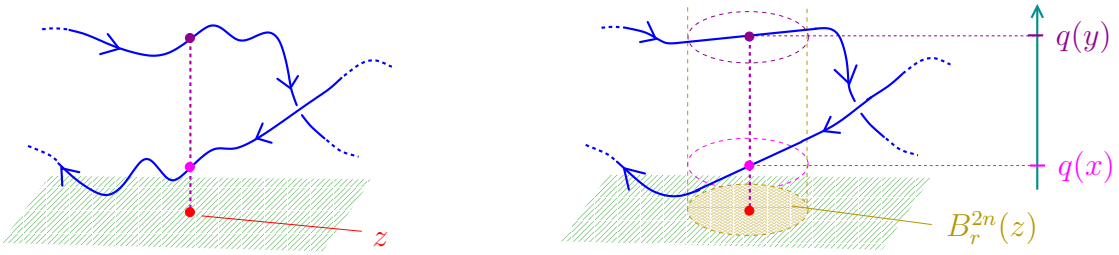
We apply the Flattening Lemma 33.15 to x, y and the homomorphism $p\colon \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$. Thus, after a smooth isotopy, we can assume that locally around x, y is contained in some $\mathbb{R}^{2n} \times \{*\}$.

Next we apply the Flattening Lemma 33.15 to x, y, to the above open neighborhoods and to the orthogonal projection onto the physical tangent spaces of these points.

We have now arranged that there exists an $r > 0$ such that[1959]

$$p^{-1}(\overline{B}^{2n}_{r_1}(z)) \;=\; (\overline{B}^{2n}_{r_1}(z_1) \cap ((\widetilde{\mathrm{T}}M_x + x) \times \{q(x)\}) \cup ((\widetilde{\mathrm{T}}M_{x_y} + y) \times \{q(y)\}).$$

We consider the subspaces $V := \mathrm{D}p_x(\widetilde{\mathrm{T}}_xM)$ and $W := \mathrm{D}p_x(\widetilde{\mathrm{T}}_xM)$. These are n-dimensional

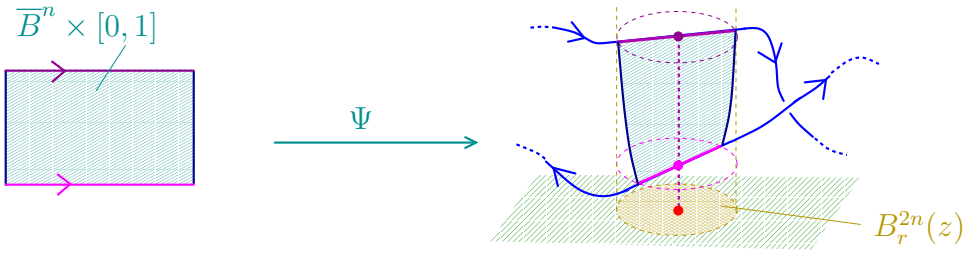


vector spaces of $\mathbb{R}^{2n}$ with $V \cap W = \{0\}$. We pick a *negative* basis $v_1, \ldots, v_n$ and we pick a *positive* basis $w_1, \ldots, w_n$ for W. Finally we consider the map

$$\begin{array}{rcl} \Psi\colon \overline{B}^n \times [0,1] & \to & \mathbb{R}^{2n+1} \\ (\underbrace{(u_1, \ldots, u_m)}_{=u}, t) & \mapsto & \Big(P + \underbrace{\sum_{j=1}^{n} u_j \cdot (v_j \cdot \cos(\tfrac{\pi}{2} \cdot t) + w_j \cdot \sin(\tfrac{\pi}{2} \cdot t))}_{\text{ball spanned by } v_j \cdot \cos(\frac{\pi}{2} \cdot t) + w_j \cdot \sin(\frac{\pi}{2} \cdot t)}, q(x) \cdot (1-t) + t \cdot q(y)\Big). \end{array}$$

[1959] If we were dealing with points $z_1, \ldots, z_m$ then we would of course pick $r_1, \ldots, r_m \in \mathbb{R}_{>0}$ such that the closed balls $\overline{B}^{2n}_{r_1}(z_1), \ldots, \overline{B}^{2n}_{r_1}(z_m)$ are disjoint.

Note that each $\Psi\colon \overline{B}^n \times [0,1] \to \mathbb{R}^{2n+1}$ is a smooth embedding. Also note that the restriction of p to $\Psi(S^{n-1} \times [0,1])$ is injective. This follows from the observation that it is injective, almost by default, on each $\Psi(S^{n-1} \times \{t\})$ and the fact that t can be obtained from consider the plane spanned by say v_1 and W_1. ■



The following corollary is the analogue of Lemma 158.8. Note that the corollary has nothing to do with the Hopf invariant.

Corollary 158.15. Let (M, v) be a framed n-dimensional submanifold of $\mathbb{R}^{2n+1}$. There exists a framed cobordism from (M, v) to a framed k-dimensional submanifold (N, w) of $\mathbb{R}^{2n+1}$ with $N \subset \mathbb{R}^{2n} \times \{0\}$.

Proof. We equip M with orientation coming from the framing. In the proof of Proposition 158.14 we constructed a cobordism from M to a k-dimensional submanifold $N \subset \mathbb{R}^{2n} \times \{0\}$. By doing surgeries along It follows from Proposition 158.10. ■

Proof of the Freudenthal Suspension Theorem 158.12 "$\subset$"-inclusion. Let $n \in \mathbb{N}$. We need to show that

$$\operatorname{Im}(\Sigma\colon \pi_{2n}(S^n) \to \pi_{2n+1}(S^{n+1})) \ \subset\ \ker(\text{Hopf}\colon \pi_{2n+1}(S^{n+1}) \to \mathbb{Z}).$$

Let (M, v) be a framed $2n$-dimensional submanifold in $\mathbb{R}^{4n+1}$ with Hopf invariant zero. We need to show that there exists a framed $2n$-dimensional submanifold $(\widetilde{M}, \widetilde{v})$ in $\mathbb{R}^{4n}$ such that there exists a framed cobordism from the suspension of $(\widetilde{M}, \widetilde{v})$ to (M, v).

By Corollary 158.15 we know that there exists a framed cobordism from (M, v) to a framed k-dimensional submanifold (N, w) of $\mathbb{R}^{2n+1}$ with $N \subset \mathbb{R}^{2n} \times \{0\}$.

As in Lemma 158.9 we need to show that w is equivalent to a framing such that the last vector is e_{2n+1}.

Here we are supposed to use that the Hopf invariant is zero.

Putman argues roughly as follows. Since $N \subset \mathbb{R}^{2n} \times \{0\}$ we know that e_{2n+1} is orthogonal to all tangent spaces of N. Next recall that $w\colon N \times \mathbb{R}^{n+1} \to \mathbb{R}^{2n+1}$ is an orthonormal framing. This means that for each $x \in N$ there exists a $g(x) \in S^{n+1}$ with $w(g(x)) = e_{2n+1}$. This defines a map $g\colon N \to S^n$. The degree of this map is supposedly the Hopf invariant of (N, w).

Since we assume that the Hopf invariant is zero it follows from the Hopf Degree Theorem that the map $N \to S^n$ is null-homotopic. Now proceed as in page 8 of Putman ■

Exercise for Chapter 158.

Exercise 158.1. In the proof of Proposition 159.4 we used the Freudenthal Suspension Theorem 158.6 to show that the map

$$\begin{array}{rcl} \mathbb{Z} & \to & \Omega_1^{\mathrm{fr}}(\mathbb{R}^4) \\ [k] & \mapsto & k \cdot [(\mathrm{C}_4, h_4)] \end{array}$$

is an epimorphism. Use Exercise **??** to give an alternative argument, which does not rely on the Freudenthal Suspension Theorem 158.6.

159. Thom-Pontryagin Theory III: The groups $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1})$ and $\Omega_2^{\mathrm{fr}}(\mathbb{R}^{n+2})$

In this chapter we will calculate the framed cobordism groups $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1})$ and $\Omega_2^{\mathrm{fr}}(\mathbb{R}^{n+2})$ "by hand" for $n \geq 3$. By the Thom-Pontryagin Theorem 157.11 and Proposition 158.4 this means that we calculate the homotopy groups $\pi_{n+1}(S^n)$ and $\pi_{n+2}(S^n)$ for $n \geq 3$ By now there might be more modern and efficient ways for computing these homotopy groups. But the geometric approach using Thom-Pontryagin theorem is still very charming, in particular it lets us use many techniques we have developed so far, and it gives us an opportunity to introduce a few new tricks. In particular we will get to introduce quadratic forms and the Arf invariant, which are purely algebraic objects, and are of great interest in the own right.

159.1. **The calculation of the framed cobordism groups $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1})$ for $n \geq 3$.** In Theorem 155.10 (2) (and alternatively in Theorem 157.14) we saw that $\pi_3(S^2) \cong \mathbb{Z}$ and we saw that the Hopf map represents a generator of $\pi_3(S^2)$. Furthermore, by the Freudenthal Suspension Theorem 158.6 we know that for every $n \geq 3$ the suspension homomorphism $\pi_3(S^2) \to \pi_{n+1}(S^n)$ is an epimorphism. Thus we know that $\pi_{n+1}(S^n)$ is isomorphic to a quotient of $\mathbb{Z}$ and that $\pi_{n+1}(S^n)$ is generated by the $(n-2)$-fold suspension of the Hopf map. The goal of this section is to prove the following theorem which determines this quotient.

Theorem 159.1. For every $n \geq 3$ there exists an isomorphism $\pi_{n+1}(S^n) \cong \mathbb{Z}_2$.

Remark.

(1) In Theorem 151.13 we already saw that for every $n \geq 3$ we have $\pi_{n+1}(S^n) \neq 0$. The proof of Theorem 159.1 below is independent of Theorem 151.13. In particular we will not make use of Steenrod operations.

(2) By the Thom-Pontryagin Theorem 157.11 and Proposition 158.4 the theorem is equivalent to showing that for every $n \geq 3$ we have an isomorphism $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1}) \cong \mathbb{Z}_2$. This is exactly what we will do below. The approach that we take below to calculate $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1})$ differs, to the best of my knowledge, from the other proofs in the literature. We refer to [**Pon1959**, Chapter 14] for Pontryagin's original proof, which makes for challenging reading for the modern reader. A third, quite readable approach is presented in [**Put**].

(3) There are various alternative proofs of Theorem 159.1. The first proof was given by Hans Freudenthal [**Freu1937**, p. 301] in 1937. Another proof is given in [**Hat**, p. 549] using "spectral sequences".

(4) The discussion preceding the theorem shows that the suspension of the Hopf map defines a non-trivial element in $\pi_4(S^3)$. This gives in particular an affirmative answer to Question 71.20.

As we had just mentioned above, proving Theorem 159.1 is equivalent to showing that for every $n \geq 3$ there exists an isomorphism $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1}) \cong \mathbb{Z}_2$. We will perform this calculation in the remainder of this section.

We start out our calculation of $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1})$ for $n \geq 3$ with the following definition.

Definition.

(1) We refer to $C := S^1$ as the standard circle in $\mathbb{R}^2$.
(2) We refer to the framing of $C \subset \mathbb{R}^2$ that is given by

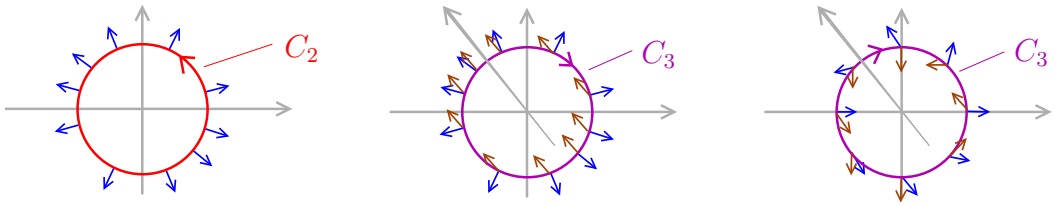
$$\begin{array}{rcl} t\colon C & \to & \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} x \\ y \end{pmatrix} \end{array}$$

as the trivial framing t of C.
(3) Let $n \in \mathbb{N}_{\geq 2}$. We adopt the convention from page 3369 to define the standard standard circle C_n in $\mathbb{R}^n$. Furthermore, using the definition from page 3369 we define the trivial framing t_n for C_n.
(4) We refer to the framing of $\mathrm{C}_3 \subset \mathbb{R}^3$ that is given by the map[1960]

$$\begin{array}{rcl} h\colon \mathrm{C}_3 & \to & \mathbb{R}^3 \\ \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} & \mapsto & t_3 \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} = \begin{pmatrix} \cos(\varphi) & 0 \\ \sin(\varphi) & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \end{array}$$

$\uparrow$ don't overlook the t_3

as the Hopf framing h of C_3.
(5) Let $n \in \mathbb{N}_{\geq 3}$. As in (3) we use the convention from page 3369 to define the Hopf framing h_n of C_n. These definitions are illustrated in the figure below.



trivial framing t_2 trivial framing t_3 Hopf framing h_3

Throughout this section we also use the following notation:

Notation.

(1) Given $g \in \mathbb{N}_0$ we denote by $\Sigma_{g,1}$ the surface of genus g minus one open disk. We pick once and for all an orientation for $\Sigma_{g,1}$.
(2) For any $n \in \mathbb{N}_{\geq 2}$ we denote by

$$\begin{array}{rcl} \rho\colon \mathbb{R}^n & \to & \mathbb{R}^n \\ (x_1, x_2, x_3, \ldots, x_n) & \mapsto & (x_1, -x_2, x_3, \ldots, x_n) \end{array}$$

the reflection in the $(x_2 = 0)$-hyperplane.
(3) Given $n \in \mathbb{N}_0$ we make the identification $\mathbb{R}^n = \{0\} \times \mathbb{R}^n \subset [0,1] \times \mathbb{R}^n$.

The following two lemmas justify the names "trivial framing" and "Hopf framing".

Lemma 159.2.

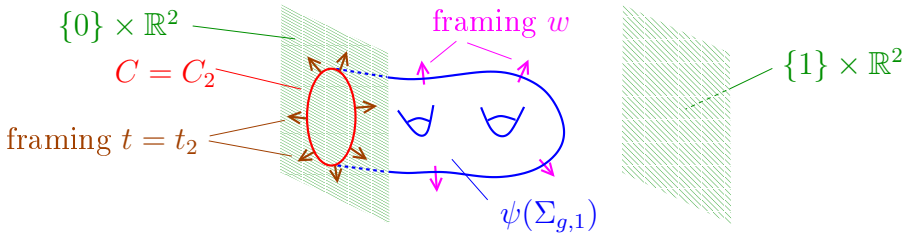
[1960] We leave the task of showing that this map is smooth to the every-ready reader.

(1) Given any $g \in \mathbb{N}_0$ there exists a proper smooth embedding $\psi\colon \Sigma_{g,1} \to [0,1]\times\mathbb{R}^2$ and a framing w for $\psi(\Sigma_{g,1})$ such that $\partial\psi(\Sigma_{g,1}) = \mathrm{C}_2$ and such that the restriction of w to C_2 equals t.
(2) For every $n \geq 2$ we have $[(C_n, t_n)] = 0 \in \Omega^{\mathrm{fr}}_1(\mathbb{R}^n)$.

Sketch of proof.

(1) By now the seasoned reader should be fairly convinced that there exists a proper smooth embedding $\psi\colon \Sigma_{g,1} \to [0,1] \times \mathbb{R}^2$ with $\psi(\partial\Sigma_{g,1}) = \mathrm{C}_2$ and such that the smooth embedding is a "product near the boundary". More precisely, such that for some $\epsilon \in (0,1)$ we have the equality $\psi(\Sigma_{g,1}) \cap ([0,\epsilon]\times\mathbb{R}^2) = [0,\epsilon]\times\mathrm{C}_2$. Since $\psi(\Sigma_{g,1})$ is an oriented connected submanifold of $[0,1]\times\mathbb{R}^2$ of codimension two we see that $\psi(\Sigma_{g,1})$ has precisely two normal fields of length one. We refer to the figure below for an illustration.

It follows almost immediately from the fact that $\psi(\Sigma_{g,1})$ is a product near $\{0\}\times\mathbb{R}^2$ that one of the above two normal vectors fields agrees with t on $\{0\}\times S^1$.
(2) It follows from (1) that $[(\mathrm{C}_2, t_2)] = 0 \in \Omega^{\mathrm{fr}}_1(\mathbb{R}^2)$. From Lemma 158.5 we obtain that $[(C_n, t_n)] = 0 \in \Omega^{\mathrm{fr}}_1(\mathbb{R}^n)$ for any $n \in \mathbb{N}_{\geq 2}$. ■



Lemma 159.3.

(1) Under the isomorphism $\Omega^{\mathrm{fr}}_1(\mathbb{R}^3) \cong \pi_3(S^2, *)$ the element $[(\mathrm{C}_3, h_3)]$ corresponds, up to a sign, to the element represented by the Hopf map $S^3 \to S^2$.
(2) Let $n \geq 4$. Under the isomorphism $\Omega^{\mathrm{fr}}_1(\mathbb{R}^n) \cong \pi_n(S^{n-1}, *)$ the element $[(C_n, h_n)]$ corresponds, up to a sign, to the element represented by the $(n-3)$-fold suspension of the Hopf map $S^3 \to S^2$.

Proof. The first statement follows easily from Lemma 157.4. The second statement is a straightforward consequence of Lemma 158.5 (2). ■

The following proposition is the first key ingredient in the calculation of $\Omega^{\mathrm{fr}}_1(\mathbb{R}^n)$ for $n \geq 4$.

Proposition 159.4. Let $n \in \mathbb{N}_{\geq 4}$. The map

$$\begin{array}{ccc} \mathbb{Z}_2 & \to & \Omega^{\mathrm{fr}}_1(\mathbb{R}^n) \\ [k] & \mapsto & k \cdot [(C_n, h_n)] \end{array}$$

is well-defined and it is an epimorphism.

Our proof of Proposition 159.4 requires one definition and two lemmas.

Definition. Let $n \in \mathbb{N}_{\geq 2}$ and let $\gamma\colon S^1 \to \mathbb{R}^n$ be a smooth embedding. Furthermore let $f\colon \gamma(S^1) \to \mathrm{GL}(n, n-1)$ be a framing for $\gamma(S^1)$ and let $\alpha\colon S^1 \to \mathrm{GL}_+(n-1, \mathbb{R})$ be a

smooth map. We define

$$\begin{array}{rcl} f(\alpha)\colon \gamma(S^1) & \to & \mathrm{GL}(n,n-1) \\ \gamma(z) & \mapsto & \underbrace{f(\gamma(z))}_{\in \mathrm{GL}(n,n-1)} \cdot \underbrace{\alpha(z)}_{\in \mathrm{GL}(n-1,\mathbb{R})}. \end{array}$$

It follows easily from the fact that α takes values in $\mathrm{GL}_+(n-1,\mathbb{R})$ that $f(\alpha)$ is again a framing of $\gamma(S^1)$.

The first lemma that goes into the proof of Proposition 159.4 is really just a crucial example masquerading as a lemma.

Lemma 159.5. Let $n \in \mathbb{N}_{\geq 3}$ and let $\gamma\colon S^1 \to C_n$ be the obvious diffeomorphism. We consider the loop

$$\begin{array}{rcl} \alpha\colon S^1 & \to & \mathrm{GL}_+(n-1,\mathbb{R}) \\ z = \exp(\mathrm{i}\varphi) & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-3} \end{pmatrix}. \end{array}$$

As above let t_n be the trivial framing for C_n and let h_n be the Hopf framing for C_n. We have the following two equalities of framings of C_n:

(1) We have $t_n(\alpha) = h_n$.
(2) We have $t_n(\overline{\alpha}) = \rho_*(h_n)$.[1961]

Proof. The first statement follows immediately from the definitions. We turn to the proof of the second statement. As we will see, the proof of the second statement is given by a slightly tedious calculation. To simplify the notation we only deal with the case $n = 3$.

At a given point $P = \gamma(\exp(\mathrm{i}\varphi)) = \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} \in \mathrm{C}_3$ we have

$$\begin{aligned} t_n(\overline{\alpha}) \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} &= t_n \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} \cdot \underbrace{\alpha(\exp(-\mathrm{i}\varphi))}_{=\overline{\alpha}(\exp(\mathrm{i}\varphi))} = \begin{pmatrix} \cos(\varphi) & 0 \\ \sin(\varphi) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(-\varphi) & -\sin(-\varphi) \\ \sin(-\varphi) & \cos(-\varphi) \end{pmatrix} \\ &= \rho \left(\begin{pmatrix} \cos(-\varphi) & 0 \\ \sin(-\varphi) & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(-\varphi) & -\sin(-\varphi) \\ \sin(-\varphi) & \cos(-\varphi) \end{pmatrix} \right) = \rho \left(h_n \begin{pmatrix} \cos(-\varphi) \\ \sin(-\varphi) \\ 0 \end{pmatrix} \right) \\ &\quad \uparrow \text{ since } \rho \text{ is reflection in the second coordinate together with the symmetries of sin and cos} \qquad \uparrow \text{ definition of } h_n, \text{ see page 3383} \\ &= \rho \left(h_n \left(\rho^{-1} \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} \right) \right) = \rho_*(h_n) \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix}. \\ &\quad \uparrow \text{ definition of } \rho \qquad \uparrow \rho_*(h_n) \text{ as defined on page 3366} \end{aligned}$$

This concludes the mind-numbing verification of the second statement. ∎

[1961] Here $\overline{\alpha}$ is the "inverse of the loop α", i.e. it is the loop given by $\overline{\alpha}(\exp(\mathrm{i}\varphi)) = \alpha(\exp(-\mathrm{i}\varphi))$. As above $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ is the reflection in the $(x_2 = 0)$-hyperplane. Finally the framing $\rho_*(h_n)$ is defined as on page 3366.

Lemma 159.6. Let $n \in \mathbb{N}_{\geq 2}$ and let $\gamma\colon S^1 \to \mathbb{R}^n$ be a smooth embedding. Furthermore let $f\colon \gamma(S^1) \to \mathrm{GL}(n, n-1)$ be a framing for $\gamma(S^1)$. Finally let $\alpha, \beta\colon S^1 \to \mathrm{GL}_+(n-1, \mathbb{R})$ be two smooth maps. If α and β are smoothly homotopic, then

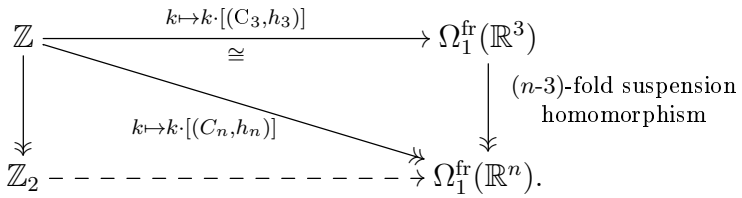
$$[(\gamma(S^1), f(\alpha))] \;=\; [(\gamma(S^1), f(\beta))] \;\in\; \Omega_1^{\mathrm{fr}}(\mathbb{R}^n).$$

Proof of Lemma 159.6. We write $M = \gamma(S^1)$. Since $\alpha, \beta\colon S^1 \to \mathrm{GL}_+(n-1, \mathbb{R})$ are smoothly homotopic we can pick a smooth homotopy $H\colon S^1 \times [0,1] \to \mathrm{GL}_+(n-1, \mathbb{R})$ with $H_0 = \alpha$ and $H_1 = \beta$. We consider the product cobordism $[0,1] \times M$ equipped with the framing

$$\begin{array}{rcl} V\colon [0,1] \times M & \to & \mathrm{GL}(n+1, n) \\ (t, \gamma(z)) & \mapsto & \begin{pmatrix} 1 & 0 \\ 0 & f(\gamma(z)) \cdot H(z,t) \end{pmatrix} \end{array}$$

It is clear that $([0,1] \times M, V)$ is a framed cobordism between $(\gamma(S^1), f(\alpha))$ and $(\gamma(S^1), f(\beta))$. ∎

Proof of Proposition 159.4. Let $n \in \mathbb{N}_{\geq 4}$. We consider the following diagram



We make the following observations:

(1) It follows quite easily from Theorem 157.14, Lemma 159.3 and Proposition 158.4 that the top horizontal map is an isomorphism.
(2) The right vertical map is given by the $(n-3)$-fold application of the suspension homomorphism defined on page 3369. Note that the right vertical map is an epimorphism by the Freudenthal Suspension Theorem 158.7.
(3) The top right triangle commutes by definition of the suspension homomorphism and by definition of (C_n, h_n).

The above diagram and the above discussion show that it remains to prove the following claim.

Claim. The element $2 \cdot [(C_n, h_n)]$ is trivial in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$.

Proof. We consider the loop

$$\begin{array}{rcl} \alpha\colon S^1 & \to & \mathrm{GL}_+(n-1, \mathbb{R}) \\ z = \exp(\mathrm{i}\varphi) & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-3} \end{pmatrix}. \end{array}$$

We have the following equalities in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$:

$$2\cdot[(C_n,h_n)] \underset{\uparrow}{=} [(C_n,h_n)]+[(C_n,t_n(\alpha))] \underset{\uparrow}{=} [(C_n,h_n)]+[(C_n,t_n(\overline{\alpha}))]$$

Lemma 159.5 (1): since $n\geq 4$ we know by Proposition 156.11 (2) that $\pi_1(\mathrm{GL}_+(n-1,\mathbb{R}))\cong\mathbb{Z}_2$, therefore we see that $[\alpha]\cdot[\alpha]=0\in\pi_1(\mathrm{GL}_+(n-1,\mathbb{R}))$, this implies by Proposition 47.6 that α is path-homotopic to $\overline{\alpha}$, in fact, by the Whitney Approximation Theorem 29.1 we see that α is actually smoothly homotopic to $\overline{\alpha}$, by Lemma 159.6 this gives us the equality $[(C_n,t_n(\alpha))]=[(C_n,t_n(\overline{\alpha}))]$

$$\underset{\uparrow}{=} [(C_n,h_n)]+[(\rho_*(\mathrm{C}_n),\rho_*(h_n))] \underset{\uparrow}{=} [(C_n,h_n)]+(-[(C_n,h_n)]) = 0.$$

Lemma 159.5 (2); Proposition 158.2 ■

Let $n\geq 4$. Note that by Proposition 159.4 we now know that there exists an epimorphism $\mathbb{Z}_2\to\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)\cong\pi_n(S^{n-1})$. In Theorem 151.13 we showed that $\pi_{n+1}(S^n)$ is non-trivial. Therefore it follows that $\pi_{n+1}(S^n)\cong\mathbb{Z}_2$, which means we have proved Theorem 159.1. The readers who studied Steenrod operations *and* who happen to be tired of Thom-Pontryagin theory can thus move on to next section.
For all others we provide a proof of Theorem 159.1 that does not rely on Theorem 151.13. More precisely, we prove the following proposition which is the main technical result of this section.

Proposition 159.7. For any $n\in\mathbb{N}_{\geq 4}$ the map

$$\begin{array}{rcl}\mathbb{Z}_2 & \to & \Omega_1^{\mathrm{fr}}(\mathbb{R}^n)\\ [k] & \mapsto & k\cdot[(C_n,h_n)]\end{array}$$

is an isomorphism.

As we had mentioned many times, Theorem 159.1 is an immediate consequence of the Thom-Pontryagin Theorem 157.11 and Proposition 158.4 together with Proposition 159.7.

Proof. Let $n\in\mathbb{N}_{\geq 4}$. By Proposition 159.4 it remains to show that (C_n,h_n) represents a non-trivial element in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$.

We start out with a more general discussion. Given any $m\in\mathbb{N}_{\geq 2}$ we make the identification $\mathrm{C}_m=\{0\}\times\mathrm{C}_m\subset[0,1]\times\mathbb{R}^m$. The following claim applies to all dimensions.

Claim. Let $m\in\mathbb{N}_{\geq 2}$. Let F be a compact orientable connected proper 2-dimensional smooth manifold in $[0,1]\times\mathbb{R}^m$ with $\partial F=\mathrm{C}_m$. At most one of the two framings t_m and h_m can be extended to a framing of F.

Proof. Suppose there exist framings T and H on F with $T|_{\mathrm{C}_m}=t_m$ and $H|_{\mathrm{C}_m}=h_m$. It follows from Lemma 158.3 that we can assume that T and H are orthonormal. We consider the map[1962]

$$\begin{array}{rcl}\Psi\colon F & \to & \mathrm{GL}(m-1,\mathbb{R})\\ P & \mapsto & \text{the unique matrix } \Psi(P)\in\mathrm{GL}(m-1,\mathbb{R}) \text{ with } T(P)\cdot\Psi(P)=H(P).\end{array}$$

We denote by $i\colon\mathrm{C}_m\to F$ the inclusion map and we denote by $\gamma\colon S^1\to\mathrm{C}_m$ the usual diffeomorphism given by $\gamma(\exp(\mathrm{i}\varphi))=(\cos(\varphi),\sin(\varphi),0,\ldots,0)$. We make the following

[1962]Let $P\in F$. The fact that T and H are orthonormal means that the columns of $T(P)$ and $H(P)$ are bases for the same vector space, namely $(\mathrm{T}_PF)^\perp$. Therefore there exists a matrix $\Psi(P)\in\mathrm{GL}(m-1,\mathbb{R})$ such that $T(P)\cdot\Psi(P)=H(P)$. We leave it to the reader to verify that the map $P\mapsto\Psi(P)$ is continuous.

key observation: it follows immediately from the definitions and Lemma 159.5 that the map $\Psi \circ i \circ \gamma \colon S^1 \to \mathrm{GL}(m-1,\mathbb{R})$ is precisely, on the nose, given by the map

$$\begin{array}{rcl} \alpha \colon S^1 & \to & \mathrm{GL}(m-1,\mathbb{R}) \\ \exp(\mathrm{i}\varphi) & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{m-3} \end{pmatrix}. \end{array}$$

Now we consider the following diagram

$$\begin{array}{ccccc} \mathrm{H}_1(\partial F;\mathbb{Z}) & \xleftarrow{\gamma_*} & \mathrm{H}_1(S^1;\mathbb{Z}) & \xleftarrow[\cong]{\text{Hurewicz isom.}} & \pi_1(S^1,1) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} \\ \mathrm{H}_1(F;\mathbb{Z}) & \xrightarrow{\Psi_*} & \mathrm{H}_1(\mathrm{GL}(m-1,\mathbb{R})) & \xleftarrow[\text{Hurewicz isom.}]{\cong} & \pi_1(\mathrm{GL}(m-1,\mathbb{R}),\mathrm{id}). \end{array}$$

We make the following observations:

(1) The left-hand square commutes by the above equality $\alpha = \Psi \circ i \circ \gamma$.
(2) The two horizontal maps to the right are the Hurewicz homomorphisms that we introduced on page 1832. The Hurewicz homomorphisms are isomorphisms by the Hurewicz Theorem 84.5 together with the fact that the fundamental groups on the right are both abelian.
(3) The right-hand square commutes by the naturality of the Hurewicz homomorphisms, see Proposition 84.2 (5).
(4) By Proposition 156.11 (2) and (3) we know that the loop $\alpha \colon S^1 \to \mathrm{GL}(m-1,\mathbb{R})$ represents a non-trivial element in $\pi_1(\mathrm{GL}(m-1,\mathbb{R}),\mathrm{id})$. This implies that the right vertical map $\alpha_* \colon \pi_1(S^1,1) \to \pi_1(\mathrm{GL}(m-1,\mathbb{R}),\mathrm{id})$ is non-trivial.
(5) It follows from (1) – (4) that the left vertical map is also non-trivial. But this contradicts Lemma 80.9 (1). ⊞

Now we start out with the actual proof of the theorem. Thus let $n \in \mathbb{N}_{\geq 4}$. As we pointed out above, we need to show that (C_n, h_n) represents a non-trivial element in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$. Suppose that (C_n, h_n) represents in fact the trivial element in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$. This means that there exists a framed cobordism $(F, v) = (F_n, v_n)$ in $[0,1] \times \mathbb{R}^n$ from (C_n, h_n) to the empty smooth manifold. In other words, $F = F_n$ is a compact orientable proper 2-dimensional submanifold of $[0,1] \times \mathbb{R}^n$ with $\partial F_n = C_n$ and $v = v_n$ is a framing with $v|_{\mathrm{C}_n} = h_n$. By throwing away all closed components of F_n we can assume that F_n is connected.

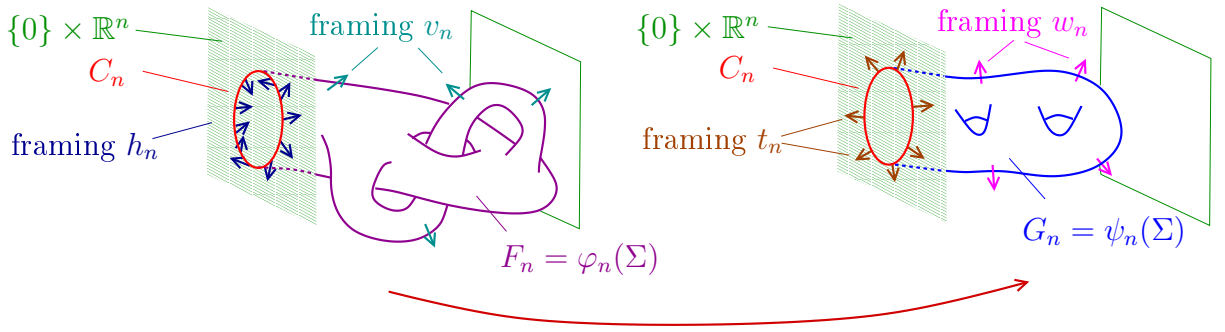
By the Surface Classification Theorem 55.4 we know that there exists a $g \in \mathbb{N}_0$ such that F is diffeomorphic to $\Sigma := \Sigma_{g,1}$. In other words, there exists a proper smooth embedding $\varphi \colon \Sigma \to [0,1] \times \mathbb{R}^n$ with $\varphi(\Sigma) = F_n$. Given any $m \geq n$ we denote by $\varphi_m \colon \Sigma \to [0,1] \times \mathbb{R}^m$ the smooth embedding that is given by the composition of the smooth embedding φ with the obvious smooth embedding $[0,1] \times \mathbb{R}^n \to [0,1] \times \mathbb{R}^m$.

By Lemma 159.2 (1) there exists a proper smooth embedding $\psi \colon \Sigma \to [0,1] \times \mathbb{R}^2$ and a framing $w_2 = w$ for $G_2 = G := \psi(\Sigma)$ such that $\partial\psi(\Sigma) = \mathrm{C}_2$ and such that the restriction of $w_2 = w$ to C_2 equals t_2. Given $m \geq 2$ we denote by $\psi_m \colon \Sigma \to [0,1] \times \mathbb{R}^m$ the smooth embedding that is given by the composition of the smooth embedding ψ with the obvious smooth embedding $[0,1] \times \mathbb{R}^2 \to [0,1] \times \mathbb{R}^m$.

Now we consider the two proper smooth embeddings $\psi_n, \varphi_n \colon \Sigma \to [0,1] \times \mathbb{R}^n$. We can arrange[1963] that $\psi_n|_{\partial\Sigma} = \varphi_n|_{\partial\Sigma}$.

It follows from Proposition **??** (2) and the Isotopy Extension Theorem 36.1 that there exists an $m \geq n$ and a diffeomorphism Ψ of the smooth manifold $[0,1] \times \mathbb{R}^m$ such that $\Psi(F_m) = G_m$ and such that the restriction of Ψ to $\{0\} \times \mathbb{R}^m$ is the identity. We refer to the figure below for an illustration.

Since (F_m, v_m) is an extension of (C_m, h_m) and since the restriction of Ψ to $\{0\} \times \mathbb{R}^m$ is the identity we see that $(\Psi(F_m), \Psi_*(v_m)) = (G_m, \Psi_*(v_m))$ is an extension of (C_m, h_m). On the other hand we saw (G_m, w_m) is an extension of (C_m, t_m). Thus both h_m and t_m can be extended over G_m. But this contradicts the above claim. ■



In summary we have the following proposition.

Proposition 159.8.

(1) The map

$$\begin{array}{ccccc} \Omega_1^{\mathrm{fr}}(\mathbb{R}^3) & \xleftarrow[\cong]{\text{given by Proposition 158.4}} & \Omega_1^{\mathrm{fat}}(\mathbb{R}^3) & \to & \mathbb{Z} \\ & & [(C,g)] & \mapsto & \text{self-linking number } \mathrm{slk}(C,g) \end{array}$$

is an isomorphism.

(2) For $n \geq 4$ the map

$$\begin{array}{ccc} \mathbb{Z}_2 & \to & \Omega_1^{\mathrm{fr}}(\mathbb{R}^n) \\ [k] & \mapsto & k \cdot [(C_n, h_n)] \end{array}$$

is an isomorphism.

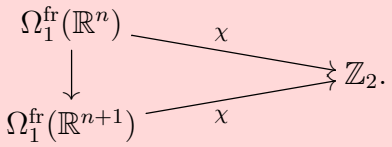
For $n \in \mathbb{N}_{\geq 3}$ we denote by

$$\chi \colon \Omega_1^{\mathrm{fr}}(\mathbb{R}^n) \to \mathbb{Z}_2$$

[1963] This sentence contains a slight technical nuisance which we cleverly hide in a footnote. First of all, it should be pretty clear by now that Σ admits an orientation-reversing self-diffeomorphism. Thus, after possibly precomposing φ_n with such a diffeomorphism we can assume that $\psi_n^{-1} \circ \varphi_n$ is orientation-preserving. Next note that it follows from the Diffeotopies-of-S^1-Classification Proposition 34.5 and Corollary 34.10 that if C and D are two oriented smooth manifolds that are diffeomorphic to S^1, then any two orientation-preserving diffeomorphisms $C \to D$ are diffeotopic. In our context this implies that, using the Smooth Collar Neighborhood Theorem 28.3, we can then arrange, after possibly precomposing φ_n by another diffeomorphism, that $\psi_n|_{\partial\Sigma} = \varphi_n|_{\partial\Sigma}$.

the corresponding epimorphism.

(3) Given any $n \geq 3$ the following diagram commutes:



Proof.

(1) This statement is the content of Proposition 157.14.
(2) This statement is precisely the content of Proposition 159.7.
(3) This statement follows immediately from Theorem 158.7. ■

159.2. **Further results obtained using the Thom-Pontryagin approach.** In this final section we summarize a few more results that have been proved using the Thom-Pontryagin approach. First of all, using the Thom-Pontryagin method Pontryagin [**Pon1950b**] proved in 1950 the following theorem.

Theorem 159.9. For any $n \geq 2$ we have $\pi_{n+2}(S^n) \cong \mathbb{Z}_2$.

Remark.

(1) Recall that by Theorem 155.10 (1) we have for every $n \geq 3$ we have an isomorphism $\pi_n(S^3) \cong \pi_n(S^2)$. Furthermore we know by Theorem 159.1 that for every $n \geq 3$ we have $\pi_{n+1}(S^n) \cong \mathbb{Z}_2$. Putting these two results together shows that $\pi_4(S^2) \cong \mathbb{Z}_2$.
(2) In Theorem 152.2 we used Steenrod operations to show that $\pi_{n+2}(S^n) \neq 0$ for all $n \geq 2$.
(3) By the Freudenthal Suspension Theorem 158.6 we know that the suspension homomorphism $\pi_5(S^3) \to \pi_6(S^4)$ is an epimorphism and we know that for $n > k+1$ the suspension homomorphism $\pi_{n+2}(S^n) \to \pi_{n+3}(S^{n+1})$ if an isomorphism for $n \geq 5$.
(4) Unfortunately (1), (2) and (3) are *not* strong enough to give an alternative proof of Theorem 159.9.

Proof. Let $n \geq 2$. First note that we have the following the isomorphisms and epimorphisms:

$$\mathbb{Z}_2 \underset{\text{Theorem 159.1}}{\cong} \pi_4(S^3) \underset{\text{Theorem 155.10 (1)}}{\cong} \pi_4(S^2) \underset{\text{Corollary 158.13}}{\twoheadrightarrow} \pi_5(S^3) \underset{\substack{\text{Freudenthal Suspension} \\ \text{Theorem 158.6}}}{\twoheadrightarrow} \pi_6(S^4) \cong \pi_{n+2}(S^n).$$

Now there are two ways to proceed:

(1) In Theorem 152.2 we used Steenrod operations to show that $\pi_{n+2}(S^n) \neq 0$ for all $n \geq 2$.

Either of these two statements shows that the maps sequence of maps are all necessarily isomorphisms. ■

Proof.

(1) As mentioned before, the theorem was proved by Lev Pontryagin [**Pon1950b, Pon1959**]. See also [**Scor2005**, p. 233] and [**WX2010**, Chapter 2.5] for outlines of the proof.

The most readable account of the Thom-Pontryagin approach to proving the theorem is surely [**Put**].

(2) At the same time as Pontryagin two very different proofs that $\pi_{n+2}(S^n) \cong \mathbb{Z}_2$ for $n \geq 2$ were given by George W. Whitehead [**WhdG1950**] and Jean-Pierre Serre [**Ser1953**, Proposition 10]. We also refer to [**Hat**, Theorem 1.40 (a)] for a proof using spectral sequences and Steenrod operations. Alternatively see also [**Hu1959**, Theorem XI.15.2]. ■

Finally Rokhlin [**Rok1952, Rok1986a**] used the Thom-Pontryagin approach to prove the following theorem.

Theorem 159.10.

(1) We have[1964] $\pi_5(S^3) \cong \mathbb{Z}_2$, $\pi_6(S^3) \cong \mathbb{Z}_{12}$, $\pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}_{12}$.

(2) For any $n \geq 5$ we have $\pi_{n+3}(S^n) \cong \mathbb{Z}_{24}$.

Remark.

(1) In Theorem 151.13 we used Steenrod operations to show the weaker statement that $\pi_{n+3}(S^n) \neq 0$ for all $n \geq 2$.

(2) Note that Theorem 159.10 (1) gives a negative answer to Question 122.15 (2).

Proof.

(a) The theorem was proved by Vladimir Rokhlin [**Rok1952**][**Rok1986a**, p. 21] in 1952, using the above Thom-Pontryagin approach.[1965]

(b) A somewhat different proof is sketched in [**WX2010**, Chapter 2.6]. Some of the results can also be found in [**Ser1953**, Proposition 10]. Other proofs are given in [**Hat**, Theorem 1.40 (b)] and [**Hu1959**, Chapter XI.16]. ■

The results of the last two chapters summarize most (or all?) of the calculations of homotopy groups of spheres that have been obtained using the Thom-Pontryagin approach.

Exercises for Chapter 159.

Exercise 159.1. Let $n \in \mathbb{N}_{\geq 2}$. We consider the loops

$$\begin{array}{ccc} \alpha\colon S^1 \to \mathrm{GL}_+(n,\mathbb{R}) & & \overline{\alpha}\colon S^1 \to \mathrm{GL}_+(n,\mathbb{R}) \\ z = \exp(\mathrm{i}\varphi) \mapsto \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-2} \end{pmatrix} & \text{and} & z = \exp(\mathrm{i}\varphi) \mapsto \begin{pmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-2} \end{pmatrix}. \end{array}$$

Recall that in the proof of Lemma 159.6 we used the fact, proved in Proposition 156.11 (2), that $\pi_1(\mathrm{GL}_+(n-1,\mathbb{R})) \cong \mathbb{Z}_2$, to show that α is path-homotopic to $\overline{\alpha}$. Give a direct proof that the loop α is path-homotopic to the loop $\overline{\alpha}$.

[1964] In fact the first isomorphism $\pi_5(S^3) \cong \mathbb{Z}_2$ is a consequence of Theorem 155.10 and Theorem 159.9.

[1965] In fact Rokhlin [**Rok1986b**] initially thought that for $n \geq 5$ the groups $\pi_{n+3}(S^n)$ are isomorphic to $\mathbb{Z}_{12}$. The key to fixing the mistake is Rokhlin's Theorem 135.5 regarding signatures of certain 4-dimensional smooth manifolds.

160. Summary of homotopy groups of spheres

The results from last chapter conclude our calculations of homotopy groups of spheres. Throughout the course we employed a long and varied list of tools to compute homotopy groups of spheres:

(1) Covering theory, see page 1573.
(2) The Cellular Approximation Theorem 70.15, see Proposition 71.10.
(3) The Simplicial Approximation Theorem 94.8, see Corollary 94.10 and Exercise 94.6.
(4) Homology theory and the Hurewicz Theorem 85.5, see Corollary 85.6.
(5) The cup product, see Corollary 122.14.
(6) Steenrod operations, see Theorems 151.13 and 152.2.
(7) The long exact sequence of a fibration, see Theorem 155.10.
(8) The Thom-Pontryagin Theory, see the Freudenthal Suspension Theorem 158.6 and Theorem 159.1.

In the following we summarize the results we have obtained on homotopy groups of spheres:

(1) On page 1573 we saw that $\pi_n(S^1) = 0$ for $n \geq 2$.
(2) Proposition 71.10, or alternatively Exercise 94.6, says that for any $n \in \mathbb{N}$ we have $\pi_1(S^n) = \cdots = \pi_{n-1}(S^n) = 0$.
(3) In Corollary 94.10 we showed that the homotopy groups of spheres are countable.
(4) Corollary 85.6 (or alternatively by Corollary 157.12) we know that for any $n \in \mathbb{N}$ we have $\pi_n(S^n) \cong \mathbb{Z}$.
(5) By Theorem 155.10 (1) we know that for every $n \geq 3$ we have an isomorphism $\pi_n(S^3) \cong \pi_n(S^2)$. In particular, together with (4) we know that $\pi_3(S^2) \cong \mathbb{Z}$. An alternative proof of the fact that $\pi_3(S^2) \cong \mathbb{Z}$ is given by the combination of the Thom-Pontryagin Theorem 157.11 and Theorem 157.14.
(6) In Corollary 122.14, and alternatively in Proposition 157.23 we saw that given any even $n \in \mathbb{N}$ there exists an epimorphism $\pi_{2n-1}(S^n) \twoheadrightarrow \mathbb{Z}$.
(7) By Theorem 155.11 (1) we know that for every $n \geq 2$ we have an isomorphism $\pi_n(S^4) \cong \pi_n(S^7) \oplus \pi_{n-1}(S^3)$. Furthermore, by Theorem 155.11 (1) we know that for every $n \geq 2$ we have an isomorphism $\pi_n(S^8) \cong \pi_n(S^{15}) \oplus \pi_{n-1}(S^7)$.
(8) By the Freudenthal Suspension Theorem 158.6 we know that for $n = k + 1$ we have an epimorphism $\pi_{n+k}(S^n) \twoheadrightarrow \pi_{n+k+1}(S^{n+1})$ and we know that for $n > k + 1$ we have an isomorphism $\pi_{n+k}(S^n) \cong \pi_{n+k+1}(S^{n+1})$.
(9) By Theorem 155.11 (2) we know that $\pi_6(S^4) \cong \pi_5(S^3)$ and by Theorem 155.14 (3) we know that $\pi_{14}(S^8) \cong \pi_{13}(S^7)$.
(10) In Theorem 151.13 we proved the following three statements:
  (a) for any $n \geq 2$ the group $\pi_{n+1}(S^n)$ is non-trivial,
  (b) for any $n \geq 4$ the group $\pi_{n+3}(S^n)$ is non-trivial,
  (c) for any $n \geq 8$ the group $\pi_{n+7}(S^n)$ is non-trivial.
(11) By Theorem 159.1 we know that for every $n \geq 3$ we have $\pi_{n+1}(S^n) \cong \mathbb{Z}_2$. This is a refinement of statement (10a). Note that it follows from this fact together with (5) that $\pi_4(S^2) \cong \mathbb{Z}_2$.
(12) In Theorem 152.2 we proved the following three statements:
  (a) for any $n \geq 2$ the group $\pi_{n+2}(S^n)$ is non-trivial,
  (b) for any $n \geq 4$ the group $\pi_{n+6}(S^n)$ is non-trivial,

(c) for any $n \geq 8$ the group $\pi_{n+14}(S^n)$ is non-trivial.

We recall that Serre's Theorem 86.17 says that all homotopy groups of spheres are finitely generated abelian groups. In the discussion above we saw that some of the homotopy groups of spheres are infinite.
The following theorem can be summarized as saying that we have picked up all elements of infinite order. In particular the theorem gives a complete answer to Question 122.16. This theorem can also be viewed as a strengthening of Serre's Theorem 86.17. In fact, the strengthening is also due to Jean-Pierre Serre.

Theorem 160.1.
(1) If n is odd, then $\pi_k(S^n)$ is finite for $k \geq n+1$.
(2) If n is even, then the following two statements hold:
  (a) The group $\pi_k(S^n)$ is finite for $k \geq n+1$ and $k \neq 2n-1$.
  (b) The group $\pi_{2n-1}(S^n)$ is isomorphic to the direct sum of $\mathbb{Z}$ with a finite group.

Proof. The theorem was first proved by Serre [**Ser1953**] using "spectral sequences". Expositions of the proof are also given in [**Hat**, Theorem 1.21] and [**Spa1995**, p. 515f]. ■

The following table, obtained from [**Tod1962**, Chapter XIV], shows the low-dimensional homotopy groups of low-dimensional spheres. The entries colored in blue have been proved in these lecture notes.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$
S^9	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$

Note that we showed in Theorem 155.10 (1) that the second and the third row, starting with π_3, are the same.

Recall that the Freudenthal Suspension Theorem 158.6 says that for any $n, k \in \mathbb{N}_0$ with $n > k+1$ the suspension homomorphism $\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ is an isomorphism. Thus we see that the groups $\pi_{n+k}(S^n)$ "stabilize" as $n \to \infty$. It is therefore perhaps more instructive to shift in the above diagram the i-th row leftward by $i-1$ terms. This way the diagonal lines become vertical. In other words we consider the following diagram where

π_{n+k} is short for $\pi_{n+k}(S^n)$.

n	π_{n+0}	π_{n+1}	π_{n+2}	π_{n+3}	π_{n+4}	π_{n+5}	π_{n+6}	π_{n+7}	π_{n+8}	π_{n+9}	π_{n+10}
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0
2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
3	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$
4	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$
5	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
6	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_8\times\mathbb{Z}_2\times\mathbb{Z}_9$
7	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_8\times\mathbb{Z}_3\times\mathbb{Z}_2$
8	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^5$	$\mathbb{Z}_8^2\times\mathbb{Z}_2\times\mathbb{Z}_3^2$
9	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_8\times\mathbb{Z}_6$
10	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2^2$	$\mathbb{Z}\times\mathbb{Z}_2^3$	$\mathbb{Z}_4\times\mathbb{Z}_6$
11	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$
11	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2\times\mathbb{Z}_3$

Here for convenience we colored the "diagonal" $\pi_n(S^{2n})$ in red, since the Freudenthal Suspension Theorem 158.6 implies that below the diagonal the groups on each column are the same.
We conclude this discussion of the precise homotopy groups of spheres with the following sobering fact:

Fact 160.2. There is no sphere S^n with $n \geq 2$ for which we[1966] know the isomorphism types of all homotopy groups.

As we had mentioned above, the Freudenthal Suspension Theorem 158.6 says that homotopy groups of spheres "stabilize". This leads us to the following definition.

Definition. Let $k \in \mathbb{N}_0$. We refer to[1967]

$$\pi_k^s := \varinjlim \pi_{n+k}(S^n)$$

as the k-th stable homotopy group of spheres.[1968]

Note that it follows from Theorem 160.1 that for any $k \geq 1$ the k-th stable homotopy group π_k^s is finite. The following table obtained from [**Wiki**] gives the stable homotopy groups up to $k = 63$. It seems like 64 is the first dimension for which we do not know the

[1966]Here "we" is defined as members of our species.
[1967]Given $k \in \mathbb{N}_0$ the groups $\{\pi_{n+k}(S^n)\}_{n\in\mathbb{N}}$ together with the suspension homomorphisms form a direct system. Thus it makes sense to consider the direct limit.
[1968]Note that for any n, k with $n > k + 1$ the Freudenthal Suspension Theorem 158.6, together with Lemma 111.2 (4) and the Limit-over-Subset Lemma 111.3, says that the natural map $\pi_{n+k}(S^n) \to \pi_k^s$ is an isomorphism.

corresponding stable homotopy groups.

n	0	1	2	3	4	5	6	7
π^s_{0+n}	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{8\cdot 3}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{16\cdot 3\cdot 5}$
π^s_{8+n}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{2\cdot 3}$	$\mathbb{Z}_{8\cdot 9\cdot 7}$	0	$\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2\,\mathbb{Z}_{32\cdot 3\cdot 5}$
π^s_{16+n}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_8\,\mathbb{Z}_2$	$\mathbb{Z}_2\,\mathbb{Z}_{8\cdot 3\cdot 11}$	$\mathbb{Z}_{8\cdot 3}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2\,\mathbb{Z}_{8\cdot 3}\,\mathbb{Z}_{16\cdot 9\cdot 5\cdot 7\cdot 13}$
π^s_{24+n}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2\,\mathbb{Z}_3$	$\mathbb{Z}_{8\cdot 3}$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{2\cdot 3}$	$\mathbb{Z}_2^2\,\mathbb{Z}_{64\cdot 3\cdot 5\cdot 17}$
π^s_{32+n}	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^5$	$\mathbb{Z}_4\,\mathbb{Z}_2^3$	$\mathbb{Z}_2^2\,\mathbb{Z}_{8\cdot 27\cdot 7\cdot 19}$	$\mathbb{Z}_{2\cdot 3}$	$\mathbb{Z}_2^2\,\mathbb{Z}_3$	$\mathbb{Z}_4\,\mathbb{Z}_{2\cdot 3\cdot 5}$	$\mathbb{Z}_2^5\,\mathbb{Z}_3\,\mathbb{Z}_{16\cdot 3\cdot 25\cdot 11}$
π^s_{40+n}	$\mathbb{Z}_2^5\,\mathbb{Z}_{4\cdot 3}$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^2\,\mathbb{Z}_{8\cdot 3}$	$\mathbb{Z}_{8\cdot 3\cdot 23}$	$\mathbb{Z}_8$	$\mathbb{Z}_2^3\,\mathbb{Z}_{16\cdot 9\cdot 5}$	$\mathbb{Z}_2^4\,\mathbb{Z}_3$	$\mathbb{Z}_2^3\,\mathbb{Z}_4\,\mathbb{Z}_3\,\mathbb{Z}_{32\cdot 9\cdot 5\cdot 7\cdot 13}$
π^s_{48+n}	$\mathbb{Z}_4\,\mathbb{Z}_2^4$	$\mathbb{Z}_2^2\,\mathbb{Z}_3$	$\mathbb{Z}_2^3\,\mathbb{Z}_3$	$\mathbb{Z}_4\,\mathbb{Z}_2^2\,\mathbb{Z}_{8\cdot 3}$	$\mathbb{Z}_2^3\,\mathbb{Z}_3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_4\,\mathbb{Z}_2$	$\mathbb{Z}_3\,\mathbb{Z}_{16\cdot 3\cdot 5\cdot 29}$
π^s_{56+n}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2\,\mathbb{Z}_{8\cdot 9\cdot 7\cdot 11\cdot 31}$	$\mathbb{Z}_4$	0	$\mathbb{Z}_2^2\,\mathbb{Z}_{4\cdot 3}$	$\mathbb{Z}_2^3\,\mathbb{Z}_{128\cdot 3\cdot 5\cdot 17}$

This concludes our discussion of the homotopy groups of spheres. We refer to [**May1999b, WX2010, IWX2001, Wiki**] and [**Hat2002**, Chapter 4] for a more detailed discussion.

Exercises for Chapter 160.

Exercise 160.1.

(a) Show that $\pi_4(\mathbb{HP}^\infty) \cong \mathbb{Z}$.
(b) Show that $\pi_n(\mathbb{HP}^\infty)$ is finite for $n \neq 4$.
(c) Show that $\pi_5(\mathbb{HP}^\infty) \cong \mathbb{Z}_2$.

Remark. Use the Hopf-Bundle Proposition 155.9 together with the results of this chapter.

161. The CW-Approximation Theorem

In the following chapters we will work a lot with CW-complexes. In this chapter we will first prove a hotchpotch of results on CW-complexes. Afterwards we will use these results to prove the CW-Approximation Theorem 161.8 which often allows us to replace a topological space X by a CW-complex that is "weakly homotopy equivalent" to X. The study of "weak homotopy equivalences" will also play an essential role in the next chapter, when we prove the Whitehead Theorem 162.1.

161.1. **Modifying CW-complexes.** As mentioned in the beginning of the chapter, CW-complexes will play a major role in this and the subsequent chapters. Thus, before we start out with proving the enticing theorems mentioned above, we intend to improve our understanding of CW-complexes.
The following, fairly self-evident convention will be in use in the coming chapters.

Convention.

(1) Given $n \in \mathbb{N}_0$ we write $* = (0, \dots, 0, 1) \in S^n$. We view S^n as a CW-complex with one 0-cell $\{*\}$ and one n-cell.
(2) We always form any wedge of spheres $\bigvee_{i \in I} S_i^{n_i}$ by identifying the points $*$. By the CW-Complex Construction Lemma 68.32 (4) we can view the wedge $\bigvee_{i \in I} S_i^{n_i}$ as a CW-complex in the obvious way, i.e. we have one 0-cell $\{*\}$ and for each $i \in I$ we have one n_i-cell.

Before we do anything else we will introduce the following definition which will accompany us in the near future.

Definition. Let $f\colon X \to Y$ be a map between topological spaces.

(1) We say the map $f\colon X \to Y$ is 0-connected if $f_*\colon \pi_0(X) \to \pi_0(Y)$ is a surjection, in other words, each path-component of Y contains a point in $f(X)$.
(2) Let $k \in \mathbb{N}$. We say the map $f\colon X \to Y$ is k-connected if the following two conditions are satisfied:
   (a) the map $f_*\colon \pi_0(X) \to \pi_0(Y)$ is a bijection,
   (b) for every $x_0 \in X$ the induced map

$$f_*\colon \pi_i(X, x_0) \to \pi_i(Y, f(x_0)) \text{ is an } \begin{cases} \text{isomorphism for } i = 1, \dots, k-1, \\ \text{epimorphism for } i = k. \end{cases}$$

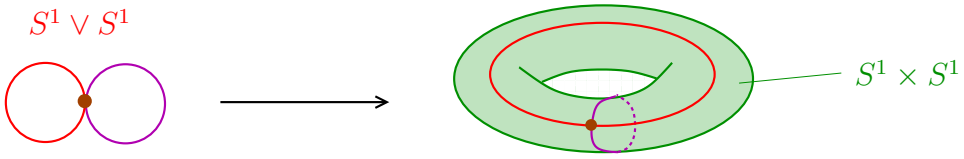
Remark. Let $f\colon X \to Y$ be a map between path-connected topological spaces and let $k \in \mathbb{N}$. It follows from the Change-of-Base Point Proposition 71.5 that to verify that f is k-connected it suffices to check the Condition (2b) for a single base point $x_0 \in X$.

Examples.

(1) The map $f\colon S^1 \vee S^1 \to S^1 \times S^1$ that is illustrated in the figure below@ is easily seen to be 1-connected, but it is not 2-connected.
(2) Let X be a CW-complex and let $k \in \mathbb{N}_0$. Corollary 153.10 says that the inclusion $X^k \to X$ of the k-skeleton[1969] is a k-connected map. In fact, the previous example

[1969]Since we did not work with CW-complexes for a while we remind the reader that for a CW-complex X and $l \in \mathbb{N}_0$ we denote by X^l the l-skeleton i.e. the union of all cells of dimension $\leq l$.

is of such type if we consider $X = S^1 \times S^1$ with the usual CW-structure, where the 1-skeleton is precisely $S^1 \vee S^1$.

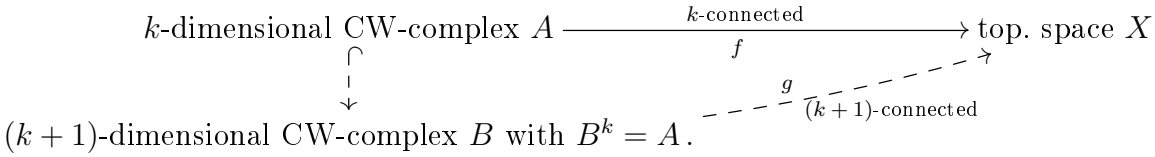


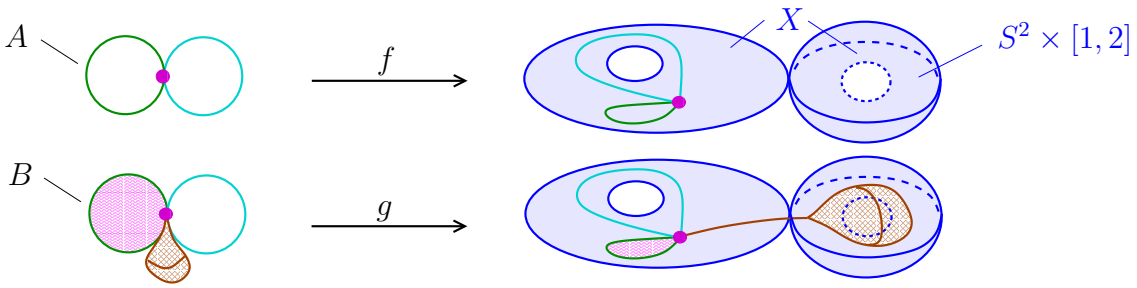
The following is the main technical result which underpins many of the later results.

Proposition 161.1. (Higher Connected Map Proposition) Let $k \in \mathbb{N}_0$, let A be a CW-complex and let $f\colon A \to X$ be a map to a topological space. If the map $f\colon A \to X$ is k-connected, then there exists CW-complex B and a map $g\colon B \to X$ such that the following statements hold:

(1) A is a subcomplex of B,
(2) the only extra cells of B are $(k+1)$-dimensional, in particular the k-skeleton of B equals the k-skeleton of A,
(3) the restriction of g to A equals f,
(4) the map $g\colon B \to X$ is $(k+1)$-connected.

Remark. In most applications of the Higher Connected Map Proposition 161.1 we start out with a k-dimensional CW-complex. In this case we obtain a $(k+1)$-dimensional CW-complex such that k-skeleton equals A and such that we can complete the map $f\colon A \to X$ to the following diagram:





Before we turn to the proof of the Higher Connected Map Proposition 161.1 let us first formulate and prove the following corollary.

Corollary 161.2. Let A be a CW-complex and let $f\colon A \to X$ be a map to a topological space. There exists a CW-complex B and a map $g\colon B \to X$ such that the following statements hold:

(1) A is a subcomplex of B,
(2) the restriction of g to A equals f,
(3) the map $g\colon B \to X$ is k-connected for every $k \in \mathbb{N}_0$.

Proof of Corollary 161.2. Let A be a CW-complex and let $f\colon A \to X$ be a map to a topological space. Without loss of generality we can and will assume that A and X are 0-connected. We set $B_0 := A$ and $g_0 := f$. We apply the Higher Connected Map Proposition 161.1 to B_0 and g_0 and iterate this procedure to obtain a sequence of CW-complexes $\{B_n\}_{n\in\mathbb{N}_0}$ and maps $\{g_n\colon B_n \to X\}_{n\in\mathbb{N}_0}$ such that for each $n \in \mathbb{N}$ the following statements hold:

(a) A is a subcomplex of B_n and B_{n-1} is a subcomplex of B_n,
(b) the only cells of B_n that do not lie in B_{n-1} are n-dimensional,
(c) the restriction of g_n to A equals f,
(d) the map $g_n\colon B_n \to X$ is $(n+1)$-connected.

We set $B := \varinjlim B_n$ and we equip with the CW-structure given by the Ascending Union-CW Complex Lemma 68.34. Furthermore we denote by $g\colon B \to X$ the map that is uniquely determined by the property that for each $n \in \mathbb{N}_0$ we have $g|_{B_n} = g_n$. We claim that (B, g) have the desired properties. In fact (1) and (2) are clear by construction. Now let $k \in \mathbb{N}_0$. For a fixed base point in B_{k+1} we consider the following commutative diagram

$$\begin{array}{ccccccc}
\pi_k(B) & \xleftarrow{\cong} & \pi_k((k+1)\text{-skeleton of } B) & \xrightarrow{\mathrm{id}} & \pi_k((k+1)\text{-skeleton of } B_{k+1}) & \xrightarrow{=} & \pi_k(B_{k+1}). \\
& {\scriptstyle g_*}\searrow & \downarrow{\scriptstyle g_*} & \swarrow{\scriptstyle g_*} & & {\scriptstyle \cong}\searrow{\scriptstyle g_*} & \\
& & \pi_k(X) & & & &
\end{array}$$

Note that by construction the $(k+1)$-skeleton of B equals the $(k+1)$-skeleton of B_{k+1}, thus the middle map is indeed the identity. It follows from the CW-Skeleton-π_n-Proposition 71.9 (1) that the outer horizontal maps are isomorphisms. Finally note that $g|_{B^{k+1}} = g_{k+1}$, which induces an isomorphism on π_k. Thus the left diagonal map is, as promised, an isomorphism. Keeping in mind the remark on page 3396 we see that the map $g\colon B \to X$ is k-connected for every $k \in \mathbb{N}_0$. ■

Now we turn to the proof of the Higher Connected Map Proposition 161.1.

Proof of the Higher Connected Map Proposition 161.1. To simplify the discussion we only deal with the case that A and X are 0-connected. We leave it to the reader to handle the case that at least one of A and Y is not 0-connected. In particular we will not deal with the case $k = 0$.

So in the following, let $k \in \mathbb{N}$, let A be a 0-connected CW-complex and let $f\colon A \to X$ be a k-connected map to a path-connected topological space. We pick a base point a_0 in the 0-skeleton of A. We write $x_0 = f(a_0)$. We note that in general the map $f\colon A \to X$ fails to be $(k+1)$-connected for the following two reasons:[1970]

(a) the map $\pi_k(A, a_0) \to \pi_k(X, x_0)$ is not a monomorphism,
(b) the map $\pi_{k+1}(A, a_0) \to \pi_{k+1}(X, x_0)$ is not an epimorphism.

[1970]Here and throughout the proof we will implicitly use the remark on page 3396 regarding k-connected maps.

Both issues will be fixed by attaching $(k+1)$-cells to A and by extending f to the $(k+1)$-cells in a suitable way. More precisely, we construct the CW-complex B and a map $g\colon B \to X$ as follows:

(a) As usual we denote by A^k the k-skeleton of A. Note that by the CW-Skeleton-π_n-Proposition 71.9 the map $\pi_k(A^k, x_0) \to \pi_k(A, a_0)$ is an epimorphism. This implies that we can pick a family $\{\varphi_i\colon (S^k, *) \to (A^k, a_0)\}_{i\in I}$ of maps that *go into the k-skeleton of A*, and that represent a generating set for $\ker(f_*\colon \pi_k(A, a_0) \to \pi_k(X, x_0))$.[1971] Note that by definition each map $f \circ \varphi_i\colon (S^k, *) \to (X, x_0)$ represents the trivial element in $\pi_k(X, x_0)$. Thus it follows from the Trivial-in-π_n-Lemma 71.4 that for each $i \in I$ where exists a map $F_i\colon \overline{B}^{k+1} \to X$ such that $F_i|S^k = f \circ \varphi_i$.

(b) Next we pick a family $\{\psi_j\colon (S^{k+1}, *) \to (X, x_0)\}_{j\in J}$ of maps that represent a generating set for $\pi_{k+1}(X, x_0)$.

(c) We set

$$B = \Big(A \sqcup \bigsqcup_{i\in I} \overline{B}_i^{k+1}\Big)/\!\sim \;\vee \bigvee_{j\in J} S_j^{k+1} \quad \text{with } \varphi_i(s) \sim s \text{ for } s \in S_i^k = \partial\overline{B}_i^{k+1}.$$

Note that B is a CW-complex in the following sense:

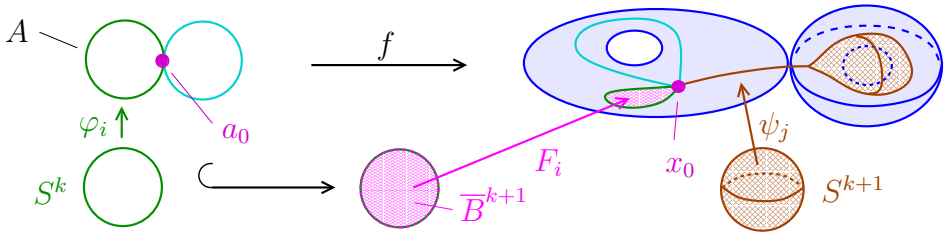
(i) We start out with the CW-complex that is given by the k-skeleton of A.

(ii) We attach one $(k+1)$-cell for each $i \in I$ with attaching map given by φ_i.

(iii) Afterwards we form the wedge with $\bigvee_{j\in J} S_j^{k+1}$. In other words, we attach one $(k+1)$-cell for each $j \in J$ where the attaching map is given by sending all points in $S_j^k = \partial\overline{B}_j^{k+1}$ to a_0.[1972]

(iv) Finally we attach the cells of A of dimension $\geq k+1$ as usual to the above.

(d) We extend the original map $f\colon A \to X$ to a map $g\colon B \to X$ by defining it to be F_i on each $\overline{B}_i^{k+1}$ and by defining it to be ψ_j on each S_j^{k+1}. It follows easily from the CW-Complex Pasting Lemma 68.20 that this map $g\colon B \to X$ is indeed continuous.



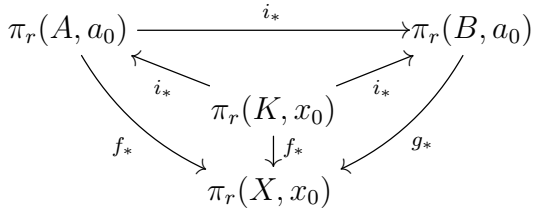
By construction A is a subcomplex of B and the only extra cells of B are $(k+1)$-dimensional. Furthermore, by construction the restriction of g to A equals f. Thus it remains to show

[1971] A generating set for the kernel is for example given by taking all elements in the kernel.

[1972] Here we implicitly use the convention from page 3396 to view the wedge of spheres, and thus also B, as a CW-complex.

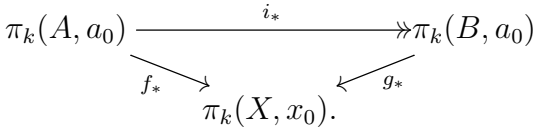
that the map $g\colon B \to X$ is $(k+1)$-connected. We denote by $K := A^k = B^k$ the k-skeleton of A which equals by construction the k-skeleton of B.

(a) First we show that the map $g\colon B \to X$ is k-connected. To do so we note that for any $r \in \mathbb{N}_0$ we have the following commutative diagram



where i always denotes an inclusion map. Since K is the k-skeleton of A and B we obtain from Corollary 153.10 that the maps $K \to A$ and $K \to B$ are k-connected. By hypothesis the map $f\colon A \to X$ is k-connected. It follows almost immediately from this discussion and the commutative diagram that $g\colon B \to X$ is also k-connected. Note that this discussion also implies that the inclusion $A \to B$ is k-connected.

(b) Next we show that $g_*\colon \pi_k(B, a_0) \to \pi_k(X, x_0)$ is a monomorphism. Similar to (a) we consider the following commutative diagram:



As we pointed out in (a), we know that the top map $i_*\colon \pi_k(A, a_0) \to \pi_k(B, a_0)$ is an epimorphism. Thus it suffices to show that

$$\ker(f_*\colon \pi_k(A, a_0) \to \pi_k(X, x_0)) \;\subset\; \ker(i_*\colon \pi_k(A, a_0) \to \pi_k(B, a_0)).$$

But this inclusion holds basically by definition of B. More precisely, it follows from the definition of B that for each $i \in I$ the map $\varphi_i\colon (S^k, *) \to (A, a_0)$ extends to a map $(\overline{B}^{k+1}, *) \to (B, a_0)$. But by the Trivial-in-$\pi_n$-Lemma 71.4 we know that this implies that the maps $\varphi_i\colon (S^k, *) \to (A, a_0)$ represent elements in the kernel of i_*. On other hand, by definition, these maps represent a generating set for $\ker(f_*\colon \pi_k(A, a_0) \to \pi_k(X, x_0))$. The desired inclusion of kernels is now a consequence of Lemma 51.3.

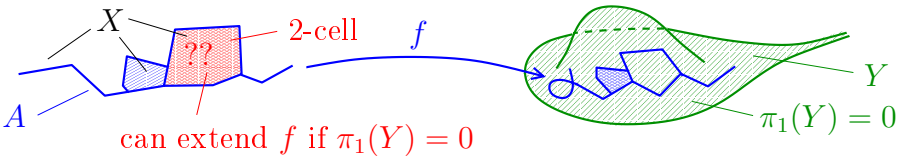
(c) It remains to show that the map $g_*\colon \pi_{k+1}(B, a_0) \to \pi_{k+1}(X, x_0)$ is an epimorphism. In fact this is true almost by definition of B. More precisely, given $j \in J$ we have $g_*([\mathrm{id}_{S^{k+1}_j}] = [g|_{S^{k+1}_j}] = [\psi_j]$. This shows that the classes $\{[\psi_j]\}_{j\in J}$ lie in the image of g_*. But since these classes generate $\pi_{k+1}(X, x_0)$ we obtain from Lemma 51.3 that g_* is an epimorphism. ∎

The following lemma gives a simple but useful criterion for extending maps from a subcomplex to the whole CW-complex.

Lemma 161.3. Let (X, A) be a pair of CW-complexes and let $f\colon A \to Y$ be a map to a path-connected topological space. Suppose the following condition holds:

(∗) For every $n \in \mathbb{N}$ such that $X \backslash A$ contains an n-cell we have the equality[1973] $\pi_{n-1}(Y) = 0$.

Then f can be extended to a map $\widetilde{f}\colon X \to Y$. This statement is illustrated in the figure below.



Proof. We start out with the following claim.

Claim. Given any $n \in \mathbb{Z}_{\geq -1}$ we can extend $f\colon A \to Y$ to a map $A \cup X^n \to Y$.

Proof. We prove the claim by induction on $n \in \mathbb{Z}_{\geq -1}$. For $n = -1$ there is nothing to prove. So suppose the statement holds for some $n \in \mathbb{Z}_{n \geq -1}$. By the slightest of all abuses we denote the given extension $A \cup X^n \to Z$ again by f. We denote by $\{\Phi_i\colon S^n \to X^n\}_{i\in I}$ the family of the attaching maps of all $(n+1)$-dimensional cells of X that are not contained in A. If $I = \varnothing$, then there is nothing to do. On the other hand if $I \neq \varnothing$, then we know by the hypothesis $(*)$ that $\pi_n(Y) = 0$. It follows from the Trivial-in-π_n-Lemma 71.4 that for each $i \in I$ there exists a map $g_i\colon \overline{B}^{n+1} \to Y$ with $g_i|_{S^n} = f \circ \Phi_i\colon S^n \to Y$. We consider the following map:

$$\begin{array}{rcl} A \cup X^{n+1} = \Big((A \cup X^n) \sqcup \bigsqcup\limits_{i\in I} \overline{B}_i^{n+1}\Big)/\!\sim & \to & Y \\ [P] & \mapsto & \begin{cases} f(P), & \text{if } P \in A \cup X^n, \\ g_i(P), & \text{if } P \in \overline{B}_i^{n+1}. \end{cases} \end{array}$$

It follows from $g_i|_{S^n} = f \circ \Phi_i$ that this map is well-defined. Furthermore we obtain from the Topological-Quotient Proposition 5.15 (1b) that the map is continuous. ⊞

The sequence of maps from the claim defines a map $\widetilde{f}\colon X = A \cup X = \bigcup\limits_{n\in\mathbb{N}_0} (A \cup X^n) \to Y$. Since the map is continuous on each $A \cup X^n$ we obtain from the CW-Complex Pasting Lemma 68.20 that $\widetilde{f}$ is also continuous. It now clear that this map $\widetilde{f}$ is the desired extension. ■

We conclude this section with the formulation of a special case Lemma 161.3.

Definition. We say a topological space Y is weakly contractible if[1974] $\pi_n(Y) = 0$ for all $n \in \mathbb{N}_0$.

Example. By Corollary 71.12 we know that the infinite-dimensional sphere S^∞ is weakly contractible.

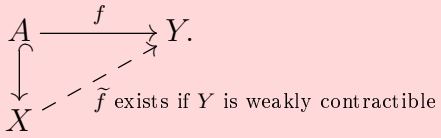
The following corollary is an immediate consequence of Lemma 161.3.

Corollary 161.4. (Weakly Contractible Domain Corollary) Let (X, A) be a pair of CW-complexes and let $f\colon A \to Y$ be a map to a topological space. If Y is weakly contractible, then f can be extended to a map $\widetilde{f}\colon X \to Y$. In other words, we have the

[1973] Since Y is path-connected we know by the Change-of-Base Point Proposition 71.5 that we do not need to worry about base points.

[1974] Recall that by the definition on page 167 the statement $\pi_0(Y) = 0$ means that Y is *non-empty* and path-connected.

following situation



161.2. **Weak homotopy equivalences and the CW-Approximation Theorem.** The definition of a k-connected map between topological spaces naturally leads to the following definition which we had encountered once before on page 2062.

Definition. We say that a map $f\colon X \to Y$ between topological spaces is a weak homotopy equivalence if f is k-connected for all $k \in \mathbb{N}_0$.

The following lemma gives in particular a practical approach to showing that a given map is a weak homotopy equivalence.

Lemma 161.5. (Weak Homotopy Equivalence Criteria Lemma) Let $f\colon X \to Y$ be a map between topological spaces.

(1) If f is a homotopy equivalence, then it is also a weak homotopy equivalence.
(2) If $g\colon X \to Y$ is a map that is homotopic to f, then

$$f \text{ is a weak homotopy equivalence} \iff g \text{ is a weak homotopy equivalence}$$

(3) If X and Y are path-connected, then the following holds:

$$f \text{ is a weak homotopy equivalence} \iff \begin{array}{c}\text{there exists an } x_0 \in X \text{ such that} \\ \text{for every } k \in \mathbb{N} \text{ the induced map} \\ f_*\colon \pi_k(X, x_0) \to \pi_k(Y, f(x_0)) \\ \text{is an isomorphism.}\end{array}$$

Proof.

(1) This statement is an immediate consequence of the Homotopy-π_n-Proposition 71.7 (2).
(2) This statement follows immediately from the Homotopy-π_n-Proposition 71.7 (1).
(3) This statement follows easily from the definitions, the fact that X and Y are path-connected and the Change-of-Base Point Proposition 86.9 (3). ■

As we just saw in the Weak Homotopy Equivalence Criteria Lemma 161.5, a homotopy equivalence is also a weak homotopy equivalence. But the following example shows that in general the converse does not hold.

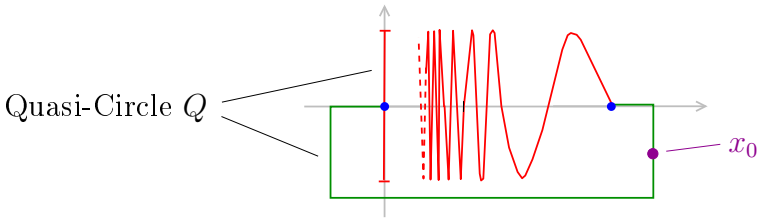
Example. Let

$$S := \left\{(0, y) \,\middle|\, y \in [-1, 1]\right\} \cup \left\{(x, \sin(\tfrac{1}{x})) \,\middle|\, x \in (0, \tfrac{1}{\pi}]\right\} \subset \mathbb{R}^2$$

be the "Topologist's Sine Curve" from page 164. We turn the Topologist's Sine curve into the *Quasi-Circle* Q by connecting the point $(0, 0)$ with $(\frac{1}{\pi}, 0)$ via the green curve shown in the figure below. We make the following observations:

(1) One can easily show that the quasi-circle is path-connected, i.e. $\pi_0(Q) = 0$.
(2) In Exercise 2.48 we showed that the Topologist's Sine Curve is not path-connected. As we saw in Exercise 71.9, this fact easily implies that all homotopy groups $\pi_n(Q)$, $n \geq 1$ are zero.

(3) Pick $x_0 \in Q$. It follows from (1) and (2) that the inclusion $i\colon \{x_0\} \to Q$ is a weak homotopy equivalence.
(4) In Exercise 15.14 we showed that Q is not contractible.[1975] By the Contractibility Criterion Lemma 15.2 this implies that the inclusion i is not a homotopy equivalence.



We conclude this discussion of weak homotopies with the following proposition which relates weak homotopy equivalences to induced maps on homology groups.

Proposition 161.6. (Weak Homotopy-vs-Homology Isomorphisms Proposition) Let $f\colon X \to Y$ be a map between topological spaces.

(1) If f is a weak homotopy equivalence, then for any abelian group G and any $n \in \mathbb{N}_0$ the induced maps $f_*\colon \mathrm{H}_n(X;G) \to \mathrm{H}_n(Y;G)$ and $f^*\colon \mathrm{H}^n(Y;G) \to \mathrm{H}^n(X;G)$ are isomorphisms.
(2) If X and Y are simply connected and if $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is an isomorphism for every $n \geq 2$, then f is a weak homotopy equivalence.

In fact we can basically generalize (2) to the following less readable statement:

(2′) Suppose the following conditions are satisfied:
  (a) X and Y are both path-connected, both are locally path-connected and both are semi-locally simply connected. (As discussed in Corollary 61.11 this means that both topological spaces admit universal coverings.)
  (b) For some $x \in X$ the induced map $f_*\colon \pi_1(X,x)) \to \pi_1(Y, f(x))$ is an isomorphism.

  We write $y = f(x)$ and we denote by $p\colon (\widetilde{X},\widetilde{x}) \to (X,x))$ and $q\colon (\widetilde{Y},\widetilde{y}) \to (Y,y)$ the universal covers of (X,x) and (Y,y). By (a) and the Map Lifting Criterion 61.2 we know that the map f lifts to a map $\widetilde{f}\colon (\widetilde{X},\widetilde{x}) \to (\widetilde{Y},\widetilde{y})$. We also demand the following:
  (c) The map $\widetilde{f}_*\colon \mathrm{H}_n(\widetilde{X}) \to \mathrm{H}_n(\widetilde{Y})$ is an isomorphism for every $n \geq 2$.

  Then f is a weak homotopy equivalence.

Proof.

[1975] The argument we provided in Exercise 15.14 is rather ad hoc. There are several other approaches. For example one can use the Čech cohomology groups $\check{\mathrm{H}}(X;\mathbb{Z})$ of a topological space X, which are for example defined in [**Dol1980**, Chapter VIII.6]. Note that

$$\underset{\text{by the Duality Theorem [Dol1980, Theorem VIII.7.2]}}{\check{\mathrm{H}}(Q;\mathbb{Z}) \cong \widetilde{\mathrm{H}}_0(\mathbb{R}^2 \setminus Q;\mathbb{Z})} \underset{\text{by Exercise 43.11 we know that } \mathbb{R}^2 \setminus Q \text{ has two path-components}}{\cong \mathbb{Z} \neq 0 \cong \mathrm{H}^1(\{*\};\mathbb{Z})} \underset{\text{by Proposition VIII.6.12 in [Dol1980]}}{\cong \check{\mathrm{H}}^1(\{*\};\mathbb{Z})}.$$

By [**Dol1980**, Proposition VIII.6.6] the Čech cohomology groups are homotopy invariants. Thus the above argument shows that Q and $\{*\}$ are not homotopy equivalent. A more direct calculation of $\check{\mathrm{H}}(Q;\mathbb{Z}) \cong \mathbb{Z}$ is provided in [**MillH2021**, Chapter 34].

(1) Let $f\colon X \to Y$ be a weak homotopy equivalence. It follows easily from Lemmas 72.14 and Lemmas 108.14 that we can assume, without loss of generality, that X and Y are path-connected. Let $x_0 \in X$. By our hypothesis we know that the map $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism for all $n \in \mathbb{N}_0$. It follows from Theorem 153.14 (1), which in turn is a consequence of the Relative Hurewicz Theorem 153.13, that the induced maps $f_*\colon \mathrm{H}_n(X;\mathbb{Z}) \to \mathrm{H}_n(Y;\mathbb{Z})$ are isomorphisms for all $n \in \mathbb{N}_0$. The desired statement is now a consequence of Corollary 108.20.
(2) This statement is just a reformulation of Theorem 153.14 (2).
(3) By (b) together with the Weak Homotopy Equivalence Criteria Lemma 161.5 it remains to show that $f_*\colon \pi_n(X, x)) \to \pi_n(Y, y)$ is an isomorphism for every $n \geq 2$. Now let $n \geq 2$. We consider the following diagram:

$$\begin{array}{ccc} \pi_n(\widetilde{X}, \widetilde{x}) & \xrightarrow{\widetilde{f}_*} & \pi_n(\widetilde{Y}, \widetilde{y}) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle q_*} \\ \pi_n(X, x) & \xrightarrow{f_*} & \pi_n(Y, y). \end{array}$$

We make the following observations:
(a) The diagram commutes by definition of $\widetilde{f}$.
(b) It follows from the Coverings-π_n-Proposition 71.13 that the vertical maps are isomorphisms.
(c) By (2) and by our hypothesis (c) we know that $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ is a weak homotopy equivalence. This implies that the top horizontal map is an isomorphism.

It follows from the above that the bottom horizontal map is an isomorphism. ■

In the remainder of this section we will study CW-approximations of topological spaces.

Definition. Let X be a topological space. A CW-approximation is a CW-complex Z together with a weak homotopy equivalence $f\colon Z \to X$.

Example. Let X be the quasi-circle. As we had just discussed, the inclusion of the CW-complex $\{x_0\}$, consisting of a single 0-cell, into the quasi-circle X is a CW-approximation of the quasi-circle X.

Before we state a general theorem we turn to the delicate topic of products of CW-complexes. More precisely, if X and Y are CW-complex with only countably many cells, then it follows from Proposition 68.23 that the product $X \times Y$ has a product CW-structure. But, as we pointed out in Lemma 68.24, in general this is not true for arbitrary CW-complexes. This leads us to the following lemma.

Lemma 161.7. Let X and Y be two CW-complexes. We define the topological space $X \otimes Y$, whose underlying *set* is the product $X \times Y$, as on page 1500. The identity map $\mathrm{id}\colon X \otimes Y \to X \times Y$ is a CW-approximation.

Proof. By Proposition 68.23 we know that $X \otimes Y$ is a CW-complex and we know that the identity map $\mathrm{id}\colon X \otimes Y \to X \times Y$ is continuous. In Proposition 68.26 we showed that the induced maps on fundamental groups are isomorphisms. Basically the same argument, with the obvious extension of Lemma 68.27 to higher dimensions, shows that the map $\mathrm{id}\colon X \otimes Y \to X \times Y$ is a weak homotopy equivalence. ■

The following theorem says that CW-approximations always exist.

Theorem 161.8. (CW-Approximation Theorem)

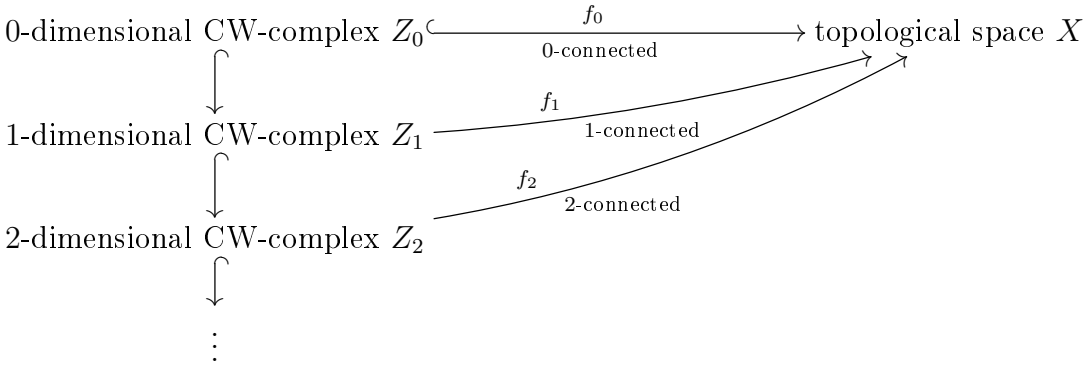
(1) Every topological space X admits a CW-approximation $f\colon Z \to X$.
(2) Let $k \in \mathbb{N}_0$. If X is a k-connected space, then there exists a CW-approximation $f\colon Z \to X$ such that Z has a single 0-cell and such that Z has no cells in dimensions $1, \ldots, k$.

In both cases, given a point $x_0 \in X$ we can find f and $z_0 \in Z^0$ such that $f(z_0) = x_0$.

Proof of the CW-Approximation Theorem 161.8. Let X be a topological space.

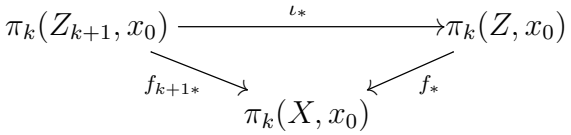
(1) We need to show that X admits a CW-approximation. By considering each path-component of X separately we can assume that X is path-connected.

Thus let X be a path-connected topological space. We pick a point $x_0 \in X$. We set $Z_0 = \{x_0\}$ and we define $f_0\colon Z_0 \to X$ as the inclusion map. By iteratively appealing to the Higher Connected Map Proposition 161.1 we obtain the following commutative diagram



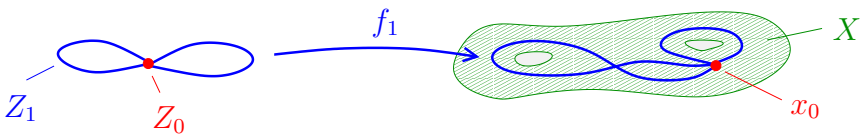
such that the $(k-1)$-skeleton of each Z_k equals Z_{k-1}. We set $Z = \bigcup_{k \in \mathbb{N}_0} Z_k$ and we define $f\colon Z \to X$ in the obvious way. Note that by construction Z is a CW-complex such that for each $k \in \mathbb{N}_0$ the k-skeleton is given by Z_k. Furthermore it follows from the CW-Complex Pasting Lemma 68.20 that f is continuous.

It remains to show that $f\colon Z \to X$ is indeed a weak homotopy equivalence. So let $k \in \mathbb{N}_0$. We consider the following commutative diagram



where $\iota\colon Z_{k+1} \to Z$ denotes the inclusion map. The left-hand diagonal map is an isomorphism by construction of $f_{k+1}\colon Z_{k+1} \to X$. Furthermore the top horizontal map is an isomorphism by the CW-Skeleton-π_n-Proposition 71.9 and the fact that Z_{k+1} is the $(k+1)$-skeleton of Z. We deduce from the commutative diagram that the right-hand diagonal map is also an isomorphism.

(2) Let $k \in \mathbb{N}_0$. If X is k-connected, then in the proof given in (1) we can start out with the CW-complexes $\{x_0\} = Z_0 = Z_1 = \cdots = Z_k$ and where $f_0 = f_1 = \cdots = f_k$ are the inclusion maps. Only beginning with f_{k+1} do we appeal to the Higher Connected Map Proposition 161.1 to provide the further CW-complexes Z_j and maps $f_j\colon Z_j \to X$.



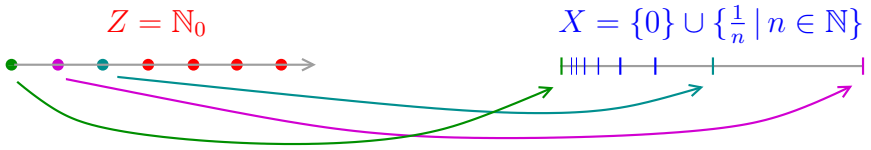
For the remainder of the proof of (2) we now proceed as in (1). Since the k-skeleton of $Z = \bigcup_{k\in\mathbb{N}_0} Z_k$ equals $Z_k = \{x_0\}$ we see that Z has the desired properties. ■

Examples.

(1) We consider the topological space $X = \{0\} \cup \{\frac{1}{n} \,|\, n \in \mathbb{N}\} \subset \mathbb{R}$ equipped with the subspace topology coming from $\mathbb{R}$. Note that X is not a CW-complex.[1976] Furthermore we consider $Z = \mathbb{N}_0 \subset \mathbb{R}$ with the discrete topology. This is a 0-dimensional CW-complex and the map

$$\begin{array}{rcl} f\colon Z = \mathbb{N}_0 & \to & X = \{0\} \cup \{\frac{1}{n} \,|\, n \in \mathbb{N}\} \subset \mathbb{R} \\ n & \mapsto & \begin{cases} 0, & \text{if } n = 0, \\ \frac{1}{n}, & \text{otherwise} \end{cases} \end{array}$$

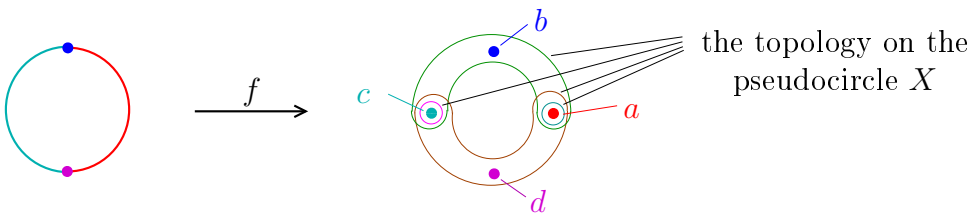
is easily seen[1977] to be a weak homotopy equivalence. Thus $f\colon Z \to X$ is a CW-approximation. We refer to the figure below for an illustration.



(2) We consider the pseudocircle that we introduced on page 141. Recall that the pseudocircle is the set $X = \{a, b, c, d\}$ where the topology is given by the set

$$\mathcal{T} \;:=\; \{\varnothing, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}.$$

On page 141 we gave an explicit map $f\colon S^1 \to X$ that is also illustrated in the figure below. In Exercise 48.15 we showed that $f_*\colon \pi_1(S^1, 1) \to \pi_1(X, a)$ is an isomorphism. A slight variation on the definition of f also shows that X is path-connected. Furthermore, by the discussion on page 1573 and by Exercise 71.5 we know that $\pi_n(S^1, 1) = \pi_n(X, a) = 0$ for $n \geq 2$. Thus we obtain from the Weak Homotopy Equivalence Criteria Lemma 161.5 that f is a weak homotopy equivalence. Since S^1 is a CW-complex we see that the map $f\colon S^1 \to X$ is a CW-approximation for the pseudocircle X.



[1976] Why is it not a CW-complex?
[1977] What is so easy about this?

Exercises for Chapter 161.

Exercise 161.1.

(a) Give an explicit CW-approximation for the Cantor set, as defined on page 130.
(b) Give an explicit CW-approximation for the Topologist's Sine Curve, as defined on page 164.

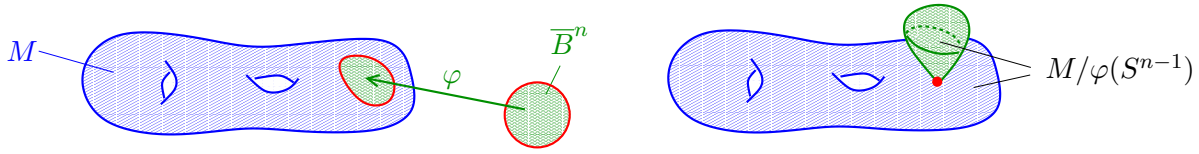
Exercise 161.2.

(1) Let M be a connected non-empty n-dimensional smooth manifold. Furthermore let $\varphi\colon \overline{B}^n \to M$ be a smooth embedding. Show that there exists a homeomorphism $\varphi\colon M/\varphi(S^{n-1}) \to M \vee S^n$ such that the map $M \to M/\varphi(S^{n-1}) \xrightarrow{\varphi} M \vee S^n \to M$ is homotopic to the identity.
(2) We consider the map

$$\Theta\colon S^1 \times S^2 \xrightarrow{\text{projection}} (S^1 \times S^2) \vee S^3 \xrightarrow{\text{as in (1)}} (S^1 \times S^2) \vee S^3 \to S^1 \times S^2.$$

(the last map: given by id on $S^1 \times S^2$ and by the Hopf map $S^3 \to S^2$ on S^3)

Show the following statements:
(a) Θ induces the identity on all homology groups.
(b) Θ induces the identity on π_1 and π_2.
(c) Θ does not induce the identity on π_3.



162. The Whitehead Theorem

In this chapter we will prove the Whitehead Theorem 162.1 which says that if a map $f\colon X \to Y$ between CW-complexes is a weak homotopy equivalence, i.e. if it induces an isomorphism of all homotopy groups, then f is in fact a homotopy equivalence. Afterwards we will study several applications of the Whitehead Theorem 162.1. In particular we will be able to complete the classification of lens spaces up to homotopy equivalence.

162.1. **The Whitehead Theorem.** Usually, in algebraic topology one goes from topological situations to algebra using various functors. For example, any map between (pointed) topological spaces gives rise to a map between homology groups and homotopy groups. Unfortunately almost invariably in the passage from topology to algebra some information gets lost. For example there are many non-homeomorphic topological spaces with isomorphic homology groups.

In the Homotopy-π_n-Proposition 71.7 (2) we saw that a homotopy equivalence between topological spaces gives rise to isomorphisms of higher homotopy groups. The following theorem shows that, rather amazingly, in this particular case the converse holds, at least if we restrict ourselves to maps between CW-complexes.

Theorem 162.1. (Whitehead Theorem)[1978] Let $f\colon X \to Y$ be a map between two connected CW-complexes.[1979]

(1) If $f\colon X \to Y$ is a weak homotopy equivalence, then f is a homotopy equivalence.

In fact, there is a more refined version dealing with base points:

(2) Let $x_0 \in X^0$ and $y_0 \in Y^0$ be base points in the 0-skeleta of X respectively Y. If $f\colon X \to Y$ is a weak homotopy equivalence with $f(x_0) = y_0$, then there exists a homotopy equivalence between the pointed topological spaces (X, x_0) and (Y, y_0).

For inclusion maps we have the following very useful variation on the Whitehead Theorem 162.1.

Proposition 162.2. (Whitehead Deformation Retract Proposition) Let Y be a connected CW-complex.

(1) Let X be a connected subcomplex and let $x_0 \in X$ be a point. We denote by $i\colon X \to Y$ the inclusion map. If the induced maps $i_*\colon \pi_n(X, x_0) \to \pi_n(Y, x_0)$ are isomorphisms for all $n \in \mathbb{N}$, then X is a deformation retract of Y.

(2) If Y is weakly contractible, then for every point $y_0 \in Y$ the one-point subset $\{y_0\}$ is a deformation retract of Y. In particular Y is contractible.

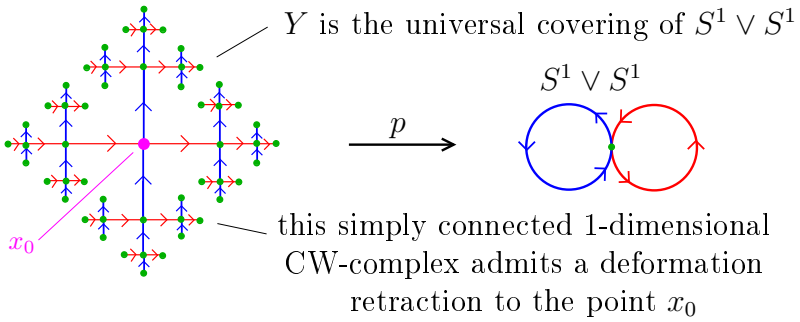
Examples.

(1) Let Y be a simply connected 1-dimensional CW-complex and let x_0 be a point in the 0-skeleton of Y. By Lemma 71.15 (or alternatively Lemma 85.8) we know that $\pi_n(Y, x_0) = 0$ for all $n \in \mathbb{N}$. It follows from the Whitehead Deformation Retract Proposition 162.2 (2) that Y admits a deformation retraction to $\{x_0\}$, in particular Y is contractible.

[1978] As we pointed out on page 3258, the name "Whitehead Theorem" is sometimes also used for Theorem 153.14. Our naming convention agrees with [**Hat2002**, Theorem 4.5] and [**Rot1988**, p. 370].

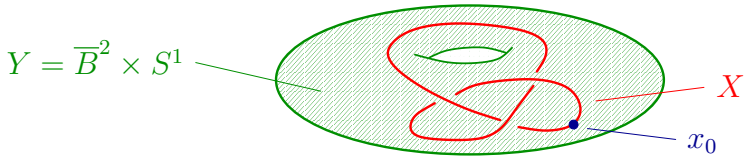
[1979] Note that we do not assume that the map f is cellular.

For example, let Y be the universal covering of $S^1 \vee S^1$. We view $S^1 \vee S^1$ as a CW-complex in the obvious way. By the CW-complex Covering Proposition 68.37 we can equip Y with the structure of a 1-dimensional CW-complex. By definition of a universal covering we know that Y is simply connected. It follows from the above discussion that any point x_0 in the 0-skeleton of Y is a deformation retract. We refer to the figure below for an illustration.



(2) On page 3401 we pointed out that Corollary 71.12 says that S^∞ is weakly contractible. Thus it follows from the Whitehead Deformation Retract Proposition 162.2 (2) that S^∞ is contractible.[1980]

(3) We consider the knotted curve X in the solid torus $Y = \overline{B}^2 \times S^1$ that is shown in the figure below. It follows from the Smooth Manifold-Simplicial Structure Theorem 96.2 (3) together with the Simplicial-Implies-CW Lemma 93.24 that we can equip Y with a CW-structure such that X is a subcomplex. It should be pretty clear that the inclusion induced map $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ is an isomorphism. Furthermore, since X and Y are homotopy equivalent to S^1 we see that the higher homotopy groups of X and Y vanish. In particular for every $n \in \mathbb{N}_{\geq 2}$ the induced maps $i_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ are isomorphisms. In other words we see that all the hypotheses of the Whitehead Deformation Retract Proposition 162.2 are satisfied. This shows that X is a retract of Y. This answers the question that we had posed on page 1061.



Remark. One might be tempted to interpret the Whitehead Theorem 162.1 as saying that if two CW-complexes have isomorphic homotopy groups, then they are homotopy equivalent. But this statement is false.

(1) For example let us consider the lens spaces $L(5, 1)$ and $L(5, 2)$. Recall that both are quotients of S^3 by some action of $\mathbb{Z}_5$ and both are closed 3-dimensional smooth manifolds. In particular both are CW-complexes by the Smooth Manifold-CW Structure

[1980] In fact in Lemma 68.4 we gave an explicit homotopy from the identity of S^∞ to a constant map. By the Contractibility Criterion Lemma 15.2 this implies that S^∞ is contractible.

Theorem 96.5 .[1981] On page 1574 we saw that the homotopy groups of these two lens spaces are isomorphic. On the other hand we seen on page 3156 that $L(5,1)$ and $L(5,2)$ are not homotopy equivalent.

(2) We consider the topological spaces S^2 and $S^3 \times \mathbb{C}\mathrm{P}^\infty$. Evidently S^2 and S^3 are CW-complexes. It follows from Lemma 68.6 and Proposition 68.23 that $S^3 \times \mathbb{C}\mathrm{P}^\infty$ is also a CW-complex. In Exercise 155.12 we saw that S^2 and $S^3 \times \mathbb{C}\mathrm{P}^\infty$ have isomorphic homotopic groups.[1982] On the other hand it follows immediately from the discussion on page 1772 and the Künneth Theorem 90.8 that the homology groups of S^2 and $S^3 \times \mathbb{C}\mathrm{P}^\infty$ are (dramatically) different. Thus the two CW-complexes are not homotopy equivalent.

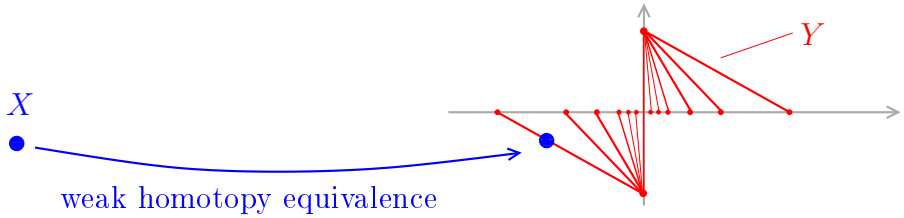
Remark. The analogous statement of the Whitehead Theorem 162.1, respectively the Whitehead Deformation Retract Proposition 162.2, does not hold if we drop the hypothesis that X and Y are CW-complexes, in other words, the statement does not hold for general topological spaces. We can provide two examples:

(1) On page 3402 we already discussed the fact that the Quasi-Circle is weakly contractible but not contractible.

(2) Since weird topological spaces are so much fun, let us consider another example. More precisely, let us consider the following subset of $\mathbb{R}^2$ which is also illustrated in the figure below:

$$\begin{aligned} Y := \text{line segment from } (0,1) \text{ to } (0,-1) \cup \bigcup_{n\in\mathbb{N}} \text{line segment from } (0,1) \text{ to } (\tfrac{1}{n},0) \\ \cup \bigcup_{n\in\mathbb{N}} \text{line segment from } (0,-1) \text{ to } (-\tfrac{1}{n},0). \end{aligned}$$

We equip Y with the subspace topology. In Exercise 162.4 (a) we will see that the homotopy groups of Y are trivial, i.e. Y is weakly contractible. On the other hand we show in Exercise 162.4 (b) that Y is not contractible. The readers who are less inclined towards doing the exercises are referred to [**Mau1980**, p. 301] instead.[1983]



(3) In Exercise 162.13 we will see that the long line $\mathbb{L}$, which we introduced on page 613, is weakly contractible but not contractible.

(4) On [**Barm2011**, p. 54] it is shown that there exists a finite topological space that is weakly contractible but not contractible.

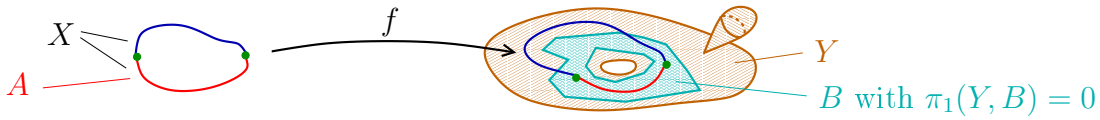
[1981] It is also a moderately amusing exercise to find an explicit CW-structure on lens spaces. One could start either from the definition of a lens space or the reinterpretation given by the Lens Space-Splitting Lemma 146.2.

[1982] In fact this follows easily from the fact that $\pi_k(S^2) = \pi_k(S^3)$ for $k \geq 3$ (see Theorem 155.10) and the fact that $\pi_k(\mathbb{C}\mathrm{P}^\infty) = \mathbb{Z}$ for $k = 2$ and $\pi_k(\mathbb{C}\mathrm{P}^\infty) = 0$ for $k \neq 2$ (see Theorem 155.12 (2)).

[1983] This shows in particular that Y is not a CW-complex. Is there a more direct way for showing that Y is not a CW-complex?

The proof of the Whitehead Theorem 162.1 relies on the following lemma.

Lemma 162.3. Let (X, A) be a pair of CW-complexes and let (Y, B) be a pair of topological spaces with $B \neq \varnothing$. We make the following assumption: For every $n \in \mathbb{N}_0$ such that $X \setminus A$ has at least one n-dimensional cell we assume that $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$. Then every map $f\colon (X, A) \to (Y, B)$ is homotopic rel A to a map $X \to B$.



Proof. We start out with the following claim.

Claim. Let $k \in \mathbb{N}$. Let $f\colon (X, A) \to (Y, B)$ be a map. If $f(X^{k-1}) \subset B$, then there exists a homotopy rel $A \cup X^{k-1}$ from f to a map g with $g(X^k) \subset B$.

Proof. If there are no k-cells in $X \setminus A$, then we have $X^k \subset X^{k-1} \cup A$, in particular we already have $f(X^k) \subset B$, and we are thus done. Now suppose that there exists at least one k-cell in $X \setminus A$. By our hypothesis this implies that $\pi_k(Y, B, y_0) = 0$ for every $y_0 \in B$. Let $\{\Phi_i\colon \overline{B}^k \to X\}_{i \in I}$ be the characteristic maps of the k-cells in $X \setminus A$. Given $i \in I$ we denote by $\phi_i\colon S^{k-1} \to X$ the corresponding attaching map. It follows from $f(X^{k-1}) \subset B$, from $\pi_k(Y, B, y_0) = 0$ and Proposition 153.8 (2) $\Rightarrow$ (1) that there exists a homotopy F_i rel S^{k-1} from the map $f \circ \Phi_i\colon \overline{B}^k \to Y$ to a map whose image is contained in B. It follows from the Homotopy Pushout Lemma 14.9, applied to $A \cup X^k = (A \cup X^{k-1}) \cup_{\bigsqcup_{i \in I} \phi_i} \left(\bigsqcup_{i \in I} \overline{B}_i^k \right)$ that these homotopies on all such cells, together with the constant homotopy on $A \cup X^{k-1}$, give rise to a homotopy H on $A \cup X^k$ rel $A \cup X^{k-1}$ with $\mathrm{H}_0|_{A \cup X^k} = f$ and with $\mathrm{H}_1(X^k) \subset B$. By the Homotopy Extension Theorem 70.1 we can extend this homotopy H to a homotopy $\widetilde{H}\colon X \times [0, 1] \to Y$ with $\widetilde{\mathrm{H}}_0 = f$. This homotopy has all the desired properties. ⊞

First suppose that X is a finite-dimensional CW-complex. It is clear that the hypothesis of the claim holds for $k = -1$ since, by definition, $X^{-1} = \varnothing$. Now we can apply the claim iteratively finitely many times. Combining these finitely many homotopies together as in the Homotopy Combination the Homotopy Combination Lemma 14.4 we obtain the desired homotopy.

In the case that X is infinite-dimensional we obtain an infinite sequence of homotopies. We can combine these homotopies using the rather tricky Lemma 70.13 and we obtain again the desired homotopy. ■

Now we are already in a position to provide the proof of the Whitehead Deformation Retract Proposition 162.2.

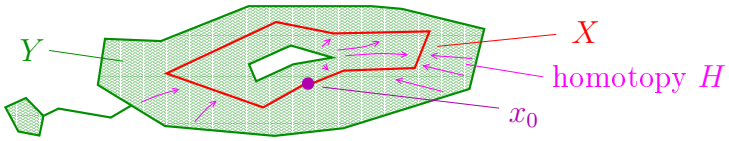
Proof of the Whitehead Deformation Retract Proposition 162.2. Let Y be a connected CW-complex.

(1) Let X be a connected subcomplex and let $x_0 \in X$. We denote by $i\colon X \to Y$ the inclusion map. We suppose that the induced maps $i_*\colon \pi_n(X, x_0) \to \pi_n(Y, x_0)$ are isomorphisms for all $n \in \mathbb{N}$. We need to show that X is a deformation retract of Y.

Recall that by our hypothesis Y is connected, in particular it is path-connected by the CW-Complex Properties Proposition 68.10 (7). Thus it follows from the Change-of-Base Point Proposition 86.9 that the induced maps $i_*\colon \pi_n(X, x_0) \to \pi_n(Y, x_0)$ are

in fact isomorphisms for all $x_0 \in X$. Next we note that it follows from our hypothesis and the long exact sequence of homotopy groups given by the Homotopy Groups-LES Proposition 153.4 that all relative homotopy groups $\pi_n(Y, X, x_0)$ are trivial for all $x_0 \in X$. This observation allows us apply Lemma 162.3 to the identity map $(Y, X) \to (Y, X)$. From Lemma 162.3 we obtain a homotopy H rel X from id_Y to a map whose image lies in X. But this means, by definition, that H is a deformation retraction from Y to X.

(2) We assume that Y is weakly contractible. Let $y_0 \in Y$. We denote by $i\colon \{y_0\} \to Y$ the inclusion. Since Y is weakly contractible we see that for each $n \in \mathbb{N}$ the map $i_*\colon \pi_n(\{y_0\}, y_0) \to \pi_n(Y, y_0)$ is an isomorphism, for the silly reason, that the groups on both sides are trivial. It follows from (1) that $\{y_0\}$ is a deformation retract of Y. The elementary the Point Deformation Retract Lemma 15.6 now tells us that Y is contractible. ■

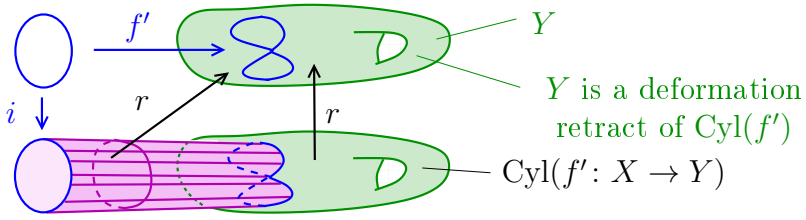


We will reduce the proof of the Whitehead Theorem 162.1 to the Whitehead Deformation Retract Proposition 162.2. The trick hereby is to use the notion of a mapping cylinder that we introduced on page 454. For the reader's convenience we recall the definition.

Definition. Given a map $f\colon X \to Y$ between topological spaces we define the corresponding mapping cylinder to be the topological space

$$\mathrm{Cyl}(f) \;:=\; \mathrm{Cyl}(f\colon X \to Y) \;:=\; ((X \times [0,1]) \sqcup Y)/\!\sim \quad \text{where } (x,1) \sim f(x) \text{ for all } x \in X.$$

We illustrate the definition in the figure below.

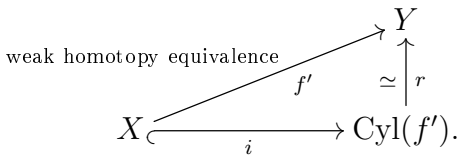


Now we can provide the actual proof of the Whitehead Theorem 162.1.

Proof of the Whitehead Theorem 162.1. Let X and Y be two connected CW-complexes. Let $x_0 \in X$ be a base point. Furthermore let $f\colon X \to Y$ be a map such that the induced maps $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ are isomorphisms for all $n \in \mathbb{N}$.

(1) By the Cellular Approximation Theorem 70.15 the map f is homotopic to a cellular map f'. We make the following two observations:
   (i) By the Homotopy-Equivalence Basics Lemma 15.1 (3) the map f is a homotopy equivalence if and only if f' is a homotopy equivalence.
   (ii) By the Weak Homotopy Equivalence Criteria Lemma 161.5 (2) the map f is a weak homotopy equivalence if and only if f' is a weak homotopy equivalence.

   As usual we denote by $\mathrm{Cyl}(f')$ the mapping cylinder of $f'\colon X \to Y$. We consider the following diagram of maps:



We make the following comments and observations:[1984]

(a) The map $r\colon \mathrm{Cyl}(f') \to Y$ is the obvious map given by the identity on Y and given by sending $[(x,t)]$ to $f'(x)$.
(b) The map $i\colon X \to \mathrm{Cyl}(f')$ is the obvious inclusion given by $x \mapsto [(x,0)]$.
(c) It is clear that the above diagram commutes.
(d) By (ii) and our hypothesis the map $f'\colon X \to Y$ is a weak homotopy equivalence.
(e) By the Mapping Cylinder Basics Lemma 16.12 (2) we know that the map $r\colon \mathrm{Cyl}(f) \to Y$ is a homotopy equivalence. In particular we obtain from the Weak Homotopy Equivalence Criteria Lemma 161.5 (1) that r is a weak homotopy equivalence.
(f) It follows from (c), (d) and (e) that i is a weak homotopy equivalence.
(g) Since $f'\colon X \to Y$ is cellular we use the CW-Complex Cone Corollary 68.35 (3) to equip $\mathrm{Cyl}(f'\colon X \to Y)$ with a natural CW-structure such that $X \times \{0\} = i(X)$ is a subcomplex.
(h) It follows immediately from (f) and (g) together with the Whitehead Deformation Retract Proposition 162.2 that the map $i\colon X \to \mathrm{Cyl}(f')$ is homotopy equivalence.
(i) We obtain from (e) and (h) and the Homotopy-Equivalence Basics Lemma 15.1 (1) that f is indeed a homotopy equivalence.

(2) Now suppose we are given $x_0 \in X^0$ and $y_0 \in Y^0$ such that $f(x_0) = y_0$. Since f is cellular on the subcomplex $\{x_0\}$ we obtain from the Cellular Approximation Theorem 70.15 that the map f is homotopic *rel* x_0 to a cellular map f'. Note that we still have $f'(x_0) = y_0$. Using this input it is not difficult to convince oneself that the argument in (1) can be modified to give the desired homotopy rel $\{x_0\}$. We leave it to the punctilious reader to go through the details. ■

In the remainder of this chapter we will discuss many applications of the Whitehead Theorem 162.1 to various problems.

162.2. **Homotopy classification of lens spaces.** One of the recurring themes of these notes is the classification of lens spaces. Using the Whitehead Theorem 162.1 we can now finish the classification of lens spaces up to homotopy equivalence. It is worth recalling the following definition from page 1075, so that we are all on the same sheet.

Definition. We consider the 3-sphere $S^3 = \{(z_1,z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\}$ as a submanifold of $\mathbb{C}^2$. Let $p \in \mathbb{N}$. Furthermore let $q \in \mathbb{Z}$ with $\gcd(p,q) = 1$. The map

$$\begin{array}{rcl} \mathbb{Z}_p \times S^3 & \to & S^3 \\ (k + p\mathbb{Z}, (z_1,z_2)) & \mapsto & (\exp(2\pi \mathrm{i} k/p) \cdot z_1, \exp(2\pi \mathrm{i} kq/p) \cdot z_2) \end{array}$$

defines a smooth, orientation-preserving and discrete action of $\mathbb{Z}_p$ on S^3. We refer to $L(p,q) := S^3/\mathbb{Z}_p$ as a lens space.[1985]

[1984] Some of the arguments below overlap with the proof of the Mapping Cylinder Basics Lemma 16.12 (2). The reason for not appealing to the Mapping Cylinder Basics Lemma 16.12 (2) is that in Statement (2) of the Whitehead Theorem 162.1 we also want to do the case rel a fixed point.

Recall that in Corollary 148.21 we showed that if two lens spaces $L(p, q_1)$ and $L(p, q_2)$ are homotopy equivalent, then there exists an $\epsilon \in \{-1, 1\}$ and an $r \in \mathbb{Z}$, coprime to p, such that $q_1 = \epsilon \cdot r^2 \cdot q_2 \bmod p$. We can now prove the converse of this result.

Proposition 162.4. (Lens Space-Homotopy Equivalence Proposition) Let $p \in \mathbb{N}$ and let $q_1, q_2 \in \mathbb{Z}$ with $\gcd(p, q_1) = \gcd(p, q_2) = 1$. If there exists an $m \in \mathbb{Z}$ with $q_1 \equiv \pm m^2 \cdot \cdot q_2 \bmod p$, then the lens spaces $L(p, q_1)$ and $L(p, q_2)$ are homotopy equivalent.

Remark. This proposition was first shown by John Whitehead [**WhdJ1941b**, Theorem 10]. A direct proof, that does not make use of the Whitehead Theorem 162.1, is also given in [**CoheM1973**, Chapter 29].

Before we turn to the proof of the Lens Space-Homotopy Equivalence Proposition 162.4 let us remind the reader that we have now proved the following theorem, which for convenience we had stated already at an earlier occasion.

Theorem 148.22. (Lens Space-Homotopy Classification Theorem) Let $p \in \mathbb{N}$. If $q, \widetilde{q} \in \mathbb{Z}$ are coprime to p, then

$$\begin{array}{c}\text{the lens spaces } L(p,q) \text{ and } L(p,\widetilde{q}) \\ \text{are homotopy equivalent}\end{array} \iff \begin{array}{c}\text{there exists an } r \in \mathbb{Z}, \text{ coprime to } p, \\ \text{such that } q \equiv \pm r^2 \cdot \widetilde{q} \bmod p.\end{array}$$

Remark. The classification of lens spaces up to homeomorphism will have to wait till the Lens Space-Homeomorphism Classification Theorem **??**.

Proof. The "$\Rightarrow$"-direction is precisely the statement of Corollary 148.21 and the "$\Leftarrow$"-direction is the content of the Lens Space-Homotopy Equivalence Proposition 162.4. ■

The proof of the Lens Space-Homotopy Equivalence Proposition 162.4 requires several preparations. The key idea is to make use of the recently acquired Whitehead Theorem 162.1. In fact the Whitehead Theorem 162.1 is the key ingredient in the proof of the following lemma.

Lemma 162.5. Let M and N be closed oriented n-dimensional smooth manifolds such that the universal covering in both cases is S^n. Let $g\colon M \to N$ be a map. If $\deg(g) = \pm 1$ and if $g_*\colon \pi_1(M) \to \pi_1(N)$ is an isomorphism, then g is a homotopy equivalence.

Proof. First note that by the Smooth Manifold-CW Structure Theorem 96.5 we know that M and N admit CW-structures.[1986] Thus, by the Whitehead Theorem 162.1, it remains to show that $g\colon M \to N$ is a weak homotopy equivalence.

Let $p\colon S^n \to M$ and $q\colon S^n \to N$ be the universal coverings. It follows from the Map Lifting Criterion 61.2 that there exists a map $\widetilde{g}\colon S^n \to S^n$ such that the following diagram commutes:

[1985] As we mentioned on page 1075, it follows from the Manifold Quotient-by-Group Action Proposition 19.32 and the Orientation-Action Proposition 25.19 that $L(p, q)$ is a closed 3-dimensional smooth manifold that comes with a natural orientation.

[1986] Alternatively it follows from Topological Manifold-Simplicial Complex Retract Theorem 104.13 together with the brand-new CW-Complex Domination Proposition 162.8 or from the Morse Functions-Cells Proposition 138.6 that M and N are homotopy equivalent to CW-complexes, which is good enough for our purpose.

$$\begin{array}{ccc} S^n & \xrightarrow{\widetilde{g}} & S^n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ M & \xrightarrow{g} & N. \end{array}$$

It follows from our hypothesis that $g_*\colon \pi_1(M) \to \pi_1(N)$ is an isomorphism together with the Weak Homotopy-vs-Homology Isomorphisms Proposition 161.6 $(2')$ that it suffices to prove the following claim.

Claim. For each $k \in \mathbb{N}_0$ the map $\widetilde{g}_*\colon \mathrm{H}_k(S^n) \to \mathrm{H}_k(S^n)$ is an isomorphism.

Proof. Evidently we only have to consider the case $k = n$. It follows from the above simple-minded commutative diagram, the Degree Multiplicativity Theorem 38.4, the hypothesis that $\deg(g) = 1$ and the observation that by the Covering Map-Degree Proposition 101.7 we have $\deg(p) = \#\pi_1(M) = \#\pi_1(N) = \deg(q)$, that $\deg(\widetilde{g}) = 1$. But this implies that $\widetilde{g}_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$ is an isomorphism. ∎

It follows from Lemma 162.5 that, in the setting of the Lens Space-Homotopy Equivalence Proposition 162.4, it suffices to show that there exists a map $L(p, q_1) \to L(p, q_2)$ of degree ± 1. The following lemma is the first step towards the construction of such a map.

Lemma 162.6. Let $p \in \mathbb{Z} \setminus \{0\}$ and let $q_1, q_2 \in \mathbb{Z}$ with $\gcd(p, q_1) = \gcd(p, q_2) = 1$. Given any $t_1, t_2 \in \mathbb{Z}$ with $q_1 \cdot t_1 \equiv q_2 \cdot t_2 \bmod p$ the map

$$\begin{array}{rcl} f\colon L(p, q_1) & \to & L(p, q_2) \\ [(z, w)] & \mapsto & [(z^{t_2}, w^{t_1})] \end{array}$$

is well-defined, it is smooth and it has degree $t_1 \cdot t_2$.

Proof. First we show that the map $f\colon L(p, q_1) \to L(p, q_2)$ is well-defined. To do so we introduce two pieces of notation. We write $\xi := \exp(2\pi \mathrm{i}/p)$ and for $i = 1, 2$ we denote by $\sim_i$ the equivalence relation on S^3 that is defined by the action of $\mathbb{Z}_p$ on S^3 with parameter q_i. In other words

$$(z, w) \sim_i (\widetilde{z}, \widetilde{w}) \iff \text{there exists a } k \in \mathbb{Z} \text{ with } (z, w) = (\xi^k \cdot \widetilde{z}, \xi^{k \cdot q_i} \cdot \widetilde{w}).$$

We need to show that if we are given $(z, w), (\widetilde{z}, \widetilde{w}) \in S^3$ with $(z, w) \sim_1 (\widetilde{z}, \widetilde{w})$, then $(f(z), f(w)) \sim_2 (f(\widetilde{z}), f(\widetilde{w}))$. This follows from the following sequence of implications:

$$\begin{array}{rcll} (z, w) \sim_1 (\widetilde{z}, \widetilde{w}) & \implies & (z, w) = (\xi^k \cdot \widetilde{z}, \xi^{k \cdot q_1} \cdot \widetilde{w}) & \text{(definition of } \sim_1) \\ & \implies & (z^{t_2}, w^{t_1}) = (\xi^{k \cdot t_2} \cdot \widetilde{z}^{t_2}, \xi^{k \cdot q_1 \cdot t_1} \cdot \widetilde{w}^{t_1}) & \text{(since take same powers)} \\ & \implies & (z^{t_2}, w^{t_1}) = (\xi^{k t_2} \cdot \widetilde{z}^{t_2}, \xi^{k t_2 \cdot q_2} \cdot \widetilde{w}^{t_1}) & \text{(since } q_1 \cdot t_1 \equiv q_2 \cdot t_2 \bmod p) \\ & \implies & (z^{t_2}, w^{t_1}) \sim_2 (\widetilde{z}^{t_2}, \widetilde{w}^{t_1}) & \text{(definition of } \sim_2) \\ & \implies & (f(z), f(w)) \sim_2 (f(\widetilde{z}), f(\widetilde{w})) & \text{(definition of } f). \end{array}$$

We have thus shown that the map $f\colon L(p, q_1) \to L(p, q_2)$ is well-defined. Next note that it follows easily from the Smooth Map-Descends-to-Quotient Corollary 19.33 that f is smooth. It remains to calculate the degree of the map. We consider the following diagram:

$$\begin{array}{ccc} S^3 & \xrightarrow{(z,w) \mapsto (z^{t_1}, w^{t_2})} & S^3 \\ {\scriptstyle (z,w) \mapsto [(z,w)]_1}\downarrow & & \downarrow{\scriptstyle (z,w) \mapsto [(z,w)]_2} \\ L(p, q_1) & \xrightarrow{f} & L(p, q_2). \end{array}$$

We make the following observations:

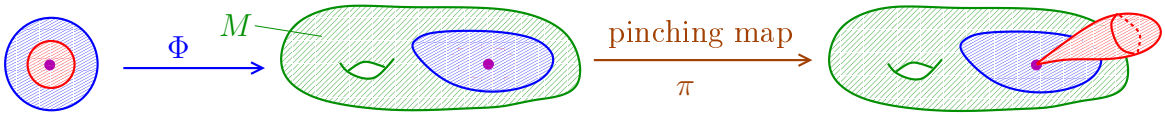
(1) It is clear that the diagram commutes.
(2) The vertical maps are covering maps of degree d. It follows from the Covering Map-Degree Proposition 101.7 that the vertical also have degree p in terms of fundamental classes.
(3) One can easily show, using the definition of the degree on page 898, that the degree of top horizontal map equals $t_1 \cdot t_2$.

It follows from these three observations together with the Degree Multiplicativity Theorem 38.4 that the degree of the bottom horizontal map f also equals $t_1 \cdot t_2$. ■

Lemma 162.7. Let M be a closed oriented connected non-empty n-dimensional smooth manifold. We pick an orientation-preserving smooth embedding $\Phi\colon \overline{B}^n_2 \to M$. Let $\Theta\colon \overline{B}^n/S^{n-1} \to S^n$ be the homeomorphism from the Ball-Quotient Sphere Lemma 5.20. We refer to the map[19871988]

$$\begin{array}{rcl} \pi\colon M & \to & M \vee S^n \\ P & \mapsto & \begin{cases} \Phi(2(t-1)\cdot x), & \text{if } P = \Phi(t\cdot x) \text{ for } t \in [1,2] \text{ and } x \in S^{n-1}, \\ \Theta(y), & \text{if } P = \Phi(y) \text{ with } y \in \overline{B}^n, \\ P, & \text{otherwise.} \end{cases} \end{array}$$

as the pinching map. This map has the property that the maps $M \xrightarrow{\pi} M \vee S^n \to M$ and $M \xrightarrow{\pi} M \vee S^n \to S^n$ both have degree 1.



Proof. This statement can be verified easily using the Local-Global Degree Proposition 101.6. We leave it to the reader to fill in the details. ■

Now we have assembled all the tools that we need for provide the proof of the Lens Space-Homotopy Equivalence Proposition 162.4.

Proof of Proposition 162.4. First let $p \in \mathbb{Z} \setminus \{0\}$. Furthermore let $q_1, q_2 \in \mathbb{Z}$ with $\gcd(p, q_1) = \gcd(p, q_2) = 1$. We suppose that there exists an $m \in \mathbb{Z}$ and an $\epsilon \in \{-1, 1\}$ with $q_1 \equiv \epsilon \cdot q_2 \cdot m^2 \bmod p$. We need to show that $L(p, q_1)$ and $L(p, q_2)$ are homotopy equivalent. By Lemma 162.5 it suffices to show that there exists a map $g\colon L(p, q_1) \to L(p, q_2)$ of degree ϵ. We will construct such g in several steps:

(1) Since $\gcd(p, q_1) = \gcd(p, q_2) = 1$ we can pick $s_1, s_2 \in \mathbb{Z}$ with $s_i \cdot q_i \equiv 1 \bmod p$.
(2) By Lemma 162.6, applied to $t_i = q_2 m \cdot s_i$, there exists a map $f\colon L(p, q_1) \to L(p, q_2)$ of degree $(q_2 m)^2 \cdot s_1 \cdot s_2$.
(3) Note that

$$(q_2 m)^2 \cdot s_1 \cdot s_2 \;\equiv\; q_2 \cdot m^2 \cdot s_1 \cdot q_2 \cdot s_2 \underset{\substack{\uparrow \\ \text{since } q_1 \equiv \epsilon \cdot q_2 \cdot m^2 \bmod p}}{\equiv} \epsilon \cdot q_1 \cdot s_1 \cdot q_2 \cdot s_2 \;\equiv\; \epsilon \mod p.$$

[1986]It follows from the Pasting Proposition 2.6 together with the Compact Image Lemma 2.13 and the Compact-Closed Lemma 1.21 that π is continuous.
[1986]In Exercise 100.16 we introduced the pinching map $M \# N \to M \vee N$. Our pinching map is homotopic to the map $M \cong M \# S^n \to (M \# S^n) \vee S^n \to M \vee S^n$.

This calculation shows that there exists a $d \in \mathbb{Z}$ with $(q_2 m)^2 \cdot s_1 \cdot s_2 = \epsilon + d \cdot p$.

(4) By the Degree Realization Proposition 38.13 there exists a map $h\colon S^3 \to S^3$ of degree $-d$.

We now consider the following map

$$L(p,q_1) \xrightarrow{\pi} L(p,q_1) \vee S^3 \xrightarrow{f \vee h} L(p,q_2) \vee S^3 \underset{\uparrow}{\to} L(p,q_2)$$

given by id on $L(p,q_2)$ and given by the covering map $S^3 \to L(p,q_2)$ on S^3

It follows from Lemma 162.7, the choice of f and h and the Covering Map-Degree Proposition 101.7 that the induced maps on homology groups are given as follows:

$$\underbrace{\mathrm{H}_3(L(p,q_1))}_{\mathbb{Z}\cdot[L(p,q_1)]} \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} \underbrace{\mathrm{H}_3(L(p,q_1)\vee S^3)}_{\mathbb{Z}\cdot[L(p,q_1)]\oplus\mathbb{Z}\cdot[S^3]} \xrightarrow{\begin{pmatrix}(q_2m)^2\cdot s_1\cdot s_2 & 0\\ 0 & -d\end{pmatrix}} \underbrace{\mathrm{H}_3(L(p,q_2)\vee S^3)}_{\mathbb{Z}\cdot[L(p,q_2)]\oplus\mathbb{Z}\cdot[S^3]} \xrightarrow{(1\ p)} \underbrace{\mathrm{H}_3(L(p,q_2))}_{\mathbb{Z}\cdot[L(p,q_2)]}.$$

In summary we see that the map $L(p,q_1) \to L(p,q_2)$ has degree $(q_2m)^2 \cdot s_1 \cdot s_2 - d \cdot p = \epsilon$. ■

162.3. **Topological spaces dominated by CW-complexes.** We move on to another application of the Whitehead Theorem 162.1. We recall the following definition from page 2307.

Definition. We say that a topological space X is dominated by a topological space Y if there exist maps $i\colon X \to Y$ and $r\colon Y \to X$ such that $r \circ i$ is homotopic to the identity id_X.

The goal of this section is to prove the following result which we already mentioned in Section 104.5.

Proposition 162.8. (CW-Complex Domination Proposition) Let X be a topological space. If X is dominated by a CW-complex, then X is homotopy equivalent to a CW-complex.

Remark.

(1) An alternative proof for the CW-Complex Domination Proposition 162.8, that does not use the Whitehead Theorem 162.1, but that uses a mapping telescope construction similar to the telescope construction from page 1546, is given in [**Hat2002**, Proposition A.11].

(2) In Section 104.5 we stated, without proofs but with detailed references, several generalizations of refinements of the CW-Complex Domination Proposition 162.8.

The following simply definition will prove convenient in the proof of the CW-Complex Domination Proposition 162.8.

Definition. Let $f\colon X \to Y$ be a map between topological spaces. A map $g\colon Y \to X$ is called a homotopy left inverse of f if $g \circ f$ is homotopic to id_X. Furthermore, a map $h\colon Y \to X$ is called a homotopy right inverse of f if $f \circ h$ is homotopic to id_Y.

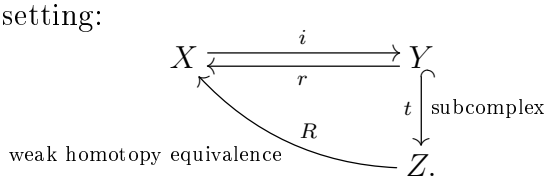
For the reader's convenience we recall the following elementary lemma.

Lemma 15.11. Let $f\colon X \to Y$ be a map between topological spaces. If f admits a homotopy left inverse g and a homotopy right inverse h, then f is in fact a homotopy equivalence and both g and h are homotopy inverses.

Now it is fairly straightforward to provide a proof of the CW-Complex Domination Proposition 162.8.

Proof of the CW-Complex Domination Proposition 162.8. Let X be a topological space that is dominated by a CW-complex Y. Recall that this means that there exist maps $i\colon X \to Y$ and $r\colon Y \to X$ such that $r \circ i$ is homotopic to the identity id_X. Without loss of generality we can and will assume that X and Y are path-connected. We pick a base point $y_0 \in Y$ and we write $x_0 := r(x_0)$. Since $r \circ i$ is homotopic to the identity we obtain from the Homotopy-π_n-Proposition 71.7 (1) that the map $r_* \circ i_* = (r \circ i)_*\colon \pi_j(X, x_0) \to \pi_j(X, x_0)$ is an isomorphism for all $j \in \mathbb{N}_0$. In particular we see that $r_*\colon \pi_j(Y, y_0) \to \pi_j(X, x_0)$ is an epimorphism for all $j \in \mathbb{N}_0$.

By Corollary 161.2 there exists a CW-complex Z which contains Y as a subcomplex and a map $R\colon Z \to X$ that extends r and that is a weak homotopy equivalence. Thus we are now in the following setting:



Here denote by $t\colon X \to Z$ the inclusion map. (Note though that we do not claim that the diagram commutes.) It remains to prove the following claim.

Claim. The map $t \circ i\colon X \to Z$ is a homotopy equivalence.

Proof. Since $R \circ (t \circ i) = (R \circ t) \circ i = r \circ i$ is homotopic to id_X we know that R is homotopy left inverse of $t \circ i$. Furthermore, we obtain from the Homotopy-π_n-Proposition 71.7 (1) that $R \circ (t \circ i)$ is a weak homotopy equivalence.[1987] Since R is a weak homotopy equivalence we see that $t \circ i$ is also a weak homotopy equivalence. Thus we see that $(t \circ i) \circ R\colon Z \to Z$ is a weak homotopy equivalence. Since Z is a CW-complex it follows from the Whitehead Theorem 162.1 that the map $(t \circ i) \circ R$ is a homotopy equivalence. This implies that there exists a map $k\colon Z \to Z$ such that $(t \circ i) \circ R \circ k$ is homotopic to id_Z. We have thus shown that $R \circ k$ is a homotopy right inverse for $t \circ i$. Since we already pointed out above that $t \circ i$ admits is a homotopy left inverse we obtain from Lemma 15.11 that $t \circ i$ is a homotopy equivalence. ∎

162.4. Highly connected CW-complexes.

In this section we want to discuss the following natural question.

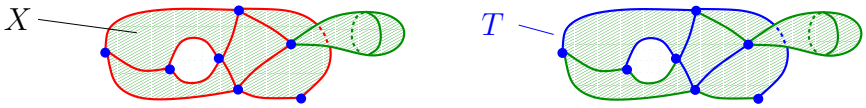
Question 162.9. Let X be a CW-complex which is n-connected for some $n \in \mathbb{N}_0$. Is X homotopy equivalent to a CW-complex Y which has one 0-cell and no cells in dimensions $1, \ldots, n$?

It turns out that we already dealt with the not entirely trivial case, that $n = 0$. For the convenience of the reader we recall Proposition 70.4. Its precise statement gives us one possible approach to answering Question 162.9.

[1987] Throughout the proof we use the characterization of weak homotopy equivalences given by the Weak Homotopy Equivalence Criteria Lemma 161.5 (3).

Proposition 70.4. Let X be a 0-connected CW-complex.

(1) There exists a contractible 1-dimensional subcomplex T which contains all 0-cells.[1988]
(2) Given any T as in (1) the quotient X/T is a CW-complex with precisely one 0-cell. Furthermore the projection $X \to X/T$ is a homotopy equivalence.

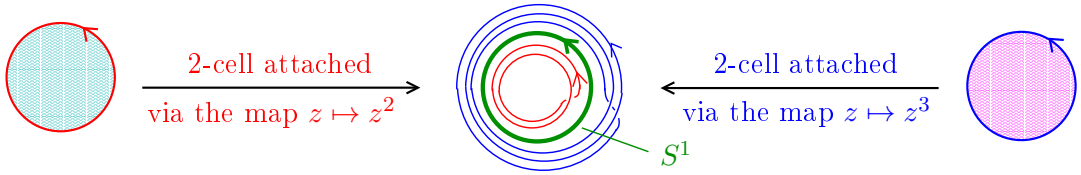


Proposition 70.4 suggests the following approach to answering Question 162.9: Given an n-connected CW-complex X we need to find a contractible subcomplex S that contains the n-skeleton X^n. The catch is, as the following example shows, in general such a subcomplex just does not exist.

Example. Let X be the 2-dimensional CW-complex corresponding to the presentation $\langle x \,|\, x^2 = e, x^3 = e\rangle$.[1989] This CW-complex is constructed by attaching two 2-cells to S^1 via the attaching maps $z \mapsto z^2$ and $z \mapsto z^3$. We refer to the figure below for an illustration. We have

$$\pi_1(X) \underset{\text{Corollary 69.12}}{\cong} \langle x \,|\, x^2 = e, x^3 = e\rangle \underset{\text{since } \langle x\rangle \cong \mathbb{Z}}{\cong} \mathbb{Z}/\text{subgroup generated by 2 and 3} \underset{\text{since 2 and 3 are coprime}}{=} 0.$$

We have thus shown that X is 1-connected. In Exercise 162.8 we will see that X does not admit a contractible subcomplex S which contains the 1-skeleton of X and which has the property that X/S is homotopy equivalent to X.



In light of the example it comes perhaps as a surprise that Question 162.9 actually does have an affirmative answer:

Proposition 162.10. Let X be a CW-complex, let $x_0 \in X$ and let $n \in \mathbb{N}_0$. If X is n-connected, then there exists a CW-complex Y which has one 0-cell y_0 and no cells in dimensions $1, \ldots, n$ such that the pair (X, x_0) is homotopy equivalent to the pair (Y, y_0).

Proof. Let X be an n-connected CW-complex and let $x_0 \in X$. By the CW-Approximation Theorem 161.8 (2) there exists a CW-complex Y which has a single 0-cell $\{y_0\}$ and no cells in dimensions $1, \ldots, n$ and a map $f\colon (Y, y_0) \to (X, x_0)$ which is a weak homotopy equivalence. Thus we can apply the Whitehead Theorem 162.1 (2) to the map f and we see that f is in fact a homotopy equivalence of pointed topological spaces. Well, that was quick. ∎

Remark. The construction of the CW-complex Y in the proof of Proposition 162.10 is quite wasteful. In particular, without any refinement of the construction, it will almost invariably provide an infinite-dimensional CW-complex Y. It turns out that often one

[1988] Such a subcomplex is basically what we called a spanning tree on page 317.

[1989] Recall that on page 1531 we showed that any finite presentation $\langle g_1, \ldots, r_k \,|\, r_1, \ldots, r_l\rangle$ naturally gives rise to a CW-complex with one 0-cell, k 1-cells and l 2-cells.

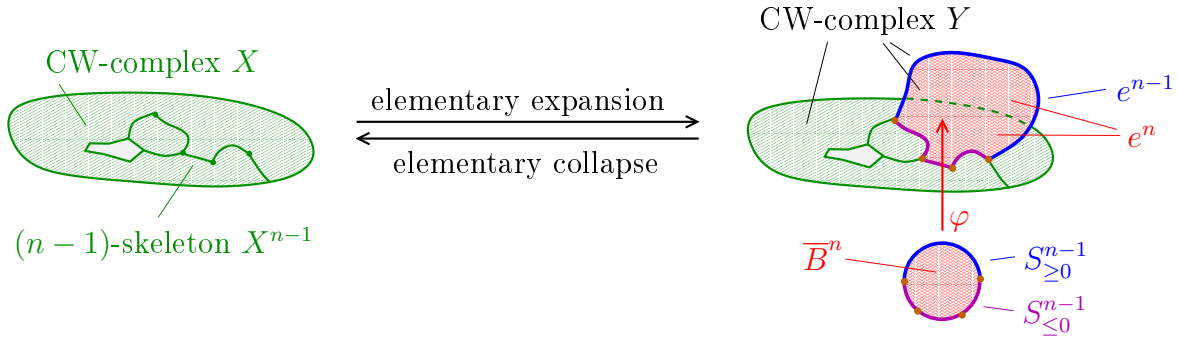
can do better. More precisely, in [**Hat2002**, Proposition 4.C.1] the following statement is shown: given a finite n-connected CW-complex X there exists a CW-approximation $f\colon Y \to X$ where Y is a finite CW–complex with a single 0-cell and such that Y has no cells in dimensions $1, \ldots, n$.

162.5. **Elementary collapses.** It is not always easy to determine whether two topological spaces are homotopy equivalent. The following definition is useful for systematizing the search for homotopy equivalences between CW-complexes.

Definition. Let X be a CW-complex. We say that a CW-complex Y is an elementary expansion of X if the following holds:

(1) $Y = X \cup e^{n-1} \cup e^n$ where e^{n-1} is an $(n-1)$-cell that is not contained in X and e^n is an n-cell that is not contained in X.
(2) There exists a map $\varphi\colon \overline{B}^n \to Y$ with the following properties:
   (a) The map φ is the characteristic map for the n-cell e^n.
   (b) The map[1990] $\overline{B}^{n-1} \xrightarrow{\cong} S^{n-1}_{\geq 0} \xrightarrow{\varphi} Y$ is the characteristic map for the $(n-1)$-cell e^{n-1}.
   (c) We have $\varphi(S^{n-1}_{\leq 0}) \subset X^{n-1}$.

Conversely we also say that the CW-complex X is an elementary collapse of Y. We refer to the figure below for an illustration.



We continue with another definition.

Definition. Let X and Y be CW-complexes. We say X and Y are simple homotopy equivalent if there exists a sequence $X = X_0, X_1, \ldots, X_k = Y$ such that for $i = 1, \ldots, k$ one of the following three statements holds:

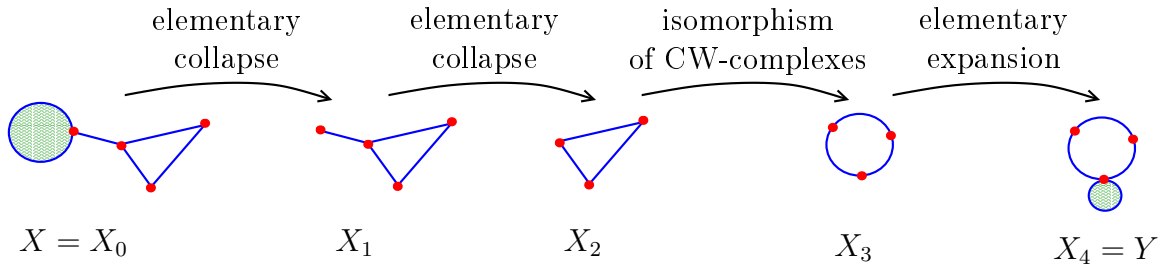
(1) X_i is an elementary expansion of X_{i-1}, or
(2) X_i is an elementary collapse of X_{i-1}, or
(3) X_i is isomorphic, as a CW-complex, to X_{i-1}.

The following lemma justifies the name "simple homotopy equivalent".

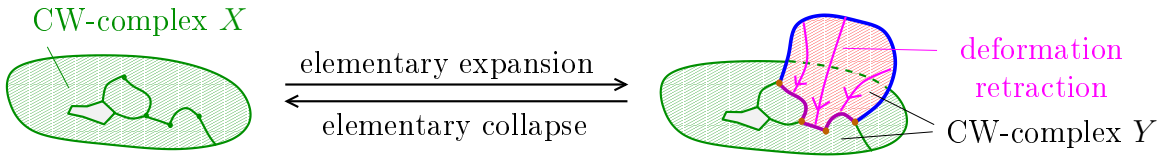
Lemma 162.11. Let X and Y be CW-complexes. If X and Y are simple homotopy equivalent, then they are homotopy equivalent.

Proof. By the Homotopy-Equivalence Basics Lemma 15.1 the notion of being homotopy equivalent is an equivalence relation. Thus it suffices to show that if a CW-complex X is an elementary collapse of a CW-complex Y, then X is homotopy equivalent to Y. In

[1990] Here we use the explicit homeomorphism $\overline{B}^{n-1} \xrightarrow{\cong} S^{n-1}_{\geq 0}$ from page 152.

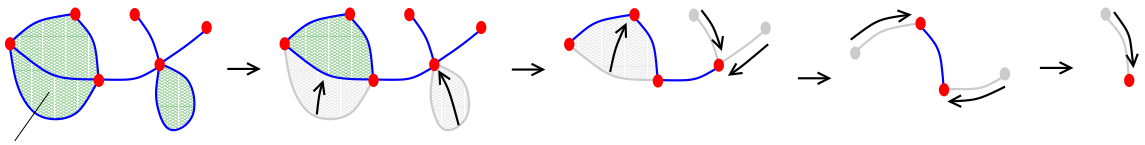


fact, by the Deformation Retract-Homotopy Equivalence Lemma 15.5 it suffices to show that X is a deformation retract of Y. We saw on page 434 that there exists a deformation retraction from $\overline{B}^n$ to $S^{n-1}_{\leq 0}$. It follows from the Homotopy Pushout Lemma 14.9 that this deformation retraction, together with the constant homotopy on X, defines a deformation retraction from Y to X. ■



It is also interesting to consider an interesting special case.

Definition. A finite CW-complex is called collapsible if there exists a sequence of elementary collapses that abuts in a CW-complex consisting of a single point.



collapsible CW-complex

The following lemma is an immediate consequence of Lemma 162.11.

Lemma 162.12. Let X be a CW-complex. If X is collapsible, then X is contractible.

It is natural to ask whether the converses of Lemmas 162.11 and 162.12 hold. More precisely, we have the following questions.

Question 162.13.

(1) Let X be a finite CW-complex. If X is homotopy equivalent to a point, does it follow that X is collapsible?
(2) Let X and Y be finite CW-complexes.
   (a) If X and Y are homeomorphic, does it follow that X and Y are simple homotopy equivalent?
   (b) If X and Y are homotopy equivalent, does it follow that X and Y are simple homotopy equivalent?

In the remainder of this chapter we will discuss Question 162.13 (1). We will answer Question 162.13 (2) when we discuss Chapman's Theorem ?? and Proposition ??. To address Question 162.13 (1) it is useful to formulate the following corollary to the Whitehead Theorem 162.1.

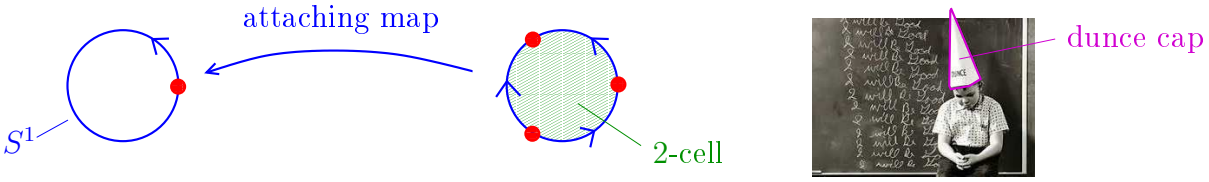
Corollary 162.14. (CW-Complex-Contractible Corollary) Let X be a CW-complex. If X is simply connected and if we have $\mathrm{H}_n(X;\mathbb{Z}) = 0$ for all $n \geq 2$, then X is contractible, i.e. X is homotopy equivalent to a point.

Proof. We let $Y = \{*\}$ be the topological space that consists of a single point and let $f\colon X \to Y = \{*\}$ be the only map there is. It follows immediately from the Weak Homotopy-vs-Homology Isomorphisms Proposition 161.6 that f is a weak homotopy equivalence. Thus the desired statement follows from the Whitehead Theorem 162.1 (1). ■

In the following we will use the CW-Complex-Contractible Corollary 162.14 to give not one, but two examples of CW-complexes that give a negative answer to Question 162.13 (1).

Examples.

(1) We equip S^1 with the usual CW-structure with one 0-cell $1 \in S^1$ and one 1-cell. Let $\gamma = \mathrm{id}\colon S^1 \to S^1$ be the identity map. The *dunce cap* is defined as the CW-complex X that is given by attaching one 2-cell to S^1 where the attaching map $S^1 \to S^1$ is given by the loop $\gamma * \overline{\gamma} * \gamma$,[1991] i.e. by the loop that is given by the composition of γ, $\overline{\gamma}$ and γ. Using the Cell Attachment-π_1-Proposition 69.11 (2) it is straightforward to show that X is simply connected. Furthermore it follows almost from the Cellular Boundary Map-via-Degrees Proposition 80.8 that the cellular homology, and thus by the Singular-Cellular H_*-Isomorphism Proposition 80.4 also the singular homology, vanishes in dimensions ≥ 2. It follows from the CW-Complex-Contractible Corollary 162.14 that the dunce cap is contractible.[1992] On the other hand it is clear that the dunce cap is not collapsible since there is only one cell in dimensions 0, 1 and 2 and no two of them can be matched to allow for an elementary collapse.



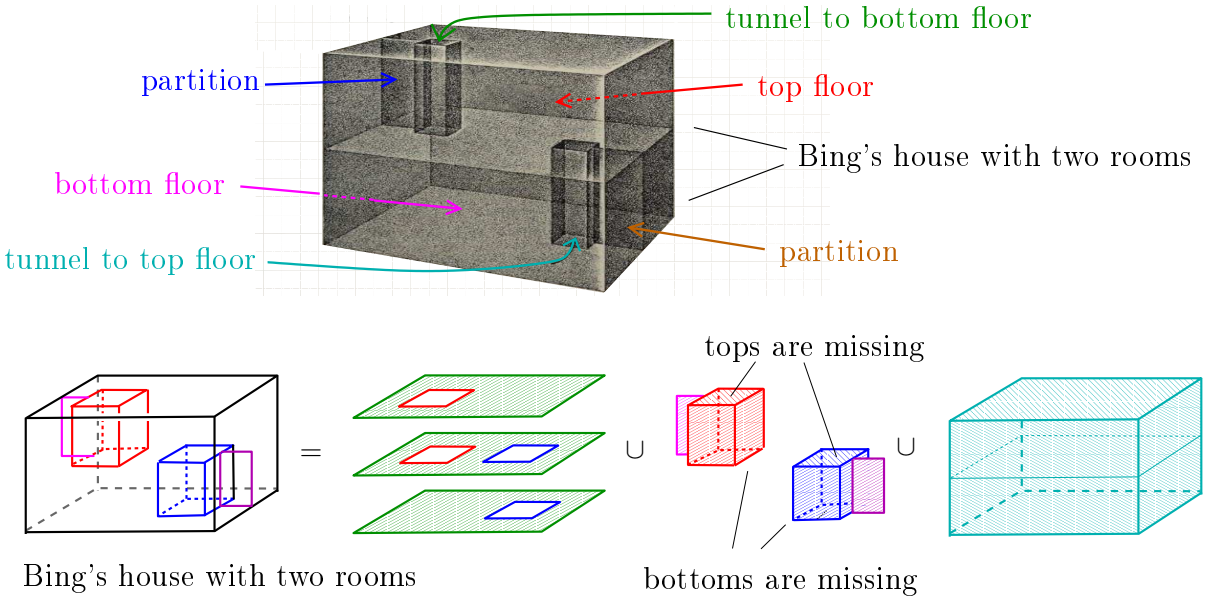
(2) It is also possible to give an example of a CW-complex that can be visualized as a subset of $\mathbb{R}^3$. More precisely, consider the topological space X that is shown in the two figures. It is commonly referred to as *Bing's house with two rooms*.[1993] We equip X with the fairly obvious CW-structure. In Exercise 162.10 we will see that X is simply connected and that $\mathrm{H}_n(X;\mathbb{Z}) = 0$ for $n \geq 2$. Thus, as in the previous example we can appeal to the CW-Complex-Contractible Corollary 162.14 to conclude that

[1991] The conscientious reader will notice that we had encountered this topological space in Exercise 70.2.

[1992] One could of course also prove this "by hand", but I am pretty sure that few people will enjoy doing this exercise. In fact I have not yet found a book which gives a down-to-earth proof that the dunce cap is contractible.

[1993] This example was given by R. H. Bing [**Bin1964**, p. 109] in 1964. In fact the picture we show in the first figure below is a slightly modified version of Bing's original drawing.

the house with two rooms is contractible.[1994] We leave the arduous task of showing that X is not collapsible to the reader.



Let X be the dunce cap. In 1963 Christopher Zeeman [**Zee1963a**, Theorem 1] showed that $X \times [0,1]$, equipped with the product CW-structure, is in fact collapsible. This result prompted Zeeman in 1963 to propose the following conjecture [**Zee1963a**, Conjecture 1].

Conjecture 162.15. (Zeeman Conjecture) If X is a finite contractible 2-dimensional CW-complex, then $X \times [0,1]$ is collapsible.

Like many other conjectures on CW-complexes, this conjecture sounds deceptively simple and elementary. In fact Zeeman was fully aware of the fact that the conjecture is almost certainly very difficult since he showed [**Zee1963a**, Theorem 2] that a proof of the Zeeman Conjecture 162.15 would lead to a new proof of the Poincaré Conjecture in dimension three, which was finally proved in 2003 by Perelman using a heavy dose of analysis, see Theorem **??**. Furthermore, Sergey Matveev [**Matv1987**] (see also [**Kupe2017a**, Proposition 4.2] and [**Barm2011**, Chapter 11]) showed that the Andrews-Curtis Conjecture 53.18 is a special case of the Zeeman Conjecture 162.15. We refer to [**MR1993, Matv2007, Kupe2017a**] for more on the Zeeman Conjecture.

Exercises for Chapter 162.

Exercise 162.1. Let $p \in \mathbb{N}$ and let $q_1, \dots, q_n \in \mathbb{Z}$ be integers that are coprime to p. We consider the following action of $\mathbb{Z}_p$ on $S^{2n-1} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \,|\, |z_1|^2 + \cdots + |z_n|^2 = 1\}$:

$$\begin{array}{rcl} \mathbb{Z}_p \times S^{2n-1} & \to & S^{2n-1} \\ ([k], (z_1, \dots, z_n)) & \mapsto & (z_1 \cdot e^{2\pi \mathrm{i} k \dot{q}_1/p}, \dots, z_n \cdot e^{2\pi \mathrm{i} k \dot{q}_n/p}). \end{array}$$

[1994] A more direct argument for showing that Bing's house with two rooms is contractible is given on [**Hat2002**, p. 4].

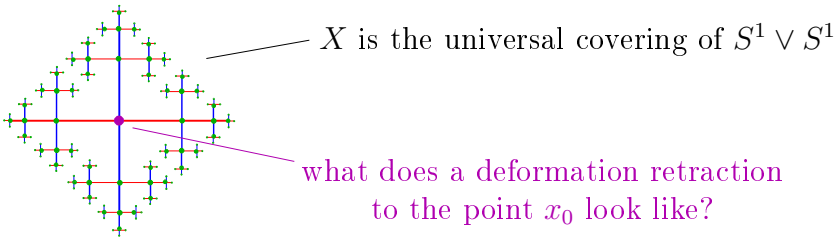
One can easily verify that this action is smooth, orientation-preserving and discrete action. Thus it follows from the Manifold Quotient-by-Group Action Proposition 19.32 and the Orientation-Action Proposition 25.19 that the quotient $L(p;q_1,\dots,q_n) := S^{2n-1}/\mathbb{Z}_p$ is a closed orientable $(2n-1)$-dimensional smooth manifold.

Let $q_1,\dots,q_n$ and $\widetilde{q}_1,\dots,\widetilde{q}_n$ be integers that are coprime to p. Show that if there exists an $m \in \mathbb{Z}$ with $q_1 \cdot \dots \cdot q_n \equiv m^n \cdot \widetilde{q}_1 \cdot \dots \cdot \widetilde{q}_n$, then $L(p;q_1,\dots,q_n)$ and $L(p;\widetilde{q}_1,\dots,\widetilde{q}_n)$ are homotopy equivalent.

Remark. For $n = 2$ this is just a slight reformulation of the Lens Space-Homotopy Equivalence Proposition 162.4.

Exercise 162.2. Let $X = \mathbb{R}$. We equip X with the CW-structure where the 0-skeleton is given by $\mathbb{Z}$. The proof of the Whitehead Deformation Retract Proposition 162.2 provides a fairly explicit deformation retraction from $X = \mathbb{R}$ to $\{0\}$. What does this deformation retraction look like?

Exercise 162.3. Let X be the universal covering of $S^1 \vee S^1$. On page 3409 we saw that X admits a deformation retraction to $x_0 \in X$. What does an explicit deformation retraction of X to the point x_0 look like? We refer to the figure below for an illustration.



Exercise 162.4. We consider the following subset of $\mathbb{R}^2$:

$$X := \text{line segment from } (0,1) \text{ to } (0,-1) \cup \bigcup_{n\in\mathbb{N}} \text{line segment from } (0,1) \text{ to } (\tfrac{1}{n},0) \cup \bigcup_{n\in\mathbb{N}} \text{line segment from } (0,-1) \text{ to } (-\tfrac{1}{n},0).$$

We equip X with the subspace topology coming from $\mathbb{R}^2$. We refer to the figure on page 3410 for an illustration.

(a) Show that all of the homotopy groups of X are trivial.

(b) Show that X is not contractible.
You might want to make use of the Contractibility Criterion Lemma 15.2 and Exercise 15.10.

Exercise 162.5. Let X and Y be two CW-complexes of dimension at most n. Furthermore let $f\colon X \to Y$ be a map that is $(n+1)$-connected. Show that f is a homotopy equivalence.

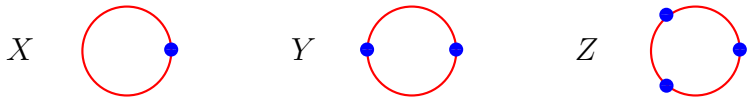
Exercise 162.6. Let $f\colon X \to Y$ be a map between path-connected CW-complexes. Let $x_0 \in X$. We assume that $f_*\colon \pi_k(X,x_0) \to \pi_k(Y,f(x_0))$ is the trivial map for every $k \in \mathbb{N}$. Does it follow that f is homotopic to a constant map?

Exercise 162.7. Let X be a connected CW-complex and let $k \leq l$ be natural numbers. We suppose that $\pi_i(X) = \mathrm{H}_i(X;\mathbb{Z}) = 0$ for $i = k,\dots,l$. Does it follow that X is homotopy

equivalent to a CW-complex which has no cells in dimensions $k, \dots, l$?
Remark. If $k = 1$, then this question is precisely Question 162.9.

Exercise 162.8. Let X be the 2-dimensional CW-complex corresponding to the presentation $\langle x \,|\, x^2 = e, x^3 = e \rangle$. We refer to the figure on page 3419 for an illustration. Show that X does not admit a contractible subcomplex S which contains the 1-skeleton of X and which has the property that X/S is homotopy equivalent to X.
Hint. Use the long exact sequence coming from Theorem 74.3.

Exercise 162.9. Show that the three CW-complexes shown in the figure below are simple homotopy equivalent.
Remark. It is possibly less confusing to first consider Y and Z.



Exercise 162.10. Let X be Bing's house with two rooms, i.e. X is the CW-complex that we defined in the figure on page 3423.

(a) Show that X is simply connected.
(b) Show that $\mathrm{H}_n(X; \mathbb{Z}) = 0$ for $n \geq 2$.
Hint. Using the Homology-of-CW-Complex Proposition 80.6 and using Euler characteristics will cut down the work considerably.
(c) Show that X is not collapsible.

Exercise 162.11. Let $n \in \mathbb{N}$ and furthermore let $f, g \colon S^n \to S^n$ be two maps. Show that if $f_* = g_* \colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$, then f and g are homotopic.

Exercise 162.12. Let $T = S^1 \times S^1$ be the 2-dimensional torus, equipped with the usual CW-structure. We attach two 2-cells along $S^1 \times \{0\}$ and along $\{0\} \times S^1$ and we obtain a CW-complex X.

(a) Show that X that is homotopy equivalent to S^2.
(b) Show that there is no embedding $S^2 \to X$.
Remark. You might want to use Exercise 82.3.

Exercise 162.13. Show that the long line $\mathbb{L}$, which we introduced on page 613, is weakly contractible but not contractible.
Hint. Use Proposition 22.14.

Exercise 162.14. Show that there exists a compact 4-dimensional smooth manifold that does not admit a smooth structure.
Hint. Use Hatcher's Theorem [**Hat1983**] that every orientation-preserving diffeomorphism of S^3 is diffeotopic to the identity.

163. Eilenberg-Maclane Spaces

In this chapter we introduce Eilenberg-Maclane spaces. Even though the definition is quite straightforward, these topological spaces will henceforth play an essential role.

163.1. **The definition and basic examples of Eilenberg-Maclane Spaces.** We recall the following definition from page 1761.

Definition. Let π be an abelian group and let $n \in \mathbb{N}$. We say that a non-empty topological space X is a Moore space of type $\mathrm{M}(\pi, n)$ if the following conditions are satisfied:

(1) X admits a CW-structure,
(2) for any $i \neq n$ we have $\widetilde{\mathrm{H}}_i(X;\mathbb{Z}) = 0$,
(3) $\mathrm{H}_n(X) \cong \pi$.
(4) If $n \geq 2$, then we demand that X is simply connected.

For the reader's convenience we recall the main result that we proved regarding Moore spaces.

Proposition 79.11. (Moore Space Existence Proposition) Given any $n \in \mathbb{N}$ and given any abelian group π there exists an $(n+1)$-dimensional CW-complex that has no cells in dimensions $1, \ldots, n-1$ and that is a Moore space of type $\mathrm{M}(\pi, n)$.

In light of the topics discussed in this chapter it is perhaps helpful to give a quick recap of the proof of the Moore Space Existence Proposition 79.11.

Sketch of proof. Since π is abelian it follows immediately from Lemma 89.14 (1) that there exists a free resolution of π of the following form:

$$0 \to \mathbb{Z}^{(A)} \xrightarrow{\varphi} \mathbb{Z}^{(B)} \xrightarrow{\rho} \pi \to 0.$$

We let $X := \bigvee_{a \in A} S^n_a$ and we let $Y := \bigvee_{b \in B} S^n_b$. We equip X and Y with the obvious CW-structures and we can make the identifications $\mathrm{H}_n(X;\mathbb{Z}) = \mathbb{Z}^{(A)}$ and $\mathrm{H}_n(Y;\mathbb{Z}) = \mathbb{Z}^{(B)}$. We construct a cellular map $h\colon X \to Y$ such that the induced map $\mathrm{H}_n(X;\mathbb{Z}) \to \mathrm{H}_n(Y;\mathbb{Z})$ agrees with φ under the above identifications. The mapping cone $\mathrm{Cone}(h\colon X \to Y)$ then has the desired properties. ■

The definition of an Eilenberg-Maclane space is almost identical to the definition of a Moore space, we just swap homology groups for homotopy groups:

Definition. Let π be a group and let $n \in \mathbb{N}$. Let X be a topological space.

(1) We say that X is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ if the following conditions hold:
   (a) X admits a CW-structure,
   (b) for any $i \neq n$ we have $\pi_i(X) = 0$,
   (c) $\pi_n(X) \cong \pi$.[1995]
(2) Let $x_0 \in X$. We say (X, x_0) is a pointed Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ if X is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ and if X admits a CW-structure such that x_0 lies in the 0-skeleton of X.

[1995] It is perhaps worth recalling that $\pi_0(X) = 0$ means, by the definition on page 167, that X is non-empty and path-connected. In particular, by the Change-of-Base Point Proposition 86.9 (2) the isomorphism type of the homotopy groups $\pi_n(X)$, $n \in \mathbb{N}$, does not depend on the choice of the base point.

Remark.

(1) Our definition of an Eilenberg-Maclane space agrees with the definition in [**DaK2001**, p. 177]. Conversely in [**WhdG1978**, p. 239], [**Spa1995**, p. 424], [**Bre1993**, p. 488] and [**Hat2002**, p. 365] it does not say (at least explicitly) that an Eilenberg-Maclane space needs to admit a CW-structure.
(2) If a topological space X is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$, then often in the literature one just says that X *is a* $\mathrm{K}(\pi, n)$. We will refrain from using this language.

It is perhaps initially not clear why the concept of an Eilenberg-Maclane space is useful, but at the very least it is a fun exercise to go through our long list of topological spaces that we have studied and to determine which ones are Eilenberg-Maclane spaces:

(1) We start with the trivial group $\{e\}$. Let $n \in \mathbb{N}$. In this case it is easy to find Eilenberg-Maclane spaces of type $\mathrm{K}(\{e\}, n)$. In fact it follows from the Homotopy-π_n-Proposition 71.7 (2) that any contractible CW-complex is an Eilenberg-Maclane space of type $\mathrm{K}(\{e\}, n)$. Thus we see in particular that for any $k \in \mathbb{N}_0$ the open ball B^k and the closed ball $\overline{B}^k$ are Eilenberg-Maclane spaces of type $\mathrm{K}(\{e\}, n)$.
(2) Let $k \in \mathbb{N}$. It follows from Corollary 48.18 and the discussion on page 1573 that the k-dimensional torus $(S^1)^k$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}^k, 1)$.
(3) Let $g \in \mathbb{N}_{\geq 2}$ and let Σ_g be a surface of genus g. On page 1575 we saw that $\pi_i(\Sigma_g) = 0$ for $i \geq 2$. Thus we see that Σ_g is an Eilenberg-Maclane space of type $\mathrm{K}(\pi_1(\Sigma_g), 1)$.
(4) Let $k \in \mathbb{N}$ and let $F = \langle x_1, \ldots, x_k \rangle$ be the free group on k generators. It follows from the discussion on page 1173 and Lemma 71.15 that the wedge $\bigvee_{i=1}^{k} S^1$ of k circles is an Eilenberg-Maclane space of type $\mathrm{K}(F, 1)$.
(5) It follows from Lemma 68.5 (3) together with the Coverings-π_n-Proposition 71.13 and Corollary 71.12 that the infinite real projective space $\varinjlim \mathbb{R}\mathrm{P}^k =: \mathbb{R}\mathrm{P}^\infty \cong S^\infty / x \sim -x^{1996}$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}_2, 1)$.



type $(\mathbb{Z}, 1)$ type $(\mathbb{Z}^2, 1)$ type $(\pi_1(\Sigma_2), 1)$ type $(\langle x, y \rangle, 1)$

We have now obtained a short but pleasant list of Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, 1)$. It is harder to find examples of Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, n)$ with $n \geq 2$ for a non-trivial group π. It turns out that we had found one interesting example of that type:

(6) Recall that by Lemma 68.6 we know that $\mathbb{C}\mathrm{P}^\infty$ admits a CW-structure. It follows from Theorem 155.12 (2) that the infinite complex projective space $\mathbb{C}\mathrm{P}^\infty$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, 2)$.

Remark.

[1996] In Lemma 68.5 (4c) we showed that there exists a homeomorphism $\mathbb{R}\mathrm{P}^\infty \cong S^\infty / x \sim -x$.

(1) One might initially hope that perhaps $\mathbb{H}\mathrm{P}^\infty$ could be an Eilenberg-Maclane space of type $\mathrm{K}(\pi, 4)$. First note that by the Quaternion-Projective Space Lemma 92.9 (7) we know that $\mathbb{H}\mathrm{P}^\infty$ is indeed a CW-complex. Furthermore, in in Exercise 160.1 we obtained the following results:
   (a) There exists an isomorphism $\pi_4(\mathbb{H}\mathrm{P}^\infty) \cong \mathbb{Z}$.
   (b) The group $\pi_n(\mathbb{H}\mathrm{P}^\infty)$ is finite for $n \neq 4$, in particular $\pi_n(\mathbb{H}\mathrm{P}^\infty) \otimes \mathbb{Q} = 0$ for $n \neq 4$.
   (c) There exists an isomorphism $\pi_5(\mathbb{H}\mathrm{P}^\infty) \cong \mathbb{Z}_2$.
   So we see that "rationally" $\mathbb{H}\mathrm{P}^\infty$ looks like an Eilenberg-Maclane space, but it is just not an Eilenberg-Maclane space.
(2) In the following chapters we will give many examples of "perfectly reasonable" topological spaces that are Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, 1)$. For $n \geq 2$ it seems like $\mathbb{C}\mathrm{P}^\infty$ is basically[1997] the only Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ for some non-trivial group π which appears "in nature".

Our next goal is to find Eilenberg-Maclane spaces of type $\mathrm{K}(\mathbb{Z}_m, 1)$ for arbitrary m. For $m = 2$ we had just seen that $\mathbb{R}\mathrm{P}^\infty$ is such an Eilenberg-Maclane space. We recall that in Lemma 68.5 (3) we saw that we can make the identification $\mathbb{R}\mathrm{P}^\infty = S^\infty/\{\pm 1\}$ where the group $\{\pm 1\} \cong \mathbb{Z}_2$ acts in the obvious way on S^∞. The idea is to generalize this example. More precisely we want to find appropriate actions of any finite cyclic group on S^∞.
At this point it is convenient to recall that given coprime $p, q \in \mathbb{N}$ we defined on page 1075 the lens spaces $L(p, q) = S^3/\mathbb{Z}_p$ as the quotient of S^3 by a suitable action of $\mathbb{Z}_p$ on S^3. If one ponders for a while about the definition of lens spaces one is led to the following lemma.

Lemma 163.1. Let $m \in \mathbb{N}$ and let $r_1, r_2, \dots \in \mathbb{N}$ be natural numbers that are coprime to m.

(1) The map[1998]
$$\begin{array}{rcl} \mathbb{Z}_m \times S^\infty & \to & S^\infty \\ ([k], \underbrace{(z_1, z_2, z_3, \dots)}_{\in \mathbb{C}^\infty}) & \mapsto & (e^{2\pi \mathrm{i} k r_1/m} \cdot z_1, e^{2\pi \mathrm{i} k r_2/m} \cdot z_2, \dots) \end{array}$$
   defines a continuous and discrete action of $\mathbb{Z}_m$ on S^∞.
(2) The quotient space $L(m, r_1, r_2, \dots) := S^\infty/\mathbb{Z}_m$ admits a CW-structure.
(3) The quotient space $L(m, r_1, r_2, \dots) := S^\infty/\mathbb{Z}_m$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}_m, 1)$.

On page 3156 we saw, using the linking pairing, that lens spaces with isomorphic fundamental groups are not necessarily homotopy equivalent. We cannot define a linking pairing on the topological spaces $L(m, r_1, r_2, \dots)$. Thus the following question arises:

Question 163.2. How can one show that given $m \in \mathbb{N}$ the homotopy type of the topological spaces $L(m, r_1, r_2, \dots)$ depends on the choice of the parameters r_i?

Proof.

[1997] The slightly hesitant "basically" refers to the fact that out of $\mathbb{C}\mathrm{P}^\infty$ one can build a few other examples of Eilenberg-Maclane spaces, e.g. one can consider $\mathbb{C}\mathrm{P}^\infty \times \overline{B}^k$ or, as we will see shortly, the product $\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}^2, 2)$.
[1998] Here we view S^∞ as a subset of $\mathbb{C}^\infty$ via the identification $\mathbb{C}^\infty = \mathbb{R}^\infty$ from page 1476.

(1) The proof of this statement is quite similar to the task performed in Exercise 68.4. We leave it to the motivated reader to prove the desired statement. As always it takes a little effort to work rigorously with the topology of $S^\infty = \varinjlim S^n$.

(2) Given $n \in \mathbb{N}$ the action of $\mathbb{Z}_m$ on S^∞ restricts to an action on $S^{2n-1} \subset S^\infty$. In [**Hat2002**, p. 144] or alternatively [**Shas2014**, p. 196] an explicit CW-structure for $S^{2n-1}/\mathbb{Z}_m$ is constructed. These CW-structures have the property that for $k < l$ the $(2k-1)$-skeleton of $S^{2l-1}/\mathbb{Z}_m$ is precisely the CW-complex $S^{2k-1}/\mathbb{Z}_m$. It follows quite easily from Lemma 56.5 that the CW-structures on the $S^{2n-1}/\mathbb{Z}_m$ give rise to a CW-structure on $S^\infty/\mathbb{Z}_m$.

(3) (a) Let $n \geq 2$. We calculate that

$$\pi_n(L(m, r_1, r_2, \dots)) \quad \cong \quad \pi_n(S^\infty/\mathbb{Z}_m) \quad \underset{\uparrow}{\cong} \quad \pi_n(S^\infty) \quad \underset{\uparrow}{=} \quad 0.$$

follows from (1), the Actions-Covering Proposition 48.9 and Coverings-π_n-Proposition 71.13 and from the fact that $n \geq 2$ — Corollary 71.12

(b) It follows immediately from (1), the Fundamental Group-via-Actions Theorem 48.16 and Corollary 71.12 that we have isomorphisms $\pi_1(L(m, r_1, r_2, \dots)) = \pi_1(S^\infty/\mathbb{Z}_m) \cong \mathbb{Z}_m$. ■

The following proposition allows us to construct more Eilenberg-Maclane spaces out of given ones.

Proposition 163.3. (Eilenberg-Maclane Construction Proposition)

(1) Let X be an Eilenberg-Maclane space of type $\mathrm{K}(A, n)$ and let Y be an Eilenberg-Maclane space of type $\mathrm{K}(B, n)$.
   (a) The CW-complex $X \otimes Y$, that is defined on page 1500, is an Eilenberg-Maclane space of type $\mathrm{K}(A \times B, n)$.
   (b) If at least one of X or Y has only countably many cells, then the product $X \times Y$ is an Eilenberg-Maclane space of type $\mathrm{K}(A \times B, n)$.

The remaining two statements only apply to the case $n = 1$.

(2) Let X be an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$. We pick a base point $x_0 \in X$ and we pick an isomorphism $\varphi\colon \pi_1(X, x_0) \to \pi$. Let Γ be a subgroup of π. We denote by $\widetilde{X}$ the covering space of X corresponding to the subgroup[1999] $\varphi^{-1}(\Gamma) \subset \pi_1(X, x_0)$. Then $\widetilde{X}$ is an Eilenberg-Maclane space of type $\mathrm{K}(\Gamma, n)$.

(3) Let X be an Eilenberg-Maclane space of type $\mathrm{K}(A, 1)$ and let Y be an Eilenberg-Maclane space of type $\mathrm{K}(B, 1)$. We pick CW-structures for X and Y. Then any wedge[2000] $X \vee Y$ is an Eilenberg-Maclane space of type $\mathrm{K}(A * B, 1)$.

Remark. The restriction in the Eilenberg-Maclane Construction Proposition 163.3 (1) on the CW-complexes X and Y is a major nuisance. Even though this condition on the number

[1999] Since X is a CW-complex we know from the CW-Complex Properties Proposition 68.10 (6) that X is locally simply connected. Therefore it follows from the Covering Existence Theorem 61.5 that the covering exists and it follows from the Covering Uniqueness Theorem 61.9 that the covering corresponding to the subgroup $\varphi^{-1}(\Gamma) \subset \pi_1(X, x_0)$ is essentially unique.

[2000] Recall our convention from page 1507: whenever we take the wedge of CW-complexes, we only use points in the 0-skeleton. The "any" now refers to the fact that, under this restriction, any wedge is in fact an Eilenberg-Maclane space.

of cells is not an issue for the Eilenberg-Maclane spaces that we have considered above, the "canonical" Eilenberg-Maclane spaces that we will construct shortly do *not* satisfy this condition. Thus one should apply the Eilenberg-Maclane Construction Proposition 163.3 (1) with great caution.

Proof.

(1) Recall that by construction and by Proposition 68.23 we know $X \otimes Y$ is a CW-complex whose underlying set is $X \times Y$. Let $\mathrm{id}\colon X \otimes Y \to X \times Y$ be the identity map. We pick $x \in X$ and $y \in Y$. Given any $k \in \mathbb{N}_0$ we have the following isomorphisms

$$\pi_k(X \otimes Y, (x,y)) \xrightarrow[\uparrow]{\cong} \pi_k(X \times Y, (x,y)) \xrightarrow[\uparrow]{\cong} \pi_k(X,x) \times \pi_k(Y,y) \xrightarrow{\cong} \begin{cases} A \times B, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 161.7 — Propositions 48.20 and 71.8

Since $X \otimes Y$ is a CW-complex we obtain from this discussion that $X \otimes Y$ is an Eilenberg-Maclane space of type $\mathrm{K}(A \times B, n)$.

If at least one of X or Y has only countably many cells, then we know by Proposition 68.23 that $X \otimes Y = X \times Y$, i.e. $X \times Y$ itself is an Eilenberg-Maclane space of type $\mathrm{K}(A \times B, n)$.

(2) First note that we know by the CW-complex Covering Proposition 68.37 that $\widetilde{X}$ is a CW-complex. It now follows immediately from the Coverings-π_n-Proposition 71.13 that $\widetilde{X}$ is an Eilenberg-Maclane space of type $\mathrm{K}(\Gamma, n)$.

(3) Let $x \in X^0$ and $y \in Y^0$ be points in the 0-skeleta of X and Y. By the CW-Complex Properties Proposition 68.10 (5) we know that the points x and y are good. Thus it follows from the Wedge-π_1-Proposition 52.3 that we have isomorphisms $\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y) \cong A * B$. Furthermore we obtain basically immediately from Exercise 85.3 that all higher homotopy groups of $X \vee Y$ vanish. Finally we know by the CW-Complex Construction Lemma 68.32 (4) that $X \vee Y$ is a CW-complex. ■

The following lemma summarizes a few basic facts about Eilenberg-Maclane spaces.

Lemma 163.4. Let $n \in \mathbb{N}$ and let π be a group. If X is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$, then given any abelian group G the following statements hold:

(1) We have $\mathrm{H}_0(X;G) \cong G$ and $\mathrm{H}^0(X;G) \cong G$.
(2) For $i = 1, \ldots, n-1$ we have $\mathrm{H}_i(X;G) = 0$ and $\mathrm{H}^i(X;G) = 0$.
(3) Given any $x_0 \in X$ the Hurewicz homomorphism $\pi_n(X,x_0)_{\mathrm{ab}} \to \mathrm{H}_n(X;\mathbb{Z})$ is an isomorphism. In particular $\mathrm{H}_n(X;\mathbb{Z}) \cong \pi_{\mathrm{ab}}$.
(4) The evaluation map $\mathrm{H}^n(X;G) \to \mathrm{Hom}(\mathrm{H}_n(X;\mathbb{Z}), G)$ is an isomorphism.

Proof. The statements follow easily from the Hurewicz Theorems 84.5 and 85.5 and from the Universal Coefficient Theorems 89.17 and 110.12. The experienced reader will have no troubles with filling in the details. ■

163.2. **Killing higher homotopy groups.** In the subsequent section we will give a general construction of Eilenberg-Maclane spaces. To streamline the discussion we outsource a few preparations to the present section.

First we point out that the following, fairly self-evident convention will be in effect in the coming chapters.

Convention.

(1) Given $n \in \mathbb{N}_0$ we write $* = (0, \ldots, 0, 1) \in S^n$. We equip S^n with the canonical CW-structure from page 1472, i.e. we view S^n as a CW-complex with one 0-cell $\{*\}$ and one n-cell.

(2) We always form any wedge of spheres $\bigvee_{i \in I} S_i^{n_i}$ by identifying the points $* \in S_i^{n_i}$. By the CW-Complex Construction Lemma 68.32 (4) we can and will view the wedge $\bigvee_{i \in I} S_i^{n_i}$ as a CW-complex in the obvious way, i.e. we have one 0-cell $\{*\}$ and for each $i \in I$ we have one n_i-cell.

(3) By a slight abuse of notation we sometimes say X is a k-dimensional CW-complex if the highest dimension of a cell is $\leq k$.

We start out with the following calculation.

Proposition 163.5. (Wedge-of-Spheres-π_n-Proposition) Let I be a set, let $n \in \mathbb{N}$ and for each $i \in I$ let S_i^n be a copy of S^n. For each $i \in I$ we denote by $\varphi_i \colon S^n \to S_i^n$ the obvious homeomorphism. By a slight abuse of notation we denote by φ_i also the composition of φ_i with the canonical inclusion map $S_i^n \to \bigvee_{i \in I} S_i^n$.

(1) For any $i \in \{1, \ldots, n-1\}$ we have $\pi_i\Big(\bigvee_{i \in I} S_i^n, *\Big) = 0$.

(2) If $n = 1$, then the homomorphism[2001]

$$\begin{array}{rcl} \overbrace{\langle \{x_i\}_{i \in I} \rangle}^{\text{free group}} & \to & \pi_1\Big(\bigvee_{i \in I} S_i^1, *\Big) \\ x_i & \mapsto & [\varphi_i] \end{array}$$

is an isomorphism.

(3) If $n \geq 2$, then the homomorphism[2002]

$$\begin{array}{rcl} \overbrace{\mathbb{Z}^{(I)} = \bigoplus_{i \in I} \mathbb{Z} \cdot i}^{\text{free abelian group}} & \to & \pi_n\Big(\bigvee_{i \in I} S_i^n, *\Big) \\ i & \mapsto & [\varphi_i] \end{array}$$

is an isomorphism.

Proof.

(1) As we pointed out above, we can view $\bigvee_{i \in I} S_i^n$ as a CW-complex with one 0-cell and no other cells outside of dimension n. It follows from the CW-Skeleton-π_n-Proposition 71.9 that all homotopy groups in dimension $1, \ldots, n-1$ vanish.

(2) In the case that I is a finite set, then, as we saw in the proof of the Fundamental Group-Wedge-of-Circles Lemma 52.4, the statement follows easily from applying the Wedge-π_1-Proposition 52.3 altogether $k - 1$ times. The general case is proved in Proposition 52.7.

[2001] By Lemma 51.15 the given data does indeed determine a unique homomorphism.

[2002] Since $n \geq 2$ we know by the Homotopy Group-Abelian Proposition 71.3 that the right-hand side is an abelian group. Thus it follows from Lemma 51.1 that the given data does indeed determine a unique homomorphism.

(3) Let $n \in \mathbb{N}_{\geq 2}$. We consider the maps

$$\mathbb{Z}^{(I)} = \bigoplus_{i\in I} \mathbb{Z}\cdot i \xrightarrow[\uparrow]{\cong} \bigoplus_{i\in I} \mathrm{H}_n(S_i^n;\mathbb{Z}) \xrightarrow[\uparrow]{\cong} \mathrm{H}_n\Big(\bigvee_{i\in I} S_i^n;\mathbb{Z}\Big) \xleftarrow[\uparrow]{\cong} \pi_n\Big(\bigvee_{i\in I} S_i^n, *\Big).$$

given by $i \mapsto [S_i^n]$ — Wedge-H$_*$-Proposition 79.9 — by the Hurewicz Theorem 85.5 which we can apply by (1)

It is straightforward to verify that this composition of isomorphisms is precisely the isomorphism under consideration. ■

We continue with the following two propositions which basically says that for a given CW-complex one can "kill" all homotopy groups beyond a chosen degree.

Proposition 163.6. (Kill-Higher-π_*-Proposition) Let $k \in \mathbb{N}$, let X be a connected $(k+1)$-dimensional CW-complex[2003] and let $x_0 \in X$ be a base point.

(1) There exists a $(k+2)$-dimensional CW-complex Y with the following properties:
   (a) the $(k+1)$-skeleton of Y agrees with X,
   (b) for $i = 1, \ldots, k$ the inclusion induced map $\pi_i(X, x_0) \to \pi_i(Y, x_0)$ is an isomorphism, and
   (c) we have $\pi_{k+1}(Y, x_0) = 0$.

(2) The CW-complex Y in (1) can be chosen canonically.[2004]

(3) If X has countably many cells, then Y as in (1) can be chosen to have countably many cells.[2005]

Proof.

(1) We pick a set $\{\varphi_j \colon (S^{k+1}, *) \to (X, x_0)\}_{j\in J}$ of maps that represent a generating set for the group $\pi_{k+1}(X, x_0)$. We set

$$Y = \Big(X \sqcup \bigsqcup_{j\in J} \overline{B}_j^{k+2}\Big)/\!\sim \qquad \text{with } \varphi_j(s) \sim s \text{ for } s \in S_j^{k+1} = \partial\overline{B}_j^{k+2}.$$

Put differently, Y is the $(k+2)$-dimensional CW-complex that is obtained from X by attaching one $(k+2)$-cell for each $j \in J$ with attaching map given by φ_j. We denote by $i \colon X \to Y$ the inclusion map. Now we verify that Y has all the desired qualities.

   (a) By construction Y is a $(k+2)$-dimensional CW-complex such that the $(k+1)$-skeleton equals X.

   (b) It follows immediately from the CW-Skeleton-π_n-Proposition 71.9 and (a) that the inclusion induced map $i_* \colon \pi_s(X, x_0) \to \pi_s(Y, x_0)$ is an isomorphism for $s = 1, \ldots, k$ and that the map $i_* \colon \pi_{k+1}(X, x_0) \to \pi_{k+1}(Y, x_0)$ is an epimorphism.

[2003] We follow the convention established on pages 1469 and 3431, i.e. when we say "n-dimensional CW-complex" we allow for the possibility that the highest dimension of a cell is $< n$.

[2004] Here "canonical" is shorthand for saying that in the proof of the proposition we provide an explicit construction of Y does not depend on any choices.

[2005] Note though that we do *not* claim that we can find a canonical Y with countably many cells.

(c) Let $j \in J$. We have the following simple but useful commutative diagram:

$$\begin{array}{ccccc} S^{k+1} & & \xrightarrow{\quad\varphi_j\quad} & & X \\ \downarrow & & & & \downarrow i \\ \overline{B}^{k+2} & \xrightarrow{\text{id}} & \overline{B}_j^{k+2} & \hookrightarrow & X \sqcup \bigsqcup_{j\in J} \overline{B}_j^{k+2} \twoheadrightarrow Y. \end{array}$$

In other words, the map $i \circ \varphi_j \colon S^{k+1} \to Y$ extends to a map $\overline{B}^{k+2} \to Y$. By the Trivial-in-π_n-Lemma 71.4 this implies that $i_*([\varphi_j]) = [i \circ \varphi_j] = 0 \in \pi_{k+1}(Y, x_0)$.

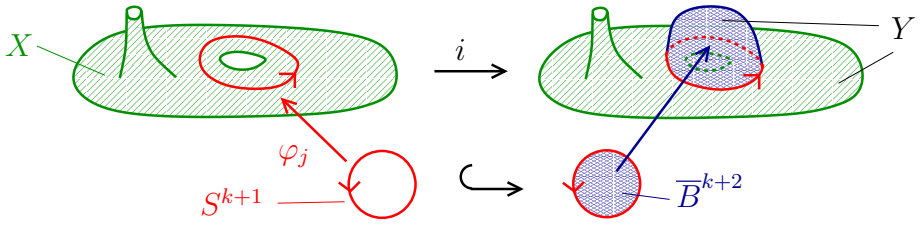
(d) Since $\{[\varphi_j]\}_{j\in J}$ is by choice a generating set for $\pi_{k+1}(X, x_0)$ we obtain from (c) that $i_* \colon \pi_{k+1}(X, x_0) \to \pi_{k+1}(Y, x_0)$ has trivial image.

(e) It follows from (b) and (d) that $\pi_{k+1}(Y, x_0) = 0$.

We have thus verified that Y has all the desired properties.

(2) It is easy to come up with a canonical Y: in the construction provided in (1) we just need to consider the set of all maps $(S^{k+1}, *) \to (X, x_0)$. Evidently this set is canonical and evidently it represents a generating set for $\pi_{k+1}(X, x_0)$. The resulting CW-complex Y is the canonical CW-complex we desired.

(3) Finally suppose that X has countably many cells. It follows from the rather non-trivial Proposition 104.21 that $\pi_{k+1}(X, x_0)$ is countable. Therefore we can pick a countable family $\{\varphi_j \colon (S^{k+1}, *) \to (X, x_0)\}_{j\in J}$ of maps that represent a generating set for the group $\pi_{k+1}(X, x_0)$. The resulting CW-complex Y evidently has only countably many cells. ■



We conclude this technical section with the following proposition that is mostly just an iterated application of the Kill-Higher-π_*-Proposition 163.6.

Proposition 163.7. (Kill-All Higher-π_n-Proposition) Let $k \in \mathbb{N}$, let Y be a connected $(k+1)$-dimensional CW-complex and let $y_0 \in Y$.

(1) There exists a CW-complex Z with the following three properties:
  (a) the $(k+1)$-skeleton of Z equals Y,
  (b) we have $\pi_i(Z) = 0$ for $i \geq k+1$,
  (c) for any $i \in \{1, \ldots, k\}$ the inclusion induced map $\pi_i(Y, y_0) \xrightarrow{\cong} \pi_i(Z, y_0)$ is an isomorphism.

(2) The CW-complex Z in (1) can be chosen canonically.

(3) If Y has countably many cells, then Z as in (1) can be chosen to have countably many cells.

Proof. Let $k \in \mathbb{N}$ and let Y be a connected $(k+1)$-dimensional CW-complex. Let $y_0 \in Y$.

(1) We define $Z_{k+1} = Y$. For $i = k+1, k+2, \ldots$ we iteratively apply the Kill-Higher-π_*-Proposition 163.6 to Z_i to obtain a sequence of CW-complexes $Z_{k+1}, Z_{k+2}, \ldots$ with the following properties:
   (i) each Z_i is an i-dimensional CW-complex,
   (ii) for each $i \geq k+1$ the i-skeleton of Z_{i+1} is given by Z_i,
   (iii) for each $i \geq k+1$ we have $\pi_i(Z_{i+1}, y_0) = 0$.

   We set $Z := \varinjlim Z_i = \bigcup_{i=k+1}^{\infty} Z_i$ and we equip it with the obvious CW-structure such that for each $i \geq k+1$ the i-skeleton of Z is given by Z_i. We claim that Z has the desired three properties:
   (a) It follows from (ii) that the $(k+1)$-skeleton of Z equals Y.
   (b) Let $i \geq k+1$. We have

$$\pi_i(Z, y_0) \underset{\substack{\uparrow \\ \text{CW-Skeleton-}\pi_n\text{-Proposition 71.9}}}{\overset{\cong}{\leftarrow}} \pi_i((i+1)\text{-skeleton } Z^{i+1} \text{ of } Z, y_0) \underset{\substack{\uparrow \\ \text{since } Z^{i+1} = Z_{i+1}}}{=} \pi_i(Z_{i+1}, y_0) \underset{\substack{\uparrow \\ \text{by (iii)}}}{=} 0.$$

   (c) This statement follows immediately from (a) and the CW-Skeleton-π_n-Proposition 71.9.

(2) This statement follows immediately from the Kill-Higher-π_*-Proposition 163.6 (2).

(3) This statement follows immediately from the Kill-Higher-π_*-Proposition 163.6 (3). ■

163.3. **Existence of Eilenberg-Maclane Spaces.** Using the examples from the first section and using the Eilenberg-Maclane Construction Proposition 163.3 we can "by hand" construct Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, 1)$ for many groups π. One can ask for which pairs (π, n) there exist corresponding Eilenberg-Maclane spaces. There is one obvious restriction: since homotopy groups $\pi_n(X)$ with $n \geq 2$ are abelian, see the Homotopy Group-Abelian Proposition 71.3, we see that for $n \geq 2$ we can only hope to find an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ if π is abelian. Perhaps somewhat surprisingly this is the only restriction on the existence of Eilenberg-Maclane spaces.

Theorem 163.8. (Eilenberg-Maclane Existence Theorem) Let $n \in \mathbb{N}$ and let π be a group. If $n \geq 2$, then we demand that π is abelian. There exists a *canonical* triple $(X, x_0, \varphi\colon \pi_n(X, x_0) \to \pi)$ with the following properties:
(1) X is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$.
(2) X is a CW-complex with one 0-cell $\{x_0\}$ and with no cells in dimensions $1, \ldots, n-1$.
(3) $\varphi\colon \pi_n(X, x_0) \to \pi$ is an isomorphism.

Definition. We refer to the canonical CW-complex from the Eilenberg-Maclane Existence Theorem 163.8 as the canonical Eilenberg-Maclane space $\mathrm{K}(\pi, n)$.

Remark. As we will see shortly, the canonical Eilenberg-Maclane spaces that we construct in the proof of the Eilenberg-Maclane Existence Theorem 163.8 are monstrously large. More precisely, the CW-complex of type $\mathrm{K}(\pi, n)$ has uncountably many cells in each dimension $\geq n+1$. The cautionary remark from page 3430 about taking products of Eilenberg-Maclane spaces thus applies to the canonical Eilenberg-Maclane spaces.

After the discussion of the previous section the approach to producing an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ is presumably pretty clear:

(1) Start out with a CW-complex with one 0-cell, no cells in dimension $1, \ldots, n-1$ and "sufficiently" many n-cells to be able to "generate" the group π.
(2) Attach $(n+1)$-cells to chop down the n-th homotopy group to obtain the desired group π.
(3) Attach cells of dimension $\geq n+2$ to kill all homotopy groups in dimensions $\geq n+1$.

We basically know how to perform (1) and (3), so it remains to do (2) carefully. The combination of steps (1) and (2) is performed in the following proposition.

Proposition 163.9. (Homotopy Group-Realization Proposition)

(1) Let π be any group. There exists a canonical triple $(Y, x_0, \varphi\colon \pi_1(Y, x_0) \to \pi)$ with the following properties:
   (a) Y is a 2-dimensional CW-complex with one 0-cell $\{x_0\}$,
   (b) $\varphi\colon \pi_1(Y, x_0) \to \pi$ is an isomorphism.
(2) Let π be an abelian group π and let $n \in \mathbb{N}_{\geq 2}$. There exists a canonical triple $(Y, x_0, \varphi\colon \pi_n(Y, x_0) \to \pi)$ with the following properties:
   (a) Y is an $(n+1)$-dimensional CW-complex with one 0-cell $\{x_0\}$ and no cells in dimensions $1, \ldots, n-1$,
   (b) $\varphi\colon \pi_n(Y, x_0) \to \pi$ is an isomorphism.

Remark. The first result is related to Corollary 69.12 which says that given any *finitely presented* group π there exists a *finite* connected 2-dimensional CW-complex Y with $\pi_1(Y) \cong \pi$.

Before we provide the proof of the Homotopy Group-Realization Proposition 163.9 we convince ourselves of the fact that this proposition, together with the Kill-All Higher-π_n-Proposition 163.7, does indeed imply the Eilenberg-Maclane Existence Theorem 163.8.

Proof of the Eilenberg-Maclane Existence Theorem 163.8. In the following we discuss the case $n \geq 2$. The proof for $n = 1$ is almost identical. Thus let $n \in \mathbb{N}_{\geq 2}$ and let π be an abelian group. By the Homotopy Group-Realization Proposition 163.9 (2) there exists a canonical triple $(Y, x_0, \varphi\colon \pi_n(Y, x_0) \to \pi)$ with the following properties:

(1) Y is an $(n+1)$-dimensional CW-complex with one 0-cell $\{0\}$ and no cells in dimensions $1, \ldots, n-1$,
(2) $\varphi\colon \pi_n(Y, x_0) \to \pi$ is an isomorphism.

We plug this data into the Kill-All Higher-π_n-Proposition 163.7 and we obtain a canonical CW-complex Z with the following three properties:

(3) the $(n+1)$-skeleton of Z equals Y (here we use that Y equals its own $(n+1)$-skeleton),
(4) we have $\pi_j(Z) = 0$ for $j \geq n+1$,
(5) the inclusion $i\colon Y \to Z$ induces an isomorphism $\pi_n(Y, x_0) \xrightarrow{\cong} \pi_n(Z, x_0)$.

Note that it follows from (1) and (3) together with the CW-Skeleton-π_n-Proposition 71.9 that the following holds:

(6) We have $\pi_j(Z, x_0) = 0$ for $j = 1, \ldots, n-1$.

It follows immediately from all of the above statements that the canonical triple

$$\big(Z, x_0, \varphi \circ i_*^{-1} \colon \pi_n(Z, x_0) \xleftarrow[\cong]{i_*} \pi_1(Y, x_0) \xrightarrow[\cong]{\varphi} \pi\big)$$

has all the desired properties. ■

Now we turn to the proof of the Homotopy Group-Realization Proposition 163.9. The proof naturally splits into the two cases $n = 1$ and $n \geq 2$.

Proof of the Homotopy Group-Realization Proposition 163.9 (1). Let π be a group. Given $g \in \pi$ we write $S^1_g := S^1 \times \{g\}$, i.e. S^1_g is a copy of S^1. Let $* = (0, 1) \in S^1$ be the usual base point of S^1. We consider the wedge $X := \bigvee_{g \in \pi} S^1_g$ which is given by identifying all the points $(*, g) \in S^1_g$. We denote by $x_0 \in X$ the wedge point. We apply the convention from page 3431 to view X as a CW-complex in the obvious way.

By the Wedge-of-Spheres-π_n-Proposition 163.5 (2) we have a canonical identification $\langle \{x_g\}_{g \in \pi} \rangle = \pi_1(X, x_0)$. Thus, by Lemma 51.15, there exists a unique homomorphism $\psi \colon \pi_1(X, x_0) = \langle \{x_g\}_{g \in \pi} \rangle \to \pi$ such that $\psi(x_g) = g$ for every $g \in \pi$. This homomorphism is, by construction, an epimorphism. We set $\Gamma = \ker(\psi)$.

Next let $\{\alpha_j \colon (S^1, *) \to (X, x_0)\}_{j \in J}$ be the family of all loops that represent an element in $\Gamma = \ker(\psi \colon \pi_1(X, x_0) \to \pi)$. We set

$$Y \;=\; \Big(X \sqcup \bigsqcup_{j \in J} \overline{B}^2_j\Big)/\!\sim \qquad \text{with } \alpha_j(s) \sim s \text{ for } s \in S^1_j = \partial \overline{B}^2_j.$$

Put differently, Y is the 2-dimensional CW-complex that is obtained from X by attaching one 2-cell for each $j \in J$ with attaching map given by α_j. By construction Y is a 2-dimensional CW-complex such that the 1-skeleton equals X. We denote by $i \colon X \to Y$ the inclusion map.

Claim. The map $i_* \colon \pi_1(X, x_0) \to \pi_1(Y, x_0)$ descends to a map

$$\varphi \colon \pi_1(X, x_0)/\Gamma \;\to\; \pi_1(Y, x_0)$$

and this map is an isomorphism.

Proof. The proof naturally breaks into several parts.

(1) Let $j \in J$. The same elementary argument as in the proof of the Kill-Higher-π_*-Proposition 163.6 shows that the map $i \circ \alpha_j \colon S^1 \to Y$ extends to a map $\overline{B}^2 \to Y$. By the Extension-to-Disk Lemma 47.1 this observation implies that $i_*([\alpha_j]) = [i \circ \alpha_j] = 0 \in \pi_1(X, x_0)$. Since $\{[\alpha_j]\}_{j \in J}$ contains every single element of Γ we see that $i_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ descends to a map

$$\varphi \colon \pi_1(X, x_0)/\Gamma \;\to\; \pi_1(Y, x_0).$$

(2) It follows from $X = Y^1$ and the CW-Skeleton-π_n-Proposition 71.9 that the map $i_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an epimorphism. This implies that φ is also an epimorphism.

(3) We still need to show that the map $\varphi \colon \pi_1(X, x_0)/\Gamma \to \pi_1(Y, y_0)$ is a monomorphism. At this stage this is equivalent to showing that $\ker(i_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0)) = \Gamma$. Thus let $g \in \ker(i_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0))$. We start out with the following claim.

Claim. There exists a finite subset $\{i_1, \dots, i_m\} \subset I$ such that g lies in the kernel of the inclusion induced map $\pi_1(X, x_0) \to \pi_1\big(\big(X \sqcup \big(\overline{B}^2_{i_1} \cup \dots \cup \overline{B}^2_{i_m}\big)\big)/\!\sim, x_0\big)$.

Proof. By Lemma 56.9 there exists a compact subset K of Y such that g lies in the kernel of the inclusion induced map $\pi_1(X, x_0) \to \pi_1(X \cup K, x_0)$. The claim now follows easily from Lemma 68.16 (2). ⊞

Let $I = \{i_1, \dots, i_m\}$ be such a finite subset. It follows from the Cell Attachment-π_1-Proposition 69.11 (2), applied altogether m times, together with Exercise 53.1 that we have the following commutative diagram

$$\begin{array}{ccc} \pi_1(X, x_0) & \longrightarrow & \pi_1\big(\big(X \sqcup \big(\overline{B}^2_{i_1} \cup \dots \cup \overline{B}^2_{i_m}\big)\big)/\!\sim, x_0\big) \\ & \searrow & \downarrow \\ & & \pi_1(X, x_0)/\langle\!\langle [\alpha_{i_1}], \dots, [\alpha_{i_m}] \rangle\!\rangle. \end{array}$$

Thus we see that $g \in \langle\!\langle [\alpha_{i_1}], \dots, [\alpha_{i_m}] \rangle\!\rangle$. Since $[\alpha_{i_1}], \dots, [\alpha_{i_m}]$ lie by definition in Γ we see that $g \in \Gamma$.

We are now basically done. Indeed, the CW-complex Y is 2-dimensional, it has a single 0-cell x_0 and it is canonical. Furthermore the map $\psi\colon \pi_1(X, x_0) \to \pi$ induces an isomorphism $\overline{\psi}\colon \pi_1(X, x_0)/\Gamma \to \pi$. Thus we see that the canonical triple

$$\Big(Y, x_0, \pi_1(Y, x_0) \underset{\cong}{\overset{\varphi}{\leftarrow}} \pi_1(X, x_0)/\Gamma \underset{\cong}{\overset{\overline{\psi}}{\to}} \pi\Big)$$

has all the desired properties. ■

Now we turn to the proof of the Homotopy Group-Realization Proposition 163.9 (2).

Proof of the Homotopy Group-Realization Proposition 163.9 (2). Let π be an abelian group and let $n \geq 2$. If we do not care about a canonical construction, then we could just use the Moore Space Existence Proposition 79.11 to obtain an $(n+1)$-dimensional CW-complex Y with one 0-cell and no cells in dimensions $1, \dots, n-1$ and with $\mathrm{H}_n(Y; \mathbb{Z}) \cong \pi$. By the Hurewicz Theorem 85.5 this implies that $\pi_n(Y) \cong \pi$. Unfortunately this construction is not canonical.

Since we are really eager to have canonical constructions we need to provide a different argument. Evidently, in the following discussion, we will try for as long as possible to roll over ideas from the proof of the Homotopy Group-Realization Proposition 163.9 (1).

For each $g \in \pi$ we write $S^n_g := S^n \times \{g\}$. We consider $* = (0, \dots, 0, 1) \in S^n$. As in (1) we consider the wedge $X := \bigvee_{g \in \pi} S^n_g$ and we view X canonically as a CW-complex. We denote by $x_0 \in X$ the wedge point.

By the Wedge-of-Spheres-π_n-Proposition 163.5 (3) we have a canonical identification $\pi_n(X, *) = \mathbb{Z}^{(\pi)}$. Since π is an abelian group we know from Lemma 51.1 that there exists a unique homomorphism $\psi\colon \pi_n(X, *) = \mathbb{Z}^{(\pi)} \to \pi$ with $\psi(g) = g$ for every $g \in \pi$. This homomorphism is, by construction, an epimorphism. We set $\Gamma = \ker(\psi)$.

Next let $\{\alpha_j\colon (S^n, *) \to (X, x_0)\}_{j \in J}$ be the family of all maps that represent an element in $\Gamma = \ker(\psi\colon \pi_n(X, *) \to \pi)$. As in the proof of (1) we set

$$Y \;=\; \Big(X \sqcup \bigsqcup_{j \in J} \overline{B}^{n+1}_j\Big)/\!\sim \qquad \text{with } \alpha_j(s) \sim s \text{ for } s \in S^n_j = \partial \overline{B}^n_j.$$

By construction Y is an $(n+1)$-dimensional CW-complex such that the n-skeleton equals X. We denote by $i\colon X \to Y$ the inclusion map.

Claim. The map $i_*\colon \pi_n(X,x_0) \to \pi_n(Y,x_0)$ descends to a map

$$\varphi\colon \pi_n(X,x_0)/\Gamma \;\to\; \pi_n(Y,x_0)$$

and this map is an isomorphism.

Proof. We consider the following diagram

$$\begin{array}{ccccccc}
 & & \Gamma & \hookrightarrow & \pi_n(X,x_0) & \xrightarrow{i_*} & \pi_n(Y,x_0) \longrightarrow 0 \\
 & & {\scriptstyle a_j\mapsto[\alpha_j]}\uparrow\!\!\uparrow & & \downarrow{=} & & \downarrow{=} \\
 & \bigoplus_{j\in J}\mathbb{Z}\cdot a_j & \xrightarrow{a_j\mapsto[\alpha_j]} & \pi_n(X,x_0) & \xrightarrow{i_*} & \pi_n(Y,x_0) & \longrightarrow 0 \\
 & \downarrow{=} & & \downarrow{\cong} & & \downarrow{\cong} & \\
 & \bigoplus_{j\in J}\mathbb{Z}\cdot a_j & \xrightarrow{a_j\mapsto\alpha_{j*}([S^n])} & \mathrm{H}_n(X;\mathbb{Z}) & \xrightarrow{i_*} & \mathrm{H}_n(Y;\mathbb{Z}) & \longrightarrow 0.
\end{array}$$

We make the following clarification and observations:

(1) The squares at the top commute basically by definition. Note that the vertical map to the top left is an epimorphism by definition of the family $\{\alpha_j\}_{j\in J}$.
(2) The vertical maps between the second and the third row, which are located in the middle and to the right, are the Hurewicz homomorphisms from page 1842.
(3) It follows from the naturality of the Hurewicz homomorphism, see Lemma 85.1 (3), that the square to the bottom right commutes.
(4) It follows basically immediately from the definition of the Hurewicz homomorphism that the square to the bottom left commutes.
(5) By construction we know that X and Y have precisely one 0-cell and no cells in dimension $1,\dots,n-1$. This implies by the CW-Skeleton-π_n-Proposition 71.9 that X and Y are $(n-1)$-connected. Now we obtain from the Hurewicz Theorem 85.5 that the vertical maps in the middle and to the right are both isomorphisms.
(6) It follows from a modest generalization of the Cell Attachment-H_*-Lemma 78.18 that the bottom horizontal sequence is exact.[2006]

It follows from all the information assembled above that the top horizontal sequence is also exact. But this fact is just a reformulation of the statement that we are supposed to prove in the claim. ⊞

As in the proof of (1) we are now basically done. Indeed, the CW-complex Y is $(n+1)$-dimensional, it has a single 0-cell x_0, it has no cells in dimensions $1,\dots,n$ and it is canonical. Furthermore the map $\psi\colon \pi_n(X,x_0) \to \pi$ induces an isomorphism $\overline{\psi}\colon \pi_n(X,x_0)/\Gamma \to \pi$. Thus we see that the canonical triple

$$\Big(Y, x_0, \pi_n(Y,x_0) \xleftarrow[\cong]{\varphi} \pi_n(X,x_0)/\Gamma \xrightarrow[\cong]{\overline{\psi}} \pi\Big)$$

has all the desired properties. ■

[2006] the Cell Attachment-H_*-Lemma 78.18 was formulated for attaching a single $(n+1)$-cell, but basically the same argument also works for attaching an arbitrary family of $(n+1)$-cells.

We have now seen that given any group there exists a CW-complex that is an Eilenberg-Maclane space. But it is pretty clear that the construction provided by the proof of the Eilenberg-Maclane Existence Theorem 163.8 is pretty wasteful, except for the trivial group the construction provides an infinite-dimensional CW-complex. In many cases, e.g. for the groups $\pi = \mathbb{Z}^m$, there are, as we have seen on page 3427, much simpler, finite-dimensional Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, 1)$. This point towards a common tension:

(1) On the one hand it is nice to have "small constructions" with few cells. For example it is much easier to visualize finite CW-complexes and it is usually much easier to calculate invariants for finite CW-complexes.
(2) On the other hand it is nice to have canonical constructions which do not depend on any choices. But such constructions are almost invariably humongous.

This situation is comparable to the discussion of free resolutions of abelian groups. On the one hand, for actual calculations one prefers "small" free resolutions. On the hand, say to give unambiguous definitions of Tor-groups and Ext-groups one needs to work with the canonical free resolution from page 1922 which is outrageously wasteful.

It is natural to ask whether "small" groups also admit "small" Eilenberg-Maclane spaces. For example the following question arises:

Question 163.10. Let $m \in \mathbb{N}_{\geq 2}$. Can we find a finite-dimensional Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}_m, 1)$?

In some instances one can also attempt to perform the three-step-process from page 3435 with more care to provide "smaller" Eilenberg-Maclane spaces. In the remainder of this section we want to discuss the construction of the Eilenberg-Maclane spaces $\mathrm{K}(\mathbb{Z}, n)$. For $n = 1$ everything that can be said had been said on page 3427.
We will now construct an explicit Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, 2)$ using the above procedure:

(1) We start with $X_2 := \mathbb{C}\mathrm{P}^1 = S^2$. We have $\pi_1(X_2 = \mathbb{C}\mathrm{P}^1) = 0$ and $\pi_2(X_2 = \mathbb{C}\mathrm{P}^1) \cong \mathbb{Z}$.
(2) By Theorem 155.12 we know that $\pi_3(\mathbb{C}\mathrm{P}^1) = [p]$ where $p\colon S^3 \to \mathbb{C}\mathrm{P}^1$ denotes the Hopf map from page 3306.
(3) We set $X_4 := \overline{B}^4 \cup_{p\colon S^3 \to \mathbb{C}\mathrm{P}^1} \mathbb{C}\mathrm{P}^1$.
(4) By the Mapping Cone-of-Hopf Map Lemma 122.2 (1) we know that X_4 is homeomorphic to $\mathbb{C}\mathrm{P}^2$.[2007]

[2007] For the convenience of the reader we repeat the argument. For the convenience of the author we prove a slight generalization of the Mapping Cone-of-Hopf Map Lemma 122.2 (1). Thus let $n \in \mathbb{N}$. We consider the maps

$$\begin{array}{rcl} f\colon \mathrm{Cone}(S^{2n+1}) & \to & \mathbb{C}\mathrm{P}^{n+1} \\ [(z_0, \ldots, z_n), r] & \mapsto & [(r \cdot z_0, \ldots, r \cdot z_n, 1 - r)] \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \mathbb{C}\mathrm{P}^n & \to & \mathbb{C}\mathrm{P}^{n+1} \\ [z_0 : \cdots : z_n] & \mapsto & [z_0 : \cdots : z_n : 0]. \end{array}$$

These maps are continuous by the Topological-Quotient Proposition 5.15 (1b) and the Topological-Quotient Proposition 5.15 (1a). It follows immediately from the definitions that these two maps define a map

$$\underbrace{\mathrm{Cone}(p\colon S^{2n+1} \to \mathbb{C}\mathrm{P}^n)}_{=(\mathrm{Cone}(S^{2n+1}) \cup \mathbb{C}\mathrm{P}^n)/\sim} \to \mathbb{C}\mathrm{P}^{n+1}$$

(5) By Theorem 155.12 we know that $\pi_3(\mathbb{C}\mathrm{P}^2) = 0$ and[2008] $\pi_4(\mathbb{C}\mathrm{P}^2) = 0$. Furthermore it tells us that $\pi_5(\mathbb{C}\mathrm{P}^2) = [p]$ where $p\colon S^5 \to \mathbb{C}\mathrm{P}^2$ denotes the Hopf map from page 3306.
(6) Similar to (3) we consider $X_6 := \overline{B}^6 \cup_{p\colon S^3 \to \mathbb{C}\mathrm{P}^1} \mathbb{C}\mathrm{P}^1$.
(7) As in (4), using Footnote 2007, we see that X_6 is homeomorphic to $\mathbb{C}\mathrm{P}^3$.

Iterating this process we obtain the sequence $\mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^4, \ldots$ and we end up with the infinite complex projective space $\mathbb{C}\mathrm{P}^\infty$. This shows, once again, that $\mathbb{C}\mathrm{P}^\infty$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, 2)$. A similar discussion, shows how to construct an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}_2, 1)$ starting with $X_2 = \mathbb{R}\mathrm{P}^2$, by imitating the above approach one once again ends up with $\mathbb{R}\mathrm{P}^\infty$.
For $n \geq 3$ there seem to be no straightforward models for $\mathrm{K}(\mathbb{Z}, n)$. But the following lemma gives us at least a little bit of extra control, which will play a crucial role at a later point.

Lemma 163.11. Given any $n \in \mathbb{N}$ there exists an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, n)$ with precisely one 0-cell, precisely one n-cell, no cell of dimension $n+1$ and only countably many cells in dimensions $\geq n+2$.

Remark. Let $n \in \mathbb{N}$. Note that the canonical Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, n)$ that we constructed in the proof of the Eilenberg-Maclane Existence Theorem 163.8 is *not* of the form as described in Lemma 163.11. We will discuss this in more detail in Exercise 163.4.

Proof. Let $n \in \mathbb{N}$. We consider $Y = S^n$. Recall that by Corollary 85.6 we know that $\pi_n(S^n) \cong \mathbb{Z}$. By the Kill-All Higher-π_n-Proposition 163.7 (3) there exists a CW-complex Z with the following four properties:

(1) the $(n+1)$-skeleton of Z equals $Y = S^n$, in particular Z has only one 0-cell, no cells in dimension $1, \ldots, n-1$, one cell in dimension n and no cell in dimension $n+1$,
(2) we have $\pi_i(Z) = 0$ for $i \geq n+1$,
(3) for any $i \in \{1, \ldots, n\}$ and any base point $x_0 \in S^n$ the inclusion induced map $\pi_i(Y, x_0) \xrightarrow{\cong} \pi_i(Z, x_0)$ is an isomorphism,
(4) the CW-complex Z has only countably many cells.

By Proposition 71.10 we know that $\pi_i(Y, x_0) = 0$ for $i = 1, \ldots, n-1$. Thus Z is the desired Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, n)$. ∎

163.4. **Uniqueness of Eilenberg-Maclane Spaces.** Next we turn to the question to what degree Eilenberg-Maclane spaces of a given type are unique. Evidently the homeomorphism type is not unique, since given an Eilenberg-Maclane space X we obtain Eilenberg-Maclane spaces of the same type by considering say the products $X \times \overline{B}^m$. Nonetheless the following theorem says that the homotopy type of Eilenberg-Maclane spaces of a given type are in fact unique:

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which is easily seen to be a bijection. It follows from the Topological-Quotient Proposition 5.15 (1b) that this map is continuous. Furthermore, using the Compact-Hausdorff Proposition 2.17 (3) and the Projective Spaces-Compact+Hausdorff Proposition 5.30 one sees that the map is in fact a homeomorphism.
[2008] Here we are lucky, we did everything to kill $\pi_3(\mathbb{C}\mathrm{P}^2)$, but without any effort on our side we also managed to dispose of $\pi_4(\mathbb{C}\mathrm{P}^2)$.

Theorem 163.12. (Eilenberg-Maclane Uniqueness Theorem) Let $n \in \mathbb{N}$ and let π be an (abelian) group[2009]. The following two statements hold:

(1) Any two Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, n)$ are homotopy equivalent.
(2) If (X, x_0) and (Y, y_0) are two pointed Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, n)$ with $x_0 \in X^0$ and $y_0 \in Y^0$ and if $\varphi\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is an isomorphism, then there exists a homotopy equivalence $f\colon (X, x_0) \to (Y, y_0)$ of pointed topological spaces with $f_* = \varphi\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$.

Before we turn to the proof of the Eilenberg-Maclane Uniqueness Theorem 163.12 it is worth, and fun, to discuss several immediate consequences thereof. For example, the Eilenberg-Maclane Uniqueness Theorem 163.12 settles Question 163.2 in a perhaps unexpected way: Given $m \in \mathbb{N}$ the homotopy type of the topological spaces $L(m, r_1, r_2, \dots)$ does *not* depend on the choice of the parameters r_i, since all of these topological spaces are Eilenberg-Maclane spaces of type $\mathrm{K}(\mathbb{Z}_m, 1)$, but these are therefore homotopy equivalent by the Eilenberg-Maclane Uniqueness Theorem 163.12.

For convenience we record the following corollary which is an immediate consequence of the Eilenberg-Maclane Uniqueness Theorem 163.12, the Homotopy Equivalence-H_*-Corollary 73.9 and the discussion on page 1914.

Corollary 163.13. Let π be a group and let $n \in \mathbb{N}$. If X and Y are two Eilenberg-Maclane spaces of the type $\mathrm{K}(\pi, n)$, then for any $i \in \mathbb{N}$ and any abelian group G we have an isomorphism $\mathrm{H}_i(X; G) \cong \mathrm{H}_i(Y; G)$.

Perhaps somewhat surprisingly we can use Corollary 163.13 to give a negative answer to Question 163.10 for $m = 2$.

Corollary 163.14. There is no finite-dimensional Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}_2, 1)$.

Proof. Let K be an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}_2, 1)$. Given any $i \in \mathbb{N}$ we calculate that

$$\mathrm{H}_i(K; \mathbb{Z}_2) \underset{\text{by Corollary 163.13}}{\uparrow} \cong \mathrm{H}_i(\text{any Eilenberg-Maclane space of type } \mathrm{K}(\mathbb{Z}_2, 1); \mathbb{Z}_2) \underset{\text{see page 3427}}{\uparrow} \cong \mathrm{H}_i(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2) \underset{\text{see page 1916}}{\uparrow} \cong \mathbb{Z}_2.$$

It follows from the discussion on page 1915 that K is of dimension $> i$ for all $i \in \mathbb{N}$, in other words, K is infinite-dimensional. ■

Evidently this suggests that the answer to Question 163.10 is negative. But right now we lack the tools to calculate the homology groups of the quotients $S^\infty/\mathbb{Z}_m$ that we constructed in Lemma 163.1. We will return to this question on page **??**

The following proposition basically says that it is easy to construct maps from CW-complexes to Eilenberg-Maclane spaces. This statement is the key to proving the Eilenberg-Maclane Uniqueness Theorem 163.12. As we will see shortly, the proposition is of interest in its own right.

Proposition 163.15. (Maps-to-$\mathrm{K}(\pi, n)$-Proposition) Let $n \in \mathbb{N}$ and let π be an (abelian) group. Furthermore let Y be an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ and

[2009] Whenever we deal with Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, n)$ with $n \geq 2$ it is understood throughout these lecture notes that π is abelian.

let $y_0 \in Y$. Finally let X be a CW-complex and let $x_0 \in X$ be a point in the 0-skeleton X^0. If X is $(n-1)$-connected, then the following two statements hold:

(1) Let $\varphi\colon \pi_n(X,x_0) \to \pi_n(Y,y_0)$ be a homomorphism. There exists a map $f\colon X \to Y$ with $f(x_0) = y_0$ such that $f_* = \varphi\colon \pi_n(X,x_0) \to \pi_n(Y,y_0)$.

(2) Suppose we are given two maps $f_0, f_1\colon X \to Y$ such that $f_0(x_0) = f_1(x_0) = y_0$. If $f_{0*} = f_{1*}\colon \pi_n(X,x_0) \to \pi_n(Y,y_0)$, then there exists a homotopy rel x_0 from f_0 to f_1.

We write

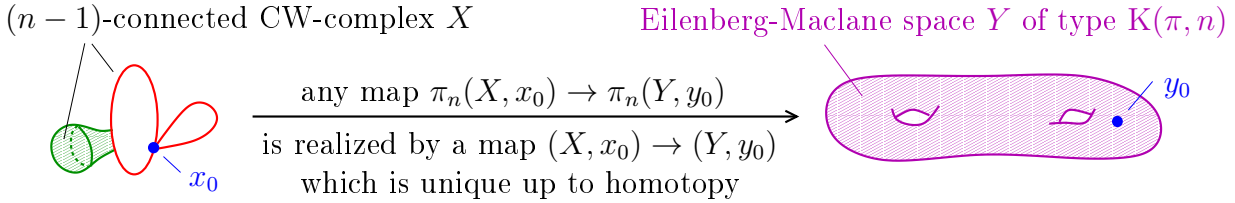
$$\langle (X,x_0),(Y,y_0)\rangle := \{\text{maps } (X,x_0) \to (Y,y_0)\}/\sim \qquad \text{where } f \sim g \text{ if there exists a homotopy rel } \{x_0\} \text{ between } f \text{ and } g.$$

With this notation Statements (1) and (2) can be summarized to obtain the following statement:

(3) The map

$$\begin{array}{ccc} \langle (X,x_0),(Y,y_0)\rangle & \to & \mathrm{Hom}(\pi_n(X,x_0),\pi_n(Y,y_0)) \\ [f] & \mapsto & [f_*] \end{array}$$

is a bijection.



Remark. In the Maps-to-S^1 Proposition 118.13, with $n = 1$ and $Y = S^1$, we proved a "smooth manifold version" of the Maps-to-K(π,n)-Proposition 163.15.

Proof of Theorem 163.12 assuming Proposition 163.15. Let $n \in \mathbb{N}$ and let π be an (abelian) group. Statement (1) is basically an immediate consequence of Statement (2). Therefore it now remains to prove Statement (2). Thus let (X,x_0) and (Y,y_0) be two pointed Eilenberg-Maclane spaces of type K(π,n) where $x_0 \in X^0$ and $y_0 \in Y^0$. Furthermore let $\varphi\colon \pi_n(X,x_0) \to \pi_n(Y,y_0)$ be an isomorphism. Since X is, as an Eilenberg-Maclane space of type K(π,n), $(n-1)$-connected, we can appeal to the Maps-to-K(π,n)-Proposition 163.15 (1) to obtain a map $f\colon X \to Y$ with $f(x_0) = y_0$ such that $f_* = \varphi\colon \pi_n(X,x_0) \to \pi_n(Y,y_0)$. It suffices to prove the following claim.

Claim. The map f is a homotopy equivalence $(X,x_0) \to (Y,y_0)$ of pointed topological spaces.

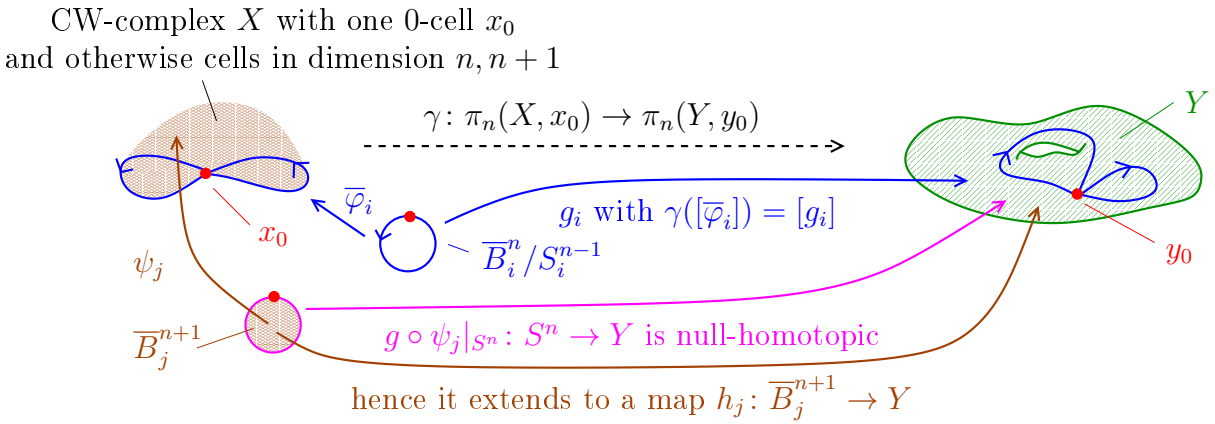
Proof. As above we obtain from the Maps-to-K(π,n)-Proposition 163.15 (1) the existence of a map $g\colon Y \to X$ with $g(y_0) = x_0$ such that $g_* = \varphi^{-1}\colon \pi_n(Y,y_0) \to \pi_n(X,x_0)$. Note that

$$(g \circ f)_* \;=\; \varphi^{-1} \circ \varphi \;=\; \mathrm{id}_{\pi_n(X,x_0)} \;=\; (\mathrm{id}_X)_*$$

and note that $(g \circ f)(x_0) = x_0 = (\mathrm{id}_X)(x_0)$. Thus it follows from the Maps-to-K(π,n)-Proposition 163.15 (2) that $g \circ f$ is homotopic rel x_0 to id_X. Basically the same argument shows that $f \circ g$ is homotopic rel y_0 to id_Y. But this means that g is indeed the desired homotopy inverse. ∎

At this point it remains to prove the Maps-to-K(π, n)-Proposition 163.15. The proof of the proposition relies on the following two lemmas. The following lemma is the key ingredient in the proof of the Maps-to-K(π, n)-Proposition 163.15.

Lemma 163.16. Let $n \in \mathbb{N}$. Let X be a CW-complex that consists of a single 0-cell x_0 and that otherwise contains only cells in dimensions n and $n+1$. Furthermore let (Y, y_0) be a pointed topological space and let $\gamma\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ be a homomorphism. There exists a map $h\colon (X, x_0) \to (Y, y_0)$ such that $h_* = \gamma$.



Proof of Lemma 163.16. We proceed in the following steps.

(1) We note that by our hypothesis we have $X^{n-1} = \{x_0\}$.

(2) Let $\{\varphi_i\colon (\overline{B}^n, S^{n-1}) \to (X^n, X^{n-1} = \{x_0\})\}_{i \in I}$ be the characteristic maps of the n-cells of the CW-complex X. Note that since $X^{n-1} = \{x_0\}$ each map φ_i descends to a map $\overline{\varphi}_i\colon \overline{B}^n / S^{n-1} \to X^n$.

(3) By the Skeleton-Quotient Lemma 68.33 (2) the maps $\overline{\varphi}_i$ induce a homeomorphism $\Phi\colon \bigvee_{i \in I} \overline{B}^n_i / S^{n-1}_i \to X^n$. By a slight abuse of notation we denote by x_0 the wedge point of the wedge.

(4) Let $i \in I$. Note that the map $\overline{\varphi}_i$ defines an element in $\pi_n(X, x_0)$. We pick a map $g_i\colon \overline{B}^n / S^{n-1} \to (Y, y_0)$ with $\gamma([\overline{\varphi}_i]) = [g_i]$.

(5) We consider the map

$$\begin{array}{rccl} g\colon \bigvee\limits_{i \in I} \overline{B}^n_i / S^{n-1}_i & \to & Y & \\ [x] & \mapsto & g_i([x]) & \text{if } x \in \overline{B}^n_i. \end{array}$$

(6) We consider the following diagram:

$$\begin{array}{ccccc} \pi_n(X^n, x_0) & \xleftarrow[\cong]{\Phi_*} & \pi_n\Big(\bigvee\limits_{i \in I} \overline{B}^n_i / S^{n-1}_i, x_0\Big) & \xrightarrow{g_*} & \pi_n(Y, y_0). \\ & {\scriptstyle i_*}\searrow & & \nearrow{\scriptstyle \gamma} & \\ & & \pi_n(X, x_0) & & \end{array}$$

where the left diagonal map is induced by the inclusion $i\colon X^n \to X$.

(7) It follows immediately from the construction of g that if we send any of the classes $[\mathrm{id}\colon \overline{B}^n \to \overline{B}_i] \in \pi_n(X^n, x)$ on a journey through the diagram, then we obtain the same result. But by the Wedge-of-Spheres-π_n-Proposition 163.5 (3) the classes

$[\mathrm{id}\colon \overline{B}^n \to \overline{B}_i]$, $i \in I$, form a generating set for $\pi_n\Big(\bigvee_{i\in I} \overline{B}^n_i/S^{n-1}_i, x_0\Big)$. Thus it follows that the diagram commutes.

(8) Let $* = (0,\dots,0,1) \in S^n \subset \overline{B}^{n+1}$ the usual base point. Let $\{\psi_j\colon \overline{B}^{n+1} \to X^{n+1}\}_{j\in J}$ be the characteristic maps of the $(n+1)$-cells of X. It follows from the Orthogonal Action Lemma 5.26 that we can arrange that for every $j \in J$ we have $\psi_j(*) = x_0$

(9) Let $j \in J$. Note that

$$[g\circ\Phi^{-1}\circ\psi_j|_{S^n}] \;=\; (g_*\circ\Phi_*^{-1})(\overbrace{[\psi_j|_{S^n}]}^{\in\pi_n(X^n,x_0)}) \;\underset{\uparrow}{=}\; \overbrace{\gamma([\psi_j|_{S^n}])}^{\in\pi_n(X,x_0)} \;\underset{\uparrow}{=}\; \gamma([0]) \;\underset{\downarrow}{=}\; 0 \in \pi_n(Y,y_0).$$

by the above commutative diagram

since $\psi_j|_{S^n}$ extends to a map $\overline{B}^{n+1} \to X$ we have $[\psi_j|_{S^n}] = 0 \in \pi_n(X,x_0)$ by Trivial-in-π_n-Lemma 71.4

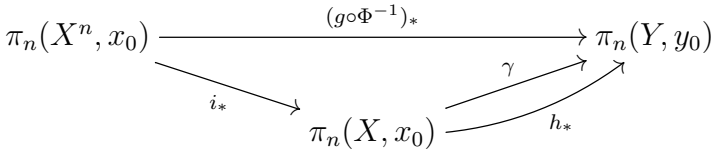
since γ is a homomorphism

Thus we obtain from the Trivial-in-π_n-Lemma 71.4 that there exists a map $h_j\colon \overline{B}^{n+1} \to Y$ such that $h_j|_{S^n} = g \circ \Phi^{-1} \circ (\psi_j|_{S^n})$.

(10) Since for each $j \in J$ we have $h_j|_{S^n} = g \circ \Phi^{-1} \circ (\psi_j|_{S^n})$ we can use the maps h_j to extend $g \circ \Phi^{-1}$ to a map

$$h\colon X = \Big(X^n \cup \bigsqcup_{j\in J} \overline{B}^{n+1}_j\Big)/\!\sim \;\to\; Y \qquad \text{where } \psi_j(x) \sim x \text{ for any } x \in S^n_j.$$

(11) We obtain the following diagram



(12) We already saw that the diagram commutes for γ. Since $h|_{X^n} = g \circ \Phi^{-1}$ we see that the diagram commutes also for the map h_*.

(13) The left diagonal map $i_*\colon \pi_n(X^n,x_0) \to \pi_n(X,x_0)$ is an epimorphism by the CW-Skeleton-π_n-Proposition 71.9.

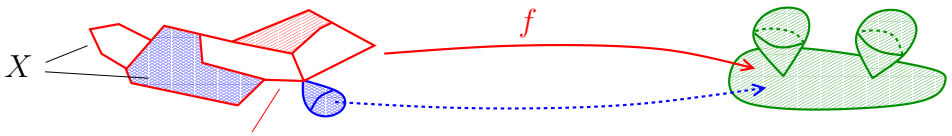
(14) It follows from (12) and (13) that the maps γ and h_* from $\pi_n(X,x_0)$ to $\pi_n(Y,y_0)$ agree. ■

The second lemma is just a convenient special case of Lemma 161.3.

Lemma 163.17. Let X be a CW-complex, let $n \in \mathbb{N}$ and let $f\colon A \to Y$ be a map from a subcomplex $A \subset X$ to a path-connected topological space with $X^n \subset A$.

(1) If $\pi_n(Y) = 0$, then we can extend f to a map $f\colon A \cup X^{n+1} \to Y$.

(2) If $\pi_i(Y) = 0$ for every $i \geq n$, then we can extend f to a map $f\colon X \to Y$.



We break the proof of the Maps-to-K(π,n)-Proposition 163.15 into two parts. In the first part we deal with the situation that the $(n-1)$-skeleton consists just of a point. Afterwards we will discuss how the general case can be reduced to this special case.

Proof of the Maps-to-K(π,n)-Proposition 163.15 if $X^{n-1}=\{x_0\}$. Let $n\in\mathbb{N}$ and let π be an (abelian) group. Furthermore let Y be an Eilenberg-Maclane space of type K(π,n) and let $y_0\in Y$. Finally let X be a CW-complex and let $n\in\mathbb{N}$. We assume that $X^{n-1}=\{x_0\}$.

(1) Let $\varphi\colon \pi_n(X,x_0)\to\pi_n(Y,y_0)$ be a homomorphism. We need to show that there exists a map $f\colon X\to Y$ with $f(x_0)=y_0$ such that $f_*=\varphi\colon \pi_n(X,x_0)\to\pi_n(Y,y_0)$. We denote by $i\colon X^{n+1}\to X$ the inclusion map. By the CW-Skeleton-π_n-Proposition 71.9 we know that $i_*\colon \pi_n(X^{n+1},x_0)\to\pi_n(X,x_0)$ is an isomorphism. The following diagram contains all the maps of interest to us:

$$\begin{array}{ccc} \pi_n(X^{n+1},x_0) & \overset{h_*}{\dashrightarrow} & \\ {\scriptstyle i_*}\downarrow{\scriptstyle \cong} & \overset{f_*}{\nearrow} & \\ \pi_n(X,x_0) & \xrightarrow[\varphi]{} & \pi_n(Y,y_0). \end{array}$$

Since $X^{n-1}=\{x_0\}$ we can apply Lemma 163.16 to the CW-complex X^{n+1}. We obtain from Lemma 163.16 that there exists a map $h\colon X^{n+1}\to Y$ with $h(x_0)=y_0$ such that $h_*=\varphi\circ i_*\colon \pi_n(X^{n+1},x_0)\to\pi_n(Y,y_0)$. Since Y is an Eilenberg-Maclane space of type K(π,n) we know that $\pi_j(Y)=0$ for $j\geq n+1$. Thus we obtain from Lemma 163.17 (2) that we can extend $h\colon X^{n+1}\to Y$ to a map $f\colon X\to Y$. It remains to show that $f_*=\varphi\colon \pi_n(X,x_0)\to\pi_n(Y,y_0)$. This is indeed the case since

$$f_* \underset{\text{since } i_* \text{ is an isomorphism}}{=} f_*\circ i_*\circ i_*^{-1} = (f\circ i)_*\circ i_*^{-1} \underset{\text{since } h \text{ extends } f}{=} h_*\circ i_*^{-1} \underset{\text{since } h_*=\varphi\circ i_*}{=} (\varphi\circ i_*)\circ i_*^{-1} = \varphi.$$

(2) Suppose we are given two maps $f_0,f_1\colon X\to Y$ such that $f_0(x_0)=f_1(x_0)=y_0$ and such that $f_{0*}=f_{1*}\colon \pi_n(X,x_0)\to\pi_n(Y,y_0)$. We need to show that there exists a homotopy rel x_0 from f_0 to f_1. We consider the product $X\times[0,1]$ and we equip it with the product CW-structure coming from Proposition 68.23. Given $k\in\mathbb{N}_0$ we write

$$Z^k = (X\times\{0\})\cup\underbrace{(X^{k-1}\times[0,1])}_{k\text{-dimensional}}\cup(X\times\{1\}) \subset X\times[0,1].$$

The idea is define the homotopy on $X\times[0,1]=\bigcup\limits_{n\in\mathbb{N}_0} Z^n$ by breaking the task into three steps.

(a) First we note that the fact that $X^0=\cdots=X^{n-1}$ implies that $Z^1=\cdots=Z^n$. Now let $h\colon Z^1=Z^n\to Y$ be the map that is given by f_0 on $X\times\{0\}$, f_1 on $X\times\{1\}$ and the constant map c_{x_0} on $\{x_0\}\times[0,1]$.

(b) We denote by $\{\varphi_i\colon \overline{B}_i^n\to X\}_{i\in I}$ the characteristic maps of the n-cells of X. Since $X^{n-1}=\{x_0\}$ we see that each φ_i descends to a map $\psi_i\colon \overline{B}_i^n/S_i^{n-1}\to X$. We denote by $*=[S_i^{n-1}]$ the obvious base point of the sphere $\overline{B}_i^n/S_i^{n-1}$. Now recall that by hypothesis we have $f_{0*}=f_{1*}\colon \pi_n(X,x_0)\to\pi_n(Y,y_0)$. This implies that for every $i\in I$ we have $[f_0\circ\psi_i]=[f_1\circ\psi_i]\in\pi_n(Y,y_0)$. In other words,

for every $i \in I$ there exists a map $F_i \colon \overline{B}_i^n / S_i^{n-1} \times [0,1] \to Y$ with $F_{i0} = f_0 \circ \psi_i$, $F_{i1} = f_1 \circ \psi_i$ and $F_i(*,t) = y_0$ for all $t \in [0,1]$. We define

$$\begin{array}{rcl} h_{n+1} \colon Z^{n+1} & \to & Y \\ P & \mapsto & \begin{cases} h_0(P), & \text{if } P \in Z^1 = Z^n, \\ F_i(Q,t), & \text{if } P = (\varphi_i(Q),t) \text{ with } Q \in \overline{B}_i^n. \end{cases} \end{array}$$

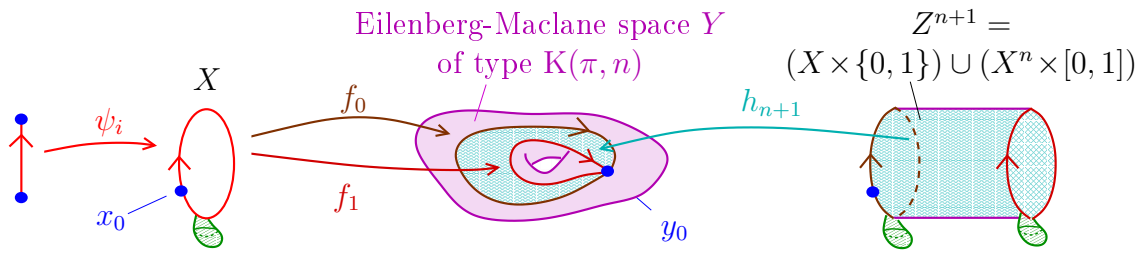
It follows immediately from the properties of the F_i that this map is well-defined. Furthermore, it follows almost immediately from the Characteristic Maps Lemma 68.7 (4) that h_{n+1} is continuous.

(c) Finally we are in the following situation:

$$\begin{array}{ccc} Z^{n+1} & \xrightarrow{h_{n+1}} & Y \\ \downarrow & \nearrow_{H} & \\ X \times [0,1]. & & \end{array}$$

By definition of the CW-structure on $X \times [0,1]$ we know that the $(n+1)$-skeleton of $X \times [0,1]$ is contained in Z^{n+1}. In other words, any cell of $X \times [0,1]$ that is not contained in Z^{n+1} is of dimension $\geq n+2$. Since Y is an Eilenberg-Maclane space of type $\mathrm{K}(\pi,n)$ we know that $\pi_k(Y,y_0) = 0$ for $k \geq n+1$. Thus we obtain from Lemma 163.17 (2) that there exists a map $H \colon X \times [0,1] \to Y$ which makes the above diagram commute. Evidently this is the desired homotopy between f_0 and f_1.

(3) The last statement is just a reformulation of Statement (1) together with Statement (2). ■



Proof of the Maps-to-K(π,n)-Proposition 163.15 for the general case. Let X be a CW-complex and let $x_0 \in X$ be a point in the 0-skeleton X^0. We suppose that X is $(n-1)$-connected. By Proposition 162.10 we know that there exists a homotopy equivalence $f \colon (X,x_0) \to (\widetilde{X},\widetilde{x}_0)$ where $\widetilde{X}$ is a CW-complex that has one 0-cell $\{\widetilde{x}_0\}$ and no cells in dimensions $1,\ldots,n$. By the Homotopy-π_n-Proposition 71.7 we know that $f_* \colon \pi_n(X,x_0) \to \pi_n(\widetilde{X},\widetilde{x}_0)$ is an isomorphism. By the above argument we know that the desired conclusions hold for $(\widetilde{X},\widetilde{x}_0)$. It is now fairly elementary to see that the statements of the Maps-to-K(π,n)-Proposition 163.15 also hold for (X,x_0). ■

Before we move on we prove an important addendum to the Homotopy-π_1-Proposition 50.3. To motivate its relevance we recall an earlier result: If $f_0, f_1 \colon (X,x_0) \to (Y,y_0)$ are two

maps between pointed topological spaces, then by the Homotopy-π_1-Proposition 50.3 (1) the following two statements hold:

(1) If f_0 and f_1 are homotopic rel x_0, then $f_{0*} = f_{1*}$.
(2) If f_0 and f_1 are homotopic (with no constraints on the base point), then there exists an $\alpha \in \pi_1(Y, y_0)$ such that $f_{0*}(z) = \alpha \cdot f_{1*}(z) \cdot \alpha^{-1}$ for all $z \in \pi_1(Y, y_0)$.

One can view the Maps-to-K(π, n)-Proposition 163.15 (2) as a partial converse to (1). The following proposition is the analogous partial converse to (2).

Proposition 163.18. Let π be a group, let Y be an Eilenberg-Maclane space of type $K(\pi, 1)$ and let $y_0 \in Y$. Furthermore let X be a 0-connected CW-complex and let $x_0 \in X$ be a point in the 0-skeleton X^0. Suppose we are given two maps $f_0, f_1 \colon X \to Y$ such that $f_0(x_0) = f_1(x_0) = y_0$. If there exists an $\alpha \in \pi_1(Y, y_0)$ such that $f_{0*}(z) = \alpha \cdot f_{1*}(z) \cdot \alpha^{-1}$ for all $z \in \pi_1(Y, y_0)$, then there exists a homotopy[2010] from f_0 to f_1.

Proof. The proof of the proposition is similar to the proof of the Maps-to-K(π, n)-Proposition 163.15 (2). In the following provide a sketch of the argument. We leave it to the reader to fill in the details. First of all we point out that by Proposition 162.10 we can assume that X has a single 0-cell $\{x_0\}$. Now we consider the product $X \times [0,1]$ and we equip it with the product CW-structure coming from Proposition 68.23. Given $k \in \mathbb{N}_0$ we write

$$Z^k \;=\; (X \times \{0\}) \cup \underbrace{(X^{k-1} \times [0,1])}_{k\text{-dimensional}} \cup (X \times \{1\}) \;\subset\; X \times [0,1].$$

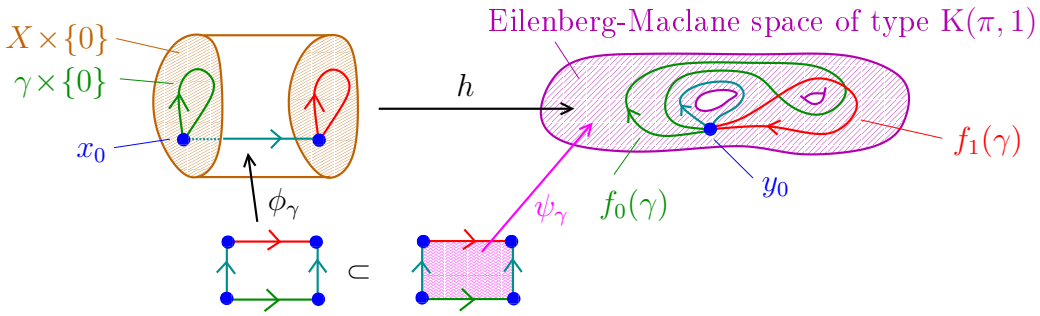
As above we define the homotopy on $X \times [0,1] = \bigcup_{n \in \mathbb{N}_0} Z^n$ by breaking the task into three friendly steps.

(1) We pick a map $\beta \colon [0,1] \to Y$ that represents $\alpha \in \pi_1(Y, y_0)$. Let $h \colon Z^1 = Z^n \to Y$ be the map that is given by f_0 on $X \times \{0\}$, f_1 on $X \times \{1\}$ and the map β on $\{x_0\} \times [0,1] = [0,1]$.
(2) Given a 1-cell γ of X we denote by $\phi_\gamma \colon S^1 = \partial([0,1]^2) \to Z^1$ the attaching map of the product cell $\gamma \times [0,1]$. We have

$$\begin{array}{rcl} & & \text{by definition of } h \\ & & \downarrow \\ h_*([\phi_\gamma]) & = & [h|_{\gamma\times 0}] \cdot [h|_{\{x_0\}\times[0,1]}] \cdot \overline{[h|_{\gamma\times 1}]} \cdot \overline{[h|_{\{x_0\}\times[0,1]}]} \\ & = & \alpha \cdot [f_{1*}(\gamma)] \cdot \alpha^{-1} \cdot \alpha \cdot [f_{1*}(\gamma)]^{-1} \cdot \alpha^{-1} \\ & & \uparrow \\ & & \text{by hypothesis and choice of } \beta \end{array} \qquad \begin{array}{rcl} & & \text{by a generous interpretation of the notation} \\ & & \downarrow \\ & = & [f_{0*}(\gamma)] \cdot [\beta] \cdot [f_{1*}(\gamma)]^{-1} \cdot [\beta]^{-1} \\ & = & e \;\in \pi_1(Y, y_0). \end{array}$$

It follows from the Extension-to-Disk Lemma 47.1 that the map $h \circ \phi_\gamma \colon \partial([0,1]^2) \to Y$ extends to a map $\psi_\gamma \colon [0,1]^2 \to Y$. We use this map to extend h over the 2-cell $\gamma \times [0,1]$. If we do this for all 1-cells of X we obtain the desired extension of h to a map $Z^2 \to Y$.
(3) We use Lemma 163.17 (2) and the fact that Y is an Eilenberg-Maclane space of type $K(\pi, 1)$ to extend the map h to a homotopy $F \colon X \times [0,1] \to Y$. ∎

[2010] Note that in contrast to the Maps-to-K(π, n)-Proposition 163.15 (2) it does *not* say "rel x_0".

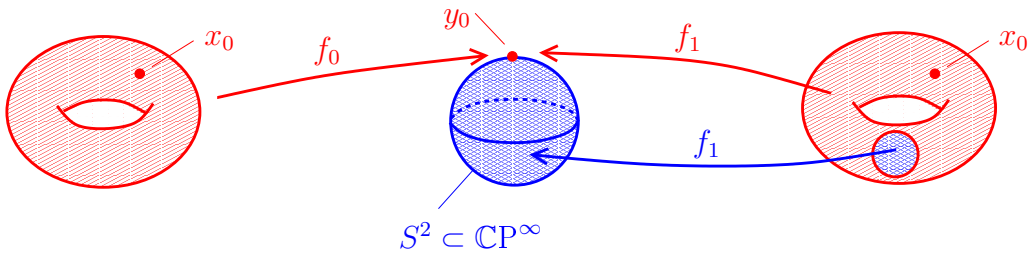


We conclude this section with a discussion of the statement of the Maps-to-K(π, n)-Proposition 163.15. It is not unreasonable to ask whether one really needs in the Maps-to-K(π, n)-Proposition 163.15 the hypothesis that X is $(n-1)$-connected. But we will now see that in neither of the two parts can we drop this condition.

(1) By Theorem 155.10 we know that there exists an isomorphism $\varphi\colon \pi_3(S^2, *) \xrightarrow{\cong} \mathbb{Z}$. Now let (Y, y_0) be any pointed Eilenberg-Maclane space of type K$(\mathbb{Z}, 3)$. At a very first glance one might hope that there exists a map $f\colon (S^2, *) \to (Y, y_0)$ such that $f_* = \varphi\colon \pi_3(S^2, *) \to \pi_3(Y, y_0) = \mathbb{Z}$. But this is not the case. Indeed, since Y is an Eilenberg-Maclane space of type K$(\mathbb{Z}, 2)$ we know that $\pi_2(Y, y_0) = 0$. This means that any map $f\colon (S^2, *) \to (Y, y_0)$ is homotopic to the constant map. In particular any such map induces the zero map on π_3. Thus we cannot realize φ by a map $(S^2, *) \to (Y, y_0)$.

(2) On page 3427 we pointed out that the infinite complex projective space $\mathbb{C}\mathrm{P}^\infty$ is an Eilenberg-Maclane space of type K$(\mathbb{Z}, 2)$. By the discussion on page 258 we can view $S^2 = \mathbb{C}\mathrm{P}^1$ as a subset of $\mathbb{C}\mathrm{P}^\infty$. We denote by $y_0 \in S^2$ the North Pole. We pick some point x_0 on the torus $S^1 \times S^1$. We consider the following two maps from the torus $S^1 \times S^1 \to S^2 \subset \mathbb{C}\mathrm{P}^\infty$:
   (a) The map f_0 is the constant map that sends all points to y_0.
   (b) The map $f_1\colon S^1 \times S^1 \to \overline{B}^2/S^1 = S^2$ is a map that sends the points of a given embedded disk to $\overline{B}^2$ and sends all other points to the single point $[S^1] = y_0$.

   (These two maps f_0 and f_1 are illustrated in the figure below.) Now we claim that we have $f_{0*} = f_{1*}\colon \pi_2(S^1 \times S^1, *) \to \pi_2(\mathbb{C}\mathrm{P}^\infty, *)$, but that f_0 and f_1 are, contrary to what one might expect, *not* homotopic. To prove this claim we recall that on page 1573 we saw that $\pi_2(S^1 \times S^1) = 0$. In particular both maps induce the same map on π_2. Nonetheless the two maps are not homotopic. This can be seen by studying the induced map on H_2. It is clear that f_0 induces the trivial map on H_2. On the other hand we saw in the proof of the Degree Realization Proposition 38.13 that f_1 induces an isomorphism $\mathrm{H}_2(S^1 \times S^1; \mathbb{Z}) \to \mathrm{H}_2(S^2; \mathbb{Z})$. By the discussion on page 1773 we know that the inclusion induced map $\mathrm{H}_2(S^2; \mathbb{Z}) \to \mathrm{H}_2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})$ is an isomorphism. Therefore we see that the map $f_{1*}\colon \mathrm{H}_2(S^1 \times S^1; \mathbb{Z}) \to \mathrm{H}_2(\mathbb{C}\mathrm{P}^\infty)$ is non-trivial. Thus we obtain from the Homotopic Maps-and-Homology Proposition 73.6 that the maps f_0 and f_1 are in fact not homotopic.



At least in terms of invariants any Eilenberg-Maclane space $\mathrm{K}(\pi, n)$ is a particularly simple topological space. Therefore it seems not totally unreasonable to hope that given a CW-complex X one can perhaps classify the maps from X to $\mathrm{K}(\pi, n)$, say up to homotopy. The above examples show that the maps are *not* classified by $\mathrm{Hom}(\pi_n(X), \pi)$. But perhaps there is some other way? Let us record this as a question:

Question 163.19. Let $n \in \mathbb{N}$ and let π be an (abelian) group. Let X be a CW-complex. Can we classify the maps $X \to \mathrm{K}(\pi, n)$ up to homotopy?

163.5. **First applications of Eilenberg-Maclane spaces.** We recall the following definition from page 2605.

Definition. Let G be a group, let Y be a topological space and let $y_0 \in Y$ be a point with an identification $\pi_1(Y, y_0) \xrightarrow{=} G$. Given another topological space X and $x_0 \in X$ we say that a homomorphism $\varphi\colon \pi_1(X, x_0) \to G = \pi_1(Y, y_0)$ is realized by a map $g\colon X \to Y$ if $g(x_0) = y_0$ and if $g_* = \varphi$. If there is no danger of confusion we will drop the base points from the notation.

Next we recall the following question that we had formulated a while ago.

Question 118.12 and and Question118.14. Let X be a connected CW-complex.[2011]

(0) Can every homomorphism $\pi_1(X) \to \mathbb{Z} = \pi_1(S^1)$ be realized by a map $X \to S^1$?

(1) Let $g \geq 1$ and let Σ_g be the surface of genus $g \geq 1$. Can every homomorphism $\pi_1(X) \to \pi_1(\Sigma_g)$ be realized by a map $X \to \Sigma_g$?

(2) Can every homomorphism $\pi_1(X) \to \pi_1\Big(\bigvee_{i=1}^{k} S^1\Big) = \langle x_1, \ldots, x_k \rangle$ be realized by a map $X \to \bigvee_{i=1}^{k} S^1$?

(3) Can every homomorphism $\pi_1(X) \to \mathbb{Z}_2 = \pi_1(\mathbb{R}\mathrm{P}^2)$ be realized by a map $X \to \mathbb{R}\mathrm{P}^2$?

In the Maps-to-S^1 Proposition 118.13 we had used de Rham cohomology to give an affirmative answer to (0) if X is a smooth manifold. Furthermore in Exercise 121.8 we gave, by considering the isomorphism $\pi_1(\mathbb{R}\mathrm{P}^3) \xrightarrow{\cong} \mathbb{Z}_2$, a negative answer to (3). Fortunately we can now give a complete answer to (0), (1) and (2).

Proposition 163.20.

[2011] If M is a smooth manifold, then we saw in the Smooth Manifold-Simplicial Structure Theorem 96.2 and the Simplicial-Implies-CW Lemma 93.24 that M admits a CW-structure.

(a) The answer to (0), (1) and (2) is "yes". Furthermore, if in the formulation of (3) we replace $\mathbb{R}\mathrm{P}^2$ by $\mathbb{R}\mathrm{P}^\infty$, then the answer to (3) is also yes.
(b) If X is a smooth manifold, then the following two statements hold:
(א) In (0) and (1) we can find smooth realizations.
(ב) In (2) we can find a realization $f\colon X \to \bigvee_{i=1}^{k} S_i^1$ which has the property that the restriction of f to the complement of f^{-1}(wedge point) is smooth.

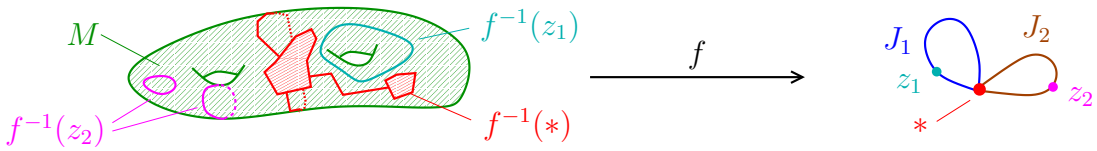
Proof.

(a) The desired statement follows immediately from the Maps-to-K(π, n)-Proposition 163.15 and the fact that on page 3427 we saw that S^1, $\bigvee_{i=1}^{k} S^1$, Σ_g and $\mathbb{R}\mathrm{P}^\infty$ are Eilenberg-Maclane spaces.
(b) (א) This statement follows from the Whitney Approximation Theorem 29.1.
(ב) As we pointed out on page 759, this statement is a consequence of Proposition 29.9. ■

Using Proposition 163.20 we obtain the following pleasant proposition.

Proposition 163.21. If M is a compact orientable connected n-dimensional smooth manifold, then the following two statements are equivalent:
(1) there exists an epimorphism $\pi_1(M) \to \langle x_1, \dots, x_k \rangle$,
(2) there exist disjoint connected non-empty $(n-1)$-dimensional submanifolds $F_1, \dots, F_k$ of M such that $M \setminus (F_1 \cup \dots \cup F_k)$ is connected.

Sketch of a proof. First note that the "(2) $\Rightarrow$ (1)" implication is precisely the content of the Epimorphism-onto-Free Group Proposition 130.8. Thus it remains to prove the "(1) $\Rightarrow$ (2)" implication. Therefore suppose that there exists an epimorphism $\pi_1(M) \to \langle x_1, \dots, x_k \rangle$. We consider the wedge $\bigvee_{i=1}^{k} S_i^1$. We denote by $*$ the wedge point and for $i = 1, \dots, k$ we write $J_i := S_i^1 \setminus \{*\}$. By Proposition 163.20 (b) we can realize this epimorphism by a map $f\colon M \to \bigvee_{i=1}^{k} S_i^1$ which is smooth on $M \setminus f^{-1}(*)$. Let $i \in \{1, \dots, k\}$. It follows from Sard's Theorem 31.1 that there exists a regular value $z_i \in J_i$. We obtain from the Preimage-of-Point-in-Ball Lemma 29.10 that the preimage $G_i := f^{-1}(z_i)$ is an orientable proper $(n-1)$-dimensional submanifold of M. In Exercise 163.3 we will prove the following two statements:
(1) each G_i is non-empty,
(2) for each $i \in \{1, \dots, k\}$ we can find a component F_i of G_i such that $M \setminus (F_1 \cup \dots \cup F_k)$ is connected. ■



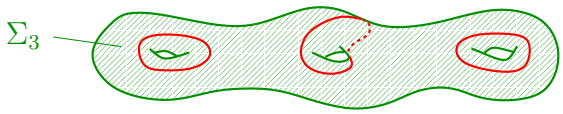
Now is a good time to recall the following question.

Question 130.9. Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g.

(1) What is the maximal number $k = k(g) \in \mathbb{N}_0$ for which we can find k disjoint curves $F_1, \dots, F_k$ on Σ_g such that $\Sigma_g \setminus (F_1 \cup \dots \cup F_k)$ is connected?
(2) What is the maximal $l = l(g) \in \mathbb{N}_0$ for which there exists an epimorphism from $\pi_1(\Sigma_g)$ onto a free group of l generators?

We make the following easy observations:

(1) As we already observed on page 2843, it follows immediately from peeking at the figure below that $k(g) \geq g$.



there exist three curves such that the complement is still connected

(2) It follows from Lemma 53.13 that we have an epimorphism

$$\overbrace{\langle x_1, y_1, \dots, x_g, y_g \,|\, x_1 y_1 x_1^{-1} y_1^{-1} \cdot \dots \cdot x_g y_g x_g^{-1} y_g^{-1} \rangle}^{\cong \pi_1(\Sigma_g) \text{ by the Fundamental Group-of-Surfaces Proposition 54.5}} \quad \begin{array}{rcl} \to & \langle x_1, \dots, x_g \rangle \\ x_i & \mapsto & x_i \\ y_i & \mapsto & e \end{array}$$

from $\pi_1(\Sigma_g)$ onto a free group on g generators.

More interestingly, by Proposition 163.21 we now know that the two questions actually give the same answer. Even better, we can now give a complete answer:

Proposition 163.22. Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g. If there exists an epimorphism Σ_g onto a free group on k generators, then $k \leq g$.

Proof. Let $g \in \mathbb{N}$, let Σ_g be the surface of genus g and let $* \in \Sigma_g$ be a base point. Let $\varphi\colon \pi_1(\Sigma_g, *) \to F_k$ be an epimorphism onto a free group on k generators. We start out with the following preparations:

(1) By the discussion on page 3427 we know that $X := \bigvee_{i=1}^{k} S^1$ is an Eilenberg-Maclane space of type $\mathrm{K}(F_k, 1)$. We pick $x_0 \in X$ and as usual we make the identification $\pi_1(X, x_0) = F_k$.
(2) It follows from the Maps-to-K(π, n)-Proposition 163.15 that there exists a map $f\colon (\Sigma_g, *) \to (X, x_0)$ with $f_* = \varphi\colon \pi_1(\Sigma_g, *) \to \pi_1(X, x_0) = F_k$. Note that the map $f_*\colon \pi_1(\Sigma_g, *) \to \pi_1(X, x_0)$ is an epimorphism.
(3) The Hurewicz Theorem 84.5, together with the Abelianization Proposition 53.20 (9), implies that the induced map $f_*\colon \mathrm{H}_1(\Sigma_g; \mathbb{Z}) \to \mathrm{H}_1(X; \mathbb{Z})$ is an epimorphism.
(4) In general, if $\alpha\colon A \to B$ is an epimorphism between two groups, then elementary algebra tells us that the induced map $\mathrm{Hom}(B, \mathbb{Z}) \to \mathrm{Hom}(A, \mathbb{Z})$ is a monomorphism. From this elementary observation, together with (3), we obtain that the induced map $f^*\colon \mathrm{Hom}(\mathrm{H}_1(X; \mathbb{Z}), \mathbb{Z}) \to \mathrm{Hom}(\mathrm{H}_1(\Sigma_g; \mathbb{Z}), \mathbb{Z})$ is a monomorphism.
(5) It follows immediately from (4) together with the First Cohomology Proposition 110.17 that the induced map $f^*\colon \mathrm{H}^1(X; \mathbb{Z}) \to \mathrm{H}^1(\Sigma_g; \mathbb{Z})$ is also a monomorphism.
(6) Since X is a 1-dimensional CW-complex we obtain almost immediately from the Singular-Cellular H^*-Isomorphism Proposition 109.4 that $\mathrm{H}^2(X; \mathbb{Z}) = 0$. In particular the cup product on $\mathrm{H}^1(X; \mathbb{Z})$ vanishes.

Now we consider the form

$$\begin{array}{rcl}\varphi\colon \mathrm{H}^1(\Sigma_g;\mathbb{Z}) \times \mathrm{H}^1(\Sigma_g;\mathbb{Z}) & \to & \mathbb{Z} \\ (\alpha,\beta) & \mapsto & \langle \alpha \cup \beta, [\Sigma_g]\rangle.\end{array}$$

Given $U \subset \mathrm{H}^1(\Sigma_g;\mathbb{Z})$ we write as usual

$$U^\perp \;=\; \{v \in \mathrm{H}^1(\Sigma_g;\mathbb{Z}) \,|\, \varphi(u,v) = 0 \text{ for all } u \in U\}.$$

Now we see that

since $X = \bigvee_{i=1}^{k} S^1$ — since i^* is a monomorphism — by the Manifold-Non Singularity Proposition 121.2 we know that φ is non-singular hence we obtain the equality from Exercise 121.3

$$\begin{array}{rcl} k & \overset{\downarrow}{=} & \mathrm{rank}(\mathrm{H}^1(X;\mathbb{Z})) \overset{\downarrow}{=} \mathrm{rank}(i^*\mathrm{H}^1(X;\mathbb{Z})) \overset{\downarrow}{=} \mathrm{rank}(\mathrm{H}^1(\Sigma_g;\mathbb{Z})) - \mathrm{rank}\left((i^*\mathrm{H}^1(X;\mathbb{Z}))^\perp\right) \\ & \underset{\uparrow}{\leq} & \mathrm{rank}(\mathrm{H}^1(\Sigma_g;\mathbb{Z})) - \mathrm{rank}(i^*\mathrm{H}^1(X;\mathbb{Z})) \underset{\uparrow}{=} 2g-k. \end{array}$$

it follows from (6) and the Cup Product-Naturality Lemma 114.10 that $i^*\mathrm{H}^1(X;\mathbb{Z}) \subset (i^*\mathrm{H}^1(X;\mathbb{Z}))^\perp$ — as above we have $\mathrm{rank}(i^*(\mathrm{H}^1(X;\mathbb{Z}))) = k$ and by the discussion on page 2422 we know that $\mathrm{rank}(\mathrm{H}^1(\Sigma_g;\mathbb{Z})) = 2g$

Thus we see that $k \leq 2g-k$, i.e. we see that $k \leq g$. ■

163.6. **Eilenberg–Maclane spaces and the social choice problem ❀.** In this section we give a fun, but perhaps not entirely serious application of Eilenberg-Maclane spaces. Let us recall the following definition from page 1864.

Definition. Let X be a topological space and let $n \in \mathbb{N}$. A social choice of type n is a map

$$f\colon X^n \;=\; \underbrace{X \times \cdots \times X}_{n \text{ times}} \;\to\; X$$

such that the following three axioms are satisfied:

(a) The map f is continuous.
(b) For every $x \in X$ we have $f(x,\ldots,x) = x$.
(c) For every permutation $\sigma \in S_n$ and any $(x_1,\ldots,x_n) \in X^n$ we have

$$f(x_{\sigma(1)},\ldots,x_{\sigma(n)}) \;=\; f(x_1,\ldots,x_n).$$

We say that a topological space is social, if for every $n \in \mathbb{N}$ it admits a social choice of type n.

We refer to Section 86.1 for a discussion why social topological spaces are interesting. Now let X be a topological space. The combination of Lemmas 86.4 and 86.5 shows that if X is social, then for every $k \in \mathbb{N}$ the homotopy group $\pi_k(X)$ is abelian and it is strongly divisible, i.e. given any $n \in \mathbb{N}$ the map $g \mapsto n \cdot g$ is an isomorphism. Furthermore, Proposition 86.2 together with the Whitehead Theorem 162.1 show that if X is in fact a CW-complex, with the property that in any dimension it has only finitely many cells, then X is contractible. Back in Section 86.1 this discussion lead us to the following question.

Question 86.7. Does there exist a non-empty CW-complex that is social but that is not contractible?

It turns out that we can now formulate a positive answer to Question 86.7.

Theorem 163.23.

(1) Any CW-complex that is homotopy equivalent to the product of a family of Eilenberg-Maclane spaces of type $\mathrm{K}(\mathbb{Q}, k_i)$, $i \in I$ is social.
(2) Conversely, if X is a connected social CW-complex, then it is homotopy equivalent to the product of a family of Eilenberg-Maclane spaces of type $\mathrm{K}(\mathbb{Q}, k_i)$, $i \in I$.

Proof. This theorem is proved in [**EGH1962**, p. 90] and [**Weinb2004**, Theorem 1.3], see also [**Eckm2004**, Theorems 4 and 7]. For the most part the proofs do not go much beyond what we have covered so far, except that they build on a result of Alexander Grothendieck [**Groth1957**, Chapter V], see also [**HZ1974**, p. 32] and [**Hirz1969**, p. 254]. ■

Remark. As always there are various variations and generalizations of Theorem 163.23. For example by [**Weinb2004**, Example 1.4] any Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}_2, 1)$, e.g. according to the discussion on page 3427 we could take $\mathbb{R}\mathrm{P}^\infty$, has the property that given any *odd* $n \in \mathbb{N}$ it admits a social choice of type n.

163.7. **An extension of the Hurewicz Theorem 85.5.** In this last section we will use Eilenberg-Maclane spaces in a surprising way to obtain an addendum to the Hurewicz Theorem 85.5. To formulate the theorem it is convenient to introduce the following definition.

Definition. Given a group π and $n \in \mathbb{N}_0$ we define $\mathrm{H}_n(\pi) := \mathrm{H}_n(\mathrm{K}(\pi, 1), \mathbb{Z})$ where $\mathrm{K}(\pi, 1)$ denotes the canonical Eilenberg-Maclane space introduced in the Eilenberg-Maclane Existence Theorem 163.8. We refer to $\mathrm{H}_n(\pi)$ as the n-th homology group of the group π.

Remark. The concept of the homology groups $\mathrm{H}_n(\pi)$ of a group π will play an important role in this section. Afterwards it takes a well-earned rest before it will take center stage in Chapter **??**.

The following lemma is an immediate consequence of Corollary 163.13.

Lemma 163.24. Let π be a group and let $n \in \mathbb{N}_0$. For any Eilenberg-Maclane space X of type $\mathrm{K}(\pi, 1)$ we have $\mathrm{H}_n(\pi) \cong \mathrm{H}_n(X; \mathbb{Z})$.

Example. Let $m \in \mathbb{N}$ and let $n \in \mathbb{N}_0$. We calculate that

$$\mathrm{H}_n(\text{the group } \mathbb{Z}^m) \underset{\text{by Lemma 163.24}}{\cong} \mathrm{H}_n(\text{any Eilenberg-Maclane space } \mathrm{K}(\mathbb{Z}^m, 1)) \underset{\substack{\text{by the discussion on page 3427 we know that } (S^1)^m \\ \text{is an Eilenberg-Maclane space of type } \mathrm{K}(\mathbb{Z}^m, 1)}}{\cong} \mathrm{H}_n((S^1)^m) \underset{\text{see page 1948}}{\cong} \mathbb{Z}^{\binom{m}{n}}.$$

We continue with the following lemma.

Lemma 163.25. Let (X, x_0) be a pointed 0-connected space. There exists a natural map

$$\eta\colon \mathrm{H}_2(X; \mathbb{Z}) \ \to\ \mathrm{H}_2(\pi_1(X, x_0))$$

which has the following property: whenever (Y, y_0) is a pointed Eilenberg-Maclane space of type $\mathrm{K}(\pi_1(X, x_0), 1)$ and $f\colon (X, x_0) \to (Y, y_0)$ is a map of pointed topological spaces that induces an isomorphism of fundamental groups, then there exists an isomorphism

$\mu\colon \mathrm{H}_2(Y;\mathbb{Z}) \to \mathrm{H}_2(\pi_1(X,x_0))$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_2(X;\mathbb{Z}) & \xrightarrow{\quad\eta\quad} & \mathrm{H}_2(\pi_1(X,x_0)) \\ & {\scriptstyle f_*}\searrow \quad \nearrow{\scriptstyle \mu}\ {\scriptstyle \cong} & \\ & \mathrm{H}_2(Y;\mathbb{Z}). & \end{array}$$

Proof. Let (X,x_0) be a pointed 0-connected space. We write $\pi = \pi_1(X,x_0)$. We denote by $(Z,z_0,\varphi\colon \pi_1(Z,z_0)\to\pi)$ the canonical triple introduced in the Eilenberg-Maclane Existence Theorem 163.8. By the CW-Approximation Theorem 161.8 there exists a CW-approximation $f\colon (Y,y_0)\to(X,x_0)$. Note that by the Weak Homotopy-vs-Homology Isomorphisms Proposition 161.6 we know that f induces isomorphisms on all homology groups.

By the Maps-to-K(π,n)-Proposition 163.15 (1) we know that there exists a map $g\colon (Y,y_0)\to(Z,z_0)$ such that $g_* = \varphi^{-1}\circ f_*\colon \pi_1(Y,y_0)\to\pi_1(Z,z_0)$. We set

$$\eta \;:=\; g_*\circ(f_*)^{-1}\colon \mathrm{H}_2(X;\mathbb{Z}) \;\to\; \mathrm{H}_2(Z;\mathbb{Z}) =: \mathrm{H}_2(\pi_1(X,x_0)).$$

Using the Maps-to-K(π,n)-Proposition 163.15 (1) it is not difficult to verify that η has the desired properties. ■

Now we can formulate the following theorem.

Theorem 163.26. (Hurewicz Theorem) Let $n\in\mathbb{N}$ and let X be an $(n-1)$-connected space and let $x_0\in X$.

(1) If $n=1$, then the sequence

$$\pi_2(X,x_0) \;\xrightarrow{\Phi_{(X,x_0)}}\; \mathrm{H}_2(X;\mathbb{Z}) \;\xrightarrow{\eta}\; \mathrm{H}_2(\pi_1(X,x_0)) \;\to\; 0$$

is exact. This implies in particular that the cokernel of the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_2(X,x_0)\to\mathrm{H}_2(X;\mathbb{Z})$ is isomorphic to $\mathrm{H}_2(\pi_1(X))$.

(2) If $n>1$, then the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_{n+1}(X,x_0)\to\mathrm{H}_{n+1}(X;\mathbb{Z})$ is an epimorphism.

Remark.

(1) We refer to [**WhdG1978**, Theorem V.7.9] for a generalization of the above theorem.

(2) Let $n>1$. Furthermore let X be an $(n-1)$-connected space. By the above Hurewicz Theorem 163.26 (2) we now know that the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_{n+1}(X,x_0)\to\mathrm{H}_{n+1}(X;\mathbb{Z})$ is an epimorphism. For $n\geq 3$ the kernel of the Hurewicz homomorphism is discussed in [**Hilt1953**, p.105] and [**WhdJ1950**, p. 72]. Furthermore, for the more tricky case $n=2$ the kernel of the Hurewicz homomorphism is determined in [**WhdJ1950**, pages 72, 76 and Theorem 20].

Example. Let X be a simply connected space and let $x_0\in X$. By Theorem 163.26, applied to the case $n=2$, the Hurewicz homomorphism $\pi_3(X,x_0)\to\mathrm{H}_3(X;\mathbb{Z})$ is in fact an epimorphism. It is natural to wonder whether Theorem 163.26 can be improved upon. For example it is a priori not unreasonable to ask whether the Hurewicz homomorphism $\pi_k(X,x_0)\to\mathrm{H}_k(X;\mathbb{Z})$ is an epimorphism for every $k\in\mathbb{N}_{\geq 3}$. But it turns out, we have a counterexample at hand, namely $X=\mathbb{C}\mathrm{P}^\infty$. By Theorem 155.12 we know that $\mathbb{C}\mathrm{P}^\infty$ is simply connected and that $\pi_k(\mathbb{C}\mathrm{P}^\infty)=0$ for $k\geq 3$. But by the discussion on page 1772 we

know that $\mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \cong \mathbb{Z}$ for every even k. In particular the Hurewicz homomorphism $\pi_4(\mathbb{C}\mathrm{P}^\infty) \to \mathrm{H}_4(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$ is not an epimorphism.

The proof of Theorem 163.26 crucially relies on the following lemma.

Lemma 163.27. Let π be an abelian group and let $n \in \mathbb{N}_{\geq 2}$. If Y is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$, then $\mathrm{H}_{n+1}(Y;\mathbb{Z}) = 0$.

Examples.

(1) The statement of Lemma 163.27 is somewhat surprising since it is emphatically wrong for $n = 1$. For example by the discussion on page 3427 we know that $S^1 \times S^1$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}^2, 1)$ and we have $\mathrm{H}_2(S^1 \times S^1;\mathbb{Z}) \neq 0$.

(2) As a reality check, let us consider the only Eilenberg-Maclane space for $n \geq 2$ that we understand, namely $\mathbb{C}\mathrm{P}^\infty$ which is by the discussion on page 3427 an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, 2)$. By the calculation on page 1772 we do know that $\mathrm{H}_3(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$ is indeed trivial.

Proof of Lemma 163.27. Let $n \in \mathbb{N}_{\geq 2}$ and let π be an abelian group. It follows from Corollary 163.13 that it suffices to show that there exists a single Eilenberg-Maclane space Z of type $\mathrm{K}(\pi, n)$ with $\mathrm{H}_{n+1}(Z;\mathbb{Z}) = 0$.

By the Moore Space Existence Proposition 79.11 we know that there exists an $(n+1)$-dimensional CW-complex that is a Moore space Y of type $\mathrm{K}(\pi, n)$. Recall that this means that

(1) Y is simply connected,
(2) $\mathrm{H}_n(Y;\mathbb{Z}) \cong \pi$,
(3) for any $i \neq n$ we have $\widetilde{\mathrm{H}}_i(Y;\mathbb{Z}) = 0$.

We pick a base point $y_0 \in Y$. By the Hurewicz Corollary 85.7 we now know that (1), (2) and (3) imply that

(4) $\pi_i(Y, y_0) = 0$ for $i = 1, \ldots, n-1$ and that $\pi_n(Y, y_0) \cong \pi$.

By the Kill-All Higher-π_n-Proposition 163.7 there exists a CW-complex Z with the following properties:

(5) The $(n+1)$-skeleton of Z equals Y.[2012]
(6) We have $\pi_i(Z) = 0$ for $i \geq n+1$.
(7) For $i = 1, \ldots, n$ the inclusion $Y \to Z$ induces an isomorphism $\pi_i(Y, y_0) \xrightarrow{\cong} \pi_i(Z, y_0)$.

It follows from (4), (6) and (7) that Z is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$.

Since Y is the $(n+1)$-skeleton of the CW-complex Z we obtain from the Skeleton-Homology Lemma 80.1 (4) that the map $\mathrm{H}_{n+1}(Y;\mathbb{Z}) \to \mathrm{H}_{n+1}(Z;\mathbb{Z})$ is an epimorphism. Together with (3) this implies that $\mathrm{H}_{n+1}(Z;\mathbb{Z}) = 0$.[2013] We have thus found the desired Eilenberg-Maclane space Z. ∎

Now we turn to the actual proof of Theorem 163.26. For convenience we break the proof into three parts:

(1) First we prove Theorem 163.26 (1) for 2-dimensional CW-complexes.

[2012] Here we use that Y is its own $(n+1)$-skeleton.

[2013] The reader will not have failed to notice that this proof shares some ideas with the proof of Lemma 163.11.

(2) Then we prove Theorem 163.26 (2) for $(n+1)$-dimensional CW-complexes.
(3) Finally we deal with the general case of Theorem 163.26.

Proof of Theorem 163.26 (1) for 2-dimensional CW-complexes. Let X be a 2-dimensional CW-complex that is 0-connected. Let $x_0 \in X$. By the Kill-All Higher-π_n-Proposition 163.7 there exists a CW-complex Y with the following properties:

(1) The 2-skeleton of Y equals X.
(2) We have $\pi_i(Y, x_0) = 0$ for $i \geq 2$.
(3) The inclusion map $i\colon X \to Y$ induces an isomorphism $i_*\colon \pi_1(X, x_0) \xrightarrow{\cong} \pi_1(Y, x_0)$.

It follows from (1), (2) and (3) that Y is an Eilenberg-Maclane space of type $\mathrm{K}(\pi_1(X, x_0), 1)$. Next we consider the following diagram:

$$\begin{array}{ccccccccc}
\pi_3(Y,X,x_0) & \xrightarrow{\partial} & \pi_2(X,x_0) & \longrightarrow & \overbrace{\pi_2(Y,x_0)}^{=0} & \longrightarrow & \pi_2(Y,X,x_0) & \xrightarrow{\partial} & \pi_1(X,x_0) \xrightarrow[\cong]{i_*} \pi_1(Y,x_0) \\
\downarrow{\scriptstyle \Phi_{(Y,X,x_0)}} & & \downarrow{\scriptstyle \Phi_{(X,x_0)}} & & \downarrow{\scriptstyle \Phi_{(Y,x_0)}} & & \downarrow{\scriptstyle \Phi_{(Y,X,x_0)}} & & \\
\mathrm{H}_3(Y,X;\mathbb{Z}) & \xrightarrow{\partial} & \mathrm{H}_2(X;\mathbb{Z}) & \twoheadrightarrow & \mathrm{H}_2(Y;\mathbb{Z}) & \longrightarrow & \underbrace{\mathrm{H}_2(Y,X;\mathbb{Z})}_{=0}. & & \\
 & & \downarrow{\scriptstyle \eta} & \overset{\cong}{\swarrow}{\scriptstyle \mu} & & & & & \\
 & & \mathrm{H}_2(\pi_1(X,x_0)) & & & & & &
\end{array}$$

We make the following clarifications and observations:

(a) The horizontal sequences are the long exact sequences of homology and homotopy groups of the pair (Y, X), given by the Topological LES-Corollary 74.14 and by the Homotopy Groups-LES Proposition 153.4.
(b) The vertical maps are the (relative) Hurewicz homomorphisms that we introduced on pages 1842 and 3254
(c) The diagram commutes by Lemma 153.11 (3).
(d) It follows from (2), (3) and the lower end of the upper long exact sequence that $\pi_j(Y, X) = 0$ for $j = 0, 1$. In other words, the pair (Y, X) is 2-connected.
(e) It follows from (d) and the Relative Hurewicz Theorem 153.13 that the left-hand vertical map is an epimorphism.[2014]
(f) The map η and the isomorphism μ are given by Lemma 163.25. In particular these maps turn the lower triangle into a commutative triangle.
(g) Since X is the 2-skeleton of Y we obtain from the Skeleton-Homology Lemma 80.1 that $\mathrm{H}_2(Y, X; \mathbb{Z}) = 0$.

From this data one can easily deduce that the sequence

$$\pi_2(X, x_0) \xrightarrow{\Phi_{(X,x_0)}} \mathrm{H}_2(X;\mathbb{Z}) \xrightarrow{\eta} \mathrm{H}_2(\pi_1(X, x_0)) \to 0$$

is exact. ∎

[2014]People with shaky memories, like your author, might be surprised that it does not say that the map is an isomorphism. The point is that the Relative Hurewicz Theorem 153.13 shows that the Hurewicz homomorphism $\pi_3(Y, X, x_0)' \to \mathrm{H}_3(Y, X; \mathbb{Z})$ is an isomorphism where $\pi_3(Y, X, x_0)'$ is the quotient of $\pi_3(Y, X, x_0)$ by a suitable action of $\pi_1(X, x_0)$. Thus the Hurewicz homomorphism $\pi_3(Y, X, x_0) \to \mathrm{H}_3(Y, X; \mathbb{Z})$ is an epimorphism, but in general it is not an isomorphism.

Proof of Theorem 163.26 (2) for $(n+1)$-dimensional CW-complexes. Let $n \in \mathbb{N}_{\geq 2}$ and let X be an $(n+1)$-dimensional CW-complex that is $(n-1)$-connected. Let $x_0 \in X$. By the Kill-All Higher-π_n-Proposition 163.7 there exists a CW-complex Y with the following properties:

(1) The $(n+1)$-skeleton of Y equals X.
(2) We have $\pi_i(Y, x_0) = 0$ for $i \geq n+1$.
(3) For every $j \in \{1, \ldots, n\}$ the inclusion map $i\colon X \to Y$ induces an isomorphism $i_*\colon \pi_j(X, x_0) \xrightarrow{\cong} \pi_j(Y, x_0)$.

It follows from (1), (2) and (3) that Y is an Eilenberg-Maclane space of type $\mathrm{K}(\pi_n(X, x_0), n)$. Next we consider the following diagram:[2015]

$$\begin{array}{ccccccccccc}
\pi_{n+2}(Y,X) & \xrightarrow{\partial} & \pi_{n+1}(X) & \longrightarrow & \overbrace{\pi_{n+1}(Y)}^{=0} & \longrightarrow & \pi_{n+1}(Y,X) & \xrightarrow{\partial} & \pi_n(X) & \xrightarrow[\cong]{i_*} & \pi_n(Y) \longrightarrow \ldots \\
\cong\downarrow{\scriptstyle \Phi_{(X,Y)}} & & \downarrow{\scriptstyle \Phi_X} & & \downarrow{\scriptstyle \Phi_Y} & & & & & & \\
\mathrm{H}_{n+2}(Y,X) & \xrightarrow{\partial} & \mathrm{H}_{n+1}(X) & \longrightarrow & \underbrace{\mathrm{H}_{n+1}(Y)}_{=0 \text{ by Prop. 163.27}}. & & & & & &
\end{array}$$

We make the following clarifications and observations:

(a) As in the previous proof, the horizontal sequences are long exact sequences of the pair (Y, X). Furthermore the vertical maps are the (relative) Hurewicz homomorphisms and the diagram commutes by Lemma 153.11 (3).
(b) It follows from (2), (3) and the lower end of the upper long exact sequence that $\pi_j(Y, X) = 0$ for $j = 0, \ldots, n+1$. In other words, the pair (Y, X) is $(n+1)$-connected.
(c) It follows immediately from (a) and the Relative Hurewicz Theorem 153.13, together with the fact that $\pi_1(X, x_0) = 0$, that the left-hand vertical map is an isomorphism.
(d) Since Y is an Eilenberg-Maclane space of type $\mathrm{K}(\pi_n(X, x_0), n)$ and since $n \geq 2$ we obtain from Lemma 163.27 that $\mathrm{H}_{n+1}(Y) = 0$.

It follows from the above that the second vertical map $\pi_{n+1}(X) \to \mathrm{H}_{n+1}(X)$ is, as promised, an epimorphism. ■

Proof of the general Theorem 163.26. First we consider the case that $n \geq 2$. In the following let X be an $(n-1)$-connected space. Let $x_0 \in X$. By the CW-Approximation Theorem 161.8 there exists a CW-complex Y together with a weak homotopy equivalence $f\colon Y \to X$. Since f is a weak homotopy equivalence we see that Y itself is $(n-1)$-connected. We denote by $i\colon Y^{n+1} \to Y$ the inclusion of the $(n+1)$-skeleton Y^{n+1} into Y. We pick $y_1 \in Y^{n+1}$ and we write $x_1 = f(y_1)$. Since X is in particular path-connected we can pick a path γ from x_0 to x_1. We consider the following diagram

$$\begin{array}{ccccccc}
\pi_{n+1}(Y^{n+1}, y_1) & \xrightarrow{i_*} & \pi_{n+1}(Y, y_1) & \xrightarrow[\cong]{f_*} & \pi_{n+1}(X, x_1) & \xrightarrow[\cong]{\gamma_*} & \pi_{n+1}(X, x_0) \\
\downarrow{\scriptstyle \Phi_{(Y^{n+1}, y_1)}} & & \downarrow{\scriptstyle \Phi_{(Y,y_1)}} & & \downarrow{\scriptstyle \Phi_{(X,x_1)}} & & \downarrow{\scriptstyle \Phi_{(X,x_0)}} \\
\mathrm{H}_{n+1}(Y^{n+1}; \mathbb{Z}) & \xrightarrow{i_*} & \mathrm{H}_{n+1}(Y; \mathbb{Z}) & \xrightarrow[\cong]{f_*} & \mathrm{H}_{n+1}(X; \mathbb{Z}) & \xrightarrow{\mathrm{id}} & \mathrm{H}_{n+1}(X; \mathbb{Z}).
\end{array}$$

[2015] For evident space reasons we drop the base points from the discussion.

We make the following clarifications and observations:

(1) The vertical maps are the Hurewicz homomorphisms.
(2) The horizontal map to the top right is the isomorphism that is induced by the path γ_*, see the Change-of-Base Point Proposition 71.5.
(3) The two squares to the left commute by Proposition 84.2.
(4) The square to the right commutes by Lemma 85.1 (2).
(5) We know from the above proof of Theorem 163.26 for $(n+1)$-dimensional CW-complexes, and the fact that Y^{n+1} is $(n-1)$-connected, that the left vertical map is an epimorphism.
(6) By the Skeleton-Homology Lemma 80.1 (4) we know that the bottom left horizontal map is an epimorphism.
(7) Since f is a weak homotopy equivalence we know that the top middle horizontal map is an isomorphism. Furthermore we obtain from the Weak Homotopy-vs-Homology Isomorphisms Proposition 161.6 that the bottom middle horizontal map is also an isomorphism.
(8) It follows from all the information assembled above that the right vertical map is an epimorphism.

The proof for $n = 1$ is quite similar, with a few minor extra complications. We leave it to the reader to fill in the details. ■

Theorem 163.26 (2) says that given a pointed 0-connected space (X, x_0) there is a curious interplay between $\pi_2(X, x_0)$, $\mathrm{H}_2(X)$ and the second homology of $\pi_1(X, x_0)$. We want to get a better intuition for what is happening by considering two corollaries to Theorem 163.26 (2).

Corollary 163.28. Let (X, x_0) be some pointed 0-connected space. If $\pi_1(X, x_0) \cong \mathbb{Z}$, then the Hurewicz homomorphism $\Phi_{(X,x_0)} \colon \pi_2(X, x_0) \to \mathrm{H}_2(X; \mathbb{Z})$ is an epimorphism.

Proof. In the discussion on page 3453 we saw that $\mathrm{H}_2(\pi_1(X, x_0)) \cong \mathrm{H}_2(\mathbb{Z}) \cong \mathrm{H}_2(S^1) = 0$. Thus it follows immediately from Theorem 163.26 (2) that the Hurewicz homomorphism $\pi_2(X, x_0) \to \mathrm{H}_2(X; \mathbb{Z})$ is an epimorphism. ■

Remark. The proof of Corollary 163.28 is in many ways beyond reproach, it is clear and rigorous. The only catch is that it is not very geometric and we do not really get a good understanding why the statement is true. To rectify this we will sketch an alternative, quite geometric, proof for Corollary 163.28 in Exercise 163.10.

We move on to the second corollary to Theorem 163.26 (2).

Corollary 163.29. If X is a 0-connected space such that $\pi_1(X) \cong \mathbb{Z}^m$ for some $m \in \mathbb{N}_{\geq 2}$, then for any $x_0 \in X$ we have

$$\operatorname{coker}\left(\Phi_{(X,x_0)} \colon \pi_2(X, x_0) \to \mathrm{H}_2(X; \mathbb{Z})\right) \underset{\text{Theorem 163.26 (1)}}{\overset{\uparrow}{\cong}} \mathrm{H}_2(\pi_1(X)) \cong \mathrm{H}_2(\mathbb{Z}^m) \underset{\text{by the example on page 3453}}{\overset{\uparrow}{\cong}} \mathbb{Z}^{\binom{m}{2}}.$$

In particular we see that $\mathrm{H}_2(X; \mathbb{Z})$ is non-trivial.

In the following we want to get a better understanding how this "extra" homology in $\mathrm{H}_2(X;\mathbb{Z})$ arises. To simplify the discussion let us consider the case $m = 2$. Thus let (X, x_0) be a pointed 0-connected space with $\pi_1(X, x_0) \cong \mathbb{Z}^2$. We proceed as follows:

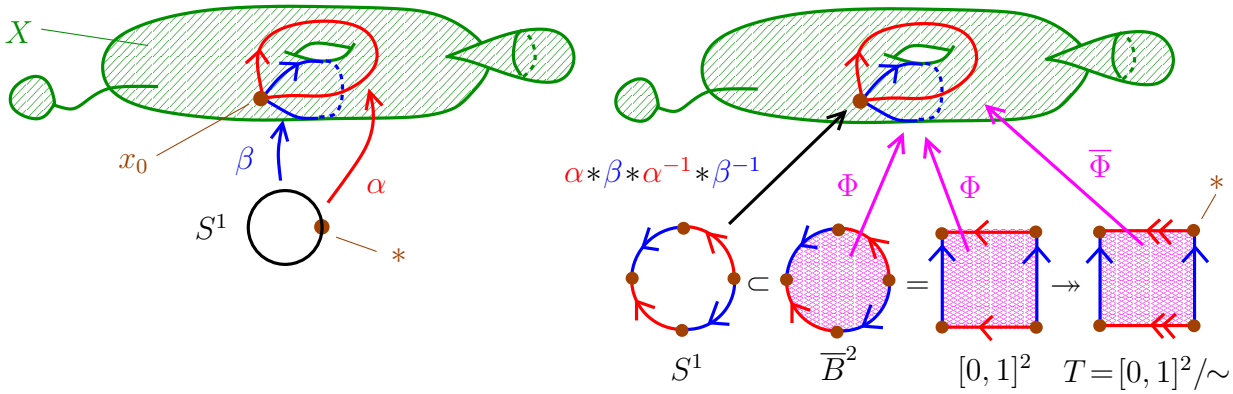
(1) We pick two loops $\alpha, \beta\colon (S^1, *) \to (X, x_0)$ that represent a generating set for the group $\pi_1(X, x_0) \cong \mathbb{Z}^2$.

We consider the loop $\varphi = \alpha * \beta * \overline{\alpha} * \overline{\beta}\colon (S^1, *) \to (X, x_0)$. Since $\pi_1(X, x_0)$ is abelian we see that the loop φ represents the trivial element in $\pi_1(X, x_0)$. By the Extension-to-Disk Lemma 47.1 this means that we can extend the loop to a map $\Phi\colon \overline{B}^2 \to X$.

(2) We make the "obvious" identification[2016] $\overline{B}^2 = [0,1]^2$ which identifies the "quartercircles" on S^1 with the four sides of the square.

(3) Let $\sim$ be the usual equivalence relation on $[0,1]^2$ given by $(x, 0) \sim (x, 1)$ and $(y, 0) \sim (y, 1)$. The quotient $T := [0,1]^2/\sim$ is the 2-dimensional torus. We equip T with the base point $*$ that is represented by any of the vertices.

(4) The map $\Phi\colon \overline{B}^2 = [0,1]^2 \to X$ descends to a map $\overline{\Phi}\colon T := [0,1]^2/\sim \to X$.



This approach feels like we are creating surfaces ex nihilo. But the following lemma shows that miraculously this approach works.

Lemma 163.30. We continue with the above notation. The map

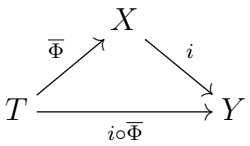
$$\begin{array}{rcl} \mu\colon \overbrace{\mathrm{H}_2(T;\mathbb{Z})}^{=\mathbb{Z}\cdot[T]} & \to & \operatorname{coker}(\pi_2(X, x_0) \to \mathrm{H}_2(X;\mathbb{Z})) \\ \sigma & \mapsto & [\overline{\Phi}_*(\sigma)] \end{array}$$

is an isomorphism.

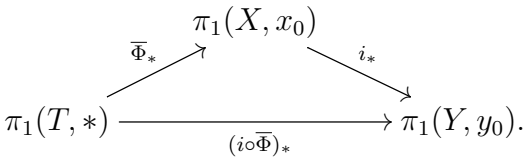
Sketch of proof. As in the proof of Theorem 163.26 (2) we can assume that X is a 3-dimensional CW-complex. Furthermore, by the proof of Theorem 163.26 (2) we know that there exists an inclusion $i\colon X \to Y$ of X into an Eilenberg-Maclane space Y of type $\mathrm{K}(\mathbb{Z}^2, 1)$ such that the inclusion induced map $i_*\colon \pi_1(X, x_0) \to \pi_1(Y, x_0)$ is an isomorphism.

[2016] The identification is given by rotation by $\frac{\pi}{4}$ followed by "outward" stretching as in the proof of the Convex-to-Ball Proposition 2.20.

We consider the following simple minded diagram



and we consider the induced diagram



We make the following observations:

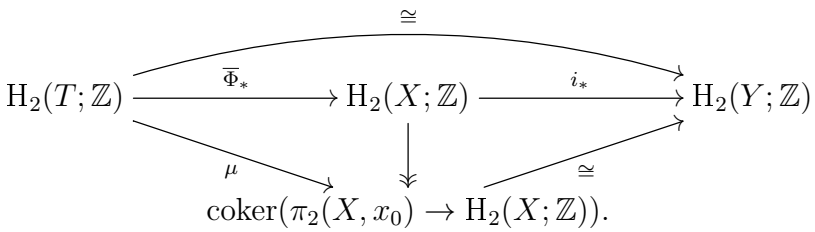
(1) The map $\overline{\Phi}_*\colon \pi_1(T,*) \to \pi_1(X,x_0)$ is by design an epimorphism. Since both groups are isomorphic to $\mathbb{Z}^2$ we obtain from Lemma 51.9 (5) that $\overline{\Phi}_*$ is in fact an isomorphism.

(2) As we mentioned above, the map $i_*\colon \pi_1(X,x_0) \to \pi_1(Y,x_0)$ is an isomorphism.

(3) By (1) and (2) we know that $(i\circ\overline{\Phi})_* = i_*\circ\overline{\Phi}_*\colon \pi_1(T,*) \to \pi_1(Y,x_0)$ is an isomorphism. By the discussion on page 3427 and by the above we know that both T and Y are Eilenberg-Maclane spaces of type $\mathrm{K}(\mathbb{Z}^2,1)$. Thus we obtain from the Whitehead Theorem 162.1 that $i\circ\overline{\Phi}\colon T \to Y$ is in fact a homotopy equivalence.

(4) Since $i\circ\overline{\Phi}$ is a homotopy equivalence we obtain from the Homotopy Equivalence-H_*-Corollary 73.9 that the induced map $(i\circ\overline{\Phi})_* = i_*\circ\overline{\Phi}_*\colon \mathrm{H}_2(T;\mathbb{Z}) \to \mathrm{H}_2(Y;\mathbb{Z})$ is an isomorphism.

It remains to consider the following diagram



By the above discussion we know that the composition of the top two horizontal maps is an isomorphism and by Theorem 163.26, together with Lemma 163.25, we know that the diagonal map to the right is an isomorphism. It follows that the diagonal map to the left is also an isomorphism. ■

Exercises for Chapter 163.

Exercise 163.1. We consider the real projective plane $\mathbb{R}\mathrm{P}^2 = \overline{B}^2/\!\sim$ where for $x \in S^1$ we have $x \sim -x$. Let $\varphi\colon \overline{B}^2 \to \mathbb{R}\mathrm{P}^2$ be a smooth embedding. We consider the map

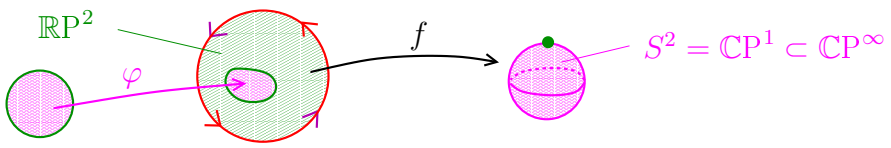
$$\begin{aligned} f\colon \mathbb{R}\mathrm{P}^2 &\to \overline{B}^2/S^1 = S^2 = \mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^\infty \\ P &\mapsto \begin{cases} [Q], & \text{if } P = \varphi(Q) \text{ for some } Q \in \overline{B}^2, \\ [S^1], & \text{otherwise.} \end{cases} \end{aligned}$$

Is the map f homotopic to the constant map?

Exercise 163.2. Let X be a connected k-dimensional CW-complex and let $k \in \mathbb{N}$.

(a) Does there exist a CW-complex Y with $Y^k = X$ and $\pi_k(Y) = 0$?

(b) Does there exist a CW-complex Y with $Y^k = X$ and $\mathrm{H}_k(Y;\mathbb{Z}) = 0$?



Exercise 163.3. Finish the proof of Proposition 163.21.
Hint. Use Lemma 130.5.

Exercise 163.4. Let $n \in \mathbb{N}$. In Lemma 163.11 we showed that there exists an Eilenberg-Maclane space X of type $\mathrm{K}(\mathbb{Z}, n)$ with precisely one 0-cell, precisely one n-cell and no cell of dimension $n+1$. What is the minimal dimension in which the canonical Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, n)$ has more cells than X?

Exercise 163.5. Let $n \in \mathbb{N}$. We consider $n \cdot \mathbb{R}\mathrm{P}^2$, i.e. we consider the connected sum of n copies of $\mathbb{R}\mathrm{P}^2$.

(a) What is the maximal number $k \in \mathbb{N}_0$ for which we can find k disjoint curves $F_1, \ldots, F_k$ on $n \cdot \mathbb{R}\mathrm{P}^2$ such that $n \cdot \mathbb{R}\mathrm{P}^2 \setminus (F_1 \cup \cdots \cup F_k)$ is connected?
(b) What is the maximal $l \in \mathbb{N}_0$ for which there exists an epimorphism from $\pi_1(n \cdot \mathbb{R}\mathrm{P}^2)$ onto a free group of l generators?

Remark. By the Fundamental Group-of-Surfaces Proposition 54.9 we know that there exists an isomorphism

$$\pi_1(n \cdot \mathbb{R}\mathrm{P}^2) \;\cong\; \langle x_1, \ldots, x_n \mid x_1^2 \cdot \cdots \cdot x_n^2 \rangle.$$

Remark. This exercise is basically a variation on Question 130.9.

Exercise 163.6. In Lemma 114.9 we proved the following statement:

($*$) Let X be a topological space. For any $\phi \in \mathrm{H}^1(X; \mathbb{Z})$ we have $\phi \cup \phi = 0$.

Our goal is to give an alternative proof of this statement:

(a) Show that ($*$) holds for $X = S^1$.
(b) Let X be a CW-complex. Use (a) and the Maps-to-$\mathrm{K}(\pi, n)$-Proposition 163.15 to provide a proof of ($*$) for CW-complexes.
(c) Use the CW-Approximation Theorem 161.8 and the Weak Homotopy-vs-Homology Isomorphisms Proposition 161.6 to show, using (b), that ($*$) holds for all topological spaces.

Exercise 163.7. Show that there exists an abelian group π and Moore spaces X and Y of type $\mathrm{K}(\pi, 1)$ that are not homotopy equivalent.
Hint. You could use the fundamental group to distinguish X and Y.
Remark. In Exercise 163.8 we will deal with Moore spaces of type $\mathrm{K}(\pi, n)$ with $n \geq 2$.

Exercise 163.8. Let π be an abelian group and let $n \in \mathbb{N}_{\geq 2}$. In this exercise we will show that any two Moore spaces of type $\mathrm{M}(\pi, n)$ are in fact homotopy equivalent. First note that by the Moore Space Existence Proposition 79.11 there exists an $(n+1)$-dimensional CW-complex X that has no cells in dimensions $1, \ldots, n-1$ and that is a Moore space of type $\mathrm{M}(\pi, n)$. It suffices to show that X is homotopy equivalent to any other Moore space Y of type $\mathrm{M}(\pi, n)$. This task is broken up into two steps:

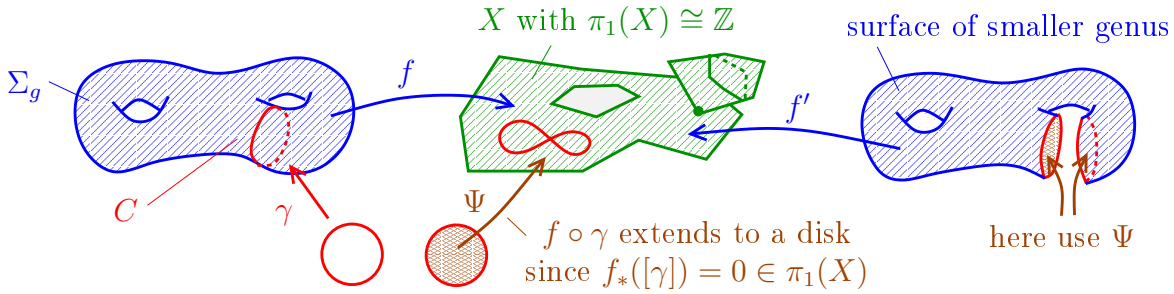
(a) Use Lemma 163.16 to show that there exists a map $f\colon X \to Y$ that induces an isomorphism $f_*\colon \pi_n(X) \to \pi_n(Y)$.
(b) Use Lemma 85.1, the Weak Homotopy-vs-Homology Isomorphisms Proposition 161.6 and the Whitehead Theorem 162.1 (1) to show that f is a homotopy equivalence.

Remark. Together with Exercise 163.7 this gives a complete answer to Question 79.14.

Exercise 163.9. Let $n \in \mathbb{N}$. We set $M = (S^1)^n$. Let $\varphi\colon \pi_1(M) \to \mathbb{Z}^n$ be an isomorphism. Show that it is not possible to extend φ over a bounding $(n+1)$-dimensional smooth manifold. More precisely, show that there is no pair $(W, \psi\colon \pi_1(W) \to \mathbb{Z}^n)$ where W is a compact orientable connected smooth manifold with $\partial W = M$ and where ψ is a homomorphism that satisfies $\varphi = \psi \circ i_*$ where $i\colon M \to W$ denotes the inclusion.
Remark. By the Smooth Manifold-CW Structure Theorem 96.5 we know that any compact smooth manifold W admits a CW-structure.

Exercise 163.10. Let X be a 1-connected space with $\pi_1(X) \cong \mathbb{Z}$. By Corollary 163.28 we know that the Hurewicz homomorphism $\pi_2(X) \to \mathrm{H}_2(X;\mathbb{Z})$ is an epimorphism. The goal of this exercise is to sketch an alternative geometric proof. Thus let $\varphi \in \mathrm{H}_2(X;\mathbb{Z})$. By the Realization-by-Surface Proposition 102.3 there exists a surface Σ_g of some genus $g \in \mathbb{N}_0$ and a map $f\colon \Sigma_g \to X$ such that $f_*([\Sigma_g]) = \varphi \in \mathrm{H}_2(X;\mathbb{Z})$. If $g = 0$, then we are done. So suppose that $g \geq 1$. We proceed with the following steps:

(a) Show that there exists a curve $C \subset \Sigma_g$ with $[C] \neq 0 \in \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ but such that $f_*([C]) = \pi_1(X)$.
Remark. You might want to use Corollary 102.17 at some point.
(b) Get inspired by the figure below.
(c) Complete the proof of the statement.



Exercise 163.11. Let π be a group, let $\varphi\colon \pi \to \pi$ be an endomorphism and let K be an Eilenberg-Maclane space of type $\mathrm{K}(\pi, 1)$. By the Maps-to-$\mathrm{K}(\pi, n)$-Proposition 163.15 we know that there exists a map $f\colon (\mathrm{K}(\pi,1), *) \to (\mathrm{K}(\pi,1))$ such that $f_* = \varphi$. Now suppose that φ is an isomorphism. Can always find such a map f which is furthermore a homeomorphism?

Exercise 163.12. Let $F_n := \langle x_1, \ldots, x_n \rangle$ the free group on n generators $x_1, \ldots, x_n$ and let $\varphi\colon F_n \to F_n$ be an isomorphism. We consider the semidirect product $\pi := F_n \rtimes \mathbb{Z}$ as defined on page 1127. Show that there exists a 2-dimensional Eilenberg-Maclane space of type $\mathrm{K}(\pi, 1)$.

Hint. Note that $X := \bigvee_{i=1}^{n} S^1_i$ is an Eilenberg-Maclane space of type $\mathrm{K}(F_n, 1)$. By the Maps-to-$\mathrm{K}(\pi, n)$-Proposition 163.15 there exists a map $f\colon (X, *) \to (X, *)$ with $f_* = \varphi$. Now

consider the mapping torus $\mathrm{Tor}(X, f)$. Use the Mapping Torus-π_1-Proposition 50.6. It might also be useful to consider Mapping Torus Covering Lemma 16.29.

Exercise 163.13. Let X be a connected aspherical CW-complex, let $U \subset X$ be a connected open subset and let $x_0 \in U$. We suppose that $\pi_1(U, x_0) \to \pi_1(X, x_0)$ is trivial. Show that the inclusion $\iota\colon U \to X$ is homotopic to a constant map.
Hint. Use the universal covering of X.

Exercise 163.14. Let $K \subset S^3$ be a knot. By the Tubular Neighborhood Theorem **??** there exists a smooth embedding $\Phi\colon \overline{B}^2 \times K \to S^3$ such that $\Phi(0, P) = P$ for all $P \in K$. As on page 2213 we refer to $X_K := S^3 \setminus \Phi(B^2 \times K)$ as the *exterior of* K.

(a) Recall that by Lemma 100.16 we know that $\mathrm{H}_1(X_K) \cong \mathbb{Z}$. Show that there exists a map $f\colon X_K \to S^1$ that induces an isomorphism on H_1.

(b) Show that there exists a map $f\colon X_K \to \overline{B}^2 \times S^1$ of degree ± 1, i.e. a map such that $f_*\colon \mathrm{H}_3(X_K, \partial X_K) \to \mathrm{H}_3(\overline{B}^2 \times S^1, S^1 \times S^1)$ is an isomorphism.
Hint. Make use of the Smooth Collar Neighborhood Theorem 28.3.

Exercise 163.15. Let $\{r_k\}_{k\in\mathbb{N}}$ be a sequence of natural numbers. We consider the topological space

$$X \;:=\; ((S^1 \times [0,1]) \times \mathbb{Z})/\sim \quad \text{where } ((z,1),k) \sim ((z^{r_{k-1}},0),k-1) \text{ for } z \in S^1 \text{ and } k \in \mathbb{N}.$$

(a) Show, say using the CW-Complex Construction Lemma 68.32, that X is a 2-dimensional CW-complex.

(b) Let Π be the direct limit of the direct system of abelian groups $\{\mathbb{Z}\}_{k\in\mathbb{N}}$ where for $k \le l$ we consider the homomorphism that is given by multiplication by $r_k \cdot \dots \cdot r_{l-1}$. Show that X is an Eilenberg-Maclane space of type $\mathrm{K}(\Pi, 1)$.
Hint. What can you say about the subcomplexes where k just runs between 1 and some $n \in \mathbb{N}$?

(c) Suppose that $r_k = k$. Show that X is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Q}, 1)$.
Remark. This statement follows from (a) and (b) together with Exercise 56.3.

(d) Suppose that $r_k = 2$. Show that X is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}[\frac{1}{2}], 1)$.

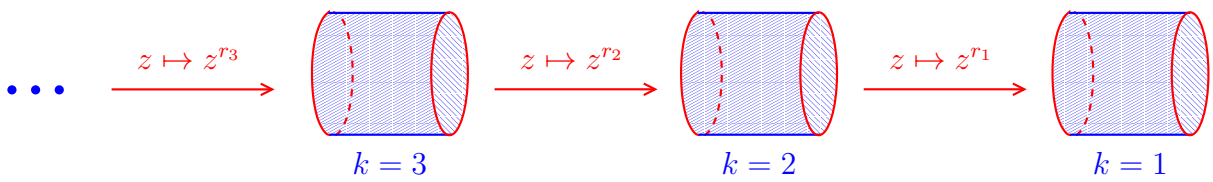
Given a group π we define its *geometric dimension*

$$\mathrm{gd}(\pi) \;:=\; \text{minimal dimension of an Eilenberg-Maclane space of type } \mathrm{K}(\pi,1) \;\in\; \mathbb{N}_0 \cup \{\infty\}.$$

the Eilenberg-Maclane Construction Proposition 163.3 (1) shows that we always have $\mathrm{gd}(A \times B) \le \mathrm{gd}(A) + \mathrm{gd}(B)$. The following exercises shows that equality does not always hold.

(e) Show that $\mathrm{gd}(\mathbb{Q}) = 2$.

(f) Show that $\mathrm{gd}(\mathbb{Q} \times \mathbb{Q}) \le 3$.
Hint. Write $\mathbb{Q} \times \mathbb{Q} = \mathbb{Q}^2$ as the direct limit of a direct system of endomorphisms of $\mathbb{Z}^2$.


$z \mapsto z^{r_3}$
$z \mapsto z^{r_2}$
$z \mapsto z^{r_1}$
$k = 3$
$k = 2$
$k = 1$

164. Invariants obtained through Eilenberg-Maclane spaces

In this chapter we will see how one can use Eilenberg-Maclane spaces to define contravariant functors from the category of CW-complexes to the category of abelian groups. In this chapter and in Chapter ?? we will relate these functors to the cohomology functors.
Before we jump into the chapter we remind the reader that given topological spaces X and Y we denote by Y^X the set of all continuous maps from X to Y. Throughout this and the coming chapters we always equip Y^X with the compact-open topology that we introduced on page 392.

164.1. **The homotopy category of topological spaces.** In this short section we recall some definitions and results regarding topological spaces and the homotopy category of topological spaces. This section partly serves as a warm-up for the discussion of pointed topological spaces in the following sections.
First we recall the following definition from page 441.

Definition. Given two topological spaces X and Y we define

$$[X,Y] := \underbrace{\{\text{continuous maps } X \to Y\}}_{=Y^X}/\sim \quad \text{where } f \sim g \text{ if there exists a homotopy between } f \text{ and } g.$$

We also recall the following definition from page 441.

Definition. The homotopy category $\mathcal{H}$ is defined to be the category where the objects are topological spaces and where the set of morphisms between two topological spaces X and Y is given by the set $[X,Y]$.[2017]

The following lemma is basically an immediate consequence of the definitions.

Lemma 164.1. Let $f\colon X \to Y$ be a map between topological spaces. The corresponding morphism $[f] \in \mathrm{Mor}_{\mathcal{H}}(X,Y)$ is invertible in the sense of the definition on page 198 if and only if f is a homotopy equivalence.

Proof. Note that by definition $[f] \in \mathrm{Mor}_{\mathcal{H}}(X,Y)$ is invertible if and only if there exists $[g] \in \mathrm{Mor}_{\mathcal{H}}(Y,X)$ with $[f \circ g] = [\mathrm{id}_Y]$ and $[g \circ f] = [\mathrm{id}_X]$. Now the statement of the lemma is indeed clear. ∎

In the following we show how some constructions of topological spaces interact with the homotopy category $\mathcal{H}$. First we have following rather dull lemma.

Lemma 164.2. Let X, Y and Z be topological spaces. The map

$$\begin{array}{rcl} \chi\colon [X,Y] \times [X,Z] & \to & [X, Y \times Z] \\ ([f\colon X \to Y], [g\colon X \to Z]) & \mapsto & [f \times g\colon X \to Y \times Z] \end{array}$$

is well-defined.

Proof. Nothing can possibly go wrong in any attempt to prove this lemma. ∎

[2017] The composition of $[f\colon X \to Y]$ and $[g\colon Y \to Z]$ is evidently defined to be $[g \circ f\colon X \to Z]$, by Lemma 15.10 this definition does not depend on the choice of representatives.

Lemma 164.3. The maps

$$\begin{array}{rcl} X & \mapsto & \text{free loop space } \Omega(X) := \{\varphi \in X^{[0,1]} \mid \varphi(0) = \varphi(1)\} \\ [f\colon X \to Y] & \mapsto & \left[\begin{array}{rcl} \Omega(f)\colon \Omega(X) & \to & \Omega(Y) \\ (\varphi\colon [0,1] \to X) & \mapsto & (f \circ \varphi\colon [0,1] \to Y) \end{array}\right] \end{array}$$

and

$$\begin{array}{rcl} X & \mapsto & \text{suspension } \Sigma(X) := (X \times [-1,1])/\sim \\ [f\colon X \to Y] & \mapsto & \left[\begin{array}{rcl} \Sigma(f)\colon \Sigma(X) & \to & \Sigma(Y) \\ [(x,t)] & \mapsto & [(f(x),t)] \end{array}\right] \end{array}$$

both define covariant functors from $\mathcal{H}$ to itself.

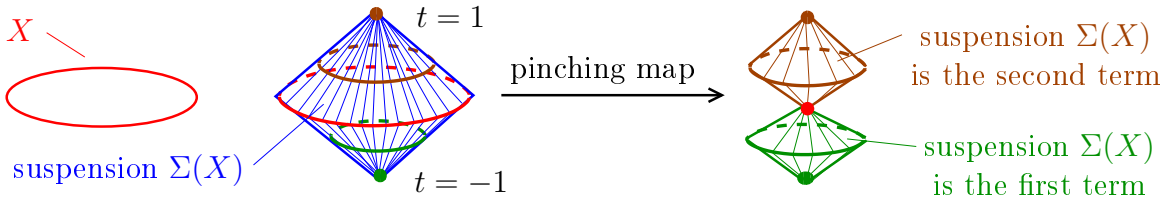
Proof. There are only two statements one needs to think about: given a map $f\colon X \to Y$ one needs to show that $\Omega(f)$ and $\Sigma(f)$ are continuous, and given homotopic maps $f, g\colon X \to Y$ one needs to show that the corresponding maps on the right hand side are also homotopic. For the free loop spaces the statements follow from the Compact-Open Functoriality Lemma 12.5 (1) and (3). For suspensions these two statements follow immediately from the Suspension Functor Lemma 16.9 (1) and (4). ■

Just for kicks we also mention the following definition, which is reminiscent of the definition that we gave on page 1565. We will make use of this definition, with a slight twist, in the near future.

Definition. Let X be a topological space. We consider the wedge $\Sigma(X) \vee \Sigma(X)$ that is given by identifying the "South Pole $[X \times \{-1\}]$" on the left with the "North Pole $[X \times \{1\}]$" on the right. We refer to the map

$$\begin{array}{rcl} \Sigma(X) & \to & \Sigma(X) \vee \Sigma(X) \\ [(x,t)] & \mapsto & \begin{cases} [(x, 2t+1)] \text{ in the first term}, & \text{if } t \in [-1,0], \\ [(x, 2t-1)] \text{ in the second term}, & \text{if } t \in [0,1]. \end{cases} \end{array}$$

as the pinching map.[2018]



In some instances the sets $[X, Y]$ have some extra structure. For example we have the following elementary but cute lemma.

Lemma 164.4. Let X be a topological space and let G be an (abelian) topological group, e.g. we could take $G = S^1$. The set $[X, G]$ together with the multiplication map

$$\begin{array}{rcl} \nu\colon [X,G] \times [X,G] & \to & [X,G] \\ [f],[g] & \mapsto & \left[\begin{array}{rcl} X & \to & G \\ x & \mapsto & f(x) \cdot g(x) \end{array}\right] \end{array}$$

is an (abelian) group.

[2018] It follows easily from the Topological-Quotient Proposition 5.15 (1b) and Pasting Proposition 2.6 (2) that the pinching map is continuous.

Remark. Let X be a topological space and let G be a topological group. In Lemma 12.12 we showed, under the extra hypothesis that G is regionally compact, that the set G^X with the obvious multiplication map $G^X \times G^X \to G^X$ is in fact a topological group. We could equip $[X, G] = G^X/\sim$ with the quotient topology, but the discussion on page 3316 shows that in general the resulting multiplication map $\nu\colon [X, G] \times [X, G] \to [X, G]$ is no longer continuous. Thus in general we cannot view $[X, G]$ as a topological group.

Proof. In fact the only thing one needs to verify is that the map ν is actually well-defined. But this follows easily from the fact that the group multiplication $G \times G \to G$ is continuous. We leave it to the reader to fill in the details. ■

There are several drawbacks to considering topological spaces which are not equipped with base points.

(1) Without a base point the homotopy groups $\pi_n(X)$ are at best defined up to isomorphism, but they are not defined "on the nose".
(2) The operation of taking wedges is not well-defined.

In the following section we will introduce the analogue of the homotopy category for pointed topological spaces. This will take care of the above issues, but it will require some adjustments at other places.

164.2. **The homotopy category of pointed topological spaces.** In the latter part of this chapter it will be essential that we always work with pointed topological spaces. To prepare for this discussion we need to rephrase all the concepts and results from the previous section in the language of pointed topological spaces.
We start out with a few definitions. Some of them we have encountered before on page 1542 and in the formulation of the Maps-to-K(π, n)-Proposition 163.15.

Definition.

(1) A pointed topological space is a pair (X, x_0) where X is a topological space and $x_0 \in X$ is a point. We refer to x_0 as the base point of the pointed topological space.
(2) A map $f\colon (X, x_0) \to (Y, y_0)$ between pointed topological spaces is a map $f\colon X \to Y$ with $f(x_0) = y_0$.
(3) We say that two maps $f, g\colon (X, x_0) \to (Y, y_0)$ between pointed topological spaces are homotopic if there exists a homotopy $F\colon X \times [0, 1] \to Y$ rel $\{x_0\}$ from f to g. We use this notion of "homotopic" to define the notion of a homotopy equivalence between pointed topological spaces.
(4) A pointed CW-complex is a pair (X, x_0) where X is a CW-complex and x_0 is a point in the 0-skeleton X^0 of X.
(5) Given two pointed topological spaces (X, x_0) and (Y, y_0) we write[2019]
$$\langle (X, x_0), (Y, y_0) \rangle := \{\text{maps } (X, x_0) \to (Y, y_0)\}/\sim$$
where $f \sim g$ if there exists a homotopy in the above sense between f and g.
Given a map $f\colon (X, x_0) \to (Y, y_0)$ we denote its equivalence class by $[f]$ as usual.
(6) Let (X, x_0) and (Y, y_0) be two topological spaces. Let $0 \in \langle (X, x_0), (Y, y_0) \rangle$ be the element that is represented by constant map $X \to Y$ given by $x \mapsto y_0$.[2020] We refer to 0 as the trivial element.

The following lemma is the analogue of Lemma 15.10.

Lemma 164.5. Let (X, x_0), (Y, y_0) and (Z, z_0) be pointed topological spaces. The map

$$\begin{array}{rcl} \langle (X,x_0), (Y,y_0)\rangle \times \langle (Y,y_0), (Z,z_0)\rangle & \to & \langle (X,x_0), (Z,z_0)\rangle \\ ([f],[g]) & \mapsto & [g \circ f] \end{array}$$

is well-defined.

Proof. The proof of this lemma is almost identical to the proof of Lemma 15.10 and it is left to the insomniac reader. ∎

Lemma 164.5 shows that the following definition makes sense.

Definition. The pointed homotopy category $\mathcal{PHomTop}$ is the category where the objects are pointed topological spaces and where the set of morphisms between two pointed topological spaces (X, x_0) and (Y, y_0) is given by the set $\langle (X, x_0), (Y, y_0)\rangle$.

The following lemma is the analogue of Lemma 164.1.

Lemma 164.6. Let (X, x_0) and (Y, y_0) be pointed topological spaces.

(1) Let $f\colon (X, x_0) \to (Y, y_0)$ be a map. The corresponding morphism[2021]

$$[f] \in \mathrm{Mor}_{\mathcal{PHomTop}}((X, x_0), (Y, y_0)) = \langle (X, x_0), (Y, y_0)\rangle$$

is invertible if and only if f is a homotopy equivalence.

(2) In the category $\mathcal{PHomTop}$ we have[2022]

$$\langle (X, x_0), (Y, y_0)\rangle_{\mathrm{inv}} = \text{all elements of } \langle (X, x_0), (Y, y_0)\rangle \text{ that are represented by a homotopy equivalence.}$$

Proof. The first statement is, as in the case of Lemma 164.1, an immediate consequence of the definitions. The second statement is a reformulation of the first statement. ∎

In the following we discuss the operations of taking products, wedges, loop spaces and suspensions in the context of the pointed homotopy category. We start out with the following convention.

Convention. Given pointed topological spaces (X, x_0) and (Y, y_0) we equip the product $X \times Y$ with the point (x_0, y_0).

For completeness we mention the following uninspiring analogue to Lemma 164.2.

Lemma 164.7. Let (X, x_0), (Y, y_0) and (Z, z_0) be pointed topological spaces. The map

$$\begin{array}{rcl} \chi\colon \langle (X,x_0), (Y,y_0)\rangle \times \langle (X,x_0), (Z,z_0)\rangle & \to & \langle (X,x_0), (Y \times Z, (y_0,z_0))\rangle \\ ([f\colon X \to Y], [g\colon X \to Z]) & \mapsto & [f \times g\colon X \to Y \times Z] \end{array}$$

is well-defined and it is natural.

Proof. Conveniently enough the proof is identical to the proof of Lemma 164.2. ∎

As discussed on page 1132, one advantage of working with pointed topological spaces is that we have an unambiguous wedge operation.

[2019] The notation $\langle X, Y\rangle$ goes back to [**Hat2002**, p. 357], where it says "the notation $\langle \ldots \rangle$ is intended to suggest pointed homotopy classes".

[2020] At the moment this notation might sound a little risqué. But it will turn out to be a useful notation.

[2021] We refer to page 198 for the definition of an invertible morphism.

[2022] On page 198 we defined $\mathrm{Mor}_{\mathcal{C}}(X, Y)_{\mathrm{inv}}$ to be the set of all invertible morphisms.

Definition. Given a family $\{(X_i, x_i)\}_{i\in I}$ of pointed topological spaces we define the wedge

$$\bigvee_{i\in I}(X_i, x_i) \;:=\; \Big(\bigcup_{i\in I} X_i\Big)/\!\sim \qquad \text{where } x_i \sim x_j \text{ for any } i, j \in I.$$

We equip this wedge with the base point given by the wedge point.

Lemma 164.8. Let (X, x_0), (Y, y_0) and (Z, z_0) be pointed topological spaces. The map

$$\begin{array}{rcl} \nu\colon \langle (X,x_0), (Z,z_0)\rangle \times \langle (Y,y_0), (Z,z_0)\rangle & \to & \langle (X,x_0) \vee (Y,y_0), (Z,z_0)\rangle \\ ([f\colon X \to Z], [g\colon Y \to Z]) & \mapsto & [f \vee g] \end{array}$$

is well-defined.

Proof. The statement of the lemma is, in contrast to Lemma 164.5 and 164.7, non-trivial. It is a reformulation of Lemma 50.12 which traces its ancestry to the formidable Whitehead Quotient Theorem 13.4. ∎

Now we would like to generalize the free loop space $\Omega(X)$ and the suspension $\Sigma(X)$ to pointed topological spaces. It is pretty clear what we do about loop spaces:

Definition. Given a pointed topological space (X, x_0) we consider the

$$\text{loop space } \Omega(X, x_0) \;:=\; \{f \in X^{[0,1]} \mid f(0) = f(1) = x_0\}$$

and we equip the loop space $\Omega(X, x_0)$ with the base point c_{x_0} which is just the constant path $[0, 1] \to X$, $t \mapsto x_0$. Often we will suppress the base point from the notation, but throughout the subsequent chapters we will always view $\Omega(X, x_0)$ as a pointed topological space.

For completeness we write down explicitly the following fairly obvious lemma which is the analogue of the first statement of Lemma 164.3.

Lemma 164.9. The maps

$$\begin{array}{rcl} (X, x_0) & \mapsto & (\Omega(X, x_0), c_{x_0}) \\ [f\colon (X, x_0) \to (Y, y_0)] & \mapsto & \left[\begin{array}{rcl} \Omega(X, x_0) & \to & \Omega(Y, y_0) \\ (\gamma\colon [0, 1] \to X) & \mapsto & (f \circ \gamma\colon [0, 1] \to Y) \end{array}\right] \end{array}$$

define a covariant functor from the category *PHomTop* to itself.

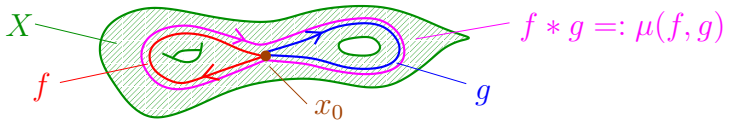
Proof. The lemma follows easily basically from the Compact-Open Functoriality Lemma 12.5 (1) and (3), one just needs to make a few minute modifications to factor in the base points. We leave this elementary task to the inexhaustible reader. ∎

The loop spaces $\Omega(X, x_0)$ have one big advantage over the free loop spaces $\Omega(X)$. Given two loops in (X, x_0) we can form the product, as defined on page 1052. As we had already hinted at in Lemma 156.22, this multiplication turns $\Omega(X, x_0)$ into a group "up to homotopy". More precisely we have the following definition and lemma which hark back to the happy days when our biggest worry was to define the fundamental group of a pointed topological space.

Definition. Let (X, x_0) be a pointed topological space. We consider the map

$$\begin{array}{rcl} \mu\colon \Omega(X,x_0)\times\Omega(X,x_0) & \to & \Omega(X,x_0) \\ (f,g) & \mapsto & f*g = \left(\begin{array}{rcl} [0,1] & \to & X \\ t & \mapsto & \begin{cases} f(2t), & \text{if } t\in[0,\frac{1}{2}], \\ g(2t-1), & \text{if } t\in(\frac{1}{2},1] \end{cases} \end{array}\right). \end{array}$$

For entertainment the definition of this map is illustrated in the figure below.



The following lemma summarizes many important properties of μ.

Lemma 164.10. Let (X, x_0) be a pointed topological space. The multiplication map

$$\mu\colon \Omega(X,x_0)\times\Omega(X,x_0) \to \Omega(X,x_0)$$

has the following properties.

(1) The map $\mu\colon \Omega(X,x_0)\times\Omega(X,x_0)\to\Omega(X,x_0)$ is continuous[2023], it is base-point preserving and it is natural in (X,x_0).[2024]

(2) The two maps

$$\begin{array}{rcl} \lambda\colon \Omega(X,x_0) & \to & \Omega(X,x_0) \\ f & \mapsto & \mu(c_{x_0},f) \end{array} \quad \text{and} \quad \begin{array}{rcl} \rho\colon \Omega(X,x_0) & \to & \Omega(X,x_0) \\ f & \mapsto & \mu(f,c_{x_0}) \end{array}$$

between pointed topological spaces are both homotopic to the identity.

(3) The map

$$\begin{array}{rcl} \eta\colon \Omega(X,x_0) & \to & \Omega(X,x_0) \\ f & \mapsto & \overline{f} := \left(\begin{array}{rcl} [0,1] & \to & X \\ t & \mapsto & f(1-t) \end{array}\right) \end{array}$$

is a homeomorphism and base point preserving.

(4) The map

$$\begin{array}{rcl} \Omega(X,x_0) & \to & \Omega(X,x_0) \\ f & \mapsto & \mu(f,\overline{f}) \end{array}$$

between pointed topological spaces is homotopic to the constant map $f\mapsto c_{x_0}$.

(5) The two maps

$$\begin{array}{rcl} \Omega(X,x_0)\times\Omega(X,x_0)\times\Omega(X,x_0) & \to & \Omega(X,x_0) \\ (f,g,h) & \mapsto & \mu(\mu(f,g),h) \end{array}$$

and

$$\begin{array}{rcl} (f,g,h) & \mapsto & \mu(f,\mu(g,h)) \end{array}$$

between pointed topological spaces are homotopic.

[2023] Recall that $\Omega(X,x_0)$ is equipped with the compact-open topology and $\Omega(X,x_0)\times\Omega(X,x_0)$ is correspondingly equipped with the product topology. Furthermore we view $\Omega(X,x_0)$ and $\Omega(X,x_0)\times\Omega(X,x_0)$ as pointed topological spaces.

[2024] Here "natural" means that if $f\colon (X,x_0)\to(Y,y_0)$ is a map between pointed topological spaces, then the following diagram commutes:

$$\begin{array}{ccc} \Omega(X,x_0)\times\Omega(X,x_0) & \xrightarrow{\mu} & \Omega(X,x_0) \\ \downarrow{\scriptstyle f_*\times f_*} & & \downarrow{\scriptstyle f_*} \\ \Omega(Y,y_0)\times\Omega(Y,y_0) & \xrightarrow{\mu} & \Omega(Y,y_0). \end{array}$$

Remark.

(1) In many ways Lemma 164.10 is quite similar to the Path-Homotopy Concatenation Proposition 47.5. In fact it is not difficult to formulate and prove a result which combines both. As it is, the two results are stated at the appropriate level of generality for what they are aimed for.

(2) Let (X, x_0) be a pointed topological space. In Lemma 164.10 (2) we just showed that the multiplication map $\Omega(X, x_0) \times \Omega(X, x_0) \to \Omega(X, x_0)$ is continuous. The fundamental group $\pi_1(X, x_0) := \Omega(X, x_0)/\sim$ is a quotient of $\Omega(X, x_0)$, hence it can be equipped with the quotient topology. It turns out, see the discussion on page 3316, that the induced multiplication map $\pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0)$ is in general *not* continuous.

Proof. Let (X, x_0) be a pointed topological space.

(1) The statement that μ is continuous is precisely the content of Lemma 156.22 (1). The statements that μ is base-point preserving and natural follows basically immediately from the definitions.

(2) This statement is a reformulation of Lemma 156.22 (2).

(3) Evidently the map $\eta\colon \Omega(X, x_0) \to \Omega(X, x_0)$ is base point preserving. It follows immediately from the Compact-Open Functoriality Lemma 12.5 (5), applied to the homeomorphism $[0, 1] \to [0, 1]$ that is given by $t \mapsto 1-t$, that the map $\eta\colon \Omega(X, x_0) \to \Omega(X, x_0)$ is continuous. Since $\eta \circ \eta = \mathrm{id}$ we see that η is in fact a homeomorphism.

(4) We consider the map

$$\begin{array}{rcl} H\colon \Omega(X, x_0) \times [0, 1] & \to & \Omega(X, x_0) \\ (f, t) & \mapsto & \left(\begin{array}{rcl} [0, 1] & \to & X \\ s & \mapsto & \begin{cases} f(2s \cdot t), & \text{if } s \in [0, \frac{1}{2}], \\ f(2t - 2s \cdot t), & \text{if } s \in \left(\frac{1}{2}, 1\right] \end{cases} \end{array} \right). \end{array}$$

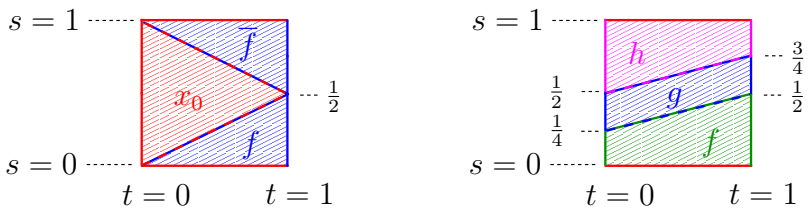
Again, using the argument in the proof of Lemma 156.22 (1) we see that H is continuous. It is now clear that H is a homotopy between the constant map $f \mapsto c_{x_0}$ and the map $f \mapsto \mu(f, \overline{f})$. We refer to the figure below for an illustration.

(5) To simplify the notation we write $\Omega(X)$ instead of $\Omega(X, x_0)$. We consider the map

$$\begin{array}{rcl} H\colon (\Omega(X) \times \Omega(X) \times \Omega(X)) \times [0, 1] & \to & \Omega(X) \\ ((f, g, h), t) & \mapsto & \left(\begin{array}{rcl} [0, 1] & \to & X \\ s & \mapsto & \begin{cases} f(\frac{4s}{t+1}), & \text{if } s \in [0, \frac{t+1}{4}), \\ g(4s - t - 1), & \text{if } s \in [\frac{t+1}{4}, \frac{t+2}{4}), \\ h\left(\frac{4s-2-t}{2-t}\right), & \text{if } s \in [\frac{t+2}{4}, 1] \end{cases} \end{array} \right). \end{array}$$

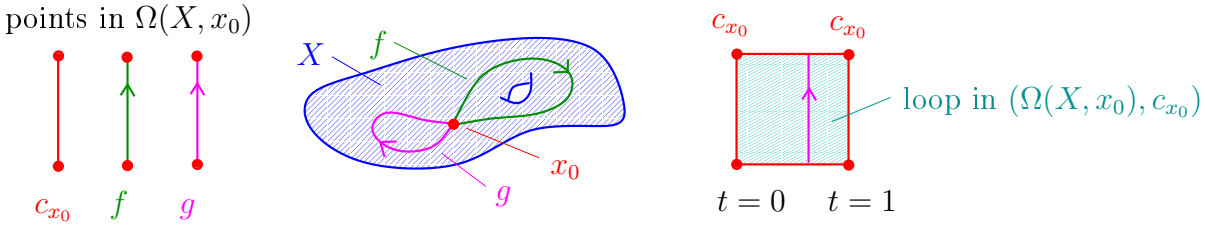
The same argument as in the proof of Lemma 156.22 (1) shows that the map H is continuous. One can now easily verify that the map H defines a homotopy between $(f, g, h) \mapsto \mu(\mu(f, g), h)$ and $(f, g, h) \mapsto \mu(f, \mu(g, h))$. We refer to the figure below for an illustration. ∎

The loop space construction is so much fun that one might as well apply it several times. This leads us to the following notation:



Notation. Let (X, x_0) be a pointed topological space. We define

$$\Omega^k(X, x_0) \;:=\; \Omega\cdots\Omega(\underbrace{\Omega(X, x_0)}_{\text{pointed topological space}}) \quad \text{where we apply the } \Omega\text{-functor } k \text{ times.}$$



In the figure above we do our best at illustrating a loop in the pointed topological space $(\Omega(X, x_0), c_{x_0})$. What should become apparent is that such a loop is basically the same as a map $([0,1]^2, \partial[0,1]^2) \to (X, x_0)$. The following proposition makes this idea precise.

Proposition 164.11. (Iterated Loop Space Proposition) Let (X, x_0) be a pointed topological space and let $k \in \mathbb{N}$. The map

$$\begin{array}{rcl} \overbrace{\{F \in X^{[0,1]^k} \mid F(\partial[0,1]^k) = \{x_0\}\}}^{\text{subset of the topological space } X^{[0,1]^k}} & \to & \Omega^k(X, x_0) \\ F & \mapsto & (t_k \mapsto (t_{k-1} \mapsto \ldots (\underbrace{t_1 \mapsto F(t_1, \ldots, t_k)}_{\in \Omega(X, x_0)}) \ldots)) \end{array}$$

is a homeomorphism.

Proof. To simplify the notation we only discuss the case $k = 2$. The general case is treated the same way. We write $I = J = [0,1]$. We start out with the following subtle observation:

(∗) The inclusion $\Omega(X, x_0) \to X^I$ induces a map $\Omega(X, x_0)^J \to (X^I)^J$. By the Compact-Open Functoriality Lemma 12.5 (2) this map is an embedding. This means that it is legitimate to view the topological space $\Omega(\Omega(X, x_0), c_{x_0})$ as a subspace of $(X^I)^J$.

This preamble implies that it suffices to show that the map

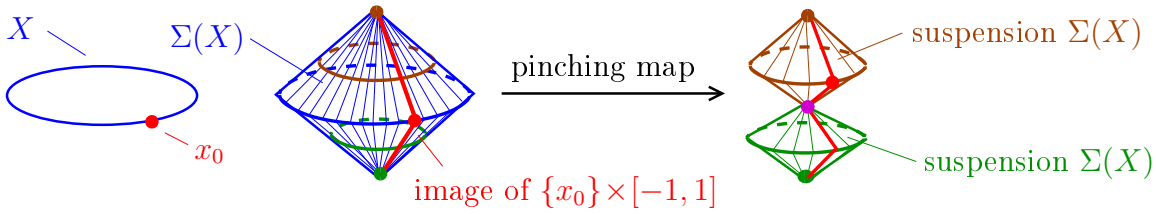
$$\begin{array}{rcl} \Phi\colon \{F \in X^{I\times J} \mid F(\partial(I \times J)) = \{x_0\}\} & \to & \Omega(\Omega(X, x_0), c_{x_0}) \subset (X^I)^J \\ F & \mapsto & \left(\begin{array}{ccc} J & \to & \Omega(X, x_0) \subset X^I \\ t & \mapsto & \left(\begin{array}{ccc} I & \to & X \\ s & \mapsto & F(s,t) \end{array} \right) \end{array} \right) \end{array}$$

is a homeomorphism. First note that the map Φ is, basically by definition, a bijection. Next note that Φ is a restriction of the map

$$\begin{array}{rcl} \Psi\colon X^{I\times J} & \to & (X^I)^J \\ (F\colon I\times J\to X) & \mapsto & \begin{pmatrix} J & \to & X^I \\ t & \mapsto & (s\mapsto F(s,t)) \end{pmatrix}. \end{array}$$

But it follows from the intricate Exponential Proposition 12.7, and the fact that $I = J = [0,1]$ are regionally compact and Hausdorff, that the map Ψ is a homeomorphism. Thus we have shown that our original map Φ is also a homeomorphism. ■

Now let us turn towards the suspension operation. Given a pointed topological space (X, x_0) ideally we would like to equip the suspension $\Sigma(X)$ with a base point in a "systematic" way. The obvious idea is to fix some $t \in [-1,1]$ once and for all and to equip every suspension $\Sigma(X)$ with the base point represented by $[(x_0,t)]$. As we will see later on, we really really want the pinching map to be a map of pointed topological spaces. But if one thinks about it, and if one keeps an eye on the figure below, one see that for every single choice of fixed t the pinching map (and note that for the wedge operation we also have to use our specified base points) will fail to be base point preserving.



There is a rather "brutal" solution to the dilemma: We solve the conundrum of which $[(x_0,t)]$ to choose by declaring all these points to be the same. More precisely, we consider the following modified version of the suspension.

Definition. Given a pointed topological space (X, x_0) we defined the reduced suspension of (X, x_0) to be the topological space[2025]

$$S(X,x_0) := \Sigma(X)/[\{x_0\}\times[-1,1]] \underset{\uparrow}{=} (X\times[-1,1])/(X\times\{-1,1\}\cup\{x_0\}\times[-1,1]).$$

strictly speaking we are using the Twice Quotient Lemma 5.23 (1)

We always equip $S(X,x_0)$ with the base point $s_{x_0} := [\{x_0\}\times[-1,1]]$. We refer to the figure below for an illustration.

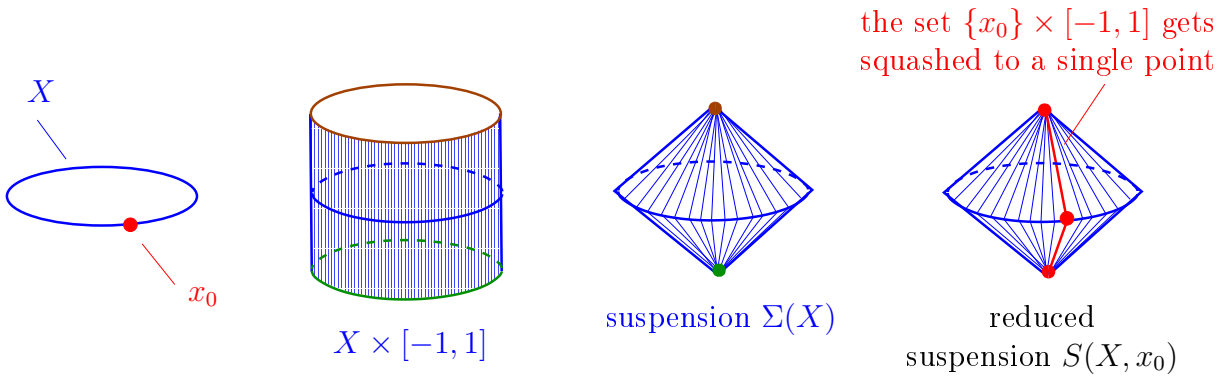
The following lemma summarizes the key properties of the reduced suspension. The lemma can partly be viewed as the "pointed" version of the second part of Lemma 164.3.

Lemma 164.12.

(1) The maps[2026]

$$\begin{array}{rcl} (X,x_0) & \mapsto & (S(X,x_0), s_{x_0}) \\ (f\colon (X,x_0)\to(Y,y_0)) & \mapsto & \begin{pmatrix} S(f)\colon S(X,x_0) & \to & S(Y,y_0) \\ [(x,t)] & \mapsto & [(f(x),t)] \end{pmatrix} \end{array}$$

[2025]Note that given a topological space X we denote the usual suspension by $\Sigma(X)$ and given a pointed topological space we denote its reduced suspension by $S(X,x_0)$. Rather confusingly this notation is the opposite of the notation used in [**Hat2002**].



define a covariant functor from the category $\mathcal{PHomTop}$ to itself.

(2) The natural projection $\Sigma(X) \to S(X, x_0)$ defines a natural transformation between the functors $(X, x_0) \mapsto \Sigma(X)$ and $(X, x_0) \mapsto S(X, x_0)$. In other words, in plain English, given a map $f\colon (X, x_0) \to (Y, y_0)$ between pointed topological spaces the following diagram commutes:

$$\begin{array}{ccc} \Sigma(X) & \xrightarrow{\Sigma(f)} & \Sigma(Y) \\ \downarrow & & \downarrow \\ S(X, x_0) & \xrightarrow{S(f)} & S(Y, y_0). \end{array}$$

(3) Given any pointed topological space (X, x_0) the pinching map

$$\begin{array}{rcl} \omega\colon S(X, x_0) & \to & S(X, x_0) \vee S(X, x_0) \\ [(x,t)] & \mapsto & \begin{cases} [(x, 2t+1)] \text{ in the first term}, & \text{if } t \in [-1, 0], \\ [(x, 2t-1)] \text{ in the second term}, & \text{if } t \in [0, 1]. \end{cases} \end{array}$$

is continuous and base point preserving.

(4) If (X, x_0) is a pointed CW-complex, then the following two statements hold:

(a) The reduced suspension $S(X, x_0)$ has a natural CW-structure such that the natural projection $\Sigma(X) \to S(X, x_0)$ is a cellular map and such that $\{s_{x_0}\}$ is a 0-cell. In particular $(S(X, x_0), s_{x_0})$ is a pointed CW-complex.

(b) Given any $t \in [-1, 1]$ the projection $\Sigma(X) \to S(X, x_0)$ is a homotopy equivalence of the pointed topological spaces $(\Sigma(X), [(x_0, t)]) \to (S(X, x_0), s_{x_0})$.

Convention. Given a pointed CW-complex (X, x_0) we always equip the reduced suspension $S(X, x_0)$ with the CW-structure coming from Lemma 164.12 (4a).

Proof.

(1) The first statement is proved basically the same way as we proved the Suspension Functor Lemma 16.9. Note that the proof of the statement that if $f, g\colon (X, x_0) \to (Y, y_0)$ are homotopic implies that $S(f), S(g)$ are homotopic is somewhat delicate, see the proof of the Suspension Functor Lemma 16.9 (4).

(2) This statement follows almost immediately from the definitions.

(3) This statement follows easily from the Topological-Quotient Proposition 5.15 (1b).

[2026]It follows from the Topological-Quotient Proposition 5.15 (1b) that the map $S(f)$ is actually continuous.

(4) The last statement is slightly more subtle. We proceed as follows:
   (a) By Proposition 68.23 (3b) we can equip $X \times [-1,1]$ with the product CW-structure.
   (b) As the CW-Complex Cone Corollary 68.35 we equip the suspension $\Sigma(X) = (X \times [-1,1])/\sim$ with the quotient CW-structure.
   (c) This CW-structure has the property that the image of $\{x_0\} \times [-1,1]$ in $\Sigma(X)$ is a subcomplex which we now denote by A.
   (d) It follows from the CW-Complex Construction Lemma 68.32 that the quotient $\Sigma(X)/A = S(X,x_0)$ admits a natural CW-structure which has the convenient property that the projection $\Sigma(X) \to \Sigma(X)/A = S(X,x_0)$ is cellular and which has the property that s_{x_0} is a 0-cell.

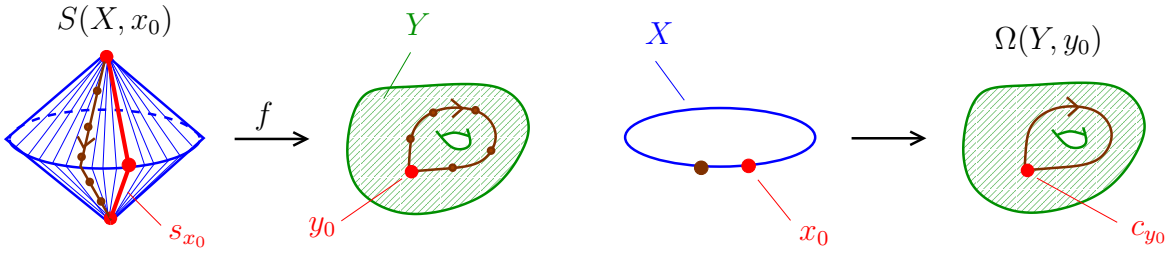
   By the Subcomplex Cofibration Theorem 17.9 we know that the inclusion map $i\colon A \to X$ is a cofibration. Since $[(x_0,t)]$ is a deformation retract of A we obtain from the Cofibration-Quotient Lemma 17.7 (3) that the natural projection $\Sigma(X) \to S(X,x_0)$ is a homotopy equivalence of the pointed topological spaces $(\Sigma(X),[(x_0,t)]) \to (S(X,x_0),s_{x_0})$. ■

The following innocuous looking proposition is one of the main building stones for the subsequent discussion.

Proposition 164.13. Let (X,x_0) and (Y,y_0) be pointed topological spaces. The map

$$\begin{array}{rcl} \Upsilon\colon \langle (S(X,x_0),s_{x_0}),(Y,y_0)\rangle & \to & \langle (X,x_0),(\Omega(Y,y_0),c_{y_0})\rangle \\ \left[\begin{array}{rcl} f\colon S(X,x_0) & \to & Y \\ [(x,t)] & \mapsto & f([(x,t)]) \end{array}\right] & \mapsto & \left[\begin{array}{rcl} X & \to & \Omega(Y,y_0) \\ x & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & Y \\ t & \mapsto & f([(x,2t-1)]) \end{array}\right) \end{array}\right] \end{array}$$

is a bijection. Furthermore this bijection is natural in (X,x_0) and (Y,y_0).[2027]



Remark. Recall that we denote by *PHomTop* the pointed homotopy category that we introduced on page 3468. In the language of the definition on page 3292 the proposition says

[2027]Put somewhat more precisely: if we are given a pointed topological space (Y,y_0), then the two maps $(X,x_0) \mapsto \langle (S(X,x_0),s_{x_0}),(Y,y_0)\rangle$ and $(X,x_0) \mapsto \langle (X,x_0),(\Omega(Y,y_0),c_{y_0})\rangle$ define contravariant functors from the category of pointed topological spaces to the category of sets. The map Υ defines a natural isomorphism between these two functors. The analogous statement holds with the roles of (X,x_0) and (Y,y_0) reversed.

that the functors

$$\begin{array}{rcl} F\colon \mathcal{PHomTop} & \to & \mathcal{PHomTop} \\ (X,x_0) & \mapsto & S(X,x_0) \\ [f] & \mapsto & [S(f)] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G\colon \mathcal{PHomTop} & \to & \mathcal{PHomTop} \\ (X,x_0) & \mapsto & \Omega(X,x_0) \\ [f] & \mapsto & [\Omega(f)] \end{array}$$

which are defined by

are adjoint.

Proof. Let (X,x_0) and (Y,y_0) be pointed topological spaces. First we consider the map

$$\begin{array}{ccc} \overbrace{\{\text{set of maps } X\times[-1,1]\to Y\}}^{=Y^{X\times[-1,1]}} & \to & \overbrace{\{\text{set of maps } X\to Y^{[0,1]}\}}^{=(Y^{[0,1]})^X} \\ \left(\begin{array}{rcl} f\colon X\times[-1,1] & \to & Y \\ (x,t) & \mapsto & f(x,t) \end{array}\right) & \mapsto & \left(\begin{array}{rcl} X & \to & Y^{[0,1]} \\ x & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & Y \\ t & \mapsto & f(x,2t-1) \end{array}\right) \end{array}\right). \end{array}$$

It follows from the Compact-Open Exponential Proposition 12.6 and the fact that $[0,1]\cong[-1,1]$ is regionally compact that this map is well-defined[2028] and that it is a bijection. Using Lemma 5.18 it is not hard to see that the given map induces a bijection

$$\{\text{set of maps } (S(X,x_0),s_{x_0})\to(Y,y_0)\} \;\to\; \{\text{set of maps } (X,x_0)\to(\Omega(Y,y_0),c_{y_0})\}.$$

Basically the same approach also deals with homotopies. More precisely, it follows from the Compact-Open Exponential Proposition 12.6 and the fact that $[0,1]\cong[-1,1]$ is regionally compact that the "obvious" map

$$Y^{[-1,1]\times(X\times[0,1])} \;\to\; \left(Y^{[0,1]}\right)^{X\times[0,1]}$$

is a bijection. Using the Homotopy Quotient Theorem 14.7 one can now show that this bijection induces a bijection between homotopies of maps $(S(X,x_0),s_{x_0})\to(Y,y_0)$ and homotopies of maps $(X,x_0)\to(\Omega(Y,y_0),c_{y_0})$. In summary this shows that the map

$$\Upsilon\colon \langle(S(X,x_0),s_{x_0}),(Y,y_0)\rangle \;\to\; \langle(X,x_0),(\Omega(Y,y_0),c_{y_0})\rangle$$

is indeed a bijection. We leave it to the reader to fill in the details.

Finally we point out that it follows immediately from the definitions that Υ is natural in (X,x_0) and (Y,y_0). ∎

164.3. **Eilenberg-Maclane spaces give rise to invariants of topological spaces.** Ideally we prefer to work, for as long as possible, with CW-complexes. Alas not all interesting constructions of new topological spaces necessarily turn CW-complexes into CW-complexes. For example, if (X,x_0) is a pointed CW-complex it is not clear whether the loop space $\Omega(X,x_0)$ is again a CW-complex.[2029]
This leads us to the following definition.

[2028] Here "well-defined" means that the map $X\to Y^{[0,1]}$ that we associate to $f\colon X\times[-1,1]\to Y$ is indeed continuous.

[2029] I could not find an example where $\Omega(X,x_0)$ does not admit a CW-structure and I could not find a theorem that says that $\Omega(X,x_0)$ always admits a CW-structure.

Notation. We denote by $\mathcal{W}$ the class of all pointed topological spaces (X, x_0) such that there exists a pointed CW-complex (Y, y_0) which is homotopy equivalent to (Y, y_0).

The following theorem was first proved by Milnor in 1958.

Theorem 164.14. If $(X, x_0) \in \mathcal{W}$, then for any $k \in \mathbb{N}$ the iterated loop space $\Omega^k(X, x_0)$ also lies in $\mathcal{W}$.

Proof. If (X, x_0) is a pointed CW-complex, then [**Miln1959a**, Corollary 3] implies that the pointed topological space $(\Omega(X, x_0), c_{x_0})$ lies in $\mathcal{W}$. (Alternatively we refer to [**FrPi1990a**, Corollary 5.3.7] for a proof of a very similar statement.) More generally, if $(X, x_0) \in \mathcal{W}$, then it follows from the Compact-Open Functoriality Lemma 12.5 (5) and the above that $(\Omega(X, x_0), c_{x_0})$ again lies in $\mathcal{W}$. Iterating this argument we also obtain the statement for every $k \geq 2$. ■

This leads us to the following definition:

Definition. Let $n \in \mathbb{N}$ and let G be an (abelian) group[2030].

(1) A weak pointed Eilenberg-Maclane space of type $\mathrm{K}(G, n)$ is a pair $(X, x_0) \in \mathcal{W}$ such that for any $i \neq n$ we have $\pi_i(X, x_0) = 0$ and such that $\pi_n(X, x_0) \cong G$.
(2) A polarized Eilenberg-Maclane space[2037] of type $\mathrm{K}(G, n)$ is a triple (X, x_0, α) where (X, x_0) is a weak pointed Eilenberg-Maclane space of type $\mathrm{K}(G, n)$ and α is an isomorphism $\pi_n(X, x_0) \xrightarrow{\cong} G$.

Examples.

(1) We equip S^1 with the base point $* = (1, 0)$. Let $\varphi\colon \pi_1(S^1, *) \xrightarrow{\cong} \mathbb{Z}$ be the standard isomorphism from page 1573. We see that $(S^1, *, \varphi)$ is a polarized Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, 1)$.
(2) We equip $\mathbb{C}\mathrm{P}^\infty$ with the base point $* = [1 : 0 : 0 \ldots]$. Let $\psi\colon \pi_2(\mathbb{C}\mathrm{P}^\infty, *) \xrightarrow{\cong} \mathbb{Z}$ be the usual isomorphism from Theorem 155.12 (2). We see that $(\mathbb{C}\mathrm{P}^\infty, *, \psi)$ is a polarized Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, 2)$.

Now we turn to more general constructions of polarized Eilenberg-Maclane spaces.

Examples. Let $n \in \mathbb{N}$ and let G be an (abelian) group.

(1) We denote by $\mathrm{K}(G, n)$ the canonical Eilenberg-Maclane space of type $\mathrm{K}(G, n)$ that was introduced in the Eilenberg-Maclane Existence Theorem 163.8. We denote its canonical base point by $*$. Furthermore we recall that the canonical Eilenberg-Maclane space $\mathrm{K}(G, n)$ comes with a canonical isomorphism $\varphi_G\colon \pi_n(\mathrm{K}(G, n), *) \xrightarrow{\varphi} G$.

[2036] As explained in Footnote 2009, whenever we deal with Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, n)$ with $n \geq 2$ it is understood that π is abelian.
[2037] Our language now allows for the slightly awkward possibility that a polarized Eilenberg-Maclane space is not an Eilenberg-Maclane space since the underlying topological space might not be a CW-complex. We are confident though that this will not become an issue.

(2) Let (X, x_0) be a pointed topological space. By the Loop Space-Homotopy Groups Corollary 154.15 we know that for any $k \in \mathbb{N}$ the map

$$\begin{array}{rcl} \partial_{k+1} \colon \pi_{k+1}(X, x_0) & \to & \pi_k(\Omega(X, x_0), c_{x_0}) \\ \left[[0,1]^{k+1} \xrightarrow{f} X\right] & \mapsto & \left[\begin{array}{rcl} [0,1]^k & \to & \Omega(X, x_0) \\ x & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & X \\ t & \mapsto & f(x,t) \end{array}\right) \end{array}\right] \end{array}$$

is a natural isomorphism. This observation, together with Theorem 164.14, shows that if (X, x_0, α) is a polarized Eilenberg-Maclane space of type $\mathrm{K}(G, n+1)$, then $(\Omega(X, x_0), c_{x_0}, \alpha \circ \partial_{n+1}^{-1})$ is a polarized Eilenberg-Maclane space of type $\mathrm{K}(G, n)$.

The discussion above shows that given any $k \in \mathbb{N}_0$ we can and will view $\Omega^k(\mathrm{K}(G, n+k))$ as a polarized Eilenberg-Maclane of type $\mathrm{K}(G, n)$.

The following lemma shows that in the based homotopy category all polarized Eilenberg-Maclane spaces of a given type are essentially the same.

Lemma 164.15. Let (K, k_0, φ) and (L, l_0, ψ) be two polarized Eilenberg-Maclane spaces of the same type. There exists a unique class[2038] $\Xi = \Xi_{(K,k_0),(L,l_0)} \in \langle (K, k_0), (L, l_0) \rangle_{\mathrm{inv}}$ such that the induced map $\Xi_* \colon \pi_n(K, k_0) \to \pi_n(L, l_0)$ agrees with $\psi^{-1} \circ \varphi$. When there is no danger of confusion we just write $\Xi = \Xi_{(K,k_0),(L,l_0)}$.

Proof. Let $n \in \mathbb{N}$, let G be an (abelian) group and let (K, k_0, φ) and (L, l_0, ψ) be two polarized Eilenberg-Maclane spaces of type $\mathrm{K}(G, n)$. Since (K, k_0) and (L, l_0) lie in $\mathcal{W}$ we know that there exist pointed CW-complexes (X, x_0) and (Y, y_0) together with homotopy equivalences $\kappa \colon (X, x_0) \to (K, k_0)$ and $\lambda \colon (Y, y_0) \to (L, l_0)$ of pointed topological spaces. We consider the following diagram

$$\begin{array}{ccc} \langle (K, k_0), (L, l_0) \rangle_{\mathrm{inv}} & \xrightarrow{[f] \mapsto f_*} & \mathrm{Hom}(\pi_n(K, k_0), \pi_n(L, l_0)) \\ \downarrow{\scriptstyle \kappa^*} & & \downarrow{\scriptstyle \kappa^*} \\ \langle (X, x_0), (L, l_0) \rangle_{\mathrm{inv}} & \xrightarrow{[f] \mapsto f_*} & \mathrm{Hom}(\pi_n(X, x_0), \pi_n(L, l_0)) \\ \downarrow{\scriptstyle \lambda_*} & & \downarrow{\scriptstyle \lambda_*} \\ \langle (X, x_0), (Y, y_0) \rangle_{\mathrm{inv}} & \xrightarrow{[f] \mapsto f_*} & \mathrm{Hom}(\pi_n(X, x_0), \pi_n(Y, y_0)). \end{array}$$

It follows almost immediately from the definitions that this diagram commutes. We obtain from Lemmas 164.6 and 3.4 that the vertical maps to the left are bijections. Furthermore we obtain from the Homotopy-π_n-Proposition 71.7 that the vertical maps to the right are bijections. It follows from the Eilenberg-Maclane Uniqueness Theorem 163.12 and the Maps-to-$\mathrm{K}(\pi, n)$-Proposition 163.15 that any isomorphism in $\mathrm{Hom}(\pi_n(X, x_0), \pi_n(Y, y_0))$ is realized by a unique element in $\langle (X, x_0), (Y, y_0) \rangle_{\mathrm{inv}}$. By the diagram this information propagates to the top. ■

Notation. Let $n \in \mathbb{N}$, let G be an abelian group and let $k \in \mathbb{N}_0$. From the discussion on page 3478 and Lemma 164.15 we obtain a canonical element $\Xi \in \langle \mathrm{K}(G, n), \Omega^k \mathrm{K}(G, n+$

[2038] To lessen the burden on the reader's eyes we do not include φ and ψ in the notation.

$k)\rangle_{\text{inv}}$. By Lemma 3.4 we know that the map

$$\begin{array}{rcl} \Xi_*\colon \langle (X,x_0),\ \overbrace{\mathrm{K}(G,n)}^{\text{viewed as a pointed topological space}}\ \rangle & \to & \langle (X,x_0),\ \overbrace{\Omega^k\,\mathrm{K}(G,n+k)}^{\text{viewed as a pointed topological space}}\ \rangle \\ [g] & \mapsto & \Xi\circ[g] \end{array}$$

is a bijection. Sometimes we use the bijection Ξ_* to identify the left-hand side with the right-hand side. We make the analogous identification if (X,x_0) is the target instead of the domain.

We formulate the next example as a lemma. Since we will not directly make use of it we take a few liberties in its formulation.

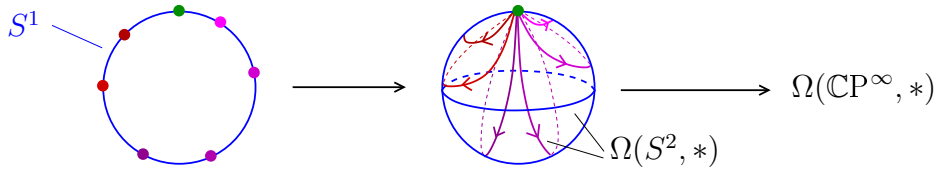
Lemma 164.16. We equip S^1 with the base point $*=(1,0)$ and we equip $\mathbb{C}\mathrm{P}^\infty$ with the base point $[1:0:0\ldots]$. As on page 3477 we consider the polarized Eilenberg-Maclane spaces $(S^1,*,\varphi)$ of type $\mathrm{K}(\mathbb{Z},1)$ and we consider the polarized Eilenberg-Maclane space $(\mathbb{C}\mathrm{P}^\infty,*,\psi)$ of type $\mathrm{K}(\mathbb{Z},2)$. We have the following homeomorphisms and inclusions maps

$$\begin{array}{ccccccccc} [0,1]^2/\partial[0,1]^2 & \overset{\cong}{\to} & \overline{B}^2/S^1 & \overset{\cong}{\to} & S^2 & \overset{\cong}{\to} & \mathbb{C}\mathrm{P}^1 & \hookrightarrow & \mathbb{C}\mathrm{P}^\infty. \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ & & \text{page 158} & & \text{Lemma 5.20} & & \text{page 258} & & \text{page 1481} \end{array}$$

We use these maps to view $[0,1]^2/\partial([0,1]^2)$ as a subset of $\mathbb{C}\mathrm{P}^\infty$.[2039] The map

$$\begin{array}{rcl} \Xi\colon S^1 & \to & \Omega(\mathbb{C}\mathrm{P}^\infty,*) \\ \exp(\mathrm{i}s) & \mapsto & \left(\begin{array}{ccc} [0,1] & \to & [0,1]^2/\partial([0,1]^2)\subset\mathbb{C}\mathrm{P}^\infty \\ t & \mapsto & [(s,t)] \end{array}\right) \end{array}$$

represents the class $\Xi\in\langle(S^1,*),\Omega(\mathbb{C}\mathrm{P}^\infty,*)\rangle$ given by Lemma 164.15.



Proof. Let $i\colon [0,1]^2/\partial([0,1]^2)\to\mathbb{C}\mathrm{P}^\infty$ be the above inclusion map. It follows immediately from the definition of $\Xi\colon S^1\to\Omega(\mathbb{C}\mathrm{P}^\infty,*)$ that the map Ξ factors through a map $\eta\colon S^1\to\Omega([0,1]^2/\partial([0,1]^2),*)$. We consider the following diagram:

$$\begin{array}{ccc} \pi_2([0,1]^2/\partial([0,1]^2),*) & \xrightarrow[\cong]{\partial_2} & \pi_1(\Omega([0,1]^2/\partial([0,1]^2),*)) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \Omega(i)_*} \\ \pi_2(\mathbb{C}\mathrm{P}^\infty,*) & \xrightarrow[\cong]{\partial_2} & \pi_1(\Omega(\mathbb{C}\mathrm{P}^\infty,*)). \end{array} \qquad \begin{array}{c} \xleftarrow{\eta_*} \\ \pi_1(S^1,*) \\ \xrightarrow{\Xi_*} \end{array}$$

We make the following observations:

(1) The horizontal maps are defined in the Loop Space-Homotopy Groups Corollary 154.15. Furthermore note that the Loop Space-Homotopy Groups Corollary 154.15 tells us that the horizontal maps are isomorphisms and that the square commutes.

[2039] Note that by the definition of the above maps the point $\partial([0,1]^2)$ corresponds to the point $*=[1\colon 0\colon 0\colon\ldots]\in\mathbb{C}\mathrm{P}^\infty$.

(2) The triangle commutes since $\Xi = \Omega(i) \circ \eta$.
(3) The left-hand vertical map is an isomorphism by Corollary 85.6.
(4) By Corollary 85.6 we know that $\pi_2([0,1]^2/\partial([0,1]^2), *) = \mathbb{Z} \cdot \mathrm{id}_{[0,1]^2}$ and by Proposition 48.17 (1) we know that $\pi_1(S^1, *) = \mathbb{Z} \cdot \mathrm{id}_{[0,1]}$.
(5) It follows easily from the definition of ∂_2, see the Loop Space-Homotopy Groups Corollary 154.15, that ∂_2 and η_* send their respective generators to the same element in $\pi_1(\Omega([0,1]^2/\partial([0,1]^2), *))$. Since all groups are isomorphic to $\mathbb{Z}$ and since ∂_2 is an isomorphism we see that η_* is an isomorphism.
(6) We obtain from the above data that Ξ_* is an isomorphism. It is now basically clear that Ξ represents the desired element in $\langle (S^1, *), \Omega(\mathbb{C}\mathrm{P}^\infty, *) \rangle$. ■

So far it seems like we have been rambling. But the following amazing definition puts us finally on an interesting track.

Definition. Let $n \in \mathbb{N}$ and let G be an abelian group. Furthermore let $X = (X, x_0)$ be a pointed topological space. To simplify the notation we suppress the base point x_0 throughout the subsequent discussion.[2040] We refer to the following map as the multiplication map on $\langle X, \mathrm{K}(G,n) \rangle$:

$$\begin{array}{ccc}
\langle X, \mathrm{K}(G,n)\rangle \times \langle X, \mathrm{K}(G,n)\rangle & \overset{\text{since } G \text{ is abelian the Eilenberg-Maclane space } \Omega\,\mathrm{K}(G,n+1) \text{ exists for every } n \text{, thus we can use the identification } \Xi_* \text{ from page 3479}}{=} & \langle X, \Omega\,\mathrm{K}(G,n+1)\rangle \times \langle X, \Omega\,\mathrm{K}(G,n+1)\rangle \\
& & \downarrow \text{map } \chi \text{ from Lemma 164.7} \\
& & \langle X, \Omega\,\mathrm{K}(G,n+1) \times \Omega\,\mathrm{K}(G,n+1)\rangle \\
& & \downarrow \mu_* \text{ defined on page 3470} \\
& & \langle X, \Omega\,\mathrm{K}(G,n+1)\rangle \\
& & \overset{\text{again using the identification } \Xi_* \text{ from page 3479}}{=} \langle X, \mathrm{K}(G,n)\rangle.
\end{array}$$

Remark.

(1) It is worth stressing what we just we did. In the definition of the multiplication on $\langle (X, x_0), \mathrm{K}(G,n) \rangle$ we used two essential facts:
   (a) We could replace $\mathrm{K}(G,n)$ by the loop space $\Omega(\mathrm{K}(G,n+1))$.
   (b) The loop space $\Omega(\mathrm{K}(G,n+1))$ comes with a multiplication.
   Later on we will use these two ideas in a more general context.
(2) It is perhaps worth pointing out that in general topological spaces are *not* even homotopy equivalent to loop spaces. For example we know by Lemma 156.22 together with Theorem 156.21, the discussion on page 3337 and Exercise 156.8 that S^n with $n \neq 1, 3, 7$ and $\mathbb{C}\mathrm{P}^n$ with $n \in \mathbb{N}$ are *not* homotopy equivalent to loop spaces.

The following proposition says that the above definition is actually meaningful.

Proposition 164.17. Let $n \in \mathbb{N}$, let G be an abelian group and let (X, x_0) be a pointed topological space. The above multiplication turns the set $\langle (X, x_0), \mathrm{K}(G,n) \rangle$ into a group.

[2040] It is though essential that all maps we consider are in fact base point preserving. In particular, recall that $\mathrm{K}(G,n)$ and $\Omega(\mathrm{K}(G,n))$ are pointed topological spaces.

Furthermore the neutral element of the group $\langle (X,x_0), \mathrm{K}(G,n)\rangle$ is given, not surprisingly, by 0.[2041]

Proof. Let $n \in \mathbb{N}$, let G be an abelian group and let (X,x_0) be a pointed topological space. Recall that we have the identification

$$\Xi_* \colon \langle (X,x_0), \mathrm{K}(G,n)\rangle \;=\; \langle (X,x_0), \Omega\,\mathrm{K}(G,n+1)\rangle$$

from page 3479. Note that $\Xi_*(0) = 0$. Thus the proposition follows from the next lemma. ∎

Lemma 164.18. Let (K,k_0) be a pointed topological space. The map

$$\begin{array}{c} \langle (X,x_0), \Omega(K,k_0)\rangle \times \langle (X,x_0), \Omega(K,k_0)\rangle \\ \big\downarrow \text{map } \chi \text{ from Lemma 164.7} \\ \langle (X,x_0), \Omega(K,k_0) \times \Omega(K,k_0)\rangle \\ \big\downarrow \mu_* \\ \langle (X,x_0), \Omega(K,k_0)\rangle. \end{array}$$

defines a group structure. The group structure has the following properties:

(1) The neutral element is given by 0.

(2) We consider the map[2042]

$$\begin{array}{rcl} \eta \colon \Omega(K,k_0) & \to & \Omega(K,k_0) \\ f & \mapsto & \overline{f}. \end{array}$$

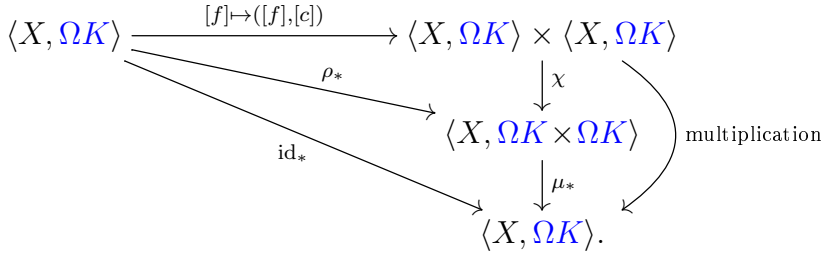
The inverse of $[f] \in \langle (X,x_0), \Omega(K,k_0)\rangle$ is given by $\eta_*([f])$.

Proof. As we will see shortly, this statement follows fairly easily from Lemma 164.10. In particular we will see that the statements have preciously little to do with (X,x_0). In an effort to lighten the notation we henceforth write X instead of the more accurate (X,x_0) and similarly we write K instead of (K,k_0). This time we denote by $c \in \Omega K$ the constant path at $k_0 \in K$. We consider the maps

$$\begin{array}{rcl} \rho \colon \Omega K & \to & \Omega K \times \Omega K \\ f & \mapsto & (f,c) \end{array} \qquad \begin{array}{rcl} \lambda \colon \Omega K & \to & \Omega K \times \Omega K \\ f & \mapsto & (c,f) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \sigma \colon \Omega K & \to & \Omega K \times \Omega K \\ f & \mapsto & (f,\overline{f}). \end{array}$$

Now we turn to the actual proofs of the various statements.

(1) Let us first show that $0 = [c]$ is a neutral element in $\langle X, \Omega K\rangle$. We consider the following diagram



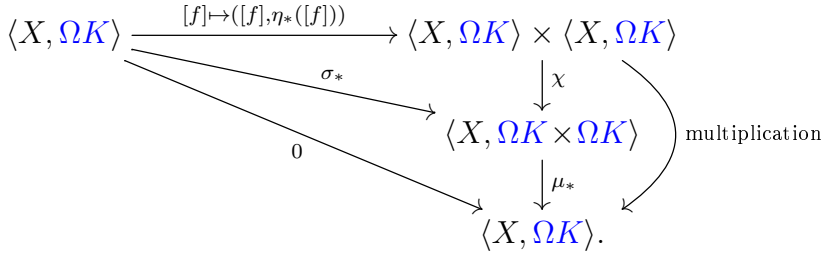
[2041] Recall that on page 3467 we defined $0 \in \langle (X,x_0), (Y,y_0)\rangle$ to be the class that is represented by the constant map $X \to Y$, $x \mapsto y_0$.

[2042] In Lemma 164.10 (3) we showed that η is continuous.

The upper triangle commutes basically by definition of the maps. Now let us turn to the lower triangle. It follows from Lemma 164.10 (2) that the map $\mu \circ \rho \colon \Omega K \to \Omega K$ is homotopic to the identity rel the base point. In other words, we have the equality $[\mu \circ \rho] = [\mathrm{id}] \in \langle \Omega K, \Omega K \rangle$. Thus it follows, if you want from Lemma 3.6, that the triangle does indeed commute.

Now we start out with $[f] \in \langle X, \Omega K \rangle$ at the top left. Since going right-down is precisely right-multiplication by $[c] = 0$ we see that 0 is indeed a right-neutral element with respect to the product structure. Basically the same argument, with ρ replaced by λ shows that 0 is also a left-neutral element.

(2) We consider the following diagram:



As in (1) we obtain from Lemma 164.10 (4) that the diagram commutes. It follows that $\eta_*([f])$ is a right-inverse to $[f]$. The same argument shows that it is also a left-inverse to $[f]$.

(3) Using Lemma 164.10 (5) one can show, with the same approach as above, that the product structure on $\langle X, \Omega K \rangle$ is associative. Filling in the details is the content of Exercise 164.3. ■

The reader of the previous proof might ask the following question:

Question 164.19. Why do we actually consider the sets $\langle (X, x_0), \mathrm{K}(G, n) \rangle$ instead of the sets $\langle (X, x_0), \Omega\, \mathrm{K}(G, n+1) \rangle$? After all, we just showed that the group structure is really defined on the latter set.

The group structure on $\langle (X, x_0), \mathrm{K}(G, n) \rangle$ is certainly rather mysterious. The following lemma hopefully elucidates the group structure in the case $\mathrm{K}(\mathbb{Z}, 1)$.

Lemma 164.20. Let $\Xi \colon S^1 \to \Omega(\mathbb{C}\mathrm{P}^\infty, *)$ be the homotopy equivalence defined in Lemma 164.16. Given any topological space X we have the following commutative diagram where the horizontal maps are bijections:

$$\begin{array}{ccc} [X, S^1] \times [X, S^1] & \xrightarrow{\Xi_* \times \Xi_*} & [X, \Omega\mathbb{C}\mathrm{P}^\infty] \times [X, \Omega\mathbb{C}\mathrm{P}^\infty] \\ \downarrow \nu & & \downarrow \mu_* \\ [X, S^1] & \xrightarrow{\Xi_*} & [X, \Omega\mathbb{C}\mathrm{P}^\infty]. \end{array}$$

Here the left vertical map is given by Lemma 164.4.

Proof. The statement of the lemma has nothing to do with X. In fact, it suffices to show that the following map agrees up to homotopy:

$$\begin{array}{ccc} S^1 \times S^1 & \xrightarrow{\Xi\times\Xi} & (\Omega\mathbb{C}\mathrm{P}^\infty, *) \times (\Omega\mathbb{C}\mathrm{P}^\infty, *) \\ \downarrow{\scriptstyle (z,w)\mapsto z\cdot w} & & \downarrow{\scriptstyle \mu} \\ S^1 & \xrightarrow{\Xi} & \Omega(\mathbb{C}\mathrm{P}^\infty, *). \end{array}$$

We outsource this verification to Exercise 164.1. ■

Perhaps the first question that arises when we introduce a group is whether the group abelian. It turns out that in this case we can answer this question in the affirmative.

Proposition 164.21. Let $n \in \mathbb{N}$ and let G be an abelian group. For every pointed topological space (X, x_0) the group $\langle (X, x_0), \mathrm{K}(G, n)\rangle$ is abelian.

Proof. Let $n \in \mathbb{N}$ and let G be an abelian group. We write $L = \mathrm{K}(G, n+1)$ and we write $M = \mathrm{K}(G, n+2)$. Finally let (X, x_0) be a pointed topological space. We make the following preparations.

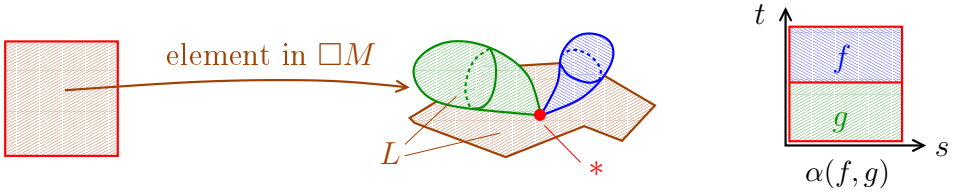
(1) By the discussion on page 3478 we know that $\mathrm{K}(G, n)$, ΩL and $\Omega(\Omega M)$ are polarized Eilenberg-Maclane spaces of type $\mathrm{K}(G, n)$. Thus we obtain from Lemma 164.15 homotopy equivalences $f\colon (\mathrm{K}(G, n), *) \to (\Omega L, *)$ and $g\colon (\Omega L, *) \to (\Omega(\Omega M), *)$.

(2) We define
$$\square M \;:=\; \{f\colon [0,1]^2 \to M \mid f(\partial[0,1]^2) = \{*\}\}.$$
As always we equip $\square M$ with the compact-open topology. Furthermore let
$$\begin{array}{rcl} \Theta\colon \square M & \to & \Omega(\Omega M) \\ (f\colon [0,1]^2 \to M) & \mapsto & (t \mapsto (s \mapsto f(s,t))) \end{array}$$
be the *homeomorphism* from the Iterated Loop Space Proposition 164.11.

(3) We consider the map
$$\begin{array}{rcl} \alpha\colon \square M \times \square M & \to & \square M \\ (f, g) & \mapsto & \left(\begin{array}{rcl} [0,1]^2 & \to & M \\ (s,t) & \mapsto & \begin{cases} f(s, 2t), & \text{if } t \in [0, \frac{1}{2}], \\ g(s, 2t-1), & \text{if } t \in (\frac{1}{2}, 1] \end{cases} \end{array} \right). \end{array}$$
The same argument as in the proof of Lemma 156.22 (1) shows that α is continuous.

(4) Since the pointed topological space (X, x_0) is really just a bystander we will abbreviate it to X.

Some of the above objects are illustrated in the figure below.



Claim. The following diagram commutes:

$$\begin{array}{ccccc}
\langle X, \mathrm{K}(G,n)\rangle \times \langle X, \mathrm{K}(G,n)\rangle & & \xrightarrow{\text{group multiplication}} & & \langle X, \mathrm{K}(G,n)\rangle \\
\downarrow{\scriptstyle (\Theta^{-1}\circ g\circ f)_* \times (\Theta^{-1}\circ g\circ f)_*} & & & & \downarrow{\scriptstyle (\Theta^{-1}\circ g\circ f)_*} \\
\langle X, \square M\rangle \times \langle X, \square M\rangle & \xrightarrow{\chi} & \langle X, \square M \times \square M\rangle & \xrightarrow{\alpha_*} & \langle X, \square M\rangle.
\end{array}$$

Furthermore the vertical maps are bijections.

Proof. To show that the above diagram commutes we expand it into an even larger diagram. More precisely, we consider the following diagram:

$$\begin{array}{ccccc}
\langle X, \mathrm{K}(G,n)\rangle \times \langle X, \mathrm{K}(G,n)\rangle & & \xrightarrow{\text{group multiplication}} & & \langle X, \mathrm{K}(G,n)\rangle \\
\downarrow{\scriptstyle f_* \times f_*} & & & & \downarrow{\scriptstyle f_*} \\
\langle X, \Omega L\rangle \times \langle X, \Omega L\rangle & \xrightarrow{\chi} & \langle X, \Omega L \times \Omega L\rangle & \xrightarrow{\mu_*} & \langle X, \Omega L\rangle \\
\downarrow{\scriptstyle g_* \times g_*} & & \downarrow{\scriptstyle (g_* \times g_*)_*} & & \downarrow{\scriptstyle g_*} \\
\langle X, \Omega(\Omega M)\rangle \times \langle X, \Omega(\Omega M)\rangle & \xrightarrow{\chi} & \langle X, \Omega(\Omega M) \times \Omega(\Omega M)\rangle & \xrightarrow{\mu_*} & \langle X, \Omega(\Omega M)\rangle \\
\downarrow{\scriptstyle \Theta_*^{-1} \times \Theta_*^{-1}} & & \downarrow{\scriptstyle (\Theta_*^{-1} \times \Theta_*^{-1})_*} & & \downarrow{\scriptstyle \Theta_*^{-1}} \\
\langle X, \square M\rangle \times \langle X, \square M\rangle & \xrightarrow{\chi} & \langle X, \square M \times \square M\rangle & \xrightarrow{\alpha_*} & \langle X, \square M\rangle.
\end{array}$$

We make the following observations:

(1) By Lemma 3.4 we know that the vertical maps between the first and second and between the second and the third row are bijections.
(2) Since Θ is a homeomorphism we see that the bottom vertical maps are also bijections.
(3) By definition of the group multiplication on $\langle X, \mathrm{K}(G,n)\rangle$ the top rectangle commutes.
(4) It follows immediately from the definition of χ that the two smaller rectangles on the left commute.
(5) It follows from Lemma 164.10 (1) that the middle right rectangle commutes.
(6) It follows immediately from the definition of the maps μ, α and Θ that

$$\alpha \circ (\Theta \times \Theta) \;=\; \Theta \circ \mu \colon \Omega(\Omega M) \times \Omega(\Omega M) \;\to\; \square M.$$

This equality implies that the lower right rectangle commutes. ⊞

The claim can be summarized as saying that it suffices to show that the bottom multiplication map on $\langle X, \square M\rangle$ is commutative. Next we consider the swap map

$$\begin{array}{rcl}
\sigma \colon \square M \times \square M & \to & \square M \times \square M \\
(f, g) & \mapsto & (g, f).
\end{array}$$

We obtain the following diagram:

$$\begin{array}{ccccc}
\langle X, \square M\rangle \times \langle X, \square M\rangle & \xrightarrow{\chi} & \langle X, \square M \times \square M\rangle & \xrightarrow{\alpha_*} & \langle X, \square M\rangle \\
\downarrow{\scriptstyle (a,b)\mapsto(b,a)} & & \downarrow{\scriptstyle \sigma_*} & & \downarrow{\scriptstyle \mathrm{id}} \\
\langle X, \square M\rangle \times \langle X, \square M\rangle & \xrightarrow{\chi} & \langle X, \square M \times \square M\rangle & \xrightarrow{\alpha_*} & \langle X, \square M\rangle
\end{array}$$

We make the following observations:

(1) The statement that the multiplication on $\langle X, \square M\rangle$ is commutative is, by definition, equivalent to the statement that the large rectangle commutes.

(2) It follows immediately from the definitions of the various maps that the smaller rectangle on the left commutes. Thus it suffices to show that the smaller rectangle on the right commutes.

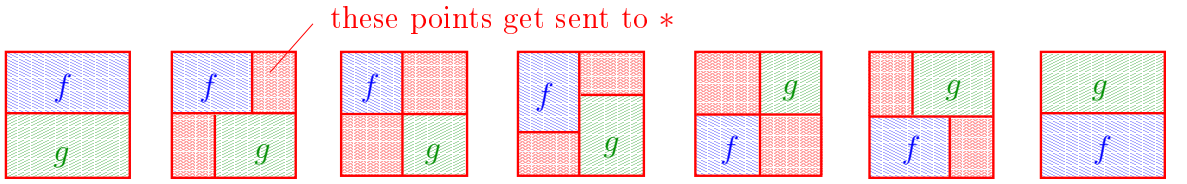
In particular, since $\alpha_* \circ \sigma_* = (\alpha \circ \sigma)_*$ it suffices to prove the following claim.[2043]

Claim. The maps α and $\alpha \circ \sigma$ from the pointed topological space $\square M \times \square M$ to the pointed topological space $\square M$ are homotopic.

Proof. In the figure below we indicate how one can define a homotopy from α to $\alpha \circ \sigma$. We leave it to the punctilious reader to write down a precise formula for a homotopy[2044]

$$(\square M \times \square M) \times [0,1] \ \to \ \square M.$$

As in the proof of Lemma 164.10 one can use the "⇒"-direction of the Compact-Open Exponential Proposition 12.6 to show that the map provided by the reader is actually continuous. ■



Remark. As the reader with a well-developed long-term memory will have noticed, the proof of Proposition 164.21 eventually is very similar to the proof of the Homotopy Group-Abelian Proposition 71.3. In particular the above figure looks almost the same as the figure on page 1564.

164.4. **Alternative description of the group structure.** Once again let $n \in \mathbb{N}$, let G be an abelian group and let (X, x_0) be a pointed topological space. In this section we give an alternative description of the group structure on

$$\langle (X,x_0), \mathrm{K}(G,n) \rangle \underset{\uparrow}{\overset{\Xi}{\to}} \langle (X,x_0), \Omega\,\mathrm{K}(G,n+1) \rangle \underset{\uparrow}{\overset{\Upsilon}{\to}} \langle S(X,x_0), \mathrm{K}(G,n+1) \rangle.$$

bijection from page 3478 — bijection from Proposition 164.13

Before we state the relevant proposition we recall some of the objects that we have introduced recently:

(1) On page 3474 we introduced the pinching map

$$\omega \colon S(X,x_0) \ \to \ S(X,x_0) \vee S(X,x_0)$$

where $S(X,x_0)$ denotes the reduced suspension.

(2) In Lemmas 164.7 and 164.8 we introduced the maps

$$\begin{array}{rcl} \chi \colon \langle (X,x_0), (Y,y_0) \rangle \times \langle (X,x_0), (Z,z_0) \rangle & \to & \langle (X,x_0), (Y \times Z, (y_0,z_0)) \rangle \\ ([f \colon X \to Y], [g \colon X \to Z]) & \mapsto & [f \times g \colon X \to Y \times Z] \end{array}$$

and

[2043] Thus once again X plays no role in the argument.

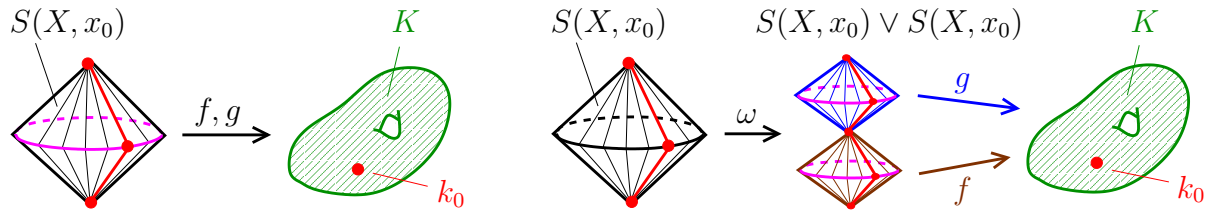
[2044] For example this could be done using the maps introduced in the proof of Lemma 71.2.

$$\begin{array}{rcl} \nu\colon \langle (X,x_0),(Z,z_0)\rangle \times \langle (Y,y_0),(Z,z_0)\rangle & \to & \langle (X,x_0)\vee(Y,y_0),(Z,z_0)\rangle \\ ([f\colon X\to Z],[g\colon Y\to Z]) & \mapsto & [f\vee g] \end{array}$$

The following proposition shows that instead of the map μ on the loop space we can also use the pinching map on the reduced suspension to define the group structure.

Proposition 164.22. Let (X,x_0) and (K,k_0) be pointed topological spaces. The following diagram commutes:

$$\begin{array}{ccc} \langle (X,x_0),\Omega(K,k_0)\rangle \times \langle (X,x_0),\Omega(K,k_0)\rangle & \xrightarrow{\Upsilon\times\Upsilon} & \langle S(X,x_0),(K,k_0)\rangle \times \langle S(X,x_0),(K,k_0)\rangle \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \nu} \\ \langle (X,x_0),\Omega(K,k_0)\times\Omega(K,k_0)\rangle & & \langle S(X,x_0)\vee S(X,x_0),(K,k_0)\rangle \\ \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \omega^*} \\ \langle (X,x_0),\Omega(K,k_0)\rangle & \xrightarrow{\qquad\Upsilon\qquad} & \langle S(X,x_0),(K,k_0)\rangle. \end{array}$$



Remark. The multiplication map on $\langle S(X,x_0),(K,k_0)\rangle$ is arguably more geometric surely more appealing to some readers. Nonetheless, any attempt to prove that this multiplication map defines an (abelian) group structure promises to be notationally somewhat messy.

Proof. Let $f,g\colon (X,x_0)\to\Omega(K,k_0)$ be two maps. One can easily verify that

$$(\Upsilon\circ\mu_*\circ\chi)([f],[g]) = \left[\begin{array}{rcl} S(X,x_0) & \to & (K,k_0) \\ [(x,t)] & \mapsto & \begin{cases} f(t+1), & \text{if } t\in[-1,0], \\ g(t), & \text{if } t\in[0,1] \end{cases} \end{array}\right] = (\omega^*\circ\nu)(\Upsilon([f]),\Upsilon([g])).$$

This shows that the diagram does indeed commute. ∎

164.5. **Functoriality.** Let $n\in\mathbb{N}$. In this section we will see, perhaps not surprisingly, that the abelian groups $\langle (X,x_0),\mathrm{K}(G,n)\rangle$ are functorial in (X,x_0) and G. To make this precise we need to introduce the following piece of notation.

Definition. Let $n\in\mathbb{N}$ and let $\varphi\colon G\to H$ be a homomorphism between two (abelian) groups. We denote by $[\varphi]\in\langle \mathrm{K}(G,n),\mathrm{K}(H,n)\rangle$ the canonical homotopy class of maps defined in the Maps-to-K(π,n)-Proposition 163.15.[2045]

We have the following lemma.

[2045]More precisely, by the Maps-to-K(π,n)-Proposition 163.15 there exists a unique $[f]\in\langle \mathrm{K}(G,n),\mathrm{K}(H,n)\rangle$ such that the map $f_*\colon G=\pi_n(\mathrm{K}(G,n))\to\pi_n(\mathrm{K}(H,n))=H$ agrees with φ. Here we used that the canonical Eilenberg-Maclane spaces $\mathrm{K}(J,n)$ are pointed and that they come with an identification $\pi_n(\mathrm{K}(J,n))=J$.

Lemma 164.23. Let $n \in \mathbb{N}$. The maps

$$\begin{array}{rcl} G & \mapsto & \mathrm{K}(G,n) \\ (\varphi\colon G \to H) & \mapsto & [\varphi] \in \langle \mathrm{K}(G,n), \mathrm{K}(H,n)\rangle \end{array}$$

define a covariant functor from the category of (abelian) groups to the pointed homotopy category.

Proof. The fact that the maps define a covariant functor is an immediate consequence of the uniqueness statement in the Maps-to-K(π,n)-Proposition 163.15. ■

If one thinks about this lemma for a second, then one realizes that this lemma is actually quite astounding. So far we spent almost all our time trying to define functors from topological spaces to (abelian) groups in the hope of being able to use algebra to understand topological spaces. But now we have functors which go the other way around. In principle this will allow us to use tools from topology to gain a better understanding of groups. In fact we already saw a glimpse of this phenomenon in Proposition 163.22. Unfortunately at this very moment we cannot develop this theme. But in Chapter **??** this idea will come back with a vengeance.

After this short digression we can formulate the following proposition.

Proposition 164.24. Let $n \in \mathbb{N}$.

(1) Let G be an abelian group. The maps

$$\begin{array}{rcl} (X,x_0) & \mapsto & \langle (X,x_0), \mathrm{K}(G,n)\rangle \\ (f\colon (X,x_0) \to (Y,y_0)) & \mapsto & \left(\begin{array}{rcl} \langle (Y,y_0), \mathrm{K}(G,n)\rangle & \to & \langle (X,x_0), \mathrm{K}(G,n)\rangle \\ [\gamma] & \mapsto & [\gamma \circ f] \end{array} \right) \end{array}$$

define a contravariant functor from the category of pointed topological spaces to the category of abelian groups.

(2) Let (X,x_0) be a pointed topological space. The maps

$$\begin{array}{rcl} G & \mapsto & \langle (X,x_0), \mathrm{K}(G,n)\rangle \\ (\varphi\colon G \to H) & \mapsto & \left(\begin{array}{rcl} \langle (X,x_0), \mathrm{K}(G,n)\rangle & \to & \langle (X,x_0), \mathrm{K}(H,n)\rangle \\ [\gamma] & \mapsto & [\varphi \circ \gamma] \end{array} \right) \end{array}$$

define a covariant functor from the category of abelian groups to the category of abelian groups.

Proof. Let $n \in \mathbb{N}$.

(1) We fix an abelian group G. First note that it follows from Lemma 3.7 that the assignment $(X,x_0) \to \langle (X,x_0), \mathrm{K}(G,n)\rangle$ is a contravariant functor from the category of pointed topological spaces to the category of sets. Thus it remains to show that this assignment defines actually a functor to the category of abelian groups. More precisely, it remains to show that for any map $f \in \langle (X,x_0),(Y,y_0)\rangle$ the induced map $\langle (Y,y_0), \mathrm{K}(G,n)\rangle \to \langle (X,x_0), \mathrm{K}(G,n)\rangle$ is actually a group homomorphism. But that is basically obvious since the above definition of the group structure on $\langle (X,x_0), \mathrm{K}(G,n)\rangle$ has nothing to do with X. We leave it to the skeptical reader to fill in the details.

(2) Let (X, x_0) be a pointed topological space. By Lemmas 3.6 and Lemma 164.23 we know that the assignment $G \mapsto \langle (X, x_0), \mathrm{K}(G, n)\rangle$ defines a covariant functor from the category of abelian groups to the category of sets. As in (1) it remains to show that the assignment actually defines a functor to the category of abelian groups. This requires a little more effort than (1). The argument breaks up into two parts, namely we have to deal with the "horizontal" and the "vertical" aspects of the definition on page 3480.

(a) Recall that in the Loop Space-Homotopy Groups Corollary 154.15 we gave an explicit *natural* isomorphism

$$\alpha_k \colon \pi_k(K, k_0) \xrightarrow{\cong} \pi_{k-1}(\Omega(K, k_0), c_{x_0}) \qquad \text{for every } k \in \mathbb{N}.$$

It follows from the naturality of these isomorphisms, and the precise definition of the maps Ξ on page 3478, that for a given homomorphism $\varphi \colon G \to H$ of (abelian) groups we obtain a commutative diagram

$$\begin{array}{ccc} \langle (X, x_0), \mathrm{K}(G, n)\rangle & \xrightarrow[\Xi_*]{=} & \langle (X, x_0), \Omega\,\mathrm{K}(G, n+1)\rangle \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \langle (X, x_0), \mathrm{K}(H, n)\rangle & \xrightarrow[\Xi_*]{=} & \langle (X, x_0), \Omega\,\mathrm{K}(H, n+1)\rangle. \end{array}$$

This takes care of the "horizontal" part.

(b) It follows from Lemma 164.7, Lemma 164.10 (1) and Lemma 164.23 that the "vertical" part of the definition on page 3480 is functorial in φ. ■

As the reader will know by now, base points are something of a nuisance. Fortunately in Proposition 70.9 we already saw that on many occasions we can stop worrying about base points, the following proposition nicely complements Proposition 70.9.

Proposition 164.25. Let (X, x_0) and (Y, y_0) be two pointed CW-complexes. If Y is path-connected, if $\pi_1(Y, y_0)$ is abelian and if $\pi_i(Y, y_0) = 0$ for $i \geq 2$,[2046] then the natural map $\langle (X, x_0), (Y, y_0)\rangle \to [X, Y]$ is also a bijection.

Remark. We equip S^1, S^2 and $S^1 \vee S^2$ with the usual base point $*$. In Exercise 164.2 we will show that the map $[(S^2, *), (S^1 \vee S^2, *)] \to [S^2, S^1 \vee S^2]$ is not an injection. This shows that in Proposition 164.25 we cannot just drop the hypothesis that $\pi_i(Y, y_0) = 0$ for $i \geq 2$.

Proof. Let (Y, y_0) be a pointed path-connected CW-complex such that the fundamental group $\pi_1(Y, y_0)$ is abelian and such that $\pi_i(Y, y_0) = 0$ for $i \geq 2$. By Proposition 70.9 (1) it remains to show that the map $\langle (X, x_0), (Y, y_0)\rangle \to [X, Y]$ is injective. Despite initial appearances, the argument will be quite different from the argument provided in the proof of Proposition 70.9 (1) (2). So let $f_0, f_1 \colon (X, x_0) \to (Y, y_0)$ be two maps that are homotopic as maps $X \to Y$. By the Homotopy-π_1-Proposition 50.3 (1) we know that there exists a $\gamma \in \pi_1(Y, y_0)$ such that $f_{0*}(g) = \gamma \cdot f_{1*}(g) \cdot \gamma^{-1} \in \pi_1(Y, y_0)$ for every $g \in \pi_1(X, x_0)$. But since $\pi_1(Y, y_0)$ is abelian we see that $f_{0*} = f_{1*}$. Since by hypothesis we have $\pi_i(Y, y_0) = 0$ for $i \geq 2$ we can appeal to the Maps-to-K(π, n)-Proposition 163.15 (2) which now shows that there exists a homotopy from f_0 to f_1 rel x_0. ■

[2046] In other words, (Y, y_0) is a pointed Eilenberg-Maclane space of type $\mathrm{K}(\pi, 1)$ where $\pi \cong \pi_1(Y, y_0)$ is an abelian group.

In the following discussion we will use Proposition 164.24 (1) to dispose of the base points.

Definition. Let $n \in \mathbb{N}$ and let G be an abelian group.

(1) Let X be a connected CW-complex. We pick a point $x_0 \in X^0$. By Propositions 70.9 and 164.25[2047] the natural map

$$\begin{array}{rccc} \Psi_{(X,x_0)} \colon & \langle (X,x_0), (\mathrm{K}(G,n),*)\rangle & \to & [X,\mathrm{K}(G,n)] \\ & [f] & \mapsto & [f] \end{array}$$

is a bijection. We use this bijection, together with the group structure on the left-hand side defined in Proposition 164.17, to turn the right-hand side $[X,\mathrm{K}(G,n)]$ into a group. In Corollary 164.26 below we will see that this definition does not depend on the choice of x_0.

(2) Now let X be any CW-complex. Let $\{X_i\}_{i\in I}$ be the set of its components. We have

$$[X,\mathrm{K}(G,n)] \underset{\uparrow}{=} \prod_{i\in I} [X_i,\mathrm{K}(G,n)]$$

by the Topological-Coproduct Proposition 5.11 (1)

We define the group structure on each $[X_i,\mathrm{K}(G,n)]$ as in (1) and for $i \neq j$ we define the multiplication on $[X_i,\mathrm{K}(G,n)] \times [X_j,\mathrm{K}(G,n)]$ to be the trivial map. One easily verifies that this defines an abelian group structure on $[X,\mathrm{K}(G,n)]$.

The following corollary to Proposition 164.24 (1) shows that this definition of the group structure on $[X,\mathrm{K}(G,n)]$ does not depend on the choice of the base point $x_0 \in X^0$.

Corollary 164.26. Let $n \in \mathbb{N}$, let G be an abelian group and let X be a connected CW-complex. The definition of the group structure on $[X,\mathrm{K}(G,n)]$ does not depend on the choice of $x_0 \in X^0$.

Proof. Let $n \in \mathbb{N}$, let G be an abelian group and let X be a connected CW-complex. Now let $x_0, x_1 \in X$ be two points in the 0-skeleton of X. Since X is connected we know by Proposition 70.12 that there exists a map $f \colon X \to X$ with the following properties:

(a) The map f is homotopic to the identity.
(b) We have $f(x_0) = x_1$.
(c) The map f defines a homotopy equivalence $(X,x_0) \to (X,x_1)$ of pointed topological spaces.

We consider the following diagram

$$\begin{array}{ccc} \langle (X,x_1), (\mathrm{K}(G,n),*)\rangle & \xrightarrow[[g]\mapsto[g\circ f]]{f^*} & \langle (X,x_0), (\mathrm{K}(G,n),*)\rangle \\ \Big\downarrow{\scriptstyle \Psi_{(X,x_1)}}\ {\scriptstyle [g]\mapsto[g]} & & \Big\downarrow{\scriptstyle \Psi_{(X,x_0)}}\ {\scriptstyle [g]\mapsto[g]} \\ [X,\mathrm{K}(G,n)] & \xrightarrow[[g]\mapsto[g\circ f]]{f^*} & [X,\mathrm{K}(G,n)]. \end{array}$$

We make the following observations:

(1) It follows basically immediately from the definitions that the diagram commutes.
(2) It follows from (c) and Lemmas 3.7 and 3.10 that the top horizontal map is a bijection.
(3) By Proposition 164.24 (1) we know that top horizontal map is in fact an isomorphism.
(4) By (a) we know that the bottom map is in fact the identity.

[2047] Note that here we use that G is an abelian group, even for $n = 1$.

It follows immediately from the above that it does not matter whether we use $\Psi_{(X,x_1)}$ or $\Psi_{(X,x_0)}$ to define the group structure on $[X, \mathrm{K}(G,n)]$. ■

164.6. **Basic calculations.** In this section we want to do a few bare-hands calculation of our new invariant $\langle (X,x_0), \mathrm{K}(G,n)\rangle$. Before we start with the calculations it is useful to introduce the following notation.

Notation. Let $n \in \mathbb{N}$ and let G be an abelian group. Furthermore let (X,x_0) be a pointed topological space. For the purpose of this section we write

$$h^n((X,x_0);G) \;:=\; \langle (X,x_0), \mathrm{K}(G,n)\rangle.$$

Before we state the second main calculation it is helpful to introduce the following convention.

Convention. Let G be an abelian group. We make the identifications

$$\mathrm{H}_1(\mathrm{K}(G,1);\mathbb{Z}) \underset{\uparrow}{=} \pi_1(\mathrm{K}(G,1),*)_{\mathrm{ab}} \underset{\uparrow}{=} G_{\mathrm{ab}} \underset{\uparrow}{=} G.$$

Hurewicz Theorem 84.5 — by the Eilenberg-Maclane Existence Theorem 163.8 — since G is abelian

Lemma 164.27. Let G be an abelian group. Let X be a connected CW-complex and let $x_0 \in X^0$ be a point. The map

$$\begin{array}{rcl} h^1((X,x_0);G) & \to & \mathrm{Hom}(\mathrm{H}_1(X;\mathbb{Z}),G) \\ [f\colon (X,x_0) \to (\mathrm{K}(G,1),*)] & \to & (f_*\colon \mathrm{H}_1(X;\mathbb{Z}) \to \mathrm{H}_1(\mathrm{K}(G,1);\mathbb{Z}) = G) \end{array}$$

is a natural isomorphism.

Proof.

(1) It follows from the Maps-to-K(π,n)-Proposition 163.15 and the Hurewicz Theorem 84.5 that the given map is a bijection.
(2) It follows immediately from the definitions that the map is natural in X.
(3) It remains to show that the given map is a homomorphism. We write $Y = \mathrm{K}(G,2)$. We make the usual identification $\langle X, \mathrm{K}(G,1)\rangle = \langle X, \Omega Y\rangle$. Furthermore we use the natural isomorphism

$$\begin{array}{rcl} \mathrm{Hom}(\mathrm{H}_1(X;\mathbb{Z}),G) & \to & \mathrm{Hom}(\pi_1(X,x_0),\pi_1(\Omega Y)) \\ \varphi & \mapsto & (\pi_1(X,x_0) \to \pi_1(X,x_0)_{\mathrm{ab}} = \mathrm{H}_1(X;\mathbb{Z}) \xrightarrow{\varphi} G = \pi_1(\Omega Y)) \end{array}$$

to identify the group on the left with the group on the right. Next we consider the following diagram

$$\begin{array}{ccc} \langle X,\Omega Y\rangle \times \langle X,\Omega Y\rangle & \xrightarrow{([f],[g])\mapsto(f_*,g_*)} & \mathrm{Hom}(\pi_1(X),\pi_1(\Omega Y)) \times \mathrm{Hom}(\pi_1(X),\pi_1(\Omega Y)) \\ \downarrow{\scriptstyle ([f],[g])\mapsto[f\times g]} & & \downarrow{\scriptstyle (\varphi,\psi)\mapsto(g\mapsto(\varphi(g),\psi(g)))} \\ \langle X,\Omega Y\times\Omega Y\rangle & \longrightarrow & \mathrm{Hom}(\pi_1(X),\pi_1(\Omega Y)\times\pi_1(\Omega Y)) \\ \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \mu_*} \\ \langle X,\Omega Y\rangle & \longrightarrow & \mathrm{Hom}(\pi_1(X),\pi_1(\Omega Y)) \end{array}$$

We make the following clarifications and observations:
(a) The upper rectangle commutes basically by definition.
(b) The lower square also commutes basically by definition. Slightly more precisely, let $[f] \in \langle X, \Omega Y \times \Omega Y \rangle$. We consider the maps $X \xrightarrow{f} \Omega Y \times \Omega Y \xrightarrow{\mu} \Omega Y$. By functoriality we have $(\mu \circ f)_* = \mu_* \circ f_*$. But this equality just means that in the lower square it does not matter whether we take the down-right route or the right-down route.
(c) It follows from Lemma 156.28 that the right vertical maps give precisely the group multiplication on $\mathrm{Hom}(\pi_1(X), \pi_1(\Omega Y))$.

This shows that the given map is indeed a group homomorphism. ∎

164.7. **Product structures.** Let (X, x_0) be a pointed CW-complex and let $m, n \in \mathbb{N}$. Our goal in this section is to define a potentially interesting map

$$\langle (X, x_0), \mathrm{K}(\mathbb{Z}, m) \rangle \times \langle (X, x_0), \mathrm{K}(\mathbb{Z}, n) \rangle \ \to \ \langle (X, x_0), \mathrm{K}(\mathbb{Z}, m+n) \rangle.$$

The key to defining such a map is the following humble construction.

Definition. Given two pointed topological spaces (X, x_0) and (Y, y_0) the smash product is defined as

$$X \wedge Y \ := \ (X \times Y)/(X \vee Y) \quad \text{where we use Exercise 50.6 to make the identification } X \vee Y := (X \times \{y_0\}) \times (\{x_0\} \times Y).$$

We equip $X \wedge Y$ with the base point given by $[X \vee Y]$.

Remark. As we discussed in Exercise 68.12 and on page 1502, in the category of pointed topological spaces the smash product is actually the categorical product.

In Exercise 50.6 we already showed that $S^m \wedge S^n$ is homeomorphic to S^{m+n}. For our purpose it is useful to have a canonical homeomorphism that we can all agree upon. This is given in the following lemma.

Lemma 164.28. Given $m, n \in \mathbb{N}$ there exists a canonical homeomorphism

$$S^m \wedge S^n \ \xrightarrow{\cong} \ S^{m+n}.$$

Convention. We will use the canonical homeomorphism from Lemma 164.28 to make the identification $S^m \wedge S^n = S^{m+n}$.

Proof. We write $I = [0, 1]$. Given any $k \in \mathbb{N}_0$ we have the following identifications of pointed topological spaces:

$$(I^k/\partial I^k, [\partial I^k]) \ \underset{\uparrow}{\xrightarrow{\cong}} \ (\overline{B}^k/S^{k-1}, [S^{k-1}]) \ \underset{\uparrow}{\xrightarrow{\cong}} \ (S^k, *).$$

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Let $m, n \in \mathbb{N}_0$. Using the above identification we obtain the following homeomorphism

homeomorphism by the Twice Quotient Lemma 5.23 (1)

$$S^m \wedge S^n \xrightarrow{\cong} I^m/\partial I^m \wedge I^n/\partial I^n \underset{[([x],[y])] \mapsto [(x,y)]}{\overset{\downarrow \cong}{\to}} (I^{m+n})/(I^m \times \partial I^n \cup \partial I^m \times I^n) = I^{m+n}/\partial I^{m+n} \xleftarrow{\cong} S^{m+n}.$$

We have thus given an explicit homeomorphism between $S^m \wedge S^n$ and S^{m+n}. ∎

The following lemma is basically just a mild reformulation of Lemma 163.11.

Lemma 164.29. Let $k \in \mathbb{N}$. There exists a polarized Eilenberg-Maclane space $(Y_k, *, \varphi)$ of type $\mathrm{K}(\mathbb{Z}, k)$ with the following properties:

(1) The k-skeleton of Y_k equals S^k with the usual CW-structure defined on page 3431.
(2) The CW-complex Y_k has no cell of dimension $k+1$.
(3) The CW-complex Y_k has only countably many cells in dimensions $\geq k+2$.

Now we turn to the actual definition of the product structure.

Definition. Let $m, n \in \mathbb{N}$. In the following discussion all topological spaces are viewed as pointed objects.

(1) We consider the polarized Eilenberg-Maclane spaces Y_m and Y_n that we introduced in Lemma 164.29. Recall that from Lemma 164.15 we obtain canonical elements $\Xi_m \in \langle \mathrm{K}(\mathbb{Z}, m), Y_m \rangle_{\mathrm{inv}}$ and $\Xi_n \in \langle \mathrm{K}(\mathbb{Z}, n), Y_n \rangle_{\mathrm{inv}}$.
(2) Since Y_m and Y_n are CW-complexes with *countably* many cells we can equip the product $Y_m \times Y_n$ with the CW-structure defined in the CW-Complex Construction Lemma 68.32.
(3) We denote by $p\colon Y_m \times Y_n \to Y_m \wedge Y_n = (Y_m \times Y_n)/\sim$ the obvious projection.
(4) It follows from the explicit description of the CW-structure defined in the CW-Complex Construction Lemma 68.32 that $Y_m \wedge Y_n = (Y_m \times Y_n)/\sim$ has precisely one cell in dimension 0, no cells in dimension $1, \ldots, m+n-1$, one cell in dimension $m+n$ and no cell in dimension $m+n+1$. Furthermore it follows immediately from the definitions that the map
$$S^{m+n} \;=\; S^m \wedge S^n \;\to\; Y_m \wedge Y_n$$
defines a homeomorphism onto the $(m+n+1)$-skeleton of $Y_m \wedge Y_n$. In particular we obtain from the CW-Skeleton-π_n-Proposition 71.9 that Y_{m+n} is $(m+n-1)$-connected and we obtain that the map $\varphi\colon \mathbb{Z} = \pi_{m+n}(S^{m+n}) \to \pi_{m+n}(Y_m \wedge Y_n)$ is an isomorphism.
(5) Since $Y_m \times Y_n$ is an $(m+n-1)$-connected CW-*complex* we obtain from the Maps-to-$\mathrm{K}(\pi, n)$-Proposition 163.15 a unique class $\mu \in \langle Y_m \wedge Y_n, \mathrm{K}(\mathbb{Z}, m+n) \rangle$ that induces the preferred isomorphism $\varphi^{-1}\colon \pi_{m+n}(Y_m \wedge Y_n, *) \xrightarrow{\cong} \mathbb{Z}$.

After these preparations we consider the following diagram

$$\begin{array}{ccccc}
\langle X, Y_m \rangle \times \langle X, Y_n \rangle & \xrightarrow{\times} & \langle X, Y_m \times Y_n \rangle & \xrightarrow{p_*} & \langle X, Y_m \wedge Y_n \rangle \\
{\scriptstyle \Xi_{m*} \times \Xi_{n*}} \uparrow & & & & \searrow^{\mu_*} \\
\langle X, \mathrm{K}(\mathbb{Z}, m) \rangle \times \langle X, \mathrm{K}(\mathbb{Z}, n) \rangle & & \xrightarrow{\text{definition of cup product } \cup} & & \langle X, \mathrm{K}(\mathbb{Z}, m+n) \rangle .
\end{array}$$

We refer to the bottom horizontal map, which we defined through the upper path of maps, as the cup product.

Theorem 164.30. Let $X = (X, x_0)$ be a pointed CW-complex and let $m, n \in \mathbb{N}$. The cup product
$$\cup\colon \langle X, \mathrm{K}(\mathbb{Z}, m) \rangle \times \langle X, \mathrm{K}(\mathbb{Z}, n) \rangle \;\to\; \langle X, \mathrm{K}(\mathbb{Z}, m+n) \rangle$$

is bilinear. Furthermore the cup product is associative and graded commutative in the sense that for $\varphi \in \langle X, \mathrm{K}(\mathbb{Z}, m)\rangle$ and $\psi \in \langle X, \mathrm{K}(\mathbb{Z}, n)\rangle$ we have

$$\varphi \cup \psi \;=\; (-1)^{m\cdot n} \cdot \psi \cup \varphi.$$

Proof. We will not make use of this theorem, thus we will also not provide a proof. Instead we refer to [**Stro2011**, Theorem 21.61] for the proof. Note that [**Stro2011**, Theorem 21.61] does not explicitly say that X needs to be a CW-complex, but implicitly the discussion on [**Stro2011**, Chapter 21] seems to assume that the setting is a convenient category in the sense of Theorems 68.28 and 68.30. We will discuss this issue in more detail right after this proof. ■

The above definition of the cup product is in a way rather awkward, it is esthetically wrong to introduce the CW-complexes Y_m and Y_n. We conclude this section with a longish but hopefully quite interesting discussion which explains why we introduced the CW-complexes Y_m and Y_n and how we can get around introducing them.

Definition.

(1) Let A and B be CW-complexes. We denote by $A \otimes B$ the product CW-complex that we defined on page 1500. Recall that the underlying set is just $A \times B$.

(2) Let (A, a_0) and (B, b_0) be pointed CW-complexes. We write

$$A \circledcirc B \;:=\; (A \otimes B)/(A \times \{b_0\} \cup \{a_0\} \times B).$$

We equip $A \circledcirc B$ with the CW-structure given by the CW-structure on $A \otimes B$ and the quotient structure defined in the CW-Complex Construction Lemma 68.32.

(3) We denote by $\eta\colon A \otimes B \to A \times B$ the identity map. Note that η is continuous by Proposition 68.26 (1).

Now let $X = (X, x_0)$ be a pointed topological space and let $m, n \in \mathbb{N}$. We introduce the following notation:

(1) Let $r \in \mathbb{N}$. We write $K_r = \mathrm{K}(\mathbb{Z}, r)$. We consider the polarized Eilenberg-Maclane spaces Y_r that we introduced in Lemma 164.29 and we denote by $\Xi_r \in \langle K_r, Y_r\rangle_{\mathrm{inv}}$ the canonical element given by Lemma 164.15.

(2) Given topological spaces A and B we denote by $p\colon A \times B \to A \wedge B$ the natural projection. Similarly, given CW-complexes A and B we denote by $p\colon A \otimes B \to A \circledcirc B$ the natural projection.

We consider the following diagram

$$\begin{array}{ccccccc}
\langle X, Y_m\rangle \times \langle X, Y_n\rangle & \xrightarrow{\times} & \langle X, Y_m \times Y_n\rangle & \xrightarrow{p_*} & \langle X, Y_m \wedge Y_n\rangle & & \\
\uparrow{\scriptstyle \Xi_{m*}\times\Xi_{n*}} & & & & & \searrow^{\mu_*} & \\
\langle X, K_m\rangle \times \langle X, K_n\rangle & \xrightarrow[(f,g)\mapsto f\times g]{\times} & \langle X, K_m \times K_n\rangle & \xrightarrow{p_*} & \langle X, K_m \wedge K_n\rangle & \xrightarrow{???} & \langle X, K_{m+n}\rangle. \\
\downarrow{\scriptstyle (f,g)\mapsto f\widetilde{\times}g} & & \uparrow{\scriptstyle \eta_*} & & & \nearrow_{\nu_*} & \\
\langle X, K_m \widetilde{\times} K_n\rangle & \xrightarrow[\Theta_*]{} & \langle X, K_m \otimes K_n\rangle & \xrightarrow{p_*} & \langle X, K_m \circledcirc K_n\rangle & &
\end{array}$$

We make the following observations:

(1) Since K_m and K_n have uncountably many cells the product $K_m \times K_n$ is not a CW-complex.

(2) It follows from (1) that $K_m \wedge K_n$ has no obvious CW-structure. Thus there is no clear path to constructing a map $K_m \wedge K_n \to K_{m+n}$.[2048]
(3) It follows from (2) that the "direttissima" approach to defining the cup product by going straight from left to right runs into a major obstacle.
(4) Since Y_m and Y_n have countably many cells we know, by Lemmas 68.32 and 68.32 that $Y_m \times Y_n$ and $Y_m \wedge Y_n$ are CW-complexes. As discussed above, we obtain from the Maps-to-K(π, n)-Proposition 163.15 a canonical class $\mu \in \langle Y_m \wedge Y_n, \mathrm{K}(\mathbb{Z}, m+n)\rangle$.
(5) One way of getting around (3) is to use (4), i.e. we take the "upper route" in the diagram. This is how in the above we defined the cup product.
(6) As for $Y_m \wedge Y_n$ one sees that $K_m \owedge K_n$ is $(m+n-1)$-connected and that we have a canonical map $\pi_{m+n}(K_m \owedge K_n) \to \mathbb{Z}$. Since $K_m \owedge K_n$ is a CW-complex we obtain from the Maps-to-K(π, n)-Proposition 163.15 a canonical element $\nu \in \langle K_m \owedge K_n, K_{m+n}\rangle$.
(7) Unfortunately the issue in (1) also means that the map $\eta\colon K_m \otimes K_n \to K_m \times K_n$ is not a homeomorphism. Thus there is no obvious way how we can take the lower route in the diagram.

Now we switch to the category of CGWH-spaces that we introduced on page 1504. In particular we now assume that X is a CGWH-space. Recall that by the CGWH-Category Theorem 68.30 (2) we know that every CW-complex is a CGWH-space.

(8) Given two CGWH-spaces A and B we know by the CGWH-Category Theorem 68.30 that there exists a product *in the category of* CGWH-*spaces*. As on page 1505 we denote this categorical product by $A \widetilde{\times} B$ and we denote by $p\colon A \widetilde{\times} B \to A$ and $q\colon A \widetilde{\times} B \to B$ the two projections maps.
(9) Let $f\colon X \to K_m$ and $g\colon X \to K_n$ be maps. Since X, K_m and K_n are CGWH-spaces we know, by definition of the product in a category, that there exists a unique map $f \widetilde{\times} g\colon X \to K_m \widetilde{\times} K_n$ such that $f = p \circ (f \widetilde{\times} g)$ and $g = q \circ (f \widetilde{\times} g)$.
(10) Since K_m and K_n are CW-complexes we know by the CGWH-Category Theorem 68.30 (5) that the identity map $\Theta\colon K_m \widetilde{\times} K_n \to K_m \otimes K_n$ is actually a homeomorphism.[2049]
(11) It follows from (9) and (10) that we have a "lower route" from $\langle X, K_m\rangle \times \langle X, K_n\rangle$ to $\langle X, K_{m+n}\rangle$. This gives us an approach to defining the cup product without introducing the slightly artificial Y_m and Y_n.
(12) It follows easily from the definitions that in the diagram the square to the lower left commutes. One can now see reasonably easily that the two definitions of the cup product agree.

The second approach to defining the cup product is much more natural than the initial approach. In particular, using a similar circle of ideas one can define a cup product

$$\langle X, \mathrm{K}(R, m)\rangle \times \langle X, \mathrm{K}(R, n)\rangle \to \langle X, \mathrm{K}(R, m+n)\rangle$$

for any commutative ring R. We refer to [**Stro2011**, Chapter 21.4] for details.

Exercises for Chapter 164.

[2048] In fact it is also not immediately clear whether $K_m \wedge K_n$ is say $(m+n-1)$-connected.
[2049] For our purpose we only need the weaker result that $\Theta\colon K_m \widetilde{\times} K_n \to K_m \otimes K_n$ is continuous.

Exercise 164.1. We consider the following diagram:

$$\begin{array}{ccc} S^1 \times S^1 & \xrightarrow{\Xi\times\Xi} & (\Omega\mathbb{C}\mathrm{P}^\infty, *) \times (\Omega\mathbb{C}\mathrm{P}^\infty, *) \\ \downarrow{\scriptstyle (z,w)\mapsto z\cdot w} & & \downarrow{\scriptstyle \mu} \\ S^1 & \xrightarrow{\Xi} & \Omega(\mathbb{C}\mathrm{P}^\infty, *). \end{array}$$

The map $\Xi\colon S^1 \to \Omega(\mathbb{C}\mathrm{P}^\infty, *)$ is defined in Lemma 164.16 and the map μ is defined on page 3470. Show that the diagram commutes up to homotopy.

Exercise 164.2. We equip S^1, S^2 and $S^1 \vee S^2$ with the usual base points which in each case we call $*$. Show that the map $\langle (S^2, *), (S^1 \vee S^2, *)\rangle \to [S^2, S^1 \vee S^2]$ is not an injection.

Exercise 164.3. Let (X, x_0) and (K, k_0) be pointed topological spaces. Show that the multiplication on $\langle (X, x_0), (K, k_0)\rangle$ defined in Lemma 164.18 is actually associative.

Exercise 164.4. Let (X, x_0) and $(K, *)$ be pointed topological spaces. We consider the double loop space $\Omega^2(K, *) = \Omega(\Omega(K, *))$. By the construction on page 3480 we have a binary operation

$$\circ\colon \langle (X, x_0), \Omega(\Omega(K, *))\rangle \times \langle (X, x_0), \Omega(\Omega(K, *))\rangle \to \langle (X, x_0), \Omega(\Omega(K, *))\rangle.$$

In Proposition 164.17 we showed in particular that this binary operation is unital. Provide another binary unital operation "$\otimes$" on $\langle (X, x_0), \Omega(\Omega(K, *))\rangle$ such that the conditions of the Eckmann-Hilton Theorem 156.27 are satisfied.
Remark. Once we have shown that we can apply the Eckmann-Hilton Theorem 156.27 we see that "$\circ$" defines a group structure on $\langle (X, x_0), \Omega^2(K, *)\rangle$ and we obtain that this group is abelian. Thus this provides an alternative approach to proving Proposition 164.21 and to proving some of the statements of Proposition 164.17.
Remark. Coming up with an idea for defining "$\otimes$" is not difficult. Providing a complete proof that Condition (b) of the Eckmann-Hilton Theorem 156.27 is satisfied might be more painful. Especially once one starts wondering why certain maps are actually continuous.

Exercise 164.5. Let $n \in \mathbb{N}$ and let G be an abelian group. Let $k \in \mathbb{N}$. We denote by $*$ the usual base point of S^k. Compute the groups $\langle (S^k, *), \mathrm{K}(G, n)\rangle$.
Remark. Once one unravels the definitions one can fairly easily write down *bijections* to familiar groups. The only catch is that one really needs to show that these bijections are in fact group isomorphisms. The Eckmann-Hilton Theorem 156.27 might once again ride to the rescue.

Exercise 164.6. Let (X, x_0) be a pointed topological space. We equip S^1 with the base point $* = (0, 1)$. Show that the smash product $X \wedge S^1$ is homeomorphic to the reduced suspension $S(X, x_0)$.
Remark. Personally I like the Twice Quotient Lemma 5.23 (1).

Exercise 164.7. To simplify the notation, given a pointed topological space (Y, y_0) we now write $SY = S(Y, y_0)$ and $\Omega Y = \Omega(Y, y_0)$.

(a) Let (Y, y_0) be a point topological space. Show that the set $\pi_0(\Omega Y)$ has a natural group structure.
(b) Let (X, x_0) be a point topological space.

(1) Show that there exists a natural isomorphism from $\pi_0(\Omega(SX))$ to the free group on the set $\pi_0(X)$.
(2) Let (X, x_0) be a point topological space. Show that there exists a natural isomorphism from $\pi_0(\Omega^2(S^2X))$ to the free *abelian* group on the set $\pi_0(X)$.
(3) How does the sequence continue?

Part XIX

(Co-) homology with twisted and local coefficients

The goal of this part of the lecture notes is to introduce and study (co-) homology with twisted coefficients and (co-) homology with local coefficients. In both settings we start out with a topological space that is equipped with extra data, namely either a module over the group ring of the fundamental group, or a local coefficient system.

The two theories are quite similar and in many settings they give isomorphic results. Even though in most settings the theories of (co-) homology with twisted coefficients and of (co-) homology with local coefficients are isomorphic both are of interest and relevant:

(1) (Co-) homology with twisted coefficients requires the choice of a base point. This makes it less flexible when it comes to considering maps between topological spaces. But the definition of (co-) homology with twisted coefficients is closer to the theory that we have developed so far and makes it easier to apply results that we have proved earlier. Furthermore there is a cellular version of (co-) homology with twisted coefficients that is very useful when it comes to doing actual calculations.

(2) (Co-) homology with local coefficients does not need the choice of a base point. It works particularly well when one wants to study maps that are induced by maps between topological spaces. But the definitions are further from our known concepts and it requires more work to make sure that all the definitions actually make sense.

We start out with a chapter on (homological) algebra over rings. We will apply these to the case where the ring is the group ring $\mathbb{Z}[\pi_1(X, x_0)]$ of the fundamental group of a based topological space (X, x_0). This allows us define in a fairly painless fashion (co-) homology with twisted coefficients. Afterwards we introduce the (co-) homology with local coefficients. The latter concept is more elegant, but it requires more of an effort to make it work and for most applications it suffices to work with (co-) homology with twisted coefficients.

165. Modules over rings

In the previous chapters we mostly worked with abelian groups or vector spaces. In the future chapters we will also work with modules over possibly non-commutative rings. To prepare the reader for these future chapters we will summarize the algebra involved in this and the following chapter.

We start out with recalling the following definitions from page 98.

Definition.

(1) A ring is a triple $(R, +\colon R \times R \to R,\ \cdot\ \colon R \times R \to R)$ where $(R, +)$ is an abelian group and such that the following conditions are satisfied:
   (a) the multiplication "$\cdot$" is associative,
   (b) the multiplication "$\cdot$" is distributive,
   (c) there exists a multiplicatively neutral element $1 = 1_R$ with $1 \neq 0$.

(2) A ring R is called commutative, if given any $r, s \in R$ we have $r \cdot s = s \cdot r$.

(3) A ring homomorphism between two rings $(R, +, \cdot)$ and $(S, +, \cdot)$ is a map $R \to S$ that preserves the additive and multiplicative structures and that sends 1_R to 1_S.

(4) A field is a commutative ring such that every non-zero element has a multiplicative inverse.

Throughout the lecture notes we developed various algebraic concepts over commutative rings. In the following chapters we will be interested in (possibly) non-commutative rings, the prime example being the group ring of a (possibly) non-abelian group.

Example. Let R be a ring and $n \in \mathbb{N}$. The matrices $\mathrm{M}(n \times n, R)$, with the usual addition and multiplication of square matrices form a ring. Furthermore, if $\varphi\colon R \to S$ is a ring homomorphism, then the induced map

$$\begin{array}{ccc} \mathrm{M}(n \times n, R) & \to & \mathrm{M}(n \times n, S) \\ (a_{ij}) & \mapsto & (\varphi(a_{ij})) \end{array}$$

is also a ring homomorphism.

Definition. Let R be a commutative ring (in most applications $R = \mathbb{Z}$) and let G be a group. We set

$$R[G] \ := \ \left\{ \sum_{i=1}^{m} r_i \cdot g_i \,\middle|\, r_1, \ldots, r_m \in R \text{ and } g_1, \ldots, g_m \in G \right\}.$$

We equip $R[G]$ with the multiplication that is given by[2050]

$$\left(\sum_{i=1}^{m} r_i \cdot g_i \right) \cdot \left(\sum_{j=1}^{n} s_j \cdot h_j \right) \ := \ \sum_{i=1}^{m} \sum_{j=1}^{n} \underbrace{r_i \cdot_R s_j}_{\in R} \cdot \underbrace{(g_i \cdot_G h_j)}_{\in G}.$$

We refer to $R[G]$ as the group ring of G with coefficients in R.

[2050] In Lemma 86.10 we showed that the group ring $R[G]$ is indeed a ring.

Convention. Let R be a commutative ring. If $G = \langle t \rangle$ is the infinite cyclic group generated by some symbol t, then we have a ring isomorphism

$$\begin{array}{rcl} R[\langle t \rangle] & \to & R[t^{\pm 1}] \\ r \cdot t^n & \mapsto & r \cdot t^n. \end{array}$$

We use this isomorphism to identify the group ring $R[\langle t \rangle]$ with the Laurent-polynomial ring $R[t^{\pm 1}]$, which is more familiar and easier on the eyes.

Definition. Given two rings R and S we denote by $R \times S$ the ring whose underlying set is the product $R \times S$ and where addition and multiplication is defined component wise.

165.1. Modules over rings.

Definition. Let R be a ring.

(1) An R-left module is an abelian group A together with a map

$$\begin{array}{rcl} R \times A & \to & A \\ (r, a) & \mapsto & r \cdot a \end{array}$$

such that the following conditions are satisfied:

(a)	$r \cdot (a+b)$	$=$	$r \cdot a + r \cdot b$	for all $r \in R$ and $a, b \in A$,
(b)	$1 \cdot a$	$=$	a	for all $a \in A$,
(c)	$(r+s) \cdot a$	$=$	$r \cdot a + s \cdot a$	for all $r, s \in R$ and $a \in A$,
(d)	$s \cdot (r \cdot a)$	$=$	$(s \cdot r) \cdot a$	for all $r, s \in R$ and $a \in A$.

(2) An R right-module is an abelian group A together with a map

$$\begin{array}{rcl} R \times A & \to & A \\ (r, a) & \mapsto & a \cdot r \end{array}$$

such that the following conditions are satisfied:

(a)	$(a+b) \cdot r$	$=$	$a \cdot r + b \cdot r$	for all $r \in R$ and $a, b \in A$,
(b)	$a \cdot 1$	$=$	a	for all $a \in A$,
(c)	$a \cdot (r+s)$	$=$	$a \cdot r + a \cdot s$	for all $r, s \in R$ and $a \in A$,
(d)	$(a \cdot r) \cdot s$	$=$	$a \cdot (r \cdot s)$	for all $r, s \in R$ and $a \in A$.

Remark.

(1) If one is not used to dealing with R-left modules and R right-modules, then it is perhaps difficult to spot the essential difference. The fact that we write multiplication by R on the left respectively right is not essential, it is just a mnemonic device. What matter is the different given by the (c). To make the difference clear we write it out in words:

Left R-module: multipl. by r followed by multipl. by s equals multipl. by $s \cdot r$.
Right R-module: multipl. by r followed by multipl. by s equals multipl. by $r \cdot s$.

In practice the notation is set up in such a way that it should be clear what is allowed and what is not.

(2) If R is a commutative ring, then any R-module in the usual sense satisfies both conditions (1d) and (2d). In other words, if R is a commutative ring, then any R-module is automatically an R-left module and also an R right-module.

Example.

(1) Let A be an abelian group. Given $n \in \mathbb{N}$ and $a \in A$ we define

$$n \cdot a := \underbrace{a + \cdots + a}_{n \text{ times}} \qquad \text{and} \qquad -n \cdot a := \underbrace{-a - \cdots - a}_{n \text{ times}}.$$

This defines $\mathbb{Z}$-left module and $\mathbb{Z}$-right module structure on A. Thus we can, and will, view any abelian group as a $\mathbb{Z}$-left module and a $\mathbb{Z}$-right module.

(2) Let R be a ring and $n \in \mathbb{N}$. The matrices $\mathrm{M}(n \times n, R)$, with the usual addition and multiplication of square matrices form a ring. The map

$$\begin{array}{ccc} \mathrm{M}(n \times n, R) \times R^n & \to & \mathrm{M}(n \times n, R) \\ (A, v) & \mapsto & A \cdot v \end{array}$$

defines a $\mathrm{M}(n \times n, R)$-left module structure on R^n.

In the remainder of these notes we will encounter a large number of left and right modules. As of right now we only give a few simple examples.

Many of our future objects have two "compatible" modules structures at the same time. This leads us to the following definition:

Definition. Let R and S be two rings.

(1) A (R, S)-bimodule is an abelian group A together with R-left module structure and an S-right module structure, such that for any $r \in R$, any $s \in S$ and any $a \in A$ we have $r \cdot (a \cdot s) = (r \cdot a) \cdot s$.

(2) A (R, S)-left left module is an abelian group A together with a R-left module structure and an S-left module structure, such that for any $r \in R$, any $s \in S$ and any $a \in A$ we have $r \cdot (s \cdot a) = s \cdot (r \cdot a)$. The same way we define an (R, S)-right right module.

Remark. Note that an (R, S)-left left module is essentially the same as an $(R \times S)$-left module.

Examples.

(1) If R is a commutative ring, then any R-left module is automatically also an (R, R)-bimodule and an (R, R)-left module.

(2) If R is a ring, then any R-left module is also automatically an $(R, \mathbb{Z})$-bimodule and an $(R, \mathbb{Z})$-left left module. This simple observation will get used frequently.

(3) Let R be a ring and let $k \in \mathbb{N}$. Using the usual left and right multiplication we can view R^k as an (R, R)-bimodule.

(4) Let $\varphi \colon R \to S$ be a ring homomorphism. The maps

$$\begin{array}{ccc} R \times S & \to & S \\ (r, s) & \mapsto & \varphi(r) \cdot s \end{array} \qquad \text{and} \qquad \begin{array}{ccc} S \times R & \to & S \\ (s, r) & \mapsto & s \cdot \varphi(r) \end{array}$$

turn S into an R-left module and into an R right-module, these satisfy the condition of a bimodule, thus we can view S as an (R, R)-bimodule. By using these maps and multiplication by S we can also view S as an (R, S)-bimodule and an (S, R)-bimodule.

Definition. Let $\varphi\colon R \to S$ be a homomorphism between two rings and let M be an S-left module. We denote by φ^*M the R-left module that is given by

$$\begin{array}{rcl} R \times M & \to & M \\ (r, m) & \mapsto & \varphi(r) \cdot M. \end{array}$$

We use analogous notation if we are dealing with right modules, bimodules, etc. Note that, with the obvious interpretation, we have $\mathrm{id}^* M = M$ and $(\varphi \circ \psi)^* M = \psi^*(\varphi^* M)$.

165.2. Homomorphisms between modules.

Notation. Let R be a ring. We denote by ${}_R\mathcal{Mod}$ the category of R-left modules and we denote by $\mathcal{Mod}_R$ the category of R-right modules.

In this very straightforward section we collect a few very elementary definitions and statements on homomorphisms between modules.

Notation. Let R be a ring. Given two R-left modules A and B we denote by $\mathrm{Hom}_{R\text{-left}}(A, B)$ the abelian group of all R-left module homomorphisms $A \to B$. Furthermore we denote by $\mathrm{Aut}_{R\text{-left}}(A)$ the abelian group of all R-left module automorphisms of R. We define $\mathrm{Hom}_{\text{right-}R}(A, B)$ and $\mathrm{Aut}_{\text{right-}R}(A)$ in the obvious way.

Lemma 165.1. Let R and S be rings, let M be an R-left module and let N be an (R, S)-left left module. Then

$$\begin{array}{rcl} S \times \mathrm{Hom}_{R\text{-left}}(M, N) & \to & \mathrm{Hom}_{R\text{-left}}(M, N) \\ (s, \varphi) & \mapsto & (m \mapsto s \cdot \varphi(m)) \end{array}$$

is a well-defined S-left module structure. The evident variations on this statement also hold. ■

Lemma 165.2. Let R be a ring, let L, M and N be R-left modules and let $f\colon L \to M$ and $g\colon M \to N$ be R-left module homomorphisms. The maps

$$\begin{array}{rcl} f^*\colon \mathrm{Hom}_{R\text{-left}}(M, N) & \to & \mathrm{Hom}_{R\text{-left}}(L, N) \\ \varphi & \mapsto & \varphi \circ f \end{array} \quad \text{and} \quad \begin{array}{rcl} g_*\colon \mathrm{Hom}_{R\text{-left}}(L, M) & \to & \mathrm{Hom}_{R\text{-left}}(L, N) \\ \varphi & \mapsto & f \circ \varphi \end{array}$$

are both well-defined and functorial in the obvious sense. ■

Lemma 165.3. Let $f\colon R \to S$ be a ring homomorphism. Let M and N be S-left modules. We consider the corresponding R-left modules f^*M and f^*N, as defined on page 3502. The map

$$\begin{array}{rcl} f_*\colon \mathrm{Hom}_{S\text{-left}}(M, N) & \to & \mathrm{Hom}_{R\text{-left}}(f^*M, f^*N) \\ \varphi & \mapsto & \varphi \end{array}$$

is well-defined and functorial in the obvious sense. ■

165.3. Rings with involution.

In this short section we introduce rings with involution. As is clear from the color of the section, this section consists only of definitions.

Definition. Let R be a ring.

(1) An involution on R is a map $R \to R$, usually written $z \mapsto \overline{z}$, with the following properties:
   (a) $\overline{z + w} = \overline{z} + \overline{w}$ for all $z, w \in R$,
   (b) $\overline{z \cdot w} = \overline{w} \cdot \overline{z}$ for all $z, w \in R$,

(c) $\overline{\overline{z}} = z$ for all $z \in R$.

(2) A homomorphism between rings with involution is a ring homomorphisms that respects the involutions.

Convention.

(1) Unless we say something else we will view the complex numbers a ring with the usual involution $\overline{x + \mathrm{i}y} := x - \mathrm{i}y$.

(2) Let R be a ring (possibly with involution). We will view the ring of Laurent polynomials $R[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ as a ring that is equipped with the involution that is given by the involution on R and $\overline{t_i} := t_i^{-1}$.

(3) Given a group π we will view the group ring $\mathbb{Z}[\pi]$ always as a ring with involution, where the involution is given by $\overline{g} := g^{-1}$.

Notation. Let R be a ring with involution (e.g. a group ring $\mathbb{Z}[\pi]$) and let M be an R-left module. We denote by $\overline{M}$ the R-right module that is given by $m \cdot_{\overline{M}} r := \overline{r} \cdot_M m$. We proceed in the analogous way if M is an R-right module.

Definition. Let R be a ring with involution, let M and N be R-left modules and let $f\colon M \to N$ be a map. We say that f is R-antilinear, if it is a homomorphism of abelian groups and if for any $m \in M$ and $r \in R$ we have $f(r \cdot m) = \overline{r} \cdot f(m)$.

165.4. **Free modules.** The following definition is modelled on the definition of a free abelian group that we gave on page 1148.

Definition. Let R be a ring.

(1) Let M be an R-left module. We say M is a free R-left module on a set $S \subset M$ if given any map $\phi\colon S \to N$ to an R-left module N there exists a unique homomorphism $\psi\colon M \to N$ of R-left modules that makes the following diagram commute

$$\begin{array}{ccc} S & \hookrightarrow & M \\ & \searrow^{\phi} & \downarrow{\scriptstyle \psi} \\ & & N. \end{array}$$

(2) We say an R-left module is free if there exists a subset $S \subset M$ such that R is a free R-left module on the set S. We refer to S also as a basis for M.

In a totally analogous fashion we also introduce the corresponding notions for R-right modules.

Definition. Let R be a ring. Given any set X we set

$$R^{(X)} := \text{ all maps from } X \text{ to } R \text{ which are non-zero for only finitely many } x \in X.$$

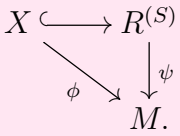
We equip $R^{(X)}$ with the obvious structure of an R-left module. We refer to $R^{(X)}$ as the free R-left module generated by X or as the free R-left module on the generating set X.

The following lemma justifies the above name.

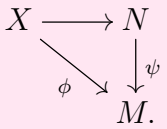
Lemma 165.4. Let R be a ring.

(1) Let X be a set and let $\phi\colon X \to M$ be a map to an R-left module M. There exists a unique homomorphism $\psi\colon R^{(X)} \to M$ of R-left modules that makes the following

diagram commute



(2) Conversely let N be an R-left module that admits a subset X with the following universal property: given any map $\phi\colon X \to M$ to an R-left module M there exists a unique R-left module homomorphism $\psi\colon N \to M$ that makes the following diagram commute



Then N is isomorphic to $R^{(X)}$.

Proof. The proof of the lemma is basically the same as the proof of the analogous Lemma 51.1. ■

Remark.

(1) It follows immediately from the definitions that a free abelian group is basically the same as a free $\mathbb{Z}$-module.
(2) In many ways free R left/right modules are quite similar to free abelian groups. There is one major difference though. By the Subgroup-of-Free Abelian Group Lemma 51.2 (1) we know that any submodule of a free $\mathbb{Z}$-module is itself a free $\mathbb{Z}$-module. The analogous statement is in general *not true* for free R-left modules. For example, consider the ring $R = \mathbb{Z}[x]$ of polynomials. We consider the ideal $I = (2, 1 - x)$ that is generated by 2 and $1 - x$. It is a submodule of $\mathbb{Z}[x]$, which is of course a free $\mathbb{Z}[x]$-module in its own right. In Exercise 165.1 we will show that I, viewed as a $\mathbb{Z}[x]$-module, is *not free*.
(3) In Section **??** we will study the question whether the cardinality of a basis of a free module is an invariant of the free module.

In the coming chapters we will work a lot with homomorphisms between free R-left modules. One of most popular bits in linear algebra is the fact that any homomorphism between vector spaces can be represented by a matrix. The same applies in the context of homomorphisms between free R-left modules, except that one is forced to multiply by matrices *on the right*. More precisely, we have the following somewhat confusing lemma.

Lemma 165.5. Let R be a ring.

(1) If A is a $(k \times l)$–matrix over R, then the map

$$\begin{array}{ccc} R^k & \to & R^l \\ v & \mapsto & v \cdot A \\ \uparrow & & \uparrow \\ \text{viewed as row vectors} & & \end{array}$$

is a homomorphism of R-left modules.
(2) Given any homomorphism $\varphi\colon R^k \to R^l$ of R-left modules there exists a $(k \times l)$–matrix over R such that φ is given by *right* multiplication by A.

Proof. Both statements are essentially clear. The reader who is unfamiliar with dealing with non-commutative rings should think for a second about the question, why left-multiplication by A is *not* a homomorphism of R-left modules. ■

On page 1715 we defined the notion of a splitting of a short exact sequence of (not necessarily abelian) groups. We now have the analogue of modules.

Definition. Let R be a ring. We say that a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

of R-left modules splits, if there exists a splitting $s\colon C \to B$, i.e. an R-left module homomorphism $s\colon C \to B$ such that $p \circ s = \mathrm{id}_C$.

The following lemma is a generalization of Lemma 78.1.

Lemma 165.6. Let R be a ring and let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of R-left modules. If C is a free R-left module, then the short exact sequence splits as a sequence of R-left modules.

Proof. The proof is basically the same as the proof of Lemma 78.1. ■

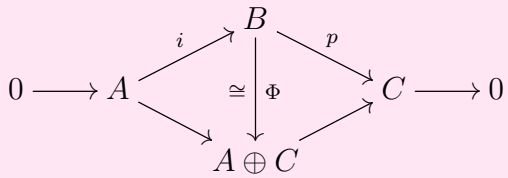
We continue with the following variation on the Splitting Lemma 78.2.

Lemma 165.7. (Splitting Lemma for Rings) Let R be a ring and let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of R-left modules. The following three statements are equivalent:

(1) There exists a homomorphism of R-left modules$t\colon B \to A$ such that $t \circ i = \mathrm{id}_A$.
(2) There exists an isomorphism $\Phi\colon B \to A \oplus C$ of R-left modules such that the following diagram commutes:



where the lower left diagonal map is the obvious inclusion and the lower right diagonal map is the natural projection.
(3) The short exact sequence splits.

Proof. The proof is a mild variation on the proof of the Splitting Lemma 78.2. We leave the details to the reader. ■

Lemma 165.8. Let R and S be rings and let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of S-left modules and let M be an (S, R)-left left module. If the short exact sequence splits (by Lemma 165.6 this holds if C is a free S-left module), then

$$0 \to \mathrm{Hom}_{S\text{-left}}(C, M) \xrightarrow{i^*} \mathrm{Hom}_{S\text{-left}}(B, M) \xrightarrow{p^*} \mathrm{Hom}_{S\text{-left}}(A, M) \to 0$$

is also a short exact sequence of R-left modules and it also splits as a short exact sequence of R-left modules.

Proof. The proof is almost identical to the proof of the Tensor-Split-SES Lemma 89.6. ■

165.5. **Modules over group rings I. Basics.** On page 248 we introduced the notion of an action of a group G on a set X. Since we are now carefully distinguish between left and right module structures it is worth renaming the notion of an left action as a "left-action".

Definition. Let X be a set and let G be a group with trivial element e.

(1) A left action of G on X is a map

$$\begin{array}{rcl} G \times X & \to & X \\ (g, x) & \mapsto & g \cdot x \end{array}$$

with the following properties

$$\begin{array}{rcll} e \cdot x & = & x, & \text{for all } x \in X, \\ g \cdot (h \cdot x) & = & (g \cdot_G h) \cdot x, & \text{for all } x \in X \text{ and } g, h \in G. \end{array}$$

($g \cdot (h \cdot x)$: $g \cdot$ group action, $h \cdot x$ group action; $g \cdot_G h$: multiplication in the group G)

A right action is defined in the hopefully obvious way.

(2) The left action is called free, if $g \cdot x = x$ for some $x \in X$ implies that $g = e$.

(3) We say G acts transitively, if for every x and y in X there exists a $g \in G$ with $g \cdot x = y$.

For modules over group rings we recall the following lemma.

Lemma 86.13. Let G be a group. Let A be an abelian group (with additive notation for the group structure). Suppose we are given a homomorphism

$$\varphi \colon G \to \mathrm{Aut}(A),$$

then the map

$$\begin{array}{rcl} \mathbb{Z}[G] \times A & \to & A \\ \left(\sum_{i=1}^m r_i \cdot g_i, a \right) & \mapsto & \sum_{i=1}^m r_i \cdot \underbrace{\varphi(g_i)(a)}_{\in A} \end{array}$$

defines a natural $\mathbb{Z}[G]$-left module structure on A.

Notation. In the setting of Lemma 86.13 we denote the corresponding $\mathbb{Z}[G]$-left module by ${}_\varphi A$.

Example. Let G be a group and let S be a set together with a G-left action. The map

$$\begin{array}{rcl} G \times \mathbb{Z}^{(S)} & \to & \mathbb{Z}^{(S)} \\ \left(g, \sum_{i=1}^k n_i \cdot s_i \right) & \mapsto & \sum_{i=1}^k n_i \cdot \underbrace{(g \cdot s_i)}_{\in S} \end{array}$$

defines a G- left action on the abelian group $\mathbb{Z}^{(S)}$. It follows from Lemma 86.13 that $\mathbb{Z}^{(S)}$ inherits a $\mathbb{Z}[G]$-left module structure in a natural way.

[2050] If the left action is continuous, then the map $x \mapsto g \cdot x$ is in fact a homeomorphism with inverse map given by $x \mapsto g^{-1} \cdot x$.

Definition. Let G be a group and let R be a ring. Given an R-left module M we denote by ${}_{\text{tr}}M$ the $(\mathbb{Z}[G], R)$-left left module where the $\mathbb{Z}[G]$-left module structure is given by the trivial g-action $g \cdot m := m$ for each $g \in G$ and $m \in M$.

We continue with a more interesting example.

Definition. Let G be a group and let R be a ring.

(1) Let M be an R-left module and let $\varphi\colon G \to \mathrm{Aut}_{R\text{-left}}(M)$ be a representation. We use this representation to define a $\mathbb{Z}[G]$-left module structure on M via $g \cdot m := \varphi(g)(m)$. We denote by ${}_{\varphi}M$ this $\mathbb{Z}[G]$-left module. Note that the two modules structures combine to define an $(\mathbb{Z}[G], R)$-left left module structure on ${}_{\varphi}M$.

(2) Let $\alpha\colon G \to \mathrm{GL}(n, R)$ be a homomorphism. We consider[2051]
$$\begin{array}{rcl} G \times R^n & \to & R^n \\ g \cdot \underbrace{(v_1, \ldots, v_n)}_{\text{row vector}} & \mapsto & \underbrace{(v_1, \ldots, v_n)}_{\text{row vector}} \cdot \alpha(g^{-1}). \end{array}$$
We view R^n as an R-left module in the obvious ways. The above left action defines a representation $G \to \mathrm{Aut}_{R\text{-left}}(R^n)$. We denote the corresponding $(\mathbb{Z}[G], R)$-left left module by ${}_{\alpha}R^d$.

(3) The case $n = 1$ corresponds to a ring homomorphism $\alpha\colon \mathbb{Z}[G] \to R$. In this case the left-multiplication by g is defined as $g \cdot r := r \underset{R}{\cdot} \alpha(g^{-1})$. We denote the corresponding $(\mathbb{Z}[G], R)$-left left module by ${}_{\alpha}R$.

The next lemma gives us a much interesting source for free modules.

Definition. Suppose that a group G admits a left action on a set S. We say $T \subset S$ is a complete set of representatives if given any $s \in S$ there exists a unique $t \in T$ such that $s = g \cdot t$.

Lemma 165.9. Let G be a group and let S be a set together with a G-left action. We suppose that the left action of G on S is free. If $T \subset S$ is a complete set of representatives[2052], then $\mathbb{Z}^{(S)}$ is a free $\mathbb{Z}[G]$-left module with generating set T.

Example. In this section figures are few and far between, so let us take this opportunity to look at one pictorial example. We consider the set $S = \mathbb{Z} \times \{A, B, C, D\}$ with the obvious $\mathbb{Z}$-action. This left action is free. There are many choices of complete sets of representatives.

Proof. Let $T \subset S$ be a complete set of representatives of the G-action on S. Recall that the hypothesis that the left action of G on S is free means that if $g \cdot s = s$ for some $s \in S$, then $g = e$. It follows easily from the axioms of an left action that this translates into the statement that given any $s, t \in S$ with $[s] = [t]$ where exists a *unique* $g \in G$ with $g \cdot s = t$.

[2051] The convention that we act on row vectors by right multiplication $\alpha(g)^{-1}$ might initially come as a surprise. But it is forced on us if we want to get an R-left module isomorphism. The case of a non-commutative ring R will occur naturally later on, for example in Proposition 168.2.

[2052] It follows from the Axiom of Choice 0.1 that such a T always exists.



Now let N be a $\mathbb{Z}[G]$-left module and let $\phi\colon T \to N$ be any map. We extend ϕ to a map $\widetilde{\phi}\colon S \to N$ as follows:

$$\begin{array}{rcl} \widetilde{\phi}\colon S & \to & N \\ s & \mapsto & g \cdot \phi(t) \text{ where } [t] = [s] \text{ and where } g \in G \text{ is} \\ & & \qquad \text{the unique element with } g \cdot t = s. \end{array}$$

By the definition of a free abelian group there exists a unique homomorphism $\psi\colon \mathbb{Z}^{(S)} \to N$ of abelian groups that makes the following diagram commute:

$$\begin{array}{ccc} S & \longrightarrow & \mathbb{Z}^{(S)} \\ & \searrow^{\widetilde{\phi}} & \downarrow \psi \\ & & N. \end{array}$$

We need to show that ψ is in fact a homomorphism of $\mathbb{Z}[G]$-left modules. Note that one can easily verify that for any $s \in S$ and any $g \in G$ we have $\psi(g \cdot s) = g \cdot \psi(s)$. Since S is a generating set for $\mathbb{Z}^{(S)}$ as an abelian group we see that $\psi\colon \mathbb{Z}^{(S)} \to N$ is in fact a homomorphism of $\mathbb{Z}[G]$-left modules.

The proof of the fact that ψ is unique can be reduced, with the same arguments as above, to the uniqueness of the homomorphism $\mathbb{Z}^{(S)} \to N$ of abelian groups. We leave it to the reader to fill in the details. ■

We continue with the following slight generalization of Lemma 86.13.

Lemma 165.10. Let G be a group. Let R be a ring and let A be an R-left module and suppose we are given a homomorphism

$$\varphi\colon G \to \mathrm{Aut}_{R\text{-left}}(A).$$

Then the map

$$\begin{array}{rcl} \mathbb{Z}[G] \times A & \to & A \\ \left(\sum_{i=1}^m r_i \cdot g_i, a\right) & \mapsto & \sum_{i=1}^m r_i \cdot \varphi(g_i)(a), \end{array}$$

together with the given R-left module structure defines a natural $(\mathbb{Z}[G], R)$-left left module structure on A.

165.6. Modules over group rings II: Finite dimensional representations.

Definition. Let π be a group, let R be a ring and let M be an R-left ring.

(1) An R-representation of π is a homomorphism $\alpha\colon \pi \to \mathrm{Aut}_{R\text{-left}}(M)$.
(2) If $R = \mathbb{Z}$, $R = \mathbb{Q}$, $R = \mathbb{R}$ or $R = \mathbb{C}$, then we also say integral, rational, real or complex representation.

(3) If R is a field and M is d-dimensional, then we say that α is a d-dimensional representation. If $d < \infty$, then we call the representation finite dimensional.

We camouflage our first example as a definition.

Definition. Let π be a group and let R be a commutative ring. The representation

$$\begin{array}{rcl} \pi & \to & \mathrm{Aut}_{R\text{-left}}(R[\pi]) \\ g & \mapsto & (f \mapsto f \cdot g^{-1}) \end{array}$$

is called the R-regular representation of π.

Definition. Let π be a group and let R be a ring.

(1) A representation of $\alpha\colon \pi \to \mathrm{Aut}_R(M)$ is called reducible, if M admits an R-submodule N with $0 \subsetneq N \subsetneq M$ such that π preserves M. Otherwise we say that the representation is irreducible.

(2) Given two representations $\alpha\colon \pi \to \mathrm{Aut}_R(M)$ and $\beta\colon \pi \to \mathrm{Aut}_R(N)$ we define the direct sum of the two representations as follows:

$$\begin{array}{rcl} \alpha \oplus \beta\colon \pi & \to & \mathrm{Aut}_R(M \oplus N) \\ g & \mapsto & (m \oplus n \mapsto \alpha(g) \cdot m \oplus \alpha(g) \cdot n)). \end{array}$$

(3) A representation of π that is the direct sum of irreducible representations is called semisimple

Example.

(1) We consider the representation

$$\begin{array}{rcl} \pi = \mathbb{Z} & \to & \mathrm{GL}(2, \mathbb{C}) \\ n & \mapsto & \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \end{array}$$

This representation is reducible, since $N = \{(z, 0) \,|\, z \in \mathbb{C}\}$ is preserved by π. But one can easily show that the representation is not semisimple.

(2) Let G be a finite group and let R be a commutative ring. The submodule

$$R \cdot \sum_{g \in G} g$$

is preserved by G. Thus we see that if G is a non-trivial group, then the R-regular representation of G is reducible.

Proposition 165.11. If π is an abelian group, then every complex representation $\pi \to \mathrm{Aut}_{\mathbb{C}}(M)$ of dimension ≥ 2 is reducible.

Proof. Let π be an abelian group and let $\varphi\colon \pi \to \mathrm{Aut}_{\mathbb{C}}(M)$ be a finite-dimensional complex representation. Given $g \in \pi$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $N \subset M$ we set

$$E^N_{g,\lambda} \;:=\; \{v \in N \,|\, \varphi(g)(v) = \lambda \cdot v\}.$$

We start out with two claims.

Claim 1. Let $g \in \pi$ and let $N \subset M$ be a non-zero complex vector subspace with $\varphi(g)(N) \subset N$. Then there exists a $\lambda \in \mathbb{C} \setminus \{0\}$ with $E^N_{g,\lambda} \neq 0$.

Proof. Let

$$p(t) \ := \ \det(\varphi(g) - t \cdot \mathrm{id}_N)$$

be the characteristic polynomial. This is a polynomial with $\deg(p(t)) = \dim_{\mathbb{C}}(N) > 0$. By the Fundamental Theorem of Algebra 26.3, which we proved multiple times, the polynomial has a zero $\lambda \in \mathbb{C}$. Elementary linear algebra shows that $E^N_{g,\lambda} \neq 0$. Since $\varphi(g)$ is an automorphism we see that $\lambda \neq 0$. ⊞

Claim 2. Given any $g, h \in \pi$ and $\lambda \in \mathbb{C}$ we have $\varphi(h)(E^N_{g,\lambda}) \subset E^N_{g,\lambda}$.

Proof. Let $v \in E^N_{g,\lambda}$. Then

$$\begin{array}{ccccccc} & & \text{since } \varphi \text{ is a representation} & & \text{since } \pi \text{ is abelian} & & \text{since } \varphi \text{ is a representation} \\ \varphi(g)(\varphi(h)(v)) & \overset{\downarrow}{=} & \varphi(g \cdot h)(v) & \overset{\downarrow}{=} & \varphi(h \cdot g)(v) & \overset{\downarrow}{=} & \varphi(h)(\varphi(g)(v) \\ & = & \varphi(h)(\lambda \cdot v) & = & \lambda \cdot \varphi(h)(v). & & \\ & & \uparrow & & \uparrow & & \\ & & \text{since } \lambda \in E^N_{g,\lambda} & & \text{since } \varphi(h) \in \mathrm{Aut}_{\mathbb{C}}(M) & & \end{array}$$

This shows that $\varphi(h)(v) \in E^N_{g,\lambda}$. ⊞

If there exists a $g \in \pi$ and a $\lambda \in \mathbb{C} \setminus \{0\}$ with $0 < \dim_{\mathbb{C}}(E^N_{g,\lambda}) < \dim_{\mathbb{C}}(M)$ then it follows from Claim 2 that $E^N_{g,\lambda}$ is a vector subspace that is preserved by π. Thus we have shown that the representation is reducible.

Otherwise each $\varphi(g)$ acts by multiplication by some $\lambda(g) \in \mathbb{C} \setminus \{0\}$. Thus we see that each vector subspace is preserved by π. Since $\dim_{\mathbb{C}}(M) \geq 2$ we see that the representation is reducible. ■

165.7. **The tensor product over rings.** On page 1907 we introduced the tensor product of two abelian groups. In this section we will introduce the tensor product of modules over a given ring R.

The following definition is an almost shameless copy-paste from page 1907.

Definition. Let R be a ring, let A be a right R-module and let B be a left R-module. We define the tensor product $A \otimes_R B$ as follows:

$$A \otimes_R B \ := \ \left\{ \begin{array}{l} \text{free abelian group which is generated by all} \\ \text{symbols of the form } a \otimes b \text{ with } a \in A,\ b \in B \end{array} \right\} / \, N(A, B)$$

where $N(A, B)$ is the subgroup generated by the elements of the form

$$\begin{array}{lll} \text{(i)} & (a + a') \otimes b - (a \otimes b + a' \otimes b) & \text{with } a, a' \in A \text{ and } b \in B, \\ \text{(ii)} & a \otimes (b + b') - (a \otimes b + a \otimes b') & \text{with } a \in A \text{ and } b, b' \in B, \\ \text{(iii)} & a \cdot r \otimes b - a \otimes r \cdot b & \text{with } a \in A, b \in B \text{ and } r \in R. \end{array}$$

For $a \in A$ and $b \in B$ we denote, by a slight abuse of notation, the image of $a \otimes b$ in $A \otimes B$ again by $a \otimes b$.

Remark.

(1) Note that in the notation we use the subscript R to denote the tensor product $A \otimes_R B$. To simplify the subsequent notation, given elements $a \in A$ and $b \in B$ we write $a \otimes b$ instead of the more logical $a \otimes_R b$. It should be clear from the context what we mean.

(2) We consider the case that $R = \mathbb{Z}$. Any abelian group is a $\mathbb{Z}$-module, in particular, any abelian group is a $\mathbb{Z}$-left module and a $\mathbb{Z}$-right module. It follows immediately

from Lemma 165.12 (2) that we have an equality

$$\underbrace{A \otimes_{\mathbb{Z}} B}_{\substack{\text{tensor product} \\ \text{of modules}}} = \underbrace{A \otimes B.}_{\substack{\text{tensor product of} \\ \text{abelian groups}}}$$

(3) In Exercise **??** we will discuss the question whether it makes sense to define the tensor product if we are given say two R-right modules.

We summarize some elementary properties of the tensor product in the following lemma.

Lemma 165.12. Let R be a ring, let A be an R-right module and let B be an R-left module.

(1) For any $a, a' \in A$ and $b, b' \in B$ we have

$$\begin{aligned} (a + a') \otimes b &= a \otimes b + a' \otimes b \in A \otimes_R B \\ a \otimes (b + b') &= a \otimes b + a \otimes b' \in A \otimes_R B. \end{aligned}$$

(2) For any $a \in A, b \in B$ and any $r \in R$ we have

$$(a \cdot r) \otimes b = a \otimes (r \cdot b).$$

(3) Any element in $A \otimes_R B$ is of the form $\sum_{i=1}^{n} a_i \otimes b_i$.

Proof. The statements are basically just notationally convenient equalities that follow immediately from the definition of the tensor product $A \otimes_R B$. ■

The tensor product can be characterized by the following universal property.

Lemma 165.13. Let R be a ring, let A be an R-right module and let B be an R-left module.

(1) The homomorphism

$$\begin{aligned} \mu \colon A \times B &\to A \otimes_R B \\ (a, b) &\mapsto a \otimes b \end{aligned}$$

has the property that for any $a \in A, b \in B$ and $r \in R$ we have $\mu(a \cdot r, b) = \mu(a, r \cdot b)$.

(2) Given any homomorphism $\varphi \colon A \times B \to Y$ of abelian groups with the property that for any $a \in A, b \in B$ and $r \in R$ we have $\varphi(a \cdot r, b) = \varphi(a, r \cdot b)$ there exists a *unique* homomorphism $\psi \colon A \otimes_R B \to Y$ of abelian groups such that the following diagram commutes:

$$\begin{array}{ccc} A \times B & \xrightarrow[\mu]{(a,b) \mapsto a \otimes b} & A \otimes_R B \\ & \searrow^{g} & \downarrow \psi \text{ unique} \\ & & Y. \end{array}$$

(3) If Y is an abelian group and $\nu \colon A \times B \to Y$ is a homomorphism that satisfies (1) and (2), then there exists an isomorphism $\Theta \colon Y \to A \otimes_R B$ with $\nu = \mu \circ \Theta$.

Proof. One can easily verify that $\psi(a \otimes b) := \varphi(a, b)$ defines a homomorphism that makes the diagram commute and that it is uniquely determined. ■

For completeness' sake we state the following analogue of Lemma 165.1.

Lemma 165.14. Let R and S be rings, let A be an (S,R)-bimodule and let B be an R-left module. Then

$$\begin{array}{rcl} S \times A \otimes_R B & \to & A \otimes_R B \\ (s, a \otimes b) & \mapsto & (s \cdot a) \otimes b \end{array}$$

is a well-defined R-left module structure. The evident variations on this statement also hold. ■

Lemma 165.15. Let S be a ring.

(1) Let $f\colon A \to A'$ a homomorphism of S-right modules and let $g\colon B \to B'$ be a homomorphism of S-left modules. The map

$$\begin{array}{rcl} f \otimes g\colon A \otimes_S B & \to & A' \otimes_S B' \\ \sum_{i=1}^m a_i \otimes b_i & \to & \sum_{i=1}^m f(a_i) \otimes g(b_i) \end{array}$$

is well-defined and it is a homomorphism of abelian groups. If R is a ring and $f\colon A \to A'$ is a homomorphism of (R,S)-bimodules, then the above map $f \otimes g$ is a homomorphism of R-left modules. Obvious variations on this observation also hold.

(2) Let $f\colon A \to A'$ and $f'\colon A' \to A''$ be two homomorphisms of S-right modules and let $g\colon B \to B'$ and $g'\colon B' \to B''$ be homomorphisms of S-left modules. Then we have

$$(f' \otimes g') \circ (f \otimes g) \;=\; (f' \circ f) \otimes (g' \otimes g).$$

(3) Given an S-right module A the maps

$$B \mapsto A \otimes_S B \quad \text{and} \quad (f\colon B \to B') \mapsto (\mathrm{id} \otimes f\colon A \otimes_S B \to A \otimes_S B')$$

form a functor from the category ${}_S\mathcal{M}od$ of S-left modules to the category $\mathcal{A}b\mathcal{G}roup$ of abelian groups.

(4) The analogous statement as in (3) holds if we start out with an S-left module B.

Proof. We leave it to the reader to verify the statements. ■

The following lemma is a crucial example for the tensor product.

Lemma 165.16. Let R and S be rings. Let A be an (R,S)-bimodule and let B be an S-left module. We view S as S-left module and as an S-right module where the module structure is given by left respectively right multiplication. The maps

$$(1) \quad \begin{array}{rcl} A \otimes_S S & \to & A \\ \sum_{i=1}^n a_i \otimes r_i & \mapsto & \sum_{i=1}^n a_i \cdot r_i \end{array} \qquad \text{and} \qquad (2) \quad \begin{array}{rcl} S \otimes_S B & \to & B \\ \sum_{i=1}^n r_i \otimes b_i & \mapsto & \sum_{i=1}^n r_i \cdot b_i \end{array}$$

are well-defined and they are natural isomorphisms of (R,S)-bimodules.

Proof. Let us show that the map (1) is in fact well-defined. By Lemma 51.1 there exists a unique homomorphism

$$\Phi\colon \text{free abelian group generated by all symbols } a \otimes r \;\to\; A, \quad a \otimes r \mapsto a \cdot r.$$

of abelian groups that satisfies

We need to show that all the elements of the types (i), (ii) and (iii) as defined on page 3510 lie in the kernel of Φ. For (i) and (ii) that is basically obvious. We consider (iii) in slightly

more detail. Let $a \in A$, let $s \in S$ and let $s \in S$. We have

$$\Phi(a \cdot r \otimes s - a \otimes r \cdot s) \underset{\text{since } \Phi \text{ is a homomorphism of abelian groups}}{=} \Phi(a \cdot r \otimes s) - \Phi(a \otimes r \cdot s) \underset{\text{by definition of } \Phi \text{ and since we view } S \text{ as a left } S\text{-module via left multiplication}}{=} (a \cdot r) \cdot s - a \cdot (r \cdot s) \underset{\text{since } A \text{ is a } S\text{-right module}}{=} 0.$$

This shows that the map (1) is in fact well-defined. We leave it to the reader to verify that an inverse is given by map

$$\begin{array}{ccc} A & \to & A \otimes_S S \\ a & \mapsto & a \otimes 1. \end{array}$$

Basically the same argument also shows that the map (2) is a well-defined isomorphism. ■

On many occasions we will also make use of the following lemma which is a partial analogue of the Tensor Lemma 89.3.

Lemma 165.17. Let R be a ring. Let A and $\{A_i\}_{i \in I}$ be R-left modules and let B and $\{B_i\}_{i \in I}$ be R-right modules.

(1) The maps

$$\begin{array}{ccc} \left(\bigoplus_{i \in I} A_i\right) \otimes_R B & \to & \bigoplus_{i \in I} A_i \otimes_R B \\ \left(\sum_{i=1}^n a_i\right) \otimes b & \mapsto & \sum_{i=1}^n a_i \otimes b \end{array} \quad \text{and} \quad \begin{array}{ccc} A \otimes_R \left(\bigoplus_{i \in I} B_i\right) & \to & \bigoplus_{i \in I} A_i \otimes_R B \\ a \otimes \left(\sum_{i=1}^n b_i\right) & \mapsto & \sum_{i=1}^n a \otimes b_i \end{array}$$

are well-defined and they are natural isomorphisms.

(2) Suppose $(I, \leq)$ is an ordered set and $(\{A_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j})$ is a direct system of R-left modules, then

$$\varinjlim (A_i \otimes_R B) = \left(\varinjlim A_i\right) \otimes_R B.$$

Proof. The proof is basically the same as the proof of the Tensor Lemma 89.3. ■

Remark. In Lemma 165.17 we saw that the tensor product commutes with forming direct sums. Note that in Exercise 89.11 we already saw that tensor products, even just over the ring $R = \mathbb{Z}$, do not commute with forming direct products.

The following lemma is an analogue of Lemma 165.8.

Lemma 165.18. Let R and S be rings and let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of S-left modules and let M be an (R, S)-bimodule. If the short exact sequence splits as an exacts sequence of S-left modules (by Lemma 165.6 this is the case if C is a free S-left module), then

$$0 \to M \otimes_S A \xrightarrow{\mathrm{id} \otimes i} M \otimes_S B \xrightarrow{\mathrm{id} \otimes p \otimes} M \otimes_S C \to 0$$

is also a short exact sequence and it also splits.

Proof. The proof is almost identical to the proof of the Tensor-Split-SES Lemma 89.6. ■

Lemma 165.19. Let $\varphi\colon R \to S$ be a ring homomorphism, let A be an S-right module and let B be an S-left module. As explained on page 3502 we obtain, via φ, an R-right

module φ^*A and an R-left module φ^*B. With these conventions the map

$$\begin{array}{rcl} \varphi^*A \otimes_R \varphi^*B & \to & A \otimes_S B \\ a \otimes b & \mapsto & a \otimes b \end{array}$$

is well-defined. ■

Lemma 165.20. Let $\varphi\colon R \to S$ be a ring homomorphism, let M be an S-right module and let N be an R-left module. The maps[2053]

$$\begin{array}{rcl} M \otimes_S (S \otimes_R N) & \to & \varphi^*M \otimes_R N \\ (m \otimes (s \otimes n)) & \mapsto & m \cdot s \otimes n \end{array} \quad \text{and} \quad \begin{array}{rcl} \varphi^*M \otimes_R N & \to & M \otimes_S (S \otimes_R N) \\ m \otimes n & \mapsto & m \otimes (1 \otimes n) \end{array}$$

are well-defined and inverses of one another, in particular they are both isomorphism. ■

Lemma 165.21. Let R be a ring and let $\varphi\colon R \to S$ be a ring homomorphism. We use this ring homomorphism to view S as an R-right module.

(1) For any $k \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} S \otimes_R R^k & \to & S^k \end{array}$$

that is determined by

$$\begin{array}{rcl} s \otimes (r_1, \ldots, r_k) & \mapsto & (s \cdot \varphi(r_1), \ldots, s \cdot \varphi(r_k)) \end{array}$$

is an isomorphism.

(2) Let X be a set. The map

$$\begin{array}{rcl} S \otimes_R R^{(X)} & \to & S^{(X)} \end{array}$$

that is determined by

$$\begin{array}{rcl} \sum_{i=1}^n s_i \otimes (r_i \cdot x_i) & \mapsto & \sum_{i=1}^n (s_i \cdot \varphi(r_i)) \cdot x_i \end{array}$$

is an isomorphism. If T is a ring and M is a (T,S)-bimodule, then the obvious map

$$\begin{array}{rcl} M \otimes_R R^{(X)} & \to & \varphi^*M \otimes_S S^{(X)} \end{array}$$

is an isomorphism.

(3) Let $A = (a_{ij})$ be a $k \times l$–matrix over R. If we write $\varphi(A) := (\varphi(a_{ij}))$, then the following diagram commutes[2054]

$$\begin{array}{ccc} S \otimes_R R^k & \xrightarrow{v \mapsto v \cdot A} & S \otimes_R R^l \\ \cong \downarrow & & \downarrow \cong \\ S^k & \xrightarrow{w \mapsto \varphi(A) \cdot w} & S^l. \end{array}$$

(4) Let R be a ring and let M free R-left module on a set X. If $\varphi\colon R \to S$ is a ring homomorphism, then $S \otimes_R M$ is a free S-left module on the set $\{1 \otimes x\}_{x \in X}$.

[2053]Here we view S as an (S,R)-bimodule, thus by Lemma 165.14 we can view $S \otimes_R N$ as an S-left module to define the tensor product $M \otimes_S (S \otimes_R N)$.

[2054]The vertical isomorphisms are of course the natural maps defined in (2).

Lemma 165.22. Let $\varphi: R \to S$ be a ring homomorphism, let be an S-left module and let M be an S-left module. The map

$$\begin{array}{rcl} \mathrm{Hom}_{R\text{-left}}(R \otimes_S C, \varphi^* M) & \to & \mathrm{Hom}_{S\text{-left}}(C, M) \\ \gamma & \mapsto & (c \mapsto \gamma(1_R \otimes c)) \end{array}$$

is well-defined. If C is a free S-left module, then the above map is in fact an isomorphism.

Remark. In Exercise 165.3 we will see that in general the conclusion of Lemma 165.22 does not hold, if C is not a free S-left module.

Proof. One can easily verify that the map is well-defined. If C is a free S-left module, then using Lemma 165.21 (2) and using the definition of a free module one can easily verify that the map is an isomorphism. ■

Finally in many cases we will be interested in the tensor product over a group ring. The following elementary lemma simplifies the discussion of such tensor products a little bit.

Lemma 165.23. Let π be a group, R be a ring, let M be a $(R, \mathbb{Z}[\pi])$-bimodule and let N be a $\mathbb{Z}[\pi]$-left module. The map

$$\begin{array}{rcl} M \otimes_{\mathbb{Z}} \mathbb{Z} \Big/ \left\{ \begin{array}{l} \text{subgroup of } M \otimes_{\mathbb{Z}} N \text{ generated by the elements} \\ m \cdot g \otimes n - m \otimes g \cdot n \text{ with } m \in M, n \in N, g \in \pi \end{array} \right\} & \to & M \otimes_{\mathbb{Z}[\pi]} N \\ [m \otimes n] & \mapsto & m \otimes n \end{array}$$

is a natural isomorphism of R-left modules.

Proof. The proof follows easily from the definitions and is left to the reader. ■

Lemma 165.24. Given any group π and given any $\mathbb{Z}[\pi]$-left module M the maps

$$\begin{array}{rcl} M \otimes_{\mathbb{Z}} {}_{\mathrm{tr}}\mathbb{Z} & \to & M \\ m \otimes n & \mapsto & n \cdot m \end{array} \quad \text{and} \quad \begin{array}{rcl} {}_{\mathrm{tr}}\mathbb{Z} \otimes_{\mathbb{Z}} M & \to & M \\ n \otimes m & \mapsto & n \cdot m \end{array}$$

are natural isomorphisms of $\mathbb{Z}[\pi]$-left modules.

Proof. The lemma can be proved easily, say using Lemma 165.23. ■

165.8. Modules over skew fields and PIDS.

Definition. A ring R is called a principal ideal domain (PID) if R is commutative[2055] and if every ideal of R is generated by a single element.

The following theorem generalizes the Subgroup-of-Free Abelian Group Lemma 51.2 (1).

Theorem 165.25. Let R be a PID. Every submodule of a free R-module is again free.

Proof. This theorem is proved in [**Rot2002**, Theorem 8.9]. ■

We now introduce skew fields. In most applications the skew fields we use are in fact just ordinary fields. On the other hand, it costs little to generalize to skew fields and working with (possibly) non-commutative skew fields actually makes some of the future arguments clearer since it prevents us from using "accidents" in the commutative setting, e.g. the fact that over a commutative ring every left-module is automatically a right module.

Definition.

[2055] One could also define a non-commutative analogue of PIDs, but we will refrain from doing so.

(1) A skew field is a ring $\mathbb{K}$ such that given any $a \in \mathbb{K}$ there exist $x, y \in \mathbb{K}$ with $a \cdot x = y \cdot a = 1$.[2056]
(2) If $\mathbb{K}$ is a skew field, then we refer to a $\mathbb{K}$-left module also as a $\mathbb{K}$-left vector space.

As we will see, many results of modules over fields (i.e. commutative skew fields) generalize in a suitable sense to modules over skew fields.

Proposition 165.26. Let $\mathbb{K}$ be a skew field.

(1) Every $\mathbb{K}$-left module M is a free $\mathbb{K}$-left module.
(2) Every epimorphism $\varphi\colon M \to N$ of $\mathbb{K}$-left modules admits a left inverse $s\colon N \to M$.
(3) Every submodule A of a $\mathbb{K}$-left module M is a summand, i.e. there exists another submodule B such that $M = A \oplus B$.
(4) Every $\mathbb{K}$-left module M has a uniquely defined dimension $\dim_{\mathbb{K}}(M) \in \mathbb{N}_0 \cup \{\infty\}$.
(5) Let $\varphi\colon M \to N$ be a $\mathbb{K}$-module homomorphism between two $\mathbb{K}$-left modules of the same finite dimension. Then the following are equivalent:
  (a) φ is an isomorphism,
  (b) φ is a monomorphism,
  (c) φ is an epimorphism.
(6) Every short exact sequence $0 \to A \to B \to C \to 0$ of $\mathbb{K}$-left modules splits and we have the equality $\dim_{\mathbb{K}}(B) = \dim_{\mathbb{K}}(A) + \dim_{\mathbb{K}}(C)$.

Proof. We leave it to the reader to dig out their books on linear algebra and verify that in the proofs of the corresponding statements over fields one never really needed to use that fields are commutative. ■

Exercises for Chapter 165.

Exercise 165.1.

(a) We consider the ring $R = \mathbb{Z}[x]$ of polynomials and we consider the ideal $I = (2, 1-x)$ that is generated by 2 and $1 - x$. Show that I is not a free $\mathbb{Z}[x]$-module.
(b) Show that the Laurent polynomial ring $R = \mathbb{Z}[x^{\pm 1}]$ admits submodules that are not free.

Exercise 165.2. Let R be a ring, let M be an R-left module let G be a group, let H be a subgroup of G and let $\varphi\colon H \to \mathrm{Aut}_{R\text{-left}}(M)$ a representation. We view $\mathbb{Z}[G]$ as a $(\mathbb{Z}[G], \mathbb{Z}[H])$-bimodule in the obvious way. We consider the tensor product

$$\mathrm{Ind}_H^G M \ := \ \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M$$

which is a $\mathbb{Z}[G]$-left module and thus also a representation of G, which is called the *induced representation*.

(a) Show that if R is a skew field, then $\dim_{\mathbb{K}}(\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M) = [G : H] \cdot \dim_{\mathbb{K}}(M)$.
(b) A representation of a group is called *faithful* if it is a monomorphism. Show that if M is a faithful representation of H, then $\mathrm{Ind}_H^G M$ is a faithful representation of G.

Exercise 165.3. We consider the polynomial ring $\mathbb{Q}[x]$.

[2056] If such that x, y exist, then as always $x = y$. Indeed, we have $x = 1 \cdot x = (y \cdot a) \cdot x = y \cdot (a \cdot x) = y \cdot 1 = y$.

(a) Given a polynomial $p(x) \in \mathbb{Q}[x]$ we view $\mathbb{Q}[x]/(p(x))$ as a $\mathbb{Q}[x]$-module in the obvious way. Show that the map

$$\begin{array}{rcl} \mathbb{Q} \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(p(x)) & \to & \mathbb{Q}/(p(1)) \\ n \otimes [f(x)] & \mapsto & n \cdot f(1) \end{array}$$

is an isomorphism.

(b) We consider the ring homomorphism $\varphi\colon \mathbb{Q} \to \mathbb{Q}[x]$ that is given by the inclusion and we consider the $\mathbb{Q}[x]$-modules $C := \mathbb{Q}[x]/(1-x)$ and $M := \mathbb{Q}[x]/(1+x)$. Show that the map

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Q}[x]}, \varphi^* M) & \to & \mathrm{Hom}_{\mathbb{Q}[x]}(C, M) \\ \gamma & \mapsto & (c \mapsto \gamma(1_{\mathbb{Q}} \otimes c)) \end{array}$$

is not an isomorphism.

Remark. This shows that in Lemma 165.22 we cannot drop the hypothesis that the module C is a free $\mathbb{Q}[x]$-module.

166. Homological algebra over rings

166.1. **Chain complexes over rings.** In this section we generalize the theory of chain complexes of abelian groups to chain complexes of R-left modules. Basically everything we do in this section is self-explanatory. First we have the following generalization of the definitions on pages 1591, and 1592

Definition. Let R be a ring.

(1) An R-left chain complex (C_*, ∂_*) is a sequence
$$\ldots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots C_1 \xrightarrow{\partial_1} C_0 \to 0$$
of R-left homomorphisms between R-left modules such that for every $i \in \mathbb{N}$ we have $\partial_{i-1} \circ \partial_i = 0$. As before we introduce the convention that $C_n = 0$ for $n \in \mathbb{Z}_{<0}$.

(2) We define the n-th homology module of the R-left chain complex as
$$\mathrm{H}_n(C) \ := \ \ker(\partial_n \colon C_n \to C_{n-1}) \,/\, \mathrm{im}(\partial_{n+1} \colon C_{n+1} \to C_n).$$
Note that $\mathrm{H}_n(C)$ inherits the structure of an R-left module.

(3) An chain map $f \colon C_* \to D_*$ between two R-left chain complexes consists of a family $\{f_n \colon C_n \to D_n\}_{n \in \mathbb{N}_0}$ of R-left homomorphisms such that for each $n \in \mathbb{N}_0$ we have $f_{n-1} \circ \partial_n \ = \ \partial_n \circ f_n$.

Lemma 166.1. Let R be a ring and let $f \colon C_* \to D_*$ be a chain map between chain complexes of R-left modules. Then the boundary map of C_* defines a chain complex structure on the R-left modules $\ker(C_k \to D_k)$ and the boundary map of D_* descends to a chain complex structure on the R-left modules $\mathrm{coker}(C_k \to D_k)$.

Proof. The statement follows easily from the definitions. ∎

Definition.

(1) Let $f, g \colon C_* \to D_*$ be chain maps between chain complexes C_* and D_*. A chain homotopy $P = \{P_n\}_{n \in \mathbb{N}_0}$ between f and g consists of family $\{P_n \colon \mathrm{C}_n \to D_{n+1}\}_{n \in \mathbb{N}_0}$ of R-left homomorphisms such that for each $n \in \mathbb{N}_0$ we have
$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \ = \ f_n - g_n.$$
If there exists a chain homotopy between chain maps f and g, then we say that f and g are chain homotopic and we write $f \simeq g$.

(2) Let C_* and D_* be two chain complexes.
   (a) A chain map $f \colon \mathrm{C}_* \to D_*$ is called a chain homotopy equivalence between C_* and D_*, if it admits a chain homotopy inverse, i.e. a chain map $g \colon D_* \to \mathrm{C}_*$ with $g \circ f \simeq \mathrm{id}_{C_*}$ and $f \circ g \simeq \mathrm{id}_{D_*}$.
   (b) If there exists a chain homotopy equivalence between C_* and D_*, then we say that the chain complexes C_* and D_* are chain homotopy equivalent and we write $C_* \simeq D_*$.

The following lemma generalizes the earlier results of Lemmas 73.1, 72.7 and 73.2 and Corollary 73.4.

Lemma 166.2. Let R be a ring and let C_* and D_* be two R-left chain complexes.

(1) For every chain map $f: C_* \to D_*$ the following map is well-defined:
$$\begin{array}{rcl} f_*: \mathrm{H}_n(C_*) & \to & \mathrm{H}_n(D_*) \\ [c] & \mapsto & [f_n(c)]. \end{array}$$
(2) Let $f, g: C_* \to D_*$ be two chain maps between chain complexes. If f and g are chain homotopic, then $f_* = g_*: \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ for all $n \in \mathbb{N}_0$.
(3) If $f: C_* \to D_*$ is a chain homotopy equivalence, then the maps $f_*: \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ are isomorphisms.

Proof. The proof of the lemma is basically the same as the, frankly rather boring, proofs of Lemmas 73.1, 72.7 and 73.2 and Corollary 73.4. ■

On page 1912 we saw that we can tensor a chain complex C_* of abelian groups with an abelian group G to obtain a new chain complex $C_* \otimes G$. Furthermore, on page 2400 we saw that we can consider the cochain complex $\mathrm{Hom}(C_*, G)$. The same constructions also in the more general settings of chain complexes of modules over rings. We summarize the key statements in the following two lemmas.

Lemma 166.3. (Tensoring Chain Complex Lemma) Let R and S be rings and let M be an (R, S)-bimodule.

(1) If (C_*, ∂_*) is an S-left chain complex, then $(M \otimes_S C_*, \mathrm{id}_M \otimes \partial_*)$ is an R-left chain complex.
(2) If $f_*: C_* \to D_*$ is a chain map (chain homotopy equivalence) of S-left chain complexes, then the maps $\mathrm{id}_M \otimes f_*: M \otimes_S C_* \to M \otimes_S D_*$ form a chain map (chain homotopy equivalence) of R-left chain complexes.
(3) If (C_*, ∂_*) is an S-left chain complex and if $\varphi: M \to N$ is a homomorphism of (R, S)-bimodules, then the maps $\varphi: \mathrm{id}: M \otimes_S C_* \to N \otimes_S C_*$ form a chain map of R-left chain complexes.

Proof.

(1) Let $n \in \mathbb{N}_0$. We calculate that
$$(\mathrm{id}_M \otimes \partial_{n-1}) \circ (\mathrm{id}_M \otimes \partial_n) \underset{\uparrow}{=} \mathrm{id}_M \otimes (\partial_{n-1} \circ \partial_n) = \mathrm{id}_M \otimes 0 = 0.$$
Lemma 165.15
This shows that $(M \otimes_{SR} C_*, \mathrm{id}_M \otimes \partial_*)$ is chain complex. It follows from Lemma 165.14 that it is in fact a chain complex of S-left modules.
(2) The proof of this statement is almost as trivial as the proof of Statement (1).
(3) See the proof of Statement (2). ■

166.2. Cochain complexes over rings.

Definition. Let R be a ring.

(1) An R-left cochain complex is a sequence
$$\leftarrow D^n \xleftarrow{\delta_{n-1}} D^{n-1} \xleftarrow{\delta_{n-2}} \ldots \xleftarrow{\delta_1} D^1 \xleftarrow{\delta_0} D^0 \leftarrow 0$$
of R-left homomorphism between R-left modules such that for all $i \in \mathbb{N}_0$ we have $\delta_{i+1} \circ \delta_i = 0$. We refer to the maps δ_* as coboundary maps.

(2) We define the n-th cohomology group of the R-left cochain complex (D^*, δ_*) to be

$$\mathrm{H}^n(D) \;:=\; \frac{\ker(\delta_n\colon D^n \to D^{n+1})}{\mathrm{im}(\delta_{n-1}\colon D^{n-1} \to D^n)}.$$

Note that $\mathrm{H}^n(D)$ inherits the structure of an R-left module.

(3) The notion of a cochain map, cochain homotopy and cochain homotopy equivalence that we introduced on page 2397 for cochain complexes of abelian groups generalize in the obvious way to the more general setting.

For the reader's convenience we recall the following slightly shortened version of the Cochain Map Lemma 108.5.

Lemma 108.5.

(1) If $f\colon C^* \to D^*$ is a cochain map between cochain complexes C^* and D^*, then the map

$$\begin{array}{rcl} f_*\colon \mathrm{H}^n(C) & \to & \mathrm{H}^n(D) \\ [c] & \mapsto & [f_n(c)] \end{array}$$

is well-defined.

(2) Two cochain maps that are cochain homotopic induce the same map on cohomology groups.

(3) Every cochain homotopy equivalence induces an isomorphism of cohomology groups.

As before the key source of cochain complexes is given by dualizing a chain complex. More precisely, we have the following generalization of the definition from page 2400.

Definition. Let S and be a ring and let

$$(C_*, \partial_*)\colon \quad \to\; C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \;\to\; 0$$

be an S-left chain complex and let M be an S-left module. For each $n \in \mathbb{N}_0$ we write

$$\delta_n \;:=\; \partial_{n+1}^*\colon \mathrm{Hom}_{S\text{-left}}(C_n, M) \;\to\; \mathrm{Hom}_{S\text{-left}}(C_{n+1}, M).$$

Note that $\delta_n \circ \delta_{n-1} = (\partial_n)^* \circ (\partial_{n-1})^* = (\partial_{n-1} \circ \partial_n)^* = 0$, thus we obtain the following cochain complex

$$\ldots \xleftarrow{\delta_2} \mathrm{Hom}_{S\text{-left}}(C_2, M) \xleftarrow{\delta_1} \mathrm{Hom}_{S\text{-left}}(C_1, M) \xleftarrow{\delta_0} \mathrm{Hom}_{S\text{-left}}(C_0, M) \leftarrow 0.$$

We refer to it as cochain complex dual to (C_*, ∂_*) and we denote it by $\mathrm{Hom}_{S\text{-left}}(C_*, M)$. We define the n-th cohomology group of the chain complex (C_*, ∂_*) with M-coefficients as the cohomology of the cochain complex $(\mathrm{Hom}(C_*, M), \delta_*)$, i.e. we define

$$\mathrm{H}^n(C; G) \;:=\; \frac{\ker(\delta_n\colon \mathrm{Hom}_{S\text{-left}}(C_n, M) \to \mathrm{Hom}_{S\text{-left}}(C_{n+1}, M))}{\mathrm{im}(\delta_{n-1}\colon \mathrm{Hom}_{S\text{-left}}(C_{n-1}, M) \to \mathrm{Hom}_{S\text{-left}}(C_n, M))}.$$

The following lemma is a generalization of the classical the Dual Cochain Map Lemma 108.8 and it can be viewed as an analogue of the Tensoring Chain Complex Lemma 166.3.

Lemma 166.4. (Dualizing Chain Complex Lemma) Let R and S be rings and let M be an (S,R)-left left module.

(1) If (C_*, ∂_*) is an S-left chain complex, then the R-left modules $\mathrm{Hom}_{S\text{-left}}(C_*, M)$ together with the maps

$$\begin{array}{rcl} \delta\colon \mathrm{Hom}_{S\text{-left}}(C_n, M) & \to & \mathrm{Hom}_{S\text{-left}}(C_{n+1}, M) \\ (\varphi\colon C_n \to M) & \mapsto & (\varphi \circ \partial\colon C_{n+1} \to M) \end{array}$$

form a cochain complex of R-left modules.

(2) If $f_*\colon C_* \to D_*$ is a chain map (chain homotopy equivalence) of R-left chain complexes, then the maps

$$\begin{array}{rcl} f^*\colon \mathrm{Hom}_{S\text{-left}}(D_n, M) & \to & \mathrm{Hom}_{S\text{-left}}(C_n, M) \\ (\varphi\colon D_n \to M) & \mapsto & (f \circ \varphi\colon C_n \to M) \end{array}$$

form a cochain map (cochain homotopy equivalence) of R-left cochain complexes.

(3) If (C_*, ∂_*) is an R-left chain complex and if $\alpha\colon M \to N$ is a homomorphism of (S,R)-bimodule, then the maps

$$\begin{array}{rcl} \alpha_*\colon \mathrm{Hom}_{S\text{-left}}(C_n, M) & \to & \mathrm{Hom}_{S\text{-left}}(C_n, N) \\ (\varphi\colon C_n \to M) & \mapsto & (\alpha \circ \varphi\colon C_N \to N) \end{array}$$

form a chain map of S-left chain complexes.

Proof. We leave the straightforward verification of this lemma to the reader. ■

166.3. **Free resolutions.** In several chapters we already developed the theory of chain complexes of free abelian groups. Many statements generalize without problems to the setting of chain complexes of free modules. More precisely, all statements that only made use of the universal property of free abelian groups generalize with basically the same proof. On several occasions we did use the fact, see the Subgroup-of-Free Abelian Group Lemma 51.2 , that any subgroup of a free abelian group is again a free abelian group. This statement does *not* generalize to rings.

The following generalizes the notion of a (free) resolution of an abelian group that we gave on page 1791.

Definition. Let R be a ring and let M be an R-left module. A (free) resolution of an abelian group M is an exact sequence

$$\ldots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$$

where the F_i are (free) R-left modules.

In the following definition, which is modelled on the definition on page 1922, we show that every R-left module admits a canonical free resolution.

Definition. Let R be a ring. We recall the following basic fact: if K is an R-left module, then by Lemma 165.4 there exists a unique homomorphism $\alpha(M)\colon R^{(K)} \to K$ such that for each $k \in K$ we have $\alpha(K)(k) = k$. Now let M be any R-left module. We consider the sequence

$$R^{(K)} \xrightarrow{\alpha(K)} K \to 0.$$

The group $R^{(K)}$ is free abelian and $\alpha(K)$ is an epimorphism. In particular the sequence is exact. We set $K_1 := \ker\left(R^{(K)} \xrightarrow{\alpha(K)} K\right)$. Then

$$K_1 \hookrightarrow R^{(K)} \xrightarrow{\alpha(K)} K \to 0$$

is exact. Now we consider

$$R^{(K_1)} \xrightarrow{\alpha(K_1)} R^{(K)} \xrightarrow{\alpha(K)} K \to 0.$$

This sequence is exact and both the groups $R^{(K_1)}$ and $R^{(K)}$ are free abelian. Now we set $K_2 := \ker\left(R^{(K_1)} \xrightarrow{\alpha(K)} K\right)$ and we iterate this procedure. This way we obtain a free resolution of K. We refer to this free resolution of K as the canonical free resolution of K.

The following theorem is a straightforward generalization of the Fundamental Theorem of Homological Algebra 81.1.

Theorem 166.5. (Fundamental Theorem of Homological Algebra over Rings) Let R be a ring and let $\alpha\colon H \to H'$ be an R-left module homomorphism between R-left modules. Furthermore let F_* be a free resolution of H and let F'_* be a resolution[2057] of H'.

(1) There exists an extension of α to the free resolutions F_* and F'_*.
(2) Any two extensions α_* and α'_*, viewed as chain maps, are chain homotopic.

Remark. In fact we only need that F_* is a projective resolution of H.

Proof. The proof is basically identical to the proof of the Fundamental Theorem of Homological Algebra 81.1. ■

Definition. Let $f\colon C_* \to D_*$ be a chain map between chain complexes. We say f is a quasi-isomorphism if for each $k \in \mathbb{N}_0$ the induced map $f_*\colon \mathrm{H}_k(C_*) \to \mathrm{H}_k(D_*)$ is an isomorphism.

We also have the following generalization of the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 81.4.

Proposition 166.6. (Quasi Isomorphisms-are-Homotopy Equivalences Proposition) Let R be a ring and let $f\colon C_* \to \mathrm{C}'_*$ be a chain map between chain complexes of free R-left modules. If f induces an isomorphism of all homology groups, then f is in fact a chain homotopy equivalence.

Proof. The proof is identical to the proof of the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 81.4. ■

The following corollary is a generalization of Corollary 166.7.

Corollary 166.7. [2058]Let R be a ring and let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \longrightarrow & B_* & \longrightarrow & C_* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A'_* & \longrightarrow & B'_* & \longrightarrow & C'_* & \longrightarrow & 0 \end{array}$$

[2057]Note that we do not demand that F'_* is a *free* resolution.

be a commutative diagram of short exact sequences of chain complexes of free R-left modules. If two of the vertical maps are chain homotopy equivalences, then the third is also a chain homotopy equivalence.

Proof. The same way that we deduced the Two-Out-Of-Three-Homotopy Equivalence Corollary 81.6 from the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 81.4 one can also deduce Corollary 166.7 from the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 166.6. ■

Remark. It is also useful to know the results that do not generalize. In the proofs of Proposition 81.5 and the Lift-to-Chain Map-Proposition 81.13 we used the Subgroup-of-Free Abelian Group Lemma 51.2 which says that subgroups of free abelian groups are free abelian. Furthermore in the proof of the Finite-Rank Chain Complex Lemma 81.14 we used the Finitely Generated Abelian Group Classification Theorem 51.4. Thus the proofs of these results do not generalize in a one-to-one fashion to the setting of chain complexes over rings.

166.4. **Free imitations of chain complexes.** Recall that in the definition of a chain complex we did not demand that the chain groups are free modules. In this section we will see that we can replace any chain complex by a "similar looking" chain complex consisting of free modules. To make this precise we need the following definition.

Definition. Let R be a ring and let C_* be a chain complex of R-left modules. A free imitation of C_* is a chain complex F_* of R-left modules together with a chain map $\varphi\colon F_* \to C_*$ of R-left chain complexes that has the following two properties:

(1) The chain map is a quasi-isomorphism $\varphi\colon F_* \to C_*$, i.e. it induces an isomorphism on all homology groups.
(2) Each $\varphi\colon F_k \to C_k$ is an epimorphism.

We will now see that free imitations always exist and that they are unique up to chain homotopy equivalence.

Lemma 166.8. (Free Imitation Existence Lemma) Let R be a ring. Every chain complex of R-left modules admits a free imitation.

Remark. For $R = \mathbb{Z}$ the statement of the Free Imitation Existence Lemma 166.8 is the content of Exercise 72.16.

Proof. Let R be a ring and let (C_*, ∂_*) be a chain complex of R-left modules. We need to show that there exists a chain complex $(F_*, \widetilde{\partial}_*)$ of free R-left modules together with a chain map $\varphi\colon F_* \to C_*$ that is a quasi-isomorphism and that is an epimorphism at each degree. Most of the work is done by formulating the following claim.

Claim. Suppose that we are in the following setting

$$\begin{array}{ccccccccccccc}
 & & 0 & \longrightarrow & F_k & \xrightarrow{\widetilde{\partial}_k} & F_{k-1} & \xrightarrow{\widetilde{\partial}_{k-1}} & \dots & \xrightarrow{\widetilde{\partial}_2} & F_1 & \xrightarrow{\widetilde{\partial}_1} & F_0 & \xrightarrow{0} & 0 \\
 & & & & \downarrow{\scriptstyle \varphi_k} & & \downarrow{\scriptstyle \varphi_{k-1}} & & & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \\
\dots & \longrightarrow & C_{k+1} & \xrightarrow[\partial_{k+1}]{} & C_k & \xrightarrow[\partial_k]{} & C_{k-1} & \xrightarrow[\partial_{k-1}]{} & \dots & \xrightarrow[\partial_2]{} & C_1 & \xrightarrow[\partial_1]{} & C_0 & \xrightarrow[0]{} & 0
\end{array}$$

where the following statements hold:

(1) The top sequence of maps is a chain complex of free R-left modules.
(2) The vertical maps define a chain map.
(3) For $i = 0, \dots, k-1$ the map $\varphi_*\colon \mathrm{H}_i(F_*) \to \mathrm{H}_i(C_*)$ is an isomorphism.
(4) The map $\varphi_*\colon \mathrm{H}_k(F_*) \to \mathrm{H}_k(C_*)$ is an epimorphism.
(5) Each vertical map is an epimorphism.

Then we can extend the above horizontal sequence by one term to the left such that the extensions of the above statements hold.

Proof. Fortunately formulating the claim takes (almost) longer than proving it. We set

$$\begin{aligned} A &:= \{(f,c) \in \underbrace{\varphi_k^{-1}(\mathrm{im}(\partial_{k+1}\colon C_{k+1} \to C_k)) \cap \ker(\widetilde{\partial}_k)}_{\subset F_k} \times C_{k+1} \mid \varphi_k(f) = \partial_{k+1}(c)\} \\ B &:= \ker(\partial_{k+1}\colon C_{k+1} \to C_k). \end{aligned}$$

Next we set $F_{k+1} := R^{(A)} \oplus R^{(B)}$. We consider the map

$$\widetilde{\partial}_{k+1}\colon F_{k+1} = R^{(A)} \oplus R^{(B)} \to F_k$$

that is induced by sending each $(f,c) \in A$ to $f \in F_k$ and by sending each $b \in B$ to 0. Furthermore we consider the map

$$\varphi_{k+1}\colon F_{k+1} = R^{(A)} \oplus R^{(B)} \to C_{k+1}$$

that is induced by sending each $(f,c) \in A$ to c and by sending each $b \in B$ to $b \in C_{k+1}$. We now claim that F_{k+1} together with the above maps gives the desired extension:

(1) Evidently F_{k+1} is a free R-left module. Note that it follows easily from the definitions that $\partial_k \circ \partial_{k+1} = 0$. This implies that we do indeed have a chain complex of free R-left modules.
(2) For $(f,c) \in A$ we have

$$(\partial_{k+1} \circ \varphi_{k+1})(f,c) \underset{\text{definition of } \varphi_{k+1}}{\overset{\uparrow}{=}} \partial_{k+1}(c) \underset{\text{definition of } A}{\overset{\uparrow}{=}} \varphi_k(f) \underset{\text{definition of } \widetilde{\partial}_{k+1}}{\overset{\uparrow}{=}} (\varphi_k \circ \widetilde{\partial}_{k+1})(f,c).$$

Furthermore for $b \in B$ we have

$$(\partial_{k+1} \circ \varphi_{k+1})(b) = \partial_{k+1}(\varphi_{k+1}(b)) \underset{\text{definition of } \varphi_{k+1}}{\overset{\uparrow}{=}} \partial_{k+1}(b) \overset{\text{since } b \in B = \ker(\partial_{k+1})}{\overset{\downarrow}{=}} 0 \underset{\text{definition of } \widetilde{\partial}_{k+1}}{\overset{\uparrow}{=}} (\varphi_k \circ \widetilde{\partial}_{k+1})(b).$$

Thus the vertical maps form a chain map.

(3) We denote by F_* the chain complex starting at F_k and we denote by F'_* the chain complex starting at F_{k+1}. Recall that we assume that $\varphi_*\colon \mathrm{H}_k(F_*) \to \mathrm{H}_k(C_*)$ is an epimorphism. It follows almost immediately that $\varphi_*\colon \mathrm{H}_k(F'_*) \to \mathrm{H}_k(C_*)$ is also an epimorphism. It remains to show that $\varphi_*\colon \mathrm{H}_k(F'_*) \to \mathrm{H}_k(C_*)$ is also a monomorphism. Thus let $[\sigma] \in \mathrm{H}_k(F'_*)$ with $\varphi_*([\sigma]) = 0$. This means that $\varphi_k(\sigma) \in \mathrm{im}(\partial_{k+1}\colon C_{k+1} \to C_k)$. Also note that $\sigma \in \ker(\widetilde{\partial}_k)$. This shows that $\sigma \in \varphi_k^{-1}(\mathrm{im}(\partial_{k+1}\colon C_{k+1} \to C_k)) \cap \ker(\widetilde{\partial}_k)$. Evidently we can pick $c \in C_{k+1}$ with $\partial_{k+1}(c) = \varphi_k(\sigma)$. Note that $\widetilde{\partial}_{k+1}(\sigma, c) = \sigma$. In other words, we have shown that $\sigma \in \mathrm{im}(\widetilde{\partial}_{k+1}\colon F_{k+1} \to F_k)$. But this implies that $[\sigma] = 0 \in \mathrm{H}_k(F'_*)$.

(4) By construction the map $\varphi_{k+1}\colon \ker(\widetilde{\partial}_{k+1}\colon F_{k+1} \to F_k) \to \ker(\partial_{k+1}\colon C_{k+1} \to C_k)$ is an epimorphism. But this implies that $\varphi_*\colon \mathrm{H}_k(F_*) \to \mathrm{H}_k(C_*)$ is an epimorphism.
(5) Let $c \in C_{k+1}$. Since $\varphi_k\colon F_k \to C_k$ is an epimorphism there exists an $f \in F_k$ with $\varphi_k(f) = \partial_{k+1}(c)$. Note that $(f, c) \in A$ and note that $\varphi_{k+1}(f, c) = c$. Thus we have shown that $\varphi_{k+1}\colon F_{k+1} \to C_{k+1}$ is an epimorphism. ⊞

Now we start the iterative procedure with $k = -1$ by considering the chain complex consisting only of the zero group. By applying the claim iteratively we obtain the desired chain complex and the desired quasi-isomorphism consisting of epimorphisms. ■

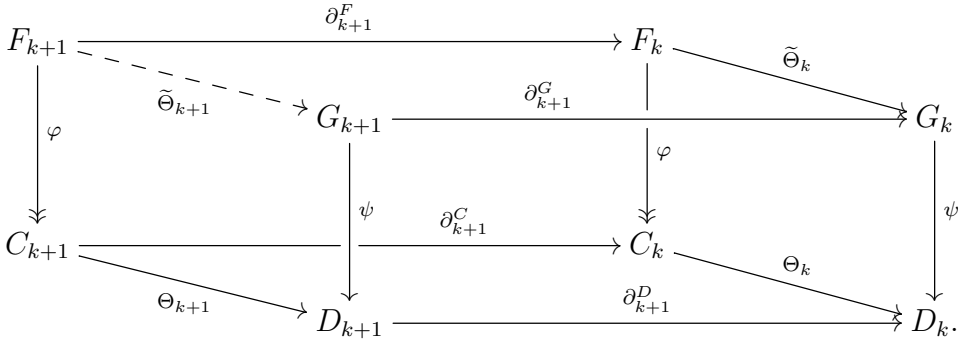
Lemma 166.9. (Homomorphism Lifting Lemma) Let R be a ring, let C_* and D_* be two chain complexes of R-left modules and let $\Theta\colon C \to D$ be a chain map of R-left chain complexes. We assume that we are given another two chain maps:

(1) Let $\varphi\colon F_* \to C_*$ be a chain map such that each F_k is a free R-left module and such that each $\varphi\colon F_k \to C_k$ is an epimorphism.
(2) Let $\varphi\colon G_* \to D_*$ be a quasi-isomorphism.

There exists a chain map $\widetilde{\Theta}\colon F \to G$ such that the following diagram commutes:

$$\begin{array}{ccc} F & \xrightarrow{\widetilde{\Theta}} & G \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ C & \xrightarrow{\Theta} & D. \end{array}$$

Proof. We construct the chain map $\widetilde{\Theta}$ inductively. We define $\widetilde{\Theta}_{-1}\colon F_{-1} = 0 \to 0 = G_{-1}$ to be, evidently, the zero map. Now suppose we have already constructed maps $\widetilde{\Theta}_i\colon F_i \to G_i$, $i = 0, \ldots, k$ which form a (partial) chain map and such that the corresponding diagram commutes. Now let us consider the following diagram:



By hypothesis F_{k+1} is a free R-left module. We pick a basis B for F_{k+1}. It suffices to define $\widetilde{\Theta}_{k+1}(b) \in G_{k+1}$ for each $b \in B$ in such that a way that the above diagram commutes for b. Thus let $b \in B$. We proceed as follows:

(a) By hypothesis (1) we know that $\psi\colon G_{k+1} \to D_{k+1}$ is an epimorphism. Therefore there exists a $g_{k+1} \in G_{k+1}$ with $\psi(g_{k+1}) = (\Theta_{k+1} \circ \varphi)(b)$.

(b) Note that the four sides of the hyperrectangle that do not involve the unknown dotted arrow commute. This implies that

$$\begin{array}{rcl} (\psi \circ \widetilde{\Theta}_k \circ \partial^F_{k+1})(b) & = & (\Theta_k \circ \varphi \circ \partial^F_{k+1})(b) \\ & = & (\partial^D_{k+1} \circ \Theta_{k+1} \circ \varphi)(b) \end{array} \begin{array}{rcl} & = & (\Theta_k \circ \partial^C_{k+1} \circ \varphi) \\ & \underset{\uparrow}{=} & (\partial^D_{k+1} \circ \psi)(g_{k+1}) \\ & & \text{definition of } g_{k+1} \end{array} \begin{array}{rcl} & = & (\psi \circ \partial^G_{k+1})(g_{k+1}). \end{array}$$

(c) It follows in particular from (b) that $\psi((\widetilde{\Theta}_k \circ \partial^F_{k+1})(b))$ and $\psi(\partial^G_{k+1}(g_{k+1}))$ are homologous in D_*.

(d) By hypothesis (2) we know that $\mathrm{H}_k(G_*) \to \mathrm{H}_k(D_*)$ is a monomorphism. This observation together with (c) implies that $(\widetilde{\Theta}_k \circ \partial^F_{k+1})(b)$ and $\partial^G_{k+1}(g_{k+1})$ are homologous in G_*.

(e) By (d) there exists an $r_{k+1} \in G_{k+1}$ with $\partial^G_{k+1}(r_{k+1}) = \partial^G_{k+1}(g_{k+1}) - (\widetilde{\Theta}_k \circ \partial^F_{k+1})(b)$.

(f) Note that

$$\partial^D_k(\psi(r_{k+1})) \underset{\substack{\uparrow \\ \text{since } \psi \text{ is a chain map}}}{=} \psi(\partial^G_k(r_{k+1})) \underset{\substack{\uparrow \\ \text{by (e)}}}{=} \psi\big(\partial^G_{k+1}(g_{k+1}) - (\widetilde{\Theta}_k \circ \partial^F_{k+1})(b)\big) \underset{\substack{\uparrow \\ \text{by (b)}}}{=} 0.$$

In other words, $\psi(r_{k+1}) \in D_{k+1}$ is a cycle in the chain complex D_*.

(g) By hypothesis (2) we know that $\mathrm{H}_{k+1}(G_*) \to \mathrm{H}_{k+1}(D_*)$ is an epimorphism. Note that it follows from this observation and (f) that there exists a *cycle* $s_{k+1} \in G_{k+1}$ with $\psi(s_{k+1}) = \psi(r_{k+1})$.

Claim. If we set $\widetilde{\Theta}_{k+1}(b) := g_{k+1} - r_{k+1} + s_{k+1}$, then the diagram commutes for b.

Proof.

- We first consider the upper parallelogram. We have the following equality in G_k:

$$(\partial^G_{k+1} \circ \widetilde{\Theta}_{k+1})(b) \underset{\substack{\uparrow \\ \text{by definition of } \widetilde{\Theta}_{k+1}(b)}}{=} \partial^G_{k+1}(g_{k+1} - r_{k+1} + s_{k+1}) \underset{\substack{\uparrow \\ \text{since } s_{k+1} \text{ is a cycle}}}{=} \partial^G_{k+1}(g_{k+1} - r_{k+1}) \underset{\substack{\uparrow \\ \text{by (e)}}}{=} \widetilde{\Theta}_k(\partial^F_{k+1}(b)).$$

- We consider the parallelogram to the left. We have the following equality in D_{k+1}:

$$(\psi \circ \widetilde{\Theta}_{k+1})(b) \underset{\substack{\uparrow \\ \text{by definition of } \widetilde{\Theta}_{k+1}(b)}}{=} \psi(g_{k+1} - r_{k+1} + s_{k+1}) \underset{\substack{\uparrow \\ \text{since } \psi(r_{k+1}) = \psi(s_{k+1})}}{=} \psi(g_{k+1}) \underset{\substack{\uparrow \\ \text{by (a)}}}{=} (\Theta_{k+1} \circ \varphi)(b).$$

■

Lemma 166.10. (Free Imitation Uniqueness Lemma) Let R be a ring and let (C_*, ∂_*) be a chain complex of R-left modules. If $\varphi\colon F_* \to C_*$ and $\varphi'\colon F'_* \to C_*$ are two free imitations, then there exists a chain homotopy equivalence $\Theta\colon F_* \to F'_*$ of R-left chain complexes such that $\varphi' \circ \Theta = \varphi$.

Proof. By the Homomorphism Lifting Lemma 166.9 there exists a chain map $\widetilde{\Theta}\colon F \to F'$ such that the following diagram commutes:

$$\begin{array}{ccc} F & \xrightarrow{\widetilde{\Theta}} & F' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ C & \xrightarrow{\mathrm{id}} & C. \end{array}$$

Note that Θ is a quasi-isomorphism. Since F and F' are chain complexes of free R-left modules we obtain from the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 166.6 that $\widetilde{\Theta}\colon F_* \to F'_*$ is in fact a chain homotopy equivalence. ■

166.5. **Connecting homomorphisms.**

Definition. Let R be a ring and let

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p_*} C_* \to 0$$

be a short exact sequence of R-left modules. Using the recipe from page 1620 we define a connecting homeomorphism

$$\partial_n\colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$$

as follows:

(1) Let $z \in \mathrm{H}_n(C)$.
(2) We pick $c \in C_n$ with $z = [c]$.
(3) By the surjectivity of p we have $c = p(b)$ for some $b \in B_n$.
(4) We have
$$p(\partial b) \underset{\text{since the diagram commutes}}{=} \partial p(b) = \partial c \underset{\text{since } c \text{ is a cycle}}{=} 0.$$
(5) It follows easily from (4) that $\partial b = i(a)$ for some uniquely determined $a \in A_{n-1}$.
(6) We put $\partial_n(z) := [a]$.

We refer to this map $\partial_n\colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$ as the connecting homomorphism.

The following lemma is the analogue of the Connecting Homomorphism Lemma 74.8 of the LES Proposition 74.9.

Lemma 166.11. (LES Lemma) Let R be a ring.

(1) Let $0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$ be a short exact sequence of chain complexes of R-left modules. The connecting homomorphism $\partial_n\colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$ is well-defined and it is a homomorphism of R-left modules
(2) The connecting homomorphism is natural.
(3) The sequence
$$\ldots \to \mathrm{H}_n(A) \xrightarrow{i_*} \mathrm{H}_n(B) \xrightarrow{p_*} \mathrm{H}_n(C) \xrightarrow{\partial} \mathrm{H}_{n-1}(A) \xrightarrow{i_*} \mathrm{H}_{n-1}(B) \xrightarrow{p_*} \ldots$$
is exact.

Proof. The proof is basically the same as the proof of the Connecting Homomorphism Lemma 74.8 and of the LES Proposition 74.9. ■

166.6. **Homological algebra over skew fields.** In this section we introduce skew fields. In most applications the skew fields we use are in fact just ordinary fields. On the other hand, it costs little to generalize to skew fields and working with (possibly) non-commutative skew fields actually makes some of the future arguments clearer since it prevents us from using "accidents" in the commutative setting, e.g. the fact that over a commutative ring every left-module is automatically a right module.

We have the following generalization of Theorems **??** and Theorem 110.20.

Theorem 166.12. Let $\mathbb{K}$ be a skew field and let (V_*, ∂_*) be a chain complex of $\mathbb{K}$-left modules. The map

$$\begin{array}{rcl} \mathrm{ev}_{\mathbb{K}} \colon \mathrm{H}^k(V;\mathbb{K}) & \to & \mathrm{Hom}_{\mathbb{K}\text{-left}}(\mathrm{H}_k(V),\mathbb{K}). \\ (\varphi \colon V_k \to \mathbb{K}) & \mapsto & \begin{pmatrix} \mathrm{H}_k(V) & \to & \mathbb{K} \\ [c] & \mapsto & \varphi(c) \end{pmatrix} \end{array}$$

is well-defined and it is an isomorphism of $\mathbb{K}$-right modules.

The proof of Theorem 166.12 is almost verbatim the same as the proof of Theorem **??**. For the reader's convenience we nonetheless provide the proof.

Auxiliary Lemma 166.13. Let $\mathbb{K}$ be a skew field and let $0 \to B \xrightarrow{i} Z \xrightarrow{p} H \to 0$ be a short exact sequence of $\mathbb{K}$-left modules. Then the sequence

$$0 \to \mathrm{Hom}_{\mathbb{K}\text{-left}}(H,\mathbb{K}) \xrightarrow{p^*} \mathrm{Hom}_{\mathbb{K}\text{-left}}(Z,\mathbb{K}) \xrightarrow{i^*} \mathrm{Hom}_{\mathbb{K}\text{-left}}(B,\mathbb{K}) \to 0$$

is also exact.

Proof of Lemma 166.13. This statement follow easily from the fact, see Proposition 165.26, that any short exact sequence of $\mathbb{K}$-left modules splits. We leave it to the reader to fill in the few remaining details. ∎

Proof of Theorem 166.12. Let $\mathbb{K}$ be a skew field and let $(\mathrm{C}_*, \partial_*)$ be a chain complex of $\mathbb{K}$-left modules. For each $n \in \mathbb{N}_0$ we write as always $Z_n := \ker(\partial_n)$ and $B_n := \mathrm{im}(\partial_{n+1})$. We consider the following short exact sequence of vertical chain complexes:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_n & \longrightarrow & \mathrm{C}_n & \xrightarrow{\partial_n} & B_{n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & Z_{n-1} & \longrightarrow & \mathrm{C}_{n-1} & \xrightarrow{\partial_{n-1}} & B_{n-2} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

We apply the functor $\mathrm{Hom}(-,\mathbb{K}) := \mathrm{Hom}_{\mathbb{K}\text{-left}}(-,\mathbb{K})$ to the above short exact sequence of chain complexes and we obtain the following diagram:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathrm{Hom}(B_{n-1},\mathbb{K}) & \xrightarrow{\partial_n^*} & \mathrm{Hom}(C_n,\mathbb{K}) & \longrightarrow & \mathrm{Hom}(Z_n,\mathbb{K}) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle \partial_n^*} & & \uparrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & \mathrm{Hom}(B_{n-2},\mathbb{K}) & \xrightarrow{\partial_{n-1}^*} & \mathrm{Hom}(C_{n-1},\mathbb{K}) & \longrightarrow & \mathrm{Hom}(Z_{n-1},\mathbb{K}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

This is a sequence of cochain complexes. Since we work over a skew field we obtain from Lemma 166.13 that the above horizontal sequences of cochain complexes are still exact. By the obvious analogue of the Connecting Homomorphism Proposition 108.6 we obtain the following long exact sequence

$$\to \mathrm{Hom}(Z_{n-1},\mathbb{K}) \xrightarrow{d_n} \mathrm{Hom}(B_{n-1},\mathbb{K}) \to \mathrm{H}^n(C;\mathbb{K}) \xrightarrow{\partial_n} \mathrm{Hom}(Z_n,\mathbb{K}) \xrightarrow{d_{n-1}} \mathrm{Hom}(B_n,\mathbb{K}) \to \dots$$

where the maps d_n are the connecting homomorphisms of the long exact sequence. One can show easily that the connecting homomorphism $d_n \colon \mathrm{Hom}(Z_n,\mathbb{K}) \to \mathrm{Hom}(B_n,\mathbb{K})$ is the

map $d_n = i_n^*$ where $i_n \colon B_n \to Z_n$ is the inclusion map. By the Break-into-SES Lemma 78.4 we obtain the following short exact sequence

$$0 \to \underbrace{\operatorname{coker}(\operatorname{Hom}(Z_{n-1}, \mathbb{K}) \to \operatorname{Hom}(B_{n-1}, \mathbb{K}))}_{= 0 \text{ by Lemma 166.13 applied to the short exact sequence } 0 \to B_l \to Z_l \to \mathrm{H}_l(C) \to 0} \to \mathrm{H}^n(C; \mathbb{K}) \to \underbrace{\ker(\operatorname{Hom}(Z_n, \mathbb{K}) \to \operatorname{Hom}(B_n, \mathbb{K}))}_{\cong \operatorname{Hom}(\mathrm{H}_n(C), \mathbb{K}) \text{ by Lemma 166.13 applied to the short exact sequence } 0 \to B_l \to Z_l \to \mathrm{H}_l(C) \to 0} \to 0.$$

■

166.7. Tor-modules.

Definition. Let S and S be rings and let M be an (R, S)-bimodule. We take the canonical free resolution

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$$

of free (R, S)-bimodules and we define

$$\operatorname{Tor}_n^S(M, N) := \mathrm{H}_n(F_* \otimes_S N).$$

Note that each $\operatorname{Tor}_n^S(M, N)$ is naturally an R-left module.

Many, but not all, of our results on the Tor groups of abelian groups generalize to this new setting. The following lemma is a generalization of the Tor_0–Lemma 89.13.

Lemma 166.14. Let S be a ring, let M be an (R, S)-bimodule and let N be an S-left module. There exists a natural isomorphism

$$\operatorname{Tor}_0^S(M, N) \xrightarrow{\cong} M \otimes_R N$$

of R-left modules.

Proof. The proof is basically the same as the proof of the Tor_0–Lemma 89.13. We just need to replace Lemma 89.10 by Lemma **??**. ■

Here the following statements hold:

(1) As in the Tor-Free Resolution Proposition 89.12 one can show that the isomorphism type of $\operatorname{Tor}_n^S(M, N)$ can be determined using any free resolution for M.
(2) The argument of page 1923 shows that for each $n \in \mathbb{N}_0$ the S-torsion module $\operatorname{Tor}_n^S(M, N)$ is covariantly functorial in both entries.
(3) The obvious analogues of the Tor-Properties Lemma 89.15 (1) and (2) also hold in the more general context. Furthermore the obvious analogues of the Tor-Properties Lemma 89.15 (3) and (4) hold if we replace "torsion-free abelian group" by "free S-module".[2059] In general there are no analogues of the Tor-Properties Lemma 89.15 (5), (6) and (7).
(4) For general commutative rings it is no longer true though that $\operatorname{Tor}_i^S(M, N) = 0$ for $i \geq 2$. For example for $S = \mathbb{Q}[t_1, \ldots, t_k]$ the S-torsion modules can be non-zero for $i = 1, \ldots, k$ and they are only zero for $i > k$. We refer to [**Mac1975**, Chapter VII.4] for details.

Proposition 166.15. If $\mathbb{K}$ is a skew field, then for any $\mathbb{K}$-left module M and any $n \geq 1$ we have $\operatorname{Tor}_n(M, \mathbb{K}) = 0$.

Proof. This follows from Proposition 165.26. ■

[2059] Note that in general "S-torsion free module" is not going to be strong enough.

The following theorem is a generalization of the Algebraic Universal Coefficient Theorem 89.16 and of the UCT-Splitting Proposition 89.21 from the PID $\mathbb{Z}$ to general PIDs.

Theorem 166.16. (Algebraic Universal Coefficient Theorem over Rings) Let S be a PID[2060], let (C_n, ∂_n) be a chain complex of free S modules and let M be an S-module. For each $n \in \mathbb{N}_0$ there exists a natural homomorphism

$$p\colon \mathrm{H}_n(M \otimes_S C_*) \ \to\ \mathrm{Tor}_S(\mathrm{H}_{n-1}(C), M)$$

of S-modules such that the following sequence is exact:

$$0 \ \to\ \mathrm{H}_n(C) \otimes_S M \ \xrightarrow{\mu}\ \mathrm{H}_n(M \otimes_S C_*) \ \xrightarrow{p}\ \mathrm{Tor}_S(\mathrm{H}_{n-1}(C), M) \ \to\ 0.$$

This short exact sequence splits and there exists therefore an isomorphism

$$\mathrm{H}_n(M \otimes_S C_*) \ \cong\ \mathrm{H}_n(C) \otimes_S M \oplus \mathrm{Tor}_S(\mathrm{H}_{n-1}(C), M).$$

of S-modules.

Proof. The proof of Theorem 166.16 is basically identical to the proof of the Algebraic Universal Coefficient Theorem 89.16 and the UCT-Splitting Proposition 89.21. All one really has to do is to the following replacements:

(1) Replace the Subgroup-of-Free Abelian Group Lemma 51.2 (1) by Theorem 165.25.
(2) Replace Lemma 78.1 by Lemma 78.1.
(3) Replace the Tensor-Split-SES Lemma 89.6 by Lemma 165.18.
(4) Replace the LES Proposition 74.9 by the LES Lemma 166.11.
(5) Replace the Tor_0–Lemma 89.13 by Lemma 166.14.
(6) Replace the Splitting Lemma 78.2 by the Splitting Lemma 165.7 for Rings. ■

On the other hand, if R is not a principal ideal, then as we pointed out above, the higher torsion groups are in general non-zero and the problem becomes much more subtle. The torsion-terms and the homology with M-coefficients are then related by a "spectral sequence", see e.g. [**Rot2009**, Theorem 10.90] for details.

166.8. **Ext-modules.** We have the following generalization of the Hom-Left Exact Lemma 110.1.

Lemma 166.17. Let S be a ring. For any S-left module M the contravariant functor $\mathrm{Hom}_{S\text{-left}}(-, M)$ is left-exact.

Proof. The proof is basically the same as the proof of the Hom-Left Exact Lemma 110.1. ■

Definition. Let S and S be rings and let M be an (R, S)-left left module. We take the canonical free resolution

$$\cdots \to\ F_2 \ \xrightarrow{f_2}\ F_1 \ \xrightarrow{f_1}\ F_0 \ \xrightarrow{f_0}\ M \ \to\ 0$$

of free (R, S)-left left modules and we define

$$\mathrm{Ext}^n_{S\text{-left}}(M, N) \ :=\ \mathrm{H}^n(\mathrm{Hom}_{S\text{-left}}(F_*, N)).$$

Note that each $\mathrm{Ext}^n_{S\text{-left}}(M, N)$ is naturally an R-left module.

[2060] Recall that according to the definition on page 3515 a PID is commutative. So we do not need to worry about left and right module structures.

Here the following statements hold:

(1) As in the Ext-Free Resolution Proposition 110.2 one can show that the isomorphism type of $\mathrm{Ext}^n_{S\text{-left}}(M,N)$ can be determined using any free resolution for M.

Proposition 166.18. If $\mathbb{K}$ is a skew field, then for any $\mathbb{K}$-left module M and any $n \geq 1$ we have $\mathrm{Ext}^n_{\mathbb{K}\text{-left}}(M,\mathbb{K}) = 0$.

Proof. This follows from Proposition 165.26. ■

Definition. Let S be a ring, let $(\mathrm{C}_*, \partial_*)$ be a chain complex of S-left modules and let M be an S-left module. We refer to the map[2061]

$$\begin{array}{rcl} \mathrm{ev}\colon \mathrm{H}^n(C;M) & \to & \mathrm{Hom}_{S\text{-left}}(\mathrm{H}_n(C),M) \\ [\varphi\colon \mathrm{C}_n \to M] & \mapsto & \begin{pmatrix} \mathrm{H}_n(C) & \to & M \\ [c] & \mapsto & \langle \varphi(c) \end{pmatrix} \end{array}$$

as the evaluation homomorphism.

Theorem 166.19. (Universal Coefficient Theorem for Cohomology Modules) Let S be a PID, let $(\mathrm{C}_*, \partial_*)$ be a chain complex of free S-modules and let M be an S-module. For each $n \in \mathbb{N}_0$ there exists a homomorphism

$$\epsilon\colon \mathrm{Ext}_S(\mathrm{H}_{n-1}(C),M) \to \mathrm{H}^n(C;M)$$

of S-modules which is natural in the chain complex C_* and natural in the S-module M and which has the property that the following sequence is exact:

$$0 \to \mathrm{Ext}_S(\mathrm{H}_{n-1}(C),M) \xrightarrow{\epsilon} \mathrm{H}^n(C;M) \xrightarrow{\mathrm{ev}} \mathrm{Hom}_S(\mathrm{H}_n(C),M) \to 0.$$

This short exact sequence splits and there exists therefore an isomorphism[2062]

$$\mathrm{H}^n(C;M) \cong \mathrm{Ext}_S(\mathrm{H}_{n-1}(C),M) \oplus \mathrm{Hom}_S(\mathrm{H}_n(C),M)$$

of S-modules.

Proof. The proof of Theorem 166.19 is again basically identical to the proof of the Algebraic Universal Coefficient Theorem 110.11. ■

Exercises for Chapter 166.

[2061] One can easily verify that the evaluation homomorphism is well-defined.

[2062] Since this cannot be mentioned often enough, recall that in Exercise 110.1 we saw that this isomorphism is not natural.

167. Twisted (co-) homology

In the last two chapters we studied (homological) algebra over a possibly non-commutative ring. Our favorite ring in this chapter are going to be the group rings $\mathbb{Z}[\pi_1(X, x_0)]$ of the fundamental group of a pointed topological space (X, x_0). Furthermore our favorite modules are going to be the chain groups $\mathrm{C}_*(\widetilde{X}_{x_0})$ of the natural universal covering of a pointed topological space. (X, x_0). As we will see, these are naturally $\mathbb{Z}[\pi_1(X, x_0)]$-left modules.

167.1. **Basics of covering theory.** We recall some concepts and results from our earlier study of covering theory. We start out our recap with the following definition from page 1383.

Definition. Let $p\colon \widetilde{X} \to X$ be a covering in the sense of the definition on page 1067.

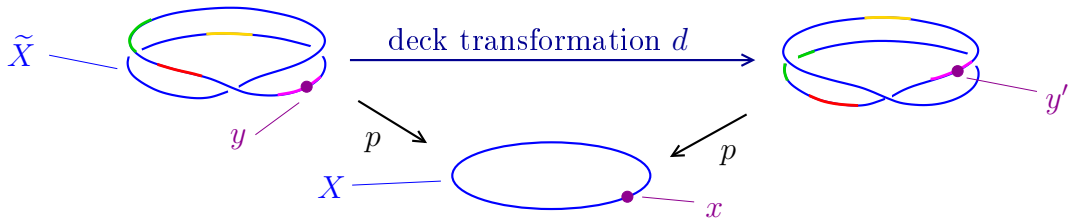
(1) A deck transformation is a homeomorphism $d\colon \widetilde{X} \to \widetilde{X}$ such that $p(d(x)) = p(x)$ for all $x \in \widetilde{X}$, i.e. such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow[\cong]{d} & \widetilde{X} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & X. & \end{array}$$

The group of deck transformations is called the deck transformation group of the covering $p\colon \widetilde{X} \to X$ and we denote it by $\mathrm{Deck}(p\colon \widetilde{X} \to X)$.

(2) We say that the covering is regular if for every $x \in X$ the deck transformation group $\mathrm{Deck}(p\colon \widetilde{X} \to X)$ acts transitively on $p^{-1}(x)$, i.e. if given any two points $y, y' \in \widetilde{X}$ with $p(y) = p(y') = x$ there exists a deck transformation d with $d(y) = y'$. Otherwise we say that the covering is irregular.

(3) Let P be a property of isomorphism types of groups, e.g. cyclic, free, nilpotent, solvable etc. We say the covering $p\colon \widetilde{X} \to X$ is a P-covering if the covering is regular and if the deck transformation group has the property P.



In the following we recall some basic statements about deck transformation groups.

Lemma 62.2. Given any covering $p\colon \widetilde{X} \to X$ of path-connected topological spaces and given any $x_0 \in X$ the deck transformation group $\mathrm{Deck}(p\colon \widetilde{X} \to X)$ acts freely on $p^{-1}(x_0)$.

The following simplified version of the Regular Covering-Characterization Corollary 62.8 gives us a criterion for a covering to be regular.

Corollary 62.8. Let $p\colon (\widetilde{X},\widetilde{x}_0) \to (X,x_0)$ be a covering of pointed topological spaces that are path-connected and locally path-connected. Then the following statements are equivalent:

(1) The covering is regular.
(2) The deck transformation group $\mathrm{Deck}(p\colon \widetilde{X} \to X)$ acts transitively on $p^{-1}(x_0) \subset \widetilde{X}$.
(3) The subgroup $p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$ is a normal subgroup of $\pi_1(X,x_0)$.

Proposition 167.1. Let $p\colon (\widetilde{X},\widetilde{x}_0) \to (X,x_0)$ be a *regular* covering of pointed topological spaces. We suppose that $\widetilde{X}$ and X are both path-connected and locally path-connected.

(1) The map[2063]

$$\begin{aligned}\Xi_{\widetilde{X},\widetilde{x}_0}\colon \mathrm{Deck}(p\colon \widetilde{X} \to X) &\to \pi_1(X,x_0)/p_*(\pi_1(\widetilde{X},\widetilde{x}_0))\\ d &\mapsto \big[p \circ (\text{path in } \widetilde{X} \text{ from } \widetilde{x}_0 \text{ to } d(\widetilde{x}_0))\big].\end{aligned}$$

is an isomorphism.

(2) Let $q\colon (\widetilde{Y},\widetilde{y}_0) \to (Y,y_0)$ be another *regular* covering of pointed topological spaces such that $\widetilde{Y}$ and Y are both path-connected and locally path-connected. Suppose we are given maps $\Phi\colon \widetilde{X} \to \widetilde{Y}$ and $\varphi\colon X \to Y$ such that $\Phi(\widetilde{x}_0) = \widetilde{y}_0$ and such that the following diagram commutes:

$$\begin{array}{ccc}\widetilde{X} & \xrightarrow{\Phi} & \widetilde{Y}\\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q}\\ X & \xrightarrow{\varphi} & Y.\end{array}$$

Then the following diagram commutes:

$$\begin{array}{ccc}\mathrm{Deck}(p\colon \widetilde{X} \to X) & \xrightarrow{\Xi_{\widetilde{X},\widetilde{x}_0}} & \pi_1(X,x_0)/p_*(\pi_1(\widetilde{X},\widetilde{x}_0))\\ \downarrow & & \downarrow{\scriptstyle \varphi_*}\\ \mathrm{Deck}(q\colon \widetilde{Y} \to Y) & \xrightarrow{\Xi_{\widetilde{Y},y_0}} & \pi_1(Y,x_0)/q_*(\pi_1(\widetilde{Y},y_0)),\end{array}$$

where the horizontal maps are the maps we introduced (1), where the left vertical map is the map given by the Induced-Deck Transformation Proposition 62.9 and the right vertical map is the projection.[2064]

Proof. The first statement is a simplified version of Proposition 62.6. The second statement is the content of Lemma 62.11. ∎

Convention. Let $p\colon (\widetilde{X},\widetilde{x}_0) \to (X,x_0)$ be a *regular* covering of pointed topological spaces. We suppose that $\widetilde{X}$ and X are both path-connected and locally path-connected. The

[2063]Since $p\colon \widetilde{X} \to X$ is a regular covering we know by the Regular Covering-Characterization Corollary 62.8 that $p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$ is a normal subgroup of $\pi_1(X,x_0)$, thus $\pi_1(X,x_0)/p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$ is a group.
[2064]It follows from $p = q \circ r$ that $p_*(\pi_1(\widetilde{X},\widetilde{x}_0)) \subset q_*(\pi_1(\widetilde{Y},y_0))$.

isomorphism

$$\begin{array}{rccl}\Xi_{\widetilde{X},\widetilde{x}_0}\colon & \mathrm{Deck}(p\colon \widetilde{X}\to X) & \to & \pi_1(X,x_0)/p_*(\pi_1(\widetilde{X},\widetilde{x}_0)) \\ & d & \mapsto & \big[p\circ(\text{path in } \widetilde{X} \text{ from } \widetilde{x}_0 \text{ to } d(\widetilde{x}_0))\big]\end{array}$$

from the above Proposition 167.1 allows us to equip $\widetilde{X}$ with a natural left action by the group $\pi_1(X,x_0)/p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$. More precisely, we equip $\widetilde{X}$ with the left action

$$\begin{array}{rcl}\pi_1(X,x_0)/p_*(\pi_1(\widetilde{X},\widetilde{x}_0))\times\widetilde{X} & \to & \widetilde{X} \\ (g,y) & \mapsto & \underbrace{((\Xi_{\widetilde{X},\widetilde{x}_0})^{-1})(g)}_{\in\mathrm{Deck}(p\colon \widetilde{X}\to X)}(y).\end{array}$$

We move on to covering that will be most relevant for us throughout the following chapters.

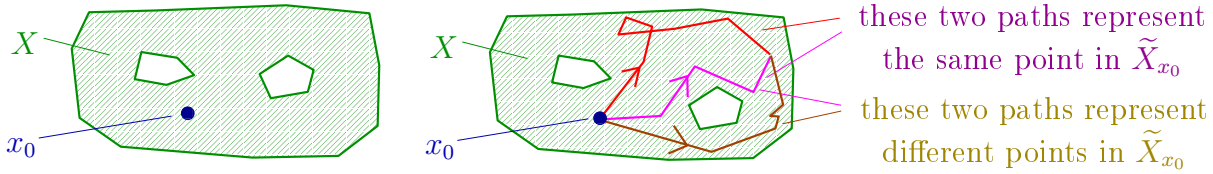
Notation. Let X be a topological space that is locally contractible and path-connected and let $x_0\in X$.

(1) We set

$$\widetilde{X}_{x_0} \;=\; \{\gamma\colon[0,1]\to X\,|\,\gamma(0)=x_0\}/\!\sim \qquad \begin{array}{l}\text{where } \alpha\sim\beta \text{ if } \alpha(1)=\beta(1) \text{ and} \\ \alpha*\overline{\beta} \text{ is a null-homotopic loop in } (X,x_0)\end{array}$$

and we equip $\widetilde{X}_{x_0}$ with the topology that we introduced in the proof of the Covering Existence Theorem 61.5.

(2) We denote by $p\colon\widetilde{X}_{x_0}\to X$ the evaluation map $[\gamma]\mapsto\gamma(1)$. In the proof of the Covering Existence Theorem 61.5 we showed that $\widetilde{X}_{x_0}$ is simply connected and we showed that $p\colon\widetilde{X}_{x_0}\to X$ is a covering map. As on page 1370 we refer to $p\colon\widetilde{X}_{x_0}\to X$ as the natural universal covering of (X,x_0). Furthermore we equip $\widetilde{X}_{x_0}$ with the natural base point $\widetilde{x}_0$ that is given by the constant path.



Remark. Let X be a topological space that is locally contractible and path-connected and let $x_0\in X$. We denote by $p\colon\widetilde{X}_{x_0}\to X$the corresponding natural universal covering. By the discussion on page 3534 we have a natural $\pi_1(X,x_0)$ left action on $\widetilde{X}_{x_0}$. As we saw in Exercise 62.7 this left action is given by

$$\begin{array}{rcl}\pi_1(X,x_0)\times\widetilde{X}_{x_0} & \to & \widetilde{X}_{x_0} \\ ([\alpha],[\gamma]) & \mapsto & [\alpha*\gamma].\end{array}$$

Remark. In the following chapters, when we discuss (co-) homology with twisted coefficients, we will only be interested in topological spaces that are locally contractible. Throughout these chapters we frequently and often subconsciously make use of the following two facts:

(1) By the Topological Manifolds-Local Properties Lemma 18.8 every topological manifold is locally contractible and by the CW-Complex Properties Proposition 68.10 (6) every CW-complex is locally contractible.

(2) If X is a locally contractible, then it follows from Lemmas 2.40 and 15.3 that X is path-connected if and only if X is connected. For no particular reason we usually write "locally contractible and path-connected" instead of the equivalent notion "locally contractible and connected".

167.2. **Definition of twisted (co-) homology.** At this stage it is worth recalling the following proposition.

Proposition 61.2. (Map Lifting Criterion) Let $p\colon (\widetilde{X},\widetilde{x}_0) \to (X,x_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected topological space and let $f\colon (Z,z_0) \to (B,b_0)$ be a map.

(1) If Z is locally path-connected, then we have the following equivalence of statements:

$$\begin{array}{ccc} & & (\widetilde{X},\widetilde{x}_0) \\ & \overset{\text{there exists a lift } \widetilde{f}}{\nearrow} & \downarrow p \\ (Z,z_0) & \xrightarrow[f]{} & (X,x_0) \end{array} \qquad \Longleftrightarrow \qquad \operatorname{im}(f_*) \subset \operatorname{im}(p_*).$$

(2) There exist at most one such lift, in other words, any two such lifts agree.

We will now apply the Map Lifting Criterion 61.2 to the study of the singular chain complex of a regular covering:

Lemma 167.2. (Regular Covering-Chain Complex Lemma) Let X be a topological space that is locally path-connected and path-connected. Furthermore let $p\colon \widetilde{X} \to X$ be a regular covering and let $A \subset X$. We write $\widetilde{A} := p^{-1}(A)$. We write $D := \operatorname{Deck}(p\colon \widetilde{X} \to X)$. Note that the natural left action of D on $\widetilde{X}$ preserves $\widetilde{A}$. This allows us to view each $\mathrm{C}_k(\widetilde{X},\widetilde{A})$ as a $\mathbb{Z}[D]$-left module.

(1) For each singular simplex $\sigma\colon \Delta^k \to X$ there exists a singular simplex $\widetilde{\sigma}\colon \Delta^k \to \widetilde{X}$ with $p \circ \widetilde{\sigma} = \sigma$.

(2) For each singular simplex $\sigma\colon \Delta^k \to X$ that is not contained in A we pick a lift $\widetilde{\sigma}\colon \Delta^k \to X$.[2065] The set of singular simplices

$$\{\widetilde{\sigma}\colon \Delta^k \to \widetilde{X} \,|\, \sigma\colon \Delta^k \to X \text{ with } \sigma(\Delta^k) \not\subset A\}$$

is a basis for the $\mathbb{Z}[D]$-left module $\mathrm{C}_k(\widetilde{X},\widetilde{A})$. In particular $\mathrm{C}_k(\widetilde{X},\widetilde{A})$ is a free $\mathbb{Z}[D]$-left module.

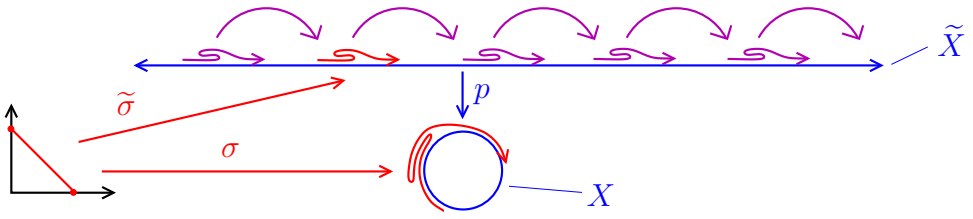
(3) The usual boundary map $\partial_k\colon \mathrm{C}_k(\widetilde{X},\widetilde{A}) \to \mathrm{C}_{k-1}(\widetilde{X},\widetilde{A})$ is a homomorphism of $\mathbb{Z}[D]$-left modules.

(4) $(\mathrm{C}_*(\widetilde{X},\widetilde{A}),\partial_*)$ is a chain complex of free $\mathbb{Z}[D]$-left modules.

Proof.

(1) First recall that a covering is by definition surjective. Since Δ^k is evidently locally path-connected and path-connected we thus obtain from the Map Lifting Criterion 61.2 (1) that we can lift any map $\sigma\colon \Delta^k \to X$ to a map $\widetilde{\sigma}\colon \Delta^k \to \widetilde{X}$.

[2065] It follows from Statement (1) and the Axiom of Choice 0.1 that we can do so.



(2) Let $k \in \mathbb{N}_0$. We start out with some notation. Given a singular simplex $\sigma\colon \Delta^k \to X$ with $\sigma(\Delta^k) \not\subset A$ we consider

$$L(\sigma) \;=\; \{\text{all lifts of } \sigma \text{ to maps } \Delta^k \to \widetilde{X}\}.$$

We make two observations:

(a) It follows easily from our hypothesis that $p\colon \widetilde{X} \to X$ is a regular covering, together with the Deck Group-Free Action Lemma 62.2 and the Map Lifting Criterion 61.2 (1), that the obvious action of D on $L(\sigma)$ is free and transitive.

(b) It follows easily from the definitions that as an abelian group $\mathrm{C}_k(\widetilde{X}, \widetilde{A})$ is generated by the union of all the $L(\sigma)$.

It follows from these two observations, together with Lemma 165.9, that $\mathrm{C}_k(\widetilde{X}, \widetilde{A})$ is a free $\mathbb{Z}[D]$-left module and that a basis is given by a choice of a lift for each such σ.

(3) By Lemma 72.9 we know that the boundary maps are natural, which implies in particular that the boundary map $\partial_k\colon \mathrm{C}_k(\widetilde{X}, \widetilde{A}) \to \mathrm{C}_{k-1}(\widetilde{X}, \widetilde{A})$ commute with the D-action. This means that ∂_k is a homomorphism of $\mathbb{Z}[D]$-left modules.

(4) This statement is just a summary of (1) and (2). ■

Convention. Let X be a topological space that is locally contractible and path-connected, let $x_0 \in X$ and let $A \subset X$.

(1) Let $\Gamma \subset \pi_1(X, x_0)$ be a normal subgroup. If $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is any path-connected covering with $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = \Gamma$[2066], then it follows from the convention on page 3534 together with the Regular Covering-Chain Complex Lemma 167.2 that we can view $\mathrm{C}_*(\widetilde{X}, \widetilde{A})$ as a chain complex of free $\mathbb{Z}[\pi_1(X, x_0)/\Gamma]$-left modules.

(2) In particular we can view $\mathrm{C}_*(\widetilde{X}_{x_0}, \widetilde{A})$ as a chain complex of free $\mathbb{Z}[\pi_1(X, x_0)]$-left modules.

the Regular Covering-Chain Complex Lemma 167.2 is very powerful since, in the setting of the above convention, we can apply all kinds of algebraic tools from the last two chapters to the chain complex $\mathrm{C}_*(\widetilde{X}_{x_0}, \widetilde{A})$ of free $\mathbb{Z}[\pi_1(X, x_0)]$-left modules to define new invariants. Before we introduce these new invariants, let us recall the following notation from page 3503.

Notation. Let G be a group and let M be a $\mathbb{Z}[G]$-left module. We denote by $\overline{M}$ the $\mathbb{Z}[G]$-right module that is given by $m \underset{\overline{M}}{\cdot} g := g^{-1} \underset{M}{\cdot} m$.

Definition. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. Furthermore let $A \subset X$. Finally let R be a ring and let M be a

[2066] It follows from the Covering Existence Theorem 61.5 that such a covering does indeed exist.

$(\mathbb{Z}[\pi_1(X_0,x_0)],R)$-left left module. We denote by $p\colon \widetilde{X}_{x_0}\to X$ the natural universal covering and we write $\widetilde{A}:=p^{-1}(A)$. Given $k\in\mathbb{N}_0$ we define the R-left modules[2067]

$$\begin{array}{lcll} \mathrm{C}_k^{\mathrm{tw}}(X,A;M) & := & \overline{M}\otimes_{\mathbb{Z}[\pi_1(X,x_0)]}\mathrm{C}_k(\widetilde{X}_{x_0},\widetilde{A}) & (\text{twisted chain complex})\\ \mathrm{C}^k_{\mathrm{tw}}(X,A;M) & := & \mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]\text{-left}}(\mathrm{C}_k(\widetilde{X}_{x_0},\widetilde{A}),M) & (\text{twisted cochain complex}).\end{array}$$

Since the boundary maps are $\mathbb{Z}[\pi_1(X,x_0)]$-left module homomorphisms we obtain from the Tensoring Chain Complex Lemma 166.3 and the discussion on page 3520 that $\mathrm{id}\otimes\partial$ and $\mathrm{Hom}(\partial,\mathrm{id})$ define (co-) boundary maps. Thus we can define the R-left modules

$$\begin{array}{lcll} \mathrm{H}_k^{\mathrm{tw}}(X,A;M) & := & \mathrm{H}_k\big(\overline{M}\otimes_{\mathbb{Z}[\pi_1(X,x_0)]}\mathrm{C}_*(\widetilde{X}_{x_0},\widetilde{A})\big) & (\text{twisted homology})\\ \mathrm{H}^k_{\mathrm{tw}}(X,A;M) & := & \mathrm{H}^k\big(\mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]\text{-left}}(\mathrm{C}_*(\widetilde{X}_{x_0},\widetilde{A}),M)\big) & (\text{twisted cohomology}).\end{array}$$

As always, if $A=\varnothing$, then we drop it from the notation.
To discuss our first example let us recall the following definition from page 3507.

Definition. Let G be a group and let R be a ring. Given an R-left module M we denote by ${}_{\mathrm{tr}}M$ the $(\mathbb{Z}[G],R)$-left left module where the $\mathbb{Z}[G]$-left module structure is given by the trivial g-action $g\cdot m:=m$ for each $g\in G$ and $m\in M$.

Let (X,A) be a pair of topological spaces and let G be an abelian group. On page 1913 we introduced the singular homology groups $\mathrm{H}_k(X,A;G)$ with coefficients in G. Furthermore from the outset we introduced on page 2402 the singular cohomology groups $\mathrm{H}^k(X,A;G)$. If G has an R-left module structure over a ring R, then it follows immediately from the definitions, see also the discussion on page 1914, that both $\mathrm{H}_k(X,A;G)$ and $\mathrm{H}^k(X,A;G)$ can be equipped with a natural R-left module structure in an obvious way.

The following little lemma shows that these two earlier types of (co-) homology groups are basically special cases of the present more general setting.

Lemma 167.3. (Twisted (co-) homology with trivial coefficients) Let X be a topological space that is locally contractible and path-connected, let $x_0\in X$ and let $A\subset X$. We write $\pi:=\pi_1(X,x_0)$ and we denote by $p\colon\widetilde{X}_{x_0}\to X$ the natural universal covering. We write $\widetilde{A}:=p^{-1}(A)$. Furthermore let R be a ring and let M be an R-left module. As above we consider the $(Z[\pi_1(X,x_0)],R)$-left left module ${}_{\mathrm{tr}}M$. The maps

$$\overbrace{\overline{{}_{\mathrm{tr}}M}\underset{\mathbb{Z}[\pi]}{\otimes}\mathrm{C}_k(\widetilde{X}_{x_0},\widetilde{A})}^{=\mathrm{C}_k^{\mathrm{tw}}(X,A;{}_{\mathrm{tr}}M)}\to\mathrm{C}_k(X,A)\underset{\mathbb{Z}}{\otimes}M \quad\&\quad \mathrm{Hom}\underset{\mathbb{Z}}{}(\mathrm{C}_k(X,A),M)\to\overbrace{\underset{\mathbb{Z}[\pi]\text{-left}}{\mathrm{Hom}}(\mathrm{C}_k(\widetilde{X}_{x_0},\widetilde{A}),{}_{\mathrm{tr}}M)}^{=\mathrm{C}^k_{\mathrm{tw}}(X,A;{}_{\mathrm{tr}}M)}$$
$$m\otimes\sigma\mapsto p_*(\sigma)\otimes m \qquad\qquad \varphi\mapsto\varphi\circ p_*$$

are natural isomorphisms of R-left module chain complexes. Thus they give rise to natural isomorphisms

$$T\colon\underbrace{\mathrm{H}_k^{\mathrm{tw}}(X,A;{}_{\mathrm{tr}}M)}_{\text{twisted coefficients}}\xrightarrow{\cong}\underbrace{\mathrm{H}_k(X,A;M)}_{\text{see page 1913}}\quad\text{and}\quad T\colon\underbrace{\mathrm{H}^k(X,A;M)}_{\text{see page 2402}}\xrightarrow{\cong}\underbrace{\mathrm{H}^k_{\mathrm{tw}}(X,A;{}_{\mathrm{tr}}M)}_{\text{twisted coefficients}}$$

of R-left modules.

[2067]It follows from Lemma 165.14 and Lemma 165.1 that $\mathrm{C}_k^{\mathrm{tw}}(X,A;M)$ and $\mathrm{C}^k_{\mathrm{tw}}(X,A;M)$ are both R-left modules.

Proof. One can easily verify that the maps are well-defined and that both maps are (co-) chain maps. It remains to show that both maps are isomorphisms.

(1) We consider the map

$$\begin{array}{rcccl} \overline{{}_{\mathrm{tr}}M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}) & \to & \overline{{}_{\mathrm{tr}}M} \otimes_{\mathbb{Z}} ({}_{\mathrm{tr}}\mathbb{Z} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})) & \to & \mathrm{C}_k(X, A) \underset{\mathbb{Z}}{\otimes} M \\ m \otimes \sigma & \mapsto & m \otimes (1 \otimes \sigma) & & \\ & & m \otimes (n \otimes \sigma) & \mapsto & n \cdot p_*(\sigma) \otimes m. \end{array}$$

The first map is an isomorphism by Lemma 165.20. Furthermore it follows easily from the Regular Covering-Chain Complex Lemma 167.2 together with Lemma 165.21 (2), that the second map is an isomorphism. The composition of these two maps is precisely the map that we were supposed to consider.

(2) Using Lemma 165.22 one can easily verify that the map on the "Hom-level" is also an isomorphism. We leave it to the reader to provide the details. ■

The definition of (co-) homology with twisted coefficients depends in many ways on the basis point. The following lemma explains how a path between different base points induces isomorphisms on (co-) homology with twisted coefficients.

In Lemmas 89.7 and 108.9 we saw that singular (co-) homology is covariantly functorial in the coefficients. The following definition generalizes this fact to our more general setting.

Definition. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the corresponding natural universal covering. Let $A \subset X$. We write $\widetilde{A} := p^{-1}(A)$. Let R be a ring and let M and N be $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left modules and let $\Theta\colon M \to N$ be a homomorphism of $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left modules. The R-left homomorphisms

$$\begin{array}{rcl} \Theta_*\colon \overbrace{\overline{M} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})}^{=\mathrm{C}_k^{\mathrm{tw}}(X,A;M)} & \to & \overbrace{\overline{N} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})}^{=\mathrm{C}_k^{\mathrm{tw}}(X,A;N)} \\ m \otimes \sigma & \mapsto & \Theta(m) \otimes \sigma \end{array}$$

$$\begin{array}{rcl} \Theta_*\colon \overbrace{\mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]\text{-left}}(\mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}), M)}^{=\mathrm{C}^k_{\mathrm{tw}}(X,A;M)} & \to & \overbrace{\mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]\text{-left}}(\mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}), N)}^{=\mathrm{C}^k_{\mathrm{tw}}(X,A;N)} \\ \gamma & \mapsto & \Theta \circ \gamma \end{array}$$

are chain maps and thus induce maps on twisted (co-) homology which we denote by Θ_*.

The following lemma now says that (co-) homology with twisted coefficients is indeed covariantly functorial in the $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left module.

Lemma 167.4. We suppose that we are in the setting of the above definition.

(1) For $\mathrm{id}\colon M \to M$ we have $\mathrm{id}_* = \mathrm{id}$ on the (co)-chain complexes and on the (co)-homology groups.

(2) For $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left module homomorphisms $\Phi\colon L \to M$ and $\Psi\colon M \to N$ we have $(\Phi \circ \Psi)_* = \Phi_* \circ \Psi_*$ on the (co)-chain complexes and on the (co)-homology groups.

Proof. Both statements are essentially trivial. ■

Using functoriality of (co-) homology with twisted coefficients we can easily state and prove the following little lemma.

Lemma 167.5. (Direct Sum-of-Coefficients Lemma) Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the corresponding natural universal covering. Let $A \subset X$. We write $\widetilde{A} := p^{-1}(A)$. Let R be a ring and let M and N be $(\mathbb{Z}[\pi_1(X,x_0)],R)$-left left modules. We consider the direct sum $M \oplus N$, which is again a $(\mathbb{Z}[\pi_1(X,x_0)],R)$-left left module. We denote by $i\colon M \to M \oplus N$ and $j\colon N \to M \oplus N$ the natural inclusions and we denote by $p\colon M \oplus N \to M$ and $q\colon M \oplus N \to N$ the natural projections. The maps

$$\begin{array}{rcl} p_* \oplus q_*\colon \mathrm{C}_k^{\mathrm{tw}}(X,A;M \oplus N) & \to & \mathrm{C}_k^{\mathrm{tw}}(X,A;M) \oplus \mathrm{C}_k^{\mathrm{tw}}(X,A;N) \\ p_* \oplus q_*\colon \mathrm{C}_{\mathrm{tw}}^k(X,A;M \oplus N) & \to & \mathrm{C}_{\mathrm{tw}}^k(X,A;M) \oplus \mathrm{C}_{\mathrm{tw}}^k(X,A;N) \end{array}$$

are isomorphisms of R-left module (co)-chain complexes with inverse given by $i_* \circ j_*$. In particular they induce isomorphism on (co-) homology with twisted coefficients.

Proof. The lemma is an immediate consequence of Lemma 167.4 since $i_* \oplus j_*$ and $p_* \oplus q_*$ are inverses of one another. ∎

(Co-) homology with twisted coefficients depends in two ways on the choice of a base point x_0: we use it to define the fundamental group $\pi_1(X,x_0)$ which in turn we need to define the corresponding group ring $\mathbb{Z}[\pi_1(X,x_0)]$, and we need the base point to define the natural universal covering $\widetilde{X}_{x_0}$. The following proposition describes how we can go from one base point to another.

Proposition 167.6. (Change of Base Point Proposition) Let X be a topological space that is locally contractible and path-connected and let $A \subset X$. Let $\gamma\colon [0,1] \to X$ be a path from $x_0 = \gamma(0)$ to $x_1 = \gamma(1)$. For $i = 0,1$ we denote by $p_i\colon \widetilde{X}_{x_i} \to X$ the natural universal covering and we write $\widetilde{A}_i := p_i^{-1}(A)$. Finally let R be a ring and let M be a $(\mathbb{Z}[\pi_1(X,x_0)],R)$-left left module. As in the Change-of-Base Point Proposition 47.10 we consider the isomorphism

$$\begin{array}{rcl} \gamma^*\colon \pi_1(X,x_1) & \to & \pi_1(X,x_0) \\ [\alpha] & \mapsto & [\gamma * \alpha * \overline{\gamma}]. \end{array}$$

The isomorphism γ^* gives rise to a ring homomorphism $\gamma^*\colon \mathbb{Z}[\pi_1(X,x_1)] \to \mathbb{Z}[\pi_1(X,x_0)]$. Thus, as defined on page 3502, we can consider the $\mathbb{Z}[\pi_1(X,x_1)]$-left module $\gamma^{**}M$. Furthermore, as in Lemma 61.10 we consider the homeomorphism

$$\begin{array}{rcl} \gamma^*\colon \widetilde{X}_{x_1} & \to & \widetilde{X}_{x_0} \\ [\beta] & \mapsto & [\gamma * \beta]. \end{array}$$

(1) With the above notation the maps

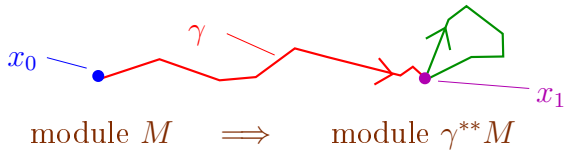
$$\begin{array}{rcl} \gamma^*\colon \overline{\gamma^{**}M} \otimes_{\mathbb{Z}[\pi_1(X,x_1)]} \mathrm{C}_k(\widetilde{X}_{x_1},\widetilde{A}_1) & \to & \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_k(\widetilde{X}_{x_0},\widetilde{A}_0) \\ m \otimes \sigma & \mapsto & m \otimes \gamma^*(\sigma) \end{array}$$

form an isomorphism of R-left module chain complexes and the maps

$$\begin{array}{rcl} \gamma^*\colon \mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_1)]-\mathrm{left}}(\mathrm{C}_k(\widetilde{X}_{x_1},\widetilde{A}_1),\gamma^{**}M) & \to & \mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]-\mathrm{left}}(\mathrm{C}_k(\widetilde{X}_{x_0},\widetilde{A}_0),M) \\ \varphi & \mapsto & \varphi \circ \gamma^* \end{array}$$

form an isomorphism of R-left module cochain complexes. In particular we get induced isomorphisms on (co-) homology with twisted coefficients.

(2) If $\gamma_0, \gamma_1 \colon [0,1] \to X$ are path-homotopic[2068], then $\gamma_0^* = \gamma_1^*$.
(3) Given paths $\alpha \colon [0,1] \to X$ and $\beta \colon [0,1] \to X$ with $\alpha(1) = \beta(0)$ we have $(\alpha * \beta)^* = \beta^* \circ \alpha^*$.



the path γ from x_0 to x_1 induces an isomorphism $\gamma_* \colon \pi_1(X, x_1) \to \pi_1(X, x_0)$

Proof.

(1) First let us show that the chain map given in the proposition is well-defined. To do so we need to consider the corresponding $\mathbb{Z}[\pi_1(X, x_0)]$-right module $\overline{M}$ and the corresponding $\mathbb{Z}[\pi_1(X, x_1)]$-right module $\overline{\gamma^{**}M}$. Note that given any $h \in \pi_1(X, x_1)$ and given any $m \in M$ we have the following equalities

$$m \underset{\overline{\gamma^{**}M}}{\cdot} h \;=\; h^{-1} \underset{\gamma^{**}M}{\cdot} m \;=\; \gamma^*(h^{-1}) \underset{M}{\cdot} m \;=\; m \underset{\overline{M}}{\cdot} \gamma^*(h).$$

Now we turn to the map γ^* on the chain level. Given any $m \in M$, given any singular simplex $\sigma \colon \Delta^k \to \widetilde{X}_{x_1}$ and given any $h \in \pi_1(X, x_1)$ we have the following equalities:

$$\begin{aligned} \gamma^*\big(m \underset{\overline{\gamma^{**}M}}{\cdot} h \otimes \sigma\big) \;=\; m \underset{\overline{\gamma^{**}M}}{\cdot} h \otimes \gamma^*(\sigma) \;&\overset{\text{by the above observation}}{=}\; m \underset{\overline{M}}{\cdot} \gamma^*(h) \otimes \sigma \;=\; m \otimes \gamma^*(h) \cdot \gamma^*(\sigma) \\ &=\; m \otimes \gamma^*(h \cdot \sigma) \;=\; \gamma^*(m \otimes h \cdot \sigma). \end{aligned}$$

This shows that the map on the chain level is well-defined. Furthermore it follows easily from the definitions that these maps form a chain map. Similarly one can show easily that the given map form a cochain map. It follows easily from Statements (2), which we will discuss shortly, that the (co-) chain maps are isomorphisms.

(2) If γ_0 and γ_1 are path-homotopic, then it follows easily from the definitions that $\gamma_0^* = \gamma_1^* \colon \pi_1(X, x_1) \to \pi_1(X, x_0)$ and $\gamma_0^* = \gamma_1^* \colon \widetilde{X}_{x_1} \to \widetilde{X}_{x_0}$. This observation implies Statement (2).

(3) This statement follows easily from the definitions. ∎

Remark. We suppose that we are in the setting of the Change of Base Point Proposition 167.6 with $x_0 = x_1$. If M is a $\mathbb{Z}[\pi_1(X, x_0)]$-module that has the property that the map $\pi_1(X, x_0) \to \mathrm{Aut}_{R\text{-left}}(M)$ factors through the abelianization of $\pi_1(X, x_0)$, then one can easily see that for any loop γ in (X, x_0) we have $\gamma^* M = M$. It follows from Change of Base Point Proposition 167.6 (2) and (3) that the map

$$\begin{array}{rcl} \pi_1(X, x_0) \times \mathrm{H}^k_{\mathrm{tw}}(X, A; M) & \to & \mathrm{H}^k_{\mathrm{tw}}(X, A; M) \\ ([\gamma], h) & \mapsto & \overline{\gamma}^*(h) \end{array}$$

defines a left action by $\pi_1(X, x_0)$ on $\mathrm{H}^k_{\mathrm{tw}}(X, A; M)$. In Exercise 167.2 we will see that this left action is in general, e.g. for $S^1 \vee S^2$, non-trivial.

[2068] Recall that according to the definition on page 1045 two paths maps $\gamma_0, \gamma_1 \colon [0,1] \to X$ are called *path-homotopic* if γ_0 and γ_1 are homotopic rel $\{0,1\}$.

167.3. **Functoriality.** In this section we will consider the functoriality of (co-) homology with twisted coefficients. The formulations of the statements require some care, since we have to deal with maps between *pointed* topological spaces and since we also have to worry about the module structures we are dealing with. The discussion of module structures leads us to the following definition.

Definition. Let $f\colon (X,x_0) \to (Y,y_0)$ be a map between pointed topological spaces. Furthermore let M be a $\mathbb{Z}[\pi_1(Y,y_0)]$-left module. We denote by f^*M the $\mathbb{Z}[\pi_1(X,x_0)]$-left module that is induced by the ring homomorphism $f_*\colon \mathbb{Z}[\pi_1(X,x_0)] \to \mathbb{Z}[\pi_1(Y,y_0)]$.[2069]

Definition. Let $f\colon (X,x_0) \to (Y,y_0)$ be a map between pointed topological spaces. We assume that X is locally path-connected and path-connected. Let $p\colon (\widehat{X},\widehat{x}_0) \to (X,x_0)$ and $q\colon (\widehat{Y},\widehat{y}_0) \to (Y,y_0)$ be two path-connected coverings with $f_*(p_*(\pi_1(\widehat{X},\widehat{x}_0))) \subset q_*(\pi_1(\widehat{Y},\widehat{y}_0))$. By the Map Lifting Criterion 61.2, applied to the map $f\circ p\colon (\widehat{X},\widehat{x}_0) \to (Y,y_0)$ and the covering $q\colon (\widehat{Y},\widehat{y}_0) \to (Y,y_0)$, there exists a unique map $\widetilde{f}\colon (\widehat{X},\widehat{x}_0) \to (X,x_0) \to (\widehat{Y},\widehat{y}_0)$ such that the following diagram commutes

$$\begin{array}{ccc} (\widehat{X},\widehat{x}_0) & \xrightarrow{\widetilde{f}} & (\widehat{Y},\widehat{y}_0)\ . \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ (X,x_0) & \xrightarrow{f} & (Y,y_0) \end{array}$$

We refer to $\widetilde{f}$ as the induced map.

Example. Let $f\colon (X,x_0) \to (Y,y_0)$ be a map between pointed topological spaces that are locally contractible and path-connected. We obtain an induced map

$$\widetilde{f}\colon \widetilde{X}_{x_0} \ \to\ \widetilde{Y}_{y_0}$$

between the natural universal coverings. It follows almost immediately from the definitions that it is given by

$$\underbrace{[\alpha\colon [0,1] \to X]}_{\in \widetilde{X}_{x_0}} \ \mapsto\ \underbrace{[f\circ\alpha\colon [0,1] \to Y]}_{\in \widetilde{Y}_{y_0}}.$$

Lemma 167.7. Let $f\colon (X,x_0) \to (Y,y_0)$ and $g\colon (Y,y_0) \to (Z,z_0)$ be maps between pointed topological spaces. We assume that X and Y are locally path-connected and path-connected. Let $p\colon (\widehat{X},\widehat{x}_0) \to (X,x_0)$, $q\colon (\widehat{Y},\widehat{y}_0) \to (Y,y_0)$ and $r\colon (\widehat{Z},\widehat{z}_0) \to (Z,z_0)$ be coverings with $f_*(p_*(\pi_1(\widehat{X},\widehat{x}_0))) \subset q_*(\pi_1(\widehat{Y},\widehat{y}_0))$ and $g_*(p_*(\pi_1(\widehat{Y},\widehat{y}_0))) \subset r_*(\pi_1(\widehat{Z},\widehat{z}_0))$. Then we have an equality of maps

$$\widetilde{g\circ f} \ =\ \widetilde{g}\circ\widetilde{f}\colon (\widehat{X},\widehat{x}_0) \ \to\ (\widehat{Z},\widehat{z}_0).$$

Proof. We will prove the lemma in Exercise 167.1. ∎

[2069] Let R be a ring and let M be a $(\mathbb{Z}[\pi_1(Y,y_0)],R)$-left left module. Then f^*M is naturally a $(\mathbb{Z}[\pi_1(X,x_0)],R)$ left left module.

Proposition 167.8. (Chain Complex-Equivariance Proposition) Let $f\colon (X,x_0)\to(Y,y_0)$ be a map between pointed topological spaces. We assume that X is locally path-connected and path-connected. Let $p\colon (\widehat{X},\widehat{x}_0)\to(X,x_0)$ and $q\colon(\widehat{Y},\widehat{y}_0)\to(Y,y_0)$ be two *regular* coverings with $f_*(p_*(\pi_1(\widehat{X},\widehat{x}_0)))\subset q_*(\pi_1(\widehat{Y},\widehat{y}_0))$. We consider the induced map $\widetilde{f}\colon(\widehat{X},\widehat{x}_0)\to(\widehat{Y},\widehat{y}_0)$.

(1) Suppose that we are given subsets $A\subset X$ and $B\subset Y$ with $f(A)\subset B$. We write $\widehat{A}:=p^{-1}(A)$ and $\widehat{B}:=q^{-1}(B)$. The induced map $\widetilde{f}$ has the property that given any $Q\in\widehat{X}_{x_0}$ and given any $g\in\pi_1(X,x_0)/\operatorname{im}(p_*)$ we have

$$\widetilde{f}(g\cdot Q) \quad = \quad \underbrace{f_*(g)}_{\in\pi_1(Y,y_0)/\operatorname{im}(q_*)}\cdot\widetilde{f}(Q).$$

(2) Let R be a ring and let M be a $(\mathbb{Z}[\pi_1(Y,y_0)],R)$-left left module. The maps[2070]

$$\begin{array}{rcl}\overline{f^*M}\otimes_{\mathbb{Z}[\pi_1(X,x_0)/\operatorname{im}(p_*)]}\mathrm{C}_k(\widehat{X},\widehat{A}) & \to & \overline{M}\otimes_{\mathbb{Z}[\pi_1(Y,y_0)/\operatorname{im}(q_*)]}\mathrm{C}_k(\widehat{Y},\widehat{B})\\ m\otimes(\sigma\colon\Delta^k\to\widehat{X}) & \mapsto & m\otimes(\widetilde{f}\circ\sigma)\end{array}$$

and

$$\begin{array}{rcl}\operatorname{Hom}_{\mathbb{Z}[\pi_1(Y,y_0)/\operatorname{im}(q_*)]\text{-left}}(\mathrm{C}_k(\widehat{Y},\widehat{B}),M) & \to & \operatorname{Hom}_{\mathbb{Z}[\pi_1(X,x_0)/\operatorname{im}(p_*)]\text{-left}}(\mathrm{C}_k(\widehat{X},\widehat{A}),f^*M)\\ \varphi & \mapsto & (\sigma\mapsto\varphi(\widetilde{f}\circ\sigma))\end{array}$$

are well-defined (co-) chain maps.

Proof.

(1) This statement is a slight variation on Proposition 62.13.

(2) This statement follows easily from (a) and Lemmas 165.15 and 165.19, respectively Lemmas 165.2 and 165.3. ■

Definition. Let $f\colon(X,x_0)\to(Y,y_0)$ be a map between pointed topological spaces that are locally contractible and path-connected. We write $\pi_X:=\pi_1(X,x_0)$, $\pi_Y:=\pi_1(Y,y_0)$. Furthermore let R be a ring and let M be a $(\mathbb{Z}[\pi_Y],R)$-left left module. We denote by $p\colon\widetilde{X}_{x_0}\to X$ and $q\colon\widetilde{Y}_{y_0}\to Y$ the natural universal coverings. Suppose that we are given subsets $A\subset X$ and $B\subset Y$ with $f(A)\subset B$. We write $\widetilde{A}:=p^{-1}(A)$ and $\widetilde{B}:=q^{-1}(B)$. We denote by $\widetilde{f}\colon\widetilde{X}_{x_0}\to\widetilde{Y}_{y_0}$ the map that is induced by $f\colon X\to Y$. Using the Chain Complex-Equivariance Proposition 167.8 we define

$$\begin{array}{rcl}f_*\colon \overbrace{\overline{f^*M}\otimes_{\mathbb{Z}[\pi_X]}\mathrm{C}_k(\widetilde{X}_{x_0},\widehat{A})}^{=\mathrm{C}_k^{\mathrm{tw}}(X,A;f^*M)} & \to & \overbrace{\overline{M}\otimes_{\mathbb{Z}[\pi_Y]}\mathrm{C}_k(\widetilde{Y}_{y_0},\widehat{B})}^{=\mathrm{C}_k^{\mathrm{tw}}(Y,B;M)}\\ m\otimes(\sigma\colon\Delta^k\to\widetilde{X}_{x_0}) & \mapsto & m\otimes(\widetilde{f}\circ\sigma\colon\Delta^k\to\widetilde{Y}_{y_0})\end{array}$$

and

$$\begin{array}{rcl}f^*\colon \overbrace{\operatorname{Hom}_{\mathbb{Z}[\pi_Y]\text{-left}}(\mathrm{C}_k(\widetilde{Y}_{y_0},\widehat{B}),M)}^{=\mathrm{C}^k_{\mathrm{tw}}(Y,B;M)} & \to & \overbrace{\operatorname{Hom}_{\mathbb{Z}[\pi_X]\text{-left}}(\mathrm{C}_k(\widetilde{X}_{x_0},\widehat{A}),f^*M)}^{=\mathrm{C}^k_{\mathrm{tw}}(X,A;f^*M)}\\ \varphi & \mapsto & (\sigma\mapsto\varphi(r\circ\sigma))\end{array}$$

and we define the corresponding induced maps

$$\begin{aligned} f_*\colon \mathrm{H}_k^{\mathrm{tw}}(X; f^*M) &\to \mathrm{H}_k^{\mathrm{tw}}(Y; M) \\ f^*\colon \mathrm{H}^k_{\mathrm{tw}}(Y; M) &\to \mathrm{H}^k_{\mathrm{tw}}(X; f^*M). \end{aligned}$$

The following proposition says that (co-) homology with twisted coefficients is (co-) variantly functorial, at least if we take care with the choice of coefficients we are dealing with.

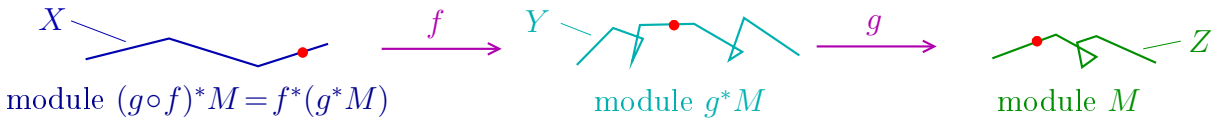
Proposition 167.9. (Twisted Homology-Functoriality Proposition) Let Z be a topological space that is locally contractible and path-connected. Furthermore let M be a $\mathbb{Z}[\pi_1(Z, z_0)]$-left module and let $C \subset Z$.

(1) We have $(\mathrm{id}_Z)^*M = M$ and the induced maps $\mathrm{id}_*\colon \mathrm{C}_k^{\mathrm{tw}}(Z, C; \mathrm{id}_Z^* M) \to \mathrm{C}_k^{\mathrm{tw}}(Z, C; M)$ and $\mathrm{id}^*\colon \mathrm{C}_k^{\mathrm{tw}}(Z; M) \to \mathrm{C}_k(Z; \mathrm{id}_Z^* M)$ are the identity.

(2) Let X and Y be topological spaces that are locally contractible and path-connected and let $A \subset X$ and $B \subset Y$. Let $f\colon (X, x_0) \to (Y, y_0)$ and $g\colon (Y, Y_0) \to (Z, Z_0)$ be maps between pointed topological spaces. We assume that $f(A) \subset B$ and $g(B) \subset C$. Then $(g \circ f)^*M = f^*(g^*M)$ and

$$\begin{aligned} (g \circ f)_* = g_* \circ f_*\colon \mathrm{C}_k^{\mathrm{tw}}(X, A; (g \circ f)^*M) &\to \mathrm{C}_k^{\mathrm{tw}}(Z, C; M) \\ (g \circ f)^* = f^* \circ g^*\colon \mathrm{C}^k_{\mathrm{tw}}(Z, C; M) &\to \mathrm{C}^k_{\mathrm{tw}}(X, A; (g \circ f)^*M). \end{aligned}$$

In both cases the identities on maps between (co-) chain complexes induce of course identities of maps between (co-) homology groups.



Proof. The proposition follows easily from Lemma 167.7. We leave the details to the reader. ∎

In the following we want to prove twisted analogues of earlier results, in particular we want to show the existence of a long exact sequence of a pair and we want to prove an excision theorem for twisted (co-) homology. In these two instances, and also many other settings, we will need the following slightly subtle proposition.

Proposition 167.10. (Change of Covering Proposition) Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. We denote by $q\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. Let R be a ring and let M be a $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left module. Let $B \subset X$ be a subspace with $x_0 \in B$. We assume that B is also locally contractible and path-connected. We write $\widetilde{B} := p^{-1}(B)$ and we denote by $p\colon \widehat{B}_{x_0} \to B$ the natural universal covering. Finally let $A \subset X$. We write $\widehat{A} := p^{-1}(A)$ and $\widetilde{A} := q^{-1}(A)$. We denote by $i\colon B \to X$ the inclusion map and we denote by $\widetilde{i}\colon \widehat{B}_{x_0} \to \widetilde{X}_{x_0}$ the induced map as defined on page 3541.

(1) The maps

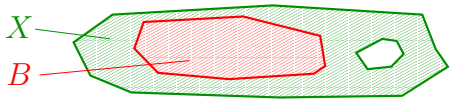
$$\begin{aligned} \overbrace{i^*M \otimes_{\mathbb{Z}[\pi_1(B,x_0)]} \mathrm{C}_k(\widehat{B}_{x_0}, \widehat{A})}^{=\mathrm{C}_k^{\mathrm{tw}}(B,A;i^*M)} &\to M \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_k(\widetilde{B}, \widetilde{A}) \\ m \otimes \sigma &\mapsto m \otimes \widetilde{i}_*(\sigma) \end{aligned}$$

are well-defined and they are natural isomorphisms of R-left module chain complexes.

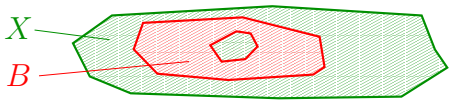
(2) For each $k \in \mathbb{N}_0$ the map

$$\overbrace{i^*M \underset{\mathbb{Z}[\pi_1(B,x_0)]}{\otimes} \mathrm{C}_k(\widehat{B}_{x_0}, \widehat{A})}^{=\mathrm{C}_k^{\mathrm{tw}}(B,A;i^*M)} \to \underset{\uparrow}{} \underset{\text{isomorphism from (1)}}{} M \underset{\mathbb{Z}[\pi_1(X,x_0)]}{\otimes} \mathrm{C}_k(\widetilde{B}, \widetilde{A}) \xrightarrow{\mathrm{id} \otimes \text{inclusion}} \overbrace{M \underset{\mathbb{Z}[\pi_1(X,x_0)]}{\otimes} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})}^{=\mathrm{C}_k^{\mathrm{tw}}(X,A;M)}$$

is on the nose the map $i_*\colon \mathrm{C}_k^{\mathrm{tw}}(B, A; i^*M) \to \mathrm{C}_k^{\mathrm{tw}}(X, A; M)$ that we defined on page 3542.



$q\colon \widetilde{B} \to B$ is an infinite covering, but in this case $\widehat{B} = B$



we have $\widetilde{B} = B$, but in this setting $p\colon \widehat{B} \to B$ is an infinite covering

Remark. We will develop the theme of the Change of Covering Proposition 167.10 further in the eagerly awaited sequel, the Change of Covering Proposition II 169.4.

Proof.

(1) To simplify the notation a little bit we assume that $A = \varnothing$. Furthermore we write $\pi_B := \pi_1(B, x_0)$ and we write $\pi_X := \pi_1(X, x_0)$. Let $\widetilde{i}\colon \widehat{B}_{x_0} \to \widetilde{X}_{x_0}$ be the induced map as defined on page 3541. Note that it follows easily from the definitions that $\widetilde{i}(\widehat{B}) \subset \widetilde{B}$. In the Chain Complex-Equivariance Proposition 167.8 we saw that the map

$$i^*M \otimes_{\mathbb{Z}[\pi_B]} \mathrm{C}_k(\widehat{B}_{x_0}) \to M \otimes_{\mathbb{Z}[\pi_X]} \mathrm{C}_k(\widetilde{X}_{x_0})$$

is well-defined. The current map under consideration is exactly the same map, except that it follows from $\widetilde{i}(\widehat{B}) \subset \widetilde{B}$ that the map we are really interested in actually takes values in $M \otimes_{\mathbb{Z}[\pi_X]} \mathrm{C}_k(\widetilde{B})$. This shows that the map in our proposition is well-defined.

Claim. The map

$$\begin{aligned} \mathbb{Z}[\pi_X] \otimes_{\mathbb{Z}[\pi_B]} \mathrm{C}_k(\widehat{B}_{x_0}) &\to \mathrm{C}_k(\widetilde{B}) \\ g \otimes \sigma &\mapsto g \cdot (\widetilde{i} \circ \sigma) \end{aligned}$$

is an isomorphism of $\mathbb{Z}[\pi_X]$-left modules.

Proof. Given any singular k-simplex σ in B we pick a lift $\widetilde{\sigma}$ to $\widehat{B}_{x_0}$. In the Regular Covering-Chain Complex Lemma 167.2 we showed that these lifts form a basis of the free $\mathbb{Z}[\pi_B]$-left module $\mathrm{C}_k(\widehat{B}_{x_0})$. Furthermore by the Regular Covering-Chain Complex Lemma 167.2 we also know that the singular simplices $\widetilde{i}_* \circ \widetilde{\sigma}$ form a basis of the free $\mathbb{Z}[\pi_X]$-left module $\mathrm{C}_k(\widetilde{B})$. It follows from this observation, together with Lemma 165.21 (2), that the map sends a basis of the free $\mathbb{Z}[\pi_X]$-left module $\mathbb{Z}[\pi_X] \otimes_{\mathbb{Z}[\pi_B]} \mathrm{C}_k(\widehat{B}_{x_0})$ to a basis of the free $\mathbb{Z}[\pi_X]$-left module $\mathrm{C}_k(\widetilde{B})$. Thus we see that the map is indeed an isomorphism of $\mathbb{Z}[\pi_X]$-left modules. ⊞

Note that the map we are supposed to study equals the composition of the following maps:

$$\begin{array}{rcccl} i^*M \otimes_{\mathbb{Z}[\pi_B]} \mathrm{C}_k(\widehat{B}_{x_0}) & \to & M \otimes_{\mathbb{Z}[\pi_X]} (\mathbb{Z}[\pi_X] \otimes_{\mathbb{Z}[\pi_B]} \mathrm{C}_k(\widehat{B}_{x_0}) & \to & M \otimes_{\mathbb{Z}[\pi_X]} \mathrm{C}_k(\widetilde{B}) \\ m \otimes \sigma & \mapsto & m \otimes (1 \otimes \sigma) & & \\ & & m \otimes (g \otimes \sigma) & \mapsto & m \otimes g \cdot (\widetilde{i}_* \circ \sigma). \end{array}$$

It follows from Lemma 165.20 that the map to the left is well-defined and an isomorphism of R-left modules. It follows from the claim that the map to the right is an isomorphism of R-left modules.

(2) This statement follows immediately from the definitions. ■

By the Topological LES-Proposition 74.13 we know that a triple (X, A, B) of topological spaces gives rise to a long exact sequence of relative singular homology groups. We now have an analogous statement in the present setting.

Proposition 167.11. (Twisted-LES Proposition) Let X be a topological space that is locally contractible and path-connected, let B be subspace of X that is also locally contractible and path-connected, let $A \subset B$ be a further subspace and let $b_0 \in B$. We denote by $i\colon B \to X$ the inclusion. Let R be a ring and let M be a $(\mathbb{Z}[\pi_1(X, b_0)], R)$-left left module. Given any $n \in \mathbb{N}_0$ there exists a natural R-left module homomorphism

$$\partial_n\colon \mathrm{H}_n^{\mathrm{tw}}(X, B; M) \to \mathrm{H}_{n-1}^{\mathrm{tw}}(B, A; i^*M)$$

such that the following sequence is exact

$$\ldots \xrightarrow{\partial_{n+1}} \mathrm{H}_n^{\mathrm{tw}}(B, A; i^*M) \xrightarrow{i_*} \mathrm{H}_n^{\mathrm{tw}}(X, A; M) \underset{\uparrow}{\to} \mathrm{H}_n^{\mathrm{tw}}(X, B; M) \xrightarrow{\partial_n} \mathrm{H}_{n-1}^{\mathrm{tw}}(B, A; i^*M) \to \ldots$$

induced by the obvious map $(X, A) \to (X, B)$

An analogous statement also holds for cohomology groups with twisted coefficients.

Proof. We denote by $p\colon \widehat{B}_{b_0} \to B$ and $q\colon \widetilde{X}_{b_0} \to X$ the natural universal coverings. We write $\widehat{A} := p^{-1}(A)$, $\widetilde{A} := q^{-1}(A)$ and $\widetilde{B} := q^{-1}(B)$. The map $i\colon B \to X$ induces a map $\widetilde{i}\colon \widehat{B} \to \widetilde{X}$. Evidently $\widetilde{i}(\widehat{B}) \subset \widetilde{B}$. First let us consider the sequence

$$0 \to \mathrm{C}_k(\widetilde{B}, \widetilde{A}) \to \mathrm{C}_k(\widetilde{X}, \widetilde{B}) \to \mathrm{C}_k(\widetilde{X}, \widetilde{A}) \to 0.$$

Note that by the Regular Covering-Chain Complex Lemma 167.2 we know that this is sequence of $\mathbb{Z}[\pi_1(X, x_0)]$-left homomorphisms between free $\mathbb{Z}[\pi_1(X, x_0)]$-left modules. Furthermore, as in the proof of the Topological LES-Proposition 74.13, we obtain from the third isomorphism theorem that the sequence is exact. Since $\mathrm{C}_k(\widetilde{X}, \widetilde{A})$ is a free $\mathbb{Z}[\pi_1(X, x_0)]$-left module we obtain from Lemma 165.18 that

$$0 \to M \otimes_{\mathbb{Z}[\pi_1(X,b_0)]} \mathrm{C}_k(\widetilde{B}, \widetilde{A}) \to M \otimes_{\mathbb{Z}[\pi_1(X,b_0)]} \mathrm{C}_k(\widetilde{X}, \widetilde{B}) \to M \otimes_{\mathbb{Z}[\pi_1(X,b_0)]} \mathrm{C}_k(\widetilde{X}, \widetilde{A}) \to 0$$

is a short exact sequence of R-left modules. From the LES Lemma 166.11 we get a connecting homomorphism and a long exact sequence of the homology modules. We obtain the desired connecting homomorphism and the desired long exact sequence by replacing the homology modules of the chain complex $M \otimes_{\mathbb{Z}[\pi_1(X,b_0)]} \mathrm{C}_k(\widetilde{B}, \widetilde{A})$ by the homology groups $\mathrm{H}_k^{\mathrm{tw}}(B, A; i^*M)$ using the isomorphism that is provided by the Change of Covering Proposition 167.10 (1). Note that the Change of Covering Proposition 167.10 (2) implies that

the resulting connecting homomorphism does indeed give rise to the desired long exact sequence.

Finally note that the cohomological analogue is proved in a very similar fashion. We leave the proof to the exercise-deprived reader. ■

167.4. **Excision for (co-) homology with twisted coefficients.** As we have seen, excision theorems play an essential role when dealing with ordinary (co-) homology groups. The reader will thus be greatly relieved to hear that analogous statements also hold in the twisted context. More precisely we have the following analogue of the Excision Theorems 74.17, 74.18 and 109.1.

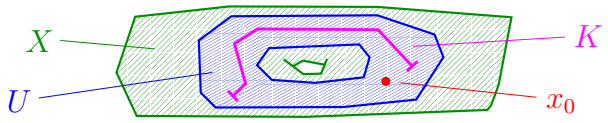
Theorem 167.12. (Excision Theorem for Twisted (Co-) Homology) Let X be a topological space that is locally contractible and path-connected, let $x_0 \in X$ and let M be a $\mathbb{Z}[\pi_1(X, x_0)]$-left module.

(1) Let $Z \subset A \subset X$ be subsets such that the closure of Z is contained in the interior of A and such that $x_0 \notin Z$. If $X \setminus Z$ is locally contractible and path-connected, then the inclusion $i\colon (X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ isomorphisms

$$\begin{aligned} i^*\colon \mathrm{H}_n^{\mathrm{tw}}(X \setminus Z, A \setminus Z; i^*M) &\xrightarrow{\cong} \mathrm{H}_n^{\mathrm{tw}}(X, A; M) \\ i^*\colon \mathrm{H}^n_{\mathrm{tw}}(X, A; M) &\xrightarrow{\cong} \mathrm{H}^n_{\mathrm{tw}}(X \setminus Z, A \setminus Z; i^*M). \end{aligned}$$

(2) Let K be a subset of X and let U be a neighborhood of the closure of K in X[2071]. If $x_0 \in U$ and if U is locally contractible and path-connected, then the inclusion $i\colon (U, U \setminus K) \to (X, X \setminus K)$ induces for each $n \in \mathbb{N}_0$ isomorphisms

$$\begin{aligned} i_*\colon \mathrm{H}_n^{\mathrm{tw}}(U, U \setminus K; i^*M) &\xrightarrow{\cong} \mathrm{H}_n^{\mathrm{tw}}(X, X \setminus K; M) \\ i^*\colon \mathrm{H}^n_{\mathrm{tw}}(X, X \setminus K; M) &\xrightarrow{\cong} \mathrm{H}^n_{\mathrm{tw}}(U, U \setminus K; i^*M). \end{aligned}$$



Proof.

(2) We denote by $o\colon \widetilde{X}_{x_0} \to X$ the natural universal covering and we write $\widetilde{U} := p^{-1}(U)$ and $\widetilde{K} := p^{-1}(K)$.

Claim. The inclusion induced map

$$i_*\colon \mathrm{C}_n(\widetilde{U}, \widetilde{U} \setminus \widetilde{K}) \to \mathrm{C}_n(\widetilde{X}, \widetilde{X} \setminus \widetilde{K})$$

is a chain homotopy equivalence of free $\mathbb{Z}[\pi_1(X, x_0)]$-left modules.

Proof. We make the following two observations:

(a) First note that

$$\text{closure of } \widetilde{K} \underset{\text{see Exercise 48.7}}{=} p^{-1}(\text{closure of } K) \underset{\text{by hypothesis}}{\subset} p^{-1}(\text{interior of } U) \underset{\text{see Exercise 48.7}}{=} \text{interior of } \widetilde{U}.$$

[2071] These conditions are satisfied for example if X is Hausdorff, if K is compact and if U is an open neighborhood of K.

Thus it follows from the original Excision Theorem 74.18 that for every $n \in \mathbb{N}_0$ the inclusion i induces an isomorphism

$$i_*\colon \mathrm{H}_n(\widetilde{U}, \widetilde{U} \setminus \widetilde{K}) \xrightarrow{\cong} \mathrm{H}_n(\widetilde{X}, \widetilde{X} \setminus \widetilde{K}).$$

(b) Next we consider the map

$$i_*\colon \mathrm{C}_n(\widetilde{U}, \widetilde{U} \setminus \widetilde{K}) \to \mathrm{C}_n(\widetilde{X}, \widetilde{X} \setminus \widetilde{K}).$$

In the Regular Covering-Chain Complex Lemma 167.2 we saw that both sides are free $\mathbb{Z}[\pi_1(X, x_0)]$-left modules. By definition of the module structure the map i_* is actually a homomorphism between two free $\mathbb{Z}[\pi_1(X, x_0)]$-left modules.

The claim now follows from (a) and (b) combined with Quasi Isomorphisms-are-Homotopy Equivalences Proposition 166.6. ⊞

Now let M be a $\mathbb{Z}[\pi_1(X, x_0)]$-left module. It follows immediately from the claim together with Lemmas 166.3 and 166.2 that the map

$$\mathrm{id}_M \otimes i_*\colon \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_n(\widetilde{U}, \widetilde{U} \setminus \widetilde{K}) \to \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_n(\widetilde{X}, \widetilde{X} \setminus \widetilde{K})$$

induces an R-left isomorphism of homology modules. It follows from this observation together with the Change of Covering Proposition 167.10 (1) and (2) that the map

$$i_*\colon \mathrm{H}_n^{\mathrm{tw}}(U, U \setminus K; i^*M) \to \mathrm{H}_n^{\mathrm{tw}}(X, X \setminus K; M)$$

is an R-left isomorphism. The statement regarding cohomology groups is proved in a very similar fashion, using the Dualizing Chain Complex Lemma 166.4 and the Cochain Map Lemma 108.5. We leave the details to the reader.

(1) This statement follows from Statement (1) by setting $U = X \setminus Z$ and $K = X \setminus A$. Again we leave the few details to the reader. ■

167.5. **Homotopic maps and (co-) homology with twisted coefficients.** In the Homotopic Maps-and-Homology Proposition 73.6 and in the Homotopic Maps-H*-Lemma 108.13 we saw that homotopic maps induce the same map on homology. A similar statement also holds in the present context, but, as we will see, one needs to be careful about the twistings because of base point issues.

First let us recall the following definition from page 439.

Definition. Let X and Y be topological spaces and let $A \subset X$ and $B \subset Y$ be subsets. Let $f, g\colon (X, A) \to (Y, B)$ be two maps of pairs of topological spaces.

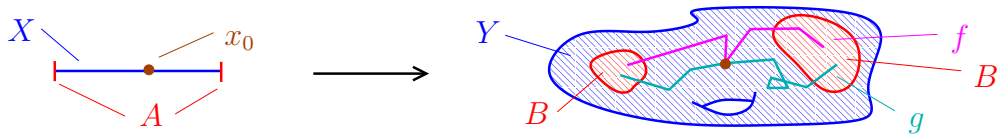
(1) A homotopy between f and g is a map

$$H\colon X \times [0, 1] \to Y$$

such that the following two conditions are satisfied:
(a) $H_0 = f$ and $H_1 = g$, and
(b) $H(a, t) \in B$ for all $a \in A$ and all $t \in [0, 1]$.

(2) Let $x_0 \in X$. We say a map $F\colon X \times [0, 1] \to Y$ is a homotopy rel x_0 if for all $s, t \in [0, 1]$ we have $F(x_0, s) = F(x_0, t)$.

Proposition 167.13. Let X and Y be topological spaces that are both locally contractible and path-connected. Let $A \subset X$ and $B \subset Y$ be subspaces. Let $f_0, f_1\colon (X, A) \to (Y, B)$ be two maps of pairs of topological spaces. Let $x_0 \in X$ and $y_0 \in Y$ with $f_0(x_0) =$



$f, g\colon (X, A) \to (Y, B)$ are homotopic rel x_0

y_0. Furthermore let R be a ring and let M be a $(\mathbb{Z}[\pi_1(Y, y_0)], R)$-left left module. Let $F\colon X \times [0,1] \to Y$ be a homotopy between f_0 and f_1. We denote by $\gamma\colon [0,1] \to Y$ the path that is given by $t \mapsto F(x_0, t)$. We have the following equality of $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left modules:

notation from the Change of Base Point Proposition 167.6

$$f_1^*(\overset{\downarrow}{\gamma^{**}}M) \;=\; f_0^*M.$$

Furthermore the maps

$$\begin{aligned} \mathrm{C}_k^{\mathrm{tw}}(X, A; f_0^*M) &\xrightarrow{f_{0*}} \mathrm{C}_k^{\mathrm{tw}}(Y, B; M) \\ \mathrm{C}_k^{\mathrm{tw}}(X, A; \underbrace{f_0^*M}_{=f_1^*(\gamma^{**}M)}) &\xrightarrow{f_{1*}} \mathrm{C}_k^{\mathrm{tw}}(X, A; \gamma^{**}M) \xrightarrow[\uparrow]{\gamma_*} \mathrm{C}_k^{\mathrm{tw}}(Y, B; M) \end{aligned}$$

isomorphism given by the Change of Base Point Proposition 167.6

are chain homotopic as chain maps between chain complexes of R-left modules. An analogous statement also holds for cochain complexes.

As the reader will have spotted, the statement becomes a little neater, if γ happens to be the constant path. This leads us to the following useful special case of Proposition 167.13.

Corollary 167.14. Let X and Y be topological spaces that are both locally contractible and path-connected. Let $A \subset X$ and $B \subset Y$ be subspaces and let $x_0 \in X$ and $y_0 \in Y$. Furthermore let R be a ring and let M be a $(\mathbb{Z}[\pi_1(Y, y_0)], R)$-left left module. Let $f, g\colon (X, A) \to (Y, B)$ be two maps of pairs of topological spaces with $f(x_0) = g(x_0) = y_0$. If f and g are homotopic rel x_0, then by the Homotopy-π_1-Proposition 50.3 we know that $f_* = g_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$, thus $f^*M = g^*M$, and we see that

$$f_*, g_*\colon \mathrm{C}_k^{\mathrm{tw}}(X, A; f^*M) \to \mathrm{C}_k^{\mathrm{tw}}(Y, B; M) \text{ and } f_*, g_*\colon \mathrm{C}_{\mathrm{tw}}^k(Y, B; M) \to \mathrm{C}_{\mathrm{tw}}^k(X, A; f^*M)$$

are (co-) chain homotopic. In particular the induced maps

$$f_*, g_*\colon \mathrm{H}_k^{\mathrm{tw}}(X, A; f^*M) \to \mathrm{H}_k^{\mathrm{tw}}(Y, B; M) \text{ and } f_*, g_*\colon \mathrm{H}_{\mathrm{tw}}^k(Y, B; M) \to \mathrm{H}_{\mathrm{tw}}^k(X, A; f^*M)$$

are equal.

Proof of Corollary 167.14. The proposition follows immediately from Proposition 167.13 and the observation that under the given hypotheses the path γ in the statement of Proposition 167.13 is just the constant path. ∎

For reader's convenience we recall the following classic lemma, which will play an essential role in the proof of Proposition 167.13.

Lemma 73.8. Let V and W be topological spaces and let $\widetilde{F}\colon V \times [0,1] \to W$ be a homotopy. Let $\Omega_n \in \mathrm{C}_{n+1}(\Delta^n \times [0,1])$, $n \in \mathbb{N}_0$, be the sequence of singular chains that we defined on page 1605.[2072] The maps

$$\begin{array}{rcl} P_n\colon \mathrm{C}_n(V) & \to & \mathrm{C}_{n+1}(W) \\ (\sigma\colon \Delta^n \to V) & \mapsto & (\widetilde{F} \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\Omega_n). \end{array}$$

satisfy

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; \widetilde{F}_{0*} - \widetilde{F}_{1*}$$

as maps from $\mathrm{C}_n(V)$ to $\mathrm{C}_n(W)$.

Proof of Proposition 167.13. To simplify the notation we assume that $A = B = \varnothing$. Mathematically this extra hypothesis makes no real difference. In the following we will only prove the statement regarding chain homotopies between chain complexes. We leave it to the reader to adapt the argument below to produce a cochain homotopy between the given cochain maps.

Without further ado, let $f_0, f_1\colon X \to Y$ be two maps of topological spaces. Let $x_0 \in X$ and $y_0 \in Y$ with $f_0(x_0) = y_0$. Furthermore let R be a ring and let M be a $(\mathbb{Z}[\pi_1(Y, y_0)], R)$-left left module. Furthermore let $F\colon X \times [0,1] \to Y$ be a homotopy between f_0 and f_1. We denote by $i_0, i_1\colon X \to X \times [0,1]$ the obvious embeddings. Note that by definition of a homotopy we have $F \circ i_0 = f_0$, $F \circ i_1 = f_1$. We denote by $\gamma\colon [0,1] \to X$ the path that is given by $t \mapsto F(x_0, t)$.

Claim 1. We have the following equality of $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left modules.

$$f_1^*(\gamma^{**}M) \;=\; f_0^*M.$$

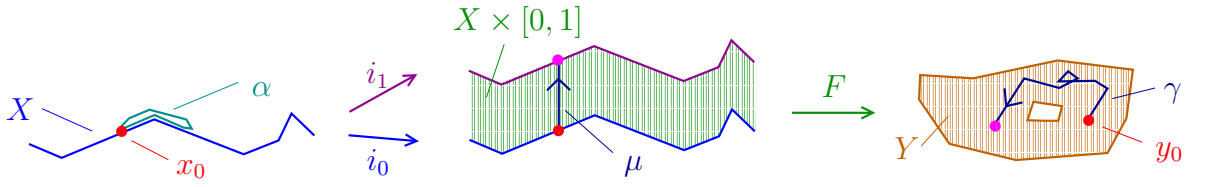
Proof. We need to show that the maps

$$\begin{array}{rccccc} & \pi_1(X,x_0) & \xrightarrow{f_{0*}} & \pi_1(Y,y_0) & & \\ \text{and} & \pi_1(X,x_0) & \xrightarrow{f_{1*}} & \pi_1(X,f_1(x_0)) & \xrightarrow{\gamma^*} & \pi_1(Y,y_0) \end{array}$$

agree. Thus let $\alpha\colon [0,1] \to X$ be a loop in x_0. We denote by $\mu\colon [0,1] \to X \times [0,1]$ the path that is given by $t \mapsto (x_0, t)$. Then we have the following equalities:

$$\begin{array}{rcccccc} & & & & & & \text{there is an explicit path homotopy between } i_0 \circ \alpha \text{ and } \mu * i_1 \circ \alpha * \overline{\mu} \\ & & & & & & \downarrow \\ f_{0*}([\alpha]) & = & [f_0 \circ \alpha] & = & [F \circ (i_0 \circ \alpha)] & = & [F \circ (\mu * i_1 \circ \alpha * \overline{\mu})] \\ & & & = & [(F \circ \mu) * (F \circ i_1 * \alpha) * (F \circ \overline{\mu})] & = & [\gamma] * f_{1*}([\alpha]) * [\overline{\gamma}] \end{array}$$

⊞



As usual we denote by $p\colon \widetilde{X}_{x_0} \to X$ and $q\colon \widetilde{Y}_{y_0} \to Y$ the natural universal coverings. One can easily see that $p \times \mathrm{id}_{[0,1]}\colon \widetilde{X}_{x_0} \times [0,1] \to X \times [0,1]$ is a universal covering.[2073] Thus it follows from the Lift-to-Universal Coverings Proposition 61.13 that there exists a unique

[2072]The precise definition of the Ω_n is completely irrelevant for what follows.

[2073]But not that this is *not* the natural universal covering of $X \times [0,1]$.

map $\widetilde{F}\colon \widetilde{X}_{x_0} \times [0,1] \to \widetilde{Y}_{y_0}$ with $\widetilde{F}(\widetilde{x}_0, 0) = \widetilde{y}_0$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X}_{x_0} \times [0,1] & \overset{\widetilde{F}}{\dashrightarrow} & \widetilde{Y}_{y_0} \\ {\scriptstyle p\times \mathrm{id}_{[0,1]}}\downarrow & & \downarrow{\scriptstyle q} \\ X \times [0,1] & \xrightarrow{\qquad F \qquad} & Y. \end{array}$$

Again we denote by $i_0, i_1\colon \widetilde{X}_{x_0} \to \widetilde{X}_{x_0} \times [0,1]$ the obvious embeddings. Below we will need the following subtle claim.

Claim 2. We write $y_1 = f_1(x_0) \in Y$. We denote by $q\colon \widetilde{Y}_{y_1} \to Y$ the corresponding natural universal covering. The following diagram commutes:

$$\begin{array}{ccc} \widetilde{X}_{x_0} & \xrightarrow{\qquad f_1 \qquad} & \widetilde{Y}_{y_1} \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle \gamma^*} \\ \widetilde{X}_{x_0} \times [0,1] & \xrightarrow{\qquad \widetilde{F} \qquad} & \widetilde{Y}_{y_0}. \end{array}$$

Proof. We start with two observations:

(a) By inspection one sees that both maps send the natural base point of $\widetilde{X}_{x_0}$ to the point of $\widetilde{Y}_{y_0}$ that is represented by the path γ.

(b) Both maps are lifts of the map $f_1 \circ p\colon \widetilde{X}_{x_0} \to Y$ to a map $\widetilde{X}_{x_0} \to \widetilde{Y}_{y_0}$.

Since $\widetilde{X}_{x_0}$ is path-connected and locally path-connected we obtain from the uniqueness statement of the Map Lifting Criterion 61.2 (2) that the two maps from $\widetilde{X}_{x_0} \to \widetilde{Y}_{y_0}$ agree. But this implies that the diagram commutes. ⊞

Now we can turn to the construction of the desired chain homotopy. Thus let $\Omega_n \in \mathrm{C}_{n+1}(\Delta^n \times [0,1])$, $n \in \mathbb{N}_0$, be the sequence of singular chains that we defined on page 1605. By Lemma 73.8 we know that the maps

$$\begin{array}{rcl} P_n\colon \mathrm{C}_n(\widetilde{X}_{x_0}) & \to & \mathrm{C}_{n+1}(\widetilde{Y}_{y_0}) \\ (\sigma\colon \Delta^n \to \widetilde{X}_{x_0}) & \mapsto & (\widetilde{F} \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\Omega_n) \end{array}$$

satisfy

$$(*) \qquad \partial \circ P_n + P_{n-1} \circ \partial \;=\; (\widetilde{F} \circ i_0)_* - (\widetilde{F} \circ i_1)_*$$

as maps from $\mathrm{C}_n(\widetilde{X}_{x_0})$ to $\mathrm{C}_n(\widetilde{Y}_{y_0})$.

Claim 3. Given any singular simplex $\sigma\colon \Delta^n \to X$ and given any $g \in \pi_1(X, x_0)$ we have

$$P_n(g \cdot \sigma) \;=\; \underbrace{(F \circ i_0)_*(g)}_{\in \pi_1(Y, y_0)} \cdot P_n(\sigma).$$

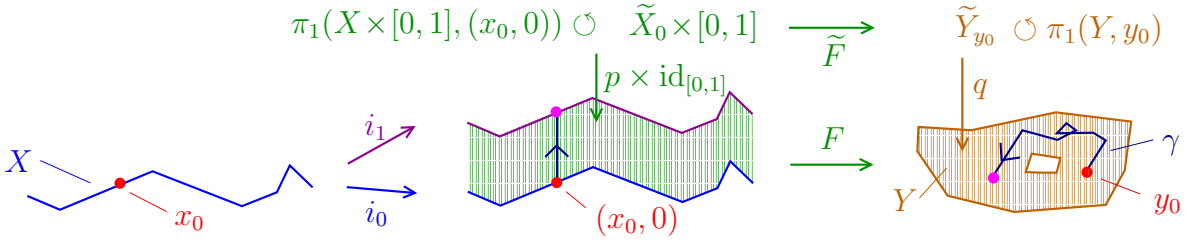
Proof. First note that by Proposition 62.13 the map $\widetilde{F}\colon \widetilde{X}_{x_0} \times [0,1] \to \widetilde{Y}$ has the following property: given any $\widetilde{z} \in \widetilde{X}_{x_0} \times [0,1]$ and given any $h \in \pi_1(X \times [0,1], (x_0, 0))$ we have

$$\widetilde{F}(h \cdot \widetilde{z}) \;=\; \underbrace{F_*(h)}_{\in \pi_1(Y, y_0)} \cdot \widetilde{F}(\widetilde{z}).$$

Now let $\sigma\colon \Delta^n \to X$ be a singular simplex and let $g \in \pi_1(X, x_0)$. Then

$$\begin{array}{rcl} P_n(g\cdot\sigma) & \overset{\substack{\text{by definition of } P_n\\ \downarrow}}{=} & (\widetilde{F}\circ((g\cdot\sigma\times \mathrm{id}_{[0,1]})))_*(\Omega_n) \\ & \underset{\substack{\uparrow\\ \text{by the above equality applied}\\ \text{to } h = i_{0*}(g)\in\pi_1(X\times[0,1],(x_0,0))}}{=} & (F_*(i_{0*}(g))\cdot(\widetilde{F}\circ(\sigma\times\mathrm{id}_{[0,1]})))_*(\Omega_n) \end{array} \quad \begin{array}{rcl} \overset{\substack{\text{by Proposition 62.13}\\ \downarrow}}{=} & (\widetilde{F}\circ(i_{0*}(g)\cdot(\sigma\times\mathrm{id}_{[0,1]})))_*(\Omega_n) \\ \underset{\substack{\uparrow\\ \text{by definition of } P_n}}{=} & (F\circ i_0)_*(g)\cdot P_n(\sigma). \end{array}$$

⊞



It follows from Claim 3 together with Lemma 165.23 that we get an induced R-left module homomorphism

$$\mathrm{id}\otimes P_n\colon \overline{(F\circ i_0)^*M}\otimes_{\mathbb{Z}[\pi_1(X,x_0)]}\mathrm{C}_n(\widetilde{X}_{x_0}) \quad\to\quad \overline{M}\otimes_{\mathbb{Z}[\pi_1(Y,y_0)]}\mathrm{C}_n(\widetilde{Y}_{y_0}).$$

It follows from $(*)$ that these maps have the property that

$$(\mathrm{id}\otimes\partial)\circ(\mathrm{id}\otimes P_n)+(\mathrm{id}\otimes P_{n-1})\circ(\mathrm{id}\otimes\partial) \;=\; \mathrm{id}\otimes(\widetilde{F}\circ i_0)_* - \mathrm{id}\otimes(\widetilde{F}\circ i_1)_*$$

as maps from $\overline{(F\circ i_0)^*M}\otimes_{\mathbb{Z}[\pi_1(X,x_0)]}\mathrm{C}_n(\widetilde{X}_{x_0})$ to $\overline{M}\otimes_{\mathbb{Z}[\pi_1(Y,y_0)]}\mathrm{C}_n(\widetilde{Y}_{y_0})$. In other words, we have found a chain homotopy between the maps $\mathrm{id}\otimes(\widetilde{F}\circ i_0)_*$ and $\mathrm{id}\otimes(\widetilde{F}\circ i_1)_*$. It remains to show that these two maps are the chain maps we are supposed to study:

(1) It follows immediately from the definitions that $\mathrm{id}\otimes(\widetilde{F}\circ i_0)_* = f_{0*}$.

(2) It follows easily from Claim 1 and 2 that $\mathrm{id}\otimes(\widetilde{F}\circ i_1)_* = \gamma_*\circ f_{1*}$. ■

In the Homotopy Equivalence-H_*-Corollary 73.9 and the Homotopic Maps-Cohomology Lemma 108.13 we saw that homotopy equivalent topological spaces have isomorphic (co-) homology groups. Now we have similar statements in the context of (co-) homology with twisted coefficients. But as usual some care needs to be taken regarding base points and modules involved.

We start out with the following, slightly restrictive proposition.

Proposition 167.15. (Homotopy Equivalence-H_*-Proposition) Let (X, x_0) and (Y, y_0) be pointed topological spaces that are both locally contractible and path-connected. Let $A\subset X$ and $B\subset Y$ be subspaces and let $x_0\in X$ and $y_0\in Y$. Furthermore let $f\colon (X,A)\to(Y,B)$ be a map with $f(x_0)=y_0$. We suppose that there exists a map $g\colon (Y,B)\to(X,A)$ such that $g\circ f\colon (X,A)\to(X,A)$ is homotopic rel x_0 to $\mathrm{id}_X\colon (X,A)\to(X,A)$ and such that $f\circ g\colon (Y,B)\to(Y,B)$ is homotopic rel y_0 to $\mathrm{id}_Y\colon (Y,A)\to(Y,B)$. Then f induces isomorphisms on twisted (co-) homology, i.e. given any $\mathbb{Z}[\pi_1(Y,y_0)]$-left module M the induced maps

$$\begin{array}{rcl} f_*\colon \mathrm{H}_k^{\mathrm{tw}}(X,A;f^*M) & \to & \mathrm{H}_k^{\mathrm{tw}}(Y,B;M) \\ f^*\colon \mathrm{H}^k_{\mathrm{tw}}(Y,B;M) & \to & \mathrm{H}^k_{\mathrm{tw}}(X,A;f^*M) \end{array}$$

are isomorphisms.

Proof. First note that by hypothesis there exists a homotopy $S\colon X\times[0,1]\to X$ rel x_0 with $S_0=g\circ f\colon (X,A)\to(X,A)$ and $S_1=\mathrm{id}_X\colon (X,A)\to(X,A)$ and there exists a homotopy $T\colon Y\times[0,1]\to Y$ rel y_0 with $T_0=f\circ g\colon (Y,B)\to(Y,B)$ and $T_1=\mathrm{id}_Y\colon (Y,B)\to(Y,B)$. It follows from Corollary 167.14 and the Twisted Homology-Functoriality Proposition 167.9 that the composition of the following two maps

$$\mathrm{H}_k^{\mathrm{tw}}(X,A;\overbrace{f^*(g^*(f^*M))}^{=(g\circ f)^*f^*M=\mathrm{id}^*\,f^*M=f^*M})\xrightarrow{f_*}\mathrm{H}_k^{\mathrm{tw}}(Y,B;\overbrace{g^*(f^*M)}^{=(f\circ g)^*M=\mathrm{id}^*\,M=M})\xrightarrow{g_*}\mathrm{H}_k^{\mathrm{tw}}(X,A;f^*M)$$

is the identity and that the composition of the following two maps

$$\mathrm{H}_k^{\mathrm{tw}}(Y,B;\underbrace{g^*(f^*M)}_{=(f\circ g)^*M=\mathrm{id}^*\,M=M})\xrightarrow{g_*}\mathrm{H}_k^{\mathrm{tw}}(X,A;f^*M)\xrightarrow{f_*}\mathrm{H}_k^{\mathrm{tw}}(Y,B;M)$$

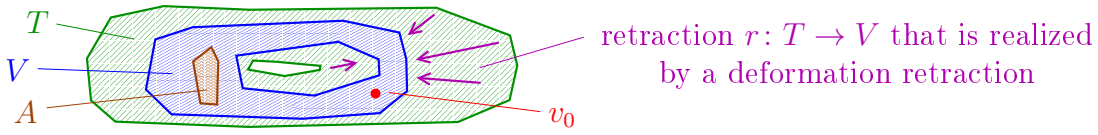
is also the identity. This shows that f_* is an isomorphism. Basically the same argument also works for twisted cohomology instead of twisted homology.[2074] ■

The following corollary considers an important special case of a homotopy equivalence.

Corollary 167.16. (Deformation Retract-Twisted Homology Corollary) Let T be a topological space and let $A\subset V\subset T$ be subspaces. We assume that T and V are both locally contractible and path-connected. Furthermore let $r\colon T\to V$ be a retraction. We assume that r is realized by a deformation retraction $F\colon T\times[0,1]\to T$, i.e. according to the definition on page 433 we assume that there exists a homotopy $F\colon T\times[0,1]\to T$ rel V with $F_0=\mathrm{id}_T$ and with $F_1=r$. For any $v_0\in V$ and any $\mathbb{Z}[\pi_1(V,v_0)]$-left module M the induced maps

$$\begin{aligned} r_*\colon \mathrm{H}_k^{\mathrm{tw}}(T,A;r^*M) &\to \mathrm{H}_k^{\mathrm{tw}}(V,A;M)\\ r^*\colon \mathrm{H}^k_{\mathrm{tw}}(V,A;M) &\to \mathrm{H}^k_{\mathrm{tw}}(T,A;r^*M)\end{aligned}$$

are isomorphisms.



Proof. We denote by $i\colon V\to T$ the inclusion. Note that $r\circ i=\mathrm{id}_V$. The desired statement follows immediately from the Homotopy Equivalence-H_*-Proposition 167.15 since the homotopy $F\colon T\times[0,1]\to T$ between $F_0=\mathrm{id}_T$ and $F_1=r=i\circ r$ and fixes the base point v_0, and since for $r\circ i$ we can use the constant homotopy, which of course also fixes the base point v_0. ■

Using the Deformation Retract-Twisted Homology Corollary 167.16 we can prove the following statement that does not bother with niceties regarding base points.

Corollary 167.17. Let X and Y be topological spaces that are both locally contractible and path-connected. We assume that X and Y are homotopy equivalent. Given any $x_0\in X$, $y_0\in Y$, given any ring R and given any $(\mathbb{Z}[\pi_1(X,x_0)],R)$-left module M there exists a $(\mathbb{Z}[\pi_1(Y,y_0)],R)$-left module N, which as an R-left module is isomorphic to M,

[2074] Note that this argument is formally the same as the argument that we used in the proof of the Homotopy Equivalence-H_*-Corollary 73.9.

such that the twisted homology groups $\mathrm{H}_k^{\mathrm{tw}}(X;M)$ and $\mathrm{H}_k^{\mathrm{tw}}(Y;N)$ are isomorphic R-left modules and such that the twisted cohomology groups $\mathrm{H}^k_{\mathrm{tw}}(X;M)$ and $\mathrm{H}^k_{\mathrm{tw}}(Y;N)$ are isomorphic R-left modules.

Proof. Let $f\colon X \to Y$ be a homotopy equivalence. As on page 454 we consider the mapping cylinder

$$\mathrm{Cyl}(f) := \mathrm{Cyl}(f\colon X \to Y) := ((X \times [0,1]) \sqcup Y)/\sim \text{ where } (x,1) \sim f(x) \text{ for all } x \in X.$$

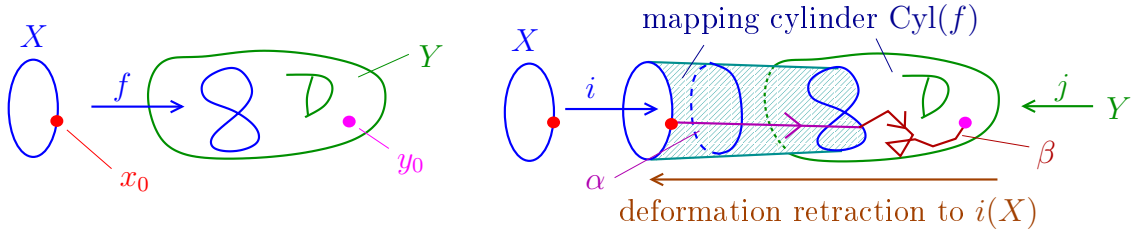
In the Mapping Cylinder Basics Lemma 16.10 we saw that the two maps

$$\begin{array}{rcl} i\colon X & \to & \mathrm{Cyl}(f\colon X \to Y) \\ x & \mapsto & [(x,0)] \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon Y & \to & \mathrm{Cyl}(f\colon X \to Y) \\ y & \mapsto & [y] \end{array}$$

are closed embeddings. Furthermore in the proof of Corollary 17.17 we saw that $i(X)$ is a deformation retract of $\mathrm{Cyl}(f)$ and we saw that $j(Y)$ is a deformation retract of $\mathrm{Cyl}(f)$. We denote by $r\colon \mathrm{Cyl}(f) \to i(X)$ the corresponding retraction. Let $\alpha\colon [0,1] \to \mathrm{Cyl}(f)$ be the path that is given by $t \mapsto [(x_0,t)]$ and let $\beta\colon [0,1] \to Y$ be any path from $f(x_0)$ to y_0. We see that[2075]

$$\begin{array}{ccccc} & & \text{given by the homeomorphism } i\colon X \to i(X) & & \text{isomorphism by (2)} \\ & & \downarrow & & \downarrow \\ \mathrm{H}_k^{\mathrm{tw}}(X;M) & \xrightarrow{\cong} & \mathrm{H}_k^{\mathrm{tw}}(i(X);(i^{-1})^*M) & \cong & \mathrm{H}_k^{\mathrm{tw}}(\mathrm{Cyl}(f);r^*(i^{-1})^*M) \\ & \cong & \mathrm{H}_k^{\mathrm{tw}}(\mathrm{Cyl}(f);\beta^{**}\alpha^{**}r^*(i^{-1})^*M) & \cong & \mathrm{H}_k^{\mathrm{tw}}(Y;j^*\beta^{**}\alpha^{**}r^*(i^{-1})^*M). \\ & & \uparrow & & \uparrow \\ & & \text{isomorphism by the Change of Base Point Proposition 167.6} & & \text{isomorphism by (2) and since } j\colon Y \to j(Y) \text{ is a homeomorphism} \end{array}$$

This concludes the arguments regarding twisted homology modules. The argument regarding twisted cohomology modules is very similar. We leave the details to the reader. ∎



Exercises for Chapter 167.

Exercise 167.1. Let $f\colon (X,x_0) \to (Y,y_0)$ and $g\colon (Y,y_0) \to (Z,z_0)$ be maps between pointed topological spaces. We assume that X and Y are locally path-connected and path-connected. Let $p\colon (\widehat{X},\widehat{x}_0) \to (X,x_0)$, $q\colon (\widehat{Y},\widehat{y}_0) \to (Y,y_0)$ and $r\colon (\widehat{Z},\widehat{z}_0) \to (Z,z_0)$ be coverings with $f_*(p_*(\pi_1(\widehat{X},\widehat{x}_0))) \subset \pi_1(\widehat{Y},\widehat{y}_0)$ and $g_*(p_*(\pi_1(\widehat{Y},\widehat{y}_0))) \subset \pi_1(\widehat{Z},\widehat{z}_0)$. We consider the induced maps $\widetilde{f}\colon (\widehat{X},\widehat{x}_0) \to (\widehat{Y},\widehat{y}_0)$, $\widetilde{g}\colon (\widehat{Y},\widehat{y}_0) \to (\widehat{Z},\widehat{z}_0)$ and $\widetilde{g \circ f}\colon (\widehat{X},\widehat{x}_0) \to (\widehat{Z},\widehat{z}_0)$ that we defined on page 3541. Show that we have an equality of maps

$$\widetilde{g \circ f} \;=\; \widetilde{g} \circ \widetilde{f}\colon (\widehat{X},\widehat{x}_0) \;\to\; (\widehat{Z},\widehat{z}_0).$$

[2075] In Exercise 16.19 we saw, using in particular an argument as in the proof of the Mapping Cylinder Basics Lemma 16.12 (1), that if X and Y are locally contractible, then the mapping cylinder $\mathrm{Cyl}(f\colon X \to Y)$ is also locally contractible. Thus it makes sense to consider the twisted homology groups of $\mathrm{Cyl}(f)$.

Exercise 167.2. We consider the topological space $X = S^1 \vee S^2$ and we take as base point the wedge point x_0. Note that $\pi_1(S^1 \vee S^2, x_0) \cong \pi_1(S^1, x_0)$ is abelian. Thus it follows from the Change of Base Point Proposition 167.6 that $\pi_1(X, x_0)$ acts on any on any twisted homology module $\mathrm{H}_k^{\mathrm{tw}}(X; M)$ and $\mathrm{H}_{\mathrm{tw}}^k(X; M)$. Given any example of a $\mathbb{Z}[\pi_1(X, x_0)]$-left module for which this left action is non-trivial.

168. Calculations and examples of twisted (co-) homology

168.1. **(Co-) homology with twisted coefficients in degree 0.** In the H_0-Proposition 72.5 and on page 1914 we saw that it is straightforward to calculate singular (co-) homology in degree 0. As we will see, it is also pretty straightforward to calculate singular (co-) homology with twisted coefficients in degree 0. To state the corresponding result we need the following definition.

Definition. Let π be a group, let R be a ring and let M be a $(\mathbb{Z}[\pi], R)$-left left module. We define

$$\begin{array}{rcll} M_\pi & := & M \Big/ \left\{ \sum_{i=1}^{k} (g_i \cdot v_i - v_i) \,\middle|\, g_i \in \pi \text{ and } v_i \in M \right\} & \pi\text{-coinvariants} \\ M^\pi & := & \{v \in M \,|\, g \cdot v = v \text{ for all } g \in \pi\} & \pi\text{-invariants} \end{array}$$

Note that M_π and M^π are both R-left modules.

Examples. Let π be a group.

(1) If π acts trivially on an abelian group M, then it follows immediately from the definitions that M_π is naturally isomorphic to M and furthermore $M^\pi = M$.
(2) We consider the case $M = \mathbb{Z}[\pi]$ and $R = \mathbb{Z}$. We equip M with the $\mathbb{Z}[\pi]$-left module structure that is given by $g \cdot m := m \cdot g^{-1}$. In Exercise 168.1 we will show that

$$\begin{array}{rcl} \mathbb{Z}[\pi]_\pi & \to & \mathbb{Z} \\ \left[\sum_{i=1}^{k} r_i \cdot g_i \right] & \mapsto & \sum_{i=1}^{k} r_i \end{array}$$

is an isomorphism and we will see that

$$\mathbb{Z}[\pi]^\pi = \begin{cases} 0, & \text{if } \pi \text{ is infinite,} \\ \mathbb{Z} \cdot \sum_{g \in \pi} g, & \text{if } \pi \text{ is finite.} \end{cases}$$

Before we can state our calculation of twisted (co-) homology in degree 0 it is convenient to recall the following notation from page 1590.

Notation. Let X be a topological space. Given a point $P \in X$ we denote by P also the singular 0-simplex $\Delta^0 \to X$ that is given by sending the single point in Δ^0 to P.

The following lemma is the analogue of the H_0-Proposition 72.5 and the discussion on page 1914.

Proposition 168.1. (Twisted H_0-Proposition) Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. We write $\pi := \pi_1(X, x_0)$. Let R be a ring and let M be a $(\mathbb{Z}[\pi], R)$-left left module. We consider the natural universal covering $p\colon \widetilde{X}_{x_0} \to X$ and we denote by $\widetilde{x}_0$ the natural base point of $\widetilde{X}_{x_0}$. The augmentation map

$$\begin{array}{rcl} \epsilon_X\colon H_0^{\mathrm{tw}}(X; M) & \to & M_\pi \\ \left[\sum_{i=1}^{k} m_i \otimes_{\mathbb{Z}[\pi]} \sigma_i \right] & \mapsto & \left[\sum_{i=1}^{k} m_i \right] \end{array}$$

and the evaluation map

$$\begin{array}{rcl} \mathrm{ev}_X\colon \mathrm{H}^0_{\mathrm{tw}}(X;M) & \to & M^\pi \\ [\varphi\colon \mathrm{C}_0(\widetilde{X}_{x_0}) \to M)] & \mapsto & \varphi(\widetilde{x}_0) \\ & & \uparrow \\ & & \text{here } \widetilde{x}_0 \text{ is short for the constant map } \Delta^0 \to \widetilde{X}_{x_0} \text{ that sends 1 to } \widetilde{x}_0 \end{array}$$

are natural isomorphisms of R-left modules

Example. We suppose that we are in the setting of the Twisted H_0-Proposition 168.1. If π acts trivially on M, then the Twisted H_0-Proposition 168.1 says that $\mathrm{H}_0^{\mathrm{tw}}(X;{}_{\mathrm{tr}}M)$ and $\mathrm{H}^0_{\mathrm{tw}}(X;{}_{\mathrm{tr}}M)$ are both naturally isomorphic to the R-left module M. This result is consistent with the fact, shown in Lemma 167.3, that $\mathrm{H}_0^{\mathrm{tw}}(X;{}_{\mathrm{tr}}M) \cong \mathrm{H}_0(X;M)$ and $\mathrm{H}^0_{\mathrm{tw}}(X;{}_{\mathrm{tr}}M) \cong \mathrm{H}^0(X;M)$ and our earlier results in the H_0-Proposition 72.5 and on page 1914.

Proof. In the following we will deal with $\mathrm{H}_0^{\mathrm{tw}}(X;M)$. We will prove the statement regarding $\mathrm{H}^0_{\mathrm{tw}}(X;M)$ in Exercise 168.3.

Claim. The map
$$\begin{array}{rcl} \epsilon_X\colon \mathrm{H}_0^{\mathrm{tw}}(X;M) & \to & M_\pi \\ \left[\sum_{i=1}^k m_i \otimes_{\mathbb{Z}[\pi]} \sigma_i\right] & \mapsto & \left[\sum_{i=1}^k m_i\right] \end{array}$$
is well-defined.

Proof. First we need to show that the map is well-defined on the twisted chain level, i.e. we need to show that the map is well-defined as a map $\overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_0(\widetilde{X}_{x_0}) = \mathrm{C}_0^{\mathrm{tw}}(X;M) \to M_\pi$. Note that for $m \in M$, $\sigma \in \mathrm{C}_0(\widetilde{X}_{x_0})$ and $g \in \pi$ we have

$$\epsilon_X([m \underset{\overline{M}}{\cdot} g \otimes \sigma]) \underset{\substack{\uparrow \\ \text{definition of } \overline{M}}}{=} \epsilon_X([g^{-1} \underset{M}{\cdot} m \otimes \sigma] = [g^{-1} \underset{M}{\cdot} m] \underset{\substack{\uparrow \\ \text{since } g^{-1}\cdot m - m = (g^{-1}-1)\cdot m = 0 \in M_\pi}}{=} [m] = \epsilon_X([m \otimes g \cdot \sigma]) \in M_\pi.$$

Note that it follows easily from this little calculation together with Lemma 165.23 that the map $\overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_0(\widetilde{X}_{x_0}) = \mathrm{C}_0^{\mathrm{tw}}(X;M) \to M_\pi$ is well-defined.

Next let $m \in M$ and let $\sigma\colon \Delta^1 \to \widetilde{X}_{x_0}$ be a singular 1-simplex. Then

$$\epsilon_X(\partial(m \otimes \sigma)) \underset{\substack{\uparrow \\ \text{see the Tensoring Chain Complex Lemma 166.3}}}{=} \epsilon_X(m \otimes \partial\sigma) \underset{\substack{\uparrow \\ \text{see page 1585}}}{=} \epsilon_X(m \otimes \sigma(0,1) - m \otimes \sigma(1,0)) = m - m = 0.$$

This shows that $\epsilon_X\colon \mathrm{H}_0^{\mathrm{tw}}(X;M) \to M_\pi$ is indeed well-defined. ⊞

It follows basically immediately from the fact that X is non-empty that the map $\epsilon_X\colon \mathrm{H}_0^{\mathrm{tw}}(X;M) \to M_\pi$ is an epimorphism. It remains to show that it is a monomorphism. Thus suppose we are given $\sum_{i=1}^k m_i \otimes Q_i \in \overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_0(\widetilde{X}_{x_0}) = \mathrm{C}_0^{\mathrm{tw}}(X;M)$ which represents an element in the kernel of ϵ_X. This means that for $j = 1, \ldots, l$ there exist $g_j \in \pi$ and $v_j \in M$ such that the following equality holds in M:

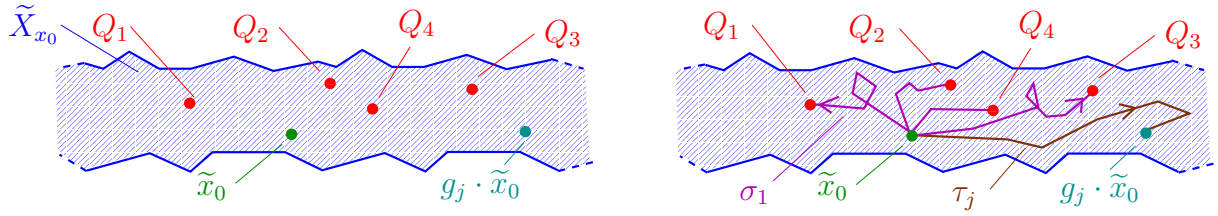
$$\sum_{i=1}^k m_i = \sum_{j=1}^l (g_j \cdot v_j - v_j).$$

Since X is path-connected there exist singular 1-simplices $\sigma_1, \ldots, \sigma_k\colon \Delta^1 \to \widetilde{X}_{x_0}$ with $\partial\sigma_i = Q_i - x_0$. Furthermore there exist singular 1-simplices $\tau_1, \ldots, \tau_l\colon \Delta^1 \to \widetilde{X}_{x_0}$ with

$\partial\tau_j = g_j \cdot x_0 - x_0$. We now see that

$$\begin{aligned}\partial_1\Big(\sum_{i=1}^{k} m_i \otimes \sigma_i \;-\; \sum_{j=1}^{l} v_j \otimes \tau_j\Big) \;&=\; \sum_{i=1}^{k}(m_i \otimes Q_i - m_i \otimes x_0) \;-\; \sum_{j=1}^{l}\big(\underbrace{v_j \otimes g_j \cdot x_0}_{=v_j\cdot g_j \otimes x_0} - v_j \otimes x_0\big)\Big)\\ &=\; \sum_{i=1}^{k} m_i \otimes Q_i - \underbrace{\Big(\sum_{i=1}^{k} m_i - \sum_{j=1}^{l}(g_j \cdot v_j - v_j)\Big)}_{=0} \otimes x_0 \;=\; \sum_{i=1}^{k} m_i \otimes Q_i.\end{aligned}$$

We have thus shown that $\sum_{i=1}^{k} m_i \otimes Q_i$ represents the zero element in $\mathrm{H}_0^{\mathrm{tw}}(X;M)$. In other words, we have shown that $\epsilon_X\colon \mathrm{H}_0^{\mathrm{tw}}(X;M) \to M_\pi$ is a monomorphism. ■



168.2. **Twisted homology and homology of covering spaces.** In this section we will see that for certain twistings we can calculate twisted homology groups in terms of ordinary homology of covering spaces. In practice this is one of the most useful tricks for calculating twisted homology groups.

Proposition 168.2. Let X be a topological space that is locally contractible and path-connected. We pick $x_0 \in X$ and we write $\pi := \pi_1(X, x_0)$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the corresponding universal covering which we equip with the natural base point $\widetilde{x}_0$. Let $A \subset X$. We write $\widetilde{A} := p^{-1}(A)$. Furthermore let $\Gamma \subset \pi$ be a subgroup. Let S be a *commutative* ring. We equip the free S-module $S[\Gamma\backslash\pi]$ that is generated by the set $\Gamma\backslash\pi$ with the $\mathbb{Z}[\pi]$-left module structure given by $h \cdot (\Gamma g) = \Gamma g h^{-1}$. Let $p\colon (Y_\Gamma, y_0) \to (X, x_0)$ be a covering of pointed topological spaces with $p_*(\pi_1(Y_\Gamma, y_0)) = \Gamma$. By Corollary 61.11 we know that there exists a unique covering $q\colon \widetilde{X}_{x_0} \to Y_\Gamma$ with $q(\widetilde{x}_0) = y_0$.

(1) The maps[2076]

$$\begin{aligned}\Omega\colon \mathrm{C}_k^{\mathrm{tw}}(X, A; S[\Gamma\backslash\pi]) = \overline{S[\Gamma\backslash\pi]} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}) \;&\to\; \mathrm{C}_k(Y_\Gamma; S) = \mathrm{C}_k(Y) \otimes S\\ s \cdot \Gamma g \otimes (\sigma\colon \Delta^k \to Y_\Gamma; S) \;&\mapsto\; q \circ (g \cdot \sigma\colon \Delta^k \to Y_\Gamma) \otimes s\end{aligned}$$

form an isomorphism of chain complexes of S-modules. In particular for each $k \in \mathbb{N}_0$ these maps induce an isomorphism

$$\Omega_*\colon \underbrace{\mathrm{H}_k^{\mathrm{tw}}(X, A; S[\Gamma\backslash\pi])}_{\text{twisted homology}} \;\to\; \underbrace{\mathrm{H}_k(Y_\Gamma; S)}_{\substack{\text{singular homology}\\ \text{with } S\text{-coefficients}}}$$

of S-modules.

(2) We now suppose that Γ is a normal subgroup of π. This allows us to equip $S[\Gamma\backslash\pi]$ with an $S[\Gamma\backslash\pi]$-left module structure that is given by left multiplication.[20772078] Furthermore we equip Y_γ, and thus also $\mathrm{C}_k(Y_\Gamma; S)$ and $\mathrm{H}_k(Y_\Gamma; S)$ with the $\Gamma\backslash\pi$-left

action given by Proposition 167.1. With these actions the above maps

$$\begin{aligned} \Omega\colon \mathrm{C}_k^{\mathrm{tw}}(X,A;S[\Gamma\backslash\pi]) &\to \mathrm{C}_k(Y_\Gamma;S) \\ \Omega_*\colon \mathrm{H}_k^{\mathrm{tw}}(X,A;S[\Gamma\backslash\pi]) &\to \mathrm{H}_k(Y_\Gamma;S) \end{aligned}$$

are isomorphisms of $S[\Gamma\backslash\pi]$-left modules.

Remark. Everybody who tries to write up the story of twisted (co-) homology groups and modules will notice that sooner or later it is necessary to make choices about the type of actions that are considered. Whichever choice one takes will sooner or later haunt the author at some other point. Our choices are designed in such a way that we get the clean statement in Proposition 168.2 about $S[\Gamma\backslash\pi]$-left modules. In particular this leads us to the popular topic of left-left modules.

Proof.

(1) Let us show that the given map

$$\begin{aligned} \Omega\colon \mathrm{C}_k^{\mathrm{tw}}(X,A;S[\Gamma\backslash\pi]) = \overline{S[\Gamma\backslash\pi]} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0},\widetilde{A}) &\to \mathrm{C}_k(Y_\Gamma;S) \\ s\cdot\Gamma g\otimes(\sigma\colon\Delta^k\to Y_\Gamma) &\mapsto q\circ(g\cdot\sigma\colon\Delta^k\to Y_\Gamma)\otimes s \end{aligned}$$

is well-defined. Given $k\in\pi$ we denote by $d_k\colon \widetilde{X}_{x_0}\to\widetilde{X}_{x_0}$ the corresponding deck transformation that is given by Proposition 167.1. First we show that the map is independent of the representative $g\in\pi$. Indeed, if $h\in\Gamma$, then

$$q\circ hg\cdot\sigma \;=\; q\circ d_h\circ d_g\circ\sigma \;\underset{\uparrow}{=}\; q\circ d_g\circ\sigma \;=\; q\circ g\cdot\sigma.$$

it follows from Proposition 167.1 that $q\circ d_h = q$

Next let us show, using Lemma 165.23, that the map Ω respects the tensor product $\overline{S[\Gamma\backslash\pi]}\otimes_{\mathbb{Z}[\pi]}\mathrm{C}_k(\widetilde{X}_{x_0})$. Thus let $\Gamma f\in\Gamma\backslash\pi$ and let $\sigma\in\mathrm{C}_k(\widetilde{X}_{x_0})$. Then given any $g\in\pi$ we have

$$\Omega((\Gamma f)\cdot g\otimes\sigma) \;\underset{\uparrow}{=}\; \Omega(\Gamma f\cdot g\otimes\sigma) \;=\; q\circ(f\cdot g)\cdot\sigma \;=\; q\circ f\cdot(g\cdot\sigma) \;=\; \Omega(\Gamma f\otimes g\cdot\sigma).$$

follows from our definition of the $\mathbb{Z}[\pi]$-left module structure on $S[\Gamma\backslash\pi]$ and definition of the $\mathbb{Z}[\pi]$-right module $\overline{S[\Gamma\backslash\pi]}$

Recall that in Lemma 72.9 we showed that the boundary maps ∂ are natural. It is now straightforward to verify that the maps Ω are chain maps. It remains to show that the maps Ω are isomorphisms. To do so we consider the map

$$\begin{aligned} \mathrm{C}_k(Y_\Gamma;S) &\to \mathrm{C}_k^{\mathrm{tw}}(X;S[\Gamma\backslash\pi]) = \overline{S[\Gamma\backslash\pi]}\otimes_{\mathbb{Z}[\pi]}\mathrm{C}_k(\widetilde{X}_{x_0}) \\ (\sigma\colon\Delta^k\to Y_\Gamma)\otimes s &\mapsto s\otimes\text{ any lift of }\sigma\text{ to a map }\Delta^k\to\widetilde{X}_{x_0}. \end{aligned}$$

It follows easily from Proposition 62.12 that this map is well-defined. It is straightforward to verify that this map is an inverse to Ω.

[2076] Note that the corresponding $\mathbb{Z}[\pi]$-right module structure on $\overline{S[\Gamma\backslash\pi]}$ is actually the obvious one given by right multiplication.

[2077] This follows from the usual observation that $(\Gamma\cdot a)\cdot(\Gamma\cdot b) := \Gamma\cdot ab$ is independent of the choice of representatives g,h, if Γ is normal. Indeed, if $g,h\in\Gamma$, then $\Gamma\cdot gahb = \Gamma\cdot g\underbrace{aha^{-1}}_{\in\Gamma}ab$, thus $\Gamma\cdot gahb = \Gamma\cdot ab$.

[2078] Note that these two left module structures turn $S[\Gamma\backslash\pi]$ into a $(\mathbb{Z}[\pi],S[\Gamma\backslash\pi])$-left left module.

(2) We only need to show that Ω is $S[\Gamma\backslash\pi]$-left linear. Thus let $\Gamma g, \Gamma h \in \Gamma\backslash\pi$ and let $\sigma \in \mathrm{C}_k(\widetilde{X}_{x_0})$. Then

$$\Omega(\Gamma h \cdot (\Gamma g \otimes \sigma)) \;=\; \Omega(\Gamma hg \otimes \sigma) \;=\; q \circ h \cdot g \circ \sigma \;\underset{\uparrow}{=}\; h \cdot q \circ g \cdot \sigma \;=\; \Gamma h \cdot \Omega(g \otimes \sigma).$$

follows from Proposition 167.1 ■

Example. Let X be a topological space that is locally contractible and path-connected. We pick $x_0 \in X$ and we write $\pi := \pi_1(X, x_0)$. If we apply Proposition 168.2 to the trivial subgroup $\Gamma = \{e\}$, then we see that for every $k \in \mathbb{N}_0$ we have an isomorphism

$$\mathrm{H}_k^{\mathrm{tw}}(X; \mathbb{Z}[\pi_1(X, x_0)]) \;\cong\; \mathrm{H}_k(\text{universal covering of } X).$$

For example, if we consider the n-dimensional torus $X = (S^1)^n$, then

$$\mathrm{H}_k^{\mathrm{tw}}((S^1)^n; \mathbb{Z}[\pi_1(X)]) \underset{\uparrow}{\cong} \mathrm{H}_k(\text{universal covering of } (S^1)^n) \underset{\uparrow}{\cong} \mathrm{H}_k(\mathbb{R}^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ 0, & \text{else.} \end{cases}$$

Proposition 168.2 — see page 1372

Example. $X = S^1 \vee S^2$.

Remark. Proposition 168.2 is very appealing since for many twistings it gives us back our intuition for what homology with twisted coefficients means. Note though that the analogous statement for *cohomology* with twisted coefficients does not hold. For example consider $X = S^1$ with base point $x_0 = 1 \in S^1$. We write $\pi := \pi_1(S^1, 1)$. Then

$$\mathrm{H}^0_{\mathrm{tw}}(S^1; \mathbb{Z}[\pi]) \underset{\uparrow}{\cong} \mathbb{Z}[\pi]^\pi \underset{\uparrow}{=} 0 \neq \mathbb{Z} \cong \mathrm{H}^0(\mathbb{R}) = \mathrm{H}^0(\text{universal covering of } S^1).$$

Twisted H_0-Proposition 168.1 — see the discussion on page 3555 here we use that π is infinite

For our next example let us recall the following definition from page 467.

Definition. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism[2079]. We refer to

$$\mathrm{Tor}(X, f) \;:=\; (X \times [0,1])/\!\sim \qquad \text{where } (x,0) \sim (f(x),1) \text{ for all } x \in X,$$

as the mapping torus of (X, f). We adopt the following language:

(1) We refer to the map

$$\begin{array}{rccc} p\colon & \mathrm{Tor}(X,f) = (X \times [0,1])/(x,0) \sim (f(x),1) & \to & S^1 \\ & [(x,t)] & \mapsto & \exp(2\pi i t) \end{array}$$

as the natural projection onto S^1. Furthermore, given any $x \in X$ we refer to the map

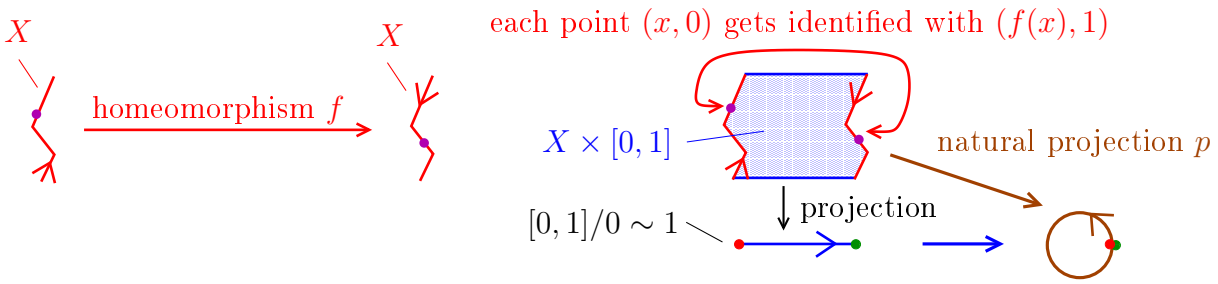
$$\pi_1(\mathrm{Tor}(X,f), [(x,0)]) \xrightarrow{p_*} \pi_1(S^1, 1) \underset{\uparrow}{\xrightarrow{\cong}} \langle t \rangle$$

unique isomorphism that sends the loop $s \mapsto \exp(2\pi i s)$ to t

as a standard epimorphism.

(2) If f is a homeomorphism, then we refer to X as fiber of the mapping torus and we refer to f as the monodromy of the mapping torus.

[2079] Before we defined the mapping torus for an arbitrary map $f\colon X \to X$, but in this section we are only interested in the case that f is a homeomorphism.



Notation. Let R be a commutative ring, let A be an R-module and let $\phi \in \mathrm{Aut}_R(A)$. We denote by $(A, t = \phi)$ the $R[t^{\pm 1}]$-module where the underlying R-module is given by A and where t acts by multiplication by ϕ, thus t^n acts by application of the automorphism $\phi^n = \phi \circ \cdots \circ \phi$.

Proposition 168.3. Let X be a topological space that is locally contractible and path-connected. Furthermore let $f\colon X \to X$ be a homeomorphism. We pick $x \in X$. We denote by $\phi\colon \pi_1(\mathrm{Tor}(X, f), [(x, 0)]) \to \langle t \rangle$ the corresponding standard epimorphism. Given any commutative ring R and given any $k \in \mathbb{N}_0$ we have a natural isomorphism[2080]

$$\text{the } R[t^{\pm 1}]\text{-module } (\mathrm{H}_k(X; R), t = f_*) \;\cong\; \mathrm{H}_k^{\mathrm{tw}}(\mathrm{Tor}(X, f); {}_\phi R[t^{\pm 1}])$$

of $R[t^{\pm 1}]$-modules.

Proof. Since f is a homeomorphism we can consider the map

$$\begin{array}{rcll} p\colon X \times \mathbb{R} & \to & \mathrm{Tor}(X, f) & \\ (x, t) & \mapsto & [(f^n(x), x - n)] & \text{where } n \in \mathbb{Z} \text{ with } x \in [n, n+1). \end{array}$$

One can easily verify that p is a covering. It follows almost immediately from the Mapping Torus-π_1-Proposition 50.6 that $p_*(\pi_1(X \times \mathbb{R})) = \ker(\varphi_*)$. We denote by $\mu\colon X \times \mathbb{R} \to X \times \mathbb{R}$ the deck transformation that is given by $(x, s) \mapsto (x, s + 1)$. By Proposition 168.2 we have a natural isomorphism

$$\Omega\colon \mathrm{H}_k(X; {}_\phi R[t^{\pm 1}]) \;\xrightarrow{\cong}\; \mathrm{H}_k(X \times \mathbb{R}; S)$$

of $R[t^{\pm 1}]$-modules where the t action is given by the map induced by the deck transformation μ. In other words, we have a natural isomorphism

$$\Omega\colon \mathrm{H}_k(X; {}_\phi R[t^{\pm 1}]) \;\xrightarrow{\cong}\; (\mathrm{H}_k(X \times \mathbb{R}; S), t = \mu_*)$$

of $R[t^{\pm 1}]$-modules. The proposition now follows from the following claim.

Claim. The $R[t^{\pm 1}]$-modules $(\mathrm{H}_k(X \times \mathbb{R}; S), t = \mu_*)$ and $(\mathrm{H}_k(X; R), t = f_*)$ are isomorphic.

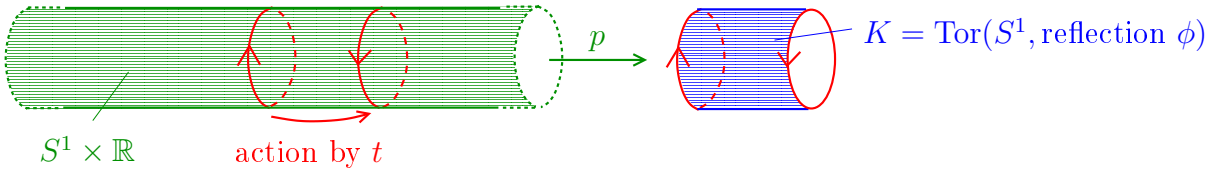
Proof. We denote by $p\colon X \times \mathbb{R} \to X$ the projection. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_k(X \times \mathbb{R}; S) & \xrightarrow{\mu_*} & \mathrm{H}_k(X \times \mathbb{R}; S) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ \mathrm{H}_k(X; R) & \xrightarrow{f_*} & \mathrm{H}_k(X; R). \end{array}$$

[2080] Here use the convention introduced on page 3500 to identify the group ring $\mathbb{Z}[\langle t \rangle]$ with the Laurent polynomial ring $\mathbb{Z}[t^{\pm 1}]$.

It follows from $p \circ \mu = f \circ p$ that the diagram commutes. Furthermore p is evidently a homotopy equivalence. Thus the two vertical maps are isomorphisms. The desired statement is an immediate consequence of these two observations. ∎

Example. We consider the Klein bottle K, which we now think of as $K := \mathrm{Tor}(S^1, \phi)$, where $\phi(z) = \overline{z}$ is the reflection in the real axis. Just for fun we mention that in this case the covering of the proof of Proposition 168.3 is given by a map $p\colon S^1 \times \mathbb{R} \to K := \mathrm{Tor}(S^1, f)$ where the t^n-action on $S^1 \times K$ is given by the map $t^n \cdot (x, s) \mapsto (x + n, (-1)^n \cdot s)$.



We want to use Proposition 168.3 to calculate the homology groups $\mathrm{H}_k^{\mathrm{tw}}(K; {}_\varphi\mathbb{Z}[t^{\pm 1}])$. We start out with the following observations:

(0) By the discussion on page 1594 we know that $\phi_*\colon \mathrm{H}_0(S^1) \to \mathrm{H}_0(S^1)$ is the identity.
(1) By the Sphere Degree Lemma 76.11 (6) we know that $\phi_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(S^1)$ is multiplication by -1.
(2) By Proposition 74.4 we have $\mathrm{H}_k(S^1) = 0$ for $k \geq 2$.

It follows that for $k \in \mathbb{N}$ we have the following isomorphisms:

$$\mathrm{H}_k^{\mathrm{tw}}(K; {}_\varphi\mathbb{Z}[t^{\pm 1}]) \underset{\text{follows from Proposition 168.3 together with (0), (1) and (2)}}{\cong} \begin{cases} (\mathbb{Z}, t = \mathrm{id}), & \text{if } k = 0, \\ (\mathbb{Z}, t = -1), & \text{if } k = 1, \\ 0, & \text{if } k \geq 2 \end{cases} \underset{\text{see Exercise 168.5}}{\cong} \begin{cases} \mathbb{Z}[t^{\pm 1}]/(t-1)\mathbb{Z}[t^{\pm 1}], & \text{if } k = 0, \\ \mathbb{Z}[t^{\pm 1}]/(t+1)\mathbb{Z}[t^{\pm 1}], & \text{if } k = 1, \\ 0, & \text{if } k \geq 2. \end{cases}$$

168.3. **Twisted cohomology and cohomology of finite-index coverings.** The following proposition is a cohomological version of Proposition 168.2. But note that it only gives something interesting for *finite index* subgroups.

Proposition 168.4. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. We write $\pi := \pi_1(X, x_0)$. Let R be a commutative ring. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the corresponding universal covering which we equip with the natural base point $\widetilde{x}_0$. Furthermore let $\Gamma \subset \pi_1(X, x_0)$ be a subgroup. Let $p\colon (Y_\Gamma, y_0) \to (X, x_0)$ be a covering of pointed topological spaces with $p_*(\pi_1(Y_\Gamma, y_0)) = \Gamma$. By Corollary 61.11 we know that there exists a unique covering $q\colon \widetilde{X}_{x_0} \to Y_\Gamma$ with $q(\widetilde{x}_0) = y_0$. As in Proposition 168.2 we equip the abelian group $R[\Gamma\backslash\pi]$ with the $\mathbb{Z}[\pi]$-left module structure given by $h \cdot (\Gamma g) = \Gamma g h^{-1}$.

(1) The maps

$$\Phi\colon \underset{\mathbb{Z}[\pi]\text{-left}}{\mathrm{Hom}}(\mathrm{C}_k(\widetilde{X}_{x_0}), R[\Gamma\backslash\pi]) \to \mathrm{Hom}_R(\mathrm{C}_k(Y_\Gamma), R)$$

$$\gamma \mapsto \left(\begin{array}{rcl} \mathrm{C}_k(Y_\Gamma) & \to & R \\ \sigma & \mapsto & \text{pick a lift } \widetilde{\sigma}\colon \Delta^k \to \widetilde{X}_{x_0} \text{ and take the} \\ & & \Gamma = \Gamma e\text{-coefficient of } \gamma(\widetilde{\sigma}) \in R[\Gamma\backslash\pi] \end{array} \right)$$

are well-defined and they form a cochain map.

(2) If Γ is a subgroup *of finite index*, then the maps

$$\begin{array}{rcl} \Psi\colon \operatorname{Hom}_R(\mathrm{C}_k(Y_\Gamma), R) & \to & \operatorname{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_k(\widetilde{X}), R[\Gamma\backslash\pi]) \\ \gamma & \mapsto & \left(\begin{array}{rcl} \mathrm{C}_k(\widetilde{X}) & \to & R[\Gamma\backslash\pi] \\ \sigma & \mapsto & \underbrace{\sum\limits_{\Gamma g \in \Gamma\backslash\pi} \gamma(g \cdot (q \circ \sigma\colon \Delta^k \to Y_\Gamma)) \cdot \Gamma g}_{\in R[\Gamma\backslash\pi] \text{ since } \Gamma \text{ is a subgroup of finite index}} \end{array} \right) \end{array}$$

are well-defined and they form a cochain map.

(3) If Γ is a subgroup of *finite index*, then the maps Φ and Ψ are both defined and they are inverses of one another. In particular we see that in this setting the maps Φ induce for each $k \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}^k_{\mathrm{tw}}(X; R[\Gamma\backslash\pi]) \ \to \ \mathrm{H}^k(Y_\Gamma; R).$$

(4) We suppose that Γ is a *normal* subgroup of π of finite index. We introduce two conventions:

- Note that $R[\Gamma\backslash\pi]$ is a $R[\Gamma\backslash\pi]$-left module in the obvious way and $R[\Gamma\backslash\pi]$-becomes a $(\mathbb{Z}[\pi], R[\Gamma\backslash\pi])$-left left module and $\mathrm{H}^k_{\mathrm{tw}}(X; R[\Gamma\backslash\pi])$ becomes a $R[\Gamma\backslash\pi]$-left module.
- Since Γ is a normal subgroup we can equip Y_Γ with the $\Gamma\backslash\pi$-left action given by Proposition 167.1. This allows us to view $\mathrm{H}^k(Y_\gamma; R)$ as a $R[\Gamma\backslash\pi]$-left module by setting $g \cdot \varphi := (g^{-1})^*(\varphi)$.[2081]

With the conventions the above maps Φ and Ψ are $R[\Gamma\backslash\pi]$-left module isomorphisms.

Proof. As with so many of these propositions, as long as the definitions are right, the proofs are dull and elementary. Thus we leave the verification of most aspects of (1) and (2) to the reader. But for peace of mind let us verify in (2) that $\Psi(\gamma)$ is indeed a homomorphism of $\mathbb{Z}[\pi]$-left modules. Thus let $\gamma \in \operatorname{Hom}_R(\mathrm{C}_k(Y_\Gamma), R)$. Then for any $\sigma \in \mathrm{C}_k(\widetilde{X})$ and any $h \in \pi$ we obtain the following equality in $R[\Gamma\backslash\pi]$:

$$\begin{array}{rcccl} \Psi(\gamma)(h \cdot \sigma) & \overset{\text{definition of } \Psi}{=} & \sum\limits_{\Gamma g \in \Gamma\backslash\pi} \gamma(g \cdot (q \circ h \cdot \sigma)) \cdot g & \overset{\text{by Proposition 167.1 we have } q \cdot h = h \cdot q}{=} & \sum\limits_{\Gamma g \in \Gamma\backslash\pi} \gamma(gh \cdot (q \circ \sigma)) \cdot g \\ & = & \sum\limits_{\Gamma g \in \Gamma\backslash\pi} \gamma(gh \cdot (q \circ \sigma)) \cdot gh \cdot h^{-1} & \overset{\text{by definition of } \Psi \text{ and since } \Gamma g \mapsto \Gamma g \cdot h \text{ is a bijection of } \Gamma\backslash\pi}{=} & \Psi(\gamma)(\sigma) \cdot h^{-1} \overset{\text{definition of left-module structure}}{=} h \cdot \Psi(\gamma)(\sigma). \end{array}$$

[2081] Since cohomology is covariant this does indeed define a $R[\Gamma\backslash\pi]$-left module structure on $\mathrm{H}^k_{\mathrm{tw}}(Y_\gamma; R)$.

Next we turn to the proof of (3). Let Γ be a normal subgroup of finite index. Let us verify that $\Psi \circ \Phi = \mathrm{id}$. Thus let $\sigma \in \mathrm{C}_k(\widetilde{X})$ and let $\mu \in \mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_k(\widetilde{X}), R[\Gamma\backslash\pi])$. Then

$$\begin{aligned}((\Psi\circ\Phi)(\mu))(\sigma) &\overset{\text{definition of } \Psi}{\underset{\downarrow}{=}} \sum_{\Gamma g\in\Gamma\backslash\pi} \Phi(\mu)(g\cdot(q\circ\sigma\colon \Delta^k\to Y_\Gamma))\cdot g \overset{\text{since } g\cdot\sigma \text{ is a lift of } g\cdot(q\circ\sigma)}{\underset{\downarrow}{=}} \sum_{\Gamma g\in\Gamma\backslash\pi} (e\text{-coefficient of } \mu(g\cdot\sigma))\cdot g\\ &= \sum_{\Gamma g\in\Gamma\backslash\pi} (g\text{-coefficient of } \mu(\sigma))\cdot g \qquad = \mu(\sigma).\\ &\underset{\uparrow}{=}\\ &\text{since } \mu \text{ is } \mathbb{Z}[\pi]\text{-left linear we have } \mu(g\cdot\sigma)=\mu(\sigma)\cdot g^{-1}\end{aligned}$$

The verification that $\Phi \circ \Psi = \mathrm{id}$ is very similar.

Finally we turn to the proof of Statement (4). Since Φ and Ψ are inverses of one another it suffices to show that Ψ is a $R[\Gamma\backslash\pi]$-left module isomorphism. Let $\Gamma h \in \Gamma\backslash\pi$. Then given $\sigma \in \mathrm{C}_k(\widetilde{X})$ and $\gamma \in \mathrm{Hom}_R(\mathrm{C}_k(Y_\Gamma), R)$ we have

$$\begin{aligned}\underbrace{\Psi(h\cdot\gamma)(\sigma)}_{=\Psi((h^{-1})^*\gamma)(\sigma)} &= \sum_{\Gamma g\in\Gamma\backslash\pi} ((h^{-1})^*\gamma)(g\cdot(q\circ\sigma))\cdot g \qquad = \sum_{\Gamma g\in\Gamma\backslash\pi} \gamma(h^{-1}g\cdot(q\circ\sigma))\cdot g\\ &= \sum_{\Gamma g\in\Gamma\backslash\pi} \gamma(h^{-1}g\cdot(q\circ\sigma))\cdot h\cdot h^{-1}g \;=\; h\cdot\sum_{k\in\Gamma\backslash\pi}\gamma(k\cdot(q\circ\sigma))\cdot k \qquad = h\cdot\Psi(\gamma)(\sigma).\end{aligned}$$

■

168.4. **Twisted (co-) homology groups of lens spaces.** We turn to our old nemesis, the lens spaces, which so far have escaped our attempts at classifying them. Recall that in Corollary 48.18 we saw that $\pi_1(L(p,q)) \cong \mathbb{Z}_p$. This shows that $\pi_1(L(p,q))$ has at least some non-trivial representations. But since $\pi_1(L(p,q))$ is abelian we know by Proposition **??** that every finite dimensional complex representation is the direct sum of one-dimensional complex representations. Thus by the Direct Sum-of-Coefficients Lemma 167.5 we only need to understand invariants corresponding to one-dimensional complex representations.

For trivial one-dimensional representations we know by Lemma 167.3 that we only recover classical invariants. Thus in the following lemma we restrict ourselves to non-trivial one-dimensional complex representations.

Proposition 168.5. Let $p, q \in \mathbb{N}$ be coprime. We consider the corresponding lens space $L(p,q)$. If $\rho\colon \pi_1(L(p,q)) \to \mathrm{GL}(1,\mathbb{C})$ is a non-trivial representation, then for all $k \in \mathbb{N}_0$ we have $\mathrm{H}_k^{\mathrm{tw}}(L(p,q);{}_\rho\mathbb{C}) = 0$ and $\mathrm{H}^k_{\mathrm{tw}}(L(p,q);{}_\rho\mathbb{C}) = 0$.

Remark.

(1) Proposition 168.5 is at first glance quite disappointing, it shows that homology with twisted complex coefficients cannot be used to distinguish lens spaces. But in Section **??** we will see, counter intuitively, that this calculation is in fact a big step towards solving the conundrum of the classification of lens spaces.

(2) The eternal optimist might hope to use twisted (co-) homology with other coefficients to distinguish lens spaces. But by Corollary 167.17 we know that the (co-) homology groups with twisted coefficients that can occur only depend on the homotopy type of lens spaces. But by Theorem 148.22 there exist lens spaces that are homotopy equivalent albeit not homeomorphic.

Proof. Let $p, q \in \mathbb{N}$ be coprime. We pick an identification $\pi_1(L(p,q), x_0) = \mathbb{Z}_p$. Let $n \in \mathbb{N}$. Given $z \in \mathbb{C}$ with $z^p = 1$ we consider the representation

$$\begin{array}{rccc} \rho_z \colon & \mathbb{Z}_p & \to & \mathrm{U}(1) \\ & [k] & \mapsto & z^k. \end{array}$$

By Proposition **??** we know the following two facts:

(1) The complex regular representation $\mathbb{C}[\mathbb{Z}_p]$ of the cyclic group $\pi_1(L(p,q), x_0) = \mathbb{Z}_p$ is isomorphic to the direct sum of the representations $\rho_{\exp(2\pi i k/p)}$, $k = 0, \dots, p-1$.
(2) Every non-trivial irreducible representation of $\mathbb{Z}_p$ is isomorphic to one of the ρ_z with $z \neq 1$.

In the following we first deal with the twisted homology modules. We see that

$$\begin{array}{ccccc} & & \text{follows from (1) and the Direct Sum-of-Coefficients Lemma 167.5, together with observation that we have } {}_{\mathrm{tr}}\mathbb{C} = \mathbb{C}_{\rho_1} & & \\ & & \downarrow & & \\ \mathrm{H}_k^{\mathrm{tw}}(L(p,q); {}_{\mathrm{tr}}\mathbb{C}) \oplus \bigoplus_{\substack{z^p=1,\\ z\neq 1}} \mathrm{H}_k^{\mathrm{tw}}(L(p,q); {}_{\rho_z}\mathbb{C}) & \cong & \mathrm{H}_k^{\mathrm{tw}}(L(p,q); \bigoplus_{z^p=1} {}_{\rho_z}\mathbb{C}) & & \\ & & \cong \uparrow & & \cong \uparrow \\ & & \mathrm{H}_k^{\mathrm{tw}}(L(p,q); \mathbb{C}[\mathbb{Z}_p]) & \cong & \mathrm{H}_k(S^3; \mathbb{C}). \\ & & \text{by Proposition ??} & & \text{by Proposition 168.2, since } S^3 \text{ is a universal covering of } L(p,q) \end{array}$$

Note that it follows from (2) and the above calculation that it suffices to show that $\mathrm{H}_k(L(p,q); {}_{\mathrm{tr}}\mathbb{C}) \cong \mathrm{H}_k(S^3; \mathbb{C})$ and that both are finite-dimensional complex vector spaces. Indeed we have

$$\begin{array}{ccccccc} \mathrm{H}_k^{\mathrm{tw}}(L(p,q); {}_{\mathrm{tr}}\mathbb{C}) & \cong & \mathrm{H}_k(L(p,q); \mathbb{C}) & \cong & \begin{cases} \mathbb{C}, & \text{if } k = 0,3 \\ 0, & \text{else} \end{cases} & \cong & \mathrm{H}_k(S^3; \mathbb{C}). \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{by Lemma 167.3} & & \text{follows from the Universal Coefficient Theorem 89.17 together with the Tor-Properties Lemma 89.15 and the calculations of } \mathrm{H}_*(L(p,q)) \text{ and } \mathrm{H}_*(S^3) \text{ in Proposition 74.4 and the Lens Space Homology Lemma 98.9} & & & & \end{array}$$

The argument for calculating the twisted cohomology groups is very similar, the main difference is that we need to replace Proposition 168.2 by Proposition 168.4. We can do so, since $S^3 \to L(p,q)$ is a finite covering. ∎

168.5. **Replacing the universal covering.** In the definition of twisted (co-) homology we work with the universal covering. In many settings the following propositions allows us to work with "smaller coverings", which in practice can simplify life quite a bit. To formulate the corresponding statement it is convenient to introduce the following convention.

Proposition 168.6. (Replace Universal Covering Proposition) Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. We denote by $p \colon \widetilde{X}_{x_0} \to X$ the corresponding universal covering. Let $A \subset X$. We write $\widetilde{A} := p^{-1}(A)$. We set $\pi := \pi_1(X, x_0)$. Let $\Gamma \subset \pi$ a normal subgroup. We denote by $\rho \colon \pi \to \pi/\Gamma$ the projection and we denote by $\rho \colon \mathbb{Z}[\pi] \to \mathbb{Z}[\pi/\Gamma]$ also the corresponding ring homomorphism. Let R be a ring and let M be a $(\mathbb{Z}[\pi/\Gamma], R)$-left left module. Let $q \colon (Y, y_0) \to (X, x_0)$ be any path-connected regular covering with $q_*(\pi_1(Y, y_0)) \subset \Gamma$. We set $B := q^{-1}(A)$. By Corollary 61.11 there exists a unique covering $r \colon (\widetilde{X}_{x_0}, \widetilde{x}_0) \to (Y, y_0)$ with $q \circ r = p$. The

natural maps[20822083]

$$\begin{array}{rcl} \overbrace{\overline{\rho^*M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})}^{\mathrm{C}_k^{\mathrm{tw}}(X,A;\rho^*M)} & \to & \overline{M} \otimes_{\mathbb{Z}[\pi/\Gamma]} \mathrm{C}_k(Y,B) \\ m \otimes (\sigma\colon \Delta^k \to \widetilde{X}_{x_0}) & \mapsto & m \otimes (r \circ \sigma\colon \Delta^k \to Y) \end{array}$$

and

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbb{Z}[\pi/\Gamma]\text{-left}}(\mathrm{C}_k(Y,B), M) & \to & \overbrace{\mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}), \rho^*M)}^{=\mathrm{C}_{\mathrm{tw}}^k(X,A;\rho^*M)} \\ \varphi & \mapsto & (\sigma \mapsto \varphi(r \circ \sigma)) \end{array}$$

are isomorphisms of (co-) chain complexes of R-left modules. In particular we get induced isomorphisms of (co-) homology modules.

Remark. Let π be a group and let Γ be a normal subgroup. We denote by $\rho\colon \pi \to \pi/\Gamma$ the projection. Let M be a $\mathbb{Z}[\pi]$-left module. If every $g \in \Gamma$ acts trivially on M, then there exists an obvious $\mathbb{Z}[\pi/\Gamma]$-left module $\widetilde{M}$ such that $\rho^*\widetilde{M} = M$. In practice that is often how the Replace Universal Covering Proposition 168.6 gets applied.

Remark. This also works for the cellular twisted (co-) homology, where it is perhaps more useful.

Proof. We denote by $\rho\colon \pi \to \pi/\Gamma$ the projection and we also denote by $\rho\colon \mathbb{Z}[\pi] \to \mathbb{Z}[\pi/\Gamma]$ the corresponding ring homomorphism.

Claim. The map

$$\begin{array}{rcl} \mathbb{Z}[\pi/\Gamma] \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}) & \to & \mathrm{C}_k(Y,B) \\ g \otimes \sigma & \mapsto & \rho(g) \cdot (r \circ \sigma) \end{array}$$

is well-defined and it is an isomorphism of $\mathbb{Z}[\Gamma\backslash\pi]$-left modules.

Proof. First note that it follows from Proposition 167.1 (2) that for any $g \in \pi_1(X, x_0)$ and any $w \in \widetilde{X}_{x_0}$ we have $r(g \cdot w) = \rho(g) \cdot r(w)$. It follows that the map is well-defined.

Given any singular simplex $\sigma\colon \Delta^k \to X$ that is not contained in A we pick a lift $\widetilde{\sigma}\colon \Delta \to \widetilde{X}$. In the Regular Covering-Chain Complex Lemma 167.2 we showed that these lifts form a basis of the free $\mathbb{Z}[\pi]$-left module $\mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})$. Furthermore by the Regular Covering-Chain Complex Lemma 167.2 we also know that the singular simplices $r \circ \widetilde{\sigma}$ form a basis of the free $\mathbb{Z}[\pi/\Gamma]$-left module $\mathrm{C}_k(Y,B)$. It follows from this observation, together with Lemma 165.21, that the map sends a basis of the free $\mathbb{Z}[\pi/\Gamma]$-left module $\mathbb{Z}[\pi/\Gamma] \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})$ to a basis of the free $\mathbb{Z}[\pi/\Gamma]$-left module $\mathrm{C}_k(Y,B)$. Thus we see that the map is indeed an isomorphism. ⊞

Note that the first map we are supposed to study equals the composition of the following maps:

$$\begin{array}{rcl} \overline{\rho^*M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}) & \to & \overline{M} \otimes_{\mathbb{Z}[\pi/\Gamma]} (\mathbb{Z}[\pi/\Gamma] \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})) \\ m \otimes \sigma & \mapsto & m \otimes (1 \otimes \sigma) \end{array} \to \overline{M} \otimes_{\mathbb{Z}[\pi/\Gamma]} \mathrm{C}_k(Y,B)$$

$$m \otimes (n \otimes \sigma) \mapsto m \otimes r_*(n \cdot \sigma).$$

[2082]Note that our hypothesis that $Y \to X$ is a regular covering is, by the Regular Covering-Characterization Corollary 62.8, equivalent to the statement that $\Gamma := q_*(\pi_1(Y, y_0))$ is a normal subgroup of $\pi = \pi_1(X, x_0)$

[2083]As we discussed on page 3534 we have a natural left action of $\pi/\Gamma = \pi_1(X, x_0)/q_*(\pi_1(Y, y_0))$ on Y.

It follows from Lemma 165.20 that the map to the left is well-defined and an isomorphism. It follows from the claim that the map to the right is an isomorphism. In summary, we are done.

The second statement regarding the cochain complexes can be proved, using Lemma 165.22, in a very similar fashion. We leave the details to the reader. ■

168.6. **The Bockstein sequence and non-orientable topological manifolds.** On page 3145 we saw that given a topological space X any short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ of abelian groups gives rise to a short exact sequence of chain complexes and thus to a long exact sequence of homology groups. We now see that an analogous statement holds for homology with twisted coefficients.

Definition. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the corresponding natural universal covering. Let $A \subset X$. We write $\widetilde{A} := p^{-1}(A)$. Let R be a ring and let

$$0 \;\to\; L \;\xrightarrow{\varphi}\; M \;\xrightarrow{\psi}\; N \;\to\; 0$$

be a short exact sequence of $(\mathbb{Z}[\pi_1(X,x_0)],R)$-left left modules. Let $n \in \mathbb{N}_0$. It follows easily from the fact, proved in the Regular Covering-Chain Complex Lemma 167.2 that $\mathrm{C}_n(\widetilde{X}_{x_0},\widetilde{A})$ is a free $\mathbb{Z}[\pi_1(X,x_0)]$-left module and from the natural isomorphisms of Lemma 165.17 that

$$0 \;\to\; \mathrm{C}_n^{\mathrm{tw}}(X,A;L) \;\xrightarrow{\varphi_*}\; \mathrm{C}_n^{\mathrm{tw}}(X,A;M) \;\xrightarrow{\psi_*}\; \mathrm{C}_n^{\mathrm{tw}}(X,A;N) \;\to\; 0$$

is also exact. Thus we obtain a short exact sequence

$$0 \;\to\; \mathrm{C}_*^{\mathrm{tw}}(X,A;L) \;\xrightarrow{\varphi_*}\; \mathrm{C}_*^{\mathrm{tw}}(X,A;M) \;\xrightarrow{\psi_*}\; \mathrm{C}_*^{\mathrm{tw}}(X,A;N) \;\to\; 0$$

of chain complexes of R-left modules. Furthermore, from the LES Lemma 166.11 we obtain natural connecting homomorphisms $\beta_n := \partial_n\colon \mathrm{H}_n^{\mathrm{tw}}(X,A;N) \to \mathrm{H}_{n-1}^{\mathrm{tw}}(X,A;L)$ that give us a long exact sequence

$$\ldots \xrightarrow[=:\beta_{n+1}]{\partial_{n+1}} \mathrm{H}_n^{\mathrm{tw}}(X,A;L) \xrightarrow{\varphi_*} \mathrm{H}_n^{\mathrm{tw}}(X,A;M) \xrightarrow{\psi_*} \mathrm{H}_n^{\mathrm{tw}}(X,A;N) \xrightarrow[=:\beta_n]{\partial_n} \mathrm{H}_{n-1}^{\mathrm{tw}}(X,A;L) \to \ldots$$

For each $n \in \mathbb{N}_0$ the connecting homomorphism $\beta = \partial_n\colon \mathrm{H}_n^{\mathrm{tw}}(X,A;N) \to \mathrm{H}_{n-1}^{\mathrm{tw}}(X,A;L)$ is called the n-th Bockstein homomorphism in homology with twisted coefficients and the long exact sequence is called the Bockstein sequence in homology with twisted coefficients.

Remark. Similar to the discussion on page 3145 there is also an analogous Bockstein sequence in twisted cohomology, but we will not make use of it.

Example. We consider the group $\langle x \,|\, x^2\rangle$, which is nothing but the group $\mathbb{Z}_2$ written multiplicatively. We denote by $\widetilde{\mathbb{Z}}$ the $\mathbb{Z}[\langle x \,|\, x^2\rangle]$-left module that is given by $x \cdot n := -n$. We leave it to the reader to verify that

$$0 \;\to\; \widetilde{\mathbb{Z}} \;\xrightarrow{n\mapsto n\cdot(1-x)}\; \mathbb{Z}[\langle x \,|\, x^2\rangle] \;\xrightarrow{x\mapsto 1}\; {}_{\mathrm{tr}}\mathbb{Z} \;\to\; 0$$

and

$$0 \;\to\; {}_{\mathrm{tr}}\mathbb{Z} \;\xrightarrow{n\mapsto n\cdot(1+x)}\; \mathbb{Z}[\langle x \,|\, x^2\rangle] \;\xrightarrow{x\mapsto -1}\; \widetilde{\mathbb{Z}} \;\to\; 0$$

are short exact sequences of $\mathbb{Z}[\langle x \,|\, x^2\rangle]$-left modules.

Proposition 168.7. Let X be a compact *non-orientable* path-connected non-empty n-dimensional topological manifold and let $x_0 \in X$. We denote by $\widetilde{X} \to X$ the natural path-connected 2-fold covering from the Orientation Covering-Proposition 105.15 where $\widetilde{X}$ is an n-dimensional topological manifold that comes with a natural orientation. We denote by

$$w_1 \colon \pi_1(X, x_0) \;\to\; \pi_1(X, x_0)/p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \;\underset{\text{unique isomorphism}}{\underset{\uparrow}{=}}\; \langle x \,|\, x^2 \rangle$$

the orientation-character that we introduced on page 2341. The following three statements hold:

(1) There exists an isomorphism $\mathrm{H}_n^{\mathrm{tw}}(X, \partial X; w_1^* \widetilde{\mathbb{Z}}) \cong \mathbb{Z}$.
(2) There exists a natural generator of $\mathrm{H}_n^{\mathrm{tw}}(X, \partial X; w_1^* \widetilde{\mathbb{Z}}) \cong \mathbb{Z}$.
(3) For each $Q \in X \setminus \partial X$ the map $\mathrm{H}_n^{\mathrm{tw}}(X, \partial X; w_1^* \widetilde{\mathbb{Z}}) \to \mathrm{H}_n^{\mathrm{tw}}(X, X \setminus \{Q\}; w_1^* \widetilde{\mathbb{Z}})$ is an isomorphism.

Proposition 168.7 shows, that for a compact non-orientable path-connected non-empty n-dimensional topological manifold we have a naturally defined generator which has, in a slightly different setup, the defining property of a fundamental class as defined on page 2350. This motivates the following definition.

Definition. Let X be a compact non-orientable path-connected non-empty n-dimensional topological manifold. We refer to the natural generator of $\mathrm{H}_n^{\mathrm{tw}}(X, \partial X; w_1^* \widetilde{\mathbb{Z}})$ as the twisted fundamental class $[X]_{\mathrm{tw}}$ of X.

Remark. A statement similar to Proposition 168.7 is proved in [**Hat2002**, p. 327].

Proof. As in the previous example we consider the following short exact sequence of $\mathbb{Z}[\langle x \,|\, x^2 \rangle]$-modules:

$$0 \;\to\; \widetilde{\mathbb{Z}} \;\xrightarrow{n \mapsto n \cdot (1-x)}\; \mathbb{Z}[\langle x \,|\, x^2 \rangle] \;\underset{=:\gamma}{\xrightarrow{x \mapsto 1}}\; {}_{\mathrm{tr}}\mathbb{Z} \;\to\; 0.$$

Via the homomorphism $w_1 \colon \pi_1(X, x_0) \to \langle x \,|\, x^2 \rangle$ we obtain a corresponding short exact sequence of $\mathbb{Z}[\pi_1(X, x_0)]$-modules. Note that $w_1^* \, {}_{\mathrm{tr}}\mathbb{Z} = {}_{\mathrm{tr}}\mathbb{Z}$. Next we consider the following diagram:

$$\begin{array}{ccccccc}
\mathrm{H}_{n+1}^{\mathrm{tw}}(X, \partial X; {}_{\mathrm{tr}}\mathbb{Z}) & \xrightarrow{\beta_{n+1}} & \mathrm{H}_n^{\mathrm{tw}}(X, \partial X; w_1^* \widetilde{\mathbb{Z}}) & \to & \mathrm{H}_n^{\mathrm{tw}}(X, \partial X; w_1^* \mathbb{Z}[\langle x \,|\, x^2 \rangle]) & \to & \mathrm{H}_n^{\mathrm{tw}}(X, \partial X; {}_{\mathrm{tr}}\mathbb{Z}) \\
\cong \big\downarrow T & & & & \cong \big\downarrow \Omega & & \cong \big\downarrow T \\
\underbrace{\mathrm{H}_{n+1}(X, \partial X; \mathbb{Z})}_{=0} & & & & \underbrace{\mathrm{H}_n(\widetilde{X}; \mathbb{Z})}_{=\mathbb{Z} \cdot [\widetilde{X}]} & & \underbrace{\mathrm{H}_n(X, \partial X; \mathbb{Z})}_{=0}
\end{array}$$

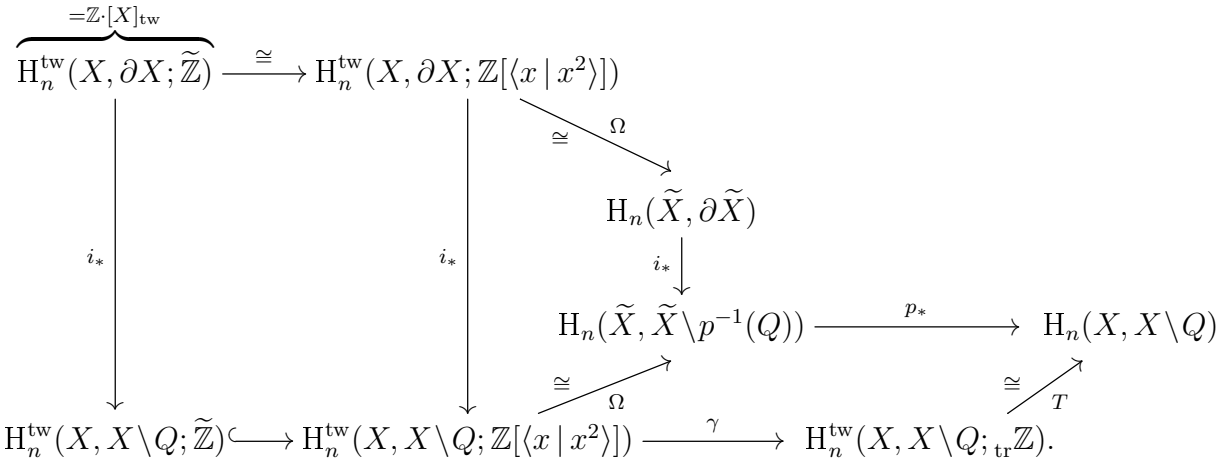
We make the following clarifications and observations:

(a) The top horizontal sequence is the Bockstein sequence corresponding to the above short exact sequence of $\mathbb{Z}[\pi_1(X, x_0)]$-left modules.
(b) The vertical map in the first and fourth column are the isomorphisms provided by Lemma 167.3.

(c) Since $\dim(X) = n$ we obtain from the Top-Homology Theorem 106.3 that $\mathrm{H}_{n+1}(X, \partial X; \mathbb{Z}) = 0$.
(d) Since X is non-orientable we obtain from the Top-Homology Theorem 106.3 that $\mathrm{H}_n(X, \partial X; \mathbb{Z}) = 0$.
(e) The vertical map in the third column is given by Proposition 168.2.
(f) Since $\widetilde{X}$ is path-connected, non-empty and naturally oriented we know by the Fundamental Class-Corollary 106.4 that $\mathrm{H}_n(\widetilde{X}, \partial\widetilde{X}; \mathbb{Z}) = \mathbb{Z} \cdot [\widetilde{X}]$.

The above discussion shows that $\mathrm{H}_n^{\mathrm{tw}}(X, \partial X; w_1^*\widetilde{\mathbb{Z}})$ is isomorphic to $\mathbb{Z}$ and we see that the natural isomorphisms and the natural generator $[\widetilde{X}]$ of $\mathrm{H}_n(\widetilde{X}, \partial\widetilde{X}; \mathbb{Z})$ give rise to a natural generator of $\mathrm{H}_n^{\mathrm{tw}}(X, \partial X; w_1^*\widetilde{\mathbb{Z}})$. This concludes the proof of Statements (1) and (2).

It remains to prove Statement (3). Thus let $Q \in X \setminus \partial X$. We consider the following diagram:



We make the following clarifications and observations:

(a) As before we see that the horizontal map at the top left is an isomorphism and the horizontal map at the bottom left is a monomorphism.
(b) All vertical maps are induced by the obvious inclusions.
(c) The rectangle to the left commutes by naturality of i_*. The middle quadrilateral commutes by the naturality of Ω. We leave it to the reader to show, by inspecting the maps, that the quadrilateral to the right commutes.
(d) The bottom sequence is exact.

We want to show that $i_*([X]_{\mathrm{tw}})$ is a generator of $\mathrm{H}_n^{\mathrm{tw}}(X, X \setminus Q; \widetilde{\mathbb{Z}})$. By the above diagram and by definition of $[X]_{\mathrm{tw}}$ it suffices to prove the following claim.

Claim. The image of $[\widetilde{X}] \in \mathrm{H}_n(\widetilde{X}, \partial\widetilde{X})$ in $\mathrm{H}_n(\widetilde{X}, \widetilde{X} \setminus p^{-1}(Q))$ is a generator of the kernel of p_*.

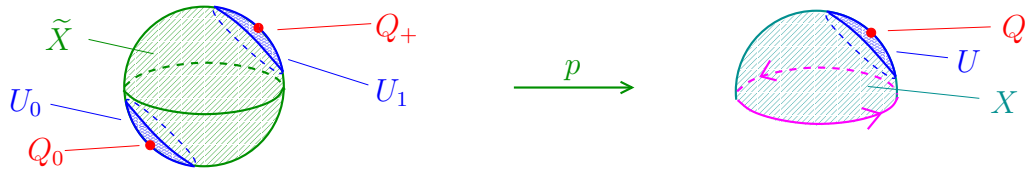
Proof. Let $\tau\colon \widetilde{X} \to \widetilde{X}$ be the orientation-reversing deck transformation of the covering $p\colon \widetilde{X} \to X$. We pick a point $Q_0 \in p^{-1}(Q)$ and we write $Q_1 := \tau(Q_1)$. We pick an open neighborhood U_0 around Q_0 such that U_0 and $U_1 := \tau(U_0)$ are disjoint. We set $\widetilde{U} := U_0 \cup U_1$

and $U := p(\widetilde{U})$. We consider the following little diagram:

$$\begin{array}{ccccc}
 & & \mathrm{H}_n(\widetilde{X},\partial\widetilde{X}) & & \\
 & & \downarrow{\scriptstyle i_*} & & \\
\ker(p_*) & \hookrightarrow & \mathrm{H}_n(\widetilde{X},\widetilde{X}\setminus p^{-1}(Q)) & \xrightarrow{p_*} & \mathrm{H}_n(X,X\setminus Q) \\
 & & \uparrow\cong \quad \searrow^{\mu_X} & & \uparrow\cong \\
 & & \mathrm{H}_n(\widetilde{X},\widetilde{X}\setminus\{Q_0\}) & & \\
 & & \uparrow\cong & & \\
 & & \mathrm{H}_n(\widetilde{U},\widetilde{U}\setminus\{Q_0\}) & & \\
 & & \uparrow\cong \quad \nearrow_{\mu_U} & & \\
\underbrace{\{\sigma\oplus-\tau_*\sigma\,|\,\sigma\in\mathrm{H}_n(U_0,U_0\setminus\{Q_0\})\}}_{\subset \mathrm{H}_n(U_0,U_0\setminus\{Q_0\})\oplus\mathrm{H}_n(U_1,U_1\setminus\{Q_1\}} & \to & \mathrm{H}_n(\widetilde{U},\widetilde{U}\setminus p^{-1}(Q)) & \xrightarrow{p_*} & \mathrm{H}_n(U,U\setminus Q)
\end{array}$$

We make the following observations:

(1) The three vertical maps that are decorated by "$\cong$" are induced by inclusions and they are isomorphisms by the Excision Theorem 74.18.
(2) The maps μ_X and μ_U are induced by the obvious map of pairs of topological spaces.
(3) The diagram commutes by functoriality of homology.
(4) Since $\mathrm{H}_n(\widetilde{U},\widetilde{U}\setminus p^{-1}(Q)) = \mathrm{H}_n(U_0,U_0\setminus p^{-1}(Q_0))\oplus\mathrm{H}_n(U_+,U_+\setminus p^{-1}(Q_+))$ one can easily verify that the bottom horizontal sequence is exact.
(5) It follows easily from (4) that the restriction of μ_U to $\ker(p_*)$ is an isomorphism.
(6) It follows from (3) and (5) that the restriction of μ_X to $\ker(p_*)$ is an isomorphism.
(7) By definition of a fundamental class, see page 2350, we know that the image of $[\widetilde{X}]$ under the map $\mu_X\circ i_*$ in $\mathrm{H}_n(\widetilde{X},\widetilde{X}\setminus\{Q_0\})$ is a generator.
(8) It follows from (6) and (7) that $i_*([\widetilde{X}])$ is indeed a generator of $\ker(p_*)$. ■



Exercises for Chapter 168.

Exercise 168.1. Let π be a group. We equip $M = \mathbb{Z}[\pi]$ with the $\mathbb{Z}[\pi]$-left module structure that is given by $g\cdot m := m\cdot g^{-1}$. We consider the π-coinvariants $\mathbb{Z}[\pi]_\pi$ and the π-invariants $\mathbb{Z}[\pi]^\pi$ as defined on page 3555.

(a) Show that the following map is an isomorphism:

$$\begin{array}{ccc} \mathbb{Z}[\pi]_\pi & \to & \mathbb{Z} \\ \left[\sum_{i=1}^k r_i\cdot g_i\right] & \mapsto & \sum_{i=1}^k r_i. \end{array}$$

(b) Show that

$$\mathbb{Z}[\pi]^{\pi} \;=\; \begin{cases} 0, & \text{if } \pi \text{ is infinite,} \\ \mathbb{Z} \cdot \sum\limits_{g \in \pi} g, & \text{if } \pi \text{ is finite.} \end{cases}$$

Exercise 168.2.

(a) Let G be a finite group that acts on the left on some $\mathbb{Q}^n$. Show that M^G and M_G are isomorphic.

(b) Let G be a finite group that acts on the left on some $\mathbb{Z}^n$. Show that M^G and M_G are in general not isomorphic.

Exercise 168.3. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. We write $\pi := \pi_1(X, x_0)$. Let R be a ring and let M be a $(\mathbb{Z}[\pi], R)$-left left module. We consider the natural universal covering $p\colon \widetilde{X}_{x_0} \to X$ and we denote by $\widetilde{x}_0$ the natural base point of $\widetilde{X}_{x_0}$. Show that the map

$$\begin{array}{rcl} \epsilon_X\colon \mathrm{H}_0^{\mathrm{tw}}(X; M) & \to & M^{\pi} \\ [\varphi\colon \mathrm{C}_0(\widetilde{X}_{x_0}) \to M)] & \mapsto & \varphi(\widetilde{x}_0) \\ & & \uparrow \\ & & \text{here } \widetilde{x}_0 \text{ is short for the constant map } \Delta^0 \to \widetilde{X}_{x_0} \text{ that sends 1 to } \widetilde{x}_0 \end{array}$$

is a natural isomorphism of R-left modules.

Exercise 168.4. We consider the group $\langle x \,|\, x^2 \rangle$, which is nothing but the group $\mathbb{Z}_2$ written multiplicatively.

(a) We denote by $\widetilde{\mathbb{Z}}$ the $\mathbb{Z}[\langle x \,|\, x^2 \rangle]$-left module that is given by $x \cdot n := -n$. Show that the short exact sequence

$$0 \;\to\; {}_{\mathrm{tr}}\mathbb{Z} \;\xrightarrow{n \mapsto n \cdot (1+x)}\; \mathbb{Z}[\langle x \,|\, x^2 \rangle] \;\xrightarrow{x \mapsto -1}\; \widetilde{\mathbb{Z}}\; 0$$

of $\mathbb{Z}[\langle x \,|\, x^2 \rangle]$-left modules does not split.

(b) Show that if in (a) replace "$\mathbb{Z}$" throughout by "$\mathbb{Q}$", then the short exact sequence of $\mathbb{Q}[\langle x \,|\, x^2 \rangle]$-left modules splits.

Exercise 168.5.

(a) Show that the $\mathbb{Z}[t^{\pm 1}]$-modules $(\mathbb{Z}, t = +1)$ and $\mathbb{Z}[t^{\pm 1}]/(t-1)\mathbb{Z}[t^{\pm 1}]$ are isomorphic.

(b) Show that the $\mathbb{Z}[t^{\pm 1}]$-modules $(\mathbb{Z}, t = -1)$ and $\mathbb{Z}[t^{\pm 1}]/(t+1)\mathbb{Z}[t^{\pm 1}]$ are isomorphic.

Exercise 168.6. Let M be the Möbius band. Determine an explicit cycle in $\mathrm{C}_2^{\mathrm{tw}}(M, \partial M; w_1^* \widetilde{\mathbb{Z}})$ that is a generator of $\mathrm{H}_2^{\mathrm{tw}}(M, \partial M; w_1^* \widetilde{\mathbb{Z}})$.

169. Universally excisive triads and Mayer-Vietoris sequences

In this short chapter we will introduce the concept of a universally excisive triad, which is a variation on the notion of an excisive triad that we introduced on page 2429. Excisive triads played roles on two different occasions. They allowed us to formulate Mayer-Vietoris type theorems and they allowed us to define relative cup and cap products. Universally excisive triads will play the same role in the setting of (co-) homology with twisted coefficients.

In this chapter we will also introduce internal twisted (co-) homology. This slightly clunky concept allows us to work with disconnected spaces and will come in handy on several occasions.

169.1. **Universally excisive triads.** We start out by recalling the following definition from page 2429.

Definition.

(1) A triad of topological spaces is a triple (X, A_1, A_2) where X is a topological space and where both A_1 and A_2 are subsets of X.

(2) A map $f\colon (X, A_1, A_2) \to (Y, B_1, B_2)$ of triads is a map $f\colon X \to Y$ with $f(A_1) \subset B_1$ and $f(A_2) \subset B_2$.

(3) Given a triad (X, A_1, A_2) of topological spaces we consider the following subgroup of $\mathrm{C}_n(A_1 \cup A_2)$:

$$\mathrm{C}_n^{\{A_1,A_2\}}(A_1 \cup A_2) := \left\{ \sum_{j=1}^{k} a_j \sigma_j \in \mathrm{C}_n(A_1 \cup A_2) \;\middle|\; \begin{array}{l} \text{for each } j \text{ there exists an } i \in \{1,2\} \\ \text{such that the image of } \sigma_j \text{ lies in } A_i \end{array} \right\}.$$

The usual boundary map on the groups $\mathrm{C}_*(A_1 \cup A_2)$ restricts to a boundary map on $\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2)$, so we can view $\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2)$ as a chain complex. We denote the corresponding homology groups by $\mathrm{H}_*^{\{A_1,A_2\}}(A_1 \cup A_2)$.

(4) We say a triad (X, A_1, A_2) is excisive if the inclusion map

$$\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2) \;\to\; \mathrm{C}_*(A_1 \cup A_2)$$

is a chain homotopy equivalence of chain complexes.

To these familiar definitions we now add a new one:

(5) Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. We say a (X, A_1, A_2) is universally excisive if the triad $(\widetilde{X}_{x_0}, p^{-1}(A_1), p^{-1}(A_2))$ is excisive.

Lemma 169.1. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. Let (X, A_1, A_2) be a universally excisive triad. We write $\widetilde{A}_1 := p^{-1}(A_1)$ and $\widetilde{A}_2 := p^{-1}(A_2)$. The following three statements hold:

(1) $\mathrm{C}_*^{\{\widetilde{A}_1,\widetilde{A}_2\}}(\widetilde{A}_1 \cup \widetilde{A}_2)$ is a free $\mathbb{Z}[\pi_1(X, x_0)]$-left module.

(2) The inclusion map

$$\mathrm{C}_*^{\{\widetilde{A}_1,\widetilde{A}_2\}}(\widetilde{A}_1 \cup \widetilde{A}_2) \;\to\; \mathrm{C}_*(\widetilde{A}_1 \cup \widetilde{A}_2)$$

is a chain homotopy equivalence of free $\mathbb{Z}[\pi_1(X, x_0)]$-left module chain complexes.

(3) The triad (X, A_1, A_2) is also excisive.

Proof.

(1) The same argument as in the Regular Covering-Chain Complex Lemma 167.2 shows that $\mathrm{C}_*^{\{\widetilde{A}_1,\widetilde{A}_2\}}(\widetilde{A}_1 \cup \widetilde{A}_2)$ is a free $\mathbb{Z}[\pi_1(X,x_0)]$-left module. For example a basis is given by picking a lift for each singular simplex that is contained in A_1 or in A_2.

(2) By definition $\mathrm{C}_*^{\{\widetilde{A}_1,\widetilde{A}_2\}}(\widetilde{A}_1 \cup \widetilde{A}_2) \to \mathrm{C}_*(\widetilde{A}_1 \cup \widetilde{A}_2)$ is a homotopy equivalence. By the super classical Corollary 73.4 this implies that the map induces an isomorphism of homology groups, i.e. it is a quasi-isomorphism. By the Regular Covering-Chain Complex Lemma 167.2 and (1) we know that we have in fact a quasi-isomorphism of chain complexes of free $\mathbb{Z}[\pi_1(X,x_0)]$-left module. Thus it follows from the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 166.6 that the map is in fact a chain homotopy equivalence of free $\mathbb{Z}[\pi_1(X,x_0)]$-left module chain complexes.

(3) We consider the following diagram:

$$\begin{array}{ccc} {}_{\mathrm{tr}}\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_*^{\{\widetilde{A}_1,\widetilde{A}_2\}}(\widetilde{A}_1 \cup \widetilde{A}_2) & \longrightarrow & {}_{\mathrm{tr}}\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_*(\widetilde{A}_1 \cup \widetilde{A}_2) \\ \cong \downarrow n\otimes\sigma \mapsto n\cdot p_*(\sigma) & & \cong \downarrow n\otimes\sigma \mapsto n\cdot p_*(\sigma) \\ \mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2) & \longrightarrow & \mathrm{C}_*(A_1 \cup A_2). \end{array}$$

Here the horizontal maps are induced by the inclusions and the vertical maps are basically the natural isomorphisms from Lemma 167.3. It is clear that the diagram commutes. It follows from (1) and (2) together with the Tensoring Chain Complex Lemma 166.3 that the top horizontal map is a chain homotopy equivalence. But then the bottom map is also a chain homotopy equivalence. That is exactly what we needed to show. ■

In the Excisive Triad-Proposition 109.12 we gave many convenient examples of excisive triads. The following proposition now says that these triads from the Excisive Triad-Proposition 109.12 are not only excisive but they are - how exciting - even universally excisive.

Proposition 169.2. Let (X, A_1, A_2) be a triad of topological spaces. We assume that X is locally contractible and path-connected. The triad is universally excisive if one of the following conditions is satisfied:

(1) $A_1 \subset A_2$ or $A_2 \subset A_1$,

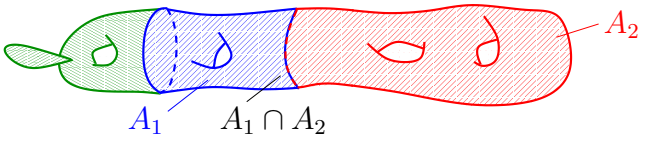
(2) $A_1 = \varnothing$ or $A_2 = \varnothing$ or $A_1 = A_2$,

(3) $A_1 \cup A_2 = \mathring{A}_1 \cup \mathring{A}_2$ where we denote by $\mathring{A}_1$ and $\mathring{A}_2$ the interiors of A_1 and A_2 in $A_1 \cup A_2$,

(4) A_1 and A_2 are both open subsets of $A_1 \cup A_2$,

(5) $A_1 \cup A_2$ is a topological manifold and A_1 and A_2 are codimension-zero submanifolds which are both closed subsets and such that each component of $A_1 \cap A_2$ is a boundary component of A_1 and a boundary component of A_2,

(6) X is a CW-complex and A_1 and A_2 are both subcomplexes.



(X, A_1, A_2) is universally excisive

Proof. Let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. We write $\widetilde{A}_i := p^{-1}(A_i)$. By the Excisive Triad-Proposition 109.12 it suffices to prove the following claim.

Claim. If (X, A_1, A_2) satisfies one of the Conditions (1)–(6), then so does $(\widetilde{X}, \widetilde{A}_1, \widetilde{A}_2)$.

Proof. Conditions (1), (2) and (4) are clear. For Condition (3) note that for a covering we say in Exercise 48.7 that taking the preimage and taking the interior commutes. With this observation Condition (3) is also clear.

Now let us consider Condition (5). By the Manifold Covering Proposition 49.2 we know that any path-connected covering of a topological manifold is again a topological manifold. Since being a submanifold is a local condition we see that $(\widetilde{X}, \widetilde{A}_1, \widetilde{A}_2)$ also satisfies Condition (5). Finally the last condition is dealt with by the CW-complex Covering Proposition 68.37. ∎

169.2. **Internal twisted (co-) homology.** In this section we introduce what we call internal twisted (co-) homology. This concept is rather technical, but it can be very useful when one wants to consider the relationship between the twisted invariants of a topological space X and of (possibly disconnected) subspaces of X. In particular we will use this concept to prove a first Mayer–Vietoris style theorem and the concept will play a key role in our proof of the Poincaré Duality Theorem 172.1 with Twisted Coefficients.

Definition. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. Let R be a ring and let M be a $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left module. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering.

(1) Given any subset $A \subset B \subset X$ we write $\widetilde{A} := p^{-1}(A)$ and $\widetilde{B} := p^{-1}(B)$ and we set

$$\begin{aligned} \mathrm{C}_k^{X,M}(B, A) &:= \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_k(\widetilde{B}, \widetilde{A}) \\ \mathrm{C}^k_{X,M}(B, A) &:= \mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]\text{-left}}(\mathrm{C}_k(\widetilde{B}, \widetilde{A}), M). \end{aligned}$$

These are (co-) chain complexes in the obvious way and we obtain the corresponding (co-) homology groups $\mathrm{H}_k^{X,M}(B, A)$ and $\mathrm{H}^k_{X,M}(B, A)$. We refer to these as internal (co-) homology groups with twisted coefficients.

(2) Given pairs (B_1, A_1) and (B_2, A_2) of subspaces of X with $B_1 \subset B_2$ and $A_1 \subset A_2$ we obtain, with the hopefully obvious notation, an inclusion induced chain map $\mathrm{C}_k(\widetilde{B_1}, \widetilde{A_1}) \to \mathrm{C}_k(\widetilde{B_2}, \widetilde{A_2})$ which is a $\mathbb{Z}[\pi_1(X, x_0)]$-left homomorphism. Thus we get induced maps

$$\begin{aligned} i_*\colon \mathrm{H}_k^{X,M}(B_1, A_1) &\to \mathrm{H}_k^{X,M}(B_2, A_2) \\ i^*\colon \mathrm{H}^k_{X,M}(B_2, A_2) &\to \mathrm{H}^k_{X,M}(B_1, A_1). \end{aligned}$$

Evidently internal (co-) homology is functorial under inclusions.

Note that in the setting of internal (co-) homology with twisted coefficients the subspaces are not required to be path-connected. This leads us to the following lemma.

Lemma 169.3. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. Let R be a ring and let M be a $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left module. Let $B \subset X$ be a subspace with path-components $\{B_j\}_{j \in J}$. Given $j \in J$ we denote by

$i_j\colon B_j \to B$ the inclusion map. For each $n \in \mathbb{N}_0$ the maps

$$\bigoplus_{j\in J} i_{j*}\colon \bigoplus_{j\in J} \mathrm{H}_n^{X,M}(B_j) \to \mathrm{H}_n^{X,M}(B) \quad \text{and} \quad \prod_{j\in J} i_j^*\colon \mathrm{H}_n^{X,M}(B) \to \prod_{j\in J} \mathrm{H}_n^{X,M}(B_j)$$

are isomorphisms.

Proof. This lemma follows easily from Lemma 72.14. We leave it to the reader to fill in the details. ■

As the reader will have noticed, the Change of Covering Proposition 167.10 relates internal (co-) homology with twisted coefficients to ordinary (co-) homology with twisted coefficients. For the reader's convenience we state the Change of Covering Proposition 167.10 with our new language.

Proposition 169.4. (Change of Covering Proposition II) Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. We denote by $q\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. Let R be a ring and let M be a $(\mathbb{Z}[\pi_1(X,x_0)],R)$-left left module. Let $B \subset X$ be a subspace with $x_0 \in B$. We assume that B is also locally contractible and path-connected. We write $\widetilde{B} := q^{-1}(B)$ and we denote by $p\colon \widehat{B}_{x_0} \to B$ the natural universal covering. Finally let $A \subset B$. We write $\widehat{A} := p^{-1}(A)$ and $\widetilde{A} := q^{-1}(A)$. We denote by the $i\colon B \to X$ the inclusion map and we denote by $\widetilde{i}\colon \widehat{B}_{x_0} \to \widetilde{X}_{x_0}$ the induced map as defined on page 3541.

(1) The maps

$$\begin{array}{rccc} \Omega_{B,A}^X\colon & \overbrace{i^*M \otimes_{\mathbb{Z}[\pi_1(B,x_0)]} \mathrm{C}_k(\widehat{B}_{x_0},\widehat{A})}^{=\mathrm{C}_k^{\mathrm{tw}}(B,A;i^*M)} & \to & \overbrace{M \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_k(\widetilde{B},\widetilde{A})}^{=\mathrm{C}_k^{X,M}(B,A)} \\ & m\otimes\sigma & \mapsto & m \otimes \widetilde{i}_*(\sigma) \end{array}$$

are well-defined and they are natural isomorphisms of R-left module chain complexes. Similarly the maps

$$\begin{array}{rccc} \Omega_{B,A}^X\colon & \overbrace{\mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]\text{-left}}(\mathrm{C}_k(\widetilde{B},\widetilde{A}),M)}^{=\mathrm{C}_{X,M}^k(B,A)} & \to & \overbrace{\mathrm{Hom}_{\mathbb{Z}[\pi_1(B,x_0)]\text{-left}}(\mathrm{C}_k(\widehat{B}_{x_0},\widehat{A}),i^*M)}^{=\mathrm{C}_{\mathrm{tw}}^k(B,A;i^*M)} \\ & \varphi & \mapsto & (\sigma \mapsto \varphi(\widetilde{i}_*(\sigma))) \end{array}$$

are well-defined (co-) chain maps and they are natural isomorphisms of R-left module cochain complexes. When $A = \varnothing$, then we just write Ω_B^X.

(2) The maps Ω from (1) are natural in the following sense. Suppose we have subspaces $S \subset T \subset X$ which are both locally contractible and path-connected and such that $x_0 \in S$. Furthermore suppose we have $A \subset S$ and $B \subset T$ with $A \subset B$. We denote by $i\colon S \to T$ and by $j\colon T \to X$ the inclusion. The following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_k^{\mathrm{tw}}(S,A;i^*j^*M) & \xrightarrow{\quad i_*\quad} & \mathrm{H}_k^{\mathrm{tw}}(T,B;j^*M) \\ {\scriptstyle\Omega_S^X}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle\Omega_T^X}\big\downarrow{\scriptstyle\cong} \\ \mathrm{H}_k^{X,M}(S,A) & \xrightarrow{\quad i_*\quad} & \mathrm{H}_k^{X,M}(T,B). \end{array}$$

The analogous statements for cohomology groups also holds.

Proof. The first statement is just the content of the Change of Covering Proposition 167.10 (1). The second statement is new. We denote by $s\colon \overline{S}_{x_0} \to S$, $t\colon \widehat{T}_{x_0} \to T$ and $p\colon \widetilde{X}_{x_0} \to X$ the natural universal coverings.

We write $k = i \circ j\colon S \to X$. By the discussion on page 3541 we get induced maps $\widetilde{i}\colon \overline{S}_{x_0} \to \widehat{T}_{x_0}$, $\widetilde{j}\colon \widehat{T}_{x_0} \to \widetilde{X}_{x_0}$ and $\widetilde{k}\colon \overline{S}_{x_0} \to \widetilde{X}_{x_0}$. One can easily verify that $\widetilde{k} = \widetilde{j} \circ \widetilde{i}$. The second statement follows in a straightforward fashion from this observation. ■

The following proposition is a refinement of the Change of Base Point Proposition 167.6 in our present context.

Proposition 169.5. (Change of Base Point Proposition II) Let X be a topological space that is locally contractible and path-connected and let $A \subset B \subset X$. Furthermore let $\gamma\colon [0,1] \to X$ be a path from $x_0 = \gamma(0)$ to $x_1 = \gamma(1)$. For $i = 0, 1$ we denote by $p_i\colon \widetilde{X}_{x_i} \to X$ the natural universal covering and we write $\widetilde{A}_i := p_i^{-1}(A)$ and $\widetilde{B}_i := p_i^{-1}(B)$. Finally let R be a ring and let M be a $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left module. As in the Change-of-Base Point Proposition 47.10 we consider the isomorphism

$$\begin{array}{rcl} \gamma^*\colon \pi_1(X, x_1) & \to & \pi_1(X, x_0) \\ [\alpha] & \mapsto & [\gamma * \alpha * \overline{\gamma}]. \end{array}$$

The isomorphism γ^* gives rise to a ring homomorphism $\gamma^*\colon \mathbb{Z}[\pi_1(X, x_1)] \to \mathbb{Z}[\pi_1(X, x_0)]$. Thus, as defined on page 3502, we can consider the $\mathbb{Z}[\pi_1(X, x_1)]$-left module $\gamma^{**}M$. Furthermore, as in Lemma 61.10 we consider the homeomorphism

$$\begin{array}{rcl} \gamma^*\colon \widetilde{X}_{x_1} & \to & \widetilde{X}_{x_0} \\ [\beta] & \mapsto & [\gamma * \beta]. \end{array}$$

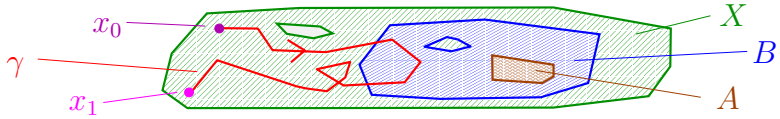
With this notation the maps

$$\begin{array}{rcl} \gamma^*\colon \overline{\gamma^{**}M} \otimes_{\mathbb{Z}[\pi_1(X,x_1)]} \mathrm{C}_k(\widetilde{B}_1, \widetilde{A}_1) & \to & \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_k(\widetilde{B}_0, \widetilde{A}_0) \\ m \otimes \sigma & \mapsto & m \otimes \gamma^*(\sigma) \end{array}$$

form an isomorphism of R-left module chain complexes and the maps

$$\begin{array}{rcl} \gamma^*\colon \mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_1)]-\mathrm{left}}(\mathrm{C}_k(\widetilde{B}_1, \widetilde{A}_1), \gamma^{**}M) & \to & \mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]-\mathrm{left}}(\mathrm{C}_k(\widetilde{B}_0, \widetilde{A}_0), M) \\ \varphi & \mapsto & \varphi \circ \gamma^* \end{array}$$

form a natural isomorphism of R-left module cochain complexes.



Proof. The proof is basically identical to the proof of the original Change of Base Point Proposition 167.6. ■

Many results on ordinary twisted (co-) homology have fairly obvious analogues to the context of internal twisted (co-) homology. For example we have the following variation on the Excision Theorem 167.12 for Twisted (Co-) Homology.

Theorem 169.6. (Excision Theorem for Internal Twisted (Co-) Homology) Let X be a topological space that is locally contractible and path-connected, let $x_0 \in X$. Suppose

we are given subsets $K \subset U \subset V$ where U is a neighborhood of the closure of K in V. The inclusion $i\colon (U, U \setminus K) \to (V, V \setminus K)$ induces for each $n \in \mathbb{N}_0$ isomorphisms

$$\begin{aligned} i_*\colon \mathrm{H}_n^{X,M}(U, U \setminus K) &\xrightarrow{\cong} \mathrm{H}_n^{X,M}(V, V \setminus K) \\ i^*\colon \mathrm{H}^n_{X,M}(V, V \setminus K; M) &\xrightarrow{\cong} \mathrm{H}^n_{X,M}(U, U \setminus K). \end{aligned}$$

Proof. The proof is similar to the proof of the Excision Theorem 167.12. We leave it to the reader to make the necessary changes. ■

169.3. **Mayer-Vietoris Theorems for (co-) homology with twisted coefficients.** Working with internal twisted cohomology one can easily formulate Mayer–Vietoris type theorems.

Theorem 169.7. (Mayer–Vietoris Theorem for Internal Twisted homology) Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. Let R be a ring and let M be a $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left module. Let (X, A, B) be a universally excisive triad with $X = A \cup B$. For every $n \in \mathbb{N}_0$ there exists a natural connecting R-left module homomorphism

$$\partial_n\colon \mathrm{H}_n^{X,M}(A \cup B) \xrightarrow{\partial_n} \mathrm{H}_{n-1}^{X,M}(A \cap B)$$

such that the following sequence

$$\begin{aligned} \cdots \to\ & \mathrm{H}_n^{X,M}(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} \mathrm{H}_n^{X,M}(A) \oplus \mathrm{H}_n^{X,M}(B) \xrightarrow{i_A + i_B} \mathrm{H}_n^{X,M}(A \cup B) \xrightarrow{\partial_n} \\ & \xrightarrow{\partial_n} \mathrm{H}_{n-1}^{X,M}(A \cap B) \to \dots \end{aligned}$$

of R-left module homomorphisms is exact.

Proof. As usual we denote by $p\colon \widetilde{A} \cup \widetilde{B} \to X$ the natural universal covering and we write $\pi := \pi_1(X, x_0)$. We write $\widetilde{A} := p^{-1}(A)$ and $\widetilde{B} := p^{-1}(B)$. Note that $\widetilde{A} \cap \widetilde{B} = p^{-1}(A \cap B)$ and $\widetilde{A} \cup \widetilde{B} = p^{-1}(A \cup B)$. In the proof of the original Mayer–Vietoris Theorem 78.5 we showed that the sequence

$$0 \to \mathrm{C}_*(\widetilde{A} \cap \widetilde{B}) \xrightarrow{i_{\widetilde{A}\cap\widetilde{B}} \oplus -i_{\widetilde{A}\cap\widetilde{B}}} \mathrm{C}_*(\widetilde{A}) \oplus \mathrm{C}_*(\widetilde{B}) \xrightarrow{i_{\widetilde{A}} + i_{\widetilde{B}}} \mathrm{C}_*^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B}) \to 0$$

of chain maps of $\mathbb{Z}[\pi]$-left chain complexes is exact. By Lemma 169.1 (1) we know that $\mathrm{C}_*^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B})$ is a free $\mathbb{Z}[\pi]$-left module. It follows from this observation together with Lemma 165.18 that the sequence

$$0 \to M \underset{\mathbb{Z}[\pi]}{\otimes} \mathrm{C}_*(\widetilde{A} \cap \widetilde{B}) \xrightarrow{i_{\widetilde{A}\cap\widetilde{B}} \oplus -i_{\widetilde{A}\cap\widetilde{B}}} M \underset{\mathbb{Z}[\pi]}{\otimes} \mathrm{C}_*(\widetilde{A}) \oplus M \underset{\mathbb{Z}[\pi]}{\otimes} \mathrm{C}_*(\widetilde{B}) \xrightarrow{i_{\widetilde{A}} + i_{\widetilde{B}}} M \underset{\mathbb{Z}[\pi]}{\otimes} \mathrm{C}_*^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B}) \to 0$$

of R-left module chain maps is also exact. We deduce from the LES Lemma 166.11 that we get natural connecting homomorphism which give rise to a long exact sequence of R-left module homomorphisms.

Recall that we assume that (X, A, B) is in fact universally excisive with $X = A \cup B$. It follows from Lemma 169.1 (2) that the inclusion map

$$\mathrm{C}_*^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B}) \to \mathrm{C}_*(\widetilde{A} \cup \widetilde{B})$$

is a chain homotopy equivalence of free $\mathbb{Z}[\pi]$-left module chain complexes. Hence for every $n \in \mathbb{N}_0$ it induces an isomorphism

$$\Xi\colon \mathrm{H}_n\big(M \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B})\big) \xrightarrow{\cong} \underbrace{\mathrm{H}_n\big(M \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{A} \cup \widetilde{B})\big)}_{=\mathrm{H}_n^{X,M}(A \cup B)}.$$

The idea now is to use Ξ to obtain the desired long exact sequence. To do so we consider the following diagram

$$\begin{array}{ccccc} \mathrm{H}_n^{X,M}(A) \oplus \mathrm{H}_n^{X,M}(B) & \xrightarrow{\mathrm{id}_M \otimes i_{\widetilde{A}} + \mathrm{id}_M \otimes i_{\widetilde{B}}} & \mathrm{H}_n\big(M \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B})\big) & \xrightarrow{\partial_n} & \mathrm{H}_n^{X,M}(A \cap B) \\ & \searrow^{i_A + i_B} & \cong \big\downarrow \Xi & \nearrow_{=:\partial_n} & \\ & & \mathrm{H}_n^{X,M}(A \cup B). & & \end{array}$$

The crucial observation is that the triangle to the left commutes. Thus going "via the bottom" we do indeed obtain the desired connecting homomorphism that gives rise to the promised long exact sequence. ■

The Mayer–Vietoris Theorem 169.7 for Internal Twisted Homology, as it is formulated, will play a major role in the proof of the Poincaré Duality Theorem 172.1 for Twisted Coefficients. But the alert reader will have noticed that the Mayer–Vietoris Theorem 169.7 for Internal Twisted Homology is not all that useful for performing calculations since it is formulated in terms of the unwieldy internal homology modules. Of course, by the Change of Covering Proposition II 169.4 together with Lemma 169.3, we know that each such internal homology module is isomorphic to an appropriate twisted homology module of each of the components. The catch is, that it becomes increasingly unclear what are the "inclusion induced maps" at this point.

The following theorem gives us a clean statement if all subspaces involved are path-connected.

Theorem 169.8. (Mayer–Vietoris Theorem for homology with twisted coefficients) Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. Let R be a ring and let M be a $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left module. Let $A, B \subset X$ be subspaces such that the following conditions are satisfied:

(1) (X, A, B) is a universally excisive triad,
(2) we have $X = A \cup B$,
(3) A, B and $A \cap B$ are locally contractible and path-connected,
(4) $x_0 \in A \cap B$.

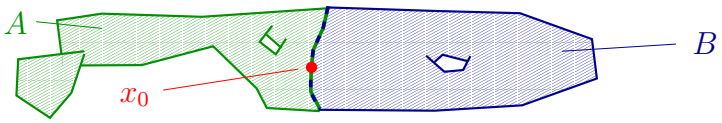
There exists a natural connecting R-left module homomorphism[2084]

$$\partial_n\colon \mathrm{H}_n^{\mathrm{tw}}(X; M) \xrightarrow{\partial_n} \mathrm{H}_{n-1}^{\mathrm{tw}}(A \cap B; i_{A\cap B}^* M)$$

such that the following sequence

$$\ldots \to \mathrm{H}_n^{\mathrm{tw}}(A \cap B; i_{A\cap B}^* M) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} \mathrm{H}_n^{\mathrm{tw}}(A; i_A^* M) \oplus \mathrm{H}_n^{\mathrm{tw}}(B; i_B^* M) \xrightarrow{i_A + i_B} \mathrm{H}_n^{\mathrm{tw}}(X; M) \xrightarrow{\partial_n} \xrightarrow{\partial_n} \mathrm{H}_{n-1}^{\mathrm{tw}}(A \cap B; i_{A\cap B}^* M) \to \ldots$$

of R-left module homomorphisms is exact.[2085]



Proof of the Mayer–Vietoris Theorem 169.8. By the Mayer–Vietoris Theorem 169.7 for Internal Twisted homology there exists a natural connecting R-left module homomorphism

$$\partial_n \colon \mathrm{H}_n^{X,M}(X) \xrightarrow{\partial_n} \mathrm{H}_{n-1}^{X,M}(A \cap B)$$

such that the following sequence is exact:

$$\cdots \to \mathrm{H}_n^{X,M}(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} \mathrm{H}_n^{X,M}(A) \oplus \mathrm{H}_n^{X,M}(B) \xrightarrow{i_A + i_B} \mathrm{H}_n^{X,M}(X) \xrightarrow{\partial_n} \xrightarrow{\partial_n} \mathrm{H}_{n-1}^{X,M}(A \cap B) \to \dots$$

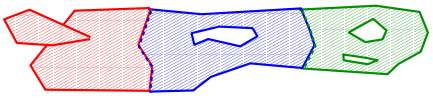
The Change of Covering Proposition 169.4 II (1) gives us isomorphisms

$$\Omega^X_{A\cap B} \colon \mathrm{H}_k^{\mathrm{tw}}(A \cap B; i^*_{A\cap B} M) \to \mathrm{H}_k^{X,M}(A \cap B),$$

and similarly for A and B. After making all these replacement we obtain a long exact sequence with the right modules at the right place. It remains to show that the maps in the resulting diagram are the maps we promised. But this is an immediate consequence of the Change of Covering Proposition 169.4 II (2). ■

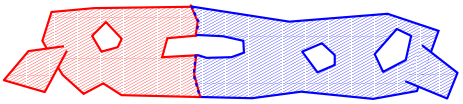
The most restrictive extra condition that we impose in the above Mayer–Vietoris Theorem 169.8 compared to the traditional Mayer–Vietoris Theorem 109.14 is that we now demand that $A \cap B$ is path-connected. If $A \cap B$ is not path-connected then sometimes we can get around this issue by applying Theorem 169.8 iteratively while moving the base point. But this approach does not always work.

decomposition of a topological space X



can iteratively apply Mayer-Vietoris for connected intersection

decomposition of a topological space X



cannot iteratively apply Mayer-Vietoris for connected intersection

The following theorem gets rid of the hypothesis in the Mayer–Vietoris Theorem 169.8 that the intersection $A \cap B$ needs to be path-connected. The price we pay for this flexibility is another layer of notation which obfuscates the statement even further. Before we can state this generalization we need to introduce one more piece of notation:

Notation. Let W be a topological space that is locally contractible and path-connected and let $w_0 \in W$. Let R be a ring and let M be a $(\mathbb{Z}[\pi_1(W, w_0)], R)$-left left module. Let $D \subset W$ be a subspace that is locally contractible and path-connected, let $d_0 \in D$ and let $\delta \colon [0,1] \to W$ be a path from w_0 to d_0. We denote by $i \colon D \to W$ the inclusion and we

[2084]Recall that hypothesis that (X, A, B) is a universally excisive triad implies in particular that A, B and $A \cap B$ are locally contractible. Since A, B and $A \cap B$ are path-connected it makes sense to consider twisted homology modules of A, B and $A \cap B$.

[2084]Here $A \cap B$, A, B and X are all equipped with the base point x_0. The various maps denoted by i are the obvious inclusion (induced) maps, with our choice of base point, all maps are base point preserving.

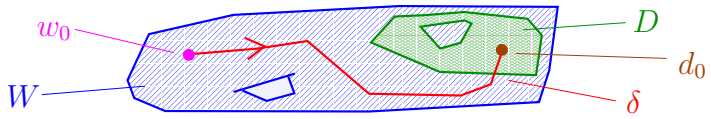
denote by i^δ the chain map

$$\mathrm{C}_k^{\mathrm{tw}}(D; i^*\delta^{**}M) \xrightarrow[\uparrow]{i} \mathrm{C}_k^{\mathrm{tw}}(W; \delta^{**}M) \xrightarrow[\uparrow]{\delta_*} \mathrm{C}_k^{\mathrm{tw}}(W; M)$$

map that is induced by the inclusion $i\colon (D, d_0) \to (W, w_0)$, see page 3542

isomorphism given by the Change of Base Point Proposition 167.6

of R-left chain complexes. Furthermore we denote by $i^\delta\colon \mathrm{H}_k^{\mathrm{tw}}(D; i^*\delta^{**}M) \to \mathrm{H}_k^{\mathrm{tw}}(W; M)$ the induced map. Finally we simplify $\delta^*{*}$ and $i^*\delta^{**}M$ to δ_*M.



Theorem 169.9. (Mayer–Vietoris Theorem for homology with twisted coefficients) Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. Let R be a ring and let M be a $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left module. Let $A, B \subset X$ be subspaces such that the following conditions are satisfied:

(1) (X, A, B) is a universally excisive triad,
(2) we have $X = A \cup B$,
(3) A and B are path-connected,
(4) A, B and $A \cap B$ are locally contractible,
(5) $A \cap B$ consists of finitely many path-components $C_1, \ldots, C_k$.

We pick $a_0 \in A$, $b_0 \in B$, we pick $c_j \in C_j$, $j = 1, \ldots, k$, we pick a path $\alpha\colon [0,1] \to X$ from x_0 to a_0, we pick a path $\beta\colon [0,1] \to X$ from x_0 to b_0, we pick paths $\gamma_{j,A}\colon [0,1] \to A$ from a_0 to c_j and we pick paths $\gamma_{j,B}\colon [0,1] \to B$ from b_0 to c_j. There exist natural R-left module homomorphisms

$$\partial_n\colon \mathrm{H}_n^{\mathrm{tw}}(X; M) \xrightarrow{\partial_n} \bigoplus_{j=1}^{k} \mathrm{H}_{n-1}^{\mathrm{tw}}(C_j; (\alpha * \gamma_{j,A})_* M)$$

and natural homomorphisms[2085]

$$\Psi_j\colon \mathrm{H}_n^{\mathrm{tw}}(C_j; (\alpha * \gamma_{j,A})_* M) \to \mathrm{H}_n^{\mathrm{tw}}(B; \beta_* M)$$

such that the following sequence

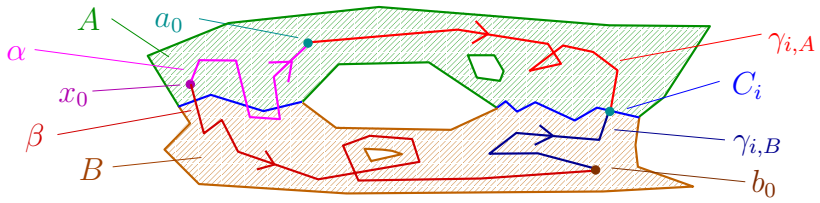
$$\to \bigoplus_{j=1}^{k} \mathrm{H}_n^{\mathrm{tw}}(C_j; (\alpha * \gamma_{j,A})_* M) \xrightarrow{\oplus_j i^{\gamma_{j,A}} \oplus -\oplus_j \Psi_j} \mathrm{H}_n^{\mathrm{tw}}(A; \alpha_* M) \oplus \mathrm{H}_n^{\mathrm{tw}}(B; \beta_* M) \xrightarrow{i^\alpha + i^\beta} \mathrm{H}_n^{\mathrm{tw}}(X; M) \xrightarrow{\partial_n}$$
$$\xrightarrow{\partial_n} \bigoplus_{j=1}^{k} \mathrm{H}_{n-1}^{\mathrm{tw}}(C_j; (\alpha * \gamma_{j,A})_* M) \to \ldots$$

of R-left module homomorphisms is exact.

Remark.

(1) At this stage it should be clear to the reader that one does not necessarily need to assume that A and B are path-connected. With even more messy notation one

[2085] These homomorphisms are described in the proof of the theorem. There does not seem to be a snappy description of these maps.



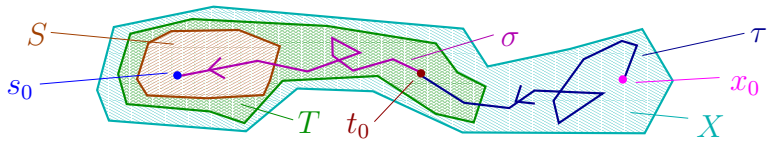
can formulate a generalization to the non path-connected case. We leave the task of texing this to the reader.

(2) In practice it can be very hard to compute honestly and rigorously the involved maps.

For the proof of the above Mayer–Vietoris Theorem 169.9 we will need the following technical lemma.

Lemma 169.10. Let X be a topological space and let $S \subset T \subset X$ be subspaces. We assume that S, T and X are locally contractible and path-connected. Let $s_0 \in S$, $t_0 \in T$ and $x_0 \in X$ be points and let $\sigma\colon [0,1] \to T$ be a path from t_0 to s_0 and let $\tau\colon [0,1] \to X$ be a path from x_0 to t_0. We denote by $i\colon S \to T$ and $j\colon T \to X$ the inclusions. Finally let R be a ring and let M be a $(\mathbb{Z}[\pi_1(X,x_0)], R)$-left left module. With all this notation, together with the notation from page 3579 the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}_n^{X,M}(S) & \xrightarrow{(\tau*\sigma)_*} & \mathrm{H}_n^{X,(\tau*\sigma)_*M}(S) & \xrightarrow{\Omega_S^X} & \mathrm{H}_n^{\mathrm{tw}}(S;(\tau*\sigma)_*M) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i^\sigma} & & \downarrow{\scriptstyle i} \\ \mathrm{H}_n^{X,M}(T) & \xrightarrow{\tau_*} & \mathrm{H}_n^{X,\tau_*M}(T) & \xrightarrow{\Omega_T^X} & \mathrm{H}_n^{\mathrm{tw}}(T;\tau_*M). \end{array}$$



Proof. We leave it to the reader to verify this lemma. ■

Proof. It follows from the Mayer–Vietoris Theorem 169.7 for Internal Twisted homology together with Lemma 169.3 that there exists a natural R-left module homomorphism

$$\partial_n\colon \mathrm{H}_n^{tw}(X;M) \xrightarrow{\partial_n} \bigoplus_{j=1}^{k} \mathrm{H}_{n-1}^{X,M}(C_j)$$

such that the following sequence

$$\cdots \to \bigoplus_{j=1}^{k} \mathrm{H}_n^{X,M}(C_j) \xrightarrow{\oplus i_{C_j} \oplus -\oplus i_{C_j}} \mathrm{H}_n^{X,M}(A) \oplus \mathrm{H}_n^{X,M}(B) \xrightarrow{i_A+i_B} \mathrm{H}_n^{\mathrm{tw}}(M) \xrightarrow{\partial_n} \xrightarrow{\partial_n} \bigoplus_{j=1}^{k} \mathrm{H}_{n-1}^{X,M}(C_j) \to \dots$$

of R-left module homomorphisms is exact. As before we want replace these internal homology groups by honest-to-G*d twisted homology groups. But this time we are in a sea of troubles since we surrendered all control over base points. Nonetheless, we will give it a

try. We have the isomorphisms

$$\begin{array}{ccccc}
\mathrm{H}_n^{X,M}(C_j) & \xrightarrow{(\alpha*\gamma_{j,A})_*} & \mathrm{H}_n^{X,(\alpha*\gamma_{j,A})_*M}(C_j) & \xrightarrow{\Omega^X_{C_j}} & \mathrm{H}_n^{\mathrm{tw}}(C_j;(\alpha*\gamma_{j,A})_*M) \\
\mathrm{H}_n^{X,M}(A) & \xrightarrow{\alpha_*} & \mathrm{H}_n^{X,\alpha_*M}(A) & \xrightarrow{\Omega^X_{A}} & \mathrm{H}_n^{\mathrm{tw}}(A;\alpha_*M) \\
\mathrm{H}_n^{X,M}(B) & \xrightarrow{\beta_*} & \mathrm{H}_n^{X,\beta_*M}(B) & \xrightarrow{\Omega^X_{B}} & \mathrm{H}_n^{\mathrm{tw}}(B;\beta_*M).
\end{array}$$

Here the isomorphisms to the left are provided by the Change of Base Point Proposition II 169.5 and the isomorphisms to the right are provided by the Change of Covering Proposition II 169.4. Fortunately Lemma 169.10 sails to our rescue. It implies that in most cases, if we use the above isomorphisms to replace our previous inclusion induced maps, we get again inclusion induced maps. For example for each $j \in \{1,\dots,k\}$ we get that the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{H}_n^{X,M}(C_j) & \xrightarrow[\cong]{(\alpha*\gamma_{j,A})_*} & \mathrm{H}_n^{X,(\alpha*\gamma_{j,A})^*M}(C_j) & \xrightarrow[\cong]{\Omega^X_{C_j}} & \mathrm{H}_n^{\mathrm{tw}}(C_j;(\alpha*\gamma_{j,A})_*M) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i^{\gamma_{j,A}}} & & \downarrow{\scriptstyle i} \\
\mathrm{H}_n^{X,M}(A) & \xrightarrow[\cong]{alpha_*} & \mathrm{H}_n^{X,\alpha_*M}(A) & \xrightarrow[\cong]{\Omega^X_{A}} & \mathrm{H}_n^{\mathrm{tw}}(A;\alpha_*M).
\end{array}$$

A similar, in fact much simpler logic, applies to i_A and i_B. But the inclusion from C_j to B gets screwed up, since the paths do not match up. In fact the map Ψ_j is the following composition of maps, for which this particular author cannot find a pleasant shortcut:

$$\begin{array}{ccccc}
\mathrm{H}_n^{X,M}(C_j) & \xrightarrow[\cong]{(\alpha*\gamma_{j,B})_*} & \mathrm{H}_n^{X,(\alpha*\gamma_{j,B})^*M}(C_j) & \xrightarrow[\cong]{\Omega^X_{C_j}} & \mathrm{H}_n^{\mathrm{tw}}(C_j;(\alpha*\gamma_{j,B})_*M) \\
\downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle =:\Psi_j} \\
\mathrm{H}_n^{X,M}(B) & \xrightarrow[\cong]{\beta_*} & \mathrm{H}_n^{X,\beta_*M}(B) & \xrightarrow[\cong]{\Omega^X_{B}} & \mathrm{H}_n^{\mathrm{tw}}(B;\beta_*M).
\end{array}$$

■

Exercises for Chapter 169.

Exercise 169.1. Use the Mayer–Vietoris theorem to show that given any lens space and given any non-trivial one-dimensional complex representation all twisted homology groups are zero.

170. Twisted cellular (co-) homology

170.1. **Twisted cellular (co-) homology.** We start out with a few preliminaries.

Notation. Let X be a CW-complex and let A be a subcomplex.

(1) Given $k \in \mathbb{N}_0$ we denote by X^k the k-skeleton of X, as defined on page 1469.
(2) As on page 1492 we say that a map $g\colon X \to Y$ between two CW-complexes is cellular if for each $k \in \mathbb{N}_0$ we have $g(X^k) \subset Y^k$.
(3) As on page 1770 we set
$$\mathrm{C}_n^{\mathrm{CW}}(X) \ := \ \mathrm{H}_n(X^n, X^{n-1})$$
and we define the cellular chain map as on page 1769. The corresponding homology groups are the cellular homology groups $\mathrm{H}_n^{\mathrm{CW}}(X)$. Furthermore we consider the relative chain complex
$$\mathrm{C}_*^{\mathrm{CW}}(X, A) \ := \ \mathrm{coker}\left(i_*\colon \mathrm{C}_*^{\mathrm{CW}}(A) \to \mathrm{C}_*^{\mathrm{CW}}(X)\right),$$
and we refer to the corresponding homology groups as the relative cellular homology groups $\mathrm{H}_*^{\mathrm{CW}}(X, A)$.
(4) By the CW-Complex Properties Proposition 68.10 we know that X is locally contractible. Thus, if X is path-connected, then given any $x_0 \in X$ we know that the natural universal covering $p\colon \widetilde{X}_{x_0} \to X$ exists.
(5) If $q\colon Y \to X$ is any covering, e.g. the natural universal covering, then it follows from the CW-complex Covering Proposition 68.37 that we can equip Y with a natural CW-structure which has the property, that each deck transformation of $q\colon Y \to X$ acts cellularly. If $A \subset X$ is a subcomplex, then $p^{-1}(A)$ is, basically by construction, a subcomplex of Y.

There are three reasons why we were and are interested in cellular chain complexes:

(1) Cellular chain complexes are incomparably smaller than the unwieldy singular chain complexes.
(2) In the Cellular Boundary Map-via-Degrees Proposition 80.8 we gave a practical way for determining the boundary maps in the cellular chain complex.
(3) In the Singular-Cellular H_*-Isomorphism Proposition 80.4 and the Singular-Cellular H^*-Isomorphism Proposition 109.4 we saw that given a CW-complex X the singular (co-) homology groups and the cellular (co-) homology groups are naturally isomorphic.

Our goal in this chapter is to repeat this success story in the present context of twisted (co-) homology.

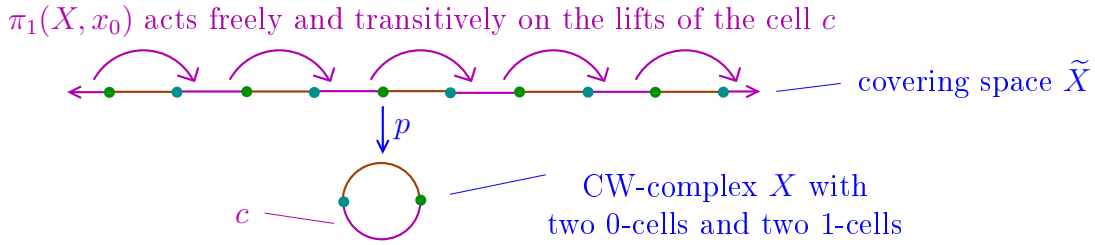
Notation. Let X be a CW-complex and let A be a subcomplex.

(1) By the CW-Complex Properties Proposition 68.10 we know that X is locally contractible. Thus, if X is path-connected, then given any $x_0 \in X$ we know that the natural universal covering $p\colon \widetilde{X}_{x_0} \to X$ exists.
(2) If $q\colon Y \to X$ is any covering, e.g. the natural universal covering, then it follows from the CW-complex Covering Proposition 68.37 that we can equip Y with a natural CW-structure which has the property, that each deck transformation of $q\colon Y \to X$

acts cellularly. If $A \subset X$ is a subcomplex, then $p^{-1}(A)$ is, basically by construction, a subcomplex of Y.

Lemma 170.1. Let X be a path-connected CW-complex, let $x_0 \in X$ and let $A \subset X$ be a subcomplex. Let $p\colon \widetilde{X} \to X$ be a regular covering. We write $\widetilde{A} := p^{-1}(A)$ and we denote by $D := \mathrm{Deck}(p\colon \widetilde{X} \to X)$ the corresponding deck transformation group.

(1) Let $k \in \mathbb{N}_0$. For each open k-cell σ in X that is not contained in A we pick a single open cell $\widetilde{\sigma} \subset \widetilde{X}$ with $p(\widetilde{\sigma}) = \sigma$. These cells $\widetilde{\sigma}$ are a basis for $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}, \widetilde{A})$ as a free $\mathbb{Z}[D]$-left module.
(2) Each cellular chain group $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}, \widetilde{A})$ is a free $\mathbb{Z}[D]$-left module where the cardinality of the basis is given by the cardinality of the set of k-cells of the CW-complex X that are not contained in A.
(3) If X is compact, then each cellular chain group $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}, \widetilde{A})$ is a finitely generated free $\mathbb{Z}[D]$-left module.
(4) The cellular boundary maps $\partial\colon \mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}, \widetilde{A}) \to \mathrm{C}_{k-1}^{\mathrm{CW}}(\widetilde{X}, \widetilde{A})$ are $\mathbb{Z}[D]$-left module homomorphisms.



Proof.

(1) This statement follows from the explicit description of the CW-structure on $\widetilde{X}$ that is described in the proof of the CW-complex Covering Proposition 68.37 together with the explicit description of the cellular chain groups in the Skeleton-Homology Lemma 80.1.
(2) This statement is just a slightly more reader friendly version of Statement (1).
(3) If X is compact, then it follows from the CW-Complex Compactness Corollary 68.15 that X has only finitely many cells. Thus this statement follows immediately from Statement (1).
(4) This statement follows almost immediately from the naturality of the cellular boundary maps, see Lemma 80.3 and the fact, observed above, that each deck transformation is a cellular self-map of $\widetilde{X}$. ■

We can now introduce the cellular analogues of the twisted (co-) homology groups that we defined on page 3537.

Definition. Let X be a path-connected CW-complex, let $x_0 \in X$ and let $A \subset X$ be a subcomplex. We write $\pi := \pi_1(X, x_0)$. Furthermore let R be a ring and let M be a $(\mathbb{Z}[\pi_1(X_0, x_0)], R)$-left left module. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal

covering and we write $\widetilde{A} := p^{-1}(A)$. Given $k \in \mathbb{N}_0$ we define the R-left modules

$$\begin{aligned} \mathrm{C}_k^{\mathrm{tw,CW}}(X, A; M) &:= \overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_{x_0}, \widetilde{A}) && \text{(twisted cellular chain complex)} \\ \mathrm{C}^k_{\mathrm{tw,CW}}(X, A; M) &:= \mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_{x_0}, \widetilde{A}), M) && \text{(twisted cellular cochain complex).} \end{aligned}$$

By Lemma 170.1 we know that the boundary maps of $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_{x_0}, \widetilde{A})$ are $\mathbb{Z}[\pi]$-left module homomorphisms. Thus we see that $\mathrm{id} \otimes \partial$ and $\mathrm{Hom}(\partial, \mathrm{id})$ induce (co-) chain maps. Thus we can define the R-left modules

$$\begin{aligned} \mathrm{H}_k^{\mathrm{tw,CW}}(X, A; M) &:= \mathrm{H}_k\big(\overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*^{\mathrm{CW}}(\widetilde{X}_{x_0}, \widetilde{A})\big) && \text{(twisted cellular homology)} \\ \mathrm{H}^k_{\mathrm{tw,CW}}(X, A; M) &:= \mathrm{H}^k\big(\mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_*^{\mathrm{CW}}(\widetilde{X}_{x_0}, \widetilde{A}), M)\big) && \text{(twisted cellular cohomology).} \end{aligned}$$

As always, if $A = \varnothing$, then we drop it from the notation.

Example. We consider the topological space $S^1 \subset \mathbb{C}$ which we equip with the CW-structure that consists of a single 0-cell $c = \{1\}$ and a single 1-cell d. Let $\varphi\colon \pi_1(S^1, 1) \to \langle x \rangle$ be the usual isomorphism. We denote by $p\colon \mathbb{R} \to S^1$ the universal covering that is given by $p(t) = \exp(2\pi \mathrm{i} t)$. We pick the lift $\widetilde{c} = \{0\} \subset \mathbb{R}$ and as a lift of the 1-cell d we pick the 1-cell $\widetilde{d}$ with attaching map $\Phi\colon (-1, 1) \to \mathbb{R}$ that is given by $\Phi(s) = \frac{1}{2}(1 + s)$. With respect to these lifts the cellular chain $\mathrm{C}_*^{\mathrm{CW}}(\mathbb{R})$ is given by

$$0 \;\to\; \mathbb{Z}[x^{\pm 1}] \cdot \widetilde{d} \;\xrightarrow{\widetilde{d} \mapsto (x-1)\cdot \widetilde{c}}\; \mathbb{Z}[x^{\pm 1}] \cdot \widetilde{c} \;\to\; 0.$$

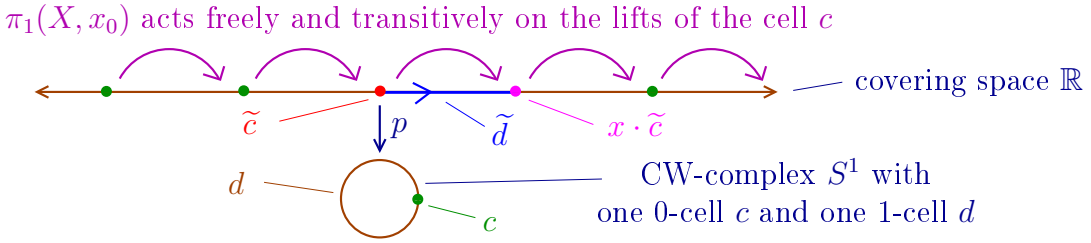
Given $\xi \in \mathbb{C} \setminus \{0\}$ we consider the representation

$$\rho_\xi\colon \pi_1(S^1, 1) \;\xrightarrow{\varphi}\; \langle x \rangle \;\xrightarrow{x \mapsto \xi}\; \mathrm{GL}(1, \mathbb{C}).$$

We see that[2086]

$$\begin{aligned} \mathrm{H}_1^{\mathrm{CW}}(S^1; {}_{\rho_\xi}\mathbb{C}) &\cong \ker\big({}_{\rho_\xi}\mathbb{C} \otimes_{\mathbb{Z}[x^{\pm 1}]} \mathbb{Z}[x^{\pm 1}] \cdot \widetilde{d} \xrightarrow{\mathrm{id} \otimes (x-1)} {}_{\rho_\xi}\mathbb{C} \otimes_{\mathbb{Z}[x^{\pm 1}]} \mathbb{Z}[x^{\pm 1}] \cdot \widetilde{c}\big) \\ &\cong \ker(\mathbb{C} \xrightarrow{\cdot(\xi - 1)} \mathbb{C}) \cong \begin{cases} 0, & \text{if } \xi \neq 1, \\ \mathbb{C}, & \text{if } \xi = 1. \end{cases} \end{aligned}$$

For $k = 1$ we consider the cokernel instead of the kernel and, lo and behold, we end up with the same result.



Example. We consider the torus

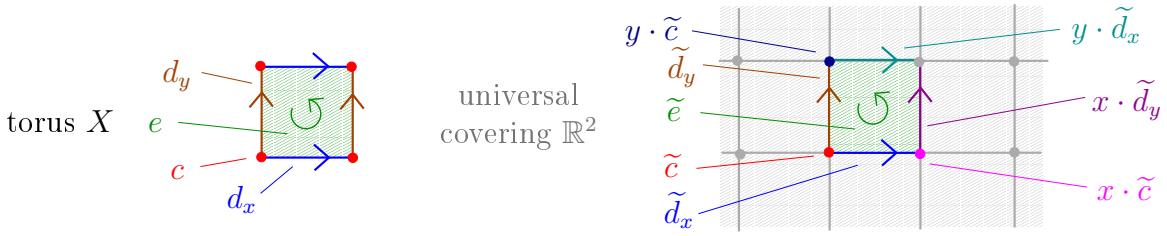
$$X := ([0, 1] \times [0, 1])/\sim \qquad \text{where } (x, 0) \sim (x, 1) \text{ and } (0, y) \sim (1, y).$$

[2086] We know by the Replace Universal Covering Proposition 168.6 that we can determine the isomorphism type of (co-) homology with twisted coefficients using any universal covering, i.e. for the isomorphism type we do not have to work with the natural universal covering. This logic also applies to twisted cellular (co-) homology. We will use this little observation on numerous occasions.

We equip X with the usual CW-structure that we introduced on page 1471 which has one 0-cell c, two 1-cells d_x and d_y and one 2-cell e. We equip each cell with the (fairly) obvious characteristic map. We make the usual identification $\pi_1(X,(0,0)) = \mathbb{Z}\cdot x \oplus \mathbb{Z}\cdot y$. Furthermore, if we make the usual identification $X = S^1 \times S^1$, then we have a universal covering $p\colon \mathbb{R}^2 \to X$. The resulting CW-structure on $\mathbb{R}^2$, and lifts of c, d_x, d_y and e are shown in the figure. Using the Cellular Boundary Map-via-Degrees Proposition 80.8 we can calculate the cellular chain complex of the universal covering and we see that it is of the form[2087]

$$0 \to \mathbb{Z}[x^{\pm1},y^{\pm1}]\cdot\widetilde{e} \xrightarrow{\begin{pmatrix}1-y\\x-1\end{pmatrix}} \begin{matrix}\mathbb{Z}[x^{\pm1},y^{\pm1}]\cdot\widetilde{d}_x\\ \mathbb{Z}[x^{\pm1},y^{\pm1}]\cdot\widetilde{d}_y\end{matrix} \to \xrightarrow{((x-1)\ (y-1))} \mathbb{Z}[x^{\pm1},y^{\pm1}]\cdot\widetilde{c} \to 0.$$

Now one can calculate all the (co-) homology modules of the torus without any problems.



Example. Let $n \in \mathbb{N}_{\geq 2}$. We consider the real projective space $\mathbb{R}\mathrm{P}^n = S^n/P \sim -P$. By Corollary 48.18 there exists an isomorphism $\pi_1(\mathbb{R}\mathrm{P}^n) \cong \langle x \,|\, x^2\rangle$. We denote by $p\colon S^n \to \mathbb{R}\mathrm{P}^n$ the projection . We equip $\mathbb{R}\mathrm{P}^n$ with the CW-structure that we introduced in Lemma 68.1, which has one cell in each of the dimensions $0,\ldots,n$. The corresponding CW-structure on S^n has precisely two cells in each of the dimensions $0,\ldots,n$. In fact the open cells are precisely

$$\begin{aligned} C_{i,+} &:= \{(x_0,\ldots,x_i,0,\ldots,0) \in S^n \,|\, x_i > 0\}\\ \text{and}\quad C_{i,-} &:= \{(x_0,\ldots,x_i,0,\ldots,0) \in S^n \,|\, x_i < 0\}. \end{aligned}$$

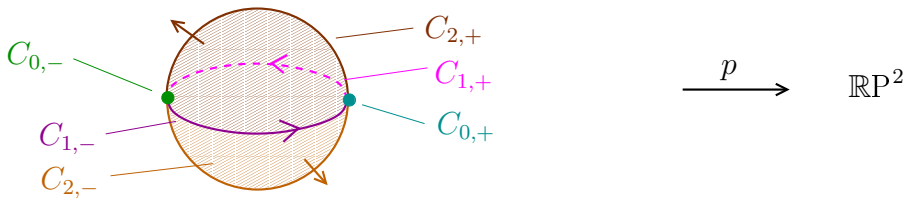
With an appropriate choice of characteristic maps we see, using the Cellular Boundary Map-via-Degrees Proposition 80.8, that the cellular chain complex of S^n is isomorphic to the chain complex

$$0 \to \mathbb{Z}[\langle x \,|\, x^2\rangle] \xrightarrow{1+(-1)^{n+1}\cdot x} \ldots \to \mathbb{Z}[\langle x \,|\, x^2\rangle] \xrightarrow{1+x} \mathbb{Z}[\langle x \,|\, x^2\rangle] \xrightarrow{1-x} \mathbb{Z}[\langle x \,|\, x^2\rangle] \to 0.$$

It we take the trivial representation $\rho\colon \pi_1(\mathbb{R}\mathrm{P}^n) = \langle x \,|\, x^2\rangle \to \mathrm{GL}(1,\mathbb{Z})$, i.e. the representation that is given by $\rho(x) = 1$, then we recover exactly the cellular chain complex that we had obtained in the proof of Proposition 80.11, when we calculated the ordinary (cellular) homology groups of $\mathbb{R}\mathrm{P}^n$.

We will now see that twisted cellular (co-) homology is functorial in a suitable sense. First we have the following variation on the definition on page 3542.

[2087] Note that it follows from the subtle sign in the boundary map ∂_2 that this is indeed a chain complex, as it should be.



Definition. Let $f\colon (X,x_0)\to(Y,y_0)$ be a map between pointed connected CW-complexes. We write $\pi_X := \pi_1(X,x_0)$, $\pi_Y := \pi_1(Y,y_0)$. Furthermore let R be a ring and let M be a $(\mathbb{Z}[\pi_Y],R)$-left left module. We denote by $p\colon \widetilde{X}_{x_0}\to X$ and $q\colon \widetilde{Y}_{y_0}\to Y$ the natural universal coverings. Suppose that we are given subsets $A\subset X$ and $B\subset Y$ with $f(A)\subset B$. We write $\widetilde{A} := p^{-1}(A)$ and $\widetilde{B} := q^{-1}(B)$. We denote by $\widetilde{f}\colon \widetilde{X}_{x_0}\to\widetilde{Y}_{y_0}$ the map that is induced by $f\colon X\to Y$. It follows easily from the fact that f is cellular that $\widetilde{f}$ is also cellular. Using the cellular analogue of the Chain Complex-Equivariance Proposition 167.8 we define

$$\begin{array}{rcl} f_*\colon \overbrace{\overline{f^*M}\otimes_{\mathbb{Z}[\pi_X]}\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_{x_0},\widehat{A})}^{=\mathrm{C}_k^{\mathrm{tw,CW}}(X,A;f^*M)} & \to & \overbrace{\overline{M}\otimes_{\mathbb{Z}[\pi_Y]}\mathrm{C}_k^{\mathrm{CW}}(\widetilde{Y}_{y_0},\widehat{B})}^{=\mathrm{C}_k^{\mathrm{tw,CW}}(Y,B;M)} \\ m\otimes(\sigma\colon\Delta^k\to\widetilde{X}_{x_0}) & \mapsto & m\otimes(\widetilde{f}\circ\sigma\colon\Delta^k\to\widetilde{Y}_{y_0}) \end{array}$$

and

$$\begin{array}{rcl} f^*\colon \overbrace{\underset{\mathbb{Z}[\pi_Y]\text{-left}}{\mathrm{Hom}}(\mathrm{C}_k^{\mathrm{CW}}(\widetilde{Y}_{y_0},\widehat{B}),M)}^{=\mathrm{C}^k_{\mathrm{tw,CW}}(Y,B;M)} & \to & \overbrace{\underset{\mathbb{Z}[\pi_X]\text{-left}}{\mathrm{Hom}}(\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_{x_0},\widehat{A}),f^*M)}^{=\mathrm{C}^k_{\mathrm{tw,CW}}(X,A;f^*M)} \\ \varphi & \mapsto & (\sigma\mapsto\varphi(r\circ\sigma)) \end{array}$$

and we define the corresponding induced maps

$$\begin{array}{rcl} f_*\colon \mathrm{H}_k^{\mathrm{tw,CW}}(X;f^*M) & \to & \mathrm{H}_k^{\mathrm{tw,CW}}(Y;M) \\ f^*\colon \mathrm{H}^k_{\mathrm{tw,CW}}(Y;M) & \to & \mathrm{H}^k_{\mathrm{tw,CW}}(X;f^*M). \end{array}$$

We leave it to the reader to formulate the obvious analogue of the Twisted Homology-Functoriality Proposition 167.9.

170.2. **The interaction between twisted cellular and singular (co-) homology.** In the Singular-Cellular H_*-Isomorphism Proposition 80.4 and the Singular-Cellular H^*-Isomorphism Proposition 109.4 we saw that given a CW-complex X the singular (co-) homology groups and the cellular (co-) homology groups are naturally isomorphic. The following theorem gives us an analogous statement for twisted (co-) homology.

Theorem 170.2. (Singular-Cellular Twisted Isomorphism Theorem) Let X be a connected CW-complex, let $A\subset X$ be a subcomplex, let $x_0\in X$, let R be a ring and let M be a $(\mathbb{Z}[\pi_1(X,x_0)],R)$-left left module. Then there exist natural isomorphisms

(1) $\Phi_{(X,A)}\colon \mathrm{H}_k^{\mathrm{tw}}(X,A;M) \to \mathrm{H}_k^{\mathrm{tw,CW}}(X,A;M)$

(2) $\Phi_{(X,A)}\colon \mathrm{H}^k_{\mathrm{tw}}(X,A;M) \to \mathrm{H}^k_{\mathrm{tw,CW}}(X,A;M)$

of R-left modules. Here natural means the following: if $f\colon (X,A)\to(Y,B)$ is a cellular map between pairs of CW-complexes, if X and Y are connected, if $x_0\in X$, if R is a ring,

and if M is a $(\mathbb{Z}[\pi_1(Y, f(x_0))], R)$-left left module, then the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}_k^{\mathrm{tw}}(X, A; f^*M) & \xrightarrow{\Phi_{(X,A)}} & \mathrm{H}_k^{\mathrm{tw,CW}}(X, A; f^*M) \\ \downarrow{\scriptstyle f_*} & & {\scriptstyle f_*}\downarrow \\ \mathrm{H}_k^{\mathrm{tw}}(Y, B; M) & \xrightarrow{\Phi_{(Y,B)}} & \mathrm{H}_k^{\mathrm{tw,CW}}(Y, B; M) \end{array}$$

and the corresponding diagram for twisted cohomology also commutes.

Remark.

(1) The proof we give of the Singular-Cellular Twisted Isomorphism Theorem **??** is based on the proof given in [**Lü1998**, Lemma 4.2] and [**Lü1989**, p. 263], see also [**Wall1966a**, Lemma 1]. An alternative proof is given in [**KwS2018**, Theorem 4.1]. It seems like the naturality statement was not dealt with in great detail in the earlier references.

(2) Let K be an *ordered* abstract simplicial complex. By the Simplicial-Singular Chain Complex Theorem 95.25 the maps

$$\begin{array}{rccl} \Theta_n \colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n(|K|) \\ s & \mapsto & (\Phi_s^{\leq} \colon \Delta^n \to |K|) \end{array}$$

are a natural chain homotopy equivalence. Note that naturality implies that if we work with the universal covering of an ordered simplicial complex, then the above map is a chain homotopy equivalence of free $\mathbb{Z}[\pi_1(K)]$-chain complexes, and thus induces an isomorphism between "twisted simplicial" (co-) homology and twisted singular (co-) homology groups. We leave it to the reader to fill in the details.

Example. Let X be a CW-complex, let $x_0 \in X$ and let M be a $\mathbb{Z}[\pi_1(X, x_0)]$-left module. If X has no k-cells, then it follows almost immediately from Theorem 170.2 that $\mathrm{H}_k^{\mathrm{tw}}(X; M) = 0$ and $\mathrm{H}^k_{\mathrm{tw}}(X; M) = 0$.

The proof of the Singular-Cellular Twisted Isomorphism Theorem 170.2 will require the remainder of this section. The idea is to generalize the proof of the Singular-Cellular H_*-Isomorphism Proposition 80.4 that we gave in Section 81.4. The key idea was to introduce a suitable intermediate chain complex $\mathrm{C}_*^{\mathrm{int}}(W)$, which allowed us to relate the singular chain complex $\mathrm{C}_*(W)$ to the cellular chain complex $\mathrm{C}_*^{\mathrm{int}}(W)$. In the following we recall and expand on this concept.

Definition. Let W be a CW-complex.

(1) Given $n \in \mathbb{N}_0$ we define the intermediate cellular chain complex

$$\mathrm{C}_n^{\mathrm{int}}(W) \ := \ \ker\big(\mathrm{C}_n(W^n) \xrightarrow{\partial} \mathrm{C}_{n-1}(W^n) \to \mathrm{C}_{n-1}(W^n, W^{n-1})\big).$$

In Lemma 81.9 we saw that $\mathrm{C}_*^{\mathrm{int}}(W)$ is a subcomplex of $\mathrm{C}_*(W)$ and that $\mathrm{C}_*^{\mathrm{int}}(W)$ is covariantly functorial under cellular maps.

(2) Let B be a subcomplex of W. We denote by $i \colon B \to W$ the inclusion. It follows easily from the definitions that the induced map $i_* \colon \mathrm{C}_n^{\mathrm{int}}(B) \to \mathrm{C}_n^{\mathrm{int}}(W)$ is a monomorphism. We set

$$\mathrm{C}_n^{\mathrm{int}}(W, B) \ := \ \mathrm{coker}\big(i_* \colon \mathrm{C}_n^{\mathrm{int}}(B) \to \mathrm{C}_n^{\mathrm{int}}(W)\big).$$

(3) Given any subcomplex B of W we have natural chain maps

$$\iota\colon \mathrm{C}_*^{\mathrm{int}}(W,B) \hookrightarrow \mathrm{C}_*(W,B) \qquad \text{and} \qquad \pi\colon \mathrm{C}_*^{\mathrm{int}}(W,B) \twoheadrightarrow \mathrm{C}_*^{\mathrm{CW}}(W,B)$$

which are induced by the obvious inclusion and the natural projection.

Proposition 170.3. Let W be a CW-complex and let B be a subcomplex of W. For each $n \in \mathbb{N}_0$ the natural maps

(1) $\quad \iota_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{int}}(W,B)) \to \mathrm{H}_n(\mathrm{C}_*(W,B))$

(2) $\quad \pi_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{int}}(W,B)) \to \mathrm{H}_n(\mathrm{C}_*^{\mathrm{CW}}(W,B))$

are isomorphisms.

Proof. In the Intermediate Chain Complex Proposition 81.11 we proved both statements in the absolute case, i.e. if $B = \varnothing$. Using generic little tricks, which we know from Exercise 74.11 and the proof of the Two-Out-Of-Three-Homotopy Equivalence Corollary 81.6, we will now deduce the relative case from the absolute case. To deal with Statement (1) let us consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{C}_*^{\mathrm{int}}(B) & \longrightarrow & \mathrm{C}_*^{\mathrm{int}}(W) & \longrightarrow & \mathrm{C}_*^{\mathrm{int}}(W,B) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \\
0 & \longrightarrow & \mathrm{C}_*(B) & \longrightarrow & \mathrm{C}_*(W) & \longrightarrow & \mathrm{C}_*(W,B) & \longrightarrow & 0.
\end{array}$$

It follows easily from the definitions that the diagram commutes and that the horizontal sequences are short exact sequences of chain complexes. Therefore, by the LES Proposition 74.9, we get a commutative diagram of long exact sequences:

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & \mathrm{H}_n(\mathrm{C}_*^{\mathrm{int}}(B)) & \longrightarrow & \mathrm{H}_n(\mathrm{C}_*^{\mathrm{int}}(W)) & \longrightarrow & \mathrm{H}_n(\mathrm{C}_*^{\mathrm{int}}(W,B)) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(\mathrm{C}_*^{\mathrm{int}}(B)) \longrightarrow \ldots \\
 & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} \\
\ldots & \longrightarrow & \mathrm{H}_n(\mathrm{C}_*(B)) & \longrightarrow & \mathrm{H}_n(\mathrm{C}_*(W)) & \longrightarrow & \mathrm{H}_n(\mathrm{C}_*(W,B)) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(\mathrm{C}_*(B)) \longrightarrow \ldots.
\end{array}$$

By the Intermediate Chain Complex Proposition 81.11 (1) we know that the two vertical maps relating the absolute cases are isomorphisms. It follows from the Five Lemma 74.10 that the vertical maps dealing with the relative case are also isomorphisms. This concludes the proof of Statement (1).

We turn to the proof of Statement (2). Using basically the same approach as above we can deduce from the Intermediate Chain Complex Proposition 81.11 (2) that the maps $\pi_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{int}}(W,B)) \to \mathrm{H}_n(\mathrm{C}_*^{\mathrm{CW}}(W,B))$ are isomorphisms. ■

We recall the following definition from page 2.

Definition. Let R be a ring and let C_* be a chain complex of R-left modules. A free imitation of C_* is a chain complex F_* of R-left modules together with a quasi-isomorphism $\varphi\colon F_* \to C_*$.[2088]

[2088] Recall that a quasi-isomorphism is a chain map that induces isomorphisms on homology groups.

We now turn the actual proof of the Singular-Cellular Twisted Isomorphism Theorem 170.2. We break it up into several steps:

(i) First we construct the isomorphisms $\Phi_{(X,A)}\colon \mathrm{H}_k^{\mathrm{tw}}(X,A;M) \to \mathrm{H}_k^{\mathrm{tw,CW}}(X,A;M)$ in twisted homology.
(ii) Secondly we show that the isomorphisms are natural.
(iii) Thirdly we quickly outline how to modify (i) and (ii) to prove the corresponding statements for twisted cohomology.

Proof of Theorem 170.2 (1) – Construction of Isomorphism. Let X be a connected CW-complex, let $A \subset X$ be a subcomplex, let $x_0 \in X$, let R be a ring and let M be a $(\mathbb{Z}[\pi_1(X,x_0)],R)$-left left module. We write $\pi := \pi_1(X,x_0)$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering and we write $\widetilde{A} := p^{-1}(A)$.

We equip $\widetilde{X}$ with the CW-structure given by the CW-complex Covering Proposition 68.37. By Proposition 167.1 we have a natural isomorphism $\pi \cong \mathrm{Deck}(p\colon \widetilde{X} \to X)$. Thus, via Lemma 170.1, we obtain a cellular and free left action by π on $\widetilde{X}$.

Note that in our present setting the Free Imitation Existence Lemma 166.8 says that there exists a chain complex (F_*,∂_*) of *free* $\mathbb{Z}[\pi]$-left modules together with a quasi-isomorphism $\varphi\colon F_* \to \mathrm{C}_*^{\mathrm{int}}(\widetilde{X},\widetilde{A})$ of $\mathbb{Z}[\pi]$-left modules. Now we consider the following little diagram:

$$\begin{array}{ccccc} & & F_* & & \\ & \swarrow^{\iota\circ\varphi} & \downarrow{\scriptstyle \varphi} & \searrow^{\pi\circ\varphi} & \\ \mathrm{C}_*(\widetilde{X},\widetilde{A}) & \xleftarrow{\ \iota\ } & \mathrm{C}_*^{\mathrm{int}}(\widetilde{X},\widetilde{A}) & \xrightarrow{\ \pi\ } & \mathrm{C}_*^{\mathrm{CW}}(\widetilde{X},\widetilde{A}). \end{array}$$

Note that the action of π on $\widetilde{X}$ turns all three chain complexes at the bottom into chain complexes of $\mathbb{Z}[\pi]$-left modules and the naturality of ι and π implies that these maps commute with the π-action, in other words, the horizontal maps ι and π are homomorphisms of $\mathbb{Z}[\pi]$-left modules. It follows from the above together with Proposition 170.3 that $i\circ\varphi\colon F_* \to \mathrm{C}_*(\widetilde{X},\widetilde{A})$ and $\pi\circ\varphi\colon F_* \to \mathrm{C}_*^{\mathrm{CW}}(\widetilde{X},\widetilde{A})$ are quasi-isomorphisms between chain complexes of free $\mathbb{Z}[\pi]$-left modules. Thus it follows from the Quasi Isomorphisms-are-Homotopy Equivalences Proposition 166.6 that $i\circ\varphi$ and $\pi\circ\varphi$ are chain homotopy equivalences.[2089] Next we consider the induced map

$$\begin{array}{ccccc} & & \mathrm{H}_k(M \otimes_{\mathbb{Z}[\pi]} F_*) & & \\ & \swarrow^{(\mathrm{id}\otimes(\iota\circ\varphi))_*} & \downarrow{\scriptstyle (\mathrm{id}\otimes\varphi)_*} & \searrow^{(\mathrm{id}\otimes(\pi\circ\varphi))_*} & \\ \underbrace{\mathrm{H}_k(M \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X},\widetilde{A}))}_{\mathrm{H}_k^{\mathrm{tw}}(X,A;M)} & \xleftarrow{(\mathrm{id}\otimes\iota)_*} & \mathrm{H}_k(M \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*^{\mathrm{int}}(\widetilde{X},\widetilde{A})) & \xrightarrow{(\mathrm{id}\otimes\pi)_*} & \underbrace{\mathrm{H}_k(M \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*^{\mathrm{CW}}(\widetilde{X},\widetilde{A}))}_{\mathrm{H}_k^{\mathrm{tw,CW}}(X,A;M)}. \end{array}$$

[2089] By Proposition 170.3 the chain maps $\iota\colon \mathrm{C}_*^{\mathrm{int}}(\widetilde{X},\widetilde{A}) \to \mathrm{C}_*(\widetilde{X},\widetilde{A})$ and $\pi\colon \mathrm{C}_*^{\mathrm{int}}(\widetilde{X},\widetilde{A}) \to \mathrm{C}_*^{\mathrm{CW}}(\widetilde{X},\widetilde{A})$ are quasi-isomorphisms. But there is no reason to believe that $\mathrm{C}_n^{\mathrm{int}}(\widetilde{X},\widetilde{A})$ is a *free* (or even just projective) $\mathbb{Z}[\pi]$-left module. Thus there is no reason why these chain maps should be chain homotopy equivalences of $\mathbb{Z}[\pi]$-left chain complexes. The case $\pi = \{e\}$ is much easier, since by the Subgroup-of-Free Abelian Group Lemma 51.2 we know that any subgroup of a free abelian group is again free abelian.

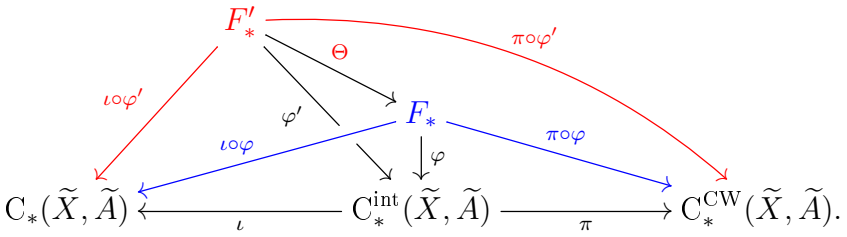
Since $\iota \circ \varphi$ and $\pi \circ \varphi$ are both chain homotopy equivalence we obtain from Lemmas 166.3 and 166.2 that the diagonal maps are isomorphisms of R-left modules. Therefore we can now define

$$\Phi_{(X,A)} := (\mathrm{id} \otimes (\pi \circ \varphi))_* \circ (\mathrm{id} \otimes (\iota \circ \varphi))_*^{-1} \colon \mathrm{H}_k^{\mathrm{tw}}(X, A; M) \ \to \ \mathrm{H}_k^{\mathrm{tw,CW}}(X, A; M).$$

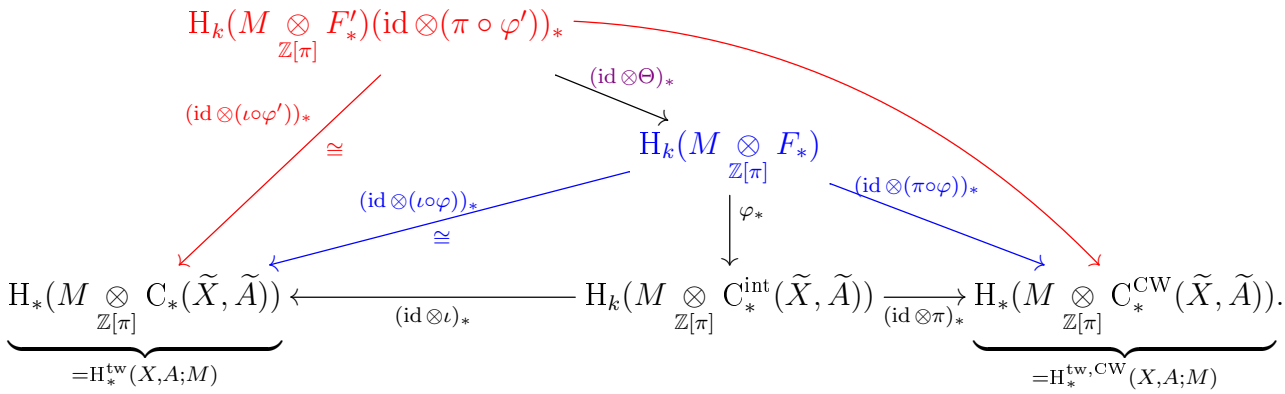
As we just discussed, the above diagonal maps are isomorphisms of R-left modules. Thus we see that $\Phi_{(X,A)} \colon \mathrm{H}_k^{\mathrm{tw}}(X, A; M) \to \mathrm{H}_k^{\mathrm{tw,CW}}(X, A; M)$ is indeed an isomorphism of R-left modules. ■

Proof of Theorem 170.2 (1) – Well-definedness. We now want to show that the isomorphism we just constructed does not depend on the choice of the free imitation (F_*, φ_*) of $\mathrm{C}_*^{\mathrm{int}}(\widetilde{X}, \widetilde{A})$.

Thus let (F'_*, φ'_*) be another free imitation of $\mathrm{C}_*^{\mathrm{int}}(\widetilde{X}, \widetilde{A})$. It follows from the Homomorphism Lifting Lemma 166.9 that there exists a chain map $\Theta \colon F_* \to F'_*$ with $\varphi = \Theta \circ \varphi'$. We now consider the following diagram:



Note that the diagram commutes by construction and the choice of Θ. Tensoring with M we obtain the following commutative diagram:



Since the diagram commutes we see that (F_*, φ) and (F'_*, φ') lead to the same definition of the isomorphism $\Phi_{(X,A)} \colon \ = \mathrm{H}_*^{\mathrm{tw}}(X, A; M) \to = \mathrm{H}_*^{\mathrm{tw,CW}}(X, A; M)$. ■

Proof of Theorem 170.2 – Naturality. To unburden the notation from unnecessary baggage we will prove naturality only for absolute twisted homology. The argument for the relative case is basically the same, one just needs to carry a little bit more data around.

Thus let $f \colon X \to Y$ be a cellular map between two connected CW-complexes. Let $x_0 \in X$, let R be a ring, and let M be a $(\mathbb{Z}[\pi_1(Y, f(x_0))], R)$-left left module. We write $\pi_X := \pi_1(X, x_0)$ and $\pi_Y := \pi_1(Y, f(x_0))$. We denote by $p \colon \widetilde{X} \to X$ and $q \colon \widetilde{Y} \to Y$ the natural coverings corresponding to the base points x_0 and $f(x_0)$. Furthermore we denote by

$\widetilde{x}_0 \in \widetilde{X}$ and $\widetilde{y}_0 \in \widetilde{Y}$ the natural base points. We denote by $\widetilde{f}\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0) \to (\widetilde{Y}, \widetilde{y}_0)$ the induced map, as defined on page 3541.

From the Free Imitation Existence Lemma 166.8 we obtain the following:

(1) A chain complex (F_{X*}, ∂_*) of free $\mathbb{Z}[\pi_X]$-left modules together with a quasi-isomorphism $\varphi_X\colon F_{X*} \to \mathrm{C}_*^{\mathrm{int}}(\widetilde{X})$ of $\mathbb{Z}[\pi_X]$-left chain complexes.
(2) A chain complex (F_{Y*}, ∂_*) of free $\mathbb{Z}[\pi_Y]$-left modules together with a quasi-isomorphism $\varphi_Y\colon F_{Y*} \to \mathrm{C}_*^{\mathrm{int}}(\widetilde{Y})$ of $\mathbb{Z}[\pi_Y]$-left chain complexes.

Claim 1. There exists a chain map

$$\Theta\colon \mathbb{Z}[\pi_Y] \otimes_{\mathbb{Z}[\pi_X]} F_X \;\to\; F_Y$$

of $\mathbb{Z}[\pi_Y]$-left chain complexes such that the following diagram commutes:

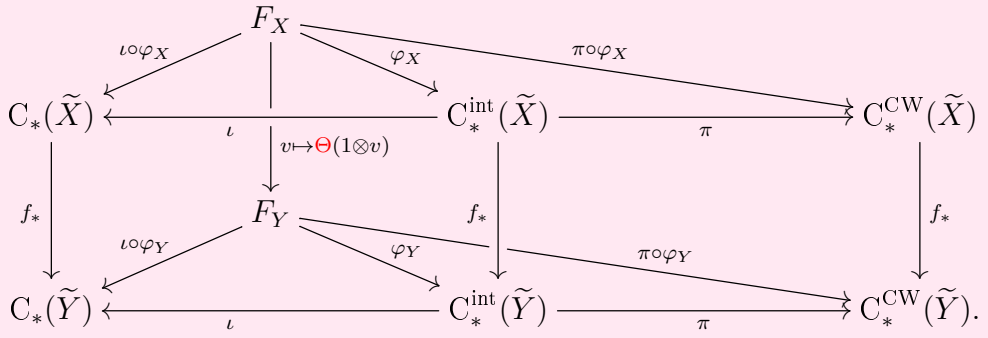
$$\begin{array}{ccc} \mathbb{Z}[\pi_Y] \otimes_{\mathbb{Z}[\pi_X]} F_X & \xrightarrow{\mathrm{id}\otimes\varphi_X} & \mathbb{Z}[\pi_Y] \otimes_{\mathbb{Z}[\pi_X]} \mathrm{C}_*^{\mathrm{int}}(\widetilde{X}) \\ \downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle g\otimes\sigma\mapsto g\cdot(f_*\sigma)} \\ F_Y & \xrightarrow{\varphi_Y} & \mathrm{C}_*^{\mathrm{int}}(\widetilde{Y}). \end{array}$$

Proof. We make the following three observations:

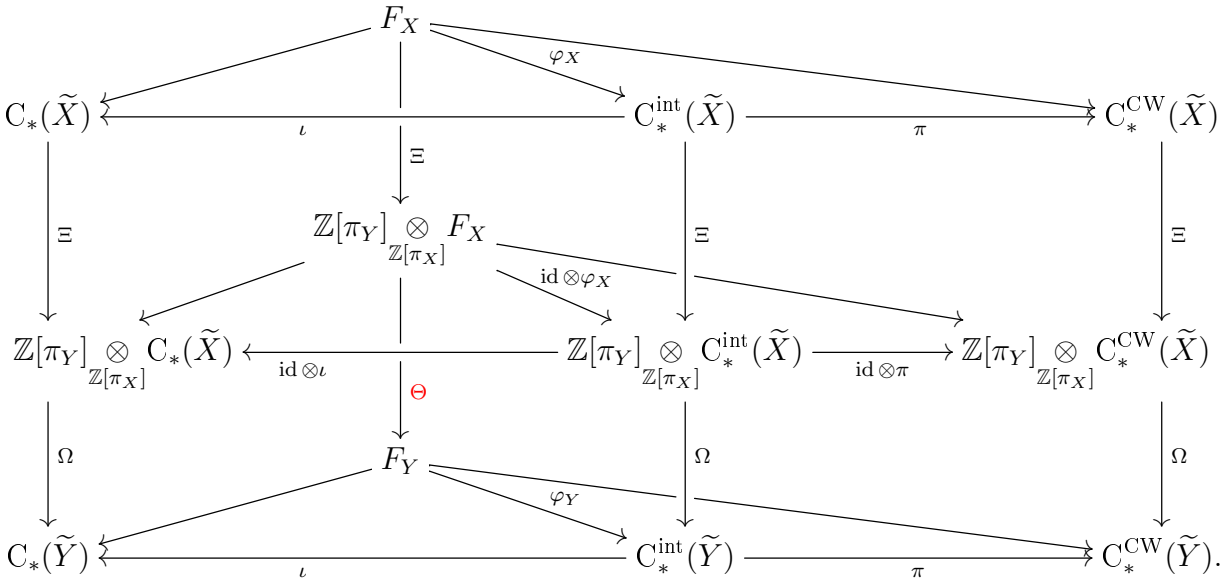
- It follows easily from the Chain Complex-Equivariance Proposition 167.8 that the right vertical map is a chain map of $\mathbb{Z}[\pi_Y]$-left chain complexes.
- By Lemma 165.21 we know that $\mathbb{Z}[\pi_Y] \otimes_{\mathbb{Z}[\pi_X]} F_X$ is a chain complex of *free* $\mathbb{Z}[\pi_Y]$-left modules.
- By construction we know that $\varphi_Y\colon F_{Y*} \to \mathrm{C}_*^{\mathrm{int}}(\widetilde{Y})$ is a quasi-isomorphism.

It follows from these observations together with the Homomorphism Lifting Lemma 166.9 that there exists a chain map $\Theta\colon \mathbb{Z}[\pi_Y] \otimes_{\mathbb{Z}[\pi_X]} F_X \to F_Y$ with the desired properties. ⊞

Claim 2. The following diagram commutes:



Proof. We consider the following even more excessive diagram:



We make the following clarifications:

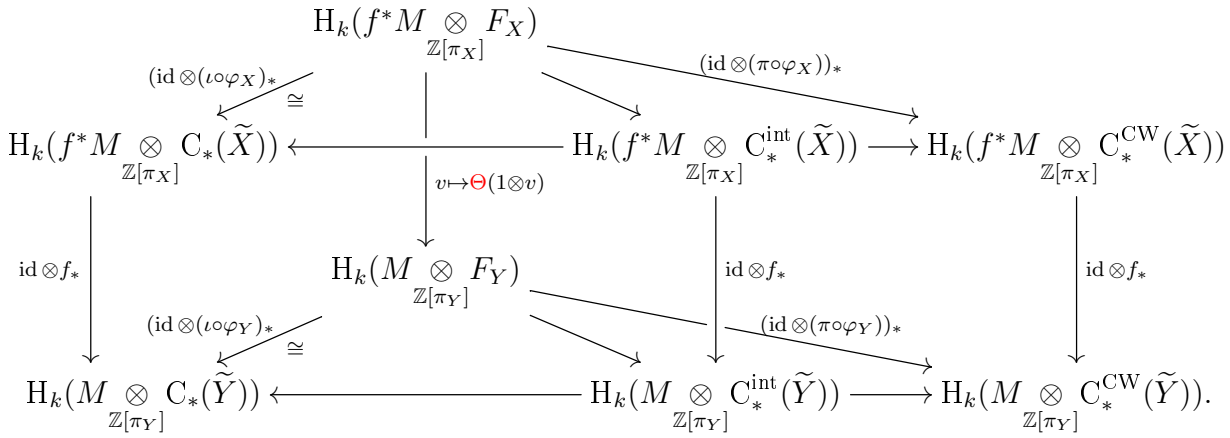
- The undecorated diagonal maps are, as before, given by the composition of the corresponding two other maps. For example the top left diagonal map is $\iota \circ \varphi_X$.
- The curved maps $\widetilde{f}_*$ are induced by the cellular map $\widetilde{f} \colon \widetilde{X} \to \widetilde{Y}$.
- The top vertical maps Ξ are the purely algebraic maps given by $\sigma \mapsto 1 \otimes \sigma$.
- The bottom vertical maps Ω are all given by $g \otimes \sigma \mapsto g \cdot f_*(\sigma)$.

Now we want to argue that the diagram commutes:

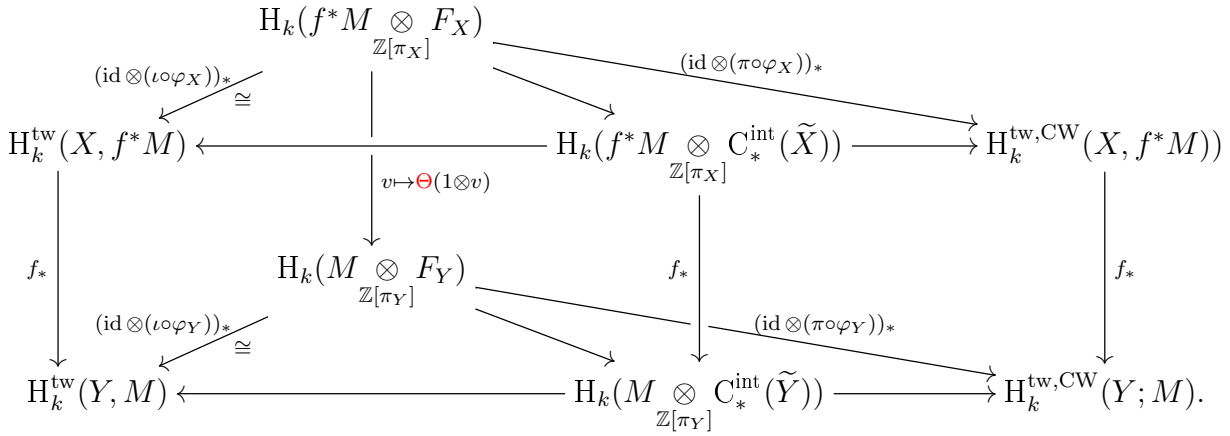
- The top part of the diagram commutes basically by definition.
- It follows immediately from the definitions that the two bottom rectangles commute.
- The bottom center parallelogram involving Θ commutes by the choice of Θ. The two other bottom parallelograms commute since they are defined in terms of the other maps.

In summary we see that the big diagram commutes. Finally note that the composition $\Omega \circ \Xi$ in each case equals, basically by definition, the map f_*. This final observation implies that the commutativity of the big diagram gives us also the commutativity of the diagram given in the claim. ⊞

Next note that it follows easily from the claim that the following diagram also commutes:



In other words, the following diagram commutes:



By definition of Φ_X and Φ_Y this just means that the following diagram does indeed commute, fulfilling our deepest wishes:

$$\begin{array}{ccc} \mathrm{H}^{\mathrm{tw}}_k(X; f^*M) & \xrightarrow{\Phi_X} & \mathrm{H}^{\mathrm{tw,CW}}_k(X; f^*M) \\ \downarrow{\scriptstyle f_*} & & {\scriptstyle f_*}\downarrow \\ \mathrm{H}^{\mathrm{tw}}_k(Y; M) & \xrightarrow{\Phi_Y} & \mathrm{H}^{\mathrm{tw,CW}}_k(Y; M). \end{array}$$

■

It remains prove the statement of Theorem 170.2 regarding twisted cohomology.

Proof of Theorem 170.2 (2). The proof of the statement regarding twisted cohomology is evidently almost identical to the proof for twisted homology. All one needs to is to replace the Tensoring Chain Complex Lemma 166.3 by the Dualizing Chain Complex Lemma 166.4. We leave it to the reader to fill in the details. ■

170.3. **Twisted (co-) homology of compact CW-complexes and manifolds.** We recall the following definition from page 1519.

Definition. Given a finite CW-complex X we define its Euler characteristic $\chi(X)$ as follows:

$$\chi(X) \;:=\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \text{number of } n\text{-cells}.$$

In the Euler Characteristic-H_*-Proposition 87.1 we saw that given a finite CW-complex X we have the following relation between the Euler characteristic X and its homology groups $\mathrm{H}_n(X)$:

$$(*) \qquad \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \operatorname{rank}(\mathrm{H}_n(X)) \;=\; \chi(X).$$

Furthermore on page 1915 we saw that a similar result holds if replace the ordinary homology groups $\mathrm{H}_n(X)$ by homology groups $\mathrm{H}_n(X;\mathbb{F})$ where $\mathbb{F}$ is a field. On page 1882 we used $(*)$ to define the Euler characteristic also for suitable topological spaces that are not CW-complexes.

We can now generalize the above results to twisted homology for suitable coefficients.

Proposition 170.4. Let X be a topological space that is locally contractible and path-connected. Let $x_0 \in X$, let $\mathbb{K}$ be a skew field and let M be a $(\mathbb{Z}[\pi_1(X,x_0)],\mathbb{K})$-left left module with $\dim_{\mathbb{K}}(M) < \infty$. If X is homotopy equivalent to a finite CW-complex, then the following two equalities hold:

$$\begin{aligned}
\sum_{n\in\mathbb{N}_0} (-1)^n \cdot \dim_{\mathbb{K}}(\mathrm{H}^{\mathrm{tw}}_n(X;M)) &= \dim_{\mathbb{K}}(M)\cdot\chi(X)\\
\sum_{n\in\mathbb{N}_0} (-1)^n \cdot \dim_{\mathbb{K}}(\mathrm{H}^{n}_{\mathrm{tw}}(X;M)) &= \dim_{\mathbb{K}}(M)\cdot\chi(X).
\end{aligned}$$

Remark. It follows from the Smooth Manifold-CW Structure Theorem 96.5, or alternatively the Smooth Manifold-CW-Homotopy Type Proposition 139.8, that Proposition 170.4 applies to compact smooth manifolds. In fact by Corollary 104.25 it also applies to compact topological manifolds. But the later statement is perhaps less satisfactory since we did not even come close to proving Corollary 104.25.

Example. Let $X = S^1 \vee \cdots \vee S^1$ be the wedge of k circles, let $x_0 \in X$ and finally let $\varphi\colon \pi_1(X,x_0) \to \mathrm{GL}(1,\mathbb{C})$ be a one-dimensional non-trivial representation. We view X as a CW-complex in the obvious way. Since X has no cells of dimension ≥ 2 we obtain from the discussion on page 3587 that $\mathrm{H}^{\mathrm{tw}}_k(X;{}_\varphi\mathbb{C}) = 0$ for $k \geq 2$. Since φ is a one-dimensional non-trivial representation we obtain easily from the Twisted H_0-Proposition 168.1 that $\mathrm{H}^{\mathrm{tw}}_0(X;{}_\varphi\mathbb{C}) = 0$. Since X has Euler characteristic $1-k$ we now obtain from Proposition 170.4 that $\mathrm{H}^{\mathrm{tw}}_k(X;{}_\varphi\mathbb{C}) \cong \mathbb{C}^{k-1}$.

Proof. In the following we only prove the equality regarding homology with twisted coefficients. The proof for cohomology with twisted coefficients is very similar. We leave it to the reader to make the necessary changes.

First assume that X itself is already a finite k-dimensional CW-complex. In this case the proof of Proposition 170.4 is almost identical to the proof of the Euler Characteristic-H_*-Proposition 87.1. But for fun, let us go through the motions. We start out with the following claim which is a variation on Lemma 87.3.

Claim. Let $\quad C_* \;:=\; 0 \;\to\; \mathrm{C}_k \xrightarrow{\partial_k} \mathrm{C}_{k-1} \xrightarrow{\partial_{k-1}} \ldots\; \mathrm{C}_1 \xrightarrow{\partial_1} \mathrm{C}_0 \;\to\; 0$

be a chain complex of finite dimensional $\mathbb{K}$-left vector spaces. The following equality holds:

$$\sum_{n=0}^{k}(-1)^n \cdot \dim_{\mathbb{K}}(\mathrm{C}_n) \;=\; \sum_{n=0}^{k}(-1)^n \dim_{\mathbb{K}}(\mathrm{H}_n(C_*)).$$

Proof. The proof of the claim is identical to the proof Lemma 87.3, except that we need to replace the Rank Additivity Lemma 51.7 by Proposition 165.26 (6). ⊞

We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. We perform the following calculation:

$$\begin{array}{ccc} & \text{follows from Lemma 170.1 (2), together with Lemma 165.17 (1) and Lemma 165.21} & \\ & \downarrow & \\ \dim_{\mathbb{K}}(M)\cdot\chi(X) = \sum_{n=0}^{k}(-1)^n\cdot\dim_{\mathbb{K}}(M)\cdot\#n\text{-cells} & = & \sum_{n=0}^{k}(-1)^n\cdot\dim_{\mathbb{K}}(\overline{M}\otimes_{\mathbb{Z}[\pi_1(X,x_0)]}\mathrm{C}_n^{\mathrm{CW}}(\widetilde{X}_{x_0})) \\ = \sum_{n=0}^{k}(-1)^n\cdot\dim_{\mathbb{K}}(\mathrm{H}_n^{\mathrm{tw,CW}}(X;M)) & = & \sum_{n=0}^{k}(-1)^n\cdot\dim_{\mathbb{K}}(\mathrm{H}_n^{\mathrm{tw}}(X;M)). \\ \uparrow & & \uparrow \\ \text{by the claim} & & \text{Proposition 170.2} \end{array}$$

Now assume that X is homotopy equivalent to a finite CW-complex Y. We pick $y_0 \in Y$. By Corollary 167.17 there exists a $(\mathbb{Z}[\pi_1(Y,y_0)],\mathbb{K})$-left left module, which as a $\mathbb{K}$-left vector space is isomorphic to M, such that for all $n \in \mathbb{N}_0$ the $\mathbb{K}$-left vector spaces $\mathrm{H}_k^{\mathrm{tw}}(X;M)$ and $\mathrm{H}_k^{\mathrm{tw}}(Y;N)$ are isomorphic. Now we see that

$$\begin{array}{ccccc} & & \text{by the choice of } N & & \\ & & \downarrow & & \\ \sum_{n\in\mathbb{N}_0}(-1)^n\cdot\dim_{\mathbb{K}}(\mathrm{H}_n^{\mathrm{tw}}(X;M)) & = & \sum_{n\in\mathbb{N}_0}(-1)^n\cdot\dim_{\mathbb{K}}(\mathrm{H}_n^{\mathrm{tw}}(Y;N)) & & \\ & = & \dim_{\mathbb{K}}(N)\cdot\chi(Y) & = \dim_{\mathbb{K}}(N)\cdot\chi(X) & = \dim_{\mathbb{K}}(M)\cdot\chi(X). \\ & & \uparrow & \uparrow & \uparrow \\ & & \text{by the above since } Y & \text{by Corollary 73.9} & \text{since } M \text{ and } N \text{ are isomorphic} \\ & & \text{is a finite CW-complex} & & \mathbb{K}\text{-left vector spaces} \end{array}$$ ■

Next recall that in Propositions 96.6 and 104.14 we showed that the ordinary homology groups of compact smooth and topological manifolds are finitely generated. In the remainder of this section we will generalize these results to (co-) homology with twisted coefficients. To control the "size" of (co-) homology of topological spaces we need control over the rings involved and we need some control over the topological spaces involved. Let us start out with a definition from the realm of algebra (see [**Lang2002b**, p. 186] or [**Lam1991**, p. 20]).

Definition. Let R be a ring. If every R-left submodule of R is finitely generated, then we call R left Noetherian. If R is commutative, then we also say Noetherian instead of left Noetherian.

The following proposition shows that many rings that are close to our heart are (left) Noetherian.

Proposition 170.5.

(1) Every principal ideal domain, in particular the ring $\mathbb{Z}$, is Noetherian.
(2) Every (skew) field is left Noetherian.
(3) Let A be a commutative Noetherian ring
  (a) If $S \subset M$ is a multiplicatively closed subset, then the ring $S^{-1}A$ is Noetherian.
  (b) Every multivariable polynomial ring $A[t_1,\ldots,t_m]$ is Noetherian.
  (c) Every multivariable Laurent polynomial ring $A[t_1^{\pm 1},\ldots,t_k^{\pm 1}]$ is Noetherian.

Proof.

(1) This statement is proved in [**Lang2002b**, Corollary III.7.2].
(2) This statement can be deduced easily from Proposition 165.26.
(3) (a) This statement is [**Lang2002b**, Proposition X.1.6].
(b) If A is Noetherian, then it follow from [**Lang2002b**, Theorem IV.4.1] that $A[t]$ is Noetherian ring. Applying this principle m times we see that $A[t_1, \ldots, t_m]$ is Noetherian.
(c) This statement follows from (a) and (b), since we obtain the multivariable Laurent polynomial ring $A[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$ by inverting the multiplicatively closed subset that is given all monomials in $t_1, \ldots, t_k$. ■

The following proposition states the key property of left Noetherian rings that we are interested in.

Proposition 170.6. Let R be a left Noetherian ring. If P is a finitely generated R-left module, then every R-left submodule is also finitely generated.

Proof. This proposition is proved in [**Lam1991**, Proposition 1.21]. The commutative case is also proved in [**Lang2002b**, Proposition X.1.1]. ■

The following proposition explains why we are suddenly interested in left Noetherian rings.

Proposition 170.7. Let X be a *compact* path-connected space, let $x_0 \in X$, let R be a left Noetherian ring and let M be a $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left module. We suppose that one of the following topological conditions is satisfied:

(T1) X is a CW-complex.
(T2) X is a smooth manifold.
(T3) X is a topological manifold.

Then for any $k \in \mathbb{N}_0$ the twisted (co-) homology modules $\mathrm{H}_k(X; M)$ and $\mathrm{H}^k_{\mathrm{tw}}(X; M)$ are finitely generated R-left modules.

Proof. We write $\pi := \pi_1(X, x_0)$ and we write $\widetilde{X} := \widetilde{X}_{x_0}$.

(T1) First we consider the case that X is a CW-complex. We see that

$$\mathrm{H}^{\mathrm{tw}}_k(X; M) \underset{\text{follows from Theorem 170.2}}{\underset{\uparrow}{=}} \mathrm{H}^{\mathrm{tw,CW}}_k(X; M) = \mathrm{H}_k(\overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}^{\mathrm{CW}}_*(\widetilde{X}))$$

By Lemma 170.1 (1) we know that each $\mathrm{C}^{\mathrm{CW}}_k(\widetilde{X})$ is a free $\mathbb{Z}[\pi]$-left module. Since X is assumed to be compact we obtain in fact from Lemma 170.1 (3) that each $\mathrm{C}^{\mathrm{CW}}_k(\widetilde{X})$ is a finitely generated free $\mathbb{Z}[\pi]$-left module. Since M is a finitely generated R-left module we see that each $\overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}^{\mathrm{CW}}_*(\widetilde{X})$ is a finitely generated R-left module.[2090] It now follows from Proposition 170.6 that each

$$\ker\left(\mathrm{id} \otimes \partial_k \colon \overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}^{\mathrm{CW}}_k(\widetilde{X}) \to \overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}^{\mathrm{CW}}_{k-1}(\widetilde{X})\right)$$

is also a finitely generated R-left module. Thus we see that $\mathrm{H}^{\mathrm{tw}}_k(X; M)$ is a quotient of finitely generated R-left module, which implies of course that $\mathrm{H}^{\mathrm{tw}}_k(X; M)$ is also

[2090] Indeed, if M is generated by $a_1, \ldots, a_r$ and if $\mathrm{C}^{\mathrm{CW}}_k(\widetilde{X})$ is generated by $b_1, \ldots, b_s$, then one can easily verify that the R-left module $\overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}^{\mathrm{CW}}_*(\widetilde{X})$ is generated by $a_i \otimes b_j$.

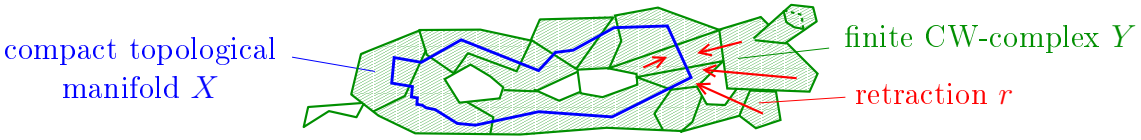
a finitely generated R-left module. A very similar argument also proves the corresponding statement for the twisted cohomology modules $\mathrm{H}^k_{\mathrm{tw}}(X;M)$. We leave it to the reader to make the necessary changes.

(T2) If X is a smooth manifold, then we know by the Smooth Manifold-CW Structure Theorem 96.5 that X admits a CW-structure and we are done by (T1).

(T3) Finally we suppose that X is a compact topological manifold. By the Topological Manifold-Simplicial Complex Retract Theorem 104.13 we know that X is a retract of a finite CW-complex. More precisely, there exists a finite CW-complex Y that contains X together with a retraction $r\colon Y \to X$. We now see that

$$\mathrm{H}^{\mathrm{tw}}_k(X;M) \underset{\text{since } r\circ i = \mathrm{id}_X}{\overset{\uparrow}{=}} (r\circ i)_*(\mathrm{H}^{\mathrm{tw}}_k(X;(r\circ i)_*M)) \underset{\text{by functoriality, see the Twisted Homology-Functoriality Proposition 167.9}}{\overset{\uparrow}{\subset}} r_*(\mathrm{H}^{\mathrm{tw}}_k(Y;r_*M)).$$

Thus we see that $\mathrm{H}^{\mathrm{tw}}_k(X;M)$ is isomorphic to a quotient of the finitely generated R-left module $\mathrm{H}^{\mathrm{tw}}_k(Y;r_*M)$. This implies that $\mathrm{H}^{\mathrm{tw}}_k(X;M)$ itself is a finitely generated R-left module.[2091] ■



Exercises for Chapter 170.

Exercise 170.1. Let X be a path-connected finite CW-complex and let $x_0 \in X$. We consider the product topological space $X \times S^1$. We denote by $p\colon X \times S^1 \to S^1$ the projection. Furthermore let $\Phi\colon \pi_1(S^1,1) \to \mathbb{Z}$ be the isomorphism given by Proposition 48.17. Given $z \in S^1$ we consider the representation

$$\rho_z\colon \pi_1(X \times S^1,(x_0,1)) \xrightarrow{p_*} \pi_1(S^1,1) \xrightarrow{\Phi} \mathbb{Z} \xrightarrow{n\mapsto z^n} \mathrm{GL}(1,\mathbb{C}).$$

Show that if $z \neq 1$, then for all $k \in \mathbb{N}_0$ we have $\mathrm{H}^{\mathrm{tw}}_k(X \times S^1;{}_{\rho_z}\mathbb{C}) = 0$.
Hint. Consider the product CW-structure on $X \times S^1$ as defined on page 1498.

Exercise 170.2. Let X be a path-connected finite CW-complex and let $x_0 \in X$ be a point in the 0-skeleton. We consider the wedge product topological space $X \vee S^1$ that is given by identifying $x_0 \in X$ with $(1,0) \in S^1$. We denote by $p\colon X \vee S^1 \to S^1$ the natural map. Furthermore let $\Phi\colon \pi_1(S^1,1) \to \mathbb{Z}$ be the isomorphism given by Proposition 48.17. Given $z \in S^1$ we consider the representation

$$\rho_z\colon \pi_1(X \vee S^1,(x_0,1)) \xrightarrow{p_*} \pi_1(S^1,1) \xrightarrow{\Phi} \mathbb{Z} \xrightarrow{n\mapsto z^n} \mathrm{GL}(1,\mathbb{C}).$$

Determine the twisted homology groups $\mathrm{H}^{\mathrm{tw}}_k(X \vee S^1;{}_{\rho_z}\mathbb{C})$ in terms of k, z and the usual homology groups of X.

[2091] In Corollary 104.25 we mentioned that every compact topological manifold is homotopy equivalent to a finite CW-complex. This would also have allowed us to prove the desired statement. But we try to avoid using Corollary 104.25 as much as possible, since we did not provide a proof .

Hint. Using the CW-Complex Construction Lemma 68.32 you can equip $X \vee S^1$ with a CW-structure.

Exercise 170.3. Let $g \in \mathbb{N}$ and let Σ_g the surface of genus g. We pick $x_0 \in \Sigma_g$. Let $\alpha\colon \pi_1(\Sigma_g, x_0) \to \mathrm{GL}(1,\mathbb{C})$ be a non-trivial one-dimensional representation. Determine the twisted homology groups $\mathrm{H}_k^{\mathrm{tw}}(\Sigma_g; {}_\alpha\mathbb{C})$.
Remark. The way the exercise is formulated already indicates, that the answer does not depend on the precise choice of the non-trivial one-dimensional representation.

Exercise 170.4. Wedge of two circles.

Exercise 170.5. Let

$$K \;=\; ([0,1]\times[0,1])/\sim \qquad \text{where } (x,0)\sim(x,1) \text{ and } (0,y)\sim(1,1-y)$$

be the Klein bottle as defined on page 265. We consider the two singular 1-cycles

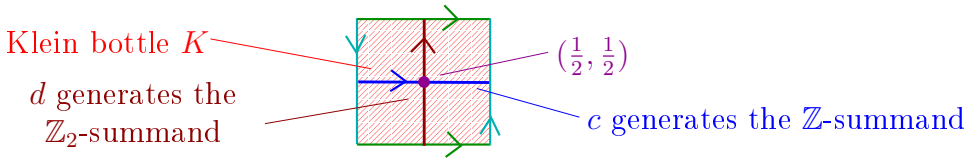
$$\begin{array}{rcl} c\colon \Delta^1 & \to & K \\ (t,1-t) & \mapsto & [(\frac{1}{2},t)] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} d\colon \Delta^1 & \to & K \\ (t,1-t) & \mapsto & [(\frac{1}{2},t)] \end{array}$$

In Lemma 78.12 we saw that $\mathrm{H}_1(K) = \mathbb{Z}\cdot[c] \oplus \mathbb{Z}_2\cdot[d]$. Let

$$\alpha\colon \pi_1(K;[(\tfrac{1}{2},\tfrac{1}{2})]) \;\underset{\uparrow}{\to}\; \mathrm{H}_1(K) \;\to\; \mathrm{GL}(1,\mathbb{C})$$

Hurewicz homomorphism as defined on page 1832

be a 1-dimensional representation. Compute the twisted homology groups $\mathrm{H}_i^{\mathrm{tw}}(K; {}_\alpha\mathbb{C})$ in terms of $\alpha([c])$ and $\alpha([d])$.



Exercise 170.6. Let X be a topological space that is locally contractible and path-connected, let $x_0 \in X$ and let $\varphi\colon \pi_1(X,x_0) \to \langle t\rangle$ be an epimorphism.

(a) Show that for each commutative ring S there exists a long exact sequence of the form

$$\ldots \to \mathrm{H}_{k+1}(X;S) \to \mathrm{H}_k^{\mathrm{tw}}(X;S[t^{\pm1}]) \xrightarrow{(t-1)\cdot} \mathrm{H}_k^{\mathrm{tw}}(X;S[t^{\pm1}]) \to \mathrm{H}_k(X;S) \to \ldots$$

(b) Suppose that X is finite CW-complex and with $\mathrm{H}_*(X;\mathbb{Q}) \cong \mathrm{H}_*(S^1;\mathbb{Q})$. Show that each $\mathbb{Q}[t^{\pm1}]$-module $\mathrm{H}_k^{\mathrm{tw}}(X;\mathbb{Q}[t^{\pm1}])$ is a torsion $\mathbb{Q}[t^{\pm1}]$-module.

Exercise 170.7. Let R be a left Noetherian ring.

(a) Show that if $0 \to A \to B \to C \to 0$ is a short exact sequence of R-left modules and if two are finitely generated, then so is the third.

(b) Let X be a compact path-connected topological manifold and let A be a union of boundary components. Furthermore let $x_0 \in X$ and let M be a $(\mathbb{Z}[\pi_1(X,x_0)], R)$-left left module that is finitely generated as an R-left module. Show that the twisted R-left modules $\mathrm{H}_k^{\mathrm{tw}}(X,A;M)$ are finitely generated.

Exercise 170.8. We consider the topological space $X = S^1$ with the base point $1 \in S^1$. Let R be a ring.

(a) Show that for any $(\mathbb{Z}[\pi_1(S^1, 1)], R)$-left left module and any $i \in \{0, 1\}$ we have an isomorphism

$$\mathrm{H}^i_{\mathrm{tw}}(S^1; M) \ \to \ \mathrm{H}^{\mathrm{tw}}_{1-i}(S^1; M)$$

of R-left modules.

Remark. This slightly mystifying result foreshadows the Poincaré Duality Theorem 172.1 with Twisted Coefficients.

(b) Show that in general $\mathrm{H}^1_{\mathrm{tw}}(S^1; M)$ and $\mathrm{H}^{\mathrm{tw}}_1(S^1; M)$ are not isomorphic.

171. The cup and cap-product on twisted (co-) homology

In this section we will introduce cup- and cap-products on twisted (co-) homology. Recall that, in its simplest form the cup- and cap-product for a topological space X together with a commutative ring R were defined as maps

$$\mathrm{H}^k(X;R) \times \mathrm{H}^l(X;R) \ \to \ \mathrm{H}^{k+l}(X;R) \qquad \text{and} \qquad \mathrm{H}^k(X;R) \times \mathrm{H}_l(X;R) \ \to \ \mathrm{H}_{l-k}(X;R).$$

If we start out with (co-) homology with twisted coefficients it is a priori not entirely clear what coefficients the target should have. To answer this questions we first need to introduce a little more algebra which we will do in our first section.

171.1. Algebraic preparations.

Lemma 171.1. Let R and S be rings. The abelian group $R \otimes_{\mathbb{Z}} S$ together with the multiplication that is defined by

$$\Big(\sum_{i=1}^{k} r_i \otimes s_i\Big) \cdot \Big(\sum_{j=1}^{l} \widetilde{r}_j \otimes \widetilde{s}_j\Big) \ := \ \sum_{i=1}^{k}\sum_{j=1}^{l} r_i \cdot \widetilde{r}_j \otimes s_i \cdot \widetilde{s}_j$$

is a ring. If R and S are commutative, then so $R \otimes_{\mathbb{Z}} S$.

Convention. Let R and S be rings. We will always use Lemma 171.1 to view $R \otimes_{\mathbb{Z}} S$ as a ring.

Proof. One can easily verify that the multiplication on $R \otimes_{\mathbb{Z}} S$ is actually well-defined and that it satisfies all the properties of a ring, as laid out in the definition on page 98. For example a multiplicatively neutral element is given by $1_R \otimes 1_S$. Furthermore it is clear that if R and S are commutative, then so is $R \otimes_{\mathbb{Z}} S$. We leave it to the reader to fill in the details. ■

Lemma 171.2. Let R and S be rings and let G be a group.

(1) If M is an R-left module and N is an S-left module, then

$$\Big(\sum_{i=1}^{k} r_i \otimes s_i\Big) \cdot \Big(\sum_{j=1}^{l} m_j \otimes n_j\Big) \ := \ \sum_{i=1}^{k}\sum_{j=1}^{l} r_i \cdot m_j \otimes s_i \cdot n_j$$

defines an $R \otimes_{\mathbb{Z}} S$-left module structure on $M \otimes_{\mathbb{Z}} N$.

(2) If M is a $\mathbb{Z}[G]$-left module and N is a $\mathbb{Z}[G]$-left module, then

$$g \cdot (m \otimes n) \ := \ g \cdot m \otimes g \cdot n$$

defines a $\mathbb{Z}[G]$-left module structure on $M \otimes_{\mathbb{Z}} N$.

(3) If M is a $(\mathbb{Z}[G], R)$-left module and N is a $(\mathbb{Z}[G], S)$-left module, then the module structures given in (1) and (2) define a $(\mathbb{Z}[G], R \otimes_{\mathbb{Z}} S)$-left module structure on $M \otimes_{\mathbb{Z}} N$.

Proof. We leave the elementary proof to the reader. ■

Convention. Let R and S be rings, let G be a group, let M be a $(\mathbb{Z}[G], R)$-left module and let N be a $(\mathbb{Z}[G], S)$-left module. We will always equip $M \otimes_{\mathbb{Z}} N$ with the $(\mathbb{Z}[G], R \otimes_{\mathbb{Z}} S)$-left module structure that we defined in Lemma 171.2.

171.2. **The cup product on twisted cohomology.** First we recall the following definition from page 2503.

Notation. For $i = 0, \dots, n$ we write

$$v_i \ := \ (0, \dots, 1, 0, \dots, 0) \ \in \ \mathbb{R}^{n+1}.$$

Furthermore for $a_0, \dots, a_k \in \Delta^n$ we consider the map

$$\begin{aligned} [a_0, \dots, a_k] \colon \Delta^k \ &\to \ \Delta^n \\ \sum_{j=0}^{k} \lambda_j \cdot v_j \ &\mapsto \ \sum_{j=0}^{k} \lambda_j \cdot a_j. \end{aligned}$$

We now adapt the definition of the cup product on singular cohomology, which we gave on pages 2513 and 3147, to the twisted setting.

Definition. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. We denote by $p \colon \widetilde{X}_{x_0} \to X$ the natural universal covering. Furthermore let M and N be $\mathbb{Z}[\pi_1(X, x_0)]$-left modules and let $p, q \in \mathbb{N}_0$. Finally let $\varphi \in \mathrm{C}^p_{\mathrm{tw}}(X; M)$ and $\psi \in \mathrm{C}^q_{\mathrm{tw}}(X; N)$ be cochains. We define the cup product of φ and ψ to be the map

$$\begin{aligned} \varphi \cup_{\otimes} \psi \colon \mathrm{C}_{p+q}(\widetilde{X}_{x_0}) \ &\to \ M \otimes_{\mathbb{Z}} N \\ (\sigma \colon \Delta^{p+q} \to \widetilde{X}_{x_0}) \ &\mapsto \ \underbrace{\varphi(\sigma \circ [v_0, \dots, v_p])}_{\in M} \otimes \underbrace{\psi(\sigma \circ [v_p, \dots, v_{p+q}])}_{\in N}. \end{aligned}$$

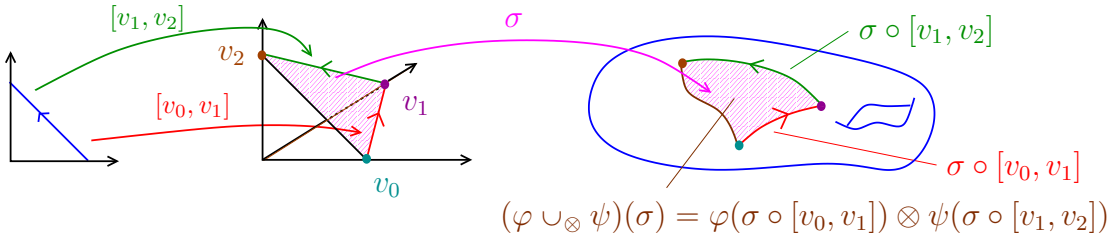
that is given by

This map is evidently a $\mathbb{Z}[\pi_1(X, x_0)]$-left module homomorphism, thus $\varphi \cup_{\otimes} \psi$ is an element in $\mathrm{C}^{p+q}_{\mathrm{tw}}(X; M \otimes_{\mathbb{Z}} N)$. In other words, the cup product defines a map

$$\cup_{\otimes} \colon \mathrm{C}^p_{\mathrm{tw}}(X; M) \times \mathrm{C}^q_{\mathrm{tw}}(X; N) \ \to \ \mathrm{C}^{p+q}_{\mathrm{tw}}(X; M \otimes_{\mathbb{Z}} N).$$

In Lemma 171.3 we will show that this map descends to a well-defined map

$$\cup_{\otimes} \colon \mathrm{H}^p_{\mathrm{tw}}(X; M) \times \mathrm{H}^q_{\mathrm{tw}}(X; N) \ \to \ \mathrm{H}^{p+q}_{\mathrm{tw}}(X; M \otimes_{\mathbb{Z}} N).$$



Lemma 171.3. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. Furthermore let M and N be $\mathbb{Z}[\pi_1(X, x_0)]$-left modules and let $p, q \in \mathbb{N}_0$.

(1) Given any cochains $\varphi \in \mathrm{C}^p_{\mathrm{tw}}(X; M)$ and $\psi \in \mathrm{C}^q_{\mathrm{tw}}(X; N)$ we have the equality

$$\delta(\varphi \cup_{\otimes} \psi) \ = \ \delta\varphi \cup_{\otimes} \psi \ + (-1)^p \cdot \varphi \cup_{\otimes} \delta\psi \ \in \ \mathrm{C}^{p+q+1}(X; M \otimes_{\mathbb{Z}} N).$$

(2) The cup product $\cup_{\otimes}$ on cochains descends to a well-defined map on cohomology with twisted coefficients.

Proof.

(1) The proof of this statement is basically the same as the proof of Coboundary-Cup Product Lemma 114.2. Throughout that long and unpleasant calculation we just need to replace multiplication "$\cdot$" in R by the tensor product $\otimes$.

(2) This statement follows, as we saw already in the corresponding Lemma 114.3, easily from Statement (1). ■

We now turn to the definition of the relative cup product with twisted coefficients, which will generalize the relative cup product on ordinary singular cohomology that we introduced in Chapter 115. As the reader surely remembers, even that definition already took some effort. In the following we will go through the key steps again and make appropriate modifications where necessary.

Definition. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. Let (X, A, B) be a triad of topological spaces. We write $\widetilde{A} := p^{-1}(A)$ and $\widetilde{B} := p^{-1}(B)$. As on page 3571 we write

$$\mathrm{C}_n^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B}) := \left\{ \sum_{j=1}^{k} n_j \cdot \sigma_j \in \mathrm{C}_n(\widetilde{A} \cup \widetilde{B}) \,\middle|\, \begin{array}{c} \text{for each } j \text{ the image of } \sigma_j \\ \text{lies in } \widetilde{A} \text{ or it lies in } \widetilde{B} \end{array} \right\}$$

Given a ring R and given a $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left module M we set

$$\mathrm{C}^n_{\mathrm{tw}}(X, \{A, B\}; M) := \ker\Big(\mathrm{C}^n_{\mathrm{tw}}(X; M) \underset{\uparrow}{\to} \mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]\text{-left}}\big(\mathrm{C}_n^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B}), M\big)\Big).$$

we restrict a homomorphism $\mathrm{C}_n(\widetilde{X}_{x_0}) \to M$ to the submodule $\mathrm{C}_n^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B})$

This is again a cochain complex of R-left modules and we denote the corresponding cohomology groups by $\mathrm{H}^*_{\mathrm{tw}}(X, \{A, B\}; M)$.

Before we continue we also need to introduce the following convention which is a slight variation on a convention that we had introduced on page 2534.

Convention. Let R be a ring. Given an R-left module C, an R-left submodule D and an R-left module M we have a natural isomorphism

$$\begin{array}{rcl} \mathrm{Hom}_{R\text{-left}}(C/D, M) & \xrightarrow{\cong} & \{f\colon C \to G \,|\, f \text{ vanishes on } D\} \\ (\varphi\colon C/D \to G) & \mapsto & (C \to C/D \xrightarrow{\varphi} G). \end{array}$$

We will use this isomorphism to identify the groups left and right.

Lemma 171.4. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. Let R be a ring and let W be a $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left module. Furthermore let $A, B \subset X$ such that (X, A, B) is a universally excisive triad of topological spaces. By the above convention we have R-left module homomorphisms

$$\mathrm{C}^*_{\mathrm{tw}}(X, A \cup B; W) \to \mathrm{C}^*_{\mathrm{tw}}(X, \{A, B\}; W).$$

These maps induce for every $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}^n_{\mathrm{tw}}(X, A \cup B; W) \xrightarrow{\cong} \mathrm{H}^n_{\mathrm{tw}}(X, \{A, B\}; W).$$

Proof. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. We write $\widetilde{A} := p^{-1}(A)$ and $\widetilde{B} := p^{-1}(B)$. We have the following commutative diagram of short exact sequences of

chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{C}_*^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A}\cup\widetilde{B}) & \longrightarrow & \mathrm{C}_*(\widetilde{X}_{x_0}) & \longrightarrow & \dfrac{\mathrm{C}_*(\widetilde{X}_{x_0})}{\mathrm{C}_*^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A}\cup\widetilde{B})} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle =\mathrm{id}} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{C}_*(\widetilde{A}\cup\widetilde{B}) & \longrightarrow & \mathrm{C}_*(\widetilde{X}_{x_0}) & \longrightarrow & \mathrm{C}_*(\widetilde{X}_{x_0},\widetilde{A}\cup\widetilde{B}) & \longrightarrow & 0 \end{array}$$

of $\mathbb{Z}[\pi_1(X,x_0)]$-left modules. It follows immediately from the Regular Covering-Chain Complex Lemma 167.2 that all but possibly the top right module are free $\mathbb{Z}[\pi_1(X,x_0)]$-left modules. But the argument of the Regular Covering-Chain Complex Lemma 167.2 shows that the top right module is also free $\mathbb{Z}[\pi_1(X,x_0)]$-left module, for example a basis is given by taking for each singular simplex in X, that is neither contained in A nor in B, a lift to $\widetilde{X}_{x_0}$. Next note that by our hypothesis the triad (X,A,B) is universally excisive. By definition this means that the left vertical map is a chain homotopy equivalence. The middle vertical map is the identity. It now follows from Corollary 166.7 that the right vertical map is also a chain homotopy equivalence of free $\mathbb{Z}[\pi_1(X,x_0)]$-left modules.

Note that by the above convention we have an identification

$$\mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]\text{-left}}\left(\frac{\mathrm{C}_n(\widetilde{X}_{x_0})}{\mathrm{C}_n^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A}\cup\widetilde{B})},W\right) \;=\; \mathrm{C}^n_{\mathrm{tw}}(X,\{A,B\};W).$$

The lemma follows from the above vertical chain homotopy equivalence, the above observation and the Dual Cochain Map Lemma 108.8. ■

Definition. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. Let M and N be $\mathbb{Z}[\pi_1(X,x_0)]$-left modules. If (X,A,B) is a universally excisive triad of topological spaces, then we refer to the composition

$$\mathrm{H}^p_{\mathrm{tw}}(X,A;M)\times\mathrm{H}^q_{\mathrm{tw}}(X,B;N) \;\underset{\uparrow}{\to}\; \mathrm{H}^{p+q}_{\mathrm{tw}}(X,\{A,B\};M\underset{\mathbb{Z}}{\otimes}N) \;\underset{\uparrow}{\overset{\cong}{\leftarrow}}\; \mathrm{H}^{p+q}_{\mathrm{tw}}(X,A\cup B;M\underset{\mathbb{Z}}{\otimes}N)$$

as in Lemma 115.2 we see that the usual formula for $\cup_\otimes$ defines such a map

isomorphism by Lemma 171.4

again as a cup product (or sometimes as a relative cup product) and we denote it as usual by the "$\cup_\otimes$"-symbol.

We continue with a frequently used variation on the cup product.

Definition. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. We write $\pi := \pi_1(X,x_0)$. Furthermore let R be a ring with (possibly trivial) involution and let $\varphi\colon \mathbb{Z}[\pi]\to R$ be a homomorphism of rings with involution.[2092] Recall that according to the definition on page 3507 we denote by ${}_\varphi R$ the $\mathbb{Z}[\pi]$-left module, where $g\in\pi$ acts on $r\in R$ via $g\cdot r := r\cdot_R \varphi(g^{-1})$. We define the multiplication map

$$\begin{array}{rccc} \mu\colon & {}_\varphi R\underset{\mathbb{Z}}{\otimes}{}_\varphi R & \to & {}_{\mathrm{tr}}R \\ & r\otimes s & \mapsto & r\cdot\overline{s}. \end{array}$$

Note that this is a homomorphism of $\mathbb{Z}[\pi]$-left modules. If (X, A, B) is a universally excisive triad of topological spaces, then we define[2093]

$$\begin{aligned} \cup_R \colon \mathrm{H}^k_{\mathrm{tw}}(X, A; {}_{\varphi}R) \times \mathrm{H}^l_{\mathrm{tw}}(X, B; {}_{\varphi}R) &\to \mathrm{H}^{k+l}_{\mathrm{tw}}(X, A \cup B; {}_{\mathrm{tr}}R) \\ (\alpha, \beta) &\mapsto \mu_*\big(\underbrace{\alpha \cup_{\otimes} \beta}_{\in \mathrm{H}^{k+l}_{\mathrm{tw}}(X, A\cup B; {}_{\varphi}R \otimes_{\mathbb{Z}} {}_{\varphi}R)} \big). \end{aligned}$$

Lemma 171.5. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. Let R and S be rings, let M be a $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left module and let N be a $(\mathbb{Z}[\pi_1(X, x_0)], S)$-left left module. Furthermore let (X, A, B) be a universally excisive triad of topological spaces. The relative cup product

$$\cup_{\otimes} \colon \mathrm{H}^p_{\mathrm{tw}}(X, A; M) \otimes_{\mathbb{Z}} \mathrm{H}^q_{\mathrm{tw}}(X, B; N) \to \mathrm{H}^{p+q}_{\mathrm{tw}}(X, A \cup B; M \otimes_{\mathbb{Z}} N)$$

is a homomorphism of $R \otimes_{\mathbb{Z}} S$-left modules.[2094]

Proof. This lemma follows immediately from the definitions. ■

The following lemma shows that we recover our classical cup product in all relevant cases from the twisted cup product.

Lemma 171.6. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$ and let (X, A, B) be a universally excisive triad of topological spaces. Let R be a commutative ring. We consider the trivial $\mathbb{Z}[\pi_1(X, x_0)]$-left module ${}_{\mathrm{tr}}R$. In other words we consider the homomorphism $\mathbb{Z}[\pi_1(X, x_0)] \to R$ that is given by $g \mapsto 1_R$. If we equip $\mathbb{Z}[\pi_1(X, x_0)]$ with the usual involution and if we equip R with the trivial involution, then this is a homomorphism between rings with involution, thus the cup product $\cup_R$ is defined. The following diagram, where the vertical maps are the natural isomorphisms T from Lemma 167.3, commutes:[2095]

$$\begin{array}{ccccc} \mathrm{H}^k(X, A; R) & \times & \mathrm{H}^l(X, B; R) & \xrightarrow[\text{as defined on page 2513}]{\cup} & \mathrm{H}^{k+l}(X, A \cup B; R) \\ \cong \big\downarrow T & & T \big\downarrow \cong & & T \big\downarrow \cong \\ \mathrm{H}^k_{\mathrm{tw}}(X, A; {}_{\mathrm{tr}}R) & \times & \mathrm{H}^l_{\mathrm{tw}}(X, B; {}_{\mathrm{tr}}R) & \xrightarrow{\cup_R} & \mathrm{H}^{k+l}_{\mathrm{tw}}(X, A \cup B; {}_{\mathrm{tr}}R). \end{array}$$

Proof. The proof follows easily from the definitions. ■

The following proposition can be viewed as an analogue of the "classical" the Cup Product-Commutativity Proposition 114.8.

[2092] As we mentioned on page 3503, we will always view the group ring $\mathbb{Z}[\pi]$ as equipped with the involution $\overline{g} := g^{-1}$.

[2093] Recall that on page 3538 we showed that a homomorphism of $\mathbb{Z}[\pi]$-left modules induces a map on twisted cohomology.

[2094] We can use Lemma 171.2 to view $\mathrm{H}^p_{\mathrm{tw}}(X, A; M) \otimes_{\mathbb{Z}} \mathrm{H}^q_{\mathrm{tw}}(X, B; N)$ and $M \otimes_{\mathbb{Z}} N$ as $R \otimes_{\mathbb{Z}} S$-left modules. Thus both sides have a natural $R \otimes_{\mathbb{Z}} S$-left module structure.

[2095] By Lemma 169.1 we know that universally excisive triads are also excisive, this shows that the bottom relative cup product is indeed defined.

Proposition 171.7. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. Let M and N be $\mathbb{Z}[\pi_1(X,x_0)]$-left modules. We denote by

$$\begin{array}{rcl} \tau\colon M_{\otimes_{\mathbb{Z}}} N & \to & N \otimes_{\mathbb{Z}} M \\ \sum_{i=1}^{k} m_i \otimes n_i & \mapsto & \sum_{i=1}^{k} n_i \otimes m_i \end{array}$$

the $\mathbb{Z}[\pi_1(X,x_0)]$-left module isomorphism that swaps the two factors. Given any subset $A \subset X$ and given any $\varphi \in \mathrm{H}^k_{\mathrm{tw}}(X,A;M)$ and $\psi \in \mathrm{H}^l_{\mathrm{tw}}(X,A;N)$ the following equality holds:

$$\tau(\underbrace{\varphi \cup_{\otimes} \psi}_{\in M\otimes_{\mathbb{Z}} N}) \;=\; (-1)^{k\cdot l} \cdot \psi \cup_{\otimes} \varphi \;\in\; \mathrm{H}^{k+l}_{\mathrm{tw}}(X,A,N \otimes_{\mathbb{Z}} M).$$

Proof. We leave it to the reader to verify that the not entirely proof of the Cup Product-Commutativity Proposition 114.8 does indeed generalize to the present setting. Again we mostly have to replace multiplication "$\cdot$" in a commutative ring by taking the tensor product "$\otimes$". ■

The following proposition is the analogue of our old and much-used friend, the Cup Product-Naturality Proposition 115.4.

Proposition 171.8. Let X and Y be topological spaces that are locally contractible and path-connected and let $x_0 \in X$ and $y_0 \in Y$. Furthermore let (X,A,B) and (Y,C,D) be universally excisive triads of topological spaces and let R be a ring.

(1) Let M and N be $(\mathbb{Z}[\pi_1(Y,y_0)],R)$-left left modules. Let $f\colon (X,A,B) \to (Y,C,D)$ be a map with $f(x_0) = y_0$. For all $k,l \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(Y,C;M) \times \mathrm{H}^l_{\mathrm{tw}}(Y,D;N) & \xrightarrow{\cup_{\otimes}} & \mathrm{H}^{k+l}_{\mathrm{tw}}(Y,C\cup D; M\otimes_{\mathbb{Z}} N) \\ \downarrow{\scriptstyle f^*} \quad \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{H}^k_{\mathrm{tw}}(X,A;f^*M) \times \mathrm{H}^l_{\mathrm{tw}}(X,B;f^*N) & \xrightarrow{\cup_{\otimes}} & \mathrm{H}^{k+l}_{\mathrm{tw}}(X,A\cup B; \underbrace{f^*M \otimes_{\mathbb{Z}} f^*N}_{=f^*(M\otimes_{\mathbb{Z}} N)}). \end{array}$$

(2) If $\varphi\colon M \to M'$ and $\psi\colon N \to N'$ are homomorphisms of $(\mathbb{Z}[\pi_1(X,x_0)],R)$-left left modules, then for all $k,l \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(X,A;M) \times \mathrm{H}^l_{\mathrm{tw}}(X,B;N) & \xrightarrow{\cup_{\otimes}} & \mathrm{H}^{k+l}_{\mathrm{tw}}(X,A\cup B; M\otimes_{\mathbb{Z}} N) \\ \downarrow{\scriptstyle \psi_*} \quad \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*\otimes\psi_*} \\ \mathrm{H}^k_{\mathrm{tw}}(X,A;M') \times \mathrm{H}^l_{\mathrm{tw}}(X,B;N') & \xrightarrow{\cup_{\otimes}} & \mathrm{H}^{k+l}_{\mathrm{tw}}(X,A\cup B; M'\otimes_{\mathbb{Z}} N'). \end{array}$$

Here the vertical maps are the maps defined on page 3538.

Proof. As in the proof of the classical the Cup Product-Naturality Proposition 115.4 one sees that the statements follow easily from the definitions. ■

171.3. **The cap product on twisted (co-) homology.** Next we want to generalize the definition of the cap-product that we gave on page 2551 for singular (co-) homology.

Definition. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. Furthermore

let M and N be $\mathbb{Z}[\pi_1(X,x_0)]$-left modules and let $p,q\in\mathbb{N}_0$. Given $\varphi\in \mathrm{C}^p_{\mathrm{tw}}(X;M)$, given $\sigma\colon \Delta^q\to X$ with $p\le q$ and given $n\in N$ we define the cap product as[2096]

$$\varphi\cap_\otimes (n\underset{\mathbb{Z}[\pi]}{\otimes}\sigma) \;:=\; \big(\underbrace{\varphi(\sigma\circ[v_0,\dots,v_p])}_{\in M}\underset{\mathbb{Z}}{\otimes} n\big)\underset{\mathbb{Z}[\pi]}{\otimes}\underbrace{\sigma\circ[v_p,\dots,v_q]}_{\text{map }\Delta^{q-p}\to\widetilde{X}_{x_0}} \;\in\; \underbrace{(\overline{M}\underset{\mathbb{Z}}{\otimes}N)\underset{\mathbb{Z}[\pi]}{\otimes}\mathrm{C}^{\mathrm{tw}}_{q-p}(\widetilde{X}_{x_0})}_{=\mathrm{C}^{\mathrm{tw}}_{q-p}(X;M\otimes_{\mathbb{Z}}N)}.$$

(We will show in Lemma 171.9 below that this definition makes sense.) We extend this definition linearly to $\mathrm{C}^{\mathrm{tw}}_p(X;N)=\overline{N}\otimes_{\mathbb{Z}[\pi_1(X,x_0)]}\mathrm{C}_q(\widetilde{X}_{x_0})$ and we obtain the cap product

$$\cap_\otimes\colon \mathrm{C}^p_{\mathrm{tw}}(X;M)\times\mathrm{C}^{\mathrm{tw}}_q(X;N)\;\to\;\mathrm{C}^{\mathrm{tw}}_{q-p}(X;M\otimes_{\mathbb{Z}}N).$$

For $p>q$ we define the cap product to be the zero map. In Lemma 171.9 we see that the above map $\cap_\otimes$ descends to a well-defined map

$$\cap_\otimes\colon \mathrm{H}^p_{\mathrm{tw}}(X;M)\times\mathrm{H}^{\mathrm{tw}}_q(X;N)\;\to\;\mathrm{H}^{\mathrm{tw}}_{q-p}(X;M\otimes_{\mathbb{Z}}N).$$

The following lemma is the analogue of Lemmas 116.1 and 116.2 in the untwisted case.

Lemma 171.9. Let X be a topological space that is locally contractible and path-connected and let $x_0\in X$. We denote by $p\colon\widetilde{X}_{x_0}\to X$ the natural universal covering. Furthermore let M and N be $\mathbb{Z}[\pi_1(X,x_0)]$-left modules and let $p,q\in\mathbb{N}_0$.

(0) The above definition of the cap product on the (co-) chain level makes sense.

(1) Given $\varphi\in\mathrm{C}^p_{\mathrm{tw}}(X;M)$, given $\sigma\colon\Delta^q\to\widetilde{X}_{x_0}$ with $p\le q$ and given $n\in N$ we have

$$\partial(\varphi\cap_\otimes n\otimes\sigma)\;=\;(-1)^p\cdot(-\delta\varphi\cap_\otimes n\otimes\sigma+\varphi\cap_\otimes n\otimes\partial\sigma)\;\in\;\mathrm{C}^{\mathrm{tw}}_{l-k-1}(X;M\otimes_{\mathbb{Z}}N).$$

(2) The cap product on the (co-) chain level descends to a map on the (co-) homological level.

Proof. Let X be a topological space that is locally contractible and path-connected and let $x_0\in X$. We write $\pi=\pi_1(X,x_0)$ and we denote by $p\colon\widetilde{X}_{x_0}\to X$ the natural universal covering. Furthermore let M and N be $\mathbb{Z}[\pi]$-left modules and let $p,q\in\mathbb{N}_0$ with $p\le q$. We start out with the following claim.

Claim. Given $\varphi\in\mathrm{C}^p_{\mathrm{tw}}(X;M)$, given $\sigma\colon\Delta^q\to\widetilde{X}_{x_0}$ with $p\le q$ and given $n\in N$ we have the following equality in $(\overline{M\otimes_{\mathbb{Z}}N})\otimes_{\mathbb{Z}[\pi_1(X,x_0)]}\mathrm{C}_{q-p}(\widetilde{X}_{x_0})$:

$$\big(\varphi(\sigma\circ[v_0,\dots,v_p])\otimes n\cdot g\big)\otimes\sigma\circ[v_p,\dots,v_q]\;=\;\big(\varphi((g\cdot\sigma)\circ[v_0,\dots,v_p])\otimes n\big)\otimes(g\cdot\sigma)\circ[v_p,\dots,v_q].$$

Proof. The claim follows from a careful little calculation in $(\overline{M\otimes_{\mathbb{Z}}N})\otimes_{\mathbb{Z}[\pi]}\mathrm{C}_{q-p}(\widetilde{X}_{x_0})$:

$$\begin{array}{ll}
\big(\varphi(\sigma\circ[v_0,..,v_p])\underset{\mathbb{Z}}{\otimes}n\cdot g\big)\underset{\mathbb{Z}[\pi]}{\otimes}\sigma\circ[v_p,..,v_q] & =\big(\varphi(\sigma\circ[v_0,..,v_p])\cdot g^{-1}\cdot g\underset{\mathbb{Z}}{\otimes}n\cdot g\big)\underset{\mathbb{Z}[\pi]}{\otimes}\sigma\circ[v_p,..,v_q]\\
\uparrow= \big(\varphi(\sigma\circ[v_0,..,v_p])\cdot g^{-1}\underset{\mathbb{Z}}{\otimes}n\big)\underset{\mathbb{Z}[\pi]}{\otimes}g\cdot(\sigma\circ[v_p,..,v_q]) & \uparrow= \big(\varphi((g\cdot\sigma)\circ[v_0,..,v_p])\underset{\mathbb{Z}}{\otimes}n\big)\underset{\mathbb{Z}[\pi]}{\otimes}(g\cdot\sigma)\circ[v_p,..,v_q].
\end{array}$$

by definition of the tensor product over $\mathbb{Z}[\pi]$ and the definition of the $\mathbb{Z}[\pi]$-right module structure on $\overline{M\otimes_{\mathbb{Z}}N}$

since $(g\cdot\sigma)([v_i,\dots,v_j])=g\cdot(\sigma\circ[v_i,\dots,v_j])$, since φ is a $\mathbb{Z}[\pi]$-left module homomorphism, and by definition of the $\mathbb{Z}[\pi]$-right module structure on $\overline{M}$ ⊞

Now we turn to the proofs of the three statements of the lemma.

(0) This statement follows immediately from the claim.

(1) The proof of this statement is almost identical to the proof of Lemma 116.1. In fact, as we already explained in the proof of Lemma 171.3, we just need to replace multiplication "$\cdot$" in $\mathbb{R}$ by the tensor product $\otimes$.

(2) As in the proof of Lemma 116.2 we see that Statement (2) is a straightforward consequence of Statement (1). ■

Before we discuss properties of the cap product with twisted coefficients let us immediately move on to the slightly more tricky definition of the relative cap product on twisted (co-) homology. As in the classical setting this requires a little bit of an effort. We start out with the following definition.

Definition. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. Let R be a commutative ring and let N be a $(\mathbb{Z}[\pi_1(X,x_0)],R)$-left left module. Let (X,A,B) be a triad of topological spaces. We write $\widetilde{A} := p^{-1}(A)$ and $\widetilde{B} := p^{-1}(B)$. We write

$$\mathrm{C}_n^{\mathrm{tw}}(X,\{A,B\};N) \;:=\; \mathrm{coker}\left(\overline{N} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_n^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A}\cup\widetilde{B}) \to \overline{N} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_n(\widetilde{X}_{x_0})\right).$$

As always the boundary map on the chain complex $\overline{N} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_n(\widetilde{X}_{x_0})$ descends to a boundary map on $\mathrm{C}_n^{\mathrm{tw}}(X,\{A,B\};N)$ which we can thus view as a chain complex of R-left modules. We denote the corresponding homology groups by $\mathrm{H}_*^{\mathrm{tw}}(X,\{A,B\};N)$.

The following lemma is the analogue of the "untwisted" Lemma 116.6.

Lemma 171.10. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. Let R be a commutative ring and let N be a $(\mathbb{Z}[\pi_1(X,x_0)],R)$-left left module. If (X,A,B) is a universally excisive triad of topological spaces, then the obvious homomorphism

$$\mathrm{C}_*^{\mathrm{tw}}(X,\{A,B\};N) \quad\to\quad \mathrm{C}_*^{\mathrm{tw}}(X,A\cup B;N)$$

of R-left modules induces for every $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n^{\mathrm{tw}}(X,\{A,B\};N) \quad\xrightarrow{\cong}\quad \mathrm{H}_n^{\mathrm{tw}}(X,A\cup B;N).$$

Proof. In the proof of Lemma 171.4 we already showed that the map

$$\mathrm{C}_*(\widetilde{X},\{\widetilde{A},\widetilde{B}\}) \;\to\; \mathrm{C}_*(\widetilde{X},\widetilde{A}\cup\widetilde{B})$$

is a chain homotopy equivalence of free $\mathbb{Z}[\pi_1(X,x_0)]$-left modules. The lemma now follows from the Tensoring Chain Complex Lemma 166.3 together with Lemma 166.2. ■

The following definition can be viewed as the twisted analogue of the definition on page 2557.

Definition. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. Furthermore let M and N be $\mathbb{Z}[\pi_1(X,x_0)]$-left modules and let $p,q \in \mathbb{N}_0$. Finally suppose that we are given a universally excisive triad (X,A,B). We refer to the map

$$\begin{array}{ccc} \mathrm{H}_{\mathrm{tw}}^k(X,A;M) \times \mathrm{H}_l^{\mathrm{tw}}(X,A\cup B;N) & \xleftarrow{\cong} & \mathrm{H}_{\mathrm{tw}}^k(X,A;M) \times \mathrm{H}_l^{\mathrm{tw}}(X,\{A,B\};N) \\ & \xrightarrow{\cap_\otimes} & \mathrm{H}_{l-k}^{\mathrm{tw}}(X,B;M\otimes_{\mathbb{Z}} N) \end{array}$$

(the left arrow: isomorphism given by Lemma 171.10; the right arrow: argument of Lemma 116.5 shows that the same explicit map as on page 3606 descends to a well-defined map)

as the cap product (sometimes we refer to it as the relative cap product) that we also denote by "$\cap_\otimes$". For $k > l$ we define the cap product to be the zero map.

Remark. As in the case of the original cap product, see the discussion on page 2558, the relative cap product

$$\cap_{\otimes}\colon \mathrm{H}^k_{\mathrm{tw}}(X,A;M)\times \mathrm{H}^{\mathrm{tw}}_l(X,A\cup B;N) \;\to\; \mathrm{H}^{\mathrm{tw}}_{l-k}(X,B;M\otimes_{\mathbb{Z}} N)$$

is given by the explicit formula if $A=\varnothing$, or $B=\varnothing$ or $A=B$.

The following lemma is the analogue of Lemma 171.5.

Lemma 171.11. Let X be a topological space that is locally contractible and path-connected and let $x_0\in X$. Let R and S be rings, let M be a $(\mathbb{Z}[\pi_1(X,x_0)],R)$-left left module and let N be a $(\mathbb{Z}[\pi_1(X,x_0)],S)$-left left module. Furthermore let (X,A,B) be a universally excisive triad of topological spaces. The relative cap product

$$\cap_{\otimes}\colon \mathrm{H}^k_{\mathrm{tw}}(X,A;M)\otimes_{\mathbb{Z}} \mathrm{H}^{\mathrm{tw}}_l(X,A\cup B;N) \;\to\; \mathrm{H}^{\mathrm{tw}}_{l-k}(X,B;M\otimes_{\mathbb{Z}} N)$$

is a homomorphism of $R\otimes_{\mathbb{Z}} S$-left modules.

Proof. The lemma follows immediately from the definitions. ■

On page 3603 we defined the cup product with coefficients in a ring. We have the following analogous definition in the context of the cap product.

Definition. Let X be a topological space that is locally contractible and path-connected and let $x_0\in X$. We write $\pi:=\pi_1(X,x_0)$. Furthermore let R be a ring with (possibly trivial) involution and let $\varphi\colon \mathbb{Z}[\pi]\to R$ be a homomorphism of rings with involution. We consider again the multiplication map

$$\begin{aligned}\mu\colon {}_{\varphi}R\underset{\mathbb{Z}}{\otimes}{}_{\varphi}R &\to {}_{\mathrm{tr}}R\\ r\otimes s &\mapsto r\cdot\overline{s}\end{aligned}$$

which is a homomorphism of $\mathbb{Z}[\pi]$-left modules. If (X,A,B) is a universally excisive triad of topological spaces, then we define[2097]

$$\begin{aligned}\cap_R\colon \mathrm{H}^k_{\mathrm{tw}}(X,A;{}_{\varphi}R)\times \mathrm{H}^{\mathrm{tw}}_l(X,A\cup B;{}_{\varphi}R) &\to \mathrm{H}^{l-k}_{\mathrm{tw}}(X,B;{}_{\mathrm{tr}}R)\\ (\alpha,\beta) &\mapsto \mu_*\Big(\underbrace{\alpha\cap_{\otimes}\beta}_{\in \mathrm{H}^{l-k}_{\mathrm{tw}}(X,B;{}_{\varphi}R\underset{\mathbb{Z}}{\otimes}{}_{\varphi}R)}\Big).\end{aligned}$$

The following, rather unsurprising analogue of Lemma 171.6 says we that we can recover our classical cap product in all relevant cases from the twisted cap product.

Lemma 171.12. Let X be a topological space that is locally contractible and path-connected and let $x_0\in X$ and let (X,A,B) be a universally excisive triad of topological spaces. Let R be a commutative ring. We consider the trivial $\mathbb{Z}[\pi_1(X,x_0)]$-left module ${}_{\mathrm{tr}}R$. In other words we consider the homomorphism $\mathbb{Z}[\pi_1(X,x_0)]\to R$ that is given by $g\mapsto 1_R$. If we equip $\mathbb{Z}[\pi_1(X,x_0)]$ with the usual involution and if we equip R with the trivial involution, then this is a homomorphism between rings with involution, thus the cup product $\cup_R$ is defined. The following diagram, where the vertical maps are the natural

[2097] Recall that on page 3538 we showed that a homomorphism of $\mathbb{Z}[\pi]$-left modules induces a map on twisted homology.

isomorphisms T from Lemma 167.3, commutes:

$$\begin{array}{ccccc} \mathrm{H}^k_{\mathrm{tw}}(X,A;{}_{\mathrm{tr}}R) & \times & \mathrm{H}^{\mathrm{tw}}_l(X,A\cup B;{}_{\mathrm{tr}}R) & \xrightarrow{\cap_R} & \mathrm{H}^{\mathrm{tw}}_{l-k}(X,B;{}_{\mathrm{tr}}R) \\ {\scriptstyle T}\uparrow\cong & & {\scriptstyle T}\downarrow\cong & & {\scriptstyle T}\downarrow\cong \\ \mathrm{H}^k(X,A;R) & \times & \mathrm{H}_l(X,A\cup B;R) & \xrightarrow[\text{as defined on page 2551}]{\cap} & \mathrm{H}_{l-k}(X,B;R). \end{array}$$

Proof. The lemma follows again easily from plugging in all the definitions. ∎

Next we want to introduce an important variation on the cap product, which allows us - perhaps counter intuitively - to cap a twisted cohomology class with an untwisted ordinary homology class.

Definition. Let X be a topological space that is locally contractible and path-connected, let $x_0 \in X$ and let (X,A,B) be a universally excisive triad of topological spaces. Furthermore let R be a ring, let M be a $(\mathbb{Z}[\pi_1(X,x_0)],R)$-left left module and let $k,l \in \mathbb{N}_0$. We consider the two natural isomorphisms

$$\begin{array}{rcl} T\colon \mathrm{H}^{\mathrm{tw}}_l(X,A\cup B;{}_{\mathrm{tr}}\mathbb{Z}) & \to & \mathrm{H}_l(X,A\cup B) \\ [n\otimes\sigma] & \mapsto & [p_*(\sigma)\cdot n] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mu\colon M\otimes_{\mathbb{Z}}{}_{\mathrm{tr}}\mathbb{Z} & \to & M \\ m\otimes n & \mapsto & n\cdot m \end{array}$$

$p\colon \widetilde{X}_{x_0}\to X$ is the natural universal covering

from Lemma 167.3 (together with the remark on page 1913) and Lemma 165.24. We define

$$\begin{array}{rcl} \cap_{\mathbb{Z}}\colon \mathrm{H}^k_{\mathrm{tw}}(X,A;M)\times \mathrm{H}_l(X,A\cup B) & \to & \mathrm{H}^{\mathrm{tw}}_{l-k}(X,B;M) \\ (\varphi,\sigma) & \mapsto & \mu_*(\underbrace{\varphi\cap_{\otimes} T^{-1}(\sigma)}_{\in \mathrm{H}^{\mathrm{tw}}_{l-k}(X,B;M\otimes_{\mathbb{Z}}{}_{\mathrm{tr}}\mathbb{Z})}). \end{array}$$

It follows easily from Lemma 171.11 that given $\sigma\in \mathrm{H}_l(X,A\cup B)$ the map $\varphi\mapsto \varphi\cap_{\mathbb{Z}}\sigma$ is a R-left homomorphism.

The following proposition is a sibling of Proposition 171.8 and it contains the twisted analogue of the Cap Product-Naturality Lemma 116.8.

Proposition 171.13. Let X and Y be topological spaces that are locally contractible and path-connected and let $x_0\in X$ and $y_0\in Y$. Furthermore let (X,A,B) and (Y,C,D) be universally excisive triads of topological spaces.

(1) Let $f\colon (X,A,B)\to(Y,C,D)$ be a map with $f(x_0)=y_0$ and let $k,l\in\mathbb{N}_0$.
   (a) Let M and N be $\mathbb{Z}[\pi_1(Y,y_0)]$-left modules. The following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(Y,C;M)\times \mathrm{H}^{\mathrm{tw}}_l(Y,C\cup D;N) & \xrightarrow{\cap_{\otimes}} & \mathrm{H}^{\mathrm{tw}}_{l-k}(Y,D;M\otimes_{\mathbb{Z}}N) \\ {\scriptstyle f^*}\downarrow \quad \uparrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{H}^k_{\mathrm{tw}}(X,A;f^*M)\times \mathrm{H}^{\mathrm{tw}}_l(X,A\cup B;f^*N) & \xrightarrow{\cap_{\otimes}} & \mathrm{H}^{\mathrm{tw}}_{l-k}(X,B;\underbrace{f^*M\otimes_{\mathbb{Z}} f^*N}_{=f^*(M\otimes_{\mathbb{Z}}N)}). \end{array}$$

(b) Let M be a $\mathbb{Z}[\pi_1(Y,y_0)]$-left module. The following diagram commutes:[2098]

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(Y,C;M) \times \mathrm{H}_l(Y,C\cup D) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{\mathrm{tw}}_{l-k}(Y,D;M) \\ {\scriptstyle f^*}\downarrow \quad\quad \uparrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{H}^k_{\mathrm{tw}}(X,A;f^*M) \times \mathrm{H}_l(X,A\cup B) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{\mathrm{tw}}_{l-k}(X,B;f^*M). \end{array}$$

(2) If $\varphi\colon M \to M'$ and $\psi\colon N \to N'$ are homomorphisms of $\mathbb{Z}[\pi_1(X,x_0)]$-left modules, then for all $k,l \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(X,A;M) \times \mathrm{H}^{\mathrm{tw}}_l(X,A\cup B;N) & \xrightarrow{\cap_{\otimes}} & \mathrm{H}^{\mathrm{tw}}_{l-k}(X,B;M\otimes_{\mathbb{Z}} N) \\ {\scriptstyle \psi_*}\downarrow & \downarrow{\scriptstyle \varphi_*} & \downarrow{\scriptstyle \varphi_*\otimes\psi_*} \\ \mathrm{H}^k_{\mathrm{tw}}(X,A;M') \times \mathrm{H}^{\mathrm{tw}}_l(X,A\cup B;N') & \xrightarrow{\cap_{\otimes}} & \mathrm{H}^{\mathrm{tw}}_{l-k}(X,B;M'\otimes_{\mathbb{Z}} N'). \end{array}$$

Here the vertical maps are the maps defined on page 3538.

Proof. All statements follow easily from the definitions. ■

The following lemma is the generalization of the Cup-Cap Lemma 116.7 to the present context.

Lemma 171.14. Let X be a topological space that is locally contractible and path-connected. Let $x_0 \in X$ and let L, M and N be $\mathbb{Z}[\pi_1(X,x_0)]$-left modules. Furthermore let $A \subset X$ be a subset and let $C,D \in \{\varnothing, A\}$. Given any $\varphi \in \mathrm{H}^k_{\mathrm{tw}}(X,C;L)$, $\psi \in \mathrm{H}^l_{\mathrm{tw}}(X,D;M)$ and $\sigma \in \mathrm{H}^{\mathrm{tw}}_n(X,C\cup D;N)$ we have[2099]

$$\varphi \cap_{\otimes} (\psi \cap_{\otimes} \sigma) \;=\; (\psi \cup_{\otimes} \varphi) \cap_{\otimes} \sigma \;\in\; \mathrm{H}^{\mathrm{tw}}_{n-k-l}(X; L\otimes_{\mathbb{Z}} M\otimes_{\mathbb{Z}} N).$$

If $\sigma \in \mathrm{H}_n(X,C\cup D)$, then

$$\varphi \cap_{\otimes} (\psi \cap_{\mathbb{Z}} \sigma) \;=\; (\psi \cup_{\otimes} \varphi) \cap_{\mathbb{Z}} \sigma \quad\in\; \mathrm{H}^{\mathrm{tw}}_{n-k-l}(X; L\otimes_{\mathbb{Z}} M).$$

Proof. As in the case of the Cup-Cap Lemma 116.7 the statement follows immediately from the definitions. ■

[2098] Statement (1b) is essentially Statement (1a) with $N = {}_{\mathrm{tr}}\mathbb{Z}$. Since this special case will be needed frequently we decided to formulate it separately.

[2099] As we discussed on page 1909, for tensor products over $\mathbb{Z}$ we decided - with a slightly guilty conscience - to drop parenthesis.

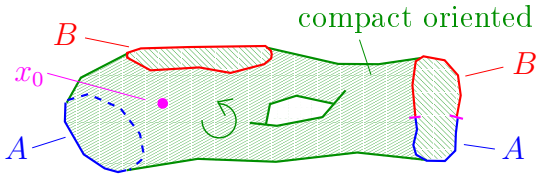
172. Poincaré Duality with twisted coefficients

172.1. **Statement of the Poincaré Duality Theorem with twisted coefficients.** The Poincaré Duality Theorem 119.1 was surely the most important result so far that we proved regarding the topology of compact topological manifolds. Now we have all the tools to formulate and prove a significant generalization of Poincaré Duality Theorem 119.1.

Theorem 172.1. (Poincaré Duality Theorem with Twisted Coefficients) Let X be a compact oriented path-connected n-dimensional topological manifold and let $x_0 \in X$. We denote by $[X] \in \mathrm{H}_n(X, \partial X)$ the fundamental class. Furthermore suppose that we are given a decomposition $\partial X = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂X such that $A \cap B = \partial A = \partial B$. For every $\mathbb{Z}[\pi_1(X, x_0)]$-left module M and each $k \in \mathbb{N}_0$ the map[2100]

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(X, A; M) & \xrightarrow{\cong} & \mathrm{H}^{\mathrm{tw}}_{n-k}(X, B; M) \\ \sigma & \mapsto & \sigma \cap_{\mathbb{Z}} [X] \end{array}$$

is an isomorphism.



Remark.

(1) For $M = {}_{\mathrm{tr}}\mathbb{Z}$ we recover the $R = \mathbb{Z}$-statement of the Poincaré Duality Theorem 119.1.

(2) The Poincaré Duality Theorem 172.1 with Twisted Coefficients gets commonly used in the literature. But proofs in the literature a few and far between. The proof we provide is a "twisted version" of our proof of the original Poincaré Duality Theorem 119.1. Another "twisted version" of a classical proof is provided in [**FNOP2019**] and in [**Sun2017**]. Somewhat different approaches are given in [**KwS2018**] and [**Wall1970**, Chapter 2].

(3) Let M be a closed oriented smooth manifold. By the Smooth Manifold-Simplicial Structure Theorem 96.2 we know that M admits a smooth simplicial structure. In Chapter 103 we used this fact to provide for such M an alternative proof of the Poincaré Duality Theorem 119.1. Using equivariant versions of the chain homotopy equivalence used in the proof of the Simplicial Poincaré Duality Theorem 103.4 one can surely provide an alternative proof of the Poincaré Duality Theorem 172.1 with Twisted Coefficients for closed oriented smooth manifolds. Our well developed taste for brevity prevents us from carrying out this program.

[2100] Note that if M is a $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left module M where R is some ring, then it follows from the discussion on page 3609 that the map

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(X, A; M) & \xrightarrow{\cong} & \mathrm{H}^{\mathrm{tw}}_{n-k}(X, B; M) \\ \sigma & \mapsto & \sigma \cap_{\mathbb{Z}} [X] \end{array}$$

is a homomorphism of R-left modules, thus in our setting it is an isomorphism of R-left modules.

Notation. The isomorphism of the Poincaré Duality Theorem 172.1 with Twisted Coefficients is called the Poincaré Duality isomorphism. We denote it and its inverse both by PD_X, i.e. with the notation of the Poincaré Duality Theorem 172.1 we have the isomorphism

$$\begin{array}{rcl} \mathrm{PD}_X\colon \mathrm{H}^k_{\mathrm{tw}}(X,A;M) & \overset{\cong}{\to} & \mathrm{H}^{\mathrm{tw}}_{n-k}(X,B;M) \\ \sigma & \mapsto & \sigma \cap_{\mathbb{Z}} [X] \end{array}$$

together with the inverse

$$\mathrm{PD}_X\colon \mathrm{H}^{\mathrm{tw}}_{n-k}(X,B;M) \overset{\cong}{\to} \mathrm{H}^k_{\mathrm{tw}}(X,A;M).$$

Note that if R is a ring and M is a $(\mathbb{Z}[\pi_1(X,x_0)],R)$-left left module, then it follows easily from Lemma 171.11 that the Poincaré Duality isomorphisms are isomorphisms of R-left modules.

Example. Could show that the Alexander polynomial is symmetric

We will prove the Poincaré Duality Theorem 172.1 in the following sections. We will try to stick as closely as possible to the earlier proof of the untwisted Poincaré Duality Theorem 119.1, and we will try to outline where the differences lie.

Before we get to the proof of the Poincaré Duality Theorem 172.1 let us formulate the following variation on the Poincaré Duality Theorem 172.1.

Theorem 172.2. (Poincaré Duality Theorem for Non-Orientable Topological Manifolds) Let X be a compact *non-orientable* path-connected n-dimensional topological manifold and let $x_0 \in X$. We denote by

$$w_1\colon \pi_1(X,x_0) \;\to\; \pi_1(X,x_0)/p_*(\pi_1(\widetilde{X},\widetilde{x}_0)) \underset{\text{unique isomorphism}}{\overset{\uparrow}{=}} \langle x \,|\, x^2\rangle$$

the orientation-character that we introduced on page 2341 and we denote by $\widetilde{\mathbb{Z}}$ the $\mathbb{Z}[\langle x \,|\, x^2\rangle]$-left module that is given by $x\cdot n := -n$. We denote by $[X]_{\mathrm{tw}} \in \mathrm{H}_n(X,\partial X; w_1^*\widetilde{\mathbb{Z}})$ the twisted fundamental class that we introduced on page 3567. Furthermore suppose that we are given a decomposition $\partial X = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂X such that $A \cap B = \partial A = \partial B$. For each $k \in \mathbb{N}_0$ and every $\mathbb{Z}[\pi_1(X,x_0)]$-left module M the map

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{tw}}(X,A;M) & \overset{\cong}{\to} & \mathrm{H}^{\mathrm{tw}}_{n-k}(X,B;M\otimes_{\mathbb{Z}} w_1^*\widetilde{\mathbb{Z}}) \\ \sigma & \mapsto & \sigma \cap_{\mathbb{Z}} [X]_{\mathrm{tw}} \end{array}$$

is an isomorphism.

Proof. The proof of the Poincaré Duality Theorem for Non-Orientable Topological Manifolds 172.2 is in principle quite similar to the proof of the Poincaré Duality Theorem 172.1. The key property of the twisted fundamental class $[X]_{\mathrm{tw}} \in \mathrm{H}^{\mathrm{tw}}_n(X,\partial X; w_1^*\widetilde{\mathbb{Z}})$ which one uses, and which we proved in Proposition 168.7, is the fact that for each $Q \in X \setminus \partial X$ the map $\mathrm{H}^{\mathrm{tw}}_n(X,\partial X; w_1^*\widetilde{\mathbb{Z}}) \to \mathrm{H}^{\mathrm{tw}}_n(X, X\setminus\{Q\}; w_1^*\widetilde{\mathbb{Z}})$ is an isomorphism. Like every author that is known to this author we will not work out the details of the proof of this Poincaré Duality Theorem. We hope that the next generation of authors will step up and finally provide a proof of this well-known folklore theorem. ■

172.2. **The proof of Theorem 172.1 for topological manifolds without boundary.** In this section we will prove the Poincaré Duality Theorem 172.1 for closed topological manifolds. More precisely, we will prove the following theorem.

Theorem 172.3. Let X be a closed oriented path-connected n-dimensional topological manifold and let $x_0 \in X$. We denote by $[X] \in \mathrm{H}_n(X)$ the fundamental class. For each $k \in \mathbb{N}_0$ and every $\mathbb{Z}[\pi_1(X, x_0)]$-left module M the map

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(X; M) & \xrightarrow{\cong} & \mathrm{H}^{\mathrm{tw}}_{n-k}(X; M) \\ \sigma & \mapsto & \sigma \cap_{\mathbb{Z}} [X] \end{array}$$

is an isomorphism.

As in the proof of the Poincaré Theorem 119.1 we want to reduce the proof eventually to the fairly trivial case of an open ball in $\mathbb{R}^n$. To do so we need a version of Poincaré Duality for every topological manifold (not necessarily compact) with empty boundary. Formulating this generalized Poincaré Duality Theorem requires some preparations.

In the proof of Poincaré Theorem 119.1 we used cohomology with compact support. But in fact we did not work with the actual definition of cohomology with compact support but with a reformulation that was given in the H^n_c-via-Direct Limits Proposition 112.4. We use this reformulation to define cohomology with compact support and twisted coefficients.

Definition.

(1) Given a topological space X we denote by $\mathcal{K}(X)$ the set of all compact subsets of X. We equip $\mathcal{K}(X)$ with the partial order that is given by inclusion.

(2) Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. Let R be a ring and let M be a $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left module. Using the functoriality of cohomology with twisted coefficients we can define the cohomology of X with compact support and twisted coefficient corresponding to M to be the following R-left module:

$$\mathrm{H}^k_{\mathrm{tw},c}(X; M) \ := \ \varinjlim_{K \in \mathcal{K}(X)} \mathrm{H}^k_{\mathrm{tw}}(X, X \setminus K; M).$$

Notation. Let X be an n-dimensional topological manifold with empty boundary. If X is equipped with an orientation $\{\mu_x \in \mathrm{H}_n(X, X \setminus \{x\})\}_{x \in X}$, then we saw in the Homology-Section Theorem 106.10 (i) that given any compact subset K of X there exists a unique class $\mu^X_K \in \mathrm{H}_n(X, X \setminus K)$ such that for any $x \in K$ the image of μ^X_K under the map

$$\mathrm{H}_n(X, X \setminus K) \ \to \ \mathrm{H}_n(X, X \setminus \{x\})$$

equals the given orientation μ_x at x. If X is understood, then we drop it from the notation, i.e. we just write μ_K instead of μ^X_K.

Lemma 172.4. Let X be an oriented path-connected n-dimensional topological manifold with $\partial X = \varnothing$ and let $x_0 \in X$. Let R be a ring and let M be a $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left module. For any $k \in \mathbb{N}_0$ there exists a unique R-left module homomorphism

$$\mathrm{PD}_X \colon \mathrm{H}^k_{\mathrm{tw},c}(X; M) \ \to \ \mathrm{H}^{\mathrm{tw}}_{n-k}(X; M)$$

such that for each compact subset $K \subset X$ the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(X, X \setminus K; M) & \xrightarrow{\cap_{\mathbb{Z}} \mu_K} & \mathrm{H}^{\mathrm{tw}}_{n-k}(X; M). \\ & \searrow \quad \nearrow_{\mathrm{PD}_X} & \\ & \mathrm{H}^k_{\mathrm{tw,c}}(X; M) & \end{array}$$

Proof. The proof is basically identical to the proof of Lemma 119.17, we only need to replace the Cap Product-Naturality Lemma 116.8 by Proposition 171.13 (1b). ■

We will now actually prove a generalization of Theorem 172.3 where we replace the hypothesis that X is closed by the hypothesis that the boundary is empty. This more general version of Poincaré Duality will not only be useful for the divide-et-impera approach to the proof of Theorem 172.3, but surprisingly it will also be of interest when we deal with Poincaré Duality for topological manifolds with non-empty boundary.

Theorem 172.5. Let X be an oriented path-connected n-dimensional topological manifold with $\partial X = \varnothing$. Let $x_0 \in X$ and let M be a $\mathbb{Z}[\pi_1(X, x_0)]$-left module. For each $k \in \mathbb{N}_0$ the map

$$\mathrm{PD}_X \colon \mathrm{H}^k_{\mathrm{tw},c}(X; M) \xrightarrow{\cong} \mathrm{H}^{\mathrm{tw}}_{n-k}(X; M)$$

from Lemma 172.4 is an isomorphism.

As a warm up for the proof of Theorem 172.5 we will deal with the case $X = B^n$ by hand.[2101] More precisely, we will prove the following lemma.

Lemma 172.6. Let X be an oriented topological manifold that is homeomorphic to the open ball B^n, let $x_0 \in X$ and let M be a $\mathbb{Z}[\pi_1(X, x_0)]$-left module. For each $k \in \mathbb{N}_0$ the map

$$\mathrm{PD}_X \colon \mathrm{H}^k_{\mathrm{tw},c}(X; M) \xrightarrow{\cong} \mathrm{H}^{\mathrm{tw}}_{n-k}(X; M)$$

is an isomorphism.

Proof. Since X is homeomorphic to the open ball B^n we see that $\pi_1(X, x_0)$ is the trivial group, which implies that M is the trivial $\mathbb{Z}[\pi_1(X, x_0)]$-module ${}_{\mathrm{tr}}M$. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw},c}(X; {}_{\mathrm{tr}}M) & \xrightarrow{\cap_{\mathbb{Z}} \mu_X} & \mathrm{H}^{\mathrm{tw}}_{n-k}(X; {}_{\mathrm{tr}}M) \\ {\scriptstyle T}\uparrow\cong & & \cong\downarrow{\scriptstyle T} \\ \mathrm{H}^k_c(X; M) & \xrightarrow{\cap_{\mathbb{Z}} \mu_X} & \mathrm{H}_{n-k}(X; M) \end{array}.$$

We make the following clarifications and observations:

(1) By Lemma 167.3 we have a natural isomorphism

$$T \colon \mathrm{H}^{\mathrm{tw}}_{n-k}(X; {}_{\mathrm{tr}}M) \to \mathrm{H}_{n-k}(X; M).$$

This gives us the vertical isomorphism on the right.

(2) Again by Lemma 167.3 we have a natural isomorphism

$$T \colon \mathrm{H}^k_{\mathrm{tw}}(X, X \setminus K; M) \to \mathrm{H}^k_{\mathrm{tw}}(X, X \setminus K; {}_{\mathrm{tr}}M).$$

[2101] We use the convention that any n-dimensional submanifold of $\mathbb{R}^n$ is equipped with the orientation induced from the standard orientation of $\mathbb{R}^n$, see the Topological Manifold-Induced Orientation Lemma 105.17.

By the naturality of these isomorphisms, these isomorphisms induce the vertical isomorphism on the left.

(3) It follows easily from the definitions, similar to Lemma 171.12, that the diagram commutes.

(4) In Lemma 119.19 we showed, using an explicit calculation, that the bottom horizontal map is an isomorphism for $X = B^n$. But since X is homeomorphic to B^n it follows immediately that the bottom horizontal map is also an isomorphism in our case. ■

One of key ideas in the proof of the Poincaré Duality Theorem 119.1, and in the intermediate the Non-Compact PD Theorem 119.18, was to reduce the general case to the rather trivial case of an open ball. To reduce the general case to the trivial case we needed to show that if Poincaré Duality holds on two open subsets U and V of a given topological manifold X, then it also holds in the union $U \cup V$ of the two open subsets. To show this result we had to state and prove in Proposition 172.9 a Mayer–Vietoris type result that relates U, V to $U \cup V$ and also the intersection $U \cap V$. The technical nuisance that arises is that there is no way that we can assume that the intersection $U \cap V$ is again path-connected, thus we cannot talk about twisted (co-) homology of $U \cap V$. This means that we have to revert to working with internal twisted (co-) homology that we introduced in Section 169.2. For the reader's convenience we recall the definitions.

Definition. Let X be a topological space that is locally contractible and path-connected, let $x_0 \in X$ and let M be a $\mathbb{Z}[\pi_1(X, x_0)]$-left module. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering.

(1) Given any subset $S \subset T \subset X$ we write $\widetilde{S} := p^{-1}(S)$ and $\widetilde{T} := p^{-1}(T)$ and we set

$$\begin{aligned} \mathrm{C}_k^{X,M}(T,S) &:= \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_k(\widetilde{T},\widetilde{S}) \\ \mathrm{C}^k_{X,M}(T,S) &:= \mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]\text{-left}}(\mathrm{C}_k(\widetilde{T},\widetilde{S}), M). \end{aligned}$$

These are (co-) chain complexes in the obvious way and we obtain the corresponding (co-) homology groups $\mathrm{H}_k^{X,M}(T,S)$ and $\mathrm{H}^k_{X,M}(T,S)$. We refer to these as internal (co-) homology groups with twisted coefficients.

(2) Given pairs (T_1, S_1) and (T_2, S_2) of subspaces of X with $T_1 \subset T_2$ and $S_1 \subset S_2$ we obtain, with the hopefully obvious notation, an inclusion induced chain map $\mathrm{C}_k(\widetilde{T}_1, \widetilde{S}_1) \to \mathrm{C}_k(\widetilde{T}_2, \widetilde{S}_2)$ which is a $\mathbb{Z}[\pi_1(X, x_0)]$-left homomorphism. Thus we get induced maps

$$\begin{aligned} i_*\colon \mathrm{H}_k^{X,M}(T_1,S_1) &\to \mathrm{H}_k^{X,M}(T_2,S_2) \\ i^*\colon \mathrm{H}^k_{X,M}(T_2,S_2) &\to \mathrm{H}^k_{X,M}(T_1,S_1). \end{aligned}$$

Evidently internal (co-) homology is functorial under inclusions.

We add some new definitions:

(3) Let (T, S) be a pair of subspaces of X. As on page 3609 we define a cap product

$$\cap_{\mathbb{Z}}\colon \mathrm{H}^k_{X,M}(T,S) \times \mathrm{H}_n(T,S) \to \mathrm{H}^{X,M}_{n-k}(T).$$

(4) Given $S \subset X$ we can use (2) to define internal cohomology with compact support and twisted coefficients

$$\mathrm{H}^k_{X,M,c}(S) := \varinjlim_{K \in \mathcal{K}(S)} \mathrm{H}^k_{X,M}(S, S \setminus K).$$

The next lemma allows us to go back and forth between internal twisted (co-) homology and usual twisted (co-) homology.

Lemma 172.7. Let X be a topological space that is locally contractible and path-connected, let $x_0 \in X$ and let M be a $\mathbb{Z}[\pi_1(X, x_0)]$-left module.

(1) Let (T, S) be a pair of subspaces of X with $x_0 \in T$. We assume that T is locally contractible and path-connected. We denote by $i\colon T \to X$ the inclusion. We have isomorphisms
$$\begin{aligned} \Omega^X_{T,S}\colon \mathrm{H}^{\mathrm{tw}}_k(T, S; i^*M) &\to \mathrm{H}^{X,M}_k(T, S) \\ \Omega^X_{T,S}\colon \mathrm{H}^k_{X,M}(T, S) &\to \mathrm{H}^k_{\mathrm{tw}}(T, S; i^*M) \end{aligned}$$
that are natural under inclusions.

(2) The following diagram commutes:
$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(T, S; i^*M) \times \mathrm{H}_n(T, S) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{\mathrm{tw}}_{n-k}(T; i^*M) \\ \cong\uparrow \Omega^X_{T,S} \quad \downarrow \mathrm{id} & & \cong\uparrow \Omega^X_T \\ \mathrm{H}^k_{X,M}(T, S) \times \mathrm{H}_n(T, S) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{X,M}_{n-k}(T). \end{array}$$

Proof. The first statement is a simplified version of the Change of Covering Proposition II 169.4. The second statement follows easily from the definitions. ∎

We will need the following reformulation of Lemma 172.4.

Lemma 172.8. Let X be an oriented path-connected n-dimensional topological manifold with $\partial X = \varnothing$, let $x_0 \in X$ and let M be a $\mathbb{Z}[\pi_1(X, x_0)]$-left module. For any open subset $U \subset X$ and any $k \in \mathbb{N}_0$ there exists a unique map
$$\mathrm{PD}_U\colon \mathrm{H}^k_{X,M,\mathrm{c}}(U) \to \mathrm{H}^{X,M}_{n-k}(U)$$
such that for each compact subset $K \subset U$ the following diagram commutes
$$\begin{array}{ccc} \mathrm{H}^k_{X,M}(U, U \setminus K) & \xrightarrow{\cap_{\mathbb{Z}}\mu_K} & \mathrm{H}^{X,M}_{n-k}(U). \\ & \searrow \quad \nearrow_{\mathrm{PD}_U} & \\ & \mathrm{H}^k_{X,M,\mathrm{c}}(U) & \end{array}$$

Proof. The proof of the lemma is basically the same as the proof of the analogous statements in Lemma 119.17 and Lemma 172.4. Alternatively one sees that the lemma follows immediately from Lemma 172.4 together with Lemma 172.7. ∎

Before we can formulate our next results we need to introduce some notation.

Notation. Let X be a topological space that is locally contractible and path-connected, let $x_0 \in X$ and let M be a $\mathbb{Z}[\pi_1(X, x_0)]$-left module.

(1) Given $A \subset X$ we denote by $i_*\colon \mathrm{H}^{X,M}_k(A) \to \mathrm{H}^{X,M}_k(X)$ and $i^*\colon \mathrm{H}^{X,M}_k(X) \to \mathrm{H}^{X,M}_k(A)$ the inclusion induced map.

(2) Suppose that X is Hausdorff and that U is an open subset of X with $x_0 \in U$. Then we denote by $i\colon \mathrm{H}^k_{X,M,c}(U) \to \mathrm{H}^k_{X,M,c}(X)$ the map that is defined by

$$\overbrace{\varinjlim_{K\in\mathcal{K}(U)} \mathrm{H}^k_{X,M}(U, U\setminus K)}^{=\mathrm{H}^k_{X,M,\mathrm{c}}(U)} \xleftarrow{\cong} \underbrace{\varinjlim_{K\in\mathcal{K}(U)} \mathrm{H}^k_{X,M}(X, X\setminus K)}_{\text{isomorphism by the Excision Theorem 169.6}} \to \overbrace{\varinjlim_{K\in\mathcal{K}(X)} \mathrm{H}^k_{X,M}(X, X\setminus K)}^{=\mathrm{H}^k_{X,M,\mathrm{c}}(X)}.$$

As in Lemma 112.5 we see that these maps $\mathrm{H}^k_{X,M,c}(U) \to \mathrm{H}^k_{X,M,c}(X)$ are covariantly functorial with respect to compositions of inclusions.

Now we can state the key technical ingredient to the proof of Theorem 172.5:

Proposition 172.9. Let X be an oriented path-connected n-dimensional topological manifold with $\partial X = \varnothing$. Let $x_0 \in X$ and let M be a $\mathbb{Z}[\pi_1(X, x_0)]$-left module. Let U and V be open subsets of X with $X = U \cup V$. Then there exists a diagram

$$\begin{array}{ccccccccc}
\dots \to \mathrm{H}^k_{X,M,\mathrm{c}}(U\cap V) & \xrightarrow{i\oplus -i} & \mathrm{H}^k_{X,M,\mathrm{c}}(U)\oplus \mathrm{H}^k_{X,M,\mathrm{c}}(V) & \xrightarrow{i+i} & \overbrace{\mathrm{H}^k_{X,M,\mathrm{c}}(X)}^{=\mathrm{H}_{\mathrm{tw},\mathrm{c}}(X;M)} & \to & \mathrm{H}^{k+1}_{X,M,\mathrm{c}}(U\cap V) & \to \dots \\
\downarrow{\scriptstyle \mathrm{PD}_{U\cap V}} & & \downarrow{\scriptstyle \mathrm{PD}_U \oplus \mathrm{PD}_V} & & \downarrow{\scriptstyle \mathrm{PD}_X} & & \downarrow{\scriptstyle \mathrm{PD}_{U\cap V}} & \\
\dots \to \mathrm{H}^{X,M}_{n-k}(U\cap V) & \xrightarrow{i\oplus -i} & \mathrm{H}^{X,M}_{n-k}(U)\oplus \mathrm{H}^{X,M}_{n-k}(V) & \xrightarrow{i+i} & \underbrace{\mathrm{H}^{X,M}_{n-k}(X)}_{=\mathrm{H}^{\mathrm{tw}}_{n-k}(X;M)} & \to & \mathrm{H}^{X,M}_{n-k-1}(U\cap V) & \to \dots
\end{array}$$

with the following properties:

(1) the first two squares commute, the third square commutes up to the sign $(-1)^{k+1}$,
(2) the lower exact sequence is the long exact Mayer–Vietoris sequence for $X = U \cup V$ that we constructed in the Mayer–Vietoris Theorem 169.7 for Internal Twisted Homology,
(3) the upper horizontal sequence is also exact.

The proof of Proposition 172.9 requires several preparations. The reader might want to put their finger on the fast-forward button. First of all we will need the following "inverted" Mayer–Vietoris Theorem, which is the twisted analogue of Theorem 119.22.

Theorem 172.10. (Inverted Mayer–Vietoris Theorem for Cohomology groups) Let X be a topological space that is locally contractible and path-connected, let $x_0 \in X$ and let M be a $\mathbb{Z}[\pi_1(X, x_0)]$-left module. Furthermore let A and B be open subsets of X.

(1) For every $n \in \mathbb{N}_0$ and every cochain $\varphi \in \mathrm{C}^n_{X,M}(X, A\cup B)$ there exist $\varphi_A \in \mathrm{C}^n_{X,M}(X, A)$ and $\varphi_B \in \mathrm{C}^n_{X,M}(X, B)$ such that $\varphi_A + \varphi_B = \varphi$.
(2) For all $n \in \mathbb{N}_0$ there exists a unique homomorphism

$$\delta\colon \mathrm{H}^n_{X,M}(X, A\cap B) \ \to\ \mathrm{H}^{n+1}_{X,M}(X, A\cup B)$$

which has the following property: for every $\varphi_A \in \mathrm{C}^n_{X,M}(X, A)$ and $\varphi_B \in \mathrm{C}^n_{X,M}(X, B)$ such that $\varphi_A + \varphi_B$ is a cocycle in $\mathrm{C}^n_{X,M}(X, A\cap B)$ we have

$$\delta([\varphi_A + \varphi_B]) \ =\ [\delta\varphi_A] \ \xleftarrow{\cong}\ -[\delta\varphi_B] \ \in \mathrm{H}^{n+1}_{X,M}(X, \{A, B\}) = \mathrm{H}^{n+1}_{X,M}(X, A\cup B).$$

(3) The sequence

$$\ldots \to \mathrm{H}^n_{X,M}(X, A\cup B) \xrightarrow{i^*\oplus -i^*} \mathrm{H}^n_{X,M}(X,A)\oplus \mathrm{H}^n_{X,M}(X,B) \xrightarrow{i^*\oplus i^*} \mathrm{H}^n_{X,M}(X, A\cap B) \xrightarrow{\delta} \mathrm{H}^{n+1}_{X,M}(X, A\cup B) \to \ldots$$

is exact.

Proof of Theorem 172.10. We start out with the following two observations:

(1) Similar to the discussion on page 3602 we set

$$\mathrm{C}^k_{X,M}(X,\{A,B\}) = \begin{array}{l}\text{all } \mathbb{Z}[\pi_1(X,x_0)]\text{-left homomorphisms } \mathrm{C}_k(\widetilde{X}_{x_0}) \to M \\ \qquad\text{that vanish on } \mathrm{C}_k(\widetilde{A}) \text{ and on } \mathrm{C}_k(\widetilde{B}).\end{array}$$

This is again a cochain complex and we denote the corresponding cohomology groups by $\mathrm{H}^k_{X,M}(X,\{A,B\})$. It follows easily from the Regular Covering-Chain Complex Lemma 167.2 together with Lemma 165.8 that the following sequence of cochain maps between cochain complexes is exact:

$$0 \to \mathrm{C}^*_{X,M}(X,\{A,B\}) \xrightarrow{i^*\oplus -i^*} \mathrm{C}^*_{X,M}(X,A)\oplus \mathrm{C}^*_{X,M}(X,B) \xrightarrow{i^*\oplus i^*} \mathrm{C}^*_{X,M}(X,A\cap B) \to 0.$$

(2) Since the subsets A and B are open we know from Proposition 169.2 (4) that the triad (X,A,B) is universally excisive. Therefore it follows from the argument of Lemma 171.4 that the obvious map $\mathrm{C}^*_{X,M}(X,A\cup B) \to \mathrm{C}^*_{X,M}(X,\{A,B\})$ induces for every $n\in\mathbb{N}_0$ an isomorphism $\mathrm{H}^n_{X,M}(X,A\cup B) \xrightarrow{\cong} \mathrm{H}^n_{X,M}(X,\{A,B\})$.

With these two observations it is straightforward to modify the proof of Theorem 78.5 to obtain the theorem. We leave it to the reader to fill in the details. ■

In the proof of Proposition 172.9 we will also need to recycle the following two technical lemmas.

Lemma 119.20. (Compact Subset Decomposition Lemma) Let X be a topological manifold and let U,V be open subsets such that $X = U\cup V$. Then given any compact subset $C\subset X$ there exist compact subsets $K\subset U$ and $L\subset V$ such that $C = K\cup L$.

Lemma 119.21. (Fundamental Cycle Decomposition Lemma) Let X be a compact oriented n-dimensional topological manifold and let U,V be open subsets such that $X = U\cup V$ and let $K\subset U$ and $L\subset V$ be compact subsets. Then there exist singular n-chains $\alpha_{U\setminus L}$, $\alpha_{U\cap V}$ and $\alpha_{V\setminus K}$ in $U\setminus L$, $U\cap V$ and $V\setminus K$ such that the following three equalities hold:

(a) $\mu^X_{K\cup L} = [\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}]$ in $\mathrm{H}_n(X, X\setminus(K\cup L))$

(b) $\mu^{U\cap V}_{K\cap L} = [\alpha_{U\cap V}]$ in $\mathrm{H}_n(U\cap V, U\cap V\setminus(K\cap L))$

(c) $\mu^U_K = [\alpha_{U\setminus L} + \alpha_{U\cap V}]$ in $\mathrm{H}_n(U, U\setminus K)$.

Proof of Proposition 172.9. Let X be an oriented path-connected n-dimensional topological manifold with $\partial X = \varnothing$. Let $x_0\in X$ and let M be a $\mathbb{Z}[\pi_1(X,x_0)]$-left module. We start with two preliminaries:

(1) Given any two subsets $A\subset B$ and $i\in\mathbb{N}_0$ we write $\mathrm{H}^i_{X,M}(B|A) := \mathrm{H}^i_{X,M}(B, B\setminus A)$.

(2) If U is an open subset of X and $K\subset U$ is compact, then, by the definition on page 3615, we obtain a map

$$-\cap_{\mathbb{Z}} \mu^U_K \colon \mathrm{H}^k_{X,M}(U|K) \to \mathrm{H}^{X,M}_{n-k}(U).$$

Now let let U and V be open subsets with $X = U \cup V$. Furthermore let $K \subset U$ and $L \subset V$ be compact subsets. We consider the following diagram

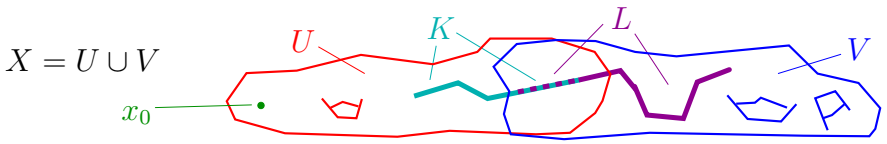
$$\begin{array}{ccccccccc}
.. \to \mathrm{H}^k_{X,M}(X|K\cap L) & \to & \mathrm{H}^k_{X,M}(X|K)\oplus \mathrm{H}^k_{X,M}(X|L) & \longrightarrow & \mathrm{H}^k_{X,M}(X|K\cup L) & \xrightarrow{\delta} & \mathrm{H}^{k+1}_{X,M}(X|K\cap L) & \to .. \\
\downarrow\cong & & \downarrow\cong & & & & \downarrow\cong & \\
\mathrm{H}^k_{X,M}(U\cap V|K\cap L) & & \mathrm{H}^k_{X,M}(U|K)\oplus \mathrm{H}^k_{X,M}(V|L) & & \downarrow\cap_{\mathbb{Z}}\mu^X_{K\cup L} & & \mathrm{H}^{k+1}_{X,M}(U\cap V|K\cap L) & \\
\downarrow\cap_{\mathbb{Z}}\mu^{U\cap V}_{K\cap L} & & \downarrow\cap_{\mathbb{Z}}\mu^U_K\oplus\cap_{\mathbb{Z}}\mu^V_L & & & & \downarrow\cap_{\mathbb{Z}}\mu^{U\cap V}_{K\cap L} & \\
.. \longrightarrow \mathrm{H}^{X,M}_{n-k}(U\cap V) & \longrightarrow & \mathrm{H}^{X,M}_{n-k}(U)\oplus \mathrm{H}^{X,M}_{n-k}(V) & \longrightarrow & \mathrm{H}^{X,M}_{n-k}(X) & \xrightarrow{\partial} & \mathrm{H}^{X,M}_{n-k-1}(U\cap V) & \longrightarrow ..
\end{array}$$

which we now explain in more detail:

(1) We write $A := X \setminus K$ and $B := X \setminus L$. The upper horizontal sequence is the inverted Mayer–Vietoris Sequence for cohomology groups from Theorem 172.10.
(2) The lower horizontal sequence is the long exact sequence coming from the Mayer–Vietoris Theorem 169.7 for Internal Twisted Homology, corresponding to the decomposition $X = U \cup V$.
(3) The vertical maps on the top are all induced by inclusions of pairs of topological spaces, they are isomorphisms by the Excision Theorem 169.6.

The following is the key claim of the proof.

Claim. The first two rectangles of the above diagram commute and the third rectangle commutes up to the sign $(-1)^{k+1}$.



Before we prove the claim we show that it implies the lemma. For each group appearing in the above diagram we take the direct limit over the directed set[2102]

$$\mathcal{K}(U,V) \;=\; \{(K,L) \mid K \text{ compact subset of } U \text{ and } L \text{ compact subset of } V\}$$

and we consider the induced maps of direct limits that were introduced on page 1753. We make the following remarks regarding this new diagram:

(1) Using the Cofinal-Isomorphism Lemma 111.4 one can show that the groups that we obtain taking the direct limit over $\mathcal{K}(U,V)$ are naturally isomorphic to the corresponding cohomology groups with compact support. We carry out the argument for two groups:

$$\varinjlim_{(K,L)\in\mathcal{K}(U,V)} \mathrm{H}^k_{X,M}(U\cap V|K\cap L) \;=\; \varinjlim_{C\in\mathcal{K}(U\cap V)} \mathrm{H}^k_{X,M}(U\cap V|C) \;=\; \mathrm{H}^k_{X,M,c}(U\cap V).$$

we apply the Cofinal-Isomorphism Lemma 111.4 to the cofinal morphism of directed sets given by $(K,L) \mapsto K \cap L$

and

[2102] We view $\mathcal{K}(U,V)$ as a directed set given by the relation $(K,L) \le (K',L')$ if $K \subset K'$ and $L \subset L'$. It is clear that each of the groups in the diagram forms a direct system with respect to this directed set.

$$\varinjlim_{(K,L)\in\mathcal{K}(U,V)} \mathrm{H}^k_{X,M}(X|K\cup L) \;=\; \varinjlim_{C\in\mathcal{K}(X)} \mathrm{H}^k_{X,M}(X|C) \;=\; \mathrm{H}^k_{X,M,c}(X).$$

we apply the Cofinal-Isomorphism Lemma 111.4 to the morphism of directed sets given by $(K,L)\mapsto K\cup L$, this morphism is cofinal by the Compact Subset Decomposition Lemma 119.20

All other cases are treated in a very similar fashion.

(2) By the functoriality of direct limits, and since the signs in the commutativity do not depend on K and L, the new diagram also commutes (up to the same signs).

(3) Using (2) and the definition of the "PD-maps" in Lemma 172.8 one can easily verify that the direct limit of the diagram does indeed give rise to the above diagram specified in the statement of the lemma.

(4) the Direct Limit-Exactness Proposition 79.2 says that the direct limit of exact sequences is again exact.

All these remarks put together show that the direct limit of the above diagram gives rise precisely to the diagram that we claimed exists and that all claimed statements are satisfied. This concludes the proof that the claim implies the lemma.

Proof. Now we turn to the actual proof of the claim. We show in three separate subclaims that the three rectangles commute. The proof of the first two subclaims consists of a clever usage of Proposition 171.13 (1b). The proof of the third subclaim is much more sophisticated since it relies on understanding the connecting homomorphisms in the Mayer-Vietoris sequences in (co-) homology.

Subclaim 1. The first rectangle of the diagram on page 3619 commutes.

So we need to show that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{H}^k_{X,M}(X, X\setminus(K\cap L)) & \xrightarrow{i\oplus -i} & \mathrm{H}^k_{X,M}(X, X\setminus K)\oplus \mathrm{H}^k_{X,M}(X, X\setminus L) \\
\downarrow\cong & \searrow^{i\oplus -i} & \downarrow\cong \\
\mathrm{H}^k_{X,M}(U\cap V, (U\cap V)\setminus(K\cap L)) & & \mathrm{H}^k_{X,M}(U, U\setminus K)\oplus \mathrm{H}^k_{X,M}(V, V\setminus L) \\
\downarrow\cap_{\mathbb{Z}}\mu^{U\cap V}_{K\cap L} & & \downarrow\cap_{\mathbb{Z}}\mu^{U}_{K}\oplus\cap_{\mathbb{Z}}\mu^{V}_{L} \\
\mathrm{H}^{X,M}_{n-k}(U\cap V) & \xrightarrow{i\oplus -i} & \mathrm{H}^{X,M}_{n-k}(U)\oplus \mathrm{H}^{X,M}_{n-k}(V).
\end{array}$$

Evidently the triangle commutes. We note that the remaining pentagon breaks up into the "direct sum" of two diagrams. By symmetry reasons it suffices to consider the left-hand summands. So let $\phi\in \mathrm{H}^k_{X,M}(X, X\setminus(K\cap L))$. We consider the following diagram of maps between pairs of topological spaces

$$\begin{array}{ccc}
(U\cap V, (U\cap V)\setminus(K\cap L)) & \xrightarrow{=:\alpha} & (U, U\setminus(K\cap L)) \\
\downarrow f\circ\alpha & \swarrow^{f} & \uparrow{\scriptstyle =:\beta} \\
(X, X\setminus(K\cap L)) & \xleftarrow[f\circ\beta]{} & (U, U\setminus K).
\end{array}$$

With this notation we need to show that

$$\alpha_*\big((f\circ\alpha)^*(\phi)\cap_{\mathbb{Z}}\mu^{U\cap V}_{K\cap L}\big) \;=\; (f\circ\beta)^*(\phi)\cap_{\mathbb{Z}}\mu^{U}_{K}.$$

But this is indeed the case, since

$$\begin{array}{ccccc}
\alpha_*\big(\underbrace{(f\circ\alpha)^*}_{=\alpha^*\circ f^*}(\phi)\cap_{\mathbb{Z}}\mu_{K\cap L}^{U\cap V}\big) & = & f^*(\phi)\cap_{\mathbb{Z}}\alpha_*(\mu_{K\cap L}^{U\cap V}) & = & f^*(\phi)\cap_{\mathbb{Z}}\mu_{K\cap L}^{U} \\
\uparrow & & \uparrow & & \uparrow \\
\text{Proposition 171.13 (1b)} & & \text{by Corollary 106.12 we have } \alpha_*(\mu_{K\cap L}^{U\cap V})=\mu_{K\cap L}^{U} & & \\
\beta_*\big((f\circ\beta)^*(\phi)\cap_{\mathbb{Z}}\mu_K^U\big) & = & (f\circ\beta)^*(\phi)\cap_{\mathbb{Z}}\mu_K^U & & \\
\uparrow & & \uparrow & & \\
\text{same argument backwards with the roles of } \alpha \text{ and } \beta \text{ exchanged} & & \text{since } \beta \text{ is on the first coordinate the identity} & &
\end{array}$$

This concludes the proof of the Subclaim 1.

Subclaim 2. The second rectangle of the diagram on page 3619 commutes.

The proof of that subclaim is very similar to the proof of the previous subclaim. Thus we leave it to the reader to fill in the details.

Subclaim 3. The third rectangle of the diagram on page 3619 commutes up to the sign $(-1)^{k+1}$.

As a reminder, we need to show that the following diagram commutes up to the sign $(-1)^{k+1}$:

$$\begin{array}{ccccc}
\mathrm{H}^k_{X,M}(X, X\setminus K\cup L) & \xrightarrow{\delta} & \mathrm{H}^{k+1}_{X,M}(X, X\setminus(K\cap L)) & \xrightarrow{\cong} & \mathrm{H}^{k+1}_{X,M}(U\cap V, (U\cap V)\setminus(K\cap L)) \\
\downarrow{\scriptstyle \cap_{\mathbb{Z}}\mu_{K\cup L}} & & & & \downarrow{\scriptstyle \cap_{\mathbb{Z}}\mu_{K\cap L}} \\
\mathrm{H}^{X,M}_{n-k}(X) & & \xrightarrow{\qquad\qquad\partial\qquad\qquad} & & \mathrm{H}^{X,M}_{n-k-1}(U\cap V).
\end{array}$$

So let $[\varphi]\in \mathrm{H}^k_{X,M}(X, X\setminus(K\cup L)) = \mathrm{H}^k_{X,M}(X, A\cap B)$. We start out with the following preparations:

(1) By Theorem 172.10 (1) we can find $\varphi_A\in \mathrm{C}^k_{X,M}(X,A)$ and $\varphi_B\in \mathrm{C}^k_{X,M}(X,B)$ such that $\varphi=\varphi_A+\varphi_B$.
(2) By the Fundamental Cycle Decomposition Lemma 119.21 there exist singular chains $\alpha_{U\setminus L}$, $\alpha_{U\cap V}$ and $\alpha_{V\setminus K}$ in $U\setminus L$, $U\cap V$ and $V\setminus K$ such that the following three equalities hold:

$$\begin{array}{lrcll}
\text{(2a)} & \mu_{K\cup L} & = & [\alpha_{U\setminus L}+\alpha_{U\cap V}+\alpha_{V\setminus K}] & \text{in } \mathrm{H}_n(X, X\setminus(K\cup L)) \\
\text{(2b)} & \mu_{K\cap L}^{U\cap V} & = & [\alpha_{U\cap V}] & \text{in } \mathrm{H}_n(U\cap V, (U\cap V)\setminus(K\cap L)) \\
\text{(2c)} & \mu_K^U & = & [\alpha_{U\setminus L}+\alpha_{U\cap V}] & \text{in } \mathrm{H}_n(U, U\setminus K).
\end{array}$$

After these preparations we obtain the following equalities in $\mathrm{H}^{X,M}_{n-k-1}(U \cap V)$:[2103]

$$\begin{array}{ccccc}
\text{maps in the rectangle applied clockwise} & & \overset{\text{Theorem 172.10 and (2b)}}{\overset{\downarrow}{=}} & & [\delta\varphi_A \cap_{\mathbb{Z}} \alpha_{U\cap V}] \\
\text{to some } [\varphi] \in \mathrm{H}^k_{X,M}(M, M\setminus(K\cup L)) & & & & \\
= & [\varphi_A \cap_{\mathbb{Z}} \partial\alpha_{U\cap V}] & = & [\varphi_A \cap_{\mathbb{Z}} \partial\alpha_{U\cap V} - \underbrace{\varphi_A \cap_{\mathbb{Z}} \partial(\alpha_{U\cap V} + \alpha_{U\setminus L})}_{=0}] & = [-\varphi_A \cap_{\mathbb{Z}} \partial\alpha_{U\setminus L}] \\
& \uparrow & & \uparrow & \\
& \text{Lemma 171.9} & & \text{note that (2c) implies that } \partial(\alpha_{U\cap V}+\alpha_{U\setminus L}) \text{ is a singular chain in } U\setminus K, & \\
& & & \text{but } \varphi_A \text{ vanishes on singular chains in } A = X\setminus K & \\
= & [-\varphi_A \cap_{\mathbb{Z}} \partial\alpha_{U\setminus L} - \underbrace{\varphi_B \cap_{\mathbb{Z}} \partial\alpha_{U\setminus L}}_{=0}] & = & [-(\varphi_A+\varphi_B) \cap_{\mathbb{Z}} \partial\alpha_{U\setminus L}] & = [-\varphi \cap_{\mathbb{Z}} \partial\alpha_{U\setminus L}] \\
\uparrow & & & & \uparrow \\
\text{since } \varphi_B \text{ vanishes on singular chains in } B = X\setminus L & & & & \text{since } \varphi = \varphi_A + \varphi_B \\
= & [(-1)^{k+1} \cdot \partial(\varphi \cap_{\mathbb{Z}} \alpha_{U\setminus L})] & = & (-1)^{k+1} \cdot \partial[\underbrace{(\varphi \cap_{\mathbb{Z}} \alpha_{U\setminus L})}_{\in \mathrm{C}_{n-k}(U)} + \underbrace{(\varphi \cap_{\mathbb{Z}} \alpha_{U\cap V} + \varphi \cap_{\mathbb{Z}} \alpha_{V\setminus K})}_{\in \mathrm{C}_{n-k}(V)}] & \\
\uparrow & & \uparrow & & \\
\text{by Lemma 171.9 and since } \delta\varphi = 0 & & \text{by the definition of the connecting homomorphism } \partial \text{ in the Mayer–Vietoris sequence, see Theorem 169.7} & & \\
= & (-1)^{k+1} \cdot \partial\big([\varphi] \cap \mu^X_{K\cup L}\big) = (-1)^{k+1} \cdot & \text{maps in the rectangle applied counterclockwise to } [\varphi]. & & \\
\uparrow & & & & \\
\text{by equality (2a)} & & & &
\end{array}$$

This concludes the proof of Subclaim 3, thus of the claim and thus also of the lemma. ■

Now we turn to the actual proof of Theorem 172.5.

Proof of Theorem 172.5. We fix a dimension n. Throughout this proof by a "manifold" we mean a path-connected oriented n-dimensional topological manifold without boundary.

We introduce two closely related notions.

(1) We say *Poincaré Duality holds for a manifold* X if for each $x_0 \in X$, each $\mathbb{Z}[\pi_1(X, x_0)]$-left module M and for each $k \in \mathbb{N}_0$ the following map is an isomorphism:

$$\mathrm{PD}_X \colon \mathrm{H}^k_{\mathrm{tw},c}(X; M) \to \mathrm{H}^{\mathrm{tw}}_{n-k}(X; M).$$

(2) Let X be a manifold and let $U \subset X$ be an open subset. We say *Internal Poincaré Duality holds for* U *within* X if for each $x_0 \in X$, each $\mathbb{Z}[\pi_1(X, x_0)]$-left module M and for each $k \in \mathbb{N}_0$ the following map is an isomorphism:

$$\mathrm{PD}_U \colon \mathrm{H}^k_{X,M,c}(U) \to \mathrm{H}^{X,M}_{n-k}(U).$$

Our goal is to show that Poincaré Duality holds for all manifolds. We try to follow the logic of the proof of Theorem 172.5. But now we need an extra initial claim.

Claim 0. Let X be a manifold and let $U \subset X$ be an open subset. If each path-component of U, viewed as a manifold in its own right, satisfies Poincaré Duality, then U satisfies Internal Poincaré Duality within X.

Proof. Let X be a path-connected manifold.

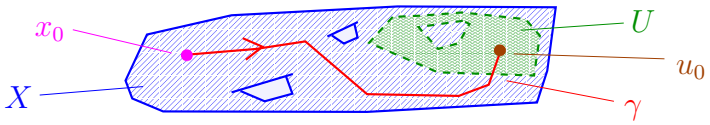
(1) First let $U \subset X$ be a path-connected open subset that, if viewed as n-dimensional topological manifold in its own right, satisfies Poincaré Duality. We need to show

[2103] The maps δ and ∂ written in blue are the connecting homomorphisms in the Mayer–Vietoris sequences coming from Theorem 169.7 and Theorem 172.10.

that U also satisfies Internal Poincaré Duality within X. Let $x_0 \in X$. We pick $u_0 \in U$. Since X is path-connected there exists a path $\gamma\colon [0,1] \to X$ from x_0 to u_0. We denote by $i\colon U \to X$ the inclusion. We consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}^k_{\mathrm{tw},c}(U; i^*\gamma_* M) & \xleftarrow[\cong]{\Omega^X_U} & \mathrm{H}^k_{X,\gamma_* M,c}(U) & \xrightarrow[\cong]{\gamma_*} & \mathrm{H}^k_{X,M,c}(U) \\
\cong\downarrow \mathrm{PD}_U & & \downarrow \mathrm{PD}_U & & \downarrow \mathrm{PD}_U \\
\mathrm{H}^{\mathrm{tw}}_{n-k}(U; i^*\gamma_* M) & \xrightarrow[\cong]{\Omega^X_U} & \mathrm{H}^{X,\gamma_* M}_{n-k}(U) & \xrightarrow[\cong]{\gamma_*} & \mathrm{H}^{X,M}_{n-k}(U).
\end{array}$$

Here the horizontal maps to the left are given by Lemma 172.7. The horizontal maps to the right are isomorphisms that are defined as in the Change of Base Point Proposition II 169.5. It follows easily from Lemma 172.7 (2) that the square to the left commutes. Furthermore it is straightforward to verify that the square to the right commutes. By hypothesis the vertical map to the left is an isomorphism. It follows from the above that the vertical map to the right is also an isomorphism.



(2) Now suppose that U has path-components $U_1, \ldots, U_k$ such that each U_i satisfies Poincaré Duality viewed as a manifold. By (1) we know that each U_i also satisfies Internal Poincaré Duality within X. As in the Cup-Cap Disjoint Union Lemma 116.9 one can show easily, using Lemma 169.3, that $U_1 \cup \cdots \cup U_k$ also satisfies Internal Poincaré Duality within X. ⊞

Claim 1. Let X be a manifold.

(A) If Poincaré Duality holds for X equipped with an orientation, then it also holds for any other orientation of X.

(B) If X is homeomorphic to a manifold for which Poincaré Duality holds, then Poincaré Duality also holds for X.

(C) Suppose that X is the union of two open subsets U and V. If Poincaré Duality holds for each path-component of U, V and $U \cap V$, then it also holds for X.

(D) If X is the union of a nested sequence $U_1 \subset U_2 \subset \ldots$ of open subsets such that Poincaré Duality holds for each path-component of U_i, then it also holds for X.

Proof. So let X be a manifold.

(A) Recall that by our convention in this proof every manifold is path-connected. By Lemma 105.10 the oriented path-connected manifold X admits only one other orientation, namely the reverse orientation $-X$. It follows from that $[-X] = -[X]$ that Poincaré Duality also holds for $-X$.

(B) If we have an orientation-preserving homeomorphism, then this statement follows immediately from the Proposition 171.8. For a non-orientation-preserving homeomorphism this statement follows from (A).

(C) We suppose that Poincaré Duality holds for each path-component of U, V and $U \cap V$. It follows from Claim 0 that Internal Poincaré Duality holds for U, V and $U \cap V$ within X. The combination of Proposition 172.9 and the Five Lemma 74.10 now

shows that also each PD_X is an isomorphism, i.e. Poincaré Duality also holds for $X = U \cup V$.

(D) Let $k \in \mathbb{N}_0$. We suppose that each path-component of each $U_i \subset X$ satisfies Poincaré Duality. We need to show that X also satisfies Poincaré Duality. Let $x_0 \in X$ and let M be a $\mathbb{Z}[\pi_1(X, x_0)]$-left module. By hypothesis and Claim 0 we know that for every $i \in \mathbb{N}$ the map $\mathrm{PD}_{U_i} \colon \mathrm{H}^k_{X,M,\mathrm{c}}(U_i) \to \mathrm{H}^{X,M}_{n-k}(U_i)$ is an isomorphism. Using the pretty obvious variation on Proposition 171.13 and using Lemma 172.8 it is straightforward to verify that for any $i \leq j$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k_{X,M,c}(U_i) & \xrightarrow{\mathrm{PD}_{U_i}} & \mathrm{H}^{X,M}_{n-k}(U_i) \\ \downarrow & & \downarrow \\ \mathrm{H}^k_{X,M,c}(U_j) & \xrightarrow{\mathrm{PD}_{U_j}} & \mathrm{H}^{X,M}_{n-k}(U_j). \end{array}$$

We obtain isomorphisms

$$\overbrace{\mathrm{H}^k_{X,M,\mathrm{c}}(X)}^{=\mathrm{H}^k_{\mathrm{tw},c}(X;M)} \underset{\uparrow}{=} \varinjlim \mathrm{H}^k_{X,M,\mathrm{c}}(U_i) \underset{\uparrow}{\xrightarrow{\varinjlim \mathrm{PD}_{U_i}}} \varinjlim \mathrm{H}^{X,M}_{n-k}(U_i) \underset{\uparrow}{=} \overbrace{\mathrm{H}^{X,M}_{n-k}(X).}^{=\mathrm{H}^{\mathrm{tw}}_{n-k}(X;M)}$$

by a straightforward generalization of Exhaustion-H^*_c-Proposition 112.6

isomorphism by our hypothesis and since the above diagram commutes

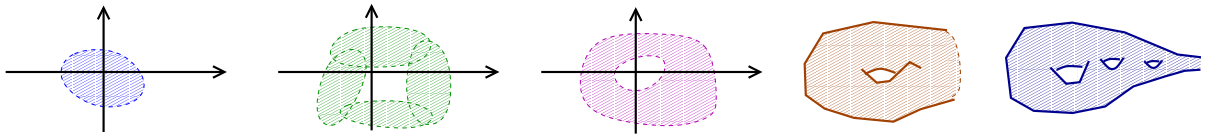
by a straightforward generalization of the Homology-via-Exhaustions Proposition 79.4

We leave it to the reader to verify that the map from left to right is indeed given by PD_X. ⊞

The following claim completes the proof of the theorem.

Claim 2.

(A) Poincaré Duality holds for every open bounded convex subset of $\mathbb{R}^n$.
(B) For every $m \in \mathbb{N}$ Poincaré Duality holds for every path-connected sum of $\mathbb{R}^n$ that is the union of m open bounded convex subsets of $\mathbb{R}^n$.
(C) Poincaré Duality holds for every path-connected open subset of $\mathbb{R}^n$.
(D) For every $m \in \mathbb{N}$ Poincaré Duality holds for all manifolds that admit an atlas with m charts.
(E) Poincaré Duality holds for all manifolds.



Proof. [2104]

(1) Let X be an open bounded convex subset of $\mathbb{R}^n$. If X is empty, then there is nothing to show. Now assume that X is non-empty. By the Convex-to-Ball Proposition 2.20

[2104] The proof is basically the same as the proof of Claim 2 on page 2638 of the untwisted case.

(1) there exists a homeomorphism $X \to B^n$. It follows from Lemma 172.6 and Claim 1 (B) that Poincaré Duality also holds for X.

(2) We prove the claim by induction on m. By (A) we know that the statement holds for $m = 1$. Now suppose it holds for all path-connected subsets of $\mathbb{R}^n$ that are unions of $m-1$ open bounded convex subsets of $\mathbb{R}^n$. Let X be a path-connected subset of $\mathbb{R}^n$ that is the union of m open bounded convex subsets $U_1, \ldots, U_m$. We set $V = U_1 \cup \cdots \cup U_{m-1}$. By our hypothesis and by induction the statement holds for U_m, for V and also for each path-component of $V \cap U_m = (U_1 \cap U_m) \cup \cdots \cup (U_{m-1} \cap U_m)$, since each path-component is the union of $(m-1)$ open bounded convex subsets. It follows from Claim 1 (C) that Poincaré Duality also holds for

$$X \;=\; V \cup U_m \;=\; (U_1 \cup \cdots \cup U_{m-1}) \cup U_m.$$

(3) Let X be an arbitrary path-connected open subset of $\mathbb{R}^n$. By the Rational Open Balls Lemma 118.4 we can write X as the union of countably many open balls $B_i, i \in \mathbb{N}$. For $i \in \mathbb{N}$ we set $U_i := \bigcup_{j \le i} B_j$. Each path-component of any U_i is evidently also the union of finitely many open balls. Thus we obtain from (B) that Poincaré Duality holds for each path-component of each $U_i \in \mathbb{N}$. But then it follows from Claim 1 (D) that Poincaré Duality also holds for X.

(4) We prove this statement by induction on m. The case $m = 1$ is taken care of by (C) and Claim 1 (B). Now suppose Poincaré Duality holds for all manifolds that admit an atlas with $m-1$ charts. Let X be a manifold that admits an atlas $\{\phi_i \colon U_i \to V_i\}_{i=1,\ldots,m}$ with m charts. We write $U = U_1 \cup \cdots \cup U_{m-1}$ and $V = U_m$. Both are open subsets of a manifold. Note that U and V admit an atlas with $m-1$ charts respectively one chart. Similarly $U \cap V = (U_1 \cap U_m) \cup \cdots \cup (U_{m-1} \cap U_m)$ admits an atlas with $m-1$ charts. It follows easily from induction that Poincaré Duality holds for each path-component of U, V and $U \cap V$. Thus it follows from Claim 1 (C) that Poincaré Duality also holds for X itself.

(5) Finally let X be any manifold. By the Countable Covering Lemma 9.2 we know that X admits a countable atlas $\{\Phi_j \colon U_j \to V_j\}_{j \in \mathbb{N}}$. For $i \in \mathbb{N}$ we set $X_i := \bigcup_{j \le i} U_j$. It follows easily from (D) that Poincaré Duality holds for each path-component of each X_i. It follows from Claim 1 (D) that Poincaré Duality also holds for X. ∎

172.3. **The proof of Poincaré Duality for topological manifolds with boundary.** Now we will finally prove the general statement of the Poincaré Duality Theorem 172.1. For the reader's convenience we recall the statement.

Theorem 172.1. (Poincaré Duality Theorem with Twisted Coefficients) Let X be a compact oriented path-connected n-dimensional topological manifold and let $x_0 \in X$. We denote by $[X] \in \mathrm{H}_n(X, \partial X)$ the fundamental class. Furthermore suppose that we are given a decomposition $\partial X = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂X such that $A \cap B = \partial A = \partial B$. For each $k \in \mathbb{N}_0$ and every $\mathbb{Z}[\pi_1(X, x_0)]$-left module M the map

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(X, A; M) & \xrightarrow{\cong} & \mathrm{H}^{\mathrm{tw}}_{n-k}(X, B; M) \\ \sigma & \mapsto & \sigma \cap_{\mathbb{Z}} [X] \end{array}$$

is an isomorphism.

In the following we will prove the Poincaré Duality Theorem 172.1 using Theorem 172.5 and formal arguments that have basically nothing to do with twisted coefficients. The proofs below are very very similar to the arguments for the untwisted case that we provided in Section 119.4. For peace of mind let us nonetheless go through the arguments.

Proof of Theorem 172.1 for $A = \partial X$ and $B = \varnothing$. Let X be a compact oriented path-connected n-dimensional topological manifold and let $x_0 \in X$. As usual we denote by $[X] \in \mathrm{H}_n(X, \partial X)$ the fundamental class. Let M be a $\mathbb{Z}[\pi_1(X, x_0)]$-left module. We want to show that for each $k \in \mathbb{N}_0$ the map

$$- \cap_{\mathbb{Z}} [X] \colon \mathrm{H}^k_{\mathrm{tw}}(X, \partial X; M) \ \to \ \mathrm{H}^{\mathrm{tw}}_{n-k}(X; M)$$

is an isomorphism. To do so we first introduce the following objects:

(a) From the Topological Collar Neighborhood Theorem 75.5 we obtain a collar neighborhood $[-1, 0] \times \partial X$ of $\partial X = \{0\} \times X$. We refer to it as the "internal collar".

(b) We define

$$W \ := \ X \cup ([0, 1) \times \partial X)/\sim \qquad \text{with } x \sim (x, 0) \text{ for every } x \in \partial X.$$

Sometimes $[0, 1) \times \partial X$ is called an *external collar of* X. Furthermore, given any $t \in [-1, 1)$ we write

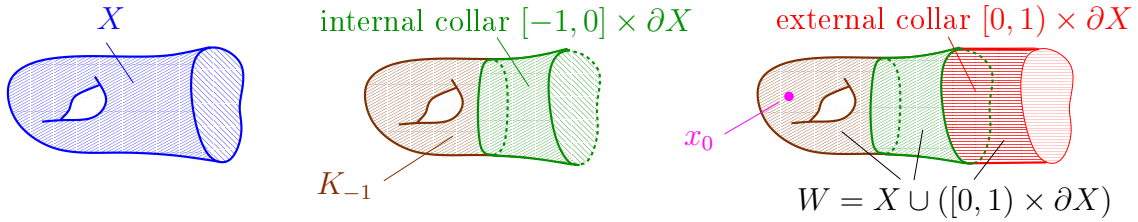
$$K_t \ := \ (X \setminus ([-1, 0] \times \partial X)) \cup ([-1, t] \times \partial X)/\sim .$$

(c) We denote by $i \colon X \to W$ the obvious map. By the Topological Pushout-Maps Lemma 5.34 (3) and (4) this map is a closed embedding. We will use this to view X as a closed subset of W. Note that it follows easily from the Homotopy Pushout Lemma 14.9 that $X = i(X)$ is a deformation retract of W.

(d) We denote by $i \colon (X, \partial X) \to (W, W \setminus K_{-1})$ the inclusion map.

(e) We equip W with the base point x_0.

(f) By the Homotopy-π_1-Proposition 50.3 we know that $i_* \colon \pi_1(X, x_0) \to \pi_1(W, x_0)$ is an isomorphism. This allows us to consider the $\mathbb{Z}[\pi_1(W, x_0)]$-left module $(i_*)^{-1}M$.



By elementary arguments and using the Topological Manifold Boundary Proposition 75.2 (1) one can easily show the following statements:

(A) W is an n-dimensional topological manifold with empty boundary.

(B) W admits an orientation such that X is a submanifold and such that $X \to W$ is orientation-preserving.

(C) Every K_t is a compact subset of W and given any compact subset K of W there exists a $t \in [-1, 1)$ with $K \subset K_t$.

We obtain the following diagram:

$$\begin{array}{ccccc} \mathrm{H}^k_{\mathrm{tw}}(X,\partial X;M) & \xleftarrow{i^*} & \mathrm{H}^k_{\mathrm{tw}}(W,W\setminus K_{-1};(i_*)^{-1}M) & \longrightarrow & \mathrm{H}^k_{\mathrm{tw,c}}(W;(i_*)^{-1}M) \\ \downarrow{\scriptstyle \cap_{\mathbb{Z}}[X]} & & \downarrow{\scriptstyle \cap_{\mathbb{Z}}\mu_{K_t}} & & \downarrow{\scriptstyle \mathrm{PD}_W} \\ \mathrm{H}^{\mathrm{tw}}_{n-k}(X;M) & \xrightarrow{i_*} & \mathrm{H}^{\mathrm{tw}}_{n-k}(W;(i_*)^{-1}M) & \xrightarrow{\mathrm{id}} & \mathrm{H}^{\mathrm{tw}}_{n-k}(W;(i_*)^{-1}M). \end{array}$$

It remains to prove the following claim.

Claim. The following three statements hold:

(1) the diagram commutes,
(2) the right vertical map is an isomorphism, and
(3) all horizontal maps are isomorphisms.

Proof.

(1) The right-hand square commutes by definition of PD_W. Using (B) one can show easily, using Proposition 171.13, that the left-hand square commutes.
(2) Theorem 172.5 implies that the right vertical map is an isomorphism.
(3) It follows easily from the discussion in (c) together with the Deformation Retract-Twisted Homology Corollary 167.16 that the left horizontal maps are isomorphisms. Since the bottom right map is the identity it remains to show that the top right horizontal map is an isomorphism. We consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(W,W\setminus K_{-1};(i_*)^{-1}M) & \longrightarrow & \overbrace{\varinjlim_{K\in\mathcal{K}(W)} \mathrm{H}^k_{\mathrm{tw}}(W,W\setminus K;(i_*)^{-1}M)}^{=\mathrm{H}^k_{\mathrm{tw,c}}(W;(i_*)^{-1}M)}. \\ & \searrow \quad \nearrow & \\ & \varinjlim_{t\in[-1,1)} \mathrm{H}^k_{\mathrm{tw}}(W,W\setminus K_t;(i_*)^{-1}M) & \end{array}$$

We make the following two observations:

- By (C) every compact subset of W is contained in some K_t. Therefore it follows from the Limit-over-Subset Lemma 111.3 that the right diagonal map is an isomorphism.
- It follows easily from the Homotopy Equivalence-H_*-Proposition 167.15 (2) and Lemma 111.2 (4) that the left diagonal map is an isomorphism.

These two observations together imply that the horizontal map is also an isomorphism.

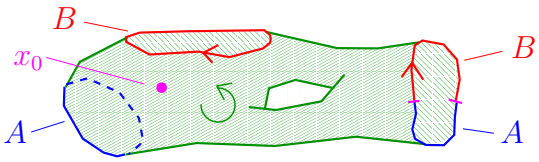
This concludes the proof of the claim and it concludes the proof of the Poincaré Duality Theorem 172.1 in the special case that $A = \partial X$ and $B = \varnothing$. ∎

Proof of Theorem 172.1 for the general case. Let X be a compact oriented path-connected n-dimensional topological manifold and let $x_0 \in X$. As always we denote by $[X] \in \mathrm{H}_n(X,\partial X)$ the fundamental class. Let M be a $\mathbb{Z}[\pi_1(X,x_0)]$-left module M. Finally suppose that we are given a decomposition $\partial X = A \cup B$ where A and B are compact

$(n-1)$-dimensional submanifolds of ∂X such that $A \cap B = \partial A = \partial B$. We want to show that for each $k \in \mathbb{N}_0$ the map

$$\cap_{\mathbb{Z}}[X] \colon \mathrm{H}^k_{\mathrm{tw}}(X, A; M) \;\to\; \mathrm{H}^{\mathrm{tw}}_{n-k}(X, B; M)$$

is an isomorphism.



We equip ∂X and then also B with the orientations coming from the Topological Manifold-Boundary Orientation Proposition 105.19 and the Topological Manifold-Induced Orientation Lemma 105.17. As before we work again with internal twisted (co-) homology. We consider the following diagram:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & \mathrm{H}^k_{X,M}(X, \partial X) & \longrightarrow & \overbrace{\mathrm{H}^k_{X,M}(X, A)}^{=\mathrm{H}^k_{\mathrm{tw}}(X,A;M)} & \longrightarrow & \mathrm{H}^k_{X,M}(\partial X, A) & \xrightarrow{\delta} & \mathrm{H}^{k+1}_{X,M}(X, \partial X) \longrightarrow \dots \\
 & & \cong \downarrow -\cap_{\mathbb{Z}}[X] & & \downarrow -\cap_{\mathbb{Z}}[X] & & \cong \downarrow i^* & & \cong \downarrow -\cap_{\mathbb{Z}}[X] \\
 & & & & & & \mathrm{H}^k_{X,M}(B, \partial B) & & \\
 & & & & & & \cong \downarrow -\cap_{\mathbb{Z}}[B] & & \\
\dots & \longrightarrow & \mathrm{H}^{X,M}_{n-k}(X) & \longrightarrow & \underbrace{\mathrm{H}^{X,M}_{n-k}(X, B)}_{=\mathrm{H}^{\mathrm{tw}}_{n-k}(X,B;M)} & \xrightarrow{\partial} & \mathrm{H}^{X,M}_{n-k-1}(B) & \longrightarrow & \mathrm{H}^{X,M}_{n-k-1}(X) \longrightarrow \dots
\end{array}$$

We make the following observations:

(1) The top and bottom horizontal sequences are the long exact sequence in internal (co-) homology with twisted coefficients of the triple $(X, \partial X, A)$ and the pair (X, B), that we implicitly constructed in the proof of Proposition 167.11.
(2) The above vertical map $i^* \colon \mathrm{H}^k_{X,M}(\partial X, A) \to \mathrm{H}^k_{X,M}(B, \partial B)$ is induced by the inclusion $i \colon (B, \partial B) \to (\partial X, A)$. It is an isomorphism by Proposition 169.2 (5) together with the obvious twisted version of the Excisive Characterization Lemma 109.13 (3).
(3) The other maps decorated with "$\cong$" are isomorphisms by the special case of the Poincaré Duality Theorem 172.1 that we have already dealt with.

By the Five Lemma 74.10 it remains to prove the following claim.

Claim. The above diagram commutes up to signs, more precisely:
(1) the first rectangle commutes,
(2) the second rectangle commutes up to the sign $(-1)^k$,
(3) the third rectangle commutes.

Proof. We consider the three rectangles separately.

(1) We consider the following diagram:

$$\begin{array}{ccc}
\mathrm{H}^k_{X,M}(X,\partial X)\times \mathrm{H}_n(X,\partial X) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{X,M}_{n-k}(X) \\
{\scriptstyle =}\uparrow \quad \downarrow{\scriptstyle =} & & \downarrow \\
\mathrm{H}^k_{X,M}(X,\partial X)\times \mathrm{H}^{X,M}_n(X,\partial X) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}_{n-k}(X,B) \\
\downarrow \quad \uparrow{\scriptstyle =} & & {\scriptstyle =}\uparrow \\
\mathrm{H}^k_{X,M}(X,A)\times \mathrm{H}^{X,M}_n(X,\partial X) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}_{n-k}(X,B).
\end{array}$$

We can convince ourselves that the diagram commutes by applying Proposition 171.13 (1b) to the maps $(X,\partial X,\varnothing)\to(X,\partial X,B)$ and $(X,A,B)\to(X,\partial X,B)$ of universally excisive triads. It follows easily from this observation that the first rectangle of the claim commutes.

Before we continue with the discussion of the other two rectangles we need to make one remark. We denote by $j\colon (B,\partial B)\to(\partial X,A)$ the inclusion map and we denote by $\partial\colon \mathrm{H}_n(X,\partial X)\to \mathrm{H}_{n-1}(\partial X,A)$ the connecting homomorphism in the long exact sequence of the triple $(X,\partial X,A)$. It follows from the Fundamental Class-of-Boundary Proposition 106.27 (1), the Codimension Zero-Fundamental Class Lemma 106.24 and the remark on page 1629 that the following equality holds

$$(*)\qquad\qquad j_*([B]) \;=\; \partial[X]\in \mathrm{H}_{n-1}(\partial X,A).$$

Now we continue with the proof of the claim.

(2) We apply the twisted analogue of the Cap Product-Connecting Homomorphism Lemma 116.10 (2) to the universally excisive triad (X,A,B). (Recall that $A\cup B=\partial X$ and $A\cap B=\partial B$.) We obtain that for any $k\in\mathbb{N}_0$ the following diagram commutes up to the sign $(-1)^k$:

$$\begin{array}{ccc}
\mathrm{H}^k_{X,M}(X,A)\times \mathrm{H}_n(X,\partial X) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{X,M}_{n-k}(X,B) \\
{\scriptstyle i^*}\downarrow \quad \partial\downarrow & & \\
\mathrm{H}^k_{X,M}(\partial X,A)\times \mathrm{H}_{n-1}(\partial X,A) & & \Big\downarrow{\scriptstyle \partial} \\
{\scriptstyle j^*}\downarrow{\scriptstyle \cong} \quad {\scriptstyle (j_*)^{-1}}\downarrow{\scriptstyle \cong} & & \\
\mathrm{H}^k_{X,M}(B,\partial B)\times \mathrm{H}_{n-1}(B,\partial B) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{X,M}_{n-k-1}(B).
\end{array}$$

Together with $(*)$ we obtain that the second rectangle of the claim does indeed commute up to the sign $(-1)^k$.

(3) We consider the following diagram:

$$\begin{array}{ccc}
\mathrm{H}^k_{X,M}(B,\partial B)\times \mathrm{H}_{n-1}(B,\partial B) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{X,M}_{n-k-1}(B) \\
{\scriptstyle i^*}\uparrow \quad \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\
\mathrm{H}^k_{X,M}(\partial X,A)\times \mathrm{H}_{n-1}(\partial X,A) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{X,M}_{n-k-1}(\partial X) \\
{\scriptstyle \delta}\downarrow \quad \uparrow{\scriptstyle \partial} & & \downarrow{\scriptstyle i_*} \\
\mathrm{H}^{k+1}_{X,M}(X,\partial X)\times \mathrm{H}_n(X,\partial X) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{X,M}_{n-k-1}(X).
\end{array}$$

The upper part of this diagram commutes by Proposition 171.13 (1b) applied to the map $(B, \partial B, \varnothing) \to (\partial X, A, \varnothing)$ between excisive triads. The lower part commutes by the twisted analogue of the Cap Product-Connecting Homomorphism Lemma 116.10 (1) applied to the triple $(X, \partial X, A)$. This discussion, together with $(*)$, implies that the third rectangle of the diagram in the claim commutes.

This concludes the proof of the claim and thus also of the Poincaré Duality Theorem 172.1. ■

Exercises for Chapter 172.

Exercise 172.1. Let Σ_g be the surface of genus g and let $x_0 \in X$. Furthermore let $\varphi\colon \pi_1(\Sigma_g, x_0) \to \mathrm{GL}(1, \mathbb{C})$ be a one-dimensional non-trivial representation. Use the Poincaré Duality Theorem with Twisted Coefficients 172.1 together with Propositions 168.1 and 170.4 to calculate the twisted homology modules $\mathrm{H}_k^{\mathrm{tw}}(\Sigma_g; {}_\varphi\mathbb{C}) \cong \mathbb{C}^{k-1}$ in a pain-free way.

173. Twisted intersection pairings

We continue our discussion of twisted invariants. In this chapter we introduce and discuss the twisted intersection pairings, which is the twisted cousin of the intersection pairings that we introduced on page 2895.

173.1. **Definition of twisted intersection pairings.** Before we turn to topology it is worth introducing some algebraic notions which are generalizations and analogues of the concepts introduced on page 2656.

Definition.

(1) Let V, W, Z be abelian groups. We refer to any homomorphism $V \otimes_{\mathbb{Z}} W \to Z$ of abelian groups as a pairing.

(2) Let G be a group, let V, W be two $\mathbb{Z}[G]$-left modules and let Z be an abelian group. We say a pairing $\varphi\colon V \otimes_{\mathbb{Z}} W \to Z$ is preserved by the G-action if φ is a homomorphism of $\mathbb{Z}[G]$-left modules where we equip $V \otimes_{\mathbb{Z}} W$ with the $\mathbb{Z}[G]$-left module structure from Lemma 171.2 and where we equip Z with the trivial $\mathbb{Z}[G]$-left module structure.

Now let R be a ring with (possibly trivial) involution, let V and W be R-left modules and let $\varphi\colon V \otimes_{\mathbb{Z}} W \to R$ be a pairing. We equip $V \otimes_{\mathbb{Z}} W$ with the $R \otimes_{\mathbb{Z}} R$-left module structure from Lemma 171.2.

(3) The pairing φ is called R-sesquilinear if $\varphi\colon V \otimes_{\mathbb{Z}} W \to R$ is a $R \otimes_{\mathbb{Z}} R$-left module homomorphism where we equip R with the $R \otimes_{\mathbb{Z}} R$-left module structure given by $(r \otimes s) \cdot t := r \cdot t \cdot \overline{s}$.

(4) Let $\varphi\colon V \otimes_{\mathbb{Z}} V \to R$ be an R-sesquilinear pairing.

(a) We say that φ is hermitian if $\varphi(v \otimes w) = \overline{\varphi(w \otimes v)}$ for all $v, w \in V$.

(b) We say that φ is anti-hermitian if $\varphi(v \otimes w) = -\overline{\varphi(w \otimes v)}$ for all $v, w \in V$.

If the involution is trivial, then usually we say R-bilinear instead of R-sesquilinear and we say (anti-) symmetric instead of (anti-) hermitian.

For readers who would rather avoid tensor products we have the following definition.

Definition. Let V and W be abelian groups and let Z be an abelian group. We say a map $\varphi\colon V \times W \to Z$ is a pairing if the map

$$\begin{array}{rcl} \varphi\colon V \otimes_{\mathbb{Z}} W & \to & Z \\ \sum_{i=1}^{k} v_i \otimes w_i & \mapsto & \sum_{i=1}^{k} \varphi(v_i, w_i) \end{array}$$

is well-defined and if it is a pairing.[2105] Using this definition one can translate the above notions into arguably more familiar language. For example if G is a group and V and W are two $\mathbb{Z}[G]$-left modules, then a pairing $\varphi\colon V \times W \to Z$ is preserved by the G-action precisely if for all $v \in V, w \in W$ and $g \in G$ we have $\varphi(g \cdot v, g \cdot w) = \varphi(v, w)$.

We also recall the following definition that we made in the Twisted H_0-Proposition 168.1.

[2105] It follows easily from the definitions that a map $\psi\colon V \times W \to Z$ is a pairing in this sense if and only if it is a pairing in the sense of the definition 2656. Using this translation many concepts that we define in this chapter correspond to the eponymous concepts of Section 121.1.

Definition. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. Let S be a ring. We consider the $\mathbb{Z}[\pi_1(X,x_0)]$-module ${}_{\mathrm{tr}}S$ that is given by the trivial $\pi_1(X,x_0)$-action. We refer to the map

$$\begin{aligned} \epsilon_X \colon \mathrm{H}_0({}_{\mathrm{tr}}S \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_*(\widetilde{X}_{x_0}) = \mathrm{H}_0^{\mathrm{tw}}(X;{}_{\mathrm{tr}}S) &\to S \\ \Big[\sum_{i=1}^k r_i \otimes \sigma_i\Big] &\mapsto \sum_{i=1}^k r_i \end{aligned}$$

as the augmentation map. By the Twisted H_0-Proposition 168.1 it is an isomorphism of S-left modules.

Now we can finally introduce twisted intersection pairings.

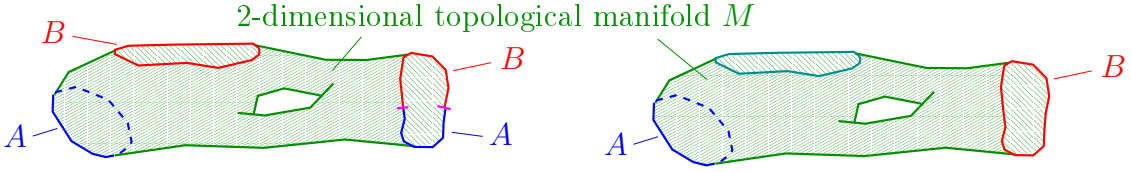
Definition. Let X be a compact oriented path-connected n-dimensional topological manifold and let $x_0 \in X$. We write $\pi = \pi_1(X,x_0)$. We denote by $[X] \in \mathrm{H}_n(X,\partial X)$ the fundamental class. Furthermore suppose that we are given compact $(n-1)$-dimensional submanifolds A and B of ∂X such that $A \cap B = \partial A = \partial B$. Let R be a ring with involution and let M be a $(\mathbb{Z}[\pi],R)$-left left module. Furthermore let

$$\Theta \colon M \otimes_{\mathbb{Z}} M \to R$$

be an R-sesquilinear pairing that is preserved by the π-action, i.e. such that given any $v,w \in M$ and given any $g \in \pi$ we have $\Theta(g\cdot v, g\cdot w) = \Theta(v,w)$. We write $C := \partial X \setminus \mathring{A}$ and $D := \partial X \setminus \mathring{B}$.[2106] For each $k \in \mathbb{N}_0$ we refer to the map[210721082109]

$$\begin{aligned} Q^M_{X,A,B} \colon \mathrm{H}_k^{\mathrm{tw}}(X,A;M) \times \mathrm{H}_{n-k}^{\mathrm{tw}}(X,B;M) &\to R \\ (v,w) &\mapsto \epsilon_X\big(\Theta_*\big(\underbrace{\underbrace{\mathrm{PD}_X(v)}_{\in \mathrm{H}^{n-k}_{\mathrm{tw}}(X,C;M)} \cup_{\otimes} \underbrace{\mathrm{PD}_X(w)}_{\in \mathrm{H}^{k}_{\mathrm{tw}}(X,D;M)}}_{\in \mathrm{H}^{n}_{\mathrm{tw}}(X,\partial X;M\otimes_{\mathbb{Z}} M)}\big) \cap_{\mathbb{Z}} \underbrace{[X]}_{\in \mathrm{H}^{\mathrm{tw}}_n(X,\partial X)}\big). \end{aligned}$$

as a twisted intersection pairing. If X is closed, then $A = B = \varnothing$ and we just write Q^M_X. Furthermore if $M = {}_{\mathrm{tr}}\mathbb{Z}$, then we drop the superscript.[2110]



[2106] Note that by Proposition 75.3 we know that C and D are also compact submanifolds of ∂M with $\partial C = \partial D = A \cap C$ and $\partial D = \partial B = B \cap D$.

[2107] We have

$$C \cup D = (\partial X \setminus \mathring{A}) \cup (X \setminus \mathring{B}) \underset{\text{Proposition 75.3 (3)}}{=} ((\partial X \setminus A) \cup \partial A) \cup ((\partial X \setminus B) \cup \partial B) = (\partial X \setminus (A \cap B)) \cup \partial A \cup \partial B = \partial X.$$

[2108] Note that in the settings (i) and (ii), it follows from Proposition 169.2 (5) and (3) that (M,C,D) is a universally excisive triad. In particular it follows, see the definition on page 3603, that the cup product is actually defined.

[2109] As we discussed above, Θ gives rise to a homomorphism $\Theta \colon M \otimes_{\mathbb{Z}} M \to {}_{\mathrm{tr}}R$ of $\mathbb{Z}[G]$-left modules.

[2110] We now have a minuscule discrepancy in our notation, on pages 2800 and 2895 we defined the intersection form on $\mathrm{FH}_l(X;\mathbb{Z}) := \mathrm{H}_l(X;\mathbb{Z})/\text{torsion}$.

Remark. If we work with the trivial ${}_{\mathrm{tr}}\mathbb{Z}$-coefficients, then it follows from Lemma 171.6 and Lemma 171.12 that the twisted intersection form is isomorphic to the usual intersection form, except that now we do not mod out the torsion on the homology groups.

With the above definition we can now summarize a few basic properties of twisted intersection pairings.

Proposition 173.1. Let X be a compact oriented path-connected n-dimensional topological manifold and let $x_0 \in X$. Furthermore suppose that we are given compact $(n-1)$-dimensional submanifolds A and B of ∂X such that $A \cap B = \partial A = \partial B$. Let R be a ring with involution and let M be a $(\mathbb{Z}[\pi_1(X, x_0)], R)$-left left module. Finally suppose that we are given an R-sesquilinear pairing

$$\Theta\colon M \otimes M \ \to \ R$$

that is preserved by the $\pi_1(X, x_0)$-action. Then the following statements hold:

(1) The pairing $Q^M_{X,A,B}$ is R-sesquilinear.
(2) If $n = 2m$, if $A = B$ and if Θ is hermitian, then the pairing

$$Q^M_{X,A,B}\colon \mathrm{H}^{\mathrm{tw}}_m(X, A; M) \times \mathrm{H}^{\mathrm{tw}}_m(X, A; M) \ \to \ R$$

is hermitian if m is even and it is anti-hermitian if m is odd. If Θ is anti-hermitian, then the statements flips in the obvious way.

Proof. We start out with the following observations:

(a) As we mentioned on page 3612, the Poincaré Duality isomorphism is an isomorphism of R-left modules.
(b) It follows from (a) and Lemma 171.5 that the map

$$\begin{aligned} \varphi\colon \mathrm{H}^{\mathrm{tw}}_k(X, A; M) \otimes_{\mathbb{Z}} \mathrm{H}^{\mathrm{tw}}_{n-k}(X, B; M) &\to \mathrm{H}^n_{\mathrm{tw}}(X; M \otimes_{\mathbb{Z}} M) \\ (v, w) &\mapsto \mathrm{PD}_X(v) \cup_{\otimes} \mathrm{PD}_X(w) \end{aligned}$$

is a homomorphism of $R \otimes_{\mathbb{Z}} R$-left modules.
(c) The augmentation isomorphism $\mathrm{H}^n_{\mathrm{tw}}(X; M \otimes_{\mathbb{Z}} M) \to M \otimes_{\mathbb{Z}} M$ from the Twisted H_0-Proposition 168.1 is an isomorphism of $R \otimes_{\mathbb{Z}} R$-left modules.
(d) By hypothesis the pairing $\Theta\colon M \otimes_{\mathbb{Z}} M \to R$ is R-sesquilinear and hermitian.

Now we turn to the proof of the proposition:

(1) This statement follows immediately from (b), (c) and (d).
(2) We now suppose that $n = 2m$, $A = B$ and that $\Theta\colon M \times M \to R$ is hermitian. We set $C := X \setminus \mathring{A}$. If follows from (a), (b) and (c) together with Proposition 171.7 that given $v, w \in \mathrm{H}^{\mathrm{tw}}_n(X, A; M)$ we have the following equality:

$$\overbrace{\epsilon_X(\Theta_*((\underbrace{\mathrm{PD}_X(v)}_{\in \mathrm{H}^m_{\mathrm{tw}}(X,C;M)} \cup_{\otimes} \underbrace{\mathrm{PD}_X(w)}_{\in \mathrm{H}^n_{\mathrm{tw}}(X,C;M)}) \cap_{\mathbb{Z}} [X]))}^{=Q^M_{X,A,A}(v,w)} = (-1)^{m\cdot m} \cdot \overbrace{\epsilon_X(\Theta_*((\underbrace{\mathrm{PD}_X(w)}_{\in \mathrm{H}^m_{\mathrm{tw}}(X,C;M)} \cup_{\otimes} \underbrace{\mathrm{PD}_X(v)}_{\in \mathrm{H}^n_{\mathrm{tw}}(X,C;M)}) \cap_{\mathbb{Z}} [X]))}^{=Q^M_{X,A,A}(w,v)}.$$

The statement follows from the observation that $(-1)^{m\cdot m} = (-1)^m$. ∎

Examples. Let X be a compact oriented path-connected non-empty n-dimensional topological manifold and let $x_0 \in X$. We write $\pi := \pi_1(X, x_0)$. Furthermore suppose that we

are given compact $(n-1)$-dimensional submanifolds A and B of ∂X such that $A \cap B = \partial A = \partial B$.

(1) Let G be a group and let $\varphi\colon \pi \to G$ be a homomorphism. Given $f \in \mathbb{Z}[G]$ and $g \in \pi$ we set $g \cdot f := f \cdot g^{-1}$. This allows us to view $\mathbb{Z}[G]$ as a $\mathbb{Z}[\pi]$-left module.[2111] This allows us to view $\mathbb{Z}[G]$ as a $(\mathbb{Z}[\pi], \mathbb{Z}[G])$-left left module. As always we view $\mathbb{Z}[G]$ as a ring with involution. The multiplication map

$$\begin{array}{rcl} \mu\colon \mathbb{Z}[G] \times \mathbb{Z}[G] & \to & \mathbb{Z}[G] \\ (r,s) & \mapsto & r \cdot \overline{s} \end{array}$$

has the required properties[2112] and we obtain the twisted intersection pairing

$$Q_X^{\mathbb{Z}[G]}\colon \mathrm{H}_k^{\mathrm{tw}}(X;\mathbb{Z}[G]) \times \mathrm{H}_{n-k}^{\mathrm{tw}}(X;\mathbb{Z}[G]) \to \mathbb{Z}[G].$$

By Proposition 173.1 (1) this pairing is $\mathbb{Z}[G]$-sesquilinear. If $A = B$, if $n = 4m$ and if $k = 2m$, then it follows from Proposition 173.1 (2) that the pairing is hermitian.

(2) Let $\alpha\colon \pi \to \mathrm{U}(d) := \{A \in \mathrm{M}(d \times d, \mathbb{C}) \,|\, \overline{A}^T A = \mathrm{id}\}$ be a unitary representation. We obtain the corresponding $\mathbb{Z}[\pi]$-left module $\mathbb{C}^d_\varphi$. We equip $\mathbb{C}$ with the standard involution $\overline{x + y\mathrm{i}} := x - y\mathrm{i}$ and we consider the $\mathbb{C}$-sesquilinear map

$$\begin{array}{rcl} \Theta\colon \mathbb{C}^d \times \mathbb{C}^d & \to & \mathbb{C} \\ (v,w) & \mapsto & v^T \cdot \overline{w}. \end{array}$$

Since α is a unitary representation we see that the corresponding left action of π on $\mathbb{C}^d$ preserves Θ. Thus we obtain the twisted intersection pairing

$$Q_X^{\mathbb{C}^d}\colon \mathrm{H}_k^{\mathrm{tw}}(X;{}_\alpha\mathbb{C}^d) \times \mathrm{H}_{n-k}^{\mathrm{tw}}(X;{}_\alpha\mathbb{C}^d) \to \mathbb{C}.$$

By Proposition 173.1 (1) this pairing is $\mathbb{C}$-sesquilinear. If $A = B$, if $n = 4m$ and if $k = 2m$, then it follows from Proposition 173.1 (2) that this pairing is hermitian.

(3) Let $\alpha\colon \pi \to \mathrm{SL}(2,\mathbb{C})$ be a unitary representation. We consider the corresponding $\mathbb{Z}[\pi]$-left module ${}_\alpha\mathbb{C}^2$. We equip $\mathbb{C}$ with the *trivial* involution and we consider the $\mathbb{C}$-bilinear map

$$\begin{array}{rcl} \Theta\colon \mathbb{C}^2 \times \mathbb{C}^2 & \to & \mathbb{C} \\ (v,w) & \mapsto & \det(v\ w). \end{array}$$

Since α takes values in $\mathrm{SL}(2,\mathbb{C})$ it follows from a straightforward argument that the corresponding left action of π on $\mathbb{C}^2$ preserves Θ. Thus we obtain the twisted intersection pairing

$$Q_X^{\mathbb{C}^2}\colon \mathrm{H}_k^{\mathrm{tw}}(X;{}_\alpha\mathbb{C}^2) \times \mathrm{H}_{n-k}^{\mathrm{tw}}(X;{}_\alpha\mathbb{C}^2) \to \mathbb{C}.$$

By Proposition 173.1 (1) this pairing is $\mathbb{C}$-bilinear. If $A = B$, if $n = 4m+2$ and if $k = 2m+1$, then it follows from Proposition 173.1 (2) that this pairing is symmetric.

Given a commutative ring we defined on pages 2656 and 2894 the notion of non-singular and non-degenerate R-bilinear pairings. We now extend these notions in the obvious way to the more general setting of R-sesquilinear pairings.

[2111] We pick this $\mathbb{Z}[\pi]$-left module structure to stay in tune with Proposition 168.2.

[2112] Indeed, given $r, s \in \mathbb{Z}[G]$ and $g \in \pi$ we have

$$\Theta(g \cdot r, g \cdot s) = \Theta(r \cdot g^{-1}, s \cdot g^{-1}) = (r \cdot g^{-1}) \cdot \overline{(s \cdot g^{-1})} r \cdot g^{-1} \cdot g \cdot \overline{s} = r \cdot \overline{s} = \Theta(r,s).$$

Definition. Let R be a ring with involution and let V and W be two R-left modules. We say that an R-sesquilinear pairing $\varphi\colon V \otimes_{\mathbb{Z}} W \to R$ is non-singular if the induced maps

$$\underbrace{\begin{array}{ccc} V & \to & \mathrm{Hom}_{R\text{-right}}(\overline{W}, R) \\ v & \mapsto & \left(\begin{array}{ccc} \overline{W} & \to & R \\ w & \mapsto & \varphi(v\otimes w)\end{array}\right)\end{array}}_{\text{homomorphism of } R\text{-left modules}} \quad \text{and} \quad \underbrace{\begin{array}{ccc} \overline{W} & \to & \mathrm{Hom}_{R\text{-left}}(V, R) \\ w & \mapsto & \left(\begin{array}{ccc} V & \to & R \\ v & \mapsto & \varphi(v\otimes w)\end{array}\right)\end{array}}_{\text{homomorphism of } R\text{-right modules}}$$

are both[21132114] isomorphisms. Otherwise we say that the pairing is non-singular. We say that the R-sesquilinear pairing is non-degenerate if the above two maps are both monomorphisms. Otherwise we say that the R-bilinear pairing is degenerate.

Proposition 173.2. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. Let $\mathbb{K}$ be a skew field with involution and let M be a $(\mathbb{Z}[\pi_1(X,x_0)],\mathbb{K})$-left left module. Finally suppose that we are given a non-singular $\mathbb{K}$-sesquilinear pairing

$$\Theta\colon M \otimes_{\mathbb{Z}} M \to \mathbb{K}$$

that is preserved by the $\pi_1(X,x_0)$-action. We denote by $\epsilon_X\colon \mathrm{H}_0^{\mathrm{tw}}(X;{}_{\mathrm{tr}}\mathbb{K}) \to \mathbb{K}$ the augmentation isomorphism from the Twisted H_0-Proposition 168.1. Then for any $A \subset X$ the $\mathbb{K}$-left linear map

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(X,A;M) & \to & \mathrm{Hom}_{\mathbb{K}\text{-left}}(\mathrm{H}^{\mathrm{tw}}_k(X,A;M), \mathbb{K}) \\ \varphi & \mapsto & (\mu \mapsto \epsilon_X(\Theta_*(\underbrace{\varphi \cap_{\otimes} \mu}_{\in \mathrm{H}_0^{\mathrm{tw}}(X;M\otimes_{\mathbb{Z}}M)}))) \end{array}$$

($\mathbb{K}$-left module via $k\cdot m := m\cdot_{\mathbb{K}} \overline{k}$)

is an isomorphism of $\mathbb{K}$-left modules. (For this statement the $\mathbb{K}$-left module structures on $\mathrm{H}^k_{\mathrm{tw}}(X,A;M)$ is given by the given $\mathbb{K}$-left module structure on M and the $\mathbb{K}$-left module structure on $\mathrm{Hom}_{\mathbb{K}\text{-left}}(\mathrm{H}^{\mathrm{tw}}_k(X,A;M),\mathbb{K})$ is given by the obvious $\mathbb{K}$-left module structure on $\mathbb{K}$.)

Proof. To simplify the notation we assume that $A = \varnothing$. It follows from the little remark on page 3608 that mathematically it makes no difference in the argument that follows. We write $\pi := \pi_1(X,x_0)$ and we write $C_* := \mathrm{C}_*(\widetilde{X}_{x_0})$.

Claim 1. The map

$$\begin{array}{ccc} \Xi\colon \mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(C_*, M) & \to & \mathrm{Hom}_{\mathbb{K}\text{-left}}(\overline{M}\otimes_{\mathbb{Z}[\pi]} C_*, \mathbb{K}) \\ \varphi & \mapsto & \left(\begin{array}{ccc} \overline{M}\otimes_{\mathbb{Z}[\pi]} C_* & \to & \mathbb{K} \\ m\otimes\sigma & \mapsto & \Theta(\varphi(\sigma)\otimes m)\end{array}\right)\end{array}$$

($\mathbb{K}$-left module via $k\cdot m := m\cdot_{\mathbb{K}} \overline{k}$)

is a well-defined chain map and it is a homomorphism of $\mathbb{K}$-left modules. (For this statement the $\mathbb{K}$-left module structures are given by the given $\mathbb{K}$-left module structure on M and the obvious $\mathbb{K}$-left module structure on $\mathbb{K}$.)

[2113] We equip the target R with the obvious (R,R)-bimodule structure.

[2114] If the pairing $\varphi\colon V\otimes_{\mathbb{Z}} W \to R$ is (anti-) hermitian, then one can easily verify that if one map is an isomorphism (respectively monomorphism), then so is the other.

Proof. As always, if the claim is correct, it is trivially correct. But since these arguments always produce anxieties for this famously anxious author, let us nonetheless provide some details. First we need to verify that the map $\Xi(\varphi)\colon \overline{M} \otimes_{\mathbb{Z}[\pi]} C_* \to \mathbb{K}$ respects the tensor product. Thus let $m \in \overline{M}$, $\sigma \in C_k$ and $g \in \pi$. Then

$$\overbrace{\Theta(\varphi(\sigma) \otimes m \cdot g)}^{=\Xi(\varphi)(mg\otimes\sigma)} \underset{\uparrow}{=} \Theta(\varphi(\sigma) \otimes g^{-1} \cdot m) \underset{\uparrow}{=} \Theta(g \cdot \varphi(\sigma) \otimes gg^{-1} \cdot m) \underset{\uparrow}{=} \overbrace{\Theta(\varphi(g \cdot \sigma) \otimes m)}^{=\Xi(\varphi)(m\otimes g\sigma)}.$$

by definition of Ξ and by definition of the $\mathbb{Z}[\pi]$-right module $\overline{M}$ — since π preserves Θ — since φ is $\mathbb{Z}[\pi]$-left linear

Next we need to verify that the map $\Xi(\varphi)\colon \overline{M} \otimes_{\mathbb{Z}[\pi]} C_* \to \mathbb{K}$ is $\mathbb{K}$-left linear. Thus let $m \in \overline{M}$, let $\sigma \in C_k$ and let $k \in \mathbb{K}$, then

$$\Xi(\varphi)(k \cdot (m \otimes \sigma)) \underset{\uparrow}{=} \Theta(\varphi(\sigma) \otimes k \cdot \sigma) \underset{\uparrow}{=} \overbrace{\Theta(\varphi(\sigma) \otimes m)}^{=\Xi(\varphi)(m\otimes\sigma)} \cdot \overline{k} \underset{\uparrow}{=} k \cdot \Xi(\varphi)(m \otimes \sigma).$$

$\mathbb{K}$-left structure on $\overline{M} \otimes_{\mathbb{Z}[\pi]} C_*$ — since Θ is $\mathbb{K}$-sesquilinear — $\mathbb{K}$-left structure on $\mathbb{K}$

Now we need to show that $\Xi\colon \mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(C_*, M) \to \mathrm{Hom}_{\mathbb{K}\text{-left}}(\overline{M} \otimes_{\mathbb{Z}[\pi]} C_*, \mathbb{K})$ is a homomorphism of $\mathbb{K}$-left modules. Fortunately, given $\varphi \in \mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(C_*, M)$ and $k \in M$ we indeed have

$$\Xi(k\cdot\varphi)(m \otimes \sigma) \underset{\uparrow}{=} \Theta((k\cdot\varphi)(s) \otimes m) \overset{\downarrow}{=} \Theta(k\cdot\varphi(s) \otimes m) \underset{\uparrow}{=} k\cdot\Theta(\varphi(\sigma) \otimes m) \overset{\downarrow}{=} k\cdot\Xi(\varphi)(m \otimes \sigma).$$

definition of Ξ — definition of the $\mathbb{Z}[\pi]$–left structure — since Θ is $\mathbb{K}$-sesquilinear — definition of Ξ

We leave it to the reader to verify that the maps Ξ are chain maps. ⊞

Our next goal is to show, using the hypothesis that Θ is non-singular, that the maps Ξ are isomorphisms. This leads us to the following claim.

Claim 2. Given $\psi \in \mathrm{Hom}_{\mathbb{K}\text{-left}}(\overline{M} \otimes_{\mathbb{Z}[\pi]} C_n, \mathbb{K})$ and given $\sigma \in C_n$ the map

$$\begin{array}{rcl} M & \to & \mathbb{K} \\ v & \mapsto & \psi(v \otimes \sigma) \end{array}$$

is $\mathbb{K}$-left linear. (Recall that $\mathbb{K}$ is here equipped with the $\mathbb{K}$-left linear structure given by $k \cdot m := m \cdot_{\mathbb{K}} m^{-1}$.) Since $\Theta\colon M \otimes_{\mathbb{Z}} M \to \mathbb{K}$ is non-singular there exists a unique $f(\psi, \sigma) \in M$ such that

$$\psi(v \otimes \sigma) \;=\; \Theta(f(\psi, \sigma) \otimes v) \qquad \text{for all } v \in V.$$

With this observation and definition we can now consider the map

$$\begin{array}{rcl} \Omega\colon \mathrm{Hom}_{\mathbb{K}\text{-left}}(\overline{M} \otimes_{\mathbb{Z}[\pi]} C_*, \mathbb{K}) & \to & \mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(C_*, M) \\ \psi & \mapsto & \begin{pmatrix} C_* & \to & M \\ \sigma & \mapsto & f(\psi, \sigma) \end{pmatrix} \end{array}$$

We claim these maps are well-defined and that they are inverses to the map Ξ in Claim 1.

Proof. For well-definedness we only have to show that for $\psi \in \mathrm{Hom}_{\mathbb{K}\text{-left}}(\overline{M} \otimes_{\mathbb{Z}[\pi]} C_k, \mathbb{K})$ the map $\Omega(\psi)\colon C_k \to M$ is $\mathbb{Z}[\pi]$-linear. So let $\sigma \in C_*$ and let $g \in \pi$. Given any $v \in V$ we have

$$\begin{array}{lcccc} \Theta(f(\psi, g\cdot\sigma)\otimes v) & \overset{\text{by definition of } f(\psi,g\cdot\sigma)}{\overset{\downarrow}{=}} & \psi(v\otimes g\cdot\sigma) & \overset{\text{tensor product over } \mathbb{Z}[\pi]}{\overset{\downarrow}{=}} & \psi(v\cdot g\otimes\sigma) \\ & & & = & \Theta(f(\psi,\sigma)\otimes g^{-1}\cdot v) \\ & & & \overset{\text{by definition of } f(\psi,\sigma)}{\uparrow} & \\ & & & \overset{\text{definition of } \pi\text{-right left action on } \overline{M}}{\overset{\downarrow}{=}} & \psi(g^{-1}\cdot v\otimes\sigma) \\ & & & \overset{\text{since } \pi \text{ preserves } \Theta}{=} & \Theta(g\cdot f(\psi,\sigma)\otimes v). \end{array}$$

Since Θ is non-singular we obtain that $f(\psi, g\cdot\sigma) = g\cdot f(\psi,\sigma)$. We have thus verified that $\sigma \mapsto f(\psi,\sigma)$ is indeed π-left linear.

It remains to show that Ξ and Ω are inverses of one another. First we show that $\Omega\circ\Xi = \mathrm{id}$. Thus let $\varphi \in \mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(C_*, M)$ and let $\sigma \in C_*$. Note that by definition of Ξ we have

$$\Xi(\varphi)(v\otimes\sigma) \;=\; \Theta(\varphi(\sigma)\otimes v) \qquad \text{for all } v\in M \text{ and } \sigma \in C_*.$$

But this means, by definition of f, that $f(\Xi(\varphi),\sigma) = \varphi(\sigma)$. Thus we see that

$$(\Omega\circ\Xi)(\varphi)(\sigma) \;=\; \Omega(\Xi(\varphi))(\sigma) \;\overset{\text{definition of } \Omega}{=}\; f(\Xi(\varphi),\sigma) \;\overset{\text{see above}}{=}\; \varphi(\sigma).$$

Thus we have shown that $\Omega\circ\Xi = \mathrm{id}$. Finally let us check that $\Xi\circ\Omega = \mathrm{id}$. Thus let $\psi \in \mathrm{Hom}_{\mathbb{K}\text{-left}}(\overline{M}\otimes_{\mathbb{Z}[\pi]} C_n, \mathbb{K})$. Given any $\sigma \in C_n$ and $v \in M$ we have

$$(\Xi\circ\Omega)(\psi)(v\otimes\sigma) \;=\; \Xi(\Omega(\psi))(v\otimes\sigma) \;\overset{\text{definition of } \Xi}{=}\; \Theta(\Omega(\psi)(\sigma)\otimes v) \;\overset{\text{definition of } \Omega}{=}\; \Theta(f(\psi,\sigma)\otimes v) \;\overset{\text{definition of } f}{=}\; \psi(v\otimes\sigma).$$

Thus we have shown that $\Omega\circ\Xi = \mathrm{id}$. ⊞

Claim 3. Given any $\varphi \in \mathrm{Hom}_{\mathbb{Z}[\pi]}(\mathrm{C}_*(\widetilde{X}), M)$ and $m\otimes\sigma \in \overline{M}\otimes_{\mathbb{Z}[\pi]} C_k(\widetilde{X})$ we have

$$\epsilon_X(\Theta_*(\varphi\cap_\otimes (m\otimes\sigma))) \;=\; \Xi(\varphi)(m\otimes\sigma).$$

Proof. Let $\varphi \in \mathrm{Hom}_{\mathbb{Z}[\pi]}(\mathrm{C}_*(\widetilde{X}), M)$ and let $m\otimes\sigma \in \overline{M}\otimes_{\mathbb{Z}[\pi]} C_k(\widetilde{X})$. Then

$$\begin{array}{lcccc} \epsilon_X(\Theta_*(\varphi\cap_\otimes (m\otimes\sigma))) & \overset{\text{see definition of } \cap_\otimes \text{ on page 3606}}{\overset{\downarrow}{=}} & \epsilon_X(\Theta_*(\varphi(\sigma)\otimes m\otimes(\sigma\circ[v_k]))) & & \\ & = & \epsilon_X(\Theta(\varphi(\sigma))\otimes m)\otimes(\sigma\circ[v_k]) & = & \Theta(\varphi(\sigma)\otimes m) \\ & \overset{\text{definition of } \Theta_* \text{ on page 3538}}{\uparrow} & & \overset{\text{Twisted H}_0\text{-Proposition 168.1}}{\uparrow} & \\ & & & & = \quad \Xi(\varphi)(m\otimes\sigma). \\ & & & & \overset{\text{definition } ⊞}{\uparrow} \end{array}$$

We can now finally conclude the argument. To do so we consider the following diagram

$$\begin{array}{ccc} \mathrm{H}^k(X;M) & \xrightarrow{\varphi\mapsto(m\otimes\sigma\mapsto\epsilon_X(\Theta_*(\varphi\cap_\otimes m\otimes\sigma)))} & \mathrm{Hom}_{\mathbb{K}\text{-left}}(\mathrm{H}_k(X;M),\mathbb{K}) \\ \cong\big\downarrow \Xi & & \big\downarrow \mathrm{id} \\ \mathrm{H}^k(\overline{M}\otimes_{\mathbb{Z}[\pi]}\mathrm{C}_*(\widetilde{X});\mathbb{K}) & \xrightarrow[\text{isomorphism by Theorem 166.12}]{\varphi\mapsto(\mu\mapsto\varphi(\mu))} & \mathrm{Hom}_{\mathbb{K}\text{-left}}(\mathrm{H}_k(\overline{M}\otimes_{\mathbb{Z}[\pi]}\mathrm{C}_*(\widetilde{X})),\mathbb{K}). \end{array}$$

We make the following observations:

(1) By Claim 2 we know that the vertical map Ξ is an isomorphism.
(2) By Claim 3 we know that the diagram commutes.

(3) Since $\mathbb{K}$ is a skew field we obtain from Theorem 166.12, applied to the chain complex $V_* := \overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X})$ of $\mathbb{K}$-left modules, that the bottom horizontal map is an isomorphism.

It follows from the above that the top horizontal map is indeed an isomorphism. ■

The following proposition is the analogue of the Intersection Form Proposition 132.4 (3).

Proposition 173.3. Let X be a compact oriented path-connected non-empty n-dimensional topological manifold and let $x_0 \in X$. Furthermore suppose that we are given compact $(n-1)$-dimensional submanifolds A and B of ∂X such that $A \cap B = \partial A = \partial B$. Let $\mathbb{K}$ be a skew field with involution and let M be a $(\mathbb{Z}[\pi_1(X, x_0)], \mathbb{K})$-left left module. Finally suppose that we are given a non-singular $\mathbb{K}$-sesquilinear pairing

$$\Theta\colon M \otimes_{\mathbb{Z}} M \ \to\ \mathbb{K}$$

such that the $\pi_1(X, x_0)$-action preserves Θ, i.e. such that given any $v, w \in M$ and given any $g \in \pi_1(X, x_0)$ we have $\Theta(g \cdot v \otimes g \cdot w) = \Theta(v \otimes w)$. If $A \cup B = \partial X$, then the twisted intersection pairing

$$Q^M_{X,A,B}\colon \mathrm{H}^{\mathrm{tw}}_k(X, A; M) \times \mathrm{H}^{\mathrm{tw}}_{n-k}(X, B; M) \ \to\ \mathbb{K}$$

is non-singular.

Proof. Note that it follows from the hypothesis that $A \cap B = \partial A = \partial B$ and $A \cup B = \partial X$ that the Poincaré Duality Theorem 172.1 with Twisted Coefficients gives us the Poincaré Duality isomorphism

$$\mathrm{PD}\colon \mathrm{H}^{\mathrm{tw}}_{n-k}(X, A; M) \ \to\ \mathrm{H}^{n-k}_{\mathrm{tw}}(X, B; M).$$

We continue with the following claim.

Claim. The following diagram commutes

$$\begin{array}{ccccc}
\mathrm{H}^{\mathrm{tw}}_{n-k}(X, A; M) \times \mathrm{H}^{\mathrm{tw}}_k(X, B; M) & \xrightarrow{(v,w)\mapsto \mathrm{PD}(w)\cup_{\otimes}\mathrm{PD}(v)} & \mathrm{H}^n_{\mathrm{tw}}(X, \partial X; M \underset{\mathbb{Z}}{\otimes} M) & \xrightarrow{\Theta_*} & \mathrm{H}^n_{\mathrm{tw}}(X, \partial X; {}_{\mathrm{tr}}\mathbb{K}) \\
\downarrow \mathrm{PD} \qquad \downarrow \mathrm{id} & & \downarrow -\cap_{\mathbb{Z}}[X] & & \downarrow -\cap_{\mathbb{Z}}[X] \\
\mathrm{H}^k_{\mathrm{tw}}(X, B; M) \times \mathrm{H}^{\mathrm{tw}}_k(X, B; M) & \xrightarrow{\cap_{\otimes}} & \mathrm{H}^{\mathrm{tw}}_0(X; M \underset{\mathbb{Z}}{\otimes} M) & \xrightarrow{\Theta_*} & \mathrm{H}^{\mathrm{tw}}_0(X; {}_{\mathrm{tr}}\mathbb{K}).
\end{array}$$

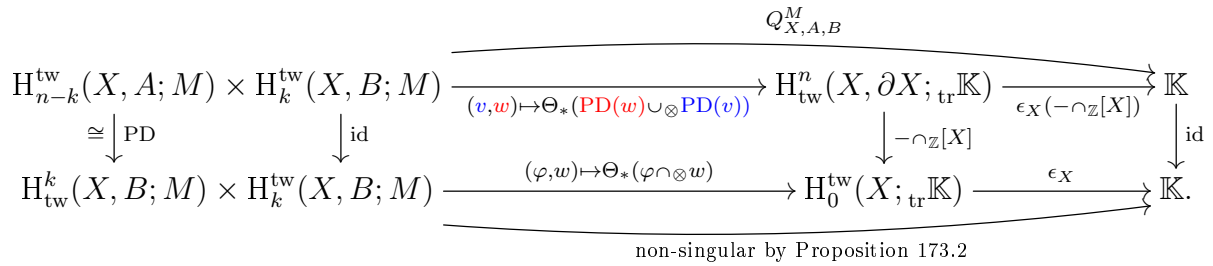
Proof. We consider the two squares separately.

(1) For the first square let $v \in \mathrm{H}^{\mathrm{tw}}_{n-k}(X, A; M)$ and $w \in \mathrm{H}^{\mathrm{tw}}_k(X, B; M)$. Then

$$(\mathrm{PD}(w) \cup_{\otimes} \mathrm{PD}(v)) \cap_{\mathbb{Z}} [X] \underset{\text{Lemma 171.14}}{=} \mathrm{PD}(v) \cap_{\otimes} (\mathrm{PD}(w) \cap_{\mathbb{Z}} [X]) \underset{\text{by definition of PD}(w)}{=} \mathrm{PD}(v) \cap_{\otimes} w.$$

(2) It follows from Proposition 171.8 (2) that the second square commutes. ⊞

Next we consider the following diagram:



We make the following observations:

(1) The top sequences is by definition the intersection pairing $Q^M_{X,A,B}$.
(2) It follows from Claim 1 that the square to the left commutes.
(3) The square to the right commutes by fiat.
(4) As we mentioned before, by the Poincaré Duality Theorem 172.1 with Twisted Coefficients the vertical map PD is an isomorphism of $\mathbb{K}$-left modules.
(5) By Proposition 173.2 we know that the $\mathbb{K}$-sesquilinear pairing at the bottom is non-singular.

All these statements combined show that the $\mathbb{K}$-sesquilinear intersection pairing $Q^M_{X,A,B}$ is indeed non-singular. ■

173.2. **Twisted signatures.** state that twisted signatures of closed manifolds are just multiples of ordinary signatures

173.3. **Twisted invariants and finite covers.**

Proposition 173.4. Let X be a closed oriented path-connected n-dimensional topological manifold, let $x_0 \in X$ and let $\varphi\colon \pi_1(X,x_0) \to G$ be an epimorphism onto a *finite* group G. We denote by $p\colon \widehat{X}_\varphi \to X$ the covering corresponding to the subgroup $\ker(\varphi)$. We consider the isomorphism

$$\begin{array}{rcl}\Omega\colon \mathrm{C}^{\mathrm{tw}}_k(X;\mathbb{Z}[G]) = \overline{\mathbb{Z}[G]} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}) & \to & \mathrm{C}_k(\widehat{X}_\varphi) \\ g \otimes (\sigma\colon \Delta^k \to Y_\Gamma) & \mapsto & q \circ (g\cdot\sigma\colon \Delta^k \to Y_\Gamma)\end{array}$$

of $\mathbb{Z}[G]$-left modules from Proposition 168.2. Given any $\mu \in \mathrm{H}^{\mathrm{tw}}_k(X;\mathbb{Z}[G])$ and given any $\nu \in \mathrm{H}^{\mathrm{tw}}_{n-k}(X;\mathbb{Z}[G])$ we have[2115]

$$Q^{\mathbb{Z}[G]}_X(\mu,\nu) \;=\; \sum_{g\in G} Q^{\mathbb{Z}}_{\widehat{X}_\varphi}\big(\underbrace{\Omega(\mu)}_{\in \mathrm{H}^{\mathrm{tw}}_k(\widehat{X}_\varphi)}, g\cdot \underbrace{\Omega(\nu)}_{\in \mathrm{H}^{\mathrm{tw}}_{n-k}(\widehat{X}_\varphi)}\big)\cdot g \;\in\; \mathbb{Z}[G].$$

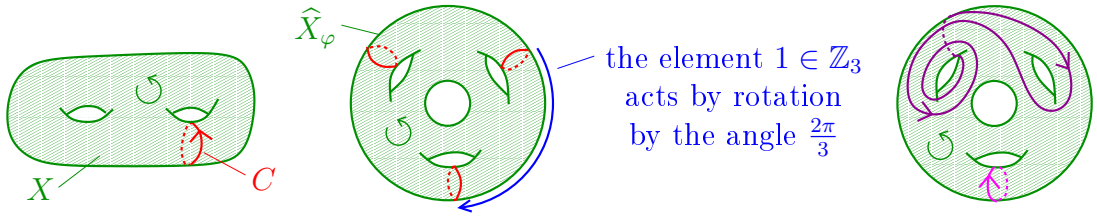
Example. Let X be the surface of genus 2and let $C \subset X$ be a non-separating closed orientable path-connected 1-dimensional submanifold. The homology class $[C] \in \mathrm{H}^{\mathrm{tw}}_1(X;\mathbb{Z})$ determines, via the isomorphisms

$$\mathrm{H}^{\mathrm{tw}}_1(X;\mathbb{Z}) \;\cong\; \mathrm{H}^1_{\mathrm{tw}}(X;\mathbb{Z}) \;\to\; \mathrm{Hom}(\mathrm{H}^{\mathrm{tw}}_1(X),\mathbb{Z}) \;=\; \mathrm{Hom}(\pi_1(X),\mathbb{Z}),$$

an epimorphism $\varphi\colon \pi_1(X) \to \mathbb{Z} \to \mathbb{Z}_3$. In the Cyclic-Covering Proposition 130.11 we saw how one can explicitly construct the covering $\widehat{X}_\varphi$ corresponding to $\ker(\varphi)$. Thus

[2115] It is a good reality check, see Exercise 173.1 to verify that the map on the right hand side is indeed $\mathbb{Z}[G]$-sesquilinear.

Proposition 173.4 allows us to compute the twisted intersection form $Q_X^{\mathbb{Z}[\mathbb{Z}_3]}$ by considering the usual intersection form on the explicit covering $\widehat{X}_\varphi$.



Proof. In the following we use the Replace Universal Covering Proposition 168.6 to replace the universal covering of X by the covering $p\colon \widehat{X}_\varphi \to X$. This approach simplifies the notation quite a bit. In particular we now consider the isomorphism

$$\begin{array}{rcl} \Omega\colon \mathrm{C}_k^{\mathrm{tw}}(X;\mathbb{Z}[G]) = \overline{\mathbb{Z}[G]} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widehat{X}_\varphi) & \to & \mathrm{C}_k(\widehat{X}_\varphi) \\ g \otimes (\sigma\colon \Delta^k \to Y_\Gamma) & \mapsto & (g \cdot \sigma\colon \Delta^k \to Y_\Gamma) \end{array}$$

of $\mathbb{Z}[G]$-left modules from Proposition 168.2. Furthermore, since G is finite we obtain from Proposition 168.4 that the maps

$$\begin{array}{rcl} \Phi\colon \mathrm{C}^k_{\mathrm{tw}}(X;\mathbb{Z}[G]) & \to & \mathrm{C}^k(\widehat{X}_\varphi;\mathbb{Z}) \\ \varphi & \mapsto & \sigma \mapsto e\text{-coefficient of } \varphi(\sigma) \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi\colon \mathrm{C}^k(\widehat{X}_\varphi;\mathbb{Z}) & \to & \mathrm{C}^k_{\mathrm{tw}}(X;\mathbb{Z}[G]) \\ \gamma & \mapsto & \sum\limits_{g\in G} g^*(\gamma)\cdot g \end{array}$$

are well-defined $\mathbb{Z}[G]$-left homomorphisms and inverses of one another.

Claim 1. Given any $\mu \in \mathrm{H}_k^{\mathrm{tw}}(X;\mathbb{Z}[G])$ we have

$$|G| \cdot \mathrm{PD}_{\widehat{X}_\varphi}(\Omega(\mu)) \;=\; |G| \cdot \Phi(\mathrm{PD}_X(\mu)) \;\in\; \mathrm{H}^{\dim(X)-k}_{\mathrm{tw}}(\widehat{X}_\varphi).$$

Proof. We start out with a little calculation. Namely let $\alpha \in \mathrm{C}^k(X;\mathbb{Z}[G])$ and let $\sigma\colon \Delta^{k+l} \to \widehat{X}_\varphi$. Then we have the following equality in $\mathrm{C}_{l-k}(\widehat{X}_\varphi)$:

$$\begin{aligned} \sum_{g\in G} g\cdot(g^*\Phi(\alpha) \cap_{\mathbb{Z}} (1\otimes\sigma)) &= \overbrace{\alpha(\sigma\circ[v_0,..,v_k])}^{\in\mathbb{Z}[G]}\cdot\sigma\circ[v_k,..,v_l]) \\ &\quad \uparrow \\ &\quad \text{follows from plugging in, as in the proof of } \Xi\circ\Phi = \mathrm{id} \\ &= \Omega(\underbrace{(\alpha(\sigma\circ[v_0,..,v_k])\otimes 1)}_{\in\mathbb{Z}[G]\otimes_{\mathrm{tr}}\mathbb{Z}} \otimes \sigma\circ[v_k,..,v_l]) \;= \Omega(\alpha\cap_\otimes 1\otimes\sigma). \end{aligned}$$

Now we turn to the actual proof of the claim. Given any $\mu \in \mathrm{H}_k^{\mathrm{tw}}(X;\mathbb{Z}[G])$ we have the following equalities in $\mathrm{H}_k^{\mathrm{tw}}(\widehat{X}_\varphi)$:

$$\begin{array}{rcll} & & \text{definition of } \mathrm{PD}_{\widehat{X}_\varphi} & \text{definition of } \mathrm{PD}_X \\ |G| \cdot \mathrm{PD}_{\widehat{X}_\varphi}(\Omega(\mu)) \cap_{\mathbb{Z}} [\widehat{X}_\varphi] & \overset{\downarrow}{=} & |G| \cdot \Omega(\mu) & \overset{\downarrow}{=} \Omega(\mathrm{PD}_X(\mu) \cap_{\mathbb{Z}} |G| \cdot [X]) \\ & = & \Omega(\mathrm{PD}_X(\mu) \cap_{\mathbb{Z}} p_*([\widehat{X}_\varphi])) & = \sum_{g \in G} g \cdot (g^* \Phi(\mathrm{PD}_X(\mu)) \cap_{\mathbb{Z}} [\widehat{X}_\varphi]). \\ & & \uparrow & \uparrow \\ & & \text{Fundamental Class-Covering Proposition 106.25 follows from the above calculation} & \\ & = & \sum_{g \in G} \Phi(\mathrm{PD}_X(\mu)) \cap_{\mathbb{Z}} g \cdot [\widehat{X}_\varphi] & = |G| \cdot \Phi(\mathrm{PD}_X(\mu)) \cap_{\mathbb{Z}} [\widehat{X}_\varphi]. \\ & & \uparrow & \uparrow \\ & & \text{by Proposition 171.13 (1b) we have in general } f_*(f^*\gamma \cap_{\mathbb{Z}} \sigma) = f_*(\sigma) & \text{since } g \text{ acts in an orientation-preserving way we see that } g \cdot [\widehat{X}_\varphi] = [\widehat{X}_\varphi] \end{array}$$

Since the map $- \cap_{\mathbb{Z}} [\widehat{X}_\varphi]$ is an isomorphism we obtain the desired statement. ⊞

Claim 2. Given any $\widetilde{\mu} \in \mathrm{C}^r(\widehat{X}_\varphi;\mathbb{Z})$ and $\widetilde{\nu} \in \mathrm{C}^s(\widehat{X}_\varphi;\mathbb{Z})$ we have

$$\Psi(\widetilde{\mu}) \underset{\mathbb{Z}[G]}{\cup} \Psi(\widetilde{\nu}) = |G| \cdot \sum_{k \in G} \Big(\underbrace{\widetilde{\mu} \cup k \cdot \widetilde{\nu}}_{\in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{C}_{r+s}(\widetilde{X}),\mathbb{Z})} \Big) \cdot k \quad \in \mathrm{Hom}_{\mathbb{Z}[G]-\mathrm{left}}(\mathrm{C}_{r+s}(\widetilde{X}),\mathbb{Z}[G]).$$

Proof. Given any $\widetilde{\mu} \in \mathrm{C}^r(\widehat{X}_\varphi;\mathbb{Z})$ and $\widetilde{\nu} \in \mathrm{C}^s(\widehat{X}_\varphi;\mathbb{Z})$ we perform the following calculation in $\mathbb{Z}[G]$:

$$\begin{array}{rcll} & & \text{definition of } \Psi & \text{definition of } \cup_{\mathbb{Z}[G]} \text{ on page 3603} \\ \Psi(\widetilde{\mu}) \underset{\mathbb{Z}[G]}{\cup} \Psi(\widetilde{\nu}) & \overset{\downarrow}{=} & \sum_{g \in G} g^*\widetilde{\mu} \cdot g \underset{\mathbb{Z}[G]}{\cup} \sum_{h \in G} h^*\widetilde{\nu} \cdot h & \overset{\downarrow}{=} \sum_{g \in G} \sum_{h \in G} (g^*\widetilde{\mu} \cup h^*\widetilde{\nu}) \cdot gh^{-1} \\ & = & \sum_{g \in G} \sum_{h \in G} (\widetilde{\mu} \cup \underbrace{(g^{-1})^* h^*(\widetilde{\nu})}_{=(hg^{-1})^*}) \cdot gh^{-1} & = \sum_{g \in G} \sum_{k \in G} (\widetilde{\mu} \cup (k^{-1})^*\widetilde{\nu}) \cdot k \\ & & \uparrow & \uparrow \\ & & \text{since by the Cup Product-Naturality Lemma 114.10 we know we rename } k = gh^{-1}, \text{ that } f^*\alpha \cup g^*\beta = f^*(\alpha \cup \beta) & \\ & = & \sum_{g \in G} \sum_{k \in G} (\widetilde{\mu} \cup k \cdot \widetilde{\nu}) \cdot k & = |G| \cdot \sum_{k \in G} (\widetilde{\mu} \cup \underbrace{(k^{-1})^* \cdot \widetilde{\nu}}_{k \cdot \widetilde{\nu}}) \cdot k \\ & & \uparrow & \uparrow \\ & & \text{by definition, see Proposition 168.2, we have } k \cdot \widetilde{\nu} = (k^{-1})^*\widetilde{\nu} & \text{since the } g\text{-summands are independent of } g \end{array}$$

⊞

Given any $\mu \in \mathrm{H}_k^{\mathrm{tw}}(X;\mathbb{Z}[G])$ and $\nu \in \mathrm{H}_{n-k}^{\mathrm{tw}}(X;\mathbb{Z}[G])$ we perform the following calculation in $\mathbb{Z}[G]$:

$$\begin{array}{rcll} & & \text{definition of } Q_X & \text{since } \Psi \circ \Phi = \mathrm{id} \\ |G|^3 \cdot Q_X(\mu,\nu) & \overset{\downarrow}{=} & \epsilon_X((|G| \cdot \mathrm{PD}_X(\mu) \underset{\mathbb{Z}[G]}{\cup} |G| \cdot \mathrm{PD}_X(\nu)) \cap_{\mathbb{Z}} |G| \cdot [X]) & \overset{\downarrow}{=} \\ & = & \epsilon_X\big(\big(\Psi(\underbrace{\Phi(|G| \cdot \mathrm{PD}_X(\mu))}_{=|G| \cdot \mathrm{PD}_{\widehat{X}_\varphi}(\Omega(\mu))}) \underset{\mathbb{Z}[G]}{\cup} \Psi(\underbrace{\Phi(|G| \cdot \mathrm{PD}_X(\nu))}_{=|G| \cdot \mathrm{PD}_{\widehat{X}_\varphi}(\Omega(\nu))}) \big) \cap_{\mathbb{Z}} |G| \cdot [X]\big) & = (*) \end{array}$$

We pick $\widetilde{\mu} \in \mathrm{C}^{n-k}(\widehat{X}_\varphi;\mathbb{Z})$ that represents $|G| \cdot \mathrm{PD}_{\widehat{X}_\varphi}(\Omega(\nu))$ and we pick $\widetilde{\nu} \in \mathrm{C}^k(\widehat{X}_\varphi;\mathbb{Z})$ that represents $|G| \cdot \mathrm{PD}_{\widehat{X}_\varphi}(\Omega(\nu))$. Furthermore we pick $\sigma \in \mathrm{C}_n(\widehat{X}_\varphi)$ that represents the fundamental class $[\widehat{X}_\varphi]$. Note that it follows from the Fundamental Class-Covering Proposition 106.25 that $p_*(\sigma)$ represents $|G| \cdot [X]$. Then we obtain the following equalities in

$\mathbb{Z}[G]$:

$$
\begin{array}{lll}
 & \text{definition of } \widetilde{\mu}, \widetilde{\nu} \text{ and } \sigma & \text{Claim 2} \\
 & \downarrow & \downarrow \\
(*) & = \epsilon_X\big((\Psi(\widetilde{\mu}) \cup_{\mathbb{Z}[G]} \Psi(\widetilde{\nu})) \cap_{\mathbb{Z}} p_*(\sigma)\big) & = |G| \cdot \epsilon_X\Big(\sum_{k \in G} ((\widetilde{\mu} \cup k \cdot \widetilde{\nu}) \cdot k) \cap_{\mathbb{Z}} p_*(\sigma)\Big) \\
 & = |G| \cdot \epsilon_X\Big(\sum_{k \in G} p_*((\widetilde{\mu} \cup k \cdot \widetilde{\nu}) \cap_{\mathbb{Z}} \sigma) \cdot k\Big) & = |G| \cdot \sum_{k \in G} \epsilon_{\widehat{X}}\big((\widetilde{\mu} \cup k \cdot \widetilde{\nu}) \cap_{\mathbb{Z}} \sigma\big) \cdot k \\
 & \uparrow & \uparrow \\
 & \text{by explicit definitions of the cap products} & \text{by explicit definition of } \epsilon_X \text{ and } \epsilon_{\widehat{X}_\varphi} \\
 & = |G| \cdot \sum_{k \in G} \epsilon_{\widehat{X}}\big(|G| \cdot \mathrm{PD}_{\widehat{X}_\varphi}(\Omega(\mu)) \cup k \cdot |G| \cdot \mathrm{PD}_{\widehat{X}_\varphi}(\Omega(\nu))) \cap_{\mathbb{Z}} [X]\big) \cdot k & \\
 & \uparrow & \\
 & \text{choice of } \widetilde{\mu} \text{ and } \widetilde{\nu} & \\
 & = |G|^3 \cdot \sum_{k \in G} Q_{\widehat{X}_\varphi}(\Omega(\mu), k \cdot \Omega(\nu)) \cdot k \ \in \ \mathbb{Z}[G]. & \\
 & \uparrow & \\
 & \text{definition of } Q_{\widehat{X}_\varphi} &
\end{array}
$$

Since multiplication by $|G|^3$ is injective in $\mathbb{Z}[G]$ we obtain the desired result.[2116] ■

Exercises for Chapter 173.

Exercise 173.1. Let G be a finite group, let M be a $\mathbb{Z}[G]$-left module and let $\Theta\colon M \times M \to \mathbb{Z}$ be a pairing. Show that

$$
\begin{array}{rcl}
M \times M & \to & \mathbb{Z}[G] \\
(r, s) & \mapsto & \sum_{g \in G} \Theta(r, g \cdot s) \cdot g \ \in \ \mathbb{Z}[G]
\end{array}
$$

is $\mathbb{Z}[G]$-bilinear.

[2116] Frankly I find it weird that this fact gets used, one would naively think that the same statement also holds if we replace $\mathbb{Z}$ by $\mathbb{F}_2$.

174. Local coefficients

In the last chapters we introduced and discussed the main properties of (co-) homology with twisted coefficients. It was fairly easy to develop this theory with off-the-shelves methods and it works quite well when we want to study a single topological spaces.

But it has the awkward feature that it depends on the choice of a base point and that it is very difficult to study maps between different topological spaces. In the next chapter we will introduced (co-) homology with local coefficients. This concept rectifies many of the issues we had with (co-) homology with twisted coefficients. But before we can talk about (co-) homology with local coefficients we need to introduce the notion of a local coefficient system over a topological space. That is what we will do in this chapter.

174.1. **Local coefficients.**

Definition. Let X be a topological space and let G be an abelian group. A G-local coefficient system on X consists of the following:

(1) a topological space C,
(2) a map $p\colon C \to X$,
(3) a group structure on each fiber $C_x := p^{-1}(x)$, with $x \in X$.

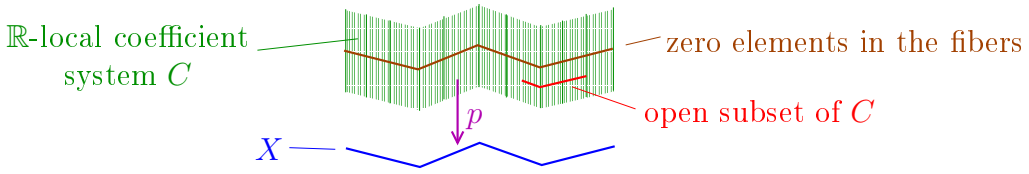
such that the following condition is satisfies: Given any $x \in X$ there exists an open neighborhood U of x and a homeomorphism $f\colon p^{-1}(U) \to G \times U$, where we equip G with the discrete topology, such that the following two conditions are satisfied:[2117]

(1) The following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow[\cong]{f} & G \times U \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (g,x)\mapsto x} & \\ & U. & \end{array}$$

(2) For each $x \in U$ the map $C_x = p^{-1}(x) \to G \times \{x\}$ is an isomorphism of groups.[2118]

Given a ring R we define in a very similar way an R-left module local coefficient system on X. If the group or the module are irrelevant, then we just talk of a local coefficient system.



Our first example of a local coefficient system gets an immediate upgrade to a definition.

Definition. Let X be a topological space and let G be an abelian group. We equip G with the discrete topology and we equip $G \times X$ with the product topology. We denote by $p\colon G \times X \to X$ the natural projection. It is clear that p defines a G-local coefficient system on X. We refer to it as the trivial G-local coefficient system and we denote it by ${}_{\mathrm{tr}}G$.

[2117] Later we will call such a map f a *local trivialization*.
[2118] Here we give $G \times \{x\}$ the obvious group structure coming from G.

The following definition shows that a local coefficient system gives to many other local coefficient systems.

Definition. Let X be a topological space, let G be an abelian group, and let $p\colon C \to X$ be a G-local coefficient system. Given any subspace $U \subset X$ we refer to

$$p|_U\colon p^{-1}(U) \;\to\; X$$

as the restriction of $p\colon C \to X$ to U. It is straightforward to verify that this is again a G-local coefficient system.

Lemma 174.1. Let X be a topological space and let G be an abelian group. Every G-local coefficient system $p\colon C \to X$ is a covering map, in the sense of the definition on page 1067.

Proof. This lemma is a special case of Lemma **??** (1). For the reader's convenience let us recall the argument. Since groups are by definition non-empty we see that $p\colon C \to X$ is surjective. Now let $x \in X$. By definition there exists an open neighborhood U of x and a homeomorphism $f\colon p^{-1}(U) \to G \times U$, where we equip G with the discrete topology, such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow[\cong]{f} & G \times U \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (g,x)\mapsto x} & \\ & U. & \end{array}$$

It follows now easily from Exercise 5.4 that U is evenly covered. ∎

The following lemma can be viewed as an analogue of Lemma 143.3.

Lemma 174.2. Let $p\colon C \to X$ be a local coefficient system and let W be a topological space.

(1) Let $f\colon W \to X$ be a map. The map

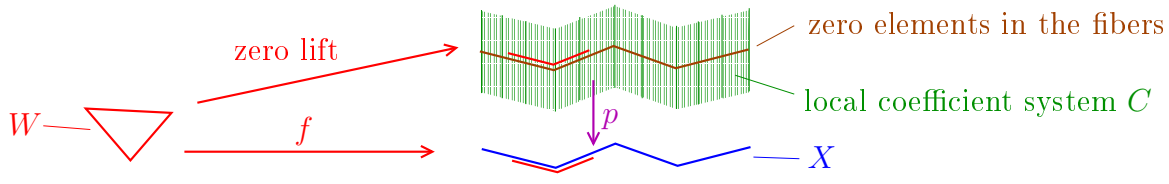
$$\begin{array}{rcl} \widetilde{f}\colon W & \to & C \\ w & \mapsto & \text{the unique zero element in } p^{-1}(f(w)) \end{array}$$

is continuous. We refer to it as the *zero-lift of* f.

(2) Let $f, g\colon W \to C$ be two maps with $p \circ f = p \circ g$. Then the maps

$$\begin{array}{rcl} W & \to & C \\ w & \mapsto & \underbrace{f(w) + g(w)}_{\substack{\text{addition in the abelian} \\ \text{group } p^{-1}(f(w))}} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} W & \to & C \\ w & \mapsto & -f(w) \end{array}$$

are also continuous.



Proof. Note that by the Pasting Proposition 2.6 we have to verify continuity only locally. But locally the statement follows easily from the definition of a local coefficient system. ∎

The following definition is the analogue of the definition of a bundle map of vector bundles that we introduced on page 3068.

Definition. Let G and H be abelian groups.

(1) Let X and Y be topological spaces, let $p\colon C \to X$ be a G-local coefficient system and let $q\colon D \to X$ be an H-local coefficient system. A morphism from the G-local coefficient system $p\colon C \to Y$ to the H-local coefficient system $q\colon D \to Y$ is a pair of maps $\Phi\colon E \to F$ and $\varphi\colon X \to Y$ such that the following two conditions are satisfied:
   (a) The following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\Phi} & D \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{\varphi} & Y. \end{array}$$

   (b) For each $x \in X$ the map $f\colon C_x = p^{-1}(x) \to q^{-1}(\varphi(x)) = D_{\varphi(x)}$ is a homomorphism of abelian groups.

(2) Let X be a topological space, let $p\colon C \to X$ be a G-local coefficient system and let $q\colon D \to X$ be an H-local coefficient system. A morphism over X from the G-local coefficient system $p\colon C \to X$ to the H-local coefficient system $q\colon D \to X$ is a morphisms as in (1) but with $\varphi = \mathrm{id}_X$. This notion turns the local coefficient systems over a given topological space into a category. In particular it also allows us to say what it means for two local coefficient systems over a topological space X to be isomorphic.

The following definition is the analogue of the definition of a trivialization of a vector bundle that we gave on page 3063.

Definition. Let $p\colon F \to X$ be a G-local coefficient system.

(1) We say p is trivializable if p is isomorphic to the trivial G-local coefficient system ${}_{\mathrm{tr}}G = G \times X$.

(2) Any choice of an isomorphism $E \to {}_{\mathrm{tr}}G = G \times X$ is called a trivialization of the G-local coefficient system.

We have the following analogue of Proposition 143.18.

Proposition 174.3. Let $p\colon C \to X$ be a G-local coefficient system over a topological space X and let $\Phi\colon C \to {}_{\mathrm{tr}}G = G \times X$ be a trivialization. We denote by $q\colon G \times X \to G$ the projection. We equip $\mathrm{Aut}(G)$ with the discrete topology.

(1) If $f\colon X \to \mathrm{Aut}(G)$ is a continuous map, then the map

$$\begin{array}{ccc} C & \to & G \times X \\ c & \mapsto & (p(c), \underbrace{f(p(c))}_{\in \mathrm{Aut}(G)}(\underbrace{q(\Phi(c))}_{\in G}) \end{array}$$

   is also a trivialization.

(2) Every trivialization of $p\colon C \to X$ is of the form described in (1).

Proof. The proof is similar to the rather terse proof of Proposition 143.18. ∎

174.2. **Local coefficient systems and modules over group rings.** In the Vector Bundles-from-Representation 143.1 we showed that representations $\pi_1(X, x_0) \to \mathrm{GL}(n, \mathbb{R})$

gives rise to n-dimensional real vector bundles. The following lemma now shows that representations also give rise to local coefficient systems.

Proposition 174.4. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. We write $\pi := \pi_1(X, x_0)$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering that defined on page 1370. Furthermore let M be a $\mathbb{Z}[\pi]$-left module, i.e. let M be an abelian group and suppose we are given a group homomorphism $\varphi\colon \pi \to \mathrm{Aut}(M)$. As on page 3503 we denote by $\overline{M}$ the $\mathbb{Z}[G]$-left module that is given by $m \underset{\overline{M}}{\cdot} g := g^{-1} \underset{M}{\cdot} m$. With slightly risque notation we set

$$ {}_{\varphi}M \;:=\; \overline{M} \times_{\pi} \widetilde{X}_{x_0} \;:=\; (\overline{M} \times \widetilde{X}_{x_0})/\!\sim \qquad \text{where } (m \cdot g, x) \sim (m, g \cdot x) $$

We equip $\overline{M}$ with the discrete topology, we equip $\overline{M} \times \widetilde{X}_{x_0}$ with the product topology and finally we equip ${}_{\varphi}M := (\overline{M} \times \widetilde{X}_{x_0})/\sim$ with the quotient topology.

(1) The map

$$ \begin{array}{rcl} q\colon \overline{M} \times_{\pi} \widetilde{X}_{x_0} & \to & X \\ [(m, x)] & \mapsto & p(x) \end{array} $$

is continuous.

(2) Let $x \in X$. We pick a path $\mu\colon [0,1] \to X$ from x_0 to x. The map

$$ \begin{array}{rcl} q^{-1}(x) & \to & M \\ [(m, [\alpha])] & \mapsto & \varphi([\underbrace{\mu * \overline{\alpha}}_{\text{loop in } x_0}])(m) \end{array} $$

$\uparrow$ path from x_0 to x

is well-defined and a bijection. We use the abelian group structure on M and this bijection to define a group structure on $q^{-1}(x)$. This group structure does not depend on the choice of μ.

All this data combined defines an M-local coefficient system on X.

Proof. The proof of the proposition is almost the same as the proof of the Vector Bundles-from-Representation 143.1. We leave it to the reader to make the necessary modifications. ■

Example. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. Furthermore let V be an abelian group. Let $\mathrm{tr}\colon \pi_1(X, x_0) \to \mathrm{Aut}(V)$ be the trivial homomorphism. One can easily verify that the map

$$ \begin{array}{rcl} {}_{\mathrm{tr}}V = \overline{V} \times_{\pi_1(X,x_0)} \widetilde{X}_{x_0} & \to & {}_{\mathrm{tr}}V = V \times X \\ (v, \widetilde{x}) & \mapsto & (v, p(\widetilde{x})) \end{array} $$

is an isomorphism of local coefficient systems, in other words, it is a trivialization of the V-local coefficient system ${}_{\mathrm{tr}}V$.

If X is locally contractible and path-connected, then the converse holds. More precisely we have the following proposition.

Proposition 174.5. Let X be a topological space that is locally contractible and path-connected and let $p\colon C \to X$ be a G-local coefficient system. Let $x_0 \in X$. We denote

by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. There exists a $\mathbb{Z}[\pi_1(X,x_0)]$-left module structure on G such that the corresponding local coefficient system is isomorphic to the given G-local coefficient system $p\colon C \to X$.

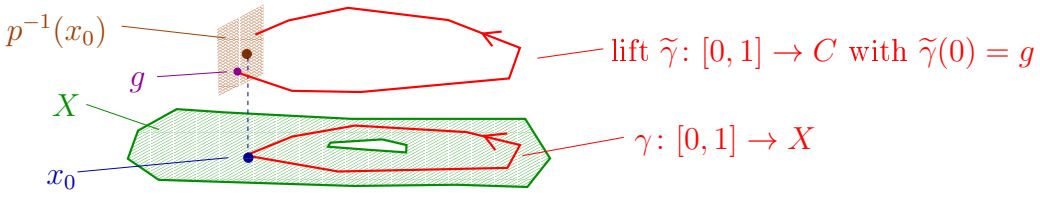
Proof. Let x_0. Since $p\colon C \to X$ is a G-local coefficient system we know that the fiber $p^{-1}(x_0)$ is isomorphic to G.

Claim. The map

$$\begin{array}{rcl} \varphi\colon \pi_1(X,x_0) & \to & \mathrm{Aut}(p^{-1}(x_0)) \\ [\gamma] & \to & \left(\begin{array}{rcl} p^{-1}(x_0) & \to & p^{-1}(x_0) \\ & & \text{by Lemma 174.1 and Proposition 48.11 there exists} \\ g & \mapsto & \text{a unique lift } \widetilde{\gamma}\colon [0,1] \to C \text{ of } \gamma\colon [0,1] \to X, \\ & & \text{with } \gamma(0) = g, \text{ we set } \varphi([\gamma])(g) := \widetilde{\gamma}(1) \end{array} \right) \end{array}$$

is well-defined.

Proof. It follows immediately from Corollary 48.13 that the definition does not depend on the choice of the representative $\gamma\colon [0,1] \to X$ of the given element in $\pi_1(X,x_0)$. Furthermore it follows easily from Lemma 174.2 that given $[\gamma] \in \pi_1(X,x_0)$ the corresponding map $\varphi([\gamma])\colon p^{-1}(x_0) \to p^{-1}(x_0)$ is indeed a group homomorphism, i.e. it defines an element in $\mathrm{End}(p^{-1}(x_0))$. But one can also easily show that φ is a homomorphism of monoids, which then also implies that each $\varphi(g)$ is actually an automorphism of $p^{-1}(x_0)$. ⊞



Now we need to show that the G-local coefficient system that is defined by φ is isomorphic to the G-local coefficient system we started out with. We consider the map

$$\begin{array}{rcl} \Theta\colon p^{-1}(x_0) \times \widetilde{X}_{x_0} & \to & C \\ & & \text{by Lemma 174.1 and Proposition 48.11 there exists a unique} \\ (g,[\gamma]) & \mapsto & \text{lift } \widetilde{\gamma}\colon [0,1] \to C \text{ of } \gamma\colon [0,1] \to X \text{ with } \gamma(0) = g, \\ & & \text{we set } \Theta([\gamma])(g) := \widetilde{\gamma}(1) \end{array}$$

It follows again almost immediately from Corollary 48.13 that the definition does not depend on the choice of the representative $\gamma\colon [0,1] \to X$ of the given point in $\widetilde{M}$. One can easily verify that the map Θ descends to a map

$$p^{-1}(x) \times_{\pi_1(X,x_0)} \widetilde{X}_{x_0} \to C.$$

This map commutes with the projections to X and one can easily verify that it is an isomorphism on the fibers. It follows easily from the definition of a local coefficient system that this implies that the map is in fact an isomorphism of local coefficient systems. ■

174.3. Pullbacks of local coefficient systems.

Definition. Let $\varphi\colon X \to Y$ be a map between topological spaces and let $q\colon C \to Y$ be a G-local coefficient system on Y. We consider the pullback

$$\varphi^*C \;:=\; \{(x,v)\,|\,x \in X \text{ and } v \in q^{-1}(\varphi(x))\} \;\subset\; X \times C,$$

which is equipped with the subspace topology and we consider the map

$$\begin{aligned} p\colon \varphi^*C \;:=\; \{(x,v)\,|\,x \in X \text{ and } v \in q^{-1}(\varphi(x))\} &\to X \\ (x,v) &\mapsto x. \end{aligned}$$

Furthermore given any $x \in X$ we equip $p^{-1}(x) = \{x\} \times q^{-1}(\varphi(x))$ with the group structure that is already given on $q^{-1}(\varphi(x))$.

The pullback of a G-local coefficient system behaves in a similar fashion as the pullback of (vector) bundles that we had studied earlier. Since these statements are at times somewhat subtle it is worth spelling them out again.

The following lemma is the analogue of Lemmas **??** and 143.8.

Lemma 174.6. Let $\varphi\colon X \to Y$ be a map between topological spaces and let $q\colon C \to Y$ be a G-local coefficient system on Y. We consider the pullback

$$\varphi^*C \;:=\; \{(x,v)\,|\,x \in X \text{ and } v \in q^{-1}(\varphi(x))\} \;\subset\; X \times C.$$

which we equip with the subspace topology, the map $p\colon \varphi^*C \to X$ and the group structure on the fibers as described above.

(1) The map $p\colon \varphi^*C \to X$ together with the group structures on the fibers is a G-local coefficient system over X.

(2) The maps $(x,v) \mapsto v$ and φ define a morphism of local coefficient systems:

$$\begin{array}{ccc} \varphi^*C & \xrightarrow{(x,v)\mapsto v} & C \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{\varphi} & Y. \end{array}$$

(3) If we are given a morphism of local coefficients

$$\begin{array}{ccc} C & \xrightarrow{\Phi} & D \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{\varphi} & Y \end{array}$$

such that for each $x \in X$ the map $\Phi|_{C_x}\colon C_x \to D_{\varphi(x)}$ is an isomorphism, then the local coefficient system $C \to D$ is naturally isomorphic to the pullback $\varphi^*D \to X$.

Proof. The proof is almost identical to the straightforward proof of Lemmas **??** and 143.8. ∎

We continue with the analogue of Lemmas **??** and 143.9.

Lemma 174.7. Let $\varphi\colon X \to Y$ be a map between topological spaces and let $q\colon F \to Y$ be a G-local coefficient system on Y. We consider the pullback

$$\varphi^*F \;:=\; \{(x,v)\,|\,x \in X \text{ and } v \in q^{-1}(\varphi(x))\} \;\subset\; X \times F.$$

which we equip with the subspace topology, the map $p\colon \varphi^*F \to X$ and the group structure on the fibers as described above.

(1) The pullbacks of isomorphic G-local coefficient systems are isomorphic.
(2) Let $q\colon G \times Y = {}_{\mathrm{tr}}G \to Y$ be the trivial G-local coefficient system. The map
$$\begin{array}{ccc} \varphi^*\,{}_{\mathrm{tr}}G & \to & {}_{\mathrm{tr}}G \\ (x,(g,y)) & \mapsto & (g,x) \end{array}$$
is a natural isomorphism from the pullback of the trivial G-local coefficient system on Y to the trivial G-local coefficient system on X.
(3) The pullback of a trivializable G-local coefficient system is trivializable.
(4) If $q\colon F \to Y$ is a G-local coefficient system and if X is a subset of Y with inclusion map $i\colon X \to Y$, then
$$\begin{array}{ccc} i^*F & \to & q^{-1}(X) \\ (x,v) & \mapsto & v \end{array}$$
is a natural isomorphism from the G-local coefficient system $i^*F \to X$ to the restriction of F to the G-local coefficient system $q^{-1}(X) \to X$.

Proof. The proof is almost identical to the straightforward proof of Lemmas **??** and 143.9. ∎

Finally the next lemma says that pullbacks are contravariantly functorial. The lemma is the analogue of Lemmas **??** and 143.10.

Lemma 174.8.

(1) Let $q\colon F \to Z$ be a G-local coefficient system. Then
$$\begin{array}{ccc} \mathrm{id}_Z^*\,F & \to & F \\ (z,v) & \mapsto & v \end{array}$$
is a natural isomorphism from the pullback G-local coefficient system $\mathrm{id}_Z^*\,F \to Z$ to the original G-local coefficient system $q\colon F \to Z$.
(2) Let $q\colon F \to Z$ be a G-local coefficient system. If $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ are maps between topological spaces, then
$$\begin{array}{ccc} \varphi^*(\psi^*F) & \to & (\psi\circ\varphi)^*F \\ (x,(y,v)) & \mapsto & (x,v) \end{array}$$
is a natural isomorphism between the double pullback G-local coefficient system $\varphi^*(\psi^*F) \to X$ to the pullback G-local coefficient system $(\psi\circ\varphi)^*F \to X$.

Proof. The proof is almost identical to the straightforward proof of Lemmas **??**, **??** and 143.8. ∎

Lemma 174.9. Let X be a topological space that is locally contractible and path-connected. Let $x_0 \in X$. We write $\pi := \pi_1(X,x_0)$. Furthermore let M be a $\mathbb{Z}[\pi]$-left module, i.e. let M be an abelian group and suppose we are given a group homomorphism $\varphi\colon \pi_1(X,x_0) \to \mathrm{Aut}(M)$. Let $A \subset X$ be a subspace that is also locally contractible and path-connected and that contains x_0. We denote by $i\colon A \to X$ the inclusion map and we write $\Gamma := \pi_1(A,a_0)$.

We equip A with the representation $\varphi \circ i_* \colon \Gamma = \pi_1(A, x_0) \to \mathrm{Aut}(M)$. Then the map

$$\begin{array}{ccc} {}_{\varphi\circ i_*}M & \to & i^*{}_{\varphi}M \\ (v, [\alpha]) & \mapsto & (\alpha(1), \underbrace{[(v, [\alpha])]}_{\in M \times_\pi \widetilde{X}_{x_0}}) \\ \uparrow & & \\ \alpha \text{ is a path in } A \text{ with } \alpha(0) = x_0 & & \end{array}$$

is a natural isomorphism from the M-local coefficient system ${}_{\varphi\circ i_*}M \to A$ corresponding to the representation $\varphi \circ i_* \colon \pi_1(A, x_0) \to \mathrm{Aut}(M)$ to the pullback M-local coefficient system $i^*{}_{\varphi}M \to A$.

Proof. We consider the following heroic map:

$$\begin{array}{ccc} i^*{}_{\varphi}M & \to & {}_{\varphi\circ i_*}M \\ (a, [(v, [\gamma])]) & \mapsto & [(\varphi([\gamma * \overline{\alpha}])(v), \alpha)] \\ \uparrow & & \uparrow \\ \gamma \text{ is a path in } M & & \text{we pick a path } \alpha \text{ in } A \\ \text{with } \gamma(0) = x_0 & & \text{with } \alpha(0) = x_0 \text{ and } \alpha(1) = a \end{array}$$

We leave it to the reader to show that the given maps ${}_{\varphi\circ i_*}M \to i^*{}_{\varphi}M$ and $i^*{}_{\varphi}M \to {}_{\varphi\circ i_*}M$ are morphisms of M-local coefficients and that they are inverses of one another. ∎

174.4. **Local coefficient systems and homotopies.** In the Vector Bundles-Homotopies Proposition 143.13 we studied pullbacks of vector bundles under homotopic maps. The following is now the analogue of the Vector Bundles-Homotopies Proposition 143.13 in our present context.

Proposition 174.10. Let G be an abelian group.

(1) Let $q \colon F \to Y$ be a G-local coefficient system, let X be a topological space and finally let $\varphi_0, \varphi_1 \colon X \to Y$ be two maps.
   (a) Every homotopy $\Phi \colon X \times [0,1] \to Y$ between φ_0 and φ_1 naturally defines an isomorphism between the G-local coefficient systems $\varphi_0^* F \to X$ and $\varphi_1^* F \to X$.
   (b) In particular, if φ_0 and φ_1 are homotopic, then the G-local coefficient systems $\varphi_0^* F \to X$ and $\varphi_1^* F \to X$ are isomorphic.

(2) Let X be a topological space. If X is contractible, then every G-local coefficient system over X is trivializable.

Definition. Let G be an abelian group.

(1) If $q \colon F \to X$ is a G-local coefficient system and if Y is a topological space, then the product topological space $F \times Y$ together with the map

$$\begin{array}{rccc} q \times \mathrm{id}_Y \colon & F \times Y & \to & X \times Y \\ & (v, y) & \mapsto & (q(v), y) \end{array}$$

defines a G-local coefficient system in the obvious way

(2) Let $p \colon F \to X \times [0,1]$ be a G-local coefficient system. Given $t \in [0,1]$ we consider the G-local coefficient system

$$p_t \colon F_t := p^{-1}(X \times \{t\}) \xrightarrow{p} X \times \{t\} \xrightarrow{(x,t)\mapsto x} X.$$

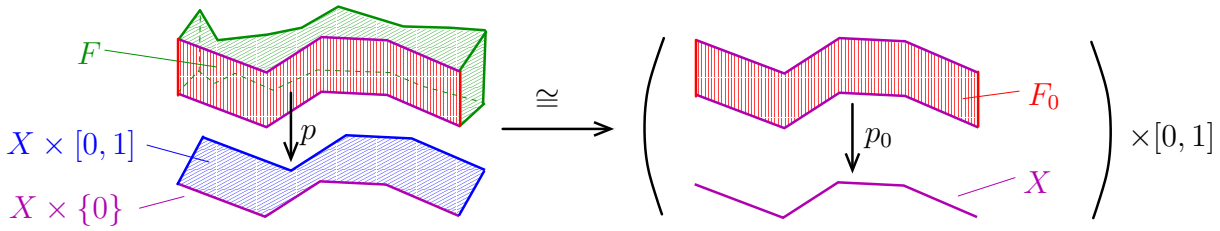
Lemma 174.11. Let X be a topological space and let $p \colon F \to X \times [0,1]$ be a G-local coefficient system. As above we consider the G-local coefficient system $p_0 \colon F_0 := p_0^{-1}(X \times$

$\{0\}) \to X$. There exists a *unique* isomorphism of G-local coefficient systems

$$\begin{array}{ccc} F & \xrightarrow[\cong]{\Phi} & F_0 \times [0,1] \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle p_0 \times \mathrm{id}_{[0,1]}} & \\ & X \times [0,1] & \end{array}$$

over $X \times [0,1]$ such that for any $x \in X$ and $v \in p^{-1}(x,0)$ we have

$$\Phi(v) \;=\; (v,0).$$



Proof of Lemma 174.11. We start out with the following claim.

Claim 1. Let $p\colon F \to Y \times [a,c]$ be a G-local coefficient system and let $b \in [a,c]$. If the restrictions of p to $Y \times [a,b]$ and to $Y \times [b,c]$ are both trivializable, then so is the restriction of p to $Y \times [a,c]$.

Proof. Since the restrictions of p to $Y \times [a,b]$ and to $Y \times [b,c]$ are both trivializable we can pick trivializations $\Phi\colon p^{-1}(Y \times [a,b]) \to G \times Y \times [a,b]$ and $\Psi\colon p^{-1}(Y \times [b,c]) \to G \times Y \times [b,c]$. We denote by $q\colon G \times Y \times [a,c] \to G$ the projection. It follows almost immediately from Proposition 174.3 that there exists a continuous map $\Theta\colon Y \times \{b\} \to \mathrm{Aut}(G)$ such that for all $v \in p^{-1}(Y \times \{b\})$ we have $q(\Phi(v)) = \Theta(p(v)) \cdot q(\Psi(v)) \in G$. It is straightforward to verify that

$$\begin{array}{rcl} p^{-1}(Y \times [a,c]) & \to & G \times Y \times [a,c] \\ v & \mapsto & \begin{cases} \Phi(v), & \text{if } p(v) \in Y \times [a,b], \\ (p(v), \Theta(p(y,b)) \cdot q(\Psi(v))), & \text{if } p(v) = (y,t) \in Y \times [b,c] \end{cases} \end{array}$$

is an isomorphism of G-local coefficient systems. ⊞

Now we turn to the actual setting of the lemma. Thus let $p\colon F \to X \times [0,1]$ be G-local coefficient system over a topological space X.

Claim 2. For every $x \in X$ there exists an open neighborhood U_x of $x \in X$ and a trivialization

$$\Theta_x\colon p^{-1}(U_x \times [0,1]) \;\to\; G \times U_x \times [0,1].$$

of the restriction of the G-local coefficient system $p\colon E \to X \times [0,1]$ to $U_x \times [0,1]$.

Proof. Let $x \in X$. By definition of a local coefficient system and by compactness of $[0,1]$ there exist neighborhoods $V_1, \ldots, V_m$ of x and $0 = t_0 < \cdots < t_m = 1$ such that the restriction of $p\colon E \to X \times [0,1]$ to each $V_i \times [t_{i-1}, t_i]$ is trivializable. It follows from Claim 1 that the restriction of p to $(V_1 \cap \cdots \cap V_m) \times [0,1]$ is also trivializable. Thus $U_x := V_1 \cap \cdots \cap V_m$ admits such a trivialization Φ_x. ⊞

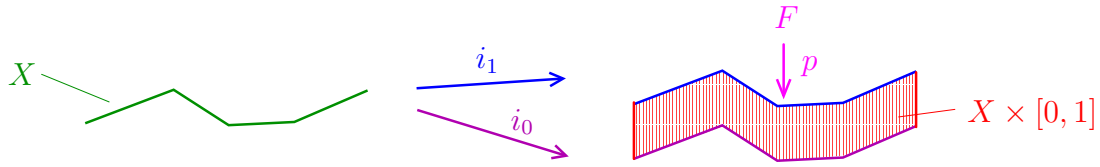
We denote by $r\colon X \times [0,1] \to X$ and $s\colon X \times [0,1] \to [0,1]$ the obvious projection. Now let $x \in X$. We define

$$\begin{array}{rcl} \Phi_{U_x}\colon p^{-1}(U_x \times [0,1]) & \to & p^{-1}(U_x \times \{0\}) \times [0,1] \\ v & \mapsto & \text{we write } \Theta_x(v) = (g,y,t) \in G \times U_x \times [0,1] \text{ and} \\ & & \text{we set } \Phi_{U_x}(v) = (\Theta_x(y,0,g),t). \end{array}$$

Note that this map is continuous and an isomorphism on the fibers. Let $y \in X$. It follows from Proposition 174.3 together with the Connected-to-Discrete Lemma 2.27 applied to the map $y \times [0,1] \to p^{-1}(\{(y,0)\})$ that this map does not depend on the choice of $x \in X$ with $y \in U_x$. Thus it follows from the Pasting Proposition 2.6 that these maps Φ_x define a continuous map $\Phi\colon F \to F_0 \times [0,1]$ that is an isomorphism of local coefficient systems. Basically by definition we $\Phi(v) = (v,0)$ for every $x \in X$ and $v \in p^{-1}(x,0)$. ■

The following corollary is somewhat similar to Lemma 143.14.

Corollary 174.12. Let X be a topological space. We denote by $i_0\colon X \xrightarrow{x \mapsto (x,0)} X \times [0,1]$ and $i_1\colon X \xrightarrow{x \mapsto (x,1)} X \times [0,1]$ the obvious embeddings. Given any G-local coefficient system $p\colon F \to X \times [0,1]$ the pullbacks i_0^*F and i_1^*F are naturally isomorphic.



Proof of Corollary 174.12. Let $\Phi\colon F \to F_0 \times [0,1]$ be the unique isomorphism of G-local coefficient systems over $X \times [0,1]$ from Lemma 174.11. Note that for $t \in [0,1]$ we have

$$i_t^*(F_0 \times [0,1]) \;=\; \{(x,v,t) \in X \times F_0 \times [0,1] \,|\, v \in p^{-1}(x,0)\}.$$

Then we see that

$$\begin{array}{ccccccc} i_0^*F & \cong & i_0^*(F_0 \times [0,1]) & \cong & i_1^*(F_0 \times [0,1]) & \cong & i_1^*F. \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{via the isomorphism } \Phi & & \text{a natural isomorphism is} & & \text{via the isomorphism } \Phi^{-1} & & \\ & & \text{given by } (x,v,0) \mapsto (x,v,1) & & & & \end{array}$$

■

The same way as we deduced the Vector Bundles-Homotopies Proposition 143.13 from Lemma 143.14 we can now deduce Proposition 174.10 from Corollary 174.12

Proof of Proposition 174.10.

(1) Let $q\colon F \to Y$ be a G-local coefficient system, let X be a topological space and let $H\colon X \times [0,1] \to Y$ be a homotopy between two maps $f_0, f_1\colon X \to Y$. We denote by $i_0, i_1\colon X \to X \times [0,1]$ the two obvious inclusions. Now we see that we have the following natural isomorphism

$$\begin{array}{ccccccccc} f_0^*F & = & (H \circ i_0)^*F & \cong & i_0^*(H^*F) & \cong & i_1^*(H^*F) & \cong & (H \circ i_1)^*F & = & f_1^*F. \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ \text{since } f_0 = H \circ i_0 & & \text{Lemma 174.7} & & \text{Corollary 174.12} & & \text{Lemma 174.7} & & \text{since } f_1 = H \circ i_1 & & \end{array}$$

(2) Let $F \to X$ be a G-local coefficient system over a topological space that is contractible. We pick $x_0 \in X$. By the Contractibility Criterion Lemma 15.2 the inclusion map $i\colon \{x_0\} \to X$ is a homotopy equivalence. We denote by $p\colon X \to \{x_0\}$

the unique map. Note that $p \circ i$ is homotopic to i. Thus we have the following isomorphisms of G-local coefficient systems on X:

$$\begin{array}{ccccccc} F & \cong & (\mathrm{id}_X)^*F & \cong & (i \circ p)^*F & \cong & p^*(i^*F). \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Lemma 174.7} & & \text{Statement (1)} & & \text{Lemma 174.7.} & \end{array}$$

Note that i^*F is a local coefficient system on a point, hence trivializable. Thus F is isomorphic to the pullback of a trivializable local coefficient system. Thus we obtain from Lemma 174.7 that F is trivializable. ■

174.5. **Tensor products of local coefficient systems.** The reader will have noticed that G-local coefficients are somewhat similar in spirit to vector bundles. In Section 143.8 we saw that many well-known constructions in linear algebra also work for vector bundles. For example we saw how to take the tensor product of two vector bundles. Similarly we can now construct new local coefficient systems out of given ones. This leads us to the following definition, which is modelled on the definition of the tensor product of vector bundles that we gave on page 3080.

Definition. Let X be a topological space, let $p\colon E \to X$ be a G-local coefficient system and let $q\colon F \to X$ be an H-local coefficient system. We consider the set

$$E \otimes_{\mathbb{Z}} F \;:=\; \bigsqcup_{x \in X} \underbrace{p^{-1}(x) \otimes_{\mathbb{Z}} q^{-1}(x)}_{\text{abelian group}} \quad \text{with the obvious map } r\colon E \otimes_{\mathbb{Z}} F \to X.$$

We equip $E \otimes_{\mathbb{Z}} F$ with the smallest topology that has the property that for any open subset $U \subset X$ and any two trivializations $\phi\colon p^{-1}(U) \to G \times U$ and $\psi\colon q^{-1}(U) \to U \times H$ the map

$$\begin{array}{ccc} r^{-1}(U) & \to & U \times \overbrace{(G \otimes_{\mathbb{Z}} H)}^{\text{discrete topology}} \\ v_x \otimes w_x & \mapsto & (x, \phi_x(v_x) \otimes \psi_x(w_x)) \end{array}$$

is a homeomorphism. One can now easily show that $r\colon E \otimes_{\mathbb{Z}} F \to X$ is a $G \otimes H$-local coefficient system. It is called the tensor product of the local coefficient systems E and F.

Lemma 174.13. Let X be a topological space and let $p\colon E \to X$ be a G-local coefficient system.

(1) The map

$$\begin{array}{ccc} \mu\colon E \otimes_{\mathbb{Z}\ {}_{\mathrm{tr}}\mathbb{Z}} & \to & E \\ (e \otimes n) & \mapsto & n \cdot e \end{array}$$

is a natural isomorphism of G-local coefficient systems.[2119]

(2) If $q\colon F \to X$ is another local coefficient system, then

$$\begin{array}{ccc} \tau\colon E \otimes_{\mathbb{Z}} F & \to & F \otimes_{\mathbb{Z}} E \\ e \otimes_{\mathbb{Z}} f & \mapsto & f \otimes_{\mathbb{Z}} e \end{array}$$

is a natural isomorphism of G-local coefficient systems.

Proof. We leave the verification of both statements to the reader. ■

[2119] Note that if G and G' are isomorphic abelian groups, then a G-local coefficient system is the same as a G'-local coefficient system. In particular every $G \otimes_{\mathbb{Z}} \mathbb{Z}$-local coefficient system is also a G-local coefficient system.

Exercises for Chapter 174.

Exercise 174.1. Let X be a topological space that is locally contractible and path-connected and let $x_0 \in X$. Let M and N be abelian groups and let $\varphi\colon \mathrm{Aut}(M)$ and $\psi\colon \mathrm{Aut}(N)$ be representations. We consider the corresponding $\mathbb{Z}[\pi_1(X, x_0)]$-left modules M and N and we consider the corresponding $\mathbb{Z}[\pi_1(X, x_0)]$-left module $M \otimes_{\mathbb{Z}} N$. As we saw in Proposition 174.4 these modules give rise to local coefficient systems that we denote by ${}_{\varphi}M$, ${}_{\varphi}N$ and ${}_{\varphi\otimes\psi}M \otimes_{\mathbb{Z}} N$. Show that the tensor product of the local coefficient systems ${}_{\varphi}M$ and ${}_{\varphi}N$ is isomorphic to the ${}_{\varphi\otimes\psi}M \otimes_{\mathbb{Z}} N$.

Exercise 174.2. Let X be a topological space and let $0 \to A \to B \to C \to 0$ be a short exact sequence of local coefficient systems over X. Show that this give rise to short exact sequences of (co-) homology groups with local coefficients.

175. (Co-) homology with local coefficients

175.1. Homology with local coefficients.

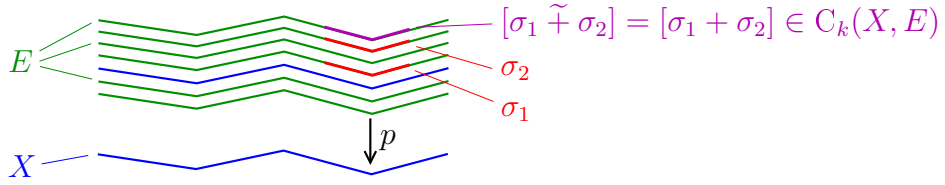
Definition. Let X be a topological space, let G be an abelian group and let $p\colon E \to X$ be a G-local coefficient system.

(1) Given two singular k-simplices $\sigma_1, \sigma_2\colon X \to E$ in C with $p \circ \sigma_1 = p \circ \sigma_2$ we now define[21202121]
$$\begin{aligned} \sigma_1 \widetilde{+} \sigma_2\colon \Delta^k &\to E \\ x &\mapsto \underbrace{\sigma_1(x) + \sigma_2(x)}_{\text{sum in the fiber } E_x}. \end{aligned}$$

(2) As usual we denote by $\mathrm{C}_k(E)$ the set of singular k-chains in the topological space E. Now we set
$$\mathrm{C}_k^{\mathrm{lc}}(X; E) \;:=\; \mathrm{C}_k(E)/\sim$$
where $\sim$ is the equivalence relation generated by the relation that for two singular k-simplices σ_1, σ_2 in E with $p \circ \sigma_1 = p \circ \sigma_2$ we set $\sigma_1 + \sigma_2 \sim \sigma_1 \widetilde{+} \sigma_2$.

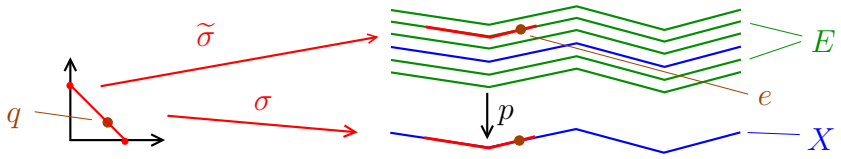
(3) Given a singular k-simplex $\sigma\colon X \to E$ we refer to corresponding element in $\mathrm{C}_k^{\mathrm{lc}}(X; E)$ as a singular k-simplex with coefficients in E.

(4) We refer to an element of $\mathrm{C}_k^{\mathrm{lc}}(X; E)$ as a singular k-chain with coefficients in E.



Convention. Let X be a topological space and let $p\colon E \to X$ be a local coefficient system. To preserve sanity in our notation we will usually not distinguish between a singular k-simplex with coefficients in E and the element it represents in $\mathrm{C}_k^{\mathrm{lc}}(X; E)$.

Lemma 175.1. (Unique Simplex Lifting Lemma) Let X be a topological space and let $p\colon E \to X$ be a local coefficient system. Furthermore let $\sigma\colon \Delta^n \to X$ be a singular simplex, let $q \in \Delta^n$ and let $e \in E$ such that $p(e) = \sigma(q)$. There exists a unique singular simplex $\widetilde{\sigma}\colon \Delta^n \to E$ with $p \circ \widetilde{\sigma} = \sigma\colon \Delta^n \to X$ and $\widetilde{\sigma}(q) = e$.



Proof. By Lemma 174.1 we know that $p\colon E \to X$ is a covering map. Thus the statement follows from general covering theory. More precisely, it follows from the Map Lifting Criterion 61.2 together with with the observation that Δ^n is simply connected and locally path-connected. ∎

[2120] By Lemma 174.2 we know that $\sigma_1 \widetilde{+} \sigma_2\colon \Delta^k \to E$ is continuous.

[2121] Since G is abelian we see that $\sigma_1 \widetilde{+} \sigma_2 = \sigma_2 \widetilde{+} \sigma_1$. It is a fun exercise to figure out where this actually gets used.

On page 1585 we introduced the boundary maps in the ordinary singular chain complexes. Since $\mathrm{C}_k^{\mathrm{lc}}(X;E)$ is a quotient of $\mathrm{C}_k(E)$ we do not need to be very creative when it comes to defining a boundary map on $\mathrm{C}_k^{\mathrm{lc}}(X;E)$:

Definition. Let X be a topological space and let $p\colon E \to X$ be a local coefficient system We refer to[2122]

$$\begin{array}{rcl} \partial_n\colon \mathrm{C}_k^{\mathrm{lc}}(X;E) & \to & \mathrm{C}_{k-1}^{\mathrm{lc}}(X;E) \\ \Big[\underbrace{\sum_{i=1}^{k}(\sigma_i\colon \Delta^n \to E)}_{\in \mathrm{C}_k(E)}\Big] & \mapsto & \Big[\sum_{i=1}^{k}\underbrace{\partial_k\sigma_i}_{\in \mathrm{C}_{k-1}(E)}\Big] \end{array}$$

as the k-th boundary map.

By Proposition 72.2 we know that each $\partial_{k-1}\circ\partial_k\colon \mathrm{C}_k(E) \to \mathrm{C}_{k-2}(E)$ is the zero map. It follows immediately that $(\mathrm{C}_*^{\mathrm{lc}}(X;E),\partial_*)$ is a chain complex. Thus we can define the corresponding homology groups.

Definition. Let X be a topological space and let $p\colon E \to X$ be a local coefficient system. Given $n \in \mathbb{N}_0$ we define the n-th singular homology group of X with E-coefficients as the quotient group

$$\mathrm{H}_n^{\mathrm{lc}}(X;E) \;:=\; \frac{\ker(\partial_n\colon \mathrm{C}_n^{\mathrm{lc}}(X;E) \to \mathrm{C}_{n-1}^{\mathrm{lc}}(X;E))}{\mathrm{im}(\partial_{n+1}\colon \mathrm{C}_{n+1}^{\mathrm{lc}}(X;E) \to \mathrm{C}_n^{\mathrm{lc}}(X;E))}.$$

Remark. The literature contains many different, but essentially equivalent, definitions of homology groups with coefficients in a local coefficient systems. We refer to [**KwS2018**, p. 284], [**DaK2001**, Chapter 5.3], [**WhdG1978**, Chapter VI.2] for slightly different definitions. Our definition is basically the same as the definition given in [**Hat2002**, Chapter 3.H]. There are few sources that study homology with local coefficients in details, the most thorough source seems to be [**WhdG1978**, Chapter VI].

From ordinary homology we are used to the fact that maps between topological spaces induces maps between homology groups. A similar statements holds for homology with local coefficients, but, as in the case of homology with twisted coefficients that we studied on page 3542, some care needs to be taken to work with matching local coefficient systems.

Definition. Let $f\colon X \to Y$ be a map between topological spaces and let $p\colon E \to Y$ be a G-local coefficient system on Y. We denote by $q\colon f^*E \to E$ the projection. We define

$$\begin{array}{rcl} f_*\colon \mathrm{C}_k^{\mathrm{lc}}(X;f^*E) & \to & \mathrm{C}_k^{\mathrm{lc}}(Y;E) \\ [\sigma\colon \Delta^k \to f^*E] & \to & [q\circ\sigma\colon \Delta^k \to E] \end{array}$$

One can easily verify that these maps are well-defined and that they form a chain map. Thus we obtain induced maps

$$f_*\colon \mathrm{H}_k^{\mathrm{lc}}(X;f^*E) \;\to\; \mathrm{H}_k^{\mathrm{lc}}(Y;E).$$

The previous definition now also allows us to introduce relative homology with local coefficients.

[2122] Let $i_j^k\colon \Delta^{k-1} \to \Delta^k$ be the face maps as defined on page 1585. Let $\sigma_1,\sigma_2\colon \Delta^k \to X$ be two singular simplices with $p\circ\sigma_1 = p\circ\sigma_2$. It follows easily from the definitions that $i_j^k\circ\sigma_1 \mathbin{\widetilde{+}} i_j^k\circ\sigma_2 = i_j^k\circ(\sigma_1 \mathbin{\widetilde{+}} \sigma_2)$. It follows from this observation that the usual boundary map ∂_n descends to a map $\mathrm{C}_k^{\mathrm{lc}}(X;E) \to \mathrm{C}_{k-1}^{\mathrm{lc}}(X;E)$.

Definition.

(1) Let X be a topological space, let $A \subset X$, let G be an abelian group and let $p\colon E \to X$ be a G-local coefficient system. We denote by $i\colon A \to X$ the inclusion. We set
$$\mathrm{C}^{\mathrm{lc}}_n(X, A; E) \;:=\; \mathrm{coker}(i_*\colon \mathrm{C}^{\mathrm{lc}}_n(A; i^*E) \to \mathrm{C}^{\mathrm{lc}}_n(X; E)).$$
By Lemma 166.1 we know that the chain map on $\mathrm{C}^{\mathrm{lc}}_n(X; E)$ descends to a chain map on $\mathrm{C}^{\mathrm{lc}}_n(X, A; E)$. Thus we can set
$$\mathrm{H}^{\mathrm{lc}}_n(X, A; E) \;:=\; \mathrm{H}_n(\mathrm{C}^{\mathrm{lc}}_*(X, A; E)).$$

(2) If we are given a map $f\colon (X, A) \to (Y, B)$ of pairs of topological spaces and if we are given a G-local coefficient system E on Y, then as before we get induced maps $f_*\colon \mathrm{C}^{\mathrm{lc}}_k(X, A; f^*E) \to \mathrm{C}^{\mathrm{lc}}_k(Y, B; E)$ and $f_*\colon \mathrm{H}^{\mathrm{lc}}_k(X, A; f^*E) \to \mathrm{H}^{\mathrm{lc}}_k(Y, B; E)$.

The following lemma, which can be viewed as a generalization of Lemma 89.7 and an analogue of Lemma 167.4 says that singular homology with local coefficients is covariantly functorial in the local coefficient system.

Lemma 175.2. Let X be a topological space and let $A \subset X$.

(1) Let $\varphi\colon E \to F$ be a morphism from a G-local coefficient system $p\colon E \to X$ to an H-local coefficient system $q\colon F \to X$. The maps
$$\begin{aligned} \varphi_*\colon \mathrm{C}^{\mathrm{lc}}_k(X, A; E) &\to \mathrm{C}^{\mathrm{lc}}_k(X, A; F) \\ [\sigma\colon \Delta^k \to E] &\to [\varphi \circ \sigma\colon \Delta^k \to F] \end{aligned}$$
are a well-defined chain map and thus they induce maps
$$\varphi_*\colon \mathrm{H}^{\mathrm{lc}}_k(X, A; E) \;\to\; \mathrm{H}^{\mathrm{lc}}_k(X, A; F).$$

(2) With the obvious interpretations we have $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$ and $\mathrm{id}_* = \mathrm{id}$.

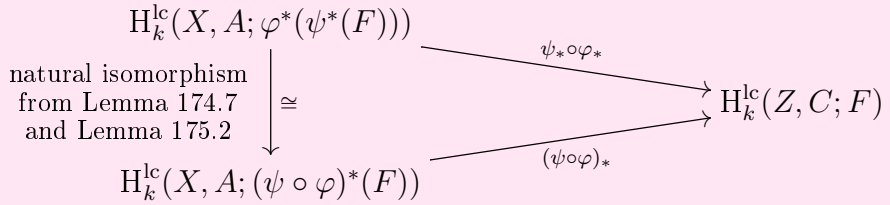
Proof. We leave the elementary proof of the lemma to the reader. ∎

The following lemma shows that homology with local coefficients is "almost" covariantly functorial.

Lemma 175.3. Let $p\colon F \to Z$ be a G-local coefficient system on a topological space Z and let $C \subset Z$.

(1) The natural isomorphism between the G-local coefficient systems $\mathrm{id}^* F$ and F that we gave in Lemma 174.7 induces the map $\mathrm{id}_*\colon \mathrm{H}^{\mathrm{lc}}_k(Z, C; \mathrm{id}^* F) \to \mathrm{H}^{\mathrm{lc}}_k(Z, C; F)$.

(2) Let $\varphi\colon (X, A) \to (Y, B)$ and $\psi\colon (Y, B) \to (Z, C)$ be maps between pairs of topological spaces We have the following commutative diagram:



Remark. Lemma 175.3 plays the same role as the Twisted Homology-Functoriality Proposition 167.9 did for homology with twisted coefficients. Note that the statement for homology with twisted coefficients is much neater though, since we do not need the awkward

vertical map to go from one coefficient system to another. On the other hand, when we are dealing with twisted coefficients we have to constantly worry about base points.

Proof. All the statements follows immediately from the definitions. We leave it to the reader to provide the details. ■

We formulate our first example as a lemma. It says that homology with local coefficients in a trivial G-local coefficient system is basically the same as our old friend, namely singular homology with G-coefficients.

Lemma 175.4. Let X be a topological space, let $A \subset X$ and let G be an abelian group. We denote by $p\colon {}_{\mathrm{tr}}G := G \times X \to X$ the corresponding trivial G-local coefficient system that we introduced on page 3643.

(1) The maps

$$\begin{array}{rcl} T\colon \mathrm{C}_k(X,A) \otimes_{\mathbb{Z}} G & \to & \mathrm{C}_k^{\mathrm{lc}}(X,A;{}_{\mathrm{tr}}G) = \mathrm{C}_k^{\mathrm{lc}}(X,A;G\times X) \\ \sigma \otimes g & \mapsto & \text{the lift } \sigma \to G\times X \text{ given by } x \mapsto (\sigma(x),g) \end{array}$$

define a chain isomorphism that is natural in the sense that given a homomorphism $\varphi\colon G \to H$ between abelian groups the following diagram commutes:

$$\begin{array}{ccc} \mathrm{C}_k(X,A;G) & \xrightarrow{\qquad\varphi_*\qquad} & \mathrm{C}_k(X,A;H) \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ \mathrm{C}_k^{\mathrm{lc}}(X,A;{}_{\mathrm{tr}}G) & \xrightarrow{\qquad\varphi_*\qquad} & \mathrm{C}_k^{\mathrm{lc}}(X,A;H\,{}_{\mathrm{tr}}H) \end{array}$$

and it is natural in the sense that given any map $f\colon (X,A) \to (Y,B)$ between pairs of topological spaces the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{C}_k(X,A;G) & & \xrightarrow{\qquad f_*\qquad} & & \mathrm{C}_k(Y,B;G) \\ {\scriptstyle T}\big\downarrow & & & & \big\downarrow{\scriptstyle T} \\ \mathrm{C}_k^{\mathrm{lc}}(X,A;{}_{\mathrm{tr}}G) & \xrightarrow[\text{Lemma 174.7}]{\text{isomorphism from}} & \mathrm{C}_k^{\mathrm{lc}}(X,A;f^*({}_{\mathrm{tr}}G)) & \xrightarrow{f_*} & \mathrm{C}_k^{\mathrm{lc}}(Y,B;{}_{\mathrm{tr}}G). \end{array}$$

(2) The natural chain isomorphisms from (1) give rise to isomorphisms

$$\mathrm{H}_k(X,A;G) \xrightarrow{\cong} \mathrm{H}_k^{\mathrm{lc}}(X,A;{}_{\mathrm{tr}}G)$$

that are natural in the above sense.

Proof. We denote by $p\colon G\times X \to X$ and $q\colon G\times X \to G$ the two projections. We consider the map

$$\begin{array}{rcl} \mathrm{C}_k^{\mathrm{lc}}(X,A;G\times X) & \to & \mathrm{C}_k(X,A)\otimes_{\mathbb{Z}} G \\ [\sigma\colon \Delta^k \to G\times X] & \mapsto & \sigma\otimes g, \text{where } g \text{ is the unique } g\in G \text{ with } q(\sigma(\Delta^k)) = \{g\}. \end{array}$$

We leave it to the reader to show that the two given maps are chain maps, that they are natural in the stated sense and that they are inverses of one another. ■

In the Homotopy Equivalence-H_*-Corollary 73.9 and Proposition 74.15 we showed that homotopic maps induce chain homotopic maps on (relative) singular chain complexes. In Corollary 167.14 we saw that an analogous statement also holds for (co-) homology with twisted coefficients. The following proposition is the "local coefficient" analogue.

[2122]By the Unique Simplex Lifting Lemma 175.1 we know that such a lift exists and that it is unique.

Proposition 175.5. Let $h_0, h_1 \colon (X, A) \to (Y, B)$ be two maps between pairs of topological spaces and let $p \colon F \to Y$ be a G-local coefficient system on Y.

(1) If $H \colon X \times [0,1] \to Y$ is a homotopy between h_0 and h_1, then we obtain from Proposition 174.10 an explicit isomorphism Θ_H between the G-local coefficient systems h_0^*F and h_1^*F. If for each $t \in [0,1]$ we have $H_t(A) \subset B$, then the maps

$$\begin{array}{rcl} (h_0)_* \colon \mathrm{C}_k^{\mathrm{lc}}(X, A; h_0^*F) & \to & \mathrm{C}_k^{\mathrm{lc}}(Y, B; F) \\ \text{and} \quad (h_1)_* \circ (\Theta_H)_* \colon \mathrm{C}_k^{\mathrm{lc}}(X, A; h_0^*F) & \to & \mathrm{C}_k^{\mathrm{lc}}(X, A; h_1^*F) \to \mathrm{C}_k^{\mathrm{lc}}(Y, B; F) \end{array}$$

are chain homotopic, in particular they induce the same map on homology.

(2) Homotopy equivalences induces isomorphisms on homology groups with local coefficients.

In the Homotopy Equivalence-H_*-Corollary 73.9 we showed the analogous statement for ordinary singular homology. The idea for the proof of Proposition 175.5 is that, by carefully reading the proof of the Homotopy Equivalence-H_*-Corollary 73.9, we can modify the proof of the Homotopy Equivalence-H_*-Corollary 73.9 to give us the statement we are presently interested in. It is thus convenient to recall the main concepts behind the proof of the Homotopy Equivalence-H_*-Corollary 73.9.

Lemma 73.8. Let W and F be topological spaces and let $\widetilde{H} \colon W \times [0,1] \to F$ be a homotopy. Let $\Omega_n \in \mathrm{C}_{n+1}(\Delta^n \times [0,1])$, $n \in \mathbb{N}_0$, be the sequence of singular chains that we defined on page 1605.[2123] The maps

$$\begin{array}{rcl} P_n \colon \mathrm{C}_n(W) & \to & \mathrm{C}_{n+1}(F) \\ (\sigma \colon \Delta^n \to W) & \mapsto & (\widetilde{F} \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\Omega_n). \end{array}$$

satisfy

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \ = \ \widetilde{H}_{0*} - \widetilde{H}_{1*}$$

as maps from $\mathrm{C}_n(W)$ to $\mathrm{C}_n(F)$.

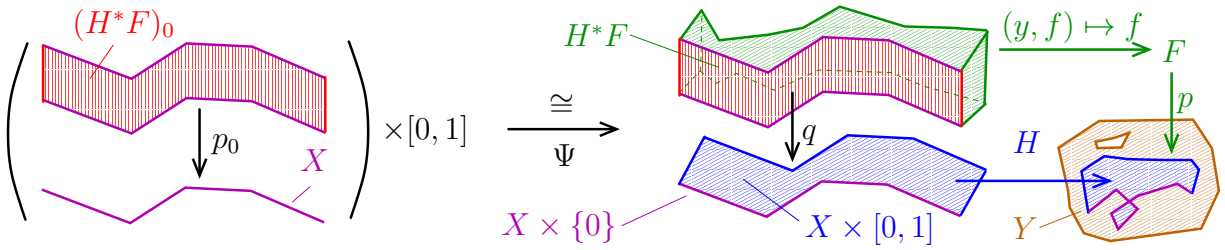
Proof of Proposition 175.5. To simplify the notation we assume that $A = B = \varnothing$. Mathematically this extra hypothesis makes no real difference. Thus let $h_0, h_1 \colon X \to Y$ be two maps between topological spaces and let $p \colon F \to Y$ be a G-local coefficient system on Y. Furthermore let $H \colon X \times [0,1] \to Y$ be a homotopy between h_0 and h_1. We consider the G-local coefficient system $q \colon H^*F \to X \times [0,1]$. As on page 3650 we consider the G-local coefficient system

$$\begin{array}{rcl} q_t \colon (H^*F)_t := q^{-1}(X \times \{t\}) & \to & X \\ v & \mapsto & \text{unique } x \in X \text{ with } q(v) = (x,t). \end{array}$$

Recall that by Lemma 174.11 there exists a unique isomorphism of G-local coefficient systems $\Psi \colon (H^*F)_0 \times [0,1] \to H^*F$ over $X \times [0,1]$ such that for any $x \in X$ and $v \in q^{-1}(x,0)$ we have $\Psi(v,0) = v$. Next we consider the map

$$\begin{array}{rcccl} \widetilde{H} \colon (H^*F)_0 \times [0,1] & \to & H^*F & \to & F \\ (v,t) & \mapsto & \Psi(v,t) & & \\ & & (z,f) & \mapsto & f. \end{array}$$

[2123] The precise definition of the Ω_n is completely irrelevant for what follows.



Claim. The explicit chain homotopy

$$P_n \colon \mathrm{C}_n((H^*F)_0) \;\to\; \mathrm{C}_{n+1}(F)$$

from Lemma 73.8 between $\widetilde{H}_{0*} \colon \mathrm{C}_n((H^*F)_0) \to \mathrm{C}_n(F)$ and $\widetilde{H}_{1*} \colon \mathrm{C}_n((H^*F)_0) \to \mathrm{C}_n(F)$ descends to a chain homotopy

$$Q_n \colon \underbrace{\mathrm{C}^{\mathrm{lc}}_n(X;(H^*F)_0)}_{=\mathrm{C}_n((H^*F)_0)/\sim} \;\to\; \underbrace{\mathrm{C}^{\mathrm{lc}}_{n+1}(Y;F)}_{=\mathrm{C}_{n+1}(F)/\sim}$$

between

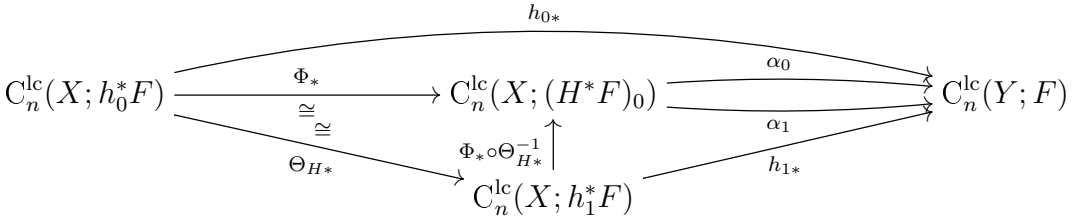
$$\begin{array}{rcl} \alpha_0 \colon \mathrm{C}^{\mathrm{lc}}_n(X;(H^*F)_0) & \to & \mathrm{C}^{\mathrm{lc}}_n(Y;F) \\ [\sigma] & \mapsto & [\widetilde{H}_{0*}(\sigma)] \end{array} \quad \text{and} \quad \begin{array}{rcl} \alpha_1 \colon \mathrm{C}^{\mathrm{lc}}_n(X;(H^*F)_0) & \to & \mathrm{C}^{\mathrm{lc}}_n(Y;F) \\ [\sigma] & \mapsto & [\widetilde{H}_{1*}(\sigma)]. \end{array}$$

Proof. We need to show that the equivalence relation that we introduced on page 3655 is respected. Thus let $\sigma_1, \sigma_2 \colon X \to (H^*F)_0$ be two singular n-simplices with $p \circ \sigma_1 = p \circ \sigma_2$. It follows easily from the explicit definitions of the maps P_n in Lemma 73.8 that $P_n(\sigma_1 \widetilde{+} \sigma_2) = P_n(\sigma_1) \widetilde{+} P_n(\sigma_2)$. ⊞

Before we come to the denouement of the argument we need to introduce the following isomorphism of local coefficient systems over X:

$$\begin{array}{rcl} \Phi \colon h_0^*F & \to & (H^*F)_0 \\ (x,v) & \mapsto & ((x,0),v). \end{array}$$

Finally we consider the following diagram



It follows easily from the definition that this diagram commutes??? By the claim the two maps $\alpha_0, \alpha_1 \colon \mathrm{C}^{\mathrm{lc}}_n(X;(H^*F)_0) \to \mathrm{C}^{\mathrm{lc}}_n(Y;F)$ are chain homotopic. Since the diagram commutes we see that the two maps $\mathrm{C}^{\mathrm{lc}}_n(X;h_0^*F) \to \mathrm{C}^{\mathrm{lc}}_n(Y;F)$ are also chain homotopic. But that is what we had to show. ■

We have the following analogue of Proposition 168.2.

Proposition 175.6. Let X be a topological space that is locally contractible and path-connected. We pick $x_0 \in X$ and we write $\pi := \pi_1(X,x_0)$. We denote by $p \colon \widetilde{X}_{x_0} \to$

X the corresponding universal covering which we equip with the natural base point $\widetilde{x}_0$. Furthermore $\Gamma \subset \pi_1(X, x_0)$ be a subgroup[2124]. Let let $p\colon (Y_\Gamma, y_0) \to (X, x_0)$ be a covering of pointed topological spaces with $p_*(\pi_1(Y_\Gamma, y_0)) = \Gamma$. By Corollary 61.11 we know that there exists a unique covering $q\colon \widetilde{X}_{x_0} \to Y_\Gamma$ with $q(\widetilde{x}_0) = y_0$. We equip the abelian group $\mathbb{Z}[\Gamma\backslash\pi]$ with the $\mathbb{Z}[\pi]$-left module structure φ that is given by $\varphi(g) \cdot \Gamma h := \Gamma h \cdot g^{-1}$. As in Proposition 174.4 we consider the coefficient system

$$\begin{array}{rcl} {}_\varphi\mathbb{Z}[\Gamma\backslash\pi] := \mathbb{Z}[\Gamma\backslash\pi] \times_\pi \widetilde{X}_{x_0} & \to & X \\ [(m,x)] & \mapsto & p(x). \end{array}$$

(1) The maps ????

$$\begin{array}{rcl} \mathrm{C}_k^{\mathrm{lc}}(X; {}_\varphi\mathbb{Z}[\Gamma\backslash\pi]) & \to & \mathrm{C}_k(Y_\Gamma) \\ \sigma\colon \Delta^k & \mapsto & \text{pick lift } (\sigma_1, \sigma_2) \text{ to } \mathbb{Z}[\Gamma\backslash\pi] \times \widetilde{X}_{x_0} \text{ and consider } \sigma_1(\Delta^k) \cdot q \circ \sigma_2 \end{array}$$

form an isomorphism of chain complexes.

(2) If Γ is normal in π, then we equip $\mathbb{Z}[\Gamma\backslash\pi]$ with the obvious $\mathbb{Z}[\Gamma\backslash\pi]$-left module structure and we equip Y_Γ with the $\Gamma\backslash\pi$- left action given by Proposition 167.1. In this setting the above isomorphism is an isomorphism of $\mathbb{Z}[\Gamma\backslash\pi])$-left modules.

(3) For each $k \in \mathbb{N}_0$ the above maps induce an isomorphism

$$\mathrm{H}_k^{\mathrm{lc}}(X; {}_\varphi\mathbb{Z}[\Gamma\backslash\pi]) \to \mathrm{H}_k(Y_\Gamma).$$

Proof. We can write down explicit inverse maps. More precisely we consider the map

$$\begin{array}{rcl} \mathrm{C}_k(Y_\Gamma) & \to & \mathrm{C}_k^{\mathrm{lc}}(X; {}_\varphi\mathbb{Z}[\Gamma\backslash\pi]) \\ (\sigma\colon \Delta^k \to Y_\Gamma) & \mapsto & \begin{array}{c} \text{pick lift } \widetilde{\sigma}\colon \Delta^k \to \widetilde{X}_{x_0} \text{ and consider the} \\ \text{map } \Delta^k \to \mathbb{Z}[\Gamma\backslash\pi] \times \widetilde{X}_{x_0} \text{ that is given by } x \mapsto [(1, \widetilde{\sigma}(x))]. \end{array} \end{array}$$

We leave it to the reader to show that the maps are both chain maps and that they are inverses of one another. ■

Example. We consider the usual covering $p\colon \mathbb{R} \to S^1$ that is given by $p(t) = \exp(2\pi \mathrm{i} t)$. The corresponding coefficient system is isomorphic to the local coefficient system ${}_\varphi\mathbb{Z}[t^{\pm 1}]$ that we introduced on page **??**. Thus we see that for each $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k^{\mathrm{lc}}(S^1; {}_\varphi\mathbb{Z}[t^{\pm 1}]) \underset{\uparrow}{\cong} \mathrm{H}_k^{\mathrm{lc}}(\mathbb{R}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 175.6

175.2. **Cohomology with local coefficients.** On page 2402 we defined singular cohomology by a purely formal method, namely by taking the dual of the singular chain complex. A similar, purely formal, approach allowed us on page 3537 to define cohomology with twisted coefficients. Perhaps somewhat surprisingly singular cohomology with local coefficients is defined independently of the chain complex with local coefficients. First let us introduce the following notation.

Notation. Given a topological space Y and given $n \in \mathbb{N}_0$ we set

$$\mathrm{S}_n(Y) := \text{the set of all singular } n\text{-simplices in } Y,$$

[2124]Note that we do not demand that the subgroup is normal.

in other words, $\mathrm{S}_n(Y)$ is the set of all maps $\Delta^n \to Y$.

Definition. Let X be a topological space and let $p\colon E \to X$ be a local coefficient system.

(1) A *n-cochain with respect to the local coefficient system $p\colon E \to X$* is a map

$$\varphi\colon \mathrm{S}_n(X) \to \mathrm{S}_n(E)$$

such that for every $\sigma\colon \Delta^n \to X$ the corresponding map $\varphi(\sigma)\colon \Delta^n \to E$ is a lift of σ, i.e. we have $p \circ \varphi(\sigma) = \sigma$.

(2) Given two n-cochains $\varphi, \psi\colon \mathrm{S}_n(X) \to \mathrm{S}_n(E)$ we define

$$\begin{aligned} \varphi + \psi\colon \mathrm{S}_n(X) &\to \mathrm{S}_n(E) \\ \sigma &\mapsto \left(\begin{array}{ccc} \Delta^n & \to & E \\ x & \mapsto & \underbrace{(\varphi(\sigma))(x) + (\psi(\sigma))(x)}_{\substack{\text{addition in the group } p^{-1}(x), \\ \text{continuous by Lemma 174.2}}} \end{array} \right). \end{aligned}$$

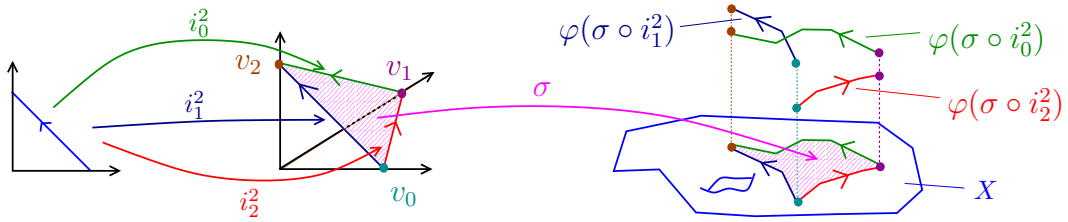
One can easily verify that this addition turns the set of cochains into an abelian group, where the trivial element is the cochain that sends every simplex to the zero-lift.

(3) We denote by $\mathrm{C}^n_{\mathrm{lc}}(X; E)$ the group of all n-cochains.

(4) Given $n \in \mathbb{N}_0$ we define the *coboundary*[2125]

$$\begin{aligned} \delta^n\colon \mathrm{C}^n_{\mathrm{lc}}(X;E) &\to \mathrm{C}^{n+1}_{\mathrm{lc}}(X;E) \\ \varphi &\mapsto \left(\begin{array}{ccc} \mathrm{S}_{n+1}(X) & \to & \mathrm{S}_{n+1}(E) \\ \sigma & \mapsto & \sum_{j=0}^{n+1} (-1)^j \cdot \text{the unique lift}^{2126}\ \widetilde{\sigma}\colon \Delta^{n+1} \to E \text{ such that } \varphi(\sigma \circ i^{n+1}_j) \text{ and } \widetilde{\sigma} \circ i^n_j \text{ agree} \end{array} \right). \end{aligned}$$

It follows easily from the definitions that δ^n is a homomorphism.



One of the most fundamental facts in algebraic topology, proved in Proposition 72.2, is the fact that the composition of two boundary maps $\partial_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X)$ and $\partial_{n-1}\colon \mathrm{C}_{n-1}(X) \to \mathrm{C}_{n-2}(X)$ is the zero map. The following proposition now says that the analogous statement also holds for the above coboundary maps.

Proposition 175.7. Let X be a topological space and let $p\colon E \to X$ be a local coefficient system. For any $n \in \mathbb{N}_0$ the composition

$$\delta^{n+1} \circ \delta^n\colon \mathrm{C}^n_{\mathrm{lc}}(X;E) \to \mathrm{C}^{n+2}_{\mathrm{lc}}(X;E)$$

is the zero map.

Proof.

[2125] We define the sum of two lifts and the inverse of a lift by the usual fiberwise operations.

[2126] By the Unique Simplex Lifting Lemma 175.1 we know that such a lift exists and that it is unique.

If we are dealing with a trivial local coefficient system, then Proposition 175.7 boils down very quickly to the setting for the usual singular cochains. For a general local coefficient system the situation is initially much more puzzling, since we cannot compare the various fibers in a coherent way. What saves the day is the observation that we only have to work with a singular simplex $\sigma\colon \Delta^{n+2} \to X$ at a time, and that the pullback of the local coefficient system E to Δ^{n+2} is trivializable, since Δ^{n+2} is contractible. Thus we can deduce the desired statement after all from the classical statement.

We start out with several preparations:

(a) Let $\mu\colon D \to X$ be a map between topological spaces and let $p\colon E \to X$ be a G-local coefficient system on X. We define

$$\begin{array}{rcl} \mu^*\colon \mathrm{C}^k_{\mathrm{lc}}(X;E) & \to & \mathrm{C}^k_{\mathrm{lc}}(D;\mu^*E) \\ \psi & \mapsto & \left(\begin{array}{rcl} \mathrm{S}_k(D) & \to & \mathrm{S}_k(\mu^*E) \\ (\sigma\colon \Delta^k \to D) & \mapsto & \left(\begin{array}{rcl} \Delta^k & \to & \mu^*E \\ z & \mapsto & (\sigma(z), \underbrace{\psi(\underbrace{\mu \circ \sigma}_{\in \mathrm{S}_k(X)})}_{\in \mathrm{S}_k(E)}(z)) \end{array} \right) \end{array} \right) \end{array}$$

We make the following two observations:

- It follows easily from the definitions that $\delta \circ \mu^* = \mu^* \circ \delta$.
- We denote by $q\colon \mu^*E \to E$ the projection. If $\sigma\colon \Delta^k \to D$ is a simplex, then it follows immediately from the definitions that $q \circ \mu^*(\psi)(\sigma) = \psi(\mu \circ \sigma)\colon \Delta^k \to E$.

(b) Let $\Theta\colon A \to B$ be a morphism from a G-local coefficient system $p\colon A \to D$ to an H-local coefficient system $q\colon B \to D$. We consider the maps

$$\begin{array}{rcl} \Theta_*\colon \mathrm{C}^k_{\mathrm{lc}}(D;A) & \to & \mathrm{C}^k_{\mathrm{lc}}(D;B) \\ (\psi\colon \mathrm{S}_k(X) \to \mathrm{S}_k(A)) & \to & (\Theta_* \circ \psi\colon \mathrm{S}_k(X) \to \mathrm{S}_k(B)). \end{array}$$

It follows easily from the definitions that $\delta \circ \Theta^* = \Theta^* \circ \delta$. If Θ is an isomorphism of local coefficient systems, then Θ_* is evidently also an isomorphism.

(c) By Proposition 174.10 we know that every G-local coefficient system over a contractible topological space D is trivializable, i.e. it is isomorphic to ${}_{\mathrm{tr}}G$.

(d) By the Standard Simplex Homeomorphism Lemma 72.1 (1) we know that each Δ^k is contractible.

(e) Let D be a topological space, let G be an abelian group and let $p\colon G \times D \to D$ be the trivial G-local coefficient system. One can easily verify that the maps

$$\begin{array}{l} \Xi\colon \mathrm{C}^k(D;G) \to \mathrm{C}^k_{\mathrm{lc}}(D;{}_{\mathrm{tr}}G) \\ (\mathrm{C}_k(D) \xrightarrow{\psi} G) \mapsto \left(\begin{array}{l} \mathrm{S}_k(D) \to \mathrm{S}_k(G \times D) \\ \quad \sigma \mapsto \text{the lift } \sigma\colon \Delta^k \to G \times D \text{ given by } x \mapsto (\sigma(x), \psi(\sigma)) \end{array} \right) \end{array}$$

define an isomorphism of cochain groups such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{C}^k(D;G) & \xrightarrow{\quad\Xi\quad} & \mathrm{C}^k_{\mathrm{lc}}(D;{}_{\mathrm{tr}}G) \\ \downarrow{\scriptstyle (\partial_{k+1})^*} & & \downarrow{\scriptstyle \delta} \\ \mathrm{C}^{k+1}(D;G) & \xrightarrow{\quad\Xi\quad} & \mathrm{C}^{k+1}_{\mathrm{lc}}(D;{}_{\mathrm{tr}}G), \end{array}$$

here $\partial_k\colon \mathrm{C}_{k+1}(D;G) \to \mathrm{C}_k(D;G)$ is the usual boundary map in the singular chain complex and the dual $\partial_k^*\colon \mathrm{C}^k(D;G) \to \mathrm{C}^{k+1}(D;G)$ is the usual coboundary map in the singular cochain complex.

Now we turn to the actual proof of the proposition. Thus let X be a topological space, let $p\colon E \to X$ be a local coefficient system and let $n \in \mathbb{N}_0$. We need to show that the composition $\delta^{n+1} \circ \delta^n\colon \mathrm{C}^n_{\mathrm{lc}}(X;E) \to \mathrm{C}^{n+2}_{\mathrm{lc}}(X;E)$ is the zero map.

By the above we have the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{C}^n_{\mathrm{lc}}(X;E) & \xrightarrow{\mu^*} & \mathrm{C}^n_{\mathrm{lc}}(\Delta^{n+2};\mu^*E) & \xrightarrow[\cong]{\Theta_*} & \mathrm{C}^n_{\mathrm{lc}}(\Delta^{n+2};{}_{\mathrm{tr}}G) & \xleftarrow[\cong]{\Xi_*} & \mathrm{C}^n(\Delta^{n+2};G) \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\partial^*_{n+1}} \\
\mathrm{C}^{n+1}_{\mathrm{lc}}(X;E) & \xrightarrow{\mu^*} & \mathrm{C}^{n+1}_{\mathrm{lc}}(\Delta^{n+2};\mu^*E) & \xrightarrow[\cong]{\Theta_*} & \mathrm{C}^{n+1}_{\mathrm{lc}}(\Delta^{n+2};{}_{\mathrm{tr}}G) & \xleftarrow[\cong]{\Xi_*} & \mathrm{C}^{n+1}(\Delta^{n+2};G) \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\partial^*_{n+2}} \\
\mathrm{C}^{n+2}_{\mathrm{lc}}(X;E) & \xrightarrow{\mu^*} & \mathrm{C}^{n+2}_{\mathrm{lc}}(\Delta^{n+2};\mu^*E) & \xrightarrow[\cong]{\Theta_*} & \mathrm{C}^{n+2}_{\mathrm{lc}}(\Delta^{n+2};{}_{\mathrm{tr}}G) & \xleftarrow[\cong]{\Xi_*} & \mathrm{C}^{n+2}(\Delta^{n+2};G).
\end{array}$$

Recall that

$$\begin{array}{ccccccc}
\partial^*_{n+1} \circ \partial^*_{n+2} & = & (\partial_{n+2} \circ \partial_{n+1})^* & = & 0^* & = & 0. \\
& & \uparrow & & \uparrow & & \\
& & \text{since } \mathrm{Hom}_{\mathbb{Z}}(-,G) \text{ is contravariant} & & \text{by Proposition 72.2} & &
\end{array}$$

Thus we see that the two composition of the two vertical maps in the fourth column, and thus also the third and the second column, is zero. In summary we see that we have the following equality of lifts of $\mu\colon \Delta^{n+2} \to X$ to a map $\Delta^{n+2} \to E$:

$$\begin{array}{ccccccc}
 & & & & & & \text{follows from the discussion in (a), applied to } \sigma = \mathrm{id}_{\Delta^{n+2}} \text{ and } \psi = (\delta\circ\delta)(\varphi) \\
 & & & & & & \text{where } q\colon \mu^*E \to E \text{ denotes the projection} \\
(\delta(\delta(\mu)))(\varphi)(\mu) & = & (\delta\circ\delta)(\varphi)(\mu) & = & (\delta\circ\delta)(\varphi)(\mu\circ\mathrm{id}_{\Delta^{n+2}}) & \overset{\downarrow}{=} & q\circ\mu^*((\delta\circ\delta)(\varphi))(\mathrm{id}_{\Delta^{n+2}}) \\
 & & & = & q\circ((\delta\circ\delta)(\mu^*\varphi))(\mathrm{id}_{\Delta^{n+2}}) & = & q(\text{zero-lift of } \mathrm{id}_{\Delta^{n+2}} \text{ to } \mu^*E) \\
 & & & & \uparrow & & \uparrow \\
 & & & & \text{since the squares to the left commute} & & \text{since the composition of the vertical} \\
 & & & & & & \text{maps in the second column is zero} \\
 & & & & = & & \text{zero-lift of } \mu\colon \Delta^{n+2} \to X \text{ to } E. \\
 & & & & \uparrow & & \\
 & & & & \text{follows easily from the definition of } \mu^*E \text{ and the definition of } q & &
\end{array}$$

■

Proposition 175.7 allows us to introduce the following definition.

Definition. Let X be a topological space and let $p\colon E \to X$ be a local coefficient system. Given $n \in \mathbb{N}_0$ we define the n-th singular cohomology group of X with E-coefficients as the quotient group

$$\mathrm{H}^n_{\mathrm{lc}}(X;E) := \frac{\ker(\delta_n\colon \mathrm{C}^n_{\mathrm{lc}}(X;E) \to \mathrm{C}^{n+1}_{\mathrm{lc}}(X;E))}{\mathrm{im}(\delta_{n-1}\colon \mathrm{C}^{n-1}_{\mathrm{lc}}(X;E) \to \mathrm{C}^n_{\mathrm{lc}}(X;E))}.$$

On page 3656 we saw that maps between topological spaces induce maps between homology groups with local coefficients. We now have the analogous statement for cohomology with local coefficients.

Definition. Let $f\colon X \to Y$ be a map between topological spaces and let $p\colon E \to Y$ be a G-local coefficient system on Y. We define

$$\begin{array}{rcl} f^*\colon \mathrm{C}^n_{\mathrm{lc}}(Y;E) & \to & \mathrm{C}^n_{\mathrm{lc}}(X;f^*E) \\ \varphi & \mapsto & \left(\begin{array}{rcl} \mathrm{S}_n(X) & \to & \mathrm{S}_n(f^*E) \\ (\sigma\colon \Delta^n \to X) & \mapsto & \left(\begin{array}{rcl} \Delta^n & \to & f^*E \\ z & \mapsto & (\sigma(z), \underbrace{\varphi(\underbrace{f\circ\sigma}_{\in \mathrm{S}_n(Y)})}_{\in \mathrm{S}_n(E)}(z)) \end{array} \right) \end{array} \right) \end{array}$$

One can easily verify that these maps are well-defined and that they form a chain map. Thus we obtain induced maps

$$f^*\colon \mathrm{H}^n_{\mathrm{lc}}(Y;E) \to \mathrm{H}^n_{\mathrm{lc}}(X;f^*E).$$

The previous definition now also allows us to introduce relative cohomology with local coefficients.

Definition. Let X be a topological space, let $A \subset X$, let G be an abelian group and let $p\colon E \to X$ be a G-local coefficient system. We denote by $i\colon A \to X$ the inclusion. We set

$$\mathrm{C}^n_{\mathrm{lc}}(X,A;E) := \ker(i^*\colon \mathrm{C}^{\mathrm{lc}}_n(X;E) \to \mathrm{C}^n_{\mathrm{lc}}(A;i^*E)).$$

One can easily show that the cochain map on $\mathrm{C}^*_{\mathrm{lc}}(X;E)$ descends to a cochain map on $\mathrm{C}^*_{\mathrm{lc}}(X,A;E)$. Thus we can set

$$\mathrm{H}^n_{\mathrm{lc}}(X,A;E) := \mathrm{H}^n(\mathrm{C}^*_{\mathrm{lc}}(X,A;E)).$$

In Lemma 108.9 we saw that singular homology is covariantly functorial in the coefficients. The following lemma generalizes this statement to the setting of cohomology with local coefficients.

Lemma 175.8. Let X be a topological space and let $A \subset X$.

(1) Let $\Theta\colon E \to F$ be a morphism from a G-local coefficient system $p\colon E \to X$ to an H-local coefficient system $q\colon F \to X$. The maps

$$\begin{array}{rcl} \Theta_*\colon \mathrm{C}^k_{\mathrm{lc}}(X,A;E) & \to & \mathrm{C}^k_{\mathrm{lc}}(X,A;F) \\ (\varphi\colon \mathrm{S}_k(X) \to \mathrm{S}_k(E)) & \to & (\Theta_* \circ \varphi\colon \mathrm{S}_k(X) \to \mathrm{S}_k(F)) \end{array}$$

are a well-defined cochain map and thus they induce maps

$$\varphi_*\colon \mathrm{H}^k_{\mathrm{lc}}(X,A;E) \to \mathrm{H}^k_{\mathrm{lc}}(X,A;F).$$

(2) With the obvious interpretations we have $(\psi\circ\varphi)_* = \psi_* \circ \varphi_*$ and $\mathrm{id}_* = \mathrm{id}$.

Proof. Once again we leave the elementary proof of the lemma to the reader. ∎

The following lemma is the cohomological analogue of Lemma 175.4.

Lemma 175.9. Let X be a topological space, let $A \subset X$, let G be an abelian group and let $p\colon G \times X \to X$ be the trivial G-local coefficient system. The maps

$$\begin{array}{rcl} \mathrm{C}^k(X,A;G) & \to & \mathrm{C}^k_{\mathrm{lc}}(X,A;{}_{\mathrm{tr}}G) \\ (\psi\colon \mathrm{C}_k(X,A) \to G) & \mapsto & \left(\begin{array}{rcl} \mathrm{S}_k(X) & \to & \mathrm{S}_k(G\times X) \\ \sigma & \mapsto & \text{the lift } \sigma\colon \Delta^k \to G\times X \text{ given by } x \mapsto (\sigma(x), \psi(\sigma)) \end{array} \right) \end{array}$$

defines an isomorphism of cochain complexes that is natural in the sense of Lemma 175.4. Furthermore, as for homology groups, these natural chain isomorphisms give rise to a natural isomorphism

$$\mathrm{H}^k(X, A; G) \xrightarrow{\cong} \mathrm{H}^k_{\mathrm{lc}}(X, A; G_{\mathrm{lc}})$$

Proof. The lemma follows easily from the definitions. The less said, the clearer. ■

In the Homotopic Maps-H*-Lemma 108.13 we saw that singular cohomology is contravariantly functorial and in the Twisted Homology-Functoriality Proposition 167.9 we saw that cohomology with twisted coefficients is in a suitable sense contravariantly functorial. In Lemma 175.3 we showed that homology with local coefficients is close to being covariantly functor. In a similar vein we now have the following lemma.

Lemma 175.10. Let $p\colon F \to Z$ be a G-local coefficient system on a topological space Z and let $C \subset Z$.

(1) The natural isomorphism between the G-local coefficient systems $\mathrm{id}^* F$ and F that we gave in Lemma 174.7 induces the map $\mathrm{id}^*\colon \mathrm{H}^k_{\mathrm{lc}}(Z, C; F) \to \mathrm{H}^k_{\mathrm{lc}}(Z, C; \mathrm{id}^* F)$.

(2) Let $\varphi\colon (X, A) \to (Y, B)$ and $\psi\colon (Y, B) \to (Z, C)$ be maps between topological spaces We have the following commutative diagram:

$$\begin{array}{ccc} & & \mathrm{H}^{\mathrm{lc}}_k(X, A; \varphi^*(\psi^*(F))) \\ & \overset{\varphi^*\circ\psi^*}{\nearrow} & \Big\downarrow \cong \text{ natural isomorphism from Lemma 174.7 and Lemma 175.2} \\ \mathrm{H}^{\mathrm{lc}}_k(Z, C; F) & & \\ & \underset{(\psi\circ\varphi)^*}{\searrow} & \mathrm{H}^{\mathrm{lc}}_k(X, A; (\psi \circ \varphi)^*(F)) \end{array}$$

Proof. As so often, the lemma follows almost immediately from the definitions. ■

175.3. **Cup product with local coefficients.** Given a topological space X and a commutative ring R we defined on page 2513 the cup product $\cup\colon \mathrm{H}^k(X; R) \times \mathrm{H}^l(X; R) \to \mathrm{H}^{k+l}(X; R)$. The cup product, together with its sibling the cap product, gave us many of the most interesting applications of cohomology groups. It is thus natural to try to introduce a cup product on cohomology with local coefficients.

First we recall the following definition from page 2503.

Notation. For $i = 0, \ldots, n$ we write

$$v_i \;:=\; (0, \ldots, 1, 0, \ldots, 0) \in \mathbb{R}^{n+1}.$$

Furthermore for $a_0, \ldots, a_k \in \Delta^n$ we consider the map

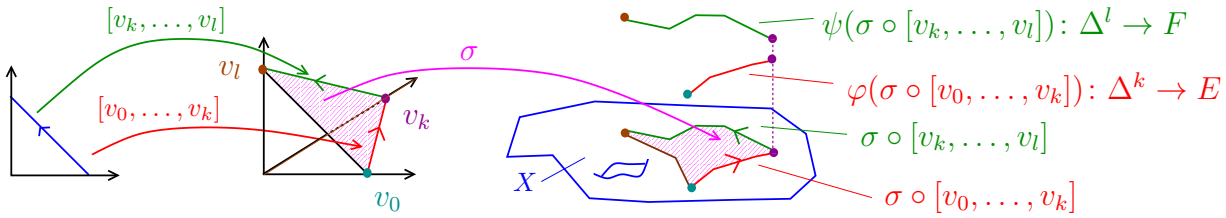
$$\begin{array}{rccc} [a_0, \ldots, a_k]\colon & \Delta^k & \to & \Delta^n \\ & \sum_{j=0}^{k} \lambda_j \cdot v_j & \mapsto & \sum_{j=0}^{k} \lambda_j \cdot a_j. \end{array}$$

Now we can provide the definition of the cup product on cohomology with local coefficients.

Definition. Let X be a topological space, let $p\colon E \to X$ be a G-local coefficient system and let $q\colon F \to X$ be an H-local coefficient system. We consider the $G \otimes_{\mathbb{Z}} H$-local coefficient system $E \otimes_{\mathbb{Z}} F$ that we defined on page 3653. Given $\varphi \in \mathrm{C}^k_{\mathrm{lc}}(X; E)$ and

$\psi \in \mathrm{C}^l_{\mathrm{lc}}(X;F)$ we define $\varphi \cup_\otimes \psi \in \mathrm{C}^{k+l}_{\mathrm{lc}}(X; E \otimes_{\mathbb{Z}} F)$ to be the following map:[2127]

$$\begin{array}{rcl} \varphi \cup_\otimes \psi\colon \mathrm{S}_{k+l}(X) & \to & \mathrm{S}_{k+l}(E \otimes_{\mathbb{Z}} F) \\ (\sigma\colon \Delta^{k+l} \to X) & \mapsto & \text{the unique lift of } \sigma \text{ to a map } \widetilde{\sigma}\colon \Delta^{k+l} \to E \otimes_{\mathbb{Z}} F \text{ such that} \\ & & \widetilde{\sigma}(v_k) = (\underbrace{\varphi(\sigma \circ [v_0,\dots,v_k])}_{\Delta^k \to E})(v_k) \otimes (\underbrace{\psi(\sigma \circ [v_k,\dots,v_l])}_{\Delta^l \to F})(v_0) \end{array}$$



As for the cup product on singular homology we would like to show that the above definition on the cochain level descends to a map on cohomology. This leads us to the following analogue of Coboundary-Cup Product Lemma 114.2.

Lemma 175.11. Let X be a topological space and let $p\colon E \to X$ and $q\colon F \to X$ be local coefficient systems. Given any $\varphi \in \mathrm{C}^p_{\mathrm{lc}}(X;E)$ and given any $\psi \in \mathrm{C}^q_{\mathrm{lc}}(X;F)$ we have

$$\delta(\varphi \cup_\otimes \psi) \;=\; \delta\varphi \cup_\otimes \psi + (-1)^p \cdot \varphi \cup_\otimes \delta\psi \;\in\; \mathrm{C}^{p+q+1}_{\mathrm{lc}}(X; E \otimes_{\mathbb{Z}} F).$$

Proof. In the proof of Proposition 175.7 we used a very general approach to reducing proofs on (co-) homology with local coefficients to the well-known case of ordinary singular (co-) homology. Entirely the same approach allows us to reduce the present setting to the our previous original Coboundary-Cup Product Lemma 114.2. ■

Lemma 175.12. Let X be a topological space, let $p\colon E \to X$ be a G-local coefficient system and let $q\colon F \to X$ be an H-local coefficient system. The map

$$\begin{array}{rcl} \cup\colon \mathrm{H}^k_{\mathrm{lc}}(X;E) \times \mathrm{H}^l_{\mathrm{lc}}(X;F) & \to & \mathrm{H}^{k+l}_{\mathrm{lc}}(X; E \otimes_{\mathbb{Z}} F) \\ ([\varphi],[\psi])] & \mapsto & [\varphi] \cup_\otimes [\psi] := [\varphi \cup_\otimes \psi] \end{array}$$

is well-defined.

Proof. The proof of Lemma 175.12 is basically the same as the rather elementary proof of Lemma 114.3. We just need to replace Coboundary-Cup Product Lemma 114.2 by Lemma 175.11. ■

Definition. We refer to the pairing of Lemma 175.12 as the cup product on cohomology with local coefficients.

Proposition 175.13. Let X be a topological space and let $p\colon E \to X$ and $q\colon F \to X$ be local coefficient systems. Let $\tau\colon E \otimes_{\mathbb{Z}} F \to F \otimes_{\mathbb{Z}} E$ be the natural isomorphism from Lemma 174.13 (2). Then given any $\varphi \in \mathrm{C}^p_{\mathrm{lc}}(X;E)$ and given any $\psi \in \mathrm{C}^q_{\mathrm{lc}}(X;F)$ we have

$$\varphi \cup \psi \;=\; (-1)^{kl} \cdot \tau_*(\psi \cup \varphi).$$

Proof. Recall that we proved Lemma 175.11 by modifying the "classical" proof of Coboundary-Cup Product Lemma 114.2 in a suitable way. The same approach can also be applied

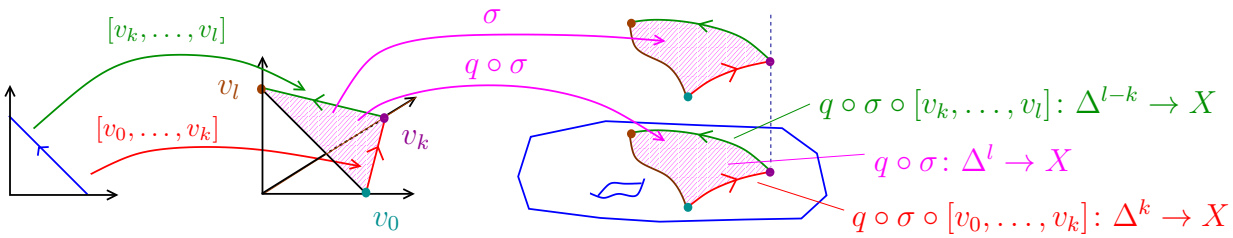
[2127] Implicitly we are using once again the Unique Simplex Lifting Lemma 175.1.

to obtain a of Proposition 175.13 by suitably modifying the proof of the Cup Product-Commutativity Proposition 114.8. We leave the unpleasant task of turning this slogan into a proper proof to the reader. ■

175.4. Cap product with local coefficients.

Definition. Let X be a topological space and let $p\colon E \to X$ be a G-local coefficient system and let $q\colon F \to X$ be an H-local coefficient system. We consider the $G \otimes_{\mathbb{Z}} H$-local coefficient system $E \otimes_{\mathbb{Z}} F$ that we defined on page 3653. Given $\varphi \in \mathrm{C}^k_{\mathrm{lc}}(X;E)$ and $\sigma\colon \Delta^l \to F$ we define $\varphi \cap_{\otimes} \sigma\colon \Delta^{k-l} \to E \otimes_{\mathbb{Z}} F$ to be the following map:[2128]

$$\varphi\cap_{\otimes}\sigma \;:=\; \text{the unique lift of } q\circ\sigma\circ[v_k,\dots,v_l]\colon \Delta^{l-k}\to X \text{ to a map } \widetilde{\mu}\colon \Delta^{l-k}\to E\otimes_{\mathbb{Z}} F$$
$$\text{such that } \widetilde{\mu}(v_0) = \underbrace{(\varphi(q\circ\sigma\circ[v_0,\dots,v_k]))}_{\Delta^k\to E}(v_k)\otimes(\underbrace{\sigma}_{\Delta^k\to F})(v_k)$$



Lemma 175.14. Let X be a topological space and let $p\colon E \to X$ and $q\colon F \to X$ be local coefficient systems.

(1) The map $\cap_{\otimes}$ descends to a well-defined map
$$\cap_{\otimes}\colon \mathrm{C}^k_{\mathrm{lc}}(X;E)\times \mathrm{C}^{\mathrm{lc}}_l(X;F) \;\to\; \mathrm{C}^{\mathrm{lc}}_{l-k}(X;E\otimes_{\mathbb{Z}} F).$$

(2) Given any $\varphi \in \mathrm{C}^k_{\mathrm{lc}}(X;E)$ and given any $\sigma \in \mathrm{C}^{\mathrm{lc}}_l(X;F)$ we have
$$\partial(\varphi\cap_{\otimes}\sigma) \;=\; (-1)^k\cdot(-\delta\varphi\cap_{\otimes}\sigma + \varphi\cap_{\otimes}\partial\sigma) \;\in\; \mathrm{C}_{l-k-1}(X;E\otimes_{\mathbb{Z}} F).$$

(3) The map $\cap_{\otimes}$ descends to a well-defined map
$$\cap_{\otimes}\colon \mathrm{H}^k_{\mathrm{lc}}(X;E)\times \mathrm{H}^{\mathrm{lc}}_l(X;F) \;\to\; \mathrm{H}^{\mathrm{lc}}_{l-k}(X;E\otimes_{\mathbb{Z}} F).$$

Proof.

(1) One can easily show that if $\sigma_1,\sigma_2\colon X\to F$ are two simplices with $q\circ\sigma_1 = q\circ\sigma_2$, then $\varphi\cap_{\otimes}(\sigma_1 \mathbin{\widetilde{+}} \sigma_2) = \varphi\cap_{\otimes}\sigma_1 \mathbin{\widetilde{+}} \varphi\cap_{\otimes}\sigma_2$. From this observation it follows that we do indeed get a well-defined map $\mathrm{C}^k_{\mathrm{lc}}(X;E)\times \mathrm{C}^{\mathrm{lc}}_l(X;F)\to \mathrm{C}^{\mathrm{lc}}_{l-k}(X;E\otimes_{\mathbb{Z}} F)$.

(2) Using the approach of the proof of Proposition 175.7 one can reduce the proof of Statement (2) to Lemma 116.1. We leave the details to the reader.

(3) This statement is a straightforward consequence of Statement (2). ■

The following variation of the cap product is the analogue of the definition of $\cap_{\mathbb{Z}}$ that we gave on page 3609.

Definition. Let X be a topological space and let $p\colon E\to X$ be a local coefficient system. Let $\varphi \in \mathrm{H}^k_{\mathrm{lc}}(X;E)$ and let $\sigma \in \mathrm{H}_l(X;\mathbb{Z})$. We denote by $T\colon \mathrm{H}_l(X;\mathbb{Z})\to \mathrm{H}^{\mathrm{lc}}_l(X;\mathbb{Z}_{\mathrm{tr}}H)$ the

[2128] Implicitly we are using once again the Unique Simplex Lifting Lemma 175.1.

natural isomorphism from Lemma 175.4 and we denote by $\mu\colon E\otimes_{\mathbb{Z}}\mathbb{Z}_{\mathrm{tr}}\mathbb{Z}\to E$ the natural isomorphism of local coefficient systems from Lemma 174.13. We define

$$\varphi\cap_{\mathbb{Z}}\sigma \;:=\; \mu(\underbrace{\varphi\cap_{\otimes}T(\sigma)}_{\in \mathrm{H}^{\mathrm{lc}}_{l-k}(X;E\otimes_{\mathbb{Z}}\mathbb{Z}_{\mathrm{tr}}\mathbb{Z})})\;\in\;\mathrm{H}^{\mathrm{lc}}_{l-k}(X;E).$$

175.5. **Excision with local coefficients.** We recall some definitions from page 2429.

Definition.

(1) A triad of topological spaces is a triple (X,A_1,A_2) where X is a topological space and where both A_1 and A_2 are subsets of X.

(2) Given a triad (X,A_1,A_2) of topological spaces and given a local coefficient system $p\colon E\to X$ we consider the following subgroup of $\mathrm{C}^{\mathrm{lc}}_n(A_1\cup A_2;E)$:

$$\mathrm{C}^{\mathrm{lc},\{A_1,A_2\}}_n(A_1\cup A_2)\;:=\;\left\{\sum_{j=1}^k\sigma_j\;\middle|\;\begin{array}{c}\text{for each } j \text{ there exists an } i\in\{1,2\}\\ \text{such that } \sigma_j \text{ lies in } \mathrm{C}^{\mathrm{lc}}_n(A_i;E)\end{array}\right\}.$$

The usual boundary map on the groups $\mathrm{C}^{\mathrm{lc}}_*(A_1\cup A_2;E)$ restricts to a boundary map on $\mathrm{C}^{\mathrm{lc},\{A_1,A_2\}}_*(A_1\cup A_2;E)$, so we can view $\mathrm{C}^{\mathrm{lc},\{A_1,A_2\}}_*(A_1\cup A_2;E)$ as a chain complex.

(3) We say a triad (X,A_1,A_2) is excisive with respect to local coefficients if if for any local coefficient system $p\colon E\to X$ the inclusion map

$$\mathrm{C}^{\{A_1,A_2\}}_*(A_1\cup A_2;E)\;\to\;\mathrm{C}^{\mathrm{lc}}_*(A_1\cup A_2;E)$$

is a chain homotopy equivalence of chain complexes.

In the Excisive Triad-Proposition 109.12 we gave many convenient examples of excisive triads. The following proposition now says that these triads are not only excisive but they are universally excisive and in Proposition 169.2 we saw that the examples of the Excisive Triad-Proposition 109.12 are also universally excisive.

Proposition 175.15. Let (X,A_1,A_2) be a triad of topological spaces. We assume that X is locally contractible and path-connected. The triad is excisive with respect to local coefficients if one of the following conditions is satisfied:

(1) $A_1\subset A_2$ or $A_2\subset A_1$,
(2) $A_1=\varnothing$ or $A_2=\varnothing$ or $A_1=A_2$,
(3) $A_1\cup A_2=\mathring{A}_1\cup\mathring{A}_2$ where we denote by $\mathring{A}_1$ and $\mathring{A}_2$ the interiors of A_1 and A_2 in $A_1\cup A_2$,
(4) A_1 and A_2 are both open subsets of $A_1\cup A_2$,

Remark. In the Excisive Triad-Proposition 109.12 and Proposition 169.2 we also proved the following two statements:

(1) if $A_1\cup A_2$ is a topological manifold and A_1 and A_2 are codimension-zero submanifolds which are both closed subsets and such that each component of $A_1\cap A_2$ is a boundary component of A_1 and a boundary component of A_2, then (X,A_1,A_2) is excisive and universally excisive.
(2) If X is a CW-complex and A_1 and A_2 are both subcomplexes, then (X,A_1,A_2) is excisive and universally excisive.

Currently we cannot show that these are excisive with respect to local coefficients. Presumably one can prove it using Proposition 169.2 (4).

Proof. The first two statements are clear and Statement (4) is just a special case of Statement (3). So let us deal with Statement (3).

Let $p\colon E \to X$ be a local coefficient system over X. For each $n \in \mathbb{N}_0$ we denote by

$$u_n\colon \mathrm{C}_n(E) \ \to \ \mathrm{C}_n(E)$$

the subdivision map that we introduced on page 1640. We recall the following statements:

(1) In the Chain Subdivision Lemma 74.24 we saw that the maps u_n form a chain map.
(2) In Lemma 74.25 we saw that the maps u_n are natural.
(3) In Lemma 74.26 we saw the u_n are chain homotopic to the identity. Even though we did not say so explicitly, the proof of Lemma 74.26 shows that we can find a natural chain homotopy inverse $v_n\colon \mathrm{C}_(X) \to \mathrm{C}_n(X)$ to u_n.

The constructions of u_n and v_n show that they preserve $\widetilde{+}$, i.e. they descend to maps $\mathrm{C}_n^{\mathrm{lc}}(X;E) \to \mathrm{C}_n^{\mathrm{lc}}(X;E)$. Now the statement should be fairly obvious. ■

175.6. **The interaction between twisted (co-) homology and (co-) homology with local coefficients.** Recall that in Proposition 174.4 and ???? we saw that for a locally contractible and path-connected space we have a natural bijection between the class of representations and the class of local systems.

Proposition 175.16. Let X be a path-connected space and let $x_0 \in X$. There exists a natural isomorphism between (co-) homology with twisted coefficients and (co-) homology with local coefficients.

Let M be a connected smooth manifold and let $x_0 \in M$. Let $w_1\colon \pi_1(M, x_0) \to \{\pm 1\}$ be the orientation character that we defined on page 1114. Furthermore let $\wedge^n \mathrm{T}M^*$ be orientation bundle as defined on page 3082. The vector bundle defined by the orientation character is basically the orientation bundle.

Need to explain that vector bundles with *discrete* fibers are in one-to-one correspondence with representations, but not true if look at them without discrete fibers. See e.g. Hopf bundle over S^2.

175.7. **Poincaré Duality with local coefficients.**

Notation. Let X be a compact oriented non-empty n-dimensional topological manifold. We denote by $[X] \in \mathrm{H}_n^{\mathrm{lc}}(X, \partial X; \mathbb{Z}_{\mathrm{tr}}\mathbb{Z})$ the image of the fundamental class $\mathrm{H}_n(X, \partial X; \mathbb{Z})$ under the natural isomorphism $\mathrm{H}_n(X, \partial X; \mathbb{Z}) \to \mathrm{H}_n^{\mathrm{lc}}(X, \partial X; \mathbb{Z}_{\mathrm{tr}}\mathbb{Z})$ from Lemma 175.4.

Theorem 175.17. (Poincaré Duality Theorem with Local Coefficients) Let X be a compact oriented non-empty n-dimensional topological manifold. We denote by $[X] \in \mathrm{H}_n^{\mathrm{lc}}(X, \partial X; \mathbb{Z}_{\mathrm{tr}}\mathbb{Z})$ the fundamental class. Furthermore suppose that we are given a decomposition $\partial X = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂X such that $A \cap B = \partial A = \partial B$. Then for each $k \in \mathbb{N}_0$ and every local coefficient system $p\colon E \to X$ the map

$$\begin{array}{ccc} \mathrm{H}^k(X, A; E) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(X, B; E) \\ \sigma & \mapsto & \sigma \cap_{\mathbb{Z}} [X] \end{array}$$

is an isomorphism.

Example. We consider the topological manifold S^1 with the local coefficient system ${}_\varphi\mathbb{Z}[t^{\pm 1}]$ that we introduced on page **??**. We see that

$$\mathrm{H}^k_{\mathrm{lc}}(S^1; {}_\varphi\mathbb{Z}[t^{\pm 1}]) \;\cong\; \mathrm{H}^{\mathrm{lc}}_{1-i}(S^1; {}_\varphi\mathbb{Z}[t^{\pm 1}])$$

Proof. The theorem follows immediately from the Poincaré Duality Theorem 172.1 with Twisted Coefficients and the various propositions translating between (co-) homology with twisted coefficients and (co-) homology with local coefficients. ■

Proposition 175.18. Let X be an n-dimensional topological manifold with $\partial X = \varnothing$. We consider the set

$$C \;:=\; \bigsqcup_{x\in X} \mathrm{H}_n(X, X\setminus\{x\}).$$

Given an open subset $U \subset X$ and an element $\varphi \in \mathrm{H}_n(X, X\setminus U)$ we consider

$$M(U,\varphi) \;:=\; \bigsqcup_{x\in u} \text{image of } \varphi \text{ in } \mathrm{H}_n(X, X\setminus\{x\}).$$

(1) The subsets $M(U,\varphi)$ of C have the basis property that we introduced on page 131.
(2) We equip C with the topology that is defined by the sets $M(U,\varphi)$ and the Topology-from-Basis Lemma 1.27. The natural projection $C \to X$ is a local coefficient system for X.
(3) The local coefficient system in (2) is isomorphic to the local coefficient system defined by the orientation-character $w_1(\pi_1(M,x_0)) \to \mathbb{Z}_2$ from page 2341, combined with the unique isomorphism $\mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$.

Definition. Let X be an n-dimensional topological manifold with $\partial X = \varnothing$. We refer to the local coefficient system from Proposition 175.18 as the orientation local coefficient system.

Proof. Should be straightforward. ■

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Nomenclature

$(M\#N, K\#L)$	connected sum of pairs (M, K) and (N, L)
$*\mathbb{C}\mathrm{P}^2$	fake $\mathbb{C}\mathrm{P}^2$ with non-trivial Kirby-Siebenmann invariant
$-M$	smooth manifold equipped with the opposite orientation
1_K	multiplicatively neutral element in $\mathrm{H}^*_{\mathrm{simp}}(K; R)$
1_X	multiplicatively neutral element in $\mathrm{H}^*(X; R)$
$[\widetilde{M}]_{\mathrm{tw}}$	fundamental class of a compact non-orientable path-connected topological manifold
$[K]$	fundamental class of a closed oriented pseudomanifold
$[K]_{\mathbb{F}_2}$	$\mathbb{F}_2$-fundamental class of a closed pseudomanifold
$[M]$	fundamental class of a compact oriented smooth manifold
$[M]$	fundamental class of a compact oriented topological manifold
$[M]^*$	dual fundamental class of a compact oriented topological manifold
$[M]_{\mathbb{R}}$	real fundamental class of a compact oriented smooth manifold
$[M]_{\mathbb{F}_2}$	$\mathbb{F}_2$-fundamental class of a compact smooth manifold
$[M]_{\mathbb{F}_2}$	$\mathbb{F}_2$ fundamental class of a compact oriented smooth manifold
$[M]_{\mathbb{F}_2}$	$\mathbb{F}_2$-fundamental class of a compact topological manifold
$[s]$	equivalence class of $s \in S$
$[S^n]$	standard generator of $\mathrm{H}_n(S^n)$
$[X, Y]$	maps $X \to Y$ up to homotopy
$[X, Y]$	set of homotopy classes of maps $X \to Y$
$[x, y]$	commutator $xyx^{-1}y^{-1}$
$[z_0, \ldots, z_k]$	map $\Delta^k \to \mathbb{R}^n$ defined by $z_0, \ldots, z_k \in \mathbb{R}^n$
$\#S$	cardinality of a set
$\cap$	cap product
$\cap_\otimes$	cap product with twisted coefficients
$\alpha * \beta$	concatenation of paths
$\angle M$	corner of a smooth manifold with corner
$\mathcal{AbGroup}$	category of abelian groups
$\mathcal{AbsGraph}$	category of abstract graphs
$\mathcal{CGWH}$	category of CGWH-spaces
$\mathcal{ChCplx}$	category of chain complexes
$\mathcal{CplxMfd}$	category of complex manifolds
$\mathcal{CRing}$	category of commutative rings
$\mathcal{CW}$	category of CW-complexes
$\mathcal{FullCW}$	full category of CW-complexes
$\mathcal{FullPairCW}$	full category of pairs of CW-complexes

$\mathcal{G}roup$	category of groups
$\mathcal{G}rad\mathcal{A}b$	category of S-graded abelian groups
$\mathcal{G}rad\mathcal{M}od_R$	category of S-graded R-modules
$\mathcal{G}r\mathcal{R}ing$	category of graded rings
$\mathcal{H}om\mathcal{T}op$	homotopy category of topological spaces
$\mathcal{M}ap\mathcal{T}op$	category of maps between topological spaces
${}_R\mathcal{M}od$	category of R-left modules
$\mathcal{M}od_R$	category of R-right modules
$\mathcal{P}air\mathcal{S}et$	category of pairs of sets
$\mathcal{P}air\mathcal{T}op$	category of pairs of topological spaces
$\mathcal{P}\mathcal{S}m\mathcal{M}fd$	category of pointed smooth manifolds
$\mathcal{P}\mathcal{T}op$	category of pointed topological spaces
$\mathcal{R}ing$	category of rings
$\mathcal{S}et$	category of sets
$\mathcal{S}impCplx$	category of abstract simplicial complexes
$\mathcal{S}impCplx$	category of simplicial complexes
$\mathcal{S}m\mathcal{M}fd_{\partial=\varnothing}$	category of smooth manifolds with empty boundary
$\mathcal{S}m\mathcal{M}fd$	category of smooth manifolds
$\mathcal{S}m\mathcal{M}fd_{\mathbb{R}^n}$	category of smooth submanifolds of some $\mathbb{R}^n$
$\mathcal{T}op$	category of topological spaces
$\mathcal{T}op\mathcal{G}r$	category of topological groups
$\mathcal{T}op\mathcal{P}rop\mathcal{M}ap$	category of topological maps where morphisms are proper maps
$\mathcal{U}ndir\mathcal{A}bs\mathcal{G}raph$	category of undirected abstract graphs
$\mathcal{V}ec_{\mathbb{K}}$	category of $\mathbb{K}$-vector spaces
$\chi(X)$	Euler characteristic of a CW-complex
$\chi(X)$	Euler characteristic of a CW-complex
$\chi(X)$	Euler characteristic of a topological space
$\chi^{\mathrm{sm}}(M)$	smooth Euler characteristic of a smooth manifold
$\mathrm{C}^*(X, A; G)$	relative cochain complex
$\mathrm{C}^*_n(X; \mathbb{R})$	bounded cochain complex
$\mathrm{C}^*_{\mathrm{smooth}}(M; G)$	smooth singular cochain complex
Δ^n	standard n-simplex
Δ_M	diagonal of a smooth manifold
δ_{ij}	Kronecker delta
$\varinjlim F$	limit of a diagram
$\varnothing$	the empty set
ϵ_X	augmentation map $\mathrm{C}_0(X) \to \mathbb{Z}$
$\Gamma^A_R(M)$	set of all R-sections along A
$\mathbb{H}^n$	upper half-space in $\mathbb{R}^n$
$\mathring{\Delta}^n$	$\Delta^n \setminus \partial\Delta^n$
$\mathrm{St}(\mathring{K}, s)$	open star of the simplex s
$\mathring{A}$	interior of a subset A
$\varprojlim F$	colimit of a diagram
$\mathbb{L}$	long line

$\mathbb{L}^+$	open long ray
$\Lambda(\varphi)$	Lefschetz number of a map φ
$\Lambda(\varphi, \mathbb{F})$	$\mathbb{F}$-Lefschetz number of a map φ
$\langle (X, x_0), (Y, y_0) \rangle$	maps $(X, x_0) \to (Y, y_0)$ up to homotopy
$\langle (X, x_0), (Y, y_0) \rangle$	maps $(X, x_0) \to (Y, y_0)$ up to homotopy
$\langle\ ,\ \rangle$	Kronecker pairing with R-coefficients
$\langle X \mid R \rangle$	presentation with generating set X and relator set R
$\langle x_1, \dots, x_k \mid r_1, \dots, r_l \rangle$	presentation with generators $x_1, \dots, x_k$ and relators $r_1, \dots, r_l$
$\langle\ ,\ \rangle$	Kronecker pairing
$\triangle$	planar n-simplex
я	map from ordinary to relative homology
ж	map from reduced to ordinary homology
ж	map from reduced to relative homology
ю	map from relative homology to homology of the quotient
$\mathbb{Z}_n$	$\mathbb{Z}/n\mathbb{Z}$
$\mathbb{C}\mathrm{P}^\infty$	infinite-dimensional complex projective space
$\mathbb{C}\mathrm{P}^n$	complex projective space
$\mathbb{H}\mathrm{P}^\infty$	infinite-dimensional quaternionic projective space
$\mathbb{H}\mathrm{P}^n$	quaternionic projective space
$\mathbb{N}$	the natural numbers $1, 2, \dots$
$\mathbb{N}_0$	the natural numbers $0, 1, 2, \dots$
$\mathbb{O}\mathrm{P}^1$	octonionic projective line
$\mathbb{O}\mathrm{P}^2$	octonionic projective plane
$\mathbb{R}\mathrm{P}^\infty$	infinite-dimensional real projective space
$\mathbb{R}\mathrm{P}^n$	real projective space
$\mathbb{Z}^{(S)}$	free abelian group generated by the set S
$\mathcal{N}_n$	unoriented cobordism group
$\mathcal{P}(X)$	power set of the set X
∇f	gradient vector field of f
$\Omega(X)$	free path space
$\Omega(x, x_0)$	loop space
$\Omega_k^{\mathrm{fr}}(\mathbb{R}^m)$	framed cobordism group
$\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$	thickened cobordism group
Ω_n	n-th cobordism group
$\mathrm{Sp}(n)$	compact symplectic group
$\mathrm{Aut}(\pi)$	automorphism group of a group π
$\mathrm{Conf}_k(M)$	configuration space of k points in M
$\mathrm{Ind}(X)$	large inductive dimension of a topological space
$\mathrm{ind}(X)$	small inductive dimension of a topological space
$\mathrm{Inn}(\pi)$	inner automorphism group of a group π
$\mathrm{Out}(\pi)$	outer automorphism group of a group π
$\mathrm{cat}(X)$	Lusternik-Schnirelmann category of a topological space X
$\mathrm{cdim}(X)$	covering dimension of a topological space
$\mathrm{cl}(X)$	cup length of a topological space X
$\mathrm{Cone}(f\colon A \to X)$	mapping cone of a map $f\colon A \to X$

$\mathrm{Cone}(K)$	simplicial cone on an abstract simplicial complex K
$\mathrm{Cone}(X)$	cone of a topological space X
$\mathrm{Cyl}(f\colon A \to X)$	mapping cylinder of a map $f\colon A \to X$
$\mathrm{C}^*_{\mathrm{tw,CW}}(X, A; M)$	twisted cellular cochain complex
$\mathrm{C}^*_{\mathrm{tw}}(X, A; M)$	twisted cochain complex
$\mathrm{C}^n_{\mathrm{lc}}(X, A; E)$	relative n-cochain group with local coefficients
$\mathrm{C}_*^{\mathrm{tw,CW}}(X, A; M)$	twisted cellular chain complex
$\mathrm{C}_*^{\mathrm{tw}}(X, A; M)$	twisted chain complex
$\mathrm{C}_n(X)$	n-th singular chain group of X
$\mathrm{C}_n^{\mathrm{lc}}(X, A; E)$	relative n-chain group with local coefficients
$\mathrm{C}_n^{\mathrm{lc}}(X; E)$	n-chain group with local coefficients
$\mathrm{C}^n_{\mathrm{lc}}(X; E)$	n-cochain group with local coefficients
$\mathrm{Deck}(p\colon X \to B)$	deck transformation group
$\deg(\Phi)$	degree of an automorphism of a group isomorphic to $\mathbb{Z}$
$\deg(f)$	degree of a map $f\colon S^n \to S^n$
$\deg(f, x)$	local degree of a map f at a point x
$\mathrm{diam}(A)$	diameter of a subset A of a metric space
$\mathrm{Diff}(M, N)$	set of diffeomorphisms
$\mathrm{Diff}^+(M, N)$	set of orientation-preserving diffeomorphisms
$\mathrm{Emb}(M, N)$	set of diffeomorphisms
$\mathrm{Emb}^+(M, N)$	set of orientation-preserving diffeomorphisms
$\mathrm{ev}\colon \mathrm{H}^n(C; G) \to \mathrm{Hom}(\mathrm{H}_n(C), G)$	evaluation map
$\mathrm{ev}\colon \mathrm{H}^n(X; \mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_n(X; \mathbb{F}), \mathbb{F})$	evaluation map with $\mathbb{F}$-coefficients
$\mathrm{Ext}(H, G)$	G-Ext-group of H
$\mathrm{Ext}^n(H, G)$	n-th G-Ext-group of H
$\mathrm{Ext}^n_{S\text{-left}}(M, N)$	n-th N-Ext-module of M
$\mathrm{F}\,H$	maximal torsion-free quotient of H
$\mathrm{F}\,H$	maximal torsion-free quotient of an abelian group
$\mathrm{genus}(M)$	genus of a surface M
$\mathrm{GL}_+(n, \mathbb{R})$	real matrices of positive determinant
$\mathrm{Homeo}(X)$	group of self-homeomorphism of X
$\mathrm{Hom}_{\text{right-}R}(A, B)$	group of R-left module homomorphisms $A \to B$
$\mathrm{H}^n_{\mathrm{lc}}(X, A; E)$	n-th singular cohomology group with local coefficients
$\mathrm{H}_n^{\mathrm{lc}}(X, A; E)$	n-th singular homology group with local coefficients
$\mathrm{H}_n^{\mathrm{lc}}(X; E)$	n-th singular homology group with local coefficients
$\mathrm{H}_n^{\mathrm{tw,CW}}(X, A; M)$	twisted cellular cohomology group
$\mathrm{H}_n^{\mathrm{tw,CW}}(X, A; M)$	twisted cellular homology group
$\mathrm{H}_n^{\mathrm{tw}}(X, A; M)$	twisted cohomology group
$\mathrm{H}_n^{\mathrm{tw}}(X, A; M)$	twisted homology group
$\mathrm{H}^n_{\mathrm{lc}}(X; E)$	n-th singular cohomology group with local coefficients
id_X	identity morphism
$\mathrm{index}(f, g, P)$	index of an intersection point of two maps f and g
$\mathrm{index}(f, z)$	index of a map f at a fixed point z
$\mathrm{index}(v, x)$	index of a vector field v at a zero x

$\operatorname{index}(v, x)$	index of a vector field v at a zero x
$\operatorname{int}(A)$	interior of a subset A
$\operatorname{Isom}(M, g)$	group of isometries of the Riemannian manifold (M, g)
$\operatorname{Isom}^+(M, g)$	group of orientation-preserving isometries of the Riemannian manifold (M, g)
$\mathrm{K}(\pi, n)$	canonical Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$
$\lim^1$	lim one of an inverse system of abelian groups
$\operatorname{link}(K, s)$	link of the simplex s
$\operatorname{lk}(K, J)$	linking number of oriented knots K and J
$\operatorname{mesh}(Y)$	the largest diameter of a simplex
$\operatorname{Mor}_{\mathcal{C}}(X, Y)_{\mathrm{inv}}$	set of invertible morphisms
$\mathrm{M}(\pi, n)$	Moore space of type $\mathrm{M}(\pi, n)$
$\mathrm{M}(\varphi)$	algebraic mapping cone of $\varphi\colon C_* \to D_*$
O	infinite orthogonal group
PD_M	Poincaré Duality isomorphism
$\operatorname{rank}(A)$	rank of an abelian group A
$\operatorname{rank}(A)$	rank of an abelian group
$\operatorname{rank}(M)$	rank of a module over a commutative domain R.
$\operatorname{sd}(K)$	barycentric subdivision of the simplicial complex K
$\operatorname{sign}(M)$	signature of a $4n$-dimensional topological manifold
$\operatorname{Spin}(n)$	spin group
$\operatorname{St}(K, s)$	star of the simplex s
$\operatorname{supp}(f)$	support of a map $f\colon X \to \mathbb{R}$
$\operatorname{supp}(f\colon X \to \mathbb{R})$	support of the map f
$\mathrm{S}_n(X)$	set of all singular n-simplices in Y
$\operatorname{topval}(x, X)$	topological valence of a point
$\operatorname{Tor}(A)$	torsion subgroup of an abelian group A
$\operatorname{Tor}(H, G)$	G-torsion of H
$\operatorname{Tor}(X, f)$	mapping torus of a map $f\colon X \to X$
$\operatorname{Tor}_n(H, G)$	n-th G-torsion group of H
$\operatorname{Tor}_n^S(M, N)$	n-th N-torsion module of M
$\operatorname{tr}(\varphi)$	trace of the endomorphism φ
U	infinite unitary group
$\operatorname{Vect}_{\mathbb{C}}^n(X)$	set of isomorphism classes of n-dimensional $\mathbb{C}$-vector bundles over X
$\operatorname{Vect}_{\mathbb{R}}^n(X)$	set of isomorphism classes of n-dimensional $\mathbb{R}$-vector bundles over X
$\operatorname{Vect}_{\mathbb{R}}^{n,+}(X)$	set of isomorphism classes of oriented n-dimensional $\mathbb{R}$-vector bundles over X
$\operatorname{sign}(\sigma)$	sign of the permutation σ
Sp	infinite symplectic group
$\operatorname{Sp}(2g, \mathbb{Z})$	symplectic group
$\overline{\mathbb{L}_{\geq 0}}$	closed long ray
$\overline{\mathbb{L}}$	long ray
$\overline{A}$	closure of a subset A
$\overline{B}^n$	n-dimensional closed ball in $\mathbb{R}^n$

$\overline{B}_r^n(y)$	closed r-ball around $y \in \mathbb{R}^n$
$\partial\Delta^n$	all points on Δ^n where at least one coordinate is zero
∂M	boundary of a smooth manifold with corner
∂M	boundary of a smooth manifold
∂X	boundary of a topological manifold
$\partial_0 X$	$\partial_0 X = \partial X \setminus \partial M$ of a submanifold X of a manifold M
∂_0	∂_0 of a submanifold with corner
$\partial_1 X$	$\partial_1 X = \partial X \cap \partial M$ of a submanifold X of a manifold M
∂_1	∂_1 of a submanifold with corner
$\partial_{\mathrm{c}} M$	corner set of a submanifold M with corner
$\Phi_s^{\leq}$	characteristic map in a totally ordered simplicial complex
$\pi_0(X)$	the set of path-components of a topological space X
$\pi_1(X, x_0)$	fundamental group of (X, x_0)
π_k^s	stable homotopy group
$\pi_n(X, A, x_0)$	relative homotopy group
$\pi_n(X, x_0)$	n-th homotopy group of (X, x_0)
${}_{\varphi}A$	$\mathbb{Z}[G]$-left module corresponding to $\varphi\colon G \to \mathrm{Aut}(A)$
$\mathbb{R}^n_{++}$	the upper right quadrant of $\mathbb{R}^n$.
$\prod_{i \in I} X_i$	product of sets
$\coprod_{i \in I} X_i$	product of sets
$\Sigma(K)$	simplicial suspension of an abstract simplicial complex K
$\Sigma(X)$	Suspension of X
$\Sigma(X)$	suspension of a topological space X
Σ_g	surface of genus g
$\Sigma_{g,n}$	surface of genus g minus n open disks
$\theta_{\mathbb{R}}$	the canonical singular cochain in $\mathrm{C}^1; \mathbb{R})$
$\theta_{\mathbb{R}}$	the canonical singular cochain in $\mathrm{C}^1; \mathbb{Z})$
$\varprojlim$	inverse limit of an inverse system
$\varinjlim$	direct limit of a direct system
$\mathrm{C}_*^{\mathrm{CW}}(X, A)$	cellular chain complex
$\mathrm{H}_n^{\mathrm{CW}}(X, A)$	n-th relative cellular homology group
$\mathrm{H}_n^{\mathrm{lf}}(X; G)$	locally finite homology
$\mathrm{H}_{\mathrm{dR}}^k(M, A)$	relative de Rham cohomology groups
$\mathrm{C}_{\mathrm{c}}^*(X; G)$	chain complex with compact support
$\mathrm{C}_{\mathrm{simp},\leq}^*(K; G)$	simplicial cochain complex
$\mathrm{C}_{\mathrm{simp}}^*(K; G)$	simplicial cochain complex
$\mathrm{C}_*^{\mathrm{CW}}(X)$	cellular chain complex
$\mathrm{C}_*^{\mathrm{CW}}(X)$	cellular cochain complex
$\mathrm{C}_*^{\mathrm{CW}}(X, A)$	cellular chain complex
$\mathrm{C}_*^{\mathrm{han}}(M)$	handle chain complex
$\mathrm{C}_*^{\mathrm{simp},\leq}(K)$	simplicial chain complex of an ordered abstract simplicial complex
$\mathrm{C}_*^{\mathrm{simp}}(K)$	simplicial chain complex of an abstract simplicial complex

$\mathrm{C}_n(X, A)$	relative singular chain group
$\mathrm{C}_n^{\mathrm{simp},\leq}(K)$	n-th simplicial chain group of an ordered abstract simplicial complex K
$\mathrm{C}_n^{\mathrm{simp}}(K)$	n-th simplicial chain group of the abstract simplicial complex K
$\mathrm{D}_W M$	double of a topological (smooth) manifold M along W
$\mathrm{D} f_P$	differential of a smooth map $f\colon M \to N$ at $P \in M$
$\mathrm{D} M$	double of a topological (smooth) manifold M
$\mathrm{H}^k_{\mathrm{simp},\leq}(K; G)$	k-th simplicial cohomology group
$\mathrm{H}^k_{\mathrm{simp}}(K; G)$	k-th simplicial cohomology group
$\mathrm{H}^n(X, A; G)$	n-th relative singular cohomology
$\mathrm{H}^n_{\mathrm{CW}}(X)$	n-th cellular cohomology group
$\mathrm{H}^n_{\mathrm{smooth}}(M; G)$	n-th smooth singular cohomology group
$\mathrm{H}^{\mathrm{han}}_k(M)$	n-th handle homology group
$\mathrm{H}_n(\pi)$	n-th homology group of a group π
$\mathrm{H}_n(X)$	n-th singular homology group
$\mathrm{H}_n(X; G)$	singular homology with G-coefficients
$\mathrm{H}^{\mathrm{CW}}_n(X)$	n-th cellular homology group
$\mathrm{H}^{\mathrm{CW}}_n(X, A)$	n-th relative cellular homology group
$\mathrm{H}^{\mathrm{simp},\leq}_n(K)$	n-th simplicial homology group of an ordered abstract simplicial complex
$\mathrm{H}^{\mathrm{simp}}_n(K)$	n-th simplicial homology group of an abstract simplicial complex
$\mathrm{H}^{\mathrm{simp}}_n(K; G)$	simplicial homology with G-coefficients
$\mathrm{H}_{\mathbb{H}}$	quaternionic Hopf map
$\mathrm{H}_{\mathbb{O}}$	octonionic Hopf map
$\mathrm{H}^n_{\mathrm{b}}(X; \mathbb{R})$	n-th bounded cohomology group
$\mathrm{H}^n_{\mathrm{c}}(X; G)$	cohomology with compact support
$\mathrm{T}_P M$	tangent space of a smooth manifold M at P
$\varphi \cup \psi$	cup product
$\varphi \cap \sigma$	cap product on (co-) homology
$\varphi \cup \psi$	relative cup product on cohomology
$\varphi \cup \psi$	relative cup product
$\varphi \cup_{\otimes} \psi$	relative cup product on cohomology with twisted coefficients
$\varphi \times \psi$	cross product on cohomology
$\widetilde{\mathrm{Diff}}(M, N)$	set of diffeomorphisms up to diffeotopy
$\widetilde{\mathrm{Diff}}^+(M, N)$	set of orientation-preserving diffeomorphisms up to diffeotopy
$\widetilde{\mathrm{Emb}}(M, N)$	set of diffeomorphisms up to diffeotopy
$\widetilde{\mathrm{Emb}}^+(M, N)$	set of orientation-preserving diffeomorphisms up to diffeotopy
$\widetilde{\mathrm{T}}_P M$	physical tangent space of a smooth submanifold of some $\mathbb{R}^n$
$\widetilde{C}_*(X)$	reduced chain complex
$\widetilde{p}_I(M)$	I-Pontryagin number of a smooth manifold M
$\widetilde{X}_{x_0}$	natural universal covering corresponding of (X, x_0)
$\widetilde{X}^{\Gamma}_{x_0}$	natural covering corresponding to $\Gamma \subset \pi_1(X, x_0)$
$\widetilde{\mathrm{H}}_n(X)$	n-th reduced homology group
$A \otimes B$	tensor product of two abelian groups A and B

$A \otimes_{\mathbb{K}} B$	tensor product of two real vector spaces A and B
$A \subset B$	A is a subset of B
$A \subsetneq B$	A is a subset of B with $A \neq B$
B^n	n-dimensional open ball in $\mathbb{R}^n$
$B^n_r(y)$	open r-ball around $y \in \mathbb{R}^n$
$b_n(X)$	the n-th Betti number of a topological space
$b_n(X, A)$	the n-th Betti number of a pair of topological space
$C^\infty(M, N)$	set of all smooth maps between smooth manifolds M and N
$C_* \otimes D_*$	tensor product of chain complexes
c_y	the constant map that sends all points to y
$d(A, B)$	distance between two subsets of a metric space
E_8	the 8×8-matrix E_8
$f \cdot g$	algebraic intersection of two maps that intersect nicely
$f^{-1}(y)$	preimage of the one-point subset $\{y\}$
$G \rtimes_\varphi \pi$	semidirect product of two groups G and π
H	Hopf map
i^n_j	the j-th face map $\Delta^{n-1} \to \Delta^n$
K^{inv}	inverse of a knot
K^{mir}	mirror image of a knot K
K^{mir}	mirror of a knot
K^{rev}	reverse of a knot
$K_1 * \cdots * K_m$	simplicial join of abstract simplicial complexes
$L(p, q)$	lens space
L^{cl}	closure of a long knot L
$M \# N$	connected sum of topological manifolds M and N
$M \setminus\setminus F$	the smooth manifold M cut along a codimension-one submanifold F
M^π	π-invariants
$M^{\mathrm{h}}(\varphi)$	handle attachment to M along φ
M_π	π-coinvariants
$N \rtimes_\varphi \mathbb{Z}_k$	semidirect product of groups N and $\mathbb{Z}_k$
$N(\mathcal{U})$	nerve complex of a cover $\mathcal{U}$
N_g	non-orientable surface of genus g
$N_{g,n}$	non-orientable surface of genus g minus n open disks
$p(E)$	total Pontryagin class of a vector bundle E
$P(x, x_0)$	path space
$p_i(E)$	i-th Pontryagin class of a vector bundle E
$p_I(M)$	I-Pontryagin class of a smooth manifold M
$p_i(M, \mathbb{Q})$	i-th rational Pontryagin class of a smooth manifold
$Q^{\mathrm{as}}_{M,A,B}$	asymmetric intersection pairing
$Q^{\mathrm{red}}(M)$	reduced intersection form
Q_M	intersection form of an even dimensional topological manifold
$Q^{\mathbb{F}_2}_M$	$\mathbb{F}_2$-value intersection form of an even dimensional topological manifold
$R^{(X)}$	free R-left module generated by the set X
$S(X, x_0)$	reduced suspension

$S/\sim$	set of equivalence classes
$S^3_r(L)$	Dehn surgery along the framed link (L, r)
$S^3_r(K)$	Dehn surgery along K
S^∞	infinite-dimensional sphere
S^n	n-dimensional sphere in $\mathbb{R}^{n+1}$
$S^n_r(y)$	n-dimensional sphere of radius r around y
$S^n_{=0}$	equator of S^n
$S^n_{\geq 0}$	upper hemisphere of S^n
$S^n_{\leq 0}$	lower hemisphere of S^n
$s^\dagger$	dual chain of a simplex s
w_1	orientation character of a smooth manifold
w_1	orientation character of a topological manifold
$X * X$	join of topological spaces X and Y
$X * Y * Z$	join of topological spaces X, Y and Z
X/A	quotient of topological space X by subset A
$X \cdot Y$	algebraic intersection number of X and Y
$X \cdot f$	algebraic intersection number of X and f
$X \cdot X$	algebraic self-intersection number of a submanifold
$X \cdot Y$	algebraic intersection number of X and Y
$X \cdot_{\mathbb{Z}_2} f$	$\mathbb{Z}_2$-valued intersection number of X and f
$X \cdot_{\mathbb{Z}_2} Y$	$\mathbb{Z}_2$-valued intersection number of X and Y
$X \otimes Y$	product of two CW-complexes
$X \pitchfork f$	X and Y intersect transversally
$X \pitchfork f$	X and f intersect transversally
$X \vee Y$	wedge of the topological spaces X and Y
$X \wedge Y$	smash product of X and Y
$X \widetilde{\times} Y$	product in the category of CGWH-spaces
X^Y	set of continuous maps $Y \to X$
$Y \cup_\varphi Z$	Y glued to Z via φ
$Y \cup_Z X$	pushout of $Y \leftarrow Z \to X$

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- Cones are contractible, see 16.7.
- the topological space $\mathbb{R}^{\mathbb{N}}$ is metrizable, see 9.5.

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- Dimension of a non-empty smooth manifold is well-defined, see 19.28.
- Points in path-connected smooth manifolds can be connected via smooth paths, 19.29 and 19.29.
- Smooth manifolds are homogeneous, see 36.4 and 33.14.
- Smooth map is homotopic to its linearization, see 76.15.
- Smooth embeddings of smooth manifolds into some $\mathbb{R}^n$, see 27.1 (1), 30.7, **??**, **??**, **??**.
- Smooth manifolds are diffeomorphic to smooth submanifolds of some $\mathbb{R}^n$, see 27.2.
- Map between smooth manifolds is homotopic to map without triple points, see **??**.
- Existence of partitions of unity, see 21.1.
- Existence of a locally finite atlas, see 21.3.
- Continuous maps are homotopic to smooth maps, see 19.29, 29.1.
- Smooth orientation-preserving embeddings of collection of balls are unique up to smooth isotopy, see 37.8.
- For $n \geq 4$ every finitely presented group is the fundamental group of an n-dimensional smooth manifold, see 54.12 and page 3031.
- Smooth manifolds admit simplicial structures, see 95.37 and 96.2.
- The Hauptvermutung holds for smooth simplicial structures, see 96.4.
- Compact smooth manifolds admit Morse functions, see the Morse Function-Existence Proposition 137.6.
- Compact smooth manifold admit handle decompositions, see 138.5.
- Handle decompositions give rise to CW-complex, see 136.18.
- CW-structures on smooth manifolds, see 95.37, 96.5, 98.14, 98.16, 104.20, 138.6, 139.8, 136.18.
- There exist only countably many diffeomorphism types of compact n-dimensional smooth manifolds, see 60.3.
- The diffeomorphism problem is unsolvable in dimension ≥ 4, see 60.4.
- Continuous maps between smooth manifolds are homotopic to smooth maps, see 29.1.
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- Closed smooth manifolds of Euler characteristic zero admit nowhere-vanishing vector field, see 129.6.
- Homology groups of compact smooth manifolds are finitely generated and fundamental groups of compact smooth manifold are finitely presented, see 96.6, **??**, 170.7 and page 2997.
- De Rham cohomology groups of compact smooth manifolds are finite dimensional, see **??**.
- Homology groups and fundamental groups of smooth manifolds are countable, see 49.1 and 96.7.
- Homotopy balls of dimension $n \geq 6$ are diffeomorphic to $\overline{B}^n$, see **??**.
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- Isotopies of submanifolds can be extended, see 36.1.
- The Lefschetz Fixed Point Theorem for smooth manifolds, see 97.10.

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- Topological manifolds are homogeneous, see 36.4.
- Topological manifolds can be embedded in some $\mathbb{R}^n$, see 27.1 (2).
- Existence of partitions of unity, see 21.1.
- Topological manifolds are paracompact, see **??**.
- Topological manifolds are normal, see **??**.
- Topological manifolds are completely metrizable, see **??**.
- Existence of a locally finite atlas, see 21.3.
- Dimension of a non-empty topological manifold is well-defined, see 75.4.
- There exist only countably many homeomorphism types of compact n-dimensional topological manifolds, see 60.3.
- Every compact topological manifold is a retract of a finite simplicial complex, see 104.13.
- Every compact topological manifold is homotopy equivalent to a CW-complex, see 104.20, 104.26 and 104.27.
- CW-structures on topological manifolds, see 104.29, 104.30 and 104.32
- Homology groups of compact topological manifolds are finitely generated and fundamental group of compact topological manifold are finitely presented, see 104.14 and 170.7.
- Homology groups and fundamental groups of topological manifolds are countable, see 49.1 and 104.15.
- The homeomorphism problem is unsolvable in dimension ≥ 4, see 60.4.
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– Simplicial structures give rise to CW-structures, see 93.24.
– Simplicial (co-) homology is naturally isomorphic to singular (co-) homology, see 95.22, 95.25, 95.27, 95.30, 99.6, 99.11 and 99.17.
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– Every point on a CW-complex is good, see 68.10.
– CW-complexes have the homotopy extension property, see 70.1.
– Maps are homotopic to cellular maps, see 70.15.
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– Every finite CW-complex can be embedded in some $\mathbb{R}^n$, see 104.10.
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– Every CW-complex is homotopy equivalent to a simplicial complex, see 94.12.
– Singular and cellular (co-) homology are naturally isomorphic, see 80.4, 81.12, 109.4, and 170.3
– The singular and cellular chain complexes are chain homotopy equivalent, see 81.12 and ??.
– A topological space that is dominated by a CW-complex is homotopy equivalent to a CW-complex, see 162.8.
– The higher homotopy groups of any countable 0-connected CW-complex are countable, see 104.21.
– Every k-connected CW-complex is homotopy equivalent to a CW-complex with a single 0-cell and no cells in dimension $1, \ldots, k$, see 70.4 and 162.10.
– Taking the quotient by a contractible subcomplex is a homotopy equivalence, see 69.8 and 70.3.
– The Lefschetz Fixed Point Theorem for CW-complexes, see 104.12.
– Every finite CW-complex admits a thickening, i.e. it is homotopy equivalent to a compact smooth manifold, see 141.7.

Results and statements about surfaces, knots and links:

– Classification of curves on surfaces, 55.16.
– Calculation of the fundamental group of the complement of the trefoil, see 59.5.
– Every knot, after a rotation, admits a projection that is a knot diagram, see 59.19.
– Two knot diagrams for a given knot are related by Reidemeister moves, 59.21.
– Characterization of the meridian of a knot, see 58.10.

- Definition of the connected sum of two oriented knots, see 1319.
- Knots of genus zero are trivial, see 58.8.
- Calculation of the fundamental group of any knot complement, see 59.23.
- Abelianization of the fundamental group of a knot complement, see 59.25.
- Homology groups of a knot complement, see 100.16.
- Knots admit Seifer surfaces, see 131.6.
- Knots of genus zero are trivial, see 58.8.
- Linking numbers are symmetric, see 131.12.
- Linking numbers can be calculated from Seifert surfaces, see 131.13.
- Linking numbers can be calculated from link diagrams, see 131.14.
- 4-dimensional interpretation of linking numbers, see 131.15.
- Different definitions of linking numbers agree, see 131.16
- Any two knots in $\mathbb{R}^3$ become smoothly isotopic in $\mathbb{R}^4$, see **??**.
- The trefoil is chiral, see 131.5 and 149.10.
- Existence of a knot that is neither amphichiral, nor reversible nor invertible, see 131.4.
- Existence of non-trivial high-dimensional knots, see 59.38.
- Wild knots exist, see 124.13.

Results and statements about groups:

- Classification of finitely generated abelian groups, see 51.4.
- Retract of a finitely presented group is again finitely presented, see 53.16.
- Maps in amalgamated products and HNN-extensions are monomorphisms, see 53.24 and 57.2.
- The word problem, the conjugacy problem and the isomorphism problem are unsolvable for finitely presented groups, see 60.2.
- Subgroups of free groups are free groups, see 64.4 and 69.10.
- Subgroups of free abelian groups are free abelian groups, see 51.2.
- The Ping-Pong Lemma, see 51.13.
- Any two finite presentations are related by Tietze transformations, see 53.7.
- Free groups are residually finite, see 51.15 and 64.15.
- Surface groups are linear and residually finite, see 66.19.
- Surface groups are not free, see 64.17.
- Finite-index subgroups of finitely presented groups are finitely presented, see 69.15.

Results and statements about (co-) homology groups:

- Isomorphisms between homology groups are induced by chain maps, see 81.13.
- Maps that induce isomorphisms on homology are chain homotopy equivalences, see 81.4 and 81.5.
- Singular and cellular (co-) homology are naturally isomorphic, see 80.4, 81.12 and 109.4.
- Singular and cellular chain complexes are chain homotopy equivalent, see 81.12 and 99.1.
- Singular and simplicial (co-) homology are naturally isomorphic, see 95.22, 95.25, 95.27 and 99.6.
- Singular and simplicial chain complexes are chain homotopy equivalent, see 95.25.

– Singular and handle homology are isomorphic, see 140.3.
– Relationship between relative homology and homology of quotient space, see 74.20 and 78.16.
– Explicit representatives of homology classes, see 76.2, 76.3 and page 2054.
– Linking pairing is (anti-) symmetric, see 148.16.

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